Generalized LKF transformations for $N$-point fermion correlators in QED

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(Dated: February 1, 2022)

Within the worldline approach to quantum electrodynamics (QED), a change of the photon’s covariant gauge parameter $\xi$ is investigated to analyse the non-perturbative gauge dependence of the configuration space fermion correlation functions, deriving a generalization of the Landau-Khalatnikov-Fradkin transformations (LKF). These transformations reveal how the non-perturbative gauge dependence of position space amplitudes can be absorbed into a multiplicative exponential factor.

Keywords: Quantum Field Theory, LKF Transformations, Gauge Symmetry, Worldline Formalism

I. INTRODUCTION

The LKFt, derived by Landau and Khalatnikov [1], and independently by Fradkin [2], are non-perturbative transformations which relate field theory propagators (or vertices) calculated in different covariant gauges, originally phrased in position space. The transformation was derived first in QED and then generalized to correlation functions in Quantum Chromodynamics (QCD) [3, 4]. In particular cases, as in scalar QED in 3-dimensions, it is possible to obtain exact LKFt in momentum space [3].

Their applications range from non-perturbative analyses, implying constraints for a non-perturbative ansatz for the fermion-boson vertex [6] in the context of the Schwinger-Dyson equations; to perturbative ones, such as to obtain information on Feynman diagrams at higher loop orders, starting from an amplitude at a fixed loop level [4, 7]. The extra information at higher loops is always gauge dependent, and is such that the propagator at fixed loop order calculated in some gauge (e.g. Landau gauge, with gauge parameter $\xi = 0$), determined to order $O(\alpha^n)$ in the fine structure constant, fixes – through the LKFt – the coefficients of the contributions at order $\alpha^i \mu \xi^j$, with $i = 0, 1, ..., j = 0, 1, ..., n$. [8].

In 2016, a generalized LKFt was derived for the case of $N (= 2n)$-point correlator functions [9] in scalar QED, applying the worldline formalism (for reviews see [10, 11]), an alternative formulation of quantum field theory based on first quantised particle path integrals [12]. In this contribution we recap more recent work extending and improving this to spinor QED first presented in [13].

II. THE DRESSED PROPAGATOR

In section II of [14], also in these proceedings, the worldline representation of the fermion propagator, $S^{\tau \bar{\tau}} = \langle x' | [m - iD]^{-1} | x \rangle$, in an electromagnetic field, $A_\mu$, is discussed – see also [15, 16]. In the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ the field is specialized to plane waves,

$$A_\mu^\tau(x) = \sum_{i=1}^N \varepsilon_{i\mu} e^{ik_i \cdot x},$$

(1)

to describe external photons scattering off the fermion line, with definite momenta, $\vec{k}_i$, and polarizations, $\varepsilon_{i\mu}$.

A. supersymmetric invariance in the wordline

Noting that the worldline action $S[x, \psi, A^\tau]$ described in (3) of [14] is invariant under the supersymmetric transformations on the worldline (for Grassmann variable $\zeta$)

$$\delta x^\mu = - 2 \zeta \psi^\mu, \quad \delta \psi^\mu = \zeta \partial^\mu,$$

(2)

we can rewrite that action in a more compact form in terms of the superfield and superderivative

$\Xi^\mu(\tau, \theta) = x^\mu(\tau) + \sqrt{2} \theta (\psi^\mu(\tau) + \eta^\mu),$ \quad $\mathbb{D} = \partial_\theta - \theta \partial_\tau.$

(3)

Here $\theta$ is a Grassmann parameter which extends the parameter domain, $\tau \rightarrow \tau \theta$. Then the action is written:

$$S[x, \psi, A^\tau] = \int_0^T d\tau \int d\theta \left[ - \frac{1}{4} \Xi \cdot \mathbb{D} \Xi - ieA[\Xi] \cdot \mathbb{D} \Xi \right] ,$$

(4)
B. Interaction with virtual photons

The interaction with a quantum field, \( \hat{A}_\mu \), can be included by splitting the field \( A = A^{\pm} + \hat{A} \), and integrating over the field \( \hat{A} \) (the background field method [14])

\[
Z_{\hat{A}} = \int D\hat{A} e^{-\int d^Dx(-\frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) - S_{gf}(\xi)} \tag{5}
\]

with \( A^{\pm} \) the external photon field and where \( S_{gf}(\xi) \) is the gauge-fixing action depending on gauge parameter \( \xi \)

\[
S_{gf} = -\int d^4x \frac{(\partial \cdot \hat{A})}{2\xi} \tag{6}
\]

The inclusion of the quantum field makes the propagator in the covariant gauge \( \xi \) become

\[
\mathcal{S}^{x|x}(\xi) = \left[ m + i \hat{D}' + i \frac{\delta}{\delta \bar{J}[x\xi]} \right] \left[ e^{\int d^Dx J(x)\hat{A}[x]} K^{x|x} \right] \bar{\hat{A}}(\xi) = 0, \tag{7}
\]

where the insertion of \( \bar{A} \) in the pre-factor of the kernel, in (1) of [14], is now generated by the functional derivative of the source, \( J[x] \), in \( e^{\int d^Dx J(x)\hat{A}[x]} \). After evaluating the expression on \( J = 0 \), the term \( S_t \) represents the interaction with virtual photons, and is given by

\[
S_t = \frac{e^2}{2} \int d^Dy d^Dy' J(y) \cdot G(y - y'; \xi) \cdot J(y'), \tag{8}
\]

\[
J = J^\mu(y) + \int_0^T dt \int d\theta \, \delta^D(y - X_t) \Xi, \tag{9}
\]

where \( G_{\mu\nu}(y; \xi) \) is the configuration space photon propagator in dimensions in covariant gauge \( \xi \).

III. N-POINT FERMION CORRELATOR

The \( N(=2n) \)-point correlation function with covariant gauge parameter \( \xi \), defined by \( S(x_1 \ldots x_n; x'_1 \ldots x'_n) = \langle \psi(x_1) \ldots \psi(x_n) \psi(x'_1) \ldots \psi(x'_n) \rangle \) relates to the partial one, \( S_{\pi} \), by summing over all pairings (contractions) of the fields \( \psi(x_i) \) with the \( \psi(x'_j) \):

\[
S(x_1 \ldots x_n; x'_1 \ldots x'_n) = \sum_{\pi \in S_n} S_{\pi}(x_1 \ldots x_n; x'_{\pi(1)} \ldots x'_{\pi(n)}) \tag{10}
\]

where the fermion lines in \( S_{\pi} \) go from \( x_i \) to \( x'_{\pi(i)} \). The partial \( N \)-point function in the presence of an electromagnetic field, \( A^{\mu}_{\pm} \), and including virtual photons, is

\[
S_{\pi}(x_1 \ldots x_n; x_{\pi(1)} \ldots x_{\pi(n)}) = \left\langle \prod_{i=1}^n S^x_{\pi(i)\pi(i)} \right\rangle \bar{A}_{\xi}, \tag{11}
\]

with the fermion propagator, \( S^x_{\pi(i)\pi(i)} \), as in (1) of [14], but now with \( A^{\mu}_{\pm} \) instead of just \( A^{\mu}_{\pi} \). The worldline representation of the \( N \)-point partial amplitude is

\[
S_{\pi}(x_1 \ldots x_n; x_{\pi(1)} \ldots x_{\pi(n)}) = \prod_{j=1}^n \left[ \begin{array}{c} m + i \delta j' + i \frac{\delta}{\delta \bar{J}[x\xi]} \end{array} \right] \prod_{j=1}^n \left( \begin{array}{c} K_{\pi(i)\pi(j)} \end{array} \right) \bar{A}_{\xi} = 0 \tag{12}
\]

\[
= \prod_{j=1}^n \left( \begin{array}{c} m + i \delta j' + i \frac{\delta}{\delta \bar{J}[x\xi]} \end{array} \right) \prod_{j=1}^n 2^{-\frac{D}{2}} \text{symb}^{-1} \int_0^\infty dT_j e^{-m^2 T_j} \int_{x_j(0) = x_j}^{x_j(T_j) = x_j} D\bar{x}_j(\tau_j) \int_{\text{APC}} D\psi_j(\tau_j) e^{-\sum_{i} S^{(i)}[\xi_i]} - S_{\pi} \bar{A}_{\xi} \tag{13}
\]

where \( S^{(i)}[\xi_i] \) is the action [11] of the fermion line \( l \) and \( S_{\pi} \) as expressed in [8], but with a sum over lines

\[
J^\mu(y) = J^\mu(y) + \sum_{l=1}^T \int dt_1 \int d\theta_1 D^D(y - X_t) \Xi, \tag{14}
\]

where we used (2) of [14] for the kernel \( K^{x|x}[k_1, \xi_1; \ldots; k_N, \xi_N | A^{\mu} + \hat{A}] \).

IV. THE GENERALIZED LKFT IN SPINOR QED

Interactions with external photons can be expressed in terms of the vertex operator in (5) of [14]. Then, a gauge transformation can be done by making the replacement \( \epsilon_{\mu} \rightarrow \epsilon_{\mu} + \xi_{\mu} \) in the vertex operator and the on-shell invariance of the amplitude is well understood by the Ward-Takahashi identity. So the non-trivial transformation properties of the \( N \)-point correlation functions only require analysis of the gauge transformation of virtual photons under a variation in gauge parameter \( \xi \).

A gauge transformation of virtual photons can be realised by sending \( \xi \rightarrow \xi + \Delta \xi \) in the photon propagator \( G_{\mu\nu}(y; \xi) \). The transformation properties of the \( N \)-point correlation functions are then determined by how \( S_{\pi} \) transforms under this change. The derivation of this generalized LKFT for spinor QED can be found in [13]. Essentially, the idea is to analyse the expectation value of the product of kernels, \( \prod_{i=1}^n K^{x|x}(A_{\xi}) \), given by

\[
\prod_{j=1}^n 2^{-\frac{D}{2}} \text{symb}^{-1} \int_0^\infty dT_j e^{-m^2 T_j} \int_{x_j(0) = x_j}^{x_j(T_j) = x_j} D\bar{x}_j(\tau_j) \int_{\text{APC}} D\psi_j(\tau_j) e^{-\sum_{i} S^{(i)}[\xi_i]} - S_{\pi} \bar{A}_{\xi} \tag{15}
\]

with \( \xi_{(k,l)} \) as defined in [8], but with \( J = 0 \). Its variation
due to a gauge transformation of virtual photons is
\[
\Delta \xi S^{(k,l)}_{\pi} = \Delta \xi \frac{\epsilon^2}{32\pi^2} \Gamma \left(\frac{D}{2} - 2\right) \left\{ \left[(x_k - x_l)^2 - D/2\right] - \left[(x_k - x'_\pi(t))^2 - D/2\right] - \left[(x'_\pi(k) - x_l)^2 - D/2\right] + \left[(x'_\pi(k) - x'_\pi(t))^2 - D/2\right] \right\}.
\]
(16)

We highlight that this does not depend on the spinor degrees of freedom, since their interaction is already gauge invariant, and expanding about $D = 4 - 2\epsilon$, at leading order it involves only conformal cross ratios of the $N$ worldline endpoints. Hence this is the same as in scalar QED \[9\] and so the quenched approximation of the QED is sufficient to derive the LKFt, since $\Delta \xi S^{(k,l)}_{\pi}$ is zero for a virtual photon attached to a closed fermion loop $(x_l = x'_\pi(t))$. Then, the product of kernels transforms as
\[
\langle K^{x'_\pi(t), x_1} \cdots K^{x'_\pi(n)-x_2} \rangle_{A, \xi + \Delta \xi}
= \langle K^{x'_\pi(t), x_1} \cdots K^{x'_\pi(n)-x_2} \rangle_{A, \xi} e^{-\sum_{k,l} \Delta \xi S^{(k,l)}_{\pi}},
\]
(17)
where the exponential can be factorised since it only depends on the endpoints of the lines. This transformation shares the multiplicative form of the original LKFt.

The next step is to include the pre-factors of $[m + iD'_{\pi} - eA_\pi]$ in \[13\] it was found that the partial derivatives of the exponential in \[17\] from transforming the kernels cancel with other terms coming from the expected values involving insertions of $A$. So at the end, the transformation rule which arises from this analysis is
\[
S^{\text{LKF}}_{\pi}(x_1 \cdots x_n; x'_\pi(1) \cdots x'_\pi(n)) |\xi + \Delta \xi\rangle,
\]
(18)
where the label “LKF” indicates that the LHS involves the $N$-point amplitude in the gauge $\xi + \Delta \xi$ plus extra gauge dependent parts of higher order diagrams generated by the exponent acting on the original amplitude.

\section{V. Conclusion}

Since $\sum_{k,l} \Delta \xi S^{(k,l)}_{\pi}$ is independent of the permutation, i.e., equal for each partial amplitude, the identical transformation rule is valid for the total amplitude, which defines the complete, generalized LKFt for spinor QED. It has turned out to be the same as in the scalar case (the original LKFt are recovered for $N = 2$). In ongoing work these transformations are being analysed in the context of the Schwinger model, extended to propagation in external electromagnetic fields and applied to derive the analogous transformations of the interaction vertex.

[1] L. Landau and I. Khalatnikov, The gauge transformation of the Green function for charged particles, Sov. Phys. JETP 2, 69 (1956).
[2] E. S. Fradkin., Zh. Eksp. Teor. Fiz. 29, 258261 (1955).
[3] M. J. Aslam, A. Bashir, and L. Gutierrez-Guerrero, Local Gauge Transformation for the Quark Propagator in an SU(N) Gauge Theory, Phys. Rev. D 93, 076001 (2016).
[4] P. Dall'Olio, T. De Meerleer, D. Dudal, S. Sorella, and A. Bashir, Landau-Khalatnikov-Fradkin transformations for the two loop massless quark propagator, Nuclear Physics B 973, 115606 (2021).
[5] V. M. Villanueva-Sandoval, Y. Concha-Sánchez, L. X. G. Guerrero, and A. Raya, On How the Scalar Propagator Transforms Covariantly in Spinless Quantum Electrodynamics, J. Phys. Conf. Ser. 1208, 012001 (2019).
[6] C. J. Burden and C. D. Roberts, Gauge covariance and the fermion - photon vertex in three-dimensional and four-dimensional, massless quantum electrodynamics, Phys. Rev. D 47, 5581 (1993) [arXiv:hep-th/9303098]
[7] A. Bashir and A. Raya, Landau-Khalatnikov-Fradkin transformations and the fermion propagator in quantum electrodynamics, Phys. Rev. D 66, 105005 (2002).
[8] A. Bashir and A. Raya, Fermion propagator in quenched QED3 in the light of the Landau-Khalatnikov-Fradkin transformation, Nucl. Phys. B. 141, 259 (2005).
[9] N. Ahmadiniaz, A. Bashir, and C. Schubert, Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED. Phys. Rev. D 93, 045023 (2016). arXiv:1511.05087 [hep-ph]
[10] C. Schubert, Perturbative quantum field theory in the string inspired formalism, Phys. Rept. 355, 73 (2001) arXiv:hep-th/0101036 [hep-th]
[11] J. P. Edwards and C. Schubert, Quantum mechanical path integrals in the first quantised approach to quantum field theory (2019) arXiv:1912.10004 [hep-th]
[12] M. J. Strassler, Field theory without Feynman diagrams: One loop effective actions, Nucl. Phys. B 385, 145 (1992).
[13] N. Ahmadiniaz, J. P. Edwards, J. Nicasio, and C. Schubert, Generalized Landau-Khalatnikov-Fradkin transformations for arbitrary N-point fermion correlators, Phys. Rev. D 104, 025014 (2021) arXiv:2012.10536 [hep-th]
[14] N. Ahmadiniaz, V. M. Banda Guzmán, C. Schubert, F. Bastianelli, O. Corradini, and J. P. Edwards, Obtaining fully polarised amplitudes in gauge invariant form (2022) To be published in Proc. 20th Lomonosov Conference, arXiv:2201.10852 [hep-th]
[15] N. Ahmadiniaz, V. M. Banda Guzmán, F. Bastianelli, O. Corradini, J. P. Edwards, and C. Schubert, Worldline master formulas for the dressed electron propagator. Part I. Off-shell amplitudes, JHEP 08 (2008), 049. arXiv:2001.03191
[16] N. Ahmadiniaz, V. M. Banda Guzmán, F. Bastianelli, O. Corradini, J. P. Edwards, and C. Schubert, Worldline master formulas for the dressed electron propagator. Part 2. On-shell amplitudes, JHEP 01, 050 arXiv:2107.00199
[17] L. Abbott, Introduction to the Background Field Method, Acta Phys. Polon. B 13, 33 (1982).