A toy model for low-energy nuclear fusion

K RAMKUMAR¹, HARISH YAM KUMAR¹ and PANKAJ JAIN²∗

¹Department of Physics, Indian Institute of Technology, Kanpur 208 016, India
²Department of Space Science and Astronomy, Indian Institute of Technology, Kanpur 208 016, India
∗Corresponding author. E-mail: pkjain@iitk.ac.in

MS received 1 November 2022; revised 13 February 2023; accepted 15 March 2023

Abstract. We study the fusion of a proton with a nucleus with the emission of two photons at low incident energy of the order of eV or smaller. We use a step model for the repulsive potential between the proton and the nuclei. We consider the reaction both in free space and inside a medium. We make a simple model for the medium by assuming a hard wall potential beyond a certain length scale. This essentially leads to discretisation of the energy spectrum which is expected inside a medium and is seen both for a crystalline lattice structure and for amorphous materials. We use second-order perturbation theory to compute the transition rate. We find that the rate in free space is very small. However, in the medium, the rate may be substantial. Hence, we conclude that nuclear fusion reactions may take place at low energies at observable rates.

Keywords. Low-energy nuclear fusion; photon emission; nuclear fusion in medium.

PACS Nos 25.20.−x; 24.10.−i; 25.10.+s

1. Introduction

The nuclear fusion processes are expected to be strongly suppressed at low energies [1]. There have been experimental claims that such processes may be occurring at observable rates in a medium, see for example [2–5]. However, despite considerable effort [6–10], so far there does not exist any reliable theoretical model of how this can happen. A useful review of the shortcomings of a wide range of theoretical proposals is given in [11].

In [12,13], the authors have explored the possibility that low-energy fusion may arise at second order in perturbation theory. A similar idea has also been proposed in [14]. At this order, we need to sum over all the intermediate state energies and hence the Coulomb barrier may not be very prohibitive. The process considered in [12,13] involves two interactions or vertices and involves emission of two photons, one at each vertex. The process may be expressed as

\[^1\text{H} + ^A\text{X} \rightarrow ^{A+1}\text{Y} + \gamma(\omega_1) + \gamma(\omega_2),\]  

(1)

where \(^1\text{H}\) denotes hydrogen, \(^A\text{X}\) a nucleus with atomic number \(Z\) and mass number \(A\) and \(^{A+1}\text{Y}\) a nucleus with atomic number \(Z + 1\) and mass number \(A + 1\). The emitted photons have frequencies \(\omega_1\) and \(\omega_2\). This may be compared to the related first-order process,

\[^1\text{H} + ^A\text{X} \rightarrow ^{A+1}\text{Y} + \gamma(\omega),\]  

(2)

which is the standard fusion process with the emission of a photon.

At second order in perturbation theory, the two-photon emission process, eq. (1) gets contributions from two amplitudes which are shown in figure 1. At the first interaction or vertex, the proton or the \(X\) nucleus emits a photon forming an intermediate state consisting of a proton and \(X\). We work in the centre of mass and relative coordinates and only the relative coordinates are relevant [12,13]. In the intermediate state, we need to sum over states of all energy, without imposing energy conservation at either of the two vertices. Of course, the total energy has to be conserved. At the second vertex, the proton gets captured by the nucleus \(X\) with the emission of another photon. The process, therefore, involves two matrix elements, one for each vertex. In earlier papers [12,13], the authors refer to the first matrix element as the molecular matrix element since it gets dominant contributions from distances of order 1 a.u., while the second matrix elements gets contributions dominantly from nuclear distances and is called nuclear matrix element.

The important point is that we need to sum over intermediate states of all energies. Since the initial state has very small energy \(E_i\) and momentum, the molecular
The matrix element is appreciable only when the energy of the intermediate state $E_n$ is such that the corresponding momentum $P_n$ closely balances the photon momentum $P_γ$. One might expect that the dominant contribution to the entire amplitude would come from $P_n \approx P_γ$ with the corresponding energy $E_n \gg E_i$, thereby leading to a rather large amplitude. However, explicit calculations in [12,13] show that this fails. The problem is that here we are dealing with particles in a potential and hence the energy eigenstates are not eigenstates of momentum. Due to this, the molecular matrix element does not select a unique value of $P_n$ and a large range of values of $P_n = |P_n|$ contribute. Explicit calculations show that these cancel among one another leading to a very small amplitude.

In [12,13], the authors suggested several possibilities which might evade the acute cancellation described above. One suggestion was that in a medium, the energy eigenvalues are discrete and do not form a continuous set as in free space. In such a case, the cancellation may not be complete and we may get a substantial contribution. In the present paper, we test this possibility in detail assuming a step function potential for the tunnelling barrier. This captures the essential details of the process while facilitating mathematical calculations. Calculation with Coulomb is postponed to future work. Another suggestion in [12,13] was the presence of a resonant nuclear state at energy $E_R \gg E_i$. In this case, the dominant contribution will arise from energies very close to $E_R$ which may not cancel out. We briefly comment on this possibility also in the present paper.

We also point out that in our paper we have confined ourselves to one process to demonstrate that low-energy nuclear reactions are possible. There may be other processes which may not involve emission of photons and might proceed at higher rates. In future, this needs to be studied in detail. While demonstrating the theoretical possibility of such reactions to occur at observable rates, our work paves the way for studying other related processes.

2. Potential model and wave functions

In this section, we give the potential model used for the calculations and the corresponding energy eigenstates.

As mentioned in the Introduction, we use a step function for the potential barrier instead of the standard Coulomb potential. This has the advantage that the wave functions can be computed analytically.

We assume a spherically symmetric potential which can be written as

$$V(r) = \begin{cases} -V_0, & r < L_n, \\ V_1, & L_n \leq r \leq L_b, \\ 0, & L_c \geq r > L_b, \\ V_2, & r > L_c, \end{cases}$$

where $V_0$ is the nuclear potential, $V_1$ represents our model for the tunnelling barrier and we shall take $V_2$ to be infinitely large. The potential is shown schematically in figure 2. A possible set of parameters are $V_0 = 50$ MeV, $V_1 = 100$ in atomic units (Hartree), the nuclear length scale $L_n = 0.9566 \times 10^{-4}$ a.u. and the barrier length scale $L_b = 0.1$ a.u. We point out that 1 Hartree $\sim 27.2$ eV and the atomic unit for length is Bohr radius. The cut-off length scale $L_c$ is not shown in figure 2. We assume that the potential $V_2$ beyond this point is infinitely large. Hence, all the wave functions are set to zero at this radius.

The potential model represents a mathematically well-defined quantum system. It also approximately describes a physical situation in which we have a small
spherically symmetric cavity in a solid material composed of element with large atomic mass. In the cavity, we assume the presence of a gas composed of hydrogen and a relatively heavier element X. We are considering fusion of hydrogen with X. The effect of solid matter is to simply discretise the energy eigenvalues of the H–X system. Each lattice site of the solid matter presents a very high potential barrier to H–X fusion of hydrogen with X. The effect of solid matter is posed of element with large atomic mass. In the cavity, we assume the presence of a gas composed of hydrogen and a relatively heavier element X. We are considering the presence of a gas composed of hydrogen and a relatively heavier element X.

For simplicity, we assume that this barrier height is very high potential barrier to H–X. The effect of solid matter is to simply discretise the energy eigenvalues of the H–X system. Each lattice site of the solid matter presents a very high potential barrier to H–X fusion of hydrogen with X. The effect of solid matter is posed of element with large atomic mass. In the cavity, we assume the presence of a gas composed of hydrogen and a relatively heavier element X. We are considering the presence of a gas composed of hydrogen and a relatively heavier element X.

Pramana – J. Phys. (2023) 97:109 Page 3 of 9 109

\[ U(r) = 0, \quad \text{for } r > L_c \]

where

\[ K = \sqrt{\frac{2mE}{\hbar^2}}, \]

\[ K_1 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}, \quad K_2 = \sqrt{\frac{2m(V_1 - E)}{\hbar^2}}. \]

\[ N'_L = KN_L, \quad N_L = \sqrt{D_L^2 + F_L^2}, \]

\[ B_L = \frac{b}{2K_2e^{-\tilde{K}_2L_n}}, \]

\[ C_L = \frac{c}{2K_2e^{\tilde{K}_2L_n}}, \]

\[ b = -K_1\cos(K_1L_n) + K_2\sin(K_1L_n), \]

\[ c = K_1\cos(K_1L_n) + K_2\sin(K_1L_n) \]

and

\[ D_L = (B_Le^{-\tilde{K}_2L_b} + Ce^{\tilde{K}_2L_b})\sin(KL_b) \]

\[ -\frac{K_2}{K}(B_Le^{-\tilde{K}_2L_b} - C_Le^{\tilde{K}_2L_b})\cos(KL_b) \]

\[ F_L = (B_Le^{-\tilde{K}_2L_b} + C_Le^{\tilde{K}_2L_b})\cos(KL_b) \]

\[ +\frac{K_2}{K}(B_Le^{-\tilde{K}_2L_b} - C_Le^{\tilde{K}_2L_b})\sin(KL_b). \]

Furthermore, we have

\[ D_L\sin(KL_c) + F_L\cos(KL_c) = 0. \]

Due to the boundary condition at \( L_c \) we get discrete energy eigenvalues and each eigenfunction is normalised to unity.

For \( E \geq V_1 \), the eigenfunctions for \( r < L_n \) and \( r > L_b \) have the same form as for \( E < V_1 \) with \( N' \), \( D_L \) and \( F_L \) replaced by \( N' \), \( D_H \) and \( F_H \), respectively. In the region \( L_n \leq r \leq L_b \), it takes the form,

\[ U(r) = \frac{1}{N_L'} [B_H\sin(\tilde{K}_2r) + C_H\cos(\tilde{K}_2r)], \]

with

\[ \tilde{K}_2 = \sqrt{\frac{2m(E - V_1)}{\hbar^2}}. \]

The coefficients are given by

\[ B_H = \frac{K_1}{\tilde{K}_2} \cos(K_1L_n)\cos(\tilde{K}_2L_n) \]

\[ + \sin(K_1L_n)\sin(\tilde{K}_2L_n) \]

\[ C_H = \frac{K_1}{\tilde{K}_2} \sin(K_1L_n)\cos(\tilde{K}_2L_n) \]

\[ - \cos(K_1L_n)\sin(\tilde{K}_2L_n). \]

\[ \tilde{U}(r) = \frac{U(r)}{r}. \]

where \( Y_0^0 = 1/\sqrt{4\pi} \). For energy eigenvalue \( E < V_1 \), we obtain

\[ U(r) = \frac{1}{N_L'} \sin(K_1r), \quad \text{for } r < L_n \]

\[ U(r) = \frac{1}{N_L'} \left[ B_Le^{-\tilde{K}_2r} + C_Le^{\tilde{K}_2r} \right], \]

\[ \text{for } L_n \leq r \leq L_b \]

\[ U(r) = \frac{1}{N_L} \left[ D_L\sin(Kr) + F_L\cos(Kr) \right], \]

\[ \text{for } L_b < r \leq L_c \]
where \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are the kinetic energies of the \(^1\text{H}\) and the \(^A\text{X}\) nucleus, respectively, and \( V(r) \) is the potential given in eq. (3). Here, we focus on the relative motion relevant for the fusion process. The relative coordinate is denoted by \( r \), such that \( r = r_2 - r_1 \), where \( r_1 \) and \( r_2 \) are the coordinates of the \(^1\text{H}\) and \(^A\text{X}\) nucleus. The potential is assumed to be spherically symmetric and depends only on the magnitude \( r = |r| \). We express the kinetic energies in terms of the centre of mass and relative momenta and focus on the relative momentum.

The perturbation \( H_1 \) can be expressed as [18,19]

\[
H_1(t) = -\frac{Z_1e}{cm_1}A(r_1,t) \cdot p_1 - \frac{Z_2e}{cm_2}A(r_2,t) \cdot p_2
\]

\[
+ \frac{ehg_p}{2m_1c} \sigma \cdot B + \cdots ,
\]

where \( Z_1, m_1 \) and \( p_1 \) are the atomic number, mass and momentum of \(^1\text{H}\). The corresponding quantities for \(^A\text{X}\) are \( Z_2, m_2 \) and \( p_2 \). Furthermore, \( \sigma_i \) are the Pauli matrices, \( g_p \) is the proton g factor, \( A \) is the vector potential,

\[
A(r,t) = \frac{1}{\sqrt{\Omega}} \sum_k \sum_\beta c_v \sqrt{\frac{\hbar}{2\omega}} \left[ a_{k,\beta}(t) \epsilon_\beta e^{ik \cdot r} + a_{k,\beta}^*(t) \epsilon_\beta^* e^{-ik \cdot r} \right]
\]

and \( B = V \times A \) is the magnetic field. The vector potential is expressed in terms of the photon polarisation vector \( \epsilon_\beta \), the wave vector \( k \), the frequency \( \omega \) and the total volume \( \Omega \).

We next compute the rate for \( l = 0 \) to \( l = 1 \) transition at first order in perturbation theory. We consider a particular process in which the initial state proton is in the spin-up state. The final state is taken to be \( j = 3/2 \) and \( j_c = 3/2 \). The corresponding rate is given by

\[
\Gamma_1 = \frac{4\alpha\xi^2 E_p^3}{3\hbar^3 c^2} |\langle \psi_f | r | \psi_i \rangle|^2,
\]

where \( E_p \) is the energy of the photon emitted as given by the conservation of energy,

\[
E_p = E_f - E_i
\]

and \( \xi = (Z_1 m_2 - Z_2 m_1) / (m_1 + m_2) \). We evaluate the reaction rate by setting the cut-off length scale \( L_c = 10 \) atomic units. For such a length scale, we find an eigenstate with energy equal to \( E_i = 0.0903 \) eV. Furthermore, we set the parameter \( \xi = 1 \). A more realistic value of this parameter is about 1/2. However, we are only interested in approximate values and such changes of order unity make no difference in our conclusions. Furthermore, we are only interested in comparison between the processes shown in eqs (1) and (2). Since the same parameter appears in both, the ratio of the two rates is
two photons. This gets dominant contribution at second order of eq. (19) contributing at the first interaction and the transition with the first two terms on the right-hand side we take the initial state to be
\begin{equation}
|\psi_i\rangle = \frac{i e \hbar g_{\mu k}}{2 m_1 \sqrt{\Omega_2}} \chi_1 |k\rangle e^{-i k \cdot r_1 i \omega_1 t} |i\rangle.
\end{equation}
We choose the quantisation axis to be the z-axis. The matrix element at the first vertex is given by
\begin{equation}
\langle n | H_I(t) | i \rangle = \frac{i e \hbar g_{\mu k}}{2 m_1 \sqrt{\Omega_2}} \chi_1 |k\rangle e^{-i k \cdot r_1 i \omega_1 t} |i\rangle.
\end{equation}
The centre of mass dependence of the initial and intermediate functions can be expressed as $\chi_i \propto \exp(i k_i \cdot R_{cm})$ and $\chi_n \propto \exp(i k_0 \cdot R_{cm})$, respectively. Hence, the centre of mass integral in the matrix element simply leads to a delta function $\delta(K_m + k_1 - K_i)$. The relative coordinate dependence of the exponent in eq. (23) is given by $k_1 \cdot r_{M} / M$ which is approximately equal to $k_1 \cdot r$ in the limit $M \gg m_1$. Let us express the unit vector $\hat{k}$ as
\begin{equation}
\hat{k} = \cos \theta \hat{z} + \sin \theta \left[ \cos \phi \hat{x} + \sin \phi \hat{y} \right].
\end{equation}
We take the two-photon polarisation vectors as
\begin{align*}
\hat{\epsilon}_1 &= - \sin \theta \hat{z} + \cos \theta \left[ \cos \phi \hat{x} + \sin \phi \hat{y} \right] \\
\hat{\epsilon}_2 &= - \sin \phi \hat{x} + \cos \phi \hat{y}.
\end{align*}
We consider the contribution only from $\hat{\epsilon}_1$. This is sufficient for our purpose since the contribution from $\hat{\epsilon}_2$ will add incoherently and can only change the result by a factor of order unity. In any case, the other polarisation does not contribute to the particular transition being considered. Using $\hat{k} \times \hat{\epsilon}_1 = \hat{\epsilon}_2$, we obtain
\begin{equation}
s \cdot (\hat{k} \times \hat{\epsilon}_1) = \cos \phi_1 s_y - \sin \phi_1 s_x,
\end{equation}
where $s = \sigma / 2$.
The matrix element in the right-hand side of eq. (23) can now be written as
\begin{equation}
\langle n | \sigma \cdot (\hat{k} \times \hat{\epsilon}^*) a^\dagger e^{-i k \cdot r + i \omega_1 t} |i\rangle.
\end{equation}
\[ \langle n | e^{-i \mathbf{k}_1 \cdot \mathbf{r} + i \omega t} | i \rangle = (i \cos \phi_1 - \sin \phi_1) \mathcal{M}_{cm}, \]  
\tag{28}

where the spin part has been evaluated and \( \mathcal{M}_{cm} = \delta^3(\mathbf{K}_m + \mathbf{k}_1 - \mathbf{K}_i) / \Omega \) represents the contribution due to the integral over the centre of mass coordinate. In the spatial part of the matrix element, only the \( l = 0 \) term of the photon wave function contribute. Hence, using plane-wave expansion and doing the angular integral, we obtain
\[ \langle n | e^{-i \mathbf{k}_1 \cdot \mathbf{r} + i \omega t} | i \rangle = \epsilon e^{i \omega t} \int dr \frac{\sin(k_1 r)}{k_1 r} U_i. \]  
\tag{29}

In this evaluation, we have assumed that the photon with frequency \( \omega_1 \) is emitted at the first vertex. As shown in figure 1, there are two contributions and either of the photons can be emitted at the first vertex. However, for the range of parameters considered, the second contribution corresponding to the photon of frequency \( \omega_2 \) emitted at the first vertex turns out to be much smaller. This is because of our choice of \( k_1 \) and \( k_2 \) values and is discussed later in §4.

We next consider the matrix element from intermediate to final state. The emitted photon has wave vector \( \mathbf{k}_2 \) and frequency \( \omega_2 \). We denote the polar coordinates of the unit vector \( \mathbf{k}_2 \) by \( \theta_2 \) and \( \phi_2 \). The polarisation vectors are denoted by \( \epsilon'_1 \) and \( \epsilon'_2 \). These three unit vectors are given by eqs (25) and (26) with \( (\theta_1, \phi_1) \) replaced by \( (\theta_2, \phi_2) \). We consider the nuclear final state with total angular momentum \( J = 3/2 \), \( j_z = 3/2 \) and orbital angular momentum \( l = 1 \), \( l_z = 1 \). Hence, its wave function takes the form
\[ \psi_f = Y_1^1(\theta, \phi) \frac{U_f(r)}{r} \]
along with the spin part equal to \( |s = 1/2, s_z = 1/2 \). The other states will add incoherently and including them will only produce a change of order unity. Hence, we ignore them here to focus on the main result. Furthermore, we take the polarisation vector of the photon produced at this vertex to be \( \epsilon'_2 \). The matrix element evaluates to
\[ \langle f | H_1(t) | n \rangle = -\epsilon' e^{i \omega_2 t} \frac{1}{2 \Omega \omega_2 \hbar} \sqrt{\frac{1}{2 \Omega \omega_2 \hbar}} \times (E_f - E_n) \langle f | \epsilon' \cdot \mathbf{r} | n \rangle \mathcal{M}_c, \]  
\tag{30}

After integrating over the angular variables, we obtain
\[ \langle f | H_1(t) | n \rangle = -\epsilon' e^{i \omega_2 t} \frac{1}{2 \Omega \omega_2 \hbar} \sqrt{\frac{1}{2 \Omega \omega_2 \hbar}} \times (E_f - E_n) \sin \phi_2 + i \cos \phi_2) \mathcal{M}_c \times \int dr U_f' r U_n. \]  
\tag{31}

Here \( \mathcal{M}_c = \delta^3(\mathbf{K}_f + \mathbf{k}_2 - \mathbf{K}_m) / \Omega \) is the contribution due to the centre of mass coordinate. The angular dependence of the emitted photons can be extracted from eqs (28) and (31). In determining \( |\langle f | T(t_0, t) | i \rangle|^2 \) we find that the angular distribution of both the photons is isotropic, i.e. independent of angles, for the particular transition being considered. The centre of mass dependence simply enforces the overall momentum conservation, i.e. \( \mathbf{K}_f + \mathbf{k}_2 + \mathbf{k}_1 = \mathbf{K}_i \). Furthermore, in the limit \( n_2 \gg m_1 \), the recoil is negligible. Assuming that the initial centre of mass momentum \( \mathbf{K}_i = 0 \), the two-photon momentum, which is distributed isotropically, is balanced by the final nuclear momentum. We next focus on the relative coordinate.

The reaction rate can be expressed as
\[ \frac{dP}{dt} = \frac{1}{\Delta T} \int dE_1 dE_2 \rho_1 \rho_2 |\langle f | T(t_0, t) | i \rangle|^2, \]  
\tag{32}

where \( E_1 = \hbar \omega_1 \), \( E_2 = \hbar \omega_2 \) and \( \rho_1 \) is the photon density of state factor, given by
\[ \rho_1 = \frac{\Omega \omega_1^2}{(2\pi)^3 \frac{\hbar e^3}{m_p}} \frac{d\phi_1}{d\cos \theta_1}, \]  
\tag{33}

along with a corresponding formula for \( \rho_2 \). This leads to the following formula for the reaction rate:
\[ \frac{dP}{dt} = \frac{\alpha^2 \xi^2 \xi^2 g_p^2}{12 \pi \hbar^3 e^6 m_p^2} \times \int dE_1 E_1^3 E_2 |I|^2, \]  
\tag{34}

\[ I = \sum_n I_1 I_2 E_f - E_n - E_i + E_i, \]  
\tag{35}

\[ I_1 = \int dr U_f' \sin(k_1 r) U_i, \]  
\tag{36}

\[ I_2 = \int dr' U'_{f'} r' U_n, \]  
\tag{37}

where \( k_1 \) is the photon wave number. In eq. (35), the sum over intermediate states runs over all the energy values from zero to infinity.

We next turn to the calculation of the integrals \( I_1, I_2 \) and the sum over energies. We point out that it is the sum over energies, leading to the acute cancellation observed in [12,13].

3.2 Calculational details

In this section, we provide some details of the computation of the molecular matrix element, i.e. the integral \( I_1 \), given in eq. (36). Both the wave functions \( U_1 \) and \( U_n \) correspond to \( l = 0 \). For \( r < L_0 \) the wave function \( U_i \)
decays very sharply. Hence, the dominant contribution is obtained from $L_c > r > L_b$, although in our calculation we include contribution from all regions. The wave functions in the region $L_c > r > L_b$ can be expressed as

$$U_n = \frac{1}{K_n N_n} [D_n \sin(K_n r) + F_n \cos(K_n r)]$$

$$U_i = \frac{1}{K_i N_i} [D_i \sin(K_i r) + F_i \cos(K_i r)].$$

Using these equations and simplifying $I_1$ we get a sum of four sine terms and four cosine terms as follows:

$$T_1 = \sin ((k_1 + K_n - K_i) r) [D_n D_i + F_n F_i]$$
$$T_2 = \sin ((k_1 - K_n + K_i) r) [D_n D_i + F_n F_i]$$
$$T_3 = \sin ((k_1 + K_n + K_i) r) [F_n F_i - D_n D_i]$$
$$T_4 = \sin ((k_1 - K_n - K_i) r) [F_n F_i - D_n D_i]$$
$$T_5 = -\cos ((k_1 + K_n - K_i) r) [D_n F_i - F_n D_i]$$
$$T_6 = -\cos ((k_1 - K_n + K_i) r) [F_n D_i - D_n F_i]$$
$$T_7 = \cos ((k_1 + K_n + K_i) r) [-D_n F_i - F_n D_i]$$
$$T_8 = \cos ((k_1 - K_n - K_i) r) [F_n D_i + D_n F_i]$$

each with a pre-factor,

$$\frac{1}{4 N_i K_i N_n K_n k_1 r}.$$ 

For small $K_i$ the dominant contribution comes from the kinematic region $K_n \approx k_1$ due to terms which involve $K_n - k_1$. The sum of these two wave numbers is very large and leads to negligible values of $I_1$.

The integrals are facilitated by the following formulas:

$$\int_{L_b}^{L_c} \frac{\sin(K r)}{r} dr = [\text{Si}(\vert K \vert L_c) - \text{Si}(\vert K \vert L_b)] \text{sgn}(K),$$

where

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$$ 

Similarly, the cosine integral gives,

$$\int_{L_b}^{L_c} \frac{\cos(K r)}{r} dr = \text{Ci}(\vert K \vert L_c) - \text{Ci}(\vert K \vert L_b),$$

where $\text{Ci}(x)$ can be expressed as

$$\text{Ci}(x) = \gamma + \ln x - \int_0^x \frac{1 - \cos t}{t} dt.$$ 

We similarly compute the integral in the region $r < L_b$.

We next consider the integral $I_2$ which appears in the nuclear matrix element. This integral gets dominant contribution from very small values of $r$. It shows a very mild dependence on intermediate state energy eigenvalue $E_n$ for $E_n < V_1$, except for the factor $N_n$ in the denominator. This essentially corresponds to a barrier penetration factor and shows an increase with $E_n$. For $E_n \gg V_1$, $I_2$ starts to oscillate with $E_n$.

4. Results

We first consider the case $L_c = \infty$. In this case, the potential $V_2$ plays no role and we obtain a continuous energy spectrum. We take the initial energy eigenvalue $E_i$ to be equal to 0.1 eV. As found earlier [12,13], the amplitude becomes large in the regime when $K_n \approx k_1$. However, adding over all intermediate eigenstates leads to a very small result. It is difficult to compute the precise value numerically, but we find a cancellation up to six significant digits, suggesting that the rate in free space is very small. This result is in agreement with what was found in [12,13]. We also test the possibility that rate may be enhanced if the photon wave number $k_1$ is set equal to the wave number $K_n$ of an intermediate state which corresponds to a nuclear resonance. We find that even in this case the rate turns to be very small due to a delicate cancellation.

We next compute the rate for a finite value of $L_c$ which would be applicable in a medium. We first take a relatively small value of $L_c = 10$ a.u. This will lead to a relatively large separation in energy among different states. We may compare this value with the level spacing in the case of Kronig–Penney model which is a one-dimensional periodic potential. We find that the level spacing in this case is determined by the width of the potential wells [18]. This in our case would represent the distance between the lattice sites in the medium and hence would be of order of a few atomic units. Hence the choice $L_c = 10$ is not unphysical. For this value, we find an energy eigenvalue at $E_i = 0.0903$ eV which we take to be the initial state energy. We take the photon wave number $k_1$ close to 600 a.u. In this case, the amplitude is quite large. In figure 4 we show the amplitude $I$, defined in eq. (35) as a function of the upper limit on the intermediate state energy $E_n$ for photon wave number $k_1 = 635$ a.u. We see that the amplitude settles to a finite value as $E_n \rightarrow \infty$. The dependence of rate on the photon wave number is shown in figure 5. We find a rate of the order of $10^{-18}$ per second for the photon wave number close to 600 a.u. The rate in this case is much larger than that is found for the first-order transition. If we set the chosen photon wave number equal to intermediate state wave number $K_n$, the corresponding intermediate energy comes out to be close to 100 a.u., which is equal to the barrier height $V_1$. From figure 5, it is clear that this choice of wave number leads to maximum rate. We
Figure 4. The amplitude $I$ (eq. (35)) as a function of the upper limit on the energy $E_n$ of the intermediate state. Here the photon wave number is set equal to 635 a.u. and $L_c = 10$.

Figure 5. The reaction rate as a function of wave number of the photon emitted at the first vertex for $L_c = 10$. The rate shows very rapid fluctuations which are apparent in the figure.

Figure 6. The reaction rate as a function of the cut-off length scale $L_c$ for photon wave number $k_1 = 635$ a.u. We find that the rate drops roughly as $1/L_c^3$.

also check the contribution due to the second term in figure 1 with $k_1$ interchanged with $k_2$. For $k_1 = 600$, $k_2$ is close to 3000. For these two interchanged the amplitude is found to be four orders of magnitude smaller and, hence, it is negligible. We can, therefore, ignore the second amplitude for this choice of parameters.

As we increase the value of $L_c$ we find that the dominant change occurs due to the normalisation factors in the wave functions. Both the $U_i$ and $U_n$ wave functions contain a factor $\sqrt{L_c}$ in the denominator. Hence, the rate shows an approximate decrease as $1/L_c^3$. In figure 6 we show the dependence of rate on $L_c$ for $k_1 = 635$ a.u. for $L_c$ up to 100. We have explicitly checked the dependence on $L_c$ up to $L_c = 1000$ and the trend shown in figure 6 continues up to this value. At some sufficiently large value of $L_c$ we expect a much sharper decrease, due to a delicate cancellation that was seen in the continuum case. However, the numerical work to obtain this value of $L_c$ becomes prohibitively time intensive and we do not pursue this in the current paper.

5. Conclusions

We conclude that nuclear fusion reactions may be possible at low energies inside a medium through the mechanism of second-order perturbation theory [12–14]. We find that the rate in free space turns out to be very small, which is in agreement with earlier results [12,13]. The amplitude in this case does become relatively large for intermediate state wave number values $K_n$ close to the photon wave number $k_1$. However, as we sum over all intermediate wave numbers the amplitude becomes very small due to a delicate cancellation. In a medium, however, we expect that the energy eigenvalues would not be continuous. This is seen, for example, in a crystalline lattice which leads to a band structure with energy levels being separated by forbidden bands [15]. In a disordered system we expect to see localised states, analogous to Anderson localisation [16,17]. In either cases, we expect discontinuous energy eigenvalues. In the present paper, we use a simple model by imposing a hard wall cut-off beyond a certain length scale. The essential feature of this model is that it leads to discretisation of energy levels. This model leads to substantial rates for fusion reaction to take place at second order in perturbation theory. We obtain dominant contribution from intermediate states whose wave number $K_n$ in the region $L_b < r < L_c$ is close to the photon wave number $k_1$. The contribution is maximal if the intermediate states have energy close to the height of the potential barrier.

Our results suggest that, in favourable conditions, nuclear fusion reactions can take place at low energies at observable rates. However, so far we have only
presented a toy model to test this phenomenon. Considerable effort is needed to meet the observations. This will require realistic models of the solid structure. Furthermore, photon emission may not be the dominant process and it would be interesting to consider other mechanisms.

References

[1] Donald D Clayton, *Principles of stellar evolution and nucleosynthesis* (The University of Chicago Press, Chicago, 1968)
[2] Steven B Krivit, *Development of low-energy nuclear reaction research* (John Wiley & Sons, Ltd, 2011) Chapter 41, pp. 479–496
[3] Mahadeva Srinivasan, George Miley and Edmund Storms, *Low-energy nuclear reactions: Transmutations* (John Wiley & Sons, Ltd., 2011) Chapter 43, pp. 503–539
[4] Edmund Storms, *Curr. Sci.* 108, 535 (2015)
[5] Jean-Paul Biberian, *J. Cond. Matter Nucl. Sci.* 29, 211 (2019)
[6] K P Sinha, *Curr. Sci.* 108, 516 (2015)
[7] Francesco Celani, Antonino Tommaso and Giorgio Vassallo, *J. Cond. Matter Nucl. Sci.* 24, 32 (2017)
[8] C Spitaleri, C A Bertulani, L Fortunato and A Vitturi, *Phys. Lett. B* 755, 275 (2016)
[9] Peter Hagelstein, *J. Cond. Matter Nucl. Sci.* 29, 392 (2019)
[10] Jean-Luc Paillet and Andrew Meulenberg, *J. Cond. Matter Nucl. Sci.* 29, 472 (2019)
[11] V A Chechin, V A Tsarev, M Rabinowitz and Y E Kim, *Int. J. Theor. Phys.* 33(3), 617 (1994)
[12] P Jain, A Kumar, R Pala and K P Rajeev, *Pramana – J. Phys.* 96, 96 (2022)
[13] P Jain, A Kumar, K Ramkumar, R Pala and K P Rajeev, *JCMNS* 35, 1 (2021)
[14] Péter Kálmán and Tamás Keszthelyi, *Phys. Rev. C* 99, 054620 (2019)
[15] N W Ashcroft and N D Mermin, *Solid state physics* (Holt-Saunders, 1976)
[16] Elíhu Abrahams, *50 Years of Anderson localization* (World Scientific, 2010)
[17] Patrick A Lee and T V Ramakrishnan, *Rev. Mod. Phys.* 57, 287 (1985)
[18] E Merzbacher, *Quantum mechanics* (Wiley, 1998)
[19] J J Sakurai, *Advanced quantum mechanics*, Always learning (Pearson Education Incorporated, 1967)