Equilibrium of Surfaces in a Vertical Force Field

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Abstract

In this paper we study \( \varphi \)-minimal surfaces in \( \mathbb{R}^3 \) when the function \( \varphi \) is invariant under a two-parametric group of translations. Particularly those which are complete graphs over domains in \( \mathbb{R}^2 \). We describe a full classification of complete flat embedded \( \varphi \)-minimal surfaces if \( \varphi \) is strictly monotone and characterize \( \varphi \)-minimal bowls by its behavior at infinity when \( \varphi \) has a quadratic growth.

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1 Introduction

The equilibrium of a flexible, inextensible surface \( \Sigma \) in a force field \( \mathcal{F} \) was given by Poisson [12, pp. 173-187] and when the intrinsic forces of the surface are assumed to be equal, the external force must have a potential

\[ \mathcal{P} = e^\varphi, \]

that is, \( \mathcal{F} = \nabla \mathcal{P} \), for some smooth function \( \varphi \) on a domain of \( \mathbb{R}^3 \) which contains \( \Sigma \). In this case, the equilibrium condition is given in terms of the mean curvature vector \( \mathbf{H} \) of \( \Sigma \) as follows:

\[ \mathbf{H} = \left( \nabla \varphi \right)^\perp \]

(1.1)

where \( \nabla \) is the gradient operator in \( \mathbb{R}^3 \) and \( \perp \) denotes the projection to the normal bundle of \( \Sigma \).

A surface satisfying (1.1) is called \( \varphi \)-minimal and it can be also viewed either as a critical point of the weighted volume functional

\[ V_\varphi(\Sigma) := \int_\Sigma e^\varphi \, dA_\Sigma, \]

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where $dA_\Sigma$ is the volume element of $\Sigma$, or as a minimal surface in the conformally changed metric

\begin{equation}
G_\varphi := e^{\varphi} \langle \cdot, \cdot \rangle.
\end{equation}

From this property of minimality, a tangency principle can be applied and any two different $\varphi$-minimal surfaces cannot “touch” each other at one interior or boundary point (see [3, Theorem 1 and Theorem 1a]).

In this paper, we are interested in the case that $\varphi$ is invariant under a two-parameter group of translations in $\mathbb{R}^3$. Up to a motion in $\mathbb{R}^3$, we can assume that the external force field $\mathcal{F}$ is always a vertical field, that is,

\[ \mathcal{F} \wedge \vec{e}_3 = e^{\varphi} \nabla \varphi \wedge \vec{e}_3 \equiv 0, \]

where $\varphi$ only depends on the third coordinate in $\mathbb{R}^3$ and the mean curvature vector of $\Sigma$ satisfies

\begin{equation}
H = \dot{\varphi} \vec{e}_3^\perp,
\end{equation}

where $(\dot{\cdot})$ denotes derivative respect to the third coordinate.

A $\varphi$-minimal surface $\Sigma$ satisfying (1.4) will be called $[\varphi, \vec{e}_3]$-minimal surface and if $\Sigma$ is the graph of a function $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, we say that $\Sigma$ is a $[\varphi, \vec{e}_3]$-minimal graph, in this case, we also refer to $u$ as $[\varphi, \vec{e}_3]$-minimal. Hence, $u$ is $[\varphi, \vec{e}_3]$-minimal if and only if it solves the $[\varphi, \vec{e}_3]$-minimal equation:

\begin{equation}
(1 + u_x^2)u_{yy} + (1 + u_y^2)u_{xx} - 2u_yu_xu_{xy} = \dot{\varphi}(u) \left( 1 + u_x^2 + u_y^2 \right).
\end{equation}

Some interesting examples of $[\varphi, \vec{e}_3]$-minimal surfaces are:

- **Minimal surfaces**, if $\varphi$ is constant.
- **Translating solitons**, if $\varphi(p) = \langle p, \vec{e}_3 \rangle$.
- **Singular minimal surfaces** (also called cupolas), if $\varphi(p) = \alpha \log (\langle p, \vec{e}_3 \rangle)$, where $\alpha \in \mathbb{R}$.

Our objective in this paper is to develop a general theory of $[\varphi, \vec{e}_3]$-minimal surfaces taking as a starting point some of the recent and important progress in theory of translating solitons and singular minimal surfaces in $\mathbb{R}^3$ (see for instance [2, 6, 7, 9, 11, 13, 16]).

Nonetheless, the class of $[\varphi, \vec{e}_3]$-minimal surfaces is indeed very large and much richer in what refers to examples and geometric behaviors. Although new ideas are needed for its study, it will be necessary, in order to get classification results, to impose some additional conditions to the function $\varphi$. Here, as a general assumption we will consider $\varphi$ strictly monotone, that is,

\begin{equation}
\varphi : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R} \text{ is a strictly increasing (or decreasing) function and } \Sigma \subset \mathbb{R}^2 \times [a, b].
\end{equation}
Despite these difficulties the results we present in this paper are given for \( \varphi \) in very general classes of regular functions.

The paper is organized as follows, in Section 2 we show some fundamental equations related to our family of surfaces and as a consequence we prove the non-existence of closed examples and two results about strictly convexity and mean convexity of \([\varphi, \vec{e}_3]\)-minimal surfaces.

Section 3 is devoted to the study and classification of embedded complete flat \([\varphi, \vec{e}_3]\)-minimal surfaces. We describe the so called \([\varphi, \vec{e}_3]\)-grim reapers and tilted \([\varphi, \vec{e}_3]\)-grim reapers and characterize them as the unique examples of embedded complete flat \([\varphi, \vec{e}_3]\)-minimal surfaces.

In Section 4 we study the existence and classification of rotational examples. We construct for \( \varphi \) in a very general class of functions (strictly increasing and convex) a family of \([\varphi, \vec{e}_3]\)-minimal bowls (which are strictly convex graphs) and \([\varphi, \vec{e}_3]\)-minimal catenoids with a wing-like shape (which resemble the usual translating catenoids in \( \mathbb{R}^3 \)).

Finally, Sections 5 and 6 are devoted to study \([\varphi, \vec{e}_3]\)-minimal surfaces when \( \varphi \) has a quadratic growth. We provide the asymptotic behavior of rotationally symmetric examples and characterize \([\varphi, \vec{e}_3]\)-bowls by their behavior at infinity.

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2 Some relevant equations

Here, we will give some local fundamental equations related to \([\varphi, \vec{e}_3]\)-minimal surfaces.

Let \( \psi: M \rightarrow \mathbb{R}^3 \) be a 2-dimensional \([\varphi, \vec{e}_3]\)-minimal immersion (maybe with a non empty boundary) with Gauss map \( N \), induced metric \( g \) and second fundamental form \( A \). We shall denote by \( \nabla, \Delta \) and \( \nabla^2 \), respectively, the Gradient, Laplacian and Hessian operators of \( g \).

The mean curvature vector of \( \psi \) is defined by \( H = \text{trace}_g A \) and the symmetric bilinear form \( A \) given by \( A(X,Y) = -\langle A(X,Y), N \rangle, \ X, Y \in T\Sigma, \) is called scalar second fundamental form. The mean curvature function \( H \) will be the trace of \( A \) with respect to \( g \). With this notation, (1.4) is equivalent to

\[
H := -\varphi \langle N, \vec{e}_3 \rangle.
\]

We will assume that \( \varphi \) satisfies (1.6) and let us introduce the height and angle functions, respectively, by:

\[
\mu := \langle \psi, \vec{e}_3 \rangle, \quad \eta := \langle N, \vec{e}_3 \rangle.
\]
In the next result we show some relations involving $H$, $\mu$ and $\eta$:

**Lemma 2.1.** The following relations hold

1. $\nabla \mu = e_3^T$, $\langle \nabla \eta, \cdot \rangle = A(\nabla \mu, \cdot)$,
2. $\dot{\varphi}^2 = \varphi^2 |\nabla \mu|^2 + H^2$,
3. $\varphi \nabla^2 \mu = HA$,
4. $\nabla^2 \eta = (\nabla A)(\nabla \mu, \cdot, \cdot) + \frac{H}{\varphi^2} A^{[2]}$,
5. $\Delta \mu = \dot{\varphi}(1 - |\nabla \mu|^2)$,
6. $\Delta N + \varphi \nabla \eta + \varphi \eta \nabla \mu + |A|^2 N = 0$,
7. $\nabla^2 H = -\eta \nabla^2 \varphi - (\nabla A)(\nabla \varphi, \cdot, \cdot) - HA^{[2]} + B$
8. $\Delta A + (\nabla A)(\nabla \varphi, \cdot, \cdot) + \eta \nabla^2 \varphi + |A|^2 A - B = 0$,

where $A^{[2]}$ and $B$ are the symmetric 2-tensors given by the following expressions:

$$A^{[2]}(X,Y) = \sum_k A(X,E_k)A(E_k,Y)$$
$$B(X,Y) = (\nabla \varphi, X)A(\nabla \mu, Y) + (\nabla \varphi, Y)A(\nabla \mu, X) = 0.$$

for any vector fields $X, Y \in T\Sigma$ and any orthonormal frame $\{E_1, E_2\}$ of $T\Sigma$.

**Proof.** (1) Differentiating $\mu$ and $\eta$ respect to any $X \in T\Sigma$, we get,

$$\langle \nabla \mu, X \rangle = d\mu(X) = \langle e_3^T, X \rangle,$$
$$\langle \nabla \eta, X \rangle = d\eta(X) = \langle dN(X), e_3^T \rangle = A(X, e_3^T).$$

(2) From (2.1) and (1), it is clear that

$$1 = |\nabla \mu|^2 + \frac{H^2}{\varphi^2}.$$

(3) From definition of the Hessian operator,

$$\nabla^2 \mu(X,Y) = XY(\mu) - (\nabla_X Y)(\mu) = \langle A(X,Y), e_3 \rangle = -A(X,Y)\eta.$$

So (3) follows from (2.1).

(4) From Codazzi equation and (2.1):

$$\nabla^2 \eta(X,Y) = \sum_k (\nabla A)(E_k, X, Y)E_k(\mu) - \sum_k A(X, E_k)A(Y, E_k)\eta =$$
$$= (\nabla A)(\nabla \mu, X, Y) + \frac{H}{\varphi^2} A^{[2]}(X,Y).$$
(5) From (2) and (3),
\[ \Delta \mu = \sum_k \nabla^2 \mu(E_k, E_k) = \frac{H^2}{\dot{\varphi}} = \dot{\varphi}(1 - |\nabla \mu|^2). \]

(6) As \( H = -\dot{\varphi} \eta \), we have
\[ \nabla H = -\ddot{\varphi} \eta \nabla u - \dot{\varphi} \nabla \eta, \]
and (6) follows from the well known fact that \( \Delta N = \nabla H - |A|^2 N. \)

(7) From (2.1) and (4) we obtain
\[ \nabla^2 H(X,Y) = XY(H) - (D_X Y)H = \]
\[ = \eta \nabla^2 \dot{\varphi}(X,Y) + \dot{\varphi} \nabla^2 \eta(X,Y) + \langle \nabla \dot{\varphi}, X \rangle \langle \nabla \eta, Y \rangle = \]
\[ = \eta \nabla^2 \dot{\varphi}(X,Y) - \langle \nabla A \rangle \nabla \phi - H A^2(X,Y) + B(X,Y). \]

which give the proof of (7).

(8) Using the well known Simon's identity:
\[ \Delta A = \nabla^2 H - |A|^2 A + H A^2 \]
and (7) we obtain (8).

From this Lemma we have,

**Corollary 2.2.** If \( \varphi : [a,b] \to \mathbb{R} \), is a strictly increasing (or decreasing) function, then, the height function \( \mu \) of \( \psi \) cannot attain a local maximum (or local minimum) at any interior point.

**Corollary 2.3.** There is no any closed 2-dimensional \([\varphi, \vec{e}_3]\)-minimal immersion \( \psi : M \to \mathbb{R}^2 \times [a,b] \).

About the sign of the curvatures of \( \psi \) we have,

**Theorem 2.4.** Let \( \varphi : [a,b] \to \mathbb{R} \) be a strictly increasing function satisfying
\[ \dot{\varphi} + \lambda \dot{\varphi}^2 \geq 0, \]
for some constant \( \lambda > 0 \), and let \( \psi : \Sigma \to \mathbb{R}^2 \times [a,b] \) be a 2-dimensional \([\varphi, \vec{e}_3]\)-minimal immersion with \( H \geq 0 \). If \( H \) vanishes anywhere, then \( H \) vanishes everywhere and \( \psi(\Sigma) \) lies in a vertical plane.

**Proof.** By using (2.1) and the equations (1), (2), (5) and (6) in Lemma 2.1, we have
\[ \Delta(e^{-\lambda \ddot{\varphi}} + \lambda e^{-\lambda \ddot{\varphi}} |\nabla \mu|^2 + H^2 - \lambda \dot{\varphi}^2 |\nabla \mu|^2) = 0, \]
\[ \Delta \eta + \dot{\varphi} \eta \nabla \mu + (|A|^2 + \dot{\varphi} |\nabla \mu|^2) \eta = 0. \]
Thus, we obtain
\[
\Delta(e^{-\lambda \varphi} \eta) + (2\lambda + 1)\langle \nabla(e^{-\lambda \varphi} \eta), \nabla \varphi \rangle =
\]
\[
= -\eta e^{-\lambda \varphi}((\lambda + 1)(\ddot{\varphi} + \lambda \dot{\varphi}^2)|\nabla \mu|^2 + \lambda H^2 + |A|^2).
\]

But, by hypothesis, \(\eta\) is a nonpositive function, and so, from the strong maximum principle, if it vanishes anywhere then it vanishes everywhere, which concludes the proof. \(\square\)

**Theorem 2.5.** Let \(\varphi : [a, b] \to \mathbb{R}\) be a strictly increasing function satisfying \(\dot{\varphi} \leq 0\), and let \(\psi : M \to \mathbb{R}^2 \times [a, b]\) be a 2-dimensional locally convex \([\varphi, \vec{e}_3]\)-minimal immersion. If the Gauss curvature \(K\) vanishes anywhere, then \(K\) vanishes everywhere and \(\psi(\Sigma)\) lies in a vertical plane.

**Proof.** By hypothesis, the Gauss map \(N\) can be chosen such that \(A\) is a positive semi-definite bilinear form and from (8), we have
\[
\Delta A + (\nabla A)(\nabla \varphi, \ldots) + G(A) = 0
\]
where
\[
G(A) = \eta \nabla^2 \dot{\varphi} + |A|^2 A - B.
\]
But, from Lemma 2.1 if \(\dot{\varphi} \leq 0\) we obtain \(G(A)(v,v) = \eta \dot{\varphi}(\nabla \mu, v)^2 \geq 0\) for each null vector \(v\) of \(A\). So, we can apply the maximum principle of Hamilton (see [14, Section 2]) and if there is an interior point of \(\Sigma\) where \(A\) has a null-eigenvalue then \(A\) must have a null-eigenvalue everywhere, which concludes the proof of the theorem. \(\square\)

### 3 Complete flat \([\varphi, \vec{e}_3]\)-minimal surfaces

#### 3.1 Vertical graphs invariant by horizontal translations

Consider the \([\varphi, \vec{e}_3]\)-minimal vertical graph given by a function \(u\) which only depend on one variable, \(u = u(x)\), from (1.5) \(u\) must be a solution of the following ODE:
\[
(3.1) \quad u''(x) = \ddot{\varphi}(u)(1 + u'(x)^2)
\]
In order to look for complete examples we will consider that
\[
\varphi : [a, \infty[ \to \mathbb{R}
\]
is either a strictly increasing (or decreasing) function. Then, by taking \(z = \varphi(u)\) and \(u' = \tan(v)\), we obtain that (3.1) is equivalent to
\[
(3.2) \quad \begin{cases}
\quad v' = h(z), \\
\quad z' = h(z) \tan(v),
\end{cases}
\]
where \(h(z) = \ddot{\varphi}(\varphi^{-1}(z))\).
Figure 3.1: Phase portrait of (3.2)

It is clear that \( e^z \cos(v) \) is constant along the solutions of (3.2) and from Figure 3.1 for each solution \( u \) of (3.1) there exists a unique \( x_0 \in \mathbb{R} \) such that \( v(x_0) = 0 \) (it is not a restriction to assume that \( x_0 = 0 \)).

By taking the initial conditions

\[
(3.3) \quad u(0) = u_0, \quad u'(0) = 0,
\]

we have that for each \( x \geq 0 \), \( u(x) \) is given by

\[
(3.4) \quad u(x) := (\mathcal{X} \circ \varphi)^{-1}(x), \quad \text{with} \quad \mathcal{X}(z) = \int_{z_0}^z \frac{d\tau}{|h(\tau)|\sqrt{e^{2(\tau-z_0)}-1}},
\]

where \( z_0 = \varphi(u_0) \). Thus, from (3.1) and (3.3), we obtain,

**Proposition 3.1.** The solution \( u \) of (3.1)-(3.3) is even and it is defined in the interval \( ]-\Lambda_{u_0}, \Lambda_{u_0}[ \), where

\[
(3.5) \quad \Lambda_{u_0} = \lim_{u \to \infty} \int_{\varphi(u_0)}^{\varphi(u)} \frac{d\tau}{|h(\tau)|\sqrt{e^{2(\tau-z_0)}-1}}.
\]

**Theorem 3.2.** If \( \varphi : ]a,\infty[ \to \mathbb{R} \) is a strictly increasing function, then,

- \( \Lambda_{u_0} < \infty \) if and only if \( \int_{\lambda_0}^{\infty} e^{-\varphi(\lambda)} d\lambda < \infty \). So, if \( \Lambda_{\lambda_0} < \infty \) for some \( \lambda_0 \in ]a,\infty[ \), then \( \Lambda_\lambda < \infty \) for all \( \lambda \in ]a,\infty[ \).
• If $\Lambda_\lambda < \infty$ and $\dot{\varphi}$ is increasing (respectively, decreasing), then $\Lambda_\lambda$ is decreasing (respectively, increasing) in $\lambda$.

Proof. As

$$
\lim_{\tau \to \infty} \frac{\sqrt{e^{2(\tau - z_0)} - 1}}{e^{\tau - z_0}} = 1 \neq 0,
$$

the first item follows from (3.5).

On the other hand, by assuming that $\dot{\varphi}$ is increasing and $\Lambda_\lambda < \infty$ for all $\lambda \in [a, \infty[$, we have from (3.5), that, if $\lambda_1 \leq \lambda_2$,

$$
\Lambda_{\lambda_1} \geq \Lambda_{\lambda_2} + \lim_{z \to \infty} \int_{z - \varphi(\lambda_2)}^{z - \varphi(\lambda_1)} \frac{d\tau}{h(\tau + \varphi(\lambda_1)) \sqrt{e^{2\tau} - 1}} = \Lambda_{\lambda_2}.
$$

A similar discussion can be done when $\dot{\varphi}$ is decreasing.

From (3.1), (3.2), (3.3), (3.4), (3.5) and Theorem 3.2, we can prove the following properties of the solutions.

**Theorem 3.3.** Let $\varphi : ]a, \infty[ \to ]b, c[\, a, b \in \mathbb{R} \cup \{-\infty\}, c \in \mathbb{R} \cup \{\infty\}$ be a strictly increasing diffeomorphism, then the solution $u$ of (3.1)-(3.3) is defined in $]-\Lambda_{u_0}, \Lambda_{u_0}[$, $\Lambda_{u_0} \in \{\mathbb{R}^+, \infty\}$, it is convex, symmetric about the $y$-axis and has a minimum at $x = 0$. Moreover,

- if $c < \infty$, then $\Lambda_{u_0} = \infty$ and,
  
  $$
  \lim_{x \to \pm \infty} u(x) = \infty, \quad \lim_{x \to \pm \infty} u'(x) = \pm \sqrt{e^{2(c - z_0)} - 1}.
  $$

- if $c = \infty$,
  
  $$
  \lim_{x \to \pm \Lambda_{u_0}} u(x) = \infty, \quad \lim_{x \to \pm \Lambda_{u_0}} u'(x) = \pm \infty.
  $$

In particular, if $\Lambda_{u_0} < \infty$, the graph of $u$ is asymptotic to two vertical lines.

**Theorem 3.4.** Let $\varphi : ]a, \infty[ \to ]b, c[\, a, b \in \{\mathbb{R}, -\infty\}, c \in \{\mathbb{R}, \infty\}$ be a strictly decreasing diffeomorphism, then the solution $u$ of (3.1)-(3.3) is defined in $]-\Lambda_{u_0}, \Lambda_{u_0}[$, $\Lambda_{u_0} \in \{\mathbb{R}^+, \infty\}$, it is concave, symmetric about the $y$-axis and has a maximum at $x = 0$. Moreover,

- if $c < \infty$, then $\Lambda_{u_0} < \infty$ and,
  
  $$
  \lim_{x \to \pm \Lambda_{u_0}} u(x) = a, \quad \lim_{x \to \pm \Lambda_{u_0}} u'(x) = \pm \sqrt{e^{2(c - z_0)} - 1}.
  $$

- if $c = \infty$,
  
  $$
  \Lambda_{u_0} < \infty \iff \int_a^{u_0} e^{-\varphi(\lambda)} d\lambda < \infty,
  $$

and,

$$
\lim_{x \to \pm \Lambda_{u_0}} u(x) = a, \quad \lim_{x \to \pm \Lambda_{u_0}} u'(x) = \pm \infty.
$$
Remark 3.5. In the hypothesis of Theorem 3.4, the graph of $u$ is complete when $a = -\infty$. But in this case, by changing $\varphi$ by $-\varphi$, we can also apply Theorem 3.3.

Definition 3.6. For each solution $u$ of (3.1)-(3.3) we refer $T := \text{Graph}(u) \times \mathbb{R}$ as a $[\varphi, \vec{e}_3]$-grim reaper surface.

Figure 3.2: Grim reapers with $\dot{\varphi} = 1$ and $\dot{\varphi} = 1/u^2$, respectively.

3.2 Tilted $[\varphi, \vec{e}_3]$-grim reapers

Let $\psi := (x, y, u(x)), x \in [\Lambda_{u_0}, \Lambda_{u_0}]$ be a $[\varphi, \vec{e}_3]$-grim reaper with $u$ satisfying (3.3) and Gauss map,

$$N = \frac{1}{\sqrt{1 + u'^2}}(u', 0, -1).$$

If we rotate the surface by an angle $\theta \in [0, \pi/2]$ about the $x$-axis and dilate by $1/\cos \theta$, the resulting surface may be written as follows,

$$\tilde{\psi} = \psi + \frac{1 - \cos \theta}{\cos \theta} \langle \psi, \vec{e}_1 \rangle \vec{e}_1 + (\tan \theta) \vec{e}_1 \wedge \psi,$$

where $\vec{e}_1 = (1, 0, 0)$ and whose Gauss map is given by,

$$\tilde{N} = \cos \theta \ N + (1 - \cos \theta) \langle N, \vec{e}_1 \rangle \vec{e}_1 + \sin \theta \ \vec{e}_1 \wedge N.$$

The mean curvature $\tilde{H}$ of $\tilde{\psi}$ verifies

$$\tilde{H} = \cos \theta \ H = -\cos \theta \ \dot{\varphi} \langle \vec{e}_3, N \rangle = -\dot{\varphi} \langle \vec{e}_3, \tilde{N} \rangle.$$

Consequently, $\tilde{\psi}$ is also $[\varphi, \vec{e}_3]$-minimal and we are going to refer these examples as tilted $[\varphi, \vec{e}_3]$-Grim reapers.
Observe that,

\[
\tilde{\psi}(x, y) := \left( \frac{x}{\cos \theta}, y - u(x) \tan \theta, u(x) + y \tan \theta \right),
\]

and it is the graph of the function

\[
\mathcal{T}_\theta : \left\{ \frac{\Lambda u_0}{\cos \theta}, \frac{\Lambda u_0}{\cos \theta} \times \mathbb{R} \right\} \to \mathbb{R}
\]

\[
\mathcal{T}_\theta(x, y) = \frac{u(x \cos \theta)}{\cos^2 \theta} + y \tan \theta
\]

Figure 3.3: Titled Grim reapers with \( \dot{\varphi} = 1 \) and \( \dot{\varphi} = 1/u^3 \), respectively.

**Theorem 3.7.** Let \( \Sigma \subset \mathbb{R}^3 \) be a complete flat embedded \([\varphi, \vec{e}_3]\)-minimal surface. If \( \varphi : \mathbb{R} \to \mathbb{R} \) is a strictly increasing diffeomorphism, then \( \Sigma \) is either a vertical plane or a \([\varphi, \vec{e}_3]\)-grim reaper (maybe tilted) surface.

**Proof.** From basic differential geometry, \( \Sigma = \alpha \times \Pi^\perp \) is a ruled surface and its Gauss map is constant along the rules, where \( \alpha \) is a complete regular curve in a plane \( \Pi \subset \mathbb{R}^3 \).

CLAIM: Let \( \mathcal{L} \) be a straight line of \( \Sigma \) and \( \mathcal{V}_L \) be the unit normal vector along \( \mathcal{L} \). If \( \langle \mathcal{V}_L, \vec{e}_3 \rangle \neq 0 \), then there exists a \([\varphi, \vec{e}_3]\)-grim reaper \( \mathcal{T}_\mathcal{L} \) (tilted, if \( \mathcal{L} \) is not horizontal) containing \( \mathcal{L} \) and tangent to \( \Sigma \) along \( \mathcal{L} \).

Then, up to an appropriate rotation and dilatation, \( \Sigma \) is tangent to a \([\varphi, \vec{e}_3]\)-grim reaper along a rule. The result follows from standard theory of uniqueness of solution for the ODE (3.1).
Proof of the claim. If \( \mathcal{L} \) is horizontal then, after a rotation about the axis \( \vec{e}_3 \), we may assume that

\[
\mathcal{L} = \{(x_0, 0, u_0) + s(0, 1, 0) \mid s \in \mathbb{R}\}
\]

and there exists \( \phi \in ]-\pi/2, \pi/2[ \) such that \( \mathcal{V}_\mathcal{L} = (-\sin \phi, 0, \cos \phi) \). Then, as \( \varphi : \mathbb{R} \to \mathbb{R} \) is a strictly increasing diffeomorphism, from (3.1), there exists a solution \( u_\mathcal{L} \) of (3.1)-(3.3) and \( x_1 \in \mathbb{R} \), such that \( u_\mathcal{L}(x_1) = u_0 \) and \( u'_\mathcal{L}(x_1) = \tan \phi \).

The \([\varphi, \vec{e}_3]\)-grim reaper we are looking for is just a translation in the \( \vec{e}_1 \)-axis of the grim reaper \( T_{u_\mathcal{L}} \) associated to \( u_\mathcal{L} \).

If \( \mathcal{L} \) is not horizontal and \( p = \mathcal{L} \cap \{z = 0\} \), then by rotation of center \( p \) and axis \( \vec{e}_3 \) we may assume there exists \( \theta \in ]-\pi/2, 0[ \) and \( \alpha \in \mathbb{R} \), such that \( \mathcal{V}_\mathcal{L} = \frac{1}{\sqrt{\alpha^2 + 1}} (-\alpha, -\sin \theta, \cos \theta) \).

So, from (3.6) and (3.7), if we take the solution \( u_\mathcal{L} \) of (3.1)-(3.3) satisfying

\[
u_\mathcal{L}(x_1) = \langle p, \vec{e}_2 \rangle \cos \theta \sin \theta, \quad u'_\mathcal{L}(x_1) = \alpha,
\]

for some \( x_1 \in \mathbb{R} \), we conclude that our tilted \([\varphi, \vec{e}_3]\)-grim reaper is a translation in the \( \vec{e}_1 \)-axis of the tilted \([\varphi, \vec{e}_3]\)-grim reaper obtained after rotation of angle \( \theta \) around the \( \vec{e}_2 \)-axis and dilation of \( 1/\cos \theta \) the \([\varphi, \vec{e}_3]\)-grim reaper associated to \( u_\mathcal{L} \).

\[
\square
\]

4 \([\varphi, \vec{e}_3]\)-minimal surfaces of revolution

In this section and in a similar way to the case of translating solitons (see [8, 9, 11]), we are going to study the existence \([\varphi, \vec{e}_3]\)-minimal surfaces of bowl-type and catenoid-type.

4.1 The singular case

In the rotationally symmetric case, the equation (1.5) reduces to the following ordinary differential equation for \( u = u(r), \ r = \sqrt{x^2 + y^2} \):

\[
u'' = (1 + u^2) \left( \varphi(u) - \frac{u'}{r} \right),
\]

where (') denotes derivative respect to \( r \) and \( \varphi : ]a, b[ \subseteq \mathbb{R} \to \mathbb{R} \) is a regular \((C^\infty)\) function.

Since (4.1) is degenerated, the existence and uniqueness of solution at \( r = 0 \) is not assured by standard theory. Multiplying by \( r \) we obtain that (4.1) also writes as,

\[
\left( \frac{r \ u'}{\sqrt{1 + u^2}} \right)' = \frac{r \varphi(u)}{\sqrt{1 + u^2}}.
\]
But, from [15, Theorem 2], a solution of (1.5) cannot possess isolated non-removable singularities, hence, it is not a restriction to look for the existence of solutions of (4.2) with the following initial conditions:

\[(4.3) \quad u(0) = u_0 \in ]a, b[, \quad u'(0) = 0.\]

In this sense and by using a similar argument to [13, Proposition 2] we can assert

**Proposition 4.1.** The problem (4.1)-(4.3) has a unique solution \(u \in C^2([0, R])\) for some \(R > 0\) which depends continuously on the initial data and such that

\[u'(0) = \frac{\dot{\varphi}(u_0)}{2}.\]

The following result allows us to compare rotational symmetric \([\varphi, \vec{e}_3]\)-minimal graphs,

**Proposition 4.2.** Let \(\varphi_1, \varphi_2 : ]a, b[ \to \mathbb{R}\) be strictly increasing and convex functions satisfying that \(\dot{\varphi}_1 > \dot{\varphi}_2\) on \([0, R]\) and denote by \(u_{\varphi_1}\) and \(u_{\varphi_2}\) the \([\varphi_i, \vec{e}_3]\)-minimal graphs solutions to the corresponding problem (4.1)-(4.3). Then

\[u_{\dot{\varphi}_1} > u_{\dot{\varphi}_2}, \quad \text{on } ]0, r_0[.\]

**Proof.** If we take the function \(d := u_{\dot{\varphi}_1} - u_{\dot{\varphi}_2}\), then \(d(0) = 0\) and

\[d'(0) = u_{\varphi_1}'(0) - u_{\varphi_2}'(0) = \left(\frac{\dot{\varphi}_1(u_0)}{2} - \frac{\dot{\varphi}_2(u_0)}{2}\right) > 0.\]

Hence, there exists \(\epsilon > 0\) such that \(d = u_{\dot{\varphi}_1} - u_{\dot{\varphi}_2} > 0\) on \([0, \epsilon[\). If there exists \(r_1 > 0\) satisfying \(d(r_1) \leq 0\), we can take \(r^* := \inf\{r > 0 : d(r) < 0\}\) so that \(d(r^*) = 0\) and \(d'(r^*) \leq 0\). But, from (4.1) and having in mind that \(\int_0^{r^*} d > 0\), we get

\[0 \geq d'(r^*) = (1 + u_{\varphi_1}'(r^*)^2) [\dot{\varphi}_1(u_{\varphi_1}(r^*)) - \dot{\varphi}_2(u_{\varphi_2}(r^*))] > (1 + u_{\varphi_1}'(r^*)^2) [\dot{\varphi}_1(u_{\varphi_2}(r^*)) - \dot{\varphi}_2(u_{\varphi_2}(r^*))] > 0,
\]

which is a contradiction. \(\square\)

**Remark 4.3.** The above Proposition also holds if we assume that \(\varphi_1, \varphi_2 : ]a, b[ \to \mathbb{R}\) are regular functions so that

\[\inf \dot{\varphi}_1 > \sup \dot{\varphi}_2, \quad \text{on } ]a, b[.\]

As consequence of Proposition 4.2 and the asymptotic behavior of rotational solitons proved in [2] we have

**Corollary 4.4.** Let \(\varphi : ]a, +\infty[ \to \mathbb{R}\) be strictly increasing regular function and \(u\) be an entire solution of (4.1). If there exists \(\alpha > 0\) such that \(\dot{\varphi} > \alpha\), then

\[u'(r) \geq \alpha r - \frac{1}{\alpha r},\]

for \(r\) large enough.
4.2 Bowls-type and catenoid-type examples

Now, we want to describe $[\varphi, \vec{e}_3]$-minimal surfaces that are invariant under the one-parameter group of rotations that fix the $\vec{e}_3$ direction. A such surface with generating curve the arc-length parametrized curve

$$\gamma(s) = (x(s), 0, z(s)), \quad s \in I \subset \mathbb{R}$$

is given by,

$$\psi(s, t) = (x(s) \cos(t), x(s) \sin(t), z(s)), \quad (s, t) \in I \times \mathbb{R}. \quad (4.4)$$

The inner normal of $\psi$ writes as

$$N(s, t) = (-z'(s) \cos(t), -z'(s) \sin(t), x'(s)), \quad (4.5)$$

and the coefficients of the first and second fundamental form,

$$\langle \psi_s, \psi_s \rangle = 1, \quad \langle \psi_s, N_s \rangle = -\kappa, \quad (4.6)$$

$$\langle \psi_t, \psi_t \rangle = x'^2, \quad \langle \psi_t, N_t \rangle = -xz', \quad (4.7)$$

$$\langle \psi_s, \psi_t \rangle = 0, \quad \langle \psi_s, N_t \rangle = 0, \quad (4.8)$$

where $\kappa$ is the curvature of $\gamma$ and by $'$ we denote derivative respect to $s$.

From (4.6), the mean curvature vector of $\psi$ is given by

$$H = -\left( \kappa + \frac{z'}{x} \right) N. \quad (4.9)$$

Consequently, from (2.1), (4.4) and (4.5), the surface $\psi$ is a $[\varphi, \vec{e}_3]$-minimal surface if and only if

$$\left\{ \begin{array}{l}
x' = \cos(\theta) \\
y' = \sin(\theta), \\
\theta' = \dot{\phi}(z) \cos(\theta) - \frac{\sin(\theta)}{x},
\end{array} \right. \quad (4.10)$$

where $\theta(s) = \int_0^s \kappa(t) dt$.

Along this section we will consider that $\varphi : ]a, \infty[ \longrightarrow \mathbb{R}$ is a strictly increasing and convex function, that is

$$\dot{\varphi} > 0, \quad \ddot{\varphi} \geq 0, \quad \text{on } ]a, \infty[. \quad (4.11)$$

4.2.1 Bowl-type examples

Here, we want to study the solutions of (4.10) with the following initial conditions,

$$x(0) = 0, \quad z(0) = z_0 \in ]a, \infty[, \quad \theta(0) = 0. \quad (4.12)$$

In this case $G$ intersects orthogonally the rotation axis and we have the following result:
Theorem 4.5 (Existence of Bowl-type Examples). If \( x_0 = 0 \), then \( \gamma \) is the graph of a strictly convex symmetric function \( u(x) \) defined on a maximal interval \( ]-\omega_+ , \omega_+ [ \) which has a minimum at 0 and
\[
\lim_{x \to \pm \omega_+} u(x) = \infty.
\]

Proof. First of all, we remark that the existence of \( \gamma \) around \( s = 0 \) is guaranteed from Proposition 4.1.

Moreover, it is easy to see that \( x(s) = -x(-s) \), \( z(s) = z(-s) \), and \( \bar{\theta}(s) = -\bar{\theta}(-s) \) are also solutions of the same initial value problem (4.10)-(4.12). Hence, \( \gamma \) is symmetric respect to \( \vec{e}_3 \) direction and we may consider only the case \( s \geq 0 \).

By application of L’Hôpital’s rule, we have that \( 2\theta'(0) = \dot{\varphi}(z_0) > 0 \) and \( \gamma \) is a strictly locally convex planar curve around of \( s = 0 \). We assert that \( \theta'(s) > 0 \) for \( s \geq 0 \), otherwise from (4.12), there exists a first value \( s_0 > 0 \) such that \( \theta'(s_0) = 0 \) and \( \theta''(s_0) \leq 0 \). As \( \theta' > 0 \) on \( [0 , s_0] \), from (4.10) we have that \( 0 < 2\theta(s_0) < \pi \) and by differentiation of (4.10), we get,
\[
\theta''(s_0) = \frac{\sin(2\theta(s_0))}{2} \left( \ddot{\varphi}(z(s_0)) + \frac{1}{x(s_0)^2} \right) > 0,
\]
getting a contradiction.

In the same way, as \( \theta' > 0 \) for \( s > 0 \), we have that \( 0 < 2\theta(s) < \pi \) for \( s > 0 \) and \( \gamma \) is the graph of a strictly convex function \( u = u(x) \) which is a \( C^2 \) solution of
\[
\begin{align*}
(4.14) \\
\left\{ 
\begin{array}{ll}
  u''(x) = (1 + u^2) \left( \ddot{\varphi}(u) - \frac{u'}{x} \right) > 0, \\
  u(0) = z_0, \\
  u'(0) = 0,
\end{array}
\right.
\end{align*}
\]
on the maximal interval of existence \( ]-\omega_+ , \omega_+ [ \).

If \( \lim_{x \to \pm \omega_+} u(x) = h_0 < \infty \), then the standard theory of prolongation of solutions, gives that \( \omega_+ = +\infty \) which is also a contradiction by the convexity of \( u \).

Definition 4.6. If \( \gamma \) is a graph as in Theorem 4.5, we are going to say that the revolution surface with generating curve \( \gamma \) is a \( [\varphi , \vec{e}_3 ] \)-minimal bowl.

4.2.2 Catenoid-type examples

Now, we want to study the solutions of (4.10) with the following initial conditions,
\[
(4.15) \quad x(0) = x_0 > 0, \quad z(0) = z_0 \in ]a , \infty [ , \quad \theta(0) = 0.
\]

From standard theory, the existence and uniqueness of solution to the problem (4.10)-(4.15) is guaranteed.
Let \([-s_-, s_+]\) be the maximal interval of existence and consider \(\gamma^+ := \gamma|_{[0, s_+]}\) the right branch of \(\gamma\). Arguing as in Theorem 4.5 we can prove that \(\gamma^+\) is the graph of a convex function \(u = u(x)\) defined on a maximal interval \([x_0, \omega_]\), such that
\[
\lim_{x \to \omega_+} u(x) = \infty.
\]

For studying the left branch of \(\gamma\) we are going to consider, \(\gamma^- := \gamma|_{[0, s_-]}\) for \(s \in [0, s_-]\). Then, by taking \(\overline{\varphi}(s) = \varphi(-s), \overline{\varphi}(s) = z(-s)\) and \(\overline{\theta}(s) = \theta(-s) + \pi\) for \(s \in [0, s_-]\), we have that \(\{\overline{x}, \overline{z}, \overline{\theta}\}\) is a solution of (4.10) on \([0, s_-]\) satisfying (4.16)
\[
\overline{x}(0) = x_0 > 0, \quad \overline{z}(0) = z_0 \in ]a, \infty[, \quad \overline{\theta}(0) = \pi.
\]

Lemma 4.7. There exists \(s_0 \in [0, s_-]\) such that \(2\overline{\theta}(s_0) = \pi\).

Proof. Assume on the contrary, \(\overline{\theta}(s) \in ]0, \pi]\) for all \(s \in [0, s_-]\) and, from (4.10)-(4.16), we have that \(\overline{\varphi}' < 0, \overline{\theta}' < 0\) and \(\overline{z}' > 0\) on \([0, s_-]\). Hence, there exist
\[
\overline{x} = \lim_{s \to s_-} \overline{x}(s), \quad \overline{z} = \lim_{s \to s_-} \overline{z}(s), \quad \overline{\theta} = \lim_{s \to s_-} \overline{\theta}(s),
\]
and, as \([-s_-, s_+]\) is the maximal interval of existence of \(\gamma\), we have that either \(\overline{x} = 0\) or \(\overline{z} = \infty\). So, \(\gamma^-\) is the graph of a convex function \(\overline{\varphi} = \overline{\varphi}(\overline{x})\) on \([\overline{x}, x_0]\) such that either \(\overline{x} = 0\) or \(\lim_{\overline{x} \to \overline{x_-}} \overline{\varphi}(\overline{x}) = +\infty\).

In the first case, if \(\lim_{\overline{x} \to \overline{x_-}} \overline{\varphi}(\overline{x}) = +\infty\), from the convexity of \(\overline{\varphi}\) we get that \(\overline{\theta} = \frac{\pi}{2}\) and there exists a sequence \(\{s_n\} \to s_-\) satisfying \(\overline{\theta}'(s_n) \to 0\), but then,
from (4.10),

\[
0 = \lim_{n \to \infty} \frac{\theta'(s_n)}{\theta(s_n)} = \cos(\theta(s_n)) \frac{1}{\theta(s_n)} \phi'(z(s_n)) - \cos(\theta(s_n)) \phi'(z(s_n)) \leq 0.
\]

Thus,

\[
0 = \lim_{n \to \infty} \cos(\theta(s_n)) \phi'(z(s_n)) = \lim_{n \to \infty} \frac{\sin(\theta(s_n))}{\theta(s_n)} = \frac{1}{\theta_-} \neq 0,
\]

which is a contradiction.

If \( \theta_- = 0 \) then, from [15, Theorem 2], \( \lim_{x \to 0} \theta(x) = +\infty \) and arguing as above we also obtain a contradiction.

**Lemma 4.8.** If \( s \in ]s_0, s_-[ \), then \( 0 < 2\theta(s) < \pi \).

**Proof.** It is clear because \( \theta' < 0 \) on \( \theta \left( \frac{\pi}{2} \right) \) and \( \theta' > 0 \) on \( \theta(0) \).

**Lemma 4.9.** \( \theta \) has a minimum at a point \( s_1 \in ]s_0, s_-[ \) and \( \theta' > 0 \) on \( ]s_1, s_-[ \).

**Proof.** Assume that \( \theta' < 0 \) on \( ]s_0, s_-[ \). Then, from Lemma [4.8, \( \theta \left( \frac{\pi}{2} \right), \pi \left( \frac{\pi}{2} \right) \) and \( \theta \left( \frac{\pi}{2} \right) \) when \( s \to s_- \). In particular, there is a sequence \( \{s_n\} \to s_- \) satisfying \( \lim_{n \to \infty} \theta'(s_n) = 0 \).

Under this assumption, we assert that \( \theta_- \neq 0 \) and \( \theta_- < +\infty \), otherwise

\[
0 = \lim_{n \to \infty} \frac{\theta'(s_n)}{\theta(s_n)} = \cos(\theta(s_n)) \phi'(z(s_n)) - \cos(\theta(s_n)) \phi'(z(s_n)) \geq \cos(\theta_-) \phi'(z_0) > 0,
\]

which is a contradiction. Thus, \( \gamma_- \) is the graph of a concave function \( \theta = \theta(x) \) on a bounded interval \( ]\theta(s_0), \theta_-[ \) satisfying \( \lim_{x \to \theta_-} \theta(x) = +\infty \) but this is also a contradiction because \( \theta \) is strictly decreasing on \( ]s_0, s_-[ \).

Hence there exists \( s_1 \in ]s_0, s_-[ \) such that \( \theta'(s_1) = 0 \). Moreover, from (4.10),

\[
\theta''(s_1) = \sin(\theta(s_1)) \cos(\theta(s_1)) \phi'(z(s_1)) + \frac{1}{\theta(s_1)} > 0,
\]

and \( s_1 \) is a local minimum of \( \theta \). Now, arguing as in Theorem 4.5, we can prove that, on the interval \( ]\theta(s_1), \theta_-[ \), \( \gamma_- \) is the graph of a convex function satisfying \( \lim_{x \to \theta_-} \theta(x) = +\infty \).

\[ \square\]
Theorem 4.10 (Existence of Catenoid-type Examples). For every $x_0 > 0$, there exists a complete rotational $[\varphi, \vec{e}_3]$-minimal, see Figure 4.2 (right) with the annulus topology whose distance to axis of revolution is $x_0$ and whose generating curve $\gamma$ is of winglike type see Figure 4.2 (left).

These examples will be called $[\varphi, \vec{e}_3]$-minimal catenoids.

Proof. It follows from Lemma 4.7, Lemma 4.8 and Lemma 4.9.

Proposition 4.11. Under the above conditions, the following statements hold:

1. If $\dot{\varphi}$ has at most a linear growth, then $\omega_+ = +\infty$ and $\varpi_- = +\infty$.
2. If $\dot{\varphi}$ grows as $u^\alpha$ for some $\alpha > 1$, then $\omega_+, \varpi_- \in \mathbb{R}$.

Proof. If $\dot{\varphi}$ has at most a linear growth, then there must be a constant $c > 0$ such that $\dot{\varphi}(u)/u \leq c$ outside a compact set. Thus, from the inequality (4.14), when $x$ is large enough the following inequalities hold,

\begin{equation}
(4.17) \quad x \geq \frac{u'}{\dot{\varphi}(u)}(x) \geq \frac{1}{c} \frac{u'}{u}(x).
\end{equation}

Integrating both members of the inequality (4.17), we get that,

\begin{equation}
(4.18) \quad \frac{x_0^2}{2} - \frac{x_0^2}{2} \geq \frac{1}{c} \log \left( \frac{u(x)}{u(x_0)} \right) \text{ for some } x_0 > 0.
\end{equation}

Hence, $\omega_+ = +\infty$ and $\varpi_- = +\infty$. 

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Let’s go to consider now that
\[
\lim_{u \to +\infty} \frac{\dot{\varphi}(u)}{u^\alpha} = M \neq 0 \quad \text{for some } \alpha > 1,
\]
and suppose that \(\omega_+ = +\infty\). Then, from the Theorem 4.5 and Theorem 4.10 the real function \(f\) given by,
\[
f(r) := \frac{u'(r)}{M u^\alpha(r)}
\]
has, for \(r\) large enough, a bounded and strictly monotone primitive \(F(u)(r)\). Hence, there exists a sequence \(\{r_n\} \uparrow +\infty\) such that
\[
(4.19) \quad \lim_{n \to \infty} f(r_n) = 0.
\]

**Claim 4.12.** The function \(f\) satisfies that \(\lim_{r \to \infty} \frac{f(r)}{r} = 0\).

**Proof of Claim 4.12.** Assuming on the contrary, there exists \(\delta > 0\) and a sequence \(\{s_n\} \uparrow +\infty\) such that
\[
f(s_n) > \frac{f(s_n)}{s_n} > \delta,
\]
which together \((4.19)\) says that \(f^{-1}(\delta)\) is unbounded real subset containing a divergent sequence to \(+\infty\).

But, from the equation \((4.1)\), the function \(f\) satisfies the following differential equation
\[
(4.20) \quad f' = \left( \frac{\dot{\varphi}(u)}{M u^\alpha} - \frac{f(r)}{r} \right) + M^2 f^2 u^{2\alpha} \left( \frac{\dot{\varphi}(u)}{M u^\alpha} - \frac{f}{r} - \frac{\alpha}{M} u^{-\alpha+1} \right)
\]
and we obtain that there exists \(\hat{r} \in f^{-1}(\delta)\) such that \(f'(r) > 1\) for any \(r \in f^{-1}(\delta), r \geq \hat{r}\), which is impossible because \(f^{-1}(\delta)\) is unbounded. \(\square\)

From \((4.20)\), Claim 4.12 and using that \(u\) diverges to \(+\infty\) we get that, for \(r\) sufficiently large, the following inequality holds,
\[
(4.21) \quad \frac{2f'}{1 + f^2} > 1.
\]
By integration of this expression, we conclude that \(\omega_+ < +\infty\). \(\square\)

**Remark 4.13.** Notice that \(\omega_+ = +\infty\) does not imply that \(\dot{\varphi}\) has at most a linear growth. For example, by taking \(\dot{\varphi}(u) = u \log(u)\) with \(u \geq 1\) and by the integration of both members in \((4.14)\), we get that,
\[
\frac{x^2}{2} - \frac{x_0^2}{2} \geq \log \left( \log \left( \frac{u(x)}{u(x_0)} \right) \right) \quad \text{for some } x_0 > 0.
\]
Thus, \(\omega_+ = +\infty\) but the function \(\log(u)\) is not bounded.
5 Asymptotic behavior of rotational examples

Clutterbuck, Schn"{u}rer and Schulze studied in [2] the asymptotic behavior of solitons rotationally symmetric. They proved that the problem

\[
\begin{align*}
\begin{cases}
  u'' = (1 + u^2) \left( 1 - \frac{u'}{r} \right), & r > R, \\
  u(R) = u_0 \in \mathbb{R}, \\
  u'(R) = u_1 \in \mathbb{R}.
\end{cases}
\end{align*}
\]

has a unique $C^\infty$-solution $u$ on $[R, \infty[$. Moreover, as $r \to \infty$, $u$ has the following asymptotic expansion

\[
u(r) = \frac{v^2}{2} - \log(r) + O(r^{-2}).
\]

Due to the arbitrariness of the problem (4.1), it is impossible to find a general asymptotic behavior of their solutions because if you consider any strictly convex smooth function $u = u(r)$, $r > R$, one can find a function $\varphi$ such that $u$ is a solution of (4.1).

Proposition 4.11 motivates to consider $\varphi : [a, +\infty[ \to \mathbb{R}$ a regular function satisfying (4.11) and with a quadratic growth, that is, with the following asymptotic behavior,

\[
\begin{align*}
\begin{cases}
  u'' = (1 + u^2) \left( \varphi(u) - \frac{u'}{r} \right), & r > r_0 \geq 0, \\
  u(r_0) = u_0 > a, \\
  u'(r_0) = u_1 \geq 0,
\end{cases}
\end{align*}
\]

with $\varphi : [a, +\infty[ \to \mathbb{R}$ satisfying (4.11) and (5.2).

Remark 5.1. Observe that if $\alpha > 0$, then $u$ is solution of (4.1) if and only if $v = u + \frac{\beta - \tilde{\beta}}{\alpha}$ is solution of

\[
\begin{align*}
\begin{cases}
  v'' = (1 + v^2) \left( \tilde{\psi}(v) - \frac{v'}{r} \right), & r > r_0 \geq 0, \\
  v(r_0) = u_0 > a, \\
  v'(r_0) = u_1 \geq 0,
\end{cases}
\end{align*}
\]

where $\psi(v) = \varphi \left( v - \frac{\beta - \tilde{\beta}}{\alpha} \right)$ satisfies

\[
\begin{align*}
\lim_{v \to \infty} \tilde{\psi}(v) = \alpha \geq 0 \quad \text{and} \quad \lim_{v \to \infty} (\psi(v) - \alpha v) = \tilde{\beta}.
\end{align*}
\]

It is also clear that $\frac{\nu'}{\psi(u)} = \frac{u'}{\varphi(u)}$. 

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Theorem 5.2 (Case $\alpha > 0$). Assume that $\dot{\varphi}(u_0) r_0 \geq u_1$ and $\alpha > 0$. Then the problem \eqref{eq:5.3} has an unique strictly convex $C^\infty$-solution $u$ on $[r_0, \infty]$. Moreover, as $r \to \infty$, we have the following asymptotic expansion:

\begin{align}
\dot{\varphi}(u)(r) &= e^{\frac{1}{2} \alpha r^2 + o(r^2)} \\
u'(u)(r) &= r - \alpha r \dot{\varphi}(u)^{-2}(r) + o(r \dot{\varphi}(u)^{-2}(r)) \tag{5.4}
\end{align}

Proof. First of all, arguing as in Theorem 4.5, Theorem 4.10 and Proposition 4.11, \eqref{eq:5.3} has a unique $C^\infty$-solution $u$ on $[r_0, \infty]$ which is strictly convex function satisfying that $\lim_{r \to \infty} u(r) = \infty$. Hence, from \eqref{eq:4.1},

\begin{align}
r \dot{\varphi}(u) > u', \quad r \geq r_0. \tag{5.5}
\end{align}

From Remark 5.1 in order to study the asymptotic behavior of $\frac{u'}{\varphi(u)}$, it is not a restriction to assume that $\beta > 0$.

Take $\epsilon > 0$ such that $\beta > 2\epsilon$, from \eqref{eq:5.2} there exists $r_\epsilon$ such that if $r \geq r_\epsilon$,

\begin{align}
-\epsilon < \dot{\varphi}(u)(r) - \alpha u(r) - \beta < \epsilon, \\
-\epsilon < \ddot{\varphi}(u)(r) - \alpha < \epsilon. \tag{5.7}
\end{align}

Claim 5.3. Consider for any $R > r_0$ the function,

\begin{align}
\zeta_R(r) := g_\epsilon \left( u(R) + \int_R^r t \dot{\varphi}(u)(t) dt \right), \quad r \geq R, \quad g_\epsilon = \frac{\beta - 2\epsilon}{\beta + \epsilon}. \tag{5.8}
\end{align}

Then there exists $r_1 \in \mathbb{R}$, depending only on $\epsilon$, such that for any $R \geq r_1$, $\zeta_R$ satisfies the following inequality,

\begin{align}
\zeta_R'' < 1 + \zeta_R^2 \left( \dot{\varphi}(\zeta_R) - \frac{\zeta_R}{r} \right), \quad r \geq R. \tag{5.9}
\end{align}

Proof of Claim 5.3. From the inequality \eqref{eq:5.9}, $\zeta_R(r) > u(r) g_\epsilon$. Hence, from \eqref{eq:5.7}, when $r$ is large enough, we have

\begin{align}
\dot{\varphi}(\zeta_R)(r) > \alpha g_\epsilon u(r) + \beta - \epsilon. \tag{5.10}
\end{align}

Using \eqref{eq:5.6}, \eqref{eq:5.7} and by a straightforward computation,

\begin{align}
\zeta_R''(r) < g_\epsilon \dot{\varphi}(u)(r)(1 + (\alpha + \epsilon)r^2), \quad r \geq r_\epsilon. \tag{5.11}
\end{align}

On the other hand, from \eqref{eq:5.9} and \eqref{eq:5.7}, when $r \geq r_\epsilon$, the following inequality holds,

\begin{align}
(1 + \zeta_R^2) \left( \dot{\varphi}(\zeta_R) - \frac{\zeta_R}{r} \right) > \epsilon (1 + \dot{\varphi}(u)^2 r^2 g_\epsilon^2).
\end{align}

Thus, \eqref{eq:5.8} follows from \eqref{eq:5.9}, \eqref{eq:5.10}, \eqref{eq:5.11} having in mind that $u \to +\infty$ when $r \to +\infty$. \qed
Claim 5.4. For any $R \geq r_0$ there exists $r_R \geq R$ such that $u'(r_R) - \zeta_R(r_R) > 0$.

Proof of Claim [5.4] Assuming on the contrary, if $u'(r) - \zeta_R(r) \leq 0$ for any $r > R$, then the following inequalities hold,

$$\frac{u''(r)}{1 + u^2(r)} \geq \frac{3\varepsilon}{\beta + \varepsilon} \dot{\varphi}(u)(r) > \frac{3\varepsilon}{\beta + \varepsilon} \dot{\varphi}(u)(r_0),$$

Integrating, we can find a finite radius $\tau$ such that $u' \to +\infty$ as $r \to \tau$, getting a contradiction since the solution $u$ is defined for all $r > r_0$.

Let’s consider the function $d = u' - \zeta_R$ on $[R, \infty[$. From Claims 5.3 and 5.4 we can find $R \gg r_0$ verifying $u(R) > 0$, $d(R) > 0$ and such that the inequality (5.8) holds. Hence, if there exists a first $s \geq R$ such that $d(s) = 0$ and $d'(s) < 0$, we have

$$0 > d'(s) = (1 + u'(s)^2)(\dot{\varphi}(u(s)) - \dot{\varphi}(\zeta_R(s))).$$

On the other hand, as $d'(r) > 0$ for any $r \in]R, s[$ we have by integration of $d'$ that,

$$u(s) > \zeta_R(s) + u(R) - \zeta_R(R) = \zeta_R(s) + \frac{3\varepsilon}{\beta + \varepsilon} u(R) > \zeta_R(s),$$

and (4.11) gives that $d'(s) > \dot{\varphi}(u(s)) - \dot{\varphi}(\zeta_R(s)) > 0$ which is a contradiction.

Thus, $d(r) > 0$ for $r$ large enough and by using the inequality (5.6), we get,

$$u'(r) \varphi(u)(r) = r + \mathcal{V}_1(r), \quad \text{with} \quad \lim_{r \to +\infty} \frac{\mathcal{V}_1(r)}{r} = 0. \tag{5.12}$$

Moreover, from the previous formula (5.12) and L’Hôpital’s rule, we also get that,

$$\lim_{r \to +\infty} \frac{\log(\varphi^2(u(r)))}{\alpha r^2} = 1$$

and $\varphi(u)$ has the following asymptotic expansion,

$$\dot{\varphi}(u)(r) = e^{\frac{1}{2} \alpha r^2 + o(r^2)}. \tag{5.13}$$

Claim 5.5. $\mathcal{V}_1 \to 0$ as $r \to +\infty$.

Proof of Claim [5.5] As $\mathcal{V}_1$ is sublinear we have that for $r$ large enough, $|\mathcal{V}_1(r)| < cr$ for all $c > 0$. Moreover, from (5.3) and the inequality (5.6), $\mathcal{V}_1$ is a non-positive function and it satisfies the following differential equation,

$$\mathcal{V}_1(r) = - \frac{\mathcal{V}_1(r)}{r} (1 + \varphi^2(r)(r + \mathcal{V}_1(r))^2) - 1 - \dot{\varphi}(r)(r + \mathcal{V}_1(r))^2. \tag{5.14}$$

Take $\varepsilon > 0$ and $R \gg r_0$. If $r \geq R$ and $\mathcal{V}_1(r) \leq -\varepsilon$, from the sublinearity, we can suppose that $-r/2 < \mathcal{V}_1(r)$ and,

$$\frac{r^2}{4} < (r + \mathcal{V}_1(r))^2 < (c + 1)^2 r^2. \tag{5.15}$$
Now, choosing \( R \) large enough, the equation (5.14) and the inequalities (5.7) and (5.15) give,
\[
V_1(r) \geq -1 \frac{\varepsilon}{r} + r \left( \frac{\varepsilon}{4} \varphi'(r)^2 - (\alpha + \varepsilon)(c + 1)^2r \right).
\]
Using the conditions (4.11) and the asymptotic behavior (5.13), \( R \) may be chosen large enough so that
\[
\dot{\varphi}'(u) \geq \frac{4}{\varepsilon} \left( (\alpha + \varepsilon)(c + 1)^2r + \frac{1}{r} \left( c + 1 - \frac{\varepsilon}{r} \right) \right), \quad r \geq R.
\]
Thus, if \( R \) is large enough and \( r \geq R \) where \( V_1(r) \leq -\varepsilon \), then
\[
V_1(r) \geq -\varepsilon > 0.
\]
Hence, \( V_1(r) \geq -\varepsilon \) for \( r \) large enough and we conclude the proof.

**Claim 5.6.** \( \lim_{r \to +\infty} \frac{1}{r} \varphi'(u) V_1(r) = -\alpha \).

**Proof of Claim 5.6.** If \( \lambda(r) = \frac{1}{r} \varphi'(u) V_1(r) \), then from (5.3) and (5.12) we have,
\[
\lambda'(r) = \varphi'(u)(r) \left( 2V_1(r) \left( \frac{\varphi(u)(r)}{r} \left( 1 + \frac{V_1(r)}{r} \right) - \frac{1}{r^2} \right) - \frac{1}{r} \right) + \varphi^2(u)(r) \left( \frac{(r + V_1(r))^2}{r} (\varphi(u)(r) - \lambda(r)) \right).
\]
Fix \( \varepsilon > 0 \) and \( R \) large enough. Consider points \( r \geq R \) where \( \lambda(r) \geq -\alpha + \varepsilon \), then
\[
-\varphi'(u)(r) - \lambda(r) \leq -\varphi'(u)(r) + \alpha - \varepsilon
\]
and, if \( R \) is large enough, from (5.2) and (5.17), we also get that,
\[
-\varphi'(u)(r) - \lambda(r) \leq -\frac{\varepsilon \alpha}{2} < 0
\]
and then \( \lambda(r) < -1 \) when \( R \) is chosen sufficiently large. Hence, we obtain that \( \lambda(r) \leq -\alpha + \varepsilon \) for \( r \) large enough.

In a similar way we may prove that \( \lambda(r) \leq -\alpha - \varepsilon \) for \( r \) sufficiently large.

Now (5.5) follows from (5.12), (5.13) and Claims 5.5 and 5.6.

**Theorem 5.7 (Case \( \alpha = 0 \)).** Assume that \( \varphi(u_0) r_0 \geq u_1 \), \( \alpha = 0 \) and \( \beta > 0 \). Then the problem (5.3) has an unique strictly convex \( C^\infty \)-solution \( u \) on \([r_0, \infty] \). Moreover, if
\[
\lim_{u \to +\infty} u \varphi(u) = 0,
\]
we have the following asymptotic expansion:
\[
\frac{u'}{\varphi(u)}(r) = r - \frac{1}{\beta^2 r} + o \left( r^{-1} \right),
\]
Proof. Arguing as in Theorem 4.5, Theorem 4.10 and Proposition 4.11, (5.3) has a unique \( C^\infty \)-solution \( u \) on \([r_0, \infty)\) which is strictly convex function satisfying that \( \lim_{r \to \infty} u(r) = \infty \). Moreover, as Claims 5.3 and 5.4 also work in this case, we have the following asymptotic expansion

\[
\frac{u'}{\varphi(u)}(r) = r + \mathcal{V}_1(r),
\]

where \( \mathcal{V}_1 \) verifies the same differential equation (5.14), is also nonpositive and \( \mathcal{V}_1(r) \to 0 \). Moreover, from (5.2), \( \varphi \) writes as

\[
\varphi(u)(r) = \beta + o(1).
\]

Consider now the new function \( \mathcal{V}_2(r) = r \varphi^2(u)(r)\mathcal{V}_1(r) \). Then

\[
\mathcal{V}_2 = r \varphi^2 \left( 2\varphi\mathcal{V}_1(r + \mathcal{V}_1) - 1 + \frac{(r + \mathcal{V}_1)}{r^2}(-r^2\varphi - \mathcal{V}_2) \right).
\]

From the expressions (5.18), (5.20) and L’Hôpital’s rule, we have

\[
\lim_{r \to +\infty} \frac{\varphi(u(r))}{r} = 0 \quad \text{and} \quad \lim_{r \to +\infty} \frac{\varphi(u(r))}{r^2} = 0,
\]

and working as in Claim 5.6 we can prove that \( \mathcal{V}_2(r) \to -1 \). Finally, the theorem follows from the expansion (5.20) as \( r \to +\infty \).

\[\square\]

**Corollary 5.8.** Assume that \( \varphi \) has the following expansion

\[
\varphi(u)(r) = \alpha u + \beta + \sum_{n=1}^{\infty} \frac{a_n}{u^n}, \quad a_n \in \mathbb{R}, \tag{5.23}
\]

where either \( \alpha > 0 \) and the first non-vanishing \( a_k \) is positive or \( \alpha = 0 \), \( \beta > 0 \) and the first non-vanishing \( a_k \) is negative. Then, for any solution \( u \) of the problem (5.1) we have the following asymptotic behavior,

\[
\varphi(u)(r) = C e^{\alpha r^2} + O(r^2), \quad C > 0, \tag{5.24}
\]

if \( \alpha > 0 \) and, up to a constant

\[
\mathcal{G}(u)(r) = \frac{r^2}{2} - \frac{1}{\beta^2} \log(r) + O(r^{-2}), \tag{5.25}
\]

where \( \mathcal{G} \) is the strictly increasing function given by \( \mathcal{G}(u) = \int_{u_0}^{u} \frac{d\xi}{\varphi(\xi)} \), if \( \alpha = 0 \).

Proof. If \( \alpha > 0 \), from (5.12) and (5.13) we can write,

\[
\log(\varphi(u))(r) = \frac{\alpha r^2}{2} + \Upsilon(r), \tag{5.26}
\]
where \( \Upsilon = (\tilde{\varphi} - \alpha)r + \tilde{\varphi}V_1 \). Hence, as the first non-vanishing \( a_k \) is positive, for \( r \) large enough \( \Upsilon \) is a decreasing function in \( r \) such that \(-\infty < c = \lim_{r \to +\infty} \Upsilon(r) \) otherwise from Claim 5.6, (5.23), (5.26) and by using L'Hôpital's rule, we have that,
\[
+\infty = \lim_{r \to +\infty} \dot{\varphi}^2(u(r)) = \lim_{r \to +\infty} \frac{e^{2\Upsilon}}{e^{-\alpha r^2}} = \lim_{r \to +\infty} \frac{(e^{2\Upsilon})'}{(e^{-\alpha r^2})'},
\]
\[
= -\lim_{r \to +\infty} \frac{\dot{\varphi}(u)^2(r)((\dot{\varphi}(u)(-\alpha)r + \ddot{\varphi}(u)V_1(r)))}{\alpha r} = \alpha a_1,
\]
which is a contradiction.

Applying again L'Hôpital's rule to \( \lim_{r \to +\infty} \frac{e^{2\Upsilon} - e^{2c}}{e^{-\alpha r^2}} \), we have
\[
\dot{\varphi}^2(u(r)) = e^{\alpha r^2 + 2c} + O(1) \quad \text{and} \quad \lim_{r \to +\infty} O(1) = \alpha a_1.
\]
Thus, from Claim 5.6 and Theorem 5.2
\[
\varphi(u)^2(r) = re^{\alpha r^2 + 2c} + \alpha a_1 r + o(r),
\]
and (5.24) follows by integration of the above expression.

If \( \alpha = 0 \) then, the condition (5.18) follows from (5.20) and we have that
\[
(5.27) \quad \frac{u'}{\varphi(u)}(r) = r - \frac{1}{\beta^2 r} + o \left(r^{-1}\right).
\]
Now, by taking \( V_3(r) = (V_2(r) + 1)r^2 \) we get
\[
V_3' = \frac{2V_3}{r} + r^3 \dot{\varphi}^2 \left( 2\ddot{\varphi}V_1(r + V_1) - 1 + \frac{(r + V_1)^2}{r^2}(-r^2 \ddot{\varphi} + 1 - V_3) \right)
\]
\[
= r\dot{\varphi}^2 \left( \frac{2V_3}{\dot{\varphi}^2 r^2} + 2r^4 \ddot{\varphi} \frac{V_1(r + V_1)}{r^2} - r^2 + \frac{(r + V_1)^2}{r^2}(-r^4 \ddot{\varphi} + r^2 - V_3) \right)
\]
\[
= r\dot{\varphi}^2 \left( \frac{(r + V_1)^2}{r^2} \left( -r^4 \ddot{\varphi} + r^2 \left( 1 - \frac{r^2}{(r + V_1)^2} \right) - V_3 \right) \right)
\]
\[
+ r\dot{\varphi}^2 \left( \frac{2V_3}{\dot{\varphi}^2 r^2} + 2r^4 \ddot{\varphi} \frac{V_1(r + V_1)}{r^2} \right).
\]
But, from (5.20) and L'Hôpital's rule, we obtain
\[
\lim_{r \to +\infty} \frac{\ddot{\varphi}(u(r))}{r^4} = -\frac{4a_1}{\beta^2},
\]
\[
\lim_{r \to +\infty} r^2 \left( 1 - \frac{r^2}{(r + V_1)^2} \right) = -\frac{2}{\beta^2}
\]
thus, by working as in Claim 5.6 we prove that
\[ \lim_{r \to \infty} V_3(r) = \frac{-2 + 4a_1}{\beta^2}. \]
Hence,
\[ \frac{u'}{\varphi}(r) = r - \frac{1}{\beta^2 r} - \frac{2 - 4a_1}{\beta^4 r^5} + o(r^{-3}), \]
and [5.25] follows from integration in the above expression.

6 Uniqueness of bowl-type’s solutions

Along this section \( \varphi : [a, +\infty[ \mapsto \mathbb{R} \) will be a regular function satisfying the expansion (5.23).

For any \( \theta \in [0, 2\pi[ \) we consider \( \vec{v} = (\cos \theta, \sin \theta, 0) \) and denote by \( \Pi_{\vec{v}}(t) \) the vertical plane
\[ \Pi_{\vec{v}}(t) = \{ p \in \mathbb{R}^3 : \langle p, \vec{v} \rangle = t \}, \]

**Definition 6.1.** Let \( M_1 \) and \( M_2 \) be two arbitrary subsets of \( \mathbb{R}^3 \). We say that \( M_1 \) is on the right hand side of \( M_2 \) respect to \( \Pi_{\vec{v}}(t) \) if and only if for every point \( q \in \Pi_{\vec{v}}(t) \) such that, \( \pi^{-1}(q) \cap M_1 \neq \emptyset \) and \( \pi^{-1}(q) \cap M_2 \neq \emptyset \), we have the following inequality,
\[ \inf\{ \langle p, \vec{v} \rangle : p \in \pi^{-1}(q) \cap M_1 \} \geq \sup\{ \langle p, \vec{v} \rangle : p \in \pi^{-1}(q) \cap M_2 \}, \]
where \( \pi : \mathbb{R}^3 \to \Pi_{\vec{v}}(t) \) denotes the orthogonal projection on \( \Pi_{\vec{v}}(t) \).

For an arbitrary subset \( M \) of \( \mathbb{R}^3 \) we also consider the following subsets:
\[ M_+(t) := \{ p \in M : \langle p, \vec{v} \rangle \geq t \}, \]
\[ M_-(t) := \{ p \in M : \langle p, \vec{v} \rangle \leq t \}, \]
\[ M_+^*(t) := \{ p + 2(t - \langle p, \vec{v} \rangle)\vec{v} \in \mathbb{R}^3 : p \in M_+(t) \}, \]
\[ M_-^*(t) := \{ p + 2(t - \langle p, \vec{v} \rangle)\vec{v} \in \mathbb{R}^3 : p \in M_-(t) \}. \]

From Corollary 5.8 it is natural to study \( [\varphi, e_3^\delta] \)-minimal surfaces whose behavior at infinity is of rotational type. To be more precise,

**Definition 6.2.** We say that a \( [\varphi, e_3^\delta] \)-minimal end \( \Sigma \) is *smoothly asymptotic* to a rotational-type example if \( \Sigma \) can be expressed outside a ball as a vertical graph of a function \( u_\Sigma \) so that, according to \( \alpha \) is either positive or zero, one of the following expressions holds
\[ \varphi(u_\Sigma)(x) = Ce^{\alpha|x|^2} + O(|x|^2), \quad \text{if} \quad \alpha > 0, \]
where $C$ is a positive constant or up to a constant,

\begin{equation}
G(u_\Sigma)(x) = \frac{|x|^2}{2} - \frac{1}{\beta^2} \log(|x|) + O\left(|x|^{-2}\right),
\end{equation}

if $\alpha = 0$ and $\beta > 0$.

Let $\Sigma$ be an embedded $[\varphi, \vec{e}_3]$-minimal surface $\Sigma$ with a single end smoothly asymptotic to a bowl-type example. Then, there exists $R > 0$ large enough such that $\Sigma \cap (\mathbb{R}^3 \setminus B(0, R))$ is the vertical graph of a function $u_\Sigma$ verifying either (6.2) if $\alpha > 0$ or (6.3) if $\alpha = 0$ and $\beta > 0$.

**Lemma 6.3.** There exists $r_1 > R$ such that if $t > r_1$ then $\Sigma_+(t)$ is a graph over $\Pi_\vec{v}(t)$.

**Proof.** It is clear that when $t > R$, $\Sigma_+(t)$ has only one component which is unbounded. Moreover, if $\alpha > 0$ then from (6.2),

$$
\varphi(u_\Sigma)(x)(du_\Sigma)_x(\vec{v}) \geq 2\alpha e^{\alpha|x|^2} \langle x, \vec{v} \rangle \left(C + e^{-\alpha|x|^2}g(|x|)\right),
$$

where

$$
\lim_{|x| \to \infty} \frac{g(|x|)}{|x|^2} = 0.
$$

Hence, there exists $r_1$ large enough such that if $\langle x, \vec{v} \rangle \geq r_1$, then $(du_\Sigma)_x(\vec{v}) > 0$ and, in this case, the Lemma follows because $\Sigma$ is embedded and $\Sigma_+(r_1) \cup \pi(\Sigma_+(r_1))$ bounds a domain in $\mathbb{R}^3$.

When $\alpha = 0$ a similar argument with (6.3) also works. \qed

From Lemma 6.3, fixed $t > r_1$, $\Sigma^*_+(t) \cap \{p \in \mathbb{R}^3 : \langle p, \vec{e}_3 \rangle > R\}$ is the vertical graph of the function satisfying

\begin{equation}
u^*_t(x) = u_\Sigma(x + 2(t - \langle x, \vec{v} \rangle)\vec{v})\end{equation}

**Lemma 6.4.** Consider $a > 0$ not depending on $R$ and $\epsilon_0 > 0$. Then, for $R$ large enough and $t > a + \langle x, \vec{v} \rangle$, we have

$$
u^*_t(x) - u_\Sigma(x) > \epsilon_0 > 0.
$$

**Proof.** If $\alpha > 0$ then, from (6.2) and (6.4), we obtain

$$
\varphi(u^*_t)(x) - \varphi(u_\Sigma)(x) \geq C e^{\alpha|x|^2} \left(e^{4at\langle t - \langle x, \vec{v} \rangle \rangle} - 1\right) - M \left(2|x|^2 + 4t(t - \langle x, \vec{v} \rangle)\right),
$$

for some positive constant $M$. Hence, taking $\lambda$ such that

$$\frac{1 + \sqrt{1 + \lambda}}{\lambda} < \frac{R}{2t},$$

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and $R > \alpha^{-1}$, we have that $4t(t - \langle x, \vec{v} \rangle) \leq \lambda |x|^2$ and

$$\varphi(u_t^*) - \varphi(u_\Sigma) > C e^{\alpha R^2} \left( e^{4\alpha R^2} - 1 - Me^{-\alpha R^2} (\lambda + 2) R^2 \right) > 0$$

for $R$ large enough. The result follows because $\varphi$ is strictly increasing.

When $\alpha = 0$, we can estimate $G(u_t^*)(x) - G(u_\Sigma)(x)$ as in [9, Claim 1, Step 3] and to use that $G$ is a strictly increasing function. 

**Theorem 6.5.** Let $\Sigma$ be a complete properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$ with a single end that is smoothly asymptotic to a bowl-type example. Then the surface $\Sigma$ is a $[\varphi, \vec{e}_3]$-minimal bowl.

**Proof.** The main idea is to use the Alexandrov’s reflection principle, [4], for proving that $\Sigma$ is symmetrical with respect to $\Pi_{\vec{v}}(0)$. For proving that, it is not difficult to see that Lemma 6.3 and Lemma 6.4 are the fundamental facts we need to check that all the steps in the proof of Theorem A in [9] can be adapted to our case and for getting to prove that $0 \in \mathcal{A}$ were

$$\mathcal{A} := \{ t \geq 0 : \Sigma(t) \text{ is a graph over } \Pi_{\vec{v}}(t) \text{ and } \Sigma^*_+ (t) \geq \Sigma_-(t) \}.$$ 

A symmetrical argument gives that $\Sigma^*_+ (0) \leq \Sigma_+ (0)$. Hence, $\Sigma^*_+ (0) = \Sigma_- (0)$ and $\Sigma$ is symmetric respect to the plane $\Pi_{\vec{v}}(0)$. As $\vec{v} = (\cos \theta, \sin \theta, 0)$ represents any unit horizontal vector, $\Sigma$ would be a revolution surface touching the axis of revolution, that is, a $[\varphi, \vec{e}_3]$-minimal bowl. 

### 7 Concluding remarks

(i) It would be interesting to give a classification of $[\varphi, \vec{e}_3]$-maximal surfaces in the Lorentz-Minkowski space $L^3$ using the Calabi’s Type correspondence of [10].

(ii) From the minimality of these surfaces in the conformally changed metric $G_\varphi$ it is reasonable to think whether classical theorems on minimum surfaces are true. For example: if we consider an one-parametric family of winglikes $\{W\}_R$ and taking the size of the neck $R$ converging to zero, then they converging to a double recovering of a punctured bowl. Thus, it is possible that a result as the half-space theorem holds for $[\varphi, \vec{e}_3]$-minimal surfaces.

(iii) Other related problem with the theory of minimal surfaces, that we could be study is the Jenkins-Serrin problem for $[\varphi, \vec{e}_3]$-minimal graphs when the metric $e^{\varphi}\langle \cdot, \cdot \rangle$ is complete. For this purpose [24, 5] are interesting references.
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