Numerical Study on a Crossing Probability for the Four-State Potts Model: Logarithmic Correction to the Finite-Size Scaling

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Abstract

A crossing probability for the critical four-state Potts model on the $L \times M$ rectangle on the square lattice is numerically studied. The crossing probability here denotes the probability that spin clusters cross from one side of the boundary to the other. First, using a Monte-Carlo simulation, we show that the spin cluster interfaces for the fluctuating boundary condition are described by the Schramm-Loewner Evolution (SLE) with $\kappa = 4$. Then, we compute the crossing probability of this spin cluster interface for various system sizes and aspect ratios. Furthermore, comparing with the analytical results for the scaling limit, which have been previously obtained by a combination of the SLE and conformal field theory, we numerically find that the crossing probability exhibits a logarithmic correction $\sim 1/\log(LM)$ to the finite-size scaling.

1. Introduction

The geometrical description of critical phenomena has brought a renewed interest in theoretical study of phase transitions. Various theoretical tools have been developed especially in the 2D case where the structure of critical phenomena is expected to be strongly constrained by an infinite-dimensional conformal symmetry.

In particular, the Schramm-Loewner evolution (SLE) [1, 2, 3, 4, 5] describes the random fractals arising in 2D critical phenomena as a growth process defined by a stochastic evolution of conformal maps: the SLE generates a random curve on a planar domain from a 1D Brownian motion on the boundary. For instance, the SLE defined on the complex upper half-plane $\mathbb{H}$, which is conventionally called the chordal SLE, is described by the evolution of the following conformal map:

$$dg_t(z) = \frac{2dt}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z \in \mathbb{H}, \quad (1)$$

where $B_t$ is the 1D standard Brownian motion living on $\mathbb{R}$, i.e., its expectation value and variance are, respectively, given by $\mathbb{E}[dB_t] = 0$ and $\mathbb{E}[dB_t dB_t] = dt$. The tip $\gamma_t$ of the
random curve evolves as \( \gamma_t = \lim_{\epsilon \to +0} g_t^{-1}(z + i\epsilon) \). Namely the SLE gives a direct description of non-local geometrical objects which are difficult to describe within traditional approaches. The SLE has only one parameter \( \kappa \) associating with the diffusion constant of the Brownian motion. This parameter \( \kappa \) qualitatively and quantitatively characterizes the random curves generated by the SLE. For example, the fractal dimension \( d_f \) of the SLE curve is expressed as the formula:

\[
d_f = \min\left(2, 1 + \frac{\kappa}{8}\right).
\]

(2)

On the other hand, the universal properties of 2D critical systems are explained by conformal field theory (CFT) \([6, 7]\). In contrast to the SLE, CFT determines correlation functions among local operators. The connection between the SLE and CFT has been established \([8, 9]\):

\[
c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa},
\]

(3)

where \( c \) is the central charge characterizing the universality class of 2D critical systems. By combining the SLE with CFT, we can systematically calculate geometrical objects for 2D critical statistical mechanics models.

Crossing probabilities in statistical mechanics models, which are mainly discussed in this letter, are one of these non-local geometrical objects. The crossing probability in this letter denotes the probability that a cluster composed by local variables such as spins connects two disjoint segments on the boundary of a simply connected planer domain. Thanks to the conformal invariance expected in the critical systems, this problem can be mapped to that of the upper half-plane \( \mathbb{H} \). For the statistical models whose cluster boundaries described by the SLE for \( \kappa > 4 \), the crossing probability can be computed by use of the single SLE curve \([11]\). The result contains the one for the critical percolation (\( \kappa = 6; c = 0 \)) which is well-known as the Cardy formula \([10]\), and for the Fortuin-Kasteley clusters in the Ising model (\( \kappa = 16/3; c = 1/2 \)). On the other hand, the crossing probability for the interfaces characterized by the SLE for \( \kappa \leq 4 \), which includes the spin-cluster boundaries of the Ising model (\( \kappa = 3; c = 1/2 \)), may be evaluated by considering the multiple (three) SLE curves constructed in \([11]\).

There still exists, however, a non-trivial problem: what kinds of cluster boundaries in a statistical model do SLE curves actually correspond to? This problem becomes serious for the systems possessing multiple (typically more than two) local variables, such as the three- or four-state Potts model (see for instance \([12]\) for a treatment of the three-state Potts model), where various kinds of cluster boundaries can be defined.

In this letter, by use of a Monte-Carlo method, we numerically compute the crossing probability for certain spin clusters of the four-state Potts model on the \( L \times M \) rectangle. From high-precision Monte-Carlo data, we find that the spin-cluster interface under the ‘fluctuating’ boundary condition gives the crossing probability described by the multiple SLE with \( \kappa = 4 \). Moreover, comparing the results with those obtained in the scaling limit \([11]\), we numerically show that the crossing probability exhibits a logarithmic correction \( \sim 1/\log(LM) \) to the finite-size scaling.

2. **Potts Model** We shall study the four-state Potts model on a rectangle on the square lattice. The \( q \)-state Potts model is an interacting spin model with the spin variables taking
on the values 1, 2, . . . , q. The partition function \( Z \) of the system is given by

\[
Z = \sum_{\{\sigma_j \in Q\}} \exp \left[ J \sum_{\langle j,k \rangle} \delta_{\sigma_j,\sigma_k} \right], \quad Q = \{1, 2, \ldots, q\},
\]

where \( J > 0 \) (i.e., ferromagnetic interaction) and \( \langle j,k \rangle \) denotes adjacent sites on the square lattice. For \( q \to 1 \) and \( q = 2 \), the model is equivalent to the bond-percolation model and the Ising model, respectively. The Potts model exhibits a second-order phase transition for \( q \leq 4 \), while for \( q > 4 \) the transition is of first order. Note that, for \( q \leq 4 \), the scaling behavior is governed by CFT with

\[
c = 1 - \frac{6}{s(s+1)}, \quad s = -1 + \frac{\pi}{\text{arcsec}(2/\sqrt{q})},
\]

The border value \( q = 4 \) is the most crucial: though the model undergoes a second-order transition, the power-law behavior of local operators is expected to be modified by logarithmic factors from marginally irrelevant operators. For our purpose, hereafter we set \( J = J_c := \log(\sqrt{q} + 1) \) where \( J_c \) denotes the critical temperature, and mainly focus on the case \( q = 4 \), i.e., \( \kappa = 4 \) unless otherwise specified.

3. \textit{Multiple SLE and Crossing Probabilities} The SLEs can be generalized to multiple versions generating \( N \) random curves evolving from \( N \) points on the boundary as in \cite{11, 20, 21, 22, 23, 24}. For the chordal case, the multiple SLE is given by the following form \cite{11}:

\[
d g_t(z) = \sum_{j=1}^{N} \frac{2dt}{g_t(z) - X_t^{(j)}}, \quad dX_t^{(j)} = \sqrt{\kappa} dB_t^{(j)} + dF_t^{(j)},
\]

where \( \{B_t^{(j)}\} \) is an \( \mathbb{R}^N \)-valued Brownian motion whose expectation value and variance are given by \( \mathbb{E}[dB_t^{(j)}] = 0 \) and \( \mathbb{E}[dB_t^{(j)}dB_t^{(k)}] = \delta_{j,k} dt \), respectively. Reflecting interactions among curves, the driving term \( X_t^{(j)} \) has an additional drift term \( dF_t^{(j)} \) given by

\[
d F_t^{(j)} = \kappa dt (\partial_{X_t^{(j)}} \log Z_t) + \sum_{k \neq j} \frac{2dt}{X_t^{(j)} - X_t^{(k)}}.
\]

Here \( Z_t \) is a correlation function of boundary condition changing (bcc) operators:

\[
Z_t = \left\langle \psi(\infty)\psi_{2,1}(X_{t}^{(1)}) \cdots \psi_{2,1}(X_{t}^{(N)}) \right\rangle,
\]

where \( \psi_{2,1}(X_{t}^{(j)}) \)'s are bcc operators inserted at the positions \( z = X_{t}^{(j)} \)'s and are degenerate at level two: the conformal weight \( h \) of \( \psi_{2,1} \) is given by \( h = h_{2,1} = (6 - \kappa)/(2\kappa) \). Also \( \psi(\infty) \) is a bcc operator inserted at \( z = \infty \). The tip \( \gamma_t^{(j)} \) of \( j \)th curve is given by \( \gamma_t^{(j)} = \lim_{t \to +0} g_t^{-1}(X_t^{(j)} + i\epsilon) \). Most importantly, the conformal weight of \( \psi(\infty) \) (denoted by \( h_\infty \)) characterizes the topological configurations of the SLE curves \cite{11}. This indicates that some non-local geometrical properties can be determined by the expectation value of products of local operators, which can be exactly computed by CFT.

For instance, for \( N = 3 \) and \( h_\infty = h_{2,1} \), the two topologically inequivalent configurations
Figure 1: (a) The two topologically inequivalent configurations of the three SLE curves starting from the points \( z = 0, x (0 < x < 1), 1 \). \( C_1 \) (resp. \( C_2 \)) denotes the two curves evolving from \( z = 0 \) (resp. \( z = 1 \)) and \( z = x \) hit each other, and the curve starting from \( z = 1 \) (resp. \( z = 0 \)) eventually converges to infinity. (b) The two configurations \( C_1 \) and \( C_2 \) in (a) are mapped by the Schwarz-Christoffel transformation to those defined on a rectangle with the aspect ratio \( r = L/M \) as in \( C_1' \) and \( C_2' \), respectively. The relation between \( x \) in (a) and \( r \) is determined by (12).

are allowed. See Fig. 1(a) for these configurations where the three curves are assumed to start from the points \( X^{(1)}_{t=0} = 0, X^{(2)}_{t=0} = x (0 < x < 1) \) and \( X^{(3)}_{t=0} = 1 \). Each configuration is characterized by the correlation function

\[
Z_0 = \langle \psi_{2,1}(\infty)\psi_{2,1}(0)\psi_{2,1}(x)\psi_{2,1}(1) \rangle = Z_{C_1}(x) + Z_{C_2}(x).
\]  

(9)

Namely, the probability \( P[C_1] \) (resp. \( P[C_2] \)) of the occurrence of the configuration \( C_1 \) (resp. \( C_2 \)) as in the left (resp. right) panel of Fig. 1(a) is expressed as

\[
P[C_1] = \frac{Z_{C_1}(x)}{Z_{C_1}(x) + Z_{C_2}(x)}, \quad P[C_2] = \frac{Z_{C_2}(x)}{Z_{C_1}(x) + Z_{C_2}(x)}.
\]  

(10)

where up to an overall factor \( Z_{C_1}(x) \) and \( Z_{C_2}(x) \) are, respectively, written as \( \text{[11]} \):

\[
Z_{C_1}(x) = (1-x)^{2/\kappa}x^{2/\kappa}2F_1 \left( \frac{4}{\kappa}, \frac{12-\kappa}{\kappa}; \frac{8}{\kappa}; 1-x \right),
\]

\[
Z_{C_2}(x) = Z_{C_1}(1-x).
\]  

(11)

For the four-state Potts model \((\kappa = 4)\), the probability reduces to the more simple form \( P[C_1] = 1 - x \) and \( P[C_2] = x \). By the Schwarz-Christoffel transformation, the two configurations \( C_1 \) and \( C_2 \) defined on \( \mathbb{H} \) can be mapped to those on a rectangle (see Fig. 1(b)) with the aspect ratio \( r = L/M \) as in \( C_1' \) and \( C_2' \), respectively. The relation between \( x \) in (a) and \( r \) is determined by (12):

\[
r = \frac{K(1-k^2)}{2K(k^2)}, \quad k = \frac{\sqrt{x} - 1}{\sqrt{x} + 1},
\]  

(12)

where \( K(k^2) \) is the complete elliptic integral of the first kind with the modulus \( k \) (see for instance \[7\] for a detailed derivation, and see also the arguments in \[18, 19\] for more general polygons). The probability of the occurrence of the configuration in the left panel of Fig. 1(b) gives a crossing probability \( P(r) = P[C_1] \). Here we have assumed that the probabilistic measure is invariant under the conformal transformation. Thus, the analytical expression of the crossing probability in the scaling limit of the four-state Potts model explicitly reads

\[
P(r) = 1 - x \quad (0 < x < 1) \text{ for } \kappa = 4.
\]  

(13)
Figure 2: Upper panel: Typical snapshots of the Ising model where the boundary spins on the top and bottom edges (resp. left and right edges) are fixed to 1 (colored yellow) (resp. 2 (colored purple)). The spin-cluster boundaries starting from the corners are described by the multiple SLE curves as schematically shown in Fig. 1(b). The configurations $C'_1$ and $C'_2$ correspond to (a) and (b), respectively. Lower panel: Numerical calculation of the crossing probability of the occurrence of configuration (a) for the case $L \times M = 1,600$. The data coincides with the one in the scaling limit depicted as the thick line, which is analytically given by (10) and (11) for $\kappa = 3$.

4. **Numerical Results** Now let us numerically calculate a crossing probability of the four-state Potts model ($q = 4$) on a rectangle on the square lattice by a Monte-Carlo simulation. First we detect the interface and the boundary condition giving the above $P(\lambda)$ (13) determined by the SLE in the scaling limit. In fact, several types of spin-cluster boundaries can be defined in the Potts model for $q \geq 3$, which is contrast to the Ising model ($q = 2$) where there exists only one type of spin-cluster boundaries: the interfaces between clusters of spin type 1 and those of spin type 2. For the Ising model, we therefore can easily set up the boundary condition of the rectangle which is compatible to the multiple SLE with $\kappa = 3$. Namely the boundary spins on the top and bottom edges are set to type 1, and those on the left and right edges are fixed to type 2 and vice versa. Indeed, as in Fig. 2, the numerical results calculated by a Monte-Carlo simulation (the Swendsen-Wang method) for the crossing probability well agree with the ones in the scaling limit, which is obtained from (10) and (11) for $\kappa = 3$.

Let us go back to the case of the four-state Potts model ($q = 4; \kappa = 4$). The situation is
Figure 3: Upper panel: (a) A typical snapshot of the four-state Potts model where the boundary spins on the top and bottom edges (resp. left and right edges) are fixed to 1 (colored yellow) (resp. 3 (colored purple)). The other spins are colored red and blue. The spin-cluster boundary starting from one corner splits into two different types of spin-cluster interfaces, if it encounters the other spin clusters in the bulk. (b) A snapshot of the four-state Potts model with the fluctuating boundary condition. On the top and bottom edges (resp. left and right edges), the spins of type 1 and 2 (resp. 3 and 4) are randomly assigned. Lower panel: Snapshots of the four-state Potts model under the fluctuating boundary condition where the spins of $\{1, 2\}$ are colored yellow, while the spins of $\{3, 4\}$ are colored purple. The configurations (c) and (d), respectively, correspond to $C'_1$ and $C'_2$ in Fig. 1 (b).

Quite different from the Ising case. For instance, if we fix the boundary spins as for the Ising model (see Fig. 3 (a)), the spin-cluster boundaries starting from the corners no longer join any corners, due to the existence of clusters consisting of spins different from the ones on the boundaries. Namely the spin-cluster boundary starting from one corner splits into two different types of spin-cluster interfaces, when it meets the other spin clusters in the bulk, which is not compatible to the SLE.

Alternatively, we adopt a ‘fluctuating’ boundary condition as defined in [12, 19]. The fluctuating boundary condition in our case is defined as follows. We divide the four types of spin 1, 2, 3, 4 into two parts, say $\{1, 2\}$ and $\{3, 4\}$ (see [19] for another choice of the fluctuating boundaries). As the boundary spins on the top and bottom edges (resp. left and right edges), the type 1 and 2 spins (resp. the type 3 and 4 spins) are randomly assigned and vice versa. See Fig. 3 (b) as an example. Under this configuration, the spin-cluster boundary is defined as the interface between the spin cluster consisting of $\{1, 2\}$ and the one consisting of $\{3, 4\}$. See Fig. 3 (c) and (d) where the type $\{1, 2\}$ is colored yellow, while the type $\{3, 4\}$ is colored purple. Consequently, we can construct the spin-cluster boundary compatible to the SLE.

Indeed, by numerically analyzing the fractal dimension, we can confirm that this spin-
cluster boundary can be described by the SLE with \( \kappa = 4 \). Here the fractal dimension \( d_f \) is evaluated in the following scaling law:

\[
l \sim a \left( \frac{L}{a} \right)^{d_f} (L/a \gg 1),
\]

where \( l \) is the total length of the curve, \( L \) and \( a \) are the system size and the lattice spacing, respectively. As shown in Fig. 4, we numerically confirm that the fractal dimension of the above defined spin-cluster boundary is \( d_f = 1.504 \pm 0.020 \), which is obtained by the least-square method. The result well agrees with the prediction from the SLE \( d_f = 3/2 \) obtained by the substitution of \( \kappa = 4 \) into the formula (2).

Next we examine the crossing probability for the four-state Potts model on the rectangle with the fluctuating boundary condition by use of a Monte-Carlo method (the Wolf algorithm). Fig. 5 shows the numerical results for various system sizes and aspect ratios. In comparison with the Ising model in Fig. 2 (c), the finite-size data converges much more slowly to the one in the scaling limit given by (13). This slow convergence might be explained in terms of logarithmic corrections caused by the existence of marginally irrelevant operators, since the crossing probability is essentially governed by the correlation function of local operators (9). In fact, as shown in Fig. 6, the crossing probability for the \( L \times M \) rectangle exhibits a logarithmic correction \( \sim 1/\log(LM) \) to the finite-size scaling. This behavior also might be analytically confirmed by analyzing the correlation function (9).

5. Conclusion In this letter, we have numerically investigated the crossing probability of the Potts model on the \( L \times M \) rectangle for the fluctuating boundary condition. Comparing the fractal dimension with the prediction from the SLE, we found that the spin cluster interfaces under the fluctuating boundary condition are described by the SLE with \( \kappa = 4 \). We also have numerically shown that the crossing probability of this spin cluster boundary exhibits
Figure 5: Numerical computation of the crossing probability of the spin-cluster boundary for the four-state Potts model on the rectangle with the fluctuating boundary condition. Compared with the Ising model shown in Fig. 2(c), the numerical data slowly converges to the analytical result (13) depicted as the thick line.

A logarithmic correction $\sim 1/\log(LM)$ to the finite-size scaling. This logarithmic correction might be explained by the existence of marginally irrelevant operators. It remains a crucial problem to show this logarithmic behavior analytically by analyzing the correlation function of the local operators.

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Figure 6: The crossing probability $P(r)$ depicted against $1/\log(LM)$ for the fixed aspect ratio of $r = L/M = 0.55$ (a) and $r = 0.65$ (b), which are fitted to a line calculated by the linear least square method. The logarithmic behavior can be more clearly confirmed by the figure depicted in (c) and (d).

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