Three magnons in an isotropic $S = 1$ ferromagnetic chain as an exactly solvable non-integrable system

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Dedicated to the memory of P P Kulish.

Abstract. It is shown that a generalization of the Bethe ansatz based on a utilization of degenerative discrete-diffractive wave functions solves the three-magnon problem for the $S = 1$ isotropic ferromagnetic infinite chain. The four-magnon problem is briefly discussed.

Keywords: integrable spin chains (vertex models), spin chains, ladders and planes (theory)
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1. Introduction

Quantum integrable models are usually solved by various versions of the Bethe ansatz [1–3]. However, solvability does not imply integrability [4]. In fact, the latter results in a non-ergodic physical behavior, while the former gives the possibility to obtain the spectrum of physical states. An interplay between these conceptions may be well illustrated for models whose particles (elementary excitations) are flat waves excited from the ground state $|\varnothing\rangle$ by some creation operators $\tilde{\Psi}^j_n$. Here the index $j = 1, \ldots, d$ parameterizes internal degrees of freedom such as polarization of triplons in spin ladders ($d = 3$) [5], or electrons in the $t$–$J$ model ($d = 2$) [6]. Namely, one- and two-particle states are

$$|k\rangle^j = \sum_n e^{ikn}\tilde{\Psi}^j_n|\varnothing\rangle,$$

$$|k_1, k_2\rangle^{j_1j_2} = \sum_{n_1 < n_2} [A_{12}^{j_1j_2}e^{i(k_{1n_1}+k_{2n_2})} - A_{21}^{j_1j_2}e^{i(k_{1n_2}+k_{2n_1})}]\tilde{\Psi}^j_{n_1}\tilde{\Psi}^j_{n_2}|\varnothing\rangle.$$

As vectors in $\mathbb{C}^d \otimes \mathbb{C}^d$ the two amplitudes $A_{12}^{j_1j_2}$ and $A_{21}^{j_1j_2}$ are related by the formula

$$A_{21} = S(k_1, k_2)A_{12},$$

where the $d^2 \times d^2$ matrix $S(k_1, k_2)$ is the two-particle scattering matrix. If it satisfies the Yang–Baxter equation [7]

$$S_{12}(k_a, k_c)S_{23}(k_a, k_c)S_{12}(k_b, k_c) = S_{23}(k_a, k_b)S_{12}(k_a, k_c)S_{23}(k_b, k_c),$$

where $(I$ is the finite-dimensional matrix unit)

$$S_{12}(k, \bar{k}) = S(k, \bar{k}) \otimes I, \quad S_{23}(k, \bar{k}) = I \otimes S(k, \bar{k}),$$

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then the wave function of a three-particle state

$$|k_1, k_2, k_3\rangle = \sum_{n_1 < n_2 < n_3} a_{n_1,n_2,n_3}^{j_1,j_2,j_3} \prod_{i=1}^{3} \Phi_{j_i}^{n_i} \Phi_{j_i}^{n_i} \Phi_{j_i}^{n_i} \langle \emptyset |,$$ (6)

may be obtained in the Bethe form (we have omitted the polarization indices, and used the Levi–Civita tensor)

$$a_{n_1,n_2,n_3}(k_1, k_2, k_3) = \sum_{a,b,c=1}^{3} \varepsilon_{abc} A_{abc}(k_1, k_2, k_3) e^{i(k_{n_1} + k_{n_2} + k_{n_3})},$$ (7)

which expresses both solvability and integrability. The former is evident, while the latter follows from the fact that according to (7) an initial set of incoming wave numbers does not change under the scattering.

Equation (4) was first treated as an integrability condition for quantum gas with delta-function interaction [8] and then used as an integrability test for other models such as the above-mentioned spin ladder and $t$–$J$ model [5, 6]. For a 1D system, equation (4) together with an absence of particle production are necessary conditions for factorization of the multi-particle scattering or equivalently for its reduction to a succession of space-time-separated two-particle collisions [9]. For identical particles, due to the energy and momentum conservation laws, such a collision reduces to an exchange of wave numbers between scattering particles and multiplication of amplitudes on an appropriate two-particle $S$-matrix (see equation (3)). That is why an $m$-particle wave function is a sum of $m!$ exponential terms, each of them corresponding to a permutation of wave numbers in a set $\{k_1, \ldots, k_m\}$. Since the distribution of wave numbers (momentums) does not alter under the collisions, the system possesses a non-ergodic behavior.

For a non-integrable quantum system the situation is drastically different. One- and two-particle states may be obtained as previously [5, 6]. However, a three-particle wave function should have a diffractive form [3, 10, 11]

$$a_{n_1,n_2,n_3} = \int dk_1 dk_2 dk_3 \delta \left( \prod_{l=1}^{3} e^{i k_l} - e^{i \vec{k}} \right) \delta \left( \sum_{l=1}^{3} E(k_l) - E \right)$$

$$\cdot \sum_{a,b,c=1}^{3} \varepsilon_{abc} A_{abc}(k_1, k_2, k_3) e^{i(k_{n_1} + k_{n_2} + k_{n_3})},$$ (8)

corresponding to changes of an incoming triple of wave numbers and hence to an ergodic behavior. Unfortunately a substitution of (8) into (6) does not reduce the spectral problem to a simple form as it did in the case (7).

In the present paper we suggest a discrete analog of the diffractive form (8), namely

$$a_{n_1,n_2,n_3} = \sum_{m=1}^{M} \sum_{a,b,c=1}^{3} \varepsilon_{abc} A_{abc}^{(m)} e^{i(k_{n_1}^{(m)} + k_{n_2}^{(m)} + k_{n_3}^{(m)})},$$ (9)

Here $M > 1$ and

$$\prod_{l=1}^{3} e^{i k_l^{(m)}} = \text{const}, \quad \sum_{l=1}^{3} E(k_l^{(m)}) = \text{const}.$$ (10)
As will be shown here for a degenerative case $M < \infty$, the ansatz (9) results in essential simplifications. Namely for

$$1 < M < \infty,$$

(11)

the system is still ergodic and non-integrable but the three-particle wave functions (9) may be derived exactly as in the usual version of the Bethe ansatz.

Magnons (spin waves with $\Delta S = 1$) in an $S = 1$ isotropic ferromagnetic chain (see the Hamiltonian (12)–(13) of the present paper) have no internal degrees of freedom. Hence the two-magnon $S$-matrix is a scalar function and equation (4) is satisfied automatically. However, the two-magnon scattering results in the creation of the quadruplon resonance [12] (the spin wave with $\Delta S = 2$). In the su(3)-symmetric point [13] the quadruplons are stable particles with the same energy as magnons. For a broken su(3)-symmetry the quadruplons become unstable and decay into magnon pairs, at the same time creating resonances in two-magnon collisions. Due to these processes a three-magnon scattering has the following channel. First of all, two neighboring incoming magnons create a quadruplon resonance which then collides with the third incoming magnon. Under this collision the resonance decays on two magnons. One of them goes to infinity while the other forms a new resonance with the third magnon. Finitely this new resonance again decays on two outgoing magnons. As will be shown in section 4, only this process cannot be accounted for by the wave functions of the form (7). However, the $M = 2$ wave functions of the form (9) will be derived.

2. Hamiltonian and one-magnon spectrum

In the present paper we study the general model of the isotropic $S = 1$ ferromagnet with the Hamiltonian

$$\hat{H} = \sum_{n=-\infty}^{\infty} H_{n,n+1},$$

(12)

where [12, 14]

$$H_{n,n+1} = \hat{I} - (S_n \cdot S_{n+1}) + J(\hat{I} - (S_n \cdot S_{n+1})^2).$$

(13)

Here $S_n$ is the standard triple of $S = 1$ spin operators associated with the $n$-th site. The constant terms proportional to the infinite-dimensional matrix unit $\hat{I}$ are added only for the relation

$$\hat{H}|\varnothing\rangle = 0,$$

(14)

where the state

$$|\varnothing\rangle = \prod_{n=-\infty}^{\infty} |+\rangle_n,$$

(15)

(the states $|j\rangle_n$ with $j = -1, 0, 1$ form the standard $S = 1$ triple associated with the $n$-th site) will be treated as the pseudovacuum. It may be readily proved by an analysis of the spectrum of the $9 \times 9$ Hamiltonian density matrix related to $H_{n,n+1}$ that for $J < 1$
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equation (15) gives the ground state of the model. For $J > 1$ it will be the ground state under the saturating magnetic field.

Using the representation
\[
(S_n \cdot S_{n+1}) = S^i_n S^i_{n+1} + \frac{1}{2} (S^+_n S^-_{n+1} + S^-_n S^+_{n+1}),
\]
and relations
\[
S^\pm_n | \pm 1 \rangle_n = 0, \quad S^\pm_n | 0 \rangle_n = \sqrt{2} | \pm 1 \rangle_n, \quad S^\pm_n | \mp 1 \rangle_n = \sqrt{2} | 0 \rangle_n, \quad S^2_n | j \rangle_n = | j \rangle_n,
\]
one readily gets [14]
\[
H_{n+1} | \pm 1 \rangle_n \otimes | \pm 1 \rangle_{n+1} = 0
\]
\[
H_{n+1} | 0 \rangle_n \otimes | 0 \rangle_{n+1} = | 0 \rangle_n \otimes | 0 \rangle_{n+1} - | 0 \rangle_n \otimes | 1 \rangle_{n+1},
\]
\[
H_{n+1} | 1 \rangle_n \otimes | 0 \rangle_{n+1} = (1 - J)(| 0 \rangle_n \otimes | 0 \rangle_{n+1} - | 1 \rangle_n \otimes | 1 \rangle_{n+1}),
\]
\[
H_{n+1} | \mp 1 \rangle_n \otimes | 1 \rangle_{n+1} = (2 - J)(| 0 \rangle_n \otimes | 1 \rangle_{n+1} + (J - 1)(| 0 \rangle_n \otimes | 0 \rangle_{n+1}
\]
\[
- J | \mp 1 \rangle_n \otimes | 1 \rangle_{n+1}.
\]
A one-magnon state is the flat wave [12, 14]
\[
| 1, k \rangle = \sum_n e^{ik_n} S^+_n | 0 \rangle,
\]
whose energy
\[
E_{\text{magn}}(k) = 2(1 - \cos k)
\]
may be readily obtained from (18).

3. Two-magnon scattering

A two-magnon state should have the form [14]
\[
| 2 \rangle = \left[ \sum_{n_1 < n_2} a_{n_1, n_2} S^{-}_{n_1} S^{-}_{n_2} + \sum_n b_n (S^+_n)^2 \right] | 0 \rangle.
\]
According to (18) the corresponding Schrödinger equation on the combined wave function \{a_{n_1, n_2}, b_n\} has the form of the following system [14]
\[
4a_{n_1, n_2} - a_{n_1-1, n_2} - a_{n_1, n_2-1} - a_{n_1, n_2+1} = Ea_{n_1, n_2}, \quad n_2 - n_1 > 1,
\]
\[
(3 - J)a_{n_1, n_2+1} - a_{n_1, n_2-1} - a_{n_1+1, n_2} + (J - 1)(b_n + b_{n+1}) = Ea_{n_1, n_2+1},
\]
\[
2(2 - J)b_n - J(b_{n-1} + b_{n+1}) + (J - 1)(a_{n_1, n_2} + a_{n_1+1, n_2}) = Eb_n.
\]
In the $\text{su}(3)$ invariant point $J = 1$ the system (22)–(23) splits into two independent sub-systems and a solution

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$$b_n = \epsilon^{nk},$$

(24)

with the same energy, as (20) corresponds to the above-mentioned quadrupole wave [12, 13] (quadruplon). Turning to the general case $J \neq 1$ and suggesting the following ansatz

$$a_{m, n}(k_1, k_2) = A_{12}e^{i(k_1n_1 + k_2n_2)} - A_{21}e^{i(k_2n_1 + k_1n_2)}; \quad b_n(k_1, k_2) = Be^{i(k_1 + k_2)n}$$

(25)

we readily get from (22) the energy

$$E(k_1, k_2) = E_{\text{magn}}(k_1) + E_{\text{magn}}(k_2),$$

(26)

and reduce (23) to

$$[1 + e^{i(k_1 + k_2)} - (1 + J)e^{i k_1}] A_{12} - [1 + e^{i(k_1 + k_2)} - (1 + J)e^{i k_1}] A_{21}$$

$$+ (J - 1)(1 + e^{i(k_1 + k_2)})B = 0,$$

$$A_{12} = A(k_1, k_2), \quad A_{21} = A(k_2, k_1), \quad B = B(k_1, k_2),$$

(27)

System (27) has the following solutions

$$A_{12} = A(k_1, k_2), \quad A_{21} = A(k_2, k_1), \quad B = B(k_1, k_2),$$

(28)

where

$$A(k, \tilde{k}) = e^{i k} + e^{i(k+2\tilde{k})} - (1 + J)e^{2ik} - (1 + 3J)e^{i(k+\tilde{k})} + J(3e^{ik} + 3e^{i(k+\tilde{k})}) - 1 - e^{2i(k+\tilde{k})},$$

(29)

and

$$B(k, \tilde{k}) = (1 - J)(e^{i k} - e^{i \tilde{k}})(1 + e^{i(k+\tilde{k})}).$$

(30)

As we see the two-magnon problem may be solved in a non-diffractive way for all values of $J$. In other words it is insensitive to non-integrability. In fact, equation (26) together with the condition $k_1 + k_2 = k$ (k is the total momentum of the state) define the pair of wave numbers $k_1$ and $k_2$ up to a permutation of them.

4. The three-magnon problem

A three-magnon state should have the form

$$|3\rangle = \sum_{n_1 < n_2 < n_3} a_{n_1, n_2, n_3} S_{-n_1} S_{-n_2} S_{-n_3} + \sum_{n_1 < n_2} (b_{n_1, n_2}^{(1)} S_{-n_1}^2 S_{-n_2}^2 + b_{n_1, n_2}^{(2)} S_{-n_1} S_{-n_2}^2 S_{-n_3}^2) \mid \varnothing \rangle.$$

(31)

The corresponding Shrödinger equation on the combined wave function $\{a_{n_1, n_2, n_3}, b_{n_1, n_2}^{(1)}, b_{n_1, n_2}^{(2)}\}$ splits into four groups of equations

$$6a_{n_1, n_2, n_3} - a_{n_1 + 1, n_2, n_3} - a_{n_1, n_2 + 1, n_3} - a_{n_1, n_2, n_3 + 1} - a_{n_1 - 1, n_2, n_3} - a_{n_1, n_2 - 1, n_3} - a_{n_1, n_2, n_3 - 1} = Ea_{n_1, n_2, n_3},$$

(32)

at $n_2 - n_1 > 1, n_3 - n_2 > 1$.

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\[(5 - J)a_{m-1,n,n} - a_{m-1,n,n+1} - a_{m-1,n,n+1} - a_{m-2,n,n} - a_{m-1,n,n+1} + (J - 1)[b^{(1)}_{m,n} + b^{(1)}_{m,n+1}] = Ea_{m-1,n,n},\]
\[(5 - J)a_{m,n,n+1} - a_{m,n,n+2} - a_{m,n,n+1} - a_{m-1,n,n+1} - a_{m-1,n,n+1} + (J - 1)[b^{(2)}_{m,n} + b^{(2)}_{m,n+1}] = Ea_{m,n,n+1},\]
\[2(3 - J)b^{(1)}_{m,n} - b^{(1)}_{m,n+1} - b^{(1)}_{m,n-1} - J[b^{(1)}_{m-1,n} + b^{(1)}_{m+1,n}] + (J - 1)[a_{m,m+1,n} + a_{m-1,m,n}] = Eb^{(1)}_{m,n},\]
\[2(3 - J)b^{(2)}_{m,n} - b^{(2)}_{m+1,n} - b^{(2)}_{m-1,n} - J[b^{(2)}_{m,n-1} + b^{(2)}_{m,n+1}] + (J - 1)[a_{m,n,n+1} + a_{m,n,n+1}] = Eb^{(2)}_{m,n},\]

at $n - m > 1$,

\[2(2 - J)a_{n-1,n,n+1} - a_{n-1,n,n+2} - a_{n-2,n,n+1} + (J - 1)[b^{(1)}_{n-1,n+1} + b^{(1)}_{n,n+1} + b^{(2)}_{n-1,n,n} + b^{(2)}_{n-1,n+1}] = Ea_{n-1,n,n+1},\]

and

\[(4 - J)b^{(1)}_{n,n+1} - b^{(1)}_{n,n+2} - Jb^{(1)}_{n,n+1} - b^{(2)}_{n,n+1} + (J - 1)a_{n-1,n,n+1} = Eb^{(1)}_{n,n+1},\]
\[(4 - J)b^{(2)}_{n-1,n} - b^{(2)}_{n-2,n} - Jb^{(2)}_{n-1,n} - b^{(1)}_{n-1,n} + (J - 1)a_{n-1,n,n+1} = Eb^{(2)}_{n-1,n}.\]

The following substitution

\[a_{n_1,n_2,n_3}(k_1, k_2, k_3) = \sum_{a,b,c=1}^{3} \varepsilon_{abc}A(k_a, k_b)A(k_a, k_c)A(k_b, k_c)e^{i(k_{n_1} + k_{n_2} + k_{n_3})},\]
\[b^{(1)}_{n_1,n_2}(k_1, k_2, k_3) = \frac{1}{2} \sum_{a,b,c=1}^{3} \varepsilon_{abc}B(k_a, k_b)A(k_a, k_c)A(k_b, k_c)e^{i(k_{n_1} + k_{n_2})},\]
\[b^{(2)}_{n_1,n_2}(k_1, k_2, k_3) = \frac{1}{2} \sum_{a,b,c=1}^{3} \varepsilon_{abc}A(k_a, k_b)A(k_a, k_c)B(k_b, k_c)e^{i(k_{n_1} + k_{n_2})},\]

where $A(k, \tilde{k})$ and $B(k, \tilde{k})$ are given by equations (29) and (30), solves equations (32)–(34), giving the energy

\[E(k_1, k_2, k_3) = \sum_{j=1}^{3} E_{\text{magn}}(k_j).\]

At the same time, equation (35) turns into

\[X^{(j)}(k_1, k_2, k_3)e^{i(k_{1} + k_{2} + k_{j})} = 0, \quad j = 1, 2,\]

where

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\[
X^{(1)}(k_1, k_2, k_3) = \frac{1}{2} \sum_{a, b, c = 1}^{3} \varepsilon_{abc} e^{ik_a} A(k_a, k_c) \left[ (E(k_1, k_2, k_3) - 4 + e^{ik_b}) + J(1 + e^{-ik_a + k_c}) \right] \\
\cdot B(k_a, k_b) A(k_a, k_c) + e^{ik_a} A(k_a, k_b) B(k_b, k_c) + 2(1 - J)e^{-ik_a} A(k_a, k_b) A(k_b, k_c), \\
X^{(2)}(k_1, k_2, k_3) = \frac{1}{2} \sum_{a, b, c = 1}^{3} \varepsilon_{abc} e^{-ik_a} A(k_a, k_c) \left[ (E(k_1, k_2, k_3) - 4 + e^{-ik_a}) + J(1 + e^{ik_b + k_c}) \right] \\
\cdot A(k_a, k_b) B(k_a, k_c) + e^{-ik_a} B(k_a, k_b) A(k_b, k_c) + 2(1 - J)e^{ik_a} A(k_a, k_b) A(k_b, k_c). 
\]

(39)

An evaluation of the sums in (39) with the use of the computer algebra system MAPLE gives

\[
X^{(1)}(k_1, k_2, k_3) = (1 - J^2)\varphi(k_1, k_2, k_3) \left[ (E(k_1, k_2, k_3) - 5)e^{ik} - 1 + J(2 + 3e^{ik} + e^{2ik}) \right], \\
X^{(2)}(k_1, k_2, k_3) = (1 - J^2)\varphi(k_1, k_2, k_3) \left[ e^{ik} - E(k_1, k_2, k_3) + 5 - J(3 + 2e^{ik} + e^{-ik}) \right],
\]

where $k = k_1 + k_2 + k_3$ and

\[
\varphi(k_1, k_2, k_3) = (e^{ik_1} - e^{ik_2}) (e^{ik_2} - e^{ik_3}) (e^{ik_3} - e^{ik_1}) \cdot \prod_{j=1}^{3} (1 - e^{ik_j}).
\]

(41)

As we see from (40), $X^{(j)}(k_1, k_2, k_3) \equiv 0, (j = 1, 2)$ only in two integrable cases [13, 15] $J = 1$ and $J = -1$. However, even in the general case the whole system (32)–(35) will be obviously satisfied for the following wave functions $(j = 1, 2)$

\[
a_{n_1, n_2, n_3}(k_1, k_2, k_3, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = \varphi(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) a_{n_1, n_2, n_3}(k_1, k_2, k_3) - \varphi(k_1, k_2, k_3) a_{n_1, n_2, n_3}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3), \\
b_{n_1, n_2}^{(j)}(k_1, k_2, k_3, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = \varphi(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) b_{n_1, n_2}^{(j)}(k_1, k_2, k_3) - \varphi(k_1, k_2, k_3) b_{n_1, n_2}^{(j)}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3),
\]

(42)

where

\[
\prod_{l=1}^{3} e^{ik_l} = \prod_{l=1}^{3} e^{ik_l}, \quad \sum_{l=1}^{3} E_{\text{magn}}(\tilde{k}_l) = \sum_{l=1}^{3} E_{\text{magn}}(k_l).
\]

(43)

5. Remark on the four-magnon problem

An evaluation of four-magnon states for our model is a problem of the next level of complexity. In order to see this, let us recall an evaluation of equation (42). First of all we take a triple $\{k_1, k_2, k_3\}$ and then construct the wave function (36) which satisfies equations (32)–(34) but does not satisfy equation (35). In order to solve the latter we add the term related to a new triple $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$. The resulting wave function has the form (42).

For a four-magnon state

\[
|4\rangle = \sum_{n_1 < n_2 < n_3 < n_4} a_{n_1, n_2, n_3, n_4} S_{n_1}^- S_{n_2}^- S_{n_3}^- S_{n_4}^- + \sum_{n_1 < n_2 < n_3} (b_{n_1, n_2, n_3}^{(1)} S_{n_1}^-)^2 S_{n_2}^- S_{n_3}^- \\
+ b_{n_1, n_2, n_3}^{(2)} S_{n_1}^- S_{n_2}^- S_{n_3}^- + b_{n_1, n_2, n_3}^{(3)} S_{n_1}^- (S_{n_1}^-)^2 + \sum_{n_1 < n_2} c_{n_1, n_2} (S_{n_1}^-)^2 (S_{n_2}^-)^2 \bigg| \varnothing \bigg),
\]

(44)

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the corresponding Schrödinger equation should again split into several systems of equations related to different processes in the four-magnon system. We shall study only one of them related to an extreme right magnon being separated from the others and hence not interacting with them.

As in the previous case we take at once \( a_{n_1,n_2,n_3,n_4} \) as a linear combination of \( 24 = 4! \) Bethe exponents related to different permutations of four wave numbers \( k_1, k_2, k_3 \) and \( k_4 \). Let us first consider six of them proportional to \( e^{ik_n a_i} \). From an account of the interaction between the triple of left magnons, it follows that these six terms should be added to another six, proportional to the same exponent \( e^{ik_n a_i} \) but with a new triple \( \{k_1, k_2, k_3\} \) of the left magnon wave numbers. The same picture will apply for the other three groups of exponents proportional to \( e^{ik_n a_i} \) (\( j = 1, 2, 3 \)).

Now we may explain the cardinal difference between three- and four-magnon problems. In the former case for a given triple \( \{k_1, k_2, k_3\} \) it is sufficient to add only a single triple \( \{k_1, k_2, k_3\} \); however, in the latter, for a given quartet \( \{k_1^{(0)}, k_2^{(0)}, k_3^{(0)}, k_4^{(0)}\} \), it is necessary to add at least four different new quartets

\[
\{k_1^{(j)}, \ldots, k_4^{(j)}\}, \quad k_j^{(j)} = k_j^{(0)}, \quad j = 1, \ldots, 4,
\]  

(45)

related to the same energy and total wave number. Each of the four induced quartets has similar rights to the initial one and hence should be in the same correspondence with four other quartets (one of them is the initial quartet). As a result, the total set of quartets may be represented as a graph, where each vertex (related to its own quartet) is connected to four other vertices.

Let us study in detail a construction of the simplest example of such a set of quartets. As explained above, first of all we take an initial quartet \( \{k_1^{(0)}, \ldots, k_4^{(0)}\} \) which induces the four new ones according to equation (45). Taking now the quartet \( \{k_1^{(1)}, \ldots, k_4^{(1)}\} \) and applying the same argumentation as for \( \{k_1^{(0)}, \ldots, k_4^{(0)}\} \), we see that it must also be connected with four different quartets. One of them is already known: it is \( \{k_1^{(0)}, \ldots, k_4^{(0)}\} \). Hence, we should present the other three quartets. The simplest way to do this is to use the quartets (45) with \( j = 2, 3, 4 \) (otherwise we have to introduce new quartets \( \{k_1^{(j)}, \ldots, k_4^{(j)}\}, j = 5, 6, 7 \). Under this choice, any pair \( \{k_1^{(1)}, \ldots, k_4^{(1)}\} \) and \( \{k_1^{(j)}, \ldots, k_4^{(j)}\} \) \( (j = 2, 3, 4) \) should have a common wave number. According to equation (45) the wave numbers \( k_j^{(j)} \) \( (j = 1, \ldots, 4) \) are already utilized. Hence without lost of generality we may put \( k_j^{(1)} = k_j^{(j)} \) \( (j = 2, 3, 4) \). Turning to the quartet \( \{k_1^{(2)}, \ldots, k_4^{(2)}\} \) we see that since it has already been connected with \( \{k_1^{(0)}, \ldots, k_4^{(0)}\} \) and \( \{k_1^{(1)}, \ldots, k_4^{(1)}\} \) we have to connect it only with two quartets. Again, the simplest way to do this is to use \( \{k_1^{(j)}, \ldots, k_4^{(j)}\} \) \( (j = 3, 4) \). Since the wave numbers \( k_j^{(2)} \) with \( j = 1, 2 \) are already utilized we may postulate (without any lost of generality) that \( k_j^{(2)} = k_j^{(2)} \) \( (j = 3, 4) \). Finitely we consider the quartet \( \{k_1^{(3)}, \ldots, k_4^{(3)}\} \) and connect it with \( \{k_1^{(4)}, \ldots, k_4^{(4)}\} \) by the formula \( k_4^{(3)} = k_4^{(4)} \). As a result we have the system of five quartets (geometrically it may be represented as a graph with five vertices connected to each other)

\[
\{k_1^{(j)}, \ldots, k_4^{(j)}\}, \quad j = 0, \ldots, 4,
\]  

(46)

and 10 relations; namely (45) and

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$$k_{l}^{(j)} = k_{j}^{(l)}, \quad j, l = 1, \ldots, 4, \quad j \neq l.$$ (47)

According to equations (45) and (47), only 10 of the 20 wave numbers $k_{l}^{(j)}$ ($j = 0, \ldots, 4, \quad l = 1, \ldots, 4$) are independent. According to the energy and quasimomentum conservation laws they satisfy 10 independent equations

$$\sum_{l=1}^{4} E_{\text{magn}}(k_{j}^{(l)}) = \sum_{j=1}^{4} E_{\text{magn}}(k_{j}^{(1)}) = E_{\text{magn}}(k_{j}^{(1)}) + \sum_{j=2}^{4} E_{\text{magn}}(k_{j}^{(2)})$$

$$= \sum_{j=1}^{3} E_{\text{magn}}(k_{j}^{(3)}) + E_{\text{magn}}(k_{j}^{(4)}) = \sum_{j=1}^{4} E_{\text{magn}}(k_{j}^{(j)}) = E.$$ (49)

Correspondingly, a general system of $M$ quartets contains $4M$ wave numbers. Since each of them is common to two different quartets, only half of them (namely $2M$) are independent. Equations

$$\prod_{l=1}^{4} e^{ik_{l}^{(j)}} = e^{ik}, \quad \prod_{l=1}^{4} E_{\text{magn}}(k_{l}^{(j)}) = E, \quad j = 1, \ldots, M,$$ (50)

where $k$ is the quasimomentum (total wave number) and $E$ is the energy, give $2M$ conditions on these $2M$ wave numbers. Hence the existence of finite-$M$ four-magnon Bethe wave functions is in question even without an analysis of the pure four-magnon collisions.

6. Summary and discussion

In the present paper we have studied the three-magnon problem for a general isotropic $S=1$ ferromagnetic infinite chain. Except for the two integrable cases [13, 15], the corresponding wave functions cannot be represented in the Bethe form (7) but only as a non-integrable modification (9), (42) (which we call the degenerative, discrete-diffractive form). Since the presented set of states is highly overloaded, a complete description of the three-magnon scattering [16] may be obtained only after an extraction of a non-overloaded complete system of the three-magnon states. However, it is not clear how to represent such a system. In fact, for an integrable spin chain the three-magnon eigenstates may be parameterized by their energy $E$, quasimomentum $k$ and the eigenvalue of an additional first integral. The latter belongs to an infinite set of commuting first integrals which may be obtained by the standard procedure [1–3, 15]. The system studied in the present paper is however non-integrable. Probably there exists an operator $\hat{Q}$ which commutes both with the Hamiltonian and the shift operator. If its eigenvalue $Q$ is independent from $E$ and $k$, it may be used for a parametrization of the spectrum. Otherwise the parametrization procedure seems unclear and probably may be developed.

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on the base of non-commutative geometry [17]. Nevertheless, it seems evident that a scattering of three incoming magnons with wave numbers $\{k_1, k_2, k_3\}$ should result in the creation of all possible outgoing three-magnon states with the same $E$ and $k$.

We also have shown that the corresponding four-magnon problem is much more difficult.

We suggest that the obtained result in its future development may be useful for a derivation of low-temperature expansions for thermodynamical quantities in the gapped regime [18–20].

Finally we notice that although equation (4) on the $S$-matrix has the same form as the equation on the so-called $R$-matrix (the Yang–Baxter equation in the braid group form [1, 2, 7, 15]), the two subjects are not directly connected to each other. The $S$-matrix characterizes a two-magnon scattering and its dimension $d^2 \times d^2$ depends on the number of elementary excitations (that is $d$ in our notations). On the other hand, the $R$-matrix has the same dimension $\tilde{d}^2 \times \tilde{d}^2$ as the Hamiltonian density matrix. Here $\tilde{d}$ is the dimension of the Hilbert space associated with each site of the chain (for example $\tilde{d} = 4$ for spin ladders [5] and $\tilde{d} = 3$ for the $t$–$J$ model [6]). Usually $d < \tilde{d}$. Moreover, as shown in the present paper, the Yang–Baxter equation for the $S$-matrix may be satisfied even in non-integrable cases when the $R$-matrix formalism is irrelevant.

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