Determinantal identity for multilevel systems and finite determinantal point processes

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Abstract

We give a simple algebraic derivation of a useful determinantal identity for multilevel systems such as random matrix chains and finite determinantal point processes, with applications to the calculation of point correlators, gap probabilities and Janossy densities.
1 Multilevel determinantal ensembles

1.1 Joint probability distributions

The following common setting underlies various multilevel determinantal ensembles, including chains of random matrices \[2, 3, 8\], and finite determinantal point processes such as Dyson processes \[10\] and polynuclear growth \[6\]. Let \(\{(\Gamma_j, d\mu_j)\}_{j=1,\ldots,m}\) be a set of measure spaces and \(\{\tilde{H}_j := L^2(\Gamma_j, d\mu_j)\}_{j=1,\ldots,m}\) the Hilbert spaces of square integrable functions on them. Suppose we are given \(m-1\) functions \(\{w_{j+1,j}\}_{j=1,\ldots,m-1}\) on the product spaces \(\Gamma_{j+1} \times \Gamma_j\), such that the corresponding integral operators

\[
w_{j+1,j} : \tilde{H}_j \rightarrow \tilde{H}_{j+1} \quad w_{j+1,j}(f)(x^{(j+1)}) := \int_{\Gamma_j} w_{j+1,j}(x^{(j+1)}, x^{(j)}) f(x^{(j)}) d\mu_j(x^{(j)}),
\]

(1.1)

and their transposes

\[
w_{j+1,j}^* : \tilde{H}_{j+1} \rightarrow \tilde{H}_j \quad w_{j+1,j}^*(f)(x^{(j)}) := \int_{\Gamma_{j+1}} w_{j+1,j}(x^{(j+1)}, x^{(j)}) f(x^{(j+1)}) d\mu_j(x^{(j+1)}),
\]

(1.2)

are well defined injective maps on a sequence of dense subspaces \(\tilde{H}_1 \subset \hat{H}_1\), \(\tilde{H}_2 = w_{21}(\hat{H}_1) \subset \hat{H}_2\), ..., \(\tilde{H}_m = w_{m,m-1}(\hat{H}_{m-1}) \subset \hat{H}_m\), as are their composites:

\[
w_{kj} := w_{k,k-1} \circ \ldots \circ w_{j+1,j}, \quad m \geq k > j \geq 1.
\]

(1.3)

Let \(H_1 \subset \hat{H}_1\) be an \(N\) dimensional subspace with basis \(\{\psi_a^{(1)}\}_{a=1,\ldots,N}\), and \(H_j := w_{j1}(H_1) \subset \hat{H}_j\) for \(j = 1,\ldots,m\), the corresponding subspaces obtained by applying the operators \(\{w_j\}\) to \(H_1\), with bases

\[
\{\psi_a^{(j)} := w_{j1}(\psi_a^{(1)})\}_{a=1,\ldots,N}.
\]

(1.4)

Now choose the basis \(\{\phi_a^{(m)}\}_{a=1,\ldots,N}\) for \(H_m\) dual to \(\{\psi_a^{(m)}\}_{a=1,\ldots,N}\), and let

\[
\phi_a^{(j)} := w_{mj}^*(\phi_a^{(m)}), \quad a = 1,\ldots,N
\]

(1.5)

be the corresponding dual bases for the \(H_j\)'s. Thus

\[
\int_{\Gamma_j} \psi_a^{(j)}(x^{(j)}) \phi_b^{(j)}(x^{(j)}) d\mu_j(x^{(j)}) = \delta_{ab}.
\]

(1.6)
Assume that
\[ \det(\psi^{(1)}_a(x^{(1)}_b)) \det(\phi^{(m)}_a(x^{(m)}_b)) \prod_{j=1}^{m-1} \det(w_{j+1,j}(x^{(j+1)}_a, x^{(j)}_b)) \prod_{j=1}^{m-1} \prod_{a=1}^{N} d\mu_j(x^{(j)}_a) \]  
(1.7)
is a positive Borel measure on \( \prod_{j=1}^{m} (\Gamma_j)^N \). It follows from the duality relations (1.6) that this is a normalized probability measure
\[ \prod_{j=1}^{m} \left( \int_{\Gamma_j}^{\Gamma_{N,j}} (d\mu_j(x^{(j)}_a)) \right) \det(\psi^{(1)}_a(x^{(1)}_b)) \det(\phi^{(m)}_a(x^{(m)}_b)) \times \prod_{j=1}^{m-1} \det(w_{j+1,j}(x^{(j+1)}_a, x^{(j)}_b)) = 1 \]  
(1.8)
so that
\[ P^{N,m}(x^{(j)}_a) := \det(\psi^{(1)}_a(x^{(1)}_b)) \det(\phi^{(m)}_a(x^{(m)}_b)) \prod_{j=1}^{m-1} \det(w_{j+1,j}(x^{(j+1)}_a, x^{(j)}_b)) \]  
(1.9)
may be interpreted as a joint probability density for points in \( \prod_{j=1}^{m} (\Gamma_j)^N \).

A typical example of such probability measures arises in the theory of random matrices. We start with a chain of \( N \times N \) Hermitian matrices \( \{M_j \in \mathbb{H}^N\}_{j=1,...,m} \) with nearest neighbor couplings of exponential type \( e^{\text{tr}(M_j M_{j+1})} \), and each site in the chain contributing a conjugation invariant factor \( e^{-\text{tr}(V_j(M_j))} \) to the joint probability density (where the \( V_j \)'s are e.g. polynomials in the matrices), and the product gives a normalizable positive measure. Integrating over the “angular variables”, using the Harish-Chandra-Itzykson-Zuber (HCIZ) identity [5], leads to a reduced joint probability measure of the form (1.9) on the space of eigenvalues, where the \( \Gamma_j \)'s are all taken as the real line,
\[ w_{j+1,j}(x^{(j+1)}, x^{(j)}) := e^{\text{tr}(V_j(x^{(j)})) + V_{j+1}(x^{(j+1)})} \]  
(1.10)
\[ \psi^{(1)}_a(x^{(1)}) := p_{a-1}(x^{(1)}) e^{-\frac{1}{2} V_1(x^{(1)})}, \quad \phi^{(m)}_a(x^{(m)}) := s_{a-1}(x^{(m)}) e^{-\frac{1}{2} V_m(x^{(m)})} \]  
(1.11)
and \( \{p_a(x^{(1)}), s_a(x^{(m)})\}_{a=0,1,...} \) are a sequence of pairs of polynomials of degrees \( a \) satisfying the biorthogonality relations
\[ \prod_{j=1}^{m} \left( \int_{\Gamma_j}^{\Gamma_{N,j}} d\mu_j(x^{(j)}) \right) \psi^{(1)}_a(x^{(1)}) \phi^{(m)}_b(x^{(m)}) \prod_{j=1}^{m-1} w_{j+1,j}(x^{(j+1)}, x^{(j)}) = \delta_{ab}. \]  
(1.12)

If all the \( \Gamma_j \)'s are identified as the same space \( \Gamma \), (1.9) may also be interpreted as a measure on the space of \( m \)-step paths of \( N \)-tuples of points in \( \Gamma \), defining a point process, in which \( j \) is viewed as a discrete time parameter. Examples of this kind lead to Dyson processes [10], describing diffusion of eigenvalues, and Polynuclear Growth [6].
1.2 Point correlators and gap probabilities

What characterizes such multilevel determinantal ensembles is that multi-point correlators (marginal distributions), gap probabilities and expectation values may all be expressed as determinants, either finite, or determinants of Fredholm integral operators, in terms of a single $m \times m$ matrix kernel function $\mathcal{K}_{ij}(x^{(i)}, x^{(j)})_{i,j=1,...,m}$ defined as follows:

$$\mathcal{K}_{ij}(x^{(i)}, x^{(j)}) := K_{ij}(x^{(i)}, x^{(j)}) - w_{ij}(x^{(i)}, x^{(j)}), \quad (1.13)$$

where

$$K_{ij}(x^{(i)}, x^{(j)}) := \sum_{a=1}^{N} \psi_a^{(i)}(x^{(i)}) \phi_a^{(j)}(x^{(j)}). \quad (1.14)$$

and $w_{ij}(x^{(i)}, x^{(j)})$ is defined as in (1.3) for $i > j$. In terms of this, the joint probability density (1.9) may be expressed as

$$P_{N,m}(x^{(j)}) = \det(\mathcal{K}_{ij}(x^{(i)}, x^{(j)})_{1 \leq i, a \leq k_i, 1 \leq b \leq k_j}). \quad (1.16)$$

where $\mathcal{K}_{ij}(x^{(i)}, x^{(j)})$ is viewed as the $(i, a, j, b)$ element of a matrix of dimension $Nm \times Nm$, labelled by pairs of double indices $1 \leq i, j \leq m, 1 \leq a, b \leq N$. By integrating eq. (1.16) over some of the variables while setting the rest equal to fixed values, it follows that the correlation function giving the probability density for finding $k_j$ elements in $\Gamma_j$ at the points $\{x^{(j)}_1, ... x^{(j)}_{k_j}\}$ for $j = 1 ... m$ is similarly given, within a combinatorial factor, by the determinant

$$P_{k_1,...,k_m}^{N,m}(\{x^{(j)}_1, ... x^{(j)}_{k_j}\}_{j=1,...,m}) := \det(\mathcal{K}_{ij}(x^{(i)}_a, x^{(j)}_b))_{1 \leq i, a \leq k_i, 1 \leq b \leq k_j}. \quad (1.17)$$

Alternatively, choosing a measurable subset $J_j \subset \Gamma_j$ of each $\Gamma_j$, the probability $E(0, \mathbf{J})$ of finding no points within the set \( \mathbf{J} := J_1 \times J_2 \times \ldots J_m \) (1.18) is given, within a normalization constant, by the Fredholm determinant ([8], [10])

$$E_{N,m}^{N,m}(0, \mathbf{J}) = C_{Nm} \det(\mathcal{I} - \mathcal{K} \circ \chi_{\mathbf{J}}), \quad (1.19)$$
where $\mathcal{K}$ is the $m \times m$ matrix integral operator with kernel $K_{ij}$ defined in (1.13) and $\chi_{J}$ is the direct sum of the operators of multiplication by the characteristic function $\{\chi_{J_{j}}\}_{j=1,...,m}$ on each of the factors in the direct sum

$$\hat{H} := \oplus_{j=1}^{m} \hat{H}_{j}. \quad (1.20)$$

More refined statistics, giving the probability of finding any specified number of points within disjoint subintervals $\{J_{jl}\}_{l=1,m_{j}}$ of the $J_{j}$’s may be similarly computed by replacing the characteristic functions $\chi_{J_{j}}$ by weighted ones $\sum_{l=1}^{m_{j}} z_{jl} \chi_{J_{jl}}$ and evaluating the coefficients of the monomials $\prod_{j=1}^{m} \prod_{l=1}^{m_{j}} z_{jl}^{k_{jl}}$.

A determinantal expression similar to (1.17) may also be found for the so-called “Janossy densities” [3, 8], which give the probabilities of finding elements at the given set of points $\{x_{1}^{(j)},...,x_{k_{j}}^{(j)}\}$ within the subsets $(J_{1}, J_{2},..., J_{m})$, while all others are on the outside, normalized to the probability of all points being on the outside. In that case, the integral kernel $K_{ij}(x_{a}^{(i)}, x_{b}^{(j)})$ appearing in (1.17) needs simply to be replaced by the corresponding kernel $R_{ij}\chi_{J}(x_{a}^{(i)}, x_{b}^{(j)})$ of the Fredholm resolvent operator:

$$R_{ij}\chi_{J} := (1 - \mathcal{K}\chi_{J})^{-1} \circ \mathcal{K}\chi_{J}, \quad (1.21)$$

where

$$\mathcal{K}\chi_{J} := \mathcal{K} \circ \chi_{J}. \quad (1.22)$$

These results may all be obtained as consequences of a single determinantal identity for multilevel ensembles, which can be expressed in a purely algebraic form. In the next section this algebraic identity will be stated, and a very simple proof given. All the above expressions for correlators and gap probabilities may be deduced from it, as can similar expressions for arbitrary multilevel determinantal ensembles.

This result is not at all new; it was derived e.g. in refs. [2], [3], [8], [10] in a number of particular cases, using a variety of different methods. The point of giving a new proof of the underlying identity is just to unify and simplify the argument, showing its universal applicability. By suitably particularizing to the various cases of interest, all previous results of this type follow, and determinantal expressions may be similarly derived for further cases that have not previously been studied. Some examples, consisting of chains of Lie algebra valued matrices, including those having various nearest neighbour couplings that admit reductions similar to that of Harish-Chandra-Itzykson-Zuber, are given in section 3.

Following Tracy and Widom [9, 10], the relevant integrals are first obtained in a uniform way by introducing a set of $m$ measurable functions or distributions $\{\rho_{1},...,\rho_{m}\}$
on \{\Gamma_1, \ldots, \Gamma_m\} and considering the integral obtained by replacing each \(d\mu_j\) by \(d\mu_j(1 - \rho_j)\).

\[
P_{\rho_1, \ldots, \rho_m}^{N,m} := \prod_{j=1}^m \left( \int_{\Gamma_j} \prod_{a=1}^N (d\mu_j(x_a^{(j)})(1 - \rho_j(x_a^{(j)}))) \det(\psi_a^{(1)}(x_b^{(1)})) \det(\phi_a^{(m)}(x_b^{(m)})) \times \prod_{j=1}^{m-1} \det(w_{j+1,j}(x_a^{(j+1)}, x_b^{(j)})) \right)
\]

(1.23)

We then apply a multi-level version of the Andreief identity \[\prod\] to express this integral as an \(N \times N\) determinant.

\[
P_{\rho_1, \ldots, \rho_m} := (N!)^m \det G,
\]

(1.24)

where

\[
G_{ab} := \prod_{j=1}^m \left( \int_{\Gamma_j} (d\mu_j(x^{(j)})(1 - \rho_j(x^{(j)}))) \psi_a^{(1)}(x^{(1)}) \phi_b^{(m)}(x^{(m)}) \prod_{j=1}^{m-1} w_{j+1,j}(x^{(j+1)}, x^{(j)}) \right).
\]

(1.25)

Like the one-level version, this follows easily from the invariance of the integrand under permutations in the various integration variables at each level.

By choosing the factors \{\rho_j\} in different ways, Tracy and Widom \[9, 10\] showed that point correlation functions, gap probabilities and Janossy densities may all be obtained from the formula (1.23). For example, choosing the \(\rho_j\)'s as:

\[
\rho_j := - \sum_{l=1}^{k_j} z_{jl} \delta_{X_j^{(l)}}
\]

(1.26)

the coefficient of the monomial factor \(\prod_{j=1}^m \prod_{l=1}^{k_j} z_{jl}\) in \(P_{\rho_1, \ldots, \rho_m}\) gives the point correlator \(P_{k_1, \ldots, k_m} (\{X_1^{(j)}, \ldots, X_{k_j}^{(j)}\}_{j=1, \ldots, m})\). Choosing instead

\[
\rho_j := \sum_{l=1}^{k_j} z_{jl} \chi_{X_j^{(l)}}
\]

(1.27)

gives the generating function for probabilities of finding the various numbers of points within disjoint subintervals \(J_{jl} \subset J_j\). Choosing

\[
\rho_j := \chi_{X_j} - \sum_{l=1}^{k_j} z_{jl} \delta_{X_j^{(l)}}
\]

(1.28)
and evaluating the coefficient of the monomial factor $\prod_{j=1}^{m} \prod_{l=1}^{k_j} z_{jl}$ in $P_{\rho_1,\ldots,\rho_m}$ gives the corresponding Janossy densities.

In order to apply the abstract form of the determinantal identity derived in the following section, it is convenient to interpret the matrix elements of $G$ as scalar products. To express integrals involving the functions $\psi_a^{(j)}$ and $\phi_a^{(j)}$ as scalar products, these will be represented as vectors or linear forms, using the bra and ket notation: $\langle \psi_a^{(j)} |$, $\langle \phi_a^{(j)} |$, $| \psi_a^{(j)} \rangle$ and $| \phi_a^{(j)} \rangle$. Thus, $G_{ab}$ may equivalently be written as:

$$G_{ab} = \langle \psi_a^{(1)} | (1 - \rho_1) w_{2,1}^* (1 - \rho_2) \cdots (1 - \rho_{m-1}) w_{m+1,m}^* (1 - \rho_m) | \phi_b^{(m)} \rangle$$

(1.29)

where

$$\rho_j : \hat{H}_j \to \hat{H}_j$$

(1.30)

is the operator of multiplication by $\rho_j$, which is symmetric ($\rho_j = \rho_j^*$) with respect to the scalar product

$$\langle v^{(j)} | u^{(j)} \rangle_{j} = \int_{\Gamma_j} v^{(j)}(x^{(j)}) u^{(j)}(x^{(j)}) d\mu_j(x^{(j)})$$

(1.31)

on $\hat{H}_j$.

## 2 Multilevel determinantal identity

To provide a uniform algebraic setting for all the previous examples, we assume that we have $m$ Euclidean inner product spaces $\{ \hat{H}_1, \ldots, \hat{H}_m \}$, and denote their direct sum by $\hat{H}$ as in (1.20). We also assume that there are $m - 1$ injective linear maps:

$$w_{j+1,j} : \hat{H}_j \to \hat{H}_{j+1}, \quad j = 1, \ldots, m - 1$$

(2.1)

with their duals

$$w_{j+1,j}^* : \hat{H}_{j+1} \to \hat{H}_j$$

(2.2)

as well as $m$ endomorphisms of the spaces $\hat{H}_j$

$$\rho_j : \hat{H}_j \to \hat{H}_j$$

(2.3)

that are symmetric $\rho_j = \rho_j^*$. The map $\rho : \hat{H} \to \hat{H}$ is defined to be the diagonal one leaving each subspace $\hat{H}_j$ invariant and equal to $\rho_j$ when restricted to $\hat{H}_j$. We also define composite maps

$$w_{kj} : \hat{H}_j \to \hat{K}_k, \quad m \geq k > j \geq 1$$

(2.4)
as in (1.3), with \( w_{kj} := 0 \) for \( k \leq j \). Together, these define an operator

\[
\mathcal{W} : \hat{H} \rightarrow \hat{H}
\]

that is lower triangular with respect to the decomposition (1.20) and can be written in matrix form as

\[
\mathcal{W} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
w_{21} & 0 & 0 & \cdots & 0 & 0 \\
w_{31} & w_{32} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
w_{m1} & w_{m2} & w_{m3} & \cdots & w_{m,m-1} & 0
\end{pmatrix}
\]

(2.6)

This may be expressed as

\[
\mathcal{W} = \mathcal{N} \circ (I - \mathcal{N})^{-1}
\]

(2.7)

where

\[
\mathcal{N} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
w_{21} & 0 & 0 & \cdots & 0 & 0 \\
0 & w_{32} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & w_{m,m-1} & 0
\end{pmatrix}
\]

(2.8)

We also denote the corresponding dual upper triangular operators as

\[
\mathcal{W}^* = \begin{pmatrix}
0 & w_{21}^* & w_{31}^* & \cdots & \cdots & w_{m1}^* \\
0 & 0 & w_{32}^* & \cdots & \cdots & w_{m2}^* \\
0 & 0 & 0 & \cdots & \cdots & w_{m3}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & w_{m,m-1}^* \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(2.9)

and

\[
\mathcal{N}^* = \begin{pmatrix}
0 & w_{21}^* & 0 & \cdots & \cdots & 0 \\
0 & 0 & w_{32}^* & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & w_{m,m-1}^* \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(2.10)

As previously, let \( H_1 \subset \hat{H}_1 \) be an \( N \) dimensional subspace with basis \( \{ \psi_a^{(1)} \}_{a=1,\ldots,N} \), and \( \{ H_j := w_{j1}(H_1) \subset \hat{H}_j \}_{j=1,\ldots,m} \), the corresponding subspaces obtained by applying the
operators \{w_j\} to \(H_1\), with bases \(\{\psi^{(j)}_a := w^{(j)}_1(\psi^{(1)}_a)\}_{a=1,\ldots,N}\). Let \(\{\phi^{(m)}_a\}_{a=1,\ldots,N}\) be the basis for \(H_m\) dual to \(\{\psi^{(m)}_a\}_{a=1,\ldots,N}\), and \(\{\phi^{(j)}_a := w^{(j)}_m(\phi^{(m)}_a)\}_{a=1,\ldots,N}\) the corresponding dual bases for the \(H_j\)'s, so that

\[
<\psi^{(j)}_a|\phi^{(j)}_b> = \delta_{ab}, \quad j = 1, \ldots, m. \tag{2.11}
\]

For \(a = 1, \ldots, N\), we introduce the \(m\)-component row and column vectors

\[
\Psi_a := \begin{pmatrix}
|\psi^{(1)}_a>
|\vdots|
|\psi^{(m)}_a>
\end{pmatrix}, \quad \Phi_a := \begin{pmatrix}
|\phi^{(1)}_a>
|\vdots|
|\phi^{(m)}_a>
\end{pmatrix} \tag{2.12}
\]

\[
\Psi^T_a := (<\psi^{(1)}_a|, \ldots, <\psi^{(m)}_a|), \quad \Phi^T_a := (<\phi^{(1)}_a|, \ldots, <\phi^{(m)}_a|), \tag{2.13}
\]

which may be viewed as elements of \(\hat{H}\) and \(\hat{H}^*\):

\[
\Psi_a = \sum_{j=1}^m \ell_j |\psi^{(j)}_a>, \quad \Phi_a = \sum_{j=1}^m \ell_j |\phi^{(j)}_a>
\]

\[
\Psi_a^T = \sum_{j=1}^m <\psi^{(j)}_a|\pi_j, \quad \Phi_a^T = \sum_{j=1}^m <\phi^{(j)}_a|\pi_j, \tag{2.14}
\]

where

\[
\ell_j : \mathcal{H}_j \rightarrow \oplus_{j=1}^m \mathcal{H}_j \quad \pi_j : \oplus_{j=1}^m \mathcal{H}_j \rightarrow \mathcal{H}_j \tag{2.15}
\]

are the standard injection and projection maps. These may also be expressed as

\[
\Psi_a = (\mathcal{I} - \mathcal{N})^{-1}\ell_1 |\psi^{(1)}_a> = (\mathcal{I} + \mathcal{W})\ell_1 |\psi^{(1)}_a>, \tag{2.16}
\]

\[
\Phi_a = (\mathcal{I} - \mathcal{N}^*)^{-1}\ell_m |\phi^{(m)}_a> = (\mathcal{I} + \mathcal{W}^*)\ell_m |\phi^{(m)}_a>, \tag{2.17}
\]

\[
\Psi_a^T = <\psi^{(1)}_a|\pi_1(\mathcal{I} - \mathcal{N}^*)^{-1} = <\psi^{(1)}_a|\pi_1(\mathcal{I} + \mathcal{W}^*), \tag{2.18}
\]

\[
\Phi_a^T = <\phi^{(m)}_a|\pi_m(\mathcal{I} - \mathcal{N}^*)^{-1} = <\phi^{(m)}_a|\pi_m(\mathcal{I} + \mathcal{W}^*). \tag{2.19}
\]

The duality relations (1.6) may equivalently be expressed as

\[
<\psi^{(1)}_a|\pi_1(\mathcal{I} - \mathcal{N}^*)^{-1}|\phi^{(m)}_b> = \delta_{ab}. \tag{2.20}
\]

In terms of the elements \(\Psi_a, \Phi_a \in \hat{H}\), define two maps

\[
\Psi : \mathbb{C}^N \rightarrow \hat{H}, \quad \Phi : \mathbb{C}^N \rightarrow \hat{H}, \\
\Psi : e_a \mapsto \Psi_a, \quad \Phi : e_a \mapsto \Phi_a \tag{2.21}
\]
and their transposes

\[ \Psi^T : \hat{H} \to \mathbb{C}^N, \]

\[ \Psi^T : \begin{pmatrix} |v^{(1)}> \\ \vdots \\ |v^{(m)}> \end{pmatrix} \mapsto \sum_{a=1}^N \sum_{j=1}^m <\psi^{(j)}_a|v^{(j)}> e_a \] (2.22)

\[ \Phi^T : \hat{H} \to \mathbb{C}^N, \]

\[ \Phi^T : \begin{pmatrix} |v^{(1)}> \\ \vdots \\ |v^{(m)}> \end{pmatrix} \mapsto \sum_{a=1}^N \sum_{j=1}^m <\phi^{(j)}_a|v^{(j)}> e_a, \] (2.23)

where \( \{e_a\}_{a=1,...,N} \) is the standard basis for \( \mathbb{C}^N \).

Denote by

\[ \mathcal{K} := \Psi \circ \Phi^T \] (2.24)

the linear operator \( \mathcal{K} : \hat{H} \to \hat{H} \) which is the composite map. This may be expressed in \( m \times m \) matrix form as

\[ \mathcal{K} = \begin{pmatrix} \mathcal{K}_{11} & \cdots & \mathcal{K}_{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{K}_{m1} & \cdots & \mathcal{K}_{mm} \end{pmatrix} \] (2.25)

where

\[ \mathcal{K}_{ij} := \sum_{a=1}^N |\psi^{(i)}_a> <\phi^{(j)}_a|. \] (2.26)

We also define the linear operator \( \tilde{\mathcal{K}} : \hat{H} \to \hat{H} \) as

\[ \tilde{\mathcal{K}} := \mathcal{K} - \mathcal{W} \] (2.27)

and let \( \mathcal{I} : \hat{H} \to \hat{H} \) denote the identity map. Finally define, as in (1.29) the \( N \times N \) matrix \( G \) whose components \( G_{ab} \) are the scalar products

\[ G_{ab} := <\psi^{(1)}_a|(1 - \rho_1)w^{*}_{2,1}(1 - \rho_2) \cdots (1 - \rho_{m-1})w^{*}_{m+1,m}(1 - \rho_m)|\phi^{(m)}_b>. \] (2.28)

The main result of this section is the following identity

**Proposition 2.1**

\[ \det(G) = \det(\mathcal{I} - \tilde{\mathcal{K}} \circ \rho). \] (2.29)
Proof: We begin by defining \( m - 1 \) modified linear maps

\[
\tilde{w}_{j+1,j} : \hat{H}_j \to \hat{H}_{j+1}, \quad j = 1, \ldots, m - 1
\]  

(2.30)

by composition with \((1 - \rho_j) : \hat{H}_j \to \hat{H}_j\)

\[
\tilde{w}_{j+1,j} := w_{j+1,j} \circ (1 - \rho_j)
\]  

(2.31)

We also define, analogously to \( N \) and \( W \), the operators \( \tilde{N} \) and \( \tilde{W} \) by

\[
\tilde{N} := N \circ (I - \rho) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\check{w}_{21} & 0 & 0 & \cdots & 0 & 0 \\
0 & \check{w}_{32} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & \check{w}_{m,m-1} & 0
\end{pmatrix}
\]  

(2.32)

\[
\tilde{W} := \tilde{N} \circ (I - \tilde{N})^{-1} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\check{w}_{21} & 0 & 0 & \cdots & 0 & 0 \\
\check{w}_{31} & \check{w}_{32} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
\check{w}_{m1} & \check{w}_{m2} & \check{w}_{m3} & \cdots & \check{w}_{m,m-1} & 0
\end{pmatrix},
\]  

(2.33)

where

\[
\tilde{w}_{kj} := \tilde{w}_{k,k-1} \circ \cdots \circ \tilde{w}_{j+1,j}, \quad m \geq k > j \geq 1.
\]  

(2.34)

The dual operators are

\[
\tilde{N}^* = \begin{pmatrix}
0 & \check{w}_{21}^* & 0 & \cdots & \cdots & 0 \\
0 & 0 & \check{w}_{32}^* & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & \check{w}_{m,m-1}^* & 0
\end{pmatrix}
\]  

(2.35)

and

\[
\tilde{W}^* = \begin{pmatrix}
0 & \check{w}_{21}^* & \check{w}_{31}^* & \cdots & \cdots & \check{w}_{m1}^* \\
0 & 0 & \check{w}_{32}^* & \cdots & \cdots & \check{w}_{m2}^* \\
0 & 0 & 0 & \cdots & \cdots & \check{w}_{m3}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & \check{w}_{m,m-1}^* & 0
\end{pmatrix}.
\]  

(2.36)
Note now that
\[
\left( (I + \tilde{W}^*) \circ (I - \rho) \right)_{1m} = \pi_1(I + \tilde{W}^*) \circ (I - \rho) \iota_m = \tilde{w}_{2,1}^* \circ \cdots \circ \tilde{w}_{m,m-1}^* \circ (1 - \rho_m)
\]
\[
= (1 - \rho_1)w_{2,1}^*(1 - \rho_2) \cdots (1 - \rho_{m-1})w_{m+1,m}^*(1 - \rho_m),
\]
and hence
\[
G_{ab} = \psi^{(1)}_a \pi_1(I + \tilde{W}^*) \circ (I - \rho) \iota_m | \phi^{(m)}_b >
\]
\[
= \psi^{(1)}_a \pi_1(I - (I - \rho) \circ N^*)^{-1} \circ (I - \rho) \iota_m | \phi^{(m)}_b >.
\]
(2.38)

From (2.20) it follows that
\[
G_{ab} = \delta_{ab} - \gamma_{ab},
\]
(2.39)

where
\[
\gamma_{ab} := - \psi^{(1)}_a | \pi_1((I - (I - \rho) \circ N^*)^{-1} \circ (I - \rho) - (I - N^*)^{-1}) \iota_m | \phi^{(m)}_b >
\]
\[
= \psi^{(1)}_a \pi_1((I - (I - \rho) \circ N^*)^{-1} \circ \rho \circ (I - N^*)^{-1}) \iota_m | \phi^{(m)}_b >
\]
\[
= \Psi_T((I - N^*) \circ (I - (I - \rho) \circ N^*)^{-1} \circ \rho) \Phi
\]
(2.40)

(where (2.17), (2.18) have been used in the last line). Now applying the well-known identity
\[
\det(I - A \circ B) = \det(I - B \circ A)
\]
(2.41)
to the operators
\[
B := \rho \circ \Phi : C^N \to \hat{H}
\]
\[
A := \Psi_T \circ ((I - N^*) \circ (I - (I - \rho) \circ N^*)^{-1}) : \hat{H} \to C^N,
\]
(2.42)
we obtain
\[
\det(G) = \det \left( (I - \rho \circ \Phi \circ \Psi_T(I - N^*) \circ (I - (I - \rho) \circ N^*)^{-1}) \right)
\]
\[
= \det \left( (I - (I - N \circ (I - \rho))^{-1} \circ (I - N) \circ K \circ \rho) \right)
\]
\[
= \det \left( (I - N \circ (I - \rho) - (I - N) \circ K \circ \rho) \right)
\]
\[
= \det \left( (I - (I - N)^{-1} \circ N \circ \rho) - K \circ \rho) \right)
\]
\[
= \det \left( I - (K - W) \circ \rho \right) = \det \left( I - \hat{K} \circ \rho \right),
\]
(2.43)

where in the third and fourth lines we have used the fact that
\[
\det (I - N \circ (I - \rho)) = \det (I - N) = 1.
\]
(2.44)
3 Examples

3.1 Classical Lie algebra chains

As further applications of the above identity, we consider coupled chains of \( m \) random matrices having values in the Lie algebras of the compact forms of the classical Lie groups \([7]\), with Itzykson-Zuber nearest neighbour exponential couplings. The partition function is of the form

\[
Z^m_N = \prod_{j=1}^{m} \left( \int dA_j \right) \prod_{j=1}^{m} e^{-\text{tr} V_j(A_j)} \prod_{j=1}^{m-1} e^{\text{tr} A_j A_{j+1}} \tag{3.1}
\]

where \( \{A_j \in \mathfrak{g}\}_{j=1,...,m} \) for any of the classical Lie algebras \( \mathfrak{g} = \mathfrak{u}(N), \mathfrak{o}(2N), \mathfrak{o}(2N+1) \) or \( \mathfrak{sp}(N) \) and \( dA_j \) denotes Lebesgue measure on each of these. Using the Harish-Chandra identity to integrate out the angular variables reduces these to integrals over the Cartan subalgebras. For the case \( \mathfrak{u}(N) \), identified as Hermitian matrix chains, this gives the usual Itzykson-Zuber reduced integral \([5, 2]\)

\[
Z^m_N \propto \prod_{j=1}^{m} \left( \int_{\mathbb{R}^N} \prod_{a=1}^{N} dx_a^{(j)} e^{-V_j(x_a^{(j)})} \right) \Delta(x_1^{(1)}, \ldots, x_N^{(1)}) \Delta(x_1^{(m)}, \ldots, x_N^{(m)}) \prod_{j=1}^{m-1} \det(e^{x_a^{(j)} x_b^{(j)}}) \tag{3.2}
\]

where

\[
\Delta(x_1, \ldots, x_N) := \det(x_{N-a}^{N-a})|_{1 \leq a, b \leq N} = \prod_{1 \leq a < b \leq N} (x_a - x_b) \tag{3.3}
\]

is the Vandermonde determinant. For the other classical Lie algebras it gives (see \([7]\) for the case \( m = 2 \))

\( \mathfrak{o}(2N) \):

\[
Z^m_N \propto \prod_{j=1}^{m} \left( \int_{\mathbb{R}^N} \prod_{a=1}^{N} dx_a^{(j)} e^{-V_j(x_a^{(j)})} \right) \Delta((x_1^{(1)})^2, \ldots, (x_N^{(1)})^2) \Delta((x_1^{(m)})^2, \ldots, (x_N^{(m)})^2) \times \prod_{j=1}^{m-1} \det(2\cosh(x_a^{(j)} x_b^{(j)})) \tag{3.4}\]

\( \mathfrak{o}(2N+1) \) or \( \mathfrak{sp}(N) \):

\[
Z^m_N \propto \prod_{j=1}^{m} \left( \int_{\mathbb{R}^N} \prod_{a=1}^{N} dx_a^{(j)} e^{-V_j(x_a^{(j)})} \right) \prod_{a=1}^{N} x_a^{(1)} \Delta((x_1^{(1)})^2, \ldots, (x_N^{(1)})^2)
\]
where the potentials \( \{V_j\}_{j=1,...,m} \) must be chosen as even functions of their arguments.

Replacing the monomial elements \((x_1^{(1)})^{N-a}, (x_b^{(m)})^{N-a}\) in the Vandermonde determinants in (3.2) by 
\[
\psi_a(x_b^{(1)}) = p_{a-1}(x_b^{(1)})e^{-\frac{1}{2}V_1(x_b^{(1)})},
\phi_a(x_b^{(m)}) = s_{a-1}(x_b^{(m)})e^{-\frac{1}{2}V_{m}(x_b^{(m)})}, \quad a = 1, \ldots, N
\]
(3.6)
and \(p_a(x^{(1)}), s_a(x^{(m)})\) are polynomials of degree \(a \in \mathbb{N}\) satisfying the biorthogonality conditions
\[
\prod_{j=1}^{m} \left( \int_{\Gamma_j} dx^{(j)} e^{-V_j(x^{(j)})} \right) p_a(x^{(1)}) s_b(x^{(m)}) \prod_{j=1}^{m-1} e^{x^{(j+1)}x^{(j)}} = \delta_{ab},
\]
(3.7)
we obtain the joint probability density (1.9) with \((\Gamma_j, d\mu_j) = (\mathbb{R}, dx^{(j)}), j = 1, \ldots, m\) and
\[
w_{j+1,j}(x^{(j+1)}, x^{(j)}) := e^{x^{(j)}x^{(j+1)}}
\]
(3.8)
Thus, applying Proposition 2.1 to this case, and choosing the functions \(\rho_j\) to be the characteristic functions \(\chi_{J_j}\) of unions of subintervals \(J_j = \bigcup_l J_{jl} \subset \mathbb{R}\) gives the gap probability
\[
P_{N,m}^{\chi_{J_1},...,\chi_{J_m}} := E_{N,m}^{0,0} = C_{N,m}\det(\mathcal{I} - \tilde{\mathcal{C}} \circ \chi_J)
\]
(3.9)
where the \(m \times m\) matrix Fredholm operator \(\tilde{\mathcal{C}}\) is defined as in (1.13), (1.14) with \((\psi^{(i)}_a, \phi^{(i)}_a)\) defined in (1.3), (1.4), (1.5).

The other two cases are nearly identical, the only difference being that, for \(o(2N)\), the potentials \(V_j(x^{(j)})\) should be taken as even functions, and the biorthogonal polynomials \((p_{a-1}(x^{(1)})s_{a-1}(x^{(m)}))\) in (3.6) replaced by even polynomials of even degrees
\[
\psi_a(x^{(1)}) := p_{2(a-1)}(x^{(1)})e^{-\frac{1}{2}V_1(x^{(1)})},
\phi_a(x^{(m)}) := s_{2(a-1)}(x^{(m)})e^{-\frac{1}{2}V_{m}(x^{(m)})}, \quad a = 1, \ldots, N
\]
(3.10)
satisfying the biorthogonality conditions
\[
\prod_{j=1}^{m} \left( \int_{\Gamma_j} dx^{(j)} e^{-V_j(x^{(j)})} \right) p_{2a}(x^{(1)}) s_{2b}(x^{(m)}) \prod_{j=1}^{m-1} 2 \cosh(x^{(j+1)}x^{(j)}) = \delta_{ab},
\]
(3.11)
The functions $w_{j+1,j}$ entering in (1.3), (1.4), (1.5) are

$$w_{j+1,j}(x^{(j+1)}, x^{(j)}) := 2 \cosh(x^{(j+1)}x^{(j)}).$$

(3.12)

For $\mathfrak{o}(2N + 1)$ and $\mathfrak{sp}(N)$ the potentials $V_j(x^{(j)})$ must again be taken as even functions, the biorthogonal polynomials $(p_{a-1}(x^{(1)})s_{a-1}(x^{(m)}))$ in (3.6) are replaced by odd polynomials of odd degrees

$$\psi_a(x^{(1)}) := p_{2a-1}(x^{(1)})e^{-\frac{1}{2}V_1(x^{(1)})},$$
$$\phi_a(x^{(m)}) := s_{2a-1}(x^{(m)})e^{-\frac{1}{2}V_m(x^{(1)})}, \quad a = 1, \ldots, N$$

(3.13)

satisfying the biorthogonality conditions

$$\prod_{j=1}^{m} \left( \int_{F_j} dx^{(j)} e^{-V_j(x^{(j)})} \right) p_{2a-1}(x^{(1)})s_{2b-1}(x^{(m)}) \prod_{j=1}^{m-1} 2 \sinh(x^{(j+1)}x^{(j)}) = \delta_{ab},$$

(3.14)

and the functions $w_{j+1,j}$ entering in (1.3), (1.4), (1.5) are

$$w_{j+1,j}(x^{(j+1)}, x^{(j)}) := 2 \sinh(x^{(j+1)}x^{(j)}).$$

(3.15)

**Remark 3.1** In view of the fact that the joint probability densities for the cases $\mathfrak{o}(2N)$, $\mathfrak{o}(2N + 1)$ and $\mathfrak{sp}(N)$ are all even functions of their arguments, if averages only over even functions are considered, as would be the case, for example, in computing expectation values of conjugation invariant functions of the matrices, the coupling terms $2 \cosh(x^{(j)}x^{(j+1)})$ and $2 \sinh(x^{(j)}x^{(j+1)})$ appearing may equivalently by replaced by the usual exponential couplings $e^{x^{(j)}x^{(j+1)}}$.

### 3.2 Nonexponential couplings

As explained in [4], Appendix A, another generalization of the Hermitian matrix chain can be made by choosing a set of functions $f_j(z)$ having a Taylor series expansion about the origin of the form

$$f_j(x) = 1 + \sum_{k=1}^{\infty} r_j(1) \cdots r_j(k)x^k, \quad j = 1, \ldots, m - 1.$$  

(3.16)

The exponential coupling term $e^{trA_jA_{j+1}}$ in (3.1) may be replaced by the more general set of conjugation invariant functions

$$\tau_j(A_jA_{j+1}) := \sum_{\lambda} \left( \prod_{i,k \in \lambda} r_j(N + k - i) \right) d_{\lambda,N}s_{\lambda}(A_jA_{j+1}),$$

(3.17)
where $s_\lambda$ is the Schur function corresponding to partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)$ of length $\ell(\lambda)$, viewed as a character, and $d_{\lambda,N}$ is the dimension of the irreducible tensor representation of $\mathfrak{gl}(N)$ corresponding to $\lambda$. The joint probability density for the eigenvalues on the chain is then again of the form (1.9) with the functions $(\psi_a(x^{(1)}), \phi_a(x^{(m)})$ defined as in (1.11) satisfying the biorthogonality relations (1.12), but with the functions $w_{j+1,j}$ defined as

$$w_{j+1,j}(x^{(j+1)}, x^{(j)}) := f_j(x^{(j)} x^{(j+1)}).$$  \hspace{1cm} (3.18)

An example of such a coupling is obtained by choosing a set of $m-1$ pairs of constants $\{(a_1, z_1), \ldots, (a_{m-1}, z_{m-1})\}$ and setting

$$f_j(x) := (1 - z_j x)^{N-a_j-1}, \quad j = 1, \ldots, m-1. \hspace{1cm} (3.19)$$

Hence

$$w_{j+1,j}(x^{(j+1)}, x^{(j)}) = (1 - z_j x^{(j)} x^{(j+1)})^{N-a_j-1}, \hspace{1cm} (3.20)$$

which corresponds to

$$r_j(i) = \frac{z_j(a_j - N + i)}{i}. \hspace{1cm} (3.21)$$

This choice gives

$$\tau_j(A_j A_{j+1}) = \text{det}(I - z_j A_j A_{j+1})^{-a_j} \hspace{1cm} (3.22)$$

and hence the partition function (3.1) for the coupled random matrix chain is

$$Z_N^m = \prod_{j=1}^{m} \left( \int dA_j \right) \prod_{j=1}^{m} e^{-\text{tr}V_j(A_j)} \prod_{j=1}^{m-1} \text{det}(I - z_j A_j A_{j+1})^{-a_j}. \hspace{1cm} (3.23)$$

All the above formulae for multi-point correlators, gap probabilities and Janossy densities are applicable to these cases, simply by adapting the general formulæ to these choices for the coupling functions $w_{j+1,j}$.

Acknowledgements. The authors would like to thank A. Prats-Ferrer for helpful comments regarding the case of coupled classical Lie algebra chains.
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