NOTE ON THE MODIFIED $q$-BERNSTEIN POLYNOMIALS

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Abstract. In the present paper, we propose the modified $q$-Bernstein polynomials of degree $n$, which are different $q$-Bernstein polynomials of Phillips (see [4]). From these the modified $q$-Bernstein polynomials of degree $n$, we derive some interesting recurrence formulae for the modified $q$-Bernstein polynomials.

§1. Introduction

Let $C[0, 1]$ denote the set of continuous function on $[0, 1]$. In [7], Bernstein introduced the following well-known linear positive operators.

$$B_n(f : x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}$$

$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad (1)$$

for $f \in C[0, 1]$. $B_n(f : x)$ is called the Bernstein operator for $f$. The Bernstein polynomial of degree $n$ is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad (2)$$

for $k, n \in \mathbb{Z}_+$, where $x \in [0, 1]$ and $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. It is easy to show that

$$B_{0,1}(x) = 1 - x, B_{1,1}(x) = x,$$

$$B_{0,2}(x) = (1 - x)^2, B_{1,2}(x) = 2x(1 - x), B_{2,2}(x) = x^2,$$

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\[ B_{0,3}(x) = (1 - x)^3, \quad B_{1,3}(x) = 3x(1 - x)^2, \quad B_{2,3}(x) = 3x^2(1 - x), \]
\[ B_{3,3}(x) = x^3, \cdots. \]

Many researchers have studied the Bernstein polynomials in the area of approximation theory (see [1-8]). For \( k \in \mathbb{Z}_+ \), it is easy to show that
\[
\frac{t^k e^{(1-x)t} x^k}{k!} = \frac{x^k}{k!} \left( t^k \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{n!} \right) = \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n t^{n+k}}{n!} = x^k \sum_{n=k}^{\infty} \frac{(1-x)^{n-k} (n+k) t^n}{n!} = x^k \sum_{n=k}^{\infty} \frac{(1-x)^{n-k} (n+k) t^n}{n!} = x^k \frac{B_{k,n}(x) t^n}{n!},
\]
and \( B_{k,0}(x) = B_{k,1}(x) = \cdots = B_{k,k-1}(x) = 0 \). Thus, we obtain the generating function for \( B_{k,n}(x) \) as follows:
\[
F^{(k)}(t, x) = \frac{x^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} \frac{B_{k,n}(x) t^n}{n!}, \quad (3)
\]
where \( k \in \mathbb{Z}_+ \) and \( x \in [0,1] \). From (3), we can derive
\[
B_{k,n}(x) = \begin{cases} 
\binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \geq k \\
0 & \text{if } n < k,
\end{cases}
\]
for \( n, k \in \mathbb{Z}_+ \).

Let \( q \) be regarded as a real number with \( 0 < q < 1 \) and let us define the \( q \)-number as follows:
\[
[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad (\text{see [1-7] }).
\]
Note that \( \lim_{q \to 1} [x]_q = x \). In [4], Phillips introduced the \( q \)-extension of Bernstein polynomials. Recently, Simsek and Acikgoz have also studied the \( q \)-extension of Bernstein type polynomials ([5]). Their \( q \)-Bernstein type polynomials are given by
\[
Y_n(k; x : q) = \binom{n}{k} (-1)^k k! \sum_{m,l=0}^{n-k} \sum_{j=0}^{k+l-1} \binom{k + l - 1}{l} \binom{n-k}{k} \times \left( -1 \right)^j q^{j+k-l-1} S(m,k)(x \ln q)^m.
\]
where \( S(m,k) \) are the second kind stirling number.
In this paper we consider the $q$-extension of the generating function of Bernstein polynomials (see Eq. (3)). From these $q$-extension of generating function for the Bernstein polynomials, we propose the modified $q$-Bernstein polynomials of degree $n$, which are different $q$-Bernstein polynomials of Phillips. By using the properties of the modified $q$-Bernstein polynomials, we can obtain some interesting recurrence formulae for the modified $q$-Bernstein polynomials of degree $n$.

§2. The modified $q$-Bernstein polynomials

For $q \in \mathbb{R}$ with $0 < q < 1$, let us consider the $q$-extension of Eq. (3) as follows:

$$F_q^{(k)}(t, x) = \frac{t^k e^{[1-x]q t} [x]^k}{k!} = \frac{[x]^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n_q}{n!} t^{n+k}$$

$$= \frac{[x]^k}{k!} \sum_{n=0}^{\infty} \left( \begin{array}{c} n + k \end{array} \right)_q [1-x]^n_q \frac{t^{n+k}}{(n+k)!}$$

$$= \sum_{n=k}^{\infty} \left( \begin{array}{c} n \end{array} \right)_q [x]^k_q [1-x]^{n-k} \frac{t^n}{n!}, \quad (4)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. Note that $\lim_{q \to 1} F_q^{(k)}(t, x) = F(t, x)$. By (4), we can define the modified $q$-Bernstein polynomials as follows:

$$F_q^{(k)}(t, x) = \frac{t^k e^{[1-x]q t} [x]^k}{k!} = \sum_{n=k}^{\infty} B_{k,n}(x, q) \frac{t^n}{n!}, \quad (5)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. By comparing the coefficients on the both sides of (4) and (5), we obtain the following theorem.

**Theorem 1.** For $k, n \in \mathbb{Z}_+, x \in [0, 1]$, we have

$$B_{k,n}(x, q) = \left\{ \begin{array}{ll} \left( \begin{array}{c} n \end{array} \right)_q [x]^k_q [1-x]^{n-k}_q & \text{if } n \geq k \\ 0 & \text{if } n < k. \end{array} \right.$$ 

For $0 \leq k \leq n$, we have

$$[1-x]_q B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q)$$

$$= [1-x]_q \left( \begin{array}{c} n-1 \end{array} \right)_k [x]_q^k [1-x]^{n-1-k}_q + [x]_q \left( \begin{array}{c} n-1 \end{array} \right)_{k-1} [x]_{q}^{k-1} [1-x]^{n-k}_q$$

$$= \left( \begin{array}{c} n-1 \end{array} \right)_k [x]^k_q [1-x]^{n-k}_q + \left( \begin{array}{c} n-1 \end{array} \right)_{k-1} [x]^k_q [1-x]^{n-k}_q$$

$$= \left( \begin{array}{c} n \end{array} \right)_k [x]^k_q [1-x]^{n-k}_q,$$
and the derivative of the modified $q$-Bernstein polynomials of degree $n$ are also polynomials of degree $n - 1$. That is,

\[
\frac{d}{dx}B_{k,n}(x,q) = \binom{n}{k} k[x]_q^{k-1} [1-x]_q^{n-k} \frac{\ln q}{q-1} q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} \left( -\frac{\ln q}{q-1} \right) q^{1-x}
\]

\[
= \frac{\ln q}{q-1}\left\{ \binom{n}{k} k[x]_q^{k-1} [1-x]_q^{n-k} q^x - \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} q^{1-x} \right\}
\]

\[
= n (q^x B_{k-1,n-1}(x,q) - q^{1-x} B_{k,n-1}(x,q)) \frac{\ln q}{q-1}.
\]

Therefore, we obtain the following recurrence formulæ.

**Theorem 2 (Recurrence formulæ for $B_{k,n}(x,q)$).** For $k, n \in \mathbb{Z}_+, x \in [0, 1]$, we have

\[
[1-x]_q B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k} = B_{k,n}(x,q),
\]

and

\[
\frac{d}{dx} B_{k,n}(x,q) = n (q^x B_{k-1,n-1}(x,q) - q^{1-x} B_{k,n-1}(x,q)) \frac{\ln q}{q-1}.
\]

Let $f$ be a continuous function on $[0, 1]$. Then the modified $q$-Bernstein operator is defined by

\[
B_{n,q}(f : x) = \sum_{j=0}^{n} \frac{j}{n} B_{j,n}(x,q), \tag{6}
\]

where $0 \leq x \leq 1, n \in \mathbb{Z}_+$. By Theorem 1 and (6), we see that

\[
B_{n,q}(f : x) = B_{n,q} f(x) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} [x]_q^k [1-x]_q^{n-k}
\]

\[
= [x]_q (1 - [1-x]_q [x]_q (q-1))^{n-1},
\]

where $f(x) = x$. Thus, we have

\[
B_{n,q}(f : x) = f([x]_q) (1 + (1 - q)[x]_q [1-x]_q)^{n-1}. \tag{7}
\]

From Theorem 1, we note that

\[
\sum_{k=0}^{n} B_{k,n}(x,q) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k [1-x]_q^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} [x]_q^k (1 - q^{1-x} [x]_q)^{n-k}
\]

\[
= (1 + [x]_q [1-x]_q (q-1))^n = B_{n,q}(1 : x).
\]
The modified $q$-Bernstein polynomials are symmetric polynomials. That is, by the definition of the modified $q$-Bernstein polynomials of degree $n$, we see that

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$  

Thus, we have

$$B_{n-k,n}(1-x, q) = \binom{n}{n-k} [1-x]_q^{n-k} [x]_q^k = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$  

Therefore, we obtain the following theorem.

**Theorem 3.** For $k, n \in \mathbb{Z}_+, x \in [0, 1]$, we have

$$B_{n-k,n}(1-x, q) = B_{k,n}(x, q),$$

and

$$\sum_{k=0}^{n} B_{k,n}(x, q) = (1 + [x]_q[1-x]_q(1-q))^n = B_{n,q}(1:x).$$

For $t \in \mathbb{C}, x \in [0, 1],$ and $n \in \mathbb{Z}_+$, we consider

$$\frac{n!}{2\pi i} \oint_{C} \frac{([x]_q t)^k}{k!} e^{(1-x)_q t} \frac{dt}{t^{n+1}}, \quad (9)$$

where $C$ is a circle around the origin and integration is in the positive direction. By the definition of the modified $q$-Bernstein polynomials and Laurent series, we see that

$$\oint_{C} \frac{([x]_q t)^k}{k!} e^{(1-x)_q t} \frac{dt}{t^{n+1}} = \sum_{m=0}^{\infty} \oint_{C} B_{k,n}(x, q) t^m \frac{dt}{m!} \frac{t^{n+1}}{t^{n+1}} = \frac{B_{k,n}(x, q)}{n!} 2\pi i \quad (10)$$

From (9) and (10), we note that

$$\frac{n!}{2\pi i} \oint_{C} \frac{([x]_q t)^k}{k!} e^{(1-x)_q t} \frac{dt}{t^{n+1}} = B_{k,n}(x, q). \quad (11)$$

Also, we see that

$$\oint_{C} \frac{([x]_q t)^k}{k!} e^{(1-x)_q t} \frac{dt}{t^{n+1}} = [x]_q^k \sum_{m=0}^{\infty} \oint_{C} t^{m-n-1+k} \frac{[1-x]_q^m}{m!} \frac{dt}{t^{n+1}}$$

$$= \frac{x}{k!} \sum_{m=0}^{\infty} \oint_{C} t^{m-n-1+k} \frac{[1-x]_q^m}{m!} \frac{dt}{t^{n+1}}$$

$$= \frac{x}{k!} \left( \frac{(1-x)_q^{n-k}}{(n-k)!} \right) 2\pi i$$

$$= \frac{[x]_q^k [1-x]_q^{n-k}}{k!(n-k)!} 2\pi i. \quad (12)$$
From (9) and (12), we have
\[
\frac{n!}{2\pi i} \oint_C \frac{([x]_q t)^k}{k!} e^{(1-x)_q t} \frac{dt}{i^{n+1}} = \binom{n}{k} [x]_q [1 - x]_q^{n-k}.
\] (13)

By (11) and (13), we easily see that
\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q [1 - x]_q^{n-k}.
\] (14)

From (14), we derive
\[
\left( \frac{n-k}{n} \right) B_{k,n}(x, q) + \left( \frac{k+1}{n} \right) B_{k+1,n}(x, q)
= \frac{n-k}{n} \binom{n}{k} [x]_q [1 - x]_q^{n-k} + \frac{k+1}{n} \binom{n}{k+1} [x]_q [1 - x]_q^{n-k+1}
= \frac{(n-1)!}{k!(n-k-1)!} [x]_q [1 - x]_q^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x]_q [1 - x]_q^{n-k+1}
= [1 - x]_q B_{k,n-1}(x, q) + [x]_q B_{k,n-1}(x, q)
= B_{k,n-1}(x, q) + [x]_q (1 - q^{1-x}) B_{k,n-1}(x, q)
= B_{k,n-1}(x, q) + (1 - q) [x]_q [1 - x]_q B_{k,n-1}(x, q).
\]

Therefore, we can write the modified \(q\)-Bernstein polynomials as a linear combination of polynomials of higher order as follows:

**Theorem 4.** For \(k \in \mathbb{Z}_+, n \in \mathbb{N}, \) and \(x \in [0, 1], \) we have
\[
\left( \frac{n-k}{n} \right) B_{k,n}(x, q) + \left( \frac{k+1}{n} \right) B_{k+1,n}(x, q)
= B_{k,n-1}(x, q) + (1 - q) [x]_q [1 - x]_q B_{k,n-1}(x, q).
\]

By (14), we easily see that
\[
\left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{1-x]_q} \right) B_{k-1,n}(x, q)
= \left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{1-x]_q} \right) \binom{n}{k-1} [x]_q^{k-1} [1 - x]_q^{n_k+1}
= \frac{n!}{k!(n-k)!} [x]_q^k [1 - x]_q^{n-k} = \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}.
\]

Thus, we obtain the following corollary.
Corollary 5. For \( n, k \in \mathbb{N} \), and \( x \in [0,1] \), we have
\[
\left( \frac{n - k + 1}{k} \right) \left( \frac{[x]_q}{[1 - x]_q} \right) B_{k-1,n}(x, q) = B_{k,n}(x, q).
\]

From the definition of the modified \( q \)-Bernstein polynomials and binomial theorem, we note that
\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1 - x]_q^{n-k} = \left( \frac{n}{k} \right) [x]_q^k (1 - q^{1-x} [x]_q)^{n-k}
\]
\[
= \left( \frac{n}{k} \right) [x]_q^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l q^l (1-x) [x]_q^l
\]
\[
= \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l q^l (1-x) [x]_q^{l+k}
\]
\[
= \sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} (-1)^l q^l (1-x) [x]_q^{l+k}
\]
\[
= \sum_{i=k}^{n} \binom{i}{k} \binom{n}{i} (-1)^{i-k} q^{i-x} [i]_q^i.
\]

Therefore, we obtain the following theorem.

Theorem 6. For \( k, n \in \mathbb{Z}_+ \), and \( x \in [0,1] \), we have
\[
B_{k,n}(x, q) = \sum_{i=k}^{n} \binom{i}{k} \binom{n}{i} (-1)^{i-k} q^{i-x} [i]_q^i.
\]

It is possible to write each power basis element of \([x]_q^k\), in the linear combination of the modified \( q \)-Bernstein polynomials by using the degree evaluation formulae and induction method in mathematics. From the property of the modified \( q \)-Bernstein polynomials, we easily see that
\[
\sum_{k=0}^{n} \frac{k}{n} B_{k,n}(x, q) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}
\]
\[
= \sum_{k=1}^{n} \binom{n-1}{k-1} [x]_q^k [1 - x]_q^{n-k}
\]
\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_q^{k+1} [1 - x]_q^{n-1-k}
\]
\[
= [x]_q ([x]_q + [1 - x]_q)^{n-1},
\]
\[
\sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x, q) = \sum_{k=1}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} [1-x]_{q}^{n-k} x_{q}^{k} \\
= \sum_{k=2}^{\infty} \frac{k(k-1)}{n(n-1)} \binom{n}{k} [x_{q}]^{k} [1-x]_{q}^{n-k} \\
= \sum_{k=2}^{n} \frac{n-2}{k-2} [x_{q}]^{k} [1-x]_{q}^{n-k} \\
= \sum_{k=0}^{n-2} \frac{n-2}{k} [x_{q}]^{k+2} [1-x]_{q}^{n-2-k} \\
= [x_{q}^{2} ([x_{q}] + [1-x]_{q})^{n-2}.
\]

Continuing this process, we obtain

\[
\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = [x_{q}^{i}] ([x_{q}] + [1-x]_{q})^{n-i},
\]

for \(i \in \mathbb{N}\). Therefore we obtain the following theorem.

**Theorem 7.** For \(n \in \mathbb{Z}_{+}, i \in \mathbb{N}\) and \(x \in [0, 1]\), we have

\[
\frac{1}{([1-x]_{q} + [x]_{q})^{n-i}} \sum_{k=i-1}^{n} \left( \frac{k}{n} \right) \binom{n}{i} B_{k,n}(x, q) = [x_{q}^{i}],
\]

The Bernoulli polynomials of order \(k(\in \mathbb{N})\) are defined as

\[
\left( \frac{t}{e^{t} - 1} \right)^{k} e^{xt} = \left( \frac{t}{e^{t} - 1} \right)^{k} \times \cdots \times \left( \frac{t}{e^{t} - 1} \right)^{k} e^{xt} = \sum_{n=0}^{\infty} B_{n}^{(k)}(x) \frac{t^{n}}{n!}, \quad (15)
\]

and \(B_{n}^{(k)} = B_{n}^{(k)}(0)\) are called the \(n\)-th Bernoulli numbers of order \(k\). It is well known that the second kind Stirling numbers are defined as

\[
\frac{(e^{t} - 1)^{k}}{k!} = \sum_{n=0}^{\infty} S(n, k) \frac{t^{n}}{n!}, \quad (16)
\]
for \( k \in \mathbb{N} \). From (5), we note that

\[
\frac{([x]_q)^k e^{[1-x]_q t}}{k!} = \frac{[x]_q^k (e^t - 1)^k}{k!} \left( \frac{t}{e^t - 1} \right)^k e^{[1-x]_q t}
\]

\[
= [x]_q^k \left( \sum_{m=0}^{\infty} S(m, k) \frac{t^m}{m!} \right) \left( \sum_{n=0}^{\infty} B_n^{(k)} ([1-x]_q) \frac{t^n}{n!} \right)
\]

\[
= [x]_q^k \sum_{l=0}^{\infty} \left( \sum_{n=0}^{l} B_n^{(k)} ([1-x]_q) S(l-n, k) \binom{l}{n} \frac{t^n}{n!(l-n)!} \right) \frac{t^l}{l!}.
\] (17)

By (5) and (17), we have

\[
B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^{l} B_n^{(k)} ([1-x]_q) S(l-n, k) \binom{l}{n},
\]

and \( B_{k,0}(x, q) = B_{k,1}(x, q) = \cdots = B_{k,k-1}(x, q) = 0 \), where \( B_n^{(k)} ([1-x]_q) \) are called the \( n \)-th Bernoulli polynomials of order \( k \).

Let \( \Delta \) be the shift difference operator with \( \Delta f(x) = f(x + 1) - f(x) \). By iterative method, we easily see that

\[
\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k),
\] (18)

for \( n \in \mathbb{N} \). From (16) and (18), we note that

\[
\sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt}
\]

\[
= \sum_{n=0}^{\infty} \left\{ \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} t^n \right\} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{\Delta^k 0^n t^n}{k!} \frac{t^n}{n!}.
\]

By comparing the coefficients on the both sides, we have

\[
S(n, k) = \frac{\Delta^k 0^n}{k!},
\] (19)

for \( n, k \in \mathbb{Z}_+ \). Thus, we note that

\[
B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^{l} B_n^{(k)} ([1-x]_q) \binom{l}{n} \frac{\Delta^k 0^{l-n}}{k!}.
\] (20)
Let \((Eh)(x) = h(x+1)\) be the shift operator. Then the \(q\)-difference operator is defined by
\[
\Delta_q^n = \Pi_{i=0}^{n-1} (E - q^i I), \quad \text{(see [2])},
\]
where \(I\) is an identity operator. For \(f \in C[0,1]\) and \(n \in \mathbb{N}\), we have
\[
\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n-k),
\]
where \(\binom{n}{k}_q\) is Gaussian binomial coefficient.

Let \(F_q(t)\) be the generating function of the \(q\)-extension of the second kind stirling number as follows:
\[
F_q(t) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} e^{[j]_q t}
\]
\[
= \sum_{n=0}^{\infty} S(n, k : q) \frac{t^n}{n!}, \quad \text{(see [2])}. \quad \text{(21)}
\]
From (21), we have
\[
S(n, k : q) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^{j} q^{\binom{k}{2}} \binom{k}{j}_q [k-j]_q^n
\]
\[
= \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^n, \quad \text{(22)}
\]
where \([k]_q! = [k]_q [k-1]_q \cdots [2]_q [1]_q\). It is not difficult to show that
\[
[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(n, k : q), \quad \text{(see [2])}, \quad \text{(23)}
\]
Thus, we have
\[
\sum_{k=0}^{i} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(i, k : q) = \frac{1}{([1-x]_q + [x]_q)^{n-i}} \sum_{k=i+1}^{n} \binom{k}{i}_q [k]_q! B_{k,n} (x, q). \quad \text{(24)}
\]
Therefore, we obtain the following theorem.

**Theorem 8.** For \(n \in \mathbb{Z}_+, \ i \in \mathbb{N} \text{ and } x \in [0,1]\), we have
\[
\frac{1}{([1-x]_q + [x]_q)^{n-i}} \sum_{k=i+1}^{n} \binom{k}{i}_q [k]_q! B_{k,n} (x, q) = \sum_{k=0}^{i} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(i, k : q),
\]
where \(\binom{x}{k}_q = \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!}\).
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