Exact tail asymptotics for fluid models driven by an $M/M/c$ queue

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Abstract
In this paper, we investigate exact tail asymptotics for the stationary distribution of a fluid model driven by the $M/M/c$ queue, which is a two-dimensional queueing system with a discrete phase and a continuous level. We extend the kernel method to study tail asymptotics of its stationary distribution, and a total of three types of exact tail asymptotics are identified from our study and reported in the paper.

Keywords Fluid queue driven by an $M/M/c$ queue · Kernel method · Exact tail asymptotics · Stationary distribution · Asymptotic analysis

Mathematics Subject Classification 60K25 · 60J27 · 30E15 · 05A15

1 Introduction
Fluid flows have been widely used for modelling information flows in performance analysis of packet telecommunication systems. In this area, fluid queues, with Markov-modulated input rates, have played an important role. In such a fluid model, the rate of information change is modulated according to a Markov process that is evolving in the background. Several references on Markov-modulated fluid queues can be found in various studies, such as [11,21,23]. In these studies, the state space $N$ of the modulating
Markov process is assumed to be finite, which puts a restriction on applications. On the other hand, in this paper we consider an infinite-capacity fluid model driven by the $M/M/c$ queue, which is a specific type of birth–death process. First, let $Z(t)$ be the state, at time $t$, of a continuous-time Markov chain, or the background process, which has a countable state space, and let $X(t)$ be the fluid level in the queue at time $t$. Let $r_Z(t)$ denote the rate of change of the fluid level (or the net input rate) at time $t$. Then, the dynamics of the fluid level $X(t)$ are given by

$$\frac{dX(t)}{dt} = \begin{cases} r_Z(t), & \text{if } X(t) > 0 \text{ or } r_Z(t) \geq 0, \\ 0, & \text{if } X(t) = 0 \text{ and } r_Z(t) < 0. \end{cases}$$

Fluid queues driven by infinite-state Markov chains have been considered in the past by several authors. For instance, van Doorn and Scheinhardt, in [25], considered the stationary distribution of the fluid queue driven by a birth–death process, used orthogonal polynomials to solve, under certain boundary conditions, an infinite system of differential equations and provided the same integral expression obtained by Virtamo and Norros in [26] and by Adan and Resing in [1], in the case driven by the $M/M/1$ queue. Parthasarathy and Vijayashree, in [22], provided expressions, via an integral representation of Bessel functions, for the stationary distributions of the buffer occupancy and the buffer content, respectively, for a fluid queue driven by an $M/M/1$ queue. By using the Laplace transform, they obtained a system of differential equations, which led to a continued fraction and the solution of the stationary distribution. In [3], Barbot and Sericola provided an analytic expression for the stationary distribution of the fluid queue driven by an $M/M/1$ queue through the generating function technique. Analysis for the transient distribution of the fluid queue driven by an $M/M/1$ queue was reported by Sericola, Parthasarathy Vijayashree in [24].

Unfortunately, the closed integral expressions for stationary performance measures in the above-mentioned references are usually cumbersome and difficult to use directly for the purpose of analysing properties of the stationary distribution. Hence, it is meaningful to investigate either augmented truncation approximations (for example, [15–18]) or tail asymptotics (for example, [14,20]) of invariant probability measures. In this paper, we will focus on the latter and extend the kernel method so that we can characterize the exact tail asymptotic behaviour for the stationary distribution of the fluid model driven by an $M/M/c$ queue. The main contributions include:

1. An extension of the kernel method. The key idea of the kernel method was proposed by Knuth in [12] and further developed by Banderier et al. in [2]. The method has been recently extended to study the exact tail behaviour for two-dimensional stochastic networks (or random walks with reflective boundaries) for both discrete and continuous random walks in the quarter plane; for examples, see Li and Zhao [14] and Dai, Dawson and Zhao [4], and references therein. Compared to other methods, the kernel method, which has been successfully used to study the tail behaviour of models with both level and background either discrete or continuous, does not require a determination or characterization of the entire unknown function in order to characterize the exact tail asymptotic properties in stationary distributions. It is worthwhile to point out that the application of the kernel method to the fluid model driven by an $M/M/c$ queue is not straightforward and requires
significant effort, since in this case the level is continuous and the background is discrete.

2. An extension of results for the tail asymptotic behaviour in the stationary distribution of a fluid queue driven by a Markov chain. We show, in Sect. 5, that for the fluid model driven by an $M/M/c$ queue, a total of three types of exact tail asymptotic properties exist, whereas Govorun, Latouche and Remiche, in [11], showed that, for a fluid model driven by a finite state Markov chain, there is only one type of tail asymptotic property. This is also an extension of the tail asymptotic behaviour in the stationary distribution of a fluid queue driven by an $M/M/1$ queue, since the tail asymptotic property given in Case (iii) of Theorems 4 and 5 does not exist for the case $c = 1$.

The rest of the paper is organized as follows: in Sect. 2, we describe the fluid model, define the notation and present the system of partial differential equations satisfied by the joint probability distribution function of the buffer level and of the state of the driving process. In this section, we also establish the fundamental equation based on the differential equations. Section 3 is devoted to discussion of properties of the branch points in the kernel equation and the analytic continuation of the unknown functions in terms of the kernel method. In Sect. 4, an asymptotic analysis of the two unknown functions is carried out. In Sect. 5, a characterization on the exact tail asymptotic of the stationary distribution for the model is presented. We show that there exist three types of tail asymptotic properties for the boundary, joint and marginal distributions, respectively. These results are an extension of the single type of behaviour found in [11] for the stationary density of the fluid queue driven by a finite state Markov chain. In Sect. 6, two special cases ($c = 1$ and $c = 2$) are further considered. Finally, in Sect. 7, we make some concluding remarks to complete the paper.

2 Model description and fundamental equation

We consider the fluid model driven by an $M/M/c$ queueing system $\{Z(t), t \geq 0\}$, where $Z(t)$ denotes the queue length of the $M/M/c$ queue at time $t$. It is known that $Z(t)$ is a special birth–death process with the state space $E = \{0, 1, 2, \ldots\}$. Let $\lambda_i$ be the arrival rate and $\mu_i$ be the service rate in state $i$ for $\{Z(t), t \geq 0\}$. Then,

$$\lambda_i = \lambda > 0 \text{ for any } i \geq 0,$$

and, with $\mu > 0$,

$$\mu_i = \begin{cases} i\mu, & \text{for } 0 \leq i \leq c - 1, \\ c\mu, & \text{for } i \geq c. \end{cases}$$

Suppose $\lambda < c\mu$. Then, the unique stationary distribution $\xi = (\xi_i)_{i \in E}$ of $\{Z(t)\}$ exists, and is given by

Supplementary Figure
\begin{align*}
\xi_i = \begin{cases} 
\frac{\rho^i}{i!}, & \text{for } 1 \leq i \leq c, \\
\xi_c \left( \frac{\rho}{c} \right)^{i-c}, & \text{for } i > c,
\end{cases}
\end{align*}

where \( \xi_0 = \left( \sum_{i=0}^{c-1} \frac{\rho^i}{i!} + \frac{\rho^c}{(c-1)!} (c-\rho) \right)^{-1} \) and \( \rho = \frac{\lambda}{\mu} \).

According to [24], we may regard the fluid model driven by an \( M/M/c \) queue \( \{Z(t), t \geq 0\} \) as a fluid commodity, which is referred to as credit. The credit accumulates in an infinite capacity buffer during the full busy period of an \( M/M/c \) queue (i.e., whenever a customer arrives and finds all servers busy) at a positive rate \( r_{Z(t)} \), defined as \( r_i = r > 0 \) for any \( i \geq c \). The credit depletes the fluid during the partial busy period of an \( M/M/c \) queue (i.e., whenever an arriving customer finds fewer than \( c \) customers in the queue) at a negative rate \( -r_{Z(t)} \). It is reasonable to assume that the negative rate \( r_i \) increases in \( i \). Without loss of generality, we assume that the net input rate is \( r_i = i - c \) for any \( 0 \leq i \leq c - 1 \).

In order for the stationary distribution of \( X(t) \) to exist, we shall assume throughout the paper that

\[
\sum_{i \in E} \xi_i r_i < 0,
\]

which is equivalent to

\[
(r + 1)\lambda < c\mu + (c\mu - \lambda) \sum_{i=0}^{c-2} \frac{(c-i)\lambda^{i+1-c} \cdot (c-1)!}{\mu^{i+1-c} \cdot i!}.
\]

Now, we denote \( F_i(t, x) = P\{Z(t) = i, X(t) \leq x\} \) for any \( t \geq 0 \), \( x \geq 0 \) and \( i \in E \). It is well known (see, for example, [25]) that the joint distribution \( F_i(t, x) \) satisfies the following partial differential equations:

\[
\frac{\partial F_0(t, x)}{\partial t} = c \frac{\partial F_0(t, x)}{\partial x} - \lambda F_0(t, x) + \mu F_1(t, x),
\]

\[
\frac{\partial F_i(t, x)}{\partial t} = (c - i) \frac{\partial F_i(t, x)}{\partial x} + \lambda F_{i-1}(t, x) - (\lambda + i\mu) F_i(t, x) + (i + 1)\mu F_{i+1}(t, x), i \leq c - 1,
\]

\[
\frac{\partial F_i(t, x)}{\partial t} = -r \frac{\partial F_i(t, x)}{\partial x} + \lambda F_{i-1}(t, x) - (\lambda + c\mu) F_i(t, x) + c\mu F_{i+1}(t, x), i \geq c.
\]

Let \( Z \) and \( X \) be the stationary states of \( Z(t) \) and \( X(t) \), respectively. Then, the stationary distribution is given by

\[
\Pi_i(x) = \lim_{t \to \infty} F_i(t, x) = P\{Z = i, X \leq x\}.
\]
Define $\pi_i(x) = \frac{\partial \Pi_i(x)}{\partial x}$ for any $x > 0$ and $\pi_i(0) = \lim_{x \to 0^+} \pi_i(x)$. From the above partial differential equations, we have the following equations:

\begin{align*}
&-c\pi_0(x) = \mu \Pi_1(x) - \lambda \Pi_0(x), \quad (1) \\
&-(c-i)\pi_i(x) = \lambda \Pi_{i-1}(x) - (\lambda + i\mu)\Pi_i(x) + (i+1)\mu \Pi_{i+1}(x), \quad 1 \leq i \leq c - 1, \quad (2) \\
&r\pi_i(x) = \lambda \Pi_{i-1}(x) - (\lambda + c\mu)\Pi_i(x) + c\mu \Pi_{i+1}(x), \quad i \geq c. \quad (3)
\end{align*}

The initial condition of (1), (2) and (3) is given by

$$\Pi_i(0) = 0, \quad i \geq c.$$

In addition, for any $i \in E$, we have

$$\Pi_i(\infty) = \lim_{x \to \infty} \Pi_i(x) = \xi_i.$$

**Remark 1** The system of equations in (1) – (3), together with the initial conditions, completely characterizes the queueing system of interest. Instead of directly analysing this system, we will consider a functional equation, based on the system of equations (1) – (3). In other words, we will deal with a functional equation satisfied by the transformations (introduced below) defined through using $\pi_i(x)$. This is a standard technique often employed in the analysis of queueing systems, or in applied probability in general.

Now, let $\phi_i(\alpha)$ be the Laplace transform for $\pi_i(x)$, i.e.,

$$\phi_i(\alpha) = \int_0^\infty \pi_i(x)e^{\alpha x} \, dx.$$

Then, for any $i \in E$, we have

$$\int_0^\infty \Pi_i(x)e^{\alpha x} \, dx = \int_0^\infty \left[ \Pi_i(0) + \int_0^x \pi_i(s) \, ds \right] e^{\alpha x} \, dx = -\frac{1}{\alpha} \Pi_i(0) - \frac{1}{\alpha} \phi_i(\alpha).$$

Thus, taking the Laplace transforms of $\Pi_i(x)$ and $\pi_i(x)$ in (2) and (3), we get

\begin{align*}
-\phi_{c-1}(\alpha) &= -\frac{\lambda}{\alpha} [\Pi_{c-2}(0) + \phi_{c-2}(\alpha)] + \frac{\lambda + (c - 1)\mu}{\alpha} [\Pi_{c-1}(0) + \phi_{c-1}(\alpha)] \\
&\quad - \frac{c\mu}{\alpha} [\Pi_{c}(0) + \phi_{c}(\alpha)],
\end{align*}

and, for any $i \geq c$,

\begin{align*}
r\phi_i(\alpha) &= -\frac{\lambda}{\alpha} [\Pi_{i-1}(0) + \phi_{i-1}(\alpha)] \\
&\quad + \frac{\lambda + c\mu}{\alpha} [\Pi_{i}(0) + \phi_{i}(\alpha)] - \frac{c\mu}{\alpha} [\Pi_{i+1}(0) + \phi_{i+1}(\alpha)].
\end{align*}
It then follows that
\[
\sum_{i=0}^{\infty} \left[ -\lambda z^2 + (-\alpha r + \lambda + c\mu)z - c\mu \right] \phi_i(\alpha) z^i
= \lambda \phi_{c-2}(\alpha) z^c + [(\mu - \alpha - \alpha r)z - c\mu] \phi_{c-1}(\alpha) z^c
+ \sum_{i=0}^{\infty} \left[ \lambda z^2 - (\lambda + c\mu)z + c\mu \right] \Pi_i(0) z^i
+ \lambda \Pi_{c-2}(0) z^c + (\mu z - c\mu) \Pi_{c-1}(0) z^{c-1}.
\]

Denote \( \psi(\alpha, z) = \sum_{i=0}^{\infty} \phi_i(\alpha) z^i \) and \( \psi(z) = \sum_{i=0}^{\infty} \Pi_i(0) z^i \). Then, we can obtain the following fundamental equation, which connects the bivariate unknown function \( \psi(\alpha, z) \) to the univariate unknown functions \( \phi_{c-2}(\alpha) \), \( \phi_{c-1}(\alpha) \) and \( \psi(z) \), as follows:

\[
H(\alpha, z) \psi(\alpha, z) = \lambda z^c [\phi_{c-2}(\alpha) + \Pi_{c-2}(0)] + H_1(\alpha, z) \phi_{c-1}(\alpha) + H_2(\alpha, z) \psi(z) + H_0(\alpha, z) \Pi_{c-1}(0),
\]

where

\[
H(\alpha, z) = -\lambda z^2 + (-\alpha r + \lambda + c\mu)z - c\mu,
\]
\[
H_1(\alpha, z) = (\mu - \alpha - \alpha r)z^c - c\mu z^{c-1},
\]
\[
H_2(\alpha, z) = \lambda z^2 - \lambda z - c\mu z + c\mu,
\]
\[
H_0(\alpha, z) = \mu z^c - c\mu z^{c-1}.
\]

By establishing a relation between \( \phi_{c-2}(\alpha) \) and \( \phi_{c-1}(\alpha) \), we obtain the following theorem.

**Theorem 1** The fundamental equation can be rewritten as

\[
H(\alpha, z) \psi(\alpha, z) = \hat{H}_1(\alpha, z) \phi_{c-1}(\alpha) + H_2(\alpha, z) \psi(z) + \hat{H}_0(\alpha, z),
\]

where

\[
\hat{H}_1(\alpha, z) = \lambda z^c A_{c-2}(\alpha) + H_1(\alpha, z),
\]
\[
\hat{H}_0(\alpha, z) = H_0(z) \Pi_{c-1}(0) + \lambda z^c \Pi_{c-2}(0) + \lambda z^c \sum_{n=0}^{c-2} \left[ k_n \lambda^{c-2-n} \prod_{m=n}^{c-2} \frac{A_m(\alpha)}{(m + 1)\mu} \right],
\]

with

\[
k_0 = \mu \Pi_1(0) - \lambda \Pi_0(0),
\]
\[
k_i = \lambda \Pi_{i-1}(0) - (\lambda + i\mu) \Pi_i(0) + (i + 1)\mu \Pi_{i+1}(0), \quad 1 \leq i \leq c - 2,
\]

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and

\[ A_i(\alpha) = \frac{(i + 1)\mu}{\alpha + \lambda + i\mu - \lambda A_{i-1}(\alpha)}, \quad 0 \leq i \leq c - 2, \quad A_{-1}(\alpha) = 0. \]

**Proof** Taking the Laplace transform for \( \Pi_i(x) \) and \( \pi_i(x) \) in (1) and (2) leads to the following system of linear equations:

\[
\begin{align*}
(\alpha + \lambda)\phi_0(\alpha) - \mu\phi_1(\alpha) &= k_0, \\
-\lambda\phi_0(\alpha) + (\alpha + \lambda + \mu)\phi_1(\alpha) - 2\mu\phi_2(\alpha) &= k_1, \\
&\vdots \\
-\lambda\phi_{c-3}(\alpha) + [\alpha + \lambda + (c - 2)\mu]\phi_{c-2}(\alpha) &= (c - 1)\mu\phi_{c-1}(\alpha) + k_{c-2}.
\end{align*}
\]

Since \( A'_0(\alpha) = -\frac{\mu}{(\alpha + \lambda)^2} < 0 \), we assume that \( A'_{k-1}(\alpha) < 0 \) for any \( \alpha \geq 0 \), as the inductive hypothesis, to show that

\[ A'_k(\alpha) = -\frac{(k + 1)\mu[1 - \lambda A'_{k-1}(\alpha)]}{[\alpha + \lambda + k\mu - \lambda A_{k-1}(\alpha)]^2} < 0. \]

Thus, \( A_i(\alpha) \) is a decreasing function about \( \alpha \) for any \( 0 \leq i \leq c - 2 \). For any \( \alpha > 0 \) and \( 0 \leq i \leq c - 2 \), we obtain

\[ A_i(\alpha) < A_i(0) = \frac{(i + 1)\mu}{\lambda}, \]

which implies that \( A_{i+1}(\alpha) = \frac{(i+2)\mu}{\alpha + \lambda + (i+1)\mu - \lambda A_i(\alpha)} > 0 \). Hence, \( 0 < A_i(\alpha) < \frac{(i+1)\mu}{\lambda} \) for any \( 0 \leq i \leq c - 2 \) and \( \alpha > 0 \).

From the linear equations and the definition of \( A_i(\alpha) \), we have, for any \( 0 \leq i \leq c - 2 \),

\[
\phi_i(\alpha) = \sum_{n=0}^{i} \left[ k_n\lambda^{i-n} \prod_{m=n}^{i} \frac{A_m(\alpha)}{(m+1)\mu} \right] + A_i(\alpha)\phi_{i+1}(\alpha).
\]

Specifically, for the case \( c = 1 \), we have \( \hat{H}_1(\alpha, z) = H_1(\alpha, z) \) and \( \hat{H}_0(\alpha, z) = H_0(z)\Pi_0(0) \). Hence, the theorem is proved. \( \square \)

**Remark 2** It is worthwhile to note that the fluid model considered in this paper is a mixed type of random walk in the quarter plane: discrete in the state of the underlying process and continuous in the buffer content, and it has multiple boundaries (a single boundary for the fluid model driven by the \( M/M/1 \) queue). The functional equation presented in (4) is not considered as the standard fundamental form, like the equation given in (1.3.4) in [8] for discrete random walks. The standard fundamental form completely characterizes the dynamics of the whole system and therefore consists of the unknown functions at all boundary levels, which makes the analysis of the system much more difficult, while in (4) only the boundary level \( c - 1 \) is involved, which significantly simplified our analysis.
3 Kernel equation and branch points

The tail asymptotic behaviour of the stationary distribution for the fluid queue relies on properties of the kernel function $H(\alpha, z)$ and the functions $\hat{H}_1(\alpha, z)$ and $H_2(z)$. Let us now consider the kernel equation $H(\alpha, z) = 0$, which can be written as a quadratic form in $z$ as follows:

$$H(\alpha, z) = az^2 + b(\alpha)z + d = 0, \quad (6)$$

where $a = -\lambda$, $b(\alpha) = -\alpha r + \lambda + c\mu$ and $d = -c\mu$.

Let $\Delta(\alpha) = b^2(\alpha) - 4ad$ be the discriminant of the quadratic form in (6). For each $\alpha \in \mathbb{C}$, the two solutions to (6) are given by

$$Z_\pm(\alpha) = \frac{-b(\alpha) \pm \sqrt{\Delta(\alpha)}}{2a}. \quad (7)$$

When $\Delta(\alpha) = 0$, $\alpha$ is called a branch point of $Z(\alpha)$.

Symmetrically, for each $z \in \mathbb{C}$, the solution to (6) is given by

$$\alpha(z) = \frac{-\lambda z^2 + (\lambda + c\mu)z - c\mu}{zr}. \quad (8)$$

Note that all functions and variables are treated as complex throughout the paper. The next lemma gives us some information on the branch points of $Z(\alpha)$.

**Lemma 1** $\Delta(\alpha)$ has two positive zero points $\alpha_1 = \left(\sqrt{\frac{c\mu}{r}} - \sqrt{\lambda}\right)^2 r$ and $\alpha_2 = \left(\sqrt{\frac{c\mu}{r}} + \sqrt{\lambda}\right)^2 r$.

Moreover, $\Delta(\alpha) > 0$ in $(-\infty, \alpha_1) \cup (\alpha_2, \infty)$ and $\Delta(\alpha) < 0$ in $(\alpha_1, \alpha_2)$.

For convenience, define the cut plane $\tilde{\mathbb{C}}_\alpha$ by

$$\tilde{\mathbb{C}}_\alpha = \mathbb{C}_\alpha \setminus \{[\alpha_1, \alpha_2]\}.$$ 

In the cut plane $\tilde{\mathbb{C}}_\alpha$, denote the two branches of $Z(\alpha)$ by $Z_0(\alpha)$ and $Z_1(\alpha)$, where $Z_0(\alpha)$ is the one with the smaller modulus and $Z_1(\alpha)$ is the one with the larger modulus. Hence, we have

$$Z_0(\alpha) = Z_-(\alpha) \text{ and } Z_1(\alpha) = Z_+(\alpha) \text{ if } \Re(\alpha) > \frac{\lambda + c\mu}{r},$$

$$Z_0(\alpha) = Z_+(\alpha) \text{ and } Z_1(\alpha) = Z_-(\alpha) \text{ if } \Re(\alpha) \leq \frac{\lambda + c\mu}{r}.$$ 

**Remark 3** The key idea of the kernel method is to consider the zeros (or $\alpha$, $Z(\alpha)$) of the kernel function (or $H(\alpha, z)$) such that, after replacing $z$ by $Z(\alpha)$, the right-hand side of the fundamental form [or Eq. (4)] equals zero. This provides a relationship between the two unknown functions $\phi_{c-1}$ and $\psi$, which is the starting point of most advanced methods for characterizing the solution of the unknown transformation functions for
random walks in the quarter plane, such as the formulation of boundary-value problems (for example, [7]), the uniformization method (for example, [10]) and the algebraic method (for example, [8]). This is also the starting point for the kernel method used in this paper. Comparing to the above-mentioned methods, the main advantage of using the kernel method for the exact stationary tail asymptotic behaviour is that it does not require a determination of the unknown functions. Instead, it is enough to know the detailed asymptotic properties at the dominant singularity (or singularities) of the unknown function.

For this reason, we will discuss the analytic continuation of the two unknown functions \( \phi_c - 1 \) and \( \psi \) and then characterize the asymptotic behaviour, at the dominant singularity, of these two functions, respectively. For the analytic continuation, we first discuss the analyticity of the two branches, defined by the algebraic function \( Z(\alpha) \), then the continuation of the given functions \( \hat{H}_1, H_2 \) and \( \hat{H}_0 \), and finally the continuation of the unknown functions \( \phi_c - 1 \) and \( \psi \) that will be discussed in the next section. The asymptotic behaviour of these two unknown functions will be discussed as well in the next section.

**Lemma 2** The functions \( Z_0(\alpha) \) and \( Z_1(\alpha) \) are analytic in \( \tilde{C}_\alpha \). Similarly, \( \alpha(z) \) is meromorphic in \( C_z \) and \( \alpha(z) \) has two zero points and one pole.

**Proof** We first give a proof for \( Z_0(\alpha) \). The proof for \( Z_1(\alpha) \) can be given in the same fashion.

Let \( \alpha = a + bi \) with \( a, b \in \mathbb{R} \) and \( \arg(\alpha) \in (-\pi, \pi] \), and write \( \Delta(\alpha) = \Re(\Delta(\alpha)) + \Im(\Delta(\alpha))i \). We then have

\[
\Re(\Delta(\alpha)) = R(a, b) = (a^2 - b^2)r^2 - 2(\lambda + c\mu)ra + (\lambda - c\mu)^2, \\
\Im(\Delta(\alpha)) = I(a, b) = 2abr^2 - 2(\lambda + c\mu)br.
\]

Letting \( \Im(\Delta(\alpha)) = 0 \), we obtain that either \( a = \frac{\lambda + c\mu}{r} \) or \( b = 0 \). For \( b = 0 \), from Lemma 1, we know that \( R(a, b) \leq 0 \) and \( I(a, b) = 0 \) along the curve \( \mathcal{C}_1 = \{ \alpha = a + bi : \alpha_1 \leq a \leq \alpha_2, b = 0 \} \). According to properties of the square root function, if we take \( \mathcal{C}_1 \) as a cut of \( \sqrt{\Delta(\alpha)} \), then the function \( Z_0(\alpha) \) cannot be analytic on the curve \( \mathcal{C}_1 \). Thus, we will consider the analyticity of \( Z_0(\alpha) \) on the cut plane \( \tilde{C}_\alpha = C_\alpha \setminus \mathcal{C}_1 \) for what follows.

For \( a = \frac{\lambda + c\mu}{r} \) and any \( b \in \mathbb{R} \), we obtain

\[
R\left(\frac{\lambda + c\mu}{r}, b\right) = R\left(\frac{\lambda + c\mu}{r}, 0\right) - b^2r^2 < R\left(\frac{\lambda + c\mu}{r}, 0\right) < 0.
\]

Therefore, along the curve \( \mathcal{C}_2 = \{ \alpha = a + bi : a = \frac{\lambda + c\mu}{r} \} \), we have \( R(a, b) \leq 0 \) and \( I(a, b) = 0 \), which implies that either \( \sqrt{\Delta(\alpha)} \) or \( (-\sqrt{\Delta(\alpha)}) \) cannot be analytic on \( \mathcal{C}_2 \). However, from the definition of \( Z_0(\alpha) \), we have that the branch \( Z_0(\alpha) = Z_+(\alpha) \) is analytic in the domain \( \{ \alpha \in \tilde{C}_\alpha : \Re(\alpha) < \frac{\lambda + c\mu}{r} \} \) and \( Z_0(\alpha) = Z_- (\alpha) \) is analytic in the
complementary domain of the closure of this set in \( \tilde{C}_\alpha \). From the choice of the square root, we know that the function \( Z_0(\alpha) \) is continuous on the curve \( \tilde{C}_2 \), which separates the two above domains. Thus, by Morera’s theorem, we have that the function \( Z_0(\alpha) \) is analytic in the cut plane \( \tilde{C}_\alpha \).

From (8), \( \alpha(z) \) is analytic in \( C_z \) except at the pole \( z = 0 \), which implies that \( \alpha(z) \) is meromorphic in \( C_z \). It also follows from (8) that \( \alpha(z) \) has two zero points.

Based on Lemma 2, we have the analytic continuation of \( \hat{H}_1(\alpha, Z_0(\alpha)) \) and \( H_2(z) \).

Lemma 3 The function \( \hat{H}_1(\alpha, Z_0(\alpha)) \) is analytic on \( \tilde{C}_\alpha \) and \( H_2(z) \) is analytic on \( C_z \).

Proof From Theorem 1, we have

\[
\hat{H}_1(\alpha, Z_0(\alpha)) = [\lambda A_{c-2}(\alpha) + \mu - \alpha r - \alpha]Z_0(\alpha)c - c\mu Z_0(\alpha)c^{-1}.
\]

The analytic property is immediate from Lemma 2.

From the definition of \( H_2(z) \), we can easily get the assertion.

\[ \square \]

4 Asymptotic analysis of \( \phi_{c-1}(\alpha) \) and \( \psi(z) \)

In order to characterize the exact tail asymptotics for the stationary distribution \( \Pi_i(x) \), we need to study the asymptotic behaviour of the two unknown functions \( \phi_{c-1}(\alpha) \) and \( \psi(z) \) at their dominant singularities. There are three steps in the asymptotic analysis of \( \phi_{c-1}(\alpha) \) and \( \psi(z) \): (i) analytic continuation of the functions \( \phi_{c-1}(\alpha) \) and \( \psi(z) \); (ii) singularity analysis of the functions \( \phi_{c-1}(\alpha) \) and \( \psi(z) \); and (iii) applications of a Tauberian-like theorem. In this section, we give details of the first and second steps, and the details for the third step will be given in the Appendix.

We first introduce the following lemma, which is a transformation of Pringsheim’s theorem for a generating function (see, for example, Dai and Miyazawa [5]).

Lemma 4 Let \( g(x) = \int_0^\infty e^{xt} f(t) dt \) be the moment generating function with real variable \( x \). The convergence parameter of \( g(x) \) is given by

\[
C_p(g) = \sup \{ x \geq 0 : g(x) < \infty \}.
\]

Then, the complex-valued function \( g(\alpha) \) is analytic on \( \{ \alpha \in \mathbb{C} : \Re(\alpha) < C_p(g) \} \).

We will now provide detailed information on the extended generator for the fluid queue, which will be used later on to investigate the analytic continuation of \( \phi_{c-1}(\alpha) \). Instead of focusing on the case where the modulated process is an M/M/c queue, we will consider a general setting, whose background process is a general continuous-time Markov chain with an irreducible, conservative and countable (finite or infinitely countable) generator \( Q = (q_{ij}) \).

We first recall some related definitions. Let \( \Phi \) be a continuous-time Markov process with a locally compact, separable metric space \( X \) and transition function \( P^t(i, j) \). We
denote by $D(\mathcal{A})$ the set of all functions $f$ for which there exists a measurable function $g$ such that the process $C_t^f$, defined by

$$C_t^f = f(\Phi_t) - f(\Phi_0) - \int_0^t g(\Phi_s)ds,$$

is a local martingale. We write $\mathcal{A} f = g$ and call $\mathcal{A}$ the extended generator of the process $\Phi$.

Secondly, consider a general fluid model $(X(t), Z(t))$ and define its weakly infinitesimal generator $B$ of the fluid queue by

$$B g(x, i) = \lim_{t \to 0} \frac{E_{(x,i)}[g(X(t), Z(t))] - g(x, i)}{t}.$$

Now, we can present the following lemma.

**Lemma 5** Let $(X(t), Z(t))$ be the general fluid queue with the generator $Q = (q_{ij})$, and let $g(x, i)$ be a function such that $g$ is partially differentiable about $x$ and, for any $x \geq 0$,

$$\sum_{j \in E} g(x, j)q_{ij} < \infty.$$

Moreover, we assume that $\sup_{i \in E} |r_i| < \infty$.

(i) For $x > 0$, $i \in E$ or $x = 0$, $i \in E^+$, we have

$$B g(x, i) = r_i \frac{dg(x, i)}{dx} + \sum_{j \in E} g(x, j)q_{ij},$$

and for $x = 0$, $i \in E^- \cup E^\circ$, we have

$$B g(0, i) = \sum_{j \in E} g(0, j)q_{ij},$$

where $E^+ = \{i \in E | r_i > 0\}$, $E^- = \{i \in E | r_i < 0\}$ and $E^\circ = \{i \in E | r_i = 0\}$.

(ii) If the partial derivative $\frac{dg(x, i)}{dx}$ is continuous in $x$, then $f \in D(\mathcal{A})$ and $\mathcal{A} f = B f$.

**Proof** This proof is similar to the proof of Lemma 3.1 in [17], and we will omit the detail here. It is worth noting that the phase process in [17] is a finite continuous-time Markov chain, which is different from the phase process $\{Z(t)\}$ in this paper. In order to extend the result in [17], we need to impose the assumption that $\sup_{i \in E} |r_i| < \infty$.

According to Lemma 5, we can state the following lemma, which is crucial for the analytic continuation of $\phi_{c-1}(\alpha)$ and $\psi(z)$. 

\[ \text{Springer} \]
Lemma 6 \( \phi_{c-1}(\alpha) \) is analytic on \( \{ \alpha : \Re(\alpha) < \alpha^* \} \), where \( \alpha^* = C_p(\phi_{c-1}) > 0 \), and \( \psi(z) \) is analytic on the disc \( \Gamma_{z^*} = \{ z : |z| < z^* \} \), where \( z^* = \frac{c\mu}{\lambda} \). Moreover, the following equation is satisfied in the domain \( D_{\alpha,z} = \{ (\alpha, z) : H(\alpha, z) = 0 \text{ and } \psi(\alpha, z) < \infty \} \):

\[
\hat{H}_1(\alpha, z)\phi_{c-1}(\alpha) + H_2(z)\psi(z) + \hat{H}_0(\alpha, z) = 0. \tag{9}
\]

Proof First, we prove that \( \alpha^* > 0 \). It follows from Lemma 5 that the extended generator is given by

\[
\mathcal{A}V(x, i) = -e^{\alpha x} z^i \left[ \lambda + c\mu - \alpha r - \lambda z - \frac{c\mu}{z} \right],
\]

for \( x \geq 0, i \geq c \), and

\[
\mathcal{A}V(x, i) = -e^{\alpha x} z^i \left[ (c - i)\alpha + \lambda + i\mu - \lambda z - \frac{i\mu}{z} \right],
\]

for \( x > 0, 0 \leq i \leq c - 1 \), and

\[
\mathcal{A}V(0, i) = -z^i \left[ \lambda + i\mu - \lambda z - \frac{i\mu}{z} \right],
\]

for \( 0 \leq i \leq c - 1 \).

In order to find some constant \( s > 0 \) such that

\[
\mathcal{A}V(x, i) \leq -sV(x, i),
\]

for \( x > 0, i \geq 0 \), we need to choose an appropriate \( \alpha \) and \( z \) such that, for any \( 0 \leq i \leq c - 1 \),

\[
\begin{align*}
\lambda + c\mu - \alpha r - \lambda z - \frac{c\mu}{z} &> 0, \\
(c - i)\alpha + \lambda + i\mu - \lambda z - \frac{i\mu}{z} &> 0.
\end{align*}
\]

For \( \alpha = \frac{c\mu - \lambda}{r} > 0 \), we have

\[
\frac{\lambda + c\mu - r\alpha - \sqrt{\Delta_1}}{2\lambda} < 1, \quad \frac{\lambda + i\mu + (c - i)\alpha - \sqrt{\Delta_2}}{2\lambda} < 1,
\]

and

\[
\frac{\lambda + c\mu - r\alpha + \sqrt{\Delta_1}}{2\lambda} > 1, \quad \frac{\lambda + i\mu + (c - i)\alpha + \sqrt{\Delta_2}}{2\lambda} > 1,
\]

where \( \Delta_1 = (\lambda + c\mu - r\alpha)^2 - 4c\lambda\mu \) and \( \Delta_2 = [\lambda + i\mu + (c - i)\alpha]^2 - 4i\lambda\mu \). Thus, there must exist some \( z \in B_1 \cap B_2 \cap (1, \infty) \) such that \( s > 0 \), and then

\[
\mathcal{A}V(x, i) \leq -sV(x, i) + bI_{L_0}, \tag{10}
\]
where
\[
B_1 = \left( \frac{\lambda + c\mu - r\alpha - \sqrt{\Delta_1}}{2\lambda}, \frac{\lambda + c\mu - r\alpha + \sqrt{\Delta_1}}{2\lambda} \right),
\]
\[
B_2 = \left( \frac{\lambda + i\mu + (c - i)\alpha - \sqrt{\Delta_2}}{2\lambda}, \frac{\lambda + i\mu + (c - i)\alpha + \sqrt{\Delta_2}}{2\lambda} \right).
\]
\[
s = \min\left\{ \lambda + c\mu - cr - \lambda z - \frac{c\mu}{z}, \lambda + i\mu + \alpha - \lambda z - \frac{i\mu}{z} \right\} > 0,
\]
\[L_0 = \{(x, i) : x = 0, 0 \leq i \leq c - 1\}.\]

Since the drift condition (10) holds, from Theorem 7 in [19] we know that
\[
\phi_{c-1} \left( \frac{c\mu - \lambda}{r} \right) z^{c-1} < \psi \left( \frac{c\mu - \lambda}{r}, z \right) < \sum_{i=0}^{\infty} \int_0^\infty \pi_i(x) V(x, i) dx < \infty.
\]

Thus, from Lemma 4, we obtain that \( \alpha^* \geq \frac{c\mu - \lambda}{r} > 0. \)

For \( \psi(z) = \sum_{i=c-1}^{\infty} \Pi_i(0) z^i \), we have
\[
\psi(z) \leq \Pi_{c-1}(0) z^{c-1} + \sum_{i=c}^{\infty} \xi_i z^i = \Pi_{c-1}(0) z^{c-1} + \xi_c z^c \sum_{i=0}^{\infty} \left( \frac{\lambda}{c\mu} \right)^i z^i.
\]

Since \( \sum_{i=0}^{\infty} \left( \frac{\lambda}{c\mu} \right)^i z^i \) is convergent in \( |z| < \frac{c\mu}{\lambda} \), we can conclude that \( \psi(z) \) is analytic in the disc \( \Gamma_{\frac{c\mu}{\lambda}} \).

Now we prove the second assertion. From Eq. (4), we can show that if both \( \phi_{c-1}(\alpha) \) and \( \beta(z) \) are finite, then \( \beta(\alpha, z) \) is finite as long as \( H(\alpha, z) \neq 0 \). Assume that \( H(\alpha_0, z_0) = 0 \) for some \( \alpha_0 > 0 \) and \( 1 < z_0 < \frac{c\mu}{\lambda} \), and \( \phi_{c-1}(\alpha_0) < \infty, \psi(z_0) < \infty \). Then, for small enough \( \varepsilon > 0 \), we have \( \psi(\alpha_0, z_0) < \psi(\alpha_0, z_0 + \varepsilon) < \infty \), and thus (9) holds for such a pair \( \alpha_0, z_0 \). \( \square \)

**Remark 4** Actually, according to [6], we know that Eq. (10) implies that the fluid model driven by an \( M/M/c \) queue is \( V \)-uniformly ergodic.

We now present another relationship between \( \phi_{c-1}(\alpha) \) and \( \beta(z) \) and extend their analytic domains.

**Lemma 7** (i) \( \phi_{c-1}(\alpha) \) can be analytically continued to the domain \( D_{\alpha} = \{ \alpha \in \mathbb{C}_H : H_1(\alpha, Z_0(\alpha)) \neq 0 \} \cap \{ \alpha \in \mathbb{C}_H : |Z_0(\alpha)| < \frac{c\mu}{\lambda} \} \), and
\[
\phi_{c-1}(\alpha) = -\frac{H_2(Z_0(\alpha)) \psi(Z_0(\alpha)) + \hat{H}_0(\alpha, Z_0(\alpha))}{\hat{H}_1(\alpha, Z_0(\alpha))}. \tag{11}
\]

(ii) \( \beta(z) \) can be analytically continued to the domain \( D_z = \{ z \in \mathbb{C} : H_2(z) \neq 0 \} \cap \{ z \in \mathbb{C} : \Re(\alpha(z)) < \alpha^* \} \) and
\[
\beta(z) = -\frac{\hat{H}_1(\alpha(z), z) \phi_{c-1}(\alpha(z)) + \hat{H}_0(\alpha(z), z)}{H_2(z)}. \tag{12}
\]
Proof (i) For any \((\alpha, z)\) such that \(H(\alpha, z) = 0\) and \(\psi(\alpha, z) < \infty\), we can get Eq. (9). Using \(z = Z_0(\alpha)\) leads to (11). Then, from Lemma 6, we know that the right-hand side of the above equation is analytic except for the points where \(\hat{H}_1(\alpha, Z_0(\alpha)) = 0\) or \(|Z_0(\alpha)| \geq \frac{c\mu}{\lambda}\).

Similarly, we can prove assertion (ii).

Based on the above arguments, we have the following lemma.

**Lemma 8** The convergence parameter \(\alpha^*\) satisfies \(0 < \alpha^* \leq \alpha_1\). If \(\alpha^* < \alpha_1\), then \(\alpha^*\) is necessarily a zero point of \(\hat{H}_1(\alpha, Z_0(\alpha))\).

**Proof** From Lemma 7(i), we know that \(\phi_{c-1}(\alpha)\) is analytic on \(D_\alpha\), and thus, \(\alpha^* \leq \alpha_1\).

For the case \(\alpha^* < \alpha_1\), we can deduce from Lemma 7(i) that \(\alpha^*\) is either a zero point of \(\hat{H}_1(\alpha, Z_0(\alpha))\) or a point such that \(|Z_0(\alpha)| \geq \frac{c\mu}{\lambda}\) for \(\alpha \in (0, \alpha_1)\).

For \(\alpha \leq \alpha_1\), we have

\[
Z_0(\alpha) = Z_+(\alpha) = \frac{-\alpha r + \lambda + c\mu - \sqrt{(-\alpha r + \lambda + c\mu)^2 - 4c\lambda\mu}}{2\lambda},
\]

which is a strictly increasing function of \(\alpha\). Thus, for any \(\alpha \in (0, \alpha_1)\), we have

\[
1 = Z_0(0) < Z_0(\alpha) < Z_0(\alpha_1) = \frac{\sqrt{c\mu}}{\lambda} < \frac{c\mu}{\lambda}. \tag{13}
\]

Remark 5 (i) For any set of model parameters \(c, \lambda\) and \(\mu\), (i), (ii) and (iii) of Assumption 1 can be easily checked numerically.

(ii) In many cases, these assumptions are not necessary. For example, if \(c\mu > \lambda(r + 1)\), we can derive from the expression for \(\hat{H}_1(\alpha, Z_0(\alpha))\) that the unique zero point \(\tilde{\alpha}\) must be a simple zero point. In this case, (iii) of Assumption 1 is redundant. Moreover, we will show in Sect. 6 that all (i), (ii) and (iii) of Assumption 1 are redundant for the special cases \(c = 1\) and \(c = 2\). Actually, our extensive numerical calculations (for many sets of \(\lambda, \mu\) and \(r\) values) suggest that all (i), (ii) and (iii) are redundant in the general case, but a rigorous proof is still not available at this moment.

The next lemma, which follows from Lemma 7(i), provides more details about the convergence parameter \(\alpha^*\).
Lemma 9 Suppose that (i) and (ii) of Assumption 1 hold. Then,

(i) if the zero point $\tilde{\alpha}$ exists and $\tilde{\alpha} < \alpha_1$, we have $\alpha^* = \tilde{\alpha}$,
(ii) if the zero point $\tilde{\alpha}$ exists and $\tilde{\alpha} = \alpha_1$, we have $\alpha^* = \tilde{\alpha} = \alpha_1$,
(iii) if $\hat{H}_1(\alpha, Z_0(\alpha))$ has no real zero points in $(0, \alpha_1)$, we have $\alpha^* = \alpha_1$.

Based on the above analysis, we can provide the following tail asymptotic properties for $\phi_{c-1}(\alpha)$ and $\psi(z)$, which are key for characterizing exact tail asymptotics in the stationary distribution of the fluid queue.

Theorem 2 Suppose that (i) and (ii) of Assumption 1 hold. For the function $\phi_{c-1}(\alpha)$, a total of three types of asymptotics exist as $\alpha$ approaches $\alpha^*$, based on the properties of $\alpha^*$ stated in Lemma 9.

Case (i) If (i) of Lemma 9 and (iii) of Assumption 1 hold, then

$$\lim_{\alpha \to \alpha^*} (\alpha^* - \alpha)^k \phi_{c-1}(\alpha) = c_1,$$

where

$$c_1 = \frac{H_2(Z_0(\alpha^*))\psi(Z_0(\alpha^*)) + \hat{H}_0(\alpha^*, Z_0(\alpha^*))}{\hat{H}_1^{(k)}(\alpha^*, Z_0(\alpha^*))},$$

and $\hat{H}_1^{(k)}(\alpha^*, Z_0(\alpha^*))$ is the $k$th derivative of $\alpha^*$.

Case (ii) If (ii) of Lemma 9 holds, then

$$\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \phi_{c-1}(\alpha) = c_2,$$

where

$$c_2 = \frac{2\lambda[H_2(Z_0(\alpha^*))\psi(Z_0(\alpha^*)) + \hat{H}_0(\alpha^*, Z_0(\alpha^*))]}{\frac{\partial \hat{H}_1(\alpha^*, Z_0(\alpha^*))}{\partial Z_0(\alpha^*)} \cdot \sqrt{\alpha^* - \alpha}}.$$

Case (iii) If (iii) of Lemma 9 holds, then

$$\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \phi'_{c-1}(\alpha) = c_3,$$

where

$$c_3 = \frac{\partial L(\alpha, z)}{\partial z}_{(\alpha^*, Z_0(\alpha^*))} \frac{\sqrt{\alpha_2 - \alpha_1}}{2\lambda},$$

and $L(\alpha, z) = -\frac{H_2(\alpha, z)\psi(z) + \hat{H}_0(\alpha, z)}{H_1(\alpha, z)}$. 
Proof (i) In this case, \( \alpha^* = \tilde{\alpha} \) is a multiple zero, with degree \( k \), of \( \hat{H}_1(\alpha, Z_0(\alpha)) \). From (11), we have

\[
(\tilde{\alpha} - \alpha)^k \phi_{c-1}(\alpha) = -\frac{H_2(Z_0(\alpha))\psi(Z_0(\alpha)) + \hat{H}_0(\alpha, Z_0(\alpha))}{\hat{H}_1(\alpha, Z_0(\alpha)) / (\tilde{\alpha} - \alpha)^k}.
\]

It follows that

\[
\lim_{\alpha \to \tilde{\alpha}} (\tilde{\alpha} - \alpha)^k \phi_{c-1}(\alpha) = \frac{H_2(Z_0(\tilde{\alpha}))\psi(Z_0(\tilde{\alpha})) + \hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha}))}{\hat{H}_1^{(k)}(\tilde{\alpha}, Z_0(\tilde{\alpha}))} = c_1.
\]

Moreover, as stated in Remark 5, we can show that \( c_1 \neq 0 \). Similarly, we also have \( c_2, c_3 \neq 0 \) in the following proof.

(ii) In this case, \( \alpha^* = \tilde{\alpha} = \alpha_1 \), which implies that \( \alpha_1 \) is not only a zero point of \( \Delta(\alpha) \) but also the zero point of \( \hat{H}_1(\alpha, Z_0(\alpha)) \). Suppose that \( \alpha_1 \) is a zero of degree \( m \geq 2 \). Then, we have

\[
\lambda A'_{c-2}(\alpha_1) = \frac{3}{2(c-1)\mu} r + \frac{1}{(c-1)\mu} > 0,
\]

which conflicts with the fact that \( \lambda A'_{c-2}(\alpha_1) < 0 \). Hence, \( \alpha_1 \) is a simple zero point of \( \hat{H}_1(\alpha, Z_0(\alpha)) \). Thus, we have

\[
\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \phi_{c-1}(\alpha) = \lim_{\alpha \to \alpha^*} -\frac{H_2(Z_0(\alpha))\psi(Z_0(\alpha)) + \hat{H}_0(\alpha, Z_0(\alpha))}{\hat{H}_1(\alpha, Z_0(\alpha)) / \sqrt{\alpha^* - \alpha}}
\]

\[
= \lim_{\alpha \to \alpha^*} \frac{H_2(Z_0(\alpha))\psi(Z_0(\alpha)) + \hat{H}_0(\alpha, Z_0(\alpha))}{\sqrt{\alpha^* - \alpha} \cdot [\frac{\partial \hat{H}_1(\alpha, Z_0(\alpha))}{\partial \alpha^*} + Z_0'(\alpha^*) \cdot \frac{\partial \hat{H}_1(\alpha, Z_0(\alpha))}{\partial Z_0(\alpha^*)}]},
\]

where

\[
\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \frac{\partial \hat{H}_1(\alpha, Z_0(\alpha))}{\partial \alpha^*} = 0,
\]

and

\[
\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} Z_0'(\alpha^*) = \lim_{\alpha \to \alpha^*} \frac{Z_0(\alpha^*) - Z_0(\alpha)}{\sqrt{\alpha^* - \alpha}}
\]

\[
= \lim_{\alpha \to \alpha^*} \left[ \frac{b(\alpha^*) - b(\alpha)}{2\lambda \sqrt{\alpha^* - \alpha}} + \frac{\sqrt{\alpha - \alpha^*}(\alpha - \alpha_2)}{2\lambda \sqrt{\alpha^* - \alpha}} \right]
\]

\[
= \frac{\sqrt{\alpha_2 - \alpha^*}}{2\lambda}.
\]
It follows that
\[
\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \phi_{c-1}(\alpha) = 2\lambda \left[ H_2(Z_0(\alpha^*))\psi(Z_0(\alpha^*)) + \hat{H}_0(\alpha^*, Z_0(\alpha^*)) \right] = c_2.
\]

(iii) In this case, \( \alpha^* = \alpha_1 \). Let
\[
L(\alpha, z) = -\frac{H_2(z)\psi(z) + \hat{H}_0(\alpha, z)}{\hat{H}_1(\alpha, z)}.
\]
From (11), we have
\[
\phi'_{c-1}(\alpha) = \frac{\partial L(\alpha, z)}{\partial \alpha} + \frac{\partial L(\alpha, z)}{\partial z} \cdot Z'_0(\alpha).
\]
It follows that
\[
\lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \phi'_{c-1}(\alpha) = \lim_{\alpha \to \alpha^*} \sqrt{\alpha^* - \alpha} \cdot \left[ \frac{\partial L(\alpha, z)}{\partial \alpha} + \frac{\partial L(\alpha, z)}{\partial z} \cdot Z'_0(\alpha) \right]
\]
\[
= \lim_{\alpha \to \alpha^*} \frac{\partial L(\alpha, z)}{\partial z} \cdot \sqrt{\alpha^* - \alpha} \cdot Z'_0(\alpha)
\]
\[
= \frac{\partial L(\alpha, z)}{\partial z} \big|_{(\alpha^*, Z_0(\alpha^*))} \sqrt{\alpha^*_2 - \alpha_1} = c_3.
\]
\[\square\]

The asymptotic property of \( \psi(z) \) can be stated in the next theorem.

**Theorem 3** For the function \( \psi(z) \), we have the following asymptotic property as \( z \) approaches \( \tilde{z} = \frac{c_\mu}{\lambda} \):
\[
\lim_{z \to \tilde{z}} (\tilde{z} - z)\psi(z) = d_{\tilde{z}},
\]
where
\[
d_{\tilde{z}} = \frac{\hat{H}_1(\alpha(\tilde{z}), \tilde{z})\phi_{c-1}(\alpha(\tilde{z})) + \hat{H}_0(\alpha(\tilde{z}), \tilde{z})}{\lambda(\tilde{z} - 1)}.
\]

**Proof** From (12), we have
\[
\lim_{z \to \tilde{z}} (\tilde{z} - z)\psi(z) = \lim_{z \to \tilde{z}} \frac{\hat{H}_1(\alpha(z), z)\phi_{c-1}(\alpha(z)) + \hat{H}_0(\alpha(z), z)}{\lambda(z - 1)}
\]
\[
= \frac{\hat{H}_1(\alpha(\tilde{z}), \tilde{z})\phi_{c-1}(\alpha(\tilde{z})) + \hat{H}_0(\alpha(\tilde{z}), \tilde{z})}{\lambda(\tilde{z} - 1)}.
\]
Moreover, we can show that \( \hat{H}_1(\alpha(\tilde{z}), \tilde{z}) > 0 \), which implies that \( d_{\tilde{z}} > 0 \). \[\square\]
5 Exact tail asymptotics for $\Pi_i(x)$ and $\Pi(x)$

Lemmas 10 and 11 specify exact tail asymptotic properties for the density function $\pi_{c-1}(x)$ and boundary probabilities $\Pi_i(0)$, respectively, which are direct consequences of the detailed asymptotic behaviour of $\phi_{c-1}(\alpha)$ and $\psi(z)$, and the Tauberian-like theorem, given in the Appendix. Furthermore, the tail asymptotics for the joint probability $\Pi_i(x)$, the density function $\pi_i(x)$, the marginal distribution $\Pi(x) = \sum_{i=0}^{\infty} \Pi_i(x)$ and the density function $\pi(x) = \frac{d\Pi(x)}{dx}$, for $x > 0$, are also provided in this section.

Lemma 10 Suppose that (i) and (ii) of Assumption 1 hold. For the density function $\pi_{c-1}(x)$ of the fluid queue, we have the following tail asymptotic properties for large enough $x$:

Case (i) If (i) of Lemma 9 and (iii) of Assumption 1 hold, then

$$\pi_{c-1}(x) \sim C_1 e^{-\alpha^* x} x^{k-1}. $$

Case (ii) If (ii) of Lemma 9 holds, then

$$\pi_{c-1}(x) \sim C_2 e^{-\alpha^* x} x^{-\frac{1}{2}}. $$

Case (iii) If (iii) of Lemma 9 holds, then

$$\pi_{c-1}(x) \sim C_3 e^{-\alpha^* x} x^{-\frac{3}{2}}, $$

where $C_1 = \frac{c_1}{\Gamma(k)}$, $C_2 = \frac{c_1}{\sqrt{\pi}}$, $C_3 = \frac{-c_3}{2\sqrt{\pi}}$ and $c_i$, $i = 1, 2, 3$, are defined in Theorem 2.

Lemma 11 For the boundary probabilities $\Pi_i(0)$ of the fluid queue, we have the following tail asymptotic properties for large enough $i$:

$$\Pi_i(0) \sim d_{\tilde{z}} \cdot \left(\frac{1}{\tilde{z}}\right)^{i+1}, $$

where $\tilde{z} = \frac{c\mu}{\lambda}$ and $d_{\tilde{z}}$ is defined in Theorem 3.

We now provide additional details for the exact tail asymptotic characterization in the (general) joint probabilities $\Pi_i(x)$ for any $i \geq c - 1$.

Theorem 4 Suppose that (i) and (ii) of Assumption 1 hold. For the joint probabilities $\Pi_i(x)$ of the fluid queue, we then have the following tail asymptotic properties for any $i \geq c - 1$ and large enough $x$:

Case (i) If (i) of Lemma 9 and (iii) of Assumption 1 hold, then

$$\Pi_i(x) \sim \frac{\lambda}{c\mu} \left(\frac{\lambda}{c\mu}\right)^{i-c} \sim -\frac{C_1}{\alpha^*} e^{-\alpha^* x} x^{k-1} \left(\frac{1}{\tilde{z}^n}\right)^{i-c}. $$ (14)
\[ \pi_i(x) \sim C_1 e^{-\alpha^* x} x^{k-1} \left( \frac{1}{z^*} \right)^{i-c}. \]

**Case (ii)** If (ii) of Lemma 9 holds, then

\[ \Pi_i(x) - \xi_c \left( \frac{\lambda}{c \mu} \right)^{i-c} \sim -\frac{C_2}{\alpha^*} e^{-\alpha^* x} x^{-\frac{1}{2}} \left( \frac{1}{z^*} \right)^{i-c}, \]

and

\[ \pi_i(x) \sim C_2 e^{-\alpha^* x} x^{-\frac{1}{2}} \left( \frac{1}{z^*} \right)^{i-c}. \]

**Case (iii)** If (iii) of Lemma 9 holds, then

\[ \Pi_i(x) - \xi_c \left( \frac{\lambda}{c \mu} \right)^{i-c} \sim -\frac{C_3}{\alpha^*} e^{-\alpha^* x} x^{-\frac{3}{2}} \left( \frac{1}{z^*} \right)^{i-c}, \]

and

\[ \pi_i(x) \sim C_3 e^{-\alpha^* x} x^{-\frac{3}{2}} \left( \frac{1}{z^*} \right)^{i-c}, \]

where \( z^n = Z_0(\alpha^*) \) and \( C_1, C_2, C_3 \) are defined in Lemma 10.

**Proof** We only prove (i), since (ii) and (iii) can be proved in a similar fashion. For \( i = c - 1 \), we have

\[
\lim_{x \to \infty} \frac{C_1 e^{-\alpha^* x} x^{k-1}}{\xi_{c-1} - \Pi_{c-1}(x)} = \lim_{x \to \infty} \frac{\alpha^* C_1 e^{-\alpha^* x} x^{k-1} - (k - 1)C_1 e^{-\alpha^* x} x^{k-2}}{\pi_{c-1}(x)} = \alpha^*,
\]

where the first equality follows from L’Hospital’s rule and the second equality follows from Lemma 10. Hence, we have \( \Pi_{c-1}(x) - \xi_{c-1} \sim -\frac{C_1}{\alpha^*} e^{-\alpha^* x} x^{k-1} \) as \( x \to \infty \).

Now, suppose that (14) is true for any \( i = m > c - 1 \). Thus, for \( i = m + 1 \), it follows from (3) that

\[ c \mu \Pi_{m+1}(x) = -\lambda \Pi_{m-1}(x) + (\lambda + c \mu) \Pi_{m}(x) + r \pi_m(x), \]

which leads to
\[
\lim_{x \to \infty} \frac{\Pi_{m+1}(x) - \xi_c \left( \frac{\lambda}{c\mu} \right)^{m+1-c}}{\frac{C_1}{\alpha^*} e^{-\alpha^* x} x^{k-1}} \\
= \lim_{x \to \infty} \left[ -\frac{\lambda}{c\mu} \cdot \frac{\Pi_{m-1}(x) - \xi_c \left( \frac{\lambda}{c\mu} \right)^{m-c-1}}{\frac{C_1}{\alpha^*} e^{-\alpha^* x} x^{k-1}} + \frac{\lambda + c\mu}{c\mu} \cdot \frac{\Pi_m(x) - \xi_c \left( \frac{\lambda}{c\mu} \right)^{m-c}}{\frac{C_1}{\alpha^*} e^{-\alpha^* x} x^{k-1}} + \frac{r\alpha^*}{c\mu} \cdot \frac{\pi_m(x)}{C_1 e^{-\alpha^* x} x^{k-1}} \right] \\
= -\left( \frac{1}{z^*} \right)^{m-c} \left[ -\frac{\lambda}{c\mu} z^* + \frac{\lambda + c\mu}{c\mu} - \frac{r\alpha^*}{c\mu} \right] \\
= -\left( \frac{1}{z^*} \right)^{m+1-c},
\]

where the last equation follows from the fact that \( H(\alpha^*, z^*) = 0 \) and \( z^* = Z_0(\alpha^*) \). This completes the proof. \( \square \)

**Remark 6** According to (5), we can derive tail asymptotic properties of \( \phi_{c-2}(\alpha) \) from \( \phi_{c-1}(\alpha) \) and thus the tail asymptotic properties for \( \Pi_{c-2}(x) \). Similarly, a relationship can be established between \( \phi_i(\alpha) \) and \( \phi_{i-1}(\alpha) \) for any \( 1 \leq i \leq c - 2 \), and thus tail asymptotic properties for the joint probability \( \Pi_i(x) \) can be obtained for any \( 0 \leq i \leq c - 1 \).

In the following theorem, we provide exact tail asymptotics for the marginal distribution \( \Pi(x) \).

**Theorem 5** Suppose that (i) and (ii) of Assumption 1 hold. For the marginal distribution \( \Pi(x) \) of the fluid queue, we have the following tail asymptotic properties:

**Case (i)** If (i) of Lemma 9 and (iii) of Assumption 1 hold, then

\[
\Pi(x) - 1 \sim -\frac{\tilde{C}_1}{\alpha^*} e^{-\alpha^* x} x^{k-1},
\]

and

\[
\pi(x) \sim \tilde{C}_1 e^{-\alpha^* x} x^{k-1}.
\]

**Case (ii)** If (ii) of Lemma 9 holds, then

\[
\Pi(x) - 1 \sim -\frac{\tilde{C}_2}{\alpha^*} e^{-\alpha^* x} x^{-\frac{1}{2}},
\]

and

\[
\pi(x) \sim e^{-\alpha^* x} x^{-\frac{1}{2}}.
\]
Case (iii) If (iii) of Lemma 9 holds, then

$$
\Pi(x) - 1 \sim -\frac{\tilde{C}_3}{\alpha^*} e^{-\alpha^* x} x^{-\frac{3}{2}},
$$

and

$$
\pi(x) \sim \tilde{C}_3 e^{-\alpha^* x} x^{-\frac{3}{2}},
$$

where $\tilde{C}_i = \left[ \frac{\hat{H}_1(\alpha^*, 1)}{H(\alpha^*, 1)} + \sum_{k=0}^{c-2} A_k(\alpha^*) A_{k+1}(\alpha^*) \cdots A_{c-2}(\alpha^*) \right] C_i$, $i = 1, 2, 3, C_1, C_2$ and $C_3$ are defined in Lemma 10 and $A_i(\alpha)$ is defined in Theorem 1.

**Proof** Let $z = 1$. It follows from (4) that

$$
H(\alpha, 1) \psi(\alpha, 1) = \hat{H}_1(\alpha, 1) \phi_{c-1}(\alpha) + H_2(1) \psi(1) + \hat{H}_0(\alpha, 1).
$$

Thus,

$$
H(\alpha, 1) \int_0^\infty \sum_{i=0}^\infty \pi_i(x) e^{\alpha x} \, dx = \hat{H}_1(\alpha, 1) \int_0^\infty \pi_{c-1}(x) e^{\alpha x} \, dx + \hat{H}_0(\alpha, 1),
$$

since $H_2(1) = 0$. From (15), we have

$$
\int_0^\infty \sum_{i=0}^\infty \pi_i(x) e^{\alpha x} \, dx = \frac{\hat{H}_1(\alpha, 1)}{H(\alpha, 1)} H(\alpha, 1) \int_0^\infty \pi_{c-1}(x) e^{\alpha x} \, dx + \frac{\hat{H}_0(\alpha, 1)}{H(\alpha, 1)}.
$$

Now, we want to show that

$$
\int_0^\infty \sum_{i=0}^\infty \pi_i(x) e^{\alpha x} \, dx = \int_0^\infty \pi(x) e^{\alpha x} \, dx,
$$

where for any $x > 0$, $\pi_i(x) = \frac{\partial \Pi_i(x)}{\partial x}$ and $\pi(x) = \frac{d\Pi(x)}{dx}$.

For any fixed $x$, we obtain

$$
\sum_{i=0}^\infty \Pi_i(x) = P \{ X < x \} \leq 1,
$$

which implies that $\sum_{i=0}^\infty \Pi_i(x)$ is convergent for any $x$. From (3), we have, for $i \geq c$,

$$
\pi_i(x) = \frac{\lambda}{r} \Pi_{i-1}(x) - \frac{\lambda + c\mu}{r} \Pi_i(x) + \frac{c\mu}{r} \Pi_{i+1}(x) \leq \frac{\lambda}{r} \xi_{i-1} + \frac{c\mu}{r} \xi_{i+1}.
$$
Since
\[
\sum_{i=c}^{\infty} \frac{\lambda}{r} \xi_{i-1} + \sum_{i=c}^{\infty} \frac{c\mu}{r} \xi_{i+1} < \frac{\lambda + c\mu}{r} < \infty,
\]
according to the Weierstrass criterion, we know that \( \sum_{i=0}^{\infty} \pi_i(x) \) is convergent uniformly in \( x \). Thus, we can get Eq. (16). From (15), we have
\[
\int_{0}^{\infty} \pi(x)e^{\alpha x} \, dx = \frac{\hat{H}_1(\alpha, 1)}{H(\alpha, 1)} \phi_{c-1}(\alpha) + \frac{\hat{H}_0(\alpha, 1)}{H(\alpha, 1)} + \sum_{i=0}^{c-2} \phi_i(\alpha).
\]
From (5), we can establish the relationship between \( \phi_i(\alpha) \) and \( \phi_{c-1}(\alpha) \) for any \( 0 < i \leq c-2 \), and thus,
\[
\sum_{i=0}^{c-2} \phi_i(\alpha) = \left( \sum_{k=0}^{c-2} A_k(\alpha) A_{k+1}(\alpha) \cdots A_{c-2}(\alpha) \right) \phi_{c-1}(\alpha) + H_{c-1}(\alpha).
\]
Here \( H_{c-1}(\alpha) \) is an analytic function about \( \alpha \), which can be determined explicitly by (5). Hence, according to the Tauberian-like theorem and the asymptotic behaviour of \( \phi_{c-1}(\alpha) \), we can obtain the tail asymptotic properties of \( \pi(x) \) and thus attain the tail asymptotic properties of \( \Pi(x) \).

6 Special cases

In this section, we consider two important special cases: \( c = 1 \) and \( c = 2 \), for which exact asymptotic properties for the stationary distribution can be obtained without Assumption 1. The analysis of these two cases is feasible. However, the arguments for the cases \( c \geq 3 \) are rather complex since the expression for \( \hat{H}_1(\alpha, Z_0(\alpha)) \) is intractable for any \( c \geq 3 \).

6.1 Fluid queue driven by an \( M/M/1 \) queue

In this case, the unique zero point of \( \hat{H}_1(\alpha, Z_0(\alpha)) \) can be obtained explicitly as follows.

Lemma 12 Let
\[
\tilde{\alpha} = \frac{\mu}{r+1} - \lambda,
\]
then \( \tilde{\alpha} \) is the only possible zero point of \( H_1(\alpha, Z_0(\alpha)) \). Moreover, \( \tilde{\alpha} \) must be a simple zero point of \( H_1(\alpha, Z_0(\alpha)) \).
Proof We rationalize $\hat{H}_1(\alpha, Z_0(\alpha))$ by
\begin{equation}
  g(\alpha) = 2a \hat{H}_1(\alpha, Z_0(\alpha)) \hat{H}_1(\alpha, Z_1(\alpha)).
\end{equation}
(17)

Then, it follows from the definition of $\hat{H}_1(\alpha, z)$ in Theorem 1 and (7) that
\begin{align*}
g(\alpha) &= -2\lambda Z_0(\alpha) Z_1(\alpha) [(\mu - \alpha r - \alpha) Z_0(\alpha) - \mu] \quad (\mu - \alpha r - \alpha) Z_1(\alpha) - \mu] \\
        &= \frac{-2\alpha \mu^2}{\lambda} \{ (r + 1)\alpha - \mu + \lambda(r + 1) \}.
\end{align*}

It is obvious that $\tilde{\alpha} = \frac{\mu}{r+1} - \lambda$ is the only possible zero point of $H_1(\alpha, Z_0(\alpha))$ with a modulus greater than 0.
\hfill\Box

Lemma 13 The unique zero point $\tilde{\alpha}$ satisfies the following inequality:
\begin{equation*}
  H_2(Z_0(\tilde{\alpha})) \psi(Z_0(\tilde{\alpha})) + \hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha})) \neq 0.
\end{equation*}

Proof From the initial condition $\Pi_i(0) = 0$, for any $i \geq 1$, we have
\begin{equation*}
  \psi(Z_0(\tilde{\alpha})) = \sum_{i=0}^{\infty} \Pi_i(0) Z_i(\tilde{\alpha}) = \Pi_0(0).
\end{equation*}

For $\tilde{\alpha} = \frac{\mu}{r+1} - \lambda$, we have $Z_0(\tilde{\alpha}) = \min\{1 + r, \frac{\mu}{\lambda(1+r)}\}$. Thus, from the definitions of $\hat{H}_0(\alpha, z)$ and $H_2(z)$ in Theorem 1, and the fact that $\frac{\mu}{\mu - \lambda Z_0(\tilde{\alpha})} \neq 1$, we can obtain
\begin{equation*}
  \frac{-\hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha}))}{H_2(Z_0(\tilde{\alpha}))} = \frac{\mu \Pi_0(0)}{\mu - \lambda Z_0(\tilde{\alpha})} \neq \Pi_0(0),
\end{equation*}
which implies that $H_2(Z_0(\tilde{\alpha})) \psi(Z_0(\tilde{\alpha})) + \hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha})) \neq 0$. \hfill\Box

Since the following inequality always holds:
\begin{equation*}
  \tilde{\alpha} = \frac{\mu}{r + 1} - \lambda \leq \alpha_1 = \frac{(\sqrt{\mu} - \sqrt{\lambda})^2}{r},
\end{equation*}
we only have two tail asymptotic properties for the stationary distribution $\Pi_i(x)$ and the marginal distribution $\Pi(x)$ of the fluid queue in this case. Here we omit the details and only present the asymptotic properties of the marginal distribution.

Theorem 6 For the marginal distribution $\Pi(x)$ of the fluid queue, we have the following tail asymptotic properties for large enough $x$:

Case (i) If (i) of Lemma 9 holds (i.e., $\tilde{\alpha} = \frac{\mu}{r+1} - \lambda < \alpha_1$), then
\begin{equation*}
  \Pi(x) - 1 \sim \frac{(r + 1)\tilde{\alpha} c_1}{r} e^{-\tilde{\alpha} x},
\end{equation*}

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\end{center}
Case (ii) If (ii) of Lemma 9 holds (i.e., \( \tilde{\alpha} = \frac{\mu}{r+1} - \lambda = \alpha_1 \)), then

\[
\Pi(x) - 1 \sim -\frac{(r+1)\tilde{\alpha}c_2}{r\sqrt{\pi}} e^{-\tilde{\alpha}x} (1 - \frac{1}{2}).
\]

Here \( c_1 \) and \( c_2 \) are defined in Theorem 2.

6.2 Fluid queue driven by an \( M/M/2 \) queue

In this case, we can obtain the following lemma.

Lemma 14 The function \( \hat{H}_1(\alpha, Z_0(\alpha)) \) has at most one real zero point in \((0, \alpha_1]\). Moreover, this unique zero point, denoted by \( \tilde{\alpha} \), if it exists, must be a simple zero point.

Proof Let \( g(\alpha) = 0 \), where \( g(\alpha) \) is defined in (17). We can obtain the following equation:

\[
(r + 1)\alpha^3 + [3\lambda(r + 1) + \mu r]\alpha^2 + [3\lambda^2(r + 1) + \mu \lambda r - \lambda \mu - \mu^2] \alpha \\
+ \lambda^2(r + 1) - \lambda^2 \mu - 2\lambda \mu^2 = 0.
\]

Denote the left-hand side of the above equation by \( \tilde{g}(\alpha) \). For any \( \alpha > 0 \), we have

\[
\tilde{g}''(\alpha) = 6(r + 1)\alpha + 6\lambda(r + 1) + 2\mu r > 0,
\]

which implies that \( \tilde{g}(\alpha) \) is a convex function for any \( \alpha > 0 \).

If \( \tilde{g}(0) = \lambda^3(r + 1) - \lambda^2 \mu - 2\lambda \mu^2 < 0 \), we can conclude that \( \tilde{g}(\alpha) = 0 \) has only one real solution in \((0, \infty)\), which implies that there exists at most one real solution in \((0, \alpha_1]\), denoted by \( \tilde{\alpha} \), if it exists. Moreover, according to properties of convex functions, we have \( \tilde{g}'(\tilde{\alpha}) > 0 \), which implies that \( \tilde{\alpha} \) is a simple zero point.

If \( \tilde{g}(0) \geq 0 \), we have \( \tilde{g}'(0) = 3\lambda^2(r + 1) + \mu \lambda r - \lambda \mu - \mu^2 > 0 \) and thus \( \tilde{g}(\alpha) = 0 \) has no real solution in \((0, \infty)\). \( \square \)

Lemma 15 The unique zero point \( \tilde{\alpha} \), if it exists, satisfies the following inequality:

\[
H_2(Z_0(\tilde{\alpha}))\psi(Z_0(\tilde{\alpha})) + \hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha})) \neq 0.
\]

Proof From the definitions of \( H_2(z) \) and \( \hat{H}_0(\alpha, z) \) in Theorem 1, for \( c = 2 \), we have

\[
H_2(Z_0(\alpha))\psi(Z_0(\alpha)) + \hat{H}_0(\alpha, Z_0(\alpha)) \\
= (\lambda Z_0(\alpha) - 2\mu)Z_0(\alpha) - 1\psi(Z_0(\alpha)) \\
+ \left[ \mu Z_0(\alpha)^2 - 2\mu Z_0(\alpha) + \frac{\lambda \mu Z_0(\alpha)}{\alpha + \lambda} \right] \Pi_1(0) + \frac{\lambda \alpha Z_0(\alpha)}{\alpha + \lambda} \Pi_0(0) \\
= Z_0(\alpha)^2 \left[ \lambda(Z_0(\alpha) - 1)\Pi_1(0) + \frac{\alpha(\lambda \Pi_0(0) - \mu \Pi_1(0))}{\alpha + \lambda} \right].
\]
where the second equality follows from the fact that $\psi(z) = \Pi_1(0)z$.

Actually, from (1), we can obtain that $\lambda \Pi_0(0) - \mu \Pi_1(0) \geq 0$. Moreover, from (13), we have $Z_0(\alpha) > 1$ for any $\alpha \in (0, \alpha_1]$. Hence, from (18) we get

$$H_2(Z_0(\tilde{\alpha}))\psi(Z_0(\tilde{\alpha})) + \hat{H}_0(\tilde{\alpha}, Z_0(\tilde{\alpha})) > 0.$$

\[\square\]

From the above lemmas, we can obtain the following theorem.

**Theorem 7** For the marginal distribution $\Pi(x)$ of the fluid queue, we have the following tail asymptotic properties:

**Case (i)** If (i) of Lemma 9 holds, then

$$\Pi(x) - 1 \sim -\frac{\tilde{C}_1}{\alpha^*} e^{-\alpha^* x};$$

**Case (ii)** If (ii) of Lemma 9 holds, then

$$\Pi(x) - 1 \sim -\frac{\tilde{C}_2}{\alpha^*} e^{-\alpha^* x} x^{-\frac{1}{2}};$$

**Case (iii)** If (iii) of Lemma 9 holds, then

$$\Pi(x) - 1 \sim -\frac{\tilde{C}_3}{\alpha^*} e^{-\alpha^* x} x^{-\frac{3}{2}}.$$

Here $\tilde{C}_i$, $i = 1, 2, 3$, are defined in Theorem 5.

**Remark 7** Compared with the case $c = 1$, a new asymptotic behaviour, Case (iii), appears in the case $c = 2$. We now give an example to illustrate that this new asymptotic exists for the case $c \geq 3$. For example, let $c = 3$, and if we take $r = 10$, $\lambda = 20$, $\mu = 30$, we can obtain four zero points $4$, $-67$ and $-15 + 5i$ of $\hat{H}_1(\alpha, Z_0(\alpha))$. Thus, we have $\tilde{\alpha} = 4 > \alpha_1 = 0.5$, which implies that Case (iii) holds.

### 7 Concluding remarks

In this paper, we applied the kernel method to investigate the exact tail asymptotic behaviour of the joint stationary probabilities and the marginal distribution of the fluid queue driven by an $M/M/c$ queue. Different from the model studied in [14] and [13], for which the tail asymptotic properties are symmetric between the level and the phase, since both level and phase processes are discrete, the tail asymptotics for $\phi_{c-1}(\alpha)$ and $\psi(z)$ are asymmetric in this paper, since the phase process is discrete and the level process is continuous.

There exist a total of three different types of exact tail asymptotics for the stationary probabilities of the fluid queue in this paper. However, we may see in [11] that the
stationary probabilities of the fluid queue, driven by a finite Markov chain, are always exactly geometric, which corresponds to Case (i) of Theorem 4. This implies that the infinite phase causes new phenomena.

In Sect. 6, we showed that Case (iii) of Theorem 4 does not appear in the case \( c = 1 \), but exists for the case \( c \geq 2 \). This implies that the asymptotic behaviour for \( c \geq 2 \) can be significantly different from that for the case \( c = 1 \).

From the arguments given in this paper, we have seen that Assumption 1 is redundant in the special cases \( c = 1 \) and \( c = 2 \). Based on our numerical calculations for a broad range of selections of parameter values, we hypothesize that Assumption 1 is redundant for all cases \( c \geq 3 \).

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Appendix: Tauberian-like theorem

Denote

\[ \Delta_1(\phi, \varepsilon) = \left\{ x : |x| \leq |x_0| + \varepsilon, \ |\arg(x - x_0)| > \phi, \varepsilon > 0, 0 < \phi < \frac{\pi}{2} \right\}. \]

Let \( f_n \) be a sequence of numbers with the generating function

\[ f(x) = \sum_{n \geq 1} f_n x^n. \]

Lemma 16 (Flajolet and Odlyzko 1990) Assume that \( f(x) \) is analytic in \( \Delta_1(\phi, \varepsilon) \) except at \( x = x_0 \) and

\[ f(x) \sim K(x_0 - x)^s \text{ as } x \to x_0 \text{ in } \Delta_1(\phi, \varepsilon). \]

Then, as \( n \to \infty \), (i) If \( s \notin \{0, 1, 2, \ldots\} \),

\[ f_n = \frac{K}{\Gamma(-s)} n^{-s-1} x_0^{-n}. \]

where \( \Gamma(\cdot) \) is the Gamma function.

(ii) If \( s \) is a non-negative integer, then

\[ f_n = o(n^{-s-1} x_0^{-n}). \]

For the continuous case, let

\[ g(x) = \int_0^\infty e^{xt} f(t) dt. \]
Denote
\[ \Delta_2(\phi, \varepsilon) = \{ x : \Re(x) \leq |x_0| + \varepsilon, x \neq x_0, \varepsilon > 0, |\arg(x - x_0)| > \phi \}. \]

The following lemma is shown in Theorem 2 in [4].

**Lemma 17** Assume that \( g(x) \) satisfies the following conditions:

(i) The leftmost singularity of \( g(x) \) is \( x_0 \), with \( x_0 > 0 \). Furthermore, we assume that as \( x \to x_0 \),
\[ g(x) \sim (x_0 - x)^{-s} \]
for some \( s \in \mathbb{C} \setminus \mathbb{Z}_- \).

(ii) \( g(x) \) is analytic on \( \Delta_2(\phi_0, \varepsilon) \) for some \( \phi_0 \in (0, \frac{\pi}{2}] \).

(iii) \( g(x) \) is bounded on \( \Delta_2(\phi_1, \varepsilon) \) for some \( \phi_1 > 0 \).

Then, as \( t \to \infty \),
\[ f(t) \sim e^{-x_0t} \frac{t^{s-1}}{\Gamma(s)}, \]
where \( \Gamma(\cdot) \) is the Gamma function.

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