Input-to-State Stability, integral Input-to-State Stability, and $\mathcal{L}_2$-Gain Properties: Qualitative Equivalences and Interconnected Systems

Christopher M. Kellett and Peter M. Dower

Abstract—Input-to-state stability (ISS) and $\mathcal{L}_2$-gain are well-known robust stability properties that continue to find wide application in the analysis and control of nonlinear dynamical systems and their interconnections. We investigate the relationship between ISS-type and $\mathcal{L}_2$-gain properties, demonstrating several qualitative equivalences between these two approaches. We subsequently present several new sufficient conditions for the stability of interconnected systems derived by exploiting these qualitative equivalences.

I. INTRODUCTION

Historically, there have been two dominant approaches to the study of interconnected dynamical systems via a modular, stability-of-subsystems approach. The first was pioneered by Zames in the 1960’s and employs $\mathcal{L}_2$-gain from the input to the output of a (sub-)system [31] (see also [7]). The second approach developed from the introduction of the Input-to-State Stability (ISS) concept by Sontag in 1989 [25] which extended classical state-space stability notions for systems described by ordinary differential equations to include inputs.

The $\mathcal{L}_2$-gain input-output approach, derived largely from frequency domain considerations, led to the highly successful optimal control techniques for linear time-invariant systems first suggested in [32] where it is possible to design feedback controllers in a systematic way to achieve a desired closed-loop $\mathcal{L}_2$-gain from disturbance input to some suitably weighted penalty variable (regarded as an output) (see [36] and the references therein). The state space formulation and solution of the $\mathcal{H}_\infty$ optimal control problem for linear systems [11] paved the way for extending these techniques to the study of nonlinear systems where the design goal remained the design of a closed-loop system with a linear $\mathcal{L}_2$-gain from disturbance input to the penalty variable. This line of research is referred to as nonlinear $\mathcal{H}_\infty$ optimal control (see for example [5] [14] [30]).

In contrast to the explicit quantitative $\mathcal{L}_2$-gain design goal above, ISS was formulated as a qualitative robust stability property explicitly for nonlinear systems. While there has been recent work on computing ISS gains [15], the feedback design techniques to achieve ISS typically rely on Lyapunov-based techniques such as control-Lyapunov functions [22], [24]. Consequently, while the design techniques for ISS are more easily applied to nonlinear systems, they generally lack the pre-specified gain limits of nonlinear $\mathcal{H}_\infty$ control.

Sontag [26] investigated integral variants of the ISS property and demonstrated that ISS is equivalent to an integral-to-integral ISS-type estimate (stated here as (5)). He termed this an “$\mathcal{L}_2$ to $\mathcal{L}_2$ property” as, by a particular choice of the scaling functions involved, one exactly recovers the standard definition of linear $\mathcal{L}_2$-gain. In addition, Sontag observed that by taking an integral of the input, but not the state, one obtains a fundamentally different stability property, which he termed integral ISS (usually abbreviated to iISS). In [26] iISS is referred to as an “$\mathcal{L}_2$ to $\mathcal{H}_\infty$ property” and is shown to be strictly weaker than ISS; i.e., all ISS systems are iISS but there exist iISS systems that are not ISS.

Inspired by the nonlinear gains used in ISS-type estimates, we explicitly considered the notion of nonlinear $\mathcal{L}_2$-gain [8], where the energy of the state or output penalty variable is bounded from above by a nonlinear scaling of the energy of the input. This generalization of linear $\mathcal{L}_2$-gain is intuitively appealing as one would not a priori expect a linear bound for nonlinear systems. In principle, the nonlinear $\mathcal{L}_2$-gain property applies to a wider class of systems than does the linear $\mathcal{L}_2$-gain property. Furthermore, when dealing with quantitative results, nonlinear gains allow for tighter gain bounds, allowing for more precise results. We subsequently developed verification [10] and synthesis [35] tools for the nonlinear $\mathcal{L}_2$-gain property. These tools can be seen as an extension of nonlinear $\mathcal{H}_\infty$ control.

The ISS and $\mathcal{L}_2$-gain approaches developed in parallel and largely independent of each other. A rare point of contact is the work of Grüne, Sontag, and Wirth [12] where, for systems of dimension different from 4 or 5, a certain qualitative equivalence was demonstrated between global asymptotic stability of the origin and global exponential stability of the origin (and hence $\mathcal{L}_2$-stability of the associated system). Additionally, a similar qualitative equivalence between ISS and linear $\mathcal{L}_2$-gain was demonstrated. To be precise, Grüne, et al. showed that a system with linear $\mathcal{L}_2$-gain is always ISS and that given an ISS system it is possible to find a nonlinear change of coordinates so that, in the new coordinates, the system has linear $\mathcal{L}_2$-gain [12, Theorems 3, 4]. In this context, nonlinear
$H^\infty$ control can be seen as a method to design ISS systems with a prescribed ISS gain. Design tools for attaining pre-specified ISS gains for nonlinear discrete-time systems were presented in [15]. In Section III we present a result similar to that of [12, Theorems 3, 4] demonstrating a qualitative equivalence between ISS and linear $L_2$-gain (we will make the differences precise in Theorem 2.) We then present a qualitative equivalence between iISS and nonlinear $L_2$-gain (Theorem 3). As a consequence, the synthesis and verification results for nonlinear $L_2$-gain from [35] and [10] can be seen as design tools for iISS systems. To date, design tools for iISS systems are largely unavailable.

A natural approach to analyzing large-scale dynamical systems involves separating the large-scale system into several smaller interconnected subsystems, analyzing the subsystems, and then investigating overall system behavior on the basis of subsystem behavior and their interconnections. Both ISS and $L_2$-gain have been widely used in this context. Consequently, it is of interest to discuss how both cascade and feedback interconnections behave for the different stability properties. It is immediately obvious that the cascade connection of two systems with linear $L_2$-gain results in an overall system with linear $L_2$-gain. A small-gain condition [31, Theorem 1] is sufficient to guarantee that the feedback interconnection of systems with linear $L_2$-gain results in an overall system with linear $L_2$-gain. Similarly, the cascade connection of two ISS systems is ISS [25] and a small-gain condition is sufficient to guarantee that the feedback interconnection of two ISS systems is also ISS [19].

As we show in Section IV, when considering system interconnections, the nonlinear $L_2$-gain property shares many similarities with ISS and linear $L_2$-gain. In Section IV-A we show that the cascade of two systems with nonlinear $L_2$-gain also has nonlinear $L_2$-gain. In Section IV-B we show that if a small-gain condition is satisfied then the feedback interconnection of two systems with nonlinear $L_2$-gain also has nonlinear $L_2$-gain. By contrast, it is known that a cascade interconnection of iISS systems is not necessarily iISS [4] and, even if a small gain condition is satisfied, a feedback interconnection of iISS systems is not necessarily iISS. Consequently, the aforementioned qualitative equivalence between nonlinear $L_2$-gain and iISS (Theorem 3) and the results of section IV-B appear to contradict known results on interconnections of iISS systems.

This apparent contradiction is resolved in Section V by recognizing that, while nonlinear $L_2$-gain and iISS are qualitatively equivalent, the relationship is asymmetric in the sense that all systems with the nonlinear $L_2$-gain property are iISS, while there exists a coordinate transformation for a given iISS system so that, in the new coordinates, the system satisfies the nonlinear $L_2$-gain property. Consequently, studying interconnections of iISS systems via the asserted nonlinear $L_2$-gain qualitative equivalence requires careful consideration of the state and input transformations used. This consideration leads to several sufficient conditions for stability of interconnected iISS systems similar to those found in [4], [6], and [18].

The paper is organized as follows. In Section II we present some necessary mathematical preliminaries including precise definitions of the stability concepts of interest, as well as two key lemmas on nonlinear changes of coordinates. In Section III we demonstrate essential qualitative equivalences between the six different stability properties of interest. In Section IV we present sufficient conditions for the interconnection (cascade or feedback) of systems with nonlinear $L_2$-gain to also have nonlinear $L_2$-gain while in Section V we present several sufficient conditions for $L_2$-stability, ISS, or iISS of interconnected systems by drawing on the qualitatively equivalent properties developed in Section III. In Section VI we provide some concluding remarks.

II. Preliminaries

We consider systems described by ordinary differential equations of the form

$$\frac{d}{dt} x(t) = f(x(t)), \quad x(0) \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz. We also consider systems with inputs described by

$$\frac{d}{dt} x(t) = f(x(t), w(t)), \quad x(0) \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz in its first argument, locally uniformly in its second argument. We take as the class of inputs, $\mathcal{W}^m$, those functions $w : \mathbb{R}_0^+ \rightarrow \mathbb{R}^m$ that are measurable and locally essentially bounded. We make use of the standard function classes $K_{\infty}$ and $KL$ (see [13] or [20]). For a measurable, locally essentially bounded function $y : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ we denote the squared two-norm by $\|y\|_{L_2^2[0,t]}^2 \doteq \int_0^t |y(\tau)|^2 d\tau$.

**Remark 1:** When considering $L_2$-type properties, it is standard to take inputs from the space of locally $L_2$ functions, denoted by $L_2^2$ (i.e., those functions whose truncation to any finite time horizon is in $L_2$). However, a subspace of $L_2^2$, consisting of all measurable and locally essentially bounded functions is sufficient for what follows. With respect to the two-norm, we exclusively use the truncated two-norm above which is finite for any fixed $t \in \mathbb{R}_0^+$ and any $w \in \mathcal{W}^m$. Furthermore, this class of inputs is commonly used in the ISS literature since, again for any fixed $t \in \mathbb{R}_0^+$, it guarantees that the integral of nonlinearly scaled versions of the input is finite over $[0,t]$ (see [26] or Lemma 2 below).

A. Stability Properties

There are six stability properties that will be of interest in the sequel. The first two properties are for systems without inputs (1), while the final four are for systems with inputs (2).

**Definition 1:** System (1) is $\alpha$-integrable if there exists $\alpha, \beta \in K_{\infty}$ so that

$$\int_0^t \alpha(|x(\tau)|) d\tau \leq \beta(|x(0)|), \quad \forall x(0) \in \mathbb{R}^n, \ t \in \mathbb{R}_0^+. \tag{3}$$

In [29] it was observed that $\alpha$-integrability is equivalent to uniform global asymptotic stability of the origin for (1).
**Definition 2:** System (1) is $\mathcal{L}_2$-stable if there exists $\beta \in \mathcal{K}_{\infty}$ so that
\[
|x|_{\mathcal{L}_2[0,T]}^2 \leq \beta(|x(0)|), \quad \forall x(0) \in \mathbb{R}^n, \ t \in \mathbb{R}_{\geq 0}.
\] (4)

We observe that $\mathcal{L}_2$-stability is a special case of $\alpha$-integrability where $\alpha(s) = s^2$ for all $s \in \mathbb{R}_{\geq 0}$.

We now define four stability properties for systems with inputs. The first two are the well-known properties of Input-to-State Stability (ISS) [25] and integral Input-to-State Stability (iISS) [26].

**Definition 3:** System (2) is Input-to-State Stable (ISS) if there exist $\alpha, \beta, \sigma \in \mathcal{K}_{\infty}$ so that the estimate
\[
\int_0^t \alpha(|y(\tau)|) d\tau \leq \max \left\{ \beta(|x(0)|), \int_0^t \sigma(|w(\tau)|) d\tau \right\}
\] (5)
holds for all $x(0) \in \mathbb{R}^n, w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$.

**Definition 4:** System (2) is integral Input-to-State Stable (iISS) if there exist $\alpha, \beta, \gamma, \sigma \in \mathcal{K}_{\infty}$ so that the estimate
\[
\int_0^t \alpha(|y(\tau)|) d\tau \leq \max \left\{ \beta(|x(0)|), \gamma \left( \int_0^t \sigma(|w(\tau)|) d\tau \right) \right\}
\] (6)
holds for all $x(0) \in \mathbb{R}^n, w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$.

The final two stability properties are based on the $\mathcal{L}_2$-norm.

**Definition 5:** System (2) has linear $\mathcal{L}_2$-gain with transient and gain bound $\beta \in \mathcal{K}_{\infty}, \gamma \in \mathbb{R}_{\geq 0}$ if the estimate
\[
|x|_{\mathcal{L}_2[0,T]}^2 \leq \max \left\{ \beta(|x(0)|), \gamma^2 \|w\|_{\mathcal{L}_2[0,T]}^2 \right\}
\] (7)
holds for all $x(0) \in \mathbb{R}^n, w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$.

**Definition 6:** System (2) has nonlinear $\mathcal{L}_2$-gain with transient and gain bound $\beta, \gamma \in \mathcal{K}_{\infty}$ if the estimate
\[
|x|_{\mathcal{L}_2[0,T]}^2 \leq \max \left\{ \beta(|x(0)|), \gamma \left( \|w\|_{\mathcal{L}_2[0,T]}^2 \right) \right\}
\] (8)
holds for all $x(0) \in \mathbb{R}^n, w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$.

With the standing assumption that (2) is forward complete, the original definitions of ISS [25] and iISS [26] were shown to be equivalent to Definition 3 and Definition 4 in [26] and [3], respectively.

**B. Changes of Coordinates**

In [27], Sontag asserted that “notions of stability should be invariant under (nonlinear) changes of variables.” In part, this derives from the fact that in order to apply various nonlinear control design methods the system equations are usually required to be in a certain normal form. When a system model as given is not in the necessary normal form, a common technique is to search for a change of coordinates such that, in the new coordinates, the normal form is achieved and a stabilizing control design can be undertaken. However, unless invariance of the stability property is guaranteed under changes of coordinates, such a stabilizing design in the new coordinates may fail to be stabilizing in the original, possibly physically meaningful, coordinates.

**Definition 7** ([27]): A change of coordinates is a homeomorphism $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$ that fixes the origin. In other words, $T(\cdot)$ is continuous with a well-defined and continuous inverse $T^{-1} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and such that $T(0) = 0$.

If it is desirable to express the system differential equations (2) in new coordinates, then the change of coordinates $T(\cdot)$ must be differentiable, at least away from the origin. However, the results and discussion in this paper relate to trajectory based properties, and so do not require differentiability of $T(\cdot)$.

The following fact was observed in [27] and is a useful tool for analyzing the effect of changes of coordinates on stability properties.

**Lemma 1:** Given any change of coordinates $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$, there exist $\alpha, \beta \in \mathcal{K}_{\infty}$ such that
\[
\alpha(|\zeta|) \leq |T(\zeta)| \leq \beta(|\zeta|), \quad \forall \zeta \in \mathbb{R}^p.
\] (9)

**Proof:** Simply take $\alpha(r) \doteq \min_{|x| \leq r} |T(x)|$ and $\beta(r) \doteq \max_{|x| \leq r} |T(x)|$.

An immediate consequence of the above lemma is that integrability is preserved under changes of coordinates.

**Lemma 2:** Let $\xi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ be measurable and locally essentially bounded, $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be any change of coordinates, and take any $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$. For any $t \in \mathbb{R}_{\geq 0}$, the integrals
\[
\int_0^t \alpha_1(|\xi(\tau)|) d\tau
\] (10)
and
\[
\int_0^t \alpha_2(|T(\xi(\tau))|) d\tau
\] (11)
exist and are finite.

**Proof:** From Lemma 1, there exists $\overline{\alpha}_T \in \mathcal{K}_{\infty}$ so that $|T(\xi(\tau))| \leq \overline{\alpha}_T(|\xi(\tau)|)$ for all $\tau \in \mathbb{R}_{\geq 0}$. Therefore, proving (11) exists and is finite reduces to proving (10) exists and is finite. To prove (10) exists and is finite we simply note that $\alpha_1 \in \mathcal{K}_{\infty}$ and the norm are both continuous functions on $\mathbb{R}_{\geq 0}$ and $\mathbb{R}^n$, respectively. Consequently, if $\tau \mapsto \xi(\tau)$ is measurable and essentially bounded then $\tau \mapsto \alpha_1(|\xi(\tau)|)$ is also measurable and locally essentially bounded, yielding the desired result.

With Lemma 1 available, it is straightforward to see that $\alpha$-integrability, ISS, and iISS satisfy Sontag’s assertion that stability notions should be invariant under changes of coordinates. Using ISS as an example, given any change of coordinates on the state $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, let the functions $\alpha_T, \overline{\alpha}_T \in \mathcal{K}_{\infty}$ come from Lemma 1, and define $\mathcal{K}_{\infty}$ functions $\alpha_T \doteq \alpha_T \circ T^{-1}$ and $\beta_T \doteq \beta_T \circ T^{-1}$ where $\alpha, \beta \in \mathcal{K}_{\infty}$ are from the ISS estimate (5). Furthermore, let $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$, be any change of coordinates on the input space, let $\alpha_S, \overline{\alpha}_S \in \mathcal{K}_{\infty}$ come from Lemma 1, and define the $\mathcal{K}_{\infty}$ function $\sigma_T \doteq \sigma_T \circ S^{-1}$ with $\sigma \in \mathcal{K}_{\infty}$ from the ISS estimate (5). Define $v(t) \doteq S(w(t))$ for all $t \in \mathbb{R}_{\geq 0}$ and $\psi(t) \doteq T(x(t))$ for all $t \in \mathbb{R}_{\geq 0}$ Then the bounds from
Lemma 1 and the ISS estimate (5) yield
\[
\int_0^t \dot{\alpha}(\psi(\tau)) \, d\tau = \int_0^t \dot{\alpha}(|T(x(\tau))|) \, d\tau \leq \int_0^t \dot{\alpha}(\sqrt{T(x(\tau)))} \, d\tau
\]
\[
= \max \left\{ \beta(0), \int_0^t \sigma(|w(\tau)|) \, d\tau \right\}
\]
\[
\leq \max \left\{ \tilde{\beta}(\psi(0)), \int_0^t \hat{\sigma}(|v(\tau)|) \, d\tau \right\}.
\]
In other words, the system in the new coordinates also satisfies an ISS estimate (5) with functions \( \dot{\alpha}, \tilde{\beta}, \hat{\sigma} \in K_\infty \) in place of \( \alpha, \beta, \sigma \in K_\infty \).

Note that precisely the same argument as above holds for systems that are \( \alpha \)-integrable or which satisfy an ISS estimate (6). Hence, \( \alpha \)-integrability, ISS, and iISS are invariant under changes of coordinates in the input and state variables. However, none of \( L_2 \)-stability, linear \( L_2 \)-gain, or nonlinear \( L_2 \)-gain satisfy this property. (See Examples 1 and 2 in Sections III-B and III-C, respectively.)

A. Qualitative Equivalence

In the result of (12), while the comparison functions differ between the original ISS estimate (5) and the ISS estimate for the new coordinates (12), the form of the inequality is clearly the same, and consequently (5) and (12) are said to be qualitatively equivalent. Where the comparison functions are the same, this equivalence is said to be quantitative.

The notions of qualitative and quantitative equivalence can be extended to pairs of properties of different forms. In particular, if a given property implies a second property of a different form which, in turn, implies a third property of the same form as the first, and if the first and third properties are qualitatively equivalent, then we refer to all three properties as being qualitatively equivalent.

For example, bounds defined by sums and maximums are qualitatively equivalent. That these provide qualitatively equivalent properties follows from the fact that for any \( a, b \in \mathbb{R}_{\geq 0} \)
\[
a + b \leq \max\{2a, 2b\} \quad \text{and} \quad \max\{a, b\} \leq a + b.
\]

Applied to the ISS estimate (5), the above inequalities yield
\[
\int_0^t \alpha(|x(\tau)|) \, d\tau \leq \max \left\{ \beta(|x(0)|), \int_0^t \sigma(|w(\tau)|) \, d\tau \right\}
\]
\[
\leq \beta(|x(0)|) + \int_0^t \sigma(|w(\tau)|) \, d\tau
\]
\[
\leq \max \left\{ 2\beta(|x(0)|), 2 \int_0^t \sigma(|w(\tau)|) \, d\tau \right\}.
\]
Clearly (17) and (19) are qualitatively equivalent and, by our extended notion of qualitative equivalence, (18) is qualitatively equivalent to (17). We note that the nature of this equivalence is qualitative rather than quantitative since the comparison function bounds are not the same due to the factor of 2 involved in the first relation in (16).

Similarly, the definitions of ISS (Definition 3) and iISS (Definition 4) are qualitatively equivalent to the original definitions proposed in the literature. In particular, under the assumption of forward completeness of (2), [26, Theorem 1]...
demonstrated that (5) is qualitatively equivalent to the original definition of ISS in [25]; i.e., there exists $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that

$$|x(t)| \leq \max \left\{ \beta(|x(0)|, t), \sup_{\tau \in [0,t]} \gamma(|w(\tau)|) \right\}$$

(20)

holds for all $x(0) \in \mathbb{R}^n$, $w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$. Similarly, again under the assumption of forward completeness of (2), [3, Theorem 1] showed that (6) is qualitatively equivalent to the existence of $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that

$$\alpha(|x(t)|) \leq \max \left\{ \beta(|x(0)|, t), \int_0^t \gamma(|w(\tau)|)d\tau \right\}$$

(21)

holds for all $x(0) \in \mathbb{R}^n$, $w \in \mathcal{W}^m$, and $t \in \mathbb{R}_{\geq 0}$, as defined in [26].

The original definitions of ISS, (20), and iISS, (21), possess some appealing intuitive properties. For example, ISS involves bounds on system trajectories at a given time that depend on a decaying transient term due to the initial condition ($\beta \in \mathcal{KL}$) as well as an additional term due to the worst-case input up to the current time ($\gamma \in \mathcal{K}_\infty$). This desired property is more obvious in (20) than in the qualitatively equivalent definition (5). Similarly, the fact that the input is treated in a fundamentally different manner for integral ISS than it is for ISS is more obvious in the difference between (20) and (21) than it is in the difference between (5) and (6). On the other hand, the fact that iISS is a strictly weaker property than ISS is more obvious when examining (5) and (6) than it is when examining (20) and (21). Indeed, all ISS systems are iISS since the identity is simply one possible choice of the function $\gamma \in \mathcal{K}_\infty$ of (6). Furthermore, since there are many $\mathcal{K}_\infty$ functions which are not the identity, iISS possibly encompasses a larger class of systems. That this is in fact the case was shown in [26]. Therefore, we see that by examining qualitatively equivalent ISS properties and their relationships to qualitatively equivalent iISS properties, we gain a clearer understanding of the relationship between ISS and iISS systems. Furthermore, as is evident in the sequel, the ISS and iISS definitions given by (5) and (6), respectively, are better suited to clarifying the relationship between these properties and the $L_2$-gain properties (7) and (8).

With the notion of qualitative equivalence established, the remainder of this section is concerned with establishing that $\alpha$-integrability, ISS, and iISS are qualitatively equivalent, via a change of coordinates, to $L_2$-stability, linear $L_2$-gain, and nonlinear $L_2$-gain, respectively. It is important to note that these equivalences are not quantitative; e.g., the iISS-gain $\gamma \in \mathcal{K}_\infty$ of (6) is not, in general, the nonlinear $L_2$-gain $\gamma \in \mathcal{K}_\infty$ of (8).

### B. $L_2$-stability and $\alpha$-integrability

**Theorem 1:** If system (1) is $L_2$-stable then it is $\alpha$-integrable. Conversely, if system (1) is $\alpha$-integrable then there exists a change of coordinates for the state such that the system in the new coordinates is $L_2$-stable.

**Proof:** That $L_2$-stability implies $\alpha$-integrability is obvious by inspection since $\alpha(s) = s^2$ for all $s \in \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$.

In order to prove the converse, assume we have $\alpha, \beta \in \mathcal{K}_\infty$ so that (3) is satisfied. Apply Lemma 3 to $\alpha^\frac{1}{2} \in \mathcal{K}_\infty$ with $p = n$ to obtain $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that $|T(x)| \leq \alpha^\frac{1}{2}(|x|)$, for all $x \in \mathbb{R}^n$. Lemma 1 implies the existence of $\alpha_T \in \mathcal{K}_\infty$ such that $\alpha_T(|x|) \leq |T(x)|$ for all $x \in \mathbb{R}^n$. Defining the new coordinates $\xi = T(x)$, we then see that, for all $\xi(0) \in \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$,

$$\|\xi(t)\|_{L_2[0,t]} = \int_0^t |\xi(\tau)|^2 d\tau = \int_0^t |T(x(\tau))|^2 d\tau$$

$$\leq \int_0^t \alpha(|x(\tau)|) d\tau \leq \beta(|x(0)|)$$

$$\leq \beta \circ \alpha_T^{-1}(|T(x(0))|) = \beta \circ \alpha_T^{-1}(|\xi(0)|)$$

so that, in the new coordinates, system (1) is $L_2$-stable. □

**Example 1:** The origin can be shown to be globally asymptotically stable for

$$\frac{d}{dt} x(t) = -x(t)^3, \quad x(0) \in \mathbb{R}$$

(22)

by using the Lyapunov function $V(x) = \frac{1}{2} x^2$. Consequently, by the observation in [29], (22) is $\alpha$-integrable. The solution of (22) is

$$x(t) = \frac{x(0)}{\sqrt{1 + 2x(0)^2} t}, \quad \forall x(0) \in \mathbb{R}, t \in \mathbb{R}_{\geq 0}$$

so that

$$|x|^2_{L_2[0,t]} = \frac{1}{2} \log (1 + 2x(0)^2 t)$$

and hence there is no $\beta \in \mathcal{K}_\infty$ such that $|x|^2_{L_2[0,t]} \leq \beta(|x(0)|)$. In other words, while (22) is $\alpha$-integrable, it is not $L_2$-stable. However, Theorem 1 states that there exists a change of coordinates so that, in the new coordinates, the system is $L_2$-stable. Let

$$z = T(x) = x \exp \left( -\frac{1}{2x^2} \right), \quad x \in \mathbb{R} \setminus \{0\},$$

and $T(0) = 0$. This change of coordinates is a homeomorphism on $\mathbb{R}$ and a diffeomorphism on $\mathbb{R} \setminus \{0\}$. It is straightforward to write the system equation in the new coordinates as

$$\frac{d}{dt} z(t) = -z(t) \left( 1 + |T^{-1}(z(t))|^2 \right), \quad z(0) \in \mathbb{R} \setminus \{0\}$$

(24)

from which it follows that $|z(t)| \leq |z(0)| \exp(-t)$ for all $z(0) \in \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$. Consequently, $\|z(t)\|^2_{L_2[0,t]} \leq \frac{1}{2} \|z(0)\|^2$, and hence the system in the new coordinates is $L_2$-stable.

We also observe that this example demonstrates that $L_2$-stability is not invariant under changes of coordinates. This follows from the fact that $T^{-1}(\cdot)$ is a change of coordinates that transforms the $L_2$-stable system (24) to the system (22) that is not $L_2$-stable.

### C. Linear $L_2$-gain and ISS

The following theorem is similar to [12, Theorem 4] demonstrating a qualitative equivalence between ISS and (linear) $L_2$-gain, with a few key differences.

**Theorem 2:** If system (2) has linear $L_2$-gain then system (2) is ISS. Conversely, for any $\tilde{s}^2 \in \mathbb{R}_{\geq 0}$, if system (2) is ISS
then there exist changes of coordinates for both the input and state such that the system in the new coordinates has linear $L_2$-gain $\gamma^2$.

In [12, Theorem 4], a change of coordinates is constructed such that the bounding term related to the initial condition in (7) can be taken as $\beta = 1$. Obtaining this result relies on the level sets of an appropriate Lyapunov function being homeomorphic (or diffeomorphic) to spheres and, as a consequence, [12, Theorem 4] does not hold for dimensions $n = 4, 5$. Theorem 2 above has no such restriction at the expense of not being able to choose a priori the function $\beta \in K_\infty$.

We observe that one can arbitrarily set the gain parameter by appropriate choice of the change of coordinates on the input variable. The ability to fix the $L_2$-gain parameter in Theorem 2 is analogous to the ability to fix the decay rate in Sontag’s lemma on $KL$-estimates [26, Proposition 7] (also [20, Lemma 7]). That is, for a given function $\beta \in KL$ and a desired decrease rate $\lambda \in \mathbb{R}_{> 0}$, there exist $\alpha_1, \alpha_2 \in K_\infty$ such that $\alpha_1(\beta(s, t)) \leq \alpha_2(s) \exp(-\lambda t)$ for all $s, t \in \mathbb{R}_{\geq 0}$.

**Proof of Theorem 2:** The proof of the first statement in Theorem 2 is straightforward since linear $L_2$-gain (7) is a special case of the ISS estimate (5) where the comparison functions in the ISS definition are simply $\alpha(s) = s^2$ and $\sigma(s) = \tilde{\gamma}^2 s^2$, for all $s \in \mathbb{R}_{\geq 0}$.

To show the converse statement of Theorem 2, suppose that system (2) is ISS so that (5) holds. Suppose $\alpha, \beta, \gamma \in K_\infty$. For the function $\alpha \frac{2}{m} \in K_\infty$, with $p = m$ Lemma 3 yields the existence of a change of coordinates $T: \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(x) \leq \frac{1}{2}(|x|), \quad \forall x \in \mathbb{R}^n. \quad (25)$$

For the change of coordinates $T(\cdot)$, let $\alpha \in K_\infty$ come from Lemma 1 so that

$$\alpha(|x|) \leq |T(x)|, \quad \forall x \in \mathbb{R}^n. \quad (26)$$

For the function $\tilde{\gamma}^{-1} \sigma \frac{2}{m} \in K_\infty$, with $p = m$ Lemma 3 yields the existence of a change of coordinates $S: \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\tilde{\gamma}^{-1} \sigma \frac{2}{m}(|w|) \leq |S(w)|, \quad \forall w \in \mathbb{R}^m. \quad (27)$$

Combining (25), (5), (26), and (27) we have, for all $x(0) \in \mathbb{R}^n$, $w \in \mathbb{W}^m$, and $t \in \mathbb{R}_{\geq 0}$,

$$|T(x)|^2_{L_2[0, t]} = \int_0^t |T(x(\tau))|^2 d\tau \leq \int_0^t \alpha(|x(\tau)|) d\tau \leq \max \left\{ \beta(|x(0)|), \frac{\tilde{\gamma}^2}{2} \int_0^t \tilde{\gamma}^{-2} \sigma(|w(\tau)|) d\tau \right\} \leq \max \left\{ \beta \left( \alpha^{-1}(|T(x(0))\rangle) \right), \frac{\tilde{\gamma}^2}{2} \int_0^t |S(w(\tau))|^2 d\tau \right\} = \max \left\{ \beta(|T(x(0))\rangle), \frac{\tilde{\gamma}^2}{2} |S(w)|^2_{L_2[0, t]} \right\}. \quad (28)$$

where $\tilde{\beta} := \beta \circ \alpha^{-1} \in K_\infty$. Since the input $w(\cdot)$ is measurable and locally essentially bounded, Lemma 2 yields that $|S(w(\cdot))|^2_{L_2[0, t]}$ is also measurable and locally essentially bounded, and hence the final two input-dependent terms above are well-defined. In other words, in the state coordinates defined by $T(\cdot)$ and the input coordinates defined by $S(\cdot)$, the system has linear $L_2$-gain with transient and gain bound $\tilde{\beta} \in K_\infty$.

**Example 2:** Consider the system (22) of the previous example augmented with an input; i.e.,

$$\frac{d}{dt} x(t) = -x(t)^3 + w(t), \quad x(0) \in \mathbb{R}, \quad w \in \mathbb{W}^1. \quad (29)$$

Define $V(x) = x_1^2 - \frac{1}{2} x_1^2$ for all $x \in \mathbb{R}$ and observe that $|x| > |w|^{1/3}$ implies $\frac{d}{dt} V(x(t)) < 0$. Therefore, $V(\cdot)$ is an ISS-Lyapunov function and, consequently, (28) is ISS [28, Theorem 1]. However, by setting $w \equiv 0$, we can repeat the argument of Example 1 to see that (28) cannot have linear (or in fact nonlinear) $L_2$-gain. However, as indicated by Theorem 2, there exists a change of coordinates so that, in the new coordinates, the system (28) has linear $L_2$-gain. Using the same change of coordinates (23) as in Example 1, we see that (28) becomes

$$\frac{d}{dt} z(t) = -z(t) \left( 1 + T^{-1}(z(t))^2 \right) + \exp \left( -\frac{1}{2x^2(t)} \right) \left( 1 + \frac{1}{x^2(t)} \right) w(t). \quad (30)$$

The inequality $1 - \frac{1}{s^2} \leq \log s, \quad s \in \mathbb{R}_{> 0}$, implies that $\exp(-\frac{1}{2s^2}) \leq \frac{2s^2}{2s^2+1}$ and hence the term multiplying the input is bounded from above by 2. Consequently,

$$\frac{d}{dt} |z(t)| \leq -|z(t)| \left( 1 + |T^{-1}(z(t))|^2 \right) + 2|w(t)| \leq -|z(t)| + 2|w(t)|$$

and hence the system in the new coordinates has a linear $L_2$-gain of 2.

As in Example 1, the change of coordinates $T^{-1}(\cdot)$ takes a system with linear $L_2$-gain to one that is ISS but which has neither linear nor nonlinear $L_2$-gain. In this regard, with respect to Sontag’s assertion that stability notions should be invariant under nonlinear changes of coordinates [27], linear $L_2$-gain is not a “good” notion of robust stability. In this case, maintaining robust stability is not an issue but achieving robust performance may be. In particular, if a feedback design is performed in transformed coordinates in order to achieve a particular linear $L_2$-gain, there is no guarantee that system in the original, probably physically meaningful, coordinates will satisfy any linear $L_2$-gain. This is not to say that $L_2$-gain is somehow an inappropriate design goal in general, as the literature demonstrates it has been highly successful, but that care must be taken when $L_2$-gain techniques are coupled with the use of coordinate transformations.

Finally, we note that Theorem 2 suggests a method for designing Input-to-State Stabilizing controllers based on finding a change of coordinates so that, in the new coordinates, one can construct a feedback stabilizer achieving a linear $L_2$-gain. In the original coordinates, this then provides a feedback stabilizer rendering the system ISS.

**D. Nonlinear $L_2$-gain and iISS**

The relationship between iISS and nonlinear $L_2$-gain is similar to that between ISS and linear $L_2$-gain.

**Theorem 3:** If system (2) has nonlinear $L_2$-gain then system (2) is iISS. Conversely, if system (2) is iISS then there
exist changes of coordinates for both the input and state such that the system expressed in the new coordinates has nonlinear $L_2$-gain.

One critical difference between Theorem 2 and Theorem 3 is in the reverse statement where, in Theorem 2, one can choose the linear $L_2$-gain, $\gamma \in \mathbb{R}_{>0}$, arbitrarily. By contrast, it is not possible to set the $L_2$-gain function in the reverse statement of Theorem 3. However, we can introduce a scaling factor inside the gain function as follows:

**Proposition 1:** Fix $\lambda \in \mathbb{R}_{>0}$. If the system (2) is iISS then there exist changes of coordinates $S: \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ for the input and state, respectively, such that the system in the new coordinates satisfies

$$\|T(x(t))\|_{L_2[\lambda]} \leq \max \left\{ \hat{\beta}(|T(x(0))|), \gamma \left( \lambda \|S(w)\|_{L_2[\lambda]} \right) \right\}$$

for all $x(0) \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \in \mathbb{R}_{\geq 0}$.

**Proof of Theorem 3 and Proposition 1:** As with Theorem 2, the proof of the first statement in Theorem 3 is straightforward since the nonlinear $L_2$-gain estimate (8) is an iISS estimate (6) with the functions $\alpha, \sigma \in \mathcal{K}_\infty$ given by $\alpha(s) = \sigma(s) = s^2$ for all $s \in \mathbb{R}_{\geq 0}$.

The proof of the converse statement of Theorem 3 is a special case of the proof of Proposition 1 with $\lambda = 1$ and follows the same argument as above for the converse statement of Theorem 2. With the function $\alpha \in \mathcal{K}_\infty$ from (6) we again use the state change of coordinates (25) and the bound (26). From Lemma 3, with $p = m$, we obtain a change of coordinates for the input satisfying $\lambda \frac{1}{2} \sigma \frac{1}{2} (|w|) \leq |S(w)|$, for all $w \in \mathbb{R}^m$. We then obtain a nonlinear $L_2$-gain estimate as follows:

$$\|T(x(t))\|_{L_2[\lambda]} = \int_0^t |T(x(\tau))|d\tau \leq \int_0^t \alpha(|x(\tau)|)d\tau$$

$$\leq \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \int_0^t \lambda |x(\tau)|d\tau \right) \right\}$$

$$\leq \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \lambda \int_0^t |S(w(\tau))|d\tau \right) \right\}$$

$$= \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \lambda \|S(w)\|_{L_2[\lambda]} \right) \right\}$$

for all $x(0) \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \in \mathbb{R}_{\geq 0}$.

**Example 3:** Consider the scalar bilinear system

$$\frac{d}{dt}x(t) = -x(t) + x(t)w(t), \quad x(0) \in \mathbb{R}, \quad w \in \mathcal{W}^1.$$  (31)

The iISS-Lyapunov function $V(x) = \log(1 + x^2)$ can be used to show that (31) is iISS [2, Theorem 1]. That (31) is not ISS can be seen by taking the constant input $w(t) = 2$ for all $t \in \mathbb{R}_{\geq 0}$. We now proceed to demonstrate that (31) satisfies the nonlinear $L_2$-gain property (8) but not the linear $L_2$-gain property (7). As a consequence, just as there are iISS systems which are not ISS, there are systems with nonlinear $L_2$-gain which do not admit a linear $L_2$-gain.

Setting $Q(x) \doteq \frac{1}{2} x^2$, (31) implies that

$$\frac{1}{Q(x(\sigma))} \frac{dQ(x(\sigma))}{d\sigma} = -2 + 2w(\sigma)$$  (32)

for all $\sigma \in [0, s]$, $s \in \mathbb{R}_{\geq 0}$. Fix any $x(0) \in \mathbb{R}$, $t \in \mathbb{R}_{\geq 0}$, and any $w \in \mathcal{W}^1$. Integrating (32) over $[0, s]$, $s \in [0, t]$, yields

$$\log \left( \frac{Q(x(s))}{Q(x(0))} \right) = -2s + 2 \int_0^s w(\sigma) d\sigma$$

or $|x(s)|^2 = |x(0)|^2 \exp \left( -2s + 2 \int_0^s w(\sigma) d\sigma \right)$. Hence,

$$\|x\|^2 \leq |x(0)|^2 \exp \left( -2s + 2 \int_0^s w(\sigma) d\sigma \right)$$

$$\leq |x(0)|^2 \exp \left( -2s + 2 \int_0^s |w(\sigma)|^2 d\sigma \right)$$

$$= |x(0)|^2 \left( 1 - \exp(-t) \right) \exp \left( \|w\|^2 \right)$$

$$\leq |x(0)|^2 \exp \left( \|w\|^2 \right)$$

$$= |x(0)|^2 \left( 1 - \exp(-t) \right) \exp \left( \|w\|^2 \right)$$

$$\leq |x(0)|^2 \left( 1 - \exp(-t) \right) \exp \left( \|w\|^2 \right)$$

$$\leq |x(0)|^2 \left( 1 - \exp(-t) \right) \exp \left( \|w\|^2 \right)$$

The proof of the converse statement is straightforward since the nonlinear $L_2$-gain estimate (8) is an iISS estimate (6) with the functions $\alpha, \sigma \in \mathcal{K}_\infty$ given by $\alpha(s) = \sigma(s) = s^2$ for all $s \in \mathbb{R}_{\geq 0}$.

The proof of the converse statement of Theorem 3 is a special case of the proof of Proposition 1 with $\lambda = 1$ and follows the same argument as above for the converse statement of Theorem 2. With the function $\alpha \in \mathcal{K}_\infty$ from (6) we again use the state change of coordinates (25) and the bound (26). From Lemma 3, with $p = m$, we obtain a change of coordinates for the input satisfying $\lambda \frac{1}{2} \sigma \frac{1}{2} (|w|) \leq |S(w)|$, for all $w \in \mathbb{R}^m$. We then obtain a nonlinear $L_2$-gain estimate as follows:

$$\|T(x(t))\|_{L_2[\lambda]} = \int_0^t |T(x(\tau))|d\tau \leq \int_0^t \alpha(|x(\tau)|)d\tau$$

$$\leq \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \lambda \int_0^t \sigma(|w(\tau)|)d\tau \right) \right\}$$

$$\leq \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \lambda \int_0^t |S(w(\tau))|d\tau \right) \right\}$$

$$= \max \left\{ \hat{\beta}(|x(0)|), \gamma \left( \lambda \|S(w)\|_{L_2[\lambda]} \right) \right\}$$

for all $x(0) \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \in \mathbb{R}_{\geq 0}$.

**Example 3:** Consider the scalar bilinear system

$$\frac{d}{dt}x(t) = -x(t) + x(t)w(t), \quad x(0) \in \mathbb{R}, \quad w \in \mathcal{W}^1.$$  (31)

The iISS-Lyapunov function $V(x) = \log(1 + x^2)$ can be used to show that (31) is iISS [2, Theorem 1]. That (31) is not ISS can be seen by taking the constant input $w(t) = 2$ for all $t \in \mathbb{R}_{\geq 0}$. We now proceed to demonstrate that (31) satisfies the nonlinear $L_2$-gain property (8) but not the linear $L_2$-gain property (7). As a consequence, just as there are iISS systems which are not ISS, there are systems with nonlinear $L_2$-gain which do not admit a linear $L_2$-gain.

Setting $Q(x) \doteq \frac{1}{2} x^2$, (31) implies that

$$\frac{1}{Q(x(\sigma))} \frac{dQ(x(\sigma))}{d\sigma} = -2 + 2w(\sigma)$$  (32)
and consequently
\[ \int_0^t \alpha(|x(\tau)|) d\tau < \infty. \]

Finally, since \( \alpha \in \mathcal{K}_\infty \), we see that, for all \( x(0) \in \mathbb{R}^n \), \( x(t) \to 0 \) as \( t \to \infty \). With this fact and Theorem 3 we immediately see that if system (2) has the nonlinear \( \mathcal{L}_2 \)-gain property (8) then, for all \( x(0) \in \mathbb{R}^n \), system trajectories satisfy \( x(t) \to 0 \) as \( t \to \infty \).

Remark 3: The notion of \( \mathcal{L}_2 \)-gain is usually stated as an input-output stability property. If (2) is augmented with a continuous output mapping \( h : \mathbb{R}^n \to \mathbb{R}^p \) for some \( p \in \mathbb{Z}_{>0} \) such that there exist \( \underline{\alpha}_h, \bar{\alpha}_h \in \mathcal{K}_\infty \) satisfying
\[ \underline{\alpha}_h(|\xi|) \leq |h(\xi)| \leq \bar{\alpha}_h(|\xi|), \quad \forall \xi \in \mathbb{R}^n \]
then the previous equivalences in Theorems 2 and 3 can be shown to hold in an input-output sense.

It is unknown if the ISS equivalences, (5) and (20), and iISS equivalences, (6) and (21), still hold in the case of outputs that do not satisfy (36). However, in [21] it was shown that dissipative-form and implication-form ISS-Lyapunov functions are not equivalent in the absence of (36). As this equivalence is used in the proof of [26, Theorem 1], it is possible that in the input-output case Definitions 3 and 4 are not qualitatively equivalent to the original definitions of ISS (20) and iISS (21), respectively. Consequently, in the absence of (36) the results of Theorems 2 and 3 may not generalize to the input-output case.

IV. INTERCONNECTIONS OF SYSTEMS WITH NONLINEAR \( \mathcal{L}_2 \)-GAIN

In this section we will show that the cascade interconnection of two systems with nonlinear \( \mathcal{L}_2 \)-gain itself has nonlinear \( \mathcal{L}_2 \)-gain. We also present a small-gain theorem for the feedback interconnection of systems with nonlinear \( \mathcal{L}_2 \)-gain that guarantees nonlinear \( \mathcal{L}_2 \)-gain for the interconnected system. Later, in Section V we relate these results to those known to hold for iISS systems. Here, we specifically consider two systems
\[
\begin{align*}
\Sigma_1 : & \quad \frac{d}{dt} x_1(t) = f_1(x_1(t), w_1(t)) \quad (36) \\
\Sigma_2 : & \quad \frac{d}{dt} x_2(t) = f_2(x_2(t), w_2(t)) \quad (37)
\end{align*}
\]
where \( x_i(0) \in \mathbb{R}^{n_i}, w_i \in \mathcal{W}^{m_i}, i = 1, 2 \), and each satisfying
\[
\begin{align*}
\|x_1\|_{\mathcal{L}_2[0,t]}^2 & \leq \max \{ \beta_1(|x_1(0)|), \gamma_1 \left( \|w_1\|^2_{\mathcal{L}_2[0,t]} \right) \}, \quad (38) \\
\|x_2\|_{\mathcal{L}_2[0,t]}^2 & \leq \max \{ \beta_2(|x_2(0)|), \gamma_2 \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \}, \quad (39)
\end{align*}
\]
for all \( x_i(0) \in \mathbb{R}^{n_i}, w_i \in \mathcal{W}^{m_i}, \) and \( t \in \mathbb{R}_{\geq 0} \).

A necessary prerequisite for our results on interconnection systems is the following weak triangle inequality from [19] (see also [20, Lemma 4]):

Lemma 4: For any \( \gamma \in \mathcal{K}_\infty \), any \( \rho \in \mathcal{K}_\infty \) such that \( \rho - \text{Id} \in \mathcal{K}_\infty \), and \( a, b \in \mathbb{R}_{\geq 0} \),
\[ \gamma(a + b) \leq \max \{ \gamma \circ \rho(a), (\rho \circ (\rho - \text{Id})^{-1})(b) \}. \quad (40) \]

We note that the above inequality is a generalization of the weak triangle inequality in [25]; i.e., for any \( \gamma \in \mathcal{K}, a, b \in \mathbb{R}_{\geq 0} \),
\[ \gamma(a + b) \leq \max \{ \gamma(2a), \gamma(2b) \}. \quad (41) \]

Lemma 5: Given \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \), there exists \( \alpha \in \mathcal{K}_\infty \) so that, for all \( s_1, s_2 \in \mathbb{R}_{\geq 0}, \alpha(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2). \)

Proof: Define \( \alpha(s) = \min \{ \alpha_1 \left( \frac{s}{2} \right), \alpha_2 \left( \frac{s}{2} \right) \} \) for all \( s \in \mathbb{R}_{\geq 0} \). Then,
\[
\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)
\]
\[ = \min \{ \alpha_1(s_1), \alpha_2(s_2) \} + \min \{ \alpha_1(s_2), \alpha_2(s_1) \}
\]
\[ \leq \alpha_1(s_1) + \alpha_2(s_2). \]

The following is a consequence of the definition of the \( \mathcal{L}_2 \)-norm, the triangle inequality, and Young’s inequality.

Lemma 6: For any \( \varepsilon > 0 \) and for all \( a, b \in \mathcal{W}^m \),
\[ |a + b|^2_{\mathcal{L}_2[0,t]} \leq (1 + \varepsilon^2) |a|^2_{\mathcal{L}_2[0,t]} + (1 + \frac{1}{\varepsilon^2}) |b|^2_{\mathcal{L}_2[0,t]}.
\]

A. Cascade Interconnection

We first examine the cascade interconnection of (36) and (37) with the interconnection \( w_1 = x_2 \) (requiring \( m_1 = n_2 \)) as shown in Figure 1.

Fig. 1. Cascade Interconnection

Proposition 2: Suppose that systems (36) and (37) satisfy the nonlinear \( \mathcal{L}_2 \)-gain properties (38) and (39), respectively. Then there exist \( \beta, \gamma \in \mathcal{K}_\infty \) such that the system given by the cascade interconnection defined by \( w_1 = x_2 \) satisfies
\[ \|x_1\|_{\mathcal{L}_2[0,t]}^2 \leq \max \{ \beta((x(0))), \gamma \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \} \quad (42) \]
for all \( x(0) \in \mathcal{X}_1, \gamma_1 \left( \|w_1\|^2_{\mathcal{L}_2[0,t]} \right) \), and \( t \in \mathbb{R}_{\geq 0} \), and where \( x(t) = x_1(t) \), \( x_2(t) \) for all \( t \in \mathbb{R}_{\geq 0} \).

Proof: Using bounds (38) and (39), and the interconnection constraint \( w_1 = x_2 \) we have
\[ \|x_1\|_{\mathcal{L}_2[0,t]}^2 \leq \max \{ \beta_1(|x_1(0)|), \gamma_1 \left( \|w_1\|^2_{\mathcal{L}_2[0,t]} \right) \}
\]
\[ = \max \left\{ \beta_1(|x_1(0)|), \gamma_1 \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \right\}
\]
\[ \leq \max \left\{ \beta_1(|x_1(0)|), \gamma_1 \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \right\}
\]
\[ = \max \left\{ \beta_1(|x_1(0)|), \gamma_1 \circ \beta_2(|x_2(0)|), \gamma_2 \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \right\}
\]
\[ \leq \max \left\{ \beta_1(|x_1(0)|), \gamma_1 \circ \beta_2(|x_2(0)|), \gamma_2 \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \right\}. \quad (43) \]

For all \( s \in \mathbb{R}_{\geq 0} \), define \( \beta \in \mathcal{K}_\infty \) by \( \beta(s) = \frac{1}{2} \max \{ \beta_1(s), \beta_2(s) \} \), and \( \gamma \in \mathcal{K}_\infty \) by \( \gamma(s) = \frac{1}{2} \max \{ \gamma_1(s), \gamma_2(s) \} \). Combining (39) and (43) yields
\[ \|x_1\|_{\mathcal{L}_2[0,t]}^2 = \|x_1\|^2_{\mathcal{L}_2[0,t]} + \|x_2\|^2_{\mathcal{L}_2[0,t]}
\]
\[ \leq \max \left\{ \beta(|x(0)|), \gamma \left( \|w_2\|^2_{\mathcal{L}_2[0,t]} \right) \right\}
\]
and therefore the cascade connection of (36)-(37) with interconnection \( w_1 = x_2 \) has the nonlinear \( \mathcal{L}_2 \)-gain property.
B. Feedback Interconnection

We consider two feedback interconnections; one without external inputs (Figure 2 with $\eta_1 \equiv \eta_2 \equiv 0$) and one with external inputs (Figure 2, as shown). We include the former as it has a much simpler small-gain condition than the latter and gives rise to an interesting sufficient condition for the stability of feedback interconnections of ISS systems, which we will discuss in Section V (see Theorem 10).

![Fig. 2. Feedback Interconnection](image)

In the diagram of Figure 2, we obviously require that $\eta_i \in \mathcal{W}^{m_i}$, $i = 1, 2$, and, so that the input/output dimensions are consistent, we also require that $m_2 = n_1$ and $n_1 = n_2$.

**Theorem 4:** Suppose systems (36) and (37) satisfy the nonlinear $L_2$-gain bounds (38) and (39), respectively, and the interconnection constraints $w_1 = x_2$ and $w_2 = x_1$. If the small-gain conditions

$$
\text{Id} - \gamma_i \circ \gamma_j \in \mathcal{K}_\infty
$$

are satisfied for $i, j = 1, 2$, $i \neq j$, then the system is $L_2$-stable.

**Proof:** For $i, j = 1, 2$, $i \neq j$, using (38), (39), and the interconnection constraints $w_i = x_j$, we see that

$$
\|x_i\|^2_{L_2[0,t]} \leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \|x_j\|^2_{L_2[0,t]} \right) \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \max \left\{ \beta_j(|x_j(0)|), \gamma_j \left( \|x_i\|^2_{L_2[0,t]} \right) \right\} \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \circ \beta_j(|x_j(0)|) \right\} + \gamma_i \circ \gamma_j \left( \|x_i\|^2_{L_2[0,t]} \right).
$$

Therefore, if $\text{Id} - \gamma_i \circ \gamma_j \in \mathcal{K}_\infty$, we can derive an upper bound on $\|x_i\|^2_{L_2[0,t]} \leq \|x_i^T x_j^T\|^2_{L_2[0,t]}$ that depends only on the initial condition $x(0) = [x_1^T(0), x_2^T(0)]^T$. Specifically, let $\beta_i \in \mathcal{K}_\infty$ for $i, j = 1, 2$, $i \neq j$, be given by

$$
\tilde{\beta}_i(s) = \max \left\{ (\text{Id} - \gamma_i \circ \gamma_j) \circ \beta_i(s), \left( (\text{Id} - \gamma_i \circ \gamma_j) \circ \gamma_i \circ \beta_j(s) \right), \quad \forall s \in \mathbb{R}_{\geq 0}. \right\}
$$

Furthermore, define $\beta \in \mathcal{K}_\infty$ by $\beta \equiv \tilde{\beta}_1 + \tilde{\beta}_2$. Then,

$$
\|x_2\|^2_{L_2[0,t]} = \|x_1\|^2_{L_2[0,t]} + \|x_2\|^2_{L_2[0,t]} \leq \tilde{\beta}_1(|x_1(0)|) + \tilde{\beta}_2(|x_2(0)|) \leq \beta(|x(0)|)
$$

demonstrating that the interconnected system is $L_2$-stable.

**Theorem 5:** Suppose systems (36) and (37) satisfy the nonlinear $L_2$-gain bounds (38) and (39), respectively, and the interconnection constraints $w_1 = x_2 + \eta_1$ and $w_2 = x_1 + \eta_2$. Fix $\varepsilon \in \mathbb{R}_{>0}$. Let $\rho \in \mathcal{K}_\infty$ be such that $(\rho - \text{Id}) \in \mathcal{K}_\infty$ and define

$$
\tilde{\gamma}_i(s) = \gamma_i \circ \rho((1 + \varepsilon^2)s)
$$

for all $s \in \mathbb{R}_{\geq 0}$, $i = 1, 2$. If the small-gain conditions

$$
\text{Id} - \tilde{\gamma}_i \circ \tilde{\gamma}_j \in \mathcal{K}_\infty
$$

are satisfied for $i, j = 1, 2$, $i \neq j$, then the interconnected system satisfies the nonlinear $L_2$-gain property from input $\eta = [\eta_1^T \eta_2^T]^T$ to state $x = [x_1^T x_2^T]^T$.

**Proof:** Let $\mu \in \mathcal{K}_\infty$ be given by $\mu \equiv \rho \circ (\rho - \text{Id})^{-1}$. We derive an upper bound on $\|x_1\|^2_{L_2[0,t]}$ using the nonlinear $L_2$-gain property (38), the interconnection condition $w_i = x_j + \eta_i$, Lemma 6, and Lemma 4 as

$$
\|x_1\|^2_{L_2[0,t]} \leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \|w_i\|^2_{L_2[0,t]} \right) \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \|x_j + \eta_i\|^2_{L_2[0,t]} \right) \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \|x_j\|^2_{L_2[0,t]} + \|\eta_i\|^2_{L_2[0,t]} \right) \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \gamma_i \left( \|x_j\|^2_{L_2[0,t]} \right) \right\}.
$$

Repeating the same arguments, we can derive an upper bound on $\|x_2\|^2_{L_2[0,t]}$ that we then substitute into (47) to obtain

$$
\|x_1\|^2_{L_2[0,t]} \leq \max \left\{ \beta_i(|x_i(0)|), \tilde{\gamma}_i \circ \beta_j(|x_j(0)|), \gamma_i \circ \mu \left( \left( 1 + \frac{1}{\varepsilon^2} \right) \|\eta_i\|^2_{L_2[0,t]} \right) \right\}
$$

$$
\leq \max \left\{ \beta_i(|x_i(0)|), \tilde{\gamma}_i \circ \beta_j(|x_j(0)|), \gamma_i \circ \gamma_j \circ \mu \left( \left( 1 + \frac{1}{\varepsilon^2} \right) \|\eta_i\|^2_{L_2[0,t]} \right) \right\}.
$$

With the small gain condition (46), we see that we can derive an upper bound on $\|x_2\|^2_{L_2[0,t]}$ by terms depending solely on initial conditions $x_1(0) \in \mathbb{R}^{n_1}$, $x_2(0) \in \mathbb{R}^{n_2}$ and inputs $\eta_1 \in \mathcal{W}^{m_1}$ and $\eta_2 \in \mathcal{W}^{m_2}$. With the derived bounds on $\|x_1\|^2_{L_2[0,t]}$ and $\|x_2\|^2_{L_2[0,t]}$ we may bound $\|\eta\|^2_{L_2[0,t]}$ as in the conclusion of the proofs of Proposition 2 and Theorem 4 and we omit the details.

**Remark 4:** We note that by choosing $\rho(s) = (1 + \varepsilon^2)s$ with $\varepsilon^2 \ll 1$, the small gain condition (46) approaches $\text{Id} - \gamma_i \circ \gamma_2, \text{Id} - \gamma_2 \circ \gamma_1 \in \mathcal{K}_\infty$. This is (44) and the obvious analogue of the classical linear small-gain condition given by $\gamma_1 \gamma_2 < 1$, with $\gamma_1, \gamma_2 \in \mathcal{K}_{\infty}$. A further consequence of choosing $\varepsilon^2 < 1$ is that the related functions or constants in Lemma 4 and Lemma 6 become large; i.e.,

$$
\mu(s) \equiv \rho \circ (\rho - \text{Id})^{-1}(s) = \frac{1 + \varepsilon^2}{\varepsilon^2} (1 + \varepsilon^2) s
$$

and $1 + \frac{1}{\varepsilon^2}$, respectively. As can be seen in the proof above, the function $\mu$ and the constant $1 + \frac{1}{\varepsilon^2}$ being large correspond to large bounds on external inputs.
We observe that by using (41) (i.e., \( \rho(s) = 2s \)) and \( \varepsilon^2 = 1 \) the function in the small-gain condition reduces to \( s - \hat{\gamma}_i \circ \hat{\gamma}_j(s) = s - \gamma_i \circ 4\gamma_j(4s) \in K_{\infty} \).

Though we generally adhere to the maximum formulation of gain properties, rather than using the qualitatively equivalent summation formulation, we state here a small-gain theorem for the \( L_2 \)-gain property given by
\[
\|x_i\|^2_{L_2[0,t]} \leq \beta_i(|x_i(0)|) + \gamma_i \left( \|w_i\|^2_{L_2[0,t]} \right) \quad (48)
\]

Despite the qualitative equivalence between (48) and (38), if one is interested in quantitative results, for example in looking to compute tight gain bounds (e.g., \([33],[34]\)), the form of the small-gain condition below is useful.

**Theorem 6:** Suppose systems (36) and (37) satisfy the nonlinear \( L_2 \)-gain bounds (48) for \( i = 1, 2, \) respectively, and the interconnection constraints \( w_1 = x_2 + \eta_1 \) and \( w_2 = x_1 + \eta_2 \). Fix \( \varepsilon \in \mathbb{R}_{>0} \). Let \( \rho \in K_{\infty} \) be such that \( (\rho - \text{Id}) \in K_{\infty} \) and define
\[
\begin{align*}
\hat{\gamma}_i(s) &\equiv \gamma_i \circ \rho \circ \rho(1 + \varepsilon^2)s \quad (49) \\
\hat{\gamma}_j(s) &\equiv \gamma_j \circ \rho(1 + \varepsilon^2)s \\
\end{align*}
\]

for all \( s \in \mathbb{R}_{\geq 0}, \ i = 1, 2 \). If the small-gain conditions
\[
\text{Id} - \hat{\gamma}_i \circ \rho \circ \hat{\gamma}_j \in K_{\infty} \\
\]

are satisfied for \( i, j = 1, 2, \ i \neq j \), then the interconnected system satisfies the nonlinear \( L_2 \)-gain property from input \( \eta = [\hat{\eta}_1^T \hat{\eta}_2^T]^T \) to state \( x = [x_1^T x_2^T]^T \).

**Remark 5:** In order to avoid a proliferation of unnecessary notation in Theorems 5 and 6, we fixed a single constant \( \varepsilon \in \mathbb{R}_{>0} \) and used a single function \( \rho \in K_{\infty} \) such that \( (\rho - \text{Id}) \in K_{\infty} \). In fact, each of the instances of these elements in (45), (49), and (50) may be chosen independently. For example, rather than a single \( \varepsilon \in \mathbb{R}_{>0} \) in (45), it is possible to choose two constants, say \( \varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0} \). This follows from the fact that we could, in principle, fix a different \( \varepsilon \in \mathbb{R}_{>0} \) each time we apply Lemma 6 in the proof of Theorem 5. A similar remark holds with regard to the two appearances of \( \rho \in K_{\infty} \) corresponding to the two applications of Lemma 4 in Theorem 5, as well as for the constants \( \varepsilon \in \mathbb{R}_{>0} \) and functions \( \rho \in K_{\infty} \) in Theorem 6.

V. INTERCONNECTIONS OF (I)ISS SYSTEMS

Cascade and feedback interconnections of ISS and iISS systems have been extensively studied in the literature (see \([6],[17],[18],[23]\) and references therein). In the case of iISS systems, it is known that iISS of the individual subsystems alone is insufficient to guarantee desired properties such as zero-input global asymptotic stability (0-GAS) or iISS of interconnected systems. The results on interconnections of systems with nonlinear \( L_2 \)-gain in Section IV do not appear to require additional conditions and, in light of the relationship between nonlinear \( L_2 \)-gain and iISS described in Theorem 3, we now turn to the relationship between interconnections of systems with nonlinear \( L_2 \)-gain and known results for the interconnection of ISS and iISS systems. We first examine the feedback interconnection of ISS systems.

**Theorem 7:** Suppose systems (36) and (37) are ISS with functions \( \alpha_i, \beta_i, \sigma_i \in K_{\infty}, \ i = 1, 2, \) as in (5) and that the systems are connected in feedback with \( w_1 = x_2 \) and \( w_2 = x_1 \). Let \( \hat{\gamma}_i = 1 \) and let \( T_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i} \) and \( S_i : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i} \) be the changes of coordinates from Theorem 2 that yield new coordinates in which \( \Sigma_1 \) and \( \Sigma_2 \) satisfy linear \( L_2 \)-gain bounds (7) with \( \hat{\beta}_i \in K_{\infty} \) and \( \hat{\gamma}_i = 1 \). If there exist \( c_1, c_2 \in \mathbb{R}_{>0} \) such that, for \( i = 1, 2, \ i \neq j \),
\[
|S_j(\xi_j)| \leq \sqrt{c_i} |T_i(\xi_i)|, \ \forall \xi \in \mathbb{R}^{n_i}, \\
\]

and if \( c_1c_2 < 1 \), then the feedback interconnection is \( \alpha \)-integrable.

**Proof:** With the changes of coordinates for \( \Sigma_1 \) and \( \Sigma_2 \) that yield linear \( L_2 \)-gain with \( \hat{\gamma}_i = 1 \),
\[
\|T_i(x_i)\|^2_{L_2[0,t]} \leq \max \left\{ \hat{\beta}_i(|T_i(x_i(0))|), |S_i(w_i)|^2_{L_2[0,t]} \right\} \\
\]

for \( i = 1, 2 \). Let \( \psi_i = T_i(x_i) \) and \( \xi_i = \psi_i(0) = T_i(x_i(0)) \). Using the bounds (52), and the interconnection conditions, we obtain
\[
\|\psi_i\|^2_{L_2[0,t]} \leq \max \left\{ \hat{\beta}_i(|T_i(x_i(0))|), |S_i(w_i)|^2_{L_2[0,t]} \right\} \\
\]

\[
\leq \max \left\{ \hat{\beta}_i(|\xi_i|), c_2|T_2(x_2)|^2_{L_2[0,t]} \right\} \\
\]

\[
\leq \max \left\{ \hat{\beta}_i(|\xi_i|), c_2\hat{\beta}_2(|\xi_2|), c_2|S_2(x_2)|^2_{L_2[0,t]} \right\} \\
\]

\[
\leq \max \left\{ \hat{\beta}_i(|\xi_i|), c_2\hat{\beta}_2(|\xi_2|) \right\} + c_2c_1|\psi_i|^2_{L_2[0,t]}. \\
\]

Let \( \xi \in [\xi_1 \xi_2]^T \in \mathbb{R}^{n_1+n_2} \) and \( \hat{\beta}_i \in K_{\infty} \) for \( i = 1, 2, \ i \neq j \), be defined by
\[
\hat{\beta}_i(s) = \max \left\{ \hat{\beta}_i(s), c_j\hat{\beta}_j(s) \right\}, \ \forall s \in \mathbb{R}_{\geq 0}. \\
\]

Then
\[
\|\psi_i\|^2_{L_2[0,t]} \leq \frac{1}{1 - c_1c_2} \hat{\beta}_i(|\xi|) \quad \text{and a similar argument yields} \quad |\psi_i|^2_{L_2[0,t]} \leq \frac{1}{1 - c_1c_2} \hat{\beta}_i(|\xi|). \\
\]

Let \( \alpha_i, \alpha_i^\dagger \in K_{\infty} \) come from Lemma 1 applied to the change of coordinates \( T_i(\cdot), \ i = 1, 2, \) and let \( \sigma_i \in K_{\infty} \) come from Lemma 5 applied to \( \alpha_i^\dagger \in K_{\infty} \). Let \( \rho \in K_{\infty} \) be such that \( \rho - \text{Id} \in K_{\infty} \) and define \( \mu = \rho \circ (\rho - \text{Id})^{-1} \in K_{\infty} \). Define \( \hat{\beta}, \beta \in K_{\infty} \) by
\[
\hat{\beta}(s) = \frac{1}{1 - c_1c_2} (\beta(\sigma) + \beta_2(s)) \\
\beta(s) = \max \left\{ \beta \circ \rho \circ \alpha_i(\sigma), \beta \circ \mu \circ \alpha_i^\dagger(\sigma) \right\} \\
\]

for all \( s \in \mathbb{R}_{\geq 0}. \) Then
\[
\int_0^t \alpha(|\dot{x}(\tau)|) d\tau \leq \int_0^t \alpha(|x_1(\tau)| + |x_2(\tau)|) d\tau \\
\leq \int_0^t \alpha_i^\dagger(|x_1(\tau)|) d\tau + \int_0^t \alpha_i^\dagger(|x_2(\tau)|) d\tau \\
\leq \|\psi_1\|^2_{L_2[0,t]} + \|\psi_2\|^2_{L_2[0,t]} \\
\leq \frac{1}{1 - c_1c_2} (\beta(\sigma) + \beta_2(\sigma)) = \hat{\beta}(\sigma) \\
\leq \max \left\{ \beta \circ \rho(\sigma), \beta \circ \mu(\sigma) \right\} \\
\leq \max \left\{ \beta \circ \rho \circ \alpha_i(\sigma), \beta \circ \mu \circ \alpha_i^\dagger(\sigma) \right\} \\
= \beta(|\dot{x}(\sigma)|). \\
\]
The condition \( c_1 c_2 < 1 \) is analogous to the classical \( L_2 \) small-gain theorem [30]. While at first glance the above theorem may appear to provide a much simpler condition to check than, for example, [19, Theorem 2.1], finding the appropriate changes of coordinates so that an arbitrary ISS system exhibits (linear) \( L_2 \)-gain appears to be a difficult task.

When attempting to prove results on interconnected ISS systems directly using the ISS estimates of Definition 3 it is sometimes necessary to impose additional assumptions beyond those already known in the literature. For example, the cascade interconnection of ISS systems is always ISS (e.g., [25, Proposition 7.2]). Attempting to prove this directly using the ISS estimates of Definition 3 requires being able to compare the state scaling, \( \alpha_2 \in \mathcal{K}_\infty \), of the driving system with the input scaling, \( \sigma_1 \in \mathcal{K}_\infty \), of the driven system. A similar assumption is required to prove a small-gain theorem using the ISS estimates of Definition 3. Since such results are less general than those available in the literature we do not present them here.

A. Cascade Interconnections of iISS Systems

In [4], [6], and [17], sufficient conditions are given guaranteeing ISS of a cascade connection of ISS systems. Generally, these conditions involve a relationship between the decay rate of the driving system and the gain of the driven system (\( \Sigma_2 \) and \( \Sigma_1 \), respectively, of Figure 1).

The following sufficient condition for iISS of a cascade interconnection of iISS systems makes use of the qualitative equivalence between iISS systems and those with nonlinear \( L_2 \)-gain as described in Theorem 3.

**Theorem 8:** Suppose systems (36)-(37) are iISS with functions \( \alpha_i, \beta_i, \gamma_i, \sigma_i \in \mathcal{K}_\infty \), \( i = 1, 2 \), as in (6) and that the systems are connected in cascade with \( w_1 = x_2 \). Let \( T_i : \mathbb{R}^m_i \to \mathbb{R}^n_i \) and \( S_i : \mathbb{R}^m_i \to \mathbb{R}^m_i \) be the changes of coordinates from Theorem 3 that yield new coordinates in which \( \Sigma_1 \) and \( \Sigma_2 \) satisfy nonlinear \( L_2 \)-gain bounds (8) with \( \hat{\beta}_i, \hat{\gamma}_i \in \mathcal{K}_\infty \). If there exists \( c \in \mathbb{R}_{>0} \) such that

\[
|S_1(\zeta)| \leq \sqrt{c}|T_2(\zeta)|, \quad \forall \zeta \in \mathbb{R}^{n_2}
\]

then the cascade connection is iISS.

**Proof:** Let \( \psi_i = T_i(x_i), \xi_i = \psi_i(0) = T_i(x_i(0)) \), and \( v_1 = S_i(w_1) \). Using the bounds (38) and (39) with \( \hat{\beta}_i, \hat{\gamma}_i \in \mathcal{K}_\infty \), the interconnection condition \( w_1 = x_2 \), the bound (54), and Lemma 4, we obtain

\[
\|\psi_1\|_{L_2[0,t]} \leq \max \left\{ \hat{\beta}_1(|\xi_1|), \hat{\gamma}_1 \left( \|S_2(x_2)\|_{L_2[0,t]} \right) \right\} 
\]

\[
\leq \max \left\{ \hat{\beta}_1(|\xi_1|), \hat{\gamma}_1 \left( c \|T_2(x_2)\|_{L_2[0,t]} \right) \right\} 
\]

\[
\leq \max \left\{ \hat{\beta}_1(|\xi_1|), \hat{\gamma}_1 \left( c \hat{\beta}_2(|\xi_2|) \right), \hat{\gamma}_1 \left( c \hat{\gamma}_2 \left( \|v_2\|_{L_2[0,t]} \right) \right) \right\} .
\]

For \( i = 1, 2 \), let \( \alpha_i \in \mathcal{K}_\infty \) come from Lemma 1 applied to \( T_i(\cdot) \) and let \( \hat{\alpha} \in \mathcal{K}_\infty \) come from Lemma 5 applied to \( \alpha_i \in \mathcal{K}_\infty \). This yields the following bound:

\[
\int_0^t \hat{\alpha}(|x(t)|)dt \leq \int_0^t \left( \alpha_1^2(|x_1(t)|) + \alpha_2^2(|x_2(t)|) \right) dt 
\]

\[
\leq \|\psi_1\|^2_{L_2[0,t]} + \|v_2\|^2_{L_2[0,t]} 
\]

\[
\leq 2 \max \left\{ \hat{\beta}_1(|\xi_1|), \hat{\gamma}_1 \left( c \hat{\beta}_2(|\xi_2|) \right), \hat{\gamma}_2 \left( \|v_2\|^2_{L_2[0,t]} \right) \right\} .
\]

(55)

For all \( s \in \mathbb{R}_{>0} \), define \( \hat{\beta} = \mathcal{K}_\infty \) by \( \hat{\beta}(s) = 2 \max \left\{ \hat{\beta}_1 \circ \alpha_1^2(s), \hat{\gamma}_1(c \hat{\beta}_2 \circ \alpha_2^2(s)), \hat{\beta}_2 \circ \alpha_2^2(s) \right\} \) and \( \hat{\gamma} \in \mathcal{K}_\infty \) as \( \hat{\gamma}(s) = 2 \max \left\{ \hat{\gamma}_1(c \hat{\gamma}_2(s)), \hat{\gamma}_2(s) \right\} \). Finally, Lemma 1 allows us to upper bound \( |S_2(w_2)| \) by \( \hat{\gamma}^{1/2} \in \mathcal{K}_\infty \). With the above definitions, (55) becomes

\[
\int_0^t \hat{\alpha}(|x(t)|)dt \leq \max \left\{ \hat{\beta}(|x(0)|), \hat{\gamma} \left( \int_0^t \sigma(|w_2(t)|)dt \right) \right\} 
\]

which demonstrates that the cascade is iISS from input \( w_2 \) to state \( x \).

From Theorem 8 we note that while iISS and nonlinear \( L_2 \)-gain are qualitatively equivalent properties (Theorem 3), there is no contradiction between the extra conditions required to guarantee iISS of cascaded systems in [4], [6] and the lack of such extra conditions in Proposition 2 for the cascade of systems with nonlinear \( L_2 \)-gain. In fact, the result of Theorem 8 is similar to the results of [4], [6] in that the sufficient condition requires a relationship between the state change of coordinates of the driving system and the input change of coordinates of the driven system.

The qualitatively equivalent definition of iISS in Definition 4 ([13]) gives rise to the following similar sufficient condition for iISS of cascaded iISS systems.

**Theorem 9:** Suppose systems (36)-(37) are iISS with functions \( \alpha_i, \beta_i, \gamma_i, \sigma_i \in \mathcal{K}_\infty \), \( i = 1, 2 \), as in (6) and that the systems are connected in cascade with \( w_1 = x_2 \). If there exists a \( c \in \mathbb{R}_{>0} \) so that

\[
\sigma_1(s) \leq c \sigma_2(s), \quad \forall s \in \mathbb{R}_{>0},
\]

then the cascade connection is iISS.

We note that the simplicity of the condition (56) and the following proof, as compared with the results of [4] or [6], stems from the fact that the iISS property defined by (6) treats the input and the state in the same manner; i.e., (6) is an integral-to-integral estimate. By contrast, (21) does not treat the input and state in the same manner and consequently relating the input of the driven system to the state of the driving system requires more involved arguments.

**Proof:** The iISS estimate for system (37) is given by

\[
\int_0^t \alpha_2(|x_2(t)|) 
\]

\[
\leq \max \left\{ \beta_2(|x_2(0)|), \gamma_2 \left( \int_0^t \sigma_2(|w_2(t)|)dt \right) \right\} .
\]

(57)
With the iISS estimate (6) for systems (36)-(37), the interconnection condition $w_1 = x_2$, and the condition (56), the following calculation is straightforward:

$$
\int_0^t \alpha_1(|x_1(\tau)|) d\tau \leq \max \left\{ \beta_1(|x_1(0)|), \gamma_1 \left( c_1 \sigma_2(|w_2(\tau)|) \right) \right\}, \quad \gamma_1 \left( c_1 \sigma_2(|w_2(\tau)|) \right). \quad (58)
$$

Let $\alpha \in K_\infty$ come from the application of Lemma 5 to $\alpha_1, \alpha_2 \in K_\infty$ so that (16) together with the bounds (57) and (58) implies

$$
\int_0^t \alpha_1(|x(t)|) d\tau \leq \max \left\{ \beta_1(|x(0)|), \gamma_1 \left( \int_0^t \sigma_2(|w_2(\tau)|) d\tau \right) \right\}
$$

where, for all $s \in \mathbb{R}_{\geq 0}$, $\beta, \gamma \in K_\infty$ are given by $\beta(s) = 2 \max \{ \beta_1(s), \gamma_1(c_1 \beta_2(s)), \beta_2(s) \}$ and $\gamma(s) = 2 \max \{ \gamma_1(c_1 \gamma_2(s)), \gamma_2(s) \}$. Therefore, the cascade system is iISS from input $w_2$ to state $x = [x_1 \, x_2]^T$.

B. Feedback Interconnections of iISS Systems

Similar to the extra conditions required for cascades of iISS systems to be iISS, sufficient conditions for iISS of feedback interconnections have been shown to require more than iISS of the subsystems and a small-gain condition (see [16], [18]). We now turn to the relationship between iISS systems, coordinates in which these systems satisfy the nonlinear $L_2$-gain property, and the small-gain result of Theorem 6.

**Theorem 10:** Suppose systems (36)-(37) are iISS with functions $\alpha_i, \beta_i, \gamma_i, \sigma_i \in K_\infty$, $i = 1, 2$, as in (6) and that the systems are connected in feedback with $w_1 = x_2$ and $w_2 = x_1$. Let $T_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $S_i : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $i = 1, 2$, be the changes of coordinates from Theorem 3 that yield new coordinates in which $\Sigma_1$ and $\Sigma_2$ satisfy nonlinear $L_2$-gain bounds (8) with $\beta_i, \gamma_i \in K_\infty$. Fix $\varepsilon \in \mathbb{R}_{\geq 0}$ and let $\rho \in K_\infty$ be such that $\rho - \text{Id} \in K_\infty$. Suppose there exist $c_{S_i}, c_{T_i} \in \mathbb{R}_{>0}$, $i = 1, 2$, such that

$$
|S_i(\zeta)| \leq \sqrt{c_{S_i}} |T_i(\zeta)|, \quad \forall \zeta \in \mathbb{R}^{m_i}, \quad (59)
$$

and if the small-gain conditions

$$
\text{Id} - \gamma_i(c_{j} \gamma_j(c_{j})) \in K_\infty \quad (60)
$$

hold, then the feedback interconnection is $\alpha$-integrable.

**Remark 6:** We observe that, with the exception of the constants $c_1$ and $c_2$, the small-gain condition in Theorem 10 is the same as that of Theorem 4. However, to obtain a stability result for general iISS systems requires the additional conditions on state and input changes of coordinates given in (59).

**Proof:** With the changes of coordinates for $\Sigma_1$ and $\Sigma_2$ that yield nonlinear $L_2$-gain with $\beta_i, \gamma_i \in K_\infty$, we have

$$
\|T_i(x_i)\|_{L_2[0,t]} \leq \max \left\{ \beta_i(|T_i(x_i(0))|), \gamma_i \left( \|S_i(w_i)\|_{L_2[0,t]} \right) \right\}
$$

for $i = 1, 2$. Let $\psi_i \triangleq T_i(x_i)$ and $\xi_i \triangleq \psi_i(0) = T_i(x_i(0))$. Using the bounds (59), and the interconnection conditions we obtain

$$
\|\psi_i\|_{L_2[0,t]}^2 \leq \max \left\{ \beta_1(|\xi_i|), \gamma_1 \left( \|S_i(x_i)\|_{L_2[0,t]}^2 \right) \right\}
$$

Then the small-gain condition (60) yields an upper bound on $\|\psi_i\|_{L_2[0,t]}^2$ in terms of the initial condition $\xi_i \in \mathbb{R}^{n_1+n_2}$. A similar argument yields a similar bound for $\|\psi_2\|_{L_2[0,t]}^2$. From here we follow the argument in (53) to obtain that the feedback interconnection is $\alpha$-integrable.

When we additionally allow external inputs we obtain the following result.

**Theorem 11:** Suppose systems (36)-(37) are iISS with functions $\alpha_i, \beta_i, \gamma_i, \sigma_i \in K_\infty$, $i = 1, 2$, as in (6) and that the systems are connected in feedback with $w_1 = x_2 + \eta_1$ and $w_2 = x_1 + \eta_2$. Let $T_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $S_i : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $i = 1, 2$, be the changes of coordinates from Theorem 3 that yield new coordinates in which $\Sigma_1$ and $\Sigma_2$ satisfy nonlinear $L_2$-gain bounds (8) with $\beta_i, \gamma_i \in K_\infty$. Fix $\varepsilon \in \mathbb{R}_{\geq 0}$ and let $\rho \in K_\infty$ be such that $\rho - \text{Id} \in K_\infty$. Suppose there exist $c_{S_i}, c_{T_i} \in \mathbb{R}_{>0}$, $i = 1, 2$, such that

$$
|S_i(\zeta)| \leq \sqrt{c_{S_i}} |T_i(\zeta)|, \quad \forall \zeta \in \mathbb{R}^{m_i}, \quad (61)
$$

$$
|\zeta| \leq \sqrt{c_{T_i}} |T_i(\zeta)|, \quad \forall \zeta \in \mathbb{R}^{m_i}, \quad (62)
$$

and define $\hat{\gamma}_i \in K_\infty$, $i, j = 1, 2$, $i \neq j$, by

$$
\hat{\gamma}_i(s) \triangleq \gamma_0 \circ \gamma_1 \circ (\gamma_2, \gamma_3)(s), \quad \forall s \in \mathbb{R}_{\geq 0}. \quad (63)
$$

If the small-gain conditions

$$
\text{Id} - \hat{\gamma}_i \circ \hat{\gamma}_j \in K_\infty \quad (64)
$$

hold for $i = 1, 2$, $i \neq j$, then the feedback interconnection is iISS.

The proof is similar to that of Theorem 10 with two essential differences. The first is in the need to appeal to Lemma 4 to overbound $K_\infty$ functions of sums. The second difference comes from the need to use Lemma 6 and conditions (61) and (62) to overbound $L_2$-norms of sums. In particular, the sector bounds (61) and (62) are used to derive bounds in the following way:

$$
|S_i(w_i)|_{L_2[0,t]}^2 = |S_i(x_j + \eta_i)|_{L_2[0,t]}^2
$$

$$
\leq c_{S_i} |x_j + \eta_i|_{L_2[0,t]}^2
$$

$$
\leq c_{S_i} (1 + \varepsilon^2) |x_j|_{L_2[0,t]}^2 + c_{S_i} (1 + \frac{1}{\varepsilon^2}) |\eta_i|_{L_2[0,t]}^2
$$

$$
\leq c_{S_i} (1 + \varepsilon^2) c_{T_j} |T_j(x_j)|_{L_2[0,t]}^2 + c_{S_i} (1 + \frac{1}{\varepsilon^2}) |\eta_i|_{L_2[0,t]}^2.
$$

We then appeal to the fact that, in the coordinates defined by the change of coordinates $T_j(\cdot)$, the system has the nonlinear
\(L_2\)-gain property. With this calculation, the proof then closely follows that of Theorem 10 and we omit further details.

Similar to Theorem 9, the qualitatively equivalent definition of iISS in Definition 4 ([3]) yields the following novel sufficient condition for iISS of the feedback interconnection of iISS systems.

**Theorem 12:** Suppose systems (36)-(37) are iISS with functions \(\alpha_i, \beta_i, \gamma_i, \sigma_i \in K_\infty, \ i = 1, 2\), as in (6) and that the systems are connected in feedback with \(w_1 = x_2 + \eta_1\) and \(w_2 = x_1 + \eta_2\). If there exist \(c_i \in \mathbb{R}_{>0}\) and \(\rho_i \in K_\infty\) with \(\rho_i - \text{Id} \in K_\infty\) so that

\[
\sigma_i \circ \rho_i (s) \leq c_j \alpha_j (s), \quad \forall s \in \mathbb{R}_{\geq 0}
\]

(65)

for \(i, j = 1, 2, i \neq j\), and if there exists \(\rho \in K_\infty\) with \(\rho - \text{Id} \in K_\infty\) so that

\[
\text{Id} - \gamma_i \circ \rho \circ c_j \gamma_j (c_i) \in K_\infty,
\]

(66)

then the feedback interconnection is iISS.

The proof of Theorem 12 involves deriving several lengthy but entirely straightforward upper bounds similar to the proofs of Theorems 4, 5, and 9. We omit the details.

**VI. CONCLUSION**

In this paper we have clarified the relationship between various ISS-type and \(L_2\)-type stability properties. In particular, we have demonstrated the qualitative equivalence between \(L_2\)-stability and \(\alpha\)-integrability, between linear \(L_2\)-gain and ISS, and between nonlinear \(L_2\)-gain and integral ISS. Demonstrating these qualitative equivalences is done by considering nonlinear changes of coordinates.

We further presented several new sufficient conditions for stability of systems connected in cascade or feedback. These conditions are derived using various qualitatively equivalent versions of the desired stability properties. In particular, we have clarified the relationship between cascade and feedback stability results for systems with nonlinear \(L_2\)-gain and results in the literature for cascade and feedback stability results for iISS systems, where the latter systems are known to require extra conditions to guarantee the desired stability results.

It appears unlikely that the results of Theorems 8, 10, and 11 will be useful in the sense of providing easily checkable conditions for stability of cascade or feedback interconnected systems due to the fact that finding the appropriate changes of coordinates would seem to be a challenging task. However, these theorems serve to illustrate that there is no contradiction between the stability results for systems already in coordinates such that the nonlinear \(L_2\)-gain property holds, such as Proposition 2 and Theorem 6, and the results available in the literature on interconnections of iISS systems.

**ACKNOWLEDGEMENTS**

The authors would like to thank Hiroshi Ito, Fabian Wirth, and Huan Zhang for helpful discussions on this work.

**REFERENCES**

[1] D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. Systems & Control Letters, 38(4-5):209–217, 1999.

[2] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. IEEE Transactions on Automatic Control, 45(6):1082–1097, 2000.

[3] D. Angeli, E. D. Sontag, and Y. Wang. Further equivalences and semiglobal versions of integral input to state stability. Dynamics and Control, 10(2):127–149, 2000.

[4] M. Arcak, D. Angeli, and E. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. SIAM J. Control and Optimization, 40(6):1888–1904, 2002.

[5] T. Basar and P. Bernhard. \(H_\infty\)-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach. Birkhäuser, 2nd edition, 2008.

[6] A. Chaillot and D. Angeli. Integral input to state stable systems in cascade. Systems & Control Letters, 57:519–527, 2008.

[7] C. A. Desoer and M. Vidyasagar. Feedback Systems: Input-Output Properties. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2009. Originally published 1975 by Academic Press.

[8] P. M. Dower and C. M. Kellett. A dynamic programming approach to the approximation of nonlinear \(L_2\)-gain. In Proceedings of the 47th IEEE Conference on Decision and Control, pages 1–6, Cancun, Mexico, December 2008.

[9] P. M. Dower, C. M. Kellett, and H. Zhang. A weak \(L_2\)-gain property for nonlinear systems. In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, Hawaii, December 2012.

[10] P. M. Dower, H. Zhang, and C. M. Kellett. Nonlinear \(L_2\)-gain verification for nonlinear systems. Systems & Control Letters, 61:563–572, 2012.

[11] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State-space solutions to standard \(H_2\) and \(H_\infty\) control problems. IEEE Transactions on Automatic Control, 34(8):831–847, 1989.

[12] L. Grune, E. D. Sontag, and F. R. Wirth. Asymptotic stability equals exponential stability, and ISS equals finite energy gain – if you twist your eyes. Systems and Control Letters, 38(2):127–134, 1999.

[13] W. Hahn. Stability of Motion. Springer-Verlag, 1967.

[14] J. W. Helton and M. R. James. Extending \(H_\infty\) Control to Nonlinear Systems: Control of Nonlinear Systems to Achieve Performance Objectives. Society for Industrial and Applied Mathematics, 1999.

[15] L. Huang, M. R. James, D. Nedić, and P. M. Dower. A unified approach to controller design for achieving ISS and related properties. IEEE Transactions on Automatic Control, 50(11):1681–1697, November 2005.

[16] H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. IEEE Transactions on Automatic Control, 51(10):1626–1643, 2006.

[17] H. Ito. A Lyapunov approach to cascade interconnection of integral input-to-state stable systems. IEEE Transactions on Automatic Control, 55(3):702–708, March 2010.

[18] H. Ito and Z.-P. Jiang. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective. IEEE Transactions on Automatic Control, 54(10):2389–2404, 2009.

[19] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for input-to-state stability and related properties. IEEE Transactions on Automatic Control, 48(11):1663–1682, 2003.

[20] M. Krstić, I. Kanellakopoulos, and P. Kokotović. Nonlinear and Adaptive Control Design. John Wiley and Sons, Inc., 1995.

[21] B. S. Ruffer, C. M. Kellett, and S. R. Weller. Connection between cooperative positive systems and integral input-to-state stability of large-scale systems. Automatica, 47(7):1188–1200, July 2010.

[22] R. Sepulchre, M. Jankovič, and P. Kokotović. Constructive Nonlinear Control. Springer-Verlag, 1997.

[23] E. D. Sontag. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control, 34(4):433–443, April 1989.

[24] E. D. Sontag. Comments on integral variants of ISS. Systems and Control Letters, 34(1-2):93–100, 1998.
[27] E. D. Sontag. Input to state stability: Basic concepts and results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, pages 163–220. Springer, 2007.

[28] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Letters*, 24:351–359, 1995.

[29] A. R. Teel, E. Panteley, and A. Loria. Integral characterizations of uniform asymptotic stability and exponential stability with applications. *Mathematics of Control, Signals, and Systems*, 15:177–201, 2002.

[30] A. van der Schaft. *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer, 2nd edition, 2000.

[31] G. Zames. On the input-output stability of time-varying nonlinear feedback systems part I: Conditions derived using concepts of loop gain, conicity, and passivity. *IEEE Transactions on Automatic Control*, 11:228–238, 1966.

[32] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26:301–320, 1981.

[33] H. Zhang and P. M. Dower. Performance bounds for nonlinear systems with a nonlinear $L_2$-gain property. *International Journal of Control*, 85:1293–1312, 2012.

[34] H. Zhang and P. M. Dower. Computation of tight integral input-to-state stability bounds for nonlinear systems. *Systems & Control Letters*, 62:355–365, 2013.

[35] H. Zhang, P. M. Dower, and C. M. Kellett. State feedback controller synthesis to achieve a nonlinear $L_2$-gain property. In *Proceedings of the 2011 IFAC World Congress*, Milan, Italy, August 2011.

[36] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.