MOORHOUSE’S QUESTION ON LOCALLY FINITE GENERALIZED QUADRANGLES, PART 1
—THE COUNTABLE CASE—

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ABSTRACT. We settle a question posed by G. Eric Moorhouse on the model theory and existence of locally finite generalized quadrangles. In this paper, we completely handle the case in which the generalized quadrangles have a countable number of elements.

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1. INTRODUCTION

Consider a thick generalized \(n\)-gon \(\Gamma\) [27] with \(s + 1\) points on each line and \(t + 1\) lines through each point. If \(n\) is odd, then it is easy to show that \(s = t\), see [27, section 1.5.3]. If \(n\) is even though, there are examples where \(s \neq t\), a most striking example being \(n = 8\) in which case a theorem of Feit and Higman [8] implies that if \(s t\) is finite, \(2st\) is a perfect square and so \(s\) is never equal to \(t\). If both \(s\) and \(t\) are finite, they are bounded by each other; to be more specific, \(s \leq t^2 \leq s^4\) for \(n = 4\) and \(n = 8\) by results of Higman [10] (1975), and \(s \leq t^3 \leq s^9\) for \(n = 6\) by Haemers and Roos [9] (1981). These results can also be found in [27, section 1.7.2]. For other even values of \(n\), \(\Gamma\) cannot exist by a famous result of Feit and Higman which also appeared in their 1964 paper [8]. In loc. cit., several necessary divisibility conditions involving the parameters \(s\) and \(t\) of a generalized \(n\)-gon can be found (with \(n \in \{3, 4, 6, 8\}\)), in order for a generalized \(n\)-gon with these parameters to exist.

The very first question which arises about parameters of generalized polygons — before one is ready to formulate prime power conjectures in the finite case — obviously is: “Do thick generalized polygons exist with one finite parameter and one non-finite parameter?” First posed by Tits in the 1960s, this question remains a mystery in our knowledge about the fundamentals of generalized polygons.

By definition, we call such polygons locally finite. Note that in Van Maldeghem’s book [27], such generalized polygons are called semi-finite.

Not much is known on Tits’s question. All the known results comprise the case \(n = 4\). Here they are.

- P. J. Cameron [4] showed in 1981 that if \(n = 4\) and \(s = 2\), then \(t\) is finite.
• In [1] A. E. Brouwer shows the same thing for \( n = 4 \) and \( s = 3 \), and the proof is purely combinatorial (unlike a nonpublished but earlier proof of Kantor [12]).
• More recently, G. Cherlin used Model Theory (in [5]) to handle the generalized 4-gons with five points on a line.
• For other values of \( n \) and \( s \) (where \( n \) of course is even), nothing is known without any extra assumptions.

Apart from the aforementioned results, there is only one other “general result” on parameters of generalized polygons (without invoking additional structure through, e.g., the existence of certain substructures or the occurrence of certain group actions):

**Theorem 1.1** (Bruck and Ryser [2], 1949). If \( \Gamma \) is a finite projective plane of order \( m \), \( m \in \mathbb{N}^\times \), and \( m \equiv 1, 2 \mod 4 \), then \( m \) is the sum of two perfect squares.

Note that it is not hard to construct generalized quadrangles (= generalized 4-gons) with parameters \((\alpha, \beta)\), where \( \alpha \) and \( \beta \) are different cardinal numbers both not a finite positive integer. Let \( I \) be an infinite uncountable set, and consider the vector space \( \mathbb{Q}[I] \) consisting of all \(|I|\)-tuples \((q_i)_{i \in I}, q_i \in \mathbb{Q}\), with only a finite number of nonzero entries. We assume without loss of generality that \( I \) contains the symbols 0 and 1. Define the following quadratic form:

\[
\ell : X_0^2 + X_1^2 - \sum_{i \in I \setminus \{0, 1\}} X_i^2.
\]

Then \( \ell \) has Witt index 2, and the corresponding classical orthogonal quadrangle \( \mathbb{Q}(|I|, \mathbb{Q}, \ell) \) (cf. [27, chapter 2]) is a Moufang generalized quadrangle with \(|Q|\) points per line and \(|I|\) lines on a point. It is fully embedded in the projective space \( \mathbb{P}(\mathbb{Q}[I]) \) (cf. [27, chapter 2]). Other examples of buildings with “mixed parameters” can be found in [20].

**This paper.** This note is inspired by G. Eric Moorhouse’s notes [13] on a paper of Cherlin [5] which deals with locally finite quadrangles with 5 points per line. Moorhouse’s text ends with two basic questions (stated in section 5 of this paper) on the model theory of hypothetical locally finite generalized quadrangles, and answering these questions is the main purpose of this paper.

In [5], Cherlin observes that if a locally finite generalized quadrangle would happen to exist, then a generalized quadrangle with the same parameters would exist with an infinite and independent set of points or lines (that is, an infinite partial ovoid or partial spread) which are in a precise sense coordinatized over some totally ordered set \((S, \leq)\), and with extreme symmetry properties relative to the automorphism group of the quadrangle. Such sets are called “indiscernible” over \((S, \leq)\). It should be emphasized that one is allowed to choose \((S, \leq)\) arbitrary in this result. Moorhouse’s questions amount to considering the extremal cases for these sets: ovoids and spreads.

In this note, we will show that locally finite generalized quadrangles with ovoids or spreads which are indiscernible over any given infinite totally ordered set \((S, \leq)\) with \( S \) countable, cannot exist. This result should be seen as a very first step in attacking Tits’s problem through Model Theory, after Cherlin’s seminal note [5]. A first version of the proof benefited greatly from the proof of an old result by Dushnik and Miller [7] on similarity transformations of totally ordered sets, which dates back to 1940. When the paper was almost completed, the author noticed a geometrically much simpler proof which only requires the existence of certain isomorphisms that appear to be naturally present in abundance under Moorhouse’s conditions.

**ACKNOWLEDGMENT.** I would like to thank Gregory Cherlin, Alex Kruckman and Eric Moorhouse for several valuable communications on the topic of this paper.

2. SOME NOTES ON GENERALIZED QUADRANGLES

In this paper, we solely concentrate on generalized quadrangles.

A **generalized 4-gon**, also **generalized quadrangle** (abbreviated as “GQ”), is a point-line incidence geometry \( \Gamma = (\mathcal{P}, \mathcal{B}, \ell) \) for which the following axioms are satisfied:

(i) \( \Gamma \) contains no ordinary \( k \)-gon (as a subgeometry), for \( 2 \leq k < 4 \);
(ii) any two elements \( x, y \in \mathcal{P} \cup \mathcal{B} \) are contained in some ordinary 4-gon in \( \Gamma \);
(iii) there exists an ordinary 5-gon in $\Gamma$.

The value 4 is called the gonalit of the GQ.

By (iii), generalized quadrangles have at least three points per line and three lines per point. So by this definition, we do not consider “thin” quadrangles. Note that points and lines play the same role in the defining axioms; this is the principle of “duality”.

It can be shown that generalized quadrangles have an order; there exist constants $s, t$ such that the number of points incident with a line is $s + 1$, and the number of lines incident with a point is $t + 1$, cf. [27, section 1.5.3].

A sub generalized quadrangle or subquadrangle of a GQ $\Gamma = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a GQ $\Gamma' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ for which $\mathcal{P} \subseteq \mathcal{P}', \mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{I}' \subseteq \mathcal{I}$. It is full if for any line $L$ of $\Gamma'$, we have that $x \Gamma' L$ if and only if $x \mathcal{I} L$. Dually, we define ideal subquadrangles.

An automorphism of a generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a bijection of $\mathcal{P} \cup \mathcal{B}$ which preserves $\mathcal{P}$, $\mathcal{B}$ and incidence. The full set of automorphisms of a GQ forms a group in a natural way — the automorphism group of $\Gamma$, denoted $\text{Aut}(\Gamma)$. It is one of its most important invariants. If $B$ is an automorphism group of a generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, and $R$ is a subset of $\mathcal{P}$, $B_{[R]}$ is the subgroup of $B$ fixing $R$ pointwise (in this notation, a line is also considered to be a point set).

Generalized quadrangles and polygons were introduced by Tits in a famous work on triality [24], in order to propose an axiomatic and combinatorial treatment for semisimple algebraic groups (including Chevalley groups and groups of Lie type) of relative rank 2. The most important generalized polygons are those with gonality 4 — the generalized quadrangles. Standard reference works are [14], [19, 21] and [18]. Note that projective planes are nothing else than generalized 3-gons. Also, the class of generalized polygons coincides with the class of Tits buildings of rank 2; so they are the rank 2 residues for Tits buildings of higher rank.

Let us proceed with the model theoretic approach to locally finite quadrangles, first initiated by Gregory Cherlin in [5].

3. INDISCERNIBLES AND GENERALIZED QUADRANGLES

Let $\Gamma$ be a generalized quadrangle, and let $(S, \leq)$ be a totally ordered set. A set $L = \{M_s \mid s \in S\}$ of lines is indiscernible over $(S, \leq)$ if for any two increasing sequences $M_{s_1}, M_{s_2}, \ldots, M_{s_n}$ and $M'_{s_1}, M'_{s_2}, \ldots, M'_{s_n}$ (of the same length $n$) of lines in $L$, there is an automorphism of $\Gamma$ mapping $M_{s_i}$ onto $M'_{s_i}$ for each $i$. Here, by “increasing” we mean that in $S$, $s_1 < s_2 < \ldots < s_n$ and $s'_1 < s'_2 < \ldots < s'_n$.

By combining the Compactness Theorem and Ramsey’s Theorem [11] (in a theory which has a model in which a given definable set is infinite, one can prove the following.

**Theorem 3.1** (G. Cherlin [5]). Suppose there is an infinite locally finite generalized quadrangle with finite lines. Then there is an infinite locally finite generalized quadrangle $\Gamma$ containing an indiscernible sequence $L$ of parallel (= mutually skew) lines, of any specified order type.

**Remark 3.2.** In Theorem 3.1, $\Gamma$ may supposed to be generated by $L \cup \{L\}$ [6] (where $L$ is seen as a point set). “Generated by $L \cup \{L\}$” here, means that $\Gamma$ is the only subquadrangle which contains $L$, the lines of $L$, and the points of $L$.

**Moorhouse’s text.** We strongly advise the reader who is interested in understanding the model theoretic details behind Cherlin’s paper [5], to consult the excellent notes of G. Eric Moorhouse [13]. Moorhouse’s text contains all the details a geometer needs to start on this topic.

4. SOME FURTHER CONVENTIONS

Let $\Gamma$ be a locally finite GQ, and let $X$ be a set of points or lines (not both), which is indiscernible over the totally ordered set $(S, \leq)$. By $A$, we denote the subgroup of $\text{Aut}(\Gamma)$ which globally stabilizes $X$. So $A = \text{Aut}(\Gamma)_X$. 
Let \((S, \leq)\) be a totally ordered set, and let \(S'\) be a subset of \(S\); then by \((S', \leq)\) we denote the totally ordered set which arises by restricting \(\leq\) to \(S'\).

5. Moorhouse’s questions

Let \(S\) be a GQ, and let \(X\) be a line set in \(S\). If no two distinct elements in \(X\) intersect, then we call \(X\) a partial spread. If every point of \(S\) is incident with a line of \(X\), then we call \(X\) a spread. Dually, we speak of partial ovoids and ovoids.

\(\text{IND}_{ov}\) Suppose \(\Gamma\) is a locally finite GQ of order \((\omega, k)\) with \(k\) finite, which contains an ovoid \(O\), which is indiscernible over some totally ordered set \((S, \leq)\). What can we say about \(\Gamma\)?

\(\text{IND}_{sp}\) Suppose \(\Gamma\) is a locally finite GQ of order \((\omega, k)\) with \(k\) finite, which contains a spread \(L\), which is indiscernible over some totally ordered set \((S, \leq)\). What can we say about \(\Gamma\)?

Relative to the assumption that \(\Gamma\) has order \((\omega, k)\), we will see that \(\text{IND}_{ov}\) is easier than \(\text{IND}_{sp}\). Both questions will have the same answer, though: such quadrangles \(\Gamma\) cannot exist.

In this paper, we consider all generalized quadrangles \(\Gamma\) for which the totally ordered set \((S, \leq)\) is countable, that is, for which \(O\) (case \(\text{IND}_{ov}\)) and \(L\) (case \(\text{IND}_{sp}\)) have a countable number of elements.

6. Quadrangles with an indiscernible ovoid or spread

Let \(\Gamma\) be a thick generalized quadrangle of order \((\omega, k)\), with \(k\) finite and \(\omega\) not finite.

6.1. Hypothesis (\(\text{IND}_{ov}\)). First suppose that \(\Gamma\) satisfies \(\text{IND}_{ov}\): we let \(O\) be an ovoid of indiscernible points, over the totally ordered set \((S, \leq)\). Let \(w\) be a point which is not in \(O\), and let \(X, Y, Z\) be three different lines incident with \(w\); since \(O\) is an ovoid, each of these lines intersects \(O\) in precisely one point, say (respectively) \(x, y\) and \(z\). Without loss of generality, we suppose that

\[
x < y < z.
\]

The group \(A_{x,y}\) fixes \(x\) and \(y\), and acts transitively on the points of \(\{\epsilon \mid \epsilon \in O, \epsilon > z\}\). As each of the points of this set is collinear with some point of \(\{x, y\}^\perp\), the fact that \(|\{x, y\}^\perp| = k + 1 \in \mathbb{N}\) leads to the fact that some point of \(\{x, y\}^\perp\) must be incident with an infinite number of lines, contradiction.

This solves the first question.

**UNCOUNTABLE CASE.** Note that the proof does not use the fact that \(S\) is countable. So in general, case \(\text{IND}_{ov}\) is resolved.

6.2. Hypothesis (\(\text{IND}_{sp}\)). We now turn to the second hypothesis, Hypothesis (\(\text{IND}_{sp}\)). This case is more delicate, and the previous proof does not apply.

Let \(L\) be a spread of lines of \(\Gamma\), indiscernible over the totally ordered set \((S, \leq)\). In this paper, we suppose that \(S\) is countable.

A first direct observation is that directly due to \(\text{IND}_{sp}\), we know that \((S, \leq)\) is a dense linear order without end points, so by a result of Cantor, \((S, \leq) \simeq (\mathbb{Q}, \leq)\).

Let \(u\) be any point in \(\Gamma\), and let \(V\) be any line incident with \(u\), not contained in \(L\). Let \(S_V \subset S\) be the set of elements \(s\) in \(S\) which correspond to the lines in \(L\) that meet \(V\).

Now introduce an equivalence relation on \(S_V\) as follows: \(a \sim b\) if only a finite number (including \(0\)) of elements in \(S_V\) are contained between \(a\) and \(b\). Denote the set of equivalence classes in \(S_V\) by \(\overline{S_V}\), and note that each equivalence class trivially has a finite or countable number of elements. Also, \(\leq\) naturally induces an order on \(\overline{S_V}\), and we will keep using the same symbol “\(\leq\)” for that order. Putting \(\overline{S_V} := \mathbb{Q} \setminus S_V\), we similarly introduce \(\overline{S_V}\).

Now note the following property:

\(\overline{S_V}, \leq\) is a dense total order.

(Indeed, if \(r, s \in \overline{S_V}\) and \(r < s\), then if no \(t \in \overline{S_V}\) would exist for which \(r < t < s\), we would have that \(r = s\).)
We will call elements of $\tilde{S}_V$, resp. $\tilde{\hat{S}}_V$, “singletons” if they only contain one element as an equivalence class in $S_V$, resp. $\tilde{S}_V$. We distinguish a number of possibilities, depending on properties of “non-singletons.”

**How to proceed.** Suppose $\tilde{S}_V$ is a subset in $S$ such that there is some element $\alpha \in A$ for which $\alpha(S_V) = \tilde{S}_V$, and such that $S_V \subset \tilde{S}_V$ (strict inclusion).

Then we have obtained a contradiction, since $\{V, V^\alpha\}^\perp = \{L_s \mid s \in S_V\}$, while $V^\alpha$ is incident with points which are not incident with lines of $\{V, V^\alpha\}^\perp$.

In each of the next paragraphs, we will search for such automorphisms $\alpha$. Only if $S_V = S$ for all choices of $u$ and $V$, no contradiction arises. And this is precisely the case where $\Gamma$ is a grid.

Below, we say that $U$ is a dense subset of the linear order $(V, \leq)$ if for all $a < b$ in $V$, there is a $c \in U$ such that $a < c < b$.

**There are only singletons (in both $\tilde{S}_V$ and $\tilde{\hat{S}}_V$)**

This means that $S_V = \tilde{S}_V$ (modulo bracket notation), and that $(S_V, \leq)$ is a dense linear order, possibly with boundaries. Moreover, $(\tilde{S}_V, \leq)$ is also a dense linear order (with $\tilde{S}_V := \mathbb{Q} \setminus S_V$).

We assume without loss of generality that there are no intervals in $S$ with only interior points in $\tilde{S}_V$; if intervals would occur both with only points in $S_V$ and only points in $\tilde{S}_V$, the same argument as in subsection [6.4.1](#) proves our main statement. Also, if additionally there would be no intervals with only interior points in $S_V$, then both $S_V$ and $\tilde{S}_V$ are dense subsets of $S$, and we can proceed with a particular case of the Back & Forth argument at the end of the section.

Let $q < q'$ be points in $S$: then we write $q \asymp q'$ if each point $r \in S$ for which $q < r < q'$, is contained in $S_V$. If $q = q'$, then also $q \asymp q'$. We will assume without loss of generality that “$\asymp$” defines an equivalence relation on $S$ (and $S_V$), and we denote the equivalence class containing some element $v \in S$ by $[v]$. For $\asymp$ not to be an equivalence relation, we need a violation of transitivity, and hence we must have distinct points $a, b, c$ in $S$ such that $a \asymp b$ and $b \asymp c$ while $a \not\asymp c$. But in that case, we can immediately turn to section [6.3](#). If $(u, v)$ is an open interval in $S$, then we will, at various points, also consider the induced equivalence classes of $\asymp$ inside $(u, v)$. Call points in $S$ (or in $(u, v) \subseteq S$) isolated if their class only has one element, and of “type 2” otherwise. In the latter case, an equivalence class $[v]$ is an interval of which the endpoints cannot be both elements of $\tilde{S}_V$ (by the main assumption in this section). We have a number of different types of such intervals: open with both endpoints in $\tilde{S}_V$; half-open with both endpoints in $\tilde{S}_V$ (type $(a, a')$ and $[a, a')$); open with one endpoint in $\tilde{S}_V$ and one endpoint in $\tilde{\hat{S}}_V$ (type $(a, q)$ and $(q, a)$), etc. Each of these is considered as a different type 2 equivalence class (but we will still say that their “main type” is 2). Usually we do not consider intervals with non-finite endpoints (they are usually not needed in our arguments). The type of an interval which corresponds to an equivalence class, is by definition the type of its class. For the sake of convenience, denote the set of all types with main type 2 by $\mathcal{T}$.

Let $a < c$ points in $S$. By the expression “$a \leq I < c$, I with an interval of type $T \in \mathcal{T}$, we mean that $a < e$ and $e' < c$, with $e < e'$ the endpoints of $I$.

Proceed as follows.

First consider some interval $(q, q')$, and let $I_1$ and $I_2$ be two distinct intervals of main type 2 in $(q, q')$; this means they arise from $\asymp$ restricted to $(q, q')$. Suppose they are maximal in $(q, q')$ with respect to being main type 2. Suppose first that they share an endpoint $e_2 = e_1'$, where $e_2$ is the largest endpoint of $I_1$ and $e_1'$ is the smallest endpoint of $I_2$. Then since both intervals are maximal, we have that $e_2$ cannot be in $S_V$; so it is a point of $\tilde{S}_V$. In that case, proceed as in subsection [6.3](#) (where $e_2$ is the point that is moved).

Henceforth, we may and will suppose that if such distinct intervals exist, they do not share endpoints.

(A) If there exist $q_1 < q_2$ in $S$ such that $q_1 \not\asymp q_2$, and such that for no element $I$ of type $T$ (with main type 2), we have that $q_1 < I < q_2$, then we replace $S$ by $(q_1, q_2)$. Now repeat the same argument with $S$ replaced by $(q_1, q_2)$, and $I$ replaced by $I \setminus \{T\}$. After having considered all possible types, we end up with an open interval $(q_1^*, q_2^*)$ (which might still equal $S$ but which otherwise has finite endpoints) with $q_1^* \not\asymp q_2^*$, and such that for all $q_1 < q_2$ in this interval for which $q_1 \not\asymp q_2$, we have an interval $I$ of type $T \in \mathcal{T}'$ such that $q_1 < I < q_2$. Here $\mathcal{T}'$ is a subset of $\mathcal{T}$ of types with main type 2; in principle, the set $\mathcal{T}'$ can be empty. In that case, there are no intervals of
type $T \in \mathcal{T}$ inside $(q_1^*, q_2^*)$, except, possibly, intervals containing $q_1^*$ or $q_2^*$.

(B) Now suppose there is no isolated point of $S_V$ in some $(q_1, q_2) \subseteq (q_1^*, q_2^*)$ with $q_1 \neq q_2$. Note that intervals of type 2 must exist inside $(q_1, q_2)$, since we otherwise have nontrivial intervals whose interior points are solely in $\tilde{S}_V$. Replace $(q_1^*, q_2^*)$ by $(q_1, q_2)$ (while keeping the same notation). If there would be $q_1^* < q_2^*$ inside $(q_1^*, q_2^*)$ with $q_1^* \neq q_2^*$ and no isolated point of $\tilde{S}_V$ in between, then we replace once again $(q_1^*, q_2^*)$ by $(q_1, q_2)$ while keeping the same notation; otherwise, the isolated points of $\tilde{S}_V$ define a dense set inside the equivalence classes of $\sim$ in $(q_1^*, q_2^*)$. Let $I = [u, v]$ be an equivalence class in $(q_1^*, q_2^*)$ such that it does not contain $q_1^*$; without loss of generality, we will suppose that $u \in S_V$ (all the other variations are completely similar as this case). If $q_2^*$ would be an element of $I$, then $I = [u, v]$ instead of $[u, v]$. Now let $\alpha'$ be an order automorphism of $S$ such that for all $r \geq v$ and $r \leq q_1^*$, we have that, $\alpha'(r) = r$, and such that $\alpha'(u) < u$ be such that $\alpha'(u) \in (q_1^*, q_2^*)$. Define

$$\mathcal{D} := \{\alpha'(t) \mid t \in \tilde{S}_V, t \in (q_1^*, u)\}.$$  

Enumerate the elements of $\mathcal{D}$ as $d_1, d_2, \ldots$. We will construct a new order automorphism $\alpha$ by first defining it on the points $t \in \tilde{S}_V \cap (q_1^*, u)$. It is sufficient to define the action $d \to d'$ on the points of $\mathcal{D}$. Suppose $m$ is the first index for which $d_m = \alpha'(t)$ does not have a well-defined image. Suppose $d_a < d_m < d_b$ for the well-defined images $d_a$ and $d_b$. If $d_m \in \tilde{S}_V$, then we put $d_m = d_m$. If not, then we take an element $s_m \in \tilde{S}_V \cap (d_a, d_b)$ which is isolated if $t$ is isolated, and a (left resp. right) boundary point if $t$ is a (left resp. right) boundary point — by our definition of $(q_1^*, q_2^*)$, this is always possible. Now put $\alpha(t) = d_m$. Put $\alpha = \alpha'$ for all $r \leq q_1^*$ and $r \geq u$, and put $\alpha(t) = s_m$ for all $t \in (q_1^*, u)$ with $d_m = \alpha'(t)$. One now completes the construction of $\alpha$ by defining images for the points in $\tilde{S}_V \cap (q_1^*, u)$ while respecting the order of the already defined images, and this is done through a simple Back & Forth argument. For such $\alpha$, we have that $\alpha(S_V) \supset S_V \neq \alpha(S_V)$ (as $[u, v] \subseteq [\alpha(u), \alpha(v) = v]$).

So in what follows, we may and will assume that for all $q_1 < q_2$ in $(q_1^*, q_2^*)$ with $q_1 \neq q_2$, there is an $r \in S_V$ with $[r] = \{r\}$ such that $q_1 < r < q_2$.

(C) Suppose that $q_1 < q_2$ in $(q_1^*, q_2^*)$ are such that $q_1 \neq q_2$. Then we may assume (through an argument similar as in the previous point) that we have points $r$ in $\tilde{S}_V$ for which $[r] = \{r\}$ such that $q_1 < r < q_2$.

Similarly as above, we can narrow down $(q_1^*, q_2^*)$ and $T'$ (while keeping the same notation), such that for all $q_2 \in (q_1^*, q_2^*)$ for which $q_2 \neq q_2^*$, we can find isolated points $r$ in $S_V$, respectively $\tilde{S}_V$, respectively intervals $I$ of type $T$ in $T'$ such that $q_2 < r < q_2^*$, resp. $q_2 < r < q_2^*$, resp. $q_2 < 1 < q_2^*$. And if necessary, we can also narrow down $(q_1^*, q_2^*)$ and $T'$ (again keeping the same notation), such that for all $q_1 \in (q_1^*, q_2^*)$ for which $q_1 \neq q_1^*$, we can find isolated points $r$ in $S_V$, resp. $\tilde{S}_V$, resp. intervals $I$ of type $T$ in $T'$ such that $q_1^* < r < q_1^*$, resp. $q_1^* < r < q_1$, resp. $q_1^* < q_1 < q_2$. Note that if $1 < 1 \geq 1$ (11, 12 intervals), then for $q_1 \in 11$ and $q_2 \in 12$, we have that $q_1 \neq q_2$ and the above properties apply.

We now construct the desired $\alpha \in A$.

**BACK & FORTH argument.** Enumerate the equivalence classes of $\sim$ in $(q_1^*, q_2^*)$: $\alpha_1, \alpha_2, \ldots$. We do not include the classes of $q_1^*$ and/or $q_2^*$ in the enumeration; below, once we define $\alpha$, we put $\alpha(L_r) = L_r$ for each element $r$ inside $[q_1^*]$ and/or $[q_2^*]$.

Now let $r$ be an element in $(q_1^*, q_2^*) \cap \tilde{S}_V$ with $r$ not in $[q_1^*]$ nor $[q_2^*]$ and such that $[r] = \{r\}$, and define: $S'_V := S_V \cup \{r\}, \hat{\tilde{S}}_V := \tilde{S}_V \setminus \{r\}$. Relative to these new sets $S'_V$ and $\hat{\tilde{S}}_V$, we define $\sim'$ similarly as $\sim$, and adapt the definition of types. Note that $[r]' = \{r\}$. Enumerate the equivalence classes of $\sim'$ as $s_1', s_2', \ldots$.

Let $(E, \leq)$ be the set of equivalence classes in $(q_1^*, q_2^*)$ endowed with the induced order relation; the reason why the Back & Forth argument below to construct $\alpha$ works, is the fact that the equivalence classes of each fixed type (including singletons) are dense in $E$.

Suppose that $m$ is the smallest index such that $s_m$ is not yet paired with any member of $s_1', s_2', \ldots$. Let $s_k'$ be an element that is not yet paired, and which is of the same type as $s_m$, while respecting the order of the elements. Pair $s_m$ to $s_k'$. Now let $n$ be the smallest index for which $s_n'$ is not yet paired, and pair it similarly with an element of $s_1, s_2, \ldots$ that is not yet paired. One uses the density properties of the previous paragraph to show that the pairings can be done in each step without violating the order. (For instance, if $s_{l_1} < s_m < s_{l_2}$ with $l_1, l_2$ indexes which
were already taken care of in previous steps, and \( s_m = \{ \rho \} \) with \( r \) in \( \hat{S}_V \), and \( s_{l_1} \) was paired to \( s'_{l_1} \), and \( s_{l_2} \) to \( s'_{l_2} \), then \( s'_{l_1} < s'_{l_2} \) and we can find a singleton \( s'_n \) in \( \hat{S}_V \) such that \( s'_{l_1} < s'_n < s'_{l_2} \). This means that \( \{ s'_n \} \). \( \rho' \in \hat{S}_V \). That is the point to which \( s_m \) is paired. All the other cases are similar.

As such, we have constructed a bijection between \( s_1, s_2, \ldots, s'_1, s'_2, \ldots \); call it \( \alpha \). Now we define the desired element of \( A \) as follows: \( \alpha \) acts trivially on lines indexed by elements outside \((q_1^1, q_2^1)\); if \( [s_i] = \{s_i\} \), then 
\[ \alpha(L_{s_i}) := L_{q_{s_j}} \] if \( s_j \) is an interval, then choose an arbitrary order bijection \( \tilde{\alpha} \) between \( s_j \) and \( q_{s_j} \) which also defines the images for the endpoints (such bijections exist since \( s_j \) and \( q_{s_j} \) are of the same type), and set \( \alpha(L_s) = L_{q(s)} \) for the points in \( s_j \), and for its endpoints.

Then \( \alpha \in A \) is such that \( \alpha(S_V) = S'_V = S_V \cup \{r\} \).

**Alternative argument.** One can also define \( \alpha \) through the method presented in the penultimate item (B) in the enumeration above (which essentially solely concentrates on the images of \( S_V \), starting from an order automorphism \( \alpha' \) as above). The essence is the same.

**There is at least one non-singleton which is not finite**

Call it \( s \). Note that it is a discrete set (for each element \( r \) in this class, there exists a positive rational number \( \delta \) such that \((r - \delta, r + \delta) \cap s = \{r\} \).

Now we construct an element of \( A \).

First of all, let \( \gamma(t) = t \) for all \( t \in S \) such that \( t < s \) (by which we mean that \( t < s' \) for all \( s' \in s \)). Similarly, let \( \gamma(t) = t \) for all \( t > s \). Now consider consecutive elements \( u, v \) in \( s \), and let \( w \in S \) be strictly contained between \( u \) and \( v \). Define \( s' = s \cup \{w\} \). Then \( (s, \leq) \simeq (s', \leq) \) and we assume that \( \gamma \) induces an order isomorphism between \( (s, \leq) \) and \( (s', \leq) \).

Now we extend \( \gamma \) to an element \( \overline{\gamma} = \alpha \in A \) (by a standard Back & Forth argument). Then \( \overline{S_V} := \overline{\alpha}(S_V) \) strictly contains \( S_V \). Now proceed as before.

Remark 6.1. The reasoning which we will see below in subsection 6.3 can also be applied to the case considered in this section.

**There is at least one non-singleton, and all of them are finite**

We consider two sub-cases.

6.3. **At least one non-singleton has size \( \geq 3 \).** Let \( s \) be such a non-singleton, and let \( r \) be a point of \( s \) which is not the largest nor smallest element of \( s \). Now consider an element \( \alpha \) in \( A \) such that all elements of \( S_V \setminus \{r\} \) are fixed, and such that \( r \) is moved inside \( s \). Such \( \alpha \) are easily seen to exist, since \( r \) has a positive distance to each of its neighbors in \( S_V \).

Then \( V^\alpha \) necessarily is a line different from \( V \), and \( \{V, V^\alpha\} \) contains all the lines of \( \mathcal{L} \) which are indexed by an element of \( S_V \setminus \{r\} = \alpha(S_V) \setminus \{\alpha(s)\} \). The same property is true for all elements in the infinite set

\[ \sigma(V) := \{\alpha^m(V) \mid m \in \mathbb{N} \setminus \{0\}\}. \]

It is straightforward to see that there is one extra line in \( \Gamma \) which meets all lines of \( \{V \cup \sigma(V) \), and hence \( \{V \cup \sigma(V) \subseteq \{V, \alpha(V)\} \). \)

Now consider a point \( \omega \) not contained in the set of points incident with lines in \( \{V \cup \sigma(V) \); by projecting \( \omega \) on the lines in the latter line set, we immediately obtain a contradiction since there are only a finite number of lines incident with \( r \). (If \( U, W \) are different lines in \( \{V \cup \sigma(V) \), then the projection line from \( \omega \) to \( U \) is different than the projection line from \( \omega \) to \( W \).)

6.4. **All non-singletons have size \( 2 \).** We need to consider two sub-cases.
6.4.1. \( \hat{S}_V \) has non-singletons. Let \((a, a')\) be an open interval for which \(a, a' \neq a \in S_V\), and all interior elements are in \( \hat{S}_V \), and let \((q, q')\) be an open interval for which \(q, q' \neq q \in \hat{S}_V\), and all the interior elements are in \( \hat{S}_V \). By assumption, these intervals exist.

We suppose without loss of generality that \( q < q' < a < a' \) (the other case is handled by a symmetrical argument).

Now define an automorphism \( \alpha \in A \) with the following properties:

- \( \alpha(q) = q \) and \( \alpha(a') = a' \);
- if \( r < q \) or \( r > a' \), then \( \alpha(r) = r \) (\( r \in \mathbb{Q} \); \( \alpha(q') \in (a, a') \));
- \( \alpha(q') < \alpha(a) < a' \).

It is easy to see that such \( \alpha \) exist in \( A \), and it is obvious that \( \alpha(S_V) \) strictly contains \( S_V \). Now proceed as before. Note that if \( q = -\infty \) and/or \( a' = +\infty \), the approach above trivially adapts just fine.

6.4.2. \( \hat{S}_V \) has no non-singletons. This case can be handled through a similar Back & Forth method as the case without non-singletons.

The finally completes the proof of the answers to both questions. ■

**FORTHCOMING.** In a forthcoming sequel to this paper [23] (which we hope to have finished rather sooner than later), we will handle the uncountable version of (IND).
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