SYMMETRIC BILINEAR FORM ON A LIE ALGEBRA

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Abstract. Let \( g \) be the finite dimensional simple Lie algebra associated to an indecomposable and symmetrizable generalized Cartan matrix \( C = (a_{ij})_{n \times n} \) of finite type and let \( \mathfrak{d} \) be a finite dimensional Lie algebra related to a quantum group \( D_{q,p}^{-1}(g) \) obtained by Hodges, Levasseur and Toro \( \text{[1]} \) by deforming the quantum group \( U_q(g) \). Here we see that \( \mathfrak{d} \) is a generalization of \( g \) and give a \( \mathfrak{d} \)-invariant symmetric bilinear form on \( \mathfrak{d} \).

Let \( C = (a_{ij})_{n \times n} \) be an indecomposable and symmetrizable generalized Cartan matrix of finite type and let \( g \) be the finite dimensional simple Lie algebra associated to \( C \). (Refer to \( \text{[2, Chapter 2]} \) and \( \text{[4, Chapter 1, 2, 4]} \) for details.) Hodges, Levasseur and Toro constructed a quantum group \( D_{q,p}^{-1}(g) \) in \( \text{[1, Theorem 3.5]} \) by deforming the quantum group \( U_q(g) \), which is considered as a generalization of \( U_q(g) \), and obtained a Hopf dual \( \mathbb{C}_{q,p}[G] \) of \( D_{q,p}^{-1}(g) \) in \( \text{[1, §3]} \) that is a generalization of the Hopf algebra \( \mathbb{C}_q[G] \) studied in \( \text{[3]} \) and \( \text{[7, §3]} \). The second author constructed a finite dimensional Lie algebra \( \mathfrak{d} \) in \( \text{[6, Theorem 1.3]} \) by using a skew symmetric bilinear form \( u \) on a Cartan subalgebra \( \mathfrak{h} \) of \( g \). He showed in \( \text{[6, Proposition 1.4]} \) that \( \mathfrak{d} \) is a generalized Lie bialgebra of the standard Lie bialgebra given in \( g \). Moreover he studied in \( \text{[6, §3 and §5]} \) the Poisson structure of the Hopf dual \( \mathbb{C}[G] \) of the universal enveloping algebra \( U(\mathfrak{d}) \) that is considered as a Poisson version of \( \mathbb{C}_{q,p}[G] \). In this note we see that \( \mathfrak{d} \) is a generalization of \( g \) (Proposition \( \text{[3]} \)) and find a \( \mathfrak{d} \)-invariant symmetric bilinear form on \( \mathfrak{d} \) (Theorem \( \text{[4]} \)).

We begin with explaining the notations in \( \text{[6, 1.1]} \). Let \( C = (a_{ij})_{n \times n} \) be an indecomposable and symmetrizable generalized Cartan matrix of finite type. Hence there exists a diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \), where all \( d_i \) are positive integers, such that the matrix \( DC \) is symmetric positive definite. (Each \( d_i \) is denoted by \( s_i \) and \( \epsilon_i^{-1} \) in \( \text{[2, §2.3]} \) and \( \text{[4, Chapter 2]} \) respectively.) Throughout the paper, we denote by

\[
\begin{align*}
g &= (g, [\cdot, \cdot]_g) \quad \text{the finite dimensional simple Lie algebra over the complex number field } \mathbb{C} \text{ associated to } C, \\
\mathfrak{h} &= \text{a Cartan subalgebra of } g \text{ with simple roots } \alpha_1, \ldots, \alpha_n.
\end{align*}
\]

Choose \( h_i \in \mathfrak{h}, 1 \leq i \leq n \), such that

\[
(1) \quad \alpha_j : \mathfrak{h} \rightarrow \mathbb{C}, \quad \alpha_j(h_i) = a_{ij} \quad \text{for all } j = 1, \ldots, n.
\]
Then \( \{h_i\}_{i=1}^{n} \) forms a basis of \( \mathfrak{h} \), since \( C \) has rank \( n \), and \( \mathfrak{g} \) is generated by \( h_i \) and \( x_{\pm \alpha_i} \), \( i = 1, \ldots, n \), with relations

\[
[h_i, h_j]_{\mathfrak{g}} = 0, \quad [h_i, x_{\pm \alpha_i}]_{\mathfrak{g}} = \pm a_{ij} x_{\pm \alpha_j}, \quad [x_{\alpha_i}, x_{-\alpha_j}]_{\mathfrak{g}} = \delta_{ij} h_i,
\]

\[
(\text{ad}_{x_{\alpha_i}})^{1-a_{ij}}(x_{\pm \alpha_j}) = 0, \quad i \neq j
\]

by [2] Definition 2.1.3. (In [2] Definition 2.1.3, \( x_{\alpha} \) and \( x_{-\alpha} \) are denoted by \( e_i \) and \( f_i \) respectively.) Denote by

- \( n^+ \) the subspace of \( \mathfrak{g} \) spanned by root vectors with positive roots
- \( n^- \) the subspace of \( \mathfrak{g} \) spanned by root vectors with negative roots

and set

\[
n = n^- \oplus n^+.
\]

Hence

\[
\mathfrak{g} = \mathfrak{h} \oplus n = n^- \oplus \mathfrak{h} \oplus n^+.
\]

Henceforth we mean by \( x_{\alpha} \) that \( x_{\alpha} \) is a root vector of \( n \) with root \( \alpha \).

By [2] §2.3, there exists a nondegenerate symmetric bilinear form \((\cdot|\cdot)\) on \( \mathfrak{h}^* \) given by

\[
(\alpha|\alpha_j) = d_i a_{ij}
\]

for all \( i, j = 1, \ldots, n \). This form \((\cdot|\cdot)\) induces the isomorphism

\[
\mathfrak{h}^* \rightarrow \mathfrak{h}, \quad \lambda \mapsto h_{\lambda},
\]

where \( h_{\lambda} \) is defined by

\[
(\alpha_i|\lambda) = \alpha_i(h_{\lambda}) \quad \text{for all } i = 1, \ldots, n.
\]

(The isomorphism \( \mathfrak{h}^* \rightarrow \mathfrak{h}, \lambda \mapsto h_{\lambda} \), is denoted by \( \nu^{-1} \) in [2] §2.3 and [4] Chapter 2.) Identifying \( \mathfrak{h}^* \) to \( \mathfrak{h} \) via \( \lambda \mapsto h_{\lambda} \), \( \mathfrak{h} \) has a nondegenerate symmetric bilinear form \((\cdot|\cdot)\) given by

\[
(\lambda|\mu) = (h_{\lambda}|h_{\mu}) = \lambda(h_{\mu}).
\]

This is extended to a nondegenerate \( \mathfrak{g} \)-invariant symmetric bilinear form on \( \mathfrak{g} \) by [4] Theorem 2.2 and its proof] and [2] (2.7) and Proposition 2.3.6;

\[
(h_i|h_j) = d^{-1}_j a_{ij}, \quad (h|x_{\alpha}) = 0, \quad (x_{\alpha}|x_{\beta}) = 0 \text{ if } \alpha + \beta \neq 0, \quad (x_{\alpha_i}|x_{\alpha_j}) = d^{-1}_i \delta_{ij}
\]

for \( h \in \mathfrak{h} \), root vectors \( x_{\alpha}, x_{\beta} \) and \( i = 1, \ldots, n \).

**Lemma 1.** [6] Lemma 1.1] For each positive root \( \alpha \),

\[
[x_{\alpha}, x_{-\alpha}]_{\mathfrak{g}} = (x_{\alpha}|x_{-\alpha}) h_{\alpha}.
\]

Let \( u = (u_{ij}) \) be a skew symmetric \( n \times n \)-matrix with entries in \( \mathbb{C} \). Then \( u \) induces a skew symmetric bilinear (alternating) form \( u \) on \( \mathfrak{h}^* \) given by

\[
u(\lambda, \mu) := \sum_{i,j} u_{ij} \lambda(h_i) \mu(h_j)
\]

for any \( \lambda, \mu \in \mathfrak{h}^* \). Hence there exists a unique linear map \( \Phi : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \) such that

\[
u(\lambda, \mu) = (\Phi(\lambda)|\mu)
\]

for any \( \lambda, \mu \in \mathfrak{h}^* \) since the form \((\cdot|\cdot)\) on \( \mathfrak{h}^* \) is nondegenerate. Set

\[
\Phi_+ = \Phi + I, \quad \Phi_- = \Phi - I,
\]
where \( I \) is the identity map on \( \mathfrak{h}^* \). Thus
\[
(\Phi_+(\lambda|\mu)) = u(\lambda, \mu) + (\lambda|\mu) \\
(\Phi_-(\lambda|\mu)) = u(\lambda, \mu) - (\lambda|\mu)
\]
for all \( \lambda, \mu \in \mathfrak{h}^* \).

Fix a vector space \( \mathfrak{k} \) isomorphic to \( \mathfrak{h} \) and let
\[
\varphi : \mathfrak{h} \rightarrow \mathfrak{k}
\]
be an isomorphism of vector spaces. For each \( \lambda \in \mathfrak{h}^* \), denote by \( k_\lambda \in \mathfrak{k} \) the element \( \varphi(h_\lambda) \). Let
\[
g' := \mathfrak{k} \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{k} \oplus \mathfrak{n}^+
\]
be the Lie algebra isomorphic to \( \mathfrak{g} \) such that each element \( k_\lambda \in \mathfrak{k} \) corresponds to the element \( h_\lambda \) and each root vector \( x_\alpha \) corresponds to \( x_\alpha \). That is, \( g' \) is the Lie algebra with Lie bracket
\[
[k_\lambda, k_\mu]_{g'} = 0, \\
[x_\alpha, x_\beta]_{g'} = [x_\alpha, x_\beta]_\mathfrak{g} \quad (\alpha \neq -\beta), \\
[x_\alpha, x_{-\alpha}]_{g'} = \varphi([x_\alpha, x_{-\alpha}]_\mathfrak{g}),
\]
where \( \lambda, \mu \in \mathfrak{h}^* \) and \( x_\alpha, x_{-\alpha}, x_\beta \) are root vectors with roots \( \alpha, -\alpha, \beta \) respectively.

**Definition 2.** [6, Theorem 1.3] The vector space \( \mathfrak{d} := \mathfrak{n}^- \oplus \mathfrak{k} \oplus \mathfrak{n}^+ \) is a Lie algebra with Lie bracket
\[
[h_\lambda, h_\mu] = 0, \quad [h_\lambda, k_\mu] = 0, \quad [k_\lambda, k_\mu] = 0, \\
[h_\lambda, x_\alpha] = - (\Phi_-(\lambda|\alpha)) x_\alpha, \\
[k_\lambda, x_\alpha] = (\Phi_+\lambda|\alpha) x_\alpha, \\
x_\alpha, x_{-\alpha} = 2^{-1}([x_\alpha, x_\beta]_\mathfrak{g} + [x_\alpha, x_{-\alpha}]_\mathfrak{g})
\]
for all \( h_\lambda, h_\mu \in \mathfrak{h}, k_\lambda, k_\mu \in \mathfrak{k} \) and root vectors \( x_\alpha, x_\beta \in \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{n}^+ \).

The Lie algebra \( \mathfrak{d} \) is a generalization of \( \mathfrak{g} \) as seen in the following proposition.

**Proposition 3.** (a) Let \( \mathfrak{l} \) be the subspace of \( \mathfrak{d} \) spanned by all \( h_\lambda - k_\lambda, \lambda \in \text{rad}(u) \), where
\[
\text{rad}(u) = \{ \lambda \in \mathfrak{h}^* | u(\lambda, \mathfrak{h}^*) = 0 \}.
\]
Then \( \mathfrak{l} \) is a solvable ideal of \( \mathfrak{d} \). In particular, if \( u = 0 \) then \( \mathfrak{l} \) is the maximal solvable ideal of \( \mathfrak{d} \) and \( \mathfrak{d}/\mathfrak{l} \) is isomorphic to \( \mathfrak{g} \).

(b) Let \( \mathfrak{m} \) be the subspace of \( \mathfrak{d} \) spanned by all root vectors \( x_\alpha \) and all \( h_\lambda + k_\lambda, \lambda \in \mathfrak{h}^* \). Then \( \mathfrak{m} \) is an ideal of \( \mathfrak{d} \) isomorphic to \( \mathfrak{g} \).

**Proof.** (a) Note that
\[
[h_\lambda, k_\lambda, x_\alpha] = -2u(\lambda, \alpha) x_\alpha
\]
for any root vector \( x_\alpha \). Hence if \( \lambda \in \text{rad}(u) \) then \( [h_\lambda, k_\lambda, x_\alpha] = 0 \) for all root vectors \( x_\alpha \). It follows that \( \mathfrak{l} \) is an ideal of \( \mathfrak{d} \). Clearly \( \mathfrak{l} \) is solvable by \( [3] \).

Suppose that \( u = 0 \). Then \( \text{rad}(u) = \mathfrak{h}^* \) and thus the ideal \( \mathfrak{l} \) is the subspace spanned by all \( h_\lambda - k_\lambda, \lambda \in \mathfrak{h}^* \), which is solvable. It is easy to see that \( \mathfrak{g} \cong \mathfrak{d}/\mathfrak{l} \) by \( [11] \) and \( [7] \). Hence \( \mathfrak{d}/\mathfrak{l} \) is simple and thus \( \mathfrak{l} \) is the unique maximal solvable ideal of \( \mathfrak{d} \).

(b) Note that
\[
[h_\lambda + k_\lambda, x_\alpha] = 2(\lambda|\alpha) x_\alpha
\]
for any root vector $x_\alpha$ and
\[
[x_\alpha, x_{-\alpha}] = 2^{-1} (x_\alpha | x_{-\alpha}) (h_\alpha + k_\alpha)
\]
for any positive root $\alpha$ by Lemma 11 and 111. Hence $\mathfrak{m}$ is an ideal of $\mathfrak{d}$ by (81-111). Moreover $\mathfrak{m}$ is isomorphic to $\mathfrak{g}$ since the linear map from $\mathfrak{m}$ into $\mathfrak{g}$ defined by
\[
h_\lambda + k_\lambda \mapsto 2h_\lambda, \quad x_\alpha \mapsto x_\alpha \quad (\lambda \in \mathfrak{h}^*)
\]
is a Lie algebra isomorphism. □

We give a $\mathfrak{d}$-invariant symmetric bilinear form on $\mathfrak{d}$ as in the following theorem.

**Theorem 4.** Define a bilinear form $(\cdot | \cdot)_{\mathfrak{d}}$ on $\mathfrak{d}$ by
\[
\begin{align*}
(12) \quad (h|h')_{\mathfrak{d}} & = 2(h|h'), \quad (h|x)_{\mathfrak{d}} = (h|x) = 0, \\
(13) \quad (x|h)_{\mathfrak{d}} & = (x|h) = 0, \quad (x|x')_{\mathfrak{d}} = (x|x'), \\
(14) \quad (h_\lambda | k_\mu)_{\mathfrak{d}} & = -2u(\lambda, \mu), \quad (k_\mu | h_\lambda)_{\mathfrak{d}} = 2u(\mu, \lambda), \\
(15) \quad (k|x)_{\mathfrak{d}} & = (\varphi^{-1}(k)|x) = 0, \quad (x|k)_{\mathfrak{d}} = (x|\varphi^{-1}(k)) = 0, \\
(16) \quad (k|k')_{\mathfrak{d}} & = 2(\varphi^{-1}(k)|\varphi^{-1}(k'))
\end{align*}
\]
for $h, h', h_\lambda \in \mathfrak{h}$, $x, x' \in \mathfrak{n}^+ \oplus \mathfrak{n}^-$, $k, k', k_\mu \in \mathfrak{k}$, where $\varphi$ is the isomorphism given in (6). Then $(\cdot | \cdot)_{\mathfrak{d}}$ is a $\mathfrak{d}$-invariant symmetric bilinear form.

**Proof.** Since $(\cdot | \cdot)$ is a symmetric bilinear form on $\mathfrak{g}$ and $u$ is a skew symmetric bilinear form on $\mathfrak{h}^*$, $(\cdot | \cdot)_{\mathfrak{d}}$ is clearly symmetric by (12)-(16).

Let us show that $(\cdot | \cdot)_{\mathfrak{d}}$ is $\mathfrak{d}$-invariant, that is,
\[
(a|b, c)_{\mathfrak{d}} = ([a, b]|c)_{\mathfrak{d}}
\]
for all $a, b, c \in \mathfrak{d}$.

Case I. $a = x_\alpha, b = x_\beta, c = x_\gamma$:

(a) $\alpha + \beta + \gamma \neq 0$: Note that $(x_\alpha | x_\beta')_{\mathfrak{d}} = (x_\alpha | x_\beta') = 0$ for root vectors $x_\alpha', x_\beta'$ of $\mathfrak{n}$ with $\alpha' + \beta' \neq 0$ by (3) and 111. Hence
\[
([x_\alpha, x_\beta]|x_\gamma)_{\mathfrak{d}} = 0
\]
since $[x_\alpha, x_\beta] \in \mathfrak{n}$ if $\alpha + \beta \neq 0$ and $[x_\alpha, x_\beta] \in \mathfrak{h} \oplus \mathfrak{k}$ if $\alpha + \beta = 0$ by (12) and 115. Similarly
\[
(x_\alpha|[x_\beta, x_\gamma])_{\mathfrak{d}} = 0
\]
since $[x_\beta, x_\gamma] \in \mathfrak{n}$ if $\beta + \gamma \neq 0$ and $[x_\beta, x_\gamma] \in \mathfrak{h} \oplus \mathfrak{k}$ if $\beta + \gamma = 0$ by (13) and 115. Hence
\[
([x_\alpha, x_\beta]|x_\gamma)_{\mathfrak{d}} = 0 = (x_\alpha|[x_\beta, x_\gamma])_{\mathfrak{d}}.
\]

(b) $\alpha + \beta + \gamma = 0$: Since $\alpha + \beta = -\gamma$, we have $[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_{\mathfrak{g}}$ by (7) and 111. Similarly, since $\beta + \gamma = -\alpha$, we also have $[x_\beta, x_\gamma] = [x_\beta, x_\gamma]_{\mathfrak{g}}$ by (7) and 111. Hence we have that
\[
([x_\alpha, x_\beta]|x_\gamma)_{\mathfrak{d}} = ([x_\alpha, x_\beta]_{\mathfrak{g}}|x_\gamma) = (x_\alpha|[x_\beta, x_\gamma]_{\mathfrak{g}}) = (x_\alpha|[x_\beta, x_\gamma])_{\mathfrak{d}}
\]
by (13).

Case II. $a = h_\lambda \in \mathfrak{h}$, $b = x_\beta, c = x_\gamma$:

(a) $\beta + \gamma = 0$: Note that $x_\gamma = x_{-\beta}$. Since $[h_\lambda, x_\beta] = -(\Phi_{-\lambda}(\beta))x_\beta$ by (9) and
\[
[x_\beta, x_{-\beta}] = 2^{-1} ([x_\beta, x_{-\beta}]_{\mathfrak{g}} + [x_\beta, x_{-\beta}]_{\mathfrak{g}'} + 2^{-1} (x_\beta | x_{-\beta}) h_\beta + 2^{-1} (x_\beta | x_{-\beta}) k_\beta
\]
by (11) and (41), we have that
\[
([h_\lambda, x_\beta]|x_\gamma) = -((\Phi - \lambda)|\beta)(x_\beta|x_\gamma) = (\lambda|\beta)(x_\beta|x_\gamma) - u(\lambda, \beta)(x_\beta|x_\gamma)
\]
(by (9))
\[
= 2^{-1}(h_\lambda)(x_\beta|x_\gamma) + 2^{-1}(h_\lambda)(x_\beta|x_\gamma)k_\beta
\]
(by (12), (14))
\[
= 2^{-1}(h_\lambda)(x_\beta|x_\gamma) + (x_\beta, x_\gamma)_{\Phi,}\beta
\]
(by (14))
\[
= (h_\lambda)(x_\beta, x_\gamma)_{\Phi,}\beta.
\]
(by (11))

(b) \( \beta + \gamma \neq 0 \): Since \([x_\beta, x_\gamma] \in \mathfrak{g} \), we have that
\[
([h_\lambda, x_\beta]|x_\gamma) = -((\Phi - \lambda)|\beta)(x_\beta|x_\gamma) = 0 = (h_\lambda)(x_\beta, x_\gamma)_{\Phi,}\beta
\]
by (3), (9), (12) and (13).

Case III. \( a = k_\lambda \in \mathfrak{f}, b = x_\beta, c = x_\gamma \): This case is to be shown as in Case II. We repeat it for completion.

(a) \( \beta + \gamma = 0 \): Note that \( x_\gamma = x_\beta \). Since
\[
[x_\beta, x_\beta] = 2^{-1}(x_\beta, x_\gamma) = 2^{-1}(x_\beta, x_\gamma)h_\beta + 2^{-1}(x_\beta, x_\gamma)k_\beta
\]
by (11) and (41), we have that
\[
([k_\lambda, x_\beta]|x_\beta) = (\Phi + \lambda)|\beta)(x_\beta|x_\gamma) = 0 = (k_\lambda)(x_\beta, x_\gamma)_{\Phi,}\lambda
\]
by (3), (10), (13) and (15).

Case IV. \( a = x_\beta, b = x_\gamma, c = h_\lambda \in \mathfrak{h} \): Since \( (\cdot)|\lambda \) is symmetric, we have that
\[
([x_\beta, x_\gamma]|h_\lambda) = -(h_\lambda)(x_\gamma, x_\beta) = -(h_\lambda)(x_\gamma, x_\beta)_{\Phi,}\lambda = (x_\beta, x_\gamma)|h_\lambda)_{\Phi,}\lambda
\]
by Case II.

Case V. \( a = x_\beta, b = x_\gamma, c = k_\lambda \in \mathfrak{f} \): Since \( (\cdot)|\lambda \) is symmetric, we have that
\[
([x_\beta, x_\gamma]|k_\lambda) = -(k_\lambda)(x_\gamma, x_\beta) = -(k_\lambda)(x_\gamma, x_\beta)_{\Phi,}\lambda = (x_\beta, x_\gamma)|k_\lambda)_{\Phi,}\lambda
\]
by Case II.

Case VI. \( a = x_\beta, b = h_\lambda \in \mathfrak{h}, c = x_\gamma \): Since \( (\cdot)|\lambda \) is symmetric, we have that
\[
([x_\beta, h_\lambda]|x_\gamma) = -(h_\lambda)(x_\beta, x_\gamma) = -(h_\lambda)(x_\beta, x_\gamma)_{\Phi,}\lambda = (x_\beta, h_\lambda)|x_\gamma)_{\Phi,}\lambda
\]
by Case II.

Case VII. \( a = x_\beta, b = k_\lambda \in \mathfrak{f}, c = x_\gamma \): Since \( (\cdot)|\lambda \) is symmetric, we have that
\[
([x_\beta, k_\lambda]|x_\gamma) = -(k_\lambda)(x_\beta, x_\gamma) = -(k_\lambda)(x_\beta, x_\gamma)_{\Phi,}\lambda = (x_\beta, k_\lambda)|x_\gamma)_{\Phi,}\lambda
\]
by Case III.

Case VIII. \( a \in \mathfrak{h} \oplus \mathfrak{k}, \ b \in \mathfrak{h} \oplus \mathfrak{k}, \ c = x_\gamma \): Clearly we have that
\[
([a, b]|x_\gamma) = 0 = (a|[[b, x_\gamma]])
\]
by (8), (12) and (15).

Case IX: \( a \in \mathfrak{h} \oplus \mathfrak{k}, \ b = x_\gamma, \ c \in \mathfrak{h} \oplus \mathfrak{k} \): Clearly we have that
\[
([a, x_\gamma]|c) = 0 = (a|[x_\gamma, c])
\]
by (12), (13) and (15).

Case X. \( a = x_\gamma, \ b \in \mathfrak{h} \oplus \mathfrak{k}, \ c \in \mathfrak{h} \oplus \mathfrak{k} \): Clearly we have that
\[
([x_\gamma, b]|c) = 0 = (x_\gamma|[b, c])
\]
by (13), (15) and (8).

Case XI. \( a \in \mathfrak{h} \oplus \mathfrak{k}, \ b \in \mathfrak{h} \oplus \mathfrak{k}, \ c \in \mathfrak{h} \oplus \mathfrak{k} \): Clearly we have that
\[
([a, b]|c) = 0 = (a|[b, c])
\]
by (8). \( \square \)

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