The spin-statistics theorem — did Pauli get it right?

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Abstract

In this article, we begin with a review of Pauli’s version of the spin-statistics theorem and then show, by re-defining the parameter associated with the Lie-Algebra structure of angular momentum, that another interpretation of the theorem may be given. It will be found that the vanishing commutator and anticommutator relationships can be associated with independent and dependent probability events respectively, and not spin value. Consequently, it gives a more intuitive understanding of quantum field theory and it also suggests that the distinction between timelike and spacelike events might be better described in terms of local and non-local events. Pacs: 3.65, 5.30, 3.70.+k

1 Introduction

In Pauli’s paper of 1940 [5] the distinction between bose-einstein and fermi-dirac statistics is made according as to whether the commutator or anti-commutator relationships vanish in expression (20) of his paper. In modern terminology, we would claim that bosons cannot be second quantized as fermions and vice-versa. This can also be seen in modern formulations of the spin-statistics theorem where $[\psi_i(x), \psi_j(x')] = 0$ and $\{\psi_i(x), \psi_j(x')\} = 0$ distinguish the two types of statistics, according as to whether they are vector or spinor fields [5]. Moreover, as direct calculation shows, once $\{\psi_i(x), \psi_j(x')\} = 0$ then $[\psi_i(x), \psi_j(x')] \neq 0$ unless $\psi(x)_i\psi(x')_j = 0$.

This theorem is then applied directly to particle systems by associating particles of integral spin with a vanishing commutator relationship and particles of half-integral spin with the vanishing anticommutator relationship, from which we conclude that all particles of integral spin are bosons and all particles of half-integral spin are fermions. However on further investigation, we realize that it is the identification of half-integer spin particles with the spinor representation of the rotation group that effectively forces the distinction. In other words, if we were able to find a spinor representation for particles of integral spin then they too would have the characteristics of fermions, in accordance with the spin-statistics theorem.

I would like to suggest that by re-defining the parameter of the Lie Algebra associated with angular momentum, this in fact can be done. Moreover, I would also like to suggest that this is justified for photons, since the expression “spin
angular momentum” is a misnomer when applied to photons but serves rather to distinguish two different polarized states. Consequently, in this formalism, it is not spin-value that determines whether the particles obey fermi-dirac or bose-einstein statistics but rather the probability relationship between them.

2 Angular Momentum Theory

In the usual development of angular momentum theory, we define

$$L^\pm = L_x \pm i L_y.$$  
(1)

We then write

$$L^2 = L_x^2 + L_y^2 + L_z^2$$
(2)

$$= L^- L^+ + L_z^2 + L_z,$$
(3)

from which it follows that

$$L^2 |l\rangle = l(l + 1) |l\rangle,$$
(4)

and that $|l\rangle$ is an eigenvector of $L^2$. Similarly, the basis vectors $|l\rangle \cdots |l-n\rangle$ are eigenvectors of $L^2$ and $L_z$. Because of this, we denote the basis vectors of $L^2, L_z$ by $|l, m\rangle$ with $-l \leq m \leq l$. Now consider the operator $L = L_1 + L_2$, where $L_1$ and $L_2$ are angular momentum operators as defined above. Denote the basis vectors of $L^2, L_z$ by $|l_1 l_2 LM\rangle$, where $|l_1 - l_2| \leq L \leq |l_1 + l_2|$. In particular, when $l_1 = l_2$ denote the joint state by $|llLM\rangle$, then we can write

$$|llLM\rangle = \frac{1}{2} \sum_{mm'} (|lm\rangle_1 |lm'\rangle_2 + (-1)^{l-2l} |lm'\rangle_1 |lm\rangle_2) \langle llmm'|LM\rangle.$$  
(5)

In the case of electrons and deuterons, it is sufficient to assign respective spin values of 1/2 and 1 in formula (5) above to obtain the correct probability distribution, associated with pairs of these particles. For example, a calculation of the Clebsch-Gordan (C-G)coefficients for a pair of deuterons gives

$$|2, 0\rangle = \sqrt{2 \over 3} |0, 0\rangle + \frac{1}{\sqrt{6}} |1, -1\rangle + \frac{1}{\sqrt{6}} |1, 1\rangle.$$  
(6)

On the other hand, if we assign a value of spin 1 to a photon, the above formula breaks down, since the probability of observing the $|0, 0\rangle$ of two photons (given that $|L = 2, M = 0\rangle$ has occurred) is 0 while the C-G prediction would be 2/3. Briefly put, the spin 0 state of a photon is never observed.
How then can the above theory be extended to include spin 1 particles like the photon? A simple remedy can be found provided we agree to re-define the angular momentum operator by

\[ S_i = nL_i \]

where \( n \) is an integer, and define the subsequent Lie Algebra by

\[ [S_i, S_j] = in\epsilon_{ijk}S_k. \] (7)

Note immediately that when \( n = 1 \) we obtain the usual relationship for spin \( \frac{1}{2} \) particles, whereas for \( n = 2 \) we obtain the usual properties for a spin 1 photon, and for \( n = 4 \) we obtain the properties of the spin 2 gravitons. Using this modified algebraic structure, it is instructive to work out the theory of angular momentum in the usual way. We then quickly find that

\[ S_2 \ket{l} = n^2l(l + 1) \ket{l} = s(s + n) \ket{l}. \] (8)

Also \( S^zS^z - S^zS^z = n^2(L^zL^z - L^zL^z) = \mp n^2L^z = \mp nS^z \) and it then follows that

\[ S_zS^- \ket{l} = n^2(L_-L_z \ket{l} - L_- \ket{l}) \] (9)

\[ = n^2(l-1)L_- \ket{l} \] (10)

\[ = (s-n)S_- \ket{l} \] (11)

Also by the usual arguments, there exists an \( r \), such that \( L^- \ket{l-r} = 0 \) or equivalently \( S^- \ket{l-r} = 0 \). Hence

\[ S_2 \ket{l-r} = (S^+S^- + S^z_n - nS_z) \ket{l-r} \] (12)

\[ = (S^z_n - nS_z) \ket{l-r} \] (13)

\[ = \{ (s-nr)^2 - n(s-nr) \} \ket{l-r} \] (14)

But \( S_2 \ket{l-r} = s(s+n) \ket{l-r} \) and therefore,

\[ s(s+n) = (s-nr)^2 - n(s-nr). \] (15)

Solving for \( s \) gives \( s = \frac{nR}{r} \). In particular when \( n = 2 \), we obtain the spin structure of a photon, with only two permissible states, which are denoted by \( (1,0) \) and \( (0,1) \) respectively. Also, if we let \( n = 2 \) in the commutator relations of equation (6), then the rotational properties of a polarized photon can be modeled very effectively by the SU(2) group with parameter vector \( 2\theta \), in contrast to similar properties of an electron (or neutrino) which are associated with the SU(2) group with parameter \( \theta \).

In other words, for photons and electrons the group representations are given by

\[ U_{2\theta} = e^{i\theta,\sigma} \quad \text{and} \quad U_{\theta} = e^{(i/2)\theta,\sigma} \]

respectively, where \( \sigma \) represents the Pauli spin matrices. Moreover, an SU(2) representation for the photon predicts a full angle formula, in contrast to the half-angle formula associated with the spin of the electron (or neutrino). Furthermore, a C-G calculation based on the commutator relations (6) applied to photons (see next section), naturally gives rise to the triplet and singlet state decomposition associated with photons (without any need of a helicity argument), in contrast to the usual decomposition associated with two spin 1 particles with an observable spin-0 state.
3 Clebsch Gordan Coefficients for paired photons and deuterons

Before reinterpreting the spin-statistics from the perspective of the generalized commutator relations of angular momentum, it is useful to work out the respective joint states $|llSM\rangle$ for pairs of photons and pairs of deuterons.

Consider two photons. Let $S = S_1 + S_2$ represent their joint spins, and denote their joint state by $|llSM\rangle$. Then three possible states emerge for $S = 2$:

\[
|2, 2\rangle = |1, 1\rangle \quad (16)
\]
\[
|2, 0\rangle = \langle 1, -1|2, 0\rangle |1, -1\rangle + \langle -1, 1|2, 0\rangle |-1, 1\rangle \quad (17)
\]
\[
|2, -2\rangle = |-1, -1\rangle \quad (18)
\]

which defines the triplet state. Also for $S = 0$ we obtain

\[
|0, 0\rangle = \langle 1, -1|0, 0\rangle |1, -1\rangle + \langle -1, 1|0, 0\rangle |-1, 1\rangle \quad (19)
\]

which defines the singlet state. Note that the $|2, 0\rangle = \langle 1, -1|2, 0\rangle |1, -1\rangle$ state can be calculated directly by means of C-G coefficients, by observing that

\[
\langle L, M_l|S^\pm S^\mp L, M_L\rangle = 4 \langle L, M_l|L^\mp L^\pm L, M_L\rangle
\]

from which it follows that

\[
S^\pm |L, M_l\rangle = 2\hbar[(l \mp m)(l \pm m + 1)]^{1/2} |L, M_L\pm 2\rangle.
\]

In particular, if we set $S^- = S_1^- + S_2^-$ then

\[
2L^- |2, 2\rangle = 2\hbar[(2 + 2)(2 - 2 + 1)]^{1/2} |2, 0\rangle
\]

and

\[
|2, 0\rangle = \frac{1}{\sqrt{2}} |-1, 1\rangle + \frac{1}{\sqrt{2}} |1, -1\rangle.
\]

Also if we assume orthogonality of the different states then $\langle 2, 0|0, 0\rangle = 0$ implies

\[
|0, 0\rangle = \frac{1}{\sqrt{2}} |1, -1\rangle - \frac{1}{\sqrt{2}} |-1, 1\rangle. \quad (20)
\]

The deuteron is likewise a spin-1 particle. However in this case, the spin 0 case can be observed. Moreover, a calculation of the C-G coefficients for a pair of
deuterons says a lot about the probability weightings associated with the \( |1\rangle, |0\rangle, |−1\rangle \) states of an individual deuteron. Direct calculation gives:

\[
\begin{align*}
|2, 2\rangle &= |1, 1\rangle \\
|2, 1\rangle &= \frac{1}{\sqrt{2}} |1, 0\rangle + \frac{1}{\sqrt{2}} |0, 1\rangle \\
|2, 0\rangle &= \frac{\sqrt{2}}{3} |0, 0\rangle + \frac{1}{\sqrt{6}} |1, −1\rangle + \frac{1}{\sqrt{6}} |−1, 1\rangle \\
|2, −1\rangle &= \frac{1}{\sqrt{2}} |−1, 0\rangle + \frac{1}{\sqrt{2}} |0, −1\rangle \\
|2, −2\rangle &= |−1, −1\rangle.
\end{align*}
\]

It is worth asking, if the probabilities associated with the C-G coefficients of the states can be calculated directly from conditional probability theory? The answer is yes, provided the spectral distribution of an individual deuteron has a probability distribution of \( 1/4, 1/2, 1/4 \) and not \( 1/3, 1/3, 1/3 \) which is the current belief. In particular, let \( M_i \) where \( i = 1, 2 \) be a random variable associated with the spin of two independent deuterons such that

\[
P(M_i = 1) = P(M_i = 0) = P(M_i = −1) = \frac{1}{3} \quad (26)
\]

Let \( M = M_1 + M_2 \) be the sum of the spins. Note that \( M \) too is a random variable with values \( 2, 1, 0, −1, −2 \). Then the conditional distribution for the state \( |2, 0\rangle \) associated with the two independent deuterons gives

\[
P(M_1 = 0, M_2 = 0| M = 0) = P(M_1 = 1, M_2 = −1| M = 0) = P(M_1 = −1, M_2 = 1| M = 0) = \frac{1}{3}
\]

However, this distribution is clearly different from the C-G calculations which gives

\[
(⟨0, 0|2, 0⟩)^2 = \frac{2}{3}, (⟨1, −1|2, 0⟩)^2 = \frac{1}{6}, (⟨−1, 1|2, 0⟩)^2 = \frac{1}{6}.
\]

On the other hand, if

\[
P(M_i = 1) = P(M_i = −1) = \frac{1}{4} \quad P(M_i = 0) = \frac{1}{2} \quad (27)
\]

then direct calculation shows that

\[
P (M_1 = 0, M_2 = 0| M = 0) = \frac{2}{3} \quad (28)
\]

\[
P (M_1 = 1, M_2 = −1| M = 0) = P(M_1 = −1, M_2 = 1| M = 0) = \frac{1}{6} \quad (29)
\]

\[\text{1}^1 \text{Recall that for two events } A \text{ and } B \text{ defined on a finite sample space } S, \text{ the conditional probability of } A \text{ given } B \text{ is denoted by } P(A|B) \text{ and } P(A|B) = P(A \cap B)/P(B) \text{ provided } P(B) \neq 0.\]
which coincides with the C-G calculation. Similarly, we find the remaining paired deuteron states are given by:

$$\begin{align*}
\left| 1, 1 \right> &= \frac{1}{\sqrt{2}} \left| 1, 0 \right> - \frac{1}{\sqrt{2}} \left| 0, 1 \right> \quad (30)
\left| 1, 0 \right> &= \frac{1}{\sqrt{2}} \left| 1, -1 \right> - \frac{1}{\sqrt{2}} \left| -1, 1 \right> \quad (31)
\left| 1, -1 \right> &= \frac{1}{\sqrt{2}} \left| -1, 0 \right> - \frac{1}{\sqrt{2}} \left| 0, -1 \right> \quad (32)
\end{align*}$$

and

$$\left| 0, 0 \right> = \frac{1}{\sqrt{3}} \left| 1, -1 \right> + \frac{1}{\sqrt{3}} \left| -1, 1 \right> - \frac{1}{\sqrt{3}} \left| 0, 0 \right> . \quad (34)$$

Note that \( \left| 0, 0 \right> \) case can be explained by assuming equation (26).

### 4 A probability interpretation

Thus far, we have given an algebraic interpretation of quantum angular momentum and also pointed out that by reparametrizing the \( SU(2) \) group the spin value no longer serves to distinguish fermions from bosons. However, a simple shift of emphasis from numerical value to correlated and non-correlated states (particles), not only allow fermions and bosons to be better understood in terms of non-local and local events, or equivalently in terms of singlet and independent states but is also very consistent with the usual quantum field theory formulation of the spin-statistics theorem.

Indeed, we have already noted that the C-G calculations for two independent deuterons, suggests that in general, the triplet state associated with \( S = n \) (\( n = 1 \) for electrons and \( n = 2 \) for photons) can be understood intuitively in terms of two independent coin-flips. Moreover, it is completely symmetric under the exchange of spin values, with the states \( \left| n, n \right> \), \( \left| n, 0 \right> \) and \( \left| n, -n \right> \) having probability weightings 1/4, 1/2, 1/4 respectively. In other words, not only do the states obey bose-einstein statistics, but the above analysis suggests that such a statistics is a consequence of statistical independence. Moreover, since the observation on one spin state has no influence on the observation of the other state, by definition of independent probability, it further suggests that any spin observable can always be represented by \( \sigma_z(x) = \sigma_z(x') = \sigma_z \). It follows, trivially, that \( [\sigma_z(x), \sigma_z(x')] = 0 \) and consequently, statistical independent states need to be quantized according to the commutator rules, in keeping with a scalar quantum field.
On the other hand, in contrast to the triplet state, the singlet $|0,0\rangle$ defines a rotationally invariant state and obeys a fermi-dirac statistic. It is a pure state and cannot be decomposed, nor transformed into any other state without destroying the rotational invariance. More specifically, consider two particles in the spin-singlet state:

\[ |\psi\rangle = |+\rangle |-\rangle - |-\rangle |+\rangle. \]

and let $\vec{\sigma}(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x))$ and $\vec{\sigma}(x') = (\sigma_1(x'), \sigma_2(x'), \sigma_3(x'))$ represent spin observables at $x$ and $x'$ respectively. It then follows, because of the non-local interaction at $x$ and $x'$, that the spin-singlet correlation permits the identification of the $\sigma_i(x)$ operator with the $\sigma_i(x')$ operator. Therefore, $[\sigma_i(x), \sigma_j(x')] = i\epsilon_{ijk}\sigma_k$ but $\{\sigma_i(x), \sigma_j(x')\} = 0$ and as a consequence, non-local events in the form of spin-singlet states need to be quantized according to the anticommutator rule. Moreover, $\sigma_i \sigma_j \neq 0$ and therefore events quantized according to the anticommutator rule CANNOT be quantized with commutators (cf opening paragraph of introduction), or equivalently, particles cannot be simultaneously in non-local and local states; cannot be simultaneously subjected to the rules of conditional and independent probability; cannot be simultaneously rotationally invariant and non-invariant. However, once a measurement is performed, the fermi-dirac state can be undone and changed into a bose-einstein state; the singlet-state can become two independent states.

5 Conclusion

The paper has tried to put together the various elements composing the spin-statistics theorem and has shown that the interpretations of Pauli’s original theorem can be generalized in a coherent way if the $SU(2)$ group is parametrized in different ways, with Pauli’s own theorem serving as a special case.

In the process, we have also linked quantum statistics to classical probability theory (cf equations (23), (28) and (29)) and made the prediction that a spectral decomposition of a deuteron beam will have probability intensities $1/4$, $1/2$, $1/4$ and not $1/3$, $1/3$, $1/3$ which is the current quantum prediction. Hopefully, some type of Stern-Gerlach experiment using deuteron atoms can be done to confirm or negate this prediction. Moreover, it would seem that the prediction of $1/4$, $1/2$, $1/4$ is the only one compatible with a C-G decomposition of angular momentum.

Finally, we note that the above analysis compliments the usual commutator and anticommutator relations of quantum field theory, associated with the scalar and Dirac fields respectively and gives a more intuitive understanding of these fields in terms of independent and dependent probability. It also further clarifies the analogy between the Pauli spin matrices and the creation and annihilation operators of quantum field theory, referred to by Bjorken and Drell. Furthermore, this generalized spin-statistics theorem gives a more natural and intuitive interpretation of
microcausality. In the case of vanishing commutators, microcausality implies that local events, even at space-like differences, have simultaneous eigenstates and hence can be in the same state, in contrast to the microcausality associated with a vanishing anticommutator which implies that two operators can never commute, can never have simultaneous eigenstates and hence can never be in the same state. It then follows, from Pauli’s (reparametrized) spin-statistics theorem that the spin-singlet state obeys the fermi-dirac statistic and statistically independent spin states obey the bose-einstein statistic. Moreover, this generalized result is in full agreement with another formulation of the spin-statistics theorem to be found in Rotational invariance and the Pauli exclusion principle.\[4\]

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