Existence and stability of periodic solution of impulsive neural systems with complex deviating arguments

Yong Zhao\textsuperscript{a}, Zhaosheng Feng\textsuperscript{b*} and Wei Ding\textsuperscript{c}

\textsuperscript{a}School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, People’s Republic of China; \textsuperscript{b}Department of Mathematics, University of Texas–Pan American, Edinburg, TX 78539, USA; \textsuperscript{c}Department of Mathematics, Shanghai Normal University, Shanghai 200234, People’s Republic of China

(Received 11 March 2014; accepted 5 October 2014)

This paper discusses a class of impulsive neural networks with the variable delay and complex deviating arguments. By using Mawhin’s continuation theorem of coincidence degree and the Halanay-type inequalities, several sufficient conditions for impulsive neural networks are established for the existence and globally exponential stability of periodic solutions, respectively. Furthermore, the obtained results are applied to some typical impulsive neural network systems as special cases, with a real-life example to show feasibility of our results.

Keywords: periodic solution; spectral theory; impulsive network; coincidence degree; matrix theory; global exponential stability

2000 Mathematics Subject Classification: 92B20; 93C40

1. Introduction

In the recent years, dynamics and applications of the cellular neural networks (CNNS) have been extensively studied [1–5, 8, 10, 13–16, 24]. It is shown that the CNNS have been successfully applied to signal and image processing, pattern recognition and associative memories. Considerable attention has been dedicated to the dynamics of CNNS without impulses [1–5, 7, 8, 10, 11, 13–16, 18, 24]. For example, Cao and Wang [2] considered a class of BAM networks with delays and explored exponential stability of periodic solution. Guo and Huang [7] investigated the existence, uniqueness and global exponential stability of periodic solutions by using the M-matrix and the spectrum theory.

Although the non-impulsive systems have been well studied in theory and in practice, in many cases the qualitative theory of impulsive differential equations is not only being recognized to be richer than the corresponding theory of differential equations without impulse, but also represents a more natural framework for mathematical modelling of many real-world phenomena such as...
population dynamics and neural networks. So impulsive differential equations and systems are continuously attracting attention from a diversity of groups [9, 12, 17, 19–21, 22, 23, 25–29]. In some cases the models involve deviating arguments because it reflects effects of fluctuation factors. Recently, Liu and Huang [11] discussed the existence of periodic solution of a class of CNNS with complex deviating arguments. In this paper, we are concerned with the existence and globally exponential stability of periodic solution for the following impulsive neural networks with variable delay and complex deviating arguments:

\[
\dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} [a_{ij}(t)f_j(x_j(t)) + b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + c_{ij}(t)f_j(x_j(x_j))] \\
+ I_i(t), \quad t \geq 0, \ t \neq t_k, \\
\Delta x_i(t_k) = -\gamma_{ik}x_i(t_k), \quad i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, (1)
\]

where \(I_i(t)\) denotes the external input, \(n\) corresponds to the number of neurons in \(X\)-layer, \(x_i(t)\) represents the activation of the \(i\)th neuron, the time delay \(\tau_{ij}(t)\) corresponds to the finite speed of the axonal transmission of signal satisfying \(0 \leq \tau_{ij}(t) \leq \tau_{ij}\), the function \(c_i(t) > 0\) represents the rate with which the \(i\)th neuron will rest its potential to the resting state in isolation when it is disconnected from the network and external inputs, \(f_j (j = 1, 2, \ldots, n)\) are signal transmission functions. Here \(\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)\) is the impulse at the moment \(t_k\) and \(t_1 < t_2 < \cdots\) is a strictly increasing sequence satisfying \(\lim_{k \to \infty} t_k = +\infty\).

As usual, in the theory of impulsive differential equations, at the points of discontinuity \(t_k\) of the solution \(t \mapsto x_i(t)\) we assume that \(x_i(t_k) \equiv x_i(t_k^-)\) and \(x_i'(t_k) \equiv x_i'(t_k^-)\). We also suppose that system (1) is supplemented with the initial values given by \(y_i(s) = \phi_i(s), \ s \in [-\tau_{ij}, 0]\), where \(\phi_i\) is bounded and continuous on \([-\tau_{ij}, 0]\).

Throughout this paper, we assume that:

\(H_1\) the delays \(0 \leq \tau_{ij}(t) \leq \tau_{ij}\), \(c_i(t), a_{ij}(t), b_{ij}(t)\) and \(c_{ij}(t)\) are bounded continuous \(\omega\)-periodic functions;

\(H_2\) \(I_i(t)\) are continuous \(\omega\)-periodic functions;

\(H_3\) \(f_j (j = 1, \ldots, n)\) are Lipschitzian with the positive Lipschitz constants \(L_j\) such that

\[|f_j(x_j) - f_j(\bar{x}_j)| \leq L_j|x_j - \bar{x}_j|;\]

\(H_4\) There exists a positive constant \(M > 0\) such that

\[|f_j(\cdot)| \leq M, \quad j = 1, \ldots, n;\]

\(H_5\) There exists a positive integer \(m\) such that

\[t_{k+m} = t_k + \omega, \quad \gamma_{i(k+m)} = \gamma_{ik} < 1,\]

for \(k = 1, 2, \ldots, i = 1, \ldots, n;\)

\(H_6\) \(\prod_{0 \leq \gamma_{ik} < 1}(1 - \gamma_{ik}) (i = 1, \ldots, n)\) are periodic functions with the period \(\omega\).

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and preliminary results. In Section 3, we deal with the existence of periodic solutions of system (1) by means of the continuation theorem of coincidence degree theory. Section 4 is dedicated to sufficient conditions of the global exponential stability of periodic solutions of system (1) by using the Halanay-type inequalities. Section 5 provides an illustrative example.
2. Preliminaries

In this section, we introduce some notations, definitions and some preliminaries.

Let

$$\varepsilon_j < c_i(t) < \bar{c}_i,$$

$$\|x(t)\|^2 = \sum_{i=1}^{n} (x_i(t))^2,$$

$$\bar{I}_i = \sup\{|I_i(t)|, \ t \in [0, \omega]\}, \quad N_i = \prod_{0 \leq k \leq t} (1 - \gamma_{ik})^{-1},$$

$$\bar{a}_{ij} = \sup\{|a_{ij}(t)|, \ t \in [0, \omega]\}, \quad \bar{b}_{ij} = \sup\{|b_{ij}(t)|, \ t \in [0, \omega]\},$$

$$\bar{c}_{ij} = \sup\{|c_{ij}(t)|, \ t \in [0, \omega]\}, \quad i, j = 1, 2, \ldots, n.$$

Consider the following system

$$x'(t) = x(t)f(t, x(t - \tau_1(t)), x(t - \tau_2(t))), \quad t \neq t_k, \ k = 1, 2, \ldots$$

$$\Delta x(t)|_{t=t_k} = I_k(x(t_k^-)),$$ \hspace{1cm} (2)

where \(x \in \mathbb{R}^n, f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous, and

$$f(t + \omega, x(t - \tau_1(t)), \ldots, x(t - \tau_n(t))) = f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_n(t))),$$

and

$$I_k: \mathbb{R}^n \to \mathbb{R}^n, \quad k = 1, 2, \ldots$$

are continuous. \(\tau_i(t) \in C([0, \infty))\) is a Lebesgue integrable periodic function with the period \(\omega\) and satisfies

$$t - \tau_i(t) \to \infty, \quad \text{as} \ t \to \infty, \ i = 1, \ldots, n.$$

There exists a positive integer \(q\) such that

$$t_{k+q} = t_k + \omega, \quad I_{k+q}(x) = I_k(x) \text{ with } t_k \in \mathbb{R} \text{ and } t_{k+1} > t_k,$$

and

$$\lim_{k \to +\infty} t_k = +\infty, \quad \Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_k^-).$$

For \(t_k \neq 0 (k = 1, 2, \ldots)\), it has \([0, \omega] \cap \{t_k\} = \{t_1, t_2, \ldots, t_q\}\). As we know, \(\{t_k\}\) is called the point of jump.

For any \(\delta \geq t_0\), let \(r_\delta = \min_{1 \leq i \leq n} \inf_{t \geq t_\delta} \{t - \tau_i(t)\}\) and let \(PC_\delta\) denote the set of functions \(\phi: [r_\delta, \delta] \to \mathbb{R}\) which are real-valued absolutely continuous in \([t_k, t_{k+1}] \cap (r_\delta, \delta)\) and at \(t_k\) situated in \((r_\delta, \delta)\) it may have discontinuity of the first kind.

**Definition 1** For any \(\delta \geq 0\) and \(\phi \in PC_\delta\), a function \(x \in PC_\delta([r_\delta, \infty), \mathbb{R})\) denoted by \(x(t, \delta, \phi)\) is said to be a solution of system (2) on \([\delta, \infty)\) satisfying the initial value condition

$$x(t) = \phi(t), \quad \phi(0) > 0, \ t \in [r_\delta, \delta]$$ \hspace{1cm} (3)

if the following conditions are satisfied

(i) \(x(t)\) is absolutely continuous on each interval \((t_k, t_{k+1}) \subset [r_\delta, \infty)\);

(ii) for any \(t_k \in [\delta, \infty), k = 1, 2, \ldots\), there exist \(x(t_k^+\) and \(x(t_k^-)\) such that \(x(t_k^-) = x(t_k)\).
(iii) $x(t)$ satisfies system (2) for almost everywhere in $[\delta, \infty)$ and at the impulsive point $t_k$ situated in $[\delta, \infty)$ it may have discontinuity of the first kind.

Consider the following non-impulsive delayed differential system

$$
\dot{y}_i(t) = -c_i(t)y_i(t) + \prod_{0 \leq \tau < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t)f_j \left( \prod_{0 \leq \tau < t} (1 - \gamma_{jk})y_j(t) \right) \\
+ b_{ij}(t)f_j \left( \prod_{0 \leq \tau < t} (1 - \gamma_{ik})y_j(t - \tau_j(t)) \right) \\
+c_{ij}(t)f_j \left( \prod_{0 \leq \tau < t} (1 - \gamma_{ik})y_j \left( \prod_{0 \leq \tau < t} (1 - \gamma_{jk})y_j(t) \right) \right) \\
+ \prod_{0 \leq \tau < t} (1 - \gamma_{ik})^{-1}I_i(t) = G_i(t), \quad t \geq 0, \tag{4}
$$

with the initial condition $y_i(t) = \phi_i(t), \ t \leq 0, i = 1, \ldots, n, k = 1, 2, \ldots$, where $\phi_i(t)$ is defined as the above.

Following Xia et al. [21] and Yan and Zhao [22], we obtain the following result.

**Theorem 1** Assume $(H_6)$ holds. (i) If $y = (y_1, \ldots, y_n)^T$ is a solution of system (2), then

$$
x = \left( \prod_{0 \leq \tau < t} (1 - \gamma_{1k})y_1, \ldots, \prod_{0 \leq \tau < t} (1 - \gamma_{nk})y_n \right)
$$

is a solution of system (1).

(ii) If $x = (x_1, \ldots, x_n)^T$ is a solution of system (2), then

$$
y = \left( \prod_{0 \leq \tau < t} (1 - \gamma_{1k})^{-1}x_1, \ldots, \prod_{0 \leq \tau < t} (1 - \gamma_{nk})^{-1}x_n \right)
$$

is a solution of system (4).

**Proof** For part (i), it is easy to verify that $x_i = \prod_{0 \leq \tau < t} (1 - \gamma_{ik})y_i$ (i = 1, ..., n) is absolutely continuous on the interval $(t_k, t_{k+1}]$ and for any $t \neq t_k, k = 1, 2, \ldots$,

$$
x = \left( \prod_{0 \leq \tau < t} (1 - \gamma_{1k})y_1, \ldots, \prod_{0 \leq \tau < t} (1 - \gamma_{nk})y_n \right)
$$

satisfies system (1).
On the other hand, for every $t_k \in \{t_k\}$ we have

\[
x_i(t_k^+) = \lim_{t \to t_k^-} \prod_{0 \leq j < t} (1 - \gamma_{ij})y_i(t)
\]

\[
= \prod_{0 \leq j \leq t} (1 - \gamma_{ij})y_i(t_k), \quad i = 1, \ldots, n,
\]

and

\[
x_i(t) = \prod_{0 \leq j < t} (1 - \gamma_{ij})y_i(t_k), \quad i = 1, \ldots, n.
\]

Thus, for every $k = 1, 2, \ldots$, it holds

\[
x_i(t_k^+) = (1 - \gamma_{ik})y_i(t_k), \quad i = 1, \ldots, n. \tag{5}
\]

For part (ii), since $x_i(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, in view of Equation (5) it follows that for any $k = 1, 2, \ldots$, there holds

\[
y_i(t_k^+) = \prod_{0 \leq j \leq t_k} (1 - \gamma_{ij})^{-1}x_i(t_k^+)
\]

\[
= \prod_{0 \leq j < t_k} (1 - \gamma_{ij})^{-1}x_i(t_k) = y_i(t_k), \quad i = 1, \ldots, n,
\]

and

\[
y_i(t_k^-) = \prod_{0 \leq j \leq t_{k-1}} (1 - \gamma_{ij})^{-1}x_i(t_k^-) = y_i(t_k), \quad k = 1, 2, \ldots; \quad i = 1, \ldots, n.
\]

This implies that $y_i(t)$ is continuous on $[0, \infty)$. It indicates that $y_i(t)$ is absolutely continuous on $[0, \infty)$ and a straightforward calculation shows that

\[
y = \left( \prod_{0 \leq t \leq t} (1 - \gamma_{ik})^{-1}x_1, \ldots, \prod_{0 \leq t \leq t} (1 - \gamma_{nk})^{-1}y_n \right)
\]

is the solution of system (2).

Assume that $x(t) = (x_1(t), \ldots, x_n(t))^T$ is any solution of system (1) and $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))^T$ is a periodic solution of system (1). Then we define

**Definition 2** The periodic solution $x^*(t)$ of system (1.3) is said to be globally exponential stable, if there exist some constants $\alpha > 0$ and $M \geq 1$ such that

\[
\|x(t) - x^*(t)\|_2^2 \leq M \|\phi(t) - x^*(t)\|_2^2 e^{-\alpha t}
\]

for all $t \geq 0$, where

\[
\|\phi(t) - x^*(t)\|_2^2 = \sup_{-t < s < 0} \sum_{i=1}^n (\phi_i(s) - x^*(s))^2.
\]
Lemma 1 (Halanay’s inequality). Let $a > b > 0$ and $v(t)$ be a non-negative continuous function on $[t_0 - \tau, t_0]$, and satisfy the following inequality:

$$D^+ v(t) \leq -av(t) + b \left( \sup_{s \in [t-\tau, t]} v(s) \right), \quad t \geq t_0,$$

where $\tau$ is a non-negative constant, then there exist constants $\lambda > 0$ and $k > 0$ such that

$$v(t) \leq k e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

where

$$k = \sup_{s \in [t_0 - \tau, t_0]} v(s),$$

and $\lambda$ is a unique positive solution of the equation

$$\lambda = a - b e^{\lambda \tau}.$$

3. Existence of periodic solution

Let $X, Y$ be real Banach spaces, $L: \text{Dom} L \subset X \to Y$ be a linear mapping, and $N: X \to Y$ be a continuous mapping. A mapping $L$ is called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \text{im} L < +\infty$ and $\text{im} L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P: X \to X$ and $Q: Y \to Y$ such that $\text{im} P = \ker L$ and $\ker Q = \text{im}(I - Q)$, the mapping $L|_{\text{Dom} L \cap \ker P}: (I - P)X \to \text{im} L$ is invertible. We denote the inverse of the mapping by $K_P$. If $\Omega$ is an open bounded subset of $X$, a mapping $N$ is called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N: \bar{\Omega} \to X$ is compact. Since $\text{im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J: \text{im} Q \to \ker L$.

Let us recall the Mawhin’s continuation theorem [6].

Lemma 2 Let $\Omega \subset X$ be an open bounded set and $N: X \to Y$ be a continuous operator which is $L$-compact on $\Omega$. Assume the following two statements are true:

(a) for each $\lambda \in (0, 1), x \in \partial \Omega \cap \text{Dom} L, Lx \neq \lambda Nx$;

(b) for each $x \in \partial \Omega \cap \ker L, QNx \neq 0$, and $\text{deg}(JQN, \Omega \cap \ker L, 0) \neq 0$.

Then, $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom} L$.

We are in a position to state and prove the existence of periodic solutions of system (1).

Theorem 2 In addition to conditions $(H_1), (H_2), (H_4) - (H_6)$, we further suppose that

$$c_i > \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})\bar{N}_i, \quad i = 1, \ldots, n$$

hold, then system (1) has at least one $\omega$-periodic solution.

Proof From the discussion in the preceding section, it suffices to show that the non-impulsive delay differential system (3) has an $\omega$-periodic solution.
In order to use the continuation theorem of coincidence degree to establish the existence of the solution of system (4), we take

\[ X = Z = \{ x(t) \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t), \ t \in \mathbb{R}, \ x = (x_1, \ldots, x_2)^T \} \]

with the norm

\[ \| x \| = \sum_{k=1}^{n} |x_k|_0, \ |x_k|_0 = \sup_{t \in [0, \omega]} |x_k(t)|, \ k = 1, 2, \ldots, \]

then both \( X \) and \( Z \) are Banach spaces.

Set

\[ Lx = x', \ Px = \frac{1}{\omega} \int_0^{\omega} x(t) \, dt, \ x \in X; \]

\[ Qz = \frac{1}{\omega} \int_0^{\omega} z(t) \, dt, \ z \in Z \text{ and } Ny = (G_1(t), \ldots, G_n(t))^T, \ y \in X. \]

It gives

\[ \text{Ker} \ L = \{ y \in X, \ y = h, \ h \in \mathbb{R}^n \}, \]
\[ \text{Im} \ L = \left\{ x \in X, \int_0^{\omega} x(s) \, ds = 0 \right\}, \]
\[ \dim \text{Ker} \ L = n = \text{codim} \text{Im} \ L. \]

So \( \text{Im} \ L \) is closed in \( Z \) and \( L \) is the Fredholm mapping of index zero. It is easy to see that both \( P \) and \( Q \) are continuous projectors satisfying

\[ \text{Im} \ P = \text{Ker} \ L, \ \text{Im} \ L = \text{Ker} \ Q = \text{Im}(I - M). \]

Furthermore, a straightforward calculation shows that the inverse \( K_P : \text{Im} \ L \to \text{Ker} \ P \cap \text{dom} \ L \) of \( L_P \) can be expressed as

\[ K_P(z) = \int_0^{t} z(s) \, ds - \frac{1}{\omega} \int_0^{\omega} \int_0^{t} z(s) \, d(s) \, dt. \]

Thus, when \( y \in X \), we have

\[ QNy = \left( \frac{1}{\omega} \int_0^{\omega} G_1(t) \, dt, \ldots, \frac{1}{\omega} \int_0^{\omega} G_n(t) \, dt \right)^T, \]

and

\[ K_P(I - Q)Ny = \left( \int_0^{t} G_i(s) \, ds \right)_{n \times 1} - \left( \frac{1}{\omega} \int_0^{\omega} \int_0^{t} G_i(s) \, d(s) \, dt \right)_{n \times 1} \]
\[ - \left( \frac{1}{\omega} - \frac{1}{2} \right) \int_0^{\omega} G_i(s) \, d(s) \, dt \right)_{n \times 1}. \]

Apparently, \( QN \) and \( K_P(I - Q)N \) are continuous. By virtue of the Arzela–Ascoli theorem, we know that \( QN(\bar{\Omega}) \) and \( K_P(I - Q)N(\bar{\Omega}) \) are relatively compact for any open bounded set \( \bar{\Omega} \subset X \). Thus, \( N \) is \( L \)-compact on \( \bar{\Omega} \) for any open bounded set \( \bar{\Omega} \subset X \).
Now we reach the position to search for an appropriate open, bounded subset $\Omega_1$ for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$x'_i(t) = \lambda G_i(t), \quad x \in X, \ i = 1, \ldots, n. \quad (6)$$

Suppose that $x(t) = (x_1(t), \ldots, x_i(t))^T \in X$ is a solution of system (6) for some $\lambda \in (0, 1)$. Integrating Equation (6) over the interval $[0, \omega]$ gives

$$\int_0^{\omega} c_i(t)x_i(t) \, dt = \int_0^{\omega} \prod_{0 \leq t_i < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t) \right)$$

$$+ b_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t)) \right)$$

$$+ c_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t) \right) \right) \, dt$$

$$+ \int_0^{\omega} \prod_{0 \leq t_i < t} (1 - \gamma_{ik})^{-1} I_i(t) \, dt, \quad i = 1, \ldots, n. \quad (7)$$

Denote

$$D_i = \prod_{0 \leq t_i < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t) \right)$$

$$+ b_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t)) \right)$$

$$+ c_{ij}(t)f_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j \left( \prod_{0 \leq t_i < t} (1 - \gamma_{jk})y_j(t) \right) \right) + \prod_{0 \leq t_i < t} (1 - \gamma_{ik})^{-1} I_i(t).$$

From Equation (7), we have

$$\left| \int_0^{\omega} c_i(t)x_i(t) \, dt \right| = \left| \int_0^{\omega} D_i(t) \, dt \right|. \quad (8)$$

Note that

$$\left| \int_0^{\omega} c_i(t)x_i(t) \, dt \right| \geq c_i |x_i| - \omega$$

and

$$\left| \int_0^{\omega} D_i(t) \, dt \right| \leq \int_0^{\omega} \sum_{j=1}^{n} ((\hat{a}_{ij} + \hat{b}_{ij} + \hat{c}_{ij})M + \bar{I})N_i \, dt$$

$$= \left( \sum_{j=1}^{n} (\hat{a}_{ij} + \hat{b}_{ij} + \hat{c}_{ij})M + \bar{I} \right) \omega \bar{N}_i.$$
Thus, it follows from Equation (8) that

\[ |x_i| - \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i. \]

Let \( x_i(t_0) = |x_i| - \) for some \( t_0 \in [0, \omega] \). Then, for \( t \in [t_0, t_0 + \omega] \) we have

\[ |x_i(t)| \leq |x_i(t_0)| + \int_{t_0}^{t} |\dot{x}_i(s)| \, ds \]

\[ \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i + \int_{t_0}^{t} c_i(t)|x_i(s)| \, ds + \int_{t_0}^{t} |D_i(t)| \, dt \]

\[ \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i + \int_{t_0}^{t} c_i(t)|x_i(s)| \, ds + \int_{t_0}^{t + \omega} |D_i(t)| \, dt \]

\[ \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i(1 + c_i \omega) + \int_{t_0}^{t} c_i(t)|x_i(s)| \, ds. \] (9)

By using the Gronwall’s inequality and inequality (9), we deduce that

\[ |x_i(t)| \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i(1 + c_i \omega) \times \exp \left( \int_{t_0}^{t} c_i(t) \, ds \right) \]

\[ \leq \frac{1}{c_i} \left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i(1 + c_i \omega) \times \exp(c_i \omega). \] (10)

Let

\[ \frac{\left( \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})M + \bar{I} \right) \bar{N}_i(1 + c_i \omega) \times \exp(c_i \omega)}{\min\{c_i, c_i - \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) \bar{N}_i\}} := H_i. \]

Clearly, \( H_i (i = 1, \ldots, n) \) are independent of \( \lambda \). Let

\[ H = \sum_{j=1}^{n} H_i + K, \]

where \( K \) is a sufficiently large positive constant. Let

\[ \Omega = \{ y \in Y : \|y\| < H \}. \]

Then, there are no \( \lambda \in (0, 1) \) and \( y \in \partial \Omega \) such that \( Ly = \lambda Ny \).

Note that \( QN y = JQ N y \) when \( y \in \text{Ker} \, L \). That is

\[ QN y = \left( -c_i y_i + \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) \bar{N}_j(y_j) + \bar{N}_i \bar{I}_i \right)_{n \times 1}. \]
Then for any \( u \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap \mathbb{R}^n, u = h = (h_1, \ldots, h_n) \) and \( \|h\| = H \). We have \( QNh \neq 0 \), since \( QNh = 0 \) means \( \|h_i\| \leq H_i, i = 1, \ldots, n \). When \( \|h\| \leq \sum_{j=1}^{n} H_j < H \), it is easy to see that \[(JQN)^{-1}(0) \cap (\Omega \cap \text{Ker } L) \neq \phi.\]

Consider the homotopy \( F: (\Omega \cap \text{Ker } L) \times [0, 1] \rightarrow \Omega \cap \text{Ker } L \), defined by

\[
F(y_1, \ldots, y_n, \mu) = \left( -\zeta_j y_1 + \sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) \tilde{N}_j y_i + I_i \right)_{2 \times 1} + \mu \left( \sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) \tilde{N}_j f_j(y_j) - \tilde{c}_i \tilde{A}_i g_i(x_i, x_j) - \tilde{b}_i \tilde{A}_i x_i - \tilde{c}_i \tilde{A}_i x_i \right)_{n \times 1}.
\]

Notice that \( F(x_1, x_2, 1) = JQN x \). If \( F(x_1, x_2, \mu) = 0 \), we get

\[
\|x\| \leq \frac{(\sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) M + I) \tilde{N}_j (1 + \zeta_j \omega) e^{\tilde{c}_j \omega}}{\zeta_j - (1 - \mu) \sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) \tilde{N}_j} < H.
\]

Hence, we conclude \( F(y_1, \ldots, y_n, \mu) \neq 0 \) for \( (y_1, \ldots, y_n, \mu) \in (\partial \Omega \cap \text{Ker } L) \times [0, 1] \). In view of hypotheses of Theorem 2, it follows from the property of invariance under a homotopy that

\[
\text{degree}[JQN, \Omega \cap \text{Ker } L, 0] = \text{degree}[F(y_1, \ldots, y_n, 0), \Omega \cap \text{Ker } L, 0] = \text{sign} \left| \begin{array}{cccc}
\sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) \tilde{N}_1 - \zeta_1 & 0 & \cdots & 0 \\
0 & \sum_{j=1}^{n} (\tilde{a}_{2j} + \tilde{b}_{2j} + \tilde{c}_{2j}) \tilde{N}_2 - \zeta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_{j=1}^{n} (\tilde{a}_{nj} + \tilde{b}_{nj} + \tilde{c}_{nj}) \tilde{N}_n - \zeta_n
\end{array} \right|
\geq \text{sign} \prod_{i=1}^{n} \left( \sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij}) \tilde{N}_i - \zeta_j \right) \neq 0.
\]

Consequently, all conditions of Lemma 2 are fulfilled. According to Lemma 2, \( Lx = Nx \) has at least one \( \omega \)-periodic solution on \( \text{Dom } L \cap \tilde{\Omega} \). \( \Box \)

### 4. Global exponential stability of periodic solution

Suppose that \( x^\ast(t) = (x_1^\ast(t), \ldots, x_n^\ast(t)) \) is an \( \omega \)-periodic solution of system (1). In what follows, we apply the Halanay-type inequalities to explore sufficient conditions to ensure the global exponential stability of periodic solution.
Theorem 3 Assume that conditions \((H_1)\)–\((H_7)\) are satisfied and

\[
c_j > \sum_{j=1}^{n} (\bar{a}_j + \tilde{b}_j + \bar{c}_j)N_j, \quad i = 1, 2, \ldots, n,
\]

hold. Then the \(\omega\)-periodic solution of system (1) is globally exponentially stable.

Proof According to Theorem 2, we know that system (1) has an \(\omega\)-periodic solution \(x^*(t) = (x_1^*(t), \ldots, x_n^*(t))\). For an arbitrary solution \(x(t) = (x_1(t), \ldots, x_n(t))\) of system (1), we let \(y_i(t) = x_i(t) - x_i^*(t)\), then system (1) can be rewritten as

\[
\dot{y}_i(t) = -c_i(t)y_i(t) + \sum_{j=1}^{n} [a_{ij}(t)g_j(y_j(t)) + b_{ij}(t)g_j(y_j(t - \tau_j(t))) + c_{ij}(t)g_j(y_j(t))],
\]

\(t \geq 0, \ t \neq t_k, \ \Delta y_i(t_k) = -\gamma_{ik}y_i(t_k), \quad i = 1, 2, \ldots, n, \ k = 1, 2, \ldots,\)

where \(g_j(y_j(t)) = f_j(x_j(t)) - f_j(x_j^*(t)), \ j = 1, 2, \ldots, n, \) Due to the assumption of \((H_3)\), we have

\[0 \leq \lvert g_j(y_j(t)) \rvert \leq L_j \lvert y_j(t) \rvert, \quad i = 1, 2, \ldots, n.\]

Considering the initial condition of system (1), we have

\[\varphi(s) = \phi(s) - x^*(s), \quad s \in [-\tau, 0].\]

In view of Lemma 2, we consider the non-impulsive system:

\[
\dot{z}_i(t) = -c_i(t)z_i(t) + \prod_{0 \leq \tau_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t)f_j \left( \prod_{0 \leq \tau_k < t} (1 - \gamma_{jk})z_j(t) \right)
\]

\[
+ b_{ij}(t)f_j \left( \prod_{0 \leq \tau_k < t_j} (1 - \gamma_{jk})z_j(t - \tau_j(t)) \right)
\]

\[
+ c_{ij}(t)f_j \left( \prod_{0 \leq \tau_k < t_j} (1 - \gamma_{jk}) \left( \prod_{0 \leq \tau_k < t_j} (1 - \gamma_{jk})z_j(t) \right) \right) \quad (12)
\]

with the initial condition

\[z_i(s)\varphi(s) = \phi(s) - x^*(s), s \in [-\tau, 0].\]

Choosing \(V(t) = \sum_{i=1}^{n} z_i(t)^2\), from system (12) we have

\[D^+ V(t) = \sum_{i=1}^{n} 2z_i \dot{z}_i(t) \leq \sum_{i=1}^{n} 2z_i \left[ -c_i(t)z_i(t) + \prod_{0 \leq \tau_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^{n} a_{ij}(t)f_j \left( \prod_{0 \leq \tau_k < t} (1 - \gamma_{jk})z_j(t) \right) \right] \]
\[ + b_{ij}(t)f_j \left( \prod_{0 \leq t_k < t_j(t)} (1 - \gamma_{jk})z_j(t - \tau_{ij}(t)) \right) + c_{ij}(t)f_j \left( \prod_{0 \leq t_k < t_j(t)} (1 - \gamma_{jk})z_j(t) \right) \left( \prod_{0 \leq t_k < t_j(t)} (1 - \gamma_{jk})z_j(t) \right) \] \]

\[ \leq -2 \left( \phi_i + \sum_{j=1}^{n} (\bar{a}_{ij} + \tilde{b}_{ij})N_iL_j \right) V + 2 \sum_{j=1}^{n} \bar{a}_{ij}N_iL_j \sup_{t-\tau \leq s \leq t} V(s). \]

That is

\[ D^+ V(t) \leq -2(c_i^+ + n \sum_{j=1}^{n} (\bar{a}_{ij} + \tilde{b}_{ij})N_iL_j) V + 2 \sum_{j=1}^{n} \bar{a}_{ij}N_iL_j \sup_{t-\tau \leq s \leq t} V(s). \]

By using Lemma 1, there exist constants \( \lambda > 0 \) and \( k > 0 \) such that

\[ V(t) \leq k e^{-\lambda t}, \quad t \geq 0, \]

where \( k = \sup_{s \in [-\tau, 0]} V(s) \). This gives

\[ \| x(t) - x^*(t) \|^2 \leq \| \phi_1(t) - x^*(t) \|^2 e^{-\lambda t}. \]

Consequently, from Definition 2, the \( \omega \)-periodic solution of system (1) is globally exponentially stable.

It is notable that when \( c_{ij} = 0 \), system (1) can be rewritten the following system Li and Xing [9]

\[ \dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} [a_{ij}(t)f_j(x_j(t)) + b_{ij}(t)f_j(x_j(t - \tau_{ij}(t)))] + I_i(t), \quad t \geq 0, \quad t \neq t_k, \]

\[ \Delta x_i(t_k) = -\gamma_{ik}x_i(t_k), \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots. \]

From Lemma 2 and Theorem 3, we can also obtain the related existence and globally exponential stability of periodic solution. When the impulsive jumps are absent, namely, \( N_i = 1 \), our system can be rewritten as

\[ \dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} [a_{ij}(t)f_j(x_j(t)) + b_{ij}(t)f_j(x_j(t - \tau_{ij}(t)))] + c_{ij}(t)f_j(x_j(t)) + I_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n. \]

Consequently, we can obtain the following corollaries.

**Corollary 1** In addition to conditions \((H_1)\) and \((H_2)\), we further suppose that

\[ \phi_i > \sum_{j=1}^{n} (\bar{a}_{ij} + \tilde{b}_{ij} + \bar{c}_{ij}), \quad i = 1, \ldots, n, \]

hold. Then system (14) has at least one \( \omega \)-periodic solution.
COROLLARY 2  In addition to conditions \((H_1)-(H_3)\), we further suppose that
\[
c_j > \sum_{j=1}^{n} (\tilde{a}_{ij} + \tilde{b}_{ij} + \tilde{c}_{ij})L_j, \quad i = 1, \ldots, n,
\]
hold. Then the \(\omega\)-periodic solution of system (14) is globally exponentially stable.

It is notable that when all coefficients are constants, the periodic attractor degenerates into an equilibrium point. When \(b_{ij} = 0\) and \(c_i(t) = c_i\), system (14) can reduce to the associated system described in [11]. When \(c_{ij} = 0\), system (14) can be reduced to the associated system described in [7].

5. An illustrative example

A logarithmic population network model with multi-delays is described by the following system [27]

\[
\dot{N}_i(t) = N_i(t) \left\{ r_i(t) - \sum_{j=1}^{n} a_{ij}(t) \ln[N_j(t)] - \sum_{j=1}^{n} b_{ij}(t) \ln[N_j(t - \tau_{ij}(t))] \right\}, \quad i = 1, 2, \ldots, n,
\]

where \(N_i\) represents the density of the \(i\)th species, \(r_i(t), a_{ij}(t), b_{ij}(t)\) and \(\tau_{ij}(t)\), are continuously positive periodic functions defined on \(t \in [0, +\infty)\) with a common period \(\omega > 0\). Taking account of the birth and the harvesting of some species as instantaneous impulsive processes and fluctuation factors, we need to add the variable delay, complex deviating arguments and impulsive effects in system (15). For simplicity, we only consider the delayed logarithmic population network model with impulses of two species in this case, that is,

\[
\dot{N}_i(t) = N_i(t) \left\{ r_i(t) - \sum_{j=1}^{2} a_{ij}(t) \ln[N_j(t)]
\right.
\]

\[
- \sum_{j=1}^{2} b_{ij}(t) \ln[N_j(t - \tau_{ij}(t))] - \sum_{j=1}^{2} c_{ij}(t) \ln[N_j(\ln(N_j(t)))] \right\}, \quad t \geq 0, \ t \neq t_k,
\]

\[\Delta N_i(t_k) = I_k(N_i(t_k)), \quad i = 1, 2, \quad k = 1, 2, \ldots,\]

where \(I_k(N_i(t_k))\) is \(\omega\)-periodic, \(r_i(t), a_{ij}(t), b_{ij}(t)\) and \(\tau_{ij}(t)\) are the same as those in system (15).

By making the change of variables \(N_i(t) = e^{u_i(t)}, i = 1, 2\), system (10) becomes

\[
\dot{x}_i(t) = -a_{ii}(t)x_i(t) - \sum_{j=1, j\neq i}^{2} a_{ij}(t)x_j(t) - \sum_{j=1}^{2} b_{ij}(t)x_j(t - \tau_{ij}(t)) - \sum_{j=1}^{2} c_{ij}(t)x_j(t) + r_i(t), \quad t \geq 0, \ t \neq t_k,
\]

\[\Delta x_i(t_k) = J_k(x_i(t_k)), \quad i = 1, 2, \quad k = 1, 2, \ldots,\]

where \(J_k(x_i(t_k)) = I_k(\ln(N_i(t_k)))\). As an example, we take \(a_{11}(t) = 3 + \frac{1}{4} \sin t, \ a_{22}(t) = 3 + \frac{1}{2} \sin t, \ a_{12}(t) = \frac{1}{2}(1 - \cos t), \ b_{11}(t) = \frac{1}{10}(1 - \frac{1}{2} \sin t), \ b_{12} = \frac{1}{4} + \frac{1}{4} \cos t, \ c_{11}(t) = \frac{1}{10} - \frac{1}{10} \sin t, \ c_{12} = \frac{1}{10} - \frac{1}{10} \cos t, \ a_{21}(t) = \frac{1}{3}(1 - \cos t), \ b_{21}(t) = \frac{1}{4} + \frac{1}{4} \cos t, \ b_{22}(t) = \frac{1}{4} - \frac{1}{4} \sin t,\)
\[ c_{21}(t) = \frac{1}{12} + \frac{1}{12} \cos t, \quad c_{22}(t) = \frac{1}{12} - \frac{1}{12} \sin t, \quad \tau_{ij}(t) = 1 \quad \text{and} \quad r_i(t) = 1 + \frac{1}{2} \cos t, \quad i = 1, 2. \]

Then we have

\[ J_k(x_i(t_k)) = -\gamma_{ik} x(t_k), \quad \gamma_{ik} = -2 - \frac{1}{2} \cos k, \quad i = 1, 2 \]

and

\[ a_{ii} > \sum_{j=1, j \neq i}^{2} \bar{a}_{ij} + \sum_{j=1}^{2} (\bar{b}_{ij} + \bar{c}_{ij}) N_i, \quad i = 1, 2. \]

Figure 1. Trajectory $x_1$ of the example.

Figure 2. Trajectory $x_2$ of the example.
Figure 3. Phase portrait of the example.

By virtue of Theorem 2, system (17) has at least one $\omega$-periodic solution $x^*(t)$. Furthermore, it is easy to verify $a_{ii} > \sum_{j=1,j\neq i}^{2} \tilde{a}_{ij} + \sum_{j=1}^{2} (\tilde{b}_{ij} + \tilde{c}_{ij})N_iL_j (i = 1, 2)$. Then the $\omega$-periodic solution of system (17) is globally exponentially stable. Equivalently, system (16) has a positive $\omega$-periodic solution $N^*(t)$, which is globally exponentially stable too. The simulation results show that there is a periodic solution (see Figures 1 and 2) to system (17). Figure 3 shows that all solutions converge to this periodic solution.

Funding

This work is supported by National Science Foundation of China under Grant [No. 11326125] and Scientific Research Fund of Henan Provincial Education Department [No.14A110004] and Doctoral Foundation of Henan Polytechnic University [No. B2012-107].

References

[1] S. Arik and V. Tavanoglu, *Equilibrium analysis of delayed CNNs*, IEEE Trans. Circuits Syst. I Regul. Pap. 45 (1998), pp. 168–171.
[2] J.D. Cao and L. Wang, *Exponential stability and periodic oscillatory solution in BAM networks with delays*, IEEE Trans. Neural Netw. 13 (2002), pp. 457–463.
[3] J.D. Cao and J. Wang, *Absolute exponential stability of recurrent neural networks with Lipschitz-continuous activation functions and time delays*, Neural Netw. 17 (2004), pp. 379–390.
[4] L.Q. Chua and L. Yang, *Cellular neural networks: Theory*, IEEE Trans. Circuits Syst. I Regul. Pap. 35 (1988), pp. 1257–1272.
[5] Y.G. Fang and T.G. Kincaid, *Stability analysis of dynamical neural networks*, IEEE Trans. Neural Netw. 7 (1996), pp. 996–1006.
[6] R.E. Gains and J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1987.
[7] S.J. Guo and L.H. Huang, *Periodic oscillation for a class of neural networks with variable coefficients*, Nonlinear Anal. (RWA) 6 (2005), pp. 545–561.
[8] C. Huang, L. Huang, and Z. Yuan, *Global stability analysis of a class of delayed cellular neural networks*, Math. Comput. Simulation 70 (2005), pp. 133–148.
[9] Y.K. Li and W.Y. Xing, *Existence and global exponential stability of periodic solution of a class of neural networks with impulses*, Chaos Solitons Fractals 27 (2006), pp. 437–445.
[10] X.B. Liang, Global exponential stability and application of the Hopfield neural networks, Sci. China 25 (1995), pp. 523–532.
[11] B.W. Liu and L.H. Huang, Existence of periodic solutions for cellular neural networks with complex deviating arguments, Appl. Math. Lett. 20 (2007), pp. 103–109.
[12] X. Liu, K.L. Teo, and B.J. Xu, Exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays, IEEE Trans. Neural Netw. 16 (2005), pp. 1329–1339.
[13] H.T. Lu, R.M. Shen, and F.L. Chung, Global exponential convergence of Cohen-Grossberg neural networks with time delays, IEEE Trans. Neural Netw. 16 (2005), pp. 1694–1696.
[14] S. Mohamad and K. Gopalsamy, Exponential stability of continuous-time and discrete-time cellular neural networks with delays, Appl. Math. Comput. 135 (2003), pp. 17–38.
[15] Z. Wang, Y. Liu, and X. Liu, On global asymptotic stability of neural networks with discrete and distributed delays, Phys. Lett. A 345 (2005), pp. 299–308.
[16] Z. Wang, Y. Liu, L. Yu, and X. Liu, Exponential stability of delayed recurrent neural networks with Markovian jumping parameters, Phys. Lett. A 356 (2006), pp. 346–352.
[17] G.T. Wang, L.H. Zhang, and G.X. Song, Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions, Nonlinear Anal. 74 (2011), pp. 974–982.
[18] Y.H. Xia, Periodic solutions of certain nonlinear differential equations: Via topological degree theory and matrix special theory, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22 (2012), pp. 1250196–1250217.
[19] Y.H. Xia and P.J.Y. Wong, Global exponential stability of a class of retarded impulsive differential equations with applications, Chaos Solitons Fractals 39 (2009), pp. 440–453.
[20] Y.H. Xia, J.D. Cao, and S.S. Cheng, Global exponential stability of delayed cellular neural networks with impulses, Neurocomputing 70 (2007), pp. 2495–2501.
[21] Y.H. Xia, J.D. Cao, and M. Lin, Existence and global exponential stability of periodic solutions of a class of impulsive networks with infinite delays, Int. J. Neural Syst. 17 (2007), pp. 35–42.
[22] J.R. Yan and A. Zhao, Oscillation and stability of linear impulsive delay differential equation, J. Math. Anal. Appl. 227 (1998), pp. 187–194.
[23] X.F. Yang, X.F. Liao, D.J. Evans, and Y.Y. Tang, Existence and stability of periodic solution in impulsive Hopfield neural networks with finite distributed delays, Phys. Lett. A 343 (2005), pp. 108–116.
[24] X.S. Yang, F. Li, Y. Long, and X.Z. Cui, Existence of periodic solution for discrete-time cellular neural networks with complex deviating, J. Franklin Inst. 347 (2010), pp. 559–566.
[25] Y. Zhang and J.T. Sun, Stability of impulsive neural networks with time delays, Phys. Lett. A 348 (2005), pp. 44–50.
[26] Y. Zhao, Y.H. Xia, and Q.S. Lu, Stability analysis of a class of general periodic neural networks with delays and impulses, Int. J. Neural Syst. 19 (2009), pp. 375–386.
[27] Y. Zhao, Q.S. Lu, and Z.S. Feng, Stability for the mix-delayed Cohen Grossberg neural networks with nonlinear impulse, J. Syst. Sci. and Complex. 23 (2010), pp. 665–680.
[28] J. Zhen, Z.E. Ma, and M.A. Han, The existence of periodic solutions of the n-species Lotka-Volterra competition systems with impulse, Chaos Solitons Fractals 22 (2004), pp. 181–188.
[29] J. Zhou, T.P. Chen, L. Xiang, and M.C. Liu, Global synchronization of impulsive coupled delayed neural networks, Lecture Notes in Comput. Sci. 3971 (2006), pp. 303–308.