ASYMPTOTIC EXPANSIONS FOR CERTAIN MATHEMATICAL CONSTANTS AND SPECIAL FUNCTIONS

Chao-Ping Chen

For fixed real $b > 1$ and $\alpha > 0$, let $S_{b}^{[\alpha]}(n) = \sum_{k=1}^{n} \frac{b^k}{k^{\alpha}}$. Abel proved that $S_{b}^{[\alpha]}(n) \sim b^{n} \sum_{k=0}^{\infty} c_{k} n^{-(k+\alpha)}$ ($n \to \infty$), and gave an explicit formula for determining the coefficients $c_{k} \equiv c_{k}(b, \alpha)$ in terms of Stirling numbers of the second kind. We here provide a recurrence relation for determining the coefficients $c_{k}$, without Stirling numbers. We also consider asymptotic expansions concerning Somos’ quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes $G$-function.

1. INTRODUCTION

Abel [1] derived a complete asymptotic expansion for a sequence of the following sum

$$\begin{equation}
S_{b}^{[\alpha]}(n) = \sum_{k=1}^{n} \frac{b^k}{k^{\alpha}}
\end{equation}$$

as $n \to \infty$, for fixed real $b > 1$ and $\alpha > 0$. More precisely, Abel [1] established the following asymptotic expansion:

$$\begin{equation}
S_{b}^{[\alpha]}(n) \sim b^{n} \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} \frac{a_{k}(b)}{n^{\alpha+k}} \quad (n \to \infty),
\end{equation}$$

2010 Mathematics Subject Classification. 41A60, 40A05.
Keywords and Phrases. Asymptotic expansion, Somos’ quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, Barnes $G$-function.
where \( a_k(b) \) are given by

\[
a_k(b) = b \sum_{j=0}^{k} \frac{j! \sigma(k,j)}{(b-1)^{j+1}},
\]

and \( \sigma(k,j) \) denote Stirling numbers of the second kind. Stirling numbers of the second kind can be computed by the formula (see, e.g., [54, p. 99])

\[
\sigma(k,j) = \frac{1}{j!} \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} i^k.
\]

**Remark 1.** Hassani [34] proposed the following problem: Prove that

\[
\sum_{k=1}^{n} \frac{2^k - 1}{k} = \frac{2^{n+1}}{n}(1 + R_n), \quad \text{where} \quad R_n \sim \frac{1}{n} \quad (n \to \infty).
\]

This problem has been proved by Giuliano [31] Simic [51]. Abel [1] gave a complete asymptotic expansion for the sequence \( \{S_b(n)\}_{n \in \mathbb{N}} \) of sums

\[
S_b(n) = \sum_{k=1}^{n} \frac{b^k - 1}{k} \sim b^n \sum_{k=0}^{\infty} \frac{a_k(b)}{n^{k+1}} \quad (n \to \infty),
\]

as \( n \to \infty \), for fixed real \( b > 1 \). The choice \( b = 2 \) yields a solution of the problem [34] by Hassani. Also in [1], Abel derived a complete asymptotic expansion for \( S_b^{(a)}(n) \), and pointed out that \( S_b(n) \) is asymptotically equivalent to \( S_b^{(1)}(n) \) as \( n \to \infty \).

Alzer et al. [3] applied a classical series identity involving the psi function with a view to deriving series representations for a number of known mathematical constants. Chen and Srivastava [18] obtained several properties associated with inequalities and the logarithmically complete monotonicity of functions related to the gamma and psi functions and the Barnes \( G \)-function. Chen et al. [21] presented some properties associated with the monotonicity and the complete monotonicity of the psi function and establish the higher-order estimate for the familiar Euler-Mascheroni constant. Chen and Srivastava [19] established new analytical representations for the Euler-Mascheroni constant in terms of the psi function. Chen and Srivastava [20] established several further analytical representations for the Euler-Mascheroni constant in terms of the psi function.

In this paper, we aim to provide a recurrence relation for determining the coefficients of \( n^{-(\alpha+k)} \) in Abel’s expansion (2), without help of Stirling numbers of the second kind. We also consider asymptotic expansions concerning Somos’ quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes \( G \)-function.
2. COEFFICIENTS OF ABEL’S EXPANSION

Theorem 1. For reals $\alpha > 0$ and $b > 1$, the following asymptotic formula holds

\[
S_{b}[\alpha](n) \sim b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \quad (n \to \infty),
\]

where the coefficients $c_k \equiv c_k(b, \alpha)$ are given by the recurrence relation

\[
c_0 = \frac{b}{b-1}, \quad c_k = \frac{1}{b-1} \sum_{j=0}^{k-1} c_j \binom{k + \alpha - 1}{k - j} \quad (k \geq 1).
\]

Namely,

\[
S_{b}[\alpha](n) \sim b^n \left\{ \frac{1}{b-1} + \frac{\alpha}{(b-1)^2 n} + \frac{\alpha(1 + \alpha)(b+1)}{2(b-1)^3 n^2} 
\right.
\]

\[
+ \frac{\alpha(1 + \alpha)(2 + \alpha)(b^2 + 4b + 1)}{6(b-1)^4 n^3} + \ldots \right\} \quad (n \to \infty).
\]

Proof. Denote $S_n \equiv S_{b}[\alpha](n)$ and $T_n = b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}}$. In view of (2), we can let $S_n \sim T_n$ and

\[
\Delta S_n := S_n - S_{n-1} \sim T_n - T_{n-1} =: \Delta T_n
\]

as $n \to \infty$, where $c_k$ are real numbers to be determined.

It is easy to see that

\[
\Delta S_n = b^n n^{-\alpha}.
\]

Direct computation yields

\[
\sum_{k=0}^{\infty} \frac{c_k}{(n-1)^{k+\alpha}} = \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \left\{ 1 - \frac{1}{n} \right\}^{k+\alpha} = \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} (-1^j \frac{k+\alpha-1}{j}) \frac{1}{n^j}
\]

\[
= \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} \binom{k + \alpha + j - 1}{j} \frac{1}{n^j} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} c_j \binom{k + \alpha - 1}{k - j} \frac{1}{n^{k+\alpha}}.
\]

We then obtain

\[
\Delta T_n = \sum_{k=0}^{\infty} \left\{ b^n c_k - \sum_{j=0}^{k} b^{n-1} c_j \binom{k + \alpha - 1}{k - j} \right\} \frac{1}{n^{k+\alpha}}
\]

\[
= \frac{b^n(1 - \frac{1}{b}) c_0}{n^{\alpha}} + b^n \sum_{k=1}^{\infty} \left\{ c_k - \sum_{j=0}^{k} c_j \frac{b}{b} \binom{k + \alpha - 1}{k - j} \right\} n^{-(k+\alpha)}.
\]
Equating coefficients of the term \( n^{-(k+\alpha)} \) \((k = 0, 1, 2, \ldots)\) on the right-hand sides of (8) and (9) yields \( c_0 = \frac{1}{b} \) and

\[
k_k = \sum_{j=0}^{k} \frac{c_j}{b} \binom{k + \alpha - 1}{k - j} = \sum_{j=0}^{k-1} \frac{c_j}{b} \binom{k + \alpha - 1}{k - j} + \frac{c_k}{b} \quad (k \geq 1),
\]

which gives the desired formula (5). The proof is complete.

3. SOMOS’ QUADRATIC RECURRENCE CONSTANT

Somos’ quadratic recurrence constant is defined by

\[
\sigma = \sqrt{1 \sqrt{2} \sqrt{3} \cdots} = \prod_{n=1}^{\infty} n^{1/2^n} = 1.66168794\ldots
\]

The constant \( \sigma \) arises in the study of the asymptotic behaviour of the sequence

\[
g_0 = 1, \quad g_n = n g_{n-1}^2, \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\},
\]

with first few terms

\[
g_0 = 1, \quad g_1 = 1, \quad g_2 = 2, \quad g_3 = 12, \quad g_4 = 576, \quad g_5 = 1658880, \ldots.
\]

This sequence behaves as follows (see [30, p. 446]):

\[
g_n \sim \sigma^{2^n} \left( 1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \cdots \right)^{-1} \quad (n \to \infty).
\]

The constant \( \sigma \) appears in important problems from pure and applied analysis, which is a motivation of a large number of papers (see, e.g., [12, 14, 16, 33, 35, 39, 40, 41, 43, 47, 48, 52]).

Nemes [43] studied the coefficients in the asymptotic expansion (12) and developed some recurrence relations. Chen [12] presented the following asymptotic expansion:

\[
g_n \sim \frac{\sigma^{2^n}}{n} \left( 1 - \frac{2}{n} + \frac{5}{n^2} - \frac{16}{n^3} + \frac{66}{n^4} - \frac{348}{n^5} + \cdots \right),
\]

which yields

\[
g_n \sim \frac{\sigma^{2^n}}{n} \exp \left\{ -\frac{2}{n} + \frac{3}{n^2} - \frac{26}{3n^3} + \frac{75}{2n^4} - \frac{1082}{5n^5} + \cdots \right\}.
\]

In this section, we provide a recurrence relation for determining the coefficients of \( n^{-k} \) in the expansion (14), asserted in Theorem 2.
Theorem 2. As \( n \to \infty \), the following asymptotic expansion holds

\[
g_n \sim \frac{\sigma^2}{n} \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{n^k} \right),
\]

where the coefficients \( p_k \) are given by the recurrence relation

\[
p_1 = -2, \quad p_k = (-1)^k \left\{ \frac{2}{k} + \sum_{j=1}^{k-1} (-1)^j p_j \left( \frac{k-1}{k-j} \right) \right\} \quad (k \geq 2).
\]

Proof. In view of (14), we can let

\[
g_n \sim \frac{\sigma^2}{n} \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{n^k} \right)
\]

as \( n \to \infty \), where \( p_k \) (\( k \in \mathbb{N} \)) are real numbers to be determined. Denote

\[
U_n = \frac{\ln k}{2^k} + \frac{\ln n}{2^n} - \ln \sigma \quad \text{and} \quad V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{p_k}{n^k}.
\]

Noting that \( g_n = \prod_{k=1}^{n} k^{2^{n-k}} \), we can let \( U_n \sim V_n \) and

\[
\Delta U_n := U_{n+1} - U_n \sim V_{n+1} - V_n =: \Delta V_n \quad \text{as} \quad n \to \infty.
\]

It is easy to see that

\[
\Delta U_n = \frac{1}{2^n} \ln \left( 1 + \frac{1}{n} \right) = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{(-1)^{-1}}{k} n^{-k}.
\]

Direct computation yields

\[
\sum_{k=1}^{\infty} \frac{p_k}{(n+1)^k} = \sum_{k=1}^{\infty} \frac{p_k}{n^k} \left( \frac{1}{n} + \frac{1}{n^2} \right)^{-k} = \sum_{k=1}^{\infty} \frac{p_k}{n^k} \sum_{j=0}^{\infty} \left( \frac{-1}{n^j} \right) \frac{1}{n^j} = \frac{1}{2^n} \sum_{j=0}^{\infty} \left( k + j - 1 \right) \frac{1}{n^j}.
\]

We then obtain

\[
\Delta V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} \frac{p_j}{2} (k-j) \left( \frac{k-1}{k-j} \right) - p_k \right\} \frac{1}{n^k}.
\]
Equating coefficients of the term $n^{-k}$ on the right-hand sides of (18) and (20) yields

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^{k} \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - p_k \quad (k \geq 1).$$

For $k = 1$ we obtain $p_1 = -2$, and for $k \geq 2$ we have

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^{k-1} \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - \frac{1}{2} p_k \quad (k \geq 2),$$

which gives the desired formula (16). The proof is complete.

4. BARNES $G$-FUNCTION

The double gamma function $\Gamma_2$ and the multiple gamma functions $\Gamma_n$ were introduced and investigated by Barnes in a series of papers [4, 5, 6, 7]. Barnes applied these functions in the theory of elliptic functions and theta functions. Nonetheless, except possibly for the citations of $\Gamma_2$ in the exercises by Whittaker and Watson [57, p.264] and also by Gradshteyn and Ryzhik [32, p.661, Entry 6.441 (4); p.937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of determinants of Laplacians on the $n$-dimensional unit sphere $S^n$ (see, e.g., [24, 38, 44, 49, 55, 56]). The theory of the double gamma function has indeed found interesting applications in many other recent investigations (see, for details, [53, 54]).

Barnes [4] defined the double gamma function (or Barnes $G$-function) $\Gamma_2 = 1/G$ satisfying each of the following properties:

(i) $G(z + 1) = \Gamma(z)G(z)$, for all complex $z$,

(ii) $G(1) = 1$,

(iii) As $n \to \infty$,

$$\ln G(n + 2) = \frac{n + \frac{1 + z}{2}}{2} \ln(2\pi) + \left( \frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n + 1)z \right) \ln n$$

$$- \frac{3n^2}{4} - n - nz - \ln A + \frac{1}{12} + O(n^{-1}),$$

(21)

where $\Gamma$ is the gamma function and $A$ is called the Glaisher-Kinkelin constant defined by

$$\ln A = \lim_{n \to \infty} \left\{ \ln \left( \prod_{k=1}^{n} k^k \right) - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \right\},$$

(22)

the numerical value of $A$ being $1.28242713\ldots$. 

The Glaisher-Kinkelin constant $A$ can be expressed as follows (see [29])

\[
A = \lim_{n \to \infty} n^{-n^2/2-n/2-1/12}e^{n^2/4} \prod_{k=1}^{n} k^{k},
\]

and (see [23, p. 129, Eq. (3.22)])

\[
e^{1/12}A = \lim_{n \to \infty} \frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}},
\]

where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [27]). The constant $A$ has drawn attention in many works (for example) [13, 17, 23, 26, 27] as well as in [4]. Finch shared a section in his book [30, pp. 135–138] to introduce this Glaisher-Kinkelin constant $A$. The Glaisher-Kinkelin constant $A$ plays an important role in the study of Barnes $G$-function (for details, see, e.g., [54, Section 1.4]).

The following integral representation for Barnes $G$-function was established by Ferreira and López [29, Theorem 1]: For $|\text{Arg}(z)| < \pi$,

\[
\ln G(z+1) = \frac{1}{4} z^2 + z \ln \Gamma(z+1) - \left(\frac{1}{2} z^2 + \frac{1}{2} z + \frac{1}{12}\right) \ln z - \ln A \\
+ \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}} + R_N(z) \quad (N = 1, 2, \ldots),
\]

where $B_{2k+2}$ are the Bernoulli numbers. The remainder $R_N(z)$ is for $\Re(z) > 0$ given by

\[
R_N(z) = \int_{0}^{\infty} \left(\frac{t}{e^t - 1} - \sum_{k=0}^{2N} \frac{B_k}{k!} t^k\right) e^{-zt} \frac{t}{t^3} dt.
\]

Estimates for $|R_N(z)|$ are also found by Ferreira and López [29], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative real axis. Pedersen [45, Theorem 1.1] proved that for any $N \geq 1$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on $(0, \infty)$. Other asymptotic relations (avoiding the $\ln \Gamma$ term) has been obtained by Ruijsenaars [50] and investigated by Pedersen [46], Koumandos [36] and Koumandos and Pedersen [37]. Some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions, see [8, 9, 10, 18]. Chen [11] and Mortici [42] established the inequalities and asymptotic expansions for $\ln A$ in (22). Chen and Lin [17] and Chen [13] presented a class of asymptotic expansions related to Glaisher-Kinkelin constant and the Barnes $G$-function.
As $x \to \infty$, the Stirling formula for Barnes $G$-function can be found (see [53, p. 26]):

$$
\ln G(x + 1) = \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right) \ln x + O(x^{-1}).
$$

(28)

In this section, we develop the formula (28) to produce a complete asymptotic expansion given by Theorem 3.

**Theorem 3.** As $x \to \infty$, the following asymptotic expansion holds

$$
\ln G(x + 1) = \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right) \ln x + \sum_{k=1}^{\infty} \frac{q_k}{x^k},
$$

(29)

where the coefficients $q_k$ are given by the recurrence relation

$$
q_1 = 0, \quad q_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=1}^{k-1} q_j (-1)^j \left( \frac{k}{k-j+1} \right) + \frac{(-1)^k B_{k+2}}{(k+1)(k+2)} \right\} \quad \text{for} \quad k \geq 2.
$$

(30)

Here $B_n$ denote Bernoulli numbers. Namely,

$$
\ln G(x + 1) \sim \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right) \ln x
$$

$$
- \frac{1}{240x^2} + \frac{1}{1008x^3} - \frac{1}{1440x^4} + \frac{1}{1056x^5} - \frac{691}{32760x^6} + \cdots
$$

(31)

**Proof.** Denote

$$
P(x) = \ln G(x + 1) - \left\{ \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right) \ln x \right\}
$$

and

$$
Q(x) = \sum_{k=1}^{\infty} \frac{q_k}{x^k}.
$$

In view of (28), we can let $P(x) \sim Q(x)$ and

$$
\Delta P(x) := P(x + 1) - P(x) \sim Q(x + 1) - Q(x) =: \Delta Q(x)
$$

(32)

as $x \to \infty$, where $q_k$ are real numbers to be determined.

Noting that $\ln G(x + 1) - \ln G(x) = \ln \Gamma(x)$ and

$$
\ln \Gamma(x + 1) \sim x \ln x - x + \ln \sqrt{2\pi x} + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)x^k}.
$$

(33)
Asymptotic expansions . . .

(see [2, p. 257, Eq. (6.1.40)]), we obtain

\[
\Delta P(x) = \ln \Gamma(x + 1) - x \ln x + x - \ln \sqrt{2\pi x} + \frac{2x + 3}{4} - \left(\frac{x^2}{2} + x + \frac{5}{12}\right) \ln \left(1 + \frac{1}{x}\right)
\]

\[
\sim \sum_{k=3}^{\infty} \left\{ \frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} \right\} \frac{1}{x^k}
\]

By (19), we have

\[
\Delta Q(x) = \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} q_j (-1)^{k-j} \binom{k-1}{k-j} - q_k \right\} \frac{1}{x^k}
\]

\[
= \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j} \right\} \frac{1}{x^k}
\]

\[
= \frac{q_1}{x^2} + \sum_{k=3}^{\infty} \left\{ \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j} \right\} \frac{1}{x^k}
\]

Equating coefficients of the term \(x^{-k}\) on the right-hand sides of (34) and (35) yields

\[
q_1 = 0 \quad \text{and for } k \geq 3,
\]

\[
\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j},
\]

\[
\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-2} q_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1)q_{k-1},
\]

\[
q_{k-1} = \frac{(-1)^k}{k-1} \left\{ \sum_{j=1}^{k-2} q_j (-1)^j \binom{k-1}{k-j} + \frac{(-1)^{k+1}B_{k+1}}{k(k+1)} + \frac{(k+5)(k-2)}{12(k+2)(k+1)k} \right\},
\]

which can be written as (30). The proof is complete.

5. GLAISHER-KINKELIN AND CHOI-SRIVASTAVA CONSTANTS

Choi and Srivastava (see [26, p.102] and [27]) introduced two mathematical constants \(B\) and \(C\) (analogous to the Glaisher-Kinkelin constant \(A\)) defined by

\[
\ln B = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \ln n + \frac{n^3}{9} - \frac{n}{12} \right\}
\]
and

\[
\ln C = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} k^3 \ln k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \right\}
\]

for which the approximate numerical values are given by

\[ B = 1.03091675 \ldots \quad \text{and} \quad C = 0.97955746 \ldots \]

Like the expression of the Glaisher-Kinkelin constant \( A \) in (25), the constants \( B \) and \( C \) are also known to be expressible in terms of special values of the derivative of the Riemann zeta function \( \zeta(s) \) as follows (see [27] and [28, Eq. (1.9)]):

\[
\ln B = -\zeta'(-2) \quad \text{and} \quad \ln C = -\frac{11}{720} - \zeta'(-3).
\]

As the Euler-Mascheroni constant \( \gamma \) is involved in the classical gamma function \( \Gamma \), the constants \( A, B \) and \( C \) have appeared naturally in the theory of the multiple gamma functions \( \Gamma_n \) (see, e.g., [54, Section 1.4]) and play their respective roles, for example ([53, p. 39, p. 247], [25, p. 523, Eq. (2.50)], [23]).

Chen [11] established the asymptotic expansions related to the Glaisher-Kinkelin constant \( A \) and Choi-Srivastava constants \( B \) and \( C \). Mortici [42] dealt with the same problem. Recently, Cheng and Chen [22] and Chen and Choi [15] established new asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants. For example, by using Bernoulli numbers, Chen [11] established the asymptotic expansions related to the constants \( A, B \) and \( C \).

In this section, we provide a recurrence relation for determining the coefficients of each asymptotic expansion related to the constants \( A, B \) and \( C \), without Bernoulli numbers.

**Theorem 4.** As \( n \to \infty \), the following asymptotic expansion holds

\[
\sum_{k=1}^{n} k \ln k - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \sim \ln A - \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k},
\]

where the coefficients \( \lambda_k \) are given by the recurrence relation

\[
\lambda_2 = -\frac{1}{720}, \quad \lambda_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=2}^{k-1} \lambda_j (-1)^j \left( \frac{k}{k-j+1} \right) + \frac{k(k-1)}{12(k+1)(k+2)(k+3)} \right\}
\]

for \( k \geq 3 \). Namely,

\[
\sum_{k=1}^{n} k \ln k - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \sim \ln A + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \ldots.
\]
We can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n$$

as $n \to \infty$, where $\lambda_k$ are real numbers to be determined.

We obtain, after some elementary transformations, that

$$\Delta I_n = \frac{n}{2} + \frac{1}{4} - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right) \ln \left(1 + \frac{1}{n}\right) = -\sum_{k=3}^{\infty} \frac{(-1)^{k-1}(k-1)(k-2)}{2k(k+1)(k+2)} n^{-k}. $$

Direct computation yields

$$\sum_{k=2}^{\infty} \frac{\lambda_k}{(n+1)^k} = \sum_{k=2}^{\infty} \lambda_k \left(1 + \frac{1}{n}\right)^{-k} = \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \sum_{j=0}^{\infty} \binom{k-1}{j} 1 \frac{1}{n^j} = \sum_{k=2}^{\infty} \lambda_k \left(\frac{1}{n}\right)^{k-1} \binom{k-1}{j} \frac{1}{n^j}.$$  

We then obtain

$$\Delta J_n = -\sum_{k=3}^{\infty} \left\{ \sum_{j=2}^{\infty} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - \lambda_k \right\} \frac{1}{n^k}$$

$$= -\sum_{k=3}^{\infty} \left\{ \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \right\} n^{-k}.$$  

Equating coefficients of the term $n^{-k}$ on the right-hand sides of (42) and (43) yields

$$\frac{(-1)^{k-1}(k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \quad (k \geq 3).$$  

For $k = 3$ we obtain $\lambda_2 = -\frac{1}{120}$, and for $k \geq 4$ we have

$$\frac{(-1)^{k-1}(k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-2} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1) \lambda_{k-1} \quad (k \geq 4),$$

$$\lambda_{k-1} = \frac{(-1)^k}{k-1} \left\{ \sum_{j=2}^{k-2} \lambda_j (-1)^j \binom{k-1}{k-j} + \frac{(k-1)(k-2)}{12k(k+1)(k+2)} \right\} \quad (k \geq 4),$$

which can be written as (40). The proof is complete.

Following the same method as was used in the proof of Theorem 4, we can prove the following Theorems 5 and 6. We here omit the proofs.
**Theorem 5.** As \(n \to \infty\), the following asymptotic expansion holds

\[
\sum_{k=1}^{n} k^2 \ln k - \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B + \sum_{k=1}^{\infty} \frac{\mu_k}{n^k},
\]

where the coefficients \(\mu_k\) are given by the recurrence relation

\[
\mu_1 = -\frac{1}{360}, \quad \mu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=1}^{k-1} \mu_j (-1)^j \left( \frac{k}{k-j+1} \right) - \frac{k}{6(k+2)(k+3)(k+4)} \right\}
\]

for \(k \geq 2\). Namely,

\[
\sum_{k=1}^{n} k^2 \ln k - \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B - \frac{1}{360n^2} + \frac{1}{7560n^3} - \frac{1}{25200n^4} + \frac{1}{33264n^5} - \frac{691}{16216200n^6} + \ldots
\]

**Theorem 6.** As \(n \to \infty\), the following asymptotic expansion holds

\[
\sum_{k=1}^{n} k^3 \ln k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \sum_{k=2}^{\infty} \frac{\nu_k}{n^k},
\]

where the coefficients \(\nu_k\) are given by the recurrence relation

\[
\nu_2 = \frac{1}{5040} \quad \text{and} \quad \nu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=2}^{k-1} \nu_j (-1)^j \left( \frac{k}{k-j+1} \right) - \frac{k(k-1)(k+13)}{120(k+1)(k+3)(k+4)(k+5)} \right\}
\]

for \(k \geq 3\). Namely,

\[
\sum_{k=1}^{n} k^3 \ln k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \frac{1}{5040n^2} + \frac{1}{33600n^3} - \frac{1}{66528n^4} + \frac{691}{43243200n^5} - \frac{1}{34320n^6} + \ldots
\]
REFERENCES

1. U. Abel: A complete asymptotic expansion for a sequence of certain sums. Appl. Math. Comput., 217 (2010), 4302–4305.
2. M. Abramowitz, I.A. Stegun (eds.): Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, Applied Mathematics Series, vol. 55. 9th printing, Dover, New York, 1972.
3. H. Alzer, D. Karayannakis, H.M. Srivastava: Series representations for some mathematical constants. J. Math. Anal. Appl. 320 (2006), 145–162.
4. E. W. Barnes: The theory of G-function. Quart. J. Math., 31 (1899), 264–314.
5. E. W. Barnes: Genesis of the double gamma function. Proc. Lond Math. Soc., 31 (1900), 358–381.
6. E. W. Barnes: The theory of the double gamma function. Philos. Trans. R. Soc. Lond. Ser. A, 196 (1901), 265–388.
7. E. W. Barnes: On the theory of the multiple gamma functions. Trans. Cambridge Philos. Soc., 19 (1904), 374–439.
8. N. Batir: Inequalities for the double gamma function. J. Math. Anal. Appl., 351 (2009), 182–185.
9. N. Batir, M. Cancan: A double inequality for the double gamma function. Int. J. Math. Anal., 2 (2008), 329–335.
10. C.-P. Chen: Inequalities associated with Barnes G-function. Expo. Math., 29 (2011), 119–125.
11. C.-P. Chen: Glaisher-Kinkelin constant. Integral Transforms Spec. Funct., 23 (2012), 785–792.
12. C.-P. Chen: New asymptotic expansions related to Somos’ quadratic recurrence constant. C. R. Acad. Sci. Paris, Ser. I, 351 (2013), 9–12.
13. C.-P. Chen: Asymptotic expansions for Barnes G-function. J. Number Theory, 135 (2014), 36–42.
14. C.-P. Chen: Sharp inequalities and asymptotic series related to Somos’ quadratic recurrence constant. J. Number Theory, 172 (2017), 145–159.
15. C.-P. Chen, J. Choi: Unified treatment of several asymptotic expansions concerning some mathematical constants. Appl. Math. Comput., 305 (2017), 348–363.
16. C.-P. Chen, X.-F. Han: On Somos’ quadratic recurrence constant. J. Number Theory, 166 (2016), 31–40.
17. C.-P. Chen, L. Lin: Asymptotic expansions related to Glaisher-Kinkelin constant based on the Bell polynomials. J. Number Theory, 133 (2013), 2699–2705.
18. C.-P. Chen, H. M. Srivastava: Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function. Integral Transforms Spec. Funct., 22 (2011), 1–15.
19. C.-P. Chen, H. M. Srivastava: New representations for the Lugo and Euler-Mascheroni constants. Appl. Math. Lett. 24 (2011), 1239–1244.
20. C.-P. Chen, H. M. Srivastava: New representations for the Lugo and Euler-Mascheroni constants. II. Appl. Math. Lett. 25 (2012), 333–338.
21. C.-P. Chen, H. M. Srivastava, L. Li, S. Manyama: Inequalities and monotonicity properties for the psi (or digamma) function and estimates for the Euler–Mascheroni constant. Integral Transforms Spec. Funct. 22 (2011), 681–693.

22. J.-X. Cheng, C.-P. Chen: Asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants. J. Number Theory, 144 (2014), 105–110.

23. J. Choi: Some mathematical constants. Appl. Math. Comput., 187 (2007), 122–140.

24. J. Choi: Determinant of Laplacian on $S^3$. Math. Japon., 40 (1994), 155–166.

25. J. Choi, Y.J. Cho, H.M. Srivastava: Series involving the zeta function and multiple Gamma functions. Appl. Math. Comput., 159 (2004), 509–537.

26. J. Choi, H.M. Srivastava: Certain classes of series involving the zeta function. J. Math. Anal. Appl., 231 (1999), 91–117.

27. J. Choi, H.M. Srivastava: Certain classes of series associated with the zeta function and multiple Gamma functions. J. Comput. Appl. Math., 118 (2000), 87–109.

28. J. Choi, H. M. Srivastava: Asymptotic formulas for the triple gamma function $\Gamma_3$ by means of its integral representation. Appl. Math. Comput., 218 (2011), 2631–2640.

29. C. Ferreira, J. L. López: An asymptotic expansion of the double gamma function. J. Approx. Theory, 111 (2001), 298–314.

30. S. R. Finch: Mathematical Constants. Cambridge Univ. Press, 2003.

31. R. Giuliano: A possible solution to the problem of Hassani. RGMIA mailing list, 2008.

32. I. S. Gradshteyn, I. M. Ryzhik: Tables of Integrals, Series, and Products (Corrected and Enlarged Edition prepared by A. Jeffrey). Academic Press, New York, 1980.

33. J. Guillera, J. Sondow: Double integrals and infinite products for some classical constants via analytic continuations of Lerch’s transcendent. Ramanujan J., 16 (2008), 247–270.

34. M. Hassani: Problem. RGMIA mailing list, 2008.

35. M.D. Hirschhorn: A note on Somos’ quadratic recurrence constant. J. Number Theory, 131 (2011), 2061–2063.

36. S. Koumandos: On Ruijsenaars’ asymptotic expansion of the logarithm of the double gamma function. J. Math. Anal. Appl., 341 (2008), 1125–1132.

37. S. Koumandos, H. L. Pedersen: Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function. J. Math. Anal. Appl., 355 (2009), 33–40.

38. H. Kumagai: The determinant of the Laplacian on the n-sphere. Acta Arith., 91 (1999), 199–208.

39. V. Lampret: Approximation of Sondow’s generalized-Euler-constant function on the interval $[-1, 1]$. Ann. Univ. Ferrara, 56 (2010), 65–76.

40. D. Lu, Z. Song: Some new continued fraction estimates of the Somos’ quadratic recurrence constant. J. Number Theory, 155 (2015), 36–45.

41. C. Mortici: Estimating the Somos’ quadratic recurrence constant. J. Number Theory, 130 (2010), 2650–2657.
42. C. Mortici: Approximating the constants of Glaisher-Kinkelin type. J. Number Theory, **133** (2013), 2465–2469.

43. G. Nemes: On the coefficients of an asymptotic expansion related to Somos’ quadratic recurrence constant. Appl. Anal. Discrete Math., **5** (2011), 60–66.

44. B. Osgood, R. Phillips, P. Sarnak: Extremals of determinants of Laplacians. J. Funct. Anal., **80** (1988), 148–211.

45. H. L. Pedersen: On the remainder in an asymptotic expansion of the double gamma function. Mediterr. J. Math., **2** (2005), 171–178.

46. H. L. Pedersen: The remainder in Ruijsenaars’ asymptotic expansion of Barnes double gamma function. Mediterr. J. Math., **4** (2007), 419–433.

47. K.H. Pilehrood, T.H. Pilehrood: Vacca-type series for values of the generalized Euler constant function and its derivative. J. Integer Sequences, **13** (2010), Article 10.7.3

48. K.H. Pilehrood, T.H. Pilehrood: Arithmetical properties of some series with logarithmic coefficients. Math. Z., **255** (2007), 117–131.

49. J. R. Quine, J. Choi: Zeta regularized products and functional determinants on spheres. Rocky Mountain J. Math., **26** (1996), 719–729.

50. S. N. M. Ruijsenaars: On Barnes’ multiple zeta and gamma functions. Adv. Math., **156** (2000), 107–132.

51. S. Simic: A simpler solution of the recent problem. RGMIA mailing list, 2008.

52. J. Sondow, P. Hadjicostas: The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos’s quadratic recurrence constant. J. Math. Anal. Appl., **332** (2007), 292–314.

53. H. M. Srivastava, J. Choi: Series Associated with the zeta and related functions. Kluwer Academic Publishers, Dordrecht, 2001.

54. H. M. Srivastava and J. Choi: Zeta and $q$-zeta functions and associated series and integrals. Elsevier Science Publishers, Amsterdam, London and New York, 2012.

55. I. Vardi: Determinants of Laplacians and multiple gamma functions. SIAM J. Math. Anal., **19** (1988), 493–507.

56. A. Voros: Special functions, spectral functions and the Selberg Zeta function. Comm. Math. Phys., **110** (1987), 439–465.

57. E. T. Whittaker, G. N. Watson: A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, 4th ed. (Reprinted), Cambridge University Press, Cambridge, London and New York, 1963.

---

**Chao-Ping Chen**

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454000, Henan Province, China

E-mail: chenchaoping@sohu.com

(Received 08.04.2018)  (Revised 28.08.2018)