THE COVARIANT SCATTERING AND COHOMOLOGY
OF $W_3$ STRINGS

Michael Freeman
&
Peter West

Department of Mathematics
King’s College
Strand
London WC2R 2LS

February 1993

Abstract

A general formalism for covariant $W_3$ string scattering is given. It is found necessary to use screening charges that are constructed from the $W_3$ fields including ghosts. The scattering amplitudes so constructed contain within them Ising model correlation functions and agree with those found previously by the authors. Using the screening charge and a picture changing operator, an infinite number of states in the cohomology of $Q$ are generated from only three states. We conjecture that, apart from discrete states, these are all the states in the cohomology of $Q$. 
Introduction

The existence of \( W \)-algebra \([1,2] \) extensions of the Virasoro algebra has led to a number of new developments. One of these has been the discovery of new string theories based on these extensions of the (super) Virasoro algebra. An important early result was the construction of the BRST charge for the \( W_3 \) algebra, which was found to be nilpotent for a matter central charge of \( c = 100 \) and to imply intercept 4 for standard ghost number states \([3]\). This work was used to discuss the possible properties of \( W_3 \) string theories in references \([4,5]\). Unlike Virasoro representations, two \( W_N \) algebra representations cannot be added to obtain a third representation. The most commonly used representations of the \( W_N \) algebras are those of reference \([6]\), which involve \( N - 1 \) free scalar fields and can have any value of the central charge. A variant of these representations was found \([7]\) which replaced one of these scalar fields by any number of scalar fields \( x^\mu, \mu = 0, 1, \ldots, D - 1 \).

These representations with \( c = 100 \) were then used to construct \( W_3 \) strings \([8,9,10]\). The physical states of the bosonic string were found long ago\([11]\), but we now realize that they can be thought of as being the cohomology of the BRST operator \( Q \), subject to a ghost number constraint. We recall that the cohomology of \( Q \) \([12]\) consists of the states \( |\psi^x\rangle |\downarrow\rangle \) and \( |\psi^x\rangle c_0 |\downarrow\rangle \), where \( |\psi^x\rangle \) depends only on \( x^\mu \) and \( |\downarrow\rangle = c_1 |0\rangle \), with \( |0\rangle \) being the \( SL(2,\mathbb{R}) \) invariant vacuum. There are also two further states with zero momentum.

For a \( W_3 \) string we also regard the physical states as being given by the cohomology of \( Q \), subject to a suitable ghost number constraint. It was pointed out in references \([4,5]\) that if the physical states had a ghost vacuum of standard type, that is they are of the form \( |\psi^{x,\varphi}\rangle |\downarrow\rangle \) where \( |\downarrow\rangle = c_1 e_1 e_2 |0\rangle \), then the \( x^\mu-\varphi \) dependent part \( |\psi^{x,\varphi}\rangle \) would be subject to the constraints

\[
(L_0^m - 4)|\psi^{x,\varphi}\rangle = 0, \quad W_0^m |\psi^{x,\varphi}\rangle = 0,

L_n^m |\psi^{x,\varphi}\rangle = 0, \quad W_n^m |\psi^{x,\varphi}\rangle = 0, \quad n \geq 1.
\]

We give details of our conventions in section 2.

Included amongst the states satisfying the conditions of equation (1.1) are physical states having the form \([8,9,10]\)

\[
|\psi^{x,\varphi}\rangle = |\psi^x\rangle |0, \beta\rangle |\downarrow\rangle,
\]

\( 1 \).
where $|0, \beta\rangle$, is a state with $\varphi$ momentum equal to $\beta$ and no $\varphi$ oscillators. Such states will satisfy the conditions (1.1) provided that the state $|\psi^x\rangle$, which depends on $x^\mu$ alone, satisfies the conditions

$$L^x_n|\psi^x\rangle = 0, \quad n \geq 1, \quad (L^x_0 - a)|\psi^x\rangle = 0,$$

(1.3)

where $a = 1$ for $\beta = 8iQ/7$ and $6iQ/7$, and $a = 15/16$ for $\beta = iQ$. The above values of $\beta$ are in fact the only ones allowed by the on-shell conditions of equation (1.1). We will refer to such states as intercept 1 or $15/16$ states. A systematic study of the physical states of equation (1.1) at levels up to and including 2 was undertaken in reference [13]. It was found that any state that had the form given in equation (1.1) and that contained $\varphi$ oscillators was null, and, by examining all other null states, the count of physical degrees of freedom at these levels was found. It thus became clear that the non-null physical states were of the form of equation (1.2) and that the open $W_3$ string had only one massless particle, a photon.

Already it had been noticed for the $W_N$ string, as a matter of phenomenological number matching, that the allowed values of the intercepts $a$ referred to above were related to the weights of some of the primary fields of the minimal conformal models [8,9], and also that the fields and ghosts of the $W_N$ string, with the exception of $x^\mu$ and the $b,c$ ghosts, had the central charge of these models [8]. For $W_3$ this observation amounts to the fact that $0 = 1 - 1$ and $1/16 = 1 - 15/16$ were weights of Ising primary fields, and that the $\varphi,d,e$ system has a central charge $c = 1/2$.

For the bosonic string it is only necessary to consider physical states built on the standard ghost vacuum $|\downarrow\rangle$, since, in addition to these states, the cohomology of $Q$ consists of only a copy of these states built on $c_0|\downarrow\rangle$ and two so-called discrete states having fixed momenta. For the $W_3$ string, however, it emerged in two ways that the situation was not so simple. Physical states having ghost number 2 were first found in reference [14] in the context of the two scalar $W_3$ string. These authors realized that such states were, in the sense of number matching discussed above, associated with Ising weight 1/2 states. It is known [15] that the two-dimensional bosonic strings have discrete physical states, that is states with fixed momenta, occurring at a variety of different ghost numbers. The analogue of this phenomenon was discussed for the $W_3$ string in reference [16], and these
authors also found two examples of ghost number 1, level one physical states having continuous momenta in the 3 scalar $W_3$ string.

In a separate development [17] it was found that scattering of $W_3$ strings was not consistent with states corresponding to intercepts 1 and $15/16$ alone, since factorization of the amplitude for 4 of the $15/16$ intercept strings revealed an infinite number of intermediate states associated with intercept $1/2$. In reference [18] all the physical states of the type of equation (1.3) were classified by constructing a spectrum generating algebra. This consisted of the operators $B_n, 1 = 1, \ldots, D - 1$ and $B_n$, where the latter obeyed a Virasoro algebra with central charge $1/2$. It was shown [18] that the count of states was given by the Ising model characters $\chi_h$, where $h = 1 - a$, and that the states had positive norm and so obeyed a no-ghost theorem. These results allowed a discussion of modular invariance. It emerged [18] that the $W_3$ string was not modular invariant for the intercept $15/16$ and 1 states alone, but required states with a count given by a character $\chi_{1/2}$. It was realised that states of the form of equation (1.3) for $a = 1/2$ would have precisely this count and that such states did indeed belong [18] to the cohomology of $Q$ at ghost number 1. It also emerged that the cohomology of $Q$ contained complete copies of the intercept $15/16$ and $1/2$ states.

Taking the above facts into account, it was conjectured [18] that the cohomology of $Q$ contained only intercept $1, 15/16$ and $1/2$ states, together with copies of them and possible discrete states. We will refer to this conjecture as the spectrum conjecture. A review of $W_3$ string theory including its ghosts and physical states is given in section 2.

In reference [17] a formula for the scattering of any $W_3$ strings was derived using the group theoretic approach. This approach stemmed from a desire to formulate a method of computing string amplitudes that utilized the minimum amount of machinery and assumptions. It was this that enabled us to work with a reduced subspace of the full Hilbert space using properties such as the null state structure of the states and $W_3$ properties of the vertices that followed from the full $W_3$ theory. These results [17], which showed that $W_3$ string scattering amplitudes contain within them Ising model correlators, are reviewed in section 3. Further explicit evaluations of particular scattering amplitudes are given in section 3, so that they may be compared with other derivations.
It was suggested in reference [17] that it would be worthwhile to recover these results from a covariant formalism. This is the central task of this paper. The covariant formalism is taken to mean that which is often embodied in the conformal field theory approach to string scattering. This method [19] constructs the amplitudes as vacuum expectation values, with respect to the $SL(2, \mathbb{R})$-invariant vacuum, of BRST-invariant vertex operators (such as $ce^{ipx}$ for the bosonic string) and of BRST-invariant integrated vertices obtained by acting with the operator \( \int dz \oint z \, dv_b(v) \) on a BRST-invariant operator $V(z)$; such integrated vertices have ghost number one less than the original vertex $V$. In the case of the superstring it is also necessary to use picture changing operators constructed from the bosonized ghosts. These building blocks must be assembled in such a way that the final amplitude is BRST invariant. The bosonic and superstring theory tree amplitudes have been known since the earliest days of string theory, and the rules used originally proved a very useful guide to constructing rules of the covariant formalism.

For a new string theory such as a $W_3$ string the bosonic and superstring rules provide a useful guide and we will need all the above building blocks. There is, however, an element of guesswork in this, and as such it is important to verify that the final scattering amplitudes are consistent as a check on a given set of rules. We find in section 4 that for the $W_3$ string one must extend the usual covariant scattering rules to include screening charges, that is, objects that are integrals of a vertex operator that is not associated with an external string. A departure of some kind is perhaps to be expected in view of the role which will be played by the bosonic $W$-moduli, once this phenomenon is understood.

The screening charges, which are constructed from the fields of the $W_3$ string, play a role analogous to the screening operators used in the Feigin-Fuchs [20] formulation of minimal conformal models. The scattering of 4 intercept $15/16$ strings is explicitly evaluated in section 4 and found to be in agreement with the result of reference [17], where it was found that it contained the Ising model correlation function for 4 weight $1/16$ primary fields.

In section 5 the cohomology of $Q$ is examined. We show that, starting from one BRST vertex operator $V(a,0)$ for each intercept $a = 1, 15/16, 1/2$, we can act with the screening operator $S$ and picture changing operator $P$ in suitable ways to obtain whole families of BRST invariant vertex operators $V(15/16, m)$, $V(1, m)$,
\( \bar{V}(1, m), V(1/2, m), \bar{V}(1/2, m) \) for \( m \in \mathbb{Z} \), together with extra states given by the action of \( P \) on these. For example \( V(15/16, m) \) is given by

\[
V(15/16, m) = (S^2 P)^m V(15/16, 0).
\]

We extend our spectrum conjecture in the light of this construction.

In section 6 we give a complete set of rules for covariant \( W_3 \) string scattering. Any scattering amplitude can be built from the three vertices \( V(a, 0) \), the screening charge \( S \), the picture charging operator \( P \) and the operation of \( \int dz \oint z dv b(z) \) on a vertex. We explain how to assemble these building blocks and give a number of examples.

In references [21] and [22] an attempt was made to give a formulation of covariant \( W_3 \) scattering using only the usual rules found in the bosonic string and superstring. In this attempt many scattering amplitudes vanished, including that for four intercept \( 15/16 \) tachyonic strings. The authors noted that this disagreed with the results of reference [17] and concluded that reference [17] was incorrect. The factorization and duality properties of the amplitude for four intercept \( 15/16 \) tachyonic strings were discussed in detail in reference [17] and the amplitude was shown to factorize into two three point couplings consistent with the Ising model fusion rules. In section 7 we explain why the rules of references [21,22] lead to string amplitudes violating unitarity and the assumptions of S-matrix theory, as well as assumptions more specific to string theory such as duality

2. The covariant \( W_3 \) string

In this section we summarize those features of the \( W_3 \) string, including its ghosts, that will be required later in the paper.

The covariant formulation of the \( W_3 \) string involves the scalar fields \( \varphi, x^\mu \) for \( \mu = 0, 1, ..., D-1 \), reparameterization ghosts \( b \) and \( c \), and \( W_3 \) transformation ghosts \( d \) and \( e \). The latter ghosts have spins 3 and \(-2\) respectively. The corresponding energy-momentum tensor \( T^{tot} \) and \( W_3 \)-current \( W^{tot} \) are of the form

\[
T^{tot} = T^m + T^{gh}, \quad W^{tot} = W^m + W^{gh}
\]

where

\[
T^m = T^\varphi + T^x
\]
Here the background charge $Q$ is given by $Q^2 = 49/8$, and $\alpha$ is such that $T^x$ has central charge $51/2$. The BRST charge $Q$ is given by 

$$Q = \int dz j^{BRST},$$

where

$$j^{BRST} = c(T^m + \frac{1}{2} T^{gh}) + e(W^m + \frac{1}{2} W^{gh}). \quad (2.9)$$

(There will be no confusion between the background charge $Q$ and the BRST charge $Q$.) Some useful relations are

$$T^{tot}(z) = \{Q, b(z)\}, \quad W^{tot}(z) = \{Q, d(z)\}, \quad (2.10)$$

as a consequence of which

$$[Q, T^{tot}(z)] = [Q, W^{tot}(z)] = 0. \quad (2.11)$$

It will be useful to discuss the various possible vacua associated with the ghosts. The natural vacuum, $|0\rangle$, of the ghost system is that for which $e(z) = \sum_n e_{-n} z^{n+2}$ and $d(z) = \sum_n d_{-n} z^{n-3}$ are well defined at $z = 0$. This requires

$$e_n |0\rangle = 0, \quad n \geq 3, \quad d_n |0\rangle = 0, \quad n \geq -2 \quad (2.12)$$

We can construct other vacua by acting on $|0\rangle$ with $e_n$ for $n = 0, \pm 1, \pm 2$ and with $c_n$ for $n = 0, \pm 1$. One of the most useful is

$$c_1 e_1 e_2 |0\rangle \equiv \downarrow, \quad (2.13)$$
which is annihilated by $e_n, c_n$ for $n \geq 1$ and $b_n, d_n$ for $n \geq 0$. In terms of the conformal fields we may express the relation between the two states as

$$c \partial e e|0\rangle = |\downarrow\rangle,$$  \hspace{1cm} (2.14)

where $c \partial e e$ is understood to be evaluated at $z = 0$. Similar formulae hold for the other vacuum states.

In order to gain a non-zero vacuum expectation value with respect to the state $|0\rangle$ we must have 3 factors of $c$ and 5 of $e$. We set

$$\langle 0| c_{-1} c_{0} c_{1} e_{-2} e_{-1} e_{0} e_{1} e_{2} |0\rangle = \frac{1}{576} \langle 0| \partial^2 c \partial c c \partial^4 e \partial^3 e \partial^2 e \partial e e |0\rangle = 1$$  \hspace{1cm} (2.15)

It is often useful to bosonize the ghosts. We adopt the well known rules [19]

$$c = e^{i\sigma}, \quad b = e^{-i\sigma}, \quad e = e^{i\rho}, \quad d = e^{-i\rho},$$  \hspace{1cm} (2.16)

where $\sigma(z)\sigma(w) = -\ln(z-w)$ and similarly for $\rho$ in order to give the correct operator product relations for the ghosts. The fields $\sigma$ and $\rho$ have energy-momentum tensors

$$T^{\sigma} = -\frac{1}{2} (\partial \sigma)^2 + \frac{3i}{2} \partial^2 \sigma, \quad T^{\rho} = -\frac{1}{2} (\partial \rho)^2 + \frac{5i}{2} \partial^2 \rho.$$  \hspace{1cm} (2.17)

We recognise that these background charges are compatible with equation (2.15) once we use formulae such as

$$\partial c e =: \partial e^{i\sigma} e^{i\sigma} : (z) = \oint_z dw \frac{\partial_w (e^{i\sigma(w)} e^{i\sigma}(z))}{w-z} = e^{2i\sigma}(z).$$  \hspace{1cm} (2.18)

We take the vacuum $|\downarrow\rangle$ to have ghost number 0, $e$ and $c$ to have ghost number +1 and $d$ and $b$ to have ghost number −1.

We now summarize some of the states known to be in the cohomology of $Q$. The first such states that were found [4,5,8,9] were those of so-called standard ghost type, which were based on the $|\downarrow\rangle$ vacuum and were of the form

$$|\psi\rangle |\downarrow\rangle,$$

where $|\psi\rangle$ is made from the $\alpha_n$ and $\alpha^\mu_n$ oscillators of $\varphi$ and $x^\mu$ respectively acting on their usual vacua. These states are annihilated by $Q$ if

$$(L^{m}_0 - 4)|\psi\rangle = 0 = W^m_0 |\psi\rangle, \quad L^m_n |\psi\rangle = W^m_n |\psi\rangle = 0, \quad n \geq 1.$$  \hspace{1cm} (2.20)
It was shown [13] that such physical states, up to level 2 at least, were either null or of the form
\[ e^{i\beta\varphi(0)}|\psi_a^{x}\rangle|\downarrow\rangle, \] (2.21)
where we use the notation that \( |\psi_a^{x}\rangle \) contains only \( \alpha_n^a \) oscillators and obeys
\[ L_n^{x}|\psi_a^{x}\rangle = 0, \quad n \geq 1, \quad (L_0^{x} - a)|\psi_a^{x}\rangle = 0. \]
The intercept \( a \) can take only the two values 1 and 15/16. When \( a = 1 \), \( \beta \) can have the values \( 6iQ/7 \) or \( 8iQ/7 \), and when \( a \) is 15/16 \( \beta \) takes the value \( iQ \). In reference [18] the spectrum generating algebra for these states was found and used to classify them. It was shown that the count of states was described by Ising model characters \( \chi_h \) where \( h = 1 - a \).

A few examples of physical states that were of non-standard type were found in references [14,16]. It also emerged, however, that the \( W_3 \) string must contain more than the states of standard ghost type to be consistent. It was shown that the scattering of four states of intercept 15/16 led to an infinite number of intermediate states with intercept 1/2 [17], and that modular invariance required states whose count was given by the Ising character \( \chi_{1/2} \) [18]. It was further pointed out [18] that states of the form \( |\psi_{1/2}^{x}\rangle \) would indeed have such a count.

By using a vanishing null-state argument, the cohomology of \( Q \) was found in reference [18] to contain such states having the form
\[ \left(d_{-1} + \frac{i}{\sqrt{522}}b_{-1}\right)e^{i\beta_1\varphi(0)}|\psi_{1/2}^{x}\rangle|\downarrow\rangle, \] (2.22)
where \( \beta_1 = 4iQ/7 \). The same argument also lead to the states [18]
\[ \left(d_{-1} - \frac{i}{\sqrt{522}}b_{-1}\right)e^{i\beta_2\varphi(0)}|\psi_{15/16}^{x}\rangle|\downarrow\rangle \] (2.23)
and
\[ \left(\frac{2}{261}b_{-2} + \frac{9}{522}L_{-1}^{tot}b_{-1} + W_{-1}^{tot}d_{-1}\right)e^{i\beta_3\varphi(0)}|\psi_{1/2}^{x}\rangle|\downarrow\rangle, \] (2.24)
where \( \beta_2 = 3iQ/7 \) and \( \beta_3 = 2iQ/7 \). Thus it became apparent that the cohomology of \( Q \) contained copies of the same states at different ghost number. Since the theory was modular invariant with only one copy of states from each of the sectors 1, 1/2 and 15/16, and the cohomology of \( Q \) where investigated contained only copies of
these states, it was conjectured [18] that the cohomology of $Q$ consisted of the states of equations (2.21) and (2.22) corresponding to intercepts $1, 15/16$ and $1/2$, as well as copies of these states and possible discrete states.

Using equations (2.13) and (2.14) it is possible to rewrite the states of equations (2.21) as

$$\begin{align*}
|1, 0\rangle &= c \partial ee^{i\beta(1;0)} \psi_1^x |0\rangle \\
|1, 0\rangle &= c \partial ee^{i\bar{\beta}(1;0)} \psi_1^x |0\rangle,
\end{align*}$$

(2.25)

(2.26)

where we have introduced the notation

$$\beta(1; n) = (8 - 8n) \frac{iQ}{I} \quad \text{and} \quad \bar{\beta}(1, n) = (6 - 8n) \frac{iQ}{I},$$

(2.27)

and

$$|15/16, 0\rangle = c \partial ee^{i\beta(15/16, 0)} \psi_{15/16}^x |0\rangle$$

(2.28)

where $\beta(15/16, n) = (7 - 4n)iQ/7$.

Similarly, up to a constant of proportionality, the states of equations (2.22), (2.23) and (2.24) become respectively

$$\begin{align*}
|1/2, 0\rangle &= \left( ce - \frac{i}{\sqrt{522}} \partial ee \right) e^{i\beta(1/2; 0)} \psi_{1/2}^x |0\rangle \\
|15/16, 1\rangle &= \left( ce + \frac{i}{\sqrt{522}} \partial ee \right) e^{i\beta(15/16; 1)} \psi_{15/16}^x |0\rangle
\end{align*}$$

(2.29)

(2.30)

and finally

$$\begin{align*}
|1/2, 0\rangle &= \left( -\frac{4}{3\sqrt{58}} bce e - \frac{4}{3\sqrt{58}} \partial^2 ee \\
&\quad + \frac{1}{\sqrt{29}} \partial e e e + i\sqrt{2} c e \partial e - \frac{3i}{2} c \partial e \right) e^{i\bar{\beta}(1/2, 0)} \psi^{1/2} |0\rangle
\end{align*}$$

(2.31)

where

$$\beta(1/2, m) = (4 - 8m) \frac{iQ}{I} \quad \text{and} \quad \bar{\beta}(1/2, m) = (2 - 8m) \frac{iQ}{I}.$$  

(2.32)

The reason for this notation will emerge once we discuss the cohomology of $Q$ in section 5.
The simplest such states are those that have no $\alpha_n^\mu$ oscillators and so are tachyonic. In this case
\[ |\psi_a^x\rangle = e^{ip \cdot x} |0,0\rangle, \]
with $\frac{1}{2} p \cdot (p - 2i\alpha) = a$. The next simplest possibility is
\[ |\psi_a^x\rangle = \xi \cdot \partial_x e^{ip \cdot x} |0,0\rangle \]
where $p \cdot \xi = 0$ and $\frac{1}{2} p \cdot (p - 2i\alpha) = a - 1$. We will often write $|\psi_a^x\rangle$ in vertex operator form as $|\psi_a^x\rangle = V^x(a)|0,0\rangle$. Further low level states belonging to the cohomology of $Q$ have been found in references [21] and [22].

3. Explicit results of $W_3$ scattering.

In a recent paper [17] it was shown that the scattering, at tree level, of $N$ $W_3$ string states is given by
\[ \int \prod_i dz_i V f(z_i). \]
Here $V$ is the usual scattering vertex in the presence of a background charge, and $f$ is an Ising model correlation function that depends on the intercepts of the external states. To be specific, if the $N$ external states have effective intercepts $a_i, \quad i = 1, ..., N$, which can take only the values 1, 15/16 or 1/2, then $f = \langle \prod_{i=1}^N \varphi_i(z_i) \rangle$ where $\varphi_i$ is the Ising field of weight $h_i = 1 - a_i$.

This result followed from an application of the group theoretic approach to string theory. The essential steps in this process are the computation of the vertex $V$ using overlap relations, and then the determination of the measure $f$ by demanding that null states decouple. Using this technique it was possible to work with the reduced subspace of the full $W_3$ Fock space in which the physical states sit. It is important to understand that the properties of the vertices and null states used in this reduced Hilbert space are those inherited from the full Fock space of the $W_3$ string.

We found that the decoupling of the null states of the $W_3$ string implied that $f$ obeyed the differential equations satisfied by the Ising model correlators, which are
\[ \frac{4}{3} \frac{\partial^2 f}{\partial z_j^2} - \sum_{i=1}^N \left\{ \frac{\partial f}{\partial z^i} \frac{1}{(z^i - z^j)^2} + \frac{1}{16} \frac{f}{(z^j - z^i)^2} \right\} = 0, \]
\[
\frac{3}{4} \frac{\partial^2 f}{\partial z_j^2} - \sum_{i=1}^{N} \left\{ \frac{\partial f}{\partial z_i} \left( \frac{1}{z_j - z_i} \right) + \frac{1}{2} \frac{f}{(z_j - z_i)^2} \right\} = 0 \quad (3.3)
\]

and
\[
\frac{\partial f}{\partial z_j} = 0, \quad (3.4)
\]

when the \(j^{th}\) leg or string has intercept 15/16, 1/2 and 1 respectively.

The measure \(f\) must also satisfy the usual \(sl(2, \mathbb{R})\) relations
\[
\sum_i \frac{\partial}{\partial z_i} f = \sum_i (z_i \frac{\partial}{\partial z_i} + h_i) f = (\sum_i z_i^2 \frac{\partial}{\partial z_i} + 2h_i z_i) f = 0 \quad (3.5)
\]
as a result of the corresponding relations for the vertex of equation (3.1) (see equations (9) and (10) of reference [17]).

We refer the reader to reference [17] for further details and in particular for a discussion of how the two solutions to the above equations must be combined to ensure duality.

It is straightforward to evaluate the \(W_3\) string scattering given by equation (3.1) whenever the Ising model correlation functions, or equivalently the solutions to the above differential equations, are known. In reference [1] we carried this out for certain cases. In this paper we extend these results so that they can be compared with the covariant approach to \(W_3\) string scattering.

Clearly, when a leg is from the intercept 1 sector, the measure \(f\) does not depend upon that Koba-Nielsen coordinate. Consequently, the form of \(f\) is the same as that for the scattering process with only states of intercept 15/16 and 1/2. Having found \(f\) it is straightforward to evaluate the scattering for any external physical states, but in order to be concrete we will often evaluate the scattering of tachyon states. Applying such states to the vertex \(V\) leads to the expression
\[
\prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \quad (3.6)
\]

For four tachyon scattering, with the choice of Koba-Nielsen coordinates \(z_1 = \infty\), \(z_2 = 1\), \(z_3 = x\) and \(z_4 = 0\), this reduces to
\[
x^{-\alpha' s - a_3 - a_4} (1 - x)^{-\alpha' t - a_2 - a_3}, \quad (3.7)
\]
where \( s = -(p_1 + p_2) \cdot (p_1 + p_2 - 2i\alpha) \), \( t = -(p_2 + p_3) \cdot (p_2 + p_3 - 2i\alpha) \) and a factor of \((z_1)^{-2a_1}\), which is cancelled by other such factors, has been removed.

In the case of three string scattering the measure \( f \) is determined by the usual \( sl(2,\mathbb{R}) \) relations to be of the form

\[
C = \frac{1}{(z_1 - z_2)^{h_1 + h_2 - h_3} (z_2 - z_3)^{h_2 + h_3 - h_1} (z_1 - z_3)^{h_1 + h_3 - h_2}}
\tag{3.8}
\]

As is well known, the values that \( C \) can take are equivalent to the fusion rules of the Ising model.

As explained above, the same formula of equation (3.8) can be used for four \( W_3 \) string scattering whenever one of the legs has intercept 1. We now give some examples of such scattering. If we take legs one and two to be intercept \( \frac{15}{16} \) states, leg 3 an intercept 1 state and leg 4 an intercept \( \frac{1}{2} \) state then

\[
f = \frac{1}{(z_1 - z_2)^{-3/8}} \frac{1}{(z_2 - z_4)^{1/2}} \frac{1}{(z_4 - z_1)^{1/2}}.
\tag{3.9}
\]

Using equation (3.7) we find that such tachyon scattering is given by

\[
\int dx \ x^{-\alpha's-3/2} (1 - x)^{-\alpha't-31/16}.
\tag{3.10}
\]

To give one other example let us consider strings with intercepts \( 1, \frac{1}{2}, \frac{1}{2} \) and 1. On the basis of \( sl(2,\mathbb{R}) \) invariance,

\[
f = \frac{1}{(z_2 - z_3)},
\tag{3.11}
\]

and tachyon scattering is given by

\[
\int dx \ x^{-\alpha's-3/2} (1 - x)^{-\alpha't-2}.
\tag{3.12}
\]

We now consider the scattering of strings all having the same intercept. If the intercept is 1 then \( f = 1 \), and the scattering is the same as in the ordinary bosonic string with a background charge [17]. The Ising correlation function of \( 2N \) weight \( 1/2 \) states is well known [24] and hence [17]

\[
f = P f \frac{1}{z_{ij}} \equiv \frac{1}{2^N N!} \sum_{\sigma} (-1)^{\sigma} \frac{1}{z_{\sigma_1 \sigma_2}} \cdots \frac{1}{z_{\sigma_2 \sigma_{N-1} \sigma_{2N}}}
\tag{3.13}
\]
where \( z_{ij} = z_i - z_j \). For \( 2N = 4 \) this becomes

\[
f = \left( \frac{1}{z_{12} z_{34}} - \frac{1}{z_{13} z_{24}} + \frac{1}{z_{14} z_{23}} \right).
\] (3.14)

Using equation (3.7) we find that four tachyon intercept 1/2 strings scatter according to

\[
\int dx x^{-\alpha's-2} (1 - x)^{-\alpha't-2} (1 - x + x^2),
\] (3.15)

while for \( 2N \) such tachyonic states we obtain

\[
\int \prod_i' dz_i \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} Pf \frac{1}{z_{ij}}.
\] (3.16)

The \( \prod_i' \) means we omit three indices, say \( k,l,m \), from the product, but instead include the factor \( z_{kl} z_{km} z_{lm} \).

Let us now consider \( 2N \) intercept 15/16 strings. The correlation function for \( 2N \) weight 1/16 Ising states can be computed from the Feigin-Fuchs construction [20].

Given a scalar field \( \phi \) with an energy momentum tension \( T = -\frac{1}{2} (\partial \phi)^2 - \alpha_0 \partial^2 \phi \), the central charge is \( c = 1 + 12\alpha_0^2 \). Hence if \( c = 1/2 \) we require \( \alpha_0 = i/2\sqrt{6} \).

The Ising fields of weights 0, 1/2 and 1/16 are represented by the fields \( V_\alpha = e^{i\alpha \phi} \) where \( \alpha \) takes the values \( 0, -4i\alpha_0, 3i\alpha_0 \) (or \( 2i\alpha_0 - \alpha \)) respectively. As is well known [20] one requires screening charges in order to obtain the correct Ising model correlation functions since one cannot balance the momentum to be \( 2i\alpha_0 \).

These screening charges are given by \( S_\pm = \oint dw e^{i\alpha \pm \phi} \), where \( \frac{1}{2} \alpha_\pm (\alpha_\pm - 2i\alpha_0) = 1 \), and so \( \alpha_+ = -6i\alpha_0 \), \( \alpha_- = 8i\alpha_0 \). For \( 2N \) weight 1/16 fields we can achieve such a balance by taking the fields

\[
\langle 0 | V_{2i\alpha_0 - \alpha}(z_1) V_\alpha(z_2) V_\alpha(z_3) \cdots V_\alpha(z_{2N}) S_+^{N-1} | 0 \rangle,
\] (3.17)

which results in the correlation function

\[
\langle \prod_j \sigma(z_j) \rangle = \int \left( \prod_{i=1}^{N-1} dw_i \right) \left\langle e^{i(2i\alpha_0 - \alpha) \phi(z_1)} \prod_{j=2}^{2N} e^{i\alpha \phi(z_j)} \prod_{k=1}^{N-1} e^{i\alpha_+ \phi(w_k)} \right\rangle
\] (3.18)

\[
= \int \left( \prod_{i=1}^{N-1} dw_i \right) \prod_{j=2}^{2N} (z_1 - z_j)^{-1/8} \prod_{k=1}^{N-1} (z_1 - w_k)^{1/4} \prod_{i,j=2}^{2N} (z_i - z_j)^{3/8}
\]

\[
\prod_{k,l} (w_k - w_l)^{3/2} \prod_{i,k} (z_i - w_k)^{-3/4}.
\]
For 4 such states this becomes

$$\int dw \ (z_1 - w)^{1/4} \prod_{i=2}^{4} (z_1 - z_i)^{-1/8} \prod_{i>j}^{4} (z_i - z_j)^{3/8} \prod_{i=2}^{4} (z_i - w)^{-3/4}, \quad (3.19)$$

which, using the canonical choice for the $z_i$, becomes

$$\left( x(1 - x) \right)^{3/8} \int dw \ (w(1 - w)(x - w))^{-3/4}$$

$$= \left( (1 - x)x \right)^{3/8} \left[ a F(3/4, 5/4, 3/2; z) + b z^{-1/2} F(3/4, 1/4, 1/2; z) \right] \quad (3.20)$$

where $a$ and $b$ are constants and where we have removed a factor of $z_1^{1/8}$ that cancels with other such factors elsewhere. Using results from appendix A we find the result for the correlation function of four weight $1/16$ fields to be

$$\frac{1}{(x(1 - x))^{1/8} \left( a' \sin \theta \ 2 + b \cos \theta \ 2 \right)}, \quad (3.21)$$

where $x = \sin^2 \theta$. This is the well known result [24]. An extensive discussion of four $W_3$ tachyonic intercept $15/16$ scattering was given in reference [17]†.

As we shall see the Feigin Fuchs construction provides a useful model, some of whose features we will exploit in the covariant formulation of $W_3$ scattering. One could continue to compute more examples of $W_3$ string scattering, but the above will more than suffice.

4. Screening charges and the scattering of 4 intercept $15/16$ $W_3$ strings

In this section we wish to give an alternative construction of $W_3$ string scattering to that of reference [17]. We intend to mimic for the $W_3$ string what has become known as the conformal field theory approach to string theory, which involves constructing amplitudes from the ghost fields and vertex operators. This generalizes the original approach to string scattering which used only the latter objects. These procedures [19] for the usual bosonic and superstring theories are well known and can act as a useful guide in formulating a set of rules. We must also ensure that the amplitudes are BRST invariant, which corresponds in the old method to the

† There is an obvious misprint in equations (19) and (20) of reference [17], but the subsequent equations are recorded correctly.
decoupling of null states. However, given a new theory such as $W_3$ string theory one must essentially guess a good set of rules and then check the consistency of the amplitudes with themselves and with the same amplitudes found in other formalisms.

The vertex operators can be read off from the states that occur in the cohomology of $Q$. As was explained in Section 2 such states are associated with one of three intercepts $1, 1/2$ and $15/16$. Using the notation employed in section 2 for the states, we find the following vertex operators:

\[
V(1, 0) = c \partial e e^{i \beta(1,0)} V^x(1)
\]

\[
\bar{V}(1, 0) = c \partial e e^{i \beta(1,0)} \bar{V}^x(1),
\]

\[
V(1/2, 0) = \left(c e - \frac{i}{\sqrt{522}} \partial e e\right) e^{i \beta(1/2,0)} V^x(1/2)
\]

\[
\bar{V}(1/2, 0) = \left(-\frac{4}{3\sqrt{58}} b c \partial e e - \frac{4}{3\sqrt{58}} \partial^2 e e
\right.
\]

\[
+ \frac{1}{\sqrt{29}} \partial \varphi \partial e e + i \sqrt{2} c e \partial \varphi - \frac{3i}{2} c \partial e\right) e^{i \beta(1/2,0)} \bar{V}^x(1/2),
\]

and

\[
V(15/16, 0) = c \partial e e^{i \beta(15/16,0)} V^x(15/16)
\]

\[
V(15/16, 1) = \left(c e + \frac{i \partial e e}{\sqrt{522}} \right) e^{i \beta(15/16,1)} V^x(15/16).
\]

Here $V^x(a)$ is any vertex operator, constructed from $x^\mu$ alone, that has conformal weight $a$ with respect to $T^x(z)$. The simplest is $V^x(a) = e^{i p \cdot x}$, where $\frac{1}{2} p \cdot (p - 2i \alpha) = a$. It is straightforward if tedious to verify that $Q$ commutes with all these vertex operators, as expected.

Given the states or vertex operators we may find new states or vertex operators by acting with the operators $[\varphi, Q]$ and $[x^\mu, Q]$. It is well known that, although these operators are formed from commutators with $Q$, they do not lead to BRST trivial operators when acting on vertex operators, since $\varphi$ and $x^\mu$ are not well-defined conformal fields. Extensive use has been made of these operators in two dimensional gravity theories [23] and in the analogue of these states in the two scalar $W_3$ string [16]. They were also used in references [21] and [22]. Such
operators also occur in the superstring, but there $\varphi$ and $x^\mu$ are replaced by one of the bosonized ghost fields. There they lead to the phenomenon of picture changing [19], and in view of the similar role in the played by these operators in the $W_3$ string we shall refer to them here as picture changing operators.

It is convenient to work with some linear combination $P$ of $[x^\mu, Q]$ and $[\varphi, Q]$, chosen such that the action of $P$ on a vertex operator having the form of a function of ghosts and $\partial \varphi$ multiplied by $V^x(a)$ leads to another vertex operator of the same form. Since the operator $P$ commutes with $Q$, the product $P(z)V(w)$ will also commute with $Q$ if $V(w)$ does. Consequently all terms in the operator product expansion, including the term independent of $z$,

$$ : PV : (w) = \oint_dz \frac{P(z)V(w)}{(z-w)} ,$$

(4.10)

will also commute with $Q$.

One could attempt to use the above vertex operators alone to construct string scattering amplitudes. Such an approach would be in complete analogy with other well understood string theories. This was the approach adopted in references [21,22]. However, in the $W_3$ string we have a background charge, and the vertex operators can possess only certain values of the $\varphi$ momentum. These values are discussed more fully in the next section. It is readily apparent that one is unable, in general, to use the vertex operators alone in such a way that their total momentum adds up to the required value $2iQ$. This is particularly apparent for $2N$ intercept $15/16$ states, since the allowed $\varphi$ momenta are $\beta(15/16, n) = (7-4n)iQ/7$, $n \in \mathbb{Z}_+$, which would require the condition $-2 \sum_i n_i = 7(N-2)$ with $n_i \in \mathbb{Z}_+$. In reference [17], however, these amplitudes were found not to vanish, and in section 5 we shall give a physical argument for why they must be non-vanishing.

The way out of this apparent paradox is to use screening charges, that is, objects of the form

$$ \int dw \ e^{i\beta\varphi} f(b, c, d, e, \partial\varphi) $$

(4.11)

that commute with $Q$. If we can find such charges that do not involve the field $x^\mu$, then the insertion of these charges into a correlation function will not change the effective space-time interpretation of the correlation function.
Such charges will commute with $Q$ only if the integrand has weight 1, which, since $f$ is a function of the ghosts and $\partial \varphi$, means that $\frac{1}{2} \beta (\beta - 2iQ) = n$ for $n \in \mathbb{Z}$. If we write $\beta = isQ/7$ then $s^2 - 14s - 16n = 0$, which implies that

$$s = 7 \pm \sqrt{49 - 16n}.$$  \hfill (4.12)

For $s$ to be an integer, as it appears always to be, we must have $49 - 16n = m^2, m \in \mathbb{Z}$. This can only be the case if $m = 8N \pm 1$ and $n = -(4N \mp 3)(N \pm 1)$, with $N \in \mathbb{Z}$. The values of $n$ so found are $3, 0, -2, -11, -15, \ldots$. The function $f$ must have weight $1 - n$, that is $-2, 1, 3, 12, 16, \ldots$. There are some obvious candidates for weights $-2$ and $3$, namely $f = e + \ldots$ and $f = d + \ldots$ respectively.

To evaluate the commutator of $Q$ with the screening charge, we use the formula

$$[Q, (f e^{i\beta \varphi})(z)] = \oint_z dw j^{BRST}(w) (f e^{i\beta \varphi})(z) \hfill (4.13)$$

We first consider $f = e$, in which case we need the formula

$$[Q, e e^{i\beta \varphi}(z)] = \partial [c e^{i\beta \varphi}] + \frac{i}{6} \beta (\beta - iQ)(\beta - 2iQ) \partial [\partial e^{i\beta \varphi}] + \beta (\beta - iQ) \frac{\partial e \partial \varphi}{12} \left[ \frac{1}{2} \beta (\beta - 2iQ) - 3 \right] \hfill (4.14)$$

In the derivation of this we used the result

$$W^m(z) e^{i\beta \varphi}(w) = \frac{i}{3} \frac{\beta (\beta - 2iQ)(\beta - iQ)}{(z-w)^3} e^{i\beta \varphi}(w) - \frac{\beta (\beta - iQ)}{(z-w)^2} \partial \varphi \ e^{i\beta \varphi}(w) + \frac{1}{z-w} (-i\beta (\partial \varphi)^2 - 2\beta T^x - \beta^2 \partial^2 \varphi) \ e^{i\beta \varphi}(w). \hfill (4.15)$$

In the case we are considering, $n \equiv \frac{1}{2} \beta (\beta - 2iQ) = 3$, and so $\beta$ takes the values $\bar{\beta}_1 = 6iQ/7$ or $\bar{\beta}_2 = 8iQ/7$. Then the last term of equation (4.14) vanishes, and it is trivial to see that

$$[Q, \int dz \ e^{i\beta_1 \varphi}(z)] = 0 = [Q, \int dz \ e^{i\beta_2 \varphi}(z)] \hfill (4.16)$$
The next interesting possibility is \( n \equiv \frac{1}{2} \beta(\beta - 2iQ) = -2 \), so that \( \beta = -2iQ/7 \) or \( 16iQ/7 \). Taking \( f \) to have terms of ghost number \(-1\) and weight \(3\) one finds after a lengthy calculation that

\[
[S, Q] = 0
\]

where

\[
S = \oint dz \left\{ d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial b e - \frac{4i}{3} \frac{1}{\sqrt{58}} d b e \right\} e^{i\beta_s \phi} \tag{4.17}
\]

and \( \beta^s = -2iQ/7 \).

We are now in a position to construct \( W_3 \) string scattering, with the building blocks being the vertices and the screening charges above. It is possible to construct all amplitudes using only the vertices \( V(1, 0) \), \( V(1/2, 0) \) and \( V(15/16, 0) \) together with the picture changing operator \( P \) and the screening charge \( S \). The reason for this will become apparent in section 6. For our initial example, however, we will use the additional vertex operator \( V(15/16, 1) \) [18].

A useful standard manoeuvre for reducing the number of \( c \) ghosts contained in a vertex is to realise that if \( V(z) \) is any vertex that commutes with \( Q \) then

\[
\int dz \oint z \, dv b(v) V(z) \tag{4.18}
\]

also commutes with \( Q \). This follows from the relation

\[
[Q, \oint z \, dv b(v) V(z)] = \oint z \, dv \, T^{tot}(v) V(z) = [L_{-1}^{tot}, V(z)] = \partial_z V(z). \tag{4.19}
\]

To illustrate this procedure, we compute the scattering of 4 sector 15/16 states. We can balance the \( \phi \) momentum by taking one vertex operator \( V(15/16, 0) \) with momentum \( \beta(15/16, 0) = 7iQ/7 \), three vertices \( V(15/16, 1) \) with momentum \( \beta(15/16, 1) = 3iQ/7 \), and one screening charge \( S \) with momentum \( \beta^s = -2iQ/7 \), since then

\[
\beta(15/16, 0) + 3\beta(15/16, 1) + \beta^s = 2iQ. \tag{4.20}
\]

To obtain the correct number of \( c \) ghosts, namely 3, we make use of equation (4.18) applied to one of the \( V(15/16, 1) \), i.e.

\[
\int dz \oint z \, dv b(v) V(15/16, 1)(z) = \int dz \, e^{i\beta(15/16, 1) \phi} V^x(15/16)(z). \tag{4.21}
\]
We also require 5 factors of $e$ and so we must apply one picture changing operator to, say, $V(15/16, 0)$. This gives

$$PV(15/16, 0) = c \partial^2 e \partial e e^{i\beta(15/16, 0)\varphi} V^x(15/16)$$

The amplitude for 4 intercept 15/16 $W_3$ string scattering is thus given by

$$\langle 0 | PV(15/16, 0)(z_1) V(15/16, 1)(z_2) \int dz_3 \int dv b(v) V(15/16, 1)(z_3) V(15/16, 1)(z_4) S | 0 \rangle = \int dz_3 \int dw \langle 0 | \left( c \partial^2 e \partial e e^{i\beta(15/16, 0)\varphi} V^x(15/16) \right) (z_1) \left( c e^{-i\beta(15/16, 1)\varphi} V^x(15/16) \right) (z_2) \left( e^{-i\beta(15/16, 1)\varphi} V^x(15/16) \right) (z_3) \left( c e^{-i\beta(15/16, 1)\varphi} V^x(15/16) \right) (z_4) \left( d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial b e - \frac{4i}{3} \frac{1}{\sqrt{58}} d b e e^{i\beta \varphi} \right) (w) | 0 \rangle$$

$$= - \int dz_3 \int dw \langle 0 | \left( \prod_{i=1}^{4} c(z_i) \right) \left( \partial^2 e \partial e \right) (z_1) e(z_2) e(z_3) e(z_4) d(w)$$

$$e^{i\beta(15/16, 0)\varphi}(z_1) \prod_{i=2}^{4} e^{i\beta(15/16, 1)\varphi}(z_i) e^{i\beta \varphi}(w) \prod_{i=1}^{4} V^x(15/16)(z_i) | 0 \rangle$$

By bosonizing the ghosts we find the $c$ ghosts give a factor

$$(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

and the $e$-$d$ ghosts a factor

$$(z_1 - w) \prod_{i=2}^{4} (z_1 - z_i)^3(z_i - w)^{-1} \prod_{i,j=2}^{4} (z_i - z_j).$$

Evaluating the exponential factors in the usual way we find that the amplitude for tachyonic states is proportional to

$$\int dw \int dx x(1-x)^{-1/8}[w(1-w)(x-w)]^{-1/4}(1-x)^{p_2 \cdot p_3 \cdot p_4}$$

$$= \int dw \int dx x^{s/2-2}(1-x)^{t/2-2}[(1-w)(x-w)]^{-1/4}$$

(4.26)
where we have chosen $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$.

Using appendix (A) we find the result

$$
\int dw \int dx \frac{x^{5/2-2}(1-x)^{t/2-2}}{2} (a \cos \theta/2 + b \sin \theta/2) \tag{4.27}
$$

where $a$ and $b$ are constants and $x = \sin^2 \theta$. This agrees with that found in reference [17]. This reference also contains detailed discussion of how crossing allows one to determine the coefficients $a$ and $b$; the factorization and duality properties of this result are also discussed.

5. The cohomology of $Q$

It was discovered in reference [18] that the cohomology of $Q$ contained, in addition to the intercept $15/16$ and $1$ states of ghost number zero, states associated with intercept $1/2$ with ghost number $-1$ and also copies of these states and those with intercept $15/16$ at other ghost numbers. It was also shown [18] that in order to build a modular invariant theory it was sufficient to take only one copy of each of the states with intercepts $1$, $1/2$ and $15/16$.

This leads one to suspect that there may be some simple relations connecting states in the cohomology of $Q$ associated with the same intercept. This is indeed the case; we will now show that such states are related by the actions of the screening operator $S$ of equation (4.17) and the picture changing operator $P$.

Since $S$ commutes with $Q$, it follows that whenever the action of $S$ on a physical state is well-defined and non-zero it will produce another physical state. Before giving a general discussion, let us consider an example. We shall show how the vertices $V(1/2, 0)$ and $\bar{V}(1/2, 0)$ can be related by the action of a single screening charge, after a suitable picture change. Given only one screening charge $S$ with $\phi$ momentum $\beta^s$, its action on a vertex operator with $\phi$ momentum $\beta$ will be well defined if $\beta^s \beta$ is an integer.

For the case under consideration $\beta^s \beta(1/2, 0) = 1$ and so the $\phi$ exponentials combine to provide one power of $(z - w)$. Hence, to gain a non-zero result the ghosts must provide at least a factor of $(z - w)^{-2}$. A short calculation shows that although $SV(1/2, 0)$ is well defined it vanishes. We therefore introduce further powers of
the ghost $e$ into the vertex using a picture changing operator. We have

$$PV(1/2, 0) = (5 \partial^2 e e - \frac{24Q}{7} \partial \phi \partial e e$$

$$\quad - \frac{19i}{3\sqrt{58}} \partial^2 e \partial e e) e^{i\beta(1/2, 0)\phi} V^x(1/2),$$

and acting on this with the screening operator we find the vertex $\bar{V}(1/2, 0)$,

$$SPV(1/2, 0) \propto \bar{V}(1/2, 0).$$

Let us now consider when two screening operators have a well defined action on a vertex operator with $\phi$ momentum $\beta$. Since this has not, to our knowledge, been clearly discussed in the literature let us consider the more general case of two screening operators with $\phi$ momentum $\beta_i$, $i = 1, 2$. Since the factors in front of the exponentials are made from either ghosts or $\partial \phi$ they can only contribute integer powers of the coordinates, and consequently do not affect whether or not the action of the screening operators is well defined. Consequently, we must focus our attention on the factors

$$\oint_{C_1} dw_1 \oint_{C_2} dw_2 e^{i\beta_1^* \phi(w_1)} e^{i\beta_2^* \phi(w_2)} e^{\beta \phi(z)}$$

$$= \oint_{C_1} dw_1 \oint_{C_2} dw_2 (w_1 - w_2)^{\beta_1^* \beta_2^*} (w_1 - z)^{\beta_1^* \beta} (w_2 - z)^{\beta_2^* \beta}$$

Since $|w_1 - z| \gg |w_2 - z|$ we arrange the $w_1$ and $w_2$ contours to be around the point $z$ in such a way that the above condition is satisfied.

Let us consider the substitution $(w_1, w_2)$ to $(w_1, \tau)$ given by $(w_2 - z) = \tau(w_1 - z)$. The above condition implies that $|\tau| < 1$, but we also demand that $\tau = 1$ at one and only point in other words the $w_1$ and $w_2$ contours touch one another. This latter condition ensures that the value of the integral is dependent on only the one place where the contours cross the branch cut. We are to regard $w_2$ as fixed and consider the $\tau$ integration. Substituting $dw_2 = d\tau(w_1 - z)$ we find the above integrals become

$$\oint_z dw_1 \oint d\tau \tau^{\beta_1^* \beta}(1 - \tau)^{\beta_1^* \beta_2^*} (w_1 - z)^P$$

$$\quad \exp(i\beta_1^* \phi(z + (w_1 - z)) + i\beta_2^* \phi(z + \tau(w_1 - z)) + i\beta \phi(z))$$
where \( P = 1 + \beta_1^s \beta^* + \beta_1^s \beta_2^s + \beta_2^s \beta \). This integral is well defined if \( P \) is an integer.

The generalization to \( n \) screening charges with \( \varphi \) momenta \( \beta_i^s \) is straightforward. Their action contains the term

\[
\prod_{i=1}^{n} \left( \oint d\omega_i e^{i\beta_i^s(\omega_i)} \right) e^{i\beta \varphi(z)} = \oint dw_i \prod_{i<j}^{n} (w_i - w_j) \beta_i^s \beta_j^s \prod_{j=1}^{n} (w_j - z) \exp i\{\sum_{j=1}^{n} \beta_j^s \varphi(w_j) + \beta \varphi(z)\} \tag{5.5}
\]

We now replace \( w_j, j = 2, \ldots, n \) by \( w_1, \tau_j \) \( j = 2, \ldots, n \) using the formula \((w_j - z) = \tau_j(w_1 - z)\). The above expression becomes

\[
\oint dw_1 \prod_{i=1}^{n-1} \oint d\tau_i \ f(\tau_i) \ (w_1 - z)^P \ \exp i\left\{\sum_{j=1}^{n} \beta_j^s \varphi(z + \tau_j(w_n - z)) + \beta \varphi(z)\right\} \tag{5.6}
\]

where \( f(\tau_i) \) is a function of \( \tau_i \) and

\[
P' = (n - 1) + \sum_{i<j}^{N} \beta_i^s \beta_j^s + \sum_{j=1}^{N} \beta_j^s \beta. \tag{5.7}
\]

Thus the integrals are well defined if \( P' \) is an integer.

Let us apply this general discussion to the case of interest, namely the action of \( n \) screening charges \( S \) with momentum \( \beta^s = -2iQ/7 \). Taking \( \beta = isQ/7 \) with \( s \) an integer, their action is well defined if

\[
\frac{n}{4}[-(n - 1) + s] \in \mathbb{Z}, \tag{5.8}
\]

and then the momentum of the resulting vertex is

\[
\frac{iQ}{7}(s - 2n). \tag{5.9}
\]

For the action on \( V(15/16, 0) \) we have \( s = 7 \). We then find that \( n \) must be even and that the vertices so constructed have momenta

\[
\beta(15/16, m) = \frac{iQ}{7}(7 - 4m), \quad m \in \mathbb{Z}_+. \tag{5.10}
\]
For the action on $V(1,0)$, for which $s = 8$, we find that $n = 4m$ or $4m + 1$ with $m \in \mathbb{Z}_+$, which leads to the vertices with momenta

$$\beta(1, m) = (8 - 8m) \frac{i Q}{\ell}, \quad m \in \mathbb{Z}_+, \quad (5.11)$$

and

$$\bar{\beta}(1, m) = (6 - 8m) \frac{i Q}{\ell}, \quad m \in \mathbb{Z}_+. \quad (5.12)$$

We can also consider the action on $\bar{V}(1,0)$, but in this case we find the same series of momenta.

Finally the action of $n$ screening charges on $V(1/2,0)$ is well defined if $n = 4m$ or $4m + 1$ with $m \in \mathbb{Z}_+$, which leads to vertices with momenta

$$\beta(1/2, m) = \frac{i Q}{\ell} (4 - 8m) \quad (5.13)$$

and

$$\bar{\beta}(1/2, m) = \frac{i Q}{\ell} (2 - 8m). \quad (5.14)$$

It can and does happen that applying $S$’s alone leads to a vanishing result. This can be avoided, however, by the judicious use of the picture changing operator $P$. Before giving a general discussion of this point we give another example. Let us consider the intercept 15/16 vertices and show how to go from $V(15/16, 0)$ to $V(15/16, 1)$. As is clear from the above, we require two screening charges. However, their action will vanish unless we first act with a picture changing operator.

We find that

$$PV(15/16, 0) \propto c \partial^2 e \partial e e^{i \beta(15/16,0)} \varphi V^x(15/16), \quad (5.15)$$

and acting with $S^2$ we find

$$S^2(PV(15/16, 0)) = \oint dw_2 \oint dw_1$$

$$\left( d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial b_1 e - \frac{4i}{3} \frac{1}{\sqrt{58}} db e)(w_1)e^{i\beta_e \varphi(w_1)} \right)$$

$$\left( d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial b_1 e - \frac{4i}{3} \frac{1}{\sqrt{58}} db e)(w_2)e^{i\beta_e \varphi(w_2)} \right)$$

$$(c \partial^2 e \partial e e^{i \beta(15/16,0)} \varphi e^{ip \cdot x})(z) V^x(15/16) \quad (5.16)$$

24
Bosonizing the ghosts, the term with two $d$'s is proportional to

$$c(z) \, V^x(15/16)(z)$$

$$\oint dw_2 \oint dw_1 e^{-i\rho(w_1)} e^{-i\rho(w_2)} e^{i\beta^s \varphi(w_1)} e^{i\beta^s \varphi(w_2)} e^{3i\rho(z)} e^{i\beta(15/16,0)\varphi(z)}$$

$$= c(z) V^x(15/16)(z) \oint dw_2 \oint dw_1 (w_1 - z)^{-5/4} (w_2 - z)^{5/4}$$

$$\exp \left( -\frac{5i}{4} (w_2 - z) - \frac{1}{2} \exp \left( i \beta^s \varphi(w_1) + \beta^s \varphi(w_2) + \beta(15/16,0)\varphi(z) \right) \right)$$

(5.17)

We now make the substitution $w_1 - z = \tau(w_2 - z)$, $|\tau| < 1$, to find that the above becomes

$$S^2(PV(15/16,0)) \propto c(z) \, V(15/16)(z) \oint dw_2 \int d\tau \, \tau^{-5/4}$$

$$\left( \tau - 1 \right)^{1/2} (w_2 - z)^{-1} \exp \left( -\rho(z + \tau(w_2 - z)) \right)$$

$$- \rho(z + (w_2 - z)) + 3\rho(z) \exp \left( i\beta^s \varphi(z + \tau(w_2 - z)) \right)$$

$$+ \beta^s \varphi(z + w_2 - z) + \beta(15/16,0)\varphi(z)$$

$$= c(z) V^x(15/16)(z) \exp e^{i\beta(15/16,1)\varphi(z)} \int d\tau \, \tau^{-5/4} (\tau - 1)^{1/2}$$

$$\propto c e^e e^{i\beta(15/16,1)\varphi} V^x(15/16)$$

(5.18)

which we recognise as the first term in the vertex $V(15/16,1)$. The additional terms are calculated in a similar way. For example, the term with $d(w_1)(-5i/(3\sqrt{58}) \partial b(w_2))$ provides us with a ghost contribution of $-\tau^{-3}(w_2 - z)^5$, and so leads to a contribution $(-5i/(2.3\sqrt{58}))de e$ multiplied by exponentials. Calculating the three other terms one finds

$$V(15/16,1) = S^2(PV(15/16,0)).$$

(5.19)

It is straightforward, if tiresome, to show that we can repeat this operation to find the vertex $V(15/16,2)$, which is given by

$$V(15/16,2) = S^2 \, PV(15/16,1) = S^2 P S^2 PV(15/16,0)$$

(5.20)

In this way the general pattern for the vertices in the sector 15/16 emerges. We have the vertices

$$V(15/16,m) = (S^2 P)^m V(15/16,0)$$

(5.21)
and in addition
\[ P V(15/16, m). \] (5.22)

Let us turn our attention to the intercept 1 vertices. Although \( S \) gives zero when acting on \( V(1, 0) \), we can act with \( P \) first to find the vertex
\[ P V(1, 0) = c \partial^2 e \partial e e^{i\beta(1,0)} V^x(1). \] (5.23)

The action of \( S \) on this vertex does give a non-zero result, namely
\[ \bar{V}(1, 0) = c \partial e e^{i\bar{\beta}(1,0)} V^x(1) = SPV(1, 0). \] (5.24)

To find the next vertex \( V(1, 1) \) we must act with \( P \) and then three screening charges. Since this provides us with a simple example with three screening charges we will give a few of the details. We act on the state
\[ PV(1, 0) = c \partial^2 e e^{i\bar{\beta}(1,0)} V^x(1) \] (5.25)

with three screening charges
\[
\prod_{i=1}^{3} \left( \int dw_i (d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial b e - \frac{4i}{3} \frac{1}{\sqrt{58}} db e) e^{i\beta^s \varphi(w_i)} \right) P\bar{V}(1, 0).
\]

The leading term with three \('d's is given by
\[
\begin{align*}
&= c V^x(1)(z) \prod_{i=1}^{3} \int dw_i \prod_{i=1}^{3} (e^{-\rho(w_i)}) e^{3i\rho(z)} e^{i\bar{\beta}(1,0) \varphi(z)} \\
&= c V^x(1)(z) \prod_{i} \int dw_i \prod_{i<j} (w_i - w_j)^{1/2} \prod_{j=1}^{3} (w_j - z)^{-3/2} \\
&\exp i \left( - \sum_j \rho(w_j) + 3\rho(z) \right) \exp i \left( \sum_i \beta^s \varphi(w_j) + \beta \varphi(z) \right) \\
&= c V^x(1)(z) \int d\tau_1 \int d\tau_2 [(\tau_1 - \tau_2)(\tau_1 - 1)(\tau_2 - 1)]^{1/2} \\
&\left( (\tau_1 - \tau_2)^{-3/2} \exp i(\beta(1,1) \varphi(z)) \right) \\
&= c e^{i\beta(1,1) \varphi(z)} V^x(1)(z).
\end{align*}
\] (5.26)

Evaluating the remaining terms we find the result
\[
\left( c + \frac{7i}{3\sqrt{58}} \partial e - \frac{8}{261} b \partial e e - \frac{4i}{3\sqrt{29}} \partial \varphi e \right) e^{i\beta(1,1) \varphi} V^x(1),
\]
which can be written as

\[ V(1,1) = S^3P\bar{V}(1,0) = S^3PSPV(1,0). \]  \hspace{1cm} (5.27)

It is straightforward to compute further vertices, but the general pattern now emerges. We have the vertices

\[ V(1,m) \quad \text{and} \quad \bar{V}(1,m) \]  \hspace{1cm} (5.28)

as well as

\[ PV(1,m) \quad \text{and} \quad P\bar{V}(1,m). \]  \hspace{1cm} (5.29)

These vertices are defined by the relations

\[ \bar{V}(1,m) = SPV(1,m), \]  \hspace{1cm} (5.30)

\[ V(1,m) = S^3P\bar{V}(1,m-1). \]  \hspace{1cm} (5.31)

The vertices with intercept 1/2 have a similar pattern to those of intercept 1. Starting from the vertex \( V(1/2,0) \) we can, by acting with \( P \) and \( S \), create the vertices

\[ V(1/2,m) \quad \text{and} \quad \bar{V}(1/2,m) \]  \hspace{1cm} (5.32)

as well as

\[ PV(1/2,m) \quad \text{and} \quad P\bar{V}(1/2,m) \]  \hspace{1cm} (5.33)

by using the relations

\[ V(1/2,m) = S^3P\bar{V}(1/2,m-1) \]  \hspace{1cm} (5.34)

and

\[ \bar{V}(1/2,m) = SPV(1/2,m). \]  \hspace{1cm} (5.35)

To summarise, we have found that given the basic vertices \( V(a,0) \) for \( a = 1, 1/2 \) and 15/16, we can use \( S \) and \( P \) to create the BRST invariant vertices \( V(a,m) \); for \( a = 1,1/2 \) we obtain in addition the vertices \( \bar{V}(a,m) \). We can also obtain further vertices by the action of \( P \) on these. Leaving aside the question of discrete states and the further action of picture changing operators it would seem most likely that these are the only states in the cohomology of \( Q \), thus extending the
spectrum conjecture of reference [18]. We have now amassed considerable evidence for this extended conjecture. One copy of the states from each sector is known to be sufficient for modular invariance [18] and is consistent with factorization of tree level scattering [17]. Further, any states that arise with this effective intercept can be of positive norm only for the intercept vertices \(1, 1/2\) and \(15/16\) [18]. These facts make it rather unlikely that the cohomology of \(Q\) could contain states not of the above type. Finally, the vertices above lead to all vertices known to belong to the cohomology of \(Q\) and in particular to the \(V(1/2, 0), V(1/2, 1)\) and \(V(15/16, 1)\) found in reference [18] and the additional vertices found in references [21, 22].

It was shown in reference [18] that the physical states for a given intercept have a spectrum generating algebra involving the operators \(B_n^i, i = 1, \ldots D - 2\) and \(B_n\). Consequently, the conclusion given above is that states in the cohomology of \(Q\) are generated by \(B_n^i, B_n, S\) and \(P\).

6. General Formalism for \(W_3\) String Scattering

The \(W_3\) string scattering amplitudes are to be constructed from the building blocks

\[
V(a, 0) ; a = 1, 15/16, 1/2 ; S, P
\]  

(6.1)

and the operation

\[
\int dz \oint z dv b(v).
\]  

(6.2)

The latter is a standard operation used for the \(b - c\) ghost system. We must, however, assemble the building blocks so that the \(\varphi\) momentum sums to \(2iQ\). This tells us the required number \(N_s\) of screening charges. We must also have, after carrying out all the operator product expansions, 3 \(c\) ghosts and 5 \(e\) ghosts, otherwise the correlator will vanish. As usual, we require for an \(N\) string amplitude \(N - 3\) of the operators of equation (6.2). The blocks \(V(a, 0)\) with \(a = 1, 15/16, V(1/2, 0), S\) and \(P\) have ghosts number 3, 2, \(-1\) and 1 respectively. The ghost number requirement gives us the number \(N_p\) of picture changing operators \(P\). To be precise if we have a scattering of \(N_1\) intercept \(1\), \(N_{15/16}\) intercept \(15/16\) and \(N_{1/2}\) intercept \(1/2\) strings, then \(\varphi\) momentum conservation demands that

\[
8N_1 + 7N_{15/16} + 4N_{1/2} - 2N_p = 14
\]  

(6.3)
while the ghost number count yields the relations

\[ 3N_1 + 3N_{15/16} + 2N_{1/2} - N_S + N_P - (N_1 + N_{15/16} + N_{1/2} - 3) = 8 \quad (6.4) \]

These equations imply that

\[ N_S = 4N_1 + \frac{7}{2}N_{15/16} + 2N_{1/2} - 7 \]
\[ N_P = 2N_1 + \frac{3}{2}N_{15/16} + N_{1/2} - 2 \quad (6.5) \]

We must now distribute all these factors of \( P \) and some of the factors of \( S \) among the vertices so as to gain a non-zero result. To be concrete let us consider the scattering of \( N_{15/16} = 2n \geq 6 \) intercept \( 15/16 \) states, in which case \( N_S = 7(n - 1) \) and \( N_P = 3n - 2 \).

We must assign these to the vertices of equations (5.21) and (5.22). One way to do this is to take \( n - 3 \) of the vertices \( V(15/16, 2) = (S^2P)^2V(15/16, 0) \) and \( n + 3 \) of the vertices \( V(15/16, 1) = S^2PV(15/16, 0) \). This leaves over one \( P \) factor and \( n - 1 \) \( S \) factors and so leads to the correlator

\[
\langle 0 | \prod_{i=4}^{2n} \left\{ \oint dz_i \oint_{z_i} dv_i \ b(v_i) \right\} \prod_{i=1}^{n+2} V(15/16, 1)(z_i) \\
P \ V(15/16, 1)(z_{n+3}) \prod_{j=n+4}^{2n} V(15/16, 2)(z_j) S^{n-1} |0 \rangle
\]

(6.6)

We have chosen to assign the final \( P \) factor to the \( n + 3 \) vertex, but any other vertex is just as good. We could also use two of the final screening charges on the \( PV(15/16, 1) \) vertex to yield the correlator

\[
\langle 0 | \prod_{i=4}^{2n} \left\{ \oint dz_i \oint_{z_i} dv_i \ b(v_i) \right\} \prod_{i=1}^{n+2} V(15/16, 1)(z_i) \\
\prod_{j=n+3}^{2n} V(15/16, 2)(z_j) S^{n-3} |0 \rangle
\]

(6.7)

Clearly there are many ways to write such a correlation using also the vertices \( V(15/16, m), m > 2 \).
For the scattering of $N_{1/2} = 2n$ intercept $1/2$ states, $N_S = 4n - 7$ and $N_P = 2(n - 1)$. In this case we can assign all the factors of $S$ to the vertices, for example by the choice of $n - 2$ of the vertices $V(1/2, 1)$, one of $\bar{V}(1/2, 0)$ and $n + 1$ of the vertices $V(1/2, 0)$ and then place on one of these vertices the required extra $P$. The correlator is then

$$\langle 0 | \prod_{i=4}^{2n} \left\{ \int dz_i \oint dv_i b(v_i) \right\} \prod_{i=1}^{n+1} V(1/2, 0)(z_i) \prod_{j=n+2}^{2n-1} V(1/2, 1)(z_j) P \bar{V}(1/2, 0)(z_{2n}) | 0 \rangle$$

(6.8)

Finally for $N_1$ intercept $1$ states, $N_S = 4N_1 - 7$ and $N_P = 2N_1 - 2$, which we can assign as $N_1 - 2$ vertices $V(1, 1)$, one of $V(1, 0)$ and one of $\bar{V}(1, 0)$, with one additional factor of $P$ to give the correlator

$$\langle 0 | \prod_{i=4}^{N_1} \int_{z_i} dz_i \int_{z_i} dv_i b(v_i) \prod_{j=1}^{N_1-2} V(1; 1)(z_j) PV(1, 0)(z_{N_1-1}) \bar{V}(1, 0)(z_{N_1}) | 0 \rangle$$

(6.9)

The many ways of constructing the above correlators lead to the same results. The correlator is independent of the place where the picture changing operator is applied, since

$$P(z_1) - P(z_2) = [Q, \varphi(z_1) - \varphi(z_2)] = [Q, \int_{z_1}^{z_2} \partial \varphi],$$

and this is a BRST trivial operator as it contains $\partial \varphi$ and not $\varphi$. Further, we can change on which vertex the screening charges act by deforming the $w$-contours. In fact there are different choices for the contours for the residual screening charges and these lead to different results. As explained in section 4, in the context of the $4$ intercept $15/16$ scattering, we require these different solutions since only a particular combination of the contours gives a crossing symmetric amplitude.

7. Unitarity

Unitarity of any theory requires that the $S$ matrix satisfy the constraint $S^+ S = 1$ and that individual probabilities must be positive. The former condition implies
the well known optical theorem [25]. It is an obvious consequence of this theorem that if the non-trivial part of the 2 particle to 2 particle scattering amplitude were to vanish then the amplitude for the scattering of 2 particles to any number of particles would vanish. This would include the 2 particle to one particle scattering amplitude.

In references [21] and [22] a covariant approach to $W_3$ string scattering was given. The authors adopted the usual rules of the covariant scattering formalism for the bosonic string and superstring, and applied them to the $W_3$ string. They found that many scattering amplitudes, particularly those involving intercept $15/16$ states, vanished. They stressed the significance of this fact and remarked that it contradicted the results of previous work [17] and concluded that this former work, which gave all $W_3$ scattering amplitudes, was incorrect. In particular, references [21] and [22] claimed that the scattering of 4 intercept $15/16$ string vanished, but that in agreement with reference [17], the amplitude for 2 intercept $15/16$ strings to both intercept $1/2$ and $1$ were non-vanishing. This contradicts the optical theorem and so the set of rules given in references [21] and [22] do not lead to a unitary theory.

One might argue that, since the particles under discussion are unstable, one could avoid this conclusion. As one might expect, however, this is not the case. It can be shown [26], using only rather innocuous assumptions, that a 2 particle to 2 particle amplitude factorizes, near the energy of the mass for a single intermediate state, into the product of the two couplings of the 2 particles into the intermediate state multiplied by the intermediate state propagator. The assumption of duality [27] in string theory goes much further; it states that not only can the entire amplitude be written as a sum over 3 particle couplings multiplied by the propagator of single particle intermediate states, but that it can be written in this way for either the $s$ or $u$ channel exchanges. Indeed it was this factorization procedure that enabled the early pioneers of string theory to deduce all the scattering amplitudes from the tachyon amplitudes. As explained in this paper it is not sufficient just to mimic the covariant rules for the usual bosonic and superstrings, but one should check whether a given set of string amplitudes factorizes correctly. The covariant scattering formalism given in this paper leads to a non-vanishing amplitude for four intercept $15/16$ strings which is in agreement with that found earlier [17]. In this reference it was indeed verified that the four $15/16$ intercept string amplitude does
factorize correctly to give three point functions that are consistent with the fusion rules. Thus the covariant scattering formalism given in this paper is compatible with unitarity and with the assumptions of S-matrix theory, as well as with duality and factorization. This is not the case for the covariant formalism of references [21] and [22], which violate all of these principles.

As with any approach to string theory, other than gauge covariant string field theory, unitarity must be used to fix the weights of the individual amplitudes.

Although we have shown here that the approach advocated in references [21,22] is not, in general, adequate to compute $W_3$ string scattering, nevertheless the amplitude for the scattering of 4 tachyonic intercept $1/2$ states, does, as these authors observed, agree with that found in reference [17]. In fact, all of the non-vanishing scattering amplitudes found in references [21] and [22] can readily be seen to agree with the general formula for $W_3$ string scattering of reference [17]. In section 3 some further explicit evaluations of the general formula of reference [17] were given to facilitate this comparison which largely applies to the scattering of 4 tachyon strings.

The positivity of probabilities implies in particular that the norm of physical states is positive. It was shown in reference [18] that the effective physical states for $c = 51/2$ with intercepts $1$, $1/2$ and $15/16$ had positive definite norm. Further, it was shown [18] that only for these three values of the intercepts was this the case. It was conjectured in reference [18] that the cohomology of $Q$ consisted of these states, copies of them and possible discrete states. Considerable additional evidence for this spectrum conjecture was found in section 6. It follows from the conjecture and reference [18] that all physical states of the $W_3$ string have positive norm and so satisfy a no-ghost theorem.

It has been claimed in references [21] and [22] that one can prove a no-ghost theorem by using the fact that the effective states with $c = 51/2$ at levels one and two have positive norm if the intercept $a$ satisfies the bounds

$$15/16 \leq a \leq 1 \quad \text{or} \quad a \leq 1/2,$$

and that a standard result from ordinary string theory is that the unitarity bounds derived from level 1 and level 2 states are sufficient to ensure unitarity at all excited levels.
In fact in the early days of string theory positivity of the norm up to quite high levels was investigated, but it required the classic and beautiful work of reference [28] to establish the no-ghost theorem, which is one of the cornerstones of string theory. Although the original arguments have been simplified, as far as we are aware the shortest such arguments are in essence the same as those given in the original paper.

In fact, at least as far as the evidence given in their paper is concerned, the authors of references [21] and [22] have shown that demanding unitarity at levels 1 and 2 does not imply unitarity at all levels since the former holds for values of $a$ other than $a = 1, 15/16$ and $1/2$. They do note, however, that the bounds are saturated for the actual values of $a$. In reference [18] it was shown that an intercept not equal to these values led to negative norm states at higher levels; it might perhaps be possible also to derive this result by exploiting the fact that the bounds are saturated for these values.

8. Discussion

In this paper we have given a general formalism for covariant $W_3$ string scattering; any amplitude can be built from three vertices $V(a, 0)$, with $a = 1, 15/16$ or $1/2$, together with a screening charge $S$, a picture changing operator $P$, and the usual $b$ ghost insertions. The amplitudes are found to contain Ising correlations and are in agreement with general results for $W_3$ scattering found previously [17]. They are in disagreement, however, with the results of references [21,22], which are shown to violate the assumptions of S-matrix theory and the string assumption of duality.

Starting from three states, one for each of the intercepts 1, 15/16 and 1/2, the screening and picture-changing operators are shown to generate an infinite number of states in the cohomology of $Q$. It is conjectured that these are all the elements of the cohomology of $Q$, apart from discrete states and further states resulting from the action of picture-changing operators.

It would be interesting to give a path-integral derivation of the scattering results that we have obtained. This would require a knowledge of $W$-moduli. Any such derivation would, however, have to reproduce the results found in this paper, and this could provide a clue to our understanding of $W$-moduli. We observe that the number of $W$-moduli is $2N - 5$ for the scattering of $N$ strings, and this number emerges from our results in the guise of $N_S - N_P + N_{1/2}$.
Although the scattering formalism given in references [21,22] was not correct, these authors did propose an interesting field redefinition. Many of the calculations given in this paper are considerably simplified when carried out in terms of these new fields.

Although the string amplitudes we have found do contain the Ising model correlation functions and we do use screening charges, these correlation functions do not, at least at first sight, use the Feigin-Fuchs representation. It would be of interest further to investigate this phenomenon.

It would seem apparent that the pattern found for $W_3$ will also occur for $W_N$. The physical states will be effective space-time states with intercepts given by weights in the corresponding unitary minimal models [18], the scattering will contain the corresponding minimal model correlation functions and the cohomology of $Q$ will be generated by acting with screening charges and picture-changing operators on a set of states which are in one to one correspondence with the weights in the corresponding minimal models.

Acknowledgements

We wish to thank N. Berkovits and G. Duke for discussions. M. Freeman is grateful to SERC for financial support.

Appendix A

In this appendix we summarize some of the known technical results [29] used to compute the scattering amplitudes of the $W_3$ string which involve the hypergeometric function $F(a, b, c; z) = F(b, a, c; z)$. This function is known to possess the integral representation

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \quad (A1)$$

By the change of variable $w = 1/t$ and redefinition of $a, b$ and $c$ we find that

$$\int_1^\infty w^c(w-1)^b(w-z)^a dw = \frac{\Gamma(1+b-c-a)\Gamma(b+1)}{\Gamma(-c-a)} F(-a, -b - c - a - 1; c - a, z) \quad (A2)$$

34
Similarly, one finds that

\[
\int_0^z dw w^a (1-w)^b (z-w)^c = z^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} F(-b, a+1, a+c+2; z) \tag{A3}
\]

For the four intercept 15/16 strings one finds certain hypergeometric functions that can be given in terms of elementary functions by the following identities

\[
\cos a \theta = F\left(\frac{a}{2}, -\frac{a}{2}; \frac{1}{2}; \sin^2 \theta \right) \\
\quad = \cos \theta F\left(\frac{a+1}{2}, \frac{1-a}{2}; \frac{1}{2}; \sin^2 \theta \right) 
\]

\[
\sin a \theta = a \sin \theta F\left(\frac{a+1}{2}, \frac{1-a}{2}; 3/2; \sin^2 \theta \right) \\
\quad = a \sin \theta \cos \theta F\left(1 + \frac{a}{2}, 1 - \frac{a}{2}; 3/2; \sin^2 \theta \right) 
\tag{A4}
\]

References

[1] A. B. Zamolodchikov, Theor. Math. Phys 65 (1989) 1205.
[2] V.A. Fateev and S. K. Lukyanov, Int. J. Mod. Phys. A3 (1988) 507.
[3] J. Thierry-Mieg, Phys. Lett. B197 (1987) 368.
[4] A. Bilal and J. L. Gervais, Nucl. Phys. B326 (1989) 222.
[5] P. Howe and P. West, unpublished
[6] V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. B280 [FS18] (1987) 644.
[7] L. J. Romans, Nucl. phys B352 (1991) 829.
[8] S. Das, A. Dhar and S. Kalyana Rama, Mod. Phys. Lett. 268B (1991); Mod. Phys. Lett. 269B (1991) 167; Int. J. Mod. Phys. A7 (1992) 2295.
[9] C. N. Pope, L. J. Romans, K. S. Stelle, Phys. Lett. 268B (1991) 167; Phys. Lett. 269B (1991) 287.
[10] H. Lu, C. N. Pope, S. Schrans and K. W. Xu, Texas A & M preprint CTP TAMU-5/92.
[11] E. del Giudice and P. di Vecchia, Nuovo Cimento 5A (1971) 90.
[12] M. Kato and K. Ogawa, Nucl. Phys. B212 (1983) 443. M. D. Freeman and D. I. Olive, Phys. Lett. B 175 (1986) 151.
[13] H. Lu, B. E. W. Nilsson, C. N. Pope, K. S. Stelle and P. West, “The low level spectrum of the W_3 string,”
[14] S. Kalyana Rama, Mod. Phys. Lett A6 (1991) 3531.
[15] B. H. Lian and G. J. Zuckerman, Phys. Lett. B 254 (1991); Phys. Lett. B 266 (1991) 21. A. M. Polyakov, Mod. Phys. Lett. A6 (1991) 635. E. Witten, Nucl. Phys. B 373 (1992) 187. N. Ohta, “Discrete states in two-dimensional quantum gravity” and references therein.

[16] C. N. Pope, E. Sezgin, K. S. Stelle and X. J. Wang, “Discrete states in the $W_3$ string,” CTP-TAMU-64/92, Imperial/TP/91-92/40.

[17] M. Freeman and P. West; ”$W_3$ string scattering”, KCL-TH-92-4, NI-92007, Phys. Lett. B to be published.

[18] P. West, “On the spectrum, no-ghost theorem and modular invariance of $W_3$ strings,” KCL-th-92-7, to be published in Int. J. Mod. Phys.

[19] For a review see D. Lust and S. Theisen, “Lectures on string theory,” Springer-Verlag 1989.

[20] F. L. Feigin and D. B. Fuchs, Moscow preprint 1983. Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240 (1984) 312.

[21] H. Lu, C. N. Pope, S. Schrans and X-J. Wang, “The interacting $W_3$ string,” CTP-TAMU-86/92.

[22] H. Lu, C. N. Pope, S. Schrans and X-J. Wang, “On the spectrum and scattering of $W_3$ strings,” CTP-TAMU-4/93.

[23] E. Witten and B. Zwieback, Nucl. Phys. B377 (1992) 644.

[24] For a review see C. Itzykson and M. Drouffe, “Statistical Field Theory,” Cambridge University Press.

[25] See, for example, C. Itzykson and J-B. Zuber, “Quantum Field Theory,” McGraw Hill.

[26] R. Eden, P. Landshoff, D. Olive and J. Polkinghorne, “The Analytic S-matrix,” Cambridge University Press.

[27] R. Dolan, D. Horn and C. Schmid, Phys. Rev. Lett. 19 (1967) 402; Phys. Rev. Lett. 166 (1968) 1768.

[28] R. C. Brower and P. Goddard, Nucl. Phys. B40 (1972) 437; R. C. Brower, Phys. Rev. D6 (1972) 1655; P. Goddard and C. Thorn, Phys. Lett. 40B (1972) 235.

[29] “Bateman Manuscript Project,” edited A. Erdelyi, McGraw Hill.