Cosmological Models with Variable Constants.
Their Solution Through Similarity Methods.

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(Dated:)

Abstract

In this work we compile a few differential equations (ODEs) that arise from the relativistic equations in cosmological models that consider the “constants” as scalars functions dependent on time and they are described as perfect as well as viscous fluids. The general idea of the paper is to show how to solve the equations of the models through dimensional techniques (self-similarity). The results are compared with those obtained by other authors and new solutions are introduced.

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I. INTRODUCTION.

The purpose of this work is to study different cosmological models that envisage the classical constant as scalar functions dependent on time. To outline the differential equations that govern such models and to compare their solution through the traditional method (integration of ODEs, this is the way followed by other authors) with the Dimensional Method (self-similarity). The latter method will consist in the reduction of the number of variables and the obtainment of ODEs easily integrable (see\textsuperscript{1, 2} and in special\textsuperscript{3} for a review of self-similarity in General Relativity).

In all the studied cases, the equations are very similar, there will only be differences when the term \( p \) (pressure) is defined, depending on if it is considered a perfect or viscous fluid. The modified field equations are: (the “constants” \( G, c \) and \( \Lambda \) are functions on \( t \)).

\[
R_{ij} - \frac{1}{2} g_{ij} R - \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij} \tag{1}
\]

where \( \Lambda \) represents the cosmological “constant” and we impose that the second Bianchi identity is verified: (after raising an index)

\[
\left( R^i_{\ j} - \frac{1}{2} \delta^i_{\ j} R \right)_{\ ij} = \left( \frac{8\pi G}{c^4} T^i_{\ j} + \Lambda \delta^i_{\ j} \right)_{\ ij} \tag{2}
\]

as well as the so-called conservation principle (bad so-called\textsuperscript{1} see\textsuperscript{4}) for the energy-momentum tensor i.e. the covariant divergence of the stress-energy tensor:

\[
div(T^j_{\ i}) = 0 \tag{3}
\]

this condition will be considered in some cases since we shall see we that can avoid it.

1. The line element is defined by:

\[
ds^2 = -c^2 dt^2 + f^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]
\]

we only consider the case \( k = 0 \), here.

\textsuperscript{1} we understand that this is only an expression since in GR the conservation of the energy and momentum is only approximate.
2. The energy-momentum tensor is defined by:

\[ T_{ij} = (\rho + p)u_i u_j - pg_{ij} \]

where \( \rho \) is the energy density and \( p \) represents pressure (to be defined in a generical way), always verifying \([\rho] = [p] \).

The paper is organized as follows: In the second section, a model will be studied that only envisages the “constants” \( G \) and \( \Lambda \) as variable and whose energy-momentum tensor is described by a perfect fluid. As in all the studied cases, this section will begin outlining the equations that govern such model, going next to study its detailed solution through the two techniques commented previously. The traditional one, will consist in the math integration of the equations (non dimensional method). The other is based on dimensional techniques. The latter will be studied in two ways, the first one, that we shall designate as the “simplest method” (naive), will consist in the dimensional study of the problem (model) analyzing its set of governing quantities. This set will lead us to the solution of the outlined equations through the application of the Pi theorem. While the other dimensional technique that we shall study and that we designate as not so simple method, is a mixture of the classic dimensional analysis (obtainment of \( \pi \)-monomials) with the direct integration of the equations. The pi monomials will reduce the number of variables in the differential equations and therefore it will bring us to obtain a simpler differential equation (ODEs) that will be directly integrated. The efficiency of Dimensional Analysis will enable us to provide new solutions even in the cases in which we do not envisage some of the departure hypotheses as for example \( \text{div}(T^i_j) = 0 \). In the third section we shall study a model that envisages \( G \) as well as \( \Lambda \) as scalar functions and whose tensor is characterized by a bulk viscous fluid. As in the second paragraph, the classic solution of the problem will be studied and it will be compared to the one obtained through the dimensional techniques, providing new solutions in this case. In the fourth section, constant \( G, c \) and \( \Lambda \) are envisaged as dependent functions on time within a model described by a bulk viscous fluid. Only in this case they are provided dimensional solutions since, at the moment, they are the only ones known. In the fifth section a model will be studied that may be understood as a generalization of the previous one upon considering in it mechanisms of adiabatic creation of matter. In this case, it is observed that the governing equations of the model are reduced.
to the case above therefore no further commentaries are needed. Finally it will be ended with some brief conclusions.

II. CASE $G$ AND A VARIABLE FOR A PERFECT FLUID.

Attending to the specifications made in the introduction in this case $p$ (pressure) verifies the following state equation:

$$p = \omega \rho \quad \omega = \text{const.} \quad (4)$$

where $\omega \in [0, 1]$ (i.e. it is a pure number) so that the energy-momentum tensor verifies the energy conditions. Under these circumstances the equations are:

$$2 \frac{f''}{f} + \frac{(f')^2}{f^2} = - \frac{8\pi G(t)}{c^2} p + c^2 \Lambda(t) \quad (5)$$

$$3 \frac{(f')^2}{f^2} = \frac{8\pi G(t)}{c^2} \rho + c^2 \Lambda(t) \quad (6)$$

from the expressions (5) and (6) we obtain the following relationship

$$G\rho' + 3(1 + \omega)\rho G\frac{f'}{f} + \rho G' + \frac{\Lambda'c^4}{8\pi} = 0 \quad (7)$$

furthermore the following law is taken into account

$$\text{div}(T^i_j) = 0 \iff \rho' + 3(\rho + p) \frac{f'}{f} = 0 \quad (8)$$

Now, we go on to see how this model is solved through two methods, one of them, by traditional integration and the other by dimensional approach.

A. Non Dimensional method.

Deriving the equation (6) and simplifying with (5) we obtain the following relationship. This relationship was used by Lau who reconciled the LNH of Dirac with the GR (see$^5$)

$$G\rho' + 3(1 + \omega)\rho G\frac{f'}{f} + \rho G' + \frac{\Lambda'c^4}{8\pi} = 0 \quad (9)$$

regrouping terms and taking into account $\text{div}(T^i_j) = 0$ we obtain:

$$\rho' + 3(\rho + p) \frac{f'}{f} = - \left[ \frac{\Lambda'c^4}{8\pi G} + \rho \frac{G'}{G} \right] \quad (10)$$
From the equations (9) and (8) the following equation that relates $G$ with $\Lambda$ is obtained.

$$ G' = -\frac{N'c^4}{8\pi\rho} $$  \hspace{1cm} (11)

From all these relationships the following differential equation is obtained which is not immediately integrable: combining (6) with (8) (here we are continuing Kalligas et al’s work see$^6$)

$$ \left(\frac{\rho'}{\rho}\right)^2 = 9(1+\omega)^2 \left(\frac{8\pi G}{3c^2} \rho + \frac{\Lambda c^2}{3}\right) $$

deriving with respect to $t$ then we obtain:

$$ \frac{\rho'\rho''}{\rho^2} - \left(\frac{\rho'}{\rho}\right)^3 = 12\pi(\omega + 1)^2 \frac{G\rho'}{c^2} $$  \hspace{1cm} (12)

that is to say:

$$ \rho'\rho'' - (\rho')^2 = 12\pi(\omega + 1)^2 \frac{G\rho^3}{c^2} $$  \hspace{1cm} (13)

To integrate this equation Kalligas et al made the following hypothesis on the behavior of the function $G : G \propto t^\alpha$ with $\alpha \in \mathbb{R}$ (see$^6$) this is an unfounded hypothesis in our opinion leading to $G = Ct^\alpha$ (where $C$ represents certain constant of proportionality). The obtained results are:

$$ \rho(t) = \frac{\alpha + 2}{12\pi(\omega + 1)^2 C} \frac{1}{t^{\alpha+2}} $$

eq...

(see$^6$). We believe that this solution, even though correct, under the outlined hypotheses presents certain degree of arbitrariness, since we will know neither the behavior of $G$ nor the one of any other quantity in function of the equation of state that we are imposing i.e. we should give any value to $\alpha$ and “guess” which model is describing i.e. what state equation belongs to, furthermore they introduce a new dimensional constant, $C$, of doubtful physical meaning.

**B. Dimensional Method.**

In this section, two dimensional tactics are studied. The first of them, designated as “naive method” (since it is always the simplest method as well as effective) studies the set of governing quantities of the problem and through the application of the Pi theorem a solution to the equations except the numerical constant ones is obtained. The second method, the one designated as “not so simplest method”, combines the classic dimensional analysis with
the direct integration of the differential equations. This method has been developed by Prof. M. Castaños for ordinary differential equations (see\textsuperscript{7} and\textsuperscript{8}).

1. \textit{Naive Method.}

With the dimensional method that we have followed we are not forced to impose similar condition and obviously the results that we obtain are as good or even better since our results depend on the state equation that is imposed (see\textsuperscript{9} and\textsuperscript{10}). The trick is based in integrating the equation \(\rho' + 3(\rho + p)\frac{f'}{f} = 0\) from which we obtain a dimensional constant \(A_\omega\) indispensable for our trifling count. With this constant the set of governing parameters is: \(\mathfrak{M} = \mathfrak{M}(t, c, A_\omega)\). With these quantities in a dimensional base \(\mathfrak{B} = \{L, M, T\}\) the problem remains perfectly determined with not further conditions (see\textsuperscript{9} and\textsuperscript{10}).

2. \textit{Not so simple Method}

In this section we shall study several of the possibilities that may arise.

\textbf{a. Considering} \(\text{div}(T^j_i) = 0\). We note in the equation

\[
\rho' + 3(\omega + 1)\rho \frac{f'}{f} = - \left[ \frac{\Lambda' c^4}{8\pi G} + \frac{\rho G'}{G} \right]
\]

if the conservation principle for the energy-momentum tensor is taken into account, from the part \((A1)\) of the equation we obtain the well-known relationship \(\rho = A_\omega f^{-3(\omega + 1)}\). Regarding to the second term of the equation \((A2)\) from it, we can extract a \(\pi\)-monomial \(\pi_1 = \Lambda c^2 t^2\) that we may express through the following equality \(\Lambda = \frac{d}{c^2 t^2}\) where \(d \in \mathbb{R}\) i.e. it is a pure number. This \(\pi\)-monomial is replaced into the equation in the following way:

\[
\frac{dc^2}{4\pi t^3} = \frac{A_\omega G'}{f^{3(\omega + 1)}}
\]

From this equation we can obtain another \(\pi\)-monomial, \(\pi_2 = f c^{-1} t^{-1}\) that we may express as \(f = act\) where \(a \in \mathbb{R}\) is a numerical constant. With this new relationship we simplify our last equation, yielding:

\[
\frac{dc^2}{4\pi t^3} = \frac{A_\omega G'}{(act)^{3(\omega + 1)}}
\]

that also reads:

\[
G' = \frac{a^{3(\omega + 1)} dc^{5 + 3\omega}}{4\pi A_\omega t^{3\omega}}
\]
whose trivial integration is:

\[ G = \frac{a^{3(\omega+1)} d}{4\pi} \frac{c^{5+3\omega}}{A_\omega} t^{3\omega+1} \]  

This result has already been obtained through the simplest technique, except the numerical constant \( \frac{a^{3(\omega+1)} d}{4\pi} \).

\[ \text{b. Regardless of } \text{div}(T_{ij}) = 0. \] If we do not take into account the condition \( \text{div}(T_{ij}) = 0 \) we shall tackle the problem in the following way:

\[ \rho' + 3(\omega + 1) \rho \frac{f'}{f} + \frac{N' c^4}{8\pi G} + \rho \frac{G'}{G} = 0 \]  

(18)

directly we obtain two pi-monomials: \( \pi_1 = \Lambda c^2 t^2 \) and \( \pi_2 = f c^{-1} t^{-1} \) that we replace into the equation:

\[ \rho' + 3(\omega + 1) \rho \frac{1}{t} - \frac{dc^2}{4\pi t^3 G} + \rho \frac{G'}{G} = 0 \]  

(19)

that we cannot simplify more. At this time, we note that with these hypothesis we do not really get anything important.

\[ \text{c. Considering the conditions } G = \beta_\alpha t^\alpha \text{ and } \text{div}(T_{ij}) = 0. \] If we take into account the hypothesis imposed by Kalligas et al (see\(^6\)) on the behavior of \( G = C t^\alpha \) with \( \alpha \in \mathbb{R} \) the dimensional treatment would be this: first, we define correctly the (dimensional) constant that establishes the proportionality between \( G \) and \( t \), \( G = \beta_\alpha t^\alpha \) in such a way that \( \beta_\alpha \) has the following dimensions: [\( \beta_\alpha \)] = \( L^3 M^{-1} T^{-2-\alpha} \). As above we shall tackle the equation in the same way:

\[ \rho' + 3(\omega + 1) \rho \frac{f'}{f} = - \left[ \frac{N' c^4}{8\pi G} + \rho \frac{G'}{G} \right] \]  

(20)

from (A1) we obtain \( \rho = A_\omega f^{-3(\omega+1)} \) and from (A2) taking into account \( \pi_1 = \Lambda c^2 t^2 \) and \( G = \beta_\alpha t^\alpha \) we obtain the following relationship

\[ \rho G' = \frac{dc^2}{4\pi t^3} \quad \rho = \frac{dc^2}{4\pi \alpha \beta_\alpha} \frac{1}{t^{\alpha+2}} \]  

(21)

and therefore \( f \)

\[ f = \left( \frac{A_\omega \beta_\alpha}{c^2} \right)^{\frac{1}{\pi+1}} t^{\frac{2+\alpha}{\pi+1}} \]  

(22)

as we have seen, through the dimensional method the same results can be obtain but in an easier way.
d. **Considering only the condition** \( G = \beta_\alpha t^\alpha \). Under this hypothesis we can also outline the problem without considering the condition \( \text{div}(T_{ij}) = 0 \). With these suppositions the equation to solve is:

\[
\rho' + 3(\omega + 1)\rho \frac{f'}{f} + \rho \frac{\alpha}{t} + \frac{\Lambda^4}{8\pi \beta_\alpha t^\alpha} = 0
\]  

(23)

if furthermore, \( \pi_1 = \Lambda c^2 t^2 \) and \( \pi_2 = \rho_\beta_\alpha t^{\alpha+2} c^{-2} \) are taken into account then:

\[
-\frac{(2 + \alpha)gc^2}{\beta_\alpha t^{\alpha+3}} + 3(\omega + 1) \left( \frac{gc^2}{\beta_\alpha t^{\alpha+2}} \right) \frac{f'}{f} + \left( \frac{gc^2}{\beta_\alpha t^{\alpha+2}} \right) \frac{\alpha}{t} - \frac{dc^2}{4\pi \beta_\alpha t^{\alpha+3}} = 0
\]

(24)

where \( g \) and \( d \) are numerical constants. Simplifying

\[
-\frac{(2 + \alpha)g}{t} + 3(\omega + 1)gH + \frac{\alpha g}{t} - \frac{d}{4\pi t} = 0
\]

integrating

\[
f = K_\chi t^\chi
\]

(25)

where \( \chi = \frac{d + 4\pi g(\alpha + 1)}{3(\omega + 1)g} \) and \( K_\chi \) is a constant of proportionality with dimensions \( [K_\chi] = LT^{-\chi} \).

As we see, the recipe is always the same. First, consider the equation to be integrated obtaining from it the highest number of pi-monomials, in order to decrease the number of variables, in such a way that an easily integrable equation is obtained.

### III. CASE \( G \) AND \( \Lambda \) VARIABLE FOR A VISCOUS FLUID.

This problem was posed by Arbab (see\( ^{11} \)). The basic ingredients of the model are:

The momentum-energy tensor defined by:

\[
T_{ij} = (\rho + p^*)u_iu_j - pg_{ij}
\]

where \( \rho \) is the energy density and \( p^* \) represents pressure \([\rho] = [p^*]\). The effect of viscosity is seen in:

\[
p^* = p - 3\xi H
\]

(26)

where: \( p \) is the thermostatic pressure, \( H = (f'/f) \) and \( \xi \) is the viscosity coefficient that follows the law:

\[
\xi = k_\gamma \rho^\gamma
\]

(27)
where $k, \gamma$ makes the equation be homogeneous i.e. it is a constant with dimensions and where the constant $\gamma \in [0, 1]$. And $p$ also verifies the next state equation:

$$p = \omega \rho \quad \omega = \text{const.} \quad (28)$$

where $\omega \in [0, 1]$ (i.e. it is a pure number) so that the momentum-energy tensor verifies the so-called energy conditions.

The field equations are:

$$2\frac{f''}{f} + \left(\frac{f'}{f}\right)^2 = -\frac{8\pi G(t)}{c^2}p^* + c^2\Lambda(t) \quad (29)$$

$$3\frac{(f')^2}{f^2} = \frac{8\pi G(t)}{c^2}c^2\Lambda(t) \quad (30)$$

deriving (30) and simplifying with (29) it yields

$$\rho' + 3(\omega + 1)\rho H - 9k, \gamma H^2 + \frac{Nc^4}{8\pi G} + \rho \frac{G'}{G} = 0 \quad (31)$$

and at the moment we consider this equation.

$$\text{div}(T^i_j) = 0 \Leftrightarrow \rho' + 3(\rho + p^*)\frac{f'}{f} = 0 \quad (32)$$

if we develop the equation (32) we get:

$$\rho' + 3(\omega + 1)\rho H - 9k, \gamma H^2 = 0 \quad (33)$$

As in the case above, we shall study the model in two ways, one of them analytic (non dimensional) and the other dimensional.

A. Non Dimensional Method.

In this section we will mainly follow Singh et al work (see\(^{12}\)). If we take the equation (31) regrouped, we get:

$$\underbrace{\rho' + 3(\omega + 1)\rho H - 9k, \gamma H^2}_{A_1} = - \left[ \rho \frac{G'}{G} + \frac{Nc^4}{8\pi G} \right] \quad (34)$$

if take into account the conservation principle

$$\rho' + 3(\omega + 1)\rho H - 9k, \gamma H^2 = 0 \quad (35)$$
then we solve this equation by solving the equation $A^2$ in (34), in such a way that the equation to be solved is now:

$$\left[ \rho \frac{G'}{G} + \frac{Nc^4}{8\pi G} \right] = 0$$  \hspace{1cm} (36)

this equation is tried to be solved like this (see12). $\Lambda = \frac{3\beta H^2}{c^2}$ is defined (hypothesis by Arbab (see11) as well as by Singh et al (see12), condition that as we shall see, is not necessary to impose in the solution through D.A.) and from the equation (30) the following relationship is obtained: $8\pi G \rho = 3(1 - \beta)H^2$. Therefore if all the equalities are replaced in the equation (36) it yields:

$$\frac{2}{(1 - \beta)} \frac{H'}{H} = \frac{\rho'}{\rho}$$  \hspace{1cm} (37)

which is easily integrated.

$$H = C_1 \rho^{1/d} \quad d = \frac{2}{(1 - \beta)}$$  \hspace{1cm} (38)

we get to the equation (35) with all these results

$$\rho' + 3(\omega + 1)\rho H - 9k\gamma H^2 = 0$$

we arrive to the next equation:

$$\rho' + 3C_1(\omega + 1)\rho^{\frac{d+1}{d-\gamma}} - 9C_1^2k\gamma \rho^{\frac{d+2}{d-\gamma}} = 0$$  \hspace{1cm} (39)

which has got a particular solution in the case $\gamma = d^{-1}$ obtaining:

$$\rho(t) = \frac{1}{(a_0t)^d} \quad / \quad a_0 = (3C_1(\omega + 1) - 9k\gamma C_1^2) \quad d^{-1}$$

and obtaining from it:

$$f(t) = C_2t (3(\omega + 1) - 3k\gamma) (1 - \gamma)$$

This is the most developed solution reached by Singh et al (see12) which is slightly different from the one by Arbab (see11).

B. Dimensional Method.

We shall explore two dimensional methods in this section. The first one, probably the simplest one, has the inconvenience of having to depend on Einstein criterion(see13 and Barenblatt2), while the second one is more powerful and more elaborated. We shall finish showing an equation obtained without having to impose the condition $\text{div}(T^j_1) = 0$. 

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1. **Naive Method.**

We summarized here the method addressing to the reference \(^{(14)}\) to know more deeply the followed method. In this model the set of quantities and constants \(\mathfrak{M}\) remain reduced to \(\mathfrak{M} = \mathfrak{M}(t, c, A, k_\gamma)\) while the dimensional base continues being \(\mathfrak{B} = \{L, M, T, \theta\}\). The followed dimensional method leads us to the obtainment of two dimensionless \(\pi\)-monomials \(\pi_1 = \varphi(\pi_2)\) where \(\varphi\) represents a unknown function. To arrive to a satisfactory solution i.e. to get rid of such function \(\varphi\), we shall have to take into account the criteria of Einstein or Barenblatt. With this criterion we obtain the same results as the already existing ones in the literature, but let us say, in a trivial way (for more details see\(^{(14)}\)).

2. **Not so simple Method**

In this section we shall combine dimensional techniques with standard techniques of ODEs integration. With the dimensional method we shall obtain dimensionless monomials, which will be replaced in the equations. Thus, the number of variables will be reduced in such way that the resulting equation is integrable in a trivial way. We study two cases, the first one in which we consider \(\text{div}(T_i^j) = 0\), while in the other, as we shall see, such hypothesis is not needed.

a. **Considering the condition** \(\text{div}(T_i^j) = 0\). In this case we shall pay attention into the equation:

\[
\rho' + 3(\omega + 1)\rho H - 9k_\gamma\rho^2H^2 + \rho' G' + \frac{N'c^4}{8\pi G} = 0
\]

taking into account the relationship \(\text{div}(T_i^j) = 0\) The following equality is brought up:

\[
\rho' + 3(\omega + 1)\rho H - 9k_\gamma\rho^2H^2 = -\left[\rho\frac{G'}{G} + \frac{N'c^4}{8\pi G}\right]_{A1} \quad \text{and} \quad \rho' + 3(\omega + 1)\rho H - 9k_\gamma\rho^2H^2 = -\left[\rho\frac{G'}{G} + \frac{N'c^4}{8\pi G}\right]_{A2}
\]

(40)

The idea is the following: By using D.A. we obtain two \(\pi\)-monomials, which are replaced in the equation, obtaining a huge simplification of it. On the other hand we integrate \((A1)\) and \((A2)\), solving the problem completely in this way, now without Barenblatt’s criterion. Let see. The monomials obtained are: \(\pi_1 = \rho\frac{1}{k_\gamma}t^{-\frac{1}{\gamma}}\) and \(\pi_2 = \Lambda c^2t^2\) i.e.

\[
\rho = ak_\gamma^{-\frac{1}{\gamma}}t^{-\frac{1}{\gamma}} \quad \Lambda = \frac{d}{c^2t^2}
\]
where \( a \) and \( d \) are numerical constants. In a generic way the solution is of the following way: 
\[
\rho = a k_{\gamma}^{b-1} t^{\frac{1}{\gamma-1}}
\]
if we define \( b = \frac{1}{1-\gamma} \) then \( \rho = a k_{\gamma}^{b} t^{-b} \) where \( a = \text{const.} \in \mathbb{R} \) then 
\[
\rho' = -b a k_{\gamma}^{b} t^{-b-1}
\]
(paying attention only to the term \((A1)\) of the equation) it yields:
\[
-b a k_{\gamma}^{b} t^{-b-1} + 3(\omega + 1) a k_{\gamma}^{b} t^{-b} H - 9 k_{\gamma} (a k_{\gamma}^{b} t^{-b})^\gamma H^2 = 0
\]
that simplifying it is reduced to:
\[
9 a^{(\gamma-1)} (f')^2 - 3 \omega t^{-1} f f' + b t^{-2} f^2 = 0
\]
(42)
\[
f' = \frac{f}{t} \left[ \frac{1}{6a^{\gamma-1}} \left( w \pm (w^2 - 4ba^{\gamma-1})^{\frac{1}{2}} \right) \right]
\]
(43)
where \( w = (\omega + 1) \), if define
\[
D = \left[ \frac{1}{6a^{\gamma-1}} \left( w \pm (w^2 - 4ba^{\gamma-1})^{\frac{1}{2}} \right) \right]
\]
(44)
then, the solution has the following form:
\[
f = l B t^D
\]
(45)
where \( l \) is a certain numerical constant and \( B \) is a integration constant with dimensions, that can be identified with our result by making \( B = A_{\omega} k_{\gamma} \).

Now we shall solve the other term of the equation (the \( A2 \)). the equation \( \left[ \rho \frac{G'}{G} + \frac{\Lambda c^4}{8\pi G} \right] = 0 \) (36)) can be solved in a trivial way if we follow the next results. If we replace the monomials \( \pi_1 = \rho k_{\gamma}^{b-1} t^{-b} \) and \( \pi_2 = \Lambda c^2 t^2 \) in such equation the integration of it becomes trivial:
\[
\rho k_{\gamma}^{b-1} t^{-b} \left( \frac{G'}{G} \right) - \frac{dc^2}{4\pi G t^3} = 0
\]
\[
G' = \frac{dc^2}{a 4\pi k_{\gamma}^{b-3}} \implies G(t) = g \frac{dc^2}{a 4\pi k_{\gamma}^{b-2}} t^{b-2}
\]
(46)
where \( a, d \) and \( g \in \mathbb{R} \) (they are pure numbers). We can also observe that this integral needs not be solved since a more careful analysis on the number of \( \pi \)–monomials that we can obtain from the equation, leads us to obtain a solution of the type:
\[
G = G(k_{\gamma}, c, t)
\]
which brings us to:
\[
G(t) = g k_{\gamma}^{-b} c^2 t^{b-2}
\]
(47)
This method, as we have seen, is more prepared (strived) and the solution, therefore, finer though coincident with the previous one.
b. **Case in which** div$\left(T^j_i\right) = 0$ **is not considered.** Let see now how we can tackle (broach) this problem from the D.A. point of view, without imposing the condition div$\left(T^j_i\right) = 0$. The base $\mathfrak{B}$ as before, is still $\mathfrak{B} = \{L, M, T\}$ while the fundamental set of quantities and constant this time is $\mathfrak{M} = \mathfrak{M}(t, c, k_{\gamma})$, with these data we can obtain the following monomials from the equation

$$
\rho' + 3(\omega + 1)\rho H - 9k_{\gamma}\rho^7H^2 + \rho \frac{G'}{G} + \frac{N'c^4}{8\pi G} = 0
$$

(48)

considering that:

$$
\rho = ak_{\gamma}^{\frac{1}{7}}t^{\frac{1}{7}} \quad \Lambda = \frac{d}{c^2t^2}
$$

(49)

this two monomials are replaced into the equation, which is quite simplified:

$$
-bak_{\gamma}^b t^{-b-1} + 3(\omega + 1)ak_{\gamma}^b t^{-b} H - 9k_{\gamma} (ak_{\gamma}^b t^{-b})^7 H^2 + ak_{\gamma}^b t^{-b} \frac{G'}{G} - \frac{dc^2}{4\pi Gt^3} = 0
$$

(50)

simplifying this equation, it yields:

$$
-9a^{(\gamma-1)} tH^2 + 3wH - bt^{-1} + \frac{G'}{G} - \frac{dc^2}{4\pi ak_{\gamma}^b} \frac{t^{b-3}}{G} = 0
$$

(51)

that along with the field equations (29) and (30) carry us to the next set of equations. For example we see:

$$
3H^2 = a \frac{8\pi G}{c^2} k_{\gamma}^b t^{-b} + \frac{d}{t^2}
$$

that we replaced into the equation that we are treating (dealing with), resulting:

$$
-bt^{-1} + 3w \left( \frac{a8\pi k_{\gamma}^b}{3c^2} Gt^{-b} + \frac{d}{3t^2} \right)^{\frac{1}{2}} -
$$

$$
-9a^{(\gamma-1)} \left( \frac{a8\pi k_{\gamma}^b}{3c^2} Gt^{-b} + \frac{d}{3t^2} \right) t + \frac{G'}{G} - \frac{dc^2}{4\pi ak_{\gamma}^b} \frac{t^{b-3}}{G} = 0
$$

that solving it results:

$$
G = gk_{\gamma}^{-b}c^2t^{b-2}
$$

(52)

where $g \in \mathbb{R}$ represents a numerical constant. We finally observe that as in the previous section we could have taken into account the three monomials obtained from the equation i.e.

$$
\rho = ak_{\gamma}^b t^{-b} \quad \Lambda = \frac{d}{c^2t^2} \quad G = g \frac{c^2t^{b-2}}{k_{\gamma}^b}
$$
replace them into the equation

$$\rho' + 3(\omega + 1)\rho H - 9k\gamma\rho^\gamma H^2 + \rho \frac{G'}{G} + \frac{N'c^4}{8\pi G} = 0$$

and calculate $f$, arriving at the same solution obtained in the section above i.e.

$$f = lBt^D$$

We have proved that it is not necessary to impose the condition $\text{div}(T^j_i) = 0$ since, in this case, the same solution is obtained as imposing it.

**IV. CASE $G, C$ AND $\Lambda$ VARIABLE AND VISCOS FLUID.**

This model has been developed exclusively through dimensional techniques, therefore, at this time no other solution is known, except the one showed in section 4.2. We can consider that this model is a natural generalization of the previous one (see\textsuperscript{15}).

Following the Alberch and Magueijo work and therefore all the assumptions considered there, we arrive to the requirement that the standard equations still retain their form but with $G(t), c(t)$ and $\Lambda(t)$ varying (see\textsuperscript{16}).

The field equations that describe our viscous model are (in this section we shall take into account the hypothesis made in the above section):

$$2f'' f + \left(\frac{f'}{f}\right)^2 = -\frac{8\pi G(t)}{c^2(t)} p^* + c^2(t)\Lambda(t)$$

(53)

$$3\left(\frac{f'}{f}\right)^2 = \frac{8\pi G(t)}{c^2(t)} \rho + c^2(t)\Lambda(t)$$

(54)

developing equation (2) i.e. calculating $(\frac{8\pi G}{c^2}T^j_i + \Lambda\delta^j_i)_{,j}$ in this case it yields:

$$\rho' + 3(\omega + 1)\rho H - 9k\gamma\rho^\gamma H^2 + \frac{N'c^4}{8\pi G} + \rho \frac{G'}{G} - 4\frac{c'}{c^3}\rho = 0$$

(55)

besides the so-called conservation principle for the energy-momentum tensor is taken into account

$$\text{div}(T^j_i) = 0 \iff \rho' + 3(\rho + p^*) \frac{f'}{f} = 0$$

(56)

Now, if we develop the equation (56) it is obtained:

$$\rho' + 3(\omega + 1)\rho H - 9k\gamma\rho^\gamma H^2 = 0$$

(57)
Under these circumstances the equation to be treated is:

$$\rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^7 H^2 = - \left[ \frac{N'c^4}{8\pi G} + \frac{G'}{G} - \frac{4c'}{c}\rho \right]$$  \hspace{1cm} (58)

We cannot play the same trick as before since in this case the “constant” \(c \mapsto c(t)\) varies and therefore the equation are much more complicated. We shall show here two methods: the naive and the elaborated one (not so simple).

### A. Naive Method.

Since in this case we are considering the “constant” \(c\) as variable, the set of governing quantities remains reduced to \(\mathcal{M} = \mathcal{M}(t, k_\gamma, A_\omega)\) while the dimensional base is still \(\mathcal{B} = (L, M, T)\). The obtained solutions in this case are perfectly definite since a single \(\pi\)–monomial is obtained. To see more details about this technique we address to reference\(^{15}\).

### B. Not so simple Method.

In this occasion the equation to solve is (considering the condition \(\text{div}(T_{ij}) = 0\), as we cannot get rid of it):

$$\rho' + 3(\omega + 1)\rho H - 9k_\gamma \rho^7 H^2 = - \left[ \frac{N'c^4}{8\pi G} + \frac{G'}{G} - \frac{4c'}{c}\rho \right]$$  \hspace{1cm} (59)

This equation, is very similar to equation (40). We shall follow in this paragraph the tactics developed in section 3.2.2. On the one hand, we already know how to tackle the left side of equation (59) i.e. the \(A1\) side. If we take into account these results and imposing the following condition \(\Lambda = \frac{d}{c^2(t)t^2}\) (that might be unfounded) we can simplify this equation:

$$-bt^{-1} + 3wH - 9a(\gamma^{-1})tH^2 + \frac{G'}{G} - \frac{2dc'[c't + c]}{8\pi ak_b^b} \frac{t^{b-3}}{G} - \frac{4c'}{c} = 0$$

with the usual notation that we are following. From one of the field equations expression for \(G\) has been obtained.

$$G = \frac{c^2}{8\pi ak_b^b t^{b-5}} \left[ 3H^2 - \frac{d}{t^2} \right]$$

also \(f\) is a previous result from last section:

$$f = tBt^{D}$$
We should not forget that (\(A1\)) from equation (59) is the same as the one studied in section 3.2.2. (40) and \(\rho\) makes no difference. With this expressions we continue our simplification.

Since \(H^2 = \frac{D^2}{t^2}\) then

\[
G = \frac{(3D^2 - d)c^2}{8\pi ak^b} t^{b-2}
\]

(60)

We observe again that \(c = c(t)\).

\[
G' = \frac{2(3D^2 - d)cc'}{8\pi ak^b} t^{b-2} + (b - 2) \frac{(3D^2 - d)c^2}{8\pi ak^b} t^{b-3}
\]

\[
G' = 2 \frac{c'}{c} + \frac{b - 2}{t}
\]

with this results we go on to simplify the term (\(A2\)) in equation (59) obtaining

\[
\frac{G'}{G} + \frac{2dc [c't + c]}{8\pi ak^b} \frac{t^{b-3}}{G} - 4 \frac{c'}{c} = 0
\]

\[
2 \frac{c'}{c} + \frac{b - 2}{t} + \left[ \frac{-2d}{3(D^2 - d)} \right] \left( \frac{c'}{c} + \frac{1}{t} \right) - 4 \frac{c'}{c} = 0
\]

\[
\frac{c'}{c} = \left( \frac{b}{2} - 1 - \frac{bd}{6D^2} \right) \frac{1}{t}
\]

\[
c = K_\chi t^\chi
\]

(61)

where \(\chi = \left( \frac{b}{2} - 1 - \frac{bd}{6D^2} \right)\), and \(K_\chi\) is the proportionality constant that relates \(c\) with \(t\) with dimensions \([K_\chi] = LT^{-1-\chi}\). From this result the expression of \(G\) is obtained:

\[
G = \frac{(3D^2 - d)K^2_\chi}{8\pi ak^b} t^{b+2+2\chi}
\]

(62)

With this solution, it is observed for the case \(\gamma = \frac{1}{2}\) what corresponds to \(b = 2\)

\[
\frac{G}{c^2} = \frac{t^{b-2+2\chi}}{t^{2\chi}} = \text{const.}
\]

(63)

i.e. the covariance principle is verified. In particular if \(b = 2\) then \(\chi = \left( \frac{-d}{3D^2} \right)\).

While the cosmological “constant” yields \(\left( \Lambda = \frac{d}{c^2(t)t^2} \right)\):

\[
\Lambda = \frac{d}{c^2(t)t^2} = \frac{d}{K^2_\chi t^{2\chi+2}}
\]

(64)

In this way the remaining quantities are calculated.
V. CASE OF $G, C$ AND $\Lambda$ VARIABLE WITH ADIABATIC MATTER CREATION.

The momentum-energy tensor is defined by:

$$T_{ij} = (\rho + p^* + p_c)u_iu_j - (p^* + p_c)g_{ij} \quad (65)$$

where $\rho$ is the energy density and $p^*$ represents pressure $[\rho] = [p^*]$. The effect of the viscosity is given by:

$$p^* = p - 3\xi H$$

where: $p$ is the thermostatic pressure, $H = (f'/f)$ and $\xi$ is the viscosity coefficient that follows the law:

$$\xi = k_\gamma \rho^{\gamma} \quad (66)$$

The field equations are as it follows:

$$2\frac{f''}{f} + \frac{(f')^2}{f^2} = -8\pi G(t)\frac{(p^* + p_c)}{c^2(t)} + c^2(t)\Lambda(t) \quad (67)$$

$$\frac{(f')^2}{f^2} = \frac{8\pi G(t)}{3c^2(t)}\rho + c^2(t)\Lambda(t) \quad (68)$$

$$n' + 3nH - \psi = 0 \quad (69)$$

where $n$ measures the particles number density, $\psi$ is the function that measures the matter creation, $H = f'/f$ represents the Hubble parameter ($f$ is the scale factor that appears in the metric), $p$ is the thermostatic pressure, $\rho$ is energy density and $p_c$ is the pressure that generates the matter creation.

The creation pressure $p_c$ depends on the function $\psi$. For adiabatic matter creation this pressure takes the following form:

$$p_c = -\left[\frac{\rho + p}{3nH\psi}\right] \quad (70)$$

The state equation that we next use is the well-known expression

$$p = \omega \rho \quad (71)$$

where $\omega = const. \; \omega \in [0, 1]$ physically realistic equations, making in this way that the energy-momentum tensor $T_{ij}$ verifies the energy conditions.
We need to know the exact form of the function $\psi$, which is determined from a more fundamental theory that involves quantum processes. We assume that this function follows the law:

$$\psi = 3\beta nH$$  \hspace{1cm} (72)

where $\beta$ is a dimensionless constant (if $\beta = 0$ then there is no matter creation since $\psi = 0$) presumably given by models of particles physics of matter creation.

The generalized principle of conservation brings us to:

$$\rho' + 3(\omega + 1)(1 - \beta)\rho H - 9k_\gamma \rho^\gamma H^2 = -\left[\frac{\Lambda' c^4}{8\pi G} + \rho\frac{G'}{G} - 4\frac{c'}{c}\rho\right]$$  \hspace{1cm} (73)

The conservation principle for the momentum-energy tensor is expressed through the following law:

$$\rho' + 3(\rho + p + p_c - 3\xi H)H = 0$$

$$\rho' + 3(\omega + 1)(1 - \beta)\rho H - 9k_\gamma \rho^\gamma H^2 = 0$$  \hspace{1cm} (74)

This equation has been solved in the above section since the only difference between this one and the one exposed there is the term $(1 - \beta)$ and the solution will be very similar. The dimensional solution can be found in the reference (17).

VI. CONCLUSIONS.

The purpose of this work has been to show how to solve this type of models through dimensional techniques. We have been able to prove that this technic is applied in a comparatively easy way enabling us to obtain almost trivial solutions and allowing to avoid hypotheses needed when using other methods.

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