Abstract

Uniqueness is proven for two 3-d inverse problems of the determination of the spatially distributed sound speed in the frequency dependent acoustic PDE. The main new point is the assumption that only the modulus of the scattered complex valued wave field is measured on a certain set.

1 Introduction

When considering Coefficient Inverse Problems (CIPs) in the frequency domain, it is usually assumed that both modulus and phase of the complex valued function representing the wave field is known on a certain set, see, e.g. [27, 28] for global uniqueness results and reconstruction methods. However, it is impossible to measure the phase in many applications. In these applications only the modulus of the scattered complex valued wave field can be measured (section 1.3). Therefore, it is worthy to investigate CIPs in the frequency domain, assuming that only the modulus of the scattered wave field is known on a certain set.

In the recent work [21] the author has proven uniqueness theorems for four inverse scattering problems of determining the compactly supported potential \(q(x), x \in \mathbb{R}^3\) in the Schrödinger equation in the case when only the modulus of the complex valued wave field is measured on a certain set and the phase is unknown. The goal of the current publication is to extend the result of [21] to the case of the 3-d acoustic equation with the unknown spatially varying sound speed. The author is unaware about previous similar results for the acoustic equation in \(n-d, n = 1, 2, 3\).

Below \(C^{s+\alpha}\) are Hölder spaces, where \(s \geq 0\) is an integer and \(\alpha \in (0, 1)\). Let \(\Omega, G \subset \mathbb{R}^3\) be two bounded domains. Let \(G_1 \subset \mathbb{R}^3\) be a convex bounded domain with its boundary \(S \in C^1\). Let \(\varepsilon \in (0, 1)\) be a number. Below

\[
\Omega \subset G_1 \subset G, S \cap \partial G = \emptyset.
\]

(1)

\[\text{dist} (S, \partial \Omega) > \varepsilon,\]

(2)

where \(\text{dist} (S, \partial \Omega)\) is the Hausdorff distance between \(S\) and \(\partial \Omega\). Let \(c(x)\) be the variable sound speed satisfying the following conditions

\[c \in C^5 (\mathbb{R}^3), c(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus G,\]

(3)

\[c(x) \geq c_0 = \text{const.} > 0, \forall x \in G.\]

(4)
In addition, we assume that there exists a point \( x_0 \in \Omega \) such that

\[
(x - x_0, \nabla e^{-2}(x)) \geq 0, \forall x \in \overline{G}.
\] (5)

Note that usually the minimal smoothness of unknown coefficients is of a minor concern of uniqueness theorems for multidimensional CIPs, see, e.g. \([27, 28]\) and Theorem 4.1 in \([29]\). The \( C^5 \)-smoothness condition of \( c(x) \) is imposed because we need to use Theorem 3.1 of \([20]\). This theorem, in turn requires the \( C^4 \)-smoothness of the solution of the Cauchy problem for the acoustic equation in the time domain, see section 2 for this problem. To establish that \( C^4 \)-smoothness, we refer in section 2 to Theorem 2.2 of \([19]\), which requires \( c \in C^5(\mathbb{R}^3) \).

The survey \([20]\) is about the method of proofs of global uniqueness and stability theorems for multidimensional non-over-determined CIPs for PDEs, which was originally proposed in \([5]\). This method is based on Carleman estimates.

**Lemma 1.** Assume that conditions (3)-(5) are in place. Then the family of geodesic lines generated by the function \( c(x) \) holds the non-trapping property in \( \mathbb{R}^3 \).

**Proof.** The validity of this lemma follows immediately from formulae (3.23') and (3.24) of section 2 of chapter 3 of the book \([29]\). \(\square\)

Consider the function \( g(x) \) satisfying the following conditions

\[
g \in C^7(\mathbb{R}^3), g(x) = 0 \text{ in } \mathbb{R}^3 \setminus G, \tag{6}
g(x) \neq 0, x \in S. \tag{7}
\]

It would be probably better to assume in Inverse Problems 1, 2 below that \( g(x) = \delta(x - x_0) \) for a certain source position \( x_0 \in \mathbb{R}^3 \). However, even if the entire wave field, rather than only its modulus, would be measured, still uniqueness theorems for corresponding CIPs for the 3-d acoustic equation in the case \( g(x) = \delta(x - x_0) \) are currently known only if the data are over-determined ones, see, e.g. \([22, 25]\). This is the case of infinitely many measurements when the number of free variables in the data exceeds the number of free variables in the unknown coefficient. The above mentioned technique of \([5, 20]\) is currently the only one, which enables to prove global uniqueness for multidimensional CIPs with the data resulting from a single measurement event. The data in this case are non-over-determined ones. On the other hand, this technique requires that

\[
\Delta g(x) \neq 0, \forall x \in \overline{G}_1. \tag{8}
\]

To mitigate the concern about \( \delta(x - x_0) \), consider an analog of examples of \([20, 21]\). Let the function \( \chi(x) \in C^\infty(\mathbb{R}^3) \) be such that \( \chi(x) = 1 \) in \( G_1 \) and \( \chi(x) = 0 \) for \( x \notin G \). The existence of such functions \( \chi(x) \) is well known from the Real Analysis course. Let the point \( x_0 \in \overline{G}_1 \). For a number \( \sigma > 0 \) consider the function \( \delta_\sigma(x - x_0) \),

\[
\delta_\sigma(x - x_0) = C \frac{\chi(x)}{(2\sqrt{\pi\sigma})^3} \exp\left(-\frac{|x - x_0|^2}{4\sigma}\right),
\]

where \( C = \frac{\sqrt{\pi}}{2} \) if \( \chi(x) = 1 \) in \( G_1 \).
\[ \int_G \delta_\sigma(x - x_0) \, dx = 1, \quad (9) \]

where the number \( C > 0 \) is chosen such that (9) holds. The function \( \delta_\sigma(x - x_0) \) approximates the function \( \delta(x - x_0) \) in the distribution sense for sufficiently small values of \( \sigma \). The function \( \delta_\sigma(x - x_0) \) is acceptable in Physics as a proper replacement of \( \delta(x - x_0) \), since there is no “true” delta-function in the physical reality. On the other hand, the above mentioned method of \([5, 20]\) is applicable to the case when \( \delta(x - x_0) \) is replaced with \( \delta_\sigma(x) \). Therefore, it is reasonable from the Physics standpoint to impose condition (8).

### 1.1 Main results

Consider the following problem

\[ \Delta u + \frac{k^2}{c^2(x)} u = -g(x), \quad x \in \mathbb{R}^3, \quad (10) \]

\[ \sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} u(x, k) -iku(x, k) = o(1), \quad |x| \to \infty. \quad (11) \]

We now refer to Theorem 6 of Chapter 9 of the book \([32]\), Theorem 3.3 of the paper \([31]\) as well as to Theorem 6.17 of the book \([11]\). Combining these results with Lemma 1, we obtain that for each \( k \in \mathbb{R} \) there exists unique solution

\[ u(x, k) \in C^{6+\alpha}(\mathbb{R}^3), \quad \forall \alpha \in (0, 1) \]

of the problem \((10), (11)\).

**Inverse Problem 1 (IP1).** Suppose that the function \( c(x) \) satisfying conditions \((3)-(5)\) is unknown for \( x \in \Omega \) and known for \( x \in \mathbb{R}^3 \setminus \Omega \). Assume that the following function \( f_1(x, k) \) is known

\[ f_1(x, k) = |u(x, k)|, \quad \forall x \in S, \forall k \in (a, b). \quad (12) \]

Determine the function \( q(x) \) for \( x \in \Omega \).

**Theorem 1.** Consider IP1. Let conditions \((4)\) and \((2)\) hold. Let the function \( g(x) \) satisfies conditions \((4)-(8)\). Consider two functions \( c_1(x), c_2(x) \) satisfying conditions \((3)-(5)\) and such that \( c_1(x) = c_2(x) = c(x) \) for \( x \in \mathbb{R}^3 \setminus \Omega \). For \( j = 1, 2 \) let \( u_j(x, k) \in C^{6+\alpha}(\mathbb{R}^3), \forall \alpha \in (0, 1) \) be the solution of the problem \((10), (11)\) with \( c(x) = c_j(x) \). Assume that

\[ |u_1(x, k)| = |u_2(x, k)|, \quad \forall x \in S, \forall k \in (a, b). \quad (13) \]

Then \( c_1(x) \equiv c_2(x) \).

IP1 is about the case when the modulus of the total wave field is measured for \( x \in S, k \in (a, b) \). Consider the function \( u_0(x, k) \),

\[ u_0(x, k) = \int_G \frac{\exp(ik|x - \xi|)}{4\pi |x - \xi|} g(\xi) \, d\xi. \]
This function is the solution of the problem (10), (11) with \( c(x) \equiv 1 \). Since \( c(x) = 1 \) for \( x \in \mathbb{R}^3 \setminus G \), then one can consider \( u_0(x, k) \) as the solution of the problem (10), (11) for the background medium. Hence, the function \( u_i(x, k) = u(x, k) - u_0(x, k) \) can be considered as the wave, which is scattered due to the inhomogeneous structure of the coefficient \( c(x) \) for \( x \in G \). This is our motivation for posing Inverse Problem 2.

**Inverse Problem 2 (IP2).** Suppose that the function \( c(x) \) satisfying conditions (3)-(5) is unknown for \( x \in \Omega \) and known for \( x \in \mathbb{R}^3 \setminus \Omega \). Let \( u_s(x, k) = u(x, k) - u_0(x, k) \). Assume that the following function \( f_2(x, k) \) is known

\[
f_2(x, k) = |u_s(x, k)|, \forall x \in S, \forall k \in (a, b).
\]

Determine the function \( c(x) \) for \( x \in \Omega \).

**Theorem 2.** Consider IP2. Assume that

\[
c^2(x) \neq 1, \forall x \in S.
\]

Let all conditions of Theorem 1 hold, except that (7) is not imposed. In addition, let (13) be replaced with

\[
|u_{s,1}(x, k)| = |u_{s,2}(x, k)|, \forall x \in S, \forall k \in (a, b),
\]

where \( u_{s,j}(x, k) = u_j(x, k) - u_0(x, k), j = 1, 2 \). Then \( c_1(x) \equiv c_2(x) \).

### 1.2 The main difficulty

We now outline the main difficulty of proofs of Theorems 1,2. Although the same difficulty was described in [21], we briefly present it here for reader’s convenience. Below for any number \( a \in \mathbb{C} \) its complex conjugate is denoted as \( \overline{a} \). For an arbitrary number \( \beta > 0 \) denote

\[
\mathbb{C}^\beta = \{ k \in \mathbb{C} : \text{Im} k > -\beta \}.
\]

Also, denote \( \mathbb{C}_+ = \{ k \in \mathbb{C} : \text{Im} k \geq 0 \} \).

Consider an arbitrary point \( x_0 \in S \). By Lemma 2 (section 2) there exists a number \( \beta > 0 \) such that the function \( u(x', k) \) admits the analytic continuation from the real line \( \mathbb{R} \) in the half-plane \( \mathbb{C}^\beta \). Since \( |u(x_0, k)|^2 = u(x_0, k) \overline{u(x_0, k)}, \forall k \in \mathbb{C}^\beta \), then the function \( |u(x_0, k)|^2 \) is analytic as the function of the real variable \( k \in \mathbb{R} \). Hence, (12) implies that the function \( |u(x_0, k)| \) is known for all \( k \in \mathbb{R} \). The main difficulty is linked with zeros of the function \( u(x_0, k) \) in the upper half-plane \( \mathbb{C}_+ \setminus \mathbb{R} \). Indeed, let the number \( a = a(x_0) \in \mathbb{C}_+ \setminus \mathbb{R} \) be such that \( u(x_0, a) = 0 \). Consider the function \( \tilde{u}(x_0, k) \),

\[
\tilde{u}(x_0, k) = \frac{k - \overline{a}}{k - a} u(x_0, k).
\]

Since

\[
\left| \frac{k - \overline{a}}{k - a} \right| = 1, \forall k \in \mathbb{R},
\]

then \( |\tilde{u}(x_0, k)| = |u(x_0, k)|, \forall k \in \mathbb{R} \). In addition, the function \( \tilde{u}(x_0, k) \) is analytic in \( \mathbb{C}^\beta \). Therefore, it is necessary in proofs of Theorems 1,2 to use a linkage between the function \( u(x, k) \) and the differential operator in (10).
1.3 Published results

Phaseless inverse problems have a central importance in those applications where only the amplitude of the scattered signal can be measured, while the phase either cannot be measured or can be measured only with a poor precision. Some examples are specular reflection of neutrons [4], x-ray crystallography [23] and astronomical imaging [8], also see [9] for other applied examples.

The first uniqueness result for the phaseless inverse scattering problem for the 1-d Schrödinger equation

\[ y'' + k^2y - q(x)y = 0, \quad x \in \mathbb{R} \]

was proven in [16]. Next, it was extended in [26] to the case of the discontinuous impedance. Also, see [1] for a relevant result. A survey can be found in [17].

There is also a reach literature about the reconstruction of a compactly supported complex valued function from the modulus of its Fourier transform. Uniqueness results for this problem were proven in [15, 18]. The majority of works about this problem is dedicated to numerical methods, see, e.g. [7, 8, 9, 12, 30]. Recently regularization algorithms were developed for a similar, the so-called “autocorrelation problem” [6, 10]. In addition, numerical methods were developed for the phaseless inverse problem of the determination of obstacles [13, 14]. Related problems of synthesis were considered in [2, 3, 7].

In section 2 we formulate Lemmata 2-8. In section 3 we prove Theorem 1. Theorem 2 is proven in section 4.

2 Lemmata 2-8

Consider the following Cauchy problem for the acoustic equation in the time domain

\[ v_{tt} = c^2(x) \nabla^2 v, \quad x \in \mathbb{R}^3, \quad t \in (0, \infty), \]

\[ v(x, 0) = 0, \quad v_t(x, 0) = g(x). \]  

(16)  

(17)

For any appropriate function \( f(t) \) such that \( f(t) = 0 \) for \( t < 0 \) let \( \mathcal{F}(f)(k) \) denotes its Fourier transform,

\[ \mathcal{F}(f)(k) = \int_0^\infty f(t) e^{ikt} dt, \quad k \in \mathbb{R}. \]

(19)

Lemma 2. Assume that conditions [3]-[6] hold. Then there exists unique solution of the problem (16), (17) such that \( v, v_t \in C^4 (\mathbb{R}^3 \times (0, T)) \), \( \forall T > 0 \). Also, for any bounded domain \( \Phi \subset \mathbb{R}^3 \) there exist constants \( B = B(\Phi, c, g) > 0 \) and \( b = b(\Phi, c, g) > 0 \) depending only on \( \Phi, c \) and \( g \) such that the following estimates hold

\[ |D^\gamma_{x_t} v(x, t)| \leq Be^{-bt}, \quad \forall x \in \Phi, \quad \forall t > 0; |\gamma| \leq 3. \]

(18)

Furthermore,

\[ u(x, k) = \mathcal{F}(v)(x, k), \quad \forall x \in \mathbb{R}^3, \quad \forall k \in \mathbb{R}, \]

(19)
where the function \( u(x,k) \in C^{6+\alpha}(\mathbb{R}^3), \forall \alpha \in (0,1), \forall k \in \mathbb{R} \) is the unique solution of the problem (10), (11). For every point \( x \in \Phi \) the function \( u(x,k) \) admits the analytic continuation with respect to \( k \) from the real line in the half-plane \( \mathbb{C}^b \).

**Proof.** Existence and uniqueness of the solution \( v \in H^2(\mathbb{R}^3 \times (0,T)), \forall T > 0 \) of the Cauchy problem (16), (17) follows from corollary 4.2 of chapter 4 of the book [24]. Consider the function \( w(x,t) = v_t(x,t) \). Then \( w \in H^1(\mathbb{R}^3 \times (0,T)), \forall T > 0 \) and this function is the weak solution of the following problem

\[
\frac{\partial w}{\partial t} = c^2(x) \Delta w, \quad x \in \mathbb{R}^3, \quad t \in (0,\infty),
\]

\[
w(x,0) = g(x), \quad w_t(x,0) = 0,
\]

see chapter 4 of [24]. Applying again corollary 4.2 of chapter 4 of the book [24], we obtain that \( w \in H^2(\mathbb{R}^3 \times (0,T)), \forall T > 0 \). Hence, Theorem 2.2 of [19] implies that \( w \in C^4(\mathbb{R}^3 \times (0,T)), \forall T > 0 \). Since

\[
v(x,t) = \int_0^t w(x,\tau) \, d\tau,
\]

then the function \( v \) has at least the same smoothness as the function \( w \).

To prove (18), we refer to well known results of Vainberg about the asymptotic behavior of solutions of Cauchy problems for hyperbolic equations. More precisely, we refer to Lemma 6 in chapter 10 of the book [32] as well as to Remark 3 after this lemma. To apply these results, we need the non-trapping property of geodesic lines generated by the function \( c(x) \).

Since Lemma 1 guarantees this property, then (18) is true.

To prove connection (19) between the solution of the problem (10), (11) and the Fourier transform of the function \( v(x,t) \), we again refer to Lemma 1, Theorem 6 of Chapter 9 of the book [32], Theorem 3.3 of the paper [31] and to Theorem 6.17 of the book [11]. The assertion about the analytic continuation follows from (18) and (19). □

The integration by parts in the integral (19) of the Fourier transform immediately implies Lemma 3.

**Lemma 3.** Assume that conditions (3)-(6) hold. Then the following asymptotic formulae are valid uniformly for \( x \in \mathbb{C} \)

\[
u(x,k) = -\frac{1}{k^2} \left[ g(x) + O\left(\frac{1}{k}\right) \right], \quad |k| \to \infty, \quad k \in \mathbb{C}_+,
\]

(20)

\[
u_s(x,k) = \frac{1}{k^4} \left[ (c^2(x) - 1) \Delta g(x) + o(1) \right], \quad |k| \to \infty, \quad k \in \mathbb{C}_+.
\]

(21)

Lemmata 4,5 follow immediately from Lemma 3.

**Lemma 4.** Assume that conditions (3)-(6) hold. In addition, assume that there exists a point \( x' \in \mathbb{C}_1 \) such that \( g(x') \neq 0 \). Then the function \( v(x',k) \) has at most finite number of zeros in \( \mathbb{C}_+ \).
Lemma 5. Assume that conditions (3)–(5) hold. In addition, assume that there exists a point $x' \in \mathbb{C}_1$ such that $(c^2(x') - 1) \Delta g(x') \neq 0$. Then the function $u_s(x', k)$ has at most finite number of zeros in $\mathbb{C}_+$. Lemma 6 follows immediately from Proposition 4.2 of [17].

Lemma 6. Let $\beta > 0$ be a number. Let the function $d(k)$ be analytic in $\mathbb{C}_\beta$ and does not have zeros in $\mathbb{C}_+$. Assume that

$$d(k) = \frac{C}{k^n}[1 + o(1)] \exp(ikL), \quad |k| \to \infty, k \in \mathbb{C}_+,$$

where $C \in \mathbb{C}$ and $n,L \in \mathbb{R}$ are some numbers and also $n \geq 0$. Then the function $d(k)$ can be uniquely determined for $k \in \mathbb{C}_\beta$ by the values of $|d(k)|$ for $k \in \mathbb{R}$.

Lemma 7 was actually proven in section 1.2, since the analyticity of the function $|u(x', k)|^2$ for $k \in \mathbb{R}$ was proven there.

Lemma 7. Let the function $d(k)$ be analytic for all $k \in \mathbb{R}$. Then the function $|d(k)|$ can be uniquely determined for all $k \in \mathbb{R}$ by values of $|d(k)|$ for $k \in (a, b)$.

Lemma 8 is one of versions of the well known principle of the finite speed of propagation for hyperbolic equations. The proof of this lemma follows immediately from the standard energy estimate of §2 in chapter 4 of the book [24].

Lemma 8. Let $c_1(x)$ and $c_2(x)$ be two functions satisfying conditions (3)–(5). Also, let conditions (1), (2) and (6) hold. Assume that $c_1(x) = c_2(x) = c(x)$ for $x \in \mathbb{R}^3 \setminus \Omega$. For $j = 1, 2$ let $v_j \in C^4(\mathbb{R}^3 \times (0, T)), \forall T > 0$ be the solution of the problem (16), (17) with $c(x) = c_j(x)$. Then there exists a sufficiently small number $\xi = \xi(c, g, \varepsilon) > 0$ such that

$$v_1(x,t) = v_2(x,t), \forall x \in S, \forall t \in (0, \xi).$$

3 Proof of Theorem 1

Consider an arbitrary point $x_0 \in S$. Denote

$$q_1(k) = u_1(x_0, k), q_2(k) = u_2(x_0, k).$$

(22)

By Lemma 2 there exists a number $\theta > 0$ such that each of functions $q_1(k)$ and $q_2(k)$ admits the analytic continuation in the half-plane $\mathbb{C}_\theta$. It follows from (13) and (22) that $|q_1(k)| = |q_2(k)|, \forall k \in (a, b)$. Hence, using Lemma 7, we obtain

$$|q_1(k)| = |q_2(k)|, \forall k \in \mathbb{R}. \quad (23)$$

First, we prove that sets of real zeros of functions $q_1(k)$ and $q_2(k)$ coincide. Let $a \in \mathbb{R}$ be a real zero of the multiplicity $r_1 > 0$ of the function $q_1(k)$. Suppose that $a$ is also one of zeros of the function $q_2(k)$ of the multiplicity $r_2 \geq 0$. Lemma 4 implies that both numbers $r_1, r_2 < \infty$. By (23)

$$|(k - a)^{r_1}| \cdot |\tilde{q}_1(k)| = |(k - a)^{r_2}| \cdot |\tilde{q}_2(k)|, \forall k \in \mathbb{R}, \quad (24)$$

$$\frac{|q_1(k)|}{|q_2(k)|} = \frac{|\tilde{q}_1(k)|}{|\tilde{q}_2(k)|}, \forall k \in \mathbb{R}.$$
where
\[ \tilde{q}_1(a) \tilde{q}_2(a) \neq 0. \] (25)

Assume, for example that \( r_2 < r_1 \). Dividing (24) by \( |(k - a)^2| \) and setting \( k \to 0 \), we obtain \( \tilde{q}_2(a) = 0 \), which contradicts (25). Hence, functions \( q_1(k) \) and \( q_2(k) \) have the same real zeros.

We now focus on complex zeros in \( \mathbb{C}_+ \setminus \mathbb{R} \). Since by Lemma 4 each of functions \( q_1(k) \), \( q_2(k) \) has at most finite number of zeros in \( \mathbb{C}_+ \), then let \( \{ \eta_s \}_{s=1}^n \subset (\mathbb{C}_+ \setminus \mathbb{R}) \) and \( \{ \sigma_p \}_{p=1}^{m'} \subset (\mathbb{C}_+ \setminus \mathbb{R}) \) be zeros of functions \( q_1(k) \) and \( q_2(k) \) respectively. Also, let \( \{ a_r \}_{r=1}^m \subset \mathbb{R} \) be real zeros for both functions \( q_1(k) \), \( q_2(k) \). Here each zero is counted as many times as its multiplicity is.

Consider functions \( \widehat{q}_1(k) \), \( \widehat{q}_2(k) \) defined as
\[
\widehat{q}_1(k) = q_1(k) \left( \prod_{s=1}^{n} \frac{k - \eta_s}{k - \eta_s} \right) \left( \prod_{r=1}^{m'} \frac{1}{k - a_r} \right), \quad k \in \mathbb{C},
\] (26)
\[
\widehat{q}_2(k) = q_2(k) \left( \prod_{p=1}^{m} \frac{k - \sigma_p}{k - \sigma_p} \right) \left( \prod_{r=1}^{m'} \frac{1}{k - a_r} \right), \quad k \in \mathbb{C}. \] (27)

Hence, \( \widehat{q}_1(k) \) and \( \widehat{q}_2(k) \) are analytic functions in \( \mathbb{C}^\theta \). In addition, it follows from (20), (22), (23), (26) and (27) that
\[
\widehat{q}_j(k) = -\frac{1}{k^{m'+2}} \left[ g(x_0) + O \left( \frac{1}{k} \right) \right], \quad |k| \to \infty, \quad k \in \mathbb{C}_+, \quad j = 1, 2, \] (28)
\[
\widehat{q}_j(k) \neq 0, \quad \forall k \in \mathbb{C}_+, \quad j = 1, 2, \] (29)
\[
|\widehat{q}_1(k)| = |\widehat{q}_2(k)|, \quad \forall k \in \mathbb{R}. \] (30)

Combining (28), (29) and (30) with Lemma 6, we obtain
\[
\widehat{q}_1(k) = \widehat{q}_2(k), \quad \forall k \in \mathbb{R}. \]

Hence, (26) and (27) lead to
\[
q_1(k) \left( \prod_{s=1}^{n} \frac{k - \eta_s}{k - \eta_s} \right) = q_2(k) \left( \prod_{p=1}^{m} \frac{k - \sigma_p}{k - \sigma_p} \right). \] (31)

Or
\[
q_1(k) \left( \prod_{p=1}^{m} \frac{k - \sigma_p}{k - \sigma_p} \right) = q_2(k) \left( \prod_{s=1}^{n} \frac{k - \eta_s}{k - \eta_s} \right). \]

Or
\[
q_1(k) Y_1(k) + q_1(k) = q_2(k) Y_2(k) + q_2(k), \] (32)
where
\[ Y_1(k) = \prod_{p=1}^{m} \frac{k - \sigma_p}{k - \bar{\sigma}_p} - 1, \quad (33) \]
\[ Y_2(k) = \prod_{s=1}^{n} \frac{k - \eta_s}{k - \bar{\eta}_s} - 1. \quad (34) \]

We now calculate the inverse Fourier transform \( \mathcal{F}^{-1} \) of functions \( Y_1(k) \), \( Y_2(k) \). It follows from (33) that the function \( Y_1(k) \) can be represented as
\[ Y_1(k) = P_1(k) \prod_{p=1}^{m} \frac{1}{k - \sigma_p}, \]
where \( P_1(k) \) is a polynomial of the degree less than \( m \). Using the partial fraction expansion, we obtain
\[ Y_1(k) = \sum_{j=1}^{\tilde{m}} \frac{C_j}{(k - \bar{\sigma}_j)^{r_j}}, \]
where \( C_j \in \mathbb{C} \) are certain numbers, \( r_j \geq 1 \) are some integers and \( \sigma_{j_1} \neq \sigma_{j_2} \) if \( j_1 \neq j_2 \). The straightforward calculation shows that
\[ \frac{1}{(k - \bar{\sigma}_j)^{r_j}} = B_j \int_0^\infty t^{r_j-1} \exp(-it\bar{\sigma}_j) \exp(ikt) \, dt, \]
where \( B_j \in \mathbb{C} \) is a certain number. Hence,
\[ \mathcal{F}^{-1}(Y_1) := y_1(t) = H(t) \sum_{j=1}^{\tilde{m}} K_j^{(1)} t^{r_j-1} \exp(-it\bar{\sigma}_j), \quad (35) \]
where \( K_j^{(1)} \in \mathbb{C} \) are certain numbers and \( H(t) \) is the Heaviside function,
\[ H(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \]

Similarly, using (34), we obtain
\[ \mathcal{F}^{-1}(Y_2) := y_2(t) = H(t) \sum_{j=1}^{\tilde{n}} K_j^{(2)} t^{r_j-1} \exp(-itm_j), \quad (36) \]
with certain numbers \( K_j^{(2)} \in \mathbb{C} \).
Next, we apply the operator $F^{-1}$ to both sides of (32). Using (19), (22), (32), (35), (36) and the convolution theorem, we obtain

$$v_1(x_0, t) + \int_0^t v_1(x_0, t - \tau) y_1(\tau) \, d\tau = v_2(x_0, t) + \int_0^t v_2(x_0, t - \tau) y_2(\tau) \, d\tau, t > 0. \quad (37)$$

Denote

$$y(\tau) = y_1(\tau) - y_2(\tau). \quad (38)$$

By Lemma 8 $v_1(x_0, t) = v_2(x_0, t) := h(x_0, t)$ for $t \in (0, \xi)$. Hence, (37) and (38) imply that

$$\int_0^t h(x_0, t - \tau) y(\tau) \, d\tau = 0, t \in (0, \xi). \quad (39)$$

Differentiating equality (39) twice with respect to $t$ and using (7) and (17), we obtain

$$y(t) + \frac{1}{g(x_0)} \int_0^t h_{tt}(x_0, t - \tau) y(\tau) \, d\tau = 0, t \in (0, \xi). \quad (40)$$

This is a homogeneous Volterra integral equation of the second kind. Hence,

$$y(t) = 0, t \in (0, \xi). \quad (41)$$

It follows from (35), (36) and (38) that the function $y(t)$ is analytic for $t > 0$ as the function of real variable. Hence, (41) implies that $y(t) = 0, \forall t > 0$. Hence, by (38) $y_1(t) = y_2(t), \forall t > 0$. Therefore, functions $q_1(k)$ and $q_2(k)$ have the same sets of zeros in $\mathbb{C}_+ \setminus \mathbb{R}$, i.e. \{\eta_s\}_{s=1}^n = \{\sigma_p\}_{p=1}^m. Thus, (31) implies that

$$q_1(k) = q_2(k), \forall k \in \mathbb{R}.$$ 

Therefore, (19) and (22) imply that

$$v_1(x_0, t) = v_2(x_0, t), \forall t > 0.$$ 

Denote $S_\infty = S \times (0, \infty)$. Since $x_0 \in S$ is an arbitrary point, then

$$v_1(x, t) = v_2(x, t) := p(x, t), \forall (x, t) \in S_\infty. \quad (42)$$

Hence, it follows from (16) and (17) that both functions $v_1, v_2$ are solutions of the following initial boundary value problem outside of the domain $G_1$

$$\partial_t^2 v_j = c^2(x) \Delta v_j, x \in \mathbb{R}^3 \setminus G_1, t \in (0, \infty), j = 1, 2,$$

$$v_j(x, 0) = 0, \partial_t v_j(x, 0) = g(x), x \in \mathbb{R}^3 \setminus G_1,$$
where \( v = v(x) \) is the unit normal vector at the point \( x \in S \), which points outside of the domain \( G_1 \). Hence, using (16), (17), (12) and (13), we obtain inside of the domain \( G_1 \)

\[
\partial_t^2 v_j = c_j^2(x) \Delta v_j, x \in G_1, t \in (0, \infty), j = 1, 2,
\]

(44)

\[
v_j(x, 0) = 0, \partial_t v_j(x, 0) = g(x), x \in G_1,
\]

(45)

\[
v_j \big|_{s_\infty} = p(x, t), \partial_t v_j(x, t) \big|_{s_\infty} = \bar{p}(x, t).
\]

(46)

By Lemma 2 \( v_j \in C^4 \left( \overline{G_1} \times [0, T] \right) \), \( \forall T > 0 \). In addition, condition (5) guarantees the validity of the Carleman estimate for the operator \( \partial_t^2 - c^2(x) \Delta \), see Theorem 2.6 in [20]. Thus, it follows from Theorem 3.1 of [20] that conditions (8), (14), (15) and (16) imply that \( c_1(x) = c_2(x) \) in \( G_1 \). Finally, since one of conditions of this theorem is that \( c_1(x) = c_2(x) \) for \( x \in \mathbb{R}^3 \setminus \Omega \), then \( c_1(x) \equiv c_2(x) \).

Note that Theorem 3.1 of [20] can also be applied in the case when \( t \in (0, \infty) \) in (14) and (46) is replaced with \( t \in (0, T) \) for a certain finite number \( T > 0 \). The proof of Corollary 1 follows immediately from the proof of Theorem 1: the part, which is before (12).

**Corollary 1.** Let conditions (1) and (2) hold. Let the function \( g(x) \) satisfies conditions (6). Let \( x_0 \in S \) be an arbitrary point. Assume that \( g(x_0) \neq 0 \). Consider two functions \( c_1(x), c_2(x) \) satisfying conditions (3)-(7) and such that \( c_1(x) = c_2(x) = c(x) \) for \( x \in \mathbb{R}^3 \setminus \Omega \). For \( j = 1, 2 \) let \( u_j(x, k) \in C^{6+\alpha}(\mathbb{R}^3), \forall \alpha \in (0, 1) \) be the solution of the problem (10), (11) with \( c(x) = c_j(x) \). Assume that

\[
|u_1(x_0, k)| = |u_2(x_0, k)|, \forall k \in (a, b).
\]

Then \( u_1(x_0, k) = u_1(x_0, k), \forall k \in \mathbb{R}. \)

## 4 Proof of Theorem 2

Let the function \( v_0(x, t) \) be the solution of the Cauchy problem (16), (17) with \( c(x) \equiv 1 \). Denote \( v_s(x, t) = v(x, t) - v_0(x, t) \). Let \( x_0 \in S \) be an arbitrary point of the surface \( S \). By Lemma 8

\[
v_{s,1}(x_0, t) = v_{s,2}(x_0, t) := h_s(x_0, t), \forall t \in (0, \xi),
\]

(47)

where \( v_{s,j}(x, t) \) is the function \( v_s(x, t) \) for the case when \( c(x) = c_j(x), j = 1, 2 \). It follows from (8), (14) and (21) that we can apply the same technique as the one in section 3 before (39). Hence, (39) is replaced now with

\[
\int_0^t h_s(x_0, t - \tau) y(\tau) d\tau = 0, t \in (0, \xi).
\]

(48)
By (16) and (17) \( \partial_t^l v_s (x, 0) = 0, l = 0, 1, 2 \) and also
\[
\partial_t^2 v_s (x, 0) = \left( c^2 (x) - 1 \right) \Delta g (x) := \tilde{g} (x).
\] (49)

By (8), (14) and (49)
\[
\tilde{g} (x) \neq 0, \forall x \in S.
\] (50)

Differentiate equality (48) three times and use (47), (49) and (50). We obtain the following integral equation of the Volterra type
\[
y (t) + \frac{1}{\tilde{g} (x_0)} \int_0^t \partial_t^3 h_s (x_0, t - \tau) y (\tau) d\tau = 0, t \in (0, \xi).
\]
The rest of the proof is the same as the one in section 3 after (40). □

**Corollary 2.** Let conditions (1) and (2) hold. Let the function \( g (x) \) satisfies conditions (6). Let \( x_0 \in S \) be an arbitrary point. Assume that \( (c^2 (x_0) - 1) \Delta g (x_0) \neq 0 \). Consider two functions \( c_1 (x), c_2 (x) \) satisfying conditions (3)-(5) and such that \( c_1 (x) = c_2 (x) = c (x) \) for \( x \in \mathbb{R}^3 \setminus \Omega \). For \( j = 1, 2 \) let \( u_j (x, k) \in C^{0+\alpha} (\mathbb{R}^3), \forall \alpha \in (0, 1) \) be the solution of the problem (10), (11) with \( c (x) = c_j (x) \). Assume that
\[
|u_1 (x_0, k)| = |u_2 (x_0, k)|, \forall x \in S, \forall k \in (a, b).
\]
Then \( u_1 (x_0, k) = u_1 (x_0, k), \forall k \in \mathbb{R} \).

The proof of Corollary 2 follows immediately from the proof of Theorem 2.

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