Power-law exponent in multiplicative Langevin equation with temporally correlated noise

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Power-law distributions are ubiquitous in nature. Random multiplicative processes are a basic model for the generation of power-law distributions. It is known that, for discrete-time systems, the power-law exponent decreases as the autocorrelation time of the multiplier increases. However, for continuous-time systems, it has not yet been elucidated as to how the temporal correlation affects the power-law behavior. Herein, we have analytically investigated a multiplicative Langevin equation with colored noise. We show that the power-law exponent depends on the details of the multiplicative noise, in contrast to the case of discrete-time systems.

I. INTRODUCTION

Fluctuations following power-law distributions have been found not only in natural systems but also in social systems [13]. For instance, city sizes [4, 5], firm sizes [6, 7], stock returns [8, 9] and personal incomes [10, 11] follow the power-law distribution over large scales. This expression is widely known as Pareto’s law [12] or Zipf’s law [13], and it has been investigated using various models. A well-known mechanism that generates power-law distributions is the random multiplicative process [14–22]. For example, consider the case of personal income: if a person invests his/her money, he/she will get a certain percentage return that varies over time. With repeated investments, the evolution of income approaches

\[ x(t+1) = \lambda(t)x(t) + b, \]  

where a small positive term \( b \) is added to introduce a lower bound on the value of \( x(t) \). In general, the multiplier \( \lambda(t) \) is a stochastic variable. If \( \langle \log \lambda \rangle < 0 \) and \( \lambda(t) \) sometimes takes values larger than one, the asymptotic distribution of \( x(t) \) has a power law tail [16, 17]. Although the added term \( b \) had been a stochastic variable in the previous works [16, 17], we have set \( b \) as a constant for this investigation because fluctuations of \( b \) will not affect the power-law tail of the distribution when it is sufficiently small. If the multiplier is uncorrelated, the power-law exponent is given by a solution of eq. (4) [18, 19].

\[ \langle \lambda^\gamma \rangle = 1. \]  

If the multiplier is temporally correlated, the power-law exponent \( \gamma \) decreases as the autocorrelation time increases [24, 25]. This relation can be explained intuitively by pointing out that the temporal correlation of the multiplier tends to influence the size of its fluctuations, i.e., the denominator of eq. (4).

In contrast, for continuous-time systems, the effect of a temporally correlated multiplier on the power-law exponent has not yet been investigated sufficiently. This may be because results for discrete-time systems can be applied directly for continuous-time systems. In this paper, we investigate the relationship between the power-law exponent and the temporal correlation of the multiplier. First, we introduce a continuous-time version of eq. (2). Next, we analytically estimate the power-law exponent for this model on the condition that the autocorrelation time is small. Finally, we perform numerical simulations to confirm the predictions.

II. MODEL

We consider a Langevin equation (stochastic differential equation)

\[ \frac{dx(t)}{dt} = (-r + \xi(t))x(t) + \epsilon, \]  

where \( r \) and \( \epsilon \) are positive constants, and \( \xi(t) \) is a temporally correlated noise term whose mean value is zero. Since \(-r+\xi(t)\) represents the growth rate, \(-r\) is the mean growth rate and \( \xi(t) \) is the deviation from the mean. Here, we have assumed that \( \xi(t) \) is characterized by intensity \( D \) and autocorrelation time \( \tau \), where the autocorrelation function of \( \xi(t) \) is given as

\[ \langle \xi(t)\xi(t') \rangle = \frac{D}{\tau}e^{-|t-t'|/\tau}. \]  

Thus, the power spectrum of \( \xi(t) \) is

\[ S(\omega) = \frac{2D}{\tau^2\omega^2 + 1}. \]
For \( \omega = 0 \), \( S(0) = \int_{-\infty}^{\infty} \xi(t)^2 dt = 2D \). In the limit of \( \tau \to 0 \), eq. (13) converges to \( 2D \delta(t - t') \), which indicates that \( \xi(t) \) becomes white noise. For these reasons, \( D \) is referred to as the noise intensity [24–26].

If we neglect quadratic and higher order terms for infinitesimal small \( \Delta t \), the Langevin equation in (10) can be rewritten as

\[
x(t + \Delta t) = [1 + (-r + \xi(t))\Delta t]x(t) + \epsilon\Delta t.
\]

(8)

If we substitute \( \lambda(t) = 1 + (-r + \xi(t))\Delta t \) and \( b = \epsilon\Delta t \), eq. (8) is equivalent to eq. (3). Note that for eq. (3), the noise term is proportional to \( \Delta t \), in contrast that noise term is proportional to \( \sqrt{\Delta t} \) for usual stochastic differential equations. Consequently, in calculating Eq. (2), there is no problem which calculis, Ito or Stratonovich, is used [20].

The noise term that satisfies condition (6) is not determined uniquely. Herein, we focus on a simple case. We assume that \( \xi(t) \) has a stationary distribution \( \rho(\xi) \). The value of \( \xi(t) \) changes at each point in time and remains constant over the subsequent moments. The points in time follow the Poisson process with a rate of 1. At each point in time, a new value of \( \xi(t) \) is chosen at random from \( \rho(\xi) \). Consider the case in which \( \rho(\xi) \) is an \( n \)-point discrete distribution, i.e., \( \xi \) is \( \xi_i \) with probability \( \rho_i \) for \( i = 1, 2, \ldots, n \). In this case, the evolution of the probability \( p(\xi_i, t) \) of \( \xi = \xi_i \) at time \( t \) is described by

\[
\frac{d}{dt} \begin{pmatrix} p(\xi_1, t) \\ p(\xi_2, t) \\ \vdots \\ p(\xi_n, t) \end{pmatrix} = \frac{1}{\tau} A \begin{pmatrix} p(\xi_1, t) \\ p(\xi_2, t) \\ \vdots \\ p(\xi_n, t) \end{pmatrix}.
\]

(9)

Here, \( \frac{1}{\tau} A \) represents the transition rate matrix; the matrix \( A \) is defined as

\[
A_{ij} = -\delta_{ij} + \rho_i,
\]

(10)

where \( \delta_{ij} \) is Kronecker’s delta. Eq. (9) delivers the correlation function (10) from the fact that the dominant eigenvalue of \( A \) is 0 and all the rest eigenvalues are –1.

By neglecting the quadratic and higher order terms of \( \Delta t \), we obtain a discrete-time version of eq. (3)

\[
\begin{pmatrix} p(\xi_1, t + \Delta t) \\ p(\xi_2, t + \Delta t) \\ \vdots \\ p(\xi_n, t + \Delta t) \end{pmatrix} = B \begin{pmatrix} p(\xi_1, t) \\ p(\xi_2, t) \\ \vdots \\ p(\xi_n, t) \end{pmatrix},
\]

(11)

where \( B \) represents the transition matrix, which is given as

\[
B_{ij} = \delta_{ij} + \frac{\Delta t}{\tau} A_{ij} = \left(1 - \frac{\Delta t}{\tau}\right) \delta_{ij} + \frac{\Delta t}{\tau} \rho_i.
\]

(12)

If \( \rho(\xi) \) is a continuous distribution, then the operators \( A \) and \( B \) can be defined similarly.

### III. Calculation of Power-Law Exponent

The stationary distribution of \( x \) for stochastic process described by eq. (10) or (13) has a power-law tail. Let \( \bar{x}^\alpha(\xi, t) \) be defined as the expected value of \( x^\alpha(t) \) on the condition that \( \xi(t) = \xi \). Thus, the expected value of \( x^\alpha(t) \) is given as

\[
\bar{x}^\alpha(t) = \int \bar{x}^\alpha(\xi, t) \rho(\xi).
\]

(13)

The power-law exponent \( \gamma \) is determined by the boundary between growth and decay of \( \bar{x}(t)^\alpha \) [12, 16]. In the case of \( n \)-point discrete distribution, neglecting the added term \( \epsilon \), we have

\[
\begin{pmatrix} \bar{x}^\alpha(\xi_1, t + \Delta t) \\ \bar{x}^\alpha(\xi_2, t + \Delta t) \\ \vdots \\ \bar{x}^\alpha(\xi_n, t + \Delta t) \end{pmatrix} = BC \begin{pmatrix} \bar{x}^\alpha(\xi_1, t) \\ \bar{x}^\alpha(\xi_2, t) \\ \vdots \\ \bar{x}^\alpha(\xi_n, t) \end{pmatrix},
\]

(14)

where the matrix \( C \) is defined as

\[
C_{ij} = [1 + \Delta t(-r + \xi_i)]^\alpha \delta_{ij}.
\]

(15)

As a result, the growth rate of \( \bar{x}(t)^\alpha \) is given by the dominant eigenvalue of the matrix \( BC \).

Thus, the power-law exponent \( \gamma \) can be derived by the condition that the eigenvalue of \( BC \) equals one, i.e.,

\[
\det(BC - I) = 0,
\]

(16)

where \( I \) is the unit matrix. Neglecting the quadratic and higher order terms of \( \Delta t \) again, we obtain

\[
[BC - I]_{ij} = \Delta t \left\{ \left(-r + \xi_i\right)\alpha - \frac{1}{\tau} \delta_{ij} + \frac{1}{\tau} \rho_i \right\}.
\]

(17)

By simple algebra, the equation can be rearranged as

\[
\det(BC - I) = \Delta t^n \left[ 1 + \sum_{i=1}^{n} \frac{\rho_i}{\tau(-r + \xi_i)\alpha - 1} \right] \prod_{i=1}^{n} \left[ \frac{(-r + \xi_i)\alpha - 1/\tau}{\tau(-r + \xi_i)\gamma - 1} \right].
\]

(18)

Consequently, the power-law exponent \( \gamma \) can be determined by solving eq. (19).

\[
\sum_{i=1}^{n} \frac{\rho_i}{\tau(-r + \xi_i)\gamma - 1} = -1.
\]

(19)

Expanding the result of eq. (19) to the case of a continuous distribution \( \rho(\xi) \), we obtain

\[
\int \frac{\rho(\xi)}{\tau(-r + \xi)\gamma - 1} d\xi = -1.
\]

(20)

From eq. (19), the variance of \( \rho(\xi) \) is \( D/\tau \). Using the rescaled variable \( \xi' = \xi/\sqrt{D/\tau} \), whose distribution is

\[
\rho'(\xi') = \rho(\xi')\sqrt{D/\tau}/\sqrt{D/\tau},
\]

(21)
eq. (19) can be rewritten as
\[
\int \frac{\rho'(\xi')}{(r \tau + \xi' \sqrt{D \tau})^{\gamma-1}} \, d\xi' = -1. \tag{22}
\]
Here, the variance of \(\rho'(\xi')\) is one. Performing Maclaurin series expansion with respect to \(\tau^{1/2}\) to the left-hand side of eq. (22), the condition of eq. (22) can be rewritten as
\[
(r \gamma - D)^2 - SD^{3/2} \gamma^{3/2} \tau^{1/2} - \gamma^2 (r^2 - 3r \gamma D + \gamma^2 D^2 (K + 3)) \tau + O(\tau^{3/2}) = 0,
\tag{23}
\]
where \(S\) and \(K\) are the skewness and the kurtosis of \(\rho(\xi)\), respectively.

\[
S = \langle \xi^3 \rangle / \langle \xi^2 \rangle^{3/2}, \tag{24}
\]
\[
K = \langle \xi^4 \rangle / \langle \xi^2 \rangle^2 - 3. \tag{25}
\]
In the limit \(\tau \to 0\) (the white noise limit), the first term on the left-hand side of (23) must be zero, so that we can obtain
\[
\gamma = \frac{r}{D}. \tag{26}
\]
Eq. (26) coincides with the result of the previous work \cite{21} in which the multiplier was the white noise for Stratonovich stochastic differential equation.

Consider the case of \(\tau \ll 1\). In the case when \(\rho(\xi)\) is asymmetric \((S \neq 0)\), we obtain an approximation formula \cite{27}, taking the first and second terms of on the left-hand side of (23) into account.

\[
\gamma \simeq \frac{r}{D} \left(1 - r S \sqrt{\frac{r}{D}}\right). \tag{27}
\]
Since the leading term of (27) is the square root of \(r\), the skewness \(S\) seriously affects the dependence of the power-law exponent \(\gamma\) on the autocorrelation time \(\tau\). If \(S < 0\), the power-law exponent \(\gamma\) increases quickly with the autocorrelation time \(\tau\), whereas if \(S > 0\), the opposite is true.

In the case when \(\rho(\xi)\) is symmetric \((S = 0)\), we obtain another approximation formula, while taking the linear term of \(\tau\) and the constant term into account.

\[
\gamma \simeq \frac{r}{D} \left[1 - r^2 (K + 1) \frac{\tau}{D}\right]. \tag{28}
\]
If \(K > -1\), the power-law exponent \(\gamma\) decreases when the autocorrelation time \(\tau\) increases. If \(K < -1\), the opposite is true.

\section*{IV. EXAMPLES}

Next, we examine two simple examples of the distribution \(\rho(\xi)\). First, consider the case of two-point discrete distribution:

\[
\xi = \begin{cases} 
\sqrt{D \tau} \sqrt{\frac{1 - p}{p}} & \text{(probability } p\text{)} \\
-\sqrt{D \tau} \sqrt{\frac{p}{1 - p}} & \text{(probability } 1 - p\text{)}
\end{cases}. \tag{29}
\]

In this case, \(S = \frac{1 - 2p}{\sqrt{p(1-p)}}\) and \(K = \frac{1}{p(1-p)} - 6\). \tag{30}

By solving eq. (19) analytically, the power-law exponent can be expressed as follows:

\[
\gamma = \frac{r}{D + S \tau \sqrt{D \tau - r^2}}. \tag{31}
\]
In fig. 1, curves for eq. (31) are plotted for \(p = 0.7, 0.5, \text{ and } 0.3\) \((S = -0.87 \cdots, 0, 0.87 \cdots, \text{ respectively})\) when we set \(r = 0.1\) and \(D = 0.05\). If \(S \neq 0\), the curves have a parabola-like shape. On the contrary, if \(S = 0\) \((p = 0.5)\), the power-law exponent \(\gamma\) increases almost linearly with the autocorrelation time \(\tau\), because \(K = -2\). These results agree with the predictions we made above. To confirm our analytical results, we perform numerical simulations of the Langevin equation \cite{5}. Here, we apply the Euler method (eq. (5)) for \(\Delta t = 0.001\). Fig. 1 shows the consistency of numerical results with the analytical results.

Second, consider the case of Gaussian distribution for the stationary distribution

\[
\rho(\xi) = \frac{1}{\sqrt{2 \pi D / \tau}} \exp\left(-\frac{\xi^2}{2D / \tau}\right). \tag{32}
\]
In this case, we cannot obtain an explicit expression for \(\gamma\). In fig. 1, the curve is plotted by calculating eq. (22).
The power-law exponent $\gamma$ decreases almost linearly as the autocorrelation time $\tau$ increases, because $S = 0$ and $K = 0$ for the Gaussian distribution. Also in this case, the numerical results are consistent with the analytical results (see the triangles markers in fig. 1).

V. CONCLUSION

In summary, we showed that the power-law exponent $\gamma$ for a stochastic differential equation depends on the stationary distribution of the multiplier term even if the autocorrelation function is the same. Particularly, in the case when the skewness $S$ of the stationary distribution is nonzero, a slight change to the autocorrelation time can have a dramatic effect on the power-law exponent $\gamma$. If $S = 0$, the relation between the power-law exponent and the autocorrelation time is determined by whether kurtosis $K$ is larger than $-1$ or not. In practice, for continuous distributions which have the same tails as the Gaussian distribution or longer tails ($K \geq 0$), the power-law exponent $\gamma$ would decrease gently as the autocorrelation time increases. For example, empirical works for the sales of American companies [27] and the national GDPs [28] reported that the growth rates follow symmetric exponential distributions ($S = 0$ and $K = 3$). Another report finds that the growth rates for the income of Japanese companies follow an asymmetric exponential distribution ($S < 0$) [29]. The latter case is very interesting because the temporal correlation may significantly affect power-law behavior. Future works will need to address practical data to further explore this topic.

Our results are seemingly inconsistent with previous studies reporting that the power-exponent is proportional to the inverse of the autocorrelation time $\tau$ for large values of $\tau$ [22, 23]. These studies assumed that the autocorrelation function can be described with

$$\langle \xi(t)\xi(t') \rangle = D e^{-|t-t'|/\tau}$$

instead of eq. (6). Although eq. (33) is suitable for discrete-time systems, eq. (6) is more appropriate for describing continuous-time systems because $\int_{-\infty}^{\infty} \xi(t)^2 dt = 2D$ for eq. (6) and the autocorrelation function converges to $2D\delta(t-t')$ in the limit of $\tau \to 0$, as is mentioned above. On the other hand, eq. (33) cannot converges to white noise in the limit $\tau \to 0$. Finally it should be noted that we have focused on a simple case that satisfies eq. (10). Generally, for an operator $A$ when the dominant eigenvalue is 0 and all the other eigenvalues are $-1$, eq. (10) produces noise with an exponential autocorrelation function [10]. In this case, the power-law exponent $\gamma$ can be calculated by solving eq. (10) at least in principle. An alternative method used to generate temporally correlated noise is the Ornstein-Uhlenbeck process

$$d\xi(t) = -\frac{1}{\tau}\xi(t)dt + \sqrt{2D}\,dW(t),$$

where $W(t)$ denotes the Wiener process. In this case, the operator $B$ is given by using the Fokker-Planck equation. However, the derivation of the operator $B$ and solution for eq. (10) are quite difficult. Calculating the power-law exponent for such cases remains an open problem to be addressed in future work.

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