ON THE ENUMERATION OF FANO BOTT MANIFOLDS

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Abstract. Fano Bott manifolds bijectively correspond to signed rooted forests with some equivalence relation. Using this bijective correspondence, we enumerate the isomorphism classes of Fano Bott manifolds and the diffeomorphism classes of indecomposable Fano Bott manifolds. We also observe that the signed rooted forests with the equivalence relation bijectively correspond to rooted triangular cacti.

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1. Introduction

A Bott manifold of complex dimension $n$ is a smooth projective toric variety whose fan is the normal fan of a polytope combinatorially equivalent to the cube $[0, 1]^n$. A family of Bott manifolds was first considered by Grossberg and Karshon [8] in the context of toric degenerations of Bott–Samelson varieties. Since then, topological or geometric properties of Bott manifolds have been intensively studied in [7, 5, 2, 14]. Recently, motivated by Suyama’s work [16], we showed the $c_1$-cohomological rigidity for Fano Bott manifolds, which means that two Fano Bott manifolds are isomorphic if and only if there is a cohomology ring isomorphism between them preserving their first Chern classes [4].

It is known that there are only finitely many smooth Fano toric varieties up to isomorphism in each dimension (cf. [15]), and therefore there are also only finitely many Fano Bott manifolds up to isomorphism in each dimension. Higashitani and Kurimoto [11] associate signed rooted forests with Fano Bott manifolds to classify Fano Bott manifolds up to diffeomorphism. In this paper, we enumerate the isomorphism classes of Fano Bott manifolds and the diffeomorphism classes of indecomposable Fano Bott manifolds using this correspondence.

To introduce our main result, we prepare some terminologies. Recall that a fan associated to a Bott manifold of complex dimension $n$ is the normal fan of a polytope combinatorially equivalent to $[0, 1]^n$ and so it has $2n$ rays. We denote the primitive ray generators by $v_1, \ldots, v_n, w_1, \ldots, w_n$, where $v_i$ and $w_i$ are pairwise normal vectors of opposite facets.

If a Bott manifold is Fano, then it is known that the sum $v_i + w_i$ is either the zero vector or another ray generator, say $v_{\varphi(i)}$ or $w_{\varphi(i)}$, where $\varphi$ is a permutation on $[n]$. Accordingly, one may associate a signed rooted forest with a Fano Bott manifold as follows: the vertices are $[n] = \{1, \ldots, n\}$, the parent of $i$ is $\varphi(i)$, and the edge $\{i, \varphi(i)\}$ is signed by $+$ if $v_i + w_i = v_{\varphi(i)}$; and by $-$ otherwise. Indeed, whenever the sum $v_i + w_i$ is the zero vector, the vertex $i$ is a root (see Section 2 for more precise definition).

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For each vertex \( i \) of a signed rooted forest, we obtain another signed rooted forest by changing the signs of all the edges connecting \( i \) and its children simultaneously. By considering this operation for all vertices, we obtain an equivalence relation \( \sim \) on the isomorphism classes \( SF_n \) of signed rooted forests with vertices \( [n] \). It is observed in [11, Remark 5.8] that the isomorphism classes in Fano Bott manifolds of complex dimension \( n \) bijectively correspond to the equivalence classes \( SF_n/\sim \) of signed rooted forests with \( n \) vertices. Now we state our main theorem.

**Theorem 1.1** (Corollary 4.4). The generating function \( \mathcal{F}(x) = \sum_{n=1}^\infty |SF_n/\sim |x^n \) satisfies

\[
\mathcal{F}(x) = \exp \left( \sum_{k=1}^\infty \frac{x^k}{2k} \left( \mathcal{F}(x^{2k}) + \mathcal{F}(x^k)^2 \right) \right).
\]

This functional equation determines \( \mathcal{F}(x) \). Indeed, a straightforward computation shows

\[
\mathcal{F}(x) = 1 + x + 2x^2 + 5x^3 + 13x^4 + 37x^5 + 111x^6 + 345x^7 + 1105x^8 + 3624x^9 + \cdots
\]

The generating function \( \Delta(x) = 1 + \sum_{n=1}^\infty \Delta_n x^n \) for the number \( \Delta_n \) of rooted triangular cacti with \( 2n+1 \) vertices and \( n \) triangles satisfies the same functional equation ([9], [10]). It turns out that there is a bijective correspondence between \( SF_n/\sim \) and rooted triangular cacti with \( 2n+1 \) vertices and \( n \) triangles.

The result [11] by Higashitani and Kurimoto implies that the diffeomorphism classes of indecomposable Fano Bott manifolds of complex dimension \( n \) bijectively correspond to \( SF_{n-1}/\sim \). Here, we say a Fano Bott manifold is *indecomposable* if it is not isomorphic to a product of lower dimensional Fano Bott manifolds. This provides an enumeration of the diffeomorphism classes of indecomposable Fano Bott manifolds.

This paper is organized as follows. In Section 2, we provide the definition of Bott manifolds and their Fano conditions. Moreover, we recall the association of signed rooted forests with Fano Bott manifolds. In Section 3, we show that the association induces a bijection between the isomorphism classes in Fano Bott manifolds and the equivalence classes \( SF_n/\sim \) of signed rooted forests with \( n \) vertices. In Section 4, we enumerate the equivalence classes \( SF_n/\sim \) of signed rooted forests. In Section 5, we give a bijective correspondence between \( SF_n/\sim \) and rooted triangular cacti with \( 2n+1 \) vertices and \( n \) triangles.

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2. **Fano Bott manifolds and signed rooted forests**

In this section we review the definition of Bott manifolds and their fans. We also recall the relation between Fano Bott manifolds and signed rooted forests from [11].

**Definition 2.1** ([8, §2.1]). A Bott tower \( \mathcal{B}_* \) is an iterated \( CP^1 \)-bundle starting with a point:

\[
\begin{array}{cccc}
\mathcal{B}_n & \longrightarrow & \mathcal{B}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{B}_0, \\
\bigg| & & & & & & \bigg| & & \bigg| \\
P(\mathbb{C} \oplus \xi_n) & & & & & & \mathbb{C}P^1 & & \{ \text{a point} \}
\end{array}
\]

where each \( \mathcal{B}_i \) is the complex projectivization of the Whitney sum of a holomorphic line bundle \( \xi_i \) and the trivial line bundle \( \mathbb{C} \) over \( \mathcal{B}_{i-1} \). The total space \( \mathcal{B}_n \) is called a *Bott manifold.*
A Bott manifold $B_n$ is a smooth projective toric variety by the construction. Its fan $\Sigma$ has $2n$ rays. We denote by $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ the ray generators, where a pair of $v_i$ and $w_i$ does not span a cone for each $i$. A subset $S$ of ray generators having $n$ elements form a maximal cone of $\Sigma$ if and only if
\[ \{v_i, w_i\} \not\subset S \quad \text{for any } i \in [n]. \]
Because of this description, one may see that the fan $\Sigma$ is the normal fan of a polytope combinatorially equivalent to the cube $[0,1]^n$.

For a fan $\Sigma$ and its ray generators $\{u_\rho \mid \rho \in \Sigma(1)\}$ in $\Sigma$, we call a subset $P \subset \{u_\rho \mid \rho \in \Sigma(1)\}$ a primitive collection if
\[ \text{Cone}(P) \not\subset \Sigma \quad \text{but} \quad \text{Cone}(P \setminus \{x\}) \in \Sigma \quad \text{for every } x \in P. \]
We denote by $PC(\Sigma)$ the set of primitive collections of $\Sigma$. We briefly review Batyrev’s criterion [1, Proposition 2.3.6], which determines whether a given toric variety is Fano or not. Let $\Sigma$ be a smooth complete fan. For each primitive collection $P = \{u_1, \ldots, u_r\}$, there exists a unique cone $\sigma$ such that $u_1 + \cdots + u_r$ is in the relative interior of $\sigma$. Let $v_1, \ldots, v_\ell$ be the primitive generators of $\sigma$. Then
\[ u_1 + \cdots + u_r = a_1v_1 + \cdots + a_\ell v_\ell \]
for some positive integers $a_1, \ldots, a_\ell$. If the sum of primitive generators is the zero vector, then the cone $\sigma$ is of zero-dimensional and the set $\{v_1, \ldots, v_\ell\}$ is assumed to be empty. We call this relation the primitive relation for $P$ and we define the degree of $P$ by
\[ \text{deg}(P) := r - (a_1 + \cdots + a_\ell). \]
Here, we note that if the sum of primitive generators of $P$ is the zero vector, then $\text{deg}(P) = r$.

**Proposition 2.2** ([1, Proposition 2.3.6]). Let $X_\Sigma$ be a nonsingular projective toric variety and $PC(\Sigma)$ be the primitive collection of the fan $\Sigma$. Then the toric variety $X_\Sigma$ is Fano if and only if $\text{deg}(P) > 0$ for every $P \in PC(\Sigma)$.

Now we apply Batyrev’s criterion to Bott manifolds. Let $\Sigma$ be the fan of a Bott manifold $B_n$. Then the set of primitive collection is
\[ (2.1) \quad PC(\Sigma) = \{\{v_i, w_i\} \mid i \in [n]\}. \]
Using Proposition 2.2, we can see that $B_n$ is Fano if and only if each primitive collection $P = \{v_i, w_i\}$ satisfies one of the following:

1. $v_i + w_i = 0$ (that is, $\text{deg}(P) = 2 > 0$);
2. $v_i + w_i = v_{\varphi(i)}$ (that is, $\text{deg}(P) = 2 - 1 = 1 > 0$); or
3. $v_i + w_i = w_{\varphi(i)}$ (that is, $\text{deg}(P) = 2 - 1 = 1 > 0$).

Here, $\varphi : [n] \setminus Z \to [n]$, where $Z := \{i \mid v_i + w_i = 0\}$. We also define a sign map $\sigma : [n] \setminus Z \to \{+, -\}$ by
\[ \sigma(i) = \begin{cases} + & \text{if } v_i + w_i = v_{\varphi(i)}, \\ - & \text{if } v_i + w_i = w_{\varphi(i)}. \end{cases} \]
This leads us to the following definition.

**Definition 2.3** ([11, Definition 4.1]). Let $\Sigma$ be the fan of a Fano Bott manifold having (ordered) ray generators $S = (v_1, \ldots, v_n, w_1, \ldots, w_n)$ with the primitive collections as in (2.1). Let $\varphi$ and $\sigma$ be as above. We define the associated signed rooted forest $(T, s) = (T(\Sigma, S), s(\Sigma, S))$ (i.e., rooted forest $T$ with the sign map $s : E(T) \to \{+, -\}$) to be

- $V(T) = [n]$;
- $E(T) = \{i, \varphi(i)\} \mid i \in [n] \setminus Z$ and $s(i, \varphi(i)) = \sigma(i)$.

From the definition, one can see that for a signed rooted forest $(T, s)$, the set of roots is $Z$ and the parent of each vertex $i \in [n] \setminus Z$ is $\varphi(i)$. We denote the assignment provided in Definition 2.3 by $\Phi$, that is, $\Phi(\Sigma, S) = (T(\Sigma, S), s(\Sigma, S))$ is the associated signed rooted forest.

**Remark 2.4.** The association $\Phi$ is surjective, that is, for each signed rooted forest $(T, s)$ with $n$ vertices, there exists a Fano Bott manifold of dimension $n$ whose fan defines $(T, s)$. 
Example 2.5. We will present ray generators of the fan of a Bott manifold using a matrix, i.e., the columns of an $n \times 2n$ matrix are ray generators. Consider the following two matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$ 

Let $\mathcal{B}$ be the Bott manifold such that the ray generators $(v_1, v_2, v_3, w_1, w_2, w_3)$ of the fan are the column vectors of $A$. Then, we have

$$v_1 + w_1 = v_2,$$
$$v_2 + w_2 = 0,$$
$$v_3 + w_3 = w_2.$$ 

Therefore, the Bott manifold $\mathcal{B}$ is Fano, and moreover, $\varphi(1) = 2, \varphi(3) = 2$, and $\sigma(1) = +, \sigma(3) = -$. The associated signed rooted tree is given in Figure 1 (without vertex labeling).

Let $\mathcal{B}'$ be the Bott manifold such that ray generators $(v_1, v_2, v_3, w_1, w_2, w_3)$ of the fan are the column vectors of $A'$. Consider a primitive collection $P = \{v_1, w_1\}$. The sum of ray generators is

$$v_1 + w_1 = v_2 + v_3,$$

so $\deg(P) = 2 - 2 = 0 \neq 0$ and this primitive collection does not satisfy the Fano condition. Therefore, the Bott manifold $\mathcal{B}'$ is not Fano.

We provide all signed rooted forests having three vertices in Figure 1.

![Signed rooted forests with 3 vertices](https://example.com/signs.png)

Figure 1. Signed rooted forests with 3 vertices.

Remark 2.6. We say that a signed rooted forest is binary if each vertex has at most two children and when the vertex has two children, the edges connecting the vertex and its children have different signs. In Figure 1, all but (5) and (7) are binary. Binary rooted forests appear in connection with a certain family of Fano toric Richardson varieties (called of Catalan type) in the full flag variety ([12]).

3. Classification of Fano Bott manifolds

We say that signed rooted forests $(T, s)$ and $(T', s')$ with vertices $[n]$ are isomorphic if there is a permutation $\pi \in \mathfrak{S}_n$ which sends the roots of $T$ to the roots of $T'$ and induces a bijection between the edges preserving the signs. Let $\mathcal{SF}_n$ be the isomorphism classes of signed rooted forests with vertices $[n]$. For each vertex $i \in [n]$, we define an operation

$$r_i : \mathcal{SF}_n \to \mathcal{SF}_n$$

which changes the signs of all edges connecting the vertex $i$ and its children simultaneously. Denote by $\sim$ the equivalence relation on $\mathcal{SF}_n$ generated by the operations $r_i$ for all $i \in [n]$. The following is mentioned in [11, Remark 5.8], but we include its proof for readers' convenience.

Theorem 3.1 (cf. [11, Remark 5.8]). The isomorphism classes in Fano Bott manifolds of complex dimension $n$ bijectively correspond to $\mathcal{SF}_n/\sim$. 


Proof. Let $\mathcal{B}$ be a Fano Bott manifold of complex dimension $n$, $\Sigma = \Sigma_\mathcal{B}$ a fan defining $\mathcal{B}$. We fix an ordering on ray generators $\mathcal{S} = (v_1, \ldots, v_n, w_1, \ldots, w_n)$ of $\Sigma$. For $A \in \text{GL}(n, \mathbb{Z})$, we denote by $A \cdot \Sigma$ the fan consisting of cones $A \cdot \sigma$’s for $\sigma \in \Sigma$ and denote by $A \cdot \mathcal{S}$ the ordered ray generators of $A \cdot \Sigma$ given by

$$A \cdot \mathcal{S} := (Av_1, \ldots, Av_n, Aw_1, \ldots, Aw_n).$$

By [4, Proposition 3.4], any other pair $(\Sigma', \mathcal{S}')$ defines a Fano Bott manifold isomorphic to $\mathcal{B}$ if and only if $\Sigma' = A \cdot \Sigma$ for some $A \in \text{GL}(n, \mathbb{Z})$ and the set $\mathcal{S}'$ is obtained from $A \cdot \mathcal{S}$ by performing the following two operations on $A \cdot \mathcal{S}$:

(1) swapping $A v_i$ with $A w_i$, that is, $S'_i := (Av_1, \ldots, Av_{i-1}, Aw_i, Av_{i+1}, \ldots, Av_n, Aw_{i-1}, Av_i, Aw_{i+1}, \ldots, Aw_n)$;

(2) reordering $A v_i$ (as well as $A w_i$’s), that is, for a permutation $\pi \in S_n$, $S'_\pi := (Av_{\pi(1)}, \ldots, Av_{\pi(n)}, Aw_{\pi(1)}, \ldots, Aw_{\pi(n)})$.

For the ordered ray generators $S'_i$ obtained by applying (Op.1), we have $\Phi(\Sigma', S'_i) = r_i(\Phi(\Sigma_\mathcal{B}, \mathcal{S}))$.

For the ordered ray generators $S'_\pi$ obtained by applying (Op.2), $\Phi(\Sigma', S'_\pi)$ is obtained from $\Phi(\Sigma_\mathcal{B}, \mathcal{S})$ by changing the numbering of the vertices by the permutation $\pi$, so they are isomorphic as signed rooted forests. This finishes the proof.

Example 3.2. Consider $\mathcal{S}F_3$ described in Figure 1. Then we obtain the five equivalence classes

$$(T_1, s_1) \sim (T_2, s_2) \sim (T_3, s_3) \sim (T_4, s_4),$$

$$(T_5, s_5), \quad (T_6, s_6), \quad (T_7, s_7), \quad (T_8, s_8), \quad (T_9, s_9), \quad (T_{10}, s_{10}).$$

All signed rooted forests in $\mathcal{S}F_4$ are illustrated in Figure 2. Roots of the forests are the top vertices. We omit plus signs on edges and put a minus sign on an edge. We also write ID numbers of the corresponding Fano Bott manifolds according to the list of ‘Smooth toric Fano varieties’ [15] in the Graded Ring Database [3].

![Figure 2](image-url)

**Figure 2.** Representatives of $\mathcal{S}F_4/\sim$. Numbers are ID’s by Øbro.

Higashitani and Kurimoto [11] provide another equivalence relation $\approx$ on the set of signed rooted forests which is used to consider the diffeomorphism classes in Fano Bott manifolds. The equivalence relation $\approx$ is induced from the relation $\sim$ by neglecting signs on the edges incident on the roots. Using this relation, we recall the following.
Theorem 3.3 ([11, Theorem 1.8 and Remark 6.4]). The diffeomorphism classes in Fano Bott manifolds of complex dimension \( n \) bijectively correspond to \( SF_n/\sim \).

We say that a Bott manifold \( B \) is indecomposable if it is not isomorphic to a product of lower dimensional Bott manifolds (as toric varieties). Otherwise, we say that \( B \) is decomposable.\(^1\)

Corollary 3.4. The diffeomorphism classes of indecomposable Fano Bott manifolds of complex dimension \( n \) bijectively correspond to \( SF_{n-1}/\sim \).

Proof. We first notice that by Theorem 3.1, a Fano Bott manifold is indecomposable if and only if the corresponding signed rooted forest is a signed rooted tree, that is, it has only one root vertex. Since the equivalence relation \( \approx \) is induced from the relation \( \sim \) by neglecting the signs on the edges incident on the root, we obtain the desired bijection by erasing the root vertex. \( \Box \)

Example 3.5. In Figure 2, three pairs ((5),(6)), ((8),(9)), ((12),(13)) are diffeomorphic to each other but (10) and (11) are not diffeomorphic to each other. Considering signed rooted trees in Figure 2, we obtain the five equivalence classes

\[ \{ (7), (8), (9), (10), (11), (12), (13) \}/\sim = \{ [(7)], [(8)], [(10)], [(11)], [(12)] \}. \]

By erasing the root vertex, each of which is associated to an element in \( SF_3/\sim \).

\[ [(7)] \leftrightarrow [(T_1, s_1)], \quad [(8)] \leftrightarrow [(T_8, s_8)], \quad [(10)] \leftrightarrow [(T_5, s_5)], \quad [(11)] \leftrightarrow [(T_6, s_6)], \quad [(12)] \leftrightarrow [(T_{10}, s_{10})]. \]

4. Counting signed rooted forests in terms of signed rooted trees

We denote by \( SF_n/\sim \) the set of signed rooted trees in \( SF_n/\sim \). We set \( t_n = |SF_n/\sim| \) and \( f_n = |SF_n/\sim| \). Now we let \( T(x) \) and \( F(x) \) be the generating functions of the sequences \( \{t_n\} \) and \( \{f_n\} \), respectively, that is,

\[ T(x) = \sum_{n=1}^{\infty} t_n x^n \quad \text{and} \quad F(x) = 1 + \sum_{n=1}^{\infty} f_n x^n. \]

In this section, we compute the generating functions \( T(x) \) and \( F(x) \), and study their relations.

Proposition 4.1. The generating function \( T(x) \) satisfies

\[ T(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-t_k}. \]

Proof. Note that from the generalized binomial theorem, for any positive integer \( m \), we have

\[ (1 - x)^{-m} = \sum_{p=0}^{\infty} \binom{-m}{p} (-x)^p = \sum_{p=0}^{\infty} \frac{(-m)(-m-1)\cdots(-m-(p-1))}{p!} (-x)^p = \sum_{p=0}^{\infty} \binom{m-1+p}{p} x^p. \]

Then, since

\[ (1 - x^k)^{-t_k} = \sum_{p_k=0}^{\infty} \binom{t_k - 1 + p_k}{p_k} x^{kp_k}, \]

\( \Box \)

The notion of indecomposability can have different meanings in different contexts. Especially, in [6], they consider another family of smooth manifolds, called real Bott manifolds, and say that a real Bott manifold is indecomposable if it is not diffeomorphic to a product of lower dimensional real Bott manifolds.
the coefficient of $x^n$ in the product $\prod_{k=1}^{\infty} (1 - x^k)^{-t_k}$ is given by

\[(4.2) \sum \prod_{(p_1, \ldots, p_n) k=1}^{n} \frac{(t_k - 1 + p_k)}{p_k},\]

where $(p_1, \ldots, p_n)$ runs over all $n$-tuples of nonnegative integers with $\sum_{k=1}^{n} kp_k = n$. Here $(t_k - 1 + p_k)$ is the number of signed rooted trees with $p_k$ components such that each component is a signed rooted tree with $k$ vertices by (4.1) and the sum in (4.2) counts all decompositions of elements in $\mathcal{SF}_n/\sim$ into connected components, so the proposition follows. \qed

The following is an easy consequence of the above proposition.

**Corollary 4.2.** The generating functions $\mathcal{F}(x)$ and $\mathcal{T}(x)$ satisfy

$$\mathcal{F}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{\mathcal{T}(x^n)}{n} \right).$$

**Proof.** Taking logarithm on both sides of

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-t_k},$$

we obtain

$$\log \mathcal{F}(x) = - \sum_{k=1}^{\infty} t_k \log(1 - x^k) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t_k x^{kn}}{n} = \sum_{n=1}^{\infty} \frac{\mathcal{T}(x^n)}{n},$$

which implies the corollary. \qed

**Lemma 4.3** (cf. (1) in [10]). The generating functions $\mathcal{F}(x)$ and $\mathcal{T}(x)$ satisfy

$$\mathcal{T}(x) = \frac{x}{2} (\mathcal{F}(x^2) + \mathcal{T}(x)^2).$$

**Proof.** We set

$$\mathcal{SF}/\sim = \bigsqcup_{n=0}^{\infty} \mathcal{SF}_n/\sim \text{ and } \mathcal{SF}/\sim = \bigsqcup_{n=1}^{\infty} \mathcal{SF}_n/\sim,$$

where $\mathcal{SF}_0/\sim$ is understood to be the empty set. Given an unordered pair \{A, B\} of $\mathcal{SF}/\sim$, we obtain an element $AB$ of $\mathcal{SF}/\sim$ by joining the roots of A and B to a new root $v$ and assign all the new edges joining the roots of A to $v$, say + sign, and all the new edges joining the roots of B to $v$, say - sign. We may assign - sign to the former and + sign to the latter. In any case, $AB$ is well-defined in $\mathcal{SF}/\sim$. Conversely, given an element $T$ of $\mathcal{SF}/\sim$, there is a unique unordered pair \{A, B\} of $\mathcal{SF}/\sim$ such that $AB = T$.

This implies the lemma. Indeed, $\mathcal{T}(x)^2$ counts unordered pairs \{A, B\} twice when $A$ and $B$ are different but once when $A = B$. This is why we add $\mathcal{T}(x^2)$ in the formula. Multiplication by $x$ corresponds to the new vertex $v$. \qed

Combining Corollary 4.2 and Lemma 4.3, we obtain the following functional equation mentioned in the introduction.

**Corollary 4.4** (cf. (3) in [10]). The generating function $\mathcal{F}(x)$ satisfies

$$\mathcal{F}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{2n} (\mathcal{F}(x^{2n}) + \mathcal{T}(x^n)^2) \right).$$

This functional equation determines $\mathcal{F}(x)$. Using Lemma 4.3 and Corollary 4.4, we obtain the following table.

The numbers $t_n$ and $f_n$ in Table 1 satisfy $f_n < 2t_n < 4f_{n-1}$ for $n \leq 10$ and the sequences $\{t_n/t_{n-1}\}_{n=2}^{10}$ and $\{f_n/f_{n-1}\}_{n=2}^{10}$ are both increasing.

**Question 4.5.** Are the sequences $t_n/t_{n-1}$ and $f_n/f_{n-1}$ increasing and bounded above by 4?
Table 1. The numbers of signed rooted trees and signed rooted forests

| $n$ | $t_n$ | $f_n$ |
|-----|-------|-------|
| 1   | 1     | 1     |
| 2   | 1     | 2     |
| 3   | 3     | 5     |
| 4   | 7     | 13    |
| 5   | 21    | 37    |
| 6   | 60    | 111   |
| 7   | 189   | 345   |
| 8   | 595   | 1105  |
| 9   | 1948  | 3624  |
| 10  | 6455  | 12099 |

The formula in Corollary 4.4 also holds for the generating function of the number of rooted triangular cacti with $2n+1$ vertices and $n$ triangles. In this section, we give a bijective correspondence between the equivalence classes of signed rooted forests $SF_n/\sim$ and rooted triangular cacti with $2n+1$ vertices and $n$ triangles.

**Definition 5.1.** A cactus (or a cactus tree) is a connected graph in which any two simple cycles have at most one vertex in common, equivalently, no line lies on more than one cycle. A triangular cactus (or a 3-cactus) is a cactus such that every cycle has length three. A rooted triangular cactus is a triangular cactus having a root vertex.

We sometimes call a 3-cycle in a 3-cactus a triangle. In Figure 3, we present rooted 3-cacti having nine vertices and four triangles. The sequence of numbers of rooted 3-cacti with $2n+1$ vertices and $n$ triangles is Sequence A003080 in [13].

![Figure 3. Rooted triangular cacti](image)

**Remark 5.2.** A cactus is also called a Husimi tree (see [9, 10]).

**Proposition 5.3.** There is a bijective correspondence between $SF_n/\sim$ and the set of rooted 3-cacti with $2n+1$ vertices and $n$ triangles.

**Proof.** Let $(T, s)$ be a signed rooted forest. For each root vertex of $(T, s)$, we draw a triangle and decorate the top vertex of the triangle with a double circle to indicate the root of the triangular cacti. For each child of the root of $T$, we draw a triangle as follows. If the sign of the edge incident on the root is positive, we attach the new triangle to the left bottom vertex; if the sign is negative, we attach the new triangle to the right bottom vertex. Continuing this process to each child vertex, we get a bunch of rooted triangular cacti. Finally, we merge all the root vertices of
rooted triangular cacti to one root vertex so we obtain one rooted triangular cacti. See Figure 4. Obviously, the rooted triangular cacti corresponding to $\langle T, s \rangle$ and $r_i \langle T, s \rangle$ are isomorphic to each other. This proves the proposition. \hfill \qed

Figure 4. Construction of triangular cacti from signed rooted forests

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