Non-perturbative renormalization of the average color charge and multi-point correlators of color charges from a non-Gaussian small-x action

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Gluons dominate the hadronic wave function @ small-x

Small-x is accessible at high-energies: $x \sim p_T / \sqrt{s}$
QCD matter @ high gluon densities

**Color Glass Condensate:** E.F.T. of QCD @ small-x

McLerran, Venugopalan: PRD 49 (1994), 2233, ibid, 3352, PRD 50 (1994), 2225

Also: McLerran, Venugopalan, Jalilian-Marian, Kovner, Weigert, Iancu, Leonidov, Kovchegov, Balitski

Hadrons described as **classical systems** @ high gluon densities

**Large-x partons** → static randomly distributed color charges

**Small-x gluon fields** obtained from CYM equations for **fixed** color charge

\[
\langle \mathcal{O}[\rho] \rangle_Y = \frac{\int [d\rho] W_Y[\rho] \mathcal{O}[\rho]}{\int [d\rho] W_Y[\rho]}
\]

Here: \( Y = 0 \)
Weight functions for color charge average

\[ W_{MV}[\rho_x] = \exp \left\{ - \int d^2x \frac{\rho_x^a \rho_x^a}{2\mu^2} \right\} \]

\( a = 1, \ldots, N_c^2 - 1 \)

\( \mu = \text{constant} \)

McLerran-Venugopalan (MV) model

Only non-trivial correlator:

\[ \langle \rho_x^a \rho_y^b \rangle_{MV} = \delta^{ab} \delta^{(2)}(x - y) \mu^2 \]

\[ \langle \rho_x^a \rho_y^b \rho_u^c \rho_v^d \rangle_{MV} = \langle \rho_x^a \rho_y^b \rangle \langle \rho_u^c \rho_v^d \rangle + \text{permutations} \]
Weight functions for color charge average

\[ W_{NG}[\rho_x] \simeq \exp \left\{ - \int \, d^2 x \left[ \frac{\rho_x^a \rho_x^a}{2\bar{\mu}^2} + \frac{3(\rho_x^a \rho_x^a)^2}{\kappa_4} \right] \right\} \]

Dumitru, Jalilian-Marian, Petreska, PRD 84 (2011), 014018

Deviations from MV model:

\[ Z = \frac{\mu^2}{\bar{\mu}^2} \]

\[ \langle \rho_x^a \rho_y^b \rho_u^c \rho_v^d \rangle_{NG} \propto \delta^{(2)}(x - y)\delta^{(2)}(u - v)[1 - C_{NG} \delta^{(2)}(x - u)] + \text{perm.} \]

non-factorizable & local

May explain initial condition for dipole evolution w/ \[ \gamma > 1 \]

Dumitru, Petreska, NP A879 (2012) 59-76

Multi-point Wilson line correlators w/ NG ensemble: ongoing calculation w/ Y. Nara

preferred by fits of HERA inclusive cross-section data
Non-perturbative calculation on lattice

\[ W_{NG}[\rho] \rightarrow W_r = \frac{a^2 Z r^2}{2\mu^2} + \frac{3a^2 r^4}{\kappa_4} \]

\[ r^2 \equiv \sum \rho^a \rho^a \]
\[ \int d^2 x \rightarrow a^2 \]

No expansion of \( e^{-W_r} \)

\[ \langle \rho_x^a \rho_y^b \rangle_{NG} \quad \& \quad \langle \rho_x^a \rho_y^b \rho_u^c \rho_v^d \rangle_{NG} \quad \text{to all orders in} \quad \frac{1}{\kappa_4} \]

Perturbative results in the limit of small & large non-Gaussian fluctuations

\[ Z \sim 1 \quad \rightarrow \quad Z \sim 0 \]
Renormalization schemes (RS)

\[ W_r = \frac{a^2 Z}{2 \mu^2} r^2 + \frac{3 a^2 r^4}{\kappa_4} \]

\[ \langle \rho_x^a \rho_y^b \rangle_{\text{NG}} = \langle \rho_x^a \rho_y^b \rangle_{\text{MV}} \]

- **RS 1:**
  \[ \frac{\kappa_4}{\bar{\mu}^6} = \text{constant} \rightarrow \kappa_4 \propto \mu^6/Z^3 \rightarrow Z(a) \]
  - No control over non-Gaussian fluctuations

- **RS 2:**
  \[ \kappa_4 = \text{constant} \rightarrow Z(a) \]

- **RS 3:**
  \[ \bar{\mu}^2 = \text{constant} \rightarrow Z = \text{constant} \rightarrow \kappa_4(a) \]
  - Can control deviations from MV model!
Ratio \( R \equiv \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{MV}}} \) in RS3

\[ 0.905 \lesssim R \lesssim 1 \quad SU(3) \]

\[ 0.823 \lesssim R \lesssim 1 \quad SU(2) \]

Ratio decreases by moving away from MV model.
Ratio $R \equiv \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{NG}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{MV}}$ in RS3

Fixed deviation from MV model as $a \rightarrow 0$
Conclusions & Remarks

- Studied non-perturbative effects of first (even C-parity) non-Gaussian correction to the Gaussian approximation of the CGC.

- Considered different renormalization schemes for parameters appearing in the non-Gaussian small-\(x\) action. Found a way to control deviations from MV model.

- 1\(^{\text{st}}\) step to have lattice calculations beyond the Gaussian limit of CGC.

- May lead to better characterization of small systems in CGC.

- **Gluon field fluctuations \(\neq\) Gaussian: affects multiplicity distributions.**
  Dumitru, Petreska 1209.4105

- **Includes contributions missed by Gaussian averaging.**
  Kovner, Lublinsky, PRD83, 034017 (2011)  
  Dumitru, Jalilian-Marian, Petreska, PRD 84 (2011), 014018

- May impact particle production @ LHC as well as @ Electron – Ion.
Thank you for your attention!

Questions / comments are welcome!
Weight functions

$\nu \alpha \approx 0.024 \quad \& \quad N_s = 128$

**MV model**

$W_r = \frac{a^2 r^2}{2\mu^2}$

**Non-Gaussian**

$W_r = \frac{a^2 r^2}{2\bar{\mu}^2} + \frac{r^4}{\kappa_4}$

**Gaussian envelope**

$W_r = \frac{a^2 r^2}{2\bar{\mu}^2}$
Color charge correlators: main expressions

When $\mathcal{O}$ is local:

$$
\langle \mathcal{O} \rangle = \frac{\int (\prod_x \prod_a d\rho_x^a) \mathcal{O} e^{-\sum_y W_y}}{\int (\prod_x \prod_a d\rho_x^a) e^{-\sum_y W_y}} = \frac{\int dr \, r^{N_c^2-2} \mathcal{O}_r e^{-W_r}}{\int dr \, r^{N_c^2-2} e^{-W_r}},
$$

$$
\langle \rho_x^a \rho_x^a \rangle_{\text{NG}} = (N_c^2 - 1) \frac{\bar{\mu}^2 \sqrt{X}}{a^2} \frac{U \left( \frac{1}{4} \left( N_c^2 + 1 \right), \frac{1}{2}, X \right)}{U \left( \frac{1}{4} \left( N_c^2 - 1 \right), \frac{1}{2}, X \right)}
$$

$$
X = \frac{a^2 \kappa_4}{48 \bar{\mu}^4}
$$

$$
\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}} = (N_c^4 - 1) \frac{\bar{\mu}^4 \sqrt{X} U \left( \frac{1}{4} \left( N_c^2 + 3 \right), \frac{1}{2}, X \right)}{a^4} \frac{U \left( \frac{1}{4} \left( N_c^2 - 1 \right), \frac{1}{2}, X \right)}
$$

$$
X \rightarrow \infty \quad \text{limit: small non-Gaussian fluctuations,} \quad Z \rightarrow 1
$$

$$
X \rightarrow 0 \quad \text{limit: large non-Gaussian fluctuations,} \quad Z \rightarrow 0
$$

$$
U(\alpha, \beta, \omega) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\omega t} t^{\alpha-1} (1 + t)^{\beta-\alpha-1} dt \quad \text{Tricomi's hypergeometric function}
$$
Renormalization eq.: \( \langle \rho_x^a \rho_y^b \rangle_{NG} = \langle \rho_x^a \rho_y^b \rangle_{MV} \)

Small non-Gaussian fluctuations (Z~1)

\[
\mu^2 \equiv \bar{\mu}^2 \left( 1 - 12 \frac{\bar{\mu}^4}{\kappa_4} \frac{N_c^2 + 1}{a^2} \right)
\]

\[\bar{\mu}^2 > \mu^2 \quad \rightarrow \quad 0 < Z < 1\]

Large non-Gaussian fluctuations (Z~0)

\[
\frac{\Gamma \left( \frac{1}{4} (N_c^2 + 1) \right)}{4 \sqrt{3} \Gamma \left( \frac{1}{4} (N_c^2 + 3) \right)} \frac{\sqrt{\kappa_4}}{a} + \left( \frac{\Gamma \left( \frac{1}{4} (N_c^2 + 1) \right)^2}{\Gamma \left( \frac{1}{4} (N_c^2 - 1) \right) \Gamma \left( \frac{1}{4} (N_c^2 + 3) \right)} - 1 \right) \frac{Z \kappa_4}{24 \mu^2} = \frac{\mu^2}{a^2}
\]
Renormalization eq. in RS1 ($\kappa_4 / \bar{\mu}^6 = \gamma / g^2$ const.)

$$\bar{\mu}^2 \sim \frac{g^2 A}{\pi R^2} \quad \kappa_4 \sim \frac{g^4 A^3}{(\pi R^2)^3} \quad \Rightarrow \quad \kappa_4 = \frac{\gamma}{g^2 \bar{\mu}^6}$$

Small non-Gaussian fluctuations ($Z \sim 1$)

$$Z(a) = \frac{\mu^2 a^2}{12(N_c^2 + 1) g^2 / \gamma + \mu^2 a^2}$$

Large non-Gaussian fluctuations ($Z \sim 0$)

$$Z^3(a) = \frac{\gamma \Gamma \left(\frac{1}{4} \left(N_c^2 + 1\right)\right)^2}{48 g^2 \Gamma \left(\frac{1}{4} \left(N_c^2 + 3\right)\right)^2} a^2 \mu^2$$

$$Z(a) \to 0 \quad \text{as} \quad a \to 0$$

SU(2)

$\gamma = 48$

SU(3)

$\gamma = 144$
Increasing deviations from MV model as $a \rightarrow 0$
4-point function in RS1

\[ \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{MV}}} = 1 - \frac{24 \left( \frac{g^2}{\gamma} \right) \mu^4 a^4}{\left[ \mu^2 a^2 + 12 \left( N_c^2 + 1 \right) g^2 / \gamma \right]^3} \]

Ratio equal to 1 when \( a \to 0 \) (inconsistent with previous result)

\[ \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{MV}}} = \begin{cases} 0.822504, & \text{for SU}(2) \\ 0.905415, & \text{for SU}(3) \end{cases} \]

Maximum deviation from Gaussian theory (for this ratio!)
Ratio $\frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{NG}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{MV}}$ in RS1

Increasing deviations from MV model as $\mu_L \to 0$

Similar results for SU(2)
Renormalization eq. in RS2 ($\kappa_4$ const.)

When $Z \sim 1$:

$$Z^2(1 - Z) = \frac{12 \left( N_c^2 + 1 \right) \mu^4}{\kappa_4 a^2}$$

$$Z(a) = \frac{1}{3} \left[ 1 - (\nu(a) - 1)^{-1/3} - (\nu(a) - 1)^{1/3} \right]$$

$$\nu(a) \equiv \frac{18 \mu^2 \left( \sqrt{N_c^2 + 1} \left( 81 (N_c^2 + 1) \mu^4 - a^2 \kappa_4 \right) + 9 (N_c^2 + 1) \mu^2 \right)}{a^2 \kappa_4}$$

$Z(a) < 0$ as $a \to 0$

RS2: pert. calculation for $Z \sim 1$ not applicable!
Z(a) vs a in RS2

Dashed line: \( Z(a, \kappa_4) = \frac{2 \mu^2 \Gamma \left( \frac{1}{4} (N_c^2 - 1) \right)}{\Gamma \left( \frac{1}{4} (N_c^2 - 1) \right) \Gamma \left( \frac{1}{4} (N_c^2 + 3) \right) - \Gamma \left( \frac{1}{4} (N_c^2 + 1) \right)^2} a^2 \kappa_4 \)

Result for Z~0

Result for Z~1

Cannot take a \( \to 0 \) limit

\( K_4^{1/6} = 100 \text{ GeV} \)
Renormalization eq. RS3 (fixed Z)

Small non-Gaussian fluctuations (Z~1)

\[ a^2 \kappa_4 = 12 \frac{\mu^4}{Z^2 - Z^3} (N_c^2 + 1) \]

Large non-Gaussian fluctuations (Z~0)

\[ \kappa_4(a, Z) \equiv \frac{\kappa_4(Z)}{a^2} = \frac{6 \mu^4 [\alpha(Z) - \beta(Z)]}{a^2 Z^2 \Gamma^2_{1/4,1} \left( \Gamma^2_{1/4,1} - \Gamma_{1/4,-1} \Gamma_{1/4,3} \right)^2} \]

\[ \Gamma_{k,m} \equiv \Gamma \left( k (N_c^2 + m) \right) \]

\[ a^2 \kappa_4 = \text{constant} \quad \text{for fixed Z} \]
4-point function in RS3

\[ Z \sim 1: \quad \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{MV}}} = 1 - \frac{2 Z^2 (1 - Z)}{(N_c^2 + 1)} \]

\[ Z \sim 0: \quad \frac{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{NG}}}{\langle \rho_x^a \rho_x^a \rho_x^c \rho_x^c \rangle_{\text{MV}}} = Z^{-2} \left[ 4.72132 - 0.505056 \Psi(Z) - 3.55162 \epsilon(Z) + 0.37993 \Psi(Z) \epsilon(Z) + Z (2.45171 \Psi(Z) + 0.532404 \Psi(Z) + 1.71781 Z - 8.23612) \right] \]

\[ \Psi(Z) \equiv \sqrt{-14.0948 \epsilon(Z) - 19.7514 Z + 18.7368} \quad \epsilon(Z) \equiv \sqrt{1.76715 - 3.72567 Z} \]

Independent of lattice spacing → Fixed deviation from MV model
$4$-point function $\langle \rho_x^a \rho_y^a \rho_u^b \rho_v^b \rangle$

\[
\langle \rho_x^a \rho_y^b \rho_u^c \rho_v^d \rangle_{NG} = \delta^{ab} \delta^{cd} \frac{\delta_{xy}}{a^2} \frac{\delta_{uv}}{a^2} \mu^4 \left( 1 - C_{NG} \frac{\mu^4}{\kappa_4} \frac{\delta_{ux}}{a^2} \right) + \text{permutations}
\]

$Z\sim1$: $C = 24$

$Z\sim0$: $C = 48 \frac{\Gamma\left(\frac{1}{4}(N_c^2 + 3)\right)^2}{\Gamma\left(\frac{1}{4}(N_c^2 + 1)\right)^2} \left[ 1 - \frac{\Gamma\left(\frac{1}{4}(N_c^2 + 3)\right)^2}{\Gamma\left(\frac{1}{4}(N_c^2 + 1)\right) \Gamma\left(\frac{1}{4}(N_c^2 + 5)\right)} \right]$

\[
\langle \rho^4 \rangle_{NG} = \langle \rho^4 \rangle_{MV} \quad \text{when} \quad x = y, \ u = v, \ x \neq u
\]

\[
\langle \rho^4 \rangle_{NG} < \langle \rho^4 \rangle_{MV} \quad \text{when} \quad x = y = u = v
\]
Lattice regularization

2D transverse space \sim lattice with \( N_s^2 \) sites of length \( a \)

Continuum recovered by: \( a \to 0 \) with fixed \( L = N_s a \)

\[
\begin{align*}
\int d^2 x & \to a^2 \\
\mu & \to \mu_L = \mu a \\
\rho & \to \rho_L = \rho a^2 \\
\delta^{(2)}(x - y) & \to \frac{\delta_{xy}}{a^2}
\end{align*}
\]

\[
\langle \rho_x^a \rho_y^b \rangle = \delta^{ab} \delta^2(x - y) \mu^2 \\
\langle \rho_x^a \rho_y^b \rangle = \frac{\delta^{ab} \delta_{xy}}{a^2} \mu^2
\]

Krasnitz, Venugopalan: Nucl. Phys. B 557 (1999), 237