Character Formulae of $\hat{sl}_n$-Modules and Inhomogeneous Paths

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Abstract

Let $B(l)$ be the perfect crystal for the $l$-symmetric tensor representation of the quantum affine algebra $U'_q(\hat{sl}_n)$. For a partition $\mu = (\mu_1, \ldots, \mu_m)$, elements of the tensor product $B(\mu_1) \otimes \cdots \otimes B(\mu_m)$ can be regarded as inhomogeneous paths. We establish a bijection between a certain large $\mu$ limit of this crystal and the crystal of an (generally reducible) integrable $U_q(\hat{sl}_n)$-module, which forms a large family depending on the inhomogeneity of $\mu$ kept in the limit. For the associated one dimensional sums, relations with the Kostka-Foulkes polynomials are clarified, and new fermionic formulae are presented. By combining their limits with the bijection, we prove or conjecture several formulae for the string functions, branching functions, coset branching functions and spinon character formula of both vertex and RSOS types.

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0 Introduction

Probably, the Kostka-Foulkes polynomial ranks among the most important polynomials in combinatorics and representation theory. Let \( \lambda, \mu \) be partitions with the same number of nodes. The Kostka-Foulkes polynomial \( K_{\lambda \mu}(q) \) is defined as the transition matrix which expresses the Schur function \( s_\lambda(x) \) in terms of the Hall-Littlewood polynomials \( P_\mu(x; q) \): 

\[
   s_\lambda(x) = \sum_{\mu} K_{\lambda \mu}(q) P_\mu(x; q).
\]

(See [30] for details.)

Let us consider the affine Lie algebra \( \widehat{\mathfrak{sl}}_n \). We denote by \( V(l \Lambda_0) \) the integrable highest weight \( \widehat{\mathfrak{sl}}_n \)-module with highest weight \( l \Lambda_0 \). Let \( \lambda \) be a partition whose depth is less than or equal to \( n \). We further assume \(|\lambda|\) (= the number of nodes in \( \lambda \)) is divisible by \( n \). \( \lambda \) can also be viewed as a level \( l \) integral weight by \((l + \lambda_n - \lambda_1)\Lambda_0 + (\lambda_1 - \lambda_2)\Lambda_1 + \cdots + (\lambda_{n-1} - \lambda_n)\Lambda_{n-1} \). In [20], A.N. Kirillov conjectured the following identity.

\[
   \lim_{N \to \infty} q^{-EN} K_{(lN-|\lambda|/n)n+\lambda,(lN)}(q) = \sum_j (\dim M^{l \Lambda_0}_{\lambda-j\delta}) q^j,
\]

\[\begin{align*}
   M^{l \Lambda_0}_{\mu} = \{ v \in V(l \Lambda_0) \mid e_i v = 0 \ (i \neq 0), \text{wt} v = \mu \}.
\end{align*}\]

Here \( E_N \) is a known constant. For the definition of \((k^n) + \lambda\), see the beginning of Section 2.2. It had not been long before Nakayashiki and Yamada [31] solved this conjecture. Their idea was to relate Lascoux-Schützenberger’s charge of a tableau with the so-called energy of a path. Once this correspondence is established, the conjecture is found to be a corollary of the theory of perfect crystals [17, 18].

The purpose of this paper is to extend their result to more general setting and elucidate an interplay among the theory of crystals, the Kostka-Foulkes polynomials, one dimensional sums, their fermionic formulae and affine Lie algebra characters. In a sense this is a far reaching application of the corner transfer matrix method [1] and the Bethe ansatz [5] in solvable lattice models where many important ideas came from. Let us give below an overview of the main contents and results.

In Section 1 we recall the definition of the energy in crystal base theory based on [31].

In Section 2 we prove our first main Theorem 2.4 which establishes a bijection of crystals related to a large family of (generally reducible) integrable highest weight \( U_q(\widehat{\mathfrak{sl}}_n) \)-modules. To explain it more concretely, let us introduce some notations. Let \( B(\lambda) \) be the crystal base of the integrable highest weight \( U_q(\widehat{\mathfrak{sl}}_n) \)-module with highest weight \( \lambda \). The symmetric tensor representation of \( U_q^l(\widehat{\mathfrak{sl}}_n) \) of degree \( l \) also has a crystal base, which is denoted by \( B(lj) \). Let \( \mu \) be a partition with signature \((\mu_1, \mu_2, \cdots, \mu_m)\). We consider the tensor product

\[
   B(\mu_1) \otimes B(\mu_2) \otimes \cdots \otimes B(\mu_m).
\]

(0.1)
An element of this tensor product is to be called \textit{inhomogeneous} path, since the degrees $\mu_i$ are not necessarily equal. From the viewpoint of crystal base theory, Kirillov’s conjecture corresponds to the fact that if $\mu$ is of shape $(lnN)$, the crystal (1) in the limit $N \to \infty$ is bijective to $B(l\Lambda_0)$. In contrast to this, we consider in this paper the case when $\mu$ has the form:

$$\begin{array}{cccc}
1 & 2 & \cdots & l \\
L_1 & L_2 & \cdots & L_s \\
\mu^1 & \mu^2 & \cdots & \mu^s \\
\end{array}$$

We shall consider the limit $L_J - L_{J+1} \to \infty$ ($1 \leq J \leq s, L_{s+1} = 0$) with $l_J, \mu^J$ and $L_J \equiv r_J$ mod $n$ fixed. Our Theorem 2.4 together with Proposition 2.10 assert that such limit of (1) is bijective to

$$\bigotimes_{J=1}^{s} \left( \bigoplus_{p \in \mathcal{H}(l_j \Lambda_{r_j}, \mu^J)} B(l_j \Lambda_{r_j} + af(wt p) - (E(p) - E(l_j \Lambda_{r_j}, \mu^J))\delta) \right)$$

as affine weighted crystals. Here $E(p)$ is the energy described in Section 1.2, $\mathcal{H}(l \Lambda_r, \mu^*)$ is a set of restricted paths (2.2) and $E(l \Lambda_r, \mu^*)$ is a ground state energy (2.7). The theorem implies a “factorization” into a tensor product of the pieces $J = 1, \ldots, s$, and each piece itself is a direct sum of crystals of certain integrable highest weight modules. In the module language it corresponds to

$$\mathcal{V} = \bigotimes_{J=1}^{s} \left( \bigoplus_{p \in \mathcal{H}(l_j \Lambda_{r_j}, \mu^J)} V(l_j \Lambda_{r_j} + af(wt p) - (E(p) - E(l_j \Lambda_{r_j}, \mu^J))\delta) \right).$$

See also (3.17). With various choices of $\{l_J, r_J, \mu^J\}_{J=1}^{s}$, this $\mathcal{V}$ covers a large family of (generally reducible) $U_q(\widehat{\mathfrak{sl}}_n)$-modules.

In Section 3 we introduce three kinds of paths and the associated $q$-polynomials by extending those in [27] naturally to the inhomogeneous case. We call them the unrestricted, classically restricted and (level $l$) restricted one dimensional sums (1dsums) and denote by $g_{\mu}(\lambda), X_{\mu}(\lambda)$ and $X_{\mu}^{(l)}(\lambda)$, respectively. (Their analogues $g'_{\mu}(\lambda), X'_{\mu}(\lambda)$ and $X'_{\mu}^{(l)}(\lambda)$ for the antisymmetric tensor case $B(1^{\nu}) \otimes$
• • • ⊗ B_{(1 \mu_m)} are also introduced.) By definition they all have an expression \( \sum p q^{E(p)} \) where the sum runs over the weight \( \lambda \) subset of the corresponding set of paths. For \( \mu \) finite, Proposition 3.3 relates the 1dsums to the Kostka-Foulkes polynomials as

\[
 g_{\mu}(\lambda) = \sum_{\eta(\eta) \leq n} K_{\eta \lambda}(1) K_{\eta \mu}(q),
 X_{\mu}(\lambda) = K_{\lambda \mu}(q),
\]

where the latter is due to [31]. On the other hand, in the large \( \mu \) limit we have Proposition 3.6 as a corollary of Theorem 2.4. It identifies the limits of the 1dsums \( g_{\mu}(\lambda) \), \( X_{\mu}(\lambda) \) and \( X_{\mu}(L) \) with the string function \( c^{V}_{\lambda}(q) \), the classical branching function \( b^{V}_{\lambda}(q) \) and the coset branching function \( a^{V \otimes V}_{\lambda}(q) \), which are defined in (3.10)–(3.12) and detailed in (3.18)–(3.20). (\( l_0 = l - \sum_{j=1}^{\mu} l_j \).)

Thus it is an important clue to investigate the limiting behaviour of the Kostka-Foulkes and related polynomials for the study of these characters.

Until the end of Section 3 the paper only concerns the bijection of crystals and its general consequences on the 1dsums, which are independent of the concrete expressions. The rest of the paper is devoted to our second theme, explicit formulae of the 1dsums and their limits in fermionic forms. By fermionic forms we roughly mean those polynomials or series which are free of signs, admit a quasi-particle interpretation or have an origin in the Bethe ansatz, etc. Thanks to the absence of signs they are suitable for studying the limiting behaviour and serve as a key to establish various formulae for the characters related to the affine Lie algebras and Virasoro algebra. The main aim of Sections 4 and 5 is to illustrate this thesis on several examples.

In Section 4.1 we consider the 1dsums. As a prototype example we quote a Bethe ansatz type fermionic formula for the Kostka-Foulkes polynomials obtained by Kirillov and Reshetikhin [24] in Proposition 4.5. One of our main results in Section 4.1 is the fermionic formulae for the unrestricted 1dsums \( g_{\mu}(\lambda) \) and \( g'_{\mu}(\lambda) \) in Propositions 4.1 and 4.3. The former reads

\[
 g_{\mu}(\lambda) = \sum_{\eta(\eta) \leq n} K_{\eta \lambda}(1) K_{\eta \mu}(q) = \sum_{\nu} q^{\phi(\nu)} \prod_{1 \leq a \leq n - 1, 1 \leq i \leq \mu_1} \left[ \frac{\nu_i^{(a+1)} - \nu_i^{(a)}}{\nu_i^{(a)} - \nu_i^{(a+1)}} \right]
\]

(0.2)

\[
 \phi(\nu) = \sum_{a=0}^{n-1} \sum_{i=1}^{\mu_1} \left( \frac{\nu_i^{(a+1)} - \nu_i^{(a)}}{2} \right),
\]

where the sum (0.2) runs over all sequences of diagrams \( \nu^{(1)} , \cdots , \nu^{(n-1)} \) such that

\[
 \emptyset =: \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu^{(n)} := \mu',
\]

\[
 |\nu^{(a)}| = \lambda_1 + \cdots + \lambda_a \quad \text{for} \ 1 \leq a \leq n - 1.
\]
The formula (0.2) is a far generalization of the corresponding results obtained in [7, 8, 13, 20, 33].

In Section 4.2 we calculate the limits of the fermionic forms. Proposition 4.11 provides a fermionic formula of the string function $c_V^V(\lambda(q))$ for the tensor product module $V = \otimes_{j=1}^{s} V(l_j, \Lambda_j)$. This is obtained by computing the limit of (0.2) with $\forall \mu^J = \emptyset$. When $s = 1, r_1 = 0$ it reduces to the one conjectured in [28], announced in [9] and proved in [11]. Part of some other results have also been obtained by G. Georgiev [10, 11] and for $\widehat{sl}_2$ by Schilling and Warnaar [33]. Another important result is Proposition 4.12, which shows that the limit $L \to \infty$ ($L \equiv 0 \mod n$) of $K_{\lambda_1}^\mu(q)$ is expressed as a sum involving a bilinear product of the Kostka-Foulkes polynomial and its restricted analogue. Under the conjecture (4.29) this proves the $\Lambda = l\Lambda_0$ case of the spinon character formula conjectured in [32]:

$$b_V^{\lambda}(\Lambda)(q) = \sum_{\eta} \frac{X_{\eta}(\lambda) X_{\eta}(\Lambda)}{(q)_{\xi_1} \cdots (q)_{\xi_{s-1}}},$$

where the sum $\sum_{\eta}$ runs over the partitions $\eta = ((n-1)^{\xi_{s-1}}, \cdots, 1^{\xi_1})$ satisfying $|\eta| \equiv |\lambda| \mod n$. The numerators are the 1dsums associated with the antisymmetric tensors defined in Section 3. Proposition 4.14 is a similar result related to an RSOS version of the spinon character formula.

Section 5 contains further generalizations. In particular we have Conjecture 5.1 on the fermionic formula of the string function of the module $V(l_1 \Lambda_0 \otimes \cdots \otimes V(l_s \Lambda_0)$ for arbitrary non-twisted affine Lie algebra $X^{(1)}_n$.

Let us close with a few more comments on the limiting behaviour of the Kostka-Foulkes polynomials. Its study was initiated by R. Gupta [12] and R. Stanley [34], (see also [21]) in connection to investigation of the stable behaviour of some characters of the special linear group $SL(n)$. In the context of integrable systems, it was initiated by A.N. Kirillov [20] and continued by Nakayashiki and Yamada [31]. The $s = 1$ case of the large $\mu$ limit in this paper corresponds to the so-called thermodynamical Bethe Ansatz limit [20], when for all $l_i \lambda_i \to \infty$ and $\mu'_i \to \infty$, but all differences $l_i - l_{i+1}$ and $\mu'_i - \mu'_{i+1}$ are fixed. Mathematically, there are yet other interesting limits. For example, it is known [20] that the 1dsums $g^\mu_\lambda(\lambda), X^\mu_\lambda(\lambda)$ and $X^{(\mu')}_{\eta}(\lambda)$ based on the antisymmetric tensors yield just level 1 characters in the large $\mu$ limit under the replacement $q \to q^{-1}$. On the other hand from Proposition 3.3 they should emerge also from the limit of $\sum_{\eta(l_{1} \leq \eta_{s})} K_{\eta'}(q^{-1})$ or $K_{\lambda}(q^{-1})$, etc. Starting from their fermionic forms one can verify this easily by using the Durfee rectangle identity at most. Another interesting limit is the so-called thermodynamical limit, when $l_i \lambda_i \to \infty$ and $\mu'_i \to \infty$, but $(\lambda_2, \ldots, \lambda_n)$ and $(\mu'_2, \ldots, \mu'_m)$ are fixed. In this limit the Kostka-Foulkes polynomial $K_{\lambda_\mu}(q)$ tends to some rational function, see, e.g., [20].
1 Crystals

1.1 Preliminaries

We recapitulate necessary facts and notations concerning crystals of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$. Let $\alpha_i, h_i, \Lambda_i$ ($i = 0, 1, \cdots, n - 1$) be the simple roots, simple coroots, fundamental weights for the affine Lie algebra $\hat{\mathfrak{sl}}_n$. For our convenience we set $\Lambda_i' = \Lambda_i$ for any $i' \in \mathbb{Z}$ such that $i' \equiv i \text{ mod } n$. Let $(\cdot | \cdot)$ be the standard bilinear form normalized by $(\alpha_i | \alpha_i) = 2$. The following value will be used later: $(\Lambda_i | \Lambda_j) = \min(i, j) - ij/n$ $(0 \leq i, j < n)$. Let $\delta = \sum_{i=0}^{n-1} \alpha_i$ denote the null root, and $c = \sum_{i=0}^{n-1} h_i$ the canonical central element. Let $P = \oplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ be the weight lattice. We define the following subsets of $P$: $P^+ = \sum_{i=0}^{n-1} \mathbb{Z}_{\geq 0} \Lambda_i$, $P_i^+ = \{ \lambda \in P^+ | (\lambda, c) = 1 \}$, $\overline{P} = \sum_{i=1}^{n-1} \mathbb{Z}\Lambda_i$. Here $\Lambda_i = \Lambda_i - \Lambda_0$ is the classical part of $\Lambda_i$. This map $\overline{\cdot}$ is extended to a map on $P$ so that it is $\mathbb{Z}$-linear. To consider finite dimensional $U_q'(\hat{\mathfrak{sl}}_n)$-modules, the classical weight lattice $P_{cl} = \overline{P}/\mathbb{Z}\delta$ is also needed. We further define the following subsets of $P_{cl}$: $P_{cl}^+ = \{ \lambda \in P_{cl} | (\lambda, h_i) \geq 0 \text{ for any } i \}$, $(P_{cl}^+)_l = \{ \lambda \in P_{cl}^+ | (\lambda, c) = l \}$. We introduce an element $\Lambda_i^l \in P_{cl}$ by $\Lambda_i^l = \Lambda_i \mod \mathbb{Z}\delta$, and fix the map $af : P_{cl} \to P$ by $af(\Lambda_i^l) = \Lambda_i$. See Section 3.1 of [17] for the details of $P_{cl}, af$, etc.

The irreducible highest weight module $V(\lambda)$ with highest weight $\lambda \in P^+$ has a crystal base $(L(\lambda), B(\lambda))$ [13]. We denote the highest weight vector in $B(\lambda)$ by $u_\lambda$. On the crystal $B = B(\lambda)$, the actions of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i = 0, 1, \cdots, n - 1$) are given:

\begin{align*}
    \tilde{e}_i : B &\rightarrow B \sqcup \{0\}, \\
    \tilde{f}_i : B &\rightarrow B \sqcup \{0\}.
\end{align*}

For $b, b' \in B$, $\tilde{f}_i b = b'$ is equivalent to $b = \tilde{e}_i b'$. Setting $\varepsilon_i(b) = \max\{ n \in \mathbb{Z}_{\geq 0} | \tilde{e}_i^n b \neq 0 \}$, $\varphi_i(b) = \max\{ n \in \mathbb{Z}_{\geq 0} | \tilde{f}_i^n b \neq 0 \}$, we have $\varphi_i(b) - \varepsilon_i(b) = (h_i, wt b)$.

Crystals from the category of finite dimensional modules are also important. Let $V(l_i)$ be the symmetric tensor representation of $U_q'(\hat{\mathfrak{sl}}_n)$ of degree $l_i$. $(U_q'(\hat{\mathfrak{sl}}_n))$ is the subalgebra of $U_q(\hat{\mathfrak{sl}}_n)$ generated by $\tilde{e}_i, \tilde{f}_i, q^h$ ($h \in (P_{cl})^*$). $V(l_i)$ also has a crystal base $(L(l_i), B(l_i))$. We note that $B(l_i)$ is a $P_{cl}$-weighted crystal. As a set, $B(l_i)$ is described as

$$B(l_i) = \{ (x_1, \cdots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid x_1 + \cdots + x_n = l_i \}.$$  

It can be identified with the set of semi-standard tableaux of shape $(l_i)$ with letters from $\{1, 2, \cdots, n\}$. The crystal structure of $B(l_i)$ is given by

\begin{align*}
    \tilde{f}_0(x_1, \cdots, x_n) &= (x_1 + 1, \cdots, x_n - 1), \\
    \tilde{f}_i(x_1, \cdots, x_i, x_{i+1}, \cdots, x_n) &= (x_1, \cdots, x_i - 1, x_{i+1} + 1, \cdots, x_n) \quad (i \neq 0) \quad (1.4)
\end{align*}
Suppose Assume the following conditions: Using this function, they represented the Kostka polynomial. We review the energy function introduced by Nakayashiki and Yamada [31]. Here \( \otimes \) is understood as 0. For \( b = (x_1, \ldots, x_n) \in B_{(l)} \), we have \( \varepsilon_i(b) = x_{i+1}, \varphi_i(b) = x_i(i \neq 0), = x_n(i = 0) \) and \( \text{wt} b = \sum_{i=1}^{n} x_i(\Lambda_i - \Lambda_{i-1}) \). It is sometimes convenient to write \( x_i(b) \) for a component \( x_i \) of \( b \in B_{(l)} \).

We also review the \( P_{cl} \)-weighted crystal \( B_{(1^l)} \) of the anti-symmetric tensor representation of \( U'_q(\mathfrak{sl}_n) \) of degree \( l (l < n) \). As a set, it is described as

\[
B_{(1^l)} = \{(x_1, \ldots, x_n) \in \{0, 1\}^n \mid x_1 + \cdots + x_n = l\}
\]

It can be identified with the set of semi-standard tableaux of shape \( (1^l) \) with letters from \( \{1, 2, \ldots, n\} \). The crystal structure is given similarly.

For two crystals \( B_1 \) and \( B_2 \), the tensor product \( B_1 \otimes B_2 \) is defined.

\[
B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}
\]

The actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) are defined by

\[
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}, \tag{1.5}
\]

\[
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{1.6}
\]

Here \( 0 \otimes b \) and \( b \otimes 0 \) are understood to be 0. \( \varepsilon_i, \varphi_i \) and \( \text{wt} \) are given by

\[
\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \tag{1.7}
\]

\[
\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \tag{1.8}
\]

\[
\text{wt}(b_1 \otimes b_2) = \text{wt} b_1 + \text{wt} b_2. \tag{1.9}
\]

### 1.2 Nakayashiki-Yamada’s energy function

We review the energy function introduced by Nakayashiki and Yamada [31]. Using this function, they represented the Kostka polynomial \( K_{\mu\nu}(q) \) as a sum over paths \( b_1 \otimes \cdots \otimes b_m \in B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)} \) or \( b_1 \otimes \cdots \otimes b_m \in B_{(1^{\nu_1})} \otimes \cdots \otimes B_{(1^{\nu_m})} \) (\( \mu = (\mu_1, \ldots, \mu_m) \)) satisfying certain conditions.

Let us consider crystals \( B_1 \) and \( B_2 \) of finite dimensional \( U'_q(\mathfrak{sl}_n) \)-modules. Assume the following conditions:

\[
B_1 \otimes B_2 \text{ is connected.} \tag{1.10}
\]

\[
B_1 \otimes B_2 \text{ is isomorphic to } B_2 \otimes B_1. \tag{1.11}
\]

Suppose \( b_1 \otimes b_2 \in B_1 \otimes B_2 \) is mapped to \( b'_2 \otimes b'_1 \in B_2 \otimes B_1 \) under the isomorphism. A \( \mathbb{Z} \)-valued function \( H \) on \( B_1 \otimes B_2 \) is called an energy function if for any \( i \) and \( b_1 \otimes b_2 \in B_1 \otimes B_2 \) such that \( \tilde{e}_i(b_1 \otimes b_2) \neq 0 \) it satisfies

\[
H(\tilde{e}_i(b_1 \otimes b_2)) = H(b_1 \otimes b_2) + 1 \quad \text{if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_2), \varphi_0(b'_2) \geq \varepsilon_0(b'_1),
\]

\[
= H(b_1 \otimes b_2) - 1 \quad \text{if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2), \varphi_0(b'_2) < \varepsilon_0(b'_1),
\]

\[
= H(b_1 \otimes b_2) \quad \text{otherwise.} \tag{1.12}
\]
Explicit descriptions of the isomorphism (1.11) and energy function in the cases of \((B_1, B_2) = (B_{(k)}, B_{(l)})\) and \((B_{(k^*)}, B_{(l^*)})\) \((k \geq l)\) are given in the next subsection.

Let \(B_i\) \((i = 1, \ldots, m)\) be finite crystals such that \(B_i\) and \(B_j\) satisfy both (1.11) and (1.14) for any \(i, j\) \((i < j)\). Using the isomorphism (1.11), we define \(b_j^{(i)}\) \((i < j)\) by

\[
B_i \otimes \cdots \otimes B_{j-1} \otimes B_j \simeq B_i \otimes \cdots \otimes B_j \otimes B_{j-1} \simeq \cdots
\]

\[
b_i \otimes b_{j-1} \otimes b_j \mapsto b_i \otimes \cdots \otimes b_j^{(j-1)} \otimes b_{j-1} \mapsto \cdots
\]

\[
\cdots \simeq B_i \otimes B_j \otimes \cdots \otimes B_{j-1}
\]

\[
\cdots \mapsto b_j^{(i)} \otimes b_j' \otimes \cdots \otimes b_{j-1},
\]

and set \(b_j^{(i)} = b_i\). Consider an element \(p = b_1 \otimes \cdots \otimes b_m\) of \(B_1 \otimes \cdots \otimes B_m\). We call the following quantity the energy of \(p\).

\[
E(p) = \sum_{i<j} H_{ij}(b_i \otimes b_j^{(i+1)}).
\]

Here \(H_{ij}\) is the energy function on \(B_i \otimes B_j\) defined previously.

Consider the case when \(B_i = B_{(\mu)}\) with \(\mu = (\mu_1, \ldots, \mu_m)\) a partition. The set of highest weight crystals of weight \(\lambda\) in \(B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)}\) with respect to \(U_q(\mathfrak{sl}_n)\) is known to be bijective to the set of semi-standard tableaux of shape \(\lambda\) and weight \(\mu\). It was shown in [31] that the energy of \(p\) coincides with the charge of the corresponding tableau in the sense of Lascoux-Schützenberger [29].

Let \(p\) be as above, and consider the following condition for \(b_1 \in B_1\).

For any \(j \neq 1\) and \(b_j \in B_j\),

\[
\text{if } \bar{e}_0(b_1 \otimes b_j) = b_1 \otimes \bar{e}_0 b_j, \text{ then } \bar{e}_0(b_j' \otimes b_j') = b_j' \otimes \bar{e}_0 b_j'.
\]

(1.13)

Here \(b_1 \otimes b_j\) is mapped to \(b_j' \otimes b_j'\) under the isomorphism \(B_1 \otimes B_j \simeq B_j \otimes B_1\).

We have the following representation-theoretic interpretation of energy.

**Proposition 1.1** Let \(p\) be as above. If \(i \neq 0\) and \(\bar{e}_i p \neq 0\), then

\[
E(\bar{e}_i p) = E(p).
\]

If \(\bar{e}_0 p = b_1 \otimes \cdots \otimes \bar{e}_0 b_k \otimes \cdots \otimes b_m \neq 0\) with \(k \neq 1\) and \(b_1\) satisfies the condition (1.11), then

\[
E(\bar{e}_0 p) = E(p) - 1.
\]

The case of \(i \neq 0\) is clear. The case of \(i = 0\) reduces to the following.

**Lemma 1.2** If \(\bar{e}_0 p = b_1 \otimes \cdots \otimes \bar{e}_0 b_k \otimes \cdots \otimes b_m \neq 0\) with \(k \neq 1\) and \(b_1\) satisfies the condition (1.11), then

\[
E^{(j)}(\bar{e}_0 p) = E^{(j)}(p) - 1(j = k)
\]

\[
= E^{(j)}(p) \quad (j \neq k).
\]

(1.14)  \(1.15\)
Here $E^{(j)}(p) = \sum_{i=1}^{j-1} H_{ij}(b_i \otimes b_j^{(i+1)})$.

Proof. Set $\tilde{b}_i = \tilde{e}_0 b_i (i = k), b_i (i \neq k)$. In the case of $j < k$, we have $\tilde{b}_i = b_i, \tilde{b}_j^{(i+1)} = b_j^{(i+1)}$ for $1 \leq i \leq j - 1$. This shows (1.15) when $j < k$.

To show it in the case of $j \geq k$, we rewrite (1.12) in the following manner.

$$H(\tilde{e}_0(b_1 \otimes b_2)) = H(b_1 \otimes b_2) + 1 \text{ if } \tilde{e}_0(b_1 \otimes b_2) = \tilde{e}_0 b_1 \otimes b_2 \text{ and}$$
$$\tilde{e}_0(b'_i \otimes b'_1) = \tilde{e}_0 b'_i \otimes b'_1,$$

$$= H(b_1 \otimes b_2) - 1 \text{ if } \tilde{e}_0(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0 b_2 \text{ and}$$
$$\tilde{e}_0(b'_i \otimes b'_1) = b'_2 \otimes \tilde{e}_0 b'_1,$$

$$= H(b_1 \otimes b_2) \text{ otherwise.}$$

Let $k'$ be the largest integer such that

\begin{align*}
B_1 \otimes \cdots \otimes B_k \otimes \cdots \otimes B_j \\
= b_1 \otimes \cdots \otimes \tilde{e}_0 b_k \otimes \cdots \otimes b_j \\
\simeq B_1 \otimes \cdots \otimes B_j \otimes B_{j+1} \otimes \cdots \otimes B_j^{-1} \\
\implies b_1 \otimes \cdots \otimes \tilde{e}_0 b_k \otimes b_j^{(k+1)} \otimes \cdots \otimes b_j^{-1} \\
\simeq B_1 \otimes \cdots \otimes B_j \otimes B_{k'} \otimes B_j \otimes \cdots \otimes B_j^{-1} \\
\implies b_1 \otimes \cdots \otimes b_{k'} \otimes \tilde{e}_0 b_j^{(k+1)} \otimes \cdots \otimes b_j^{-1} \\
\simeq B_1 \otimes \cdots \otimes B_j \otimes B_{k'} \otimes \cdots \otimes B_j^{-1} \\
\implies b_1 \otimes \cdots \otimes b_{k'} \otimes \tilde{e}_0 b_j^{(k+1)} \otimes \cdots \otimes b_j^{-1}.
\end{align*}

Note that $1 \leq k' < k$ if $j = k$ and $1 \leq k' \leq k$ if $j > k$. The existence of such $k'$ is guaranteed by (1.13). If $k' = k$, the 3rd and 4th terms should be omitted. From the above property of $H$, we get the following. If $j = k$, $H_{ij}(\tilde{b}_i, \tilde{b}_j^{(i+1)}) - H_{ij}(b_i, b_j^{(i+1)}) = -1 (i = k'), = 0 (otherwise)$. If $j > k$ and $k' \neq k$, $H_{ij}(\tilde{b}_i, \tilde{b}_j^{(i+1)}) - H_{ij}(b_i, b_j^{(i+1)}) = 1 (i = k'), -1 (i = k'), = 0 (otherwise)$. If $j > k$ and $k' = k$, $H_{ij}(\tilde{b}_i, \tilde{b}_j^{(i+1)}) - H_{ij}(b_i, b_j^{(i+1)}) = 0$. These imply (1.14) and (1.13) respectively.

1.3 Evaluation of the energy

Here we give an explicit procedure to obtain the energy function $H : B_1 \otimes B_2 \rightarrow \mathbb{Z}$ and the isomorphism $\iota : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$ in the case of symmetric tensor representations $(B_1, B_2) = (B_{(k)}, B_{(l)}) (k \geq l)$. 


Let \( b_1 \otimes b_2 \) be an element in \( B_1 \otimes B_2 \) such as \( b_1 = (x_1, \ldots, x_n) \) and \( b_2 = (y_1, \ldots, y_n) \). We represent \( b_1 \otimes b_2 \) by the two column diagram. Each column has \( n \) rows, enumerated as 1 to \( n \) from the top to the bottom. We put \( x_i \) (resp. \( y_i \)) dots • in the \( i \)-th row of the left (resp. right) column. The labels of rows are sometimes extended to \( \mathbb{Z}/n\mathbb{Z} \).

**Proposition 1.3** The rule to obtain the energy function \( H \) and the isomorphism \( \iota \) is as follows.

1. Pick any dot, say •, in the right column and connect it with a dot • in the left column by a line (which we call \( H \)-line). The partner • is chosen from the dots which are in the lowest row among all dots whose positions are higher than that of •. If there is no such dot, we return to the bottom and the partner • is chosen from the dots in the lowest row among all dots. In the latter case, we call such a pair or line “winding”.

2. Repeat the procedure (1) for the remaining unconnected dots \((l-1)\)-times.

3. The isomorphism \( \iota \) is obtained by sliding the remaining \((k-l)\) unpaired dots in the left column to the right.

4. The value of the energy function is the number of the “winding” pairs.

The \( H \) and \( \iota \) obtained by this rule have the correct property as the energy function and isomorphism. This fact has been proved in [31] Section 3, where the following two lemmas are also proved.

**Lemma 1.4** The map \( \iota \) determined by the above rule is independent of the order of drawing lines.

**Proof.** Suppose there exists a dot (say •) in the left column that is unpaired in one ordering (say A) and paired in another (say B). Let • be the partner of • in B, and let • be the partner of • in A. Then • must be paired with some dot (say •) in B since the \( R_1 \) line already passes through •. This process of determining \( R_1, R_2, \ldots \) does not stop and \( R_1, R_2, \ldots \) are all distinct. This is a contradiction, since the number of dots in the right column is finite (\( =l \)). Therefore the set of end points of the \( H \)-lines is independent of the order in which they are drawn and the lemma is proved. 

\[ \square \]
Lemma 1.5 The value of the function $H$ determined by the above rule is independent of the order of drawing lines.

Proof. For $j$ ($1 \leq j \leq n$) we assign a non-negative integer $h_j(A)$ as the number of lines passing the $j$-th row. Here the word “passing” is defined as follows. Let $\alpha$ be a line starting from the $i$-th row and ending at the $j$-th row, then the line $\alpha$ passes the $k$-th row if and only if (1) $i > k > j$ (for non-winding $\alpha$) or (2) $k < i$ or $k > j$ (for winding $\alpha$). $h_j(A)$ may depend on the order $A$ of drawing $H$-lines. $h_j(A)$ is subject to the following relation:

$$h_{j+1}(A) = h_j(A) + e_j - s_{j+1},$$  \hspace{1cm} (1.16)

where $e_j$ is the number of dots in the left column which are end points of the $H$-lines and sitting in the $j$-th row. Similarly, $s_j$ is the number of dots in the right column (starting points of the $H$-lines) sitting in the $j$-th row.

The value of the energy function $H$ with respect to the order $A$ is given by $h_1(A) + s_1$. Note that the set of end points of the $H$-lines is independent of the order $A$, hence $e_j$ and $s_j$ are independent of the order $A$.

We prove that $h_j(A)$ does not depend on $A$. To this end, it is sufficient to prove that for any $A$ there exists $j$ such that $h_j(A) = 0$. In fact, for two orders $A$ and $B$, there exists an integer $m$ such that

$$h_j(A) = h_j(B) + m$$

for any $j$ by (1.16). By exchanging $A$ and $B$ if necessary, we can assume $m \geq 0$. The existence of $j$ for which $h_j(A) = 0$ means $m = 0$, since $h_j(B) \geq 0$. The existence of such $j$ can be proved as follows. Suppose that such $j$ does not exist. Then there exists a sequence of $H$-lines $\alpha_1, \alpha_2, \ldots, \alpha_k (= \alpha_0)$ such that the end point of $\alpha_i$ is passed through by the line $\alpha_{i-1}$. If such a situation occurs, however, the order of drawing $H$-lines should satisfy

order of $\alpha_1 >$ order of $\alpha_2 > \ldots >$ order of $\alpha_k >$ order of $\alpha_1$

which is a contradiction. ■

The following is just a corollary of Proposition 1.3.
Proposition 1.6 Let $\mu = (\mu_1, \ldots, \mu_m)$ be a partition and let $p = b_1 \otimes \ldots \otimes b_m$ be a path in $B_{(\mu_1)} \otimes \ldots \otimes B_{(\mu_m)}$. The rule to evaluate the energy

$$E(p) = \sum_{j=1}^{m} E^{(j)}(p),$$

$$E^{(j)}(p) = \sum_{i=1}^{j-1} H_{(\mu_i)(\mu_j)}(b_1 \otimes b_j^{i+1}) \quad (1 \leq j \leq m)$$

is given as follows.

1. Pick any dot in $b_m$. According to the $H$-line rule, connect the dot in $b_m$ with a dot in $b_{m-1}$ and connect the dot in $b_{m-1}$ with a dot in $b_{m-2}$ and continue this until we come to a dot in $b_1$. We call this line $E$-line.

2. Repeat the procedure (1) for the remaining unconnected dots $\mu_m - 1$-times.

3. Forget about the connected dots and repeat the procedures (1) and (2) from a rightmost unconnected dot. Eventually, all the dots in $p$ are decomposed into a disjoint union of $E$-lines. We call it $E$-line decomposition of $p$.

4. $E^{(j)}(p)$ is given as the sum of winding numbers between $b_1$ and $b_j$ of all the $E$-lines starting from $b_i$ with $i \geq j$.

Example 1.7 Let $n = 3$ and $p = (1,0,2) \otimes (0,2,0) \otimes (0,1,1) \otimes (0,1,0) \in B_{(3)} \otimes B_{(2)} \otimes B_{(2)} \otimes B_{(1)}$. The energy $E(p) = 0 + 1 + 2 + 1 = 4$ of this path $p$ is evaluated by the following diagram.

\begin{center}
\includegraphics[width=0.3\textwidth]{example_diagram.png}
\end{center}

Remark 1.8 Let $p$ be as in Proposition 1.4. We can check any $b_1 \in B_{(\mu_1)}$ satisfies the condition (1.13). Just note that $\tilde{e}_0(b_1 \otimes b_j) = b_1 \otimes \tilde{e}_0 b_j$ implies $x_n(b_1) < x_1(b_j)$, which is sufficient for $x_n(b_j') < x_1(b_j')$ by Proposition 1.3.

Remark 1.9 For the anti-symmetric case $(B_1, B_2) = (B_{(1^k)}, B_{(1^l)})$ ($k \geq l$), the rule is almost the same as in the symmetric case. The only differences are in (1) and (4).
(1) The partner •′ₐ is a dot that has the highest position among all dots whose positions are not higher than •. If there is no such dot, we return to the top. We call such a pair “winding”.

(4) The value of the energy function is \((-1)\) times the number of “winding” pairs.

The independence of this rule with respect to the order of drawing \(H\)-lines can be proved similarly. There also exists a similar rule to obtain the energy \(E(p)\) for the anti-symmetric case.

**Remark 1.10** The energy functions \(H_{(\mu_i)(\mu_j)}\) and \(H_{(\lambda_i^\prime)(\lambda_j^\prime)}\) can also be evaluated by using the “nonmovable tableaux” as in [23].

## 2 Tensor product of crystals

In this section, we consider \(B(\lambda)\) as a \(P_{cl}\)-weighted crystal except Section 2.4.

### 2.1 Decomposition of \(B(\lambda) \otimes B(l)\)

The crystal \(B(l)\) is known to be perfect of level \(l\) [17, 18]. It means that for any \(\lambda \in (P_{cl}^+)_l\), we have an isomorphism

\[
B(\lambda) \otimes B(l) \simeq B(\lambda')
\]

for some \(\lambda' \in (P_{cl}^+)_l\). If the level \(k\) of \(\lambda\) is greater than \(l\), it is known that \(B(\lambda) \otimes B(l)\) decomposes into a disjoint union of crystals \(B(\mu)\) with \(\mu\) being a dominant integral weight of level \(k\). (See the beginning of Section 5 of [10].)

More precisely, we have

**Theorem 2.1** Let \(\lambda \in (P_{cl}^+)_k\). If \(k \geq l\), then

\[
B(\lambda) \otimes B(l) \simeq \bigoplus_{b \in B_{(l)}^{\leq \lambda}} B(\lambda + \text{wt} b),
\]

where

\[
B_{(l)}^{\leq \lambda} = \{ b \in B(l) \mid \varepsilon_i(b) \leq \langle h_i, \lambda \rangle \text{ for all } i \}.
\]

**Corollary 2.2** Let \(\mu = (\mu_1, \ldots, \mu_m)\) be a partition and \(\lambda \in (P_{cl}^+)_l\). If \(l \geq \mu_1\), then

\[
B(\lambda) \otimes B(\mu_1) \otimes \cdots \otimes B(\mu_m) \simeq \bigoplus_{p \in \mathcal{H}(\lambda, \mu)} B(\lambda + \text{wt} p),
\] (2.1)
The correspondence is given by 

\[ \nu \] row of 
to a sequence of Young diagrams

appear in later sections. Each path \( p \)

Example 2.3 We give examples of \( B(\lambda, \mu) \).

(1) Let \( \lambda \) be of level \( l \) and \( \mu = (l^m) \). Then the perfectness of \( B(l) \) fixes \( p \) to be unique, and we have

[2.1]

(2) Consider the case when \( \lambda = l\Lambda_r^{cl}, \mu = (s) \) \((0 \leq r \leq n - 1, l \geq s) \). In this case, we have

[2.2]

(3) Consider the case when \( \lambda = l\Lambda_0^{cl}, \mu = (s, t) \) \((l \geq s \geq t) \). In this case, we have

\[ \begin{align*}
\mathcal{H}(l\Lambda_r^{cl}, (s, t)) &= \left\{ (s, 0, \ldots, 0) \otimes b \mid x_1(b) \leq l - s, x_2(b) \leq s, x_i(b) = 0 \quad (i \geq 3) \right\}, \\
\mathcal{B}(l\Lambda_0^{cl}, (s, t)) &= \bigoplus_{i=0}^{r} B((l - s - i)\Lambda_0^{cl} + (s - t + 2i)\Lambda_r^{cl} + (t - i)\Lambda_2^{cl}),
\end{align*} \]

where \( r = \min(l - s, t) \).

We give a characterization of the set \( \mathcal{H}(l\Lambda_r^{cl}, \mu) \) \((0 \leq r < n) \), which will often appear in later sections. Each path \( p = b_1 \otimes \cdots \otimes b_m \) in \( \mathcal{H}(l\Lambda_r^{cl}, \mu) \) corresponds to a sequence of Young diagrams \( \nu^{(a)} \) \((a = 0, \ldots, m) \) such that

(1) \( \nu^{(0)} = (l^r) \),

(2) \( \nu^{(a)}/\nu^{(a-1)} \) is horizontal strip of length \( \mu_a \) for \((a = 1, \ldots, m) \),

(3) depth of \( \nu^{(a)} \leq n \) and \( \nu^{(a)}_1 - \nu^{(a)}_n \leq l \).

The correspondence is given by \( x_i(b_a) = \) the number of nodes in the \( i \)-th row of \( \nu^{(a)}/\nu^{(a-1)} \). Under this correspondence, we have \( l\Lambda_r^{cl} + wt\ p = l\Lambda_0^{cl} + \sum_{k \geq 1} \nu^{(m)'}_k(\Lambda_r^{cl} - \Lambda_0^{cl}) \), where \( \nu^{(m)'} \) denotes the transpose of \( \nu^{(m)} \).
2.2 Limit

For given partitions $\lambda$ and $\mu$, we define two operations $\lambda \cup \mu$ and $\lambda + \mu$. Let $\lambda = (\lambda_1, \cdots, \lambda_l)$, $\mu = (\mu_1, \cdots, \mu_m)$. Assuming $\lambda_l \geq \mu_1$, $\lambda \cup \mu$ is defined by $(\lambda_1, \cdots, \lambda_l, \mu_1, \cdots, \mu_m)$. Appending 0's if necessary, we now assume $l = m$. Then $\lambda + \mu$ is defined by $(\lambda_1 + \mu_1, \cdots, \lambda_l + \mu_l)$. Associated to a partition $\mu = (\mu_1, \cdots, \mu_m)$, we next define a special element $\overline{p}$ of $B(\mu_1) \otimes \cdots \otimes B(\mu_m)$ called the ground state path. It is given by

$$\overline{p} = \overline{b}_1 \otimes \overline{b}_2 \otimes \cdots \otimes \overline{b}_m,$$

(2.3)

where the position $r$ is determined by $r \equiv j \pmod{n}$, $1 \leq r \leq n$.

Let $l_J, L_J$ be positive integers and $\mu^J$ be a partition such that $l_J \geq \mu^J_1$ for $J = 1, \cdots, s$. It is convenient to set $l = \sum_{J=1}^s l_J$ and $L_{s+1} = 0$. We consider the following partition.

$$\mu = \mu((l_1, L_1, \mu^1), \cdots, (l_s, L_s, \mu^s))$$

(2.5)

$$= ((l^1_1 \cup \mu^1) + \cdots + (l^s_1 \cup \mu^s)).$$

(2.6)

We would like to consider the limit when all $L_J - L_{J+1}$ ($J = 1, \cdots, s$) go to infinity with the residue of $L_J$ mod $n$ fixed. For $r_J$ ($0 \leq r_J < n, 1 \leq J \leq s$), let us define $P_{<\infty}((l_1, r_1, \mu^1), \cdots, (l_s, r_s, \mu^s))$ to be the set of elements $p$ such that

1. $p$ is an element of the limit of $B(\mu_1) \otimes \cdots \otimes B(\mu_m)$ when all $L_J - L_{J+1}$ ($J = 1, \cdots, s$) go to infinity with the residue of $L_J$ mod $n$ fixed as $r_J$.

2. $E(p) - E(\overline{p})$ is finite.

**Theorem 2.4** With the definitions as above, we have the following statements.
Lemma 2.5 Let \( \mu, \rho, \nu \) be partitions such that \( \mu = (l_1^{(1)}) \cup \nu, l_1 \geq \nu_1 \) and the depth of \( \mu^2 \) is smaller than \( L_1 \). Let \( p \) be a path corresponding to \( \mu = \mu^1 + \mu^2 \). Then we have
\[
E(p) \geq \min(E(p^1) + E(p^2)),
\]
where the minimum is taken over all \( (\mu^1, \mu^2) \)-partitions \( (p^1, p^2) \) of \( p \).

Proof. First fix any \( E \)-line decomposition \( \ell \) of \( p \). Define \( \ell_1 \) to be the set of \( E \)-lines penetrating the \( L_1 \)-th tensor component, and set \( \ell_2 = \ell \setminus \ell_1 \). Next define \( p_0^1 = b_1^1 \otimes b_2^1 \otimes \cdots (J = 1, 2) \) in such a way that \( x_k(b_j^1) \) is the number of dots on \( \ell_j \) at position \( k \) in the \( j \)-th tensor component. Then we have
\[
E(p) = E(p_0^1) + E(p_0^2) \geq \min_{(p^1, p^2)} (E(p^1) + E(p^2)).
\]

Here the first equality is due to Proposition 1.6. \( \square \)

In what follows, we set \( \delta_{ij}^{(n)} = 1 \) \( (i \equiv j \mod n) \), 0 (otherwise).

Lemma 2.6 Let \( \mu, \rho, \nu \) and \( p \) be as above. Let \( (p^1, p^2) \) be a \( (\mu^1, \mu^2) \)-partition of \( p \). We further assume that by setting \( p^1 = b_1^1 \otimes b_2^1 \otimes \cdots \), we have \( x_k(b_j^1) = l_1 \delta_{ij}^{(n)} \) if \( j \leq L_1 \). Then we have
\[
E(p) = E(p^1) + E(p^2).
\]
Proof. It is clear by Proposition 1.6. ■

Lemma 2.7 Let \( \mu \) be any partition. Let \( p \) (resp. \( \overline{p} \)) be a path (resp. the ground state path) corresponding to \( \mu \). Then we have

\[
E(p) \geq E(\overline{p}).
\]

Proof. Decompose \( \mu \) into \( \mu = \mu^1 + \cdots + \mu^s \) such that each \( \mu^j \) is of rectangular shape of distinct depth. From Lemma 2.5, we get

\[
\min_{(p^1, \ldots, p^s)} \sum_j E(p^j),
\]

where the minimum is taken over all \( (\mu^1, \ldots, \mu^s) \)-partitions of \( p \). Let \( \overline{p} \) be the ground state path corresponding to \( \mu \). It is known in the theory of perfect crystals that \( E(p^j) \geq E(\overline{p}^j) \) for any path \( p^j \) corresponding to \( \mu^j \). Combining with Lemma 2.6, we get

\[
E(p) \geq \sum_j E(\overline{p}^j) = E(\overline{p}).
\]

Lemma 2.8 Let \( L \) be a positive integer and fix \( r \) to be \( 0 \leq r < n \). Let \( p = \cdots \otimes b_2 \otimes b_1 \) (resp. \( \overline{p} = \cdots \otimes \overline{b}_2 \otimes \overline{b}_1 \)) be a path (resp. the ground state path) corresponding to the partition \( (1^L) \). When \( L \to \infty \) with \( L \equiv r \mod n \), \( E(p) - E(\overline{p}) \to \infty \) unless the set \( Y(p) = \{ j \mid b_j \neq \overline{b}_j \} \) is bounded.

Proof. Recall that if a path \( p \) corresponding to a single column \( (1^L) \), the energy \( E(p) \) is given by \( E(p) = \sum_{j=1}^{L-1} j \theta(b_{j+1} - b_j) \). Here \( \theta(z) = 1 \) \((z \geq 0)\), \( = 0 \)(\(z < 0\)) and \( b_j \) is identified with a number from \( \{1, \ldots, n\} \). Fix \( N_0 \) such that \( b_{N_0+r+1-j} \neq \overline{b}_{N_0+r+1-j} \) for some \( j \) satisfying \( 1 \leq j \leq n \), and set \( L_0 = N_0 + r \). Then there exists \( j_0 \) such that \( 1 \leq j_0 < n \), \( \theta(b_{L_0+1-j_0} - b_{L_0-j_0}) = 1 \). It is easy to see that in this case

\[
E(p) - E(\overline{p}) \geq \sum_{k=0}^{\infty} ([L_0 - kn - j_0]_+ - [L_0 - kn - n]_+)
\]

\[
\geq (N_0 - 1)(n - j_0).
\]

Here \([z]_+ = z \) \((z \geq 0)\), \( = 0 \)(\(z < 0\)). Since \( N_0 \) can be arbitrarily large, \( E(p) - E(\overline{p}) \to \infty \).

Lemma 2.9 Let \( l, N \) be positive integers and \( \mu \) be a partition such that \( l \geq \mu_1 \). Set \( \mu^{(N)} = (1^{Nn}) \cup \mu \). Let \( p = \cdots \otimes b_2 \otimes b_1 \) (resp. \( \overline{p} = \cdots \otimes \overline{b}_2 \otimes \overline{b}_1 \)) be a path (resp. the ground state path) corresponding to \( \mu^{(N)} \). When \( N \to \infty \), \( E(p) - E(\overline{p}) \) remains finite if and only if the set \( Y(p) = \{ j \mid b_j \neq \overline{b}_j \} \) is bounded.
Proof. The “if” part is trivial. To show the “only if” part, we decompose \( \mu^{(N)} \) into \( \mu^{(N)} = \mu^1 + \cdots + \mu^s (s = p_1^{(N)}) \) such that each \( \mu^j \) is of shape \((1^{L_j})\). Let \((p^1, \ldots, p^s)\) be a \((\mu^1, \ldots, \mu^s)\)-partition of \( p \) and \( \vec{\pi}' \) be the ground state path corresponding to \( \mu^j \). Assume now that \( J(p) \) is not bounded. The previous lemma and inequality

\[
E(p) - E(\vec{\pi}) \geq \min_{(p', \ldots, p^r)} \sum J(E(p^j) - E(\vec{\pi}'))
\]

shows \( E(p) - E(\vec{\pi}) \to \infty \), which completes the proof.

Proof of Theorem 2.4. We only give proofs when \( s = 2 \), since the case when \( s > 2 \) is essentially the same.

Let \( L_1, L_2 \) be finite at first. From Lemma 2.3 and 2.6, we have an inequality

\[
E(p) - E(\vec{\pi}) \geq \min_{(p', p'')} \sum_{J=1}^2 (E(p^j) - E(\vec{\pi}')).
\]

Here the minimum is taken over all \(( (L_1^1) \cup \mu^1, (L_2^2) \cup \mu^2)\)-partitions of \( p \). Set \( p^j = \cdots \otimes b_j' \otimes b_j' \) for \( J = 1, 2 \). Now consider the limit when both \( L_1 \) and \( L_2 \) go to infinity, and let \( p \) be an element of \( \mathcal{P}_{<\infty}((l_1, r_1, \mu^1), (l_2, r_2, \mu^2)) \). If there is no partition \((p^1, p^2)\) of \( p \) such that \( Y(p^j) = \{ j \mid b_j^l = b_j'^l \} \) are bounded both for \( J = 1, 2 \), we have \( E(p) - E(\vec{\pi}) \to \infty \) by Lemma 2.4. Thus there exists a partition \((p^1, p^2)\) of \( p \) such that \( E(p^j) - E(\vec{\pi}') \) remains finite for \( J = 1, 2 \). Since \( Y(p^1) \) is bounded, this \((p^1, p^2)\) is unique in the limit. This shows (1).

(2) is clear from Lemma 2.4. To see (3) note that from Lemma 2.4 we have \( \mathcal{P}_{<\infty}((l, r, \mu)) = \mathcal{P}_{<\infty}((l, r, \emptyset)) \otimes B(\mu_1) \otimes \cdots \otimes B(\mu_m) (m = l(\mu)) \). From the same lemma and the theory of perfect crystals \( [17] \), we also have \( \mathcal{P}_{<\infty}((l, r, \emptyset)) \simeq B(lN_r^{cl}) \), which proves (3) by the definition (2.1).  

2.4 Affine weight

So far we only considered the weight of crystal in the set \( P_{cl} = P/\mathbb{Z} \delta \). In this subsection, we take the degree of \( \delta \) into account by using the energy of path. This is of crucial importance in the following sections.

By Theorem 2.4, we reduce our consideration to the set \( \mathcal{P}_{<\infty}((l, r, \mu)) \). Let \( \vec{\pi} \) be the ground state path of \( \mathcal{P}_{<\infty}((l, r, \mu)) \). In view of Proposition 1.1 it is natural to define the affine weight of \( p \in \mathcal{P}_{<\infty}((l, r, \mu)) \) by

\[
\text{affine wt } p = a f (\text{wt } p) - (E(p) - E(\vec{\pi})) \delta \in P.
\]

For the map \( a f \), see Section 1.1. We want to know the difference of the degree of \( \delta \) of each connected component of \( \mathcal{P}_{<\infty}((l, r, \mu)) \). Since any highest weight
element \( p \) in \( \mathcal{P}_{<\infty}((l, r, \mu)) \) is of the form
\[
p = p^\infty \otimes p^\mu,
\]
where \( p^\infty \) is the ground state path of \( \lim_{N \to \infty} B^{(Nn+r)} \), \( p^\mu \in \mathcal{H}(l \Lambda^cl_r, \mu) \), we have
\[
E(p) - E(p^\infty) = E(p^\mu) - E(p^\infty),
\]
where \( p^\mu \in B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)} \) is defined from \( p = p^\infty \otimes p^\mu \). Here we note that the first component of \( p^\mu \) is fixed from the condition for \( p^\mu \) to be in \( \mathcal{H}(l \Lambda^cl_r, \mu) \).

By abuse of notation we set \( \mathcal{H}(a \lambda(\lambda), \mu) = \mathcal{H}(\lambda, \mu) \) for \( \lambda \in \mathcal{P}^+ \). Calculating \( E(p^\mu) \) explicitly, we get

**Proposition 2.10** As a \( P \)-weighted crystal, \( \mathcal{P}_{<\infty}((l, r, \mu)) \) is isomorphic to
\[
\bigoplus_{p \in \mathcal{H}(l \Lambda_r, \mu)} B(l \Lambda_r + a \lambda(\lambda p)) - (E(p) - E(l \Lambda_r, \mu) \delta),
\]
where
\[
E(l \Lambda_r, \mu) = \frac{1}{2} \sum_{j=1}^{\mu_1} \left( \frac{t_j^2}{n} - t_j + (\Lambda t_j | \Lambda t_j) \right), \quad t_j = \mu_j' + r. \tag{2.7}
\]

## 3 One dimensional sums

In [27], three kinds of paths, unrestricted, classically restricted and restricted ones, are studied, which are associated with the tensor power of a single perfect crystal. Here we shall introduce their inhomogeneous versions associated to \( B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)} \) and \( B_{(1 \mu_1)} \otimes \cdots \otimes B_{(1 \mu_m)} \), and state the consequence of Theorem 2.4.

### 3.1 Unrestricted, classically restricted and restricted paths

Let \( \mathcal{F}_i = \Lambda_i - \Lambda_0 \) be the classical part of \( \Lambda_i \). We set \( \mathcal{P} = \oplus_{i=1}^{n-1} \mathbf{Z} \mathcal{F}_i \) and \( \mathcal{P}^+ = \oplus_{i=1}^{n-1} \mathbf{Z}_{\geq 0} \mathcal{F}_i \) as in Section 1.1. For any partition \( \mu = (\mu_1, \ldots, \mu_m(>0)) \) and \( \lambda \in \mathcal{P} \), we define
\[
\mathcal{P}_\mu(\lambda) = \{ p = b_1 \otimes \cdots \otimes b_m \in B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)} \mid a \lambda(\lambda p) = \lambda \},
\]
\[
\mathcal{P}_\mu'(\lambda) = \{ p = b_1 \otimes \cdots \otimes b_m \in B_{(1\mu_1)} \otimes \cdots \otimes B_{(1\mu_m)} \mid a \lambda(\lambda p) = \lambda \}. \tag{3.1}
\]
Elements of \( \mathcal{P}_\mu(\lambda) \) and \( \mathcal{P}_\mu'(\lambda) \) will be called the unrestricted paths of weight \( \lambda \).

For the latter we assume \( \mu_1 \leq n - 1 \). For \( \lambda \in \mathcal{P}^+ \subset \mathcal{P} \), we set
\[
\mathcal{P}_\mu(\lambda) = \left\{ b_1 \otimes \cdots \otimes b_m \in \mathcal{P}_\mu(\lambda) \mid \varepsilon_i(b_j) \leq \langle h_i, \lambda b_1 + \cdots + \lambda b_{j-1} \rangle \right\} \quad \text{for } 1 \leq i \leq n - 1 \quad \text{and} \quad j = 1, \ldots, m \}. \tag{3.2}
\]
We define \( P'(\mu) \) similarly by the above equation by replacing \( P(\mu) \) with \( P'(\mu) \). Elements of \( P(\mu) \) and \( P'(\mu) \) will be called the classically restricted paths of weight \( \lambda \). Finally for \( \lambda \in P^+_l \) we set
\[
P(\lambda) = \{ b_1 \otimes \cdots \otimes b_m \in P(\lambda) \mid \varepsilon_i(b_j) \leq \langle h_i, l\Lambda_0^d + \text{wt } b_1 + \cdots + \text{wt } b_{j-1} \rangle \}
\]
for \( 0 \leq i \leq n - 1 \) and \( j = 1, \ldots, m \).
\( \text{(3.3)} \)

We define \( P'(\lambda) \) similarly by the above equation by replacing \( \mu \rightarrow \mu' \). Elements of \( P(\lambda) \) and \( P'(\lambda) \) will be called the (level \( l \)) restricted paths of weight \( \lambda \). For the former we assume \( \mu_1 \leq l \). Notice that \( H(l\Lambda_0, \mu) = \bigsqcup_{\lambda \in P^+_l} P(\lambda) \). See (2.2). For \( \lambda \in P^+_l \) such that \( \lambda + l\Lambda_0 \in P^+_l \) one has the relations
\[
P(\lambda + l\Lambda_0) \subset P(\lambda) \subset P(\lambda),
P'(\lambda + l\Lambda_0) \subset P'(\lambda) \subset P'(\lambda).
\]

Having defined the paths we now introduce the associated one dimensional sums (1dsums) by
\[
g(\lambda) = \sum_{p \in P(\lambda)} q^{E(p)} \lambda \in P, \]
\[
g'(\lambda) = \sum_{p \in P'(\lambda)} q^{-E(p)} \lambda \in P',
\]
\[
X(\lambda) = \sum_{p \in P(\lambda)} q^{E(p)} \lambda \in P^+_l,
\]
\[
X'(\lambda) = \sum_{p \in P'(\lambda)} q^{-E(p)} \lambda \in P^+_l,
\]
\[
X^{(l)}(\lambda) = \sum_{p \in P^{(l)}(\lambda)} q^{E(p)} \lambda \in P^+_l,
\]
\[
X^{(l)'}(\lambda) = \sum_{p \in P^{(l)'}(\lambda)} q^{-E(p)} \lambda \in P^+_l,
\]
\( \text{(3.4)} \)

where the energy \( E(p) \) of a path \( p \) is given in Section 3. We note that when \( \forall \mu_i = 1 \), the energy functions in the two cases are different by an additive constant.

The functions \( g(\lambda), X(\lambda), X^{(l)}(\lambda) \) (and their analogues for \( B_{(1\mu_1)} \otimes \cdots \otimes B_{(1\mu_m)} \)) are called the unrestricted, the classically restricted and the (level \( l \)) restricted 1dsum, respectively. They are polynomials in \( q \) with non-negative integer coefficients. For rectangular \( \mu \), the unrestricted and the restricted 1dsums
have the origin in the studies of solvable vertex and RSOS models in statistical mechanics \([1], [7], [8], [13]\). In \([27]\), paths associated with \(B \otimes \cdots \otimes B\) were studied from the Demazure module viewpoint. They are homogeneous but \(B\) can be any perfect crystal. Let \(g_j(b, \lambda)_B, X_j(b, \xi, \eta)_B\) and \(X_j(b, \xi, \eta)_B\) be the unrestricted, classically restricted and restricted 1dsums introduced there, respectively, where we have exhibited the \(B\)-dependence explicitly. They are related with the 1dsums in \([14]\) as

\[
\begin{align*}
g_{(i)}(\lambda) &= q^{-j}g_j(b^+, \lambda)_{B(i)}, \\
X_{(i)}(\lambda) &= q^{-j}X_j(b^+, 0, \lambda)_{B(i)}, \\
X_{(i)}^{(l+t)}(\lambda) &= q^{-j}X_j(b^+, l \Lambda_0, \lambda)_{B(i)}.
\end{align*}
\]

where \(b^+ = (0, \ldots, 0, l) \in B(i)\) and \(b^- = (0, \ldots, 0, 1, \ldots, 1) \in B(1^l)\).

**Example 3.1** Let \(\lambda = (321), \mu = (2211)\) and \(n = 3\). All the elements \(p\) of \(\mathcal{P}_\mu(\lambda), \mathcal{P}_\mu(\lambda)\) and \(\mathcal{P}_\mu^{(2)}(\lambda)\) are listed with their energy \(E(p)\). The elements of \(\mathcal{P}_\mu(\lambda)\) and \(\mathcal{P}_\mu^{(2)}(\lambda)\) are labelled \(c\) and \(r\), respectively. The element \((x_1, x_2, x_3)\) of each crystal \(B(k)\) is denoted by \(1^{x_1}2^{x_2}3^{x_3}\).

| \(p\) | \(E\) | \(p\) | \(E\) |
|-----|-----|-----|-----|
| \(11 \otimes 12 \otimes 2 \otimes 3\) | 3 | \(11 \otimes 12 \otimes 3 \otimes 2\) | 2 |
| \(11 \otimes 13 \otimes 2 \otimes 2\) | 4 | \(11 \otimes 22 \otimes 1 \otimes 3\) | 2 |
| \(11 \otimes 22 \otimes 3 \otimes 1\) | 1 | \(11 \otimes 23 \otimes 1 \otimes 2\) | 2 |
| \(11 \otimes 23 \otimes 2 \otimes 1\) | 3 | \(12 \otimes 11 \otimes 2 \otimes 3\) | 4 |
| \(12 \otimes 11 \otimes 3 \otimes 2\) | 5 | \(12 \otimes 12 \otimes 1 \otimes 3\) | 3 |
| \(12 \otimes 12 \otimes 3 \otimes 1\) | 2 | \(12 \otimes 13 \otimes 1 \otimes 2\) | 3 |
| \(12 \otimes 13 \otimes 2 \otimes 1\) | 4 | \(12 \otimes 23 \otimes 1 \otimes 1\) | 3 |
| \(13 \otimes 11 \otimes 2 \otimes 2\) | 5 | \(13 \otimes 12 \otimes 1 \otimes 2\) | 3 |
| \(13 \otimes 12 \otimes 2 \otimes 1\) | 4 | \(13 \otimes 22 \otimes 1 \otimes 1\) | 4 |
| \(22 \otimes 11 \otimes 1 \otimes 3\) | 6 | \(22 \otimes 11 \otimes 3 \otimes 1\) | 5 |
| \(22 \otimes 13 \otimes 1 \otimes 1\) | 4 | \(23 \otimes 11 \otimes 1 \otimes 2\) | 6 |
| \(23 \otimes 11 \otimes 2 \otimes 1\) | 5 | \(23 \otimes 12 \otimes 1 \otimes 1\) | 7 |

**Example 3.2** Let \(\lambda = (322), \mu = (2221)\) and \(n = 3\). All the elements \(p\) of \(\mathcal{P}_\mu(\lambda), \mathcal{P}_\mu(\lambda)\) and \(\mathcal{P}_\mu^{(2)}(\lambda)\) are listed with their energy \(E(p)\). The elements of \(\mathcal{P}_\mu(\lambda)\) and \(\mathcal{P}_\mu^{(2)}(\lambda)\) are labelled \(c\) and \(r\), respectively. The element \((x_1, x_2, x_3)\) of each crystal \(B(1^k)\) is denoted by \(1^{x_1}2^{x_2}3^{x_3}\).
such that

\[
\text{view of this we shall identify }
\]

\[
\lambda
\]

\[
\text{empty hence the 1dsums (3.4) are zero, unless } |\mu| - \sum_{j=1}^{n-1} jh_j, \lambda) \in n\mathbb{Z}_{\geq 0}. \text{ In view of this we shall identify } \lambda \in P \text{ with the composition } (\lambda_1, \ldots, \lambda_n) \in (\mathbb{Z}_{\geq 0})^n \text{ such that } \lambda = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})X_i \text{ and } \sum_{i=1}^{n} \lambda_i = |\mu|. \text{ This is independent of the level of } \lambda. \]

If \( \lambda \in \mathcal{P}_l \), the composition \((\lambda_1, \ldots, \lambda_n)\) is in fact a partition; \( \lambda_1 \geq \cdots \geq \lambda_n \). If \( \lambda \in \mathcal{P}_l^+ \), it is a \textit{(level l) restricted} partition, by which we mean \( l + \lambda_n \geq \lambda_1 \geq \cdots \geq \lambda_n \). We regard a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) as a Young diagram in which the length of the \( i \)-th row is \( \lambda_i \). The depth of \( \lambda \) will be denoted by \( l(\lambda) \). The conjugate partition \( \lambda' \) is obtained from \( \lambda \) by transposing the diagram. Given partitions \( \xi \) and \( \eta \) let \( K_{\xi\eta}(q) \) be the associated Kostka-Foulkes polynomial and \( K_{\xi\eta} = K_{\xi\eta}(1) \) the Kostka number \([30]\). We extend the definition of the latter to compositions \( \eta \in (\mathbb{Z}_{\geq 0})^n \) by assuming the invariance under any permutation of the components of \( \eta \). Under the above identification of \( \lambda \in P \) with the compositions the 1dsums are expressed as

**Proposition 3.3**

\[
g_{\mu}(\lambda) = \sum_{\eta} K_{\eta\lambda}K_{\eta\mu}(q) \quad \lambda \in \mathcal{P}, \tag{3.5}
\]

\[
g'_{\mu}(\lambda) = \sum_{\eta} K_{\eta'\lambda}K_{\eta\mu}(q) \quad \lambda \in \mathcal{P}, \tag{3.6}
\]

\[
X_{\mu}(\lambda) = K_{\lambda\mu}(q), \quad X'_{\mu}(\lambda) = K_{\lambda'\mu}(q) \quad \lambda \in \mathcal{P}_l^+, \tag{3.7}
\]

where the sum in (3.5) (resp. (3.6)) runs over all partitions \( \eta \) of \( |\mu| \) with \( l(\eta) \leq n \) (resp. \( \eta_1 \leq n \)).

**Proof.** (3.7) is due to [1]. To see (3.5) consider the character of the energy and the \( s_\xi \) weight over \( B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_n)} \). It can be counted either weight vector-wise or \( s_\xi \) component-wise, leading to \( \sum_{\lambda} g_{\mu}(\lambda)m_\lambda = \sum_{\eta} K_{\eta\mu}(q)s_\eta \). Here \( m_\lambda \) and \( s_\eta \) denote the monomial symmetric function and the Schur function \([20]\), respectively and the both sums run over the partitions of \(|\mu|\) with depth \( \leq n \). Substituting the expansion \( s_\eta = \sum_{\lambda} K_{\eta\lambda}m_\lambda \) into this and comparing the coefficients of \( m_\lambda \) one has (3.5). (3.7) can be verified similarly.  

|     |   |   |     |   |
|-----|---|---|-----|---|
| 12  | 12 | 13 | 3   | −2 |
| 12  | 13 | 12 | 3   | 1  |
| 12  | 13 | 12 | 2   | −3 |
| 12  | 23 | 12 | 3   | 1  |
| 13  | 12 | 13 | 2   | −2 |
| 13  | 12 | 13 | 2   | 0  |
| 23  | 12 | 13 | 3   | −1 |
| 23  | 12 | 13 | 3   | 0  |
| 12  | 13 | 13 | 3   | −4 |
| 13  | 13 | 13 | 2   | −2 |
| 13  | 12 | 12 | 1   | −1 |
| 13  | 12 | 12 | 1   | 0  |
| 12  | 13 | 12 | 3   | c  |
| 13  | 13 | 13 | 2   | c  |
| 12  | 13 | 23 | 0   | r  |
| 13  | 12 | 23 | 0   | r  |
Example 3.4  Let $\lambda = (321)$, $\mu = (2211)$ and $n = 3$ as in Example 3.1. The RHS of (3.3) is calculated from the data

| $\eta$ | $K_{\eta}(321)$ | $K_{\eta}(2211)(q)$ |
|--------|----------------|---------------------|
| (6)    | 1              | $q^2$               |
| (51)   | 2              | $q^4 + q^6 + q^8$   |
| (42)   | 2              | $2q^3 + q^4 + q^5$  |
| (412)  | 1              | $q^2 + q^3 + q^4$   |
| (32)   | 1              | $q^2 + q^4$         |
| (321)  | 1              | $q + 2q^2 + q^3$    |

as

$$\text{RHS} = \sum_{\eta \ (\ell(\eta) \leq 3)} K_{\eta}(321)K_{\eta}(2211)(q)$$

$$= q + 4q^2 + 6q^3 + 6q^4 + 4q^5 + 2q^6 + q^7,$$

(3.8)
in agreement with Example 3.4.

Example 3.5  Let $\lambda = (322)$, $\mu = (2221)$ and $n = 3$ as in Example 3.2. The RHS of (3.6) is calculated from the data

| $\eta$ | $K_{\eta}(322)$ | $K_{\eta}(2221)(q)$ |
|--------|----------------|---------------------|
| (321)  | 1              | $q^2 + q^4 + q^6$   |
| (322)  | 1              | $q + q^3$           |
| (321^2)| 2              | $q + q^2$           |
| (31^2) | 2              | 1                   |

as

$$\text{RHS} = \sum_{\eta \ (\ell(\eta) \leq 3)} K_{\eta}(322)K_{\eta}(2221)(q)$$

$$= 2 + 3q + 4q^2 + 2q^3 + q^4,$$

(3.9)
in agreement with Example 3.5.

By analogy with (3.7) the 1dsums $X^{(l)}_\mu(\lambda)$ and $X^{(l)'}_\mu(\lambda)$ are also called the (level $l$) restricted Kostka-Foulkes polynomials.

3.3 Limit of 1dsoms

Given an $U_q(\hat{sl}_n)$-module $M$ and $\lambda \in P$, we define

$$M_\lambda = \{ v \in M \mid \text{wt} \, v = \lambda \},$$

$$[M : \lambda]_{cl} = \dim \{ v \in M_\lambda \mid e_i v = 0 \text{ for } 1 \leq i \leq n - 1 \},$$

$$[M : \lambda] = \dim \{ v \in M_\lambda \mid e_i v = 0 \text{ for } 0 \leq i \leq n - 1 \}.$$
Suppose that $M$ is a level $l$ module (not necessarily irreducible). We prepare the following notations for the relevant branching functions:

\[ c^M_\lambda(q) = \sum_i (\dim M_{\lambda + i\Lambda_0 - i\delta}) q^i \quad \lambda \in \mathcal{P}, \]  

\[ b^M_\lambda(q) = \sum_i [M : \lambda + i\Lambda_0 - i\delta] q^i \quad \lambda \in \mathcal{P}^+, \]  

\[ a^M_\lambda(q) = \sum_i [M : \lambda - i\delta] q^i \quad \lambda \in P^+. \]  

Up to an overall power of $q$, (3.10) is the string function \cite{15}, (3.12) is a branching coefficient of the irreducible $U_q(\mathfrak{sl}_n)$ module with highest weight $\lambda \in \mathcal{P}^+$, where $U_q(\mathfrak{sl}_n)$ stands for the subalgebra of $U_q(\mathfrak{sl}_n)$ generated by $e_i, f_i, t_i (1 \leq i \leq n-1)$.

The $U_q(\mathfrak{sl}_n)$-module corresponding to Theorem 2.4 (1) is given by

\[ V = \bigotimes_{J=1}^s \left( \bigoplus_{p \in \mathcal{H}(l_j\Lambda_{r_j}, \mu_j)} V(l_j\Lambda_{r_j} + af(wt p) - \mathcal{E}(l_j\Lambda_{r_j}, \mu_j')\delta) \right). \]  

As a consequence of Theorem 2.4 and Proposition 2.11, the large $\mu$ limit (2.3) of the 1dsums $g_\mu(\lambda), X_\mu(\lambda)$ and $X^{(j)}_\mu(\lambda)$ gives rise to the branching functions related to $V$. We summarize them in

**Proposition 3.6** Let $l_0 = l - \sum_{j=1}^s l_j$ and assume that $l_0 \geq 0$. Then we have

\[ \lim_{\mu} q^{-\mathcal{E}(l_0\Lambda_0, \mu)} g_\mu(\lambda) = c^V_\lambda(q) \quad \lambda \in \mathcal{P}, \]  

\[ \lim_{\mu} q^{-\mathcal{E}(l_0\Lambda_0, \mu)} X_\mu(\lambda) = b^V_\lambda(q) \quad \lambda \in \mathcal{P}^+, \]  

\[ \lim_{\mu} q^{-\mathcal{E}(l_0\Lambda_0, \mu)} X^{(j)}_\mu(\lambda) = a_\lambda \otimes V(l_0\Lambda_0)(q) \quad \lambda \in P^+. \]  

The $U_q(\mathfrak{sl}_n)$-module $V$ (3.13) is reducible in general and has the form

\[ V = \bigoplus (V(\xi_1 - d_1\delta) \otimes \cdots \otimes V(\xi_s - d_s\delta)), \]  

where the direct sum runs over the $s$-tuple of the restricted paths $(p^1, \ldots, p^s) \in \mathcal{H}(l_1\Lambda_{r_1}, \mu_1^1) \times \cdots \times \mathcal{H}(l_s\Lambda_{r_s}, \mu_1^s)$. $\xi_j \in P^+_l$ and $d_j \in \mathbb{Z}$ are specified as $\xi_j = l_j\Lambda_{r_j} + af(wt p^j)$ and $d_j = E(p^j) - \mathcal{E}(l_j\Lambda_{r_j}, \mu_j^j)$ for each summand. Correspondingly the branching functions appearing in (3.14)-(3.16) are in fact equal to the linear combinations:

\[ c^V_\lambda(q) = \sum q^{d_1 + \cdots + d_s} c_\lambda^{V(\xi_1) \otimes \cdots \otimes V(\xi_s)}(q), \]  

\[ b^V_\lambda(q) = \sum q^{d_1 + \cdots + d_s} b_\lambda^{V(\xi_1) \otimes \cdots \otimes V(\xi_s)}(q), \]  

\[ a_\lambda \otimes V(l_0\Lambda_0)(q) = \sum q^{d_1 + \cdots + d_s} a_\lambda^{V(\xi_1) \otimes \cdots \otimes V(\xi_s) \otimes V(l_0\Lambda_0)}(q), \]
where the three sums are taken in the same way as explained above. In this way our construction by means of the inhomogeneous paths naturally gives rise to the finite linear combinations of the "usual" branching functions \(a, b\) and \(c\). By choosing \(l, r, \mu\) \((1 \leq J \leq s)\) variously, one can let \(\mathcal{V}\) cover a large family of (generally reducible) \(U_q(\mathfrak{sl}_n)\)-modules. In general the corresponding sum (3.18)–(3.20) is complicated depending especially on \(\mu^1, \ldots, \mu^s\). We shall therefore calculate the limits in Proposition 3.6 explicitly only for \(s = 1\) or \(\mu_1 = \cdots = \mu_s = \emptyset\) in Sections 4.2 and 5. These sample calculations already lead to several character formulae, which appear to be new. To exploit the full variety of such formulae for general \(\mathcal{V}\) will be an interesting problem. Now we proceed to another essential input for our calculation, fermionic forms of the 1dsums.

4 Fermionic formulae

Here we shall present explicit formulae for the 1dsums. They all have a fermionic form, by which we roughly mean the polynomials or series that are free of signs or possibly have some relevance to the Bethe ansatz. Such formulae for \(g_\mu(\lambda)\) and \(g^{(l)}_\mu(\lambda)\) are new, while for \(X_\mu(\lambda)\) and \(X^{(l)}_\mu(\lambda)\) they have been known or conjectured in various cases in \([3, 20, 21, 24]\). We shall also present their limiting forms as \(|\mu| \to \infty\) as in (2.6). By virtue of Proposition 3.6 this establishes several character formulae concerning the tensor products of \(U_q(\mathfrak{sl}_n)\)-modules. It turns out that the limit of \(g_\mu(\lambda)\) gives rise to a generalization of the formulae in \([9, 11, 28]\) and those of \(X_\mu(\lambda), X^{(l)}_\mu(\lambda)\) yield generalizations of a spinon character formulae in \([2, 4, 6, 32]\).

In the working below we shall employ the notations:

\[
\begin{align*}
\left[ \frac{m}{k} \right] &= \left\{ \begin{array}{ll}
\frac{(q)_m}{(q)_k(q)_{m-k}} & \text{if } 0 \leq k \leq m \\
0 & \text{otherwise}
\end{array} \right., \\
(q)_m &= \prod_{i=1}^{m} (1 - q^i) & \text{for } m \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

We shall also use

\[
n(\nu) = \sum_{i \geq 1} \left( \frac{\nu_i}{2} \right), \quad \left( \frac{x}{2} \right) = \frac{x(x-1)}{2}
\]

according to \([30]\). The symbol \(n(\nu)\) should not be confused with \(n\) from \(\mathfrak{sl}_n\). Elements of the Cartan matrix of \(\mathfrak{sl}_k\) and its inverse will be written as (1 \( \leq i, j \leq k - 1\))

\[
C_{ij}^{(k)} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}, \quad C_{ij}^{(k)^{-1}} = \min(i, j) - \frac{ij}{k}. \quad (4.1)
\]
We assume that $C_{ij}^{(k)-1} = 0$ if $i$ or $j$ is equal to 0 or $k$.

4.1 Fermionic formulae of 1dsums

A fermionic formula of the unrestricted 1dsum $g_{\mu}(\lambda)$ (3.5) is given by

**Proposition 4.1** For any partition $\mu$ and composition $\lambda \in (\mathbb{Z}_{\geq 0})^n$ such that $|\lambda| = |\mu|$ we have

$$
\sum_{\{\nu\} \in \mathcal{Y}} K_{\lambda \mu}(q) = \sum_{\{\nu\}} q^{\hat{\phi}(\{\nu\})} \prod_{1 \leq a \leq n-1, 1 \leq i \leq \mu_1} \left[ \frac{\nu_i^{(a+1)} - \nu_i^{(a)}}{\nu_i^{(a+1)} - \nu_i^{(a+1)}} \right],
$$

(4.2)

where the sum $\sum_{\{\nu\}}$ runs over the sequences of Young diagrams $\nu^{(1)}, \ldots, \nu^{(n-1)}$ such that

$$
\emptyset =: \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu^{(n)} := \mu',
$$

$|\nu^{(a)}| = \lambda_1 + \cdots + \lambda_a$ for $1 \leq a \leq n-1.

(4.4)

When $n = 2$ and $\mu$ is rectangular the result $g_{\mu}(\lambda) = \text{RHS of (4.2)}$ was known in [7], and the above formula is reduced to the one in [20]. For $n = 2$ and general $\mu$ it agrees with [33]. The case $\mu = (1^n)$ was known for general $n$ in [13] and [8]. See also (4.33)–(4.36) for an equivalent expression in terms of “TBA-like” variables and Remark 4.9 concerning it. The proof of Proposition 4.1 will be given after the proof of Proposition 4.3. Here we shall illustrate the formula by an example.

**Example 4.2** Let $\lambda = (321)$, $\mu = (2211)$ and $n = 3$ as in Example 3.1 and 3.4. The relevant $\{\nu^{(1)}, \nu^{(2)}, \nu^{(3)} = (42)\}$'s and their contributions are

- $q^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $q^3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$
- $q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $q \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

By summing them up, we obtain (3.8).
A fermionic formula of the unrestricted 1dsum $g'_\mu(\lambda)$ \(...\) is given by

**Proposition 4.3** For any partition $\mu$ and composition $\lambda \in (\mathbb{Z}_{\geq 0})^n$ such that $|\lambda| = |\mu|$ and $\mu_1 \leq n - 1$ we have

$$
\sum_{\eta_{(n_1 \leq n)}} K_{\eta, \lambda} K_{\eta, \mu}(q) = \sum_{\{\nu\}} \prod_{1 \leq a \leq n} \left[ \frac{\nu_{(a+1)} - \nu_{(a+1)}}{\nu_{(a)} - \nu_{(a+1)}} \right],
$$

(4.5)

where the sum $\sum_{\{\nu\}}$ runs over the sequences of Young diagrams $\nu^{(1)}, \ldots, \nu^{(n-1)}$ such that

$0 =: \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu^{(n)} := \mu'$,

$\nu^{(a)}/\nu^{(a-1)}$: horizontal strip of length $\lambda_a$.

(4.6)

**Example 4.4** Let $\lambda = (322)$, $\mu = (2221)$ and $n = 3$ as in Example 3.2 and 3.5. The relevant $\{\nu^{(1)}, \nu^{(2)}, \nu^{(3)} = (43)\}$’s and their contributions are

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By summing them up, we obtain (3.9).

**Proof.** (Proposition 4.3) Let us note that the left hand side of the equation (4.3) is nothing but the transition coefficient from elementary symmetric function basis to Hall-Littlewood basis

$$
e_\lambda(x) = e_{\lambda_1}(x) \cdots e_{\lambda_n}(x) = \sum_{\eta_{(n_1 \leq n)}} \sum_{\mu} K_{\eta, \lambda} K_{\eta, \mu}(q) P_\mu(x; q),
$$

which follows from $e_\lambda(x) = \sum_\eta K_{\eta, \lambda} s_\eta(x)$ and $s_\eta(x) = \sum_\mu K_{\eta, \mu} P_\mu(x; q)$. Then by using the formula (see, e.g., \[eq.(3.2) p215\])

$$
e_m(x) P_\nu(x; q) = \sum_\mu f^\mu_{\nu(1^m)} P_\mu(x; q),
$$

$$f^\mu_{\nu(1^m)} = \prod_{i \geq 1} \left[ \frac{\mu'_i - \nu'_{i+1}}{\mu'_i - \nu'_i} \right],
$$

(4.7)

and therefore $f^\mu_{\nu(1^m)} = 0$ unless $\mu/\nu$ is a vertical $m$-strip, we obtain

$$
e_\lambda(x) = \sum_{\{\mu\}} f^{\mu^{(n)}}_{\mu(n-1)(1^\lambda_n)} \cdots f^{\mu^{(2)}}_{\mu(1^2)} f^{\mu^{(1)}}_{\mu(0)(1^\lambda_1)} P_{\mu^{(n)}}(x; q),
$$

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where the sum is taken over the partitions \( \{ \mu^{(0)} = \emptyset, \mu^{(1)}, \ldots, \mu^{(n)} \} \). By putting \( \mu^{(a)} = \nu^{(a)} \) we have the desired formula (4.5).

**Proof.** (Proposition 4.1) The proof is essentially the same as the proof of Proposition 4.3. The relevant transition coefficients are those from complete symmetric function basis to Hall-Littlewood basis

\[
h_{\lambda}(x) = h_{\lambda_1}(x) \cdots h_{\lambda_n}(x) = \sum_{\eta \in (\mathcal{N}_n)} \sum_{\mu} K_{\eta \lambda} K_{\eta \mu}(q) P_{\mu}(x; q),
\]

which follows from \( h_{\lambda}(x) = \sum_{\eta} K_{\eta \lambda} s_{\eta}(x) \) and \( s_{\eta}(x) = \sum_{\mu} K_{\eta \mu}(q) P_{\mu}(x; q) \). In this case, the formula corresponding to (4.7) is the following,

\[
h_{m}(x) P_{\nu}(x; q) = \sum_{\mu} g_{\nu}^\mu(x; q), \quad g_{\nu}^\mu = \sum q^{i \geq 1} \left( \frac{\mu_i - \nu_i}{2} \right) \prod_{i \geq 1} \left[ \frac{\mu_i - \mu_{i+1}}{\nu_i - \nu_{i+1}} \right], \tag{4.8}
\]

and \( g_{\nu}^\mu = 0 \) unless \( \nu \subset \mu \) and \( |\mu/\nu| = m \). The equation (4.8) can be proved as follows.

Because of the relation \( \sum_{m=0}^{m=1} (-1)^l e_{m-l}(x) h_l(x) = \delta_{m,0} \), we have

\[
\sum_{\mu} f_{\nu}^\mu g_{\mu(l)}(-1)^l = \delta_{\lambda, \nu}, \tag{4.9}
\]

where \( k = |\mu/\nu|, l = |\lambda/\mu| \) and the sum is taken for all \( \mu \) such that \( \nu \subset \mu \subset \lambda \). Since \( \{g_{\nu}^\mu(l)\} \) are uniquely determined from \( \{f_{\nu}^\mu(l)\} \) by this relation (4.9), it is enough to show that the explicit formulae for \( g_{\nu}^\mu(l) \) (4.8) and \( f_{\nu}^\mu(l) \) (4.7) indeed satisfy this relation. To this end, we rewrite the left hand side of the relation (4.9) as follows

\[
\sum_{\nu \subset \mu \subset \lambda} (-1)^{\sum (\lambda_i' - \mu_i')} \sum_{\nu'} q^{\sum (\lambda_i' - \mu_i')} \prod_{i \geq 1} \left[ \frac{\lambda_i' - \mu_{i+1}'}{\lambda_i' - \mu_i'} \right] \left[ \frac{\mu_i' - \mu_{i+1}'}{\mu_i' - \mu_i'} \right] = \prod_{i \geq 1} \Phi_i(q), \tag{4.10}
\]

where

\[
\Phi_i(q) = \sum_{\nu' \leq \mu_i \leq \lambda_i} (-1)^{\lambda_i' - \mu_i'} q^{\sum (\lambda_i' - \mu_i')} \frac{(q)_{\lambda_i' - \mu_i'} (q)_{\mu_i' - \nu_i'} (q)_{\nu_i' - \mu_i'}}{(q)_{\lambda_i' - \mu_i'} (q)_{\mu_i' - \nu_i'} (q)_{\nu_i' - \mu_i'}},
\]

and \( (q)_{\lambda_i' - \mu_i'}/(q)_{\nu_i' - \mu_i'} = 1 \). Precisely speaking, the sum in equation \( \Phi_i \) should be taken over the \( \mu_i' \) such as \( \max(\mu_{i+1}', \nu_i') \leq \mu_i' \leq \lambda_i' \), since \( \mu \) is a partition. However, if \( \mu_{i+1}' > \nu_i' \), one can consider \( \Phi_{i+1} \) vanishes by understanding \( 1/(q)_{\nu_i' - \mu_i'} = 0 \), hence one can simply write as above.
Consider at first $\Phi_1(q)$. We have
\begin{align*}
(q)\lambda'_1 - \nu'_1 \Phi_1(q) &= \sum_{\nu'_1 \leq \mu'_1 \leq \lambda'_1} (-1)^{\lambda'_1 - \mu'_1} q^{\frac{\lambda'_1 - \mu'_1}{2}} \left[ \frac{\lambda'_1 - \nu'_1}{\lambda'_1 - \mu'_1} \right] \\
&= \sum_{m=0}^{\lambda'_1 - \nu'_1} (-1)^m q^{\frac{m}{2}} \left[ \frac{\lambda'_1 - \nu'_1}{m} \right] = \delta_{\lambda'_1, \nu'_1}. \tag{4.11}
\end{align*}

The last equality follows from the $q$–binomial theorem
\begin{align*}
\sum_{m=0}^{N} (-z)^m q^{\frac{m}{2}} \left[ \frac{N}{m} \right] = \prod_{i=1}^{N} (1 - q^{i-1} z).
\end{align*}
Thus we have $\Phi_1 = \delta_{\lambda'_1, \nu'_1}$. Similarly, under the condition $\lambda'_1 = \nu'_1$, we obtain $\Phi_2 = \delta_{\lambda'_2, \nu'_2}$, since
\begin{align*}
(q)\lambda'_2 - \nu'_2 \Phi_2(q) &= \sum_{m=0}^{\lambda'_2 - \nu'_2} (-1)^m q^{\frac{m}{2}} \left[ \frac{\lambda'_2 - \nu'_2}{m} \right] = \delta_{\lambda'_2, \nu'_2}.
\end{align*}
Repeating these arguments we see that the product (4.10) is equal to $\delta_{\lambda, \nu}$. This proves (4.9) and hence the Proposition 4.1.

Essentially the same proof is also available in [22] as well as discussions on several other aspects.

The classically restricted 1dsum $X_{\mu}(\lambda)$ (3.7) is nothing but the Kostka-Foulkes polynomial. Its fermionic formula reads as

**Proposition 4.5 ([24])** For any partitions $\lambda$ and $\mu$ such that $|\lambda| = |\mu|$ and $l(\lambda) \leq n$ we have
\begin{align*}
K_{\lambda \mu}(q) &= \sum_{\{m\}} q^{c(\{m\})} \prod_{i \geq 1}^{1 \leq a \leq n-1} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right], \tag{4.12}
\end{align*}
\begin{align*}
c(\{m\}) &= n(\mu) + \frac{1}{2} \sum_{1 \leq a, b \leq n-1} C_{ab}^{(n)} \sum_{j, k \geq 1} \min(j, k) m_j^{(a)} m_k^{(b)} \\
&\quad - \sum_{j, k \geq 1} \min(j, \mu_k) m_j^{(1)}, \tag{4.13}
\end{align*}
\begin{align*}
p_i^{(a)} &= \delta_{a1} \sum_{k \geq 1} \min(i, \mu_k) - \sum_{b=1}^{n-1} C_{ab}^{(n)} \sum_{k \geq 1} \min(i, k) m_k^{(b)}, \tag{4.14}
\end{align*}
29
where the sum $\sum_{(m)}$ is taken over \(\{m_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n - 1, i \geq 1\}\) satisfying $p_i^{(a)} \geq 0$ for $1 \leq a \leq n - 1, i \geq 1$, and
\[
\sum_{i \geq 1} i m_i^{(a)} = \lambda_{a+1} + \lambda_{a+2} + \cdots + \lambda_n \quad \text{for} \quad 1 \leq a \leq n - 1.
\] (4.15)

This formula is originated in the Bethe ansatz. When $q = 1$ it counts the multiplicity of the $\lambda$-representation in the tensor product of the $\mu$-fold symmetric tensor representations ($1 \leq i \leq l(\mu)$). Kostka-Foulkes polynomials $K_{\xi\eta}(q)$ can also be realized through the dual picture corresponding to the 1dsum $X_\eta^l(\xi')$, namely, the $(q)$-multiplicity of the $\xi'$-representation in the tensor product of the $\eta$-fold antisymmetric tensor representations ($1 \leq i \leq l(\eta)$). Such a duality will be discussed in more general setting in \[22\]. From the results therein we have another fermionic formula as

**Proposition 4.6 \([22]\)** For any partitions $\xi$ and $\eta = ((n-1)\xi_1 \cdots \xi_i)$ such that $|\xi| = |\eta|$ and $\xi_1 \leq n$, we have
\[
K_{\xi\eta}(q) = \sum_{(\hat{m})} q^{\hat{c}(\hat{m})} \prod_{1 \leq a \leq n-1} \left[ \frac{p_i^{(a)} + \hat{m}_i^{(a)}}{\hat{m}_i^{(a)}} \right],
\] (4.16)
\[
\hat{c}(\hat{m}) = \frac{1}{2} \sum_{1 \leq a,b \leq n-1} C_{ab}^{(n)} \sum_{j,k \geq 1} \min(j,k) \hat{m}_j^{(a)} \hat{m}_k^{(b)},
\] (4.17)
\[
p_i^{(a)} = \zeta_a - \sum_{b=1}^{n-1} C_{ab}^{(n)} \sum_{k \geq 1} \min(i,k) \hat{m}_k^{(b)},
\] (4.18)

where the sum $\sum_{(\hat{m})}$ is taken over \(\{\hat{m}_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n - 1, i \geq 1\}\) satisfying $p_i^{(a)} \geq 0$ for $1 \leq a \leq n - 1, i \geq 1$, and
\[
\sum_{i \geq 1} i \hat{m}_i^{(a)} = \sum_{b=1}^{n-1} \min(a,b) \xi_b - (\xi_1 + \cdots + \xi_i) \quad \text{for} \quad 1 \leq a \leq n - 1.
\] (4.19)

As for the restricted 1dsums $X_{\mu}^{(l)}(\lambda)$ and $X_{\mu}^{(l)'}(\lambda)$, fermionic formulae are yet conjectural in general and not yet available for arbitrary $\lambda$. (sl$_2$ case is the exception, see \[3, 22\] and references therein.) Here we shall only deal with the vacuum case $\lambda = l\Lambda_0$ corresponding to the partitions ($((\frac{\mu}{n})^n)$ and $(n \frac{\mu}{n})$, respectively. See Section 5.2 for more general $\lambda$ cases. To present our conjecture we prepare two expressions $F_{\mu}^{(l)}(q)$ and $F_{\eta}^{(l)'}(q)$. The first one is defined for partitions $\mu$ satisfying $|\mu| \equiv 0 \mod n$, $\mu \leq l$ and reads
\[
F_{\mu}^{(l)}(q) = \sum_{(m)} q^{\hat{c}(\hat{m})} \prod_{1 \leq a \leq n-1} \prod_{1 \leq i \leq l-1} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right],
\] (4.20)
\[ c_i(\{m\}) = n(\mu) + \frac{1}{2} \sum_{1 \leq a, b \leq n-1} C_{ab}^{(n)} \sum_{1 \leq j, k \leq l} \min(j, k)m_j^{(a)}m_k^{(b)} \]
\[ - \sum_{k \geq 1} \sum_{j=1}^{l} \min(j, \mu) m_j^{(1)}, \] (4.21)
\[ p_i^{(a)} = \delta_{i1} \sum_{k \geq 1} \min(i, \mu) - n-1 \sum_{b=1}^{l} C_{ab}^{(n)} \sum_{k=1}^{l} \min(i, k)m_k^{(b)}, \] (4.22)

where the sum \(\sum_{\{m\}}\) is taken over \(\{m_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n-1, 1 \leq i \leq l\}\) satisfying \(p_i^{(a)} \geq 0\) for \(1 \leq a \leq n-1, 1 \leq i \leq l-1\), and

\[ \sum_{i=1}^{l} i m_i^{(a)} = \frac{n-a}{n} |\mu| \quad \text{for} \quad 1 \leq a \leq n-1. \] (4.23)

This is very similar to the fermionic form appearing in Proposition 4.3 with \(\lambda = ((\frac{|\mu|}{n})^n)\). In fact (4.20)-(4.23) correspond to truncating the indices of \(m_i^{(a)}\) and \(p_i^{(a)}\) in (4.12)-(4.15) to the range \(1 \leq i \leq l\). The second one \(F_{\eta}^{(l)'(q)}\) is a similar analogue of the fermionic form in Proposition 4.6. It is defined for the partitions of the form \(\eta = ((n-1)^{i-1} \cdots 1^i)\) such that \(|\eta| = 0 \mod n\).

\[ F_{\eta}^{(l)'(q)} = \sum_{\{\tilde{m}\}} \tilde{q}_{\zeta_i(\{\tilde{m}\})} \prod_{1 \leq a, b \leq n-1, 1 \leq i \leq l-1} \left[ \frac{p_i^{(a)} + \tilde{m}_i^{(a)}}{\tilde{m}_i^{(a)}} \right], \] (4.24)

\[ \tilde{c}_i(\{\tilde{m}\}) = \frac{1}{2} \sum_{1 \leq a, b \leq n-1} C_{ab}^{(n)} \sum_{1 \leq j, k \leq l} \min(j, k)m_j^{(a)}m_k^{(b)}, \] (4.25)

\[ \tilde{p}_i^{(a)} = C_{1i}^{(l)-1} \zeta_a - \sum_{b=1}^{n-1} C_{ab}^{(n)} \sum_{k=1}^{l-1} C_{ik}^{(l)-1} \tilde{m}_k^{(b)}, \] (4.26)

where the sum \(\sum_{\{\tilde{m}\}}\) is taken over \(\{\tilde{m}_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n-1, 1 \leq i \leq l\}\) satisfying \(\tilde{p}_i^{(a)} \geq 0\) for \(1 \leq a \leq n-1, 1 \leq i \leq l-1\), and

\[ \sum_{i=1}^{l} i \tilde{m}_i^{(a)} = \sum_{b=1}^{n-1} C_{ab}^{(n)-1} \zeta_b \quad \text{for} \quad 1 \leq a \leq n-1. \] (4.27)

This is a truncation of (4.16)-(4.19) with \(\xi = (\frac{n}{m})\). Note that the variables \(\tilde{m}_i^{(a)} (1 \leq a \leq n-1)\) are specified and appear only in (4.27) to impose the condition \(\tilde{m}_i^{(a)} \in \mathbb{Z}_{\geq 0}\). Now our conjecture is stated as
Conjecture 4.7 For a partition $\mu$ such that $|\mu| \equiv 0 \text{ mod } n$ and $\mu_1 \leq l$ we have

$$X_{\mu}^{(l)}(\lambda_0) = F_{\mu}^{(l)}(q).$$  \hspace{1cm} (4.28)

For a partition $\eta$ such that $\eta_1 \leq n - 1$ and $|\eta| \equiv 0 \text{ mod } n$ we have

$$X_{\eta}^{(l)}(\lambda_0) = F_{\eta}^{(l)}(q).$$  \hspace{1cm} (4.29)

This is also based on the Bethe ansatz for RSOS models, in which an analogous conjecture on $X_{\eta}^{(l)}(\lambda)$ for $\lambda$ non vacuum type.

4.2 Fermionic formulae of the limits

By taking the large $\mu$ limit of the 1dsums in Section 1.1, we obtain explicit $q$-series formulae of the characters in Proposition 3.6. For the unrestricted 1dsum $g_{\mu}(\lambda)$ we consider two particular limits. The first one is $\mu = (L) \cup \nu$, where $L \to \infty$ under the condition $L \equiv r \ (0 \leq r \leq n - 1) \text{ mod } n$ and $\nu$ is a fixed finite partition with $\nu_1 < l$. By (8.13) this corresponds to the module $V = \oplus_{p \in H(\Lambda_r, \nu)} V(\Lambda_r + af(\nu p) - (E(p) - E(\nu(p)))\delta)$. See Example 2.3. By calculating the above limit of (4.2) one can show

Proposition 4.8 Let $0 \leq r \leq n - 1$ and $V = \oplus_{p \in H(\Lambda_r, \nu)} V(\Lambda_r + af(\nu p) - (E(p) - E(\nu(p)))\delta)$. For $\lambda = (\lambda_1 - \lambda_2)\Sigma_1 + \cdots + (\lambda_{n-1} - \lambda_n)\Sigma_{n-1} \in \mathcal{F}^l$, we have

$$q^\Delta C\mathcal{V}(q) = \frac{1}{(q)^{n-1}} \sum_{\{m\}} \prod_{1 \leq \alpha \leq n-1} (q^{C_{\alpha}}(m))^{-1/2},$$

$$C\{\{m\}\} = \frac{1}{2} \sum_{a,b=1}^{n-1} \sum_{j,k=1}^{l-1} C^{m(a)}_{ab} C_{jk}^{l-1} m_{j}^{(a)} m_{k}^{(b)}$$

$$- \sum_{j=1}^{l-1} C_{jk}^{l-1} (\nu_j' - \nu_{j+1}') m_{k}^{(n-1)},$$

where $\Delta = \frac{1}{2} \sum_{j=1}^{l-1} (\lambda_r + \nu_j'|\lambda_r + \nu_j) - (\lambda_l|\lambda_l) + \frac{n-1}{2l} \sum_{j,k=1}^{l-1} C_{jk}^{l-1} (\nu_j' - \nu_{j+1}') (\nu_k' - \nu_{k+1}')$. The sum $\sum_{\{m\}}$ is taken over $\{m_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n-1, 1 \leq i \leq l-1\}$ satisfying

$$\sum_{i=1}^{l-1} i m_i^{(a)} \equiv \sum_{b=1}^{a} \left( \lambda_b - \frac{|\lambda| - |\nu| - lr}{n} \right) \text{ mod } l \quad \text{for } 1 \leq a \leq n-1.$$  \hspace{1cm} (4.32)
Proof. Let us rewrite (4.32) in terms of the variables $m_i^{(a)} = \nu^{(a)}_i - \nu^{(a)}_{i+1}$, \(1 \leq i \leq l = m_1:

\[
\sum_{\eta(l(\eta) \leq n)} K_{\eta \lambda} K_{\mu \eta}(q) = \sum_{\{m\}} q^{\psi(m)} \prod_{1 \leq a \leq n-1, 1 \leq i \leq l} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right], \tag{4.33}
\]

\[
p_i^{(a)} = \sum_{j=i}^{l} (m_j^{(a+1)} - m_j^{(a)}), \tag{4.34}
\]

\[
\psi(\{m\}) = n(\mu) + \frac{(\lambda|\lambda)}{2l} - \frac{n-1}{2nl}|\mu|^2 + \frac{1}{2} \sum_{a,b=1}^{n-1} \sum_{j,k=1}^{l-1} C_{ab}(n)^{(l)} C_{jk}^{-1} m_j^{(a)} m_k^{(b)} - \sum_{j,k=1}^{l-1} C_{jk}^{(l)} (\mu_j' - \mu_{j+1}') m_k^{(n-1)}, \tag{4.35}
\]

where $m_i^{(n)} = \mu_i' - \mu_{i+1}'$. In view of (4.4) the sum $\sum_{\{m\}}$ in (4.33) runs over $\{m_i^{(a)} \in \mathbb{Z}_{\geq 0} | 1 \leq a \leq n-1, 1 \leq i \leq l\}$ obeying the condition

\[
\sum_{i=1}^{l} i m_i^{(a)} = \lambda_1 + \cdots + \lambda_a. \tag{4.36}
\]

In deriving (4.35) we have eliminated $m_i^{(a)}$ for $1 \leq a \leq n-1$ by (4.33) and used $(\lambda|\lambda) = \sum_{a=1}^{n} \lambda_a^2 - |\lambda|^2/n$. The limit $L \to \infty$ of (4.33) is to be expanded from such $m_i^{(a)}$ that attains the minimum of $\psi$ (4.35). It occurs around $m_i^{(a)} = \frac{a}{n}(L - r) \delta_{ij}$. Thus $\forall p_i^{(a)} \to \infty$ in the limit $L \to \infty$ hence the product of the $q$-binomial coefficients tends to $\left((q)^{n-1} \prod_{a=1}^{n-1} \prod_{i=1}^{l-1} (q)^{m_i^{(a)}}\right)^{-1}$. The powers of $q$ can be adjusted by noting $\mu_j' - \mu_{j+1}' = \nu_j' - \nu_{j+1}'$ for the shape $\mu = (l^2) \cup \nu$ in (4.33) and verifying $-E(l \Lambda_0, \mu) + \Delta + \psi(\{m\})|_{m_i^{(a)} \to m_i^{(a)}} + \Theta(L-r) \delta_{ij} = C(\{m\})$ by using the ground state energy (2.7).

For $r = 0$ and $\nu = 0$ one has $V = V(l \Lambda_0)$, in which case the above formula was conjectured in [28], announced in [1] and proved in [11]. When $r = n-1$ and $\nu = (s)$ it agrees with $j = n$ case of eq.(5.7) in [11]. However the above result does not cover the $V((l-s) \Lambda_0 + s \Lambda_1)$ case for general $j$ obtained in [11].

Remark 4.9 Although the fermionic form (4.33) looks similar to that in Proposition 4.4, its interpretation in terms of the Bethe ansatz is yet unknown to us.

For an $U_q(\widehat{\mathfrak{sl}_n})$-module $M$, let $\text{ch} M$ denote its character:

\[
\text{ch} M = \sum_{\lambda} (\text{dim} M_{\lambda}) e^\lambda.
\]

33
where \( q = e^{-\delta} \) and the sum runs over all the affine weights \( \lambda \in \mathcal{P} \). Then we make

**Remark 4.10** Substituting \( r = 0, \nu = 0 \) case of Proposition 4.8 into eq. (12.7.12) in \([4]\), one can express the character of the level \( l \) vacuum module \( \mathcal{V} = \mathcal{V}(lA_0) \) as

\[
e^{-lA_0} ch\mathcal{V}(lA_0) = \frac{1}{(q)^{(n-1)}} \sum_{\{m\}} \frac{q^{C_i(m)}}{\prod_{1 \leq a \leq n-1} (q)_{m_a}^{(a)}} e^\delta(m), \tag{4.37}
\]

\[C_i(m) = \frac{1}{2} \sum_{a,b=1}^{n-1} \sum_{j,k=1}^{l} C_{ab}^{(n)} \min(j,k) m_a^{(a)} m_b^{(b)}, \tag{4.38}\]

\[\beta_i(m) = \sum_{a=1}^{n-1} \sum_{i=1}^{l} i m_i^{(a)} \alpha_a. \tag{4.39}\]

Here the sum \( \sum_{\{m\}} \) runs over \( \{m_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n - 1, 1 \leq i \leq l - 1 \} \) and \( \{m_i^{(a)} \in \mathbb{Z} \mid 1 \leq a \leq n - 1 \} \). This agrees with a formula announced in \([3]\) and proved in \([4]\).

The second limit we consider of \( g_{r_1}(\lambda) \) given by (4.5) and Proposition 4.1 is 
\[
\mu = (l_1^{r_1}) + \cdots + (l_s^{r_s}) \text{ in the notation of Section 2.2, in which } L_J = L_{J+1} \rightarrow \infty \text{ for } 1 \leq J \leq s \text{ keeping } L_J = r_J \text{ mod } n \text{ fixed (} 0 \leq r_J \leq n - 1, L_{s+1} = r_{s+1} = 0 \).
\]

A similar calculation to Proposition 4.8 leads to

**Proposition 4.11** Fix \( l_1, \ldots, l_s \in \mathbb{Z}_{\geq 0} \) and \( 0 \leq r_1, \ldots, r_s \leq n - 1 \). Set \( |r| = r_1 + \cdots + r_s \), \( l = l_1 + \cdots + l_s \) and \( M_J = l_1 + l_2 + \cdots + l_J \). For \( \lambda = (\lambda_1 - \lambda_2)A_1 + \cdots + (\lambda_{n-1} - \lambda_n)A_{n-1} \in \mathcal{F} \) and the tensor product module \( \mathcal{V} = \otimes_{J=1}^s \mathcal{V}(l_JA_{r_J}) \), we have

\[
q^{\Delta} c^\lambda_{J}(q) = \frac{1}{(q)^{(n-1)}} \sum_{\{m\}} \frac{q^{C(m)}}{\prod_{1 \leq a \leq n-1} (q)_{m_a}^{(a)}} \tag{4.40}
\]

\[C(m) = \frac{1}{2} \sum_{a=1}^{n-1} \sum_{j,k=1}^{l} C_{ab}^{(n)} C_{jk}^{(l)} m_a^{(a)} m_j^{(b)} \]

\[+ \sum_{j=1}^{l-1} \sum_{k=1}^{s-1} \sum_{M_J}^{(l-1)} (r_J - r_{J+1}) m_j^{(n-1)}, \tag{4.41}\]

where \( \Delta = \sum_{J=1}^{s} \frac{1}{2} (A_{r_J} | A_{r_J}) - (\lambda | \lambda) \frac{n-1}{2n} \sum_{J,K=1}^{s} C_{JK}^{(l)} (r_J - r_{J+1})(r_K - r_{K+1}) \). The sum \( \sum_{\{m\}} \) runs over \( \left\{ m_i^{(a)} \mid 1 \leq a \leq n - 1, 1 \leq i \leq l - 1 \right\} \) satisfying

\[m_i^{(a)} \in \left\{ \begin{array}{ll}
\mathbb{Z} & \text{otherwise} \\
\mathbb{Z}_{\geq 0} & \text{otherwise}
\end{array} \right. \tag{4.42}\]
\[
\sum_{i=1}^{l-1} i m_i^{(a)} = \sum_{b=1}^{a} \left( \lambda_b - \frac{|\lambda| - \sum_{j=1}^{s} l_j r_j}{n} \right) \mod l \text{ for } 1 \leq a \leq n - 1. \tag{4.43}
\]

Proof. Again we start with the expression \([4.33]-[4.35]\) with \(\mu = (l_1^{i_1}) + \cdots + (l_s^{i_s})\). This time the minimum of \(\psi \tag{4.35}\) is attained around \(m_i^{(a)} = \frac{a}{n}(L_j - L_{j+1} - r_J + r_{J+1})\) for \(i = M_j (1 \leq J \leq s), = 0\) for \(i \notin \{M_1, \ldots, M_s\}\). In the limit \(L_j - L_{j+1} \to \infty\), \(m_i^{(a)}\) tends to infinity for \(i \in \{M_1, \ldots, M_s\}\) and so does \(\forall i^{(a)}\). Thus the product of the \(q\)-binomial coefficients converges to \(\left(\left(q\right)^{(n-1)s} \prod_{n=1}^{a-1} \prod_{i \in \{1, \ldots, l-1\}\setminus\{M_1, \ldots, M_s-1\}} \left(q\right)_{m_i^{(a)}}\right)^{-1}\). After the replacement \(m_i^{(a)} \to m_i^{(a)} + m_i^{(a)}\), the new variables \(m_i^{(a)}\) are to satisfy \([4.42]-[4.43]\). It remains to check \(-E(l\Lambda_0, \mu) + \Delta + \psi(\{m\})\big|_{m_i^{(a)} \to m_i^{(a)} - m_i^{(a)}} = C(\{m\})\) for \(\mu = (l_1^{i_1}) + \cdots + (l_s^{i_s})\). This is straightforward by using the explicit forms \([2.7]\) and \([1.33]\).

By the definition the module \(V\) remains unchanged under any permutations of \((l_j, r_j), (J = 1, \ldots, s)\). Note however that such symmetry is not manifest in the RHS of \([4.44]-[4.42]\). When \(r_1 = \cdots = r_s = 0\), Proposition \([4.11]\) can also be shown by decomposing the product of \(e^{-L_{\lambda_0}} V(l\Lambda_0)\) over \(1 \leq J \leq s\) given in Remark \([4.41]\). This will be explained more precisely in Section \([5.1]\) together with a conjectural extension of Proposition \([4.11]\) to an arbitrary non twisted affine Lie algebra \(X_n^{(1)}\).

Let us proceed to the limit of the classically restricted 1dsum \(X_{\mu}(\lambda)\). Here we shall exclusively consider the situation \(\mu = (l^L)\), in which \(L \to \infty\) under the condition \(L \equiv 0 \mod n\). From Example \([2.3]\) \((1)\) this is related to the vacuum module \(V = V(l\Lambda_0)\). Our task is to compute the limit of the Kostka-Foulkes polynomial given in Proposition \([4.3]\). Combining the result with \([3.7]\) and \([3.13]\) we obtain

**Proposition 4.12** For any \(l \in \mathbb{Z}_{\geq 1}\) and \(\lambda \in \mathcal{P}^+\) such that \(|\lambda| \equiv 0 \mod n\), we have

\[
\mathcal{K}_\lambda^V(l\Lambda_0)(q) = \sum_{\eta} K^{\xi} q F^{(l)}(q) \frac{(q)_{\xi_1} \cdots (q)_{\xi_{n-1}}}{(q)_{\xi}} , \tag{4.44}
\]

\[
\xi = \left(n - \frac{|\lambda|-\Delta}{n}\right) \cup \lambda', \quad \eta = (n - 1)^{\xi_{n-1}} \cdots 1^{\xi_1}, \tag{4.45}
\]

where the sum \(\sum_{\eta}\) runs over the partitions \(\eta\) satisfying \(|\eta| \leq n - 1\) as above and \(|\eta| \equiv 0 \mod n\).

Proof. We start with the expression \([4.12]-[4.15]\) with \(\mu = (l^L)\). The limit is to be expanded around the minimum of \(c(\{m\})\). This takes place at \(m_i^{(a)} = \frac{a-s}{n} L\delta_{it}\), which is tending to infinity. Thus under the identification \(\xi_a = \ldots = \ldots = \ldots\)
\( p_i^{(a)} \) the factor \( \prod_{a=1}^{n-1} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right] \) in (4.12) gives rise to \( \left( \frac{\epsilon_1}{(q)_{\epsilon_i} \cdots (q)_{\epsilon_{n-1}}} \right)^{-1} \) as \( L \to \infty \). After the shift \( m_i^{(a)} \to m_i^{(a)} + m_{i,0} \), the relations (4.13), (4.14) and its \( i = l \) case become

\[
\sum_{k \geq 1} k m_k^{(a)} = \frac{a L}{n} - \lambda_1 - \cdots - \lambda_a, \quad (4.46)
\]

\[
p_i^{(a)} = - \sum_{b=1}^{n-1} C^{(n)}_{ab} \sum_{k \geq 1} \min(i, k) m_k^{(b)}, \quad (4.47)
\]

\[
\sum_{b=1}^{n-1} C^{(n)}_{ab}^{-1} \zeta_b = - \sum_{k \geq 1} \min(l, k) m_k^{(a)}. \quad (4.48)
\]

Eliminating \( m_i^{(a)} \) with (4.48), one rewrites (4.47) as

\[
p_i^{(a)} = \begin{cases} 
\zeta - \sum_{b=1}^{n-1} C^{(n)}_{ab} \sum_{k \geq l} \min(i-l, k-l) m_k^{(b)} & i > l \\
C^{(l)}_{1-i} \zeta_a - \sum_{b=1}^{n-1} C^{(n)}_{ab} \sum_{k=1}^{l-1} C^{(l)}_{ik}^{-1} m_k^{(b)} & 1 \leq i < l.
\end{cases}
\]

By setting \( p_i^{(a)} = \tilde{p}_i^{(a)} \) and \( m_i^{(a)} = \tilde{m}_i^{(a)} \), the \( i > l \) case in the above coincides with (4.15). So does the \( 1 \leq i < l \) case with (4.26) under \( p_i^{(a)} = \tilde{p}_i^{(a)} \) and \( m_i^{(a)} = \tilde{m}_i^{(a)} \). Thus we are going to extract \( K_{\eta, l}^{(a)} \) in Proposition 4.13 from \( \prod_{a=1}^{n-1} \prod_{i \geq l} \) part” and \( F_{\eta, l}^{(a)} \) in \( (4.24) \) from “\( \prod_{a=1}^{n-1} \prod_{i \leq l} \) part” and \( E_{\eta, l} \) in (4.25). Actually the equality \( -E(l \Lambda_0, (l^L)) + c\{m_i^{(a)} + m_{i,0}^{(a)}\} = \tilde{c}\{\tilde{m}_i\} + \tilde{c}\{\tilde{m}\} \) is valid among the quadratic forms (4.13), (4.14) and (4.25). It remains to check (i) (4.19) for Conjecture 4.13 (spinon character formula, [32]). For any \( \Lambda \in (P^+)_l \) and \( \lambda \in \mathbb{Z}^l \) of

**Conjecture 4.13** (spinon character formula, [32]) For any \( \Lambda \in (P^+)_l \) and \( \lambda \in \mathbb{Z}^l \) of
such that $\Lambda \equiv \lambda + l\Lambda_0 \mod Q$, we have

$$b^V(\Lambda)(q) = \sum_{\eta}^{\Lambda} X^\prime_\eta(\lambda) X^\prime_\eta(\Lambda),$$

(4.49)

where the sum $\sum_{\eta}$ runs over the partitions of the form $\eta = ((n-1)^{s_{n-1}}, \ldots, 1^{s_1})$ satisfying $|\eta| \equiv |\lambda| \mod n$.

Finally we turn to a limit related to the restricted 1dsum $X^i_{\mu}(l\Lambda_0)$. By a parallel calculation with Proposition 4.12 one can derive

**Proposition 4.14** For any integers $1 \leq t \leq l - 1$ we have

$$\lim_{L \to \infty} q^{-E(l\Lambda_0, (tL))} F^{(t)}_{(tL)}(q) = \sum_{\eta}^{l-t} F^\prime_{\eta}(l-t)(q) F^\prime_{\eta}(q),$$

(4.50)

$$\eta = ((n-1)^{s_{n-1}}, \ldots, 1^{s_1}),$$

(4.51)

where the ground state energy is $E(l\Lambda_0, (tL)) = \frac{tL(L-n)}{2n}$ and the sum runs over the partitions $\eta$ satisfying $\eta_1 \leq n-1$ as above and $|\eta| \equiv 0 \mod n$.

By Example 2.1 (1) the same limit of $X^i_{\mu}(l\Lambda_0)$ corresponds the the choice $V = V(t\Lambda_0)$ in (3.13). Thus under the assumption (4.29), Proposition 4.14 and (3.16) implies

$$a^V_{t\Lambda_0 \odot V((l-t)\Lambda_0)}(q) = \sum_{\eta}^{l-t} X^\prime_{\eta}(l-t)(l-t\Lambda_0) X^\prime_{\eta}(t\Lambda_0),$$

(4.52)

where the $\eta$-sum and the relation with $\zeta$ is the same as in Proposition 4.14. This is an RSOS analogue of the spinon character formula conjectured in [32].

5 Discussion

Let us discuss further generalizations of the results in Section 4.2.

5.1 Fermionic string function for arbitrary $X_n^{(1)}$

The $q$-series formulae in Proposition 4.8, Remark 4.10 and Proposition 4.11 are all originated in some limits of the unrestricted 1dsum $g_\mu(\lambda)$. Let us discuss their possible extensions to an arbitrary non-twisted affine Lie algebra $X_n^{(1)}$. Our argument in this subsection will only concern infinite series in $q$. To seek their finite ($q$-polynomial) versions as in Section 4.1 is an important open problem. So far only level 1 cases have been studied for $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ in [8] and $A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ in [27]. We hope to report higher level cases in near future.
Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of the classical subalgebra $X_n$. For $1 \leq a \leq n$ set $t_a = 2/|\alpha_a|^2$ in the normalization $|\text{long root}|^2 = 2$. Thus one has $\forall t_a = 1$ for simply laced algebras and $t_a \in \{1, 2, 3\}$ in general. The root and coroot lattices are denoted by $Q = \sum_{a=1}^{n} \mathbb{Z} \alpha_a$ and $Q^\vee = \sum_{a=1}^{n} \mathbb{Z} t_a \alpha_a$, respectively. Fix $s \in \mathbb{Z}_{\geq 1}$ arbitrarily. For any positive integers $l_1, \ldots, l_s$ we set $l = l_1 + \cdots + l_s$ and $M^{(a)}_J = t_a (l_1 + \cdots + l_j)$ for $1 \leq j \leq s$ and $1 \leq a \leq n$. Consider the tensor product of the vacuum modules of $X_n^{(1)}$, namely, $\mathcal{V} = \otimes_{j=1}^{s} V(l_j \Lambda_0)$. For its character we have

**Conjecture 5.1** Let $l, l_J, M^{(a)}_J (1 \leq J \leq s)$ and $\mathcal{V}$ be as above. For $\lambda \in Q$ we have

\[
q^{-\langle \lambda, \lambda \rangle} e_\lambda(q) = \frac{1}{(q)_{\infty}^{ns}} \sum_{\{m\} \subset \mathcal{C}_l} \prod_{a=1}^{n} \prod_{i \notin M^{(a)}_1, \ldots, M^{(a)}_{l-1}} \frac{q^{C(i)}(m)}{q^{C(i)}(q)_{m_i^{(a)}}},
\tag{5.1}
\]

where $\sum_{\{m\} \subset \mathcal{C}_l}$ is taken over $(a, j), (b, k) \in G_l$ and the sum $\sum_{\{m\} \subset \mathcal{C}_l}$ is taken over $\{m^{(a)}_j \in \mathbb{Z}_{\geq 0} \mid (a, j) \in G_l\}$ satisfying the condition $\sum_{(a, i) \in G_l} i m_i^{(a)} \alpha_a \equiv \lambda \mod l Q^\vee$.

\[
e^{-\lambda_0} \text{ch}\mathcal{V} = \frac{1}{(q)_{\infty}^{ns}} \sum_{\{m\} \subset \mathcal{C}_l} \prod_{a=1}^{n} \prod_{i \notin M^{(a)}_1, \ldots, M^{(a)}_{l-1}} \frac{q^{C(i)}(m)}{q^{C(i)}(q)_{m_i^{(a)}}} e_{\beta_i^{(a)}(m)},
\tag{5.4}
\]

where $\sum_{\{m\} \subset \mathcal{C}_l}$ is taken over $(a, j), (b, k) \in G_l$ and the sum $\sum_{\{m\} \subset \mathcal{C}_l}$ is taken over $\{m^{(a)}_j \in \mathbb{Z}_{\geq 0} \mid (a, j) \in G_l\}$ and $\{m_i^{(a)} \in \mathbb{Z} \mid 1 \leq a \leq n\}$.

For $X_n^{(1)} = A_n^{(1)}$, (5.1)–(5.3) reduce to $\forall r_j = 0$ case of Proposition 4.11. For $s = 1$ the conjecture (5.1)–(5.3) goes back to [28]. In fact for any $X_n^{(1)}$ the two conjectures (5.1)–(5.3) and (5.6)–(5.7) are equivalent. Moreover they follow from the $s = 1$ cases by noting that $e^{-\lambda_0} \text{ch}\mathcal{V} = \prod_{j=1}^{s} e^{-l_j \lambda_0} \text{ch}V(l_j \Lambda_0)$. Let us explain these facts more
partitions \( \eta \) in (5.2) can be replaced with \( \sum \). Subsequent argument explains, it is intimately related to the spinor character \( \sum \). Precisely, to see the equivalence recall the decomposition of characters in terms of theta functions (cf. eq. (12.7.12) in [14]):

\[
e^{-l\Lambda_0} \text{ch} V = \sum_{\lambda \in \mathbb{Q}/l\mathbb{Q}} \theta_{\lambda, l} q^{-\langle \lambda, \lambda \rangle} c_{\lambda}^V(q), \quad (5.8)
\]

\[
\theta_{\lambda, l} = \sum_{\xi \equiv \lambda \mod l\mathbb{Q}} q^{\langle \xi, \xi \rangle} \xi.
\]

Substitute (5.1) and (5.9) into (5.8). The resulting double sum \( \sum_{\lambda, \xi} \) is equivalent to the single one \( \sum_{\xi \in Q} \). Moreover, \( \sum_{\xi \in Q} \sum_{\{m\}_{G_l}} \) can further be replaced by \( \sum_{\{m\}_{G_l}} \), by identifying \( \xi \) with \( \beta_l(\{m\}) \) in (5.6). On the other hand \( \sum_{G_l} \) in (5.2) can be replaced with \( \sum_{G_l} \) without causing any change. Therefore \( C(\{m\}) = \tilde{C}_l(\{m\}) - \langle \tilde{\beta}_l(\{m\}), \tilde{\beta}_l(\{m\}) \rangle \) holds between (5.4) and (5.5). Combining these facts we obtain (5.4)–(5.7) from (5.1)–(5.3). Obviously the converse of this argument is also valid. To explain that Conjecture 5.1 reduces to \( s = 1 \), it suffices essentially to verify \( s = 2 \) case by induction. Let us do so for (5.4)–(5.7).

Taking the product of (5.4) with \( (s, l) = (1, l_1) \) and \( (1, l_2) \), we get the following expression for \( e^{-(l_1 + l_2)\Lambda_0^*} \text{ch} V(l_1\Lambda_0) \text{ch} V(l_2\Lambda_0) \):

\[
\frac{1}{(q)^{2m}} \sum_{\{m\}_{G_l_1}} \sum_{\{m\}_{G_l_2}} q^{\tilde{C}_l_1(\{m\})+\tilde{C}_l_2(\{m\})} \prod_{(a, j) \in G_{l_1}} (q)^{m_{a(j)}^{(a)}} \prod_{(a, j) \in G_{l_2}} (q)^{m_{a(j)}^{(a)}} e^{\tilde{\beta}_l_1(\{m\})+\tilde{\beta}_l_2(\{m\})}, \quad (5.10)
\]

For each color \( 1 \leq a \leq n \), replace here as \( \tilde{m}_{\lambda, l_1}^{(a)} \rightarrow \tilde{m}_{\lambda, l_1}^{(a)} + \sum_{j=1}^{l_2} m_{a(j)}^{(a)} \) and introduce the new variables \( \{m_{a(j)}^{(a)} \mid (a, j) \in G_{l_1+l_2} \} \) by \( m_{a(j)}^{(a)} = \tilde{m}_{a(j)}^{(a)} \) for \( 1 \leq j \leq t_{a} l_{1} \), \( \tilde{m}_{a(j)}^{(a)} \), for \( t_{a} l_{1} < j \leq t_{a} (l_{1} + l_{2}) \). Then it is straightforward to check \( \tilde{\beta}_l(\{\tilde{m}\})|_{\text{replacement}} + \tilde{\beta}_l(\{\tilde{m}\}) = \tilde{\beta}_{l_1+l_2}(\{m\}) \) and \( \tilde{C}_l(\{\tilde{m}\})|_{\text{replacement}} + \tilde{C}_l(\{\tilde{m}\}) = \tilde{C}_{l_1+l_2}(\{m\}) \). Thus (5.10) indeed yields (5.4)–(5.7) with \( s = 2 \) and \( l = l_1 + l_2 \).

5.2 Fermionic form of \( X^{(l)}_\eta(\lambda) \) for \( \lambda \) non vacuum type

In [14], we conjectured a fermionic formula of the restricted Idsum \( X^{(l)}_\eta(\lambda) \) only for \( \lambda = l\Lambda_0 \). Here we present a conjecture involving more general \( \lambda \). As the subsequent argument explains, it is intimately related to the spinor character formula (Conjecture 4.13) for general \( \Lambda \).

First we introduce a fermionic expression \( F^{(l, r)}_{\eta, \mu}(q) \) (0 \( \leq r \leq n - 1 \)) for partitions \( \eta = ((n - 1)^{\alpha - 1}, \ldots, 1^\alpha) \) and \( \mu \) satisfying \( \mu_l \leq l - 1 \) and \(|\eta| \equiv
for any partition \( \eta \) limit of the Kostka-Foulkes polynomial as \( p \) of only one element

Remark 5.3

For any \( \eta \), we have

\[ \sum_{\eta, \mu} C_{\eta, \mu}^{(n)} \prod_{1 \leq a \leq n - 1, 1 \leq l \leq l - 1} \left( \tilde{p}_{\eta}^{(a)}(l) \tilde{\tilde{m}}_{(a)}^{(l)} \right), \]

(5.11)

Here the sum \( \sum \tilde{m}_{\eta}^{(a)} \) runs over \( \tilde{m}_{\eta}^{(a)} \in \mathbb{Z}_{\geq 0} \) for \( 1 \leq a \leq n - 1, 1 \leq i \leq l \) satisfying \( \tilde{p}_{\eta}^{(a)} \geq 0 \) for \( 1 \leq a \leq n - 1, 1 \leq i \leq l - 1 \), and

\[ \sum_{i=1}^{l} \tilde{m}_{\eta}^{(a)} = \sum_{b=1}^{n-1} C_{\eta, \mu}^{(n)} \tilde{\xi}_{b} + l \mu_{\eta}^{j} - \frac{n - a}{n} |\mu| + \frac{a r}{n} \quad \text{for} \quad 1 \leq a \leq n - 1. \quad (5.14) \]

Note that \( F_{\eta, \mu}^{(l, r)}(q) \) is a generalization of \( F_{\eta}^{(l)}(q) \) in (4.24)–(4.27) in that \( F_{\eta}^{(l)}(q) = F_{\eta, \emptyset}^{(l, 0)}(q) \). Now our generalized conjecture reads

**Conjecture 5.2** For any \( 0 \leq r \leq n - 1 \) and partitions \( \eta, \mu \) satisfying \( \eta_{l} \leq n - 1, \mu_{l} \leq l - 1 \) and \( \kappa = |\mu| + lr \) mod \( n \), we have

\[ \sum_{p \in \mathcal{H}(l \Lambda_{r}, \mu)} q^{E(p)} X_{\eta}^{(l)}((l \Lambda_{r} + a f(wt)) = F_{\eta, \mu}^{(l, r)}(q). \]

(5.15)

**Remark 5.3** When \( \mu = (s) \) for some \( 0 \leq s \leq n - 1 \), the set \( \mathcal{H}(l \Lambda_{r}, \mu) \) consists of only one element \( p \) as in Example 2.3 (2), for which \( l \Lambda_{r} + a f(wt) = (l - s) \Lambda_{r} + s \Lambda_{r+1} \) and \( E(p) = 0 \) hold. Thus the above conjecture reduces to

\[ X_{\eta}^{(l)}((l - s) \Lambda_{r} + s \Lambda_{r+1}) = F_{\eta, (s)}^{(l, r)}(q) \]

(5.16)

for any partition \( \eta \) satisfying \( \eta_{l} \leq n - 1 \) and \( \kappa \equiv lr + s \) mod \( n \).

We explain briefly how the above conjecture emerged. Let us consider the limit of the Kostka-Foulkes polynomial as
Proposition 5.4 For finite partitions $\lambda$ and $\mu$ satisfying $l(\lambda) \leq n-1$, $\mu_1 \leq l-1$ and $|\lambda| = |\mu| + lr$, define $\bar{\mu} = (l^L) \cup \mu$, $\bar{\lambda} = (\lfloor \frac{|\mu|-|\lambda|}{n} \rfloor)^n + \lambda$. Then, we have

$$\lim_{L \to \infty} q^{-E(l(A_0, \bar{\mu}))} K_{\bar{\lambda}, \bar{\mu}}(q) = q^{-E(l(A_r, \bar{\mu}))} \sum_{\eta} K_{\eta \mu}(q) F_{\eta \mu}(l, r)'(q).$$

(5.17)

$$\xi = (n^{-|\lambda|}) \cup \lambda', \quad \eta = ((n-1)^{\xi_{n-1}} \cdot \cdots \cdot 1^{\xi_1}) \quad (5.18)$$

where the sum $\sum_{\eta}$ runs over the partitions $\eta$ of the above form obeying $|\eta| \equiv |\lambda|$ mod $n$.

This is obtained by taking the limit of $[4, 12]$, which is a very parallel calculation with Proposition 4.12. Here we only mention that in its proof, the “minimum point” $m^{(a)}_{i, 0}$ should be replaced by $m^{(a)}_{i, \frac{|\eta|}{r}} = L(r)\delta_i$ and the relation $E(l(A_0, \bar{\mu})) = E(l(A_r, \mu)) + \frac{1}{2n}(L^2 - r^2) + (\frac{m}{n} - \frac{i}{2})(L - r)$ is used. See (5.7).

From (5.7), (5.12) and (5.19), we know that

$$\text{LHS of (5.17)} = \sum_{p \in H(l(A_r, \mu))} q^{E(p)} \sum_{\eta} X_\mu(\lambda) X_\mu(l(A_r + af(wt(p)))) (q| \zeta | \cdots | \zeta |_{n-1}) \quad (5.20)$$

Admitting Conjecture 4.13, we rewrite this as

$$q^{-E(l(A_r, \mu))} \sum_{p \in H(l(A_r, \mu))} q^{E(p)} \sum_{\eta} X_\mu(\lambda) X_\mu(l(A_r + af(wt(p)))) (q| \zeta | \cdots | \zeta |_{n-1}) \quad (5.20)$$

where the sum $\sum_{\eta}$ and $\zeta$ are specified as in Proposition 5.4. From (5.7) we have $X_\mu(\lambda) = K_{\xi \eta}(q)$ for $\xi$ given in (5.18). Thus by comparing (5.20) with (5.17), we arrive at Conjecture 5.2 naturally.

Example 5.5 Let $n = 3$, $r = 1$, $l = 3$, $\eta = (2, 2, 1, 1)$, $\mu = (2, 1)$. Then, $H(l(A_r, \mu)) = \{22 \otimes 2, 22 \otimes 3\}$, where the element $(x_1, x_2, x_3)$ of each crystal $B_k$ is denoted by $1^{x_1}2^{x_2}3^{x_3}$. The LHS of (5.17) is calculated from the data

| $p \in H(l(A_1, \mu))$ | $E(p)$ | $3\Lambda_1 + wt p$ | $p_0 \in P_0^{(3)}(3\Lambda_1 + wt p)$ | $E(p_0)$ |
|-------------------|--------|---------------------|-------------------------|--------|
| $22 \otimes 2$   | 1      | $3\Lambda_2$        | $12 \otimes 12 \otimes 1 \otimes 2$ | $-1$   |
| $22 \otimes 3$   | 0      | $3\Lambda_0 + \Lambda_1 + \Lambda_2$ | $12 \otimes 12 \otimes 1 \otimes 3$ | $-1$   |
|                   |        |                     | $12 \otimes 12 \otimes 3 \otimes 1$ | $-2$   |
|                   |        |                     | $12 \otimes 13 \otimes 1 \otimes 2$ | $-2$   |
|                   |        |                     | $12 \otimes 13 \otimes 2 \otimes 1$ | $-3$   |

as

$$\text{LHS} = q^1 q + q^0 (q + 2q^2 + q^3) = q + 3q^2 + q^3. \quad (5.21)$$
On the other hand, $F_{\eta,\mu}^{(3,1)}(q)$ defined by (5.11)-(5.14) is calculated from the data below, which illustrates all the admissible values of $\{\bar{m}_j^{(a)}\}$ for $\sum_1^{(a)}$ in (5.11) and the corresponding $\{\bar{p}_j^{(a)}\}$, etc.

| $\bar{m}_1^{(1)}$ | $\bar{p}_1^{(1)}$ | $\bar{m}_2^{(1)}$ | $\bar{p}_2^{(1)}$ | $\bar{m}_3^{(1)}$ | $\bar{p}_3^{(1)}$ | $\bar{m}_1^{(2)}$ | $\bar{p}_1^{(2)}$ | $\bar{m}_2^{(2)}$ | $\bar{p}_2^{(2)}$ | $\bar{m}_3^{(2)}$ | $\bar{p}_3^{(2)}$ | $c_3(\{\bar{m}\})$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0                 | 2                 | 2                 | 0                 | 1                 | 1                 | 0                 | 1                 | 1                 | 0                 | 1                 | 0                 | 2                 |
| 1                 | 1                 | 0                 | 1                 | 2                 | 0                 | 2                 | 0                 | 1                 | 3                 | 1                 | 0                 | 2                 |
| 2                 | 0                 | 1                 | 0                 | 1                 | 1                 | 1                 | 1                 | 0                 | 3                 | 1                 | 0                 | 2                 |

as

$$F_{\eta,\mu}^{(3,1)}(q) = q^2 + q(1 + q) + q^2(1 + q) = q + 3q^2 + q^3.$$  \hspace{1cm} (5.22)

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