Random walker derivation of Archie’s law

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Theoretical justification is provided for Archie’s law. This phenomenological equation, having the form of a power law, relates the measured electrical resistivity of electrolyte-saturated rock samples to their connected porosity. Historically it has been important for oil and gas exploration.

I. INTRODUCTION

The empirical Archie’s law \cite{archie1942} came out of the laboratory, where electrical resistivity \( \rho \) and porosity \( \phi \) values were obtained for samples of a particular rock type fully saturated by brine or other electrolyte. The pairs of measured values \( \phi \) and \( \rho / \rho_e \) (\( \rho_e \) is the resistivity of the electrolyte) were plotted on a ln–ln plot and fit to a straight line, with slope \( -m \).

Thus Archie’s law has the form

\[
F = a \phi^{-m}
\]

where \( F \) is the so-called resistivity formation factor,

\[
F = \frac{\rho}{\rho_e} > 1.
\]

Like the exponent \( m \), the prefactor \( a \) characterizes a collection of rock samples that exhibits Archie’s law. Thus Eq. (1) shows that the value \( F \to a \) as the sample porosities \( \phi \to 1 \). [That limit is never reached since a rock sample with \( \phi = 1 \) is nonsensical.] Then Eq. (2) reveals that the prefactor \( a > 1 \).

The utility of Archie’s law follows from the circumstance that the points fall on, or very near, that best-fit line. In that case Archie’s law can be used in conjunction with resistivity logs (measurements) taken in the field, to obtain an estimate of the (conductive) porosity of the rock formation in question (the rock matrix itself is not conductive). An anomalously high measured resistivity value may indicate the presence of non-conductive oil or gas in the pore space.

Here it is more convenient to consider the relation

\[
\frac{\sigma}{\sigma_e} \propto \phi^\mu
\]

where \( \sigma \) and \( \sigma_e \) are the electrical conductivities of a saturated rock sample and the electrolyte, respectively, and \( \mu \) is the cementation exponent for that sample. (In contrast, the cementation exponent \( m \) is obtained graphically from a collection of samples of the rock type.)

The formula for the exponent \( \mu \) is obtained below in a manner that makes clear its value is determined by the particular heterogeneity of the porous rock. The derivation relies on the walker diffusion method (WDM), which is introduced in the following section. Section III then presents the derivation of \( \mu \), and Sec.IV discusses and gives meaning to the parameters that emerge from the derivation. Section V gives a brief discussion of the Menger and Sierpinski sponges, which are recursive—not “natural”—fractals, mainly to show example sets of the parameter values. Section VI shows that adherence to Archie’s law signifies that the rock formation is (statistically) self-similar, over the length scales represented in the collection of samples. Section VII applies the model developed in this paper to the artificial rock that is an electrolyte-saturated assemblage of glass beads. Its porosity can be reduced in steps by sintering. Final comments are made in Sec.VIII.

Note that, in the remainder of this paper, the symbol \( \phi \) will refer to the volume fraction composed of the conductive domain—the electrolyte—that spans the sample. That lets the model apply to porous rock that is not fully saturated with electrolyte.

II. WALKER DIFFUSION METHOD

This application of the WDM \cite{wagner2002} utilizes the relation

\[
\sigma = \langle \sigma(\mathbf{r}) \rangle \, D_{w}
\]

between the effective conductivity \( \sigma \) of a composite material and the (dimensionless) diffusion coefficient \( D_w \) obtained from walkers diffusing through a virtual replica of the composite. The factor \( \langle \sigma(\mathbf{r}) \rangle \) is the volume average of the constituent conductivities; the vector \( \mathbf{r} \) locates a point in that volume.

The phase domains that make up the composite are host to walker populations, where the walker density of a population is proportional to the conductivity value of its host domain. The principle of detailed balance ensures that the population densities are maintained, by providing the following rule for walker diffusion over the digitized composite: a walker at site (or pixel/voxel) \( i \) attempts a move to a randomly chosen adjacent site \( j \) during the time interval \( \tau = (4d)^{-1} \), where \( d \) is the Euclidean dimension of the system; this move is successful with probability \( p_{ij} = \sigma_j / (\sigma_i + \sigma_j) \), where \( \sigma_i \) and \( \sigma_j \) are the conductivities of sites \( i \) and \( j \), respectively. The path of the walker thus reflects the composition and morphology of the domains that are encountered.
The diffusion coefficient $D_w$ is calculated by use of the equation
\[
D_w = \frac{\langle R(t)^2 \rangle}{2 dt}
\]  
(5)
where the set $\{R\}$ of walker displacements, each occurring over the time interval $t$, must have a Gaussian probability distribution that is necessarily centered well beyond a displacement value $\xi$. The correlation length $\xi$ is identified as the length scale above which a composite material attains its "effective", or macroscopic, value of a scalar transport property, such as electrical conductivity.

For displacements $R < \xi$, the walker diffusion is anomalous rather than Gaussian, due to the heterogeneity of the composite at length scales less than $\xi$. Importantly, $\xi$ is expressed in units of $\xi_0$, which may be considered the resolution, or size of the smallest feature, of the system. Then a walker displacement $\xi_0$, requiring a travel time $t_\xi = \xi^2 / (2 d D_w)$, is produced by a walk comprising $(\xi/\xi_0)^{d_w}$ segments of length $\xi_0$, each requiring a travel time of $t_0 = \xi_0^2 / (2 d D_0)$, where $D_0$ is the (dimensionless) walker diffusion coefficient associated with displacements $R = \xi_0$. Thus $t_\xi = (\xi/\xi_0)^{d_w} t_0$, which gives the relation
\[
D_w = D_0 \left( \frac{\xi}{\xi_0} \right)^{2-d_w}
\]  
(6)
between the walker diffusion coefficient $D_w$, the correlation length $\xi$ (expressed in units of $\xi_0$), and the parameter $d_w$ associated with walker paths of displacement $\xi$.

Note that a random walk through an infinite, homogeneous, conducting space produces values $d_w = 2$ and $D_w = D_0 = 1$.

### III. DERIVATION OF THE EXPONENT $\mu$

As porous rock is heterogeneous, the linear dimension $L$ of a sample is the correlation length $\xi$.

Then by combining Eqs. (2) and (4), the effective conductivity $\sigma$ of a rock sample of linear dimension $\xi$ is shown to be
\[
\sigma = \sigma_e D_0 \left( \frac{\xi}{\xi_0} \right)^{2-d_w}.
\]  
(7)

Let the volume of the conductive domain spanning the sample be designated by $V$, expressed in units of $(\xi_0)^3$. Then the volume fraction $\phi$ is
\[
\phi = \frac{V}{\xi^3} = \frac{V}{(\xi_0)^3} \left( \frac{\xi}{\xi_0} \right)^3
\]  
(8)
A more useful expression of this relation is
\[
\phi = \left( \frac{\xi}{\xi_0} \right)^D \left( \frac{\xi}{\xi_0} \right)^D
\]  
(9)
with the exponent $D$ given by
\[
D = 3 + \frac{\ln \phi}{\ln \left( \frac{\xi}{\xi_0} \right)}
\]  
(10)
Note that the value $D < 3$ reflects the relationship between sample size and porosity. Thus $D$ can be regarded as a fractal “mass dimension”, indicating the extent, or volume fraction, of the conductive domain within the sample.

Then combining Eqs. (7) and (9) gives the power-law relation
\[
\frac{\sigma}{\sigma_e} = \phi D_0 \left( \frac{\xi}{\xi_0} \right)^{2-d_w} = D_0 \phi^\mu
\]  
(11)
with the cementation exponent $\mu$ being
\[
\mu = 1 + \frac{d_w - 2}{3 - D}
\]  
(12)
The fraction in this expression is always positive, so imposing the lower bound $\mu > 1$.

### IV. UNDERSTANDING $\mu$, $d_w$ AND $D_0$

Evidently the conductive domain of a rock sample is parameterized by $D$, which is directly related to the volume fraction occupied by that domain. In contrast, the parameter $d_w$, which appears in Eq. (11), is a measure of the resistance to current flow through the conductive domain. A larger $d_w$ value indicates greater resistance. Note that a $d_w$ value is not unique to a particular domain morphology.

The values of $d_w$ and $D_0$ cannot be determined by physical inspection. However, comparative values can be inferred by considering the behavior of walkers confined to a virtual replica of the domain.

The parameter $d_w$, as introduced in Sec. II, is determined by random walks that begin at one end of the rock sample and end at the opposite end. Then the value of $d_w$ is obtained from the relation
\[
\left( \frac{\xi}{\xi_0} \right)^{d_w} = \langle n \rangle
\]  
(13)
where $n$ is the number of steps of size $\xi_0$, comprising a walk, and the average value $\langle n \rangle$ is obtained from many such walks. The more tortuous and complex (e.g., presence of dead ends) the domain morphology, the more steps are taken and so the larger the $d_w$ value. In any case a finite domain is characterized by a value $d_w > 2$.

The parameter $D_0$ is the (dimensionless) walker diffusion coefficient obtained from random walks of displacement $\xi_0$ (such as comprise the displacement $\xi$). Walks confined to the conductive domain produce $D_0 < 1$, due to the attempted moves by the walkers to exit the domain (a consequence of the “blind ant” rule imposed by
the principle of detailed balance). Thus the value $D_0$ is smaller for more-complex domains.

There is no algebraic relation between $d_w$ and $D_0$, but logically a smaller value for $D_0$ will occur with a greater value of $d_w$. For infinite, homogeneous, conductive media, $d_w + D_0 = 3$. This relation may persist as a reasonable expectation for heterogeneous and fractal media, with the conditions that $d_w > 2$ and $D_0 < 1$.

Note that the exponent $\mu > 2$ when $d_w + D > 5$, with the condition that $D < 3$.

In any case, the value of $\mu$ is determined by the ratio in its expression, Eq. (12). The numerator accounts for the shape or morphology of the conductive domain, and the denominator accounts for the volume fraction $\phi$ that domain occupies.

Note that Eq. (13) does not impose an upper limit on the value of $d_w$. Thus there is no upper limit for the value $\mu$.

High values for $\mu$ or $m$ obtained experimentally can be understood by noting that the parameter $D$ for a given sample is determined solely by the volume of the conductive domain. That domain may have any one of many possible configurations. Thus high $\mu$ and $m$ values indicate a more complex (high $d_w$ value) domain structure.

### V. SELF-SIMILAR FRACTALS

A heuristic example is given by the Menger Sponge [4], which is a fractal object in 3D space often used as a model for porous media.

It is a recursive, self-similar fractal, such that sponges produced by different numbers of iterations can be regarded as different “samples”. Equation (14) is reproduced by combining Eqs. (6)–(8) in Ref. [4], where $v_i$ is the porosity $\phi$ of the $i$-th iteration sponge, and the fractal dimension $H$ is $D$. And Eq. (12) giving the exponent $\mu$ is identical to Eq. (11) in Ref. [4], giving the exponent $t$. Note that the Menger Sponge has the characteristic length $\xi_0 = 1$.

Table I shows the calculated values of the parameters and the exponent $\mu$, for the Menger Sponge [4] and, for comparison, the Sierpinski Sponge [5]. Note that the prefactor $D_0 < 1$ reflects the condition that walkers are confined at all porosities.

| Table I. Calculated parameters         | Menger Sponge | Sierpinski Sponge |
|-----------------------------------------|---------------|-------------------|
| $D_0$                                   | 0.65564       | 0.035312          |
| $d_w$                                   | 2.16326       | 2.02026           |
| $D$                                     | 2.72683       | 2.96565           |
| $\mu$                                   | 1.59744       | 1.58976           |

An extreme example comes from percolation theory. At the percolation threshold, there are conductive clusters of all sizes attached to the conducting “backbone” that carries the electric current. The mass dimension of that “incipient infinite cluster” in 3D space is $D = 2.52295$ [6], and the corresponding walker path dimension is $d_w = 3.84331$ [6]. Putting these values into Eq. (12) gives $\mu = 4.86398$.

Interestingly, Eq. (12), combined with Eqs. (18) and (20) from Ref. [6], gives the relation $\mu = t/\beta$, where $t$ and $\beta$ are the critical exponents in the asymptotic expressions for the conductivity $\sigma$, and the fraction of sites comprising the percolating cluster, respectively.

### VI. ARCHIE FORMATIONS

A geological formation is self-similar if samples of different size $\xi$ have $d_w$ and $D$ values in common. Then they have a $\mu$ value in common, and so satisfy Archie’s law [in the form of Eq. (1)] with cementation exponent $m = \mu$, and prefactor $a = 1/D_0 > 1$, reflecting the fact that walkers are confined to the conductive domains.

More likely the formation is statistically self-similar, meaning that the variation in $d_w$ and $D$ values is small.

Possibly this graphical method for detecting self-similarity could be extended to ostensibly non-self-similar (that is, heterogeneous) formations, to look for length scales at which the formation is self-similar. Self-similarity may be indicated when adjacent $(\phi, F)$ points on the ln–ln plot line up on a straight line that intercepts the $\phi$ axis at $\phi > 1$.

### VII. SINTERING EXPERIMENTS

An artificial system, mimicking an unconsolidated sandstone, is a dense, randomly packed assemblage of glass beads saturated with an electrolyte. A consolidated sandstone is obtained by fusing the glass beads to a lesser or greater degree by heating the assemblage above its softening temperature for a lesser or greater time duration. Experiments [7] over a porosity range $0.399 > \phi > 0.023$ find $\mu \approx 1.5$ for $\phi > 0.2$ and $\mu \approx 2$ for $\phi < 0.2$.

Other experiments of this sort [8, 9] have obtained similar results; in particular, finding that $\mu$ increases from 1.5 as the porosity $\phi$ decreases.

The increase in $\mu$ must be due to the increased densification of the glass-bead complex. That reduces both the linear dimension $\xi$ of the sample and its porosity $\phi$. Consequently there may be little change in the value $D$. However the value $d_w$ will certainly increase, due to the further constriction of the walker paths. Under this scenario the value $\mu$ increases.

Note that the value $\mu = 1.5$ found for the higher porosity complex indicates that $d_w$ and $D$ satisfy the relation $2d_w + D = 7$. As $d_w > 2$ and $D < 3$, the value $d_w$ must be slightly larger than 2 and the value $D$ must be slightly less than 3.
VIII. CONCLUDING REMARKS

Archie’s law has been derived, and shown to be a consequence of the statistical self-similarity of particular geological formations. (Not all heterogeneity is self-similar.) This required a random-walk approach, as self-similarity, and heterogeneity in general, are indicated by a value \( d_w > 2 \) that is obtained from random walks over the conductive domain.

Other electrical conductivity models have been developed using fractal, percolation, and effective medium theories. Those are reviewed or demonstrated in Refs. [10], [11], [12], [13]. The variety of approaches is a response to the difficulty in relating measurable pore-space properties to the conductivity value of an electrolyte-saturated rock sample.

It is clear that the parameter \( d_w \) is responsive to the morphology of the conductive domain. Interestingly, the walker behavior within the domain also determines the electrical potential field \([14]\). At equilibrium (no field), the principle of detailed balance ensures that walkers spend equal time at every location (site) within the domain. Then the potential field induced by application of a potential drop across the sample is obtained by injecting random walkers at one end of the conductive domain and immediately removing them when they reach the opposite end. In that case the accumulated residence times at the sites comprising the domain give the potential field. That is, the accumulated residence time at a location within the conductive domain is proportional to the electrical potential at that location.

The electric current flows through the sample as directed by the potential field. While the walkers visit every location within the domain, and so determine the potential at every location, there may be conductive regions where current does not flow. This occurs, for example, in dead-end features, where a walker can only exit where it enters. Thus the parameter \( d_w \) in Eq. (12) accounts for the tortuosity of the electric field lines, and for any “dead space”, within the conductive domain.

Therefore the value of the parameter \( d_w \) reflects both the physical morphology of the conductive domain, and its consequent electrical properties.

In contrast to the parameter \( D \), which in principle or practice can be calculated by the box-counting method \([15]\), there is no means to calculate \( d_w \) for rock samples. Note that \( d_w \) could be calculated for the self-similar objects considered in Sec. V only because those objects are infinite in size, so allowing use of the infinitesimally small value \( \xi_0 = 1 \) in calculations.

Nevertheless, it should be appreciated how remarkable it is that just two parameters, \( d_w \) and \( D \), are sufficient to determine the electrical properties of very complex conductive domains within pore spaces.

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