Abstract. Quasivelocity techniques were applied to derive the dynamics of a Differential Wheeled Mobile Robot (DWMR) in the companion paper. The present paper formulates a control system design for trajectory tracking of this class of robots. The method develops a feedback linearization technique for the nonlinear system using dynamic extension algorithm. The effectiveness of the nonlinear controller is illustrated with simulation example.

1. Introduction

Maggi’s quasivelocity method was used to derive the dynamical equations of motion of a Differential Wheeled Mobile Robot (DWMR) and the derivation is shown in the companion paper [1]. The system has two inputs, however, it can be transformed into a “single input” system by noticing that by taking the difference between the first and second equations of (8) in [1], duplicated here for completeness, and by taking time derivative of the resulting equation, together with row 8\(^{th}\) of (2) one can transform the system into ”single input system”. Thereafter one can use single-input single-output techniques developed for example in [2] to simultaneously compute the input torques. The equations resulting from the transformation are given in the first part of the work presented here. In the second part of the controller design, a multi-input multi-output feedback linearization technique is considered. The first difficulty encountered is that there is no relative degree with respect to the output vector considered. Dynamic extension algorithm is used to obtain relative degree with respect to each output in the output vector.

2. Dynamic transformation

The kinematic constraints of the mobile robot derived in the companion paper is shown below for completeness

\[
\begin{align*}
\dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) - b\dot{\phi} &= r\dot{\theta}_l \\
\dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) + b\dot{\phi} &= r\dot{\theta}_r \\
-\dot{x}_p \sin(\phi) + \dot{y}_p \cos(\phi) &= 0.
\end{align*}
\]
The dynamical equation of motion of the mobile robot derived in [1] is given by
\[
\dot{q} = \begin{bmatrix}
q_6 \\
q_7 \\
q_8 \\
q_9 \\
q_{10} \\
f_6(q_3, q_6, q_7, q_8) \\
f_7(q_3, q_6, q_7, q_8) \\
f_8(q_3, q_6, q_7, q_8) \\
f_9(q_3, q_6, q_7, q_8) \\
f_{10}(q_3, q_6, q_7, q_8)
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
g_61(q_3) & g_61(q_3) \\
g_71(q_3) & g_71(q_3) \\
g_81(q_3) & -g_81(q_3) \\
g_91 & g_{92} \\
g_{10} & g_{11}
\end{bmatrix} \begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
\] (2)

By taking the difference between the first and second equations of (1) in [1], we have
\[
\dot{\phi} = \frac{r}{2b} (\dot{\theta}_r - \dot{\theta}_l) .
\] (3)

Taking time derivative of (3), the resulting equation is equal to row 8 of (2). From this, we can solve for \(\tau_1\) which is given by
\[
\tau_1 = \frac{r}{2bg_81} (\dot{\theta}_r - \dot{\theta}_l) .
\] (4)
Upon substitution in (2) and noticing that the substitution does not affect the upper half equation in (2), the lower half of (2) is given by

\[
\begin{bmatrix}
1 & 0 & 0 & -g_6K & g_6K \\
0 & 1 & 0 & -g_7K & g_7K \\
0 & 0 & 1 & -\frac{g}{g_81} & \frac{g}{g_81} \\
0 & 0 & 0 & 1 - g_{91}K & g_{91}K \\
0 & 0 & 0 & -g_{10}K & 1 + g_{10}K
\end{bmatrix}
\begin{bmatrix}
\dot{q}_6 \\
\dot{q}_7 \\
\dot{q}_8 \\
\dot{q}_9 \\
\dot{q}_{10}
\end{bmatrix}
= \begin{bmatrix}
f_6 - \frac{g_{91}}{g_{g_81}}f_8 \\
f_7 - \frac{g_{91}}{g_{g_81}}f_9 \\
f_8 \\
f_9 - \frac{g_{10}f_{g_81}}{g_{g_81}} \\
f_{10} - \frac{g_{10}f_{g_81}}{g_{g_81}}
\end{bmatrix}
+ \begin{bmatrix}
2g_{61} \\
g_{71} \\
0 \\
g_{91} + g_{92} \\
g_{10} + g_{11}
\end{bmatrix}
\tau_2 ,
\] (5)

where $K = \frac{1}{2g_{91}}$. $g_{81}$ is a constant so the division by it is well defined. The first half of (2) together with (5) give a system of “single input”. Upon computing the control law for $\tau_2$ one can compute $\tau_1$ and $\dot{\phi}$ explicitly. Because of limited space we do not pursue the design of this control law.

The control law derived in the rest of the paper involves the two inputs system shown in (2) using dynamic extension.

3. Tracking Control Design of the Two Inputs System

The dynamics of the mobile robot are formulated in the state space as dynamics of a standard nonlinear system as

\[
\dot{q} = f(q) + G(q)u
\] (6)

Immediate calculations show that $\text{rank}\{g_1, g_2\} = 2$ i.e. the vectors $g_1, g_2$ are linearly independent. The distribution $G = \{g_1, g_2\}$ is also involutive because simple calculations show that $\text{rank}\{g_1, g_2, [g_1, g_2]\} = 2$ for all $q$. Where $[g_1, g_2]$ is the Lie bracket of $g_1$ and $g_2$ defined by

\[
[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2
\] (7)

Thus by Frobenius’ Theorem, we deduce the existence of 8 real-valued functions that span the space orthogonal to the space spanned by $\{g_1, g_2\}$. However, it is not always very easy to find these functions because it involves solving partial differentials equations. It is well known that systems with at least one nonholonomic constraints are not input-state linearizable [3] and since the system cannot be made asymptotically stable by a smooth feedback [4], we seek feedback control which achieves input-output stability. A natural choice of outputs are the coordinates of the reference point $P$ figure 1 [1] which defines a position of the mobile robot in the plan. The output vector is defined as

\[
y = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}
\] (8)

With respect to these outputs, the system does not have a relative degree because immediate calculations show that

\[
L_y h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\] (9)

and

\[
A(q) = L_y L_y h(q) = \begin{bmatrix} g_{91}(q_3) & g_{91}(q_3) \\ g_{71}(q_3) & g_{71}(q_3) \end{bmatrix}
\] (10)

which has rank 1 for all $q$. We apply the dynamic extension algorithm. Relative degree $r=\{4,2\}$ is achieved after two iterations of Dynamic Extension Algorithm when the following compensator is cascaded with the system. (Refer for example to [2][5] for Dynamic Extension).
\[
\begin{align*}
\tau_1 & = \frac{1}{g_{61}(q_3)} \left(-f_6(q) + \eta\right) - v_2 \\
\dot{\eta} & = \zeta \\
\dot{\zeta} & = v_1 \\
\tau_2 & = v_2
\end{align*}
\]  \hspace{1cm} (11)

4. Simulation
A simulation is developed to demonstrate the effectiveness of the control law designed above. As figures 2 and 3 demonstrate, the tracking control which takes the dynamics of the mobile robot into account perform good tracking. One should also note that the control law is valid for \(g_{16}(q_3) \neq 0\), i.e., \(q_3 \neq \frac{\pi}{2}(2k + 1)\), where \(k\) is an integer.

Figure 2. \(x_p\) and the desired trajectory.

Figure 3. \(y_p\) and the desired trajectory.

Figure 4. Circular path.

Figure 5. Straight line path.

5. Conclusion
We consider the modeling of the dynamics of a differential mobile robot and the derivation of a nonlinear control law to track a predefined trajectory. Maggi’s method, a quasivelocity technique was used to derive the dynamic. This method eliminates the Lagrange multipliers used to enforce the nonholonomic constraints from the start as most of the time one is not interested in the Lagrange multipliers. It thus reduce the number of the variables and the number of equations.
to solve. Feedback linearization is the technique employed in this paper to derive a trajectory control law for the wheeled mobile robot. It is well known that one of the key points to start feedback linearization is the selection of the output vector. Often, some state variables are not available for measurement and the choice is limited. With respect to the output vector selected in this paper, there is no relative degree. Dynamic extension is then used to obtain a relative degree vector relative to each component of the output vector. Computer simulations are then added to demonstrate the effectiveness of the control law implemented for this class of wheeled mobile robots.

References
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