CASES OF EQUALITY
IN CERTAIN MULTILINEAR INEQUALITIES
OF HARDY-RIESZ-BRASCAMP-LIEB-LUTTINGER TYPE

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ABSTRACT. Cases of equality in certain Hardy-Riesz-Brascamp-Lieb-Luttinger rearrangement inequalities are characterized.

1. STATEMENT OF RESULT

Let $m \geq 2$ and $n \geq m + 1$ be positive integers. For $j \in \{1, 2, \cdots, n\}$ let $E_j \subset \mathbb{R}$ be Lebesgue measurable sets with positive, finite measures, and let $L_j$ be surjective linear maps $\mathbb{R}^m \rightarrow \mathbb{R}$. This paper is concerned with the nature of those $n$–tuples $(E_1, \cdots, E_n)$ of measurable sets that maximize expressions

$$I(E_1, \cdots, E_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{n} \mathbb{1}_{E_j}(L_j(x)) \, dx,$$

among all $n$–tuples with specified Lebesgue measures $|E_j|$. Our results apply only in the lowest-dimensional nontrivial case, $m = 2$, but apply for arbitrarily large $n$.

Definition 1. A family $\{L_j\}$ of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}$ is nondegenerate if for every set $S \subset \{1, 2, \cdots, n\}$ of cardinality $m$, the map $x \mapsto \{(L_j(x) : j \in S)$ from $\mathbb{R}^m$ to $\mathbb{R}^S$ is a bijection.

For any Lebesgue measurable set $E \subset \mathbb{R}$ with finite Lebesgue measure, $E^*$ denotes the nonempty closed interval centered at the origin satisfying $|E| = |E^*|$. Brascamp, Lieb, and Luttinger \cite{1} proved that among sets with specified measures, the functional $I$ attains its maximum value when each $E_j$ equals $E_j^*$, that is,

\begin{equation}
I(E_1, \cdots, E_n) \leq I(E_1^*, \cdots, E_n^*).
\end{equation}

In this paper we study the uniqueness question and show that these are the only maximizing $n$–tuples, up to certain explicit symmetries of the functional, in those situations in which a satisfactory characterization of maximizers can exist.

Inequalities of this type can be traced back at least to Hardy and to Riesz \cite{8}. In the 1930s, Riesz and Sobolev independently showed that

$$\int_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(y) \mathbb{1}_{E_3}(x + y) \, dx \, dy \leq \int_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1^*}(x) \mathbb{1}_{E_2^*}(y) \mathbb{1}_{E_3^*}(x + y) \, dx \, dy$$

for arbitrary measurable sets $E_j$ with finite Lebesgue measures. Brascamp, Lieb, and Luttinger \cite{1} later proved the more general result indicated above, and in a yet more general

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\footnote{A more common convention is that $E^*$ should be open, but this convention will be convenient in our proofs. If $E = \emptyset$ then $E^* = \{0\}$, rather than the empty set, under our convention.}
form in which the target spaces $\mathbb{R}^1$ are replaced by $\mathbb{R}^k$ for arbitrary $k \geq 1$, satisfying an appropriate equivariance hypothesis.

The first inverse theorem in this context, characterizing cases of equality, was established by Burchard [3], [2]. The cases $n \leq m$ are uninteresting, since $I(E_1, \ldots, E_n) = \infty$ for all $(E_1, \ldots, E_n)$ when $n < m$, and equality holds for all sets when $n = m$. The results of Burchard [2] apply to the smallest nontrivial value of $n$ for given $m$, that is to $n = m+1$, but not to larger $n$. We are aware of no further progress in this direction since that time. This paper treats a situation at the opposite extreme of the spectrum of possibilities, in which $m = 2$ is the smallest dimension of interest, but the number $n \geq 3$ of factors can be arbitrarily large.

Burchard’s inverse theorem has more recently been applied to characterizations of cases of equality in certain inequalities for the Radon transform and its generalizations the $k$–plane transforms $\mathcal{A}$, [7]. Cases of near but not exact equality for the Riesz-Sobolev inequality have been characterized still more recently [3], [6].

As was pointed out by Burchard [3], a satisfactory characterization of cases of equality is possible only if no set $E_j$ is too large relative to the others. This is already apparent for the trilinear expression associated to convolution,

$$I(E_1, E_2, E_3) = \int \int \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(y) \mathbb{1}_{E_3}(x + y) \, dx \, dy;$$

if $|E_3| > |E_1| + |E_2|$ and if $E_1, E_2$ are intervals, then equality holds whenever $E_3$ is the union of an arbitrary measurable set with the algebraic sum of those two intervals.

Consider any expression $I(E_1, \ldots, E_n)$ where the integral is taken over $\mathbb{R}^n$, $E_j \subset \mathbb{R}^1$, and $L_j : \mathbb{R}^m \to \mathbb{R}^1$ are linear and surjective. Set $S_j = \{x \in \mathbb{R}^m : L_j(x) \in E_j\}$. Then $I(E_1, \ldots, E_n)$ is equal to the $m$–dimensional Lebesgue measure of $\bigcap_j S_j$. Define also

$$(2) \quad S^*_j = \{x \in \mathbb{R}^m : L_j(x) \in E^*_j\}.$$

**Definition 2.** Let $(L_j : 1 \leq j \leq n)$ be an $n$–tuple of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}$. An $n$–tuple $(E_j : 1 \leq j \leq n)$ of subsets of $\mathbb{R}^1$ is admissible relative to $(L_j)$ if each $E_j$ is Lebesgue measurable and satisfies $0 < |E_j| < \infty$, and if there exists no index $k$ such that $S^*_k$ contains a neighborhood of $\bigcap_{j \neq k} S^*_j$.

$(E_j)$ is strictly admissible relative to $(L_j)$ if each set $E_j$ is Lebesgue measurable, $0 < |E_j| < \infty$ for all $j$, and there exists no index $k$ such that $S^*_k$ contains $\bigcap_{j \neq k} S^*_j$.

Once the maps $L_j$ are specified, admissibility of $(E_1, \ldots, E_n)$ is a property only of the $n$–tuple of measures $(|E_1|, \ldots, |E_n|)$. Its significance is easily explained. Suppose that $(e_1, \ldots, e_n)$ is a sequence of positive numbers such that an $n$–tuple of sets with these measures is not admissible. The sets $E^*_j, S^*_j$ are determined by $e_j$. Choose an index $k$ such that $S^*_k \supset \bigcap_{j \neq k} S^*_j$. For $j \neq k$ set $E_j = E^*_j$. Choose the unique closed interval $I$ centered at 0 such that the strip $S = \{x : L_k(x) \in I\}$ contains $\bigcap_{j \neq k} S^*_j$, but $|I|$ is as small as possible among all such intervals. Choose $E_k$ to be the disjoint union of $I$ with an arbitrary set of measure $|E_k| - |I|$. Then $I(E_1, \ldots, E_n) = I(E^*_1, \ldots, E^*_n)$, yet $E_k \setminus I$ is an arbitrariness set of the specified measure. Thus without admissibility, extremizing $n$–tuples are highly nonunique.

Admissibility and strict admissibility manifestly enjoy the following invariance property. Let $\Phi$ be an affine automorphism of $\mathbb{R}^m$, and for $j \in \{1, 2, \ldots, n\}$ let $\Psi_j$ be affine automorphisms of $\mathbb{R}^1$. Each composition $\Psi_j \circ L_j \circ \Phi$ is affine mapping from $\mathbb{R}^m$ to $\mathbb{R}^1$.

Write $\Psi_j \circ L_j \circ \Phi(x) = \tilde{L}_j(x) + a_j$ where $\tilde{L}_j : \mathbb{R}^m \to \mathbb{R}^1$ is linear. Define $\tilde{E}_j = \Psi_j(E_j)$ for all $j$. Then $(E_j : 1 \leq j \leq n)$ is admissible relative to $(L_j : 1 \leq j \leq n)$ if and only
Likewise, strict inequality is equivalent to inclusion of $\{ \tilde{L}_j : 1 \leq j \leq n \}$. Strict admissibility is invariant in the same sense.

$A \triangle B$ will denote the symmetric difference of two sets. $|E|$ will denote the Lebesgue measure of a subset of either $\mathbb{R}^1$ or $\mathbb{R}^2$. We say that sets $A, B$ differ by a null set if $|A \triangle B| = 0$.

The following theorem, our main result, characterizes cases of equality, in the situation in which $I(E_1, \ldots, E_n)$ is defined by integration over $\mathbb{R}^2$ and $E_j \subset \mathbb{R}^1$.

**Theorem 1.** Let $n \geq 3$. Let $(L_i : 1 \leq i \leq n)$ be a nondegenerate $n$-tuple of surjective linear maps $L_i : \mathbb{R}^2 \to \mathbb{R}^1$. Let $(E_i : 1 \leq i \leq n)$ be an admissible $n$-tuple of Lebesgue measurable subsets of $\mathbb{R}^1$. If $I(E_1, \ldots, E_n) = I(E_1^*, \ldots, E_n^*)$ then there exist a point $z \in \mathbb{R}^2$, and for each index $i$ an interval $J_i \subset \mathbb{R}$, such that $|E_i \triangle J_i| = 0$ and the center point of $J_i$ equals $L_i(z)$. Conversely, $I(E_1, \ldots, E_n) = I(E_1^*, \ldots, E_n^*)$ in all such cases.

We conjecture that Theorem 1 extends to arbitrary $m \geq 2$.

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**2. On Admissibility Conditions**

For maps $L_j$ from $\mathbb{R}^m$ to the simplest target space $\mathbb{R}^1$, which is the subject of this paper, the most general case treated by Burchard concerns

\[
\int_{\mathbb{R}^m} \mathbb{1}_{E_0}(x_1 + x_2 + \cdots + x_m) \prod_{j=1}^m \mathbb{1}_{E_j}(x_j) \, dx_1 \cdots dx_m,
\]

where $m$ is any integer greater than or equal to 2. Cases of equality are characterized under the admissibility condition

\[
|E_i| \leq \sum_{j \neq i} |E_j| \text{ for all } i \in \{0, 1, 2, \ldots, m\}.
\]

Strict admissibility is the same condition, with inequality replaced by strict inequality for all $i$. This single case subsumes many cases, in light of the invariance property discussed above.

**Lemma 1.** For the expression (3), admissibility in the sense (4) is equivalent to admissibility in the sense of Definition 2. Likewise, the two definitions of strict admissibility are mutually equivalent.

Proof. $S_0^* = \{ x : |\sum_{j=1}^n x_j | \leq \frac{1}{2} |E_0| \}$, while for $j \geq 1$, $S_j^* = \{ x : |x_j| \leq \frac{1}{4} |E_j| \}$. Thus $|E_0| \geq \sum_{j=1}^n |E_j|$ if and only if

$S_0^* \supset \{ x : |x_j| \leq \frac{1}{4} |E_j| \text{ for all } 1 \leq j \leq n \} = \cap_{j=1}^n S_j^*.$

Likewise, strict inequality is equivalent to inclusion of $\cap_{j=1}^n S_j^*$ in the interior of $S_0^*$.

For any $i \in \{1, \ldots, n\}$,

$\cap_{j \neq i} S_j^* = \{ x : |x_k| \leq \frac{1}{2} |E_k| \text{ for all } k \neq i \in \{1, 2, \ldots, n\} \} \cap \{ x : \sum_{l=1}^n x_j \leq \frac{1}{2} |E_0| \}$

while

$S_i^* = \{ x : |x_i| \leq \frac{1}{4} |E_i| \}$.

Therefore $|E_i| \geq \sum_{0 \leq j \neq i} |E_j|$ if and only if $S_i^* \supset \cap_{0 \leq j \neq i} S_j^*$, and strict inequality is equivalent to inclusion of $\cap_{0 \leq j \neq i} S_j^*$ in the interior of $S_i^*$. □
The case $m = 2, n = 3$ of Theorem 1 says nothing new. Indeed, let $(L_j : 1 \leq j \leq 3)$ be a nondegenerate family of linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^1$. By making a linear change of coordinates in $\mathbb{R}^2$ we can make $L_1(x, y) = x$ and $L_2(x, y) = y$, so that

$$I(E_1, E_2, E_3) = \epsilon \int_{\mathbb{R}^2} 1_{E_1}(x) 1_{E_2}(y) 1_{E_3}(ax + by) \, dx \, dy$$

where $a, b$ are both nonzero. This equals

$$c' \int_{\mathbb{R}^2} 1_{E_1}(x/a) 1_{E_2}(y/b) 1_{E_3}(x + y) \, dx \, dy = c' \int_{\mathbb{R}^2} 1_{E_1}(x) 1_{E_2}(y) 1_{E_3}(x + y) \, dx \, dy$$

where $E_j$ are appropriate dilates and reflections of $E_j$.

We will need the following simple result concerning the stability of strict admissibility.

**Lemma 2.** Let $(L_j : 1 \leq j \leq n)$ be a nondegenerate family of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^1$. Let $(E_1, \cdots, E_n)$ be a strictly admissible $n$-tuple of Lebesgue measurable subsets of $\mathbb{R}^1$. There exists $\varepsilon > 0$ such that any $n$-tuple $(E_1', \cdots, E_n')$ of Lebesgue measurable subsets of $\mathbb{R}^1$ satisfying $|E_j' - |F_j|| < \varepsilon$ for all $j \in \{1, 2, \cdots, n\}$ is strictly admissible.

**Proof.** Suppose that no $\varepsilon$ satisfying the conclusion exists. Then there exists a sequence of $n$-tuples $((E_{j, \nu}) : \nu \in \mathbb{N})$ such that $|E_{j, \nu}| \to |E_j|$ as $\nu \to \infty$, for each $j \in \{1, 2, \cdots, n\}$, and such that for each $\nu \in \mathbb{N}$, $(E_{n, \nu} : 1 \leq j \leq n)$ is not admissible.

Let $E_{j, \nu}^\ast \subset \mathbb{R}^1$ be the associated closed intervals centered at 0. Let

$$S_{j, \nu} = \{x \in \mathbb{R}^m : L_j(x) \in E_{j, \nu}^\ast\}$$

be the associated closed strips. The failure of strict admissibility means that for each $\nu$ there exists $J(\nu)$ such that $S_{j, \nu} \cap \bigcap_{j \neq J(\nu)} S_{j, \nu}'$. By passing to a subsequence we may assume that $J(\nu) \equiv J$ is independent of $\nu$.

Since $|E_{j, \nu}| \to |E_j|$, the closed strips $S_{j, \nu}$ converge to the closed strips $S_j^\ast$ as $\nu \to \infty$, in such a way that it follows immediately that $S_j^\ast \supset \bigcap_{j \neq J} S_j^\ast$. Therefore $(E_1, \cdots, E_n)$ is not strictly admissible.

\section{Truncation}

**Definition 3.** Let $E \subset \mathbb{R}^1$ have finite measure. Let $\alpha, \beta > 0$. If $\alpha + \beta \leq |E|$ then the truncation $E(\alpha, \beta)$ of $E$ is

$$E(\alpha, \beta) = E \cap [a, b]$$

where $a, b \in \mathbb{R}$ are respectively the minimum and the maximum real numbers that satisfy

$$|E \cap (-\infty, a]| = \alpha \text{ and } |E \cap [b, \infty)| = \beta.$$

In the degenerate case in which $\alpha + \beta = |E|$, $E(\alpha, \beta)$ has Lebesgue measure equal to zero, and may be empty or nonempty. According to our conventions, $E(\alpha, \beta)^\ast = \{0\}$ in this circumstance, in either case. This convention will be convenient below.

**Lemma 3.** Let $k \geq 1$. Let $\{E_i : i \in \{1, 2, \cdots, k\}\}$ be a finite collection of Lebesgue measurable subsets of $\mathbb{R}^1$ with positive, finite Lebesgue measures. Let $\alpha, \beta > 0$, and suppose that $|E_i| \geq \alpha + \beta$ for each index $i$. If $\bigcap_{i=1}^k E_i(\alpha, \beta) \neq \emptyset$ then

$$\int_{\mathbb{R}} \prod_{i=1}^k 1_{E_i}(y) \, dy \leq \alpha + \beta + \int_{\mathbb{R}} \prod_{i=1}^k 1_{E_i(\alpha, \beta)}(y) \, dy.$$

If $E_i$ are closed intervals and if $\bigcap_{i=1}^k E_i(\alpha, \beta) \neq \emptyset$ then equality holds in inequality (6).
This generalizes a key element underpinning the work of Burchard \[3\], which in turn is related, but not identical, to the construction employed by Riesz \[8\].

Proof. For each index \(i\), let \(a_i, b_i \in \mathbb{R}\) respectively be the smallest and the largest real numbers satisfying \(|E_i \cap (-\infty, a_i)| = \alpha\) and \(|E_i \cap [b_i, \infty)| = \beta\). Thus \(E_i = [a_i, b_i]\). Let \(a = \max_{i} a_i\) and \(b = \min_{i} b_i\). Then \(\bigcap_i E_i(\alpha, \beta) = (\bigcap_i E_i) \cap [a, b]\). It is given that \(\bigcap_i E_i(\alpha, \beta)\) is nonempty, so \(a \leq b\).

Thus
\[
\int \prod_{i=1}^{k} 1_{E_i(\alpha, \beta)}(y) \, dy = |\bigcap_i E_i(\alpha, \beta)| = |(\bigcap_i E_i) \cap [a, b]|.
\]

Therefore
\[
\int \prod_{i=1}^{k} 1_{E_i}(y) \, dy - \int \prod_{i=1}^{k} 1_{E_i(\alpha, \beta)}(y) \, dy = |(\bigcap_i E_i) \setminus [a, b]|
\]
\[
= |(\bigcap_i E_i) \cap (-\infty, a)| + |(\bigcap_i E_i) \cap (b, \infty)|.
\]

Choose \(l\) such that \(a_l = a\). Then \((\bigcap_i E_i) \cap (-\infty, a) \subset E_l \cap (-\infty, a)\) and hence
\[
|(\bigcap_i E_i) \cap (-\infty, a)| \leq |E_l \cap (-\infty, a)| = \alpha.
\]

Similarly \(|(\bigcap_i E_i) \cap (b, \infty)| \leq \beta\).

For the converse, suppose that the \(E_i\) are closed intervals, and that \(\bigcap_i E_i(\alpha, \beta) \neq \emptyset\).

Then \(\bigcap_i E_i(\alpha, \beta) = [a, b]\) where \(a \leq b\), as above. In the same way, \(\bigcap_i E_i = [a^*, b^*]\) where \(a^*\) is the maximum of the left endpoints of the intervals \(E_i\), and \(b^*\) is the minimum of their right endpoints. Obviously \(a^* = a - \alpha\) and \(b^* = b + \beta\). \(\square\)

The next lemma is evident.

Lemma 4. Let \(0 \leq \alpha, \beta < \infty\). Let \(\{I_k\}\) be a collection of closed bounded subintervals of \(\mathbb{R}\) satisfying \(|I_k| \geq \alpha + \beta\). Suppose that \(\bigcap_k I_k(\alpha, \beta) \neq \emptyset\), and that \(J(\alpha, \beta) \supset \bigcap_k I_k(\alpha, \beta)\). Then \(J \supset \bigcap_k I_k\).

4. Deformation

We change notation: The number of sets \(E_j\) will be \(n + 1\), and the index \(j\) will run through \(\{0, 1, \cdots, n\}\). The index \(j = 0\) will have a privileged role.

Consider a functional
\[
I(E_0, \cdots, E_n) = \int_{\mathbb{R}^2} \prod_{j=0}^{n} 1_{E_j}(L_j(x)) \, dx,
\]
with \(\{L_j : 0 \leq j \leq n\}\) nondegenerate. The invariance under changes of variables noted above, together with this nondegeneracy, make it possible to bring this functional into the form
\[
I(E_0, \cdots, E_n) = c \int_{\mathbb{R}} 1_{E_0}(x) \int_{\mathbb{R}} \prod_{j=1}^{n} 1_{E_j}(y + t_j x) \, dy \, dx
\]
where \(c\) is a positive constant, and the \(t_j\) are pairwise distinct. This is accomplished by means of a linear change of variables in \(\mathbb{R}^2\) together with linear changes of variables in each of the spaces \(\mathbb{R}_j^1\) in which the sets \(E_j\) lie. The sets \(E_j\) which appear here are images

\[\footnote{Riesz considers only the case of three sets, truncates all three in this fashion, uses only the case \(\alpha = \beta\), and works directly with the integral over \(\mathbb{R}^2\) which defines \(I(E_1, \cdots, E_n)\), rather than with one-dimensional integrals.} \]
of the original sets $E_j$ under invertible linear mappings of $\mathbb{R}_1^j$, but equality holds in the inequality (1) for this rewritten expression $I(E_0, \ldots, E_n)$ if and only if it holds for the original expression, and the property of admissibility is preserved.

With $I(E_0, \ldots, E_n)$ written in this form,

\[
S_0^* = \{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{r} |E_0|\}
\]

\[
S_j^* = \{(x, y) \in \mathbb{R}^2 : |y + t_j x| \leq \frac{1}{r} |E_j|\} \quad \text{for } 1 \leq j \leq n.
\]

Let $\pi : \mathbb{R}^2 \to \mathbb{R}^1$ be the projection $\pi(x, y) = x$. Define

\[
E_j(r) = E_j \left( \frac{1}{2} r, \frac{1}{2} r \right) \quad \text{for } j \geq 1 \text{ and } 0 \leq r \leq |E_j|,
\]

\[
E_j(0) = E_j, \quad E_0(r) = E_0.
\]

Thus $|E_j(r)| = |E_j| - r$ for $j \geq 1$. Let $S_j^*(r)$ be the associated strips; $S_0^*(r) = S_0^*$ while for $j \geq 1$,

\[
S_j^*(r) = \{(x, y) \in \mathbb{R}^2 : |y + t_j x| \leq \frac{1}{r} |E_j| - \frac{1}{r} r\}
\]

for $0 \leq r \leq \min_j |E_j|$. Thus if $j \geq 1$ and $r = |E_j|$ then $S_j^*(r)$ is a line in $\mathbb{R}^2$.

The cases $n \geq 3$ of the next lemma will later be used to prove Theorem 6 by induction on $n$.

**Lemma 5.** Let $n \geq 2$. Let $\{E_j : 0 \leq j \leq n\}$ be a strictly admissible family of $n + 1$ Lebesgue measurable subsets of $\mathbb{R}^1$. Then there exists $\bar{r} \in (0, \min_{1 \leq j \leq n} |E_j|)$ such that

\[
(E_j(\bar{r})) : 0 \leq j \leq n \quad \text{is admissible}
\]

\[
S_0^* \supset \cap_{j \geq 1} S_j^*(\bar{r}).
\]

The second conclusion says in particular that $(E_j(\bar{r}) : 0 \leq j \leq n)$ fails to be strictly admissible. Because admissibility is a property of the measures of sets only with no reference to their geometry, Lemma 5 concerns deformations of intervals centered at 0 and of associated strips, not of more general sets.

**Proof.** Define $\bar{r}$ to be the infimum of the set of all $r \in [0, \min_{k \geq 1} |E_k|]$ for which $(E_j(r) : 0 \leq j \leq n)$ fails to be strictly admissible. If $r = \min_{k \geq 1} |E_k| = |E_i|$ then $|E_i(r)| = 0$ and therefore $(E_j(r) : 0 \leq j \leq n)$ is not strictly admissible. Thus $\bar{r}$ is defined as the infimum of a nonempty set, and $0 \leq \bar{r} \leq \min_{k \geq 1} |E_k|$.

Since $(E_0, \ldots, E_n) = (E_0(0), \ldots, E_n(0))$ is strictly admissible, and since strict admissibility is stable under small perturbations of the type under consideration, the $(n + 1)$-tuple $(E_0(r), \ldots, E_n(r))$ is strictly admissible for all sufficiently small $r \geq 0$. Therefore $\bar{r} > 0$.

Consequently the definition of $\bar{r}$ implies one of two types of degeneracy: Either $|E_i^*(\bar{r})| = 0$ for some $l \geq 1$, or there exists $i \in \{0, 1, \ldots, n\}$ such that

\[
S_i^*(\bar{r}) \supset \cap_{j \neq i} S_j^*(\bar{r}).
\]

**Claim 1.** The inclusion (7) must hold for at least one index $i \in \{0, 1, \ldots, n\}$.

**Proof.** If not, then the other alternative must hold; there exists an index $l$ such that $|E_l^*(\bar{r})| = 0$. In that case, $S_l^*(\bar{r})$ is by definition equal to the line $\{(x, y) : y + t_l x = 0\}$, which contains 0. For each index $j \neq l$, the intersection of $S_j^*(\bar{r})$ with $L$ is a nonempty closed interval of finite nonnegative length, centered at 0. Choose $i \neq l$ for which the length of $S_i^*(\bar{r}) \cap L$ is maximal. Then $S_i^*(\bar{r})$ contains $S_i^*(\bar{r}) \cap L$, which in turn contains $S_j^*(\bar{r}) \cap L$ for every $j \notin \{i, l\}$. Therefore (7) holds for this index $i$. $\square$
Let

\[ K = \cap_{j=1}^{n} S_j^\star (\bar{r}), \]

which is a nonempty balanced convex subset of \( \mathbb{R}^2 \). \( K \) is compact, by the nondegeneracy hypothesis, since \( E_j^\star \) are compact intervals.

\( \pi(K) \subset \mathbb{R} \) is a compact interval centered at 0, as is \( E_0^\star \). Therefore \( \pi(K) \subset E_0^\star \), or \( E_0^\star \subset \pi(K) \).

**Claim 2.** If \( \pi(K) \supset E_0^\star \) and if an index \( i \) satisfies (7), then \( i = 0 \).

**Proof.** Suppose that \( \pi(K) \supset E_0^\star \) and that \( i \neq 0 \) satisfies (7). For \( 1 \leq j \leq n \) define the closed intervals

\[ J(x, j, r) = \{ y \in \mathbb{R} : (x, y) \in S_j^\star (r) \} \subset \mathbb{R}. \]

For any \( x \in \pi(K) \), these intervals have at least one point in common. Since \( S_i^\star (\bar{r}) \supset \cap_{j \neq i} S_j^\star (\bar{r}) \),

\[ J(x, i, \bar{r}) \supset \cap_{j \neq i} J(x, j, \bar{r}) \]

for any \( x \in E_0^\star \).

Therefore by Lemma 4

\[ J(x, i, 0) \supset \cap_{1 \leq j \neq i} J(x, j, 0) \]

for all \( x \in E_0^\star \).

Since \( S_0^\star = \pi^{-1} (E_0^\star) \) it then follows that

\[ S_i^\star \supset S_i^\star \cap \pi^{-1} (E_0^\star) \supset \cap_{1 \leq j \neq i} S_j^\star \cap \pi^{-1} (E_0^\star) = \cap_{0 \leq j \neq i} S_j^\star, \]

contradicting the hypothesis that \( (E_0, \cdots, E_n) \) is strictly admissible. \( \square \)

**Claim 3.** \( \pi(K) \) cannot properly contain \( E_0^\star \).

**Proof.** Suppose that \( \pi(K) \) properly contains \( E_0^\star \). By the preceding Claim, (7) holds for \( i = 0 \). Let \( x \in \pi(K) \setminus E_0^\star \). There exists \( y \in \mathbb{R} \) such that \((x, y) \in K\). Since \( x \notin E_0^\star \), \((x, y) \notin S_0^\star = \pi^{-1} (E_0^\star) \). Therefore \( K = \cap_{j \geq 1} S_j^\star (\bar{r}) \) is not contained in \( S_0^\star = S_0^\star (\bar{r}) \), contradicting (7). \( \square \)

**Claim 4.** \( \pi(K) \) is not properly contained in \( E_0^\star \).

**Proof.** If \( \pi(K) \) is properly contained in \( E_0^\star \), then it is contained in the interior of \( E_0^\star \), since each of these sets is a closed interval centered at 0. Consequently \( K \) is contained in the interior of \( \pi^{-1} (E_0^\star) = S_0^\star = S_0^\star (\bar{r}) \); that is, \( \cap_{j \geq 1} S_j^\star (\bar{r}) \) is contained in the interior of \( S_0^\star \). Therefore for every \( r' \prec \bar{r} \) sufficiently close to \( \bar{r} \), \( \cap_{j \geq 1} S_j^\star (r') \) is contained in \( S_0^\star \). Thus \( (E_0 (r'), \cdots, E_n (r')) \) fails to be strictly admissible. This contradicts the definition of \( \bar{r} \) as the infimum of the set of all \( r \) for which \( (E_0 (r), \cdots, E_n (r)) \) fails to be strictly admissible. \( \square \)

Combining the above four claims, we conclude that (7) holds for \( i = 0 \) and for no other index, and that \( \pi(K) = E_0^\star \).

**Claim 5.** \( |E_j (\bar{r})| > 0 \) for every index \( j \in \{0, 1, \cdots, n\} \).

**Proof.** If \( |E_l (\bar{r})| = 0 \) then since \( E_0 (\bar{r}) = E_0 \), the index \( l \) cannot equal 0. \( S_j^\star (\bar{r}) \) is the line \( \mathcal{L} = \{(x, y) : y + t_i x = 0\} \). For each \( j \neq l \), \( S_j^\star (\bar{r}) \cap \mathcal{L} \) is a closed subinterval of \( \mathcal{L} \) centered at 0. Therefore \( K \) is equal to the smallest of these subintervals.

Since \( \pi(K) = E_0^\star \), and since \( \pi : \mathcal{L} \to \mathbb{R} \) is injective, \( K \) must equal \( \mathcal{L} \cap S_0^\star = S_0^\star (\bar{r}) \cap \mathcal{L} \). Therefore \( S_j^\star (\bar{r}) \cap \mathcal{L} \supset S_0^\star (\bar{r}) \cap \mathcal{L} \). Therefore every \( i \notin \{0, l\} \) satisfies (7). Since \( n \geq 2 \) there are at least three indices \( 0 \leq i \leq n \), so there exists at least

\footnote{This apparently innocuous step is responsible for the restriction \( m = 2 \) in our main theorem.}
one index \( i \notin \{0, l\} \). But we have shown that the only such index is \( i = 0 \), so this is a contradiction. \( \square \)

To conclude the proof of Lemma 5 it remains to show that \( (E_0(\bar{r}), \cdots, E_n(\bar{r})) \) must be admissible. We have shown that \(|E_j(\bar{r})| > 0 \) for all \( j \). The failure of admissibility is a stable property for sets with positive measures, so if \( (E_0(\bar{r}), \cdots, E_n(\bar{r})) \) were not admissible then there would exist \( 0 < r < \bar{r} \) for which \( (E_0(r), \cdots, E_n(r)) \) was not admissible, contradicting the minimality of \( \bar{r} \). \( \square \)

5. Conclusion of the Proof

The proof of Theorem 1 proceeds by induction on the degree of multilinearity of the form \( I \), that is, on the number of sets appearing in \( I(E_1, \cdots, E_n) \). The base case \( n = 3 \) is a restatement of the one-dimensional case of Burchard’s theorem, in its invariant form, since the two definitions of admissibility are equivalent.

Assuming that the result holds for expressions involving \( n \) sets \( E_j \), we will prove it for expressions involving \( n + 1 \) sets. Let \( (E_0, \cdots, E_n) \) be any admissible \( n + 1 \)-tuple of sets satisfying \( I(E_0, \cdots, E_n) = I(E_0^*, \cdots, E_n^*) \).

Consider first the case in which \( (E_j : 0 \leq j \leq n) \) is not strictly admissible. Then there exists \( i \) such that \( S_i^* \supset \cap_{j \neq i} S_j^* \). By permuting the indices, we may assume without loss of generality that \( i = 0 \). Then

\[
I(E_0, \cdots, E_n) \leq I(\mathbb{R}, E_1, \cdots, E_n) \leq I(\mathbb{R}, E_1^*, \cdots, E_n^*) = I(E_0^*, \cdots, E_n^*),
\]

so \( I(\mathbb{R}, E_1, \cdots, E_n) = I(\mathbb{R}, E_1^*, \cdots, E_n^*) \).

Defining

\[
J(E_1, \cdots, E_n) = I(\mathbb{R}, E_1, \cdots, E_n),
\]

we have \( J(E_1, \cdots, E_n) = J(E_1^*, \cdots, E_n^*) \). Now \( (E_1, \cdots, E_n) \) is admissible relative to \( \{L_j : 1 \leq j \leq n\} \). For if not, then there would exist \( k \in \{1, 2, \cdots, n\} \) for which \( S_k^* \) properly contained \( \cap_{1 \leq j \neq k} S_j^* \). Since \( S_k^* \supset \cap_{j \geq 1} S_j^* \),

\[
\cap_{1 \leq j \neq k} S_j^* = S_k^* \cap (\cap_{1 \leq j \neq k} S_j^*).
\]

so \( S_k^* \) would properly contain \( \cap_{0 \leq j \neq k} S_j^* \), contradicting the hypothesis that \( (E_0, \cdots, E_n) \) is admissible.

By the induction hypothesis, equality in the rearrangement inequality for \( J \) can occur only if \( E_j \) differs from an interval by a null set, for each \( j \geq 1 \). Moreover, there must exist a point \( z \in \mathbb{R}^2 \) such that for every \( j \in \{1, 2, \cdots, n\} \), \( L_j(z) \) equals the center of the interval corresponding to \( E_j \).

For \( j = 1 \), replace \( E_j \) by the unique closed interval which differs from \( E_j \) by a null set. By an affine change of variables in \( \mathbb{R}^2 \), we can write \( I(E_0, \cdots, E_n) \) in the form

\[
c \int E_0(x) \int \prod_{j=1}^{n} \mathbb{1}_{E_j}(y + t_j x) \, dy \, dx
\]

where \( c \in (0, \infty) \) and \( t_j \in \mathbb{R} \), and now for each \( j \geq 1 \), \( E_j \) is an interval centered at 0. The inner integral defines a nonnegative function \( F \) of \( x \in \mathbb{R} \) which is continuous, nonincreasing on \([0, \infty)\), even, and has support equal to a certain closed bounded interval centered at 0. The condition that \( (E_0, \cdots, E_n) \) is admissible but \( S_k^* \supset \cap_{j=1}^{n} S_j^* \) means that this support is equal to the closed interval \( E_0^* \). Among sets \( E \) satisfying \(|E| = |E_0|\), \( \int_E F < \int_{\mathbb{R}} F \) unless \( E \) differs from \( E_0^* \) by a null set. We have thus shown that in any case of nonstrict admissibility, all the sets \( E_j \) differ from intervals by null sets, and the centers
$c_j$ of these intervals are coherently situated, in the sense that $c_j = L_j(z)$ for a common point $z \in \mathbb{R}^2$.

Next consider the case in which $(E_0, \cdots, E_n)$ is strictly admissible. Change variables to put $I(E_0, \cdots, E_n)$ into the form (10). This replaces the sets $E_j$ by their images under certain invertible linear transformations, but does not affect the validity of the two conclusions of the theorem.

Let $\vec{r}$ be as specified in Lemma 3. Set $\tilde{E}_j = E_j(\vec{r})$, and recall that $\tilde{E}_0 = E_0$. Let $\tilde{S}_j$ be the strips in $\mathbb{R}^2$ associated to the rearrangements $\tilde{E}_j$. By Lemma 3

\[
\int_{\mathbb{R}} \prod_{j=1}^n 1_{E_j}(y + t_j x) \, dy \leq \vec{r} + \int_{\mathbb{R}} \prod_{j=1}^n 1_{\tilde{E}_j}(y + t_j x) \, dy
\]

for each $x \in E_0$. Multiplying both sides by $1_{E_0}(x)$ and integrating with respect to $x$ gives

\[
\int_{\mathbb{R}} 1_{E_0}(x) \int_{\mathbb{R}} \prod_{j=1}^n 1_{E_j}(y + t_j x) \, dy \, dx \leq \vec{r}|E_0| + \int_{\mathbb{R}} 1_{E_0}(x) \int_{\mathbb{R}} \prod_{j=1}^n 1_{\tilde{E}_j}(y + t_j x) \, dy \, dx.
\]

Thus

\[
I(E_0, \ldots, E_n) \leq \vec{r}|E_0| + I(E_0, \tilde{E}_1, \ldots, \tilde{E}_n).
\]

By the general rearrangement inequality applied to the $n + 1$–tuple $(E_0, E_1, \ldots, E_n)$,

\[
\vec{r}|E_0| + I(E_0, \tilde{E}_1, \ldots, \tilde{E}_n) \leq \vec{r}|E_0| + I(E_0^*, \tilde{E}_1^*, \ldots, \tilde{E}_n^*).
\]

Since $(\tilde{E}_j : 0 \leq j \leq n)$ is admissible, for each $x \in E_0$ there exists $y$ such that $(x, y) \in \bigcap_{j \geq 1} \tilde{S}_j$. Therefore by the second conclusion of Lemma 3,

\[
\int_{\mathbb{R}} \prod_{i=1}^n 1_{E^*_i}(y + t_j x) \, dy = \vec{r} + \int_{\mathbb{R}} \prod_{i=1}^n 1_{\tilde{E}^*_i}(y + t_j x) \, dy.
\]

Integrating both sides of this inequality with respect to $x \in E_0^*$ gives

\[
I(E_0^*, E_1^*, \ldots, E_n^*) = \vec{r}|E_0^*| + I(E_0^*, \tilde{E}_1^*, \ldots, \tilde{E}_n^*).
\]

Combining (11), (12), and (13) yields

\[
I(E_0, \ldots, E_n) \leq \vec{r}|E_0| + I(E_0, \tilde{E}_1, \ldots, \tilde{E}_n) \leq \vec{r}|E_0| + I(E_0^*, \tilde{E}_1^*, \ldots, \tilde{E}_n^*).
\]

We are assuming that $I(E_0, E_1, \ldots, E_n) = I(E_0^*, \tilde{E}_1^*, \ldots, \tilde{E}_n^*)$, so equality holds in each inequality in this chain. Hence

\[
I(E_0, \tilde{E}_1, \ldots, \tilde{E}_n) = I(E_0^*, \tilde{E}_1^*, \ldots, \tilde{E}_n^*).
\]

Thus the $n + 1$–tuple $(E_0, \tilde{E}_1, \ldots, \tilde{E}_n)$ is admissible but not strictly admissible, and achieves equality in the inequality (11). This situation was analyzed above. Therefore we conclude that $E_0$ coincides with an interval, up to a null set.

The same reasoning can be applied to $E_j$ for all $j$, by permuting the indices, so each of the sets $E_j$ is an interval up to a null set. In this case (returning to the above discussion in which the index $j = 0$ is singled out), each interval $E_j$ has the same center as $E_j(\vec{r})$. The discussion above has established that the centers of the intervals $E_j(\vec{r})$ are coherently situated.
REFERENCES

[1] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, A general rearrangement inequality for multiple integrals, J. Functional Analysis 17 (1974), 227–237

[2] A. Burchard, Cases of equality in the Riesz rearrangement inequality, Thesis (Ph.D.) Georgia Institute of Technology. 1994. 94 pp, ProQuest LLC

[3] _________, Cases of equality in the Riesz rearrangement inequality, Ann. of Math. (2) 143 (1996), no. 3, 499–527

[4] M. Christ, Extremizers of a Radon transform inequality, preprint math.CA arXiv:1106.0719 to appear in proceedings of Princeton symposium in honor of E. M. Stein

[5] _________, An approximate inverse Riesz-Sobolev rearrangement inequality, preprint, math.CA arXiv:1112.3715

[6] _________, Near equality in the Riesz-Sobolev inequality, in preparation.

[7] T. Flock, Uniqueness of extremizers for an endpoint inequality of the k-plane transform, preprint, math.CA arXiv:1307.6551

[8] F. Riesz, Sur une inégalité intégrale, Journal London Math. Soc. 5 (1930), 162–168

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