NEAR OPTIMAL SAMPLE COMPLEXITY FOR MATRIX AND TENSOR NORMAL MODELS VIA GEODESIC CONVEXITY

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The matrix normal model, the family of Gaussian matrix-variate distributions whose covariance matrix is the Kronecker product of two lower dimensional factors, is frequently used to model matrix-variate data. The tensor normal model generalizes this family to Kronecker products of three or more factors. We study the estimation of the Kronecker factors of the covariance matrix in the matrix and tensor models. We show nonasymptotic bounds for the error achieved by the maximum likelihood estimator (MLE) in several natural metrics. In contrast to existing bounds, our results do not rely on the factors being well-conditioned or sparse. For the matrix normal model, all our bounds are minimax optimal up to logarithmic factors, and for the tensor normal model our bound for the largest factor and overall covariance matrix are minimax optimal up to constant factors provided there are enough samples for any estimator to obtain constant Frobenius error. In the same regimes as our sample complexity bounds, we show that an iterative procedure to compute the MLE known as the flip-flop algorithm converges linearly with high probability. Our main tool is geodesic strong convexity in the geometry on positive-definite matrices induced by the Fisher information metric. This strong convexity is determined by the expansion of certain random quantum channels. We also provide numerical evidence that combining the flip-flop algorithm with a simple shrinkage estimator can improve performance in the undersampled regime.

CONTENTS
1 Introduction ......................................................... 2
  1.1 Our contributions ........................................... 4
  1.2 Outline .......................................................... 5
  1.3 Notation .......................................................... 6
2 Model and main results .............................................. 6
  2.1 Matrix and tensor normal model ............................... 6
  2.2 Results on the MLE ........................................... 7
  2.3 Flip-flop algorithm ........................................... 9
  2.4 Results on the flip-flop algorithm ............................ 10
3 Sample complexity for the tensor normal model ................. 10
  3.1 Geodesic convexity .......................................... 11
  3.2 Sketch of proof ............................................... 12
  3.3 Bounding the gradient ....................................... 15
  3.4 Strong convexity from expansion ............................ 18

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1. Introduction. Covariance matrix estimation is an important task in statistics, machine
learning, and the empirical sciences. We consider covariance estimation for matrix-variate
and tensor-variate Gaussian data, that is, when individual data points are matrices or ten-
sors. Matrix-variate data arises naturally in numerous applications like gene microarrays,
spatio-temporal data, and brain imaging. A significant challenge is that the dimensionality
of these problems is frequently much higher than the number of samples, making estimation
information-theoretically impossible without structural assumptions.

To remedy this issue, matrix-variate data is commonly assumed to follow the matrix normal
distribution (Dutilleul, 1999; Werner, Jansson and Stoica, 2008). Here the matrix follows a
multivariate Gaussian distribution and the covariance between any two entries in the matrix
is a product of an inter-row factor and an inter-column factor. In spatio-temporal statistics
this is referred to as a separable covariance structure. Formally, if a matrix normal random
variable $X$ takes values in the $d_1 \times d_2$ matrices, then its covariance matrix $\Sigma$ is a $d_1 d_2 \times d_1 d_2$
matrix that is the Kronecker product $\Sigma_1 \otimes \Sigma_2$ of two positive-semidefinite matrices $\Sigma_1$ and $\Sigma_2$
of dimension $d_1 \times d_1$ and $d_2 \times d_2$, respectively. This naturally extends to the tensor normal
model, where $X$ is a $k$-dimensional array, with covariance matrix equal to the Kronecker
product of $k$ many positive semidefinite matrices $\Sigma_1, \ldots, \Sigma_k$. In this paper we consider the
estimation of $\Sigma_1, \ldots, \Sigma_k$ from $n$ samples of a matrix or tensor normal random variable $X$.

A great deal of research has been devoted to estimating the covariance matrix for the
matrix and tensor normal models, but gaps in rigorous understanding remain. In unstructured
covariance matrix estimation, i.e., $k = 1$, it is well-known that the maximum likelihood
estimator exists whenever $d \geq n$ and achieves mean-squared Frobenius norm error $O(d^2/n)$
and mean-squared operator norm error $O(d/n)$, which are both minimax optimal. This fact
is the starting point for a vibrant area of research attempting to estimate the covariance or
precision matrix with fewer samples under structural assumptions. Particularly important is
the study of graphical models, which seeks to better estimate the precision matrix under the
assumption that it is sparse (has few nonzero entries).
For the matrix and tensor normal models, much of the work (apart from an initial flurry of work on the asymptotic properties of the maximum likelihood estimator) has approached the sparse case directly. In contrast to the unstructured problem above, the fundamental problem of determining the optimal rates of estimation without sparsity assumptions is still largely open. We study this basic question in order to provide a firmer foundation for the large body of work studying its many variants, including the sparse case. We begin by discussing the related work in detail.

In the asymptotic arena, Dutilleul (1999) and later Werner, Jansson and Stoica (2008) proposed an iterative algorithm, known as the flip-flop algorithm, to compute the maximum likelihood estimator (MLE). In the latter work, the authors also showed that the MLE is consistent and asymptotically normal, and showed the same for the estimator obtained by terminating the flip-flop after three steps. For the tensor normal model, a natural generalization of the flip-flop algorithm was proposed to compute the MLE (Mardia and Goodall, 1993; Manceur and Dutilleul, 2013), but its convergence was not proven. Here we will be interested in non-asymptotic rates.

For the matrix normal model, treating the covariance matrix $\Sigma$ as unstructured and estimating it by the sample covariance matrix (the MLE in the unstructured case) yields a mean-squared Frobenius norm error of $O(d_1^2 d_2^2/n)$ assuming $n \geq C d_1 d_2$ for a large enough constant $C$. The matrix normal model has only $\Theta(d_1^2 + d_2^2)$ parameters, so it should be possible to do much better. The state of the art towards optimal rates for the matrix normal model without sparsity assumptions is the work of Tsiligkaridis, Hero and Zhou (2013), which showed that a three-step flip-flop estimator has mean-squared Frobenius error of $O((d_1^2 + d_2^2)/n)$ for the full matrix $\Sigma$. However, their result requires the individual factors have constant condition number and that $n$ is at least $\tilde{\Omega}(\max\{d_1, d_2\})$. They also did not state a bound for the individual factors $\Sigma_1, \Sigma_2$, and did not state bounds for estimation in the operator norm. For the tensor normal model, Sun et al. (2015) present an estimator with tight rates assuming constant condition number of the true covariances and foreknowledge of initializers within constant Frobenius distance of the true precision matrices. In both the matrix and tensor case, no estimator for the Kronecker factors has been proven to have tight rates without additional assumptions on the factors’ structure.

Regarding the sparse case, simply setting $\Sigma_2 = I_{d_2}$ or $\Sigma_1 = I_{d_1}$, in which case the matrix normal model reduces to standard covariance estimation with $d_2 n$ (resp. $d_1 n$) samples, shows the necessity of additional assumptions like sparsity or well-conditionedness if $n < \max\{d_1 / d_2, d_2 / d_1\}$. Tsiligkaridis, Hero and Zhou (2013) also propose a penalized estimator which obtains tighter rates that hold even for $n \ll d_i$ under the additional assumption that the precision matrices $\Sigma_i^{-1}$ are sparse. In the extremely undersampled regime, Zhou (2014) demonstrated a single-step penalized estimator that converges even for a single matrix ($n = 1$) when the precision matrices have constant condition number, are highly sparse, and have bounded $\ell_1$ norm off the diagonal. Allen and Tibshirani (2010) also considered penalized estimators for the purpose of missing data imputation.

Even characterizing the existence of the MLE for the matrix and tensor normal model has remained elusive until recently, in contrast to the unstructured case ($k = 1$). Améndola et al. (2021) recently noted that the matrix normal and tensor MLEs are equivalent to algebraic problems about a group action called the left-right action and the tensor action, respectively. In the computer science literature these two problems are called operator and tensor scaling, respectively. Independently from Améndola et al. (2021), it was pointed out by Franks and Moitra (2020) that the Tyler’s M estimator for elliptical distributions (which is the MLE for the matrix normal model under the additional promise that $\Sigma_2$ is diagonal) is a special case of operator scaling. Using the connection to group actions, exact sample size thresholds for the existence of the MLE were recently determined in Derksen and Makam (2021) for the matrix
normal model and subsequently for the tensor normal model in Derksen, Makam and Walter (2020). In the context of operator scaling, Gurvits (2004) showed much earlier that the flip-flop algorithm converges to the matrix normal MLE whenever it exists. Recently it was shown that the number of flip-flop steps to obtain a gradient of magnitude $\varepsilon$ in the log-likelihood function for the tensor and matrix normal model is polynomial in the input size and $1/\varepsilon$ (Garg et al., 2019; Bürgisser et al., 2018, 2019).

1.1. Our contributions. We take a geodesically convex optimization approach to provide rigorous nonasymptotic guarantees for the estimation of the precision matrices, without any assumptions on their structure. For the matrix normal model we provide high probability bounds on the estimator that are tight up to logarithmic factors. For the tensor normal model, our bounds are tight up to factors of $k$ (the number of Kronecker or tensor factors) whenever it is information-theoretically possible to recover the factors to constant Frobenius error.

In the current literature on matrix normal and tensor models, typically the estimators are assessed using Frobenius or spectral norm distance between the estimated parameter and the truth. However, neither of these metrics bound statistical dissimilarity measures of interest such as the total variation or Kullback-Leibler divergence between the true distribution and that corresponding to the estimated parameter, or the Fisher-Rao distance. The latter statistical measures enjoy an invariance property for multivariate normals – namely, they are preserved under acting on both random variables by the same invertible linear transformation. Such transformations only change the basis in which the data is represented; ideally the performance of estimators should not depend on this basis and hence should not require the covariance matrix to be well-conditioned.

Here we consider the relative Frobenius error $D_F(A\|B) = \|I - B^{-1/2}AB^{-1/2}\|_F$ of the precision matrices. This dissimilarity measure is invariant under the linear transformations discussed above, whereas the the Frobenius norm distance is not. Moreover, the relative Frobenius error $D_F(\Theta_1\|\Theta_2)$, the total variation distance $D_{TV}(\mathcal{N}(0,\Theta_1^{-1}),\mathcal{N}(0,\Theta_2^{-1}))$, the square root of the KL-divergence $D_{KL}(\mathcal{N}(0,\Theta_1^{-1}),\mathcal{N}(0,\Theta_2^{-1}))^{1/2}$, and the Fisher-Rao distance between $\Theta_1$ and $\Theta_2$ all coincide “locally.” That is, if any of them is at most a small constant then they are all on the same order. The estimation of precision and covariance matrices under $D_{KL}$ was suggested by James and Stein (1992) due to its natural invariance properties, and has been studied extensively (e.g., Ledoit and Wolf (2012)). To obtain the sharpest possible results, we also consider the relative spectral error $D_{op}(A\|B) = \|I - B^{-1/2}AB^{-1/2}\|_{op}$, which has been studied in the context of spectral graph sparsification. The dissimilarity measure $D_F(A\|B)$ (resp. $D_{op}(A\|B)$) can be related to the usual norm $\|A - B\|_F$ (resp. $\|A - B\|_{op}$) by a constant factor assuming the spectral norms of $B, B^{-1}$ are bounded by a constant. Though we caution that $D_F$ and $D_{op}$ are not truly metrics, we will call them distances because they approximately (or “locally”) obey symmetry and the triangle inequality. See Appendix E for a discussion of these properties.

Informally, our contributions are as follows:

1. Consider the matrix normal model for $d_1 \leq d_2$. We show that for some $n_0 = \tilde{O}(d_2/d_1)$, if $n \geq n_0$ then the MLE for the precision matrices $\Theta_1, \Theta_2$ has error $\tilde{O}(\sqrt{d_1/n d_2})$ for $\Theta_1$ and $\tilde{O}(\sqrt{d_2/n d_1})$ for $\Theta_2$ in $D_{op}$ with probability $1 - e^{-\Omega(d_i)}$. For estimating $\Theta_1$ alone, we obtain the error $\tilde{O}(\sqrt{d_1/n \min\{d_2, n d_1\}})$ for any $n$. The $\tilde{O}$ notation suppresses logarithmic factors in $d_1$ and $d_2$.

2. In the tensor normal model, for $k$ fixed we show that for some $n_0 = O(\max\{d_i^3/\prod_{i=1}^k d_i\})$, if $n \geq n_0$ then the MLE for the precision matrix $\Theta$ has error $O(\frac{d_{max}}{\sqrt{n}})$ in $D_F$ with probability $1 - (n D/ \max\{d_i^2\})^{-\Omega(\min\{d_i\})}$. We also give bounds for growing $k$ and for the individual Kronecker factors $\Theta_i$. Our bound for the error of the largest Kronecker factor of largest dimension is tight.
3. Under the same sample requirements as above in each case, the flip-flop algorithm of (Mardia and Goodall, 1993; Manceur and Dutilleul, 2013) converges exponentially quickly to the MLE with high probability. As a consequence, the output of the flip-flop algorithm with $O_k(d_{\max}(1 + \kappa(\Theta)) + \log n)$ iterations is an efficiently computable estimator that enjoys statistical guarantees at least as strong (up to constant factors) as those we show for the MLE. Here $\kappa(\Theta)$ denotes the condition number, the ratio between the largest and smallest eigenvalues of $\Theta$.

4. To handle the undersampled case, we introduce a shrinkage-based estimator that is much simpler to compute than the LASSO-type estimators of Tsiligkaridis, Hero and Zhou (2013); Sun et al. (2015); Zhou (2014) and give empirical evidence that it improves on them in a generative model for dense precision matrices.

We now discuss the tightness of our results. Our first result is tight up to logarithmic factors in the sense that it is information-theoretically impossible to obtain an error bound that is smaller by a polylogarithmic factor and holds with constant probability. We also show that our results for estimating $\Theta_1$ alone are tight up to logarithmic factors; i.e., that it is impossible to obtain a rate better than $O(\sqrt{d_1/n}\min\{nd_1, d_2\})$. Similarly, for the second result, provided $n \geq n_0$ it is impossible to obtain an error bound that is smaller than ours by a factor tending to infinity that holds with constant probability. For $n \ll n_0$, no constant error bound on the $D_F$ error of the largest Kronecker factor can hold with constant probability. Apart from the lower bound for estimating $\Theta_1$ alone, which to our knowledge is novel, these tightness results follow by reduction to known results on the Frobenius and operator error for covariance estimation; see Section 6.

For interesting cases of the tensor normal model such as $d \times d \times d$ tensors we just require that $n$ is at least a large constant. For the matrix normal model, our first result removes the added constraint $n \geq C\max\{d_1, d_2\}$ in Tsiligkaridis, Hero and Zhou (2013). We leave extending the $D_{\text{op}}$ bounds for the matrix normal model to the tensor normal model as an open problem.

We now briefly discuss our estimator for the undersampled case, which is described in detail in Section 7. The MLE is a function of the sample covariance matrix, but in the undersampled case the MLE need not exist. To remedy this, we replace the sample covariance matrix by a shrinkage estimator for it (in particular, by taking a convex combination with the identity matrix) and then compute the MLE for the “shrunk” covariance matrix. Though our estimator is empirically outperformed by Zhou (2014) for sparse precision matrices, it empirically outperforms Zhou (2014) in a natural dense generative model of random approximately low-rank Kronecker factors which we refer to as the “spiked” model. Moreover, we found that on average our estimator is faster to compute than the estimator of Zhou (2014) by a factor of 500 for the matrix model with $d_1 = 100, d_2 = 200$. Given this empirical evidence, we view our shrinkage-based estimator as a potentially useful tool for the undersampled tensor normal model which merits further theoretical attention.

1.2. **Outline.** In the next section, Section 2, we precisely describe the model and formally state our results. In Section 3, we prove our main sample complexity bound for the tensor normal model (Theorem 2.4). In Section 4 we prove our improved bound for the matrix normal model (Theorem 2.7). Our results on the flip-flop algorithm for tensor and matrix normal models (Theorems 2.9 and 2.10, respectively) are proven in Section 5. Section 6 contains the proofs of our lower bound for the matrix normal model, and Section 7 contains empirical observations about the performance of our regularized estimator.
1.3. Notation. We write $\text{Mat}(d)$ for the space of $d \times d$ matrices with real entries, $\text{PD}(d)$ for the convex cone of $d \times d$ positive definite symmetric matrices; and $\text{SPD}(d)$ for the $d \times d$ positive definite symmetric matrices with unit determinant; $\text{GL}(d)$ denotes the group of invertible $d \times d$ matrices with real entries. We write $\succeq$ for the Loewner order. For a matrix $A$, $\|A\|_{\text{op}}$ denotes the operator norm, $\|A\|_F = (\text{Tr} A^T A)^{\frac{1}{2}}$ the Frobenius norm, and $\langle A, B \rangle = \text{Tr} A^T B$ the Hilbert-Schmidt inner product. We also denote by $\kappa(A) = \|A\|_{\text{op}} \|A^{-1}\|_{\text{op}}$ the condition number of $A$. For functions $f, g: S \to \mathbb{R}$ for any set $S$, we say $f = O(g)$ if there is a constant $C > 0$ such that $f(x) \leq C g(x)$ for all $x \in S$, and similarly $f = \Omega(g)$ if there is a constant $c > 0$ such that $f(x) \geq cg(x)$ for all $x \in S$. If $f = O(g)$ and $g = O(f)$ we write $f \asymp g$. In case the constants $C, c$ depend on another parameter $k$, we write $O_k$ and $\Omega_k$, respectively. We abbreviate $[k] = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. All other notation is introduced in the remainder.

2. Model and main results. In this section we define the matrix and tensor normal models and we state our main technical results.

2.1. Matrix and tensor normal model. The tensor normal model, of which the matrix normal model is a particular case, is formally defined as follows.

**Definition 2.1.** For dimensions $d_1, \ldots, d_k \in \mathbb{N}$, we define the tensor normal model as the family of centered multivariate Gaussian distribution with covariance matrix given by a Kronecker product $\Sigma = \Sigma_1 \otimes \cdots \otimes \Sigma_k$ of positive definite matrices with $\{\Sigma_a \in \text{PD}(d_a)\}_{a \in [k]}$. For $k = 2$, this is known as the matrix normal model.

Note that if each $\Sigma_a$ is a $d_a \times d_a$ matrix then $\Sigma$ is a $D \times D$-matrix, where $D = d_1 \cdots d_k$. Our goal is to estimate $k$ Kronecker factors $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_k$ such that $\Sigma_a \approx \hat{\Sigma}_a$ for each $a \in [k]$ given access to $n$ i.i.d. random samples $x_1, \ldots, x_n \in \mathbb{R}^D$ drawn from the model. A weaker requirement is for $\Sigma \approx \hat{\Sigma}_1 \otimes \cdots \otimes \hat{\Sigma}_k$.

One may also think of each random sample $x_j$ as taking values in the set of $d_1 \times \cdots \times d_k$ arrays of real numbers. There are $k$ natural ways to “flatten” $x_j$ to a matrix: for example, we may think of it as a $d_1 \times d_2 d_3 \cdots d_k$ matrix whose column indexed by $(i_2, \ldots, i_k)$ is the vector in $\mathbb{R}^{d_1}$ with $i_1^{th}$ entry equal to $(x_j)_{i_1, \ldots, i_k}$. In the tensor normal model, the $d_2 d_3 \cdots d_k$ many columns are each distributed as a Gaussian random vector with covariance proportional to $\Sigma_1$. In an analogous way we may flatten it to a $d_2 \times d_1 d_3 \cdots d_k$ matrix, and so on. As such, the columns of the $a^{th}$ flattening can be used to estimate $\Sigma_a$ up to a scalar. However, doing so naïvely (e.g., using the sample covariance matrix of the columns) can result in an estimator with very high variance. This is because the columns of the flattennings are not independent. In fact they may be so highly correlated that they effectively constitute only one random sample rather than $d_2 \cdots d_k$ many. The MLE decorrelates the columns to obtain rates like those one would obtain if the columns were independent.

The MLE is easier to describe in terms of the precision matrices, the inverses of the covariance matrices.

**Definition 2.2 (Precision matrices).** For a $D \times D$-covariance matrix $\Sigma$ arising in the tensor normal model, we refer to $\Theta = \Sigma^{-1}$ as the precision matrix. We also define the Kronecker factor precision matrices $\Theta_1, \ldots, \Theta_k$ as the unique positive-definite matrices such that $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$ and $(\det \Theta_a)^{1/d_a}$ is the same for each $a \in [k]$. In other words, we choose $\Theta_a = \lambda \Theta_a'$ where $\det \Theta_a' = 1$ and $\lambda > 0$ is a constant (not depending on $a \in [k]$). We make this choice because the Kronecker factors of $\Theta$ are determined only up to a scalar.
Let \( P \) denote the manifold of all precision matrices \( \Theta \) for the tensor normal model of fixed format \( d_1 \times \cdots \times d_k \), i.e.,
\[
P = \{ \Theta = \Theta_1 \otimes \cdots \otimes \Theta_k : \Theta_a \in \text{PD}(d_a) \}.
\]
Given a tuple \( x \) of samples \( x_1, \ldots, x_n \in \mathbb{R}^D \), the following function is proportional to the negative log-likelihood:
\[
f_x(\Theta) = \frac{1}{nD} \sum_{i=1}^n x_i^T \Theta x_i - \frac{1}{D} \log \det \Theta.
\]
(2.1)

The maximum likelihood estimator (MLE) for \( \Theta \) is then
\[
\hat{\Theta} := \arg \min_{\Theta \in P} f_x(\Theta)
\]
whenever the minimizer exists and is unique. We write \( \hat{\Theta} = \hat{\Theta}(x) \) when we want to emphasize the dependence of the MLE on the samples \( x \), and we say \( (\hat{\Theta}_1, \ldots, \hat{\Theta}_k) \) is an MLE for \( (\Theta_1, \ldots, \Theta_k) \) if \( \otimes_{a=1}^k \hat{\Theta}_a = \hat{\Theta} \). Note that \( P \) is not a convex domain under the Euclidean geometry on the \( D \times D \) matrices.

2.2. Results on the MLE. We may now state our result for the tensor normal models precisely. As mentioned in the introduction, we use the following natural distance measures.

**Definition 2.3 (Relative error).** For positive definite matrices \( A, B \), define their relative Frobenius error (or Mahalanobis distance) as
\[
\text{DF}(A \parallel B) = \| I - B^{-1/2} AB^{-1/2} \|_F.
\]
(2.3)

Similarly, define the relative spectral error as
\[
\text{Dop}(A \parallel B) = \| I - B^{-1/2} AB^{-1/2} \|_{\text{op}}.
\]
(2.4)

To state our results, and throughout this paper, we write \( d_{\text{min}} = \min_a d_a, d_{\text{max}} = \max_a d_a \). Recall also that \( D = \prod_{i=1}^k d_a \). Recall that we identify \( \Theta_1, \ldots, \Theta_k \) from \( \hat{\Theta} \) using the convention \( \det \Theta_1^{1/d_1} = \cdots = \det \Theta_k^{1/d_k} \), and likewise for the MLE \( \hat{\Theta} \). For the reader interested only in the behavior for constant \( k \), we display this special case afterwards in Corollary 2.5.

**Theorem 2.4 (Tensor normal Frobenius error).** There is a universal constant \( C > 0 \) such that the following holds. Suppose \( t \geq 1 \) and
\[
n \geq Ck^2 d_{\text{max}}^2 \frac{d_{\text{min}}^2}{D} \max\{k, d_{\text{max}}\} t^2.
\]
(2.5)

Then the MLE \( \hat{\Theta} = \hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_k \) for \( n \) independent samples of the tensor normal model with precision matrix \( \Theta = \Theta_1 \otimes \cdots \otimes \Theta_k \) satisfies
\[
\text{DF}(\hat{\Theta}_a \parallel \Theta_a) = O \left( tk^{1/2} d_{\text{max}} \sqrt{\frac{d_a}{nD}} \right)\quad \text{for all } a \in [k]
\]
and
\[
\text{DF}(\hat{\Theta} \parallel \Theta) = O \left( tk^{3/2} d_{\text{max}} \frac{1}{\sqrt{n}} \right),
\]
with probability at least
\[
1 - ke^{-\Omega(t^2d_{\text{max}})} - k^2 \left( \frac{\sqrt{nD}}{kd_{\text{max}}} \right)^{-\Omega(d_{\text{min}})}.
\]
The error for the precision matrix $\Theta_a$ with $d_a = d_{\text{max}}$ matches that of the MLE for the precision matrix of a single Gaussian with $D/d_{\text{max}}$ samples, which is the special case when all the other Kronecker factors are the identity. We also state our result for a constant number of Kronecker factors $k$, as is often the case in applications.

**Corollary 2.5 (Tensor normal Frobenius error, constant $k$).** For any fixed $k$, there is a constant $C = C(k) > 0$ such that the following holds. Suppose $t \geq 1$ and $n \geq C d_{\text{max}}^{d_1} t^2$. Then the MLE $\hat{\Theta} = \hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_k$ for $n$ independent samples of the tensor normal model with precision matrix $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$ satisfies

$$D_F(\hat{\Theta}_a \| \Theta_a) = O \left( t d_{\text{max}} \sqrt{\frac{d_a}{nD}} \right) \text{ for all } a \in [k]$$

and

$$D_F(\hat{\Theta} \| \Theta) = O \left( \frac{t d_{\text{max}}}{\sqrt{n}} \right),$$

with probability at least

$$1 - e^{-\Omega(t^2 d_{\text{max}})} - \left( \frac{\sqrt{nD}}{d_{\text{max}}} \right)^{-\Omega(d_{\text{min}})} .$$

**Remark 2.6 (Geodesic and Fisher-Rao distance).** Our bounds on $D_F$ in the above theorem follow from a stronger bound on the geodesic distance

$$d(\hat{\Theta}, \Theta) = \frac{1}{\sqrt{D}} \| \log \Theta^{-1/2} \hat{\Theta} \Theta^{-1/2} \|_F ,$$

which we discuss and motivate in Section 3.1. With the same hypotheses and failure probability as Theorem 2.4, we have $d(\hat{\Theta}, \Theta) = O(\sqrt{k} d_{\text{max}} t / \sqrt{nD})$; see Proposition 3.23. The geodesic distance is a constant multiple of the Fisher-Rao distance

$$d_{\text{FR}}(\hat{\Theta}, \Theta) = \frac{1}{\sqrt{2}} \| \log \Theta^{-1/2} \hat{\Theta} \Theta^{-1/2} \|_F ,$$

which arises naturally from the Fisher information metric for centered Gaussians parameterized by their covariance matrices (Skovgaard, 1984). Up to a constant factor $1/\sqrt{2}$, it is also the same as the metric considered in (Bhatia, 2009; Skovgaard, 1984).

For the matrix normal model ($k = 2$), we obtain a stronger result. Recall that we identify $\Theta_1, \Theta_2$ from $\Theta$ using the convention $\det \Theta_1^{1/d_1} = \det \Theta_2^{1/d_2}$.

**Theorem 2.7 (Matrix normal spectral error).** There is a universal constant $C > 0$ with the following property. Suppose $1 < d_1 \leq d_2$, $t \geq 1$, and

$$n \geq C d_2 d_{\text{max}} \max\{\log d_2, t^2 \log^2 d_1\}.$$

Then the MLE $\hat{\Theta} = \hat{\Theta}_1 \otimes \hat{\Theta}_2$ for $n$ independent samples from the matrix normal model with precision matrix $\Theta = \Theta_1 \otimes \Theta_2$ satisfies

$$D_{\text{op}}(\hat{\Theta}_1 \| \Theta_1) = O \left( t \sqrt{\frac{d_1}{nd_2} \log d_1} \right) \text{ and } D_{\text{op}}(\hat{\Theta}_2 \| \Theta_2) = O \left( t \sqrt{\frac{d_2}{nd_1} \log d_1} \right)$$

with probability at least $1 - e^{-\Omega(t^2)}$. 
In applications such as brain fMRI, one is interested only in $\Theta_1$, and $\Theta_2$ is treated as a nuisance parameter. Assuming the nuisance parameter $\Theta_2$ is known, we can compute $(I \otimes \Theta_2^{1/2})X$, which is distributed as $nd_2$ independent samples from a Gaussian with precision matrix $\Theta_1$. In this case, one can estimate $\Theta_1$ in operator norm with an RMSE rate of $O(\sqrt{d_1/nd_2})$ no matter how large $d_2$ is. One could hope that this rate holds for $\Theta_1$ even when $\Theta_2$ is not known. In Section 6 we show that, to the contrary, the rate for $\Theta_1$ cannot be better than $O(\sqrt{d_1/n \min(nd_1, d_2)})$. Thus, for $d_2 > nd_1$, it is impossible to estimate $\Theta_1$ as well as one could if $\Theta_2$ were known. Note that in this regime there is no hope of recovering $\Theta_2$ even if $\Theta_1$ is known. As the random variable $Y_i$ obtained by ignoring all but $d_2'$ columns of each $X_i$ is distributed according to the matrix normal model with covariance matrix $\Sigma_1 \otimes \Sigma_2'$ for some $\Sigma_2 \in \mathcal{P}(d_2')$, the MLE for $Y$ obtains a matching upper bound.

**Corollary 2.8 (Estimating only $\Theta_1$).** There is a universal constant $C > 0$ with the following property. Let $X$ be distributed according to the matrix normal model with precision matrix $\Theta = \Theta_1 \otimes \Theta_2$ and suppose that $1 < d_1 \leq d_2$ and $t \geq 1$. Let $Y = (Y_1, \ldots, Y_n)$ be the random variable obtained by removing all but $d_2' = \min\left\{d_2, \frac{nd_1}{C \max\{\log n, t^2 \log^2 d_1\}}\right\}$ columns of $X_i$ for each $i \in [n]$. Then the MLE $\widehat{\Theta} = \widehat{\Theta}_1 \otimes \widehat{\Theta}_2$ for $Y$ satisfies

$$D_{\text{op}}(\widehat{\Theta}_1 \| \Theta_1) = O\left(t \sqrt{\frac{d_1}{nd_2'} \log d_1}\right),$$

with probability $1 - e^{-\Omega(d_1t^2)}$. This rate is tight up to factors of $\log d_1$ and $t^2 \log^2 d_1$.

### 2.3. Flip-flop algorithm.

The MLE can be computed by a natural iterative procedure known as the flip-flop algorithm (Dutilleul, 1999; Gurvits, 2004). For simplicity, we describe it for the matrix normal model $(k = 2)$, so that the samples $x_i$ can be viewed as $d_1 \times d_2$ matrices which we denote by $X_i$. The general flip-flop algorithm is described in Algorithm 2 in Section 5.

**Algorithm 1:** Flip-flop algorithm specialized to the matrix normal model.

**Input:** Samples $X = (X_1, \ldots, X_n)$, where each $X_i$ is a real $d_1 \times d_2$ matrix. Parameters $T \in \mathbb{N}$ and $\delta > 0$.

**Output:** An estimate $\widehat{\Theta} = \widehat{\Theta}_1 \otimes \widehat{\Theta}_2 \in \mathcal{P}$ of the MLE.

**Algorithm:**

1. Set $\widehat{\Theta}_1 = I_{d_1}$ and $\widehat{\Theta}_2 = I_{d_2}$.

2. For $t = 1, \ldots, T$, repeat the following:
   - If $t$ is odd, set $a = 1$ and $\Upsilon = \frac{1}{nd_2} \sum_{i=1}^n X_i \widehat{\Theta}_2 X_i^T$. If $t$ is even, set $a = 2$ and $\Upsilon = \frac{1}{nd_1^2} \sum_{i=1}^n X_i^T \widehat{\Theta}_1 X_i$.
   - If $t > 1$ and $D_{\text{F}}(\Upsilon \| \Sigma_a^{\text{op}}) \leq \sqrt{\delta d_a}$, output $\widehat{\Theta}$ and return.
   - Update $\Upsilon_a \leftarrow \Upsilon^{-1}$.

We can motivate this procedure by noting that if in the first step we already have $\widehat{\Theta}_2 = \Theta_2$, then $\frac{1}{nd_2} \sum_{i=1}^n X_i \widehat{\Theta}_2 X_i^T$ is simply a sum of outer products of $nd_2$ many independent random vectors with covariance $\Sigma_1 = \Theta_1^{-1}$; as such the inverse is a good estimator for $\Theta_1$. As we don’t know $\Theta_2$, the flip-flop algorithm instead uses $\widehat{\Theta}_2$ as our current best guess. We will use a slightly different stopping criterion in our formal description of the algorithm.
For the general tensor normal model, in each step the flip flop algorithm chooses one of the dimensions $a \in [k]$ and uses the $a^{th}$ flattening of the samples $x_i$ (which are just $X_i$ and $X_i^T$ in the matrix case) to update $\Theta_a$.

2.4. Results on the flip-flop algorithm. Our next results show that the flip-flop algorithm can efficiently compute the MLE with high probability when the hypotheses of Theorem 2.4 or Theorem 2.7 hold. We first state our result for the general tensor normal model and then give an improved version for the matrix normal model. In the theorems that follow, we use the same convention for choosing the Kronecker factors as in the preceding section.

**Theorem 2.9 (Tensor normal flip-flop).** There are universal constants $C, c > 0$ such that the following holds. Suppose $x = (x_1, \ldots, x_n)$ are $n \geq C k^2 d_{\max}^3 / D$ independent samples from the tensor normal model with precision matrix $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$. Then, with probability at least $1 - k e^{-\Omega(n D^2 / kd_{\max})}$, the MLE $\hat{\Theta}$ exists, and for every $0 < \delta < c/\sqrt{(k+1)d_{\max}}$, Algorithm 2 within

$$T = O \left( k^2 d_{\max} (1 + \log \kappa(\Theta)) + k \log \frac{1}{\delta} \right)$$

iterations outputs $\Theta$ with $d(\Theta, \hat{\Theta}) \leq 2\delta$ and $D_F(\Theta_a \| \hat{\Theta}_a) \leq 4 \sqrt{d_a} \cdot \delta$ for all $a \in [k]$.

**Theorem 2.10 (Matrix normal flip-flop).** There are universal constants $C, c > 0$ such that the following holds. Let $1 < d_1 \leq d_2$. Suppose $x_1, \ldots, x_n \in \mathbb{R}^{d_1 \times d_2}$ are

$$n \geq C \frac{d_2}{d_1} \max \{ \log d_2, \log^2 d_1 \}$$

independent samples from the matrix normal model with precision matrix $\Theta = \Theta_1 \otimes \Theta_2$. Then, with probability at least $1 - e^{-\Omega(n d_2^3 / d_1 \log^2 d_1)}$, the MLE $\hat{\Theta}$ exists, and for every $0 < \delta < c/\sqrt{d_2}$, Algorithms 1 and 2 within

$$T = O \left( d_2 (1 + \log \kappa(\Theta)) + \log \frac{1}{\delta} \right)$$

iterations outputs $\Theta$ with $d(\Theta, \hat{\Theta}) = O(\delta)$ and $D_F(\Theta_a \| \hat{\Theta}_a) = O(\sqrt{d_a} \cdot \delta)$ for all $a \in \{1, 2\}$.

Plugging in the error rates for the MLE from Theorems 2.4 and 2.7 into Theorems 2.9 and 2.10 (with $\varepsilon = 1$) shows that the output of the flip-flop algorithm with $O \left( k^2 d_{\max} (1 + \kappa(\Theta)) + k \log(n) \right)$ iterations is an efficiently computable estimator with the same statistical guarantees as we have shown for the MLE.

3. Sample complexity for the tensor normal model. It was observed by Wiesel (2012) that the negative log-likelihood exhibits a certain variant of convexity known as geodesic convexity. In this section, we use geodesic convexity, following a strategy similar to Franks and Moitra (2020), to prove Theorem 2.4. Our improved result for the matrix normal model, Theorem 2.7, requires additional tools and will be proved later in Section 4.
3.1. Geodesic convexity. We now discuss the geodesic convexity used here and outline the strategy for our proof. We start by introducing a Riemannian metric on the manifold $\text{PD}(D)$ of positive-definite real symmetric $D \times D$ matrices. Rather than simply considering the metric induced by the Euclidean metric on the symmetric matrices, we consider the Riemannian metric whose geodesics starting at a point $\Theta \in \text{PD}(D)$ are of the form $t \mapsto \Theta^{1/2} e^{Ht} \Theta^{1/2}$ for $t \in \mathbb{R}$ and a symmetric matrix $H$. This metric arises from the Hessian of the log-determinant (Bhatia, 2009) and also as the Fisher information metric on centered Gaussians parametrized by their covariance matrices (Skovgaard, 1984). If $\Theta$ is positive definite and $A$ an invertible matrix then $A\Theta A^T$ is again positive definite. The transformation $\Theta \mapsto A\Theta A^T$ is an isometry with respect to this metric, i.e., it preserves the geodesic distance, as well as the statistical distances $D_F$ and $D_{\text{op}}$ that we use (Definition 2.3). This invariance is natural because changing a pair of precision matrices in this way does not change the statistical relationship between the corresponding Gaussians; in particular the total variation distance, Fisher-Rao, and Kullback-Leibler divergence are unchanged.

As observed by Wiesel (2012), the negative log-likelihood is convex as the precision matrix moves along the geodesics of the Fisher information metric, and in particular for the tensor normal model it is convex along geodesics in $\mathbb{P} = \{ \Theta_1 \otimes \ldots \otimes \Theta_k \in \text{PD}(d_1) \times \cdots \times \text{PD}(d_k) \}$. This is because the geodesics in $\text{PD}(D)$ between elements of the manifold

$$\mathbb{P} = \{ \Theta = \Theta_1 \otimes \ldots \otimes \Theta_k : \Theta_a \in \text{PD}(d_a) \}$$

remain in $\mathbb{P}$. That is, $\mathbb{P}$ is a totally geodesic submanifold of $\text{PD}(D)$. Its tangent space can be identified with the real vector space

$$\mathbb{H} = \{(H_0; H_1, \ldots, H_k) : H_0 \in \mathbb{R}, H_a \text{ a symmetric traceless } d_a \times d_a \text{ matrix } \forall a \in [k]\},$$

equipped with the norm and inner product

$$\|H\|_F := \langle H, K \rangle^{1/2}, \quad \langle H, K \rangle := H_0 K_0 + \sum_{a=1}^k \text{Tr} H_a^T K_a.$$

The direction $(1; 0, \ldots, 0)$ changes $\Theta$ by an overall scalar, and tangent directions supported only in the $a$th component for $a \in [k]$ only change $\Theta_a$, subject to its determinant staying fixed. The geodesics on $\mathbb{P}$ are simply the geodesics of the Fisher-information metric on $\text{PD}(D)$, but we will define them precisely in terms of the tangent space $\mathbb{H}$ as follows to fix our conventions.

**Definition 3.1 (Exponential map and geodesics).** The exponential map $\exp_{\Theta} : \mathbb{H} \to \mathbb{P}$ at $\Theta = \Theta_1 \otimes \ldots \otimes \Theta_k \in \mathbb{P}$ is defined by

$$\exp_{\Theta}(H) = e^{H_0} \cdot (\Theta_1^{1/2} e^{\sqrt{\pi} H_1} \Theta_1^{1/2}) \otimes \cdots \otimes (\Theta_k^{1/2} e^{\sqrt{\pi} H_k} \Theta_k^{1/2}).$$

By definition, the geodesics through $\Theta$ are the curves $t \mapsto \exp_{\Theta}(tH)$ for $t \in \mathbb{R}$ and $H \in \mathbb{H}$. Up to reparameterization, there is a unique geodesic between any two points of $\mathbb{P}$.

We take the convention that the geodesics have unit speed if $\|H\|_F^2 = 1$. The geodesic distance $d(\Theta, \Theta')$ between two points $\Theta$ and $\Theta' = \exp_{\Theta}(H)$ is therefore equal to $\|H\|_F$, which can also be computed as $D^{-1/2}\| \log \Theta^{-1/2} \Theta' \Theta^{-1/2} \|_F$. To summarize:

**Definition 3.2 (Geodesic distance and balls).** The geodesic distance $d(\Theta, \Theta')$ between two points $\Theta$ and $\Theta'$ is given by

$$d(\Theta, \Theta') = \frac{1}{\sqrt{D}} \| \log \Theta^{-1/2} \Theta' \Theta^{-1/2} \|_F = \sqrt{\frac{2}{D}} \cdot d_{\text{FR}}(\Theta, \Theta').$$

(3.1)
where \( \log \) denotes the matrix logarithm and \( d_{FR} \) is the Fisher-Rao distance defined in Eq. (2.6). We choose the normalization in Eq. (3.1) because it will make the negative log-likelihood functions typically \( \Omega(1) \)-strongly convex, while ensuring that the gradients are \( O(1) \).

The closed \( ( \text{geodesic} ) \) ball of radius \( r > 0 \) about \( \Theta \) is defined as

\[
B_r(\Theta) = \{ \exp(\Theta)(H) : \|H\|_F \leq r \},
\]

The manifold \( \text{PD}(D) \), and hence \( \mathbb{P} \), is a Hadamard manifold, i.e., a complete, simply connected Riemannian manifold of non-positive sectional curvature (Bacák, 2014). Thus geodesic balls are geodesically convex subsets of \( \mathbb{P} \), that is, if \( \gamma(t) \) is a geodesic such that \( \gamma(0), \gamma(1) \in B_r(\Theta) \) then \( \gamma(t) \in B_r(\Theta) \) for all \( t \in [0, 1] \).

Using our definition of geodesics, we obtain the following notion of geodesic convexity of functions.

**Definition 3.3 (Geodesic convexity).** Given a geodesically convex domain \( D \subseteq \mathbb{P} \), we say that a function \( f \) is (strictly) geodesically convex on \( D \) if, and only if, the function \( t \mapsto f(\gamma(t)) \) is (strictly) convex on \( [0, 1] \) for any geodesic \( \gamma(t) \) with \( \gamma(0), \gamma(1) \in D \). It is called \( \lambda \)-strongly geodesically convex if \( t \mapsto f(\gamma(t)) \) is \( \lambda \)-strongly convex along every unit-speed geodesic \( \gamma \) with endpoints in \( D \).

For a twice differentiable function \( f : \mathbb{P} \to \mathbb{R} \), we can ensure that it is \( \lambda \)-strong geodesically convex on \( D \) by requiring that \( f \) is \( \lambda \)-strong geodesically convex at \( \Theta \), i.e.

\[
\partial^2_{t=0} f(\exp(\Theta)(tH)) \geq \lambda \|H\|_F^2,
\]

for all \( H \in \mathbb{H} \), for every \( \Theta \in D \).

**Example 3.4.** It is instructive to consider the case \( k = 1 \), or \( \mathbb{P} = \text{PD}(D) \). The geodesics through \( \Theta \) are the curves \( t \mapsto \sqrt{\Theta} e^{Ht} \sqrt{\Theta} \). As an example of a geodesically convex function, consider the likelihood for the precision matrix of a Gaussian with data \( x_1, \ldots, x_n \).

Let \( \rho := \frac{1}{nD} \sum_i x_i x_i^T \) denote the matrix of “second sample moments” of the data. Then we can rewrite the objective function (2.1) as

\[
f_x(\Theta) = \operatorname{Tr} \rho \Theta - \frac{1}{D} \log \det \Theta.
\]

We claim that \( f_x(\Theta) \) is always geodesically convex, and in fact strictly geodesically convex whenever \( \rho \) is invertible. Indeed,

\[
\partial^2_{t=0} f_x(\sqrt{\Theta} e^{Ht} \sqrt{\Theta}) = \operatorname{Tr} \sqrt{\Theta} \rho \sqrt{\Theta} H^2 \geq 0
\]

with strict inequality whenever \( \rho \) is invertible (and \( H \) nonzero).

The invariance properties described above for \( \text{PD}(D) \) are directly inherited to \( \mathbb{P} \). The manifold \( \mathbb{P} \) carries a natural action by the group

\[
\mathbb{G} = \{ A = A_1 \otimes \cdots \otimes A_k : G_a \in \text{GL}(d_a) \}
\]

Namely, if \( \Theta \in \mathbb{P} \) and \( A \in \mathbb{G} \) then the \( A \Theta A^T \) is in \( \mathbb{P} \). Moreover, as discussed above, the mapping \( \Theta \mapsto A \Theta A^T \) is an isometry of the Riemannian manifold \( \mathbb{P} \) and it preserves the statistical distances \( D_F \) and \( D_{op} \).

3.2. Sketch of proof. With these definitions in place, we are able to state a proof plan for Theorem 2.4. The proof is a Riemannian version of the standard approach using strong convexity.
1. **Reduce to identity:** We can obtain \( n \) independent samples from \( N(0, \Theta^{-1}) \) as \( x'_i = \Theta^{-1/2} x_i \), where \( x_1, \ldots, x_n \) are distributed as \( n \) independent samples from a standard Gaussian. The MLE \( \Theta(x') \) for the former is exactly \( \Theta^{1/2} \Theta(x) \Theta^{1/2} \). By invariance of the relative Frobenius error, \( D_F(\Theta(x'))\Theta = D_F(\Theta(x)||I_D) \); the same is true for \( D_{op} \). This shows that to prove Theorem 2.4 it is enough to consider the case that \( \Theta = I_D \), i.e., the standard Gaussian.

2. **Bound the gradient:** Show that the gradient \( \nabla f_x(I_D) \) (defined below) is small with high probability.

3. **Show strong convexity:** Show that, with high probability, \( f_x \) is \( \Omega(1) \)-strongly geodesically convex near \( I \).

These together imply the desired sample complexity bounds – as in the Euclidean case, strong convexity in a suitably large ball about a point implies the optimizer cannot be far. Moreover, it happens that under alternating minimization \( f_x \) obeys a descent lemma (similar to what is shown in Bürgisser et al. (2018)); as such the flip-flop algorithm must converge exponentially quickly by the strong geodesic convexity of \( f_x \).

To make this discussion more concrete, we now define the gradient and Hessian formally, and state the lemma that we will use to relate the gradient and strong convexity to the distance to the optimizer as in the plan above.

**Definition 3.5 (Gradient and Hessian).** Let \( f : \mathbb{P} \to \mathbb{R} \) be a once or twice differentiable function and \( \Theta \in \mathbb{P} \). The (Riemannian) gradient \( \nabla f(\Theta) \) is the unique element in \( \mathbb{H} \) such that

\[
\langle \nabla f(\Theta), H \rangle = \partial_{t=0} f(\exp_{\Theta}(tH)) \quad \forall H \in \mathbb{H}.
\]

Similarly, the (Riemannian) Hessian \( \nabla^2 f(\Theta) \) is the unique linear operator on \( \mathbb{H} \) such that

\[
\langle H, \nabla^2 f(\Theta)K \rangle = \partial_{s=0, \partial_{t=0}} f(\exp_{\Theta}(sH + tK)) \quad \forall H, K \in \mathbb{H}.
\]

We abbreviate \( \nabla f = \nabla f(I_D) \) and \( \nabla^2 f = \nabla^2 f(I_D) \) for the gradient and Hessian, respectively, at the identity matrix, and we write \( \nabla_a f \) and \( \nabla_{ab} f \) for the components. As block matrices,

\[
\nabla f = \begin{bmatrix}
\nabla_0 f \\
\nabla_1 f \\
\vdots \\
\nabla_k f
\end{bmatrix}, \quad \nabla^2 f = \begin{bmatrix}
\nabla_{00}^2 f & \nabla_{01}^2 f & \cdots & \nabla_{0k}^2 f \\
\nabla_{10}^2 f & \nabla_{11}^2 f & \cdots & \nabla_{1k}^2 f \\
\vdots & \vdots & \ddots & \vdots \\
\nabla_{k0}^2 f & \nabla_{k1}^2 f & \cdots & \nabla_{kk}^2 f
\end{bmatrix}.
\]

Here, \( \nabla_0 f \in \mathbb{R} \) and each \( \nabla_a f(\Theta) \) is a \( d_a \times d_a \) traceless symmetric matrix. Similarly, for \( a, b \in [k] \) (i.e., for the blocks of the submatrix to the lower-right of the lines) the components \( \nabla_{ab}^2 f(\Theta) \) of the Hessian are linear operators from the space of traceless symmetric \( d_b \times d_b \) matrices to the space of traceless symmetric \( d_a \times d_a \) matrices, while \( \nabla_{00}^2 f \) is a linear operator from \( \mathbb{R} \) to the space of traceless symmetric \( d_a \times d_a \) matrices (hence can itself be viewed as such a matrix), \( \nabla_{0a} f \) is the adjoint of this linear operator, and \( \nabla_{00}^2 f(\Theta) \) is a real number.

We note that the Hessian is symmetric with respect to the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{H} \). Just like in the Euclidean case, the Hessian is convenient to characterize strong convexity. Indeed, \( \langle H, \nabla^2 f(\Theta)H \rangle = \partial_{t=0}^2 f(\exp_{\Theta}(tH)) \) for all \( H \in \mathbb{H} \). Thus, \( f \) is geodesically convex if and only if the Hessian is positive semidefinite, that is, \( \nabla^2 f(\Theta) \succeq 0 \). Similarly, \( f \) is \( \lambda \)-strongly geodesically convex if and only if \( \nabla^2 f(\Theta) \succeq \lambda I_\mathbb{H} \), i.e., the Hessian is positive definite with eigenvalues larger than or equal to \( \lambda \).

We can now state and prove the following lemma, which shows that strong convexity in a ball about a point where the gradient is sufficiently small implies the optimizer cannot be far.
LEMMA 3.6. Let \( f : \mathbb{P} \to \mathbb{R} \) be geodesically convex and twice differentiable. Assume the gradient at some \( \Theta \in \mathbb{P} \) is bounded as \( \| \nabla f(\Theta) \|_F \leq \delta \), and that \( f \) is \( \lambda \)-strongly geodesically convex in a ball \( B_r(\Theta) \) of radius \( r > \frac{2\delta}{\lambda} \). Then the sublevel set \( \{ \Upsilon \in \mathbb{P} : f(\Upsilon) \leq f(\Theta) \} \) is contained in the ball \( B_{2\delta/\lambda}(\Theta) \), \( f \) has a unique minimizer \( \hat{\Theta} \), this minimizer is contained in the smaller ball \( B_{\delta/\lambda}(\Theta) \), and

\[
(3.2) \quad f(\hat{\Theta}) = f(\Theta) - \frac{\delta^2}{2\lambda}.
\]

PROOF. We first show that the sublevel set of \( f(\Theta) \) is contained in the ball of radius \( \frac{2\delta}{\lambda} \). Consider \( g(t) := f(\exp_{\Theta}(tH)) \), where \( H \in \mathbb{H} \) is an arbitrary vector of unit norm \( \| H \|_F = 1 \). Then, using the assumption on the gradient,

\[
(3.3) \quad g'(0) = \frac{d}{dt} f(\exp_{\Theta}(tH)) = \langle \nabla f(\Theta), H \rangle \geq -\| \nabla f(\Theta) \|_F \| H \|_F \geq -\delta.
\]

Since \( f \) is \( \lambda \)-strongly geodesically convex on \( B_r(\Theta) \), we have \( g''(t) \geq \lambda \) for all \( |t| \leq r \). It follows that for all \( 0 \leq t \leq r \) we have

\[
(3.4) \quad g(t) \geq g(0) - \delta t + \frac{1}{2} \lambda t^2.
\]

Plugging in \( t = r \) yields \( g(r) \geq g(0) + \left( \frac{\lambda r}{2} - \delta \right) r > g(0) \). Since \( g \) is convex due to the geodesic convexity of \( f \), it follows that, for any \( t \geq r \),

\[
g(0) < g(r) \leq \left( 1 - \frac{r}{t} \right) g(0) + \frac{r}{t} g(t),
\]

hence

\[
f(\Theta) = g(0) < g(t) = f(\exp_{\Theta}(tH)).
\]

Thus, since \( H \) was an arbitrary unit norm tangent vector, the sublevel set of \( f(\Theta) \) is contained in the ball of radius \( r \) about \( \Theta \). By replacing \( r \) with any smaller \( r' > \frac{2\delta}{\lambda} \), we see that the sublevel set is in fact contained in the closed ball of radius \( \frac{2\delta}{\lambda} \). In particular, the minimum of \( f \) is attained and any minimizer \( \hat{\Theta} \) is contained in this ball. Moreover, as the right-hand side of Eq. (3.4) takes a minimum at \( t = \frac{\lambda}{2\delta} \), we have \( g(t) \geq g(0) - \frac{\delta t^2}{2\lambda} \) for all \( 0 \leq t \leq r \). By definition of \( g \), this implies Eq. (3.2).

Next, we prove that any minimizer of \( f \) is necessarily contained in the ball of radius \( \frac{\delta}{\lambda} \). To see this, take an arbitrary minimizer \( \hat{\Theta} \) and write it in the form \( \hat{\Theta} = \exp_{\Theta}(TH) \), where \( H \in \mathbb{H} \) is a unit vector and \( T > 0 \).

As before, we consider the function \( g(t) = f(\exp_{\hat{\Theta}}(tH)) \). Then, using Eq. (3.3), the convexity of \( g(t) \) for all \( t \in \mathbb{R} \) and the \( \lambda \)-strong convexity of \( g(t) \) for \( |t| \leq r \), we have

\[
0 = g'(T) = g'(0) + \int_0^T g''(t) \, dt \geq \lambda \min(T, r) - \delta.
\]

If \( T > r \) then we have a contradiction as \( \lambda r - \delta > \lambda r/2 - \delta > 0 \). Therefore we must have \( T \leq r \) and hence \( \lambda T - \delta \leq 0 \), so \( T \leq \frac{\delta}{\lambda} \). Thus we have proved that any minimizer of \( f \) is contained in the ball of radius \( \frac{\delta}{\lambda} \).

We still need to show that the minimizer is unique; that this follows from strong convexity is convex optimization “folklore,” but we include a proof nonetheless. Indeed, suppose that \( \hat{\Theta} \) is a minimizer and let \( H \in \mathbb{H} \) be arbitrary. Consider \( h(t) := f(\exp_{\hat{\Theta}}(tH)) \). Then the function \( h(t) \) is convex, has a minimum at \( t = 0 \), and satisfies \( h''(0) > 0 \), since \( f \) is \( \lambda \)-strongly geodesically convex near \( \hat{\Theta} \), as \( \hat{\Theta} \in B_r(\Theta) \) by what we showed above. It follows that \( h(t) > h(0) \) for any \( t \neq 0 \). Since \( H \) was arbitrary, this shows that \( f(\Upsilon) > f(\hat{\Theta}) \) for any \( \Upsilon \neq \hat{\Theta} \).
Using the geodesic distance bounds from the previous lemma allows us to obtain bounds on the statistical distance measures $D_F$ and $D_{op}$ of the overall precision matrix as well as of the individual Kronecker factors, assuming strong convexity.

**Lemma 3.7 (From geodesic distance to $D_F$, $D_{op}$).** Suppose the geodesic distance between $\hat{\Theta}, \Theta \in \mathcal{P}$ satisfies $d(\hat{\Theta}, \Theta) \leq \delta$ for $\delta \leq 1/\sqrt{d_{\max}}$. Writing $\hat{\Theta} = \hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_k$ and $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$ with $(\det \hat{\Theta}_1)^{1/d_1} = \cdots = (\det \hat{\Theta}_k)^{1/d_k}$ and $(\det \Theta_1)^{1/d_1} = \cdots = (\det \Theta_k)^{1/d_k}$, we have

$$D_{op}(\hat{\Theta}_a \| \Theta_a) \leq D_F(\hat{\Theta}_a \| \Theta_a) \leq 2\sqrt{d_a} \cdot \delta$$

and

$$D_{op}(\hat{\Theta} \| \Theta) \leq D_F(\hat{\Theta} \| \Theta) \leq 2k \sqrt{D} \cdot \delta e^{2k\delta}.$$

**Proof.** It suffices to prove the bounds for $D_F$. By the invariance of $D_F$ and the geodesic distance, we may assume that $\Theta = I_D$, i.e., $\Theta_a = I_a$ for all $a \in [k]$. By our assumption on the geodesic distance, we may write $\hat{\Theta} = \exp_{I_a}(H)$ for $H \in \mathbb{H}$ and $\|H\|_F \leq \delta$. Then by our convention, the Kronecker factors are given by

$$\hat{\Theta}_a = e^{\mu_a/\sqrt{d_a}H_a} = e^{\mu_a}I_{a} + \sqrt{d_a}H_a$$

for all $a \in [k]$, since $H_a$ is traceless. Note that for each $a \in [k]$ we have

$$\left\| \frac{H_0}{k}I_{a} + \sqrt{d_a}H_a \right\|_F^2 = \frac{1}{k^2}\|H_0\|^2d_a + d_a\|H_a\|_F^2 \leq d_a\|H\|_F^2 \leq d_a\delta^2.$$

By assumption, the above is at most one, so by Fact B.1 we obtain

$$D_F(\hat{\Theta}_a \| I_a) = \|I_a - \hat{\Theta}_a\|_F = \|I_a - e^{\mu_a}I_{a} + \sqrt{d_a}H_a\|_F$$

$$\leq 2\|\frac{H_0}{k}I_{a} + \sqrt{d_a}H_a\|_F \leq 2\sqrt{d_a} \cdot \delta,$$

which establishes the first bound. The second now follows easily by a telescoping sum:

$$D_F(\hat{\Theta} \| I_D) = \|I_{d_1} \otimes \cdots \otimes I_{d_k} - \hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_k\|_F$$

$$\leq \sum_{a=1}^{k}\|\hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_{a-1}\|_F \|I_{d_a} - \hat{\Theta}_a\|_F \|I_{d_{a+1}} \otimes \cdots \otimes I_{d_k}\|_F$$

$$\leq 2\sqrt{D} \cdot \delta \sum_{a=1}^{k}(1 + 2\delta)^{a-1} \leq 2k \sqrt{D} \cdot \delta e^{2k\delta}.$$ \hfill \Box

### 3.3. Bounding the gradient.

Proceeding according to step 2 of the plan outlined in Section 3.2, we now compute the gradient of the objective function and bound it using matrix concentration results.

To calculate the gradient, we need a definition from linear algebra. Recall that our data comes as an $n$-tuple $x = (x_1, \ldots, x_n)$ of $k$-tensors. As in Example 3.4, let $\rho := \frac{1}{nD} \sum_i x_i x_i^T$ denote the “second sample moments”. and rewrite the objective function (2.1) as

$$f_x(\Theta) = \text{Tr} \rho \Theta - \frac{1}{D} \log \det \Theta.$$

We may also consider the “second sample moments” of a subset of the coordinates $J \subseteq [k]$. For this the following definition is useful.
DEFINITION 3.8 (Partial trace). Let \( \rho \) be an operator on \( \mathbb{R}^{d_1} \otimes \ldots \otimes \mathbb{R}^{d_k} \), and \( J \subseteq [k] \) an ordered subset. Define the partial trace \( \rho^{(J)} \) as the \( d_J \times d_J \)-matrix, \( d_J = \prod_{a \in J} d_a \), that satisfies the property that

\[
\text{Tr} \rho^{(J)} H = \text{Tr} \rho H_J
\]

for any \( d_J \times d_J \) matrix \( H \), where \( H_J \) denotes the operator on \( \mathbb{R}^{d_1} \otimes \ldots \otimes \mathbb{R}^{d_k} \) that acts as \( H \) on the tensor factors labeled by \( J \) (in the order determined by \( J \)) and as the identity on the rest. This property uniquely determines \( \rho^{(J)} \). We write \( \rho^{(a)} \) and \( \rho^{(ab)} \) when \( J = \{a\} \) and \( J = \{a, b\} \), respectively.

If \( \rho \) is positive (semi)definite then so is \( \rho^{(J)} \). Moreover, \( \text{Tr} \rho = \text{Tr} \rho^{(J)} \) and \( (\rho^{(J)})^{(K)} = \rho^{(K)} \) for \( K \subseteq J \).

Concretely, the partial trace \( \rho^{(J)} \) can be calculated as follows: Analogously to the discussion in Section 2.1, “flatten” the data \( x \) by regarding it as a \( d_J \times N_J \) matrix \( x^{(J)} \), where \( N_J = \frac{nD}{d_J} \); then \( \rho^{(J)} = \frac{1}{nD} x^{(J)} (x^{(J)})^T \).

The components of the gradient can be readily computed in terms of partial traces.

**LEMMA 3.9 (Gradient).** Let \( \rho = \frac{1}{nD} \sum_{i=1}^n x_i x_i^T \). Then the components of the gradient \( \nabla f_x \) at the identity are given by

\[
\nabla_a f_x = \sqrt{d_a} \left( \rho^{(a)} - \frac{\text{Tr} \rho}{d_a} I_{d_a} \right)
\]

for \( a \in [k] \),

\[
\nabla_0 f_x = \text{Tr} \rho - 1.
\]

**PROOF.** For all \( a \in [k] \) and any traceless symmetric \( d_a \times d_a \) matrix \( H \), we have

\[
\langle \nabla_a f_x (I_D), H \rangle = \partial_{t=0} f_x (e^{t \sqrt{\text{Tr}} H^{(a)}}) = \partial_{t=0} \left( \text{Tr} \rho e^{t \sqrt{\text{Tr}} H^{(a)}} - \frac{1}{D} \log \det (e^{t \sqrt{\text{Tr}} H^{(a)}}) \right)
\]

\[
= \sqrt{d_a} \text{Tr} \rho H^{(a)} = \sqrt{d_a} \text{Tr} \rho^{(a)} H
\]

using Eqs. (3.5) and (3.6) and that \( \text{Tr} H^{(a)} = 0 \) since \( \text{Tr} H = 0 \). Since \( \nabla_a f_x \) is traceless and symmetric by definition, while \( \rho^{(a)} \) is symmetric, this implies that \( \nabla f_x \) is the orthogonal projection of \( \rho^{(a)} \) onto the traceless matrices, i.e.,

\[
\nabla_a f_x = \sqrt{d_a} \left( \rho^{(a)} - \frac{\text{Tr} \rho^{(a)}}{d_a} I_{d_a} \right) = \sqrt{d_a} \left( \rho^{(a)} - \frac{\text{Tr} \rho}{d_a} I_{d_a} \right).
\]

Finally,

\[
\nabla_0 f_x = \partial_{t=0} \left( \text{Tr} \rho e^{t} - \frac{1}{D} \log \det (e^{t I_D}) \right) = \partial_{t=0} \left( \text{Tr} \rho e^{t} - t \right) = \text{Tr} \rho - 1.
\]

\( \square \)

**REMARK 3.10 (Gradient at other points from equivariance).** In the previous lemma we only computed the gradient at the identity. However, this is without loss of generality, since from the calculations above one easily obtains \( \nabla f_x (\Theta) = \nabla f_{\Theta^{1/2} x} (I) \). That is, the gradient \( \nabla f_x (\Theta) \) is given by Eqs. (3.7) and (3.8) with \( \rho \) replaced by \( \Theta^{1/2} \rho \Theta^{1/2} \).

Having calculated the gradient of the objective function, we are ready to state our bound:
PROPOSITION 3.11 (Gradient bound). Let \( x = (x_1, \ldots, x_n) \) consist of independent standard Gaussian random variables in \( \mathbb{R}^D \). Suppose that \( 0 < \varepsilon < 1 \) and \( n \geq \frac{D}{\varepsilon^2} \). Then, the following occurs with probability at least \( 1 - 2(1 + k)e^{-\varepsilon^2nD/8d_{\max}} \):
\[
\|\nabla_a f_x\|_{op} \leq \frac{9\varepsilon}{\sqrt{d_a}} \quad \text{for all } a \in [k],
\]
\[
|\nabla_0 f_x| \leq \varepsilon.
\]
As a consequence,
\[
\|\nabla f_x\|_F^2 \leq (1 + 81k)\varepsilon^2 \leq 82k\varepsilon^2.
\]

To prove this we will need a standard result in matrix concentration. By the discussion below Definition 3.8, when the samples \( x = (x_1, \ldots, x_n) \) are independent standard Gaussians in \( \mathbb{R}^D \), then \( \rho^{(a)} \) is distributed as \( \frac{1}{nD}YY^T \), where \( Y \) is a random \( d_a \times N_a \) matrix with independent standard Gaussian entries, where \( N_a = \frac{nD}{d_a} \). The following result bounds the singular values of such random matrices.

THEOREM 3.12 (Corollary 5.35 of Vershynin (2012)). Let \( Y \in \mathbb{R}^{d \times N} \) have independent standard Gaussian entries where \( N \geq d \). Then, for every \( t > 0 \), the following occurs with probability at least \( 1 - 2e^{-t^2/2} \):
\[
\sqrt{N} - \sqrt{d} - t \leq \sigma_d(Y) \leq \sigma_1(Y) \leq \sqrt{N} + \sqrt{d} + t,
\]
where \( \sigma_j \) denotes the \( j \)-th largest singular value.

We will also need to bound \( \text{Tr} \rho = \frac{1}{nD}\|x\|_2^2 \). Because \( \|x\|_2^2 \) is simply a sum of \( nD \) many \( \chi^2 \) random variables, the next proposition follows from standard concentration bounds.

PROPOSITION 3.13 (Example 2.11 of Wainwright (2019)). Let \( x = (x_1, \ldots, x_n) \) consist of independent standard Gaussian random variables in \( \mathbb{R}^D \). Then, for \( 0 < t < 1 \), the following occurs with probability at least \( 1 - 2e^{-t^2nD/8} \):
\[
(1 - t)nD \leq \|x\|_2^2 \leq (1 + t)nD.
\]

Equipped with these results we now prove our gradient bound.

PROOF OF PROPOSITION 3.11. For any fixed \( a \in [k] \), recall that \( \rho^{(a)} \) has the same distribution as \( \frac{1}{nD}YY^T \), where \( Y \) is a random \( d_a \times N_a \) matrix with standard Gaussian entries where \( N_a = \frac{nD}{d_a} \).

By Theorem 3.12, we have the following bound with failure probability at most \( 2e^{-t^2/2} \):
\[
\sqrt{N_a} - \sqrt{d_a} - t \leq \sigma_d(Y) \leq \sigma_1(Y) \leq \sqrt{N_a} + \sqrt{d_a} + t.
\]
This event tells us that the eigenvalues of \( d_a\rho^{(a)} \) are in the range \( (1 - \frac{\sqrt{d_a} + t}{\sqrt{N_a}})^2, (1 + \frac{\sqrt{d_a} + t}{\sqrt{N_a}})^2 \).

Let \( t = \varepsilon\sqrt{nD/d_a} = \varepsilon\sqrt{N_a} \). Because \( n \geq d_{\max}^2/D\varepsilon^2 \) and \( 0 < \varepsilon \leq 1 \), we have \( \sqrt{d_a} \leq t \leq \sqrt{N_a} \). Hence, the eigenvalues of \( d_a\rho^{(a)} \) are contained in \( (1 - \frac{\sqrt{d_a}}{\sqrt{N_a}})^2, (1 + \frac{\sqrt{d_a}}{\sqrt{N_a}})^2 \), and so the eigenvalues of \( d_a\rho^{(a)} - I_{d_a} \) are bounded in absolute value by \( 8\varepsilon \) with failure probability at most \( 2e^{-\varepsilon^2nD/2d_a} \). Moreover, by Proposition 3.13, we have that \( |\text{Tr} \rho - 1| \leq \varepsilon \) with failure probability at most \( 2e^{-\varepsilon^2nD/8d_{\max}} \). The formulae in Lemma 3.9 and the union bound imply
\[
\|\nabla_a f_x\|_{op} \leq \frac{1}{\sqrt{d_a}}\|d_a\rho^{(a)} - I_{d_a}\|_{op} + \frac{|\text{Tr} \rho - 1|}{\sqrt{d_a}} \leq \frac{8\varepsilon}{\sqrt{d_a}} + \frac{\varepsilon}{\sqrt{d_a}} \leq \frac{9\varepsilon}{\sqrt{d_a}}
\]
for all \( a \in [k] \), as well as
\[
|\nabla_0 f_x| = |\text{Tr} \rho - 1| \leq \varepsilon,
\]
with failure probability at most \( 2(k + 1)e^{-\varepsilon^2nD/8d_{\max}} \). \( \square \)
3.4. Strong convexity from expansion. In this section, we prove our strong convexity result, Proposition 3.21, in order to carry out step 3 of the plan from Section 3.2. The theorem states that, with high probability, \( f_x \) is strongly convex near the identity. We will prove it by first establishing strong convexity at the identity, Proposition 3.19, using quantum expansion techniques, and then giving a bound on how the Hessian changes away from the origin, Lemma 3.20. We then combine these results to prove Proposition 3.21 at the end of this subsection.

Similarly as for the gradient, we can compute the components of the Hessian in terms of partial traces, but now we also need to consider two coordinates at a time.

**Lemma 3.14 (Hessian).** Let \( \rho = \frac{1}{nD} \sum_{i=1}^{n} x_i x_i^T \). Then the components of the Hessian \( \nabla^2 f_x \) at the identity are given by

\[
\langle H, (\nabla^2_{aa} f_x) H \rangle = d_a \Tr \rho^{(a)} H^2
\]

\[
\langle H, (\nabla^2_{ab} f_x) K \rangle = \sqrt{d_a d_b} \Tr \rho^{(ab)} (H \otimes K)
\]

for all \( a \neq b \in [k] \) and traceless symmetric \( d_a \times d_a \) matrices \( H \), \( d_b \times d_b \) matrices \( K \), and

\[
\nabla^2_{aa} f_x \equiv \sqrt{d_a} \left( \rho^{(a)} - \frac{\Tr \rho}{d_a} I_{d_a} \right) \equiv \nabla^2_{a0} f_x,
\]

\[
\nabla^2_{00} f_x = \Tr \rho.
\]

for all \( a \in [k] \).

Here we use the conventions discussed in Definition 3.5. In particular, we identify \( \nabla^2_{a0} f_x \), which is a linear operator from the real numbers to the traceless symmetric matrices, with a traceless symmetric matrix, and similarly for its adjoint \( \nabla^2_{0a} f_x \). The notation \( \equiv \) reminds us of these identifications.

**Proof.** Note that the Hessian of \( f_x \) coincides with the one of \( \Tr \rho \Theta \). This follows from Eq. (3.5), since the Hessian of \( \log \det \Theta \) vanishes identically. Accordingly, we will compute the Hessian of \( \Tr \rho \Theta \). For \( a \in [k] \) and any traceless symmetric \( d_a \times d_a \) matrix \( H \), we have

\[
\langle H, (\nabla^2_{aa} f_x) H \rangle = \partial_{s=0} \partial_{t=0} \Tr \rho e^{(s+t)\sqrt{d_a} H(a)} = d_a \Tr \rho H^2(a) = d_a \Tr \rho^{(a)} H^2
\]

using Eq. (3.6). Similarly, for \( a \neq b \in [k] \), any traceless symmetric \( d_a \times d_a \) matrix \( H \), and any traceless symmetric \( d_b \times d_b \) matrix \( K \), we find that

\[
\langle H, (\nabla^2_{ab} f_x) K \rangle = \partial_{s=0} \partial_{t=0} \Tr \rho e^{s \sqrt{d_a} H(a) + t \sqrt{d_b} K(b)}
\]

\[
= \sqrt{d_a d_b} \Tr \rho H(a) K(b) = \sqrt{d_a d_b} \Tr \rho^{(ab)} (H \otimes K)
\]

using Eq. (3.6). Next, for \( a \in [k] \) and any traceless symmetric \( d_a \times d_a \) matrix \( H \), we have

\[
\langle H, \nabla^2_{0a} f_x \rangle = \partial_{s=0} \partial_{t=0} \Tr \rho e^{s \sqrt{d_a} H(a) + t} = \sqrt{d_a} \Tr \rho H(a) = \sqrt{d_a} \Tr \rho^{(a)} H.
\]

As we identify \( \nabla^2_{a0} f_x \) with a traceless symmetric \( d_a \times d_a \) matrix; this shows that

\[
\nabla^2_{a0} f_x = \sqrt{d_a} \left( \rho^{(a)} - \frac{\Tr \rho}{d_a} I_{d_a} \right),
\]

and similarly for the transpose. Finally,

\[
\nabla^2_{00} f_x = \partial_{s=0} \partial_{t=0} \Tr \rho e^{s+t} = \Tr \rho.
\]

\(\square\)
REMARK 3.15 (Hessian at other points from equivariance). Analogously to Remark 3.10, we can compute the Hessian at other points using \( \nabla^2 f_x(\Theta) = \nabla^2 f_{x^{1/2}} \). That is, the Hessian \( \nabla^2 f_x(\Theta) \) is given by Lemma 3.14 with \( \rho \) replaced by \( \Theta^{1/2} \rho \Theta^{1/2} \).

The most interesting part of the Hessian are the off-diagonal components for \( a \neq b \in [k] \), which up to an overall factor \( \sqrt{d_a d_b} \) can be seen as the restrictions of the linear maps

\[
(3.9) \quad \Phi^{(ab)} : \operatorname{Mat}(d_b) \to \operatorname{Mat}(d_a), \quad (H, \Phi^{(ab)}(K)) = \text{Tr} \rho^{(ab)} (H \otimes K)
\]

to the traceless symmetric matrices. Equation (3.9) is a special case of a completely positive map, which is a linear map of the form

\[
(3.10) \quad \Phi_A : \operatorname{Mat}(d_b) \to \operatorname{Mat}(d_a), \quad \Phi_A(X) = \sum_{i=1}^N A_i X A_i^T
\]

for \( d_a \times d_b \) matrices \( A_1, \ldots, A_N \). To see the connection, note that since \( \rho^{(ab)} \) is positive semidefinite, it can be written in the form \( \sum_{i=1}^N \text{vec}(A_i) \text{vec}(A_i)^T \); then \( \Phi^{(ab)} = \Phi_A \) follows. The matrices \( A_1, \ldots, A_N \) are known as Kraus operators. Equation (3.10) can also be written as

\[
(3.11) \quad \text{vec}(\Phi_A(X)) = \sum_{i=1}^N (A_i \otimes A_i) \text{vec}(X).
\]

We denote by \( \Phi^* : \operatorname{Mat}(d_a) \to \operatorname{Mat}(d_b) \) the adjoint of a completely positive map \( \Phi \) with respect to the Hilbert-Schmidt inner product; this is again a completely positive map, with Kraus operators \( A_1^T, \ldots, A_N^T \). In our proof of strong convexity, we will show that strong convexity follows if the completely positive maps \( \Phi^{(ab)} \) are good quantum expanders.

DEFINITION 3.16 (Quantum expansion). Let \( \Phi : \operatorname{Mat}(d_b) \to \operatorname{Mat}(d_a) \) be a completely positive map. Say \( \Phi \) is \( \varepsilon \)-doubly balanced if

\[
(3.12) \quad \left\| \frac{\Phi(I_{d_a}) - I_{d_a}}{\text{Tr} \Phi(I_{d_a})} \right\|_{\text{op}} \leq \frac{\varepsilon}{d_a} \quad \text{and} \quad \left\| \frac{\Phi^*(I_{d_a}) - I_{d_b}}{\text{Tr} \Phi^*(I_{d_a})} \right\|_{\text{op}} \leq \frac{\varepsilon}{d_b}.
\]

The map \( \Phi \) is an \( (\varepsilon, \eta) \)-quantum expander if \( \Phi \) is \( \varepsilon \)-doubly balanced and

\[
(3.13) \quad \| \Phi \|_0 := \max_{H \text{ traceless symmetric}} \frac{\| \Phi(H) \|_F}{\| H \|_F = 1} \leq \eta \frac{\text{Tr} \Phi(I_{d_a})}{\sqrt{d_a d_b}}.
\]

A \( (0, \eta) \)-quantum expander is called a \( \eta \)-quantum expander.

Quantum expanders originate in quantum information theory and quantum computation Hastings (2007). There one typically takes \( d_a = d_b \) and \( \varepsilon = 0 \), so that Eq. (3.13) simplifies to \( \| \Phi \|_0 \leq \eta \). Definition 3.16 is invariant under rescaling \( \Phi \mapsto c \Phi \) for \( c > 0 \). Here we follow the definitions of Kwok, Lau and Ramachandran (2019); Franks and Moitra (2020), who recognized the connection between the quantum expansion and spectral gaps of the Hessian for operator scaling (but we note that some of the following can be slightly simplified if one opts for a non-scale invariant definition). For us, the following lemma will allow us to translate quantum expansion properties into strong convexity.

LEMMA 3.17 (Strong convexity from expansion). If the completely positive maps \( \Phi^{(ab)} \) defined in Eq. (3.9) are \( (\varepsilon, \eta) \)-quantum expanders for every \( a \neq b \in [k] \), then

\[
\left\| \frac{\nabla^2 f_x}{\text{Tr} \rho} - I_H \right\|_{\text{op}} \leq (k - 1) \eta + (\sqrt{k} + 1) \varepsilon.
\]

Assuming \( k \geq 2 \), the right-hand side is at most \( k(\eta + \varepsilon) \).
It suffices to verify the hypothesis for $a < b$. Indeed, since $\text{Tr} \, \Phi^*(I_{da}) = \text{Tr} \, \Phi(I_{da})$, any $\Phi$ is an $(\varepsilon, \eta)$-quantum expander if and only if this is the case for the adjoint $\Phi^*$, but note that the adjoint of $\Phi^{(ab)}$ is $\Phi^{(ba)}$. To prepare the proof, we also note that

$$\Phi^{(ab)}(I_{da}) = \rho^{(a)} \quad \text{and} \quad (\Phi^{(ab)})^*(I_{da}) = \Phi^{(ba)}(I_{da}) = \rho^{(b)},$$

hence in particular $\text{Tr} \, \Phi^{(ab)}(I_{da}) = \text{Tr} \, \rho$. 

**Proof.** We wish to bound the operator norm of $M = \sum_{a \neq b} f_{a \rightarrow b} - I_H$, which we consider as a block matrix as in Definition 3.5. For this, we use the following basic estimate of the norm of a block matrix in terms of the norm of the matrix of block norms:

$$\|M\|_{op} \leq \|m\|_{op}, \quad \text{where} \quad m = (\|M_{a,b}\|_{op})_{a,b \in \{0, 1, \ldots, k\}}.$$ 

We first bound the individual block norms, using that the blocks can be computed using Lemma 3.14. Recall that the off-diagonal blocks of the Hessian, $\nabla^2 f_a f_x$ for $a \neq b \in [k]$, are given by the restriction of $\sqrt{d_a d_b} \Phi^{(ab)}$ to the traceless symmetric matrices. Since $\Phi^{(ab)}$ is an $(\varepsilon, \eta)$-quantum expander, we have

$$\|M_{a,b}\|_{op} = \frac{\|\nabla^2 f_{a \rightarrow b}\|_{op}}{\text{Tr} \, \rho} = \frac{\sqrt{d_a d_b}}{\text{Tr} \, \Phi^{(ab)}(I_{da})} \|\Phi^{(ab)}\|_{0} \leq \eta,$$

using that $\text{Tr} \, \Phi^{(ab)}(I_{da}) = \text{Tr} \, \rho$. The remaining off-diagonal blocks can be bounded as

$$\|M_{00}\| = \left\|\frac{\nabla^2 f_{0 \rightarrow 0}}{\text{Tr} \, \rho} \right\|_{op} = \left\|\sqrt{d_a} \left( \frac{\rho^{(a)}}{\text{Tr} \, \rho} - I_{da} \right) \right\|_F = \sqrt{d_a} \left\| \frac{\Phi^{(ab)}(I_{da})}{\text{Tr} \, \Phi^{(ab)}(I_{da})} - I_{da} \right\|_F \leq d_a \left\| \frac{\Phi^{(ab)}(I_{da})}{\text{Tr} \, \Phi^{(ab)}(I_{da})} - I_{da} \right\|_{op} \leq \varepsilon,$$

using the fact that the operator norm of a linear functional $\langle K, \cdot \rangle$ is the same as the Frobenius norm of $K$, and Eq. (3.14). On the other hand, the diagonal blocks for $a \in [k]$ can be bounded by observing that, for any traceless Hermitian $H$,

$$\|H, M_{a,a} H\| = \left| \langle H, \left( \frac{\nabla^2 f_{a \rightarrow a}}{\text{Tr} \, \rho} - I \right) H \rangle \right| = d_a \left| \text{Tr} \left( I_{da} \frac{\rho^{(a)}}{\text{Tr} \, \rho} - I_{da} \right) H^2 \right|$$

$$\leq d_a \left\| \frac{\rho^{(a)}}{\text{Tr} \, \rho} - I_{da} \right\|_{op} \|H\|_F^2 \leq \varepsilon \|H\|_F^2,$$

hence $\|M_{a,a}\|_{op} \leq \varepsilon$, while $|M_{00}| = |\frac{\nabla^2 f_{0 \rightarrow 0}}{\text{Tr} \, \rho} - 1| = 0$. To conclude the proof, decompose

$$m = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & m_{12} & \cdots & m_{1k} \\ 0 & m_{21} & 0 & \cdots & m_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m_{k1} & m_{k2} & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & m_{11} & 0 & \cdots & 0 \\ 0 & 0 & m_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{kk} \end{bmatrix} + \begin{bmatrix} 0 & m_{01} & m_{02} & \cdots & m_{0k} \\ m_{10} & 0 & 0 & \cdots & 0 \\ m_{20} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{k0} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$ 

The nonzero entries of the first matrix are bounded by $\eta$, hence its operator norm is at most $(k - 1)\eta$. The second matrix is diagonal with diagonal entries bounded by $\varepsilon$, hence its operator norm is at most $\varepsilon$. The third matrix has nonzero entries bounded by $\varepsilon$, hence its operator norm is bounded by $\sqrt{k}\varepsilon$. Using Eq. (3.15) we obtain the desired bound. \qed
We are concerned with $\Phi^{(ab)}$ that arise from random Gaussians. Just like random graphs can give rise to good expanders, it is known that random completely positive maps, namely $\Phi$ constructed by choosing Kraus operators at random from well-behaved distributions, yield good quantum expanders. When the Kraus operators are chosen to be standard Gaussian we have the following result:

**Theorem 3.18** (Pisier (2012, 2014)). Let $A_1, \ldots, A_N$ be independent $d_a \times d_b$ random matrices with independent standard Gaussian entries. Then, for every $t \geq 2$, with probability at least $1 - t^{-\Omega(d_a + d_b)}$, the completely positive map $\Phi_A$, defined as in Eq. (3.10), satisfies

$$\|\Phi_A\|_0 \leq O\left(t^2 \sqrt{N} (d_a + d_b)\right).$$

Pisier’s actual result is slightly different. We present the details in Appendix A.

When the samples $x = (x_1, \ldots, x_n)$ are independent standard Gaussians in $\mathbb{R}^D$, the random completely positive maps $\Phi^{(ab)}$ we are interested in have the same distribution as $\frac{1}{d_a} \Phi_A$, where the Kraus operators $A_1, \ldots, A_N$ are $d_a \times d_b$ matrices with independent standard Gaussian entries and $N = \frac{nD}{d_a d_b}$. Accordingly, strong convexity at the identity follows quite easily from Theorem 3.18 once double balancedness can be controlled. For the latter, observe that

$$\|\Phi^{(ab)}(I_{d_a}) - \frac{I_{d_a}}{d_a}\|_{\text{op}} = \frac{1}{\text{Tr} \rho} \|\rho^{(a)} - \frac{\text{Tr} \rho}{d_a} I_{d_a}\|_{\text{op}} = \frac{1}{1 + \nabla_0 f_x} \frac{1}{\sqrt{d_a}} \|\nabla_a f_x\|_{\text{op}},$$

by Lemma 3.9, and similarly for the adjoint. Therefore, the completely positive maps $\Phi^{(ab)}$ are $\varepsilon$-doubly balanced if and only if, for all $a \in [k]$,

$$(3.16) \quad \sqrt{d_a} \|\nabla_a f_x\|_{\text{op}} \leq \varepsilon \text{Tr} \rho = (1 + \nabla_0 f_x) \varepsilon,$$

hence double balancedness can be controlled using the gradient bounds in Proposition 3.11.

We now state and prove our strong convexity result at the identity:

**Proposition 3.19** (Strong convexity at identity). There is a universal constant $C > 0$ such that the following holds. Let $x = (x_1, \ldots, x_n)$ be independent standard Gaussian random variables in $\mathbb{R}^D$, where $n \geq Ck \frac{d_{\text{max}}}{d_{\text{min}}}$. Then, with probability at least $1 - k^2 (\frac{nD}{d_{\text{max}}})^{\Omega(d_{\text{min}})}$,

$$\|\nabla^2 f_x - \nabla^2 f_{\text{min}}\|_{\text{op}} \leq \frac{1}{4};$$

in particular, $f_x$ is $\frac{3}{4}$-strongly convex at the identity.

**Proof.** By Lemma 3.17, it is enough to prove that with the desired probability all $\Phi^{(ab)}$ are $(\varepsilon, \eta) := \left(\frac{1}{40k^{1/2}}, \frac{1}{20k}\right)$-quantum expanders for $a \neq b \in [k]$ and $\text{Tr} \rho \in \left(\frac{7}{8}, \frac{9}{8}\right)$. If that is the case, then

$$\|\nabla^2 f_x - \nabla^2 f_{\text{min}}\|_{\text{op}} \leq \text{Tr} \rho \cdot \left(\frac{\|\nabla^2 f_x - \nabla^2 f_{\text{min}}\|_{\text{op}}}{\text{Tr} \rho} + |1 - \text{Tr} \rho|\right) \leq \left((k - 1)\eta + (\sqrt{k} + 1)\varepsilon\right) \text{Tr} \rho + |1 - \text{Tr} \rho| \leq \frac{1}{4}.$$

Firstly, $\text{Tr} \rho = \frac{1}{nD} \|X\|^2$ is in $\left(\frac{7}{8}, \frac{9}{8}\right)$ with failure probability $e^{-\Omega(nD)}$ by Proposition 3.13.

Next, we describe an event that implies the $\Phi^{(ab)}$ are all $\varepsilon$-doubly balanced for $\varepsilon = \frac{\eta}{\sqrt{k^{1/2}}}$. By Eq. (3.16), this is equivalent to the condition $\sqrt{d_a} \|\nabla_a f_x\|_{\text{op}} \leq \varepsilon \text{Tr} \rho$ for all $a \in [k]$. By Proposition 3.11, and assuming the bound $\text{Tr} \rho \geq \frac{7}{8}$ from above, the latter occurs with failure probability $ke^{-\Omega(nD)}$ provided $n \geq Ck \frac{d_{\text{max}}}{d_{\text{min}}}$. For a universal constant $C > 0$. 
Finally, we describe an event that ensures that \( \| \Phi^{(ab)} \|_0 \leq \eta \frac{\text{Tr} \rho}{\sqrt{d_a d_b}} \) for any fixed \( a \neq b \), which is the other condition needed for quantum expansion. Recall that each \( \Phi^{(ab)} \) is distributed as \( \frac{1}{\sqrt{D}} \Phi_A \), where \( A \) is a tuple of \( \frac{n D}{d_a d_b} \) many \( d_a \times d_b \) matrices with independent standard Gaussian entries. Thus, taking \( t^2 = O(\frac{\sqrt{n D}}{d_a + d_b}) \) and again assuming that \( \text{Tr} \rho \geq \frac{7}{8} \), we have \( \| \Phi^{(ab)} \|_0 \leq \eta \frac{\text{Tr} \rho}{\sqrt{d_a d_b}} \) by Theorem 3.18, with failure probability at most \( (\frac{\sqrt{n D}}{k d_{\max}})^{-\Omega(d_{\min})} \).

By the union bound, we conclude that all \( \Phi^{(ab)} \) for \( a \neq b \) are \((\varepsilon, \eta)\)-quantum expanders and that \( \text{Tr} \rho \in (\frac{7}{8}, \frac{9}{8}) \), up to a failure probability of at most

\[
e^{-\Omega(n D)} + k e^{-\Omega(\frac{n D}{k d_{\max}})} + k^2 \left( \frac{\sqrt{n D}}{k d_{\max}} \right)^{-\Omega(d_{\min})}.
\]

The final term dominates, which implies the desired failure probability. To see that the final term dominates compare exponents: it suffices to show that \( n D / k d_{\max} \geq d_{\min} \log(\frac{\sqrt{n D}}{k d_{\max}}) \) by our assumption on \( n \), which states that \( \alpha := n D / k d_{\max} \geq C \). Writing the desired inequality in terms of \( \alpha \), we need \( d_{\max} \alpha \geq d_{\min} \log(\sqrt{\alpha / k}) \). This holds for \( C \) large enough.

We now show our second strong convexity result, namely that if our function is strongly convex at the identity then it is also strongly convex in an operator norm ball about the identity. For this, we define

\[
(3.17) \quad d_{\text{op}}(A, B) := \| \log B^{-1/2} A B^{-1/2} \|_{\text{op}},
\]

which is a metric on \( \text{PD}(D) \) and hence also on \( \mathbb{P} \) (cf. Eq. (3.1)). Then we have the following result, which we prove in Appendix B.

**Lemma 3.20 (Robustness of strong convexity).** There is a universal constant \( 0 < \varepsilon_0 < 1 \) such that if \( \| \nabla_f x(I_D) \|_{\text{op}} \leq \varepsilon_0 / \sqrt{d_a} \) for all \( a \in [k] \) and \( |\nabla_0 f_x(I_D)| \leq \varepsilon_0 \), then

\[
\| \nabla^2 f_x(\Theta) - \nabla^2 f_x \|_{\text{op}} = O(\delta)
\]

for any \( \Theta \in \mathbb{P} \) such that \( \delta := d_{\text{op}}(\Theta, I_D) \leq \varepsilon_0 \). In particular, for any \( \lambda > 0 \), if \( f_x \) is \( \lambda \)-strongly convex at \( I_D \) then \( f_x \) is \((\lambda - O(\delta))\)-strongly convex at \( \Theta \).

Finally we obtain our strong convexity result near the identity.

**Proposition 3.21 (Strong convexity near identity).** There are constants \( C, c > 0 \) such that the following holds. Let \( x = (x_1, \ldots, x_n) \) be independent standard Gaussian random variables in \( \mathbb{R}^D \), where \( n \geq Ck \frac{d_{\max}}{D} \). Then, with probability at least \( 1 - k^2 \left( \frac{\sqrt{n D}}{k d_{\max}} \right)^{-\Omega(d_{\min})} \), the function \( f_x \) is \( \frac{1}{2} \)-strongly convex at any point \( \Theta \in \mathbb{P} \) such that \( d_{\text{op}}(\Theta, I_D) \leq c \).

**Proof.** We can choose \( C > 0 \) such that both Propositions 3.11 and 3.19 apply (the former with \( \varepsilon \leq \varepsilon_0 / 9 \), where \( \varepsilon_0 \) is the universal constant from Lemma 3.20). Then the assumptions of Lemma 3.20 are satisfied for \( \lambda = \frac{3}{4} \) with failure probability at most

\[
2(k + 1) e^{-\varepsilon^2 \frac{n D}{8 d_{\max}}} + k^2 \left( \frac{\sqrt{n D}}{k d_{\max}} \right)^{-\Omega(d_{\min})},
\]

where the latter term dominates, and there exists a constant \( 0 < c \leq \varepsilon_0 \) such that \( f \) is \( \frac{1}{2} \)-strongly convex at any point \( \Theta \) such that \( d_{\text{op}}(\Theta, I_D) \leq c \). \( \Box \)
Remark 3.22. While Proposition 3.21 uses $d_{op}$ to quantify closeness to the identity, we can easily translate it into a statement in terms of the geodesic distance. Namely, under the same hypotheses it holds that $f_x$ is $\frac{1}{2}$-strongly convex on the geodesic ball $B_r(I_D)$ of radius

$$r = \frac{c}{\sqrt{(k+1)d_{\max}}}.$$ 

where $c > 0$ is the universal constant from Proposition 3.21. Indeed, if $\Theta = \exp_{I_D}(H)$, then

$$\|\log \Theta\|_{op} \leq |H_0| + \sum_{a=1}^{k} \sqrt{d_a} \|H_a\|_{op} \leq \sqrt{d_{\max}} \left(|H_0| + \sum_{a=1}^{k} \|H_a\|_{F}\right) \leq \sqrt{d_{\max}} \sqrt{k+1} \|H\|_{F},$$

so if $d(\Theta, I_D) = \|H\|_{F} \leq r$, then $d_{op}(\Theta, I_D) = \|\log \Theta\|_{op} \leq c$.

3.5. Proof of Theorem 2.4. We are now ready to prove the main result of this section according to the plan outlined in Section 3.2. We first state a result that bounds the geodesic distance between the precision matrix and the MLE. The theorem then follows from bounds for $D_{op}$ and $D_{F}$ in terms of geodesic distance.

Proposition 3.23 (Tensor normal geodesic error). There is a universal constant $C > 0$ such that the following holds. Suppose that $t \geq 1$ and $n \geq Ck^2 d_{\max}^2 t^2$. (3.18)

Then the MLE $\hat{\Theta}$ for $n$ independent samples of the tensor normal model with precision matrix $\Theta$ satisfies

$$d(\hat{\Theta}, \Theta) = O\left(\frac{\sqrt{k} d_{\max}}{\sqrt{nD}} t\right),$$

with probability at least

$$1 - ke^{-\Omega\left(t^2 d_{\max}\right)} - k^2 \left(\frac{\sqrt{nD}}{kd_{\max}}\right)^{-\Omega(d_{\min})}.$$ 

Proof. By step 1 in Section 3.2, it is enough to prove the theorem assuming $\Theta = I_D$. Assuming this, we now show that the minimizer of $f_x$ exists and is close to $\Theta = I_D$ with high probability. Let $c > 0$ be the constant from Proposition 3.21. Consider the following two events:

1. $\|\nabla f_x\|_{F} \leq \delta := \sqrt{82k} d_{\max} \frac{D}{\sqrt{nD}} t$.
2. $f_x$ is $\lambda$-strongly convex on the geodesic ball $B_r(I_D)$, where $\lambda = \frac{1}{2}$ and radius $r := \frac{c}{\sqrt{(k+1)d_{\max}}}$. 

Now, Proposition 3.11 (with $\varepsilon = \frac{\delta}{\sqrt{82k}}$) shows that, assuming $\frac{\delta}{\sqrt{82k}} < 1$ and $n \geq \frac{d_{\max}^2}{D(\sqrt{82k})^2}$, that is, $n > \frac{d_{\max}^2}{D} t^2$ and $t \geq 1$, the first event holds up to a failure probability of at most $2(k+1)e^{-\left(\frac{\delta}{\sqrt{82k}}\right)^2 \frac{nD}{d_{\max}}} = ke^{-\Omega(t^2 d_{\max})}$. 


On the other hand, Proposition 3.21 and Remark 3.22 show that, assuming \( n \geq Ck^{d_\text{max}} t^2 \frac{\sqrt{d_\text{max}}}{D} \) for a universal constant \( C > 0 \), the second event holds up to a failure probability of at most
\[
k^2 \left( \frac{\sqrt{nD}}{kd_{\text{max}}} \right)^{-\Omega(d_{\text{min}})}.
\]

Note that both assumptions follow from Eq. (3.18) for some appropriate choice of \( C \). Thus, by the above and the union bound, both events hold simultaneously with the advertised success probability. For \( C \) large enough, Eq. (3.18) moreover implies \( r > \frac{2\delta}{\lambda} \), since the latter is equivalent to
\[
n > \frac{16 \cdot 82}{c^2} k(k + 1) \frac{d_{\text{max}}^3}{D} t^2.
\]
Thus, if the above two events hold, Lemma 3.6 applies (with our choice of \( \delta \) and \( \lambda \)) and shows that the MLE \( \hat{\Theta} \) exists, is unique, and has geodesic distance at most \( \frac{2\delta}{\lambda} = 2\delta \) from \( \Theta = I_D \).

The theorem now follows as a corollary. We restate it for convenience.

**THEOREM 2.4 (Tensor normal Frobenius error, restated).** There is a universal constant \( C > 0 \) such that the following holds. Suppose \( t \geq 1 \) and
\[
n \geq Ck^2 \frac{d_{\text{max}}^2}{D} \max\{k, d_{\text{max}}\} t^2.
\]
Then the MLE \( \hat{\Theta} = \hat{\Theta}_1 \otimes \cdots \otimes \hat{\Theta}_k \) for \( n \) independent samples of the tensor normal model with precision matrix \( \Theta = \Theta_1 \otimes \cdots \otimes \Theta_k \) satisfies
\[
D_F(\hat{\Theta}_a \parallel \Theta_a) = O\left( tk^{1/2} d_{\text{max}} \sqrt{\frac{d_a}{nD}} \right) \quad \text{for all } a \in [k]
\]
and
\[
D_F(\hat{\Theta} \parallel \Theta) = O\left( tk^{3/2} \frac{d_{\text{max}}}{\sqrt{n}} \right),
\]
with probability at least
\[
1 - ke^{-\Omega\left( t^2 d_{\text{max}} \right)} - k^2 \left( \frac{\sqrt{nD}}{kd_{\text{max}}} \right)^{-\Omega(d_{\text{min}})}.
\]

**PROOF.** By Proposition 3.23, noting that Eq. (2.5) implies Eq. (3.18), we have with the desired failure probability that \( \hat{\Theta} \) is at most geodesic distance \( \delta \) from \( \Theta \), where \( \delta = O\left( \sqrt{k d_{\text{max}} t / \sqrt{nD}} \right) \). Moreover, Eq. (3.18) also implies \( \delta \leq 1/\sqrt{d_{\text{max}}} \) if we choose \( C \) large enough. Thus, Lemma 3.7 applies, and we have that for each \( a \in [k] \),
\[
D_F(\hat{\Theta}_a \parallel \Theta_a) \leq 2\sqrt{d_a} \cdot \delta = O\left( tk^{1/2} d_{\text{max}} \sqrt{\frac{d_a}{nD}} \right)
\]
as well as
\[
D_F(\hat{\Theta} \parallel \Theta) \leq 2k \sqrt{D} \cdot \delta e^{2k\delta} = O\left( tk^{3/2} \frac{d_{\text{max}}}{\sqrt{n}} \right).
\]
In the last step, we used that \( \delta k = O(1) \), which also follows from Eq. (2.5). Indeed, it is equivalent to \( n \geq C'k^3 \frac{d_{\text{max}}^2}{D} t^2 \) for some \( C' > 0 \).
4. Improvements for the matrix normal model. We now prove Theorem 2.7, which improves over Theorem 2.4 in the case of the matrix normal model \((k = 2)\). Throughout this section we assume without loss of generality that \(d_1 \leq d_2\). Our results for the matrix normal model are stronger in that:

1. the MLE is shown to be close to the truth in spectral norm \(D_{op}\) rather than the looser Frobenius norm \(D_F\),
2. the errors are tight for the individual factors, and
3. the failure probability is inverse exponential in the number of samples rather than inverse polynomial.

4.1. Spectral gap and expansion. The proof plan is similar to that in Section 3.2, the main difference being that we now work directly with quantum expansion instead of translating into strong convexity. An important tool will be a bound by Kwok, Lau and Ramachandran (2019) which uses the notion of a spectral gap, which is closely related to quantum expansion.

**Definition 4.1 (Spectral gap).** Let \(\Phi : \text{Mat}(d_b) \to \text{Mat}(d_a)\) be a completely positive map. Say \(\Phi\) has spectral gap \(\gamma > 0\) if

\[
\sigma_2(\Phi) \leq (1 - \gamma) \frac{\text{Tr} \, \Phi(I_{d_b})}{\sqrt{d_ad_b}}
\]

where \(\sigma_2\) denotes the second largest singular value of \(\Phi\). Note that \(\gamma \leq 1\). Moreover, the definition is invariant under rescaling \(\Phi \mapsto c\Phi\) for \(c > 0\).

Recall that by the variational formula for singular values, if we let \(K \in \text{Mat}(d_b)\) be the first (right) singular vector of \(\Phi\), we can rewrite the above condition as

\[
\sigma_2(\Phi) = \max_{\langle H, K \rangle = 0} \frac{\|\Phi(H)\|_F}{\|H\|_F} \leq (1 - \gamma) \frac{\text{Tr} \, \Phi(I_{d_b})}{\sqrt{d_ad_b}}.
\]

On the other hand, the definition of an \((\varepsilon, \eta)\)-quantum expander is given in Eq. (3.13) as

\[
\|\Phi\|_0 := \max_{\langle H, I_{d_b} \rangle = 0} \frac{\|\Phi(H)\|_F}{\|H\|_F} \leq \eta \frac{\text{Tr} \, \Phi(I_{d_b})}{\sqrt{d_ad_b}}.
\]

Due to the \(\varepsilon\)-doubly balanced condition in Eq. (3.12), it turns out that these two notions are closely related, as the following lemma shows.

**Lemma 4.2 (Lemma A.3 in Franks and Moitra (2020)).** There exists a universal constant \(c > 0\) with the following property. If \(\Phi\) is an \((\varepsilon, \eta)\)-quantum expander and \(\varepsilon \leq c(1 - \eta)\), then \(\Phi\) has spectral gap \(1 - \eta - O(\varepsilon)\).

We now state the bound of Kwok, Lau and Ramachandran (2019) in our language. Because \(k = 2\), the gradient and Hessian are completely described by the single completely positive map \(\Phi^{(12)}\) (compare the formulas in Lemmas 3.9 and 3.14 with Eqs. (3.9) and (3.14)). Suppose we are given samples \(y_1, \ldots, y_n\), which we can identify with \(d_1 \times d_2\) matrices \(Y_1, \ldots, Y_n\). Then \(\Phi^{(12)} = \frac{1}{nD} \Phi_Y\), as discussed below Theorem 3.18. Moreover, the double balancedness and spectral gap are invariant under rescaling. This explains why the following bound can be purely stated in terms of \(\Phi_Y\). Recall that \(\text{SPD}(d)\) denotes the \(d \times d\) positive definite matrices of unit determinant.
Moreover,

therefore, for \( f \) conclude that hence in particular enough, we obtain

By our assumptions on \( \gamma \) also have

By Theorem 4.3 and using the assumption that \( P \)

This completes the proof.

We can immediately translate this into a statement about the MLE.

**Corollary 4.4 (Spectral gap implies MLE nearby).** There is a universal constant \( C > 0 \) such that the following holds. Let \( \varepsilon, \gamma \in (0, 1) \), \( 1 < d_1 \leq d_2 \), and suppose that the completely positive map \( \Phi_Y \) is \( \varepsilon \)-doubly balanced and has spectral gap \( \gamma \), where \( \gamma^2 \geq C \varepsilon \log d_1 \). Further assume that \( \| y \|_2 = nD \). Then the MLE \( \hat{\Theta} = \hat{\Theta}_1 \otimes \hat{\Theta}_2 \) exists, is unique, and satisfies (using our conventions)

\[
\max \left\{ \| \hat{\Theta}_1 - I_d \|_{\text{op}}, \| \hat{\Theta}_2 - I_d \|_{\text{op}} \right\} = O \left( \frac{\varepsilon \log d_1}{\gamma} \right).
\]

**Proof.** To compute the MLE, we reparametrize by \( \hat{\Theta}_1 = \lambda P_1 \) and \( \hat{\Theta}_2 = \lambda P_2 \) where \( P_1 \in \text{SPD}(d_1), P_2 \in \text{SPD}(d_2) \), and \( \lambda \in \mathbb{R}_{\geq 0} \). Plugging this reparametrization into the formula (2.1) for \( f_y \) shows that \((\lambda, P_1, P_2)\) solve

\[
\arg \min_{\lambda, P_1, P_2} \lambda^2 f_x(P_1 \otimes P_2) - \log(\lambda^2).
\]

In particular, the MLE \( \hat{\Theta}_1, \hat{\Theta}_2 \) exists uniquely if \( f_y \) has a unique minimizer \( P = P_1 \otimes P_2 \) when restricted to \( \text{SPD}(d_1) \otimes \text{SPD}(d_2) \). Such unique minimizers exist by Theorem 4.3. Given \( P_1, P_2 \), solving the simple one-dimensional optimization problem for \( \lambda \) yields

\[
\lambda = \frac{1}{\sqrt{f_y(P_1)}}.
\]

By Theorem 4.3 and using the assumption that \( \text{Tr} \rho = \| y \|_2^2 / nD = 1 \), \( f_y(P) \geq 1 - 4\varepsilon^2 \gamma \), and we also have \( f_y(P) \leq f_y(I_D) = \text{Tr} \rho = 1 \) since \( P \) is the minimizer in \( \text{SPD}(d_1) \otimes \text{SPD}(d_2) \). Therefore,

\[
1 \leq \lambda \leq \left( 1 - \frac{4\varepsilon^2}{\gamma} \right)^{-1/2}.
\]

By our assumptions on \( \gamma \) and \( \varepsilon \), we have \( \frac{\varepsilon^2}{\gamma} \leq \frac{\varepsilon}{\gamma} \leq \frac{1}{C \log d_1} \). Thus, choosing \( C > 0 \) large enough, we obtain

\[
|\lambda - 1| = O \left( \frac{\varepsilon^2}{\gamma} \right) \leq O \left( \frac{\varepsilon \log d_1}{\gamma} \right).
\]

hence in particular \( \lambda = O(1) \). Since also \( \| P_a - I_{d_a} \|_{\text{op}} = O(\varepsilon \log d_1 / \gamma) \) by Theorem 4.3, we conclude that

\[
\| \hat{\Theta}_a - I_{d_a} \|_{\text{op}} \leq \lambda \| P_a - I_{d_a} \|_{\text{op}} + |\lambda - 1| = O \left( \frac{\varepsilon \log d_1}{\gamma} \right)
\]

for \( a \in \{1, 2\} \). This completes the proof. \( \square \)
Lemma 4.2 and Theorem 4.3, along with what we have shown so far, already imply a preliminary version of Theorem 2.7. Indeed, similarly to the proof of Proposition 3.19, one can use Propositions 3.11 and 3.13 to show that under suitable assumptions on $n$, $t$, the completely positive map $\Phi^{(12)}$ is a $(t \sqrt{d_2/nd_1}, \eta)$-quantum expander for some universal constant $\eta \in (0, 1)$ with failure probability
\[ e^{-\Omega(d_2t^2)} + \left( \frac{\sqrt{nD}}{d_2} \right)^{-\Omega(d_1)}. \]
By Theorem 4.3, we immediately have that with the above failure probability the MLEs satisfy
\[ D_{\text{op}}(\Theta_a' \| \Theta_a) = O \left( t \sqrt{\frac{d_2}{nd_1}} \log d_1 \right), \]
which matches Theorem 2.7 for the larger Kronecker factor.

One of the main results of this section is the following theorem, which shows that the expansion constant $\eta$ of $\Phi$ can be made constant with exponentially small failure probability. Recall that for the matrix model, the samples $x_i$ can be viewed as $d_1 \times d_2$-matrices, which we denote by $X_i$.

**THEOREM 4.5 (Expansion).** There are universal constants $C > 0$ and $\eta \in (0, 1)$ such that the following holds. For $d_1 \leq d_2$, $d_2 > 1$, let $X = (X_1, \ldots, X_n)$ be random $d_1 \times d_2$ matrices with independent standard Gaussian entries, where $n \geq C \frac{d_2}{d_1} \max \{ \log d_2, t^2 \}$ and $t \geq 1$. Then, $\Phi_X$ is a $(t \sqrt{\frac{d_2}{nd_1}}, \eta)$-quantum expander with probability at least $1 - e^{-\Omega(d_2t^2)}$.

We will prove Theorem 4.5 in Appendix C using techniques similar to Franks and Moitra (2020) using Cheeger’s inequality. This also improves our result on strong convexity (Proposition 3.21), which will be useful in the analysis of the flip-flop algorithm. Indeed, for $k = 2$, using Theorem 4.5 (in place of Theorem 3.18) with Lemma 3.17 in the proof of Proposition 3.19 improves the failure probability in Proposition 3.19 to $1 - e^{-\Omega(d_2t^2)}$. As in the proof of Proposition 3.21, combining this failure probability bound with Lemma 3.20 yields the next corollary.

**COROLLARY 4.6.** There are universal constants $C, c > 0$ and $\lambda \in (0, 1)$ such that the following holds. For $d_1 \leq d_2$, let $x = (x_1, \ldots, x_n)$ be independent standard Gaussian random variables in $\mathbb{R}^{d_1 \times d_2}$, where $n \geq C \frac{d_2}{d_1} \max \{ \log d_2, t^2 \}$ and $t \geq 1$. Then, with probability at least $1 - e^{-\Omega(d_2t^2)}$, the function $f_x$ is $\lambda$-strongly convex at any point $\Theta \in \mathbb{P}$ such that $d_{\text{op}}(\Theta, I_D) \leq c$.

### 4.2. Proof of Theorem 2.7.
We now use Theorem 4.5 as well as some more refined concentration inequalities to prove Theorem 2.7. The additional concentration is required to obtain the tighter bounds on the smaller Kronecker factor. Throughout this section, we still assume without loss of generality that $d_1 \leq d_2$.

The idea of the proof is to apply one step of the flip-flop algorithm to “renormalize” the samples such that the second partial trace is proportional to $I_{d_2}$. This has the effect of making the second component of the gradient $\nabla f_x$ equal to zero. We will show that the first component still enjoys the same concentration exploited in Proposition 3.11 even after the step of flip-flop – thus the total gradient has become smaller, but only the second component of the MLE estimate has changed. Thus, intuitively, the total change in the first component will be small. Using Lemma 3.20 to control the change induced in the minimum eigenvalue of the Hessian by the first step of the flip-flop and applying Lemma 3.6 results in a geodesic distance error...
proportional to the new gradient after flip-flop, which gives us the tighter bound. To obtain a relative spectral norm error bound, we employ a similar strategy but with Corollary 4.4 instead of Lemma 3.6.

We now discuss the concentration bound. Let \( X_1, \ldots, X_n \) be random \( d_1 \times d_2 \) matrices with independent standard Gaussian entries. Consider new random variables \( Y_1, \ldots, Y_n \) obtained by one step of the flip-flop algorithm applied to the second, larger Kronecker factor (cf. Algorithm 1). That is, for \( i \in [n] \):

\[
Y_i = X_i \left( \frac{1}{n d_1} \sum_{i=1}^{n} X_i^T X_i \right)^{-1/2}.
\]

The completely positive map \( \Phi^{(12)} \) corresponding to the “renormalized” samples \( Y_1, \ldots, Y_n \) is \( \frac{1}{nD} \Phi_Y \). By construction, it satisfies

\[
\frac{1}{nD} \Phi_Y(I_{d_2}) = \frac{1}{d_2} \sum_{i=1}^{n} X_i \left( \frac{n}{d_2} \sum_{i=1}^{n} X_i^T X_i \right)^{-1} X_i^T \quad \text{and} \quad \frac{1}{nD} \Phi_Y^*(I_{d_1}) = \frac{I_{d_2}}{d_2}.
\]

Note also that \( \text{Tr} \Phi_Y(I_{d_2}) = \text{Tr} \Phi_Y^*(I_{d_1}) = \| Y \|^2 = nD \). Thus \( \Phi_Y \) is \( \delta \)-doubly balanced if and only if \( \| \frac{1}{nD} \Phi_Y(I_{d_2}) - \frac{I_{d_2}}{d_2} \|_{op} \leq \frac{\delta}{d_1} \).

**Lemma 4.7 (Concentration after flip-flop).** There is a universal constant \( C' > 0 \) such that the following holds. Let \( X_1, \ldots, X_n \) be random \( d_1 \times d_2 \) matrices with independent standard Gaussian entries, where \( d_1 \leq d_2 \). If \( n \geq \frac{C'}{\delta^2} \) and \( t \geq C' \), then for \( \Phi_Y \) with \( Y \) as in Eq. (4.2) we have, with probability at least \( 1 - e^{-\Omega(d_1 t^2)} \),

\[
\left\| \frac{1}{nD} \Phi_Y(I_{d_2}) - \frac{I_{d_1}}{d_1} \right\|_{op} \leq t \sqrt{\frac{1}{nD}}.
\]

By the remarks preceding the lemma, this condition implies \( \Phi_Y \) is \( t \sqrt{\frac{d_1}{nd_2}} \)-doubly balanced.

The proof of this lemma uses a result of Hayden, Leung and Winter (2006) on the overlap of two random projections, combined with a standard net argument. The details can be found in Appendix D.

We need as one final ingredient the following robustness result for quantum expansion which will play a role analogous to our Lemma 3.20.

**Lemma 4.8 (Robustness of quantum expansion, Lemma 4.4 in Franks and Moitra (2020)).** There exists a universal constant \( c > 0 \) with the following property: Let \( X = (X_1, \ldots, X_n) \), \( Y = (Y_1, \ldots, Y_n) \) be tuples of \( d_1 \times d_2 \) matrices such that \( Y_i = X_i R \) for some \( R \in \text{GL}(d_2) \). Let \( 0 < \epsilon, \eta < 1 \). If \( \Phi_X \) is an \( (\epsilon, \eta) \)-quantum expander and \( \| R^T R - I_{d_2} \|_{op} \leq \delta \) for some \( \delta \leq c \), then \( \Phi_Y \) is an \( (\epsilon + O(\delta), \eta + O(\delta)) \)-quantum expander.

With the above tools in hand, we may now prove Theorem 2.7.

**Theorem 2.7 (Matrix normal spectral error, restated).** There is a universal constant \( C > 0 \) with the following property. Suppose \( 1 < d_1 \leq d_2 \), \( t \geq 1 \), and

\[
n \geq C \frac{d_2}{d_1} \max\{ \log d_2, t^2 \log^2 d_1 \}.
\]
Then the MLE $\widehat{\Theta} = \widehat{\Theta}_1 \otimes \widehat{\Theta}_2$ for $n$ independent samples from the matrix normal model with precision matrix $\Theta = \Theta_1 \otimes \Theta_2$ satisfies

$$D_{op}(\widehat{\Theta}_1 \| \Theta_1) = O \left( t \sqrt{\frac{d_1}{nd_2}} \log d_1 \right) \quad \text{and} \quad D_{op}(\widehat{\Theta}_2 \| \Theta_2) = O \left( t \sqrt{\frac{d_2}{nd_1}} \log d_1 \right)$$

with probability at least $1 - e^{-\Omega(d_1 \varepsilon^2)}$.

**Proof.** As discussed in Section 3.2, we may assume without loss of generality that $\Theta_a = I_a$ for $a \in \{1, 2\}$. Let $x = (x_1, \ldots, x_n)$ be our tuple of samples, which we can identify with a tuple $X = (X_1, \ldots, X_n)$ independent random $d_1 \times d_2$ matrices with independent standard Gaussian entries. Define $Y = (Y_1, \ldots, Y_n)$ as in Eq. (4.2). Consider the following three events:

1. The operator $\Phi_X$ is a $(t \sqrt{d_2/nd_1}, \eta)$-quantum expander for $\eta \in (0, 1)$ as in Theorem 4.5.
2. The operator $\Phi_Y$ is $t \sqrt{d_1/nd_2}$-doubly balanced.
3. $|\frac{\|x\|_2^2}{\sqrt{d_1}} - 1| \leq t \sqrt{\frac{d_2}{nd_1}}$.

By Theorem 4.5 and our assumptions, the first event occurs with probability at least $1 - e^{-\Omega(d_1 \varepsilon^2)}$ provided we choose $C$ large enough. By Lemma 4.7 and our assumptions, the second event occurs with probability at least $1 - e^{-\Omega(d_1 \varepsilon^2)}$ assuming $t \geq C'$. Finally, the third event occurs with probability at least $1 - e^{-\Omega(d_1 \varepsilon^2)}$ by Proposition 3.13 and our assumptions. By the union bound, all three events occur simultaneously with probability at least $1 - e^{-\Omega(d_1 \varepsilon^2)}$, which is the desired success probability.

We now show that the three events together imply the desired properties. We first want to use Lemma 4.8 to relate the quantum expansion of $\Phi_X$ and $\Phi_Y$. By definition, $Y_i = X_i R$ for $R := (\frac{1}{nd_2} \sum_{i=1}^n X_i^T X_i)^{-1/2} = R^T$. Now note that

$$R^{-2} - I_{d_2} = \frac{1}{nd_1} \sum_{i=1}^n X_i^T X_i - I_{d_2} = \frac{\|x\|_2^2}{nD} \left( d_2 \frac{\Phi_X(I_{d_1})}{\text{Tr} \Phi_X(I_{d_1})} - I_{d_2} \right) + \left( \frac{\|x\|_2^2}{nD} - 1 \right) I_{d_2}.$$ 

Therefore, by the first and the third event,

$$\|R^{-2} - I_{d_2}\|_{op} = O \left( t \sqrt{\frac{d_2}{nd_1}} \right),$$

noting that $t \sqrt{\frac{d_2}{nd_1}} \leq \frac{1}{\sqrt{c}}$ can be made smaller than any constant by choosing $C$ large enough. This also implies that

$$\|R^T R - I_{d_2}\|_{op} = \|R^2 - I_{d_2}\|_{op} = O \left( t \sqrt{\frac{d_2}{nd_1}} \right).$$

Noting again that the right-hand side can be made smaller than any universal constant, we can now apply Lemma 4.8 to see that $\Phi_Y$ is a $(t \sqrt{d_1/nd_2}, \eta')$-quantum expander for some universal constant $\eta' \in (0, 1)$ (the double balancedness follows from the second event!), and then Lemma 4.2, which shows that $\Phi_Y$ has spectral gap $\gamma$ for a universal constant $\gamma \in (0, 1)$.

Finally, noting that $\|y\|_2^2 = \sum_{i=1}^n \text{Tr} Y_i^T Y_i = nD$ and using our assumption on $n$, provided we choose $C$ large enough we may apply Corollary 4.4 with $\varepsilon = t \sqrt{d_1/nd_2}$. We obtain:

$$\max \left\{ \|\widehat{\Theta}_1(Y) - I_{d_1}\|_{op}, \|\widehat{\Theta}_2(Y) - I_{d_2}\|_{op} \right\} = O \left( t \sqrt{\frac{d_1}{nd_2} \log d_1} \right).$$
where $\hat{\Theta}_a(Y)$ denotes components the MLE for the samples $Y = (Y_1, \ldots, Y_n)$. By equivariance, the components of the MLE for the samples $X = (X_1, \ldots, X_n)$ are then given by $\hat{\Theta}_1(X) = \hat{\Theta}_1(Y)$ and $\hat{\Theta}_2(X) = R \hat{\Theta}_2(Y) R$. This immediately yields the bound

$$D_{\text{op}}(\hat{\Theta}_1(X) \mid \Theta_1) = D_{\text{op}}(\hat{\Theta}_1(X) \mid I_{d_1}) = O\left(t \sqrt{\frac{d_1}{n \log d_1}}\right).$$

To bound $D_{\text{op}}(\hat{\Theta}_2(X) \mid \Theta_2)$, we use invariance of $D_{\text{op}}$ and the approximate triangle inequality (Lemma E.1) to write

$$D_{\text{op}}(\hat{\Theta}_2(X) \mid \Theta_2) = D_{\text{op}}(\hat{\Theta}_2(X) \mid I_{d_2}) = D_{\text{op}}(R \hat{\Theta}_2(Y) R \mid I_{d_2}) = D_{\text{op}}(\hat{\Theta}_2(Y) \mid R^{-2})$$

$$= O\left(D_{\text{op}}(\hat{\Theta}_2(Y) \mid I_{d_2}) + D_{\text{op}}(I_{d_2} \mid R^{-2})\right)$$

$$= O\left(t \sqrt{\frac{d_1}{n \log d_1}}\right) + O\left(t \sqrt{\frac{d_2}{n \log d_1}}\right) = O\left(t \sqrt{\frac{d_1}{n \log d_1}}\right)$$

using Eqs. (4.3) and (4.4); by choosing $C$ large enough we can ensure that the right-hand side is smaller than any universal constant, which justifies the application of Lemma E.1. Reparametrizing $t$ by $t \leftarrow t/C'$ allows us to assume $t \geq 1$ rather than $t \geq C'$.

5. Convergence of the flip-flop algorithm. In this section we prove that the flip-flop algorithms for the matrix and tensor normal models converge quickly to the MLE with high probability. The general flip-flop algorithm is stated in Algorithm 2. It generalizes Algorithm 1 described in Section 2.3 from the matrix normal model to the general situation. We first analyse it for the general tensor normal model and then give improved guarantees for the matrix normal model.

**Algorithm:**

1. **Set** $\Theta_a = I_{d_a}$ for each $a \in [k]$.

2. **For** $t = 1, \ldots, T$, repeat the following:
   - **Compute** $\rho_t = \frac{1}{nD} \sqrt{\frac{1}{2} \left( \sum_{i=1}^{n} x_i x_i^T \right) \Theta^{1/2} \Theta^{1/2}}$, where $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$.
   - **Compute** each component of the gradient using the formula $\nabla_a f_\xi(\Theta) = \sqrt{d_a} (\rho_t(a) - \text{Tr}(\rho_t(a) I_{d_a} \rho_t(a) \rho_t(a)^T) \frac{I_{d_a}}{d_a})$, where $\rho_t(a)$ denotes the partial trace (Definition 3.8), and find the index $a \in [k]$ for which $\| \nabla_a f_\xi(\Theta) \|_F$ is largest.
   - **If** $\| \nabla_a f_\xi(\Theta) \|_F \leq \delta$, output $\Theta$ and return.
   - **Update** $\Theta_a \leftarrow \frac{1}{d_a} \Theta_a \rho_t(a) \Theta_a^{1/2} \rho_t(a)^{-1} \Theta_a^{1/2}$.

Algorithm 2: Flip-flop algorithm for the tensor normal model ($k \geq 2$). See Theorems 2.9 and 2.10 for guarantees on the estimate $\Theta$ for appropriate choice of parameters $T$, $\delta$.

**Remark 5.1 (Matrix flip-flop).** To see how Algorithm 1 arises from Algorithm 2, note that if we update $\Theta_a$ in the $t$-th iteration, then the corresponding gradient component vanishes in the subsequent iteration (we discuss why this is so below Lemma 5.3). Since for the matrix normal model there are only two gradient components to consider, this means that the...
Therefore, Algorithm 1 agrees with Algorithm 2 except that in the first iteration we skip the stopping condition and always update \( \overline{\Theta}_1 \). This will not impact the analysis; see the proof of Lemma 5.6.

Before we analyze its convergence (for suitable choices of the parameters \( T \) and \( \delta \)), we discuss straightforward generalizations of standard convergence results for descent methods under strong convexity to the geodesically convex setting.

The next lemma shows that any descent method which manages to decrease the value of the function with respect to the gradient by a constant (more precisely the parameter \( \alpha \)), if starting from a sublevel set where the function is strongly convex, will converge quickly to the optimum. The proof of the lemma is a straightforward translation of a corresponding proof in Franks and Moitra (2020); we give it here for completeness.

**Lemma 5.2** (Proof of Lemma 4.11 in Franks and Moitra (2020)). Let \( f : \mathbb{P} \rightarrow \mathbb{R} \) be \( \lambda \)-strongly geodesically convex on a sublevel set \( S \). Let \( x_0 \in S \) and let \( \alpha, \beta > 0 \) such that \( \| \nabla f(x_0) \|_F^2 \leq \beta \) and \( \{ x_t \}_{t \in [T]} \) be a sequence satisfying

\[
(5.1) \quad f(x_t) \leq f(x_{t-1}) - \alpha \cdot \min \{ \beta, \| \nabla f(x_{t-1}) \|_F^2 \},
\]

for \( t \in [T] \). Then,

\[
\min_{0 \leq t \leq T} \| \nabla f(x_t) \|_F^2 \leq \| \nabla f(x_0) \|_F^2 \cdot 2^{-T \lambda}.
\]

**Proof.** Let \( f^* \) be the minimum value of the function \( f \). Since \( f \) is \( \lambda \)-strongly geodesically convex on \( S \), we have

\[
(5.2) \quad f^* \geq f(x) - \frac{1}{2 \lambda} \| \nabla f(x) \|_F^2
\]

for any \( x \in S \). Since \( \{ x_t \} \) is a descent sequence, i.e., \( f(x_t) \leq f(x_{t-1}) \) for all \( t \in [T] \), we know that each \( x_t \in S \). Therefore, Eq. (5.2) holds for any \( x_t, 0 \leq t \leq T \).

We claim that for any \( x_t \) such that \( \epsilon := \| \nabla f(x_{t+\ell}) \|_F^2 \leq \beta \), there exists \( \ell \leq 1 / \alpha \lambda \) such that \( \| \nabla f(x_{t+\ell}) \|_F^2 \leq \epsilon / 2 \). This is enough to conclude the proof of the lemma, as with this claim we see that we have the squared norm of the gradient at every sequence of \( 1 / \alpha \lambda \) steps.

To prove the claim, we assume that \( \| \nabla f(x_{t+\ell}) \|_F^2 \geq \epsilon / 2 \) for all \( \ell \in [m] \) (this is also true for \( \ell = 0 \)). We wish to show that \( m \leq 1 / \alpha \lambda \). To see this, note that from Eq. (5.1) we have

\[
f(x_{t+\ell}) \leq f(x_{t+\ell-1}) - \alpha \cdot \min \{ \beta, \| \nabla f(x_{t+\ell-1}) \|_F^2 \} \leq f(x_{t+\ell-1}) - \frac{\alpha \epsilon}{2}
\]

for all \( \ell \in [m] \), and therefore

\[
f(x_{t+m}) \leq f(x_t) - \frac{\alpha \epsilon m}{2}.
\]
Together, we find that
\[ f(x_{t+m}) \geq f^* \geq f(x_t) - \frac{1}{2\lambda} \| \nabla f(x_t) \|^2_F \geq f(x_t) - \frac{\varepsilon}{2\lambda}. \]

We now consider the flip-flop algorithm in Algorithm 2. Note that at the end of each iteration, we update only a single Kronecker factor $\Theta_a$. This update has the following property.

**Lemma 5.3 (Flip-flop update).** Let $t \in \{1, \ldots, T - 1\}$ and assume the flip-flop algorithm has not terminated before the $(t+1)$-st iteration. Then $\rho_{t+1}^{(a)} = \frac{I_a}{d_a}$, where $a \in [k]$ denotes the index chosen in the $t$-th iteration. As a consequence, $\text{Tr} \rho_t = 1$ for $t = 2, \ldots, T$.

In view of Lemma 3.9 and Remark 3.10, the above means that in each iteration but the first, $\nabla_0 f_x(\Theta) = 0$ as well as $\nabla_a f_x(\Theta) = 0$ for the $a \in [k]$ chosen in the preceding iteration. Thus the flip-flop algorithm can be understood as carrying out an alternating minimization or coordinate descent of the objective function $f_x$.

**Proof.** Let $\Theta$ denote the precision matrix at the beginning of the $t$-th iteration. Then,
\[
\rho_{t+1}^{(a)} = \left( \frac{1}{d_a} \Theta_a^{1/2} \left( \rho_t^{(a)} \right)^{-1} \Theta_a^{1/2} \right) \left( \frac{1}{d_a} \Theta_a^{1/2} \left( \rho_t^{(a)} \right)^{-1} \Theta_a^{1/2} \right)^{1/2} = \frac{1}{d_a} \left( \Theta_a^{1/2} \left( \rho_t^{(a)} \right)^{-1} \Theta_a^{1/2} \right)^{1/2} = \frac{1}{d_a} I_a.
\]

We now show that the flip-flop algorithm produces a descent sequence as in Eq. (5.1).

**Lemma 5.4 (Descent).** Let $k \geq 2$ and $t \in \{2, \ldots, T - 1\}$. Assume that the flip-flop algorithm has not terminated before the $(t+1)$-st iteration. Let $\Theta^{(t)}$, $\Theta^{(t+1)}$ denote the precision matrices at the beginning of the $t$-th and the $(t+1)$-st iteration, respectively. Then,
\[
f_x(\Theta^{(t+1)}) \leq f_x(\Theta^{(t)}) - \frac{1}{6(k-1)} \min \left\{ \frac{k-1}{d_{\text{max}}}, \| \nabla f_x(\Theta^{(t)}) \|^2_F \right\}.
\]

If $\text{Tr} \rho_1 = 1$ then this estimate also holds for $t = 1$.

**Proof.** Recall that
\[
f_x(\Theta^{(t)}) = \text{Tr} \rho_t - \frac{1}{D} \log \det \Theta^{(t)}.
\]

and similarly for $f_x(\Theta^{(t+1)})$. By Lemma 5.3, we have $\text{Tr} \rho_t = \text{Tr} \rho_{t+1} = 1$. Moreover, by definition of the update step
\[
\frac{1}{D} \log \det \Theta^{(t+1)} = \frac{1}{D} \log \det \Theta^{(t)} - \frac{1}{d_a} \log \det \left( d_a \rho_t^{(a)} \right).
\]

It follows that
\[
f_x(\Theta^{(t+1)}) = f_x(\Theta^{(t)}) + \frac{1}{d_a} \log \det \left( d_a \rho_t^{(a)} \right).
\]
Lemma 5.1 in Garg et al. (2019) states that for any $d \times d$ positive semidefinite matrix $Z$ of trace $d$, the following inequality holds:

$$\log \det(Z) \leq -\frac{1}{6} \min \left\{ \|Z - I_d\|_{F}^2, 1 \right\}.$$ 

Applying this with $Z = d a \rho_t^{(a)}$, we obtain

$$\frac{1}{d a} \log \det(d a \rho_t^{(a)}) \leq -\frac{1}{6} \min \left\{ \|\rho_t^{(a)} - \frac{I_d}{d a}\|_{F}^2, \frac{1}{d a} \right\} \leq -\frac{1}{6} \min \left\{ \|\nabla f_x(\Theta^{(t)})\|_{F}^2, \frac{1}{d a} \right\} \leq -\frac{1}{6} \min \left\{ \|\nabla f_x(\Theta^{(t)})\|_{F}^2, \frac{1}{d a} \right\} \leq -\frac{1}{6} \min \left\{ \frac{\|\nabla f_x(\Theta^{(t)})\|_{F}^2}{k - 1}, \frac{1}{d_{\text{max}}} \right\}.$$ 

The equality follows from Lemma 3.9 and Remark 3.10. In the last inequality we used that $\nabla f_x(\Theta^{(t)}) = 0$ and at least one other component of the gradient is zero, as follows from Lemma 5.3, and that $\rho_t$ is the index where the gradient has largest norm.

To prove that flip-flop converges once the initial conditions are satisfied, we need the following general lemma on strongly geodesically convex functions, which tells us that once the gradient is small then the point must be inside a sublevel set of our function which is contained in a ball where our function is strongly convex. This result is stated in Franks and Moitra (2020) for the manifold of positive definite matrices of determinant one, but the proof uses no specific properties of this manifold beyond the fact that it is a Hadamard manifold. Thus it holds for the manifold $\mathbb{P}$ as well.

**Lemma 5.5 (Proof of Lemma 4.7 in Franks and Moitra (2020)).** Let $f : \mathbb{P} \to \mathbb{R}$ be a geodesically convex function with optimizer $x \in \mathbb{P}$ (i.e. $\nabla f(x) = 0$), and further assume that $f$ is $\lambda$-strongly geodesically convex on the ball $B_r(x)$. If $y \in \mathbb{P}$ is such that $\|\nabla f(y)\|_{F} < \frac{\lambda r}{8}$, then $y$ is contained in a sublevel set $S$ of $f$ which in turn is contained in $B_r(x)$. In particular, $f$ is $\lambda$-strongly geodesically convex on $S$.

Now we must show that the flip-flop algorithm reaches a point with small enough gradient relatively quickly. This is given by the following lemma, which follows the analysis given by Garg et al. (2019); Bürgisser et al. (2018):

**Lemma 5.6 (Flip-flop reduces gradient).** For any $\gamma > 0$, Algorithm 2 reaches some $\Theta$ such that $\|\nabla f_x(\Theta)\|_{F} < \gamma$ within the first

$$T_0 = \left\lceil 3 + 6(k + 1)(1 + \log f_x(I_D) - f^*_{x}) \max \left\{ \frac{d_{\text{max}}}{k - 1}, \frac{1}{\gamma^2} \right\} \right\rceil$$

iterations, where $f^*_{x} := \inf_{\Theta \in \mathbb{P}} f_x(\Theta)$ (if it has not terminated earlier).

**Proof.** We denote by $\Theta^{(t)}$ the precision matrices at the beginning of the $t$-th iteration of the flip-flop algorithm. We will analyze the flip-flop algorithm starting at $\Theta^{(1)} = \frac{1}{f_x(I_D)} I_D$ rather than $I_D$, so that $\text{Tr} \rho_1 = 1$, and skipping the stopping criterion in the first iteration. Note that the iterates $\Theta^{(t)}$ for $t = 2, \ldots, T_0$ from this initialization agree with the iterates initialized at the identity. This is because multiplying $\rho_1$ by a scalar leads to the same update at the end of the first flip-flop iteration, hence to the same $\Theta^{(2)}$. 


By Lemma 5.4, using that Tr $\rho_1 = 1$, we have that

$$f^*_x \leq f_x(\Theta(T_0)) \leq f_x(\Theta(1)) - \frac{1}{6(k - 1)} \sum_{t=2}^{T_0-1} \min \left\{ \frac{k - 1}{d_{\max}}, \|\nabla f_x(\Theta(t))\|_F^2 \right\}$$

(we omit the summand for $t = 1$). Therefore, if $\|\nabla f_x(\Theta(t))\|_F \geq \gamma$ for $t = 2, \ldots, T_0 - 1$, then

$$\frac{T_0 - 2}{6(k - 1)} \min \left\{ \frac{k - 1}{d_{\max}}, \gamma^2 \right\} \leq f_x(\Theta(1)) - f^*_x = 1 + \log f_x(I_D) - f_x$$

This implies the desired bound. $\square$

We now have all the tools we need to prove that, given appropriate initial conditions on the samples (which we later show to hold when the samples are drawn from the tensor normal model with precision matrix $\Theta$), the flip-flop algorithm will converge quickly to the MLE. Namely, we show that the MLE is in a constant size operator norm ball around the true precision matrix, where the negative log-likelihood is strongly geodesically convex. This implies that the log-likelihood is strongly geodesically convex in a small geodesic ball around the MLE. Hence, by Lemma 5.5, any point with sufficiently small gradient of the log-likelihood function is contained in a sublevel set on which the log-likelihood is strongly geodesically convex. By Lemma 5.6, such a point is found in polynomially many iterations of the flip-flop algorithm, at which point Lemma 5.2 applies to yield an $\delta$-minimizer in $O(\log(1/\delta))$ further iterations. To state our result, we use the metric $d_{op}(A, B) = \|\log B^{-1/2}AB^{-1/2}\|_op$ defined in Eq. (3.17).

**Theorem 5.7 (Convergence from initial conditions).** Let $\Theta \in \mathbb{P}$, $x_1, \ldots, x_n \in \mathbb{R}^D$, and $\lambda, \zeta > 0$ be such that

1. $f_x$ is $\lambda$-strongly geodesically convex at any $\Theta' \in \mathbb{P}$ such that $d_{op}(\Theta', \Theta) \leq \zeta$, where $\zeta \leq \min\{1, 16 \sqrt{(k+1)(k-1)/\lambda}\}$.
2. $|\nabla_0 f_x(\Theta)| \leq 1/2$.
3. The MLE $\hat{\Theta}$ exists and satisfies $d_{op}(\hat{\Theta}, \Theta) \leq \zeta/2$.

Then, for every $0 < \delta < \lambda \zeta / 16 \sqrt{(k+1)d_{\max}}$, Algorithm 2 within

$$T = \left[ 3 + 1536(1 + \log \kappa(\Theta)) \left( \frac{k+1}{2d_{\max}} \right)^2 \right] + \left[ \frac{18(k - 1)}{\lambda} \log \frac{\lambda \zeta}{16 \sqrt{(k+1)d_{\max}}} \cdot \frac{1}{\delta} \right]$$

iterations outputs $\Theta$ with $d(\Theta, \hat{\Theta}) \leq \delta / \lambda$ and $D_F(\Theta_a || \hat{\Theta}_a) \leq 2 \sqrt{d_a} \cdot \delta / \lambda$ for all $a \in [k]$.

**Proof.** By the triangle inequality for $d_{op}$, the first and the third assumptions imply that $f_x$ is $\lambda$-strongly geodesically convex at all $\Theta' \in \mathbb{P}$ such that $d_{op}(\Theta', \hat{\Theta}) \leq \zeta/2$. In particular, it is $\lambda$-strongly geodesically convex on the geodesic ball $B_r(\hat{\Theta})$ of radius

$$r = \frac{\zeta}{2 \sqrt{(k+1)d_{\max}}}$$

by Remark 3.22.

First note that our error bounds on the MLE follow if Algorithm 2 reaches the stopping criterion within $T$ iterations, that is, if we reach a precision matrix $\Theta = \Theta^{(t)}$ such that $\|\nabla f_x(\Theta)\|_F \leq \delta$. Indeed, since $\delta < \frac{\lambda \zeta}{16 \sqrt{(k+1)d_{\max}}}$ by our assumption on $\delta$, Lemma 5.5 applies (with $x = \Theta^{(t)}$) and we get

$$\frac{T_0 - 2}{6(k - 1)} \min \left\{ \frac{k - 1}{d_{\max}}, \gamma^2 \right\} \leq f_x(\Theta(1)) - f^*_x = 1 + \log f_x(I_D) - f_x$$

This implies the desired bound. $\square$
We can use Lemma 5.5 (with \(x = \nabla f_x(I_D)\)) since \(d_\Theta = \log \det \Theta \geq \log \det \Theta Linkage \geq \log \det \Theta Linkage = 1 - \log ||\Theta||_{op}.\)

Therefore, Lemma 5.4 shows that each subsequent step of the algorithm will decrease the value of the objective function in accordance with the requirements of Lemma 5.2, with parameters \(\alpha = \frac{1}{6(k-1)}\) and \(\beta = \frac{1}{6(k-1)}\) as defined above. Thus, for any \(\delta > 0\), within at most

\[
T_0 = \left[ 3 + 6(k+1)(1 + \log f_x(I_D)) - f_x^* \right] \max \left\{ \frac{d_{\text{max}}}{k-1}, \frac{64}{r^2 \lambda^2} \right\}
\]

iterations, where \(f_x^* = f_x(\hat{\Theta})\), the algorithm reaches a point \(\Theta(t_0)\) such that

\[
\|\nabla f_x(\Theta(t_0))\|_F < \frac{\lambda r}{8}.
\]

We can use Lemma 5.5 (with \(x = \hat{\Theta}, y = \Theta(t_0)\)) to see that \(\Theta(t_0)\) is contained in a sublevel set of \(f_x\) on which \(f_x\) is \(\lambda\)-strongly geodesically convex.

Note also that \(\|\nabla f_x(\Theta(t_0))\|_F \leq \beta := \frac{k-1}{d_{\text{max}}}\) because \(\lambda r / 8 = \zeta / 16\sqrt{(k+1)d_{\text{max}}} \leq \sqrt{(k-1)/d_{\text{max}}} \) by our assumption that \(\zeta \leq 16\sqrt{(k+1)(k-1)/\lambda}\).

Therefore, Lemma 5.4 shows that each subsequent step of the algorithm will decrease the value of the objective function in accordance with the requirements of Lemma 5.2, with parameters \(\alpha = \frac{1}{6(k-1)}\) and \(\beta = \frac{1}{6(k-1)}\) as defined above. Thus, for any \(\delta > 0\), within at most

\[
T_1 := \left[ 6(k-1) \frac{\log \lambda r}{\delta^2} \right] \leq \left[ \frac{12(k-1)}{\lambda} \log \frac{\lambda r}{8\delta} \right] \leq \left[ \frac{18(k-1)}{\lambda} \log \frac{\lambda r}{8\delta} \right] = \left[ \frac{18(k-1)}{\lambda} \log \frac{\lambda r}{16\sqrt{(k+1)d_{\text{max}}}} \right]
\]

further iterations we will encounter a point \(\Theta = \Theta(t)\) such that \(\|\nabla f_x(\Theta(t))\|_F \leq \delta\), i.e., such that the algorithm will stop.

Finally, we bound our expression for \(T_0\) in Eq. (5.3) using the assumptions. On the one hand,

\[
f_x^* = f_x(\hat{\Theta}) = \text{Tr} \hat{\Theta} \rho_1 - \frac{1}{D} \log \det \hat{\Theta} = 1 - \frac{1}{D} \log \det \left( \Theta^{1/2} \Theta^{-1/2} \Theta \Theta^{-1/2} \Theta^{1/2} \right)
\]

\[
\geq 1 - \frac{\zeta}{2} - \frac{1}{D} \log \det \Theta \geq 1 - \frac{\zeta}{2} - \log ||\Theta||_{op}.
\]

where the third equality follows since \(\nabla_0 f_x(\hat{\Theta}) = \text{Tr} \hat{\Theta} \rho_1 - 1 = 0\) at the MLE; the final inequality holds because \(d_{\text{op}}(\hat{\Theta}, \Theta) \leq \zeta/2\) by our third assumption, hence \(\Theta^{-1/2} \hat{\Theta} \Theta^{-1/2} \leq e^{\zeta/2} I_D\). On the other hand,

\[
f_x(I_D) = \text{Tr} \rho_1 = \text{Tr} \Theta^{-1} \Theta \rho_1 \leq ||\Theta^{-1}||_{op} \text{ Tr} \Theta \rho_1 \leq \frac{3}{2} ||\Theta^{-1}||_{op},
\]
using the second assumption, which states that $|\nabla f_x(\Theta)| = |\Tr \Theta \rho_1 - 1| \leq \frac{1}{2}$. Together,

$$
\log f_x(I_D) - f_x^* \leq \log \frac{3}{2} + \log \|\Theta^{-1}\|_{op} - 1 + \frac{\zeta}{2} + \log \|\Theta\|_{op} \leq \log \kappa(\Theta),
$$

using the assumption that $\zeta \leq 1$. Finally, we can simply the maximum in Eq. (5.3) using again that $\lambda r/8 = \lambda \zeta/16 \sqrt{(k + 1)d_{max}} \leq \sqrt{(k - 1)/d_{max}}$. We obtain

$$
T_0 \leq \left[ 3 + 6(k + 1)(1 + \log \kappa(\Theta)) \right] \frac{64}{r^2 \lambda^2} = \left[ 3 + 1536(1 + \log \kappa(\Theta)) \frac{(k + 1)^2 d_{max}}{\zeta^2 \lambda^2} \right].
$$

Adding the bound on $T_1$ concludes the proof.

5.1. Proofs of Theorem 2.9 and 2.10. We are now ready to prove the fast convergence of the flip-flop algorithm to the MLE.

**Theorem 2.9** (Tensor flip-flop convergence, restated). There are universal constants $C, c > 0$ such that the following holds. Suppose $x = (x_1, \ldots, x_n)$ are $n \geq Ck^2 d_{max}/D$ independent samples from the tensor normal model with precision matrix $\Theta = \Theta_1 \otimes \cdots \otimes \Theta_k$. Then, with probability at least

$$
1 - k e^{-\Omega(\frac{nD}{\zeta^2 d_{max}})} - k^2 \left( \frac{\sqrt{nD}}{kd_{max}} \right)^{-\Omega(d_{min})},
$$

the MLE $\hat{\Theta}$ exists, and for every $0 < \delta < c/\sqrt{(k + 1)d_{max}}$, Algorithm 2 within

$$
T = O \left( k^2 d_{max} (1 + \log \kappa(\Theta)) + k \log \frac{1}{\delta} \right)
$$

iterations outputs $\hat{\Theta}$ with $d(\hat{\Theta}, \Theta) \leq 2\delta$ and $D_F(\hat{\Theta}_a||\Theta_a) \leq 4\sqrt{d_a} \cdot \delta$ for all $a \in [k]$.

**Proof.** For $\lambda = \frac{1}{2}$, $r = \zeta/\sqrt{(k + 1)d_{max}}$, and $\zeta$ a universal constant, consider the following two events:

1. $f_x$ is $\lambda$-strongly geodesically convex at any $\Theta' \in \mathbb{P}$ such that $d_{op}(\Theta', \Theta) \leq \zeta$, where $\zeta \leq \min\{1, 16 \sqrt{(k + 1)(k - 1)/\lambda}\}$. In particular, $f_x$ is $\lambda$-strongly geodesically convex on the geodesic ball $B_r(\Theta)$.

2. $\|\nabla f_x(\Theta)\|_F < \frac{\lambda r}{2}$. In particular, $|\nabla f_x(\Theta)| < \frac{1}{2}$.

We first bound the success probability of these events similarly to the proof of Proposition 3.23. For this, we may assume without loss of generality that $\Theta = I_D$ by Remarks 3.10 and 3.15. Then the first event holds with probability at least $1 - k^2 \left( \frac{\sqrt{nD}}{kd_{max}} \right)^{-\Omega(d_{min})}$ by Proposition 3.21 and Remark 3.22, provided we choose $C$ large enough and $\zeta$ small enough. For the second event, we apply Proposition 3.11 with

$$
\varepsilon = \frac{1}{10 \sqrt{k}} \frac{r \lambda}{2} = \frac{\zeta}{40 \sqrt{k(k + 1)d_{max}}}.
$$

which satisfies $\varepsilon < 1$ and $n \geq \frac{d_{max}}{\varepsilon^2} \cdot \frac{\sqrt{nD}}{d_{max}}$ provided we choose $\zeta$ sufficiently small and $C$ sufficiently large. We find that the second event holds with probability at least

$$
1 - 2(k + 1) e^{-\varepsilon^2 \frac{nD}{\zeta^2 d_{max}}} = 1 - k e^{-\Omega(\frac{nD}{\zeta^2 d_{max}})}.
$$

Thus, the two events hold simultaneously with the desired success probability by the union bound. Moreover, by Lemma 3.6, the two events together also imply the following:
The MLE $\hat{\Theta}$ exists and satisfies $d(\hat{\Theta}, \Theta) \leq \frac{r}{2}$. In particular, $d_{\text{op}}(\hat{\Theta}, \Theta) \leq \frac{\zeta}{2}$.

Accordingly, the theorem follows from Theorem 5.7, noting that 

$$T = \left[ 3 + 1536(1 + \log \kappa(\Theta)) \left( \frac{(k + 1)^2 d_{\max}}{\zeta^2 \lambda^2} \right) \right] + \left[ \frac{18(k - 1)}{\lambda} \log \frac{\lambda \zeta}{16 \sqrt{(k + 1)d_{\max} \cdot \delta}} \right]$$

and taking $c = \frac{\lambda \zeta}{16} = \frac{\zeta}{32}$.

We now give an improved analysis for the matrix flip-flop algorithm with exponentially small failure probability under the hypotheses of Theorem 2.7.

**Theorem 2.10 (Matrix flip-flop convergence, restated).** There are universal constants $C,c > 0$ such that the following holds. Let $1 < d_1 \leq d_2$. Suppose $x_1, \ldots, x_n \in \mathbb{R}^{d_1 \times d_2}$ are independent samples from the matrix normal model with precision matrix $\Theta = \Theta_1 \otimes \Theta_2$. Then, with probability at least $1 - e^{-\Omega(nd_1^2/d_2 \log^2 d_1)}$, the MLE $\hat{\Theta}$ exists, and for every $0 < \delta < c/\sqrt{d_2}$, Algorithms 1 and 2 within

$$T = O \left( d_2 \left( 1 + \log \kappa(\Theta) \right) + \log \frac{1}{\delta} \right)$$

iterations outputs $\widehat{\Theta}$ with $d(\widehat{\Theta}, \hat{\Theta}) = O(\delta)$ and $D_F(\widehat{\Theta}_a \| \hat{\Theta}_a) = O(\sqrt{d_a} \cdot \delta)$ for all $a \in \{1, 2\}$.

**Proof.** Consider the following three events for universal constants $\lambda, \zeta > 0$:

1. $f_x$ is $\lambda$-strongly geodesically convex at any $\Theta' \in \mathcal{P}$ such that $d_{\text{op}}(\Theta', \Theta) \leq \zeta$, where $\zeta \leq \min\{1, 16\sqrt{3}/\lambda\}$.
2. $|\nabla_0 f_x(\Theta)| \leq 1/2$.
3. The MLE $\hat{\Theta}$ exists and satisfies $d_{\text{op}}(\hat{\Theta}, \Theta) \leq \zeta/2$.

We first bound the success probability of these events. For this, we may assume without loss of generality that $\Theta = I_D$ by Remarks 3.10 and 3.15. Then, if $\lambda \in (0, 1)$ is a suitable constant, $C$ is large enough, and $\zeta$ is small enough, then by Corollary 4.6 with $t^2 = nd_1/d_2$, the first event holds with probability at least $1 - e^{-\Omega(nd_1)} \geq 1 - e^{-\Omega(nd_1^2/d_2 \log^2 d_1)}$ in view of our assumption on $n$.

The second event holds with probability at least $1 - e^{-\Omega(nD)}$ by Proposition 3.13. Finally, by Theorem 2.7 with $t^2 = nd_1/d_2 \log^2 d_1$ (which can be made larger than 1 by our assumption on $n$ assuming $C$ is large enough), the third event holds with probability at least $1 - e^{-\Omega(nd_1^2/d_2 \log^2 d_1)}$. To obtain the second event from Theorem 2.7 we use Lemma E.3 to relate $D_{\text{op}}$ to $d_{\text{op}}$ and that $d_{\text{op}}(\Theta_1 \otimes \Theta_2, \Theta_1 \otimes \Theta_2) \leq d_{\text{op}}(\Theta_1, \Theta_1) + d_{\text{op}}(\Theta_2, \Theta_2)$. Thus, all three events hold simultaneously with probability at least $1 - e^{-\Omega(nd_1^2/d_2 \log^2 d_1)}$ by the union bound. Accordingly, the theorem follows from Theorem 5.7 with $k = 2$, noting that

$$T = \left[ 3 + 1536(1 + \log \kappa(\Theta)) \left( \frac{(k + 1)^2 d_{\max}}{\zeta^2 \lambda^2} \right) \right] + \left[ \frac{18(k - 1)}{\lambda} \log \frac{\lambda \zeta}{16 \sqrt{(k + 1)d_{\max} \cdot \delta}} \right]$$
\[
\begin{align*}
&= \left[ 3 + 13824 \left( 1 + \log \kappa(\Theta) \right) \frac{d_2}{\zeta^2 \lambda^2} \right] + \left[ \frac{18}{\lambda} \log \frac{\lambda \zeta}{16 \sqrt{3d_2} \cdot \delta} \right] \\
&= O \left( d_2 \left( 1 + \log \kappa(\Theta) \right) + \log \frac{1}{\delta} \right)
\end{align*}
\]

and taking \( c = \lambda \zeta / 16 \sqrt{3} \).

6. Lower bounds. In this section we discuss known lower bounds for estimating unstructured precision matrices (i.e., the case \( k = 1 \) of the tensor normal model). Afterwards we prove a new lower bound on the matrix normal model.

6.1. Lower bounds for unstructured precision matrices. Here we briefly recall and, for completeness, prove well-known lower bounds on the accuracy of any estimator for the precision matrix in the Frobenius and operator error from independent samples of a Gaussian. The lower bounds follow from Fano’s method and the relationship between the Frobenius error and the relative entropy (which is proportional to Stein’s loss). Informally, both bounds imply that no estimator for a \( d \times d \) precision matrix from \( n \) samples can have accuracy smaller than \( d / \sqrt{n} \) (resp. \( \sqrt{d/n} \)) in Frobenius error or relative Frobenius error (resp. operator norm error or relative operator norm error) with probability more than 1/2.

**Proposition 6.1** (Frobenius and operator error). There is \( c > 0 \) such that the following holds. Let \( X \in \mathbb{R}^{d \times n} \) denote \( n \) independent random samples from a Gaussian with precision matrix \( \Theta \in \text{PD}(d) \). Consider any estimator \( \hat{\Theta} = \hat{\Theta}(x) \) for the precision matrix \( \Theta \), and let \( B \subset \text{PD}(d) \) denote the ball about \( I_d \) of radius \( 1/2 \) in the operator norm.

1. Let \( \delta^2 = c \min \left\{ 1, d^2/n \right\} \). Then,

\[
\sup_{\Theta \in B} \Pr \left[ \| \hat{\Theta} - \Theta \|_F \geq \delta \right] \geq \frac{1}{2}.
\]

2. Let \( \delta^2 = c \min \left\{ 1, d/n \right\} \). Then,

\[
\sup_{\Theta \in B} \Pr \left[ \| \hat{\Theta} - \Theta \|_{\text{op}} \geq \delta \right] \geq \frac{1}{2}.
\]

As a consequence, we have

\[
\sup_{\Theta \in B} \mathbb{E}[\| \hat{\Theta} - \Theta \|_F^2] = \Omega \left( \min \left\{ \frac{d^2}{n}, 1 \right\} \right) \quad \text{and} \quad \sup_{\Theta \in B} \mathbb{E}[\| \hat{\Theta} - \Theta \|_{\text{op}}^2] = \Omega \left( \min \left\{ \frac{d}{n}, 1 \right\} \right).
\]

The proof uses Fano’s inequality with mutual information bounded by relative entropy, as in Yang and Barron (1999).

**Lemma 6.2** (Fano’s inequality). Let \( \{P_i\}_{i \in [m]} \) be a finite set of probability distributions over a set \( \mathcal{X} \), and let \( T : \mathcal{X} \to [m] \) be an estimator for \( i \) from a sample of \( P_i \). Then

\[
\max_{i \in [m]} \Pr_{X \sim P_i} [T(X) \neq i] \geq 1 - \frac{\log 2 + \max_{i,j \in [m]} D_{\text{KL}}(P_i \| P_j)}{\log m}.
\]

**Proof of Proposition 6.1.** We first prove Eq. (6.1), the lower bound on estimation in the Frobenius norm. We begin by the standard reduction from estimation to testing. Let \( V_0 \) be a 1-separated set in the Frobenius ball \( B_F \) of radius 1 in the \( d \times d \) symmetric matrices, i.e., the set \( B_F = \{ A : A \text{ Symmetric}, \| A \|_F \leq 1 \} \).
We may take \( V_0 \) to have cardinality \( m \geq 2^{d(d+1)/2} \) because \( B_F \) is a Euclidean ball of radius 1 in the linear subspace of \( d \times d \) symmetric matrices, which has dimension \( d(d+1)/2 \), and hence any maximal Frobenius 1/2-packing (collection of disjoint radius 1/2 Frobenius balls) in \( B_F \) has cardinality at least \( 2^{d(d+1)/2} \). Let \( 0 \leq \delta \leq 1/2 \), and let \( V = I_d + \delta v : v \in V_0 \). Write \( V = \{ \Theta_1, \ldots, \Theta_m \} \). Note that \( V \) is contained within the operator norm ball \( B \).

Let \( P_i = \mathcal{N}(0, \Theta_i^{-1}) \otimes n \) for \( i \in [m] \), and define the estimator \( T \) by

\[
T(x) = \arg\min_{i \in [m]} \| \Theta_i - \hat{\Theta}(x) \|_F.
\]

Then, because \( V \) is 2\( \delta \)-separated,

\[
\Pr_{X \sim P_i} [T(X) = i] \geq \Pr \left[ \| \Theta - \Theta_i \|_F \leq \delta \right].
\]

In order to apply Fano’s inequality, we use the well-known fact that \( D_{KL}(P_i \| P_j) = nD_{KL}(\mathcal{N}(0, \Theta_i^{-1}) \| \mathcal{N}(0, \Theta_j^{-1})) = O(nD_F(\Theta_j \| \Theta_i)^2) \) when \( \Theta_i^{-1} \Theta_j \) has eigenvalues uniformly bounded away from zero by the proof of Lemma E.2. This condition on the eigenvalues holds because \( I_d/2 \leq \Theta_j, \Theta_j \geq 3I_d/2 \) for \( i, j \in [m] \) by our assumption that \( \delta \leq 1/4 \).

Moreover, for \( i \in [m] \), we have \( \| \Theta_i^{-1} \|_{op} \leq 2 \) and so \( D_{F}(\Theta_j \| \Theta_i) = O(\| \Theta_i - \Theta_j \|_F) = O(\delta) \) by Eq. (E.1). Thus we have \( D_{KL}(P_i \| P_j) \leq Cn\delta^2 \) for some absolute constant \( C \). Then, by Lemma 6.2,

\[
\min_{i \in [m]} \Pr_{X \sim P_i} [T(X) = i] \leq \frac{\log 2 + Cn\delta^2}{d(d+1)(\log 2)/2}.
\]

If \( \delta^2 = c \min\{ \frac{d}{m}, 1 \} \), the right-hand side of the inequality above is bounded by \( \frac{1}{2} \) and the assumption \( \delta \leq 1/4 \) is satisfied provided \( c \) is a small enough absolute constant. In view of Eq. (6.3), it follows that

\[
\min_{i \in [m]} \Pr \left[ \| \Theta - \Theta_i \|_F \leq \delta \right] \leq 1/2.
\]

Because \( V \subset B \), this proves Eq. (6.1).

To obtain Eq. (6.2), the lower bound in operator norm, instead start with a packing \( V_0 \) of the unit operator norm ball of cardinality \( m \geq 2^{d(d+1)/2} \) and define \( V = \{ \Theta_1, \ldots, \Theta_m \} \) as above. We modify the proof by bounding \( D_{KL}(P_i \| P_j) = O(n\| \Theta_i - \Theta_j \|_F^2) = O(nd\| \Theta_i - \Theta_j \|_{op}) \leq Cn\delta^2 \). Proceeding as before, we find that for \( \delta = c \min\{ \frac{d}{m}, 1 \} \),

\[
\min_{i \in [m]} \Pr \left[ \| \Theta - \Theta_i \|_{op} \leq \delta \right] \leq 1/2.
\]

Again, we have \( V \subset B \), so Eq. (6.2) follows.

We remark that the above proof shows the necessity of a scale-invariant dissimilarity measure to obtain error bounds that are independent of the ground truth precision matrix \( \Theta \). Indeed, replacing the packing \( V \) by \( \kappa V \) for \( \kappa \to \infty \) in the proof shows that \( \sup_{\Theta \in \kappa B} \Pr[\| \Theta - \hat{\Theta} \|_F \geq \kappa \delta] \geq \frac{1}{2} \). That is, no fixed bound can be obtained with probability \( 1/2 \).

We now use the result just obtained to prove bounds on the relative Frobenius and operator error. Because \( I_d/2 \leq \Theta \leq 3I_d/2 \) for \( \Theta \in B \), the bounds \( \| \Theta - \hat{\Theta} \|_F \leq \| \Theta \|_{op} D_F(\Theta, \hat{\Theta}) \) and \( \| \Theta - \hat{\Theta} \|_{op} \leq \| \Theta \|_{op} D_{op}(\Theta, \hat{\Theta}) \) together with Proposition 6.1 imply the following corollary.

**Corollary 6.3** (Relative Frobenius and operator error). There is \( c > 0 \) such that the following holds for \( X, \hat{\Theta}, B \) as in Proposition 6.1.
1. Let $\delta^2 = c \min \left\{ 1, d^2/n \right\}$. Then

$$\sup_{\Theta \in B} \Pr \left[ D_F(\hat{\Theta} \parallel \Theta) \geq \delta \right] \geq \frac{1}{2}. \quad (6.4)$$

2. Let $\delta^2 = c \min \{ 1, d/n \}$. Then

$$\sup_{\Theta \in B} \Pr \left[ D_{op}(\hat{\Theta} \parallel \Theta) \geq \delta \right] \geq \frac{1}{2}. \quad (6.5)$$

As a consequence, we have

$$\sup_{\Theta \in B} \mathbb{E}[D_F(\hat{\Theta} \parallel \Theta)^2] = \Omega \left( \min \left\{ \frac{d^2}{n}, 1 \right\} \right) \quad \text{and} \quad \sup_{\Theta \in B} \mathbb{E}[D_{op}(\hat{\Theta} \parallel \Theta)^2] = \Omega \left( \min \left\{ \frac{d}{n}, 1 \right\} \right).$$

6.2. Lower bounds for the matrix normal model. If $\Theta_2$ is known, then we can compute $(I \otimes \Theta_2^{1/2})X$, which is distributed as $nd_2$ independent samples from a Gaussian with precision matrix $\Theta_1$. In this case, one can estimate $\Theta_1$ in operator norm with an RMSE rate of $O(\sqrt{d_1/n \min(d_1, d_2)})$. One could hope that this rate holds for $\Theta_1$ even when $\Theta_2$ is not known. Here we show that, to the contrary, the rate for $\Theta_1$ cannot be better than $O(\sqrt{d_1/n \min(d_1, d_2)})$. Thus, for $d_2 > nd_1$, it is impossible to estimate $\Theta_1$ as well as one could if $\Theta_2$ were known. Note that, in this regime, there is no hope of recovering $\Theta_2$ even if $\Theta_1$ is known.

**Theorem 6.4** (Lower bound for matrix normal models). There is $c > 0$ such that the following holds. Let $\hat{\Theta}_1$ be any estimator for $\Theta_1$ from a tuple $X$ of $n$ samples of the matrix normal model with precision matrices $\Theta_1, \Theta_2$. Let $B \subset \mathcal{PD}(d_1)$ denote the ball about $I_{d_1}$ of radius $1/2$ in the operator norm.

1. Let $\delta^2 = c \min \left\{ 1, \frac{d^2}{n \min\{nd_1, d_2\}} \right\}$. Then

$$\sup_{\Theta_1 \in B, \Theta_2 \in \mathcal{PD}(d_2)} \Pr \left[ D_F(\hat{\Theta}_1 \parallel \Theta_1) \geq \delta \right] \geq \frac{1}{2}. \quad (6.6)$$

2. Let $\delta^2 = c \min \left\{ 1, \frac{d_1}{n \min\{nd_1, d_2\}} \right\}$. Then

$$\sup_{\Theta_1 \in B, \Theta_2 \in \mathcal{PD}(d_2)} \Pr \left[ D_{op}(\hat{\Theta}_1 \parallel \Theta_1) \geq \delta \right] \geq \frac{1}{2}. \quad (6.7)$$

As a consequence, we have

$$\sup_{\Theta_1 \in B, \Theta_2 \in \mathcal{PD}(d_2)} \mathbb{E}[D_F(\hat{\Theta}_1 \parallel \Theta_1)^2] = \Omega \left( \min \left\{ \frac{d^2}{n \min\{nd_1, d_2\}}, 1 \right\} \right)$$

and

$$\sup_{\Theta_1 \in B, \Theta_2 \in \mathcal{PD}(d_2)} \mathbb{E}[D_{op}(\hat{\Theta}_1 \parallel \Theta_1)^2] = \Omega \left( \min \left\{ \frac{d_1}{n \min\{nd_1, d_2\}}, 1 \right\} \right).$$

Intuitively, the above theorem holds because we can choose $\Sigma_2$ to zero out all but $nd_1$ columns of each $X_i$, which allows access to at most $n \cdot nd_1$ samples from a Gaussian with precision $\Theta_1$. However, this does not quite work because $\Sigma_2$ would not be invertible and hence the precision matrix $\Theta_2$ would not exist. We must instead choose $\Sigma_2$ to be approximately equal to a random projection of rank $nd_1$. The resulting construction allows us to deduce the same lower bounds for estimating $\Theta_1$ as the Gaussian case with at most $n \min\{d_2, nd_1\}$ independent samples.
One might ask why the rank of the random projection cannot be taken to be even less than \( nd_1 \), yielding an even stronger bound. If the rank is less than \( nd_1 \), then the support of \( \Sigma_2 \) can be estimated. This would allow one to approximately diagonalize \( \Sigma_2 \) so that the \( n \) samples can be treated as \( nd_2 \) independent samples in \( \mathbb{R}^{d_1} \), yielding the rate \( \sqrt{d_1/nd_2} \) using, e.g., Tyler’s M estimator. We now state the lower bound.

**Lemma 6.5.** Let \( X \) denote a tuple of \( n \) samples from the matrix normal model with precision matrices \( \Theta_1, \Theta_2 \). Let \( Y \) be a tuple of \( n \min\{nd_1, d_2\} \) Gaussians on \( \mathbb{R}^{d_1} \) with precision matrix \( \Theta_1 \). Let \( \Theta_1(X) \) be any estimator for \( \Theta_1 \). For every \( \delta > 0 \), there is a distribution on \( \Theta_2 \) and an estimator \( \hat{\Theta}(Y) \) such that the distribution of \( \hat{\Theta}_1(X) \) and the distribution of \( \hat{\Theta}(Y) \) differ by at most \( \delta \) in total variation distance.

**Proof.** If \( d_2 \leq nd_1 \), then setting \( \Theta_2 = I_{d_2} \) shows that \( \hat{\Theta}_1 \) has access to precisely \( nd_2 \) samples from a Gaussian \( \mathbb{R}^{d_1} \) with precision matrix \( \Theta_1 \). Thus we may take \( \hat{\Theta} = \hat{\Theta}_1 \) in that case, completing the proof. The harder case is \( d_2 > nd_1 \).

For intuition, let \( B \) be any \( d_2 \times d_2 \) matrix such that the last \( d_2 - nd_1 \) columns are zero. Given access to the tuple \( X \) of \( n \) samples \( \sqrt{\Sigma}Z_iB^T \), where \( Z_i \) are i.i.d standard Gaussian \( d_1 \times d_2 \) matrices, clearly \( \hat{\Theta}_2 \) has access to at most \( n^2d_1 \) samples of the Gaussian on \( \mathbb{R}^{d_1} \) with precision matrix \( \Theta_1 \) because \( Z_iB^T \) depends only on the first \( d_1 \) columns of each \( Z_i \).

However, we must supply invertible \( B \) in order for \( \Theta_2 = (BB^T)^{-1} \) to exist. Let \( \delta \geq 0 \). Let \( B_\delta \) be the random matrix obtained by choosing the first \( nd_1 \) columns of \( B_\delta \) uniformly at random among the collections of \( nd_1 \) orthonormal vectors in \( \mathbb{R}^{d_2} \). Let the remaining entries be i.i.d uniform in \( [-\delta, \delta] \) (the precise distribution of the remaining entries does not matter as long as they are independent, continuous, and small). Let \( Y_\delta := (\sqrt{\Sigma}Z_1B_\delta^T, \ldots, \sqrt{\Sigma}Z_nB_\delta^T) \) denote the resulting random variable with \( B_\delta \) and \( X \) chosen independently. If \( \delta = 0 \), then, by the argument above, with access to the random variable \( Y_\delta := (\sqrt{\Sigma}Z_1B^T, \ldots, \sqrt{\Sigma}Z_nB^T) \), the estimator \( \hat{\Theta}_1 \) has access to at most \( n^2d_1 \) samples of a Gaussian on \( \mathbb{R}^{d_1} \) with precision matrix \( \Theta_1 \). We claim that as \( \delta \to 0 \), the distribution of \( Y_\delta \) tends to that of \( Y_0 \) in total variation distance. Thus the distribution of \( \Theta_1(Y_\delta) \) converges to that of \( \Theta_1(Y_0) \) in total variation. Since \( Y_0 \) only depends on \( n^2d_1 \) samples to the Gaussian on \( \mathbb{R}^{d_1} \) with precision matrix \( \Theta_1 \), which we call \( Y \), defining \( \hat{\Theta}(Y) = \hat{\Theta}_1(Y_0) \) proves the theorem. \(^1\)

It remains to prove that \( Y_\delta \) converges to \( Y_0 \) in total variation distance. First note that \( Y_\delta = Y_0 + \delta W \) where \( W_i = \sqrt{\Theta_1}Z_iC^T \), where \( C \) is a random matrix where the first \( nd_1 \) columns are zero and the last \( d_2 - nd_1 \) columns have entries i.i.d uniform on \( [-1, 1] \). Because of the zero patterns of \( B_\delta \) and \( C \) and the fact that the entries of \( Z \) are i.i.d., the random variables \( Y_\delta \) and \( W \) are independent. If we can show that \( Y_0 \) has a density with respect to the Lebesgue measure on \( \mathbb{R}^{nd_1,d_2} \), then \( Y_0 + \delta W \) converges to \( Y_0 \) in total variation distance as \( \delta \to 0 \). This follows because \( Y_0 + \delta W \) has a density obtained by convolving the density of \( Y_0 \), an \( L_1 \) function, by the law of \( \delta W \). The density of \( Y_0 + \delta W \) then converges to that of \( Y_0 \) in \( L_1 \) by the continuity of the translation operator in \( L_1 \). \(^2\)

By invertibility of \( \Sigma_1 \), it is enough to show that \( Y_0 \) has a density when \( \Sigma_1 = I_{d_1} \). Consider \( Y_0 = (Z_1B_0^T, \ldots, Z_nB_0^T) \). We may think of \( Y_0 \) as the \( d_2 \times nd_1 \) random matrix obtained by horizontally concatenating the matrices \( B_0Z_i^T \). \(^3\)

---

\(^1\) Actually, as \( B \) has a probability zero chance of being singular, the final family of densities \( Y_\delta' \) we will use is \( Y_\delta \) conditioned on \( B \) being invertible. As \( B \) is invertible with probability 1 for \( \delta > 0 \), the total variation distance between \( Y_\delta', Y_\delta \) is zero for all \( \delta > 0 \) and hence \( Y_\delta \) converges to \( Y_0 \) in total variation distance provided \( Y_\delta \) does.

\(^2\) We thank Oliver Diaz for communicating a proof of this fact.

\(^3\) Almost every matrix of these dimensions has rank \( nd_1 \), but if we had set even more of the columns of \( B_0 \) to zero then \( Y_0 \) would have rank less than \( nd_1 \) with probability 1 and hence would not have a density. This is why we cannot push this argument any further.
Now consider the $nd_1$ random vectors in $\mathbb{R}^{d_2}$ that are the columns of the matrices $B_0 Z_i^T$, for $i \in \{1, \ldots, n\}$. Because $B_0$ is supported only in its first $nd_1$ columns, the joint distribution of these random vectors may be obtained by sampling $nd_1$ independent standard Gaussian vectors $v_j$ on $\mathbb{R}^{nd_1}$ and then multiplying them by the $d_2 \times nd_1$ matrix $B'$ that is the restriction of $B_0$ to its first $nd_1$ columns. We have chosen $B'$ such that it is an isometry into a uniformly random subspace of $\mathbb{R}^{d_2}$ of dimension $nd_1$. Thus $Bv_j/\|v_j\|$ are $nd_1$ many independent, random unit vectors in $\mathbb{R}^{d_2}$. As the $\|v_j\|$ are also independent, $Bv_j$ are thus independent. Each marginal $Bv_i$ has a density; one may sample it by choosing a uniformly random vector and then choosing the length $\|v_i\|$, hence the density is a product density in spherical coordinates. The joint density of the $Bv_j$ is then the product density of the marginal densities. □

The above lemma combined with Corollary 6.3 immediately implies Theorem 6.4. We remark that the below proof uses no properties about $D_F$; a lower bound on any error metric for estimating a Gaussian with $n \min\{nd_1, d_2\}$ samples will transfer to the matrix normal model. In particular, Theorem 6.4 holds true when $D_F$ is replaced by the Frobenius error and $D_{op}$ replaced by the operator norm error.

**Proof of Theorem 6.4.** To show Item 1, let $\delta^2 \leq c \min\left\{1, \frac{d_2^2}{n \min\{nd_1, d_2\}}\right\}$. Let $\Theta_2$ be the distribution as in Lemma 6.5 so that, as guaranteed by Lemma 6.5 there is an estimator $\tilde{\Theta}$ with access to a tuple $Y$ of $n \min\{nd_1, d_2\}$ samples of a Gaussian on $\mathbb{R}^{d_1}$ with precision matrix $\Theta_1$ satisfying $D_{TV}(\tilde{\Theta}(X), \tilde{\Theta}(Y)) \leq \delta_0$. Here $X$ is distributed according to the normal model with precision matrices $\Theta_1, \Theta_2$. Corollary 6.3 implies

$$\sup_{\Theta \in D} \Pr \left[ D_F(\tilde{\Theta}(Y) \| \Theta_1) \geq \delta \right] \geq \frac{1}{2}.$$

Clearly we have

$$\sup_{\Theta_1 \in B, \Theta_2 \in PD(d_2)} \Pr \left[ D_F(\tilde{\Theta}_1(X) \| \Theta_1) \geq \delta \right] \geq \sup_{\Theta_1 \in B, \Theta_2 \in X} \Pr \left[ D_F(\tilde{\Theta}_1(X) \| \Theta_1) \geq \delta \right].$$

On the other hand, the total variation distance bound implies

$$\sup_{\Theta_1 \in B, \Theta_2 \in X} \Pr \left[ D_F(\tilde{\Theta}_1(X) \| \Theta_1) \geq \delta \right] \geq \sup_{\Theta_1 \in B} \Pr \left[ D_F(\tilde{\Theta}(Y) \| \Theta_1) - \delta_0 \geq \delta \right]$$

$$\geq \frac{1}{2} - \delta_0.$$

Allowing $\delta_0 \to 0$ implies the theorem. The proof of Item 2 is the same but with $D_F$ replaced by $D_{op}$. □

7. **Numerics and regularization.** In the undersampled regime, most effort so far has focused on the sparse case. Existing estimators, such as the Gemini estimator Zhou (2014) and KGlasso estimator Tsiligkaridis, Hero and Zhou (2013) enforce sparsity by adding a regularizer proportional to the $\ell_1$ norm of the precision matrices to encourage sparsity. We refer to these as Glasso-type estimators. We propose a new, shrinkage-based estimator that is simple to compute and experimentally outperforms Gemini and KGlasso in a natural generative model.

To describe our estimator, we remind the reader that the maximum likelihood estimator as defined in Eq. (3.5) is the optimum of a function depending on the sample covariance matrix. Namely, we choose $\Theta \in P$ optimizing $\text{Tr} \Theta \rho - \frac{1}{D} \log \det \Theta$ where $\rho = \frac{1}{n D} \sum_{i=1}^n x_i x_i^T$
is the sample covariance matrix.\footnote{or, more accurately, the matrix of second moments} In our estimator, $\rho$ is replaced by a shrinkage estimator for the covariance matrix. Namely, we replace $\rho$ by $\tilde{\rho} := (1 - \alpha)\rho + \alpha \frac{\text{Tr}\rho}{D} I$ for some $\alpha \in [0, 1]$. Formally, define $\hat{\Theta}_x^\alpha = \arg \min_{\Theta \in \mathbb{P}} f_x^\alpha(\Theta)$ where

$$f_x^\alpha(\Theta) := \text{Tr} \left( (1 - \alpha)\rho + \alpha \frac{\text{Tr}\rho}{D} I \right) - \frac{1}{D} \log \det \Theta.$$ 

This is known as a shrinkage estimator with ridge regularization (Warton, 2008). The estimator, which we call the \textit{shrinkage-based flip-flop estimator}, or ShrinkFlop for short, is closely related to the shrinkage estimator considered in Goes et al. (2020) and the Frobenius penalty considered in Tang and Allen (2018).

We consider a generative model in which the covariance matrices $\Theta_i$ are distributed as a rank one Wishart matrix plus a small multiple of the identity matrix to ensure invertibility; we refer to them as \textit{spiked} covariance matrices. We also show that, even when $\Theta_1$ is sparse, our shrinkage-based estimator can outperform Gemini and KGlasso when $\Sigma_2$ is spiked. Moreover, we observe that our regularized estimator is significantly faster to compute than the Glasso-type estimators. All three estimators require parameter tuning, so when possible we compare throughout a plausible range of parameters for each of them. We leave determination of $\alpha$ by cross-validation for future work.

7.1. \textit{ShrinkFlop estimator}. We now discuss some properties of the ShrinkFlop estimator. As above, set $\tilde{\rho} := (1 - \alpha)\rho + \alpha \frac{\text{Tr}\rho}{D} I$. If $\alpha > 0$, then $\tilde{\rho}$ is invertible and therefore $\hat{\Theta}_x^\alpha$ exists uniquely for every $x$. This is because, whenever $\tilde{\rho}$ is invertible, the function $f_x^\alpha$ is strictly geodesically convex even on the larger domain $\text{PD}(D)$ (see Example 3.4). Thus it has a unique minimizer over the geodesically convex domain $\mathbb{P}$.

The estimator $\hat{\Theta}_x^\alpha$ can be approximately computed using the flip-flop algorithm (as in Algorithm 2) with $\rho$ replaced by $\tilde{\rho}$. Due to the special form of $\tilde{\rho}$, one need not look at the entirety of $\tilde{\rho}$ to perform each step of the flip-flop algorithm. The resulting algorithm is summarized in Algorithm 3. The only modification as compared to the original algorithm is that at each flip-flop step the estimate of the covariance is replaced by a convex combination with a scaled identity matrix, as well as the computation of the gradient.

7.2. \textit{Scenario 1: Spiked, dense covariances}. Here we compare the performance of our shrinkage based estimator with Zhou’s single step estimator (Gemini) for the matrix normal model assuming $\Sigma_1, \Sigma_2$ are dense, spiked covariance matrices. Fig. 1 was generated by setting $d_1 = 25, d_2 = 50$ and $n = 1$, and for each choice of regularization parameter independently generating 5 different pairs $\Sigma_1 \sim I_{d_1} + 10v_1 v_1^T$ and $\Sigma_2 \sim I_{d_2} + 10v_2 v_2^T$ where the $v_i$ are standard $d_i$-dimensional Gaussian vectors and then normalizing each to have trace $d_i$, respectively. As in Zhou (2014), we always normalize the trace of $\hat{\Theta}_1$ to match that of $\hat{\Theta}_1$ to focus on the core difficulties rather than estimating the overall scale of the data (which is easy to do). Though our main focus is estimation in geodesic distance, we also considered the Frobenius distance due to its prevalence in the literature. For each of the 5 pairs, following Tsiligkaridis, Hero and Zhou (2013) we computed the normalized squared error (the squared Frobenius distance from $\hat{\Theta}_1$ to $\hat{\Theta}_1$ divided by the squared Frobenius norm of $\hat{\Theta}_1$ for samples drawn from this distribution 5 times. Finally we averaged all 25 errors. Note that with $d_1 = 25, d_2 = 50$ and $n = 1$, the MLE never exists and so we cannot compare with flip-flop without shrinkage.

We also computed the error in geodesic distance. We used geodesic distance in place of the relative Frobenius error (Definition 2.3) because these quantities are comparable (see
Appendix E). To compute the error in geodesic distance, we compute the squared geodesic distance between the estimator and the truth and divide by the geodesic distance between the truth and the identity.

Fig. 1 demonstrates that in the spiked case, for all choices of regularization parameter, the Gemini and KGlasso estimator were outperformed by the “trivial” estimator which always outputs the identity matrix. For a broad range of regularization parameters, our regularized estimator outperforms both the trivial estimator and Gemini. The poor performance of Gemini and KGlasso in this case are to be expected because the true precision matrices are dense.

7.3. Scenario 2: Sparse and partially sparse precision matrices. We now compare the performance of our estimator with other leading estimators in the case when one or more of the precision matrices $\Theta_1, \Theta_2$ is sparse. We find that when both $\Theta_1$ and $\Theta_2$ are sparse, the Gemini estimator outperforms the regularized Sinkhorn algorithm in Frobenius error; see Fig. 2. However, when $\Theta_2$ is spiky, we find that the shrinkage-based flip-flop estimator outperforms the Gemini estimator; see Fig. 3. In practice, $\Theta_2$ is often considered a nuisance parameter and $\Theta_1$ is the object that should be interpretable (e.g., sparse). Thus ill-conditioned and dense nuisance parameters $\Theta_2$ can break Glasso-type estimators even when $\Theta_1$ is sparse.

The figures were generated in the same manner as Fig. 1, apart from the generative model. The sparse matrices $\Theta_i$ were generated by adding $\frac{1}{2} I_{d_i}$ to the Laplacian of a random multigraph with $0.4 d_i$ edges and normalizing to have trace $d_i$, and the spiked covariance matrix $\Sigma_2$ for Fig. 3 was drawn according to $\Sigma_2 \sim I_{d_2} + 100 v_2 v_2^T$ where $v_2$ has i.i.d. Gaussian coordinates, and then normalized to have trace $d_2$.

7.4. Performance as a function of the number of samples. In this subsection we examine how the number of samples affects the error. We found best performance when the shrinkage parameter scales inverse exponentially with the number of samples, in this case $2^{-1.1n}$. The regularization parameter for Gemini was chosen to scale as $\sqrt{\log d_1}/d_2 n$ as suggested in Zhou (2014).
Figure 1: Average Frobenius error with $d_1 = 25, d_2 = 50, n = 1$ for spiked, dense covariance matrices. “Gemini” refers to the Gemini estimator of Zhou (2014), “KGlasso” refers to the Kronecker Glasso algorithm Tsiligkaridis, Hero and Zhou (2013), “ShrinkFlop” refers to our shrinkage-based flip-flop estimator, and “Trivial” refers to the estimator that always outputs the identity matrix. The choice of regularizer $\alpha$ for ShrinkFlop is given by $\alpha = \frac{2}{\pi} \arctan x$, where $x$ is the value on the $x$-axis in the figures above.

Figure 2: Average error with $d_1 = 25, d_2 = 50, n = 1$ for both precision matrices sparse. Labels and choice of regularizer as in Fig. 1.

In Fig. 4 we see that Gemini outperforms flip-flop with a single sample, and shrinkage-based flip-flop outperforms both. When the number of samples are increased, the error for shrinkage-based flip-flop approaches that of flip-flop from below.

7.5. Computational aspects. Here we experimentally demonstrate that our shrinkage-based estimator is much faster to compute than the Glasso-type estimators. We considered $d_1 = 100, d_2 = 200$ with $n = 1$ samples. We considered regularized flip-flop and Gemini on spiked data, and chose the best regularization parameters for both using the computation from Fig. 1. We did not consider KGLasso because it consistently took far longer than Gemini. Out of sixteen total runs (4 instances with 4 draws of data each), the average time of completion was 0.081 seconds for regularized flip-flop and 33.61 seconds for Gemini. Both computations were done in R using a MacBook Pro with an Apple M1 chip with 16 gigabytes of RAM.
Figure 3: Average error with $d_1 = 25, d_2 = 50, n = 1$ for $\Theta_1$ sparse and $\Theta_2$ spiked. Interestingly, for Frobenius distance the shrinkage-based estimator performs best at the bottom of the regularization path and for the geodesic distance it performs best for larger regularization parameters. Labels and choice of regularizer as in Fig. 1. All algorithms perform poorly in geodesic distance under this model; as expected, ShrinkFlop approaches the trivial algorithm as the regularizer becomes large.

Figure 4: Average error with $d_1 = 25, d_2 = 25$, and number of samples $n$ ranging from 1 to 50 for both precision matrices spiked. It was necessary to choose $d_1 = d_2$ so that the flip-flop estimator converged for $n = 1$. The regularization for ShinkFlop was given by $\alpha = 2^{-1.1n}$. We did not include KGlasso here because its running time was prohibitive.

8. Conclusion and open problems. Though there has been a large volume of work on estimating the covariance in the matrix and tensor normal models under further assumptions like sparsity and well-conditionedness, some fundamental questions concerning estimation without further assumptions were still open prior to our work. Contrary to the state of the art for unstructured covariance estimation (i.e., $k = 1$), all previous existing results depended on the condition number of the true covariance matrices (in the case of the tensor normal model under Frobenius norm) or had suboptimal sample complexity (the matrix normal model under operator norm). Using strong convexity in the geometry induced by the Fisher information metric, we are largely able to remedy these issues and obtain nearly optimal estimates in the strongest possible metrics, namely the relative operator and Frobenius norms.
As a consequence, we can also control other equivariant statistical distances such as relative entropy, Fisher-Rao distance, and total variation distance.

In particular, we showed that the maximum likelihood estimator (MLE) for the covariance matrix in the matrix normal models has optimal sample complexity up to logarithmic factors in the dimensions. We showed that the MLE for tensor normal models with a constant number of tensor factors has optimal sample complexity in the regime where it is information-theoretically possible to recover the covariance matrix to within a constant Frobenius error. Whenever the number of samples is large enough for either of the aforementioned statistical results to hold, we show that the flip-flop algorithm converges to the MLE exponentially quickly. Hence, the output of the flip-flop algorithm with $O(d_{\text{max}}(1 + \kappa(\Theta)) + \log n)$ iterations (see the discussion after Theorem 2.10) is an efficiently computable estimator with statistical guarantees comparable to those we show for the MLE.

We also observed empirically that under a certain natural generative model of ill-conditioned, dense covariance matrices, the flip-flop algorithm combined with a very simple shrinkage technique can outperform existing estimators designed for the sparse case (Gemini and KGlasso). We view our empirical results as evidence that, in some cases, flip-flop combined with shrinkage provides the fastest, simplest and most statistically accurate known method to estimate the covariances. More work is needed in the future to rigorously understand the statistical guarantees for flip-flop with shrinkage.

Our main theoretical open question is whether the assumption $n = \Omega(d_{\text{max}}^3/D)$ for Theorem 2.4 can be weakened to $n = \Omega(d_{\text{max}}^2/D)$ for $k \geq 3$. Equivalently, do the guarantees of Theorem 2.4 hold even when one cannot hope to estimate the Kronecker factors to constant Frobenius error, but only to constant operator norm error? In the case $k = 1$ (i.e., unstructured covariance estimation) the weaker assumption is well-known to suffice, and for $k = 2$ the same follows (up to logarithmic factors) by our Theorem 2.7. Filling in this final gap will place the tensor normal model on the same tight theoretical footing as unstructured covariance estimation.

**APPENDIX A: PISIER’S PROOF OF QUANTUM EXPANSION**

In this appendix we prove the spectral condition for random Gaussian completely positive maps given in Theorem 3.18. This follows from the work of Pisier (2012), whose original theorem dealt with square matrices and gave slightly weaker probabilistic guarantees than Theorem A.1 stated below. We adapt this result to give exponentially small error probability in the setting of rectangular matrices. We emphasize that these are minor modifications, which follow readily from Pisier (2014, 2012). Therefore, we state the proof below for completeness and claim no originality.

**THEOREM A.1 (Pisier).** Let $A_1, \ldots, A_N, A$ be independent $n \times m$ random matrices with independent standard Gaussian entries. For any $t \geq 2$, with probability at least $1 - t^{-\Omega(m+n)}$,

$$\left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_{\text{op}} \leq O \left(t^2 \sqrt{N(m+n)}\right),$$

where $\Pi$ denotes the orthogonal projection onto the traceless subspace of $\mathbb{R}^m \otimes \mathbb{R}^m$, that is, onto the orthogonal complement of $\text{vec}(I_m)$.

We first explain how Theorem A.1 implies Theorem 3.18.

**PROOF OF THEOREM 3.18.** Choose $n = d_a$ and $m = d_b$. Observe that

$$\|\Phi_A\|_0 = \max_{H \text{ traceless symmetric}} \frac{\|\Phi(H)\|_F}{\|H\|_F} \leq \max_{H \in \text{Mat}(m)} \frac{\|\Phi(\Pi(H))\|_F}{\|H\|_F} = \|\Phi \circ \Pi\|_{\text{op}}.$$
Then the random variable \(\|\langle \xi, A \rangle\|_F\) where we first used that \(\|\Pi(H)\|_F \leq \|H\|_F\), since \(\Pi\) is an orthogonal projection. Using Eq. (3.11), the result now follows from Theorem A.1.

In the remainder we discuss the proof of Theorem A.1. The proof proceeds by a symmetrization trick, followed by the trace method. We first state some relevant bounds on Gaussian random variables and then give the proof of Theorem A.1.

We will often use the following estimate of the operator norm of a standard Gaussian \(n \times m\) random matrix \(A\) (see Theorem 5.32 in Vershynin (2012)),

\[
E\|A\|_{\text{op}} \leq \sqrt{n} + \sqrt{m}.
\]

**Theorem A.2.** Let \(A\) be a centered Gaussian random variable that takes values in a separable Banach space with norm \(\|\cdot\|\). Then \(\|A\|\) satisfies the following concentration and moment inequalities with parameter \(\sigma^2 := \sup\{E\langle \xi, A \rangle^2 \mid \|\xi\|_* \leq 1\}\), where \(\|\cdot\|_*\) denotes the dual norm:

\[
\forall t > 0: \quad \Pr\left(\|A\| - E\|A\| \geq t\right) \leq 2\exp\left(-\frac{\Omega(t^2)}{\sigma^2}\right), \quad \text{and}
\]

\[
\forall p \geq 1: \quad E\|A\|^p \leq (2E\|A\|)^p + O(\sigma\sqrt{p})^p.
\]

**Proof.** The first statement on concentration is exactly Theorem 1.5 in Pisier (1986). For the second, we consider the random variable \(X := \frac{1}{\sigma}(\|A\| - E\|A\|)\). Then the equivalence in Lemma 5.5 of Vershynin (2012) gives the moment bound

\[
\left(\mathbb{E}|X|^p\right)^{1/p} = \frac{1}{\sigma}\left(\mathbb{E}\|A\| - E\|A\|\right)^{1/p} \leq O(\sqrt{p}).
\]

The moment bound in the theorem now follows by rearranging as

\[
E\|A\|^p = E\left(E\|A\| + \sigma X\right)^p \leq 2p\left((E\|A\|)^p + O(\sigma\sqrt{p})^p\right),
\]

where the last step was by the simple inequality \((a + b)^p \leq 2^p(|a|^p + |b|^p)\).

Below, we calculate the \(\sigma^2\) parameter in Theorem A.2 with regards to our random matrix setting.

**Corollary A.3.** Let \(A\) be an \(n \times m\) matrix with independent standard Gaussian entries. Then the random variable \(\|A\|_{\text{op}}\) satisfies the conclusions of Theorem A.2 with \(\sigma^2 = 1\).

**Proof.** Note that the dual norm is the trace norm \(\|\cdot\|_{\text{tr}}\), hence the concentration parameter can be estimated as

\[
\sigma^2 = \sup\{E\langle \xi, A \rangle^2 \mid \|\xi\|_{\text{tr}} \leq 1\} = \sup\{\|\xi\|_F^2 \mid \|\xi\|_{\text{tr}} \leq 1\} = 1,
\]

where we first used that \((\xi, A)\) is distributed the same as \(\|\xi\|_{F} A_{11}\) by orthogonal invariance, and then that the trace norm dominates the Frobenius norm, with equality attained for example by \(\xi = E_{11}\).

We will also use the the Schatten \(p\)-norms \(\|A\|_p = (\text{Tr} \left( (A^T A)^{\frac{p}{2}} \right)^{\frac{1}{p}}\)\), which generalize the trace, Frobenius, and operator norms. They satisfy the following Hölder inequality for \(p \geq 1\):

\[
\text{Tr} \prod_{i=1}^p A_i \leq \prod_{i=1}^p \|A_i\|_p,
\]

Here we identified \(\text{Mat}(m) \cong \mathbb{R}^m \otimes \mathbb{R}^m\), so \(\Pi\) identifies with the orthogonal projection onto the traceless matrices, and we used that \(\|\Pi(H)\|_F \leq \|H\|_F\), since \(\Pi\) is an orthogonal projection. Using Eq. (3.11), the result now follows from Theorem A.1.

In the remainder we discuss the proof of Theorem A.1. The proof proceeds by a symmetrization trick, followed by the trace method. We first state some relevant bounds on Gaussian random variables and then give the proof of Theorem A.1.

We will often use the following estimate of the operator norm of a standard Gaussian \(n \times m\) random matrix \(A\) (see Theorem 5.32 in Vershynin (2012)),

\[
E\|A\|_{\text{op}} \leq \sqrt{n} + \sqrt{m}.
\]

**Theorem A.2.** Let \(A\) be a centered Gaussian random variable that takes values in a separable Banach space with norm \(\|\cdot\|\). Then \(\|A\|\) satisfies the following concentration and moment inequalities with parameter \(\sigma^2 := \sup\{E\langle \xi, A \rangle^2 \mid \|\xi\|_* \leq 1\}\), where \(\|\cdot\|_*\) denotes the dual norm:

\[
\forall t > 0: \quad \Pr\left(\|A\| - E\|A\| \geq t\right) \leq 2\exp\left(-\frac{\Omega(t^2)}{\sigma^2}\right), \quad \text{and}
\]

\[
\forall p \geq 1: \quad E\|A\|^p \leq (2E\|A\|)^p + O(\sigma\sqrt{p})^p.
\]

**Proof.** The first statement on concentration is exactly Theorem 1.5 in Pisier (1986). For the second, we consider the random variable \(X := \frac{1}{\sigma}(\|A\| - E\|A\|)\). Then the equivalence in Lemma 5.5 of Vershynin (2012) gives the moment bound

\[
\left(\mathbb{E}|X|^p\right)^{1/p} = \frac{1}{\sigma}\left(\mathbb{E}\|A\| - E\|A\|\right)^{1/p} \leq O(\sqrt{p}).
\]

The moment bound in the theorem now follows by rearranging as

\[
E\|A\|^p = E\left(E\|A\| + \sigma X\right)^p \leq 2p\left((E\|A\|)^p + O(\sigma\sqrt{p})^p\right),
\]

where the last step was by the simple inequality \((a + b)^p \leq 2^p(|a|^p + |b|^p)\).

Below, we calculate the \(\sigma^2\) parameter in Theorem A.2 with regards to our random matrix setting.

**Corollary A.3.** Let \(A\) be an \(n \times m\) matrix with independent standard Gaussian entries. Then the random variable \(\|A\|_{\text{op}}\) satisfies the conclusions of Theorem A.2 with \(\sigma^2 = 1\).

**Proof.** Note that the dual norm is the trace norm \(\|\cdot\|_{\text{tr}}\), hence the concentration parameter can be estimated as

\[
\sigma^2 = \sup\{E\langle \xi, A \rangle^2 \mid \|\xi\|_{\text{tr}} \leq 1\} = \sup\{\|\xi\|_F^2 \mid \|\xi\|_{\text{tr}} \leq 1\} = 1,
\]

where we first used that \((\xi, A)\) is distributed the same as \(\|\xi\|_{F} A_{11}\) by orthogonal invariance, and then that the trace norm dominates the Frobenius norm, with equality attained for example by \(\xi = E_{11}\).

We will also use the the Schatten \(p\)-norms \(\|A\|_p = (\text{Tr} \left( (A^T A)^{\frac{p}{2}} \right)^{\frac{1}{p}}\)\), which generalize the trace, Frobenius, and operator norms. They satisfy the following Hölder inequality for \(p \geq 1\):
PROOF OF THEOREM A.1. The operator we want to control has entries which are dependent in complicated ways. We first begin with a standard symmetrization trick to linearize (compare the proof of Lemma 4.1 in Pisier (2014)). A single entry of $A_i \otimes A_i$ is either a product $gg'$ of two independent standard Gaussians, or the square $g^2$ of a single standard Gaussian. In expectation, we have $E gg' = 0, E g^2 = 1$, and so the expected matrix is

$$E \left( \sum_{i=1}^{N} A_i \otimes A_i \right) = N \text{vec}(I_n) \text{vec}(I_m)^T.$$ 

Accordingly, after projection we have

$$E \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi = 0.$$

Therefore we may add an independent copy: Let $B_1, \ldots, B_N$ be independent $n \times m$ random matrices with standard Gaussian entries, that are also independent from $A_1, \ldots, A_N$. Then,

$$\left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi = E_B \left( \sum_{i=1}^{N} A_i \otimes A_i - \sum_{i=1}^{N} B_i \otimes B_i \right) \circ \Pi$$

and hence, for any $p \geq 1$,

$$E \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_p \leq E \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i - \sum_{i=1}^{N} B_i \otimes B_i \right) \circ \Pi \right\|_p$$

by Jensen’s inequality, as $\| \cdot \|_p$ is convex as the composition of the norm $\| \cdot \|_p$ with the convex and nondecreasing function $x \rightarrow x^p$. Now note $(A_i, B_i)$ has the same distribution as $(A_i + B_i, A_i - B_i)$, so the right-hand side is equal to

$$E \left\| \frac{1}{2} \left( \sum_{i=1}^{N} (A_i + B_i) \otimes (A_i + B_i) - \sum_{i=1}^{N} (A_i - B_i) \otimes (A_i - B_i) \right) \circ \Pi \right\|_p$$

$$= E \left\| \sum_{i=1}^{N} A_i \otimes B_i + \sum_{i=1}^{N} B_i \otimes A_i \right\|_p \leq 2^p E \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_p$$

Thus, we have proved that

$$E \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_p \leq 2^p E \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_p.$$ 

(A.4) Note that we have lost the projection and removed the dependencies. Next we use the trace method to bound the right-hand side of Eq. (A.4). That is, we approximate the operator norm by the Schatten $p$-norm for a large enough $p$ and control these Schatten norms using concentration of moments of Gaussians (compare the proof of Theorem 16.6 in Pisier (2012)). For any $q \geq 1$,

$$E \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_{2q}^{2q} = E \text{Tr} \left[ \left( \sum_{i,j \in [N]} A_i^T A_j \otimes B_i^T B_j \right)^q \right]$$

$$= \sum_{i,j \in [N]^q} E \text{Tr} \left( A_i^T A_j \cdots A_i^T A_j \otimes B_i^T B_j \cdots B_i^T B_j \right)$$
where we used the independence of \( \{A_i\} \) and \( \{B_i\} \) in the last step. Now, the expectation of a monomial of independent standard Gaussian random variables is always nonnegative. Thus the same is true for \( \mathbb{E} \operatorname{Tr}(A_i^T A_j \cdots A_q^T A_j) \), so we can upper bound the sum term by term as

\[
\sum_{i,j \in [N]} \mathbb{E} \operatorname{Tr}(A_i^T A_j \cdots A_q^T A_j) \leq \sum_{i,j \in [N]} \mathbb{E} \operatorname{Tr}(A_i^T A_j \cdots A_q^T A_j) \mathbb{E}(\|B_i\|_{op}\|B_j\|_{op} \cdots \|B_q\|_{op})
\]

In the first step we used Hölder’s inequality (A.3) for the Schatten norm. The second step holds since \( \mathbb{E}\|B_i\|_{op} \leq (\mathbb{E}\|B_i\|^{2q}_{2q})^{\frac{1}{2q}} \) by Jensen’s inequality, so we can collect like terms together. Next, we used that the \( B_i \) have the same distribution as \( A \). In the last step, we used that \( \sum_i A_i \) has the same distribution as \( \sqrt{NA} \). Accordingly, we obtain for \( q \geq 1 \),

\[
\mathbb{E} \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_{2q}^{2q} \leq N^q \left( \mathbb{E}\|A\|^{2q}_{2q} \right)^2,
\]

and hence

\[
\mathbb{E} \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_{op}^{2q} \leq 4^q \mathbb{E} \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_{op}^{2q} \leq 4^q \mathbb{E} \left\| \sum_{i=1}^{N} A_i \otimes B_i \right\|_{2q}^{2q} \leq (4N)^q \left( \mathbb{E}\|A\|^{2q}_{2q} \right)^2 \leq (4N)^q m^2 \left( \mathbb{E}\|A\|^{2q}_{op} \right)^2.
\]

The first inequality is Eq. (A.4), and in the last inequality we used that \( A \in \text{Mat}(n,m) \) has rank \( \leq m \), and therefore \( \|A\|^{2q}_{2q} \leq m \|A\|^{2q}_{op} \). To bound the right-hand side, we use Theorem A.2, applied to the random variable \( A \) in the Banach space \( \text{Mat}(n,m) \) with the operator norm \( \|\cdot\|_{op} \). Then, \( \sigma^2 = 1 \), as computed in Corollary A.3, and we find that

\[
\mathbb{E} \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_{op}^{2q} \leq (4N)^q m^2 \left( (2\mathbb{E}\|A\|_{op})^{2q} + (C\sqrt{q})^{2q} \right)^2.
\]

where \( C > 0 \) is a universal constant implied by the big-\( O \) notation in Eq. (A.2). We can bound the first term \( \mathbb{E}\|A\|_{op} \leq \sqrt{m} + \sqrt{n} \) by Eq. (A.1), so for \( q = 2(m+n) \), we can upper bound the mean by

\[
\mathbb{E} \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_{op}^{2q} \leq 4m^2 \left( (\max\{2, C\})^2 \cdot q \cdot \sqrt{4N} \right)^{2q}.
\]
Finally, we can use Markov’s inequality to see that, for $C' = \sqrt{2} \max\{2, C\}$, the event
\begin{equation}
(A.5) \quad \left\| \left( \sum_{i=1}^{N} A_i \otimes A_i \right) \circ \Pi \right\|_{\text{op}} \leq (C't)^2 \cdot (m + n) \cdot \sqrt{4N}
\end{equation}
holds up to failure probability at most
\[4m^2 \left( \frac{\max\{2, C\}^2 \cdot q \cdot \sqrt{4N}}{(C't)^2 \cdot (m + n) \cdot \sqrt{4N}} \right)^{2q} \leq 4m^2 t^{-2q} \leq t^{-\Omega(m+n)},\]
where the first step was by our choice of $q = 2(m + n)$ and of $C' = \sqrt{2} \max\{2, C\}$, and the final inequality was by the fact that $t \geq 2$, so the prefactor $4m^2$ can be absorbed at the cost of slightly changing the constant in the exponent. \hfill \square

APPENDIX B: PROOF OF THE ROBUSTNESS LEMMA

In this appendix we give a proof of Lemma 3.20, which shows that strong convexity at a particular point implies strong convexity nearby. First note that by Remark 3.15, we have $\nabla^2 f_x(\Theta) = \nabla^2 f_{x'}$ where $x' = \Theta^{1/2} x$. Thus we need only bound the difference between $f_x$ and $f_{x'}$ for $\|\log \Theta\|_{\text{op}}$ small. $\Theta \in \mathbb{P}$. For a matrix $\delta_a$ in $\text{Mat}(d_a)$, we use $\epsilon^{(\delta_a)_{(a)}}$ to denote
\[e^{(\delta_a)_{(a)}} = I_{d_a} \otimes \cdots \otimes I_{d_a-1} \otimes \delta \otimes I_{d_a+1} \otimes \cdots \otimes I_{d_a},\]
as in Definition 3.8. We will write $\Theta^{1/2}$ as $e^\delta$, where $\delta = \sum_{a=1}^{k} (\delta_a)_{(a)}$. We now have $\Theta^{1/2} = e^\delta = \otimes_{a=1}^{k} e^{\delta_a}$, and $\frac{1}{2} \|\log \Theta\|_{\text{op}} = \|\delta\|_{\text{op}} = \sum_{a=1}^{k} \|\delta_a\|_{\text{op}}$. To bound the difference between $\nabla^2 f_{x'}$ and $\nabla^2 f_x$, we will show each component of the Hessian $\nabla f_{x'}$ (as presented in Lemma 3.14) only changes (from $\nabla f_x$) by a small amount under the perturbation $x \rightarrow x' := e^\delta x$. In particular we will give bounds on each block under each component-wise perturbation $x \rightarrow e^{(\delta_a)_{(a)}} x$, and write the overall perturbation as a sequence of such component-wise perturbations. For convenience, we adopt the short-hand
\[\rho_x := \frac{1}{nD} xx^T.\]

We begin with an easy fact relating the exponential map and matrix norms.

FACT B.1. \textit{For all symmetric $d \times d$ matrices $A$ such that $\|A\|_{\text{op}} \leq 1$, we have}
\[\|e^A - I\|_{\text{op}} \leq 2\|A\|_{\text{op}}\]
\textit{and}
\[\|e^A - I\|_{F} \leq 2\|A\|_{F}.\]

The 00 component of the Hessian is a scalar $\nabla^2_{00} f = \text{Tr}[\rho]$, and for $a \geq 1$ we think of each 0a component as a vector:
\[\sum_{a} \langle z_a, (\nabla^2_{0a} f) Z_a \rangle = z_0 \langle \rho, \sum_{a} \sqrt{d_a} Z_{(a)} \rangle \]

The diagonal components involve only one-body marginals of $\rho$:
\[\langle Z_a, (\nabla^2_{aa} f) Z_a \rangle = \langle d_a \rho_{(a)}, Z_a^2 \rangle \]
And the off-diagonal components involve two-body marginals:
\[\langle Z_a, (\nabla^2_{ba} f) Z_b \rangle = \langle \sqrt{d_a d_b} \rho_{(ab)}, Z_a \otimes Z_b \rangle.\]
Therefore in Lemma B.2 and Lemma B.3, we will prove perturbation bounds on one-body marginals, and in Lemma B.6 we will prove perturbation bounds on two-body marginals. This will allow us to bound the change in the $0a$ components, diagonal components, and the off-diagonal components, respectively. Then, following the structure of the proof of Proposition 3.19, we will collect all the term-wise bounds to prove an overall bound at the end of the section.

**Lemma B.2.** For $x \in \mathbb{R}^{D \times n}$ and a symmetric matrix $\delta \in \text{Mat}(d_a)$ such that $\|\delta\|_{\text{op}} \leq 1$, if we denote $x' := e^{\delta^{(a)} x}$ then

$$\|\rho_x^{(a)} - \rho_x^{(a)}\|_{\text{op}} \leq 8\|\delta\|_{\text{op}}\|\rho_x^{(a)}\|_{\text{op}}.$$  

**Proof.** By definition, $\|\rho_x^{(a)} - \rho_x^{(a)}\|_{\text{op}} = \sup_{\|Z\|_1 \leq 1} \langle Z, \rho^{(a)} - \rho_x \rangle$. Let $\delta' := \delta^a - I_a$. Note that $\|\delta'\|_{\text{op}} \leq 2\|\delta\|_{\text{op}}$ by Fact B.1 and our assumption $\|\delta\|_{\text{op}} \leq 1$. Now

$$\langle Z, \rho^{(a)} - \rho_x \rangle = \langle Z, (I + \delta') \rho_x (I + \delta') - \rho_x \rangle$$

$$= \langle Z, \delta' \rho_x^{(a)} \rangle + \langle Z, \rho_x^{(a)} \delta' \rangle + \langle Z, \delta' \rho_x^{(a)} \delta' \rangle$$

$$\leq (2\|\delta'\|_{\text{op}} + \|\delta'\|_{\text{op}}^2)\|\rho_x^{(a)}\|_{\text{op}}\|Z\|_1$$

$$\leq 8\|\delta\|_{\text{op}}\|\rho_x^{(a)}\|_{\text{op}}.$$

\[\square\]

**Lemma B.3.** For $x \in \mathbb{R}^{D \times n}$ and symmetric matrix $\delta \in \text{Mat}(d_b)$ such that $\|\delta\|_{\text{op}} \leq 1$, if we denote $x' := e^{\delta^{(b)} x}$ then for $b \neq a$:

$$\|\rho_x^{(a)} - \rho_x^{(a)}\|_{\text{op}} \leq 2\|\delta\|_{\text{op}}\|\rho_x^{(a)}\|_{\text{op}}.$$  

**Proof.** By definition, $\|\rho_x^{(a)} - \rho_x^{(a)}\|_{\text{op}} = \sup_{\|Z\|_1 \leq 1} \langle Z, \rho^{(a)} - \rho_x \rangle$. Let $\delta' := \delta^a - I_b$. Note that $\|\delta'\|_{\text{op}} \leq 2\|\delta\|_{\text{op}}$ by Fact B.1 and our assumption $\|\delta\|_{\text{op}} \leq 1$. Now

$$\|\langle Z, \rho^{(a)} - \rho_x \rangle\|_{\text{op}} = \|\langle Z, e^{\delta^{(b)} \rho_x^{(a)}} - \rho_x \rangle\|_{\text{op}}$$

$$= \|\langle Z, \delta^{(b)} \rho_x^{(a)} \rangle\|_{\text{op}} = \|\langle Z \otimes \delta', \rho_x^{(a)} \rangle\|_{\text{op}}$$

$$\leq \|\delta'\|_{\text{op}}\|Z \otimes I_b, \rho_x^{(a)}\|_{\text{op}}$$

$$\leq \|\delta'\|_{\text{op}}\|Z\|_1\|\rho_x^{(a)}\|_{\text{op}}.$$  

\[\square\]

This is already enough to prove a bound on $0a$ and $aa$ terms:

**Corollary B.4.** Let $x \in \mathbb{R}^{D \times n}$ be such that $\|d_a \rho_x^{(a)}\|_{\text{op}} \leq 1 + \frac{1}{2^m}$, and for $b \in [k]$ let $\delta_b \in \text{Mat}(d_b)$ be a symmetric matrix such that $\sum_b \|\delta_b\|_{\text{op}} \leq \frac{1}{8}$. Denoting $\delta^{(b)} := (\delta_b)^{(b)}$, $\delta = \sum_b \delta^{(b)}$ and $x' = e^{\delta^{(b)} x}$, for $a \geq 1$ we have

$$\|\nabla_{aa}^2 f(e^{2\delta}) - \nabla_{aa}^2 f(I)\|_{\text{op}} \leq 25\|\delta\|_{\text{op}}.$$
We can use Lemma B.2 to handle $e^8$ and hence by induction the $j^{th}$ bound the inner product with any traceless matrix. In the last step we used our bound on $\sum_{i=1}^{k} |\langle \delta \rangle_{\text{op}}(\rho_{x(i-1)}^{(a)})| \leq 8\|\delta\|_{\text{op}} \|\rho_{x(i-1)}^{(a)}\|_{\text{op}} \|Z\|_1.

Where the last inequality is due to Lemmas B.2 and B.3. To bound each term in the right-hand side, note that by Lemmas B.2 and B.3 we have

$$\|\rho_{x(i)}^{(a)}\|_{\text{op}} \leq \|\rho_{x(i)}^{(a)} - \rho_{x(i-1)}^{(a)}\|_{\text{op}} + \|\rho_{x(i-1)}^{(a)}\|_{\text{op}} \leq (1 + 8\|\delta\|_{\text{op}})\|\rho_{x(i-1)}^{(a)}\|_{\text{op}}$$

and hence by induction the $j^{th}$ term in the sum is at most

$$8\|\delta\|_{\text{op}} \left( \prod_{l=1}^{k} (1 + 8\|\delta\|_{\text{op}}) \right) \|\rho_{x}^{(a)}\|_{\text{op}} \|Z\|_1.$$ 

By our assumption that $\sum_{i=1}^{k} |\delta_{\text{op}}| \leq 1/8$, this is at most $8\|\delta\|_{\text{op}} e^{8\sum_{k=0}^{n} |\delta_{k}|_{\text{op}}} \|\rho_{x}^{(a)}\|_{\text{op}} \|Z\|_1 \leq 8\|\delta\|_{\text{op}} \|\rho_{x}^{(a)}\|_{\text{op}} \|Z\|_1$. Adding up the terms and using that $\|\delta\|_{\text{op}} = \sum_{k} |\delta_k|_{\text{op}}$, the overall sum is then at most $8\|\delta\|_{\text{op}} \|\rho_{x}^{(a)}\|_{\text{op}} \|Z\|_1$. Using our assumption on $\|\rho_{x}^{(a)}\|_{\text{op}}$ completes the proof.

**Corollary B.5.** Let $x \in \mathbb{R}^{D \times n}$ be such that $\|d_a \rho_{x}^{(a)}\|_{\text{op}} \leq 1 + 1/20$, and for $b \in [k]$ let $\delta_b$ be symmetric matrices such that $\|\sum_{b} \delta_{b}\|_{\text{op}} \leq 1/8$, where once again we denote $\delta_{(b)} := \delta_{b}$ and $\delta := \sum_{b} \delta_{b}$. Denoting $x' := e^{\delta} x$, for $a \geq 1$ we have

$$|\nabla_{00}^2 f_{x'} - \nabla_{00}^2 f_x| \leq 5\|\delta\|_{\text{op}}$$

and $$\|\nabla_{0a}^2 f_{x'} - \nabla_{0a}^2 f_x\|_{\text{op}} \leq 25\|\delta\|_{\text{op}}.$$ 

**Proof.** Recall from Lemma 3.14 that the 00 component of the Hessian is just the scalar $\text{Tr} \rho$. The assumption that $\|d_a \rho_{x}^{(a)}\|_{\text{op}} \leq 1 + 1/20$ implies $\text{Tr}[\rho_{x}^{(a)}] = \text{Tr} \rho_{x}^{(a)} \leq 1 + 1/20$. Now we can use the approximation for $e^\delta$ in Fact B.1:

$$|\text{Tr}[\rho_{x'} - \rho_{x}]| = |\langle \rho_{x}, e^{2\delta} - I \rangle| \leq \text{Tr}[\rho_{x}^{(a)}] \|e^{2\delta} - I\|_{\text{op}} \leq 5\|\delta\|_{\text{op}}$$

In the last step we used our bound on $\text{Tr}[\rho_{x}^{(a)}]$. The 0a component is a vector, so it is enough to bound the inner product with any traceless matrix $Z$ of unit Frobenius norm:

$$|\langle \rho_{x'}^{(a)} - \rho_{x}^{(a)}, \sqrt{d_a Z} \rangle| \leq \|\rho_{x'}^{(a)} - \rho_{x}^{(a)}\|_{\text{op}} \|\sqrt{d_a Z}\|_1.$$ 

In the proof of Corollary B.4 we showed under the same assumptions we have $\|\rho_{x'}^{(a)} - \rho_{x}^{(a)}\|_{\text{op}} \leq 25\|\delta\|_{\text{op}}/d_a$, from which it follows that the above is at most $25\|\delta\|_{\text{op}} \|Z\|_F$. \qed

The off-diagonal components require the following two lemmata on bipartite marginals:
LEMMA B.6. For \( x \in \mathbb{R}^{D \times n} \) and a symmetric matrix \( \delta \in \text{Mat}(d_c) \) such that \( \| \delta \|_{\text{op}} \leq \frac{1}{8} \); if we denote \( x' := e^{\delta(x)} \), then for \( c \in \{a, b\} \) we have

\[
\sup_{Y \in S^0_d, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_{x'}, \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F} \leq 3\|\delta\|_{\text{op}} \sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F}.
\]

Note that \( S^0_d \) are traceless symmetric matrices, whereas \( S^0_{d_b} \) are symmetric matrices.

**Proof.** By taking adjoints, we can assume w.l.o.g. that \( c = b \). Let \( R : \text{Mat}(d_b) \to \text{Mat}(d_b) \) be defined as \( R(Z) := e^{\delta} Ze^{\delta} \). Then

\[
|\langle \rho^{(ab)}_x, Y \otimes Z \rangle| = |\langle \rho^{(ab)}_x, Y \otimes (R(Z) - Z) \rangle|
\]

The subspace \( S^0_{d_b} \) is not invariant under \( R \), but we show \( R \approx I \). Let \( \delta' := e^{\delta} - I \); by Fact B.1, \( \| \delta' \|_{\text{op}} \leq \frac{1}{4} \). Now

\[
\|R(Z) - Z\|_F \leq 2\|\delta' Z\|_F + \|\delta' Z\|_F \leq (2\|\delta'\|_{\text{op}} + \|\delta'\|_{\text{op}}^2)\|Z\|_F \leq 3\|\delta\|_{\text{op}}\|Z\|_F.
\]

We combine these inequalities and apply a change of variables \( R(Z) - Z \leftarrow Z' \) to finish the proof.

\[
\sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_{x'}, \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F} \leq 3\|\delta\|_{\text{op}} \sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F}.
\]

\( \boxdot \)

LEMMA B.7. For \( x \in \mathbb{R}^{D \times n} \) and a symmetric matrix \( \delta \in \text{Mat}(d_c) \) such that \( \| \delta \|_{\text{op}} \leq \frac{1}{8} \); if we denote \( x' := e^{\delta(x)} \), then for \( c \notin \{a, b\} \) we have

\[
\sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_{x'}, \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F} \leq 4\|\delta\|_{\text{op}} \sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{|\langle \rho^{(ab)}_x, Y \otimes Z \rangle|}{\|Y\|_F \|Z\|_F}.
\]

**Proof.** Let \( \delta' := e^{2\delta} - I_c \) so that \( |\langle \rho^{(ab)}_x - \rho^{(ab)}_{x'}, Y \otimes Z \rangle| = |\langle \rho^{(ab)}_x, Y \otimes Z \otimes \delta' \rangle| \). We first assume \( Y, Z \succeq 0 \), and without loss of generality we assume that \( \|Y\|_F = \|Z\|_F = 1 \). Because \( \rho^{(abc)}_x \), \( Y, Z \succeq 0 \), and \( \delta' \succeq \| \delta' \|_{\text{op}} \cdot I_c \), we have

\[
|\langle \rho^{(abc)}_x, Y \otimes Z \otimes \delta' \rangle| \leq \langle \rho^{(abc)}_x, Y \otimes Z \otimes \| \delta' \|_{\text{op}} \cdot I_c \rangle \leq \| \delta' \|_{\text{op}} \langle \rho^{(ab)}_x, Y \otimes Z \rangle \leq 2\| \delta' \|_{\text{op}} \langle \rho^{(ab)}_x, Y \otimes Z \rangle,
\]

where the last inequality is by Fact B.1. To finish the proof we decompose \( Y = Y_+ - Y_- \), \( Z = Z_+ - Z_- \), where \( Y_+, Y_-, Z_+, Z_- \) are all positive semidefinite, and bound

\[
|\langle \rho^{(ab)}_x - \rho^{(ab)}_{x'}, Y \otimes Z \rangle| \leq \sum_{s,t \in \{+,-\}} |\langle \rho^{(ab)}_x - \rho^{(ab)}_{x'}, Y \otimes Z \rangle| \leq \sum_{s,t \in \{+,-\}} 2\| \delta' \|_{\text{op}} \langle \rho^{(ab)}_x, Y_s \otimes Z_t \rangle \leq 2 \left( \sup_{Y \in S^0_{d_a}, Z \in S^0_{d_b}} \frac{\langle \rho^{(ab)}_x, Y \otimes Z \rangle}{\|Y\|_F \|Z\|_F} \right) \cdot \| \delta' \|_{\text{op}} \sum_{s,t \in \{+,-\}} \|Y_s\|_F \|Z_t\|_F.
\]
The Cauchy Schwarz inequality allows us to bound the summation:

\[
\sum_{s,t \in \{+,-\}} \| Y_s \|_F \| Z_t \|_F \leq (2\| Y_+ \|_F^2 + 2\| Y_- \|_F^2)^{1/2} (2\| Z_+ \|_F^2 + 2\| Z_- \|_F^2)^{1/2} = 2\| Y \|_F \| Z \|_F.
\]

Plugging this bound in to the supremum on the left-hand side in the statement of the lemma completes the proof.

The following definition will be helpful for translating the above lemmas into statements about the Hessian.

**Definition B.8.** For a linear map \( M : \text{Mat}(d) \to \text{Mat}(d') \), we let \( \| M \|_0 \) denote the \( F \to F \) norm of its restriction to the traceless subspaces \( S_0^d \to S_0^{d'} \), i.e.

\[
\| M \|_0 = \sup_{Z \in S_0^d} \frac{\| M(Z) - \frac{\text{Tr}(M(Z))}{d'} I_{d'} \|_F}{\| Z \|_F}.
\]

The following lemma will be helpful.

**Lemma B.9 (Kwok, Lau and Ramachandran (2019)).** For \( x \in \mathbb{R}^{d \times n} \),

\[
\| \nabla^2_{ab} f_x \|_{F \to F}^2 \leq \| d_a \rho_x^{(a)} \|_{op} \| d_b \rho_x^{(b)} \|_{op}.
\]

Analogously to the proof of Corollary B.4, we can now combine Lemma B.6 and Lemma B.7 to bound the effect of a perturbation with more than one nontrivial tensor factor.

**Corollary B.10.** Let \( x \in \mathbb{R}^{d \times n} \) be such that \( \| d_a \rho_x^{(a)} \|_{op}, \| d_b \rho_x^{(b)} \|_{op} \leq 1 + \frac{1}{20d} \), and for \( c \in [k] \) let \( \delta_c \) be a symmetric matrix such that \( \| \sum_c \delta_c \|_{op} = \sum_c \| \delta_c \|_{op} \leq \frac{1}{8} \). Denoting \( x' := e^{\delta_x} \), we have

\[
\| \nabla^2_{ab} f_{x'} - \nabla^2_{ab} f_x \|_0 \leq 21 \| \delta \|_{op}
\]

**Proof.** First, using Lemma 3.14, we write the left-hand and right-hand sides of the inequalities in Lemma B.6 and Lemma B.7 in terms of the Hessian:

\[
\sup_{Y \in S_{a,b}^d, Z \in S_{a,b}^d} \frac{\langle \rho_x^{(ab)} - \rho_x^{(ab)}, Y \otimes Z \rangle}{\| Y \|_F \| Z \|_F} = \frac{\| \nabla^2_{ab} f_{x'} - \nabla^2_{ab} f_x \|_0}{\sqrt{d_a d_b}},
\]

and

\[
\sup_{Y \in S_{a,b}^d, Z \in S_{a,b}^d} \frac{\langle \rho_x^{(ab)}, Y \otimes Z \rangle}{\| Y \|_F \| Z \|_F} = \frac{\| \nabla^2_{ab} f_x \|_{F \to F}}{\sqrt{d_a d_b}}.
\]

Using the same iterative strategy as in the proof of Corollary B.4 for the left-hand sides of the above identities, we have

\[
\| Y, (\nabla^2_{ab} f_{x'} - \nabla^2_{ab} f_x)Z \| \leq 20\| \delta \|_{op} \| \nabla^2_{ab} f_x \|_{F \to F} \| Y \|_F \| Z \|_F,
\]

using Lemma B.6 for \( a \) and \( b \) and Lemma B.7 for the rest. Finally, we may rewrite Lemma B.9 using Lemma 3.14 to find \( \| \nabla^2_{ab} f_x \|_{F \to F} \leq \| d_a \rho_x^{(a)} \|_{op} \| d_a \rho_x^{(a)} \|_{op} \). Using our assumption that \( \| d_a \rho_a \|_{op}, \| d_b \rho_b \|_{op} \leq 1 + \frac{1}{20d} \) completes the proof.

Now we can combine the above term-by-term bounds to bound the change in the Hessian.
The above corollaries (B.5, B.4, B.10) require \( \|d_a \rho^{(a)}\|_{op} \leq 1 + \frac{1}{20} \), which are implied by our assumption on the gradient:

\[
\|d_a \rho^{(a)}\|_{op} \leq 1 + |\text{Tr} \rho - 1| + \|d_a \rho^{(a)} - (\text{Tr} \rho) I_d\|_{op} \\
= 1 + |\nabla_0 f| + \|\sqrt{d_a \nabla_a f}\|_{op} \leq 1 + 2 \varepsilon_0,
\]

so choosing \( \varepsilon_0 \leq \frac{1}{40} \) suffices. Recall the expression of the Hessian as a quadratic form evaluated on \( Z = (z_0, Z_1, \ldots, Z_k) : \)

\[
\langle Z, (\nabla^2 f) Z \rangle = z_0 (\nabla^2_{00} f) z_0 + 2 \sum_a \langle z_0, (\nabla^2_{0a} f) Z_a \rangle + \sum_a \langle Z_a, (\nabla^2_{aa} f) Z_a \rangle + \sum_{a \neq b} \langle Z_a, (\nabla^2_{ab} f) Z_b \rangle.
\]

Let \( x' := e^{\delta x} \). Then by Corollary B.5 we have a bound on the \( 0a \) terms:

\[
|z_0 (\nabla^2_{00} f) x' - (\nabla^2_{00} f) x| + 2 \sum_a |z_0, (\nabla^2_{0a} f) x' - (\nabla^2_{0a} f) x) Z_a | \leq 5 \|\delta\|_{op} |z_0|^2 + (2|z_0|)25 \|\delta\|_{op} \sum_a \|Z_a\|_F \leq \|\delta\|_{op} (17k z_0^2 + 25 \sum_a \|Z_a\|_F^2)
\]

In the last step we used Young’s inequality \((2pq \leq p^2 + q^2)\) for each term with \( p = z_0, q = \|Z_a\|_F \).

By Corollary B.4 we have a bound on the diagonal terms, and by Corollary B.10 we have a bound on the off-diagonal terms:

\[
|\sum_{ab} \langle Z_a, (\nabla^2_{ab} f) x' - (\nabla^2_{ab} f) x) Z_b \rangle | \leq \|\delta\|_{op} \left( 25 \sum_a \|Z_a\|_F^2 + 21 \sum_{a \neq b} \|Z_a\|_F \|Z_b\|_F \right) \leq (25 + 21(k - 1)) \|\delta\|_{op} \left( \sum_a \|Z_a\|_F^2 \right)
\]

So combining all three terms we see:

\[
|\langle Z, (\nabla^2 f) x' - (\nabla^2 f) x) Z \rangle | \leq \|\delta\|_{op} \left( 17k z_0^2 + (25 + 21(k - 1)) \sum a \|Z_a\|_F^2 \right) \leq 50k \|\delta\|_{op} \left( z_0^2 + \sum a \|Z_a\|_F^2 \right) = 50k \|\delta\|_{op} \|Z\|_2^2.
\]

Note that this also gives an upper bound for \( \|\nabla^2 f\|_{op} \).

---

**APPENDIX C: THE CHEEGER CONSTANT OF A RANDOM OPERATOR**

In this appendix we prove Theorem 4.5, which asserts that a random completely positive map with sufficiently many Kraus operators is an almost quantum expander with exponentially small failure probability. To prove the theorem, we first define the Cheeger constant of completely positive map. This is similar to a concept defined in Hastings (2007).
\textbf{Definition C.1.} Let $\Phi \colon \text{Mat}(d_2) \to \text{Mat}(d_1)$ be a completely positive map. The Cheeger constant of $\Phi$ is given by
\[
\chi(\Phi) := \min_{\Pi_1, \Pi_2} \phi(\Pi_1, \Pi_2)
\]
where $\Pi_1 : \mathbb{C}^{d_1} \to \mathbb{C}^{d_1}$ and $\Pi_2 : \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}$ are orthogonal projections, and the conductance $\phi(\Pi_1, \Pi_2)$ of the “cut” $\Pi_1, \Pi_2$ is defined to be
\[
\phi(\Pi_1, \Pi_2) := \frac{\text{cut}(\Pi_1, \Pi_2)}{\text{vol}(\Pi_1, \Pi_2)},
\]

\[
\text{cut}(\Pi_1, \Pi_2) := \text{Tr} \Phi(\Pi_2) + \text{Tr} \Phi^*(\Pi_1),
\]
\[
\text{vol}(\Pi_1, \Pi_2) := \text{Tr} \Phi(\Pi_2) + \text{Tr} \Phi^*(\Pi_1),
\]

where
\[
\text{vol}(\Pi_1, \Pi_2) := \text{Tr} \Phi(\Pi_2) + \text{Tr} \Phi^*(\Pi_1),
\]
\[
\text{cut}(\Pi_1, \Pi_2) := \text{Tr} \Phi(\Pi_2) + \text{Tr} \Phi^*(\Pi_1)(I_{d_2} - \Pi_2)
\]
\[
= \text{Tr} \Phi(\Pi_2) + \text{Tr} \Phi^*(\Pi_1)(I_{d_2} - \Pi_2).
\]

The key connection that we will leverage to prove Theorem 4.5 is that a large Cheeger constant implies quantum expansion:

\textbf{Lemma C.2 (Franks and Moitra (2020), Remark 5.5).} There exist absolute constants $c, C > 0$ such that if $\Phi$ is a completely positive map that is $\varepsilon$-doubly balanced for some $\varepsilon < c \chi(\Phi)^2$, then $\Phi$ is an $(\varepsilon, \eta)$-quantum expander, where
\[
\eta = \max \left\{ \frac{1}{2}, 1 - \chi(\Phi)^2 + C \frac{\varepsilon}{\chi(\Phi)^2} \right\}.
\]

For intuition, consider a weighted bipartite graph $G$ on $\{d_1\} \cup \{d_2\}$. The projections $\Pi_1$ and $\Pi_2$ are analogous to subsets $A \subseteq [d_1]$ and $B \subseteq [d_2]$, respectively. The quantity $\text{vol}(\Pi_1, \Pi_2)$ is analogous to the total mass of the edges adjacent to $A$ plus that of the edges adjacent to $B$, which is the volume of $A \cup B$ considered as a cut of $G$. The quantity $\text{cut}(\Pi_1, \Pi_2)$ corresponds to the total mass of the edges between $A \cup B$ and its complement, that is, to the weight of the cut defined by $A \cup B$. In fact, if the Cheeger constant were defined with $\Pi_1$ and $\Pi_2$ restricted to be coordinate projections, it would be exactly the Cheeger constant of the bipartite graph on $[d_1]$ and $[d_2]$ with edge $(i, j)$ weighted by $\text{Tr} e_i e_j^T \Phi(e_j e_i^T)$, and the volume and the cut would be the same as the volume and the cut on that bipartite graph.

For the remainder of this section let $\Phi = \Phi_X$ where $X_1, \ldots, X_n$ are random $d_1 \times d_2$ matrices with independent standard Gaussian entries. In this case, each edge-weight $\text{Tr} e_i e_j^T \Phi(e_j e_i^T)$ of the bipartite graph is an independent $\chi^2$ random variable with $n$ degrees of freedom. Accordingly:

\textbf{Lemma C.3.} Let $\Pi_1 : \mathbb{C}^{d_1} \to \mathbb{C}^{d_1}$, $\Pi_2 : \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}$ be orthogonal projections of rank $r_1$ and $r_2$, respectively. Then $\text{cut}(\Pi_1, \Pi_2)$, $\text{vol}(\Pi_1, \Pi_2)$, $\text{vol}(I_{d_1}, I_{d_2})$ are jointly distributed as
\[
R_1, R_1 + 2R_2, 2R_1 + 2R_2 + 2R_3,
\]
where $R_1$, $R_2$, $R_3$ are independent $\chi^2$ random variables with
\[
F_1 := nr_1(d_2 - r_2) + nr_2(d_1 - r_1),
\]
\[
F_2 := nr_1r_2, \quad \text{and} \quad F_3 := n(d_1 - r_1)(d_2 - r_2) \text{ degrees of freedom, respectively.}
\]

\textbf{Proof.} As the distribution of $\Phi_X$ is invariant under the action $(U_1, U_2) \cdot \Phi_X(Y) = U_1^T \Phi_X(U_2^T Y U_2) U_1^T$ of unitary matrices $U_1, U_2$, the joint distribution of $\text{cut}(\Pi_1, \Pi_2)$, $\text{vol}(\Pi_1, \Pi_2)$, $\text{vol}(I_{d_1}, I_{d_2})$ depends only on the rank of $\Pi_1, \Pi_2$. Thus we may compute in the case that $\Pi_1$ and $\Pi_2$ are coordinate projections, in which case one may verify the fact straightforwardly; see the discussion above.
We next show a sufficient condition for the Cheeger constant being bounded away from 1 that is amenable to the previous lemma.

**Lemma C.4.** Let $\Phi$ be a completely positive map and $\delta < 0.005$ such that the following hold for all orthogonal projections $\Pi_1: \mathbb{C}^{d_1} \to \mathbb{C}^{d_1}, \Pi_2: \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}$, not both zero, where we denote by $r_1$, $r_2$ the ranks of $\Pi_1$ and $\Pi_2$, respectively, and abbreviate $F_1 := nr_1(d_2 - r_2) + nr_2(d_1 - r_1)$, and $F_2 := nr_1r_2$ as in Lemma C.3:

1. If $F_2 \geq \frac{1}{4} nd_1 d_2$, then
   \[
   \text{vol}(\Pi_1, \Pi_2) \geq \left(\frac{101}{200} - \delta\right) \text{vol}(I_{d_1}, I_{d_2}) = (1.01 - 2\delta) \text{Tr} \Phi(I_{d_2}).
   \]

2. If $F_2 < \frac{1}{4} nd_1 d_2$ and $\text{vol}(\Pi_1, \Pi_2) > 0$, then
   \[
   \text{vol}(\Pi_1, \Pi_2) \leq \left(\frac{4}{3} + \delta\right)(F_1 + 2F_2) \quad \text{and} \quad \text{cut}(\Pi_1, \Pi_2) \geq \left(\frac{2}{3} - \delta\right) F_1.
   \]

Then $\text{ch}(\Phi) \geq \frac{1}{6} - O(\delta)$.

**Proof.** The first assumption implies we only need to reason about the case that $F_2 < \frac{1}{4} nd_1 d_2$. This is because the minimization in the definition of the Cheeger constant is over $\Pi_1$, $\Pi_2$ such that $\text{vol}(\Pi_1, \Pi_2) \leq \text{Tr} \Phi(I_{d_2})$. Therefore, the second assumption implies that

\[
\text{ch}(\Phi) \geq \frac{4}{3} + \delta - \frac{1}{2} - O(\delta) = \frac{1}{2} - O(\delta).
\]

It suffices to show that $F_1/(F_1 + 2F_2) \geq 1/3$ provided $r_1r_2 < \frac{1}{3} d_1 d_2$. Indeed, if either $r_1 = 0$ or $r_2 = 0$, then $F_2 = 0$ and $F_1 > 0$ and the claim holds. Otherwise, if $r_1 > 0$ and $r_2 > 0$, then

\[
\frac{F_1}{F_1 + 2F_2} = \frac{r_1d_2 + r_2d_1 - 2r_1r_2}{r_1d_2 + r_2d_1} = 1 - \frac{2r_1r_2}{r_1d_2 + r_2d_1}
\]

\[
= 1 - \frac{r_1r_2}{d_2} \left(\frac{2}{r_1d_1 + r_2d_2} \right) \geq 1 - \frac{4}{9} = \frac{1}{3}.
\]

In the last inequality we used that $a + a^{-1} \geq 2$ for all $a > 0$ and that $r_1r_2 < \frac{1}{3} d_1 d_2$. $\square$

Next we use Lemma C.3 to show that for a random completely positive map, the events in Lemma C.4 hold with high probability for any fixed $\Pi_1$ and $\Pi_2$. We also need a third bound which will use to transfer properties of a $\delta$-net to the whole space of projections.

**Lemma C.5.** Suppose $d_1 \leq d_2$. Let $\Pi_1: \mathbb{C}^{d_1} \to \mathbb{C}^{d_1}, \Pi_2: \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}$ be orthogonal projections of rank $r_1$ and $r_2$, respectively such that $r_1 + r_2 > 0$. Let $F_1 := nr_1(d_2 - r_2) + nr_2(d_1 - r_1)$ and $F_2 = nr_1r_2$. Then, the following holds for the random completely positive map $\Phi = \Phi_X$:

1. If $F_2 \geq \frac{1}{2} nd_1 d_2$, then Eq. (C.1) holds with $\delta = 0$ with probability at least $1 - e^{-\Omega(nd_1 d_2)}$.
2. If $F_2 < \frac{1}{2} nd_1 d_2$, then Eq. (C.2) holds with $\delta = 0$ with probability at least $1 - e^{-\Omega(F_1)}$.
3. Finally, $\text{vol}(\Pi_1, \Pi_2) \geq \frac{1}{4\sqrt{e}} \text{vol}(I_{d_1}, I_{d_2})$ with probability at least $1 - e^{-\Omega(F_1 + 2F_2)}$.

**Proof.** Recall from Lemma C.3 that $\text{cut}(\Pi_1, \Pi_2), \text{vol}(\Pi_1, \Pi_2), \text{vol}(I_{d_1}, I_{d_2})$ are jointly distributed as $R_1, R_1 + 2R_2, 2R_3 + 2R_2 + 2R_3$ for $R_1, R_2, R_3$ independent $\chi^2$ random variables with $F_1, F_2$, and $F_3 = n(d_1 - r_1)(d_2 - r_2)$ degrees of freedom, respectively. In view of Eqs. (C.1) and (C.2) with $\delta = 0$, it is thus enough to show that
1. If $F_2 \geq \frac{3}{4}nd_1d_2$, then $R_2 \geq \frac{1}{99}R_1 + \frac{101}{99}R_3$ with probability $1 - e^{-\Omega(nd_1d_2)}$.
2. If $F_2 < \frac{3}{4}nd_1d_2$, then $R_1 + 2R_2 \leq \frac{4}{3}(F_1 + 2F_2)$ and $R_1 \geq \frac{2}{3}F_1$ hold with probability $1 - e^{-\Omega(F_1)}$.
3. $R_1 + 2R_2 \geq \frac{2}{3}(F_1 + F_2)$ and $R_1 + R_2 + R_3 \leq \frac{2}{3}(F_1 + F_2 + F_3)$ with probability $1 - e^{-\Omega(F_1 + 2F_2)}$.

Indeed, the first (resp. second) item above implies the first (resp. second) item in the lemma by substituting the expressions for $F_1$ and $F_2$ of freedom. so $R_1$, $R_2$, $R_3$. The last item follows from the same reasoning combined with the inequality $F_1 + 2F_2 \geq \frac{1}{d_1^2}(F_1 + F_2 + F_3)$ for $r_1, r_2$ not both zero and the fact that $d_1 \leq d_2$.

All three follow from standard results for concentration of $\chi^2$ random variables; see e.g. Wainwright (2019). We only prove the first item; the second and third items are straightforward. To prove the first item, we first reason about the case when one of $r_1 = 0$.

Note that $F_1 + 2F_2 \geq \frac{4}{3}(F_1 + F_2 + F_3)$, because

$$\frac{F_1 + 2F_2}{F_1 + F_2 + F_3} = \frac{r_1d_2}{d_1} + \frac{r_2}{d_2} = \sqrt{\frac{r_1d_2}{d_1} \frac{d_1r_2}{d_2} + \frac{d_1r_2}{d_1d_2}} \geq \sqrt{\frac{4}{9} \cdot 2} = \frac{4}{3}.$$  

In particular, $F_2 \geq \frac{2}{3}(F_2 + F_3)$ and $F_2 \geq \frac{1}{6}(F_1 + F_2)$.

We first reason about the ratio between $R_2$ and $R_3$ using the first inequality. With probability $1 - e^{-cF_2} \geq 1 - e^{-\Omega(nd_1d_2)}$, it holds that $R_2 \geq \frac{3}{5}F_2$ and $R_2 + R_3 \leq \frac{10}{9}(F_2 + F_3)$. The latter holds because $R_2 + R_3$ is a $\chi^2$ random variable with $F_2 + F_3 \geq F_2$ degrees of freedom. so $R_2 \geq \frac{8}{9} \cdot \frac{9}{10}(R_2 + R_3) = \frac{8}{10}(R_2 + R_3)$, or $R_2 \geq \frac{8}{9}R_3$. We next apply the same reasoning with the inequality $F_2 \geq (F_1 + F_2)/6$ to estimate the ratio between $R_1$ and $R_2$. With probability $1 - e^{-cF_2}$, we have $R_2 \geq \frac{8}{9}F_2$ and $R_1 + R_2 \leq \frac{10}{9}(F_1 + F_2)$. Thus $R_1 \geq \frac{8}{9} \cdot \frac{9}{10}(R_1 + R_2) = \frac{4}{10}(R_1 + R_2)$, or $R_2 \geq \frac{8}{13}R_1$. Together, we obtain that $R_2 \geq \frac{1}{99}R_1 + \frac{101}{99}R_3$ with probability $1 - e^{-\Omega(nd_1d_2)}$.

Finally, we show using a net argument that the Cheeger constant is large for all projections.

**Lemma C.6 (Franks and Moitra (2020), Lemma 5.18).** For any $\varepsilon > 0$, there is an operator norm $\varepsilon$-net of the rank-$r$ orthogonal projections $\Pi : \mathbb{C}^d \rightarrow \mathbb{C}^d$ with cardinality $e^{O(dr|\log \varepsilon|)}$.

As a corollary, the set of pairs of projections $\Pi_1, \Pi_2$ of rank $r_1$ and $r_2$, respectively, has an (elementwise) operator norm $\varepsilon$-net of cardinality $e^{O((r_1d_1 + r_2d_2)|\log \varepsilon|)}$.

**Lemma C.7 (Continuity of cut and volume).** Let $\Pi_1, \Pi'_1 : \mathbb{C}^{d_1} \rightarrow \mathbb{C}^{d_1}$ and $\Pi_2, \Pi'_2 : \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_2}$ be orthogonal projections such that $\|\Pi_1 - \Pi'_1\|_{op} \leq \varepsilon$ and $\|\Pi_2 - \Pi'_2\|_{op} \leq \varepsilon$. Then:

$|\text{cut}(\Pi'_1, \Pi'_2) - \text{cut}(\Pi_1, \Pi_2)| \leq 2\varepsilon \text{vol}(I_{d_1}, I_{d_2})$

and

$|\text{vol}(\Pi'_1, \Pi'_2) - \text{vol}(\Pi_1, \Pi_2)| \leq 2\varepsilon \text{vol}(I_{d_1}, I_{d_2})$.

**Proof.** We begin with the first inequality:

$$|\text{cut}(\Pi'_1, \Pi'_2) - \text{cut}(\Pi_1, \Pi_2)| \leq |\text{Tr} \Pi'_1 \Phi(I_{d_2} - I_{d_2'}) - \text{Tr} \Pi_1 \Phi(I_{d_2} - I_{d_2'})| + |\text{Tr} \Pi_{d_1} \Phi(I_{d_2'} - I_{d_2'}) - \text{Tr} \Pi_{d_1} \Phi(I_{d_2'} - I_{d_2'})|.$$  

Consider the first term:

$$|\text{Tr} \Pi'_1 \Phi(I_{d_2} - I_{d_2'}) - \text{Tr} \Pi_1 \Phi(I_{d_2} - I_{d_2'})| = |\text{Tr} (\Pi'_1 - \Pi_1) \Phi(I_{d_2} - I_{d_2'}) + \text{Tr} \Pi_1 \Phi(I_{d_2'} - I_{d_2'})|$$
we estimate

where the first inequality is Lemma C.7, the second uses the first item above, and finally

to see that this suffices, we only need to show that it implies the hypotheses of Lemma C.4

\[3.\]

again using Lemma C.7, the second and third item above. In the last step we used the fact that

\[\delta \geq \frac{1}{2} \sum_{r}^{d} \log(d_{2})\]

to establish the lemma it suffices to show that with probability \(1 - e^{-\Omega(\text{nd}_{1})}\), the following is simultaneously true for all \(r_{1}, r_{2}\) and for all \((\Pi_{1}, \Pi_{2}) \in \mathcal{N}(r_{1}, r_{2})\):

1. If \(F_{2} := \|\Pi_{1} - \Pi_{2}\|_{\text{op}} \geq \frac{1}{3} \text{nd}_{1} d_{2}\), then Eq. (C.1) holds with \(\delta = 0\).
2. If \(F_{2} < \frac{1}{3} \text{nd}_{1} d_{2}\), then Eq. (C.2) holds with \(\delta = 0\).
3. \(\text{vol}(\Pi_{1}, \Pi_{2}) \geq \frac{1}{d_{2}} \text{vol}(I_{d_{1}}, I_{d_{2}})\).

To see that this suffices, we only need to show that it implies the hypotheses of Lemma C.4
for \(\delta = 32c\). Let \((\Pi'_{1}, \Pi'_{2})\) be an arbitrary pair of projections, not both zero. Let \(r_{1}\) and \(r_{2}\) denote their ranks. Then there exists a pair \((\Pi_{1}, \Pi_{2}) \in \mathcal{N}(r_{1}, r_{2})\) such that \(\|\Pi'_{1} - \Pi_{1}\|_{\text{op}} \leq \epsilon\) and \(\|\Pi'_{2} - \Pi_{2}\|_{\text{op}} \leq \epsilon\). If \(F_{2} \geq \frac{1}{3} \text{nd}_{1} d_{2}\),

\[
\text{vol}(\Pi'_{1}, \Pi'_{2}) \geq \text{vol}(\Pi_{1}, \Pi_{2}) - 2\epsilon \text{vol}(I_{d_{1}}, I_{d_{2}}) \\
\geq \left(\frac{101}{200} - 2\epsilon\right) \text{vol}(I_{d_{1}}, I_{d_{2}}) \geq \left(\frac{101}{200} - 2c\right) \text{vol}(I_{d_{1}}, I_{d_{2}}),
\]

where the first inequality is Lemma C.7, the second uses the first item above, and finally we estimate \(\epsilon \leq c\). Thus we have verified that Eq. (C.1) holds for \((\Pi'_{1}, \Pi'_{2})\), that is, the first hypothesis of Lemma C.4. If \(F_{2} < \frac{1}{3} \text{nd}_{1} d_{2}\), then

\[
\text{vol}(\Pi'_{1}, \Pi'_{2}) \leq \text{vol}(\Pi_{1}, \Pi_{2}) + 2\epsilon \text{vol}(I_{d_{1}}, I_{d_{2}}) \leq (1 + 8\epsilon d_{2}) \text{vol}(\Pi_{1}, \Pi_{2}) \\
\leq (1 + 8\epsilon d_{2}) \frac{4}{3} (F_{1} + 2F_{2}) = \left(\frac{4}{3} + \frac{32}{3} c\right) (F_{1} + 2F_{2}),
\]

where \(F_{1} := \|r_{1}(d_{2} - r_{2}) + r_{2}(d_{1} - r_{1})\), the first inequality is Lemma C.7, the second inequality uses the third item above, and the third inequality uses the second item above. On the other hand,

\[
\text{cut}(\Pi'_{1}, \Pi'_{2}) \geq \text{cut}(\Pi'_{1}, \Pi'_{2}) - 2\epsilon \text{vol}(I_{d_{1}}, I_{d_{2}}) \geq \text{cut}(\Pi'_{1}, \Pi'_{2}) - 8\epsilon d_{2} \text{vol}(\Pi_{1}, \Pi_{2}) \\
\geq \frac{2}{3} F_{1} - 8\epsilon d_{2} \frac{4}{3} (F_{1} + 2F_{2}) = \frac{2}{3} F_{1} - 8c \frac{4}{3} (F_{1} + 2F_{2}) \\
\geq \frac{2}{3} F_{1} - 32c F_{1} = \left(\frac{2}{3} - 32c\right) F_{1},
\]

again using Lemma C.7, the second and third item above. In the last step we used the fact that \(F_{1} \geq \frac{1}{3} (F_{1} + 2F_{2})\) provided \(F_{2} < \frac{1}{3} \text{nd}_{1} d_{2}\), which we established in the proof of Lemma C.4.

\[\square\]
Thus we have verified that Eq. (C.2) holds for \((\Pi_1', \Pi_2')\), which is the remaining hypotheses of Lemma C.4.

To prove the lemma we still need to show that the three conditions above hold with the desired probability. We show that for fixed \(r_1\) and \(r_2\), each condition holds with probability at least \(1 - e^{\Omega(n(r_1d_2 + d_1r_2))}\). By the union bound, this implies that the conditions hold simultaneously for all \(r_1 \leq d_1\) and \(r_2 \leq d_2\), not both zero, with the desired probability, because the sum of \(e^{-\Omega(n(r_1d_2 + d_1r_2))}\) over all such \(r_1\) and \(r_2\) is \(e^{-\Omega(nd_1)}\), using that \(d_1 \leq d_2\). Thus fix \(r_1\) and \(r_2\) as above. We first bound the probability for the first item. By Lemma C.5 and the union bound, if \(F_2 \geq \frac{4}{3}nd_1d_2\) then Eq. (C.1) holds for every \((\Pi_1, \Pi_2) \in \mathcal{N}(r_1, r_2)\) with probability
\[
1 - |\mathcal{N}(r_1, r_2)| e^{-\Omega(nd_1d_2)} \leq 1 - e^{(d_1r_1 + d_1r_2)\log \varepsilon} e^{-\Omega(n(r_1d_2 + d_1r_2))} \leq 1 - e^{-\Omega(n(r_1d_2 + d_1r_2))}.
\]
The last step follows by our assumption on \(n\) (for a large enough universal constant \(C > 0\)), since
\[
(d_1r_1 + d_1r_2)\log \varepsilon \leq \frac{d_2}{d_1} (r_1d_2 + d_1r_2) (\log(d_2) + |\log c|) = O \left( \frac{d_2}{d_1} \log(d_2) \right) \cdot (r_1d_2 + d_1r_2).
\]
Next we bound the probability for the second item. By Lemma C.5 and the union bound, if \(F_2 \geq \frac{4}{3}nd_1d_2\), Eq. (C.2) holds for every \((\Pi_1, \Pi_2) \in \mathcal{N}(r_1, r_2)\) with probability
\[
1 - |\mathcal{N}(r_1, r_2)| e^{-\Omega(F_1)} \leq 1 - |\mathcal{N}(r_1, r_2)| e^{-\Omega(n(r_1d_2 + d_1r_2))} \leq 1 - e^{-\Omega(n(r_1d_2 + d_1r_2))},
\]
where the first step holds since \(F_1 \geq \frac{4}{3}(F_1 + 2F_2) = \frac{1}{3}n(r_1d_2 + d_1r_2)\) whenever \(F_2 \leq \frac{4}{3}nd_1d_2\), as already used earlier in the proof, and the second step follows as above by our assumption on \(n\) (for large enough \(C > 0\)). The probability for the third item can be bounded completely analogously.

**Proof of Theorem 4.5.** Let \(\Phi := \Phi_X\). Since \(n \geq C\frac{d_2}{d_1} \log d_2\), Proposition C.8 shows that \(\text{ch}(\Phi) = \Omega(1)\) with failure probability \(e^{-\Omega(nd_1)}\). The latter is \(e^{-\Omega(d_2t^2)}\) using our assumption that \(n \geq C\frac{d_2}{d_1} t^2\).

Now let \(\varepsilon := \frac{t}{\sqrt{\frac{d_2}{nd_1}}}\), which by the same assumption satisfies \(\varepsilon \leq \frac{1}{\sqrt{C}}\). Moreover, \(n \geq \frac{d_2}{D\varepsilon^2}\), since this is equivalent to our assumption that \(t \geq 1\). Therefore, if we choose \(C\) sufficiently large then, similarly to the proof of Proposition 3.19, we find using Propositions 3.11 and 3.13 that \(\Phi\) is \(\varepsilon\)-doubly balanced with failure probability \(e^{-\Omega(nD)} + O(e^{-\Omega(nd_1\varepsilon^2)}) \leq e^{-\Omega(d_2\varepsilon^2)}\).

By making \(C\) larger, we can ensure that \(\varepsilon\) is less than any absolute constant. Then Lemma C.2 applies (with balancedness \(\varepsilon\)) and shows that \(\Phi\) is an \((\varepsilon, \eta)\)-quantum expander for some absolute constant \(\eta < 1\).

**Appendix D: Proof of Concentration for Matrix Normal Model**

In this section we prove Lemma 4.7, which shows concentration after one step of the flip-flop algorithm. For convenience, we consider the differently randomized normalized random variable \(Z = Y/\sqrt{nd_1}\). Note that these satisfy \(Z_i = X_i \Phi^*_X (I_{d_1})^{-1/2} = X_i (\sum_{i=1}^n X_i^T X_i)^{-1/2}\). Thus we wish to prove that
\[
\| \sum_{i=1}^n Z_i Z_i^T \|_{op} - \frac{d_2}{d_1} I_{d_1} \| \leq t \sqrt{\frac{d_2}{nd_1}}.
\]
Since we are interested in the spectral norm, we will consider the random variable \(\langle \xi, \sum_{i=1}^n Z_i Z_i^T \xi \rangle\) for a fixed unit vector \(\xi \in \mathbb{R}^{d_1}\). We will show that this variable is highly concentrated, and apply a union bound over a net of the unit vectors. To show the concentration, we first cast \(\langle \xi, \sum_{i=1}^n Z_i Z_i^T \xi \rangle\) as the inner product between a random orthogonal projection
and a fixed one. Since each $Z_i$ is a $d_1 \times d_2$ matrix, we can consider $Z$ as an $nd_1 \times d_2$ matrix by vertically concatenating the $Z_i$. By definition of the flip-flop step, $Z^T Z = \sum_{i=1}^{n} Z_i^T Z_i = I_{d_2}$, so $ZZ^T$ is an orthogonal projection onto a $d_2$-dimensional subspaces of $\mathbb{R}^{nd_1}$. In fact, $ZZ^T$ is a uniformly random such projection. This is because $X$, considered as a $nd_1 \times d_2$ random matrix with i.i.d. Gaussian entries, is invariant under left multiplication $X \mapsto OX$ by orthogonal transformations $O \in O(nd_1)$, hence the same is true for $Z = (X^T X)^{-1/2}$. We can now write
\[
\langle \xi, \sum_{i=1}^{n} Z_i Z_i^T \xi \rangle = \langle ZZ^T, \xi \xi^T \otimes I_n \rangle.
\]
The matrix $\xi \xi^T \otimes I_n$ is a fixed rank $n$ projection on $\mathbb{R}^{nd_1}$. We now use the following result on the inner product of random projections.

**Theorem D.1** (Lemma III.5 in Hayden, Leung and Winter (2006)). Let $P$ be a uniformly (Haar) random orthogonal projection of rank $a$ on $\mathbb{R}^m$, let $Q$ be a fixed orthogonal projection of rank $b$ on $\mathbb{R}^m$, and let $\varepsilon > 0$. Then,
\[
\Pr \left[ \langle P, Q \rangle \notin (1 \pm \varepsilon) \frac{ab}{m} \right] \leq 2e^{-\Omega(ab\varepsilon^2)}.
\]

We apply this result with $P = ZZ^T$, $Q = \xi \xi^T \otimes I_n$, $a = d_2$, $b = n$, and $m = nd_1$ to obtain
\[
\Pr \left[ \langle \xi, \left( \sum_{i=1}^{n} Z_i Z_i^T - \frac{d_2}{d_1} I_{d_1} \right) \xi \rangle > \frac{d_2}{d_1} \varepsilon \right] \leq 2e^{-\Omega(nd_2\varepsilon^2)}
\]
for any fixed unit vector $\xi \in \mathbb{R}^{d_2}$.

Next we apply a standard net argument for the unit vectors over $\mathbb{R}^{nd_1}$. We use the following lemma.

**Lemma D.2** (Lemma 5.4 Vershynin (2012)). Let $A$ be a symmetric $d \times d$ matrix, and let $N$ be a $\delta$-net of the unit sphere of $\mathbb{R}^d$ for some $\delta \in [0, 1)$. Then,
\[
\|A\|_{op} \leq (1 - 2\delta)^{-1} \sup_{\xi \in N} |\langle \xi, A\xi \rangle|.
\]

We apply the above lemma with $A = \sum_{i=1}^{n} Z_i Z_i^T - \frac{d_2}{d_1} I_{d_1}$, $d = d_1$, and a net $N$ for $\delta = 1/4$. By standard packing bounds (e.g., Lemma 4.2 in Vershynin (2012)) we may take $|N| \leq 9^{d_1}$. By Eq. (D.2) and the union bound, with failure probability $2 \cdot 9^{d_1} e^{-\Omega(nd_2\varepsilon^2)}$ we have that $|\langle \xi, A\xi \rangle| \leq \frac{d_2}{d_1} \varepsilon$ for all $\xi \in N$, and by Lemma D.2 this event implies $\|A\|_{op} \leq 2\frac{d_2}{d_1} \varepsilon$. Setting
\[
\varepsilon = t \sqrt{\frac{d_1}{4nd_2}},
\]
we obtain Eq. (D.1), i.e.,
\[
\left\| \sum_{i=1}^{n} Z_i Z_i^T - \frac{d_2}{d_1} I_{d_1} \right\|_{op} \leq 2\frac{d_2}{d_1} t \sqrt{\frac{d_1}{4nd_2}} = t \sqrt{\frac{d_2}{nd_1}},
\]
with failure probability at most $2 \cdot 9^{d_1} e^{-\Omega(d_1 t^2)}$, which is at most $e^{-\Omega(d_1 t^2)}$, provided $t$ is bounded from below by a large enough constant $C' > 0$. This concludes the proof of Lemma 4.7. □
APPENDIX E: RELATIVE ERROR METRICS

In this section we discuss the properties of our relative error metrics $D_F$, $D_{op}$ (Eqs. (2.3) and (2.4)). First note that they can be related to the usual norms by following inequalities:

\begin{align}
(E.1) & \quad \|B^{-1}\|_{op}^{-1} D_F(A\|B) \leq \|A - B\|_F \leq \|B\|_{op} D_F(A\|B) \\
(E.2) & \quad \|B^{-1}\|_{op}^{-1} D_{op}(A\|B) \leq \|A - B\|_{op} \leq \|B\|_{op} D_{op}(A\|B).
\end{align}

Next we state the approximate triangle inequality, also called a local triangle inequality in Yang and Barron (1999), and approximate symmetry for our relative error metrics.

**Lemma E.1.** Let $A, B, C \in \text{PD}(d)$. Let $D \in \{D_{op}, D_F\}$. Provided $D(A\|B), D(B\|C)$ are at most an absolute constant $c > 0$, we have

\begin{align}
(E.3) & \quad D(A\|C) = O\big(D(A\|B) + D(B\|C)\big), \\
(E.4) & \quad D(B\|A) = O\big(D(A\|B)\big), \text{ and} \\
(E.5) & \quad D(A^{-1}\|B^{-1}) = O\big(D(A\|B)\big).
\end{align}

For $D_{op}$, the result is (Franks and Moitra, 2020, Lemma C.1). For $D_F$, the result holds because $D_F(A\|B) \asymp d_{FR}(A, B)$ if either is at most some absolute constant (as shown below), where $d_{FR}(A, B)$ denotes the Fisher-Rao distance from Eq. (2.6). Because $d_{FR}(A, B)$ is a metric, it automatically satisfies Eqs. (E.3) and (E.4). Furthermore, $d_{FR}(A, B) = d_{FR}(A^{-1}, B^{-1})$ by direct calculation. We next consider the relationship between $D_F$ and other dissimilarity measures in the statistics literature.

**Lemma E.2** (Relationships between dissimilarity measures). There is a constant $c > 0$ such that the following holds. If any of $D_F(\Theta_1\|\Theta_2)$, $D_{TV}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1}))$, $D_{KL}(\mathcal{N}(0, \Theta_1^{-1})\|\mathcal{N}(0, \Theta_2^{-1}))$, or the Fisher-Rao distance $d_{FR}(\Theta_1, \Theta_2)$ is at most $c$, then

\[ D_F(\Theta_1\|\Theta_2) \asymp d_{FR}(\Theta_1, \Theta_2) \asymp D_{TV}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1})) \lesssim \sqrt{D_{KL}(\mathcal{N}(0, \Theta_1^{-1})\|\mathcal{N}(0, \Theta_2^{-1}))}. \]

**Proof.** For the relationship between $D_F(\Theta_1\|\Theta_2)^2$ and $d_{FR}(\Theta_1, \Theta_2)^2$, observe that the former is $\sum_{i=1}^d (\lambda_i - 1)^2$ and the latter is $\sum_{i=1}^d (\log \lambda_i)^2$ for the eigenvalues $\lambda_i$ of $\Theta_2^{-1}\Theta_1$. Because $\lambda - 1 \asymp \log \lambda$ in any fixed interval not containing 0, there is some absolute constant $c$ such that if $D_F(\Theta_1\|\Theta_2) \leq c$ or $d_{FR}(\Theta_1, \Theta_2) \leq c$ then $D_F(\Theta_1\|\Theta_2) \asymp d_{FR}(\Theta_1, \Theta_2)$.

To relate $D_F$ to the total variation distance, we use the following bound from Devroye, Mehrabian and Reddah (2018):

\[ 0.01 \leq \frac{D_{TV}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1}))}{D_F(\Theta_1\|\Theta_2)} \leq 1.5. \]

This implies that if either the numerator or denominator is a small enough constant, then they are on the same order.

Finally we relate $D_F$ to the relative entropy. We claim that if $D_{KL}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1})) \leq c$ for some small enough constant $c$ or if $D_F(\Theta_2\|\Theta_1) \leq 1/2$, then we have

\[ D_{KL}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1})) \asymp D_F(\Theta_2\|\Theta_1)^2. \]
This bound can be seen explicitly from the formula

\[ D_{\text{KL}}(\mathcal{N}(0, \Theta_1^{-1}), \mathcal{N}(0, \Theta_2^{-1})) = \frac{1}{2} \text{Tr} \Theta_1^{-1} \Theta_2 - \frac{1}{2} \log \det(\Theta_1^{-1} \Theta_2) - \frac{d}{2} \]

\[ \approx \frac{1}{2} \sum_{i=1}^{d} (\lambda_i - 1 - \log \lambda_i), \]

where \( \lambda_i \in 1 \pm D_{\text{op}}(\Theta_1 \| \Theta_2) \) are the eigenvalues of \( \Theta_1^{-1} \Theta_2 \), and the fact that \( \lambda - 1 - \log \lambda \approx \frac{1}{2}(\lambda - 1)^2 \) on \([1/2, 3/2]\). To complete the argument, choose \( c \) small enough that \( \frac{1}{2}(\lambda - 1 - \log \lambda) \leq c \) implies \( \lambda \in [1/2, 3/2] \).

The argument used to prove the relationship between \( D_F \) and \( d_{FR} \) also implies the following version relating \( D_{\text{op}} \) and \( d_{\text{op}} \).

**Lemma E.3.** There is a constant \( c > 0 \) such that the following holds. If \( D_{\text{op}}(\Theta_1 \| \Theta_2) \) or \( d_{\text{op}}(\Theta_1, \Theta_2) \) is at most \( c \), then \( D_{\text{op}}(\Theta_1 \| \Theta_2) \cong d_{\text{op}}(\Theta_1, \Theta_2) \).

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