An Introduction to 2d Gravity and Solvable String Models

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Continuum and discrete approaches to 2d gravity coupled to $c < 1$ matter are reviewed.

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1. Introduction

Two-dimensional gravity has undergone a thorough examination over the last few
years, especially with the emergence of efficient calculational techniques stemming from
the matrix model approach. Two dimensions is an arena where rather complex phenom-
ena (confinement, chiral symmetry breaking, integrability, nonperturbative phenomena,
solitons; the list is extensive) can be stripped of complications of higher dimensional kine-
matics and dynamics while hopefully retaining many of the physical features of the problem
of interest. Thus we study two dimensional gravity as a model for exploring the structure
and formalism of quantum gravity: the wavefunction of the universe, the Wheeler-DeWitt
equation[1], the meaning of measurement in quantum gravity, the statistical properties of
the metric and matter in a fluctuating geometry, etc. Indeed the collection of solvable 2d
gravity-matter systems provides a rich laboratory for the investigation of these issues. In
addition, since unified strings are by definition coordinate invariant 2d quantum field the-
ories these systems are equally well regarded as solvable models of string theory, where one
might begin to formulate a useful string field theory[2], study nonperturbative effects[3],
strong coupling phenomena, etc.

There are several ways one might approach the continuum theory of 2d gravity. One
is to write down a field theory on 2d metrics and matter[4], regulate it covariantly, and
try to find a consistent renormalization to the continuum limit[5][6]. Another is to dis-
cretize the theory and study the fluctuations of the discrete geometry[7], then try to take
the continuum limit of the random lattice model (a method that has also been used for
strings embedded in higher dimensional spacetimes[8]). Both should yield the same results

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if indeed 2d gravity is universal, \textit{i.e.} if there is a second order phase transition for the fluctuating surfaces in some region of the space of coupling constants (cosmological constant, topological coupling, boundary cosmological constant, etc.). The former line of attack adheres more closely to the standard conceptual framework of gravity, yet in two dimensions the latter formulation has yielded more success to date. This success comes about largely because in two dimensions it is not difficult to enumerate the lattice configurations, which are gauge invariant.

These lecture notes introduce the present understanding of 2d gravity/solvable string models, beginning in section 2 with a review of the continuum formulation in terms of the functional integral over metrics. While there is little that can be computed exactly, we argue that much of the classical structure should persist upon quantization; and that the familiar structures of rational conformal field theory might appear, albeit in a much more subtle and complicated way. In section 3 we discuss the behavior of the theory under rescalings of the 2d metric\cite{9} as well as the related dynamics in the zero mode sector\cite{3} [10] \cite{11} \cite{12}. In section 4 we switch to the discrete formulation. Here the partition function of random surfaces can be recast as an integral over $N \times N$ matrices\cite{7} where in the continuum limit $N \to \infty$\cite{13}. Both the lattice\cite{14} and continuum\cite{13} \cite{15} \cite{16} theories exhibit a rich integrability structure. The scaling behavior predicted by the continuum approach is recovered as well as the zero mode dynamics\cite{10} \cite{12}.

2. The effective action for gravity and its quantisation

A crucial feature of 2d quantum gravity is that it must be scale invariant as well as reparametrization invariant\cite{17}; local shifts $g_{ab} \to e^\epsilon g_{ab}$ of the metric scale factor are simply shifts of an integration variable in the functional integral over metrics. The effect of such a shift is to produce the trace of the stress tensor, which we can generally expand in terms of scaling fields

$$\langle T_{zz} \rangle = \frac{\delta}{\delta \epsilon(z)} Z = \int e^{-S} \beta_i O_i .$$

(2.1)

The $\beta_i$ are the beta functions of the theory. The result (2.1) must vanish apart from possible contact terms with local operators. Often we are interested in gravity coupled to some number $d$ of scalar matter fields

$$S = \frac{1}{4\pi} \int \sqrt{g} \left[ T(X) + R^{(2)} D(X) + \partial_a X^\mu \partial_b X^\nu (G_{\mu\nu}(X) g^{ab} + B_{\mu\nu}(X) e^{ab}) \right] .$$

(2.2)
in which case the vanishing trace of the stress tensor enforces a set of conditions on the spacetime fields which are the equations of motion of string theory. One may adopt either of two points of view: In ‘critical’ string theory, one sets \( \frac{\delta}{\delta \epsilon} Z_{\text{matter}} = 0 \); then one has an additional symmetry – Weyl invariance – which one can quotient by, so that one integrates over the gravitational measure \( \frac{\mathcal{D} g_{ab} \mathcal{D} X}{\text{Diff} \times \text{Weyl}} \). In this way one regards all components of the 2d metric as gauge degrees of freedom. On the other hand, from the ‘noncritical’ string point of view, one can trivially achieve scale invariance by integrating over all possible scale factors \( \frac{\mathcal{D} g_{ab} \mathcal{D} X}{\text{Diff}} \). This is equivalent to a particular class of solutions to critical string theory if we regard the local scale factor as another scalar field like the \( X \)'s; perhaps it is completely equivalent if we complexify the space of metrics. In either case the object is to compute and solve the equations of vanishing stress tensor. There are two common methods of calculation: the spacetime weak field expansion[19] and the 2d loop expansion[20]. In the former one expands around a known fixed point (e.g. free field theory in \( d \) spacetime dimensions)

\[
S = \frac{1}{4\pi} \int \partial X \bar{\partial} X + \delta T(X) + \partial X^\mu \bar{\partial} X^{\nu} (G_{\mu\nu} \delta D(X) + \delta G_{\mu\nu}(X) + \delta B_{\mu\nu}(X)) .
\]

Bringing down powers of the composite operator perturbations will reliably find nearby fixed points (unless one has chosen a singular parametrization of the coupling space). The origin of the beta function is the overall scale divergence

\[
\int \frac{d^2 \lambda}{|\lambda|^2} \lambda^{L_0 + \bar{L}_0} \int_{N-1}^{N} \left\langle \prod_{i=1}^{N} O_i(\lambda z) \right\rangle
\]

The similarity of this expression to the Koba-Nielsen prescription for the calculation of the string S-matrix is not coincidental[21]. Indeed, the effective action of string theory (actually any field theory) is determined from the S-matrix by subtracting the contributions of intermediate on-shell poles. In the Koba-Nielsen formula these are the logarithmic subdivergences in the integrations over the locations of composite operators; the beta functions are the equations of motion following from this effective action. Note that the kinetic operator \( \partial^2 S/\partial g_i \partial g_j = \partial \beta_i/\partial g_j \) for small fluctuations is the anomalous dimension operator – the linearized scaling operator \( L_0 + \bar{L}_0 - 2 \). The disadvantage of the spacetime weak field expansion is that one must work in a particular coordinate basis in 2d field space, so spacetime general coordinate invariance (invariance under 2d field redefinitions of the \( X \)'s) is not manifest. This is an advantage of the 2d loop expansion; rather than
working in a specific basis of tensor fields on the spacetime manifold, one expands in small fluctuations of the coordinates \( X \), which can be made manifestly generally covariant[20]. This perturbation series is an expansion in spacetime variation of the string coordinates relative to the 2d Planck constant, the inverse string tension \( \alpha' \). A superposition of these two methods yields the beta functions[20]

\[
\begin{align*}
\beta^G_{\mu\nu} &= R_{\mu\nu} - 2\nabla_\mu \nabla_\nu D + \nabla_\mu T \nabla_\nu T = 0 \\
\beta^D &= \frac{26 - d}{3\alpha'} + R + 4(\nabla D)^2 - 4\nabla^2 D + (\nabla T)^2 + V(T) = 0 \\
\beta^T &= -2\nabla^2 T + 4\nabla D \nabla T + V'(T) = 0
\end{align*}
\]  

in a double expansion in field strength and \( \alpha' \); \( V(T) \) is a generic potential \( V(T) = \frac{1}{2} T^2 + O(T^3) \). Both expansions are required here; the loop expansion for general covariance, and the weak field expansion to incorporate the tachyon.

From the critical string viewpoint, we are interested in strings in \( d \) spacetime dimensions. The equations (2.3) are solved by (\( \phi = X^0 \), say)

\[
\begin{align*}
T &= \frac{1}{2\gamma^2} e^{\gamma \phi} \\
D &= \frac{1}{\gamma} \phi \\
G_{\mu\nu} &= \delta_{\mu\nu}
\end{align*}
\]  

(2.4)

Although each equation of motion in (2.3) is satisfied at its leading nontrivial order in powers of \( T \), none is solved at subleading order. However we are only considering the lowest order equations; one might hope that higher orders correct the problem. After all, the \( \alpha' \) (loop) expansion is not valid here since \( \gamma \) is not small; the kinetic term in the tachyon beta function is only found after a resummation of loops, but then we have no reason to ignore terms involving the gradient of the dilaton field. Fortunately we are looking for a solution with rather special properties: it is (2.4) at lowest order and its renormalization involves only \( \phi \)-dependent fields. Since \( \nabla_\phi T \propto T \), the exact solutions \( \hat{G}, \hat{D}, \hat{T} \) are power series in \( T \) (assuming the weak field expansion is summable). Therefore we can find a field redefinition – the reversion of the power series \( \hat{G}(T), \hat{D}(T), \hat{T}(T) \) – such that (2.4) is the exact solution[1]. The importance of \( \nabla T \propto T \) is that the field redefinition required is local in spacetime.

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1 This argument is due to T. Banks.
The noncritical string viewpoint is somewhat more helpful in this situation. Here we have $d - 1$ string coordinates coupled to a dynamical metric $g_{ab} = e^{\gamma \phi} \hat{g}_{ab}$ where $\hat{g}_{ab}$ is a fixed metric (up to moduli). For $d - 1$ free scalar fields we have

$$ \beta = \frac{26-d}{3\alpha'} \sqrt{g} R^{(2)} + \frac{\mu^2}{2\gamma^2} e^{\gamma \phi} $$

Integrating $\delta S_{eff}/\delta \phi = \beta$ gives an effective classical action, the Liouville action

$$ S_{eff} = \frac{1}{4\pi} \int \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{4} \phi R^{(2)} + \frac{\mu^2}{2\gamma^2} e^{\gamma \phi} \right] $$

for the dynamics of the metric. The integral over metrics restores scale invariance if we can succeed in correctly performing the functional integral or otherwise quantizing the theory. From critical string considerations, one might expect a problem since the tachyon stress-energy appears to feed into the metric beta function. There is in fact a renormalization prescription such that $S_{eff}$ is conformally invariant\[17\]. Since the metric is dynamical, the coordinate regularization scale and the covariant scale are related by a fluctuating field: $\delta = e^{\gamma \phi/2} \epsilon$. Therefore a regulated action should be, e.g.

$$ S^{Reg}_{eff} = \frac{1}{8\pi} \int d^2 \bar{z} \partial \bar{\phi} \left( e^{\gamma \phi} \partial^2 \right) \bar{\partial} \phi + Q R^{(2)} \phi + \frac{\mu^2}{\gamma^2} e^{\gamma \phi}, $$

and we should calculate $\delta S_{eff}/\delta \phi = \beta$ incorporating the effects of the regulator. In the loop expansion, normal ordering an exponential means resumming self-contractions; this introduces cutoff-dependence, and therefore $\phi$-dependence. One finds that the entire effect of imposing $\beta_{eff} = 0$ is to renormalize the parameters in the Liouville lagrangian: $\mu \rightarrow \mu_{ren}$ (which can be absorbed in a shift of $\phi$), and $Q \rightarrow Q_{ren} = \frac{2}{\gamma} + \gamma \frac{Q}{2}$\[17\]. Essentially we are allowed to “normal order” the Liouville term, and provided its quantum scale dimension is (1,1) and $\gamma \leq \frac{Q}{2}$ all is well. Physically, at short distances ($\phi \rightarrow -\infty$) the exponential potential term is small, so divergences should be those of free field theory. It seems that the complicated field redefinition of the critical string theory amounts to the change from coordinate to covariant regulator on the world sheet; recall that spacetime field redefinitions are equivalent to a change of renormalization scheme in the 2d sigma model. It is intriguing that in this low-dimensional context there can be nontrivial solutions of the string equations of motion whose stress-energy does not back-react on the spacetime metric.

One indication of the conformal invariance of quantum Liouville theory is the degree of singularity of the operator product expansion of the Liouville perturbation; assuming free field operator products (but see \[11\] \[10\])

$$ e^{\gamma \phi(z)} e^{\gamma \phi(w)} \sim |z - w|^{-2\gamma} e^{2\gamma \phi(z)} + \ldots $$

(2.6)
In the weak coupling regime $\gamma \ll 1$ the singularity is integrable and the beta function calculated above will continue to vanish at first order in the perturbed theory. This reasoning does not give the correct upper bound on $\gamma$ however, indicating that problems set in for $\gamma \geq 1$ ($c \geq -2$) rather than the observed upper bound $\gamma = \sqrt{2}$ ($c = 1$). Ordinarily in conformal field theory, an operator that produces the identity in its operator product with itself has

$$O(z)O(w) \sim |z - w|^{-4h} \mathbb{1}$$

from which one can read off the dimension of that operator. It seems to be that, even though it does not produce the identity, in its self-product the singularity of $e^{\gamma \phi}$ is four times its ‘effective dimension’. Then from (2.4) we would conclude that $e^{\gamma \phi}$ becomes effectively irrelevant at $c = 1$, or more generally $\mathcal{O}e^{\alpha \phi}$ becomes effectively irrelevant for $\alpha > Q/2$. This behavior is rather similar to what happens when attempting to incorporate massive string states in the sigma model lagrangian of the critical string. The on-shell vertex operator for such a perturbation has $h = 1$, but only by virtue of an irrelevant spatial operator coupled to a negative dimension temporal operator. The disease of irrelevant operators, namely nonrenormalizability, shows up in the appearance of many low dimension operators with highly singular coefficients in the self-OPE. Such a similar explosion occurs for operators $\mathcal{O}e^{\alpha \phi}$ with $\alpha > Q/2$ (for instance the ‘black hole perturbation’ $\partial X \bar{\partial} X e^{Q \phi}$, hence it is in no sense a ‘perturbation’ of flat spacetime Liouville theory).

To recapitulate, we have

$$T_{zz} = 0 \quad \text{(Liouville e.o.m.)}$$
$$T_{zz} = -\frac{1}{2} (\partial \phi)^2 + \frac{Q}{2} \partial^2 \phi \quad Q = \frac{2}{\gamma} + \gamma$$
$$h_{\exp[\alpha \phi]} = -\frac{1}{2} \alpha (\alpha - Q)$$

These look like free field results, but it must be stressed that $\langle \phi \phi \rangle$ is not the free field propagator. The Liouville equation is geometrically the equation for constant (negative for $\mu > 0$) curvature surfaces $R = -\mu$. The semiclassical expansion is valid for $\gamma \ll 1$; rescaling $\phi \rightarrow \frac{2}{\gamma} \phi$ puts an overall factor of $1/\gamma^2$ in front of (2.5). If we plot the potential for the zero mode
we see that a stable solution exists only for surfaces of genus $g > 1$ (in the absence of point 

sources of curvature; see below). The stable point $\langle \phi \rangle$ increases with genus; in fact there 
is a scaling relation\cite{9}: let $\phi \to \phi + \frac{1}{g} \log a$, then

$$Z(g_{str}, \mu) = Z(g_{str}, a\mu) a^{-(2-2g)\frac{\phi}{2g}},$$

implying

$$Z_g(g_{str}, \mu) = C_g \mu^{(2-2g)\frac{\phi}{2g}}. \tag{2.7}$$

The full classical solution to the Liouville equation of motion is discovered through 

its geometrical role; $e^{\gamma \phi}$ must be a density under coordinate transformations, $e^{\gamma \phi(z')} = |\frac{\partial z'}{\partial z}|^2 e^{\gamma \phi(z)}$. Since the standard constant negative curvature metric on the upper half 
plane (UHP) is $ds^2 = dzd\bar{z}/(\text{Im } z)^2$, $\phi$ must look locally like

$$\phi = \frac{1}{\gamma} \log \left[ \frac{16 A(z) A^*(\bar{z})}{\mu (A - A^*)^2} \right],$$

where $A$, $A^*$ are local coordinates on the surface; i.e. $A(z)$ is the map from the UHP to 
the Riemann surface $\Sigma$

$$\frac{\mu}{16} e^{\gamma \phi} dzd\bar{z} = \frac{dA dA^*}{(A - A^*)^2}.$$

The line element on the UHP is invariant under $\text{SL}(2,\mathbb{R})$ transformations $A \to \frac{aA + b}{cA + d} \equiv g(A)$. This transformation must leave $\phi$ invariant, but may do so in a nontrivial way, e.g.
by making a circuit of a nontrivial closed path on Σ (see fig. 2). That is, the monodromy of the local coordinate \( A \) is a set of \( \text{SL}(2, \mathbb{R}) \) transformations which cover the surface \( \Sigma \) onto the UHP. There are several classes of monodromies:

1. **Elliptic monodromy** – \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is conjugate to a rotation, i.e., there exists \( h \) such that \( hgh^{-1} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \). The surface \( \Sigma \) has a conical singularity of deficit angle \( \theta \). Deficit angles \( \theta = \frac{2\pi j}{n} \), \( j \in \mathbb{Z} \), are ‘nice’ since they are covered by the UHP; \( j = \frac{m}{n} \) requires an \( n \)-fold branched cover of the UHP at the fixed point of the rotation.

2. **Parabolic monodromy** – \( h \) exists such that \( hgh^{-1} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \), a translation (in terms of deficit angles, \( \theta = \pi \) and the surface has a cusp).

3. **Hyperbolic monodromy** – \( hgh^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) is a dilation. The identification of the UHP under \( g \) makes a handle.

A complete solution of the Liouville equation consists of representing the surface \( \Sigma \) by its fundamental group, the discrete subgroup (Fuchsian group) of \( \text{SL}(2, \mathbb{R}) \) that covers the surface onto the UHP; then one constructs the automorphic function \( A(z) \) covariant under this group.

The three types of monodromy above are continuously related, as can be seen by pinching a handle on \( \Sigma \):
In the classical theory, a deficit angle $\theta$ is a delta-function source of curvature:

$$\frac{1}{4\pi} \partial^2 \phi + \frac{\mu}{8\pi \gamma} e^{\gamma \phi} = \frac{Q}{8\pi} \hat{R} = \sum_i \frac{\theta_i}{\pi \gamma} \delta^{(2)}(z - z_i) . \quad (2.8)$$

Integrating both sides, one sees that there is a solution whenever $2 - 2g + \theta_i/\gamma < 0$. Eq. $(2.8)$ is the saddle point of the functional integral

$$\int \mathcal{D} \phi \, e^{-S_{\text{eff}}} \prod_i e^{\frac{\theta_i}{\pi \gamma} \phi(z_i)} . \quad (2.9)$$

On the other hand, there is no local source for hyperbolic monodromy (the fixed points of the SL(2, $\mathbb{R}$) transformation $g$ on the UHP are not on the surface $\Sigma$). In terms of Liouville dynamics, the initial field configuration for $\phi$ (along some closed loop generating the monodromy) never propagates to $\phi = -\infty$ where the field configuration can be interpreted as localized at a point in the covariant metric. Note also that there is no classical geometrical interpretation for deficit angle $\theta > \pi$; i.e. when the source contributes half the curvature contribution $\frac{1}{8\pi} Q \int R$ of the sphere in the functional integral. Two parabolic cusps turn a sphere into a cylinder; we cannot go beyond this while maintaining the geometrical interpretation of $\phi$. Quantum mechanically this means $\alpha = \frac{\theta}{\pi \gamma} \leq \frac{Q}{2} = \frac{1}{\gamma} + \frac{\gamma}{2}$. This does not mean that operators with such Liouville charge ‘don’t exist’, rather merely that we cannot give them a geometrical interpretation. From the discussion of section (2), we conclude that gravitationally dressed operators with $\alpha > Q/2$ are ‘effectively irrelevant’. If we perturb the action by them, new dimension one operators will have to be added to the action to subtract singularities, a procedure that rapidly snowballs; as vertex operators they cannot be renormalized simply by normal ordering because the loop expansion is not well-behaved. It is doubtful that KPZ scaling (see below) can be maintained.

The above considerations motivate a brief review of SL(2, $\mathbb{R}$) representation theory $[23]$. Representations are labelled by their values of the quadratic Casimir $C_2 = j(j-1)$...
and \( J^3 = m + E_0, \ m \in \mathbb{Z} \). All representations can formally be built from the two-dimensional representation \( (w_1 \ w_2) \to \left( \frac{g(w_1)}{g(w_2)} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \). Realizing the \( SL(2, \mathbb{R}) \) algebra on differential operators \( J^+ = \frac{1}{\sqrt{2}} w_1 \partial_2, \ J^- = \frac{1}{\sqrt{2}} w_2 \partial_1, \ J^3 = \frac{1}{2} (w_1 \partial_2 - w_2 \partial_1) \), the monomial \( N_m w_1^a w_2^b = N_m (w_1 w_2)^j (w_1/w_2)^{E_0+m} \) transforms as part of the spin \( j \) representation, where \( N_m \) is a normalization. There are several cases:

1. The trivial representation \( j = E_0 = m = 0 \).
2. The finite dimensional representations \(-2j \in \mathbb{N}, E_0 = 0, m = -j, ..., j\); these are not unitary.
3. Discrete series representations \( D_{\pm} \): For \( D_+ \), we have \(-j + E_0 = 0, -2j \notin \mathbb{N}, J_3 + E_0 = 0, -1, -2, ...; \) for \( D_- \), \( j + E_0 = 0, -2j \notin \mathbb{N}, J_3 + E_0 = 0, 1, 2, ... \). These are unitary if \( E_0 \in \mathbb{R} \) and \( j > 0 \). They are related to the elliptic monodromy conjugacy classes of \( SL(2, \mathbb{R}) \) and have \( C_2 > 0 \).
4. Continuous series representations: \(-\frac{1}{2} < \Re E_0 \leq \frac{1}{2}, -j + E_0 \neq 0, \pm 1, \pm 2, ..., J_3 + E_0 = 0, \pm 1, \pm 2, ... \). These are unitary if \( E_0 \in \mathbb{R}, j - \frac{1}{2} \in i\mathbb{R} \); or \( E_0, j \in \mathbb{R}, |j + \frac{1}{2}| < \frac{1}{2} - |E_0| \). These are related to hyperbolic conjugacy classes in \( SL(2, \mathbb{R}) \) and have \( C_2 < 0 \).

How are these representations related to Liouville theory? Recall that classically \( T_{zz} = -\frac{1}{2} (\partial \phi)^2 + \frac{1}{\gamma} \partial^2 \phi \). From this follows [8][24]

\[
(\partial^2 + \frac{\gamma^2}{2} T(z)) e^{-\frac{2}{\gamma} \phi} = 0 , \tag{2.10}
\]

and from the classical solution we have

\[
e^{-\frac{2}{\gamma} \phi_{cl}} = \sqrt{\frac{16}{\mu}} \frac{(A - A^*)}{\sqrt{\partial A \partial A^*}} \]

\[
= \sqrt{\frac{16}{\mu}} \sum_{m=1,2} \psi_{j,m}(z) \psi^*_{j,m}(\bar{z}) \tag{2.11}
\]

for \( j = -\frac{1}{2} \), where

\[
w_1 \equiv \psi_{-\frac{1}{2}, \frac{1}{2}} = \frac{1}{\sqrt{\partial A}} ; \quad w_2 \equiv \psi_{-\frac{1}{2}, -\frac{1}{2}} = \frac{A}{\sqrt{\partial A}} .
\]

are the two solutions to \( (2.10) \). The expressions \( (2.11) \) can be regarded as a classical version of conformal field theory where each measurement is a sum of holomorphic times antiholomorphic ‘chiral vertices’, glued together in a monodromy invariant way to make
physical single-valued expressions. From the two basic solutions one can in principal build all exponentials corresponding to finite-dimensional representations, *e.g.*

\[ e^{j\gamma\phi} = \sqrt{\frac{16}{\mu}} \sum_{m=-j,...,j} \psi_{j,m}(z)\psi^{*}_{j,m}(\bar{z}) \quad , \quad -2j \in \mathbb{N} \]

An important issue is whether such expressions generalize to other classes of representations for which the sum over \( m \) is infinite. In principle one should be able to extract the answer from the classical solution \( A(z) \) to (2.8), the saddle point of the correlation (2.9); this is in turn determined by its monodromy \( \Gamma \in \text{SL}(2, \mathbb{R}) \).

The quantum theory may have a structure similar to that of known conformal quantum field theories: the correlation functions would be (generically infinite) sums of holomorphic times antiholomorphic conformal blocks glued together to make a monodromy invariant object\[ ^{25} \]

\[ \langle \prod O_i(z, \bar{z}) \rangle = \sum_\alpha \mathcal{F}_\alpha(z)\mathcal{F}^*_\alpha(\bar{z}) \quad . \quad (2.12) \]

For instance, in current algebra conformal field theories the conformal blocks \( \mathcal{F}_\alpha \) carry two sets of indices, an ‘external’ index (like \( i \) in (2.12)) which is a representation label for the current algebra, and an ‘internal’ index which is a quantum group index, since the monodromy acts on the blocks like the \( R \)-matrix of a quantum group\[ ^{26} \]. In the case of Liouville, there is no ‘external’ symmetry group other than Virasoro, and the index is continuous; however there is an ‘internal’ symmetry group \( SL_q(2) \) which is, in some sense yet to be understood, the ‘quantization’ of the classical monodromy action on \( A(z) \)\[ ^{27} \]. In this respect Liouville is like other coset conformal field theories (Liouville can be thought of as the coset conformal field theory based on \( SL(2, R)/N \), where \( N \) is the Borel subgroup\[ ^{27} \]), where the external symmetry is gauged and the internal quantum group symmetry remains but is ‘confined’ – only appearing when one pulls the theory apart into holomorphic constituents. In terms of the parameter \( \gamma \) in the Liouville lagrangian, the quantum group parameter is \( q = \exp[i\pi \gamma^2] \)\[ ^{28} \]. The translation into \( \text{SL}(2, \mathbb{R}) \) notation of the condition \( \alpha \leq Q/2 \) means that \( e^{j\gamma\phi} \) must have \( j \leq \frac{Q}{2\gamma} = \frac{1}{\gamma^2} + \frac{1}{2} \); the quantum dimension is \( h_j = -j - \frac{\gamma^2}{2} j(j-1) \). In these expressions the first term is the classical value; the second is the quantum correction. Note that \( L_0 \) is also real for \( j = \frac{Q}{2\gamma} + i\lambda \), \( \lambda \in \mathbb{R} \), for which \( h_j = \frac{\gamma^2}{2} [ (\frac{Q}{2\gamma})^2 + \lambda^2 ] \).

An analogy with the \( SU(2) \) WZW theory might be helpful here. The classical solution to the theory is

\[ g_{ab}(z, \bar{z}) = \sum_c g_{ac}(z)\bar{g}_{cb}(\bar{z}) \]
where the holomorphic part \(g(z)\) transforms under left multiplication by the loop group of SU(2) and the Virasoro algebra, and also by right multiplication under SU(2). The holomorphic constituents are determined by their monodromy \(g(z) \rightarrow g(z) \cdot h\) around nontrivial closed loops on \(\Sigma\). \(g_{ab}(z)\) intertwines with \(g_{cd}(z')\) via a classical \(r\)-matrix; upon quantization this internal index labels an \(SU_q(2)\) quantum group representation with \(q = \exp(\frac{2\pi i}{k+2})\) \[26\], and the intertwining is via a quantum \(R\)-matrix. For SL(2) and for Liouville a major complication is that the fusion of representations closes on the continuous series representations and it is not clear that one has a useful or effective way of decomposing the correlations on intermediate states.

Although the above analysis is rather appealing as it places the Liouville theory within the customary conceptual framework of conformal field theory, it has not proven to be sufficiently powerful to enable the calculation of correlation functions. It is a useful route in the case of rational conformal field theories because the monodromy representations are finite, related to a closed system of differential equations that one may derive from various Ward identities\[25\]. In the case of Liouville theory, and also noncompact current algebra, the monodromy representations are infinite-dimensional; and the Ward identities are not sufficiently powerful to give a closed system of differential equations that will determine the correlation functions. Although finite dimensional representations \(e^{-j\gamma \phi(z)}\) lead to finite order differential equations in \(z\), general matter operators require infinite dimensional representations which don’t satisfy any simple equations; such representations are required in intermediate states and therefore upon factorization, hence the dependence of correlations on the position of these operators is an open question. It seems that at least in the case of Liouville theory, other analytic techniques are available (see the lectures of D. Kutasov at this school, and references therein).

The above analysis points to a picture of ‘quantum Riemann surfaces’ where the classical solution \(A(z)\) of Liouville theory is deformed into some kind of quantum conformal block, its classical \(SL(2, \mathbb{R})\) monodromy deforming into an \(SL_q(2)\) quantum group structure with \(q\) related to the coupling constant \(\gamma\) of Liouville theory.

3. Scaly behavior

Regardless of our understanding of how to compute Liouville correlation functions, there are several general statements we can make about those correlations assuming the
quantization preserves free-field ultraviolet behavior even in the presence of the exponential interaction term. The strongest ‘theoretical’ reason for such a presumption (made implicitly by the authors of \cite{[4]} is that physical short distance as measured in the metric $d\bar{s}^2 = e^{2\phi}dz^2$ is where the Liouville potential is exponentially small, and so should not affect the free field results. The ‘experimental’ evidence for the validity of this assumption is the agreement of predicted scaling relations\cite{[3]}, and more recently tree-level S-matrix elements, with the continuum limit of discretized random surfaces in the matrix model\cite{[7]}.

What are the Liouville predictions? Suppose we wish to study conformal matter with central charge $c(=d$ above$)$ coupled to 2d gravity. The matter theory will contain a set of scaling operators $O_{k}^{\text{matter}}$ of scale dimension $h_k$. A generally covariant theory must make $O_{k}^{\text{matter}}$ into a coordinate density of scale dimension $(h_{\text{tot}}, \bar{h}_{\text{tot}}) = (1, 1)$; for instance,

$$O_{k}^{\text{grav}} = e^{\alpha_k \phi} O_{k}^{\text{matter}} \quad (3.1)$$

with

$$h_{\text{total}} = -\frac{1}{2} \alpha_k (\alpha_k - Q) + h_k = 1 \quad (3.2)$$

Then the integrated correlation functions of $O_{k}^{\text{grav}}$ are 2d general coordinate invariant. The exponential Liouville dependence of such operators will modify KPZ scaling in correlation functions, cf (2.9). The string theory interpretation of this gravitational dressing is that the Liouville field $\phi$ is the (Euclidean) time component of the string’s position in spacetime; the time ($\phi$) dependence of (3.1) is just that of a linearized solution of the (Euclidean) spacetime equations of motion with energy $\alpha_k = j_k \gamma$; and the relation $-\frac{1}{2} (\alpha_k - Q)^2 + h_k = \frac{1-d}{24}$ between $\alpha_k$ and $h_k$ is the mass shell condition for linear perturbations, i.e. $h_k$ is the eigenvalue of the spatial Laplacian $L_0^{\text{matter}}$. This makes it clear that conformal matter is from the spacetime point of view the special class of stationary solutions of the string equations. From the previous analysis of Liouville theory, if $1 - h_k > \frac{Q^2}{8}$ then there does not exist a local generally covariant measurement corresponding to the matter operator $O_k$; i.e. $O_{k}^{\text{matter}}$ looks local in coordinates $(z, \bar{z})$, but its gravitational dressing requires hyperbolic monodromy $j_k = \frac{Q}{2\gamma} + i\lambda$ which has no local interpretation in the covariant theory. Seiberg\cite{[10]} has dubbed such operators tachyonic since their spacetime mass shell condition (3.2) implies imaginary mass for the corresponding string state in spacetime. The reason that some perfectly sensible local operators may not have a local gravitational dressing is that each measurement perturbs the local geometry by making a small deficit angle $\alpha_k$ in the surface at the point of measurement; there are no probes ‘outside’ the
universe that can make such a measurement without perturbing the geometry. It can and does happen that when gravity is switched on the geometry is perturbed too violently by some operators to have a good, local continuum limit.

In conclusion, we can take away the following main lessons about Liouville theory coupled to conformal matter:

1. KPZ scaling:
   \[ Z_g = C_g \mu^{(2-2g)Q/2\gamma} \]

2. Operator scaling dimensions \( O_{grav}^k = e^{a_k \phi} O_{matter}^k \) that shift the KPZ scaling relation by \( \alpha_k \) in correlation functions. 
   \[-\frac{1}{2} (\alpha_k - \frac{Q}{2})^2 + h_k = \frac{1-d}{24} .\]

3. A ‘phase transition’ at \( d = 1 \) where the gravitationally dressed identity operator becomes tachyonic in the sense described above. There is no obvious reason why tree amplitudes might not be analytically continued as in \( d = 26 \) to yield a sensible classical string S-matrix, but (as at \( d = 26 \)) loop amplitudes are infinite because tachyons cause infrared divergences in loop integrals even in Euclidean spacetime. As emphasized by Seiberg[10], this phase transition is not always at \( c = 1 \), but occurs whenever the spectrum has physical tachyons \( h_{min} < \frac{d-1}{24} \).

Properties (1) and (2) are special to gravitationally dressed conformal matter; (3) is expected to be a generic feature persisting even when the matter theory is massive. One advantage of the matrix model, to which we turn next, is the ability to calculate the partition function and correlations even for massive matter.

The KPZ scaling relations suggest that a large part of Liouville dynamics is accounted for by the zero modes. Also \( c < 1 \) minimal models coupled to gravity have the tachyon as the only physical state (there are some additional physical states at nonstandard values of the ghost number[29] whose meaning is less clear); reparametrization invariance cancels the fluctuations of the longitudinal modes, leaving only the center of mass motion of the string when the spacetime is 1+1 dimensional. Indeed, it has been shown that free string propagation is accurately described by the quantum mechanics of the zero modes [12].

One can imagine solving the reparametrization invariance constraints \( T_{00} = T_{01} = 0 \) (or equivalently the BRST invariance constraints in conformal gauge) to show that physical states contain no excitations of the string’s nonzero modes; the remaining constraint is

\[ T_{11} = 0 \]

2 String interactions are not saturated by the zero modes, which has been interpreted in [12] as due to the violation of the single string physical state conditions by contact interactions in the vertices.
the Wheeler-deWitt equation $T_{00}^{\text{zero mode}} = 0$ on the zero modes. In a conformal matter theory coupled to gravity this separates into the dynamics of the matter zero modes, whose spectrum of scaling dimensions couples to the Liouville zero mode equation

$$\left[ -(\ell \frac{\partial}{\partial \ell})^2 + 4\mu \ell^2 + \nu^2 \right] \Psi_\mathcal{O}(\ell) = 0$$

with $\ell = e^{\gamma \phi/2}$ and $\nu = \frac{2}{\gamma}(\alpha - \frac{Q}{2})$. The appropriate solutions to this equation are the modified Bessel functions

$$\Psi_\mathcal{O}(\ell) = (\nu \sin \pi \nu)^{1/2} K_\nu(2\sqrt{\mu} \ell)$$

(3.3)

Physically, the function $K_\nu = \frac{\pi}{2\sin \nu \pi} [I_{-\nu} - I_{\nu}]$ is the linear combination of ‘incoming’ and ‘outgoing’ waves $I_{\pm \nu}(2\sqrt{\mu} \ell)$ which is exponentially damped in the infrared $\ell \to \infty$, indicating total reflection. Some of the strongest evidence for the viability of the conformal field theory approach to 2d gravity is the appearance of these wavefunctions within the matrix model.

4. The matrix model

The Hilbert space of Liouville is functionally infinite dimensional, however the subspace of physical states is much smaller – at most a countable number for $c \leq 1$. Naively each gauge invariance removes one canonical pair of variables, one by a choice of gauge and another by the gauge constraint (i.e. that the generator of gauge transformations annihilate the physical subspace – the analogue of Gauss’ law in QED). In 2d gravity there are two reparametrization degrees of freedom $\xi^a \to \tilde{\xi}^a(\xi)$. The time components of the metric are the Lagrange multipliers of the gauge constraints $T_{00} = T_{01} = 0$; the canonical degrees of freedom are the spatial metric $e^{\gamma \phi}$ and its conjugate momentum $\pi_\phi$. Thus we have the possibility to gauge away up to one scalar matter fields’ worth of local degrees of freedom. As usual in quantum gravity the difficulty in imposing the constraints lies in the Hamiltonian constraint $T_{00} = 0$, since this involves the way that the 2d worldsheet is carved up into spacelike hypersurfaces and therefore involves the (coordinate) time development. This makes it difficult to find nice global gauge invariant states. Nevertheless, the

\[3\] Note that in this counting the Ising model has $d = \frac{1}{2}$ because $\pi_\psi \equiv \psi$ is ‘half’ a canonical pair.
lesson to be drawn is that it should pay to reformulate the path integral on the space of physical configurations because it is expected to be much smaller than the space of metrics plus matter field configurations. The simplest way to enumerate physical configurations is to discretize the 2d surfaces. Consider a 2d surface built by gluing together uniform squares each with sides of length $\epsilon$ (or triangles, pentagons, etc. – it turns out not to matter which\[7\] unless one artificially tunes couplings\[30\]).

This lattice spacing replaces the covariant cutoff of Liouville theory. The only local freedom in pure gravity resides in how many squares meet at each vertex, which determines the local curvature discretized in units of $\pi/2$ (see fig.4). Note that each surface is dual to a $\Phi^4$ graph (see fig.4), each square being dual to a $\Phi^4$ vertex, each side of the square being dual to a ‘propagator’ of the $\Phi^4$ Feynman graph. Thus counting all graphs with $A$ vertices counts all surfaces with area $A\epsilon^2$; we can call this the partition function for discrete 2d Euclidean quantum gravity. The continuum limit consists of taking $\epsilon \to 0$, $A \to \infty$ such that the physical area $A_{phys} = A\epsilon^2$ is finite, assuming such a limit exists. In other words we concentrate on surfaces with a very large number of triangles; the statistics of these surfaces is governed by the large order asymptotics of graphical perturbation theory. We expect to be able to generate any local curvature in the continuum by coarse-graining over a large number of 3-, 4-, and 5+-coordinated vertices on the dual tesselation, of positive, zero and negative curvature, respectively.

In the pure gravity case, the generating function for the $\Phi^4$ graphs dual to the discretization is the integral

$$\int d\Phi \ e^{-\frac{1}{2}\Phi^2 + g\Phi^4},$$

(4.1)
i.e. the coefficient of $g^4$ in the asymptotic expansion at small $g$ is the number of surfaces with area $A$ in lattice units, since this asymptotic expansion is the enumeration of $\Phi^4$ Feynman graphs. The coupling $g$ is to be identified with the bare 2d cosmological constant $g = \exp[-\mu_{\text{bare}}]$. Note that $g$ is positive in order that each surface is counted with positive weight, so the generating function (4.1) diverges badly; we can interpret this through the asymptotic expansion as the statement that the entropy of large surfaces diverges uncontrollably. To cut down on this entropy we can try to count only surfaces of fixed genus, hoping that although the sum over topology is infinite, the individual terms in the series might be finite. This turns out to be the case for $d \leq 1$ matter.

Matter can be incorporated by introducing a label set for the dummy variable $\Phi$; then configurations are enumerated not only by the connectivity of the graph but also the element of the label set (which we think of as the value(s) of scalar field(s) or of discrete spin variables) at each point on the graph. For instance in the Ising model one considers two matrices $M, N$ with integrand

$$\exp(-\text{tr}[abM^2 + \frac{a}{b}N^2 + cMN + gdM^4 + \frac{g}{d}N^4]).$$

The couplings $a, b$ can be set to unity by a rescaling of $M, N$ (however such redundant couplings can have physical effects through contact terms in loop correlations). That leaves $c$, related to the Ising temperature because it controls the probability of transitions between up-spin ($M$) and down-spin ($N$) subgraph domains in the diagrammatic expansion; $d$ is related to the magnetic field since it preferentially weights one of the two spins; and $g$ is our friend the surface cosmological constant. The trick in this and similar cases is to take the continuum limit cleverly so that both graph connectivity and label fluctuations approach criticality. Fine-tuning more complicated potentials yields a tower of critical points.

To count the surfaces of fixed genus consider the $N \times N$ Hermitian matrix field $\Phi_{ab}$, and generating function

$$\int d\Phi \ e^{-\text{tr}[(\frac{1}{2}\Phi^2 - (g/N)\Phi^4)].}$$

The reason to make $\Phi$ Hermitian is that the Feynman graphs carry an orientation and so the partition function counts orientable surfaces only. For the generalization to unoriented surfaces, see . Looking at simple graphs
shows us that those which can be laid out smoothly on a surface of genus \( g \) come in the generating function with a factor \( N^{2-2g} \). Each closed index loop traces over the indices in that loop and gives a factor \( N \); the coupling constant of the generating function is chosen to be \( g/N \) so that adding a square without changing the topology keeps the \( N \) counting fixed (the extra \( 1/N \) cancels the additional index trace). We have

\[
Z_{\text{sft}} = \int d\Phi \ e^{-N \text{tr}[\frac{1}{2}(\Phi/\sqrt{N})^2-g(\Phi/\sqrt{N})^4]} = \exp[-\sum_g N^{2-2g}C_g].
\]  

(4.2)

We have the right to call this generating function the partition function of (discretized) string field theory in this low-dimensional situation since it is indeed the object whose asymptotic expansion in \( N \) is the sum over surfaces of some 2d field theory describing the string background. The limit \( N \to \infty \) picks out the sphere contribution \( C_0 \), the classical limit of the associated string theory; we can evaluate this from the integral (4.2) by saddle point techniques even though the full integral doesn’t make sense (the saddle is not a global minimum) since the leading contribution is just the value of the integrand at the saddle. To find the saddle, decompose the matrix in terms of ‘matrix polar coordinates’ as

\[
\Phi = U\Lambda U^{-1}
\]

where \( U \) is a unitary matrix and \( \Lambda \) is the diagonal matrix of eigenvalues of \( \Phi \). The partition function becomes

\[
Z_{\text{sft}} = \int U^{-1}dU \int d\Lambda \left| \frac{\partial \Phi}{\partial (U, \Lambda)} \right| \exp \left[ -N \sum_{i=1}^{N} \left( (\frac{\lambda_i}{\sqrt{N}})^2 - g(\frac{\lambda_i}{\sqrt{N}})^4 \right) \right].
\]

The Jacobian is easily evaluated by noting that a) it vanishes whenever any two eigenvalues coincide; b) it is symmetric under permutations of the eigenvalues; and c) it must scale like
\[ \lambda^{N(N-1)}; \text{ the unique function with these properties is } \prod_{i<j} (\lambda_i - \lambda_j)^2. \] The justification for a) is analogous to the vanishing of the Jacobian from Cartesian to polar coordinates in ordinary multidimensional integrals: the origin is a fixed point of the symmetry group of rotations. The transformation is singular there so the measure must vanish. Similarly, the coincidence of two eigenvalues is a fixed point for an SU(2) subgroup of U(N), so the Jacobian is singular. The partition function becomes

\[ Z_{sft} = \int \prod d\lambda e^{-N \sum_i V(\lambda_i/\sqrt{N}) + 2 \sum_{i<j} \log|\lambda_i - \lambda_j|/\sqrt{N}}. \]

The mental picture to adopt is to consider the eigenvalues as a set of particles lying in a metastable well, interacting through a logarithmically repulsive ‘Coulomb’ force. As \( N \to \infty \) we can replace the set \( \{\lambda_i\} \) by an eigenvalue density (which we might regard as a collective coordinate or string field[2]) and the saddle point equation is

\[ \frac{1}{2} \lambda - 2g\lambda^3 = \int_{-a}^{a} \frac{\rho(\mu)d\mu}{\lambda - \mu}, \quad \int_{-a}^{a} \rho(\mu)d\mu = 1. \]

Consideration of analytic properties provides the solution[3]

\[ \rho(\lambda) = \frac{1}{\pi} \left[ \frac{1}{2} - ga^2 - 2g\lambda^2 \right] \sqrt{a^2 - \lambda^2} \quad \lambda \leq a = \left( \frac{1}{6g} \right) \left( 1 - 48g \right)^{1/2}. \]

Near the endpoint \( a \) of the distribution, \( \rho(\lambda) \) behaves as \( \rho(\lambda) \sim \sqrt{a^2 - \lambda^2} \); but when \( g \to g_c = -\frac{1}{48} \) the analytic behavior changes to \( \rho(\lambda) \sim (a^2 - \lambda^2)^{3/2} \). What is happening? The edge of the eigenvalue density becomes softer because the last eigenvalue approaches an instability where the Coulomb repulsion of the rest of the eigenvalues overcomes the external potential force and pushes it out of the metastable well. This instability reflects a phase transition in the surface dynamics on the sphere: recall that the entropy of surface configurations grows with the number of plaquettes, but there is an energy cost exp\([-\mu A]\). At \( g \sim g_c = e^{-\mu_c} \) from above these balance, and \( Z \) is dominated by surfaces controllably large compared to the cutoff \( \epsilon \). Inserting \( \rho(\lambda) \) into the saddle point action gives

\[ \frac{1}{N^2} S_{saddle} = \int_{-a}^{a} \rho(\lambda)V(\lambda) - \int \int \rho(\mu)\rho(\lambda)\log|\lambda - \mu| \]

\[ = \frac{1}{24}(a^2 - 1)(9 - a^2) - \frac{1}{2}\log a^2 \]

\[ \sim C_0(g - g_c)^{5/2} \]
where $\epsilon^2 \mu_{\text{ren}} = g - g_c$. This saddle point action is the partition function of 2d gravity on the sphere, $S_{\text{saddle}} = Z_{\text{sphere}}^{2d} \sim C_0 N^2 (\epsilon^2 \mu_{\text{ren}})^{5/2}$. Letting

$$g_{\text{str}} = \frac{1}{N^2 \epsilon^{5/2}}$$

gives $Z_{\text{sphere}} = C_0 (\mu_{\text{ren}}^{5/2} / g_{\text{str}}^2) \equiv C_0 \mu^{5/2}$; in general the full string partition function depends on the couplings $\mu_{\text{ren}}$ and $g_{\text{str}}$ only through the ratio $\mu$. Note that to get a sensible continuum result we must let $N$ scale with the surface cutoff $\delta \sim e^{\gamma \phi/2}$, i.e. $g_{\text{str}}$ is dynamical – just what KPZ scaling predicts! Plugging numbers into the Liouville formulae (2.4), (2.7) we find $Q = 5\sqrt{3}$, $\gamma = 2/\sqrt{3}$ for $d = 0$, hence Liouville theory predicts

$$Z_{\text{sphere}}^{\text{Liou}} = C_0 \mu^{Q^2 / \gamma} = C_0 \mu^{5/2}$$

as observed!

Note that the important feature of the phase transition used to take the continuum limit is the quadratic maximum in the effective eigenvalue potential (linear vanishing of the force on the last eigenvalue at $g = g_c$), so it shouldn’t matter whether we use $\Phi^4$ squares or $\Phi^3$ triangles in the microscopic theory. In other words, only the rate of vanishing of $\rho$ near its endpoint is universal; the details of the eigenvalue distribution far from this region as well as the value of $g_c$ are irrelevant quantities. By fine tuning the relative weight of different polygonal simplices one can however reach other sorts of critical points[30][36].

The advance embodied in the discrete approach is the ease with which one may calculate the coefficients $C_g$ in the partition function[13], as well as all correlation functions. To date it has only been possible in the Liouville approach to calculate (after much effort) correlation functions on the sphere. To go beyond the sphere one needs to among other things incorporate corrections to $\rho(\lambda)$ due to the discreteness of eigenvalues; $\rho$ is not a smooth function at the $1/N$ level but rather a sum of delta functions. In fact the whole methodology above is rather cumbersome, and it proves simpler to reformulate the problem. It is apparent that the main difficulty is the Coulomb interaction among eigenvalues, so we might try to find variables that diagonalize this interaction as much as possible. This happens[34] when one writes the square root of the Jacobian as a Slater determinant

$$\prod_{i<j} (\lambda_i - \lambda_j) = \det_{ij} [\psi_i(\lambda_j)] \equiv |\Psi(\lambda)|,$$
where \( \psi_i(\lambda) = \lambda^i - 1 + c_{i-2}\lambda^{i-2} + \ldots + c_0 \). The \( c_i \) are arbitrary since they correspond to \( \psi_i \to \psi_i + \psi_k, k < i \), which cancels from the determinant. A convenient choice is to orthogonalize \( \psi \) with respect to the measure \( e^{-V(\lambda)}d\lambda \)

\[
Z_{sft} = \int \prod_i d\lambda \left< \Psi_t(\vec{\lambda}) \right| e^{-\sum_i V(\lambda_i)} \left| \Psi_t(\vec{\lambda}) \right> \equiv \left< \Psi_t(\vec{\lambda}) \left| \Psi_t(\vec{\lambda}) \right> \right>
\]

\[
V(\lambda) = \sum_k t_k \lambda^k
\]

Letting

\[
h_n \delta_{nm} = \int d\lambda \, \psi_n(\lambda)e^{-V(\lambda)}\psi_m(\lambda),
\]

we find

\[
Z_{sft} = \prod_{i=1}^N h_i
\]

up to a \( t \)-independent normalization. The computation of the partition function reduces to the determination of the \( h_i(\vec{t}) \). To this end, define

\[
h_n Q_{nm} = \int d\lambda \, \psi_n(\lambda)e^{-V(\lambda)}\lambda \psi_m(\lambda)
\]

\[
h_n P_{nm} = \int d\lambda \, \psi_n(\lambda)e^{-V(\lambda)} \frac{d}{d\lambda} \psi_m(\lambda).
\]

Considering different matrix elements yields

\[
Q = \begin{pmatrix}
b_1 & a_1 & & 0 \\
1 & b_2 & a_2 & \\
& 1 & b_3 & a_3 \\
0 & & & & & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

(4.6)

This also tells us (by acting \( \lambda \) to the left) that \( a_i = h_{i+1}/h_i \); a little thought shows \( b_i = 0 \) if \( V \) is an even function (since then we have the symmetry \( \psi_j(\lambda) = (-)^j \psi(-\lambda) \)). Integration by parts for \( P \) gives

\[
P = V'(Q)_+,
\]

(4.7)

where the + subscript means taking only the upper triangular part, setting the lower triangular and diagonal entries to zero, since \( d/d\lambda \) must lower the order of \( \psi \); similarly a − subscript refers to the lower triangular part. Now

\[
[\frac{d}{d\lambda}, \lambda] = 1 \implies [P, Q] = 1
\]

(4.8)
is an equation for $a, b$ which solves our problem, \textit{i.e.}
\[
Z_{sft} = \prod_{i=1}^{N} h_i = h_0^N \exp \left[ \sum_{i=1}^{N} (N - i) \log a_i \right].
\]

Afficionados of integrable systems will recognize that if $V = \frac{1}{2} \lambda^2$ then $P, Q$ is the Lax pair of the Toda chain; actually $\text{tr} \{ Q^{k+1} \}$ is the $k^{\text{th}}$ conserved charge of the Toda problem and $(Q^k)_+$ is the $k^{\text{th}}$ Toda Hamiltonian. In other words,
\[
\frac{\partial Q}{\partial t_k} = [Q^k_+, Q] \tag{4.9}
\]
are commuting flows. One easily checks that these flows preserve the string equation (4.8) Finally one can show that $Z_{sft}$ is a particular type of ‘tau-function’ of the Toda system; this conveys little information but sounds impressive, the content being primarily in equations (4.8) and (4.9). Commuting flows in this problem should not surprise us; we must have $\frac{\partial^2 Z}{\partial t_i \partial t_j} = \frac{\partial^2 Z}{\partial t_j \partial t_i}$, etc., so of course all the derivatives commute. The nontrivial fact is that we can represent $Z$ as an inner product in a Hilbert space
\[
Z_{sft}(t) = \langle \Psi_\hat{t} | e^{-\sum (t_k - \hat{t}_k) \hat{Q}^k} | \Psi_\hat{t} \rangle \tag{4.10}
\]
where the derivatives with respect to the couplings act as operators. The different flows in coupling space then form an integrable system in this Hilbert space.

Again we need to take the continuum limit. To this end it is convenient to absorb a factor $e^{-V/2}$ in $\psi$, normalize $\psi_k \to \frac{1}{\sqrt{h_k}} \psi_k$ and take $V$ even (although the latter is not essential). Then the Lax pair $P, Q$ are transformed into
\[
\tilde{Q} = Q_+ - Q_- \quad \tilde{P} = V'(Q)_+ - V'(Q)_-. 
\]
The scaling limit consists of taking $a_i \to \text{const.} + \epsilon^2 u(x)$, where $\frac{1}{h} = 1 - \epsilon^2 x$. The second order difference operator $\tilde{Q}$ becomes (in the sense of matrix elements) a second order differential operator $\partial_x^2 + u(x)$; $a_N \approx \frac{Z_{N+1}Z_{N-1}}{Z_N^2}$ yields $u(x) = \partial_x^2 F$ where $Z_{sft} = e^{-F}$. Finally $\tilde{P}$ scales to some odd order differential operator depending on the details of the potential (we made the rescaling in the wavefunctions in order to make this manifest). Generically $\tilde{P} \to \partial_x$ which is boring, \textit{i.e.}
\[
[\tilde{P}, \tilde{Q}] = 1 = u' = F''' \implies F = x^3.
\]
which corresponds to $\rho \sim \sqrt{a^2 - \lambda^2}$. Fine tuning as before should yield $\rho \sim (a^2 - \lambda^2)^{3/2}$ corresponding to $\tilde{P} \rightarrow \partial^3 + v_1(x)\partial + v_0$. Then the equation $[\tilde{P}, \tilde{Q}] = 1$ is the continuum statement

$$u''' + uu' = 1,$$

the Painlevé I equation. This equation embodies through its asymptotic expansion in $x = \mu$ the entire perturbation series for $Z_{sft}$ about this background. The leading solution at large $x$ is $u^2 = x$, which gives $F \sim x^{5/2}$ which is the famous KPZ scaling relation! Plugging this solution into the differential equation and iterating yields

$$F = \sum_{g=0}^{\infty} C_g \mu^{\frac{5}{2} (2-2g)}$$

so we see $Q/\gamma = 5/2$. Note that derivative terms in the string equation come with factors of the string loop coupling $g_{str} \sim 1/N$, which is why the genus expansion expands in powers of derivatives of $u(x)$. Douglas\[16\][37] has generalized this construction for an arbitrary pair of differential operators $P, Q$ of integer order $p, q$ satisfying (4.8), which produces BPZ minimal matter [25] coupled to 2d gravity. Changing the bare matrix potential scales to continuum perturbations of $P, Q$ by lower or higher order differential operators for relevant and irrelevant perturbations, respectively\[37][38]. In the two (and more) matrix case, we can again eliminate the angle variables in the partition function\[34]. To ‘diagonalize’ the eigenvalue interaction, choose independently the left and right wavefunctions $|\Psi_t\rangle$ and $\langle \Psi_t|$. Defining matrices $M, P_M$ and $N, P_N$ analogous to (4.4) and (4.5), we have the variational equations

$$P_M = N + V_M'(M)$$

$$P_N = M + V_N'(N).$$

The number of nonzero diagonals of $M, N$ are determined by the degrees of $V_N, V_M$ respectively. Tuning these potentials, we can make $M, P_M$ scale to a pair of differential operators

$$Q = \partial^q + u_{q-2} \partial^{q-2} + \ldots + u_0$$

$$P = \partial^p + v_{p-1} \partial^{p-1} + \ldots + v_0$$

of orders $(p, q)$. The Heisenberg relation $[P_M, M] = 1$ again determines the coefficients $u_n(x, t), v_m(x, t)$ and hence the free energy $\partial_x^2 F = u_{q-2}$\[16][37]. These critical points

\[^4\] Of course any fixed lattice operator is a sum over continuum operators with cutoff dependent coefficients.
have been identified with \((p,q)\) conformal matter coupled to 2d gravity. The flows in the couplings \(t\) are governed by the 2d Toda hierarchy on the lattice\([14]\), which scales to the \(q\)-reduced KP hierarchy in the continuum\([16]\,[37]\,[38]\). The coefficients \(C_g \sim (2h)!\) for large \(g\), reflecting the divergence of the sum of the perturbation series\([13]\,[13]\). It is very interesting that these coefficients can be associated with an auxiliary, higher-action saddle of the matrix integral\([3]\) corresponding to the eigenvalue configuration where the last eigenvalue is moved to its unstable equilibrium

which might be interpreted as a ‘string instanton’ mediating some kind of vacuum decay, although the precise meaning of this configuration remains unclear.

An interesting and calculable set of correlation functions in these systems are correlations of the loop operator \(W(\ell)\) which cuts a hole of boundary length \(\ell\) in the surface. In the matrix model this operator is \(\frac{1}{L} \text{tr}\Phi^l\), which in graphs inserts a source of \(l\) external lines graphically dual to a loop or hole of boundary length \(l\epsilon\). For instance, fig.4 is a contribution to \(l = 24\). In the continuum limit \(Q \to \text{const.} + \epsilon(-\partial^2 + u)\), \(l \to \ell/\epsilon\) this becomes the heat operator \(e^{-\ell(-\partial^2 + U)}\)

\[
\text{tr}\Phi^l \sim (\text{const.} + \epsilon(-\partial^2 + u))^{\ell/\epsilon} \sim e^{-\ell(-\partial^2 + u)} \equiv W(\ell)
\]

up to an unimportant (nonuniversal) normalization. The calculation of loop correlation functions thus reduces to a set of convolutions of heat kernels\([15]\), e.g.

\[
\langle W(\ell_1)W(\ell_2) \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{-\mu} dy \langle x| e^{-\ell_1 Q} |y\rangle \langle y| e^{-\ell_2 Q} |x\rangle . \quad (4.11)
\]

For the general \((p,q)\) model we simply use the operator \(Q\) determined by the string equations \((1.8)\). The loop correlations serve as a kind of generating function of local operator
correlations, in the sense that the $\ell \rightarrow 0$ limit of $W(\ell)$ can be recast as a sum (integral) over the spectrum of local operators\[12\]. Geometrically, the zero mode $\ell = \oint e^{\gamma \phi/2}$ is dual (in the sense of Fourier transform) to its conjugate Liouville momentum $p = \oint \pi_\phi$ whose eigenvalue is the deficit angle; hence it should be no surprise that a Green’s function at fixed small $\ell$ should be an integral transform over local curvatures. The small $\ell$ expansion of (4.11) is the asymptotic expansion of the heat kernel, which is evaluated in terms of formal fractional powers of $Q\[10\][15][12]$. For instance, (4.11) expands as

$$
\langle W(\ell)W(\ell') \rangle \sim 0 \sum_{n=0}^\infty \ell'^{n/q} \langle O_n^{KP} W(\ell) \rangle
$$

$$
\langle O_n^{KP} W(\ell) \rangle = \int dx [Q_n^{n/q}, \langle x | e^{-tQ} | x \rangle] .
$$

For the definition of fractional powers of differential operators and their uses, see e.g. \[38\][37][41]. The basis of operators defined by isolating individual terms on the RHS do not coincide with the dressed operators $O_\alpha$ of conformal field theory\[12\]. Instead, one must take linear combinations of $O_{n}^{KP}$ with coefficients defined by the ‘incoming wave’ $I_{n/q}(2\sqrt{\mu} \ell')$. Then one finds the two-point function on the sphere of a CFT scaling operator and a loop operator is

$$
\langle O_\alpha W(\ell) \rangle = (2\sqrt{\mu})^\alpha K_\alpha(2\sqrt{\mu} \ell) . \quad (4.12)
$$

Thus one can justifiably claim that the matrix and Liouville approaches coincide where calculations can be done in both. Moreover, according to the ideas of Hartle and Hawking\[1\], one expects that the correlation function of a scaling operator $O_k$ with a loop operator $W(\ell)$ is the corresponding wavefunction of that operator – the operator creates the appropriate state in the Hilbert space of physical states, and $W(\ell)$ is the probability of that state propagating on a Euclidean surface of average curvature $-\mu$ to a boundary of length $\ell$. This expectation is indeed borne out by the explicit calculations \[10\][12] outlined above.

The matrix model is an effective tool for the calculation of correlations of integrated local scaling operators. While this is an important class of measurements it by no means exhausts the list of interesting questions one can try to ask in 2d gravity. These integrated correlations are what might be called extensive measurements, things like the area of the surface, or the magnetization or average energy density in the Ising model. Truly local measurements, where one tries to ‘build a laboratory’ on a patch of the surface and make measurements in it, have not been addressed yet because they are intrinsically more complicated. One must describe the laboratory in a coordinate invariant way, e.g. by
prescribing the geodesic distances and relative orientation of the objects in it. Even the simplest such ‘local’ measurement, the Hausdorff dimension of the surface, turns out to be very difficult to investigate[12]. It is of course possible that, like the scaling operators of large positive Liouville momentum, such measurements in two dimensions are plagued by strong fluctuations that render local questions meaningless. In such a situation the simple model does not retain the flavor of its higher dimensional cousin. Another interesting issue is whether we can define Minkowski 2d gravity independent of working in Minkowski string theory. Usually we induce the continuation to Minkowski signature world sheet through the analytic continuation of string amplitudes from Euclidean spacetime and its associated $i\epsilon$ prescription. Nevertheless it would be odd if the beautiful physics of $1+1$-dimensional scattering theory could not be put on a fluctuating geometry. Can we not discuss the S-matrix of the Ising model, or Potts or sine-gordon models, in deSitter 2d gravity? This also may hinge on how to define localized asymptotic states in a fluctuating background geometry.
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