FUNCTIONAL INTEGRATION AND GAUGE AMBIGUITIES IN GENERALIZED ABELIAN GAUGE THEORIES

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Abstract. We consider the covariant quantization of generalized abelian gauge theories on a closed and compact \( n \)-dimensional manifold whose space of gauge invariant fields is the abelian group of Cheeger-Simons differential characters. The space of gauge fields is shown to be a non-trivial bundle over the orbits of the subgroup of smooth Cheeger-Simons differential characters. Furthermore each orbit itself has the structure of a bundle over a multi-dimensional torus. As a consequence there is a topological obstruction to the existence of a global gauge fixing condition. A functional integral measure is proposed on the space of gauge fields which takes this problem into account and provides a regularization of the gauge degrees of freedom. For the generalized \( p \)-form Maxwell theory closed expressions for all physical observables are obtained. The Green’s functions are shown to be affected by the non-trivial bundle structure. Finally the vacuum expectation values of circle-valued homomorphisms, including the Wilson operator for singular \( p \)-cycles of the manifold, are computed and selection rules are derived.

1. Introduction

The importance of antisymmetric tensor fields for string theory and for some supergravity models has been recognized for many years [1,2]. In its original version, the corresponding configuration space of abelian gauge fields of rank \( p \), denoted by \( \mathcal{A}^p \), consists of differential \( p \)-forms on a manifold \( \mathcal{M} \) on which the abelian group \( \text{im}d_{p-1} \) of exact \( p \)-forms acts by translation. To any gauge field \( A \in \mathcal{A}^p \) a gauge invariant \((p+1)\)-form field strength \( F_A := d_A A \) is associated which has vanishing magnetic flux by construction. Examples are the neutral scalar field for \( p = 0 \), the photon field for \( p = 1 \), and in the low energy limit of type II string theory the Kalb-Ramond field for \( p = 2 \) as well as the Ramond-Ramond field, whose allowed rank \( p \) depends on whether type IIA or type IIB string theory is considered.

Recently, field configurations with non-trivial fluxes gained a prominent role in the dynamics of string theoretical models. From a mathematical point of view neither the definition of the field strength \( F_A \) as an exact differential form nor the interpretation of \( \text{im}d_{p-1} \) as the underlying symmetry group are appropriate concepts to describe such topologically non-trivial configurations.

The so-called generalized abelian gauge theories provide a suitable mathematical framework for the description of such fields. In principle, a generalized abelian gauge theory is a field theory whose gauge invariant field configurations belong to a (generalized) differential cohomology group. In brief, (generalized) differential cohomology can be viewed as a specific combination of ordinary cohomology or even generalized cohomology - meaning that the Eilenberg-Steenrod axioms except for the dimension axiom are satisfied - with the algebra of closed differential forms on smooth manifolds. An exposition of the underlying concept and some applications in quantum field theory can be found in [3]. The mathematical theory of this new topic has been introduced and elaborated in [4]. Differential cohomology groups with physical relevance are in particular the group of Cheeger-Simons differential characters [5] and the (twisted) differential \( K \)-theory [3,4]. In this respect we would like to mention the generalized Maxwell-theory, which has been analyzed in the Hamiltonian approach quite recently [6,7] and the
generalized Maxwell-Chern-Simons theory [8]. Furthermore the 2-form $B$-field in type II-superstring theory, the differences of the so-called 3-form $C$-gauge fields of $M$-theory [9] and the gauge invariant classes of the Ramond-Ramond (RR) fields [10,11] can be naturally interpreted in terms of differential cohomology.

The present paper is devoted to a study of covariant quantization of generalized abelian gauge theories on closed and compact manifolds of dimension $n$. Our aims are to clarify the underlying geometrical structure, to construct an appropriate measure for the functional integral and to determine the vacuum expectation values of physical observables. In our setting a generalized abelian gauge field is described by a $p$-form gauge field $A \in \mathcal{A}^p$ and a cohomology class $c \in H^{p+1}(M; \mathbb{Z})$, which characterizes its topological type. Yet the corresponding gauge group is the abelian group $\Omega^2_Z(M; \mathbb{R})$ of differential $p$-forms with integral periods, which acts naturally on $\mathcal{A}^p$. The space of inequivalent generalized gauge fields is identified with the abelian group of Cheeger-Simons differential characters of rank $p$. Let us remind that a differential character $\hat{u}$ is a specific group homomorphism from singular $p$-cycles of $M$ to the 1-torus $\mathbb{T}^1$. A closed but non-exact differential $(p+1)$-form with integer periods, denoted by $\delta_1(\hat{u})$, is assigned to $\hat{u}$, which in physical terms is regarded as the field strength in the corresponding generalized abelian gauge theory.

We will prove explicitly that the space of gauge fields admits the structure of a non-trivial flat principal fiber bundle over the orbits which are generated by the subgroup of smooth Cheeger-Simons differential characters. In physical terms this implies that it is impossible to obtain a global smooth gauge fixing condition; the theory is said to suffer from gauge ambiguities. Topologically, the non-triviality of the bundle is related to the free part of $H^p(M; \mathbb{Z})$. Moreover, each orbit of the subgroup of smooth differential characters can be proven to be a trivializable bundle over the torus $\mathbb{T}^{b_p}$, whose dimension is the $p$-th Betti number $b_p = \dim H^p(M; \mathbb{R})$ of $M$.

In order to elucidate the general concept in a concrete field theoretical model, we will consider the covariant quantization of the so-called generalized $p$-form Maxwell theory. This model generalizes the (conventional) $p$-form Maxwell theory in so far as $F_A = d_p A$ is replaced by the generalized field strength $\delta_1(\hat{u}) \in \Omega^{p+1}_Z(M; \mathbb{R})$ in the classical action $S_{\text{inv}} = \frac{1}{2} \int_M F_A \wedge \star F_A$, where $\star$ denotes the Hodge star operator.

What could be a guiding principle for the construction of a functional integral measure for the generalized abelian gauge theory? Let us briefly review the topologically trivial case: The partition function for the $p$-form Maxwell theory is defined as the functional integral

$$Z^{(p)} = \frac{1}{\text{Vol}(\text{imd}_{p-1})} \int_{\mathcal{A}^p} \text{vol}_{\mathcal{A}^p} \ e^{-S_{\text{inv}}},$$

(1.1)

over the field space $\mathcal{A}^p$. Here $\text{vol}_{\mathcal{A}^p}$ is the formal volume form on $\mathcal{A}^p$ and $\text{Vol}(\text{imd}_{p-1})$ denotes the infinite volume of the gauge group $\text{imd}_{p-1}$. Due to the gauge invariance of the classical action $S_{\text{inv}}(A)$, the integrand in the numerator of (1.1) is constant along the orbits of the gauge group, which have infinite measure. It is argued that by separating the divergent gauge dependent part from the integrand and dividing by $\text{Vol}(\text{imd}_{p-1})$, which is infinite as well, the partition function (1.1) can be rendered finite. According to a modified Faddeev-Popov approach [12-15], which takes the reducibility of the algebra of gauge transformations into account using the so called ”ghost-for-ghost” procedure, this separation can be provided by selecting gauge fixing conditions in all dimensions up to the rank of the gauge fields. The resulting functional integral is evaluated over a global gauge fixing submanifold in $\mathcal{A}^p$ with a weight factor given by the Jacobians of the Faddeev-Popov operators associated with the given gauge fixing conditions. However, the Faddeev-Popov procedure fails if it is impossible to fix the gauge globally.

An alternative way to quantize theories which are governed by degenerate action functionals has been introduced by Schwarz [16-18]. This method of invariant integration relies on the reduction of the functional integral in (1.1) over $\mathcal{A}^p$ to an integral over the corresponding gauge orbit space $\mathcal{A}^p/\text{imd}_{p-1}$ times the volume of the gauge group modified by a ghost-for-ghost contribution. In the infinite dimensional case this extraction of the gauge group volume is ill-defined. However, Schwarz proposed to omit this infinite factor and take the remaining functional integral over the gauge orbit
space as the correct partition function of the theory.

One could raise the question if it is possible to include the gauge degrees of freedom in a reasonable way and to circumvent the gauge ambiguities.

Instead of constructing a measure on the abelian group of Cheeger-Simons differential characters we search for a functional integral formulation of generalized abelian gauge theory directly on $A^p$ including the different topological sectors in $H^{p+1}(M;\mathbb{Z})$. At first glance this approach seems to be of limited use due to the gauge ambiguities and the infinite dimensional gauge group. We propose a functional integral measure that resolves these problems and provides a mathematically reasonable treatment of the gauge degrees of freedom. For that we will apply a concept, which has been originally developed in the context of stochastic quantization of gauge theories [19,20]. In principle, the construction of this functional integral measure relies on the following three steps:

1. A regularizing measure is introduced for the gauge degrees of freedom yielding a finite volume of the gauge group.

2. A prescription is given to select a unique representative along each gauge orbit. The set of these representatives generates a gauge fixing submanifold. It is the correct region over which the functional integral has to be taken. In topologically non-trivial situations, like we encounter in the generalized abelian theory, the occurrence of gauge ambiguities prevents the existence of any smooth global gauge fixing. Hence the construction of gauge fixing submanifolds can be done locally only.

3. A family of measures is selected on $A^p$, whose domains of definition are determined by the local gauge fixing submanifolds and which are integrable along the orbits of the gauge group. Finally these local measures are glued together in such a way that the physical relevant objects become independent of the chosen gauge group regularization and of the particular way this gluing was provided. Hence the problem of gauge ambiguities can be circumvented, guaranteeing the existence of an integrable partition function on $A^p$.

This paper is structured as follows: In section 2 we will review briefly the concept of the Cheeger-Simons differential characters. The geometrical structure of the configuration space of generalized gauge theories will be studied in section 3. Some of our results regarding the geometrical structure of the inequivalent generalized gauge fields have been obtained in a different way in [6] and [21]. Section 4 is devoted to the construction of the partition function and the vacuum expectation value (VEV) for generalized abelian gauge theories. We derive closed expressions for the generalized $p$-form Maxwell theory. In the topologically trivial case, the partition function for the $p$-form Maxwell theory can be recovered, yet the gauge group volume is sufficiently regularized. In section 5 the one-point and two-point functions are explicitly computed showing non-trivial effects due to the topology of the gauge orbit space and the regularized gauge group volume. In section 6 the VEV is elaborated for circle-valued homomorphisms, which represent a natural class of gauge invariant observables. First, we study the so-called smooth circle-valued homomorphisms, which can be characterized equivalently in terms of the Poincaré - Pontrjagin duality of differential characters. Second, the Wilson operator, which is a multi-dimensional generalization of the Wilson operator for loops, is considered for singular $p$-cycles of $M$. In both cases, we will find that the corresponding VEVs vanish unless specific topological conditions are satisfied.

2. Setting the stage - Definition of the Cheeger-Simons differential characters

In this section we want to recall the concept of the abelian group of differential characters which has been introduced by Cheeger and Simons [5]. In the present paper, let $M$ be a $n$-dimensional closed, connected, oriented and compact Riemannian manifold and let us denote the complex of smooth singular chains of $M$ with coefficients in $\Lambda = \mathbb{Z} \otimes \mathbb{R}$ by $C_\ast(M;\Lambda)$ and its subcomplex of all smooth singular cycles by $Z_\ast(M;\Lambda)$. Furthermore the boundary and coboundary operators will be denoted by $\partial$ and $\delta$, respectively. There is a natural map between the complex of differential forms $\Omega^\ast(M;\mathbb{R})$ and
$C^*(M; \mathbb{R})$ given by integration of differential forms over smooth singular chains. Let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the reduction of $\mathbb{R}$ modulo $\mathbb{Z}$, then there is an induced map

$$i: \Omega^*(M; \mathbb{R}) \to C^*(M; \mathbb{R}/\mathbb{Z}), \quad i(A) = q(\int_A), \quad \sigma \in C_*(M; \mathbb{Z}).$$

(2.1)

We identify $\mathbb{R}/\mathbb{Z}$ with the 1-torus $\mathbb{T}^1$ and take $q(.) = e^{2\pi \sqrt{-1} .}$. Let us introduce the abelian group

$$\Omega_p^p(M, \mathbb{R}) = \{ A \in \Omega^p(M; \mathbb{R}) | \quad d\alpha = 0, \quad \int_{\Sigma} \alpha \in \mathbb{Z} \quad \forall \Sigma \in Z_p(M; \mathbb{Z}) \}$$

(2.2)

of closed $\mathbb{R}$-valued differential $p$-forms with integer periods.

**Definition 2.1.** The abelian group of Cheeger-Simons differential characters of degree $p$ is defined by

$$\hat{H}^p(M; \mathbb{R}/\mathbb{Z}) = \{ \hat{u} \in \text{Hom}(Z_p(M; \mathbb{Z}); \mathbb{R}/\mathbb{Z}) | \quad \hat{u} \circ \partial \} \Omega_p^p(M, \mathbb{R}) \}.$$

For any differential character $\hat{u} \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})$ one can always find a cochain $u \in C^p(M; \mathbb{R})$ whose restriction to cycles is just $\hat{u}$. There exist a differential form with integral periods $F \in \Omega_p^{p+1}(M; \mathbb{R})$ and a singular cochain $c \in C^{p+1}(M; \mathbb{Z})$, such that $\delta u = F - c$. Here $F$ satisfies the relation $(\hat{u})(\partial \sigma) = q(\Sigma) F$ for all $\sigma \in C^{p+1}(M; \mathbb{Z})$. Evidently, $c$ is a cocycle (i.e. $\delta c = 0$), so that one can construct the following two homomorphisms on $\hat{H}^p(M; \mathbb{R}/\mathbb{Z})$, namely $\delta_1(\hat{u}) = F$ and $\delta_2(\hat{u}) = [c].$

Set $R^{p+1}(M, \mathbb{Z}) = \{(F, c) \in \Omega_p^{p+1}(M, \mathbb{R}) \times H^{p+1}(M, \mathbb{Z}) | \quad r_*(c) = [F]\}$, where $r_*: H^{p+1}(M, \mathbb{Z}) \to H^{p+1}(M, \mathbb{R})$ is induced by the inclusion $\mathbb{Z} \to \mathbb{R}$ and $[F]$ is the cohomology class of $F$ in $H^{p+1}(M, \mathbb{R})$.

The abelian group of Cheeger-Simons differential characters is characterized by the following result:

**Theorem 2.2 [5].** The following sequences are exact

$$0 \rightarrow H^{p}(M; \mathbb{R}; \mathbb{Z}) \xrightarrow{j_1} \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_1} \Omega_p^{p+1}(M, \mathbb{R}) \rightarrow 0$$

$$0 \rightarrow \Omega_p^p(M, \mathbb{R})/\Omega_p^p(M, \mathbb{R}) \xrightarrow{j_2} \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_2} H^{p+1}(M, \mathbb{Z}) \rightarrow 0$$

$$0 \rightarrow H^{p}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{r_*} \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{(\delta_1, \delta_2)} R^{p+1}(M, \mathbb{Z}) \rightarrow 0,$$

where $j_1([\Sigma]) := v|_{Z_p(M; \mathbb{Z})}$ is the restriction map and $j_2([A])(\Sigma) := q(\int_{\Sigma} A)$ for $\Sigma \in Z_p(M; \mathbb{Z})$. The third sequence follows by combining the first two. □

Let us consider the long exact cohomology sequence

$$\cdots \rightarrow H^{p}(M; \mathbb{R}) \xrightarrow{j_2} H^{p}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta^*} H^{p+1}(M, \mathbb{Z}) \xrightarrow{r_*} H^{p+1}(M; \mathbb{R}) \rightarrow \cdots$$

(2.3)

induced by the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \to \mathbb{T}^1 \to 0$, where $\delta^*$ denotes the Bockstein operator, then

(1) $\delta_1|_{\Omega_p^p(M; \mathbb{R})/\Omega_p^p(M, \mathbb{R})} := \delta_1 \circ j_2 = d_p$

(2) $\delta_2|_{H^{p}(M; \mathbb{R}/\mathbb{Z})} := \delta_2 \circ j_1 = -\delta^*.$

Additionally, the differential characters can be equipped with an associative, graded commutative ring structure

$$*: \hat{H}^{p_1}(M; \mathbb{R}/\mathbb{Z}) \times \hat{H}^{p_2}(M; \mathbb{R}/\mathbb{Z}) \to \hat{H}^{p_1+p_2+1}(M; \mathbb{R}/\mathbb{Z}),$$

(2.4)

which according to [5] satisfies the following relations (in "multiplicative" notation):
(1) \((\hat{u}_1 \ast \hat{u}_2) \ast \hat{u}_3 = \hat{u}_1 \ast (\hat{u}_2 \ast \hat{u}_3)\)
(2) \(\hat{u}_1 \ast \hat{u}_2 = (\hat{u}_2 \ast \hat{u}_1)^{-(p+1)(n_2+1)}\), where \(\hat{u}_1 \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})\) and \(\hat{u}_2 \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})\),
(3) \(\delta_1(\hat{u}_1 \ast \hat{u}_2) = \delta_1(\hat{u}_1) \wedge \delta_1(\hat{u}_2)\),
(4) \(\delta_2(\hat{u}_1 \ast \hat{u}_2) = \delta_2(\hat{u}_1) \cup \delta_2(\hat{u}_2)\), where \(\cup\) denotes the cup product in singular cohomology,
(5) \(\hat{u} \ast j_2([A]) = (j_2([\delta_1(\hat{u}) \wedge A]))^{-(p+1)}\), with \(\hat{u} \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})\) and \([A] \in \Omega^p(M; \mathbb{R})/\Omega^p_c(M; \mathbb{R})\),
(6) \(\hat{u} \ast j_1([v]) = (j_1(\delta_2(\hat{u}) \cup [v]))^{-(p+1)}\), where \(\hat{u} \in \hat{H}^1(M; \mathbb{R}/\mathbb{Z})\), \([v] \in \hat{H}^2(M; \mathbb{R}/\mathbb{Z})\).

Via theorem 2.2, the abelian group \(\hat{H}^p(M; \mathbb{R}/\mathbb{Z})\) carries a natural topology coming from the \(C^\infty\)-topology on differential \((p+1)\)-forms and the standard topology on \(H^p(M, \mathbb{R})/r_*(\hat{H}^p(M, \mathbb{Z}))\). With respect to this topology, \(\ker\delta_2\) is the connected component of the identity of this group and consists of all so-called smooth differential characters, those which can be represented by smooth differential \(p\)-forms.

Furthermore there exists a Poincaré–Pontrjagin duality of differential characters [22]: Let \([M]\) denote the fundamental cycle and let \(<,>\) be the evaluation map. For each \(p\) with \(0 \leq p < n\), the pairing

\[
\hat{H}^{n-p-1}(M; \mathbb{R}/\mathbb{Z}) \times \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \to \mathbb{T}^1, \quad (\hat{v}, \hat{u}) \mapsto <\hat{v} \ast \hat{u}, [M]> \tag{2.5}
\]
is non-degenerate. The induced injective map

\[
\mathcal{D}: \hat{H}^{n-p-1}(M; \mathbb{R}/\mathbb{Z}) \to \hat{H}^p(M; \mathbb{R}/\mathbb{Z})^* := \text{Hom}(\hat{H}^p(M; \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}), \quad \mathcal{D}(\hat{v})(\hat{u}) := <\hat{v} \ast \hat{u}, [M]> \tag{2.6}
\]
has a dense range in the group of continuous homomorphisms into the 1-torus \(\mathbb{T}^1\). This range consists exactly of those homomorphisms \(\varrho \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})^*\), so that there exists a \(\hat{\varrho} \in \Omega^{n-p}_c(M; \mathbb{R})\) with \(\varrho([A]) = \hat{\varrho}(\int_M A \wedge \hat{\varrho})\) for all \(A \in \Omega^p(M; \mathbb{R})\). These circle-valued homomorphisms are called smooth. For a detailed discussion on the various topological issues we refer to [22].

3. The geometrical structure of the configuration space of generalized abelian gauge fields

Let \(\mathcal{A}^p := \Omega^p(M; \mathbb{R})\) denote the configuration space of the \(p\)-form gauge fields on the manifold \(M\). A generalized abelian gauge field of rank \(p\) is characterized by a \(p\)-form gauge field \(A \in \mathcal{A}^p\) and a cohomology class \(c \in \hat{H}^{p+1}(M; \mathbb{Z})\). \(\mathcal{A}^p\) can be equipped with the flat Riemann structure

\[
< A_1, A_2 > := \int_M A_1 \wedge A_2, \tag{3.1}
\]
where \(\ast\) is the Hodge star operator with respect to a fixed metric on \(M\), satisfying \(\ast^2 = (-1)^p\delta^{n-p}\) on differential \(p\)-forms. The co-differential \(d^p = (-1)^{n(p+1)+1} \ast d_{n-p} \ast: \Omega^p(M; \mathbb{R}) \to \Omega^{p-1}(M; \mathbb{R})\) gives rise to the Laplace operator \(\Delta_p = d^p_{p+1} + d_{p-1} d^p_p\). Let \(\text{Harm}^p(M)\) denote the space of \(\mathbb{R}\)-valued harmonic \(p\)-forms and let \(\text{Harm}^p(M)^\perp\) be its orthogonal complement. The Green’s operator [23] is defined by

\[
G_p: \Omega^p(M; \mathbb{R}) \to \text{Harm}^p(M)^\perp, \quad G_p = (\Delta_p|_{\text{Harm}^p(M)^\perp})^{-1} \circ \Pi^{\text{Harm}^p(M)^\perp}, \tag{3.2}
\]
where \(\Pi^{\text{Harm}^p(M)^\perp}\) is the projection of \(\Omega^p(M; \mathbb{R})\) onto \(\text{Harm}^p(M)^\perp\). By construction \(\Delta_p \circ G_p = G_p \circ \Delta_p = \Pi^{\text{Harm}^p(M)^\perp}\).

The gauge group of the generalized abelian gauge theory is taken to be the abelian group \(\mathcal{G}^p := \Omega^{p-1}(M; \mathbb{R}) \times \text{Harm}^p_2(M)\), where \(\text{Harm}^p_2(M) := \Omega^p_2(M; \mathbb{R}) \cap \text{Harm}^p(M)\). There is a natural action of \(\mathcal{G}^p\) on \(\mathcal{A}^p\), given by

\[
(A, (\alpha_{p-1}, \lambda_p)) \mapsto A \cdot (\alpha_{p-1}, \lambda_p) := A + d_{p-1} \alpha_{p-1} + \lambda_p, \quad (\alpha_{p-1}, \lambda_p) \in \mathcal{G}^p. \tag{3.3}
\]
Evidently this action is not free, possessing the abelian group \( \ker d_{p-1} \) as its isotropy group. Since there is only one orbit type, the abelian group of restricted gauge transformations \( G_p^0 := G_p / \ker d_{p-1} \) acts freely on \( \mathcal{A}^p \). Using the Hodge decomposition theorem it is easy to see that \( G_p^0 \) is isomorphic to \( ind_p^* \times Harm^0_p(M) \) provided by the map \( \alpha \mapsto (d_p G_p d_{p-1} \alpha, \lambda_p) \).

Let \( D_{n-p}: H^{n-p}(M; \mathbb{Z}) \to H^p(M; \mathbb{Z}) \), \( D_{n-p}(\nu) = \nu \cap [M] \) be the Poincaré duality isomorphism, where \( \cap \) is the cup product [24]. Since the homology of \( M \) is finitely generated with rank \( b_p \) we shall choose a set of \( p \)-cycles \( \gamma_i^{(p)} \in Z_p(M; \mathbb{Z}), i = 1, \ldots, b_p \), whose homology classes \( [\gamma_i^{(p)}] \) provide a Betti basis, thus generating the free part \( H_p(M; \mathbb{Z}) / Tor H_p(M; \mathbb{Z}) \) of \( H_p(M; \mathbb{Z}) \), where \( Tor H_p(M; \mathbb{Z}) \) denotes the torsion part of the \( p \)-th homology group. According to the following isomorphisms

\[
H^{n-p}(M; \mathbb{Z}) / Tor H^{n-p}(M; \mathbb{Z}) \cong H^n(Z; \mathbb{Z}) \cong Harm^0(M),
\]

a basis \( (\rho_i^{(p)})_{i=1}^{b_n} \in Harm^0_p(M) \) can be selected from the basis \( (D_{n-p}^{-1}([\gamma_i^{(p)}]))_{i=1}^{b_n-p} \) for the free part of \( H^{n-p}(M; \mathbb{Z}) \).

Using the Poincaré duality and the Universal Coefficient Theorem it follows that the product

\[
H^p(M; \mathbb{Z}) / Tor H^p(M; \mathbb{Z}) \times H^{n-p}(M; \mathbb{Z}) / Tor H^{n-p}(M; \mathbb{Z}) \to \mathbb{Z}
\]

\[
(\mu, \nu) \mapsto \mu \cdot D_{n-p}(\nu) \mapsto \mu \cup \nu, [M],
\]

(3.5)

gives a perfect pairing [24]. Hence a basis \( (\rho_i^{(p)})_{i=1}^{b_n} \in Harm^0_p(M) \) can be adjusted so that

\[
\int \gamma_j^{(p)} \rho_i^{(p)} = \int_M \rho_i^{(p)} \wedge \rho_j^{(n-p)} = \delta_{ij}
\]

(3.6)

holds, implying \( \int \gamma_j^{(p)} \alpha = \int_M \alpha \wedge \rho_j^{(n-p)} \) for any \( [\alpha] \in H^p(M; \mathbb{R}) \). Finally there exists an induced metric

\[
h_{ij}^{(p)} = \langle \rho_i^{(p)}, \rho_j^{(p)} \rangle
\]

(3.7)

on \( Harm^p(M) \). For \( p = 0 \), one has \( \rho^{(0)} = 1 \) and therefore \( h^{(0)} = Vol(M) \). There is an equivalent characterization of the restricted gauge group:

**Lemma 3.1.** There exists an isomorphism \( G_p^0 \cong \Omega_p^p(M; \mathbb{R}) \).

**Proof.** Using the Hodge decomposition theorem one verifies easily that the map \( \kappa_p : \Omega^p(M; \mathbb{R}) \to G_p^0 \), given by \( \kappa_p(\beta_p) := (d_p G_p \beta_p, \sum_{j=1}^{b_p} (\int \gamma_j^{(p)} \beta_p) \rho_j^{(n)}} \) is an isomorphism. \( \square \)

According to theorem 2.2, the gauge orbit space \( \mathcal{M}^p := \mathcal{A}^p / G_p^0 \) can be identified with the abelian subgroup of smooth differential characters. Let \( \pi_{\mathcal{A}^p} : \mathcal{A}^p \to \mathcal{M}^p \) with \( \pi_{\mathcal{A}^p}(A) := [A] \) be the natural projection. Now we will state the two main results of this section:

**Theorem 3.2.** The abelian group \( \mathcal{A}^p \) admits the structure of a non-trivial flat principle \( G_p^0 \)-bundle over \( \mathcal{M}^p \) with projection \( \pi_{\mathcal{A}^p} \). This bundle is trivializable if \( H^p(M; \mathbb{Z}) = 0 \).

**Proof.** We are going to construct a bundle atlas explicitly. For this we have to define an open cover of the gauge orbit space and a family of local sections. Let us consider the exact sequence of abelian groups

\[
0 \to \mathbb{Z}^{b_p} \to \mathbb{R}^{b_p} \to \mathbb{R}^{\sqrt{-1}} \to \mathbb{R}^{b_p} \to 1,
\]

(3.8)

which geometrically describes the universal covering of the \( b_p \)-dimensional torus. An open cover \( \mathcal{V}^{(p)} \) of \( \mathbb{T}^{b_p} \) is given by the following family of open sets

\[
\mathcal{V}^{(p)} = \{ V_a^{(p)} | a := (a_1, \ldots, a_j, \ldots, a_{b_p}), a_j \in \mathbb{Z}_2 = \{1, 2\} \},
\]

(3.9)
where each \( V_a^{(p)} = V_{a_1} \times \cdots \times V_{a_j} \times \cdots \times V_{a_k} \) is a open set in \( T^k \). Here \( V_1 = T^1\setminus\{\text{northernpole}\} \) for \( a_1 = 1 \) and \( V_2 = T^1\setminus\{\text{southernpole}\} \) for \( a_2 = 2 \) provide an open cover for each 1-torus \( T^1 \). Let us choose the following two local sections of the universal covering \( \mathbb{R}^1 \to T^1 \)

\[
\begin{align*}
\phi_{a_j}(z) &= \begin{cases} 
\frac{1}{2\pi} \arccos |(0,z)| \mathbb{R}z & \text{if } z \geq 0, \ a_j = 1 \\
\frac{1}{2\pi} \arccos |(\pi,2\pi)| \mathbb{R}z & \text{if } z < 0, \ a_j = 1 \\
\frac{1}{2\pi} \arccos |(\pi,2\pi)| \mathbb{R}z & \text{if } z \geq 0, \ a_j = 2,
\end{cases}
\end{align*}
\]

(3.10)

where \( z = \mathbb{R}z + \sqrt{-1} \mathbb{Z} z \in T^1 \). The locally constant transition functions \( g_{a_j,a_j'}^{(n)}: V_{a_j} \cap V_{a_j'} \to \mathbb{Z} \) are defined by

\[
\begin{align*}
g_{a_j,a_j'}^{(n)}(z_j) &= g_{a_j,a_j'}^{(n)}(z_j).
\end{align*}
\]

(3.11)

Thus we can generate a family of \( 2^{b_p} \) local sections \( s_a: V_a \subset T^p \to \mathbb{R}^{b_p}, \ s_a = (s_{a_1}, \ldots, s_{a_{b_p}}) \) with transition functions \( g_{a,a'}^{(n)} = (g_{a_1,a_1'}, \ldots, g_{a_{b_p},a_{b_p}}) \). These local sections will be used to construct a bundle atlas as follows: Let us define the following smooth surjective map \( \pi_{M^p}: M^p \to T^p \) by

\[
\pi_{M^p}([A]) = (e^{2\pi \sqrt{-1} f_M A \wedge \xi_1^{(n-p)}}, \ldots, e^{2\pi \sqrt{-1} f_M A \wedge \rho_{b_p}^{(n-p)}}).
\]

(3.12)

Then the family of open sets \( U_a^{(p)} := (\pi_{M^p})^{-1}(V_a^{(p)}) \) provides a finite open cover \( U = \{U_a^{(p)}\} \) of the infinite dimensional manifold \( M^p \). A bundle atlas is given by \( \varphi_a: U_a^{(p)} \times G^p_a \to (\pi_{M^p}^{-1}(U_a^{(p)}), \varphi_a([A], (\xi_1, \lambda_1)) = A \cdot (\omega_a(A))^{-1} - d_{p-1} \xi_{p-1} - \lambda_1 \) Its inverse reads \( \varphi_a^{-1}(A) = ([A], \omega_a(A)) \), where

\[
\begin{align*}
\omega_a^{(p)}: \pi_{A^p}(U_a^{(p)}) \to G^p_a, \quad \omega_a^{(p)}(A) &= (d_{p}^{*} G_{p} A, \sum_{j=1}^{b_p} \epsilon_a^{(p)}(A) \rho_{j}^{(p)}), \\
\epsilon_a^{(p)}: \pi_{A^p}(U_a^{(p)}) \to \mathbb{Z}, \quad \epsilon_a^{(p)}(A) &= \int_{M} (A \wedge \rho_{j}^{(n-p)}) - s_{a_j}(e^{2\pi \sqrt{-1} f_M A \wedge \rho_{j}^{(n-p)})}.
\end{align*}
\]

(3.13)

The locally constant transitions functions \( \tilde{g}_{a,a'}: U_a^{(p)} \cap U_{a'}^{(p)} \to G^p_a \) are finally given by

\[
\tilde{g}_{a,a'}([A]) = \left( 0, \sum_{j=1}^{b_p} g_{a_j,a_j'}^{(n)}(e^{2\pi \sqrt{-1} f_M A \wedge \rho_{j}^{(n-p)})}) \rho_{j}^{(p)} \right) \in G^p_a
\]

(3.14)

showing explicitly that the obstruction to trivialize the bundle belongs to \( H^p_7(M; \mathbb{R}) \). With respect to the basis \( (\rho_{j}^{(p)})_{j=1}^{b_p} \) the orthogonal projector onto \( Harm^p(M) \) becomes

\[
\Pi_{Harm^p(M)}(A) = \sum_{j,k=1}^{b_p} (h_{j,k}^{(p)})^{-1} < A, \rho_{j}^{(p)} > \rho_{k}^{(p)}, \ \forall A \in A^p.
\]

(3.15)

From \( \epsilon_a^{(p)}(A \cdot (\xi_{p-1}, \lambda_1)) = \epsilon_a^{(p)}(A) + \int_{\gamma_j^{(p)}} \lambda_1 \) and \( \Pi_{Harm^p(M)}(\lambda_1) = \sum_{j=1}^{b_p} (\int_{\gamma_j^{(p)}} \lambda_1) \rho_{j}^{(p)} \) one finally gets

\[
\omega_a^{(p)}(A \cdot (\xi_{p-1}, \lambda_1)) = \omega_a^{(p)}(A) \cdot (\xi_{p-1}, \lambda_1).
\]

Thus we have shown that it is impossible to choose a global gauge fixing condition, if the free part of \( H^p(M; \mathbb{Z}) \) is non-vanishing. Furthermore the subgroup of smooth differential characters has the following bundle structure:
Theorem 3.3. The manifold $\mathcal{M}^p$ admits the structure of a trivializable vector bundle over $\mathbb{T}^{b_p}$ with typical fiber $\text{ind}^{p+1}_{p+1}$ and projection $\pi_{\mathcal{M}^p}$.

Proof. A bundle atlas is provided by the diffeomorphisms $\chi_a: V_a^{(p)} \times \text{ind}^{p+1}_{p+1} \to \mathcal{M}^p$

$$\chi_a(z_1, \ldots, z_\beta, \tau_p) = \sum_{j=1}^{\beta} s_{a_j}(z_j) \rho_j^{(p)} + \tau_p$$

$$\chi_a^{-1}([A]) = ([A], d_{p+1}^{p} G_{p+1} = d_p A) \quad (3.16)$$

On each fiber $(\pi_{\mathcal{M}^p})^{-1}((z_1, \ldots, z_\beta))$ there is a unique structure of a real vector space induced by the bundle chart $\chi_a$, giving rise to a natural vector bundle structure on $\mathcal{M}^p$. $\Box$

The abelian group $\mathcal{A}^p$ acts freely on $\hat{H}^p(M; \mathbb{R}/\mathbb{Z})$ by $\hat{v} \mapsto \hat{v} \cdot j_2([A])$ for $[A] \in \mathcal{M}^p$. Let us denote by $\mathcal{M}^p_a$ the corresponding orbit (i.e. the $\mathcal{M}^p$-torsor) through $\hat{u} \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})$. Then the homomorphism $\delta_2$ is constant on $\mathcal{M}^p_a$. We define the projections $\pi_{\mathcal{M}^p_a}: \mathcal{A}^p \to \mathcal{M}^p_a$, by $\pi_{\mathcal{M}^p_a}(A) = \hat{u} \cdot j_2([A])$ and $\pi_{\mathcal{M}^p_a}: \mathcal{M}^p_a \to \mathbb{T}^{b_p}$ by $\pi_{\mathcal{M}^p_a}(\hat{v}) = \pi_{\mathcal{M}^p}([A])$, where $\hat{v} = \hat{u} \cdot j_2([A])$.

Corollary 3.4. For any fixed $\hat{u} \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})$, the following holds:

1. The abelian group $\mathcal{A}^p$ admits the structure of a non-trivial flat principal $\mathcal{G}_p^{\hat{a}}$-bundle over $\mathcal{M}^p_a$ with projection $\pi_{\mathcal{A}^p}^\hat{a}$.
2. The manifold $\mathcal{M}^p_a$ admits the structure of a trivializable vector bundle over $\mathbb{T}^{b_p}$ with typical fiber $\text{ind}^{p+1}_{p+1}$ and projection $\pi_{\mathcal{M}^p_a}$.

Proof. Let $U(\hat{u}) = \{U^\hat{a}_\beta \}$ be the induced open cover of $\mathcal{M}^p_a$, where $U^\hat{a}_\beta = \{ \hat{u} : j_2([A]) \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) | [A] \in U^{(p)}_a \}$. A bundle atlas for $\mathcal{A}^p$ is given by $\varphi^\hat{a}_\beta : U^\hat{a}_\beta \times \mathcal{G}_p^\hat{a} \to (\pi_{\mathcal{A}^p}^\hat{a})^{-1}(U^\hat{a}_\beta)$,

$$\varphi^\hat{a}_\beta(\hat{v}, (\xi_{p-1}, \lambda_p)) = A \cdot (\omega^p_a(A))^{-1} + d_{p-1} \xi_{p-1} + \lambda_p, \quad (\varphi^\hat{a}_\beta)^{-1}(A) = (\hat{u} \cdot j_2([A]), \omega^p_a(A)), \quad (3.17)$$

for $\hat{v} = \hat{u} \cdot j_2([A])$. A bundle atlas for $\mathcal{M}^p_a$ is provided by the diffeomorphisms $\chi^\hat{a}_\beta: V_a^{(p)} \times \text{ind}^{p+1}_{p+1} \to \mathcal{M}^p_a$ namely

$$\chi^\hat{a}_\beta(z_1, \ldots, z_\beta, \tau_p) = \hat{u} \cdot j_2(z_1, \ldots, z_\beta, \tau_p), \quad (\chi^\hat{a}_\beta)^{-1}(\hat{v}) = \left( \pi_{\mathcal{M}^p_a}(\hat{v}), d_{p+1}^{p} G_{p+1}(\delta_1(\hat{v}) - \delta_1(\hat{u})) \right) \quad (3.18)$$

On each fiber $(\pi_{\mathcal{M}^p_a})^{-1}(z_1, \ldots, z_\beta)$ there is a unique structure of a real vector space induced by the bundle chart $\chi^\hat{a}_\beta$, giving rise to a natural vector bundle structure on $\mathcal{M}^p_a$. $\Box$

In fact, for any fixed $\hat{u}$, the bundle $\mathcal{A}^p \to \mathcal{M}^p_a$ can be regarded equivalently as pull-back of $\mathcal{A}^p \to \mathcal{M}^p$ via the diffeomorphism $\Upsilon_{\hat{u}}: \mathcal{M}^p_a \to \mathcal{M}^p$ defined by $\Upsilon_{\hat{u}}([A]) := [A]$, where $\hat{v} = \hat{u} \cdot j_2([A])$. Fields belonging to $\mathcal{M}^p_a$ may be called inequivalent generalized gauge fields with topological type $\delta_2(\hat{u}) \in H^{p+1}(M; \mathbb{Z})$.

Corollary 3.5. For any $\hat{u}$, the manifold $\mathcal{M}^p_a$ possesses the following topological information:

$$H^k(\mathcal{M}^p_a, \mathbb{Z}) = H^k(\mathbb{T}^{b_p}, \mathbb{Z}) = \mathbb{Z}^{(p)}$$

$$\pi_1(\mathcal{M}^p_a) = \pi_0(\mathcal{G}^p_a) = \mathbb{Z}^{b_p}$$

$$\pi_k(\mathcal{M}^p_a) = \pi_{k-1}(\mathcal{G}^p_a) = 0 \quad k \geq 2 \quad (3.19)$$

$\Box$

There is one remaining question: How does the choice of the fixed differential character $\hat{u}$ affect the vector bundle structure of $\mathcal{M}^p_a$?
Proposition 3.6. For any two fixed differential characters \( \hat{u}_1, \hat{u}_2 \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \) one gets:

1. \( \mathcal{A}^p \xrightarrow{\pi \mathcal{A}^p_{\hat{u}_1}} \mathcal{M}^p_{\hat{u}_1} \) and \( \mathcal{A}^p \xrightarrow{\pi \mathcal{A}^p_{\hat{u}_2}} \mathcal{M}^p_{\hat{u}_2} \) are isomorphic as principal \( \mathcal{G}^p \)-bundles,
2. \( \mathcal{M}^p_{\hat{u}_1} \xrightarrow{\pi \mathcal{M}^p_{\hat{u}_1}} \mathbb{T}^{bp} \) and \( \mathcal{M}^p_{\hat{u}_2} \xrightarrow{\pi \mathcal{M}^p_{\hat{u}_2}} \mathbb{T}^{bp} \) are isomorphic with respect to their vector bundle structures.

Proof. Let us consider the invertible map \( \Upsilon^{\hat{u}_1, \hat{u}_2} : \mathcal{M}^p_{\hat{u}_1} \to \mathcal{M}^p_{\hat{u}_2} \), with \( \Upsilon^{\hat{u}_1, \hat{u}_2}(\hat{u}) := \hat{u}_2 \cdot j_2([A]) \), for \( \hat{u}_2 = \hat{u}_1 \cdot j_2([A]) \). Then \( \Upsilon^{\hat{u}_1, \hat{u}_2} \) fits into the following commutative diagram of bundles

\[
\begin{array}{ccc}
\mathcal{A}^p & \xrightarrow{\pi \mathcal{A}^p_{\hat{u}_1}} & \mathcal{A}^p \\
\downarrow \pi_{\mathcal{A}^p_{\hat{u}_1}} & & \downarrow \pi_{\mathcal{A}^p_{\hat{u}_2}} \\
\mathcal{M}^p_{\hat{u}_1} & \xrightarrow{\Upsilon^{\hat{u}_1, \hat{u}_2}} & \mathcal{M}^p_{\hat{u}_2} \\
\downarrow \pi_{\mathcal{M}^p_{\hat{u}_1}} & & \downarrow \pi_{\mathcal{M}^p_{\hat{u}_2}} \\
\mathbb{T}^{bp} & \xrightarrow{} & \mathbb{T}^{bp}
\end{array}
\]

(3.20)

proving that both the principal \( \mathcal{G}^p \)-bundle structure as well as the vector bundle structure are compatible. □

4. The partition function for the generalized abelian gauge theory

In this section we want to introduce a well-defined partition function for generalized abelian gauge theories. Furthermore we will study the vacuum expectation value (VEV) of gauge invariant quantities.

A gauge invariant observable is any complex-valued (continuous) function on \( \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \). For every arbitrary but fixed differential character \( \hat{u} \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z}) \) and any gauge invariant observable \( f \), there is an induced map \( f^\hat{u} : \mathcal{M}^p \to \mathbb{C} \) defined by \( f^\hat{u}([A]) := f(\hat{u} \cdot j_2([A])) \).

Our proposal for constructing an appropriate functional integral takes up a method which has been applied successfully to the stochastic quantization of Yang-Mills theory [19,20]. The main idea is to define an integrable partition function on the original field space \( \mathcal{A}^p \) by implementing a regularization of the volume of the gauge group, however, without affecting the VEV of gauge invariant observables. It will be shown that this requirement is related to the problem of gauge fixing.

Now we follow the three steps, which have been indicated in section 1. Let \( S^{(p)}_{reg} \) denote a real-valued function on \( \mathcal{G}^p \) such that the volume of the restricted gauge group

\[
Vol(\mathcal{G}^p; e^{-S^{(p)}_{reg}}) := \int_{\mathcal{G}^p} e^{-S^{(p)}_{reg}} = \sum_{\lambda_p \in Harm^p(M)} \int_{\text{im}d^p} D\tau_{p-1} e^{-S^{(p)}_{reg}(\tau_{p-1}, \lambda_p)}
\]

(4.1)

becomes finite. Analogously we assume the existence of a functional \( Q^{(p)} \), which renders the volume

\[
Vol(\mathcal{G}^p; Q^{(p)}) = \int_{\mathcal{G}^p} e^{-Q^{(p)}} = \sum_{\lambda_p \in Harm^p(M)} \int_{\Omega^{p-1}(M)} D\alpha_{p-1} Q^{(p)}(\alpha_{p-1}, \lambda_p)
\]

(4.2)

of the total gauge group \( \mathcal{G}^p \) finite. Below we will give explicit expressions for \( S^{(p)}_{reg} \) and \( Q^{(p)} \). For \( p = 0 \), \( \mathcal{G}^0 = \mathcal{G}^0 = \mathbb{Z} \) and thus one can set \( Q^{(0)} = S^{(0)}_{reg} \).

Let \( S_{\text{inv}}(\hat{u}) \) denote the gauge invariant classical action of the generalized abelian gauge theory. We introduce the following family of locally defined volume forms on the open sets \( \pi_{\mathcal{A}^p_{\hat{u}_0}}^{-1}U^p(\hat{u}_0) \)

\[
\Xi_\alpha(\hat{u}_0) := Vol_{\mathcal{A}^p_{\hat{u}_0}} e^{-S^{(p)}_{\text{inv}}(\omega^{(p)}_{\hat{u}_0})} S^{(p)}_{reg},
\]

(4.3)
where $\text{vol}_{\mathcal{A}^p}$ denotes the (formal) volume form on $\mathcal{A}^p$ induced by (3.1). Each of these local volume forms is damped along the gauge orbits and yields a VEV of gauge invariant observables which is independent of the explicit form for $S_{\text{reg}}$. Due to the gauge ambiguities, $\Xi_u$ cannot be extended to a global differential form on the whole configuration space $\mathcal{A}^p$ in a natural way. However, in order to obtain a global integrand for the functional integral, we will glue these local forms together using a partition of unity $\{\tilde{g}_a^{(p)}\}$ on $\mathcal{A}^p$ subordinate to the open cover $\mathcal{U}$. In fact, let $\{\tilde{g}_{aj}, a_j \in \mathbb{Z}_2, j\}$ be a partition of unity on the $j$-th 1-torus $T^1$ within $T^b_{\mathcal{A}^p}$ subordinate to the open cover $\{V_{aj}\}$.

By means of the projection $q_j: T^b_{\mathcal{A}^p} \to T^1$, $q_j(z_1, \ldots, z_j, \ldots, z_{b_p}) = z_j$ onto the $j$-th one-torus $T^1$, the functions $g^{(p)}_a := \prod_{j=1}^{b_p} g^*_j a_j$, induce a partition of unity of $T^b_{\mathcal{A}^p}$ subordinate to $\{V^{(p)}_a\}$. Finally $\hat{g}^{(p)}_a := (\pi_{\mathcal{A}^{p+1}})^* g^{(p)}_a$ is the sought-after partition of unity subordinate to the open cover $\mathcal{U}$ of the gauge orbit space $\mathcal{M}^p$.

**Lemma 4.1.** To every $c \in H^{p+1}(M;\mathbb{Z})$, one can assign a differential character $\hat{\delta}_0 \in \tilde{H}^p(M;\mathbb{R}/\mathbb{Z})$ — hereafter called background differential character — satisfying $\delta_2(\hat{\delta}_0) = c$ in such a way that $d_{p+1}^* \delta_1(\hat{\delta}_0) = 0$. A background differential character is unique up to the those smooth differential characters which are induced by closed differential p-forms.

**Proof.** Let $\hat{\delta}_0 \in \tilde{H}^p(M;\mathbb{R}/\mathbb{Z})$. Then the modified differential character $\hat{\delta}_0 = \hat{\delta}_0 \cdot j_2([-G,P_d^* \delta_1(\hat{\delta}_0)])$ satisfies the requested equation. Moreover, $\delta_2(\hat{\delta}_0) = \delta_2(\hat{\delta}_0) = c$. A direct calculation shows that any other background differential character relates to $\hat{\delta}_0$ by $\hat{\delta}_0 \cdot j_2([B])$, where $d_B = 0$. □

Let $\{\hat{\delta}_0 | c \in H^{p+1}(M;\mathbb{Z})\}$ be a family of background differential characters. We introduce the following functional on gauge invariant observables $f$, by

$$
\mathcal{I}^{(p)}(f) := \sum_{c \in H^{p+1}(M;\mathbb{Z})} \frac{1}{\text{Vol}(\mathcal{G}^p;\mathcal{Q}^{(p)})} \int_{\mathcal{A}^p} \sum_{a \in \mathbb{Z}_2^{b_p}} (\pi_{\mathcal{A}^{p+1}})^* g^{(p)}_a(\hat{\delta}_0)(\pi_{\mathcal{A}^{p+1}} f \hat{\delta}_0),
$$

where $a = (a_1, \ldots, a_{b_p}) \in \mathbb{Z}_2^{b_p}$ is a multi-index. Here the finite volume of the gauge group $\mathcal{G}^p$ is factored out in order to eliminate all unphysical degrees of freedom. Now we can state the main definition of this paper:

**Definition 4.2.** The partition function of the generalized abelian gauge theory is defined by

$$
\mathcal{Z}^{(p)} := \mathcal{I}^{(p)}(1).
$$

The VEV of a gauge invariant observable $f$ is defined by

$$
\mathcal{E}^{(p)}(f) := \frac{\mathcal{I}^{(p)}(f)}{\mathcal{Z}^{(p)}}.
$$

In order to make our concept explicit, we will consider the generalized p-form Maxwell theory. This field theory is governed by the classical action

$$
S_{\text{inv}}(\hat{u}) = \frac{1}{2} \| \delta_1(\hat{u}) \|^2 = \frac{1}{2} \int_M \delta_1(\hat{u}) \wedge * \delta_1(\hat{u}).
$$

**Examples.**

1. If $H^{p+1}(M;\mathbb{Z}) = 0$, every differential character $\hat{u}$ is smooth; i.e. $\hat{u} = j_2([A])$ for some $A \in \mathcal{A}^p$. Since $\delta_1(\hat{u}) = d_p A = F_A$, (4.7) reduces to the classical action for the p-form Maxwell theory.

2. In the case $p = 0$, one has $\tilde{H}^0(M;\mathbb{R}/\mathbb{Z}) \cong C^\infty(M;\mathbb{T}^1)$. It follows from theorem 2.2 that $\delta_1(\hat{u}) = \hat{u} \ast \vartheta$, where $\vartheta$ is the Maurer Cartan form on $\mathbb{T}^1$. So we recover the action $S_{\text{inv}}(\hat{u}) = \frac{1}{2} \| \hat{u} \ast \vartheta \|^2$ for circle-valued scalar fields on $M$.

The explicit calculation of (4.4) will be done in five steps: The integration over $\mathcal{A}^p$ is separated into an integration over the base manifold $T^b_{\mathcal{A}^p}$ and an integration over the fiber $\mathcal{G}^p \times \text{im}d_{p+1}^*$. Second, the division by the gauge group volume is carried out. In the third step it is shown that (4.4) does
not depend on the concrete choice for the background connections. The summation over the topological sectors is performed in step four. Finally we construct realizations for $S^{(p)}_reg$ and $Q^{(p)}$ in order to obtain an explicit expression for the partition function in the original configuration space $A^p$.

Split of the integral over $A^p$. Let us introduce the following family of local diffeomorphisms, $\psi_a = \varphi_a \circ \left( \chi_a \times \mathbb{I} \right) : V_a^{(p)} \times \text{im}d_{p+1}^* \times \mathbb{C}^2 \to (\pi_{A^p} \circ p_{A^p})^{-1}(V_a^{(p)})$, using the results of theorems 3.2 and 3.3. The induced local metrics are

$$
((\psi_a)^* \langle \cdot, \cdot \rangle)(z_1, \ldots, z_b, \tau_p, (\tau_{p-1}, \lambda_p)) \left( (w_1^1, \ldots, w_{b_p}^1, u_p^1, (v_{p-1}^1, 0)), (w_1^2, \ldots, w_{b_p}^2, u_p^2, (v_{p-1}^2, 0)) \right) = \\
\frac{1}{(2\pi)^{2}} \sum_{k=1}^{b_p} \psi_{z_j}(w_j)^2 \psi_z(w_j^2) h_j^p + <u_p^1, u_p^2> + <v_{p-1}^1, \Delta_{p-1} | \text{im} d_{p}^* v_{p-1}^2>,
$$

where $z_j \in \mathbb{T}^1$, $w_j \in T_z \mathbb{T}^1$ for $j = 1, \ldots, b_p$, $u_p \in T_{\tau_p} \text{im} d_{p+1}^*$, $(v_{p-1}, 0)) \in \mathcal{T}_{\tau_{p-1}, \lambda_p} \mathbb{C}^2$ and $\bar{\mathcal{D}}$ denotes the complex conjugate of the Maurer Cartan form on $\mathbb{T}^1$. The corresponding volume form is given by

$$
\psi^*_{a} \text{vol}_{A^p} = \begin{cases} \\
\frac{1}{2\pi}(\text{vol}(M))^{1/2} \text{vol}_{\mathcal{M}_a^1} |_{V_a^{(p)}} \wedge \text{vol}_{\text{im}d_{p+1}^*} & \text{if } p = 0 \\
\frac{1}{(2\pi)^{2}} \text{(det}(h^p))^{1/2} (\text{det}(\Delta_{p-1} | \text{im} d_{p}^* v_{p-1}^2))^{1/2} \text{vol}_{\mathcal{M}_a^1} |_{V_a^{(p)}} \wedge \text{vol}_{\text{im}d_{p+1}^*} & \text{if } p \neq 0,
\end{cases}
$$

where $\text{vol}_{\mathcal{M}_a^1}$, $\text{vol}_{\text{im}d_{p+1}^*}$ denotes the flat metric on $\text{im} d_{p+1}^*$, which is induced by (3.1). Since the bundle $A^p \to \mathcal{M}^p$ is flat, the local volume forms in (4.3) can be glued together to yield a global volume form on the product space $\mathbb{T}^b \times \text{im} d_{p+1}^*$, which is understood in terms of zeta-regularization [16]: For any non-negative self-adjoint elliptic operator $\mathcal{B}$ its regularized determinant can be defined by

$$
\text{det} \mathcal{B} = \exp \left( -\frac{d}{ds} |_{s=0} \zeta(s|\mathcal{B}) \right),
$$

where $\zeta(s|\mathcal{B})$ is the zeta-function of the operator $\mathcal{B}$, namely

$$
\zeta(s|\mathcal{B}) = \sum_{\nu_j \neq 0} \nu_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\mathcal{B}} - \Pi^B) dt.
$$

Here $\nu_j$ are the non-vanishing eigenvalues of $\mathcal{B}$ and $\Pi^B$ is the orthonormal projector onto the kernel of $\mathcal{B}$. The $\zeta$-function is analytic at the origin and possesses a meromorphic extension over $\mathbb{C}$.

Let $f$ be an arbitrary gauge invariant observable. Due to its gauge invariance, $f_{\pi_a} := (\chi_a)^* f_{\tilde{\mathcal{A}}}^a$ can be extended to a globally defined function $f_{\tilde{\mathcal{A}}}^a$ on $\mathbb{T}^b \times \text{im} d_{p+1}^*$.

Let $e_{(m_1, \ldots, m_{b_p})}(z_1, \ldots, z_{b_p}) := z_1^{m_1} \cdots z_{b_p}^{m_{b_p}}$ be an orthonormal basis of $L^2(\mathbb{T}^b; \mathbb{C})$ with respect to the inner product $\langle f_1, f_2 \rangle := \frac{1}{(2\pi)^b} \int_{\mathbb{T}^b} \text{vol}_{\text{im} d_{p+1}^*} f_1 f_2$, where $z_j \in \mathbb{T}^1$ and $m_j \in \mathbb{Z}$. So $f_{\pi_a}^{\tilde{\mathcal{A}}}(\cdot, \tau_p)$ can be rewritten in terms of a Fourier series expansion as

$$
f^{\tilde{\mathcal{A}}}(z_1, \ldots, z_{b_p}, \tau_p) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_{b_p} \in \mathbb{Z}} f^{\tilde{\mathcal{A}}}_{(m_1, \ldots, m_{b_p})}(\tau_p, z_1^{m_1} \cdots z_{b_p}^{m_{b_p}}),
$$

where $z_j = e^{2\pi \sqrt{-1} w_j}$ and the Fourier coefficients read

$$
f^{\tilde{\mathcal{A}}}_{(m_1, \ldots, m_{b_p})}(\tau_p) = \int_0^1 \cdots \int_0^1 dw_1 \cdots dw_{b_p} f^{\tilde{\mathcal{A}}}_{\pi_a}(e^{2\pi \sqrt{-1} w_1}, \ldots, e^{2\pi \sqrt{-1} w_{b_p}}, \tau_p) e^{-2\pi \sqrt{-1} \sum_{j=1}^{b_p} m_j w_j}.
$$
Using (4.9) and (4.13) the non-normalized VEV of gauge invariant observables (4.4) can be rewritten in the form

\[ T^0(f) = \text{Vol}(M)^{1/2} \sum_{c \in H^2(M;\mathbb{Z})} e^{-\frac{1}{2} \|([\omega]_0)^*c\|^2} \int_{\text{imd}\_p} D\tau_0 \tilde{\tau}^{\omega}_0(\tau_0) e^{-\frac{1}{2} \langle \tau_0, \Delta_{[\text{imd}\_p]} \tau_0 \rangle}, \tag{4.14} \]

for \( p = 0 \) and

\[ T^p(f) = (\det h^{(p)})^{1/2}(\det \Delta_{p-1}[\text{imd}\_p])^{1/2} \frac{\text{Vol}(G^p; e^{-S^{(p)}_{reg}})}{\text{Vol}(G^p; Q^{(p)})} \sum_{c \in H^{p+1}(M;\mathbb{Z})} e^{-\frac{1}{2} \|c([\omega]_0)^*c\|^2} \times \int_{\text{imd}\_p} D\tau_0 \tilde{\tau}^{\omega}_0(\tau_0) e^{-\frac{1}{2} \langle \tau_0, \Delta_{[\text{imd}\_p]} \tau_0 \rangle}, \tag{4.15} \]

for \( p \neq 0 \).

**Geometry and regularization of the gauge group.** The next task is to calculate the factor \( \frac{\text{Vol}(G^p; e^{-S^{(p)}_{reg}})}{\text{Vol}(G^p; Q^{(p)})} \) in (4.15). We will prove that this fraction equals the inverse of the regularized volume of the isotropy group of the action of \( G^p \) on \( A^p \). Let us consider the following two families of abelian groups \( G^k := \Omega_{k}^{-1}(M) \times \text{Harm}^{k}_{\mathbb{Z}}(M) \) and \( G^k_{\text{reg}} := \text{imd}_{k} \times \text{Harm}^{k}_{\mathbb{Z}}(M) \) with \( k = 1, \ldots, p-1 \).

**Proposition 4.3.** For any \( k, 1 \leq k \leq p \), \( \text{ker}d_{k-1} \) admits the structure of a non-trivial flat principal \( G^{k}_{\text{reg}}^{-1} \)-bundle over \( T^{k_{b-1}} \) with projection \( \tilde{\pi}_{k-1}(\beta_{k-1}) = (e^{2\pi \sqrt{-1} f_{j(k-1)}^{(k-1)} \beta_{k-1}}, \ldots, e^{2\pi \sqrt{-1} f_{j(k-1)}^{(k-1)} \beta_{k-1}}) \).

**Proof.** This result follows directly from theorems 3.2 and 3.3. Like in (3.9) we choose the open cover \( \mathcal{V}^{(k-1)} \) of \( T^{k_{b-1}} \) and the following bundle atlas

\[ \tilde{\psi}^{(k-1)}_a: V^{(k-1)}_a \times G^{k}_{\text{reg}}^{-1} \to \tilde{\pi}_{k-1}^{-1}(V^{(k-1)}_a) \]

\[ \tilde{\psi}^{(k-1)}_a(z_1, \ldots, z_{b_{k-1}}, \xi_{k-2}, \lambda_{k-1}) = \left( \sum_{j=1}^{b_{k-1}} \beta_j^{(k-1)} + k_{b-2}\xi_{k-2}, \lambda_{k-1} \right) \]

\[ (\tilde{\psi}^{(k-1)}_a)^{-1}(\beta_{k-1}) = (\tilde{\pi}_{k-1}(\beta_{k-1}), \omega^{(k-1)}_{(\beta_{k-1})}), \tag{4.16} \]

where \( \omega^{(k-1)}_a \) is taken as in (3.13) yet \( p = k-1 \).

For each \( k = 1, \ldots, p \), let us take a regularizing function \( S^{(k)}_{reg} \) for the abelian group \( G^{k}_{\text{reg}} \), such that \( \text{Vol}(G^{k}_{\text{reg}}; e^{-S^{(k)}_{reg}}) \) becomes finite. Then \( \tilde{Q}^{(k-1)} := \sum_{a \in \mathbb{Z}^{k_{b-1}}} (\tilde{\pi}_{k-1}^{-1}(a) \omega^{(k-1)}_{a}) e^{-\omega^{(k-1)}_{a} S^{(k)}_{reg}} \) provides a regularization for the isotropy group \( \text{ker}d_{k-1} \), since

\[ \text{Vol}(\text{ker}d_{k-1}; \tilde{Q}^{(k-1)}) = \int_{\text{ker}d_{k-1}} v_{\text{ker}d_{k-1}} \tilde{Q}^{(k-1)} \]

\[ = (\det h^{(k-1)})^{1/2}(\det \Delta_{k-2}[\text{imd}_{k-1}])^{1/2} \text{Vol}(G^{k}_{\text{reg}}^{-1}; e^{-S^{(k-1)}_{reg}}), \tag{4.17} \]

is finite. Here \( v_{\text{ker}d_{k-1}} \) is the volume form on \( \text{ker}d_{k-1} \) induced by the flat metric (3.1).

**Proposition 4.4.** For any fixed \( k, 1 \leq k \leq p \), the abelian group \( G^{k} \) admits the structure of a non-trivial flat principal \( G^{k}_{\text{reg}}^{-1} \)-bundle over the base manifold \( T^{k_{b-1}} \times G^{k}_{\text{reg}} \) with projection \( \pi_{k-1}(\alpha_{k-1}, \lambda_{k}) := (\pi_{A^{k}} \circ \pi_{A^{k}}(\alpha_{k-1}), d_{k}^{0}G_{\beta}d_{k-1}\alpha_{k-1}, \lambda_{k}) \).
**Proof.** The free right action is given by \((\alpha_{k-1}, \lambda_k) \cdot (\eta_{k-2}, \sigma_k) := \langle \alpha_{k-1} + d_{k-2} \eta_{k-2} + \sigma_k, \lambda_k \rangle\), where \((\alpha_{k-1}, \lambda_k) \in G^k\) and \((\eta_{k-2}, \sigma_k) \in G^{k-1}\). With respect to the open cover \(V^{(k-1)}\) of \(\mathbb{T}^{b_{k-1}}\) (3.9), the bundle atlas is

\[
\psi_{a}^{(k-1)}: V^{(k-1)} \times G^{k} \times G^{k-1} \rightarrow \pi_{k-1}^{-1}(V^{(k-1)} \times G^{k}) \leq G^{k}
\]

\[
\psi_{a}^{(k-1)}(z_{1}, \ldots, z_{b_{k-1}}, \xi_{k-1}, \lambda_{k}, \eta_{k-2}, \sigma_{k-1}) = \sum_{j=1}^{b_{k-1}} \delta_{a_{j}}(z_{j}) \rho_{j}^{(k-1)} + \xi_{k-1} + d_{k-2} \eta_{k-2} + \sigma_{k-1}, \lambda_{k}
\]

\[
(\psi_{a}^{(k-1)})^{-1}(\alpha_{k-1}, \lambda_{k}) = (\pi_{k-1}(\alpha_{k-1}, \lambda_{k}), \omega_{a}^{(k-1)}(\alpha_{k-1})).
\]

\[\square\]

One obtains for the induced volume form

\[\left(\psi_{a}^{(k-1)}\right)^{*} vol_{G^{k}} = \frac{1}{(2\pi)^{b_{k-1}}} \det(h^{(k-1)})^{1/2} \det(\Delta_{k-2} \det_{1-1})^{1/2} vol_{\mathbb{T}^{b_{k-1}}} \wedge vol_{G^{k}} \wedge vol_{G^{k-1}}.\]

(4.19)

Finally, \(Q^{(k)} := \sum_{a \in \sigma^{k-1}_{o}} \left(pr_{1} \circ \pi_{k-1}^{(1)}\right) g_{a}^{(k-1)} e^{-\left(pr_{2} \circ \pi_{k-1}^{(1)}\right) \left(S^{(k)}_{reg} - \left(\omega^{(k-1)}\right)_{reg}^{(k-1)}\right)}\) regularizes the volume of \(G^{k}\) for any \(k = 1, \ldots, p\), where \(pr_{1}\) and \(pr_{2}\) denote the projections from \(\mathbb{T}^{b_{k-1}} \times G^{k}\) onto the first and the second factor, respectively. The volume of \(G^{k}\), \(k = 1, \ldots, p\), reads

\[Vol(G^{k}; Q^{(k)}) = (\det(h^{(k)}))^{1/2} (\det(\Delta_{k-2} \det_{1-1}))^{1/2} Vol(G^{k}; e^{-S^{(k)}_{reg}})Vol(G^{k-1}; e^{-S^{(k-1)}_{reg}}).\]

(4.20)

Thus the volume of the restricted gauge group \(G^{k}\) can be expressed in terms of the volume of the restricted gauge group \(G^{k-1}\) of one degree lower. By induction on (4.20) and using (4.17), one finds for \(p \neq 0\)

\[\frac{Vol(G^{p}; e^{-S^{(p)}_{reg}})}{Vol(G^{p}; Q^{(p)})} = \left(Vol(kerd_{p-1}, \tilde{Q}^{(p-1)})^{-1}\right) = \prod_{j=0}^{p-1} (\det(h^{(j)}))^{\frac{1}{2}} (\det(\Delta_{j} \det_{j+1}))^{\frac{1}{2}} (\det(\Delta_{j} \det_{j+1}))^{\frac{1}{2}} \prod_{j=0}^{p-1} Vol(G^{j}; Q^{(j)})^{(-1)}(p-1)^{p-1}.\]

(4.21)

For \(p = 0\), one has \(Vol(G^{0}; e^{-S^{(0)}_{reg}}) = Vol(G^{0}; Q^{(0)})\).

Eq. (4.21) can be interpreted as generalization of the ghost-for-ghost contribution, which has been derived in the topologically trivial context based on either the Faddeev-Popov approach [12-15] or the technique of resolvents of differential operators [17]. Geometrically, this contribution traces back to the non-free action of the gauge group \(G^{p}\) on \(A^{p}\). In our analysis, (4.21) is the result of subsequent fiber integrations. In the following we choose the functionals \(Q^{(j)}\) such that \(Vol(G^{j}; Q^{(j)}) = 1\).

**Independence of the background connection.** For the moment the expression \(I^{(p)}(f)\) in (4.4) seems to depend on the choice for the family of background differential characters. Let us take a different family \(\tilde{v}_{0}^{a}\) with \(\delta_{s}(\tilde{v}_{0}^{a}) = c\). Hence there exists a family of classes \([B^{w}] \in \mathcal{M}^{p}\) with \(d_{p}B^{w} = 0\) such that \(\tilde{v}_{0}^{a} = \tilde{u}_{0}^{a} (B^{w})\). According to theorem 3.3 one can find a \(w := (w_{1}, \ldots, w_{b_{p}}) \in \mathbb{T}^{b_{p}}\) such that \([B^{w}] = \chi_{a}(w_{1}, \ldots, w_{b_{p}})\). If we define the left translation \(l_{w}\) on \(\mathbb{T}^{b_{p}}\) by \(l_{w}(z) := (w_{1}z_{1}, \ldots, w_{b_{p}}z_{b_{p}})\), then \(f^{\tilde{v}_{0}^{a}} = l_{w}^{*}f^{\tilde{v}_{0}^{a}}\) which finally yields \(f^{\tilde{v}_{0}^{a}}(0, \ldots, 0) = \tilde{f}_{0}^{\tilde{v}_{0}^{a}}(0, \ldots, 0)\). Together with \(\tilde{S}_{inv}^{\tilde{v}_{0}^{a}} = \tilde{S}_{inv}^{\tilde{v}_{0}^{a}}\), we get \(\Xi_{a}(\tilde{v}_{0}^{a}) = \Xi_{a}(\tilde{u}_{0}^{a})\) and so we have proved that \(I^{(p)}(f)\) is indeed independent of the chosen set of background differential characters.

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Sum over topological sectors. Since the cohomology of $M$ is finitely generated, $c \in H^{p+1}(M; \mathbb{Z})$ admits the following (non-canonical) decomposition

$$ c = \sum_{j=1}^{b_{p+1}} m_j f_j^{(p+1)} + \sum_{k=1}^{r} y_k l_k^{(p+1)} ,$$

where $(f_j^{(p+1)})_{j=1}^{b_{p+1}}$ denotes a Betti basis of $H^{p+1}(M; \mathbb{Z})$ and $m_j \in \mathbb{Z}$. Furthermore $\text{Tor}^H H^{p+1}(M; \mathbb{Z})$ is generated by a basis $(l_k^{(p+1)})_{k=1}^{r}$ and $y_k \in Z_{t_k}$. By definition there exists a series of elements $l_1, \ldots, l_r \in \mathbb{N}$ such that $l_k t_k^{(p+1)} = 0$ for each $k = 1, \ldots, r$. Evidently, the order of the torsion subgroup, denoted by $\text{ord}(\text{Tor}^H H^{p+1}(M; \mathbb{Z}))$, is given by $\prod_{k=1}^{r} l_k$. Let $\rho_j^{(p+1)} \in \text{Harm}_{\mathbb{Z}}^{p+1}(M; \mathbb{R})$, for $j = 1, \ldots, b_{p+1}$, be a basis of harmonic $(p+1)$-forms on $M$ with integer periods, and let $h_j^{(p+1)} = \langle \rho_j^{(p+1)} , \rho_j^{(p+1)} \rangle$ denote the induced metric on $\text{Harm}^{p+1}(M; \mathbb{R})$. By lemma 4.1 one has $\delta_1(\tilde{u}_0) \in \text{Harm}_{\mathbb{Z}}^{p+1}(M; \mathbb{R})$, so that

$$ \delta_1(\tilde{u}_0) = \sum_{k=1}^{b_{p+1}} m_k \rho_k^{(p+1)} , \quad m_k = \sum_{j=1}^{b_{p+1}} (h_j^{(p+1)})^{-1} < \delta_1(\tilde{u}_0) , \rho_j^{(p+1)} > \in \mathbb{Z}. $$

According to the Hodge decomposition theorem and the fact that $\det \Delta_{p+1|\text{imd}} = \det \Delta_{p|\text{imd}^*_{p+1}}$ [25], the determinant of the restricted Laplacian $\Delta_p|\text{Harm}^{p+1}(M)\perp$ factorizes into

$$ \det (\Delta_p|\text{Harm}^p(M)\perp) = \det (\Delta_p|\text{imd}^*_{p+1}) \cdot \det (\Delta_{p-1|\text{imd}^*_p}). $$

By induction one obtains

$$ \prod_{j=0}^{p} (\det \Delta_j|\text{imd}^*_{p+1})^{1/2} (1-p) = \prod_{j=0}^{p} (\det \Delta_j|\text{Harm}^j(M)\perp)^{1/2} (1-p)^{(p+1-j)(p+1-j)} . $$

Till now there is one step left, namely the summation over the cohomology classes in $M^{p+1}(M; \mathbb{Z})$. According to (4.22) this sum is split into two parts, one over the components of the free part, the other one over the torsion part of $c$. In order to perform this sum, let us recall the definition of the Riemann Theta function: For $\Lambda$ being a symmetric complex $k \times k$ dimensional square matrix whose imaginary part is positive definite, $b \in \mathbb{C}^k$ the $k$-dimensional Theta function is defined by

$$ \Theta_k(b; \Lambda) = \sum_{n \in \mathbb{Z}^k} \exp \left\{ \pi \sqrt{-1} n^\dagger \cdot \Lambda \cdot n + 2\pi \sqrt{-1} n^\dagger \cdot b \right\}, $$

where the superscript $\dagger$ denotes the transpose. Evidently, the Theta function possesses the symmetry $\Theta_k(b + m; \Lambda) = \Theta_k(b; \Lambda)$ for all $m \in \mathbb{Z}^k$.

**Theorem 4.5.** For the generalized $p$-form Maxwell theory, the partition function $\mathcal{Z}^{(p)}$, $0 \leq p \leq n$, is given by

$$ \mathcal{Z}^{(p)} = \prod_{j=0}^{p} \left( \frac{\det \Delta_j|\text{Harm}^j(M)\perp)^{(1-p)j}}{\det h_j^{(1)}(j)} \right)^{1/2} \Theta_{b_{p+1}} \left( 0 | -\frac{h_j^{(p+1)}}{2\pi \sqrt{-1}} \right) \text{ord}(\text{Tor}^H H^{p+1}(M; \mathbb{Z})). $$

The VEV of a gauge invariant observable $f$ admits the following form:

1. For $p = 0$:

$$ \mathcal{E}^{(0)}(f) = \frac{(\det \Delta_0|\text{imd}^*_0)^{1/2} \sum_{c \in H^1(M; \mathbb{Z})} e^{-\frac{1}{2}||\tilde{u}_0^c||^2} \int_{\text{imd}_0^c} D\tau_0 f_{\tilde{0}_0^c}(\tau_0) \cdot e^{-\frac{1}{2} < \tau_0 , \Delta_0|\text{imd}^*_0> \tau_0 > \text{ord}(\text{Tor}^H H^{p+1}(M; \mathbb{Z}))}}{\Theta_{b_{1}} \left( 0 | -\frac{h_j^{(1)}}{2\pi \sqrt{-1}} \right)} .$$

(4.28a)
(2) For \( p \neq 0 \):

\[
\mathcal{E}^{(p)}(f) = \frac{(\det \Delta_p|_{\text{Harm}^p(M) \perp})^{1/2}}{\Theta_{p+1}^p} \sum_{\mathbf{c} \in \mathbb{H}^{p+1}(M; \mathbb{Z})} e^{-\frac{1}{2}||\mathbf{c}||^2} \int_{\text{wd}\mathbf{c}} e^{-\frac{1}{2} <\tau_p, \Delta_p|_{\text{imd}_{p+1}^\star} \tau_p>} \cdot D\tau_p \mathcal{F}^{(p)}_{(0, \ldots, 0)}(\tau_p) \cdot (\det \Delta_{p-1}|_{\text{imd}_{p-1}^\star})^{1/2} \Theta_{p+1} \cdot (0| - \frac{k^{p+1}}{2\pi} \frac{1}{\Gamma}) \text{ord}(\text{TorH}^{p+1}(M; \mathbb{Z}))
\]

\[\text{(4.28a)}\]

\[\blacksquare\]

Due to the invariance under unimodular transformations the partition function is independent of the chosen basis of harmonic forms.

For manifolds which are acyclic for all \( j \) satisfying \( 0 < j \leq p + 1 < n \), one can choose \( \hat{u}_0^p = 1 \). It follows from theorem 3.3 that \( \mathcal{M}^p \cong \text{imd}_{p+1}^\star \), implying that the gauge orbit space coincides with the space of transversal fields. Then (4.27) reduces to

\[
\mathcal{Z}^{(p)} = \text{Vol}(M)\frac{1}{2}(n-1)^{p} \prod_{j=0}^{p} (\det \Delta_j|_{\text{imd}_{j+1}^\star})^{\frac{1}{2}(n+1-j)}.
\]

\[\text{(4.29)}\]

which up to the finite volume of \( M \) agrees with the known result for the partition function of the \( p \)-form Maxwell theory [18]. The reason is that the gauge groups are \( \text{imd}_{j-1}^\star \times \text{Harm}_{\perp}^j(M) \) in our setting compared with \( \text{imd}_{j-1} \) used in [15,18] for \( j = 0, \ldots, p \). These groups agree for \( j \neq 0 \) in the acyclic case.

Additionally, we recover the duality relation between the partition functions in various degrees: Let us begin with the definition of the Ray-Singer analytic torsion \( \tau(M) \) of the Riemannian manifold \( M \) [26]

\[
\tau(M) := \exp \left( \frac{1}{2} \sum_{k=0}^{n} (-1)^k k \frac{d}{ds} |_{s=0} \zeta(s) |_{\text{Harm}^k(M) \perp} \right) \prod_{k=0}^{n-1} (\det \Delta_k|_{\text{Harm}^k(M) \perp})^{(-1)^{k+1} \frac{1}{2}},
\]

\[\text{(4.30)}\]

which by (4.24) can be rewritten into the product \( \tau(M) = \prod_{k=0}^{n-1} (\det \Delta_k|_{\text{imd}_{k+1}^\star})^{\frac{1}{2}(n-k)} \). Using the fact that \( *\Delta_p = \Delta_{p-1}^\star \) holds, one can verify easily that

\[
\det \Delta_j|_{\text{imd}_{j+1}^\star} = \det \Delta_{n-j-1}|_{\text{imd}_{n-j}^\star},
\]

\[\text{(4.31)}\]

From (4.29) we finally obtain the following duality relation [18],

\[
\frac{\mathcal{Z}^{(p)}}{\mathcal{Z}^{(n-p-2)}} = \left( \frac{\tau(M)}{\text{Vol}(M)\frac{1}{2}(n+1)} \right)^{(n-p+1)}. \]

\[\text{(4.32)}\]

In even dimension \( \tau(M) = 1 \) and thus the partition functions \( \mathcal{Z}^{(p)} \) and \( \mathcal{Z}^{(n-p-2)} \) coincide.

In summary, a modified functional integral has been introduced for the quantization of generalized abelian gauge theories. It was then shown that the VEV of gauge invariant observables

(1) is independent of the choice for the regularization of the gauge groups,
(2) is independent of the local trivialization,
(3) is independent of the choice for the partition of unity,
(4) reproduces the conventional result for acyclic manifolds, yet the volume of the gauge group can be absorbed into a finite normalization constant.

**An explicit choice for the regularizing functions.** The results (4.27) and (4.28) rely on the assumption that appropriate regularizing functions \( S^{(k)}_{reg} \) and \( Q^{(k)} \) do really exist. Let us define:
(1) For $k=0$:

$$
e^{-S_{req}(\lambda_0)} = Q^{(0)}(\lambda_0) = \Theta_1 \left( 0 - \frac{Vol(M)}{2\pi \sqrt{-1}} \right)^{-1} e^{\frac{1}{2\pi \sqrt{-1}} \lambda_0^2 \cdot Vol(M)}, \quad \lambda_0 \in \mathbb{Z}. \quad (4.33)$$

(2) For $k = 1, \ldots, p$:

$$
e^{-S_{req}(\xi_k, \lambda_k)} = \prod_{l=0}^{k-1} \left( \frac{\det \Delta_l |imd^*_l+1}{\det h^{(l)}} \right)^{\frac{1}{2}} \frac{\det (\Delta_{k-1} |imd^*_k)^{1/2}}{\Theta_{b_k} \left( 0 - \frac{h^{(k)}}{2\pi \sqrt{-1}} \right)} \cdot \frac{\Theta^{(-1)^{k+1}-1} \cdot (\det \Delta_{k-1} |imd^*_k)^{\frac{1}{2}}}{\Theta_{b_k} \left( 0 - \frac{h^{(k)}}{2\pi \sqrt{-1}} \right)}. \quad (4.34)$$

By a direct calculation one finds that

$$\sum_{\lambda_k \in Harm^k(M)} \int D\xi_k e^{-\frac{1}{2\pi \sqrt{-1}} \|\Delta_{k-1} |imd^*_k \xi_k - \frac{1}{2}\lambda_k\|^2} = (\det \Delta_{k-1} |imd^*_k)^{-1} \Theta_{b_k} \left( 0 - \frac{h^{(k)}}{2\pi \sqrt{-1}} \right). \quad (4.35)$$

which in summary leads to the following finite volumes of the gauge groups

$$Vol(G^k_\epsilon, e^{-S_{req}(\xi_k)}) = \begin{cases} 1 & \text{if } k = 0 \\ (\det (\Delta_{k-1} |imd^*_k)^{-\frac{1}{2}} \prod_{l=0}^{k-1} \left( \frac{\det \Delta_l |imd^*_l+1}{\det h^{(l)}} \right)^{\frac{1}{2}} \Theta_{b_k} \left( 0 - \frac{h^{(k)}}{2\pi \sqrt{-1}} \right) & \text{if } k \neq 0. \end{cases} \quad (4.36)$$

Eq. (4.20) implies that $Vol(G^k_\epsilon, Q^{(k)}) = 1$ for all $k = 0, \ldots, p$. Given this choice, we are able to write down the partition function (4.5) in the original field space $A^p$. Using (3.13), one finds

$$Z^{(p)} = \int_{A^p} vol_{A^p} F^{(p)}(A) e^{-\frac{1}{4} <A, \Delta_v A>}, \quad (4.37)$$

with a non-negative functional $F^{(p)}(A)$, which for $p = 0$ yields

$$F^{(0)}(A) = \frac{\Theta_{b_1} \left( 0 - \frac{h^{(1)}}{2\pi \sqrt{-1}} \right)}{\Theta_1 \left( 0 - \frac{Vol(M)}{2\pi \sqrt{-1}} \right)} \sum_{a_1=1}^{2} \tilde{g}_{a_1} (e^{2\pi \sqrt{-1} \int_{M} A \wedge \rho^{(n)}(A)}) e^{-\frac{1}{4} \int_{M} Vol(M)(\tau_0^{(0)}(A))^2}, \quad (4.38)$$

and for $p \neq 0$ this functional is given by

$$F^{(p)}(A) = \frac{\Theta_{b_{p+1}} \left( 0 - \frac{h^{(p+1)}}{2\pi \sqrt{-1}} \right)}{\Theta_{b_p} \left( 0 - \frac{h^{(p)}}{2\pi \sqrt{-1}} \right)} \prod_{j=0}^{p-1} \left( \frac{\det \Delta_j |imd^*_j+1}{\det h^{(j)}} \right)^{\frac{1}{2}(-1)^{p+1-j}} \times \sum_{a_1=1}^{2} \cdots \sum_{a_{b_p}=1}^{2} \tilde{g}_{a_1} (e^{2\pi \sqrt{-1} \int_{M} A \wedge \rho^{(n-p)}}(A)) \cdots \tilde{g}_{a_{b_p}} (e^{2\pi \sqrt{-1} \int_{M} A \wedge \rho^{(n-p)}}(A)) e^{-\frac{1}{4} \sum_{j,k=1}^{b_p} h^{(j)} \rho^{(p)}(A) \tau_{a_k}^{(p)}(A)} \times \text{ord}(TorH^{p+1}(M; \mathbb{Z})). \quad (4.39)$$

If $M$ is acyclic in dimension $0 < k \leq p + 1 < n$, the bundle $A^p \to A^p/imd_{p-1}$ is trivializable and the bundle chart (3.14) gives the Hodge decomposition of $A \in A^p$. This guarantees the existence of a global smooth gauge fixing submanifold in $A^p$. The corresponding partition function is

$$Z^{(p)} = \left( Vol(M) \right)^{\frac{1}{2}(-1)^{p}} \prod_{j=0}^{p-1} \left( \det \Delta_j |imd^*_j+1 \right)^{\frac{1}{2}(-1)^{p+1-j}} \int_{A^p} vol_{A^p} e^{-\frac{1}{4} <A, \Delta_v A>}. \quad (4.40)$$
For $p = 0$, (4.37) reduces simply to the partition function for (real-valued) scalar fields. Up to the factor $\text{Vol}(M)$, (4.40) compares to [15], where the partition function for the $p$-form Maxwell theory has been derived based on the Faddeev-Popov technique in the original field space $\mathcal{A}^p$.

In the topologically non-trivial case, the functional $\mathcal{F}^p(A)$ guarantees finiteness of (4.37). In fact, if one considered the conventional gauge fixing term $\frac{1}{2}||d_A^* A||^2$ coming from the Faddeev-Popov approach, then the functional integral $\int_{\mathcal{A}^p} \exp (-\frac{1}{2} A, \Delta_{\rho} A >)$ would become infinite. This can be easily seen by rewriting this integral in terms of the local trivialization of $\mathcal{A}^p \rightarrow \mathcal{M}^p$ and is a consequence of the fact that the integrand is not damped along gauge transformations not connected to unity. Our method solves this problem by introducing an appropriate regularization, however, without affecting the VEV of gauge invariant observables.

### 5. The Green’s functions for the generalized $p$-form Maxwell theory

In this section we want to determine the one-point- and two-point functions for the gauge field $A \in \mathcal{A}^p$ in the generalized $p$-form Maxwell theory. Since these functions are not gauge invariant, one could expect additional contributions resulting from the non-trivial structure of configuration space and the regularization of the gauge group.

Like in the $p$-form Maxwell theory, the Green’s functions are generated by the the vacuum-to-vacuum transition amplitude in the presence of a source $J \in \Omega^p(M; \mathbb{R})$ which in our approach takes the form

$$Z^{(p)}[J] = \frac{1}{\text{Vol}(\mathcal{G}^p; Q^{(p)})} \sum_{c \in H^{p+1}(M; \mathbb{Z})} \int_{\mathcal{A}^p} \sum_{a \in \mathbb{Z}^b_2} (((\pi_{\mathcal{A}^p})^* g_a^{(p)}) \Xi_a(u^0_A, A) \exp <J, A>).$$

(5.1)

The $q$-point Green’s functions $S_q^{(p)}$ are defined by

$$S_q^{(p)}(v_1, \ldots, v_q) := \frac{\partial^q}{\partial t_1 \cdots \partial t_q} |_{t_1=\ldots=t_q=0} \frac{Z^{(p)}[\sum_{i=1}^q t_i v_i]}{Z^{(p)}[0]},$$

(5.2)

for $v_1, \ldots, v_q \in \Omega^p(M; \mathbb{R})$. By construction, the Green’s functions are independent of the chosen background differential character. In order to derive an explicit expression for (5.1), we take the choice (4.30) and (4.31) for $S^{(k)}_1$, $k = 0, \ldots, p$. In terms of the local trivialization $\{ \psi_a \}$ a lengthy calculation gives

$$Z^{(p)}[J] = \left( \frac{1}{(2\pi)^{p}} \prod_{j=0}^{p} \left( \frac{\det \Delta_j |_{imd_{p+1}}}{\det h^{(j)}} \right) \right)^{\frac{1}{2}(p-1)^{i}} \cdot e^{-\frac{1}{2} <J, G_{\rho} J>}

\times \Theta_{b_p} \left( K^{(p)}(J) - \frac{h^{(p)}}{2\pi \sqrt{1}} \right) \Theta_{b_p+1} \left( 0 - \frac{h^{(p+1)}}{2\pi \sqrt{1}} \right) \cdot \text{ord}(\text{Tor} H^{p+1}(M; \mathbb{Z}))

\times \int_{\mathbb{T}^p} \text{vol}^{b_p}_{\mathbb{T}} \sum_{a_1=1}^{2} \cdots \sum_{a_{b_p}=1}^{2} q_1^* \tilde{g}_{a_1} \cdots q_{b_p}^* \tilde{g}_{a_{b_p}} \cdot e^{2\pi \sum_{j=1}^{b_p} q_j^* s_j} <J, \rho_j^{(p)}>.$$

(5.3)

where $K^{(p)}(J) := <J, \rho_j^{(p)}> \; \text{with} \; j = 1, \ldots, b_p$ is regarded as $b_p$-dimensional vector, denoted by $K^{(p)}(J)$. Defining the two field independent factors

$$\varepsilon_j^{(1)} := \frac{1}{2\pi} \int_{\mathbb{T}^1} \text{vol}^{b_{1}} \sum_{a_j=1}^{2} \tilde{g}_{a_j} s_{a_j},$$

$$\varepsilon_{j,k}^{(2)} := \frac{1}{(2\pi)^{b_p}} \int_{\mathbb{T}^{b_p}} \text{vol}^{b_p}_{\mathbb{T}} \sum_{a_j=1}^{2} \cdots \sum_{a_{b_p}=1}^{2} q_j^* \tilde{g}_{a_1} \cdots q_{b_p}^* \tilde{g}_{a_{b_p}} \cdot q_j^* s_{a_j} \cdot q_k^* s_{a_k}.$$

(5.4)

one finally ends up with the following result:
Proposition 5.1. Let us regularize the gauge group by (4.33) and (4.34). Then the following holds for the Green’s functions:

1) One-point function:

\[ S_1^{(p)}(v) = \sum_{j=1}^{b_p} \varepsilon_j^{(1)} < v, \rho_j^{(p)} > + \frac{d}{dt}|_{t=0} \ln \Theta_{b_p}(K^{(p)}(tv)) - \frac{h^{(p)}}{2\pi \sqrt{-1}}. \]  

\[ (5.5) \]

2) Two-point function:

\[ S_2^{(p)}(v_1, v_2) = < v_1, G_p v_2 > + \sum_{j=1}^{b_p} \varepsilon_j^{(1)} < v_1, \rho_j^{(p)} > + \frac{d}{dt}|_{t=0} \ln \Theta_{b_p}(K^{(p)}(tv_2)) - \frac{h^{(p)}}{2\pi \sqrt{-1}} \]

\[ + \sum_{j=1}^{b_p} \varepsilon_j^{(1)} < v_2, \rho_j^{(p)} > + \frac{d}{dt}|_{t=0} \ln \Theta_{b_p}(K^{(p)}(tv_1)) - \frac{h^{(p)}}{2\pi \sqrt{-1}} \]

\[ + \Theta_{b_p}(0| - \frac{h^{(p)}}{2\pi \sqrt{-1}})^{-1} \frac{\partial^2}{\partial t_1 \partial t_2}|_{t_1 = t_2 = 0} \Theta_{b_p}(K^{(p)}(\sum_{l=1}^{2} t_lv_l)) - \frac{h^{(p)}}{2\pi \sqrt{-1}} \]  

\[ (5.6) \]

\[ \Box \]

On manifolds with vanishing \( p \)-th Betti number, the one-point function vanishes, whereas the two-point function reduces to the Green’s operator \( G_p \).

The numerical factors in (5.4) can be easily calculated in terms of a natural partition of unity for \( \mathbb{T}^1 \). Taking the following local coordinate system of \( \mathbb{T}^1 \)

\[ v_1: V_1 \rightarrow (0, 1) \quad v_1^{-1}(t) = (\cos 2\pi t, \sin 2\pi t) \]

\[ v_2: V_2 \rightarrow (-\frac{1}{2}, \frac{1}{2}) \quad v_2^{-1}(t) = (\cos 2\pi t, \sin 2\pi t). \]  

\[ (5.7) \]

A partition of unity subordinate to \( V_j \subset \mathbb{T}^1 \) is induced by the periodic functions \( \hat{g}_1(t) = \sin^2(\pi t) \) and \( \hat{g}_2(t) = \cos^2(\pi t) \). A calculation yields for (5.4)

\[ \varepsilon_j^{(1)} = \frac{3}{4} \]

\[ \varepsilon_{j,k}^{(2)} = \left\{ \begin{array}{ll}
(2\pi)^{-1}(\frac{17\pi}{12} - \frac{1}{2}) & \text{for } j = k \\
(2\pi)^{-1}(\frac{7\pi}{12}) & \text{for } j \neq k.
\end{array} \right. \]  

\[ (5.8) \]

How do these factors depend on the choice of the local section \( s_a \) in (3.10)? Any other local section \( s'_a \) of (3.8) is connected with the section \( s_a \) by \( s'_a := (s'_a_1, \ldots, s'_a_{b_p}) = (s_{a_1} + m_{a_1}^1, \ldots, s_{a_p} + m_{a_p}^1) \) with \( m_{a_j}^j \in \mathbb{Z} \) for \( j = 1, \ldots, b_p \). In terms of these new sections, the factors (5.4) read

\[ \varepsilon_j^{(1)} = \frac{1}{2}(m_j^1 + m_j^2 + \frac{3}{2}) \]

\[ \varepsilon_{j,k}^{(2)} = \left\{ \begin{array}{ll}
(2\pi)^{-1}(\frac{17\pi}{12} - \frac{1}{2} + \pi(m_j^1 + 1 + m_k^1 + 1 + m_j^2 + m_k^2 + 2)) & \text{for } j = k \\
\frac{1}{4}(m_j^1 + m_j^2 + \frac{3}{2})(m_k^1 + m_k^2 + \frac{3}{2}) & \text{for } j \neq k.
\end{array} \right. \]  

\[ (5.9) \]

Thus it is not possible to arrange a local trivialization of \( \mathcal{A}^p \) in such a way that the topological contributions in the Green’s functions would vanish. The non-vanishing of the one-point function and the occurrence of additional contributions in the two-point function are caused by the non-trivial geometric structure of the configuration space and the finiteness of the volume of the gauge degrees of freedom.
One-point functions in special dimensions. Let us conclude with two simple examples for the one-point function for \( p = 0 \) and \( p = n \), respectively. Since \( \rho^{(0)} = 1 \) and \( \rho^{(n)} = v_0 \delta_{\alpha M} \), the corresponding one-point functions read

\[
\begin{align*}
S^{(0)}_1(w) &= \frac{3}{4} \int_M *w, \quad w \in \Omega^0(M; \mathbb{R}), \\
S^{(n)}_1(v) &= \frac{3}{4} \frac{1}{Vol(M)} \int_M v, \quad v \in \Omega^n(M; \mathbb{R}).
\end{align*}
\]

(5.10)

6. The VEV of special gauge invariant observables

6.1. Smooth homomorphisms and the Poincaré-Pontrjagin duality

The Poincaré-Pontrjagin duality for differential characters induces a specific class of gauge invariant observables in a natural way. In fact, each \( \hat{v} \in H^{n-p-1}(M; \mathbb{R}/\mathbb{Z}) \) gives rise to a homomorphism \( \mathcal{D}(\hat{v}) \in H^p(M; \mathbb{R}/\mathbb{Z})^* \). Now we will study the VEV of these gauge invariant observables.

By theorem 2.2 we choose a set of differential characters for \( f_j^{(p)} \), with \( j = 1, \ldots, b_{p+1} \), and \( \hat{\tau}_k^{(p)} \), with \( k = 1, \ldots, r \), in \( H^p(M; \mathbb{R}/\mathbb{Z}) \) such that

\[
\begin{align*}
\delta_1(f_j^{(p)}) &= \rho_j^{(p+1)}, \\
\delta_2(\hat{\tau}_k^{(p)}) &= 0, \\
\delta_2(\hat{\tau}_k^{(p+1)}) &= 1.
\end{align*}
\]

(6.1)

The last line of (6.1) implies that there exists \( [v_k'] \in H^p(M; \mathbb{R}/\mathbb{Z}) \) such that \( \hat{\tau}_k^{(p)} = j_1([v_k']) \) for \( k = 1, \ldots, r \). Moreover, since \( \delta_2((\hat{\tau}_k^{(p)})^{l_k}) = 0 \) with torsion coefficients \( l_k \), there exists a family of closed differential forms \( B_k \in \Omega^p(M; \mathbb{R}) \) such that \( (\hat{\tau}_k^{(p)})^{l_k} = j_2([B_k]) \). Hence \( \hat{\tau}_k^{(p)} := \hat{\tau}_k^{(p)} \cdot j_2([-\frac{1}{l_k} B_k]) \) fulfills (6.1) but satisfies \( (\hat{\tau}_k^{(p)})^{l_k} = 1 \) for each \( k = 1, \ldots, r \). Hence there are cohomology classes \( [v_k] \in H^p(M; \mathbb{R}/\mathbb{Z}) \) such that \( \hat{\tau}_k^{(p)} = j_1([v_k]) \). In summary we will take the following choice for the background differential character

\[
\hat{v}_0 = \prod_{j=1}^{b_{p+1}} (\hat{f}_j^{(p)})^{m_j} \prod_{k=1}^{r} (\hat{\tau}_k^{(p)})^{n_k}.
\]

(6.2)

Coming back to (4.28) the zeroth Fourier coefficient of \( \mathcal{D}(\hat{v}) \) is given by

\[
\mathcal{D}(\hat{v})^{\hat{v}_0}_{(0, \ldots, 0)}(\tau_p) = \mathcal{D}(\hat{v})(\hat{v}_0) e^{2\pi \sqrt{-1}(-1)^{n-p} \int_M \delta_1(\hat{v}) \wedge \tau_p} \prod_{j=1}^{b_p} \int_0^1 dw_j e^{2\pi \sqrt{-1}((-1)^{n-p} w_j \int_M \delta_1(\hat{v}) \wedge \rho_j^{(p)} - \int_M \delta_1(\hat{v}) \wedge \rho_j^{(p)})}
\]

\[
= \begin{cases} 
0 & \text{if } \int_M \delta_1(\hat{v}) \wedge \rho_j^{(p)} = 0 \\
\mathcal{D}(\hat{v})(\hat{v}_0) e^{2\pi \sqrt{-1}(-1)^{n-p} \int_M \delta_1(\hat{v}) \wedge \tau_p} & \text{if } \int_M \delta_1(\hat{v}) \wedge \rho_j^{(p)} \neq 0
\end{cases}
\]

(6.3)

where \( j = 1, \ldots, b_p \). In order to get a non-vanishing \( \mathcal{I}^{(p)}(\mathcal{D}(\hat{v})) \) one has to demand that \( [\delta_1(\hat{v})] = 0 \) in \( H^{n-p}(M; \mathbb{R}) \), implying that \( \delta_2(\hat{v}) \in ker r_\alpha \), where the third exact sequence of theorem 2.2 has been used. According to the exact sequence in (2.3) there exists a class \( [w] \in H^{n-p-1}(M; \mathbb{R}/\mathbb{Z}) \), such that \( \delta^*([w]) = \delta_2(\hat{v}) \). From the exact sequences in theorem 2.2 we can finally conclude that there exists a \( \hat{B} \in \Omega^{n-p-1}(M; \mathbb{R}) \) such that \( \hat{v} = j_1([w]^{-1} j_2([\hat{B}])) \). Inserting this expression for \( \hat{v} \) and using the properties of the product in (2.4), we find after a straightforward calculation
\[
\mathcal{D}(\hat{v})(\hat{u}^\delta) = < j_1([w]^{-1} \cup c), [M] > e^{\frac{2\pi \sqrt{-1}}{2} j_1(\hat{u}^\delta)}.
\] (6.4)

Together with the decomposition (4.22) and (4.23) we are now ready to perform the summation over the cohomology classes \(c \in H^{p+1}(M; \mathbb{Z})\) in (4.28), namely

\[
\sum_{c \in H^{p+1}(M; \mathbb{Z})} \mathcal{D}(\hat{v})(\hat{u}^\delta) e^{-\frac{1}{2} \sum_{\delta_1} \|\delta_1(\hat{u}^\delta)\|^2} = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_r \in \mathbb{Z}} b_{p+1} \prod_{j=1}^{b_{p+1}} < [w]^{-1} \cup f_j^{(p+1)}, [M] >^{m_j}
\]

\[
\times e^{-\frac{1}{2} \sum_{i,j=1}^{b_{p+1}} h_{ij} m_i m_j + 2\pi \sqrt{-1} \sum_{j=1}^{b_{p+1}} m_j \int_M B \wedge \rho_j^{(p+1)}}
\]

\[
\times \sum_{y_1=1}^{l_1-1} \cdots \sum_{y_r=1}^{l_r-1} \prod_{k=1}^{r} < [w]^{-1} \cup t_k^{(p+1)}, [M] >^{y_k}
\]

(6.5)

where

\[
\sum_{y_1=1}^{l_1-1} \cdots \sum_{y_r=1}^{l_r-1} \prod_{k=1}^{r} < [w]^{-1} \cup t_k^{(p+1)}, [M] >^{y_k} = \begin{cases} 0, & \text{if } < [w]^{-1} \cup t_k^{(p+1)}, [M] > \neq 1 \\ \text{ord}(\text{Tor} H^{p+1}(M; \mathbb{Z})), & \text{if } < [w]^{-1} \cup t_k^{(p+1)}, [M] > = 1, \end{cases}
\]

(6.6)

where \(k = 1, \ldots, r\). In order to get a non-vanishing VEV, one has to demand that

\[
< [w]^{-1} \cup f_j^{(p+1)}, [M] > = < [w]^{-1}, \mathcal{D}(t_k^{(p+1)}) >= 1,
\]

(6.7)

implying that \([w]\) vanishes on \(\text{Tor} H_{n-p-1}(M; \mathbb{Z})\). Let us define the map \(\iota([w]) \in \text{Hom}(H_p(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z})\) by \(\iota([w])(\sigma) := < [w], \sigma > \) for all \(\sigma \in H_{n-p-1}(M; \mathbb{Z})\). Since \(\iota([w])\) vanishes on torsion classes and \(H_{n-p-1}(M; \mathbb{Z})/\text{Tor} H_{n-p-1}(M; \mathbb{Z})\) is a free abelian group, it can be extended to a homomorphism \(\hat{\iota} \in \text{Hom}(H_{n-p-1}(M; \mathbb{Z}), \mathbb{R})\). By the universal coefficient theorem [24] there exist a \([\beta] \in H^{n-p-1}(M; \mathbb{R})\) such that \(\hat{\iota}(\sigma) = < [\beta], \sigma >\). But then \([q \circ \beta]([w]) = < [w], \sigma >\) for all \(\sigma \in H_{n-p-1}(M; \mathbb{Z})\). Since \(\iota\) is an isomorphism between \(H^{n-p-1}(M; \mathbb{R}/\mathbb{Z})\) and \(\text{Hom}(H_{n-p-1}(M; \mathbb{Z}), \mathbb{R}/\mathbb{Z})\) it follows that \(q_*([\beta]) = [q \circ \beta] = [w]\). Let us represent the cohomology class \([\beta]\) by a closed differential form \(\beta \in \Omega^{n-p-1}(M; \mathbb{R})\), then we get \(\hat{v} = j_1(q_*([-\beta]))j_2(B) = j_2([B] - \beta)\). In summary, we have verified:

**Proposition 6.1.** The VEV of the Poincaré dual differential character \(\mathcal{D}(\hat{v}) \in \hat{H}^p(M; \mathbb{R}/\mathbb{Z})^*\) vanishes, unless \(\hat{v} \in \text{im}j_2\). \(\square\)

If we represent the real cohomology classes \(f_j^{(p+1)}\) by harmonic forms, then the first factor on the right hand side of (6.5) becomes

\[
< [w]^{-1} \cup f_j^{(p+1)}, [M] > = q(< -[\beta] \cup f_j^{(p+1)}, [M] >) = q(-\int_M \beta \wedge \rho_j^{(p+1)}).
\]

(6.8)

Let us write \(C := B - \beta\), then \(\hat{v} = j_2([C])\) and \(\delta_1(\hat{v}) = d_{n-p-1}C\). Inserting all our findings into (4.28) and performing the Gaussian integral over \(\tau_p \in \text{im} \delta_{p+1}\) finally gives

\[
\mathcal{E}^{(p)}(\mathcal{D}(\hat{v})) = \Theta_{b_{p+1}} \left( K^{(p+1)}(C) - \frac{h^{(p+1)}}{2\pi \sqrt{-1}} \right) e^{-\frac{1}{2} \int_M C \wedge \rho_j^{(p+1)}}.
\]

(6.9)

where \(K_j^{(p+1)}(C) := \int_M C \wedge \rho_j^{(p+1)}\) with \(j = 1, \ldots, b_{p+1}\) is regarded as \(b_{p+1}\)-dimensional vector. The choice of \(C\) is unique up to elements in \(\Omega_Z^{n-p-1}(M; \mathbb{R})\). Due to the invariance property of the Theta function, any different choice for \(C\) would give the same result for \(\mathcal{E}^{(p)}(\mathcal{D}(\hat{v}))\).
6.2. The Wilson operator

The Wilson operator is a prominent example for a gauge invariant observable. For any p-cycle \( \Sigma \in Z_p(M; \mathbb{Z}) \) with induced homology class \([\Sigma] \in H_p(M; \mathbb{Z})\) the Wilson operator is defined by \( W: Z_p(M; \mathbb{Z}) \to H^p(M; \mathbb{R}/\mathbb{Z})^*, \ W(\Sigma)(\hat{u}) := \hat{u}(\Sigma) \). In contrast to the previous subsection, \( W(\Sigma) \) is not in the smooth dual of \( H^p(M; \mathbb{R}/\mathbb{Z}) \), i.e. does not belong to the range of the Poincaré–Pontryagin map \( D \) [22]. We begin with the calculation of the zeroth Fourier coefficient of \( W(\Sigma) \)

\[
\hat{W}(\Sigma) \frac{\partial}{\partial y_0} (\tau_p) = \hat{u}_0(\Sigma) e^{2\pi \sqrt{-1} \int_{\Sigma} \tau_p} \prod_{j=1}^{b_p} dw_j \ e^{2\pi \sqrt{-1} \int_{Y_j} \rho_j(p)},
\]

\[
= \begin{cases} 
0, & \text{if } \int_{\Sigma} \rho_j(p) \neq 0 \text{ for any } j = 1, \ldots, b_p \\
\hat{u}_0(\Sigma) e^{2\pi \sqrt{-1} \int_{\Sigma} \tau_p}, & \text{if } \int_{\Sigma} \rho_j(p) = 0 \text{ for any } j = 1, \ldots, b_p
\end{cases} \ (6.10)
\]

Using (6.2) the sum over the cohomology classes \( c \) in (4.28) can be easily calculated

\[
\sum_{c \in H^{p+1}(M, \mathbb{Z})} \hat{u}_0(\Sigma) e^{-\frac{1}{2} \|\delta_1(\hat{u}_0)\|^2} = 
\sum_{m_1 \in \mathbb{Z}} \ldots \sum_{m_{p+1} \in \mathbb{Z}} b_{p+1} \prod_{j=1}^{b_{p+1}} (\hat{f}_j(p)(\Sigma))^{m_j} e^{-\frac{1}{2} \sum_{j=1}^{b_{p+1}} h_j(p+1)(p+1) \sum_{y_1=1}^{l_1} \ldots \sum_{y_r=1}^{l_r} \prod_{k=1}^{r} (f_k(p)(\Sigma))^{y_k}} \ (6.11)
\]

where

\[
\sum_{y_1=1}^{l_1} \ldots \sum_{y_r=1}^{l_r} \prod_{k=1}^{r} (f_k(p)(\Sigma))^{y_k} = \begin{cases} 
0, & \text{if } f_k(p)(\Sigma) \neq 1 \text{ for any } k = 1, \ldots, r \\
\text{ord}(Tor_{p+1}(M; \mathbb{Z})) & \text{if } f_k(p)(\Sigma) = 1 \text{ for any } k = 1, \ldots, r
\end{cases} \ (6.12)
\]

In order to obtain a non-vanishing VEV of the Wilson operator, (6.1) implies that the free part of \([\Sigma]\) vanishes in \( H_p(M; \mathbb{Z}) \) and by (6.12) that \( f_k(p)(\Sigma) = 1 \) for all \( k \). Let us now assume that \([\Sigma] \in Tor_{p+1}(M; \mathbb{Z})\) then the condition

\[
f_k(p)(\Sigma) = v_k |Z_p(M; \mathbb{Z})| = <[v_k], [\Sigma]> = 1 \ \forall k = 1, \ldots, r
\]

implies that \( [v_k] \) vanishes on the torsion elements in \( H_p(M; \mathbb{Z}) \) for all \( k \). Repeating the arguments which lead to proposition 6.1 and using (2.3), it can be shown that \( [v_k] \in \text{ker } \delta^* \). However, according to (6.1) we have \( f_k(p+1) = \delta_2(f_k(p)) = -\delta^* [v_k] \neq 0 \). So one has to draw the following conclusion:

**Proposition 6.2.** The VEV of the Wilson operator \( W(\Sigma) \) vanishes, unless \([\Sigma] = 0 \) in \( H_p(M; \mathbb{Z}) \). \( \square \)

Let us now turn to an explicit computation of \( E(p)(W(\Sigma)) \), where \( \Sigma = \partial \Sigma' \) for \( \Sigma' \in C_{p+1}(M; \mathbb{Z}) \). This implies \( f_j(p)(\partial \Sigma') = q_j(p+1) \) for all \( j = 1, \ldots, b_{p+1} \). Concerning (4.28) it remains to perform an integration over \( \tau \in \text{ind}_{p+1} \). The mapping \( J_{\Sigma; \tau_p} \to J_{\Sigma; \tau_p} \), which appears in the exponent of (6.10), can be regarded as a continuous linear functional on \( \Omega^p(M; \mathbb{R}) \), i.e. \( J_{\Sigma} \) is a current of degree \((n-p)\).

We will briefly recall the main facts concerning the theory of currents [27]. Let us denote the space of p-currents by \( C^p(M) \). Every \( \beta \in \Omega^p(M; \mathbb{R}) \) gives rise to a p-current \( \delta(\phi) := \int_M \beta \wedge \phi \) for all \( \phi \in \Omega^{n-p}(M; \mathbb{R}) \). The differential of a p-current \( T \) is the \((p+1)-\)current \( dT \), which is defined by \( dT(\phi) := (1)_{p+1} T(d\phi) \) for \( \phi \in \Omega^{n-p-1}(M; \mathbb{R}) \). The Hodge star operator naturally extends to an operator \( \star C^p(M) \to C^{n-p}(M) \) by \( \star T(\phi) := (-1)^{p(n-p)} T(\star \phi) \) for all \( \phi \in \Omega^p(M; \mathbb{R}) \). Subsequently one can introduce the co-differential \( d^* := (-1)^{n(p+1)+1} \star d \star C^p(M) \to C^{p-1}(M) \) on currents. Let

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\[<T, \phi> := T(\star \phi)\] denote the scalar product of a \(p\)-current \(T\) with a differential form \(\phi \in \Omega^p(M; \mathbb{R})\). The Hodge decomposition theorem extends to currents [27], so that every current \(T\) can be uniquely written as
\[
T = \tilde{d}_{p-1}^* \tilde{d}_p G_p T + \tilde{d}_{p+1}^* \tilde{d}_p G_p T + \tilde{H}T,
\]
where the Green’s operator \(\tilde{G}_p\) on currents is defined by \(<\tilde{G}_p T, \phi> = <T, G_p \phi>\) and \(\tilde{H}T \in Harm^p(M)\) fulfills \(<\tilde{H}T, \rho> = <T, \rho>\) for all \(\rho \in Harm^p(M)\).

Let us now introduce the dual current \(j_\Sigma := \star J_\Sigma \in \mathcal{C}^p(M)\). Since \(\Sigma = \partial \Sigma'\) one gets \(j_\Sigma = \tilde{d}_{p+1}^* j_{\Sigma'}\), so that
\[
J_{\Sigma}(\tau_p) = \int_{\Sigma} \tau_p = <j_{\Sigma}, \tau_p> = <j_{\Sigma'}, d_p \tau_p>.
\]
Performing formally the Gaussian functional integration over \(imd_{p+1}^*\) and using (6.10) and (6.11) one obtains the VEV of the Wilson operator
\[
\mathcal{E}^{(p)}(W(\Sigma)) = \Theta_{b_{p+1}} \left( \tilde{K}^{(p+1)}(\Sigma') - \frac{b_{p+1}^2}{2\pi \sqrt{1}} \right) \Theta_{b_{p+1}} \left( 0 - \frac{b_{p+1}^2}{2\pi \sqrt{1}} \right) e^{-\frac{(2\pi)^2}{2} <j_{\Sigma'}, \tilde{d}_p G_p \tilde{d}_{p+1}^* j_{\Sigma'}>},
\]
where \(\tilde{K}^{(p+1)}(\Sigma') := \int_\Sigma \rho_j^{(p+1)}\) with \(j = 1, \ldots, b_{p+1}\) is regarded as \(b_{p+1}\)-dimensional vector. Let us remark that one has to regularize \(<j_{\Sigma'}, \tilde{d}_p G_p \tilde{d}_{p+1}^* j_{\Sigma'}>\), since the product of two currents is ill-defined in general. Making precise sense of this inner product lies beyond the scope of this paper. However, in the example below we will give an explicit and finite expression.

Before closing this section, we want to notice that the value of \(\mathcal{E}^{(p)}(W(\Sigma))\) does not depend on the choice for \(\Sigma'\). In fact, suppose that there exists a second \((p+1)\)-chain \(\Sigma''\), with \(\Sigma = \partial \Sigma''\), then \(K^{(p+1)}(\Sigma'') - K^{(p+1)}(\Sigma') \in \mathbb{Z}\). The properties of \(\Theta_{b_{p+1}}\) and the fact that \(\tilde{d}_{p+1}^* j_{\Sigma'} = \tilde{d}_{p+1}^* j_{\Sigma'}\) proves this statement.

**The Wilson operator in codimension 1.** As an example we will consider the VEV of \(W(\Sigma)\) in codimension 1. We realize the 0-current \(J_{\Sigma'}\) by the characteristic function \(\mathcal{V}_{\Sigma'}\) for \(\Sigma' \subset M\). Hence \(j_{\Sigma'}(\phi) = \int_M vol_M \phi \mathcal{V}_{\Sigma'}\) for \(\phi \in C^\infty(M)\). The main point is that \(j_{\Sigma'}\) is square summable \(n\)-current [27], hence its norm \(\|j_{\Sigma'}\|^2 = \int_\Sigma vol := Vol(\Sigma')\) is finite. With respect to the normalized basis \(\phi^{(n)} = \frac{vol_M}{\sqrt{vol(\Sigma)}}\) of \(Harm^n(M)\), one finds \(H j_{\Sigma'} = \frac{Vol(\Sigma')}{Vol(M)} vol_M\). Using the Hodge decomposition (6.14), the exponent in (6.16) becomes
\[
<j_{\Sigma'}, \tilde{d}_{n-1} G_{n-1} \tilde{d}_n j_{\Sigma'}> = \|j_{\Sigma'}\|^2 - <H j_{\Sigma'}, H j_{\Sigma'}> = Vol(\Sigma') \left( 1 - \frac{Vol(\Sigma')}{Vol(M)} \right).
\]
Applying the Poisson formula \(\sum_{m=-\infty}^{\infty} G(m) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dy G(y)e^{-2\pi \sqrt{-1}ky},\) where \(G(m)\) is an arbitrary function, one obtains
\[
\Theta_1 \left( \frac{Vol(\Sigma')}{Vol(M)} - \frac{1}{2\pi \sqrt{-1} Vol(M)} \right) = \Theta_1 \left( 1 - \frac{Vol(\Sigma')}{Vol(M)} - \frac{1}{2\pi \sqrt{-1} Vol(M)} \right).
\]
Using (6.17) and (6.18) we finally get from (6.16)
\[
\mathcal{E}^{(n-1)}(W(\Sigma)) = \Theta_1 \left( 1 - \frac{Vol(\Sigma')}{Vol(M)} - \frac{1}{2\pi \sqrt{-1} Vol(M)} \right) \Theta_1 \left( \frac{Vol(\Sigma')}{Vol(M)} - \frac{1}{2\pi \sqrt{-1} Vol(M)} \right) \exp \left( -\frac{(2\pi)^2}{2} Vol(\Sigma')(1 - \frac{Vol(\Sigma')}{Vol(M)}) \right),
\]
showing that the VEV of $W(\Sigma)$ depends on the volumes of $\Sigma'$ and $M$ only. Furthermore the result (6.19) is invariant under the replacement of $\text{Vol}(\Sigma')$ by $\text{Vol}(M) - \text{Vol}(\Sigma')$. This symmetry can be traced back to the ambiguity in defining the boundary $\Sigma'$ of $\Sigma$, since one could alternatively replace $\Sigma'$ by its complement $M \setminus \Sigma'$.

Acknowledgments

I would like to express my gratitude to H. Hüffel for his encouragement and his comments.

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