On the Restricted Lie Algebra Structure for the Witt Lie Algebra in Finite Characteristic

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1. Introduction. Let $p$ be a prime, $\mathbb{F}$ be a field of characteristic $p$, and $\mathbb{F}$ be the algebraic closure of $\mathbb{F}$. Let $A = \mathbb{F}[x]/(x^p - 1)$. Notice that $\mathbb{F}[x]/(x^p - 1) \cong \mathbb{F}[x]/(x^p)$, an isomorphism may be established be the formula $x \leftrightarrow x - 1$. Notice also that $\frac{d}{dx}(x^p - 1) \subset (x^p - 1)$ and $\frac{d}{dx}(x^p) \subset (x^p)$, hence, the operator $\frac{d}{dx}$ is also defined in $A$; the isomorphism described above commutes with $\frac{d}{dx}$. Below, we abbreviate the notation $\frac{d}{dx}$ to $\partial$.

The Lie algebra $W = \text{Der} A$ is called the Witt algebra. It consists of “vector fields” $f\partial$, $f \in A$. In particular, $\text{dim} W = \text{dim}_\mathbb{F} A = p$.

As any Lie algebra of derivations of a commutative algebra over $\mathbb{F}$, $W$ has a canonical structure of a restricted Lie algebra. Recall that a restricted Lie algebra is a Lie algebra over $\mathbb{F}$ with an additional unary (in general, non-linear) operation $g \mapsto g^{[p]}$ satisfying the conditions

\[(\lambda g)^{[p]} = \lambda^p g^{[p]} (\lambda \in \mathbb{F}), \text{ad}(g^{[p]}) = (\text{ad} g)^p, \]

\[(g + h)^{[p]} = g^{[p]} + h^{[p]} + \sum_{i=1}^{p-1} s_i(g, h)\]

where $s_i(g, h)$ is the coefficient of $\lambda^{i-1}$ in $(\text{ad}(\lambda g + h))^{p-1}(h)$ times $i^{-1}$ modulo $p$; in particular, $[g^{[p]}, g] = 0$ for any $g$ (see details in [1]). In $\text{Der} A$, $g^{[p]} = g^p$ (obviously, if $g \in \text{Der} A$, then $g^p = g \circ \ldots \circ g \in \text{Der} A$).

Although the operation $g \mapsto g^{[p]}$ does not need to be linear, it is fully determined by its values on any basis of the Lie algebra. In particular, in $W$,

\[(x\partial)^{[p]} = x\partial, \ (x^k\partial)^{[p]} = 0 \text{ for } k = 0, 2, 3, \ldots, p - 1\]

(These formulas hold whether $A$ is regarded as $\mathbb{F}[x]/(x^p - 1)$ or $\mathbb{F}[x]/(x^p)$; the same is true for Theorem 1 below). We will give, however, a more detailed description of the operation $g \mapsto g^{[p]}$ in $W$.

**Theorem 1.** (a) For any $f \in A$, $(f\partial)^{[p]} = C(f)f\partial$, where $C(f)$ is a constant (depending on $f$).

(b) For any $f \in A$, $\partial(f\partial(\ldots(f\partial(f)\ldots)))$ with $p - 1 \partial$’s (and $p - 1$ $f$’s) is a constant, and this constant is equal to $C(f)$.

(c) For any $f \in A$, $\partial^{p-1}(f^{p-1})$ is a constant, and this constant is equal to $-C(f)$.

Since Parts (a) and (b) of Theorem 1 are very simple (see Section 2), it is Part (c), or rather the equivalence between (b) and (c), that is the main result of this paper. Moreover,
this result is majorated by three (actually equivalent) combinatorial theorems. We state these theorems in Section 3 and prove them in Section 4.

2. Proof of Parts (a) and (b) of Theorem 1. The Lie algebra $W$ has rank one, in the sense that there exists a non-empty Zariski open subset $U \subset W$ such that if $g \in U$ and $[h, g] = 0$, then $h \in \mathbb{F} g$. Since $[g^{[p]}, g] = 0$, $g^{[p]} \in \mathbb{F} g$, at least for $g \in U$. But since $g \mapsto g^{[p]}$ is an algebraic map, this implies that $g^{[p]} \in \mathbb{F} g$ for all $g \in W$. (Strictly speaking, for this purpose $\mathbb{F}$ should be infinite, but, if necessary, we can extend $\mathbb{F}$ to $\overline{\mathbb{F}}$ and take $U$ in the extended $W$.) This proves (a): $(f \partial)^{[p]} = C(f) f \partial$ for some algebraic function $C: W \to \mathbb{F}$. To prove (b), apply the both sides of this equality to $x \in A$. We have: $f \partial(f \partial(...(f \partial(...(f \partial f)\ldots))) = C(f) f$, which shows that $\partial(f \partial(...(f \partial f)\ldots)) = C(f)$, at least if $f \in A$ is not a zero divisor. Hence, the last equality holds for any $f$ in a non-empty Zariski open subset of $A$ (for example, if $\bar{f} \in \mathbb{F}[x]$ has a non-zero constant term, then the image of $\bar{f}$ under the projection $\mathbb{F}[x] \to \mathbb{F}[x]/(x^p) = A$ is not a zero divisor in $A$). Since the set of those $f \in A$ for which our equality does not hold is also open, it should be empty: any two non-empty Zariski open subsets of an affine space over an infinite field overlap, and $\overline{\mathbb{F}}$, if not $\mathbb{F}$ itself, is infinite. Thus, the equality holds for all $f \in A$, which is the statement of (b).

3. Three combinatorial theorems. In this section, we state Theorems 2, 3, and 4, and show that each of them implies Theorem 1 (c). In addition to that, Theorems 2 and 3 are equivalent to each other and imply Theorem 4.

First, we consider arbitrary finite words in a two-letter alphabet ($\partial, f$) ending with $f$. (We do not specify, what $f$ and $\partial$ are; for example, $f$ may be a $C^\infty$ function in one variable and $\partial$ the derivative.) This word may be regarded as an integral linear combination of differential monomials $f^{(k_1)} \ldots f^{(k_m)}$. For example, $\partial f \partial f \partial f = (f^{[2]} \partial)^{[2]} = f'' f' + f'' f$.

**Theorem 2.** For any prime $p$,

\[(\partial f)^{p-1} \equiv -\partial^{p-1} f^{p-1} \mod p.\]

**Example:**

\[
\begin{align*}
(\partial f)^4 &= \partial f \partial f \partial f \partial f = (f')^4 + 11 f(f')^2 f'' + 4 f^2 (f'')^2 + 7 f^2 f' f''' + f^3 f^{(4)}, \\
\partial^4 f^4 &= 24(f')^4 + 144 f(f')^2 f'' + 36 f^2 (f'')^2 + 48 f^2 f' f''' + 4 f^3 f^{(4)}. 
\end{align*}
\]

We see that $(\partial f)^4 \equiv -\partial^4 f^4 \mod 5$, as stated in Theorem.

Theorem 2 (with Theorems 1(a), (b)) implies Theorem 1(c): take $f \in \mathbb{F}[x], \partial = \frac{d}{dx}$, and project the equality $(\partial f)^{p-1} = -\partial^{p-1} f^{p-1}$ (which holds if char $\mathbb{F} = p$) onto $A$.

Theorem 2 may be reformulated as a congruence of symmetric polynomials, in the following way.

**Theorem 3.** In $\mathbb{Z}[t_1, \ldots, t_{p-1}]$,

\[
\sum_{\sigma \in S_{p-1}} t_{\sigma(1)}(t_{\sigma(1)} + t_{\sigma(2)}) \ldots (t_{\sigma(1)} + \ldots + t_{\sigma(p-1)}) \equiv (t_1 + \ldots + t_{p-1})^{p-1} \mod p,
\]
where \( p \), as usual, is a prime.

(No minus sign is this congruence, it is not a misprint!)

Obviously, if

\[
t_1(t_1 + t_2) \ldots (t_1 + \ldots + t_{p-1}) = \sum n_{k_1 \ldots k_{p-1}} t_1^{k_1} \ldots t_{p-1}^{k_{p-1}},
\]

then

\[
\partial f_{p-1} \partial f_{p-2} \ldots \partial f_1 = \sum n_{k_1 \ldots k_{p-1}} f_1^{(k_1)} \ldots f_{p-1}^{(k_{p-1})}.
\]

Similarly, if

\[
(t_1 + \ldots + t_{p-1})^{p-1} = \sum m_{k_1 \ldots k_{p-1}} t_1^{k_1} \ldots t_{p-1}^{k_{p-1}},
\]

then

\[
\partial^{p-1}(f_1 \ldots f_{p-1}) = \sum m_{k_1 \ldots k_{p-1}} f_1^{(k_1)} \ldots f_{p-1}^{(k_{p-1})}.
\]

Hence, the congruence in Theorem 3 is equivalent to

\[
\sum_{\sigma \in S_{p-1}} \partial f_{\sigma(p-1)} \partial f_{\sigma(p-2)} \ldots \partial f_{\sigma(1)} \equiv \partial^{p-1}(f_1 \ldots f_{p-1}) \mod p
\]

which becomes, after substituting \( f_1 = \ldots = f_{p-1} = f \),

\[
(p-1)!(\partial f)^{p-1} \equiv \partial^{p-1}f^{p-1} \mod p
\]

(the last two congruences are, actually, equivalent.) Since \((p-1)! \equiv -1 \mod p\), the last congruence is that of Theorem 2. Thus, Theorems 2 and 3 are equivalent.

Our last combinatorial theorem concerns a certain function on Young diagrams. To avoid drawing, we use the term Young diagram for a finite sequence \((j_1, \ldots, j_m)\) of integers with \(j_1 \geq \ldots \geq j_m > 0\). The sequence may be empty \((m = 0)\). For a Young diagram \(J = (j_1, \ldots, j_m)\), we put \(N(J) = j_1 + \ldots + j_m\), \(m(J) = m\), and \(n_k(J) = \#\{s | j_s = k\}\). Define a function \(d\) on Young diagrams recursively: \(d(\emptyset) = 1\), and if \(J = (j_1, \ldots, j_m), N(J) = N, \) and \(d(K)\) has been already defined for all Young diagrams \(K\) with \(N(K) = N - 1\), then

\[
d(J) = \sum_{s, j_s > j_{s+1}} (N - j_s + 1) n_{j_s}(J) d(j_1, \ldots, j_{s-1}, j_s - 1, j_{s+1}, \ldots, j_m).
\]

(Here we put \(j_{m+1} = 0\) and if \(s = m\) and \(j_s = 1\), then \(j_s - 1\) is zero, and we simply delete this zero.)

**Theorem 4.** If \(N(J) = p - 1\) (where \(p\) is prime), then

\[
d(J) \equiv 1 \mod p.
\]
Examples:

\[ d(\emptyset) = 1; \]
\[ d(1) = 1 \cdot 1 \cdot d(\emptyset) = 1; \]
\[ d(1, 1) = 2 \cdot 2 \cdot d(1) = 4, \quad d(2) = 1 \cdot 1 \cdot d(1) = 1; \]
\[ d(1, 1, 1) = 3 \cdot 3 \cdot d(1, 1) = 36, \quad d(2, 1) = 2 \cdot 1 \cdot d(1, 1) + 3 \cdot 1 \cdot d(2) = 11, \]
\[ d(3) = 1 \cdot 1 \cdot d(2) = 1; \]
\[ d(1, 1, 1, 1) = 4 \cdot 4 \cdot d(1, 1, 1) = 576, \quad d(2, 1, 1) = 3 \cdot 1 \cdot d(1, 1, 1) + 4 \cdot 2 \cdot d(2, 1) = 196, \]
\[ d(2, 2) = 3 \cdot 2 \cdot d(2, 1) = 66, \quad d(3, 1) = 2 \cdot 1 \cdot d(2, 1) + 4 \cdot 1 \cdot d(3) = 26, \]
\[ d(4) = 1 \cdot 1 \cdot d(3) = 1. \]

We see that if \( N(J) = 2 \), then \( d(J) = 4, 1 \equiv 1 \mod 3 \), and if \( N(J) = 4 \), then \( d(J) = 576, 196, 66, 26, 1 \equiv 1 \mod 5 \).

Theorem 4 is equivalent to Theorem 2 restricted to the case, when \( f \) is a monic polynomial of degree \( p - 1 \) (this case of Theorem 2 is sufficient for proving Theorem 1(c)).

Indeed, let \( f(x) = (x - \alpha_1) \ldots (x - \alpha_{p-1}) \) (where \( \alpha_1, \ldots, \alpha_{p-1} \in \mathbb{P} \)). We put \( x - \alpha_i = u_i \); thus, \( f = u_1 \ldots u_{p-1} \) and \( \partial u_i = 1 \). Let \( n \leq p - 1 \). Then \( (\partial f)^n = \partial f \partial f \ldots \partial f \) is a symmetric polynomial in \( u_1, \ldots, u_{p-1} \) of total degree \( n(p - 2) \) and of degree \( \leq p - 1 \) with respect to each variable \( u_i \). Let \( J = (j_1, \ldots, j_m) \) be a Young diagram with \( N(J) = n \). Then an obvious induction based on the equality \((\partial f)^n = \partial (u_1 \ldots u_{p-1} (\partial f)^{n-1})\) shows that the coefficient at
\[ u_1^{n-j_1} \ldots u_m^{n-j_m} u_{m+1} \ldots u_{p-1} \]
in the polynomial \( (\partial f)^n \) is \( d(J) \).

On the other hand, the coefficient at the same monomial in the polynomial \( \partial^n(f^n) \) is
\[
\frac{n!}{j_1! \ldots j_m!} \prod_{i=1}^{m} n(n-1) \ldots (n-j_i+1) = \frac{n!}{j_1! \ldots j_m!} \frac{n!}{(n-j_1)!} \ldots \frac{n!}{(n-j_m)!} = n! \left( \frac{n}{j_1} \right) \ldots \left( \frac{n}{j_m} \right).
\]

Since \( (p - 1)! \equiv -1 \mod p \) and \( \left( \begin{array}{c} p - 1 \\ j \end{array} \right) \equiv (-1)^j \mod p \), the last quantity for \( n = p - 1 \) is
\[ (p - 1)! \left( \begin{array}{c} p - 1 \\ j_1 \end{array} \right) \ldots \left( \begin{array}{c} p - 1 \\ j_m \end{array} \right) \equiv (-1) \cdot (-1)^{j_1} \ldots (-1)^{j_m} \mod p, \]
and \( (-1) \cdot (-1)^{j_1} \ldots (-1)^{j_m} = (-1)^p = -1 \) (if \( p \) is odd; if \( p = 2 \), then \(-1 \equiv 1 \mod p \)).

Thus, Theorem 2 with \( f = u_1 \ldots u_{p-1} \) is equivalent to Theorem 4.

We conclude this section with three remarks concerning Theorem 4. First, we will never mention this theorem again; certainly, it follows from the other theorems of this section, but we do not have any direct proof for it. Still we think that it deserves to be stated as one of the results of this paper. Second, this Theorem may have some meaning
in the representation theory of symmetric groups, but this meaning evades us. Third, it is not hard to deduce from Theorem 4 that the congruence \( d(J) \equiv 1 \mod p \) holds also if \( N(J) = p - 2 \) (check this for \( p = 3 \) and 5 using the example after the statement of Theorem 4). We leave this to the reader as an exercise.

4. Proofs. We will prove Theorem 3 (using its relations to propositions similar to Theorems 1 and 2). As we know, this will imply all the other theorems of this paper.

Let \( \tilde{W} = \text{Der} \mathbb{F}[x] \). This is an infinite dimensional restricted Lie algebra. Elements of \( \tilde{W} \) are “vector fields” \( f \partial, f \in \mathbb{F}[x] \). The \( p \)-th power of a derivation \( f \partial \) is also a derivation:

\[
(f \partial)^p = F \partial, \ F \in \mathbb{F}[x].
\]

Raising \( f \partial \) to the power \( p \), we get

\[
F_1 \partial + F_2 \partial^2 + \ldots + F_p \partial^p = F \partial
\]

(where \( F_1 = f \cdot (\partial (f \partial (\ldots (f \partial f) \ldots ))) \), \( F_p \). Applying both sides of this equality to \( (x - a)^k \), where \( 1 < k < p \) and \( a \in \mathbb{F} \), and then setting \( x = a \), we get

\[
F_k(a) \cdot k! = 0,
\]

which shows that \( F_2 = \ldots = F_{p-1} = 0 \). Since \( \partial^p = 0 \) on \( \mathbb{F}[x] \), we see that

\[
F = F_1 = fg, \ g = (\partial f)^{p-1} = \partial f \partial f \ldots \partial f.
\]

But \( [(f \partial)^p, f \partial] = 0 \); hence, \( [fg \partial, f \partial] = (fgf' - fff'g - f^2g') \partial = -f^2g' \partial = 0 \), that is, \( g' = 0 \) (for \( f \neq 0 \), and therefore for any \( f \)). (Actually, this means that \( g \) is a polynomial in \( x^p \), but we will not need this.)

Consider differential expression

\[
g(f) = \partial \partial f \partial f \ldots \partial f \ (p \ \partial \text{s, } p - 1 \ \text{f' s}).
\]

Polarize the restriction of the form \( f \mapsto (g(f)) (a) \), \( a \in \mathbb{F} \) of degree \( p - 1 \) to the vector space of polynomials of degree \( < p \). We get a symmetric \( (p - 1) \)-linear form

\[
G(f_1, \ldots, f_{p-1}) = \sum_{\sigma \in S_{p-1}} (\partial \partial f_{\sigma(1)} \partial f_{\sigma(2)} \ldots \partial f_{\sigma(p-1)})(a)
\]

which is equal to 0, since \( g'(f) = 0 \). As a differential expression, the right hand side of the last equality is a linear combination of monomials \( f_1^{(j_1)} f_2^{(j_2)} \ldots f_{p-1}^{(j_{p-1})}(a) \) with \( j_1 + j_2 + \ldots + j_{p-1} = p \), but plugging \( f_1 = (x - a)^{i_1}, \ldots, f_{p-1} = (x - a)^{i_{p-1}} \) with all \( i_1, \ldots, i_{p-1} \) between 0 and \( p - 1 \) and equating the results to 0, we see that all the monomials with \( j_1 < p, \ldots, j_{p-1} < p \) have zero coefficients in \( \mathbb{F} \) (that is, they are 0 modulo \( p \)). Since the coefficient at \( f_1 \ldots f_{s-1} f_s^{(p)} f_{s+1} \ldots f_{p-1} \) is obviously \( (p - 2)! \equiv 1 \mod p \), we arrive at the conclusion:

\[
\sum_{\sigma \in S_{p-1}} \partial \partial f_{\sigma(1)} \ldots \partial f_{\sigma(p-1)} \equiv \sum_{s=1}^{p-1} f_1 \ldots f_{s-1} f_s^{(p)} f_{s+1} \ldots f_{p-1} \mod p,
\]
which may be rewritten as

\[ \sum_{\sigma \in S_{p-1}} t_{\sigma(1)}(t_{\sigma(1)} + t_{\sigma(2)}) \ldots (t_{\sigma(1)} + \ldots + t_{\sigma(p-1)})(t_1 + \ldots + t_{p-1}) \equiv t_1^p + \ldots + t_{p-1}^p \mod p \]

(the last factor in the left hand side of the last formula arises from the first \( \partial \) in the left hand side of the previous formula). But \( t_1^p + \ldots + t_{p-1}^p \equiv (t_1 + \ldots + t_{p-1})^p \mod p \). Canceling \( t_1 + \ldots + t_{p-1} \), we obtain the congruence of Theorem 3.

References

[1] JACOBSON N. Lie Algebras. John Wiley, NY, 1962.