K-theory of Rotation Algebra Crossed Products by Amalgamated Products of Finite Cyclic Groups

S. Walters

ABSTRACT. The $K$-groups of the crossed product of the rotation C*-algebra $A_\theta$ by free and amalgamated products of the cyclic groups $\mathbb{Z}_n$, for $n = 2, 3, 4, 6$, are calculated. The actions here arise from the canonical actions of these groups on the rotation algebra under the flip, cubic, Fourier, and hexic automorphisms, respectively. An interesting feature in this study is that although the inclusion $A_\theta \to A_\theta \times \mathbb{Z}_n$ induces injective maps on their $K_0$-groups, the same is not the case for the inclusions $A_\theta \times \mathbb{Z}_d \to A_\theta \times \mathbb{Z}_n$ for $2 \leq d < n \leq 6$ and $d|n$, which we endeavor to calculate. Further, while for free products $K_1(A_\theta \rtimes [\mathbb{Z}_m \star \mathbb{Z}_n]) = 0$, for amalgamated products $K_1(A_\theta \rtimes [\mathbb{Z}_m \star \mathbb{Z}_d \mathbb{Z}_n]) = \mathbb{Z}^k$ is non-vanishing ($k = 1, 2$).

CONTENTS

1. Introduction 2
2. The Natsume Exact Sequence 4
3. $K$-theory of $A_\theta \rtimes \mathbb{Z}_{4,4,2}$ 6
4. $K$-theory of $A_\theta \rtimes \mathbb{Z}_{4,6,2}$ 10
5. $K$-theory of $A_\theta \rtimes \mathbb{Z}_{6,6,2}$ 13
6. $K$-theory of $A_\theta \rtimes \mathbb{Z}_{6,6,3}$ 13
References 15
1. Introduction

The rotation C*-algebra $A_\theta$ is the universal C*-algebra generated by two unitaries $U, V$ satisfying the commutation relation $VU = e^{2\pi i \theta}UV$. There are canonical actions of the finite cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ on $A_\theta$ (where $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$). These actions are given, respectively, by the flip $\phi$, cubic $\alpha$, Fourier $\sigma$, and hexic $\rho$, transforms:

$$
\phi(U) = U^{-1}, \quad \phi(V) = V^{-1} \quad (1.1)
$$

$$
\alpha(U) = e^{-\pi i \theta}U^{-1}V, \quad \alpha(V) = U^{-1} \quad (1.2)
$$

$$
\sigma(U) = V^{-1}, \quad \sigma(V) = U \quad (1.3)
$$

$$
\rho(U) = V, \quad \rho(V) = e^{-\pi i \theta}U^{-1}V. \quad (1.4)
$$

Of course, $\phi = \sigma^2 = \rho^3$, and $\alpha = \rho^2$. The C* and K-theoretic structure of these automorphism have been extensively studied in [2], [3], [4], [5], [7], [9], [11], [12], [13], [14], [15].

For convenience, we shall introduce the notation

$$
\mathbb{Z}_{m,n} = \mathbb{Z}_m \ast \mathbb{Z}_n, \quad \mathbb{Z}_{m,n,d} = \mathbb{Z}_m \ast_{\mathbb{Z}_d} \mathbb{Z}_n
$$

for the free and amalgamated products. It is well-known that $\mathbb{Z}_{2,2} \cong \mathbb{Z} \times \mathbb{Z}_2$ (the usual semi-direct product). It is also known that $\mathbb{Z}_{2,3} \cong \text{PSL}(2, \mathbb{Z})$ and that $\mathbb{Z}_{4,6,2} \cong \text{SL}(2, \mathbb{Z})$, both nonamenable groups. (See [8], II.28, III.14.)

**Convention 1.1.** We make free use of the notation $e(x) := e^{2\pi i x}$ for real $x$.

In joint work with Echterhoff, Lück, and Phillips [7], it was shown that for all real parameters $\theta$ one has

$$
K_0(A_\theta \rtimes \mathbb{Z}_3) = \mathbb{Z}^8, \quad K_0(A_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^9, \quad K_0(A_\theta \rtimes \mathbb{Z}_6) = \mathbb{Z}^{10},
$$

and that $K_1(A_\theta \rtimes \mathbb{Z}_n) = 0$ for $n = 3, 4, 6$. Before that, Kumjian [9] already showed that for the flip case $K_0(A_\theta \rtimes \mathbb{Z}_2) = \mathbb{Z}^6$ and $K_1(A_\theta \rtimes \mathbb{Z}_2) = 0$.

For simplicity, we shall write

$$
K_0(A_\theta \rtimes \mathbb{Z}_n) = \mathbb{Z}^{r(n)}
$$

where $r(n)$ is the corresponding rank. It was also shown in [7] that the crossed product $A_\theta \rtimes \mathbb{Z}_n$ is approximately finite (AF) dimensional for $n = 3, 4, 6$ and any irrational $\theta$. The AF result for the $n = 2$ case was previously proved by Bratteli and Kishimoto [4]. The AF result and the $K$-groups for the Fourier case ($n = 4$) was proved by the author in 2004 [14] for a dense $G_\delta$ set of parameters $\theta$, and in [7] this is shown to hold for all irrational $\theta$.

In this paper we prove Theorems 1.2 and 1.3, which give the $K$-groups of the crossed product of the rotation algebra by the canonical actions of the free products $\mathbb{Z}_m \rtimes \mathbb{Z}_n$ and the amalgamated products

$$
\mathbb{Z}_4 \rtimes_{\mathbb{Z}_2} \mathbb{Z}_4, \quad \mathbb{Z}_4 \rtimes_{\mathbb{Z}_2} \mathbb{Z}_6, \quad \mathbb{Z}_6 \rtimes_{\mathbb{Z}_2} \mathbb{Z}_6, \quad \mathbb{Z}_6 \rtimes_{\mathbb{Z}_3} \mathbb{Z}_6.
$$

This is done by proving Theorems 1.4 and 1.5 and using the results of [7] and applying Natsume’s exact sequence [10].
Since the free product groups involved here are nonnuclear (and non-amenable), with the exception of $\mathbb{Z}_2 \ltimes \mathbb{Z}_2$, all the crossed products considered here, both the unreduced $C^*(A_\theta, \mathbb{Z}_{m,n})$ and the reduced $C^*_r(A_\theta, \mathbb{Z}_{m,n}) \equiv A_\theta \rtimes \mathbb{Z}_{m,n}$, are not nuclear since $C^*_r(\mathbb{Z}_{m,n})$ is not nuclear for $(m,n) \neq (2,2)$. However, all these crossed products are K-nuclear on account of $\mathbb{Z}_{m,n}$ being K-amenable and $A_\theta$ nuclear (see [1], 20.10.2). Further, the K-amenability of $\mathbb{Z}_{m,n}$ implies that one has an isomorphism $K_0(C^*(A_\theta, \mathbb{Z}_{m,n})) \cong K_0(C^*_r(A_\theta, \mathbb{Z}_{m,n}))$, so that the K-groups obtained here are the same for the reduced and unreduced crossed products alike, thanks to a result of Cuntz [6]. The same applies to crossed products by the amalgamated groups $C^*(A_\theta, \mathbb{Z}_{m,n,d})$.

We now state our main results.

**Theorem 1.2.** Under the canonical actions of the groups $\mathbb{Z}_n$ ($n = 2, 3, 4, 6$) of the rotation C*-algebra $A_\theta$, and for all parameters $\theta$ one has the $K_0$-groups

\[
\begin{align*}
K_0(A_\theta \rtimes \mathbb{Z}_{2,2}) &= \mathbb{Z}^{12}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{2,3}) &= \mathbb{Z}^{14}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{2,4}) &= \mathbb{Z}^{15}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{2,6}) &= \mathbb{Z}^{16}
\end{align*}
\]

and $K_1(A_\theta \rtimes \mathbb{Z}_{m,n}) = 0$ in each case.

**Theorem 1.3.** Under the canonical actions of the groups $\mathbb{Z}_n$ ($n = 2, 3, 4, 6$) of the rotation C*-algebra $A_\theta$, and for all parameters $\theta$, the $K$-groups of the crossed product algebra $A_\theta \rtimes \mathbb{Z}_{m,n,d}$ by the amalgamated product groups, are as follows

\[
\begin{align*}
K_0(A_\theta \rtimes \mathbb{Z}_{4,4,2}) &= \mathbb{Z}^{13}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{4,6,2}) &= \mathbb{Z}^{14}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{6,6,2}) &= \mathbb{Z}^{16}, \\
K_0(A_\theta \rtimes \mathbb{Z}_{6,6,3}) &= \mathbb{Z}^{14}, \\
K_1(A_\theta \rtimes \mathbb{Z}_{4,4,2}) &= \mathbb{Z}, \\
K_1(A_\theta \rtimes \mathbb{Z}_{4,6,2}) &= \mathbb{Z}, \\
K_1(A_\theta \rtimes \mathbb{Z}_{6,6,2}) &= \mathbb{Z}^2, \\
K_1(A_\theta \rtimes \mathbb{Z}_{6,6,3}) &= \mathbb{Z}^2.
\end{align*}
\]

In particular, it is interesting that unlike the free products case, the amalgamated products actions involve nonzero $K_1$'s.

The above theorems require the following result regarding the three canonical inclusions

\[
\begin{align*}
i & : A_\theta \rtimes \mathbb{Z}_2 \to A_\theta \rtimes \mathbb{Z}_4 \\
i' & : A_\theta \rtimes \mathbb{Z}_2 \to A_\theta \rtimes \mathbb{Z}_6 \\
k & : A_\theta \rtimes \mathbb{Z}_3 \to A_\theta \rtimes \mathbb{Z}_6
\end{align*}
\]

and their induced $K_0$-maps, all of which are shown to be noninjective in a precise manner (in contrast with Theorem 1.5 below).
**Theorem 1.4.** We have the exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(A \times \mathbb{Z}_2) = \mathbb{Z}^6 & \longrightarrow & K_0(A \times \mathbb{Z}_4) = \mathbb{Z}^9 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & K_0(A \times \mathbb{Z}_2) = \mathbb{Z}^6 & \longrightarrow & K_0(A \times \mathbb{Z}_6) = \mathbb{Z}^{10} & \longrightarrow & \mathbb{Z}^6 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & K_0(A \times \mathbb{Z}_3) = \mathbb{Z}^8 & \longrightarrow & K_0(A \times \mathbb{Z}_6) = \mathbb{Z}^{10} & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & 0
\end{array}
\]

where the images of \(i_*\), \(t_*', \kappa_*\) are direct summand subgroups of rank 5, 4, and 6, respectively, and the free group injections into \(K_0\) (in the left sides) are onto direct summands as well.

With \(i : A \theta \rightarrow A \theta \times \mathbb{Z}_n\) denoting the canonical inclusion, we have the following.

**Theorem 1.5.** For \(n = 2, 3, 4, 6\), the canonical morphism \(i_* : K_0(A \theta) \rightarrow K_0(A \theta \times \mathbb{Z}_n)\) maps injectively onto a direct summand.

The proofs for Theorem 1.5 are given at the end of Section 3 for \(n = 2, 4\) (evident also from [11], [12], [13]), at the end of Section 4 for \(n = 6\), and at the end of Section 6 for \(n = 3\).

## 2. The Natsume Exact Sequence

Recall from [10] that the Natsume six-term exact sequence for crossed products by free and amalgamated actions is

\[
\begin{array}{cccccc}
K_0(A \times N) & \xrightarrow{i_1 - i_2} & K_0(A \times G) \oplus K_0(A \times H) & \xrightarrow{j_1 + j_2} & K_0(A \times (G \star_N H)) \\
\uparrow & & \uparrow & & \uparrow \\
K_1(A \times (G \star_N H)) & \xrightarrow{j_1 + j_2} & K_1(A \times G) \oplus K_1(A \times H) & \xrightarrow{i_1 - i_2} & K_1(A \times N)
\end{array}
\]

where \(G, H, N\) are groups acting on the C*-algebra \(A\), \(N\) is a common subgroup of \(G\) and \(H\), and

\[
i_1 : A \times N \rightarrow A \times G, \quad i_2 : A \times N \rightarrow A \times H,
\]

\[
j_1 : A \times G \rightarrow A \times (G \star_N H), \quad j_2 : A \times H \rightarrow A \times (G \star_N H)
\]

are the canonical inclusions. (All crossed products here are reduced.)

Applied to the cyclic groups \(\mathbb{Z}_n, \mathbb{Z}_m\) acting on the rotation algebra \(A \theta\), one gets the six-term exact sequence (so \(N = 1\), the trivial group)

\[
\begin{array}{cccccc}
\mathbb{Z}^2 & \cong & K_0(A \theta) & \xrightarrow{i_1 - i_2} & K_0(A \theta \times \mathbb{Z}_m) \oplus K_0(A \theta \times \mathbb{Z}_n) & \xrightarrow{j_1 + j_2} & K_0(A \theta \times \mathbb{Z}_{m,n}) \\
\uparrow & & \uparrow & & \uparrow \\
K_1(A \theta \times \mathbb{Z}_{m,n}) & \xrightarrow{j_1 + j_2} & K_1(A \theta \times \mathbb{Z}_m) \oplus K_1(A \theta \times \mathbb{Z}_n) & \xrightarrow{i_1 - i_2} & K_1(A \theta) \cong \mathbb{Z}^2
\end{array}
\]

By [7], \(K_1(A \theta \times \mathbb{Z}_m) = 0\) and \(K_0(A \theta \times \mathbb{Z}_n) \cong \mathbb{Z}^{r(n)}\) for all \(\theta\), so the sequence becomes

\[
\begin{array}{cccccc}
\mathbb{Z}^2 & \cong & K_0(A \theta) & \xrightarrow{i_1 - i_2} & \mathbb{Z}^{r(m)} \oplus \mathbb{Z}^{r(n)} & \xrightarrow{j_1 + j_2} & K_0(A \theta \times \mathbb{Z}_{m,n}) \\
\uparrow & & \uparrow & & \uparrow \\
K_1(A \theta \times \mathbb{Z}_{m,n}) & \xrightarrow{j_1 + j_2} & 0 \oplus 0 & \xrightarrow{i_1 - i_2} & K_1(A \theta) \cong \mathbb{Z}^2
\end{array}
\]
Here, $i_{1*} - i_{2*}$ maps a basis of $K_0(A_0)$ injectively onto a direct summand of $K_0(A_0 \times \mathbb{Z}_m) \oplus K_0(A_0 \times \mathbb{Z}_n)$ in each case in view of Theorem 1.5 and Lemma 2.1 below. This gives $K_1(A_0 \times \mathbb{Z}_m, n) = 0$. As the map $j_{1*} + j_{2*}$ has a 2-dimensional kernel, and the vertical map on the right is onto a free abelian group, we obtain $K_0(A_0 \times \mathbb{Z}_m, n) = \mathbb{Z}^{r(m)+r(n)-2} \oplus \mathbb{Z}^2$. From $r(2) = 6$, $r(3) = 8$, $r(4) = 9$, $r(6) = 10$, one obtains all the $K$-groups for the free products listed in Theorem 1.2.

For the crossed product of $A_0$ by any of the amalgamated products $\mathbb{Z}_{m,n,d} = \mathbb{Z}_{4,4,2}, \mathbb{Z}_{4,6,2}, \mathbb{Z}_{6,6,2}, \mathbb{Z}_{6,6,3},$ where $d = 2, 3$, the Natsume exact sequence is

\[ K_0(A_0 \times \mathbb{Z}_d) \xrightarrow{i_{1*}-i_{2*}} K_0(A_0 \times \mathbb{Z}_m) \oplus K_0(A_0 \times \mathbb{Z}_n) \xrightarrow{j_{1*}+j_{2*}} K_0(A_0 \times \mathbb{Z}_{m,n,d}) \]

\[ K_1(A_0 \times \mathbb{Z}_{m,n,d}) \xrightarrow{j_{1*}+j_{2*}} K_1(A_0 \times \mathbb{Z}_m) \oplus K_1(A_0 \times \mathbb{Z}_n) \xrightarrow{i_{1*}-i_{2*}} K_1(A_0 \times \mathbb{Z}_d) \]

Since $K_1(A_0 \times r, \mathbb{Z}_n) = 0$ for $n = 2, 3, 4, 6$ (see [7]), one gets

\[ \mathbb{Z}^{r(d)} = K_0(A_0 \times \mathbb{Z}_d) \xrightarrow{i_{1*}-i_{2*}} \mathbb{Z}^{r(m)} \oplus \mathbb{Z}^{r(n)} \xrightarrow{j_{1*}+j_{2*}} K_0(A_0 \times \mathbb{Z}_{m,n,d}) \]

\[ K_1(A_0 \times \mathbb{Z}_{m,n,d}) \xleftarrow{0} 0 \oplus 0 \xrightarrow{0} 0 \]

from which one gets

\[ K_0(A_0 \times \mathbb{Z}_{m,n,d}) = \mathbb{Z}^{r(m)+r(n)-s} \]

\[ K_1(A_0 \times \mathbb{Z}_{m,n,d}) = \ker(i_{1*} - i_{2*}) \]

where $s = r(d) - \text{rk}(\ker(i_{1*} - i_{2*}))$ is the rank of the image of $i_{1*} - i_{2*}$, which will need to be calculated and shown to be a direct summand of $\mathbb{Z}^{r(m)} \oplus \mathbb{Z}^{r(n)}$ in all four amalgamated cases. This, together with the kernel of $i_{1*} - i_{2*}$, are calculated in this paper.

It may be worthwhile and interesting to compare our results with similar results for the group $\mathbb{C}^*$-algebra of $\mathbb{Z}_{m,n,d}$ obtained by Natsume, $K_0(\mathbb{C}^*(\mathbb{Z}_{m,n,d})) = \mathbb{Z}^{m+n-d}$ and $K_1(\mathbb{C}^*(\mathbb{Z}_{m,n,d})) = 0$ (see [10], Section 6). The contrasting features seem more apparent with the nonvanishing of the $K_1$-groups in the amalgamated case.

We end this section with a lemma needed in our calculations.

**Lemma 2.1.** Let $f : \mathbb{Z}^p \to \mathbb{Z}^m$ and $g : \mathbb{Z}^p \to \mathbb{Z}^n$ be group morphisms with direct summand images such that $\ker(f) \subseteq \ker(g)$. Then the morphism

\[ h : \mathbb{Z}^p \to \mathbb{Z}^m \oplus \mathbb{Z}^n, \quad h(x) = (f(x), g(x)) \]

has direct summand image.

**Proof.** Let $K = \ker(f) = \ker(h)$. As $K$ is a direct summand subgroup of $\mathbb{Z}^p$, the quotient group $\mathbb{Z}^p_K \cong \mathbb{Z}^q$ for some $q$. One then gets the induced morphisms

\[ F : \mathbb{Z}^q = \frac{\mathbb{Z}^p}{K} \to \mathbb{Z}^m, \quad F[x] = f(x) \]

\[ G : \mathbb{Z}^q = \frac{\mathbb{Z}^p}{K} \to \mathbb{Z}^n, \quad G[x] = g(x) \]
\[ H : \mathbb{Z}^q = \mathbb{Z}^p / K \rightarrow \mathbb{Z}^m \oplus \mathbb{Z}^n, \quad H[x] = (f(x), g(x)) \]
in which \( F, G, H \) have the same images as \( f, g, h \), respectively, and \( H(z) = (F(z), G(z)) \) for \( z \in \mathbb{Z}^q \). Both \( H \) and \( F \) are injective. (Of course, \( G, H \) are well-defined by the hypothesis \( \ker(f) \subseteq \ker(g) \).) As \( \ker(G) \) is also a direct summand of \( \mathbb{Z}^q \), one can pick a basis for \( \mathbb{Z}^q \) of the form
\[ e_1, \ldots, e_r, e'_1, \ldots, e'_s \]
where \( e_1, \ldots, e_r \) is a basis for \( \ker(G) \). The image of \( H \) has as basis the vectors
\[ H(e_i) = (F(e_i), 0), \quad H(e'_j) = (F(e'_j), G(e'_j)) \tag{2.1} \]
where \( i = 1, \ldots, r \) and \( j = 1, \ldots, s \). Our task is to show that the vectors in \( \ker(G) \) are contained in some basis of \( \mathbb{Z}^m \oplus \mathbb{Z}^n \).

By the hypothesis on the images of \( f \) and \( g \) being direct summands (and which are, respectively, the same as the images of \( F \) and \( G \)), there is a basis of \( \mathbb{Z}^m \) of the form
\[ F(e_1), \ldots, F(e_r), F(e'_1), \ldots, F(e'_s), x_1, \ldots, x_\ell, \]
and there is a basis of \( \mathbb{Z}^n \) of the form
\[ G(e'_1), \ldots, G(e'_s), y_1, \ldots, y_\ell. \]

From these, we obtain the following basis for \( \mathbb{Z}^m \oplus \mathbb{Z}^n \):
\[ (F(e_1), 0), \ldots, (F(e_r), 0), (F(e'_1), 0), \ldots, (F(e'_s), 0), (x_1, 0), \ldots, (x_\ell, 0) \]
\[ (0, G(e'_1)), \ldots, (0, G(e'_s)), (0, y_1), \ldots, (0, y_\ell). \]

It is now evident that by adding the vectors \((0, G(e'_1)), \ldots, (0, G(e'_s))\), respectively, to the vectors \( (F(e'_1), 0), \ldots, (F(e'_s), 0) \) one obtains a basis for \( \mathbb{Z}^m \oplus \mathbb{Z}^n \) that contains exactly the vectors in \( \ker(G) \) (which are a basis for the image of \( H \)). It follows that the image of \( h \) is a direct summand. \( \blacksquare \)

### 3. K-theory of \( A_\Theta \rtimes \mathbb{Z}_{4,4,2} \)

In this section we will calculate the morphism
\[ \iota : \mathbb{Z}^6 = K_0(\mathbb{Z} \times \mathbb{Z}_2) \rightarrow K_0(\mathbb{Z} \times \mathbb{Z}_4) = \mathbb{Z}^9 \]
and show that its kernel is \( \mathbb{Z} \) and its image is a direct summand subgroup isomorphic to \( \mathbb{Z}^5 \). The canonical inclusion \( \iota : A_\Theta \rtimes \mathbb{Z}_2 \rightarrow A_\Theta \rtimes \mathbb{Z}_4 \) of crossed products is given by
\[ \iota(x + yW) = x + yZ^2 \]
\((x, y \in A_\Theta)\) where \( W \) and \( Z \) are the canonical orders 2 and 4 unitaries of the crossed products \( A_\Theta \rtimes \mathbb{Z}_2, A_\Theta \rtimes \mathbb{Z}_4 \), respectively.

It is known from Proposition 3.2 of [11] that \( K_0(\mathbb{Z} \times \mathbb{Z}_2) = \mathbb{Z}^6 \) has six basis generators \([1] = \xi_1, \xi_2, \ldots, \xi_6\) with Chern-Connes characters given by the vectors
\[ T_2(\xi_1) = (1; 0, 0, 0, 0), \quad T_2(\xi_2) = (1; 2, 0, 0, 0), \quad T_2(\xi_3) = (1/2; 0, 2, 0, 0) \]
\[ T_2(\xi_4) = (1/2; 0, 0, 2, 0), \quad T_2(\xi_5) = (1/2; 0, 0, 0, 2), \quad T_2(\xi_6) = (1/2; 1, c, -c, -1) \tag{3.1} \]
where \( c = -1 \) if \( 0 < \theta < \frac{1}{2} \) and \( c = 1 \) if \( \frac{1}{2} < \theta < 1 \).

Recall that the Chern-Connes character for the flip automorphism \( \phi \) is defined by

\[
T_2 : K_0(A_\theta \rtimes \mathbb{Z}_2) \to \mathbb{R}^5,
T_2 = (\tau; \tau_{00}, \tau_{01}, \tau_{10}, \tau_{11})
\]

where

\[
\tau_{jk}(x + yW) = 4\phi_{jk}(y)
\] (3.2)

for \( x, y \in A_\theta \), and

\[
\phi_{jk}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta_2^{m-j}\delta_2^{n-k}
\] (3.3)

where \( \delta_2^r = 1 \) if \( r|s \), and 0 otherwise, is the divisor delta function. Further, the canonical trace \( \tau \) on the crossed product is given by \( \tau(x + yW) = \tau(x) \) where \( \tau(x) \) is the canonical normalized trace on the rotation algebra \( A_\theta \).

We also remind ourselves, from [12] (see also [13]), that the Chern-Connes map \( T_4 \) associated to the Fourier transform is

\[
T_4 : K_0(A_\theta \rtimes \mathbb{Z}_4) \to \mathbb{C}^6,
T_4 = (\tau; T_{10}, T_{11}; T_{20}, T_{21}T_{20})
\]

and is injective, where, for general elements \( x = x_0 + x_1Z + x_2Z^2 + x_3Z^3 \) in \( A_\theta \rtimes \mathbb{Z}_4 \) (where \( x_j \in A_\theta \)), one has

\[
T_{10}(x) = \psi_{10}(x_3),
T_{11}(x) = \psi_{11}(x_3),
T_{20}(x) = \psi_{20}(x_2),
T_{21}(x) = \psi_{21}(x_2),
T_{22}(x) = \psi_{22}(x_2)
\]

and \( \psi_{ij} \) are the unbounded linear functionals (see [12], Proposition 2.2)

\[
\psi_{10}(U^mV^n) = e(\frac{\theta}{4}(m-n)^2)\delta_2^{m-n},
\psi_{11}(U^mV^n) = e(\frac{\theta}{4}(m-n)^2)\delta_2^{m-n-1}
\]

and

\[
\psi_{20}(U^mV^n) = \phi_{00}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta_2^{m}\delta_2^{n},
\psi_{21}(U^mV^n) = \phi_{11}(U^mV^n) = e(-\frac{\theta}{2}mn)\delta_2^{m-1}\delta_2^{n-1},
\psi_{22}(U^mV^n) = (\phi_{01} + \phi_{10})(U^mV^n) = e(-\frac{\theta}{2}mn)\delta_2^{m-n-1}.
\]

The \( T_{1j} \) are concentrated on the \( Z^3 \)-component of \( x \), and \( T_{2j} \) are concentrated on the \( Z^2 \) part, and that the \( T_{jk} \) are 0 elsewhere. The canonical trace \( \tau \) on the Fourier crossed product is

\[
\tau(x_0 + x_1Z + x_2Z^2 + x_3Z^3) = \tau(x_0)
\]

where \( \tau(x_0) \) is the normalized canonical trace on the rotation algebra \( A_\theta \).
From the character table on page 645 of [12], the range of $T_4$ has the $\mathbb{Z}$-basis vectors

\[
\begin{align*}
\eta_1 &= (\frac{1}{2}; 0, 0; \frac{1}{2}, 0, 0) \\
\eta_2 &= (\frac{1}{2}; \frac{1-i}{4}, 0; 0, 0, 0) \\
\eta_3 &= (\frac{1}{2}; \frac{1+i}{4}, 0; \frac{1}{2}, 0, 0) \\
\eta_4 &= (\frac{1}{2}; 0, 0; \frac{1}{2}, 0, 0) \\
\eta_5 &= (\frac{1}{2}; 0, \frac{1-i}{4}; 0, 0, 0) \\
\eta_6 &= (\frac{1}{2}; 0, \frac{1}{2}; 0) \\
\eta_7 &= (\frac{\theta}{4}, \frac{1+i}{8}, \frac{1+i}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}) \\
\eta_8 &= (\frac{\theta}{4}, -\frac{1+i}{8}, -\frac{1+i}{8}, -\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}) \\
\eta_9 &= (\frac{\theta}{4}, -\frac{1-i}{8}, -\frac{1-i}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}) \\
\end{align*}
\]

Since the compositions of $T_{10}$ and $T_{11}$ with the inclusion $\iota: A_\theta \times \mathbb{Z}_2 \to A_\theta \times \mathbb{Z}_4$ are identically 0, it follows that the induced $K_0$ map $\iota_*$ on all the basis elements $\xi_1, \ldots, \xi_6$ has 0’s in the $T_{10}$ and $T_{11}$ components. Further, from

\[
T_{20}\iota_*(\xi) = \frac{1}{5} \xi_{00}(\xi), \quad T_{21}\iota_*(\xi) = \frac{1}{5} \tau_{11}(\xi), \quad T_{22}\iota_*(\xi) = \frac{1}{5}(\tau_{01} + \tau_{10})(\xi)
\]

for any $K_0$-class $\xi$ in $K_0(A_\theta \times \mathbb{Z}_2)$, we obtain

\[
T_4\iota_*(\xi) = (\tau(\xi); 0, 0; T_{20}(\xi), T_{21}(\xi), T_{22}(\xi)) = (\tau(\xi_j); 0, 0; \frac{1}{5}\tau_{00}(\xi), \frac{1}{5}\tau_{11}(\xi), \frac{1}{5}(\tau_{01} + \tau_{10})(\xi)).
\]

From this we obtain the image of the generators $\xi_j$ in the span of the group generated by the $\eta_k$’s, namely

\[
\begin{align*}
\xi_1’ &= T_4\iota_*(\xi_1) = (1; 0, 0; 0, 0, 0) \\
\xi_2’ &= T_4\iota_*(\xi_2) = (\frac{1}{2}; 0, 0; \frac{1}{2}, 0, 0) = \eta_1 \\
\xi_3’ &= T_4\iota_*(\xi_3) = (\frac{1}{2}; 0, 0; 0, 0, \frac{1}{2}) \\
\xi_4’ &= T_4\iota_*(\xi_4) = (\frac{1}{2}; 0, 0; 0, 0, \frac{1}{2}) \\
\xi_5’ &= T_4\iota_*(\xi_5) = (\frac{1}{2}; 0, 0; 0, \frac{1}{2}, 0) = \eta_4 \\
\xi_6’ &= T_4\iota_*(\xi_6) = (\frac{\theta}{4}; 0, 0; \frac{1}{4}, -\frac{1}{4}, 0).
\end{align*}
\]

Two of the vectors here are the same, $\xi_3’ = \xi_4’$, giving the kernel

\[
\ker(\iota_*) = \mathbb{Z}(\xi_3 - \xi_4),
\]

and the image of $\iota_*$ is generated by the 5 independent vectors associated to $\xi_1’, \xi_2’, \xi_3’, \xi_5’, \xi_6’$ which we need to show are a subset of a basis for the integral span of the $\eta$ vectors in (3.4). Since $\xi_3’ = \eta_1$ and $\xi_5’ = \eta_4$, it remains for us to express the vectors $\xi_1’, \xi_3’, \xi_6’$ as linear combinations of the $\eta$’s. Indeed, one
checks that we have the following integral combinations
\[
\xi_1' = \eta_1 + \eta_2 - 2\eta_3 + \eta_4 + \eta_5 - 2\eta_6 + \eta_7 - \eta_9 \\
\xi_2' = \eta_1 \\
\xi_3' = \eta_1 + \eta_2 - 3\eta_3 + \eta_4 + \eta_5 - 3\eta_6 + 2\eta_7 - \eta_8 - \eta_9 \\
\xi_5' = \eta_4 \\
\xi_6' = \eta_3 - \eta_4 + \eta_6 + \eta_8 + \eta_9.
\]
From these, one obtains a basis for the range of $T_4$ containing the vectors \{$\xi_1', \xi_2', \xi_3', \xi_5', \xi_6'$\} as follows. Using the $\xi_6'$ equation, one solves to eliminate $\eta_9$:
\[
\eta_9 = \xi_6' - \eta_3 + \eta_4 - \eta_6 - \eta_8.
\]
Inserting this into the $\xi_1'$ equation one gets
\[
\xi_1' = \eta_1 + \eta_2 - \eta_3 + \eta_5 - \eta_6 + \eta_7 - \xi_6' + \eta_8
\]
and into the $\xi_3'$ equation to get
\[
\xi_3' = \eta_1 + \eta_2 - 2\eta_3 + \eta_5 - 2\eta_6 + 2\eta_7 - \xi_6'.
\] 
Solving the first of these for $\eta_8$ gives
\[
\eta_8 = \xi_1' + \xi_6' - \eta_1 - \eta_2 + \eta_3 - \eta_5 + \eta_6 - \eta_7
\]
and from (3.5) one could eliminate $\eta_5$ to obtain a basis consisting of the following nine vectors
\[
\eta_1 = \xi_2', \quad \eta_2, \quad \eta_3, \quad \eta_4 = \xi_5', \quad \eta_6, \quad \eta_7, \quad \xi_1', \quad \xi_3', \quad \xi_6'
\]
for $K_0(A_\theta \rtimes \mathbb{Z}_4)$ containing \{$\xi_1', \xi_2', \xi_3', \xi_5', \xi_6'$\}. This shows that the image of the canonical map
\[
\mathbb{Z}^6 = K_0(A_\theta \rtimes \mathbb{Z}_2) \xrightarrow{1_\theta} K_0(A_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^9
\]
is a direct summand of $K_0(A_\theta \rtimes \mathbb{Z}_4)$ isomorphic to $\mathbb{Z}^5$, and its kernel isomorphic to $\mathbb{Z}$, giving the first exact sequence in Theorem 1.4.
Therefore, in the Natsume sequence
\[
\mathbb{Z}^6 = K_0(A_\theta \rtimes \mathbb{Z}_2) \xrightarrow{1_\theta - i_{2*}} \mathbb{Z}^9 \oplus \mathbb{Z}^9 \xrightarrow{j_{1*} + j_{2*}} K_0(A_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^{18}
\]
the map
\[
i_{1*} - i_{2*:} K_0(A_\theta \rtimes \mathbb{Z}_2) \rightarrow K_0(A_\theta \rtimes \mathbb{Z}_4) \oplus K_0(A_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^{18}
\]
has direct summand image, by Lemma 2.1 (as ker $i_{1*}$ = ker $i_{2*}$). Since its rank is $6 - \text{rk}(\ker i_*) = 5$ we obtain
\[
K_0(A_\theta \rtimes \mathbb{Z}_4; 2) = \mathbb{Z}^{18-5} = \mathbb{Z}^{13}, \quad K_1(A_\theta \rtimes \mathbb{Z}_4; 2) = \mathbb{Z}
\]
as in the statement of Theorem 1.3. The $K_1$-group here follows since the kernel of $i_{1*} - i_{2*}$ is the same as the kernel of $i_*$, which is $\mathbb{Z}(\xi_3 - \xi_4) \cong \mathbb{Z}$.

In the remainder of this section we prove the injectivity of the map $i_* : K_0(A_\theta) \rightarrow K_0(A_\theta \rtimes \mathbb{Z}_2)$ as stated in Theorem 1.5. Recall, $K_0(A_\theta) = \mathbb{Z}^2$ is generated
We note that also the map in terms of the basis listed in (3.1),
\[ T_{2i_e}[e_\theta] = (\theta; 0, 0, 0, 0) = 2T_2(\xi_6) - T_2(\xi_2) + T_2(\xi_3) - T_2(\xi_4) + T_2(\xi_5) \]
so that clearly \( i_* \) maps \( K_0(A_\theta) \) to a direct summand of \( K_0(A_\theta \times \mathbb{Z}_2) \) isomorphic to \( \mathbb{Z}^2 \).

We now show that the canonical map \( i_* : K_0(A_\theta) \to K_0(A_\theta \times \mathbb{Z}_2) \) is injective and its image is a direct summand (also part of Theorem [1.5]). For the identity, we have \( T_{4i_e}[e_\theta] = (\theta; 0, 0, 0, 0) \), and for the Rieffel projection we have \( T_{4i_e}[e_\theta] = (\theta; 0, 0, 0, 0) \) (since \( e_\theta \) being in \( A_\theta \) has zero \( \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3 \) components). Therefore, in terms of the \( \eta \)-basis listed in (3.4),
\[ T_{4i_e}[1] = \xi'_1 = \eta_1 + \eta_2 - 2\eta_3 + \eta_4 + \eta_5 - 2\eta_6 + \eta_7 - \eta_9 \]
and
\[ T_{4i_e}[e_\theta] = (\theta; 0, 0, 0, 0) = 2\xi'_6 - (\xi'_2 - \xi'_5) \]
using the above expressions for \( \xi'_2, \xi'_5, \xi'_6 \). The latter can be used to eliminate \( \eta_1 \), and from the former one eliminates \( \eta_2 \) to obtain the basis
\[ T_{4i_e}[1], \; T_{4i_e}[e_\theta], \; \eta_3, \; \eta_4, \; \eta_5, \; \eta_6, \; \eta_7, \; \eta_8, \; \eta_9 \]
so that \( i_* \) maps \( K_0(A_\theta) \) onto a direct summand of \( K_0(A_\theta \times \mathbb{Z}_2) \) isomorphic to \( \mathbb{Z}^2 \).

4. \( K \)-THEORY OF \( A_\theta \times \mathbb{Z}_{4,6;2} \)

We now summarize the analogous framework for the hexic transform \( \rho \). The Chern-Connes character map \( T_6 \) for the crossed product \( A_\theta \times \mathbb{Z}_6 \) is
\[ T_6 : K_0(A_\theta \times \mathbb{Z}_6) \to \mathbb{C}^6, \quad T_6 = (\tau; H_{10}; H_{20}, H_{21}; H_{30}, H_{31}) \]
(see [5]), where, for general elements
\[ x = x_0 + x_1X + x_2X^2 + x_3X^3 + x_4X^4 + x_5X^5 \]
in \( A_\theta \times \mathbb{Z}_6 \), with \( X \) denoting the canonical unitary of \( A_\theta \times \mathbb{Z}_6 \) for \( \rho, \; X^6 = 1 \), and \( x_j \in A_\theta \), one has
\[ H_{jk}(x) = \Psi_{jk}(x_{6-j}) \]
and, in view of Theorem 3.1 [5], the unbounded linear functionals \( \Psi_{jk} \) on \( A_\theta \) are defined by
\begin{align*}
\Psi_{10}(u^{mn}) &= e^{\frac{\theta}{2}(m^2 + n^2)} \quad (4.1) \\
\Psi_{20}(u^{mn}) &= e^{\frac{\theta}{6}(m - n)^2} \delta_3^{m-n} \quad (4.2) \\
\Psi_{21}(u^{mn}) &= e^{\frac{\theta}{6}(m - n)^2} \quad (4.3) \\
\Psi_{30}(u^{mn}) &= e^{-\frac{\theta}{2}mn} \delta_2^m \delta_2^n \quad (4.4) \\
\Psi_{31}(u^{mn}) &= e^{-\frac{\theta}{2}mn} \quad (4.5)
\end{align*}
We note that \( \Psi_{30} = \phi_{00} \) and \( \Psi_{31} = \phi_{00} + \phi_{01} + \phi_{10} + \phi_{11} \) (with \( \phi_{jk} \) given by [3.3]). Also, the map \( H_{10} \) is concentrated on the \( X^5 \) term of \( x \), \( H_{2k} \) is concentrated on
the $x^4$ term, $H_{3k}$ concentrated on the $X^3$, and of course, $H_{jk}$ is 0 elsewhere. The canonical trace $\tau$ is given by

$$\tau(x_0 + x_1X + x_2X^2 + x_3X^3 + x_4X^4 + x_5X^5) = \tau(x_0).$$

From the character table on page 37 of [5], the range of $T_6$ has $\mathbb{Z}$-basis

$$\begin{align*}
\mu_1 &= (1; 0; 0, 0, 0, 0) \\
\mu_2 &= (\frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}; \frac{1}{6}) \\
\mu_3 &= (\frac{1}{6}; -\frac{1}{6}\omega; \frac{\omega-1}{6}; \frac{\omega-1}{6}; \frac{1}{6}; \frac{1}{6}) \\
\mu_4 &= (\frac{1}{6}; -\frac{1}{6}\omega; -\frac{1}{6}\omega; \frac{\omega-1}{6}; \frac{1}{6}; \frac{1}{6}) \\
\mu_5 &= (\frac{1}{6}; -\frac{1}{6}; \frac{1}{6}; \frac{1}{6}; -\frac{1}{6}; \frac{1}{6}) \\
\mu_6 &= (\frac{1}{6}; \frac{\omega-1}{6}; -\frac{1}{6}\omega; -\frac{1}{6}\omega; \frac{1}{6}; \frac{1}{6}) \\
\mu_7 &= (\frac{1}{6}; 0; 0, \frac{1}{3}; 0, 0) \\
\mu_8 &= (\frac{1}{6}; 0; 0, -\frac{1}{3}\omega; 0, 0) \\
\mu_9 &= (\frac{1}{6}; 0; 0, 0, 0, \frac{1}{2}) \\
\mu_{10} &= (\frac{\theta}{6}; \frac{1}{6}\omega; \frac{1+\omega}{18}; \frac{1+\omega}{18}; \frac{1}{12}; \frac{1}{12})
\end{align*}$$

where $\omega := e^{i\pi/3} = \frac{1}{2}(1 + i\sqrt{3})$.

Since the inclusion $t' : A_\theta \times \mathbb{Z}_2 \to A_\theta \times \mathbb{Z}_2$ is given by $t'(x + yW) = x + yX^3$, its composition with $H_{10}, H_{20}, H_{21}$ are 0. It follows that the induced $K_0$ map $t'_*'$ on all the basis elements $\xi_1, \ldots, \xi_6$ of $K_0(A_\theta \times \mathbb{Z}_2)$ has 0’s in the $H_{10}, H_{20}, H_{21}$ components. Further, from

$$H_{30}t' = \frac{1}{2}\tau_{00}, \quad H_{31}t' = \frac{1}{4}(\tau_{00} + \tau_{11} + \tau_{01} + \tau_{10}),$$

(where the $\tau_{jk}$ are given by [3.2]), we obtain, for any $K$-class $\xi$ in $K_0(A_\theta \times \mathbb{Z}_2)$,

$$T_6t'_*(\xi) = (\tau(\xi); 0; 0, 0; H_{30}(\xi), H_{31}(\xi))$$

$$= (\tau(\xi); 0; 0, 0; \frac{1}{4}\tau_{00}(\xi), \frac{1}{4}(\tau_{00} + \tau_{11} + \tau_{01} + \tau_{10})(\xi)) .$$

From this we get the corresponding images of the generators $\xi_j$ in the span of the group generated by the $\mu_k$’s as follows:

$$\begin{align*}
\xi_1'' &= T_6t'_*(\xi_1) = (1; 0, 0, 0, 0) = \mu_1 \\
\xi_2'' &= T_6t'_*(\xi_2) = (\frac{1}{2}; 0; 0, 0, \frac{1}{2}, \frac{1}{2}) \\
\xi_3'' &= T_6t'_*(\xi_3) = (\frac{1}{2}; 0; 0, 0, 0, \frac{1}{2}) = \mu_9 \\
\xi_4'' &= T_6t'_*(\xi_4) = (\frac{1}{2}; 0; 0, 0, 0, \frac{1}{2}) = \mu_9 \\
\xi_5'' &= T_6t'_*(\xi_5) = (\frac{1}{2}; 0; 0, 0, 0, \frac{1}{2}) = \mu_9 \\
\xi_6'' &= T_6t'_*(\xi_6) = (\frac{\theta}{6}; 0; 0, 0; \frac{1}{4}, 0)
\end{align*}$$

Note that now three of these vectors are the same as $\mu_9$, so that they give 4 independent vectors

$$\xi_1'' = \mu_1, \quad \xi_2'' = \mu_9, \quad \xi_3'' = \mu_9, \quad \xi_6'' = \mu_9.$$
spanning the image of $T_6t'_s$. We want to show that these 4 vectors are contained inside some basis for the integral span of the $\mu$ vectors in (4.6). It is easy to check that
\begin{align*}
\xi_2'' & = \mu_2 + \mu_4 + \mu_6 \\
\xi_6'' & = 2\mu_3 + 2\mu_4 + \mu_5 + \mu_6 - \mu_7 + \mu_8 - 2\mu_9 + 3\mu_{10}
\end{align*}
from which it is now clear that the induced map

$$t'_s : K_0(A_\theta \times \mathbb{Z}_2) \to K_0(A_\theta \times \mathbb{Z}_6) = \mathbb{Z}^{10}$$

has kernel

$$\ker(t'_s) = \mathbb{Z}(\xi_3 - \xi_4) + \mathbb{Z}(\xi_4 - \xi_5) \cong \mathbb{Z}^2.$$ 

In addition, this shows that the image of $t'_s$ is a direct summand isomorphic to $\mathbb{Z}^4$. This establishes the exact sequence in Theorem 1.4 associated to $t'_s$. Therefore, in the Natsume sequence

$$\mathbb{Z}^6 = K_0(A_\theta \times \mathbb{Z}_2) \xrightarrow{t_s - t'_s} \mathbb{Z}^9 \oplus \mathbb{Z}^{10} \xrightarrow{j_1 + j_2} K_0(A_\theta \times_\theta \mathbb{Z}_{4,6;2}) \oplus 0 \xrightarrow{0} 0$$

the map $t_s - t'_s$ has, by Lemma 2.1, direct summand image (the condition $\ker t_s \subset \ker t'_s$ of the lemma being met). Its rank is $6 - \text{rk}(\ker t_s) = 5$ so therefore obtain

$$K_0(A_\theta \times \mathbb{Z}_{4,6;2}) = \mathbb{Z}^{9+10-5} = \mathbb{Z}^{14}, \quad K_1(A_\theta \times \mathbb{Z}_{4,6;2}) = \mathbb{Z}$$

as in the statement of Theorem 1.3, as $\ker(t_s - t'_s) \cong \mathbb{Z}$.

We now check that the canonical map $i_* : K_0(A_\theta) \to K_0(A_\theta \times \mathbb{Z}_6)$ is injective and its image is a direct summand. Recall, $K_0(A_\theta) = \mathbb{Z}^2$ is generated by the classes of the identity 1 and a Rieffel projection $e_\theta$ of trace $\theta$. For the identity, we have $T_6i_*[1] = (1; 0; 0; 0; 0; 0)$ and for the Rieffel projection we have $T_6i_*[e_\theta] = (\theta; 0; 0; 0; 0; 0)$ (since $e_\theta$ being in $A_\theta$ has zero $X^j$-components). Therefore, in terms of the $\mu$-basis listed in (4.6),

$$T_6i_*[1] = \mu_1$$

and

$$T_6i_*[e_\theta] = (\theta; 0; 0; 0; 0; 0) = 2\xi_6'' - (\xi_2'' - \mu_9)$$

$$= -\mu_2 + 4\mu_3 + 3\mu_4 + 2\mu_5 + \mu_6 - 2\mu_7 + 2\mu_8 - 3\mu_9 + 6\mu_{10}$$

first by using (4.7) and then using (4.8). The latter can be used to eliminate $\mu_2$ to get the basis

$$\mu_1 = T_6i_*[1], \quad T_6i_*[e_\theta], \quad \mu_3, \quad \mu_4, \quad \mu_5, \quad \mu_6, \quad \mu_7, \quad \mu_8, \quad \mu_9, \quad \mu_{10},$$

so that $i_*$ maps $K_0(A_\theta)$ onto a direct summand of $K_0(A_\theta \times \mathbb{Z}_6)$ isomorphic to $\mathbb{Z}^2$. 

5. $K$-THEORY OF $A_\theta \rtimes \mathbb{Z}_{6,6;2}$

In the current case, again using Lemma 2.1, the Natsume sequence is

$$Z^6 = K_0(A_\theta \rtimes \mathbb{Z}_2) \xrightarrow{t'_* - t'_*} Z^{10} \oplus Z^{10} \xrightarrow{j_{1*} + j_{2*}} K_0(A_\theta \rtimes Y \mathbb{Z}_{6,6;2})$$

where the inclusion maps $t'_*, t'_*$ into each summand were calculated in the previous section. Since the kernel of $t'_* - t'_*$ is the same as the kernel of $t'_*$, which is isomorphic to $\mathbb{Z}^2$, the rank of $t'_* - t'_*$ is 4, whence

$$K_0(A_\theta \rtimes \mathbb{Z}_{6,6;2}) = Z^{10+10-4} = Z^{16}, \quad K_1(A_\theta \rtimes \mathbb{Z}_{6,6;2}) = Z^2$$

as in Theorem 1.3.

6. $K$-THEORY OF $A_\theta \rtimes \mathbb{Z}_{6,6;3}$

Here we will need to calculate the $K_0$-map

$$\kappa_* : Z^8 = K_0(A_\theta \rtimes \mathbb{Z}_3) \to K_0(A_\theta \rtimes \mathbb{Z}_6) = Z^{10}$$

induced by the canonical inclusion $\kappa : A_\theta \rtimes \mathbb{Z}_3 \to A_\theta \rtimes \mathbb{Z}_6$ given by

$$\kappa(x_0 + x_1Y + x_2Y^2) = x_0 + x_1X^2 + x_2X^4$$

where $Y$ is the canonical unitary of the crossed product $A_\theta \rtimes \mathbb{Z}_3$ (with $Y^3 = I$), and $x_j \in A_\theta$. (As above, $X^6 = I$.)

From Section 4 of [5], the Chern-Connes character map $T_3$ takes the form

$$T_3 : K_0(A_\theta \rtimes \mathbb{Z}_3) \to \mathbb{C}^4, \quad T_3 = (\tau; S_{10}, S_{11}, S_{12})$$

where

$$S_{1k}(x_0 + x_1Y + x_2Y^2) = \Phi_{1k}(x_2)$$

($k = 0, 1, 2$) and, in view of Theorem 3.3 [5], the unbounded functionals $\Phi_{1k}$ on $A_\theta$ are defined by

$$\Phi_{10}(U^mV^n) = e(\theta)(m-n)^2)3^{m-n} \quad (6.1)$$

$$\Phi_{11}(U^mV^n) = e(\theta)(m-n)^2)3^{m-n-1} \quad (6.2)$$

$$\Phi_{12}(U^mV^n) = e(\theta)(m-n)^2)3^{m-n-2}. \quad (6.3)$$

Further, the canonical trace on the cubic crossed product $A_\theta \rtimes \mathbb{Z}_3$ is, as before, $\tau(x_0 + x_1Y + x_2Y^2) = \tau(x_0)$. Observe that the $\Phi$ and $\Psi$ maps are related by

$$\Phi_{10} = \Psi_{20}, \quad \Phi_{10} + \Phi_{11} + \Phi_{12} = \Psi_{21}.$$
From the character table on page 37 of [5], the range of $T_3$ has the following vectors as a $\mathbb{Z}$-basis

\[
\begin{align*}
\lambda_1 &= (1; 0,0,0) \\
\lambda_2 &= (\frac{1}{2}; \frac{1}{3},0,0) \\
\lambda_3 &= (\frac{1}{2}; -\frac{\omega}{3},0,0) \\
\lambda_4 &= (\frac{1}{4}; 0,\frac{1}{3},0) \\
\lambda_5 &= (\frac{1}{4}; 0,-\frac{\omega}{3},0) \\
\lambda_6 &= (\frac{1}{4}; 0,0,\frac{1}{3}) \\
\lambda_7 &= (\frac{1}{4}; 0,0,-\frac{\omega}{3}) \\
\lambda_8 &= (\frac{1}{4}; \frac{1}{9},\frac{1+\omega}{9},\frac{1+\omega}{9})
\end{align*}
\]  

(6.4)

where $\omega := e^{\frac{1}{3}(1+i\sqrt{3})}$ as before.

Since the maps $H_{10}, H_{30}, H_{31}$ are clearly zero on the range of the inclusion $\kappa$, it follows that the induced $K_0$ map $\kappa_*$ on all the basis elements $\lambda_1, \ldots, \lambda_8$ of $K_0(A_0 \rtimes \mathbb{Z}_3)$ has O’s in the $H_{10}, H_{30}, H_{31}$ components. Further, from

\[
H_{20} \kappa = S_{10}, \quad H_{21} \kappa = S_{10} + S_{11} + S_{12},
\]

we obtain (for any $K$-class $\xi$)

\[
\begin{align*}
T_6\kappa_*(\xi) &= (\tau(\xi); 0; H_{20}(\kappa\xi), H_{21}(\kappa\xi); 0,0) \\
&= (\tau(\xi); 0; S_{10}(\xi), (S_{10} + S_{11} + S_{12})(\xi); 0,0).
\end{align*}
\]

From this we get the image of the generators $\lambda_j$ in the span of the group generated by the $\mu_k$’s as follows:

\[
\begin{align*}
\lambda_1' &= T_6\kappa_*(\lambda_1) = (1; 0,0,0,0,0) = \mu_1 \\
\lambda_2' &= T_6\kappa_*(\lambda_2) = (\frac{1}{4}; 0,\frac{1}{3},\frac{1}{3},0,0) = \mu_2 + \mu_5 \\
\lambda_3' &= T_6\kappa_*(\lambda_3) = (\frac{1}{4}; 0,-\frac{\omega}{3},-\frac{\omega}{3},0,0) = \mu_3 + \mu_6 \\
\lambda_4' &= T_6\kappa_*(\lambda_4) = (\frac{1}{4}; 0,0,\frac{1}{3},0,0) = \mu_7 \\
\lambda_5' &= T_6\kappa_*(\lambda_5) = (\frac{1}{4}; 0,0,-\frac{\omega}{3},0,0) = \mu_8 \\
\lambda_6' &= T_6\kappa_*(\lambda_6) = (\frac{1}{4}; 0,0,\frac{1}{3},0,0) = \mu_7 \\
\lambda_7' &= T_6\kappa_*(\lambda_7) = (\frac{1}{4}; 0,0,-\frac{\omega}{3},0,0) = \mu_8 \\
\lambda_8' &= T_6\kappa_*(\lambda_8) = (\frac{1}{4}; 0,\frac{1+\omega}{9},\frac{1+\omega}{9},0,0).
\end{align*}
\]

Here we see that two pairs are the equal $\lambda_4' = \lambda_6' = \mu_7$ and $\lambda_5' = \lambda_7' = \mu_8$. Further, it can be checked that

\[
\lambda_8' = \mu_3 + \mu_4 + \mu_5 - \mu_9 + 2\mu_{10}
\]  

(6.5)

which means that the vectors

\[
\lambda_1', \lambda_2', \lambda_3', \lambda_4', \lambda_5', \lambda_6', \lambda_8'
\]  

(6.6)

form a basis for the image of $T_6\kappa_*$. (One can easily check that they are integrally independent.) Replacing $\mu_4$ using (6.5), $\mu_5 = \lambda_2' - \mu_2$, and $\mu_6 = \lambda_3' - \mu_3$,.
the 10 vectors
\[
\mu_1 = \lambda'_1, \quad \mu_2, \quad \mu_3, \quad \lambda'_2, \quad \lambda'_3, \quad \mu_7 = \lambda'_4, \quad \mu_8 = \lambda'_5, \quad \lambda'_6, \quad \mu_9, \quad \mu_{10}
\]
constitute a basis for \( K_0(A_\theta \times \mathbb{Z}_6) \) containing the basis \([6.6]\) for the image of \( T_6 \kappa_* \). Therefore, it follows that the image of \( \kappa_* \) is a direct summand of \( K_0(A_\theta \times \mathbb{Z}_6) = \mathbb{Z}^{10} \) isomorphic to \( \mathbb{Z}^6 \), and its kernel is
\[
\ker(\kappa_*) = \mathbb{Z}(\lambda_4 - \lambda_6) + \mathbb{Z}(\lambda_5 - \lambda_7) \cong \mathbb{Z}^2.
\]
This establishes the exact sequence in Theorem [1.4] related to \( \kappa_* \). The Natsume sequence in this case then becomes
\[
\mathbb{Z}^8 = K_0(A_\theta \times \mathbb{Z}_3) \xrightarrow{\kappa_*} \mathbb{Z}^{10} \oplus \mathbb{Z}^{10} \xrightarrow{j_1 + j_2} K_0(A_\theta \times T\mathbb{Z}_{6,6,3})
\]
giving, again using Lemma [2.1],
\[
K_0(A_\theta \times \mathbb{Z}_{6,6,3}) = \mathbb{Z}^{10+10-6} = \mathbb{Z}^{14}, \quad K_1(A_\theta \times \mathbb{Z}_{6,6,3}) = \mathbb{Z}^2.
\]
This proves the corresponding part of Theorem [1.3].

We now check that the canonical map \( i_* : K_0(A_\theta) \rightarrow K_0(A_\theta \times \mathbb{Z}_3) \) is injective and its image is a direct summand. Recall, \( K_0(A_\theta) = \mathbb{Z}^2 \) is generated by the classes of the identity 1 and a Rieffel projection \( e_\theta \) of trace \( \theta \). We have \( T_3 i_*[1] = (1;0,0,0) \) and for the Rieffel projection we have \( T_3 i_*[e_\theta] = (\theta;0,0,0) \) (since \( e_\theta \) being in \( A_\theta \) has zero \( Y \) and \( Y^2 \) components). Therefore, in terms of the \( \lambda \)-basis listed in \([6.4]\),
\[
T_3 i_*[1] = (1;0,0,0) = \lambda_1
\]
and it is straightforward to verify that
\[
T_3 i_*[e_\theta] = (\theta;0,0,0) = 3 \lambda_8 - (\lambda_2 - \lambda_3) - (\lambda_4 - \lambda_5) - (\lambda_6 - \lambda_7).
\]
The latter can be used to eliminate \( \lambda_2 \) to get the basis
\[
\lambda_1 = T_3 i_*[1], \quad \lambda_3, \quad \lambda_4, \quad \lambda_5, \quad \lambda_6, \quad \lambda_7, \quad \lambda_8,
\]
so that \( i_* \) maps \( K_0(A_\theta) \) injectively onto a direct summand of \( K_0(A_\theta \times \mathbb{Z}_3) \).

Acknowledgements. Thank you, Jesus!

References

[1] B. Blackadar, K-theory for Operator Algebras, Cambridge Univ. Press, MSRI pubs 5 (2nd ed. (1998).
[2] O. Bratteli, G. A. Elliott, D. E. Evans, A. Kishimoto, Non-commutative spheres I, Internat. J. Math. 2 (1999), no. 2, 139–166.
[3] O. Bratteli, G. A. Elliott, D. E. Evans, A. Kishimoto, Non-commutative spheres II: rational rotation, J. Operator Theory 27 (1992), 53–85.
[4] O. Bratteli and A. Kishimoto, Non-commutative spheres III. Irrational Rotations, Comm. Math. Phys. 147 (1992), 605–624.
[5] J. Buck and S. Walters, Connes-Chern characters of hexit and cubic modules, J. Operator Theory 57 (2007), 35–65.
[6] J. Cuntz, K-theoretic amenability for discrete groups, J. Reine Angew. Math. 344 (1983), 180–195.
[7] S. Echterhoff, W. Lück, N. C. Phillips, S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of SL_2(Z)*, J. Reine Angew. Math. (Crelle’s Journal) **639** (2010), 173-221.

[8] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics (2000).

[9] A. Kumjian, *On the K-theory of the symmetrized non-commutative torus*, C. R. Math. Rep. Acad. Sci. Canada **12**, No. 3 (1990), 87–89.

[10] T. Natsume, *On K₀(C*(SL_2(Z))),* J. Operator Theory **13** (1985), 103–118.

[11] S. G. Walters, *Projective modules over the non-commutative sphere*, J. London Math. Soc. **51**, No. 2 (1995), 589–602.

[12] S. G. Walters, *Chern characters of Fourier modules*, Canad. J. Math. **52**, No. 3 (2000), 633–672.

[13] S. G. Walters, *K-theory of non commutative spheres arising from the Fourier automorphism*, Canad. J. Math. **53**, No. 3 (2001), 631–672.

[14] S. Walters, *The AF structure of non commutative toroidal Z/4Z orbifolds*, J. Reine Angew. Math. (Crelle’s Journal) **568** (2004), 139–196; arXiv: math.OA/0207239

[15] S. Walters, *Toroidal orbifolds of Z₃ and Z₆ symmetries of noncommutative tori*, Nuclear Physics B **894** (2015), 496-526; [http://dx.doi.org/10.1016/j.nuclphysb.2015.03.008](http://dx.doi.org/10.1016/j.nuclphysb.2015.03.008)

**Department of Mathematics & Statistics, University of Northern B.C., Prince George, B.C. V2N 4Z9, Canada.**

_E-mail address:_ walters@unbc.ca

_URL:_ [http://hilbert.unbc.ca](http://hilbert.unbc.ca)