Dyonic Dilaton Black Holes

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ABSTRACT

The properties of static spherically symmetric black holes, which are both electrically and magnetically charged, and which are coupled to the dilaton in the presence of a cosmological constant, \(\Lambda\), are considered. It is shown that apart from the Reissner-Nordström-de Sitter solution with constant dilaton, such solutions do not exist if \(\Lambda > 0\) (in arbitrary spacetime dimension \(\geq 4\)). However, asymptotically anti-de Sitter dyonic black hole solutions with a non-trivial dilaton do exist if \(\Lambda < 0\). Both these solutions and the asymptotically flat (\(\Lambda = 0\)) solutions are studied numerically for arbitrary values of the dilaton coupling parameter, \(g_0\), in four dimensions. The asymptotically flat solutions are found to exhibit two horizons if \(g_0 = 0, 1, \sqrt{3}, \sqrt{6}, \ldots, \sqrt{n(n+1)/2}, \ldots\), and one horizon otherwise. For asymptotically anti-de Sitter solutions the result is similar, but the corresponding values of \(g_0\) are altered in a non-linear fashion which depends on \(\Lambda\) and the mass and charges of the black holes. All dyonic solutions with \(\Lambda \leq 0\) are found to have zero Hawking temperature in the extreme limit, however, regardless of the value of \(g_0\).

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1. Introduction

In effective low-energy theories of gravity derived from string theory Einstein gravity is supplemented by additional fields such as the axion, gauge fields, and the scalar dilaton which couples in a non-trivial way to the other fields. The observation that black holes with non-trivial couplings to the dilaton field have properties which sometimes differ quite dramatically from those of the corresponding classical solutions of the standard Einstein theory \[1,2\] has provided fertile ground over the past few years for the development of new ideas concerning quantum gravitational phenomena, such as pair creation of black holes \[3\], black hole evaporation \[4\] and the nature of black hole entropy \[5\].

The aim of the present paper is to extend the known dilaton black hole solutions to include analogues of the Reissner-Nordström-(anti)-de Sitter metrics with both electric and magnetic charges. In addition we will also discuss the case of asymptotically flat dyonic solutions for values of the dilaton coupling constant for which exact solutions are not known. In considering spacetimes which are not asymptotically flat in the context of low-energy string inspired gravity models the first question one should ask is what is the appropriate “cosmological term” one should consider. Instead of simply taking this term to be a cosmological constant, one should conceivably take some sort of dilaton potential. An appropriate action might be

\[
S = \int d^Dx \sqrt{-g} \left\{ \frac{\mathcal{R}}{4} - \frac{1}{D - 2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \mathcal{V}(\phi) - \frac{1}{4} \exp \left( \frac{-4g_0 \phi}{D-2} \right) F_{\mu \nu} F^{\mu \nu} \right. \\
\left. - \frac{1}{2(D-2)!} \exp \left( \frac{-4g_0 \phi}{D-2} \right) F_{\mu_1 \mu_2 \ldots \mu_{D-2}} F^{\mu_1 \mu_2 \ldots \mu_{D-2}} \right\},
\]

which includes gravity, an electromagnetic field \(F_{\mu \nu}\), the \((D-2)\)-form field strength of an additional\(^1\) abelian gauge field \(F_{\mu_1 \mu_2 \ldots \mu_{D-2}}\), and the dilaton, \(\phi\), with a non-trivial dilaton potential, \(\mathcal{V}(\phi)\), which might realistically be expected to take a form such as

\[
\mathcal{V} = \mathcal{V}_{\text{exp}} + \mathcal{V}_{\text{susy}}
\]

where

\[
\mathcal{V}_{\text{exp}} = \frac{\Lambda}{2} \exp \left( \frac{-4g_1 \phi}{D-2} \right),
\]

\(^1\) Our inclusion of the \((D-2)\)-form is not demanded by the requirements of string theory, but is included as to allow for generalised dyon solutions in the case of arbitrary spacetime dimension. We are of course primarily interested in the case of four dimensions when the \((D-2)\)-form coincides with the electromagnetic field.
\[
V_{\text{susy}} = \frac{1}{4} \exp \left[ -\alpha e^{-2\phi} \right] \left\{ A_1 e^{2\phi} + A_2 + A_3 e^{-2\phi} \right\}. \tag{4}
\]

Here \(g_0, g_1, \Lambda, \alpha, A_1, A_2\) and \(A_3\) are constants. Eqns. (1)–(4) are somewhat more general than is demanded by string theory. However, if we set \(g_0 = 1\) we obtain the standard tree-level coupling between the dilaton and the electromagnetic field, while setting \(g_1 = -1, \Lambda = (D_{\text{crit}} - D_{\text{eff}})/(3\alpha')\), in the Liouville-type term (3) yields the case of a potential corresponding to a central charge deficit. The term (4), on the other hand, is the type of potential which arises in four dimensions from supersymmetry breaking via gaugino condensation in the hidden sector of the string theory\(^2\) [6].

It is generally believed that a term such as \(V_{\text{susy}}\) should be present since long-range scalar forces would be present if the dilaton were massless, which is phenomenologically unacceptable, although Damour and Polyakov have argued that a massless but very weakly coupled dilaton can be reconciled with present observational bounds [7]. If one accepts that the dilaton should be massive then the absence of a dilaton mass term is a major physical defect with most dilaton black hole solutions studied to date. However, Gregory and Harvey [8] and Horne and Horowitz [9] have recently made some investigation of black hole models which include a mass term, in the form of a simple quadratic potential [8,9], \(V = 2m^2 (\phi - \phi_0)^2\), or alternatively of the form [8] \(V = 2m^2 \cosh^2 (\phi - \phi_0)\). While a rigorous proof of the existence of black hole solutions in these models has still to be given, the arguments of Horne and Horowitz [9] are nonetheless compelling. Furthermore, it is clear from the arguments of [8,9] that the properties of the solutions with a massive dilaton are essentially the same as those with a massless dilaton in the case of black holes which are small with respect to the Compton wavelength of the dilaton, which does provide some further justification for studying the solutions with a massless dilaton.

In the present paper we will not consider potentials such as \(V_{\text{susy}}\) or other potentials \(V(\phi)\) which give the dilaton a mass, but will restrict our attention to cosmological terms of the type \(V_{\exp}\). As we will see, the solutions we will discuss here do nonetheless have some similar features to those of [8,9], although there are also some important differences. Apart from our own recent papers [10,11] and that of Okai [12], there has been relatively little work on dilaton black hole solutions with Liouville-type potentials or a simple cosmological potential.

\(^2\) The particular potential given in (4) is relevant for one gaugino condensation.
constant. Kastor-Traschen type [16] cosmological multi-black hole solutions have been discussed by Horne and Horowitz [17], and by Maki and Shiraishi [18]. However, exact solutions have been constructed only for certain special values of the dilaton coupling and for special powers of a Liouville-type dilaton potential [18]: this included the case $g_0 = 1$, $g_1 = -1$ relevant to a string theoretic model with a central charge deficit, but excluded the case $g_1 = 0$ for which the potential is simply a cosmological constant. Furthermore, all solutions with non-zero dilaton couplings obtained by Maki and Shiraishi involve a dilaton which depends on certain special powers of the time-dependent cosmic scale factor, and thus in particular they possess no static limit.

Although a Liouville type potential (3) with $g_1 = -1$ might be the most natural choice from the point of view of string theory, such a model unfortunately does not possess static spherically symmetric solutions with “realistic” asymptotics, namely solutions which are either asymptotically flat or asymptotically of constant curvature. This conclusion is based on an analysis of the relevant phase space for solutions which are either electrically or magnetically charged [10]. It was found in [10] that the only case in which “realistic” asymptotics are admitted, is the case $g_1 = 0$ corresponding to a simple cosmological constant, even allowing for the relaxed regularity assumptions on the dilaton which are consistent with the weak coupling limit of string theory. Although dyonic solutions were not explicitly considered in [10] there is no reason to expect the conclusion to be any different for dyons than for solutions which carry a single charge. We will therefore henceforth simply consider the case $g_1 = 0$ in which the action (1) contains a cosmological constant.

It does not seem likely that there is any role for a pure cosmological constant in the context of string theory. However, if a non-perturbative dilaton potential, $V(\phi)$, is present and if this potential has a minimum, $\phi_0$, for which $V(\phi_0) \neq 0$, then asymptotic regions for which $\phi \to \phi_0$ would correspond to a background universe with a non-zero (hopefully small) vacuum energy. In the case of certain potentials with a minimum for which $V(\phi_0) = 0$ it has been shown [8,9] that the appropriate dilaton black hole solutions are well approximated by the solutions with a massless dilaton in various regimes. We would hope that the solutions which we discuss here would similarly provide a useful approximation for corresponding models in which $V(\phi_0) \neq 0$. Thus although a pure cosmological constant is not favoured by string theory, we believe that the models investigated here could nevertheless be of considerable physical interest as a limiting case of more physically motivated stringy black hole solutions.

Uncharged solutions in such models [13,14] generically either possess naked singularities or have unusual asymptotics, or both. Amongst the class of solutions with unusual asymptotics one can find solutions for which the curvature vanishes asymptotically even though the solutions are not asymptotically flat according to the conventional definition [10,15]. It is possible that the subset of such solutions which also possess a regular horizon could have a consistent physical interpretation as black holes. However, we will not consider such solutions here.
The properties of dilaton black holes in the presence of a cosmological constant which carry either an electric or a magnetic charge were dealt with in our recent paper [11]. Here we will consider the extension of this work to solutions which carry both charges. To be specific, let us assume that the spacetime metric is static and spherically symmetric, taking the form

\[ ds^2 = -f dt^2 + f^{-1} dr^2 + R^2 d\Omega^2_{D-2}, \]

where \( f = f(r) \) and \( R = R(r) \), and \( d\Omega^2_{D-2} \) is the standard round metric on a \((D-2)\)-sphere, with angular coordinates \( \theta_i, i = 1 \ldots D-2 \). The electromagnetic field and the \((D-2)\)-form field will then be assumed to have components given in an orthonormal frame by

\[ F_{\hat{t}\hat{r}} = -F_{\hat{r}\hat{t}} = \exp \left( \frac{4g_0 \phi}{D - 2} \right) \frac{Q}{R^{D-2}}, \]

and

\[ F_{\hat{\theta}_1 \hat{\theta}_2 \ldots \hat{\theta}_{D-2}} = \frac{P}{R^{D-2}} \epsilon_{\hat{\theta}_1 \hat{\theta}_2 \ldots \hat{\theta}_{D-2}} \]

respectively. The ansatz (6) is appropriate to an isolated electric charge, and in four dimensions (7) becomes the ansatz for an isolated magnetic monopole, and thus with both fields present we obtain a generalised dyon ansatz \(^4\).

2. Non-existence of non-trivial dyonic dilaton black holes with a positive cosmological constant

Let us firstly consider the case of a positive cosmological constant \( V \equiv \Lambda/2 \), with \( \Lambda > 0 \). It is quite straightforward to show that apart from one trivial case dyonic black holes with a positive cosmological constant do not exist in the presence of the dilaton field. For this purpose it is convenient to define a constant

\[ a = \frac{2g_0}{(D - 2)}, \]

and to rescale the dilaton by adding a constant

\[ \Phi \equiv \phi + \frac{1}{2a} \ln \left| \frac{Q}{P} \right|. \]

\(^4\) In the context of dilaton gravity the term dyon is sometimes applied to solutions with both electric (or magnetic) and axionic charges [19]. We are not considering axionic dyons here.
The field equations derived from (1) may then be written compactly as

\[ \frac{1}{R^{D-2}} \left[ R^{D-2} f \Phi' \right]' = a(D - 2)|QP| \frac{\sinh(2a\Phi)}{R^{2(D-2)}}, \]  

\[ R'' - \frac{4\Phi'^2}{(D - 2)^2} = -4\Phi \]  

\[ \frac{1}{R^{D-2}} \left[ f \left( R^{D-2} \right)' \right]' = (D - 2)(D - 3) \frac{1}{R^2} - 2\Lambda - \frac{4|QP| \cosh(2a\Phi)}{R^{2(D-2)}}, \]  

\[ \frac{1}{R^{2D-6}} \left[ f R^{2D-6} \right]'' - (D - 4) \frac{R'}{R^{2D-5}} \left[ f R^{2D-6} \right]' = 2(D - 3)^2 - 4\Lambda, \]  

with \( ' \equiv d/dr \). Eqn. (13) follows from (10)–(12) by virtue of the Bianchi identity if \( \Phi' \neq 0 \).

It is possible to have asymptotically flat solutions to these field equations if \( \Lambda = 0 \), while asymptotically de Sitter (\( \Lambda > 0 \)) or asymptotically anti-de Sitter (\( \Lambda < 0 \)) solutions are obtained otherwise. Furthermore, de Sitter asymptotics are obtained only in the region in which the Killing vector \( \partial/\partial t \) is spacelike, and consequently any black hole solutions in such a model must possess at least two horizons. It is quite straightforward to show that in fact there are no such solutions. We prove the result by contradiction.

Suppose that black hole solutions do exist with at least two regular horizons, and let the two outermost horizons be labelled \( r_\pm \), with \( r_- < r_+ \). The requirement of regularity at the horizon means that near \( r = r_+ \), \( f \propto (r - r_+) \) and \( \phi(r_+) \) and \( R(r_+) \) are bounded with \( R(r_+) \neq 0 \), and similarly for \( r_- \). We may multiply (10) by \( \Phi \) to obtain

\[ \left[ R^{D-2} f \Phi \Phi' \right]' = R^{D-2} f \Phi'^2 + (D - 2)|QP| \frac{a\Phi \sinh(2a\Phi)}{R^{D-2}}. \]  

If we integrate (14) with respect to \( r \) between the two horizons, \( r_- \) and \( r_+ \), then the integral of the l.h.s. vanishes. Provided that the Killing vector \( \partial/\partial t \) is timelike in the region between the two horizons – which must be the case for solutions with de Sitter-like asymptotics (\( \Lambda > 0 \)) – then \( f(r) > 0 \) on the interval \( (r_-, r_+) \) and consequently the r.h.s. of (14) is positive-definite. A contradiction is thus obtained if \( \Lambda > 0 \) in all cases apart from that of a constant dilaton field with \( \Phi = 0 \), or equivalently \( e^{2a\phi} = |P|/|Q| \). To retrieve the standard normalisation for the electromagnetic field we require that \( \phi \to 0 \) at spatial infinity, and thus \( |Q| = |P| \) in the case of the solutions with a trivial constant dilaton. Furthermore, in the trivial case eqn. (10) is satisfied identically and the remaining equations reduce to those obtained in the absence of a dilaton, and thus the spacetime geometry is simply that of the
standard Reissner-Nordström-(anti)-de Sitter solution

\[ R(r) = r, \]
\[ f(r) = \frac{-2\Lambda r^2}{(D-1)(D-2)} + 1 - \frac{2M}{r^{D-3}} + \frac{2(Q^2 + P^2)}{(D-2)(D-3)r^{2D-6}}, \]  

(15)

with equal electric and magnetic charges, \(|Q| = |P|\).

In addition, there are also Robinson-Bertotti-type solutions of the form \( \Phi = 0 \), \( R = R_{\text{ext}} \) and

\[ f = \left[ (D-3)^2R_{\text{ext}}^{-2} - 4\Lambda \right] + c_1 r + c_2 \]  

(16)

where \( c_1 \) and \( c_2 \) are arbitrary constants, and the constant \( R_{\text{ext}} \) is a solution of the polynomial

\[-2\Lambda R_{\text{ext}}^{2D-4} + (D-2)(D-3)R_{\text{ext}}^{2D-6} - 4|QP| = 0. \]  

(17)

The argument we have given above for the non-existence of \( \Lambda > 0 \) black holes with a non-trivial dilaton of course only applies to dyonic solutions, as the transformation (9) used in writing the field equations in the form (10)–(12) becomes singular if either \( Q = 0 \) or \( P = 0 \). In the case of solutions which carry only an electric charge or a magnetic charge one finds that black hole solutions can have at most one horizon, for either sign of \( \Lambda \), as was shown in [11]. Since asymptotically de Sitter black holes must have at least two horizons, one immediately has the result that no singly charged dilaton black holes exist if \( \Lambda > 0 \). In this case even a trivial (constant) dilaton solution is not admitted. Solutions with a single horizon do exist, of course – however, the horizon is a cosmological one and the solutions possess a naked singularity as viewed by observers in the region where \( \partial/\partial t \) is timelike.

3. Asymptotically flat dyonic dilaton black holes

Before considering solutions in the presence of a cosmological constant, we will begin by discussing the corresponding asymptotically flat (\( \Lambda = 0 \)) dyonic dilaton black holes, so as to be able to make some comparison to the solutions with \( \Lambda < 0 \). Exact dyonic solutions are only known in the cases \( g_0 = \sqrt{3(D-3)} \) [1,20–22]\(^5\) and \( g_0 = \sqrt{D-3} \) [1,21]. In four

\(^5\) In fact, Gibbons and Maeda [1] leave the case \( g_0 = \sqrt{3(D-3)} \) as an “exercise for the reader”. For reference we list the solution to this exercise in the Appendix.
dimensions the solutions for $g_0 = 1, \sqrt{3}$ may be written\footnote{Our choice of the origin of $r$ agrees with \cite{22}, but differs from Gibbons and Maeda \cite{1}, whose corresponding radial coordinate is obtained by $r \to r + M$ above. The signs of $\phi$ and $\Sigma$ in \cite{22} are opposite to those here.}

\begin{align}
R(r) &= [A(r)B(r)]^{1/(g_0^2+1)}, \\
f(r) &= \frac{(r - M)^2 - [M^2 + \Sigma^2 - Q^2 - P^2]}{R^2}, \\
\exp(2g_0 \Phi) &= \left| \frac{Q}{P} \right| \left[ \frac{A(r)}{B(r)} \right]^{2g_0^2/(g_0^2+1)}
\end{align}

where for $g_0 = 1$ (appropriate to string theory)

\begin{align}
A(r) &= r + \Sigma, \\
B(r) &= r - \Sigma,
\end{align}

while for $g_0 = \sqrt{3}$ (appropriate to 5-dimensional Kaluza-Klein theory)

\begin{align}
A(r) &= (r - r_{A-}) \left( r - r_{A+} \right), \quad r_{A\pm} = -\Sigma/\sqrt{3} \pm \left[ \frac{2P^2\Sigma}{\Sigma + \sqrt{3}M} \right]^{1/2}, \\
B(r) &= (r - r_{B-}) \left( r - r_{B+} \right), \quad r_{B\pm} = \Sigma/\sqrt{3} \pm \left[ \frac{2Q^2\Sigma}{\Sigma - \sqrt{3}M} \right]^{1/2},
\end{align}

\begin{align}
\frac{Q^2}{\Sigma - \sqrt{3}M} + \frac{P^2}{\Sigma + \sqrt{3}M} &= \frac{2}{3} \Sigma.
\end{align}

In both cases the scalar charge, $\Sigma$, is seen to depend on the other charges, $M, Q$ and $P$, of the theory through relations (22) and (24), and thus does not constitute an independent black hole "hair".

Both solutions have the property of having two horizons if both the electric and magnetic charges are non-zero, and thus are qualitatively much like the ordinary Reissner-Nordström solution. In particular, the extreme solutions have an event horizon of finite area, and also
zero Hawking temperature. If one of the charges is zero, however, then there exists only a single horizon and in the extreme limit the area of the event horizon pinches off to zero, while the temperature is finite for the $g_0 = 1$ solution [1], and infinite for the $g_0 = \sqrt{3}$ solution [22]. The fact that the temperature is formally infinite in the second case merely signals the breakdown of the semiclassical limit if one is considering the Hawking evaporation process. Holzhey and Wilczek [4] have in fact demonstrated that in the case of the $g_0 > 1$ solutions an infinite mass gap develops for quanta with a mass less than that of the black hole so that the Hawking radiation slows down and comes to an end at the extremal limit, despite the infinite temperature.

It is worthwhile commenting on the properties of asymptotically flat dyonic solutions for values of $g_0$ other than 1 or $\sqrt{3}$. Some aspects of the global properties of the solutions for arbitrary $g_0$ and $D$ can be derived by studying the phase space using an approach similar to that of [10,13]. However, we will simply study the solutions numerically here, as we will adopt an identical approach for the $\Lambda < 0$ solutions. The behaviour of the solutions at large $r$ can be determined by making the expansions \( \Phi = \sum_{i=0}^{\infty} \Phi_i r^{-i} \), \( f = \sum_{i=0}^{\infty} f_i r^{-i} \), \( R = r + \sum_{i=0}^{\infty} R_i r^{-i} \), and substituting in the field equations (10)–(12) with $\Lambda = 0$. If we use the freedom of translating the origin in the radial direction to set $R_0 = 0$, which accords with the choice made in (19), (21), (23), we find

\[
\Phi = \Phi_0 + \frac{\Sigma}{r^{D-3}} + \left[ M \Sigma + \frac{a(D-2)|QP| \sinh(2a\Phi_0)}{2(D-3)^2} \right] \frac{1}{r^{2D-6}} + O \left( \frac{1}{r^{2D-5}} \right),
\]

\[
f = 1 - \frac{2M}{r^{D-3}} + \frac{4|QP| \cosh(2a\Phi_0)}{(D-2)(D-3)r^{2D-6}} + O \left( \frac{1}{r^{2D-5}} \right),
\]

\[
R = r - \frac{2(D-3)\Sigma^2}{(2D-7)(D-2)^2 r^{2D-7}} + O \left( \frac{1}{r^{2D-6}} \right).
\]

The constants $M$, $\Phi_0$ and $\Sigma$ are free, $M$ being proportional to the ADM mass. We can use the freedom of adding a constant to the dilaton to set

\[
\Phi_0 = \frac{1}{2a} \ln \left| \frac{Q}{P} \right|,
\]

so that $\phi = \Sigma/r^{D-3} + O \left( r^{-(2D-6)} \right)$ by (9), which yields the standard normalisation for the electromagnetic field, and accords with the choice made in the exact solutions above.

Although the scalar charge $\Sigma$ is a free parameter as far as the asymptotic series is concerned, if we further demand that a particular solution with an asymptotic expansion (25)
corresponds to a black hole, then we can integrate equation (10) between the outermost horizon, \( r_+ \), and infinity to obtain an integral relation

\[
\Sigma = -\frac{a(D-2)|QP|}{2(D-3)} \int_{r_+}^{\infty} dr \frac{\sinh(2a\Phi)}{R^{D-2}}. \tag{27}
\]

Consequently, for black hole solutions \( \Sigma \) is constrained to depend on the other charges of the theory, which is of course what we would have anticipated from the exact solutions. The sign of the scalar charge is fixed to be \( \Sigma < 0 \) if \( |Q| > |P| \) (electrically dominated solutions), or \( \Sigma > 0 \) if \( |Q| < |P| \) (magnetically dominated solutions). To see this one may observe that since the r.h.s. of (14) is positive definite on the interval \((r_+, \infty)\), if one integrates (14) from the outermost horizon \( r_+ \) to an arbitrary point \( r_o \) in the domain of outer communications then \( \Phi(r_o)\Phi'(r_o) > 0 \) if the dilaton field is non-trivial. Thus either: (i) \( \Phi > 0 \) and \( \Phi \) is monotonically increasing on the interval \((r_+, \infty)\); or (ii) \( \Phi < 0 \) and \( \Phi \) is monotonically decreasing on the interval \((r_+, \infty)\). The asymptotic expansions (25) then fix the sign of \( \Sigma \) as above, and by (26) cases (i) and (ii) are seen to correspond to electrically- and magnetically-dominated solutions respectively. This argument will not be changed if \( \Lambda < 0 \).

It is possible to derive a further constraint which must be satisfied by the charges in the case of the extreme solutions. In particular, if we multiply (10) by \( 2R^{2(D-2)}f^2/(D-2) \) we obtain

\[
\frac{1}{D-2} \left[ R^{2(D-2)}f^2\Phi'^2 \right]' = 2a|QP|\Phi f \sinh(2a\Phi), \tag{28}
\]

while the difference of \( 2(D-3) \) times (12) and \( (D-2) \) times (13) may be multiplied by \( (D-2)R^{2(D-2)}f'/[8(D-3)] \) to give

\[
\frac{1}{16} \left( \frac{D-2}{D-3} \right) \left[ R^{2(D-2)}f'^2 \right]' = |QP|f' \cosh(2a\Phi). \tag{29}
\]

If we add (28) to (29) the resulting equation may be directly integrated to give

\[
\left[ \frac{f^2\Phi'^2}{D-2} + \frac{(D-2)f'^2}{16(D-3)} \right] R^{2(D-2)} = |QP|f \cosh(2a\Phi) + c_0 \tag{30}
\]

where \( c_0 \) is an arbitrary constant. If we evaluate the integral between spatial infinity and a degenerate horizon at which \( f = 0 \) and \( f' = 0 \), and use the asymptotic series (25) we find that

\[
M^2 + \frac{4(D-3)\Sigma^2}{(D-2)^2} = \frac{2(Q^2 + P^2)}{(D-2)(D-3)^3} \tag{31}
\]
in the extreme limit.
For the purpose of numerical integration we will follow Horne and Horowitz [9] and make a change of coordinates to use $R$ as the radial variable, so that the metric becomes

$$ds^2 = -f dt^2 + h^{-1} dR^2 + R^2 dΩ_{D-2}$$

(32)

where $h(R) \equiv f \left( \frac{4R^2}{f^2} \right)^2$, and now $f = f(R)$. The advantage of working with these coordinates is that by suitably combining the appropriate differential equations one can solve for $f$ in terms of $h$ and $\phi$. One finds

$$f = h \exp \left[ \frac{8}{(D-2)^2} \int R \phi^2 \right],$$

(33)

where $\dot{.} \equiv \frac{d}{dR}$. There are then just two independent field equations remaining, viz.

$$-R \ddot{h} + (D-3)(1-h) = \frac{4R^2h\phi^2}{(D-2)^2} + \frac{2\Delta R^2}{D-2} + \frac{4|QP|\cosh(2a\Phi)}{(D-2)R^{2D-6}},$$

(34)

$$Rh\ddot{\Phi} + s(h, \Phi, \dot{\Phi}) = 0,$$

(35)

where

$$s(h, \Phi, \dot{\Phi}) \equiv \dot{\Phi} \left[ D - 3 + h - \frac{2\Delta R^2}{D-2} - \frac{4|QP|\cosh(2a\Phi)}{(D-2)R^{2D-6}} \right] - (D-2)a|QP| \frac{\sinh(2a\Phi)}{R^{2D-5}}$$

(36)

Although we are interested in solutions with $\Lambda = 0$ at present, we have left $\Lambda$ explicitly in (34)–(36), as the same equations will be used in the next section with $\Lambda < 0$. The asymptotic series (25) become

$$\Phi = \frac{1}{2a} \ln \left| \frac{Q}{P} \right| + \frac{\Sigma}{R^{D-3}} + O \left( \frac{1}{R^{D-2}} \right)$$

$$h = 1 - \frac{2M}{R^{D-3}} + \frac{2(Q^2 + P^2)}{(D-2)(D-3)R^{2D-6}} + O \left( \frac{1}{R^{2D-5}} \right)$$

(37)

in terms of the new variables. For numerical integration we will fix $D = 4$. 

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Equations (34), (35) are equivalent to three first order ordinary differential equations and thus generally have a three parameter set of solutions. However, many of these will correspond to naked singularities. The requirement that solutions have at least one regular horizon reduces the three parameters to two, which may be taken to be the radial position of the outermost horizon, \( R_+ \), and \( \Phi_+ \equiv \Phi(R_+) \), if we treat the integration as an initial value problem. Since the equations are singular on the horizon, we start the integration a small distance from \( R_+ \), the initial values of \( h \), \( \Phi \) and \( \dot{\Phi} \) being determined in terms of \( R_+ \) and \( \Phi_+ \) by solving for the coefficients \( \bar{h}_i \) and \( \bar{\Phi}_i \) in the power series expansions,

\[
h = \sum_{i=1}^{\infty} \bar{h}_i (R - R_+)^i,
\]

\[
\Phi = \Phi_+ + \sum_{i=1}^{\infty} \bar{\Phi}_i (R - R_+)^i.
\]  

(38)

Since the solutions are rather cumbersome we will not list them here.

From §2 it follows that if there is more than one regular horizon then \( h < 0 \) between any two horizons for solutions with a non-trivial dilaton, and thus black hole solutions with \( \Lambda \leq 0 \) can have at most two horizons. Since two horizons are possible we have a choice of starting on an inner horizon, \( R_- \), with initial data \( \dot{h}(R_-) < 0 \), or an outer horizon, \( R_+ \), with initial data \( \dot{h}(R_+) > 0 \). From the discussion above it follows that initial data with \( \Phi(R_+) > 0 \) and \( \dot{\Phi}(R_+) > 0 \) will yield electrically dominated solutions, and initial data with \( \Phi(R_+) < 0 \) and \( \dot{\Phi}(R_+) < 0 \) magnetically dominated solutions.

Although the exact solutions (18)–(23) both have two horizons there is no guarantee that black holes possess two horizons for other values of \( g_0 \). It is not straightforward to test for the existence of double horizon solutions if one uses an initial value integration method, as we have chosen to do, since the integration routine is usually halted when \( h \) approaches zero. The problem occurs since the numerical routine determines \( \ddot{\Phi} \) by setting \( \ddot{\Phi} = s/(hR) \), where the quantity \( s \) is defined by (36). The numerical integration is also singular near \( h = 0 \) for the cases \( g_0 = 1, \sqrt{3} \), when the appropriate exact solutions are known to exist. To distinguish a true second horizon therefore, we need to test whether \( \ddot{\Phi} \to 0 \), or equivalently whether \( s \to 0 \) sufficiently quickly, as the second horizon is approached, either integrating outwards or inwards. Given that the inner horizon often occurs at very small values of \( R \), where \( \dot{h} \) is large, to reduce numerical errors it is convenient to set initial data at a regular inner horizon and to integrate outwards. Having found a second horizon for which \( s(R_+) = 0 \), we must also check that the solution is indeed a black hole by resuming the integration just beyond the second horizon and checking that it does eventually match the series (37).
As a result of such an analysis we have found that a second horizon is obtained only if \( g_0 \) takes on certain discrete values. In fact, our numerical results are consistent with \( g_0 \) being the square root of a triangle number:

\[
g_0 = 0, 1, \sqrt{3}, \sqrt{6}, \sqrt{10}, \ldots, \sqrt{n(n+1)/2}, \ldots
\]  

(39)

For other values of \( g_0 \) the dyonic black hole solutions only have one horizon. As \( g_0 \) is varied the following pattern is observed for initial data with \( \Phi(R) < 0 \): for \( 0 < g_0 < 1 \) the function \( s(R) \) decreases from zero to a local minimum, and then increases monotonically, crossing zero a second time before the second zero of \( h(R) \) is reached. For \( g_0 = 1 \) the pattern is the same but the zero of \( s(R) \) coincides with the second zero of \( h(R) \). For \( 1 < g_0 < \sqrt{3} \) \( s(R) \) has both a local minimum and a local maximum, and for values for which the local maximum of \( s(R) \) is positive there is one additional zero before \( h(R) \) reaches its second zero. For \( g_0 = \sqrt{3} \) the third zero of \( s(R) \) coincides with the second zero of \( h(R) \). For \( \sqrt{3} < g_0 < \sqrt{6} \) the number of turning points and possible zeros of \( s(R) \) increases by one, and again the final zero of \( s(R) \) coincides with the second zero of \( h(R) \). For \( g_0 = \sqrt{6} \) the pattern is repeated with \( s(R) \) oscillating between \( R_- \) and \( R_+ \), and the number of oscillations increasing by one every time \( g_0 \) attains a value in the series \( \sqrt{n(n+1)/2}, n \in \mathbb{Z}^+ \). The behaviour of \( s(R) \) is displayed in Fig. 1 for the lowest critical values of \( g_0 \), and in Fig. 2 for some non-critical values. If we choose initial data with \( \Phi(R) > 0 \), then similar results are obtained since the equations are invariant under the duality transformation \( Q \leftrightarrow P, \Phi \rightarrow -\Phi \).

The critical values of \( g_0 \) are independent of \( |QP| \) and the initial values \( R_- \) and \( \Phi(R_-) \), with the proviso that for each \( R_- \) there is an upper bound for \( |\Phi(R_-)| \), beyond which solutions are not asymptotically flat although two regular horizons exist. In the case of larger values of \( g_0 \), however, we find that the precise numerical value of \( g_0 \) begins to show some dependence on the initial data for large values of \( \Phi(R_-) \). As the problem increases with increasing \( g_0 \) it would appear to be merely the result of numerical errors which increase rapidly for large \( g_0 \) due to the exponential dependence on \( g_0 \Phi \). Even for large \( g_0 \) the largest values of \( \Phi(R_-) \) consistent with asymptotically flat solutions yield numerical values of \( g_0 \) which agree with the series (39) to 0.1%. We have numerically checked terms up to \( g_0 = \sqrt{21} \) in the series. Since zeros of \( s(R) \) correspond to points with \( \ddot{\Phi} = 0 \) we also find that \( \Phi(R) \) oscillates between the two horizons for solutions with critical values of \( g_0 \). This is illustrated in Fig. 3.

If \( g_0 \) is not equal to one of the critical eigenvalues then black holes solutions do exist, but they have only one horizon. Physically the spacetime geometry is no different to that of the solutions with two horizons as far as observers in the domain of outer communications are
The behaviour of $R^3 s$, where $s$ is defined by (36) with $\Lambda = 0$, as a function of $R$, between regular horizons for critical values of the dilaton coupling parameter $g_0$. What appears to be occurring in the general case is that the function $h(R)$ has two zeros which would both correspond to regular horizons if it were not for the fact that in general the dilaton becomes singular at one of the potential horizons. Only for the critical values of $g_0$ is the dilaton regular at both zeros of $h$. In the case that the zero of $h$ is degenerate, i.e., $h = 0$ and $\dot{h} = 0$, the dilaton is always regular at the horizon, however. The spacetime geometry of the extreme solutions approaches that of the Robinson-Bertotti-type solution (16), (17) (with $\Lambda = 0$) in the neighbourhood of the degenerate horizon.

Since the horizon of the extreme solutions occurs at a finite value of $R$ it follows that the entropy – at least, the entropy naively defined as a quarter of the area of the event horizon – is finite, while the temperature, which is given in general by $T = \dot{h}(R_+)/4\pi$, is zero in

![Graph showing the behaviour of $R^3 s$ as a function of $R$ for different values of $g_0^2$.]
The behaviour of $R^3 s$, where $s$ is defined by (36) with $\Lambda = 0$, as a function of $R$, in some typical cases when the second zero of $h$ does not correspond to a regular horizon.

These contour plots are based on a large number ($\sim 5 \times 10^3$) of numerical integrations. The integration procedure we adopted was to fix $|QP|$ and to determine regular initial data $R_+$, $\Phi_+$ such that $\dot{h}(R_+) > 0$ at a regular horizon, using solutions obtained from substituting the expansions (38) into the field equations. We then began integrations a short distance outside $R_+$ and integrated outwards until solutions matched the asymptotic series (37) to some specified accuracy – five significant figures in our case. Not all initial values of $R_+$ and $\Phi_+$ yield asymptotically flat solutions. However, for those that do we are able to identify $\Phi_0$ according to (26). In order to fix the value of $|Q|/|P|$ so as to obtain plots such as those shown in Fig. 4, we then repeated the integration using a Newton-Raphson algorithm to vary

Fig. 2: The behaviour of $R^3 s$, where $s$ is defined by (36) with $\Lambda = 0$, as a function of $R$, in some typical cases when the second zero of $h$ does not correspond to a regular horizon.
The behaviour of $\Phi$ for $\Lambda = 0$, as a function of $R$ between the horizons $R_-$ and $R_+$, in cases that two regular horizons exist.

Although the extreme limit appears to be approximately linear over the range over the range of values shown in Fig. 4, this is not strictly the case. In general the relation between $M$, $P$ and $Q$ which defines the extreme solutions is a very complicated one. This can already be seen in the case of the $g_0 = \sqrt{3}$ exact solutions, where the extreme limit is given by $M^2 + \Sigma^2 = Q^2 + P^2$, with $\Sigma$ being a solution of the cubic equation (24). Only for $g_0 = 1$ does the extreme limit correspond to a linear relationship between $M$, $P$ and $Q$, namely

$$\sqrt{2}M = |P| + |Q|.$$ 

(40)

It is nevertheless possible to approximate the curve of extreme solutions in the limit that the asymptotic series (37) are valid up to the degenerate horizon. For example, for $D = 4$
we find from the first three terms in the series

$$\frac{\sum}{M} \approx \frac{1}{1 + \sqrt{2qp}} \left[ \frac{qp}{g_0} \ln \left| \frac{q}{p} \right| + \frac{g_0}{2} \left( p^2 - q^2 \right) \right]$$

where $q \equiv Q/M$ and $p \equiv P/M$. If we substitute (41) into (31) and solve for $q$ the result agrees with the gradient of the curve for the largest values of $M$ found numerically in Figs. 4 to within 1%.

4. Asymptotically anti-de Sitter black holes

Let us now turn to the case of a negative cosmological constant, $\Lambda < 0$. Since $\partial/\partial t$ is timelike in the asymptotic region, there are many qualitative similarities between these solutions and those discussed in the last section. There are also some important differences, however. The asymptotic expansions are given by

$$\Phi = \frac{1}{2a} \ln \left| \frac{Q}{P} \right| + \frac{\Phi_{D-1}}{r^{D-1}} + O \left( \frac{1}{r^{D-2}} \right),$$

$$f = \frac{-2\Lambda r^2}{(D-1)(D-2)} + 1 - \frac{2M}{r^{D-3}} + \frac{2(Q^2 + P^2)}{(D-2)(D-3)r^{2D-6}}$$

$$+ \frac{8\Lambda \Phi_{D-1}}{(2D-3)(D-2)^3r^{2D-4}} + O \left( \frac{1}{r^{2D-3}} \right),$$

$$R = r - \frac{2(D-1)\Phi_{D-1}^2}{(2D-3)(D-2)^2r^{2D-3}} + O \left( \frac{1}{r^{2D-2}} \right),$$

in terms of the radial coordinate $r$, where we have once again chosen to set $R_0 = 0$ and to
normalise $\Phi_0$ according to (26), or by

$$\Phi = \frac{1}{2a} \ln \left| \frac{Q}{P} \right| + \frac{\Phi_{D-1}}{R^{D-1}} + O \left( \frac{1}{R^{D-2}} \right),$$

$$f = \frac{-2\Lambda R^2}{(D-1)(D-2)} + 1 - \frac{2M}{R^{D-3}} + \frac{2(Q^2 + P^2)}{(D-2)(D-3)R^{2D-6}} - \frac{8\Lambda \Phi_{D-1}^2}{(D-2)^3 R^{2D-4}} + O \left( \frac{1}{R^{2D-3}} \right),$$

in terms of the radial coordinate $R$. In four dimensions the dilaton is effectively short-range, with $\phi \sim \Phi_3 r^{-3}$ at spatial infinity, rather than long-range as for the asymptotically flat solutions. Indeed, this bears some resemblance to the solutions with a massive dilaton [8,9], although there the dependence is one power of $r$ weaker.

The scalar charge, $\Phi_{D-1}$, is once again constrained to depend on the other charges of the theory, and if we integrate (14) between the outermost horizon, $r_+$, and infinity we obtain

$$\Phi_{D-1} = \frac{a(D-2)^2 |QP|}{2\Lambda} \int_{r_+}^{\infty} dr \frac{\sinh(2a\Phi)}{R^{D-2}}.$$  

which by an argument similar to that after (27) is seen to be negative for electrically dominated solutions ($|Q| > |P|$), and positive for magnetically dominated solutions ($|Q| < |P|$).

The next significant difference we have found is that there are no longer universal discrete eigenvalues of $g_0$ which yield black hole solutions with two horizons in four dimensions. As in the last section, it is possible numerically to find values of $g_0$ for which the function $s(R)$ defined by (36) has zeros at both zeros of $h$. However, these values now always depend on the initial data and change if one varies any of the parameters $R_-, |QP|$ or $\Lambda$. Thus the constraint for obtaining solutions with two regular horizons would appear to involve $M$, $|QP|$ and $\Lambda$ in addition to $g_0$, in some complicated non-linear fashion. Nevertheless, for the wide range of initial values we have studied we find that the critical values of $g_0$ obtained are fairly close to, but always slightly greater than the corresponding eigenvalues found for the $\Lambda = 0$ solutions.

Similarly to the case of the asymptotically flat solutions, to observers in the domain of outer communications the spacetime geometry will appear to be qualitatively the same as that of the Reissner-Nordström-anti-de Sitter solution – (15) with $\Lambda < 0$ – independently
Fig. 5: Contours of constant temperature, $T$, for asymptotically anti-de Sitter black holes (with $\Lambda = -1$) in four dimensions, as obtained by numerical integration in the case $g_0 = 1$: (a) $|P| = 0.5|Q|$; (b) $|P| = 2|Q|$.

Fig. 6: Contours of constant temperature, $T$, for the exact Reissner-Nordström-anti-de Sitter black hole solution (15) with $\Lambda = -1$ and equal charges $|Q| = |P|$ in four dimensions.

of whether the solutions have one or two horizons. The extreme solutions occur at a finite value of $R$ given by (17), which in four dimensions takes the value

$$R_{\text{ext}} = \frac{1}{2\Lambda} \left[ \sqrt{1 - 8\Lambda|QP|} - 1 \right].$$

Once again one can approximate the curve which relates $M$ to $Q$ and $P$ for the extreme solutions for large values of $M$ and $|QP|$ when the series (43) hold up to the degenerate horizon. However, this is somewhat more complicated than the asymptotically flat case as it is not possible to obtain a simple relationship analogous to (31) in closed form in this case.
The temperature of the solutions is given by

\[ T = \frac{1}{4(D-2)\pi R_+^{2D-5}} \left[-2\Lambda R_+^{2D-4} + (D-2)(D-3)R_+^{2D-6} - 4|QP| \cosh(2a\Phi_+)\right] \] (46)

where \( R_+ \) is the outermost horizon and \( \Phi_+ = \Phi(R_+) \), and in the extreme limit \( \Phi_+ = 0 \), \( R_+ = R_{\text{ext}} \) (as given by (17)) this is zero. We have once again verified these conclusions by explicit numerical integrations in four dimensions. The plots shown are for the values \( g_0 = 1 \), \( \Lambda = -1 \). However, other values of \( g_0 \neq 0 \) and \( \Lambda < 0 \) lead to results which are qualitatively the same. The numerically calculated isotherms are shown in Fig. 5 for two values of \(|QP|\). For comparison the isotherms of the special case with \(|Q| = |P|\) and a trivial dilaton are shown in Fig. 6, as determined from the exact solutions (15). As expected, Figs. 5 and 6 are all qualitatively the same, the only difference being the position of the curve which defines the extreme limit. By contrast, for asymptotically anti-de Sitter dilaton black holes with a single charge the extreme limit corresponds to \( R \to 0 \) and \( T \to \infty \) [11], leading to a pattern of isotherms as shown in Fig. 7. In all cases the Q = 0 axis corresponds to the Schwarzschild-anti-de Sitter solution, which has a minimum temperature at \( T_{\text{cr}} = \sqrt{-\Lambda}/(2\pi) \). This critical isotherm is indicated in the plots.

**Fig. 7:** Contours of constant temperature, \( T \), for the singly charged asymptotically anti-de Sitter black hole solution, as obtained numerically in the case \( g_0 = 1 \) (with \( \Lambda = -1 \)) in four dimensions.
5. Conclusion

We have shown that the previously known exact solutions [1,21–22] for dyonic black holes in four dimensions with a coupling to a non-trivial dilaton are in fact special cases when one varies the dilaton coupling parameter, $g_0$. For most values of $g_0$ dyonic solutions have a single horizon, the exceptions being $g_0 \in \{\sqrt{n(n+1)/2}, n \in \mathbb{Z}^+\}$. It is quite possible that exact solutions could found for the remaining values of $g_0$ in the series. Despite the absence of an inner horizon, the spacetime geometry is qualitatively the same as the standard Reissner-Nordström solution as far as observers in the domain of outer communications are concerned, even in the limit of extreme solutions. The extreme solutions have zero Hawking temperature.

We have further shown that no dyonic black hole solutions with a non-trivial dilaton exist if $\Lambda > 0$. This is similar to the result obtained for singly charged dilaton black holes [11], although here there is a solution with a trivial constant dilaton in the case of equal charges, $|Q| = |P|$. Asymptotically anti-de Sitter black holes with a non-trivial dilaton do exist if $\Lambda < 0$, however. These solutions have one or two horizons, the cases of two horizons being given by a complicated non-linear relationship between $g_0$, $M$, $|QP|$ and $\Lambda$. We are unable to determine this relationship analytically, but we have some numerical evidence for its existence. When combined with the results concerning singly charged black holes [11] our results here indicate that the horizon structure of asymptotically anti-de Sitter dilaton black holes is qualitatively the same as the corresponding (singly charged or dyonic) asymptotically flat solutions. Once again the physical properties of the dyonic black holes with $\Lambda < 0$ are the same regardless of the value of $g_0$, and as in the case of the asymptotically flat dyonic solutions they have zero Hawking temperature.

Although a pure cosmological constant may not be the most natural cosmological term in the context of stringy dilaton gravity, we believe that the solutions we have studied here may nonetheless provide a useful approximation in some circumstances. In particular, if we have a dilaton potential generated by supersymmetry breaking or some other mechanism, and if the minimum of that potential which corresponds to the groundstate of the dilaton has a value which is not precisely zero, then the universe would contain some (hopefully small) vacuum energy. If this vacuum energy is negative then the black hole solutions studied here (with $\Lambda < 0$) might provide a useful approximation to the complete solutions, just as the solutions of [8,9] are well approximated by the solutions with a massless dilaton in certain regimes. Phenomenologically a small positive vacuum energy is currently favoured by astronomers in order to reconcile the value of the Hubble constant and the age of the universe as measured by the oldest stars. However, we find no black hole solutions with a non-trivial dilaton in the case of positive vacuum energy. It is quite conceivable that this conclusion would be
altered in the presence of a non-trivial dilaton potential, $\mathcal{V}(\phi)$. In particular, the dilaton equation (10) upon which the non-existence arguments are based – through (14) here or by an alternative argument for singly charged solutions [11] – acquires an additional $\frac{d\mathcal{V}}{d\phi}$ term which provides an obstruction to these arguments for suitable potentials. However, we will leave such considerations to future work.

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Appendix

We list here the asymptotically flat dyonic black hole solution in arbitrary spacetime dimension with $g_0 = \sqrt{3(D - 3)}$. The solution is readily generated from the $D = 4$ case (18)–(20), (23) if one observes that by using a new radial coordinate

$$\eta = \int R^{D-4} dr,$$

the field equations can be transformed into the $D = 4$ equations after a suitable rescaling of parameters. The solution is

$$R^{D-3} = (D - 3) [A(\eta)B(\eta)]^{1/4},$$

$$f = \left(\frac{D - 3}{R}\right)^{2(D-3)} [(\eta - M)^2 - \eta_0^2],$$

$$\exp\left(\frac{4\sqrt{3(D - 3)} \Phi}{D - 2}\right) = \left|\frac{Q}{P}\right| \left[\frac{A(\eta)}{B(\eta)}\right]^{3/2},$$

(A.2)
where

\[ A(\eta) = (\eta - \eta_{A-})(\eta - \eta_{A+}), \]
\[ B(\eta) = (\eta - \eta_{B-})(\eta - \eta_{B+}), \]
\[ \eta_{A\pm} = -\frac{2\sqrt{D - 3}\Sigma}{(D - 2)\sqrt{3}} \pm \left[ \frac{8\sqrt{D - 3}P^2\Sigma}{(D - 2)(D - 3)^3 \left[ 2\sqrt{D - 3}\Sigma + (D - 2)\sqrt{3M} \right]} \right]^{1/2} \]
\[ \eta_{B\pm} = \frac{2\sqrt{D - 3}\Sigma}{(D - 2)\sqrt{3}} \pm \left[ \frac{8\sqrt{D - 3}Q^2\Sigma}{(D - 2)(D - 3)^3 \left[ 2\sqrt{D - 3}\Sigma - (D - 2)\sqrt{3M} \right]} \right]^{1/2}, \] (A.3)
\[ \eta_0^2 = M^2 + \frac{4(D - 3)\Sigma^2}{(D - 2)^2} - \frac{2}{(D - 2)(D - 3)^3} \left( \frac{Q^2 + P^2}{\sqrt{D - 3}\Sigma - (D - 2)\sqrt{3M}} + \frac{P^2}{\sqrt{D - 3}\Sigma + (D - 2)\sqrt{3M}} \right) = \frac{2(D - 3)^{7/2}}{(D - 2)} \Sigma. \]

The scalar charge \( \Sigma \) has been normalised so as to correspond to the same charge in the series (25) when expressed in terms of the radial coordinate \( r \).

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