Extended observables
in Hamiltonian theories with constraints.

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Abstract

In a classical Hamiltonian theory with second class constraints the phase space functions on the constraint surface are observables. We give general formulas for extended observables, which are expressions representing the observables in the enveloping unconstrained phase space. These expressions satisfy in the unconstrained phase space a Poisson algebra of the same form as the Dirac bracket algebra of the observables on the constraint surface. The general formulas involve new differential operators that differentiate the Dirac bracket. Similar extended observables are also constructed for theories with first class constraints which, however, are gauge dependent. For such theories one may also construct gauge invariant extensions with similar properties. Whenever extended observables exist the theory is expected to allow for a covariant quantization. A mapping procedure is proposed for covariant quantization of theories with second class constraints.

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1 Introduction.

In this paper we present new results at a very basic level for general classical Hamiltonian theories with constraints. We introduce the concept of extended observables defined in a very precise way. Roughly speaking if the original coordinates on the constraint surface are viewed as observables then extended observables are functions of the original coordinates defined on the unconstrained phase space with similar properties to the observables on the constraint surface. For theories with second class constraints in Dirac’s classifications [1] the appropriate extended observables are defined in section 3. For theories with first class constraints [1] (general gauge theories) we find three different possible definitions of extended observables. In section 5 they are defined in an analogous way to the ones in section 3. The resulting extended observables are gauge dependent. However, the general consensus is that observables in gauge theories should be gauge invariant. In section 6 we present general forms for gauge invariant extensions. Such gauge invariant observables are well known in the literature (see e.g. [2, 3]). Now gauge theories in a particular gauge are theories with second class constraints. When the construction of section 3 for second class constraints is applied to such systems we obtain one very particular gauge invariant extension which seems to have special importance.

Apart from new very precise definitions of extended observables we also provide simple algorithms for their constructions. These formulas involve a new differential operator, $V_\alpha$, which differentiates the Dirac bracket (proved in the appendix). The definition of this operator together with other basic formulas for Poisson structures of theories with second class constraints is given in section 2.

The original purpose of the present work was to develop new tools for covariant quantization of theories with second class constraints. Although these aspects are not developed here we make some important remarks on quantization in section 8. First of all we believe the existence of extended observables to be necessary for a covariant quantization. Notice e.g. that the gauge invariant extension for the bosonic string in [2] are after quantization the DDF operators of its covariant quantization. From the very simple models treated in this paper in section 4 it seems as if a covariant quantization of second class constraints may be understood from the more conventional splitting of the constraints into gauge generators and gauge fixing conditions [4, 5]. However, in section 8 we also give a simple mapping procedure for covariant quantization which directly makes use of the extended observables. This method is exemplified by a particle on a sphere which in this way is quantized in a very simple manner. The paper is then summarized in section 9.

2 Poisson structures in theories with second class constraints.

Let $x^i$, $i = 1, \ldots, 2n$, be bosonic coordinates in a symplectic manifold $\mathcal{M}$, $\dim \mathcal{M} = 2n$. Let, furthermore, there be a nondegenerate two-form $\omega$ on $\mathcal{M}$:

$$\omega = \omega_{ij}(x)dx^i \wedge dx^j, \quad \det \omega_{ij} \neq 0,$$

which is required to be closed ($\partial_i = \partial/\partial x^i$):

$$d\omega = 0 \iff \partial_i \omega_{jk}(x) + \text{cycle}(i, j, k) = 0.$$

(2.1)

(2.2)
Since \( \omega \) is nondegenerate there exists an inverse \( \omega^{ij} \) in terms of which the Poisson bracket is defined by

\[
\{ f(x), g(x) \} = \omega^{ij}(x) \partial_i f(x) \partial_j g(x), \quad \omega^{ij}(x) \omega_{jk}(x) = \delta^i_j. \tag{2.3}
\]

On \( \mathcal{M} \) we have the natural differential operators

\[
\nabla^i \equiv \{ x^i, \cdot \} = \omega^{ij} \partial_j
\]

known as skew gradients. They satisfy a closed algebra

\[
[\nabla^i, \nabla^j] = \partial_k \omega^{ij} \nabla^k, \tag{2.5}
\]

and Leibniz’ rule

\[
\nabla^i \{ f(x), g(x) \} = \{ \nabla^i f(x), g(x) \} + \{ f(x), \nabla^i g(x) \}, \tag{2.6}
\]

both of which follow from the Jacobi identities

\[
\omega^{il} \partial_l \omega^{jk} + \text{cycle}(ijk) = 0, \tag{2.7}
\]

which in turn follow from (2.2). \( \nabla^i \) are linearly independent and form a basis in the tangent space. Every vector field \( A \) is spanned by \( \nabla^i \), i.e. we have \( A = a_i \nabla^i \). \( A \) differentiates the Poisson bracket (2.3) if it satisfies Leibniz’ rule, i.e.

\[
A \{ f(x), g(x) \} = \{ Af(x), g(x) \} + \{ f(x), Ag(x) \}. \tag{2.8}
\]

This is the case if \( a_i = \partial_i a(x) \) implying that \( A \) then is a Hamiltonian vector field i.e. \( A = \{ a(x), \cdot \} \). (In invariant terms, the one-form \( a_i dx^i \) is then closed.)

We turn now to the constrained Hamiltonian theory. On \( \mathcal{M} \) we have then a dynamical theory with the Hamiltonian \( H(x) \) and the constraints \( \theta^\alpha(x) = 0, \alpha = 1, \ldots, 2m < 2n \), which we require to be of second class in Dirac’s classification \([1]\), i.e. they satisfy

\[
\det C^{\alpha\beta} \big|_{\theta=0} \neq 0, \quad C^{\alpha\beta} \equiv \{ \theta^\alpha, \theta^\beta \}. \tag{2.9}
\]

These constraints determine a hypersurface \( \Gamma \) in \( \mathcal{M} \). Notice that

\[
\theta^\alpha(x) = 0, \quad \theta^\alpha(x) = S^\beta_\alpha(x) \theta^\beta(x), \quad \det S^\beta_\alpha(x) \big|_{\theta=0} \neq 0 \tag{2.10}
\]

determine the same constraint surface \( \Gamma \). The set of constraint variables \( \{ \theta^\alpha \} \) and \( \{ \theta^\alpha \} \) are therefore equivalent.

The two-form \( \omega \) in (2.1) restricted to \( \Gamma \) remains a symplectic two-form, i.e. \( \Gamma \) is a symplectic manifold. The Poisson bracket on \( \Gamma \) may be written in terms of the coordinates \( x^i \) of the enveloping manifold \( \mathcal{M} \). This so called Dirac bracket \([1]\) is given by

\[
\{ f, g \}_D = \{ f, g \} - \{ f, \theta^\alpha \} C^{\alpha\beta} \{ \theta^\beta, g \}, \tag{2.11}
\]

where \( C^{\alpha\beta} \) is the inverse of \( C^{\alpha\beta} \) in (2.9), i.e. \( C^{\alpha\beta} C_{\beta\gamma} = \delta^\alpha_\gamma \). The expression (2.11) as written is defined on the enveloping manifold \( \mathcal{M} \). However, on \( \mathcal{M} \) it is a degenerate Poisson bracket since \( \{ f, \theta^\alpha \}_D = 0 \), i.e. \( \theta^\alpha \) are Casimir functions for the Dirac bracket.
(Notice that \( g = g_\alpha \theta^\alpha \) in (2.11) yields zero on \( \Gamma \) but \( \{ f, g_\alpha \} D \theta^\alpha \) on \( \mathcal{M} \).) The Dirac bracket (2.11) satisfies the Jacobi identities
\[
\{ f, \{ g, h \}_D \}_D + \text{cycle}(f, g, h) = 0 \tag{2.12}
\]
both on \( \Gamma \) and \( \mathcal{M} \). Every set of equivalent constraints lead to the same Poisson bracket on the constraint surface \( \Gamma \). However, the Dirac bracket (2.11) for different choices of equivalent constraints may be different on \( \mathcal{M} \). Our constructions in the following will be for a fixed constraint basis. The transformation properties between equivalent sets will be studied elsewhere.

In correspondence with (2.4) we may introduce Dirac skew gradients defined by
\[
D^i \equiv \{ x^i , \cdot \}_D \equiv \omega_D^{ij} \partial_j , \quad \omega_D^{ij} \equiv \{ x^i , x^j \}_D . \tag{2.13}
\]
They are linearly dependent since they satisfy the 2\( m \) relations
\[
\partial_\alpha \theta^\alpha D^i = 0 , \quad \alpha = 1, \ldots , 2m . \tag{2.14}
\]
\( D^i \) satisfy a closed algebra and differentiate the Dirac bracket according to Leibniz’ rule, i.e. we have
\[
[D^i , D^j ] = (\partial_k \omega_D^{ij}) D^k . \tag{2.15}
\]

Due to the Jacobi identities (2.12) for the Dirac bracket. In addition they satisfy
\[
D^i \theta^\alpha = 0 , \tag{2.17}
\]
\( D^i \) may, therefore, be said to differentiate parallel to the hypersurface \( \Gamma \). The same may be said about the Dirac vector field \( A_D \) defined by \( A_D \equiv a_i D^i \). \( A_D \) differentiate the Dirac bracket if \( a_i = \partial_i a + \gamma_\alpha \partial_i \theta^\alpha \) for any \( a \) and \( \gamma_\alpha \) (\( \gamma_\alpha \) does not contribute to \( A_D \)). Notice that the Dirac bracket (2.11) may be written as
\[
\{ f, g \}_D = \omega_D^{ij} D^i f D^j g . \tag{2.18}
\]
(In [6] this form of the Dirac bracket was applied by Batalin and Ogievetsky in their attempt to construct a star product on the second class surface \( \Gamma \) which, however, was successful only for a special set of constraints.)

On \( \mathcal{M} \) arbitrary vector fields are spanned by \( \nabla^i \) in (2.4). Since there are vector fields not spanned by \( D^i \) we expect the existence of more operators differentiating the Dirac bracket. Indeed, in addition to \( D^i \) there is another set of 2\( m \) operators, \( V_\alpha \), that satisfy Leibniz’ rule with respect to the Dirac bracket. The operators \( V_\alpha \) are defined by
\[
V_\alpha \equiv C_{\alpha \beta} \{ \theta^\beta , \cdot \} \equiv C_{\alpha \beta} \{ \theta^\beta , x^i \}_D \partial_i = C_{\alpha \beta} \partial_i \theta^\beta \nabla^i , \tag{2.19}
\]
where \( C_{\alpha \beta} \) is the inverse of (2.3) which also involved in the Dirac bracket (2.11). \( V_\alpha \) are linearly independent and satisfy the properties
\[
V_\alpha \theta^\beta = \delta_\alpha^\beta , \tag{2.20}
\]
\[ V_{\alpha}\{f, g\}_D = \{V_{\alpha}f, g\}_D + \{f, V_{\alpha}g\}_D. \] (2.21)

The proof of (2.21) is nontrivial and is given in Appendix A. As far as we know this property has not been noticed before. Notice also that

\[ V_{\alpha}f = 0 \iff \{f, \theta^\alpha\} = 0. \] (2.22)

The following commutation relations are straight-forward to derive

\[ [V_{\alpha}, V_{\beta}] = \partial_k C_{\alpha\beta} D^k \equiv \{C_{\alpha\beta}, \cdot\}_D, \]
\[ [D^i, V_{\alpha}] = -\partial_k (\{x^i, \theta^\beta\} C_{\beta\alpha}) D^k. \] (2.23)

The Dirac bracket (2.11) or equivalently (2.18) may also be written in terms of \( V_{\alpha} \):

\[ \{f, g\}_D = \omega_{ij} \partial_i f \partial_j g - C^{\alpha\beta} V_{\alpha} f V_{\beta} g = \omega_{ij} \nabla^i f \nabla^j g - C^{\alpha\beta} V_{\alpha} f V_{\beta} g. \] (2.24)

In fact, the skew gradient \( \nabla^i \) in (2.4) may be decomposed as follows

\[ \nabla^i = D^i + (\nabla^i \theta^\alpha) V_{\alpha} , \quad \partial_k = \omega_{ik} D^k + \partial_i \theta^\alpha V_{\alpha} \] (2.25)

\( V_{\alpha} \) may therefore be viewed as normal derivatives with respect to the constraint surface \( \Gamma \). Eq.(2.25) is then a decomposition of the derivative in parallel and normal parts.

Any vector field \( D \) differentiating the Dirac bracket is decomposed as follows:

\[ D\{f, g\}_D = \{Df, g\}_D + \{f, Dg\}_D \iff D = N^\alpha(\theta) V_{\alpha} + \{H, \cdot\}_D \] (2.26)

The coefficients \( N^\alpha \) of the normal projections of \( D \) depend on \( \theta \) only, whereas the parallel part is always reduced to the action of the Dirac bracket with some function \( H \). The normal vector fields of \( N^\alpha(\theta) V_{\alpha} \) form a closed algebra iff the constraints \( \theta^\alpha \) form a Poisson subalgebra in the phase space. The question of the structure of the differentiation of a degenerate regular Poisson bracket has been studied in the book by Karasev and Maslov [7]. The Dirac bracket is a special case of a degenerate Poisson bracket, which admits the explicit representation (2.26) for its differentiations. This representation seems to be unknown before.

### 3 Extended Observables

The equations \( \theta^\alpha(x) = 0 \) may be locally solved by expressing \( 2m \) coordinates in terms of the remaining \( 2(n - m) \) independent coordinates. These solutions, \( x^{*i} \), belong to the hypersurface \( \Gamma \) which as mentioned above is a symplectic manifold locally spanned by \( 2(n - m) \) independent coordinates. \( x^{*i} \) represent \( x^i \) in \( M \) on \( \Gamma \). Their Poisson brackets satisfy the relations \( \{x^{*i}, x^{*j}\}_D = \{x^i, x^j\}_D \big|_{x \to x^*} \). We view functions of \( x^{*i} \) as observables.

By extended observables we mean expressions \( \tilde{x}^i(x) \) (or functions \( f(\tilde{x}^i(x)) \)) which are defined on the original symplectic manifold \( M \) and which on \( M \) satisfy the properties of \( x^{*i} \) on \( \Gamma \). More precisely we define extended observables \( \tilde{x}^i(x) \in M \) to be functions satisfying the following three properties:

1) \[ \tilde{x}^i(x^*) = x^{*i} \] (3.1)
for whatever choice of solution $x^{*i} \in \Gamma$.

\[
2) \quad \theta^\alpha(\bar{x}^i(x)) = 0, \quad \alpha = 1, \ldots, 2m, \tag{3.2}
\]

and

\[
3) \quad \{\bar{x}^i(x), \bar{x}^j(x)\} = \{x^i, x^j\}_D|_{x \to \bar{x}(x)}, \tag{3.3}
\]

where the bracket on the left-hand side is the original Poisson bracket (2.3) on $\mathcal{M}$. Notice that $\bar{x}^i(x)$ both represents $x^{*i}$ on $\mathcal{M}$ and reduces to $x^{*i}$ on $\Gamma$. The expression (3.1) implies that $\bar{x}^i(x)$ must be of the general form

\[
\bar{x}^i(x) = x^i + \Delta^i(x), \quad \Delta^i(x) \equiv \sum_{k=1}^{\infty} X^{i}_{\alpha_1 \cdots \alpha_k}(x) \theta^{\alpha_1}(x) \cdots \theta^{\alpha_k}(x), \tag{3.4}
\]

where the expansion (3.4) is understood as a formal power series in constraints $\theta^\alpha$ with the coefficient functions $X^{i}_{\alpha_1 \cdots \alpha_k}(x)$ determined by the conditions (3.2) and (3.3). The general solutions to these conditions are derived below.

### 3.1 Solving (3.2)

First we show that (3.2) always have solutions of the form (3.4). To prove this consider the formal Taylor expansion

\[
\theta^\alpha(\bar{x}) = \theta^\alpha(x + \Delta(x)) = \theta^\alpha(x) + \Delta^i(x) \partial_i \theta^\alpha + \frac{1}{2} \Delta^i(x) \Delta^j(x) \partial_i \partial_j \theta^\alpha + \ldots \tag{3.5}
\]

By means of this expression condition (3.2) may be solved order by order in powers of $\theta^\alpha$. To first order we get the equation

\[
\theta^\alpha + \theta^\beta X^i_\beta \partial_i \theta^\alpha = 0, \tag{3.6}
\]

and from the properties (2.17) and (2.20) we find that the vector field

\[
X_\beta \equiv X^i_\beta \partial_i = -V_\beta + f_{\beta j} D^j \tag{3.7}
\]

solves (3.6) for arbitrary functions $f_{\beta j}(x)$. Thus, we have

\[
X^i_\beta = X_\beta x^i = -\{x^i, \theta^\gamma\} C_{\gamma \beta} + f_{\beta j} \{x^i, x^j\}_D. \tag{3.8}
\]

To second order in $\theta^\alpha$ (3.2) and (3.3) yield the equation

\[
\left( X^i_\beta \partial_i \theta^\alpha + \frac{1}{2} X^i_\beta X^j_\gamma \partial_i \partial_j \theta^\alpha \right) \theta^\beta \theta^\gamma = 0. \tag{3.9}
\]

Again by means of (2.17) and (2.20) we find that the vector field

\[
X_{\beta \gamma} \equiv X^j_\beta \partial_j = -\frac{1}{2} X^m_\beta X^m_\gamma \partial_n \partial_m \theta^\rho V_\rho + f_{\beta \gamma j} D^j \tag{3.10}
\]

solves (3.4) for arbitrary functions $f_{\beta \gamma j}$. It is now obvious that (3.2) may be solved by means of the Taylor expansion (3.7). To the nth order the solution has the form

\[
X^i_{\alpha_1 \cdots \alpha_n} = -\Gamma^i_{\alpha_1 \cdots \alpha_n} \{x^i, \theta^\lambda\} C_{\lambda \rho} + f_{\alpha_1 \cdots \alpha_n j} \{x^i, x^j\}_D, \tag{3.11}
\]
where \( f_{\alpha_1\cdots\alpha_n} \) are arbitrary functions of \( x^i \) which are symmetric in \( \alpha_k \), and where \( \Gamma^\rho_{\alpha_1\cdots\alpha_n} \) are sums of powers of \( X^i_{\alpha_1\cdots\alpha_k} \) for \( k \leq n-1 \) with coefficients involving derivatives of \( \theta^\rho \) up to order \( n \).

The above expressions may be considerably simplified. First one may remove the derivatives of \( \theta^\rho \) in \( X^i_{\alpha_1\cdots\alpha_n} \) by means of the properties of the first order vector field \( X_\alpha \) in (3.7). From (3.7) we have

\[
X_\alpha \theta^\gamma = X_\alpha \partial_i \theta^\gamma = -\delta^\gamma_\alpha,
\]

which implies

\[
\partial_k X^i_\alpha \partial_j \theta^\gamma + X^i_\alpha \partial_k \partial_j \theta^\gamma = 0,
\]

This relation allows us now to rewrite (3.10) as follows

\[
X_{\alpha_1 \alpha_2} = \frac{1}{4} \left( X^i_{\alpha_1} \partial_n X^m_{\alpha_2} + X^n_{\alpha_2} \partial_n X^m_{\alpha_1} \right) \partial_m \theta^\gamma V_\gamma + f_{\alpha_1 \alpha_2 k} D_k,
\]

The relation (2.25) yields then

\[
X_{\alpha_1 \alpha_2} = \frac{1}{4} \left( X^i_{\alpha_1} \partial_n X^m_{\alpha_2} + X^n_{\alpha_2} \partial_n X^m_{\alpha_1} \right) \partial_m + g_{\alpha_1 \alpha_2 k} D_k,
\]

\[
g_{\alpha_1 \alpha_2 k} = f_{\alpha_1 \alpha_2 k} - \frac{1}{4} \left( X^i_{\alpha_1} \partial_n X^m_{\alpha_2} + X^n_{\alpha_2} \partial_n X^m_{\alpha_1} \right) \omega_{mk},
\]

which is much simpler than (3.10). The same procedure may be used also for the higher order vector fields. The third order coefficient function is e.g. determined by the equation

\[
\left( X^i_{\alpha_1 \alpha_2 \alpha_3} \partial_\gamma \theta^\gamma + X^i_{\alpha_1} X^j_{\alpha_2 \alpha_3} \partial_i \partial_j \theta^\gamma + \frac{1}{6} X^i_{\alpha_1} X^j_{\alpha_2} X^k_{\alpha_3} \partial_i \partial_j \partial_k \theta^\gamma \right) \theta^\alpha_1 \theta^\alpha_2 \theta^\alpha_3 = 0,
\]

which follows from (3.2) and (3.3). By means of (3.12), (3.13) and

\[
\partial_k \partial_j X^i_\alpha \partial_i \theta^\gamma + \partial_k X^i_\alpha \partial_j \theta^\gamma + \partial_k X^i_\alpha \partial_i \partial_j \theta^\gamma + \partial_k X^i_\alpha \partial_i \partial_j \partial_k \theta^\gamma = 0,
\]

which follows from (3.13), (3.16) may be reduced to

\[
\left( X^i_{\alpha_1 \alpha_2 \alpha_3} - X^j_{\alpha_1 \alpha_2} \partial_j X^i_{\alpha_3} + \frac{1}{3} X^j_{\alpha_1} \partial_j X^k_{\alpha_2} \partial_k X^i_{\alpha_3} - \frac{1}{6} X^j_{\alpha_1} X^k_{\alpha_2} \partial_j \partial_k X^i_{\alpha_3} \right) \times \partial_\gamma \theta^\alpha_1 \theta^\alpha_2 \theta^\alpha_3 = 0.
\]

After insertion of the solution (3.15) for the second order we find

\[
X^i_{\alpha_1 \alpha_2 \alpha_3} = \frac{1}{6} \left( X_{\alpha_1} X_{\alpha_2} X_{\alpha_3} \right)_{\text{sym} \alpha} + \left( g_{\alpha_1 \alpha_2 k} D_k X^i_{\alpha_3} \right)_{\text{sym} \alpha} + g_{\alpha_1 \alpha_2 \alpha_3 k} \{x^k, x^i\} D,
\]

where “sym \( \alpha \)” means symmetrization in the \( \alpha \)-indices. \( g_{\alpha_1 \alpha_2 \alpha_3 k} \) is an arbitrary symmetric function. At an arbitrary order \( n \) we have in a similar fashion

\[
X^i_{\alpha_1 \cdots \alpha_n} = \frac{1}{n!} \left( X_{\alpha_1} \cdots X_{\alpha_{n-1}} X^i_{\alpha_n} \right)_{\text{sym} \alpha} + \cdots + g_{\alpha_1 \cdots \alpha_n k} \{x^k, x^i\} D,
\]

where the dots indicates terms involving the functions \( g_{\alpha_1 \cdots \alpha_m} \) for \( m = 2, \ldots, n-1 \). Now these arbitrary functions may be absorbed by a redefinition of \( f_{\alpha k} \) in \( X^i_\alpha \). In other words,
we may without restrictions set $g_{\alpha_1 \cdots \alpha_m k} = 0$ for $m \geq 2$, and consider the functions $f_{\alpha k}$ to be of the form

$$f_{\alpha k}(x) = \sum_{n=0}^{\infty} f^{(n)}_{\alpha_1 \cdots \alpha_n k}(x) \theta_{\beta_1} \cdots \theta_{\beta_n}. \quad (3.21)$$

(The same function $\bar{x}^i(x)$ may be obtained for different choices of the coefficient functions $X^i_{\alpha_1 \cdots \alpha_m}$ in (3.4).) This redefinition implies that the $n^{th}$ order coefficient function (3.20) reduces to

$$X^i_{\alpha_1 \cdots \alpha_n} = \frac{1}{n!} (X_{\alpha_1} \cdots X_{\alpha_n})_{\text{sym}} x^i, \quad (3.22)$$

which also follows from the recurrence relation

$$X^i_{\alpha_1 \cdots \alpha_n} = \frac{1}{n} (X^j_{\alpha_1 \cdots \alpha_{n-1}} \partial_j X^i_{\alpha_n})_{\text{sym}}, \quad (3.23)$$

which may be derived from (3.2) and (3.5). Eq.(3.22) implies now that possible extended observables have the following simple form in terms of the first order vector field (3.7)

$$\bar{x}^i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \theta_{\alpha_1} \cdots \theta_{\alpha_n} (X_{\alpha_1} \cdots X_{\alpha_n})_{\text{sym}} x^i \equiv e^{\xi^\alpha X_\alpha} x^i |_{\xi=0}, \quad (3.24)$$

where $\xi^\alpha$ are parameters. We have the relations

$$\theta^\alpha(\bar{x}) = \theta^\alpha (e^{\xi^\beta X_\beta} x^i |_{\xi=0}) = e^{\xi^\beta X_\beta \theta^\alpha(x)} |_{\xi=0} = 0, \quad (3.25)$$

which are valid for arbitrary $f_{\alpha k}$.

Notice that the higher order coefficients in (3.22) have to be symmetrized in the $\alpha$-indices since $X_\alpha$ in general do not commute. We have

$$[X_\alpha, X_\beta] = \{K_{\alpha \beta}, x^i\}_D \partial_i + h_{\alpha \beta k} D^k, \quad (3.26)$$

where

$$K_{\alpha \beta} = C_{\alpha \beta} + f_{\alpha i} \{x^i, x^j\}_D f_{\beta j} + C_{\beta \gamma} \{\theta^\gamma, x^i\}_D f_{\alpha i} - C_{\alpha \gamma} \{\theta^\gamma, x^i\}_D f_{\beta i},$$

$$h_{\alpha \beta k} = (f_{\alpha i} \{x^i, x^j\}_D - C_{\alpha \gamma} \{\theta^\gamma, x^j\}_D)(\partial_k f_{\beta j} - \partial_j f_{\beta k}) - (f_{\beta i} \{x^i, x^j\}_D - C_{\beta \gamma} \{\theta^\gamma, x^j\}_D)(\partial_k f_{\alpha j} - \partial_j f_{\alpha k}). \quad (3.27)$$

### 3.2 Solving (3.3)

The general solution (3.24) of the condition (3.2) represents a large class of solutions since the functions $f_{\alpha k}$ are completely unconstrained so far. However, when we now require the solution (3.24) also to satisfy a closed Poisson algebra in terms of the original Poisson bracket (2.3) then the arbitrariness in $f_{\alpha k}$ will be considerably reduced. Notice that if $\bar{x}^i(x)$ satisfy a closed Poisson algebra on $\mathcal{M}$ then this algebra must coincide with the Dirac bracket algebra, i.e. we must have the relation (3.3), simply since $\{\theta^\alpha(\bar{x}), \bar{x}^i\} = 0$ always is true. Now already at the zeroth order in $\theta^\alpha$ we find a restriction from (3.3). We find

$$\{\bar{x}^i(x), \bar{x}^j(x)\}_{\theta=0} = \{x^i, x^j\} + \{x^i, \theta^\alpha\} X^j_\alpha + X^i_\alpha \{\theta^\alpha, x^j\} + X^i_\alpha \{\theta^\alpha, \theta^\beta\} X^j_\beta =$$

$$= \{x^i, x^j\}_D + f_{\alpha k} \{x^k, x^i\}_D C_{\alpha \beta} \{x^l, x^j\}_D f_{\beta l}, \quad (3.28)$$
which should be \{x^i, x^j\}_D according to (3.3).

Consider now the right-hand side of the condition (3.3). Taylor expansion of the Dirac bracket yields

\[
\{x^i, x^j\}_D|_{x=\bar{x}(x)} = \{x^i, x^j\}_D + X_k^a \partial_k \{x^i, x^j\}_D \theta^a + 
\]

\[
+ X_{\alpha \beta} \partial_k \{x^i, x^j\}_D \theta^\alpha \theta^\beta + \frac{1}{2} X_m^a X_n^b \partial_m \partial_n \{x^i, x^j\}_D \theta^a \theta^b + \ldots 
\]

\[
= \sum_{n=0}^{\infty} \frac{(n)}{n!} A^{ij}_n \theta^{\alpha_1} \ldots \theta^{\alpha_n}. \quad (3.29)
\]

To the first four orders we have

\[
A^{ij} = \{x^i, x^j\}_D, \quad A^{ij}_a = X_k^a \partial_k \{x^i, x^j\}_D, \quad (3.30)
\]

\[
A^{ij}_{\alpha_1 \alpha_2} = X_{\alpha_1 \alpha_2} \partial_k \{x^i, x^j\}_D + \frac{1}{2} X_m^a X_n^b \partial_m \partial_n \{x^i, x^j\}_D, \quad (3.31)
\]

\[
A^{ij}_{\alpha_1 \alpha_2 \alpha_3} = X_{\alpha_1 \alpha_2 \alpha_3} \partial_k \{x^i, x^j\}_D + \frac{1}{3} (X_m^a X_{\alpha_2 \alpha_3} + X_{\alpha_2} X_{\alpha_3} \partial_m \partial_n \{x^i, x^j\}_D + 
\]

\[
+ \frac{1}{6} X_m^a X_m^b X_{\alpha_3} \partial_m \partial_n \{x^i, x^j\}_D, \quad (3.32)
\]

and the nth-order terms are symbolically given by (\alpha_i indices and their symmetrization are e.g. suppressed)

\[
A^{ij}_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + r\lambda_n = n} \frac{1}{\lambda_1! \ldots \lambda_r!} (X_{\lambda_1}^{(1)} \ldots (X_{\lambda_r}^{(r)} (\partial_{k_1})^{\lambda_1} (\partial_{k_2})^{\lambda_2} \ldots (\partial_{k_r})^{\lambda_r} \{x^i, x^j\}_D. \quad (3.33)
\]

Insertion of the solutions (3.22) imply now

\[
A^{ij}_n = \frac{1}{n!} (X_{\alpha_1} \ldots X_{\alpha_n})_{\text{sym} \alpha} \{x^i, x^j\}_D, \quad (3.34)
\]

which means that

\[
\{x^i, x^j\}_D|_{x=\bar{x}(x)} = e^{\xi^a X_{\alpha}} \{x^i, x^j\}_D|_{\xi=\theta}, \quad (3.35)
\]

The question now is under which conditions this is equal to \{\bar{x}^i, \bar{x}^j\}. We notice then that this definitely requires \(X_{\alpha}\) to differentiate the Dirac bracket. (We have explicitly checked this for the first three orders.) Thus, \(f_{\alpha k}\) must be of the form

\[
f_{\alpha k} = \sum_{n=0}^{\infty} \partial_k b_{\alpha \beta_1 \ldots \beta_n} \theta^{\beta_1} \ldots \theta^{\beta_n}, \quad (3.36)
\]

where \(b_{\alpha \beta_1 \ldots \beta_n}\) are arbitrary functions of \(x^i\). This implies

\[
X_{\alpha} = -V_{\alpha} + \{f_{\alpha}, \cdot\}_D, \quad f_{\alpha} = \sum_{n=0}^{\infty} b_{\alpha \beta_1 \ldots \beta_n} \theta^{\beta_1} \ldots \theta^{\beta_n}. \quad (3.37)
\]
A further consequence of \((3.36)\) is that \((3.26)\) reduces to
\[
\left[X_\alpha, X_\beta\right] = \{K_{\alpha\beta}, \cdot\}_D, \tag{3.38}
\]
where
\[
K_{\alpha\beta} = C_{\alpha\beta} + \{f_\alpha, f_\beta\}_D + C_{\beta\gamma}\{\theta^\gamma, f_\alpha\} - C_{\alpha\gamma}\{\theta^\gamma, f_\beta\}. \tag{3.39}
\]
The property \((3.38)\) is consistent with the fact that \(X_\alpha\) in \((3.37)\) differentiates the Dirac bracket. The commutator of differentiations is a differentiation, thus it must have the form \((2.26)\). \(K_{\alpha\beta}\) \((3.39)\) gives the explicit expression for the potential \(H\) in the parallel part of the derivative in \((2.26)\).

For \(X_\alpha\) of the form \((3.37)\) we have
\[
e^{\xi^\alpha X_\alpha}\{x^i, x^j\}_D\bigg|_{\xi=\theta} = \{e^{\xi^\alpha X_\alpha}x^i\bigg|_{\xi=\theta}, e^{\xi^\alpha X_\alpha}x^j\bigg|_{\xi=\theta}\}_D = \{\bar{x}^i(x), \bar{x}^j(x)\}_D\tag{3.40}
\]
From \((2.24)\) condition \((3.3)\) requires now
\[
V_\alpha\bar{x}^i(x)C^{\alpha\beta}V_\beta\bar{x}^j(x) = 0. \tag{3.41}
\]
From the general expression \((3.24)\) of \(\bar{x}^i(x)\) we have
\[
V_\alpha\bar{x}^i(x) = \{f_\alpha, \bar{x}^i(x)\}_D - X_\alpha\bar{x}^i(x),
\]
\[
X_\alpha\bar{x}^i(x) = \frac{1}{2}\theta^\beta[X_\alpha, X_\beta]x^i + O(\theta^2). \tag{3.42}
\]
Hence, to the zeroth order \((3.41)\) requires
\[
\{f_\alpha, x^i\}_D C^{\alpha\beta}\{f_\beta, x^j\}_D = 0, \tag{3.43}
\]
which is consistent with the result \((3.28)\). The condition \((3.41)\) becomes to the first order
\[
\left(\{f_\alpha, X_\gamma x^i\}_D + \frac{1}{2}(K_{\alpha\gamma}, x^i)_D\right) C^{\alpha\beta}\left(\{f_\beta, X_\delta x^j\}_D + \frac{1}{2}(K_{\beta\delta}, x^j)_D\right)\bigg|_{\text{sym} \gamma\delta} = 0. \tag{3.44}
\]
This condition as well as all the higher order conditions from \((3.41)\) are intricate conditions on \(f_\alpha\) and \(K_{\alpha\beta}\) in \((3.39)\). If we are able to choose \(f_\alpha\) such that \(K_{\alpha\beta}\) are expressed in terms of \(\theta^\alpha\) and constants then the \(K_{\alpha\beta}\)-dependence in these conditions will disappear. This is exactly the condition for \(X_\alpha\) to commute in \((3.38)\). For commuting \(X_\alpha\) we have also
\[
X_\alpha\bar{x}^i(x) = 0 \iff V_\alpha\bar{x}^i = \{f_\alpha, \bar{x}^i\}_D, \tag{3.45}
\]
which makes the conditions \((3.41)\) equivalent to
\[
\{f_\alpha, \bar{x}^i\}_D C^{\alpha\beta}\{f_\beta, \bar{x}^j\}_D = 0, \tag{3.46}
\]
which seems to be a simple condition on \(f_\alpha\). In fact, all explicit solutions considered in the following have commuting \(X_\alpha\). Maybe this is a general feature. Notice that we always have the property
\[
\bar{x}^i(x) = \bar{x}^i(x), \tag{3.47}
\]
which trivially follows if $X_\alpha$ commute due to (3.45).

Whether or not there exist functions $f_\alpha$ satisfying both (3.46) and $\{K_{\alpha\beta}, g(x)\}_D = 0$ for arbitrary functions $g(x) \in M$ in all theories with second class constraints is unclear. However, we expect that the class of theories for which this is possible to be large. (In subsection 4.2 and section 7 we give simple examples with a nontrivial $C_{\alpha\beta}$ which satisfy these properties.) One may notice that all theories for which $C_{\alpha\beta}$ is a function of only the constraints (i.e., in this case, the constraints $\theta^\alpha$ constitute Poisson subalgebra in the phase space) so that $V_\alpha$ commute are contained in this class since $f_\alpha$ then may be chosen to be zero and we have

$$V_\alpha \bar{x}^i(x) = 0 \iff \{\bar{x}^i(x), \theta^\alpha\} = 0. \quad (3.48)$$

In this special case, all the extended observables commute with the constraints. If the constraints do not form a Poisson subalgebra they can, of course, not commute with $\bar{x}^i(x)$.

### 4 Examples

#### 4.1 A simple example

Consider a dynamical theory defined on a phase space, $M$, on which $x^A$ and $p_A$ are globally defined canonically conjugate variables. We have the fundamental Poisson bracket relations

$$\{x^A, p_B\} = \delta^A_B, \quad \{x^A, x^B\} = \{p_A, p_B\} = 0. \quad (4.1)$$

The indices A, B are assumed to be raised (and lowered) by a constant regular symmetric metric $g^{AB}$ ($g_{AB}$).

On the phase space $M$ we have two constraints, $\theta^\alpha = 0$, where

$$\theta^1 = x^Ax_A - R^2, \quad \theta^2 = p_Ax^A, \quad (4.2)$$

where $R$ is a positive constant. These constraints satisfy

$$C^{12} \equiv \{\theta^1, \theta^2\} = 2x^Ax_A = 2\theta^1 + 2R^2. \quad (4.3)$$

Hence, the constraints are of second class and they form closed Poisson algebra. The Dirac bracket is

$$\{A, B\}_D = \{A, B\} - \{A, \theta^1\}C_{12}\{\theta^2, B\} - \{A, \theta^2\}C_{21}\{\theta^1, B\}, \quad (4.4)$$

where ($x^2 \equiv x^Ax_A$)

$$C_{12} = -\frac{1}{x^2} = -C_{21}. \quad (4.5)$$

Explicitly we find

$$\{x^A, x^B\}_D = 0, \quad \{x^A, p_B\}_D = \delta^A_B - \frac{x^Ax_B}{x^2},$$

$$\{p_A, p_B\}_D = \frac{1}{x^2}(p_Ax_B - p_Bx_A). \quad (4.6)$$
A general ansatz for extended observables satisfying the conditions (3.4) and (3.2) is

$$\bar{x}^A = \frac{R}{\sqrt{x^2}} x^A, \quad \bar{p}_A = p_A - p \cdot x A x^2 + M_{AB}(x, p) x^B, \quad (4.7)$$

where $M_{AB}(x, p)$ is an arbitrary antisymmetric tensor function which vanishes for $\theta^\alpha = 0$. The condition (3.3), i.e. the correct Poisson bracket algebra fixes $M_{AB}(x, p)$. We get the solution $(p \cdot x = p_A x^A)$

$$\bar{x}^A = \frac{R}{\sqrt{x^2}} x^A, \quad \bar{p}_A = \frac{\sqrt{x^2}}{R} (p_A - p \cdot x A x^2). \quad (4.8)$$

Only these expressions satisfy

$$\{ \bar{x}^A, \bar{x}^B \} = 0, \quad \{ \bar{x}^A, \bar{p}_B \} = \delta^A_B - \frac{x^A x_B}{x^2} = \delta^A_B - \frac{\bar{x}^A \bar{x}_B}{x^2} = \delta^A_B - \frac{\bar{x}^A \bar{x}_B}{R^2},$$

$$\{ \bar{p}_A, \bar{p}_B \} = \frac{1}{x^2} (\bar{p}_A \bar{x}_B - \bar{p}_B \bar{x}_A) = \frac{1}{R^2} (\bar{p}_A \bar{x}_B - \bar{p}_B \bar{x}_A). \quad (4.9)$$

Since $\{ A, C^{12} \}_D = 0$ according to (4.3), the $V_\alpha$-operators commute. We have

$$V_1 \equiv C_{12} \{ \theta^2, \cdot \} = \frac{1}{2x^2} (x^A \partial^A_\eta - p^A \partial^\eta_A), \quad V_2 \equiv C_{21} \{ \theta^1, \cdot \} = \frac{x^A}{x^2} \partial^\eta_A, \quad (4.10)$$

where $\partial^\eta_A = \partial / \partial x^A$ and $\partial^\eta_A = \partial / \partial p^A$. One may easily check that $[V_1, V_2] = 0$. Since we have $f_\alpha = 0$ here the extended observables must satisfy $V_\alpha \bar{x}^A = V_\alpha \bar{p}_A = 0$ according to (3.48) or equivalently

$$\{ \bar{x}^A, \theta^\alpha \} = \{ \bar{p}_A, \theta^\alpha \} = 0. \quad (4.11)$$

This condition on the general ansatz (4.7) yields again the expressions (4.8). Thus, the expressions (4.8) are possible to write as

$$\bar{x}^A = e^{-\xi^\alpha V_\alpha} x^A \bigg|_{\xi = \theta}, \quad \bar{p}_A = e^{-\xi^\alpha V_\alpha} p_A \bigg|_{\xi = \theta}. \quad (4.12)$$

According to (2.24) the properties (4.11) imply

$$\{ \bar{x}^A, \bar{x}^B \} = \{ \bar{x}^A, \bar{p}_B \}_D, \quad \{ \bar{x}^A, \bar{p}_B \} = \{ \bar{p}_A, \bar{p}_B \}_D, \quad \{ \bar{p}_A, \bar{p}_B \} = \{ \bar{p}_A, \bar{p}_B \}_D. \quad (4.13)$$

### 4.2 A simple but nontrivial example

Consider a $2n$-dimensional phase space $(n \geq 2)$, $\mathcal{M}$, spanned by the canonical coordinates $x^\mu$ and $p_\mu$ satisfying the Poisson algebra

$$\{ x^\mu, p_\nu \} = \delta^\mu_\nu, \quad \{ x^\mu, x^\nu \} = \{ p_\mu, p_\nu \} = 0. \quad (4.14)$$

On $\mathcal{M}$ we impose two constraints, $\theta^\alpha = 0$, where

$$\theta^1 \equiv p^2 - m^2, \quad \theta^2 \equiv x^2 - a^2, \quad (4.15)$$

where $a$ and $m$ are two real constants. $x^\mu$ and $p_\nu$ are considered to be $n$-dimensional Lorentz vectors and all inner products are Lorentz products. Thus, in (4.15) $p^2 = p_\mu p_\nu \eta^{\mu\nu}$
and \( x^2 = x^\mu x^\nu \eta_{\mu\nu} \) where \( \eta^{\mu\nu}(\eta_{\mu\nu}) \) is a time-like Minkowski metric in \( n \) dimensions. The constraints \( \theta^\alpha = 0 \) are of second class since

\[
C^{12}|_{\theta=0} \neq 0, \quad C^{12} = \{\theta^1, \theta^2\} = -4p \cdot x,
\]

where \( p \cdot x = p_\mu x^\mu \). The Dirac bracket is given by (4.4) with

\[
C_{12} = \frac{1}{4p \cdot x} = -C_{21}.
\]

Explicitly we get

\[
\{x^\mu, p_\nu\}_D = \delta^\mu_\nu - \frac{p^\mu x_\nu}{p \cdot x}, \quad \{x^\mu, x^\nu\}_D = \{p_\mu, p_\nu\}_D = 0.
\]

In order to construct appropriate extended observables we first construct general solutions of (3.2) for the ansatz (3.4). There are several solutions. Three of them are given below.

1) \( \bar{x}^\mu = x^\mu - \frac{p^\mu(p \cdot x)}{p^2} \left(1 - \sqrt{A_x}\right), \quad \bar{p}^\mu = p^\mu - \frac{x^\mu(p \cdot x)}{x^2} \left(1 - \sqrt{A_p}\right), \)

where

\[
A_x = 1 - \frac{p^2(x^2 - a^2)}{(p \cdot x)^2}, \quad A_p = 1 - \frac{x^2(p^2 - m^2)}{(p \cdot x)^2}.
\]

2) \( \bar{x}^\mu = \sqrt{\frac{p^2}{m^2}} \left(x^\mu - \frac{p^\mu(p \cdot x)}{p^2} \left(1 - \sqrt{B}\right)\right), \quad \bar{p}^\mu = \sqrt{\frac{m^2}{p^2}} p^\mu, \)

where

\[
B = 1 - \sqrt{\frac{p^2a^2 - m^2a^2}{(p \cdot x)^2}}.
\]

3) \( \bar{x}^\mu = \sqrt{\frac{a^2}{x^2}} x^\mu, \quad \bar{p}^\mu = \sqrt{\frac{x^2}{a^2}} \left(p^\mu - \frac{x^\mu(p \cdot x)}{x^2} \left(1 - \sqrt{B}\right)\right), \)

All three expressions (4.19), (4.21) and (4.23) satisfy

\[
\bar{x}^2 = a^2, \quad \bar{p}^2 = m^2.
\]

It is straightforward but tedious to check that the expressions (4.21) and (4.23) satisfy

\[
\{\bar{x}^\mu, \bar{p}_\nu\} = \delta^\mu_\nu - \frac{\bar{p}^\mu \bar{x}_\nu}{\bar{p} \cdot \bar{x}}, \quad \{\bar{x}^\mu, \bar{x}^\nu\} = \{\bar{p}_\mu, \bar{p}_\nu\} = 0,
\]

which are the correct expressions required by (3.3) due to (4.18). However, (4.19) does not satisfy (4.25). The reason why we have found more than one correct solution will be explained in section 8. (There might be more than two solutions.)
We consider now the above solutions as expansions in the constraint variables:

\[
\bar{x}^\mu = x^\mu + \sum_{k=1}^\infty X^\mu_{\alpha_1 \cdots \alpha_k} \theta^{\alpha_1} \cdots \theta^{\alpha_k},
\]

\[
\bar{p}^\mu = p^\mu + \sum_{k=1}^\infty P^\mu_{\alpha_1 \cdots \alpha_k} \theta^{\alpha_1} \cdots \theta^{\alpha_k},
\] (4.26)

The solution (4.19) yields to first order e.g.

\[
X^\mu_1 = 0, \quad X^\mu_2 = -\frac{p^\mu}{2(p \cdot x)}, \quad P^\mu_1 = -\frac{x^\mu}{(p \cdot x)}, \quad P^\mu_2 = 0.
\] (4.27)

Other choices which differ from these by terms involving powers of constraint variables are also possible and are considered to be equivalent. Comparison with the general expression (3.37), i.e.

\[
X^\mu_\beta = -\{x^\mu, \theta^\gamma\} C^\gamma_\beta + \{f^\beta, x^\mu\} D^\mu,
\]

\[
P^\mu_\beta = -\{p^\mu, \theta^\gamma\} C^\gamma_\beta + \{f^\beta, p^\mu\} D^\mu
\] (4.28)
yields \(f^\beta = 0\), which means that the fundamental first order vector fields \(X_\alpha\) do not commute for (4.19). \(X_\alpha\) are given by

\[
X_1 = X^\mu_0 \partial_\mu + P^\mu_1 \partial_\mu, \quad X_2 = X^\mu_2 \partial_\mu + P^\mu_2 \partial_\mu,
\] (4.29)

where \(\partial_\mu = \partial/\partial x_\mu\) and \(\partial_\mu = \partial/\partial p_\mu\). Although the condition (3.46) is satisfied it is not equivalent to (3.41). In fact, we have here

\[
X_1 \bar{p}^\mu = X_2 \bar{x}^\mu = 0, \quad X_1 \bar{x}^\mu \neq 0, \quad X_2 \bar{p}^\mu \neq 0,
\]

\[
\Rightarrow \quad X_{\alpha} \bar{x}^\mu C^\alpha_\beta X^\beta_\nu \bar{p}^\nu \neq 0,
\] (4.30)

which violates (3.41) since \(X_\alpha = -V_\alpha\) here. This explains why (4.19) does not satisfy (4.25).

Consider now the solution (4.21). It yields to first order e.g.

\[
X^\mu_1 = \frac{x^\mu}{2m^2} - \frac{a^2}{2p^2(p \cdot x)} p^\mu, \quad X^\mu_2 = -\frac{p^\mu}{2(p \cdot x)}, \quad P^\mu_1 = -\frac{p^\mu}{2m^2}, \quad P^\mu_2 = 0.
\] (4.31)

Comparison with the general expression (4.28) yields then the possible choices for \(f^\alpha\). One possible choice is

\[
f_1 = -\frac{(p \cdot x)}{2m^2}, \quad f_2 = 0.
\] (4.32)

This choice makes (3.46) zero and yields for (3.33)

\[
K_{12} = C_{12} \left(\frac{m^2 - p^2}{m^2}\right) = -K_{21}.
\] (4.33)

Let us now see if we can make another choice of \(f_\alpha\) such that the first order vector fields (4.29) commute by requiring \(K_{\alpha\beta}\) in (3.39) to be functions of the constraint variables \(\theta^\alpha\).
only. The allowed forms of $f_\alpha$ are given in (3.37). We may therefore replace $f_1$ in (4.32) by

$$f_1 = -\frac{(p \cdot x)}{2m^2} + b_{11}(p^2 - m^2), \quad f_2 = 0. \quad (4.34)$$

This expression makes $K_{\alpha \beta}$ zero for the choice

$$b_{11} = \frac{(p \cdot x)}{2m^2} \Rightarrow f_1 = -\frac{(p \cdot x)}{2p^2}. \quad (4.35)$$

This $f_1$ inserted into (4.28) yields the first order coefficients

$$X_\mu^1 = \frac{x^\mu}{2p^2} - \frac{x^2(x^\mu)}{2p^2(p \cdot x)} p^\mu, \quad X_2^\mu = -\frac{p^\mu}{2(p \cdot x)}, \quad P_1^\mu = -\frac{p^\mu}{2p^2}, \quad P_2^\mu = 0, \quad (4.36)$$

which differ from (4.31) by a power expansion in the constraint variables. One may easily check that ($X_\alpha$ are the first order vector field defined by (4.29))

$$X_\alpha \bar{x}^\mu = 0, \quad X_\alpha \bar{p}^\mu = 0 \quad (4.37)$$

as required by (3.45). The solutions (4.21) are therefore of the exponential form (3.24). The solutions (4.23) satisfy (4.25) since (3.46) is satisfied.

For the solutions (4.23) we find the first order coefficients

$$X_1^\mu = 0, \quad X_2^\mu = -\frac{x^\mu}{2x^2}, \quad P_1^\mu = -\frac{p^\mu}{2(p \cdot x)}, \quad P_2^\mu = \frac{p^\mu}{2x^2} - \frac{p^2}{2x^2(p \cdot x)} x^\mu. \quad (4.38)$$

The general expression (4.28) yields here

$$f_1 = 0, \quad f_2 = \frac{(p \cdot x)}{2x^2}, \quad (4.39)$$

which also satisfies the condition (3.46). One may easily check that this choice make $K_{\alpha \beta}$ in (3.39) zero, which means that the vector fields (4.29) commutes. Even the solutions (4.23) satisfy the properties (4.37) which means that also they are of the exponential form (3.24). Note that $\{\theta^1, f_1\} = 1$ for the solution (4.21) and $\{\theta^2, f_2\} = 1$ for the solution (4.23). These properties are probably not accidental as will be explained in section 8.

5 Extended observables in theories with first class constraints

Consider a dynamical system defined on a $2n$-dimensional symplectic manifold $\mathcal{M}$. Let

$$\phi_a(x) = 0, \quad a = 1, \ldots, m < n \quad (5.1)$$

be first class constraints, i.e. let $\phi_a(x)$ satisfy the Poisson algebra

$$\{\phi_a(x), \phi_b(x)\} = f_{ab}^c(x)\phi_c(x), \quad (5.2)$$

where $f_{ab}^c(x)$ are structure functions. Even here we may define extended observables along the lines of section 3. For this we need $m$ functions $\chi^a(x) = 1, \ldots, m$, satisfying the properties

$$\{\chi^a(x), \chi^b(x)\} = 0, \quad \det{\chi^a(x), \phi_b(x)}|_{\varphi=0} \neq 0. \quad (5.3)$$
Since $\chi^a(x)$ and $\phi_b(x)$ together act like the second class constraint variables $\theta^a$ in section 3, the previous analysis applies. Thus, we may define a Dirac bracket where both $\chi^a(x)$ and $\phi_b(x)$ are Casimir functions. By means of the formula (2.24) we may define the Dirac bracket by

$$\{f, g\}_D = \{f, g\} + M^a_bV^b fV_ag - M^a_bV^b f - f_a b^e c^e V^a fV^b g,$$

(5.4)

where

$$M^a_b(x) \equiv \{\chi^a(x), \phi_b(x)\},$$

(5.5)

and

$$V^a = (M^{-1})^a_b(x)\{\chi^b, \cdot\}, \quad (M^{-1})^b_a(x)M^b_c(x) = \delta^a_c,$$

$$V_a = -(M^{-1})_a^b(x)\{\phi_b, \cdot\} + (M^{-1})^c_a(x)(M^{-1})^d_b(x)f_{de}^e \phi_e\{\chi^b, \cdot\},$$

(5.6)

which are defined according to formula (2.15), i.e. they are the $V_a$-operators of section 3 for $\theta^a = (\chi^a, \phi_a)$, and they differentiate the Dirac bracket (5.4).

Let $x^{*\gamma}$ be any solution of $\phi_a(x^*) = 0$ where $m$ of the coordinates $x^i$ are made dependent variables. $x^{*\gamma}$ are viewed as observables here. By extended observables $\bar{x}^i(x)$ we then mean expressions defined on $\mathcal{M}$ satisfying the properties

$$\bar{x}^i(x^*) = x^{*\gamma},$$

(5.7)

$$\phi_a(\bar{x}) = 0,$$

(5.8)

and

$$\{\bar{x}^i(x), \bar{x}^j(x)\} = \{x^i, x^j\}_D|_{x \rightarrow \bar{x}(x)}.$$ (5.9)

The first condition (5.4) requires $\bar{x}^i(x)$ to be of the form

$$\bar{x}^i(x) = x^i + \sum_{n=1}^{\infty} \{x^{i_{a_1 \cdots a_n}}(x)\phi_{a_1}(x) \cdots \phi_{a_n}(x).$$

(5.10)

It is easily seen that

$$\bar{x}^i(x) = e^{-\xi a V^a} x^i|_{\xi \rightarrow 0}$$

(5.11)

are solutions of (5.8) of the form (5.10). As in section 3 we have also the property

$$\{x^i, x^j\}_D|_{x \rightarrow \bar{x}(x)} = e^{-\xi a V^a}\{x^i, x^j\}_D|_{\xi \rightarrow 0} = \{\bar{x}^i, \bar{x}^j\}_D,$$

(5.12)

where the last equality follows since $V^a$ differentiate the Dirac bracket. Now since the $\chi^a$-variables are chosen to satisfy (5.3) the $V^a$-operators commute, i.e.

$$[V^a, V^b] = 0.$$ (5.13)

This implies that

$$V^a \bar{x}^i(x) = 0 \iff \{\chi^a, \bar{x}^i(x)\} = 0,$$

(5.14)

which for the Dirac bracket (5.4) implies

$$\{\bar{x}^i, \bar{x}^j\}_D = \{\bar{x}^i, \bar{x}^j\}.$$ (5.15)

This together with (5.12) shows that the condition (5.9) is satisfied. Notice that in distinction to the extended observables $\bar{x}^i(x)$ in the second class case, $\bar{x}^i(x)$ are much more ambiguous due to the large freedom how to choose the “gauge fixing” variables $\chi^a$. In other words there is a large gauge freedom in $\bar{x}^i(x)$.
5.1 Example: The free relativistic particle

Let $x^\mu$ and $p_\mu$ be coordinates and momenta for a free relativistic particle. The momenta satisfy then the mass shell condition (we use timelike metric)

$$\phi \equiv p^2 - m^2 = 0.$$  \hspace{1cm} (5.16)

Let us choose as “gauge fixing” variable

$$\chi \equiv \eta \cdot x - \tau,$$  \hspace{1cm} (5.17)

where $\tau$ is a parameter and $\eta^\mu$ a constant four-vector. The condition (5.3) requires $\eta^2 \geq 0$.

The extended observables are here

$$\tilde{x}^\mu = e^{-\xi V} x^\mu \bigg|_{\xi \to \phi}, \quad \tilde{p}_\mu = e^{-\xi V} p_\mu \bigg|_{\xi \to \phi}, \quad V \equiv \frac{1}{2 \eta \cdot p} \{ \eta \cdot x, \cdot \}. \hspace{1cm} (5.18)$$

Explicitly we have

$$\tilde{x}^\mu = x^\mu, \quad \tilde{p}_\mu = p_\mu - \frac{\eta^\mu \eta \cdot p}{\eta^2} \left( 1 - \sqrt{1 - \frac{\eta^2 (p^2 - m^2)}{(\eta \cdot p)^2}} \right), \quad \eta^2 > 0.$$  \hspace{1cm} (5.19)

$$\tilde{x}^\mu = x^\mu, \quad \tilde{p}_\mu = p_\mu - \frac{\eta^\mu}{2 \eta \cdot p} (p^2 - m^2), \quad \eta^2 = 0.$$  \hspace{1cm} (5.20)

If the particle is massive, $m \neq 0$, we may also choose the proper time gauge fixing

$$\chi \equiv p \cdot x - \tau.$$  \hspace{1cm} (5.21)

In this case we have the expressions (5.18) for

$$V = \frac{1}{2 p^2} \{ p \cdot x, \cdot \}, \hspace{1cm} (5.22)$$

which explicitly yield

$$\tilde{x}^\mu = \sqrt{\frac{p^2}{m^2}} x^\mu, \quad \tilde{p}_\mu = \sqrt{\frac{m^2}{p^2}} p_\mu.$$  \hspace{1cm} (5.23)

All solutions (5.19), (5.20), and (5.23) satisfy $\tilde{p}^2 = m^2$. One may easily check that property (5.9) is valid.

6 Comparisons with gauge invariant extensions

The extended observables constructed in the previous section are not what one usually considers to be observables in a gauge theory. Normally they are gauge invariant quantities. Indeed there is something called gauge invariant extensions in the literature (see e.g. [2, 3]). They are quantities $\tilde{x}^i(x)$ defined on $\mathcal{M}$ with the following properties: They are gauge invariant, i.e. they satisfy

$$\{ \tilde{x}^i(x), \phi_a(x) \} = C^{aib}(x) \phi_b(x),$$  \hspace{1cm} (6.1)
where $\phi_a(x)$ are the first class constraint variables satisfying the algebra (5.2), and $C^b_a(x)$ are unspecified coefficient functions. $\hat{x}^i(x)$ also satisfy

$$\chi^a(\hat{x}) = 0,$$

(6.2)

where $\chi^a$ are “gauge fixing” variables satisfying the properties (5.3), and

$$\hat{x}^i(x^*) = x^{*i},$$

(6.3)

where $x^{*i}$ is any solution of $\chi^a(x^*) = 0$ with $m$ dependent coordinates. Furthermore, we have

$$\{\hat{x}^i(x), \hat{x}^j(x)\} = \{x^i, x^j\}_D |_{x \rightarrow \hat{x}(x)} + C^{ij}_b(x)\phi_b(x),$$

(6.4)

where $C^{ij}_b(x)$ are unspecified coefficient functions and where the Dirac bracket is the one in (5.4). For Lie group theories the following general formula was given in [3]

$$\hat{x}^i(x) = \int d^m\Omega | \det(\chi^a(x_\Omega), \phi_b(x_\Omega))| \delta^m(\chi(x_\Omega))x^i_{\Omega},$$

(6.5)

where $x^i_{\Omega}$ is a gauge transformed $x^i$ and $d^m\Omega$ is the volume element of the group.

From the analysis of the previous sections we are now able to make a more careful analysis of these gauge invariant extensions. From the property (6.3) it is clear that $\hat{x}^i(x)$ must be of the general form

$$\hat{x}^i(x) = x^i + \sum_{n=1}^{\infty} X^i_{a_1...a_n}(x)\chi^{a_1}(x)\cdots\chi^{a_n}(x).$$

(6.6)

In fact, the obvious solution of this form is

$$\hat{x}^i(x) = e^{-\xi^a V_a x^i}|_{\xi \rightarrow \chi},$$

(6.7)

where the differential operator $V_a$ is given in (5.6). This expression of $\hat{x}^i(x)$ satisfies (6.2) due to the properties

$$V_a \chi^b = \delta^b_a, \quad V_a \phi_b = 0.$$

(6.8)

The properties (6.4) and (6.6) follow then from the following commutation relations

$$[V_a, V_b] = \phi_e(x)\{f_{ae}^c(x)(M^{-1})^c_d(x)(M^{-1})^d_b(x), \cdot \}_D,$$

(6.9)

where the Dirac bracket is the one in (5.4). Thus, the $V_a$-operators only commute if the structure functions $f_{ab}^c(x)$ and $M^c_d(x)$ are functions of only the constraint variables $\phi_a$ and/or $\chi^b$. In this case the last terms in (6.4) vanish. Strict gauge invariance, i.e.

$$\{\hat{x}^i(x), \phi_a(x)\} = 0$$

(6.10)

is only valid for abelian gauge theories.
6.1 Example: The free relativistic particle

Let as in the previous section $x^\mu$ and $p_\mu$ be coordinates and momenta for a free relativistic particle where the momenta satisfy

$$\phi \equiv p^2 - m^2 = 0. \quad (6.11)$$

The gauge invariant extensions are easily calculated by means of formula (6.5). In the gauge $\chi = \eta \cdot x - \tau$ we find

$$\hat{p}^\mu = p^\mu, \quad \hat{x}^\mu = \frac{\eta \nu J^{\mu \nu}}{\eta \cdot p} + \tau \frac{p^\mu}{p^2} + \frac{\tau}{p^2}, \quad J^{\mu \nu} \equiv x^\mu p^\nu - x^\nu p^\mu, \quad (6.12)$$

and in the proper time gauge $\chi = x \cdot p - \tau$ we have

$$\hat{p}^\mu = p^\mu, \quad \hat{x}^\mu = \frac{p \nu J^{\mu \nu}}{p^2} + \frac{\tau}{p^2}. \quad (6.13)$$

These expressions satisfy (6.2)-(6.3) and property (6.1) with $C_{ib}^a = 0$ and (6.4) with $C^{ijb} = 0$.

7 Extended observables in general gauge theories in a particular gauge

General gauge theories are theories with first class constraints. In the previous sections, 5 and 6, we have constructed two types of extended observables for these theories. Here we define a third type namely extended observables as defined in section 3 for second class constraints. Consider therefore again the first class constraint variables $\phi_a(x)$ in (5.1)-(5.2) and the gauge fixing variables $\chi^a(x)$ with the properties (5.3). The physical degrees of freedom are described by $x^a$ satisfying the conditions

$$\phi_a(x^*) = 0, \quad \chi^a(x^*) = 0, \quad a = 1, \ldots, m. \quad (7.1)$$

$x^a$ depends on $2(n-m)$ independent coordinates. As in section 3 we view $x^a$ as observables here. In order to define extended observables $\bar{x}^i(x)$ we need differential operators $X_\alpha$, $\alpha = 1, \ldots, 2m$, which probably must be commuting. The $V_\alpha$ - operators in (5.6) do not commute in general. However, we expect that we always may define commuting $X_\alpha$-operators defined by

$$X^a \equiv -V^a - \{ f^a, \cdot \}_D, \quad X_a \equiv -V_a - \phi_e \{ f^e, \cdot \}_D, \quad (7.2)$$

where $V^a$ and $V_a$ are the $V_\alpha$-operators in (5.6), and where $f^a$ and $f^e_a$ satisfy the condition

$$\{ f^c_a, \bar{x}^j \}_D M^b_f \{ f^b, \bar{x}^j \}_D - \{ f^c_a, \bar{x}^j \}_D M^b_f \{ f^b, \bar{x}^j \}_D = 0, \quad (7.3)$$

which follows from (3.46). The extended observables are then given by

$$\bar{x}^i(x) = e^{\xi X^a + \rho^a X_a} x^i \bigg|_{\xi \to \phi, \rho \to \chi}. \quad (7.4)$$

Even these expressions are gauge invariant in the sense of (6.3). They may therefore be called proper gauge invariant extensions since they describe exactly the right number of degrees of freedom in distinctions to the gauge invariant extensions in section 6.
7.1 Example: The free relativistic particle

Consider again the relativistic particle with coordinates $x^\mu$ and momenta $p_\mu$ satisfying the mass shell condition

$$\phi \equiv p^2 - m^2 = 0. \quad (7.5)$$

In the following we denote the $V_\alpha$-operators by $V$ and $W$ corresponding to $V^\alpha$ and $V_\alpha$ respectively. In the proper time gauge

$$\chi = p \cdot x - \tau, \quad (7.6)$$

we have then

$$V \equiv \frac{1}{2p^2} \{ p \cdot x, \cdot \} \quad W \equiv -\frac{1}{2p^2} \{ p^2, \cdot \}, \quad (7.7)$$

which do commute. The extended observables are therefore

$$\tilde{x}^\mu(x, p) = e^{-\tilde{\xi}V - \tilde{\rho}W} x^\mu \left|_{\tilde{\xi} \to \phi, \tilde{\rho} \to \chi} = \tilde{x}^\mu(\tilde{x}, \tilde{p}) = \tilde{x}^\mu(\tilde{x}, \tilde{p}) = \sqrt{p^2 + \left( \frac{p^\mu \eta_{\mu\nu} + \tau p^\mu}{p^2} \right)^2}, \quad (7.8)$$

where $\tilde{x}^\mu$ and $\tilde{p}^\mu$ are given in (5.23), and $\tilde{x}^\mu$, $\tilde{p}^\mu$ in (6.13).

In the gauge

$$\chi = \eta \cdot x - \tau, \quad \eta^2 \geq 0 \quad (7.9)$$

we have

$$V \equiv \frac{1}{2\eta \cdot p} \{ \eta \cdot x, \cdot \} \quad W \equiv -\frac{1}{2\eta \cdot p} \{ p^2, \cdot \}. \quad (7.10)$$

These operators do not commute. We have

$$[V, W] = \left\{ \frac{1}{2\eta \cdot p}, \cdot \right\}_D. \quad (7.11)$$

However, if we replace $V$ by $X$ defined by

$$X \equiv V + \left\{ \frac{\eta \cdot x}{2\eta \cdot p}, \cdot \right\}_D, \quad (7.12)$$

then $X$ and $W$ commute. This choice satisfies the condition (7.3). The extended observables are therefore for $\eta^2 > 0$

$$\tilde{x}^\mu(x, p) = e^{-\tilde{\xi}X - \tilde{\rho}W} x^\mu \left|_{\tilde{\xi} \to \phi, \tilde{\rho} \to \chi} = \tilde{x}^\mu(\tilde{x}, \tilde{p}) = \tilde{x}^\mu(\tilde{x}, \tilde{p}) = \sqrt{\eta^2 p^\mu - \eta \cdot p \eta^\mu} \left( \frac{1 - \sqrt{A}}{\eta^2 \sqrt{A}} \right), \quad A \equiv 1 - \frac{\eta^2 (p^2 - m^2)}{(\eta \cdot p)^2}, \quad (7.13)$$
where $\tilde{x}^\mu$ and $\tilde{p}^\mu$ are given in (6.12), and where $\tilde{x}^\mu$ and $\tilde{p}^\mu$ are given by

\[
\tilde{x}^\mu(x) = e^{-\xi x^\mu}_{|_{\xi \to \phi}} = x^\mu + \frac{\eta \cdot x}{\eta \cdot p} \left( \frac{1 - \sqrt{A}}{\eta^2 \sqrt{A}} \right) \left( \eta^2 p^\mu - \eta \cdot p \eta^\mu \right), \\
\tilde{p}^\mu(x) = e^{-\xi p^\mu}_{|_{\xi \to \phi}} = p^\mu - \frac{\eta^\mu \eta \cdot p}{\eta^2} \left( 1 - \sqrt{A} \right). \tag{7.14}
\]

For $\eta^2 = 0$ these expressions reduce to

\[
\tilde{x}^\mu(x, p) = \hat{x}^\mu - \tau \frac{\eta^\mu}{2(\eta \cdot p)^2} (p^2 - m^2), \\
\tilde{p}^\mu(x, p) = \hat{p}^\mu - \frac{\eta^\mu}{2\eta \cdot p} (p^2 - m^2), \tag{7.15}
\]

and

\[
\tilde{x}^\mu(x) = x^\mu - \eta^\mu \frac{\eta \cdot x}{2(\eta \cdot p)^2} (p^2 - m^2), \\
\tilde{p}^\mu(x) = p^\mu - \frac{\eta^\mu}{2\eta \cdot p} (p^2 - m^2). \tag{7.16}
\]

### 8 Quantization

We believe the existence of extended observables to be a strong indication that the given theory may be quantized in a covariant fashion. Since we know how theories with first class constraints may be quantized covariantly, our main interest is in theories with second class constraints. The question then is how the extended observables and their construction could be helpful in a covariant quantum theory. The exponential mapping to the extended observables could perhaps be used for the construction of physical symbols from the original ones (cf [6]). However, before we give any prescription for a covariant quantization of second class theories we need to understand the properties obtained so far. First it is clear that although we have succeeded to specify the properties of extended observables in details in section 3, we have not yet obtained a precise classification of all theories for which these observables actually exist. This will probably be clarified in the near future since we have obtained simple general forms of the solutions and simple conditions for their existence. From the very simple examples treated in section 4 and comparisons with corresponding properties for theories with first class constraints, it is obvious that the approach here is directly connected to the approach of splitting the second class constraints into first class ones and gauge fixing conditions [4, 5]. In principle such a splitting is always possible (a polarization). One may notice that for the simple but nontrivial example treated in subsection 4.2 we found two distinct extended observables. From the point of view of sections 5-7 it is clear that these two solutions correspond to two natural choices of gauge generators. In fact, $\theta^1$ is a gauge generator for the solution (4.21) and $\theta^2$ for (4.22). The extended observables are therefore gauge invariant from this point of view and can also be viewed as proper gauge invariant extensions of the type given in section 7. The example in 4.2 seems also to give a clue for what the $f_\alpha$-functions actually do for us. The commuting $X_\alpha$-operators for the solution (4.21) and $\theta^2$ for (4.23). The extended observables are therefore gauge invariant from this point of view and can also be viewed as proper gauge invariant extensions of the type given in section 7. The example in 4.2 seems also to give a clue for what the $f_\alpha$-functions actually do for us. The commuting $X_\alpha$-operators for the solution (4.21) and $\theta^2$ for (4.23).
same happens in the example of section 7. Thus, it seems as if the nonzero $f_\alpha$-functions replace an equal number of $\theta^\alpha$-functions in such a fashion that $X_\alpha$ will commute. These observations should play an important role in a covariant quantization.

One way to quantize theories with second class constraints covariantly in which the extended observables play a crucial role may be described as follows: Consider a theory with a Hamiltonian $H(x)$ and first and second class constraints $\phi_a$ and $\theta^\alpha$. Eliminate the second class constraints $\theta^\alpha$ by replacing $x^i$ by the extended observables $\bar{x}^i(x)$ in $H(x)$ and $\phi_a(x)$ where $\bar{x}^i(x)$ is constructed according to section 3. The theory is then transformed into an equivalent theory given by the Hamiltonian $H(\bar{x})$ and the first class constraints $\phi_a(\bar{x})$ and the gauge generators under which $\bar{x}^i(x)$ is gauge invariant.

Example: Particle on a sphere.

Consider a theory with the Hamiltonian (a free nonrelativistic particle)

$$H = \frac{p^2}{2m},$$  \hfill (8.1)

and the constraints $\theta^\alpha = 0$ where

$$\theta^1 = x^2 - R^2, \quad \theta^2 = p \cdot x,$$  \hfill (8.2)

where $R$ is a positive constant (cf subsection 4.1). The resulting theory describes a free particle on a sphere with radius $R$. The extended observables are here

$$\bar{x} = \frac{R}{\sqrt{x^2}} x, \quad \bar{p} = \frac{\sqrt{x^2}}{R} \left( p - \frac{p \cdot x}{x^2} x \right).$$  \hfill (8.3)

We may therefore eliminate $\theta^\alpha$ and consider the equivalent theory with the Hamiltonian

$$H = \frac{\bar{p}^2}{2m} = \frac{L^2}{2mR^2}, \quad L = x \times p$$  \hfill (8.4)

Quantization yields then the spectrum

$$E_l = \frac{\hbar^2 l (l + 1)}{2mR^2}.$$  \hfill (8.5)

The state space is restricted by either $\theta^1$ or $\theta^2$ as gauge generator. This procedure should be compared to the treatment of [5] in which the method of splitting the second class constraints in gauge generators and gauge fixings is used. The difference is that we here are making use of uniquely defined extended observables which precisely define the equivalent theory.

9 Summary of the results

In this paper we have obtained new results for general classical Hamiltonian theories with constraints. We have for the first time precisely defined and explicitly constructed extended observables for such theories which considerably generalizes the concept of gauge invariant extensions used in general gauge theories. (Even the properties of the latter
are further clarified.) For simplicity we have considered finite dimensional theories. (The generalization to infinite dimensional theories is in principle straightforward.) The results may be summarized as follows:

Given is a $2n$-dimensional symplectic manifold $M$ with coordinates $x^i$, $i = 1, \ldots, 2n$. Its closed, non-degenerate two-form and related Poisson bracket is defined in section 2. On this manifold we have a set of constraints. They may either be of first or of second class (or a mixture) in Dirac’s classification [1]. Consider first the case of second class constraints. We have then the constraints

$$\theta^\alpha(x) = 0, \quad \alpha = 1, \ldots, 2m < 2n,$$

satisfying the properties

$$\det C^{\alpha\beta} \big|_{\theta=0} \neq 0, \quad C^{\alpha\beta} \equiv \{\theta^\alpha, \theta^\beta\}. \quad (9.1)$$

In this case the extended observables $\bar{x}^i(x)$ are defined as follows: $\bar{x}^i(x)$ are functions on $M$ satisfying the properties

1) $\bar{x}^i(x^*) = x^{*i}$

for any solution $x^{*i}$ of $\theta^\alpha(x^*) = 0$. ($x^{*i}$ are observables.) Also the extended observables themselves are solutions of the constraints,

2) $\theta^\alpha(\bar{x}(x)) = 0, \quad \alpha = 1, \ldots, 2m. \quad (9.3)$

Furthermore, they satisfy the Poisson algebra

3) $\{\bar{x}^i(x), \bar{x}^j(x)\} = \{x^i, x^j\}_D |_{x \rightarrow \bar{x}(x)}, \quad (9.4)$

where the bracket on the left-hand side is the original Poisson bracket on $M$, while the bracket on the right-hand side is the Dirac bracket defined in (2.11). The general solutions of conditions 1)-3) we have found to be of the following form

$$\bar{x}^i(x) = e^{\xi^\alpha X_\alpha} x^i \big|_{\xi=\theta}, \quad (9.5)$$

where $\xi^\alpha$ are parameters and $X_\alpha$ vector fields of the form

$$X_\alpha = -V_\alpha + \{f_\alpha, \cdot\}_D, \quad V_\alpha = C_{\alpha\beta}\{\theta^\beta, \cdot\}. \quad (9.6)$$

$X_\alpha$ differentiates the Dirac bracket (2.11) since $V_\alpha$ also has this property as proved in the appendix. The functions $f_\alpha$ in (9.6) must be chosen such that

$$V_\alpha \bar{x}^i(x) C^{\alpha\beta} V_\beta \bar{x}^j(x) = 0 \quad (9.7)$$

is satisfied. We believe that this requires $f_\alpha$ to be chosen such that $X_\alpha$ in (9.6) commute and at the same time satisfy

$$\{f_\alpha, \bar{x}^i\}_D C^{\alpha\beta}\{f_\beta, \bar{x}^j\}_D = 0. \quad (9.8)$$

However, the last two conditions seem to be stronger than (9.7). In order for $X_\alpha$ to commute $f_\alpha$ must be chosen such that $K_{\alpha\beta}$ defined by

$$K_{\alpha\beta} = C_{\alpha\beta} + \{f_\alpha, f_\beta\}_D + C_{\beta\gamma}\{\theta^\gamma, f_\alpha\} - C_{\alpha\gamma}\{\theta^\gamma, f_\beta\} \quad (9.9)$$

would
at most depends on the constraint variables $\theta^\alpha$ apart from constants (see (3.38)).

For first class constraints $\phi_a(x) = 0$, $a = 1, \ldots, m < n$ we may define extended observables $\tilde{x}^i(x)$ analogously. $\tilde{x}^i(x)$ satisfies

1) $\tilde{x}^i(x^*) = x^i$ (9.10)

for any solution $x^i$ of $\phi_a(x^*) = 0$, and $\tilde{x}^i(x)$ are themselves solutions,

2) $\phi_a(\tilde{x}^i(x)) = 0, \ a = 1, \ldots, m$. (9.11)

Furthermore, we have the relation

3) $\{\tilde{x}^i(x), \tilde{x}^j(x)\} = \{x^i, x^j\}_D|_{x^* \rightarrow \tilde{x}(x)}$, (9.12)

where the Dirac bracket is defined in terms of $\phi_a$ and a set of gauge fixing variables $\chi^a(x)$ satisfying

$$\chi^a(x), \chi^b(x) = 0, \ \det\{\chi^a(x), \phi_b(x)\}|_{\phi=0} \neq 0.$$ (9.13)

The solution is

$$\tilde{x}^i(x) = e^{-\xi^a V^a} x^i \big|_{\xi \rightarrow \phi},$$ (9.14)

where

$$V^a = (M^{-1})^a_b(x)\{\chi^b, \cdot\}, \quad M^a_b(x) \equiv \{\chi^a(x), \phi_b(x)\}.$$ (9.15)

These $V^a$-operators commute due to (9.13). The extended observables (9.14) are not what one normally would call observables in a general gauge theory since they are not gauge invariant. In section 6 we defined gauge invariant observables $\hat{x}^i(x)$ along the lines what has been considered before (see e.g. [2, 3]). They satisfy $\chi^a(\hat{x}) = 0$ instead of 1) in (9.10) and a weak form of 3) in (9.12). If we actually make use of $\chi^a$ as gauge fixing, i.e. view $\phi_a(x) = 0$ and $\chi^a(x) = 0$ as second class constraints then we may construct the corresponding extended observables according to the rules for second class constraints. We find then the solution (9.5) with $\theta^\alpha = (\phi_a, \chi^a)$. Interestingly enough this solution is also gauge invariant and it could be called the proper gauge invariant extension. The method to construct extended observables for second class constraints seems therefore to be appropriate also for first class constraints.

In section 8 we discussed the possibility to perform a covariant quantization of theories with second class constraints using the new insights of the present paper. First we noticed that the results so far seem to indicate a connection to the method of splitting the second class constraints into first class ones and gauge fixing conditions (see [4, 5]). However, the extended observables by themselves also provide for a simple algorithm how to map the constrained theory to an equivalent unconstrained one. This mapping procedure was exemplified for a particle on a sphere and led to a very simple quantization of this system. Whether or not such a procedure eventually may compete with the standard conversion method [6] in which the second class constraints are converted to first class ones by means of additional variables are left for the future to decide.

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A Proof of Leibniz’ rule \((2.14)\) for \(V_\alpha\)

Let us prove \((2.21)\) backwards:

\[
\{V_\alpha f, g\} + \{f, V_\alpha g\} = \left\{\{f, \theta^\beta\}C_{\beta\alpha}, g\right\} + \left\{f, \{g, \theta^\beta\}C_{\beta\alpha}\right\} = \left\{\{f, \theta^\gamma\}, \{g, \theta^\beta\}\right\}C_{\beta\alpha} + + \left\{f, \theta^\gamma\right\}\{C_{\gamma\alpha}, g\} + \{C_{\lambda\alpha}, f\}\{\theta^\lambda, g\} - \left\{\{f, \theta^\gamma\}, \theta^\gamma\right\}C_{\beta\alpha}C_{\gamma\lambda}\{\theta^\lambda, g\} - \left\{f, \{g, \theta^\beta\}\right\}C_{\beta\alpha} \left\{\theta^\gamma\right\}C_{\alpha\lambda}\{\theta^\beta, g\} - \left\{f, \theta^\gamma\right\}\{C_{\gamma\alpha}, \theta^\beta\}C_{\beta\lambda}\{\theta^\lambda, g\} - \left\{f, \theta^\gamma\right\}\{\theta^\beta, C_{\lambda\alpha}\}C_{\gamma\beta}\{g, \theta^\lambda\}. \tag{A.1}\]

Inserting

\[
\{C_{\gamma\alpha}, g\} = -C_{\gamma\lambda}\{\theta^\beta, g\}C_{\beta\alpha} = |\text{Jac. id.}| = C_{\gamma\lambda}\{\theta^\beta, g\}C_{\beta\alpha} + C_{\gamma\lambda}\{g, \theta^\lambda\}\theta^\betaC_{\beta\alpha}, \tag{A.2}\]

and the corresponding expression for \(\{C_{\lambda\alpha}, f\}\) into \((A.1)\) we find

\[
\{V_\alpha f, g\} + \{f, V_\alpha g\} = \left\{\{f, \theta^\gamma\}, \theta^\gamma\right\}C_{\beta\alpha}C_{\gamma\lambda}\{\theta^\lambda, g\} - \left\{f, \{g, \theta^\beta\}\right\}C_{\beta\alpha} \left\{\theta^\gamma\right\}C_{\alpha\lambda}\{\theta^\beta, g\} - \left\{f, \theta^\gamma\right\}\{C_{\gamma\alpha}, \theta^\beta\}C_{\beta\lambda}\{\theta^\lambda, g\} - \left\{f, \theta^\gamma\right\}\{\theta^\beta, C_{\lambda\alpha}\}C_{\gamma\beta}\{g, \theta^\lambda\}. \tag{A.3}\]

where we also have made use of a Jacobi identity for the first terms. Now we have

\[
\{C_{\gamma\alpha}, \theta^\beta\}C_{\beta\lambda} + cycle(\gamma, \alpha, \lambda) = 0, \tag{A.4}\]

which is easily proved by means of Jacobi identities. \((A.4)\) in \((A.3)\) yields then

\[
\{V_\alpha f, g\} + \{f, V_\alpha g\} = \left\{\{f, \theta^\gamma\}, \theta^\gamma\right\}C_{\beta\alpha}C_{\gamma\lambda}\{\theta^\lambda, g\} - \left\{f, \{g, \theta^\beta\}\right\}C_{\beta\alpha} \left\{\theta^\gamma\right\}C_{\alpha\lambda}\{\theta^\beta, g\} - \left\{f, \theta^\gamma\right\}\{C_{\gamma\alpha}, \theta^\beta\}C_{\beta\lambda}\{\theta^\lambda, g\} - \left\{f, \theta^\gamma\right\}\{\theta^\beta, C_{\lambda\alpha}\}C_{\gamma\beta}\{g, \theta^\lambda\}. \tag{A.5}\]

References

[1] P.A.M. Dirac, \textit{Lectures on Quantum Mechanics}, Belfer Graduate School of Science, Yeshiva University, New York, (1964)

[2] R. Marnelius, \textit{Nuclear Physics}, \textbf{B104}, 477 (1976)

[3] R. Marnelius, \textit{Acta Physica Polonica}, \textbf{B13}, 669 (1982)

[4] L. Faddeev and S. Shatashvili, \textit{Phys. Lett.} \textbf{167B}, 225 (1986)

[5] K. Harada and H. Mukaida, \textit{Z. Phys.} \textbf{C48}, 151 (1990)

[6] I.A. Batalin and O.V. Ogievetsky, \textit{Nuovo Cimento}, \textbf{90B}, 29 (1985)

[7] M.V. Karasev and V.P. Maslov, \textit{Nonlinear Poisson brackets: geometry and quantization}, Providence, Rhode Island: American Math. Soc. (1993)
[8] I. A. Batalin and E. S. Fradkin, *Nucl. Phys.* **B279**, 514 (1987);
*Phys. Lett.* **B180**, 157 (1986); *ibid* **B236**, 528 (1990);
I. A. Batalin, E. S. Fradkin, and T. E. Fradkina, *Nucl. Phys.* **B314**, 158 (1989); *ibid* **B323**, 734 (1989); *ibid* **B332**, 723 (1990)
I. A. Batalin and I. V. Tyutin, *Int. J. Mod. Phys.* **A6**, 3599 (1991)