Multipole Radiation in Lorentz Gauge

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Abstract: The multipole expansion for electromagnetic radiation, valid for all wave-lengths and all distances from bounded sources, is presented in Lorentz gauge, rather than the usual Coulomb gauge. This gauge is likely to be preferred in applications where one wishes to maintain manifest Lorentz invariance. The presentation also serves as a useful exercise in the use of vector spherical harmonics.

The multipole expansion is a standard issue for the description of electromagnetic radiation. In many applications one may employ suitable simplifications such as the long-wave limit, or restriction to just the asymptotic behavior of the fields. It is of interest in other contexts to describe the multipole expansion, valid for all wave-lengths and all distances from bounded sources. Typically this expansion is presented in Coulomb gauge. However for applications where one wishes to maintain manifest Lorentz invariance, the Lorentz gauge is preferable. Of course, the resulting electric and magnetic fields do not depend on gauge choice. But in some applications, the multipole expansion in terms of Lorentz covariant potentials is particularly useful.

It is the purpose of these notes to describe the multipole expansion of electromagnetic radiation in Lorentz gauge valid for all wave-lengths and all distances from bounded sources, as we are not aware that this is available elsewhere. Other useful discussions, with other aspects of the subject, are to be found in refs. [1–5]. Various useful properties of spherical Bessel functions are available in standard references on quantum mechanics, for example.

The electric and magnetic fields may be described in terms of vector and scalar potentials \( \vec{A} \) and \( \phi \),

\[
\vec{H}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t)
\]
and

\[
\vec{E}(\vec{r}, t) = -\vec{\nabla} \phi(\vec{r}, t) - \frac{1}{c} \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) .
\]

In Lorentz gauge, one has the gauge condition

\[
\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 ,
\]

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which implies the potentials satisfy the wave-equations
\[-\nabla^2 \vec{A}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A}(\vec{r}, t) = \frac{4\pi}{c} \vec{j}(\vec{r}, t)\]
and
\[-\nabla^2 \phi(\vec{r}, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(\vec{r}, t) = 4\pi \rho(\vec{r}, t) \tag{4}\]
in terms of the bounded current and charge densities \(\vec{j}\) and \(\rho\), respectively.

It is convenient to define the fourier transform in time of a function \(f(\vec{r}, t)\) by
\[f_\omega(\vec{r}) = \frac{1}{2\pi} \int dt \, e^{i\omega t} f(\vec{r}, t), \tag{5}\]
so that wave-equations are transformed to the Helmholtz equations, with \(k = \omega/c\),
\[-(\nabla^2 + k^2) \vec{A}_\omega(\vec{r}) = \frac{4\pi}{c} \vec{j}_\omega(\vec{r})\]
and
\[-(\nabla^2 + k^2) \phi_\omega(\vec{r}) = 4\pi \rho_\omega(\vec{r}) \tag{6}\]
We solve for the potentials in terms of the sources, and then reconstruct the electric and magnetic fields.

The approach described here has some overlap with that of Rose [1].

An arbitrary vector field, \(\vec{A}(\vec{r})\) can be expanded in spherical waves
\[\vec{A}(\vec{r}) = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \vec{A}(J, M, \vec{r}) \tag{7}\]
The expansion coefficients can be given in terms of vector spherical harmonics.
\[\vec{A}(J, M, \vec{r}) = \frac{1}{r} \sum_{\ell=J-1}^{J+1} \left[ f_\ell(J, M; r) \vec{Y}_{J\ell}^M \right] \tag{8}\]
where \(f_\ell(J, M; r)\) is a function of the radial coordinate \(r\), and \(\vec{Y}_{J\ell}^M\) are the vector spherical harmonics defined by
\[\vec{Y}_{J\ell M}(\theta, \phi) = \sum_{m=-\ell}^{\ell} \sum_{q=-1}^{1} (\ell m q | \ell J M) Y_\ell^m(\theta, \phi) e_1^q \tag{9}\]
where \(e_1^q\) is a spherical unit vector, \(Y_\ell^m(\theta, \phi)\) is the usual spherical harmonic satisfying \(Y_\ell^{-m} = (-1)^m Y_\ell^m\), and \((\ell m q | \ell J M)\) is a Clebsch–Gordan coefficient. If \(e_x, e_y, e_z\), are the three rectangular unit vectors, then
\[e_1^1 = -\frac{1}{\sqrt{2}} (e_x + ie_y)\]
\[e_1^{-1} = \frac{1}{\sqrt{2}} (e_x - ie_y)\]
\[e_0^0 = e_z\]
\[e_1^q = (-1)^q e_{-q} \tag{10}\]
The vector spherical harmonics obey the orthogonality

\[ \int d\Omega \left[ \nabla^M_{J\ell} (\theta, \phi) \cdot \nabla^M'_{J\ell'} (\theta, \phi) \right] = \delta_{J, J'} \delta_{\ell, \ell'} \delta_{MM'} . \]  

(11)

Using this one finds

\[ \frac{1}{r} f_\ell (J, M; r) = \int d\Omega \left[ \nabla^M_{J\ell} (\theta, \phi) \right]^* \cdot \vec{A} (\vec{r}) . \]  

(12)

In Lorentz gauge, the vector potential can be written in terms of the Green’s function

\[ \vec{A}_\omega (\vec{r}) = \frac{1}{c} \int d^3 r' \vec{j}_\omega (\vec{r}') G_k (|\vec{r} - \vec{r}'|) , \quad k = \frac{w}{c} \]  

(13)

where \( G_k (|\vec{r} - \vec{r}'|) \) = \( \frac{e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \). Hence, for (13) one has

\[ \frac{1}{r} f_\ell (J, M; r) = \frac{1}{c} \int d^3 r' \vec{j}_\omega (\vec{r}') \cdot \int d\Omega \left[ \nabla^M_{J\ell} (\theta, \phi) \right]^* G_k (|\vec{r} - \vec{r}'|) . \]  

(14)

Consider the expansion of (14) for bounded sources, for which \( r' \leq R \) and \( r \geq R \),

\[ \int d\Omega Y^M_{J\ell} G_k (|\vec{r} - \vec{r}'|) = \int d\Omega \sum_{m=-\ell}^\ell \sum_{q=-1}^1 \left( \ell m 1 q |\ell 1 J M \right)^* Y^m (\theta, \phi) e_1^q G_k (|\vec{r} - \vec{r}'|) \]

\[ = \sum_{m=-\ell}^\ell \sum_{q=-1}^1 \left( \ell m 1 q |\ell 1 J M \right)^* e_1^q \int d\Omega Y^m (\theta, \phi) \sum_{\ell'=0}^\infty \sum_{m'=-\ell'}^\ell i k h^{(1)}_{\ell'} (kr) \times \]

\[ j_\ell (kr') Y^m_{\ell'} (\theta, \phi) Y^{m*} (\theta', \phi') \]

\[ = \sum_{m=-\ell}^\ell \sum_{q=-1}^1 \left( \ell m 1 q |\ell 1 J M \right)^* e_1^q Y^{m*} (\theta', \phi') \times [i k h^{(1)}_{\ell'} (kr') j_\ell (kr')] \]

\[ = \nabla^M_{J\ell} (\theta', \phi') i k h^{(1)}_{\ell'} (kr') . \]  

(15)

In summary

\[ \left\{ \begin{array}{l}
\frac{1}{r} f_\ell (J, M; r) = i k h^{(1)}_{\ell'} (kr') \frac{1}{c} \int d^3 r' j_\ell (kr') \vec{j}_\omega (\vec{r}') \cdot \nabla^M_{J\ell'} (\theta', \phi') \\
\vec{A}_\omega (J, M; \vec{r}) = \frac{1}{r} \sum_{\ell=J-1}^{J+1} f_\ell (J, M; r) \nabla^M_{J\ell} (\theta, \phi) \\
\vec{A}_\omega (\vec{r}') = \sum_{J=0}^\infty \sum_{M=-J}^J \vec{A}_\omega (J, M; \vec{r}) .
\end{array} \right. \]  

(16)

Putting this together, we can write

\[ \vec{A}_\omega (\vec{r}) = \sum_{J=0}^\infty \sum_{M=-J}^J \sum_{\ell=J-1}^{J+1} i k h^{(1)}_{\ell'} (kr) a^{M}_{J\ell} \nabla^M_{J\ell} (\theta, \phi) . \]  

(17)
where the coefficient
\[ a_{J\ell_1}^M = \frac{1}{c} \int d^3 r' j_{\ell}(kr') \left[ j_\omega(r') \cdot Y_{J\ell_1}^M(\theta', \phi') \right] . \] (18)

Similarly, for the scalar potential
\[ \phi_\omega(r) = \int d^3 r' \rho_\omega(r') G_k(|r - r'|) \]
or
\[ \phi_\omega(r) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell ik h_\ell^{(1)}(kr) b_{\ell m} r Y_{\ell m}^\star(\theta, \phi) \] (20)
where
\[ b_{\ell m} = \int d^3 r' \rho(r') j_{\ell}(kr') Y_{\ell m}^\star(\theta', \phi'). \] (21)

The four coefficients \( b_{\ell m} \) and \( a_{J\ell_1}^M \) are not independent. They are related by the conservation of charge, which is satisfied by computing the Lorentz condition, which will also ensure the gauge invariance of our results for the fields \( \vec{E} \) and \( \vec{H} \). So one must satisfy \( \nabla \cdot \vec{A}_\omega - ik \phi_\omega = 0 \) in this gauge. Before doing this, it is convenient to tabulate some useful formulae.

For any \( \Phi(r) \)
\[
\begin{align*}
\nabla \cdot \left[ \Phi(r) Y_{J\ell_1}^M(\theta, \phi) \right] &= 0 \\
\nabla \cdot \left[ \Phi(r) Y_{J,\ell+1}^M(\theta, \phi) \right] &= -\sqrt{\ell+1} \left[ \frac{d}{dr} + \frac{\ell+2}{r} \right] \Phi(r) Y_{J,\ell+1}^M(\theta, \phi) \\
\nabla \cdot \left[ \Phi(r) Y_{J,\ell-1}^M(\theta, \phi) \right] &= \sqrt{\ell+1} \left[ \frac{d}{dr} - \frac{\ell-1}{r} \right] \Phi(r) Y_{J,\ell-1}^M(\theta, \phi) \\
\n\nabla \times \left[ \Phi(r) Y_{J,\ell+1}^M(\theta, \phi) \right] &= i \left[ \frac{d}{dr} + \frac{\ell+2}{r} \right] \Phi(r) \sqrt{\frac{J+1}{J+1}} \nabla \cdot \nabla \cdot \left[ \Phi(r) Y_{J,\ell+1}^M(\theta, \phi) \right] \\
\n\nabla \times \left[ \Phi(r) Y_{J,\ell-1}^M(\theta, \phi) \right] &= i \left[ \frac{d}{dr} - \frac{\ell-1}{r} \right] \Phi(r) \sqrt{\frac{J+1}{J+1}} \nabla \times \left[ \Phi(r) Y_{J,\ell-1}^M(\theta, \phi) \right] \\
\n\nabla \left[ \Phi(r) Y_{\ell}^m(\theta, \phi) \right] &= -\sqrt{\frac{\ell+1}{2\ell+1}} \left[ \frac{d}{dr} - \frac{\ell+1}{r} \right] \Phi(r) Y_{\ell,\ell+1}^m(\theta, \phi) \\
&\quad + \sqrt{\frac{\ell+1}{2\ell+1}} \left[ \frac{d}{dr} + \frac{\ell+1}{r} \right] \Phi(r) Y_{\ell,\ell-1}^m(\theta, \phi) .
\end{align*}
\] (22)

Now we can use the following properties for any spherical Bessel function \( z_\ell(\rho) \).
\[
\begin{align*}
z_{\ell-1}(\rho) + z_{\ell+1}(\rho) &= \frac{2\ell+1}{\rho} z_\ell(\rho) \\
d\frac{d}{dr} z_\ell(\rho) &= \frac{1}{2\ell+1} \left[ \ell z_{\ell-1}(\rho) - (\ell + 1) z_{\ell+1}(\rho) \right] .
\end{align*}
\] (23)
From these properties, we find
\[
\begin{align*}
\vec{\nabla} \cdot \left[ z_{J+1}(kr) \vec{Y}_{J+1}^M \right] &= -k \sqrt{\frac{j+1}{2j+1}} z_J(kr) Y_J^M(\theta, \phi) \\
\vec{\nabla} \cdot \left[ z_{J-1}(kr) \vec{Y}_{J-1}^M \right] &= -k \sqrt{\frac{j-1}{2j+1}} z_J(kr) Y_J^M(\theta, \phi) \\
\vec{\nabla} \times \left[ z_{J+1}(kr) \vec{Y}_{J+1}^M \right] &= ik z_J(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}_{J,J}^M \\
\vec{\nabla} \times \left[ z_{J-1}(kr) \vec{Y}_{J-1}^M \right] &= -ik z_J(kr) \sqrt{\frac{j-1}{2j+1}} \vec{Y}_{J,J}^M \\
\vec{\nabla} \times \left[ z_J(kr) \vec{Y}_{J}^M \right] &= ik \left[ -z_{J+1}(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}_{J+1}^M \right. \\
&
\left. + z_{J-1}(kr) \sqrt{\frac{j-1}{2j+1}} \vec{Y}_{J-1}^M \right]
\end{align*}
\]
\[
\vec{\nabla} [z_\ell(kr) Y_\ell^m] = k \left[ \sqrt{\frac{j+1}{2j+1}} z_{\ell+1}(kr) \vec{Y}_{\ell+1}^{m} + \sqrt{\frac{j}{2j-1}} z_{\ell-1}(kr) \vec{Y}_{\ell-1}^{m} \right]. \quad (24)
\]

Using these equations, we find that the Lorentz condition implies, writing \( a_{J\ell} \) as \( a_{J\ell} \),
\[
b_\ell^m = i \left\{ a_{\ell,\ell+1}^m \sqrt{\frac{j+1}{2j+1}} + a_{\ell,\ell-1}^m \sqrt{\frac{j}{2j-1}} \right\}. \quad (25)
\]

The magnetic field is \( \vec{H}_\omega = \vec{\nabla} \times \vec{A}_\omega \). Direct computation shows that
\[
\vec{H}_\omega = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \frac{i k^2}{4} \left\{ i h_{J}^{(1)}(kr) \vec{Y}_{J,J}^{M} \left[ \sqrt{\frac{j}{2j+1}} a_{J,J+1}^M - \sqrt{\frac{j+1}{2j+1}} a_{J,J-1}^M \right] \\
+ i a_{J,J}^M \left[ -\sqrt{\frac{j}{2j+1}} h_{J+1}^{(1)}(kr) \vec{Y}_{J+1,J+1}^{M} + h_{J-1}^{(1)}(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}_{J-1,J-1}^{M} \right] \right\}. \quad (26)
\]

The electric field is constructed from \( \vec{E}_\omega = -\vec{\nabla} \phi_\omega + ik \vec{A}_\omega \).

Again by direct computation, and using (25), we have
\[
\vec{E}_\omega = (ik)^2 \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \left\{ \left[ \sqrt{\frac{j}{2j+1}} a_{J,J+1}^M - \sqrt{\frac{j+1}{2j+1}} a_{J,J-1}^M \right] \right. \\
\times \left[ h_{J+1}^{(1)}(kr) \sqrt{\frac{j}{2j+1}} \vec{Y}_{J+1,J+1}^{M} - h_{J-1}^{(1)}(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}_{J-1,J-1}^{M} \right] \\
+ \left. a_{J,J}^M h_{J}^{(1)}(kr) \vec{Y}_{J,J}^{M} \right\}. \quad (27)
\]

One can verify that all the Maxwell equations are satisfied.

We can define the magnetic multipole \( \mu_{J}^{M} \equiv a_{J,J}^{M} \), and the electric multipole
\[
p_{J}^{M} \equiv \left[ \sqrt{\frac{j}{2j+1}} a_{J,J+1}^{M} - \sqrt{\frac{j+1}{2j+1}} a_{J,J-1}^{M} \right]. \quad (28)
\]
With these multipole coefficients, the fields take on a more compact, and symmetric appearance.

\[
\begin{align*}
\vec{H}_\omega &= (ik)^2 \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \left\{ p_J^M h_J^{(1)}(kr) \vec{Y}^M_{J,J} \right. \\
&\quad \left. + \mu_J^M \left[ - \sqrt{\frac{j}{2j+1}} h_{J+1}^{(1)}(kr) \vec{Y}^M_{J+1,J} + h_{J-1}^{(1)}(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}^M_{J,J-1} \right] \right\} \\
\vec{E}_\omega &= (ik)^2 \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \left\{ p_J^M \left[ \frac{J}{2j+1} h_{J+1}^{(1)}(kr) \vec{Y}^M_{J+1,J} - h_{J-1}^{(1)}(kr) \sqrt{\frac{j+1}{2j+1}} \vec{Y}^M_{J,J+1} \right] \\
&\quad \left. + \mu_J^M h_J^{(1)}(kr) \vec{Y}^M_{J,J} \right\} .
\end{align*}
\]

(29)

Note that if \( p_J^M \leftrightarrow \mu_J^M \), then

\[
\begin{align*}
\vec{E} \text{ (electric)} &\leftrightarrow -\vec{H} \text{ (magnetic)} \\
\vec{E} \text{ (magnetic)} &\leftrightarrow +\vec{H} \text{ (electric)}
\end{align*}
\]

where the coefficients of \( p_J^M \) and \( \mu_J^M \) are denoted (electric) and (magnetic) respectively.

One can put some of these terms in a more familiar form, if one notes that

\[
\begin{align*}
p_J^M &= \left[ \sqrt{\frac{j}{2j+1}} a_{J,J+1}^M - \sqrt{\frac{j+1}{2j+1}} a_{J,J-1}^M \right] \\
&= \frac{1}{c} \int d^3 r' \vec{j}_\omega(\vec{r}) \cdot \left\{ \sqrt{\frac{j}{2j+1}} j_{J+1}(kr') \vec{Y}^M_{J,J+1} - \sqrt{\frac{j+1}{2j+1}} j_{J-1}(kr') \vec{Y}^M_{J,J-1} \right\} \\
&= \frac{i}{c} \int d^3 r' \vec{j}_\omega(\vec{r}) \cdot \nabla \times \left[ j_{J}(kr') \vec{Y}^M_{J,J} \right] \\
&= \frac{i}{c} \int d^3 r' [j_{J}(kr') \vec{Y}^m_{J,J}] \cdot [\nabla \times \vec{j}_\omega(\vec{r})] .
\end{align*}
\]

(30)

One can prove the identity

\[
\hat{r} Y^m_{\ell} = \frac{\hat{r}}{r} Y^m_{\ell} = - \sqrt{\frac{\ell+1}{2\ell+1}} Y^m_{\ell,\ell+1} + \sqrt{\frac{\ell}{2\ell+1}} Y^m_{\ell,\ell-1} .
\]

(31)

Using this, and the previously noted identities

\[
\nabla \times \left[ j_{J}(kr) \hat{r} Y^M_{J,J} \right] = \frac{i}{r} \sqrt{J(J+1)} j_{J}(kr) \vec{Y}^M_{J,J} .
\]

(32)

Hence

\[
\begin{align*}
\mu_J^M &= \frac{i}{c} \int d^3 r' \vec{j}_\omega(\vec{r}) \cdot \frac{\vec{r}'}{\sqrt{J(J+1)}} \nabla \times \left[ j_{J}(kr') \vec{Y}^M_{J,J}(\theta', \phi') \right] \\
&= \frac{i}{c} \int d^3 r' \left[ \nabla \times (\vec{r}' \vec{j}_\omega(\vec{r}')) \right] \cdot \left[ j_{J}(kr') \frac{\vec{r}'}{r} Y^M_{J,J}(\theta', \phi') \right] \\
&= \frac{i}{c} \int d^3 r' \left[ \nabla \times \vec{j}_\omega(\vec{r}') \right] \cdot \vec{r}' \left[ j_{J}(kr') Y^M_{J,J}(\theta', \phi') \right] .
\end{align*}
\]

(33)

Note that only the transverse component of the current enters as the source, although in the longwavelength limit the longitudinal component of the current is the source for the electric multipole moments. This point is discussed in more detail by French and Shimamoto [3] and
by Snowdon [4]. Let us seek some alternate forms for our multipole expansion. Note that
\[ \nabla \times [z_J(kr) \hat{r} Y_J^M] = \frac{i}{r} \sqrt{J(J+1)} z_J(kr) \hat{r} Y_J^M, \]
so that
\[ z_J(kr) \hat{r} Y_J^M = \frac{-i r}{\sqrt{J(J+1)}} \nabla \times [z_J(kr) \hat{r} Y_J^M] \]
\[ = \frac{-\hat{L}}{\sqrt{J(J+1)}} [z_J(kr) Y_J^M] \]
where \( \hat{L} = -i \hat{r} \times \hat{\nabla} \). Also
\[ \left[ z_{J+1}(kr) \sqrt{\frac{J}{2J+1}} \hat{r} Y_{J+1}^M - z_{J-1}(kr) \sqrt{\frac{J-1}{2J+1}} \hat{r} Y_{J-1}^M \right] \]
\[ = \frac{-i}{kr} \hat{\nabla} \times \left[ z_J(kr) \hat{r} Y_J^M \right] \]
\[ = \frac{i}{kr} \hat{\nabla} \times \left\{ \frac{-\hat{L}}{\sqrt{J(J+1)}} [z_J(kr) Y_J^M] \right\} = \frac{-1}{k} \frac{\hat{\nabla} \times \hat{L}}{\sqrt{J(J+1)}} [z_J(kr) Y_J^M] \] .

Using this
\[ p_J^M = \frac{-i}{ck} \int d^3r' \sqrt{\frac{J}{J(J+1)}} \left[ j_J(kr') Y_J^M*(\theta', \phi') \right] \cdot \left[ \hat{\nabla} \times \hat{j}_\omega(\vec{r}') \right] \] (34)
that is
\[ \left\{ \begin{array}{l}
p_J^M = -\frac{i}{ck} \int d^3r' \left[ \hat{\nabla} \times \hat{j}_\omega(\vec{r}') \right] \cdot \hat{r}' \left[ j_J(kr') Y_J^M*(\theta', \phi') \right] \\
\mu_J^M = -\frac{i}{c} \int d^3r' \left[ \hat{\nabla} \times \hat{j}_\omega(\vec{r}') \right] \cdot \hat{r}' \left[ j_J(kr') Y_J^M*(\theta', \phi') \right] \end{array} \right. \] (35)
and
\[ \left\{ \begin{array}{l}
\tilde{H}_\omega = (ik)^2 \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \left\{ \frac{p_J^M \hat{L}}{\sqrt{J(J+1)}} \left[ h_J^{(1)}(kr) Y_J^M(\theta, \phi) \right] - \frac{\mu_J^M i(\hat{\nabla} \times \hat{L})}{\sqrt{J(J+1)}} \left[ h_J^{(1)}(kr) Y_J^M(\theta, \phi) \right] \right\} \\
\tilde{E}_\omega = (ik)^2 \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \left\{ \frac{p_J^M (\hat{\nabla} \times \hat{L})}{\sqrt{J(J+1)}} \left[ h_J^{(1)}(kr) Y_J^M(\theta, \phi) \right] + \frac{\mu_J^M \hat{L}}{\sqrt{J(J+1)}} \left[ h_J^{(1)}(kr) Y_J^M(\theta, \phi) \right] \right\} \right. \] (36)

Our results are now expressed in the same form as French and Shimamoto [3], as expected from the gauge invariance of the electric and magnetic fields.

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