Quantum advantage with noisy boson sampling and density of bosons

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Inevitable noise is the main problem in demonstration of computational advantage of quantum devices, such as boson sampling, over digital computers. Can a noisy realization of boson sampling be efficiently and faithfully simulated classically? It is shown how one can distinguish the output distribution of noisy N-boson sampling from that of classical approximations with mixtures of quantum interferences of up to $K \ll \sqrt{N}$ bosons, with a number of samples that depends solely on $K$, noise amplitude and density of bosons $\rho = N/M$, where $M$ is network size. The surprising result is that noisy boson sampling in a regime of finite density of bosons $\rho \leq 1$, i.e., on a small network $M = N/\rho$, retains scalable quantum advantage to arbitrary large number of bosons despite the presence of finite noise.

Introduction and problem formulation. – Quantum mechanics promises computational advantage over digital computers [1, 2]. Current technology is on the brink of building quantum devices with the promised superiority in some specific computational tasks, named the quantum supremacy [3], for which goal several quantum systems are considered [4–8]. Will noise, always present in an experimental setup, prevent demonstration of the quantum supremacy by allowing an efficient classical simulation [9]?

In boson sampling of Aaronson & Arkhipov [4], the specific classically hard computational task is sampling from many-body quantum interference of $N$ indistinguishable bosons on a unitary $M$-dimensional network (unitary $M$-port). While the formal arguments are given for the no-collision regime, when the output ports receive at most a single boson (i.e., for $M \gg N^2$ [10]), a possibility of going beyond it is not discarded. There are $N!$ quantum amplitudes contributing to many-body interference of $N$ bosons, the sum of which is the matrix permanent of an $N$-dimensional submatrix of a unitary $M$-port matrix [11, 12], hard to compute [13, 16].

Single photons [17–21] as well as Gaussian states [22–24] in optical networks, and the temporal-mode encoding [25, 26] were proposed and tested for experimental implementation with quantum optics. Alternative platforms include ion traps [27], superconducting qubits [28, 29], neutral atoms in optical lattices [30] and dynamic Casimir effect [31]

Initially it was estimated that the threshold system size for demonstration of quantum supremacy with boson sampling would be $N \approx 30$ bosons [4]. However, recent Markov Chain Monte Carlo classical simulation algorithm [32] and subsequent analytical estimate [33] pushed the threshold to $N \approx 50$ bosons. This seem to significantly affect feasibility of demonstration of quantum advantage with boson sampling with the current technology, notwithstanding the reported spectacular advances in experimental implementation [34–37]. Indeed, inevitable experimental imperfections [29, 33, 41] allow for efficient classical approximation algorithms [15, 50], which seem to prevent reaching $N \approx 50$ by combining the best of reported components [49], even assuming that the noise amplitudes remain fixed when scaling up. Moreover, current planar optical networks feature a depth-dependent transmission, thus limiting the system size for quantum advantage demonstration [47, 48].

The efficient classical algorithms [46–50], applicable to noisy boson sampling with noise amplitude independent of system size, approximate the output distribution to a small error $\epsilon$ in the total variation distance. The key point of boson sampling proposal [4] was to allow a small approximation error by showing that a classical simulation of the output probability distribution of boson sampling to an error $\epsilon$ with the computations polynomial in $N$ and $1/\epsilon$ is impossible (if some plausible conjectures are true) [4]. But for a finite-size experimental system exactly how small such an error should be? Alternatively, in the spirit of Ref. [1], we can ask if an efficient classical algorithm is possible that samples from the output distribution of a real experimental noisy quantum system in such a way that it would be impossible to tell from the sampling data whether we have the classical simulation or the quantum system? In this formulation, the abstract approximation error, to be imposed on a classical simulation, is defined (via the standard means of statistics) by the number of samples obtained from an experimental quantum device. The asymptotic (in system size) quantum advantage over classical simulations becomes a well-posed problem testable on realistic noisy finite-size quantum devices.

Boson sampling and density of bosons. – As a partial answer to the above problem of efficient classical simulation, below it is shown how one can distinguish the output of noisy quantum system, where noise has fixed amplitude as the system size scales up, and a wide range of classical algorithms capable to approximate its output distribution. We will consider the general case of quantum interference of $N$ identical single bosons on a unitary $M$-port for arbitrary $M \geq N$. As we will show below the regime of fixed density of bosons $\rho = N/M$, as $N$ scales up, is the key to our conclusions, such setup will be called the $\rho$-density boson sampling.

In the ideal (noiseless) case, the computational hard-
ness of sampling from the output distribution of $\rho$-density boson sampling depends on $N$ and $M$, or, equivalently, $N$ and $\rho$. One can sample from the output distribution by the Markov Chain Monte Carlo algorithm of Refs. [32, 33], applicable over all density regimes $0 < \rho \leq 1$. For finite $\rho$ there is bunching of bosons at the output ports [10], thus the output probabilities are given by matrix permanents of rank deficient matrices with reduced computational complexity [51, 52]. The following simple rule is applicable [53] for the sampling algorithm of Refs. [32, 33]: to sample from the $N$-boson $\rho$-density boson sampling requires at least as many classical simulations as for the no-collision boson sampling ($\rho \ll 1/N$) with $N_\rho = N/(1 + \rho)$ bosons, where $N_\rho$ is the expected number of output ports occupied by bosons. Faced by the absence of efficient (polynomial) algorithms we assume the classical hardness to hold for any $\rho \leq 1$.

The classical adversary. The above discussion of sampling complexity applies only to ideal (noiseless) boson sampling. However, our focus is on real experimental device, when noise in network [32, 40, 42], partial distinguishability of bosons [41], boson losses [43], etc. are also taken into account. There is equivalence of imperfections in their effect on classical hardness of boson sampling, e.g., noise in experimental platforms [21, 29, 39] has similar effect to that of partial distinguishability of bosons [21, 50] (below the term noise stands for all imperfections). At least in the no-collision regime, noise strongly affects many-boson quantum interference [39]. Efficient classical approximations of noisy many-body interference of $N$ bosons by a mixture of that with up to $K$ bosons were found [46, 49, 50]. Such a classical approximation becomes efficient for $K = O(1)$, since the classical computations scale exponentially only in $K$ [46] (this is also implied by the sampling algorithm of Ref. [33]). As such algorithms include previously considered (e.g., classical particles when $K = 1$), are the most powerful to-date classical approximations for noisy boson sampling, can be based on many different imperfections (such as distinguishability [40], losses [49], and errors in network [50]), we consider our classical adversary to use such an algorithm. Note that we assume that the above algorithms can actually approximate the output distribution of $\rho$-density boson sampling for all $\rho \leq 1$, to date still an open problem.

Noisy boson sampling vs. classical adversary. Consider now noisy realization of $\rho$-density boson sampling with arbitrary partially distinguishable bosons and arbitrary lossy linear network $\mathcal{U}$, $\mathcal{U}^\dagger \mathcal{U} \leq I$ (below we will also account for dark counts of detectors). An arbitrary state of partial distinguishability of $N$ bosons is described by a function on the symmetric group $S_N$ of permutations $\sigma$, defined by $J(\sigma) = \prod_{k=1}^{N} \langle \psi_{\sigma(k)} | \psi_k \rangle$, where $| \psi_k \rangle$ is the internal state of boson $k$ (by linearity, $J(\sigma)$ is extendable to arbitrary mixed internal states of bosons) [41, 54]. The probability to detect $0 \leq n \leq N$ bosons in a configuration $\mathbf{m} = (m_1, \ldots, m_M)$, $m_1 + \ldots + m_M = n$, at the output of a lossy network $\mathcal{U}$, for single bosons at input ports $1, \ldots, N$ reads [52]

$$p(\mathbf{m}) = \frac{1}{\mathbf{m}!} \sum_{\sigma \in S_N} J(\sigma) \sum_{k_1 \ldots k_n} \prod_{a=1}^{n} \mathcal{U}_\sigma(k_a).I_a \mathcal{U}_\tau(k_a).I_a \times \prod_{a=n+1} \left( I - \mathcal{U}^\dagger I_k.\sigma(I_a) \right),$$

(1)

where $I_1, \ldots, I_n$ are the output ports corresponding to $\mathbf{m}$, $\mathbf{m}' = m'_1 \ldots m'_M$, $k_1, \ldots, k_N$ is a permutation of $1, \ldots, N$, and the sum over $k_1, \ldots, k_n$ stands for summation over all $n$-dimensional subsets of $1, \ldots, N$.

We are interested in the total variation distance between the distribution $(p)$ of Eq. (1) and that of the approximation $(p^{(K)})$ accounting for noisy quantum interference of $N$ bosons by a mixture of quantum interferences with up to $K$ bosons, see Fig. [1]a). There is a general way [50] to introduce the approximate classical models used in Refs. [16, 49, 50]. To this goal one needs only to modify the $J$-function, by noting that it factorizes $J(\sigma) = J(\nu_1)J(\nu_2) \ldots J(\nu_q)$ according to the disjoint cycle decomposition $\sigma = \nu_1 \nu_2 \ldots \nu_q$ of permutation [56], where each cycle $\nu_i : k_{i_1} \rightarrow \ldots \rightarrow k_{i_m} \rightarrow k_{i_1}$ ($| \nu_i |$ being the cycle length) gets a unique factor $J(\nu_i)$ accounting for specific $| \nu_i |\text{-}boson$ interference process [57, e.g., for pure states $J(\nu_i) = \prod_{l=1}^{q} \langle \psi_{k_{i_1}} | \psi_{k_{i_2}} \rangle$. The classical approximation is obtained by imposing at least $N - K$ fixed points ($N - K$ bosons, arbitrarily chosen from $N$, do not participate in many-body interference [63]), i.e., it is given by the expression as in Eq. (1) but with the distinguishability function $J^{(K)}$ [50]:

$$J^{(K)}(\sigma) = \begin{cases} J(\sigma), & c_1(\sigma) \geq N - K, \\ 0, & c_1(\sigma) < N - K, \end{cases}$$

(2)

where $c_1(\sigma)$ is the number of fixed points (1-cycles) in $\sigma$.

To characterize the total variation distance $D(\mathbf{p}, \mathbf{p}^{(K)})$ analytically is a hard problem, whereas numerical simulations face classical hardness for $N > 1$ and absence of algorithms for noisy boson sampling similar as those for the ideal case [32, 33]. One can use, however, that $D(\mathbf{p}, \mathbf{p}^{(K)})$ is bounded from below by any difference in probability $\Delta P_{\Omega_L}$, $D(\mathbf{p}, \mathbf{p}^{(K)}) \geq |\Delta P_{\Omega_L}|$, with equality achieved for a certain probability $P = \sum_{\mathbf{m} \in \Omega_L} p(\mathbf{m})$. Consider the difference in probability to detect all output bosons in a certain subset of $M - L$ output ports, or, equivalently, no detector counts in the complementary output ports $\Omega_L = \{ l_1, \ldots, l_L \}$. This choice allows us to take into account also the dark counts of detectors, which we assume following a Poisson distribution $p_d(n) = \frac{e^{-\nu \nu}}{n!} \nu^n$ with a uniform rate $\nu$. The latter contribute to $\Delta P_L$; the factor $e^{-\nu \nu}$ equal to the probability of zero dark counts in $\Omega_L$. Setting $\Delta J = J - J^{(K)}$ we get from Eqs. (1)-(2) the
following bound  \[ \mathcal{D}(p, p^{(K)}) \geq |\Delta P_L|, \]

\[
\Delta P_L \equiv e^{-L\nu} \sum_{\sigma \in S_N} \Delta J(\sigma) \prod_{k=1}^{N} \left[ \delta_{k,\sigma(k)} - \sum_{l=1}^{L} U_{k,l} U_{\sigma(k),l}^* \right].
\]

The crucial point is that for a fixed \( L \) and large \( N \gg 1 \)

\[
\langle \mathcal{D}(p, p^{(K)}) \rangle \geq |\langle \Delta P_L \rangle| \geq W_1, \quad \text{var}(\Delta P_1) = \frac{R_1}{N},
\]

where the approximations are obtained assuming that \( K \ll \sqrt{N} \). The \( W_1 \) of Eq. (4) is also an absolute (no averaging) lower bound for all networks with one balanced output port \(|U_{k,1}| = \frac{1}{\sqrt{M}}\), if one choses \( \Omega_i \) to be that port. Such networks contain a wide class: \( U = F(1 + V) \) with the Fourier network \( F_{kl} = \frac{1}{\sqrt{M}}e^{2\pi i kl} \) and an arbitrary \((M - 1)\)-dimensional unitary network \( V \).

For multiple bosons per input port, say in a configuration \( n = (n_1, \ldots, n_M) \), \( n_1 + \ldots + n_M = N \), the output probability of Eq. (1), hence the lower bound in Eqs. (3)-(4), is divided by \( n! \) [4]. We have in this case \( n! \geq \langle n_0 \rangle^M > \langle n_0/e \rangle^M \), with \( n_0 = [p] \), i.e., the bound is exponentially small in \( N \) at least for \( \rho > e \), whereas numerical results indicate that it rapidly goes to zero when \( \rho \) grows above 1. Hence our focus is on \( \rho \leq 1 \).

In general, setting \( L > 1 \) in Eq. (3) can result in a tighter lower bound than that for \( L = 1 \). Indeed, for noiseless boson sampling, \( \xi = \eta = 1 \) and \( \nu = 0 \), for \( K = 1 \) and \( L \ll M \) the total variation distance averaged over the Haar-random network satisfies \( \langle \mathcal{D}(p, p^{(K)}) \rangle \geq (1 + \rho)^{-L} - e^{-L\rho} \), maximised for \( \rho \leq 1 \) at the value of \( L \) equal to the integer part of \( \rho^{-1} \).

Eq. (3) was simulated numerically, for different choices of \( L \), in a uniformly lossy network \( U = \sqrt{\eta}U \) (for \( \xi = 1 \) and \( \nu = 0 \)), by choosing networks \( U \) according to the Haar measure from the unitary group. The results are shown in fig. 1 where the scale-invariance (the dependence only on \((K, \rho, \eta)\)) is check by comparing the average lower bound \(|\langle \Delta P_L \rangle|\) for \( N = 12 \) and \( N = 24 \). The numerics shows that the lower bound \( W_1 \) of Eq. (4) is a very good approximation to the average \(|\langle \Delta P_L \rangle|\) for \( K \geq 3 \) (and \( N \gg K^2 \), the condition for analytical approximation).

For efficient classical approximation with \( K = O(1) \), in the no-collision regime \( \rho \ll 1/N \) the lower bound \( W_1 \) of Eq. (4) vanishes polynomially in \( 1/N \). On the other hand, for a finite \( \rho \) we get \( W_1 = O(1) \), i.e., the total variation distance remains bounded from below for all \( N \gg 1 \). Therefore noisy boson sampling is far from efficient classical approximators in the regime of a finite density of bosons. Previously noiseless boson sampling was shown [60] to be far from the uniform distribution (suggested as an approximation [61]), leaving as open problems to analyse better classical approximations and the effect of noise. Our results partially resolve open problems (2) and (4)-(6) of Ref. [60] by considering a
whole class of such approximations to a noisy realisation of boson sampling for general $M \geq N$, beyond the no-collision regime.

**Implications for quantum advantage demonstration.**– The above results on the distinguishability of noisy boson sampling from classical simulations suggest that quantum advantage may persist in a noisy experimental setup realising a finite-density regime, i.e., on a small network $M = \rho^{-1}N$, $\rho \sim 1$, rather than in the usual no-collision regime ($M \gg N^2$)\(^4\), or in any vanishing-density regime, $M \gg N$, in general. By selecting specific $\rho \leq 1$ one could optimise between the experimental coherence time (obtainable number of samples) and size of a quantum device (before noise becomes too strong), observing that the classical computational hardness decreases with $\rho$ (by the equivalent hardness rule $N_\rho = N/(1+\rho)$), whereas the resistance of the quantum advantage to noise increases with $\rho$ (as evidenced by Eq. 4 and the numerical results).

Boson bunching in the output ports of a network, not negligible for finite $\rho$, necessitates boson-number resolving detection for faithful sampling from the output distribution. For dark counts rate stronger than that of losses, using boson-number not resolving detectors can lead to efficient classical simulations in any regime of boson density $\rho$. Assume the dark counts rate being negligible. Under the condition $M \geq N \gg 1$, in a Haar-random unitary network, the average probability that boson bunching at a network output is bounded by $s$ reads 10

$$\text{Prob}(\max(m_{l}) \leq s) \approx \left[1 - \left(\frac{\rho}{1 + \rho}\right)^{s+1}\right]^{M}. \quad (5)$$

By Eq. 5 with probability $p = 1 - \delta$ the maximal boson bunching count $s$ at a network output becomes $s \approx \ln \left(\frac{N}{\rho}\right) / \ln \left(\frac{1}{r}\right)$. Therefore, the on-off detectors, sufficient in the no-collision regime, for finite $\rho$ would act similarly to nonlinear non-uniform losses of bosons at the detection stage, with the equivalent average transmission in a Haar-random network scaling (in the worst case) as $\eta \sim 1/\ln(N)$, above the currently known threshold $\eta = o(1/\sqrt{N})$ for the efficient classical simulation (48). Whether such a transmission leads to an efficient classical simulation algorithm for all $\rho \leq 1$ is currently unknown. However, the lower bound of Eq. 4 decreases with $N$ under such transmission, i.e., the number of samples for telling the quantum system from classical simulations scales up with $N$, which may be an indication of vanishing quantum advantage with the system size.

The currently used (planar) optical networks have transmission $\eta$ vanishing exponentially with size, thus limiting the size of a boson sampling device before classical simulations become efficient (48). Post selection on a fixed number of bosons is considered as means to combat such losses (36) (when the dark counts are negligible) at the expense of reducing the sampling rate by the factor equal to the respective probability. Post selection increases the effective transmission $\eta$, hence, the lower bound in Eq. 4 (as one can easily verify), allowing one to distinguish such a post-selected noisy boson sampling from classical simulations with a smaller number of samples, i.e., quantum advantage demonstration can still be possible depending on the sampling rate and specific noise amplitudes in the setup.

Shifting to a finite-density regime resolves previously reported issue with experimental boson sampling by using microwave photons and superconducting qubits\(28\), namely: the limiting number of quantum operations $M \sim 500$ (in the no-collision regime allowing to reach only $N \approx 20$). Moreover, the experimentally tested nondemolition (i.e., without loss) photon-number resolving detection \(62\) would allow for a high-fidelity photon-number resolved counting, making such platform very promising for reaching the quantum advantage threshold with $\rho$-density boson sampling.

**Conclusion.**– In conclusion, in the spirit of Ref. [1], we have considered if a realistic quantum system realising imperfect/noisy boson sampling of Ref. 2 can be efficiently and faithfully simulated classically as the system size scales up. It turns out that specifying the regime of density of bosons, defined as the ratio of the number of bosons $N$ to the network size $M$, $\rho = N/M$, is a critical parameter for the answer. It is shown how one can distinguish the output distribution of noisy boson sampling, with bounded amplitude of noise, from classical algorithms that try to emulate the distribution for any $N \gg 1$ by accounting for the many-body interference only up to a fixed $(N$-independent) order. This indicates on essential contribution from the higher orders of many-body quantum interference, the source of the classical hardness \(13, 15\), to the output of such a noisy boson sampling. Evidence is presented that noise resistance of quantum advantage with boson sampling scales up with density of bosons, with the results pointing on a phase transition as $N \to \infty$ from vanishing density regime $\rho = o(1)$ to a finite $\rho \leq 1$. Most importantly, the output distribution of noisy boson sampling with $N$ bosons on a small linear $M$-port, i.e., for $M/N = O(1)$, remains at a finite $N$\(\text{-independent}\) distance from classical simulations if noise amplitudes remain bounded as $N$ scales up.

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