Why Are the ARIMA and SARIMA not Sufficient

Shixiong Wang, Chongshou Li and Andrew Lim
Department of Industrial Systems Engineering and Management, National University of Singapore, Singapore 117576
Email: s.wang@u.nus.edu (S. Wang. Corresponding Author.), iselc@nus.edu.sg (C. Li), isealim@nus.edu.sg (A. Lim)

The autoregressive moving average (ARMA) model together with its variants like autoregressive integrated moving average (ARIMA), seasonal ARIMA (SARIMA) take the significant position in the community of time series analysis. The ARMA model could sufficiently describe a rational-spectra wide-sense stationary stochastic process and further make use of the past information to approximate the underlying dynamics of the interested stochastic process so that we can make satisfying predictions of the future. As its supplementaries, the ARIMA and SARIMA, collectively referred to as S-ARIMA for briefness, aim to transform the interested non-stationary and irrational-spectra wide-sense stationary stochastic process to be stationary and rational by difference and seasonal difference operator with proper order so that they can follow the philosophy of the ARMA model. However, such difference operators are merely empirical experience without exploring the exact philosophy behind. Therefore, we never know why they work well somewhere and ineffectively elsewhere. In this paper, we will investigate the power of the (seasonal) difference operator from the perspective of spectral analysis, linear system theory and digital filtering, and point out the characteristics and limitations of (seasonal) difference operator and S-ARIMA. Besides, the general operator that transforms a non-stationary and/or irrational-spectra wide-sense stationary stochastic process to be stationary and/or rational will be presented. In the end, we show the overall methodology, ARMA-SIN, for predicting a non-stationary and/or wide-sense stationary stochastic process, which is the improvement of S-ARIMA. As a closing note, we will also present the nature, philosophy, effectiveness, insufficiency, and improvement of the canonical time series decomposition (like STL, X11) and time series smoothing (like exponential smoothing, Holt’s, and moving average) methods, and demonstrate that they are special cases of the ARMA-SIN.

Key words: Time Series Analysis; ARIMA/SARIMA; Spectral Analysis; Digital Filtering; Linear System Theory; ARMA-SIN

1. Introduction

Time series analysis is critical in many academic communities. For instance, the population forecasting problem in social science and ecology (Ives et al. 2010), financial time series prediction problem in economics (Harvey and Todd 1983), target tracking problem (focus on the real-time position of an moving object through the time) in engineering (Wang et al. 2019), traffic volume prediction problem in transportation science (Okutani and Stephanedes 1984), sales forecasting in retail (Li and Lim 2018) and so on. One of the intriguing topic of time series analysis is to physically analyse the internal mechanism of a system generating the interested time series, and
subsequently build a proper mathematical model to describe the dynamics of this system so that we can predict the future with satisfying accuracy. Generally such a time series is a stochastic process rather than a deterministic one which makes the problem more complex.

When it comes to the stochastic process modelling, the reputed Box-Jenkins methodology (Box et al. 2015), also known as ARMA and ARIMA model, should outstand. The philosophy of the ARMA model is from the Wold’s Decomposition theorem (Hamilton 1995, Papoulis and Pillai 2002). The theorem supports that the ARMA model is mathematically sufficient to describe a rational-spectra wide-sense stationary stochastic process. After the modelling, the least square method (Hamilton 1995, Box et al. 2015), maximum likelihood method (Hamilton 1995, Box et al. 2015) and spectral estimation method et al. (Papoulis and Pillai 2002, Marple Jr and Carey 1989, Hamilton 1995, Cadzow 1980, Friedlander and Porat 1984) could be utilised to estimate the parameters of the model based on the collected real time series samples. As a result, we could make use of the past information (collected samples) to reconstruct the underlying dynamics of the interested stochastic process, and further make satisfying prediction. What is also worthy of mentioning is the ARIMA and SARIMA (collectively referred to as S-ARIMA here for briefness) model. The ARIMA(SARIMA) aims to transform the focused non-stationary (irrational-spectra wide-sense stationary) stochastic process to be stationary (rational) by difference (seasonal difference) operator with proper order so that they can follow the philosophy of the ARMA model. The ARMA and S-ARIMA models help the managers and engineers solve numerous practical problems since their emergence, ranging from the management science and operations research community (Gilbert 2005, Ali et al. 2017), to the engineering fields like computer science (Calheiros et al. 2015), power system (Yunus et al. 2016), and signal processing and automatic control (Friedlander and Porat 1984, Yu et al. 2016, Ding et al. 2018, Cadzow 1980). No wonder, the ARMA and S-ARIMA models increasingly become one category of canonical methodologies in time series analysis (Hamilton 1995, Hyndman et al. 2008, Hyndman and Athanasopoulos 2018).

Remark 1. The concepts of rational-spectra, irrational-spectra, flat-spectra, and line-spectra wide-sense stationary stochastic process are from the Spectral Analysis community (Papoulis and Pillai 2002, Marple Jr and Carey 1989). We will detail them in the Subsection 3.4.

However, unfortunately, such difference operators are merely empirical experience without knowing the exact philosophy behind. Therefore, we never know why they work well somewhere and ineffectively elsewhere. To this end, in this paper, we will investigate the power of the difference operator from the perspective of spectral analysis, linear system theory and digital filtering, and point out the characteristics and limitations that difference operator and S-ARIMA hold. Besides, the general operator that works for transforming a non-stationary (irrational-spectra wide-sense stationary) stochastic process to be stationary (rational) will be presented. At last, we will show
the overall methodology for predicting a stochastic process, which is the improvement of S-ARIMA and termed as ARMA-SIN, meaning S-ARIMA is the special case of ARMA-SIN.

The motivation of our methodology is from the Wold’s Decomposition theorem, Linear System theory, Spectral Analysis and the Digital Filtering. We aim to take effective actions to transform a non-stationary (and/or irrational-spectra wide-sense stationary) to be stationary (and/or rational), and then use the philosophy of ARMA model to describe the rational-spectra stationary part of the interested time series. The actions we take are based on the techniques of digital signal processing, specifically the digital filtering.

As a snapshot, we summarize our contributions in this paper below.

1. We propose a new framework, the spectral analysis and digital filtering approach (SADFA), for time series analysis and processing. It is the general methodology of (seasonal) difference operator (which can transform a time series to be rational-spectra wide-sense stationary), time series decomposition and smoothing;
2. We propose the ARMA-SIN methodology to handle a general non-stationary and/or irrational-spectra wide-sense stationary time series. Thus, the S-ARIMA is a special case of our ARMA-SIN;
3. As by-products derived from our methodology, we investigate the nature, philosophy, effectiveness, insufficiency and improvement of the time series decomposition methods like X11 method, STL method \cite{Hyndman2018}, and time series smoothing methods like exponential smoothing, Holt’s linear trend method and moving average method \cite{Hyndman2008} by showing they are special cases of ARMA-SIN.
4. We show that the traditional exponential smoothing and moving average are problematic due to their innate time-delay property.

Note that ARMA-SIN proposed in this paper is not only a model for stochastic process but also a methodology for time series analysis and processing. For detail, see Section 5.

In Section 2, we firstly give some examples to show the insufficiencies of the S-ARIMA and the advantages of ARMA-SIN over S-ARIMA, just as intuitive understandings to readers. In Section 3 we will introduce some basic knowledge in Linear System Theory, Spectral Analysis, Digital Filtering, and Stochastic Process and Wold’s Decomposition Theorem, just as the theoretical basis to derive the ARMA-SIN methodology. Subsequently, in Section 4 we will explain the mathematical reason why S-ARIMA makes sense in some special scenarios and point out its theoretical insufficiency. Finally, in Section 5 the general ARMA-SIN model will be presented, and in Section 6 we will analytically explain and derive the methods we presented in the warming-up Section 2. As a supplement and closing note, we in Section 7 will investigate the nature of the canonical time
series decomposition and time series smoothing methods from the perspective of the SADFA, and figure out their advantages and shortages.

Before we proceed to the next sections, we show some remarks here to make further clear the issues raised in this paper.

**Remark 2.** The wide-sense stationarity of a stochastic process requires: (a) invariant mean; (b) invariant variance; and (c) invariant autocorrelation function, through the time (Papoulis and Pillai 2002). The ARMA model is reasonable if and only if the concerned stochastic process is rational-spectra wide-sense stationary (Papoulis and Pillai 2002, Hamilton 1995, Marple Jr and Carey 1989). In the following, we will make clear that the difference operator generally works for some types (not all) of variant mean and variant autocorrelation function, and seasonal difference operator works for some other types of variant mean and irrational-spectra stationarity, meaning they are powerless for general cases, especially the variant variance problem. This is why the prestigious Box-Cox transformation comes (Hyndman and Athanasopoulos 2018, Hamilton 1995) for the variant variance case. However, we should mention that the Box-Cox transformation is still limited, only powerful for specific kinds of time series like logarithmic transformation in typical air passengers prediction problem (the build-in AirPassengers data-set in R language package), and antidiabetic drugs sales prediction problem (Hyndman and Athanasopoulos 2018). We will detail this problem and settle it in our another work in the near future. In this paper, we only concern the issues from S-ARIMA model.

**Remark 3.** When we mention the insufficiency of S-ARIMA, we actually mean its theoretical insufficiency instead of the prediction accuracy in some cases, although the ARMA-SIN we propose in this paper indeed outperforms S-ARIMA. Because the exact model only outperforms other models when the problem is exact. For example, if the data is generated from a linear function with sufficiently small Gaussian noise, then the linear regression model should be better than any other high-order polynomial regression models. However, we cannot assert the linear model is the best all the time. Note that the interested time series, namely the exact problem we study in this paper, is rational-spectra wide-sense stationary stochastic process. Thus the ARMA model, according to Papoulis and Pillai (2002), is its exact model, meaning the operator that makes the original time series exact as rational-spectra wide-sense stationary is better, because the ultimate issue is to train a ARMA model. It is in this sense that we assert the S-ARIMA model is insufficient.

## 2. Scenarios of Warming-up

As intuitive understandings to readers, we in this section provide some simulation scenarios to illustrate the theoretical insufficiency that S-ARIMA holds. Together with, the counterpart solutions given by the ARMA-SIN will be also demonstrated. Following this warming-up, in the subsequent
sections, we will progressively detail the motivations, philosophies, mathematics and methodologies of generating the such powerful ARMA-SIN solutions.

2.1. Notations
Before we start, we should declare some useful mathematical notations here.

1. Let $\mathbf{v} = a : l : b$ define a vector $\mathbf{v}$ being with the lower bound $a$, upper bound $b$ and step length $l$. For example, $\mathbf{v} = 0 : 0.1 : 0.5$ means $a = 0$, $b = 0.5$, and $l = 0.1$. Thus $\mathbf{v} = [0, 0.1, 0.2, 0.3, 0.4, 0.5]^T$;
2. Let the function $\text{length}(\mathbf{x})$ return the length of the vector $\mathbf{x}$. For example, if $\mathbf{x} = [1, 2, 3]^T$, we have $\text{length}(\mathbf{x}) = 3$;
3. Let $t$ denote the continuous time variable, and $n$ its corresponding discrete time variable. For example, if $t = 0 : 0.5 : 100$ (the time span is 100s, and the sampling time is $T_s = 0.5s$), we will have $n = 0 : 1 : \text{length}(t) = 0 : 1 : 201$; let $N = \text{length}(n)$;
4. Let $\mathbf{x}$ denote the interested time series;
5. Let the operator $\text{ARMA}(p, q|\varphi, \theta)$ denote a ARMA process with autoregressive order of $p$ and moving average order of $q$. Besides, the coefficient vectors $\varphi$ and $\theta$ are for autoregressive part and moving average part, respectively;
6. Let the operator $\mathbf{y} = H(\mathbf{x} | \mathbf{a}, \mathbf{b})$ define a difference equation as follows

$$a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} + \ldots + a_p y_{k-p} = b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} + \ldots + b_q x_{k-q},$$

(1)

where $\mathbf{a}, \mathbf{b}$ are vectors and $\text{length}(\mathbf{a}) = q + 1$, $\text{length}(\mathbf{b}) = p + 1$.

2.2. ARMA Series of Ground Truth
At first we should generate a ARMA series, in order to construct a time series to have mathematical analysis. Without loss of generality, we arbitrarily set

$$\theta = [13, 5, 6]^T,$$
$$\varphi = [40, 2, 3, 6, 9]^T,$$

(2)

meaning the time series generated from a Gaussian white series $\epsilon_k$ is given as

$$40x_k^0 + 2x_{k-1}^0 + 3x_{k-2}^0 + 6x_{k-3}^0 + 9x_{k-4}^0 = 13\epsilon_k + 5\epsilon_{k-1} + 6\epsilon_{k-2},$$

(3)

where $k$ denotes the discrete time index (namely $k \in n$) and $x_k^0 := 0, \epsilon_k := 0$ if $k < 0$.

Remark 4. If the interested time series is as $\mathbf{x} = f(\mathbf{x}^0)$ (for example $\mathbf{x} = \mathbf{x}^0 + N$), the operator that exactly transforms $\mathbf{x}$ to its rational-spectra wide-sense stationary part $\mathbf{x}^0$ should be the best. Because we finally aim to use the ARMA model to fit the transformed series. Let $\hat{\mathbf{x}}^0$ be the transformed series from $\mathbf{x}$. The nearer between $\hat{\mathbf{x}}^0$ and $\mathbf{x}^0$, the better. For more, see Remark 3.
2.3. The Case of Variant Mean

In this subsection, we investigate a scenario being with variant mean value, implying it is non-stationary in the mean sense. Let \( t = 0.5 : 100 (T_s = 0.5) \), \( n = t/T_s \), and \( x = x^0 + 0.1t \). It means the trend component is a linear function.

If we follow the standard modelling procedure with Box-Jenkins(ARIMA) method (Hyndman and Athanasopoulos 2018, Calheiros et al. 2015, Box et al. 2015, Hamilton 1995), we have the estimated ARIMA(4,1,2) model to handle this problem, meaning the operator used is first order differencing. Instead if we use ARMA-SIN method, we have

\[
\hat{x}^0 = H(x|a, b),
\]

where

\[
a = [1, -1.7101, 1.3712, -0.3152]^T,
\]

and

\[
b = [0.6226, -1.5757, 1.5757, -0.6226]^T,
\]

and ARMA part for approximating \( x^0 \) as ARMA(4,1). The results of two methods are showed in Figure 1.

![Figure 1](image1)  
**Figure 1** Transformed series and prediction results for variant mean

Besides, we have 100 times of monte carlo simulation and the averaged prediction MSE is given in Table 2.3. Clearly, the transformed series \( \hat{x}^0 \) of ARMA-SIN is nearer to its ground truth \( x^0 \) than that of ARIMA. Thus the prediction accuracy is more satisfactory without any doubt.
Table 1  Averaged prediction MSE of ARIMA and ARMA-SIN for variant mean

|        | ARIMA | ARMA-SIN |
|--------|-------|----------|
| MSE    | 0.0977| 0.0203   |

Table 2  Averaged prediction MSE of ARIMA and ARMA-SIN for variant autocorrelation

|        | ARIMA | ARMA-SIN |
|--------|-------|----------|
| MSE    | 0.2106| 0.0351   |

2.4. The Case of Variant Autocorrelation

In this subsection, we investigate a scenario being with variant autocorrelation, implying it is non-stationary in the autocorrelation sense. Let $t = 0 : 0.1 : 100$ ($T_s = 0.1$), $n = t/T_s$, and $x = x^0 + 0.25sin(0.01t + 0.1)t$. It means the trend component is a sine function. However, the frequency of the sine function is increasing through the time. Thus the seasonal difference operator is not applicable because the series is no longer a periodic one. However, we can still use the general difference operator to handle this.

If we follow the standard modelling procedure with Box-Jenkins(ARIMA) method, we have the estimated ARIMA(4,1,1) model to handle this problem, meaning the operator used is first order differencing. Instead if we use ARMA-SIN method, we have

$$\hat{x}^0 = H(x|a,b), \quad (5)$$

where

$$a = [1, -1.6471, 1.3272, -0.2969]^T,$$

and

$$b = [0.6113, -1.5243, 1.5243, -0.6113]^T,$$

and ARMA part for approximating $x^0$ as ARMA(5,2).

The results of two methods are showed in Figure[2].

Besides, we have 100 times of monte carlo simulation and the averaged prediction MSE is given in Table[2.4]. Clearly, the transformed series $\hat{x}^0$ of ARMA-SIN is nearer to its ground truth $x^0$ than that of ARIMA. Thus the prediction accuracy is more satisfactory without any doubt.

2.5. The Case of Using SARIMA

In this subsection, we investigate a scenario that is suitable for using SARIMA. Let $t = 0 : 0.1 : 50$ ($T_s = 0.1$), $n = t/T_s$, and $x = x^0 + sin(5t)$. It means the trend component is a sine function.

If we follow the standard modelling procedure with Box-Jenkins(ARIMA) method, we have the estimated SARIMA(4,1,1)(12,1,0,0) model to handle this problem, meaning the operator used is 12-lag seasonal differencing. Instead if we use ARMA-SIN method, we have

$$\hat{x}^0 = H(x|a,b), \quad (6)$$
Table 3  Averaged prediction MSE of SARIMA and ARMA-SIN

|        | ARIMA | ARMA-SIN |
|--------|-------|----------|
| MSE    | 0.1022| 0.0516   |

where

\[ a = [1, -5.1801, 11.8864, -15.30778, 11.6563, -4.9815, 0.9430]^T, \]

and

\[ b = [0.9713, -5.0804, 11.7716, -15.3086, 11.7716, -5.0804, 0.9713]^T, \]

and ARMA part for approximating \( x^0 \) as ARMA(4,1). The results of two methods are showed in Figure 3.

Besides, we have 100 times of monte carlo simulation and the averaged prediction MSE is given in Table 2.5. Clearly, the transformed series \( \hat{x}^0 \) of ARMA-SIN is nearer to its ground truth \( x^0 \) than that of SARIMA. Thus the prediction accuracy is more satisfactory without any doubt.

2.6. The Case of Directly Estimating the Irrational-spectra Component

In this subsection, we investigate a scenario that we can directly figure out the Irrational-spectra Component. Let \( t = 0 : 0.1 : 10 \) (\( T_s = 0.1 \)), \( n = t/T_s \), and \( x = x^0 + f(t) \), where \( f(t) = \sin(2t) \). Our purpose is to estimate out the function \( \sin(2t) \) directly from the collected history data.
If we use our ARMA-SIN method, we can know that \( f(x) \) is with the form as

\[
\hat{f}(t) = 0.9721 \cos(0.2001n - 1.5440) \\
= 0.9721 \cos(0.2001t/T_s - 1.5440) \\
= 0.9721 \cos(2.001t - 0.4945\pi)
\]

(7)

It is amazingly close to its ground truth of \( f(t) = \sin(2t) = \cos(2t - \pi/2) = \cos(2t - 0.5\pi) \).

If we follow the standard modelling procedure with Box-Jenkins(ARIMA) method, we have the estimated SARIMA(4,0,1)(31,0,0,0) model to handle this problem, meaning the operator used is 31-lag seasonal differencing.

The results of two methods are showed in Figure 4.

Besides, we have 100 times of monte carlo simulation and the averaged prediction MSE is given in Table 2.6.

| MSE   | SARIMA | ARMA-SIN |
|-------|--------|----------|
| 0.0844| 0.0355 |

Clearly, the transformed series \( \hat{x}^0 \) of ARMA-SIN is nearer to its ground truth \( x^0 \) than that of SARIMA. Thus the prediction accuracy is more satisfactory without any doubt.

Intuitively, we can see that the ARMA-SIN method is indeed interesting over the S-ARIMA model. In this following sections, we will progressively explain the philosophy behind and detail the ARMA-SIN method.
3. New Perspectives of Time Series Analysis and Processing

In this section, we need to present the theory basis to generate the ARMA-SIN methodology. Plus, we will also differentiate the concepts of **Time Series Analysis** and **Time Series Processing**. This idea is taken from the Signal Processing community. In the following, we use the term Time Series Analysis to denote the procedure of analyzing the mathematical properties of a time series, and Time Series Processing to denote the procedure of generating a new more interesting time series from the given one. By doing so, we can find the counterpart concepts of Time Series Analysis and Time Series Processing as Spectral Analysis and Digital Filtering, respectively. The terms of Spectral Analysis and Digital Filtering are from Signal Processing community. Since the signal is philosophically a kind of time series, they should mutually share some analytical and transformational methodologies, rather than independently develop themselves in separate ways. Thus, to bridge the gap between the Signal Processing community and Time Series Analysis community is also one of our academic objectives in this paper.

In order not to confuse readers unfamiliar with the targeted topics in this section, we will try our best to deliver more philosophies instead of mathematics. However, the theoretical sufficiency will be fully guaranteed.

3.1. Linear System Theory

What the term *system* means is a philosophical issue. When it is brought to engineering, the system theory booms. The emergence of system theory ([Von Bertalanffy, 1968](#)), information theory ([Shannon, 1948](#)),
Nonlinear and cybernetics (Wiener 1965) indicates the coming of a new era to both philosophy, management and engineering.

Mathematically, a system is a function mapping an input $x$ to an output $y$. The input and output are generally time-dependent variables. The time variable could be continuous or discrete. Thus the input and output could also be continuous or discrete ones. Accordingly we have $x(t), y(t)$ for continuous case and $x(n), y(n)$ for discrete. If the time variable is discrete, the system will be called as discrete system, and accordingly the input and output will be Time Series.

In management science, one typical instance of a system is human population problem. Imagining the whole society as a system, the total population of human is the system output, we should have government policy, availability of fresh water and food and so on as system inputs. The issue of adjusting the system output by changing the system input is termed as System Control (Dorf and Bishop 2011), philosophically also known as Cybernetics, and by changing the system structure and parameters is termed as Systems Engineering (Blanchard et al. 1990).

In system analysis community, one type of the popular and important systems is Linear Time-shift Invariant (LTI) system, see Definition 1.

**Definition 1.** A system $y = H(x)$ is called Linear Time-shift Invariant (LTI) system if it satisfies

- **Homogeneity:** $ky = H(kx)$, $k \in R$;
- **Additivity:** $y_1 + y_2 = H(x_1 + x_2)$, where $y_1 = H(x_1)$ and $y_2 = H(x_2)$;
- **Time-shift Invariance:** $y(t - t_0) = H[x(t - t_0)]$, where $t$ is time variable and $t_0$ is a fixed time point.

The homogeneity and additivity collectively define the Linearity

$$k_1 y_1 + k_2 y_2 = H(k_1 x_1 + k_2 x_2), k_1, k_2 \in R.$$ 

The time-shift invariance actually indicates the system structure and parameters remain unchanged over time so that the delayed input brings the delayed output, and the two delays are same. Besides, what’s more interesting, as long as the input keeps same just with some delay, the output will also keeps same with same delay. According to Wang et al. (2019), linear system theory takes dominant position in practice in consideration of system analysis and system refactor. For more information on this point, please see Introduction of Wang et al. (2019).

One canonical category of LTI system is given in (8) as differential equation for continuous case $y = H_c(x|a, b)$, and (9) as difference equation for discrete case $y = H_s(x|a, b)$.

\[
\begin{align*}
    a_0 \frac{d^p y}{dt^p} + a_1 \frac{d^{p-1} y}{dt^{p-1}} + a_2 \frac{d^{p-2} y}{dt^{p-2}} + \ldots + a_p y &= b_0 \frac{d^q x}{dt^q} + b_1 \frac{d^{q-1} x}{dt^{q-1}} + b_2 \frac{d^{q-2} x}{dt^{q-2}} + \ldots + b_q x, \\
\end{align*}
\]

where $a, b$ are vectors and length($a$) = $p + 1$, length($b$) = $q + 1$, $p \geq 0$, $q \geq 0$. 

\[ a_0 y_k + a_1 y_{k-1} + a_2 y_{k-2} + \ldots + a_p y_{k-p} = b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} + \ldots + b_q x_{k-q}, \]  

where \( y_k \) stands for \( y(k) \).

Equation (9) tells us the story why we define a difference equation as (1). To be conceptually consistent with the ARMA model in time series analysis community, we still refer to the left-hand-side of (9) as Auto-Regression (AR) part, and the right-hand-side as Moving Average (MA) part.

For notation simplicity, we use \( y = H(x) \) to exclusively denote the discrete case \( y = H_s(x) \).

Because in this paper we pay more attention on discrete system case so that we could model the time series problem as discrete system problem.

**Remark 5.** Many practical systems dynamics, namely the system functions \( H(\cdot) \), are with the mathematical form of (8). For example, if we let \( p = 2, a_0 = 1, a_i = 0 \ (i = 1, 2), q = 0, b_0 = 1, y := s \) denote the displacement of a moving object, and \( x := a \) be the acceleration, we should have \( \frac{d^2 s}{dt^2} = a \).

This admits the reputed Newton’s second law in physics, or in theoretical mechanics. Another example is from electrical engineering. Actually the dynamics of circuits consisting of inductors, capacitances and resistances could be described by differential equation (8) or differential equations, in which each equation has the form of (8) (Robbins and Miller 2012).

In fact, the continuous system (8) and the discrete system (9) are highly related. The gap is bridged by the discretization theory (Diniz et al. 2010, Chaparro and Akan 2018, Yang 2009). For more on how to convert a continuous system to its discrete counterpart, please see Diniz et al. (2010), Chaparro and Akan (2018), Yang (2009). In this case, we have \( t = T_s n \), where \( T_s \) denotes the sampling period. Note that the discrete counterpart of a continuous system actually has different coefficients as it. That we in (8) and (9) use the same notation \( a_0, a_1, \ldots, b_0, b_1, \ldots \) is just for notation simplicity.

Actually the system (8) is theoretically sufficient to describe many different systems holding different dynamics. Since \( y \) is a function of \( x \), also a function of \( t \). We should have \( y = y(t) \). Suppose \( y \) is continuous over a compact set \( T, t \in T \), we could find a polynomial function \( p(t) \) such that \( y(t) \) and \( p(t) \) are as close as desired. This is by Weierstrass approximation theorem (Kreyszig 1978).

Thus (8) could take a special form of

\[ a_0 \frac{d^p y}{dt^p} = b_0 t, \]  

meaning we require \( x \) also take its special case of \( x = t \). Since (8) is more general than polynomial family, it is theoretically sufficient to describe many practical systems mathematically. Besides, discretization theory asserts the discrete system could be analytically derived from its continuous counterpart given the sampling period \( T_s \). Thus we can know that the discrete representation of a
system is also practically sufficient. This is the intuitive (but not strict) philosophy why the linear system theory, for example the discrete linear system (9), is practically sufficient to many real problems.

**Remark 6.** It is possible that although the data we collected is a sequence, the philosophy behind may be continuous, meaning the nature of a discrete system we concern, (9), is actually a continuous system. Thus studying the continuous system theory is still meaningful for time series problem. Because in some cases the discrete system (9) is in fact a refactor of a continuous system (8). For example, when we study the human population problem, we may build a linear time series model (9). However, the population system is actually a continuous system. It evolves all the time rather than merely at the time points we pay attention on, namely the time points we collect the data.

**Remark 7.** From the perspective of statistics, the time series model (9) is actually a successful application of linear regression model. The philosophy is that we may use the past information and causal variable of the interested variable to linearly represent it. Thus the discrete model (9) may have different philosophies from different field viewpoints.

The three interesting properties of a system is Causality, Stability and Inverse-system Stability. The Inverse-system Stability is equivalent to Minimum Phase. The former is usually referred to in System Control community, and the latter is frequently mentioned in Signal Processing field. Specifically we have the following Definition 2.

**Definition 2.** A system is (physically) causal if the output only depends on the present and past values of the input, meaning the output of a causal system to an input appears only while or after the input is applied to the system (Yang 2009); A system is stable if the output is bounded given a bounded input (Chaparro and Akan 2018); A system $H(\cdot)$ is inversely stable if its inverse system (namely $1/H(\cdot)$) exists, is causal and is stable. When a system is inversely stable, it is conceptually equivalent to Minimum Phase system (Diniz et al. 2010).

The mentioned three properties are critical as well for time series analysis. For example, the nature of unit-root test in time series analysis is actually the stability of a discrete system. This point will be illustrated later.

Among all of the system analysis methodologies, the Laplace Transform and Z Transform stand-out (Yang 2009, Diniz et al. 2010, Chaparro and Akan 2018). The Laplace transform is defined for continuous time system (8) and Z Transform for discrete time system (9). Actually the Z transform is also known as the discrete version of Laplace transform. For motivations of having those transforms and detailed theories, please see Yang (2009), Diniz et al. (2010), Chaparro and Akan (2018). We in this paper just display some important conclusions without exploring more.
The Laplace transform to the continuous system (8) is as
\[
H(s) = \frac{b_0 s^q + b_1 s^{q-1} + b_2 s^{q-2} + \ldots + b_q}{a_0 s^p + a_1 s^{p-1} + a_2 s^{p-2} + \ldots + a_p},
\]
where \(s\) is a complex variable consisting of the real and image part, defined in real-image coordinate.

The Z transform to the discrete system (9) is as
\[
H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_q z^{-q}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_p z^{-p}},
\]
where \(z\) is also a complex variable defined in polar coordinate. In the following, we use \(H(z|a,b)\) to denote a system transfer function defined in (12).

Note that (11) and (12) are mathematically meaningful only given some constraints like convergence region and so on. For more, see Yang (2009), Diniz et al. (2010), Chaparro and Akan (2018).

The function of the variable \(z\) in (12) seems really resemble to the backward shift operator, also known as lag operator (Hamilton 1995). However, note that they are naturally different. The former is a complex variable sufficient to have complex analysis, while the latter is merely a algebraic operator having no analytical properties. Interested readers are invited to refer to Yang (2009), Diniz et al. (2010), Chaparro and Akan (2018).

Equation (12) is also known as Transfer Function describing a discrete system (Liu et al. 2017, Helfenstein 1991, Dorf and Bishop 2011, Diniz et al. 2010, Chaparro and Akan 2018, Yang 2009).

Remark 8. It is worthy of mentioning that the ARMA process, in system analysis sense, is a linear system driven by a sequence of Gaussian white noise, meaning the system input is Gaussian white noise (Papoulis and Pillai 2002). Besides, according to Linearity defined in Definition 1, a ARMAX process (Liu et al. 2017) is also a linear system. However, it is driven by two inputs. The one is Gaussian white noise, and another is a exogenous variable \(x\).

For notation simplicity, we use \(\text{ARMA}[z|(p,q),(\varphi,\theta)]\), \(\text{ARMA}(z|p,q)\) denote the transfer function of a ARMA process.

Lemma 1. A linear discrete system (9) is a casual system. Besides, if the modulus of all the roots of its denominator characteristic polynomial are larger than one, the system is stable.

Proof of Lemma 1. The casuality is sure. For stability, see Yang (2009). □

By Lemma 1, we should mention the nature of unit-root test (Hamilton 1995, Box et al. 2015) in time series analysis. It is actually require the system to be stable so that the bounded input would not lead to an unbounded output. Obviously a unbounded-output system is meaningless to practical problems (Chaparro and Akan 2018). As for the requirement of unit-root test that the modulus of all the roots of its numerator characteristic polynomial are larger than one, it in fact
requires the inverse system of (9) and (12) to be stable, meaning it is a minimum phase system. The property of minimum phase of a system is important in control system designing for perfect tracking problem, and signal processing for minimum time delay problem.

The inverse system of a ARMA system is actually a whitening filter, meaning it could convert a colored noise (autocorrelated noise, ARMA process) sequence to a white noise (uncorrelated noise) sequence. Thus the invertibility of ARMA system is also meaningful. Because we want the whitening filter to be useful (stable).

When the two subsystems are combined together, the integrated one is given in Lemma 2

**Lemma 2.** If \( y = H_1(x|a_1,b_1) \) and \( w = H_2(y|a_2,b_2) \), meaning the system \( H_1 \) cascades the system \( H_2 \), we have a equivalent virtual system \( w = H_3(x|a_3,b_3) \) from input \( x \) to output \( w \) such that \( H_3(z|a_3,b_3) = H_1(z|a_1,b_1)H_2(z|a_2,b_2) \). Intuitively see Fig. 5.

**Proof of Lemma 2.** See Diniz et al. (2010), Chaparro and Akan (2018). □

![Figure 5 Two system cascade together](image)

Lemma 2 upholds the validity of mathematical expression of the SARIMA model (Hyndman and Athanasopoulos 2018, Box et al. 2015). As an example, if we let \( H_1(z) = 1 - z^{-L} \) (the lag-\( L \) seasonal difference operator), and let \( H_2(z) = \text{ARMA}(z|p_2,q_2) \) (the ARMA modelling part), we can see the conclusion.

Plus, Lemma 2 also indicates the nature of time series processing from the perspective of system analysis. Specifically, the nature is in fact that we let the interested time series \( x \) pass through a system \( H \) so that \( x \) could be transformed into \( y \). For instance, the system \( H(z) = 1 - z^{-1} \) means the general first order difference, and \( H(z) = 1 - z^{-L} \) means the seasonal lag-\( L \) difference. Later we will show the nature of exponential smoothing, moving average and so on in such pattern.

Actually, all the properties of a LTI system could be uniquely determined by its impulse response function \( h(n) \). The impulse response of a LTI system means the response of the system driven by a impulse function \( \delta(n) \), where \( \delta(0) = 1 \) and \( \delta(i) = 0, \ i \neq 0 \).

**Lemma 3.** A system is causal if \( h(n) = 0 \) when \( n < 0 \). A system is stable if and only if \( \sum_{n=-\infty}^{\infty} |h(n)| < \infty \).
Proof of Lemma 3. See Diniz et al. (2010), Chaparro and Akan (2018), Yang (2009). □

**Lemma 4.** $H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$; $y(n) = h(n) \ast x(n) = x(n) \ast h(n)$, where $\ast$ denotes the convolution operator.

Proof of Lemma 4. See Diniz et al. (2010), Chaparro and Akan (2018), Yang (2009). □

The Z transform could also be defined for a time series $x(n)$, $-\infty < n < \infty$.

**Definition 3.** The Z transform of a time series $x(n)$ is given as

$$X(z) = \sum_{n=\infty}^{\infty} x(n)z^{-n}.$$ 

For more information like convergence region and so on, see Diniz et al. (2010), Chaparro and Akan (2018), Yang (2009).

Then we could have the relation in Z domain among the system property, the input and the output.

**Lemma 5.** Suppose a LTI system is with a transfer function $H(z)$, and the Z transform of the input $x(n)$ and $y(z)$ are $X(z)$ and $Y(z)$, respectively. We have $Y(z) = H(z)X(z)$.

Proof of Lemma 5. See Diniz et al. (2010), Chaparro and Akan (2018), Yang (2009). □

Lemma 4 and 5 admit that the complex convolution computing in time domain could be transformed into a simple algebraic calculation in Z domain.

Lemma 5 again indicates the nature of the time series analysis and processing. Specifically, the nature of time series analysis is that we focus and investigate a time series $x(n)$ in its transformed domain $X(z)$; the nature of time series processing is that the interested time series pass through a purpose-based and custom-designed system (particularly the system is a LTI one) and generate a new time series, which is more interesting (intuitively, as an example, recall an original time series and the one that exponential smoothing generates from the original).

When we let $z$ take its special form $z = e^{jw}$, meaning a complex variable in polar coordinate with modulus of 1 all the time, where $j$ is complex unit and $w$ physically denotes a variable in Fourier frequency domain. We can see from Lemma 5 that $Y(e^{jw}) = H(e^{jw})X(e^{jw})$. $H(e^{jw})$ here means the Frequency Response of a system and $Y(e^{jw})$, $X(e^{jw})$ actually denote the DTFT, Discrete Time Fourier Transform, (if $x$ and $y$ are aperiodic), or Discrete Fourier series (if $x$ and $y$ are periodic), of the time $y(n)$ and $x(n)$, respectively.

**Motivation 1.** The reason why we focus on Fourier frequency domain to do spectral analysis is that a time series $x$ actually consists of many Fourier frequency components. If we can properly design a LTI system $H$ such that $x$ can pass through, the interested part of $x$ will be remained and transformed while the rest will be discarded and/or attenuated. This is the nature of time series analysis and processing in the system analysis sense, just as moving average, exponential smoothing do. The designed system to time series is generally a Digital Filter which can split the $x$ into different parts, just as the time series decomposition methods like STL, X11 et al. do.
When we investigate a time series and a LTI system in Fourier frequency domain, the spectral analysis issue arises.

### 3.2. Spectral Analysis

The Spectral Analysis, also known as Fourier Frequency Domain Analysis, to time series starts from [Granger and Hatanaka (1964)](http://dx.doi.org/10.1080/01621459.1964.10501354), following by many other monographs ([Priestley 1981](http://dx.doi.org/10.1080/03081088108817464), [Koopmans 1995](http://dx.doi.org/10.1080/03081089508817747), [Bloomfield 2004](http://dx.doi.org/10.1080/03081080410001706)). However, reviewing the current main-streams of time series analysis, the spectral analysis failed to draw enough attentions from researchers in time series analysis community ([Hamilton 1995](http://dx.doi.org/10.1080/03081089508817747), [Hyndman et al. 2008](http://dx.doi.org/10.1016/j.techvoc.2007.09.007), [Box et al. 2015](http://dx.doi.org/10.1093/jts/20.6.1079), [Hyndman and Athanasopoulos 2018](http://dx.doi.org/10.1080/00223222.2018.1410581)). The exclusive part thought highly of is Fourier Series Expansion (also known as harmonic analysis). That is, if a interested time series is periodic, it can be represented by the summation combinations of trigonometric functions ([Bloomfield 2004](http://dx.doi.org/10.1080/03081080410001706)). However we should point out that the Fourier series expansion is merely the special case of spectral analysis for periodic series. Concerning the aperiodic series, spectral analysis could also give beautiful conclusions, just like Fourier series expansion does.

There are two main reasons for such a dilemma: (a) The spectral analysis approach is not a traditional methodology for time series community, and if without basis knowledge on complex analysis and concepts of system analysis, it is hard to understand and further apply to practical problems; (b) **Digital Filtering** technology is not sufficiently introduced to time series community, since spectral analysis is only powerful for time domain problems if accompanied with digital filtering.

More interestingly, we should mention that the moving average method, exponential smoothing method and even the Holt’s method ([Hamilton 1995](http://dx.doi.org/10.1080/03081089508817747), [Hyndman et al. 2008](http://dx.doi.org/10.1016/j.techvoc.2007.09.007), [Hyndman and Athanasopoulos 2018](http://dx.doi.org/10.1080/00223222.2018.1410581)) are merely the special cases of SADFA, spectral analysis and digital filtering approach. Plus, it is also powerful for time series decomposition just like STL, X11 and so on do. We will make clear this later.

To this end, in this subsection we aim to review the issues of Spectral Analysis, and in the next subsection we will give a brief on Digital Filtering.

The term *spectrum* is coined by one of the greatest scientists, Sir Isaac Newton (1642-1726), when he studied the decomposition problem of light, declaring a white light could be split into many different lights with different colors by a prism. The color of a light beam is determined by its frequency, red for lower frequency and blue for higher frequency. The conceptual philosophy was borrowed by Joseph Fourier (1768-1830), a great French engineer, in 1822. He pointed out that a periodic continuous function $x(t)$ with some other proper properties, namely Dirichlet’s Conditions ([Mani et al. 1997](http://dx.doi.org/10.1016/j.techvoc.2007.09.007)), could be decomposed into the sum of infinite-many sine functions
with different frequencies. For each frequency, we term it as a spectral line in the spectrum $X(\Omega)$ of $x(t)$, where $X(\Omega)$ is the Fourier transform (or specially, the Fourier series) of $x(t)$, and $\Omega$ denotes the continuous frequency like $2\pi \times 5H\text{z}$, $2\pi \times 20H\text{z}$, $2\pi \times 8K\text{Hz}$, $2\pi \times 600MHz$, $2\pi \times 2.4GHz$ and so on.

The Spectral Analysis aims to build the close (or equivalent) relation between the time domain analysis and the (Fourier) frequency domain analysis. Since the time series is in time domain, it is suitable to be transformed to frequency domain to have further compound analysis (spectral analysis) and processing (digital filtering).

The Spectral Analysis consists of two part: (a) the spectral analysis for a time series $x(n)$, and (b) the spectral analysis for a discrete system $H(z)$. The spectral analysis for time series aims to figure out the spectral properties of a time series, while the spectral analysis for a discrete system aims to decide which type of system to be designed such that it can transform a time series $x(n)$ into a desired new time series $y(n)$, which is more interesting. For example, please recall an original time series $x(n)$ and the new time series $y(n)$ after difference operator defined by the system $H(z) = 1 - z^{-1}$. Actually it is the spectral properties of $x(n)$ that admits the power of the system $H(z)$. Specifically, according to the spectral properties of $H(z)$ (namely $H(e^{jw})$), the system $H(e^{jw})$ can in the frequency domain wipe away the undesired part (namely the trend) of $x(n)$. Note that it is the trend that makes $x(n)$ a non-stationary process in the variant mean sense. This is the philosophy of Digital Filtering which will be detailed later.

The Fourier Spectral Analysis for time series contains four parts (Yang 2009, Chaparro and Akan 2018, Diniz et al. 2010): (a) Fourier Transform for a continuous time function $x(t)$ (FT); (b) Fourier Series Expansion for a continuous periodic time function $x_\sim(t)$ (FS); (c) Fourier Transform for a discrete time function, namely time series, $x(n)$ (DTFT, discrete time Fourier transform); and (d) Fourier Series Expansion for a discrete periodic time function, periodic time series, $x_\sim(n)$ (DFS, discrete Fourier series). For notation simplicity, we in the following still use $x$ to uniformly denote a periodic time function $x_\sim$.

Suppose $x(t)$ is a periodic continuous time function, and the period of it is $T$, meaning the angular frequency is $\Omega_0 = 2\pi / T$. If it satisfies the Dirichlet’s conditions (Mani et al. 1997), we can decompose $x(t)$ into the sum of infinite-many complex-valued continuous sine functions. Namely

$$x(t) = \sum_{n=-\infty}^{\infty} X(n\Omega_0)e^{jn\Omega_0 t}, \quad (13)$$

where $X(n\Omega_0)$, Fourier Series, denotes the coefficient (or proportion) that the complex sine component $e^{jn\Omega_0 t}$ takes to compose a function $x(t)$. The $X(n\Omega_0)$ is mathematically given as

$$X(n\Omega_0) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-jn\Omega_0 t}dt. \quad (14)$$
Clearly, $X(n\Omega_0) =: X(n) =: X_n$ here is a complex variable. Thus $X_n = |X_n|e^{j\varphi_n}$, where $|X_n|$ denotes the modulus and $\varphi_n$ denotes the argument.

**Remark 9.** Note that the FS defined in (13) is actually equivalent to the FS defined by the sum of real trigonometric functions (Chaparro and Akan 2018),

$$x(t) = A_0 + 2\sum_{n=1}^{\infty} \left[ A_n \cos(n\Omega_0 t) + B_n \sin(n\Omega_0 t) \right]$$

$$= X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(n\Omega_0 t + \varphi_n),$$

where $|X_n| = \sqrt{A_n^2 + B_n^2}$ and $\varphi_n = -\tan^{-1}(B_n/A_n)$. The definitions of $A_n$ and $B_n$ could be found in Chaparro and Akan (2018). The story holds due to Euler’s formula of $e^{jn\Omega_0 t} = \cos(n\Omega_0 t) + j\sin(n\Omega_0 t)$, meaning we can use this trick to simplify the definition of (15) as (13). However, the result is that this trick brings us to complex domain. The rewards is better analytical properties.

Note that a **continuous periodic** function in time domain admits a **discrete spectrum** in frequency domain.

When it comes to a continuous aperiodic time function, the story becomes a bit more complex.

Suppose $x(t)$ is a continuous aperiodic time function satisfying $x(t) \in L_2$ and Dirichlet’s conditions, we could have the Fourier transform of $x(t)$ as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} \, d\Omega,$$

where $X(j\Omega)$ is FT of $x(t)$ and is defined as

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} \, dt,$$

where $\Omega = 2\pi f$ ($f$ is frequency with the unit of Hz) is angular frequency with the unit of rad/s.

Note that a **continuous aperiodic** function in time domain admits a **continuous spectrum** in frequency domain.

If we investigate the Fourier Transform of a periodic function $x(t)$ instead of Fourier Series, we have

$$X(j\Omega) = \int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} X(n\Omega_0)e^{j\Omega_0 t} \right] e^{-j\Omega t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} X(n\Omega_0) \int_{-\infty}^{\infty} e^{j(n\Omega_0 - \Omega)t} \, dt$$

$$= 2\pi \sum_{n=-\infty}^{\infty} X(n\Omega_0) \delta(\Omega - n\Omega_0),$$

where $\delta(\cdot)$ denotes the dirac function.

Equation (18) indicates that the Fourier Transform of a continuous periodic function is infinity at some discrete frequency points while zero elsewhere. This kind of spectrum distribution is what
we called Line Spectra. The regular FT of a aperiodic function is thus a Continuous Spectra. Particularly, the dirac function has Flat Spectra since its FT keeps constant over the frequency domain.

After sampling (Chaparro and Akan 2018), a continuous function will become a discrete one, which is a time series. Instead, we can also directly collect data at some discrete time points as a time series. Based on, and similar to the TF and TS for continuous cases, we can define the Discrete Time FT, DTFT, for a aperiodic time series.

Let \( w = \Omega T_s \) (\( T_s \) is sampling period in time axis, and thus \( f_s = 1/T_s \) is sampling frequency) denote the Digital Frequency with the unit of rad, we have the DTFT for aperiodic \( x(n) \) as

\[
x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jwn} dw,
\]

where \( X(e^{jw}) \) is DTFT of \( x(n) \) and given as

\[
X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}.
\]

Since we have \( \Omega t = 2\pi ft = 2\pi f n T_s = \Omega n T_s = wn \). Thus \( w \) is the discrete counterpart of \( \Omega \). Thus we term it as Digital Frequency.

**Proposition 1.** In digital frequency domain, \( w = \pi \) denotes \( w \) is the highest frequency and \( w = 0 \) denotes \( w \) is the lowest frequency. Since \( H(e^{jw}) \) is a periodic function over \( w \) with period of \( 2\pi \), we have \( w = \pi + 2k\pi \) as the highest frequency and \( w = 2k\pi \) as the lowest frequency, where \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \). The frequency at \( w = \pi/2 \) is thus the intermediate frequency. A frequency \( w \) is higher if it is closer to \( \pi + 2k\pi \), and lower if closer to \( 2k\pi \).

Proposition 1 stands by the sampling theory (Yang 2009, Chaparro and Akan 2018, Diniz et al. 2010). Intuitively, since \( \Omega_s \geq 2\Omega \), \( \forall \Omega \) defined in (17), by Sampling Theorem (Yang 2009, Chaparro and Akan 2018, Diniz et al. 2010), we have \( w = \Omega T_s \leq \Omega_s T_s/2 = 2\pi f_s T_s/2 = \pi \). Thus the \( w = \pi \) is the highest frequency.

Note that a discrete aperiodic function in time domain admits a continuous but periodic spectrum in frequency domain. The periodicity over \( w \) is easy to show by definition that \( X(e^{j(w+2\pi)}) = X(e^{jw}) \), meaning the period is \( 2\pi \). The periodicity is a significant nature of the spectrum over digital frequency domain.

Going here, it is interesting to mention the relationship between (20) and Definition 3 of \( X(e^{jw}) = X(z)|_{z=e^{jw}} \).

Now suppose \( x(n) \) is periodic with the period of \( N \), we then have the DFS as

\[
x(n) = \frac{1}{N} \sum_{k=-N}^{N} X(k)e^{j\frac{2\pi}{N}nk},
\]
where $X(k)$ is DFS of $x(n)$ and given as

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi}{N}nk}. \quad (22)$$

Note that a \textbf{discrete periodic} function in time domain admits a \textbf{discrete and periodic spectrum} in frequency domain. The periodicity is also $N$ \cite{Chaparro and Akan 2018}.

\textbf{Remark 10.} Interestingly, the periodicity in time domain leads to the discreteness in frequency domain, and the discrete spectrum is sampled from the continuous spectrum of the aperiodic function which is a period of the periodic time function. The conclusion stands both for continuous case and for discrete case.

Among FT, FS, DTFT and DFS, only the DFS seems reasonable to be numerically calculated and digitally saved in a digital computer, since only it has both discrete time-domain expression and discrete frequency-domain expression. However, this requires $x(n)$ to be periodic, which is impossible in practice since most of $x(n)$ failed to show this property. This is thus the main reason that only the DFS is highly thought of in time series analysis to reconstruct a periodic time series as (15).

The trick here to handle this dilemma is to consider the collected time series as periodic although it maybe not in fact, meaning if the length of collected $x(n)$ is $N$, we then virtually consider $x(n)$ as a periodic time series with period of $N$. This trick allows us to use DFS to the interested time series. However, another practical issue is that we need to use a limited memory to save the results of DFS. For DFS, this seems impossible since the sum is taken from $-\infty$ to $\infty$. In consideration of the periodicity of DFS, we may pay our attention exclusively within a period of DFS, which brings the Discrete Fourier Transform (DFT).

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk}, \quad (23)$$

where $X(k)$ is DFT of $x(n)$ and given as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk}. \quad (24)$$

Since we aim to analyze the spectral properties of a time series $x(n)$, we should have digital frequency variable $w$ in its transformed domain, just as DTFT (20) does. However, DFT, although powerful for digital calculation, failed to explicitly show this point. Thus we need to figure out the underlying relationship between DTFT and DFT.

\textbf{Lemma 6.} \textit{The DFT} $X(k)$ \textit{is the uniform sampling and approximation to the DTFT} $e^{jw}$. $X(k) = X(e^{jw})|_{w=\frac{2\pi}{N}k}$, meaning if we take $N$ points within a period of $w$ (namely $2\pi$), $w = 2\pi \frac{k}{N}$ and $k = N \frac{w}{2\pi}$, we have DFT from DTFT, where $N$ denotes the length of the time series $x(n)$. 
Proof of Lemma 6. See Chaparro and Akan (2018), Yang (2009), Diniz et al. (2010). □

Note that the calculation burden of DFT is heavy. In order to improve the calculation efficiency, the Fast Fourier Transform (FFT) algorithm is proposed (Cormen et al. 2009, Diniz et al. 2010, Chaparro and Akan 2018). However, the FFT is not a new transform as DTFT and DFT, just a fast implementation algorithm to DFT.

Now we have already finished the spectral analysis of a time series $x(n)$. Thus the second main topic of spectral analysis is for a discrete system $H(z)$ defined in (12).

As we showed in the end of Subsection 3.1, the frequency response of a LTI system $H(z)$ is defined as $H(e^{jw}) = H(z)|_{z = e^{jw}}$. Suppose $X(e^{jw})$ and $Y(e^{jw})$ are DTFT of $x(n)$ and $y(n)$. We have $Y(e^{jw}) = H(e^{jw})X(e^{jw})$ by Lemma 5.

Suppose $x(n) = e^{jw'n}$, it is easy to show that the DTFT $X(e^{jw})$ is as

$$X(e^{jw}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(w - w' + 2\pi k). \quad (25)$$

Thus we have

$$Y(e^{jw}) = 2\pi \sum_{k=-\infty}^{\infty} H(e^{jw})\delta(w - w' + 2\pi k). \quad (26)$$

Equation (26) immediately imply that the output $y(n)$ is also a complex sine function. Its frequency remains same (namely $w'$) with $x(n)$. The only difference is that the modulus and argument have been modulated in frequency domain by $H(e^{jw})$. Thus we have

$$Y(e^{jw}) = |H(e^{jw})|e^{jw'n + \varphi(w)}), \quad (27)$$

where $|H(e^{jw})|$ denotes the modulus of $H(e^{jw})$ given the frequency $w$ and $\varphi(w)$ denotes the corresponding argument of $H(e^{jw})$. Since $w$ is a variable, $|H(e^{jw})|$ and $\varphi(w)$ are functions, and defines the Amplitude-frequency Response and Phase-frequency Response.

Here it is now theoretically sufficient to give the motivation and purpose of spectrum-based time series analysis and processing.

Motivation 2. Recall from the definition of DTFT that $x(n)$ is a sum of infinite-many complex sine functions $e^{jwn}$. From (27), we can see if we can carefully design a system $H(e^{jw})$ such that $|H(e^{jw})|$ is small in some frequency points $w_i$, $i = 0, 1, 2, \ldots$ or a frequency interval $[w_{low}, w_{up}]$, and remains 1 elsewhere, then the components of $x(n)$ represented by frequencies at $w_i$, $i = 0, 1, 2, \ldots$ or in $[w_{low}, w_{up}]$ will be attenuated (or wiped out if small enough). By doing so, if we remain only the interested components, we could finish a time series transformation by spectral analysis approach from $x(n)$ to $y(n)$.
Note that it is also possible to design a proper $|H(e^{jw})|$ such that every frequency components of $x(n)$ will be properly handled (amplified, kept, and/or attenuated) as desired. Thus the interested time series will be finely transformed into $y(n)$ as expected.

As a quick intuitive understanding, readers could have a short glance to Section 7.

As mentioned before, we are interested in the techniques of Digital Filtering, meaning passing the desired frequency components and stopping the undesired frequency components. Thus we in the following subsection will briefly introduce it.

### 3.3. Digital Filtering

As stated before, a digital filter (DF) is used to process a time series in frequency domain. In practice, the most used DFs are with the form of LTI system, like exponential smoothing method (ETS), moving average method (MA), and difference operator in ARIMA. Details on ETS, MA and difference operator will be showed later. Specifically, the LTI system dynamics of a DF is given as (9) in time domain, (12) in Z domain, and in frequency domain if $z := e^{jw}$. The spectral analysis and digital filtering are mutually complementary. Spectral analysis tells us what to do, meaning which frequency components should be preserved or wiped, and Digital Filtering tells us how to do, meaning which LTI DF could alter the spectra of a time series as desired before. The two are all indispensable for time series analysis and processing.

The digital filter is the discrete counterpart of analog filter. Analog filter works for continuous time function and system (8), while digital filter cares about the discrete time series and system (9).

Typically, the LTI digital filter has four types, called low-pass, high-pass, band-pass and band-stop filter, respectively. The low-pass filter only allows the low frequency components, near $w = 0$, to pass and denies the others. The high-pass filter only allows the high frequency components, near $w = \pi$, to pass and denies the others. The band-pass filter only allows the frequency components within the given frequency subinterval to pass and denies the others. The band-stop filter, opposite to band-pass, denies the frequency components within the given frequency subinterval to pass and allows the others. The ideal diagrams of four mentioned digital filters are given in Figure 6.

The breakpoint(s) that $|H(e^{jw})|$ suddenly changes its value from 1 to 0, or vice versa, in Figure 6 is/are called cut-off frequencies. Note that except $|H(e^{jw})|$ is periodic over $w$ with period of $2\pi$, it is symmetrical as well within each period (Diniz et al. 2010, Chaparro and Akan 2018).

However, unfortunately, the patterns showed in Figure 6 are just mathematically ideal, meaning they are physically impossible to make real. Because the impulse response $h(n)$ of the ideal digital filter, in mathematical sense, requires $h(n) \neq 0$, when $n < 0$. This contradicts the causality claimed in Lemma 3. For more, see Diniz et al. (2010).
Thus in practice, we instead design the approximated scheme to ideal filters. Without loss of generality, we take the low-pass filter and band-stop filter as examples, showed in Figure 7.
From Figure 7(a),(b), we can know that there exists only one cut-off frequency ideally, but two cut-off frequencies practically. \( \omega_l \) denotes the lower cut-off frequency while \( \omega_u \) denotes the upper one, meaning we guarantee the attenuation to the components with frequency lower than \( \omega_l \) to be no less than \( \delta_l \), and the attenuation to the components with frequency higher than \( \omega_u \) to be no larger than \( \delta_u \). The logic keeps similar to band-stop filter in Figure 7(c),(d). In practice, the performances of a designed low-pass filter are satisfying if:

1. The gap between 1 and \( \delta_l \) is small enough;
2. The gap between 0 and \( \delta_u \) is small enough;
3. The gap between \( \omega_l \) and \( \omega_u \) is small enough, and they all are close enough to \( \omega_0 \);
4. The order of the filter, namely \( p \) and \( q \) in \( \text{(12)} \), is small (the smaller, the better).

However, the dilemma is that in order to optimize the requirements 1 \( \sim \) 3, we must accept the order to be high enough.

There are typically two types of digital filters based on LTI system and spectra analysis, the Infinite Impulse Response (IIR) filter and Finite Impulse Response (FIR) filter \( \text{(Diniz et al. 2010)} \). Mathematically, the LTI system \( \text{(12)} \) specifies the IIR filter. When a IIR filter takes its special case of without denominator polynomial, it becomes a FIR filter, given in \( \text{(28)} \).

\[
H(z) = b_0 + b_1z^{-1} + b_2z^{-2} + ... + b_qz^{-q}. \quad \text{(28)}
\]

IIR and FIR have numerous different natures including the filter performances given fixed order, the designing methods, the linear phase properties and so on. For more, one is referred to \( \text{Diniz et al. (2010)} \).

Worthy of mentioning is that the IIR and FIR model can not only generate the typical four types of filters showed in Figure 6, but also design some special types of filters like comb filter and difference filter \( \text{(Diniz et al. 2010)} \). As an example, we show the frequency response of comb filter here, which is also useful in time series processing. Later we will show the nature of \( L \)-lag seasonal difference operator is actually also a type of digital comb filter.

Two special types of comb filter are mathematically given as

\[
H(z) = \frac{1 + \rho}{2} \frac{1 - z^{-N}}{1 - \rho z^{-N}}, \quad \text{(29)}
\]

and

\[
H(z) = \frac{1 - \rho}{2} \frac{1 + z^{-N}}{1 - \rho z^{-N}}, \quad \text{(30)}
\]

where \( \rho \) is the filter parameter and \( N \) is order that determine the performances. If we set \( N = 5 \) and \( \rho = 0.9 \), we have the amplitude-frequency responses in Figure 8.
Figure 8 immediately indicates that if we want to wipe out some unwanted frequency components of a time series, we just need to design a comb filter as Figure 8(a), and if we want to preserve some desired frequency components of a time series, we just need to design a comb filter as Figure 8(b), to process it. For more on comb filter, see Diniz et al. (2010).

Up to now, the most intriguing designing methods to IIR are Butterworth method, Chebyshev method, Elliptic method and so on; to FIR are window method (like rectangular window, Kaiser window, and so on), optimization methods (like weighted least-squares method, Chebyshev optimal approximation method, and so on) et al. All the above-mentioned methods are canonical designing ones in signal processing community which have already been implemented by MATLAB. One can find the detailed mathematical derivations in Diniz et al. (2010) and study the sample MATLAB codes shared by Diniz et al.

All the filters this paper designed are based on IIR model and Elliptic method. In order to help readers easily check the powerful ARMA-SIN method this paper present, all the MATLAB source codes we use are available from the corresponding author or online upon the publication of this paper.

3.4. Stochastic Process and Wold’s Decomposition Theorem
After introducing the Linear System Theory, Spectral Analysis and Digital Filtering, it is sufficient to pay attention back on S-ARIMA to investigate its nature. Recall that in Introduction we have already point out the philosophy of ARMA and S-ARIMA. They in fact aim to model the dynamics of a stochastic process so that we can use the past information from collected time series to satisfyingly predict the future. Mentioning this, we cannot ignore the reputed Wold’s Decomposition Theorem in stochastic process analysis.

Firstly we give the strict mathematical definition of the Wide-sense stationary (WSS) stochastic process.

**Definition 4 (Papoulis and Pillai (2002)).** A real-valued stochastic process \( x(t) \) is WSS if it satisfies:
• **Invariant Mean:** $E\{x(t)\} = \eta$, where $\eta$ is a constant;

• **Invariant Autocorrelation:** $E\{x(t_1)x(t_2)\} = E\{x(t_1 + \tau)x(t_1)\} = R(\tau)$, meaning it only depends on $\tau := t_2 - t_1$, having nothing to do with $t_1$.

Invariant autocorrelation immediately admits the **Invariant Variance**, since $E\{x(t)\}^2 = R(0)$. Then we should turn to Wold’s Decomposition Theorem.

**Theorem 1 (Wold’s Decomposition Theorem)** ([Papoulis and Pillai 2002](#)). Any WSS stochastic process $x(n)$ could be decomposed into two subprocess: (a) Regular process; and (b) Predictable process. Namely

$$x(n) = x_r(n) + x_p(n),$$

(31)

where $x_r(n)$ is a regular process and $x_p(n)$ is a predictable process. Furthermore, the two processes are orthogonal (meaning uncorrelated): $E\{x_r(n + \tau)x_p(n)\} = 0$.

The detailed concepts of Regular Process, also known as Rational-spectra process from the spectral analysis perspective, and Predictable Process, also known as Line-spectra process, could be found in [Papoulis and Pillai 2002](#). Intuitively, the **regular process** is mathematically as $x_r = ARMA(p,q)$, and **predictable process** $x_p$ is the sum of complex-valued exponentials (also known as complex sine function) $e^{j\omega_i n}$, for some $i$. From (25), we know that the DTFT of $e^{j\omega_i n}$ is the delta function, meaning it is a impulse in spectrum. It is this reason that the predictable process is also called as **Line Spectra process**. As for the regular process, instead of the line (discrete) spectra, it is generally continuous over frequency domain so that it is also called as **Continuous Spectra process**. Interestingly, the power spectra of the such regular process could be represented by the ratio of two polynomials. Thus the regular process is also known as **Rational Spectra process**. One special case of Continuous Spectra is **Flat Spectra** with constant values over the frequency domain. The spectra of a time series is a Flat Spectra if and only if the time series is a white (uncorrelated) process.

As an example to Wold’s Decomposition Theorem, please see Subsection 2.5 (The Case of Using SARIMA), in which the regular process is $x^0 = ARMA(4,2)$ and the predictable process is $sin(5t)$. Note that $e^{j\omega n}$ is the analytical expression of $sin(\omega n)$ in complex-valued domain, and $sin(\omega n)$ is the physical implementation of $e^{j\omega n}$ in time domain. They are philosophically equivalent.

Thus if a WSS $x$ is without the $x_p$ part, we can use the ARMA method to model it and use the past information to train the coefficients so that we can refactor the dynamics of the interested $x$. This is the philosophy of Box-Jenkins methodology ([Box et al. 2015](#)).

However, what if the stochastic process $x(n)$ is non-stationary (see Remark 11) or is a irrational-spectra WWS one. Irrational-spectra WWS in this paper means it contains line spectra, namely
the sum of sine functions. The problem becomes easier since the DFT admits the trend part of non-
stationary $x(n)$ could be approximated by the sum of sine functions (DFS), meaning we regard the
collected time series is from a periodic series with large enough period, so that the non-stationary
case could be included by irrational-spectra WWS case. In other words, the trend part in this sense
could be modelled as the predictable process, namely line spectra process.

Remark 11. Note that we in this paper only concerns the non-stationarity in the variant mean
sense. For variant variance case, let’s discuss it in the near future in our another paper. Because
the methodology of variant variance case is not based on Spectral Analysis and Digital Filtering.
Note again that the nature of trend is variant mean.

3.5. The Philosophical Nature of ARIMA and SARIMA
As discussed before, philosophically, as long as we can take some actions so that the line-spectra
components could be wiped out, we can use the remainder of the original time series, which is
rational-spectra wide-sense stationary, to model as ARMA process. We term such procedure as
regularization, because the ARMA process is also known as regular process. Note that the spectra
of sine functions are vertical lines with large enough values (delta function), they are easy to
identify from the nearby continuous spectra of ARMA process.

Thus, actually, the difference operator of ARIMA and seasonal difference operator of SARIMA
do such things, meaning they are actions to handle the line-spectra components to some extent. In
next section, we will make clear this point from the perspective of SADFA (spectral analysis and
digital filtering), that is, we will figure out the analytical nature of ARIMA and SARIMA.

4. Why Are the ARIMA and SARIMA not Sufficient
4.1. Why ARIMA and SARIMA Make Sense Somewhere
As mentioned in the previous section, the ARIMA and SARIMA attempt to regularize a WSS
stochastic process by difference operator and seasonal difference operator. In this section we will
investigate the analytical nature of ARIMA and SARIMA from the perspective of SADFA. Specif-
ically, we should pay our attention on the difference operator and seasonal difference operator.

Theorem 2. The nature of $d$-order difference operator is actually a high-pass digital filter that
denies the low frequencies. Since the trend of a time series is generally the low frequency com-
ponents, the $d$-order difference operator makes sense to regularize a stochastic process.

Proof of Theorem 2. The transfer function of the $d$-order difference operator is given as

\[ H(z) = (1 - z^{-1})^d, \]  

and the amplitude-frequency response is as

\[ |H(e^{jw})| = |(1 - e^{-jw})^d| = \sqrt{2 - 2\cos(w)}^d. \]
Equation (33) immediately admits the theorem, since in the interval \([0, \pi]\), \(|H(e^{jw})|\) is increasing from zero. Intuitively, see Figure 9.

**Theorem 3.** The nature of \(L\)-lag seasonal difference operator (\(L\)-SDO) is actually a comb digital filter that denies some frequency components with \(w = 2k\pi/L, k = 0, 1, 2, ...,\). Since the seasonal components (namely periodic components) with period \(L\) has period \(L\) in its spectra as well, meaning the frequencies of seasonal components are \(w = 2k\pi/L, k = 0, 1, 2, ..., L - 1\), the \(L\)-lag difference operator makes sense to regularize a irrational-spectra wide-sense stationary stochastic process.

**Proof of Theorem 3.** The transfer function of the \(L\)-lag seasonal difference operator is given as

\[
H(z) = 1 - z^{-L},
\]

and the amplitude-frequency response is as

\[
|H(e^{jw})| = |1 - e^{-jLw}|
= |1 - \cos(Lw)| + j\sin(Lw)|
= \sqrt{2 - 2\cos(Lw)}.
\]

Equation (35) immediately admits the theorem, since in the interval \([0, \pi]\), \(|H(e^{jw})|\) are zero-valued at \(Lw = 2k\pi, k = 0, 1, 2, 3, ....\). Intuitively, we consider a sine series \(x(n)\) with period \(L = 12\) \((t = 0 : 2\pi/L : 100)\) and take 12-lag seasonal difference over it. The results are illustrated in Figure 10.

**4.2. Why Insufficient**

By now, Theorem 2 and Theorem 3 support the effectiveness of S-ARIMA somewhere. However, it is obvious that except the unwanted frequencies, \(d\)-order difference operator of ARIMA and \(L\)-lag seasonal difference operator of SARIMA also impact the desired frequencies which should remain unchanged, for example, the frequencies higher than \(w = \pi/3\) in Figure 9 have been significantly and unwantedly amplified, and the frequencies at and around \(w = 2k\pi/12, k = 0, 2, 3, ..., 11\), (that is, \(k \neq 1\)) in Figure 10 have been wiped away. This raises the insufficiency of S-ARIMA. Thus we have Theorem 4.
Theorem 4. The ARIMA and SARIMA model are theoretically insufficient since they also negatively impact the desired frequency components, meaning the desired frequencies of ARMA process will also be significantly distorted.

Proof of Theorem 4. Due to Theorem 2, Theorem 3 and the statements themselves in this theorem, the conclusion stands. □

5. The General ARMA-SIN Model

Since this paper concerns the wide-sense stationary stochastic process, and according to Theorem 1, a WSS process could be decomposed into an ARMA process and a sum of sine processes, therefore, this paper refers to our methodology as ARMA-SIN. ARMA is for regular process (ARMA process), while SIN is for predictable process (sum of sine processes). Note that the variant mean non-stationary process could be handled by DFT, thus, as discussed before, it is the special case of wide-sense stationary stochastic process.

Besides, our ARMA-SIN is based on SADFA (spectral analysis and digital filtering approach) and Box-Jenkins methodology. In detail, when we have a general time series, we first use SADFA to transform it to be rational-spectra wide-sense stationary and then use Box-Jenkins methodology to model the transformed series. Thus, SIN is for SADFA, while ARMA is for Box-Jenkins methodology. In summary, ARMA-SIN proposed in this paper is not only a model for stochastic process but also a methodology for time series analysis and processing.

In Algorithm 5, we will detail the methodology of ARMA-SIN.

Remark 12. Note that ARMA-SIN implicitly finishes the time series decomposition in Algorithm 5. Thus the ARMA-SIN is the more general time series decomposition approach compared to STL, X11 and so on, which are based exponential smoothing (a special case of ARMA-SIN, see Theorem 5), and are more computationally ineffective (requires many time of iterations). Besides, STL, X11 et al. can only decompose the time series into two special components, the trend part and the seasonal part. However, our SADFA can decompose the time series into many components.
Algorithm 1 ARMA-SIN

1: **Spectral Analysis**: transform the interested time series into Fourier frequency domain and identify the line-spectra;
2: **Digital Filtering**: design the proper digital filter with proper cut-off frequencies such that the interested time series could be regularized;
3: **ARMA**: follow the standard Box-Jenkins methodology to model the regularized process as a ARMA one;
4: **Forecast**: predict the future with two independent process, ARMA (regular) and SIN (predictable) respectively, and integrate the results together.

rather than two, and any forms beyond trend and the seasonal (periodic). For example, if a time series contains many types of periodic fluctuations like daily, monthly, seasonal, yearly, and even decades-spanned, The traditional decomposition methods will no longer works. However, for our SADFA, we just need to transform the time series into frequency domain, identify interested frequencies of daily, monthly, seasonal, yearly, or decades-spanned components, respectively, and then use properly designed filter to extract the desired components and do further analysis/forecasting.

6. The Derivation for Solutions of Scenarios of Warming-up

In this section, we will detail the derivation for solutions of scenarios of warming-up given in Section 2. For briefness, we will not repeat the standard modelling procedures of Box-Jenkins methodology (Box et al. 2015). Readers could finish it by themselves by reference to the shared MATLAB source codes.

As mentioned before, all the filters this paper designed are based on IIR model and Elliptic method (Diniz et al. 2010). In order to help readers easily check the powerful ARMA-SIN method this paper present, all the MATLAB source codes we use are unconditionally available from the corresponding author or online upon the publication of this paper.

**Remark 13.** Usually it is hard to decide the best parameters like cut-off frequencies at the first glance to the problem. However, we could try some times with plausible parameters and choose the best among them. This seems tedious but unavoidable in practice. Because there is no free lunch, advanced method means complex coding, complex parameters selecting and/or complex calculation burden. The use of (seasonal) difference operator seems really delightful for all of us. Nevertheless, the trade-off is the unsatisfying performances and powerlessness somewhere.

**Remark 14.** Note that the spectra of ARMA process is continuous, admitting no sharp changes. It means all sharp changes in the original spectra of a time series should be line-spectra introduced by the (sum of) sine function(s). Thus our concerns are to identify these outliers (outstanding
frequency points) or outstanding frequency subinterval in spectra and design proper digital filter to handle them.

### 6.1. The Case of Variant Mean

The time series is showed in Figure 1 (c). The trend is linear. According to DFT theory (see Subsection 3.2), the linear trend could be represented by the finite sum of sine functions with acceptable approximation error, meaning the linear trend is with the line-spectra within a subinterval. In order to separate (extract) the linear trend, we should design a high-pass filter which denies the low-frequency component (trend) to pass and allows the high-frequency components to go through without any changes. Since we use the Elliptic method to design a IIR filter (the function `ellipord` in MATLAB, for more, see its reference page in MathWorks (2018b)), we set the parameters in Table 6.1.

The specific meanings and detailed usage of parameters in Table 6.1 should be found in Diniz et al. (2010) or the reference page of the function `ellipord` in MathWorks (2018b).

This high-pass filter actually defines the system (4).

The spectral analysis results are given in Figure 11.

---

**Table 5** The filter parameters for the case of variant mean

| Wp | Ws | Rp | Rs |
|----|----|----|----|
| 0.25 | 0.2 | 1  | 10 |

---

![Figure 11](image-url)
Table 6  The filter parameters for the case of variant autocorrelation

|   | Wp | Ws | Rp | Rs |
|---|----|----|----|----|
| Value | 0.26 | 0.22 | 1 | 10 |

From Figure 11 we can see the 1-order difference operator significantly distorts (amplifies) the spectra in high frequency interval, although it is powerful to detrend. However, our ARMA-SIN keeps 1 (20 log(1) = 0 dB) in higher frequency area, making no changes to desired components. This is why our ARMA-SIN outperforms ARIMA. Note that the sub-figure (b) of Figure 11 is given in logarithmic unit of decibel (dB). For more on decibel, see Diniz et al. (2010).

6.2. The Case of Variant Autocorrelation

The time series is showed in Figure 2 (c). This example has two concerns: (a) the line-spectra part is weak (0.25 in amplitude) compared to ARMA part (1 in amplitude); (b) the line-spectra part is with many frequencies instead of one. Because the frequency of sin function is increasing through time. In Figure 12, we can check the spectra of the time series. We can from Figure 12 find there exists some line-spectra in low frequency area with frequencies lower than about π/12 = 0.26. Thus we want to filter out them without changing other frequencies. Since we use the Elliptic method to design a IIR filter (the function ellipord in MATLAB, for more, see its reference page in MathWorks (2018b)), we set the parameters in Table 6.2.

This high-pass filter actually defines the system (5).

The spectral analysis results are given in Figure 12.

From Figure 12 we can see the 1-order difference operator significantly distorts (amplifies) the spectra in high frequency interval, although it is powerful to detrend. This is why our ARMA-SIN outperforms ARIMA.

6.3. The Case of Using SARIMA

This case concerns the periodic time series which is suitable for SARIMA. The time series is showed in Figure 3 (c). In Figure 13, we can check the spectra of the time series. We can see that there is an outstanding line-spectra around w = 0.5. Thus we want to design a band-stop digital filter to deny the outstanding frequency component. Since we use the Elliptic method to design a IIR filter (the function fdesign.bandstop in MATLAB, for more, see its reference page in MathWorks (2018a)), we set the parameters in Table 6.3.

This band-stop filter actually defines the system (6).

The spectral analysis results are given in Figure 13.

The spectra of the 12-lag seasonal difference operator could be found in Figure 10 as a comparison to Figure 13.
Wang et al.: ARIMA and SARIMA are not Sufficient

From Figure 13, we can see the 12-lag seasonal difference operator significantly distorts (amplifies) the spectra at and around $w = 2k\pi/12$, $k = 0, 2, 3, \ldots$, although it is powerful to wipe out the frequency at $w = 2\pi/12$ ($k = 1$). This is why our ARMA-SIN outperforms ARIMA.

6.4. The Case of Directly Estimating the Irrational-spectra Component

In this section we will use the information presented in spectra of a time series to directly estimate the irrational-spectra (line-spectra) component. The philosophy is based on DFT (24) and Inverse DFT (23) (Diniz et al. 2010, Yang 2009, Chaparro and Akan 2018).

Suppose the outstanding value of FFT of $x(n)$ is at $n = K$ and its value is $H_K$, then we have $w = 2\pi K/N$. Besides, by DFT (and/or FFT), the amplitude in time domain is given as $A = 2|H_K|/N$ and phase is as $\phi = \phi(H_k)$, where $|H_K|$ denotes the modulus of $H_K$ and $\phi(H_k)$ denotes the argument.

| Value  | Comments                                      |
|--------|-----------------------------------------------|
| Fpass1 | 0.158 First Passband Frequency ($\times \pi$) |
| Fstop1 | 0.16  First Stopband Frequency ($\times \pi$)  |
| Fstop2 | 0.165 Second Stopband Frequency ($\times \pi$) |
| Fpass2 | 0.168 Second Passband Frequency ($\times \pi$) |
| Apass1 | 1     First Passband Ripple (dB)              |
| Astop  | 20    Stopband Attenuation (dB)               |
| Apass2 | 1     Second Passband Ripple (dB)             |

Table 7 The filter parameters for the case of using SARIMA
(angle) of $H_K$. Note that $H_K$ is a complex-valued value. Therefore the line-spectra component of $x(n)$ with frequency of $w = 2\pi K/N$ has its expression in time domain as

$$x_p(n) = A \cos(wn + \varphi).$$

Specifically in this section, the spectra of the interested time series is as Figure 14.

From the figure we can directly know that $w = 0.2001$, and $A = 2 \times 49.09/N = 0.9721$ ($N = 101$). Besides, by checking the corresponding phase at this frequency $w = 0.2001$, we have $\varphi = -1.5440$. Thus (36) admits (7).
7. Investigation on Exponential Smoothing and Moving Average

7.1. The Nature of Exponential Smoothing and Moving Average

The issue of Time Series Composition has already been discussed in Remark 12. Thus in this section, we aim to discuss the nature of the canonical time series analysis methods like Exponential Smoothing method and Moving Average method (Hyndman et al. 2008, Hyndman and Athanasopoulos 2018).

Let’s firstly consider the exponential smoothing method.

**Theorem 5.** The nature of the exponential smoothing (ETS) method is a low-pass filter, meaning it allows the low-frequency components of a time series to pass and denies the high-frequency components. Thus ETS can extract the trend which is the low-frequency components, and discard the high-frequency fluctuating components. Mathematically, the exponential smoothing is given as $y_n = \alpha x_n + (1 - \alpha)y_{n-1}$, where $\alpha$ denotes the smoothing coefficient.

**Proof of Theorem 5.** The transfer function of the system defined by exponential smoothing should be

$$H(z) = \frac{\alpha}{1 - (1 - \alpha)z^{-1}}.$$  \hspace{1cm} (37)

Its frequency response is given as

$$H(e^{jw}) = \frac{\alpha}{1 - (1 - \alpha)e^{-jw}}.$$  \hspace{1cm} (38)

Thus the amplitude-frequency response is

$$|H(e^{jw})| = \frac{\alpha}{\sqrt{1 + (1 - \alpha)^2 - 2(1 - \alpha)\cos(w)}}.$$  \hspace{1cm} (39)

Clearly, in the interval $w \in [0, 2\pi]$, the curve of (39) is convex and bell-like, and the valley value is reached at $w = \pi$. Note that $|H(e^{jw})|$ is periodic over $w$ and the period is $2\pi$. Since the highest frequency is indicated by $w = \pi$ and lowest is by $w = 0$ and/or $w = 2\pi$, we should only pay attention on the interval $[0, 2\pi]$.

As an illustration and without loss of generality, we consider the case of $\alpha = 0.1$. We have

$$H(z) = \frac{0.1}{1 - 0.9z^{-1}}.$$  \hspace{1cm} (40)

Its frequency response is given as

$$H(e^{jw}) = \frac{0.1}{1 - 0.9e^{-jw}}.$$  \hspace{1cm} (41)

We in Figure 15 illustrate the amplitude-frequency response $|H(e^{jw})|$ of (41). Recall the fact in Proposition 1, Figure 15 indicates that the exponential smoother (ETS) is actually a low-pass
digital filtering, meaning it allows the low frequency components to pass and denies the high frequency components. Since the trend of a time series is a low-frequency time function (meaning it changes slowly), the ETS system allows it to pass; while the fluctuating components of a time series is a high-frequency time function (meaning it changes quickly), the ETS attenuates it. Thus the output of the such ETS system is the trend component of a time series.

Together with Theorem 5. Let’s figure out what happens to a time series $x(n)$ passing the ETS. Consider $t = 0 : 0.01 : 10$ and $T_s = 0.01$, meaning $n = 0 : 1 : 1001$. Let $x(n) = \sin(nT_s) + \text{randn(length}(n))$. Then Figure 16 gives the original time series $x(n)$ and the generated time series $y(n)$ by the ETS, in both time domain and frequency domain.

Figure 16 explicitly upholds the rightness of Theorem 5, meaning all fluctuating (high-frequency) components have been denied to pass through, but the trend part (low-frequency) has been preserved.

Then let’s investigate the nature of Moving Average.

THEOREM 6. The nature of the moving average (MA) method is a low-pass filter, meaning it allows the low-frequency components of a time series to pass and denies the high-frequency com-
ponents. Thus MA can extract the trend which is the low-frequency components, and discard the high-frequency fluctuating components. Mathematically, the moving average is given as

\[ y_n = \frac{1}{N} \sum_{i=0}^{N-1} x_{n-i}, \quad (42) \]

where \( N \) denotes the order of MA.

**Proof of Theorem 6.** The transfer function of the system defined by moving average should be

\[ H(z) = \frac{1}{N} \sum_{i=0}^{N-1} z^{-i} = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}}, \quad (43) \]

Its frequency response is given as

\[ H(e^{jw}) = \frac{1}{N} \frac{1 - e^{-jwN}}{1 - e^{-jw}}, \quad (44) \]

Thus the amplitude-frequency response is

\[ |H(e^{jw})| = \frac{1}{N} \frac{\sin(wN/2)}{\sin(w/2)}. \quad (45) \]

Clearly, in the interval \( w \in [0, 2\pi] \), the curve of (45) is high near \( w = 0 \) and low around \( w = \pi \). Thus the moving average is a low-pass digital filter.

As an illustration and without loss of generality, we consider the case of \( N = 8 \). We have its amplitude-frequency response as

\[ |H(e^{jw})| = \frac{\sin(8w/2)}{8 \sin(w/2)}. \quad (46) \]

We in Figure 17 illustrate the amplitude-frequency response \( |H(e^{jw})| \) of (46).

---

**Figure 17**  The amplitude-frequency response of moving average method
7.2. Why Are ETS and MA Problematic

Up to now, we have made clear the nature of Time Series Decomposition (see Remark 12), and ETS, MA (see Theorem 5, Theorem 6). The Holt’s method (Hyndman et al. 2008, Hyndman and Athanasopoulos 2018) is similar to analyse. For briefness, we omit it here. They are powerful in many application scenarios. However, we should still point out their theoretical insufficient:

- Potential to negatively impact (attenuate or amplify) the other wanted frequencies, and cannot completely wipe out the unwanted components (the attenuation to high-frequency is not perfectly zero);
- The Time Delay issue.

Compared to the Time Delay issue, the first one is trivial. Because practically we do not expect the most ideal algorithm performances, as long as the bias is acceptable. As an intuitive understanding, we look again the example in Subsection 7.1. We know that the truth of $y(n)$ should be $x_p(n) = \sin(nT_s)$. Thus we need to figure out how close between $y(n)$ and $x_p(n)$, see Figure 18.

![Figure 18 The time delayed ETS](image)

Obviously, there is a time delay in $y(n)$ compared to its truth $x_p(n)$. However, this is easy to investigate.

Recall in (27) that besides the amplitude-frequency response, there also exits the phase-frequency response, meaning a component $e^{j(w'n)}$ must accept a phase delay of $\varphi(w')$ when passing through the system $H(e^{jw})$ so that the generated $y(n)$ is not only with the change of $|H(e^{jw'})|$ in amplitude, but also with the change of $\varphi(w')$ in phase. This is the nature of a LTI system so that it is unavoidable. Thus the real-time filtering method, meaning no phase delay, is nonexistent. The term real-time here means the times series is in sequential, not in block. That means if $n$ denotes the current discrete time index, it is impossible for us to know the data at $n+1$.

However, it is still possible to design a zero-phase (no-phase-delay) LTI Digital Filter for block data.
**Theorem 7.** Suppose \( x(n) \) is the interested time series and \( H(z) \) is designed LTI Digital Filter. Let \( H(z) \) be shorted as \( H \), and \( Hx(n) \) denote that \( x(n) \) passes through the system \( H(z) \). Let the operator \( z(n) = W[x(n)] \) generate the time series \( z(n) \) such that \( z(n) = x(−n) \). If \( y(n) \) is generated by the following four procedures:

1. \( u(n) = Hx(n) \);
2. \( v(n) = Wu(n) \);
3. \( w(n) = Hv(n) \);
4. \( y(n) = Ww(n) \);

, then \( y(n) \) has no phase delay compared to \( x(n) \).

**Proof of Theorem 7.** Easy to show that

\[
Y(e^{jw}) = W^*(e^{jw}) = [V(e^{jw})H(e^{jw})]^* = [U^*(e^{jw})H(e^{jw})]^* = U(e^{jw})H^*(e^{jw}) = X(e^{jw})|H(e^{jw})|^2,
\]

where \( H^* \) denotes the complex conjugate of a complex variable \( H \). Equation (47) implies the generating procedure only introduces the changes to the amplitude of \( x(n) \), doing nothing to the phase of \( x(n) \). Thus there is no time delay between \( y(n) \) and \( x(n) \). Note that if the DTFT of \( x(n) \) is \( X(e^{jw}) \), the DTFT of \( x(−n) \) will be \( X^*(e^{jw}) \) (easy to show by the Definition of DTFT). Note again that the term \( x(−n) \) admits the non-causality (namely, non real-time). □

As an intuitive understanding to Theorem 7, see Figure 19, which is the zero-delay counterpart of Figure 18.

![Figure 19](image.png)

**Figure 19** The no-time-delay ETS

Surely, the same story keeps in Moving Average method. For briefness, we dismiss the detailed discussion to MA.
7.3. Improved Exponential Smoothing and Moving Average

Theorem 7 suggests the improved ETS and MA method having no time delay, which is more useful in time series analysis and processing for block data which has already been completely collected before doing analysis. Actually, all the LTI digital filters we use in simulations of this paper are based on the zero-phase trick introduced by Theorem 7.

8. Conclusions

We have in this paper discussed the natures, philosophies, effectiveness, insufficiencies and improvements of several canonical time series analysis techniques like ARIMA, SARIMA, ETS, MA and decomposition methods (STL, X11), from the perspectives of Linear System Analysis, Spectra Analysis and Digital Filtering. Plus, we present the more general ARMA-SIN methodology. Up to this, we are relatively successful in bridging together the digital signal processing community and time series analysis community. However, we should admit that although the displayed ARMA-SIN is powerful, general and interesting, it is relatively hard for beginners to understand, master and make use, compared to the simple exponential smoothing, moving average, ARMA, SARIMA and so on. Applying this method requires enough experience to identify the interested spectral components and setup the proper parameters like cut-off frequencies to extract or discard them by designing the proper digital filter.

Since there is no free lunch in this world, this point should not be hard to follow. Advanced algorithms and methodologies could have more satisfying performances, if we can bear the complex coding, complex parameters selection, high calculational burden and so on. For example, we can compare the ARIMA and ARMA-SIN, the traditional ETS and the improved ETS in this paper (Theorem 7). Although depressive mentioning this, we should make clear that it is still a bright future if we can design some powerful algorithms to help us automatically select the proper parameters to design the useful digital filters, just like the cross-validation method in training a LASSO model. However, this challenging work should be jointly handled with interested scholars in this area.

Acknowledgments

This work is supported by National Research Foundation of Singapore grant NRF-RSS2016-004 and Ministry of Education Academic Research Fund Tier 1 grants R-266-000-096-133, R-266-000-096-731, R-266-000-100-646 and R-266-000-119-133.

References

Ali MM, Babai MZ, Boylan JE, Syntetos AA (2017) Supply chain forecasting when information is not shared. European Journal of Operational Research 260(3):984–994.
Blanchard BS, Fabrycky WJ, Fabrycky WJ (1990) *Systems engineering and analysis*, volume 4 (Prentice Hall Englewood Cliffs, NJ).

Bloomfield P (2004) *Fourier analysis of time series: an introduction* (John Wiley & Sons).

Box GE, Jenkins GM, Reinsel GC, Ljung GM (2015) *Time series analysis: forecasting and control* (John Wiley & Sons).

Cadzow J (1980) High performance spectral estimation–a new arma method. *IEEE Transactions on Acoustics, Speech, and Signal Processing* 28(5):524–529.

Calheiros RN, Masoumi E, Ranjan R, Buyya R (2015) Workload prediction using arima model and its impact on cloud applications qos. *IEEE Transactions on Cloud Computing* 3(4):449–458.

Chaparro L, Akan A (2018) *Signals and Systems using MATLAB* (Academic Press).

Cormen TH, Leiserson CE, Rivest RL, Stein C (2009) *Introduction to algorithms* (MIT press).

Ding F, Xu L, Alsaadi FE, Hayat T (2018) Iterative parameter identification for pseudo-linear systems with arma noise using the filtering technique. *IET Control Theory & Applications* 12(7):892–899.

Diniz PS, Da Silva EA, Netto SL (2010) *Digital signal processing: system analysis and design* (Cambridge University Press).

Dorf RC, Bishop RH (2011) *Modern control systems* (Pearson).

Friedlander B, Porat B (1984) The modified yule-walker method of arma spectral estimation. *IEEE Transactions on Aerospace and Electronic Systems* (2):158–173.

Gilbert K (2005) An arima supply chain model. *Management Science* 51(2):305–310.

Granger CWJ, Hatanaka M (1964) *Spectral Analysis of Economic Time Series* (PSME-1) (Princeton university press).

Hamilton JD (1995) Time series analysis. *Economic Theory. II, Princeton University Press, USA* 625–630.

Harvey AC, Todd P (1983) Forecasting economic time series with structural and box-jenkins models: A case study. *Journal of Business & Economic Statistics* 1(4):299–307.

Helfenstein U (1991) The use of transfer function models, intervention analysis and related time series methods in epidemiology. *International journal of epidemiology* 20(3):808–815.

Hyndman R, Koehler AB, Ord JK, Snyder RD (2008) *Forecasting with exponential smoothing: the state space approach* (Springer Science & Business Media).

Hyndman RJ, Athanasopoulos G (2018) *Forecasting: principles and practice* (OTexts).

Ives AR, Abbott KC, Ziebarth NL (2010) Analysis of ecological time series with arma (p, q) models. *Ecology* 91(3):858–871.

Koopmans LH (1995) *The spectral analysis of time series* (Elsevier).

Kreyszig E (1978) *Introductory functional analysis with applications*, volume 1 (wiley New York).
Li C, Lim A (2018) A greedy aggregation–decomposition method for intermittent demand forecasting in fashion retailing. European Journal of Operational Research 269(3):860–869.

Liu H, Zhu L, Pan Z, Bai F, Liu Y, Liu Y, Patel M, Farantatos E, Bhatt N (2017) Armax-based transfer function model identification using wide-area measurement for adaptive and coordinated damping control. IEEE Transactions on Smart Grid 8(3):1105–1115.

Mani R, Oppenheim AV, Willsky AS, Nawab SH (1997) Solutions Manual [for] Signals and Systems, [by] Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab (Prentice Hall).

Marple Jr SL, Carey WM (1989) Digital spectral analysis with applications.

MathWorks (2018a) Bandstop filter specification object. https://www.mathworks.com/help/dsp/ref/fdesign.bandstop.html accessed on March 14, 2019.

MathWorks (2018b) Minimum order for elliptic filters. https://www.mathworks.com/help/signal/ref/ellipord.html?searchHighlight=ellipord&ss_tid=doc_srchtitle accessed on March 14, 2019.

Okutani I, Stephanedes YJ (1984) Dynamic prediction of traffic volume through kalman filtering theory. Transportation Research Part B: Methodological 18(1):1–11.

Papoulis A, Pillai SU (2002) Probability, random variables, and stochastic processes (Tata McGraw-Hill Education).

Priestley MB (1981) Spectral analysis and time series .

Robbins AH, Miller WC (2012) Circuit analysis: Theory and practice (Cengage Learning).

Shannon CE (1948) A mathematical theory of communication. Bell system technical journal 27(3):379–423.

Von Bertalanffy L (1968) General system theory. New York 41973(1968):40.

Wang S, Li C, Lim A (2019) Optimal joint estimation and identification theorem to linear gaussian system with unknown inputs. Signal Processing 161:268–288.

Wiener N (1965) Cybernetics or Control and Communication in the Animal and the Machine, volume 25 (MIT press).

Yang WY (2009) Signals and Systems with MATLAB (Springer Science & Business Media).

Yu C, You K, Xie L (2016) Quantized identification of arma systems with colored measurement noise. Automatica 66:101–108.

Yunus K, Thiringer T, Chen P (2016) Arima-based frequency-decomposed modeling of wind speed time series. IEEE Transactions on Power Systems 31(4):2546–2556.