Recurrence relations for off-shell Bethe vectors in trigonometric integrable models

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Abstract

The zero modes method is applied in order to get the action of the monodromy matrix entries on off-shell Bethe vectors in quantum integrable models associated with \(U_q(\mathfrak{gl}_N)\)-invariant \(R\)-matrices. The action formulas allow to get recurrence relations for off-shell Bethe vectors and for highest coefficients of the Bethe vectors scalar product.

Keywords: integrable models, algebraic Bethe ansatz, Bethe vectors

1. Introduction

The algebraic Bethe ansatz \([1]\) is a method to describe the space of states of quantum integrable models. It is applicable, in particular, to the models defined by the monodromy matrices satisfying quadratic commutation relations and possessing special vacuum vector \(|\text{vac}\rangle\) in the space of states which is annihilated by the monodromy matrix entries below the diagonal. A wide class of the quantum integrable models is related to the quantum groups \(U_q(\mathfrak{g})\) \([2]\) if the structural constants of the quadratic commutation relations for the monodromy operators are \(U_q(\mathfrak{g})\)-invariant \(R\)-matrices. In this paper we consider a class of quantum trigonometric integrable models defined by the \(U_q(\mathfrak{gl}_N)\)-invariant \(R\)-matrices. Entries of these matrices can be written as trigonometric functions of the parameters and this is why we call such models trigonometric.

Quantum affine algebras \(U_q(\hat{\mathfrak{g}})\) have several descriptions which use different sets of generators. One of the description uses \(T\)-operators \([3]\) whose commutation relations are defined by the same \(U_q(\mathfrak{g})\)-invariant \(R\)-matrices which define commutation relations of the mon-
odromy matrix entries. This coincidence opens a possibility to describe the space of states of the quantum integrable models in terms of the generators of the quantum affine algebras if one identifies the monodromy matrix of the quantum integrable model with the \( T \)-operator of \( U_q(\widehat{\mathfrak{g}}) \).

In [4] a method called zero modes method was introduced for the class of supersymmetric rational \( \mathfrak{gl}(m|n) \)-invariant quantum integrable models related to the super-Yangian double \( \mathcal{D}Y(\mathfrak{gl}(m|n)) \). This method uses the commutation relations between zero modes of the monodromy matrix entries and entries themselves in order to obtain the action of these entries on off-shell Bethe vectors in the corresponding quantum integrable model. In the present paper we develop the zero modes method to the trigonometric quantum integrable models related to the quantum affine algebra \( U_q(\mathfrak{gl}_N) \).

The action formulas of monodromy matrix entries on off-shell Bethe vectors are more fundamental than explicit expressions for these vectors in terms of monodromy matrix elements. They can be used to investigate the physical quantities in the quantum integrable models such as scalar products and form-factors of the local operators without using explicit formulas for the Bethe vectors [5]. On the other hand, the action formulas lead to the recurrent relations for the off-shell Bethe vectors, which can be solved to obtain explicit expressions for them [6–9]. The action formulas in case of \( U_q(\mathfrak{gl}_3) \)-invariant \( R \)-matrix were calculated in [10] in the framework of the projection method [11].

For the off-shell Bethe vectors in the quantum integrable models defined by \( U_q(\mathfrak{gl}_N) \)-invariant \( R \)-matrices two particular recurrence relations were obtained in [7] using method of the hierarchical Bethe ansatz. From viewpoint of this method these two types of recurrence relations are related to two different ways of embedding the \( U_q(\mathfrak{gl}_{N-1}) \) monodromy matrix the one associated to \( U_q(\mathfrak{gl}_N) \) monodromy: either in the top left corner or in the bottom right corner.

To describe off-shell Bethe vectors in terms of generators of the quantum affine algebra one has to explore the Gauss decomposition of the \( T \)-operators. Methods to express off-shell Bethe vectors in terms of the Gauss coordinates of monodromy matrix were developed in [11, 12]. There are different Gauss decompositions naturally related to the embeddings mentioned above. It was shown in [8] that each embedding leads to one type of recurrence relation and it is not an easy combinatorial problem to prove that different type recurrence relations lead to different but equivalent presentations of the same off-shell Bethe vectors. For the rational quantum integrable models defined by the \( \mathfrak{gl}(m|n) \)-invariant \( R \)-matrices this problem was solved in [13].

The zero modes method developed in the present paper allows to find the action formulas of monodromy matrix entries on off-shell Bethe vectors in trigonometric integrable models. This action can be used to get various recurrence relations including those obtained in [7]. Similar results were obtained in [14] for the case of the rational \( \mathfrak{so}_{2n+1} \)-invariant integrable models.

The paper is composed as follows. In section 2 the quantum loop algebra \( U_q(\mathfrak{gl}_N) \) in terms of fundamental \( T \)-operators is defined. Section 3 contains two main results of this paper, which include single and multiple actions of monodromy matrix entries on off-shell Bethe vectors and recurrence relations for them. Main achievement here is proposition 3.2, which is proved by zero modes method in appendix A. Section 4 is devoted to formulation of the zero modes method, which is used to obtain action formulas. The third result of the paper—the recurrence relations for the highest coefficients (HC) of the Bethe vectors scalar product—is presented in section 5. Proofs of the main propositions are gathered in two appendices.
2. Definitions and notations

In this paper we will explore the Cartan–Weyl generators of the quantum loop algebra $U_q(\tilde{\mathfrak{gl}}_N)$. This algebra is related to the quantum affine algebra by setting the central element in $U_q(\mathfrak{gl}_N)$ equal to zero.

2.1. R-matrix for $U_q(\tilde{\mathfrak{gl}}_N)$

Let $N$ be dimension of the fundamental vector representation of the algebra $U_q(\mathfrak{gl}_N)$ in $\mathbb{C}^N$. Let $e_{ij}$ be a $N \times N$ matrix unit, $e_{ij} = \delta_{ik}\delta_{jl}$ for $1 \leq i, j, k, l \leq N$. We introduce rational functions

$$f(u, v) = \frac{qu - q^{-1}v}{u - v}, \quad g(u, v) = \frac{(q - q^{-1})u}{u - v}, \quad \tilde{g}(u, v) = \frac{(q - q^{-1})v}{u - v}$$

of the formal spectral parameters $u$ and $v$. Here, $q$ is a complex number not equal to zero or a root of unity.

Define a matrix $P(u, v)$ acting in the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$ by the equality

$$P(u, v) = \sum_{1 \leq i, j \leq N} p_{ij}(u, v) e_{ij} \otimes e_{ji},$$

where the rational functions $5$ $p_{ij}(u, v)$ are defined as follows

$$p_{ij}(u, v) = \begin{cases} f(u, v) - 1, & i = j, \\ g(u, v), & i < j, \\ \tilde{g}(u, v), & i > j. \end{cases}$$

**Definition 2.1.** The quantum trigonometric $U_q(\mathfrak{gl}_N)$-invariant R-matrix acting in the tensor product of two fundamental vector representations of $U_q(\mathfrak{gl}_N)$ is

$$R(u, v) = I \otimes I + P(u, v),$$

where $I = \sum_{i=1}^N e_{ii}$ is identity matrix in $\mathbb{C}^N$.

2.2. R-matrix formulation of the algebra $U_q(\tilde{\mathfrak{gl}}_N)$

The associative algebra $U_q(\tilde{\mathfrak{gl}}_N)$ with unit $1$ over $\mathbb{C}$ is generated by the elements $T^+_{ij}[0]$, $T^-_{ij}[\pm m]$, $1 \leq i, j \leq N$, $m \in \mathbb{N}$ such that

$$T^+_{ij}[0] = T^-_{ij}[0] = 0, \quad i < j, \quad T^0_{ij}[0]T^+_{ij}[0] = T^+_{ij}[0]T^-_{ij}[0] = 1.$$ (2.4)

The generators of the algebra $U_q(\tilde{\mathfrak{gl}}_N)$ may be gathered into formal series

$$T^\pm_{ij}(u) = \sum_{m=0}^{\infty} T^\pm_{ij}[\pm m]u^m.$$ (2.5)

5 These rational functions should be always understood as formal series either over $u/v$ or $v/u$. See (2.7) and discussion after it.
and combined in the matrices
\[
T^\pm(u) = \sum_{i,j=1}^{N} e_{ij} \otimes T^\pm_{ij}(u) \in \text{End}(\mathbb{C}^N) \otimes U_q(\widetilde{gl}_\nu)[[u^{\pm 1}]]
\]
which we call \(T\)-operators.

The commutation relations in the algebra \(U_q(\widetilde{gl}_\nu)\) are given by the standard RLL commutation relations
\[
R(u, v) \cdot (T^\mu(u) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes T^\nu(v)) = (\mathbb{1} \otimes T^\nu(v)) \cdot (T^\mu(u) \otimes \mathbb{1}) \cdot R(u, v),
\]
where \(\mu, \nu = \pm\) and rational functions entering \(R\)-matrix \(R(u, v)\) (2.3) should be understood as series over \(v/u\) for \(\mu = +, \nu = -\) and as series over \(u/v\) for \(\mu = -, \nu = +\). For \(\mu = \nu\) these rational series can be either series over the ratio \(v/u\) or the ratio \(u/v\).

The commutation relations in the algebra \(U_q(\tilde{g})\) may be written in terms of matrix entries (2.2) one gets
\[
[T^\mu_{ij}(u), T^\nu_{kj}(v)] = p_{ij}(u,v) T^\nu_{kj}(v) T^\mu_{ij}(u) - p_{ik}(u,v) T^\mu_{kj}(u) T^\nu_{ij}(v).
\]

It follows from the commutation relations (2.7) or (2.8) that modes \(T^\pm_{ij}[m], m \geq 0\) form Borel subalgebras \(U^\pm_q(\tilde{gl}_\nu) \subset U_q(\tilde{gl}_\nu)\).

In section 5 we will consider the algebra \(U_{q^{-1}}(\tilde{gl}_\nu)\) which is defined in the same way as the algebra \(U_q(\tilde{gl}_\nu)\) only with \(q \to q^{-1}\). We will denote by \(\tilde{T}^\pm_{ij}(u)\) the monodromy matrix entries for the algebra \(U_{q^{-1}}(\tilde{gl}_\nu)\).

**Remark 2.1.** One can check that the restrictions to the zero mode generators (2.4) are consistent with the commutation relations (2.8).

### 2.3. Gauss coordinates and the currents

Let \(\mathcal{H}\) be a representation space of the algebra \(U_q(\tilde{gl}_\nu)\) which possesses a vector \(|\text{vac}\rangle\) with following properties
\[
T^\pm_{ij}(u)|\text{vac}\rangle = 0, \quad i > j, \quad T^\pm_{ij}(u)|\text{vac}\rangle = \lambda^\pm(u)|\text{vac}\rangle.
\]

In what follows we will use Gauss decomposition of the \(T\)-operators of the algebra \(U_q(\tilde{gl}_\nu)\) [16]
\[
T^\pm_{ij}(u) = \sum_{\ell \leq \min(i,j)} F^\pm_{ij}(u) k^\pm_\ell(u) E^\pm_\ell(u).
\]

We call series \(F^\pm_{ij}(u), E^\pm_\ell(u)\) for \(1 \leq i < j \leq N\) and \(k^\pm_\ell(u)\) for \(1 \leq \ell \leq N\) the Gauss coordinates. Properties of the vacuum vector (2.9) are translated to
\[
E^\pm_{ij}(u)|\text{vac}\rangle = 0, \quad i < j, \quad k^\pm_\ell(u)|\text{vac}\rangle = \lambda^\pm(u)|\text{vac}\rangle.
\]
In (2.10) we assume that \( F_{ij}^+(u) = E_{ij}^+(u) = 1 \) for \( 1 \leq i \leq N \). Series expansion of the \( T \)-operators (2.5) imply following expansions of the Gauss coordinates

\[
F_{ij}^+(u) = \sum_{m=0}^{\infty} F_{ij}^+[\pm m]u^{-m}, \quad E_{ij}^+(u) = \sum_{m=0}^{\infty} E_{ij}^+[\pm m]u^{-m}, \quad k_i^+(u) = \sum_{m=0}^{\infty} k_i^+[\pm m]u^{-m}.
\]

(2.11)

According to (2.4) and Gauss decomposition (2.10) \( F_{ij}^+[0] = 0 \) and \( E_{ij}^+[0] = 0 \) and we set \( F_{ij}^+[+0] \equiv F_{ij}^+[0] \) and \( E_{ij}^+[0] \equiv E_{ij}^+[0] \). We set also \( k_i^+[1] \equiv k_i^+[0] \) and according to (2.4) \( k_i^+[-0] \equiv k_i^+[0]^{-1} \).

The commutation relations between Gauss coordinates imply the commutation relations in the algebra \( U_q(\mathfrak{gl}_N) \) in terms of the currents (for \( i = 1, \ldots, N-1 \)) [16, 17]

\[
F_i(u) = F_{i+1,j}^+(u) - F_{i+1,j}^-(u) = \sum_{\ell \in \mathbb{Z}} \text{sign}(\ell)F_{i+1,j}^\ell u^{-\ell},
\]

\[
E_i(u) = E_{i+1,j}^+(u) - E_{i+1,j}^-(u) = \sum_{\ell \in \mathbb{Z}} \text{sign}(\ell)E_{i+1,j}^\ell u^{-\ell},
\]

(2.12)

where sign function \( \text{sign}(\ell) \) is defined as

\[
\text{sign}(\ell) = \begin{cases} +1, & \ell \geq 0, \\ -1, & \ell < 0. \end{cases}
\]

(2.13)

The nontrivial commutation relations between currents are

\[
k_{i+1}^+(u)F_i(v)k_{i+1}^+(u)^{-1} = \frac{q^{-1}u - qv}{u - v} F_i(v),
\]

\[
k_{i+1}^+(u)F_i(v)k_{i+1}^-(u)^{-1} = \frac{qu - q^{-1}v}{u - v} F_i(v),
\]

\[
k_i^+(u)^{-1}E_i(v)k_i^+(u) = \frac{q^{-1}u - qv}{u - v} E_i(v),
\]

\[
k_i^+(u)^{-1}E_i(v)k_{i+1}^+(u) = \frac{qu - q^{-1}v}{u - v} E_i(v),
\]

(2.14)

\[
(q^{-1}u - qv) F_i(u)F_i(v) = (qu - q^{-1}v) F_i(v)F_i(u),
\]

\[
(qu - q^{-1}v) E_i(u)E_i(v) = (q^{-1}u - qv) E_i(v)E_i(u),
\]

\[
(u - v) F_i(u)F_{i+1}(v) = (q^{-1}u - qv) F_{i+1}(v)F_i(u),
\]

\[
(q^{-1}u - qv) E_i(u)E_{i+1}(v) = (u - v) E_{i+1}(v)E_i(u),
\]

\[
[E_i(u), F_j(v)] = \delta_{ij} (q - q^{-1})\delta(u, v) \left( k_{i+1}^-(v)k_i^+(v)^{-1} - k_i^+(u)k_i^+(u)^{-1} \right)
\]

(2.15)
and Serre relations for the currents $E_i(u)$ and $F_i(u)$ which can be found in [16, 17].

**Remark 2.2.** The equalities (2.14) should be understood in a sense of equalities between formal series. It means that these commutation relations should be understood as infinite set of equalities between modes of the currents which appear after equating the coefficients at all powers $u^\ell v^{\ell'}$ for $\ell, \ell' \in \mathbb{Z}$. The rational functions in the commutation relations (2.14) should be understood as series over powers of $v/u$ in the relations containing the current $k_j^+ (u)$ and over powers of $u/v$ in the relations with the current $k_j^- (u)$. In (2.15) the multiplicative generalized function is defined by the formal series $\delta(u, v) = \sum_{\ell \in \mathbb{Z}} u^\ell / v^\ell$ which satisfy the property $\delta(u, v)G(u) = \delta(u, v)G(v)$ for any formal series $G(u)$.

### 2.4. Sets of parameters and their partitions

It is known [18] that the Cartan–Weyl generators of $U_q(\mathfrak{gl}_n)$ can be identified with the modes of the currents $F_i(u), E_i(u)$ and $k_j^\pm (u)$. Off-shell Bethe vectors which will be defined in the next section through these current generators [12] depend on the sets of the Bethe parameters. One needs $N - 1$ types of the formal parameters $t_{\ell}^a$ for $\ell = 1, \ldots, N - 1$ and $a = 1, \ldots, r_\ell$ to describe off-shell Bethe vectors for the quantum integrable model associated with $U_q(\mathfrak{gl}_n)$-invariant $R$-matrix. The superscript $\ell$ in $t_{\ell}^a$ denotes the type of the formal parameter while subscript $a$ labels the formal parameters of the same type.

We will collect formal parameters of the same type in the sets $\bar{\ell} = \{ t_{\ell}^1, \ldots, t_{\ell}^{r_\ell} \}$ with cardinalities $|\bar{\ell}| = r_\ell$ and will denote the collection of these sets as $\bar{\iota} = (\bar{\ell}_1, \ldots, \bar{\ell}_N)$. If non-negative number $r_\ell$ vanishes for some $\ell$ it means that the corresponding set $\bar{\ell}$ is empty. For any $\ell = 1, \ldots, N - 1$ and $a = 1, \ldots, r_\ell$, the set $\bar{\ell}_\ell^a$ is by definition the set $\bar{\ell} \setminus \{ \bar{\ell}_\ell^a \}$ of the cardinality $|\bar{\ell}^a| = r_\ell - 1$.

Our results are formulated as sums over partitions of the sets $\bar{\ell}$ into several nonintersecting subsets. We denote these partitions as $\bar{\ell}^\prime = \bar{\ell}^1 \vdash \bar{\ell}^N$.

For any scalar functions or commuting operators of one or two variables we will abbreviate the product of functions of elements of the sets $\bar{\iota}$ and $\bar{\bar{\iota}}$ as follows:

$$
\lambda^+ (\bar{\iota}, \bar{\bar{\iota}}) = \prod_{i=1}^{[\bar{\iota}]} \lambda^+ (u_i), \quad f (\bar{\iota}, \bar{\bar{\iota}}) = \prod_{i=1}^{[\bar{\iota}]} \prod_{p=1}^{[\bar{\bar{\iota}}]} f (u_i, v_p), \quad T^{\pm}_{i,j} (z) = \prod_{j=1}^{[\bar{\iota}]} T^{\pm}_{ij} (z_j).
$$

If any of these sets is empty then the corresponding product is equal to 1 by definition. For example, $f (\bar{\iota}, \emptyset) \equiv 1$.

### 3. Bethe vectors and main results

Monodromy matrix $T(z)$ of any quantum integrable model with $U_q(\mathfrak{gl}_n)$-invariant $R$-matrix satisfies the commutation relation (2.7) or (2.8) for $\mu = \nu$. We identify $T(z) \equiv T^+ (z)$. RTT commutation relations (2.7) imply that transfer matrix

$$
t(z) = \sum_{i=1}^{N} T_{i} (z)
$$

(3.1)

commutes for different spectral parameters. The goal of the algebraic Bethe ansatz is to describe the solution of the eigenvalue problem [1, 15]

$$
t(z) \cdot \mathbb{B} (\bar{\iota}) = \tau (z; \bar{\iota}) \mathbb{B} (\bar{\iota}),
$$

(3.2)
where

\[ \tau(z; \bar{t}) = \sum_{i=1}^{N} \lambda_i(z) f(z; \bar{t}^{i-1}) f(\bar{t}^i, z). \]  

(3.3)

Here \( \bar{t} \) is a set of Bethe parameters \( \{ \bar{t}^1, \ldots, \bar{t}^N \} \) and \( \mathbb{B}(\bar{t}) \) is a vector from the space of states of the corresponding quantum integrable model. The boundary sets \( \hat{\mathcal{P}}^0 \) and \( \hat{\mathcal{P}}^0 \) appearing in (3.3) are empty. The functions \( \lambda_i(z) \equiv \lambda^+_i(z) \) are free functional parameters of the formal parameter \( z \) and depending on the model. In each concrete integrable model these parameters are fixed to some functions of the spectral parameter. Let \( \beta(z) \) be a ratio of neighboring functional parameters

\[ \beta(z) = \frac{\lambda_{i+1}(z)}{\lambda_i(z)}, \quad i = 1, \ldots, N - 1. \]  

(3.4)

The parameters of the Bethe vectors should satisfy Bethe equations

\[ \beta_i(\bar{t}_i^j) = \frac{\lambda_{i+1}(\bar{t}_i^j)}{\lambda_i(\bar{t}_i^j)} = \frac{f(\bar{t}_1^j, \bar{t}_2^j)}{f(\bar{t}_1^j, \bar{t}_2^j)} \frac{f(\bar{t}_1^j, \bar{t}_2^{j-1})}{f(\bar{t}_1^j, \bar{t}_2^{j-1})} \]  

(3.5)

in order to fulfill eigenvalue problem (3.2). Such Bethe vectors are called \textit{on-shell}.

If (3.5) is not imposed then Bethe vectors are called \textit{off-shell} and have the structure described by hierarchical Bethe ansatz. Off-shell Bethe vectors can be described in terms of the Cartan–Weyl generators of the quantum loop algebra \( U_q(\hat{\mathfrak{gl}}_N) \). We will concentrate on the off-shell case and now discuss a method to define vectors \( \mathbb{B}(\bar{t}) \in \mathcal{H} \) in the representation space \( \mathcal{H} \) of the algebra \( U_q(\hat{\mathfrak{gl}}_N) \).

To describe this relation one has to consider Borel subalgebras of \( U^+_q(\hat{\mathfrak{gl}}_N) \subset U_q(\hat{\mathfrak{gl}}_N) \) formed by the modes of \( T^- \)-operators \( T^+(u) \) and an alternative Borel subalgebras \( U_F^E \) and \( U_E^F \) formed by the modes of the currents \( F_i(u), k_j^+(u) \) and \( E_i(u), k_j^-(u) \) respectively. Consider the intersections \( U_F^E = U_F \cap U^+_q(\hat{\mathfrak{gl}}_N) \) and \( U_E^F = U_E \cap U^+_q(\hat{\mathfrak{gl}}_N) \). There are projections \( P_E^+ : U_F \rightarrow U_F^E \) and \( P_F^+ : U_E \rightarrow U_E^F \).

Detailed investigation of these projections for the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \) was performed in [11]. For the purpose of the present paper we may understand projections \( P^+ \) acting on a product of the simple roots currents \( F_i(t) \) as follows. To calculate the projection from the product of these currents one has substitute each current by the difference of the Gauss coordinates \( \beta_j(t) \) and then use commutation relations between them to ‘normal’ order all monomials such that all ‘negative’ Gauss coordinates \( F^+_{\beta_j}(t) \) are on the left of all ‘positive’ coordinates \( F^+_{\beta_j}(t) \). Although the original expression is a collection of monomials composed from the Gauss coordinates \( F^+_{\beta_{i+1}}(t) \) only, the higher Gauss coordinates \( F^+_{\beta_j}(t) \) for \( j > i + 1 \) will appear due to this normal ordering process. Then application of the projection \( P^- \) means removing of all monomials which have at least one ‘negative’ coordinate on the left. Analogously, application of the projection \( P^- \) means removing of all monomials which have at least one ‘positive’ coordinate on the right. Projections \( P^+ \) acting on a product of the currents \( E_i(t) \) can be understood similarly according to the cyclic ordering of the Cartan–Weyl generators (see details in [11, 19]). In [12] more effective methods to calculate such projections were developed. We point an interested reader to this paper and references therein.
Let us introduce the ordered products of the simple root currents $\mathcal{F}_i(\vec{t})$

$$\mathcal{F}_i(\vec{t}) = \prod_{i \in \vec{t}} f(t'_i, t''_i) F_i(t'_i) F_i(t''_i) \ldots F_i(t''_{\vec{N}}).$$  \hfill (3.6)

Each $\mathcal{F}_i(\vec{t})$ is obviously symmetric with respect to permutations of the elements in the set $\vec{t}$ due to the commutation relations \((2.14)\). In order to express off-shell Bethe vectors in terms of the Cartan–Weyl generators of the algebra $U_q(\mathfrak{gl}_n)$ we define the normalized ordered product

$$\mathbb{F}(i) = \prod_{i=1}^{N-2} f(i^{i+1}, i^{-1})^{-1} \mathcal{F}_1(i^1) \mathcal{F}_2(i^2) \ldots \mathcal{F}_{N-1}(i^{N-1}).$$  \hfill (3.7)

We call the projection of this product of currents $P_f^+ (\mathbb{F}(i))$ off-shell pre-Bethe vector. The off-shell Bethe vector $\mathbb{B}(i)$ itself is \cite{12}

$$\mathbb{B}(i) = \mathbb{B}(i^1, i^2, \ldots, i^{N-1}) = P_f^+ (\mathbb{F}(i)) |\text{vac}\rangle.$$  \hfill (3.8)

The commutation relations of the currents and properties of the projections imply that if any of the sets of Bethe parameters $\vec{t}$ is empty then pre-Bethe vector factorizes into a product of pre-Bethe vectors for the algebras $U_q(\mathfrak{gl}_1)$ and $U_q(\mathfrak{gl}_{N-1})$

$$P_f^+ (\mathbb{F}(i^1, \ldots, \vec{t}^{-1}, \emptyset, \vec{t}^{i+1}, \ldots, i^{N-1})) = P_f^+ (\mathbb{F}(i^1, \ldots, \vec{t}^{-1})) \cdot P_f^+ (\mathbb{F}(\vec{t}^{i+1}, \ldots, i^{N-1})).$$

In particular, when $i = 1$ or $i = N - 1$ the off-shell Bethe vectors $\mathbb{B}(\emptyset, i^2, \ldots, i^{N-1})$ and $\mathbb{B}(i^1, \ldots, i^{N-2}, \emptyset)$ are $U_q(\mathfrak{gl}_{N-1})$ Bethe vectors.

3.1. Action formulas

In addition to the rational functions \((2.1)\) we introduce the functions

$$h(u, v) = \frac{f(u, v)}{g(u, v)}, \quad \tilde{h}(u, v) = \frac{f(u, v)}{\bar{g}(u, v)}$$  \hfill (3.9)

which satisfy the properties

$$1 - \frac{q^{-1}}{f(u, v)} = \frac{1}{h(u, v)} \quad \text{i.e.} \quad h(u, v) = \frac{q^{-1}}{g(u, v)} = 1$$  \hfill (3.10)

and

$$1 - \frac{q}{f(u, v)} = \frac{1}{\tilde{h}(u, v)} \quad \text{i.e.} \quad \tilde{h}(u, v) = \frac{q}{\bar{g}(u, v)} = 1.$$  \hfill (3.11)

For two sets $\bar{x}$ and $\bar{y}$ of the same cardinalities $|\bar{x}| = |\bar{y}| = n$ one can define the Izergin determinant

$$K(\bar{x}|\bar{y}) = \prod_{i=1}^{n} \prod_{1 \leq i < j \leq n} (q x_i - q^{-1} y_j) \prod_{1 \leq i < j \leq n} (x_i - x_j)(y_j - y_i) \det \left[ \frac{q - q^{-1}}{(x_i - y_j)(q x_i - q^{-1} y_j)} \right]$$  \hfill (3.12)
and

$$\tilde{K}(\tilde{x} | \tilde{y}) = \prod_{i=1}^{n} \frac{y_i}{x_i} \tilde{K}(x | y).$$  \hspace{1cm} (3.13)$$

Note that for $n = 1$

$$K(x | y) = g(x, y) \text{ and } \tilde{K}(x | y) = \tilde{g}(x, y).$$

Recall that for each $\ell = 1, \ldots, N - 1$ we have sets $\tilde{t}^\ell = \{t^\ell_1, \ldots, t^\ell_r \}$ of formal parameters.

**Definition 3.1.** Let $\tilde{z} = \{z_1, \ldots, z_r\}$ be a set of arbitrary formal parameters of cardinality $|\tilde{z}| = r$. Let $\tilde{w}^\ell = \tilde{z} \cup \tilde{t}^\ell$ be a set for each $\ell = 1, \ldots, N - 1$ and $\tilde{w}^0 = \tilde{w}^N = \tilde{\mathcal{z}}$. Given $(i, j) \in \{1, \ldots, N\}^2$, we say that the sets $\{\tilde{w}^0, \tilde{w}^1, \ldots, \tilde{w}^{N-1}, \tilde{w}^N\}$ are partitioned into subsets obeying the $(i, j)$-condition if they obey following restrictions:

(a) Each set $\tilde{w}^\ell$ for $\ell = 0, 1, \ldots, N - 1, N$ is partitioned in subsets $\{\tilde{w}^\ell_1, \tilde{w}^\ell_2, \tilde{w}^\ell_3\} \vdash \tilde{w}^\ell$.

(b) Boundary conditions: $\tilde{w}^0_1 = \tilde{w}^N_1 = \tilde{\mathcal{z}}, \tilde{w}^0_2 = \tilde{w}^N_2 = \tilde{\mathcal{w}} = \emptyset$.

(c) Subsets $\tilde{w}^\ell_1$ are nonempty only for $0 \leq \ell < i$ and have cardinality $|\tilde{w}^\ell_1| = r$.

(d) Subsets $\tilde{w}^\ell_3$ are nonempty only for $l \leq \ell \leq N$ and have cardinality $|\tilde{w}^\ell_3| = r$.

This table demonstrates an example of the cardinalities of the partitions for all $i, j = 1, \ldots, N$ in case of $N = 3$ obeying $(i, j)$-condition

| $i = 1$ | $j = 1$ | $i = 1$ | $j = 2$ | $i = 1$ | $j = 3$ |
|---------|---------|---------|---------|---------|---------|
| $\tilde{w}^1_1 = \emptyset$ | $\tilde{w}^1_1 = \emptyset$ | $\tilde{w}^1_1 = \emptyset$ | $\tilde{w}^1_1 = \emptyset$ | $\tilde{w}^1_1 = \emptyset$ | $\tilde{w}^1_1 = \emptyset$ |
| $\tilde{w}^1_2 = \emptyset$ | $\tilde{w}^1_2 = \emptyset$ | $\tilde{w}^1_2 = \emptyset$ | $\tilde{w}^1_2 = \emptyset$ | $\tilde{w}^1_2 = \emptyset$ | $\tilde{w}^1_2 = \emptyset$ |
| $\tilde{w}^1_3 = \emptyset$ | $\tilde{w}^1_3 = \emptyset$ | $\tilde{w}^1_3 = \emptyset$ | $\tilde{w}^1_3 = \emptyset$ | $\tilde{w}^1_3 = \emptyset$ | $\tilde{w}^1_3 = \emptyset$ |

For example, let us consider the structure of the sets $\tilde{w}^i_1$ and $\tilde{w}^i_3$ for $N = 3, i = j = 2, r = 2$ and $\ell = 1, 2$. First, the sets $\tilde{w}^i_2$ and $\tilde{w}^i_3$ are empty. Second, the set $\tilde{w}^1_1$ is equal to one of the following sets

- $\{z_1, z_2\}$,
- $\{t^s_1, t^s_2\}$ for $s = 1, \ldots, r_1$ and $\ell = 1, 2$,
- $\{t^s_1, t^s_2\}$ for $s < s'$ and $(s, s') \in \{1, \ldots, r_1\}^2$. 


Proposition 3.2. For all \( i, j = 1, \ldots, N \) the single monodromy matrix entry \( T_{ij}(z) \) action on off-shell Bethe vector is given by the expression
\[
T_{ij}(z)\mathcal{B}(t) = \lambda_1(z) \sum_{\text{part}} \mathcal{B}(\bar{w}_i^p, \bar{w}_j^q) A_{ij}(\bar{w}_i^p; \bar{w}_j^q; \bar{w}_i^p) ,
\]
(3.14)
\[
\text{where}
\]
\[
A_{ij}(\bar{w}_i^p; \bar{w}_j^q; \bar{w}_i^p) = \prod_{p=1}^{i-1} f(\bar{w}_i^p, \bar{w}_j^q) \prod_{p=1}^{i-1} \frac{\beta_p(\bar{w}_i^p) f(\bar{w}_i^p, \bar{w}_j^q) h(\bar{w}_i^p, \bar{w}_j^q) f(\bar{w}_i^p, \bar{w}_j^q)}{f(\bar{w}_i^p, \bar{w}_j^q) f(\bar{w}_i^p, \bar{w}_j^q) h(\bar{w}_i^p, \bar{w}_j^q)}
\]
(3.15)
\[
\text{and sum in (3.14) goes over partitions obeying the } (i,j)\text{-condition w.r.t. the set } \{z\} \text{ given by definition 3.1 for } r = 1.
\]

Proof of proposition 3.2 can be found in appendix A.

A direct corollary of proposition 3.2 is the multiple action of the product of the same monodromy matrix entries \( T_{ij}(z) \) on the off-shell Bethe vector \( \mathcal{B}(t) \).

Proposition 3.3. For all \( i, j = 1, \ldots, N \) the multiple action of monodromy matrix elements \( T_{ij}(z) \) on the off-shell Bethe vector \( \mathcal{B}(t) \) is given by the formula
\[
T_{ij}(z)\mathcal{B}(t) = \lambda_1(z) \sum_{\text{part}} \mathcal{B}(\bar{w}_i^p) A_{ij}(\bar{w}_i^p; \bar{w}_j^q; \bar{w}_i^p) K(\bar{w}_i^p|\bar{w}_j^q) f(\bar{w}_i^p, \bar{w}_j^q) f(\bar{w}_i^p, \bar{w}_j^q)
\]
(3.16)
\[
\text{where sum in (3.16) goes over partitions obeying the } (i,j)\text{-condition w.r.t. the set } \{z\} .
\]

Proof of proposition 3.3 is similar to the proof of analogous statement given in appendix B of the paper [4] and based on the properties of the Izergin determinant (3.12). The action of monodromy entries on the off-shell Bethe vectors was calculated in [10] for the different normalization of the off-shell Bethe vectors. It coincides with (3.16) up to this normalization.

□
3.2. Recurrence relations

To formulate the second main result of this paper we introduce the notation \( \{ \bar{\ell} \} \), which means a collection of the sets \( \{ \ell, \bar{\ell} + 1, \ldots, \bar{\ell} \} \).

**Proposition 3.4.** Off-shell Bethe vectors \( B(i) \) satisfy the recurrence relations

\[
B(\{ \bar{\ell} \})_{1}^{\ell-1}, \{ z, \bar{\ell} \} , \{ \bar{\ell} \} )_{\ell+1}^{N-1} = \sum_{\text{part} i} \sum_{j=1}^{\ell} \sum_{p=\ell+1}^{N} T_{i,j}(z) B(\{ \bar{\ell} \})_{1}^{\ell-1}, \{ \bar{\ell} \} )_{j}^{\ell-1}, \{ \bar{\ell} \} )_{j+1}^{\ell-1}, (\bar{\ell} )_{j}^{(N-1)} \right) \frac{f(\bar{\ell}, \bar{\ell} - 1, z)}{\lambda(z)} \frac{f(\bar{\ell} + 1, \bar{\ell} - 1, z)}{f(z, \bar{\ell} - 1, z)} \times \prod_{p=\ell+1}^{j-1} \frac{g(\bar{\ell} - 1, \bar{\ell} - 1)}{g(\bar{\ell} - 1, \bar{\ell} - 1)} \frac{f(\bar{\ell} - 1, \bar{\ell} - 1)}{f(\bar{\ell} - 1, \bar{\ell} - 1)}.
\]

where sum goes over partitions \( \{ \bar{\ell} , \bar{\ell} \} \) \( \bar{\ell} \) for \( p = i, \ldots, \ell - 1 \) and partitions \( \{ \bar{\ell} , \bar{\ell} \} \) \( \bar{\ell} \) for \( p = \ell + 1, \ldots, j - 1 \) such that \( \bar{\ell} = 1 \) for \( p = i, \ldots, \ell \) and \( \bar{\ell} = 1 \) for \( p = \ell, \ldots, j - 1 \) with fixed boundary partitions \( \bar{\ell} = \bar{\ell} = \{ z \} \) and \( \bar{\ell} = \bar{\ell} = \emptyset \).

Proof of proposition 3.4 can be found in appendix B.

Recurrence relation (3.17) is written in assumption that all sets of the Bethe parameters \( \bar{\ell} \) for \( p = 1, \ldots, N - 1 \) are not empty. If \( \bar{\ell} = \emptyset \) for some \( \ell' \neq \ell \) then for \( \ell' > \ell \) the sum in (3.17) over \( j \) ends at \( j = \ell ' \) and for \( \ell' < \ell \) the sum over \( i \) begins at \( \ell' + 1 \).

There are two extreme cases of the recurrence relations (3.17) with respect to first and last Bethe parameters when \( \ell = 1 \) and \( \ell = N - 1 \)

\[
B(\{ z, \bar{\ell} \} )_{1}^{\ell}, \{ \bar{\ell} \} )_{2}^{N-1} = \sum_{j=1}^{N} T_{1,j}(z) \sum_{\text{part} i \neq \bar{\ell} = \{ z \} } \frac{g(\bar{\ell} - 1, \bar{\ell} - 1)}{g(\bar{\ell} - 1, \bar{\ell} - 1)} \frac{f(\bar{\ell} - 1, \bar{\ell} - 1)}{f(\bar{\ell} - 1, \bar{\ell} - 1)} \times \prod_{p=\ell+1}^{j-1} \frac{g(\bar{\ell} - 1, \bar{\ell} - 1)}{g(\bar{\ell} - 1, \bar{\ell} - 1)} \frac{f(\bar{\ell} - 1, \bar{\ell} - 1)}{f(\bar{\ell} - 1, \bar{\ell} - 1)}.
\]

and

\[
B(\{ \bar{\ell} \} )_{1}^{\ell}, \{ z, \bar{\ell} \} )_{2}^{N-1} = \sum_{j=1}^{N} T_{j,1}(z) \sum_{\text{part} i \neq \bar{\ell} = \{ z \} } \frac{g(\bar{\ell} - 1, \bar{\ell} - 1)}{g(\bar{\ell} - 1, \bar{\ell} - 1)} \frac{f(\bar{\ell} - 1, \bar{\ell} - 1)}{f(\bar{\ell} - 1, \bar{\ell} - 1)} \times \prod_{p=\ell+1}^{j-1} \frac{g(\bar{\ell} - 1, \bar{\ell} - 1)}{g(\bar{\ell} - 1, \bar{\ell} - 1)} \frac{f(\bar{\ell} - 1, \bar{\ell} - 1)}{f(\bar{\ell} - 1, \bar{\ell} - 1)}.
\]

Recurrence relations (3.18) and (3.19) were obtained in [7] in the framework of the hierarchical Bethe ansatz techniques and for a different normalization of the off-shell Bethe vectors described in appendix B. One can verify that these recurrence relations are compatible with the hierarchical relations for the Bethe vectors described in [8].

4. Zero modes method

In this section we present a trigonometric version of the zero modes method developed for the rational case in [44]. This method uses the fact that all entries of the monodromy matrix can
be obtained from one selected entry using commutation relations with zero modes. Using the action of zero modes and any particular element of monodromy matrix on Bethe vector one can obtain an action of the remaining entries by induction.

The induction step is based on relations between the action of monodromy matrix entry $T_{i,j}(z)$ on off-shell Bethe vector and the action of the entries $T_{i+1,i}(z), T_{j,i-1}(z)$ and zero modes $T_{i+1,i}[0], T_{j,i-1}[0]$ (see formulas (4.1) and (4.2) below). These relations allow to obtain the action of the entries $T_{i+1,i}(z), T_{j,i-1}(z)$ assuming the action of $T_{i,j}(z)$. Since the action of the monodromy matrix entry $T_{i,j}(z)$ can be calculated using explicit presentation of the off-shell Bethe vector $B(i)$ (3.8) one can prove by induction the action formulas of all entries $T_{i,j}(z)$ for all $1 \leq i,j \leq N$. This is done in appendix A. The action of the zero mode operators $T_{i+1,i}[0]$ on off-shell Bethe vectors is given by proposition 4.1 and the action of the entry $T_{i,j}(z)$ is calculated in proposition 4.2.

To develop the zero modes method one needs to use the commutation relations of the zero modes $T_{i+1,i}[0]$ with monodromy matrix entry $T_{i,j}(z) \equiv T_{i,j}^0(z)$ for $l = i$ and $l = j - 1$. Taking $v = 0$ in (2.8) for $\mu = +, \nu = -$ and $\{i,j,k,l\} \rightarrow \{i,j,i+1,l\}$ one obtains

$$T_{i,j}(z) T_{i+1,i}[0] - q^{\delta_{k,l}} T_{i+1,i}[0] T_{i,j}(z) = (q - q^{-1}) \left( \delta_{i,j-1} T_{i,j}^0[0] T_{i,j}(z) - T_{i+1,i}(z) T_{i,j}^0[0] \right).$$

(4.1)

Analogously, taking $u = 0$ in the commutation relation (2.8) for $\mu = -, \nu = +$ and $\{i,j,k,l\} \rightarrow \{i,j,i-1,l\}$ one gets

$$q^{-\delta_{k,l}} T_{j,i-1}[0] T_{j,i}(z) - T_{j,i}(z) T_{j,i-1}[0] = (q - q^{-1}) \left( \delta_{i,j+1} T_{i,j}^0[0] T_{i,j}(z) - T_{i,j-1}(z) T_{i,j}^0[0] \right).$$

(4.2)

Also we will use commutation relation with zero modes of diagonal entries of monodromy matrices. Taking $v = 0$ in (2.8) for $\mu = +, \nu = -$ and $\{i,j,k,l\} \rightarrow \{i,j,l,l\}$ one obtains

$$q^{\delta_{k,l}} T_{i,j}(z) T_{i,l}[0] = q^{\delta_{l,i}} T_{i,l}[0] T_{i,j}(z).$$

(4.3)

According to the Gauss decomposition (2.10) and restrictions (2.4)

$$T_{i,l}[0] = k_i^{-1}, \quad T_{i+1,l}[0] = k_i^{-1} E_i,$$

(4.4)

where $k_i = k_i^+(u) [0]$ and $E_i = E_{l+l+1}[0]$ are coefficients at $u^0$ of the formal series $k_i^+(u)$ and $E_{l+1}(u)$ respectively (see definitions (2.11)).

The action of the zero mode operators on vacuum vector $|\text{vac}\rangle$ is defined as follows

$$k_i |\text{vac}\rangle = k_i^{-1} |\text{vac}\rangle, \quad E_i |\text{vac}\rangle = 0,$$

(4.5)

where $k_i \in \mathbb{C}^\times$ are nonzero complex parameters.

**Proposition 4.1.** The action of zero mode operators $T_{i+1,i}[0]$ on off-shell Bethe vector (3.8) is given by the equality

$$T_{i+1,i}[0] B(i) = (q - q^{-1}) \sum_{l=1}^{t_i} \left( \kappa_i q^{t_i-t_l-1} \beta_l(t_i) \frac{f(t_i, t_l)}{\bar{f}(t_i, t_l)} \right)
- \kappa_{i+1} q^{t_{i+1} - t_l+1} \frac{f(t_i, t_l)}{f(t_{i+1}, t_l)} B(i(t_i)).$$

(4.6)
Proof. To prove this proposition one can use the presentation of the off-shell pre-Bethe vector \([21]\)

\[
P_f^+ (\mathcal{F}(\vec{t})) = \mathcal{F}(\vec{t}) + \sum_{i=1}^{N-1} \sum_{t_i}^{r_i} \frac{f(\vec{t}_i, \vec{t})}{f(\vec{t}^{+1}, \vec{t}_i)} P_{i+1}^+ (\mathcal{F}(\vec{t}_i \setminus \{t_i\})) + \ldots ,
\]

(4.7)

where \(\ldots\) stands for the terms which are annihilated by the projection \(P_f^+\) after the adjoint action of zero mode operators \(E_i, i = 1, \ldots , N - 1\). The commutation relations between Gauss coordinates for \(\mu, \nu = \pm\)

\[
[E_i, F_{i+1}^+(u), F_{i+1}^+(v)] = \delta_{ij} g(v, u) \left( k_{i+1}^+(v) k_i^-(u) - k_{i+1}^-(u) k_i^+(v) \right)
\]

(4.8)

 imply the commutation relations

\[
[E_i, F_{i+1}^-(v)] = (q - q^{-1}) \left( k_{i+1}^-(v) k_i^+(u) - k_{i+1}^+(u) k_i^-(v) \right)
\]

(4.9)

Formulas (4.3) yield

\[
k_i F_i(v) k_i^{-1} = q^{-1} F_i(v), \quad k_{i+1} F_i(v) k_{i+1}^{-1} = q F_i(v)
\]

(4.11)

which imply

\[
k_i^{-1} \mathcal{F}(\vec{t}) k_i = q^{n_i-1} \mathcal{F}(\vec{t}).
\]

(4.12)

Equality (4.10) together with (2.14) yields

\[
[E_i, \mathcal{F}(\vec{t})] = (q - q^{-1}) \sum_{t_i}^{r_i} \frac{f(\vec{t}_i, \vec{t})}{f(\vec{t}^{+1}, \vec{t}_i)} \mathcal{F}(\vec{t}_i \setminus \{t_i\}) k_{i+1}^+(t_i) k_i^-(t_i)^{-1}
\]

(4.13)

up to the terms proportional to \(k_{i+1}^-(t_i) k_i^+(t_i)^{-1}\) since they will vanish after restriction of the action of \(T_{i+1}^+ [0] P_f^+ (\mathcal{F}(\vec{t}))\) on the subalgebra \(U_F^+ = U_F \cap U_A^+(\mathfrak{g}_K)\). Taking into account (4.4), (4.5), (4.7), (4.9) and (4.12) we obtain the statement of the proposition.

\[\square\]

Remark 4.1. Note that unlike the rational quantum integrable models (see lemma 4.2 in [4]) the on-shell Bethe vectors are not the highest weight vectors of \(U_q(\mathfrak{g}_K)\) with respect to the action of \(E_i\).

In proving the action formulas of the monodromy matrix entries on off-shell Bethe vectors we will use induction to obtain the action of the entries \(T_{i+1}^+ (z)\) and \(T_{i+1}^+(z)\) from the induction assumption of the action \(T_i^+(z)\). To do this we will equate the terms of (4.1) and (4.2) acting on \(\mathcal{B}(\vec{t})\) at the different parameters \(\kappa_i\) using their arbitrariness.

Proposition 4.2. The monodromy matrix element \(T_{i+1}^+(z)\) is acting on off-shell Bethe vector \(\mathcal{B}(\vec{t})\) as follows

\[
T_{i+1}^+(z) \mathcal{B}(\vec{t}) = \lambda_i(z) \mathcal{B}(\vec{w}),
\]

(4.14)

where \(\vec{w}\) is a collection of sets of the variables \(\{\vec{w}^1, \ldots, \vec{w}^{N-1}\}\) such that \(\vec{w}^i = \{z, \vec{p}\}\) for \(i = 1, \ldots , N - 1\).
Proof of this proposition is based on an auxiliary lemma.

**Lemma 4.3.** The following equalities in \( \mathcal{U}_q(\mathfrak{gl}_N) \) are satisfied:

\[
F^+_i(z) = P^+_j (F_{N-1}(z) F_{N-2}(z) \cdots F_2(z) F_1(z))
\]

and

\[
P^+_j \left( T_{1,N}(z) \cdot F^+_j(t) \right) = P^+_j \left( F^+_{N,1}(z) k^+_i(z) \cdot F^+_j(t) \right) = 0, \quad \forall \, i, j.
\]

The statement of the lemma can be proved using the RTT relation (2.8) (see appendix D in [19] for details).

To calculate the action (4.14) we will use (4.15) and (4.16)

\[
T_{1,N}(z) \cdot \mathcal{B}(\bar{\omega}) = T_{1,N}(z) \cdot P^+_j (F(\bar{\omega})) |\text{vac}\rangle = P^+_j \left( T_{1,N}(z) \cdot F(\bar{\omega}) \right) |\text{vac}\rangle
\]

\[
= P^+_j (F_{N-1}(z_{N-1}) \cdots F_1(z_1) k^+_i(z) F(\bar{\omega})) |\text{vac}\rangle \bigg|_{z_i = \omega^i} = \lambda_1(z) X(z, \bar{\omega}) \mathcal{B}(\bar{\omega}), \tag{4.17}
\]

where

\[
X(z, \bar{\omega}) = f(\overline{t^1}, z_1) \prod_{j=1}^{N-1} \frac{\gamma_j(\overline{t^j})}{\gamma_j(\overline{w^j})} \prod_{j=1}^{N-2} \frac{f(\overline{u}^j, \overline{w}^j)}{f(\overline{u}^j, \overline{w}^j)} \bigg|_{z_i = \omega^i} = 1
\]

and for \( i = 1, \ldots, N - 1 \)

\[
\gamma_j(\overline{t^i}) = \prod_{1 \leq \ell < \ell' \leq |\overline{t^i}|} f(t^i_{\ell}, t^i_{\ell'}). \tag{4.18}
\]

In order to perform this calculation using the commutation relations between total currents (2.14) we split parameters \( z_i \neq z_j \) and denote \( \bar{w}' = \{z_i, \overline{t^i}\} \). In (4.17) we also use property of the projections that [11]

\[
P^+_j \left( F^+_{N,1}(z) F(\bar{\omega}) \right) = P^+_j \left( P^+_j (F_{N-1}(z) \cdots F_1(z)) F(\bar{\omega}) \right) = P^+_j (F_{N-1}(z) \cdots F_1(z))) F(\bar{\omega}).
\]

\[
\square
\]

5. **Dual Bethe vector and scalar product**

Definition of dual Bethe vectors \( \mathbb{C}(\bar{u}) \) for \( \mathcal{U}_q(\mathfrak{gl}_N) \) and scalar product between \( \mathbb{C}(\bar{u}) \) and \( \mathbb{B}(\bar{\omega}) \) was given in [7]. The recurrence relations for the HC of the scalar product corresponding to the extreme recurrence relations (3.18) and (3.19) were found in this paper. Here we extend this result to more general recurrence relation (3.17) given by proposition 3.4.

Dual Bethe vectors are the vectors of the dual representation space \( \mathcal{H}^* \) of the algebra \( \mathcal{U}_q(\mathfrak{gl}_N) \) which possesses a vector \( |\text{vac}\rangle \) with properties similar to (2.9)

\[
\langle \text{vac}\, | T_{i,j}(u) | \text{vac}\rangle = 0, \quad i < j, \quad \langle \text{vac}\, | T_{i,i}(u) | \text{vac}\rangle = \lambda_i(u) |\text{vac}\rangle. \tag{5.1}
\]

In order to define dual Bethe vectors we consider an involutive algebra antihomomorphism

\[
\Psi : \mathcal{U}_q(\mathfrak{gl}_N) \to \mathcal{U}_{q^{-1}}(\mathfrak{gl}_N), \tag{5.2}
\]
defined on the matrix entries of the $T$-operators as follows

$$\Psi(T_{j,i}(u)) = \tilde{T}_{j,i}(u^{-1}),$$  \hfill (5.3)

where $\tilde{T}_{j,i}(u) \in U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$.

The action of antimorphism (5.2) can be extended to the action on vectors from representation spaces $\mathcal{H}$ and $\mathcal{H}^*$ according to the rules

$$\Psi(\langle \text{vac} \rangle) = \langle \tilde{\text{vac}} \rangle, \quad \Psi(A \langle \text{vac} \rangle) = \langle \tilde{\text{vac}} \rangle \Psi(A),$$

$$\Psi(\langle \text{vac} \rangle | A \rangle) = | \tilde{\text{vac}} \rangle \Psi(A | \text{vac} \rangle),$$ \hfill (5.4)

Here vectors $\langle \tilde{\text{vac}} \rangle$ and $\langle \text{vac} \rangle$ are defined by (2.9) and (5.1) for the algebra $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$ and $A$ is any product of the entries $T_{j,i}(u)$ of the $T$-operator. Note also that according to the equalities

$$\Psi(T_{j,i}(u) \langle \text{vac} \rangle) = \tilde{\lambda}_i(u) \langle \tilde{\text{vac}} \rangle = \langle \tilde{\text{vac}} \rangle\tilde{T}_{j,i}(u^{-1}) = \tilde{\lambda}_i(u^{-1}) \langle \text{vac} \rangle$$

one has

$$\tilde{\lambda}_i(u^{-1}) = \lambda_i(u).$$ \hfill (5.5)

Here $\tilde{\lambda}_i(u)$ are eigenvalues of the diagonal entries of the $T$-operator for the algebra $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$.

The dual Bethe vectors $\tilde{C}(i)$ can be defined as follows (see proposition 5.1 in [6])

$$\Psi(\tilde{B}_q(i)) = C_{q^{-1}}(\tilde{t}^{-1}), \quad \Psi(\tilde{C}_q(i)) = B_{q^{-1}}(\tilde{t}^{-1}),$$ \hfill (5.6)

where notation $\tilde{t}^{-1}$ means

$$\tilde{t}^{-1} = \left\{ \frac{1}{t_1}, \ldots, \frac{1}{t_1^*}; \frac{1}{t_2}, \ldots, \frac{1}{t_2^*}; \ldots; \frac{1}{t_N}, \ldots, \frac{1}{t_N^*} \right\}$$

and subscript in notation of the Bethe vectors $\tilde{B}_q(i)$ and $\tilde{C}_q(i)$ signifies that they are constructed for the algebra $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$.

The scalar product of a generic Bethe vector $B(i)$ and a generic dual Bethe vector $C(u)$

$$S(\bar{u}|\bar{t}) = C(u)B(i)$$ \hfill (5.7)

is vanishing unless $|\bar{u}| = |\bar{t}|$ for all $i = 1, \ldots, N - 1$. This vanishing property of the scalar product for the $\mathfrak{gl}(n|m)$-invariant rational quantum integrable models was discussed in [19]. The same arguments can be applied to check this property for the trigonometric quantum integrable models.

The scalar product $S(\bar{u}|\bar{t})$ can be written in the form (see proposition 4.4 in [7])

$$S(\bar{u}|\bar{t}) = \sum_{\text{part}} W(\bar{u}, \bar{t}) \prod_{i=1}^{N-1} \beta_i(\tilde{u}_i^* \beta_i(\tilde{t}_i^*),$$ \hfill (5.8)
where sum goes over partitions \( \{ \bar{u}'_i, \bar{u}''_i \} \vdash \bar{u}' \) and \( \{ \bar{t}'_j, \bar{t}''_j \} \vdash \bar{t}'' \) such that \( |\bar{u}'_i| = |\bar{t}''_j| \) for \( i = 1, \ldots, N - 1 \). The rational function \( W(\bar{u}, \bar{u}'|\bar{t}, \bar{t}'|) \) depends only on the \( R \)-matrix of the model and does not depend on the free functional parameters \( \lambda_i(z) \). The function \( W(\bar{u}, \bar{u}'|\bar{t}, \bar{t}'|) \) has a presentation (see proposition 4.5 in [7])

\[
W(\bar{u}, \bar{u}'|\bar{t}, \bar{t}'|) = Z(\bar{u}'|\bar{t}) \bar{Z}(\bar{u}'|\bar{t}') \frac{\prod_{j=1}^{N-1} f(\bar{u}'_j, \bar{u}''_j) f(\bar{t}'_j, \bar{t}''_j)}{\prod_{j=1}^{N-1} f(\bar{u}'_{j+1}, \bar{u}'_j) f(\bar{t}'_{j+1}, \bar{t}'_j)},
\]

(5.9)

where

\[
W(\bar{u}, \circ|\bar{t}, \circ) = Z(\bar{u}|\bar{t}), \quad W(\circ, \bar{u}|\circ, \bar{t}) = Z(\bar{u}|\bar{t})
\]

(5.10)

are so called HC of the scalar product. One can check that \( Z(\bar{u}|\bar{t}) = Z(\bar{t}|\bar{u}) \) due to the symmetry \( S(\bar{u}|\bar{t}) = \Psi(S(\bar{t}|\bar{u})) = S(\bar{t}|\bar{u}) \).

Recurrence relation (3.17) implies the recurrence relation for the dual Bethe vectors

\[
\mathbb{C}(\{\bar{u}'_1\}^{\ell-1}, \{\bar{u}'_j, \bar{t}''_j\}_j, \{\bar{u}''_j\}_j)^{N-1} \sum_{i=1}^{\ell} \sum_{j=1}^{N} \mathbb{C}(\{\bar{u}'_1\}^{j-1}, \{\bar{u}'_j\}_j^{j-1}, \{\bar{u}''_j\}_j^{N-1}) \frac{T_{\ell,i}(\bar{u}'_i)}{f(\bar{u}'_i, \bar{u}'_{i-1}) f(\bar{u}'_{i+1}, \bar{u}'_i)} \lambda_i(\bar{u}'_i)
\times \prod_{p=i}^{\ell-1} \frac{\beta_p(\bar{u}'_i) q(\bar{u}'_{i+p}, \bar{t}''_{i+p}) f(\bar{u}'_{i+p}, \bar{u}''_{i+p})}{f(\bar{u}'_{i+p}, \bar{u}'_{i+p-1})}
\times \prod_{p=\ell+1}^{j-1} \frac{\tilde{g}(\bar{u}'_{i+p}, \bar{t}''_{i+p}) f(\bar{u}'_{i+p}, \bar{u}''_{i+p})}{f(\bar{u}'_{i+p}, \bar{u}'_{i+p-1})},
\]

(5.11)

where we used (5.5),

\[
f_{q^{-1}}(x^{-1}, y^{-1}) = f_q(x, y), \quad g_{q^{-1}}(x^{-1}, y^{-1}) = \tilde{g}_q(x, y)
\]

and sum over partitions is the same as in (3.17). Note that there is no summation over partitions of the set \( \bar{u}' \) in (5.11).

Let us consider the scalar product (5.7) for some fixed partition \( \{ \bar{u}'_i, \bar{u}''_i \} \vdash \bar{u}' \) with cardinality \( |\bar{u}'_i| = 1 \).

**Proposition 5.1.** Highest coefficients \( Z(\bar{u}|\bar{t}) \) and \( Z(\bar{u}'|\bar{t}') \) satisfy the recurrence relations

\[
Z(\bar{u}|\bar{t}) = \sum_{\text{part}} \frac{f(\bar{t}', \bar{u}'_1)}{f(\bar{u}'_{1}^{\ell-1}, \bar{u}'_1)} \frac{g(T_{1, \bar{u}'_1}) f(T_{1, \bar{t}'_1})}{f(T_{1, \bar{u}'_1}^{\ell-1}) f(T_{1, \bar{t}'_1})} \sum_{j=1}^{N} \prod_{p=1}^{j-1} \frac{g(\bar{u}'_{i+p}, \bar{u}'_{i+p-1}) f(\bar{u}'_{i+p}, \bar{u}'_{i+p-1})}{f(\bar{u}'_{i+p}, \bar{u}'_{i+p-1})} \times Z(\{\bar{u}'_1\}^{\ell-1}, \{\bar{u}'_1\}_{j-1}^{\ell}, \{\bar{u}'_1\}_{j-1}^{N-1} | \{\bar{u}'_1\}_1^{\ell-1}, \{\bar{t}'_1\}_{j-1}^{\ell}, \{\bar{t}'_1\}_{j-1}^{N-1})
\]

(5.12)
and
\[
\tilde{Z}(\vec{u}^I) = \sum_{\text{part}} \frac{f(\vec{u}_I, \vec{r}_I)}{f(\vec{u}_I^+, \vec{r}_I^+)} \frac{\tilde{g}(\vec{u}_I^+, \vec{r}_I^+)}{f(\vec{u}_I^+, \vec{r}_I^+)} \times \sum_{i=1}^{\ell+1} \prod_{p=1}^{i-1} g(\vec{u}_I^p, \vec{r}_I^p) f(\vec{u}_I^p, \vec{r}_I^p) \prod_{p=1}^{\ell+1} g(\vec{u}_I^p, \vec{r}_I^p) f(\vec{u}_I^p, \vec{r}_I^p)
\]
\[
\times Z(\{\vec{u}_{I+1}^{(\ell+1)}, \{\vec{u}_{I+1}^{(\ell+1)}\}_1\}, \{\vec{u}_I^{(\ell+1)}\}_1, \{\vec{u}_I^{(\ell+1)}\}_1, \{\vec{w}_I^{(\ell+1)}\}_1, \{\vec{u}_{I+1}^{(\ell+1)}\}_1).
\]

In (5.12) partition is going over \(\{\vec{u}_I, \vec{u}_I\} \vdash \vec{u}^t\) for \(\ell + 1 \leq s \leq j - 1\), \(\{\vec{r}_I, \vec{r}_I\} \vdash \vec{r}^t\) for \(\ell \leq s \leq j - 1\) and \(\{\vec{u}_I, \vec{r}_I\} = \{\vec{u}_I^0, \vec{u}_I^0\} \vdash \vec{u}^t\) for \(1 \leq s \leq \ell - 1\). In (5.13) partition is going over \(\{\vec{u}_I, \vec{u}_I\} \vdash \vec{u}^t\) for \(i \leq s \leq \ell - 1\), \(\{\vec{r}_I, \vec{r}_I\} \vdash \vec{r}^t\) for \(i \leq s \leq \ell\) and \(\{\vec{u}_I, \vec{r}_I\} = \{\vec{u}_I^0, \vec{u}_I^0\} \vdash \vec{u}^t\) for \(1 \leq s \leq N - 1\). In both formulas (5.12) and (5.13) there are no partition over set \(\vec{u}^t\). Boundary sets are defined as follows: \(\vec{u}^0 = \vec{p} = \vec{u}^N = \vec{u}^N = \varnothing\) and \(\vec{w}_I = \vec{w}_I^N = \vec{u}_I\).

Proof of this proposition consists of the three steps.

(a) First, one substitutes recurrence relation for the dual Bethe vector (5.11) in the right-hand side of the equality
\[
C(\vec{u}) B(I) = C(\{\vec{u}^{(\ell+1)}_{I+1}, \{\vec{u}_I^{(\ell+1)}\}_1, \{\vec{u}_I^{(\ell+1)}\}_1\}) B(I)
\]
for some fixed partition \(\{\vec{u}_I^t, \vec{u}_I^t\} \vdash \vec{u}^t\).

(b) Then, one uses the action of the monodromy matrix entry \(T_{\ell}(\vec{u}_I^t)\) on Bethe vector \(B(I)\) according to proposition 3.2.

(c) Finally, the statement of the proposition follows from comparing the coefficients at the products \(\prod_{i=1}^{N-1} \beta(\vec{r}_I^t)\) and \(\prod_{i=1}^{N-1} \beta(\vec{u}_I^t)\) in both sides of (5.14) in the presentation (5.8). 

The extreme case \(\ell = 1\) in (5.12) takes the form
\[
Z(\vec{u}_I) = \sum_{\text{part}}^{N} \frac{g(\vec{u}_I^1, \vec{u}_I^1)}{f(\vec{u}_I^1, \vec{u}_I^1)} \prod_{p=1}^{1} g(\vec{u}_I^p, \vec{u}_I^p) f(\vec{u}_I^p, \vec{u}_I^p) \prod_{p=1}^{1} g(\vec{u}_I^p, \vec{u}_I^p) f(\vec{u}_I^p, \vec{u}_I^p)
\]
\[
\times Z(\{\vec{u}_{I+1}^{(2)}, \{\vec{u}_{I+1}^{(2)}\}_1\}, \{\vec{u}_I^{(2)}\}_1, \{\vec{u}_I^{(2)}\}_1, \{\vec{w}_I^{(2)}\}_1, \{\vec{u}_{I+1}^{(2)}\}_1).
\]

where partition is going over \(\{\vec{u}_I, \vec{u}_I\} \vdash \vec{u}^t\) for \(2 \leq s \leq j - 1\), \(\{\vec{r}_I, \vec{r}_I\} \vdash \vec{r}^t\) for \(i \leq s \leq 1\) and \(s = 1\). One can check that this equality coincide s identically with formula (4.15) from [7].

The extreme case \(\ell = N - 1\) in (5.13) takes the form
\[
Z(\vec{u}_I) = \sum_{i=1}^{N-1} \sum_{\text{part}} g(\vec{u}_I^{(N-1), \vec{u}_I^{(N-1)}}) f(\vec{u}_I^{(N-1), \vec{u}_I^{(N-1)}}) \prod_{p=1}^{N-1} g(\vec{u}_I^p, \vec{u}_I^p) f(\vec{u}_I^p, \vec{u}_I^p) \prod_{p=1}^{N-1} g(\vec{u}_I^p, \vec{u}_I^p) f(\vec{u}_I^p, \vec{u}_I^p)
\]
\[
\times Z(\{\vec{u}_{I+1}^{(N-2)}, \{\vec{u}_{I+1}^{(N-2)}\}_1\}, \{\vec{u}_I^{(N-2)}\}_1, \{\vec{u}_I^{(N-2)}\}_1, \{\vec{w}_I^{(N-2)}\}_1, \{\vec{u}_{I+1}^{(N-2)}\}_1).
\]

where partition is going over \(\{\vec{u}_I, \vec{u}_I\} \vdash \vec{u}^t\) for \(i \leq s \leq N - 2\), \(\{\vec{r}_I, \vec{r}_I\} \vdash \vec{r}^t\) for \(i \leq s \leq N - 1\). Again, equality (5.16) coincide s identically with formula (4.16) from [7] if one takes into account equality \(Z(\vec{u}_I) = Z(I|\vec{u})\).
6. Conclusion

This paper is a continuation of the research started in [4] to develop zero modes method for the quantum integrable models defined by \( U_q(\mathfrak{gl}_N) \)-invariant \( R \)-matrices. The aim of this method is to find the action formulas of monodromy matrix entries on off-shell Bethe vectors. These action formulas can be further used to obtain various recurrence relations for the Bethe vectors itself and different physical quantities in this class of the quantum integrable models. In [4] the recurrence relations for the HCs of the scalar products of the off-shell Bethe vectors were obtained for the rational integrable models associated to super-Yangian double \( \mathcal{DY}(\mathfrak{gl}(m|n)) \). In this paper we extend these results to the quantum integrable models based on the quantum loop algebra \( U_q(\tilde{\mathfrak{gl}}_N) \) and generalize recurrence relations for the HCs of the scalar products of the off-shell Bethe vectors found in [7]. Concluding we would like to stress that the explicit form of the action of the monodromy entries on off-shell Bethe vectors is a useful tool to investigate different quantum integrable models. See for example [22, 23] where such formulas were used to investigate supersymmetric \( \mathfrak{gl}_{1|1} \)-invariant XXX spin chains and form-factors in \( \mathfrak{gl}(N) \)-invariant integrable models.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Proof of proposition 3.2

Before giving the technical details let us first explain the idea behind the induction arguments used in the proof. From proposition 4.2 it follows that the desired statement for \((i, j) = (1, N)\) is equivalent to (4.14). This is the base of induction. Then we assume that (3.14) is valid for arbitrary \(i, j = 1, \ldots, N\) and prove using

\[
\left( T_{+1, j}(z) - T_{+1, j}(z) T_{i, j}(z) T_{i+1, j}(z) \right) B(\bar{\tau}) = (q - q^{-1}) (q^{r_j - r_{i-1}} \kappa_i T_{i+1, j}(z) B(\bar{\tau}) - \delta_{i+1, 1} q^{r_j - r_i} \kappa_i T_{i, j}(z) B(\bar{\tau}))
\]

(A.1)

which follows from (4.1) due to

\[
T_{i, j}[0] B(\bar{\tau}) = \kappa_i q^{r_j - r_{i-1}} B(\bar{\tau}), \quad \text{and} \quad [T_{i+1, j+1}[0], T_{i, j}(z)] = 0,
\]

that it is valid for the action of \( T_{i+1, j}(z) \) for fixed \( j \). Two cases \( i \leq j - 1 \) and \( i > j - 1 \) should be considered separately. Below we will consider only the case of \( j \geq i + 1 \) in details, since the case \( j < i + 1 \) can be considered analogously. After this verification we conclude that the action formula (3.14) of the monodromy entry \( T_{i, j}(z) \) on the off-shell Bethe vector \( B(\bar{\tau}) \) leads to the same formula for the action of \( T_{i, j+1}(z) \). This proves that (3.14) is valid for \( T_{i, j}(z), \forall \ell \geq i \) provided it is valid for \( T_{i, j}(z) \). In particular, since proposition 4.2 claims that (3.14) is valid for \( T_{1, N}(z), \forall \ell \).
Next using
\[
(T_{i,j}(z) \, T_{j-1}^{i}[0] - q^{-\delta_{ij}} \, T_{j-1}^{i}[0] \, T_{i,j}(z)) \, \mathbb{B}(\bar{t}) \\
= (q - q^{-1}) \left( q^{\lambda_i} \lambda_{i-1}^{\prime} \right) \mathbb{B}(\bar{t}) \tag{A.2}
\]
which follows from (4.2) one can check that the action formula (3.14) is valid for the monodromy entry \( T_{i,j}(z) \) if it is valid for \( T_{i,j}(z) \). Induction then proves that (3.14) is valid for \( T_{i,j}(z) \), \( \forall \ell \leq j \). Finally, since the induction on \( i \) already proves that (3.14) is valid for \( T_{i,j}(z) \), \( \forall \ell \), this yields that it is valid for \( T_{i,j}(z) \), for all \( \ell, \ell' \).

Now we consider in details how to obtain the action \( T_{i+1,j}(z) \) on the off-shell Bethe vector \( \mathbb{B}(\bar{t}) \) from (A.1) assuming that the action formula (3.14) is valid for the action of \( T_{i,j}(z) \). To do this one can equate the terms proportional to \( \lambda_i \) and check that the terms proportional to \( \lambda_{i+1} \) cancel each other in the left-hand side of (A.1) for \( j > i + 1 \). For \( j = i + 1 \) this yields the action of \( T_{i,j}(z) \) on \( \mathbb{B}(\bar{t}) \).

For \( j \geq i + 1 \) in the action (3.14) \( \bar{w}^i = \{ z, \bar{r} \} = \bar{w}_I \) and \( \bar{w}^j = \bar{w}_I = \emptyset \) for \( s = i, \ldots, j - 1 \). Using (4.6) one gets
\[
T_{i,j}(z) \, T_{i+1,j}^{[0]} \, \mathbb{B}(\bar{t}) \bigg|_{\lambda_i = 0} = (q - q^{-1}) \lambda_i \, q^{\lambda_i - \lambda_{i+1}} \lambda_1(z) \\
\times \sum_{\ell=1}^{\lambda_i} \beta(\bar{t}_I) \, f(\bar{t}_I, \bar{t}_I) \, \mathbb{B}(\bar{w}_I) \, A_{ij}(\bar{w}_I; \bar{w}_I; \bar{w}_I), \tag{A.3}
\]
where \( \bar{w}_I = \{ z, \bar{r} \} \). Acting by the same operators in the inverse order one gets
\[
T_{i+1,j}^{[0]} \, T_{i,j}(z) \, \mathbb{B}(\bar{t}) \bigg|_{\lambda_i = 0} = (q - q^{-1}) \lambda_i \, q^{\lambda_i - \lambda_{i+1}} \lambda_1(z) \\
\times \sum_{\text{part}} \mathbb{B}(\bar{w}_I) \beta(\bar{w}_I) \, f(\bar{w}_I, \bar{w}_I) \, A_{ij}(\bar{w}_I; \bar{w}_I; \bar{w}_I), \tag{A.4}
\]
where the set \( \bar{w}^i = \{ z, \bar{r} \} \) is divided into subsets \( \{ \bar{w}_I, \bar{w}_I \} = \bar{w}^j \) such that \( |\bar{w}_I| = 1 \). In (A.4) one should take into account that \( |\bar{w}_I| = r_{p-1} \) while \( |\bar{w}_I| = r_{p+1} \).

The sum over \( \ell \) in (A.3) can be rewritten as sum over partitions. One can replace the ratio
\[
\frac{f(\bar{t}_I, \bar{t}_I)}{f(\bar{t}_I, \bar{t}_I)} = \frac{f(\bar{t}_I, \bar{w}_I)}{f(\bar{t}_I, \bar{w}_I)}
\]
where \( \bar{w}_I = \{ z, \bar{r} \} \) and add to the sum over \( \ell \) the zero term proportional to
\[
\frac{f(z, \bar{t})}{f(z, \bar{w}_I)} = 0.
\]
The sum over \( \ell \) transforms into
\[
\sum_{\ell=1}^{\lambda_i} \frac{f(\bar{t}_I, \bar{t}_I)}{f(\bar{t}_I, \bar{t}_I)} (\cdot) = \sum_{\text{part}} \frac{f(\bar{w}_I, \bar{w}_I)}{f(\bar{w}_I, \bar{w}_I)} \frac{1}{f(\bar{w}_I, \bar{w}_I)} (\cdot),
\]
where sum runs over partitions \( \{ \bar{w}_I, \bar{w}_I \} = \bar{w}^j \) such that \( |\bar{w}_I| = 1 \). Subtracting (A.3) from (A.4) and using (3.10) one gets the action (3.14) with index \( i \) replaced by \( i + 1 \).
Let us check that for \( j > i + 1 \) the terms proportional to \( \kappa_{i+1} \) in (A.1) vanish. The terms which are proportional to \( \kappa_{i+1} \) in the action of \( T_{i,z}(z) T_{i+1}^{−1}[0] \) \( \mathbb{B}(i) \) is

\[
\begin{align*}
T_{i,z}(z) T_{i+1}^{−1}[0] \mathbb{B}(i) \bigg|_{\kappa_{i}=0} &= -(q - q^{-1}) \kappa_{i+1} q^{ℓ^2-i+1} \lambda_i(z) \\
\times \sum_{\text{part}} \mathbb{B}(\tilde{w}_{\bar{i}}) A_{i,j}(\tilde{w}_{\bar{i}}; \tilde{w}_{\bar{u}}; \tilde{w}_{\bar{u}}) \frac{f(\tilde{w}_{\bar{u}}, \tilde{w}_{\bar{u}})}{f(\tilde{w}_{\bar{u}}, \tilde{w}_{\bar{u}})}, \tag{A.5}
\end{align*}
\]

where we again present the sum over \( ℓ \) after the action of the zero mode operator \( T_{i+1}^{−1}[0] \) as sum over partitions of the set \( \{\tilde{w}_{\bar{i}}, \tilde{w}_{\bar{u}}\} = \{z, \bar{r}\} \) with \( |\tilde{w}_{\bar{u}}| = 1 \). The action in the inverse order can be written in the same form as (A.5) since \( |\tilde{w}_{\bar{i}}| = r_i \) and \( |\tilde{w}_{\bar{u}}^{-1}| = r_{i+1} + 1 \) and coefficient at \( \kappa_{i+1} \) in the left-hand side of (A.1) vanishes.

In the case \( j = i + 1 \) the action \( T_{i+1}(z) T_{i+1}^{−1}[0] \mathbb{B}(i) \) is given by equality (A.5), where the set \( \tilde{w}_{\bar{u}}^{i+1} \) is divided into subsets \( \tilde{w}_{\bar{u}}^{i+1} \) and \( \tilde{w}_{\bar{u}}^{i+1} \) with cardinality \( |\tilde{w}_{\bar{u}}^{i+1}| = r_{i+1} \). The action in the inverse order \( T_{i+1}[0] T_{i+1}^{−1}(z) \mathbb{B}(i) \) is

\[
\begin{align*}
T_{i+1}^{−1}[0] T_{i+1}(z) \mathbb{B}(i) \bigg|_{\kappa_{i}=0} &= -(q - q^{-1}) \kappa_{i+1} q^{ℓ^2-i+1} \lambda_i(z) \\
\times \sum_{\text{part}} \mathbb{B}(\tilde{w}_{\bar{i}}) A_{i,i+1}(\tilde{w}_{\bar{i}}; \tilde{w}_{\bar{u}}; \tilde{w}_{\bar{u}}) \frac{f(\tilde{w}_{\bar{u}}, \tilde{w}_{\bar{u}})}{f(\tilde{w}_{\bar{u}}, \tilde{w}_{\bar{u}})}, \tag{A.6}
\end{align*}
\]

Subtracting (A.5) at \( j = i + 1 \) from (A.6) and using (3.11) one gets from (A.1) the action of the diagonal monodromy entry \( T_{i,z}(z) \) on off-shell Bethe vector \( \mathbb{B}(i) \).

\section*{Appendix B. Proof of proposition 3.4}

In this appendix we prove recurrence relations for the off-shell Bethe vectors \( \mathbb{B}(i) \) formulated in proposition 3.4. Particular cases of these relations mentioned in section 3.2 were found in [7] in the framework of hierarchical Bethe ansatz. Note that in [7] the normalization of the off-shell Bethe vectors was

\[
\prod_{i=1}^{N-1} \beta_i(\bar{r}_i)^{-1} \mathbb{B}(i). \tag{B.1}
\]

To prove proposition 3.4 one can use the action formulas (3.14) and verify that (3.17) is satisfied. Since the monodromy matrix entries \( T_{i,j}(z) \) in (3.17) are upper-triangular we substitute in the right-hand side of this equality the action given by (3.14) and specialized to the case \( i < j \):

\[
\begin{align*}
\sum_{\text{part}} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{N} \frac{\lambda_j(z)}{\lambda_i(z)} \mathbb{B}(\{\tilde{w}_{\bar{u}}^{i-1}, \{z, \bar{r}_i\}^{j-1}, \{z, \bar{r}_j\}^{j-1}, \{\tilde{w}_{\bar{u}}^{j-1}\})}{f(\bar{r}_i^{i-1}, z)f(z, \bar{r}_j^{j-1})} \\
\times \prod_{p=1}^{j-1} \beta_p(\bar{r}_i^{p-1}) \frac{\mathbb{B}(\{\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p}\})}{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})} \prod_{p=1}^{\ell-1} \beta_p(\bar{r}_j^{p-1}) \frac{\mathbb{B}(\{\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p}\})}{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})} \\
\times \prod_{p=\ell+1}^{j-1} \frac{g(\bar{r}_i^{p-1}, \bar{r}_i^{p}) f(\bar{r}_i^{p}, \bar{r}_i^{p})}{h(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})} \\
\prod_{p=\ell+1}^{j-1} \frac{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})}{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})} \prod_{p=\ell+1}^{j-1} \frac{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})}{f(\tilde{w}_{\bar{u}}^{p}, \tilde{w}_{\bar{u}}^{p})} \tag{B.2}
\end{align*}
\]
where sum over partitions of the sets \( \{\bar{p}^i\}^{f-1}_i \), \( \{\bar{p}^\ell\}^{f-1}_\ell \) are described in proposition 3.4 and of the sets \( \{\bar{w}^i\}^{i-1}_i \), \( \{\bar{w}^\ell\}^{N-1}_\ell \) in proposition 3.2 respectively. Recall that according to these rules \( \bar{w}^0_i = \bar{r}^i_i = \bar{r}^\ell_\ell = \bar{w}^N_i = \{z\} \). Our goal is to rewrite the sum over partitions of the sets \( \{\bar{p}^i\}^{f-1}_i \) and \( \{\bar{p}^\ell\}^{f-1}_\ell \) in (B.2) as sum over partitions of the sets \( \{\bar{w}^i\}^{f-1}_i \) and \( \{\bar{w}^\ell\}^{f-1}_\ell \) as follows

To do this we transform second and third lines of (B.2) as follows

\[
\prod_{p=1}^{i-1} \frac{\beta_p(\bar{w}_p^i) f(\bar{w}_p^i, \bar{w}^p_i)}{h(\bar{w}_p^i, \bar{w}^p_i)} \prod_{p=1}^{\ell-1} \frac{\beta_p(\bar{w}_p^\ell) g(\bar{w}_p^\ell, \bar{w}^p_\ell)}{f(\bar{w}_p^\ell, \bar{w}^p_\ell)} \times \prod_{p=1}^{j-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1, \ldots, \bar{w}^{p_N}_N)}{f(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1, \ldots, \bar{w}^{p_N}_N)} \]

(B.3)

Let us at the moment split the values of the boundary sets \( \bar{I}_i = \bar{r}_i = \{z\} \) and \( \bar{w}^0_i = \bar{w}^N_i = \{z\} \) with \( z \neq z' \). Then the sum over partitions in (B.2) can be written as a sum over partitions \( \{\bar{w}^i_1, \bar{w}^i_1\}'^{f-1}_i \) \( \{\bar{w}^\ell_1, \bar{w}^\ell_1\}'^{f-1}_\ell \) and \( \{\bar{w}_1, \bar{w}_1\}'^{N-1}_i \) \( \{\bar{w}_1, \bar{w}_1\}'^{N-1}_\ell \) as follows

\[
\sum_{\text{part}} \frac{\mathcal{B}(\{\bar{w}^i_1\}'^{f-1}_i, \{z', \bar{p}'\}) \{\bar{w}^\ell_1\}'^{f-1}_\ell, \{\bar{w}^i_1\}'^{N-1}_i \{\bar{w}^\ell_1\}'^{N-1}_\ell)}{f(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1, \ldots, \bar{w}^{p_N}_N)} \times \frac{\lambda_1(z)}{\lambda_1(z)} \prod_{i=1}^{\ell-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)}{h(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)} \prod_{p=1}^{f-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)}{h(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)} \times \prod_{p=1}^{j-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)}{f(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)} \]

(B.4)

where the boundary sets \( \bar{w}^i_1 = \bar{r}^i_1 = \{z\} \) and \( \bar{w}^\ell_1 = \bar{r}^\ell_1 = \{z\} \) are fixed. The terms in the sum over partitions vanishes when either \( \bar{w}^i_s = \{z\} \) for \( s = i, \ldots, \ell - 1 \) or \( \bar{w}^\ell_s = \{z\} \) for \( s = \ell + 1, \ldots, j - 1 \) because either \( f(z, \bar{w}^{i-1}_s) = 0 \) or \( f(z, \bar{w}^{\ell-1}_s) = 0 \).

Using a trivial relation between rational functions, namely

\[
\bar{g}(x, y) = g(x, y) \frac{y}{x}
\]

one can calculate the sums over \( i \) and \( j \) in (B.4) to obtain

\[
\sum_{\text{part}} \frac{\mathcal{B}(\{\bar{w}^i_1\}'^{f-1}_i, \{z', \bar{p}'\}) \{\bar{w}^\ell_1\}'^{f-1}_\ell, \{\bar{w}^i_1\}'^{N-1}_i \{\bar{w}^\ell_1\}'^{N-1}_\ell)}{f(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1, \ldots, \bar{w}^{p_N}_N)} \times \frac{\lambda_1(z)}{\lambda_1(z)} \prod_{p=1}^{f-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)}{h(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)} \prod_{p=1}^{j-1} \frac{g(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)}{f(\bar{w}^{p_1}_1, \bar{w}^{p_2}_1)} \]

(B.5)
When $z' \to z$ in this sum over partitions only the term with $\bar{w}_s = \{z\}$ for all $s = 1, \ldots, \ell - 1$ and $\bar{w}_s = \{\bar{z}\}$ for all $s = \ell + 2, \ldots, N - 1$ survives. Taking into account that according to proposition 3.2 $\bar{w}_0 = \bar{w}_N = \emptyset$ and the fact that

$$
\prod_{p=1}^{\ell-1} \beta_p(z) = \frac{\lambda_1(z)}{\lambda_1(z)}
$$

one concludes that (B.5) is equal to $B(\{\bar{w}_s\}_{1}^{\ell-1}, \{z, \bar{z}\}, \{\bar{w}_s\}_{\ell+1}^{N-1})$ so (3.17) becomes an identity. □

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