SOLVABILITY OF SOME STEFAN TYPE PROBLEMS

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ABSTRACT. In this paper, we interest on some class of Stefan type problems. We prove the existence and uniqueness of renormalized solution in anisotropic Sobolev spaces with data belongs to \( L^1 \) – data, based on the properties of the renormalized truncations and the generalized monotonicity method in the functional spaces.

1. Introduction

Anisotropic elliptic equations have received much attention in recent years (see for example, [31], [8], [13], [17], [22] and their references). Time dependent versions of these equations have been used as mathematical models to describe the spread of an epidemic disease, see [9]. such evolution models also arise in fluid dynamics when the media has different conductivities in the different directions (see [6], [6]), and electrorheological fluids (see [31] for more details) as an important class of non-Newtonian fluids.

We are interested in the study of the behavior of solutions for a class of Stefan-type problems of form:

\[
(\mathcal{E}, f) \begin{cases} \beta(u) - \text{div}(a(x, Du) + F(u)) \ni f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial \Omega. \end{cases}
\]

with \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) and \( \partial \Omega \) Lipschitz boundary if \( N \geq 2 \), a right-hand side \( f \) which is assumed to belong to \( L^\infty(\Omega) \) or \( L^1(\Omega) \) for \( (E, f) \). Furthermore, \( F : \mathbb{R} \rightarrow \mathbb{R}^N \) is locally lipschitz continuous and \( \beta : \mathbb{R} \rightarrow 2^{\mathbb{R}} \) is a set valued, maximal monotone mapping such that \( 0 \in \beta(0) \) and \( a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory function satisfying the following assumptions:

\textbf{(H₁)} – Coerciveness: there exists a positive constant \( \lambda \) such that

\[
\sum_{i=1}^{N} a_i(x, \xi) \xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}
\]

holds for all \( \xi \in \mathbb{R}^N \) and almost every \( x \in \Omega \).

\textbf{(H₂)} – Growth restriction:

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\[ |a_i(x, \xi)| \leq \gamma(d_i(x) + |\xi_i^{p_i-1}|) \]

for almost every \( x \in \Omega \), \( \gamma \) is a positive constant for \( i = 1, ..., N \), \( d_i \) is a positive function in \( L^{p_i}(\Omega) \) and every \( \xi \in \mathbb{R}^N \).

\((H_3)\) - Monotonicity in \( \xi \in \mathbb{R}^N \):

\[ (a(x, \xi) - a(x, \eta)).(\xi - \eta) \geq 0, \]

for almost every \( x \in \Omega \) and for \( \xi, \eta \in \mathbb{R}^N \).

Due to the possible jumps of \( \beta \), problem \((E, f)\) enters to class of stefan problem for which there exists a large number of references, among them [19], [24]. Here we use the notion of renormalised solution developed by by DiPerna and Lions [21], for first order equations for \( L^1 \)-data in [30], and for Radon mesure data in [20]. It was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type, we refer to [14], [15].

Our problem has been studied in variable exponents spaces and Orlicz by Wittbold et al. [33], [25] and in the weighted Sobolev spaces by Akdim and Allalou [3]. Other works in this direction can be found in [4],[11],[2]. Our objective in this work is to prove an existence result of \((E, f)\) in anisotropic Sobolev spaces, this notion were introduced by Nikolskii [29], and Troisi [32]. The main tools in our proofs are Poincaré inequality and the embedding for anisotropic Sobolev spaces. It would be interesting to refer to some Embedding theorems for anisotropic Sobolev-Orlicz spaces [27] and a fully anisotropic Sobolev Inequality established by Cianchi in [18].

The paper is organized as follows: In Section 2, we recall the standard framework of anisotropic Sobolev spaces and some notations which will be used frequently. In Section 3, we introduce the notion of weak and also renormalized solution for the problem \((E, f)\) for any \( L^1 \)-data. In Section 4, we give our main results on the existence and uniqueness of renormalized solutions and we discuss the existence of weak solutions. Section 5 is devoted to the case where \( f \in L^\infty(\Omega) \), we prove the existence of a renormalized solution. Based on this result, the existence and uniqueness of a renormalized solution in the case where \( f \in L^1(\Omega) \) is shown in Section 6. In Section 7, we will prove the existence of a weak solution. Finally, we give an example for illustrating our abstract result.

2. Function spaces

2.1. Anisotropic Sobolev spaces. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( (N \geq 2) \) and let \( 1 \leq p_1, ..., p_N < \infty \) be \( N \) real numbers, \( p^+ = \max(p_1, ..., p_N) \), \( p^- = \min(p_1, ..., p_N) \) and \( \vec{p} = (p_1, ..., p_N) \). The anisotropic spaces (see [32])

\[ W^{1, \vec{p}}(\Omega) = \{ u \in W^{1,1}(\Omega) : \partial_i u \in L^{p_i}(\Omega), i = 1, ..., N \}. \]

is a Banach space with respect to norm
\[ \|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}. \]

The space \( W^{1,p}_0(\Omega) \) is the closure of \( C^\infty_0(\Omega) \) with respect to this norm.

The dual space of anisotropic Sobolev space \( W^{1,p}_0(\Omega) \) is equivalent to \( W^{-1,p'}(\Omega), \) where \( p' = (p_1',...,p'_N) \) and \( p_i' = \frac{p_i}{p_i - 1} \) for all \( i = 1,...,N. \)

We recall now a Poincaré-type inequality:

Let \( u \in W^{1,p}_0(\Omega), \) then for every \( q \geq 1 \) there exists a constant \( C_p \) (depending on \( q \) and \( i \)) such that

\[ \|u\|_{L^q(\Omega)} \leq C_p \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \text{ for } i = 1,...,N. \] (2.1)

Moreover a Sobolev-type inequality holds. Let us denote by \( \overline{p} \) the harmonic mean of these numbers, i.e. \( \frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}. \) Let \( u \in W^{1,\overline{p}}_0(\Omega) \). It follows from [32] that there exists a constant \( C_s \) such that

\[ \|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}, \] (2.2)

where \( q = \overline{p}^* = \frac{N\overline{p}}{N - \overline{p}} \) if \( \overline{p} < N \) or \( q \in [1, +\infty) \) if \( \overline{p} \geq N. \) On the right-hand side of (2.2) it is possible to replace the geometric mean by the arithmetic mean: let \( a_1, ..., a_N \) be positive numbers, it holds

\[ \prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} a_i, \]

which implies by 2.2 that

\[ \|u\|_{L^q(\Omega)} \leq C_s \frac{N}{\overline{p}} \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}. \] (2.3)

Note that when the following inequality holds

\( \overline{p} < N, \)

inequality (2.3) implies the continuous embedding of the space \( W^{1,\overline{p}}_0(\Omega) \) into \( L^q(\Omega) \) for every \( q \in [1,\overline{p}^*]. \) On the other hand, the continuity of the embedding \( W^{1,\overline{p}}_0(\Omega) \hookrightarrow L^{p^+}(\Omega) \) with \( p^+ := \max\{p_1, ..., p_N\} \) relies on inequality 2.1.

It may happen that \( \overline{p}^* < p^+ \) if the exponents \( p_i \) are closed enough, then \( p_\infty := \max\{\overline{p}^*, p^+\} \) turns out to be the critical exponent in the anisotropic Sobolev embedding (see [32]).

**Proposition 2.1.** If the condition 2.4 holds, then for \( q \in [1,p_\infty] \) there is a continuous embedding \( W^{1,\overline{p}}_0(\Omega) \hookrightarrow L^q(\Omega). \) For \( q < p_\infty \) the embedding is compact.

\[ W^{1,\overline{p}}_0(\Omega) \hookrightarrow L^q(\Omega). \] (2.5)
2.2. Notations and functions. Before we discuss the concept of solution we introduce some notations and functions that will be frequently used.

We begin by introducing the truncature operator. For given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ as

$$T_k (r) = \begin{cases} 
-k, & \text{if } r \leq -k, \\
r, & \text{if } |r| < k, \\
k, & \text{if } r \geq k,
\end{cases}$$

**Figure 1.** Truncation function

and for $r \in \mathbb{R}$, let us define the functions: $r \rightarrow r^+ := \max(r, 0)$ and $r \rightarrow \text{sign}_0(r)$ the usual sign function which is defined by

$$r \rightarrow \text{sign}_0(r) := \begin{cases} 
-1, & \text{on } ]-\infty, 0[, \\
1, & \text{on } ]0, \infty[, \\
0, & \text{if } r = 0.
\end{cases}$$

and

$$r \rightarrow \text{sign}_0^+(r) := \begin{cases} 
1, & \text{if } r > 0, \\
0, & \text{if } r \leq 0.
\end{cases}$$

Let $h_l : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $h_l(r) := \min((l + 1 - |r|)^+, 1)$ for each $r \in \mathbb{R}$.

**Figure 2.** Function $h_l(r)$
For $\sigma > 0$, we define $H^+_\sigma : \mathbb{R} \to \mathbb{R}$ by
\[
H^+_\sigma (r) := \begin{cases} 
0 \text{, if } r < 0, \\
\frac{1}{\sigma} r \text{, if } 0 \leq r \leq \sigma, \\
1 \text{, if } r > \sigma.
\end{cases}
\]
and $H_\sigma : \mathbb{R} \to \mathbb{R}$ by
\[
H_\sigma (r) := \begin{cases} 
-1 \text{, if } r < -\sigma, \\
\frac{1}{\sigma} r \text{, if } -\sigma \leq r \leq \sigma, \\
1 \text{, if } r > \sigma.
\end{cases}
\]

3. Notion of solutions

3.1. Weak solutions.

**Definition 3.1.** A weak solution of $(E, f)$ is a pair of functions $(u, b) \in W^{1,\frac{p}{p}}_0 (\Omega) \times L^1 (\Omega)$ satisfying $F(u) \in (L^1_{loc}(\Omega))^N$, $b \in \beta(u)$ almost everywhere in $\Omega$ and
\[
b - \text{div}(a(x, Du) + F(u)) = f \text{ in } D'(\Omega).
\] (3.1)

3.2. Renormalized solutions.

**Definition 3.2.** A renormalized solution of $(E, f)$ is a pair of functions $(u, b)$ satisfying the following conditions:

\[(R_1)\] $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1 (\Omega), u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

\[(R_2)\] For each $k > 0$, $T_k (u) \in W^{1,\frac{p}{p}}_0 (\Omega)$ and
\[
\int_{\Omega} b \cdot h(u) \phi + \int_{\Omega} (a(x, Du) + F(u)) \cdot D(h(u)) \phi = \int_{\Omega} f h(u) \phi, \tag{3.2}
\]
holds for all $h \in C^1_c (\mathbb{R})$ and all $\phi \in W^{1,\frac{p}{p}}_0 (\Omega) \cap L^\infty (\Omega)$.

\[(R_3)\] $\int_{\left\{ k < |u| < k + 1 \right\}} a(x, Du) \cdot Du \to 0$ as $k \to \infty$.

4. Main results

In this section, we will first state the existence and uniqueness of renormalized solutions for $(E, f)$. Then, we will prove that the renormalized solution of $(E, f)$ is a weak solution.

**Theorem 4.1.** For $f \in L^1 (\Omega)$, there exists at least one renormalized solution $(u, b)$ of $(E, f)$.

**Theorem 4.2.** Let $\beta : \Omega \times \mathbb{R} \to 2^\mathbb{R}$ be such that $\beta(x,.)$ is strictly monotone for almost every $x \in \Omega$. For $f \in L^1 (\Omega)$, let $(u, b), (\tilde{u}, \tilde{b})$ be renormalized solutions of $(E, f)$. Then $u = \tilde{u}$ and $b = \tilde{b}$. 
Proposition 4.3. Let \((u, b)\) be a renormalized solution of \((E, f)\) for \(f \in L^\infty(\Omega)\). Then \(u \in W^{1, \overrightarrow{p}}(\Omega) \cup L^\infty(\Omega)\) and thus, in particular, \(u\) is a weak solution of \((E, f)\).

To prove Theorem 4.1, we will introduce and solve approximation problems. To this end, for \(f \in L^1(\Omega)\) and \(m, n \in \mathbb{N}\) we define \(f_{m,n} : \Omega \to \mathbb{R}\) by

\[
f_{m,n}(x) = \max(\min(f(x), m), -n)
\]

for almost every \(x \in \Omega\). Clearly, \(f_{m,n} \in L^\infty(\Omega)\) for each \(m, n \in \mathbb{N}\), \(|f_{m,n}(x)| \leq |f(x)| \text{ a.e. in } \Omega\), hence \(\lim_{n \to \infty} \lim_{m \to \infty} f_{m,n} = f\) in \(L^1(\Omega)\) and almost everywhere in \(\Omega\).

The next theorem will give us existence of renormalized solutions \((u_{m,n}, b_{m,n})\) of \((E, f_{m,n})\) for each \(m, n \in \Omega\).

5. Case where \(f \in L^\infty(\Omega)\)

Theorem 5.1. For \(f \in L^\infty(\Omega)\), there exists at least one renormalized solution \((u, b)\) of \((E, f)\).

The following section will be devoted to prove of Theorem 5.1, and we will divide it into several steps.

5.1. Approximate solution for \(L^\infty\)- data. First we will introduce the approximate problem to \((E, f)\) for \(f \in L^\infty(\Omega)\) and for which the existence can be proved by standard variational arguments. For \(0 < \varepsilon \leq 1\), let \(\beta_\varepsilon : \mathbb{R} \mapsto \mathbb{R}\) be the Yosida approximation of \(\beta\) (see [16]). We introduce the operators

\[
A_{1,\varepsilon} : W_0^{1, \overrightarrow{p}}(\Omega) \to W^{-1, \overrightarrow{p}}(\Omega),
\]

\[
u \to \beta_\varepsilon(T_{1/\varepsilon}(\nu)) + \varepsilon \arctan(\nu) - \text{div}(a(x, Du))
\]

and

\[
A_{2,\varepsilon} : W_0^{1, \overrightarrow{p}}(\Omega) \to W^{-1, \overrightarrow{p}}(\Omega),
\]

\[
u \to -\text{div}F(T_{1/\varepsilon}(\nu)).
\]

Because of \((H_2) - (H_3)\), \(A_{1,\varepsilon}\) is well-defined and monotone (see [28] for instance). Since \(\beta_\varepsilon \circ T_{1/\varepsilon}\) is bounded and continuous and thanks to the growth condition \((H_2)\) on \(a\), it follows that \(A_{1,\varepsilon}\) is hermicontinuous (see [28]). From the continuity and boundedness of \(F \circ T_{1/\varepsilon}\), it follows that \(A_{2,\varepsilon}\) is strongly continuous. Therefore the operator \(A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}\) is pseudomonotone. Using the monotonicity of \(\beta_\varepsilon\), the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term \(\int_\Omega F(T_{1/\varepsilon}(\nu))Du\), we show by using similar arguments as in [12] that \(A_\varepsilon\) is coercive and bounded. Then it follows from [28] Theorem 2.7 that \(A_\varepsilon\) is surjective, i.e., for each \(0 < \varepsilon \leq 1\) and \(f \in W^{-1, \overrightarrow{p}}(\Omega)\) there exists a solution \(u_\varepsilon \in W_0^{1, \overrightarrow{p}}(\Omega)\) of the problem

\[
(E_\varepsilon, f) \left\{ \begin{array}{l}
\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon) - \text{div}(a(x, Du_\varepsilon) + F((T_{1/\varepsilon}(u_\varepsilon)))) = f \text{ in } \Omega, \\
u_\varepsilon = 0 \text{ on } \partial\Omega,
\end{array} \right.
\]
such that the following inequality holds for all $\phi \in W^{-1,p}_{0}(\Omega)$

$$\int_{\Omega} (\beta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon})) + \varepsilon \arctan(u))\phi + \int_{\Omega} (a(x, D(u_{\varepsilon})) + F(T_{1/\varepsilon}(u_{\varepsilon})))D\phi = <f, \phi>$$

where $<.,.>$ denotes the duality pairing between $W^{-1,p}_{0}(\Omega)$ and $W^{-1,p}_{0}(\Omega)$.

**Proposition 5.2.** For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^{\infty}(\Omega)$, let $u_{\varepsilon}, \tilde{u}_{\varepsilon} \in W^{-1,p}_{0}(\Omega)$ be solutions of $(E_{\varepsilon}, f)$ and $(E_{\varepsilon}, \tilde{f})$, respectively, then, the following comparison principle holds:

$$\varepsilon \int_{\Omega} (\arctan(u_{\varepsilon}) - \arctan(\tilde{u}_{\varepsilon}))^+ \leq \int_{\Omega} (f - \tilde{f}) \text{sign} \, n_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}).$$

**Proof.** We use the test function $\varphi = H_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon})$ in the weak formulation (5.1) for $u_{\varepsilon}$ and $\tilde{u}_{\varepsilon}$. Substracting the resulting inequalities, we obtain

$$I_{1,\delta}^1 + I_{1,\delta}^2 + I_{1,\delta}^3 + I_{1,\delta}^4 = I_{1,\delta}^5,$$

where

$$I_{1,\delta}^1 = \int_{\Omega} (\beta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon})) - \beta_{\varepsilon}(T_{1/\varepsilon}(\tilde{u}_{\varepsilon})))H_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}),$$

$$I_{1,\delta}^2 = \int_{\Omega} (\varepsilon \arctan(u_{\varepsilon}) - \varepsilon \arctan(\tilde{u}_{\varepsilon}))H_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}),$$

$$I_{1,\delta}^3 = \int_{\Omega} a(x, Du_{\varepsilon}) - a(x, D\tilde{u}_{\varepsilon})DH_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}),$$

$$I_{1,\delta}^4 = \int_{\Omega} (F(T_{1/\varepsilon}(u_{\varepsilon})) - F(T_{1/\varepsilon}(\tilde{u}_{\varepsilon})))DH_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}),$$

$$I_{1,\delta}^5 = \int_{\Omega} (f - \tilde{f})H_{\delta}^+(u_{\varepsilon} - \tilde{u}_{\varepsilon}).$$

Passing to the limit with $\sigma \to 0$, (5.2) follows. \hfill \Box

**Remark 5.3.** Let $f, \tilde{f} \in L^{\infty}(\Omega)$ be such that $f \leq \tilde{f}$ almost everywhere in $\Omega$, $\varepsilon > 0$ and $u_{\varepsilon}, \tilde{u}_{\varepsilon} \in W^{-1,p}_{0}(\Omega)$ be solutions of $(E_{\varepsilon}, f)$ and $(E_{\varepsilon}, \tilde{f})$, respectively, then it is an immediate consequence of Propson 5.2 is that $u_{\varepsilon} \leq \tilde{u}_{\varepsilon}$ almost everywhere in $\Omega$. Furthermore, from the monotonocity of $\beta_{\varepsilon} \circ T_{1/\varepsilon}$ it follows that also

$$\beta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon})) \leq \beta_{\varepsilon}(T_{1/\varepsilon}(\tilde{u}_{\varepsilon}))$$

a.e. in $\Omega$. 
5.2. A priori estimates.

**Lemma 5.4.** For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W^{1,\frac{p}{p}}_0(\Omega)$ be a solution of $(E_\varepsilon, f)$. Then

i) There exists a constant $C_1 = C_1(||f||_\infty, \lambda, p, N) > 0$, not depending on $\varepsilon$, such that

$$|||u_\varepsilon||| \leq C_1.$$  (5.3)

ii) for all $0 < \varepsilon \leq 1$, we have

$$||\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))||_\infty \leq ||f||_\infty$$  (5.4)

iii) for all $0 < \varepsilon \leq 1$ and all $l, k > 0$, we have

$$\int_{\{l \leq |u| \leq k + l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq k \int_{\{|u_\varepsilon| > l\}} |f|.$$  (5.5)

**Proof.** i) Taking $u_\varepsilon$ as a test function in (5.1) we obtain

$$\int_\Omega (\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon))u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon).Du_\varepsilon dx + \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by $(H_1)$, we have

$$\lambda \sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^p dx \leq \sum_{i=1}^N \int_\Omega a_i(x, Du_\varepsilon).\frac{\partial u_\varepsilon}{\partial x_i} dx \leq \int_\Omega f u_\varepsilon dx \leq C||f||_\infty \sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^p dx)^{1/p}.$$

due to the Hölder inequality. Thus $|||u_\varepsilon|||^p \leq C_2|||u_\varepsilon|||$, where $C_2$ is a positive constant. Then we can deduce that $u_\varepsilon$ remains bounded in $W^{1,\frac{p}{p}}_0(\Omega)$ i.e.,

$$|||u_\varepsilon||| \leq C_1.$$  (5.6)

ii) Taking $\frac{1}{\sigma}[T_{k+\sigma}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))]$ as a test function in (5.1), passing to the limit with $\sigma \to 0$ and choosing $k > ||f||_\infty$, we obtain ii).

iii) For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (5.1).

Using $\int_\Omega a(x, Du_\varepsilon).DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_{\{l \leq |u_\varepsilon| \leq l + k\}} a(x, Du_\varepsilon).Du_\varepsilon dx$, and as the first term on the left-hand side is nonnegative and the convection term vanishes, we get

$$\int_{\{l \leq |u_\varepsilon| \leq k + l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx \leq \int_{\{|u_\varepsilon| > l\}} |f| dx.$$  (5.7)

**Remark 5.5.** For $k > 0$, from iii) in Lemma 5.4, we deduce that

$$|\{ |u_\varepsilon| \geq l\}| \leq \frac{C_2}{l^{1-\frac{1}{p}}}$$  (5.7)

$$\int_{\{l \leq |u_\varepsilon| \leq k + l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq k||f||_\infty |\{ |u_\varepsilon| \geq l\}| \leq \frac{C_2(k)}{l^{\frac{1}{p}-1}}$$  (5.8)

for any $0 < \varepsilon \leq 1$ and a constant $C_2(k) > 0$ not depending on $\varepsilon$. 
Indeed, let \( l > 0 \) large enough we have:

\[
|\{ |u_\varepsilon| \geq l \}| = \int_{\{ |u_\varepsilon| \geq l \}} |T_i(u_\varepsilon)| \, dx \leq C \left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_i(u_\varepsilon)}{\partial x_i} \right|^{p_i} \right)^{1/p_i} \leq C_2 l^{1/p}
\]

with implies \( |\{ |u_\varepsilon| \geq l \}| \leq C_2 l^{1/p-1} \). Then

\[
\lim_{l \to +\infty} |\{ |u_\varepsilon| \geq l \}| = 0.
\]

Therefore, (5.7) follows from (5.8).

5.3. Basic convergence results.

Lemma 5.6. For \( 0 < \varepsilon \leq 1 \) and \( f \in L^\infty(\Omega) \), let \( u_\varepsilon \in W_0^{1,p}(\Omega) \) be a solution of \((E_\varepsilon, f)\). There exist \( u \in W_0^{1,p}(\Omega), b \in L^\infty(\Omega) \) such that for a not relabeled subsequence of \((u_\varepsilon)_{0<\varepsilon \leq 1}\) as \( \varepsilon \to 0 \):

\[
u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega,
\]

\[
T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,p}(\Omega) \text{ and strongly in } L^q(\Omega),
\]

\[
\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b \quad \text{in } L^\infty(\Omega).
\]

Moreover, for any \( k > 0 \),

\[
D T_k(u_\varepsilon) \rightharpoonup D T_k(u) \quad \text{in } \prod_{i=1}^{N} L^{p_i}(\Omega),
\]

\[
a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{in } \prod_{i=1}^{N} L^{p_i'}(\Omega).
\]

Proof. By combining Lemma 5.4 and Remark 5.5, we obtain (5.11). From (5.7), (5.3) and (2.5), we deduce with a classical argument (see [1]) that for a subsequence still indexed by \( \varepsilon \), (5.9) – (5.10) and (5.12) hold as \( \varepsilon \) tends to 0, where \( u \) is a measurable function defined on \( \Omega \).

It is left to prove (5.13). For this, by \((H_2)\) and (5.3) it follows that given any subsequence of \((a(x, DT_k(u_\varepsilon)))\), there exists a subsequence, still denoted by \((a(x, DT_k(u_\varepsilon)))\), such that \( a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k \) in \( \prod_{i=1}^{N} L^{p_i'}(\Omega) \). We will prove that \( \Phi_k = a(x, DT_k(u)) \) a.e. on \( \Omega \). The proof consists of three steps.

**Step 1:** For every \( h \in W^{1,\infty}(\mathbb{R}), h \leq 0 \) and \( \text{supp}(h) \) compact, we will prove that

\[
\limsup_{\varepsilon \to 0} \int_{\Omega} a(x, DT_k(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] \, dx \leq 0.
\]

Taking \( h_\varepsilon(T_k(u_\varepsilon) - T_k(u)) \) as a test function in (5.1), we have

\[
\int_{\Omega}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon))h_\varepsilon(T_k(u_\varepsilon) - T_k(u)) + \int_{\Omega} a(x, D(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] + \int_{\Omega} F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] = \int_{\Omega} fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)).
\]

(5.15)
Using $|h_\varepsilon(T_k(u_\varepsilon) - T_k(u))| \leq 2k\|h\|_\infty$, by Lebesgue’s dominated convergence theorem we find that $\lim_{\varepsilon \to 0} \int_\Omega f(h_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0$ and then $\lim_{\varepsilon \to 0} \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0$. By using the same arguments as in [4], we can prove that

$$\limsup_{\varepsilon \to 0} \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)).[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] dx \geq 0.$$ 

Passing to the limit in (5.15) and using the above results, we obtain (5.14).

Step 2: We now prove that for every $k > 0$,

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).[D(T_k(u_\varepsilon) - DT_k(u))] dx \leq 0. \quad (5.16)$$

Indeed, for $k > l$, take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (5.1). Letting $\varepsilon \to 0$ and then $l \to \infty$, we obtain

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).[D(T_k(u_\varepsilon) - DT_k(u))] dx = E_1 + E_2 + E_3.$$ 

where 

$$E_1 = \int_{\{|u_\varepsilon| \leq k\}} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)).[DT_k(u_\varepsilon) - DT_k(u)] dx,$$

$$E_2 = \int_{\{|u_\varepsilon| > k\}} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)).(-DT_k(u)) dx,$$

$$E_3 = \int_\Omega h'_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx.$$ 

Since $l > k$, on the set $\{|u_\varepsilon| \leq k\}$ we have $h_l(u_\varepsilon) = 1$ so that we can write

$$\limsup_{\varepsilon \to 0} E_1 = \limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).(DT_k(u_\varepsilon) - DT_k(u)) dx.$$

For $E_2$, using Lebesgue’s dominated convergence theorem, we get

$$\lim_{\varepsilon \to 0} E_2 = \int_{\{|u_\varepsilon| > k\}} h_l(u_\varepsilon)\Phi_{l+1}.DT_k(u) dx = 0.$$ 

For $E_3$, we have

$$\int_\Omega h'_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx \leq 2k \int_{\{|l-\varepsilon| \leq t+1\}} a(x, Du_\varepsilon) Du_\varepsilon dx.$$ 

Using (5.8), we deduce that

$$\limsup_{l \to \infty} \limsup_{\varepsilon \to 0} (-\int_\Omega h'_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx) \leq 0.$$ 

Applying (5.14) with $h$ replaced by $h_l$, $l > k$, we get

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).[DT_k(u_\varepsilon) - DT_k(u)] dx \leq \limsup_{\varepsilon \to 0} (-\int_\Omega h'_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx).$$
In this step, we prove by monotonicity arguments that for $k > 0$, $\Phi_k = a(x, DT_k(u))$ for almost every $x \in \Omega$. Let $\phi \in D(\Omega)$ and $\alpha \in \mathbb{R}$. Using (5.16), we have

$$\alpha \lim_{\varepsilon \to 0} \int_{\Omega} a(x, DT_k(u_\varepsilon)).D\phi dx \geq \alpha \int_{\Omega} a(x, D(T_k(u) - \alpha \phi)).D\phi dx.$$ 

Dividing by $\alpha > 0$ and $\alpha < 0$ and letting $\alpha \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega} a(x, DT_k(u_\varepsilon)).D\phi dx = \int_{\Omega} a(x, DT_k(u)).D\phi dx. \quad (5.17)$$

This means that for all $k > 0$, $\int_{\Omega} \Phi_k.D\phi dx = \int_{\Omega} a(x, DT_k(u))$, and then $\Phi_k = a(x, DT_k(u))$ in $D'(\Omega)$ for all $k > 0$. Hence $\Phi_k = a(x, DT_k(u))$ a.e. in $\Omega$ and then $a(x, DT_k(u_\varepsilon)) \to a(x, DT_k(u))$ weakly in $\prod_{i=1}^{N} L^p_i(\Omega)$. 

**Remark 5.7.** As an immediate consequence of (5.16) and $(H_3)$ we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega} a(x, DT_k(u_\varepsilon) - a(x, DT_k(u)).(DT_k(u_\varepsilon) - T_k(u)) = 0. \quad (5.18)$$

Let us see finally that

$$\lim_{l \to \infty} \int_{l < |u| < l+1} a(x, DT(u))dx = 0. \quad (5.19)$$

Endeed, for any $l \geq 0$ fixed we have

$$\int_{l < |u| < l+1} a(x, D(u_\varepsilon)).D(u_\varepsilon)dx = \int_{\Omega} a(x, DT_{l+1}(u_\varepsilon).(DT_{l+1}(u_\varepsilon) - DT_l(u_\varepsilon))dx$$

$$= \int_{\Omega} a(x, DT_{l+1}(u_\varepsilon)).DT_{l+1}(u_\varepsilon)dx - \int_{\Omega} a(x, DT_l(u_\varepsilon)).DT_l(u_\varepsilon)dx.$$ 

By (5.18) and passing to the limit as $\varepsilon \to 0$ for fixed $l \geq 0$ we obtain

$$\lim_{\varepsilon \to 0} \int_{l < |u| < l+1} a(x, D(u_\varepsilon)).D(u_\varepsilon)dx = \int_{\Omega} a(x, DT_{l+1}(u)).DT_{l+1}(u)dx$$

$$- \int_{\Omega} a(x, DT_l(u)).DT_l(u)dx = \int_{\{l < |u| < l+1\}} a(x, Du).D(u)dx. \quad (5.20)$$

Therefore, taking $l \to +\infty$ in (5.20) and using the estimate (5.8) show that satisfies $(R_3)$.

**5.4. Proof of the existence result.** We are now in position to conclude the proof of our main result presented in Theorem 5.1:

**Proof.** Let $h \in C^1_c(\mathbb{R})$ and $\varphi \in W^{1,\vec{p}}_0(\Omega) \cap L^\infty(\Omega)$. Taking $h_l(u_\varepsilon)h(u)(\varphi)$ as a test function in (5.1), we obtain

$$I_{\varepsilon,l}^1 + I_{\varepsilon,l}^2 + I_{\varepsilon,l}^3 + I_{\varepsilon,l}^4 = I_{\varepsilon,l}^5 \quad (5.21)$$

where

$$I_{\varepsilon,l}^1 = \int_{\Omega} \beta_\varepsilon(T_{l/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\varphi,$$
\[ I_{\varepsilon,l}^2 = \varepsilon \int_{\Omega} \arctan(u_\varepsilon)h_t(u_\varepsilon)h(u)\varphi, \]
\[ I_{\varepsilon,l}^3 = \int_{\Omega} a(x,Du_\varepsilon).D(h_t(u_\varepsilon)h(u))\varphi, \]
\[ I_{\varepsilon,l}^4 = \int_{\Omega} F(T_{1/\varepsilon}(u_\varepsilon)).D(h_t(u_\varepsilon)h(u))\varphi, \]
\[ I_{\varepsilon,l}^5 = \int_{\Omega} f h_t(u_\varepsilon)h(u)\varphi. \]

**Step 1:** Letting \( \varepsilon \to 0 \) obviously, we have

\[ \lim_{\varepsilon \to 0} I_{\varepsilon,l}^2 = 0. \]  

Using the convergence results (5.9), (5.11) from Lemma 5.6 we can immediately calculate the following limits:

\[ \lim_{\varepsilon \to 0} I_{\varepsilon,l}^1 = \int_{\Omega} bh_t(u)h(u)\varphi, \]
\[ \lim_{\varepsilon \to 0} I_{\varepsilon,l}^5 = \int_{\Omega} fh_t(u)h(u)\varphi. \]

We write \( I_{\varepsilon,l}^3 = I_{\varepsilon,l}^{3,1} + I_{\varepsilon,l}^{3,2} \) where

\[ I_{\varepsilon,l}^{3,1} = \int_{\Omega} h_t'(u_\varepsilon)a(x,Du_\varepsilon).Du_\varepsilon h(u)\varphi, \quad I_{\varepsilon,l}^{3,2} = \int_{\Omega} h_t(u_\varepsilon)a(x,Du_\varepsilon)D(h(u))\varphi. \]

Using (5.8), we get the estimate

\[ |\lim_{\varepsilon \to 0} I_{\varepsilon,l}^{3,1}| \leq ||h||_{\infty}||\varphi||_{\infty}C_2l^{-(1-1/\beta)}. \]  

By Lebesgue’s dominated convergence theorem it follows that for any \( i \in \{1, ..., N\} \), we have

\[ h_t(u_\varepsilon)\frac{\partial}{\partial x_i} (h(u)\varphi) \to h_t(u)\frac{\partial}{\partial x_i} (h(u)\varphi) \quad \text{in} \quad L^{p_i} \quad \text{as} \quad \varepsilon \to 0. \]

Keeping in mind that \( I_{\varepsilon,l}^{3,2} = \int_{\Omega} h_t(u_\varepsilon)a(x,DT_{t+1}(u_\varepsilon)).D(h(u))\varphi \) and by using (5.13), we get

\[ \lim_{\varepsilon \to 0} I_{\varepsilon,l}^{3,2} = \int_{\Omega} h_t(u)a(x,DT_{t+1}(u)).D(h(u))\varphi. \]  

Let us write \( I_{\varepsilon,l}^4 = I_{\varepsilon,l}^{4,1} + I_{\varepsilon,l}^{4,2} \), where

\[ I_{\varepsilon,l}^{4,1} = \int_{\Omega} h_t'(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon h(u)\varphi, \]
\[ I_{\varepsilon,l}^{4,2} = \int_{\Omega} h_t(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).D(h(u))\varphi. \]
For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I_{\varepsilon,l}^{4,1} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon)).h(u)\varphi. \quad (5.27)$$

Using the Gauss-Green Theorem for Sobolev functions in (5.27), we get for all $\varepsilon < \varepsilon_0(l)$,

$$I_{\varepsilon,l}^{4,1} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)dr.D(h(u)\varphi). \quad (5.28)$$

Now, using (5.9) and the Gauss-Green Theorem, after letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} I_{\varepsilon,l}^{4,1} = \int_\Omega h'_l(u)F(u).Du(h)\varphi. \quad (5.29)$$

Choosing $\varepsilon$ small enough, we can write

$$I_{\varepsilon,l}^{4,2} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)).D(h(u)\varphi), \quad (5.30)$$

and conclude that

$$\lim_{\varepsilon \to 0} I_{\varepsilon,l}^{4,2} = \int_\Omega h_l(u)F(u).D(h(u)\varphi). \quad (5.31)$$

Step 2: Passage to the limit with $l \to \infty$.

Combining (5.21) and (5.22) – (5.31) we deduce that

$$I_1^l + I_2^l + I_3^l + I_4^l + I_5^l = I_6^l \quad (5.32)$$

where

$$I_1^l = \int_\Omega bh_l(u)h(u)\varphi, \quad I_2^l = \int_\Omega h_l(u)a(x, DT_{l+1}(u)).D(h(u)\varphi),$$

$$|I_3^l| \leq C_2 l^{-(1-1/\beta)}||h||_\infty ||\varphi||_\infty, \quad I_4^l = \int_\Omega h_l(u)F(u).D(h(u)\varphi),$$

$$I_5^l = \int_\Omega h'_l(u)F(u).Duh(u)\varphi, \quad I_6^l = \int_\Omega fh_l(u)h(u)\varphi.$$  

Obviously, we have

$$\lim_{\varepsilon \to \infty} I_3^l = 0. \quad (5.33)$$

Choosing $m > 0$ such that $supp h \subset [-m, m]$, we can replace $u$ by $T_m(u)$ in $I_1^l, I_2^l, \ldots, I_6^l$, and

$$h'_l(u) = h'_l(T_m(u)) = 0 \text{ if } l + 1 > m, \quad h_l(u) = h_l(T_m(u)) = 0 \text{ if } l > m.$$  

Therefore, letting $l \to \infty$ and combining (5.32) with (5.33) we obtain

$$\int_\Omega bh(u)\varphi + \int_\Omega (a(x, Du) + F(u)).D(h(u)\varphi) = \int_\Omega fh(u)\varphi \quad (5.34)$$

for all $h \in C_0^1(\mathbb{R})$ and all $\varphi \in W^{1,\frac{m}{0.5}}(\Omega) \cap L^\infty(\Omega)$.

Step 3: Subdifferential argument
It is left to prove that \( u(x) \in D(\beta(x)) \) and \( b(x) \in \beta(u(x)) \) for almost all \( x \in \Omega \).

Since \( \beta \) is a maximal monotone graph, there exist a convex, l.s.c and proper function \( j : \mathbb{R} \to [0, \infty] \), such that

\[
\beta(r) = \partial j(r) \quad \text{for all } r \in \mathbb{R}.
\]

According to [16], for \( 0 < \varepsilon \leq 1 \), \( j_\varepsilon : \mathbb{R} \to \mathbb{R} \) defined by

\[
j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds
\]

has the following properties as in [33]

i) For any \( 0 < \varepsilon \leq 1 \), \( j_\varepsilon \) is convex and differentiable for all \( r \in \mathbb{R} \), such that

\[
j_\varepsilon'(r) = \beta_\varepsilon(r) \quad \text{for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \leq 1.
\]

ii) \( j_\varepsilon(r) \to j(r) \) for all \( r \in \mathbb{R} \) as \( \varepsilon \to 0 \).

From i), it follows that for any \( 0 < \varepsilon \leq 1 \)

\[
j_\varepsilon(r) \geq j_\varepsilon(T_1(\varepsilon u)) + (r - T_1(\varepsilon u)) \beta_\varepsilon(T_1(\varepsilon u)) \quad (5.35)
\]

holds for all \( r \in \mathbb{R} \) and almost everywhere in \( \Omega \).

Let \( E \cup \Omega \) be an arbitrary measurable set and \( \chi_E \) its characteristic function. We fix \( \varepsilon_0 > 0 \). Multiplying (5.35) by \( h_t(\varepsilon u) \chi_E \), integrating over \( \Omega \) and using ii), we obtain

\[
j(r) \int_E h_t(u) \geq \int_E j_\varepsilon(T_{l+1}(u)) h_t(u) + (r - T_{l+1}h_t(u)) \beta_\varepsilon(T_{l+1}(u)) \quad (5.36)
\]

for all \( r \in \mathbb{R} \) and all \( 0 < \varepsilon < \min(\varepsilon_0, \frac{1}{l}) \).

As \( \varepsilon \to 0 \), taking into account that \( E \) arbitrary we obtain from (5.36)

\[
j(r)h_t(u) \geq j_\varepsilon(T_{l+1}(u)) h_t(u) + bh_t(u)(r - T_{l+1}(u)) \quad (5.37)
\]

for all \( r \in \mathbb{R} \) and almost everywhere in \( \Omega \).

Passing to the limit with \( l \to \infty \) and then with \( \varepsilon_0 \to 0 \) in (5.37) finally yields

\[
j(r) \geq j(u(x)) + b(x)(r - u(x)) \quad (5.38)
\]

for all \( r \in \mathbb{R} \) and almost everywhere in \( \Omega \), hence \( u \in D(\beta) \) and \( b \in \beta(u) \) for almost everywhere in \( \Omega \). With this last step the proof of Theorem 5.1 is concluded. \( \square \)

6. Case where \( f \in L^1(\Omega) \)

6.1. **Approximate solution for \( L^1 \)- data.** The comparison principle from proposition will be the tool in second approximation procedure. For \( f \in L^1(\Omega) \) and \( m, n \in \mathbb{N} \) let \( f_{m,n} \in L^\infty(\Omega) \) be defined as in Section 3. Using Proposition 4.3, we deduce that for any \( m, n \in \mathbb{N} \), there exists \( u_{m,n} \in W_0^{1,p} (\Omega) \), \( b_{m,n} \in L^\infty(\Omega) \), such that \( (u_{m,n}, b_{m,n}) \) is a renormalized solution of \((E, f_{m,n})\). Therefore

\[
\int_{\Omega} b_{m,n} h(u_{m,n}) \phi + \int_{\Omega} (a(x, Du_{m,n}) + F(u_{m,n})).D(h(u_{m,n})\phi) = f_{m,n} h(u_{m,n}) \phi \quad (6.1)
\]

holds for all \( m, n \in \mathbb{N}, h \in C_c^1(\mathbb{R}), \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \). In the next lemma, we give a priori estimates that will be important in the following:
Lemma 6.1. For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then,

i) For any $k > 0$ we have,

$$
\sum_{i=1}^{N} \int_{\Omega} |DT_{k}(u_{m,n})|^{p_i} \leq \frac{k}{\gamma} \|f\|_{1}
$$

(6.2)

ii) for any $k > 0$, there exists a constant $C_3(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$
\sum_{i=1}^{N} \int_{\Omega} |DT_{k}(u_{m,n})|^{p_i} \leq C_3(k).
$$

(6.3)

iii) For $m, n \in \mathbb{N}$, we have:

$$
\|b_{m,n}\|_{1} \leq \|f\|_{1}.
$$

(6.4)

Proof. For $l, k > 0$, we plug $h_{l}(u_{m,n})T_{k}(u_{m,n})$ as a test function in (6.1). Then i) and ii) follows with similar arguments as used in the proof of Lemma 5.4. To prove iii), we neglet the positve term

$$
\int_{\Omega_{|u_{m,n}|>k}} |b_{m,n}| \leq \int_{\Omega} |f|.
$$

(6.5)

and we find iii) by passing to the limit with $k \to 0$.

By definition we have

$$
f_{m,n} \leq f_{m+1,n} \quad \text{and} \quad f_{m,n+1} \leq f_{m,n}
$$

(6.7)

From Propostion 5.2 it follows that

$$
u_{m,n}^{\varepsilon} \leq u_{m+1,n}^{\varepsilon} \quad \text{and} \quad u_{m,n+1}^{\varepsilon} \leq u_{m,n}^{\varepsilon},
$$

(6.8)

almost everywhere in $\Omega$ for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$. Hence passing to the limit with $\varepsilon \to 0$ in (6.8) yields

$$
u_{m,n} \leq u_{m+1,n} \quad \text{and} \quad u_{m,n+1} \leq u_{m,n},
$$

(6.9)

almost everywhere in $\Omega$ for any $m, n \in \mathbb{N}$.

Setting $b_{\varepsilon} := \beta_{\varepsilon}(T_{1/\varepsilon}(u_{\varepsilon}))$, using (6.8), Remark 5.3 and the fact that $b_{m,n}^{\varepsilon} \to b_{m,n}$ in $L^{\infty}(\Omega)$ and since this convergence preserves order we get

$$
b_{m,n} \leq b_{m+1,n} \quad \text{and} \quad b_{m,n+1} \leq b_{m,n}
$$

(6.10)

almost everywhere in $\Omega$ for any $m, n \in \mathbb{N}$. By (6.10) and (6.4), for any $n \in \mathbb{N}$ there exist $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ and $m \to \infty$ in $L^1(\Omega)$ and almost everywhere and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost every where in $\Omega$. By (6.9), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for
any \( n \in \mathbb{N}, u_{m,n} \to u^n \) almost everywhere in \( \Omega \), where \( u^n : \Omega \to \mathbb{R} \) is a measurable function. In order to show that \( u \) is finite almost everywhere we will give an estimate on the level sets of \( u_{m,n} \) in the next lemma:

**Lemma 6.2.** For \( m, n \in \mathbb{N} \) let \( (u_{m,n}, b_{m,n}) \) be a renormalized solution of \((E, f_{m,n})\). Then, there exists a constant \( C_4 > 0 \), not depending on \( m, n \in \mathbb{N} \), such that

\[
|\{|u_{m,n}| \geq l\}| \leq C_4 l^{\frac{1}{p} - 1}
\]

for all \( l \geq 0 \).

**Proof.** With the same arguments as in remark 5.5 we obtain

\[
|\{|u_{m,n}|\}| \leq C(\bar{p}, N) \bar{p}^{-} \left( \sum_{i=1}^{N} \int_{\Omega} |DT_k(u_{m,n})|^p_i + |\Omega| \right)
\]

for all \( m, n \in \mathbb{N} \) where \( C(\bar{p}, N) \) is the constant from Sobolev embedding in (2.5).

Now we plug (6.2) into (6.12) to obtain (6.11). Note that, as \( (u_{m,n})_m \) is pointwise increasing with respect to \( m \),

\[
\lim_{m \to \infty} |\{|u_{m,n} \geq l\}| = |\{u^n \geq l\}|
\]

and

\[
\lim_{m \to \infty} |\{|u_{m,n} \leq -l\}| = |\{u^n \leq -l\}|.
\]

Combining (6.11) with (6.13) and (6.14) we get

\[
|\{u^n \leq -l\}| + |\{u^n > l\}| \leq C_4 l^{\frac{1}{p} - 1}
\]

for any \( l \geq 1 \), hence \( u^n \) is finite almost everywhere for \( n \in \mathbb{N} \). By the same arguments we get

\[
|\{u < -l\}| + |\{u > l\}| \leq C_4 l^{\frac{1}{p} - 1}
\]

from (6.15), hence \( u \) is finite almost everywhere. Now, since \( b_{m,n} \in \beta(u_{m,n}) \) almost everywhere in \( \Omega \) it follows by a subdifferential argument that \( b^n \in \beta(u^n) \) and \( b \in \beta(u) \) a.e. in \( \Omega \).

\[\square\]

**Remark 6.3.** If \( (u_{m,n}, b_{m,n}) \) is renormalized solution of \((E, f_{m,n})\), using

\[h_\nu(u_{m,n})T_k(u_{m,n} - T_l(u_{m,n}))\]

as a test function in (6.1), neglecting positive terms and passing to the limit with \( \nu \to \infty \) we obtain

\[
\int_{\{l < |u_{m,n}| < l+k\}} a(x, Du_{m,n}).Du_{m,n} \leq k \left( \int_{\{|u_{m,n}| > l\} \cap \{|f| < \sigma\}} |f| + \int_{\{|f| > \sigma\}} |f| \right)
\]

for any \( k, \sigma > 0, l \). Now applying (6.11) to (6.17), we find that

\[
\int_{\{l < |u_{m,n}| < l+k\}} a(x, Du_{m,n}).Du_{m,n} \leq \sigma kC_4 l^{\frac{1}{p} - 1} + k \int_{\{|f| > \sigma\}} |f|
\]

holds for any \( k, \sigma > 0, l \geq 0 \) uniformly in \( m, n \in \mathbb{N} \).
6.2. Basic convergence results.

**Lemma 6.4.** For \( m, n \in \mathbb{N} \) let \((u_{m,n}, b_{m,n})\) be a renormalized solution of \((E, F_{m,n})\). There exists a subsequence \((m(n))_n\) such that setting \(f_n := f_{m(n),n}, b_n := b_{m(n),n}, u_n := u_{m(n),n}\) we have

\[
u_n \to u \text{ almost everywhere in } \Omega. \tag{6.19}\]

Moreover, for any \( k > 0 \),

\[
T_k(u_n) \to T_k(u) \text{ in } W_0^{1, \overline{p}}(\Omega), \tag{6.20}
\]

\[
DT_k(u_n) \rightharpoonup DT_k(u) \text{ in } \prod_{i=1}^{N} L^{p_i}(\Omega), \tag{6.21}
\]

\[
a(x, DT_k(u_n)) \to a(x, DT_k(u)) \text{ in } \prod_{i=1}^{N} L^{p'_i}(\Omega), \tag{6.22}
\]
as \( n \to \infty \).

**Proof.** We construct a subsequence \((m(n))_n\), such that

\[
\arctan(u_{m(n),n}) \to \arctan(u),
\]

\[
b_n := b_{m(n),n} \to b,
\]

\[
f_n := f_{m(n),n} \to f
\]
as \( n \to \infty \) in \( L^1(\Omega) \) and almost everywhere in \( \Omega \). It follows that (6.19) and (6.20) hold. Combining (6.20) with (6.3) we get \( T_k(u) \in W_0^{1, \overline{p}}(\Omega), T_k(u_n) \to T_k(u) \in W_0^{1, \overline{p}}(\Omega) \) and (6.21) holds for any \( k > 0 \). From (6.2) and \((H_2)\), it follows that for fixed \( k > 0 \), given any subsequence of \((a(x, DT_k(u_{m(n)}))_n)\) there exists a subsequence, still denoted by such that \( a(x, DT_k(u_{m(n)})) \to \Phi_k \) in \( \prod_{i=1}^{N} L^{p'_i}(\Omega) \) as \( n \to \infty \).

Since \( h_i(u_n)(T_k(u_n) - T_k(u)) \) is an admissible test function in (6.1),

\[
\lim_{n \to \infty} \sup_{\Omega} \int a(x, DT_k(u_n))D(T_k(u_n) - T_k(u)) \leq 0. \tag{6.23}
\]

Then, (6.22) follows with the same arguments as in the proof of Lemma 5.6. \( \square \)

**Remark 6.5.** With the same arguments as in Remark 5.7, we have

\[
\lim_{n \to \infty} \int_{\Omega} a(x, DT_k(u_n)) - a(x, DT_k(u))(T_k(u_n) - T_k(u)) = 0, \tag{6.24}
\]

\[
\lim_{l \to \infty} \int_{\{l < |u| < l+1\}} a(x, Du)Du = 0. \tag{6.25}
\]
6.3. Conclusion of the proof of Theorem 4.1. It is left to prove that \((u, b)\) satisfies
\[
\int_{\Omega} b h(u) \phi + \int_{\Omega} (a(x, Du) + F(u)) \cdot D(h(u) \phi) = \int_{\Omega} f h(u) \phi.
\] (6.26)
for all \(h \in C^1_c(\mathbb{R})\) and \(\phi \in W^{1, \frac{p}{p}}_0(\Omega) \cap L^{\infty}(\Omega)\). To this end, we take \(h \in C^1_c(\mathbb{R})\) and \(\phi \in W^{1, \frac{p}{p}}_0(\Omega) \cap L^{\infty}(\Omega)\) arbitrary and plug \(h_l(u_n)h(u)\phi\) into (6.1) to obtain
\[
I_{n,l}^1 + I_{n,l}^2 + I_{n,l}^3 = I_{n,l}^4,
\] (6.27)
where
\[
I_{n,l}^1 = \int_{\Omega} b h_l(u_n)h(u)\phi,
\]
\[
I_{n,l}^2 = \int_{\Omega} a(x, Du_n) \cdot D(h_l(u_n)h(u)\phi),
\]
\[
I_{n,l}^3 = \int_{\Omega} F(u_n) \cdot D(h_l(u_n)h(u)\phi),
\]
\[
I_{n,l}^4 = \int_{\Omega} f h_l(u_n)h(u)\phi.
\]

**Step 1.** Passing to the limit as \(n \to \infty\), applying the convergence results from Lemma 6.4 we get
\[
\lim_{n \to \infty} I_{n,l}^1 = \int_{\Omega} b h_l(u)h(u)\phi, \quad \lim_{n \to \infty} I_{n,l}^4 = \int_{\Omega} f h_l(u)h(u)\phi.
\] (6.28)
Let us write
\[
I_{n,l}^2 = I_{n,l}^{2,1} + I_{n,l}^{2,2},
\] (6.29)
where
\[
I_{n,l}^{2,1} = \int_{\Omega} h_l(u_n) a(x, Du_n) \cdot D(h(u)\phi), \quad I_{n,l}^{2,2} = \int_{\Omega} h_l'(u_n) a(x, Du_n) \cdot Du_n h(u)\phi.
\] (6.30)
With similar arguments as in the proof of (5.26) it follows that
\[
\lim_{n \to \infty} I_{n,l}^{2,1} = \int_{\Omega} h_l(u) a(x, Du) \cdot D(h(u)\phi).
\] (6.31)
By (6.18), we get the estimate
\[
\left| \lim_{n \to \infty} I_{n,l}^{2,2} \right| \leq \|h\|_{\infty} \|\phi\|_{\infty} \left( \delta C_4 l^{\frac{1}{p} - 1} + \int_{\{|f| > \delta\}} |f| \right),
\] (6.32)
for all \(n \in \mathbb{N}\) and all \(l \geq 1, \delta > 0\). Next, we write
\[
I_{n,l}^3 = I_{n,l}^{3,1} + I_{n,l}^{3,2},
\]
where
\[
\lim_{n \to \infty} I_{n,l}^{3,1} = \int_{\Omega} h_l(u) F(u) \cdot D(h(u)\phi), \quad \lim_{n \to \infty} I_{n,l}^{3,2} = \int_{\Omega} h_l'(u) F(u) \cdot Du h(u)\phi.
\] (6.33)
follows with the same arguments as in (5.27) — (5.31).

**Step 2.** Passing to the limit as \( l \to \infty \). Combining (6.27) with (6.28)-(6.33) we get for all \( \delta > 0 \) and all \( l \geq 1 \)

\[
I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \tag{6.34}
\]

where

\[
I_l^1 = \int_{\Omega} bh_l(u)h(u)\phi, \quad I_l^2 = \int_{\Omega} h_l(u)a(x, DT_{l+1}(u)).D(h(u)\phi)
\]

\[
|I_l^3| \leq \|h\|_\infty \|\phi\|_\infty (\delta C_d l^\frac{1}{\sigma} - 1 + \int_{\{|f|>\delta\}} |f|),
\]

for any \( \delta > 0 \) and

\[
I_l^4 = \int_{\Omega} h_l'(u)F(u)h(u)\phi Du, \quad I_l^5 = \int_{\Omega} h_l(u)F(u).D(h(u)\phi), \quad |I_l^6| = \int_{\Omega} f h_l(u)h(u)\phi.
\]

Choosing \( m > 0 \) such that \( \text{supph} \subset [-m,m] \), we can replace \( u \) by \( T_m(u) \) in \( I_l^1, I_l^2, \ldots, I_l^6 \) hence

\[
\lim_{l \to \infty} I_l^1 = \int_{\Omega} bh(u)\phi, \quad \lim_{l \to \infty} I_l^2 = \int_{\Omega} a(x, Du).D(h(u)\phi), \tag{6.35}
\]

\[
\lim_{l \to \infty} |I_l^3| \leq \|h\|_\infty \|\phi\|_\infty \int_{\{|f|>\sigma\}} |f|, \quad \lim_{l \to \infty} I_l^4 = 0, \tag{6.36}
\]

\[
\lim_{l \to \infty} |I_l^5| = \int_{\Omega} F(u).D(h(u)\phi), \quad \lim_{l \to \infty} |I_l^6| = \int_{\Omega} f h(u)\phi, \tag{6.37}
\]

for all \( \delta > 0 \). Combining (6.34) with (6.35)-(6.37) we finally deduce that (6.1) holds for all \( h \in C^1_{\text{loc}}(\mathbb{R}) \) and all \( \phi \in W_0^{1,\overline{p}}(\Omega) \cap L^\infty(\Omega) \).

Hence \((u, b)\) satisfies (R1), (R2) and (R3) and the proof of the theorem is completed.

6.4. **Proof of Theorem 4.2 (Uniqueness).**

**Lemma 6.6.** For \( f, \tilde{f} \in L^1(\Omega) \) let \((u, b), (\tilde{u}, \tilde{b})\) be the renormalized solutions to (\( E, f \)) and (\( E, \tilde{f} \)) respectively, then

\[
\int_{\Omega} (b - \tilde{b})\text{Sign}_0^+(u - \tilde{u})dx \leq \int_{\Omega} (f - \tilde{f})\text{Sign}_0^+(u - \tilde{u})dx, \tag{6.38}
\]

**Proof.** For \( \delta > 0 \) let \( H_\delta^+ \) be a Lipschitz approximation of the \( \text{sign}_0^+ \) function. Since \((u, b), (\tilde{u}, \tilde{b})\) are renormalized solutions, it follows that

\[
T_{l+1}(u), T_{l+1}(\tilde{u}) \in W_0^{1,\overline{p}}(\Omega) \cap L^\infty(\Omega) \text{ for all } l > 0.
\]

Hence \( H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) \in W_0^{1,\overline{p}}(\Omega) \cap L^\infty(\Omega) \) for \( l, \delta > 0 \).

Now, we choose \( H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) \) as a test function in the renormalized formulation with \( h = h_l \) for \((u, b)\) and for \((\tilde{u}, \tilde{b})\) respectively. Subtracting the resulting equalities, we obtain

\[
I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 = I_{l,\delta}^6, \tag{6.39}
\]
where $K = \{0 < T_{l+1}(u) - T_{l+1} < \delta\}$ and

$$I_{l,\delta}^1 = \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

$$I_{l,\delta}^2 = \int_\Omega (h_l'(u)a(x, Du)Du - h_l'(\tilde{u})a(x, D\tilde{u})D\tilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

$$I_{l,\delta}^3 = \frac{1}{\delta} \int_K (h_l(u)a(x, Du) - h_l(\tilde{u})a(x, D\tilde{u})).D(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

$$I_{l,\delta}^4 = \int_\Omega (h_l'(u)F(u)Du - h_l'(\tilde{u})F(\tilde{u})D\tilde{u})H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

$$I_{l,\delta}^5 = \frac{1}{\delta} \int_K (h_l(u)F(u) - h_l(\tilde{u})F(\tilde{u})).D(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

$$I_{l,\delta}^6 = \int_\Omega (fh_l(u) - \tilde{f}h_l(\tilde{u}))H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx.$$
7. Proof of Proposition 4.3

Note that for $\varepsilon, k > 0$, $h_t(u - T_\varepsilon(u - T_k(u)))$ as a test function in (3.2). Neglecting positive terms and passing to the limit with $l \to \infty$, we obtain

$$\frac{1}{\varepsilon} \sum_{i=1}^{N} \int_{k-\varepsilon < |u| < k+\varepsilon} |Du|^{p_i} \leq \|f\|_{\mathcal{N}(\phi(k))^{(N-1)/N}}, \quad (7.1)$$

where $\phi(k) := \{|u| > k\}$ for $k > 0$. Now we use similar arguments as in [33].

We apply the continuous embedding of $W^{1,1}_0(\Omega)$ into $L^{N/N-1}(\Omega)$ and the Hölder inequality to get

$$\frac{1}{\varepsilon} C_N \|T_\varepsilon(u - T_k(u))\|_{N/N-1} \leq \left( \frac{\phi(k) - \phi(k + \varepsilon)}{\varepsilon} \right)^{1/(p^-)'}, \quad (7.2)$$

where $C_N > 0$ is the constant coming from the Sobolev embedding. Notice that

$$\frac{1}{\varepsilon} \sum_{i=1}^{N} \int_{k-\varepsilon < |u| < k+\varepsilon} |Du|^{p_i} \leq \frac{\phi(k) - \phi(k + \varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} \sum_{i=1}^{N} \int_{k-\varepsilon < |u| < k+\varepsilon} |Du|^{p_i}, \quad (7.3)$$

hence, from (7.1), (7.2) and (7.3) we deduce that

$$\frac{1}{\varepsilon} C_N \|T_\varepsilon(u - T_k(u))\|_{N/N-1} \leq \left( \frac{\phi(k) - \phi(k + \varepsilon)}{\varepsilon} \right)^{1/(p^-)'}, \quad (7.4)$$

From (7.4) and Young’s inequality with $\alpha > 0$ it follows that

$$\frac{1}{C_N C} \left( \frac{\alpha^{p^-}}{p^-} \right) \|f\|_{\mathcal{N}(\phi(k))^{(N-1)/N}} - \frac{\alpha^{p^-}}{p^-} \|f\|_{\mathcal{N}(\phi(k))^{(N-1)/N}} \leq 0, \quad (7.5)$$

where

$$C := \left( \frac{1}{\alpha^{p^-}} \frac{1}{(p^-)'} + \frac{\alpha^{p^-}}{p^-} \right) > 0.$$ 

The mapping $(0, \infty) \ni k \to \phi(k)$ is non-increasing and therefore of bounded variation, hence it is differentiable almost everywhere on $(0, \infty)$ with $\phi' \in L^1_{\text{loc}}(0, \infty)$. Since it is also continuous from the right, we can pass to the limit with $\varepsilon \downarrow 0$ in (7.5) to find

$$C''(\phi(k))^{(N-1)/N} + \phi'(k) \leq 0 \quad (7.6)$$

for almost every $k > 0$ and $\alpha > 0$ choosen small enough such that

$$C'' := \left( \frac{C_N}{C} - \frac{\alpha^{p^-}}{p^-} \|f\|_N \right) > 0.$$
Now, the conclusion of the proof follows by contradiction. We assume that $\phi(k) > 0$ for each $k > 0$. For $k > 0$ fixed, we choose $k_0 < k$. From (7.6) it follows that
\[
\frac{1}{N}C'' + \frac{d}{ds}(\phi(s))^{(1/N)} \leq 0
\] (7.7)
for almost all $s \in (k_0, k)$. The left hand side of (7.7) is in $L^1(k_0, k)$, hence we integrate (7.7) over $[k_0, k]$. Moreover, since $\phi$ is non-increasing, integrating (7.7) over $(k_0, k)$ we get
\[
(\phi(k))^{1/N} \leq \phi(k_0)^{1/N} + \frac{1}{N}C''(k_0 - k)
\] (7.8)
and from (7.8) the contradiction follows.

8. Example

This section is devoted to an example for illustrating our abstract result. Let us consider the special case:
\[
\beta(r) = (r - 1)^+ - (r - 1)^- , \quad F : \mathbb{R} \to (F_i)_{i=1,...,N} \in \mathbb{R}^N,
\]
where $F$ is locally lipshitz continuous function, and
\[
a_i(x, \xi) = \sum_{i=1}^N |\xi|^{|p_i|-1} sgn(\xi), \quad i = 1, ..., N,
\]
the $a_i(x, \xi)$ are Carathéodory function satisfying the growth condition $(H_2)$, and the coercivity $(H_1)$. On the other the monotonicity condition is verified. In fact
\[
\sum_{i=1}^N \left( a_i(x, \xi) - a_i(x, \xi') \right) (\xi_i - \xi'_i) = \sum_{i=1}^N \left( |\xi|^{|p_i|-1} sgn(\xi) - |\xi'|^{p_i-1} sgn(\xi') \right) (\xi_i - \xi'_i) \geq 0,
\]
for almost all $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$. This last inequality can not be strict, since for $\xi \neq \xi'$ with $\xi_N \neq \xi_N$ and $\xi = \xi', i = 1, ..., N - 1$. The corresponding expression is zero.

Therefore, for all $f \in L(\Omega)$, the following problem:
\[
\begin{cases}
T_k(u) \in W^{1,\tilde{p}}(\Omega) \quad \text{for} \quad (k > 0); b \in L^1(\Omega) \quad \text{and} \quad b(x) \in \beta(u(x)), \\
\lim_{l \to \infty} \int_{|u| \leq l} a(x, Du) \cdot Du dx = 0, \\
\int_{\Omega} bh(u) \varphi dx = \int_{\Omega} h(u) \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-1} sgn\left( \frac{\partial u}{\partial x_i} \right) \cdot \frac{\partial \varphi}{\partial x_i} dx \\
+ \int_{\Omega} h'(u) \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-1} sgn\left( \frac{\partial u}{\partial x_i} \right) \cdot \frac{\partial \varphi}{\partial x_i} dx + \int_{\Omega} F(h(u)) \varphi dx = \int_{\Omega} f.D(h(u)) \varphi, \quad \forall \varphi \in W^{1,\tilde{p}}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad h \in C^1_c(\mathbb{R}),
\end{cases}
\]
at least one renormalized solution.
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