The rolling ball problem on the plane revisited

Laura M. O. Biscolla, Jaume Llibre and Waldyr M. Oliva

Abstract. By a sequence of rollings without slipping or twisting along segments of a straight line of the plane, a spherical ball of unit radius has to be transferred from an initial state to an arbitrary final state taking into account the orientation of the ball. We provide a new proof that with at most 3 moves, we can go from a given initial state to an arbitrary final state. The first proof of this result is due to Hammersley (1983). His proof is more algebraic than ours which is more geometric. We also showed that “generically” no one of the three moves, in any elimination of the spin discrepancy, may have length equal to an integral multiple of $2\pi$.

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1. Introduction and statement of the results

The rollings of a spherical ball $B$ of unit radius over the plane $\mathbb{R}^2$ suggest the consideration of some special kinematic (virtual) motions. A state of the ball $B$ is defined as the pair formed by the point of contact between $B$ and $\mathbb{R}^2$ together with the positive orthonormal frame attached to $B$. So the set of all the states is identified with the manifold $\mathbb{R}^2 \times SO(3)$, where $SO(3)$ denotes the group of all orthogonal $3 \times 3$ matrices with positive determinant.

A move is a smooth path on $\mathbb{R}^2 \times SO(3)$ corresponding to a rolling of $B$ on $\mathbb{R}^2$ without slipping or twisting along a straight line of the plane $\mathbb{R}^2$. No slipping in the rolling means that at each instant the point of contact between $B$ and $\mathbb{R}^2$ has zero velocity; no twisting means that, at each instant, the axis of rotation must be parallel to the plane $\mathbb{R}^2$.

According to John M. Hammersley, the following problem was proposed by David Kendall in the 1950s: What is the number $N$ of moves necessary and sufficient to reach any final state of $\mathbb{R}^2 \times SO(3)$ starting at a given initial state? In an interesting paper written in 1983, Hammersley [4] shows that $N = 3$ after using the theory of quaternions.

The following historical considerations we quote from [4, p. 112]: The original version of the question set (by David Kendall in the 1950s) for 18-year-old schoolboys, invited candidates to investigate how two moves, each of length $\pi$, would change the ball’s orientation; and to deduce in the first place that $N \leq 11$, and in the second place that $N \leq 7$. Candidates scored bonus marks for any improvement on 7 moves. When he first set the question, Kendall knew that $N \leq 5$; but, interest being aroused amongst professional mathematicians at Oxford, he and others soon discovered that the answer must be either $N = 3$ or $N = 4$. But in the 1950s nobody could decide between these two possibilities. There was renewed interest in the 1970s, and not only amongst professional mathematicians: for example the President of Trinity (a distinguished biochemist) spent some time rolling a ball around his drawing room floor in search of empirical insight. In 1978, while delivering the opening address to the first Australasian Mathematical Convention, I posed the problem to mathematicians down under; but I have not subsequently received a solution from them. So this is an opportunity to publish the solution.
In the present paper, we shall prove that $N = 3$ but the proof provided here is more geometric than the original one, which is more algebraic and uses strongly the theory of quaternions.

A state $(P, M) \in \mathbb{R}^2 \times SO(3)$ of an oriented spherical ball on the plane $\mathbb{R}^2$ means that the contact point of the ball with the $(x, y)$-plane is the point $P$ of $\mathbb{R}^2$, and its orientation is given by the orthonormal frame $(I, J, K)$ of $\mathbb{R}^3$ where the vectors $I$, $J$ and $K$ are given by the first, second and third column of the matrix $M$, respectively. So the state $(P, M)$ is also denoted by $(P, (I, J, K))$. We denote by $(i, j, k)$ the orthonormal frame $(I, J, K)$ associated with $I$, the $3 \times 3$ identity matrix. In order to simplify the notation sometimes in what follows, if we go from the initial state $(P_0, (i, j, k))$ to a final state $(P_1, (I, J, K))$ with some moves, we simply write $(P_0, (i, j, k)) \rightarrow (P_1, (I, J, K))$.

The paper is organized as follows. In Sect. 2, we derive a formula describing a general move and prove two useful lemmas. In Sect. 3, Theorem 3.1 proves that with 3 moves one goes from the initial state $(P_0, (i, j, k))$ to $(P_1, (I, J, K))$ with some moves is called the spin discrepancy, that is we start in the point $P_0$ with the orientation $(i, j, k)$ and after some moves, we end in the same point $P_0$ but with the orientation $(I, J, k)$, which corresponds to a rotation with respect to the $k$ axis with respect to the initial orientation. So we observe that Theorem 3.1 means that we obtain the so-called elimination of the spin discrepancy. In Sect. 4, we see that $N = 3$, that is 3 moves are necessary and sufficient on $\mathbb{R}^2$. We start by proving in Theorem 4.1 that $(P_0, (i, j, k)) \rightarrow (P_0, (I, J, K))$ provided that the matrix $M$ defined by $(I, J, K)$ has its rotation axis linearly independent of the vector $k = (0, 0, 1)$. As a corollary of Theorem 4.1, we obtain that $(P_0, (i, j, k)) \rightarrow (P_0, (I, J, k))$ because the matrix corresponding to $(I, J, k)$ has its axis of rotation in the plane $(i, j)$. Now, when $P_0 \neq P_1$ we also prove Proposition 3.2 and Theorem 4.2, that is $(P_0, (i, j, k)) \rightarrow (P_1, (I, J, K))$, where as in Theorem 4.1, one assumes that the rotation axis of the matrix $M$ defined by $(I, J, K)$ is linearly independent of the vector $k$. In Sect. 5, we study, more carefully, the elimination of the spin discrepancy; we remark that in all the cases mentioned above (except in Theorem 3.1), there is at least one of the 3 moves with length given by an integral multiple of $2\pi$. Moreover, an important fact is that one can also prove (see Theorem 5.1) that “generically” no one of the 3 moves, in any elimination of the spin discrepancy, may have length equal to an integral multiple of $2\pi$.

The two main reasons for revisiting the proof of the result mentioned as “Kendall problem” are: first, the paper written by Hammersley is hard to find in the mathematical literature, and second, our proof is new, shorter and does not use quaternions.

2. A formula for a general move and two lemmas

First, we formulate explicitly a move of the spherical ball from the initial point $P_0 \in \mathbb{R}^2$ with orientation $M_0 \in SO(3)$ to the point $P_1 \in \mathbb{R}^2$, always assuming $P_1 \neq P_0$. Let $(x, y)$ be a given cartesian coordinate system of $\mathbb{R}^2$ with origin at $P_0$, that is $P_0 = (0, 0)$. Without loss of generality, we can assume throughout this section that $P_0 = (0, 0)$. We shall denote the polar coordinates of $P_1$ with origin at $P_0$ as $(r, \theta)$, where $\theta$ is measured in counterclockwise with respect to the positive $x$-axis. In what follows, we refer to $(r, \theta)$ as the polar coordinates of $P_1$ with respect to $P_0$.

**Proposition 2.1.** The initial state $(P_0, M_0)$ passes to the state $(P_1, R_3(-\theta)R_2(r)R_3(\theta)M_0)$ after the move through the segment starting at $P_0$ and ending at $P_1$ with polar coordinates $(r, \theta)$ with respect to $P_0$, where

$$R_2(r) = \begin{pmatrix} \cos r & 0 & \sin r \\ 0 & 1 & 0 \\ -\sin r & 0 & \cos r \end{pmatrix} \quad \text{and} \quad R_3(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

**Proof.** Without loss of generality, we assume that $P_0$ is the origin of coordinates, see Fig. 1. Note that $R_3(\theta)M_0$ is the orientation of $M_0$ after moving the point $P_1$ to the point $P'_1 = R_3(\theta)P_1$, which is on the $x$-axis. Then $R_2(r)R_3(\theta)M_0$ is the orientation of $M_0$ after the move, following the $x$-axis, from the origin.
Without loss of generality, it may be assumed that \( P_0 \) to the point \( P'_1 \). Finally, we go back from the point \( P'_1 \) to \( P_1 \) using the rotation \( R_3(-\theta) \), and we get the final orientation \( R_3(-\theta)R_2(r)R_3(\theta)M_0 \).

We say that the move \( R_3(-\theta)R_2(r)R_3(\theta)M_0 \) of Proposition 2.1 has length \( r \) and angle \( \theta \).

We remark that the move of Proposition 2.1 is a rotation of angle \( r \) around the vector \( e = (-\sin \theta, \cos \theta, 0) \), and we write it as

\[
\mathcal{R}(e, r),
\]

because \( R_3(-\theta)R_2(r)R_3(\theta) = \mathcal{R}(e, r) \) as it easy to check. It is well known that the rotation matrix \( \mathcal{R}(e, \delta) \) of angle \( \delta \) around an axis through the origin with direction \( e = (u, v, w) \) is

\[
\frac{1}{L^2} \begin{pmatrix}
    u^2 + (v^2 + w^2) \cos \delta & uv(1 - \cos \delta) - wL \sin \delta & uw(1 - \cos \delta) + vL \sin \delta \\
    uv(1 - \cos \delta) + wL \sin \delta & v^2 + (u^2 + w^2) \cos \delta & v(1 - \cos \delta) - uL \sin \delta \\
    uw(1 - \cos \delta) - vL \sin \delta & v(1 - \cos \delta) + uL \sin \delta & w^2 + (u^2 + v^2) \cos \delta
\end{pmatrix},
\]

where \( L = \sqrt{u^2 + v^2 + w^2} \).

The following two lemmas will be very useful. The first one, Lemma 2.2, allows to pass in 2 moves from the state \((P_0, (I, J, K))\) to an arbitrary state of the form \((P_1, (I, J, K))\) with \( P_1 \neq P_0 \).

**Lemma 2.2.** Let \( M \in SO(3) \) and \( P_0, P_1 \in \mathbb{R}^2 \) with \( P_0 \neq P_1 \). Then, we can go from the initial state \((P_0, M)\) to the final state \((P_1, M)\) with 2 moves.

**Proof.** Without loss of generality, it may be assumed that \( P_0 \) is the origin of the cartesian coordinate system and that \( P_1 \) is on the positive \( x \)-axis.

Let \((r, 0)\) be the polar coordinates of \( P_1 \) with respect to \( P_0 \). Take a positive integer \( n \) such that \( 4\pi n > r \) and let \( \overline{P} \) be the point of polar coordinates \((\tau, \overline{\theta}) = (2\pi n, \arccos(r/(4\pi n)))\) with respect to \( P_0 \). It is easy to check that

(i) \( d(\overline{P}, P_0) = d(\overline{P}, P_1) = \tau \) where \( d \) denotes the Euclidean distance of \( \mathbb{R}^2 \);

(ii) \((\tau, -\overline{\theta})\) are the polar coordinates of \( P_1 \) with respect to \( \overline{P} \) (i.e., we can think that \( \overline{P} \) is the new origin of coordinates and that \( \overline{\theta} \) is measured in counterclockwise with respect to a parallel ray to the positive \( x \)-axis starting at \( \overline{P} \)); and

(iii) \((R_3(\overline{\theta})R_2(\tau)R_3(-\overline{\theta}))(R_3(-\overline{\theta})R_2(\tau)R_3(\overline{\theta})) = \text{Id} \).

Note that from (i), the points \( P_0 \), \( \overline{P} \) and \( P_1 \) are the vertices of an isosceles triangle.

The result will be proven by two applications of Proposition 2.1. First, consider the move starting at \( P_0 \) and ending at \( \overline{P} \). Second, consider the move starting at \( \overline{P} \) and ending at \( P_1 \). According to Proposition 2.1, the ball passes from \((P_0, M)\) to \((P_1, M)\) as a result of these two moves.

As usual, we denote by \( S^1 = \mathbb{R}/(2\pi \mathbb{Z}) \) the circle. The following result describes passage in 2 moves between a state \((P_0, (i, j, k))\) and a suitable state of the form \((P_2, (I, J, K))\). This result also appears in Biscolla [2].

**Lemma 2.3.** Given \( a \in S^1 \) and a point \( P_0 \in \mathbb{R}^2 \) we consider the point \( P_1 \) with polar coordinates \(((2n + 1)\pi, b)\) with respect to \( P_0 \), and the point \( P_2 \) with polar coordinates \(((2m + 1)\pi, b - a/2)\) with respect to \( P_1 \), where \( n \) and \( m \) are non-negative integers. Then the following statements hold:
(i) The geometrical locus of all points \( P \) with \( b \in S^1 \) is a circle centered at \( P_0 \) of radius
\[
\pi \sqrt{2(1 + 2n(1 + n) + 2m(1 + m) + (1 + 2n)(1 + 2m) \cos(a/2))},
\]
which does not depend on \( b \).

(ii) For any \( b \in S^1 \) the initial state \((P_0, \text{Id})\) passes to the final state \((P_2, R_3(a))\) after using 2 moves: first the move starting at \( P_0 \) and ending at \( P_1 \), and second the move starting at \( P_1 \) and ending at \( P_2 \).

Proof. Without loss of generality, we assume that the point \( P_0 \) is at the origin of the cartesian coordinate system. Then, an easy computation shows that the cartesian coordinates of \( P_2 \) are
\[
\pi \left((1 + 2n) \cos b + (1 + 2m) \cos(b - a/2), (1 + 2n) \sin b + (1 + 2m) \sin(b - a/2)\right).
\]
Hence statement (i) follows.

Since \((R_3(a/2 - b)R_2((2m + 1)\pi)R_3(b - a/2))(R_3(-b)R_2((2n + 1)\pi)R_3(b)) = R_3(a)\) for all \( b \in S^1 \), we get statement (ii). \(\square\)

3. The elimination of the spin discrepancy

The next result allows passage with 3 moves from the state \((P_0, (i, j, k))\) to the state \((P_0, (I, J, k))\).

**Theorem 3.1.** (Elimination of the spin discrepancy) Given \( P_0 \in \mathbb{R}^2 \) and \( a \in S^1 \) we can pass from the initial state \((P_0, \text{Id})\) to the final state \((P_0, R_3(a))\) using 3 moves.

Proof. Without loss of generality, we assume that \( P_0 \) is at the origin of the cartesian coordinate system, and that \( a \in (0, 2\pi) \).

Now, we shall provide the proof that 3 moves are sufficient. We first do the move: starting at origin and ending at the point \( P_1 = (r, -\theta) \) in polar coordinates with respect to \( P_0 \), for a convenient \( \theta \in (\pi/2 - a/4, \pi/2) \) and
\[
r = \arccos(\cot \theta \tan(\theta + a/2)).
\]
The second move starts at \( P_1 \) and ends at the point \( P_2 = (2r \sin(\pi/2 - \theta), \pi) \) in polar coordinates with respect to \( P_1 \). Finally, the third move starts at \( P_2 \) and ends at \( P_0 = (r, \theta) \) in polar coordinates with respect to \( P_2 \).

These three moves follow the edges of an isosceles triangle, with \( P_1 \) being the vertex between the edges of equal length \( r \) and with angle \( \pi - 2\theta \) between these two edges (see Fig. 2).

The angle \( \theta \in (\pi/2 - a/4, \pi/2) \) is chosen satisfying the equation
\[
f(\theta) = \arccos(\csc \theta \sin(\theta + a/2))
- \cos \theta \arccos(\cot \theta \tan(\theta + a/2)) = 0.
\]

![Fig. 2. Three moves take place on the sides of an isosceles triangle. The axis through the point \( P_1 \) is parallel to the x-axis.](image-url)
It is easy to check that \( f(\pi/2 - a/4) = -\pi \sin(a/4) < 0, f(\pi/2) = a/2 > 0, \) and \( f \) is well defined in the interval \([\pi/2 - a/4, \pi/2]\), so by continuity, there exists at least one \( \theta \in (\pi/2 - a/4, \pi/2) \) satisfying Eq. (4). In fact, the derivative of \( f(\theta) \) is positive in \([\pi/2 - a/4, \pi/2]\), so there is a unique \( \theta \in (\pi/2 - a/4, \pi/2) \) satisfying Eq. (4), but this uniqueness is not necessary in the proof.

If we do these 3 moves, according with Proposition 2.1, the orientation of the ball changes as follows

\[
\begin{pmatrix}
R_3(-\theta)R_2(r)R_3(\theta) \\
R_3(\theta)R_2(r)R_3(-\theta)
\end{pmatrix} \begin{pmatrix}
2r \sin \left( \frac{\pi}{2} - \theta \right) \\
R_3(\pi)
\end{pmatrix}
\begin{pmatrix}
R_3(\pi)R_2(2r)R_3(\theta) \\
R_3(\pi)
\end{pmatrix} \text{Id.}
\]

We must show that this matrix is equal to \( R_3(a) \), and the proposition would be proved. To prove the equality between the previous two matrices is equivalent to showing the matrix

\[
A = \begin{pmatrix}
R_3(\pi)R_2(2r)R_3(\theta) \\
-\left( R_3(-\theta)R_2(-r)R_3(\theta) \right) R_3(a)
\end{pmatrix}
\]

is identically zero.

We denote by \( a_{ij} \) the element in the row \( i \) and in the column \( j \) of the \( 3 \times 3 \) matrix \( A \). Due to the moves that we did, we have

\[
a_{33} = \sin(r \cos \theta) \cos(\theta) \cos(r \cos \theta) \sin r - \cos r \sin(r \cos \theta) = 0.
\]

It is easy to check that \( \sin(r \cos \theta) \neq 0 \), then

\[
\cos \theta \cos(r \cos \theta) \sin r - \cos r \sin(r \cos \theta) = 0.
\]

From this equation, we have that

\[
\sin(r \cos \theta) = \cos \theta \cos(r \cos \theta) \tan r,
\] (5)

if \( \cos r \neq 0 \). The equation \( a_{21} = 0 \) is

\[
-\cos r \sin \theta (\cos \theta + \cos(\theta + a)) + \cos \theta (\sin \theta + \sin(\theta + a)) = 0.
\] (6)

We conclude that

\[
\cos r = \frac{\cos \theta (\sin \theta + \sin(\theta + a))}{\sin \theta (\cos \theta + \cos(\theta + a))},
\]

if \( \sin \theta (\cos \theta + \cos(\theta + a)) \neq 0 \). Now substituting \( \sin(r \cos \theta) \) and \( \cos r \) in the matrix \( A \), we get that \( a_{ij} = 0 \) for all \( i, j = 1, 2, 3 \).

In short, in order that \( A \equiv 0 \), it is sufficient to choose \( r \) and \( \theta \) satisfying (5), (6) with \( \cos r \neq 0 \) and \( \sin \theta (\cos \theta + \cos(\theta + a)) \neq 0 \). Computing \( r \) from (6), we get (3). Now substituting \( \sin(r \cos \theta) \) and \( r \) into \( a_{32} = 0 \) we obtain

\[
csc^2 \theta \ g(\theta) = 0,
\]

where

\[
g(\theta) = \cos(2\theta) \cos^2(\arccos(\cot \theta \tan(\theta + a/2)) \cos \theta) - \cos(2\theta + a) \\
+ \sin^2(\arccos(\cot \theta \tan(\theta + a/2)) \cos \theta)
\]

\[
= 2 \sin^2(\theta + a/2) - 2 \cos^2(\arccos(\cot \theta \tan(\theta + a/2) \cos \theta)) \sin^2 \theta.
\]

Since \( \theta \in (\pi/2 - a/4, \pi/2) \) we have that \( \csc \theta \neq 0 \), so \( g(\theta) = 0 \). But in order that \( g(\theta) = 0 \) it is sufficient that

\[
\sin(\theta + a/2) - \cos(\arccos(\cot \theta \tan(\theta + a/2) \cos \theta)) \sin \theta = 0.
\]

That is that \( f(\theta) = 0 \).

Therefore in order that \( A \equiv 0 \), it is sufficient to solve the two Eqs. (3) and (6) with respect to \( \theta \in (\pi/2 - a/4, \pi/2) \) and \( r > 0 \), and that the solution \((r, \theta)\) satisfies \( \cos r \neq 0 \) and \( \sin \theta (\cos \theta + \cos(\theta + a)) \neq 0 \).
From (6) $\cos r \neq 0$ is equivalent to $\cos \theta (\sin \theta + \sin(\theta + a)) \neq 0$. Since $\theta \in (\pi/2 - a/4, \pi/2)$ it is easy to check that $\sin \theta (\cos \theta + \cos(\theta + a)) \neq 0$ and $\cos \theta (\sin \theta + \sin(\theta + a)) \neq 0$. Hence, the theorem is proven. □

The next result allows passage with 3 moves from the state $(P_0, (i,j,k))$ to an arbitrary state of the form $(P_1, (I,J,K))$.

**Proposition 3.2.** Given $P_0, P_1 \in \mathbb{R}^2$ with $P_0 \neq P_1$ and $a \in S^1$ we can pass from the initial state $(P_0, \text{Id})$ to the final state $(P_1, R_3(a))$ using 3 moves.

**Proof.** Without loss of generality, we assume that $P_0$ is at the origin of the cartesian coordinate system. Then by Lemma 2.3, there are infinitely many circles centered at $P_0$ whose radii increase tending to infinity. We can pass with 2 moves from the point $P_0$ to any point of these circles, and on these points, the orientation of the spherical ball is $R_3(a)$.

Clearly we can pass with 1 move from the point $P_1$ to the points of the circles centered at $P_1$ with radii $2\ell\pi$ with $\ell = 1, 2, \ldots$, and on these points the orientation of the spherical ball is the same as in $P_1$, that is $R_3(a)$.

The family of circles centered at $P_0$ intersects the family of circles centered at $P_1$ and, therefore, we can pass from the initial state $(P_0, \text{Id})$ to the final state $(P_1, R_3(a))$ using three moves. □

### 4. Three moves are necessary and sufficient on $\mathbb{R}^2$

In the proof of the next theorem, we shall need the expression of a rotation around the $x$-axis of angle $b$, it is defined as

$$R_1(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos b & \sin b \\ 0 & -\sin b & \cos b \end{pmatrix}.$$  

We will show that we can go with 3 moves from the state $(P_0, \text{Id})$ to the state $(P_0, M)$, by assuming that the rotation axis of $M$ is linearly independent of $k = (0,0,1)$.

**Theorem 4.1.** Given $P_0 \in \mathbb{R}^2$ and $M \in SO(3)$ having rotation axis linearly independent of $k = (0,0,1)$, we can pass from the initial state $(P_0, \text{Id})$ to the final state $(P_0, M)$ using 3 moves.

**Proof.** As usual, we can suppose that $P_0$ is the origin of the cartesian coordinate system. Let $(I, J, K)$ be the orthonormal frame associated with the rotation matrix $M = R_3(c)R_1(b)R_3(a)$, with $a, c \in S^1$ and $b \in [0, \pi]$. We recall that from the definition of the Euler angles, that $b$ is the angle between the vectors $k$ and $K$ (for more details, see [3, pp. 143–148]). Moreover, we can think of $M$ as a rotation around an eigenvector $v$ with eigenvalue 1, and we know that $M$ moves $k$ into $K$, $j$ into $J$ and $i$ into $I$.

Assume that the vector $v$ is horizontal, that is $v$ is in the plane generated by the vectors $\{i,j\}$. Now after rolling the unitary ball around the vector $v$ in the direction $v^D$ (the orthogonal vector to $v$) a distance $b$, we see that the ball goes from the initial state $(P_0, \text{Id})$ to a state $(P_1, M)$ because $k$ goes to $K$ and, since the orthogonal plane to $v$ is invariant under $M$, simultaneously $i$ goes to $I$ and $j$ goes to $J$. Now applying Lemma 2.2 to the ball at $(P_1, M)$, we get the state $(P_0, M)$ with two more moves. Then we go from $(P_0, \text{Id})$ to $(P_0, M)$ with 3 moves.

In the rest of the proof, we suppose that $v = (v_1, v_2, v_3)$ is not horizontal. So $v_3 \neq 0$, and we can take $v_3 = 1$. Let $II$ be the plane through $P_0$ (the origin) orthogonal to the vector $v$. The equation of this plane $II$ is $v_1 x + v_2 y + z = 0$. We have $v_1^2 + v_2^2 \neq 0$, otherwise $v = (0,0,1)$ and $k$ are linearly dependent. We choose the coordinates $(x,y,z)$ so that the intersection line between the plane $II$ and the horizontal plane is the $x$-axis. Therefore, the plane $II$ has equation $v_2 y + z = 0$ with $v_2 \neq 0$. Hence in the new coordinates $v = (0, v_2, 1)$, and let $\phi$ be the angle of the rotation $M$ around the vector $v$. Denote by $\varphi$ the norm of $v$, that is $\varphi^2 = 1 + v_2^2$. 
Choose on the plane $\Pi$ the two unitary vectors
\[ u_1(0) = \left( \cos \frac{\phi}{4} - \frac{1}{\varphi} \sin \frac{\phi}{4}, v_2 \frac{1}{\varphi} \sin \frac{\phi}{4} \right), \]
\[ u_2(0) = \left( \cos \frac{\phi}{4}, -\frac{1}{\varphi} \sin \frac{\phi}{4}, v_2 + \varphi \sin \frac{\phi}{4} \right). \]

Note that the vectors $u_1(0)$ and $u_2(0)$ form an angle $\phi/4$ with the $x$-axis.

By rotating the vectors $u_1(0)$ and $u_2(0)$ an angle $\lambda$ on the plane $\Pi$ we get the unitary vectors
\[ u_1(\lambda) = \left( \cos l, \frac{1}{\varphi} \sin l, -v_2 \frac{1}{\varphi} \sin l \right), \]
\[ u_2(\lambda) = \left( \cos L, \frac{1}{\varphi} \sin L, -v_2 \frac{1}{\varphi} \sin L \right), \]
where $l = \lambda - \phi/4$ and $L = \lambda + \phi/4$.

We introduce now the horizontal vectors
\[ h_1(\lambda) = k \wedge u_1(\lambda) = \left( -\frac{1}{\varphi} \sin l, \cos l, 0 \right), \]
\[ h_2(\lambda) = k \wedge u_2(\lambda) = \left( -\frac{1}{\varphi} \sin L, \cos L, 0 \right), \]
together with their orthogonal vectors
\[ h_1^T(\lambda) = \left( \cos l, \frac{1}{\varphi} \sin l, 0 \right), \]
\[ h_2^T(\lambda) = \left( \cos L, \frac{1}{\varphi} \sin L, 0 \right). \]

We denote by $\alpha_i(\lambda)$ the angle from the vector $k$ to the vector $u_i(\lambda)$ for $i = 1, 2$.

The orientation $M$ is given by the orthogonal matrix obtained as the rotation of angle $\phi$ around the unitary vector $(0, v_2, 1)/\varphi$. So, according with the notation introduced in (1), we have that $M = R((0, v_2, 1)/\varphi, \phi)$.

The orientation $M$ is obtained at the point
\[ P(n, \lambda) = P_0 + (2n\pi + 2\alpha_1(\lambda)) \frac{-h_1^T(\lambda)}{||h_1^T(\lambda)||} + (2n\pi - 2\alpha_2(\lambda)) \frac{h_2^T(\lambda)}{||h_2^T(\lambda)||}, \tag{7} \]
and with this notation, we mean that two moves are used to pass from $P_0$ to $P(n, \lambda)$, first rotating an angle $2n\pi + 2\alpha_1(\lambda)$ around the vector $-h_1(\lambda)$ and then rotating an angle $2n\pi - 2\alpha_2(\lambda)$ around the vector $h_2(\lambda)$. In fact one can check that
\[ M = R \left( \frac{h_2(\lambda)}{||h_2(\lambda)||}, 2n\pi - 2\alpha_2(\lambda) \right) \cdot R \left( \frac{-h_1(\lambda)}{||h_1(\lambda)||}, 2n\pi + 2\alpha_1(\lambda) \right). \]

We claim that for a sufficiently large integer $n > 0$, the closed curve $\{P(n, \lambda) : \lambda \in [0, 2\pi]\}$ intersects, at a certain value $\lambda$ of $\lambda$, a circle centered at $P_0 = (0, 0, 0)$ (the origin) with radius $2m\pi$ for some integer $m > 0$. At the point $P(n, \lambda)$ the orientation is $M$ and rolling the ball along the segment $[P_0, P(n, \lambda)]$ with a length $2m\pi$ we get, at the origin, the same orientation $M$. Therefore, starting at the origin with the identity and using three moves, we come back to the origin with the given orientation $M$. This completes the proof of the theorem. Now we shall prove the claim.

First we shall prove that
\[ ||P(n, \lambda)||_{\text{max}} - ||P(n, \lambda)||_{\text{min}} \geq ||P(n, \pi/2)|| - ||P(n, 0)|| > 2\pi, \tag{8} \]
for \( n \) sufficiently large. The previous maximum and minimum are taken over \( \lambda \in [0, 2\pi] \). Indeed from (7), we obtain for \( ||P(n, \pi/2)|| \) and \( ||P(n, 0)|| \) the expressions

\[
\begin{align*}
||P(n, \pi/2)|| &\approx \sqrt{\frac{32 A^2 \cos^2 \left( \frac{\phi}{4} \right) + 32 n^2 \pi^2 \left( 1 + v_2^2 \right) \sin^2 \left( \frac{\phi}{4} \right)}{2 + v_2^2 - v_2^2 \cos \left( \frac{\phi}{2} \right)}}, \\
||P(n, 0)|| &\approx \sqrt{\frac{32 B^2 \sin^2 \left( \frac{\phi}{4} \right) + 32 n^2 B \left( \cos \left( \frac{\phi}{4} \right) - 1 \right) + \pi^2 \left( 32 n^2 \sin^2 \left( \frac{\phi}{4} \right) + 4 \left( 1 + v_2^2 \right) \left( 1 + \cos \left( \frac{\phi}{2} \right) \right) \right)}{2 + v_2^2 + v_2^2 \cos \left( \frac{\phi}{2} \right)}},
\end{align*}
\]

respectively, where

\[
A = \arccos \left( -\frac{v_2 \cos \left( \frac{\phi}{4} \right)}{\sqrt{v_2^2 + 1}} \right), \quad B = \arcsin \left( \frac{v_2 \sin \left( \frac{\phi}{4} \right)}{\sqrt{v_2^2 + 1}} \right).
\]

If \( n \) is sufficiently large

\[
||P(n, \pi/2)|| \approx \sqrt{\frac{32 n^2 \pi^2 \left( 1 + v_2^2 \right) \sin^2 \left( \frac{\phi}{4} \right)}{2 + v_2^2 - v_2^2 \cos \left( \frac{\phi}{2} \right)}}, \quad ||P(n, 0)|| \approx \sqrt{\frac{32 \pi n^2 \sin^2 \left( \frac{\phi}{4} \right)}{2 + v_2^2 + v_2^2 \cos \left( \frac{\phi}{2} \right)}},
\]

From

\[
||P(n, \pi/2)||^2 - ||P(n, 0)||^2 \approx \frac{32 n^2 \pi^2 v_2^2 (2 + v_2^2) \left( 1 + \cos \left( \frac{\phi}{2} \right) \right) \sin^2 \left( \frac{\phi}{4} \right)}{\left( 2 + v_2^2 - v_2^2 \cos \left( \frac{\phi}{2} \right) \right) \left( 2 + v_2^2 + v_2^2 \cos \left( \frac{\phi}{2} \right) \right)},
\]

since \( \phi \in (0, 2\pi) \), if \( n \) is sufficiently large, then we have

\[
||P(n, \pi/2)|| - ||P(n, 0)|| > 2\pi.
\]

(9)

So (8) is proven.

Let \( n_0 \) be a positive integer for which (8) holds. Assume that the closed curve \( \{P(n, \lambda) : \lambda \in [0, 2\pi]\} \) stays inside the annulus centered at the origin with inner and outer radii \( R \) and \( R + 2\pi \), respectively. Then

\[
||P(n_0, \lambda)||_{\text{max}} < R + 2\pi, \quad \text{and} \quad ||P(n_0, \lambda)||_{\text{min}} > R.
\]

Therefore

\[
||P(n_0, \lambda)||_{\text{max}} - ||P(n_0, \lambda)||_{\text{min}} < 2\pi,
\]

in contradiction to (8). In short, the closed curve \( \{P(n, \lambda) : \lambda \in [0, 2\pi]\} \) intersects one of the circles of radius \( 2m\pi \) centered at the origin and the theorem is proven.

We will show that we can go with 3 moves from the state \((P_0, \text{Id})\) to the state \((P_1, M)\) by assuming that the rotation axis of \( M \) is linearly independent of \( k = (0, 0, 1) \).

**Theorem 4.2.** Given \( P_0, P_1 \in \mathbb{R}^2 \) with \( P_0 \neq P_1 \) and \( M \in SO(3) \) having rotation axis linearly independent of \( k = (0, 0, 1) \), we can pass from the initial state \((P_0, \text{Id})\) to the final state \((P_1, M)\) using 3 moves.
**Proof.** The proof of this proposition follows essentially the same steps as in the proof of Theorem 4.1. We will describe the steps which are different.

Let $d$ be the Euclidean distance between the points $P_0 = (0,0,0)$ and the point $P_1$. Let $k$ be a positive integer such that

$$2k\pi - 2d > 2\pi. \quad (10)$$

Let $\{P(n, \lambda) : \lambda \in [0, 2\pi]\}$ be the closed curve defined in the proof of Theorem 4.1. We write $r(n, \lambda) = ||P(n, \lambda)||$, that is the distance of the point $P(n, \lambda)$ to the origin $P_0$. Working as in the proof of the second inequality of (8), we obtain

$$r(n, \pi/2) - r(n, 0) > 2k\pi, \quad (11)$$

if $n$ is sufficiently large. Note that in (8) we get $2\pi$ instead of $2k\pi$, but in the proof of (8) the difference $r(n, \pi/2) - r(n, 0)$ can be made as large as we want by increasing $n$, showing that this difference can be greater than $2k\pi$.

Let $r^*(n, \lambda)$ be the Euclidean distance from the point $P(n, \lambda)$ to the point $P_1$. By the triangle inequality, we get

$$r(n, \lambda) - d \leq r^*(n, \lambda) \leq r(n, \lambda) + d.$$ 

Therefore, we obtain

$$r^*(n, \lambda)|_{\text{max}} \geq r(n, \pi/2) - d, \quad \text{and} \quad r^*(n, \lambda)|_{\text{min}} \leq r(n, 0) + d,$$

where the maximum and minimum are taken over $\lambda \in [0, 2\pi]$. From these last two inequalities, (10) and (11) we have

$$r^*(n, \lambda)|_{\text{max}} - r^*(n, \lambda)|_{\text{min}} \geq r(n, \pi/2) - r(n, 0) - 2d > 2k\pi - 2d > 2\pi.$$

So we have obtained an inequality similar to the inequality (8) but this time with respect to the point $P_1$ instead of the origin.

Now working as in the last part of the proof of Theorem 4.1, we get that the closed curve $\{P(n, \lambda) : \lambda \in [0, 2\pi]\}$ intersects one of the circles of radius $2m\pi$ centered at $P_1$, and the theorem follows as in the proof of Theorem 4.1. \hfill \Box

Clearly, the results from Theorem 3.1 until Proposition 4.2 show that we can go with at most 3 moves from the state $(P_0, (i,j,k))$ to any other state $(P_1, (I,J,K))$. Hence, we have proved that $N \leq 3$.

**Remark 4.3.** Since it is easy to see that we cannot reach $(P_0, R_3(a))$, $0 < a < \pi$, from $(P_0, \text{Id})$ with two moves, one concludes that $N = 3$.

### 5. More on the elimination of the spin discrepancy

In all the moves necessary for proving that with 3 moves, we can go from an arbitrary initial state to any arbitrary final state we use some move of length an integral multiple of $2\pi$ with the exception of the case called elimination of the spin discrepancy, see Theorem 3.1. Now we shall prove that “generically” we cannot find three moves in the spin discrepancy having one of them length a multiple of $2\pi$.

**Theorem 5.1.** Let $P_0 \in \mathbb{R}^2$, $a \in (0, 2\pi)$ and $k$ a positive integer.

(a) We cannot go with 3 moves from the state $(P_0, \text{Id})$ to the state $(P_0, R_3(a))$ if the second move has length $2\pi k$ and $\cos(a/2) \notin \mathbb{Q}$.

(b) We cannot go with 3 moves from the state $(P_0, \text{Id})$ to the state $(P_0, R_3(a))$ if the first or third move has length $2\pi k$ and $\cos a \notin \mathbb{Q}$. 

Fig. 3. Three moves with the second move of length $2\pi k$

**Proof.** We separate the proof in two cases.

**Case a:** Assume that we can go with 3 moves from the state $(P_0, \text{Id})$ to the state $(P_0, R_3(a))$ and that the second move has length $2\pi k$. More precisely,

(i) $P_0$ is the origin of coordinates and with 3 moves we pass from $P_0$ to $P_1$, from $P_1$ to $P_2$, and finally from $P_2$ to $P_0$;

(ii) $P_1$ has polar coordinates $(r, \theta)$ with respect to $P_0$ with $\theta \in (0, \pi)$;

(iii) $P_2$ has polar coordinates $(2\pi k, -\alpha)$ with respect to $P_1$ with $\alpha \in (0, \pi)$; and

(iv) $P_2$ is on the positive $x$-axis and the distance between $P_0$ and $P_2$ is $R$.

See Fig. 3.

From Proposition 2.1, we can write the composition of the three moves as follows

$$R_3(a) = R_2(-R)(R_3(\alpha)R_2(2\pi k)R_3(-\alpha))(R_3(-\theta)R_2(r)R_3(\theta))\text{Id} = R_2(-R)(R_3(-\theta)R_2(r)R_3(\theta))\text{Id}.$$ (12)

This equality provides 9 polynomial equations in the 6 variables $\cos r$, $\sin r$, $\cos R$, $\sin R$, $\cos \theta$ and $\sin \theta$. We have the additional 4 polynomial equations

$$\cos^2 r + \sin^2 r = 1,$$

$$\cos^2 R + \sin^2 R = 1,$$

$$\cos^2 \theta + \sin^2 \theta = 1.$$  

Computing the Gröbner basis of these 12 polynomial equations with respect to the mentioned 6 variables, we get an equivalent system of polynomial equations (equivalent in the sense that both systems have the same solutions) containing the following two equations

$$\sin R \sin a = 0,$$

$$\sin r \sin a = 0.$$  

For more details on the Gröbner basis, see for instance [1]. Since $\cos(a/2) \notin \mathbb{Q}$ we have that $\sin a \neq 0$, so $\sin R = 0$ and $\sin r = 0$, or equivalently $r = \ell \pi$ and $R = m \pi$ with $\ell$ and $m$ positive integers.

Now evaluating the equation of the third row and third column of (12), we get $1 = (-1)^{\ell + m}$. Therefore, $\ell$ and $m$ have the same parity. Assume that $\ell$ and $m$ are even, then evaluating the equation of the first file and second column of (12), we get $\sin a = 0$, a contradiction with the assumption that $\cos(a/2) \notin \mathbb{Q}$. Hence $\ell$ and $m$ are odd. Then evaluating the second file and first column of (12), we get the equation

$$\sin a = 2 \cos \theta \sin \theta.$$  

Applying the cosine formula to the triangle with vertices $P_i$ for $i = 0, 1, 2$, we obtain $4k^2 = \ell^2 + m^2 - 2\ell m \cos(a/2)$. Or equivalently

$$\cos \frac{a}{2} = \frac{\ell^2 + m^2 - 4k^2}{2\ell m}.$$  

In contradiction with the assumption that $\cos(a/2) \notin \mathbb{Q}$.

**Case b:** Assume that we can go with 3 moves from the state $(P_0, \text{Id})$ to the state $(P_0, R_3(a))$ and that the third move has length $2\pi k$. Note that this case also covers the situation when the move having length $2\pi k$ is the first one. Without loss of generality, we can assume that
(i) $P_0$ is the origin of coordinates and that with the 3 moves we pass from $P_0$ to $P_1$, from $P_1$ to $P_2$, and finally from $P_2$ to $P_0$;
(ii) $P_1$ has polar coordinates $(r, \theta)$ with respect to $P_0$ with $\theta \in (0, \pi)$;
(iii) $P_2$ has polar coordinates $(R, -\alpha)$ with respect to $P_1$ with $\alpha \in (0, \pi)$; and
(iv) $P_2$ is on the positive $x$-axis and the distance between $P_0$ and $P_2$ is $2\pi k$.

See Fig. 4.

From Proposition 2.1, we can write the composition of the three moves as follows

$$R_3(a) = R_2(-2\pi k) \left( R_3(\alpha) R_2(R) R_3(-\alpha) \right) \left( R_3(-\theta) R_2(r) R_3(\theta) \right) \text{Id.}$$  \hspace{1cm} (13)

This equality provides 9 polynomial equations in the 8 variables $\cos r$, $\sin r$, $\cos R$, $\sin R$, $\cos \alpha$, $\sin \alpha$, $\cos \theta$ and $\sin \theta$. We have the additional 4 polynomial equations

$$\cos^2 r + \sin^2 r = 1,$$
$$\cos^2 R + \sin^2 R = 1,$$
$$\cos^2 \alpha + \sin^2 \alpha = 1,$$
$$\cos^2 \theta + \sin^2 \theta = 1.$$

Computing the Gröbner basis of these 13 polynomial equations with respect the mentioned 8 variables, we get an equivalent system of polynomial equations (equivalent in the sense that both systems have the same solutions) containing again the following two equations

$$\sin R \sin a = 0,$$
$$\sin r \sin a = 0.$$

Since $\cos a \notin \mathbb{Q}$ we have that $\sin a \neq 0$, so $\sin R = 0$ and $\sin r = 0$, or equivalently $r = \ell \pi$ and $R = m \pi$ with $\ell$ and $m$ positive integers.

Evaluating the equation of the third file and third column of (13) we get $1 = (-1)^{\ell+m}$. Therefore $\ell$ and $m$ have the same parity. Assume that $\ell$ and $m$ are even, then evaluating the equation of the first file and second column of (13) we get $\sin a = 0$, a contradiction with the assumption that $\cos a \notin \mathbb{Q}$. Hence $\ell$ and $m$ are odd.

Now applying two times the cosine formula to the triangle with vertices $P_i$ for $i = 0, 1, 2$ we obtain

$$\cos \theta = \frac{4k^2 + \ell^2 - m^2}{4k\ell}, \quad \cos \alpha = \frac{4k^2 - \ell^2 + m^2}{4km}.$$

Then, from the equation of the first file and first column of (13), we get that $\cos a$ must be one of the following two values

$$\pm \frac{16k^4 - 8\ell^2 k^2 - 8m^2 k^2 + \ell^4 + m^4}{2\ell^2 m^2},$$

in contradiction that $\cos a \notin \mathbb{Q}$. This completes the proof of the theorem. \hfill \Box
Since the rational numbers are numerable in the interval \([-1, 1]\), it follows from Theorem 5.1 that generically we cannot go with 3 moves from the state \((P_0, \text{Id})\) to the state \((P_0, R_3(a))\) with one of the moves of length a multiple of \(2\pi\).

**Remark 5.2.** Following the proof of Theorem 5.1 [case (a)] if \(\cos(a/2) = (\ell^2 + m^2 - 4k^2)/(2\ell m)\) with \(\ell, k, m\) positive integers and \(\ell\) and \(m\) odd, one sees that it is possible with 3 moves to pass from the state \((P_0, \text{Id})\) to the state \((P_0, R_3(a))\) and the second move has length \(2\pi k\). For instance if \(\ell = m = 3\) and \(k = 1\), we get \(\cos(a/2) = 7/9\).

Similarly from the proof of Theorem 5.1 [case (b)] if \(\cos a = \pm(16k^4 - 8\ell^2k^2 - 8m^2k^2 + \ell^4 + m^4)/(2\ell^2m^2)\) with \(\ell, k, m\) positive integers and \(\ell\) and \(m\) odd, one sees that it is possible with 3 moves to pass from the state \((P_0, \text{Id})\) to the state \((P_0, R_3(a))\) and the third move has length \(2\pi k\).

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Waldyr M. Oliva  
CAMGSD  
ISR, Instituto Superior Técnico  
UTL  
Av. Rovisco Pais  
1049-001 Lisbon  
Portugal  

and  

Departamento de Matemática Aplicada  
Instituto de Matemática e Estatística  
USP  
Rua do Matão 1010  
São Paulo CEP 05508-900  
Brazil  
e-mail: wamoliva@math.ist.utl.pt  

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