PERIODIC SOLUTION OF THE NONLINEAR JERK OSCILLATOR CONTAINING VELOCITY TIMES ACCELERATION-SQUARED: AN ITERATION APPROACH

B. M. Ikramul Haque¹, Md. Zaidur Rahman², Md. Iqbal Hossain³

¹Professor, Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203.
²Professor and Head, Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203.
³M Phil Student, Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203.

¹b.m.haque75@gmail.com, ²mzrahman1968@gmail.com, ³hossainiqbal3k@gmail.com

Corresponding Author: B. M. Ikramul Haque

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Abstract

Haque’s iteration approach has been applied to obtain analytical solution of the nonlinear jerk equation containing velocity times acceleration-squared. We have used truncated Fourier series by taking different numbers of harmonics for different iteration step. The obtained solutions give more accurate result than others and very nearer to the exact solution.

Keywords: Jerk equation; nonlinear oscillator; Iteration Method; Truncated Fourier series

I. Introduction

Differential equation is a strong mathematical engine, which has its application in many branches of our real life. Several physical, mathematical, economical, chemical, biological, biochemical and many other relations build their scientific view via mathematically in the help of differential equations. Already many physical situations have been by differential equations such as spring-mass systems, nonlinear resistor-capacitor-inductor circuits, bending of beams, chemical oscillation, space dynamics and so forth. Hence, the solution of certain problems lies basically in solving the corresponding differential equations. Differential equations are linear or nonlinear, autonomous or non-autonomous. Basically, huge number of differential equations involving physical phenomena is nonlinear.

Solution methods of nonlinear differential equations are relatively complicated and not extremely upgrade. An initial concept is that, the nonlinear problems are solved by

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converting into linear equations imposing some conditions, but such linearization is also not always feasible or possible. In this case there are some access to obtain approximate analytical solutions of nonlinear problems, such as Perturbation \cite{XX}, \cite{XXI}, \cite{VI}; Standard and modified Linstedt-Poincare \cite{XXII}, \cite{XXIII}, \cite{XXIV}; Harmonic Balance \cite{XVII}, \cite{III}, \cite{IV}, \cite{V}, \cite{XXVI}, \cite{I}, \cite{XVI}; Homotopy \cite{II}, \cite{XVIII}; Iterative \cite{XV}, \cite{XXVII}, \cite{X}, \cite{XI}, \cite{XII}, \cite{XIII}, \cite{XIV}, \cite{V} methods etc.

Among them the most commonly used method is the perturbation method. In perturbation method, the nonlinear term is small. The Harmonic Balance (HB) method is developed for solving the problems with strong nonlinearities. Recently, some authors developed an iteration method to obtain the approximate frequency, corresponding periodic solution and as well as period of small as well as large nonlinear problems.

The term “jerk” was first introduced by Schot in 1978 \cite{XXV} in place of the third-order derivative of the displacement. We realize maximum of the attempts on dynamical systems are correlated to second-order differential equations, however some dynamical systems can be described by third-order nonlinear differential. This type of equations is said to be nonlinear jerk equation. General form of jerk equations is

$$\frac{d^3 x}{dt^3} = J(x, \frac{dx}{dt}, \frac{d^2 x}{dt^2}) = J(x, \dot{x}, \ddot{x})$$
(1)

Where $J(x, \dot{x}, \ddot{x})$ denotes the jerk functions.

To find periodic analytical solutions for jerk-oscillators definitelimitations and constraints have to be placed on the mathematical structure of the oscillator. We only consider the following cubic nonlinear functions investigated by Gottlieb \cite{IV}

$$\ddot{x}x^2$$
(2)

The most general jerk function with invariance of time- and space-reversal and which has only cubic nonlinearities may be written in the form \cite{IV}

$$\ddot{x} = \alpha x\ddot{x} - \beta \dddot{x} - \gamma \dot{x} - \delta x^2 \dot{x} - \epsilon x^3,$$
(3)

Where the parameters $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$ are given real constants. The corresponding initial conditions are

$$x(0) = 0, \dot{x}(0) = A \text{ and } \ddot{x}(0) = 0.$$ 
(4)

These three initial conditions in Eq. (4) are to satisfy the periodic requirement. Here, at least one of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$ should be non-zero.

In this article, we present the Haque’s approach \cite{X} of the Mickens’ iteration method \cite{XIX} owing to the determination of approximate solutions of the nonlinear jerk equation where the jerk function contains velocity times acceleration-squared. It is
stated that our method is valid for second and higher order period of oscillations and display effective agreement compared to other existing solutions.

II. The Method

Let us consider a nonlinear oscillator modeled by

\[
d^2x + f(\frac{dx}{dt}, \frac{d^2x}{dt^2}) = \ddot{x} + f(\dot{x}, \ddot{x}) = 0, \quad x(0) = 0, \quad \dot{x}(0) = A
\]

(5)

where over dots denote differentiation with respect to time, t over dash denotes integration with respect to time, we choose the frequency \( \Omega \) of this system. Then adding \( \Omega^2x \) to both sides of Eq. (5), we obtain

\[
\ddot{x} + \Omega^2x = \Omega^2x - f(\dot{x}, \ddot{x}) = G(\dot{x}, \ddot{x})
\]

(6)

According to Mickens’ law [VIII], we formulate the iteration scheme as

\[
\ddot{x}_{k+1} + \Omega_k^2x_{k+1} = G(\dot{x}_k, \ddot{x}_k); k = 0, 1, 2, .......
\]

(7)

Together with

\[
x_0(t) = A\cos(\Omega_0 t).
\]

(8)

Herein satisfies the conditions

\[
x_k(0) = A, \dot{x}_k(0).
\]

(9)

At each iterative stage, \( \Omega_k \) is determined by the condition that secular terms[XIX] should be removed from the solution. This successive scheme gives us the sequence of solutions: \( x_0(t), x_1(t), \ldots \) The technique can be continued to any order of approximate solution; but due to budding algebraic difficulty, the solution is restricted to a lower order normally the second [XIX].

III. Solution Procedure

In this research we have considered the function \( \dddot{x} \) i.e. Jerk function containing velocity times acceleration-squared.

Now, we take into account the nonlinear jerk oscillator

\[
\dddot{x} + \dot{x} = -\dddot{x}^2.
\]

(10)

Inserting the space variable \( y(t) \) into the equation (10) by the relation \( \dot{x} = \dot{y} \), we have

\[
\dddot{y} + y = -y\dddot{y}^2.
\]

(11)

Equation (11) can be written as

\[
\dddot{y} + \Omega^2y = \Omega^2y - (y + y\dddot{y}^2).
\]

(12)
Now the iteration scheme is according to

\[ \ddot{y}_{k+1} + \Omega^2_{k} \dot{y}_{k+1} = \Omega^2_{k} \dot{y}_k - (y_k + y_k \dot{y}_k^2) . \]  

(13)

Equation Eq. (8) is rewritten as

\[ y_0 = y_0(t) = A \cos x , \]  

(14)

Where \( x = \Omega t \), for \( k=0 \) the equation (13) becomes

\[ \ddot{y}_1 + \Omega^2_{0} \dot{y}_1 = \Omega^2_{0} y_0 - (y_0 + y_0 \dot{y}_0^2) . \]  

(15)

Substituting initial guess from equation (14) into the equation (15) and expanding in a truncated Fourier cosine series up to third harmonics, we obtain

\[ \ddot{y}_1 + \Omega^2_{0} \dot{y}_1 = a_{1,1} \cos x + a_{1,3} \cos 3x , \]  

(16)

Where

\[ x = \Omega_0 t \]

\[ a_{1,1} = \frac{1}{4} (-4A + 4A\Omega_0^2 - A^2\Omega_0^2) \]

\[ a_{1,3} = \frac{1}{4} A^3\Omega_0^2 \]  

(17)

After applying the removing law for secular terms from the solution of equation (16), we have

\[ \Omega_0 = \left( \frac{4}{4 - A^2} \right)^{1/2} \]  

(18)

This \( \Omega_0 \) represents the first approximate frequency of the oscillator (11).

Then solving equation (16) and satisfying the initial condition \( y_1(0) = A \) we have,

\[ y_1 = \{ A - \frac{1}{32} (-4 + A^2)a_{1,3} \} \cos x + \frac{1}{32} (-4 + A^2)a_{1,3} \cos 3x \]  

(19)

This \( y_1 \) represents the first approximate solution of equation (11) and corresponding frequency \( \Omega_1 \) is to be determined. The value of \( \Omega_1 \) will be obtained from the solution.

\[ \ddot{y}_2 + \Omega^2_{1} \dot{y}_2 = \Omega^2_{1} y_1 - (y_1 + y_1 \dot{y}_1^2) \]  

(20)

Substituting \( y_1 \) from equation (19) into the equation (20) and then expanding in a truncated Fourier cosine series up to nine harmonics we obtain

\[ \ddot{y}_2 + \Omega^2_{1} y_1 = a_{2,1} \cos x + a_{2,3} \cos 3x + a_{2,5} \cos 5x + a_{2,7} \cos 7x + a_{2,9} \cos 9x , \]  

(21)

Where

\[ x = \Omega_2 t \]
After applying the removing law for secular terms from the solution of equation (21), we have

$$\Omega_2 = \left(\frac{131072 + 4096A^2}{131072 - 28672A^2 + 2048A^4 - 352A^6 - 14A^8}\right)^{1/2}$$

This $\Omega_2$ represents the second approximate frequency of the oscillator (11).

Then solving equation (21) and satisfying the initial condition $y_2(0) = A$, we have

$$y_2 = \left(A + \frac{1}{\Omega_2^2} \sum_{n=1}^{4} \frac{a_{2,2n+1}}{(2n+1)^2 - 1}\right) \cos x - \frac{1}{\Omega_2^2} \sum_{n=1}^{4} \frac{a_{2,2n+1}}{(2n+1)^2 - 1} \cos(2n+1)x$$

This $y_2$ represents the second approximate solution of equation (11) and corresponding frequency $\Omega_2$ is to be determined. The value of $\Omega_2$ will be obtained from the solution.

$$\ddot{y}_3 + \Omega_2^2 y_3 = \Omega_2^2 y_2 - (y_2 + y_2 \dot{y}_2^2)$$

Substituting $y_3$ from equation (24) into the equation (25) and then expanding in a truncated Fourier cosine series up to twenty seventh harmonics we obtain,

$$\ddot{y}_3 + \Omega_2^2 y_3 = a_{3,1} \cos x + a_{3,5} \cos 5x + a_{3,7} \cos 7x + a_{3,9} \cos 9x + a_{3,11} \cos 11x + a_{3,13} \cos 13x + a_{3,15} \cos 15x + a_{3,17} \cos 17x + a_{3,19} \cos 19x + a_{3,21} \cos 21x + a_{3,23} \cos 23x + a_{3,25} \cos 25x + a_{3,27} \cos 27x$$

Where

$$x = \Omega_2 t$$
\[ a_{3,1} = -\frac{1}{32} A'^2 + \frac{5A'}{256} + \frac{53A'^3}{3072} + \frac{245A'^4}{786432} - \frac{523A'^5}{18064A^6} \]
\[ + 14959A'^4\Omega^2_1 - 69847A'^3\Omega^2_2 - 221477A'^2\Omega^2_3 - 506173A'\Omega^2_4 \]
\[ - 2013265920 \times 88599345920 + 3092376453120 - 158329674399744 \]
\[ - 5629499534213120 + 90071992547409920 - 25940733853654056960 \]
\[ + \frac{332591A^{27}}{864691128455135232000} \]

\[ a_{3,5} = \frac{A^5}{32} + \frac{5A^5}{256} + \frac{5A^5}{16384} + \frac{5A^5}{524288} + \frac{9A^5}{1572864} + \frac{925A^5}{5797A^{11}} + \frac{33415A^{13}}{3271207A^{15}} + \frac{3074959A^{17}}{62914560} + \frac{2415919104}{2139282339840} + \frac{24739011624960}{14296031A^{19}} + \frac{147282323A^{21}}{1979120929996800} + \frac{759982437118771200}{354} \]

\[ a_{3,7} = \frac{5A^7}{65536} + \frac{5A^7}{2097152} + \frac{A^7}{196608} + \frac{319A^7}{2097152} + \frac{93A^7}{786432} + \frac{2079595520}{262651A^{11}} + \frac{290363A^{17}}{5729087A^{19}} + \frac{136882583A^{21}}{2319282339840} + \frac{1319413953331200}{129703669268270284800} + \frac{415051746584691136000}{304} \]

\[ a_{3,9} = \frac{9A^9}{10485760} + \frac{1A^9}{31457280} - \frac{36893A^9}{31457280} + \frac{23729A^{11}}{125829120} - \frac{257489A^{13}}{12079595520} + \frac{2743A^{15}}{1610612736} - \frac{1315477A^{17}}{12369505812480} + \frac{1319413953331200}{198715A^{23}} + \frac{25332747903959040}{324259173170675712} - \frac{129703669268270284800}{415051746584691136000} \]

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\[ a_{3,11} = \Omega_2^2 \left( \frac{40829A^{11}}{377487360} + \frac{104819A^{13}}{6039797760} + \frac{2230447A^{15}}{1159641169920} + \frac{735491A^{17}}{4947802324992} \right) + \frac{7446127A^{19}}{989560464998400} + \frac{7860989A^{21}}{37999121855935600} + \frac{228569A^{23}}{540431955284459520} + \frac{1239383A^{25}}{2882303761517174400} + \frac{31213627A^{27}}{124515522497539473408000} \] 

\[ a_{3,13} = \Omega_2^2 \left( \frac{94093A^{13}}{12079595520} - \frac{944419A^{15}}{773094113280} - \frac{614095A^{17}}{4947802324992} \right) - \frac{1651679A^{19}}{197912092999680} \right) - \frac{79136401A^{21}}{25332747903959040} - \frac{15002653A^{23}}{4053239664633446400} + \frac{1680691A^{25}}{2882303761517174400} - \frac{33377A^{27}}{23058430092136935200} \] 

\[ a_{3,15} = \Omega_2^2 \left( \frac{199171A^{15}}{463856467968} + \frac{1620199A^{17}}{24739011624960} + \frac{2328773A^{19}}{39582418599360} \right) + \frac{227055907A^{21}}{75982437118771200} + \frac{1068107A^{23}}{162129586585337856} + \frac{671107A^{25}}{259407338536540569600} - \frac{79529A^{27}}{31128806243848683520} \] 

\[ a_{3,17} = \Omega_2^2 \left( \frac{464887A^{17}}{24739011624960} - \frac{10318693A^{19}}{395824185993600} - \frac{5984447A^{21}}{3166593487994880} \right) - \frac{2609969A^{23}}{405323966463344640} - \frac{16505693A^{25}}{25940733853654056960} + \frac{467051A^{27}}{1297036692682702848000} \] 

\[ a_{3,19} = \Omega_2^2 \left( \frac{431791A^{19}}{659706976665600} + \frac{1581991A^{21}}{21110623253299200} + \frac{86717A^{23}}{22517998136852480} \right) + \frac{6256057A^{25}}{86469112845513523200} + \frac{28253A^{27}}{3458764513820540928000} \] 

\[ a_{3,21} = \Omega_2^2 \left( \frac{95059A^{21}}{5629499532413120} - \frac{263811A^{23}}{18014398509419840} - \frac{52863A^{25}}{1152921504606846976} \right) - \frac{56197A^{27}}{184467440737095516160} \] 

\[ a_{3,23} = \Omega_2^2 \left( \frac{261513A^{23}}{900719925474099200} + \frac{123831A^{25}}{7205759403792793600} + \frac{233811A^{27}}{233811A^{27}} + \frac{922337203685477580800} {922337203685477580800} \right) \]
\[ a_{3,25} = \Omega_2^2 \left( \frac{16767A^{25}}{5764607523034234880} - \frac{16767A^{27}}{18467440737095516160} \right) \]
\[ a_{3,27} = \frac{59049A^{27} \Omega_2^2}{461168601842738790400} \] (27)

After applying the removing law for secular terms in the solution of equation (26), we have
\[ \Omega_2 = \frac{1}{(1 + \frac{\Omega_2^2}{32} + \frac{5A^2}{3072} + \frac{53A^4}{196608} - \frac{13A^6}{786432} / (1 - \frac{7A^2}{32} + \frac{53A^4}{3072} + \frac{245A^6}{196608}) \right) / \left( \frac{523A^8}{786432} - \frac{1061A^{10}}{188743680} - 14959A^{12} - \frac{1061A^{14}}{506173A^{18}} - \frac{221477A^{16}}{3092376453120} + \frac{5629499534213120}{25940733853654056960} - \frac{332591A^{26}}{864691128455135232000} \right)^{1/2} \] (28)

This \( \Omega_2 \) represents the third approximate frequency of the oscillator (11).

Then solving equation (26) and satisfying the initial condition \( y_3(0) = A \), we have
\[ y_3 = \left( A + \frac{1}{\Omega_2^2} \sum_{n=1}^{11} \frac{a_{3,2n+1}}{(2n+1)^2 - 1} \right) \cos x - \frac{1}{\Omega_2^2} \sum_{n=1}^{11} \frac{a_{3,2n+1}}{(2n+1)^2 - 1} \cos(2n+1)x \] (29)

This \( y_3 \) represents the third approximate solution of equation (11) and corresponding frequency \( \Omega_2 \) is to be determined. The value of \( \Omega_2 \) will be obtained from the solution.
\[ \ddot{y}_4 + \Omega_2^2 y_4 = \Omega_2^2 y_3 - (y_3 \dot{y}_3 \dot{x}) \] (30)

After substituting the third approximate solution of equation (11) from the equation (29) into the equation (30) and avoiding secular terms in the solution, we obtain
\[ \Omega_2 = \left( 48638875975601356800 + 6079859469501696000 \right)^{1/2} \]
\[ + \left( 601652762190272000 \right)^{1/2} \]
\[ + \left( 10398733225164800 \right)^{1/2} \]
\[ + \left( 54809044705280 \right)^{1/2} \]
\[ + \left( 33271389426697 \right)^{1/2} \]
\[ + \left( 508987963819105763 \right)^{1/2} \]
\[ + \left( 6801567252480 \right)^{1/2} \]
\[ + \left( 3109287886840 \right)^{1/2} \]
\[ + \left( 23216016221798400 \right)^{1/2} \]

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\[
\begin{align*}
71857669038900618179172841648370018111374913^d &
+ 415254267542175530804210733095793915790422296627200000000
+ 2838107060818738469482981512606179801119079193^d &
+ 465084779650643639450071602106728918568507137222246400000000
+ 6628112399053033505466827263308320261144021^d &
+ 335894563081020420713940601521526441188366265771622400000000
+ 709583282186903706990968238272488756908704463^d &
+ 123824117341873678919870634489550727967934021405088153600000000
+ 1447555914169973390302254089709283614440283877^d &
+ 99059337387349894313589650675916405823743742171240705228800000000
+ 79035575575547505427323454013111475165567254053^d &
+ 253591903711615729424787950730345999890878328875837620538572800000000
+ 158194562999739648442111307562206537479890123^d &
+ 300553367361914938598861636421158018135252093482474216934604800000000
+ 300024548058051332629308156308498791285904981^d &
+ 432796849001157511582360756446457171470990146146762873285830912000000000
+ 1337239599228503397635878564011518954661319^d &
+ 236071008546085915408560412607158457165994625171433849377259520000000000
+ 53220221740879267787709131240760394179950657^d &
+ 13295519201315558755810122438035164307588817289655154396272526166400000000
+ 25217461046385292249087371446262543486321077^d &
+ 2127283072210489400929619590805626289214210766344824703508360986624000000000
+ 248530953604670420042419550972716846552429^d &
+ 436365758402156171985562928380771875311201571985384007196637921280000000000
+ 17623467019144936442719884201668759726816389^d &
+ 680730583107356608297478268827400412548547445230343905122675515719680000000000
+ 1746219419873265199616181129449900502204831^d &
+ 27229232242426433189913075309601650194189780921375620490702062878720000000000
+ 1020957911062783045228286178024107057616013^d &
+ 1742670292754832917241544368198145056124281459789680397114049320242380800000000000
+ 884540246727811106624083993339348035431^d &
+ 35747082928304264969057320373925283205493632777370337896548578511257600000000000
+ 491180723467230636971075827973099807^d &
+ 1047533566459384524710096320008208298975805517296323344162595707289600000000000
+ 492067961158370725632206396205383^d &
+ 18130388636502585783122901288230912866888941645513288480281415647232000000000000000000
\end{align*}
\]
This $\Omega_3$ represents the fourth approximate frequency of the oscillator (11).

In a similar manner this technique of iteration can be continued to higher order approximations. But to avoid growing algebraic difficulties, most of the existing methods are applied up to second or third approximations.

IV. Results and Discussions

Haque’s iterative technique [X] is presented based on Mickens iteration method [XIX] to obtain approximate analytic solution of the nonlinear jerk equation containing “velocity times acceleration-squared”. The presented approach is very simple and straightforward for solving algebraic equations analytically. For this, we have found the approximate analytical solution of the oscillator up to fourth iteration step without any complexity.

Here, we present the accuracy of the solutions obtained from the modified iteration method by comparing with other results based on various methods and with the exact results of the nonlinear jerk equation containing “velocity times acceleration-squared”.

To verify the correctness, we have considered the percentage errors (denoted by Error (\%)) by the definitions

$$\text{Error} = \left| \frac{T_e - T_k}{T_e} \right| \times 100\%$$

Where $T_k; k = 0, 1, 2, \cdots$ illustrate the different approximate periods obtained by the presented modified method and $T_e$ represents the corresponding exact period of the oscillator.

Recently, Gottlieb [IV] has found approximate analytical solutions of the nonlinear jerk oscillator containing “velocity times acceleration-squared” by harmonic balance method. The solution obtained by Gottlieb was not accurate enough. He used the lower order harmonic balance method; it is very difficult to construct the higher order approximation by harmonic balance method because it requires analytical solutions of a set of complicated nonlinear algebraic equations. Zheng et al. [XXVII] has examined the comparison of two iteration procedure for a class of nonlinear jerk equations; but sometimes their modification is invalid. It has been displayed that, in most of the incident our solution gives meaningfully improved result than other results.

Herein we have calculated the first, second, third and fourth approximate frequencies $\Omega_0$, $\Omega_1$, $\Omega_2$ and $\Omega_3$ respectively and corresponding periods are $T_0$, $T_1$, $T_2$ and $T_3$. All the results are given in the following Table 1 and to compare the approximate periods we have also given the existing results determined separately by Gottlieb [IV] and Zheng et al. [XXVII] respectively in following Table 2.
Table 1: The approximate periods obtained by presents technique of \( \ddot{x} + \dot{x} = -\dot{x} x^2 \) with percentage errors:

| A  | \( T_0 \) Er(%) | \( T_1 \) Er(%) | \( T_2 \) Er(%) | \( T_3 \) Er(%) |
|----|----------------|----------------|----------------|----------------|
| 0.1| 6.27532618 e^{-4} | 6.2753342.39 e^{-7} | 6.2753342.39 e^{-7} |
| 0.2| 6.251690190 e^{-3} | 6.2518093.39 e^{-7} | 6.2518093.39 e^{-7} |
| 0.5| 6.0836687.85 e^{-2} | 6.0882463.24 e^{-5} | 6.0884495.63 e^{-6} |
| 1  | 5.4413981.55 | 5.5135432.47 e^{-1} | 5.5275115.62 e^{-1} | 5.5276598.30 e^{-3} |
| 1.5| 4.15593611.39 | 4.5187401.51 e^{-1} | 4.6831875.31 e^{-1} |

\( T_0, T_1, T_2 \) and \( T_3 \) respectively denote first, second, third and fourth modified approximate periods obtained by present method. Er(%) denotes percentage error.

Table 2: Comparison of the approximate periods with exact periods \( T_e \) [IV] of \( \ddot{x} + \dot{x} = -\dot{x} x^2 \):

| A  | \( T_e \) | \( T_G \) Er(%) | \( T_Z \) Er(%) | \( T \) Er(%) |
|----|------------|----------------|----------------|----------------|
| 0.1| 6.275338 | 6.27532641.18 e^{-4} | 6.2753382.75 e^{-7} | 6.2753342.39 e^{-7} |
| 0.2| 6.251809 | 6.2516901.90 e^{-3} | 6.2518082.68 e^{-6} | 6.2518093.39 e^{-7} |
| 0.5| 6.08449 | 6.0836687.85 e^{-2} | 6.0884165.41 e^{-4} | 6.2518093.39 e^{-7} |
| 1  | 5.527200 | 5.4413981.55 | 5.5255702.95 e^{-2} | 5.5275115.62 e^{-3} |
| 1.5| 4.690247 | 4.15593611.39 | 4.6721293.86 e^{-1} | 4.6831871.51 e^{-1} |
\(T\) denotes the modified approximate periods, \(T_G\) denotes the approximate periods obtained by Gottlieb [IV] and \(T_Z\) denotes the approximate periods obtained by Zheng et al [XXVII]. Er(\%) denotes percentage error.

V. Convergence and Consistency Analysis

The elementary concept of iteration methods is to construct a sequence of solutions \(x_k\) (as well as frequencies \(\Omega_k\)) under an initial supposition that have the property of convergence

\[
x_e = \lim (x_k) \quad \text{or,} \quad \Omega_e = \lim (\Omega_k) \quad \text{or,} \quad T_e = \lim (T_k), k \to \infty
\]

Here \(x_e\) is considered as the exact solution, \(\Omega_e\) denotes the frequencies and \(T_e\) denotes the corresponding periods of the nonlinear oscillator. In the modified technique, it has been shown that the obtained solution indicate the less error in each step of iteration compared to the priorstep of iteration and lastly \(|T_3 - T_e| < \varepsilon\), where \(\varepsilon\) is a small positive number. Thus, it is clear that the modified method is convergent.

On the other hand, we know that the iterative method will be consistent if

\[
\lim |x_k - x_e| = 0 \quad \text{or,} \quad \lim |\Omega_k - \Omega_e| = 0 \quad \text{or,} \quad \lim |T_k - T_e| = 0, k \to \infty
\]

From our modified approach for the oscillator, we can say

\[
\lim |T_k - T_e| = 0, \quad \text{as} \quad |T_3 - T_e| = 0, k \to \infty
\]

Therefore, the method is consistent.

VI. Conclusion

Most of the cases, the results have been improved significantly by modifying the method. In this paper we used a simple but effective modification of the direct iteration method to solve nonlinear differential equation. The obtained results show that the modification of the iteration method is more accurate than others method and is valid for large region. The percentage of error between exact periods and our third approximate periods are very negligible. Moreover, the present work can be found widely applicable in engineering and science.

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