An Efficient Analytical Evaluation of the Electromagnetic Cross-Correlation Green’s Function in MIMO Systems

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Abstract—In this paper, we completely eliminate all numerical integrations needed to compute the far-field envelope cross-correlation (ECC) in multiple-input-multiple-output (MIMO) systems by deriving accurate and efficient analytical expressions for the frequency-domain cross-correlation Green’s functions (CGFs), the most fundamental electromagnetic kernel needed for understanding and estimating spatial correlation metrics in multiple-antenna configurations. The analytical CGF is derived for the most general three-dimensional case, which can be used for fast CGF-based correlation matrix calculations in MIMO systems valid for arbitrary locations and relative polarizations of the constituent elements.

Index Terms—MIMO arrays, Cross-correlation Green’s Function, Infinitesimal Dipole Model (IDM).

I. INTRODUCTION

Fifth generation (5G) wireless networks aiming at high data-rate (> 10 Gbps) and efficient interference suppression between multiple users in ultra-dense networks (UDNs), deploy large antennas arrays/massive multiple-input-multiple-output (MIMO) systems as key enabling technology [1]-[6]. One crucial aspect of such multiple-antenna systems in MIMO transceivers is the latter’s “spatial correlation matrix”, which accounts for mutual interaction (conventionally, only far-field is taken into account) between the constituent antenna element-pairs [7]-[11]. Despite the pivotal role of this spatial correlation matrix in shaping the overall channel matrix and consequent capacity/interference-suppression issues (see [12]-[15] for details), the impact of pure antenna effects and various core electromagnetic aspects in MIMO channel modelling are often not emphasized adequately.

Traditionally, this antenna spatial correlation performance is determined from a formula involving the radiation patterns of individual elements [10], which makes it very cumbersome to perform antenna current level optimization aiming at a desired diversity performance. To relate the antenna correlation directly to the radiating antenna current distribution (i.e., essentially bypassing altogether the original far-field pattern route), the concept of cross-correlation Green’s functions (CGFs) was first introduced in [16],[17] and later elaborated for antenna design applications [20],[18]. A preliminary step of this CGF-based correlation calculation is to construct a suitable infinitesimal dipole model (IDM) for the radiating MIMO antenna current distribution (see [19]-[28] for detailed theory and applications of IDM). The next step is to employ suitable CGFs in order to compute individual ID-pair interactions systematically and then combine them together to construct the global correlation matrices of the system [16]. Application of CGFs to realize high diversity gain MIMO antenna arrays as well as dual-polarized massive MIMO systems has been reported in recent past [29]-[32]. By efficient integration of the CGFs with finite-difference-time-domain (FDTD) computational paradigm, one can also perform wideband time-domain correlation analysis for arbitrary antennas [33],[34]. The CGFs are also extended to deal with radiators involving both electric/magnetic current sources [35]. Possible application of the CGF methodology for near-field stochastic systems [36] and antenna directivity analysis [37] are also being actively explored presently.

However, calculation of the CGF tensor components requires numerical integration involving elevation (θ) and azimuth (φ) angle dependent terms in the argument of complex exponential functions [16],[33]. Therefore, it becomes extremely difficult to efficiently embed these CGFs in fast optimization routines aiming at finding optimum current distributions for desired diversity performance. This necessitates a robust analytical evaluation scheme for the determination of the CGF tensor components and the possibility of achieving this was in fact already suggested in [16]. Although some specialized approximation formulas of the time-domain CGFs were attempted in [38], IDM-synthesis of MIMO antennas strictly requires frequency-domain CGFs, and the analytical evaluation of the latter at a very general level has been so far an open problem. Mitigating this shortcoming in the literature will be the main contribution of the present work.

In this paper, we first employ a series-expansion approach to approximate the complex exponential functions in CGF tensors (some preliminary ideas were briefly suggested [39]). Next, by deploying carefully selected mathematical properties enjoyed by the Beta and Gamma functions, we analytically evaluate the full angular space integration involving oscillatory terms. In this way, analytical expressions of CGF tensors are presented here for the most general three-dimensional case.
II. ANALYTICAL CGF DETERMINATION IN ONE/TWO/THREE DIMENSIONAL ARRAYS

A. Review of the CGF Tensor

In standard MIMO literature, the complex correlation coefficient \( \rho \) between the far-field patterns \( \mathbf{E}_1 = \mathbf{E}_1(\theta, \phi) \) and \( \mathbf{E}_2 = \mathbf{E}_2(\theta, \phi) \), respectively generated by complex current distributions \( \mathbf{J}_1 = \mathbf{J}_1(r') \) and \( \mathbf{J}_2(r'') \), is traditionally calculated by [10]

\[
\rho = \frac{\int_{4\pi} \mathbf{E}_1 \cdot \mathbf{E}_2^* \ d\Omega}{\sqrt{\int_{4\pi} \mathbf{E}_1 \cdot \mathbf{E}_1^* \ d\Omega \int_{4\pi} \mathbf{E}_2 \cdot \mathbf{E}_2^* \ d\Omega}},
\]

(1)

where \( d\Omega \) is the solid angle element given by \( d\Omega = \sin \theta d\theta d\phi \). That is, only the radiation fields appear in the original definition. This makes the process of evaluating \( \rho \) and designing optimum antennas for spatial diversity applications very challenging since the 3D computation of the far-field pattern is demanding. Moreover, the geometrical details of the radiator, e.g., shape, orientations, excitations, do not directly manifest themselves in the far field. For those reasons, in [16] \( \rho \) is expressed directly in terms of \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \) as:

\[
\rho = \frac{\int d^3r' \int d^3r'' J_1 \cdot \bar{C} \cdot J_2^*}{\sqrt{\left[ \int d^3r' \int d^3r'' J_1 \cdot \bar{C} \cdot J_1^* \right] \left[ \int d^3r' \int d^3r'' J_2 \cdot \bar{C} \cdot J_2^* \right]}},
\]

(2)

where all integrals are performed over the entire antenna radiating surface (the support of the current distribution functions \( \mathbf{J}_1(r) \) and \( \mathbf{J}_2(r) \)). Here, \( \bar{C} = \bar{C}(r', r'') \) stands for the CGF tensor in uniform propagation environment given by [16]:

\[
\bar{C}(r', r'') = \int_{0}^{\pi} \int_{0}^{2\pi} [\hat{I} - \hat{r}\hat{r}] e^{jk(r'-r'')} \sin \theta d\theta d\phi.
\]

(3)

The quantity \( \hat{I} \) is the unit dyad, while \( r' \) and \( r'' \) denote the spatial dependencies of \( \mathbf{J}_1 \) and \( \mathbf{J}_2 \), respectively. Moreover, \( k = kr \), with \( k = 2\pi/\lambda \) (\( \lambda \) = operating wavelength) and \( r \) being the radial unit-vector in spherical coordinate system given by

\[
\hat{r}(\theta, \phi) = \hat{r}(\Omega) := \hat{x} \cos \varphi \sin \theta + \hat{y} \sin \varphi \sin \theta + \hat{z} \cos \theta.
\]

(4)

The nine components \( C_{pq} \) (where \( p = x, y, z \) and \( q = x, y, z \) ) of the CGF tensor \( \bar{C}(r', r'') \) in (3) can be derived after using [4] and elementary dyadic arithmetic rule. The results are expressed as:

\[
C_{pq} = \int_{0}^{\pi} \int_{0}^{2\pi} f_{pq} \exp[jkr_d] \ d\theta d\phi,
\]

(5)

where \( r_d = \hat{r} \cdot (r' - r'') = x_d \sin \theta \cos \phi + y_d \sin \theta \sin \phi + z_d \cos \theta \), with \( x_d = x' - x'' \), \( y_d = y' - y'' \), and \( z_d = z' - z'' \) and values of \( f_{pq} \) are given by [16, 33]:

\[
\begin{align*}
    f_{xx} &= \sin \theta \left(1 - \sin^2 \theta \cos^2 \phi\right), \\
    f_{yy} &= \sin \theta \left(1 - \sin^2 \theta \sin^2 \phi\right), \\
    f_{zz} &= \sin^3 \theta, \\
    f_{xy} &= f_{yx} = -\sin^3 \theta \cos \phi \sin \phi, \\
    f_{yz} &= f_{zy} = -\sin^2 \theta \cos \theta \sin \phi, \\
    f_{xz} &= f_{zx} = -\sin^2 \theta \cos \theta \cos \phi.
\end{align*}
\]

(6)

Consequently, the CGF-based technique completely eliminates the requirement of going through the conventional pattern-based route of [1] by focusing instead on the total radiating antenna current distribution, i.e., only current points reflecting the radiator excitation and geometry are needed, and these are considerably smaller in number than the far-field points [16, 33, 34]. Clearly, all components of the CGF involve two-dimensional angular integration operations over the entire sphere. In general, for every position pair \( r', r'' \), these integrals must be computed again. Therefore, for large antenna arrays the net number of numerical computations becomes large.

B. General Idea of CGF Tensor Approximation

While evaluation of \( C_{pq} \) via (5) requires computing an angular-space integration of \( f_{pq} \exp(jkr_d) \), it was pointed out in [16] that CGF might be computed or at least well-approximated analytically. Various approaches may be pursued here. For example, it is possible to expand the integrand of every integral into orthogonal functions then evaluate angular integrations using exact orthogonality relations. However, the orthogonal expansion itself often requires numerical integrations to obtain the needed Fourier coefficients (the weight of every orthogonal function) and hence it may not lead to efficient algorithm. Moreover, because each of the nine integrals entering into the determination of the full \( 3 \times 3 \) dyad \( \bar{C} \) may involve a distinct angular function in the integrand, performing an orthogonal function expansion here becomes cumbersome and very tedious.

Another idea is to simplify (5) by making use of a Bessel’s function utilizing the Jacobi-Anger expansion [42]. However, an alternative approach is proposed in this paper, where we simply utilize the familiar Taylor series expansion of the exponential functions

\[
\exp[jkr_d] = \sum_{n=0}^{\infty} \frac{(jkr_d)^n}{n!},
\]

(8)

after which \( C_{pq} \) in (5) can be approximated by truncating into finite number of \( N \) terms giving

\[
C_{pq} \approx \sum_{n=0}^{N} (jk)^n I_{pq,n},
\]

(9)

where,

\[
I_{pq,n} = \frac{1}{n!} \int_{0}^{\pi} \int_{0}^{2\pi} f_{pq} (r_d)^n \ d\theta d\phi.
\]

(10)

The subsequent sections will build fast and robust algorithm for the evaluation of the quantity \( I_{pq,n} \) for various infinitesimal dipole (ID) array configurations (one/two/three dimensional topologies will be considered).

We also note at this juncture that the formal limit \( N \to \infty \) in (9) will give the exact \( C_{pq} \) value as obtained from (5). However, soon we will discover that opting for \( N \to \infty \) is not required in our correlation matrix computation algorithm. The proposed method then provides a tradeoff between exactness and computational efficiency. While the use of \( N \) makes our evaluation less exact than using complete orthogonal function series expansion, Nevertheless the final algorithm turns out to...
be very efficient for reasonably finite values of $N$, while it avoids the mathematical complexity of the angular spherical function approach.

To answer this question, we probe further into $C_{pq}$, by expressing the inter-element spacing $y_d$ in terms of the operating wavelength $\lambda$. Writing $y_d = p\lambda$ and using $k = 2\pi/\lambda$, one can put $C_{pq}$ via \((9)\), \((11)\) and \((12)\) in the following form

\[
C_{pq} \approx \sum_{n=0}^{N} j^n G_{pq,n} \left[ \left( \frac{2\pi p}{n!} \right)^n \right].
\]  

At this point, we use the well known Stirling’s formula for $n!$, which is very accurate for large values of $n \quad [40, 41]$:

\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.
\]

This further reduces \((13)\) to:

\[
C_{pq} \approx \sum_{n=0}^{N} j^n \left( \frac{G_{pq,n}}{\sqrt{2\pi n}} \right) \left( \frac{2\pi p}{n} \right)^n.
\]

Note that, the factor $G_{pq,n}/\sqrt{2\pi n}$ in \((15)\) decreases asymptotically with $n$, and is ignored for the time-being. By careful observation of the next term involving $n$-th power of $(2\pi p/n)$ and using $2\pi p \approx 17.08$, we provide a rule-of-thumb to determine the necessary $n = N$ for a given value of $p$ required to truncate the series in \((9)\) yet while not compromising correlation calculation accuracy:

\[
N > 17p.
\]

For the example $p = 5$, i.e. if the two IDs are placed $5\lambda$ apart, one should need approximately 85 terms to accurately determine $C_{pq}$ from \((9)\). A numerical verification of this rule-of-thumb formula \((16)\) can be found in Fig. \(1a\) and Fig. \(1b\), where variations of $|\rho|$ with respect to inter-element spacing $p = d/\lambda$ (where $y_d = d$) are shown respectively for a $y$-directed and $z$-directed ID pair, placed along the $y$-axis. One can observe from Fig. \(1a\) and Fig. \(1b\) that for $d/\lambda > N/17$, the $|\rho|$ value using \((9)\) quickly deviates from that determined via the exact formula \((5)\). Observing Fig. \(1a\) and Fig. \(1b\), it can be said that for inter-element spacing $> 6\lambda$, spatial correlation magnitude $|\rho|$ is sufficiently small ($|\rho| < 0.1$), and may be ignored for practical application purpose. This fact will also come in handy in formulating a general spatial correlation determination algorithm in the subsequent section.

The next challenge is to determine $G_{zz,n}$ analytically, therefore completely eliminating the need for numerical integration routines, as emphasized before. We demonstrate the derivation for $G_{zz,n}$ to start with. Using $f_{zz}$ from \((9)\) in \((12)\), we obtain:

\[
G_{zz,n} = \int_0^\pi \int_0^{2\pi} f_{zz} (\sin \theta \sin \phi)^n d\theta d\phi
= \left[ \int_0^\pi \sin^{n+3} \theta d\theta \right] \left[ \int_0^{2\pi} \sin^n \phi d\phi \right].
\]

Note that, $n$ can be either odd or even. For odd values of $n$, the integral with $\phi$-dependent term vanishes, i.e. we have:

\[
\int_0^{2\pi} \sin^n \phi d\phi = 0,
\]
Therefore, \( G_{zz,n} = 0 \) for odd values of \( n \). On the other hand, for even values of \( n \), we have:

\[
\int_0^{2\pi} \sin^n \phi d\phi = 2B \left( \frac{1}{2}, \frac{n+1}{2} \right) = \frac{2\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+2}{2} \right)}
\]

\[
= \frac{2\sqrt{\pi}}{\left( \frac{n}{2} \right)!} \frac{n!}{\left( \frac{n}{2} \right)!} \left( \frac{n}{2} \right)^n = \frac{n!}{2^{n-1} \left( \frac{n}{2} \right)!} \left( \frac{n}{2} \right)^n \frac{n!}{\pi},
\]

(19)

with \( B \) and \( \Gamma \) standing for Beta and Gamma functions respectively [42]. Also, when \( n \) is even, \( n + 3 \) is odd, yielding:

\[
\int_0^{\pi} \sin^{n+3} \theta d\theta = B \left( \frac{1}{2}, \frac{n+1}{2} \right) = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+2}{2} \right)}
\]

\[
= \sqrt{\pi} \left( \frac{n}{2} + 1 \right) \left[ \frac{2^{n+4} \left( \frac{n+2}{2} \right)!}{\left( n+4 \right) \left( n+3 \right) \left( n+2 \right) \left( n+1 \right)!} \right]
\]

\[
= \left( \frac{n}{2} + 1 \right)^{n+2} \left( n+1 \right) \left( n+3 \right) \left( n+2 \right) \left( n+1 \right)!
\]

(20)

Therefore, using (17), (19) and (20), we obtain for even values of \( n \):

\[
G_{zz,n} = \left( \frac{n}{2} + 1 \right) W_n,
\]

(21)

where,

\[
W_n = \frac{8\pi}{\left( n+1 \right) \left( n+3 \right)}.
\]

(22)

\( W_n \) is a “general weighing factor” for this one-dimensional ID array scenario, which would soon prove to be readily applicable for the more general three dimensional case. The several relevant formulas used to simplify the integrations are collected in the Appendix.

The rest of the derivations for \( G_{pq,n} \) follows a similar route, and consequently will not be elaborately shown here. With the help of a symbolic computer package (e.g., the symbolic toolbox of MATLAB or Mathematica), further verification of the detailed derived expressions for \( G_{pq,n} \) were conducted by the authors. The following general observations can be drawn from the results:

1) The following condition holds always true:

\[
G_{pq,n} = 0 \quad \text{for odd values of } n \quad \text{for all } p, q.
\]

(23)

This is very significant, since it literally halves the number of integrations to be solved.

2) The coefficients for mutually orthogonal ID pairs all vanish, i.e. for all values of \( n \):

\[
G_{xy,n} = G_{yx,n} = G_{yz,n} = G_{zy,n} = G_{zz,n} = G_{zz,n} = 0.
\]

(24)

3) The coefficients for the ID pairs orthogonal to the placement axes (i.e. for \( x \)-directed or \( z \)-directed ID pairs) are identical, and can be expressed as:

\[
G_{xz,n} = G_{zz,n} = \left( \frac{n}{2} + 1 \right) G_{yy,n}.
\]

(25)

where \( G_{yy,n} = W_n \).

These analytical results and the various details about the behaviour of various terms indexed by \( n \) will be fully exploited in what follows to build efficient and robust cross-correlation computation algorithms for massive MIMO.

D. Analytical CGF Estimation for Two-dimensional Planar Dipole Arrays

In the last section, we considered that placement of IDs is restricted along \( y \)-axis, i.e. \( x_d = z_d = 0 \), which significantly simplified the scenario. Next, let us take the case of a two-dimensional/planar array of IDs placed in the \( yz \)-plane. Since here \( x_d = 0 \), we have \( r_d = y_d \sin \theta \sin \phi + z_d \cos \theta \).

By deploying the binomial series to expand \( r_d \) and following some algebraic manipulations, we get:

\[
(r_d)^n = (y_d \sin \theta \sin \phi + z_d \cos \theta)^n
\]

\[
= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} y_d^m z_d^{n-m} \sin^m \theta \cos^{n-m} \theta \sin^m \phi.
\]

(26)

Therefore, following (10), the expression for \( I_{pq,n} \) becomes:

\[
I_{pq,n} = \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} V_{pq,m} y_d^m z_d^{n-m},
\]

(27)

where,

\[
V_{pq,m} = \int_0^{\pi} \int_0^{2\pi} f_{pq} \sin^m \theta \cos^{n-m} \theta \sin^m \phi d\phi d\theta.
\]

(28)

To solve for \( V_{pq,m} \), which will finally lead to \( I_{pq,n} \), we need to carefully choose \( f_{pq} \) expressions from [6] and apply suitable properties of Beta and Gamma functions. We demonstrate the solution for \( I_{zz,n} \) here.

\[
V_{zz,m} = \int_0^{\pi} \sin^{m+3} \theta \cos^{n-m} \theta d\theta \int_0^{2\pi} \sin^m \phi d\phi.
\]

(29)

Note that, the integral with \( \phi \) vanishes for odd values of \( m \). Similar to the 1D case, we have for even \( m \), we have:

\[
\int_0^{2\pi} \sin^m \phi d\phi = 2B \left( \frac{1}{2}, \frac{m+1}{2} \right) = \frac{\pi}{2^{m-1} \left( \frac{m}{2} \right)!} m!.
\]

(30)

Now, we consider the two scenarios of \( n \). When \( n \) is odd with \( m \) being even, both the quantities \( m+3 \) and \( n-m \) are odd. Therefore using the fact that cosine function \( \cos \theta \) is odd with respect to \( \pi/2 \) we have:

\[
\int_0^{\pi} \sin^{m+3} \theta \cos^{n-m} \theta d\theta = 0.
\]

(31)

Once again, we have \( I_{zz,n} = 0 \) for odd values of \( n \). On the other hand, when \( n \) is even with \( m \) also being even, \( m+3 \) is odd while \( n-m \) is even. Therefore,

\[
\int_0^{\pi} \sin^{m+3} \theta \cos^{n-m} \theta d\theta = B \left( \frac{n-m}{2}, \frac{1}{2}, \frac{m}{2} + 2 \right)
\]

\[
= \frac{\Gamma \left( \frac{n-m}{2} + \frac{1}{2} \right) \Gamma \left( \frac{m}{2} + 2 \right)}{\Gamma \left( \frac{n+1}{2} + \frac{1}{2} \right)}
\]

\[
= \left( \frac{m}{2} + 1 \right) \frac{(n-m)!}{\left( \frac{m}{2} - \frac{n}{2} \right)!} \frac{\left( \frac{n}{2} + 1 \right)!}{\left( \frac{n}{2} + 2 \right)!} \frac{\left( \frac{n}{2} + 2 \right)!}{\left( \frac{n}{2} + 3 \right)!}.
\]

(32)
Using (32) and (30) in (28), we obtain:

\[
V_{zz,m}^n = 2^n \pi \left[ \frac{m!(n-m)!}{n!} \right] \left[ \frac{\left( \frac{n}{2} \right)!}{\left( \frac{n}{2} - m \right)! \left( \frac{n}{2} + m \right)!} \right] 
\times \left[ \frac{\left( \frac{n}{2} + 1 \right) (\frac{n}{2} + 2) (\frac{n}{2} + 1)}{(n+4)(n+3)(n+2)(n+1)} \right] 
= \frac{8\pi (\frac{n}{2} + 1)}{(n+3)(n+1)} \left[ \frac{m!(n-m)!}{n!} \right] \left[ \frac{\left( \frac{n}{2} \right)!}{\left( \frac{n}{2} - m \right)! \left( \frac{n}{2} + m \right)!} \right].
\]

(33)

When \(V_{zz,m}^n\) is substituted in (27) and the expression for \(W_n\) is recognized from (22), the expression for \(I_{zz,n}\) becomes:

\[
I_{zz,n} = \frac{W_n}{n!} \sum_{m=0}^{n} \left[ \frac{\left( \frac{n}{2} + 1 \right) \left( \frac{n}{2} \right)!}{\left( \frac{n}{2} - m \right)! \left( \frac{n}{2} + m \right)!} \right] y_d^m z_d^{n-m},
\]

(34)

At this point, we notice that \(I_{zz,n}\) is actually a sum of two series-summations as follows:

\[
\sum_{m=0}^{n} \left[ \frac{\left( \frac{n}{2} + 1 \right) \left( \frac{n}{2} \right)!}{\left( \frac{n}{2} - m \right)! \left( \frac{n}{2} + m \right)!} \right] y_d^m z_d^{n-m} = (y_d^2 + z_d^2)^n,
\]

(35)

\[
\sum_{m=0}^{n} \left[ \frac{\left( \frac{n}{2} + 1 \right) \left( \frac{n}{2} \right)!}{\left( \frac{n}{2} - m \right)! \left( \frac{n}{2} + m \right)!} \right] y_d^m z_d^{n-m} = \frac{n^2 d^2 (y_d^2 + z_d^2)^{n-1}}{2 d^2}. \]

(36)

After substituting these series summation values in (34), the final expression for \(I_{zz,n}\) reduces to:

\[
I_{zz,n} = \frac{W_n}{n!} \left( y_d^2 + z_d^2 \right)^{\frac{n-1}{2}} \left[ \frac{n}{2} + 1 \right] y_d^3 + z_d^3.
\]

(37)

In a similar fashion, the expressions for other \(I_{pq,n}\) for even values of \(n\) can be derived as follows:

\[
I_{xx,n} = \frac{W_n}{n!} \left( y_d^2 + z_d^2 \right)^{n-1},
\]

(38)

\[
I_{yy,n} = \frac{W_n}{n!} \left( y_d^2 + z_d^2 \right)^{\frac{n-1}{2}} \left[ \frac{n}{2} + 1 \right] y_d^3 + z_d^3.
\]

(39)

\[
I_{yz,n} = \frac{W_n}{n!} \left( y_d^2 + z_d^2 \right)^{\frac{n-1}{2}} \left[ \frac{n}{2} + 1 \right] y_d^3 + z_d^3.
\]

(40)

\[
I_{xy,n} = I_{yx,n} = I_{zx,n} = I_{xz,n} = 0.
\]

(41)

Note that, the condition \(I_{pq,n} = 0\) for odd values of \(n\) holds true. Furthermore, (41) suggests that the DSs oriented orthogonal to the plane of arrangement (i.e. \(x\)-directed DSs) do not have any correlation with the DSs oriented along the plane of arrangement (i.e. \(y\)-directed or \(z\)-directed DSs).

**E. Analytical CGF Estimation for Three-Dimensional Dipole Arrays: Generalized Case**

Finally, we consider the most general case of three-dimensional arrays with no restrictions imposed on the dipole locations, i.e. in general, \(x_d \neq y_d \neq z_d \neq 0\). Here, it turns out we have to deal with a trinomial expansion or successive binomial expansions of \((r_d)^n\) where

\[
r_d = x_d \sin \theta \cos \phi + y_d \sin \theta \sin \phi + z_d \cos \theta.
\]

(42)

Therefore, the analytical integrations needed to evaluate \(I_{pq,n}\) (see (10)) become slightly more complicated. Performing integration both by-hand using the properties of Beta and Gamma functions as before, we determine the following general formula for \(I_{pq,n}\) for even values of \(n\):

\[
I_{xx,n} = \frac{W_n}{n!} \left[ x_d^2 + \left( \frac{n}{2} + 1 \right) y_d^2 + \left( \frac{n}{2} + 1 \right) z_d^2 \right] d^{n-2},
\]

(43)

\[
I_{yy,n} = \frac{W_n}{n!} \left[ \left( \frac{n}{2} + 1 \right) x_d^2 + y_d^2 + \left( \frac{n}{2} + 1 \right) z_d^2 \right] d^{n-2},
\]

(44)

\[
I_{zz,n} = \frac{W_n}{n!} \left[ \left( \frac{n}{2} + 1 \right) x_d^2 + \left( \frac{n}{2} + 1 \right) y_d^2 + z_d^2 \right] d^{n-2},
\]

(45)

\[
I_{xy,n} = I_{yx,n} = \frac{-W_n}{n!} \left[ \frac{n x_d y_d d^2}{2} \right] d^{n-2},
\]

(46)

\[
I_{yz,n} = I_{zy,n} = \frac{-W_n}{n!} \left[ \frac{n y_d z_d d^2}{2} \right] d^{n-2},
\]

(47)

\[
I_{xz,n} = I_{zx,n} = \frac{-W_n}{n!} \left[ \frac{n x_d z_d d^2}{2} \right] d^{n-2},
\]

(48)

where,

\[
d = \sqrt{x_d^2 + y_d^2 + z_d^2}.
\]

(49)

However the results are further validated by use of the symbolic toolbox in MATLAB (see appendix). It is observed that for odd values of \(n\), \(I_{pq,n} = 0\). Also note that, the expressions for one-dimensional and two-dimensional ID arrays can be easily computed back from (43)-(48), substituting \(x_d, y_d\) and \(z_d\) accordingly. Consequently, the results of this subsection (43)-(45) are the most general, but we opted for presenting the one- and two-dimensional cases for convenience since the mathematical treatment is considerably more complex in three dimensional arrays while lower-dimensional MIMO systems tend to be more commonly encountered in practice.

Now, similar to our approach for the linear array (or 1D case), it is crucial to predict the maximum number of terms \(N\) needed to truncate the series in (9). With that objective in mind, we start by carefully examining \(I_{xx,n}\), where we note that for \(n > 0\):

\[
I_{xx,n} = \frac{W_n}{n!} \left[ x_d^2 + \left( \frac{n}{2} + 1 \right) y_d^2 + \left( \frac{n}{2} + 1 \right) z_d^2 \right] d^{n-2} < \frac{W_n}{n!} \left( \frac{n}{2} + 1 \right) \left[ x_d^2 + y_d^2 + z_d^2 \right] d^{n-2}.
\]

(50)

Using \(d\) from (49), following the same procedure for all all \(p = x, y, z\), and applying the Stirling’s approximation (14), the upper-bound of \(I_{pp,n}\) can be expressed as:

\[
I_{pp,n}|_{u.b.} = W_n \left( \frac{n}{2} + 1 \right) \frac{d^n}{n!} = W_n \left( \frac{n+2}{2\sqrt{2\pi n}} \right) \left( \frac{e p \lambda}{n} \right)^n,
\]

(51)

where \(d = p\lambda\). Therefore, when this \(I_{pp,n}\) is used in (9) containing the \((jk)^n\) term where \(k = 2\pi / \lambda\), we would obtain a term \((2\pi ep/n)^n\), very similar to (15). Therefore, it can be deduced that the maximum \(N\)-value for the three-dimensional array situation also follows the same guideline given by (16).
involving trigonometric functions is established using

$$\int_0^\pi \sin^{2m+1} \theta \cos^{2n+1} \theta d\theta = 0.$$  

(59)

Class-IV: Power of $\sin$ even, Power of $\cos$ even:

$$\int_0^\pi \sin^{2m} \theta \cos^{2n} \theta d\theta = \frac{1}{2} \int_0^{2\pi} \sin^{2m} \theta \cos^{2n} \theta d\theta = B \left( m+1 \right).$$  

(60)

REFERENCES

[1] J. R. Hampton, *Introduction to MIMO Communications*, Cambridge, U.K.: Cambridge Univ. Press, 2014.
[2] Y. Yang, J. Xu, G. Shi, and Cheng-Xiang Wang, *5G Wireless Systems: Simulation and Evaluation Techniques*, Springer 2017.
[3] C. X. Wang, J. Bian, J. Sun, W. Zhang and M. Zhang, “A Survey of 5G Channel Measurements and Models,” *IEEE Communications Surveys and Tutorials*, vol. 20, no. 4, pp. 3142-3168, Fourth Quarter, 2018.
[4] T. L. Marzetta, E. G. Larsson, H. Yang, and H. Q. Ngo. *Fundamentals of Massive MIMO*, Cambridge, U.K.: Cambridge Univ. Press, 2016.
[5] J. du Preez and S. Sinha, *Millimeter-Wave Antennas: Configurations and Applications*, Springer, 2018.
[6] A. Chockalingam and B. S. Rajan, *Large MIMO Systems*, Cambridge, U.K.: Cambridge Univ. Press, 2014.
[7] A. Paulraj and C. Papadias, “Space-time processing for wireless communications,” *IEEE Signal Processing Magazine*, vol. 14, no. 6, pp. 49-83, 1997.
[8] J. Dmochowski, J. Benesty and S. Affes, “Direction of Arrival Estimation Using the Parameterized Spatial Correlation Matrix,” *IEEE Transactions on Audio, Speech and Language Processing*, vol. 15, no. 4, 2007.
[9] K. Zheng, S. Ou and X. Yin, “Massive MIMO Channel Models: A Survey,” *International Journal of Antennas and Propagation* (Hindawi), Article ID 848071, pp. 1-10, 2014.
[10] M. P. Karaboikis, V. C. Papamichael, G. F. Tsachtsiris, C. F. Soras, and V. T. Makios, “Integrating Compact Printed Antennas Onto Small Diversity/MIMO Terminals,” *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 7, pp. 2067-2078, 2008.
[11] A. E. Ampoma, H. Zhang, Y. Huang, G. Wen, and O. G. Kwame, “Three dimensional spatial fading correlation of uniform rectangular array using total power of angular distribution,” *IEEE Antennas and Wireless Propagation Letters*, vol. 16, pp. 21342132, 2017.
[12] J. W. Wallace and M. A. Jensen, “Electromagnetic Considerations for Communicating on Correlated MIMO Channels with Covariance Information,” *IEEE Transactions on Wireless Communications*, vol. 7, no. 2, pp. 543-551, February 2008.
[13] R. Vaughan and J. B. Andersen, *Channels, Propagation, and Antennas for Mobile Communications*, IET, London, 2003.
[14] C. T. Neil, M. Shafi, P. J. Smith, and P. A. Dmochowski, “On the impact of antenna topologies for massive MIMO systems,” *Proc. IEEE Int. Conf. Commun. (ICC)*, London, U.K., Jun. 2015, pp. 20502055.
[15] R. Janaswamy, “Effect of Element Mutual Coupling on the Capacity of Fixed Length Linear Arrays,” *IEEE Antennas and Wireless Propagation Letters*, vol. 1, pp. 157-159, 2002.
[16] S. M. Mikki and Y. M. M. Antar, “On Cross Correlation in Antenna Arrays With Applications to Spatial Diversity and MIMO Systems,” *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 4, pp. 1708-1810, 2015.
[17] S. Mikki and Y. M. M. Antar, *New Foundations for Applied Electromagnetics: Spatial Structures of Electromagnetic Field*, Norwood, MA, USA: Artech House, 2016.
[18] S. M. Mikki, S. Clauzier and Y. M. M. Antar, “Empirical Geometrical Bounds on MIMO Antenna Arrays for Optimum Diversity Gain Performance: An Electromagnetic Design Approach,” *IEEE Access*, vol. 6, pp. 39876-39894, 2018.
[20] S. M. Mikki and Y. M. M. Antar, “Near-Field Analysis of Electromagnetic Interactions in Antenna Arrays Through Equivalent Dipole Models,” *IEEE Transactions on Antennas and Propagation*, vol. 60, no. 3, pp. 1381-1389, 2012.

[21] S. Karimkashi, A. A. Kishk, and D. Kajfez, “Antenna array optimization using dipole models for MIMO applications,” *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 8, pp. 31123116, Aug. 2015.

[22] J. F. Izquierdo, J. Rubio, and J. Zapata, “Antenna-Generalized Scattering Matrix in Terms of Equivalent Infinitesimal Dipoles: Application to Finite Array Problems,” *IEEE Transactions on Antennas and Propagation*, vol. 60, no. 10, pp. 4601-4609, 2012.

[23] S. Karimkashi, A. Kishk, and G. Zhang, “Modelling of aperiodic array antennas using infinitesimal dipoles,” *IET Microwaves, Antennas & Propagation*, vol. 6, no. 7, p. 761, 2012.

[24] J. F. Izquierdo, J. Rubio, J. Crocles, R. Gmez-Alcal, “Efficient Radiation Antenna Modeling via Orthogonal Matching Pursuit in Terms of Infinitesimal Dipoles,” *IEEE Antennas and Wireless Propagation Letters*, vol. 15, pp. 444-447, June 2015.

[25] S. Clauzier, S. Mikki, A. Shamim and Y. M. M. Antar, “A New Method for the Design of Slot Antenna Arrays: Theory and Experiment,” *Proceedings of 10th European Conference on Antennas and Propagation, Davos, Switzerland*, pp. 1-4, April 2016.

[26] S. J. Yang, D. J. Yun, H. J. Kim, J. I. Lee, W. Y. Yang, and N.H. Myung, “Antenna far-field prediction using restricted IDM based convex optimization,” *Proceedings of IEEE Asia Pacific Microwave Conference (APMC)*, pp. 1-4, 2017.

[27] I. A. Baratta, and C. B. de Andrade, “Installed Performance Assessment of Blade Antenna by means of the Infinitesimal Dipole Model,” *IEEE Latin America Transactions*, vol. 14, no. 2, pp. 569-574, 2016.

[28] S. J. Yang, Y. D. Kim, D. J. Yun, D. W. Yi, and N. H. Myung, “Antenna Modeling Using Sparse Infinitesimal Dipoles Based on Recursive Convex Optimization,” *IEEE Antennas and Wireless Propagation Letters*, vol. 17, no. 4, pp. 662-665, 2018.

[29] S. Clauzier, S. M. Mikki and Y. M. M. Antar, “Design of high diversity gain MIMO antenna arrays through surface current optimization,” *Proceedings of IEEE International Symposium on Antennas and Propagation and USNC/URSI National Radio Science Meeting*, pp. 9-10, 2015.

[30] S. Clauzier, S. M. Mikki and Y. M. M. Antar, “Generalized methodology for antenna design through optimal infinitesimal dipole model,” *Proceedings of International Conference on Electromagnetics in Advanced Applications (ICEAA)*, pp. 1264-1267, 2015.

[31] S. Clauzier, S. M. Mikki, and Y. M. M. Antar, “A Generalized Methodology for Obtaining Antenna Array Surface Current Distributions With Optimum Cross-Correlation Performance for MIMO and Spatial Diversity Applications,” *IEEE Antennas and Wireless Propagation Letters*, vol. 14, pp. 1451-1454, 2015.

[32] D. Sarkar, S. M. Mikki and Y. M. M. Antar, “Eigenspace Structure Estimation for Dual-Polarized Massive MIMO Systems Using an IDM-CGF Technique,” to appear in *IEEE Antennas and Wireless Propagation Letters*, 2019.

[33] D. Sarkar and K. V. Srivastava, “Application of Cross-correlation Greens Function along with FDTD for Fast Computation of Envelope Correlation Coefficient over Wideband for MIMO Antennas,” *IEEE Transactions on Antennas and Propagation*, vol. 65, no. 2, pp. 730-740, 2017.

[34] D. Sarkar and K. V. Srivastava, “Modified Cross-correlation Green’s Function with FDTD for Characterization of MIMO Antennas in Non-uniform Propagation Environment,” *IEEE Transactions on Antennas and Propagation*, vol. 66, no. 7, pp. 3798-3803, 2018.

[35] D. Sarkar, S. M. Mikki, K. V. Srivastava and Y. M. M. Antar, “Cross-Correlation Greens Function for Interaction Between Electric and Magnetic Current Sources,” *Proceedings of 2018 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting (APS-URSI)*, Boston, Massachusetts, USA, 2018.

[36] S. M. Mikki and J. Aulin, “The stochastic electromagnetic theory of antenna-antenna cross-correlation in MIMO systems,” *Proceedings of European Conference on Antennas and Propagation (EuCAP 2018)*, London, UK, April 9-13, 2018.

[37] S. M. Mikki, S. Clauzier and Y. M. M. Antar, “A Correlation Theory of Antenna Directivity With Applications to Superdirective Arrays,” *IEEE Antennas and Wireless Propagation Letters*, 2019. (Early Access)

[38] D. Sarkar, S. M. Mikki, K. V. Srivastava and Y. M. M. Antar, “Analytical Approximation of the Time-dependent Antenna Cross-correlation Greens Function,” *Proceedings of European Conference on Antennas and Propagation (EuCAP 2018)*, London, UK, pp. 1-4, 2018.

[39] D. Sarkar, S. M. Mikki and Y. M. M. Antar, “Estimation of the Cross-Correlation Greens Function for MIMO Systems,” submitted to 2019 *IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting (APS-URSI)*, Atlanta, Atlanta, USA, 2019.

[40] J. Dutka, “The early history of the factorial function,” *Archive for History of Exact Sciences (Springer)*, vol. 43, no. 3, pp. 225249, 1991.

[41] H. Robbins, “A Remark on Stirling’s Formula,” *The American Mathematical Monthly*, vol. 62, no. 1, pp. 2629, 1955.

[42] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, First Edition, Washington D.C., New York, 1983.