Bäcklund transformation for the Krichever-Novikov equation

V.E. Adler

18 July 1997

The Krichever-Novikov equation

\[ u_t = u_{xxx} - \frac{3}{2u_x}(u^2_{xx} - r(u)) + cu_x, \quad r^{(5)} = 0 \]  \hspace{1cm} (1)

appeared (up to change \( u = p(\tilde{u}), \ \dot{p}^2 = r(p) \)) in [1] for the first time in connection with study of finite-gap solutions of the Kadomtsev-Petviashvili equation. The distinctive feature of the equation (1) is that, accordingly to [2], no differential substitution exists connecting it with other KdV-type equations. This property impedes the construction of the Bäcklund transformation (BT) which in other cases can be obtained by composition of two differential substitutions. Nevertheless, we demonstrate that (1) admits BT which connects it with other equation of the same form

\[ v_t = v_{xxx} - \frac{3}{2v_x}(v^2_{xx} - R(v)) + cv_x, \quad R^{(5)} = 0. \]  \hspace{1cm} (2)

**Theorem 1.** Let \( h(u, v) \) be arbitrary polynomial of degree less than 3 on each variable, and \( r(u) \) and \( R(v) \) be its discriminants as quadratic trinomial on \( v \) and \( u \) correspondingly:

\[ h_{uuu} = h_{vvv} = 0, \quad r(u) = h_v^2 - 2hh_{vv}, \quad R(v) = h_u^2 - 2hh_{uu}. \]  \hspace{1cm} (3)

Then formula

\[ u_xv_x = h(u, v) \]  \hspace{1cm} (4)

defines BT between equations (1) and (2).

It is easy to see that relation \( h(u, v) = 0 \) gives a birational transformation of the curves

\[ y^2 = r(u), \quad Y^2 = R(v). \]  \hspace{1cm} (5)

The form of the relations (1) — (4) does not change under arbitrary nondegenerate linear-fractional substitutions

\[ u = \frac{k\tilde{u} + \ell}{m\tilde{u} + n}, \quad v = \frac{K\tilde{v} + L}{M\tilde{v} + N}. \]

It is clear that polynomials \( r \) and \( R \) cannot be chosen independently (indeed, they contain 10 coefficients, while \( h \) only 9). From the other hand, if equations (3) are solvable with respect to \( h \), then their solution can be not unique and then question arises about permutability of two different BT.

We consider only the case \( r(u) = R(u) \) and assume that zeroes of \( r \) are simple. Accordingly to [3] it is possible if and only if the polynomial \( h(u, v) \) is symmetric and irreducible. It should be noted, that reducible cases are also rather meaningful, for example the formula (4) where
$r = 0$, $h = (v - \mu u)^2$ defines the BT for Schwartz-KdV equation, from which the BT for other KdV-type equations can be derived by standard substitutions.

The equation $h(u, v) = 0$ is known as Euler-Chasles correspondence (see e.g. [6]) and is equivalent to the shift on the elliptic curve, which we take, without loss of generality, in the Weierstrass form: $r(u) = 4u^3 - g_2u - g_3$. The corresponding polynomial $h$ depends on arbitrary parameter $\mu$ and is of the form $h = H(u, v, \mu)/\sqrt{r(\mu)}$, where

$$H(u, v, \mu) = (uv + \mu u + \mu v + \frac{g_2}{4})^2 - (u + v + \mu)(4\mu uv - g_3).$$

The relation $H = 0$ where $u = \varphi(z)$, $v = \varphi(z \pm a)$, $\mu = \varphi(a)$ is nothing but Euler form of the addition theorem for Weierstrass $\varphi$-function. Hence it is evident, that composition of the correspondences $H(u, v, \mu) = 0$, $H(v, w, \nu) = 0$ coincides with composition $H(u, \bar{v}, \nu) = 0$, $H(\bar{v}, w, \mu) = 0$. Indeed, in the both cases $w$ takes 4 possible values $\varphi(z \pm a \pm b)$.

The Euler-Chasles correspondence is formally a particular case of (4) for $u = \text{const}$ (but it should be noted, that the constant solutions of the equation (1) are exhausted by zeroes of $r$). It turns out that analogous commutativity property is valid for the BT (4) itself. Let $u, v, \bar{v}, w$ are related by the BT as on the diagram above, that is

$$u_xv_x = H(u, v, \mu)/\alpha, \quad v_xw_x = H(v, w, \nu)/\beta, \quad \alpha^2 = r(\mu),$$
$$u_x\bar{v}_x = H(u, \bar{v}, \nu)/\beta, \quad \bar{v}_xw_x = H(\bar{v}, w, \mu)/\alpha, \quad \beta^2 = r(\nu).$$

Let us remind that the constraint $P(u, v, \bar{v}, w) = 0$ is called the nonlinear superposition principle if it is nondegenerate on each variable and its derivative on $x$ vanishes in virtue of itself: $D_x(P)|_{P=0} = 0$.

**Theorem 2.** Nonlinear superposition principle for the BT (4), (5) is given by formula

$$P = k_0uvwvw - k_1(uvw + vvw + wuv + \bar{v}v) + k_2(uw + \bar{v}v) - k_3(uv + \bar{v}w) - k_4(uv + vw) + k_5(u + v + w + \bar{v}) + k_6 = 0$$

where

$$k_0 = \alpha + \beta, \quad k_1 = \alpha \nu + \beta \mu, \quad k_2 = \alpha \nu^2 + \beta \mu^2, \quad k_5 = \frac{g_3}{2}k_0 + \frac{g_2}{4}k_1, \quad k_6 = \frac{g_2}{16}k_0 + g_3k_1,$$
$$k_3 = \frac{\alpha \beta (\alpha + \beta)}{2(\nu - \mu)} - \alpha \nu^2 + \beta (2\mu^2 - \frac{g_2}{4}), \quad k_4 = \frac{\alpha \beta (\alpha + \beta)}{2(\mu - \nu)} - \beta \mu^2 + \alpha (2\nu^2 - \frac{g_2}{4}).$$

Note that elimination of derivatives from (4) yields the relation

$$S = \alpha^2 H(u, \bar{v}, \nu)H(v, w, \nu) - \beta^2 H(u, v, \mu)H(\bar{v}, w, \mu) = 0,$$

which is not the nonlinear superposition principle by itself. It is explained by reducibility of the polynomial $S = PQ$, where $P$ is written above and $Q$ is obtained from $P$ by substitution $\beta$ to $-\beta$. In contrast to the constraint $P = 0$ the constraint $Q = 0$ is not compatible with dynamics on $x$.

The work was supported by grants INTAS-93-166-Ext and RFBR-96-01-00128.
References

[1] I.M. Krichever, S.P. Novikov, Holomorphic fibering over algebraic curves and nonlinear equations, Uspekhi mat. nauk 35(6) (1980) 47-68

[2] S.I. Svinolupov, V.V. Sokolov, R.I. Yamilov, On the Bäcklund transformations for integrable evolution equations, Dokl. Acad. Nauk SSSR 271(4) (1983) 802-805

[3] V.M. Buchstaber and A.P. Veselov, Integrable correspondences and algebraic representations of multivalued groups, IMRN 8 (1996) 381-400

[4] V.E. Adler, A.B. Shabat, On the one class of Toda lattices, Teor. i Mat. Fiz. 111(3) (1997) 323-334