Implementation of planar 3D hydraulic fracture model in rock with layered compressive stress

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Abstract. Plane 3D model of hydraulic fracture propagation is implemented. Fluid flow inside the fracture, leak-off, rock deformation and breaking are taken into account. Asymptotic solution for tip of semi-infinite plane fracture is used to set boundary conditions for fluid flow problem and to calculate fracture front propagation velocity. Elastic and fluid flow equations are united in one system of nonlinear equations and solved simultaneously by Newton method with analytically calculated Jacoby matrix. The implemented model may be used as a start point for testing various methods of solution of “hydrodynamic-elasticity” problem and improving their convergence speed. Also model can be used for developed hydraulic fracture simulation.

1. Introduction
Hydraulic fracturing technology is one of the most effective method of oil and gas reservoir stimulation that is now impossible without support of mathematical modeling. There are a lot of papers focused on developing mathematical models of this process (see, e.g., modern reviews [4, 9, 14]). One of the main questions during the model development is the balance between complexity and accuracy of the model and it’s computational cost. On one hand fully 3D models [1], [2], [3], [4], [5] take into account a lot of features and peculiarities of fracture geometry, but they are quite slow and usually are used to identify mechanisms and features of one or the other sub-processes. On the other hand pseudo 3D models are still popular [6], [7] because of their efficiency but they are based on additional geometry simplifications. Planar 3D models are fast enough to be used in industrial simulators such as MFrac [8], FracCade [9], TRIFRAC [10], etc. and at the same time can take into account main features of developed fracture. This type of models continue to evolve in both complexity and performance.

One way of computational acceleration is improving the convergence of methods applied for “hydrodynamic-elasticity” problem – the coupled system of nonlinear equations appeared after the approximation of elastic and fluid flow equations for fixed fracture front position. Some researches reduce the system size by writing the system in terms of one variable: pressure [11] or width [12]. The others [13], [14], [15] keep both variables to simplify the method development and Jacobi matrix calculation, that can be obtained analytically. Here the last approach is chosen and analytical Jacobi matrix calculation is used for Newton method applied for hydrodynamic-elasticity problem. Another way is to use explicit scheme for front propagation instead of implicit one [14] to reduce the amount of iteration cycles. As it shown in [16] «explicit version of the scheme introduces a slight error in the solution» and so such approach is used here.
The current work consists of mathematical model description (section 2), numerical method explanation (section 3) and some model verification results (section 4).

2. Mathematical model

Plane fracture in uniform, isotropic, brittle, linear elastic rock is considered. Elastic and brittle properties of the rock are Young modulus $E$, Poisson coefficient $\nu$ and fracture toughness $K_{fc}$. Rock is compressed by stress $\sigma_c$ in the direction orthogonal to the fracture plane. Fracture front form is assumed arbitrary. Newtonian incompressible fluid with viscosity $\mu$ is pumped into the fracture through perforation with pump rate $Q_{in}$.

Three mane processes that govern the fracture propagation are rock deformation, fluid flow inside the fracture with leak off into the rock and fracture front propagation caused by rock breaking. Proppant transport at the current moment is outside the scope of modeling. The fracture is assumed to be developed one so in contrast with early stage 3D model [11] no fluid lag between fluid and fracture fronts is present. So the only variables that has to be calculated at each propagation step are pressure $p$ and width $w$ distributions (2D) and fracture front $x_f$ (1D) that bounds the fracture.

Zero point of introduced Cartesian coordinate system is placed at perforation entrance and and $xy$ plane coincides with the fracture plane. Due to the fracture wings are assumed symmetrical only the one wing is considered as it shown in figure 1. All directions are equivalent except the case of compressive stress $\sigma_c(y)$ that depends on $y$ coordinate only.

\[ p(x, y) = -\frac{E'}{\pi} \int_{S} \frac{w(x', y')dx'dy'}{((x' - x)^2 + (y' - y)^2)^{3/2}} + \sigma_c(y). \]  

(1)

Here $E' = \frac{E}{1-\nu^2}$ and $S$ is region occupied by the fracture.
2.2. Fluid flow inside the fracture
Newtonian fluid flow inside the fracture is described by Reynolds equation (lubrication equation) for pressure

\[ \nabla \cdot (a \nabla p) = f, \quad (2) \]

where

\[ a = \frac{W^3}{12\mu}, \quad f = \frac{\partial W}{\partial t} + Q(x, y)\delta(x, y). \quad (3) \]

Term \( Q(x, y, t)\delta(x, y) \) here includes both fluid inflow into fracture at the point \((x, y)\) with rate \( Q_{in} \) and fluid leak-off \( Q_L \) that is calculated by Carter formula [19]

\[ Q_L = -\frac{C_L}{(t - t_{exp})^{0.5}} \quad (4) \]

with leak-off coefficient \( C_L \) and time moment \( t_{exp} \) when the fracture front reaches the point \((x, y)\).

Boundary conditions of two types supplement the equations (2)

(i) Fluid flux condition (the boundary is denoted as \( \Gamma_q \) in figure 1)

\[ \Gamma_q : q = (a \nabla p) = 0, \quad (5) \]

that comes from the symmetry of fracture wings and

(ii) pressure condition at fracture front \( x_f \) points.

\[ x_f : p(x, y) = p_{out}(x, y). \quad (6) \]

It should be noted that other cases are possible. For example in [20] fluid flux is set at the fracture front instead of pressure. With such condition the fluid flow equation can not be solved independently because of solution non-uniqueness. Of course if fluid flow problem is solved as a part of fracture model, then the mass balance condition for volume of whole fracture is applied and the fluid pressure cannot be arbitrary. But here this case is not under consideration to provide the independent elastic and fluid problems solution possibility.

2.3. Fracture propagation and conditions at the front
To close the model it is necessary to set condition for pressure \( p_{out} \) at fracture front and calculate front propagation velocity \( V_f \). Asymptotic solution [21] obtained for plane semi-infinite fracture propagated in elastic brittle material with constant velocity is used for this purpose. Using the asymptotic solution one can obtain relations for fracture parameters for any point \( x_s \) placed at small distance \( s \) from the fracture front. So there is no need to solve equations (1), (3), (5), (6) in the front vicinity shown in figure 2.

For each near front point \( x_s \) there is expression for fracture width [22] that can be written as

\[ w \approx w_a(s, V_f). \quad (7) \]

Also at each front point front velocity is equal to fluid velocity (Stefan condition). Lets rewrite it in terms of distance \( s \) from the stationary point \( x_s \) to moving front point \( x_f \) as

\[ \frac{ds}{dt} = \frac{dx_f}{dt} = V_f. \quad (8) \]

Fracture tip volume between the fracture front and point \( x_s \) is a function of width \( V = V(w(s), s) \) and obey mass balance condition

\[ \frac{dV}{dt} = v_s. \quad (9) \]
where $v_s$ is normal to front component of fluid velocity $v_x$ at $x_s$ point.

Equations (7)–(9) together can be used to obtain fracture front velocity $V_f$ and fracture width $w_a$ at the small distance $s$ from the front using the value of fluid velocity $v_s$ at a point $x_s$ placed near the front.

Figure 2. Fracture width near the front at two propagation steps.

3. Numerical method

3.1. Mesh and cell types

Here stationary Cartesian computational mesh with $n_x \times n_y$ cells of size $h_x \times h_y$ is used. Pressure and width assumed to be constant inside the cells and their values are hold at cell centers, whereas fluid flux are placed at cell edges. Like in [14], [20] cells inside the fracture are divided into two types: near front ($n$) and channel ($c$) as it shown in figure 1.

Each cell of computational domain (regardless if it is covered by fracture or not) has 2D index $i, j$, that doesn’t change while fracture propagates. For the cells that are fully covered by fracture new 1D indexation $1 < k < K$ is introduced. So pressure and width can be treated as 1D vectors $P = (p_1, \ldots, p_K)$, $W = (w_1, \ldots, w_K)$ at each propagation step. This 1D numeration helps to use and describe linear algebra procedures applied.

3.2. Approximation of rock deformation approximation

Integral in (1) can be calculated by simple summation by cells. For the cells that are not contain the point $(x, y)$ Gauss formula is used. Hypersingular integrals along the cell with point $(x, y)$ are calculated using singularity extraction techniques applied to the similar case in fully 3D fracture [23]. System of linear equation with constant coefficients is obtained as a result

$$P = TW + \sigma \text{ or } \sum_{l=1}^{K} T_l^k W_l - P_k = -\sigma_k,$$

where vector $\sigma$ contains values of compressive stresses at current depth $y_k$.

Thus the only one wing is considered the symmetrical fracture part is taken into account in (10) by doubling the the integral in right side of (1) with replacing $x$ by $-x$ in integrand.
3.3. Approximation of fluid flow equations

Equation (2) approximated by finite volume method can be written in the following form:

\[ \begin{align*}
-\frac{\tau}{h_x} & \left( a_{i+1/2,j} \frac{p_{i+1,j} - p_{i,j}}{h_x} - a_{i-1/2,j} \frac{p_{i,j} - p_{i-1,j}}{h_x} \right) - \\
-\frac{\tau}{h_z} & \left( a_{i,j+1/2} \frac{p_{i,j+1} - p_{i,j}}{h_y} - a_{i,j-1/2} \frac{p_{i,j} - p_{i,j-1}}{h_y} \right) + w_{ij}^{n+1} = w_{ij}^n - \tau Q_{i,j} \delta(i,j),
\end{align*} \tag{11} \]

where

\[ a_{i+1/2,j} = \frac{(w_{i+1,j} + w_{ij})^3}{96\mu}, \quad a_{i,j+1/2} = \frac{(w_{ij+1} + w_{ij})^3}{96\mu}. \tag{12} \]

The equation (11) is regrouped to have all unknown variables in left side and multiplied on \(-\tau\) to obtain positive-definite matrix multiplied on pressure vector.

Fluid flux boundary conditions (5) applied to the cell placed near the \(y\)-axis makes the corresponded term \((a_{i-1/2,j} \frac{p_{i,j} - p_{i-1,j}}{h_x})\) in (11) equal to zero. For the cells where pressure boundary conditions (6) is applied the equation (11) is replaced by the condition

\[ P_k = p_{out}(x_k, y_k). \tag{13} \]

Equation (11) and conditions (13) are rewritten using 1D numeration in matrix form

\[ A(W)P + DW = B_{Pc} \quad \text{or} \quad \sum_{l=1}^{K} A(W)^l_k P_l + D_k W_k = (B_{Pc})_k. \tag{14} \]

Here sparse positive defined matrix \(A\) depends on \(W\) because coefficients \(a\) in (12) depend on width. Diagonal matrix \(D\) is almost unit one. For the cells \(k\) where equation (11) is applied it has 1 at diagonal. For the cells where pressure condition is applied it has completely zero row. Right hand side vector \((B_{Pc})_k = D_k W_k^n - \tau Q_k \delta(k)\) contains source terms (leak-off and inflow rate) and width \(W^n\) at previous time step \(n\).

3.4. Approximation of fracture front conditions and fracture front propagation

Asymptotic solution described in section 2.3 is used to have fracture front velocity and boundary conditions for fluid flow problem. Asymptotic solutions can be used in various ways such as interpolation of the numerically calculated and tabulated values [20] or closed form approximation [24]. Here, the asymptotic solutions are written in the form of nonlinear equations, the their solution gives an explicit expression for the front increment and the width at a given point. This procedure is proposed and tested for the case of 1D radial model [25] and is generalized here for the Planar 3D model.

Since the computational mesh has been introduced the arbitrary point \(x_s\) in the fracture front vicinity below denotes the center of “near front” cell. Particular expression (7) is implicit but in [24] the short and convenient way of this formula interpretation is proposed. Following [22], [24] one can

- calculate dimensionless front distance

\[ \tilde{s} = \left( \frac{s}{l} \right)^{0.5}, \quad l = \left( \frac{K_{Ic}^3}{12\mu E^2 V_f} \right)^{2}, \quad K' = \left( \frac{32}{\pi} \right)^{0.5} K_{Ic}, \tag{15} \]
solve numerically nonlinear equation obtained from the zeroth-order approximate solution [22], [24]

\[
\tilde{w}^3 - 1 - 1.5b_0(\tilde{w}^2 - 1) + 3b_0(\tilde{w} - 1) - 3b_0^3 \ln \left( \frac{b_0 + \tilde{w}}{b_0 + 1} \right) = \beta_m^3 \tilde{s}
\]  

(16)

for dimensionless fracture width \(\tilde{w}\) and

• calculate dimension fracture width \(w\) as

\[
w_a = s^{0.5} \frac{K'}{E'} \tilde{w}
\]  

(17)

Here

\[
b_0 \approx 0.9912 \frac{4C_L E'}{V_f^{0.5} K'}, \quad \beta_m = 2^{1/3} 3^{-5/6}.
\]  

(18)

When the particular expression (7) has been numerically obtained it is possible to use it to solve equations (9), (8) numerically to have fracture front velocity \(V_f\) and width at point \(x_s\) as functions of fluid velocity \(v_s\) at this point. Approximation of (8) gives

\[
s^{n+1} - s^n = \Delta t V_f.
\]  

(19)

and approximation of (9) looks like

\[
\int_0^{s^{n+1}} w_a(s')ds' - \int_0^{s^n} w_a(s')ds' - w_a(s)v_s \Delta t = 0.
\]  

(20)

Integrals in (20) can be replaced by their explicit form presented in [24]

\[
\int_0^{s} w_a(s')ds' = \frac{2w_a(s)s}{3 + \frac{1}{6}}
\]  

(21)

and term \(w_a(s)\) in (20) can be taken implicitly for simplicity. Then using (19) one can obtain nonlinear equation for front velocity \(V_f\)

\[
\frac{2w_a(s^n + \Delta t V_f, V_f)(s^n + \Delta t)}{3 + \frac{1}{6}} - \frac{2w_a(s^n, V_f)s^n}{3 + \frac{1}{6}} - w_a(s^n + \Delta t V_f, V_f)v_s \Delta t = 0,
\]  

(22)

where \(w_a(s, V_f)\) is calculated using implicit expression (7) and the numerical procedure described above.

Finally for all centers of near front cells one have the implicit expression

\[
w_a = w_a(s, v_s(s))
\]  

(23)

to calculate fracture width at each near front cell \(x_f\) and nonlinear equation (22) to obtain the front velocity at the front point that is nearest to \(x_f\).

Because the asymptotic solution gives equation for width at near front cells not for pressure, the boundary conditions (6) are replaced by combination of linear equations (10) and the expression (23). Since the front velocity is obtained at the points of the current front \(x_f^n\) the front configuration \(x_f^{n+1}\) at the next step is calculated using eikonal equation solution procedure described in [24].
3.5. Coupled solution of hydrodynamic–elasticity problem

When fracture front position is known and fracture width at near front cells is set one can solve “hydrodynamic–elasticity problem” (10), (11) to find pressure and width vectors $P, W$.

For both channel “c” and near front “n” cells the elastic equation (10) can be written in matrix form

$$
\begin{pmatrix}
    T^c_c & T^c_n \\
    T^n_c & T^n_n
\end{pmatrix}
\begin{pmatrix}
    W_c \\
    W_n
\end{pmatrix}
- \begin{pmatrix}
    P_c \\
    P_n
\end{pmatrix}
+ \begin{pmatrix}
    \sigma_c \\
    \sigma_n
\end{pmatrix} = 0.
$$

(24)

Fluid flow equation (11) is written for channel cells only

$$
D^c_c W_c + A^c_c(W)P_c + A^n_c(W)P_n - B_Pc = 0,
$$

(25)

because the condition (23) is applied for “near front” cells.

Combining (24), (25) and (23) one can write system of nonlinear equation in following form

$$
F(X) = M(X)X - B = 0,
$$

(26)

where

$$
M = \begin{pmatrix}
    T^c_c & T^c_n & -I & 0 \\
    0 & I & 0 & 0 \\
    D^c_c & 0 & A^c_c(W) & A^n_c(W) \\
    -T^n_c & -T^n_n & 0 & I
\end{pmatrix},
X = \begin{pmatrix}
    W_c \\
    W_n \\
    P_c \\
    P_n
\end{pmatrix},
B = \begin{pmatrix}
    -\sigma_c \\
    w_a \\
    B_Pc \\
    \sigma_n
\end{pmatrix}.
$$

(27)

System of nonlinear equation (27) is solved by Newton method

$$
X^{m+1} = X^m - \left(\frac{\partial F}{\partial X}\right)^{-1} F(X^m)
$$

(28)

Matrix $M$ depends on vector $X$ only in $A^c_c, A^n_c$ terms, so it is simple to calculate Jacobi matrix ($\partial F/\partial X$) analytically.

4. Numerical results

4.1. Comparison with plane radial fracture solution

Simple problem of plane radial (penny-shaped) fracture propagation has been chosen to test the proposed numerical algorithm implementation. There is the analytical solution of the problem [26] that makes possible to check numerical convergence. Process parameters are $E = 20GPa, \nu = 0.25, Q_{in} = 0.042m^3/s, \mu = 0.42Pa \cdot s$. Three quadratic meshes with cell sizes $h_x = h_y = h = 0.5, 1$ and 2 m and three time steps $\Delta t = 1, 2$ and 4 s was used for this purpose. Left part of the figure 3 shows time evolution of the one fracture wing length $L$ and the fracture half height $H/2$ calculated numerically on these meshes (colored 1–3) in comparison with penny-fracture radius $R$ obtained analytically (black 0). The fracture sizes in $x$ and $y$ direction ($L$ and $H/2$) are almost the same and the fracture is really circular. Evolution of maximal fracture width in time is shown on right part of the figure 3, one can see the convergence of numerical solution $w$ (colored 1–3) to analytical one $w_a$ (black 0). Dashed lines shows the evolution of numerical errors obtained using the meshes

$$
Err_R = \left| \frac{0.5(L + H/2) - R}{R} \right| \cdot 100\%,
Err_w = \left| \frac{w - w_{an}}{w_{an}} \right| \cdot 100%.
$$

(29)

It is clear that relative errors are quite small (about 5%) and reduces with time. The big error obtained using fine mesh should be fixed further.
Figure 3. Time evolution of the fracture half sizes $L$, $H/2$ (left) and width $w$ (right) – solid lines and numerical errors for these values – dashed lines: 0 – analytical solution [26]; 1 – $\Delta t = 1s$, $h = 0.5m$; 2 – $\Delta t = 2s$, $h = 1m$; 3 – $\Delta t = 4s$, $h = 2m$.

4.2. Comparison with PKN fracture solution
Propagation of plane fracture with constant height (PKN fracture) with analytical solution [27] is considered. To prevent the planar 3D fracture from propagating in vertical $y$ direction the compressive stress in the interval $[-H/2 < y < H/2]$ is set to zero and outside the interval is set to $\sigma_c = 100MPa$. The fixed fracture height is set to $H = 20m$ and others parameters are equal to ones in previous subsection 4.1, except that only one wing is considered here. Time evolution of the fracture length and maximal width are shown in figure 4. Length evolution on the left part of the figure 4 is shown in logarithmic scale, to demonstrate that for interval of PKN solution applicability $L >> H = 20m$ the proposed planar 3D model implementation captures the power law of length $L \sim t^{4/5}$. Figure 5 shows the pressure distribution at tree time moments $t = 40, 160, 320s$ to show that the solution is really 1D and the fracture does not propagate in vertical $y$ direction due to layered compressive stress.

Figure 4. Time evolution of the fracture length $L$ (left) and maximal width $w$ (right): 0 – analytical solution [27]; 1 – $\Delta t = 2s$, $h = 1m$; 2 – $\Delta t = 4s$, $h = 2m$.

5. Conclusion
Plane 3D model of hydraulic fracture propagation is implemented. Width and pressure distributions are connected through Reynolds equation for fluid flow inside the fracture and
hypersingular equation obtained from elastic equilibrium equations solution. Fracture front propagation and boundary conditions at near front points are obtained from asymptotic solution of plane semi-infinite fracture problem. Model implementation features are following: the coupled system for pressure and width are solved by Newton method with analytical calculation of Jacobi matrix; front velocity is calculated as a solution of nonlinear equations derived from near tip asymptotic solution; explicit method is used to obtain fracture front position. Comparisons with analytical penny shaped fracture solution and PKN fracture solution have been performed.

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