AN UNCOUNTABLE FAMILY OF 3-GENERATED GROUPS 
WITH ISOMORPHIC PROFINITE COMPLETIONS

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Abstract. We construct an uncountable family of 3-generated residually finite just-infinite groups with isomorphic profinite completions. We also show that word growth rate is not a profinite property.

1. Introduction

Profinite completion $\hat{G}$ of a group $G$ is the limit of the inverse system of all finite quotients $G/N$ with respect to canonical epimorphisms $G/N_1 \to G/N_2$ induced by inclusions $N_1 \subseteq N_2$.

If $K$ is the intersection of all normal subgroups of finite index of $G$, then $\hat{G}/H$ is naturally isomorphic to $\hat{G}$. Therefore, we may restrict ourselves to residually finite groups, i.e., assume that intersection of finite index normal subgroups of $G$ is trivial.

To what extend does the profinite completion $\hat{G}$ determine the structure of $G$? Which group-theoretic properties are preserved if we replace a residually finite group by a residually finite group with the same profinite completion? (Such properties are said to be profinite.)

One of motivations of these question comes from the paper [11] of A. Grothendieck, where the following questions was asked in relation with representation theory of groups. Let $u : G \to H$ be a homomorphism of finitely presented residually finite groups such that the induced homomorphism $\hat{u} : \hat{G} \to \hat{H}$ is an isomorphism. Is $u$ an isomorphism? This question was negatively answered by M. Bridson and F. Grunewald in [5]. Finitely generated examples (without the condition of being finitely presented) were constructed before in [24, 4, 25].

The “flexibility” of a group $G$ in the sense of its relation to the profinite completion is formalized by the notion of its genus. It is known that two finitely generated groups have isomorphic completions if and only if the sets of their finite quotients are equal, see [26 Corollary 3.2.8]. The genus of a group $G$ is the set of isomorphism classes of residually finite finitely generated groups $H$ (or groups $H$ belonging to some other fixed class) such that the sets of isomorphism classes of finite quotients of $G$ and $H$ are equal, see [21, 13]. Groups whose genus contains only one element are completely determined by their profinite completion. Groups that have finite genus also can be considered as “rigid”.

It was shown by P. F. Pickel that genus of a virtually nilpotent finite generated group is always finite [22]. Later it was shown by F. J. Grunewald, P. F. Pickel, and D. Segal [12] that the same result holds for virtually polycyclic groups. See a survey of similar rigidity results in [13].
Examples of infinite (countable) genera for metabelian groups was given by P. F. Pickel in [23].

L. Pyber constructed examples of groups of uncountable genus in [25] using direct products of alternating groups.

In our note we construct a new family of finitely generated groups of uncountable genus. Namely, we construct an uncountable family of 3-generated groups $D_w$, $w \in \{0, 1\}^\infty$ with the following properties.

1. The isomorphism classes in the family $\{D_w\}$ are countable, hence there are uncountably many different isomorphism classes among the groups $D_w$.
2. Each group $D_w$ is residually finite, and every proper quotient of $D_w$ is a finite 2-group.
3. The profinite completions of $D_w$ are pairwise isomorphic for all sequences $w \in \{0, 1\}^\infty$ that have infinitely many zeros.
4. There are uncountably many different word-growth types among the groups $D_w$.

The family $\{D_w\}$ appears naturally in the study of group-theoretic properties of iterations of the complex polynomial $z^2 + i$ and the associated map on the moduli space of 4-punctured sphere, see [2]. It was for the first time defined in [17]. Later, in [19], it was shown that one of the groups in this family has non-uniform exponential growth, and then it was used in [20] to study the structure of the Julia set of an endomorphism of $\mathbb{C}P^2$.

We have tried to make these notes reasonably self-contained. Proofs of all results of [17] necessary for the main result (existence of three-generated residually finite groups of uncountable genus) are included, except for some facts that require straightforward computation.

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2. Definition of the family and its properties

2.1. Topological definition. Let $f_1, f_2, \ldots$ be a sequence of complex polynomials, which we arrange in an inverse sequence

$$
\mathbb{C} \leftarrow f_1 \mathbb{C} \leftarrow f_2 \mathbb{C} \leftarrow f_3 \mathbb{C} \ldots
$$

Suppose that for every $n \geq 0$ there exists a set $\{A_n, B_n, \Gamma_n\} \subset \mathbb{C}$ of pairwise different numbers such that $A_n$ is the critical value of $f_n$, and

$$
f_n(A_n) = B_{n-1}, \quad f_n(B_n) = \Gamma_{n-1}, \quad f_n(\Gamma_n) = B_{n-1},
$$

for all $n \geq 1$, see Figure 1. An example of such a sequence is the constant sequence $f_n(z) = z^2 + i$, for $A_n = i$, $B_n = i - 1$, $\Gamma_n = -i$.

Denote $\mathcal{M}_n = \mathbb{C} \setminus \{A_n, B_n, \Gamma_n\}$. The restriction $f_n : \mathcal{M}'_n \to \mathcal{M}_{n-1}$, where $\mathcal{M}'_n = f^{-1}_n(\mathcal{M}_{n-1}) \subset \mathcal{M}_n$, is a degree two covering map.

Choose a basepoint $t \in \mathcal{M}_0$. The union of its backward images:

$$
T_t = \{t\} \sqcup \bigsqcup_{n \geq 1} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)
$$

has a natural structure of a rooted tree with the root $t$ in which a vertex $z \in (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)$ is connected to $f_n(z)$. 
The fundamental group $\pi_1(M_0, t)$ acts naturally by the monodromy action on the levels $L_n = (f_1 \circ \cdots \circ f_n)^{-1}(t)$ of the tree $T_i$. These actions are automorphisms of $T_i$. Denote by $\operatorname{IMG}(f_1, f_2, \ldots)$ the quotient of $\pi_1(M_0, t)$ by the kernel of the action. See more on the iterated monodromy groups of this type in [18].

2.2. Automorphisms of a rooted tree. Let us give a more explicit description of the groups $\operatorname{IMG}(f_1, f_2, \ldots)$ as groups acting on rooted trees.

Let $X$ be a finite alphabet. Denote by $X^*$ the tree of finite words over $X$. Two vertices of this tree are connected by an edge if and only if they are of the form $v, vx$, for $v \in X^*$ and $x \in X$. The root of the tree $X^*$ is the empty word $\emptyset$.

Denote by $\operatorname{Aut}(X^*)$ the automorphism group of the rooted tree $X^*$. Let $g \in \operatorname{Aut}(X^*)$, and let $\pi \in \operatorname{Symm}(X)$ be the permutation $g$ induces on the first level $X \subset X^*$ of the tree. Define then automorphisms $g|_x \in \operatorname{Aut}(X^*)$ for $x \in X$ by the rule

$g(xw) = g(x)g|_x(w)$

for all $w \in X^*$. It is easy to see that $g|_x$ are well defined elements of $\operatorname{Aut}(X^*)$. The map

$g \mapsto \pi \cdot (g|_x)_{x \in X} \in \operatorname{Symm}(X) \ltimes \operatorname{Aut}(X^*)^X$

is an isomorphism $\operatorname{Aut}(X^*) \to \operatorname{Symm}(X) \ltimes \operatorname{Aut}(X^*)^X$ called the wreath recursion. We will identify the elements of $\operatorname{Aut}(X^*)$ with their images in $\operatorname{Symm}(X) \ltimes \operatorname{Aut}(X^*)^X$ and write $g = \pi \cdot (g|_x)_{x \in X}$.

Consider the case $X = \{0, 1\}$. Let us write the elements of $\operatorname{Aut}(X^*)^X$ as pairs $(h_0, h_1)$, and denote by $\sigma$ the only non-trivial element of $\operatorname{Symm}(X)$. Then the following equalities uniquely determine elements $g_0, g_1, g_2 \in \operatorname{Aut}(X^*)$:

$g_0 = \sigma, \quad g_1 = (g_0, g_2), \quad g_2 = (1, g_1),$

where an element of either $\operatorname{Symm}(X)$ or $\operatorname{Aut}(X^*)^X$ is not written if it is trivial. See Figure 2 for a graphical representations of the automorphisms $g_0, g_1, g_2$.

Let us modify the definition of the elements $g_0, g_1, g_2$ by using either $g_2 = (1, g_1)$ or $g_2 = (g_1, 1)$ depending on the level of the tree. More precisely, consider the space $X^\omega$ of the right-infinite sequences $x_1 x_2 \ldots$ over the alphabet $X = \{0, 1\}$ together with the shift map $s(x_1 x_2 \ldots) = x_2 x_3 \ldots$, and define for every $w \in X^\omega$ the
automorphisms \( \alpha_w, \beta_w, \gamma_w \) by the rules

\[
\alpha_w = \sigma,
\beta_w = (\alpha_{s(w)}, \gamma_{s(w)}),
\gamma_w = \begin{cases} 
(\beta_{s(w)}, 1) & \text{if the first letter of } w \text{ is 0}, \\
(1, \beta_{s(w)}) & \text{if the first letter of } w \text{ is 1}.
\end{cases}
\]

Denote \( D_w = \langle \alpha_w, \beta_w, \gamma_w \rangle \). It is easy to see (e.g., using the criterion from [8]) that \( \beta_w \) and \( \gamma_w \) are (independently) conjugate to \( g_1 \) and \( g_2 \) (and, of course, \( \alpha_w = g_0 \)).

The following is shown in [17, Proposition 7.1].

**Proposition 2.1.** For every sequence of polynomials \( f_1, f_2, \ldots \) as in Subsection 2.1 there exists a sequence \( w \in X^\omega \) such that \( \text{IMG} (f_1, f_2, \ldots) \) is conjugate (as a group acting on a binary rooted tree) to \( D_w \). Conversely, for every sequence \( w \in X^\omega \) there exists a sequence of polynomials \( f_1, f_2, \ldots \), satisfying the conditions of 2.1 and such that \( \text{IMG} (f_1, f_2, \ldots) \) is conjugate to \( D_w \).

The idea of the proof is as follows. Connect the points \( A_0, B_0, \Gamma_0 \) by disjoint simple paths \( l_{A_0}, l_{B_0}, l_{\Gamma_0} \) to infinity. Then define inductively paths \( l_{A_n}, l_{B_n}, l_{\Gamma_n} \) as lifts of the paths \( l_{B_{n-1}} \) and \( l_{\Gamma_{n-1}} \) connecting \( A_n, B_n, \Gamma_n \) to infinity. The lifts of the path \( l_{A_{n-1}} \) separate the plane into two connected components, which we label by \( S_0 \) and \( S_1 \), so that the first one contains \( A_n \), and the second one contains \( \Gamma_n \), see Figure 3. Then the vertices of the tree \( T_t = \bigcup_{n \geq 0} (f_1 \circ \cdots \circ f_n)^{-1} (t) \) are labeled by words over \( \{0, 1\} \) according to the itinerary with respect to the defined partitions. Define the generators \( \alpha, \beta, \gamma \) by small simple loops around \( A_0, B_0, \Gamma_0 \) connected to the basepoint by paths disjoint with \( l_{A_0}, l_{B_0}, l_{\Gamma_0} \). It is easy to see then that \( \alpha = \alpha_w, \beta = \beta_w, \gamma = \gamma_w \), where \( w \) is the sequence recording in which halves (\( S_0 \) or \( S_1 \)) of the plane the points \( B_n \) lie.

In the other direction, for every given \( w \in X^\omega \) it is easy to construct a sequence of branched coverings \( f_1, f_2, \ldots \) of plane such that \( \text{IMG} (f_1, f_2, \ldots) = \langle \alpha_w, \beta_w, \gamma_w \rangle \) (using the above construction with paths). Then we can put a complex structure on the first plane, and pull it back by the maps \( f_1 \circ \cdots \circ f_n \). Then \( f_n \) will become polynomials satisfying the conditions of Subsection 2.1.

**Proposition 2.2.** Let \( w_1 \neq w_2 \) be elements of \( X^\omega \). Then there does not exist \( g \in \text{Aut}(X^\omega) \) such that \( \alpha_{w_1} = g^{-1} \alpha_{w_2} g, \beta_{w_1} = g^{-1} \beta_{w_2} g, \) and \( \gamma_{w_1} = g^{-1} \gamma_{w_2} g \).
and since $\alpha$ and $\beta$ start with the first level. On the other hand, suppose that $\gamma$.

Proof. As in the previous paragraph, considering $\beta$ and $\gamma$, we conclude that $g$ acts trivially on the first level, i.e., is of the form $g = (h_0, h_1)$ for some $h_0, h_1 \in \text{Aut}(X^*)$. The equality $\alpha_1 = g^{-1} \alpha_2 g$ implies that $h_0 = h_1$. It follows then that $h_0$ conjugates the triples $(\alpha_s(w_1), \beta_s(w_1), \gamma_s(w_1))$ and $(\alpha_s(w_2), \beta_s(w_2), \gamma_s(w_2))$, which will eventually lead us to a contradiction.

In fact, it is proved in [16, Proposition 3.1] that for any triple $(h_0, h_1, h_2)$ such that $h_0, h_1, h_2$ are (independently) conjugate to $g_0, g_1, g_2$, respectively, there exists a unique sequence $w \in X^\omega$ such that $(h_0, h_1, h_2)$ are simultaneously conjugate to $(\alpha_w, \beta_w, \gamma_w)$. We proved uniqueness in Proposition [2.2].

Corollary 2.3. For every given $w_0 \in X^\omega$ the set of sequences $w \in X^\omega$ such that the group $\mathcal{D}_w$ is conjugate in $\text{Aut}(X^*)$ to $\mathcal{D}_{w_0}$ is at most countable.

Proof. If $\mathcal{D}_w$ is conjugate to $\mathcal{D}_{w_0}$, then $\mathcal{D}_{w_0}$ is generated by $\alpha' = g^{-1} \alpha_w g$, $\beta' = g^{-1} \beta_w g$, and $\gamma' = g^{-1} \gamma_w g$ for some $g \in \text{Aut}(X^*)$. By Proposition [2.2] the sequence $w$ is uniquely determined by $\alpha', \beta', \gamma'$. It follows that the cardinality of the set of possible sequences $w$ is not greater than the cardinality of the set of generating sets of size 3 of the group $\mathcal{D}_{w_0}$, which is at most countable.

2.3. A two-dimensional rational map. Let $f_1, f_2, \ldots$, and $A_n, B_n, \Gamma_n$ satisfy the conditions of Subsection [2.1]. Applying affine transformations $u_n : \mathbb{C} \to \mathbb{C}$, and replacing $f_n$ by $u_n^{-1} f_n u_n$, we may assume that $A_n = 0$, $B_n = 1$. Denote $p_n = \Gamma_n$. Since 0 is the critical value of $f_n$, we have $f_n(z) = (az + b)^2$ for some $a, b \in \mathbb{C}$. Since $f_n(A_n) = B_{n-1}$, we have $b^2 = 1$, and we may assume that $b = 1$. Since $f_n(\Gamma_n) = B_{n-1}$, we get $(ap_n + 1)^2 = 1$, hence $ap_n + 1 = -1$ (since $\Gamma_n \neq A_n$). It follows that $a = -2/p_n$. Then the condition $f_n(B_n) = \Gamma_{n-1}$ implies that $p_{n-1} = (1 - 2/p_n)^2$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Computation of IMG $(f_1, f_2, \ldots)$}
\end{figure}
v in the previous subsection. Denote $P$.

It is easy to check that the map $\tilde{D}$.

Further properties of the groups $D_w$. Let $\tilde{D}$.

Consider the map $P : \{1, 2, 3, 4\}^* \to X^*$ that applies to each letter of a word $v \in \{1, 2, 3, 4\}^*$ the map

$$1 \mapsto 0, \quad 2 \mapsto 0, \quad 3 \mapsto 1, \quad 4 \mapsto 1.$$
Note that it follows directly from the recursive definition of the generators \(\alpha, \beta, \gamma, a, b, c\), that \(\mathcal{D}\) belongs to the kernel of the action of \(\tilde{\mathcal{D}}\) on \(X^*\), and that the action of \(a, b, c\) on \(X^*\) are defined by the recurrent rules

\[
(1) \quad a = \sigma, \quad b = (a, c), \quad c = (b, b).
\]

Let \(H\) be the quotient of \(\tilde{\mathcal{D}}\) by the kernel of its action on \(X^*\).

Direct computations show that the following relations hold in \(\tilde{\mathcal{D}}\):

\[
\begin{align*}
\alpha^a &= \alpha, & \alpha^b &= \alpha, & \alpha^c &= \alpha, \\
\beta^a &= \beta, & \beta^b &= \beta, & \beta^c &= \beta', \\
\gamma^a &= \gamma^\alpha, & \gamma^b &= \gamma^\beta, & \gamma^c &= \gamma,
\end{align*}
\]

which implies that \(\mathcal{D}\) is a normal subgroup of \(\tilde{\mathcal{D}}\) (see [17 Proposition 4.4]). In fact, it is proved in [17 Proposition 4.8] (but we will not need it in our paper) that \(\mathcal{D}\) coincides with the kernel of the action of \(\tilde{\mathcal{D}}\) on \(X^*\), i.e., that \(\tilde{\mathcal{D}}/\mathcal{D}\) is the group \(H\) generated by the elements \(a, b, c\) defined by (1).

For an infinite sequence \(w = x_1x_2\ldots \in X^\omega\), denote by \(T_w\) the rooted subtree of \(\{1, 2, 3, 4\}^*\) equal to the inverse image \(P^{-1}(\{0, x_1, x_1x_2, x_1x_2x_3, \ldots\})\) of the corresponding path in \(X^*\). It follows directly from the definition of the generators \(\alpha, \beta, \gamma\) of \(\tilde{\mathcal{D}}\):

\[
\alpha = (12)(34), \quad \beta = (\alpha, \alpha, \alpha, \gamma), \quad \gamma = (\beta, 1, 1, \beta),
\]

that the tree \(T_w\) is \(\mathcal{D}\)-invariant, and that restriction of the action of the generators \(\alpha, \beta, \gamma\) onto \(T_w\) coincides with the action of the generators \(\alpha_w, \beta_w, \gamma_w\) of \(\mathcal{D}_w\), if we identify \(T_w\) with \(X^*\) by the map

\[
1 \mapsto 0, \quad 2 \mapsto 1, \quad 3 \mapsto 0, \quad 4 \mapsto 1.
\]

Consequently, the groups \(\mathcal{D}_w\) are obtained by restricting the action of the group \(\mathcal{D}\) onto the subtrees \(T_w\). We will denote the canonical epimorphism \(\mathcal{D} \rightarrow \mathcal{D}_w\) by \(P_w\).

For every \(g \in \tilde{\mathcal{D}}\) we have \(g(T_w) = T_{g(w)}\), just by the definition of the action of \(\tilde{\mathcal{D}}\) on \(X^\omega\). Then the next lemma follows from the fact that \(\mathcal{D}\) is normal in \(\tilde{\mathcal{D}}\).

**Lemma 2.4.** For every \(g \in \tilde{\mathcal{D}}\) the isomorphism \(g : T_w \rightarrow T_{g(w)}\) conjugates the groups \(\mathcal{D}_w\) and \(\mathcal{D}_{g(w)}\).

The group \(\tilde{\mathcal{D}}\) is transitive on the first level of the tree \(\{1, 2, 3, 4\}^*\), and the homomorphism from the stabilizer of the first level to \(\tilde{\mathcal{D}}\) given by \((g_1, g_2, g_3, g_4) \mapsto g_1\) is onto. This implies that \(\tilde{\mathcal{D}}\) is transitive on each level of the tree (it is level-transitive), see [16 Corollary 2.8.5]. Consequently \(H\) is also level-transitive.

We have finished a sketch of the proof of the following fact (see also [17 Proposition 4.6])

**Proposition 2.5.** If \(w_1, w_2 \in X^\omega\) belong to one \(H\)-orbit, then \(\mathcal{D}_{w_1}\) and \(\mathcal{D}_{w_2}\) are conjugate.

In fact, the converse is also true: if the groups \(\mathcal{D}_{w_1}\) and \(\mathcal{D}_{w_2}\) are conjugate, then \(w_1\) and \(w_2\) belong to one \(H\)-orbit, see [17 Theorem 5.1].

Note that since the action of \(H\) on \(X^*\) is level-transitive, for every \(w \in X^\omega\) the set of sequences \(w' \in X^\omega\) such that \(\mathcal{D}_w\) is conjugate to \(\mathcal{D}_{w'}\) is dense in \(X^\omega\). It also
follows that closures of \( D_w \) in the profinite group \( \text{Aut}(X^*) \) do not depend, up to isomorphism, on \( w \).

Recall, that for \( g \in \text{Aut}(X^*) \) and \( x \in X \) we denote by \( g|_x \) the element of \( \text{Aut}(X^*) \) defined by the condition \( g(xu) = g(x)g|_x(u) \). We use a similar definition of arbitrary words. Namely, for \( v \in X^* \) and \( g \in \text{Aut}(X^*) \), denote by \( g|_v \) the automorphism of \( X^* \) defined by the condition that
\[ g(vu) = g(v)g|_v(u) \]
for all \( u \in X^* \). The element \( g|_v \) is called the section of \( g \) at \( v \), and it can be computed by repeated application of the wreath recursion, since it satisfies
\[ g|_{x_1x_2\ldots x_n} = g|_{x_1}|_{x_2}\cdots|_{x_n}. \]

The following statement is proved by direct computation (see [17, Proposition 4.2]).

**Proposition 2.6.** The subgroups \( \langle \alpha, \beta \rangle, \langle \alpha, \gamma \rangle, \text{ and } \langle \beta, \gamma \rangle \) of \( D \) are isomorphic to dihedral groups of order 16, 8, and 16, respectively. Denote their union by \( N \). For every \( g \in D \) there exists \( n \) such that \( g|_v \in N \) for all words \( v \) of length at least \( n \).

In order to prove the second statement of the proposition one shows that the sections of elements of \( N : \{ \alpha, \beta, \gamma \} \) at long enough words belong to \( N \).

The next statement is also checked directly, see [17, Proposition 4.3].

**Lemma 2.7.** The canonical epimorphism \( P_w : D \to D_w \) is injective on \( N \) for all \( w \) different from 111\ldots.

The image of \( \beta_{111\ldots} \) has order 2 in \( D_{111\ldots} \).

Denote by \( X_0^w \) the set of all sequences \( w \in X^w \) with infinitely many zeros.

**Proposition 2.8.** Let \( w \in X_0^w \), and let \( A \) and \( B \) be finite subsets of \( D \) such that \( P_w(g) = 1 \) for all \( g \in A \) and \( P_w(g) \neq 1 \) for all \( g \in B \). Then there exists \( n \) such that if \( w' \in X_0^w \) has the same beginning of length \( n \) as \( w \), then \( P_{w'}(g) = 1 \) for all \( g \in A \) and \( P_{w'}(g) \neq 1 \) for all \( g \in B \).

In other words, the map \( w \mapsto \langle \alpha_w, \beta_w, \gamma_w \rangle \) is a continuous map from \( X_0^w \) to the space of three-generated groups.

**Proof.** It is enough to prove that for every \( g \in D \) there exists \( n \) such that if the length of the common beginning of sequences \( w_1, w_2 \in X_0^w \) is at least \( n \), then \( P_{w_1}(g) = 1 \) if and only if \( P_{w_2}(g) = 1 \).

Let us apply the wreath recursion defining \( D_w \) several times to \( g \). By Proposition 2.6 there exists \( n \) such that all sections \( g|_v \) belong to \( N \) for all words \( v \) of length at least \( n \). If the action of \( g \) is non-trivial on the \( n \)th level of the tree \( T_{w_1} \), then \( P_{w_1}(g) \neq 1 \). Suppose that the action is trivial. Then \( P_{w_1}(g) = 1 \) if and only if all sections \( P_{w_1}(g)|_{P_{w_2}(v)} = P_{s^n(w_1)}(g)|_v \) for \( v \in T_{w_1} \cap \{1, 2, 3, 4\}^n \) are trivial. But \( g|_v \in N \), and if \( s^n(w_1) \neq 111\ldots \), then \( g|_v \neq 1 \) if and only if \( P_{s^n(w_1)}(g)|_v \neq 1 \). It follows that triviality of \( P_{w_1}(g) \) depends only on the first \( n \) letters of \( w_1 \), provided \( s^n(w_1) \neq 111\ldots. \)

3. Profinite completion of \( D_w \)

**Definition 1.** Let \( G \) be a group acting on a rooted tree \( X^* \). The \( n \)th level rigid stabilizer \( \text{RiSt}_n(G) \) is the group generated by elements \( g \in G \) such that \( g \) acts
trivially on the nth level $X^n$ of $X^*$, and the sections $g|_v$ are trivial for all words $v \in X^n$ except for one.

Let us denote $\text{RiSt}_{n,w} = \text{RiSt}_n(D_w)$. Denote by $L_w$ the smallest normal subgroup $L_w$ of $D_w$ containing $\{[\alpha_w,\beta_w],[\gamma_w,\beta_w]\}$. Direct computations (see [17, Proposition 5.14]) show that $L_w$ (more formally, its image under the wreath recursion) contains $L_n(w) \times L_{s(w)}$. This implies that the direct product $L^X_{s^*(w)}$ is contained in $\text{RiSt}_{n,w}$. Moreover, it is also checked directly that index of $L_w$ in $D_w$ is finite. It follows that $\text{RiSt}_{n,w}$ is of finite index in $D_w$.

**Definition 2.** A group acting on rooted tree is branch if $\text{RiSt}_n(G)$ is of finite index in $G$ for all $n$. It is called weakly branch if $\text{RiSt}_n(G)$ are infinite for all $n$.

See [1] for more on branch groups.

The following theorem is proved in [14] (see also [16, Proposition 2.10.7]).

**Theorem 3.1.** Let $G_1, G_2 \leq \text{Aut}(X^*)$ be weakly branch groups, and let $\psi : G_1 \to G_2$ be an isomorphism. Suppose that there exist subgroups $H_n \leq G_1$ for all $n \geq 1$ such that the following conditions hold

1. $H_n$ and $\psi(H_n)$ belong to the stabilizer of the nth level;
2. the groups $H_n$ and $\psi(H_n)$ act level-transitively on the rooted subtrees $vX^* \subset X^*$ for $v \in X^n$.

Then the isomorphism $\psi$ is induced by conjugation in $\text{Aut}(X^*)$.

It is easy to produce subgroups $H_n$ of $D_w$ satisfying the conditions of the above theorem. For example, one can take $H_1$ to be the group generated by squares of elements $D_w$, and then define inductively $H_n$ as the group generated by squares of the elements of $H_{n-1}$, see [17, Section 6].

It follows that the isomorphism relation between groups $D_w$ coincides with conjugacy. In particular, it follows from Corollary [2.3] that the isomorphism classes in the family $D_w$ are countable. More precise information gives [17, Theorem 6.1]: two groups $D_w$ and $D_w'$ are isomorphic if and only if the sequences $w_1$ and $w_2$ are co-final, i.e., are of the form $w_1 = v_1 w$ and $w_2 = v_2 w$, where the words $v_1, v_2 \in X^*$ have equal length.

The following theorem is proved in [1] Theorem 5.2 and Lemma 5.3.

**Theorem 3.2.** Let $G$ be a level-transitive subgroup of $\text{Aut}(X^*)$. For every non-trivial normal subgroup $N$ of $G$ there exists $n$ such that $N$ contains the commutator subgroup $\text{RiSt}_n(G)'$ of $\text{RiSt}_n(G)$.

It is not hard to show that $\text{RiSt}'_{n,w}$ has finite index in $D_w$ (see [17, Proposition 5.15]). In particular, this shows that the groups $D_w$ are just-infinite, i.e., that all their non-trivial normal subgroups have finite index. (Note that there is a typo in [17, Proposition 5.15]: the word “normal” is missing.)

**Theorem 3.3.** The profinite completion $\hat{D}_w$ of $D_w$ does not depend on $w$, if $w$ has infinitely many zeros.

**Proof.** By [26, Corollary 3.2.8], it is enough to prove that the sets of all finite quotients of $D_w$ do not depend on $w \in X^*_n$. Since a group is a proper quotient of $D_w$ if and only if it is a quotient of $D_w/\text{RiSt}_{n,w}'$ for some $n$, it is enough to prove Proposition 3.4 below.
Proposition 3.4. The quotient $D_w/\text{RiSt}_{n,w}'$ does not depend, up to isomorphism, on $w \in X^m_n$.

Proof. Fix a sequence $w \in X^m_n$. The subgroups $\text{RiSt}_{n,w}$ and $\text{RiSt}_{n,w}'$ have finite index in $D_w$, hence they are finitely generated.

Let $[P_w(g_1), P_w(h_1)], [P_w(g_2), P_w(h_2)], \ldots, [P_w(g_k), P_w(h_k)]$ be a finite generating set of $\text{RiSt}_{n,w}'$, where $g_i, h_i \in D$ are such that $P_w(g_i), P_w(h_i) \in \text{RiSt}_{n,w}$. Moreover, we may assume that for every $i$ all sections of $P_w(g_i)$ and $P_w(h_i)$ at words of length $n$ are trivial except for one word $v_i$. It follows from Proposition 2.8 that there exists $m_1 \geq n$ such that if $w' \in X^m_n$ is such that $w$ and $w'$ have a common beginning of length at least $m_1$, then $P_w'(g_i), P_w'(h_i) \in \text{RiSt}_{n,w}'$ for all $i = 1, \ldots, k$.

Let $A = \{1, a_1, a_2, \ldots, a_m\} \subset D$ be such that $P_w(A)$ is a coset transversal of $D_w$ modulo $\text{RiSt}_{n,w}'$. Consider the multiplication table for $D_w/\text{RiSt}_{n,w}'$, and write its entries as relations of the form $P_w(a_i a_j a_k^{-r_{i,j}}) = 1$, where $r_{i,j}$ is a product of the generators $P_w([g_i, h_i])$ of $\text{RiSt}_{n,w}'$. We get a finite set of relations, hence, by Proposition 2.8, there exists $m_2 \geq m_1$ such that if $w' \in X^m_n$ has a common beginning with $w$ of length at least $m_2$, then the corresponding relations $P_{w'}(a_i a_j a_k^{-r})$ also hold in $D_{w'}$. It follows that $D_w/\text{RiSt}_{n,w}'$ is a homomorphic image of $D_w/\text{RiSt}_{n,w}$, where the homomorphism is induced by the map

$$P_w(g) \mapsto P_{w'}(g), \quad g \in A.$$ (2)

Choose now $w \in X^m_n$ such that $D_w/\text{RiSt}_{n,w}'$ has the smallest possible order. Then for all $w'$ with a long enough common beginning with $w$ the group $D_{w'}/\text{RiSt}_{n,w}'$ is isomorphic to $D_w/\text{RiSt}_{n,w}'$. But the orbits of $H$ on $X^m_n$ are dense, hence for every $w' \in X^m_n$ there exists $g \in D$ such that $g(w')$ is arbitrarily close to $w$, i.e., has arbitrarily long common beginning with $w$. By Lemma 2.4, $D_w/\text{RiSt}_{n,w}'$ is isomorphic to $D_{g(w')}/\text{RiSt}_{n,g(w')}$, which finishes the proof. \hfill \Box

4. Word growth

Let $G$ be a finitely generated group. For a finite generating set $S$, the word growth function if the function $r_G,S(n)$ equal to the number of elements of $G$ that can be represented as products of length at most $n$ of the generators $s \in S$ and their inverses.

For two non-decreasing positive functions $r_1, r_2 : \mathbb{N} \to \mathbb{R}$, we write $r_1 \prec r_2$ if there exists a constant $C > 1$ such that

$$r_1(n) \leq C r_2(C(n))$$

for all $n \in \mathbb{N}$. We say that $r_1$ and $r_2$ are equivalent if $r_1 \prec r_2$ and $r_2 \prec r_1$.

The growth type of a finite generated group $G$ is the equivalence class of its word growth function. It is easy to prove that the equivalence class of the growth function $r_{G,S}(n)$ does not depend on the choice of the generating set $S$. See [15] for a survey of results on word growth of groups.

It is a consequence of Gromov’s theorem on groups of polynomial growth [10] and the formula for degree of polynomial growth of a virtually nilpotent group [3] that if $G_1$ and $G_2$ are finitely generated residually finite groups such that $\hat{G_1} \cong \hat{G_2}$, and $G_1$ is of polynomial growth, then growth types of $G_1$ and $G_2$ are the same. In other words, the growth type of groups of polynomial growth is a profinite property.
In general, however, we can not reconstruct the growth type of a group from its profinite completion.

**Proposition 4.1.** The set of growth types of groups \( \{D_w : w \in X_\omega^0 \} \) is uncountable. In particular, growth type of a finitely generated residually finite group is not a profinite property.

The first statement of the proposition was proved in [19, Corollary 4.7]. The proof uses the fact that the closure of the set \( \{D_w : w \in X_\omega \} \) in the space of three-generated groups contains a group of exponential and a group of intermediate growth. The latter is \( D_{000...} \cong \text{IMG} (z^2 + i) \), sub-exponential growth of which was proved in [6]. One can use these facts and Proposition 2.8 to prove, in the same way as it is done for the family of Grigorchuk groups in [9], that the set of growth types of groups \( D_w \) is uncountable.

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