ACTIVE-SET METHODS FOR CONVEX QUADRATIC PROGRAMMING

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Abstract

Computational methods are proposed for solving a convex quadratic program (QP). Active-set methods are defined for a particular primal and dual formulation of a QP with general equality constraints and simple lower bounds on the variables. In the first part of the paper, two methods are proposed, one primal and one dual. These methods generate a sequence of iterates that are feasible with respect to the equality constraints associated with the optimality conditions of the primal-dual form. The primal method maintains feasibility of the primal inequalities while driving the infeasibilities of the dual inequalities to zero. In contrast, the dual method maintains feasibility of the dual inequalities while moving to satisfy the infeasibilities of the primal inequalities. In each of these methods, the search directions satisfy a KKT system of equations formed from Hessian and constraint components associated with an appropriate column basis. The composition of the basis is specified by an active-set strategy that guarantees the nonsingularity of each set of KKT equations. Each of the proposed methods is a conventional active-set method in the sense that an initial primal- or dual-feasible point is required. In the second part of the paper, it is shown how the quadratic program may be solved as coupled pair of primal and dual quadratic programs created from the original by simultaneously shifting the simple-bound constraints and adding a penalty term to the objective function. Any conventional column basis may be made optimal for such a primal-dual pair of shifted-penalized problems. The shifts are then updated using the solution of either the primal or the dual shifted problem. An obvious application of this approach is to solve a shifted dual QP to define an initial feasible point for the primal (or vice versa). The computational performance of each of the proposed methods is evaluated on a set of convex problems from the CUTEst test collection.

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1. Introduction

We consider the formulation and analysis of active-set methods for a convex quadratic program (QP) of the form

\[
\begin{aligned}
\text{minimize} & \quad c^T x + \frac{1}{2} x^T H x + \frac{1}{2} y^T M y \\
\text{subject to} & \quad A x + M y = b, \quad x \geq 0, \\
& \quad x, y \in \mathbb{R}^n, m,
\end{aligned}
\]  

(1.1)

where \( A, b, c, H \) and \( M \) are constant, with \( H \) and \( M \) symmetric positive semidefinite. It is assumed throughout that the matrix \( (A \ M) \) associated with the equality constraints has full row rank. (This assumption can be made without loss of generality, as shown in Proposition A.8.)

In order to simplify the theoretical discussion, the inequalities of (1.1) involve nonnegativity constraints only. However, the methods to be described are easily extended to treat all forms of linear constraints. (Numerical results are given for problems with constraints in the form \( x_L \leq x \leq x_U \) and \( b_L \leq A x \leq b_U \), for fixed vectors \( x_L, x_U, b_L \) and \( b_U \).) If \( M = 0 \), the QP (1.1) is a conventional convex quadratic program with constraints defined in standard form. A regularized quadratic program may be obtained by defining \( M = \mu I \) for some small positive parameter \( \mu \).

Active-set methods for quadratic programming solve a sequence of linear equations that involve the \( y \)-variables and a subset of the \( x \)-variables. Each set of equations constitutes the optimality conditions associated with an equality-constrained quadratic subproblem. The goal is to predict the optimal active set, i.e., the set of constraints that are satisfied with equality, at the solution of the problem. A conventional active-set method has two phases. In the first phase, a feasible point is found while ignoring the objective function; in the second phase, the objective is minimized while feasibility is maintained. A useful feature of active-set methods is that they are well-suited for “warm starts”, where a good estimate of the optimal active set is used to start the algorithm. This is particularly useful in applications where a sequence of quadratic programs is solved, e.g., in a sequential quadratic programming method or in an ODE- or PDE-constrained problem with mesh refinement. Other applications of active-set methods for quadratic programming include mixed-integer nonlinear programming, portfolio analysis, structural analysis, and optimal control.

In Section 2, the primal and dual forms of a convex quadratic program with constraints in standard form are generalized to include general lower bounds on both the primal and dual variables. These problems constitute a primal-dual pair that includes problem (1.1) and its associated dual as a special case. In Sections 3 and 4, an active-set method is proposed for each of the primal and dual forms associated with the generalized problem of Section 2. Both of these methods generate a sequence of iterates that are feasible with respect to the equality constraints associated with the optimality conditions of the primal-dual problem pair. The primal method maintains feasibility of the primal inequalities while driving the infeasibilities of the dual inequalities to zero. By contrast, the dual method maintains feasibility of the dual inequalities while moving to satisfy the infeasibilities of the primal inequalities. In each of these methods, the search directions satisfy a KKT system of equations formed from Hessian and constraint components associated with an appropriate col-
umn basis. The composition of the basis is specified by an active-set strategy that guarantees the nonsingularity of each set of KKT equations.

The methods formulated in Sections 2–4 define conventional active-set methods in the sense that an initial feasible point is required. In Section 5, a method is proposed that solves a pair of coupled quadratic programs created from the original by simultaneously shifting the simple-bound constraints and adding a penalty term to the objective function. Any conventional column basis can be made optimal for such a primal-dual pair of shifted-penalized problems. The shifts are then updated using the solution of either the primal or the dual shifted problem. An obvious application of this idea is to solve a shifted dual QP to define an initial feasible point for the primal, or vice-versa. In addition to the obvious benefit of using the objective function while getting feasible, this approach provides an effective method for finding a dual-feasible point when $H$ is positive semidefinite and $M = 0$. Finding a dual-feasible point is relatively straightforward for the strictly convex case, i.e., when $H$ is positive definite. However, in the general case, the dual constraints for the phase-one linear program involve entries from $H$ as well as $A$, which complicates the formulation of the phase-one method considerably.

Finally, in Section 7 some numerical experiments are presented for a simple MATLAB implementation of the coupled primal-dual method applied to a set of convex problems from the CUTEst test collection [33].

There are a number of alternative active-set methods available for solving a QP with constraints written in the format of problem (1.1). Broadly speaking, these methods fall into three classes defined here in the order of increasing generality: (i) methods for strictly convex quadratic programming ($H$ symmetric positive definite) [1, 22, 31, 41, 44]; (ii) methods for convex quadratic programming ($H$ symmetric positive semidefinite) [6, 26, 38, 39, 45]; and (iii) methods for general quadratic programming (no assumptions on $H$ other than symmetry) [2, 3, 9, 17, 19, 23, 28–30, 32, 35–37, 45]. Of the methods specifically designed for convex quadratic programming, only the methods of Boland [6] and Wong [45, Chapter 4] are dual active-set methods. Some existing active-set quadratic programming solvers include QPOPT [24], QPSchur [1], SQOPT [26], SQIC [30] and QPA (part of the GALAHAD software library) [34].

The primal active-set method proposed in Section 3 is motivated by the methods of Fletcher [17], Gould [32], and Gill and Wong [30], which may be viewed as methods that extend the properties of the simplex method to general quadratic programming. At each iteration, a direction is computed that satisfies a nonsingular system of linear equations based on an estimate of the active set at a solution. The equations may be written in symmetric form and involve both the primal and dual variables. In this context, the purpose of the active-set strategy is not only to obtain a good estimate of the optimal active set, but also to ensure that the systems of linear equations that must be solved at each iteration are nonsingular. This strategy allows the application of any convenient linear solver for the computation of the iterates. In this paper, these ideas are applied to convex quadratic programming. The resulting sequence of iterates is the same as that generated by an algorithm for general QP, but the structure of the iteration is different, as is the structure of the linear equations that must be solved. Similar ideas are used to formulate the new
dual active-set method proposed in Sections 4.

The proposed primal, dual, and coupled primal-dual methods use a “conventional” active-set approach in the sense that the constraints remain unchanged during the solution of a given QP. Alternative approaches that use a parametric active-set method have been proposed by Best [4, 5], Ritter [42, 43], Ferreau, Bock and Diehl [16], Potschka et al. [40] and implemented in the qpOASES package by Ferreau et al. [15]. The use of shifts for the bounds have been suggested by Cartis and Gould [11] in the context of interior methods for linear programming. Another class of active-set methods that are shown to be convergent for strictly convex quadratic programs have been considered by Curtis, Han, and Robinson [12].

Notation and terminology. Given vectors $a$ and $b$ with the same dimension, $\min(a, b)$ is a vector with components $\min(a_i, b_i)$. The vectors $e$ and $e_j$ denote, respectively, the column vector of ones and the $j$th column of the identity matrix $I$. The dimensions of $e$, $e_i$ and $I$ are defined by the context. Given vectors $x$ and $y$, the column vector consisting of the components of $x$ augmented by the components of $y$ is denoted by $(x, y)$.

2. Background

Although the purpose of this paper is the solution of quadratic programs of the form (1.1), for reasons that will become evident in Section 5, the analysis will focus on the properties of a pair of problems that may be interpreted as a primal-dual pair of QPs associated with problem (1.1).

2.1. Formulation of the primal and dual problems

For given constant vectors $q$ and $r$, consider the pair of convex quadratic programs

\[(PQP_{q,r}) \quad \begin{array}{l}
\text{minimize} \\
\quad \frac{1}{2} x^T H x + \frac{1}{2} y^T M y + c^T x + r^T x \\
\text{subject to} \\
\quad A x + M y = b, \\
\quad x \geq -q,
\end{array} \]

and

\[(DQP_{q,r}) \quad \begin{array}{l}
\text{maximize} \\
\quad -\frac{1}{2} x^T H x - \frac{1}{2} y^T M y + b^T y - q^T z \\
\text{subject to} \\
\quad -H x + A^T y + z = c, \\
\quad z \geq -r.
\end{array} \]

The following result gives joint optimality conditions for the triple $(x, y, z)$ such that $(x, y)$ is optimal for $(PQP_{q,r})$, and $(x, y, z)$ is optimal for $(DQP_{q,r})$. If $q$ and $r$ are zero, then $(PQP_{0,0})$ and $(DQP_{0,0})$ are the primal and dual problems associated with (1.1). For arbitrary $q$ and $r$, $(PQP_{q,r})$ and $(DQP_{q,r})$ are essentially the dual of each other, the difference is only an additive constant in the value of the objective function.

Proposition 2.1. Let $q$ and $r$ denote constant vectors in $\mathbb{R}^n$. If $(x, y, z)$ is a given triple in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$, then $(x, y)$ is optimal for $(PQP_{q,r})$ and $(x, y, z)$ is optimal
for \((DQP_{q,r})\) if and only if
\[
Ax + My - b = 0, \tag{2.1a}
\]
\[
Hx + c - ATy - z = 0, \tag{2.1b}
\]
\[
x + q \geq 0, \tag{2.1c}
\]
\[
z + r \geq 0, \tag{2.1d}
\]
\[
(x + q)^T(z + r) = 0. \tag{2.1e}
\]

In addition, it holds that \(\text{optval}(PQP_{q,r}) - \text{optval}(DQP_{q,r}) = -q^Tr\). Finally, (2.1) has a solution if and only if the sets
\[
\{ (x, y, z) : -Hx + ATy + z = c, z \geq -r \}\quad \text{and} \quad \{ x : Ax + My = b, x \geq -q \}
\]
are both nonempty.

**Proof.** Let the vector of Lagrange multipliers for the constraints \(Ax + My - b = 0\) be denoted by \(\tilde{y}\). Without loss of generality, the Lagrange multipliers for the bounds \(x + q \geq 0\) of \((PQP_{q,r})\) may be written in the form \(z + r\), where \(r\) is the given fixed vector. With these definitions, a Lagrangian function \(L(x, y, \tilde{y}, z)\) associated with \((PQP_{q,r})\) is given by
\[
L(x, y, \tilde{y}, z) = \frac{1}{2}x^THx + \frac{1}{2}y^TMy - \tilde{y}^T(Ax + My - b) - (z + r)^T(x + q),
\]
with \(z + r \geq 0\). Stationarity of the Lagrangian with respect to \(x\) and \(y\) implies that
\[
Hx + c + r - AT\tilde{y} - z - r = Hx + c - AT\tilde{y} - z = 0, \tag{2.2a}
\]
\[
My - M\tilde{y} = 0. \tag{2.2b}
\]
The optimality conditions for \((PQP_{q,r})\) are then given by: (i) the feasibility conditions (2.1a) and (2.1c); (ii) the nonnegativity conditions (2.1d) for the multipliers associated with the bounds \(x + q \geq 0\); (iii) the stationarity conditions (2.2); and (iv) the complementarity conditions (2.1e). The vector \(y\) appears only in the term \(My\) of (2.1a) and (2.2b). In addition, (2.2b) implies that \(My = M\tilde{y}\), in which case we may choose \(y = \tilde{y}\). This common value of \(y\) and \(\tilde{y}\) must satisfy (2.2a), which is then equivalent to (2.1b). The optimality conditions (2.1) for \((PQP_{q,r})\) follow directly.

With the substitution \(\tilde{y} = y\), the expression for the primal Lagrangian may be rearranged so that
\[
L(x, y, y, z) = -\frac{1}{2}x^THx - \frac{1}{2}y^TMy + b^Ty - q^Tz + (Hx + c - ATy - z)^Tx - q^Tr. \tag{2.3}
\]
Taking into account (2.2) for \(y = \tilde{y}\), the dual objective is given by (2.3) as \(-\frac{1}{2}x^THx - \frac{1}{2}y^TMy + b^Ty - q^Tz - q^Tr\), and the dual constraints are \(Hx + c - ATy - z = 0\) and \(z + r \geq 0\). It follows that \((DQP_{q,r})\) is equivalent to the dual of \((PQP_{q,r})\), the only difference is the constant term \(-q^Tr\) in the objective, which is a consequence of the shift \(z + r\) in the dual variables. Consequently, strong duality for convex quadratic programming implies \(\text{optval}(PQP_{q,r}) - \text{optval}(DQP_{q,r}) = -q^Tr\). In addition, the
variables $x$, $y$ and $z$ satisfying (2.1) are feasible for (PQP$_{q,r}$) and (DQP$_{q,r}$) with the difference in the objective function value being $-q^T r$. It follows that $(x, y, z)$ is optimal for (DQP$_{q,r}$) as well as (PQP$_{q,r}$). Finally, feasibility of both (PQP$_{q,r}$) and (DQP$_{q,r}$) is both necessary and sufficient for the existence of optimal solutions.  

2.2. Optimality conditions and the KKT equations

The joint optimality conditions (2.1) may be written in terms of the index sets of fixed and free primal variables. At an arbitrary point $x$, consider the index sets

$$A(x) = \{ i : x_i = -q_i \} \quad \text{and} \quad F(x) = \{ 1, 2, \ldots, n \} \setminus A(x).$$

The set $A(x)$ is the active set of bound constraints at the point $x$. The variables with indices in $F(x)$ are referred to as the free variables. If $z$ is any $n$-vector, the $n_F$-vector $z_F$ and $n_A$-vector $z_A$ denote the vectors of components of $z$ associated with $F(x)$ and $A(x)$. For the symmetric Hessian $H$, the matrices $H_{FF}$ and $H_{AA}$ denote the subset of rows and columns of $H$ associated with the sets $F(x)$ and $A(x)$, respectively. The unsymmetric matrix of components $h_{ij}$ with $i \in F(x)$ and $j \in A(x)$ will be denoted by $H_{FA}$. Similarly, $A_F$ and $A_A$ denote the matrices of free and active columns of $A$. With these definitions, the joint optimality conditions (2.1) may be written in the form

$$A_Fx_F + A_Ax_A + My - b = 0, \quad (2.4a)$$
$$H_{FF}x_F + H_{FA}x_A + c_F - A_F^Ty_F - z_F = 0, \quad x_F + q_F \geq 0, \quad z_F + r_F = 0, \quad (2.4b)$$
$$H_{FA}^Tx_F + H_{AA}x_A + c_A - A_A^Ty_A - z_A = 0, \quad x_A + q_A = 0, \quad z_A + r_A \geq 0. \quad (2.4c)$$

Eliminating $x_A$ and $z_F$ using the equalities $z_F + r_F = 0$ and $x_A + q_A = 0$ yields the equations

$$A_Fx_F + My = b + A_Aq_A,$$
$$H_{FF}x_F - A_F^Ty_F = H_{FA}q_A - c_F - r_F,$$
$$H_{FA}^Tx_F - A_A^Ty_A - z_A = H_{AA}q_A - c_A,$$

which may be expressed in matrix form as

$$\begin{pmatrix} H_{FF} & A_F^T \\ A_F & -M \end{pmatrix} \begin{pmatrix} x_F \\ -y \end{pmatrix} = \begin{pmatrix} H_{FA}q_A - c_F - r_F \\ A_Aq_A + b \end{pmatrix}, \quad (2.5)$$

with $z_A = H_{FA}^Tx_F + c_A - A_A^Ty_A - H_{AA}q_A$. If a solution of (PQP$_{q,r}$) or (DQP$_{q,r}$) exists, then the equations (2.5) are compatible, but not necessarily nonsingular.

The proposed methods are based on maintaining index sets $\mathcal{B}$ and $\mathcal{N}$ that approximate the free and active sets $F$ and $A$ at a solution. The sets $\mathcal{B}$ and $\mathcal{N}$ define a partition of the index set $I = \{ 1, 2, \ldots, n \}$, i.e., $I = \mathcal{B} \cup \mathcal{N}$ with $\mathcal{B} \cap \mathcal{N} = \emptyset$.

Following standard terminology, we refer to the subvectors $x_B$ and $x_N$ associated with an arbitrary $x$ as the basic and nonbasic variables, respectively. Let $H_{BB}$, $H_{NN}$,
$H_{BN}$, $A_B$, and $A_N$ denote submatrices of $H$ and $A$ analogous to $H_{FF}$, $H_{AA}$, $H_{FA}$, $A_F$, and $A_A$, respectively. The crucial distinction between $\mathcal{B}$ and $\mathcal{F}$ is that the basic set $\mathcal{B}$ is defined in such a way that the KKT matrix

$$K_B = \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix}$$

is nonsingular (cf. equation (2.5)). As in Gill and Wong [30], any set $\mathcal{B}$ such that $K_B$ is nonsingular is referred to as a second-order consistent basis. Methods that impose restrictions on the eigenvalues of $K_B$ are known as inertia-controlling methods. (For a description of inertia-controlling methods for general quadratic programming, see, e.g., Gill et al. [29], and Gill and Wong [30].) Given a point $(x, y, z)$ satisfying the optimality conditions (2.4), it is always possible to define a second-order consistent basis $\mathcal{B}$ such that $(x, y, z)$ satisfies the conditions

\begin{align}
A_B x_B + A_N x_N + M y &= b, \\
H_{BB} x_B + H_{BN} x_N + c_B - A_B^T y - z_B &= 0, \\
H_{BA} x_B + H_{FA} x_N + c_A - A_B^T y - z_A &= 0, \\
H_B x_B + H_N x_N + c_N - A_N^T y - z_N &= 0,
\end{align}

(2.6a) (2.6b) (2.6c)

(For simplicity, it is assumed that $\mathcal{B}$ and $\mathcal{N}$ can be defined so that $\mathcal{N} \subseteq A(x)$. In practice it may be necessary to include indices in $\mathcal{N}$ that correspond to variables that are temporarily fixed at their current values, see, e.g., Gill and Wong [30, Section 6].) Eliminating $x_N$ and $z_B$ from the equality conditions of (2.6a) and (2.6b) gives $(x_B, y)$ as the unique solution of the equations

$$\begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix} \begin{pmatrix} x_B \\ -y \end{pmatrix} = \begin{pmatrix} H_{BN} q_N - c_B - r_B \\ A_N q_N + b \end{pmatrix}.$$  

(2.7)

Once $x_B$ and $y$ have been defined, the $z_N$-variables may be computed as

$$z_N = H_{BN}^T x_B - H_{NA} q_N + c_N - A_N^T y.$$  

(2.8)

The two methods proposed in this paper generate a sequence of iterates that satisfy the equality conditions of (2.6) for some partition $\mathcal{B}$ and $\mathcal{N}$. The primal method of Section 3 imposes the restriction that $x_B + q_B \geq 0$, which implies that the sequence of iterates is primal feasible. The dual method of Section 4 imposes dual feasibility via the bounds $z_N + r_N \geq 0$.

The primal and dual methods are derived in terms of a common framework that serves to emphasize the similarities in the methods. In particular, the methods require the solution of a common set of equations defined in terms of a partition of the index set $\mathcal{I} = \{1, 2, \ldots, n\}$. At the start of each iteration, $\mathcal{I}$ is partitioned into disjoint sets $\mathcal{B}$ and $\mathcal{N}$ such that $\mathcal{B} \cup \mathcal{N} = \mathcal{I}$. This initial partition has the property that $(x, y, z)$ is uniquely defined via the equations

$$\begin{align}
H x + c - A^T y - z &= 0, \\
A x + M y - b &= 0,
\end{align}$$

with $x_N + q_N$ and $z_B + r_B$ fixed.  

(2.9)
During an iteration, $x_B + q_B$ and $z_N + r_N$ are free to move. In a primal iteration, $x$ is primal feasible, with $x_N + q_N = 0$ and $x_B + q_B \geq 0$. In a dual iteration, $z$ is dual feasible, with $z_N + r_N \geq 0$ and $z_B + r_B = 0$.

An iteration of the primal method begins with the identification of a nonbasic primal variable $x_l$ with an infeasible dual value, i.e., $z_l + r_l < 0$. The index $l$ is removed from the nonbasic set, i.e., $N \leftarrow N \setminus \{l\}$, and there follows a finite number of subiterations during which $z_l + r_l$ is driven to zero from below. During these subiterations, the index $l$ remains distinct from $B$ and $N$ while indices are moved from $B$ to $N$ in order to maintain primal feasibility. On completion of the final subiteration, $z_l + r_l = 0$ and the index $l$ is added to the basic set, i.e., $B \leftarrow B \cup \{l\}$, which defines a conventional partition $B \cup N = \{1, 2, \ldots, n\}$ for the next iteration.

At each subiteration of both the primal and dual methods, an index $l$ and index sets $B$ and $N$ are known such that $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$. A search direction $(\Delta x, \Delta y, \Delta z)$ is defined that satisfies the identities

$$H \Delta x - A^T \Delta y - \Delta z = 0, \quad (2.10a)$$
$$A \Delta x + M \Delta y = 0, \quad (2.10b)$$
$$\Delta x_N = 0, \quad (2.10c)$$
$$\Delta z_B = 0. \quad (2.10d)$$

As $l \not\in B$ and $l \not\in N$, these identities allow both $\Delta x_l$ and $\Delta z_l$ to be nonzero. The identities (2.10) may be expressed in the partitioned matrix form

$$
\begin{pmatrix}
    h_{ll} & h_{bl}^T & H_{nl}^T & a_l^T & 1 \\
    h_{bl} & H_{bb} & H_{bn} & A_b^T & I \\
    h_{nl} & H_{bn}^T & H_{nn} & A_n^T & I \\
    a_l & A_b & A_n & -M & I
\end{pmatrix}
\begin{pmatrix}
    \Delta x_l \\
    \Delta x_B \\
    \Delta x_N \\
    -\Delta y \\
    -\Delta z_l
\end{pmatrix}
= 
\begin{pmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0
\end{pmatrix},
$$

which implies that the quantities $\Delta x_l$, $\Delta x_B$, $\Delta y$ and $\Delta z_l$ satisfy the homogeneous equations

$$
\begin{pmatrix}
    h_{ll} & h_{bl}^T & a_l^T & 1 \\
    h_{bl} & H_{bb} & A_b^T & I \\
    a_l & A_b & -M
\end{pmatrix}
\begin{pmatrix}
    \Delta x_l \\
    \Delta x_B \\
    -\Delta y \\
    -\Delta z_l
\end{pmatrix}
= 
\begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix},
$$

with $\Delta z_N$ given by

$$\Delta z_N = h_{nl} \Delta x_l + H_{bn}^T \Delta x_B - A_n^T \Delta y. \quad (2.11b)$$
The properties of these equations are established in the next section.

2.3. The linear algebra framework

This section establishes the linear algebra framework that serves to emphasize the underlying symmetry between the primal and dual methods. It is shown that the search direction for the primal and the dual method is a nonzero solution of the homogeneous equations (2.11a), i.e., every direction is a nontrivial null vector of the matrix of (2.11a). In particular, it is shown that the null-space of (2.11a) has dimension one, which implies that every solution of (2.11a) is unique up to a scalar multiple. The length of the direction is then completely determined by fixing either $\Delta x_l = 1$ or $\Delta z_l = 1$. The choice of which component to fix depends on whether or not the corresponding component in a null vector of (2.11a) is nonzero. The conditions are stated precisely in Propositions 2.3 and 2.4 below.

The first result shows that the components $\Delta x_l$ and $\Delta z_l$ of any direction $(\Delta x, \Delta y, \Delta z)$ satisfying the identities (2.10) must be such that $\Delta x_l \Delta z_l \geq 0$.

\[ \text{Proposition 2.2.} \text{ If the vector } (\Delta x, \Delta y, \Delta z) \text{ satisfies the identities } \\
H \Delta x - A^T \Delta y - \Delta z = 0, \\
A \Delta x + M \Delta y = 0, \\
\text{then } \Delta x^T \Delta z \geq 0. \text{ Moreover, given an index } l \text{ and index sets } B \text{ and } N \text{ such that } \\
B \cup \{l\} \cup N = \{1, 2, \ldots, n\} \text{ with } \Delta x_N = 0 \text{ and } \Delta z_B = 0, \text{ then } \Delta x_l \Delta z_l \geq 0. \]

\[ \text{Proof.} \text{ Premultiplying the first identity by } \Delta x^T \text{ and the second by } \Delta y^T \text{ gives } \\
\Delta x^T H \Delta x - \Delta x^T A^T \Delta y - \Delta x^T \Delta z = 0, \text{ and } \Delta y^T A \Delta x + \Delta y^T M \Delta y = 0. \]

Eliminating the term $\Delta x^T A^T \Delta y$ gives $\Delta x^T H \Delta x + \Delta y^T M \Delta y = \Delta x^T \Delta z$. By definition, $H$ and $M$ are symmetric positive semidefinite, which gives $\Delta x^T \Delta z \geq 0$. In particular, if $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$, with $\Delta x_N = 0$ and $\Delta z_B = 0$, it must hold that $\Delta x^T \Delta z = \Delta x_l \Delta z_l \geq 0$.

The set of vectors $(\Delta x_l, \Delta x_B, -\Delta y, -\Delta z_l)$ satisfying the homogeneous equations (2.11a) is completely characterized by the properties of the matrices $K_B$ and $K_l$ such that

\[
K_B = \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix} \quad \text{and} \quad K_l = \begin{pmatrix} h_{ll} & h_{bl}^T & a_l^T \\ h_{bl} & H_{bb} & A_B^T \\ a_l & A_B & -M \end{pmatrix}. \tag{2.12}
\]

The properties are summarized by the results of the following two propositions.

\[ \text{Proposition 2.3.} \text{ If } K_B \text{ is nonsingular, and } \Delta x_l \text{ is a given nonnegative scalar, then the quantities } \Delta x_B, \Delta y, \Delta z_l \text{ and } \Delta z_N \text{ of (2.11) are unique and satisfy the} \]

\[ H \Delta x - A^T \Delta y - \Delta z = 0, \\
A \Delta x + M \Delta y = 0, \\
\text{then } \Delta x^T \Delta z \geq 0. \text{ Moreover, given an index } l \text{ and index sets } B \text{ and } N \text{ such that } \\
B \cup \{l\} \cup N = \{1, 2, \ldots, n\} \text{ with } \Delta x_N = 0 \text{ and } \Delta z_B = 0, \text{ then } \Delta x_l \Delta z_l \geq 0. \]

\[ \text{Proof.} \text{ Premultiplying the first identity by } \Delta x^T \text{ and the second by } \Delta y^T \text{ gives } \\
\Delta x^T H \Delta x - \Delta x^T A^T \Delta y - \Delta x^T \Delta z = 0, \text{ and } \Delta y^T A \Delta x + \Delta y^T M \Delta y = 0. \]

Eliminating the term $\Delta x^T A^T \Delta y$ gives $\Delta x^T H \Delta x + \Delta y^T M \Delta y = \Delta x^T \Delta z$. By definition, $H$ and $M$ are symmetric positive semidefinite, which gives $\Delta x^T \Delta z \geq 0$. In particular, if $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$, with $\Delta x_N = 0$ and $\Delta z_B = 0$, it must hold that $\Delta x^T \Delta z = \Delta x_l \Delta z_l \geq 0$.

The set of vectors $(\Delta x_l, \Delta x_B, -\Delta y, -\Delta z_l)$ satisfying the homogeneous equations (2.11a) is completely characterized by the properties of the matrices $K_B$ and $K_l$ such that

\[
K_B = \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix} \quad \text{and} \quad K_l = \begin{pmatrix} h_{ll} & h_{bl}^T & a_l^T \\ h_{bl} & H_{bb} & A_B^T \\ a_l & A_B & -M \end{pmatrix}. \tag{2.12}
\]

The properties are summarized by the results of the following two propositions.

\[ \text{Proposition 2.3.} \text{ If } K_B \text{ is nonsingular, and } \Delta x_l \text{ is a given nonnegative scalar, then the quantities } \Delta x_B, \Delta y, \Delta z_l \text{ and } \Delta z_N \text{ of (2.11) are unique and satisfy the} \]

the subspace of vectors satisfying (2.11a) is of dimension one, it follows that every
nontrivial solution of the homogeneous equation (2.11a) such that
value of
corresponding
eigenvector
null vector of
null vector must have a nonzero
If
particular, if
which implies that
As
nonsingular, the homogeneous equations (2.11a) may be written in the form

\[ \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_B \\ -\Delta y \end{pmatrix} = -\begin{pmatrix} h_{bl} \\ a_l \end{pmatrix} \Delta x_l, \]  \hspace{1cm} (2.13a)

\[ \Delta z_l = h_{bl} \Delta x_l + h_{bl}^T \Delta x_B - a_l^T \Delta y, \]  \hspace{1cm} (2.13b)

\[ \Delta z_N = h_{nl} \Delta x_l + H_{BN}^T \Delta x_B - A_N^T \Delta y. \]  \hspace{1cm} (2.13c)

If \( \Delta x_l = 0 \), then \( \Delta x_B = 0, \Delta y = 0, \Delta z_l = 0 \) and \( \Delta z_N = 0 \). Otherwise, \( \Delta x_l > 0 \), and either

(i) \( K_l \) is nonsingular and \( \Delta z_l > 0 \), or

(ii) \( K_l \) is singular and \( \Delta z_l = 0 \), in which case it holds that \( \Delta y = 0, \Delta z_N = 0 \), and the multiplicity of the zero eigenvalue of \( K_l \) is one, with corresponding
eigenvector \( (\Delta x_l, \Delta x_B, 0) \).

Proof. The second and third blocks of the equations (2.11a) imply that

\[ \begin{pmatrix} h_{bl} \\ a_l \end{pmatrix} \Delta x_l + \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_B \\ -\Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]  \hspace{1cm} (2.14)

As \( K_B \) is nonsingular by assumption, the vectors \( \Delta x_B \) and \( \Delta y \) must constitute the unique solution of (2.14) for a given value of \( \Delta x_l \). Furthermore, given \( \Delta x_B \) and \( \Delta y \), the quantities \( \Delta z_l \) and \( \Delta z_N \) of (2.13) are also uniquely defined. The specific value \( \Delta x_l = 0 \), gives \( \Delta x_B = 0 \) and \( \Delta y = 0 \), so that \( \Delta z_l = 0 \) and \( \Delta z_N = 0 \). It follows that \( \Delta x_l \) must be nonzero for at least one of the vectors \( \Delta x_B, \Delta y, \Delta z_l \) or \( \Delta z_N \) to be nonzero.

Next it is shown that if \( \Delta x_l > 0 \), then either (i) or (ii) must hold. For (i), it is necessary to show that if \( \Delta x_l > 0 \) and \( K_l \) is nonsingular, then \( \Delta z_l > 0 \). If \( K_l \) is nonsingular, the homogeneous equations (2.11a) may be written in the form

\[ \begin{pmatrix} h_{bl} & h_{bl}^T \\ h_{bl} & H_{BB} \\ a_l & A_B^T \\ a_l & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_B \\ -\Delta y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Delta z_l, \]  \hspace{1cm} (2.15)

which implies that \( \Delta x_l, \Delta x_B \) and \( \Delta y \) are unique for a given value of \( \Delta z_l \). In particular, if \( \Delta z_l = 0 \) then \( \Delta x_l = 0 \), which would contradict the assumption that \( \Delta x_l > 0 \). If follows that \( \Delta z_l \) must be nonzero. Finally, Proposition 2.2 implies that if \( \Delta z_l \) is nonzero and \( \Delta x_l > 0 \), then \( \Delta z_l > 0 \) as required.

For the first part of (ii), it must be shown that if \( K_l \) is singular, then \( \Delta z_l = 0 \). If \( K_l \) is singular, it must have a nontrivial null vector \( (p_l, p_B, -u) \). Moreover, every null vector must have a nonzero \( p_l \), because otherwise \( (p_B, -u) \) would be a nontrivial null vector of \( K_B \), which contradicts the assumption that \( K_B \) is nonsingular. A fixed value of \( p_l \) uniquely defines \( p_B \) and \( u \), which indicates that the multiplicity of the zero eigenvalue must be one. A simple substitution shows that \( (p_l, p_B, -u, v_l) \) is a nontrivial solution of the homogeneous equation (2.11a) such that \( v_l = 0 \). As the subspace of vectors satisfying (2.11a) is of dimension one, it follows that every
solution is unique up to a scalar multiple. Given the properties of the known solution \((p_l, p_B, -u, 0)\), it follows that every solution \((\Delta x_l, \Delta x_B, -\Delta y, -\Delta z_l)\) of (2.11a) is an eigenvector associated with the zero eigenvalue of \(K_l\), with \(\Delta z_l = 0\).

For the second part of (ii), if \(\Delta z_l = 0\), the homogeneous equations (2.11a) become
\[
\begin{pmatrix}
h_{ll} & h_{bl}^T & a_l^T \\
h_{bl} & H_{BB} & A_B^T \\
a_l & A_B & -M
\end{pmatrix}
\begin{pmatrix}
\Delta x_l \\
\Delta x_B \\
-\Delta y
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\] (2.16)

As \(K_l\) is singular in (2.16), Proposition A.1 implies that
\[
\begin{pmatrix}
h_{ll} & h_{bl}^T & a_l^T \\
h_{bl} & H_{BB} & A_B^T \\
a_l & A_B & -M
\end{pmatrix}
\begin{pmatrix}
\Delta x_l \\
\Delta x_B \\
-\Delta y
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\] (2.17)

The nonsingularity of \(K_B\) implies that \((A_B - M)\) has full row rank, in which case the second equation of (2.17) gives \(\Delta y = 0\). It follows that every eigenvector of \(K_l\) associated with the zero eigenvalue has the form \((\Delta x_l, \Delta x_B, 0)\). It remains to show that \(\Delta z_N = 0\). If Proposition A.2 is applied to the first equation of (2.17), then it must hold that
\[
\begin{pmatrix}
h_{ll} & h_{bl}^T \\
h_{bl} & H_{BB} \\
a_l & H_{BN}
\end{pmatrix}
\begin{pmatrix}
\Delta x_l \\
\Delta x_B \\
\Delta y
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

It follows from (2.13c) that \(\Delta z_N = h_Nl \Delta x_l + H_{BN}^T \Delta x_B - A_N^T \Delta y = 0\), which completes the proof.

**Proposition 2.4.** If \(K_l\) is nonsingular, and \(\Delta z_l\) is a given nonnegative scalar, then the quantities \(\Delta x_l, \Delta x_B, \Delta y\) and \(\Delta z_N\) of (2.11) are unique and satisfy the equations

\[
\begin{pmatrix}
h_{ll} & h_{bl}^T & a_l^T \\
h_{bl} & H_{BB} & A_B^T \\
a_l & A_B & -M
\end{pmatrix}
\begin{pmatrix}
\Delta x_l \\
\Delta x_B \\
-\Delta y
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \Delta z_l,
\] (2.18a)

\[
\Delta z_N = H_{Nl} \Delta x_l + H_{BN}^T \Delta x_B - A_N^T \Delta y.
\] (2.18b)

If \(\Delta z_l = 0\), then \(\Delta x_l = 0, \Delta x_B = 0, \Delta y = 0\) and \(\Delta z_N = 0\). Otherwise, \(\Delta z_l > 0\) and either

(i) \(K_B\) is nonsingular and \(\Delta x_l > 0\), or

(ii) \(K_B\) is singular and \(\Delta x_l = 0\), in which case, it holds that \(\Delta x_B = 0\) and the multiplicity of the zero eigenvalue of \(K_B\) is one, with corresponding eigenvector \((0, \Delta y)\).

**Proof.** In Proposition 2.2 it is established that \(\Delta x_l \geq 0\) if \(\Delta z_l > 0\), which implies that the statement of the proposition includes all possible values of \(\Delta x_l\).
It follows from (2.11a) that $\Delta x_l$, $\Delta x_B$, and $\Delta y$ must satisfy the equations

\[
\begin{pmatrix}
  h_{Bl}
  h_{Bl}
  a_l^T
  a_l
  A_B
  A_B
  -M
  -M
\end{pmatrix}
\begin{pmatrix}
  \Delta x_l \\
  \Delta x_B \\
  -\Delta y
\end{pmatrix} =
\begin{pmatrix}
  \Delta z_l \\
  0 \\
  0
\end{pmatrix}.
\]  

(2.19)

Under the given assumption that $K_l$ is nonsingular, the vectors $\Delta x_l$, $\Delta x_B$ and $\Delta y$ are uniquely determined by (2.19) for a fixed value of $\Delta z_l$. In addition, once $\Delta x_l$, $\Delta x_B$ and $\Delta y$ are defined, $\Delta z_N$ is uniquely determined by (2.18b). It follows that if $\Delta z_l = 0$, then $\Delta x_l = 0$, $\Delta x_B = 0$, $\Delta y = 0$ and $\Delta z_N = 0$.

It remains to show that if $\Delta z_l > 0$, then either (i) or (ii) must hold. If $K_B$ is singular, then Proposition A.1 implies that there must exist $u$ and $v$ such that

\[
\begin{pmatrix}
  H_{BB}
  A_B
\end{pmatrix}
\begin{pmatrix}
  u \\
  v
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  A_B^T
  -M
\end{pmatrix}
\begin{pmatrix}
  u \\
  v
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
\]

Proposition A.2 implies that the vector $u$ must also satisfy

\[
\begin{pmatrix}
  h_{Bl}^T
  H_{BB}
\end{pmatrix}
\begin{pmatrix}
  u
\end{pmatrix} =
\begin{pmatrix}
  0
\end{pmatrix}.
\]

If $u$ is nonzero, then $(0, u, 0)$ is a nontrivial null vector for $K_l$, which contradicts the assumption that $K_l$ is nonsingular. It follows that $(H_{BB}, A_B^T)$ has full row rank and the singularity of $K_B$ must be caused by dependent rows in $(A_B, -M)$. The nonsingularity of $K_l$ implies that $(a_l, A_B, -M)$ has full row rank and there must exist a vector $v$ such that $v^T a_l \neq 0$, $v^T A_B = 0$ and $v^T M = 0$. If $v$ is scaled so that $v^T a_l = -\Delta z_l$, then $(0, 0, -v)$ must be a solution of (2.19). It follows that $\Delta x_l = 0$, $v = \Delta y$, and $(0, \Delta y)$ is an eigenvector of $K_B$ associated with a zero eigenvalue. The nonsingularity of $K_l$ implies that $v$ is unique given the value of the scalar $\Delta z_l$, and hence the zero eigenvalue has multiplicity one.

Conversely, $\Delta x_l = 0$ implies that $(\Delta x_B, \Delta y)$ is a null vector $K_B$. However, if $K_B$ is nonsingular, then the vector is zero, contradicting (2.18a). It follows that $K_B$ must be singular.

3. A Primal Active-Set Method for Convex QP

In this section a primal-feasible method for convex QP is formulated. Each iteration begins and ends with a point $(x, y, z)$ that satisfies the conditions

\[
A_B x_B + A_N x_N + My = b, \quad x_N + q_N = 0, \quad \text{(3.1a)}
\]
\[
H_{BB} x_B + H_{BN} x_N + c_B - A_B y - z_B = 0, \quad z_B + r_B = 0, \quad \text{(3.1b)}
\]
\[
H_{BN} x_B + H_{NN} x_N + c_N - A_N^T y - z_N = 0, \quad x_N + q_B \geq 0, \quad \text{(3.1c)}
\]

for appropriate second-order consistent bases. The purpose of the iterations is to drive $(x, y, z)$ to optimality by driving the dual variables to feasibility (i.e., by driving
the negative components of $z_N + r_N$ to zero). Methods for finding $B$ and $N$ at the initial point are discussed in Section 5.

An iteration consists of a group of one or more consecutive subiterations during which a specific dual variable is made feasible. The first subiteration is called the base subiteration. In some cases only the base iteration is performed, but, in general, additional intermediate subiterations are required.

At the start of the base subiteration, an index $l$ in the nonbasic set $N$ is identified such that $z_l + r_l < 0$. The idea is to remove the index $l$ from $N$ (i.e., $N \leftarrow N \setminus \{l\}$) and attempt to increase the value of $z_l + r_l$ by taking a step along a primal-feasible direction $(\Delta x_l, \Delta x_B, \Delta y, \Delta z_l)$. The removal of $l$ from $N$ implies that $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$ with $B$ second-order consistent. This implies that $K_B$ is nonsingular and the (unique) search direction may be computed as in (2.13) with $\Delta x = 0.\alpha$

If $\Delta z_l > 0$, the step $\alpha_s = -(z_l + r_l)/\Delta z_l$ gives $z_l + \alpha_s \Delta z_l + r_l = 0$. Otherwise, $\Delta z_l = 0$, and there is no finite value of $\alpha$ that will drive $z_l + \alpha \Delta z_l + r_l$ to its bound, and $\alpha_s$ is defined to be $+\infty$. Proposition A.7 implies that the case $\Delta z_l = 0$ corresponds to the primal objective function being linear and decreasing along the search direction.

Even in the case that $\Delta z_l$ is positive, it is not always possible to take the step $\alpha_s$ and remain primal feasible. A positive step in the direction $(\Delta x_l, \Delta x_B, \Delta y, \Delta z_l)$ must increase $x_l$ from its bound, but may decrease some of the basic variables. This makes it necessary to limit the step to ensure that the primal variables remain feasible. The largest step length that maintains primal feasibility is given by

$$\alpha_{\max} = \min_{i: \Delta x_i < 0} \frac{x_i + q_i}{-\Delta x_i}.$$

If $\alpha_{\max}$ is finite, this value gives $x_k + \alpha_{\max} \Delta x_k + q_k = 0$, where the index $k$ is given by $k = \arg \min_{i: \Delta x_i < 0} (x_i + q_i)/(-\Delta x_i)$. The overall step length is then given by

$$\alpha = \min(\alpha_s, \alpha_{\max}).$$

An infinite value of $\alpha$ indicates that the primal problem (PQP$_{q,r}$) is unbounded, or, equivalently, that the dual problem (DQP$_{q,r}$) is infeasible. In this case, the algorithm is terminated. If the step $\alpha = \alpha_s$ is taken, then $z_l + \alpha \Delta z_l + r_l = 0$, the subiterations are terminated with no intermediate subiterations and $B \leftarrow B \cup \{l\}$. Otherwise, $\alpha = \alpha_{\max}$, and the basic and nonbasic sets are updated as $B \leftarrow B \setminus \{k\}$ and $N \leftarrow N \cup \{k\}$ giving an updated partition $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$ as before. In order to show that the equations associated with the new partition are well-defined, it is necessary to show that allowing $z_k$ to move does not give a singular $K_l$. Proposition A.5 shows that the submatrix $K_l$ associated with the updated $B$ and $N$ is nonsingular for the cases $\Delta z_l > 0$ and $\Delta z_l = 0$.

Because the removal of $k$ from $B$ does not alter the nonsingularity of $K_l$, it is possible to add $l$ to $B$ and thereby define a unique solution of the system (2.9). However, if $z_l + r_l < 0$, additional intermediate subiterations are required to drive $z_l + r_l$ to zero. In each of these subiterations, the search direction is computed by choosing $\Delta z_l = 1$ in Proposition 2.4. The step length $\alpha_s$ is given by $\alpha_s = -(z_l +
Similar to the base subiteration, if no constraint is added, then \( z_l + \alpha \Delta z_l + r_l = 0 \). Otherwise, the index of another blocking variable \( k \) is moved from \( B \) to \( N \). Proposition A.5 implies that the updated matrix \( K_l \) is nonsingular at the end of an intermediate subiteration. As a consequence, the intermediate subiterations can be repeated until \( z_l + r_l \) is driven to zero.

At the end of the base subiteration or after the intermediate subiterations are completed, it must hold that \( z_l + r_l = 0 \) and the final \( K_l \) is nonsingular. This implies that a new iteration may be initiated with the new basic set \( B \cup \{l\} \) defining a nonsingular \( K_B \).

The primal active-set method is summarized in Algorithm 1 below. Its convergence properties are given in Theorem 3.1.

**Theorem 3.1.** Assume that problem \((PQP_{q,r})\) is nondegenerate. Given an initial point \((x, y, z)\) satisfying conditions (3.1) for a second-order consistent basis \( B \), then Algorithm 1 finds a solution of \((PQP_{q,r})\) or determines that \((DQP_{q,r})\) is infeasible in a finite number of iterations.

**Proof.** Algorithm 1 is a special case of Algorithm 3 of Section 5, which describes a primal QP method defined as part of a primal-dual strategy for choosing appropriate nonzero shifts \( q \) and \( r \). The convergence of Algorithm 3 is established in Theorem 5.1.

4. A Dual Active-Set Method for Convex QP

Each iteration of the dual active-set method begins and ends with a point \((x, y, z)\) that satisfies the conditions

\[
\begin{align*}
Hx + c - A^T y - z &= 0, \\
Ax + My - b &= 0, \\
z_N + r_N &
\geq 0, \hspace{1cm} z_B + r_B = 0, \\
x_N + q_N &= 0,
\end{align*}
\]  

(4.1a)
(4.1b)
(4.1c)
(4.1d)

for appropriate second-order consistent bases. For the dual method, the purpose is to drive the primal variables to feasibility (i.e., by driving the negative components of \( x + q \) to zero).

An iteration begins with a base subiteration in which an index \( l \) in the basic set \( B \) is identified such that \( x_l + q_l < 0 \). The corresponding dual variable \( z_l \) may be increased from its current value \( z_l = -r_l \) by removing the index \( l \) from \( B \), and defining \( B \leftarrow B \setminus \{l\} \). Once \( l \) is removed from \( B \), it holds that \( B \cup \{l\} \cup N = \{1, 2, \ldots, n\} \). The resulting matrix \( K_l \) is nonsingular, and the unique direction \((\Delta x_l, \Delta x_B, \Delta y)\) may be computed as in Proposition 2.4 with the particular value \( \Delta z_l = 1 \).

If \( \Delta x_l > 0 \), the step \( \alpha^* = -(x_l + q_l)/\Delta x_l \) gives \( x_l + \alpha \Delta x_l + q_l = 0 \). Otherwise, \( \Delta x_l = 0 \), and there is no finite value of \( \alpha \) that will drive \( x_l + \alpha \Delta x_l + q_l \) to its bound. In this case, the result of Proposition A.7 implies that the dual objective function
Algorithm 1: A primal active-set method for convex QP.

Find \((x, y, z)\) satisfying conditions (3.1) for some second-order consistent basis \(\mathcal{B}\):

while \(\exists l : z_l + r_l < 0\) do

\(\mathcal{N} \leftarrow \mathcal{N} \setminus \{l\} \);  

PRIMAL_BASE(\(\mathcal{B}, \mathcal{N}, l, x, y, z\));  

while \(z_l + r_l < 0\) do

PRIMAL_INTERMEDIATE(\(\mathcal{B}, \mathcal{N}, l, x, y, z\));  

end while

\(\mathcal{B} \leftarrow \mathcal{B} \cup \{l\} \);  

end while

function PRIMAL_BASE(\(\mathcal{B}, \mathcal{N}, l, x, y, z\))

\(\Delta x_l \leftarrow 1\);  

Solve \(\begin{pmatrix} H_{B_B} & A_B^T \\ A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_B \\ -\Delta y \end{pmatrix} = -\begin{pmatrix} h_{B_B} \\ a_l \end{pmatrix} \);  

\(\Delta z_N \leftarrow h_{N_B}^T \Delta x_l + H_{B_B}^T \Delta x_B - A_B^T \Delta y; \)

\(\Delta z_l \leftarrow h_l^T \Delta x_l + h_{B_B}^T \Delta x_B - a_l^T \Delta y; \)

\(\alpha_s \leftarrow -(z_l + r_l) / \Delta z_l; \)

\(\alpha_{\text{max}} \leftarrow \min_{i : \Delta x_i < 0} (x_i + q_i) / (-\Delta x_i); \)

\(k \leftarrow \argmin_{i : \Delta x_i < 0} (x_i + q_i) / (-\Delta x_i); \)

\(\alpha \leftarrow \min (\alpha_s, \alpha_{\text{max}}); \)

if \(\alpha = +\infty\) then

stop;  

\([\text{QP}_{q,r} \text{ is infeasible}]\)

end if

\(x_l \leftarrow x_l + \alpha \Delta x_l; \)

\(x_B \leftarrow x_B + \alpha \Delta x_B; \)

\(y \leftarrow y + \alpha \Delta y; \)

\(z_l \leftarrow z_l + \alpha \Delta z_l; \)

\(z_N \leftarrow z_N + \alpha \Delta z_N; \)

if \(z_l + r_l < 0\) then

\(\mathcal{B} \leftarrow \mathcal{B} \setminus \{k\}; \)

\(\mathcal{N} \leftarrow \mathcal{N} \cup \{k\}; \)

end if

end function

function PRIMAL_INTERMEDIATE(\(\mathcal{B}, \mathcal{N}, l, x, y, z\))

\(\Delta z_l \leftarrow 1\);  

Solve \(\begin{pmatrix} h_l & h_{B_B}^T & a_l^T \\ h_{B_B} & H_{B_B} & A_B^T \\ a_l & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_B \\ -\Delta y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \);  

\(\Delta z_N \leftarrow h_{N_B} \Delta x_l + H_{B_B}^T \Delta x_B - A_B^T \Delta y; \)

\(\alpha_s \leftarrow -(z_l + r_l); \)

\(\alpha_{\text{max}} \leftarrow \min_{i : \Delta x_i < 0} (x_i + q_i) / (-\Delta x_i); \)

\(k \leftarrow \argmin_{i : \Delta x_i < 0} (x_i + q_i) / (-\Delta x_i); \)

\(\alpha \leftarrow \min (\alpha_s, \alpha_{\text{max}}); \)

\(x_l \leftarrow x_l + \alpha \Delta x_l; \)

\(x_B \leftarrow x_B + \alpha \Delta x_B; \)

\(y \leftarrow y + \alpha \Delta y; \)

\(z_l \leftarrow z_l + \alpha \Delta z_l; \)

\(z_N \leftarrow z_N + \alpha \Delta z_N; \)

if \(z_l + r_l < 0\) then

\(\mathcal{B} \leftarrow \mathcal{B} \setminus \{k\}; \)

\(\mathcal{N} \leftarrow \mathcal{N} \cup \{k\}; \)

end if

end function
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is linear and increasing along the search direction and $\alpha_*$ is defined to be $+\infty$. As $x_l + q_l$ is increased towards its bound, the associated dual variable $z_l$ increases from its current value, but other nonbasic dual variables may decrease and violate their bounds. This makes it necessary to limit the step by

$$
\alpha_{\text{max}} = \min_{i : \Delta z_i < 0} \frac{z_i + r_i}{-\Delta z_i}
$$

to maintain dual feasibility. If $\alpha_{\text{max}}$ is finite, then $z_k + \alpha_{\text{max}} \Delta z_k + r_k = 0$, where the index $k$ is given by $k = \arg\min_{i : \Delta z_i < 0} (z_i + r_i)/(-\Delta z_i)$. The overall step length is then $\alpha = \min (\alpha_*, \alpha_{\text{max}})$, where an infinite value of $\alpha_*$ implies that the dual problem is unbounded, or, equivalently, that the primal problem ($\text{PQP}_{q,r}$) is infeasible. If $\alpha = \alpha_{\text{max}}$, then $x_l + \alpha \Delta x_l + q_l = 0$. Otherwise $\alpha = \alpha_{\text{max}}$, and $N$ and $B$ are updated as $N = N \setminus \{k\}$ and $B = B \cup \{k\}$. Regardless of the definition of the step, the partition at the new point satisfies $B \cup \{l\} \cup N = \{1, 2, \ldots, n\}$. To ensure nonsingularity, it is necessary to show that allowing the variable $x_k$ to move does not cause singularity. In Proposition A.6 it is established that $K_B$ is nonsingular for the two possible cases $\Delta x_l > 0$ and $\Delta x_l = 0$.

As $K_B$ is nonsingular, moving $l$ into $N$ would provide second-order consistent sets $B$ and $N$ such that $B \cup N = \{1, 2, \ldots, n\}$ with the current $x$ a unique solution of (2.9). However, if $x_l + q_l < 0$, additional intermediate subiterations are necessary to drive $x_l + q_l$ to zero. The nonsingularity of $K_B$ implies that the search direction may be computed as in Proposition 2.3, with the definition $\Delta x_l = 1$. The step length is computed as $\alpha_* = -(x_l + q_l)/\Delta x_l$ as above, but in this case $\alpha_*$ is always finite because $\Delta x_l = 1$. As in the case of a base subiteration, if no constraint index is added to $B$, then $x_l + \alpha \Delta x_l + q_l = 0$. Otherwise, the index $k$ of a blocking variable is moved from $N$ to $B$. In Proposition A.6 it is shown that the updated $K_B$ is nonsingular at the end of an intermediate subiteration. Consequently, the intermediate subiterations can be repeated until the final $x_l + q_l$ is zero.

Once a zero value of $x_l + q_l$ is obtained at the end of the base subiteration or after intermediate subiterations, the resulting $K_B$ matrix is nonsingular. At this point, the iteration is complete and the index $l$ is moved to $N$. The new $K_B$ matrix is nonsingular, and a new iteration may be initiated.

The dual active-set method is summarized in Algorithm 2 below. Its convergence properties are given in Theorem 4.1.

**Theorem 4.1.** Assume that problem ($\text{DQP}_{q,r}$) is nondegenerate. Then, given an initial point $(x, y, z)$ satisfying conditions (4.1) for some second-order consistent basis $B$, Algorithm 2 either solves ($\text{DQP}_{q,r}$) or concludes that ($\text{PQP}_{q,r}$) is infeasible in a finite number of iterations.

**Proof.** The proof mirrors that of the primal active-set method of Section 5 (see Theorem 5.1).
Algorithm 2 A dual active-set method for convex QP.

Find \((x, y, z)\) satisfying conditions (4.1) for some second-order consistent basis \(\mathcal{B}\):

while \(\exists \ l : x_l + q_l < 0\) do
  \(\mathcal{B} \leftarrow \mathcal{B} \setminus \{l\}\);
  \textsc{dual}\_base(\(\mathcal{B}, \mathcal{N}, l, x, y, z\)); \hspace{1cm} \text{[Base subiteration]}
  \textbf{if} \(x_l + q_l < 0\) \textbf{do}
    \textsc{dual}\_intermediate(\(\mathcal{B}, \mathcal{N}, l, x, y, z\)); \hspace{1cm} \text{[Intermediate subiteration]}
  \textbf{end if}
end while

\begin{function}{dual\_base(\(\mathcal{B}, \mathcal{N}, l, x, y, z\))}
\(\Delta z_l \leftarrow 1\); \hspace{1cm} \text{Solve } \begin{pmatrix} h_{ll} & h_{bl} & a_l^T \\ h_{bl} & H_B & A_B^T \\ a_l & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_B \\ -\Delta y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \hspace{1cm} [\Delta x_l \geq 0]
\(\Delta z_N \leftarrow h_{Nl} \Delta x_l + H_{BN}^T \Delta x_B - A_N^T \Delta y;\)
\(\alpha_* \leftarrow -(x_l + q_l)/\Delta x_l;\) \hspace{1cm} [\alpha_* \leftarrow +\infty \text{ if } \Delta x_l = 0]
\(\alpha_{\text{max}} \leftarrow \min_{i: \Delta z_i < 0} (z_i + r_i)/(\Delta z_i);\)
\(k \leftarrow \arg\min_{i: \Delta z_i < 0} (z_i + r_i)/(\Delta z_i);\)
\(\alpha \leftarrow \min(\alpha_*, \alpha_{\text{max}});\)
\textbf{if} \(\alpha = +\infty\) \textbf{then}
  \text{stop; \hspace{1cm} \[\text{PQP}_{q,r} \text{ is infeasible}\]}\end{function}

\begin{function}{dual\_intermediate(\(\mathcal{B}, \mathcal{N}, l, x, y, z\))}
\(\Delta x_l \leftarrow 1;\) \hspace{1cm} \text{Solve } \begin{pmatrix} H_B & A_B^T \\ A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_B \\ -\Delta y \end{pmatrix} = -\begin{pmatrix} h_{bl} \\ a_l \end{pmatrix}; \hspace{1cm} [\Delta z_l \geq 0]
\(\Delta z_l \leftarrow h_{ll} \Delta x_l + h_{bl} \Delta x_B - a_l^T \Delta y;\)
\(\Delta z_N \leftarrow h_{Nl} \Delta x_l + H_{BN}^T \Delta x_B - A_N^T \Delta y;\)
\(\alpha_* \leftarrow -(x_l + q_l);\)
\(\alpha_{\text{max}} \leftarrow \min_{i: \Delta z_i < 0} (z_i + r_i)/(\Delta z_i);\)
\(k \leftarrow \arg\min_{i: \Delta z_i < 0} (z_i + r_i)/(\Delta z_i);\)
\(\alpha \leftarrow \min(\alpha_*, \alpha_{\text{max}});\)
\(x_l \leftarrow x_l + \alpha \Delta x_l;\) \hspace{1cm} \(x_B \leftarrow x_B + \alpha \Delta x_B;\)
\(y \leftarrow y + \alpha \Delta y;\) \hspace{1cm} \(z_l \leftarrow z_l + \alpha \Delta z_l;\) \hspace{1cm} \(z_N \leftarrow z_N + \alpha \Delta z_N;\)
\textbf{if} \(x_l + q_l < 0\) \textbf{then}
  \mathcal{B} \leftarrow \mathcal{B} \cup \{k\}; \hspace{1cm} \mathcal{N} \leftarrow \mathcal{N} \setminus \{k\};
\textbf{end if}
end function

end function
5. Combining Primal and Dual Active-Set Methods

The primal active-set method proposed in Section 3 may be used to solve \((\text{PQP}_{q,r})\) for a given initial second-order consistent basis satisfying the conditions (3.1). An appropriate initial point may be found by solving a conventional phase-1 linear program. Alternatively, the dual active-set method of Section 4 may be used in conjunction with an appropriate phase-1 procedure to solve the quadratic program \((\text{PQP}_{q,r})\) for a given initial second-order consistent basis satisfying the conditions (4.1). In this section a method is proposed that provides an alternative to the conventional phase-1/phase-2 approach. It is shown that a pair of coupled quadratic programs may be created from the original by simultaneously shifting the bound constraints. Any second-order consistent basis can be made optimal for such a primal-dual pair of shifted problems. The shifts are then updated using the solution of either the primal or the dual shifted problem. An obvious application of this approach is to solve a shifted dual \(\text{QP}\) to define an initial feasible point for the primal, or vice-versa. This strategy provides an alternative to the conventional phase-1/phase-2 approach that utilizes the \(\text{QP}\) objective function while finding a feasible point.

5.1. Finding an initial second-order-consistent basis

For the methods described in Section 5.2 below, it is possible to define a simple procedure for finding the initial second-order consistent basis \(B\). The required basis must define a nonsingular KKT matrix \(K_B\) such that

\[
K_B = \begin{pmatrix} H_{BB} & A_B^T \\ A_B & -M \end{pmatrix}.
\] (5.1)

The initial basis may be obtained by a finding a symmetric permutation \(\Pi\) of the “full” KKT matrix \(K\) such that

\[
\Pi^T K \Pi = \Pi^T \begin{pmatrix} H & A \\ A & -M \end{pmatrix} \Pi = \begin{pmatrix} H_{BB} & A_B^T & H_{BN} \\ A_B & -M & A_N \\ H_{BN}^T & A_N^T & H_{NN} \end{pmatrix},
\] (5.2)

where the leading principal block \(2 \times 2\) submatrix is of the form (5.1). The full row-rank assumption on \((A - M)\) ensures that the partition (5.2) is well defined, see [20, Section 6]. In practice, the permutation may be determined using any method for finding a symmetric indefinite factorization of \(K\), see, e.g., [8, 10, 18]. Such methods use symmetric interchanges that implicitly form the nonsingular matrix \(K_B\) by deferring singular pivots. In this case, \(K_B\) may be defined as any submatrix of the largest nonsingular principal submatrix obtained by the factorization (There may be further permutations within \(\Pi\) that are not relevant to this discussion; for further details, see, e.g., [13, 14, 20, 21].)

The permutation \(\Pi\) defines the initial \(B-N\) partition of the columns of \(A\). Once \(\Pi\) has been determined, the variables with indices in \(N\) are set on their bounds and the shifts are initialized.
5. Combining Primal and Dual Active-Set Methods

5.2. Initializing the shifts

Given a second-order consistent basis, it is straightforward to create a \((q^{(0)}, r^{(0)})\)-pair and corresponding \((x, y, z)\) so that \(q^{(0)} \geq 0, r^{(0)} \geq 0\) and \((x, y, z)\) are optimal for \((\text{PQP}_{q^{(0)}, r^{(0)}})\) and \((\text{DQP}_{q^{(0)}, r^{(0)}})\). First, choose nonnegative vectors \(q_N^{(0)}\) and \(r_B^{(0)}\). (Obvious choices are \(q_N^{(0)} = 0\) and \(r_B^{(0)} = 0\).) Define \(z_B = -r_B^{(0)}, x_N = -q_N^{(0)}\), and solve the nonsingular KKT-system (2.7) to obtain \(x_B\) and \(y\), and compute \(z_N\) from (2.8). Finally, let \(q_B^{(0)} \geq \max\{-x_B, 0\}\) and \(r_N^{(0)} \geq \max\{-z_N, 0\}\). Then, it follows from Proposition 2.1 that \(x, y\) and \(z\) are optimal for the problems \((\text{PQP}_{q^{(0)}, r^{(0)}})\) and \((\text{DQP}_{q^{(0)}, r^{(0)}})\), with \(q^{(0)} \geq 0\) and \(r^{(0)} \geq 0\). If \(q^{(0)}\) and \(r^{(0)}\) are zero, then \(x, y\) and \(z\) are optimal for the original problem.

5.3. Solving the original problem by removing the shifts

The original problem may now be solved as a pair of shifted quadratic programs. Two alternative strategies are proposed. The first is a “primal first” strategy in which a shifted primal quadratic program is solved, followed by a dual. The second is an analogous “dual first” strategy.

The “primal-first” strategy is summarized as follows.

(0) Find \(B, N, q^{(0)}, r^{(0)}, x, y, z\), as described in Sections 5.1 and 5.2.

(1) Set \(q^{(1)} = q^{(0)}, r^{(1)} = 0\). Solve \((\text{PQP}_{q^{(0)}, r^{(0)}})\) using the primal active-set method.

(2) Set \(q^{(2)} = 0, r^{(2)} = 0\). Solve \((\text{DQP}_{0,0})\) using the dual active-set method.

In steps (1) and (2), the initial \(B-N\) partition and initial values of \(x, y,\) and \(z\) are defined as the final \(B-N\) partition and final values of \(x, y,\) and \(z\) from the preceding step.

The “dual-first” strategy is defined in an analogous way.

(0) Find \(B, N, q^{(0)}, r^{(0)}, x, y, z\), as described in Section 5.1 and 5.2.

(1) Set \(q^{(1)} = 0, r^{(1)} = r^{(0)}\). Solve \((\text{DQP}_{0,r^{(0)}})\) using the dual active-set method.

(2) Set \(q^{(2)} = 0, r^{(2)} = 0\). Solve \((\text{PQP}_{0,0})\) using the primal active-set method.

As in the “primal-first” strategy, the initial \(B-N\) partition and initial values of \(x, y,\) and \(z\) for steps (1) and (2), are defined as the final \(B-N\) partition and final values of \(x, y,\) and \(z\) from the preceding step.

In order for these approaches to be well-defined, a simple generalization of the primal and dual active-set methods is needed.

5.4. Relaxed initial conditions for the primal QP method.

For Algorithm 1, the initial values of \(B, N, q, r, x, y,\) and \(z\) must satisfy conditions (3.1). However, the choice of \(r = r^{(1)} = 0\) in Step (1) of the primal-first strategy may give some negative components in the vector \(z_B + r_B\). This possibility may be
handled by defining a simple generalization of Algorithm 1 that allows initial points satisfying the conditions

\begin{align}
Hx + c - A^T y - z &= 0, \quad (5.3a) \\
Ax + My - b &= 0, \quad (5.3b) \\
x_B + q_B &\geq 0, \quad x_N + q_N = 0, \quad (5.3c) \\
z_B + r_B &\leq 0, \quad (5.3d)
\end{align}

instead of the conditions (3.1). In Algorithm 1, the index \( l \) identified at the start of the primal base iteration is selected from the nonbasic indices such that \( z_j + r_j < 0 \). In the generalized algorithm, the set of eligible indices for \( l \) is extended to include indices associated with negative values of \( z_B + r_B \). If the index \( l \) is deleted from \( B \), the associated matrix \( K_l \) is nonsingular, and intermediate subiterations are executed until the updated value satisfies \( z_l + r_l = 0 \). At this point, the index \( l \) is returned to \( B \). The method is summarized in Algorithm 3.

Algorithm 3

A primal active-set method for convex QP.

Find \((x, y, z)\) satisfying conditions (5.3) for some second-order consistent basis \( B \); while \( \exists l : z_l + r_l < 0 \) do

if \( l \in N \) then

\( N \leftarrow N \setminus \{l\} \);

PRIMAL_BASE\((B, N, l, x, y, z)\); \( \text{[returns} B, N, x, y, z] \)

else

\( B \leftarrow B \setminus \{l\} \);

end if

while \( z_l + r_l < 0 \) do

PRIMAL_INTERMEDIATE\((B, N, l, x, y, z)\); \( \text{[returns} B, N, x, y, z] \)

end while

\( B \leftarrow B \cup \{l\} \);

end while

Theorem 5.1. Assume that problem \((PQP_{q,r})\) is nondegenerate. Given an initial point \((x, y, z)\) satisfying conditions (5.3) for a second-order consistent basis \( B \), then Algorithm 3 finds a solution of \((PQP_{q,r})\) or determines that \((DQP_{q,r})\) is infeasible in a finite number of iterations.

Proof. Assume that \((x, y, z)\) satisfies the conditions (5.3) for the second-order consistent basis \( B \). Let \( \mathcal{B}^c \) denote the index set \( \mathcal{B}^c = \{i \in B : z_i + r_i < 0\} \), and let \( \vec{r} \) be the vector \( \vec{r}_i = r_i, i \not\in \mathcal{B}^c \), and \( \vec{r}_i = -z_i, i \in \mathcal{B}^c \). These definitions imply that \( \vec{r}_i = -z_i > -z_i + z_i + r_i = r_i \), for every \( i \in \mathcal{B}^c \). It follows that \( \vec{r} \geq r \), and the feasible region of \((DQP_{q,r})\) is a subset of the feasible region of \((DQP_{q,\vec{r}})\). In addition, if \( r \) is replaced by \( \vec{r} \) in (3.1), the only difference is that \( z_B + \vec{r}_B = 0 \), i.e., the initial point for (5.3) is a stationary point with respect to \((PQP_{q,\vec{r}})\).

The first step of the proof is to show that after a finite number of iterations of Algorithm 3, one of three possible events must occur: (i) the cardinality of the
set $\mathcal{B}^<$ is decreased by at least one; (ii) a solution of problem (PQP$_{q,r}$) is found; or (iii) (DQP$_{q,r}$) is declared infeasible. The proof will also establish that if (i) does not occur, then either (ii) or (iii) must hold after a finite number of iterations.

Assume that (i) never occurs. This implies that the index $l$ selected in the base iteration can never be an index in $\mathcal{B}^<$ because at the end of such an iteration, it would belong to $\mathcal{B}$ with $z_l + r_l = 0$, contradicting the assumption that the cardinality of $\mathcal{B}^<$ never decreases. For the same reason, it must hold that $k \notin \mathcal{B}^<$ for every index $k$ selected to be moved from $\mathcal{B}$ to $\mathcal{N}$ in any subiteration, because an index can only be moved from $\mathcal{N}$ to $\mathcal{B}$ by being selected in the base iteration. These arguments imply that $z_i = -\tilde{r}_i$, with $i \in \mathcal{B}^<$, throughout the iterations. It follows that the iterates may be interpreted as being members of a sequence constructed for solving (PQP$_{q,r}$) with a fixed $\tilde{r}$, where the initial stationary point is given, and each iteration gives a new stationary point. The nondegeneracy assumption implies that the objective value of (PQP$_{q,r}$) is strictly decreasing at each base subiteration, and nonincreasing at each intermediate subiteration. The number of intermediate subiterations is finite, which implies that a strict improvement of the objective value of (PQP$_{q,r}$) is obtained at each iteration. As there are only a finite number of stationary points, Algorithm 3 either solves (PQP$_{q,r}$) or concludes that (DQP$_{q,r}$) is infeasible after a finite number of iterations. If (PQP$_{q,r}$) is solved, then $z_N + r_N \geq 0$, because $\tilde{r}_j = r_j$ for $j \in \mathcal{N}$. Hence, Algorithm 3 can not proceed further by selecting an $l \in \mathcal{N}$, and the only way to reduce the objective is to select an $l$ in $\mathcal{B}$ such that $z_j + r_j < 0$. Under the assumption that (i) does not occur, it must hold that no eligible indices exist and $\mathcal{B}^<$ = $\emptyset$. However, in this case (PQP$_{q,r}$) has been solved with $\tilde{r} = r$, and (ii) must hold. If Algorithm 3 declares (DQP$_{q,r}$) to be infeasible, then (DQP$_{q,r}$) must also be infeasible because the feasible region of (DQP$_{q,r}$) is contained in the feasible region of (DQP$_{q,r}$). In this case (DQP$_{q,r}$) is infeasible and (iii) occurs.

Finally, if (i) occurs, there is an iteration at which the cardinality of $\mathcal{B}^<$ decreases and an index is removed from $\mathcal{B}^<$. There may be more than one such index, but there is at least one $l$ moved from $\mathcal{B}^<$ to $\mathcal{B} \setminus \mathcal{B}^<$, or one $k$ moved from $\mathcal{B}^<$ to $\mathcal{N}$. In either case, the cardinality of $\mathcal{B}^<$ is decreased by at least one. After such an iteration, the argument given above may be repeated for the new set $\mathcal{B}^<$ and new shift $\tilde{r}$. Applying this argument repeatedly gives the result the situation (i) can occur only a finite number of times.

It follows that, (ii) or (iii) must occur after a finite number of iterations, which is the required result.

5.5. Relaxed initial conditions for the dual QP method.

Analogous to the primal case, the choice of $q = q^{(1)} = 0$ in Step (1) of the dual-first strategy may give some negative components in the vector $x_N + q_N$. In the case, the
conditions on the initial values of $B, N, q, r, x, y,$ and $z$ are generalized so that

\begin{align*}
Hx + c - ATy - z &= 0, \quad (5.4a) \\
Ax + My - b &= 0, \quad (5.4b) \\
z_N + r_N &\geq 0, \quad z_B + r_B = 0, \quad (5.4c) \\
x_N + q_N &\leq 0. \quad (5.4d)
\end{align*}

Similarly, the set of eligible indices may be extended to include indices associated with negative values of $x_N + q_N$. If the index $l$ is from $N$, the associated matrix $K_B$ is nonsingular, and intermediate subiterations are executed until the updated value satisfies $x_l + q_l = 0$. At this point, the index $l$ is returned $N$. The method is summarized in Algorithm 4.

The strategies of solving two consecutive quadratic programs may be generalized to a sequence of more than two quadratic programs, where we alternate between primal and dual active-set methods, and eliminate the shifts in more than two steps.

**Algorithm 4** A dual active-set method for convex QP.

Find $(x, y, z)$ satisfying conditions (5.4) for some second-order consistent $B$;

while $\exists l : x_l + q_l < 0$ do

if $l \in B$ then

$B \leftarrow B \setminus \{l\}$;

DUAL_BASE($B, N, l, x, y, z$); \hspace{1cm} [Base subiteration]

else

$N \leftarrow N \setminus \{l\}$;

end if

while $x_l + q_l < 0$ do

DUAL_INTERMEDIATE($B, N, l, x, y, z$); \hspace{1cm} [Intermediate subiteration]

end while

$N \leftarrow N \cup \{l\}$;

end while

6. Practical Issues

6.1. Quadratic programs with upper and lower bounds

As stated, the primal quadratic program has lower bound zero on the $x$-variables. This is for notational convenience. This form may be generalized in a straightforward manner to a form where the $x$-variables has both lower and upper bounds on the primal variables, i.e., $b_L \leq x \leq b_U$, where components of $b_L$ can be $-\infty$ and components of $b_U$ can be $+\infty$. Given primal shifts $q_L$ and $q_U$, and dual shifts $r_L$ and $r_U$, we have the primal-dual pair

$$
(PQP_{q,r}) \quad \text{minimize} \quad \frac{1}{2}x^THx + \frac{1}{2}y^TMy + c^Tx + (r_L - r_U)^Tx \\
\text{subject to} \quad Ax + My = b, \quad b_L - q_L \leq x \leq b_U + q_U,
$$
and

$$\text{(DQP)}_{q,r} \quad \text{maximize} \quad \frac{1}{2} x^T H x - \frac{1}{2} y^T M y + b^T y + (b_L - q_L)^T z_L - (b_U + q_U)^T z_U$$

subject to

$$-H x + A^T y + z_L - z_U = c, \quad z_L \geq -r_L, \quad z_U \geq -r_U.$$
maximum step length at a base subiteration is infinite, it must hold that \( \Delta z_j = 0 \) for all temporary bounds \( j \), and the primal problem is infeasible.

The discussion above implies that a pair of primal and dual problems solved consecutively will terminate with \( z_j = 0 \) for all indices \( j \) associated with temporary bounds. This is because \( z_j \) is unchanged in the dual problem and driven to zero in the primal problem.

7. Numerical Examples

In this section we describe a particular formulation of the primal-dual shifted method of Section 5. In addition, some numerical experiments are presented for a simple \textsc{Matlab} implementation applied to a set of convex problems from the CUTEst test collection (see Bongartz et al. [7], and Gould, Orban and Toint [33]).

7.1. The coupled primal-dual algorithm PDQP

For illustrative purposes, a primal-dual shifted method is used in which either a “primal-first” or “dual-first” strategy is selected based on the initial point. In particular, if the point is dual feasible, then the “dual-first” strategy is used. Otherwise, the “primal-first” strategy is selected.

7.2. The implementation

Each CUTEst \textsc{QP} problem may be written in the form

\[
\begin{align*}
\text{minimize} & \quad c^T x + \frac{1}{2} x^T H x \\
\text{subject to} & \quad \ell \leq \left( \frac{x}{\bar{A} x} \right) \leq u,
\end{align*}
\]

where \( \ell \) and \( u \) are constant vectors of lower and upper bounds and \( \bar{A} \) has dimension \( m \times n \). In this format, a fixed variable or equality constraint has the same value for its upper and lower bound. Each problem was converted to the equivalent form

\[
\begin{align*}
\text{minimize} & \quad c^T x + \frac{1}{2} x^T H x \\
\text{subject to} & \quad \bar{A} x - s = 0, \quad \ell \leq \left( \begin{array}{c} x \\ s \end{array} \right) \leq u,
\end{align*}
\]

where \( s \) is a vector of slack variables. With this formulation, the \textsc{QP} problem involves simple upper and lower bounds instead of nonnegativity constraints. It follows that the matrix \( M \) is zero, but the full row-rank assumption on the constraint matrix is satisfied because the constraint matrix \( A \) takes the form \( \begin{pmatrix} \bar{A} & -I \end{pmatrix} \) and has rank \( m \). In this situation, a nonsingular \( m \times m \) submatrix \( A_B \) of \( A \) may be identified using a so-called “crash” procedure. One algorithm for doing this has been proposed by Gill, Murray and Saunders [25], who use a sparse LU factorization of \( A^T \) to identify a square nonsingular subset of the columns of \( A_B \). These factors give a matrix \( Z \) whose columns form a basis for the null space of \( A \) as

\[
Z = \begin{pmatrix} -A_B^{-1} A_N \\ I \end{pmatrix}.
\]
This $Z$ may be used to form $Z^T H Z$, and a partial Cholesky factorization with interchanges may be used to find an upper-triangular matrix $R$ that is the factor of the largest nonsingular leading submatrix of $Z^T H Z$. Let $Z_R$ denote the columns of $Z$ corresponding to $R$, and let $Z$ be partitioned as $Z = \begin{pmatrix} Z_R & Z_A \end{pmatrix}$. Then, the set $B$ given by the nonsingular $A_B$ may be augmented by the indices corresponding to $Z_R$, giving a final $B$ for which the corresponding KKT-matrix $K_B$ is nonsingular.

### 7.3. Numerical results

Numerical results from a simple MATLAB implementation of Algorithm PDQP were obtained for a set of 121 convex QPs in standard interface format (SIF). The problems were selected based on the dimension of the constraint matrix $A$ in (7.2). In particular, the test set includes all QP problems for which the smaller of $m$ and $n$ is of the order of 500 or less. This gave 121 QPs ranging in size from BQP1VAR (one variable and one constraint) to LINCONT (1257 variables and 419 constraints).

In order to judge how the proposed method compares to a conventional two-phase active-set method, the same 121 problems were solved using the convex QP solver SQOPT [26], which is a Fortran implementation of a two-phase (primal) reduced-gradient active-set method for large-scale QP. All SQOPT runs were made using the default parameter options.

Both PDQP and SQOPT are terminated at a point $(x, y, z)$ that satisfies the equations conditions of (2.6) modified to conform to the constraint format of (7.2). The feasibility and optimality tolerances are given by $\epsilon_{\text{feas}} = 10^{-6}$ and $\epsilon_{\text{opt}} = 10^{-6}$, respectively. For a given $\epsilon_{\text{opt}}$, PDQP and SQOPT terminate when

$$\max_{i \in B} |z_i| \leq \epsilon_{\text{opt}} \|y\|_{\infty}, \quad \text{and} \quad \begin{cases} z_i \geq -\epsilon_{\text{opt}} \|y\|_{\infty} & \text{if } x_i \geq -\ell_i, \; i \in N; \\ z_i \leq \epsilon_{\text{opt}} \|y\|_{\infty} & \text{if } x_i \leq \; u_i, \; i \in N. \end{cases}$$

Both PDQP and SQOPT use the EXPAND procedure of Gill et al. [27] to allow the variables $(x, s)$ to move outside their bounds by as much as $\epsilon_{\text{feas}}$.

A summary of the results is given in Table 1. The first four columns give the name of the problem, the number of linear constraints $m$, the number of variables $n$, and the optimal objective value Objective. The next two columns summarize the SQOPT result for the given problem, with Phs1 and Itn giving the phase-one iterations and iteration total, respectively. The last four columns summarize the results for the PDQP implementation. The first column gives the total number of primal and dual iterations Itn. The second column gives the order in which the primal and dual algorithms were applied, with PD indicating the “primal-first” strategy, and DP the “dual-first” strategy. The final two columns, headed by p-Itn and d-Itn, give the iterations required for the primal method and the dual method, respectively.

Of the 121 problems tested, two (LINCONT and NASH) are known to be infeasible. This infeasibility was identified correctly by both SQOPT and PDQP. In total, SQOPT solved 117 of the remaining 119 problems, but declared (incorrectly) that problems RDW2D51U and RDW2D52U are unbounded. PDQP solved the same number of problems, but failed to achieve the required accuracy for the problems RDW2D51B and RDW2D52F.
In these two cases, the final objective values computed by PDQP were 1.0947648E-02 and 1.0491239E-02 respectively, instead of the optimal values 1.0947332E-02 and 1.0490828E-02. (The five RDW2D5* problems in the test set are known to be difficult to solve, see Gill and Wong [30].)

If the failed and infeasible runs are excluded, Algorithm PDQP required the same or fewer number of iterations than SQOPT on 73 of the 115 QP problems solved to optimality by both methods. This constitutes 64% of the problems solved. Of this 64%, the “dual-first” strategy made up 44% of the cases and 56% of the improvements were associated with the “primal-first” strategy.

Table 1: Results for PDQP and SQOPT on 121 CUTEst QPs.

| Name   | m   | n   | Objective      | SQOPT | PDQP |
|--------|-----|-----|----------------|-------|------|
|        |     |     |                | PhsI  | Itn  | Order | P-Itn | D-Itn |
| ALLINQP| 50  | 100 | -9.1592833E+00| 0     | 45   | 65    | 63    | 2     |
| AUG2DQP| 100 | 223 | 1.7797215E+02  | 8     | 116  | 440   | 326   | 114   |
| AUG3D  | 27  | 156 | 8.3333333E-02  | 0     | 45   | 45    | 0     | 45    |
| AVGASA | 10  | 8   | -4.6319255E+00 | 5     | 8    | 5     | 0     | 5     |
| AVGASB | 10  | 8   | -4.4832193E+00 | 5     | 8    | 7     | 0     | 7     |
| BIGGSB1| 1   | 100 | 1.5000000E-02  | 0     | 103  | 101   | 101   | 1     |
| SQP1VAR| 1   | 1   | 0.0000000E+00  | 0     | 1    | 1     | 0     | 1     |
| SQPGBASIM | 1 | 50 | -3.7903432E-05 | 0   | 36   | 7    | 7     | 0     |
| SQPGBASIM | 1 | 50 | -6.5198140E-05 | 0   | 40   | 8    | 8     | 0     |
| CENHARK | 1   | 100 | -2.0000000E+00 | 0     | 132  | 32    | 0     | 32    |
| CVXQP1  | 1   | 100 | 2.3725000E+02  | 0     | 100  | 119   | 2     | 117   |
| CVXQP1  | 50  | 100 | 1.1590718E+04  | 5     | 67   | 91    | 1     | 90    |
| CVXQP2  | 25  | 100 | 8.1209404E+03  | 2     | 82   | 85    | 2     | 83    |
| CVXQP3  | 75  | 100 | 1.1943432E+04  | 17    | 46   | 113   | 2     | 111   |
| DEGENQP | 100 | 5   | 0.0000000E+00  | 0     | 6    | 18    | 18    | 0     |
| DTOC3   | 18  | 29  | 2.2659038E+02  | 1     | 10   | 17    | 0     | 17    |
| DUAL1   | 1   | 85  | 3.5012967E-02  | 0     | 88   | 88    | 0     | 88    |
| DUAL2   | 1   | 96  | 3.3733671E-02  | 0     | 99   | 99    | 0     | 99    |
| DUAL3   | 1   | 111 | 3.3575583E-01  | 0     | 106  | 106   | 0     | 106   |
| DUAL4   | 1   | 75  | 6.1609064E-01  | 0     | 61   | 61    | 0     | 61    |
| DUALC1  | 215 | 9   | 6.1552516E+03  | 1     | 9    | 4     | 0     | 4     |
| DUALC2  | 229 | 7   | 3.5513063E+03  | 2     | 4    | 4     | 0     | 4     |
| DUALC5  | 278 | 8   | 4.2723256E+02  | 1     | 7    | 6     | 0     | 6     |
| DUALC8  | 503 | 8   | 1.8309361E+04  | 4     | 6    | 8     | 0     | 8     |
| GENHSEA2| 8   | 10  | 9.2717369E-01  | 0     | 3    | 5     | 0     | 5     |
| GMHCASE2| 1050| 175 | -9.9444495E-01 | 18    | 99   | 91    | 0     | 91    |
| GMHCASE3| 1050| 175 | 1.5251466E+00  | 31    | 100  | 86    | 0     | 86    |
| GMHCASE4| 350 | 175 | 5.9663849E+03  | 74    | 171  | 175   | 0     | 175   |
| GOULDPQ2| 199 | 399 | 9.0045697E-06  | 0     | 213  | 419   | 0     | 419   |
| GOULDPQ3| 199 | 399 | 5.6732908E-02  | 0     | 200  | 406   | 0     | 205   |
| GRIDNETA| 100 | 180 | 9.5242163E+01  | 5     | 35   | 134   | 91    | 53    |
| GRIDNETB| 100 | 180 | 4.7268237E+01  | 0     | 81   | 97    | 0     | 97    |
| GRIDNETC| 100 | 180 | 4.8352347E+01  | 6     | 93   | 153   | 0     | 153   |
| HS3     | 1   | 2   | 0.0000000E+00  | 0     | 2    | 1     | 1     | 0     |
| HS3MOD  | 1   | 2   | 1.2325951E-32  | 0     | 2    | 1     | 1     | 0     |
| HS21    | 1   | 2   | -9.9960000E+00 | 0     | 1    | 0     | 0     | 0     |
| HS28    | 1   | 3   | 1.2325951E-32  | 0     | 2    | 0     | 0     | 0     |
| HS35    | 1   | 3   | 1.1111111E-01  | 0     | 5    | 1     | 0     | 1     |
| HS35I   | 1   | 3   | 1.1111111E-01  | 0     | 5    | 1     | 0     | 1     |
| HS35MOD | 1   | 3   | 2.5000000E-01  | 0     | 1    | 0     | 0     | 0     |
| HS44    | 6   | 4   | -1.5000000E+01 | 0     | 2    | 4     | 0     | 4     |
| HS44NEW | 6   | 4   | -1.5000000E+01 | 0     | 4    | 9     | 0     | 9     |
| HS51    | 3   | 5   | -8.8817841E-16 | 0     | 2    | 0     | 0     | 0     |
| HS52    | 3   | 5   | 5.3266475E+00  | 0     | 2    | 1     | 0     | 1     |
| HS53    | 3   | 5   | 4.0930232E+00  | 0     | 2    | 1     | 0     | 1     |
| HS78    | 3   | 4   | -4.8818181E+00 | 0     | 4    | 4     | 0     | 4     |


7. Numerical Examples

Table 1: Results for PDQP and SQOPT on 121 CUTEst QPs.
(continued)

| Name       | m  | n  | Objective       | SQOPT | PDQP |
|------------|----|----|-----------------|-------|------|
| HS76I      | 3  | 4  | -4.6818181E+00  | 0     | 4    |
| HS118      | 17 | 15 | 6.6482045E+02   | 0     | 21   |
| HS268      | 5  | 5  | 7.2759756E-12   | 0     | 0    |
| HUES-MOD   | 2  | 100| 3.4829823E+07   | 1     | 103  |
| HUESTIS    | 2  | 100| 3.4829823E+09   | 1     | 103  |
| JNLBRNG1   | 1  | 529| -1.8004556E-01  | 0     | 292  |
| JNLBRNG2   | 1  | 529| -4.1023852E+00  | 0     | 252  |
| JNLBRNGA   | 1  | 529| -3.0795806E+06  | 0     | 292  |
| JNLBRNGB   | 1  | 529| -6.5067871E+00  | 0     | 247  |
| KSPI       | 1000| 20| 5.7579792E-01   | 0     | 2847 |
| LINCONT    | 419| 1257| infeasible     | 138   | 138  |
| LISRST1    | 100| 106| 2.6072632E-01   | 0     | 52   |
| LISRST2    | 100| 106| 2.5876398E-01   | 0     | 63   |
| LISRST3    | 100| 106| 2.5876398E-01   | 0     | 64   |
| LISRST4    | 100| 106| 2.5876399E-01   | 0     | 61   |
| LISRST5    | 100| 106| 2.5876410E-01   | 0     | 58   |
| LISRST6    | 100| 106| 2.5876390E-01   | 0     | 67   |
| LISRST7    | 100| 106| 2.5896758E-01   | 0     | 68   |
| LISRST8    | 100| 106| 2.5747454E-01   | 0     | 94   |
| LISRST9    | 100| 106| 2.5843992E+00   | 0     | 102  |
| LISRST10   | 100| 106| 2.5876391E-01   | 0     | 100  |
| LISRST11   | 100| 106| 2.5876391E-01   | 0     | 100  |
| LISRST12   | 100| 106| 9.1994948E+00   | 0     | 18   |
| LOTSCHD   | 7  | 12 | 2.3984158E+03   | 4     | 8    |
| MOSARQP1   | 10 | 100| -1.5420010E+02  | 0     | 102  |
| MOSARQP2   | 10 | 100| -3.0795806E+06  | 0     | 100  |
| NASH       | 24 | 72 | 6.5193252E+00   | 0     | 5    |
| OBSTCLAE   | 1  | 529| 1.6780270E+00   | 0     | 605  |
| OBSTCLAL   | 1  | 529| 6.5193252E+00   | 0     | 263  |
| OBSTCLBM   | 1  | 529| 6.5193252E+00   | 0     | 469  |
| OBSTCLBL   | 1  | 529| 6.5193252E+00   | 0     | 469  |
| OBSTCLB1   | 1  | 529| 6.5193252E+00   | 0     | 484  |
| OBSTCLB2   | 1  | 529| 6.5193252E+00   | 0     | 303  |
| OSLBQP     | 1  | 8  | 6.2500000E+00   | 0     | 0    |
| PENTDI     | 1  | 500| 7.5000000E-01   | 0     | 2    |
| POWELL20   | 100| 100| 5.2703125E+04   | 49    | 52   |
| PRIMAL1    | 85 | 325| -3.5012967E-02  | 0     | 217  |
| PRIMAL2    | 96 | 649| -3.3733671E-02  | 0     | 407  |
| PRIMAL3    | 111| 745| -1.3575583E-01  | 0     | 1223 |
| PRIMAL4    | 75 | 1498| -7.4609064E-01  | 0     | 1264 |
| PRIMALC1   | 9  | 230| -6.1552516E+03  | 0     | 15   |
| PRIMALC2   | 7  | 231| -3.5513063E+03  | 0     | 10   |
| PRIMALC5   | 8  | 287| -7.5000000E-01  | 0     | 30   |
| PRIMALC8   | 8  | 520| -3.0795806E+06  | 0     | 15   |
| QCPCLEN0   | 74 | 74 | 5.8424525E-03   | 0     | 111  |
| QCPCCGE1   | 351| 384| -1.5039652E+07  | 0     | 415  |
| QCPCCGE2   | 166| 143| -1.8719635E+06  | 0     | 415  |
| QPCSTAIR   | 356| 467| 6.2043917E+06   | 0     | 210  |
| QUDLIN     | 1  | 420| -8.8290000E+00  | 0     | 419  |
| RDWDS1F    | 225| 578| 1.1209993E+03   | 29    | 29   |
| RDWDS1U    | 225| 578| 8.3930023E+04   | 14    | 16   |
| RDWDS2B    | 225| 578| 1.0947648E+02   | 34    | 488  |
| RDWDS2P    | 225| 578| 1.0491239E+02   | 29    | 191  |
| S268       | 5  | 5  | 7.2759756E-12   | 0     | 8    |
| SIM2BQP    | 1  | 2  | 0.0000000E+00   | 0     | 1    |
| SIMBQP     | 1  | 2  | 6.0185310E-31   | 0     | 2    |
| STCQP1     | 30 | 65 | 4.9452085E+02   | 0     | 8    |
| STCQP2     | 128| 257| 1.4294017E+07   | 80    | 215  |
| STEERBA    | 108| 432| 1.6957674E+04   | 14    | 89   |
Table 1: Results for PDQP and SQOPT on 121 CUTEst QPs. (continued)

| Name  | m  | n  | Objective         | SQOPT Phs | Itn | Itn Order | P-Itn | D-Itn |
|-------|----|----|-------------------|----------|-----|-----------|-------|-------|
| TAME  | 1  | 2  | 3.0814879E-33     | 0        | 1   | PD        | 1     | 0     |
| TORSION1 | 1 | 484 | -4.5608771E-01   | 0        | 256 | PD        | 256   | 0     |
| TORSION2 | 1 | 484 | -4.5608771E-01   | 0        | 544 | PD        | 256   | 0     |
| TORSION3 | 1 | 484 | -1.2422498E+00   | 0        | 112 | PD        | 112   | 0     |
| TORSION4 | 1 | 484 | -1.2422498E+00   | 0        | 689 | PD        | 288   | 0     |
| TORSION5 | 1 | 484 | -2.8470688E+00   | 0        | 40  | PD        | 40    | 0     |
| TORSION6 | 1 | 484 | -2.8470688E+00   | 0        | 708 | PD        | 360   | 0     |
| TORSION7 | 1 | 484 | -4.1611287E-01   | 0        | 272 | PD        | 272   | 0     |
| TORSION8 | 1 | 484 | -4.1611287E-01   | 0        | 272 | PD        | 272   | 0     |
| TORSION9 | 1 | 484 | -4.1611287E-01   | 0        | 529 | PD        | 128   | 0     |
| TORSION10 | 1 | 484 | -2.8405962E+00   | 0        | 120 | PD        | 120   | 0     |
| TORSION11 | 1 | 484 | -4.194864E+00    | 0        | 529 | PD        | 128   | 0     |
| TORSION12 | 1 | 484 | -4.194864E+00    | 0        | 280 | PD        | 280   | 0     |
| TORSION13 | 1 | 484 | -4.194864E+00    | 0        | 40  | PD        | 40    | 0     |
| TORSION14 | 1 | 484 | -2.8405962E+00   | 0        | 40  | PD        | 40    | 0     |
| TORSION15 | 1 | 484 | -2.8405962E+00   | 0        | 360 | PD        | 360   | 0     |
| TORSION16 | 1 | 484 | -2.8405962E+00   | 0        | 40  | PD        | 40    | 0     |
| TORSION17 | 1 | 484 | -2.8405962E+00   | 0        | 112 | PD        | 112   | 0     |
| TORSION18 | 1 | 484 | -2.8405962E+00   | 0        | 280 | PD        | 280   | 0     |
| TORSION19 | 1 | 484 | -2.8405962E+00   | 0        | 280 | PD        | 280   | 0     |

**Algorithm** PDQP would not be recommended for solving a linear program (LP). Nevertheless, it was applied to 16 LPs from the CUTEst test set. The results are summarized in Table 2.

Table 2: Results for PDQP and SQOPT on 16 CUTEst LPs.

| Name  | m  | n  | Objective         | SQOPT Phs | Itn | Itn Order | P-Itn | D-Itn |
|-------|----|----|-------------------|----------|-----|-----------|-------|-------|
| AGG   | 488| 183| -3.5991767e+07    | 84       | 124 | PD        | 156   | 44    |
| DEGENLPA | 15 | 20 | -7.6798369e+01   | 9        | 11  | PD        | 26    | 26    |
| DEGENLPB | 15 | 20 | -3.0742351e+01   | 9        | 10  | PD        | 37    | 32    |
| EXTRASIM | 1  | 2  | 1.0000000e+00    | 0        | 0   | PD        | 0     | 0     |
| GOFFIN | 50 | 51 | -1.2612134e+13   | 3        | 25  | PD        | 100   | 0     |
| MAKELA4 | 40 | 21 | 0.0000000e+00    | 0        | 1   | PD        | 42    | 0     |
| MODEL  | 28 | 1542| infeasible       | 12       | 12  | PD        | 0     | 44    |
| GET1  | 1002 3 | 5.3924312e-01 | 1        | 107 | PD        | 22    | 0     |
| GET3  | 1002 4 | 4.5049718e-03 | 1        | 319 | PD        | 18    | 0     |
| PT    | 501 | 1  | 1.7839423e-01    | 1        | 135 | PD        | 20    | 0     |
| S277-280 | 4  | 4  | 5.0761905e+00    | 1        | 6   | PD        | 10    | 0     |
| SIMPLPA | 2  | 2  | 1.0000000e+00    | 1        | 2   | PD        | 1     | 4     |
| SIMPLPB | 3  | 2  | 1.1000000e+00    | 1        | 6   | PD        | 6     | 0     |
| SSEBLIN | 72 | 194| 1.6170600e+07    | 26       | 136 | PD        | 8     | 254   |
| SUPERSIM | 2  | 2  | 6.6666667e-01    | 1        | 1   | PD        | 0     | 2     |
| TFI2  | 101 | 1  | 6.940311e-01     | 0        | 34  | PD        | 46    | 2     |

**Algorithm** PDQP would not be recommended for solving a linear program (LP). Nevertheless, it was applied to 16 LPs from the CUTEst test set. The results are summarized in Table 2.

8. **Summary and Conclusions**

A general framework has been proposed for solving a convex quadratic program with general equality constraints and simple lower bounds on the variables. This framework allows the definition of two methods, one primal and one dual, that generate a sequence of iterates that are feasible with respect to the equality constraints associated with the optimality conditions of a general primal–dual form. The primal method maintains feasibility of the primal inequalities while driving the infeasibil-
ities of the dual inequalities to zero. The dual method maintains feasibility of the dual inequalities while moving to satisfy the infeasibilities of the primal inequalities. In each of these methods, the search directions satisfy a KKT system of equations formed from Hessian and constraint components associated with an appropriate column basis. The composition of the basis is specified by an active-set strategy that guarantees the nonsingularity of each set of KKT equations.

Each of the proposed methods is a conventional active-set method in the sense that an initial primal- or dual-feasible point is required. In addition, it has been shown how the bounds of the primal and dual problems may be shifted so as to give a strategy for solving the original problem by solving a pair of coupled quadratic programs, one primal and one dual. An application of this approach is to solve a shifted dual QP for a feasible point for the primal (or vice versa), thereby avoiding the need for a traditional feasibility phase that ignores the properties of the objective function.

The numerical results indicate that the proposed primal, dual, and coupled primal-dual QP methods can be efficient relative to existing two-phase active-set methods. Future work will focus on the application of the proposed methods to situations in which a series of related QPs must be solved, for example, in sequential quadratic programming methods and methods for mixed-integer nonlinear programming.

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A. Appendix

The appendix concerns some basic results used in previous sections. The first result shows that the nonsingularity of a KKT matrix may be established by checking that the two row blocks \( H \quad A^T \) and \( A \quad -M \) have full row rank.

**Proposition A.1.** Assume that \( H \) and \( M \) are symmetric, positive semidefinite matrices. The vectors \( u \) and \( v \) satisfy

\[
\begin{pmatrix}
H & A^T \\
A & -M
\end{pmatrix}
\begin{pmatrix}
u \\ -v
\end{pmatrix} = \begin{pmatrix}0 \\ 0
\end{pmatrix}
\]

if and only if

\[
\begin{pmatrix}H \\ A
\end{pmatrix} u = \begin{pmatrix}0 \\ 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}A^T \\ -M
\end{pmatrix} v = \begin{pmatrix}0 \\ 0
\end{pmatrix}.
\]

**Proof.** If (A.2) holds, then (A.1) holds, which establishes the “if” direction. Now assume that \( u \) and \( v \) are vectors such that (A.1) holds. Then,

\[
u^T H u - u^T A^T v = 0, \quad \text{and} \quad v^T A u + v^T M v = 0.
\]

Adding these equations gives the identity \( u^T H u + v^T M v = 0 \). But then, the symmetry and semidefiniteness of \( H \) and \( M \) imply \( u^T H u = 0 \) and \( v^T M v = 0 \). This can hold only if \( H u = 0 \) and \( M v = 0 \). If \( H u = 0 \) and \( M v = 0 \), (A.1) gives \( A^T v = 0 \) and \( A u = 0 \), which implies that (A.2) holds.

The next result shows that when checking a subset of the columns of a symmetric positive semidefinite matrix for linear dependence, it is only the diagonal block that is of importance. The off-diagonal block may be ignored.

**Proposition A.2.** Let \( H \) be a symmetric, positive semidefinite matrix partitioned as

\[
H = \begin{pmatrix}H_{11} & H_{12} \\
H_{12}^T & H_{22}
\end{pmatrix}.
\]

Then,

\[
\begin{pmatrix}H_{11} \\ H_{12}^T
\end{pmatrix} u = \begin{pmatrix}0 \\ 0
\end{pmatrix} \quad \text{if and only if} \quad H_{11} u = 0.
\]

**Proof.** If \( H \) is positive semidefinite, then \( H_{11} \) is positive semidefinite, and it holds that

\[
\begin{pmatrix}0 \\ 0
\end{pmatrix} = \begin{pmatrix}H_{11} \\ H_{12}^T
\end{pmatrix} u = \begin{pmatrix}H_{11} & H_{12} \\
H_{12}^T & H_{22}
\end{pmatrix} \begin{pmatrix}u \\ 0
\end{pmatrix}
\]

if and only if

\[
0 = (u^T \quad 0) \begin{pmatrix}H_{11} & H_{12} \\
H_{12}^T & H_{22}
\end{pmatrix} \begin{pmatrix}u \\ 0
\end{pmatrix} = u^T H_{11} u
\]

if and only if \( H_{11} u = 0 \), as required. \( \blacksquare \)
In the following propositions, the distinct integers \( k \) and \( l \), together with integers from the index sets \( B \) and \( N \) define a partition of \( I = \{1, 2, \ldots, n\} \), i.e., \( I = B \cup \{k\} \cup \{l\} \cup N \). If \( w \) is any \( n \)-vector, the \( n_B \)-vector \( w_B \) and \( n_N \)-vector \( w_N \) denote the vectors of components of \( w \) associated with \( B \) and \( N \). For the symmetric Hessian \( H \), the matrices \( H_{BB} \) and \( H_{NN} \) denote the subset of rows and columns of \( H \) associated with the sets \( B \) and \( N \) respectively. The unsymmetric matrix of components \( h_{ij} \) with \( i \in B \) and \( j \in N \) will be denoted by \( H_{BN} \). Similarly, \( A_B \) and \( A_N \) denote the matrices of columns associated with \( B \) and \( N \).

The next result concerns the row rank of the \((A-M)\) block of the KKT matrix.

**Proposition A.3.** If the matrix \((a_l \ a_k \ A_B \ -M)\) has full row rank, and there exist \( \Delta x_l, \Delta x_k, \Delta x_B, \) and \( \Delta y \) such that \( a_l \Delta x_l + a_k \Delta x_k + A_B \Delta x_B + M \Delta y = 0 \) with \( \Delta x_k \neq 0 \), then \((a_l \ A_B \ -M)\) has full row rank.

**Proof.** It must be established that \( u^T (a_l \ A_B \ -M) = 0 \) implies that \( u = 0 \). For a given \( u \), let \( \gamma = -u^T a_k \), so that

\[
\begin{pmatrix} u^T & \gamma \end{pmatrix} \begin{pmatrix} a_l & a_k & A_B & -M \end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}.
\]

Then,

\[
0 = \begin{pmatrix} u^T & \gamma \end{pmatrix} \begin{pmatrix} a_l & a_k & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_k \\ \Delta x_B \\ -\Delta y \end{pmatrix} = \gamma \Delta x_k.
\]

As \( \Delta x_k \neq 0 \), it must hold that \( \gamma = 0 \), in which case

\[
u^T (a_l \ a_k \ A_B \ -M) = 0.
\]

As \((a_l \ a_k \ A_B \ -M)\) has full row rank by assumption, it follows that \( u = 0 \) and \((a_l \ A_B \ -M)\) must have full row rank. \( \blacksquare \)

An analogous result holds concerning the \((H \ A^T)\) block of the KKT matrix.

**Proposition A.4.** If \((H_{BB} \ A^T_{Bi})\) has full row rank, and there exist quantities \( \Delta x_N, \Delta x_B, \Delta y, \) and \( \Delta z_k \) such that

\[
\begin{pmatrix} h^T_{bk} & h^T_{bk} & a^T_{ki} \\ h^T_{bn} & H^T_{BB} & A^T_{Bi} \end{pmatrix} \begin{pmatrix} \Delta x_N \\ \Delta x_B \\ -\Delta y \\ -\Delta z_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

with \( \Delta z_k \neq 0 \), then the matrix

\[
\begin{pmatrix} h_{kk} & h^T_{bk} & a^T_{ki} \\ h_{bk} & H^T_{BB} & A^T_{Bi} \end{pmatrix}
\]

has full row rank.
Proof. Let \((\mu \ v^T)\) be any vector such that
\[
h(\mu \ v^T) \begin{pmatrix} h_{Nk}^T & h_{Bk}^T & a_k^T \\ h_{BN} & H_{BB} & A_B^T \end{pmatrix} = (0 \ 0 \ 0).
\]
The assumed identity (A.3) gives
\[
0 = (\mu \ v^T) \begin{pmatrix} h_{Nk}^T & h_{Bk}^T & a_k^T \\ h_{BN} & H_{BB} & A_B^T \end{pmatrix} \begin{pmatrix} \Delta x_N \\ \Delta x_B \end{pmatrix} = \mu \Delta z_k.
\]
As \(\Delta z_k \neq 0\) by assumption, it must hold that \(\mu = 0\). The full row rank of \((H_{BB} \ A_B^T)\) then gives \(v = 0\) and
\[
\begin{pmatrix} h_{Nk}^T & h_{Bk}^T & a_k^T \\ h_{BN} & H_{BB} & A_B^T \end{pmatrix}
\]
must have full row rank. Proposition A.1 implies that this is equivalent to
\[
\begin{pmatrix} h_{kk}^T & a_k^T \\ H_{BB} & A_B^T \end{pmatrix}
\]
having full row rank. 

The next proposition concerns the primal subiterations when the constraint index \(k\) is moved from \(\mathcal{B}\) to \(\mathcal{N}\). In particular, it is shown that the \(K_l\) matrix is nonsingular after a subiteration.

Proposition A.5. Assume that \((\Delta x_l, \Delta x_k, \Delta x_B, -\Delta y, -\Delta z_l)\) is the unique solution of the equations
\[
\begin{pmatrix} h_{ll} & h_{kl} & h_{Bl} & a_l^T \\ h_{kl} & h_{kk} & h_{Bk} & a_k^T \\ h_{Bl} & h_{Bk} & H_{BB} & A_B^T \\ a_l & a_k & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_k \\ \Delta x_B \\ -\Delta y \\ -\Delta z_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.4}
\]
and that \(\Delta x_k \neq 0\). Then, the matrices \(K_l\) and \(K_k\) are nonsingular, where
\[
K_l = \begin{pmatrix} h_{ll} & h_{Bl} & a_l^T \\ h_{Bl} & H_{BB} & A_B^T \\ a_l & A_B & -M \end{pmatrix} \quad \text{and} \quad K_k = \begin{pmatrix} h_{kk} & h_{Bk} & a_k^T \\ h_{Bk} & H_{BB} & A_B^T \\ a_k & A_B & -M \end{pmatrix}.
\]

Proof. By assumption, the equations (A.4) have a unique solution with \(\Delta x_k \neq 0\). This implies that there is no solution of the overdetermined equations
\[
\begin{pmatrix} h_{ll} & h_{kl} & h_{Bl} & a_l^T \\ h_{kl} & h_{kk} & h_{Bk} & a_k^T \\ h_{Bl} & h_{Bk} & H_{BB} & A_B^T \\ a_l & a_k & A_B & -M \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_k \\ \Delta x_B \\ -\Delta y \\ -\Delta z_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.5}
\]
Given an arbitrary matrix $D$ and nonzero vector $f$, the fundamental theorem of linear algebra implies that if $Dw = f$ has no solution, then there exists a vector $v$ such that $v^T f \neq 0$. The application of this result to (A.5) implies the existence of a nontrivial vector $(\Delta \overline{x}_l, \Delta \overline{x}_k, -\Delta \overline{y}, \Delta \overline{z}_l, \Delta \overline{z}_k)$ such that

$$
\begin{pmatrix}
   h_{ll} & h_{kl} & h_{bl} & a_l^T & 1 \\
   h_{kl} & h_{kk} & h_{bk} & a_k^T & 1 \\
   h_{bl} & h_{bk} & H_{bb} & A_B^T \\
   a_l & a_k & A_B & -M \\
   1 & 1 & 1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
   \Delta \overline{x}_l \\
   \Delta \overline{x}_k \\
   \Delta \overline{y} \\
   \Delta \overline{z}_l \\
   \Delta \overline{z}_k
\end{pmatrix} =
\begin{pmatrix}
   0 \\
   0 \\
   0 \\
   0 \\
   0
\end{pmatrix},
$$

(A.6)

with $\Delta \overline{z}_l \neq 0$. The last equation of (A.6) gives $\Delta \overline{x}_l + \Delta \overline{z}_l = 0$, in which case $\Delta \overline{x}_l \Delta \overline{z}_l = -\Delta \overline{z}_l^2 < 0$ because $\Delta \overline{z}_l \neq 0$. Any solution of (A.6) may be viewed as a solution of the equations $H \Delta \overline{x} - A^T \Delta \overline{y} - \Delta \overline{z} = 0$, $A \Delta \overline{x} + M \Delta \overline{y} = 0$, $\Delta \overline{z}_B = 0$, and $\Delta \overline{z}_i = 0$ for $i \in \{1, 2, \ldots, n\} \setminus \{l\} \setminus \{k\}$. An argument similar to that used to establish Proposition 2.2 gives

$$
\Delta \overline{x}_l \Delta \overline{z}_l + \Delta \overline{x}_k \Delta \overline{z}_k \geq 0,
$$

which implies that $\Delta \overline{x}_k \Delta \overline{z}_k > 0$, with $\Delta \overline{x}_k \neq 0$ and $\Delta \overline{z}_k \neq 0$.

As the search direction is unique, it follows from (A.4) that $(h_{bl}^T \ H_{bk} \ H_{bb} \ A_B^T)$ has full row rank, and Proposition A.2 implies that $(H_{bb}^T \ A_B^T)$ has full row rank. Hence, as $\Delta \overline{z}_l \neq 0$, it follows from (A.6) and Proposition A.4 that the matrix

$$
\begin{pmatrix}
   h_{ll} & h_{kl} & h_{bl}^T & a_l^T \\
   h_{kl} & h_{kk} & h_{bk} & a_k^T \\
   h_{bl}^T & h_{bk} & H_{bb} & A_B^T \\
   a_l & a_k & A_B & -M \\
   1 & 1 & 1 & -1
\end{pmatrix}
$$

has full row rank, which is equivalent to the matrix

$$
\begin{pmatrix}
   h_{ll} & h_{bl}^T & a_l^T \\
   h_{kl} & h_{bk} & a_k^T \\
   h_{bl}^T & H_{bb} & A_B^T \\
   a_l & A_B & -M \\
   1 & 1 & 1
\end{pmatrix}
$$

having full row rank by Proposition A.2.

Again, the search direction is unique and (A.4) implies that $(a_l \ a_k \ A_B \ -M)$ has full row rank. As $\Delta \overline{x}_k \neq 0$, Proposition A.3 implies that $(a_l \ A_B \ -M)$ must have full row rank. Consequently, Proposition A.1 implies that $K_l$ is nonsingular.

As $\Delta \overline{x}_k, \Delta \overline{x}_l, \Delta \overline{z}_k$ and $\Delta \overline{z}_l$ are all nonzero, the roles of $k$ and $l$ may be reversed to give the result that $K_k$ is nonsingular.

The next proposition concerns the situation when a constraint index $k$ is moved from $\mathcal{N}$ to $\mathcal{B}$ in a dual subiteration. In particular, it is shown that the resulting matrix $K_B$ defined after a subiteration is nonsingular.

**Proposition A.6.** Assume that there is a unique solution of the equations

$$
\begin{pmatrix}
   h_{ll} & h_{kl} & h_{bl}^T & a_l^T & 1 \\
   h_{kl} & h_{kk} & h_{bk} & a_k^T & 1 \\
   h_{bl} & h_{bk} & H_{bb} & A_B \\
   a_l & a_k & A_B & -M \\
   1 & 1 & 1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
   \Delta \overline{x}_l \\
   \Delta \overline{x}_k \\
   \Delta \overline{y} \\
   \Delta \overline{z}_l \\
   \Delta \overline{z}_k
\end{pmatrix} =
\begin{pmatrix}
   0 \\
   0 \\
   0 \\
   0 \\
   0
\end{pmatrix},
$$

(A.7)
with $\Delta z_k \neq 0$. Then, the matrices $K_l$ and $K_k$ are nonsingular, where

$$K_l = \begin{pmatrix} h_{ll} & h_{kl} & h_{kl}^T & a_l^T \\ h_{kl} & h_{kk} & h_{kl} & a_k^T \\ h_{bl} & h_{bk} & H_{bb} & A_B^T \\ a_l & a_k & A_B & -M \end{pmatrix}, \quad \text{and} \quad K_k = \begin{pmatrix} h_{kk} & h_{bk} & a_k^T \\ h_{bk} & H_{bb} & A_B^T \\ a_k & A_B & -M \end{pmatrix}.$$ 

Proof. As (A.7) has a unique solution with $\Delta z_k \neq 0$, there is no solution of

$$\begin{pmatrix} h_{ll} & h_{kl} & h_{kl}^T & a_l^T & 1 \\ h_{kl} & h_{kk} & h_{kl} & a_k^T & 1 \\ h_{bl} & h_{bk} & H_{bb} & A_B^T & 1 \\ a_l & a_k & A_B & -M & 1 \end{pmatrix} \begin{pmatrix} \Delta x_l \\ \Delta x_k \\ \Delta x_B \\ -\Delta y \\ -\Delta z_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{A.8}$$

The fundamental theorem of linear algebra applied to (A.8) implies the existence of a solution of

$$\begin{pmatrix} h_{ll} & h_{kl} & h_{kl}^T & a_l^T & 1 \\ h_{kl} & h_{kk} & h_{kl} & a_k^T & 1 \\ h_{bl} & h_{bk} & H_{bb} & A_B^T & 1 \\ a_l & a_k & A_B & -M & 1 \end{pmatrix} \begin{pmatrix} \Delta \tilde{x}_l \\ \Delta \tilde{x}_k \\ \Delta \tilde{x}_B \\ -\Delta \tilde{y} \\ -\Delta \tilde{z}_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.9}$$

with $\Delta \tilde{z}_l \neq 0$. It follows from (A.9) that $\Delta \tilde{x}_l + \Delta \tilde{z}_l = 0$. As $\Delta \tilde{z}_l \neq 0$, this implies $\Delta \tilde{x}_l \Delta \tilde{z}_l < 0$. The solution of (A.9) may be regarded as a solution of the homogeneous equations $H\Delta x - A^T\Delta y - \Delta z = 0$, $A\Delta x + M\Delta y = 0$, with $\Delta z_i = 0$, for $i \in B$, and $\Delta x_i = 0$, for $i \in \{1, \ldots, n\} \setminus \{l\}$. Hence, Proposition 2.2 gives

$$\Delta \tilde{x}_l \Delta \tilde{z}_l + \Delta \tilde{x}_k \Delta \tilde{z}_k \geq 0,$$

so that $\Delta \tilde{x}_k \Delta \tilde{z}_k > 0$. Hence, it must hold that $\Delta \tilde{x}_k \neq 0$ and $\Delta \tilde{z}_k \neq 0$.

As $\Delta \tilde{x}_k \neq 0$, $\Delta \tilde{z}_l \neq 0$, $\Delta \tilde{x}_k \neq 0$ and $\Delta \tilde{z}_l \neq 0$, the remainder of the proof is analogous to that of Proposition A.5. \[\square\]

The next result gives expressions for the primal and dual objective functions in terms of the computed search directions.

**Proposition A.7.** Assume that $(x, y, z)$ satisfies the primal and dual equality constraints

$$Hx + c - A^Ty - z = 0, \quad \text{and} \quad Ax + My - b = 0.$$ 

Consider the partition $\{1, 2, \ldots, n\} = B \cup \{l\} \cup N$ such that $x_N + q_N = 0$ and $z_B + r_B = 0$. If the components of the direction $(\Delta x, \Delta y, \Delta z)$ satisfy (2.10), then the primal and dual objective functions for $(PQP_{q,r})$ and $(DQP_{q,r})$, i.e.,

$$f_P(x, y) = \frac{1}{2} x^T H x + \frac{1}{2} y^T M y + c^T x + r^T x,$$

$$f_D(x, y, z) = -\frac{1}{2} x^T H x - \frac{1}{2} y^T M y + b^T y - q^T z,$$
satisfy the identities
\[ f_p(x + \alpha \Delta x, y + \alpha \Delta y) = f_p(x, y) + \Delta x_l(z_l + r_l)\alpha + \frac{1}{2} \Delta x_l \Delta z_l \alpha^2, \]
\[ f_D(x + \alpha \Delta x, y + \alpha \Delta y, z + \alpha \Delta z) = f_D(x, y, z) - \Delta z_l(x_l + q_l)\alpha - \frac{1}{2} \Delta x_l \Delta z_l \alpha^2. \]

**Proof.** The directional derivative of the primal objective function is given by
\[
(\Delta x^T \Delta y^T) \nabla f_p(x, y) = (\Delta x^T \Delta y^T) \begin{pmatrix} Hx + c + r \\
My \end{pmatrix} 
= (\Delta x^T \Delta y^T) \begin{pmatrix} A^T y + z + r \\
My \end{pmatrix} \tag{A.10a}
= (A\Delta x + M\Delta y)^T y + \Delta x^T(z + r) 
= \Delta x_l(z_l + r_l), \tag{A.10b}
\]
where the identity \( Hx + c = A^T y + z \) has been used in (A.10a) and the identities \( A\Delta x + M\Delta y = 0, \Delta x_N = 0 \) and \( z_b + r_B = 0 \) have been used in (A.10b).

The curvature in the direction \((\Delta x, \Delta y)\) is given by
\[
(\Delta x^T \Delta y^T) \nabla^2 f_p(x, y) (\Delta x \Delta y) = (\Delta x^T \Delta y^T) \begin{pmatrix} M \\
A^T \end{pmatrix} \begin{pmatrix} A\Delta x + M\Delta y \\
\Delta y \end{pmatrix} 
= (\Delta x^T \Delta y^T) \begin{pmatrix} A\Delta x + M\Delta y \\
\Delta y \end{pmatrix} A^T y + \Delta x^T(z + r) 
= \Delta x_l\Delta z_l, \tag{A.11a}
\]
where the identity \( H\Delta x - A^T\Delta y - \Delta z = 0 \) has been used in (A.11a) and the identities \( A\Delta x + M\Delta y = 0, \Delta x_N = 0 \) and \( \Delta z_b = 0 \) have been used in (A.11b).

The directional derivative of the dual objective function is given by
\[
(\Delta x^T \Delta y^T \Delta z^T) \nabla f_D(x, y, z) = (\Delta x^T \Delta y^T \Delta z^T) \begin{pmatrix} -Hx \\
-My + b \\
-q \end{pmatrix} \tag{A.12a}
= -\Delta x^T Hx + \Delta y^T(-My + b) - \Delta z^T q \tag{A.12b}
= -(A^T \Delta y + \Delta z)^T x + \Delta y^T(-My + b) 
- \Delta z^T q \tag{A.12c}
= -\Delta y^T(Ax + My - b) - \Delta z^T(x + q) \tag{A.12d}
= -\Delta z_l(x_l + q_l), \tag{A.12e}
\]
where the identity \( H\Delta x - A^T\Delta y - \Delta z = 0 \) has been used in (A.12c) and the identities \( Ax + My - b = 0, x_N + q_N = 0 \) and \( \Delta z_b = 0 \) have been used in (A.12e).

As \( z \) only appears linearly in the dual objective function, it follows from the structure of the Hessian matrices of \( f_p(x, y) \) and \( f_D(x, y, z) \) that
\[
(\Delta x^T \Delta y^T \Delta z^T) \nabla^2 f_D(x, y, z) \begin{pmatrix} \Delta x \\
\Delta y \\
\Delta z \end{pmatrix} = -(\Delta x^T \Delta y^T) \nabla^2 f_p(x, y) \begin{pmatrix} \Delta x \\
\Delta y \end{pmatrix}. \]
Consequently, (A.11) gives

\[
\begin{pmatrix}
\Delta x^T & \Delta y^T & \Delta z^T
\end{pmatrix} \nabla^2 f_D(x, y, z) \begin{pmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{pmatrix} = -\Delta x_1 \Delta z_1.
\]

Finally, it is shown that there is no loss of generality in assuming that \((A \quad M)\) has full row rank in \((PQP_{q,r})\).

**Proposition A.8.** There is no loss of generality in assuming that \((A \quad M)\) has full row rank in \((PQP_{q,r})\).

**Proof.** Let \((x, y, z)\) be any vector satisfying (2.1a) and (2.1b). Assume that \((A \quad M)\) has linearly dependent rows, and that \((A \quad M)\) and \(b\) may be partitioned conformally such that

\[
(A \quad M) = \begin{pmatrix}
A_1 & M_{11} & M_{12} \\
A_2 & M_{12}^T & M_{22}
\end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},
\]

with \((A_1 \quad M_{11} \quad M_{12})\) of full row rank, and

\[
\begin{pmatrix} A_2 & M_{12}^T & M_{22} \end{pmatrix} = N \begin{pmatrix} A_1 & M_{11} & M_{12} \end{pmatrix}, \quad (A.13)
\]

with \(A_1 \in \mathbb{R}^{m_1 \times n}\) and \(A_2 \in \mathbb{R}^{m_2 \times n}\) for some matrix \(N \in \mathbb{R}^{m_2 \times m_1}\). From the linear dependence of the rows of \((A \quad M)\), it follows that \(x, y\) and \(z\) satisfy (2.1a) and \((2.1b)\) if and only if

\[
A_1 x + M_{11} y_1 + M_{12} y_2 - b_1 = 0 \quad \text{and} \quad b_2 = Nb_1,
\]

\[
H x + c - A_1^T y_1 - A_2^T y_2 - z = 0.
\]

It follows from \((A.13)\) that \(M_{12} = M_{11} N^T\) and \(A_2^T = A_1^T N^T\), so that \(x, y\) and \(z\) satisfy (2.1a) and (2.1b) if and only if

\[
A_1 x + M_{11} (y_1 + N^T y_2) - b_1 = 0 \quad \text{and} \quad b_2 = Nb_1,
\]

\[
H x + c - A_1^T (y_1 + N^T y_2) - z = 0.
\]

We may now define \(\tilde{y}_1 = y_1 + N^T y_2\) and replace (2.1a) and (2.1b) by the system

\[
A_1 x + M_{11} \tilde{y}_1 - b_1 = 0,
\]

\[
H x + c - A_1^T \tilde{y}_1 - z = 0.
\]

By assumption, \((A_1 \quad M_{11} \quad M_{12})\) has full row rank. Proposition A.2 implies that \((A_1 \quad M_{11})\) has full row rank. This gives an equivalent problem for which \((A_1 \quad M_{11})\) has full row rank. \(\blacksquare\)