Elation generalised quadrangles of order \((s, p)\), where \(p\) is prime

John Bamberg, Tim Penttila, and Csaba Schneider

Abstract. We show that an elation generalised quadrangle which has \(p + 1\) lines on each point, for some prime \(p\), is classical or arises from a flock of a quadratic cone (i.e., is a flock quadrangle).

1. Introduction

A generalised quadrangle is an incidence structure of points and lines such that if \(P\) is a point and \(\ell\) is a line not incident with \(P\), then there is a unique line through \(P\) which meets \(\ell\) in a point. From this property, one can see that there are constants \(s\) and \(t\) such that each line is incident with \(t + 1\) points, and each point is incident with \(s + 1\) lines. Such a generalised quadrangle is said to have order \((s, t)\), and hence its point-line dual is a generalised quadrangle of order \((t, s)\). Of the known generalised quadrangles, most admit a group of elations (see Section 2 for a definition) and are called elation generalised quadrangles. In this paper, we will be interested in elation generalised quadrangles where the parameter \(t\) is prime.

If \(S\) is an elation generalised quadrangle of order \((p, t)\), for some prime \(p\), then the elation group \(G\) is a \(p\)-group (see \[Fro88\] Lemma 6), and note that \(s\) and \(t\) are interchanged!). In this situation, we have by a deep result of Bloemen, Thas, and Van Maldeghem \[BTVM96\], that \(S\) is isomorphic to one of the classical generalised quadrangles \(W(p)\), \(Q(4, p)\), or \(Q(5, p)\). The same is not true if we interchange points and lines. Suppose that \(S\) is an elation generalised quadrangle of order \((s, p)\) (where \(p\) is a prime). Again, by a result of Frohardt \[Fro88\] Lemma 6, we have that the elation group is a \(p\)-group, however, there exist candidates for \(S\) which are not classical but are known as flock quadrangles. These elation generalised quadrangles are obtained from a flock of \(\text{PG}(3, p)\) (a partition of the points of a quadratic cone of \(\text{PG}(3, p)\), minus its vertex, into conics) and they have order \((p^2, p)\). Such a quadrangle is classical if and only if the flock is linear; and there do exist non-linear flocks for \(p\) a prime at least 3 \[PT84, \S10.6\]. In this
paper, we prove a result that is complementary to that of Bloemen, Thas, and Van Maldeghem:

**Theorem 1.1.** If $p$ is a prime, then an elation generalised quadrangle of order $(s, p)$ is classical or a flock quadrangle.

Note that the above theorem does not hold when $p$ is replaced by a prime power since the duals of the Tits quadrangles $T_3(O)$ arising from the Tits ovoids are elation generalised quadrangles of order $(q^2, q)$ (for $q = 2^h$ and $h$ an odd number at least 3) that are not flock quadrangles, and the Roman elation generalised quadrangles of Payne are of order $(q^2, q)$ (for $q = 3^h$, $h > 2$) but are not flock quadrangles.

The proof of Theorem 1.1 relies on the following result concerning Kantor families for groups of order $p^5$ (see Section 2 for a definition of Kantor families).

**Theorem 1.2.** If $p$ is an odd prime and $G$ is a finite $p$-group of order $p^5$ that admits a Kantor family of order $(p^2, p)$, then $G$ is an extraspecial group of exponent $p$.

In Sections 2–3, we briefly revise the basic background theory and definitions needed for this paper. Kantor families for groups of order $p^5$ are then investigated in Section 4, and Theorem 1.2 is proved in Section 5. Finally in Section 6 we prove Theorem 1.1.

Though our group theoretic notation is standard, we briefly review it for the sake of a reader whose interest lies more in geometry than in group theory. If $a$ is a group element of order $p$ and $\alpha \in F_p$ then, identifying $\alpha$ with an element in $\{0, \ldots, p-1\}$, we may write $a^\alpha$. If $a$ and $b$ are group elements, then we define their commutator as $[a, b] = a^{-1}b^{-1}ab$. The properties of group commutators that we need in this paper are listed, for instance, in [Rob96 § 5.1.5]. The centre of a group $G$ consists of those elements $z \in G$ that satisfy $[g, z] = 1$ for all $g \in G$. If $H, K$ are subgroups of a group $G$, then the commutator subgroup $[H, K]$ is generated by all commutators $[a, b]$ where $a \in H$ and $b \in K$. The derived subgroup $G'$ of $G$ is defined as $[G, G]$. The symbol $\gamma_i(G)$ denotes the $i$-th term of the lower central series of $G$: that is $\gamma_1(G) = G, \gamma_2(G) = G'$, and, for $i \geq 3$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The nilpotency class of a $p$-group is the smallest $c$ such that $\gamma_{c+1}(G) = 1$. The Frattini subgroup $\Phi(G)$ of a finite group $G$ is the intersection of all the maximal subgroups. If $G$ is a finite $p$-group, then $\Phi(G) = G^{p'}$ and $\log_p |G : \Phi(G)|$ is the size of a minimal set of generators for $G$. The basic properties of the Frattini subgroup of a $p$-group can be found, for instance, in [Rob96 § 5.3]. The exponent of a finite group $G$ is the smallest positive $n$ such that $g^n = 1$ for all $g \in G$.

2. Generalised Quadrangles and Kantor Families

**The basics.** A (finite) generalised quadrangle is an incidence structure of points $\mathcal{P}$, lines $\mathcal{L}$, together with a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point lies on $t + 1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
(ii) Each line lies on $s + 1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If $P$ is a point and $\ell$ is a line not incident with $P$, then there is a unique point on $\ell$ collinear with $P$.

We say that our generalised quadrangle has order $(s, t)$ (or order $s$ if $s = t$), and the point-line dual of a generalised quadrangle of order $(s, t)$ is again a generalised quadrangle but of order $(t, s)$. Higman’s inequality states that the parameters $s$ and $t$ bound one another; that is, $t \leq s^2$ and dually, $s \leq t^2$. A collineation $\theta$ of $S$ is an elation about the point $P$ if it is either the identity collineation, or it fixes each line incident with $P$ and fixes no point not collinear with $P$. If there is a group $G$ of elations of $S$ about the point $P$ such that $G$ acts regularly on the points not collinear with $P$, then we say that $S$ is an elation generalised quadrangle with elation group $G$ and base point $P$. Necessarily, $G$ has order $s^2 t$.

The classical generalised quadrangles $W(q)$, $Q(4, q)$, $H(3, q^2)$, $Q(5, q)$ and $H(4, q^2)$, are elation generalised quadrangles and arise as polar spaces of rank 2. The first of these is the incidence structure of all totally isotropic points and totally isotropic lines with respect to a null polarity in $PG(3, q)$, and is a generalised quadrangle but of order $(t, s)$. The point-line dual of $W(q)$ is $Q(4, q)$, the parabolic quadric of $PG(4, q)$, and is therefore a generalised quadrangle of order $q$ (see [PT84 3.2.1]). The incidence structure of all points and lines of a non-singular Hermitian variety in $PG(3, q^2)$, which forms the generalised quadrangle $H(3, q^2)$ of order $(q^2, q)$, has as its point-line dual the elliptic quadric $Q(5, q)$ in $PG(5, q)$, which is a generalised quadrangle of order $(q, q^2)$ (see [PT84 3.2.3]). The remaining classical generalised quadrangle, $H(4, q^2)$, is the incidence structure of all points and lines of a non-singular Hermitian variety in $PG(4, q^2)$, and is of order $(q^2, q^3)$ (see [PT84 3.1.1]).

**Kantor families.** Now standard in the theory of elation generalised quadrangles are the equivalent objects known commonly as 4-gonal families or Kantor families (after their inventor). Let $G$ be a group of order $s^2 t$ and suppose there exist two families of subgroups $\mathcal{F} = \{A_0, \ldots, A_t\}$ and $\mathcal{F}^* = \{A^*_0, \ldots, A^*_t\}$ of $G$ such that

(a) every element of $\mathcal{F}$ has order $s$ and every element of $\mathcal{F}^*$ has order $st$;
(b) $A_i \leq A_j^*$ for all $i$;
(c) $A_i \cap A_j^* = 1$ for $i \neq j$ (the “tangency condition”);
(d) $A_i A_j \cap A_k = 1$ for distinct $i,j,k$ (the “triple condition”).

Then the triple $(G, \mathcal{F}, \mathcal{F}^*)$ is called a Kantor family, but we will also say that $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family for $G$. The pair $(s, t)$ is said to be the order of $(\mathcal{F}, \mathcal{F}^*)$. From a Kantor family as described above, we can define a point-line incidence structure as follows.
| Points                          | Lines                                      |
|--------------------------------|--------------------------------------------|
| elements $g$ of $G$            | the right cosets $A_ig$                    |
| right cosets $A_i^*g$         | symbols $[A_i]$                            |
| a symbol $\infty$.            |                                            |

**Table 1.** The points and lines of the elation generalised quadrangle arising from a Kantor family (n.b., $A_i \in \mathcal{F}$, $A_i^* \in \mathcal{F}^*$, $g \in G$).

Incidence comes in four flavours (points on the left, lines on the right):

- $g \sim A_ig$
- $A_i^*g \sim [A_i]$
- $A_i^*g \sim A_ih$, where $A_ih \subseteq A_i^*g$
- $\infty \sim [A_i]$

It turns out that this incidence structure is an elation generalised quadrangle of order $(s, t)$ with base point $\infty$ and elation group $G$. Remarkably, all elation generalised quadrangles arise this way [PT84 §8.2], and we obtain a so-called translation generalised quadrangle when $G$ is abelian [PT84 8.2.3].

### 3. Flock Quadrangles and Special Groups

**Flock generalised quadrangles.** A *flock* of the quadratic cone $\mathcal{C}$ with vertex $v$ in $\text{PG}(3, q)$ is a partition of the points of $\mathcal{C} \setminus \{v\}$ into conics. Thas [Tha87] showed that a flock gives rise to an elation generalised quadrangle of order $(q^2, q)$, which we call a *flock quadrangle*. The flocks of $\text{PG}(3, q)$ have been classified by Law and Penttila [LP03] for $q$ at most 29. A *BLT-set of lines* of $W(q)$ is a set $\mathcal{L}$ of $q + 1$ lines of $W(q)$ such that no line of $W(q)$ is concurrent with more than two lines of $\mathcal{L}$. For $q$ odd, Knarr [Kna92] gave a direct geometric construction of an elation generalised quadrangle from a BLT-set of lines of $W(q)$. The ingredients of the Knarr construction are as follows:

- a symplectic polarity $\rho$ of $\text{PG}(5, q)$;
- a point $P$ of $\text{PG}(5, q)$;
- a 3-space inducing a $W(q)$ contained in $P^\perp$, but not containing $P$;
- a BLT-set of lines $\mathcal{L}$ of $W(q)$.

For each element $\ell_i$ of $\mathcal{L}$, let $\pi_i$ be the plane spanned by $\ell_i$ and $P$. Then we construct a generalised quadrangle as follows:
ELATION GENERALISED QUADRANGLES OF ORDER \( (s, p) \), WHERE \( p \) IS PRIME

| Points | Lines |
|--------|-------|
| – points of \( \text{PG}(5, q) \) not in \( \text{P}^p \) | – totally iso. planes not contained in \( \text{P}^p \) and meeting some \( \pi_i \) in a line |
| – lines of \( \text{PG}(5, q) \) not incident with \( \text{P} \) but contained in some \( \pi_i \) | – the planes \( \pi_i \) |
| – the point \( \text{P} \) |      |

Table 2. The points and lines of the elation generalised quadrangle arising from a BLT-set of lines of \( \text{W}(q) \). Incidence is inherited from that of \( \text{PG}(5, q) \).

Kantor [Kan91] Lemma] showed that a Kantor family of the flock elation group that is constructed from a \( q \)-clan, gives rise to a BLT-set of lines of \( \text{W}(q) \). We show in Section 6, that for \( q \) prime, any Kantor family of a flock elation group gives rise to a BLT-set of lines of \( \text{W}(q) \), and the resulting flock quadrangle obtained by the Knarr construction is isomorphic to the elation generalised quadrangle arising from the given Kantor family.

**Special and extraspecial groups.** A finite \( p \)-group \( G \) is special, if its centre, its derived subgroup, and its Frattini subgroup coincide. Moreover, we say that a special group is extraspecial if its centre is cyclic of prime order. The exponent of a special group is either \( p \) or \( p^2 \). Further, the order of an extraspecial group is of the form \( p^{2m+1} \), where \( m \) is a positive integer. For each such \( m \) there are, up to isomorphism, precisely two extraspecial groups of order \( p^{2m+1} \), one with exponent \( p \), and another one with exponent \( p^2 \) [Asc00, §8]. The elation groups of the flock quadrangles of order \( (p^2, p) \) are extraspecial of exponent \( p \) (see [Pay89]).

Here we recall a few facts about extraspecial \( p \)-groups which can be readily found in [Asc00, §8]. The quotient group \( E/Z(E) \) is an elementary abelian \( p \)-group forming a vector space \( V \) over \( \text{GF}(p) \). Moreover, the map from \( V^2 \) to \( Z(E) \) defined by

\[
\langle Z(E)x, Z(E)y \rangle = [x, y]
\]
defines an alternating form on \( V \). Thus, we obtain the generalised quadrangle \( W(p) \) where the totally isotropic subspaces correspond to abelian subgroups of \( E \) properly containing \( Z(E) \).

4. **Kantor Families for \( p \)-Groups of Order \( p^5 \)**

Recall that the elation group of a generalised quadrangle of order \( (p^2, p) \), \( p \) prime, has order \( p^5 \). Thus we provide in this section some powerful tools which will enable us to prove Theorem 1.2.

**Lemma 4.1.** Let \( (G, \mathcal{F}, \mathcal{F}^*) \) be a Kantor family giving rise to an elation generalised quadrangle \( S \) of order \( (s, t) \). Suppose that \( H \) is a subgroup of \( G \) of order \( t^3 \) such that for all \( A \in \mathcal{F} \) and \( A^* \in \mathcal{F}^* \), we have

\[
|A^* \cap H| \geq t^2 \text{ and } |A \cap H| \geq t.
\]

Then

\[
(\{A \cap H : A \in \mathcal{F}\}, \{A^* \cap H : A^* \in \mathcal{F}^*\})
\]
is a Kantor family for $H$ giving rise to an elation generalised quadrangle of order $t$.

**Proof.** Suppose $A$ and $B$ are a pair of distinct elements of $\mathcal{F}$, and let $A^*$ and $B^*$ be the respective elements of $\mathcal{F}^*$ such that $A \leq A^*$ and $B \leq B^*$. Since $A$ and $B^*$ intersect trivially, we have that

$$|A^*\cap H| \geq |A^*(B \cap H)| = \frac{|A^*||B \cap H|}{|A^* \cap B \cap H|} \geq st^2.$$ 

Therefore

$$|A^* \cap H| = |A^*||H|/|A^*H| \leq t^2,$$

and so $A^*$ and $H$ intersect in $t^2$ elements, for all $A^* \in \mathcal{F}^*$. Similarly,

$$|AH| \geq |A(B^* \cap H)| = |A||B^* \cap H| \geq st^2$$

and so $|A \cap H| = t$, for all $A \in \mathcal{F}$. The “triple” and “tangency” conditions follow from those in $(G, \mathcal{F}, \mathcal{F}^*)$. 

**Theorem 4.2.** Let $p$ be an odd prime. A generalised quadrangle of order $(p^2, p)$ with an elation subquadrangle of order $p$ is isomorphic to $H(3, p^2)$. Moreover, the subquadrangle here is isomorphic to $\mathcal{W}(p)$ and so is not a translation generalised quadrangle.

**Proof.** Let $S$ be a generalised quadrangle of order $(p^2, p)$ with a subquadrangle $S'$ of order $p$. By [BTVM96], a generalised quadrangle of order $p$ is either isomorphic to $\mathcal{W}(p)$ or $Q(4, p)$. Now every line of our given generalised quadrangle of order $(p^2, p)$ induces a spread of the subquadrangle; but $Q(4, p)$ has no spreads for $p$ odd (see [PT84], 3.4.1(i)). Therefore, $S'$ is isomorphic to $\mathcal{W}(p)$. It was proved by Brown [Bro02], and independently by Brouns, Thas, and Van Maldeghem [BTVM02], that if a generalised quadrangle $S$ of order $(q, q^2)$ has a subquadrangle $S'$ isomorphic to $Q(4, q)$, and if in $S'$ each ovoid $O_X$ consisting of all of the points collinear with a given point $X$ of $S \setminus S'$ is an elliptic quadric, then $S$ is isomorphic to $Q(5, q)$. By a result of Ball, Govaerts, and Storme [BGS96], if $p$ is a prime then every ovoid of $Q(4, p)$ is an elliptic quadric. Therefore, by dualising, we have that $S$ is isomorphic to $H(3, p^2)$.

The reason why we have pointed out that the subquadrangle is not a translation generalised quadrangle will become apparent in Section 5. We obtain the following consequence of Theorem 4.2

**Lemma 4.3.** Let $p$ be a prime and let $(G, \mathcal{F}, \mathcal{F}^*)$ be a Kantor family giving rise to an elation generalised quadrangle $S$ of order $(p^2, p)$. Suppose that $H$ is a subgroup of $G$ of order $p^3$ with the property that, for all $A^* \in \mathcal{F}^*$, we have $|A^* \cap H| \geq p^2$. Then $S$ is isomorphic to $H(3, p^2)$.

**Proof.** Let $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$ such that $A \leq A^*$. The condition $|A^* \cap H| \geq p^2$ implies that $A^*H \neq G$. This gives $AH \neq G$, and so $|A \cap H| \geq p$. Now it follows from Lemma 4.1 that $H$ gives rise to an elation subquadrangle $S'$ of order $p$. The remainder follows from Theorem 4.2. 

□
For $p$ odd, $W(p)$ is not a translation generalised quadrangle, which implies in
the previous lemma that $H$ is non-abelian. The next result gives more information
about Kantor families for groups of order $p^5$.

**Lemma 4.4.** Suppose that $G$ is a group with order $p^5$ and let $(\mathcal{F}, \mathcal{F}^*)$ be a
Kantor family of order $(p^2, p)$ for $G$. Then the following hold:

(i) None of the members of $\mathcal{F}$ is normal in $G$. In particular $G$ is non-abelian.

(ii) If $G$ is not extraspecial and $H$ is a subgroup of $G$ of order $p^3$, then there
is a subgroup $U$ of $G$ such that $|U| = p^3$ and $HU = G$.

(iii) $G$ is not generated by two elements.

(iv) The nilpotency class of $G$ is two.

(v) $G'$ is elementary abelian.

**Proof.** If $G$ is an extraspecial group with order $p^5$, then properties (i), (iii),
(iv), and (v) are valid for $G$, and so we may assume, for the entire proof, that $G$ is
not extraspecial.

(i) Assume by contradiction that $A \in \mathcal{F}$ is normal, and choose distinct $B, C \in \mathcal{F} \setminus \{A\}$. Then $AB$ is a subgroup of $G$ with order $p^4$ and so $AB \cap C = 1$ is
impossible, violating the triple condition.

(ii) Let $H$ be a subgroup of $G$ with order $p^3$. Since the elation group of $H(3, p^2)$
is extraspecial with exponent $p$, Lemma 4.3 implies that there is $A^* \in \mathcal{F}^*$ such that
$|H \cap A^*| = p$, and so $HA^* = G$.

(iii) Since $G/\Phi(G)$ is not cyclic, $|\Phi(G)| \leq p^3$. Further, $\Phi(G)U = G$ implies
that $U = G$, and hence it follows from part (ii) that $\Phi(G) \neq p^3$. Therefore we
obtain that $|\Phi(G)| \leq p^2$, and so a minimal generating set of $G$ has at least three
elements.

(iv) A group of order $p^5$ has nilpotency class at most 4. If the nilpotency
class of $G$ is 4, then $|G'| = |\Phi(G)| = p^3$, which is a contradiction by the previous
paragraph. We claim that the nilpotency class of $G$ is not three. Suppose
by contradiction that it is three. In this case, as $G$ is not generated by 2 elements,$G/G' \cong C_p \times C_p \times C_p$ and $|G' : \gamma_3(G)| = |\gamma_3(G)| = p$. Choose $a, b \in G$ such that
$\langle [a, b] \gamma_3(G) \rangle = G'/\gamma_3(G)$. Let $c_1 \in G$ such that $\langle aG', bG', c_1G' \rangle = G/G'$. Then
there are $\alpha, \beta \in \mathbb{F}_p$ such that $[a, c_1] \equiv [a, b]^\alpha$ (mod $\gamma_3(G)$) and $[b, c_1] \equiv [a, b]^\beta$
(mod $\gamma_3(G)$). Set $c = c_1a^\beta b^{-\alpha}$. Then $\langle aG', bG', cG' \rangle = G/G'$ and $[a, c] \equiv [b, c] \equiv 1$ (mod $\gamma_3(G)$); that is $[a, c]$, $[b, c] \in \gamma_3(G)$. By the Hall-Witt identity,$[a, b, c] = [c, a, b][a, c, b] = 1$. As $\gamma_3(G) = \langle [a, b, a], [a, b, b], [a, b, c] \rangle$, this implies
that either $[a, b, a] \neq 1$ or $[a, b, b] \neq 1$. Hence the subgroup $\langle a, b \rangle$ has nilpotency
class 3 and order $p^4$ (see also [Sch03, Corollary 2.2(i)]).

Let $H = \langle c, G' \rangle$. Clearly, $|H| = p^3$ and $G/H = \langle aH, bH \rangle$. Let $U$ be a
subgroup of $G$ such that $HU = G$, and so $HU/H = G/H = \langle aH, bH \rangle$. This
shows that there are $h_1, h_2 \in H$ such that $ah_1, bh_2 \in U$. Since $[a, h_1]$, $[a, h_2] \in$
$\gamma_3(G)$ and $[a, b, h_1] = [a, h_2] = 1$, we obtain that $[ah_1, bh_2] \gamma_3(G) = G'/\gamma_3(G)$
and either $[ah_1, bh_2, ah_1] \neq 1$ or $[ah_1, bh_2, bh_2] \neq 1$. Thus $U$ contains $G'$$^*$ and
$U$ is a group of order at least $p^3$. This, however, is a contradiction, by part (ii).
Therefore the nilpotency class of $G$ is not three. Since, by part (i), the nilpotency
class of $G$ is not one, we obtain that the class of $G$ must be two.
(v) By (iv), we only need to show that the exponent of \( G' \) is \( p \). By \[ \text{Rob96} \], 5.2.5], the quotient \( G'/\gamma_3(G) = G' \) is abelian. We argue by contradiction and assume that 

\( G' = \Omega(G) \) is not abelian. In this case the derived subgroup \( A \) is a maximal subgroup of \( G \) and hence each \( \gamma_3(G) \) is impossible by Lemma 4.4(i). Therefore 

\[ \gamma_3(G) = \Omega(G) \]

which is impossible. 

\[ \square \]

The next lemma describes the case when either \( G' \) or \( \Phi(G) \) is small.

**Lemma 4.5.** Suppose that \( G \) is a group with order \( p^5 \) and let \( (F, F^*) \) be a Kantor family for \( G \).

(i) If \( |G'| = p \) then all members of \( F \cup F^* \) are abelian.

(ii) If \( |\Phi(G)| = p \) then all members of \( F \cup F^* \) are elementary abelian. Moreover, if \( p \) is odd, then in this case, \( G \) has exponent \( p \).

(iii) If \( p \) is odd and \( G \) is extraspecial, then \( G \) has exponent \( p \) and all members of \( F \cup F^* \) are elementary abelian.

**Proof.** (i) Let us first assume that \( |G'| = p \). It suffices to prove, for all \( A^* \in F^* \), that \( A^* \) is abelian. We argue by contradiction and assume that \( A^* \in F^* \) is not abelian. In this case the derived subgroup \( (A^*)' = A^* \) is non-trivial, and, as \( (A^*)' \leq G \), we obtain that \( (A^*)' = G' \). Let \( A \in F \) such that \( A \leq A^* \). Then \( A \) is a maximal subgroup of \( A^* \), and so \( (A^*)' = G' \leq A \). Thus \( A \) is normal in \( G \), which is impossible by Lemma 4.4(i). Therefore \( A^* \) is abelian, as claimed.

(ii) The assertion that the members of the Kantor family are elementary abelian can be proved by substituting \( \Phi(A^*) \) in the place of \( (A^*)' \) and \( \Phi(G) \) in the place of \( G' \) in the previous paragraph. Let \( p \) be an odd prime. In this case, as \( |G'| = p \), the elements of \( G \) with order \( p \) form a subgroup \( \Omega(G) \) of \( G \). Let \( A \in F \) and \( B^* \in F^* \) such that \( A \cap B^* = 1 \). In this case \( AB^* = G \) and \( A, B^* \leq \Omega(G) \). Therefore \( G = \Omega(G) \), which amounts to saying that \( G \) has exponent \( p \).

Part (iii) follows immediately from part (ii). 

\[ \square \]

The following lemma is a generalisation of \[ \text{Kan91} \] Lemma.

**Lemma 4.6.** Let \( p \) be an odd prime and let \( (F, F^*) \) be a Kantor family for an extraspecial group \( E \) of order \( p^5 \). Then the image of \( F^* \) in \( E/Z(E) \) corresponds to a BLT-set of lines of \( W(p) \).

**Proof.** First note that by Lemma 4.5(iii), all the members of \( F^* \) are abelian and hence each \( A^* \in F^* \) induces an abelian subgroup of \( E/Z(E) \), and so a totally isotropic line of the associated \( W(p) \) geometry. Therefore, every member of \( F^* \) contains \( Z(E) \). Suppose by way of contradiction that there is a line of \( W(p) \) concurrent with three elements of \( L = \{ A^*/Z(E) : A^* \in F^* \} \). Then there exists an abelian subgroup \( H \) of \( E \) of order \( p^3 \), and three elements \( A^*, B^*, C^* \) of \( F^* \) such that \( H \) intersects each of these elements in a subgroup of order \( p^2 \) properly containing \( Z(E) \) (note: \( H \) contains \( Z(E) \)). Let \( A, B, C \) be the unique elements of \( F \) contained in \( A^*, B^*, C^* \) respectively. Now \( (H \cap B)Z(E) \) is contained in \( B^* \) and so \( A \cap (H \cap B)Z(E) = 1 \). Also, we have that \( |H \cap A| = p \) as
\[
p^2 = |H \cap A^*| = |(H \cap A)Z(E)| = |H \cap A||Z(E)| \text{ (similarly, } |H \cap B| = p).\]

Thus
\[
|(H \cap A)(H \cap B)Z(E)| = \frac{|H \cap A||(H \cap B)Z(E)|}{|H \cap A||H \cap B||Z(E)|} = |H \cap A||H \cap B||H \cap A \cap (H \cap B)Z(E)| \geq p^3
\]
and so one can see that \( H = (H \cap A)(H \cap B)Z(E) \). So
\[
C^* \cap H = (C \cap (H \cap A)(H \cap B))Z(E)
\]
and by the condition \( AB \cap C = 1 \), we have that \( C^* \cap H = Z(E) \), giving us the desired contradiction. Therefore, \( \mathcal{L} \) is a BLT-set of lines of \( \mathcal{W}(p) \).

\section{The Proof of Theorem 1.2}

In this section we prove Theorem 1.2. By Lemma 4.5(iii), an extraspecial group with order \( p^5 \) and exponent \( p^2 \) does not admit a Kantor family with order \( (p^5, p) \). Hence we may assume, for a proof by contradiction, that:

\[ G \text{ is a group of order } p^5 \text{ and } (\mathcal{F}, \mathcal{F}^*) \text{ is a Kantor family for } G \text{ with order } (p^2, p). \]

Our aim is to derive a contradiction. First note that Lemma 4.4 implies that one of the following must hold:

(I) \( G/G' \cong C_p \times C_p \times C_p \times C_p \times C_p \) and \( G' \cong C_p \);

(II) \( G/G' \cong (C_p)^3 \) and \( G' \cong (C_p)^2 \);

(III) \( G/G' \cong (C_p)^4 \) and \( G' \cong C_p \).

We show, case by case, that none of the above possibilities can occur. We let \( Z \) denote the centre of \( G \).

Case (I). Using the argument in the proof of Lemma 4.4(iv), we can choose generators \( a, b, c \) of \( G \) such that \( G' = \langle [a, b] \rangle \) and \( c \in Z \). It also follows that \( Z = \langle z, \Phi(G) \rangle \), and so \( |Z| = p^3 \). By Lemma 4.5(ii), all members of \( \mathcal{F}^* \) must be abelian and so \( \mathcal{F} \subseteq \mathcal{Hac96} \). Hence case (I) cannot occur.

Case (II). First we claim that it is possible to choose the generators \( x, y, z \) of \( G \) such that \( G' = \langle [x, y], [x, z], [y, z] \rangle \) and \( [y, z] = 1 \). Let \( x, y, z \) be generators of \( G \). Then \( G' = \langle [x, y], [x, z], [y, z] \rangle \). Since \( G' \cong (C_p)^2 \) we have that there are \( \alpha, \beta, \gamma \in \mathbb{F}_p \) such that at least one of \( \alpha, \beta, \gamma \) is non-zero and \([x, y]^\alpha[x, z]^\beta[y, z]^\gamma = 1\). If \( \alpha = \beta = 0 \) then \( \gamma \neq 0 \), and \( [y, z] = 1 \) follows. If \( \alpha = 0 \) and \( \beta \neq 0 \) then \([x^\beta y^\gamma, z] = 1\). Now replacing \( x \) by \( x^\beta y^\gamma \) we find that in the new generating set \( [x, z] = 1 \) holds. Similarly, if \( \alpha \neq 0 \) and \( \beta = 0 \) then \([y, x^{-\alpha} z^\gamma] = 1 \) and replacing \( x \) by \( x^{-\alpha} z^\gamma \) we obtain that \( [x, y] = 1 \) holds in the new generating set. Finally if \( \alpha \beta \neq 0 \),
then we replace $x$ by $x^{\beta/\alpha}y^{\gamma/\alpha}$ and $y$ by $yz^{\beta/\alpha}$ to obtain that $[x, y] = 1$. Thus, after applying one of the substitutions above and possibly renaming the generators, $[y, z] = 1$ holds, and the claim is valid.

We continue by verifying the following claim: if $H$ is a subgroup in $G$ with order $p^2$ and $H \cap Z = 1$ then there are $c, d \in Z$ such that $H = \langle yc, zd \rangle$.

Assume that $H$ is a subgroup or order $p^2$ that does not intersect $Z$. Then $HZ/Z \cong H/(H \cap Z) = H$ and so $H \cong C_p \times C_p$. In particular $H$ can be generated by two elements of the form $u = x^{\alpha_1}y^{\beta_1}z^{\gamma_1}c_1$ and $v = x^{\alpha_2}y^{\beta_2}z^{\gamma_2}c_2$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}_p, c_i \in Z$ and $\langle uZ, vZ \rangle \cong C_p \times C_p$. Since $[u, v] = 1$ we obtain that

$$1 = [u, v] = [x^{\alpha_1}y^{\beta_1}z^{\gamma_1}c_1, x^{\alpha_2}y^{\beta_2}z^{\gamma_2}c_2] = [x, y]^{\alpha_1\beta_2-\alpha_2\beta_1}[x, z]^{\alpha_1\gamma_2-\alpha_2\gamma_1}.$$ 

Thus $\alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. Note that these two expressions can be viewed as determinants of suitable $(2 \times 2)$-matrices. If $\alpha_1, \alpha_2 \neq (0, 0)$ then the vectors $(\beta_1, \beta_2)$ and $(\gamma_1, \gamma_2)$ are both multiples of $(\alpha_1, \alpha_2)$ and so the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}$$

has row-rank 1. Since the row-rank of a matrix is the same as the column-rank, this also shows that the vector $(\alpha_2, \beta_2, \gamma_2)$ is a multiple of the vector $(\alpha_1, \beta_1, \gamma_1)$ and so $uZ = vZ$ which gives $HZ/Z \cong C_p$; a contradiction. Thus $(\alpha_1, \alpha_2) = (0, 0)$; thus $u = y^{\beta_1}z^{\gamma_1}c_1$ and $v = y^{\beta_2}z^{\gamma_2}c_2$. Since $\langle uZ, vZ \rangle \cong C_p \times C_p$ we must have that $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$. Also, if $\beta_1, \beta_2 = 0$ then $HZ/Z \cong C_p$ and so we may assume that $\beta_1 \neq 0$. Change $v$ to $u^{-\beta_2/\beta_1}v$; then $\langle u, v \rangle = H$ and $v$ is of the form $z^{\gamma}d'$ where $d' \in Z$. Now change $u$ to $uv^{-\gamma/\gamma_2}$. Then $\langle u, v \rangle = H$ still holds and now $u$ is of the form $y^{\beta}d'$ where $d' \in Z$. Now $u^{\beta-1}$ and $v^{\gamma-1}$ are as required.

Let us now prove that $G$ does not admit a Kantor family. We argue by contradiction and assume that $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family of order $(p^2, p)$ for $G$. If $A, B$ are distinct elements of $\mathcal{F}$ such that $A \cap Z = B \cap Z = 1$ then the claim above implies that $[A, B] = 1$, and so $AB$ is a subgroup of $G$ with order $p^3$. Thus, if $C \in \mathcal{F} \setminus \{\{A, B\}, \{A, C\}, \{B, C\}\}$, then $AB \cap C \neq 1$, which contradicts the triple condition. Thus $\mathcal{F}$ has at most one member that avoids the centre. Let us suppose now $A, B, C$ are pairwise distinct members of $\mathcal{F}$ such that $A \cap Z, B \cap Z, C \cap Z$ are non-trivial. As $A \cap B = A \cap C = B \cap C = 1$, we obtain that $|A \cap Z| = |B \cap Z| = |C \cap Z| = p$ and that $A \cap Z, B \cap Z, C \cap Z$ are three distinct subgroups of $Z$. This, however implies that $Z = (A \cap Z)(B \cap Z)$, and, in turn, that $C \cap Z \leq (A \cap Z)(B \cap Z)$, which violates the triple condition.

The argument in the last paragraph implies that at most two members of $\mathcal{F}$ can intersect $Z$ non-trivially, and at most one member of $\mathcal{F}$ can avoid the centre. Thus $|\mathcal{F}| \leq 3$, which is a contradiction as $p$ is odd and $|\mathcal{F}| = p + 1$. Therefore case (II) is impossible.

Case (III). As $G$ is not extraspecial, $|Z| = p^3$, and Lemma 3.2 (iv) implies that the members of $\mathcal{F}^*$ are abelian. In this case Theorem 3.2, Lemmas 2.1 and 2.2 show that the subgroups $A \cap Z$ and $A^* \cap Z$ (with $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$)
ELATION GENERALISED QUADRANGLES OF ORDER \((s, p)\), WHERE \(p\) IS PRIME

\(\mathcal{F}^*)\) form a Kantor family of order \(p\) for \(Z\). However, we have a contradiction to Theorem [4.2] since the associated subquadrangle of order \(p\) is not a translation generalised quadrangle.

As none of the possibilities listed at the beginning of the section can occur, Theorem [1.2] must hold.

6. The Proof of Theorem [1.1]

Here we prove Theorem [1.1] but first we show that applying the Knarr construction to a BLT-set of lines arising from a Kantor family \((\mathcal{F}, \mathcal{F}^*)\) of the flock elation group results in an elation generalised quadrangle isomorphic to that directly associated to \((\mathcal{F}, \mathcal{F}^*)\).

**Theorem 6.1.** Let \(G\) be the flock elation group of order \(p^5\), \(p\) odd, and suppose that \(G\) admits a Kantor family \((\mathcal{F}, \mathcal{F}^*)\) giving rise to an elation generalised quadrangle \(E\). Consider the BLT-set of lines \(L\) of \(W(p)\) obtained by taking the image of \(\mathcal{F}^*\) under the natural projection map from \(G\) onto \(G/Z(G)\). Then the flock quadrangle arising from \(L\) via the Knarr construction is equivalent to \(E\).

**Proof.** First note that \(G\) is extraspecial of exponent \(p\), and observe that the matrices of the form
\[
\begin{pmatrix}
1 & a & b & c & d & e \\
0 & 1 & 0 & 0 & 0 & d \\
0 & 0 & 1 & 0 & 0 & c \\
0 & 0 & 0 & 1 & 0 & -b \\
0 & 0 & 0 & 0 & 1 & -a \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
\(a, b, c, d, e \in \text{GF}(p)\)
define a representation of \(G\) into the symplectic group \(\text{PSp}(6, p)\) with its associated null polarity given by the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 1 & 0 & \\
0 & 0 & 1 & 0 & 0 & \\
0 & 0 & -1 & 0 & 0 & \\
0 & -1 & 0 & 0 & 0 & \\
-1 & 0 & 0 & 0 & 0 & \\
\end{pmatrix}
\]

Moreover, the centre of \(G\) consists only of those upper triangular matrices with zeros everywhere above the diagonal except possibly the top right corner, and \(G\) fixes the projective point \(P\) represented by \((1, 0, 0, 0, 0, 0)\). Hence \(G\) induces an action on the quotient \(P'^{\perp}/P \equiv W(p)\). It is not difficult to show that the right coset action of \(G\) on \(G/Z(G)\) is permutationally isomorphic to the action of \(G\) on \(P'^{\perp}/P\) (as a projective right-module). To be more specific, the representatives of \(G/Z(G)\) are in a bijection with matrices of the form
\[
\begin{pmatrix}
1 & a & b & c & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
\(a, b, c, d \in \text{GF}(p)\)
and \(P'^{\perp}/P\) can naturally be identified with vectors of the form \((0, a, b, c, d, 1)\). Thus we have a bijection from \(G/Z(G)\) onto \(P'^{\perp}/P\) given by
\[
\begin{pmatrix}
1 & a & b & c & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\mapsto P + (0, a, b, c, d, 1)
such that the right coset action of $G$ is equivalent to the right-module action of $G$ on $P^\perp/P$.

Let $(\mathcal{F}, \mathcal{F}^*)$ be a Kantor family for $G$ and let $\mathcal{E}$ be the associated elation generalised quadrangle with points

(i) elements of $g$,
(ii) right cosets $A_i^*g$ of elements of $\mathcal{F}^*$,
(iii) $\infty$,

and lines

(a) right cosets $A_ig$ of elements of $\mathcal{F}$,
(b) symbols $[A_i]$ where $A_i \in \mathcal{F}$.

Let $Q = (0, 0, 0, 0, 0, 1)$ and note that $Q$ is opposite to $P$. Let $\mathcal{K}$ be the flock quadrangle associated to $\mathcal{L}$ constructed from the point $P$, and define a map from $\mathcal{E}$ to $\mathcal{K}$ as follows:

$$\infty \mapsto P, \quad [A_i] \mapsto \pi_i, \quad A_i^*g \mapsto z_i^g, \quad A_i^*g \mapsto M_i^g, \quad g \mapsto Q^g.$$

We will show that this map defines an isomorphism of generalised quadrangles. Since the action of $G$ on $P^\perp/P$ is permutationally isomorphic to the right coset action of $G$ on $G/Z(G)$, we have that the stabiliser of the subspace corresponding to a subgroup $H$ containing $Z(G)$ is just $H$ itself. Therefore $A_i$ fixes $z_i$ and $A_i^*$ fixes $M_i$ (for all $i$), and so the map above is well-defined. Now we verify that the four types of incidences are compatible:

**Incidence of $\infty$ and $[A_i]$**: It is clear that $P \sim \pi_i$ for all $i$.

**Incidence of $A_i^*g$ and $[A_i]$**: We want to show that $\pi_i \sim z_i^g$ given we know that $\pi_i \sim z_i$. Now $G$ fixes every subspace of $P^\perp$ on $P$, and hence $G$ fixes $\pi_i$. Therefore $z_i^g \sim \pi_i^g = \pi$ (n.b., $g$ is a collineation).

**Incidence of $A_i^*g$ and $A_i$**: So $A_i h \subset A_i^* g$. We want to show that $M_i^h \sim z_i^g$. By definition, $z_i$ is the unique line of $\pi_i$ (not on $P$) which is on a plane $M_i$ on $Q$. We know that $M_i \sim z_i$. Since $A_i h \subset A_i Z g$, then there exists an element $e$ of $Z(G)$ such that $h g^{-1} e \in A_i$. It suffices to show that $M_i^{h g^{-1}} \sim z_i$. Now $A_i$ fixes $M_i$ and so $M_i^{h g^{-1}} = M_i^{e^{-1}}$. Now $e^{-1}$ fixes $z_i$ and so $M_i^{h g^{-1}} \sim z_i$.

**Incidence of $g$ and $A_i g$**: It is clear that $G$ acts regularly on the points opposite $P$. Since for all $i$ we have $Q \sim M_i$, it follows that $Q^g \sim M_i^g$.

Therefore, the flock quadrangle arising from $\mathcal{L}$ via the Knarr construction is equivalent to $\mathcal{E}$.

**Theorem 1.1 and its proof.**

An elation generalised quadrangle of order $(s, p)$, with $p$ prime, is a flock quadrangle, isomorphic to $Q(4, p)$, or isomorphic to $W(p)$.

**Proof.** Let $S$ be an elation generalised quadrangle of order $(s, p)$, where $p$ is prime, and suppose that $(G, \mathcal{F}, \mathcal{F}^*)$ is the corresponding Kantor family. By [BTVM96], we may assume $s = p^2$, and so $G$ has order $p^5$. By Theorem 1.2
ELATION GENERALISED QUADRANGLES OF ORDER \((s, p)\), WHERE \(p\) IS PRIME

\(G\) must be extraspecial. Now the Frattini subgroup of \(G\) has order \(p\) and so has nontrivial intersection with every subgroup of \(G\) that has order at least \(p^2\). Hence \(Z(G)\) is contained in every element of \(F^+\). Therefore, by Lemma 4.6 and Theorem 6.1 our generalised quadrangle \(S\) is a flock quadrangle. □

Acknowledgment

The authors would like to thank Stan Payne for his feedback on this work. This research formed part of an Australian Research Council Discovery Grant project that was undertaken at the University of Western Australia. The first author was supported by a Marie Curie Incoming International Fellowship within the 6th European Community Framework Programme, contract number: MIIF1-CT-2006-040360. The third author was supported by the Hungarian Scientific Research Fund (OTKA) grant F049040; he is also grateful to Hendrik Van Maldeghem for funding his visit to the workshop Groups and Buildings 2007 held in Ghent.

References

[Asc00] M. Aschbacher. *Finite group theory*, volume 10 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2000.

[BGS06] Simeon Ball, Patrick Govaerts, and Leo Storme. On ovoids of parabolic quadrics. *Des. Codes Cryptogr.*, 38(1):131–145, 2006.

[Bro02] Matthew R. Brown. A characterisation of the generalized quadrangle \(Q(5, q)\) using cohomology. *J. Algebraic Comb.*, 15(2):107–125, 2002.

[BTVM96] I. Bloemen, J. A. Thas, and H. Van Maldeghem. Elation generalized quadrangles of order \((p, t)\), \(p\) prime, are classical. *J. Statist. Plann. Inference*, 56(1):49–55, 1996. Special issue on orthogonal arrays and affine designs, Part I.

[BTVM02] Leen Brouns, Joseph A. Thas, and Hendrik Van Maldeghem. A characterization of \(Q(5, q)\) using one subquadrangle \(Q(4, q)\). *European J. Combin.*, 23(2):163–177, 2002.

[Fro88] Daniel Frohardt. Groups which produce generalized quadrangles. *J. Combin. Theory Ser. A*, 48(1):139–145, 1988.

[Hac96] Dirk Hachenberger. Groups admitting a Kantor family and a factorized normal subgroup. *Des. Codes Cryptogr.*, 8(1-2):135–143, 1996. Special issue dedicated to Hanfried Lenz.

[Kan91] William M. Kantor. Generalized quadrangles, flocks, and BLT sets. *J. Combin. Theory Ser. A*, 58(1):153–157, 1991.

[Kna92] Norbert Knarr. A geometric construction of generalized quadrangles from polar spaces of rank three. *Results Math.*, 21(3-4):332–344, 1992.

[LP03] Maska Law and Tim Penttila. Classification of flocks of the quadratic cone over fields of order at most 29. *Adv. Geom.*, (suppl.):S232–S244, 2003. Special issue dedicated to Adriano Barlotti.

[Pay89] Stanley E. Payne. An essay on skew translation generalized quadrangles. *Geom. Dedicata*, 32(1):93–118, 1989.

[PT84] S. E. Payne and J. A. Thas. *Finite generalized quadrangles*, volume 110 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[Rob96] Derek J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[Sch03] Csaba Schneider. Groups of prime-power order with a small second derived quotient. *J. Algebra*, 266(2):539–551, 2003.

[Tha87] J. A. Thas. Generalized quadrangles and flocks of cones. *European J. Combin.*, 8(4):441–452, 1987.
DEPARTMENT OF PURE MATHEMATICS, GHENT UNIVERSITY, GALGLAAN 2, B-9000 GHENT, BELGIUM.
E-mail address: bamberg@cage.ugent.be

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523, USA.
E-mail address: penttila@math.colostate.edu

INFORMATICS RESEARCH LABORATORY, COMPUTER AND AUTOMATION RESEARCH INSTITUTE, 1518 BUDAPEST PF. 63, HUNGARY.
E-mail address: csaba.schneider@sztaki.hu