Quantum Vacuum and Anomalies

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ABSTRACT

Chiral, conformal and ghost number anomalies are discussed from the viewpoint of the quantum vacuum in Hamiltonian formalism. After introducing the energy cut-off, we derive known anomalies in a new way. We show that the physical origin of the anomalies is the zero point fluctuation of bosonic or fermionic field. We first point out that the chiral U(1) anomaly is understood as the creation of the chirality at the bottom of the regularized Dirac sea in classical electromagnetic field. In the study of the (1+1) dimensional quantum vacuum of matter field coupled to the gravity, we give a physically intuitive picture of the conformal anomaly. The central charges are evaluated from the vacuum energy. We clarify that the non-Hermitian regularization factor of the vacuum energy is responsible for the ghost number anomaly.

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1. Introduction

Since the Adler-Bell-Jackiw anomaly was discovered in the Feynman diagrammatic calculation[1-3], anomalies in quantum field theory have been discussed from various viewpoints. In path-integral formalism, the anomalies are identified with the Jacobian factors under the classical symmetry transformations [4]. The path integral approach relates the chiral gauge anomalies to the topological structures of the gauge theories[4][5]. This approach gives a systematic understanding for all the anomalies.

In Hamiltonian formalism, the anomalies are discussed from the viewpoint of the quantum vacuum. The chiral U(1) anomaly is understood as the particle creation due to the change of the Fermi surface of the massless Dirac sea in electromagnetic field[6]. This viewpoint is physically intuitive. The topological analysis of the chiral anomalies is given in terms of the Berry phase associated with the adiabatic change of the Dirac sea[7-9]. However there are no systematic understanding of all the anomalies in the Hamiltonian formalism.

In this paper, chiral, conformal and ghost number anomalies are discussed from the viewpoint of the quantum vacuum in Hamiltonian formalism. We show that the physical origin of the anomalies is the zero point fluctuation of the bosonic or fermionic field.

In Sec.II, we consider the adiabatic change of the massive Dirac sea in classical electromagnetic field. We show that the chiral U(1) anomaly arises from the creation of the chirality at the bottom of the regularized Dirac sea. In this massive case, it is also shown that the chiral current conservation is restored in the adiabatic process. In Sec.III, we study the conformal anomaly in (1+1) dimensions. The energy due to the zero point fluctuation in the quantum vacuum is regularized keeping the reparametrization invariance. We give a physically intuitive picture of the conformal anomaly. The Liouville action and central charges are derived from the vacuum energy. In Sec.IV, the ghost number current conservation is evaluated from the variation of the vacuum energy under the U(1) transformation of (1+1)
dimensional $b$, $c$ system. We point out that the non-Hermitian regularization factor of the vacuum energy leads the ghost number anomaly.
2. Chiral U(1) anomaly

In this section, we show how the chiral U(1) anomaly arises in the massive Dirac sea in adiabatic process.

Let us start with the (1+1) dimensional Dirac fermion theory coupled to a uniform electric field $E = -\frac{\partial}{\partial t}A^1(t)$ in the temporal gauge. The Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\frac{\partial}{\partial t} - H[A^1(t)])\psi,$$

(2.1)

$$H[A^1(t)] = \sigma_3(-i\frac{\partial}{\partial x} + eA^1(t)) + \sigma_3m,$$

(2.2)

where the signature of the metric is $(+,-)$ with $t = x^0$, $x = x^1$, and the gamma matrices are $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$ and $\gamma^5 = \gamma^0\gamma^1 = \sigma_1$. Under the slowly varying potential $A^1(t)$, the Dirac sea changes adiabatically. We take adiabatic approximation within the order $eE/m^2$.

The Hamiltonian $H[A^1(t)]$ is diagonalized (within the order $eE/m^2$) as

$$H_{\text{eff}} = UH[A^1(t)]U^\dagger - iU\frac{\partial}{\partial t}U^\dagger$$

$$= \epsilon\sigma_3 + eE\frac{m}{2\epsilon^2}\sigma_2,$$

(2.3)

where $\epsilon = \sqrt{m^2 + \Pi^2}$, $U = (\sqrt{\epsilon + m + i\sqrt{\epsilon - m}\sigma_2})/\sqrt{2\epsilon}$, $\Pi = p + eA^1$; $p$ is the eigenvalue of the momentum $-i\partial/\partial x$. From (2.3), we obtain the eigenfunction of the energy $\omega = \pm\epsilon(\Pi)$,

$$U\psi_{+\epsilon(\Pi)} = (1,ieEm/4\epsilon^3)^t e^{ipx}/\sqrt{L},$$

$$U\psi_{-\epsilon(\Pi)} = (ieEm/4\epsilon^3,1)^t e^{ipx}/\sqrt{L},$$

(2.4)

where $L$ is the length of the system. Since the ‘momentum’ $\Pi(t)$ varies ($\dot{\Pi} = -eE$), the eigenmodes flow on the mass shell (Fig.1). From exact calculation within the
order $eE/m^2$, we see the classical conservation law of the chiral current

$$
\partial_\mu j^\mu_{[\pm\epsilon(\Pi)]} - 2imj_{5[\pm\epsilon(\Pi)]} = 0,
$$ (2.5)

with $j^\mu_{5[\pm\epsilon(\Pi)]} = \bar{\psi}_{\pm\epsilon(\Pi)}\gamma^\mu\gamma_5\psi_{\pm\epsilon(\Pi)}$, $j_{5[\pm\epsilon(\Pi)]} = \bar{\psi}_{\pm\epsilon(\Pi)}\gamma_5\psi_{\pm\epsilon(\Pi)}$.

The chiral current in the Dirac sea is the sum of all contribution from the negative-energy modes. We apparently obtain from (2.5) the conservation law of chiral current in the Dirac sea,

$$
\partial_\mu \left(L \int \frac{d\Pi}{2\pi} j^\mu_{5[-\epsilon(\Pi)]}\right) - 2im \left(L \int \frac{d\Pi}{2\pi} j_{5[-\epsilon(\Pi)]}\right) = 0.
$$ (2.6)
In this naive conservation law, however, each term in the left hand side is ill-defined. We regularize them by introducing the cut-off $\Lambda \gg m$:

$$< j_5^\mu > = L \int \frac{d\Pi}{2\pi j_5^\mu[\Pi]} \exp \left( -\frac{\Pi^2}{\Lambda^2} \right). \tag{2.7}$$

$< j_5 >$ is defined similarly. Note that the regularized Dirac sea keeps the gauge invariance as far as taking the cut-off on $\Pi$.

Now let us see the bottom of the Dirac sea (energy $\omega \sim -\Lambda$) in Fig.1. The negative energy electrons with chirality $(\Pi/\omega) \sim (-1)$ are created at the bottom $(\Pi, \omega) \sim (\Lambda, -\Lambda)$. The negative-energy electrons with chirality $(\Pi/\omega) \sim (+1)$ are annihilated at the bottom $(\Pi, \omega) \sim (-\Lambda, -\Lambda)$. From the equation of motion $\Pi = -eE$, the rate of the creation of the chirality from the bottom is evaluated as $-2(eE/2\pi)L$ per unit time. According to this anomalous effect, the current conservation (2.6) is modified to

$$\partial_\mu < j_5^\mu > - 2im < j_5 > = -\frac{eE}{\pi}. \tag{2.8}$$

This is the anomalous chiral U(1) identity as is well known.

In the massless case ($m = 0$), the Fermi surface changes and the electrons with chirality (-1) and positrons with chirality (-1) are created in the electric field(Fig.1-b). This creation breaks the chiral current conservation[6].

In the massive case ($m \neq 0$), the Fermi surface does not change, thus the chirality preserves in the adiabatic process(Fig.1-a). This is independently verified from the Feynman diagrammatic calculation of the vacuum expectation value $2im < j_5 > \sim eE/\pi$ under the uniform and constant electric field. The chiral current conservation is restored as

$$\partial_\mu < j_5^\mu > - 2im < j_5 > = -eE/\pi \sim 0 \text{ in the adiabatic process.}$$

In (1+3) dimensions, we first consider the Dirac fermion in a uniform magnetic field $B(> 0)$ along the third direction. We take the gamma matrices as $\gamma^0 = I \otimes \sigma_3$, $\gamma^1 = i \not{\partial}_1 \otimes \sigma_1$, $\gamma^2 = i \not{\partial}_2 \otimes \sigma_2$, $\gamma^3 = \not{\partial}_3 \otimes \sigma_3$, and $\gamma^\mu = \not{\partial}_\mu \otimes 1$, where $\not{\partial} = \gamma^\mu \partial_\mu$. The Dirac equation in this case is

$$\left( \not{\partial} + \gamma^5 \right) \psi = 0,$$
\[ \gamma^k = i \sigma_k \otimes \sigma_2 \] and \( \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = I \otimes \sigma_1 \). The energy levels are \( \omega = \pm \epsilon(\Pi, n) = \pm \sqrt{m^2 + 2neB + \Pi^2} \) with \( \Pi = p^3 + eA^3 \). For the non-zero mode \( (n \neq 0) \), spin-up state and spin-down state degenerate and the sum of their chirality becomes zero. For the zero mode \( (n = 0) \), only spin-down mode exists. Thus only negative-energy zero modes contribute the chirality of the Dirac sea.

Next a uniform electric field \( E = -(\partial/\partial t)A^3 \) is turned on parallel to \( B \), then \( \Pi(t) = p^3 + eA^3(t) \) varies and the eigenmodes flow on the shell \( \omega(\Pi) = \pm \sqrt{m^2 + 2neB + \Pi^2} \). Each mode satisfies the classical conservation law of chiral current. We introduce the energy cut-off for the regularization. The chirality is created from the bottom of the Dirac sea. By using the chirality of the zero mode \( -\Pi/\omega \) and the density of the zero-mode states \( (eB/2\pi)(L/2\pi) \) per length \( L \), the rate of creation of the chirality is evaluated as \( e^2EB/2\pi^2 \) per unit time and volume. This is the chiral U(1) anomaly. In this massive case, the Fermi surface does not change and the chirality is preserved in the adiabatic process. This is independently verified from the Feynman diagrammatic calculation of the vacuum expectation value \( 2im < j_5 > \sim -e^2EB/2\pi^2 \), which is similar to the (1+1) dimensional case discussed above.
3. Conformal anomaly in (1+1) dimensions

In this section, we study the conformal anomaly in (1+1) dimensions from the viewpoint of the vacuum energy in the background metric.

We start from the scalar field \( X(t, x) \) with the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \sqrt{-g(t, x)g^{\mu\nu}(t, x)\partial_\mu X(t, x)\partial_\nu X(t, x)},
\]

(3.1)

where the signature of the metric is \((+,-)\) with \( t = x^0, x = x^1 \). Under the conformal transformation \( g_{\mu\nu} \rightarrow e^{\phi(t,x)}g_{\mu\nu} \), the Lagrangian (3.1) does not change. The quantum vacuum changes however. To see this, we consider the vacuum energy due to the zero point fluctuation of the scalar field. We regularize the vacuum energy in flat space-time metric \( \eta_{\mu\nu} \) as

\[
E[0] = L \int \frac{dk}{2\pi} \frac{1}{2} |k| \exp \left( - \frac{k^2}{\Lambda^2} \right),
\]

(3.2)

where \( L \) is the length of the system.

The vacuum energy in the background metric \( e^\phi \eta_{\mu\nu} \) is determined from the reparametrization invariance in the following way (Fig.2). Under the reparametrization

\[
x^\mu \rightarrow x'^\mu = e^{-\phi/2}x^\mu,
\]

(3.3)

the momentum is transformed as \( k^\mu \rightarrow k'^\mu = e^{\phi/2}k^\mu \), where \( \phi \) is a constant. By noting \( L \rightarrow L' = e^{-\phi/2}L \) and \( \Lambda \rightarrow \Lambda' = e^{\phi/2}\Lambda \), we obtain the transformation of the vacuum energy

\[
E[0] \rightarrow E[0]' = e^{-\frac{\phi}{2}}L \int \frac{dk}{2\pi} \frac{1}{2} |k| \exp \left( - \frac{k^2}{e^\phi \Lambda'^2} \right).
\]

(3.4)

Here we rescale the length of this system as

\[
L \rightarrow e^{\phi/2}L,
\]

(3.5)
then the vacuum energy (3.4) is transformed as

\[ E[0'] \rightarrow E[\phi] = L \int \frac{dk}{2\pi} \frac{1}{2} |k| \exp \left( -\frac{k^2}{e^\phi \Lambda^2} \right). \]  \hspace{1cm} (3.6)

Under the composite transformation of (3.3) and (3.5), the metric tensor and domain of the parameter \( x \) are transformed as

\[ \eta_{\mu\nu}(0 \leq x \leq L) \rightarrow e^\phi \eta_{\mu\nu}(0 \leq x \leq e^{-\phi/2}L) \rightarrow e^\phi \eta_{\mu\nu}(0 \leq x \leq L) \]  \hspace{1cm} (3.7)

which is equivalent to the conformal transformation \( \eta_{\mu\nu} \rightarrow e^\phi \eta_{\mu\nu}(0 \leq x \leq L) \).

Thus the \( E[\phi] \) in (3.6) is the vacuum energy in the background metric \( e^\phi \eta_{\mu\nu} \).

![Figure 2](image)

Figure 2. Dispersion law for the scalar field in (1+1) dimensions. Under the reparametrization \( x^\mu \rightarrow \tilde{x}^\mu = e^{-\phi/2}x^\mu \) and rescaling the length of this system \( L \rightarrow e^{\phi/2}L \), the dispersion law is transformed as (a) \( \rightarrow \) (b) and (b) \( \rightarrow \) (c) respectively. In (a) \( \rightarrow \) (c), each mode does not change, but the cut-off level changes.

In the background metric \( e^{\phi(x)} \eta_{\mu\nu} \) which depends on the coordinate \( x \) and is independent of the time \( t \), the vacuum energy (3.6) is generalized as

\[ E[\phi(x)] = L \int \frac{dk}{2\pi} \frac{1}{2} |k| \langle f_k, \exp \left( -\frac{\Lambda^2}{\Delta} \right) f_k \rangle_{scalar}, \]  \hspace{1cm} (3.8)
where \( f_k \) is the positive-energy wave function of momentum \( k \),

\[
    f_k = e^{ikx - i|k|t / \sqrt{2|k|L}}. \tag{3.9}
\]

\( \Delta \) is the one dimensional Laplacian \( \Delta = -|g^{11}(x)|\nabla_1 \nabla_1 \) which is

\[
    \Delta^{(j)} = -\frac{1}{e^\phi} \left( \partial_1 - \frac{j + 1}{2} \partial_1 \phi \right) \left( \partial_1 - \frac{j}{2} \partial_1 \phi \right) \tag{3.10}
\]

for the field of conformal weight \( j \), and the inner product \( \langle , \rangle_{\text{scalar}} \) is given by

\[
    \langle f_l, e^{-\Delta/\Lambda^2} f_k \rangle_{\text{scalar}} = \int dx [i f_l^* \exp \left( - \frac{\Delta^{(1)}}{\Lambda^2} \right) \partial^- f_k - i \left( \frac{\partial}{\partial t} f_l^* \right) \exp \left( - \frac{\Delta^{(0)}}{\Lambda^2} \right) f_k]. \tag{3.11}
\]

Note that the regularization factor \( \exp(-\Delta/\Lambda^2) \) is reparametrization invariant in one dimension.

From the momentum integration of (3.8), we obtain the vacuum energy

\[
    E[\phi(x)] = \int dx \int \frac{dk}{2\pi} \frac{1}{2|k|} e^{-ikx} \exp \left( - \frac{\Delta^{(1)}}{\Lambda^2} \right) + \exp \left( - \frac{\Delta^{(0)}}{\Lambda^2} \right) e^{ikx} \tag{3.12}
\]

\[
    = \int dx \left[ \frac{\Lambda^2}{4\pi} e^\phi - \frac{1}{96\pi} (\partial_1 \phi)^2 - \frac{1}{48\pi} \partial_1^2 \phi + O\left( \frac{1}{\Lambda^2} \right) \right].
\]

The third term \( \partial_1^2 \phi \) is the total derivative term and gives no contribution to the vacuum energy under the periodic boundary condition. \( O(1/\Lambda^2) \) vanishes by taking \( \Lambda \to \infty \).

By noting the relation between the path-integral formalism and the Hamiltonian formalism

\[
    <0 | \exp(-i \int dt H) |0> = \int \mathcal{D}X \exp(i \int d^nx L), \tag{3.13}
\]

the partition function \( Z_X[\phi(x)] \) is written as

\[
    Z_X[\phi(x)] = \exp \left( -i \int dt E[\phi(x)] \right). \tag{3.14}
\]

Thus, under the conformal transformation \( \eta_{\mu \nu} \to e^{\phi(x)} \eta_{\mu \nu} \), the partition function
is transformed as

$$Z[0] \rightarrow Z[\phi(x)] = Z[0] \exp(iS_L), \quad (3.15)$$

$$S_L = \int dt \int dx \left[ \frac{1}{96\pi} (\partial_1 \phi)^2 - \frac{\Lambda^2}{4\pi} (e^\phi - 1) \right]. \quad (3.16)$$

This $S_L$ is the Liouville action. Equations (3.15) and (3.16) justify that the quantum vacuum changes under the conformal transformation. The Liouville action denotes the difference of the vacuum energy: $S_L = -\int dt (E[\phi(x)] - E[0])$.

From the relativistic generalization $- (\partial_1 \phi)^2 \rightarrow (\partial_0 \phi)^2 - (\partial_1 \phi)^2$ and Wick rotation $t \rightarrow -i\tau$, $\Lambda \rightarrow iM$, we obtain the Euclidean version of the Liouville action[10][11]. Here we note that the total derivative term $\partial_1^2 \phi$ in (3.12) gives no contribution to the conformal anomaly. After the relativistic generalization $-\partial_1^2 \phi \rightarrow (\partial_0^2 - \partial_1^2)\phi = -\sqrt{-g}R$ and Wick rotation, $\partial_1^2 \phi$ term becomes the Einstein term $\int d^2x \sqrt{g}R$. The Einstein term gives the Euler number and does not change under the conformal transformation.

For Majorana fermions, vacuum has negative energy and the vacuum energy formula (3.8) becomes

$$E[\phi(x)] = -L \int \frac{dk}{2\pi^2} \frac{1}{2} |k| \langle f_k, \exp \left(-\frac{\Delta^{(1/2)}}{\Lambda^2}\right) f_k \rangle_{spinor}, \quad (3.17)$$

where $f_k$ is the positive energy wave function of momentum $k$,

$$f_k = e^{ikx - i|k|t}/\sqrt{L} \quad (3.18)$$

which is right (left) handed wave function for $k > 0$ ($k < 0$). The conformal weight of the Majorana fermion is 1/2, thus the one dimensional Laplacian (3.10) becomes
\( \Delta^{(1/2)} \). The inner product \( \langle f, e^{-\Delta^{(1/2)}/\Lambda^2} f_k \rangle_{\text{spinor}} \) is given by

\[
\langle f_l, e^{-\Delta^{(1/2)}/\Lambda^2} f_k \rangle_{\text{spinor}} = \int dx f_l^* \exp \left( - \frac{\Delta^{(1/2)}}{\Lambda^2} \right) f_k. \tag{3.19}
\]

From the momentum integration of (3.17), we obtain the vacuum energy

\[
E[\phi(x)] = -\int dx \int \frac{dk}{2\pi} \left| k \right| e^{-ikx} \exp \left( - \frac{\Delta^{(1/2)}}{\Lambda^2} \right) e^{ikx} = \int dx \left[ -\frac{\Lambda^2}{4\pi} \phi - \frac{1/2}{96\pi} (\partial_1 \phi)^2 + \frac{1}{48\pi} \partial_1^2 \phi + O\left( \frac{1}{\Lambda^2} \right) \right]. \tag{3.20}
\]

The Liouville action is derived from the first and second term. In particular from the second term we obtain the central charge 1/2.
4. Ghost number anomaly in \((1+1)\) dimensions

In this section, we study the vacuum energy of the \(b, c\) system and the ghost number anomaly.

From the canonical quantization, the vacuum energy of the \(b, c\) system becomes

\[ E_{bc} = -\epsilon L \int \frac{dk}{2\pi} |k| \] with \(\epsilon = -1 (+1)\) for the bosonic (fermionic) ghost. The bases of the wave function of the \(b, c\) system are taken as

\[ f_k^{(b)} = e^{ikx - i|k|t}/L^\lambda, \quad f_k^{(c)} = e^{ikx - i|k|t}/L^{1-\lambda}, \] (4.1)

where \(\lambda\) (or \(1 - \lambda\)) is the conformal weight of \(b\) field (or \(c\) field). The inner product \(\langle \ , \ \rangle_{\text{ghost}}\) is given by

\[ \langle f_l^{(b)}, e^{-\Delta^{(1-\lambda)}/\Lambda^2} f_k^{(c)} \rangle_{\text{ghost}} = \int dx f_l^{(b)*} e^{\left(-\frac{\Delta^{(1-\lambda)}}{\Lambda^2}\right)} f_k^{(c)}. \] (4.2)

Then, the vacuum energy is regularized as

\[ E_{bc} = -\epsilon L \int \frac{dk}{2\pi} |k| \langle f_k^{(b)}, \exp \left(-\frac{\Delta^{(1-\lambda)}}{\Lambda^2}\right) f_k^{(c)} \rangle_{\text{ghost}} = \int dx E_{\epsilon, 1-\lambda}, \] (4.3)

\[ E_{\epsilon, 1-\lambda} = \frac{-\epsilon \Lambda^2}{2\pi} e^\phi - \frac{2\epsilon[6\lambda(\lambda - 1) + 1]}{96\pi} (\partial_1 \phi)^2 - \epsilon \frac{-2 + 3(1 - \lambda)}{12\pi} \partial_1^2 \phi + O\left(\frac{1}{\Lambda^2}\right). \] (4.4)

The regularization factor \(\exp(-\Delta^{(1-\lambda)}/\Lambda^2)\) is not Hermitian for \(\lambda \neq 1/2\). From the definition of Laplacian \(\Delta^{(j)}\) (3.10), it is shown that \(\Delta^{(1-\lambda)}\) and \(\Delta^{(\lambda)}\) are Hermitian conjugate. By using this, the vacuum energy (4.3) is rewritten as

\[ E_{bc} = -\epsilon L \int \frac{dk}{2\pi} |k| \langle f_k^{(b)}, \exp \left(-\frac{\Delta^{(\lambda)}}{\Lambda^2}\right) f_k^{(c)} \rangle_{\text{ghost}} = \int dx E_{\epsilon, \lambda}, \] (4.5)

\[ E_{\epsilon, \lambda} = \frac{-\epsilon \Lambda^2}{2\pi} e^\phi - \frac{2\epsilon[6\lambda(\lambda - 1) + 1]}{96\pi} (\partial_1 \phi)^2 - \epsilon \frac{-2 + 3\lambda}{12\pi} \partial_1^2 \phi + O\left(\frac{1}{\Lambda^2}\right). \] (4.6)

The difference between \(E_{\epsilon, 1-\lambda}\) and \(E_{\epsilon, \lambda}\) is the total derivative term and this gives no contribution to the vacuum energy \(E_{bc}\). From the second term of (4.4) or (4.6), we obtain the familiar central charge of the \(b, c\) system: \(C_{\text{ghost}} = -2\epsilon[6\lambda(\lambda - 1) + 1]\).
Next we consider the U(1) transformation of the \( b, c \) system

\[
\begin{align*}
    b(t, x) &\rightarrow e^{-\alpha(t,x)}b(t, x), \\
    c(t, x) &\rightarrow e^{+\alpha(t,x)}c(t, x).
\end{align*}
\]

By noting the transformation of the wave function

\[
\begin{align*}
    f_k^{(b)} &\rightarrow e^{-\alpha(t,x)}f_k^{(b)}, \\
    f_k^{(c)} &\rightarrow e^{+\alpha(t,x)}f_k^{(c)},
\end{align*}
\]

we obtain from (4.3) and (4.5) the variation of the vacuum energy

\[
\delta E_{bc} = \int dx \alpha(t, x) [\mathcal{E}_{\epsilon,\lambda} - \mathcal{E}_{\epsilon,1-\lambda}]
\]

for the infinitesimal parameter \( \alpha(t, x) \). From the relation (3.13), the infinitesimal transformation of the effective action of the \( b, c \) system is evaluated as

\[
\delta \Gamma_{bc} = -\int dt \int dx \alpha(t, x) [\mathcal{E}_{\epsilon,\lambda} - \mathcal{E}_{\epsilon,1-\lambda}].
\]

From the fact

\[
\delta \Gamma_{bc} = -\int dt \int dx \alpha(t, x) \partial_\mu < j_\mu^c(t, x) >,
\]

where \( < j_\mu^c > \) is the ghost number current, the ghost number current conservation law becomes

\[
\partial_\mu < j_\mu^c(t, x) > = \mathcal{E}_{\epsilon,\lambda} - \mathcal{E}_{\epsilon,1-\lambda}.
\]

By using (4.4), (4.6) and relativistic generalization \(-\partial_1^2 \phi \rightarrow (\partial_0^2 - \partial_1^2)\phi = -\sqrt{-g}R\), we obtain the well known anomalous conservation law

\[
\partial_\mu < j_\mu^c > = \frac{Q}{4\pi} \sqrt{-g}R,
\]

where the background charge \( Q = \epsilon(1-2\lambda) \). Thus the non-Hermitian regularization factor leads the ghost number anomaly. It is interesting that the ghost number anomaly arises from the total derivative term \( \partial_1^2 \phi \) which has given no contribution to the conformal anomaly.
For the scalar field and spinor field (Majorana fermions), the regularization factors in (3.12) and (3.20) are Hermitian, thus the vacuum energies (3.8) and (3.17) do not change under the U(1) transformation $f_k \rightarrow e^{i\alpha(t,x)} f_k$, $f^*_k \rightarrow e^{-i\alpha(t,x)} f^*_k$. Therefore the conservation law of U(1) current is anomaly free.
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