Slowly rotating Curzon-Chazy Metric

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Abstract

A new rotation version of the Curzon-Chazy metric is found. This new metric was obtained by means of a perturbation method, in order to include slow rotation. The solution is then proved to fulfill the Einstein field equations using a REDUCE program. Furthermore, the applications of this new solution are discussed.

1 Introduction

The Curzon-Chazy metric [5, 6, 20] is one of the simplest solutions of the Einstein field equations (EFE) for the Weyl metric. The original idea of Curzon [6] and Chazy [5] was the superposition of two particles at different points on the symmetry axis. This superposition exhibit a singularity between these particles along this symmetry axis. This singularity is interpreted as a strut (Weyl strut), which stress holds these particles apart and does not exert a gravitational field [2, 10].

For one particle, the Curzon-Chazy metric describes the exterior field of a finite source [10], and has a spherically symmetric Newtonian potential of a point particle located at \( r = 0 \). The resulting spacetime is not spherically symmetric and its weak limit is that of an object located at the origin with multipoles. Moreover, the singularity at \( r = 0 \) has a very interesting but

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complicated directionally dependent structure [9, 10]. There is a curvature singularity at \( \rho = 0, z = 0 \) that is not surrounded by a horizon, i.e. it is naked. Every light ray emitted from it becomes infinitely redshifted, so that it is effectively invisible [10]. Studying the principal null directions, it was found that this spacetime has an invariantly hypersurface \( \sqrt{\rho^2 + z^2} = M \), on which the Weyl invariant \( J \) (determinant of the Weyl five complex scalar functions) vanishes [1, 10]. Furthermore, this metric is Petrov type \( D \), except at two points \( (z = \pm M) \) that intersect the axis \( \rho = 0 \), where it is of Petrov type \( O \) [1].

The properties of this solution have been analyzed since its discovery such as the potential surfaces for its time-like geodesics and their variations with the change in energy [8]. A generalization of the metric to the Einstein-Maxwell equations have also been obtained [13]. A modified Curzon-Chazy metric, using a rotating reference frame as approach, has already been applied to study binary pulsar systems [21].

A slowly rotating version of the Curzon-Chazy solution could be used, for instance, to study the gravitational lens effect, because of its asymmetrical nature. Furthermore, it is important to mention Halilsoy’s research [11], in his work he obtained a rotating Curzon metric using the Ernst potential method [7], however the rotational metric component term \( (g_{03}) \) obtained in his work is non flat, therefore it could not be of astrophysical interest. Besides there is a problem with the generalization Halilsoy proposes for an arbitrary number of massive particles it fails to obtain the correct form for the case of two massive particles.

In this work, a slow rotating metric is obtained by introducing a perturbation in the metric rather than using a rotating reference frame. Our rotational metric term is Kerr like. Moreover, we discuss the possible applications of this new version.

## 2 The Curzon-Chazy Metric

The Curzon-Chazy metric in canonical cylindrical coordinates is given by [3, 5, 6] (in geometrical units \( G = c = 1 \)):

\[
\begin{align*}
\text{ds}^2 &= e^{-2\psi} dt^2 - e^{2(\psi-\gamma)} (d\rho^2 + dz^2) - e^{2\psi} \rho^2 d\phi^2
\end{align*}
\] (1)

where
\[ \psi = \frac{M}{\sqrt{\eta}}; \]
\[ \gamma = \frac{M^2 \rho^2}{2\eta^2}, \]
\[ \eta^2 = \rho^2 + z^2. \]  

(2)
The Curzon-Chazy metric in spherical coordinates can be obtained by means of the following mapping [3]:

\[ \rho = \sqrt{Z} \quad \text{and} \quad z = (r - M) \cos \theta, \]  

(3)
where \( Z = (r^2 - 2Mr) \sin^2 \theta \)

Using this transformation the Curzon-Chazy metric takes the form

\[ ds^2 = e^{-2\psi} dt^2 - e^{2(\psi - \gamma)}(X dr^2 + Y d\theta^2) - e^{2\psi} Z d\phi^2 \]  

(4)
where

\[ \psi = \frac{M}{\sqrt{\eta}}; \]
\[ \gamma = \frac{M^2 (r^2 - 2Mr) \sin^2 \theta}{2\eta^2}, \]
\[ \eta^2 = r^2 - 2Mr + M^2 \cos^2 \theta \]
\[ \Delta = r^2 - 2Mr + M^2 \sin^2 \theta, \]  

(5)
\[ X = \frac{\Delta}{r^2 - 2Mr}, \]
\[ Y = \frac{\Delta}{r^2 - 2Mr}. \]

3 The Lewis Metric

The Lewis metric is given by [15, 3]

\[ ds^2 = V dt^2 - 2W dt d\phi - e^\mu d\rho^2 - e^\nu dz^2 - \Sigma d\phi^2 \]  

(6)
where we have chosen the canonical coordinates $x^1 = \rho$ and $x^2 = z$. The potentials $V, W, \Sigma, \mu$ and $\nu$ are functions of $\rho$ and $z (\rho^2 = V\Sigma + W^2)$. Choosing $\mu = \nu$ and performing the following changes of potentials

\[ V = f, \quad W = \omega f, \quad \Sigma = \frac{\rho^2}{f} - \omega^2 f, \quad e^\nu = \frac{e^x}{f} \]  

we get the Papapetrou metric

\[ ds^2 = f(dt - \omega d\phi)^2 - \frac{e^x}{f}[d\rho^2 + dz^2] - \frac{\rho^2}{f}d\phi^2 \]  

Note that for slow rotation we neglect the second order in $\omega$, hence $\omega^2 \simeq 0 \Rightarrow W^2 \simeq 0$, and $\Sigma \simeq \rho^2/f$.

### 4 Perturbing the Curzon-Chazy Metric

To include slow rotation into the Curzon-Chazy metric we choose the Lewis-Papapetrou metric, equation (8). First of all, we choose expressions for the canonical coordinates $\rho$ and $z$. From (3) we get

\[ d\rho^2 + dz^2 = \Delta \left( \frac{dr^2}{r^2 - 2Mr} + d\theta^2 \right) = Xdr^2 + Yd\theta^2. \]  

Now, let us choose $V = f = e^{-2\psi}$ and neglect the second order in $\omega$. Then, we have

$\Sigma \simeq \frac{\rho^2}{f} = Ze^{2\psi}$ and $\frac{e^x}{f} = e^{2(\psi - \gamma)}$.

The metric takes the form

\[ ds^2 = e^{-2\psi}dt^2 - 2Wdtd\phi - e^{2(\psi - \gamma)}[Xdr^2 + Yd\theta^2] - Ze^{2\psi}d\phi^2 \]  

To obtain a slowly rotating version of the metric (4), the only potential, we have to find is $W$. In order to do this, we need to solve the Einstein equations for this metric

\[ G_{ij} = R_{ij} - \frac{R}{2}g_{ij} = 0 \]  

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where $R_{ij} (i, j = 0, 1, 2, 3)$ are the Ricci tensor components and $R$ is the curvature scalar. To find the approximated slowly version of the metric, we wrote a REDUCE program to find the Ricci tensor. The interested reader can request this program by sending us an message. Fortunately, the Ricci tensor components $R_{00}, R_{11}, R_{12}, R_{22}, R_{23}, R_{33}$ and the scalar curvature depend upon the potentials $V, X, Y, Z$ and not on $W$ (see Appendix). Hence, these components vanish. The only equation we have to solve is $R_{03} = 0$, because it depends upon $W$. The equation for this Ricci component, up to order $O(M^3, a^2)$, is

$$\sin \theta \left( \frac{\partial^2 W}{\partial \theta^2} + r^2 \frac{\partial^2 W}{\partial r^2} \right) - \cos \theta \frac{\partial W}{\partial \theta} = 0. \quad (12)$$

The solution for (12) is

$$W = \frac{K}{r} \sin^2 \theta \quad (13)$$

In order to find the constant $K$ let us see the Lense-Thirring metric which is obtained from the Kerr metric:

$$ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{4J}{r} \sin^2 \theta dtd\phi - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\Sigma^2 \quad (14)$$

where $d\Sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $J = Ma$ is the angular momentum. At first order in $M$, this metric and the Curzon-Chazy metric coincides. Then, comparing the second term of the latter metric with the corresponding term of (14), we note that $K = -2J = -2Ma$.

The new rotating Curzon-Chazy metric is

$$ds^2 = e^{-2\psi} dt^2 + \frac{4J}{r} \sin^2 \theta dtd\phi - \Delta e^{2(\psi-\gamma)} \left( \frac{dr^2}{r^2 - 2Mr} + d\theta^2 \right) \quad (15)$$

We check that the metric (15) is indeed a solution of the Einstein’s Field Equations, up to the order $O(M^3, a^2)$, using the same REDUCE program.
5 Discussion and applications of the Metric

The slow rotating solution here presented, needs to be analyzed in the same manner as the static Curzon-Chazy metric has been. The new rotating metric can be used to describe two bodies rotating around its center of mass, performing orbits. This system can then be applied to significant binary star systems.

New calculations, such as the geodesics can be performed in order to visualize the trajectories due to such gravitational field. Also, a description of the surface potentials due to these geodesics can be studied, as it was done for the non-rotating Curzon-Chazy solution [8]. Moreover, in treating with two singularities, gravitational lens calculations can be performed upon this system; equally the Lense-Thirring correction could certainly be applied, which results will lead to a more real scenario.

For Postnewtonian calculations, for example in gravitational lens theory and astrometry, we need to expand the metric (15) in a Taylor series, the result is

\[
\begin{align*}
\frac{ds^2}{dt} &= \left(1 - \frac{2M}{r} + \frac{2}{3} M^3 P_2(\cos \theta) \right) dt^2 + \frac{4J}{r} \sin^2 \theta dtd\phi \\
&- \left(1 + \frac{2M}{r} + \frac{4M^2}{r^2} + 2 \left(4 - \frac{1}{3} P_2(\cos \theta) \right) \frac{M^3}{r^3} \right) dr^2 \\
&- r^2 \left(1 - \frac{2}{3} \frac{M^3}{r^3} P_2(\cos \theta) \right) d\Sigma^2, \\
\end{align*}
\]

where \(P_2(\cos \theta) = (3 \cos^2 \theta - 1)/2\) and \(d\Sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2\). It was kept the series till the third order in \(M\) to take into account the quadrupole moment. For other purposes, for instance Astrometry, it is usual to transform this metric (16) into cartesian coordinates using the harmonic or the isotropic coordinates of Schwarzschild metric. The first one is \(r = \bar{r} + M\), and the second one is \(r = \bar{r}(1 + M/2\bar{r})^2\), where \(\bar{r}\) is a new radial coordinate [22]. Using the first one, the metric (16) becomes (dropping the bars)
\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{2M^2}{r^2} + 2 \left( \frac{1}{3} P_2(\cos \theta) - 1 \right) \frac{M^3}{r^3} \right) dt^2 + \frac{4J}{r} \sin^2 \theta dt d\phi \]
\[ - \left( 1 + \frac{2M}{r} + \frac{2M^2}{r^2} + 2 \left( 1 - \frac{1}{3} P_2(\cos \theta) \right) \frac{M^3}{r^3} \right) dr^2 \]
\[ - r^2 \left( 1 + \frac{M}{r} \right)^2 \left( 1 - \frac{2}{3} \frac{M^3}{r^3} P_2(\cos \theta) \right) d\Sigma^2. \]  

(17)

Now, the transformation into cartesian coordinates gives the following result:

\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{2M^2}{r^2} + 2 \left( \frac{1}{3} P_2(\cos \theta) - 1 \right) \frac{M^3}{r^3} \right) dt^2 \]
\[ + \frac{4J}{r} \sin^2 \theta dt (x dy - y dx) \]
\[ - \left( 1 + \frac{2M}{r} + \frac{M^2}{r^2} - \frac{M^3}{3r^3} P_2(\cos \theta) \right) dx^2 \]
\[ - \frac{M^2}{r^2} \left( 1 + \frac{2M}{r} \right) [x \cdot dx]^2, \]

where

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

(19)

The metric can be generalized for any \( J \) direction using the vector \( V \) defined by

\[ V = \frac{G}{2c^3 \mu^3} [J \times x], \]

with \( J = J e_j \) (\( e_j \) is an unit vector in the direction of \( J \)). Then, the metric (18) takes the form
\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{2M^2}{r^2} + 2 \left( \frac{1}{3} P_2(\cos \theta) - 1 \right) \frac{M^3}{r^3} \right) dt^2 + 8 \mathbf{V} \cdot d\mathbf{x} dt \\
- \left( 1 + \frac{2M}{r} + \frac{M^2}{r^2} - \frac{M^3}{3r^3} P_2(\cos \theta) \right) d\mathbf{x}^2 \\
- \frac{M^2}{r^2} \left( 1 + \frac{2M}{r} \right) [\mathbf{x} \cdot d\mathbf{x}]^2. \]

The metric (20) is ready for calculations on astrometry and gravitational lensing.

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A Appendix

The Ricci tensor components are

\[ R_{00} = \frac{e^{2(\gamma-2\psi)}}{2X^2Y^2Z} \left( -2X^2YZ \frac{\partial^2 \psi}{\partial \theta^2} - XYZ \frac{\partial \psi \partial X}{\partial \theta \partial \theta} + X^2Z \frac{\partial \psi \partial Y}{\partial \theta \partial \theta} \right. \]
\[ \left. - X^2Y \frac{\partial \psi \partial Z}{\partial \theta \partial \theta} - 2XY^2Z \frac{\partial^2 \psi}{\partial r^2} + Y^2Z \frac{\partial \psi \partial X}{\partial r \partial r} - XYZ \frac{\partial \psi \partial Y}{\partial r \partial r} \right) \]

\[ R_{01} = 0 \]

\[ R_{02} = 0 \]

\[ R_{03} = \frac{e^{2(\gamma-\psi)}}{2X^2Y^2Z} \left( -XYZ \frac{\partial X \partial W}{\partial \theta \partial \theta} + Y^2Z \frac{\partial X \partial W}{\partial r \partial r} - 2X^2YZ \frac{\partial^2 W}{\partial \theta^2} \right. \]
\[ \left. + X^2Z \frac{\partial W \partial Y}{\partial \theta \partial \theta} + X^2Y \frac{\partial W \partial Z}{\partial \theta \partial \theta} - 2XY^2Z \frac{\partial^2 W}{\partial r^2} - XYZ \frac{\partial W \partial Y}{\partial r \partial r} \right) \]

\[ R_{11} = \frac{1}{4XY^2Z^2} \left( -4X^2YZ \frac{\partial^2 \psi}{\partial \theta^2} - 2XY^2Z^2 \frac{\partial \psi \partial X}{\partial \theta \partial \theta} + 2X^2Z^2 \frac{\partial \psi \partial Y}{\partial \theta \partial \theta} \right. \]
\[ \left. - 2X^2YZ \frac{\partial \psi \partial Z}{\partial \theta \partial \theta} - 4XY^2Z^2 \frac{\partial^2 \psi}{\partial r^2} - 8XY^2Z^2 \left[ \frac{\partial \psi}{\partial r} \right]^2 + 2Y^2Z^2 \frac{\partial \psi \partial X}{\partial r \partial r} \right) \]
\[ \left. - 2X^2YZ \frac{\partial \psi \partial Y}{\partial \theta \partial \theta} - 2XY^2Z \frac{\partial \psi \partial Z}{\partial r \partial r} + 4X^2Y^2Z \frac{\partial^2 \gamma}{\partial \theta^2} + 2XY^2Z \frac{\partial \gamma \partial X}{\partial \theta \partial \theta} \right) \]
\[ \left. - 2X^2Z^2 \frac{\partial \gamma \partial Y}{\partial \theta \partial \theta} + 2X^2Y^2Z \frac{\partial \gamma \partial Z}{\partial \theta \partial \theta} + 4XY^2Z \frac{\partial^2 \gamma}{\partial r^2} - 2Y^2Z^2 \frac{\partial \gamma \partial X}{\partial r \partial r} \right) \]
\[ + 2XY^2Z \frac{\partial \gamma \partial Y}{\partial r \partial \theta} + 2XY^2Z \frac{\partial \gamma \partial Z}{\partial r \partial \theta} - 2XY^2Z^2 \frac{\partial^2 X}{\partial \theta^2} + YZ^2 \left[ \frac{\partial X}{\partial \theta} \right]^2 + 2X^2Z^2 \frac{\partial \gamma \partial Z}{\partial \theta \partial \theta} + YZ^2 \frac{\partial X \partial Y}{\partial r \partial r} \]
\[ + 2XY^2Z \frac{\partial \gamma \partial X}{\partial \theta \partial \theta} - 2X^2YZ \frac{\partial \gamma \partial Z}{\partial \theta \partial \theta} - 2XY^2Z \frac{\partial^2 X}{\partial r^2} + XY^2 \left[ \frac{\partial Z}{\partial r} \right]^2 \]
\[
R_{12} = \frac{1}{4YZ^2} \left( -8XYZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} - 2XYZ \frac{\partial \gamma}{\partial \theta} \frac{\partial Z}{\partial r} \right. \\
+ \left. YZ \frac{\partial X}{\partial \theta} \frac{\partial Z}{\partial r} + XZ \frac{\partial Y}{\partial \theta} \frac{\partial Z}{\partial r} - 2XYZ \frac{\partial^2 Z}{\partial \theta \partial r} + XYZ \frac{\partial Z}{\partial \theta} \frac{\partial Z}{\partial r} \right)
\]

\[
R_{13} = 0
\]

\[
R_{22} = \frac{1}{4X^2YZ^2} \left( -4X^2Y^2Z^2 \frac{\partial^2 \psi}{\partial \theta^2} - 8X^2YZ^2 \left[ \frac{\partial \psi}{\partial \theta} \right]^2 - 2XYZZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} \right. \\
+ \left. 2X^2Z^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} - 2X^2YZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial r} + 2Y^2Z^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} - 2X^2YZ \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} \right) \\
+ \left. 2X^2Y^2Z^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial r} + 2X^2YZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial r} - 2X^2Y^2Z^2 \frac{\partial \gamma}{\partial \theta} \frac{\partial X}{\partial \theta} \right. \\
+ \left. 2X^2YZ \frac{\partial \gamma}{\partial \theta} \frac{\partial Z}{\partial r} + XZ^2 \left[ \frac{\partial X}{\partial \theta} \right]^2 - XZ^2 \frac{\partial X}{\partial \theta} \frac{\partial X}{\partial r} + X^2Z \frac{\partial Y}{\partial \theta} \frac{\partial Z}{\partial r} \right.
\]

\[
R_{23} = 0
\]

\[
R_{33} = \frac{e^{2\gamma}}{4X^2YZ^2} \left( -4X^2YZ \frac{\partial^2 \psi}{\partial \theta^2} - 2XY^2Z^2 \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} + 2X^2Y^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} \right. \\
- \left. 2X^2YZ \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial r} - 4X^2YZ \frac{\partial^2 \psi}{\partial \theta^2} + 2Y^2Z^2 \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial r} - 2XY^2Z \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial r} \right. \\
- \left. 2XY^2Z \frac{\partial \gamma}{\partial \theta} \frac{\partial Z}{\partial r} + XY^2 \frac{\partial \gamma}{\partial \theta} \frac{\partial Z}{\partial r} + Y^2Z^2 \frac{\partial \gamma}{\partial \theta} \frac{\partial X}{\partial \theta} \right. \\
+ \left. 2XY \frac{\partial \gamma}{\partial \theta} \frac{\partial Z}{\partial r} - 2XY^2Z \frac{\partial^2 Z}{\partial \theta^2} + X^2Z \left[ \frac{\partial Z}{\partial \theta} \right]^2 - 2XY^2Z \frac{\partial^2 Z}{\partial \theta^2} \right)
\]
The scalar curvature is given by

\[
R = \frac{e^{2(\gamma - \psi)}}{2X^2Y^2Z^2} \left( 4X^2YZ^2 \frac{\partial^2 \psi}{\partial \theta^2} + 4X^2YZ^2 \left[ \frac{\partial \psi}{\partial \theta} \right]^2 + 2XYZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} \right)
\]

\[
- 2X^2Z^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} + 2X^2YZ^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial \theta} + 4XY^2Z^2 \frac{\partial^2 \psi}{\partial r^2} + 4XY^2Z^2 \left[ \frac{\partial \psi}{\partial r} \right]^2
\]

\[
- 2Y^2Z^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} + 2XY^2Z^2 \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} + 2XY^2Z \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} - 4X^2YZ^2 \frac{\partial^2 \gamma}{\partial \theta^2}
\]

\[
- 2XY^2Z \frac{\partial \gamma}{\partial r} \frac{\partial X}{\partial r} + 2XY^2Z \frac{\partial \gamma}{\partial \theta} \frac{\partial Y}{\partial \theta} - 4XY^2Z \frac{\partial^2 \gamma}{\partial r^2} + 2Y^2Z \frac{\partial \gamma}{\partial r} \frac{\partial X}{\partial r}
\]

\[
+ \frac{\partial Y}{\partial \theta} \frac{\partial Z}{\partial \theta} - Y^2Z \frac{\partial X}{\partial r} \frac{\partial Y}{\partial r} - \frac{X^2Z \frac{\partial Y}{\partial r} \frac{\partial Z}{\partial r}}{\partial \theta} + \frac{X^2 \frac{\partial Y}{\partial r} \frac{\partial Z}{\partial r}}{\partial \theta}
\]

\[
+ 2XY^2Z \frac{\partial^2 \gamma}{\partial r^2} - XZ^2 \left[ \frac{\partial \gamma}{\partial \theta} \right]^2 + \frac{\partial \gamma}{\partial r} \frac{\partial \gamma}{\partial r} + 2XY^2Z \frac{\partial^2 Z}{\partial \theta^2}
\]

\[
- X^2 \left[ \frac{\partial Z}{\partial \theta} \right]^2 + 2XY \frac{\partial^2 \gamma}{\partial \theta^2} - XY \frac{\partial \gamma}{\partial \theta} \frac{\partial \gamma}{\partial \theta}
\]