The singular limit of a haptotaxis model with bistable growth

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Abstract
We consider a model for haptotaxis with bistable growth and study its singular limit. This yields an interface motion where the normal velocity of the interface depends on the mean curvature and on some nonlocal haptotaxis term. We prove the result for general initial data after establishing a result about generation of interface in a small time.

1 Introduction
In this article we consider a model for haptotaxis with growth. Haptotaxis is the directed motion of cells by migration up a gradient of cellular adhesion sites located in the extracellular matrix (ECM). This process appears in tumor invasion and is involved in the first stage of proliferation. It also plays an important role in wound healing.

The basic mechanism involves 3 main cellular components: the tumor cells, the Extracellular Matrix (ECM), and some Matrix Degrading Enzymes (MDE). Tumor cells migrate in response to gradients of some ECM proteins. Those ECM proteins are degraded by MDE, those enzymes being produced by tumor cells themselves. Moreover, both tumor cells and MDE diffuse in the cellular medium but ECM proteins do not diffuse.

This mechanism is reminiscent of chemotaxis, which is accounting for the directed migration of biological individuals (e.g. bacteria) towards higher gradients of some chemical substance. Chemotaxis often works as an aggregating mechanism, which is reflected in the blow-up of solutions of the Keller-Segel model, a phenomenon that has been widely studied in the recent years. However there is a major difference between chemotaxis and haptotaxis: since ECM proteins do not diffuse, instead of the elliptic or parabolic coupling appearing in chemotaxis, the haptotaxis model involves an ODE coupling between the concentration of ECM proteins and the MDE

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concentration. This is also the case in angiogenesis model (cf [8]) but models for haptotaxis involve (at least) 3 equations, whereas angiogenesis is a coupled system of 2 equations.

We now give a brief review of the mathematical literature related to haptotaxis modelling. The relevant variables are the tumor cells concentration, the Extracellular Matrix concentration (ECM) the Matrix Degrading Enzymes concentration (MDE) as well as the oxygen concentration. A hybrid model using PDEs and cellular automata has been proposed by Anderson [3], involving 4 components: tumor cells, ECM, MDE + Oxygen. Global existence for Anderson’s model has been established in [14] (Walker, Webb (07)). Our model is a simpler version from this model involving 3 components where we introduce a bistable nonlinearity to model the role of changes in oxygen concentration. A similar model of haptotaxis with a logistic nonlinearity is studied in [10] (and the references therein) and global existence is proved. Finally Chaplain, Lolas (see [6] and the references therein) proposed a combined chemotaxis-haptotaxis model with logistic source. Recent results by Y. Tao, M. Wang [11] and Y. Tao [12] show global existence for this model in dimension \( N \leq 2 \). Complex patterns in haptotaxis models are obtained numerically in [14], and also in [15] and [5].

Our starting point is the haptotaxis model proposed in [14]. In this paper, the authors prove global well-posedness for a large class of initial data, a result which strongly emphasizes the difference with Keller-Segel chemotaxis model. Here we consider a different version of this model, where we do not explicitly consider the oxygen concentration as a variable. Instead we replace it by a bistable nonlinearity in the equation for the cell concentration. Next we show that in the limit \( \varepsilon \to 0 \), the solutions converge to the solutions of a free boundary problem where the interface motion is driven by mean curvature plus an haptotaxis term.

More precisely, we study the initial value Problem (\( P^\varepsilon \))

\[
\begin{align*}
\frac{u_t}{u_t} &= \Delta u - \nabla \cdot (u \nabla \chi(v)) + \frac{1}{\varepsilon^2} f(u) \quad \text{in } \mathcal{O} \times (0, T] \\
\frac{v_t}{v_t} &= -\lambda mv \quad \text{in } \mathcal{O} \times (0, T] \\
\frac{m_t}{m_t} &= \alpha \Delta m + u - m \quad \text{in } \mathcal{O} \times (0, T] \\
u(x, 0) &= u_0(x) \quad x \in \Omega \\
v(x, 0) &= v_0(x) \quad x \in \Omega \\
m(x, 0) &= m_0(x) \quad x \in \Omega \\
\frac{\partial u}{\partial \nu} = \frac{\partial m}{\partial \nu} &= 0 \quad \text{on } \partial \mathcal{O} \times (0, T],
\end{align*}
\]

where \( \mathcal{O} \) is a smooth bounded domain in \( \mathbb{R}^N (N \geq 2) \), \( \mathcal{O}_T = \mathcal{O} \times [0, T] \) with \( T > 0 \), \( \nu \) is the exterior normal vector on \( \partial \mathcal{O} \) and \( \lambda > 0, \alpha > 0 \) are strictly positive constants.
The haptotaxis sensitivity function $\chi$ is smooth and satisfies
\[ \forall v > 0, \; \chi(v) > 0, \; \chi'(v) > 0. \]

The growth term $f$ is bistable and is given by
\[ \forall u \in \mathbb{R}, \; f(u) = u(1 - u)(u - \frac{1}{2}) \]
so that $\int_0^1 f(u)du = 0$.

We make the following assumptions about the initial data.

1. $u_0$, $v_0$ and $m_0$ are nonnegative $C^2$ functions in $\Omega$ and we fix a constant $C_0 > 1$ such that
\[ ||u_0||_{C^2(\Omega)} + ||v_0||_{C^2(\Omega)} + ||m_0||_{C^2(\Omega)} \leq C_0. \] (1.1)

2. $v_0$ satisfies the homogeneous Neumann boundary condition
\[ \frac{\partial v_0}{\partial n} = 0 \text{ on } \partial \Omega. \] (1.2)

3. The open set $\Omega_0$ defined by
\[ \Omega_0 := \{ x \in \Omega, u_0(x) > 1/2 \} \]
is connected and $\Omega_0 \subset \subset \Omega$.

4. $\Gamma_0 := \partial \Omega_0$ is a smooth hypersurface without boundary.

With these assumptions $\Omega_0$ is a domain enclosed by the initial interface $\Gamma_0$ and
\[ u_0 > 1/2 \text{ in } \Omega_0, \; 0 \leq u_0 < 1/2 \text{ in } \Omega \setminus \Omega_0. \]
The existence of a unique nonnegative solution $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ to Problem $(P^\varepsilon)$ is established in Section 2. Note that it follows from (1.2) that
\[ \frac{\partial v^\varepsilon}{\partial n} = 0 \text{ on } \partial \Omega \times [0, T]. \] (1.3)

We are interested in the asymptotic behavior of $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$. The asymptotic limit of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$ is given by the following free boundary Problem $(P^0)$

\[ \begin{cases} 
    u^0(x, t) = \chi_{\Omega_t}(x) = \begin{cases} 
        1 & \text{in } \Omega_t, t \in [0, T] \\
        0 & \text{in } \Omega \setminus \Omega_t, t \in [0, T] 
    \end{cases} \\
    v_t^0 = -\lambda m^0 v^0 \\
    m_t^0 = \alpha \Delta m^0 + u^0 - m^0 \\
    V_n = -(N - 1)\kappa + \frac{\partial \chi(v^0)}{\partial n} \\
    \Gamma_{t=0} = \Gamma_0 \\
    v^0(x, 0) = v_0(x) \\
    m^0(x, 0) = m_0(x) \\
    \frac{\partial m^0}{\partial n} = 0 \\
    \end{cases} \text{ on } \Gamma_t = \partial \Omega_t, t \in (0, T] \\
\]
where \( O_t \subset \subset \Omega \) is a moving domain, \( \Gamma_t = \partial O_t \) is the limit interface, \( n \) is the exterior normal vector on \( \Gamma_t \), \( V_n \) is the normal velocity of \( \Gamma_t \) in the exterior direction and \( \kappa \) is the mean curvature at each point of \( \Gamma_t \). We first establish the well-posedness of Problem \((P^0)\) locally in time in Section 3. Our main result is to prove rigorously the convergence of \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) to \((u^0, v^0, m^0)\) for initial data satisfying the above assumptions. In a first step, we establish the following generation of interface property.

**Theorem 1.1** Assume that \((u_0, v_0, m_0)\) satisfy the hypotheses 1-2-3-4. Let \( 0 < \eta < 1/4 \) be an arbitrary constant and define \( \mu = f'(1/2) = 1/4 \). Then there exist \( \varepsilon_0 > 0 \) and \( M_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0] \) and all \( t \in [t^\varepsilon, T] \) where \( t^\varepsilon = \mu^{-1} \varepsilon^2 |\ln \varepsilon| \),

(a) for all \( x \in \Omega \), we have

\[
0 \leq u^\varepsilon(x, t) \leq 1 + \eta;
\]

(b) for all \( x \in \Omega \) such that \(|u_0(x) - \frac{1}{2}| \geq M_0 \varepsilon \), we have

if \( u_0(x) \geq \frac{1}{2} + M_0 \varepsilon \), then \( u^\varepsilon(x, t) \geq 1 - \eta \),

if \( u_0(x) \leq \frac{1}{2} - M_0 \varepsilon \), then \( 0 \leq u^\varepsilon(x, t) \leq \eta \).

The main result reads as follows.

**Theorem 1.2** Assume that \((u_0, v_0, m_0)\) satisfy the hypotheses 1-2-3-4. Let \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) be the solution of Problem \((P^\varepsilon)\) and let \((v^0, m^0, \Gamma)\) with \( \Gamma = (\Gamma_t \times \{t\})_{t \in [0, T]} \) be the smooth solution of the free boundary Problem \((P^0)\) on \([0, T]\). Then, as \( \varepsilon \to 0 \), the solution \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) converges to \((u^0, v^0, m^0)\) almost everywhere in \( \bigcup_{0 < t \leq T} ((\Omega \setminus \Gamma_t) \times t) \). More precisely,

\[
\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = u^0(x, t) \text{ a.e. in } \bigcup_{0 < t \leq T} ((\Omega \setminus \Gamma_t) \times t),
\]

and for all \( \alpha \in (0, 1) \),

\[
\lim_{\varepsilon \to 0} \|v^\varepsilon - v^0\|_{C^{1+\alpha/(1+\alpha)}(\overline{\Omega_T})} = 0,
\]

\[
\lim_{\varepsilon \to 0} \|m^\varepsilon - m^0\|_{C^{1+\alpha/(2+\alpha)}(\overline{\Omega_T})} = 0.
\]

We actually prove a stronger convergence result concerning \( u^\varepsilon \).

**Corollary 1.3** Assume that \((u_0, v_0, m_0)\) satisfy the hypotheses 1-2-3-4. Then for any \( t \in (0, T] \),

\[
\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = \chi_{\Omega_t}(x) = \begin{cases} 
1 & \text{for } x \in \Omega_t \\
0 & \text{for } x \in \Omega \setminus \Omega_t 
\end{cases}
\]  

(1.4)
Moreover like in [1], we also obtain the following estimate of the distance between the interface $\Gamma_t$ solution of Problem $(P^0)$ and the set $\Gamma^\varepsilon_t := \{ x \in \Omega, u^\varepsilon(x,t) = 1/2 \}$.

**Theorem 1.4** There exists $C > 0$ such that

$$\Gamma^\varepsilon_t \subset \mathcal{N}_{C\varepsilon}(\Gamma_t) \text{ for } 0 \leq t \leq T,$$

where $\mathcal{N}_r(\Gamma_t) := \{ x \in \Omega, \text{dist}(x, \Gamma_t) < r \}$ is the tubular neighborhood of $\Gamma_t$ of radius $r > 0$.

**Corollary 1.5** $\Gamma^\varepsilon_t \to \Gamma_t$ as $\varepsilon \to 0$, uniformly in $t \in [0,T]$ in the sense of the Hausdorff distance.

The organization of the paper is as follows. In section 2 we prove some a priori estimates, establish a comparison principle for Problem $(P^\varepsilon)$ and prove the existence of a unique global solution. In section 3 we prove the well-posedness of the free boundary problem $(P^0)$ and obtain the existence of a smooth unique solution up to some time $T > 0$. In section 4 we establish the property of generation of interface. Finally in section 5 we prove the convergence of the solution of Problem $(P^\varepsilon)$ to the solution of Problem $(P^0)$.

## 2 A priori estimates and comparison principle

### 2.1 A priori estimates

For a given $T > 0$ and a given nonnegative function $u_0 \in C^2(\Omega)$, we define

$$X_T = \{ u \in C^0(\Omega_T), \quad 0 \leq u \leq C_0 \text{ in } \Omega_T \text{ and } u(x,0) = u_0(x) \},$$

where $C_0 > 1$ is the constant defined in (1.1). It is convenient to rewrite Problem $(P^\varepsilon)$ as an evolution equation for $u$ with a nonlocal coefficient $H(u) = v$, namely

$$
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \nabla \cdot (u \nabla \chi(H(u))) + \frac{1}{\varepsilon^2} f(u) \quad \text{in } \Omega \times (0,T) \\
    u(x,0) &= u_0(x), \quad x \in \Omega \\
    \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0,T),
\end{cases}
\end{align*}
$$

(2.1)

where for a given function $u = u(x,t) \in X_T$, we define $H(u) = v$ as the first component of the unique solution $(v, m)$ of the auxiliary problem

$$
\begin{align*}
\begin{cases}
    v_t &= -\lambda mv \quad \text{in } \Omega \times (0,T) \\
    m_t &= \alpha \Delta m + u - m \quad \text{in } \Omega \times (0,T) \\
    v(x,0) &= v_0(x), \quad x \in \Omega \\
    m(x,0) &= m_0(x), \quad x \in \Omega \\
    \frac{\partial m}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0,T),
\end{cases}
\end{align*}
$$

(2.2)
The functions \( v_0 \) and \( m_0 \) are given and satisfy 1-2. We give below some a priori estimates on the solution to Problem \( (P^\varepsilon) \) and state the related properties of \( H \).

**Lemma 2.1** For \( u \in X_T \), let \((v, m)\) be the solution of Problem \( (2.2) \) and let \( H : X_T \to C^2(\overline{\Omega_T}) \) be the operator defined by \( H(u) = v \). Then there exists \( C > 0 \) only depending on \( T \) and \( \Omega \) such that

(a) for all \((u_1, u_2)\) \( \in X_T^2 \) with \( 0 \leq u_1 \leq u_2 \) in \( \Omega_T \), the solution \((v_i, m_i)\) of Problem \( (2.2) \) for \( i = 1, 2 \) satisfies

\[
0 \leq m_1 \leq m_2 \text{ and } v_2 \leq v_1 \text{ in } \Omega_T
\]

so that the operator \( H \) is nonincreasing on \( X_T \).

(b) for all \( u \in X_T \),

\[
||m||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega_T})} \leq CC_0 \text{ and } \sup_{(x,t) \in \overline{\Omega_T}} \left| \int_0^t \Delta m(x,s)ds \right| \leq CC_0.
\]

(c) for all \( u \in X_T \), the function \( v = H(u) \) satisfies

\[
||v||_{C^0(\overline{\Omega_T})} \leq C_0 \quad \text{and} \quad ||\nabla v||_{C^0(\overline{\Omega_T})} + ||\Delta v||_{C^0(\overline{\Omega_T})} \leq CC_0^3.
\]

**Proof.** To prove property (a), let \((u_1, u_2)\) \( \in X_T^2 \) with \( 0 \leq u_1 \leq u_2 \) in \( \Omega_T \). Since for \( i = 1, 2 \)

\[
(m_i)_t - \alpha \Delta m_i + m_i = u_i \geq 0 \text{ in } \Omega_T,
\]

with

\[
m_i|_{t=0} = m_0 \geq 0 \text{ and } \frac{\partial m_i}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, T),
\]

we deduce from the standard maximum principle that \( 0 \leq m_1 \leq m_2 \) in \( \Omega_T \). Next solving the equation \( v_i = -\lambda mv \) we get that

\[
v_i(x,t) = v_0(x)e^{-\lambda \int_0^t m_i(x,s)ds} \quad (2.3)
\]

for all \((x,t) \in \Omega_T \) and \( i = 1, 2 \), so that \( v_1 \geq v_2 \geq 0 \) in \( \Omega_T \), which proves that \( H \) is nonincreasing on \( X_T \).

In order to prove (b) and (c), note that \( m \) satisfies the linear parabolic equation

\[
\begin{cases}
m_t = \alpha \Delta m + u - m & \text{in } \Omega \times (0, T) \\
m(x,0) = m_0(x) & x \in \Omega \\
\frac{\partial m}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T)
\end{cases}
\quad (2.4)
\]
with $0 \leq u \leq C_0$ in $\Omega_T$ and $m_0 \geq 0 \in \Omega$. Thus it follows from the maximum principle and from standard parabolic estimates that there exists a constant $C > 0$ only depending on $T$ and $\Omega$ such that

$$0 \leq m \leq C_0 \text{ in } \Omega_T, \quad ||m||_{C^{1+\alpha, (1+\alpha)/2}(\overline{\Omega_T})} \leq CC_0. \quad (2.5)$$

In view of (2.3), $v \geq 0$ and $v_t \leq 0$ in $\Omega_T$ so that

$$0 \leq v(x, t) \leq v_0(x) \leq C_0 \text{ for all } (x, t) \in \Omega_T. \quad (2.6)$$

Since for all $(x, t) \in \Omega_T$

$$\nabla v(x, t) = \nabla v_0(x)e^{-\lambda \int_0^t m(x, s)ds} - \lambda v(x, t)\left(\int_0^t \nabla m(x, s)ds\right), \quad (2.7)$$

it follows that there exists $C > 0$ such that

$$||\nabla v(x, t)|| \leq ||\nabla v_0(x)|| + \lambda v(x, t)\left|\int_0^t \nabla m(x, s)ds\right| \leq CC_0^2 \quad (2.8)$$

Since for all $(x, t) \in \Omega_T$

$$\Delta v(x, t) = \Delta v_0(x)e^{-\lambda \int_0^t m(x, s)ds} - 2\lambda \nabla v_0(x).\left(\int_0^t \nabla m(x, s)ds\right)e^{-\lambda \int_0^t m(x, s)ds}$$

$$+ \lambda^2 v(x, t)\left|\int_0^t \nabla m(x, s)ds\right|^2 - \lambda v(x, t)\left(\int_0^t \Delta m(x, s)ds\right) \quad (2.9)$$

it follows that

$$\forall (x, t) \in \Omega_T, \quad |\Delta v(x, t)| \leq CC_0^3 + \lambda C_0\left|\int_0^t \Delta m(x, s)ds\right| \quad (2.10)$$

with $C > 0$ a suitable constant.

For any fixed $x \in \partial \Omega$, we integrate the equation $m_t - \alpha \Delta m + m = u$ on $[0, t]$ and obtain that

$$\int_0^t \Delta m(x, s)ds = \frac{1}{\alpha}[m(x, t) - m_0(x) + \int_0^t (m(x, s) - u(x, s))ds]$$

so that in view of (2.5) there exists a constant $C > 0$ such that

$$\forall (x, t) \in \Omega_T, \quad |\int_0^t \Delta m(x, s)ds| \leq CC_0 \quad (2.11)$$

which completes the proof of $(b)$. Moreover in view of (2.10) and (2.11), we conclude that there exists $C > 0$ such that

$$\forall (x, t) \in \Omega_T, \quad |\Delta v(x, t)| \leq CC_0^3$$

and obtain the property $(c)$, which completes the proof of Lemma 2.1.
2.2 Existence of a global solution to Problem \((P^\varepsilon)\)

We prove the existence of a unique solution \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) to Problem \((P^\varepsilon)\) on \(\Omega_T\) for \(\varepsilon > 0\) small enough.

**Lemma 2.2** Assume that \((u_0, v_0, m_0)\) satisfy the hypotheses 1-2-3-4. Then there exists \(\varepsilon_0 > 0\) such that for all \(0 < \varepsilon < \varepsilon_0\), Problem \((P^\varepsilon)\) has a unique solution \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) on \(\Omega \times [0, T]\) for any \(T > 0\). This solution satisfies

\[0 \leq u^\varepsilon \leq C_0\] in \(\Omega_T\).

The above lemma is similar to Lemma 4.2 in [4] and we just sketch the proof. It relies on Schauder’s fixed point theorem and on the a priori estimates on Problem \((P^\varepsilon)\) obtained in Lemma 2.1.

First let \(T > 0\) be arbitrarily fixed and for all \(u \in X_T\), let \(v = H(u)\) be defined as above. By the estimates of \(v\) in Lemma 2.1, there exists \(C_0 > 0\) such that

\[0 \leq v \leq C_0, \quad |\nabla v| + |\nabla v| \leq C C_0^3\] in \(\Omega_T\). \hfill (2.12)

Next let \(\tilde{u}\) be the unique solution of

\[
\begin{cases}
\tilde{u}_t = \Delta \tilde{u} - \nabla \cdot (\tilde{u} \nabla \chi(v)) + \frac{1}{\varepsilon^2} f(\tilde{u}) & \text{in } \Omega \times (0, T) \\
\tilde{u}(x, 0) = u_0(x) & x \in \Omega \\
\frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\] \hfill (2.13)

The key point of the proof is to show that for \(0 < \varepsilon < \varepsilon_0\) small enough, we have

\[0 \leq \tilde{u} \leq C_0\] in \(\Omega_T\).

This follows from the fact that \(C_0\) is a supersolution for equation (2.13) for \(\varepsilon > 0\) small enough. Precisely, using that \(f(C_0) < 0\) since \(C_0 > 1\) and (2.12), we have that

\[
C_0 \Delta (\chi(v)) - \frac{1}{\varepsilon^2} f(C_0)
\]

\[
= C_0 \chi'(v) \Delta (v) + \chi''(v)|\nabla (v)|^2 - \frac{1}{\varepsilon^2} f(C_0)
\]

\[
\geq -2C_0^3 \frac{1}{\varepsilon^2} f(C_0) \geq 0
\]

for \(\varepsilon > 0\) small enough. Moreover \(\tilde{u} \in C^{\alpha, \alpha/2}(\overline{\Omega_T})\) for some \(\alpha \in (0, 1)\). Hence \(u \rightarrow \tilde{u}\) maps \(X_T\) into itself and defines a compact operator. A fixed point of this operator obtained by Schauder’s theorem is then a solution to Problem \((P^\varepsilon)\). The uniqueness of solution follows from the a priori estimates on Problem \((P^\varepsilon)\). For the details of the proof, we refer to [4] and [9].
2.3 A comparison principle for Problem \((P_\varepsilon)\)

We first recall the definition of a pair of sub- and super-solutions similar to the one proposed in [4].

**Definition 2.3** Let \((u^-_\varepsilon, u^+_\varepsilon)\) be two smooth functions with \(0 \leq u^-_\varepsilon \leq u^+_\varepsilon\) in \(\Omega_T\) and \(\frac{\partial u^-_\varepsilon}{\partial \nu} \leq \frac{\partial u^+_\varepsilon}{\partial \nu}\) on \(\partial \Omega \times (0,T)\). By definition, \((u^-_\varepsilon, u^+_\varepsilon)\) is a pair of sub- and super-solutions in \(\Omega_T\) if for any \(v = H(u)\), with \(u^-_\varepsilon \leq u \leq u^+_\varepsilon\) in \(\Omega_T\), we have

\[
L_v[u^-_\varepsilon] \leq 0 \leq L_v[u^+_\varepsilon]
\]

in \(\Omega_T\), where the operator \(L_v\) is defined by

\[
L_v[\phi] = \phi_t - \Delta \phi + \nabla \cdot (\phi \nabla \chi(v)) - \frac{1}{\varepsilon^2} f(\phi).
\]

Note that in Lemma 2.2, \((0,C_0)\) is a pair of sub- and super-solutions of Problem \((P_\varepsilon)\). It is then proved in [4] that the following comparison principle holds.

**Proposition 2.4** Let a pair of sub- and super-solutions \((u^-_\varepsilon, u^+_\varepsilon)\) in \(\Omega_T\) be given. Assume that

\[
\forall x \in \Omega, \quad u^-_\varepsilon(x,0) \leq u_0(x) \leq u^+_\varepsilon(x,0),
\]

with \((u_0, v_0, m_0)\) satisfying the hypotheses 1-2. Then there exists a unique solution \((u^\varepsilon, v^\varepsilon, m^\varepsilon)\) of Problem \((P_\varepsilon)\) with

\[
\forall (x,t) \in \Omega_T, \quad u^-_\varepsilon(x,t) \leq u^\varepsilon(x,t) \leq u^+_\varepsilon(x,t).
\]

3 Well-posedness of Problem \((P^0)\)

We establish here the existence and uniqueness of a smooth solution to the free boundary Problem \((P^0)\) locally in time.

**Theorem 3.1** Let \(\Gamma_0 = \partial \Omega_0\), where \(\Omega_0 \subset \subset \Omega\) is a \(C^{2+\alpha}\) domain with \(\alpha \in (0,1)\). Then there exists a time \(T > 0\) such that Problem \((P^0)\) has a unique solution \((v^0, m^0, \Gamma)\) on \([0,T]\) with

\[
\Gamma = (\Gamma_t \times \{t\})_{t \in [0,T]} \in C^{2+\alpha,(2+\alpha)/2} \quad \text{and} \quad v^0|_\Gamma \in C^{1+\alpha, (1+\alpha)/2}
\]

This theorem is similar to Theorem 2.1 in [4] and is using a contraction fixed-point argument in suitable Hölder spaces (see Section 2 in [4]). We show here how it can actually be obtained using the result established in Theorem 2.1 in [4] and some additional properties that we state and prove below.

First we introduce some notations as in [4]. We assume that \(\Gamma_0\) is parametrized
by some smooth \((N-1)\)-dimensional compact manifold \(\mathcal{M}\) without boundaries which divides \(\mathbb{R}^N\) into two pieces. We denote by \(\vec{N}(s)\) the outward normal vector to \(\mathcal{M}\) at \(s \in \mathcal{M}\) and define

\[
X : \mathcal{M} \times (-L, +L) \to \mathbb{R}^N \quad (s, s_N) \mapsto X(s, s_N)
\]

where

\[
X(s, s_N) = s + s_N\vec{N}(s).
\]

If \(L > 0\) is chosen small enough, \(X\) is a \(C^\infty\)-diffeomorphism from \(\mathcal{M} \times (-L, +L)\) onto a tubular neighborhood of \(\mathcal{M}\) that we denote by \(\mathcal{M}^L\). We assume that \(\Gamma_0 \subset \mathcal{M}^L\) and is given by

\[
\Gamma_0 = \{X(s, s_N), s_N = \Lambda_0(s), s \in \mathcal{M}\}
\]

and that \(\Omega_0\) is the connected component of \(\Omega \setminus \Gamma_0\) which contains

\[
\{x = X(s, s_N), s_N < \Lambda_0(s), s \in \mathcal{M}\}.
\]

According to the regularity hypothesis on \(\Gamma_0\) in Theorem 3.1, \(\Lambda_0\) is a \(C^{2+\alpha}\) function with

\[
||\Lambda_0||_{C^0(\mathcal{M})} < \frac{L}{2}.
\]

Let \(T > 0\) be a fixed constant that will be chosen later. We parametrize the interface \(\Gamma = (\Gamma_t)_{t \in [0, T]}\) as follows

\[
\Gamma_t = \{X(s, s_N), s_N = \Lambda(s, t), s \in \mathcal{M}\},
\]

where \(\Lambda : \mathcal{M} \times [0, T] \to (-L, +L)\) is a function. By definition, we will say that \(\Gamma\) is \(C^{m+\alpha, \frac{m+\alpha}{2}}\) if the function \(\Lambda\) satisfies

\[
\Lambda \in C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{M} \times [0, T])
\]

For any function \(v(x, t)\) defined in \(\Omega_T\), we consider the restriction of \(v\) and of \(\nabla v\) on the interface \(\Gamma\) and we associate to \(v\) the functions \(w(s, t)\) and \(\vec{h}(s, t)\) defined on \(\mathcal{M} \times [0, T]\) by

\[
w(s, t) = v(X(s, \Lambda(s, t)), t),
\]

\[
\vec{h}(s, t) = \nabla v(X(s, \Lambda(s, t)), t).
\]

Next we split Problem \((P_0)\) into two subproblems \((p_a)\) and \((p_b)\), where Problem \((p_a)\) is given by

\[
(p_a) \quad \left\{ \begin{array}{l}
V_n = -(N-1)\kappa + \chi'(w)\vec{h} \cdot \vec{n} \text{ on } \Gamma_t = \partial\Omega_t, \ t \in (0, T] \\
\Gamma_{t=0} = \Gamma_0
\end{array} \right.
\]
and Problem \((p_b)\) is given by

\[
\begin{align*}
\begin{cases}
v^0_t &= -\lambda m^0 v^0 & \text{in } \Omega \times (0,T) \\
m^0_t - \alpha \Delta m^0 + m^0 &= u^0 & \text{in } \Omega \times (0,T) \\
\frac{\partial m^0}{\partial \nu} &= 0 & \text{on } \partial \Omega \times (0,T) \\
u^0(x,t) &= \chi_{\Omega_t}(x) = \begin{cases} 1 & \text{in } \Omega_t, t \in [0,T] \\
0 & \text{in } \Omega \setminus \Omega_t, t \in [0,T] \end{cases}
\end{cases}
\end{align*}
\]

(3.5)

Note that the difference between the free boundary problem in [4] and here concerns Problem \((p_b)\). Let us consider

\[
\forall (x,t) \in \Omega_T, \ M(x,t) = \int_0^t m^0(x,s)ds
\]

(3.6)

The restrictions of \(M\) and \(\nabla M\) on \(\Gamma\) are denoted \(a(s,t)\) and \(\vec{b}(s,t)\) and defined on \(M \times [0,T]\) by

\[
\begin{align*}
a(s,t) &= M(X(s,\Lambda(s,t)),t), \\
\vec{b}(s,t) &= \nabla M(X(s,\Lambda(s,t)),t).
\end{align*}
\]

(3.7, 3.8)

Note that using (2.3) and (2.7) we have that

\[
w(s,t) = v_0(X(s,\Lambda(s,t)))e^{-\lambda a(s,t)}
\]

and

\[
\vec{h}(s,t) = \nabla v_0(X(s,\Lambda(s,t)))e^{-\lambda a(s,t)} - \lambda w(s,t)\vec{b}(s,t),
\]

so that \(w\) has the same regularity as \(a\) and \(\vec{h}\) has the same regularity as \(\vec{b}\).

We deduce from Problem \((p_b)\) that \(M\) satisfies

\[
\begin{align*}
\begin{cases}
-\alpha \Delta M + M &= g(x,t) & \text{in } \Omega \times (0,T) \\
\frac{\partial M}{\partial \nu} &= 0 & \text{on } \partial \Omega \times (0,T),
\end{cases}
\end{align*}
\]

(3.9)

where

\[
g(x,t) = \int_0^t u^0(x,s)ds + m_0(x) - m^0(x,t).
\]

The same problem \(3.9\) has been considered in [4] but with a right-hand-side \(g = u^0\). Here the function \(g(x,t)\) is continuous in time, its regularity being the one of a time-integral of \(u^0\). Thus we can use Theorem 2.2 in [4] and obtain (at least) the same regularity for \((a,\vec{b})\) in the case considered here.

**Lemma 3.2** Let \(\Gamma = \{\Gamma_t \times \{t\}\}_{t \in [0,T]}\) be given by (3.1) with

\[
\Lambda \in C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{M} \times [0,T])
\]
for some $m \in \mathbb{N}$, $m \geq 2$ and $\alpha \in (0,1)$. Let $M$ satisfy (3.9) and let $a$ and $\bar{b}$ be associated to $M$ by (3.7) and (3.8) respectively. Then
\[ a \in C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{M} \times [0,T]) \]
and
\[ \bar{b} \in [C^{m+\alpha', \frac{m+\alpha'}{2}}(\mathcal{M} \times [0,T])]^n \text{ for all } 0 < \alpha' < \alpha. \]

By the argument in [4] we know then that Problem ($p_a$) defines a mapping $(w, \vec{h}) \mapsto \Lambda$ and Problem ($p_b$) defines a mapping $\Lambda \mapsto (w, \vec{h})$ with the proper regularity in Hölder spaces. Therefore the composition of these two mappings defines a contraction in some closed ball for $T > 0$ small enough. The unique fixed point of this contraction is the solution to Problem ($P^0$) on $[0,T]$. This completes the proof of Theorem 3.1.

4 Generation of interface

In this section we establish the rapid formation of transition layers in a neighborhood of $\Gamma_0$ within a very short time interval of order $\varepsilon^2|\ln \varepsilon|$. The width of the transition layer around $\Gamma_0$ is of order $\varepsilon$. After a short time the solution $u^\varepsilon$ becomes close to 1 or 0 except in a small neighborhood of $\Gamma_0$. It reads precisely as follows.

**Theorem 4.1** Let $u_0$ satisfy the assumptions 1-2-3-4. Let $0 < \eta < 1/4$ and define $\mu = f'(1/2) = 1/4$. Then there exist $\varepsilon_0 > 0$ and $M_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ and $t^* = \mu^{-1} \varepsilon^2 |\ln \varepsilon|$,\n
(a) for all $x \in \mathcal{O}$, we have
\[-\eta \leq u^\varepsilon(x, t^*) \leq 1 + \eta;\]

(b) for all $x \in \mathcal{O}$ such that $|u_0(x) - \frac{1}{2}| \geq M_0 \varepsilon$, we have
\[
\text{if } u_0(x) \geq \frac{1}{2} + M_0 \varepsilon, \text{ then } u^\varepsilon(x, t^*) \geq 1 - \eta, \\
\text{if } u_0(x) \leq \frac{1}{2} - M_0 \varepsilon, \text{ then } u^\varepsilon(x, t^*) \leq \eta.\]

The above theorem relies on the construction of a suitable pair of sub- and super-solutions involving the solution of the bistable ODE. We refer to the proof of Theorem 3.1 in [H] in the simple case $\delta = 0$.

5 Convergence

We split the present section into 2 parts. In a first step we establish the convergence of the $u^\varepsilon$ to $u^0$ and prove Corollary 1.3. In a second step we prove Theorem 1.2 as well as Theorem 1.1, Theorem 1.4 and Corollary 1.5.
In what follows, we construct a pair of sub- and super-solution \( u^\pm \) for Problem \( (P^\varepsilon) \) in order to control the function \( u^\varepsilon \) on \([t^*, T] \). By the comparison principle it then follows that, if \( u^-_\varepsilon(x, 0) \leq u^\varepsilon(x, t^*) \leq u^+_\varepsilon(x, 0) \), then \( u^-_\varepsilon(x, t) \leq u^\varepsilon(x, t + t^*) \leq u^+_\varepsilon(x, t) \) for all \((x, t) \in \partial T\). As a result, if both \( u^+_\varepsilon \) and \( u^-_\varepsilon \) converge to \( u^0 \), the solution \( u^\varepsilon \) also converge to \( u^0 \) for all \((x, t) \in \partial T \setminus \Gamma\).

### 5.1 Construction of sub- and super-solutions

Before the construction, we present the definition of the modified signed distance function which is essential for our construction of sub- and super-solutions. Let us first define the signed distance function.

**Definition 5.1** Let \( \Gamma = \bigcup_{0 \leq t \leq T} (\Gamma_t \times t) \) be the solution of the limit geometric motion Problem \( (P^0) \). The signed distance function \( \tilde{d}(x, t) \) is defined by

\[
\tilde{d}(x, t) = \begin{cases} 
\text{dist}(x, \Gamma_t) & \text{for } x \in \Omega \setminus \Omega_t, \\
-\text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t,
\end{cases}
\]

where \( \text{dist}(x, \Gamma_t) \) is the distance from \( x \) to the hyperface \( \Gamma_t \) in \( \Omega \).

Note that \( \tilde{d}(x, t) = 0 \) on \( \Gamma \) and that \(|\nabla \tilde{d}(x, t)| = 1 \) in a neighborhood of \( \Gamma \).

In fact, rather than working with the above signed distance function \( \tilde{d}(x, t) \), we need a modified signed distance function \( d \) defined as follows.

**Definition 5.2** Let \( d_0 > 0 \) small enough such that \( \tilde{d}(x, t) \) is smooth in

\[ \{(x, t) \in \overline{\Omega} \times [0, T], |\tilde{d}(x, t)| < 3d_0\} \]

and such that for all \( t \in [0, T] \),

\[ \text{dist}(\Gamma_t, \partial \Omega) > 4d_0. \]

We define the modified signed distance function \( d(x, t) \) by

\[ d(x, t) = \zeta(\tilde{d}(x, t)), \]

where \( \zeta(s) \) is a smooth increasing function on \( \mathbb{R}^N \) defined by

\[
\zeta(s) = \begin{cases} 
s & \text{if } |s| \leq 2d_0 \\
-3d_0 & \text{if } s \leq -3d_0 \\
3d_0 & \text{if } s \geq 3d_0.
\end{cases}
\]

Note that \(|\nabla d| = 1 \) in the region \( \{|d(x, t)| < 2d_0, (x, t) \in \overline{\Omega} \times [0, T]\} \). It follows that at \( x \in \Gamma_t \), the exterior normal vector is \( n(x, t) = \nabla d(x, t) \), the normal velocity is \( V_n(x, t) = -d_t(x, t) \) and the mean curvature is \( K = \nabla \cdot N_{\Gamma_t} \Delta d(x, t) \). Therefore the motion law on \( \Gamma^t \) given by Problem \( (P^0) \) reads

\[ d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) = 0 \text{ on } \Gamma_t = \{x \in \Omega \mid d(x, t) = 0\}. \]
By Theorem 3.1, the interface \( \Gamma_t \) is of class \( C^{2+\alpha, \frac{2+\alpha}{2}} \) and \( \nu^0 \) is of class \( C^{1+\alpha', \frac{1+\alpha'}{2}} \) for any \( \alpha, \alpha' \in (0,1) \), all the functions \( d_t, \Delta d, \nabla d \) are Lipschitz continuous near \( \Gamma_t \) and \( \nabla \chi(\nu^0) \) is continuous near \( \Gamma_t \). Therefore from the mean value theorem applied separately on both sides of \( \Gamma_t \), it follows that there exists \( N_0 > 0 \) such that

\[
\forall (x,t) \in \Omega_T, \quad |d_t - \Delta d + \nabla \chi(\nu^0)| \leq N_0|d(x,t)|.
\] (5.4)

Note also that by construction, \( \nabla d(x,t) = 0 \) in a neighborhood of \( \partial \Omega \).

As in [1], the sub- and super-solutions \( u_\pm \) are defined by

\[
u^0(x,t) = U_0\left(d(x,t) \mp \varepsilon p(t) \varepsilon\right) \pm q(t),
\] (5.5)

where \( U_0(z) \) is the unique solution of the stationary problem

\[
\begin{cases}
U_0'' + f(U_0) = 0 \\
U_0(-\infty) = 1, U_0(0) = \frac{1}{2}, U_0(+\infty) = 0
\end{cases}
\] (5.6)

and

\[
p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K
\]

\[
q(t) = \sigma(e^{-\beta t/\varepsilon^2} + \varepsilon^2 Le^{Lt})
\]

with \( L > 0 \) and \( K > 1 \) to be chosen later.

First note that \( q = \varepsilon^2 \sigma p_t \), then remark that for Problem (5.6) the unique solution \( U_0 \) has the following properties.

**Lemma 5.3** There exist the positive constants \( C \) and \( \lambda \) such that the following estimates hold:

\[
0 < U_0(z) \leq Ce^{-\lambda|z|} \quad \text{for } z \geq 0,
\]

\[
0 < 1 - U_0(z) \leq Ce^{-\lambda|z|} \quad \text{for } z \leq 0.
\]

In addition, \( U_0 \) is strictly decreasing and \( |U_0'(z)| + |U_0''(z)| \leq Ce^{-\lambda|z|} \) for all \( z \in \mathbb{R} \).

The proof of Lemma 5.4 is given in [1]. We also note that

\[
u^0(x,t) = U_0\left(d(x,t) / \varepsilon\right) \leq u_\pm(x,t)
\]

and that \( p(t) \) is bounded for all \( 0 < \varepsilon < \varepsilon_0 \) and \( t \in [0, T] \), \( \lim_{\varepsilon \to 0} q(t) = 0 \) for all \( t > 0 \). Therefore it follows from the definition of \( u_\pm(x,t) \) that for all \( t \in (0, T) \),

\[
\lim_{\varepsilon \to 0} u_\pm(x,t) = \chi_{\Omega_t}(x) = \begin{cases}
1 & \text{for all } (x,t) \in \Omega_t \\
0 & \text{for all } (x,t) \in \Omega \setminus \Omega_t
\end{cases}
\] (5.7)

The key result of this section is the following lemma.

**Lemma 5.4** There exist \( \beta > 0, \sigma > 0 \) such that for all \( K > 1 \), we can find \( \varepsilon_0 > 0 \) and \( L > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), \( (u^-_\varepsilon, u^+_\varepsilon) \) is a pair of sub- and super-solutions for Problem \((P^\varepsilon)\) in \( \overline{\Omega} \times [0,T] \).
5.2 Proof of Lemma 5.4

First note that for all \((x, t) \in \partial\Omega\),
\[
u - \varepsilon \leq U_0 \leq \nu + \varepsilon \leq u_+^\varepsilon(x, t).
\]

Next since \(\nabla d = 0\) in a neighborhood of \(\partial\Omega\), we have that \(\frac{\partial u_+^\varepsilon}{\partial \nu} = 0\) on \(\partial\Omega \times [0, T]\). Let \(v\) be such that \(v = H(u)\) with \(u_-^\varepsilon \leq u \leq u_+^\varepsilon\) in \(\partial\Omega\), we show below that
\[
L_v[u_-^\varepsilon] \leq 0 \leq L_v[u_+^\varepsilon],
\]
where the operator \(L_v\) is defined by
\[
L_v[\phi] = \phi_t - \Delta \phi + \nabla(\phi \nabla(v)) - \frac{1}{\varepsilon^2} f(\phi).
\]

Here we just consider the inequality \(L_v[u_+^\varepsilon] \geq 0\), because the proof of the other inequality \(L_v[u_-^\varepsilon] \leq 0\) is obtained by similar arguments. A direct computation gives us the following terms
\[
(u_+^\varepsilon)_t = U_0' \left( \frac{d_t}{\varepsilon} - p_t \right) + q_t,
\]
\[
\nabla u_+^\varepsilon = U_0' \nabla d, \quad \Delta u_+^\varepsilon = U_0'' \frac{|\nabla d|^2}{\varepsilon^2} + U_0' \frac{\Delta d}{\varepsilon},
\]
where the value of the function \(U_0\) and its derivatives are taken at the point \(\frac{d(x, t)}{\varepsilon} - \varepsilon p(t)\). Moreover the bistable function has the expansions
\[
f(u_+^\varepsilon) = f(U_0) + qf'(U_0) + \frac{1}{2} q^2 f''(\theta),
\]
where \(\theta(x, t)\) is a function satisfying \(U_0 < \theta < u_+^\varepsilon\). Hence, combining all the above, we obtain that
\[
L_v[u_+^\varepsilon] = (u_+^\varepsilon)_t - \Delta u_+^\varepsilon + \nabla u_+^\varepsilon \nabla(\chi(v)) + u_+^\varepsilon \Delta \chi(v) - \frac{1}{\varepsilon^2} f(u_+^\varepsilon) = E_1 + E_2 + E_3 + E_4
\]
where
\[
E_1 = -\frac{1}{\varepsilon^2} q[f'(U_0) + \frac{1}{2} q f''(\theta)] - U_0' p_t + q_t,
\]
\[
E_2 = \frac{U_0''}{\varepsilon^2} (1 - |\nabla d|^2),
\]
\[
E_3 = \frac{U_0'}{\varepsilon} (d_t - \Delta d + \nabla d \cdot \nabla \chi(v_0)),
\]
\[
E_4 = \frac{U_0'}{\varepsilon} (d_t - \Delta d + \nabla d \cdot \nabla \chi(v_0)),
\]
\[ E_4 = \frac{U_0'}{\varepsilon} \nabla d \cdot \nabla (\chi(v) - \chi(v^0)) + u_2^+ \Delta \chi(v). \]

We first need to present some useful inequalities before estimating the four terms above, this step is exactly the same as in [1].

Since \( f'(0) = f'(1) = -\frac{1}{2} \), we can find \( 0 < b < 1/2 \) and \( m > 0 \) such that

\[
\text{if } U_0(z) \in [0, b] \cup [1 - b, 1], \text{ then } f'(U_0(z)) \leq -m.
\]

Furthermore, since the region \( \{ z \in \mathbb{R}, U_0(z) \in [b, 1 - b] \} \) is compact and \( U_0' < 0 \) on \( \mathbb{R} \), there exists a constant \( a_1 > 0 \) such that

\[
\text{if } U_0(z) \in [b, 1 - b] \text{ then } U_0'(z) \leq -a_1.
\]

Now we define

\[
F = \sup_{-1 \leq z \leq 2} (|f(z)| + |f'(z)| + |f''(z)|),
\]

\[
\beta = \frac{m}{4}, \quad (5.8)
\]

and choose \( \sigma \) which satisfies

\[
0 < \sigma < \min(\sigma_0, \sigma_1, \sigma_2), \quad (5.9)
\]

where \( \sigma_0 = \frac{a_1}{m + F}, \sigma_1 = \frac{1}{\beta + 1}, \sigma_2 = \frac{4\beta}{F(\beta + 1)} \). Hence we obtain that

\[
\forall z \in \mathbb{R}, -U_0'(z) - \sigma f'(U_0(z)) \geq 4\sigma \beta.
\]

Now we have already chosen the appropriate \( \beta \) and \( \sigma \). Let \( K > 1 \) be arbitrary, next we prove that \( L \varepsilon \varepsilon_0 |u_2^+| \geq 0 \) provided that the constants \( \varepsilon_0 > 0 \) and \( L > 0 \) are appropriately chosen. From now on, we suppose that the following inequality is satisfied

\[
\varepsilon_0^2 Le^{\frac{L}{T}} \leq 1. \quad (5.10)
\]

Then given any \( \varepsilon \in (0, \varepsilon_0) \), since \( 0 < \sigma < \sigma_1 \), we have \( 0 < q(t) < 1 \) for all \( t \geq 0 \). Since \( 0 < U_0 < 1 \), it follows that for all \( (x, t) \in \partial T \)

\[
-1 < u_\varepsilon^+(x, t) < 2. \quad (5.11)
\]

We begin to estimate the four terms \( E_1, E_2, E_3 \) and \( E_4 \). The estimates of the terms \( E_1, E_2 \) and \( E_3 \) are similar to the estimates in [1] and we obtain that

\[
E_1 \geq \frac{\sigma \beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma \beta Le^{Lt} = \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1' Le^{Lt},
\]
where $C_1 = \sigma \beta^2$, $C'_1 = 2\sigma \beta$ are positive constants.

$$|E_2| \leq \frac{16C}{(e\lambda d_0)^2}(1 + ||\nabla d||^2_{\infty}) = C_2,$$

where $C$ and $\lambda$ are the constants that we choose in Lemma 5.4, so that $C_2$ is also a positive constant.

We remark that in the estimate for $E_2$ in [1], the following assumption holds:

$$|E_2| \leq 16C(e^{\lambda d_0})^2(1 + ||\nabla d||^2_{\infty}) = C_2,$$

where $C$ and $\lambda$ are the constants that we choose in Lemma 5.4, so that $C_2$ is also a positive constant.

For $E_3$, we use (5.4) and obtain that

$$|E_3| \leq C_3(e^{\lambda t} + K) + C'_3,$$

where $C_3 = N_0C$ and $C'_3 = N_0C/\lambda$ with $C$ and $\lambda$ the constants given by Lemma 5.4.

Then we consider the term $E_4$. We should know the estimates of $\nabla (\chi(v) - \chi(v^0))$ and $\Delta \chi(v)$. In fact, for this term, we have the following lemma.

**Lemma 5.5** Let $u$ be any function satisfying

$$u^-_c \leq u \leq u^+_c \text{ in } \Omega_T$$

and let $(v, m)$ be the corresponding solution of Problem (2.2) with $v = H(u)$. Then there exists $C > 0$ depending on $T$ and $\Omega$ such that for all $(x, t) \in \Omega_T$,

$$|v(x, t)| + |\nabla v(x, t)| + |\Delta v(x, t)| \leq C$$

(5.13)

$$|\int_0^t (m - m^0)(x, s)ds| + |\nabla d(x, t)\cdot \int_0^t \nabla (m - m^0)(x, s)ds| \leq C\varepsilon p(t)$$

(5.14)

$$|(v - v^0)(x, t)| + |\nabla d(x, t)\cdot \nabla (v - v^0)(x, t)| \leq C\varepsilon p(t)$$

(5.15)

where $(v^0, m^0)$ are given by the solution of Problem $P^0$.

We prove this lemma below. Let us carry on with the proof of Lemma 5.4.

We write

$$\nabla d \cdot \nabla (\chi(v) - \chi(v^0)) = \chi'(v)\nabla d \cdot \nabla (v - v^0) + (\chi'(v) - \chi'(v^0))\nabla d \cdot \nabla v^0.$$  (5.16)

Since $v^0$ is bounded in $C^{1+\alpha'}$ for any $\alpha' \in (0, 1)$, there exists $C > 0$, such that

$$||v^0||_{L^\infty(\Omega_T)} + ||\nabla v^0||_{L^\infty(\Omega_T)} \leq C,$$

which combined with (5.16), yields that

$$|\nabla d \cdot \nabla (\chi(v) - \chi(v^0))| \leq ||\chi'||_{\infty}||\nabla d \cdot \nabla (v - v^0)| + C||\nabla d||_{\infty}||\chi''||_{\infty}|v - v^0|$$

(5.17)
where the $L^\infty$-norms of $\chi'$ and $\chi''$ are considered on the interval $(-C, C)$. Therefore, since $\chi$ is smooth and $\|\nabla d\|_\infty$ is bounded, it follows from (5.17) that for all $(x, t) \in \Omega_T$, there exists $C > 0$ such that

$$|\nabla d \cdot \nabla (\chi(v) - \chi(v^0))| \leq C \varepsilon p(t). \quad (5.18)$$

Moreover, using the smoothness of $\chi$ and the first inequality of Lemma 5.6, we obtain that there exists $C > 0$ such that

$$|\Delta \chi(v)| \leq C \varepsilon. \quad (5.19)$$

Hence, by the above inequalities (5.18), (5.19) and the fact that $|u^+_x(x, t)| \leq 2$, we obtain that for all $(x, t) \in \Omega_T$,

$$|E_4| \leq \frac{C}{\varepsilon} C \varepsilon p(t) + 2C'.$$

Finally substituting the expression for $p$ and $q$, we obtain that there exist the positive constants $C_4$, $\epsilon_4$ and $\epsilon_4''$ such that

$$|E_4| \leq C_4 + \epsilon_4' e^{-\beta t/\varepsilon^2} + \epsilon_4'' e^{Lt}.$$

We collect the above four estimates of $E_1$, $E_2$, $E_3$ and $E_4$, which yield

$$L_v[u^+_x] \geq \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1' L e^{Lt} - C_2$$

$$- C_3 (e^{Lt} + K) - C_3' - C_4 - C_4' e^{-\beta t/\varepsilon^2} - C_4'' e^{Lt} \quad (5.20)$$

$$= \frac{C_1 - \varepsilon^2 C_4'}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + (L C_1' - C_3 - C_4'') e^{Lt} - C_6,$$

where $C_6 = C_2 + C_3 K + C_3' + C_4$ is a positive constant. Now we set

$$L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0},$$

where $\varepsilon_0$ is small enough and satisfies the assumptions (5.10) and (5.12), so that $L$ is large enough. It also follows that $\frac{C_1 - \varepsilon^2 C_4'}{\varepsilon^2} > 0$ and

$$L C_1' - C_3 - C_4'' \geq \frac{1}{2} L C_1',$$

therefore

$$L_v[u^+_x] \geq \frac{1}{2} L C_1' - C_6 \geq 0.$$

The proof of Lemma 5.5 is now completed, with the constants $\beta$, $\sigma$ given in (5.8), (5.9).
5.3 Proof of Lemma 5.5

Lemma 5.5 gives the key estimate and is the analogue of Lemma 4.9 in [4] and of Lemma 2.1 in [1]. However the proof is markedly different since the coupling between $u$ and $v$ is given by a system with an ODE and a parabolic equation versus an elliptic equation in the two above references. First note that (5.13) is established exactly as in Lemma 2.1 (c). Concerning the second inequality (5.14), let us recall the following properties of $U_0$ given in [1].

Lemma 5.6 For all given $a \in \mathbb{R}$ and $z \in \mathbb{R}$, we have the inequality:

$$|U_0(z + a) - \chi_{]-\infty,0]}(z)| \leq Ce^{-\lambda|z+a|} + \chi_{]-a,a]}(z)$$

Define $w(x, t) = m(x, t) - m^0(x, t)$, then $w$ satisfies

$$\begin{cases}
  w_t - \alpha \Delta w + w = h & \text{in } \Omega_T \\
  \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T) \\
  w(x, 0) = 0, & x \in \Omega
\end{cases}$$

with $h = u - u^0$ satisfying

$${u^-}^e - u^0 \leq h \leq {u^+}^e - u^0 \quad \text{in } \Omega_T.$$ 

From the definition of $u^\pm_e$ in (5.5) and from Lemma 5.6 for $z = \frac{d(x, t)}{\varepsilon}$ and $a = \pm p(t)$, we deduce that for all $(x, t) \in \Omega_T$,

$$|h(x, t)| \leq C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|}) + \chi_{\{|d(x,t)| \leq \varepsilon p(t)\}} + q(t) \quad (5.22)$$

Let us define for all $(x, t) \in \Omega_T$,

$$h_1(x, t) = q(t),$$

$$h_2(x, t) = C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|}) \chi_{\{|d(x,t)| > d_0\}},$$

and

$$h_3(x, t) = C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|}) \chi_{\{|d(x,t)| \leq \varepsilon p(t)\}} + \chi_{\{|d(x,t)| \leq \varepsilon p(t)\}}$$

and denote by $(w_i)_{i=1,2,3}$ the solutions of the three following auxiliary problems

$$\begin{cases}
  (w_i)_t - \alpha \Delta w_i + w_i = h_i & \text{in } \Omega_T \\
  \frac{\partial w_i}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T) \\
  w_i(x, 0) = 0, & x \in \Omega
\end{cases}$$
Note that in view of the definition of $p(t)$ and the inequality (5.12), we have that for all $t \in [0, T]$

$$0 < K - 1 \leq p(t) \leq \frac{d_0}{2\varepsilon_0} \quad (5.23)$$

so that the function $p$ is bounded away from 0 for all $t \in [0, T]$. It follows in particular that choosing $\varepsilon > 0$ small enough,

$$\varepsilon p(t) \leq d_0/2 \text{ for all } t \in [0, T]$$

so that $|h| \leq h_1 + h_2 + h_3$. Thus we deduce from the maximum principle that for all $x \in \Omega$ and $t \in [0, T]$,

$$|w(x, t)| \leq w_1(x, t) + w_2(x, t) + w_3(x, t).$$

We now establish estimates for $w_i$, with $i = 1, 2, 3$.

**Problem (A1)**

Set $W_1(x, t) = \int_0^t w_1(x, s) ds$, then $W_1$ satisfies

$$
\begin{cases}
(W_1)_t - \alpha \Delta W_1 + W_1 = H_1 & \text{in } \Omega_T \\
\frac{\partial W_1}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T) \\
W_1(x, 0) = 0, & x \in \Omega
\end{cases}
$$

(5.24)

with, since $q(t) = \varepsilon^2 \sigma p'(t)$,

$$H_1(x, t) = \int_0^t q(s) ds = \varepsilon^2 \sigma (p(t) - p(0))$$

so that by (5.23) we get that there exists $C > 0$ such that for all $t \in [0, T]$,

$$\sup_{(y, s) \in \partial \Omega \times [0, T]} |H_1(y, s)| \leq C \varepsilon p(t).$$

Hence by standard parabolic estimates, there exists $C > 0$ such that for all $(x, t) \in \Omega_T$, the solution $W_1$ of Problem (A1) satisfies

$$|W_1(x, t)| + |\nabla W_1(x, t)| \leq C \varepsilon p(t). \quad (5.25)$$

**Problem (A2)**

Note that by the standard parabolic estimates there exists a constant $C' > 0$ such that By definition of $h_2$, using again (5.23), we obtain that there exists $C' > 0$ such that for all $(s, t) \in [0, T]^2$

$$h_2(y, s) \leq 2Ce^{-\lambda(d_0/\varepsilon - p(s))} \leq 2Ce^{-\lambda d_0/2\varepsilon} \leq \frac{4C}{\lambda d_0 \varepsilon}.$$  

$$\leq \frac{4C}{\lambda d_0 \varepsilon} \varepsilon p(s) \leq C' \varepsilon p(t), \quad (5.26)$$

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Thus by standard parabolic estimates, we obtain that for all \((x, t) \in \Omega_T\)

\[ |w_2(x, t)| + |\nabla w_2(x, t)| \leq C'\varepsilon p(t), \]

which implies that there exists \(C > 0\) such that for all \((x, t) \in \Omega_T\)

\[ |W_2(x, t)| + |\nabla W_2(x, t)| \leq C\varepsilon p(t), \tag{5.27} \]

where we define \(W_2(x, t) = \int_0^t w_2(x, s)ds\).

**Problem \((A_3)\)**

Note that \(h_3(y, s)\) is supported in \(\{|d(y, s)| \leq d_0\}\). Moreover by linearity we may suppose that the function \(h_3\) satisfies one of the three following assumptions:

\( (H_1) \) \( |h_3(y, s)| \leq \chi_{\{|d(y, s)| \leq \varepsilon p(s)\}} \)

\( (H_2^\pm) \) \( |h_3(y, s)| \leq e^{-\lambda|d(y, s)/\varepsilon p(s)|} \)

Then under respectively assumptions \((H_1), (H_2^\pm)\), we define a function \(\tilde{h}\) on \(\mathbb{R} \times [0, T]\), respectively by

\[ \tilde{h}(r, s) = \begin{cases} \chi_{\{|r| \leq \varepsilon p(s)\}} e^{-\lambda|r/\varepsilon p(s)|} & (5.28) \end{cases} \]

Note that \(|h_3(y, s)| \leq \tilde{h}(d(y, s), s)\), and under either of the assumptions \((H_1)\) or \((H_2^\pm)\), there exists a constant \(C > 0\) such that for all \((s, t) \in [0, T]^2\)

\[ 0 \leq \int_{-d_0}^{d_0} \tilde{h}(r, s)dr \leq C\varepsilon p(t). \tag{5.29} \]

Let \(\varphi(x, t) = e^t w_3(x, t)\), then in view of Problem \((A_3)\), the function \(\varphi\) satisfies

\[ \begin{cases} \varphi_t - \alpha \Delta \varphi = f & \text{in } \Omega_T \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T) \end{cases} \tag{5.30} \]

where \(f(x, t) = e^t h_3(x, t)\) and \(\varphi(x, 0) = w_3(x, 0) = 0\) for all \(x \in \Omega\). We establish now that there exist a constant \(C > 0\) such that

\[ \forall (x, t) \in \Omega_T, \ 0 \leq \varphi(x, t) \leq C\varepsilon p(t). \tag{5.31} \]

As in \([2]\), the solution \(\varphi(x, t)\) of Problem \((5.30)\) can be expressed as

\[ \varphi(x, t) = \int_0^t \int_{|d(y, s)| \leq d_0} G(x, y, t - s)f(y, s)dyds, \]

with \(G(x, y, t)\) being the Green function associated to the Neumann boundary value problem in \(\Omega\) for the parabolic operator \(\varphi_t - \alpha \Delta \varphi\). Thus for all \((x, t) \in \Omega_T\),

\[ 0 \leq \varphi(x, t) \leq \int_0^t \int_{|d(y, s)| \leq d_0} G(x, y, t - s)e^s \tilde{h}(d(y, s), s)dyds \tag{5.32} \]
Next we recall the following important property of $G$ which is established in [2].

**Lemma 7.6, [2]:** Let $\Gamma$ be a closed hypersurface in $\Omega$ and denote by $d(x)$ the signed distance function associated with $\Gamma$. Then there exists constants $C, d_0 > 0$ such that for any function $\eta(r) \geq 0$ on $\mathbb{R}$, it holds that

$$
\int_{d \leq d_0} G(x, y, t) \eta(d(y)) dy \leq \frac{C}{\sqrt{t}} \int_{-d_0}^{d_0} \eta(r) dr \quad \text{for } 0 < t \leq T
$$

Moreover as pointed out in [2], the above inequality is uniform with respect to smooth variations of $\Gamma$ and for $t \in [0, T]$. Applying this inequality to our case, we deduce that there exists $C > 0$ such that for all $(x, y) \in \Omega^2$ and for all $0 \leq s < t \leq T$,

$$
\int_{d(y, s) \leq d_0} G(x, y, t - s) \tilde{h}(d(y, s), s) dy \leq \frac{C}{\sqrt{T - s}} \int_{-d_0}^{d_0} \tilde{h}(r, s) dr.
$$

In view of (5.32) and of (5.29), it follows that for all $x \in \Omega$ and for all $t \in [0, T]$,

$$
0 \leq \varphi(x, t) \leq C \int_0^t \int_{d(y, s) \leq d_0} G(x, y, t - s) \tilde{h}(d(y, s), s) dy ds
\leq C' \int_0^t \frac{1}{\sqrt{T - s}} \int_{-d_0}^{d_0} \tilde{h}(r, s) dr ds
\leq C' \int_0^t \frac{1}{\sqrt{T - s}} \varepsilon p(t) ds \leq 2C' \varepsilon p(t) \sqrt{T}
$$

which yields inequality (5.31).

Coming back to $w_3$, we deduce that for all $(x, t) \in \Omega_T$,

$$
|w_3(x, t)| = |e^{-t} \varphi(x, t)| \leq C \varepsilon p(t).
$$

(5.34)

Define $W_3(x, t) = \int_0^t w_3(x, s) ds$, then it follows that

$$
|W_3(x, t)| \leq C \varepsilon p(t)
$$

(5.35)

for some $C > 0$ and for all $(x, t) \in \Omega_T$. We show now that there exist $C > 0$ such that for all $(x, t) \in \Omega_T$,

$$
|\nabla d(x, t) \cdot \nabla W_3(x, t)| \leq C \varepsilon p(t).
$$

(5.36)

Time integration of the equation in Problem $(A_3)$ on $[0, t]$ gives

$$
w_3(x, t) - w_3(x, 0) - \alpha \Delta W_3(x, t) + W_3(x, t) = \int_0^t h_3(x, s) ds.
$$
Since \( w_3(x, 0) = 0 \), we obtain the following elliptic problem for any \( t \in [0, T] \),
\[
\begin{aligned}
& \left\{ \begin{array}{ll}
-\alpha \Delta W_3(., t) + W_3(., t) = \hat{H}_3(., t) & \text{in } \mathcal{O} \\
\frac{\partial W_3(., t)}{\partial \nu} = 0 & \text{on } \partial \mathcal{O}
\end{array} \right.
\end{aligned}
\tag{5.37}
\]
where for all \((x, t) \in \mathcal{O}_T\),
\[
\hat{H}_3(x, t) = \int_0^t h_3(x, s) \, ds - w_3(x, t).
\]
Let us define for any \( t \in [0, T] \) the functions \( a(., t) \) as the solution of
\[
\begin{aligned}
& \left\{ \begin{array}{ll}
-\alpha \Delta a(., t) + a(., t) = h_3(., t) & \text{in } \mathcal{O} \\
\frac{\partial a(., t)}{\partial \nu} = 0 & \text{on } \partial \mathcal{O}
\end{array} \right.
\end{aligned}
\tag{5.38}
\]
and define \( A(x, t) = \int_0^t a(x, s) \, ds \). Define similarly \( B(., t) \) as the solution of
\[
\begin{aligned}
& \left\{ \begin{array}{ll}
-\alpha \Delta B(., t) + B(., t) = -w_3(., t) & \text{in } \mathcal{O} \\
\frac{\partial B(., t)}{\partial \nu} = 0 & \text{on } \partial \mathcal{O}
\end{array} \right.
\end{aligned}
\tag{5.39}
\]
so that by linearity
\[
\forall (x, t) \in \Omega_T, \quad W_3(x, t) = A(x, t) + B(x, t).
\]
It follows from standard elliptic estimates in view of (5.34) that
\[
|B(x, t)| + |\nabla B(x, t)| \leq C\varepsilon p(t).
\]
Concerning \( a \), note that the elliptic problem appearing here is the same as for the chemotaxis-growth system studied in [4] and in [1], with the right-hand-side satisfying (5.29). Therefore the results stated in Lemma 4.2 in [1] and in Lemma 4.10 in [4] apply and prove that there exists a constant \( C > 0 \) such that for all \((x, t) \in \mathcal{O}_T\),
\[
|a(x, t)| + |\nabla a(x, t)| \leq C\varepsilon p(t)
\]
and consequently
\[
|A(x, t)| + |\nabla d(x, t).\nabla a(x, t)| \leq C\varepsilon p(t)
\]
This completes the proof of (5.36). In view of \((5.25), (5.27), (5.35)\) and \((5.36)\), inequality \((5.14)\) is now established.
In order to prove inequality \((5.15)\), note that using \((5.14)\) we obtain that for all \((x, t) \in \mathcal{O}_T\),
\[
|(v - v^0)(x, t)| = \left| v_0(x)e^{-\lambda \int_0^t m(x, s) \, ds} - v_0(x)e^{-\lambda \int_0^t m^0(x, s) \, ds} \right| \leq C|v_0(x)||\int_0^t (m - m^0)(x, s) \, ds| \leq C'\varepsilon p(t),
\tag{5.41}
\]
where \( C' \) is a constant depending only on \( \lambda \) and \( \|v_0\|_{L^p} \).
where \( C' > 0 \) is a suitable constant.

Next we have similarly that for all \((x, t) \in \mathcal{O}_T \),

\[
\begin{align*}
|\nabla d(x, t) \cdot \nabla (v - v^0)(x, t)| & \leq C|e^{-\lambda \int_0^t m(x, s)ds} - e^{-\lambda \int_0^t m^0(x, s)ds}| \\
& + C|v(x, t)\nabla d(x, t) \cdot \int_0^t \nabla m(x, s)ds - v^0(x, t)\nabla d(x, t) \cdot \int_0^t \nabla m^0(x, s)ds| \\
& \leq C' \xi p(t) + C|v(x, t)||\nabla d(x, t) \cdot \int_0^t \nabla (m - m^0)(x, s)ds| \\
& + C|v(x, t) - v^0(x, t)||\nabla d(x, t) \cdot \int_0^t \nabla m^0(x, s)ds|,
\end{align*}
\]

where \( C, C' > 0 \) are suitable constants. Using (5.41), (5.14) and upper bounds on \(|v|\) and \(|\nabla m^0|\), we deduce that (5.15) is satisfied. This completes the proof of Lemma 5.6.

5.4 Proof of Corollary 1.3 and Theorem 1.2

The pointwise convergence of \( u^\varepsilon \) to \( u^0 \) in \( \bigcup_{0 < t \leq T} \left((\Omega \setminus \Gamma) \times t\right) \) when \( \varepsilon \to 0 \) follows from Lemma 5.4 and from (5.7). Next note that \( w^\varepsilon = m^\varepsilon - m^0 \) is a solution of Problem (5.21) with the right-hand-side \( h^\varepsilon \) satisfying

\[
|h^\varepsilon(x, t)| \leq h_1(x, t) + h_2(x, t) + h_3(x, t)
\]

with \( h_i, i = 1, 2, 3 \) defined as in the proof of Lemma 5.5. This shows that there exists \( C > 0 \) such that

\[
|h^\varepsilon|_{L^1(\Omega_T)} \leq C\varepsilon.
\]

It follows then from standard parabolic estimates and Sobolev inequalities that for any \( \alpha \in (0, 1) \) there exists \( p \in (1, +\infty) \) and \( C > 0 \) such that

\[
\begin{align*}
||u^\varepsilon||_{C^{1+\alpha,1+\alpha/2}(\overline{\Omega_T})} & \leq C||u^\varepsilon - u^0||_{L^p(\Omega_T)} \\
& \leq C||h^\varepsilon||_{L^p(\Omega_T)} \leq C\varepsilon^{1/p}.
\end{align*}
\]

(5.42)

Thus for any \( \alpha \in (0, 1) \),

\[
\lim_{\varepsilon \to 0} ||m^\varepsilon - m^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega_T})} = 0.
\]

The expression of \( v^\varepsilon \) and \( \nabla v^\varepsilon \) in (2.3) and (2.7) then shows that

\[
\lim_{\varepsilon \to 0} ||v^\varepsilon - v^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega_T})} = 0
\]

which completes the proof of Theorem 1.2.
5.5 Proof of Theorem 1.1, Theorem 1.4 and Corollary 1.5

The proofs are exactly the same as the proofs of Theorem 1.3, Theorem 1.5 and Corollary 1.6 in [1] respectively, we omit the details here.

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