THE SPARSE PARITY MATRIX

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ABSTRACT. Let $A$ be an $n \times n$-matrix over $\mathbb{F}_2$ whose every entry equals 1 with probability $d/n$ independently for a fixed $d > 0$. Draw a vector $y$ randomly from the column space of $A$. It is a simple observation that the entries of a random solution $x$ to $Ax = y$ are asymptotically pairwise independent, i.e., $\sum_{i < j} E[1(x_i = s, x_j = t \mid A)] - P(x_i = s \mid A)P(x_j = t \mid A) = o(n^2)$ for $s, t \in \mathbb{F}_2$. But what can we say about the overlap of two random solutions $x, x'$, defined as $n^{-1} \sum_{i=1}^n 1(x_i = x'_i)$? We prove that for $d < e$ the overlap concentrates on a single deterministic value $a_*(d)$. By contrast, for $d > e$ the overlap concentrates on a single value once we condition on the matrix $A$, while over the probability space of $A$ its conditional expectation vacillates between two different values $a_*(d) < a^*(d)$, either of which occurs with probability $1/2 + o(1)$.

This anti-concentration result provides an instructive contribution to both the theory of random constraint satisfaction problems and of inference problems on random structures.

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1. INTRODUCTION

1.1. Motivation and background. Sharp thresholds are the hallmark of probabilistic combinatorics. The classic, of course, is the giant component threshold, below which the random graph decomposes into many tiny components but above which a unique giant emerges [25]. Its (normalised) size concentrates on a deterministic value. Similarly, once the edge probability crosses a certain threshold the random graph contains a Hamilton cycle w.h.p., while for $d < e$ the overlap concentrates on a single deterministic value $a_*(d)$. By contrast, for $d > e$ the overlap concentrates on a single value once we condition on the matrix $A$, while over the probability space of $A$ its conditional expectation vacillates between two different values $a_*(d) < a^*(d)$, either of which occurs with probability $1/2 + o(1)$.

In this paper we investigate the simplest conceivable model of a sparse random matrix. There is one single parameter, the density $d > 0$ of non-zero entries. Specifically, we obtain the $n \times n$-matrix $A = A(n, p)$ by setting every entry to one with probability $p = (d/n) \wedge 1$ independently. Remarkably, this innocuous random matrix exhibits a critical behaviour, deviant from the usual zero–one law, for all $d$ outside a small interval. The result has ramifications for random constraint satisfaction and statistical inference.

To begin with constraint satisfaction (we will turn to inference in Section 1.3), consider a random vector $y$ from the column space of $A$. The random linear system $Ax = y$ constitutes a random constraint satisfaction problem par excellence. Its space of solutions is a natural object of study. In fact, the problem is reminiscent of the intensely studied random $k$-XORSAT problem, where we ask for solutions to a Boolean formula whose clauses are XORs of $k$ random literals [2, 10, 24, 22, 28, 34, 41]. Random $k$-XORSAT is equivalent to a random linear system over $\mathbb{F}_2$ whose every row contains precisely $k$ ones.

The most prominent feature of random $k$-XORSAT is its sharp satisfiability threshold. Specifically, for any $k \geq 3$ there exists a critical value of the number of clauses up to which the random $k$-XORSAT formula possesses a solution, while for higher number of clauses no solution exists w.h.p. [22, 24, 41]. The satisfiability threshold is strictly smaller than the obvious point where the corresponding $\mathbb{F}_2$-matrix cannot have full row rank anymore because there are more rows than columns. Instead, the satisfiability threshold coincides with the threshold where due to long-range effects a linear number of variables freeze, i.e., are forced to take the same value in all solutions. Clearly, once an extensive number variables freeze, additional random constraints are apt to cause conflicts.

The precise freezing threshold can be characterised in terms of the 2-core of the random hypergraph underlying the $k$-XORSAT formula. We recall that the 2-core is what remains after recursively deleting variables of degree at most one along with the constraint that binds them (if any). If the 2-core is non-empty, then its constraints are more tightly interlocked than those of the original problem, which, depending on the precise numbers, may cause freezing. Indeed, the precise number of frozen variables can be calculated by way of a message passing process called Warning Propagation [28, 33]. The number of frozen variables concentrates on a deterministic value that

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comes out in terms of a fixed point problem. Although the $k$-XORSAT problem is conceptually far simpler than, say, the $k$-SAT problem, freezing plays a pivotal role in basically all other random constraint satisfaction problems as well [1] [23] [32] [33] [36] [38].

Surprisingly, our linear system $Ax = y$ behaves totally differently as two competing combinatorial forces of exactly equal strength engage in a tug of war. As a result, for densities $d > e$ the fraction of frozen variables fails to concentrate on a single value. Instead, that number and, in effect, the geometry of the solution space vacillate as well $[1, 23, 32, 33, 36, 38]$. We could run the process of peeling variables appearing in at most one equation of the linear system $A x = y$.

For on the other hand we could trace the number of variables that freeze because of unary equations. Indeed, such variable freezes. Substituting these frozen values into the other equations likely produces more equations of degree one, etc. Interestingly enough, the number of frozen variables that this "unary equations heuristic" predicts equals $a_\ast n$, with $a_\ast$ the least fixed point of $\Phi_d$. While for $d < e$ there is a unique fixed point and thus $a_\ast = a^\ast$, for $d > e$ the two fixed points $a_\ast, a^\ast$ are distinct. Indeed, apart from $a_\ast, a^\ast$, which are stable fixed points, there occurs a third unstable fixed point $a_\ast < a_0 < a^\ast$; see Figure

Which one of these heuristics provides the right answer? To find out we could try to assess the total number of solutions that the linear system $Ax = y$ should possess according to either prediction. Indeed, [15] Theorem 1.1] yields an asymptotic formula for the number of solutions to a sparse random linear system in terms of a parameter $\alpha$ that, at least heuristically, should equal the fraction of frozen variables. For the random matrix $A$ the formula shows that, in probability,

$$\lim_{n \to \infty} \frac{\text{nul} A}{n} = \max_{\alpha \in [0, 1]} \Phi_d(\alpha), \quad \text{where} \quad \Phi_d(\alpha) = \exp(-d \exp(-d(1-\alpha))) + (1 + d(1-\alpha)) \exp(-d(1-\alpha)) - 1$$

and where $\text{nul} A$ denotes the nullity, i.e. the dimension of the kernel, of $A$. Hence, the correct answer should be the value $\alpha \in [a_\ast, a^\ast]$ that maximises $\Phi_d$. But it turns out that $\Phi_d(a_\ast) = \Phi_d(a^\ast)$ for all $d > 0$. Accordingly, the main theorem shows that both predictions $a_\ast$ and $a^\ast$ are correct, or more precisely each of them is correct about half of the time. Formally, let

$$f(A) = |\{i \in [n] : \forall x \in \ker A : x_i = 0\}| / n$$

**Figure 1.** Left: the two fixed points $a_\ast = a_\ast(d)$ and $a^\ast = a^\ast(d)$ of $\Phi_d$. Right: the function $\Phi_d(\alpha)$ for $d = 2.5$ (blue) possesses a unique fixed point, while for $d = 3$ (red) there are two stable fixed points and an unstable one in between.
be the fraction of frozen variables.

**Theorem 1.1.** (i) For $d \leq e$ the function $\phi_d$ has a unique fixed point and

$$\lim_{n \to \infty} f(A) = \alpha_* = \alpha^*$$

in probability.

(ii) For $d > e$ we have $\alpha_* < \alpha^*$ and for all $\epsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P} \{|f(A) - \alpha_*| < \epsilon\} = \lim_{n \to \infty} \mathbb{P} \{|f(A) - \alpha^*| < \epsilon\} = \frac{1}{2}.$$  

Hence, the fraction of frozen variables fails to exhibit a zero–one behaviour for $d > e$. Instead, it shows a critical behaviour as one would normally associate only with the critical window of a phase transition.

1.3. **The overlap.** Apart from considering the linear system $Ax = y$ as a random constraint satisfaction problem, the random linear system can also be viewed as an inference problem. Indeed, we can think of the vector $y$, which is chosen randomly from the column space of $A$, as actually resulting from multiplying $A$ with a uniformly random vector $\hat{x} \in \mathbb{F}_2^n$. Then $y = A\hat{x}$ turns into a noisy observation of the ‘ground truth’ $\hat{x}$. Thus, it is natural to ask how well we can learn $\hat{x}$ given $A$ and $y$.

These two viewpoints are actually equivalent because the posterior of $\hat{x}$ given $(A, y)$ is nothing but the uniform distribution on the set of solutions to the linear system $Ax = y$. Hence,

$$\mathbb{P}[\hat{x} = x \mid A, y] = \frac{1}{|\ker A|} \sum_{x \in \ker A} R(x, \hat{x}) = \frac{1}{|\ker A|^2} \sum_{x, x' \in \ker A} R(x, x'),$$

(1.3)

Therefore, the optimal inference algorithm just draws a random solution $x$ from among all solutions to the linear system. The number of bits that this algorithm recovers correctly reads

$$R(x, \hat{x}) = \frac{1}{n} \sum_{i=1}^n 1\{|x_i| = |\hat{x}_i|\}.$$  

Adopting mathematical physics jargon, we call $R(x, \hat{x})$ the overlap of $x, \hat{x}$. Its average given $A, y$ boils down to

$$\tilde{R}(A) = \mathbb{E}[R(x, \hat{x}) \mid A, y] = \frac{1}{|\ker A|^2} \sum_{x, x' \in \ker A} R(x, x'),$$

which is independent of $y$.

Conceived wisdom in the statistical physics-inspired study of inference problems holds that the overlap concentrates on a single value given the ‘disorder’, in our case $(A, y)$ (see [13]). This property is called replica symmetry. We will verify that replica symmetry holds for the random linear system w.h.p. Additionally, in all the random inference problems that have been studied over the past 20 years the overlap concentrates on a single value that does not depend on the disorder, except perhaps at a few critical values of the model parameters where phase transitions occur [3]. This enhanced property is called strong replica symmetry. A natural question is whether strong replica symmetry holds universally. It does not. As the next theorem shows, the random linear system with $d > e$ provides a counterexample: it is replica symmetric, but not strongly so.

**Theorem 1.2.** (i) If $d < e$ then $\lim_{n \to \infty} R(x, \hat{x}) = (1 + \alpha_*)/2$ in probability.  

(ii) For all $d > e$ we have $\lim_{n \to \infty} \mathbb{E}[R(x, \hat{x}) - \tilde{R}(A)] = 0$ while

$$\lim_{n \to \infty} \mathbb{P} \left[\tilde{R}(A) - \frac{1 + \alpha_*}{2} < \epsilon\right] = \lim_{n \to \infty} \mathbb{P} \left[\tilde{R}(A) - \frac{1 + \alpha^*}{2} < \epsilon\right] = \frac{1}{2}$$  

for any $\epsilon > 0$.

The first part of the theorem posits that for $d < e$ the overlap concentrates on the single value $(1 + \alpha_*)/2$. In light of Theorem [11] this means that the optimal inference algorithm, while, unsurprisingly, capable of correctly recovering the frozen coordinates, is at a loss when it comes to the unfrozen ones. Indeed, we can get only about half the unfrozen coordinates right, no better than a random guess.

The second part of the theorem is more interesting. While the random variable $R(x, \hat{x})$ concentrates on the conditional expectation $\tilde{R}(A)$ given $A, y$, the conditional expectation $\tilde{R}(A)$ itself fails to concentrate on its mean $\mathbb{E}[\tilde{R}(A)]$. Instead it oscillates between two different values $(1 + \alpha_*)/2$ and $(1 + \alpha^*)/2$, each of which occurs with asymptotically equal probability. In fact, this failure to concentrate does not just occur at a few isolated points, but throughout the entire regime $d > e$. This behaviour mirrors the anti-concentration of the number of frozen variables from Theorem [11]. Moreover, as in the case $d < e$ the optimal inference algorithm does, of course, correctly recover the frozen variables, but cannot outperform a random guess on the unfrozen ones.
We proceed to outline the key ideas behind the proofs of Theorems \[1.1\] and \[1.2\]. Unsurprisingly, to prove the critical behaviour that these theorems assert we will need to conduct a rather subtle, accurate analysis of the random linear system and its space of solutions, far more so than one would normally have to undertake when aiming at a zero-one result. On the positive side the proofs reveal novel combinatorial insights that may have an impact on other random constraint satisfaction or inference problems as well. Let us thus survey the proof strategy.

1.4. Techniques. The main result of the paper is that for \( d > e \) the proportion \( f(A) \) of frozen variables is asymptotically equal to either of the two stable fixed points \( \alpha_*, \alpha^* \) of the function \( \phi_d \) with probability \( 1/2 + o(1) \) (see Figure [1]). Proving this statement takes three strikes.

**FIX:** \( f(A) \) concentrates on the fixed points of \( \phi_d \), either one of the two stable ones \( \alpha_*, \alpha^* \) or the third unstable fixed point \( \alpha_0 \).

**STAB:** The unstable fixed point is an unlikely outcome.

**EQ:** The two stable fixed points are equally likely.

1.4.1. Heuristics. Why are these three statements plausibly true? Let us begin with **FIX**. The random matrix \( A \) naturally induces a bipartite graph called the Tanner graph \( G(A) \). Its vertex classes are variable nodes \( v_1, \ldots, v_n \) representing the columns of \( A \) and check nodes \( a_1, \ldots, a_o \) representing the rows. There is an edge between \( a_i \) and \( v_j \) if \( A_{ij} = 1 \). The Tanner graph is distributed as a random bipartite graph with edge probability \( d/n \). As a consequence, its local structure is roughly that of a Pois(d) Galton-Watson tree.

Exploring the Tanner graph from a given variable node \( v_i \), we may view \( v_i \) as the root of such a tree. The grandchildren of \( v_i \), i.e. the variable nodes at distance two, are essentially uniformly random. Therefore, the grandchildren should each be frozen with probability \( f(A) + o(1) \) and behave very nearly independently. Further, for the obvious algebraic reason the root \( v_i \) itself is frozen iff it is parent to some check all of whose children are frozen. A few lines of calculation based on the Poisson tree structure then show that \( v_i \) ought to be frozen with probability \( \phi_d(f(A)) \). But at the same time, since \( v_i \) was itself chosen randomly, it is frozen with probability \( f(A) \). Hence, we are led to expect that \( f(A) = \phi_d(f(A)) \). In other words, **FIX** expresses that the local structure of \( G(A) \) is given by a Poisson tree, and that freezing manifests itself locally.

Apart from the two stable fixed points \( \alpha_*, \alpha^* \), Figure [1] indicates that \( \phi_d \) possesses an unstable fixed point \( \alpha_0 \) somewhere in between. How can we rule out that \( f(A) \) will take this value? The nullity formula \( \ell_2 \) suggests that \( f(A) \) should be a maximiser of the function \( \Phi_d(a) \). But its maximisers are precisely the stable fixed points \( \alpha_*, \alpha^* \), while the unstable fixed point is where the function takes its local minimum. That is why **STAB** appears plausible. However, we will see that this simplistic line of reasoning cannot be turned into a proof easily.

Finally, coming to **EQ**, we need to argue that for \( d > e \) both stable fixed points are equally likely. To this end we employ the Warning Propagation (WP) message passing scheme, where messages are sent along the edges of the Tanner graph in either direction. The message from \( v_j \) to \( a_i \) is updated at each time step according to the messages that \( v_j \) receives from its other neighbours, and similarly for the reverse message. WP does faithfully describe the local dynamics that cause freezing, but there remains a loose end: we must initialise messages somehow.

Two obvious initialisations suggest themselves. First, if we initialise assuming everything to be unfrozen, then because of **FIX** and the local geometry approximating a Galton-Watson branching tree, WP reduces to repeated application of the \( \phi_d \) function starting from 0. Since \( \lim_{t \to \infty} \phi_d^t(0) = \alpha_* \), WP then predicts \( f(A) = \alpha_* \). Second, if we initialise assuming everything to be frozen, WP mimics iterating \( \Phi_d \) from 1 and thus predicts \( f(A) = \lim_{t \to \infty} \Phi_d^t(1) = \alpha^* \).

So which initialisation is correct? Neither, unfortunately. We thus need a more nuanced version of WP, in which we describe messages and ultimately variables as “frozen”, “unfrozen” and “slush”, the last meaning uncertain. Initialising WP with either all messages frozen or all messages unfrozen still leads to the same results as before. But initialising with all messages being “slush”, WP predicts that approximately \( \alpha_* n \) variables are frozen, \((1 - \alpha^*)n \) variables are unfrozen, and \((\alpha^* - \alpha_*) n \) variables remain slush. Thus, there are actually three distinct categories.

How does this help? Since \( f(A) \) is concentrated around the stable fixed points \( \alpha_*, \alpha^* \), we know that actually the slush portion must be either (almost) entirely frozen or unfrozen; it is impossible that, say, half the slush variables freeze. To figure out whether the slush freezes, consider the minor \( A_\delta \) of \( A \) induced on the corresponding variables and constraints. If this minor has fewer rows than columns, then the corresponding linear system is under-constrained. In effect, it is inconceivable that the slush freezes completely. On the other hand, if \( A_\delta \) has more rows than columns, then by analogy to the random \( k\)-XORSAT problem we expect that the slush freezes.
Now, crucially, both the random matrix model $A$ and the WP message passing process are invariant under transposition of the matrix. Hence, $A_0$ should be over-constrained just as often as it is under-constrained. We are thus led to believe that the slush freezes with probability about half, which explains the peculiar behaviour stated in the theorems. Once again, this simple reasoning, while plausible, cannot easily be converted into an actual proof.

1.4.2. *Formalising the heuristics.* Hence, how can we corroborate these heuristics rigorously? Concerning FIX, consider the following game of “thimblerig”. The opponent generates two random graphs independently: one is simply the Tanner graph $G_1 \sim G(A)$ of $A$, the other is an independent copy $G_2 \sim G(A)$ of the Tanner graph, but with some random alterations. Specifically, the trickster generates a Po$(d)$ branching tree of height two, embeds the root and its children onto isolated variable and check nodes respectively, and embeds the remaining leaves onto variables chosen uniformly at random. The opponent then presents you with the two graphs and asks you to determine which is which. It turns out that the changes are so well-disguised that you can do no better than a random guess. To compound your misery, having told you which is the perturbed graph, your opponent asks you to guess which variable is the root of the added tree. Again, the changes are so well-disguised that you can do no better than a random guess. Not content with winning twice, your opponent wishes to assert their complete dominance and performs the same trick again, this time adding not just one tree but a slowly growing number (of order $\Omega(\sqrt{n})$). For the third time, you can only resort to a random guess.

The point of this game is to demonstrate that the root variables of the trees added behave identically to randomly chosen variables of the original graph. In particular, the proportion of variables which are frozen is distributed as $f(A)$. But we can also calculate this proportion in a different way: by considering whether the attachment variables are frozen and tracking the effects down to the roots. This tells us that the proportion of frozen roots is $\phi_d(f(A) + o(1)), \text{provided that }$ the newly added constraints do not dramatically shift the overall number of frozen variables due to long-range effects. To rule this out we use a delicate argument drawing on ideas from the study of random factor graph models and involving replica symmetry and the cut metric for discrete probability distributions from [9,17,19].

Perhaps surprisingly, it takes quite an effort to verify the claim STAB that $f(A)$ is not likely to be near the unstable fixed point. The proof employs a combinatorial construction that we call *covers*. A cover is basically a designation of the variable nodes, checks and edges of the Tanner graph that encodes which variables are frozen, and because of which constraints they freeze. We will then pursue a novel “hammer and anvil” strategy to rule out the unstable fixed point. On the one hand, we will show that if $f(A)$ is near $a_0$, then the Tanner graph $G(A)$ must contain covers that each induce a cluster of solutions with about $a_0$ frozen variables. On the other hand, we will use a moment computation to show that w.h.p. the Tanner graph $G(A)$ only contains a sub-exponential number $\exp(o(n))$ of covers. Furthermore, another moment computation shows that w.h.p. each of them only extends to about $2^{\phi_d (a_0 n)}$ solutions to the linear system $Ax = y$. As a consequence, if $f(A)$ is near $a_0$, then the random linear system $Ax = y$ would have far fewer solutions than provided by (1.2). Since the nullity of the random matrix is tightly concentrated, we conclude that the event $f(A) = a_0$ is unlikely. The novelty of this argument, and the source of its technical intricacy, is the two-step cover–solution consideration: first we verify that the set of solutions actually decomposes into clusters encoded by “covers”. Then we calculate the number of covers (corresponding to solution clusters), and finally we estimate the number of solutions inside each cluster. This two-level approach is necessary as a direct first moment calculation of the expected number of solutions with a given Hamming weight seems doomed to fail, at least for $d$ near the critical value $e$.

Coming to EQ, as indicated in the previous subsection, the “slush” portion of the matrix enjoys a symmetry property, in that it is also the slush portion of the transposed matrix. We will prove that, depending on the precise aspect ratio of the slush minor, the slush variables either do or do not freeze. But there is one subtlety: we need to show that the number of rows and the number of columns are not exactly equal w.h.p. Indeed it is not hard to show that the both numbers have standard deviation $\Theta(\sqrt{n})$. Hence, if they were independent they would differ by $\Theta(n^{1/3})$ w.h.p.. But this independence is quite clearly not satisfied. Thus, we need to argue that at least they have non-trivial covariance.

To show this, we perform a similar trick to the game of thimblerig: we show that the matrix can be randomly perturbed to decrease the number of slush columns, while preserving the number of slush rows. Furthermore, this can be achieved without an opponent being able to identify that a change has been made. Performing this trick carefully shows that it is unlikely that the slush portion of the matrix is approximately square. Symmetry then tells...
us that with probability asymptotically 1/2 it has significantly more rows than columns, and also with probability asymptotically 1/2 it has significantly more columns than rows.

It remains to prove that these two cases are likely to lead to all slush variables being frozen, or all being unfrozen respectively. Unfortunately, a simple symmetry argument does not quite suffice. Instead we first prove that it is unlikely that there are significantly, say $\omega \gg 1$, more slush variables than slush checks, but that almost all slush variables are frozen. The number of slush variables that remain unfrozen must certainly be at least $\omega$ due to elementary consideration of the nullity. We are thus left to exclude that the number is between $\omega$ and $\epsilon n$, which we establish by way of an expansion argument.

We finally need to show that it is unlikely that there are significantly more slush checks (say $m_\omega$) than slush variables ($n_\omega$), but that these slush variables remain mostly unfrozen. Crucially, thanks to replica symmetry and the cut metric we can indeed show that a “typical” kernel vector will set approximately half of the slush variables to 1 and half to 0. Of course there are approximately $2^{n_\omega}$ such vectors. On the other hand, imagine that a check with $k$ slush variable neighbours chooses these neighbours uniformly at random (this can be made formally correct by conditioning on the degree distribution and using the configuration model). Then the probability that this check is satisfied by a vector of Hamming weight approximately $n_\omega/2$ is approximately 1/2 (since e.g. based on the values of the first $k−1$ neighbours, the last must be chosen from the correct class). Therefore the expected number of kernel vectors should be approximately $2^{n_\omega−m_\omega} = o(1)$.

The problem with this basic calculation is that error terms occur which turn out to be too significant to ignore. These error terms ultimately come from check nodes of degree two in the slush minor. To deal with them, we employ a delicate percolation argument in which we contract check nodes of degree exactly two, since they just equalise their two adjacent variable nodes. Importantly, we can show that this process neither affects the number of kernel vectors nor the balance $m_\omega−n_\omega$. We can thus complete the moment calculation and show that the slush cannot have an excess of rows and still be entirely unfrozen.

1.5. Discussion. How do the techniques that we develop in this paper compare to previously known ones, and how can our techniques be extended to other problems?

The general Warning Propagation message passing scheme captures the local effects of constraint satisfaction problems; for example, in the context of satisfiability WP boils down to Unit Clause Propagation [33]. WP also yields the $k$-XORSAT threshold [28] as well as the freezing threshold in random graph colouring [36]. In addition, WP can also be used to study structural graph properties such as the $k$-core [12, 40]. In all these examples, the “correct” initialisation from which to launch WP is obvious, and the proof that random variable of interest converges to the fixed point is based on a direct and straightforward combinatorial analysis. Indeed, the standard strategy is then a two-stage one: first, show that WP quickly converges to something close to the conjectured limit; and second, show that after this initial convergence, not much else will change [11].

However, this usual technique is not enough for our purposes, essentially because of the 2-point rather than 1-point concentration of $f(A)$. Naively one might imagine that WP will converge to one of the two fixed points, each with probability 1/2. But intriguingly, the dichotomy of the random variable $f(A)$ induces a dichotomy for WP in each instance of $A$ – WP hedges its bets, identifying the two possible answers, but is unable to tell which is actually correct. As such, we are left with the “uncertain” portion of the matrix (or its Tanner graph).

To deal with this complication we enhance the WP message passing scheme to expressly identify the portion of the Tanner graph that may go either way. Along the way, we develop a versatile indirect method for proving convergence to some fixed point to replace the usual direct combinatorial argument. This technique is based on the thimblerig game that more or less justifies the WP heuristic in general. While the argument appears to be reasonably universal, it fails to identify precisely which fixed point is the correct one. As mentioned above, we follow WP up with a novel type of moment calculation based on covers to rule out the unstable fixed point. One could envisage a generalisation of this technique to other planted constraint satisfaction problems or, more generally, spin glass models. The place of the nullity formula [1,2] would then have to be filled by a formula for the leading exponential order of the partition function.

The thimble rig argument is enabled by the important observation that unfrozen variables, for the most part, behave more or less independently of each other and that the random variable $f(A)$ is fairly “robust” with respect to small numbers of local changes (see Proposition [2,9]). We establish this robustness by way of a pinning argument, in which unary checks are added that freeze certain previously unfrozen variables, and we analyse the effect that
this has on the kernel. The thimberig argument is an extension of arguments used in the study of random factor graph models [18,19,39], where the pinning operation also plays a crucial role [16,17].

Because the slush minor of the matrix displays a peculiar critical phenomenon, such as one would normally associate only with critical regimes around a phase transition, new techniques are required to study it. In particular, while it seems intuitively natural that the uncertain proportion is unfrozen if \( n_\ell - m_\ell \geq \omega \) is large and positive, but frozen if it is large and negative, proving this formally requires some significant new ideas. In particular, to prove the first statement we introduce \textit{flippers}, induced subgraphs of the uncertain portion which could confound expectations by being frozen. These flippers must satisfy various properties, and the proof consists of showing that large flippers (or more precisely, large unions of flippers) are unlikely due to expansion properties. This sort of expansion argument appears by no means restricted to the present problem. A related combinatorial structure appeared in the proof of limit theorems for cores of random graphs [13].

Proving the second statement involves a delicate moment calculation. The modification involved in contracting the checks of degree 2, which are the reason that the naive version of the argument fails, is similar to the operation to construct the kernel of a graph from its 2-core. This moment calculation is the single place where we make critical use of the fact that we are studying a problem whose variables range over a finite domain, viz. the field \( \mathbb{F}_2 \).

What are potential generalisations? The random linear system \( Ax = y \) is one case of a class of constraint satisfaction problems known as \textit{uniquely extendable problems} [20]. Such problems are characterised by the property that if all but one of the variables appearing in a constraint are fixed, there is precisely one choice for the value of the remaining variable such that the constraint is satisfied. Some of these problems are intractable, such as, for example, algebraic constraints with variables ranging over finite groups. It would be most interesting to see if and how the methods developed in this paper could be extended to uniquely extendable problems. Furthermore, since we study a critical phenomenon, namely the two-point concentration of the proportion of frozen variables, our ideas may help to understand the behaviour at the critical point of phase transitions of random constraint satisfaction problems. This type of question remains an essentially blank spot on the map.

1.6. Further related work. Perhaps surprisingly, apart from the article [15] that establishes a nullity formula for general sparse random matrices and in particular [12], there have been no prior studies of the random matrix \( A(n, p) \). However, random \( m \times n \)-matrix over finite fields \( \mathbb{F}_q \) where every row contains an equal number \( k \geq 2 \) of non-zero entries have been studied extensively. In the case \( k=q=2 \) this model is directly related to the giant component phase transition [29,30], because each row constrains two random entries to be equal. Moreover, we already saw that for \( k \geq 3 \) and \( q=2 \) the model is equivalent to random \( k \)-XORSAT. Dubois and Mandler [24] computed the critical aspect ratio \( m/n \) up to which such a matrix has full row rank for \( k=3 \). The result was subsequently extended to \( k>3 \) [22][41]. Indeed, the threshold value of \( m \) up to which the random matrix has full rank can be interpreted in terms of the Warning Propagation message passing scheme [10]. Beyond its intrinsic interest as a basic model of a random constraint satisfaction problem [33], the random \( k \)-XORSAT model has found applications in hashing and data compression [22,42].

The asymptotic rank of random matrices with a fixed number \( k \) of non-zero entries per row over finite fields has been computed independently via two different arguments by Ayre, Coja-Oghlan, Gao and Müller [8] and Cooper, Frieze and Pegden [21]. Additionally, Miller and Cohen [35] studied the rank of random matrices in which both the number of non-zero entries in each row and the number of non-zero entries in each column are fixed. However, they left out the critical case in which these two numbers are identical, which was solved recently by Huang [27]. Additionally, Bordenave, Lelarge and Salez [8] studied the rank over \( \mathbb{R} \) of the adjacency matrix of sparse random graphs. Of course, a crucial difference between the random matrix model that we study here and the adjacency matrix of a random graph is that the latter is symmetric.

A problem that appears to be inherently related to the binomial random matrix problem studied here is the matching problem on random bipartite graphs [9]. It would be interesting to see if in some form the criticality observed in Theorems 1.1 and 1.2 extends to the matching problem or, equivalently, the independent set problem on random bipartite graphs. The critical value \( d = e \) appears to be related to the uniqueness of the Gibbs measure of the latter problem [4]. In the context of the matching problem, our function \( \Phi_d(a) \) appears (as \( F(1-a) \)) in [9], in particular in the appendix where a figure shows the emergence of the two global maxima above the threshold \( d = e \). (In fact the discussion there is about the one-type graph \( G(n,d/n) \) rather than the bipartite \( G(n,n,d/n) \), which is the distribution of \( G(A) \), but since the two graphs have the same local weak limit the more general results of [9] show that the matching problem displays similar behaviour.) In some sense it is not surprising that the same
function should arise in these two problems: the Warning propagation process to determine which variables are certainly frozen in essence mimics a one-sided version of the first stage of the Karp-Sipser algorithm in which leaves and their neighbours are removed. This removal results in a remaining “core”, similar to our “slush”, of minimum degree at least 2. This is where we encounter our first fixed point of \( \Phi_d \) (or maximum of \( \Phi_d \)). For the matching problem, this first roadblock is easy to overcome: the core turns out to have an almost perfect matching w.h.p., which implies that it is always the same fixed point which gives the correct answer. By contrast, our situation is more delicate because the slush need not freeze.

2. Organisation

In this section, we state the intermediate results that lead up to the main theorems. We also detail where in the following sections the proofs of these intermediate results can be found.

2.1. The functions \( \phi_d \) and \( \Phi_d \). The formula (1.2) yields the approximate number of solutions to the linear system \( Ax = y \). We already discussed the combinatorial intuition behind the maximiser \( \alpha \) in (1.2): we will prove that the function \( \Phi_d \) attains its global maxima at the conceivable values of \( f(A) \). However, the proof of (1.2) in [15] falls short of already implying this fact as that proof strategy relies on a purely variational argument. For a start, we verify that the function \( \phi_d \) actually has a unique fixed point for \( d \leq e \) and two distinct stable fixed points for \( d > e \), and that these fixed points coincide with the local maxima of \( \Phi_d \).

**Lemma 2.1.** For all \( d > 0, d \neq e \) the local maxima of \( \Phi_d \) and the stable fixed points of \( \phi_d \) coincide. For \( d = e \) the local maximum of \( \Phi_e \) coincides with the lone fixed point, simultaneously the inflection point of \( \phi_e \).

The proof of Lemma 2.1 based on a bit of calculus, can be found in Section 3.2. Additionally, for \( d \leq e \) we define \( \alpha_0 = \alpha_* \), while for \( d > e \) we let \( \alpha_0 \) be the minimiser of \( \phi_d \) on the interval \([\alpha_*, \alpha^*] \). The following lemma, which we prove in Section 3.4 shows that the \( t \)-fold iteration \( \phi_d^t(x) \) converges to one of the stable fixed points, except if we start right at \( x = \alpha_0 \).

**Lemma 2.2.** For any \( d > 0 \) we have

\[
\lim_{t \to \infty} \Phi_d^t(x) = \alpha_* \quad \text{for any } x < \max(0, \alpha_0),
\]

\[
\lim_{t \to \infty} \Phi_d^t(x) = \alpha^* \quad \text{for any } x \in (\alpha_0, 1).
\]

The fixed point characterisation of the maximisers of \( \Phi_d \) enables us to show that the global maxima of \( \Phi_d \) occur precisely at \( \alpha_* = \alpha_*(d), \alpha^* = \alpha^*(d) \), the smallest and the largest fixed points of \( \phi_d \).

**Proposition 2.3.**

(i) If \( d \leq e \) then \( \phi_d \) has a unique fixed point, which is the unique global maximiser of \( \Phi_d \).

(ii) If \( d > e \) then the function \( \phi_d \) has precisely two stable fixed points, namely \( 0 < \alpha_* < \alpha^* < 1 \), and

\[
\Phi_d(\alpha_*) = \Phi_d(\alpha^*) > \Phi_d(\alpha) \quad \text{for all } \alpha \in [0, 1] \setminus \{\alpha_*, \alpha^*\}.
\]

In addition, \( \phi_d \) has its unique unstable fixed point at \( \alpha_0 \), which satisfies the equation

\[
1 - \alpha_0 = \exp(-d(1 - \alpha_0)).
\]

Although both the functions \( \phi_d, \Phi_d \) are explicit, the proof of Proposition 2.3 which can be found in Section 3.3 turns out to be mildly involved.

2.2. Warning Propagation. One of our principal tools is an enhanced version of the Warning Propagation message passing algorithm that identifies variables as frozen, unfrozen or slush. Specifically, we will see that WP identifies about \( n \) coordinates as positively frozen and another \( 1 - \alpha_* n \) as likely unfrozen w.h.p. Because Proposition 2.3 shows that \( \alpha_* = \alpha^* \) for \( d < e \), this already nearly suffices to establish the first part of Theorem 1.1. By contrast, in the case \( d > e \), where \( \alpha_* < \alpha^* \), we need to conduct a more detailed investigation of the \( (\alpha^* - \alpha_* + o(1))n \) coordinates that WP declares as slush.

To introduce WP, for a given \( m \times n \) matrix \( A \) over \( \mathbb{F}_2 \) we represent the matrix by its bipartite Tanner graph \( G(A) \). One of its vertex classes \( V(A) = V(G(A)) = \{v_1, \ldots, v_n\} \) represents the columns of \( A \); we refer to the \( v_i \) as the variable nodes. The second vertex class \( C(A) = C(G(A)) = \{a_1, \ldots, a_m\} \) represents the rows of \( A \); we refer to them as check nodes. There is an edge present between \( a_j \) and \( v_j \) iff \( A_{ij} = 1 \). Let \( E(A) \) denote the edge set of \( G(A) \). Moreover, let \( \partial u \) signify the set of neighbours of vertex \( u \in V(A) \cup C(A) \). Further, let \( \mathcal{F}(A) \) be the set of frozen coordinates \( i \in [n] \), i.e., coordinates such that \( x_i = 0 \) for all \( x \in \ker A \). By abuse of notation we identify \( \mathcal{F}(A) \) with the corresponding set
\[v_i : i \in \mathcal{F}(A)\] of variable nodes. Also let \(f(A) = |\mathcal{F}(A)|/n\) be the fraction of frozen coordinates. Conversely, for a given Tanner graph \(G\) we denote by \(A(G)\) the adjacency matrix induced by \(G\).

Our enhanced WP algorithm associates a pair of \(\{f, s, u\}\)-valued messages with every edge of \(G(A)\). Hence, let \(\mathcal{W}(A)\) be the set of all vectors
\[
w = (w_{v \rightarrow a}, w_{a \rightarrow v})_{v \in V(A), a \in C(A); a \in \partial v}
\]
with entries \(w_{v \rightarrow a}, w_{a \rightarrow v} \in \{f, s, u\}\).

We define the operator \(\text{WP}_A : \mathcal{W}(A) \rightarrow \mathcal{W}(A), w \rightarrow \hat{w}\), encoding one round of the message updates, by letting
\[
\hat{w}_{a \rightarrow v} = \begin{cases} f & \text{if } w_{y \rightarrow a} = f \text{ for all } y \in \partial a \setminus \{v\}, \\ u & \text{if } w_{y \rightarrow a} = u \text{ for some } y \in \partial a \setminus \{v\}, \\ s & \text{otherwise}, \end{cases}
\]
\[
\hat{w}_{v \rightarrow a} = \begin{cases} u & \text{if } \hat{w}_{b \rightarrow v} = u \text{ for all } b \in \partial v \setminus \{a\}, \\ f & \text{if } \hat{w}_{b \rightarrow v} = f \text{ for some } b \in \partial v \setminus \{a\}, \\ s & \text{otherwise}. \end{cases}
\]

as illustrated in Figure 2. Further, let \(w(A, t) = \text{WP}_A^t(s, \ldots, s)\) comprise the messages that result after \(t\) iterations of \(\text{WP}_A\) launched from the all-\(s\) message vector \(w(A, 0)\). Additionally, let \(w(A) = \lim_{t \to \infty} w(A, t)\) be the fixed point to which \(\text{WP}_A\) converges; the (pointwise) limit always exists because \(\text{WP}_A\) only updates an \(s\)-message to a \(u\)-message or to an \(f\)-message, while \(u\)-messages and \(f\)-messages will never change again.

What is the combinatorial idea behind WP? The intended semantics of the messages is that \(f\) stands for ‘frozen’, \(u\) for ‘unfrozen’ and \(s\) for ‘slush’. Since we launch from all-\(s\) messages, (2.2) shows that in the first round \(f\)-messages only emanate from check nodes of degree one, where the ‘for all’-condition on the left of (2.2) is empty and therefore trivially satisfied. Hence, if a check node \(a_i\) is adjacent to \(v_j \in V(A)\) only, then \(w_{a_i \rightarrow v_j}(A, 1) = f\). This message reflects that the \(i\)-th row of \(A\), having only one single non-zero entry, fixes the \(j\)-th entry of every vector of ker \(A\) to zero. Further, turning to the updates of the variable-to-check messages, if \(w_{a_i \rightarrow v_j}(A, 1) = f\), then \(v_j\) signals its being forced to zero by passing to all its other neighbours \(a_h \neq a_i\) the message \(w_{v_j \rightarrow a_h}(A, 1) = f\). Now suppose that check \(a_i\) is adjacent to \(v_h\) and \(w_{v_k \rightarrow a_i}(A, 1) = f\) for all \(v_k \in \partial a_i \setminus \{v_h\}\). Thus, the \(k\)-th coordinate of every vector in ker \(A\) equals zero for all neighbours \(v_k \neq v_h\) of \(a_i\). Then the only way to satisfy the \(i\)-th row of \(A\) is by setting the \(h\)-th coordinate to zero as well. Accordingly, (2.2) provides that \(w_{a_i \rightarrow v_h}(A, 2) = f\), and so on. Hence, defining
\[
V_f(A) = \{v \in V(A) : \exists a \in \partial v : w_{a \rightarrow v}(A) = f\},
\]
we see that
\[
V_f(A) \subseteq \mathcal{F}(A).
\]

The mechanics of the \(u\)-messages is similar. In the first round any variable node \(v_j\) of degree one, for which the ‘for all’ condition on the right of (2.2) is trivially satisfied, starts to send out \(u\)-messages. Subsequently, any check node \(a_i\) with an adjacent variable \(v_j\) of degree one will send a message \(w_{a_i \rightarrow v_j}(A, 2) = u\) to all its other neighbours.
Proposition 2.6. Passing algorithms from [11], which we will use in Section 4.2 to prove the following result.

\[ V_0(A) = \{ v \in V(A) : \forall a \in \partial v : w_{a \rightarrow v}(A) = u \} \tag{2.4} \]

Proposition 2.5. For any \( d > 0 \) we have \( |\mathcal{F}(A) \cap V_0(A)| = o(n) \) w.h.p.

Further, tracing WP on the random graph \( G(A) \), we will establish the following bounds.

Proposition 2.8. For any \( d > 0 \) we have \( |V_2(A)|/n \geq \alpha_* + o(1) \) and \( |V_2(A)|/n \geq 1 - \alpha^* + o(1) \) w.h.p.

The proofs of Propositions 2.4 and 2.5 can be found in Section 4.2. The proofs of Propositions 2.7–2.8 can be found in Sections 5 and 6.

### 2.3. The slush.

To this end we need to take a closer look at the inconclusive \( s \)-messages. Indeed, the \( s \)-messages naturally induce a minor \( A_\alpha \) of \( A \). Generally, for a given matrix \( A \) let

\[ V_\alpha(A) = \{ v \in V(A) : \forall a \in \partial v : w_{a \rightarrow v}(A) \neq \mathbb{1}, |\{ a \in \partial v : w_{a \rightarrow v}(A) = s \}| \geq 2 \}, \tag{2.5} \]

\[ C_\alpha(A) = \{ a \in C(A) : \forall v \in \partial a : w_{v \rightarrow a}(A) \neq u, |\{ v \in \partial a : w_{v \rightarrow a}(A) = s \}| \geq 2 \}. \tag{2.6} \]

Hence, none of the variable nodes in \( V_\alpha(A) \) receive any \( f \)-messages, but each receives at least two \( s \)-messages. Analogously, the check nodes in \( C_\alpha(A) \) do not receive \( u \)-messages but get at least two \( s \)-messages. Let \( G_\alpha(A) \) be the subgraph of \( G(A) \) induced on \( V_\alpha(A) \cup C_\alpha(A) \). Moreover, let \( A_\alpha \) be the minor of \( A \) comprising the rows and columns whose corresponding variable or check nodes belong to \( V_\alpha(A) \) and \( C_\alpha(A) \), respectively. We observe that \( G_\alpha(A) \) admits an alternative construction that resembles the construction of the 2-core of a random hypergraph.

Indeed, \( G_\alpha(A) \) results from \( G(A) \) by repeating the following peeling operation:

while there is a variable or check node of degree at most one, remove that node along with its neighbour (if any). \tag{2.7}

To determine the size and the degree distribution of \( G_\alpha(A) \) we employ a general result about WP-like message passing algorithms from [11], which we will use in Section 4.2 to prove the following result.

Proposition 2.9. Define

\[ \lambda = \lambda(d) = d(\alpha^* - \alpha_*), \quad \nu = \nu(d) = \exp(-d\alpha_*) - \exp(-d\alpha^*)(1 + d(\alpha^* - \alpha_*)). \] \tag{2.8}

For any \( d > e \) we have \( \nu > 0 \) and

\[ \lim_{n \to \infty} \frac{|V_\alpha(A)|}{n} = \lim_{n \to \infty} \frac{|C_\alpha(A)|}{n} = \nu \text{ in probability.} \tag{2.9} \]

Moreover, for any integer \( \ell \geq 2 \) we have, in probability,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{a \in V_\alpha(A)} 1 \{ |\partial a \cap C_\alpha(A)| = \ell \} = \lim_{n \to \infty} \frac{1}{n} \sum_{a \in C_\alpha(A)} 1 \{ |\partial a \cap V_\alpha(A)| = \ell \} = \mathbb{P} [\mathcal{P}_{\geq 2}(\lambda) = \ell]. \tag{2.10} \]

Based on what we have learned about Warning Propagation, we are now in a position to establish items \textbf{FIX} and \textbf{STAB} from the outline from Section 1.4.

Proposition 2.7. For all \( d \in (e, \infty) \) we have \( \lim_{n \to \infty} \mathbb{E} \left[ |f(A) - \alpha_*| \wedge |f(A) - \alpha_0| \wedge |f(A) - \alpha^*| \right] = 0. \)

Proposition 2.8. For any \( d \in (e, \infty) \) there exists \( \varepsilon > 0 \) such that \( \lim_{n \to \infty} \mathbb{P} \left[ |f(A) - \alpha_0| < \varepsilon \right] = 0. \)

The proofs of Propositions 2.7, 2.8 can be found in Sections 5 and 6.
2.4. The aspect ratio. We are left to deliver on item EQ from the proof outline. Thus, we need to show that \( f(A) \) takes either value \( \alpha_* \), \( \alpha^* \) with about equal probability if \( d > e \). The description (2.7) of \( G_\theta(A) \) in terms of the peeling process underscores that \( |V_\theta(A)| \) and \( |C_\theta(A)| \) are identically distributed. Yet in order to prove the second part of Theorem [1.1] we need to know that w.h.p. the slush matrix is not close to square. In Section [7] we prove the following.

**Proposition 2.9.** For any \( d_0 > e \) there exists a function \( \omega = \omega(n) \gg 1 \) such that for all \( d > d_0 \) we have

\[
\lim_{n \to \infty} \mathbb{P}(|V_\theta(A)| - |C_\theta(A)| \geq \omega) = \lim_{n \to \infty} \mathbb{P}(|C_\theta(A)| - |V_\theta(A)| \geq \omega) = \frac{1}{2}.
\]

2.5. Moments and expansion. Finally, to complete step EQ in Section [8] we prove that \( f(A) \) is about equal to the higher possible value \( \alpha^* \) if \( A_\theta \) has more rows than columns, and equal to the lower value \( \alpha_* \) otherwise.

**Proposition 2.10.** For any \( d > e, \varepsilon > 0, \omega = \omega(n) \gg 1 \) we have

\[
\limsup_{n \to \infty} \mathbb{P}\left(|f(A) - \alpha^*| < \varepsilon, |V_\theta(A)| - |C_\theta(A)| \geq \omega\right) = 0, \quad \limsup_{n \to \infty} \mathbb{P}\left(|f(A) - \alpha_*| < \varepsilon, |C_\theta(A)| - |V_\theta(A)| \geq \omega\right) = 0.
\]

We now have all the ingredients in place to complete the proof of the main theorem.

**Proof of Theorem [1.1]**

(i) Suppose \( d < e \). Combining Propositions [2.4] and [2.5] with [2.3] and [2.4], we conclude that \( \alpha_* - o(1) \leq f(A) \leq \alpha^* + o(1) \) w.h.p. Since Proposition [2.3] yields \( \alpha_* = \alpha^* \), the assertion follows.

(ii) Fix \( d > e \) and \( \varepsilon > 0 \) and let \( \delta_* = \{ |f(A) - \alpha_*| < \varepsilon\}, \delta^* = \{ |f(A) - \alpha^*| < \varepsilon\} \). Then Propositions [2.7] and [2.8] imply that \( \mathbb{P}(|\delta_* \cup \delta^*| = 1 - o(1)) \). Moreover, Propositions [2.9] and [2.10] show that \( \mathbb{P}(\delta_* \leq 1/2 + o(1)) \) and \( \mathbb{P}(\delta^* \leq 1/2 + o(1)) \). Hence, we conclude that \( \mathbb{P}(\delta_*, \delta^*) = 1/2 + o(1) \), as claimed.

2.6. The overlap. Theorem [1.2] concerning the overlap follows relatively easily from Theorem [1.1]. The single additional ingredient that we need is the following statement that provides asymptotic independence of the first few coordinates \( x_1, \ldots, x_\ell \) of a vector \( x \) drawn from the posterior distribution [1.3].

**Proposition 2.11.** For every \( \ell \geq 1 \) there exists \( \gamma > 0 \) such that for all \( d > 0 \) and all \( \sigma \in \mathbb{F}_2^\ell \) we have

\[
\lim_{n \to \infty} \mathbb{E}\left[n^\gamma |\mathbb{P}_n(x_1 = \sigma_1, \ldots, x_\ell = \sigma_\ell | A) - \prod_{i=1}^\ell \mathbb{P}_n(x_i = \sigma_i | A)| = 0.\right.
\]

Proposition [2.11] whose proof we defer to Appendix [A] is a corollary to a random perturbation of the matrix \( A \) developed in [3]. As an easy consequence of Proposition [2.11] we obtain the following expression for the overlap. The proof can also be found in Appendix [A].

**Corollary 2.12.** For all \( d > 0 \) we have \( \lim_{n \to \infty} \mathbb{E}\left[R(x, x') - (1 + f(A))/2\right] = 0. \)

**Proof of Theorem [1.2]** The assertion is an immediate consequence of Theorem [1.1] and Corollary [2.12].

2.7. Preliminaries and notation. Throughout the paper, we use the standard Landau notations for asymptotic orders and all asymptotics are taken as \( n \to \infty \). Where asymptotics with respect to another additional parameter are needed, we indicate this by using an index. For example, \( g(\varepsilon, n) = o_\varepsilon(1) \) means that

\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} |g(\varepsilon, n)| = 0.
\]

We ignore floors and ceilings whenever they do not significantly affect the argument.

Any \( m \times n \mathbb{F}_2 \)-matrix \( A \) is perfectly represented by its Tanner graph \( G(A) \), as defined in Section [2.2]. We simply identify \( A \) with its Tanner graph \( G(A) \). For instance, we take the liberty of writing \( f(G(A)) \) instead of \( f(A) \). Conversely, a bipartite graph \( G \) with designated sets of check nodes \( C(G) \) and variable nodes \( V(G) \) induces a \( |C(G)| \times |V(G)| \) matrix \( A(G) \). Once again we tacitly identify \( G \) with this matrix. Recall that for a Tanner graph \( G \) and a node \( z \in C(G) \cup V(G) \) we let \( \delta z = \delta_G(z) \) signify the set of neighbours. We further define \( \delta^2 z = \delta_G^2(z) \) to be the set of nodes at distance exactly \( t \) from \( z \).

For a matrix \( A \) we generally denote by \( \mathcal{F}(A) = \mathcal{F}(G(A)) \) the set of frozen variables. In addition, we let \( \hat{\mathcal{F}}(A) \) be the set of frozen checks, where a check node \( a \in C(A) \) is called frozen if \( \delta a \subseteq \mathcal{F}(A) \). Let \( \hat{f}(A) = |\hat{\mathcal{F}}(A)|/|C(A)| \) be the fraction of frozen checks.

For a matrix \( A \) with Tanner graph \( G \) and a node \( z \) of \( G \) let \( d_A(z) = d_G(z) \) denote the degree of \( z \). Furthermore, let \( d_A = (d_A(z))_{z \in C(A) \cup V(A)} \) signify the degree sequence of \( G(A) \). In addition, let \( d_{A, b} = (d_{A, b}(z))_{z \in C(A) \cup V(A)} \) encompass
the degrees of the subgraph \( G_s(A) \). Note that this sequence includes degrees of vertices which are not actually in \( G_s(A) \), whose degree in \( G_s(A) \) we define to be 0.

Returning to the random matrix \( A \), let \( G_s \) be a random multigraph drawn from the pairing model with degree distribution \( d_{A,s} \).

**Lemma 2.13.** The probability that \( G_s \) is a simple graph is bounded away from 0. Furthermore, conditioned on being simple the graph \( G_s \) has exactly the same distribution as \( G_s(A) \).

The proof of this lemma is a standard exercise, which we include in Appendix B for completeness. We further need a routine estimate of the degree distribution of the random bipartite graph \( G(A) \), whose proof can be found in Appendix C.

**Lemma 2.14.** Let \( d \in \mathbb{Z} \). Moreover, upper bounds on the cut distance carry over to upper bounds on the marginal distributions, i.e.,

\[
\max_{\sigma \in \Omega} |\partial \sigma| \leq \log n, \quad \frac{1}{n} \sum_{x \in V(A)} \left( \frac{|\partial x|}{\ell} \right) \leq (2d)^\ell \quad \text{for any integer } \ell \geq 1.
\]

Throughout the paper all logarithms are to the base e.

The entropy of a probability distribution \( \mu \) on a finite set \( \Omega \neq \emptyset \) is denoted by

\[
H(\mu) = -\sum_{\omega \in \Omega} \mu(\omega) \log \mu(\omega).
\]

As a further important tool we need the cut metric for probability measures on \( F^2_2 \). Following [14], we define the cut distance of two probability measures \( \mu, \nu \) on \( F^2_2 \) as

\[
\Delta_\square(\mu, \nu) = \frac{1}{n} \min_{\sigma, \tau} \max_{\bar{\sigma}, \bar{\tau}} \frac{1}{|I|} \sum_{i \in I} \left| \mathbb{P}[\sigma_i = 1] - \mathbb{P}[\tau_i = 1] \right|.
\]

In words, we first minimise over couplings \( (\sigma, \tau) \) of the probability measures \( \mu, \nu \). Then, given such a coupling an adversary points out the largest remaining discrepancy. Specifically, the adversary puts their finger on the event \( U \) and the set of coordinates \( I \) where the frequency of 1-entries in \( \sigma, \tau \) differ as much as possible.

The cut metric is indeed a (very weak) metric. We need to point out a few of its basic properties. For a probability measure \( \mu \) on \( F^2_2 \) let \( \sigma^{(\mu)} \) denote a sample from \( \mu \). Moreover, let \( \tilde{\mu} \) be the product measure with the same marginals, i.e.,

\[
\tilde{\mu}(\sigma) = \prod_{i=1}^n \mu\left( \left\{ \sigma_i^{(\mu)} = \sigma_i \right\} \right) \quad (\sigma \in F^2_2).
\]

It is easy to see that upper bounds on the cut distance of \( \mu, \nu \), carry over to \( \tilde{\mu}, \tilde{\nu} \), i.e.,

\[
\Delta_\square(\tilde{\mu}, \tilde{\nu}) \leq \Delta_\square(\mu, \nu).
\]

Moreover, upper bounds on the cut distance carry over to upper bounds on the marginal distributions, i.e.,

\[
\frac{1}{n} \sum_{i=1}^n \mu\left( \left\{ \sigma_i^{(\mu)} = 1 \right\} \right) \leq \Delta_\square(\mu, \nu).
\]

The distribution \( \mu \) is \( \varepsilon \)-extremal if \( \Delta_\square(\mu, \tilde{\mu}) < \varepsilon \). Furthermore, µ is \( \varepsilon \)-symmetric if

\[
\sum_{1 \leq i < j \leq n} \left| \mu\left( \left\{ \sigma_i^{(\mu)} = 1 \right\} \right) - \mu\left( \left\{ \sigma_j^{(\mu)} = 1 \right\} \right) \right| < \varepsilon n^2.
\]

Hence, for most pairs \( i, j \) the entries \( \sigma_i, \sigma_j \) are about independent. More generally, \( \mu \) is \( (\varepsilon, \ell) \)-symmetric if

\[
\sum_{\tau \in F_2} \sum_{1 \leq i < j \leq \ell} \mu\left( \left\{ \forall j \leq \ell : \sigma_j^{(\mu)} = \tau_j \right\} \right) \leq \varepsilon n^\ell.
\]

The following statement summarises a few results about the cut metric from [5,14].

**Proposition 2.15.** For any \( \ell, \varepsilon > 0 \) there exist \( \delta > 0 \) and \( n_0 > 0 \) such that for all \( n > n_0 \) and all probability measures \( \mu \) on \( F^2_2 \) the following statements hold.

(i) If \( \mu \) is \( \delta \)-extremal, then \( \mu \) is \( (\varepsilon, \ell) \)-symmetric.

(ii) If \( \mu \) is \( \delta \)-symmetric, then \( \mu \) is \( \varepsilon \)-extremal.
Furthermore, extremality of measures carries over to conditional measures so long as we do not condition on events that are too unlikely. More generally, we call two probability measures \( \mu, \nu \) on \( \mathbb{F}_2^n \) mutually \( c \)-contiguous if \( c^{-1} \mu(\sigma) \leq \nu(\sigma) \leq c \mu(\sigma) \) for all \( \sigma \in \mathbb{F}_2^n \).

**Proposition 2.16** ([19]). For any \( \epsilon > 0 \) there exist \( \delta > 0 \) and \( n_0 > 0 \) such that for all \( n > n_0 \), any \( \delta \)-extremal probability measure \( \mu \) on \( \mathbb{F}_2^n \) and any probability measure \( \nu \) on \( \mathbb{F}_2^n \) such that \( \mu, \nu \) are mutually \((1/\epsilon)\)-contiguous, we have \( \Delta \square(\mu, \nu) < \epsilon \).

Moreover, we need an elementary observation about the kernel of \( \mathbb{F}_2 \)-matrices.

**Fact 2.17** ([3] Lemma 2.3). Let \( A \) be an \( m \times n \)-matrix over \( \mathbb{F}_2 \) and choose \( \xi = (\xi_1, \ldots, \xi_n) \in \ker A \) uniformly at random. Then for any \( i, j \in [n] \) we have \( P[\xi_i = 0] \in [1/2, 1] \) and \( P[\xi_i = 1] \in [1/2, 1] \).

Finally, in Appendix D we will prove the following auxiliary statement about weighted sums.

**Lemma 2.18.** For any \( c_0, c_1 > 0 \) there exists \( c_2 > 0 \) such that for all \( n > 0 \) the following is true. Suppose that \( w : [n] \to (0, \infty) \) is any function such that

\[
\frac{1}{n} \sum_{i=1}^{n} w_i \left[ |w_i| > t \right] \leq c_0 \exp(-c_1 t)
\]

for any \( t \geq 1 \).

Moreover, assume that \( \mathcal{D} = (P_1, \ldots, P_\ell) \) is any partition of \([n]\) into pairwise disjoint sets such that

\[
\frac{1}{n} \sum_{j=1}^{\ell} \left| P_j \right| \left[ |P_j| > t \right] \leq c_0 \exp(-c_1 t)
\]

for any \( t \geq 1 \).

Then \( \frac{1}{n} \sum_{j=1}^{\ell} \left( \sum_{i \in P_j} w_i \right)^2 \leq c_2 \).

3. Fixed Points and Local Maxima

In this section we prove Lemma 2.1 and Proposition 2.3. We begin with a bit of trite calculus.

3.1. Getting started. We introduce \( D_d(a) = \exp(-d(1-a)) \) so that

\[
\phi_d(a) = 1 - \exp(-d \exp(-d(1-a))) = 1 - D_d(1-D_d(a)), \quad \Phi_d(a) = D_d(1-D_d(a)) + (1+d(1-a))D_d(a) - 1. \quad (3.1)
\]

We need two derivatives of \( \Phi_d(a) \) and \( \phi_d(a) \):

\[
\Phi'_d(a) = d^2 D_d(a) (\phi_d(a) - a), \quad \phi'_d(a) = d^2 D_d(1-D_d(a)) D_d(a), \quad (3.2)
\]

\[
\Phi''_d(a) = d^2 D_d(a) \left( \phi'_d(a) - a \right) + d^2 D_d(a) \left( \phi'_d(a) - 1 \right), \quad \phi''_d(a) = d^2 D_d(1-D_d(a)) D_d(a) (1 - dD_d(a)). \quad (3.3)
\]

Since \( D_d(a) \) is strictly increasing for all \( d > 0 \), so is \( \phi_d(a) \) due to (3.1). Thus,

\[
\phi'_d(a) > 0 \quad \text{for all } a \in [0, 1]. \quad (3.4)
\]

Moreover, (3.3) shows that the sign of \( \phi''_d(a) \) only depends on the last term, denoted by

\[
\psi_{d, \text{sign}}(a) = 1 - d D_d(a). \quad (3.5)
\]

We denote the unique zero of \( \psi_{d, \text{sign}}(a) \) by \( \bar{a} = 1 - \frac{\log d}{d} \). The following claim comes down to an exercise in calculus.

**Claim 3.1.**

(i) \( \bar{a} \) is a fixed point of \( \phi_d \) iff \( d = e \).

(ii) \( \phi''_d(0) > 0 \).

(iii) \( \phi''_d(a) \) has one zero at \( \bar{a} \) in the interval \([0, 1]\) if \( d \geq 1 \), none otherwise.

(iv) \( \phi'_d(\bar{a}) = 1 \) and \( \phi''_d(\bar{a}) = 0 \).

(v) \( \bar{a} \) is the only fixed point of \( \phi_e(a) \).

(vi) The fixed points of \( \phi_d \) coincide with the stationary points of \( \Phi_d \).

(vii) \( \Phi'_d(0) > 0 > \Phi'_d(1) \).

(viii) For any \( d > 0 \) the function \( \phi_d \) has at least one stable fixed point.

(ix) For any \( d > 0 \) the function \( \phi_d \) has at most three fixed points, no more than two of which are stable.

(x) For \( d < e \), we have \( \phi'_d(a) < 1 \) for all \( a \in [0, 1] \).

(xi) For \( d < e \), the function \( \Phi_d \) attains a unique local maximiser \( \alpha_d \in (0, 1) \).

(xii) For \( d > e \), if \( a \in (0, 1) \) is a fixed point of \( \phi_d \) then so is \( \bar{a} = 1 - \exp(-d(1-a)) \in (0, 1) \).
Proof. (i) Observe that $\phi_d(\bar{a}) = 1 - 1/e$, which is a fixed point iff $\bar{a} = 1 - \log d - 1 - 1/e$, i.e. iff $d = e$.

(ii) Recall that the sign of $\phi''_d(\alpha)$ is determined by the sign of $\psi_{d,\sign}(\alpha)$, and we have $\psi_{d,\sign}(0) = 1 - d\exp(-d) > 0$ for all $d > 0$.

(iii) Since $\psi_{d,\sign}(\alpha) = -d^2\exp(-d(1 - \alpha)) < 0$, we see that $\psi_{d,\sign}$ is a decreasing function that has its unique zero at $\bar{a}$. Furthermore, $\bar{a} < 1$ iff $d > 1$.

(iv) By (3.3), when $d = e$ and $\alpha = \bar{a}$, Equation (3.3) reduces to $\Phi''_e(\bar{a}) = e^2D_e(\bar{a})\left(\phi'_e(\bar{a}) - 1\right)$. Since also $D_e(\bar{a}) = 1/e$, by (3.2) we have $\phi'_e(\bar{a}) = 1$, and therefore also $\Phi''_e(\bar{a}) = 0$.

(v) Due to (i) $\bar{a}$ is a fixed point, and $\phi'_e(\bar{a}) = 1$ by (3.4). Since $\phi_e(\alpha)$ is convex for $\alpha < \bar{a}$ and concave for $\alpha > \bar{a}$ by (3.3), we deduce that $\phi_e(\alpha) > \alpha$ for $\alpha < \bar{a}$ and $\phi_e(\alpha) < \alpha$ for $\alpha > \bar{a}$, so $\bar{a}$ is the unique fixed point of $\phi_e(\alpha)$.

(vi) Since $d^2D_d(\alpha) > 0$, (3.2) implies that $\Phi''_d(\alpha) = 0$ iff $\phi'_d(\alpha) = \alpha$.

(vii) This follows from (3.2) since $\phi_d(0) > 0$ and $\phi_d(1) < 1$.

(viii) Since $\phi_d(0) > 0$ and $\phi_d(1) < 1$, and since $\phi_d$ is a continuous function, there must be at least one fixed point in $(0,1)$. Setting $\alpha_1 := \sup\{\alpha : \phi_d(\alpha) > \alpha\}$, we have that $\alpha_1$ is a fixed point by continuity. Furthermore, $\alpha_1$ is stable since there are points $\alpha < \alpha_1$ arbitrarily close to $\alpha_1$ for which $\phi_d(\alpha) > \alpha$, but also for any $\alpha > \alpha_1$ we have $\phi_d(\alpha) < \alpha$, and therefore $\phi'_d(\alpha_1) < 1$.

(ix) This is a consequence of (iii): between any two fixed points there must be a point with $\phi'(\alpha) = 1$, and between any two such points there must be a point with $\phi'(\alpha) < 0$; furthermore, between any two stable fixed points, there must be an unstable fixed point.

(x) If $d < 1$, (iii) and (iii) imply that $\phi''(\alpha) > 0$ on $[0,1]$. Therefore $\phi''(\alpha) \leq \phi''(1) = d^2e^{-d} < 1$. For $1 \leq d < e$, Property (iii) proves that for all $\alpha \in [0,1]$ we have $\phi'(\alpha) < \phi'(\bar{a}) = d/e < 1$.

(xi) By (3.4), we may consider stable fixed points of $\phi_d$ rather than maximisers of $\phi_d$. The difference $h(\alpha) := \phi_d(\alpha) - \alpha$ is a decreasing function since $h'(\alpha) = \phi'_d(\alpha) - 1 < 0$ by (3.4). Since $h(0) > 0$ and $h(1) < 0$, $h(\alpha)$ has only one zero for $d < e$. This shows that the stable fixed point from (viii) is the unique fixed point.

(xii) Using $\alpha = \phi_d(\alpha) = 1 - \exp(-d \exp(-d(1 - \alpha)))$, we obtain

$$\exp(-d(1 - \bar{a})) = \exp(-d \exp(-d(1 - \alpha))) = 1 - \alpha = -\log(1 - \bar{a})/d.$$ 

Rearranging this inequality shows that $\bar{d} = \phi_d(\bar{a})$.

\[\Box\]

3.2. Proof of Lemma 2.1. At a fixed point $\alpha$ of $\phi_d$, (3.3) simplifies to

$$\Phi''_d(\alpha) = d^2D_d(\alpha)(\phi'_d(\alpha) - 1).$$

This shows $\Phi''_d(\alpha) < 0$ iff $\phi'_d(\alpha) < 1$. Hence, for $d > 0, d \neq e$, (3.4) and Claim 3.1 (vi) imply that the stable fixed points of $\phi_d$ are precisely the local maximisers of $\phi_d$. Claim 3.1 (vi) proves the second assertion in the case $d = e$.

3.3. Proof of Proposition 2.3. We make further observations on the existence and stability of fixed points of $\phi_d$.

Lemma 3.2. If $d > e$ then $\Phi_d$ attains its unique local minimum $a_0 \in [\alpha, \alpha^*]$ at the root of $1 - \alpha - \exp(-d(1 - \alpha))$.

Proof. The concave function $\alpha \in [0,1] \mapsto 1 - \exp(-d(1 - \alpha))$ has a unique fixed point $\beta = \beta(d) \in (0,1)$, which satisfies

$$\phi_d(\beta) = 1 - \exp(-d(1 - \beta)) = \beta, \quad \phi'_d(\beta) = d^2\exp(-d(1 - \beta)) \exp(-d(1 - \beta)) = d^2(1 - \beta)^2.$$ 

Hence, Claim 3.1 (vi) and (3.6) yield

$$\Phi''_d(\beta) = 0, \quad \Phi''_d(\beta) = d^2\exp(-d(1 - \beta))(d^2(1 - \beta)^2 - 1).$$

In order to determine the sign of the last expression we differentiate with respect to $d$, keeping in mind that $\beta = \beta(d)$ is a function of $d$. Rearranging the fixed point equation $\beta = 1 - \exp(-d(1 - \beta))$, we obtain $d = -(1 - \beta)^{-1}\log(1 - \beta)$. The inverse function theorem therefore yields

$$\frac{d\beta}{dd} = \frac{(1 - \beta)^2}{1 - \log(1 - \beta)}.$$

Combining the chain rule with the fixed point equation $\beta = 1 - \exp(-d(1 - \beta))$, we thus obtain

$$\frac{d}{dd}d^2(1 - \beta)^2 = 2d(1 - \beta)^2 - 2d^2(1 - \beta)\frac{d\beta}{dd} = 2d(1 - \beta)^2 \left(1 - \frac{d(1 - \beta)}{1 - \log(1 - \beta)}\right) = 2d(1 - \beta)^2 > 0.$$ 

\footnote{Note that at this point we could also have observed that $\Phi_d$ attains its maximum in the interior of $(0,1)$ and then applied Lemma 4.1 to prove the existence of a stable fixed point. This would be permissible since the proof of Lemma 4.1 only uses earlier points from this Claim and not (viii) or any later points, therefore the argument is not a circular one.}
As in Claim 3.1 at $d = e$ we obtain $\hat{\alpha} = 1 - 1/e$ and thus $d^2(1 - \hat{\beta})^2 = 1$. Therefore, (3.8) implies that $d^2(1 - \hat{\beta})^2 > 1$ for all $d > e$, and thus (3.7) shows that $\Phi_d$ attains its local minimum $a_0$ precisely at the point $\hat{\beta}$. Finally, by Claim 3.1 (vi) and (ix) there is precisely one local minimum in the interval $[\alpha_*, \alpha^*]$.

**Corollary 3.3.** For $d > e$ the function $\Phi_d$ attains its local maxima at the fixed points $0 < \alpha_* < \alpha^* < 1$ of $\Phi_d$. Moreover, $\Phi_d(\alpha_*) = \Phi_d(\alpha^*)$.

**Proof.** Since by Claim 3.1 (vii) we have $\Phi_d'(0) = 0 > \Phi_d'(1)$, the existence of the local minimiser $a_0 \in (0, 1)$ provided by Lemma 3.2 implies that $\Phi_d$ has at least two local maximisers $0 < a_1 < a_2 < 1$. Lemma 2.1 and Claim 3.1 (vi) show that $a_0, a_1, a_2$ are fixed points of $\Phi_d$. Hence, Claim 3.1 (x) implies that $a_1 = a_*$ is the smallest fixed point of $\Phi_d$ and that $a_2 = \alpha^* > a_*$ is the largest fixed point. Additionally, Lemma 2.1 and Claim 3.1 (ix) imply that $\alpha_*, \alpha^*$ are the only local maximisers of $\Phi_d$.

It remains to prove that $\Phi_d(\alpha_*) = \Phi_d(\alpha^*)$. Claim 3.1 (xii) implies that

$$\hat{\alpha}_* = 1 - \exp(-d(1 - \alpha_*)) \quad \text{and} \quad \hat{\alpha}^* = 1 - \exp(-d(1 - \alpha^*))$$

are fixed points of $\Phi_d$. Because $a_0 \neq \alpha_*, \alpha^*$ is the unique root of $1 - \alpha - \exp(-d(1 - \alpha))$, we conclude that $\hat{\alpha}_* = \alpha^*$ and $\hat{\alpha}^* = a_*$. Hence,

$$1 - \alpha^* = \exp(-d(1 - \alpha_*)), \quad 1 - \alpha_* = \exp(-d(1 - \alpha^*))$$

Consequently,

$$1 - \alpha_* \exp(-d(1 - \alpha_*)) = (1 - \alpha^*) \exp(-d(1 - \alpha^*)) \quad \text{and} \quad 1 - \alpha_* + \exp(-d(1 - \alpha_*)) = 1 - \alpha^* + \exp(-d(1 - \alpha^*))$$

Finally, combining (3.10)–(3.11) with the fixed point equations $\Phi_d(\alpha_*) = \alpha_*, \Phi_d(\alpha^*) = \alpha^*$, we obtain

$$\Phi_d(\alpha^*) - \Phi_d(\alpha_*) = \exp(-d \exp(-d(1 - \alpha^*))) + \exp(-d(1 - \alpha^*)) - \left[\exp(-d \exp(-d(1 - \alpha_*))) + \exp(-d(1 - \alpha_*))\right]$$

$$+ d \left[(1 - \alpha^*) \exp(-d(1 - \alpha^*)) - (1 - \alpha_*) \exp(-d(1 - \alpha_*))\right]$$

$$= 1 - \alpha^* + \exp(-d(1 - \alpha^*)) - (1 - \alpha_* + \exp(-d(1 - \alpha_*))) = 0,$$

thereby completing the proof. □

**Proof of Proposition 2.3.** The first part follows immediately from Lemma 2.1 and Claim 3.1 (x). The second assertion follows from Lemma 2.1, Lemma 3.2 and Corollary 3.3. □

3.4. **Proof of Lemma 2.2.** By a straightforward computation, we get that $\phi_d(0) > 0$ and $\phi_d(1) < 1$ for all $d > 0$. Moreover, $\phi_d(\alpha)$ is a continuously differentiable function. For $d < e$, by Claim 3.1 (vi) and (x) (or Proposition 2.3 (i)) there is one fixed point $\alpha_* = a_0 = \alpha^*$. This implies $\phi_d(\alpha) > \alpha$ for $\alpha \in [0, \alpha_*)$ and $\phi_d(\alpha) < \alpha$ for $\alpha \in (\alpha_*, 1]$. By Equation (3.4), $\phi_d(\alpha)$ is strictly increasing so $\phi_d(\phi_d(\alpha)) > \phi_d(\alpha)$ for $\alpha \in [0, \alpha_*)$ and $\phi_d(\phi_d(\alpha)) < \phi_d(\alpha)$ for $\alpha \in (\alpha_*, 1]$. By induction, for all $t > 0$, $\phi_d^{t+1}(\alpha) > \phi_d^t(\alpha)$ for $\alpha \in [0, \alpha_*)$ and $\phi_d^{t+1}(\alpha) < \phi_d^t(\alpha)$ for $\alpha \in (\alpha_*, 1]$. In addition, the fact that $\alpha_*$ is a fixed point of $\phi_d$ implies that $\alpha_* = \phi_d(\alpha_*) > \phi_d^{t+1}(\alpha)$ for $\alpha \in [0, \alpha_*)$ and $\alpha_* = \phi_d(\alpha_*) < \phi_d^t(\alpha)$ for $\alpha \in (\alpha_*, 1]$. Hence, for $\alpha \in (0, \alpha_*)$, the sequence $(\phi_d^t(\alpha))_{t \geq 0}$ is monotonically increasing and bounded above by $\phi_d(\alpha_*) = \alpha_*$, and therefore $\lim_{t \to \infty} \phi_d^t(\alpha)$ exists. Furthermore, since $\phi_d$ is continuous, this limit must be a fixed point of $\phi_d$. Since $\alpha_*$ is the smallest fixed point, we must have $\lim_{t \to \infty} \phi_d^t(\alpha) = \alpha_*$, as required. Similarly, for $\alpha \in (\alpha_*, 1]$, the sequence $(\phi_d^t(\alpha))_{t \geq 0}$ is monotonically decreasing and bounded below thus $\lim_{t \to \infty} \phi_d^t(\alpha) = \alpha^*$.

For $d > e$, by Proposition 2.3 (i), there are three fixed points, $\alpha_* < a_0 < \alpha^*$ where $\alpha_*, \alpha^*$ are stable fixed points and $a_0$ is unstable. For the intervals $[0, \alpha_*)$, $(\alpha_*, 1]$, the proof is exactly the same as in the case $d < e$. Similarly, $(\alpha_*, a_0)$ comes down to the case of a monotonically decreasing sequence converging to $\alpha_*$ while $(a_0, \alpha^*)$ comes down to the case of a monotonically increasing sequence converging to $\alpha^*$.

4. **Tracing Warning Propagation**

In this section we will analyse the local structure of $G(A)$ together with WP messages, and show that locally the graph has a rather simple structure. For this argument we will make use of the results of [11][2] The study of WP messages will enable us to prove Propositions 2.4, 2.5 and 2.6.

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2The article [11] deals with the standard binomial random graph $G(n, d/n)$, whereas in our situation we have the bipartite graph $G(n, n, d/n)$ – however, the proofs in that paper generalise in an obvious way to this setting.
4.1. Message distributions and the local structure. To investigate the link between the local graph structure and the WP messages we need a few definitions. Let us first define a message distribution to be a vector
\[ q = (q^{(v)}, q^{(c)}) \] with
\[ q^{(v)} = \left( q_{f}^{(v)}, q_{s}^{(v)}, q_{u}^{(v)} \right), \quad q^{(c)} = \left( q_{f}^{(c)}, q_{s}^{(c)}, q_{u}^{(c)} \right) \in [0,1]^3 \] s.t.
\[ \sum_{s \in \{f,s,u\}} q^{(v)}_s = \sum_{s \in \{f,s,u\}} q^{(c)}_s = 1. \]
Intuitively, \( q^{(v)}, q^{(c)} \) model the probability distribution of an incoming message at a check/variable node, so for example \( q^{(v)}_f \) is the probability that an incoming message at a variable node is \( f \).

Given a message distribution \( q \), we define \( \text{Po}(d|q) \) to be a distribution of half-edges with incoming messages. Specifically, at a variable node, this generates \( \text{Po}(d|q_f) \) half-edges whose in-message is \( f \) and similarly (and independently) generates half-edges whose in-message is \( s \) or \( u \). At a check node, the generation of half-edges with incoming messages is analogous. Let us define the message distribution
\[ q_s := (q_s^{(v)}, q_s^{(c)}) \] with
\[ q_s^{(v)} = (q_{s,f}, q_{s,s}, q_{s,u}) : (1 - \alpha, \alpha - \alpha_s, \alpha_s), \]
\[ q_s^{(c)} = (q_{s,f}, q_{s,s}, q_{s,u}) : (\alpha_s, \alpha^* - \alpha_s, 1 - \alpha^*). \]
which is our conjectured limiting distribution of a randomly chosen message after the completion of WP, which motivates the following definitions.

Definition 4.1. We define branching processes \( \mathcal{T}, \hat{T} \) which will generate rooted trees decorated with messages along edges towards the root.

(i) The root of the first process \( \mathcal{T} \) is a variable node \( v_0 \). The root spawns \( \text{Po}(d) \) children, which are check nodes. The edges from the children to the root independently carry an \( f \)-message with probability \( 1 - \alpha^* \), an \( s \)-message with probability \( \alpha^* - \alpha_s \), and a \( u \)-message with probability \( \alpha_s \). The process then proceeds such that each check node spawns variable nodes and each variable node spawns check nodes as its offspring such that the messages sent from the children to their parents abide by the rules from Figure 2. To be precise, a check node \( a \) that sends its parent message \( z \in \{ f, s, u \} \) has offspring
\[ z = f: \text{Po}(\alpha, d) \text{ children that send an } f \text{-message}. \]
\[ z = s: \text{Po}(\alpha, d) \text{ children that send an } s \text{-message and } \text{Po}_{\geq 1}(d(\alpha^* - \alpha_s)) \text{ children that each send an } s \text{-message.} \]
\[ z = u: \text{Po}(\alpha, d) \text{ children that send an } u \text{-message and } \text{Po}_{\geq 1}(d(\alpha^* - \alpha_s)) \text{ children that each send an } u \text{-message.} \]
Analogously, a variable node \( v \) that sends its parent message \( z \in \{ f, s, u \} \) has offspring
\[ z = f: \text{Po}_{\geq 1}((1 - \alpha_s)d) \text{ children that send an } f \text{-message, } \text{Po}(d(\alpha^* - \alpha_s)) \text{ children that send an } s \text{-message, and } \text{Po}(\alpha, d) \text{ children that send a } u \text{-message.} \]
\[ z = s: \text{Po}(\alpha, d) \text{ children that each send a } u \text{-message and } \text{Po}_{\geq 1}(d(\alpha^* - \alpha_s)) \text{ children that each send an } u \text{-message.} \]
\[ z = u: \text{Po}(\alpha, d) \text{ children that send a } u \text{-message.} \]

(ii) The root of the second process \( \hat{T} \) is a check node \( a_0 \). The root spawns \( \text{Po}(d) \) children, which are variable nodes. They independently send messages \( f, s, u \) with probabilities \( \alpha_s, \alpha^* - \alpha_s, 1 - \alpha^* \). Apart from the root, the nodes have offspring as under (i).

Let us note that the processes \( \mathcal{T}, \hat{T} \), when truncated at depth \( t \in \mathbb{N} \), are equivalent to the following: generate a 2-type branching tree up to depth \( t \) from the appropriate type of root in which each variable node has \( \text{Po}(d) \) children which are check nodes and vice versa, generate messages from the leaves at depth \( t \) at random according to \( q_s \) and generate all other messages up the tree from these according to the WP update rule.

The following is the critical lemma describing the local structure. Given an integer \( t \), let us define \( \mathcal{T}_t \) to be the set of messaged trees rooted at a variable node and with depth at most \( t \), and similarly \( \hat{T}_t \) for trees rooted at a check node. For any \( T \in \mathcal{T}_t \) and matrix \( A \), let us define
\[ \xi_T(A) := \frac{1}{n} \sum_{v \in \mathcal{V}(A)} 1 \{ \delta_{G(A)}^t[v] = T \} \]
to be the empirical fraction of variable nodes whose rooted depth \( t \) neighbourhood \( G(A) \) with edges towards the root annotated by the WP messages \( (w_{u \rightarrow y}(A), w_{y \rightarrow u}(A)) \) is isomorphic to \( T \). For \( \hat{T} \in \hat{T}_t \), the parameter \( \xi_{\hat{T}}(A) \) is defined similarly. We also define \( \xi_T := \mathbb{P} \{ \mathcal{T}_t \equiv T \} \) and \( \xi_{\hat{T}} := \mathbb{P} \{ \hat{T}_t \equiv \hat{T} \} \) to be the probabilities that the appropriate branching process is isomorphic to \( T \) or \( \hat{T} \) respectively.
Lemma 4.2. For any constant $t$ and any trees $T \in \mathcal{F}_t$ and $\hat{T} \in \hat{\mathcal{F}}_t$ we have

$$\lim_{n \to \infty} |\xi_T(A) - \xi_{\hat{T}}| = 0 \quad \text{and} \quad \lim_{n \to \infty} |\xi_T^{\infty}(A) - \xi_{\hat{T}}^{\infty}| = 0 \quad \text{in probability.}$$

In other words, picking a random vertex and looking at its local neighbourhood gives asymptotically the same result as generating a $\text{Po}(d)$ branching tree to the appropriate depth and initialising messages at the leaves according to $q_\ast$.

Lemma 4.2 states that messages at the end of WP are roughly distributed according to $q_\ast$, but of course, $q_\ast$ does not reflect the messages at the start of the WP algorithm; our initialisation, in which all messages are $s$, is represented by the message distribution $q_0 = (q_0^{(v)}, q_0^{(c)}) := ((0, 1, 0), (0, 1, 0))$, but as the WP algorithm proceeds, the distribution will change, which motivates the following definition of an update function on message distributions.

Definition 4.3. Given a message distribution $q = \left(q_2^{(v)}, q_2^{(c)}, q_1^{(v)}, q_1^{(c)}\right)$, let us define the message distribution $\varphi(q)$ by setting

$$\varphi(q)_2^{(v)} := P \left[ d \left(q_2^{(v)} + q_1^{(v)}\right) = 0 \right], \quad \varphi(q)_2^{(c)} := P \left[ d q_2^{(c)} \geq 1 \right],
$$

$$\varphi(q)_1^{(v)} := P \left[ d q_1^{(v)} = 0 \right], \quad \varphi(q)_1^{(c)} := P \left[ d q_1^{(c)} = 0 \right] \cdot P \left[ d q_2^{(c)} \geq 1 \right].$$

We further recursively define $\varphi^{(t)}(q) := \varphi(\varphi^{(t-1)}(q))$ for $t \geq 2$, and define $\varphi^\infty(q) := \lim_{t \to \infty} \varphi^{(t)}(q)$ if this limit exists.

The function $\varphi$ represents an update function of the WP message distributions in an idealised scenario, but it turns out that this idealised scenario is close to the truth. The following lemma is critical in order to be able to apply the results of [11]. Let us define the total variation distance between message distributions $q_1, q_2$ by

$$d_{TV}(q_1, q_2) := d_{TV}(q_1^{(v)}, q_2^{(v)}) + d_{TV}(q_1^{(c)}, q_2^{(c)}).$$

Lemma 4.4. We have $\varphi^\infty(q_0) = q_\ast$. Furthermore, there exist $\epsilon, \delta > 0$ such that for any message distribution $q$ which satisfies $d_{TV}(q, q_\ast) \leq \epsilon$, we have $d_{TV}(\varphi(q), q_\ast) \leq (1 - \delta) d_{TV}(q, q_\ast)$.

In the language of [11], this lemma states that $q_\ast$ is the stable limit of $q_0$. Before proving this lemma, we first show how to use it to prove Lemma 4.2. We begin with the critical application of the main result of [11]. Recall that $w(A, t)$ denote the messages after $t$ iterations of WP on the Tanner graph $G(A)$ with all initial messages set as $s$, and $w(A) = \lim_{t \to \infty} w(A, t)$.

Lemma 4.5. For any $d, \delta > 0$ there exists $t_0 \in \mathbb{N}$ such that w.h.p. $w(A)$ and $w(A, t_0)$ are identical except on a set of at most $\delta n$ edges.

Proof. Since $q_\ast$ is the stable limit of $q_0$, this follows directly from [11 Theorem 1.5].

Using Lemma 4.5 we can determine the local limit of the graph with final WP messages.

Proof of Lemma 4.2. Fix $t_0$ sufficiently large, and in particular large enough that Lemma 4.5 can be applied. Since the local structure of the graph $G(A)$ is that of a $\text{Po}(d)$ branching tree, after $t_0$ iterations of WP for some sufficiently large $t_0$, the local structure with incoming messages is approximately as $\mathcal{F}_{t_0}$ and $\hat{\mathcal{F}}_{t_0}$. Subsequently, Lemma 4.5 implies that almost all messages at time $t_0$ are the final ones, and in particular there are very few vertices whose depth $t_0$ neighbourhood will change.

Proof of Lemma 4.4. For convenience, we will actually prove that $q_\ast$ is the stable limit of $q_0$ under the operator $\varphi^2$ rather than $\varphi$ – the advantage is that this 2-step operator acts on the coordinates (corresponding to variable and check nodes) independently of each other. The analogous statement for $\varphi$ follows from that for $\varphi^2$ due to continuity.

Furthermore, by symmetry we may prove the appropriate statements just for the first coordinate, i.e. for $q_1^{(v)}$ – the corresponding proof for $q_2^{(c)}$ is essentially identical.

As a final reduction, let us observe that since for any message distribution we have $q_2^{(v)} + q_1^{(v)} + q_u^{(v)} = 1$, it is sufficient to consider just two of the three coordinates. In this case it will be most convenient to consider $q_2^{(v)}$ and $q_u^{(v)}$, so let us restate what we are aiming to prove.
Consider the operator $\tilde{\varphi} : [0, 1]^2 \to [0, 1]^2$ defined by $\tilde{\varphi}(x_1, x_2) := (\varphi_1(x_1), \varphi_2(x_2))$, where

$$
\varphi_1(x_1) := \exp(-d \exp(-d x_1)), \quad \varphi_2(x_2) := 1 - \exp(-d \exp(-d (1 - x_2))).
$$

This corresponds precisely to the action of $\varphi^{\circ 2}$ on $(q^{(0)}_t, q^{(0)}_u)$. Thus our goal is to prove that $(1 - \alpha^*, \alpha_*)$ is the stable limit of $(0, 0)$ under $\tilde{\varphi}$.

Now observe that $\tilde{\varphi}_1(x_1) = 1 - \varphi_d(1 - x_1)$ and recall that $\varphi_d$ was defined in (1.1). By Lemma 2.2 and Proposition 2.3 $\varphi_d$ is a contraction on $[\alpha^*, 1]$ with unique fixed point $\alpha^*$, and so correspondingly $\tilde{\varphi}_1$ is a contraction on $[0, 1 - \alpha^*]$ with unique fixed point $1 - \alpha^*$.

On the other hand, $\tilde{\varphi}_2$ is exactly the function $\varphi_d$. Therefore, similarly, by Lemma 2.2 and Proposition 2.3 $\tilde{\varphi}_2$ is a contraction on $[0, \alpha_*]$ with unique fixed point $\alpha_*$. It follows that $(1 - \alpha_*, \alpha_*)$ is the limit $\tilde{\varphi}^* (0, 0)$.

To show that it is the \textit{stable} limit, we simply observe that $\tilde{\varphi}_1(1 - \alpha^*) = \varphi'_d(\alpha^*) < 1$ by Proposition 2.3 and similarly $\tilde{\varphi}_2(\alpha_*) = \varphi'_d(\alpha_*) < 1$. This implies that each coordinate function is a contraction in the neighbourhood of the corresponding limit point, and therefore so is $\tilde{\varphi}$.

\textbf{4.2. Proof of Proposition 2.5.} To determine the asymptotic proportion of vertices in $V_t(A)$, by Lemma 4.2 it suffices to determine the probability that in $\mathcal{F}$ the root receives at least one $\ell$-message. This event has probability

$$
P\left[\text{Po}(d(q_s^{(0)})) \geq 1\right] = 1 - \exp(-d(1 - \alpha^*)) = \alpha_*
$$

since $q_s^{(0)} = 1 - \alpha^*$ and by (8.9).

An analogous argument yields the statement for $V_u(A)$.

\textbf{4.3. Proof of Proposition 2.6.} To determine the asymptotic proportion of vertices in $V_s(A)$, by Lemma 4.2 it suffices to determine the probability that in $\mathcal{T}$ the root receives at least two $s$-messages and no $\ell$-messages. This occurs with probability

$$
P\left[\text{Po}(d(\alpha^* - \alpha_*)) \geq 2\right] \cdot P\left[\text{Po}(d\alpha_*) = 0\right] = \left(1 - \exp(-d(\alpha^* - \alpha_*)) - d(\alpha^* - \alpha_*) \exp(-d(\alpha^* - \alpha_*))\right) \cdot \exp(-d\alpha_*) = \exp(-d\alpha_*)(1 + d(\alpha^* - \alpha_*)),
$$

as claimed. The analogous statement for $C_u(A)$ can be proved similarly, or follows from the statement for $V_u(A)$ by symmetry.

The statement on degree distributions follows directly from the approximation using $\mathcal{F}$ or $\mathcal{F}$: conditioned on a node lying in $V_s$ or $C_s$, it must certainly receive at least two $s$-messages from its neighbours. Furthermore, a neighbour is in $C_u$ or $V_u$ respectively if and only if it sends an $s$-message to this vertex. The distribution of neighbours sending $s$ is $\text{Po}(\lambda)$ without the conditioning (where recall that $\lambda = d(\alpha^* - \alpha_*)$), therefore with the conditioning it is $\text{Po}_{\geq 2}(\lambda)$, as required.

\textbf{4.4. Proof of Proposition 4.4.} For a matrix $A$ we let

$$
V_t(A, t) = \{ v \in V(A) : \exists a \in \partial v : w_{u \to v}(A, t) = t \}, \quad V_u(A, t) = \{ v \in V(A) : \forall a \in \partial v : w_{u \to v}(A, t) = u \},
$$

$$
C_t(A, t) = \{ a \in C(A) : \exists v \in \partial a : w_{u \to v}(A, t) = t \}, \quad C_u(A, t) = \{ a \in C(A) : \exists v \in \partial a : w_{u \to v}(A, t) = u \}
$$

be the sets of nodes of $G(A)$ classified as frozen or unfrozen after $t$ iterations of WP. Furthermore, let $B(v, t)$ denote the nodes that are within distance $t$ of $v$. Let $\mathcal{B}_t$ be the set of variable nodes $v$ such that $B(v, t)$ contains at least one cycle.

\textbf{Claim 4.6.} Let $t_0 \geq 1$. If $v_0 \in V_u(A, t_0)$ and $v_0 \notin \mathcal{B}_{t_0}$, then $v_0 \notin \mathcal{F}(A)$.

\textbf{Proof.} Let $v_0 \in V_u(A, t_0)$. We will consider a subtree $T$ of $G(A)$ rooted at $v_0$ which we produce in the following way. All of the neighbours of $v_0$ are added to $T$ as children of $v_0$. Furthermore, since each such neighbour $a$ is a check node which sends $v_0$ a $u$-message at time $t_0$, the check node $a$ has at least one further neighbour (apart from $v_0$) from which it receives a $u$-message at time $t_0 - 1$ – we choose one such neighbour arbitrarily and add it to $T$ as a child of $a$. We continue recursively, for each variable node adding all neighbours (apart from the parent) if there are any, and for each check node at depth $i$ adding one neighbour (distinct from the parent) from which it receives message $u$ at time $t_0 - i$.

Since the leaves at depth $t_0$ send out $u$-messages at time 1, they must be unary variables (if they exist at all which is not the case if, for example, $t_0$ is odd). Therefore $T$ has the property that for any of its variable nodes, all its neighbours are also in $T$, while all checks have precisely two neighbours in $T$. 
Figure 3. An instance of the randomly generated trees added to $G(A)$ to produce $G'(A)$ in Definition 5.1: the variable and check root sets $\tilde{V}, \tilde{C}$ are shown in blue; the attachment nodes $\tilde{U}$ in green; the thick red edges are those in the trees, which are added to $G(A)$; the thin black edges were already present in $G(A)$; all explicitly drawn nodes were already present but, apart from possibly the attachment nodes (i.e. those in $\tilde{U}$), were previously isolated in $G(A)$.

Therefore we can obtain a vector in the kernel of $A$ that sets $x_{v_0}$ to 1 by simply setting all the variable nodes in $T$ to 1 and all other variables to zero. This shows that $v_0 \notin \mathcal{F}(A)$. □

Proof of Proposition 2.4. First observe that Claim 4.6 implies $V_u(A, t_0) \cap \mathcal{F}(A) \subseteq B_{t_0}$. Calculating the expectation of the number of vertices lying on cycles of length up to $2t_0$ and applying Markov inequality gives us that indeed $|B_{t_0}| = o(n)$. By choosing $t_0$ sufficiently large according to Lemma 4.5 we have $|V_u(A, t_0)| = |V_u(A)| + o(n)$ w.h.p. which concludes the proof. □

5. The standard messages

In this section we prove Proposition 2.7 which states that the proportion of frozen variables is likely close to one of the fixed points of $\phi_d$. Along the way we will establish auxiliary statements that will pave the way for the proof of Proposition 2.8 (which rules out the unstable fixed point) in Section 6 as well.

5.1. Perturbing the Tanner graph. A key observation toward Proposition 2.7 is that if we make some minor alterations to $G(A)$, the resulting graph $G'(A)$ is essentially indistinguishable from $G(A)$. Let $T = T(d)$ be the tree generated by a Galton-Watson process with the two types ‘variable node’ and ‘check node’. The root is a variable node $v_0$. Each variable node spawns $Po(d)$ check nodes as offspring. Similarly, the offspring of a check node consists of $Po(d)$ variable nodes. In addition, let $\tilde{T} = \tilde{T}(d)$ be the tree generated by a Galton-Watson process with the same offspring distribution whose root is a check node $a_0$. Given an integer $t$, we obtain $T_t$ and $\tilde{T}_t$ from $T$ and $\tilde{T}$, respectively, by deleting all nodes whose distance from the root exceeds $t$, so these are trees of depth (at most) $t$.

(Unlike the branching processes from Definition 4.1, the trees $T, \tilde{T}$ do not incorporate messages.)

Definition 5.1. Let $0 \leq \omega_1 = \omega_1(n) = o(\sqrt{n})$, $0 \leq \omega_2 = \omega_2(n) = n^{1/2-\Omega(1)}$ and obtain $G'(A)$ from $G(A)$ as follows.

(i) Generate $\omega_1$ many $T_2$ trees and $\omega_2$ many $\tilde{T}_1$ trees independently.

(ii) For each node $v$ in the final layer of these trees (which is a variable node), embed $v$ onto a variable node of $G(A)$ chosen uniformly at random and independently.

(iii) Embed the remaining nodes of the trees randomly onto nodes which were previously isolated such that variable nodes are embedded onto variable nodes and checks onto checks.

Let $G'(A)$ denote the resulting graph and let $A'$ be its adjacency matrix. (Thus $G'(A) = G(A')$ is the Tanner graph of $A'$.)
Let $\tilde{V}, \tilde{C}$ denote the set of variable and check nodes of $G'(A)$ respectively onto which the roots of the $T_2$ and $\hat{T}_1$ branching trees from Definition 5.1 (i) are embedded. Similarly, let $\tilde{U} = (\hat{\partial}C \cup \hat{\partial} \tilde{V}) \setminus \tilde{V}$ be the set of variable nodes of $G(A)$ where the checks from Definition 5.1 attach to the bulk of the Tanner graph in Step (ii). An example is shown in Figure [B].

Note that it is possible that this process fails, for example if there are not enough isolated nodes available, in which case we simply set $G' = G(A)$. However, since w.h.p. the total size of all trees is $O(\omega_1 + \omega_2)$, and w.h.p. there are $\Omega(n)$ isolated variable and check nodes available, the failure probability is $\exp(-\Omega(n))$ and thus negligible for our purposes. For the same reason w.h.p. no two nodes from the trees are embedded onto the same node of $G(A)$.

**Fact 5.2.** If $\omega_1 + \omega_2 = n^{1/2 - \Omega(1)}$, then $d_{TV}(G(A), G'(A)) = n^{-\Omega(1)}$.

This routine observation simply follows from the fact that w.h.p. we only added $n^{1/2 - \Omega(1)}$ edges attached to isolated nodes in such a way that the expected degrees are bounded, and the attachment variables were chosen uniformly at random. In particular the number of changes is of lower order than the standard deviation in the number of nodes of each type which has changed.

We point out that $\tilde{V}, \tilde{C}$ are representative of $G'(A)$ as a whole.

**Fact 5.3.** Let $\Lambda : (G, u) \to \Lambda(G, u) \in [0, 1]$ be any function that maps a pair consisting of a graph and a node to a number. If $1 \ll \omega_1, \omega_2 = n^{1/2 - \Omega(1)}$, then

$$\mathbb{E} \left[ \frac{1}{n} \sum_{v \in \tilde{V}} \Lambda(G'(A), v) - \frac{1}{|\tilde{V}|} \sum_{v \in \tilde{V}} \Lambda(G(A), v) \right] = o(1), \quad \mathbb{E} \left[ \frac{1}{n} \sum_{a \in \tilde{C}} \Lambda(G'(A), a) - \frac{1}{|\tilde{C}|} \sum_{a \in \tilde{C}} \Lambda(G(A), a) \right] = o(1).$$

**Proof.** The statement for $\tilde{V}$ follows since the graph obtained from $V(G'(A))$ and therefore also of $G'(A)$ by Fact 5.2, is that of a Po(d) branching tree, and this is clearly also the case at the variables of $\tilde{V}$. Formally, if $v$ is a variable node chosen uniformly at random from $V(G'(A))$ and $\tilde{v}$ is a random element of $\tilde{V}$, then Fact 5.2 implies that $(G'(A), \tilde{v})$ and $(G'(A), v)$ have total variation distance $o(1)$ given $G'(A)$ w.h.p. Therefore, the empirical average of $\Lambda$ on the entire set $V(G'(A))$ is well approximated by the average on $\tilde{V}$ w.h.p. The second statement concerning $\tilde{C}$ follows similarly. $\square$

**5.2. Construction of the standard messages.** In Section 2.2 we defined Warning Propagation messages via an explicit combinatorial construction that captured our intuition as to the causes of freezing. In the following we pursue a converse path. We define a set of messages implicitly, purely in terms of algebraic reality. We call these standard messages $\{f, u\}$-valued messages the standard messages. The battle plan is to ultimately match this implicit definition with the explicit construction from Section 2.2.

The standard messages can be defined for any $m \times n$-matrix $A$. Given a subset $U$ of nodes of a graph $G$, we denote by $G - U$ the graph obtained from $G$ by deleting $U$ and all incident edges. For a node $x$, we write $G - x$ instead of $G - \{x\}$. For each adjacent variable/check pair $(v, a)$ of $G(A)$ we define

$$m_{v \to a}(A) = \begin{cases} f, & \text{if } v \text{ is frozen in } G(A) - a, \\ u, & \text{otherwise}, \end{cases} \quad m_{a \to v}(A) = \begin{cases} f, & \text{if } v \text{ is frozen in } G(A) - (\partial v \setminus \{a\}), \\ u, & \text{otherwise}. \end{cases}$$

Hence, $m_{v \to a}(A) = f$ iff $v$ is frozen in the matrix obtained from $A$ by deleting the $a$-row. Moreover, $m_{a \to v}(A) = f$ iff $v$ is frozen in the matrix obtained by removing the rows of all $b \in \partial v$ except $a$. Let $m(A) = (m_{v \to a}(A), m_{a \to v}(A))_{v \in \partial a}$.

Further, we define $\{f, *, u\}$-valued marks for the variables and checks by letting

$$m_v(A) = \begin{cases} f, & \text{if } m_{a \to v}(A) = f \text{ for at least two } a \in \partial v, \\ *, & \text{if } m_{a \to v}(A) = f \text{ for precisely one } a \in \partial v, \\ u, & \text{otherwise}, \end{cases}$$

$$m_a(A) = \begin{cases} f, & \text{if } m_{v \to a}(A) = f \text{ for all } v \in \partial a, \\ *, & \text{if } m_{v \to a}(A) = f \text{ for all but precisely one } v \in \partial a, \\ u, & \text{otherwise}. \end{cases}$$

The intended semantics is that $f$ and $*$ both represent frozen variables/checks, meaning that a variable $v$ is frozen if $m_v(A) \neq u$ while for any check $a$ we have $m_a(A) \neq u$ if all variables $v \in \partial a$ are frozen. But for checks or variables with mark $*$, freezing hangs by a thread since, for instance, a variable $v$ with $m_v(A) = *$ receives just a single 'freeze'
message. We will see in Corollary 5.6 below how this manifests itself in the messages sent out by ∗-variables or checks.

We consider a dumbed-down version of the Warning Propagation operator WP_A from Section 2.2 that "updates" the messages from (5.1) to messages \( \hat{m}_{\nu \rightarrow a}(A) \) as follows:

\[
\hat{m}_{\nu \rightarrow a}(A) = \begin{cases} 
\mathbb{f} & \text{if } m_{\nu \rightarrow a}(A) = \mathbb{f} \text{ for some } b \in \partial \nu \setminus \{a\}, \\
\mathbb{u} & \text{otherwise,}
\end{cases}
\]

(5.4)

\[
\hat{m}_{a \rightarrow \nu}(A) = \begin{cases} 
\mathbb{f} & \text{if } m_{\nu \rightarrow a}(A) = \mathbb{f} \text{ for all } y \in \partial a \setminus \{\nu\}, \\
\mathbb{u} & \text{otherwise.}
\end{cases}
\]

(5.5)

We next show that the standard messages constitute an approximate fixed point of the WP_A operator and that the marks mostly match their intended semantics w.h.p.

**Lemma 5.4.** For all \( d > 0 \) we have

\[
\mathbb{E} \sum_{v \in \Gamma(A)} \sum_{a \in \partial v} \mathbb{1} \{ m_{\nu \rightarrow a}(A) \neq \hat{m}_{\nu \rightarrow a}(A) \} + \mathbb{1} \{ m_{a \rightarrow \nu}(A) \neq \hat{m}_{a \rightarrow \nu}(A) \} = o(n),
\]

(5.6)

\[
\mathbb{E} \{ v \in V(A) : m_v(A) \neq \mathbb{f} \} \Delta \mathcal{F}(A) = o(n), \quad \mathbb{E} \{ a \in C(A) : m_a(A) \neq \mathbb{u} \} \Delta \hat{\mathcal{F}}(A) = o(n).
\]

(5.7)

We prove Lemma 5.4 by way of the perturbation from Section 5.1. Specifically, in light of Fact 5.3 it suffices to consider \( G'(A) \) and the sets of variables/checks \( \tilde{V}, \tilde{C} \) onto which the roots of the \( T_2 \) and \( \tilde{T}_1 \) branching trees from Definition 5.1 are embedded. The following lemma summarises the main step of the argument. Recall that \( \tilde{U} \) is the set of variable nodes where the trees from Definition 5.1 attach to the bulk of the Tanner graph in Step (ii) (see Figure 3).

**Claim 5.5.** There exists \( 1 < \omega^* = \omega^*(n) \leq n^{1/2-\Omega(1)} \) such that for all \( \omega_1, \omega_2 \leq \omega^* \) and every \( d > 0 \) w.h.p. we have

\[
\hat{m}_{\nu \rightarrow a}(A) = \mathbb{f} \iff y \in \mathcal{F}(A)
\]

for all \( a \in \tilde{C} \cup \partial \tilde{V}, y \in \tilde{U} \cap \partial a \).

(5.8)

Furthermore, w.h.p. a random vector \( x \in \ker A \) satisfies

\[
\mathbb{P} \{ \forall y \in \tilde{U} \setminus \mathcal{F}(A) : x_y = \sigma_y | G(A), G'(A) \} = 2^{-|\tilde{U} \setminus \mathcal{F}(A)|}
\]

for all \( \sigma \in \mathbb{F}_2^{\tilde{U} \setminus \mathcal{F}(A)} \).

(5.9)

Finally, \( \mathcal{F}(A) \subseteq \mathcal{F}(A') \) and w.h.p. we have \( f(A') = f(A) + o(1) \).

**Proof.** Let us begin with the last statement. The inclusion \( \mathcal{F}(A) \subseteq \mathcal{F}(A') \) is deterministically true because \( A' \) is obtained from \( A \) by effectively adding checks (viz. "activating" formerly dormant isolated checks). Moreover, Proposition 2.11 shows that the distribution of a random \( x \in \ker A \) is \( n^{-\Omega(1)} \)-symmetric w.h.p. Since \( A' \) is obtained from \( A \) by adding no more than \( O(\omega^*) \) checks w.h.p. and since any additional check reduces the nullity by at most one, the distributions of a uniformly random \( x' \in \ker A' \) and of \( x \) are mutually \( 2^{O(\omega^*)} \)-contiguous w.h.p. Therefore, Proposition 2.16 implies that w.h.p.

\[
\Delta_{\mathbb{C}}(x, x') = o(1)
\]

(5.10)

provided that \( \omega^*(n) \) grows sufficiently slowly. Finally, since the marginals of the individual entries \( x_i, x'_i \) are either uniform or place all mass on zero by Fact 2.17 and 5.10 yield

\[
f(A') - f(A) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1} \{ v \in \mathcal{F}(A') \} - \mathbb{1} \{ v \in \mathcal{F}(A) \} \leq 2 \sum_{i=1}^{n} d_{\mathcal{V}}(x_i, x'_i) \leq 4\Delta_{\mathbb{C}}(x, x') = o(1).
\]

(5.11)

The other two assertions (5.8) and (5.9) follow from similar deliberations. Indeed, to prove (5.9) we observe that given \( G(A) \) the set \( \tilde{U} \) of variable nodes where the bottom layers of the trees from Definition 5.1 attach in Step (ii) is just a uniformly random set of \( O(\omega^*) \) variable nodes of \( G(A) \). Therefore, providing \( \omega^* \to \infty \) sufficiently slowly, Proposition 2.11 shows that w.h.p.

\[
\mathbb{P} \{ \forall y \in \tilde{U} \setminus \mathcal{F}(A) : x_y = \sigma_y | G(A), G'(A) \} = O(n^{-\Omega(1)})
\]

(5.12)

for any \( \sigma \in \mathbb{F}_2^{\tilde{U} \setminus \mathcal{F}(A)} \). Now, the projections of the vectors \( x \in \ker A \) onto the coordinates in \( \tilde{U} \setminus \mathcal{F}(A) \) form a subspace of \( \mathbb{F}_2^{\tilde{U} \setminus \mathcal{F}(A)} \). Assuming that \( |\tilde{U}| = O(\omega^*) \) and that \( \omega^* \to \infty \) sufficiently slowly, (5.12) implies that the dimension of this subspace equals \( |\tilde{U} \setminus \mathcal{F}(A)| \). Hence we obtain (5.9).
Regarding (5.8), fix some check $a \in \mathcal{C} \cup \partial \mathcal{V}$ and think of $G'(A)$, and therefore also its adjacency matrix $A'$, as being constructed in two steps. In the first step we embed all the other new checks $b \in (\mathcal{C} \cup \partial \mathcal{V}) \setminus \{a\}$ and insert the edges that join them to the variable nodes of $G(A)$. Let $G''(A)$ be the outcome of this first step and let $A''$ be its adjacency matrix. Subsequently we independently choose the set of neighbours $\partial a \setminus \mathcal{V}$ among the variable nodes of $G(A)$ to obtain $G'(A)$. Let $x'$ be a random element of $\ker A''$. Repeating the argument towards (5.10) we see that $\Delta_{\mathcal{C}}(x, x') = o(1)$ w.h.p. Hence, repeating the steps of (5.11) we conclude that $|\mathcal{F}(A)\Delta \mathcal{F}(A'')| = o(n)$ w.h.p. In since our two-round exposure $\partial a \setminus \mathcal{V}$ is independent of $A''$, we thus conclude that $\partial a \cap \mathcal{F}(A') \cap \partial a \cap \mathcal{F}(A) \cap \mathcal{V}$ w.h.p. Hence, the definition (5.1) of the standard messages implies (5.8).

\[\Box\]

**Proof of Lemma 5.4.** By Fact 5.3 it suffices to prove the fixed point conditions for the variables and checks $\hat{V}, \hat{C}$ of $G'(A)$ which are the roots of the $T_2$ and $T_1$ branching processes added in Definition 5.1. Hence, with $\omega^*$ from in Claim 5.5 let $\omega_1 = \omega_*, \omega_2 = 0$ and assume that (5.8)–(5.9) are satisfied. We may also assume that the subgraph of $G'(A)$ induced on $\mathcal{E} = \hat{V} \cup \hat{U} \cup \hat{\partial} \mathcal{V}$ is acyclic. Pick a variable $v \in \hat{V}$ and an adjacent check $a \in \partial v$. We will show that under the assumptions that the fixed point property is satisfied deterministically.

The definition (5.1) of the standard messages provides that $m_{a \rightarrow v}(A') = \# \text{ if } v \text{ is frozen in } G' - (\partial v \setminus \{a\})$. A sufficient condition is that $\partial a \setminus \{v\} \subseteq \mathcal{F}(A)$. Conversely, if $\partial a \setminus (\{v\} \cup \mathcal{F}(A)) \neq \emptyset$, then (5.9) shows that $v$ is unfrozen in $G'(A) - (\partial v \setminus \{a\})$. For there exists $\sigma \in \ker A$ such that $\sum_{y \in \partial a \setminus \{v\}} \sigma_y = 1$, and because the subgraph induced on $\mathcal{E}$ is acyclic this vector $\sigma$ extends to a vector $\sigma' \in \ker A'$ with $\sigma'_v = 1$. Hence, $v \not\in \mathcal{F}(A')$. Furthermore, (5.8) ensures that $\partial a \setminus \{v\} \subseteq \mathcal{F}(A)$ iff $m_{y \rightarrow a}(A') = \# \text{ for all } y \in \partial a \setminus \{v\}$. Hence, $m_{a \rightarrow v}(A') = \# \iff m_{y \rightarrow a}(A') = \#$ for all $y \in \partial a \setminus \{v\}$. In other words, we obtain

$$m_{a \rightarrow v}(A') = \hat{m}_{a \rightarrow v}(A') \quad \text{ for all } v \in \hat{V}, a \in \partial v. \quad (5.13)$$

A similar argument shows that

$$m_{v \rightarrow a}(A') = \hat{m}_{v \rightarrow a}(A') \quad \text{ for all } v \in \hat{V}, a \in \partial v. \quad (5.14)$$

Indeed, (5.1) guarantees that $m_{v \rightarrow a}(A') = \#$ if there is a check $b \in \partial v \setminus \{a\}$ such that $\partial b \setminus \{v\} \subseteq \mathcal{F}(A)$. Such a check satisfies $m_{b \rightarrow v}(A') = \#$, and thus (5.4) shows that $m_{v \rightarrow a}(A') = \#$. Conversely, suppose that $m_{v \rightarrow a}(A') = \hat{u}$. Then (5.1) shows that $v$ is unfrozen in $G'(A) - a$. Hence, the kernel of the matrix obtained from $A'$ by deleting the arrow contains a vector $\sigma''$ with $\sigma''_a = 1$. Therefore, any check $b \in \partial v \setminus a$ features a variable $y \in \partial b \setminus (\{v\} \cup \mathcal{F}(A))$. Consequently, because the subgraph induced on $\mathcal{E}$ is acyclic, (5.9) implies that $v$ is unfrozen in the subgraph of $G'(A) - (\partial v \setminus b)$ where the only check adjacent to $v$ is $b$. Thus, $m_{b \rightarrow v}(A') = \hat{u}$. Finally, (5.4) shows that $\hat{m}_{v \rightarrow a}(A') = \hat{u}$.

The proof of (5.7) proceeds along similar lines. Indeed, $v \in \hat{V}$ is frozen in $A'$ if there exists a check $a \in \partial v$ such that $\partial a \setminus \{v\} \subseteq \mathcal{F}(A)$. Hence, (5.9) shows that the existence of a check $a \in \partial v$ with $m_{v \rightarrow a}(A') = \#$ is a sufficient condition for $v \not\in \mathcal{F}(A')$. Conversely, (5.9) shows that the absence of such a check is a sufficient condition for $v \not\in \mathcal{F}(A')$. Thus, recalling the definition (5.5), we obtain the first part of (5.7).

To prove the second part we combine (5.6)–(5.7) with (5.14) to see that $a \in \hat{F}(A')$ iff there is at most one $y \in \partial a$ with $m_{y \rightarrow a}(A') = \hat{u}$. For clearly $a \in \hat{F}(A')$ if no such $y$ exists, while if there is precisely one such $y$ the presence of the check $a$ will freeze this variable. Conversely, if at least two $y, y' \in \partial a$ satisfy $m_{y \rightarrow a}(A'), m_{y' \rightarrow a}(A') \neq \#$, then $a \not\in \hat{F}(A')$ due to (5.9). Thus, a glance at the definition (5.5) of $m_{a}(A')$ completes the proof of (5.7).

Proposition 2.7 is a statement about the proportion of variables identified as frozen by WP; in order to prove this result, we will need to analyse the distribution of the numbers of incoming and outgoing messages of each type at a node. This motivates the following definitions.

Given a vector $L = (\ell_{uu}, \ell_{uf}, \ell_{fu}, \ell_{ff}) \in N_d^4$ and $z \in \{\#, *, u\}$, let

$$\Delta_A(z, L) = \sum_{v \in \mathcal{V}(A)} \sum_{x \in \{\#, *, u\}} 1 \left\{ m_v(A) = x \right\} \prod_{y \in \partial v} \sum_{1 \left\{ x, y \in \mathcal{F}(A) \right\}} 1 \left\{ \{a \in \partial v \setminus \{v\} : m_{a \rightarrow v}(A) = x \text{ and } m_{v \rightarrow a}(A) = y\} = \ell_{xy}\right\},$$

$$\Gamma_A(z, L) = \sum_{a \in \mathcal{A}(A)} \sum_{x \in \{\#, *, u\}} 1 \left\{ m_a(A) = z \right\} \prod_{y \in \partial a} \sum_{1 \left\{ x, y \in \mathcal{F}(A) \right\}} 1 \left\{ \{a \in \partial v \setminus \{a\} : m_{v \rightarrow a}(A) = x \text{ and } m_{a \rightarrow v}(A) = y\} = \ell_{xy}\right\}.$$

These random variables count variables/checks with certain marks and given numbers of edges with specific incoming/outgoing messages. For instance, $\ell_{uf}$ provides the number of edges with an incoming $u$-message and an outgoing $f$-message. Of course, for some choices of $z$ and $L$ the variables $\Delta_A(z, L)$ and $\Gamma_A(z, L)$ may equal zero deterministically. We can think of $\Delta$ and $\Gamma$ as generalised degrees, giving information not just about the number
Proof of Proposition 2.7. Finally, the identity

\[ \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ |\Delta_d(z, L) - 0|, |\Gamma_A(z, L) - g(f(A), z, L)| \right] = 0, \]  

(5.21)

Corollary 5.6. Let \( d > 0 \). For any \( z \in \{f, *, u\} \) and \( L = (\ell_{uu}, \ell_{uf}, \ell_{ft}) \in \mathbb{N}_d \) we have

\[ \mathbb{P} \left[ \frac{1}{N_d} \sum_{u \in \mathbb{N}_d} \mathbb{E} \left[ |\Delta_d(z, L) - 0|, |\Gamma_A(z, L) - g(f(A), z, L)| \right] \right] = 0, \]  

(5.22)

Indeed, (5.3) ensures that \( m_u(A') = u \) only if \( a \) receives at least two \( u \)-messages. Furthermore, as Fact 5.5 shows that \( f(A') = f(A) + o(1) \) w.h.p., we can rewrite (5.22) as

\[ \mathbb{P} \left[ \frac{1}{N_d} \sum_{u \in \mathbb{N}_d} \mathbb{E} \left[ |\Delta_d(z, L) - 0|, |\Gamma_A(z, L) - g(f(A), z, L)| \right] \right] = 0. \]  

(5.23)

Since by the fixed point property from Lemma 5.4 the reverse messages sent out by \( a \) are determined by the incoming ones via (5.5), all messages returned by a check with mark \( u \) are \( u \) w.h.p. Therefore, (5.23) implies the second part of (5.21). Finally, we observe that the identity \( \lim_{x \to 0} \frac{1}{x} \mathbb{E} \left[ \hat{f}(A) - (1 + d(1 - f(A))) \exp \left( -d(1 - f(A)) \right) \right] = 0 \) is equivalent to the statement that w.h.p. \( \hat{f}(A) = (1 + d(1 - f(A))) \exp \left( -d(1 - f(A)) \right) + o(1) \), which actually follows from (5.7), (5.18) and (5.21) by summing over \( L \in \mathbb{N}_d \). More precisely, (5.7) implies that w.h.p. \( \hat{f}(A) = n^{-1} \sum_{a : \text{m}_u(A') \neq u} |a| : \text{m}_u(A') \neq u | + o(1) \). Furthermore, by (5.21), w.h.p. for all but \( o(n) \) check nodes \( a \) we have \( m_u(A') \neq u \) if and only if \( a \) is adjacent to no edge along which both messages are \( u \). A glance at (5.19) shows that the sum over all \( L \in \mathbb{N}_d \) of \( g(a, u, L) \) is simply \( \mathbb{P} \left[ \frac{1}{N_d} \sum_{u \in \mathbb{N}_d} \mathbb{E} \left[ |\Delta_d(z, L) - 0|, |\Gamma_A(z, L) - g(f(A), z, L)| \right] \right] = 0. \) Considering the complement and substituting \( a = f(A) \), the result follows.

The first part of (5.21) also follows from similar deliberations. For example, for \( x \in \hat{V} \) we have \( m_u(A') = u \) iff \( m_{a \rightarrow x}(A') = u \) for all \( a \in \partial x \). Furthermore, the fixed point property from Lemma 5.6 shows that w.h.p. \( m_{a \rightarrow x}(A') = f \) iff \( y \in \mathbb{F}(A) \) for all \( y \in \partial a \setminus \hat{V} \). Since the variables \( y \) are chosen randomly and independently, we see that \( \mathbb{P} \left[ m_{a \rightarrow x}(A') = f | A \right] = \mathbb{P} \left[ \text{m}(d(1 - f(A))) = 0 \right] + o(1) = \exp \left( -d(1 - f(A)) \right) + o(1) = \hat{f}(A) + o(1) \) w.h.p. Because \( x \) has a total of \( \text{Po}(\hat{f}(A)) \) independent adjacent checks, we obtain (5.21) for \( z = u \); the cases \( z = f \) and \( z = * \) are analogous. Finally, the identity \( f(A) = \phi_d(f(A)) + o(1) \) w.h.p. follows from Fact 5.3, (5.7) and (5.21) by summing on \( \ell_{uu} \).

Proof of Proposition 2.7. Fix a small \( \varepsilon > 0 \) and let \( U(e) = \{a \in \{0, 1\}^n : \alpha - a_+ \land \alpha - a_- \land \alpha - a^* > \varepsilon \} \). Then Lemma 2.2 shows that there exists an integer \( t > 0 \) such that \( |\phi_d^t(\alpha) - a_+ | \land |\phi_d^t(\alpha) - a^- | < \varepsilon/2 \) for all \( \alpha \in U(e) \). Hence,

\[ |\alpha - \phi_d^t(\alpha) | > \varepsilon/2 \quad \text{for all } \alpha \in U(e). \]  

(5.24)

By contrast, Corollary 5.6 shows that \( |f(A) - \phi_d(f(A))| = o(1) \) w.h.p. Since \( \phi_d(\cdot) \) is uniformly continuous on \([0, 1] \), this implies that \( |f(A) - \phi_d(f(A))| = o(1) \) w.h.p. Hence, (5.24) shows that \( \mathbb{P} \left[ f(A) \in U(e) \right] = o(1) \). Because this holds for arbitrarily small \( \varepsilon > 0 \), the assertion follows.
6. The unstable fixed point

Proposition 2.7 shows that \( f(A) \) is close to one of the fixed points of the function \( \phi_d \) w.h.p. The aim in this section is to prove Proposition 2.8 by using the “hammer and anvil” strategy described in Section 1.4.2 to rule out the unstable fixed point \( \alpha_0 \). The proof is subtle and requires three steps. First we show that a random \( x \in \ker A \) sets about half the unfrozen variables to one. Indeed, even if we weight the variable nodes by their degrees the overall weight of the one-entries comes to about half w.h.p. Therefore, (1.2) implies that \( \ker A \) contains \( 2^{\Phi_d(\alpha_*) n + o(n)} \) such balanced vectors w.h.p. This is the “anvil” part of the argument.

The “hammer” part consists of the next two steps showing that the existence of that many balanced solutions is actually unlikely if \( f(A) \sim \alpha_0 \). We proceed by way of a sophisticated moment computation. Specifically, we estimate the number of fixed points of the operator from (5.4)–(5.5) that mark about \( \pi \) unstable fixed point \( \alpha \)-estimate the number of fixed points of the operator from (5.4)–(5.5) that mark about \( \pi \) unstable fixed point from (5.4)–(5.5) that mark about \( \pi \) unstable fixed point. The answer turns out to be 2 \( \Phi_d(\alpha_*) n + o(n) \). Subsequently we compute the expected number of actual balanced solutions compatible with such a WP fixed point. The answer turns out to be \( 2^{\Phi_d(\alpha_*) n + o(n)} \).

Since \( \Phi_d(\alpha_*) < \Phi_d(\alpha_+) = \max_0 \Phi_d(\alpha) \), we conclude that a random matrix with \( f(A) \sim \alpha_0 \) would have far fewer “balanced” vectors in its kernel than the anvil part of the argument demands. Consequently, the event \( f(A) \sim \alpha_0 \) is unlikely.

6.1. Degree-weighted solutions. Let us now carry this strategy out in detail. A vector \( x \in \ker A \) is called \( \delta \)-balanced if

\[
\left| \sum_{v \in \mathcal{F}(A)} d_A(v) (1 \{ x_v = 1 \} - 1/2) \right| < \delta n.
\]

The following observation is a simple consequence of Proposition 2.11.

**Lemma 6.1.** W.h.p. the random matrix \( A \) has \( 2^{\Phi_d(\alpha_+) n + o(n)} \) many \( o(1) \)-balanced solutions.

**Proof.** Since (1.2) and Proposition 2.3 show that \( \text{null } A = \Phi_d(\alpha_+) n \) w.h.p., it suffices to prove that a random \( x \in \ker A \) is \( o(1) \)-balanced w.h.p. To see this, fix any integer \( \ell > 0 \). Proposition 2.11 implies together with Proposition 2.15 that the distribution of a random \( x \in \ker A \) is \( o(1) \)-extremal w.h.p. Moreover, Fact 2.17 shows that the event \( \{ x_v = 1 \} \) has probability \( 1/2 \) for all \( v \notin \mathcal{F}(A) \). Therefore, the definition (2.12) of the cut metric implies that for any \( \ell \in \mathbb{N} \), w.h.p. over the choice of \( A \) we have

\[
\mathbb{E} \left[ \left| \sum_{v \notin \mathcal{F}(A)} 1 \{ d_A(v) = \ell \} (1 \{ x_v = 1 \} - 1/2) \right| \right] = o(n).
\]  

(6.1)

As this is true for every fixed \( \ell \) and the Poisson degree distribution of \( G(A) \) has sub-exponential tails, the assertion follows from (6.1) by summing over \( \ell \). \( \square \)

6.2. Counting WP fixed points. Proceeding to the next step of our strategy, we now estimate the expected number of approximate WP fixed points that leave about \( a_0 n \) variables unfrozen. We call such fixed points \( a_0 \)-covers. The precise definition, in which we condition on the degree sequence \( d_A \) of \( G(A) \), reads as follows.

**Definition 6.2.** Given \( d_A \) let

\[
\mathcal{D} = \bigcup_{i=1}^n \{ v_i \} \times \{ d_A(v_i) \} \quad \text{and} \quad \mathcal{C} = \bigcup_{i=1}^n \{ a_i \} \times \{ d_A(a_i) \}
\]

be sets of variable/check clones. An \( \alpha \)-cover is a pair \((m, \pi)\) consisting of a map \( m : \mathcal{D} \cup \mathcal{C} \to \{ f, u \}^2, (u, j) \mapsto (m_1(u, j), m_2(u, j)) \) and a bijection \( \pi : \mathcal{D} \to \mathcal{C} \) such that the following conditions are satisfied.

**COV1:** For all \( i \in [n] \) and \( j \in \{ d_A(v_i) \} \) we have \( \left( m_1(\pi(v_i, j)), m_2(\pi(v_i, j)) \right) = \left( m_2(v_i, j), m_1(v_i, j) \right) \).

**COV2:** For all but \( o(n) \) pairs \((i, j)\) with \( i \in [n] \) and \( j \in \{ d_A(v_i) \} \) we have

\[
m_2(v_i, j) = \begin{cases} f & \text{if } m_1(v_i, h) = f \text{ for some } h \in \{ d_A(v_i) \} \setminus \{ j \}, \\ u & \text{otherwise.} \end{cases}
\]

**COV3:** For all but \( o(n) \) pairs \((v_i, j)\) with \( i \in [n] \) and \( j \in \{ d_A(a_i) \} \) we have

\[
m_2(a_i, j) = \begin{cases} f & \text{if } m_1(a_i, h) = f \text{ for all } h \in \{ d_A(a_i) \} \setminus \{ j \}, \\ u & \text{otherwise.} \end{cases}
\]

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\textbf{COV4:} For any \( z \in \{\mathcal{f}, \star, u\} \) and \( L = (\ell_{uv}, \ell_{uf}, \ell_{uf}, \ell_{ff}) \in \mathbb{N}_0^4 \) let

\[
m(v_i) = \begin{cases} f & \text{if } m_1(v_i, j) = \mathcal{f} \text{ for at least two } j \in \{d_A(v_i)\}, \\ \star & \text{if } m_1(v_i, j) = \mathcal{f} \text{ for precisely one } j \in \{d_A(v_i)\}, \\ u & \text{otherwise}, \end{cases}
\]

\[
m(a_i) = \begin{cases} f & \text{if } m_1(a_i, j) = \mathcal{f} \text{ for all } j \in \{d_A(a_i)\}, \\ \star & \text{if } m_1(a_i, j) = \mathcal{f} \text{ for all but precisely one } j \in \{d_A(a_i)\}, \\ u & \text{otherwise}, \end{cases}
\]

\[
\Delta(z, L) = \sum_{i=1}^{n} \mathbb{1}(m(v_i) = z) \prod_{x,y \in \{u,f\}} \mathbb{1}\left\{ \{ j \in \{d_A(v_i)\} : m_1(v_i, j) = x, m_2(v_i, j) = y \} \right\} = \ell_{xy},
\]

\[
\Gamma(z, L) = \sum_{i=1}^{n} \mathbb{1}(m(a_i) = z) \prod_{x,y \in \{u,f\}} \mathbb{1}\left\{ \{ j \in \{d_A(a_i)\} : m_1(a_i, j) = x, m_2(a_i, j) = y \} \right\} = \ell_{xy}.
\]

Then with \( \delta(\cdot), g(\cdot) \) from \( \text{(5.15)} - \text{(5.20)} \) we have

\[
\Delta(z, L) = n\delta(1 - \alpha_0, z, L) + o(n), \quad \Gamma(z, L) = ng(\alpha_0, z, L) + o(n).
\]

Let \( \mathcal{Z}(\alpha) \) be the number of \( \alpha \)-covers. The main result in this section is the proof of the following proposition.

**Proposition 6.3.** For any \( d > e \) w.h.p. over the choice of the degree sequence \( d_A \) we have

\[
\frac{\mathcal{Z}(\alpha)}{(dn)! \prod_{i=1}^{d_A} d_A(v_i)! d_A(a_i)!} = \exp(o(n)) .
\]

The rest of this section is devoted to the proof of Proposition 6.3. The following lemma decomposes \( \mathcal{Z}(\alpha_0) \) into a few factors that we will subsequently calculate separately.

**Lemma 6.4.** W.h.p. over the choice of \( d_A \) we have \( \mathcal{Z}(\alpha_0) = \exp(o(n)) \tilde{\mathcal{Z}}^{2} \mathcal{L}^{2} \mathcal{E} \) where

\[
\tilde{\mathcal{Z}} = \left( n((1 - \alpha_0, z, L))_{z \in \{f, \star, u\}, L \in \mathbb{N}_0^4} \right)^{n(1 - \alpha_0, z, L)} .
\]

\[
\mathcal{L} = \prod_{z \in \{f, \star, u\}} \left( \ell_{uu}, \ell_{uf}, \ell_{uf}, \ell_{ff} \right)^{n((g(\alpha_0, z, L))_{z \in \{f, \star, u\}, L \in \mathbb{N}_0^4})} .
\]

\[
\mathcal{E} = (dn^{2})^2 (dn \alpha_0 (1 - \alpha_0))^2 (dn(1 - \alpha_0)^2)! .
\]

**Proof.** The first factor \( \tilde{\mathcal{Z}} \) simply accounts for the number of ways of partitioning the \( n \) variable nodes and the \( n \) check nodes into the various types as designated by \( \text{(6.4)} - \text{(6.5)} \). Since we need to select a type for each variable and check node, the number of possible designation actually reads

\[
\left( n((1 - \alpha_0, z, L))_{z \in \{f, \star, u\}, L \in \mathbb{N}_0^4} \right)^{n(1 - \alpha_0, z, L)} .
\]

the \( \exp(o(n)) \) error term accounts for the \( o(n) \) error terms in \( \text{(6.6)} \). But a glimpse at \( \text{(5.15)} - \text{(5.20)} \) reveals that these two multinomial coefficients coincide. Hence, \( \text{(6.7)} \) is equal to \( \tilde{\mathcal{Z}} \exp(o(n)) \). Furthermore, the factor \( \mathcal{L} \) accounts for the number of ways of selecting, for each variable/check node, the clones along which messages of the four types \( \{f, u\} \) travel. Finally, \( \mathcal{E} \) counts the number of ways of matching up these clones so that \textbf{COV2-COV3} are satisfied. To be precise, since \textbf{COV2-COV3} only provide asymptotic estimates rather than precise equalities, we incur an \( \exp(o(n)) \) error term; hence \( \mathcal{Z}(\alpha_0) = \exp(o(n)) \tilde{\mathcal{Z}}^{2} \mathcal{L}^{2} \mathcal{E} \). \( \square \)

**Lemma 6.5.** We have \( \frac{1}{n} \log \mathcal{E} = l' + l'' + o(1) \), where

\[
l' = \exp(-d) \sum_{\ell = 0}^{\infty} \frac{d^\ell}{\ell!} \log(\ell!), \quad l'' = - \sum_{z \in \{f, \star, u\}} \delta(1 - \alpha_0, z, L) \log(\ell_{uv}, \ell_{uf}, \ell_{uf}, \ell_{ff}) .
\]

**Proof.** Choose \( z \in \{f, \star, u\} \) along with non-negative vector \( L \in \mathbb{N}_0^4 \) from the distribution

\[
P(z = z, L = L) = \delta(1 - \alpha_0, z, L) \quad (z \in \{f, \star, u\}, L \in \mathbb{N}_0^4) .
\]
Lemma 6.6. \( \frac{1}{n} \log \mathcal{L} = \mathbb{E} \left[ \log(\ell_{u_0} + \cdots + \ell_{f_2}) \right] - \mathbb{E} \left[ \log(\ell_{u_0} \cdots \ell_{f_2}) \right] + o(1) = \mathbb{E} \left[ \log(\ell_{u_0} + \cdots + \ell_{f_2}) \right] - \ell'' + o(1). \) (6.8)

Moreover, (5.15)–(5.17) show that \( \ell_{u_0} + \cdots + \ell_{f_2} \) has distribution \( \text{Po}(d) \). Therefore, \( \mathbb{E} \left[ \log(\ell_{u_0} + \cdots + \ell_{f_2}) \right] = \ell' \). Hence, the assertion follows from (6.8). \( \square \)

Lemma 6.6. We have \( \frac{1}{n} \log \mathcal{J} = d \left( 1 - \log(d) - \alpha_0 \log \alpha_0 - (1 - \alpha_0) \log(1 - \alpha_0) \right) - \ell''. \)

Proof. This is a straightforward computation. For the sake of brevity we introduce \( q(\lambda, i) = P[\text{Po}(\lambda) = i] \). Using Stirling’s formula, we approximate \( \mathcal{J} \) in terms of entropy as

\[
\frac{1}{n} \log \mathcal{J} = H((\delta(1 - \alpha_0, z, L))_{z \in \{f, \star, u\}, L \in \mathbb{N}}) + o(1).
\] (6.9)

Depending on the choice of \( z \in \{f, \star, u\} \), the definitions (5.15)–(5.17) of the \( \delta(1 - \alpha_0, z, L) \) constrain some of the values \( \ell_{u_0}, \cdots, \ell_{f_2} \) to be zero. Hence, using the identity (2.1), we can spell the right hand side of (6.9) out as

\[
H((\delta(1 - \alpha_0, z, L))_{z \in \{f, \star, u\}, L \in \mathbb{N}}) = - \sum_{z, L} \delta(1 - \alpha_0, z, L) \log \delta(1 - \alpha_0, z, L)
\]

\[
= - \sum_{\ell_{u_0} \geq 0} q(d(1 - \alpha_0), 0) q(da_0, \ell_{u_0}) \log(q(d(1 - \alpha_0), 0) q(da_0, \ell_{u_0}))
\]

\[
- \sum_{\ell_{u} \geq 0} q(d(1 - \alpha_0), 1) q(da_0, \ell_{u}) \log(q(d(1 - \alpha_0), 1) q(da_0, \ell_{u}))
\]

\[
- \sum_{\ell_{f_2} \geq 2} q(d(1 - \alpha_0), \ell_{f_2}) q(da_0, \ell_{f_2}) \log(q(d(1 - \alpha_0), \ell_{f_2}) q(da_0, \ell_{f_2}))
\]

\[
= d(1 - \alpha_0)^2 - (1 - \alpha_0) \sum_{\ell_{u_0} \geq 0} q(da_0, \ell_{u_0}) \left[ \ell_{u_0} \log(da_0) - da_0 \right]
\]

\[
- d(1 - \alpha_0)^2 \log(d(1 - \alpha_0)^2) - d(1 - \alpha_0)^2 \sum_{\ell_{u} \geq 0} q(da_0, \ell_{u}) \left[ \ell_{u} \log(da_0) - da_0 \right]
\]

\[
- (\alpha_0 - (1 - \alpha_0)^2) \sum_{\ell_{f_2} \geq 2} q(d(1 - \alpha_0), \ell_{f_2}) \left[ \ell_{f_2} \log(d(1 - \alpha_0)) - d(1 - \alpha_0) \right]
\]

\[
= - \ell'' + d(1 - \alpha_0)^2 + da_0(1 - \alpha_0) - da_0(1 - \alpha_0) \log(da_0)
\]

\[
- d(1 - \alpha_0)^2 \log(d(1 - \alpha_0)^2) + d^2 \alpha_0(1 - \alpha_0)^2 - d^2 \alpha_0(1 - \alpha_0)^2 \log(da_0)
\]

\[
+ d(1 - \alpha_0) - (1 - \alpha_0) \log(d(1 - \alpha_0)) + (1 - \alpha_0) \log(1 - \alpha_0) + d(1 - \alpha_0)^2 \log(d(1 - \alpha_0)^2)
\]

\[
+ da_0(d(1 - \alpha_0)^2) - da_0(1 - \alpha_0)^2 \log(da_0)
\]

\[
= - \ell'' - d \log d + da_0 \log(1 - \alpha_0) \log(1 - \alpha_0) + d + (1 - \alpha_0) \log(1 - \alpha_0) + d(1 - \alpha_0)^2. \] (6.10)

Since \( 1 - \alpha_0 = \exp(-d(1 - \alpha_0)) \), the assertion is immediate from (6.10). \( \square \)

Lemma 6.7. W.h.p. over the choice of \( \delta \) we have \( \frac{1}{n} \log \left( \frac{\mathcal{J}}{\mathcal{P}} \right) = 2 \log da_0 \log \alpha_0 + 2d(1 - \alpha_0) \log(1 - \alpha_0). \)

Proof. This follows immediately from Stirling’s formula. \( \square \)

Proof of Proposition 6.3. The proposition is an immediate consequence of Lemmas 6.4–6.7. \( \square \)

6.3. Extending covers. While in the previous section we just estimated the number of covers, here we also count actual solutions to the random linear system encoded by a cover. The following definition captures assignments \( \sigma \) that, up to \( o(n) \) errors, comply with the frozen/unfrozen designations of a cover \( (m, \pi) \) and also satisfy the checks, again up to \( o(n) \) errors. We extend \( \sigma : \{v_1, \ldots, v_n\} \to \mathbb{F}_2 \) to the set of \( \mathcal{S} \) of clones by letting \( \sigma'(v_j, k) = \sigma(v_j) \).

Definition 6.8. An \( \alpha \)-extension consists of an \( \alpha \)-cover \( (m, \pi) \) together with an assignment \( \sigma : \{v_1, \ldots, v_n\} \to \mathbb{F}_2 \) such that the following conditions are satisfied.

- **EXT1:** We have \( \sum_{i=1}^n (1 + d_A(v_i)) 1\{\sigma(v_i) = 1, m(v_i) \neq u\} = o(n) \).

- **EXT2:** We have \( \sum_{i=1}^n d_A(v_i) 1\{\sigma(v_i) = 1, m(v_i) = u\} = o(n) + \frac{1}{2} \sum_{i=1}^n d_A(v_i) 1\{m(v_i) = u\}. \)

- **EXT3:** We have \( \sum_{i=1}^n 1\{1 \{\sum_{j \in h(u_i)} \sigma(v_i) \neq 0\} = o(n) \). \)
The first condition EXT1 posits that, when weighted according to their degrees, all but \( o(n) \) variables that are deemed frozen under \( m \) are set to zero under \( \sigma \). EXT2 provides that about half the variables that ought to be unfrozen according to \( m \) are set to one, if we weight variables by their degrees. Finally, EXT3 ensures that all but \( o(n) \) checks are satisfied.

Let \( X(\alpha) \) be the total number of \( \alpha \)-extensions. The main result of this section reads as follows.

**Proposition 6.9.** Let \( d > e \). W.h.p. over the choice of the degree sequence \( d_A \) we have

\[
\frac{X(\alpha_0)}{\prod_{i=1}^{n} d_A(v_i)! d_A(a_i)!} = \exp(n \Phi_d(\alpha_0) + o(n)).
\]

The following lemma summarises the key step toward the proof of Proposition 6.9. For a fixed \( n \) let \( \pi \) be a random matching of the clones \( \mathcal{B}, \mathcal{C} \) such that \( (m, \pi) \) is an \( \alpha_0 \)-cover.

**Lemma 6.10.** For a \( o(1) \)-balanced \( \sigma \) let \( p(m, \sigma) \) be the probability that \( \sigma \) satisfies all but \( o(n) \) checks. Then w.h.p. over the choice of \( d_A \) we have

\[
p(m, \sigma) \leq 2^{-|\{i \in [n] : m(a_i) = u\}| + o(n)}.
\]

**Proof.** Given \( m \) the precise matching \( \pi \) of the frozen/unfrozen clones remains random subject to conditions COV1–COV3. We will expose this matching in two steps. First we expose the degree-weighted fraction of occurrences of frozen/unfrozen variables set to one. Specifically, let \( r_u \sim 1/2 \) be the precise degree-weighted fraction of occurrences of unfrozen variables that are set to zero under \( \sigma \); in formulae,

\[
r_u = \frac{\sum_{i=1}^{n} \{j \in [d_A(a_i)] : m_1(a_i, j) = u, \sigma(\pi(a_i, j)) = 0\}}{\sum_{i=1}^{n} \{j \in [d_A(a_i)] : m_1(a_i, j) = u\}}.
\]

Similarly, let \( r_f \sim 1 \) be the degree-weighted fraction of frozen clones set to zero:

\[
r_f = \frac{\sum_{i=1}^{n} \{j \in [d_A(a_i)] : m_1(a_i, j) = f, \sigma(\pi(a_i, j)) = 0\}}{\sum_{i=1}^{n} \{j \in [d_A(a_i)] : m_1(a_i, j) = f\}}.
\]

Once we condition on \( r_u, r_f \), the precise matching of the various clones remains random. To study the conditional probability that \( \sigma \) satisfies all but \( o(n) \) checks, we set up an auxiliary probability space. To be precise, let \( \chi = \{ \chi_{ij} \}_{i \in [n], j \in [d_A(a_i)]} \) be a random sequence of mutually independent field elements \( \chi_{ij} \in \mathbb{F}_2 \) such that

\[
P[\chi_{ij} = 0] = \begin{cases} r_u & \text{if } m_1(a_i, j) = u, \\ r_f & \text{if } m_1(a_i, j) = f. \end{cases}
\]

Further, consider the events

\[
\mathcal{R} = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{d_A(a_i)} \mathbb{1}\{\chi_{ij} = 0, m_1(a_i, j) = z\} = r_u \sum_{i=1}^{n} d_A(a_i) \right\},
\]

\[
\mathcal{S} = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{d_A(a_i)} \chi_{ij} = o(n) \right\}.
\]

Then because the matching \( \pi \) of the clones is random subject to COV1–COV3 we obtain

\[
p(m, \sigma) = E[P[\mathcal{S} | \mathcal{R}, r_u]]
\]

Hence, we are left to calculate \( P[\mathcal{S} | \mathcal{R}, r_f, r_u] \). Calculating the unconditional probabilities is easy. Indeed, the choice (6.11)–(6.12) of \( r_u, r_f \) and the definition (6.13) of \( \chi \) and the local limit theorem for the binomial distribution ensure that

\[
P[\mathcal{R}] = \Omega(1/n).
\]

Furthermore, we claim that

\[
P[\mathcal{S}] = 2^{-|\{i \in [n] : m(a_i) = u\}| + o(n)}.
\]

Indeed, consider a check \( a_i \) such that \( m(a_i) = u \). Then there exists \( j \in [d_A(a_i)] \) such that \( m_1(a_i, j) = u \). Therefore, the choice (6.11) of \( r_u \) ensures that the event \( \chi_{ij} \neq 0 \) occurs with probability \( 1/2 + o(1) \). Similarly, if \( m(a_i) \neq u \), then by the choice of \( r_f \) the event \( \chi_{ij} \neq 0 \) has probability at most \( o(d_A(a_i)) \). Since the definition (6.13) of the
\( \chi_{ij} \) ensures that these events are independent for the different checks \( a_i \), we obtain (6.16). Finally, combining (6.14)–(6.16) with Bayes' rule, we obtain

\[
p(m, \sigma) = \mathbb{E} \left[ \mathbb{P} [ \mathcal{S} | \mathcal{R}, r_t, r_u ] \right] = \mathbb{E} \left[ \mathbb{P} [ \mathcal{S} | r_t, r_u ] \cdot \mathbb{P} [ \mathcal{R} | \mathcal{S}, r_t, r_u ] / \mathbb{P} [ \mathcal{R} | r_t, r_u ] \right] \leq 2^{-\left( \frac{1}{3} + \epsilon \right) n \log a_0 + o(n)},
\]
as desired. □

To complete the proof of Proposition 6.9, we combine Lemma 6.10 with the following statement about the numbers of variables/checks of the various types. Given \( z \in \{ f, u \} \), let us define \( \epsilon_z := 1 \{ z = u \} \).

**Lemma 6.11.** Let \( (m, \pi) \) be an \( a_0 \)-cover. Then w.h.p. over the choice of \( d_A \),

\[
\frac{1}{dn} \sum_{i=1}^{n} a_i (v_i) \sum_{j=1}^{n} 1 \{ m(v_i, j) = (x, y) \} \sim a_0^{1+\epsilon_x - \epsilon_y} (1 - a_0)^{1 - \epsilon_x - \epsilon_y} (x, y) \in \{ f, u \},
\]

(6.17)

\[
\frac{1}{dn} \sum_{i=1}^{n} a_i (a_i) \sum_{j=1}^{n} 1 \{ m(a_i, j) = (x, y) \} \sim a_0^{1 - \epsilon_x + \epsilon_y} (1 - a_0)^{1 + \epsilon_x - \epsilon_y} (x, y) \in \{ f, u \},
\]

(6.18)

\[
\frac{1}{n} \sum_{i=1}^{n} 1 \{ m(v_i) = f \} \sim a_0 - d(1 - a_0)^2,
\]

\[
\frac{1}{n} \sum_{i=1}^{n} 1 \{ m(v_i) = u \} \sim 1 - a_0,
\]

(6.19)

\[
\frac{1}{n} \sum_{i=1}^{n} 1 \{ m(a_i) = f \} \sim 1 - a_0,
\]

(6.20)

**Proof.** We observe that **COV4** implies the estimate

\[
\frac{1}{n} \sum_{i=1}^{n} a_i (v_i) \sum_{j=1}^{n} 1 \{ m(v_i, j) = (x, y) \} \sim d a_0^{1 - \epsilon_x} (1 - a_0)^{1 - \epsilon_x} \exp(-d \epsilon_y (1 - a_0)(1 - \exp(-d(1 - a_0))^1 - \epsilon_y).
\]

Using the identity (2.1), we obtain (6.17). The second identity follows from (6.17) and **COV1**. Equations (6.19)–(6.20) follow from the identity \( a_0 = 1 - \exp(-d(1 - a_0)) \) and **COV2** by summing on \( L \). □

**Proof of Proposition 6.9.** Lemmas 6.10 and 6.11 imply that w.h.p. over the choice of \( d_A \),

\[
p(m, \sigma) \leq 2^{\left( \frac{1}{3} + \epsilon \right) n \log a_0 + o(n)} \leq 2^{n(1 - 2a_0 + d(1 - a_0)^2 + o(1))}.
\]

(6.21)

Further, using the identity (2.1), we verify that \( 1 - 2a_0 + d(1 - a_0)^2 = \Phi_d(a_0) \). Thus, the assertion follows from (6.21) and Proposition 6.3.

**Proof of Proposition 2.3.** We can generate a random Tanner graph \( G(A) \) with a given degree sequence \( d_A \) by way of the pairing model. Specifically, we generate a random pairing \( \pi \) of the sets \( \mathcal{S}, \mathcal{C} \) of clones and condition on the event \( \mathcal{S} \) that the resulting graph \( G(\pi) \) is simple. W.h.p. over the choice of the degree sequence \( d_A \) we have \( \mathbb{P} [ \mathcal{S} | d_A ] = \Omega(1) \); but in fact, for the purposes of the present proof the trivial estimate

\[
\mathbb{P} [ \mathcal{S} | d_A ] = \exp(o(n)) \quad \text{w.h.p.}
\]

(6.22)

suffices. Now, let \( \mathcal{E} \) be the event that \( G(\pi) \) has at least \( 2^{\Phi_d(a_0) + o(n)} \) \( a_0 \)-extensions. Recall that w.h.p. over the choice of \( d_A \) there are \( \left( \sum_{i=1}^{n} d_A(v_i) \right)! = (dn)! \exp(o(n)) \) possible matchings of the \( 2 \sum_{i=1}^{n} d_A(v_i) \) clones in total, and that each Tanner graph extends to \( \prod_{i=1}^{n} d_A(v_i)!d_A(a_i)! \) pairings. Therefore, Propositions 2.3 and 6.9 and Markov's inequality show that w.h.p. over the choice of \( d_A \),

\[
\mathbb{P} [ \mathcal{E} \cap \mathcal{S}, d_A ] \leq 2^{-\Phi_d(a_0) + o(n)} \leq 2^{n(\Phi_d(a_0) - \Phi_d(a_0) + o(n)) = \exp(-\Omega(n))} \quad \text{w.h.p.}
\]

(6.23)

To complete the proof, assume that \( \mathbb{P} [ f(A) = a_0 + o(1) ] > \epsilon \) for some \( \epsilon > 0 \). Then (1.2), Lemma 5.4, Corollary 5.6, and Lemma 6.1 show that \( \mathbb{P} [ A \in \mathcal{E} | f(A) = a_0 + o(1) ] = 1 - o(1) \). Hence, \( \mathbb{P} [ A \in \mathcal{E} | d_A ] > \epsilon / 2 \) with probability at least \( \epsilon / 2 \), in contradiction to (6.23). □
The aim in this section is to prove Proposition 2.9 which states that w.h.p. the numbers of variables and checks in the slush are not almost equal. Thus, we study the subgraph \( G_n(A) \) induced on \( V_n(A) \cup C_n(A) \). We use the notation \( n_b := |V_n(A)| \) and \( m_b := |C_n(A)| \). We exploit the symmetry of the distribution of \( A \) by considering the transpose of the matrix. While symmetry automatically implies that events are equally likely for \( A \) and \( A^\top \), we would like to be able to deduce that the event \(|V_n(A)| - |C_n(A)| \approx \omega\) occurs with probability asymptotically \(1/2\) for some \( \omega = o(n) \gg 1\). The main step is to prove the following.

**Lemma 7.1.** There exists some \( \omega_0 \) such that w.h.p. \( |n_b - m_b| \geq \omega_0 \).

As indicated above, Proposition 2.9 follows from this Lemma and symmetry considerations. We first describe the symmetry property more explicitly.

**Lemma 7.2.** For any matrix \( A \) we have \( V_n(A^\top) = C_n(A) \) and \( C_n(A^\top) = V_n(A) \).

**Proof.** We can show by induction on \( t \in \mathbb{N} \) that the messages at time \( t \) in the Tanner graphs of \( A, A^\top \) are symmetric. More precisely, the Tanner graphs are identical except that variable nodes become check nodes and vice versa. At time \( 0 \) all messages are \( s \) in both graphs, while it can be easily checked that the update rules remain identical if we switch checks and variables and also switch the symbols \( f \) and \( u \). Therefore, introducing

\[
\begin{align*}
V_n(A, t) &= \left\{ v \in V(A) : \exists a \in \partial v : w_{v-a}(A, t) \neq 1 \right\} \text{ and } |\{ a \in \partial v : w_{v-a}(A, t) = 1 \}| \geq 2, \\
C_n(A, t) &= \left\{ a \in C(A) : \forall v \in \partial a : w_{v-a}(A, t) \neq u \right\} \text{ and } |\{ v \in \partial a : w_{v-a}(A, t) = 1 \}| \geq 2,
\end{align*}
\]

we conclude that \( V_n(A, t) = C_n(A^\top, t) \) and \( C_n(A, t) = V_n(A^\top, t) \) for all \( t \). Recalling (2.5)–(2.6), we see that \( V_n(A) = \bigcap_{t \geq 0} V_n(A, t) \) and \( C_n(A) = \bigcap_{t \geq 0} C_n(A, t) \), whence the assertion follows.

**Proof of Proposition 2.9.** We apply Lemma 7.2 to deduce that

\[
\mathbb{P}\left[ |V_n(A)| - |C_n(A)| \geq \omega_0 \right] = \mathbb{P}\left[ |C_n(A^\top)| - |V_n(A^\top)| \geq \omega_0 \right] = \mathbb{P}[|C_n(A)| - |V_n(A)| \geq \omega_0],
\]

where for the second equality we used the fact that \( A, A^\top \) have identical distributions. Furthermore Lemma 7.1 implies that \( \mathbb{P}[|V_n(A)| - |C_n(A)| \geq \omega_0] + \mathbb{P}[|C_n(A)| - |V_n(A)| \geq \omega_0] = 1 - o(1) \), and the desired statement follows.

The proof strategy for Lemma 7.1 is similar to (but rather simpler than) the standard approach to proving a local limit theorem: we will show that \( n_b - m_b \) is almost equally likely to hit any value in a range much larger than \( \omega_0 \), and therefore the probability of hitting the much smaller interval \([-\omega_0, \omega_0]\) is negligible. We begin by estimating the sizes of some special sets of vertices. Recall \( \lambda \) from (2.3).

**Definition 7.3.** (i) Let \( R = R(A) \) be the set of check nodes \( a \) of degree two such that \( w_{v-a}(A) = s \) for all \( v \in \partial a \).

(ii) Let \( S = S(A) \) be the set of isolated variable nodes.

(iii) Let \( T = T(A) \) be the set of check nodes \( a \) of degree three such that \( w_{v-a}(A) = s \) for all \( v \in \partial a \).

(iv) Let \( U = U(A) \) be the set of variable nodes which have precisely two neighbours, both in \( T \).

(v) Let

\[
\begin{align*}
r &= r(A) := |R|/n, & s &= s(A) := |S|/n, & u &= u(A) := |U|/n, \\
\tilde{r} := \exp(-d)\lambda^2/2, & \tilde{s} := \exp(-d), & \tilde{u} &:= \left(\exp(-d)\lambda^2/2\right) \cdot \left(\exp(-d\lambda^2/2)\right)^2.
\end{align*}
\]

**Lemma 7.4.** W.h.p.

\[
r = (1 + o(1))\tilde{r}, \quad s = (1 + o(1))\tilde{s}, \quad u = (1 + o(1))\tilde{u}.
\]

In particular, there exists some \( \omega_1 \to \infty \) such that

\[
r = \left(1 + o\left(1/\omega_1\right)\right)\tilde{r}, \quad s = \left(1 + o\left(1/\omega_1\right)\right)\tilde{s}, \quad u = \left(1 + o\left(1/\omega_1\right)\right)\tilde{u}.
\]

**Proof.** Since whether a node lies in each of these sets is a fact about its depth (at most) 2 neighbourhood (with messages), by Lemma 4.2 it is enough to look at the probabilities that \( \mathcal{T}_2 \) (for \( S, U \)) and \( \mathcal{T}_2 \) (for \( R \)) have the appropriate structure. (Indeed, the statement for \( S \) could be proved directly using a Chernoff bound and without appealing to Lemma 4.2.) An elementary check verifies that these probabilities are \( \tilde{r}, \tilde{s}, \tilde{u} \), as appropriate. \( \square \)
Let $1 \ll \omega_1 \ll n^{1/2}$ be a function such that Lemma [lemma7.4] holds. For the remainder of this section, we will fix further functions $\omega_0, \omega_2$ such that

$$1 \ll \omega_0 \ll \omega_1 \ll n^{1/2}$$

and such that $\omega_2$ is chosen uniformly at random from the interval $[\omega_1/2, \omega_1]$ independently of $A$. In particular, we will prove Lemma [lemma7.1] with this $\omega_0$.

**Claim 7.5.** If $|U| = \Theta(n)$, then for all but $o\left(\binom{|U|}{\omega_1}\right)$ subsets $U' \subseteq U$ of size $\omega_1$, no node has more than one neighbour in $U'$.

**Proof.** It is a simple exercise to check that if a subset $U' \subseteq U$ of size $\omega_1$ is chosen uniformly at random, then the expected number of nodes of $T$ for which two of their three neighbours are chosen to be in $U'$ is $O\left(|T|\omega_2^2/n^2\right) = o(1)$. Therefore by Markov's inequality, w.h.p. this does not occur for any check node. \hfill \Box

We will use the following notation for the remainder of the section. Given a Tanner graph $G$ and a set of variable nodes $W$, let $G(W)$ denote the graph obtained from $G$ by deleting the set of edges incident to $W$. Note that this amounts to replacing the columns of the matrix corresponding to nodes of $W$ with 0 columns.

**Claim 7.6.** Let $G$ be any Tanner graph and $U' \subseteq U(G)$ be any subset whose nodes lie at distance greater than 2. Let $U'' \subseteq U'$ be any subset of $U'$. Then $V_\omega\left(G(U')\right) = V_\omega(G) \setminus U''$.

In other words, removing $U''$ from $G$ does not have any knock-on effects on the slush.

**Proof.** Let $G' := G(U'')$, and let us run WP on both $G'$ and $G$ simultaneously, initialising with all messages being $s$. We verify by induction on $\ell$ that the messages on the common edge set (those in $G'$) are identical in both processes, since a discrepancy can only enter at edges incident to a deleted edge (i.e. in $G \setminus G'$), but our choice of $U'' \subseteq U$ is such that the messages emanating from the vertices of $T$ incident to $U''$ remain $s$. \hfill \Box

For any $r, s, u$, let $G_{r,s,u}$ denote the class of graphs with the appropriate parameters, i.e. with $r(G) = r$, with $s(G) = s$ and with $u(G) = u$, and let

$$G'_{r,s,u} = G'_{r,s,u,\omega_2} := G'_{r',s',u'}, \quad \text{where } r' := r + \frac{2\omega_2}{n}, \quad s' := s + \frac{\omega_2}{n}, \quad u' := u - \frac{\omega_2}{n}.$$  

The intuition behind this definition is that if we delete a set $U'' \subseteq U'$ of size $\omega_2$ to obtain $G'$, then by Claim 7.5 no remaining messages are changed, and therefore

- $|R(G')| = |R(G)| + 2\omega_2$ (for each vertex of $U''$, its two neighbours are moved into $R$);
- $|S(G')| = |S(G)| + \omega_2$ (the vertices of $U''$ are moved into $S$);
- $|U(G')| = |U(G)| - \omega_2$.

Furthermore, for any integer $\ell \in \mathbb{Z}$, let $G_{r,s,u}(\ell) \subseteq G_{r,s,u}$ be the subset consisting of graphs such that $n_a - m_a = \ell$, and similarly define $G'_{r,s,u}(\ell) \subseteq G'_{r,s,u}$ to be the subset consisting of graphs such that $n_a - m_a = \ell' := \ell - \omega_2$.

**Proposition 7.7.** Suppose that we have parameters $r, s, u$ satisfying

$$r = \left(1 + o\left(\frac{1}{\omega_1}\right)\right) \bar{r}, \quad s = \left(1 + o\left(\frac{1}{\omega_1}\right)\right) \bar{s}, \quad u = \left(1 + o\left(\frac{1}{\omega_1}\right)\right) \bar{u}.$$  

Then for any integer $\ell \in \mathbb{Z}$ we have $\mathbb{P}\left[G(A) \in G_{r,s,u}(\ell)\right] = (1 + o(1))\mathbb{P}\left[G(A) \in G'_{r,s,u}(\ell)\right]$.  

**Proof.** We construct an auxiliary bipartite graph $H$ with classes $G_{r,s,u}(\ell), G'_{r,s,u}(\ell)$, and with an edge between $G \in G_{r,s,u}(\ell)$ and $G' \in G'_{r,s,u}(\ell)$ if $G'$ can be obtained from $G$ by deleting the edges incident to a set $U'' \subseteq U(G)$ of size $\omega_2$. (Note that by Claim 7.5 $G'$ satisfies $n_a' = n_a - \omega_2$ and $m_a' = m_a$, so $n_a' - m_a' = (n_a - m_a) - \omega_2 = \ell$.)

By Claim 7.5 (and the fact that $\omega_2 \leq \omega_1$), every graph $G \in G_{r,s,u}(\ell)$ is incident to $(1 + o(1))\binom{n}{\omega_2}$ edges of $H$, since almost every choice of $\omega_2$ nodes from $U$ will result in a graph from $G'_{r,s,u}(\ell)$.

On the other hand, given a graph $G' \in G'_{r,s,u}(\ell)$, we may construct a graph $G \in G_{r,s,u}(\ell)$ by picking any set of $\omega_2$ nodes within $S(G')$, any set of $2\omega_2$ nodes within $R(G)$ and adding $2\omega_2$ edges between them in the appropriate way. Thus we may double-count the edges of $H$ and obtain

$$|G_{r,s,u}(\ell)| \binom{n}{\omega_2} = (1 + o(1)) |G'_{r,s,u}(\ell)| \binom{sn}{\omega_2} \frac{rn}{2\omega_2} \frac{(2\omega_2)!}{2^{\omega_2}}.$$
Since \( r, s, u \) are very close to their idealised values \( \hat{r}, \hat{s}, \hat{u} \), some standard approximations lead to
\[
\left| \frac{\mathcal{G}_{r,s,u}(\ell)}{\mathcal{G}_{r,s,u}^\prime(\ell)} \right| = (1 + o(1)) \left( \frac{\hat{\ell}^2 n^2}{2u} \right)^{\omega_2}.
\]
(7.2)

Substituting in the definitions of \( \hat{r}, \hat{s}, \hat{u} \), some elementary calculations and (3.3) show that \( \hat{\ell}^2 n^2 \approx \frac{1}{d^r} = \frac{1}{p^{r-1}} \). Substituting this into (7.2), we obtain
\[
\left| \frac{\mathcal{G}_{r,s,u}(\ell)}{\mathcal{G}_{r,s,u}^\prime(\ell)} \right| = (1 + o(1)) \left| \frac{\mathcal{G}_{r,s,u}^\prime(\ell)}{\mathcal{G}_{r,s,u}(\ell)} \right| p^{-2\omega_2}.
\]
(7.3)

On the other hand, let us observe that for any graph \( G \in \mathcal{G}_{r,s,u}(\ell) \) and any graph \( G' \) constructed from \( G \) as above, \( G' \) has precisely 2\( \omega_2 \) edges fewer than \( G \), and therefore
\[
\mathbb{P} \left[ G(A) = G' \right] = \mathbb{P} \left[ G(A) = G \right] p^{-2\omega_2} (1 - p)^{2\omega_2} = (1 + o(1)) \mathbb{P} \left[ G(A) = G \right] p^{-2\omega_2}.
\]
(7.4)

Combining (7.3) and (7.4), we deduce the statement of the proposition. \( \square \)

**Proof of Lemma 8.1** For any \((r, s, u) = (1 + o(\omega_{-1}))((\hat{r}, \hat{s}, \hat{u}))\) and for any \( G \in \mathcal{G}_{r,s,u}(\ell) \), pick an arbitrary subset \( U'' \subseteq U' \) of size \( \omega_2 \), where \( U' \) is as in Claim 7.5 and let \( G' := G(U'') \).

Let us define the set \( \mathcal{S} = \{ (r, s, u) : \frac{r}{\omega_1} = \frac{s}{\omega_2} = \frac{u}{\omega_2} = 1 + o(1) \} \). Observe that since \( \omega_2 \leq \omega_1 = o(n) \) we have
\[
(r, s, u) \in \mathcal{S} \Leftrightarrow \left( r + \frac{2\omega_2}{n}, s + \frac{\omega_2}{n}, u - \frac{\omega_2}{n} \right) \in \mathcal{S}.
\]
Using this fact, we obtain
\[
\mathbb{P} \left[ |n_a - m_a| \leq \omega_0 \right] = \left( \sum_{(r, s, u) \in \mathcal{S}} \sum_{|a| = 0} \mathbb{P} \left[ G(A) \in \mathcal{G}_{r,s,u}(\ell) \right] \right) + o(1)
\]
\[
\leq \left( \sum_{(r, s, u) \in \mathcal{S}} \sum_{|a| = 0} \mathbb{P} \left[ G(A) \in \mathcal{G}_{r,s,u}^\prime(\ell) \right] \right) + o(1) = \mathbb{P} \left[ |n_a - m_a + \omega_2| \leq \omega_0 \right] + o(1).
\]
However, since \( \omega_2 \) is chosen uniformly at random from the interval \([\omega_1/2, \omega_1]\), and in particular independently of \( A \), we may change our point of view and say that
\[
\mathbb{P} \left[ |n_a - m_a + \omega_2| \leq \omega_0 \right] = \mathbb{P} \left[ \omega_2 = |m_a - n_a| \leq \omega_0 \right] \leq \frac{2\omega_0 + 1}{\omega_1/2} = o(1),
\]
as required. \( \square \)

8. MOMENTS AND EXPANSION

8.1. Overview. In this section we prove Proposition 2.10. The proofs of the two statements of the proposition proceed via two rather different arguments. First we show that it is unlikely that \( |V_a(A)| = |C_a(A)| \) is large and at the same time \( f(A) \sim \alpha^a \), which would imply that the slush is almost entirely frozen. The proof relies on the fact that \( G(A) \) is unlikely to contain a moderately large, relatively densely connected subgraph. Specifically, let \( A \) be a matrix. A flipper of \( A \) is a set of variable nodes \( U \subseteq V(A) \) such that for all \( a \in \partial U \) we have \( |a \cap U| \geq 2 \). Let \( \mathcal{F}_\ell(A) \) be the set of all flippers \( U \) of \( A \) of size \( |U| \leq \ell n \). Moreover, let \( F_\ell(A) = \sum_{U \in \mathcal{F}_\ell(A)} |U| \) be the total size of all flippers of \( A \) which individually each have size at most \( \ell n \).

**Lemma 8.1.** For any \( d > 0 \) there exists \( \epsilon > 0 \) such that for any function \( \omega = \omega(n) \gg 1 \) we have \( F_\ell(A_a) \leq \omega \omega \text{ w.h.p.} \)

The proof of Lemma 8.1 can be found in Section 8.2. We will combine Lemma 8.1 with the following statement to bound the size of \( V_a(A) \setminus \mathcal{F}_\ell(A_a) \).

**Lemma 8.2.** The set \( U = V_a(A) \setminus \mathcal{F}_\ell(A_a) \) is a flipper of \( A_a \) of size \( |U| \geq |V_a(A)| - |C_a(A)| \) and \( U \cap \mathcal{F}(A) = \emptyset \).

**Proof.** Clearly, \( \text{null } A_a \geq |V_a(A)| - |C_a(A)| \) and thus
\[
2^{|V_a(A)| - |C_a(A)|} \leq \text{null } A_a = |\ker A_a| \leq \left| \left\{ \xi \in F_2^{V_a(A)} : \forall v \in \mathcal{F}(A) : \xi_v = 0 \right\} \right| = 2^{|U|}.
\]
Hence, \( |U| \geq |V_a(A)| - |C_a(A)| \).

To show that \( U \) is a flipper of \( A \) we consider a variable node \( v \in U \) and an adjacent check node \( a \in C_a(A) \). Assume for a contradiction that \( \partial a \cap U = \{v\} \). Then for all other variable nodes \( u \in \partial a \cap V_a(A) \) we have \( u \in \mathcal{F}(A_a) \). Hence, the only way to satisfy check \( a \) is by setting \( v \) to zero, too. Thus, \( v \in \mathcal{F}(A_a) \), which contradicts \( v \in U \).
Finally, to show that $U \cap \mathcal{F}(A) = \emptyset$ it suffices to prove that any vector $\xi \in \ker A$ extends to a vector $\xi \in \ker A$. To see this we recall the peeling process \cite{2.7} that yields $V_\ell(A)$. Let us actually run this peeling process in two stages. In the first stage we repeatedly delete check nodes of degree one or less from $G(A)$: 

while there is a check node of degree one or less, remove it along with its adjacent variable (if any). 

The set of variable nodes that this process removes is precisely $V_\ell(A)$ and we extend $\xi_s$ by setting $\xi_v = 0$ for all $v \in V_\ell(A)$. Next we repeatedly delete variable nodes of degree one or less: 

while there is a variable node of degree one or less, remove it along with its adjacent check (if any).

Let $y_1, \ldots, y_\ell$ be the variable nodes that this process deletes, and suppose that they were deleted in this order. Then we inductively extend $\xi_s$ by assigning the variables in the reverse order $y_\ell, \ldots, y_1$ as follows. At the time $y_k$ was deleted, where $1 \leq k \leq \ell$, this variable node either had no adjacent check node at all, in which case we define $\xi_{y_k} = 0$, or there was precisely one adjacent check node $b_k$. In the latter case we set $\xi_{y_k}$ to the (unique) value that satisfies $s_k$ given the previously defined entries of $\xi$. The construction ensures that $\xi \in \ker A$.

Second, we bound the probability that $|C_\ell(A)| - |V_\ell(A)|$ is large and at the same time $f(A) \sim \alpha_s$. The proof of the following lemma, which we postpone to Section \textit{8.3}, is based on a delicate moment calculation.

\textbf{Lemma 8.3.} For any $d > e$ there exists $\epsilon > 0$ such that for any $\omega = \omega(n) \gg 1$ we have

$$P \left( |C_\ell(A)| - |V_\ell(A)| \geq \omega \text{ and } |V_\ell(A) \cap \mathcal{F}(A)| < \epsilon n \right) = o(1).$$

\textbf{Proof of Proposition \textit{2.10.}} Fix a small enough $\epsilon > 0$ and suppose that $\omega = \omega(n) \gg 1$. To prove the first statement let $\mathcal{E} = \{|V_\ell(A)| - |C_\ell(A)| \geq \omega\}$ and $\mathcal{E}^\prime = \{|V_\ell(A)| < \omega\}$. Lemma \textit{8.2.2} shows that if the event $\mathcal{E} = \mathcal{E}^\prime$ occurs, then the set $U = V_\ell(A) \setminus \mathcal{F}(A)$, being a flipper of size at least $\omega$ (by $\mathcal{E}$), cannot be included in $\mathcal{F}_\ell(A)$ (because of $\mathcal{E}^\prime$) and therefore has size at least $\epsilon n$. Additionally, we have $U \cap \mathcal{F}(A) = \emptyset$ while $U \subseteq V_\ell(A) \subseteq V \setminus V_\ell(A)$. Hence, Proposition \textit{2.4} implies $f(A) \leq |V(A) \setminus V_\ell(A)|/n + o(1) - \epsilon$. Consequently, Proposition \textit{2.5} and Lemma \textit{6.1} yield

$$P \left( \mathcal{E} \cap \{f(A) > \alpha^* - \epsilon/2\} \right) \leq P \left( |F_\ell(A) > \omega| \cup \{|V(A) \setminus V_\ell(A)|/n > \alpha^* + \epsilon/3\} \right) = o(1).$$

Thus, Propositions \textit{2.7} and \textit{2.8} show that $P \left( \mathcal{E} \cap \{|f(A) - \alpha_s| > \epsilon\} \right) = o(1)$.

With respect to the second statement, let $\mathcal{A} = \{|C_\ell(A)| - |V_\ell(A)| \geq \omega\}$ and $\mathcal{A}^\prime = \{|V_\ell(A) \cap \mathcal{F}(A)| < \epsilon n\}$. Then Lemma \textit{6.3} shows that

$$P \left( \mathcal{A} \cap \mathcal{A}^\prime \right) = o(1). \hspace{1cm} (8.1)$$

Moreover, Proposition \textit{2.5} and \textit{2.7} show that

$$P \left( \{f(A) \leq \alpha_s + \epsilon/2\} \setminus \mathcal{A}^\prime \right) = o(1), \hspace{1cm} (8.2)$$

and the assertion is immediate from \textit{(6.1)}, \textit{(6.2)} and Propositions \textit{2.7} and \textit{2.8}. \hfill \Box

\textbf{8.2. Proof of Lemma \textit{8.1.}} A $(u, c, m)$-flipper of $A_u$ consists of a set $U \subseteq V_\ell(A)$ of size $|U| = u$ whose neighbourhood $C = \partial U \cap C_s(A)$ has size $|C| = c$ such that the number the number of $U$-$C$-edges in $G_\ell(A)$ is equal to $m$. Let $Z(u, c, m)$ be the number of $(u, c, m)$-flippers. As a first step we deal with flippers whose average variable degree exceeds two.

\textbf{Claim 8.4.} For any $d > 0, \delta > 0$ there exists $\epsilon > 0$ such that

$$E \left[ \sum_{U \in \mathcal{F}_\ell(A)} |U| \left\{ \sum_{x \in U} \left| \partial x \cap C_s(A) \right| \geq (2 + \delta) |U| \right\} \right] = o(1).$$

\textbf{Proof.} Recalling $p = d/n \wedge 1$, we write the simple-minded bound

$$E[Z(u, c, m)] \leq n \binom{c}{m} \binom{u}{c} p^m; \hspace{1cm} (8.3)$$

here $\binom{c}{m}$ counts the number of choices for $U$, $\binom{u}{c}$ accounts for the number of possible sets of $c$ check nodes, $\binom{u}{m}$ bounds the number of bipartite graphs on the chosen variable and check sets, and $p^m$ bounds the probability that the chosen subgraph is actually contained in $G(A)$. We aim to bound the r.h.s. of \textit{(8.3)} subject to the constraints

$$m \geq \max \{2c, (2 + \delta) u\}, \hspace{1cm} 1 \leq u \leq \epsilon n \hspace{1cm} \text{ for a small enough } \epsilon > 0. \hspace{1cm} (8.4)$$

We consider three separate cases.
Case 1: $c \leq u$: we estimate
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{en}{u}\right)^2 \left(\frac{eud}{2n}\right)^m \leq \left(\frac{en}{u}\right)^2 \left(\frac{e^2d}{2n}\right)^{u(2+\delta)c} \leq \left(\frac{e^4d^2}{2}\right)^u \left(\frac{u}{n}\right)^{\delta u}.
\] (8.5)
Combining (8.3)–(8.5), we obtain
\[
\sum_{1 \leq u \leq n} \mathbb{E}[u] \mathcal{Z}(u, c, m) \leq \sum_{1 \leq u \leq n} u^2 \left(\frac{e^4d^2}{2}\right)^u \left(\frac{u}{n}\right)^{\delta u} = o(1).
\] (8.6)
Case 2: $u \leq c \leq 100u$: due to (8.4) we obtain
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{en}{u}\right)^u \left(\frac{en}{c}\right)^c \left(\frac{eud}{2n}\right)^c \left(\frac{eud}{2n}\right)^{m/2} \leq \left(\frac{en}{u}\right)^u \left(\frac{e^2d}{2}\right)^c \left(\frac{eud}{2n}\right)^{u(1+\delta/2)} \leq \left(\frac{e^2d}{2}\right)^{400u} \left(\frac{u}{n}\right)^{\delta u/2}.
\] (8.7)
Combining (8.3) and (8.8), we get
\[
\sum_{1 \leq u \leq n} \sum_{u \leq c \leq 100u} \mathbb{E}[u] \mathcal{Z}(u, c, m) \leq \sum_{1 \leq u \leq n} \sum_{100u \leq c \leq n} u^2 \left(\frac{e^2d}{2}\right)^{400u} \left(\frac{u}{n}\right)^{\delta u/2} = o(1).
\] (8.8)
Case 3: $100u \leq c \leq n$: the condition (8.4) yields
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{100en}{c}\right)^{1.1c} \left(\frac{edu}{n}\right)^{2c} \leq \left(\frac{edu}{n}\right)^{c/2}.
\]
Hence,
\[
\sum_{1 \leq u \leq n} \sum_{100u \leq c \leq n} \mathbb{E}[u] \mathcal{Z}(u, c, m) \leq \sum_{1 \leq u \leq n} u \sum_{100u \leq c \leq n} \left(\frac{edu}{n}\right)^{c/2} \leq \sum_{1 \leq u \leq n} u \left(\frac{edu}{n}\right)^u = o(1).
\] (8.9)
Finally, the assertion follows from (8.6), (8.8) and (8.9).

Complementing Claim 8.4, we now estimate the sizes of flippers of average check degree greater than two.

Claim 8.5. For any $d > 0$, $\delta > 0$ there exists $\varepsilon > 0$ such that
\[
\mathbb{E}\left[\sum_{U \in \mathcal{A}(A)} |U| \left\{ \sum_{a \in \partial U \cap \mathcal{A}(A)} |\partial a \cap U| \geq (2 + \delta)|U| \right\} \right] = o(1).
\]
Proof. The proof is rather similar to the proof of the previous claim, except that we swap the roles of $u$ and $c$. Once more we start from the naive bound (8.3), but this time $m$ satisfies $m \geq \max \{2u, (2 + \delta)c\}$ and $1 \leq u \leq \varepsilon n$.

Case 1: $u \leq c$: we have
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{en}{u}\right)^{2c} \left(\frac{eud}{2n}\right)^{2c} \leq \left(\frac{eud}{2n}\right)^{u(1+\delta)c} \leq \left(\frac{e^2d}{2}\right)^{u(1+\delta)} \left(\frac{u}{n}\right)^{\delta u/200}.
\] (8.10)
Case 2: $c \leq u \leq 100c$: we estimate
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{en}{u}\right)^{2c} \left(\frac{eud}{2n}\right)^{2c} \leq \left(\frac{eud}{2n}\right)^{u(1+\delta)c} \leq \left(\frac{e^2d}{2}\right)^{u(1+\delta)} \left(\frac{u}{n}\right)^{\delta u/200}.
\] (8.11)
Case 3: $100c \leq u$: we have
\[
\binom{n}{u} \binom{uc}{m} p^m \leq \left(\frac{en}{u}\right)^{u(1+\delta)c} \left(\frac{eud}{n}\right)^{2u} \leq \left(\frac{eud}{n}\right)^{u(1+\delta)c}.
\] (8.12)
Summing (8.10), (8.11) and (8.12) on $u, c, m$ such that $m \geq (2 + \delta)c$, we obtain $\sum_{u, c, m} \mathbb{E}[u] \mathcal{Z}(u, c, m) = o(1)$.

Finally, we need to deal with flippers of average variable and constraint degree about two.

Claim 8.6. For any $d > e$ there exists $\varepsilon > 0$ such that for any $\omega = \omega(n) \gg 1$ we have
\[
\mathbb{P}\left[\sum_{U \in \mathcal{A}(A)} |U| \left\{ \sum_{x \in U} |\partial x \cap \mathcal{A}(A)| \leq (2 + \varepsilon)|U|, \sum_{a \in \partial U \cap \mathcal{A}(A)} |\partial a \cap U| \leq (2 + \varepsilon)|C| \right\} > \omega \right] = o(1).
\]
Proof. Choose \( L = L(d) > 0 \) sufficiently large and subsequently \( \epsilon > 0 \) sufficiently small. Moreover, for a vertex \( u \) of \( G_\beta(A) \) let \( d_\beta(u) \) signify the degree of \( u \) in \( G_\beta(A) \). Further, with \( \nu, \lambda \) from (2.3) let \( \mathcal{D} \) be the event that the graph \( G_\beta(A) \) enjoys the following four properties:

- **D1:** \( |V_\beta(A)| = (\nu + o(1))n \) and \( |C_\beta(A)| = (\nu + o(1))n \).
- **D2:** For any \( 2 \leq \ell \leq L \) we have \( \sum_{x \in V_\beta(A)} \mathbb{1} \left( d_\beta(x) = \ell \right) = \mathbb{P} \left[ \text{Po}_2(\lambda) = \ell \right] \nu n + o(n) \).
- **D3:** For any \( 2 \leq \ell \leq L \) we have \( \sum_{a \in C_\beta(A)} \mathbb{1} \left( d_\beta(a) = \ell \right) = \mathbb{P} \left[ \text{Po}_2(\lambda) = \ell \right] \nu n + o(n) \).
- **D4:** The bounds from (2.11) hold for the degree sequence of \( G(A) \).

Then Proposition 2.6 and Lemma 2.14 imply that

\[
\mathbb{P} \left[ \mathcal{D} \right] = 1 - o(1). \tag{8.13}
\]

We aim to count \((u, c, m)\)-flippers \( U \subseteq V_\beta(A) \) with neighbourhoods \( C = \partial U \cap C_\beta(A) \) of size \( |C| = c \) such that

\[
m = \sum_{x \in U} |\partial x \cap C| = \sum_{a \in C} |\partial a \cap U| \leq (2 + \epsilon) (u \wedge c), \quad \text{and, of course,} \quad \min \{|\partial a \cap U| \geq 2 \}. \tag{8.14}
\]

To estimate the number \( Z(u, c, m) \) we recall from Proposition 2.6 that the graph \( G_\beta(A) \) is uniformly random given the degrees. Therefore, according to Lemma 2.13 it suffices to bound the number of \((u, c, m)\)-flippers of a random graph chosen from the pairing model with the same degree sequence. Thus, let \( \Gamma_\beta \) be a random perfect matching of the complete bipartite graph on the vertex sets

\[
\mathcal{V} = \bigcup_{v \in V_\beta(A)} \{ v \} \times \{ d_\beta(v) \}, \quad \mathcal{C} = \bigcup_{a \in C_\beta(A)} \{ a \} \times \{ d_\beta(a) \}.
\]

Furthermore, let \( \mathcal{D}_\beta \) be the multigraph obtained from \( \Gamma_\beta \) by contracting the clones \( \{ v \} \times \{ d_\beta(v) \} \) and \( \{ a \} \times \{ d_\beta(a) \} \) of the variable and constraint nodes into single vertices for all \( v \in V_\beta(A) \), \( a \in C_\beta(A) \). Due to (8.13) it suffices to establish the bound

\[
\sum_{u, c, m} \mathbb{E} \left[ Z(u, c, m) \mid \mathcal{D}_\beta \right] = O(1). \tag{8.15}
\]

To prove (8.15) we first count viable choices of \( U \). Since (8.14) implies that \( 2u \leq m \leq (2 + \epsilon)u \), no more than \( \delta u \) of the vertices in the set \( U \) have degree greater than two. Further, D1 and D2 show that there are no more than

\[
\left( \frac{(\nu + o(1))n}{u} \right)^{u} \leq \left( \frac{eL}{\epsilon} \right) \left( \frac{e(\nu + o(1))n}{u} \right)^{u} \left( \frac{\lambda^2 + o(1)}{2(\exp(\lambda) - \lambda - 1)} \right)^{u} \tag{8.16}
\]

such sets \( U \).

By a similar token, most check nodes in \( C \) have precisely two neighbours in \( U \). Thus, we estimate the number of choices of \( C \subseteq C_\beta(A) \) of size \( c \) along with a set \( \mathcal{C} \) of \( m \) clones of these checks as follows. Summing on all vectors \( k = (k_1, \ldots, k_c) \) of integers \( k_i \geq 2 \) with \( \sum_i k_i = m \) and on all sequences \( (b_1, \ldots, b_c) \in C_\beta(A)^c \), we obtain the bound

\[
\frac{1}{c!} \sum_{k \in \mathcal{C}} \sum_{i=1}^c \frac{d_\beta(b_i)}{k_i} \leq \frac{1}{c!} \sum_{k \in \mathcal{C}} \sum_{i=1}^c \frac{d_\beta(b_i)}{k_i}. \tag{8.17}
\]

Now, (8.14) implies that \( \sum_{i \in \mathcal{C}} (k_i > 2) \leq 3\epsilon c \). Therefore, D3 and D4 ensure that for any \( k \),

\[
\sum_{i=1}^c \frac{d_\beta(b_i)}{k_i} \leq L^{3\epsilon c} \left( \frac{(\nu + o(1))n}{c} \right)^c \left( \frac{\lambda^2 \exp(\lambda) + o(1)}{2(\exp(\lambda) - \lambda - 1)} \right)^c. \tag{8.18}
\]

Furthermore, there are no more than \( \binom{m-c-1}{c-1} = \binom{m-c-1}{m-2c} \) possible vectors \( k \) and thus (8.14) yields

\[
\left( \frac{m - c - 1}{m - 2c} \right) \leq \left( \frac{2e^{L^3}}{\epsilon} \right)^{c}. \tag{8.19}
\]

Combining (8.17)–(8.19) with D1, we see that the number of possible \( C, \mathcal{C} \) is bounded by

\[
\sum_{x \in V_\beta(A)} d_\beta(x) = (1 + o_c(1)) \nu n \mathbb{E} \left[ \text{Po}_2(\lambda) \right] = (1 + o_c(1)) \frac{\nu n \lambda(\exp(\lambda) - 1)}{\exp(\lambda) - \lambda - 1}, \tag{8.20}
\]

Finally, since D2 and D4 imply that

\[
\sum_{x \in V_\beta(A)} d_\beta(x) = (1 + o_c(1)) \nu n \mathbb{E} \left[ \text{Po}_2(\lambda) \right] = (1 + o_c(1)) \frac{\nu n \lambda(\exp(\lambda) - 1)}{\exp(\lambda) - \lambda - 1},
\]
the probability that $\Gamma_a$ matches the designated variable/check clones comes to
\[
\frac{m! \sum_{x \in V_s(A)} d_a(x) - m!}{(\sum_{x \in V_s(A)} d_a(x))!} = \left( \sum_{x \in V_s(A)} d_a(x) \right)^{-1} \left( \frac{e(\lambda(\exp(\lambda) - 1) - o_\varepsilon(1)) n}{m(\exp(\lambda) - \lambda - 1)} \right)^m.
\] (8.21)

Combining (8.16), (8.20) and (8.21) (and dragging all $o(1)$-error terms into the $o_\varepsilon(1)$), we obtain
\[
\mathbb{E}[Z(u, c, m) | \mathcal{G}] \leq \left( \frac{e^\nu u}{u} \right)^c \left( \frac{\varepsilon}{m(\exp(\lambda) - \lambda - 1)} \right)^m \left( \frac{\lambda^2 \exp(\lambda)}{2(\exp(\lambda) - \lambda - 1)^2} \right)^u.
\]
Hence, (8.14) yields
\[
\mathbb{E}[Z(u, c, m) | \mathcal{G}] \leq \left( \frac{u}{n} \right)^{m-c} \left( \frac{\lambda^2 \exp(\lambda) + o_\varepsilon(1)}{(\exp(\lambda) - 1)^2} \right)^u.
\] (8.22)

Since $\lambda > 0$ we have $\lambda^2 \exp(\lambda)/((\exp(\lambda) - 1)^2) < 1$. Therefore, (8.22) implies (8.15) for small $\varepsilon > 0$.

**Proof of Lemma 8.1.** The lemma follows from Claims 8.4, 8.5 and 8.6. More precisely, let given $d > e$, let $\varepsilon_1$ be the $\varepsilon$ given by Claim 8.4 and subsequently set $\delta := \varepsilon_1$ and let $\varepsilon_2, \varepsilon_3$ be the $\varepsilon$ given by Claims 8.4 and 8.5 respectively. Then let set $\varepsilon_0 := \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$.

Now Claims 8.4 and 8.5 imply that w.h.p. there is no $U \in \tilde{G}_e(A)$ with $\sum_{x \in U} |\delta x \cap C_e(A)| \geq (2 + \delta)|U|$ or with $\sum_{x \in \partial U \cap C_e(A)} |\delta a \cap U| \geq (2 + \delta)|U|$. On the other hand, conditioning on this event, since $\varepsilon_0 \leq \varepsilon_1 = \delta$ we have $\tilde{G}_e(A) \subset \mathcal{G}$, and therefore Claim 8.6 implies that w.h.p. $F_{\varepsilon_0}(A) \leq \omega$ for any function $\omega = \omega(n) \gg 1$, as required.

**8.3. Proof of Lemma 8.3** The proof is based on a somewhat delicate moment calculation. Suppose that $|V_s(A) \cap \mathcal{F}(A)| < \varepsilon n$, i.e., very few coordinates in the slush are frozen. Then Fact 2.1 implies that for most $v \in V_s(A)$ the corresponding entry $x_{b,v}$ of a random vector $x_b \in \ker A_b$ takes the value 0 with probability precisely 1/2. Furthermore, since $|V_s(A)| = \Omega(n)$ w.h.p., Proposition 2.11 implies that for most pairs $u, v \in V_s(A)$ the entries $x_{b,u}, x_{b,v}$ are stochastically independent. Therefore, w.h.p. the random vector $x_b$ has Hamming weight $(1/2 + o_\varepsilon(1))|V_s(A)|$. Hence, a tempting first idea toward the proof of Lemma 8.3 might be to simply calculate the expected number of vectors of Hamming weight $(1/2 + o_\varepsilon(1))|V_s(A)|$ in the kernel of $A_b$.

This strategy would work if we could replace the $o_\varepsilon(1)$ error term above by $O(n^{-1/2})$. Indeed, there are $2^{\bar{V}_s(A)}$ candidate vectors of Hamming weight $|V_s(A)|/2 + O(\sqrt{n})$. Moreover, it is not very hard to verify that a given such vector satisfies all checks with probability $\Theta(2^{-|C_s(A)|})$. As a consequence, the expected number of vectors in $\ker A_b$ of Hamming weight $|V_s(A)|/2 + O(\sqrt{n})$ tends to zero if $|C_s(A)| \gg 1$. But unfortunately this simple calculation does not extend to larger $\varepsilon$ as required by Lemma 8.3. The reason is that for larger $\varepsilon$ a second order term pop up, i.e., the probability that all checks are satisfied reads
\[
2^{-|C_s(A)| + O(\varepsilon^2|C_s(A)|)}.
\]

This quadratic term is due to the presence of checks of degree two. We deal with this problem by observing that a check node of degree two simply imposes an equality constraint on its two adjacent variables. Thus, any two variable nodes that appear in a check node of degree two can be contracted into a single variable node and then the check node can be eliminated. A variant of the moment calculation, without the quadratic error term, can then be applied to the matrix that the multigraph resulting from the contraction procedure induces.

To carry out this programme we first investigate the subgraph $G_b(A)$ obtained from $G_s(A)$ by deleting all checks of degree greater than two. More precisely, invoking Lemma 2.13 for the apparent technical reason we will instead analyse the random multigraph $\mathcal{G}_b$ that results by applying the contraction procedure to the random multigraph $\mathcal{G}_b$ chosen from the pairing model with the same degrees as $G_s(A)$. The proof of the following lemma can be found in Section 8.3.

**Lemma 8.7.** For any $d > e$ there exists $b > 0$ such that for any $\omega = \omega(n) \gg 1$ the random graph $\mathcal{G}_b$ enjoys the following properties w.h.p.

(i) The largest component of $\mathcal{G}_b$ has size at most $\omega \log n$.

(ii) $\mathcal{G}_b$ contains no more than $\omega$ cycles.

(iii) For any $t > 0$ no more than $|V_b(A)| \exp(-bt)$ variable nodes belong to components of size at least $t$.  

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Indeed, let $K$ and let $t$ we claim that for $1 \leq \varepsilon > 0$ such that
\[
\frac{1}{2} - \frac{\sum_{x \in \mathcal{V}_G} d_n(x) 1 \{\xi(x) = 0\}}{\sum_{x \in \mathcal{V}_G} d_n(x)} < \varepsilon.
\]

In Section 8.5 we will prove the following statement.

**Lemma 8.8.** For any $d \geq e$ there exists $\varepsilon > 0$ such that for any $\omega = \omega(n) \gg 1$ we have
\[
P\left[ |\mathcal{E}_G| \geq |\mathcal{V}_G| + \varepsilon \text{ and } \mathcal{X}_c'' \neq \emptyset \right] = o(1).
\]

In addition, we observe the following.

**Lemma 8.9.** For any $d \geq e$, $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
P\left[ |\mathcal{V}_G \setminus \mathcal{F}(A)| > (1 - \delta)|\mathcal{V}_G(A)| \text{ and } \mathcal{X}_c'' = \emptyset \right] = o(1).
\]

The proof of Lemma 8.9 can be found in Section 8.6

**Proof of Lemma 8.3** The assertion is an immediate consequence of Lemmas 8.7, 8.8 and 8.9.

8.4. **Proof of Lemma 8.7** We apply a branching process argument to a random graph chosen from the pairing model, not unlikely the one from [37]. Specifically, let $(d_n(v))_{v \in \mathcal{V}_G(A)}$ be the degree sequence of the graph $G_n(A)$ and let $m'_n$ be the number of check of degree two in $G_n(A)$. Let us write $\{m'_n\}$. Let $\Gamma = \{\alpha_1, \ldots, \alpha_{m'_n}\}$.

To be precise, let $\Delta = \sum_{v \in \mathcal{V}_G} d_n(v)$ and let $\Gamma'_n$ be a random perfect matching of the complete bipartite graph with vertex sets
\[
\Gamma = \bigcup_{v \in \mathcal{V}_G} \{v\} \times [d_n(v)] \quad \text{and} \quad \mathcal{C} = \{\alpha_1, \ldots, \alpha_{m'_n}\} \times [2] \cup \{\beta_1, \ldots, \beta_{\Delta - 2m'_n}\}.
\]

As always, $\{v\} \times [d_n(v)]$ and $\{\alpha_i\} \times [2]$ represent sets of clones of the variable node $v$ and the check node $\alpha_i$, respectively. The ‘ballast’ clones $\beta_1, \ldots, \beta_{\Delta - 2m'_n}$ are included so that both sides of the bipartition have the same size.

Further, deleting $\beta_1, \ldots, \beta_{\Delta - 2m'_n}$ and contracting the other clones into single vertices, we obtain a random multigraph $\mathcal{G}(\Gamma)$ from the matching $\Gamma$. This multigraph is identical in distribution to $\mathcal{G}_0''$.

**Claim 8.10.** Wh.p. all connected components of $\mathcal{G}(\Gamma)$ have size $O(\log n)$.

**Proof.** To trace the set of nodes reachable from $(\alpha_1, 1)$, we classify each clone as either unexplored, active or inactive. At the start of the process only $(\alpha_1, 1)$ is active and all other clones are unexplored; thus,
\[
\mathcal{A}_0 = \{(\alpha_1, 1)\}, \quad \mathcal{U}_0 = \{(\alpha_1, 2), (\alpha_2, 1), (\alpha_2, 2), \ldots, (\alpha_{m'_n}, 1), (\alpha_{m'_n}, 2)\} \setminus \mathcal{A}_0, \quad \mathcal{F}_0 = \emptyset.
\]

The classification determines the order in which the edges of the matching $\Gamma$ are exposed. Specifically, if at some time $t \geq 1$ no active check clone remains, the process stops and we let $T_0 = t - 1$. Otherwise, at time step $t \geq 1$ an active clone $(\alpha_i, h_i) \in \mathcal{A}_t$ is chosen uniformly at random and we let $\mathcal{F}_t = \mathcal{F}_{t-1} \cup \{(\alpha_i, h_i)\}$. If the second clone $(\alpha_j, 3 - h_i)$ of the same check is either active or inactive, we let $\mathcal{U}_t = \mathcal{U}_{t-1} \setminus \{(\alpha_i, h_i)\}$. Otherwise, we expose the edge of $\Gamma$ incident with the other clone $(\alpha_j, 3 - h_i)$ of check $\alpha_i$. Let $y_t$ be the variable node on the other end of this edge. We then declare all as yet inactive clones of checks $\alpha_i$, $i \in [m'_n]$, that are adjacent to clones of $y_t$, active. Formally, let
\[
\mathcal{A}_t = \mathcal{A}_{t-1} \cup \{(\alpha_i, 1), (\alpha_i, 2)\}, \quad \mathcal{U}_t = \{\alpha_j, 3 - h_i\} \setminus \mathcal{A}_t \setminus \mathcal{F}_t \quad \text{and} \quad \mathcal{U}_t = \mathcal{U}_{t-1} \setminus \{(\alpha_i, 1), (\alpha_i, 2)\} \setminus \mathcal{F}_t.
\]

The aim is to investigate the stopping time $T_0$. We may condition on the event $d_n(v) \leq \log^2 n$ for all $v$. Moreover, we claim that for $1 \leq t \leq T_0 \wedge \log^3 n$,
\[
\mathbb{E}[[\mathcal{A}_t] - |\mathcal{A}_{t-1}| | \mathcal{A}_{t-1}] < 0.
\]

Indeed, $|\mathcal{A}_t| - |\mathcal{A}_{t-1}|$ is trivially negative if $(h_i, 3 - h_i) \in \mathcal{U}_{t-1}$. Further, if $(\alpha_i, 3 - h_i) \in \mathcal{U}_{t-1}$, then $\Gamma$ matches this clone to a random vacant variable clone. Because $t \leq \log^3 n$ and $\max_v d_n(v) \leq \log^2 n$ while the slush has size

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\[|V_b(A)| = \Omega(n),\] the distribution of \(d_y(y_i)\) is within \(O(n^{-0.99})\) in total variation of the distribution \((d_y(v)/\Delta)_{v \in V_b(A)}\) of the degree of the variable node of a random variable clone. We subsequently expose all edges of \(\Gamma\) incident with a clone of \(y_i\) that was unexplored at time \(t-1\). Once more because \(t \leq \log^2 n\) and \(\max_y d_y(v) \leq \log^2 n\), the conditional probability that a specific unexplored clone of \(y_i\) links to an unexplored clone from the set \(\{(a_i,1),(a_i,2) : i \in |m'_b|\}\) is bounded by \(2m'_b/\Delta + O(n^{-0.99})\). Therefore, we obtain the bound
\[
E[|\alpha_{t-1}| - |\alpha_{t-1}|/|\alpha_{t-1}|] \leq o(1) - 1 + \frac{2m'_b}{\Delta^2} \sum_{v \in V_b(A)} d_y(v)(d_y(v) - 1) \leq \frac{\lambda^2 \exp(\lambda)}{\exp(\lambda) - 1} - 1 + o(1).
\]

Moreover, it is easy to check that \(\lambda > 0\) for all \(d > e\) and that
\[
\frac{z^2 \exp(z)}{(\exp(z) - 1)^2} < 1\quad \text{for any } z > 0.
\]

Thus, (8.23) follows from (8.24) and (8.25). Finally, (8.23) implies that \(|\alpha_{t-1}|\) is dominated by a random walk with a negative drift. Consequently, \(P[T_0 > c \log n] = o(n^{-1})\) for a suitable \(c > 0\). The assertion follows from the union bound.

**Claim 8.11.** There exists \(b = b(d) > 0\) such that w.h.p. for all \(t > 0\) the number of variable nodes of \(\varphi'_b\) that belong to components of size at least \(t\) is bounded by \(|V_b(A)| \exp(-bt)\).

**Proof.** Let \(Z_t\) be the number of variable nodes of \(\varphi'_b\) that belong to components of size at least \(t\). Tracing the same exploration process as in the previous proof and using (8.24), we find \(\zeta = \zeta(d) > 0\) such that
\[
E[Z_t] \leq |V_b(A)| \exp(-2\zeta t).
\]

If \(t > \log\log n\), say, then the assertion simply follows from (8.26) and Markov's inequality. Thus, suppose that \(t \leq \log\log n\) and \(|V_b(A)| = \Omega(n)\) and that the largest component of \(\varphi'_b\) contains no more than \(\log n \log n\) variable nodes. Then adding to or removing from \(\varphi'_b\) a single edge can alter \(Z_t\) by at most \(2t\). Therefore, the assertion follows from (8.26) and Azuma's inequality.

As a next step we need to estimate the number of short cycles.

**Claim 8.12.** The expected number of nodes on cycles of \(\varphi'_b\) of size at most \(\log^2 n\) is bounded.

**Proof.** Let \(\ell \leq \log^2 n\), let \(y = (y_1, \ldots, y_\ell) \in V_b(A)^\ell\) be a sequence of variables, let \(i = (i_1, i_2, \ldots, i_\ell, i'_\ell)\) be a sequence that contains two clones of each variable \(y_1, \ldots, y_\ell\) and let \(\alpha = (a_1, \ldots, a_\ell)\) be a sequence of \(\ell\) distinct checks of degree two. Let \(\mathcal{E}(y, i, \alpha)\) be the event that \(\Gamma\) connects the two clones of \(\alpha_h\) with \((y_h, i'_h)\) and \((y_{h+1}, i_{h+1})\). Since Proposition 2.6 shows that \(\Delta = \Omega(n)\) and \(\ell \leq \log^2 n\), we obtain
\[
P[\mathcal{E}(y, i, \alpha) | (d_{x}), m'_b] \sim \left(\frac{2\Delta^2}{\lambda^2}\right)^\ell.
\]

Furthermore, we have
\[
E \left[ \sum_{x \in V_b(A)} \frac{d_y(d_y - 1)}{|V_b(A)|} \right] \sim \frac{\lambda^2 \exp(\lambda)}{\exp(\lambda) - 1}, \quad E \left[ \frac{\Delta}{|V_b(A)|} \right] \sim \frac{\lambda(\exp(\lambda) - 1)}{\exp(\lambda) - 1}, \quad E \left[ \frac{m'_b}{|V_b(A)|} \right] = \frac{\lambda^2}{2(\exp(\lambda) - 1)}.
\]

Consequently, the expected number of nodes on cycles of length \(\ell\) works out to be
\[
\frac{1}{2\ell} \sum_{y,i,\alpha} 2\ell P[\mathcal{E}(y, i, \alpha) | (d_{x})) \sim \left(\frac{\lambda^2 \exp(\lambda)}{(\exp(\lambda) - 1)^2}\right)^\ell = \exp(-\Omega(\ell)).
\]

Summing on \(\ell\) completes the proof.

**Proof of Lemma 8.7.** The statement follows from Claims 8.10 and 8.12.
8.5. **Proof of Lemma 8.8.** To simplify the notation we introduce $N = |\mathcal{V}_s|$, $M = |\mathcal{E}_s|$. Moreover, we write $d_1, \ldots, d_N$ for the degrees of the variable nodes of $\mathcal{G}_s''$ and $k_1, \ldots, k_M \geq 3$ for the degrees of the constraints. We need the following facts about $M, N$ and the degrees.

**Claim 8.13.** W.h.p. we have

\[
M, N = \Omega(n), \quad \max_{1 \leq i \leq N} d_i \leq \log^3 N, \quad \max_{1 \leq i \leq M} k_i \leq \log^2 N, \quad \sum_{i=1}^M k_i^2 = O(M), \quad \sum_{i=1}^N d_i^2 = O(N). \tag{8.27}
\]

**Proof.** The first estimate follows immediately from Proposition 2.6 and Lemma 8.7. The second statement follows from Lemma 8.7(i) and the fact that the maximum degree of $G(A)$ is of order $\log n$ w.h.p., which also implies the third bound. Similarly, the sum of the squares of the check degrees of $G(A)$ is bounded w.h.p. due to routine bounds on the tails of the binomial distribution. This implies that $\sum_{i=1}^M k_i^2 = O(M)$ because $M = \Omega(n)$ w.h.p. by Proposition 2.6. To obtain the final bound we apply the Chernoff bound to conclude that for any $d > 0$ there exists $b > 0$ such that w.h.p.

\[
\frac{1}{n} \sum_{i=1}^n \mathbb{I} \{d_{G(A)} v_i \geq t\} \leq \exp(-bt)/b. \tag{8.28}
\]

In other words, the degree sequence of $G(A)$ has an exponentially decaying tail w.h.p. Assuming $N = \Omega(n)$, we see that (8.28) implies the bound

\[
\frac{1}{N} \sum_{i=1}^N \mathbb{I} \{d_i \geq t\} \leq \exp(-b't)/b' \tag{8.29}
\]

for some $b' > 0$. Furthermore, Lemma 8.7(iii) implies an exponentially decaying tail for the component sizes of $\mathcal{G}_s''$. Since $\mathcal{G}_s''$ is obtained by contracting the components of $\mathcal{G}_s'$, the desired bounds follow from (8.29) and Lemma 2.16.

In the following we will condition on the event $\mathcal{D}$ that the conditions (8.27) are satisfied. Let $\sigma \in \mathbb{F}_2^N$ be a uniformly random vector. We will prove Lemma 8.8 by estimating the probability that $\sigma \in \mathcal{X}_s''$. To this end, let

\[
W = \frac{\sum_{i=1}^N d_i \mathbb{I} \{\sigma_i = 1\}}{\sum_{i=1}^N d_i}
\]

count the degree-weighted one-entries of $\sigma$. The following claim bounds the probability that $W$ deviates significantly from 1/2.

**Claim 8.14.** For any $d > e$ there is $s = s(d) > 0$ such that $\mathbb{P} \left[ |W - 1/2| \geq t \mid \mathcal{D} \right] \leq 2 \exp(-st^2 N)$.

**Proof.** This is an immediate consequence of (8.27) and Azuma’s inequality.

As a next step we calculate the probability that $\sigma \in \ker \mathcal{A}_s''$ given $W$.

**Claim 8.15.** For any $d > e$ there exist $\varepsilon > 0, \gamma > 0$ such that uniformly for every $w \in (1/2 - \varepsilon, 1/2 + \varepsilon)$ for which $w \sum_{i=1}^M k_i$ is an even integer we have

\[
\log \mathbb{P} \left[ \mathcal{A}_s'' \sigma = 0 \mid W = w, \mathcal{D} \right] \leq -M \log 2 - \gamma M (w - 1/2)^3 + O(1).
\]

**Proof.** Consider a random vector $\xi = (\xi_{ij})_{i \in [M], j \in [k_i]}$ where we choose every entry $\xi_{ij} \in \mathbb{F}_2$ to be a one with probability $w$ independently. Let $\mathcal{F}$ be the event that $\sum_{j \in [k_i]} \xi_{ij} = 0$ for all $i \in [M]$. Moreover, let

\[
\mathcal{R} = \left\{ \sum_{i=1}^M k_i \mathbb{I} \{\xi_{ij} = 1\} - w = 0 \right\}.
\]

Because $\mathcal{G}_s''$ is drawn from the pairing model, we have

\[
\mathbb{P} \left[ \mathcal{X}_s'' \sigma = 0 \mid W = w, \mathcal{D} \right] = \mathbb{P} [\mathcal{F} \mid \mathcal{R}]. \tag{8.30}
\]
We will calculate the probability on the r.h.s. of (8.30) via Bayes’ rule. The unconditional probabilities are computed easily. Indeed, for every \( i \in [M] \) we have
\[
P \left[ \sum_{j \in \{k_i\}} \xi_{ij} = 0 \right] = \frac{k}{1 \{ j \text{ even} \}} \binom{k}{j} w^j (1 - w)^{k-j}
\]
\[
= \frac{1}{2} \left[ \sum_{j=0}^{k} \binom{k}{j} w^j (1 - w)^{k-j} + \sum_{j=0}^{k} \binom{k}{j} (-w)^j (1 - w)^{k-j} \right] = \frac{1 + (1 - 2w)^k}{2}.
\]
Hence,
\[
P [\mathcal{E}] = \prod_{i=1}^{M} \frac{1 + (1 - 2w)^{k_i}}{2}. \tag{8.31}
\]
Furthermore, the local limit theorem for the binomial distribution shows that
\[
P [\mathcal{R}] = \Theta(M^{-1/2}). \tag{8.32}
\]
In addition, (8.27) and the local limit theorem for sums of independent random variables yield
\[
P [\mathcal{R} | \mathcal{E}] = \Theta(M^{-1/2}). \tag{8.33}
\]
Combining (8.31)–(8.33) and recalling that the \( \xi_{ij} \) are independent, we obtain
\[
\log P [\mathcal{R} | \mathcal{E}] = \sum_{i=1}^{M} \log \frac{1 + (1 - 2w)^{k_i}}{2} + O(1) = -M \log 2 + \sum_{i=1}^{M} \log(1 + (1 - 2w)^{k_i}) + O(1). \tag{8.34}
\]
To complete the proof we compute the derivatives of the last expression, keeping in mind that \( k_i \geq 3 \) for all \( i \):
\[
\frac{\partial \log P [\mathcal{R} | \mathcal{E}]}{\partial w} = \sum_{i=1}^{M} -2k_i (1 - 2w)^{k_i - 1},
\]
\[
\frac{\partial^2 \log P [\mathcal{R} | \mathcal{E}]}{\partial w^2} = \sum_{i=1}^{M} 4k_i (k_i - 1)(1 - 2w)^{k_i - 2} - \frac{4k_i^2 (1 - 2w)^{2k_i - 2}}{(1 + (1 - 2w)^{k_i})^2},
\]
\[
\frac{\partial^3 \log P [\mathcal{R} | \mathcal{E}]}{\partial w^3} = \sum_{i=1}^{M} -8k_i (k_i - 1)(k_i - 2)(1 - 2w)^{k_i - 3} + \frac{8k_i^2 (1 - 2w)^{k_i - 2}(1 - 2w)^{k_i - 1}}{(1 + (1 - 2w)^{k_i})^2} + \frac{16k_i^2 (1 - 2w)^{2k_i - 3}}{(1 + (1 - 2w)^{k_i})^2} - \frac{16k_i^3 (1 - 2w)^{3k_i - 2}}{(1 + (1 - 2w)^{k_i})^3}.
\]
Evaluating these derivatives at \( w = 1/2 \), we obtain
\[
\frac{\partial \log P [\mathcal{R} | \mathcal{E}]}{\partial w} \bigg|_{w=1/2} = \frac{\partial^2 \log P [\mathcal{R} | \mathcal{E}]}{\partial w^2} \bigg|_{w=1/2} = 0, \quad \frac{\partial^3 \log P [\mathcal{R} | \mathcal{E}]}{\partial w^3} \bigg|_{w=1/2} = -48 \sum_{i=1}^{M} 1 \{k_i = 3\}. \tag{8.35}
\]
Finally, combining (8.30), (8.34), and (8.35) with Taylor’s formula completes the proof.

\[\square\]

**Proof of Lemma 8.8** Choose \( \epsilon = \epsilon(d) > 0 \) small enough. Summing over \( w \in (1/2 - \epsilon, 1/2 + \epsilon) \) such that \( \omega \sum_{i=1}^{N} d_i \) is an even integer, we obtain
\[
P [\mathcal{K} \neq \emptyset | \mathcal{D}, M \geq N + \omega] \leq 2^N P \left[ \mathcal{R}'' \sigma = 0, |W - 1/2| < \epsilon | \mathcal{D}, M \geq N + \omega \right] \leq 2^N \sum_{w} P \left[ W = w | \mathcal{D}, M \geq N + \omega \right] P \left[ \mathcal{R}'' \sigma = 0 | W = w, \mathcal{D}, M \geq N + \omega \right].
\]
Combining this bound with Claims 8.14, 8.15 we obtain
\[
P [\mathcal{K} \neq \emptyset | \mathcal{D}, M \geq N + \omega] \leq 2^N \sum_{h=1}^{\lfloor \sqrt{N} \rfloor} \sum_{w: h-1 \leq w} \sum_{N < h} P \left[ W = w | \mathcal{D}, M \geq N + \omega \right] P \left[ \mathcal{R}'' \sigma = 0 | W = w, \mathcal{D}, M \geq N + \omega \right] \leq 2^N \exp \left( -\Omega(h^3) + O(h^3 MN^{-3/2}) \right) = O(2^N M) = o(1),
\]
provided that \( M \geq N + \omega \) and \( \epsilon > 0 \) is small enough. \[\square\]
8.6. **Proof of Lemma 8.9.** The following observation is an easy consequence of the construction of $A_\epsilon^\prime$.  

**Claim 8.16.** If $v, y \in V(A_\epsilon^\prime)$ are variables such that $\xi_v = \xi_y$ for all $\xi \in \ker A_\epsilon$, then $\xi_v = \xi_y$ for all $\xi \in \ker A$.

**Proof.** By construction the matrix $A_\epsilon$ is the minor of $A$ induced on $V_\epsilon(A) \times C_\epsilon(A)$. Although some of the checks $a \in C_\epsilon(A)$ may contain variables $v \not \in V_\epsilon(A)$, all such $v$ are frozen in $A$. Therefore, any $\xi \in \ker A$ induces a vector $\xi_{A_\epsilon} \in \ker A_\epsilon$.

We now combine Claim 8.16 with Proposition 2.11 to prove the lemma. Hence, let $\mathcal{W}$ be the event that $|V_\epsilon(A) \setminus \mathcal{F}(A)| > (1 - \delta)|V_\epsilon(A)|$. Provided that $\delta = \delta(d, \epsilon) > 0$ is chosen small enough, routine tail bounds for the binomial distribution imply that the event

$$\mathcal{E} = \left\{ \sum_{v \in V_\epsilon(A) \setminus \mathcal{F}(A)} d_\epsilon(v) < \frac{\epsilon}{4} \sum_{v \in V_\epsilon(A)} d_\epsilon(v) \right\}$$

satisfies $\mathbb{P} [\mathcal{W} \setminus \mathcal{E}] = o(1)$. (8.36)

Further, with $x_a = (x_{a,y})_{y \in V_\epsilon(A) \setminus \ker A_\epsilon}$ chosen randomly, Proposition 2.11 and Claim 8.16 ensure that the event

$$\mathcal{R} = \left\{ \sum_{y, y' \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) d_\epsilon^\prime(y') \left| \mathbb{P} \left[ x_{a,y} = x_{a,y'} = 0 \mid A \right] - \frac{1}{4} \right| < \left( \sum_{y \in V_\epsilon(A)} d_\epsilon(y) \right)^2 \log^{-3} n \right\}$$

has probability $1 - o(1)$. As a consequence, since all degrees of $G_\epsilon(A)$ are bounded by $\log n$ w.h.p., the event

$$\mathcal{S} = \left\{ \sum_{y, y' \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) d_\epsilon^\prime(y') \left| \mathbb{P} \left[ x_{a,y} = x_{a,y'} = 0 \mid A \right] - \frac{1}{4} \right| < \left( \sum_{y \in V_\epsilon(A)} d_\epsilon(y) \right)^2 \log^{-3} n \right\}$$

satisfies $\mathbb{P} [\mathcal{W} \setminus \mathcal{E} \setminus \mathcal{R}] = 1 - o(1)$. Hence, (8.36) yields $\mathbb{P} [\mathcal{W} \setminus (\mathcal{E} \cap \mathcal{R})] = o(1)$. In effect, it suffices to prove that on the event $\mathcal{W} \cap \mathcal{E} \cap \mathcal{R}$ we have $\mathcal{K}_\epsilon \neq \emptyset$.

To verify this we recall that any variables $y, y'$ that get contracted in the course of the construction of $G_\epsilon^\prime(A)$ deterministically satisfy $x_{a,y} = x_{a,y'}$. As a consequence, for a random $x_{a,y}^\prime \in \ker A^\prime_\epsilon$ we have

$$\sum_{y, y' \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) d_\epsilon^\prime(y') \left| \mathbb{P} \left[ x_{a,y} = x_{a,y'} = 0 \mid A \right] - \frac{1}{4} \right| = \sum_{y \in V_\epsilon(A) \setminus \mathcal{F}(A)} d_\epsilon(y) d_\epsilon(y') \left| \mathbb{P} \left[ x_{a,y} = x_{a,y'} = 0 \mid A \right] - \frac{1}{4} \right| .$$

Therefore, if $\mathcal{W} \cap \mathcal{E} \cap \mathcal{R}$ occurs, then so does the event

$$\mathcal{S} = \left\{ \sum_{y, y' \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) d_\epsilon^\prime(y') \left| \mathbb{P} \left[ x_{a,y} = x_{a,y'} = 0 \mid A \right] - \frac{1}{4} \right| < \left( \sum_{y \in V_\epsilon(A) \setminus \mathcal{F}(A)} d_\epsilon(y) \right)^2 \log^{-3} n \right\}.$$  

To complete the proof, consider the random variable

$$X = \frac{\sum_{y \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) 1 \left\{ x_{a,y} = 0 \right\}}{\sum_{y \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y)} .$$

Then on $\mathcal{W} \cap \mathcal{E} \cap \mathcal{R}$ we have $\mathbb{E}[X \mid A] \sim 1/2$ because $x_{a,y}^\prime = 0$ with probability $1/2$ for every $y \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)$. Moreover, because $\mathcal{W} \cap \mathcal{E} \cap \mathcal{R} \subseteq \mathcal{S}$ the conditional second moment works out to be $\mathbb{E}[X^2 \mid A] \sim 1/4$. Hence, Chebyshev’s inequality shows that $\mathbb{P} \left[ |X - 1/2| < \epsilon/4 \mid A \right] = 1 - o(1)$. In particular, on $\mathcal{W} \cap \mathcal{E} \cap \mathcal{R}$ there exists a vector $\xi \in \ker A^\prime_\epsilon$ such that

$$\frac{\sum_{y \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y) 1 \left\{ \xi_{a,y} = 0 \right\}}{\sum_{y \in V_\epsilon^\prime(A) \setminus \mathcal{F}(A^\prime)} d_\epsilon^\prime(y)} - \frac{1}{2} < \frac{\epsilon}{4} .$$

Recalling the definition of the event (8.36), we conclude that $\xi \in \mathcal{K}_\epsilon$ and thus $\mathcal{K}_\epsilon \neq \emptyset$.

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APPENDIX A. THE PINNING OPERATION AND THE OVERLAP

A.1. Proof of Proposition 2.11. Let \( A \) be an \( m \times n \)-matrix over \( \mathbb{F}_2 \) and let \( s_1, s_2, \ldots \in [n] \) be a sequence of uniformly distributed random variables, mutually independent and independent of all other sources of randomness. Further, for an integer \( t \geq 0 \) let \( A[t] \) be the matrix obtained by adding \( t \) more rows to \( A \) such that the \( j \)-th new row contains precisely one non-zero entry in position \( s_j \). The proof of Proposition 2.11 is based on the following fact.

Lemma A.1 (Lemma 3.1). For \( \varepsilon > 0, \ell > 0 \) let \( T = T(\varepsilon, \ell) = [4\ell^3/\varepsilon^4] + 1 \). Then for all \( m, n > 0 \) and all \( m \times n \)-matrices \( A \) over \( \mathbb{F}_2 \) the following is true. Draw \( t \in [T] \) uniformly and choose \( x \in \text{ker} A[t] \) randomly. Then

\[
\sum_{i_1, \ldots, i_\ell \in [n]} \mathbb{P} \left[ x_{i_1} = \sigma_1, \ldots, x_{i_\ell} = \sigma_\ell \, | \, A[t] \right] \leq (\ell^3)/\varepsilon^4 \leq \varepsilon n^\ell.
\]

To prove Proposition 2.11 we will combine Lemma A.1 with the observation that the random matrix \( A \) is essentially invariant under the random perturbation required by Lemma A.1. To be precise, let \( \mathcal{I} \) be the set of all indices \( i \in [n] \) such that \( A_{i,j} = 0 \) for all \( j \in [n] \). Further, for an integer \( t \geq 0 \) let \( A(t) \) be the matrix obtained from \( A \) as follows. Let \( \mathcal{I} \subseteq [n] \) be the set of all indices \( i \in [n] \) such that \( A_{i,j} = 0 \) for all \( j \in [n] \). Then \( A(t) \) is obtained by replacing \( i_h \) with \( j_h \) for all \( h = 1, \ldots, t \), where \( i_h \) is chosen uniformly at random and \( j_h \) is chosen uniformly at random from \( [n] \) independently for each \( h \in [t] \). Thus, instead of attaching \( t \) new rows as in Lemma A.1 we simply insert a single non-zero entry into \( t \) random all-zero rows of \( A \).

Lemma A.2. Let \( d > 0 \), \( T = O(\sqrt{n}) \) be an integer and choose \( t \in [T] \) uniformly. Then \( d_{TV}(A, A(t)) = o(1) \).

Proof. Because each entry of \( A \) is non-zero with probability \( d/n \) independently, the number \( X \) of rows of \( A \) with at most one non-zero entry has distribution \( \text{Bin}(n, (1-d/n)^n + d(1-d/n)^{n-1}) \). Further, given \( X \) the number \( X_0 \) of all-zero rows has a binomial distribution

\[
X_0 \sim \text{Bin} \left( X, \frac{(1-d/n)^n}{d(1-d/n)^{n-1}} \right).
\]

Let \( A \mid (X, X_0) \) denote the distribution of \( A \) given \( X, X_0 \). We have \( X \geq \exp(-d)n \) w.h.p. Given \( X \geq \exp(-d)n \) the conditional variance satisfies \( \text{Var}(X_0 \mid X) = \Omega(n) \). Therefore, the local limit theorem for the binomial distribution implies that \( A \mid (X, X_0) \) and \( A \mid (X, X_0 - t) \) have total variation distance \( o(1) \). Furthermore, \( A \mid (X, X_0 - t) \) is distributed precisely as \( A(t) \).

Proof of Proposition 2.11 The proposition is an immediate consequence of Lemmas A.1 and A.2.

A.2. Proof of Corollary 2.12. Due to Proposition 2.11 we may assume that \( A \) satisfies

\[
\frac{1}{n^2} \sum_{i=1}^n \mathbb{P} \left[ x_i = \sigma_1, \ldots, x_i = \sigma_2 \, | \, A \right] = \mathbb{P} \left[ x_i = \sigma_1 \, | \, A \right] \mathbb{P} \left[ x_i = \sigma_2 \, | \, A \right] = o(1)
\]

for all \( \sigma_1, \sigma_2 \in \mathbb{F}_2 \). (A.1)

Hence, fix \( x \in \text{ker} A \). For \( \sigma \in \mathbb{F}_2 \) let \( \mathcal{J}(x, \sigma) = \{ i \in [n] \setminus \mathcal{J}(A) : x_i = \sigma \} \). Further, define

\[
R_{\sigma}(x, x') = \frac{1}{n} \sum_{i \notin \mathcal{J}(x, \sigma)} \mathbf{1}\{ x'_i = \sigma \}.
\]

Then Fact 2.17 implies that

\[
\mathbb{E} \left[ R_{\sigma}(x, x') \mid A \right] = \frac{\mathcal{J}(x, \sigma)}{2n}.
\]

Moreover, (A.1) implies that \( \text{Var} \left[ R_{\sigma}(x, x') \mid A \right] = o(1) \). Combining this bound with (A.2) and applying Chebyshev’s inequality, we conclude that

\[
\mathbb{E} \left[ \left| R_{\sigma}(x, x') - \frac{\mathcal{J}(x, \sigma)}{2n} \right| \mid A \right] = o(1).
\]

Further, since \( R(x, x') = f(A) + \sum_{\sigma \in \mathbb{F}_2} R_{\sigma}(x, x') \), (A.3) shows that

\[
\mathbb{E} \left[ \left| R(x, x') - \left( f(A) + (1 - f(A))/2 \right) \right| \mid A \right] = o(1)
\]

for every \( x \in \text{ker} A \). (A.4)

Averaging (A.4) on \( x \in \text{ker} A \) completes the proof.
APPENDIX B. PROOF OF LEMMA 2.13

We first note that since in the pairing model we must connect variable nodes with check nodes, certainly \( G_a \) cannot contain any loops. We therefore need to show that there is at least a constant probability of creating no double-edges.

Suppose that \( d_1, \ldots, d_n \) are the degrees of variable nodes in \( G_a(A) \) (where we set \( d_i = 0 \) if the corresponding node is not in \( G_a(A) \)), and similarly let \( \hat{d}_1, \ldots, \hat{d}_n \) be the degrees of check nodes. Let \( m := \sum_{i=1}^n d_i = \sum_{i=1}^n \hat{d}_i \). It follows from Proposition 2.6 that w.h.p. \( m = \Theta(n) \). It also follows from the fact that the degree of a node in \( G_a(A) \) are necessarily at most its degree in \( G(A) \) that w.h.p. \( \sum_{i=1}^n d_i^2, \sum_{i=1}^n \hat{d}_i^2 = O(n) \). In what follows, we will implicitly condition on these high probability events.

Let \( X = (d_1, \ldots, d_n, \hat{d}_1, \ldots, \hat{d}_n) \) be the random variable counting the number of double-edges in \( G_a \). Then we have
\[
E[X] = \sum_{i,j=1}^n \frac{d_i d_j}{m(m-1)} = O(1).
\]
Similarly, it is an easy exercise to show that for any integer \( \ell \in \mathbb{N} \) the \( \ell \)-th moment of \( X \) satisfies \( E[(X)^\ell] = (1 + o(1))E[X]^\ell \). Therefore \( X \) is asymptotically distributed as a Po \( (\mathbb{E}[X]) \) random variable, and we have \( P[X = 0] \to \exp(-\mathbb{E}[X]) > 0 \), as required.

To show that \( G_a \) conditioned on being simple has the same distribution as \( G_a(A) \), we simply need to observe that every simple bipartite graph with the appropriate distribution is equally likely to be \( G_a(A) \). To see this, consider two Tanner graphs \( S, S' \) with the same degree distribution, and a Tanner graph \( H \) such that \( H_a = S \). Let \( H' \) be the Tanner graph obtained from \( H \) by replacing \( S \) with \( S' \), but otherwise leaving edges unchanged. Then the peeling process used to obtain the slush is completely identical on \( H \setminus S \) and \( H' \setminus S' \), and therefore \( H_a' = S' \). Since \( H, H' \) have the same number of edges, both are equally likely to be \( G(A) \). Summing over all possibilities for \( H \) such that \( H_a = S \), we deduce that \( S, S' \) are equally likely to be \( G_a(A) \).

APPENDIX C. PROOF OF LEMMA 2.14

For the first part of the lemma, notice that \( |\delta v| \) is distributed as a binomial random variable with parameter \( n \) and \( p \) for any \( v \in V(A) \cup C(A) \). Suppose \( v \in V(A) \) and let \( c = \lceil \log(n)/2 \rceil \). Then we have
\[
P[|\delta v| \geq c] \leq n \left( \begin{array}{c} n \vspace{1mm} \\ c \end{array} \right) p^c \leq n \left( \begin{array}{c} n \vspace{1mm} \\ \frac{d}{n} \vspace{1mm} \end{array} \right)^c \leq n \left( \frac{ed}{c} \right)^c \leq \exp \left( - \frac{\log^2 2}{2} \right) \log n - \frac{\log(n)}{2} \cdot (\log \log(n)) + O(\log \log(n)) = o(1).
\]
Similarly, for a constraint \( a \in C(A) \) we have
\[
P[|\delta a| \geq c] = o(1).
\]
Combining (C.1) and (C.2) completes the proof of the first part. For the second part, let \( x_0 \) be an arbitrary variable node. Then,
\[
E \left[ \sum_{x \in V(A)} \frac{1}{\ell!} \prod_{j=1}^\ell (|\delta x| - j + 1) \right] = \frac{1}{\ell!} E \left[ \prod_{j=1}^\ell (|\delta x_0| - j + 1) \right] = \frac{n}{\ell!} \cdot \frac{n!}{(n-\ell)!} p^\ell \leq \frac{d^\ell n}{\ell!}.
\]
Hence, the assertion follows from Markov's inequality.

APPENDIX D. PROOF OF LEMMA 2.18

Assume, without loss of generality, that \( 0 < c_1 < 10^{-5} \). Moreover, let \( c_0 > 0 \), define \( a = \exp(c_1) > 1 \) and \( \log_a^{(m)} n := \log_a \ldots \log_a n \), where the logarithm with basis \( a \) is taken \( m \) times. For any \( m \in \mathbb{N} \) (or more precisely for any \( m \) such that we have \( s_m > 0 \)), define
\[
s_m := 6 \log_a^{(m)} n.
\]
Let us set \( q_j := \max \{ w_i : i \in P_j \} \), and define the event
\[
\mathcal{E}_{j,m} := \{ s_{m+1} < \max \{ q_j, |P_j| \} \leq s_m \}
\]
and the set
\[
E_m := \{ j : \mathcal{E}_{j,m} \text{ holds} \}.
\]
Note in particular that \( \bigcup_{m' \geq m} \mathcal{E}_{j,m'} \) is the event that \( |P_j| \leq s_m \) and \( w_i \leq s_m \) for all \( i \in P_j \), i.e. both the partition class and all associated weights are at most \( s_m \). We also observe that \( \bigcup_{m=1}^{\infty} E_m = \ell \). We further define

\[
x_m := \frac{1}{n} \sum_{j \in E_m} \left( \sum_{i \in P_j} w_i \right)^2,
\]

so in particular we have

\[
x = \sum_{m=1}^\infty x_m.
\] (D.1)

We therefore aim to bound each \( x_m \). Let \( m_0 = m_0(n) \) be the largest integer such that \( s_{m_0} \geq \frac{100 \log(1/c_1)}{c_1} \).

We first consider the case when \( m \leq m_0 \). Observe that if \( j \in E_m \), then we have \( |P_j| \leq s_m \) and for all \( i \in P_j \) we have \( w_i \leq s_m \), and therefore

\[
\left( \sum_{i \in P_j} w_i \right)^2 \leq s_m.
\] (D.2)

On the other hand, we can bound \( |E_m| \) from above by making a case distinction. Let us define

\[
E_m^{(1)} := \{ j : \mathcal{E}_{j,m} \text{ holds and } q_j \geq |P_j| \},
\]

\[
E_m^{(2)} := \{ j : \mathcal{E}_{j,m} \text{ holds and } q_j \leq |P_j| \}.
\]

**Case 1: \( q_j \geq |P_j| \).**

Then we have \( w_i \geq s_{m+1} \) for some \( i \in P_j \), but since this can hold for at most \( c_0 a^{-s_{m+1}} n \leq c_0 s_m^{-5} n \) values of \( i \), we have

\[
|E_m^{(1)}| \leq c_0 s_m^{-5} n.
\]

**Case 2: \( q_j \leq |P_j| \).**

Then we have \( |P_j| \geq s_{m+1} \), which can also only hold for at most \( c_0 a^{-s_{m+1}} n \leq c_0 s_m^{-5} n \) values of \( j \), so

\[
|E_m^{(2)}| \leq c_0 s_m^{-5} n.
\]

Thus we have \( |E_m| \leq 2 c_0 s_m^{-5} n \) and together with (D.2) we deduce that \( x_m \leq 2 c_0 s_m^{-1} \). Thus (D.1) gives

\[
x \leq 2 c_0 \sum_{m=1}^{m_0} \frac{1}{s_m} + \sum_{m=m_0+1}^\infty x_m.
\] (D.3)

We further observe that for any \( m \leq m_0 \) we have

\[
\frac{s_m}{s_{m-1}} = \frac{6 \log_a \left( \frac{s_{m-1}}{b} \right)}{s_{m-1}} \leq \frac{6 \log_a s_{m-1}}{s_{m-1}} \leq \frac{6 \log_a s_{m_0}}{s_{m_0}}.
\]

We have

\[
\frac{6 \log_a s_{m_0}}{s_{m_0}} = \frac{6}{100 \log(1/c_1)} \left( \log 100 + \log(1/c_1) + \log \log(1/c_1) \right).
\]

In order to bound the ratio \( \frac{6 \log_a s_{m_0}}{s_{m_0}} \), we define the function

\[
g(c_1) = \frac{6}{10} \left( \log(100) + \log \left( \frac{1}{c_1} \right) + \log \log \left( \frac{1}{c_1} \right) \right) - \log \left( \frac{1}{c_1} \right).
\]

We have \( \lim_{c_1 \to 0} g(c_1) = -\infty \) and \( g(10^{-5}) < -0.375985860 \). Also,

\[
g'(c_1) = \frac{2}{5 c_1} - \frac{3}{5 c_1 \log(1/c_1)} > 0,
\]

so \( g \) is increasing in that interval and \( g(c_1) < 0 \). Thus, we have \( \frac{6 \log_a s_{m_0}}{s_{m_0}} < 1/10 \) because \( \frac{6 \log_a s_{m_0}}{s_{m_0}} < 1/10 \) is equivalent to \( g(c_1) < 0 \). Therefore,

\[
\sum_{m=1}^{m_0} \frac{1}{s_m} \leq \frac{1}{s_{m_0}} \left( 1 + \frac{1}{10} + \frac{1}{100} + \ldots \right) \leq 10^{-9}.
\] (D.4)
It remains to estimate \( \sum_{m=m_0+1}^{\infty} x_m \), for which we now restrict attention to \( i \) and \( j \) such that \( w_i, |P_j| \leq s_{m_0+1} \leq 100 \log(1/c) \). Then we have \( \left( \sum_{i \in P_j} w_i \right)^2 \leq 10^8 \left( \log(1/c) \right)^4 \), and we trivially have \( |\bigcup_{m \geq m_0+1} E_m| \leq \ell \leq n \), therefore
\[
\sum_{m=m_0+1}^{\infty} x_m \leq 10^8 \left( \log(1/c) \right)^4 \tag{D.5}
\]
and substituting (D.4) and (D.5) into (D.3) gives
\[
x \leq 2 \cdot c_0 \cdot 10^{-9} + 10^8 \left( \frac{\log(1/c_1)}{c_1} \right)^4.
\]
For the case \( c_1 \geq 10^{-5} \), choose \( c'_1 \) such that \( c'_1 \leq 10^{-5} \), then \( c_0 \exp(-c_1 t) \leq c_0 \exp(-c'_1 t) \). Thus, by considering the pair \( (c_0, c'_1) \) and the above reasoning we get
\[
c_2 = 2 \cdot c_0 \cdot 10^{-9} + 10^8 \left( \frac{\log(1/c'_1)}{c'_1} \right)^4.
\]