The fuzzy Henstock–Kurzweil delta integral on time scales

Dafang Zhao, Guoju Ye, Wei Liu and Delfim F. M. Torres

Abstract We investigate properties of the fuzzy Henstock–Kurzweil delta integral (shortly, FHK Δ-integral) on time scales, and obtain two necessary and sufficient conditions for FHK Δ-integrability. The concept of uniformly FHK Δ-integrability is introduced. Under this concept, we obtain a uniformly integrability convergence theorem. Finally, we prove monotone and dominated convergence theorems for the FHK Δ-integral.

Key words: fuzzy Henstock–Kurzweil integral, convergence theorems, time scales.
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1 Introduction

The Lebesgue integral, with its convergence properties, is superior to the Riemann integral. However, a disadvantage with respect to Lebesgue’s integral, is
that it is hard to understand without substantial mathematical maturity. Also, the Lebesgue integral does not inherit the naturalness of the Riemann integral. Henstock \[23\] and Kurzweil \[26\] gave, independently, a slight, yet powerful, modification of the Riemann integral to get the now called Henstock–Kurzweil (HK) integral, which possesses all the convergence properties of the Lebesgue integral. For the fundamental results of HK integral, we refer to the papers \[7, 23, 38, 39, 43, 45\] and monographs \[20, 27, 34\]. As an important branch of the HK integration theory, the fuzzy Henstock–Kurzweil (FHK) integral has been extensively studied in \[8, 13, 18, 19, 22, 30, 35, 36, 41, 42\].

In 1988, Hilger introduced the theory of time scales in his Ph.D. thesis \[24\]. A time scale \( T \) is an arbitrary nonempty closed subset of \( \mathbb{R} \). The aim is to unify and generalize discrete and continuous dynamical systems, see, e.g., \[2–6, 9, 15, 21, 31\]. In \[32\], Peterson and Thompson introduced a more general concept of integral, i.e., the HK \( \Delta \)-integral, which gives a common generalization of the Riemann \( \Delta \) and Lebesgue \( \Delta \)-integral. The theory of HK integration for real-valued and vector-valued functions on time scales has been developed rather intensively, see, e.g., the papers \[1, 11, 16, 28, 29, 33, 37, 40, 44\] and references cited therein.

In 2015, Fard and Bidgoli introduced the FHK delta integral and presented some of its basic properties \[14\]. Nonetheless, to our best knowledge, there is no systematic theory for the FHK delta integral on time scales. In this work, in order to complete the FHK delta integration theory, we give two necessary and sufficient conditions of FHK delta integrability (see Theorems 3 and 7). Moreover, we obtain some convergence theorems for the FHK delta integral, in particular Theorem 9 of dominated convergence and Theorem 10 of monotone convergence.

After Section 2 of preliminaries, in Section 3 the definition of FHK delta integral is introduced, and our necessary and sufficient conditions of FHK delta integrability proved. We also obtain some convergence theorems. Finally, in Section 4 we give conclusions and point out some directions that deserve further study.

2 Preliminaries

A fuzzy subset of the real axis \( u : \mathbb{R} \rightarrow [0, 1] \) is called a fuzzy number provided that

1. \( u \) is normal: there exists \( x_0 \in \mathbb{R} \) with \( u(x_0) = 1 \);
2. \( u \) is fuzzy convex: \( u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\} \) for all \( x_1, x_2 \in \mathbb{R} \) and all \( \lambda \in (0, 1) \);
3. \( u \) is upper semi-continuous;
4. \( [u]^0 = \{x \in \mathbb{R} : u(x) > 0\} \) is compact.

Denote by \( \mathbb{R}_\mathcal{F} \) the space of fuzzy numbers. We define the \( \alpha \)-level set \( |u|^\alpha \) by

\[
|u|^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}, \quad \alpha \in (0, 1].
\]

From conditions (1)-(4), \( |u|^\alpha \) is denoted by \( |u|^\alpha = [a^\alpha, b^\alpha] \). For \( u_1, u_2 \in \mathbb{R}_\mathcal{F} \) and \( \lambda \in \mathbb{R} \), we define
\[ [u_1 + u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha \] and \[ \lambda \odot u_1 = \lambda [u_1]^\alpha \]

for all \( \alpha \in [0, 1] \). The Hausdorff distance between \( u_1 \) and \( u_2 \) is defined by

\[
D(u_1, u_2) = \sup_{\alpha \in [0, 1]} \max \left\{ \frac{|u_1^\alpha - u_2^\alpha|}{|u_1^\alpha|}, \frac{|u_1^\alpha - u_2^\alpha|}{|u_2^\alpha|} \right\}.
\]

Then, the metric space \((\mathbb{R}, \varnothing, D)\) is complete. Let \( a, b \in T \). We define the half-open interval \([a, b]_T\) by

\[ [a, b]_T = \{ x \in T : a \leq x < b \}. \]

The open and closed intervals are defined similarly. For \( x \in T \), we denote by \( \sigma(x) := \inf \{ y > x : y \in T \} \) and by \( \rho \) the backward jump operator, i.e., \( \rho \) is the forward jump operator, i.e., \( \sigma(\inf T) = \inf T \) and \( \rho \) is the backward jump operator, i.e., \( \rho \) is the forward jump operator, i.e., \( \sigma(\sup T) = \sup T \) and \( \rho(\inf T) = \inf T \) where \( \sup T \) and \( \inf T \) are finite. In this situation, \( T^\times := T \setminus \{ \sup T \} \) and \( T_\kappa := T \setminus \{ \inf T \} \), otherwise, \( T^\times := T \) and \( T_\kappa := T \). If \( \sigma(x) > x \), then we say that \( x \) is right-scattered, while if \( \rho(x) < x \), then we say that \( x \) is left-scattered. If \( \sigma(x) = x \) and \( \rho(x) = x \) or \( x > \sup T \), then \( x \) is called right-dense, and if \( \rho(x) = x \) and \( x < \inf T \), then \( x \) is left-dense. The graininess functions \( \mu \) and \( \eta \) are defined by \( \mu(x) := \sigma(x) - x \) and \( \eta(x) := x - \rho(x) \), respectively.

In what follows, all considered intervals are intervals in \( T \). A division \( D \) of \([a, b]_T\) is a finite set of interval-point pairs \( \{([x_{i-1}, x_i]_T, \xi_i)\}_{i=1}^n \) such that

\[
\bigcup_{i=1}^n [x_{i-1}, x_i]_T = [a, b]_T
\]

and \( \xi_i \in [a, b]_T \) for each \( i \). We write \( \Delta x_i = x_i - x_{i-1} \). We say that

\[
\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))
\]

is a \( \Delta \)-gauge on \([a, b]_T\) if \( \delta_L(\xi) > 0 \) on \([a, b]_T\), \( \delta_R(\xi) > 0 \) on \([a, b]_T\), \( \delta_L(a) \geq 0 \), \( \delta_R(b) \geq 0 \) and \( \delta_R(\xi) \geq \mu(\xi) \) for any \( \xi \in [a, b]_T \). The symbol \( \Gamma(\Delta, [a, b]_T) \) stands for the set of \( \Delta \)-gauge on \([a, b]_T\). Let \( \delta^1(\xi) \) and \( \delta^2(\xi) \) be \( \Delta \)-gauges such that

\[
0 < \delta^1_L(\xi) < \delta^2_L(\xi)
\]

for any \( \xi \in [a, b]_T \) and \( 0 < \delta^1_R(\xi) < \delta^2_R(\xi) \) for any \( \xi \in [a, b]_T \). Then we call \( \delta^1(\xi) \) finer than \( \delta^2(\xi) \) and write \( \delta^1(\xi) < \delta^2(\xi) \). We say that \( D = \{([x_{i-1}, x_i]_T, \xi_i)\}_{i=1}^n \) is a \( \delta \)-fine HK division of \([a, b]_T\) if \( \xi_i \in [x_{i-1}, x_i]_T \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))_T \) for each \( i \). Let \( \mathcal{D}(\delta, [a, b]_T) \) be the set of all \( \delta \)-fine HK divisions of \([a, b]_T\). Given an arbitrary \( D \in \mathcal{D}(\delta, [a, b]_T) \), \( D = \{([x_{i-1}, x_i]_T, \xi_i)\}_{i=1}^n \), we write

\[
S(f, D, \delta) = \sum_{i=1}^n f(\xi_i) \Delta x_i
\]

for integral sums over \( D \), whenever \( f : [a, b]_T \to \mathbb{R} \).
Lemma 1 (See [25]). Suppose that \( u \in \mathbb{R}_\preceq \). Then,

1. the interval \([u]^{\alpha}\) is closed for \( \alpha \in [0, 1]\);
2. \([u]^{\alpha_1} \supset [u]^{\alpha_2}\) for \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1\);
3. for any sequence \( \{\alpha_n\} \) satisfying \( \alpha_n \leq \alpha_{n+1} \) and \( \alpha_n \rightarrow \alpha \in (0, 1]\), we have \( \bigcap_{n=1}^{\infty} [u]^{\alpha_n} = [u]^{\alpha} \).

Conversely, if a collection of subsets \( \{u^\alpha : \alpha \in [0, 1]\} \) verify (1)–(3), then there exists a unique \( u \in \mathbb{R}_\preceq \) such that \([u]^{\alpha} = u^\alpha\) for \( \alpha \in (0, 1]\) and \([u]^0 = \bigcup_{\alpha \in [0, 1]} u^\alpha \subset u^0\).

Lemma 2 (See [17]). Suppose that \( u \in \mathbb{R}_\preceq \). Then,

1. \( u^\alpha\) is bounded and nondecreasing;
2. \( \overline{u^\alpha}\) is bounded and nonincreasing;
3. \( u_1 \leq u_2\);
4. for \( e \in (0, 1]\), \( \lim_{\alpha \rightarrow -e} \overline{u^\alpha} = \overline{u}^e\) and \( \lim_{\alpha \rightarrow -e} \underline{u^\alpha} = \underline{u}^e\);
5. \( \lim_{\alpha \rightarrow 0^+} \overline{u^\alpha} = u^0\) and \( \lim_{\alpha \rightarrow 0} \underline{u^\alpha} = \overline{u}^0\).

Conversely, if \( \overline{u^\alpha}\) and \( \underline{u^\alpha}\) satisfy items (1)–(5), then there exists \( u \in \mathbb{R}_\preceq \) such that

\[
[u]^\alpha = \overline{u^\alpha, \underline{u^\alpha}} = \overline{[u^\alpha, u^\alpha]}.
\]

3 The fuzzy Henstock–Kurzweil delta integral

We introduce the concept of fuzzy Henstock–Kurzweil (FHK) delta integrability.

Definition 1. A function \( f : [a, b]_T \rightarrow \mathbb{R}_\preceq \) is called FHK \( \Delta\)-integrable on \([a, b]_T\) with the FHK \( \Delta\)-integral \( \tilde{A} = (\text{FHK}) \int_{[a, b]_T} f(x) \Delta x \), if for each \( \varepsilon > 0 \) there exists a \( \delta \in \Gamma(\Delta, [a, b]_T) \) such that \( D(S(f, D, \delta), \tilde{A}) < \varepsilon \) for each \( D \in D(\delta, [a, b]_T) \). The family of all FHK \( \Delta\)-integrable functions on \([a, b]_T\) is denoted by \( \mathcal{F}_\mathcal{H}_\mathcal{K}[a, b]_T\).

Remark 1. It is clear that Definition 1 is more general than the HK \( \Delta\)-integral introduced by Peterson and Thompson in [32] and more general than the FH integral introduced by Wu and Gong in [41, 42].

The proofs of Theorems 1 and 2 are straightforward and are left to the reader.

Theorem 1. The FHK \( \Delta\)-integral of \( f(x) \) is unique.

Theorem 2. If \( f(x), g(x) \in \mathcal{F}_\mathcal{H}_\mathcal{K}[a, b]_T \) and \( \alpha, \beta \in \mathbb{R} \), then

\[
\alpha f(x) + \beta g(x) \in \mathcal{F}_\mathcal{H}_\mathcal{K}[a, b]_T
\]

with

\[
(\text{FHK}) \int_{[a, b]_T} (\alpha f(x) + \beta g(x)) \Delta x = \alpha (\text{FHK}) \int_{[a, b]_T} f(x) \Delta x + \beta (\text{FHK}) \int_{[a, b]_T} g(x) \Delta x.
\]
Follows a Cauchy–Bolzano condition for the FHK \( \Delta \)-integral.

**Theorem 3 (The Cauchy–Bolzano condition).** Function \( f(x) \in \mathcal{H}(a,b) \) if and only if for each \( \varepsilon > 0 \) there exists a \( \delta \in \Gamma([a,b]) \) such that

\[
D(S(f,D_1,\delta),S(f,D_2,\delta)) < \varepsilon
\]

for any \( D_1, D_2 \in \mathcal{D}(\delta,[a,b]). \)

**Proof.** (Necessity) Let \( \varepsilon > 0 \). By hypothesis, there exists \( \delta \in \Gamma([a,b]) \) such that

\[
D\left(S(f,D,\delta),(FHK)\int_{[a,b]} f(x)dx\right) < \frac{\varepsilon}{2}
\]

for any \( D \in \mathcal{D}(\delta,[a,b]) \). Let \( D_1, D_2 \in \mathcal{D}(\delta,[a,b]) \). Then,

\[
D(S(f,D_1,\delta),S(f,D_2,\delta)) \\
\leq D\left(S(f,D_1,\delta),(FHK)\int_{[a,b]} f(x)dx\right) + D\left(S(f,D_2,\delta),(FHK)\int_{[a,b]} f(x)dx\right) \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

(Sufficiency) For each \( n \), choose a \( \delta_n \in \Gamma([a,b]) \) such that

\[
D(S(f,D_1,\delta_n),S(f,D_2,\delta)) < \frac{1}{n}
\]

for any \( D_1, D_2 \in \mathcal{D}(\delta_n,[a,b]) \). Replacing \( \delta_n \) by \( \bigcap_{j=1}^{n} \delta_j = \delta_n \), we may assume that \( \delta_{n+1} \subset \delta_n \). For each \( n \), fix a \( D_n \in \mathcal{D}(\delta_n,[a,b]) \). For \( j > n \), we have \( \delta_j \subset \delta_n \), so \( D_j \in \mathcal{D}(\delta_n,[a,b]) \). Thus, \( D(S(f,D_n,\delta_n),S(f,D_N,\delta_n)) < \frac{1}{N} \) and it follows that \( \{S(f,D_n,\delta_n)\} \) is a Cauchy sequence. We denote the limit of \( \{S(f,D_n,\delta_n)\} \) by \( \tilde{A} \) and let \( \varepsilon > 0 \). Choose \( N > \frac{\varepsilon}{2} \) and let \( D \in \mathcal{D}(\delta_N,[a,b]) \). Then,

\[
D\left(S(f,D,\delta_N),\tilde{A}\right) \leq D\left(S(f,D,\delta_N),S(f,D_N,\delta_N)\right) + D\left(S(f,D_N,\delta_N),\tilde{A}\right) \\
< \frac{1}{N} + \frac{1}{N} \\
< \varepsilon.
\]

Hence, \( f(x) \in \mathcal{H}(a,b) \).

**Theorem 4.** Let \( c \in (a,b) \). If \( f(x) \in \mathcal{H}(a,c) \) \( \cap \mathcal{H}(c,b) \), then

\[
f(x) \in \mathcal{H}(a,b)
\]

with

\[
(FHK)\int_{[a,b]} f(x)dx = (FHK)\int_{[a,c]} f(x)dx + (FHK)\int_{[c,b]} f(x)dx.
\]
Proof. Let $\varepsilon > 0$. By assumption, there exist $\Delta$-gauges

$$\delta^i(\xi) = (\delta^i_1(\xi), \delta^i_2(\xi)), \quad i = 1, 2,$$

such that

$$D\left( S(f, D_1, \delta^1), (FHK) \int_{(a,c)_T} f(x) \Delta x \right) < \varepsilon,$$

$$D\left( S(f, D_2, \delta^2), (FHK) \int_{(c,b)_T} f(x) \Delta x \right) < \varepsilon,$$

respectively for any $D_1 \in D(\delta^1, [a,c]_T)$, $D_1 = \{(x_{k-1}, x_k, \xi_k)\}_{k=1}^n$ and for any $D_2 \in D(\delta^2, [c,b]_T)$, $D_2 = \{(x_{k-1}, x_k, \xi_k)\}_{k=1}^n$. We define $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ on $[a,b]_T$ by setting

$$\delta_L(\xi) = \begin{cases} 
\delta^1_1(\xi), & \text{if } \xi \in [a,c]_T, \\
\delta^2_1(\xi), & \text{if } \xi = c = \rho(c), \\
\min \left\{ \delta^1_1(\xi), \frac{\eta(c)}{\rho(\xi)} \right\}, & \text{if } \xi = c > \rho(c), \\
\min \left\{ \delta^2_1(\xi), \frac{\xi - \rho(c)}{\rho(\xi)} \right\}, & \text{if } \xi \in (c,b]_T,
\end{cases}$$

and

$$\delta_R(\xi) = \begin{cases} 
\min \left\{ \delta^1_2(\xi), \max \left\{ \mu(\xi), \frac{\xi - \rho(c)}{\rho(\xi)} \right\} \right\}, & \text{if } \xi \in [a,c]_T, \\
\min \{ \delta^2_2(\xi), \}, & \text{if } \xi \in [c,b]_T.
\end{cases}$$

Now, let $D \in D(\delta, [a,b]_T)$, $D = \{(x_{k-1}, x_k, \xi_k)\}_{k=1}^n$. It follows that

(i) either $c = \xi_q$ and $t_q > c$;

(ii) or $\xi_q = \rho(c) < c$ and $t_q = c$.

The case (ii) is straightforward. For (i), one has

$$D\left( S(f, D, \delta), (FHK) \int_{(a,c)_T} f(x) \Delta x + (FHK) \int_{(c,b)_T} f(x) \Delta x \right)$$

$$= D\left( \sum_{k=1}^p f(\xi_k) \Delta x_k, (FHK) \int_{(a,c)_T} f(x) \Delta x + (FHK) \int_{(c,b)_T} f(x) \Delta x \right)$$

$$\leq D\left( \sum_{k=1}^{q-1} f(\xi_k) \Delta x_k + f(c)(c - t_{q-1}), (FHK) \int_{(a,c)_T} f(x) \Delta x \right)$$

$$+ D\left( \sum_{k=q+1}^p f(\xi_k) \Delta x_k + f(c)(t_q - c), (FHK) \int_{(c,b)_T} f(x) \Delta x \right)$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$
Corollary 1. If \( f \in \mathcal{F} \mathcal{H} \mathcal{K}_{[a,b]} \), then \( f \in \mathcal{F} \mathcal{H} \mathcal{K}_{[r,s]} \) for any \( [r,s] \subset [a,b] \).

Definition 2 (See [32]). Let \( T' \subset T \). We say \( T' \) has delta measure zero if it has Lebesgue measure zero and contains no right-scattered points. A property \( \mathcal{P} \) is said to hold \( \Delta \) a.e. on \( T \) if there exists \( T' \) of measure zero such that \( \mathcal{P} \) holds for every \( t \in T' \).

Theorem 5. Let \( f(x) = g(x) \) \( \Delta \) a.e. on \( [a,b] \). If \( f(x) \in \mathcal{F} \mathcal{H} \mathcal{K}_{[a,b]} \), then so \( g(x) \).

Moreover,
\[
(\text{FHK}) \int_{[a,b]} f(x) \Delta x = (\text{FHK}) \int_{[a,b]} g(x) \Delta x.
\]

Proof. Let \( \varepsilon > 0 \). Then there exists a \( \delta \in \Gamma(\Delta, [a,b]) \) such that
\[
|D(S(f,D,\delta), (\text{FHK}) \int_{[a,b]} f(x) \Delta x) - (\text{FHK}) \int_{[a,b]} g(x) \Delta x| < \varepsilon
\]
for any \( D \in \mathcal{D}(\delta, [a,b]) \), \( D = \{([x_{i-1}, x_i], \xi_i)\} \). Set \( E = \sum_{j=1}^\infty E_j \), where
\[
E_j = \left\{ x : j - 1 < D(f(x), g(x)) \leq j, \quad t \in [a,b] \right\}_{j=1}^\infty.
\]

For each \( j \), there exists \( F_j \) consisting of a collection of open intervals with total length less than \( \varepsilon \cdot 2^{-j} \cdot j^{-1} \), such that \( E_j \subset F_j \). Define
\[
\delta(x) = \begin{cases} 
\delta^0_1(\xi) + \delta^0_2(\xi), & \text{if } \xi \in [a,b] \setminus E, \\
\delta^1_1(\xi) + \delta^1_2(\xi), & \text{if } \xi \in E_j \text{ satisfies } (\xi - \delta^1_1(\xi), \xi + \delta^1_2(\xi)_{T} \subset F_j.
\end{cases}
\]

Then, for any \( D \in \mathcal{D}(\delta, [a,b]) \), \( D = \{([x_{i-1}, x_i], \xi_i)\} \), one has
\[
D(S(g,D,\delta), (\text{FHK}) \int_{[a,b]} f(x) \Delta x)
\]
\[
= D \left( \sum_{\xi_i \in [a,b]} g(\xi_i) \Delta x_i, (\text{FHK}) \int_{[a,b]} f(x) \Delta x \right)
\]
\[
= D \left( \sum_{\xi_i \in E} g(\xi_i) \Delta x_i + \sum_{\xi_i \in [a,b] \setminus E} g(\xi_i) \Delta x_i, (\text{FHK}) \int_{[a,b]} f(x) \Delta x \right)
\]
\[
= D \left( \sum_{\xi_i \in E} g(\xi_i) \Delta x_i + \sum_{\xi_i \in [a,b] \setminus E} f(\xi_i) \Delta x_i + \sum_{\xi_i \in E} f(\xi_i) \Delta x_i, (\text{FHK}) \int_{[a,b]} f(x) \Delta x \right)
\]
\[
= (\text{FHK}) \int_{[a,b]} f(x) \Delta x + \sum_{\xi_i \in E} f(\xi_i) \Delta x_i.
\]

Therefore,
\[ D \left( S(g, D, \delta), (FHK) \int_{[a,b]} f(x) \Delta x \right) \]
\[ \leq D \left( \sum_{\xi \in [a,b]} f(\xi) \Delta x_i, (FHK) \int_{[a,b]} f(x) \Delta x \right) \]
\[ + D \left( \sum_{\xi \in E} g(\xi) \Delta x_i, \sum_{\xi \in E} f(\xi) \Delta x_i \right) \]
\[ \leq \varepsilon + \sum_{j=1}^{\infty} \sum_{i \in E_j} D(f(\xi_i), g(\xi_i)) \Delta x_i \]
\[ \leq 2\varepsilon. \]

The proof is complete.

**Theorem 6 (See [32])**. Let \([a, b]_T\) be given. Assume

1. \( \lim_{n \to \infty} f_n(x) = f(x) \) holds \( \Delta \) a.e.;
2. \( G(x) \leq f_n(x) \leq H(x) \) holds \( \Delta \) a.e.;
3. \( f_n(x), G(x), H(x) \in \mathcal{H}_\mathcal{K}[a,b]_T \).

Then \( f(x) \in \mathcal{H}_\mathcal{K}[a,b]_T \). Moreover,

\[ \lim_{n \to \infty} (HK) \int_{[a,b]} f_n(x) \Delta x = (HK) \int_{[a,b]} f(x) \Delta x. \]

**Theorem 7.** Function \( f(x) \in \mathcal{H}_\mathcal{K}[a,b]_T \) if and only if \( f(x)^\alpha, \overline{f(x)}^\alpha \in \mathcal{H}_\mathcal{K}[a,b]_T \) for all \( \alpha \in [0, 1] \) uniformly, i.e., the \( \Delta \)-gauge in Definition 1 is independent of \( \alpha \).

**Proof.** (Necessity) Let \( \overline{\Lambda} = (FHK) \int_{[a,b]} f(x) \Delta x \). Given \( \varepsilon > 0 \), there exists a \( \delta \in \Gamma(\Delta, [a,b]_T) \) such that \( D\left(S(f, D, \delta), \overline{\Lambda}\right) < \varepsilon \) for any \( D \in \mathcal{D}(\delta, [a,b]_T) \). Then,

\[ \sup_{\alpha \in [0,1]} \max \left\{ |S(f, D, \delta) - \overline{\Lambda}^\alpha|, |S(f, D, \delta)|^\alpha - \overline{\Lambda}^\alpha \right\} \]
\[ = \sup_{\alpha \in [0,1]} \max \left\{ |S(f^\alpha, D, \delta) - \overline{\Lambda}^\alpha|, |S(f^\alpha, D, \delta)| - \overline{\Lambda}^\alpha \right\} \]
\[ < \varepsilon \]

and

\[ |S(f^\alpha, D, \delta) - \overline{\Lambda}^\alpha| < \varepsilon, \quad |S(f^\alpha, D, \delta)| - \overline{\Lambda}^\alpha | < \varepsilon \]

for any \( \alpha \in [0,1] \) and for any \( D \in \mathcal{D}(\delta, [a,b]_T) \). Thus, \( f(x)^\alpha, \overline{f(x)}^\alpha \in \mathcal{H}_\mathcal{K}[a,b]_T \) uniformly for any \( \alpha \in [0,1] \).
(Sufficiency) Let \( \varepsilon > 0 \). By assumption, there exists a \( \delta \in \Gamma(\Delta, [a, b]_T) \) such that
\[
|S(f^\alpha, D, \delta) - \overline{A^\alpha}| < \varepsilon, \quad |\overline{S(f^\alpha, D, \delta)} - \overline{\overline{A^\alpha}}| < \varepsilon
\]
for any \( D \in \mathcal{D}(\delta, [a, b]_T) \) and for any \( \alpha \in [0, 1] \), where
\[
\overline{A^\alpha} = (FHK) \int_{[a, b]_T} f^\alpha \Delta x, \quad \overline{\overline{A^\alpha}} = (FHK) \int_{[a, b]_T} \overline{f^\alpha} \Delta x.
\]

To prove that \( \{\overline{A^\alpha}, \overline{\overline{A^\alpha}}\} \) represents a fuzzy number, it is enough to check that \( \overline{A^\alpha}, \overline{\overline{A^\alpha}} \) satisfies items (1)–(3) of Lemma 1:

1. For \( \alpha \in [0, 1] \), if \( f^\alpha \leq \overline{f^\alpha} \), then \( \overline{A^\alpha} \leq \overline{\overline{A^\alpha}} \), i.e., the interval \( [\overline{A^\alpha}, \overline{\overline{A^\alpha}}] \) is closed.

2. \( f^\alpha \) and \( \overline{f^\alpha} \) are, respectively, nondecreasing and nonincreasing functions on \( [0, 1] \).

   For any \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \) one has
\[
(FHK) \int_{[a, b]_T} f^{\alpha_1} \Delta x \leq (FHK) \int_{[a, b]_T} f^{\alpha_2} \Delta x \leq (FHK) \int_{[a, b]_T} \overline{f^{\alpha_2}} \Delta x \leq (FHK) \int_{[a, b]_T} \overline{f^{\alpha_1}} \Delta x.
\]

   This implies \( [\overline{A^{\alpha_1}}, \overline{A^{\alpha_1}}] \supset [\overline{A^{\alpha_2}}, \overline{A^{\alpha_2}}] \).

3. For any \( \{\alpha_n\} \) satisfying \( \alpha_n \leq \alpha_{n+1} \) and \( \alpha_n \to \alpha \in (0, 1] \), we have
\[
\bigcap_{n=1}^{\infty} \overline{f^{\alpha_n}} = \overline{f^\alpha},
\]
that is,
\[
\bigcap_{n=1}^{\infty} \overline{f^{\alpha_n}, \overline{f^{\alpha_n}}} = \overline{f^\alpha, \overline{f^\alpha}},
\]
\[
\lim_{n \to \infty} \overline{f^{\alpha_n}} = \overline{f^\alpha} \quad \text{and} \quad \lim_{n \to \infty} \overline{\overline{f^{\alpha_n}}} = \overline{\overline{f^\alpha}}.
\]

Moreover,
\[
f^0 \leq \overline{f^{\alpha_n}} \leq f^1, \quad \overline{f^0} \leq \overline{f^{\alpha_n}} \leq \overline{f^1}.
\]

Thanks to Theorem 6 we have \( \overline{f^\alpha}, \overline{\overline{f^\alpha}} \in \mathcal{H}, \mathcal{H}^{[a,b]_T} \) and
\[
\lim_{n \to \infty} (HK) \int_{[a, b]_T} \overline{f^{\alpha_n}} \Delta x = (HK) \int_{[a, b]_T} f^\alpha \Delta x, \quad \lim_{n \to \infty} (HK) \int_{[a, b]_T} \overline{f^{\alpha_n}} \Delta x = (HK) \int_{[a, b]_T} \overline{f^\alpha} \Delta x.
\]
Consequently,
\[ \bigcap_{n=1}^{\infty} [\overline{A_n}, \underline{A_n}] = [\overline{A}, \underline{A}]. \]

Define \( \tilde{A} \) by \( \{ [\overline{A^\alpha}, \underline{A^\alpha}], \alpha \in [0, 1] \} \). Thus,
\[ D(S(f, D, \delta), \tilde{A}) < \varepsilon \]
for each \( D \in \mathcal{D}(\delta, [a, b]_T) \).

**Definition 3.** A sequence \( \{ f_n(x) \} \) of HK \( \Delta \)-integrable functions is called uniformly FHK \( \Delta \)-integrable on \([a, b]_T\) if for each \( \varepsilon > 0 \) there exists a \( \delta \in \Gamma(\Delta, [a, b]_T) \) such that
\[ D(S(f_n, D, \delta), (FHK) \int_{[a, b]_T} f_n(x) \Delta x) < \varepsilon \]
for any \( D \in \mathcal{D}(\delta, [a, b]_T) \) and for any \( n \).

**Theorem 8.** Let \( f_n(x) \in \mathcal{F} \mathcal{H} \mathcal{K}_{[a, b]_T}, n = 1, 2, \ldots \) satisfy:
(1) \( \lim_{n \to \infty} f_n(x) = f(x) \) on \([a, b]_T\);
(2) \( f_n(x) \) are uniformly FHK \( \Delta \)-integrable on \([a, b]_T\).

Then \( f(x) \in \mathcal{F} \mathcal{H} \mathcal{K}_{[a, b]_T} \) and
\[ \lim_{n \to \infty} (FHK) \int_{[a, b]_T} f_n(x) \Delta x = (FHK) \int_{[a, b]_T} f(x) \Delta x. \]

**Proof.** Let \( \varepsilon > 0 \). By assumption, there exists a \( \delta \in \Gamma(\Delta, [a, b]_T) \) such that
\[ D(S(f_n, D, \delta), (FHK) \int_{[a, b]_T} f_n(x) \Delta x) < \varepsilon \]
for any \( D \in \mathcal{D}(\delta, [a, b]_T) \) and for every \( n \). Fix a \( D_0 \in \mathcal{D}(\delta, [a, b]_T) \). From (1) of Theorem \( 8 \) there exists \( N \) such that
\[ D(S(f_n, D_0, \delta), S(f_m, D_0, \delta)) < \varepsilon \]
for arbitrary \( n, m > N \). Then,
\[
D \left( (FHK) \int_{[a, b]_T} f_n(x) \Delta x, (FHK) \int_{[a, b]_T} f_m(x) \Delta x \right) \\
\leq D \left( S(f_n, D_0, \delta), (FHK) \int_{[a, b]_T} f_n(x) \Delta x \right) + D \left( S(f_m, D_0, \delta), S(f_m, D_0, \delta) \right) \\
+ D \left( S(f_m, D_0, \delta), (FHK) \int_{[a, b]_T} f_m(x) \Delta x \right) \\
< 3\varepsilon
\]
for any $n, m > N$ and, hence, $\left\{ (FH \Delta) f_n(x) \Delta x \right\}$ is a Cauchy sequence. Let

$$\lim_{n \to \infty} (FH \Delta) \int_{[a,b]} f_n(x) \Delta x = \bar{A}.$$ 

We now prove that

$$\bar{A} = (FH \Delta) \int_{[a,b]} f(x) \Delta x.$$ 

Let $\varepsilon > 0$. By hypothesis, there exists a $\delta \in \Gamma(\Delta, [a,b])$ such that

$$D \left( S(f_n, D, \delta), (FH \Delta) \int_{[a,b]} f_n(x) \Delta x \right) < \varepsilon$$

for any $D \in \mathcal{D}(\delta, [a,b])$ and for all $n$. Choose $N$ that satisfies

$$D \left( (FH \Delta) \int_{[a,b]} f_n(x) \Delta x, \bar{A} \right) < \varepsilon$$

for all $n > N$. For the above $D$ and $N$, there exists $N_0 > N$ satisfying

$$D \left( (FH \Delta) \int_{[a,b]} f_{N_0}(x) \Delta x, \bar{A} \right) < 3\varepsilon$$

and the result follows.

**Definition 4 (See [10])**. A function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous on $[a,b]$, if for each $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \varepsilon$$

whenever $\bigcup_{i=1}^{n} [x_{i-1}, x_i] \subseteq [a,b]$ and $\sum_{i=1}^{n} \Delta x_i < \gamma$.

**Theorem 9 (Dominated convergence theorem)**. Let the time scale interval $[a,b]$ be given. If $f_n(x) \in \mathcal{F}(\mathcal{K})[a,b]$, $n = 1, 2, \ldots$, satisfy

1. $\lim_{n \to \infty} f_n(x) = f(x)$ $\Delta$ a.e.;
2. $G(x) \leq f_n(t) \leq H(x)$ $\Delta$ a.e. and $G(x), H(x) \in \mathcal{F}(\mathcal{K})[a,b]$,
then sequence \( \{f_n(x)\} \) is uniformly FHK \( \Delta \)-integrable. Thus, \( f(x) \in \mathcal{FHK}_{[a,b]} \) and
\[
\lim_{n \to \infty} (FHK) \int_{[a,b]} f_n(x) \Delta x = (FHK) \int_{[a,b]} f(x) \Delta x.
\]

Proof. By hypothesis, one has
\[
D(f_p(x), f_q(x)) = \sup_{\alpha \in [0,1]} \max \left\{ \left| f_p(x)^\alpha - f_q(x)^\alpha \right|, \left| f_p(x) - f_q(x)^\alpha \right| \right\}
\leq \sup_{\alpha \in [0,1]} \max \left\{ \left| H(x)^\alpha - G(x)^\alpha \right|, \left| H(x) - G(x)^\alpha \right| \right\}
= D(H(x), G(x)).
\]

Then, \( D(H(x), G(x)) \) is Lebesgue \( \Delta \)-integrable. Let
\[
D(x) = \int_{[a,x]} D(H(s), G(s)) \Delta s.
\]
From [10], \( D(x) \) is absolutely continuous on \([a,b] \). Let \( \varepsilon > 0 \). Then there exists \( \gamma > 0 \) such that
\[
\sum_{i=1}^{n} |D(x_i) - D(x_{i-1})| < \frac{\varepsilon}{b-a}
\]
whenever \( \bigcup_{i=1}^{n} [x_{i-1}, x_i] \subset [a,b] \) and \( \sum_{i=1}^{n} \Delta x_i < \gamma \). The limit \( \lim_{n \to \infty} f_n(x) = f(x) \) holds \( \Delta \) a.e. on \([a,b] \) and \( \{D(f_n(x), f(x))\} \) is a sequence of \( \Delta \)-measurable functions. Thanks to the Egorov's theorem, there exists an open set \( \Omega \) with \( m(\Omega) < \delta \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) uniformly for \( x \in [a,b] \setminus \Omega \). Thus, there exists \( N \) such that \( D(f_p(x), f_q(x)) < \frac{\varepsilon}{m(\Omega)} \) for any \( p, q > N \) and for any \( x \in [a,b] \setminus \Omega \). Suppose that \( \delta_1 \in \Gamma(\Delta, [a,b]) \) such that
\[
\left| S(D(H(x), G(x)), D, \delta_1) - \int_{[a,b]} D(H(x), G(x)) \Delta x \right| < \varepsilon
\]
and
\[
D\left( S(f_n, D, \delta_1), (FHK) \int_{[a,b]} f_n(x) \Delta x \right) < \varepsilon
\]
for \( 1 \leq n \leq N \) and for any \( D \in \mathcal{D}(\delta_1, [a,b]) \). Define \( \delta \in \Gamma(\Delta, [a,b]) \) by
\[
\delta(\xi) = \begin{cases} 
\delta_1(\xi) & \text{if } \xi \in [a,b] \setminus \Omega, \\
\min\{\delta_1(\xi), \rho(\xi, \Omega)\} & \text{if } \xi \in \Omega,
\end{cases}
\]
where \( \rho(\xi, \Omega) = \inf\{|\xi - \xi'| : \xi' \in \Omega\} \). Fix \( n > N \). One has
\[
D\left( S(f_n, D, \delta), S(f_N, D, \delta) \right) = D\left( \sum_{\xi \in [a,b]} f_n(\xi) \Delta x_i, \sum_{\xi \in [a,b]} f_N(\xi) \Delta x_i \right)
\]
\[ \sum_{\xi_i \in [a,b]_T \setminus \Omega} f_n(\xi_i) \Delta x_i, \sum_{\xi_i \in [a,b]_T \setminus \Omega} f_N(\xi_i) \Delta x_i \] 

\[ \leq \varepsilon + \sum_{\xi_i \in \Omega} D(f_n(\xi_i), f_n(\xi_i)) \Delta x_i \]

\[ \sum_{\xi_i \in \Omega} D(H(\xi_i), G(\xi_i)) \Delta x_i \]

\[ \leq 3\varepsilon \]

for any \( D \in \mathcal{D}(\delta, [a,b]_T) \). Hence,

\[ \mathcal{D}\left( S(f_n, D, \delta), (FHK) \int_{[a,b]_T} f_n(x) \Delta x \right) \]

\[ \leq \mathcal{D}(S(f_n, D, \delta), S(f_n, D, \delta)) + \mathcal{D}\left( S(f_N, D, \delta), (FHK) \int_{[a,b]_T} f_n(x) \Delta x \right) \]

\[ + \mathcal{D}\left( (FHK) \int_{[a,b]_T^2} f_n(x) \Delta x, (FHK) \int_{[a,b]_T^2} f_n(x) \Delta x \right) \]

\[ \leq 5\varepsilon. \]

Our dominated convergence theorem is proved.

As a consequence of Theorem 9, we get the following monotone convergence theorem.

**Theorem 10 (Monotone convergence theorem).** Let the time scale interval \([a,b]_T\) be given. If \( f_n(x) \in \mathcal{FHK}[a,b]_T \), \( n = 1, 2, \ldots \), satisfy

1. \( \lim_{n \to \infty} f_n(x) = f(x) \) a.e.;
2. \( \{f_n(x)\} \) is a monotone sequence and \( f_n(x) \in \mathcal{FHK}[a,b]_T \);

then \( \{f_n(x)\} \) is uniformly \( FHK \) \( \Delta \)-integrable. Consequently, \( f(x) \in \mathcal{FHK}[a,b]_T \).

Moreover,

\[ \lim_{n \to \infty} (FHK) \int_{[a,b]_T} f_n(x) \Delta x = (FHK) \int_{[a,b]_T} f(x) \Delta x. \]

4 Conclusion

We investigated the fuzzy Henstock–Kurzweil (FHK) delta integral on time scales. Our results give a common generalization of the classical FHK and HK integrals. For future researches, we will investigate the characterization of FHK delta integrable functions. Another interesting line of research consists to study the concept of fuzzy Henstock–Stieltjes integral on time scales.
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