CP violation in lepton number violating semihadronic decays of $K, D, D_s, B, B_c$

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We study the CP violation in lepton number violating meson decays $M^\pm \rightarrow \ell_1^+ \ell_2^- M'^\mp$, where $M$ and $M'$ are pseudoscalar mesons, $M = K, D, D_s, B, B_c$ and $M' = \pi, K, D, D_s$, and the charged leptons are $\ell_1, \ell_2 = e, \mu$. It turns out that the CP-violating difference $S_{\pm}(M) \equiv [\Gamma(M^- \rightarrow \ell_1^- \ell_2^+ M'^+) - \Gamma(M'^+ \rightarrow \ell_1^+ \ell_2^+ M'^-)]$ can become appreciable when two intermediate on-shell Majorana neutrinos $N_1$ ($j = 1, 2$) participate in these decays. Our calculations show that the asymmetry becomes largest when the masses of $N_1$ and $N_2$ are almost degenerate, i.e., when the mass difference $\Delta M_N$ becomes comparable with the (small) decay widths $\Gamma_N$ of these neutrinos: $\Delta M_N \gg \Gamma_N$. We show that in such a case, the CP ratio $A_{\text{CP}}(M) \equiv \Gamma(M^- \rightarrow \ell_1^- \ell_2^+ M'^+) - \Gamma(M'^+ \rightarrow \ell_1^+ \ell_2^+ M'^-) / \Gamma(M^- \rightarrow \ell_1^- \ell_2^- M'^-) + \Gamma(M'^+ \rightarrow \ell_1^+ \ell_2^+ M'^-) \Gamma(M'^+ \rightarrow \ell_1^+ \ell_2^+ M'^-) \sim 1$. The observation of CP violation in these decays would be consistent with the existence of the well-motivated $\nu$MSM model with two almost degenerate heavy neutrinos in the mass range between $M_N \sim 0.1-10^3$ GeV.

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I. INTRODUCTION

At this moment, one of the main questions in neutrino physics is unresolved: whether the neutrinos are Majorana or Dirac particles. If the neutrinos are Dirac particles, the lepton number is conserved in all processes. If the neutrinos are Majorana particles, i.e., if they are indistinguishable from their antiparticles, the lepton number in the reactions involving them can be violated. The main processes whose eventual detection would decide on the nature of neutrinos are the neutrinoless double beta decays ($0\nu\beta\beta$) in nuclei [1]. Among other processes which may reflect the character of neutrinos are specific scattering processes [2–5], and rare meson decays [6–14].

Another important question is the value of the masses of neutrinos. Neutrino oscillations were predicted a long time ago [15], under the assumption that neutrinos have masses. These oscillations were later observed [16–18], leading to the conclusion that the first three neutrinos have nonzero but very light masses. They can be produced via a seesaw mechanism [19], where the light neutrinos have masses $\sim M_D^2/M_R$ ($\lesssim 1$ eV), where $M_D$ is an electroweak scale or lower. The heavy Majorana neutrinos in these seesaw scenarios are very heavy, with typical masses $M_R \gg 1$ GeV, and their mixing with active neutrino flavors is very suppressed $\sim M_D/M_R$ ($\ll 1$). However, scenarios exist [3, 20–23] where the heavy Majorana neutrinos can have relatively low masses $\sim 1$ GeV and their mixings with active neutrino flavors can be larger than in the usual seesaw scenarios.

Another important question in neutrino physics is the strength (if any) of the CP violation in the neutrino sector. It could be measured by neutrino oscillations [24]. However, in this work we will investigate the possibility of detection of CP violation in the rare lepton number violating (LNV) semihadronic decays of charged pseudoscalar mesons.

In general CP violation is expected in both cases of neutrinos being Dirac or Majorana particles. Nonetheless, in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [13, 25] the number of possible CP-violating phases is larger when the neutrinos are Majorana particles. If $n$ is the number of neutrino generations, the number of CP-violating phases is $n(n-1)/2$ in the Majorana case, and $(n-1)(n-2)/2$ in the Dirac case, cf. Ref. [26].

In a recent work [27], we investigated the possibility of measuring the CP asymmetry in the rare leptonic decays of charged pions $\pi^\pm \rightarrow e^\pm e^\mp \mu^\mp \nu$. Both lepton number conserving (LNC) and lepton number violating (LNV) processes contribute to these decays and to the CP violation. We concluded that the CP violation is appreciable when these processes are mediated by two on-shell (Majorana or Dirac) sterile neutrinos $N_1$ and $N_2$ (i.e., with masses between 106 and 140 MeV), and that the CP violation effect is largest when these two neutrinos are almost degenerate in their masses. It is interesting that such neutrinos fall within the regime predicted by the $\nu$MSM model [20, 28]. Further, they are not ruled out by experiments [11, 29].

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The $\nu$MSM model [20, 28] contains two almost degenerate sterile Majorana neutrinos with mass between 100 MeV and a few GeV, and in addition a light sterile Majorana neutrino of mass $\sim 10^3$ keV and the three very light neutrinos. The model is well motivated because: (a) it can explain simultaneously the pattern of light neutrino masses and oscillations; (b) it can explain the baryon asymmetry of the Universe; (c) it provides a dark matter candidate. We refer to Refs. [30] for reviews, and to Refs. [31] for the determination of the allowed range of the sterile neutrinos of the $\nu$MSM model. Remarkably, the tentative evidence of a dark matter line, recently discussed in Refs. [32], falls into the regime predicted for $\nu$MSM in Refs. [31].

It is interesting that the requirement that the lightest sterile neutrino be the dark matter candidate reduces the parameters of the model in such a way as to make the two heavier neutrinos nearly degenerate in mass. This in turn, as demonstrated in Ref. [27], increases significantly the possible effects of CP violation.

Moreover, the CERN-SPS has proposed a search of such heavy neutrinos, Ref. [33], in the leptonic and semihadronic decays of $D, D_s$ mesons. As argued in [33] and in the works [6–14], such rare decays can have appreciable rates to be detected in future experiments (such as the experiment proposed at CERN-SPS).

In this work we investigate such rare semihadronic decays of charged pseudoscalar mesons $M^\pm \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp$, where $M = K, D, D_s, B, B_c$ and $M' = \pi, K, D, D_s$, and the charged leptons are $\ell_1, \ell_2 = e, \mu$. These decays are lepton number violating (LNV), hence the neutrinos mediating them must be of Majorana type. We focus on signals of CP violation in such processes, by working in scenarios with two on-shell sterile neutrinos $N_1$ and $N_2$, i.e., with masses $M_{N_j}$ in the intervals $M_{M'} + M_{\ell_2} < M_{N_j} < M_{M'} - M_{\ell_1}$. The signals of CP violation are represented by the CP-violating difference $S_-(M) \equiv |\Gamma(M^- \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp) - \Gamma(M^+ \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp)|$, and alternatively by the usual CP ratio $A_{CP}(M) \equiv \Gamma(M^- \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp)/\Gamma(M^+ \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp)$.

In Sec. [II] we describe the formalism for calculation of the various decay widths. The details of the calculation are given in Appendix [A] and the details for the total decay widths $\Gamma(NM_N)$ of the (heavy) sterile Majorana neutrinos are given in Appendix [B]. In Sec. [III] we present the expressions for the decay widths $S_+(M) \equiv |\Gamma(M^- \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp) + \Gamma(M^+ \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp)|$ and for the mentioned CP ratio $A_{CP}(M)$. Additional details are given in Appendix [C]. In Sec. [IV] we discuss the acceptance factor due to the (long) decay time of the on-shell sterile neutrinos, and the resulting effective (i.e., experimental) branching ratios $Br^{(\text{eff})}(M) \propto S_+(M)$ and $A_{CP}(M)Br^{(\text{eff})}(M) \propto S_-(M)$, and present numerical results. In Sec. [V] we summarize our results and present conclusions.

II. THE PROCESS AND FORMALISM FOR THE LNV SEMIHADRONIC DECAYS OF PSEUDOSCALARS

We consider the lepton number violating (LNV) processes, Fig. 1, $M^\pm \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp$, where the two intermediate Majorana neutrinos ($N_1, N_2$) are on-shell. The intermediate neutrinos have to be Majorana here because these processes violate lepton number.

In such a case, the topology of these tree level processes is like “$s$-channel.” The processes with (two-loop) “$t$-channel” topology are strongly suppressed [9]. The type of processes of Fig. 1 within the models with sterile neutrinos $N$ in the mass range of mesons, have been studied in several works, among them Refs. [6–14].

We denote the mixing coefficient for the heavy mass eigenstate $N_j$ with the standard flavor neutrino $\nu_\ell$ ($\ell = e, \mu, \tau$)
as \( B_{\ell N_j} \) \((j = 1, 2)\). The relevant mixing relations in our notation are

\[
\nu_{\ell} = \sum_{k=1}^{3} B_{\ell\nu_k} \nu_k + (B_{\ell N_1} N_1 + B_{\ell N_2} N_2) ,
\]

(1)

where \( \nu_k \) \((k = 1, 2, 3)\) are the light mass eigenstates, and the (unitary) PMNS matrix \( B \) is in this scenario a \( 5 \times 5 \) matrix.\(^2\)

We will use the phase conventions of the book Ref. 26, i.e., all the CP-violating phases are incorporated in the PMNS matrix of mixing elements. The sum and difference of the decay widths, \( S_{\pm}(M) \equiv [\Gamma(M^{-} \rightarrow \ell_{1}^{-}\ell_{2}^{-}M^{+}) \pm \Gamma(M^{+} \rightarrow \ell_{1}^{-}\ell_{2}^{-}M^{+})] \), of the processes of Fig. 1 will be appreciable only if the two intermediate neutrinos \( N_j \) are on-shell

\[
(M_{M'} + M_{\ell_2} < M_{N_j} < (M_{M'} - M_{\ell_1}) , \text{ or/and}
(M_{M'} + M_{\ell_1} < M_{N_j} < (M_{M'} - M_{\ell_2}) .
\]

(2)

We will often use schematic notations for the decay widths of these rare processes:

\[
\Gamma(M^{\pm}) \equiv \Gamma(M^{\pm} \rightarrow \ell_{1}^{\pm}\ell_{2}^{\pm}M^{\mp}) .
\]

(3)

These decay widths can be written in the form

\[
\Gamma(M^{\pm}) = (2 - \delta_{\ell_1,\ell_2}) \frac{1}{2!} \frac{1}{2M_{M'}} \frac{1}{(2\pi)^3} \int d_{3} |T(M^{\pm})|^2 ,
\]

(4)

where \( 1/2! \) is the symmetry factor when the two charged leptons are equal. Here, \( |T(M^{\pm})|^2 \) is the absolute square (summed over the final helicities) of the sum of amplitudes from \( N_1 \) and \( N_2 \) neutrinos in the two channels \( D \) (direct) and \( C \) (crossed). We refer to Appendix A for details. In Eq. (4), \( d_{3} \) denotes the integration over the three-particle final phase space

\[
d_{3} \equiv \frac{d^{3}\tilde{p}_{1}}{2E_{\ell_{1}}(\tilde{p}_{1})} \frac{d^{3}\tilde{p}_{2}}{2E_{\ell_{2}}(\tilde{p}_{2})} \frac{d^{3}\tilde{p}_{M'}}{2E_{M'}(\tilde{p}_{M'})} \delta^{(4)}(p_{M} - p_{1} - p_{2} - p_{M'}) .
\]

(5)

We denoted by \( p_{1} \) and \( p_{2} \) the momenta of \( \ell_{1} \) and \( \ell_{2} \) from the left and the right vertex of the direct channels, respectively (in the crossed channel \( \ell_{2} \) couples to the left vertex), cf. Fig. 1. The decay widths (4) can then be written as a double sum over the contributions of \( N_i \) and \( N_j \) exchanges \((i, j = 1, 2)\), with the mixing effects factored out

\[
\Gamma(M^{\pm}) = (2 - \delta_{\ell_1,\ell_2}) \sum_{i=1}^{2} \sum_{j=1}^{2} k_{i}^{(\pm)} k_{j}^{(\pm)} \Gamma(DD^{*})_{ij} + \Gamma(CC^{*})_{ij} + \Gamma(DC^{*})_{ij} + \Gamma(CD^{*})_{ij} ,
\]

(6)

where \( k_{j}^{(\pm)} \) are the corresponding mixing factors

\[
k_{j}^{(-)} = B_{\ell_1 N_j} B_{\ell_2 N_j} , \quad k_{j}^{(+)} = (k_{j}^{(-)})^{*},
\]

(7)

and \( \Gamma_{\pm}(XY^{*})_{ij} \) are the normalized (i.e., without the mixing) contributions of \( N_i \) exchange in the \( X \) channel and complex-conjugate of \( N_j \) exchange in the \( Y \) channel \((X, Y = C, D)\)

\[
\Gamma_{\pm}(XY^{*})_{ij} \equiv K^{2} \frac{1}{2!} \frac{1}{2M_{M'}} \frac{1}{(2\pi)^3} \int d_{3} P_{i}(X)P_{j}(Y)^{*} M_{N_i} M_{N_j} T_{\pm}(X)T_{\pm}(Y)^{*} .
\]

(8)

\(^{1}\) There exist also other notations for \( B_{\ell N} \) in the literature, e.g. \( U_{\ell N} \) in 12; \( V_{\ell N} \) in 11.

\(^{2}\) In our work, \( B \) can be a \( n \times n \) matrix with \( n \geq 5 \). If \( n > 5 \), we implicitly assume that the additional sterile neutrinos \((N_3, \text{etc.})\) have significantly less mixing than \( N_1 \) and \( N_2 \) with the active flavor (“light”) neutrino sector; one such framework is \( \nu_{\text{MSM}} \) 20, 28, 30, with \( n = 6 \).
Here, $T_\pm(X) (X = D, C)$ are the relevant parts of the amplitude in the $X$ channel which appear also in the total decay amplitudes $T_j$ (see Appendix A)\footnote{Since $|T_+(D)|^2 = |T_-(D)|^2$ and $|T_+(C)|^2 = |T_-(C)|^2$, we omitted the subscripts $\pm$ from the $DD^*$ and $CC^*$ contribution terms $\Gamma(DD^*)_{ij}$ and $\Gamma(CC^*)_{ij}$ in Eq. (6).} and $P_j(X) (X = D, C)$ are the propagators of the intermediate neutrinos $N_j$ in the two channels

$$P_j(D) = \frac{1}{(p_M - p_1)^2 - M_{N_j}^2 + i\Gamma_{N_j}M_{N_j}} , \tag{9a}$$

$$P_j(C) = \frac{1}{(p_M - p_2)^2 - M_{N_j}^2 + i\Gamma_{N_j}M_{N_j}} . \tag{9b}$$

The overall constant $K^2$ appearing in Eqs. (8) is

$$K^2 = G_F^4 f_M f_{M'}^2 |V_{Q_s q_d} V_{q_u q_d}|^2 , \tag{10}$$

where $f_M$ and $f_{M'}$ are the decay constants of $M^\pm$ and $M'^\mp$, and $V_{Q_s q_d}$ and $V_{q_u q_d}$ are the CKM elements corresponding to $M^\pm$ and $M'^\mp$ (the valence quark content of $M^\pm$ is $Q_s Q_d$; of $M'^\mp$ is $q_u q_d$).

Several symmetry relations exist among the normalized decay widths $\Gamma_{\pm}(XY^*)_{ij}$, as given in Eqs. (A6)-(A7) in Appendix A. The most important symmetry property is that the $(2 \times 2)$ matrices $\Gamma(DD^*)$ and $\Gamma(CC^*)$ are self-adjoint (and even equal if $\ell_1 = \ell_2$). The matrices $\Gamma_{\pm}(DC^*)$ and $\Gamma_{\pm}(CD^*)$, which represent the (normalized) $D-C$ channel interference contributions to the decay widths $\Gamma(M^\pm)$, will turn out to be several orders of magnitude smaller than the $\Gamma(DD^*)$ and $\Gamma(CC^*)$ matrices.

In our calculations we will also need to know the total decay width $\Gamma(N_j \to \ell \nu)$ of the two Majorana neutrinos $N_j$ as a function of the mass $M_{N_j}$, or more specifically, the corresponding mixing factor $\bar{K}_j$. The width $\Gamma_{N_j}$ can be written as

$$\Gamma_{N_j} = \bar{K}_j \Gamma(M_{N_j}) , \tag{11}$$

where

$$\bar{K}_j(M_{N_j}) \equiv \bar{K}_j = N_{eN_j} |B_{eN_j}|^2 + N_{\mu N_j} |B_{\mu N_j}|^2 + N_{\tau N_j} |B_{\tau N_j}|^2 , \quad (j = 1, 2) . \tag{13}$$

Here, $N_{\ell N}(M_N) = N_{\ell N} (\ell = e, \mu, \tau)$ are the effective mixing coefficients; they are numbers $\sim 10^0 - 10^1$ which depend on the mass $M_N$ of the Majorana neutrino $N$ ($N = N_1, N_2$). In Appendix B we write down the relevant formulas for the calculation of these coefficients. The results of these calculations are given in Fig. 2 for the here relevant neutrino mass interval $0.1 \text{ GeV} < M_N < 6.3 \text{ GeV}$. Some additional remarks are given in Appendix B.

On the other hand, the present upper bounds for the squares $|B_{\ell N}|^2$ of the heavy-light mixing matrix elements, in our range of interest $0.1 \text{ GeV} < M_N < 6.3 \text{ GeV}$, can be inferred from Ref. [11] (and references therein). The present upper bounds for $|B_{\ell N}|^2$, in the mentioned range of $M_N$, are largely determined by the neutrinoless double beta decay experiments [34-35] ($0\nu\beta\beta$). The upper bounds for $|B_{\mu N}|^2$ come from searches of peaks in the spectrum of $\mu$ in pion and kaon decays [36] and from decay searches [36-39]. The upper bounds for $|B_{\tau N}|^2$ come from CC interactions (if $\tau$ is produced) and from NC interactions [39-40]. In Table I we present the upper bounds on $|B_{\ell N}|^2$ for specific chosen values of $M_N$ in the mentioned integral. The upper bounds have in some cases strong dependence on the precise values of $M_N$, and for further details we refer to the corresponding figures in Ref. [11].

## III. THE DECAY WIDTHS AND CP ASYMMETRY FOR THE LNV SEMIHADRONIC DECAYS OF PSEUDOSCALARS

Here we will use the results of Sec. II and a combination of analytic and numerical evaluations, in order to obtain the results for the decay widths $S_{\pm}$ and the CP asymmetry ratios $A_{CP}$ of the discussed semihadronic LNV decays of
The effective mixing coefficients $N_{\ell N}$ ($\ell = e, \mu, \tau$) appearing in Eqs. (11)-(13), as a function of the mass $M_N$ of the Majorana neutrino $N$. See the text and Appendix B for details.

| $M_N$ (GeV) | $|B_{eN}|^2$ | $|B_{\mu N}|^2$ | $|B_{\tau N}|^2$ |
|------------|-------------|----------------|-------------|
| 0.1        | $(1.5 \pm 0.5) \times 10^{-8}$ | $(6.0 \pm 0.5) \times 10^{-8}$ | $(8.0 \pm 0.5) \times 10^{-8}$ |
| 0.3        | $(2.5 \pm 0.5) \times 10^{-9}$ | $(3.0 \pm 0.5) \times 10^{-9}$ | $(1.5 \pm 0.5) \times 10^{-9}$ |
| 0.5        | $(2.0 \pm 0.5) \times 10^{-8}$ | $(6.5 \pm 0.5) \times 10^{-8}$ | $(2.5 \pm 0.5) \times 10^{-8}$ |
| 0.7        | $(3.5 \pm 0.5) \times 10^{-9}$ | $(4.5 \pm 0.5) \times 10^{-8}$ | $(9.0 \pm 0.5) \times 10^{-8}$ |
| 1.0        | $(4.5 \pm 0.5) \times 10^{-7}$ | $(1.5 \pm 0.5) \times 10^{-7}$ | $(3.0 \pm 0.5) \times 10^{-7}$ |
| 2.0        | $(1.0 \pm 0.5) \times 10^{-6}$ | $(2.5 \pm 0.5) \times 10^{-6}$ | $(3.0 \pm 0.5) \times 10^{-6}$ |
| 3.0        | $(1.5 \pm 0.5) \times 10^{-6}$ | $(2.5 \pm 0.5) \times 10^{-6}$ | $(4.5 \pm 0.5) \times 10^{-6}$ |
| 4.0        | $(2.5 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ |
| 5.0        | $(3.0 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ |
| 6.0        | $(3.5 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ | $(1.5 \pm 0.5) \times 10^{-5}$ |

For example, if $\ell_1 = \ell_2 = \mu$, then $\theta_{21} = 2(\phi_{\mu 2} - \phi_{\mu 1}) = 2(\arg(B_{\mu N_2}) - \arg(B_{\mu N_1}))$. Here we will not write explicitly the $D$-$C$ channel interference contributions to the quantities $14$-$15$, as our numerical calculations give us for them contributions which are several orders of magnitude smaller than the contributions from the $D$ channel and from the $C$ channel.

The resulting sums $S_+(M) \equiv (\Gamma(M^-) + \Gamma(M^+))$ of the decay widths can then be written in terms of only the normalized decay widths $\bar{\Gamma}(XX^*)_{11}$, $\bar{\Gamma}(XX^*)_{22}$ and $\text{Re}{\Gamma}(XX^*)_{12}$ (where $X = D; C$), and in terms of the phase...
difference \( \theta_{21} \)

\[
S_+(M) \equiv (\Gamma(M^-) + \Gamma(M^+))
\]

\[
= 2(2 - \delta_{\ell_1 \ell_2})|B_{\ell_1 N_1}|^2|B_{\ell_2 N_2}|^2 \left\{ \begin{array}{l}
\Gamma(DD^*)_{11} \left[ 1 + \kappa_{\ell_1}^2 \kappa_{\ell_2}^2 \frac{\Gamma(DD^*)_{22}}{\Gamma(DD^*)_{11}} + 2\kappa_{\ell_1} \kappa_{\ell_2} \cos \theta_{21} \delta_1 \right] \\
+ \Gamma(CC^*)_{11} \left[ 1 + \kappa_{\ell_1}^2 \kappa_{\ell_2}^2 \frac{\Gamma(CC^*)_{22}}{\Gamma(CC^*)_{11}} + 2\kappa_{\ell_1} \kappa_{\ell_2} \cos \theta_{21} \delta_1 \right] + (D - C \text{ terms}) \end{array} \right\},
\]

where we used the notations (16), and the quantity \( \delta_1 \) measures the effect of \( N_1-N_2 \) overlap contributions

\[
\delta_j = \frac{\text{Re} \Gamma(XX^*)_{12}}{\Gamma(XX^*)_{jj}}, \quad (X = D; C; \; j = 1; 2).
\]

It is expected that \( \delta_j \approx 0 \) when \( \Delta M_N \gg \Gamma_N \), because in such a case the overlap (interference) effects of the \( N_1 \) and \( N_2 \) exchanges are expected to be absent due to a large distance between the two “bumps” of the neutrino propagators. Numerical evaluations confirm this expectation and confirm that \( \delta_j \) is practically independent of the channel \( X = D, C \) (see later on in this Section).

The (CP-violating) difference \( S_-(M) \equiv (\Gamma(M^-) - \Gamma(M^+)) \) of the LNV rare decays is

\[
S_-(M) = (\Gamma(M^-) - \Gamma(M^+)) = 4(2 - \delta_{\ell_1 \ell_2})|B_{\ell_1 N_1}|^2|B_{\ell_2 N_2}|^2 \left\{ \sin \theta_{21} [\text{Im}\Gamma(DD^*)_{12} + \text{Im}\Gamma(CC^*)_{12}] + (D - C \text{ terms}) \right\}.
\]

We can see that CP violation in these decays is proportional to the CP-odd phase difference \( \theta_{21} \) defined in Eq. (16c). The other factor in this CP violation is the imaginary part of \( \Gamma(DD^*)_{12} + \Gamma(CC^*)_{12} \); this factor will be investigated later on in this Section.

The decay widths \( \Gamma_{\ell_i \ell_j} \) are very small in comparison with the masses \( M_N \), due to the mixing suppression, cf. Eqs. (11) (in general \( \Gamma_{\ell_i \ell_j} \ll 1 \; \text{eV} \)). Therefore, the absolute value of the square of the intermediate neutrino propagator can be approximated to a high degree of accuracy by the delta function

\[
|P_j(D)|^2 = \left| \frac{1}{(p_M - p_1)^2 - M_{N_j}^2 + i\Gamma_{N_j} M_{N_j}} \right|^2 \\
\approx \frac{\pi}{M_{N_j} \Gamma_{N_j}} \delta((p_M - p_1)^2 - M_{N_j}^2); \quad (j = 1, 2; \; \Gamma_{N_j} \ll M_{N_j}),
\]

and analogous equation for \( |P_j(C)|^2 \). Therefore, in the integration \( d_3 \), the part of integration \( dp_N^2 \) \( (p_N = p_M - p_1 \text{ in } D \text{ channel}; p_N = p_M - p_2 \text{ in } C \text{ channel}) \) becomes a trivial integration over a delta function, and the expressions for the diagonal elements \( \Gamma(DD^*)_{jj} \) and \( \Gamma(CC^*)_{jj} \) can be calculated analytically, cf. Appendix C

\[
\Gamma(DD^*)_{jj} = \frac{K^2 M_{\ell_i} M_{\ell_j} \lambda^{1/2}(1, x_j, x_{\ell_1}) \lambda^{1/2}(1, x_{\ell_1}, x_{\ell_2})}{128 \pi^2} \left( \frac{x_j}{x_{\ell_1}} \right) \frac{x_{\ell_2}}{x_j} Q(x_j; x_{\ell_1}, x_{\ell_2}, x) \quad (j = 1 \text{ or } j = 2),
\]

and \( \Gamma(CC^*)_{jj} \) is obtained from the expression \( \ref{eq:21} \) by the simple exchange \( x_{\ell_1} \leftrightarrow x_{\ell_2} \)

\[
\Gamma(CC^*)_{jj} = \Gamma(DD^*)_{jj}(x_{\ell_1} \leftrightarrow x_{\ell_2}).
\]

In Eq. \( \ref{eq:21} \) we used the notations

\[
\lambda(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 - 2y_1 y_2 - 2y_2 y_3 - 2y_3 y_1,
\]

\[
x_j = \frac{M_{\ell_j}^2}{M_{\ell_i}^2}, \quad x_{\ell_1} = \frac{M_{\ell_1}^2}{M_{\ell_i}^2}, \quad x' = \frac{M_{\ell_j}^2}{M_{\ell_i}^2}, \quad (j = 1, 2; \; \ell_j = \ell_1, \ell_2),
\]

and the function \( Q(x_j; x_{\ell_1}, x_{\ell_2}, x) \) is given in Appendix C. In the special case \( \ell_1 = \ell_2 \), the expression for \( \Gamma(DD^*)_{jj} \) is somewhat simpler and can be deduced, e.g., from Ref. \( 13 \). The expressions \( \ref{eq:21} \) and \( \ref{eq:22} \) are used in the evaluation of the sum \( S_+(M) \), Eq. \( \ref{eq:17} \), of the rare decay widths of \( M^\pm \). In Eq. \( \ref{eq:17} \), the contributions of the \( N_1-N_2 \) overlap effects are parametrized in the function \( \delta_1 \) defined in Eq. \( \ref{eq:18} \), and will be evaluated later on numerically.
In order to evaluate the CP-violating difference \( S_\perp(M) \), Eq. \((19)\), of the rare decay widths \( M^\pm \), the evaluation of the quantity \( \text{Im}\Gamma(XX^*)_{12} (X = D, \bar{C}) \) is of central importance. In the integrand of \( \text{Im}\Gamma(XX^*)_{12} \) we have, according to Eq. \((8)\), a factor the following combination of the propagators of \( N_1 \) and \( N_2 \):

\[
\text{Im}P_1(D)P_2(D)^* = \frac{(p_N^2 - M_{N_1}^2)}{(p_N^2 - M_{N_1}^2)^2 + \Gamma_{N_1}^2 M_{N_1}^2} \frac{(p_N^2 - M_{N_2}^2)}{(p_N^2 - M_{N_2}^2)^2 + \Gamma_{N_2}^2 M_{N_2}^2} \tag{24a}
\]

\[
\approx \mathcal{P} \left( \frac{1}{p_N^2 - M_{N_1}^2} \right) \pi \delta(p_N^2 - M_{N_1}^2) - \pi \delta(p_N^2 - M_{N_1}^2) \mathcal{P} \left( \frac{1}{p_N^2 - M_{N_2}^2} \right) \tag{24b}
\]

\[
= \frac{\pi}{M_{N_2}^2 - M_{N_1}^2} \left[ \delta(p_N^2 - M_{N_1}^2) + \delta(p_N^2 - M_{N_2}^2) \right] , \tag{24c}
\]

where we have \( p_N = (p_M - p_1) \) in the direct \( (D) \) channel. In Eqs. \((24b)-(24c)\) we assumed \( \Gamma_{N_j} \ll |\Delta M_{N_j}| = M_{N_2} - M_{N_1} \).

The expression \((24a)\) has formally the same structure with Dirac delta functions as Eq. \((20)\), but the factors in front of these Dirac delta functions are different now. Hence we can perform the integration over the final particle phase space in the same way, but now under the more stringent assumption \( \Gamma_{N_j} \ll |\Delta M_{N_j}| \) (and not just: \( \Gamma_{N_j} \ll M_{N_j} \) which is always fulfilled),

\[4\] leading to the result

\[
\text{Im}\Gamma(DD^*)_{12} = \eta \frac{K^2 M_{N_1}^2}{128\pi^2} \frac{M_{N_1} M_{N_2}}{M_{N_2} + M_{N_1}} \Delta M_N \sum_{j=1}^2 \lambda^j(1, x, x_1, y_1), \frac{\lambda^j(1, x, x, x')}{Q(x_1, y_1, x_2', x_j)} , \tag{25a}
\]

\[
\text{Im}\Gamma(CC^*)_{12} = \text{Im}\Gamma(DD^*)_{12}(x_1 \leftrightarrow x_2) , \tag{25b}
\]

where we denoted \( \Delta M_N = M_{N_2} - M_{N_1} > 0 \). In Eqs. \((25)\) we introduced an overall factor \( \eta \) which accounts for the effects \( \Delta M_N \gg \Gamma_{N_j} \), i.e., for the situation when the approximation \((21)\) of \( \text{Im}P_1(D)P_2(D)^* \) in terms of Dirac delta functions is not justified. Later on in this Section, we will evaluate numerically the factor \( \eta \). When \( \Delta M_N \gg \Gamma_{N_j} \), i.e., when the identity \((24b)\) can be applied, the factor \( \eta \) is equal to unity, \( \eta = 1 \).

The normalized decay matrix elements \( \Gamma(XY^*)_{ij} \), Eq. \((6)\), were evaluated also numerically, by versions of Monte Carlo integration, independently by the two authors, using finite (small) widths \( \Gamma_{N_j} \) in the propagators. We confirmed numerically the analytic expression \((21)\) for \( \Gamma(X)^*(DD^*)_{jj} \) \((\approx 1/\Gamma_{N_j})\), and the analytic expression \((25)\) with \( \eta = 1 \) for \( \text{Im}\Gamma(DD^*)_{12} \) \((\approx 1/|\Delta M_N|) \) when \( \Delta M_N \gg \Gamma_{N_j} \).

Further, our numerical evaluations lead us to the conclusion that the direct-crossed channel \( (DC^* \text{ and } CD^*) \) interference contributions to the sum and the difference of the rare decay widths \( S_{\perp}(M) \) of \( M^\pm \) are by several orders of magnitude smaller than the corresponding direct \( (DD^*) \) and crossed \( (CC^*) \) channel contributions to these quantities, in all cases.

In addition, our numerical evaluations give us values of the parameters \( \delta_j \) of Eq. \((18)\), and of the \( \eta \) correction parameters of Eqs. \((25)\). In the cases when \( \Delta M_N \gg \Gamma_{N_j} \), these values differ appreciably from their limiting values \( \delta_j = 0 \) and \( \eta = 1 \) of the \( \Delta M_N \gg \Gamma_{N_j} \) limit. It turns out that the parameters \( \delta_j \) are practically independent of the channel contribution considered \( (DD^* \text{ or } CC^*) \) and of the type of pseudoscalar mesons \( (M^\pm, M^\mp) \) of the light leptons \( (\ell_1, \ell_2 = e, \mu) \) involved in the considered decays, and the same is true for the parameter \( \eta \). Further, numerical calculations show that, in the consiered case \( \Delta M_N \gg \Gamma_{N_j} \) (i.e., when \( N_1 \) and \( N_2 \) are almost degenerate), the parameters \( \eta \) and \( \delta \equiv (1/2)(\delta_1 + \delta_2) \) are functions of only one parameter \( y \equiv \Delta M_N/\Gamma_{N_j} \), where \( \Delta M_N = M_{N_2} - M_{N_1} > 0 \) and \( \Delta M_N = (1/2)(\Gamma_{N_1} + \Gamma_{N_2}) \)

\[
\eta = \eta(y) , \quad y \equiv \frac{\Delta M_N}{\Gamma_{N_j}} , \quad \Gamma_{N_j} = \frac{1}{2}(\Gamma_{N_1} + \Gamma_{N_2}) , \tag{26a}
\]

\[
\delta = \delta(y) , \quad \delta \equiv \frac{1}{2}(\delta_1 + \delta_2) , \quad \frac{\delta_1}{\delta_2} = \frac{\Gamma(DD^*)_{22}}{\Gamma(DD^*)_{11}} = \frac{\Gamma_{N_1}}{\Gamma_{N_2}} = \frac{\bar{K}_1}{\bar{K}_2} . \tag{26b}
\]

---

4 We note that this mechanism is central to the CP violation effects in the considered LNV semihadronic decays of charged pseudoscalar mesons. This mechanism was presented in Ref. \([27]\) and applied there to the CP violation of the rare leptonic decays of charged pions.

5 For example, when \( M^\pm = K^\pm \) and \( M^\mp = \pi^\mp \), and we choose in numerical calculation \( \Gamma_{N_j} \sim 10^{-3} \text{ GeV} \sim \Delta M_{N_j} \), the \( \Gamma(DD^*)_{ij} \) and \( \Gamma(CC^*)_{ij} \) contributions are by about two orders of magnitude larger than the \( D-C \) interference contributions \( \Gamma_{\perp}(DC^*)_{ij} \). When \( \Gamma_{N_j} \) and \( \Delta M_{N_j} \) are decreased further \((\Gamma_{N_j} \sim \Delta M_{N_j})\), the \( \Gamma(DD^*)_{ij} \) and \( \Gamma(CC^*)_{ij} \) contributions increase (they are \( \propto 1/\Gamma_{N_j} \), or \( \propto 1/\Delta M_{N_j} \)), while the \( D-C \) interference contributions \( \Gamma_{\perp}(DC^*)_{ij} \) remain approximately unchanged and become thus relatively insignificant.
The numerical integration gives us these values, which are tabulated in Table II as a function of \( y \). The uncertainties indicate the numerical uncertainties and the small variations from the various considered LNV semihadronic decays \( M^\pm \rightarrow \ell_1^+ \ell_2^- M^{\mp} \), where \( M \) and \( M' \) are pseudoscalar mesons, \( M = K,D,D_s,B,B_c \) and \( M' = \pi,K,D,D_s \), and the charged leptons are \( \ell_1,\ell_2 = e,\mu \). It is interesting that the values in Table II are almost equal to the values of the parameters \( \delta(y) \) and \( \eta(y) \) for the rare leptonic decays of the charged pions \( \pi^\pm \rightarrow e^\pm N \rightarrow e^\pm e^\pm \mu^+\nu \), Ref. [27]. The uncertainties in the present Table are in general smaller, though, because of the high statistics applied in Monte Carlo calculations which practically eliminates the numerical uncertainty part.

The rare LNV semihadronic decay widths of \( M^\pm \), cf. \( S_+ \) (M) of Eq. (17), at first sight appear to be quartic in the heavy-light mixing elements \( |B_{\ell N}| \) and thus very suppressed. However, they are proportional to the expressions \( \Gamma(D \rightarrow M_{jj}) \), Eq. (21), which are proportional to \( 1/\Gamma_{N_j} \) due to the on-shellness of the intermediate \( N_j \)'s [cf. also Eq. (20)].

This \( 1/\Gamma_{N_j} \) is proportional to \( 1/\widetilde{K}_j \sim 1/|B_{\ell N_j}|^2 \) according to Eqs. (11)-(13). Hence this on-shellness of \( N_j \)'s makes these rare process decay widths significantly less suppressed

\[
\Gamma(D \rightarrow M_{jj}) \propto 1/\Gamma_{N_j} \propto 1/\widetilde{K}_j \sim 1/|B_{\ell N_j}|^2 \quad \Rightarrow \quad S_+(M) \propto |B_{\ell N_j}|^2 \quad \text{. (27)}
\]

However, the expressions \( S_+(M) \), which appear in the CP-violating decay width difference \( S_-(M) \), are suppressed by mixings as \( \sim |B_{\ell N_j}|^4 \). This means that in general \( S_-(M) \) is much smaller than the decay width \( S_+(M) \propto |B_{\ell N_j}|^2 \). Nonetheless, Eqs. (25) show that \( S_-(M) \) is proportional to \( 1/\Delta M_N \), and it is this aspect that represents the opportunity to detect appreciable CP violation in such decays when \( \Delta M_N \) is sufficiently small. While in general we expect \( \Delta M_N \gg \Gamma_N \), there exists a well-motivated model [20, 28, 30] with two sterile almost degenerate neutrinos (where the relation \( \Delta M_N \gg \Gamma_N \) is possible) in the mass range 0.1 GeV \( \lesssim M_{\nu \ell} \lesssim 10^3 \) GeV. Our calculations thus suggest that in such a model the CP violation effects may be appreciable, namely for \( \Delta M_N \sim \Gamma_N \) we obtain \( S_-(M) \sim S_+(M) \) and thus \( A_{\text{CP}}(M) \sim 1 \).

For these reasons, from now on we consider the case of near degeneracy: \( \Delta M_N \gg \Gamma_N \) (i.e., \( \Delta M_N \sim \Gamma_N \)). In this case, several formulas written by now in this Section get even more simplified, in particular the expressions (21), (18), (25). Namely, they can be written in terms of the common canonical decay width \( S \)

\[
S(x;\ell_1,\ell_2,x') \equiv \frac{3\pi}{4} \frac{K^2 M_{M}}{G_F^2} \frac{1}{x^2} \lambda^{1/2}(1,\ell_1,x) \lambda^{1/2}(1,x,x') \left( \frac{1}{x'} \frac{x_2}{x} \right) Q(x;\ell_1,\ell_2,x') \quad \text{, (28)}
\]

where we use the notations \( \ell_1,\ell_2 \) and

\[
x \equiv \frac{M_{N_j}^2}{M_{M}^2} \equiv x_2 \approx x_1 \quad \text{, (29)}
\]

where we denoted by \( M_N \equiv M_{N_2} \approx M_{N_1} \). The function \( Q \) is the same as in Eqs. (21) and (25), and is given explicitly in Appendix C. In practice we will need two variants of this function \( S \), namely the one for the \( DD^* \) contributions \( (S^{(D)}) \) and the one of the \( CC^* \) contributions \( (S^{(C)}) \)

\[
S^{(D)}(x) \equiv S(x;\ell_1,\ell_2,x') \quad \text{, (30a)}
\]

\[
S^{(C)}(x) \equiv S(x;\ell_2,\ell_1,x') \quad \text{. (30b)}
\]

When \( \ell_1 = \ell_2 \) (e.g., when both final leptons are electrons; or both are muons), the two functions \( S^{(D)} \) and \( S^{(C)} \) coincide. It is straightforward to check that the expressions of Eqs. (21), (18), (25) can then be rewritten in the considered case of nearly degenerate \( N_1 \) and \( N_2 \) in terms of these common functions \( S^{(X)} \) \( (X = D, C) \) and of the

| \( y \equiv \frac{\Delta M_N}{\Gamma_N} \) | \( \log_{10} y \) | \( \delta(y) \) | \( \eta(y) \) |
|----------------|----------|----------|----------|
| 1.00            | 0.000    | 0.500 ± 0.004 | 0.500 ± 0.001 |
| 1.25            | 0.007    | 0.390 ± 0.003 | 0.610 ± 0.003 |
| 1.67            | 0.222    | 0.264 ± 0.003 | 0.736 ± 0.002 |
| 2.50            | 0.398    | 0.138 ± 0.001 | 0.862 ± 0.001 |
| 5.00            | 0.699    | 0.038 ± 0.001 | 0.962 ± 0.002 |
| 10.0            | 1.000    | 0.0098 ± 0.0010 | 0.990 ± 0.0020 | 0.0090 ± 2 × 10^{-4} |
heavy-light mixing expressions $\tilde{K}_j (\sim |B_{\ell N_j}|^2)$ of Eq. [13]

$$
\Gamma(DD^*)_{jj} = \frac{1}{K_j} \mathcal{S}^{(D)}(x), \quad \Gamma(CC^*)_{jj} = \frac{1}{K_j} \mathcal{S}^{(C)}(x),
$$

$$
\text{Re}\Gamma(DD^*)_{12} = \delta(y) \frac{2}{(K_1 + K_2)} \mathcal{S}^{(D)}(x), \quad \text{Re}\Gamma(CC^*)_{12} = \delta(y) \frac{2}{(K_1 + K_2)} \mathcal{S}^{(C)}(x),
$$

$$
\text{Im}\Gamma(DD^*)_{12} = \eta(y) \frac{2}{(K_1 + K_2)} \mathcal{S}^{(D)}(x), \quad \text{Im}\Gamma(CC^*)_{12} = \eta(y) \frac{2}{(K_1 + K_2)} \mathcal{S}^{(C)}(x),
$$

where the definition $y = \Delta M_N / \Gamma_N$ is kept.

After some straightforward algebra, we can rewrite the sum and difference $S_{\pm}(M)$ of decay widths, Eqs. [14], as expressions proportional to these canonical decay widths $\mathcal{S}^{(X)} (X = D, C)$. The proportionality factors involve the heavy-light mixing factors $|B_{\ell N_j}|$ and $\tilde{K}_j$ [cf. Eq. [13]], and the overlap functions $\delta(y)$ and $\eta(y)/y$ tabulated in Table [I].

The resulting expressions are

$$
S_{+}(M) \equiv \Gamma(M^- \to \ell_1^- \ell_2^- M'^+ ) + \Gamma(M^+ \to \ell_1^+ \ell_2^+ M'^- )
$$

$$
= 2(2 - \delta_{\ell_1 \ell_2}) \sum_{j=1}^{2} \frac{|B_{\ell_1 N_j}|^2 |B_{\ell_2 N_j}|^2}{K_j} + 4\delta(y) \frac{|B_{\ell_1 N_j}| |B_{\ell_2 N_j}||B_{\ell_2 N_2}|}{(K_1 + K_2)} \cos \theta_{21} \left( \mathcal{S}^{(D)}(x) + \mathcal{S}^{(C)}(x) \right),
$$

$$
S_{-}(M) \equiv \Gamma(M^- \to \ell_1^+ \ell_2^- M'^+ ) - \Gamma(M^+ \to \ell_1^- \ell_2^+ M'^- )
$$

$$
= 8(2 - \delta_{\ell_1 \ell_2}) \frac{|B_{\ell_1 N_j}| |B_{\ell_2 N_j}||B_{\ell_2 N_2}|}{(K_1 + K_2)} \sin \theta_{21} \frac{\eta(y)}{y} \left( \mathcal{S}^{(D)}(x) + \mathcal{S}^{(C)}(x) \right).
$$

The resulting CP violation ratio $A_{CP}(M)$, Eq. [15], can then be written in a form involving only the heavy-light mixing factors $|B_{\ell N_j}|$ and $\tilde{K}_j$ [cf. Eq. [13]], and the overlap functions $\delta(y)$ and $\eta(y)/y$ tabulated in Table [I].

$$
A_{CP}(M) = \frac{S_{-}(M)}{S_{+}(M)} = \frac{\Gamma(M^- \to \ell_1^+ \ell_2^- M'^+ ) - \Gamma(M^+ \to \ell_1^- \ell_2^+ M'^- )}{\Gamma(M^- \to \ell_1^+ \ell_2^- M'^+ ) + \Gamma(M^+ \to \ell_1^- \ell_2^+ M'^- )}
$$

$$
= \frac{1}{4} \sum_{j=1}^{2} \frac{|B_{\ell_1 N_j}|^2 |B_{\ell_2 N_j}|^2}{|B_{\ell_1 N_1}| |B_{\ell_1 N_2}| |B_{\ell_2 N_1}| |B_{\ell_2 N_2}|} \frac{(K_1 + K_2)}{K_j} + \delta(y) \cos \theta_{21} \right) \eta(y) / y
$$

$$
= \left\{ \frac{1}{4} \left[ \kappa_{\ell_1} \kappa_{\ell_2} \left(1 + \frac{\kappa_1}{\kappa_2}\right) + \frac{1}{\kappa_{\ell_1} \kappa_{\ell_2}} \left(1 + \frac{\kappa_2}{\kappa_1}\right) \right] + \delta(y) \cos \theta_{21} \right\} \eta(y) / y.
$$

In Eq. [33b] we used the notations [16a].

When $\ell_1 = \ell_2 (\equiv \ell)$, the formulas [32]-[33] simplify because then $\mathcal{S}^{(D)} = \mathcal{S}^{(C)} = \mathcal{S}$, and $B_{\ell_1 N_j} = B_{\ell_2 N_j} = B_{\ell N_j}$, $\kappa_{\ell_1} = \kappa_{\ell_2} = \kappa_\ell$

$$
S_{+}(M) = 4 \left[ \sum_{j=1}^{2} \frac{|B_{\ell N_j}|^4}{K_j} + 4\delta(y) \frac{|B_{\ell N_1}|^2 |B_{\ell N_2}|^2}{(K_1 + K_2)} \cos \theta_{21} \right] \mathcal{S}(x),
$$

$$
S_{-}(M) = 16 \frac{|B_{\ell N_1}|^2 |B_{\ell N_2}|^2}{(K_1 + K_2)} \sin \theta_{21} \frac{\eta(y)}{y} \mathcal{S}(x),
$$

$$
A_{CP}(M) = \frac{1}{4} \left[ \kappa_\ell^2 \left(1 + \frac{\kappa_1}{\kappa_2}\right) + \frac{1}{\kappa_\ell^2} \left(1 + \frac{\kappa_2}{\kappa_1}\right) \right] + \delta(y) \cos \theta_{21} \right] \eta(y) / y.
$$

From these expressions and Table [I] we can deduce:

1. When $y$ becomes large ($y > 10$, i.e., $\Delta M_N > 10 \Gamma_N$), the CP asymmetries [32b]-[33] become suppressed by the small $\eta(y)/y$ factor.

2. When $y$ is smaller ($y < 10$, i.e., $\Gamma_N < \Delta M_N < 10 \Gamma_N$), then the factor $\eta(y)/y$ is comparable with unity, the expressions $S_{\pm}(M)$ become $\sim |B_{\ell N_j}|^2 \mathcal{S}^{(D)}(x)$ (where $x \equiv M_N^2 / \Gamma_N^2$; $\ell = e, \mu$; note that $\tilde{K}_j \sim |B_{\ell N_j}|^2$); and the CP violation ratio $A_{CP}(M)$ becomes $\sim 1$. 

We present in Fig. 3 the numerical results of Table II for the suppression factor $\eta(y)/y$ and for the overlap factor $\delta(y)$ as a function of $y \equiv \Delta M_N/\Gamma_N$, for $1 < y < 10$.

In Ref. 13, the decay widths for these processes, in the case of one (on-shell) neutrino $N$, $\Gamma(M^+) \equiv \Gamma(M^+ \to \ell^+\ell'^+M^-)$, were considered. Since in our case $S_+(M) \approx 2\Gamma(M^+)$, the conclusions in Ref. 13 on the size and measurable of $\Gamma(M^+)$ can be taken over as the conclusions on the size and measurability of $S_+(M)$ here. If, in addition, $\Delta M_N \gg \Gamma_N$ (say: $y \equiv \Delta M_N/\Gamma_N < 5$), these conclusions are valid also for the measurability of the CP-violating decay width difference $S_-(M)$ provided that the phase difference $|\theta_{21}| \sim 1$.

IV. THE ACCEPTANCE FACTOR IN THE MEASUREMENT OF THE CONSIDERED DECAYS

In experiments which try to detect and investigate the LNV decay modes of the mesons $M^\pm$, the (expected) number $N_M \sim 10^N$ of produced mesons $M^\pm$ (per year, for example) is known. The value of the corresponding branching ratios of the LNV decay modes, $\text{Br}(M^\pm \to \ell_1^\pm \ell_2^\pm M^\mp) \equiv \Gamma(M^\pm \to \ell_1^\pm \ell_2^\pm M^\mp)/\Gamma(M^\pm \to \text{all})$, then becomes important. In principle, if $\text{Br}(M^\pm \to \ell_1^\pm \ell_2^\pm M^\mp) > 10^{-N}$, then such decay modes could be detected. Further, if an experiment produces approximately equal numbers of $M^+$ and $M^-$ mesons, then the branching ratios of experimental significance for the LNV decays $M^\pm \to \ell_1^\pm \ell_2^\pm M^\mp$ are

$$\text{Br}(M) \equiv \frac{S_+(M)}{[\Gamma(M^- \to \text{all}) + \Gamma(M^+ \to \text{all})]} \approx \frac{S_+(M)}{2\Gamma(M^- \to \text{all})} ,$$

$$\mathcal{A}_{\text{CP}}(M)\text{Br}(M) = \frac{S_-(M)}{[\Gamma(M^- \to \text{all}) + \Gamma(M^+ \to \text{all})]} \approx \frac{S_-(M)}{2\Gamma(M^- \to \text{all})} ,$$

where we use the notation of Eqs. (14) and (15) and (3). We also used the fact that in the considered cases of pseudoscalar mesons $M^\pm$ the total decay widths $\Gamma(M^- \to \text{all})$ and $\Gamma(M^+ \to \text{all})$ are practically equal. $\text{Br}(M)$ represents the average of the branching ratios of $M^+$ and $M^-$ for these LNV decays, while $\mathcal{A}_{\text{CP}}(M)\text{Br}(M)$ is the corresponding branching ratio for the (CP-violating) difference. The corresponding canonical branching fraction $\overline{\text{Br}}(M)$ is obtained by dividing the canonical decay width (28) by $2\Gamma(M^- \to \text{all})$

$$\overline{\text{Br}}(x; x_{\ell_1}, x_{\ell_2}, x') \equiv \frac{\overline{S}(x; x_{\ell_1}, x_{\ell_2}, x')}{2\Gamma(M^- \to \text{all})} = \frac{3\pi}{8} \frac{K^2 M_N}{G_F^2 \Gamma(M^- \to \text{all})} \frac{1}{x^2} \lambda^{1/2}(1, x, x_{\ell_1}) \lambda^{1/2} \left(1, \frac{x'}{x}, \frac{x_{\ell_2}}{x} \right) Q(x; x_{\ell_1}, x_{\ell_2}, x'),$$

6 when neglecting the $N_1-N_2$ overlap effects $\propto \delta(y)$ in $S_+(M)$

7 We recall that if $y < 5$, we have $\mathcal{A}_{\text{CP}}(M) \sim 1$ and thus $S_-(M) \sim S_+(M)$. 

FIG. 3: The suppression factors $\eta(y)/y$ and $\delta(y)$, due to the overlap of the propagator “resonances” of $N_1$ and $N_2$, as a function of $y \equiv \Delta M_N/\Gamma_N$, for $1 < y < 10$. 

\[ \text{Br}(M) \equiv \frac{S_+(M)}{[\Gamma(M^- \to \text{all}) + \Gamma(M^+ \to \text{all})]} \approx \frac{S_+(M)}{2\Gamma(M^- \to \text{all})} , \]

\[ \mathcal{A}_{\text{CP}}(M)\text{Br}(M) = \frac{S_-(M)}{[\Gamma(M^- \to \text{all}) + \Gamma(M^+ \to \text{all})]} \approx \frac{S_-(M)}{2\Gamma(M^- \to \text{all})} , \]
where the notations \( [23] \) and \([29] \) are used. We have two variants of this function: the one for the \( DD^* \) contributions \((\overline{BR}^{(D)})\) and the one of the \( CC^* \) contributions \((\overline{BR}^{(C)})\), which are obtained by dividing by \( 2\Gamma(M^* \to \text{all}) \) the expressions \( S^{(D)} \) and \( S^{(C)} \) of Eqs. \([30] \), respectively. When \( \ell_1 = \ell_2 \), the two functions \( \overline{BR}^{(D)} \) and \( \overline{BR}^{(C)} \) coincide (\( \equiv \overline{BR} \)).

Nonetheless, in experiments we must also take into account the acceptance (suppression) factor in the detection of these decays, which appears due to the small length of the detector in comparison to the relatively large lifetime of the (on-shell) sterile neutrinos \( N_j \). Stated otherwise, most of the on-shell neutrinos, produced in the decay \( M^\pm \to \ell \pm N_j \), are expected to survive long enough time to travel through the detector and decay (into \( \ell \pm M^\mp \)) outside the detector.\(^8\)

This effect suppresses the number of detected decays and should be taken into account, cf. Refs. \([4, 14, 27, 33, 41] \).

The acceptance (suppression) factor is the probability of the on-shell neutrino \( N \) to decay inside the detector of length \( L \)

\[
P_{N_j} \approx \frac{L}{\gamma_{N_j} \beta_{N_j}} \sim \frac{L \Gamma_{N_j}}{\gamma_{N_j}^2 N_j} \equiv \overline{A}(M_{N_j}) \overline{K}_j,
\]

where \( \gamma_{N_j} \) is the time dilation (Lorentz) factor \( \gamma_{N_j} = (1 - \beta_{N_j}^2)^{-1/2} \) (\( \sim 1\)-10) in the lab system. We took into account that the speed of neutrino is \( \beta_{N_j} \approx 1 \). The quantity \( \Gamma(M_{N_j}) \) (\( \propto M_{N_j}^3 \)) and the factor \( \overline{K}_j \) (\( \propto |B_{\ell N_j}|^2 \)) were defined in Eqs. \([12] \) and \([13] \), respectively. The quantity \( \overline{A}(M_{N_j}) \equiv (\Gamma(M_{N_j})/\gamma_{N_j}) \) can be called ”canonical acceptance,” and depends heavily on the neutrino mass: \( \overline{A} \propto M_{N_j}^2 \). In Fig. \(4 \) we present the values of this canonical acceptance as a function of the neutrino mass \( M_{N_j} \), for the choice \( L = 1 \) m \( (= 5.064 \cdot 10^{15} \text{ GeV}^{-1}) \) and \( \gamma_{N_j} = 2 \). The values of \( A \) for other cases of the values of \( L \) and \( \gamma_{N_j} \) are obtained directly from the presented curve by taking into account that \( \overline{A} \propto L/\gamma_{N_j} \). The realistic acceptance factor is then obtained by Eq. \([37] \), where \( \overline{K}_j \sim |B_{\ell N_j}|^2 \) (\( j = 1, 2 \)) are the heavy-light mixing factors defined in Eq. \([13] \) with coefficients \( N_{\ell N} \) there of \( \sim 10 \) according to Fig. \(2 \). Combining the results of Fig. \(2 \) with Eq. \([13] \), we can write rough approximations for \( \overline{K}_j \)

\[
\overline{K}_j \approx 15|B_{eN_j}|^2 + 2|B_{\mu N_j}|^2 + 2|B_{\tau N_j}|^2 \quad (K \text{ decays}) \tag{38a}
\]
\[
\overline{K}_j \approx 7|B_{eN_j}|^2 + |B_{\mu N_j}|^2 + 2|B_{\tau N_j}|^2 \quad (D, D_s \text{ decays}) \tag{38b}
\]
\[
\overline{K}_j \approx 8|B_{eN_j}|^2 + |B_{\mu N_j}|^2 + 3|B_{\tau N_j}|^2 \quad (B, B_c \text{ decays}) \tag{38c}
\]

The rough upper bounds for \( |B_{\ell N_j}|^2 \), for \( \ell = e, \mu, \tau \), are given in Table \(\[11\] \) for the typical ranges of our interest: \( M_{N_j} \) around 0.25; 1; 3 GeV – relevant for the decays of \( K \); \( (D, D_s) \); \( (B, B_c) \), respectively (see also Table \(\[8\] \) for several specific values of \( M_{N_j} \)). The corresponding values of the canonical acceptance factor \( \overline{A}(M_{N_j}) \) are also included. Combining Eqs. \([37] \) with \([38] \) and Table \(\[11\] \) we obtain for the acceptance factor \( P_{N_j} \) the following estimates and upper bounds

\[\]

\(^8\) Only when \( M = B \) or \( B_c \), a large part of the produced neutrinos \( N_j \) can decay within the detector (see the arguments later on).
relevant for the $K$ decays ($M_N \approx 0.25$ GeV), $D$ and $D_s$ decays ($M_N \approx 1$ GeV), and $B$ and $B_c$ decays ($M_N \approx 3$ GeV):

\[
P N_j (M_N \approx 0.25 \text{GeV}) \approx 1.7 |B_{eN_j}|^2 + 0.9 |B_{\mu N_j}|^2 \quad (+0.2 |B_{\tau N_j}|^2) \\
\lesssim 10^{-8} + 10^{-7} \quad (+10^{-5}),
\]

(39a)

\[
P N_j (M_N \approx 1 \text{GeV}) \approx 0.8 \cdot 10^3 |B_{eN_j}|^2 + 0.8 \cdot 10^3 |B_{\mu N_j}|^2 \quad (+2 \cdot 10^2 |B_{\tau N_j}|^2) \\
\lesssim 10^{-4} + 10^{-4} \quad (+10^0),
\]

(39b)

\[
P N_j (M_N \approx 3 \text{GeV}) \approx 3 \cdot 10^4 |B_{eN_j}|^2 + 3 \cdot 10^5 |B_{\mu N_j}|^2 \quad (+1 \cdot 10^5 |B_{\tau N_j}|^2) \\
\lesssim 10^0 + 10^0 \quad (+10^0),
\]

(39c)

The upper bounds for $P N_j$ in Eqs. (39) are written as a sum of the contributions of upper bounds from $|B_{eN_j}|^2$, $|B_{\mu N_j}|^2$ and $|B_{\tau N_j}|^2$ separately. Further, the contributions of $|B_{\tau N_j}|^2$ are included in Eqs. (39) optionally, in the parentheses, because the upper bounds of the mixings $|B_{\tau N_j}|^2$ are still very high and are expected to be reduced significantly in the foreseeable future. The upper bounds which give results higher than one are replaced by one ($10^0$), because the acceptance (decay probability) $P N_j$ can never be higher than one by definition.

From now on in this Section, we will assume the following:

\[
|B_{eN_j}|^2 \sim |B_{eN_j}|^2 \equiv |B_{eN}|^2 \\
\Rightarrow \widetilde{K}_1 \sim \widetilde{K}_2 \equiv \widetilde{K}.
\]

(40a)

(40b)

In addition, we consider that it is the flavor $\ell$ which has the dominant (largest) mixing $|B_{\ell N}|^2$. Then we have

\[
\widetilde{K} \approx N_{eN} |B_{eN}|^2 \sim 10 |B_{\ell N}|^2.
\]

(41)

The dominant branching ratios $Br(M)$ and $A_{CP}(M) Br(M)$ will then be, according to the obtained expressions (32)- (34) [together with the definitions (35)-(36)], those which have in the final state two equal charged leptons $\ell$ with dominant mixing: $M^\pm \rightarrow \ell^\pm \ell^\mp M^\mp$.

The theoretical branching ratios $Br(M)$ and $A_{CP}(M) Br(M)$, Eqs. (35), can be obtained by dividing Eqs. (34a)-(34b) by $2\Gamma(M^- \rightarrow \ell \ell)$. Using in addition Eqs. (40)- (41) and the definition (36), this gives

\[
Br(M) \sim 8 \frac{|B_{\ell N}|^4}{\widetilde{K}} Br(x) \sim Br(x)|B_{\ell N}|^2,
\]

(42a)

\[
A_{CP}(M)Br(M) \sim 8 \frac{|B_{\ell N}|^4}{\widetilde{K}} \sin \theta_2 \frac{\eta(y)}{y} Br(x) \sim Br(x)|B_{\ell N}|^2 \sin \theta_2,
\]

(42b)

where in the last relation we took into account that $\eta(y)/y \sim 1$ (since $\Delta M_N \gg \Gamma_N$ in our considered cases).

The effective (i.e., experimental) branching ratios $Br^{(eff)}(M) = P N Br(M)$ and $A_{CP}(M) Br^{(eff)}(M)$ can be estimated, in the considered case of Eqs. (40)- (41), in the following way [using Eqs. (37) and (42)]:

\[
Br^{(eff)}(M) \equiv P N Br(M) \sim \overline{A}(M_N) K Br(M) \sim \overline{A}(M_N) K \left( 8 \frac{|B_{\ell N}|^4}{\widetilde{K}} Br(x) \right)
\]

\[
= \left[ 8 \overline{A}(M_N) Br(x) \right] |B_{\ell N}|^4,
\]

(43a)

\[
A_{CP}(M)Br^{(eff)}(M) \equiv P N A_{CP}(M) Br(M) \sim \overline{A}(M_N) K Br_{-}(M) \sim \overline{A}(M_N) K \left( 8 \frac{|B_{\ell N}|^4}{\widetilde{K}} \sin \theta_2 \frac{\eta(y)}{y} Br(x) \right)
\]

\[
= 8 \overline{A}(M_N) |B_{\ell N}|^4 \sin \theta_2 \frac{\eta(y)}{y} Br(x) \sim \left[ 8 \overline{A}(M_N) Br(x) \right] |B_{\ell N}|^4 \sin \theta_2,
\]

(43b)
where in the last line of Eq. (43b) we took into account that $\eta(y)/y \sim 1$ (true when $\Delta M_N \gg \Gamma_N$). Furthermore, since $\ell_1 = \ell_2 = \ell$ in the considered case, the canonical branching fractions are equal: $\text{Br}^{(C)}(x) = \text{Br}^{(D)}(x) \equiv \text{Br}(x)$; and we recall that $x \equiv (M_N/M_M)^2$. We see that in Eqs. (43) the most important factor at $|B_{\text{LN}}|^4$ is the “effective” canonical branching ratio

$$\text{Br}_{\text{eff}}(M_N) \equiv 8\text{A}(M_N)\text{Br}(x).$$

Only in the case of $B^\pm$ and $B^\pm_c$ LNV decays we could have $P_N \sim 1$, Eq. (39c), and in such a case Eqs. (43) do not apply, but rather Eqs. (42). In Figs. 5-8 we present the effective canonical branching ratios (44) as a function of the neutrino mass $M_N$, for various considered LNV decays of the type $M^\pm \to \ell^\pm \ell^\mp M^\mp$, where: $M = K$ in Fig. 5, $M = D, D_s$ in Figs. 6(a), (b); $M = B, B_c$ in Figs. 7(a) and 8(a), respectively. In general $\ell = e, \mu$. We took $L = 1$ m and $\gamma_N = 2$. In addition, for the case when $P_N \sim 1$ and consequently the estimates Eqs. (42) apply, we present in Figs. 7(b) and 8(b) the theoretical branching ratios $\text{Br}(x)$ as a function of $M_N$ for $B^\pm$ and $B^\pm_c$ decays, respectively. For the CKM matrix elements and the meson decay constants, appearing in $K^2$ factor defined in Eq. (10), and for masses and lifetimes of the mesons, we used the values of Ref. [29]; and for the decay constants $f_B$ and $f_{B_c}$ we used the values of Ref. [42]: $f_B = 0.196$ GeV, $f_{B_c} = 0.322$ GeV.

![FIG. 5: The effective canonical branching ratio (44) for the $K^\pm \to \ell^\pm \ell^\mp \pi^\pm$ decays ($\ell = e, \mu$) as a function of the Majorana neutrino mass $M_N$.](image)

![FIG. 6: The effective canonical branching ratio (44) as a function of the Majorana neutrino mass $M_N$ for the LNV decays of: (a) $D^\pm$ mesons; (b) $D_s^\pm$ mesons. The solid lines are for $\ell = e$, and the dashed lines for $\ell = \mu$.](image)

---

9 Our formulas permit also evaluation of $\text{Br}_{\text{eff}}$ and $\text{Br}(x)$ for the decays $M^\pm \to \ell_1^\pm \ell_2^\mp M^\mp$ when $\ell_1 \neq \ell_2$. And also when the final leptons are $\tau$ leptons (and $M^\pm = B^\pm$ or $B^\pm_c$), with the values similar to those in Figs. 7 and 8 except that the range of $M_N$ is now significantly shorter: $M_M + M_\tau < M_N < M_M - M_\tau$. 
This is one of the preferred decay modes proposed at CERN-SPS \[33\].

In Table \[\text{IV}\] we display some values of the factor $\overline{\text{Br}}_{\text{eff}}$, for the representative values of $M_N$ in the decays $M^\pm \to \ell^\pm \ell^\mp M^\mp$. Let us now take, as an example, the decays $D_\pm \to \ell^\pm \ell^\mp \pi^\pm$, and let us assume that $|B_{\mu N}|^2$ is the dominant mixing (i.e., $\ell = \mu$). Then Eqs. \[\text{[13]}\] and Table \[\text{IV}\] imply that the effective (experimentally measurable) sum $P_N \text{Br}(D_s)$ and difference $P_N \mathcal{A}_{\text{CP}}(D_s) \text{Br}(D_s)$ of the branching ratios for these decays are

$$\overline{\text{Br}}(D_s) \equiv P_N \text{Br}(D_s) \sim 10^2 |B_{\mu N}|^4,$$

and

$$\mathcal{A}_{\text{CP}}(D_s) \overline{\text{Br}}(D_s) \equiv P_N \mathcal{A}_{\text{CP}}(D_s) \text{Br}(D_s) \sim 10^2 |B_{\ell N}|^4 \sin \theta_{21} \frac{\eta(y)}{y} \sim 10^2 |B_{\ell N}|^4 \sin \theta_{21}. \quad (45b)$$

Taking into account that in such decays the present rough upper bound on the mixing is $|B_{\mu N}|^2 \lesssim 10^{-7}$ (cf. Table \[\text{III}\]),

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$M^\pm$ & $K^\pm (\ell = e)$ & $K^\pm (\ell = \mu)$ & $D^\pm$ & $B^\pm$
\hline
8ABr & 13.5 (0.38) & 7.5 (0.35) & 8. (1.39) & 159. (1.47) & 1.93 (3.9) & 395. (4.7)
\hline
\end{tabular}
\caption{Values of the factor $8\overline{\text{Br}}(M_N)\overline{\text{Br}}(x)$ (with $L = 1$ m and $\gamma_N = 2$) for some of the considered LNV decays: $M^\pm \to \ell^\pm \ell^\mp \pi^\pm$. We chose $M_N$ such that the maximal value is obtained (this value of $M_N$ is given in parentheses, in GeV). For the $K$ decay, the two different values are given for $\ell = e$ and $\ell = \mu$. For all other decays $\ell = \mu$ is chosen (the values for $\ell = e$ are similar).}
\end{table}

\[10\] This is one of the preferred decay modes proposed at CERN-SPS \[32\].
Eqs. 45 imply that $P_N \text{Br}(D_s) \lesssim 10^{-12}$. The proposed experiment at CERN-SPS 33 would produce the numbers of $D$ and $D_s$ mesons by several orders higher than $10^{12}$ and would thus be able to explore whether there is a production of the sterile Majorana neutrinos $N_j$. Furthermore, if there are two almost degenerate neutrinos (as is the case in the $\nu$MSM model 20, 28), then in such a case it is possible that $y(= \Delta M_N/T_N) \ll 1$, and thus $\eta(y)/y \sim 1$. Then the estimate (45b) would imply that the CP-violating difference of effective branching ratios $P_{N_s}\mathcal{A}_{CP}(D_s)\text{Br}(D_s)$ is of the same order as the sum $P_N \text{Br}(D_s)$ (provided that the phase difference $|\theta_2| \ll 1$). This means that if experiments discover the aforementioned $\nu$MSM-type Majorana neutrinos, they will possibly discover also CP violation in the Majorana neutrino sector.

V. CONCLUSIONS

We investigated the possibility of detection of CP violation in lepton number violating (LNV) semihadronic decays $M^\pm \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp$, where $M$ and $M'$ are pseudoscalar mesons, $M = K, D, D_s, B, B_s$ and $M' = \pi, K, D, D_s$, and the charged leptons are $\ell_1, \ell_2 = e, \mu$. The decay widths of such decays, mediated by on-shell sterile Majorana neutrinos $N$ with masses $M_N \sim 1$ GeV, have been studied by various authors, cf. Refs. [6–13], with a view of a possible detection in future experiments such as the proposed CERN-SPS experiment 33. In the present work we investigated the possibility of detecting the CP-violating decay width difference $S_-(M) \equiv [\Gamma(M^+ \rightarrow \ell_1^- \ell_2^+ M'^-) - \Gamma(M^- \rightarrow \ell_1^+ \ell_2^- M'^+)]$ in such processes, in the scenarios of two on-shell sterile Majorana neutrinos $N_1, N_2$. We used the same approach as in our previous work 27 where CP violation was investigated in purely leptonic rare decays $\pi^\pm \rightarrow e^\pm e^\mp \mu^\mp \nu$: the crucial aspect is the expression for the imaginary part of the product of the propagators of two Majorana neutrinos, Eqs. (24). A central point, as in Ref. [27], is that when the difference of masses $\Delta M_N \equiv M_{N_2} - M_{N_1} (> 0)$ of the two sterile neutrinos becomes small enough, comparable to the (small) total decay widths of these neutrinos, $\Delta M_N \gg T_N$, the mentioned imaginary part becomes large and leads to a large CP-violating decay width difference $S_-(M)$. We show that in such a case, and provided that a specific CP-violating difference $\theta_{21}$ of the phases of heavy-light neutrino mixings is not very small ($|\theta_{21}| \ll 1$), the decay width difference $S_-(M)$ becomes comparable with the sum of the decay widths of the LNV decays $S_+(M) \equiv [\Gamma(M^- \rightarrow \ell_1^- \ell_2^+ M'^+) + \Gamma(M^+ \rightarrow \ell_1^+ \ell_2^- M'^-)]$, and the corresponding CP ratio $\mathcal{A}_{CP}(M) \equiv S_-(M)/S_+(M)$ thus becomes $\mathcal{A}_{CP}(M) \sim 1$. It is interesting that the requirement of the near degeneracy of the two sterile neutrinos (with $M_{N_1} \sim 1$ GeV), at which we arrive by requiring appreciable CP violation, fits well into the well-motivated $\nu$MSM model 20, 28, 30, where the near degeneracy of the two sterile neutrinos, with mass $M_{N_1} \sim 1$ GeV is obtained by requiring that the third (the lightest) sterile neutrino be the dark matter candidate. The results of our calculation can thus be interpreted in the framework of the $\nu$MSM model, namely that if the model is experimentally confirmed then it is possible that significant neutrino sector CP violation effects will be detected as well.

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Appendix A: Explicit formulas for the $M^\pm \rightarrow \ell_1^\pm \ell_2^\pm M'^\mp$ decay width

The matrix element $T(M^\pm)$ for the decay of Fig. 1 can be written in the form

$$T(M^\pm) = K_+ \sum_{j=1}^2 k_j^{(\pm)} M_{N_j} \left[ P_j(D) T_\pm(D) + P_{j'}(C) T_\pm(C) \right],$$

(A1)

where $j = 1, 2$ refer to the contributions of the exchanges of the two intermediate neutrinos $N_j$, and $X = D, C$ refer to the contribution of the direct and crossed channels, respectively, cf. Fig. 1. In Eq. (A1), $k_j^{(\pm)}$ are the heavy-light mixing factors defined in Eqs. (7); $P_j(X)$ ($j = 1, 2; X = D, C$) are the propagator functions of $N_j$ neutrino for the $D$ and $C$ channel, Eqs. (9), and $K_\pm$ are the constants coming from the vertices

$$K_- = -G_F^2 V_{Q_s Q_s} V_{d, q_1} d_{M, M'} \ , \quad K_+ = (K_-)^*,$$

(A2)
where \(f_M\) and \(f_{M'}\) are the decay constants of \(M^\pm\) and \(M'^\mp\), and \(V_{Q_4,Q_4}\) and \(V_{q,q_4}\) are the CKM elements for \(M^\pm\) and \(M'^\mp\): \(M^+\) has the valence quark content \(Q_u \bar{Q}_d\); \(M'^\mp\) has \(q_u \bar{q}_d\). The functions \(T_{\pm}(D)\) and \(T_{\pm}(C)\) appearing in the amplitude (A1) can be written as

\[
T_{\pm}(D) = \bar{\pi} \ell_2(p_2) p_M(p_1 + \gamma_5) v_\ell(p_1), \quad (A3a)
\]

\[
T_{\pm}(C) = \bar{\pi} \ell_2(p_2) p_M'(p_1 + \gamma_5) v_\ell(p_1), \quad (A3b)
\]

where the spinors are written in the helicity basis. Squaring and summing over the final helicities leads to the square \(|T(M^\pm)|^2\) of the total decay amplitude (A1) as given in Eq. (6) in conjunction with Eqs. (7)-(10), where the quadratic expressions \(T_{\pm}(X)T_{\pm}(Y)^*\) \((X,Y = D, C)\) appearing in the normalized decay widths \(\Gamma_{\pm}(XY^*)_{ij}\) in Eq. (8) are

\[
T_{\pm}(D)T_{\pm}(D)^* = 8[M_M' M'_M(p_1 \cdot p_2) - 2M_{M'}^2(p_1 \cdot p_M')(p_2 \cdot p_M') - 2M_{M'}^2(p_1 \cdot p_2 \cdot p_M')]
+ 4(p_2 \cdot p_M')(p_2 \cdot p_M')(p_2 \cdot p_M') \equiv T(D)T(D)^*, \quad (A4a)
\]

\[
T_{\pm}(C)T_{\pm}(C)^* = 8[M_M' M'_M(p_1 \cdot p_2) - 2M_{M'}^2(p_1 \cdot p_M')(p_2 \cdot p_M') - 2M_{M'}^2(p_1 \cdot p_M)(p_2 \cdot p_M)
+ 4(p_2 \cdot p_M')(p_2 \cdot p_M')(p_2 \cdot p_M') \equiv T(C)T(C)^*, \quad (A4b)
\]

\[
T_{\pm}(D)T_{\pm}(C)^* = 16\left\{M_M^2(p_1 \cdot p_M')(p_2 \cdot p_M') + M_{M'}^2(p_1 \cdot p_M)(p_2 \cdot p_M)
+ \frac{1}{2}M_{M'}^2M_{M'}^2(p_1 \cdot p_2)
+ (p_2 \cdot p_M')(p_1 \cdot p_M)(p_2 \cdot p_M') + (p_1 \cdot p_M)(p_2 \cdot p_M')(p_1 \cdot p_2)
\right\}
+ i(p_2 \cdot p_M')(p_2 \cdot p_M')(p_2 \cdot p_M') \equiv T(D)T(C)^* \equiv (T(C)T(D)^*)^*, \quad (A4c)
\]

where in these expressions the summation over the (final) helicities of the leptons \(\ell_1\) and \(\ell_2\) is implied, and we denoted

\[
e(\ell_1, \ell_2, \ell_3, \ell_4) \equiv \epsilon_{\ell_1\ell_2\ell_3\ell_4}\epsilon(q_1)(q_2)(q_3)(q_4), \quad (A5)
\]

and \(\epsilon_{\ell_1\ell_2\ell_3\ell_4}\) is the totally antisymmetric Levi-Civita tensor with the sign convention \(\epsilon^{0123} = +1\).

The expressions [A4], in conjunction with the definitions [8], imply for the normalized decay widths \(\Gamma_{\pm}(XY^*)_{ij}\) of Eq. [8] various symmetry relations, among them that \(\Gamma_{\pm}(DD^*)\) and \(\Gamma_{\pm}(CC^*)\) are both self-adjoint \((2 \times 2)\) matrices and that elements of the \(D-C\) interference matrices \(\Gamma_{\pm}(CD^*)\) and \(\Gamma_{\pm}(DC^*)\) are related

\[
\Gamma_{\pm}(DD^*)_{ij} = (\Gamma_{\pm}(DD^*)_{ji})^*, \quad \Gamma_{\pm}(CC^*)_{ij} = (\Gamma_{\pm}(CC^*)_{ji})^*, \quad (A6a)
\]

\[
\Gamma_{\pm}(CD^*)_{ij} = (\Gamma_{\pm}(CD^*)_{ji})^*, \quad (A6b)
\]

When the two final leptons are the same \((\ell_1 = \ell_2)\), we can use the fact that the integration \(dS\) over the final particle is symmetric under \((p_1 \leftrightarrow p_2)\) (because \(M_{\ell_1} = M_{\ell_2}\)), and we have additional symmetry relations

\[
\Gamma_{\pm}(DD^*)_{ij} = \Gamma_{\pm}(CC^*)_{ij}, \quad (A7a)
\]

\[
\Gamma_{\pm}(CD^*)_{ij} = \Gamma_{\pm}(DC^*)_{ij}, \quad (A7b)
\]

and the \((2 \times 2)\) \(D-C\) interference matrices \(\Gamma_{\pm}(CD^*)\) become self-adjoint, too.

**Appendix B: Partial decay widths of neutrino \(N\)**

The formulas for the leptonic decay and semimesonic decay widths of a sterile Majorana neutrino \(N\) have been obtained in Ref. [11] (Appendix C there), for the masses \(M_N \lesssim 1\) GeV. Nonetheless, for the higher values of the masses \(M_N\), the calculation of the semihadronic decay widths becomes increasingly complicated because not all the resonances are known. Therefore, in Refs. [12] \[43\] an inclusive approach was proposed for the calculation of the total contribution of the semihadronic decay width of \(N\), by replacing the various \(\) (pseudoscalar and vector) meson channels by quark-antiquark channels. This inclusive approach, based on duality, was applied for high masses \(M_N \lesssim M_{q'} \approx 0.958\) GeV. Here we summarize the formulas given in Ref. [12] for the decay width channels (see also: [11]). The leptonic channels


\[ 2\Gamma(N \rightarrow \ell^- \ell^+ \nu_{\ell'}) = |B_{\ell N}|^2 \frac{G_F^2}{96\pi^3} M_N^3 I_1(y_\ell, 0, y_{\ell'})(1 - \delta_{\ell\ell'}) , \]  

(B1a)

\[ \Gamma(N \rightarrow \ell^- \ell^+ \nu_{\ell'}) = |B_{\ell N}|^2 \frac{G_F^2}{96\pi^3} M_N^5 \left[ (g_L^{(\text{lept})} g_R^{(\text{lept})} + \delta_{\ell\ell'} g_R^{(\text{lept})}) I_2(0, y_{\ell'}, y_{\ell'}) \\ + \left((g_L^{(\text{lept})})^2 + (g_R^{(\text{lept})})^2 + \delta_{\ell\ell'} (1 + 2g_L^{(\text{lept})})\right) I_1(0, y_{\ell'}, y_{\ell'}) \right] , \]  

(B1b)

\[ \sum_{\nu_{\ell'}} \sum_{\ell'} \Gamma(N \rightarrow \nu_{\ell'} \ell' \nu_{\ell'}) = \sum_\ell |B_{\ell N}|^2 \frac{G_F^2}{96\pi^3} M_N^3 . \]  

(B1c)

In Eq. (B1a), factor 2 was included because both decays \( N \rightarrow \ell^- \ell^+ \nu_{\ell'} \) and \( N \rightarrow \ell^+ \ell^- \nu_{\ell'} \) contribute \( (\ell \neq \ell') \).

If \( M_N < M_{\eta'} \approx 0.968 \) GeV, the following semimeson decays contribute, involving pseudoscalar \( (P) \) and vector \( (V) \) mesons:

\[ 2\Gamma(N \rightarrow \ell^- P^+) = |B_{\ell N}|^2 \frac{G_F^2}{8\pi} M_N^3 f_P^2 |V_P|^2 F_P(y_\ell, y_P) , \]  

(B2a)

\[ \Gamma(N \rightarrow \nu_\ell P^0) = |B_{\ell N}|^2 \frac{G_F^2}{64\pi} M_N^3 f_P^2 (1 - q_P^2) , \]  

(B2b)

\[ 2\Gamma(N \rightarrow \ell^+ V^+) = |B_{\ell N}|^2 \frac{G_F^2}{8\pi} M_N^3 f_V^2 |V_V|^2 F_V(y_\ell, y_V) , \]  

(B2c)

\[ \Gamma(N \rightarrow \nu_\ell V^0) = |B_{\ell N}|^2 \frac{G_F^2}{2\pi} M_N f_V^2 \kappa_\ell^2 (1 - q_V^2)^2 (1 + 2q_V^2) , \]  

(B2d)

where factor 2 in the charged meson channels is taken because both decays \( N \rightarrow \ell^- M^+ \) and \( N \rightarrow \ell^+ M^- \) contribute \( (M' = P, V) \). The factors \( f_P \) and \( f_V \) are the corresponding CKM matrix elements involving the valence quarks of the mesons; and \( f_P \) and \( f_V \) are the corresponding decay constants. The pseudoscalar mesons which may contribute are: \( P^\pm = \pi^\pm, K^\pm; \) \( P^0 = \pi^0, K^0, \eta \). The vector mesons which may contribute are: \( V^\pm = \rho^\pm, K^{*\pm}; V^0 = \rho^0, \omega, K^{*0}, \bar{K}^{*0} \). When \( M_N \geq M_{\eta'} \approx 0.9578 \) GeV, the above semimeson decay modes are replaced \[12\], in the spirit of duality, with the following quark-antiquark decay modes:

\[ 2\Gamma(N \rightarrow \ell^- U D) = |B_{\ell N}|^2 \frac{G_F^2}{32\pi^3} M_N^4 |V_{UD}|^2 I_1(y_\ell, y_U, y_D) , \]  

(B3a)

\[ \Gamma(N \rightarrow \nu_\ell q \bar{q}) = |B_{\ell N}|^2 \frac{G_F^2}{32\pi^3} M_N^5 \left[ g_L^{(q)} g_R^{(q)} I_2(0, y_q, y_q) + \left((g_L^{(q)})^2 + (g_R^{(q)})^2\right) I_1(0, y_q, y_q) \right] . \]  

(B3b)

In the formulas (B1)-(B3) we denoted \( y_x \equiv M_X/M_N \) \( (X = \ell, \nu_\ell, P, V, q) \), and in Eqs. (B3) we denoted: \( U = u, c; D = d, s, b; q = u, d, c, s, b \). The values of quark masses which we used were: \( M_u = 140 \) MeV; \( M_d = 4.7 \) MeV; \( M_s = 1.27 \) GeV; \( M_b = 4.2 \) GeV. The SM neutral current couplings in Eqs. (B1b) and (B3b) are

\[ g_L^{(\text{lept})} = -\frac{1}{2} + \sin^2 \theta_W , \quad g_R^{(\text{lept})} = \sin^2 \theta_W , \]  

(B4a)

\[ g_L^{(U)} = \frac{1}{2} - \frac{2}{3} \sin^2 \theta_W , \quad g_R^{(U)} = -\frac{2}{3} \sin^2 \theta_W , \]  

(B4b)

\[ g_L^{(D)} = -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W , \quad g_R^{(D)} = \frac{1}{3} \sin^2 \theta_W . \]  

(B4c)

The neutral current couplings \( \kappa_V \) of the neutral vector mesons are

\[ \kappa_V = \frac{1}{3} \sin^2 \theta_W \quad (V = \rho^0, \omega) , \]  

(B5a)

\[ \kappa_V = -\frac{1}{4} + \frac{1}{3} \sin^2 \theta_W \quad (V = K^{*0}, \bar{K}^{*0}) . \]  

(B5b)

11 For the values of the decay constants \( f_P \) and \( f_V \), see, e.g., Table 1 in Ref. [12].
The kinematical expressions $I_1$, $I_2$, $F_P$ and $F_V$ are

\[
I_1(x, y, z) = 12 \int_{(x+y)^2} \frac{ds}{s} (s - x^2 - y^2) (1 + z^2 - s) \lambda^{1/2}(s, x^2, y^2) \lambda^{1/2}(1, s, z^2),
\]

\[
I_2(x, y, z) = 24 y z \int_{(y+z)^2} \frac{ds}{s} (1 + x^2 - s) \lambda^{1/2}(s, y^2, z^2) \lambda^{1/2}(1, s, x^2),
\]

\[
F_P(x, y) = \lambda^{1/2}(1, x^2, y^2) \left[ (1 + x^2)(1 + y^2 - 4z^2) \right],
\]

\[
F_V(x, y) = \lambda^{1/2}(1, x^2, y^2) \left[ (1 - x^2)^2 + (1 + x^2)y^2 - 2y^4 \right],
\]

where $\lambda$ function is written in Eq. \ref{triangle}. Using these formulas, the total decay width $\Gamma(N_j \to \text{all})$ can be calculated, and coefficients $N_{\ell N_j}$ of Eq. \ref{triangle} at the mixing terms $\mathcal{B}_{\ell N_j}$ can be evaluated and are presented in Fig. 2. The small kink in the curves of Fig. 2 at $M_N = M_{\ell N_j} (= 0.9578$ GeV) appears due to the replacement there (i.e., for $M_N = M_{\ell N_j}$) of the semi-hadronic decay channel contributions by the quark-antiquark channel contributions; we see that the duality works quite well there, with the exception of the case $\ell = \tau$ because of the large $\tau$ lepton mass.

### Appendix C: Explicit expression for the function $Q$

The expression \ref{Q} can be obtained by using in the integration over the phase space of three final particles [Eqs. \ref{1}–\ref{5}], for the contribution of the $N_j$ neutrino, the identity

\[
d_3 \left( M(p_M) \to \ell_1(p_1) \ell_2(p_2) M'(p_{M'}) \right)
= d_2 \left( M(p_M) \to \ell_1(p_1) N_j(p_N) \right) dp^2_N d_2 \left( N_j(p_N) \to \ell_2(p_2) M'(p_{M'}) \right)
= d_2 \left( M(p_M) \to \ell_2(p_2) N_j(p_N) \right) dp^2_N d_2 \left( N_j(p_N) \to \ell_1(p_1) M'(p_{M'}) \right),
\]

where the first identity can be used for the $DD^*$ contribution (where $p_N = p_M - p_1$) and the second for the $CC^*$ contribution (where $p_N = p_M - p_2$). Using the identity \ref{ident} in the $DD^*$ contribution, and the analogous identity for the $CC^*$ contribution, the integration over $dp^2_N$ becomes trivial, and the $d_2$-type of integrations are straightforward.\footnote{This is equivalent to the factorization approach $\Gamma(M \to \ell_1 N_j) \text{Br}(N_j \to \ell_2 M')$ valid when $N_j$ is on-shell.}

The resulting expression for $\Gamma(DD^*)_{ij}$ is then the expression Eq. \ref{Q} with the notations \ref{triangle} and \ref{triangle}, where the function $Q$ has the form

\[
Q(x; \ell_1, x \ell_2, x') = \left\{ \frac{1}{2} (x - x \ell_1)(x - x \ell_2)(1 - x - x \ell_1) \left( 1 - \frac{x'}{x} + \frac{x \ell_2}{x} \right) \right. \\
+ \left[ -x \ell_1 x \ell_2 (1 + x' + 2x - x \ell_1 - x \ell_2) - x \ell_1^2 (1 - x') + x \ell_2^2 (1 - x) \\
+ x \ell_1 (1 + x)(x - x') - x \ell_2 (1 - x)(x + x') \right] \right\}.
\]

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