Uniform continuity bounds for characteristics of multipartite quantum systems

M.E. Shirokov

Abstract

We consider universal methods for obtaining (uniform) continuity bounds for characteristics of multipartite quantum systems. We pay a special attention to infinite-dimensional multipartite quantum systems under the energy constraints.

By these methods we obtain continuity bounds for several important characteristics of a multipartite quantum state: the quantum (conditional) mutual information, the squashed entanglement, the c-squashed entanglement and the conditional entanglement of mutual information. The continuity bounds for the multipartite quantum mutual information are asymptotically tight for large dimension/energy.

The obtained results are used to prove the asymptotic continuity of the $n$-partite squashed entanglement, c-squashed entanglement and the conditional entanglement of mutual information under the energy constraints.

Contents

1 Introduction 2

2 Preliminaries 3
  2.1 Basic notations 3
  2.2 The set of quantum states with bounded energy 6

3 The main results 8
  3.1 The finite-dimensional case 8
  3.2 The infinite-dimensional case: arbitrary subsystems 11
  3.3 The infinite-dimensional case: identical subsystems 14

4 Applications 19
  4.1 Multipartite quantum (conditional) mutual information 19
  4.2 Squashed entanglement and c-squashed entanglement 23
  4.3 Conditional entanglement of mutual information 25

5 On preserving continuity bounds under local channels 27

*Steklov Mathematical Institute, Moscow, Russia, email:msh@mi.ras.ru
1 Introduction

Multipartite quantum systems are basic objects in quantum information theory [14] [25] [35]. Such systems play central role in algorithms of quantum information processing, quantum computation, cryptography, etc. Properties of states of multipartite quantum systems are described by different characteristics that are used essentially in analysis of information abilities of such systems. So, important task consists in studying analytical properties of these characteristics (as functions of a state), in particular, finding accurate upper and lower estimates, (uniform) continuity bounds (estimates for variation), conditions for asymptotic continuity (for entanglement measures), etc.

Continuity bounds for characteristics of a multipartite quantum state represented as a linear combination of the marginal entropies or conditional entropies of this state can be obtained (in the finite-dimensional settings) by applying Audenaert’s continuity bound for the entropy and Winter’s continuity bound for the conditional entropy (cf. [2] [11]) to each term of this linear combination. In the infinite-dimensional case the similar approach can be realized by means of Winter’s continuity bounds for the entropy and the conditional entropy under the energy constraints [41]. The obvious drawback of this approach is low accuracy of the resulting continuity bounds. More accurate continuity bounds for these characteristics can be obtained by direct applications of the Alicki-Fannes-Winter method ([1] [41]) and its infinite-dimensional generalizations ([29] [30]) to a characteristic of a multipartite quantum state without its decomposition.

The aim of this paper is to consider universal methods of obtaining accurate continuity bounds for characteristics of multipartite quantum systems paying a special attention to infinite-dimensional systems with the energy constraints of different forms.

Mathematically, a characteristic of a multipartite quantum state is a function $f$ on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ of states of a $n$-partite system $A_1...A_n$, $n \geq 2$ (in infinite dimensions such function is typically well defined only on some subset of $\mathcal{S}(\mathcal{H}_{A_1...A_n})$). We will assume that this function $f$ has the following property: $|f(\rho)|$ has an upper bound proportional to the sum of several marginal entropies of the state $\rho$. It means, w.l.o.g., that

$$|f(\rho)| \leq C_f \sum_{k=1}^{m} H(\rho_{A_k}), \quad m \leq n, \quad C_f \in \mathbb{R}_+,$$

for all states $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ having finite the marginal entropies $H(\rho_{A_1}),...,H(\rho_{A_m})$ (for other states $\rho$ the function $f$ may not be defined). In fact, many real correlation and entanglement measures on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ possess this property (see Sections 3,4).

In Section 3 we show that property (1) is one of the conditions that allow to obtain continuity bound for the function $f$ valid for all states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ with bounded energy corresponding to the system $A_1...A_m$. We note first that such continuity bound

---

\[1\] Audenaert’s continuity bound for the von Neumann entropy and Winter’s continuity bound for the conditional entropy are optimized versions of the Fannes and Alicki-Fannes continuity bounds for these quantities [10] [11].
can be obtained by using the modification of the Alicki-Fannes-Winter method proposed in \[29\], which is based on initial purification of quantum states with bounded energy. This approach gives simple and universal continuity bounds for wide class of characteristics of quantum systems composed of arbitrary subsystems provided that

\[
\lim_{\lambda \to 0^+} \left[ \text{Tr} e^{-\lambda H_{A_k}} \right] = 1, \quad k = 1, 2, \ldots, m, \quad (2)
\]

where \( H_{A_k} \) is the Hamiltonian of the subsystem \( A_k \) (Theorem 1).\(^2\) The main drawback of continuity bounds obtained by this way is their non-accuracy for small distance between quantum states.

More sharp universal continuity bounds can be obtained by using the two step technique based on appropriate finite-dimensional approximation of states with bounded energy followed by the Alicki-Fannes-Winter method.\(^3\) The two step technique can be applied when the single subsystems \( A_1, \ldots, A_m \) are arbitrary and their Hamiltonians satisfy condition (2), but the resulting continuity bounds are too complex in this case. So, to avoid technical difficulties and keeping in mind possible applications we apply the two step technique assuming that the single subsystems \( A_1, \ldots, A_m \) (involved in (1)) are identical (it means that the Hamiltonians \( H_{A_1}, \ldots, H_{A_m} \) of these subsystems are isomorphic). Under this assumption the construction is simplified essentially (Theorem 2). We pay a special attention to the case when each of the subsystems \( A_1, \ldots, A_m \) is (isomorphic to) a multi-mode quantum oscillator (Corollary 2).

In Section 4 we use general results of Section 3 to obtain continuity bounds for several important characteristics of a multipartite quantum state: the quantum (conditional) mutual information, the squashed entanglement, the c-squashed entanglement and the conditional entanglement of mutual information. We show that the continuity bounds for the multipartite quantum mutual information are asymptotically tight for large dimension/energy. We prove the asymptotic continuity of the \( n \)-partite squashed entanglement, c-squashed entanglement and the conditional entanglement of mutual information under the energy constraints.

In Section 5 we discuss an interesting feature of the proposed methods: the continuity bounds produced by these methods for many characteristics of multipartite quantum systems remain valid after actions of any local channels on states of these systems.

2 Preliminaries

2.1 Basic notations

Let \( \mathcal{H} \) be a separable Hilbert space, \( \mathfrak{B}(\mathcal{H}) \) the algebra of all bounded operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \) and \( \mathfrak{T}(\mathcal{H}) \) the Banach space of all trace-class operators on

\(^2\)The sense of condition (2) is described in Section 2.2.

\(^3\)In the case \( m = 1 \) this technique was used by A.Winter to obtain continuity bounds for the entropy and the conditional entropy.\(^4\)
\( \mathcal{H} \) with the trace norm \( \| \cdot \|_1 \). Let \( \mathcal{S}(\mathcal{H}) \) be the set of quantum states (positive operators in \( \mathcal{T}(\mathcal{H}) \) with unit trace) \([14, 25, 35]\).

Denote by \( I_{\mathcal{H}} \) the unit operator on a Hilbert space \( \mathcal{H} \) and by \( \text{Id}_{\mathcal{H}} \) the identity transformation of the Banach space \( \mathcal{T}(\mathcal{H}) \).

The von Neumann entropy of a quantum state \( \rho \in \mathcal{S}(\mathcal{H}) \) is defined by the formula

\[
H(\rho) = \text{Tr} \eta(\rho),
\]

where \( \eta(x) = -x \ln x \) for \( x > 0 \) and \( \eta(0) = 0 \). It is a concave lower semicontinuous function on the set \( \mathcal{S}(\mathcal{H}) \) taking values in \([0, +\infty] \) \([14, 20, 34]\). The von Neumann entropy satisfies the inequality

\[
H(p\rho + (1 - p)\sigma) \leq pH(\rho) + (1 - p)H(\sigma) + h_2(p)
\]
valid for any states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) and \( p \in (0, 1) \), where \( h_2(p) = \eta(p) + \eta(1 - p) \) is the binary entropy \([25, 35]\).

The quantum relative entropy for two states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) is defined as

\[
H(\rho \parallel \sigma) = \sum \langle i | \rho \ln \rho - \rho \ln \sigma | i \rangle,
\]
where \( \{|i\} \) is the orthonormal basis of eigenvectors of the state \( \rho \) and it is assumed that \( H(\rho \parallel \sigma) = +\infty \) if \( \text{supp}\rho \) is not contained in \( \text{supp}\sigma \) \([14, 20]\)

The quantum conditional entropy

\[
H(A|B)_{\rho} = H(\rho) - H(\rho_B)
\]
of a state \( \rho \) of a bipartite quantum system \( AB \) with finite marginal entropies is essentially used in analysis of quantum systems \([14, 35]\). It can be extended to the set of all states \( \rho \) with finite \( H(\rho_A) \) by the formula

\[
H(A|B)_{\rho} = H(\rho_A) - H(\rho \parallel \rho_A \otimes \rho_B)
\]
proposed in \([19]\). This extension possesses all basic properties of the quantum conditional entropy valid in finite dimensions \([19, 27]\).

The quantum mutual information of a state \( \rho \) of a bipartite quantum system \( AB \) is defined as

\[
I(A:B)_{\rho} = H(\rho \parallel \rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B) - H(\rho),
\]
where the second formula is valid if \( H(\rho) \) is finite \([21]\).

The quantum conditional mutual information \( (QCMI) \) of a state \( \rho \) of a tripartite finite-dimensional system \( ABC \) is defined as

\[
I(A:B|C)_{\rho} = H(\rho_{AC}) + H(\rho_{BC}) - H(\rho) - H(\rho_C).
\]

---

\(^4\)The support \( \text{supp}\rho \) of a state \( \rho \) is the closed subspace spanned by the eigenvectors of \( \rho \) corresponding to its positive eigenvalues.
This quantity plays important role in quantum information theory \cite{8, 35}, its nonnegativity is a basic result well known as strong subadditivity of von Neumann entropy \cite{22}. If system \( C \) is trivial then \( \Box \) coincides with \( \Box \).

In infinite dimensions formula \( \Box \) may contain the uncertainty ”\( ∞ \rightarrow \Box \). Nevertheless the conditional mutual information can be defined for any state \( \rho \) in \( \mathcal{S}(\mathcal{H}_{ABC}) \) by the expression

\[
I(A:B|C)_\rho = \sup_{P_A} \left[ I(A:BC)_Q - I(A:C)_Q \right], \quad Q = P_A \otimes I_{BC},
\]

where the supremum is over all finite rank projectors \( P_A \in \mathcal{B}(\mathcal{H}_A) \) and it is assumed that \( I(A:B')_Q = \lambda I(A:B')_\lambda \), where \( \lambda = \text{Tr} Q A \rho \) \cite{27}.

Expression \( \Box \) defines the lower semicontinuous nonnegative function on the set \( \mathcal{S}(\mathcal{H}_{ABC}) \) coinciding with the r.h.s. of \( \Box \) for any state \( \rho \) at which it is well defined and possessing all basic properties of the quantum conditional mutual information valid in finite dimensions \cite{27} Th.2. In particular,

\[
I(A:B|C)_\rho \leq 2 \min \{H(\rho_A), H(\rho_B), H(\rho_{AC}), H(\rho_{BC})\}
\]

for arbitrary state \( \rho \) in \( \mathcal{S}(\mathcal{H}_{ABC}) \).

The QCMI of a state \( \rho \) of a finite-dimensional multipartite system \( A_1 \ldots A_n C \) is defined as follows (cf. \cite{3, 13, 36, 37, 39})

\[
I(A_1: \ldots : A_n|C)_\rho \geq \sum_{k=1}^{n} H(A_k|C)_\rho - H(A_1 \ldots A_n|C)_\rho
\]

\[
= \sum_{k=1}^{n} H(A_k|C)_\rho - H(A_1 \ldots A_{n-1}|C)_\rho.
\]

Its nonnegativity and other basic properties can be derived from the corresponding properties of the tripartite QCMI by using the representation (cf. \cite{37})

\[
I(A_1: \ldots : A_n|C)_\rho = I(A_{n-1}: A_n|C)_\rho + I(A_{n-2}: A_{n-1} A_n|C)_\rho + \ldots
\]

\[
+ I(A_1: A_2 \ldots A_n|C)_\rho.
\]

By using representation \( \Box \) and the extended tripartite QCMI described before one can define QCMI for any state of an infinite-dimensional system \( A_1 \ldots A_n C \). The extended QCMI is a lower semicontinuous nonnegative function on the set \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n C}) \) coinciding with the r.h.s. of \( \Box \) for any state \( \rho \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n C}) \) with finite marginal entropies and possessing basic properties of QCMI \cite{27} Proposition 5.

If \( \rho \) and \( \sigma \) are states in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n C}) \) such that \( R = I(A_1: \ldots : A_n|C)_\rho \) and \( S = I(A_1: \ldots : A_n|C)_\sigma \) are finite then

\[
-h_2(p) \leq I(A_1: \ldots : A_n|C)_{pR+(1-p)S} - [pR + (1-p)S] \leq (n-1)h_2(p)
\]
for any $p \in (0, 1)$, where $h_2(\rho)$ is the binary entropy. Indeed, if $\rho$ and $\sigma$ are states with finite marginal entropies then inequality (10) can be proved by using the second expression in (S), concavity of the conditional entropy and inequality (S). The validity of (10) for arbitrary states $\rho$ and $\sigma$ with finite QCMI can be shown by approximation using Proposition 5 in [27].

2.2 The set of quantum states with bounded energy

Let $H_A$ be a positive (semi-definite) densely defined operator on a Hilbert space $\mathcal{H}_A$. We will assume that $\text{Tr} H_A \rho = \sup_n \text{Tr} P_n H_A \rho$ for any positive operator $\rho \in \mathfrak{S}(\mathcal{H}_A)$, where $P_n$ is the spectral projector of $H_A$ corresponding to the interval $[0, n]$.

Let $E_0^A$ be the infimum of the spectrum of $H_A$ and $E \geq E_0^A$. Then

$$\mathcal{C}_{H_A, E} = \{ \rho \in \mathfrak{S}(\mathcal{H}_A) \mid \text{Tr} H_A \rho \leq E \}$$

is a closed convex subset of $\mathfrak{S}(\mathcal{H}_A)$. If $H_A$ is treated as Hamiltonian of a quantum system $A$ then $\mathcal{C}_{H_A, E}$ is the set of states with the mean energy not exceeding $E$.

It is well known that the von Neumann entropy is continuous on the set $\mathcal{C}_{H_A, E}$ for any $E > E_0^A$ if (and only if) the Hamiltonian $H_A$ satisfies the condition

$$\text{Tr} e^{-\lambda H_A} < +\infty \quad \text{for all } \lambda > 0$$

and that the maximal value of the entropy on this set is achieved at the Gibbs state $\gamma_A(E) = e^{-\lambda(E) H_A}/\text{Tr} e^{-\lambda(E) H_A}$, where the parameter $\lambda(E)$ is determined by the equality $\text{Tr} H_A e^{-\lambda(E) H_A} = E \text{Tr} e^{-\lambda(E) H_A} - 1$. Condition (11) implies that $H_A$ is an unbounded operator having discrete spectrum of finite multiplicity. So, by the Lemma in [15] the set $\mathcal{C}_{H_A, E}$ is compact for any $E > E_0^A$.

We will use the function

$$F_{H_A}(E) = \sup_{\rho \in \mathcal{C}_{H_A, E}} H(\gamma_A(E)).$$

It is easy to show that $F_{H_A}$ is a strictly increasing concave function on $[E_0^A, +\infty)$ such that $F_{H_A}(E_0^A) = \ln m(E_0^A)$, where $m(E_0^A)$ is the multiplicity of $E_0^A$.

In this paper we will assume that the Hamiltonian $H_A$ satisfies the condition

$$\lim_{\lambda \to 0^+} [\text{Tr} e^{-\lambda H_A}]^\lambda = 1,$$

which is slightly stronger than condition (11). By Lemma 1 in [29] condition (13) holds if and only if

$$F_{H_A}(E) = o(\sqrt{E}) \quad \text{as } E \to +\infty,$$

while condition (11) is equivalent to $F_{H_A}(E) = o(E)$ as $E \to +\infty$ [26]. It is essential that condition (13) holds for the Hamiltonians of many real quantum systems [4, 29].

---

5 The compactness of $\mathcal{C}_{H_A, E}$ also follows from Corollary 7 in [26] which states that boundedness of the entropy on a convex set of quantum states implies relative compactness of this set.

6 In terms of the sequence $\{E_k\}$ of eigenvalues of $H_A$ condition (11) means that $\lim_{k \to \infty} E_k/\ln k = +\infty$, while condition (13) is valid if $\lim \inf_{k \to \infty} E_k/\ln^q k > 0$ for some $q > 2$ [29, Proposition 1].

7 Theorem 3 in [4] shows that $F_{H_A}(E) = O(\ln E)$ as $E \to +\infty$ if condition (11) below holds.
The function
\[
\bar{F}_{HA}(E) = F_{HA}(E + E_A^0) = H(\gamma_A(E + E_A^0))
\] (15)
is concave and nondecreasing on \([0, +\infty)\). Let \(\hat{F}_{HA}\) be a continuous function on \([0, +\infty)\) such that
\[
\hat{F}_{HA}(E) \geq \bar{F}_{HA}(E) \quad \forall E > 0, \quad \hat{F}_{HA}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty
\] (16)
and
\[
\hat{F}_{HA}(E_1) < \hat{F}_{HA}(E_2), \quad \hat{F}_{HA}(E_1)/\sqrt{E_1} > \hat{F}_{HA}(E_2)/\sqrt{E_2} \quad \forall E_2 > E_1 > 0.
\] (17)
Sometimes we will additionally assume that
\[
\hat{F}_{HA}(E) = \bar{F}_{HA}(E)(1 + o(1)) \quad \text{as} \quad E \to +\infty.
\] (18)
The existence of a function \(\hat{F}_{HA}\) with the required properties is established in the following proposition proved in [30].

**Proposition 1.**

A) If the Hamiltonian \(H_A\) satisfies condition (13) then
\[
\hat{F}_{HA}^*(E) = \sqrt{E} \sup_{E' \geq E} \hat{F}_{HA}(E')/\sqrt{E'}
\]
is the minimal function satisfying all the conditions in (16) and (17).

B) Let
\[
N_{\uparrow}[H_A](E) = \sum_{k,j:k + E_j \leq E} E_k^2 \quad \text{and} \quad N_{\downarrow}[H_A](E) = \sum_{k,j:k + E_j \leq E} E_k E_j
\]
for any \(E > E_A^0\). If
\[
\exists \lim_{E \to +\infty} N_{\uparrow}[H_A](E)/N_{\downarrow}[H_A](E) = a > 1
\] (19)
then
\begin{itemize}
  \item there is \(E_*\) such that the function \(E \mapsto \hat{F}_{HA}(E)/\sqrt{E}\) is nonincreasing for all \(E \geq E_*\) and hence \(\hat{F}_{HA}^*(E) = \hat{F}_{HA}(E)\) for all \(E \geq E_*\);
  \item \(\hat{F}_{HA}^*(E) = (a - 1)^{-1}(\ln E)(1 + o(1))\) as \(E \to +\infty\).
\end{itemize}
Condition (19) is valid for the Hamiltonians of many real quantum systems [4].

Practically, it is convenient to use functions \(\hat{F}_{HA}\) defined by simple formulae. The example of such function \(\hat{F}_{HA}\) satisfying all the conditions in (16), (17) and (18) in the case when \(A\) is a multimode quantum oscillator is considered in Section 3.2.

We will use the following simple

**Lemma 1.**

Let \(H\) be a positive operator on a Hilbert space \(\mathcal{H}\) having discrete spectrum of finite multiplicity and \(P_d\) the projector on the subspace \(\mathcal{H}_d\) corresponding
to the minimal $d$ eigenvalues $E_0, \ldots, E_{d-1}$ of $H$ (taking the multiplicity into account). Then for any state $\rho \in \mathcal{S}(\mathcal{H})$ such that $\text{Tr} H \rho \leq E$ the following inequality holds

$$\text{Tr}(I_\mathcal{H} - P_d)\rho \leq (E - E_0)/(E_d - E_0).$$

Proof. Since $\text{Tr}(I_\mathcal{H} - P_d)\rho = 1 - \text{Tr} P_d \rho$, the required inequality follows directly from the inequalities $E_0 \text{Tr} P_d \rho \leq \text{Tr} P_d H \rho$ and $E_d \text{Tr}(I_\mathcal{H} - P_d)\rho \leq \text{Tr}(I_\mathcal{H} - P_d)H \rho$. \square

3 The main results

3.1 The finite-dimensional case

Many important characteristics of states of a $n$-partite finite-dimensional quantum system $A^n = A_1 \ldots A_n$ have a form of a function $f$ on the set $\mathcal{S}(\mathcal{H}_{A^n})$ satisfying inequality (1) for some $m \leq n$ and the inequalities

$$-a_f h_2(p) \leq f(p \rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \leq b_f h_2(p)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A^n})$ and any $p \in [0,1]$, where $h_2$ is the binary entropy (defined after (3)) and $a_f, b_f \in \mathbb{R}_+$. Inequality (1) can be written in the following more accurate form:

$$-c_f^- S_m(\rho) \leq f(\rho) \leq c_f^+ S_m(\rho), \quad \text{where} \quad S_m(\rho) = \sum_{k=1}^{m} H(\rho_{A_k}), \quad m \leq n, \quad (21)$$

and $c_f^-, c_f^+ \in \mathbb{R}_+$, for any state $\rho$ in $\mathcal{S}(\mathcal{H}_{A^n})$.

Let $L_m^n(C, D)$ be the class of functions on $\mathcal{S}(\mathcal{H}_{A^n})$ satisfying inequalities (20) and (21) with the parameters $a_f, b_f$ and $c_f^+$ such that $a_f + b_f = D$ and $c_f^- + c_f^+ = C$. Denote by $\hat{L}_m^n(C, D)$ the class containing all functions in $L_m^n(C, D)$ and all functions of the form

$$f(\rho) = \sup_{\lambda} f_{\lambda}(\rho) \quad \text{and} \quad f(\rho) = \inf_{\lambda} f_{\lambda}(\rho),$$

where $\{f_{\lambda}\}$ is any family of functions in $L_m^n(C, D)$.

A noncomplete list of important entropic and information characteristics belonging to one of the classes $\hat{L}_m^n(C, D)$ includes the von Neumann entropy, the conditional entropy, the $n$-partite quantum (conditional) mutual information, the one way classical correlation, the quantum discord, the mutual information of a quantum channel, the coherent information of a quantum channel, the information gain of a quantum measurement with and without quantum side information, the $n$-partite relative entropy of entanglement, the quantum topological entropy and its $n$-partite generalization. For example, the von Neumann entropy belongs to the class $L_1^1(1,1)$, while the conditional entropy $H(A_1|A_2)$ lies in the class $L_2^1(2,1)$. This can be shown easily by using concavity of the entropy and the conditional entropy, inequality (3) and the well known
inequality $|H(A_1|A_2)\rho| \leq H(\rho_{A_1})$. It’s a little harder to show that the quantum discords $D(A_1|A_2)$ and $D(A_2|A_1)$ of a state of bipartite system $A_1A_2$ belong, respectively, to the classes $\hat{L}^2(2,2)$ and $\hat{L}^2(1,2)$ (we use the notation from [12]).

There is a general way to construct a characteristic $f$ of a $n$-partite quantum system $A_1...A_n$ via appropriate characteristic $h$ of extended $(n+l)$-partite quantum system $A_1...A_nA_{n+1}...A_{n+l}$: the value of $f$ at any state $\rho$ in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ is defined as

$$f(\rho) = \inf_{\hat{\rho} \in \mathcal{M}(\rho)} h(\hat{\rho}),$$

where $\mathcal{M}(\rho)$ is a particular subset of the set

$$\{ \hat{\rho} \in \mathcal{G}(\mathcal{H}_{A_1...A_{n+l}}) \mid \hat{\rho}_{A_1...A_n} = \rho \}$$

of all extensions of the state $\rho$ to a state of $A_1...A_{n+l}$. For given $m \leq n$ we will denote by $N^m_{n,s}(C, D)$ the class of all functions $f$ on $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ defined by formula (22) via particular function $h$ in $\hat{L}^m_{n+l}(C, D)$ for some $l > 0$ with $\mathcal{M}(\rho) = \mathcal{M}_s(\rho)$, $s = 1, 2, 3$, where:

- $\mathcal{M}_1(\rho)$ is the set (23) of all extensions of $\rho$;
- $\mathcal{M}_2(\rho)$ is the set of all extensions of $\rho$ having the form

$$\hat{\rho} = \sum_i p_i \rho_i \otimes |i\rangle \langle i|,$$

where $\{\rho_i\}$ is a collection of states in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$, $\{p_i\}$ is a probability distribution and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_{A_{n+1}}$ (in this case $l = 1$);
- $\mathcal{M}_3(\rho)$ is the set of all extensions of $\rho$ having the form (24) in which $\{\rho_i\}$ is a collection of pure states in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$.

The classes $N^m_{n,s}(C, D)$ contain important entanglement measures obtained either by the convex roof construction or by the construction called "conditional entanglement" in [38]. For example, the squashed entanglement, the c-squashed entanglement and the entanglement of formation in a bipartite system $A_1A_2$ belong, respectively, to the classes $N^1_{2,1}(1,1)$, $N^1_{2,2}(1,1)$ and $N^1_{2,3}(1,1)$. Indeed, these characteristics can be expressed by formula (22) with $h(\hat{\rho}) = \frac{1}{2} I(A_1 : A_2 | A_3)_{\hat{\rho}}$ in which the infimum is taken, respectively, over the sets $\mathcal{M}_1(\rho)$, $\mathcal{M}_2(\rho)$ and $\mathcal{M}_3(\rho)$. It suffices to note that the function $\varrho \mapsto \frac{1}{2} I(A_1 : A_2 | A_3)_{\varrho}$ belongs to the class $L^3_3(1,1)$ by inequalities (7) and (10).

In $n$-partite quantum systems, the squashed entanglement, the c-squashed entanglement, the conditional entanglement of mutual information belong to one of the classes $N^m_{n,s}(C, D)$ (see details in Section 4). This also concerns other conditional entanglement measures obtained via some function from one of the classes $\hat{L}^m_{n+l}(C, D)$, $l > 0$ [38, 39].

The following proposition gives continuity bounds for functions from the classes $\hat{L}^m_n(C, D)$ and $N^m_{n,s}(C, D)$ in the case of finite-dimensional subsystems $A_1,...,A_m$. [9]
Proposition 2. Let \( f \) be a function on the set of states of a composite quantum system \( A_1 \ldots A_m \ldots A_n \), where the subsystems \( A_1, \ldots, A_m \) are finite-dimensional \((m \leq n)\). Let \( \rho \) and \( \sigma \) be any states in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) such that \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1 \).

If the function \( f \) belongs to the class \( \tilde{L}_m^n(C, D) \) then

\[
|f(\rho) - f(\sigma)| \leq C \varepsilon \ln \dim \mathcal{H}_{A_1 \ldots A_m} + Dg(\varepsilon),
\]

where \( g(x) = (1 + x)h_2(\frac{x}{1 + x}) = (x + 1) \ln(x + 1) - x \ln x \). If the function \( f \) belongs to the class \( N_{m,n}^m(C, D) \) then \((23)\) holds with \( \varepsilon \) replaced by \( \sqrt{\varepsilon(2 - \varepsilon)} \).

It will be shown in Section 4 that the continuity bounds given by Proposition 2 for some important characteristics of multipartite quantum states are asymptotically tight (or close to tight) for large \( \dim \mathcal{H}_{A_1 \ldots A_m} \) \cite{43}.

Proof. If the function \( f \) belongs to the class \( \tilde{L}_m^n(C, D) \) then it satisfies inequality \((21)\) with the parameters \( c_f^- \) and \( c_f^+ \) such that \( c_f^- + c_f^+ = C \). Since the subsystems \( A_1, \ldots, A_m \) are finite-dimensional, it follows that

\[-c_f^- \dim \mathcal{H}_{A_1 \ldots A_m} \leq f(\rho) \leq c_f^+ \dim \mathcal{H}_{A_1 \ldots A_m} \]

for any state \( \rho \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \). So, by applying the Alicki-Fannes-Winter method (presented in the optimal form in \cite{41} and described in a full generality in the proof of Proposition 1 in \cite{27}) we obtain inequality \((25)\). Since the r.h.s. of \((25)\) depends only on the parameters \( C, D \) and \( m \), this inequality remains valid for any function \( f \) in \( \tilde{L}_m^n(C, D) \).

Assume that \( f \) is a function from the class \( N_{m,n}^m(C, D) \) defined via some function \( h \) in \( \tilde{L}_{n+1}^m(C, D) \). Then the standard arguments based on the isometrical equivalence of all purifications of a given state (see \cite{6}) show that

\[
f(\rho) = \inf_{\Lambda} h(\text{Id}_{A_1 \ldots A_n} \otimes \Lambda(\bar{\rho})),
\]

where \( \bar{\rho} \) is a given purification in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) of the state \( \rho \), i.e. a pure state such that \( \text{Tr}_R \bar{\rho} = \rho \), and the infimum is over all channels \( \Lambda : \mathcal{S}(\mathcal{H}_R) \rightarrow \mathcal{S}(\mathcal{H}_{A_{n+1} \ldots A_{n+l}}) \).

Since \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \), there exist purifications \( \bar{\rho} \) and \( \bar{\sigma} \) of the states \( \rho \) and \( \sigma \) such that \( \frac{1}{2} \| \bar{\rho} - \bar{\sigma} \|_1 \leq \delta = \sqrt{\varepsilon(2 - \varepsilon)} \) \cite{14, 35, 41}. By monotonicity of the trace norm we have

\[
\frac{1}{2} \| \text{Id}_{A_1 \ldots A_n} \otimes \Lambda(\bar{\rho}) - \text{Id}_{A_1 \ldots A_n} \otimes \Lambda(\bar{\sigma}) \|_1 \leq \delta
\]

for any channel \( \Lambda \). Thus, by applying continuity bound \((23)\) to the function \( h \) we obtain

\[
|h(\text{Id}_{A_1 \ldots A_n} \otimes \Lambda(\bar{\rho})) - h(\text{Id}_{A_1 \ldots A_n} \otimes \Lambda(\bar{\sigma}))| \leq C \delta \ln \dim \mathcal{H}_{A_1 \ldots A_m} + Dg(\delta).
\]

Since the r.h.s. of this inequality does not depend on \( \Lambda \), it follows from \((26)\) that \((25)\) holds for the function \( f \) with \( \varepsilon \) replaced by \( \delta \).

\(^8\)The basic idea of this method is proposed in \cite{1}, it is then modified in \cite{23, 32, 41}.  

10
Assume that $f$ is a function from the class $N_{n_1,2}^m(C, D)$ defined via some function $h$ in $\tilde{L}_{n_1}^m(C, D)$. We may assume, w.l.o.g., that $f(\rho) \leq f(\sigma)$. For given $\epsilon > 0$ let $\hat{\rho}$ be an extension of $\rho$ having form (24) such that

$$h(\hat{\rho}) \leq f(\rho) + \epsilon. \quad (28)$$

By the Schrodinger-Gisin–Hughston–Jozsa–Wootters theorem (cf. [31, 11, 16]) there is a q-c channel from $R$ to $A_{n+1}$ such that $\hat{\rho} = \text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\rho})$. Since $\frac{1}{2}\|\rho - \sigma\|_1 \leq \epsilon$, there exists a pure state $\bar{\sigma}$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ and a POVM $\{M_i\}$ in $R$ such that $\rho = \text{Tr}_R\bar{\rho}$ and $\rho_i\bar{\rho}_i = \text{Tr}_R[I_{A_1...A_n} \otimes M_i]\bar{\rho}$ for all $i$. So, if

$$\Lambda(\bar{\rho}) = \sum_i [\text{Tr}M_i|\bar{i}\rangle \langle i|$$

is a q-c channel from $R$ to $A_{n+1}$ then $\hat{\rho} = \text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\rho})$. Since $\frac{1}{2}\|\rho - \sigma\|_1 \leq \epsilon$, there exists a pure state $\bar{\sigma}$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ and a POVM $\{M_i\}$ such that $\sigma = \text{Tr}_R\bar{\sigma}$ and $\frac{1}{2}\|\bar{\rho} - \bar{\sigma}\|_1 \leq \delta$ [13 35 41]. Since the inequality (27) holds for the above channel $\Lambda$ by monotonicity of the trace norm, by applying continuity bound (25) to the function $h$ we obtain

$$h(\hat{\rho}) = h(\text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\rho})) \geq h(\text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\sigma})) - C\delta \ln \dim \mathcal{H}_{A_1...A_m} - Dg(\delta).$$

Since the state $\text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\sigma})$ belongs to the set $\mathcal{M}_2(\sigma)$, this inequality and (28) imply that

$$f(\rho) \geq f(\sigma) - C\delta \ln \dim \mathcal{H}_{A_1...A_m} - Dg(\delta) - \epsilon.$$

If $f$ is a function from the class $N_{n_3}^m(C, D)$ then we can repeat the above arguments by noting that in this case the POVM $\{M_i\}$ consists of 1-rank operators, and hence the state $\text{Id}_{A_1...A_n} \otimes \Lambda(\bar{\sigma})$ belongs to the set $\mathcal{M}_3(\sigma)$. $\square$

Applications of Proposition 2 to some important characteristics of multipartite finite-dimensional quantum systems can be found in Section 4.

### 3.2 The infinite-dimensional case: arbitrary subsystems

Assume now that $A_1,...,A_n$ are arbitrary infinite-dimensional quantum systems. We denote by $L_n^m(C, D)$, $m \leq n$, the class of all functions $f$ on the set

$$\mathcal{S}_m(\mathcal{H}_{A_1...A_n}) \doteq \{\rho \in \mathcal{S}(\mathcal{H}_{A_1...A_n}) \mid H(\rho_{A_1}),...,H(\rho_{A_m}) < +\infty\} \quad (29)$$

satisfying inequalities (20) and (21) with the nonnegative parameters $a_f$, $b_f$ and $c_f^\pm$ such that $a_f + b_f = D$ and $c_f^- + c_f^+ = C$. The classes $\tilde{L}_n^m(C, D)$ and $N_n^m(C, D)$, $s = 1, 2, 3$, are defined in the same way as in the finite-dimensional case (see the previous subsection).

We obtain continuity bounds for functions from the above classes under the energy constraint on the system $A^m = A_1...A_m$ assuming that the Hamiltonian of this system has the "standard" form

$$H_{A^m} = H_{A_1} \otimes I_{A_2} \otimes ... \otimes I_{A_m} + ... + I_{A_1} \otimes ... \otimes I_{A_{m-1}} \otimes H_{A_m}. \quad (30)$$
We will use the following simple observation.

**Lemma 2.** If \( H_{A_1}, \ldots, H_{A_m} \) are positive operators on the spaces \( \mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m} \) satisfying condition (13) then the operator \( H_{A^m} \) on the space \( \mathcal{H}_{A_1 \ldots A_m} \) defined in (37) satisfies condition (13) and \(^9\)

\[
\bar{F}_{H_{A^m}}(E) \leq \bar{F}_{H_{A_1}}(E) + \cdots + \bar{F}_{H_{A_m}}(E) \quad \forall E > 0.
\] (31)

If the operators \( H_{A_1}, \ldots, H_{A_m} \) are unitary equivalent to an operator \( H_A \) on \( \mathcal{H}_A \) then

\[
\bar{F}_{H_{A^m}}(E) = m \bar{F}_{H_A}(E/m) \quad \forall E > 0.
\]

**Proof.** By the equivalence of (13) and (14) it suffices to prove inequality (31).

By noting that \( E_0^{A^m} = E_0^{A_1} + \cdots + E_0^{A_m} \) we obtain

\[
\bar{F}_{H_{A^m}}(E) = F_{H_{A^m}}(E + E_0^{A^m}) \leq \max_{E_1 + \cdots + E_m \leq E + E_0^{A^m}, E_i \geq 0} [F_{H_{A_1}}(E_1) + \cdots + F_{H_{A_m}}(E_m)]
\]

\[
= \max_{E_1 + \cdots + E_m \leq E, E_i \geq 0} [\bar{F}_{H_{A_1}}(E_1) + \cdots + \bar{F}_{H_{A_m}}(E_m)] \leq \bar{F}_{H_{A_1}}(E) + \cdots + \bar{F}_{H_{A_m}}(E).
\]

If the operators \( H_{A_1}, \ldots, H_{A_m} \) are unitary equivalent to some operator \( H_A \) then \( \bar{F}_{H_{A_k}}(E) = \bar{F}_{H_A}(E), k = 1, m \). So, the concavity of the function \( \bar{F}_{H_A} \) implies that the last maximum in the above inequality is attained at the point \( E_k = E/m, k = 1, m \).

\( \square \)

The following theorem gives continuity bounds for functions from the classes \( \mathcal{L}_n^m(C, D) \) and \( N_{n,s}^m(C, D) \) under the energy constraint on the system \( A^m = A_1 \ldots A_m \).

**Theorem 1.** Let \( H_{A_1}, \ldots, H_{A_m} \) be positive operators on the spaces \( \mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m} \) satisfying condition (13) and \( H_{A^m} \) the operator on the space \( \mathcal{H}_{A_1 \ldots A_m} \) defined in (37). Let \( \rho \) and \( \sigma \) be arbitrary states in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_m}) \) such that \( \sum_{k=1}^m \Tr H_{A_k} \rho_{A_k}, \sum_{k=1}^m \Tr H_{A_k} \sigma_{A_k} \leq mE \) and \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1 \). Then

\[
|f(\rho) - f(\sigma)| \leq C\sqrt{2\varepsilon} F_{H_A}^m \left[ \frac{mE}{\varepsilon} \right] + Dg(\sqrt{2\varepsilon})
\] (32)

for any function \( f \) from the class \( \mathcal{L}_n^m(C, D) \), where \( \bar{E} = E - m^{-1} E_0^{A^m} \). Inequality (32) holds for any function \( f \) from the class \( N_{n,s}^m(C, D) \) with \( \varepsilon \) replaced by \( \sqrt{\varepsilon(2 - \varepsilon)} \). \(^{10}\)

The right hand side of (32) tends to zero as \( \varepsilon \to 0 \).

**Remark 1.** If the operators \( H_{A_1}, \ldots, H_{A_m} \) are unitary equivalent to some operator \( H_A \) then the last assertion of Lemma 2 shows that inequality (32) can be rewritten as

\[
|f(\rho) - f(\sigma)| \leq Cm\sqrt{2\varepsilon} F_{H_A} \left[ \frac{\bar{E}}{\varepsilon} \right] + Dg(\sqrt{2\varepsilon}).
\] (33)

\(^9\)Here and in what follows we use the notation introduced in Section 2.2.

\(^{10}\)The function \( g(x) \) is defined after inequality (25).
Remark 2. Replacing the function $F_{HA_m}$ by any its upper bound $\tilde{F}_{HA_m}$ such that the function $E \mapsto \tilde{F}_{HA_m}(E)/\sqrt{E}$ is non-increasing makes inequality (32) valid for any $\varepsilon > 0$ (including the case $\varepsilon > 1$).

Proof of Theorem 1. Since $\text{Tr}H_{A_m}[\rho_{A_1} \otimes \ldots \otimes \rho_{A_m}] = \sum_{k=1}^{m} \text{Tr}H_{A_k}\rho_{A_k}$, we have
\[
\sum_{k=1}^{m} H(\rho_{A_k}) = H(\rho_{A_1} \otimes \ldots \otimes \rho_{A_m}) \leq F_{HA_m}(mE) = \tilde{F}_{HA_m}(mE)
\]
for any state $\rho \in \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ such that $\text{Tr}H_{A_m}\rho_{A_m} = \sum_{k=1}^{m} \text{Tr}H_{A_k}\rho_{A_k} \leq mE$. Hence for any such state $\rho$ inequality (21) implies that
\[
-c_f \tilde{F}_{HA_m}(mE) \leq f(\rho) \leq c_f^+ \tilde{F}_{HA_m}(mE).
\]

Thus, in the case $\varepsilon < 1/2$ inequality (32) for any function $f$ from the class $L_n^m(C, D)$ follows from Theorem 1 in [29]. In the case $\varepsilon \geq 1/2$ this inequality directly follows from inequality (34). Since the r.h.s. of (32) depends only on the parameters $C,D$ and the characteristics of the operators $H_{A_1}, H_m$, this inequality remains valid for any function $f$ in $\hat{L}_n^m(C, D)$.

If $f$ is a function from the class $N_{n,A}^m(C, D)$ then the validity of inequality (32) with $\varepsilon$ replaced by $\sqrt{\varepsilon(2 - \varepsilon)}$ is proved by repeating the arguments used in the proof of Proposition 2.

The last assertion of the theorem follows from Lemma 2 □

Corollary 1. Let $H_{A_1}, \ldots, H_{A_m}$ be positive operators on the spaces $\mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m}$ satisfying condition (13). All functions from the classes $\hat{L}_n^m(C, D)$ and $N_{n,A}^m(C, D)$ are uniformly continuous on the set
\[
\left\{ \rho \in \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \mid \sum_{k=1}^{m} \text{Tr}H_{A_k}\rho_{A_k} \leq E \right\}
\]
for any $E > E_0^A$, where $E_0^A$ is the sum of minimal eigenvalues of $H_{A_1}, \ldots, H_{A_m}$.

Remark 3. In the above analysis we considered the classes $\hat{L}_n^m(C, D)$ of functions on the set $\mathcal{S}_n(H_{A_1 \ldots A_n})$ defined in [29]. In a similar way one can introduce the classes $\hat{L}_n^m(C, D | \mathcal{S}_0)$ of functions defined on arbitrary convex subset $\mathcal{S}_0$ of $\mathcal{S}_n(H_{A_1 \ldots A_n})$. By Remark 2 in [29] the above results (Theorem 1 and Corollary 1) are generalized to functions from the classes $\hat{L}_n^m(C, D | \mathcal{S}_0)$ provided that the set $\mathcal{S}_0$ has the following invariance property:
\[
\text{the states } \frac{\text{Tr}_R[\hat{\rho} - \hat{\sigma}]}{\text{Tr}[\hat{\rho} - \hat{\sigma}]} \text{ and } \frac{\text{Tr}_R[\hat{\rho} - \hat{\sigma}]}{\text{Tr}[\hat{\rho} - \hat{\sigma}]} \text{ belong to the set } \mathcal{S}_0
\]
for arbitrary purifications $\hat{\rho}$ and $\hat{\sigma}$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ of any different states $\rho$ and $\sigma$ in $\mathcal{S}_0$, where $T_-$ and $T_+$ are the negative and positive parts of a Hermitian operator $T$.

By the proof of Theorem 1 in [29] the above invariance property holds for any subset $\mathcal{S}_0$ of $\mathcal{S}_n(H_{A_1 \ldots A_n})$ consisting of states $\rho$ with finite values of given energy type functionals $\text{Tr}H_{1}\rho, \ldots, \text{Tr}H_{t}\rho$, where $H_{1}, \ldots, H_{t}$ are arbitrary positive operators on $\mathcal{H}_{A_1 \ldots A_n}$.
3.3 The infinite-dimensional case: identical subsystems

The continuity bounds given by Theorem 1 are simple and universal but non-accurate for small $\varepsilon$ because of their dependence on $\sqrt{\varepsilon}$. More sharp universal continuity bound can be obtained by using two step technique based on appropriate finite-dimensional approximation of arbitrary states $\rho$ and $\sigma$ followed by the Alicki-Fannes-Winter method.\textsuperscript{11}

We apply the two step technique assuming that the subsystems $A_1, \ldots, A_n$ (involved in (21)) are infinite-dimensional and isomorphic to a given system $A$. It means that the Hamiltonians $H_{A_1}, \ldots, H_{A_n}$ of these systems are unitary equivalent to the Hamiltonian $H_A$ of the system $A$. This assumption essentially simplifies the resulting continuity bound and seems reasonable from the point of view of potential applications.

In the following theorem we assume that the Hamiltonian $H_A$ satisfies condition\textsuperscript{13} and has minimal eigenvalue $E_0^A$. We also assume that $\hat{F}_{H_A}$ is any continuous function on $\mathbb{R}_+$ satisfying conditions (16) and (17).\textsuperscript{12}

**Theorem 2.** Let $A^n \doteq A_1 \ldots A_n$, where $A_k \cong A$ for $k = \overline{1,m}$, $m \leq n$, and $A_{m+1}, \ldots, A_n$ are arbitrary systems. Let $\rho$ and $\sigma$ be arbitrary states in $\mathcal{S} (\mathcal{H}_A^n)$ such that $\sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k}$, $\sum_{k=1}^{m} \text{Tr} H_{A_k} \sigma_{A_k} \leq mE$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$. Let $t \in (0, 1/\varepsilon)$.

Then

$$|f(\rho) - f(\sigma)| \leq Cm \left( (\varepsilon + \varepsilon^2 t^2) \hat{F}_{H_A} \left[ \frac{m\hat{E}}{\varepsilon^2 t^2} \right] + 2\sqrt{2\varepsilon t} \hat{F}_{H_A} \left[ \frac{\hat{E}}{\varepsilon t} \right] \right) + m \left( g(\varepsilon + \varepsilon^2 t^2) + 2g(\sqrt{2\varepsilon t}) \right)$$

(35)

for any function $f$ from the class $\hat{L}_m^m (C, D)$, where $\hat{E} = E - E_0^A$. Inequality (35) holds for any function $f$ from the class $N_m^m (C, D)$ with $\varepsilon$ replaced by $\sqrt{\varepsilon(2-\varepsilon)}$.\textsuperscript{13}

If conditions (18) and (19) hold\textsuperscript{13} then for given $\hat{E}$ the r.h.s. of (35) can be written as

$$Cm \left( (\varepsilon + \varepsilon^2 t^2) \ln \left[ \frac{m\hat{E}}{\varepsilon^2 t^2} \right] \frac{1 + o(1)}{a - 1} + 2\sqrt{2\varepsilon t} \ln \left[ \frac{\hat{E}}{\varepsilon t} \right] \frac{1 + o(1)}{a - 1} \right) + m \left( g(\varepsilon + \varepsilon^2 t^2) + 2g(\sqrt{2\varepsilon t}) \right), \varepsilon t \to 0^+.$$  

(36)

If, in addition, $f$ is a function from the class $\hat{L}_m^m (C, D)$ satisfying inequality (21) with the parameters $c_f^-$ and $c_f^+$ such that

$$\lim_{E \to +\infty} \left[ \inf_{\rho \in \mathcal{C}_E^+} f(\rho) \right] = \lim_{E \to +\infty} \left[ \frac{c_f^+}{mF_{H_A}(\hat{E})} - \frac{c_f^-}{mF_{H_A}(\hat{E})} \right] = 0,$$

(37)

where $\mathcal{C}_E^+ = \{ \rho \in \mathcal{S} (\mathcal{H}_A^n) \mid \sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k} \leq mE \}$ and $F_{H_A}$ is the function defined in (12), then continuity bound (35) with optimal $t$ is asymptotically tight for large $E$\textsuperscript{13}.

\textsuperscript{11}Similar technique was used by A.Winter in [11].

\textsuperscript{12}The role of $\hat{F}_{H_A}$ can be played by the function $\hat{F}_{H_A}^*$ defined in Proposition [1]

\textsuperscript{13}The function $g(x)$ is defined after inequality (30).

\textsuperscript{14}By Proposition [11] this holds, in particular, if $\hat{F}_{H_A} = \hat{F}_{H_A}^*$.
Remark 4. Since the function $F_{H_A}$ satisfies condition (16) and (17), the r.h.s. of (35) (denoted by $\forall B_t(\bar{E}, \varepsilon \mid C, D)$ in what follows) is a nondecreasing function of $\varepsilon$ and $\bar{E}$ tending to zero as $\varepsilon \to 0^+$ for each $m$ and any given $\bar{E}, C, D$ and $t \in (0, 1/\varepsilon)$.

Remark 5. The “free” parameter $t$ can be used to optimize continuity bound (35) for given values of $E$ and $\varepsilon$.

Proof. Since the Hamiltonian $H_A$ satisfies condition (13), it has discrete spectrum of finite multiplicity. So, we may assume that

$$H_{A_k} = \sum_{i=0}^{+\infty} E_i |\tau_i^k\rangle\langle \tau_i^k|, \quad k = \overline{1,m},$$

where $\{\tau_i^k\}$ is an orthonormal basis in the Hilbert space $\mathcal{H}_{A_k}$ and $\{E_i\}$ is a nondecreasing sequence of eigenvalues of $H_A$. Let

$$\bar{H}_{A_k} = H_{A_k} - E_{0} A_k = \sum_{i=0}^{+\infty} \bar{E}_i |\tau_i^k\rangle\langle \tau_i^k|,$$

where $\bar{E}_i = E_i - E_{0} A_k$, $P_d^k$ be the projector onto the subspace of $\mathcal{H}_{A_k}$ spanned by the vectors $\tau_0^k, \ldots, \tau_{d-1}^k$ and $\bar{P}_d^k = I_{A_k} - P_d^k$ the projector onto the orthogonal subspace.

For each $d$ such that $\bar{E}_d > m\bar{E}$ consider the states

$$\rho_d = r_d^{-1} Q_d \rho Q_d \quad \text{and} \quad \sigma_d = s_d^{-1} Q_d \sigma Q_d,$$

where $Q_d = P_d^1 \otimes \ldots \otimes P_d^m \otimes I_{A_{m+1}} \otimes \ldots \otimes I_{A_n}$,

$$r_d \equiv \text{Tr} Q_d \rho \geq 1 - m\bar{E}/\bar{E}_d \quad \text{and} \quad s_d \equiv \text{Tr} Q_d \sigma \geq 1 - m\bar{E}/\bar{E}_d. \quad (38)$$

To prove the first inequality in (38) note that Lemma 1 in Section 2.2 implies

$$|\text{Tr} Q_d^{k-1} \rho - \text{Tr} Q_d^k \rho| \leq \|Q_d^{k-1}\| |\text{Tr}[I_{A_1} \otimes \ldots \otimes I_{A_{k-1}} \otimes P_d^k \otimes I_{A_{k+1}} \otimes \ldots \otimes I_{A_n}] \rho |$$

$$= \text{Tr} \bar{P}_d^k \rho_{A_k} \leq \text{Tr} \bar{H}_{A_k} \rho_{A_k} / \bar{E}_d, \quad k = \overline{1,m},$$

where $Q_d^0 = I_{A^n}$ and $Q_d^k = P_d^1 \otimes \ldots \otimes P_d^k \otimes I_{A_{k+1}} \otimes \ldots \otimes I_{A_n}$, $k = \overline{1,m}$. It follows that

$$1 - r_d \leq \sum_{k=1}^{m} |\text{Tr} Q_d^{k-1} \rho - \text{Tr} Q_d^k \rho| \leq \sum_{k=1}^{m} \text{Tr} \bar{H}_{A_k} \rho_{A_k} / \bar{E}_d \leq m\bar{E}/\bar{E}_d.$$

The second inequality in (38) is proved similarly.

The condition $\bar{E}_d > m\bar{E}$ implies that

$$\sum_{k=1}^{m} \text{Tr} H_{A_k} [\rho_d]_{A_k} \leq m\bar{E} \quad \text{and} \quad \sum_{k=1}^{m} \text{Tr} H_{A_k} [\sigma_d]_{A_k} \leq m\bar{E}. \quad (39)$$
Indeed, by the assumption we have

\[ \text{Tr} \hat{H}_A^m \rho_A^m \leq m \bar{E}, \]

where \( \hat{H}_A^m = H_A^m - m E^A_0 I_A^m \) (\( H_A^m \) is the operator defined in (39)). Hence

\[ \text{Tr} \hat{H}_A^m [\rho_d]_{A^m} \leq r^{-1}_d (m \bar{E} - \text{Tr} \hat{H}_A^m T_d \rho_A^m) \leq r^{-1}_d (m \bar{E} - \bar{E}_d \text{Tr} T_d \rho_A^m) \leq m \bar{E}, \]

where \( T_d = I_A^m - P_1^d \otimes \ldots \otimes P_m^d \) and the second inequality follows from the fact that all eigenvalues of \( \hat{H}_A^m \) corresponding to the range of \( T_d \) are not less than \( \bar{E}_d \). The second inequality in (39) is proved similarly.

Winter’s gentle measurement lemma (cf. [40, 35]) implies

\[ \| \omega - \omega_d \|_1 \leq 2 \sqrt{\text{Tr} Q_d \omega} \leq 2 \sqrt{m \bar{E}/E_d}, \quad \omega = \rho, \sigma, \quad (40) \]

where \( Q_d = I_A^m - Q_d \) and the last inequality follows from (38).

Assume that \( f \) is a function from the class \( \hat{L}_m^0 (C, D) \). By using (39) and (40) we obtain from Theorem 1 with Remarks 1 and 2 that

\[ |f(\rho) - f(\rho_d)|, |f(\sigma) - f(\sigma_d)| \leq C m \sqrt{2 \delta_d \hat{F}^2_{H_A}(\bar{E}/\delta_d)} + D g(\sqrt{2 \delta_d}), \quad (41) \]

where \( \delta_d = \sqrt{m \bar{E}/E_d} \).

By using monotonicity of the trace norm under quantum operations and the inequalities in (38) we obtain

\[ \| \rho_d - \sigma_d \|_1 \leq \| Q_d \rho Q_d - Q_d \sigma Q_d \|_1 \| Q_d \rho Q_d \|_1 |1 - r^{-1}_d| + \| Q_d \sigma Q_d \|_1 |1 - s^{-1}_d| \]

\[ \leq 2 \varepsilon + (1 - r_d) + (1 - s_d) \leq 2 \varepsilon + 2m \bar{E}/E_d. \]

Thus, since the states \( [\rho_d]_{A_k} \) and \( [\sigma_d]_{A_k} \) are supported by the \( d \)-dimensional subspace \( P_d^k(\mathcal{H}_{A_k}) \) for each \( k = 1, \ldots, m \), it follows from Proposition 2 that

\[ |f(\rho_d) - f(\sigma_d)| \leq C m \varepsilon_d \ln d + D g(\varepsilon_d), \quad (42) \]

where \( \varepsilon_d = \varepsilon + m \bar{E}/E_d. \)

By using inequalities (11) and (42) we obtain

\[ |f(\rho) - f(\sigma)| \leq |f(\rho) - f(\rho_d)| + |f(\sigma) - f(\sigma_d)| + |f(\rho_d) - f(\sigma_d)| \]

\[ \leq C m \left( 2 \sqrt{2 \delta_d \hat{F}^2_{H_A}(\bar{E}/\delta_d)} + \varepsilon_d \ln d \right) + D \left( 2 g(\sqrt{2 \delta_d}) + g(\varepsilon_d) \right). \quad (43) \]

Since \( \| H_{A_k} P_d^k \| = E_{d-1} \), we have

\[ \ln d = H(d^{-1} P_d^k) \leq F_{H_{A_k}}(E_{d-1}) = F_{H_d}(E_{d-1}) = \hat{F}_{H_d}(\bar{E}_{d-1}) \leq \hat{F}_{H_d}(\bar{E}_{d-1}) \quad \forall d. \quad (44) \]
If \( m\bar{E} \geq \varepsilon^2 t^2 \bar{E}_{d_0} \) for given \( t \in (0, 1/\varepsilon) \), where \( d_0 \) is the multiplicity of \( E^A_0 \), then there is \( d_* > d_0 \) such that \( m\bar{E} < \bar{E}_{d_*} \) and
\[
\frac{m\bar{E}}{\bar{E}_{d_*}} \leq \varepsilon^2 t^2 \leq \frac{m\bar{E}}{\bar{E}_{d_*-1}}. \tag{45}
\]
By using (44), the second inequality in (45) and the monotonicity of \( \hat{F}_{H_A} \) we obtain
\[
\ln d_* \leq \hat{F}_{H_A}(m\bar{E}/(\varepsilon^2 t^2)). \tag{46}
\]
If \( m\bar{E} < \varepsilon^2 t^2 \bar{E}_{d_0} \) then by setting \( d_* = d_0 \) we obtain the first inequality in (45),
\[
m\bar{E} < \bar{E}_{d_*} \quad \text{and} \quad \ln d_* = F_{H_A}(E^A_0) = \hat{F}_{H_A}(0) \leq \hat{F}_{H_A}(0).
\]
So, by monotonicity of \( \hat{F}_{H_A} \), inequality (46) holds in this case as well.

By using the first inequality in (45), upper bound (46) and monotonicity of the functions \( E \mapsto \hat{F}_{H_A}(E)/\sqrt{E} \) and \( g(x) \), it is easy to obtain inequality (35) from the inequality (43) with \( d = d_* \).

If \( f \) is a function from the class \( N^m_{n,A}(C, D) \) then the validity of inequality (35) with \( \varepsilon \) replaced by \( \sqrt{\varepsilon(2 - \varepsilon)} \) is proved by repeating the arguments used in the proof of Proposition 2.

Assume that conditions (18) and (19) hold. Then it follows from Proposition 1B that
\[
\hat{F}_{H_A}(E) = (a - 1)^{-1} \ln(E)(1 + o(1)) \quad \text{as} \quad E \to +\infty. \tag{47}
\]
This implies the asymptotic representation (36).

Assume that \( f \) is a function from the class \( \tilde{L}^m_n(C, D) \) satisfying condition (37). Then for any \( \delta > 0 \) there exists \( E_\delta > E^A_0 \) such that for any \( E > E_\delta \) the set \( C^m_E \) contains states \( \rho \) and \( \sigma \) such that \( |f(\rho) - f(\sigma)| \geq (C - \delta)mF_{H_A}(E) \). Since \( \|\rho - \sigma\|_1 \leq 1 \), it follows that for any \( \varepsilon > 0 \) the set \( C^m_E \) contains states \( \rho_\varepsilon \) and \( \sigma_\varepsilon \) such that
\[
\frac{1}{2}\|\rho_\varepsilon - \sigma_\varepsilon\|_1 \leq \varepsilon \quad \text{and} \quad |f(\rho_\varepsilon) - f(\sigma_\varepsilon)| \geq \varepsilon(C - \delta)mF_{H_A}(E). \tag{48}
\]

By using (47) and the similar representation for the function \( F_{H_A}(E) \) (Theorem 3 in [1]) one can show that for any \( \delta > 0 \) there exists \( E_\delta > E^A_0 \) and \( \varepsilon_\delta \in (0, 1) \) such that the r.h.s. of (35) with \( t = \varepsilon^2 \) does not exceed
\[
C\varepsilon mF_{H_A}(E)(1 + \delta) + X(\varepsilon, E) \quad \text{for all} \quad E \geq E_\delta \quad \text{and} \quad \varepsilon \leq \varepsilon_\delta, \tag{49}
\]
where \( X(\varepsilon, E) \) is a bounded function.

Since \( F_{H_A}(E) \) tends to \(+\infty\) as \( E \to +\infty \), by using upper bound (49) and the states \( \rho_\varepsilon \) and \( \sigma_\varepsilon \) with the properties stated in (45) it is easy to show the asymptotical tightness of the continuity bound (35) for large \( E \).

\footnote{This can be shown by using the states \( \rho_k = \frac{k}{n}\rho + (1 - \frac{k}{n})\sigma, k = 0, 1, .., n \), for sufficiently large \( n \).}
Remark 6. By Remark 2 all arguments from the proof of Theorem 2 are valid for any function $f$ satisfying continuity bounds (25) and (33).

Assume now that the system $A$ is the $\ell$-mode quantum oscillator with the frequencies $\omega_1, \ldots, \omega_\ell$. The Hamiltonian of this system has the form

$$H_A = \sum_{i=1}^{\ell} \hbar \omega_i a_i^* a_i + E_0 I_A, \quad E_0 = \frac{1}{2} \sum_{i=1}^{\ell} \hbar \omega_i,$$

where $a_i$ and $a_i^*$ are the annihilation and creation operators of the $i$-th mode \[14\]. Note that this Hamiltonian satisfies condition (19) with $a = 1 + 1/\ell$ \[4\].

In this case the function $F H_A(E)$ defined in (12) is bounded above by the function $F_{\ell,\omega}(E) = \ell \ln \frac{E + E_0}{\ell E_*} + \ell$, $E_* = \left( \prod_{i=1}^{\ell} \hbar \omega_i \right)^{1/\ell}$, (50) and upper bound (50) is $\varepsilon$-sharp for large $E$ \[29, 30\]. So, the function

$$\bar{F}_{\ell,\omega}(E) = F_{\ell,\omega}(E + E_0) = \ell \ln \frac{E + 2E_0}{\ell E_*} + \ell,$$ (51)

is a upper bound on the function $\bar{F}_{H_A}(E) = F_{H_A}(E + E_0)$ satisfying all the conditions in (16), (17) and (18) \[30\]. By using the function $\bar{F}_{\ell,\omega}$ in the role of the function $F_{H_A}$ in Theorem 2 we obtain the following

**Corollary 2.** Let $A$ be the $\ell$-mode quantum oscillator with the frequencies $\omega_1, \ldots, \omega_\ell$. Let $A^n = A_1 \ldots A_n$, where $A_k \cong A$ for $k = 1, m$, $m \leq n$. Let $\rho$ and $\sigma$ be arbitrary states in $\mathcal{G}(H_{A^n})$ such that $\sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k}, \sum_{k=1}^{m} \text{Tr} H_{A_k} \sigma_{A_k} \leq mE$ and $1/2 \| \rho - \sigma \|_1 \leq \varepsilon$. Let $t \in (0, 1/\varepsilon)$. Then

$$|f(\rho) - f(\sigma)| \leq Cm(\varepsilon + \varepsilon^2 t^2) \ell \ln \left[ \frac{mE/(\varepsilon^2 t^2) + 2E_0}{e^{-1}\ell E_*} \right]$$

$$+ 2Cm\sqrt{2\varepsilon t} \ell \ln \left[ \frac{\bar{E}/(\varepsilon t) + 2E_0}{e^{-1}\ell E_*} \right] + D\left( g(\varepsilon + \varepsilon^2 t^2) + 2g(\sqrt{2\varepsilon t}) \right)$$ (52)

for any function $f$ from the class $\hat{L}_m^n(C, D)$, where $\bar{E} = E - E_0$. Inequality (52) holds for any function $f$ from the class $N_{n,s}^m(C, D)$ with $\varepsilon$ replaced by $\sqrt{\varepsilon(2 - \varepsilon)}$.

If $f$ is a function from the class $\hat{L}_m^n(C, D)$ satisfying condition (37) then continuity bound (52) with optimal $t$ is asymptotically tight for large $E$ \[43\].
4 Applications

4.1 Multipartite quantum (conditional) mutual information

The quantum mutual information of a state \( \rho \) of a multipartite system \( A_1 \ldots A_n \) is defined as follows (cf. [21, 13, 36, 37, 39]):

\[
I(A_1: \ldots : A_n)_\rho = H(\rho) - \sum_{k=1}^{n} H(\rho_{A_k}) - H(\rho), \tag{53}
\]

where the second formula is valid if \( H(\rho) < +\infty \). If all the marginal entropies \( H(\rho_{A_1}), \ldots, H(\rho_{A_n}) \) are finite then the second formula in (53) implies that

\[
I(A_1: \ldots : A_n)_\rho \leq \sum_{k=1}^{n} H(\rho_{A_k}). \tag{54}
\]

It follows from inequality (10) in Section 2 (with trivial system \( C \)) that the function \( f(\rho) = I(A_1: \ldots : A_n)_\rho \) satisfies inequality (20) with \( a_f = 1 \) and \( b_f = n - 1 \). The nonnegativity of the quantum mutual information and upper bound (54) show that this function satisfies inequality (21) with \( m = n, c_f^- = 0 \) and \( c_f^+ = 1 \). It follows that it belongs to the class \( L_n^n(1, n) \).

Thus, if all the subsystems \( A_1, \ldots, A_n \) are finite-dimensional then Proposition 2 implies that

\[
|I(A_1: \ldots : A_n)_\rho - I(A_1: \ldots : A_n)_\sigma| \leq \varepsilon \ln \dim \mathcal{H}_{A_1 \ldots A_n} + n g(\varepsilon) \tag{55}
\]

for any states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) such that \( \frac{1}{2}\| \rho - \sigma \|_1 \leq \varepsilon \). If all the subsystems \( A_1, \ldots, A_n \) have the same dimension \( d \) then there is a pure state \( \rho \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) such that \( \rho_{A_k} = d^{-1}I_{A_k} \) for \( k = 1, \ldots, n \). Since \( I(A_1: \ldots : A_n)_\rho = n \ln d = \ln \dim \mathcal{H}_{A_1 \ldots A_n} \), by using any product state \( \sigma \) one can show that continuity bound (55) is asymptotically tight for large \( d \) [43]. Note that continuity bound (55) cannot be obtained by applying Audenaert’s continuity bound (cf. [2]) to the summands in the second formula in (53).

In the infinite-dimensional case uniform continuity bounds for the function \( f(\rho) = I(A_1: \ldots : A_n)_\rho \) can be obtained by applying Theorems 1 and 2. In the following proposition \( \mathbb{V} \mathbb{E}^m_t(E, \varepsilon \mid C, D) \) denotes the expression in the r.h.s. of (35) defined by means of any continuous function \( F_{HA} \) on \( \mathbb{R}_+ \) satisfying conditions (16) and (17).
\textbf{Proposition 3.} Let \( n \geq 2 \) be arbitrary and \( H_{A_1}, \ldots, H_{A_n} \) the Hamiltonians of quantum systems \( A_1, \ldots, A_n \) satisfying condition (15). Let \( \rho \) and \( \sigma \) be states in \( \mathcal{G}(\mathcal{H}_{A_1} \square \ldots \square \mathcal{H}_{A_n}) \) such that \( \sum_{k=1}^{n} \text{Tr} H_{A_k} \rho_{A_k}, \sum_{k=1}^{n} \text{Tr} H_{A_k} \sigma_{A_k} \leq nE \) and \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1 \). Then

\[
|I(A_1: \ldots : A_n)_{\rho} - I(A_1: \ldots : A_n)_{\sigma}| \leq \sqrt{2\varepsilon} \bar{F}_{H} \left[ \frac{nE}{\varepsilon} \right] + ng(\sqrt{2\varepsilon}), \tag{56}
\]

where \( \bar{F}_{H} \) is the function defined in (13) with \( A = A^n = A_1 \ldots A_n \) and \( \bar{E} = E - E_0^A/n \).

If \( A_k \cong A \) for \( k = 1, n \) then

\[
|I(A_1: \ldots : A_n)_{\rho} - I(A_1: \ldots : A_n)_{\sigma}| \leq \mathcal{V}\mathcal{B}_n(\bar{E}, \varepsilon | 1, n) \tag{57}
\]

for any \( t \in (0, 1/\varepsilon) \), where \( \bar{E} = E - E_0^A \).

The right hand sides of (56) and (57) tend to zero as \( \varepsilon \to 0 \) for given \( \bar{E} \) and \( t \).

If conditions (18) and (19) hold then continuity bound (57) with optimal \( t \) is asymptotically tight for large \( E \) \cite{43}. This is true, in particular, if \( A \) is the \( \ell \)-mode quantum oscillator and \( \bar{F}_{H} = \bar{F}_{t,\omega}^{(\ell)} \). In this case (57) holds with \( \mathcal{V}\mathcal{B}_n(\bar{E}, \varepsilon | 1, n) \) replaced by the r.h.s. of (22) with \( C = 1 \) and \( D = n \).

\textbf{Proof.} Continuity bounds (56) and (57) follow, respectively, from Theorems 1 and 2. Since the Hamiltonians \( H_{A_1}, \ldots, H_{A_n} \) satisfy condition (13), Lemma 2 implies that \( \bar{F}_{H} \) is \( o(1/F) \) as \( E \to +\infty \) and hence the r.h.s. of (56) tends to zero as \( \varepsilon \to 0 \). The r.h.s. of (57) tends to zero as \( \varepsilon \to 0 \) by Remark 4.

To prove the asymptotical tightness of continuity bound (57) it suffices, by Theorem 2, to show that both relations in (57) hold for the function \( f(\rho) = I(A_1: \ldots : A_n)_{\rho} \). The first relation in (57) can be shown by considering the state \( \rho_E^2 = \gamma_{A_1}(E) \otimes \cdots \otimes \gamma_{A_n}(E) \) at which the function \( f \) is equal to zero for any \( E > 0 \). The second relation in (57) can be shown by using the pure state

\[
\rho_E^2 = \sum_{i,j} \sqrt{p_i p_j} \langle \varphi_i^1 \rangle \langle \varphi_j^1 \rangle \otimes \cdots \otimes \langle \varphi_i^n \rangle \langle \varphi_j^n \rangle,
\]

where \( \sum_i p_i \varphi_i \langle \varphi_i \rangle \) is the spectral decomposition of the Gibbs state \( \gamma_{A_k}(E) \) in \( \mathcal{G}(\mathcal{H}_{A_k}) \), since it is easy to see that \( f(\rho_E^2) = nF_{H}(E) \) for any \( E > 0 \).

The last assertion of the proposition follows from Corollary 2. \qed

The quantum conditional mutual information (QCM) of a state \( \rho \) of a finite-dimensional multipartite system \( A_1 \ldots A_n C \) is defined by conditioning the second expression in (53), i.e. by replacing all the entropies \( H(\rho_X) \) in this expression by the conditional entropies \( H(X|C)_{\rho} \).

Similar to the multipartite quantum mutual information the multipartite QCM has a nonnegative lower semicontinuous extension to the set of all states of an infinite-dimensional multipartite system \( A_1 \ldots A_n C \) possessing all basic properties of QCM. \footnote{The function \( F_{t,\omega} \) is defined in (51).}
But in contrast to the unconditional mutual information the entangled multipartite QCMI can not be expressed by a simple formula for any state in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ (see details in Section 2.1).

If the marginal entropies $H(\rho_{A_1}), ..., H(\rho_{A_{n-1}})$ of a state $\rho \in \mathcal{G}(\mathcal{H}_{A_1...A_n})$ are finite then the QCMI is given by formula (31) in which all the summands are explicitly expressed via the quantum mutual information as follows

$$I(A_{n-k}:A_{n-k+1}...A_{n}|C)_{\rho} = I(A_{n-k}:A_{n-k+1}...A_{n}C)_{\rho} - I(A_{n-k}:C)_{\rho}, \quad k = 1, n - 1.$$ 

Upper bound (32) implies that

$$I(A_1:...:A_n|C)_{\rho} \leq 2\sum_{k=1}^{n-1} H(\rho_{A_k}). \quad (58)$$

If all the marginal entropies $H(\rho_{A_1}), ..., H(\rho_{A_n})$ are finite then by using a version of inequality (58) with arbitrary $n - 1$ subsystems of $A_1...A_n$ (instead of $A_1, ..., A_{n-1}$) it is easy to show that

$$I(A_1:...:A_n|C)_{\rho} \leq 2\frac{n-1}{n} \sum_{k=1}^{n} H(\rho_{A_k}). \quad (59)$$

It follows from inequality (59) that the function $f(\rho) = I(A_1:...:A_n|C)_{\rho}$ satisfies inequality (21) with $a_f = 1$ and $b_f = n - 1$. The nonnegativity of QCMI and upper bounds (58) and (59) show that this function satisfies inequality (21) with $c_f^+ = 2$ in the case $m = n - 1$ and with $c_f^- = 0$ and $c_f^+ = 2 - 2/n$ in the case $m = n$. It follows that it belongs to the classes $L^n_{n-1}(2, n)$ and $L^n_{n}(2 - 2/n, n)$.

Thus, if all the subsystems $A_1, ..., A_{n-1}$ are finite-dimensional then Proposition 2 implies that

$$|I(A_1:...:A_n|C)_{\rho} - I(A_1:...:A_n|C)_{\sigma}| \leq 2\varepsilon \ln \dim \mathcal{H}_{A_1...A_{n-1}} + ng(\varepsilon) \quad (60)$$

for any states $\rho$ and $\sigma$ in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$. If all the subsystems $A_1, ..., A_n$ are finite-dimensional then the first summand in the r.h.s. of (60) can be replaced by $(2 - 2/n)\varepsilon \ln \dim \mathcal{H}_{A_1...A_{n}}$. It is easy to show that continuity bound (60) is asymptotically tight for large dim $\mathcal{H}_{A_1}$ in the case $n = 2$.

In the infinite-dimensional case we may directly apply Theorems 1 and 2 to the function $f(\rho) = I(A_1:...:A_n|C)_{\rho}$ in both cases $m = n - 1$ and $m = n$. This gives continuity bounds for $I(A_1:...:A_n|C)_{\rho}$ under two forms of energy constraint:

- the energy constraint on the subsystem $A_1...A_{n-1}$;
- the energy constraint on the whole system $A_1...A_{n}$. 

21
In the following proposition $\forall \mathbb{B}_i^m(E, \varepsilon | C, D)$ denotes the expression in the r.h.s. of (35) defined by means of any continuous function $\hat{F}_A$ on $\mathbb{R}_+$ satisfying conditions (16) and (17).

**Proposition 4.** Let $n \geq 2$ be arbitrary and $H_{A_1}, ..., H_{A_m}$ the Hamiltonians of quantum systems $A_1, ..., A_m$ satisfying condition (13), where either $m = n - 1$ or $m = n$. Let $\rho$ and $\sigma$ be states in $\mathcal{S}(H_{A_1}, \ldots, H_{A_m})$ such that $\sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k}$, $\sum_{k=1}^{m} \text{Tr} H_{A_k} \sigma_{A_k} \leq mE$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1$. Let $C_m = (n-1)/m$ and $A^m = A_1 \ldots A_m$. Then

$$|I(A_1: \ldots : A_n | C)_\rho - I(A_1: \ldots : A_n | C)_\sigma| \leq 2C_m \sqrt{2\varepsilon} \hat{F}_{H_A} \left[ \frac{mE}{\varepsilon} \right] + n g(\sqrt{2\varepsilon}),$$

(61)

where $\hat{F}_{H_A}$ is the function defined in (15) with $A = A^m$ and $\bar{E} = E - E_0^{A^m}/m$.

If $A_k \equiv A$ for $k = 1, \ldots, m$ then

$$|I(A_1: \ldots : A_n | C)_\rho - I(A_1: \ldots : A_n | C)_\sigma| \leq \forall \mathbb{B}_i^m(E, \varepsilon | 2C_m, n)$$

(62)

for any $t \in (0, 1/\varepsilon)$, where $\bar{E} = E - E^A$. The right hand sides of (61) and (62) tends to zero as $\varepsilon \to 0$ for given $E$ and $t$.

If conditions (18) and (19) hold then continuity bound (62) with optimal $t$ are close-to-tight for large $E$ up to the factor $2 - 2/n$ in the main term in both cases $m = n - 1$ and $m = n$. This is true, in particular, if $A$ is the $\ell$-mode quantum oscillator and $\hat{F}_{H_A} = \hat{F}_{\ell, \omega}$. In this case (62) holds with the r.h.s. replaced by the r.h.s. of (52) with $C = 2C_m$ and $D = n$.

**Proof.** By the observations before the proposition continuity bounds (61) and (62) follow, respectively, from Theorems 1 and 2. Since the Hamiltonians $H_{A_1}, ..., H_{A_m}$ satisfy condition (13), Lemma 2 implies that $\hat{F}_{H_A}(E)$ is $o(\sqrt{E})$ as $E \to +\infty$ and hence the r.h.s. of (61) tends to zero as $\varepsilon \to 0$. The r.h.s. of (62) tends to zero as $\varepsilon \to 0$ by Remark 4.

To prove the assertion concerning accuracy of continuity bound (62) assume that conditions (18) and (19) hold and that $C$ is a trivial system, i.e. $I(A_1: \ldots : A_n | C)_\rho = I(A_1: \ldots : A_n | C)_\rho$. By using the states $\rho_E$ and $\rho_E^2$ introduced in the proof of Proposition 3 and by repeating the arguments from the proof of the last assertion of Theorem 2 it is easy to show that continuity bound (62) with optimal $t$ is close-to-tight for large $E$ up to the factor $2 - 2/n$ in the main term in both cases $m = n - 1$ and $m = n$.

The last assertion of the proposition follows from Corollary 2.

**Remark 7.** If $n = 2$ and conditions (18) and (19) hold (in particular, if $A$ is the $\ell$-mode quantum oscillator) then continuity bound (62) with optimal $t$ is asymptotically tight for large $E$ in both cases $m = 1$ and $m = 2$ [13].

Continuity bound (61) in the case $m = n - 1$ implies the following

**Corollary 3.** Let $A_1, \ldots, A_n$ and $C$ be arbitrary quantum systems. If the Hamiltonians $H_{A_1}, ..., H_{A_{n-1}}$ satisfy condition (13) then the function $\rho \mapsto I(A_1: \ldots : A_n | C)_\rho$ is uniformly continuous on the set of states $\rho$ in $\mathcal{S}(H_{A_1}, \ldots, H_{A_{n-1}} | C)$ s.t. $\sum_{k=1}^{n-1} \text{Tr} H_{A_k} \rho_{A_k} \leq E$ for any $E > E_0^{A_{n-1}}$, where $E_0^{A_{n-1}}$ is the sum of minimal eigenvalues of $H_{A_1}, ..., H_{A_{n-1}}$.
4.2 Squashed entanglement and c-squashed entanglement

The **squashed entanglement** of a state $\rho$ of a finite-dimensional multipartite system $A_1...A_n$ is defined as

$$E_{sq}(\rho) = \frac{1}{2} \inf_{\hat{\rho} \in \mathcal{M}_1(\rho)} I(A_1 : ... : A_n | E_{\hat{\rho}}), \quad (63)$$

where $\mathcal{M}_1(\rho)$ is the set of all extensions $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{A_1...A_nE})$ of the state $\rho$. By using the extended multipartite QCMI described in Section 2.1 this definition can be generalized to any state $\rho$ of an infinite-dimensional $n$-partite system $A_1...A_n$. By using the arguments from [3, 37] one can show that in this case the function $E_{sq}$ defined by formula (63) possesses almost all properties of an entanglement measure, in particular, it is convex on the whole set of states and nonincreasing under LOCC.

Similar to the bipartite case, it is not clear how to show that $E_{sq}$ is equal to zero on the set of all separable states because of the existence of countably nondecomposable separable states in infinite-dimensional composite systems (see Remark 10 in [28]).

The **c-squashed entanglement** of a state $\rho$ of a finite-dimensional multipartite quantum system $A_1...A_n$ is defined as

$$E_{c sq}(\rho) = \frac{1}{2} \inf_{\hat{\rho} \in \mathcal{M}_2(\rho)} I(A_1 : ... : A_n | E_{\hat{\rho}}),$$

where $\mathcal{M}_2(\rho)$ is the set of all extensions $\hat{\rho} \in \mathcal{S}(\mathcal{H}_{A_1...A_nE})$ of the state $\rho$ having form (24) with $A_{n+1} = E$. By using the extended multipartite QCMI described in Section 2.1 this definition can be generalized to any state $\rho$ of an infinite-dimensional $n$-partite system $A_1...A_n$. The c-squashed entanglement can be also defined as the mixed convex roof of the quantum mutual information, i.e.

$$E_{c sq}(\rho) = \frac{1}{2} \inf \sum p_i I(A_1 : ... : A_n | \rho_i),$$

where the infimum is over all countable collections $\{\rho_i\}$ of states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ and probability distributions $\{p_i\}$ such that $\sum_i p_i \rho_i = \rho$. The equality

$$\sum_i p_i I(A_1 : ... : A_n | \rho_i) = I(A_1 : ... : A_n | E_{\hat{\rho}}), \quad \hat{\rho} = \sum_i p_i \rho_i \otimes |i\rangle\langle i|,$$

is easily verified if all the states $\rho_i$ have finite marginal entropies and the Shannon entropy of the probability distribution $\{p_i\}$ is finite. The validity of this equality in general case can be proved by using the approximation property for the extended multipartite QCMI stated in Proposition 5 in [27].

In Section 4.1 it is mentioned that the function $\rho \mapsto I(A_1 : ... : A_n | E_{\hat{\rho}})$ belongs to the classes $L_{n-1}^n(2, n)$ and $L_n^n(2 - 2/n, n)$. Hence the function $2E_{sq}$ belongs to the

---

18In [3, 37] two $n$-partite generalizations of the bipartite squashed entanglement are proposed: the first one is defined in [63], the second one is defined by the expression similar to (63) with the different $n$-partite version of QCMI (called dual conditional total correlation or secrecy monotones). In [7] it is proved that these $n$-partite generalizations of the bipartite squashed entanglement coincide.
classes \( N_{n,1}^{n-1}(2, n) \) and \( N_{n,1}^{n}(2 - 2/n, n) \), while the function \( 2E_{sq}^c \) belongs to the classes \( N_{n,2}^{n-1}(2, n) \) and \( N_{n,2}^{n}(2 - 2/n, n) \).

Thus, if the subsystems \( A_1, ..., A_{n-1} \) are finite-dimensional then Proposition 2 implies that

\[
2|E_{sq}^*(\rho) - E_{sq}^*(\sigma)| \leq 2\delta \ln \dim \mathcal{H}_{A_1 ... A_{n-1}} + ng(\delta), \quad E_{sq}^* = E_{sq}, E_{sq}^c, \tag{64}
\]

for any states \( \rho \) and \( \sigma \) in \( \mathcal{G}(\mathcal{H}_{A_1 ... A_n}) \) s.t. \( \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1 \), where \( \delta = \sqrt{\varepsilon(2 - \varepsilon)} \).

If all the subsystems \( A_1, ..., A_n \) are finite-dimensional then the first summand in the r.h.s. of (64) can be replaced by \((2 - 2/n)\delta \ln \dim \mathcal{H}_{A_1 ... A_n}\).

In the infinite-dimensional case Theorems 1 and 2 (along with Corollary 2) give continuity bounds for the functions \( E_{sq} \) and \( E_{sq}^c \) under two forms of energy constraint. They correspond to the cases \( m = n - 1 \) and \( m = n \) in the following proposition, in which \( \mathbb{V}^m B (\bar{E}, \varepsilon | C, D) \) denotes the expression in the r.h.s. of (35) defined by means of any continuous function \( \bar{F}_{H_A} \) on \( \mathbb{R}_+ \) satisfying conditions (16) and (17).

**Proposition 5.** Let \( n \geq 2 \) be arbitrary and \( H_{A_1}, ..., H_{A_m} \) the Hamiltonians of quantum systems \( A_1, ..., A_m \) satisfying condition (13), where either \( m = n - 1 \) or \( m = n \). Let \( \rho \) and \( \sigma \) be states in \( \mathcal{G}(\mathcal{H}_{A_1 ... A_n}) \) such that \( \sum_{k=1}^m \text{Tr} H_{A_k} \rho_{A_k} \), \( \sum_{k=1}^m \text{Tr} H_{A_k} \sigma_{A_k} \leq m\bar{E} \) and \( \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1 \). Let \( C_m = (n-1)/m \), \( A_m = A_1 ... A_m \) and \( \delta = \sqrt{\varepsilon(2 - \varepsilon)} \). Then

\[
2|E_{sq}^*(\rho) - E_{sq}^*(\sigma)| \leq 2C_m \sqrt{2\delta} \bar{F}_{H_{A_m}} \left[ \frac{m\bar{E}}{\delta} \right] + ng(\sqrt{2\delta}), \quad E_{sq}^* = E_{sq}, E_{sq}^c, \tag{65}
\]

where \( \bar{F}_{H_{A_m}} \) is the function defined in (15) with \( A = A_m \) and \( \bar{E} = E - E_0^{A_m}/m \).

If \( A_k \cong A \) for \( k = 1, m \) then

\[
2|E_{sq}^*(\rho) - E_{sq}^*(\sigma)| \leq \mathbb{V}^m B (\bar{E}, \delta | 2C_m, n), \quad E_{sq}^* = E_{sq}, E_{sq}^c, \tag{66}
\]

for any \( t \in (0, 1/\delta) \), where \( \bar{E} = E - E_0^A \).

The right hand sides of (65) and (66) tend to zero as \( \varepsilon \to 0 \) for given \( \bar{E} \) and \( t \).

If \( A \) is the \( \ell \)-mode quantum oscillator then inequality (66) holds with the r.h.s. replaced by the r.h.s. of (52) with \( \delta \) instead of \( \varepsilon \), \( C = 2C_m \) and \( D = n \) for any \( t \in (0, 1/\delta) \).

The continuity bounds in (65) imply the following

**Corollary 4.** Let \( A_1, ..., A_n \) be arbitrary quantum systems. If the Hamiltonians \( H_{A_1}, ..., H_{A_{n-1}} \) satisfy condition (13) then

A) the functions \( E_{sq} \) and \( E_{sq}^c \) are uniformly continuous on the set of states \( \rho \) in \( \mathcal{G}(\mathcal{H}_{A_1 ... A_n}) \) such that \( \sum_{k=1}^{n-1} \text{Tr} H_{A_k} \rho_{A_k} \leq E \) for any \( E > E_0^{A_{n-1}} = E_0^{A_1} + ... + E_0^{A_{n-1}} \).

B) the functions \( E_{sq} \) and \( E_{sq}^c \) are asymptotically continuous in the following sense (cf.[2]): if \{\( \rho_d \)\} and \{\( \sigma_d \)\} are any sequences of states such that \( \rho_d, \sigma_d \in \mathcal{G}(\mathcal{H}_{A_1 ... A_n}), \text{Tr} H_{B^d} \rho_{B^d}, \text{Tr} H_{B^d} \sigma_{B^d} \leq dE, \forall d, \text{ and } \lim_{d \to +\infty} \|\rho_d - \sigma_d\|_1 = 0, \)
where \( X^d \) denotes \( d \) copies of a system \( X \), \( B = A_1...A_{n-1} \) and \( H_{B^d} \) is the Hamiltonian of the system \( B^d \), then

\[
\lim_{d \to +\infty} \frac{|E_{sq}^*(\rho_d) - E_{sq}^*(\sigma_d)|}{d} = 0, \quad E_{sq}^* = E_{sq}, E_{sq}^c.
\]

Proof. The first assertion of the corollary directly follows from the continuity bounds in (65) in the case \( m = n - 1 \) (since the r.h.s. of (65) vanishes as \( \varepsilon \to 0 \)).

To prove the second assertion note that \( F_{H_{B^d}}(E) = dF_{H_0}(E/d) \) and \( E_d^B_0 = dE^B_0 \) for each \( d \) and hence \( F_{H_{B^d}}(E) = dF_{H_0}(E/d) \). So, continuity bound (65) with \( m = n - 1 \) implies that

\[
\frac{2|E_{sq}^*(\rho_d) - E_{sq}^*(\sigma_d)|}{d} \leq 2\sqrt{2\delta_d F_{H_0}(E/\delta_d)} + (n/d)g(\sqrt{2\delta_d}), \quad E_{sq}^* = E_{sq}, E_{sq}^c \quad (67)
\]

where \( \delta_d = \sqrt{\varepsilon_d(2 - \varepsilon_d)} \), \( \varepsilon_d = \frac{1}{d} \| \rho_d - \sigma_d \|_1 \) and \( E = E - E^B_0 \). Since the sequence \( \{ \varepsilon_d \} \) is vanishing by the condition and \( F_{H_0}(E) \) is \( o(\sqrt{E}) \) as \( E \to +\infty \) by Lemma 2, the r.h.s. of (67) tends to zero as \( d \to +\infty \). □

### 4.3 Conditional entanglement of mutual information

The conditional entanglement of mutual information of a state \( \rho \) of a finite-dimensional multipartite system \( A_1...A_n \) is defined as

\[
E_1(\rho) = \frac{1}{2} \inf_{\hat{\rho} \in M_1(\rho)} \{ I(A_1A'_1: ... : A_nA'_n|_{\hat{\rho}}) - I(A'_1: ... : A'_n|_{\hat{\rho}}) \},
\]

where \( M_1(\rho) \) is the set of all extensions \( \hat{\rho} \in \mathcal{S}(\mathcal{H}_{A_1...A_nA'_1...A'_n}) \) of the state \( \rho \) [38 39]. This definition can be extended to an arbitrary state \( \rho \) of an infinite-dimensional multipartite system \( A_1...A_n \) by noting that the function

\[
\Delta(\varrho) = I(A_1A'_1: ... : A_nA'_n|_{\varrho}) - I(A'_1: ... : A'_n|_{\varrho})
\]

well defined for any state \( \varrho \) with finite \( I(A'_1: ... : A'_n|_{\varrho}) \) has a nonnegative lower semicontinuous extension to the set of all states of the infinite-dimensional system \( A_1...A_nA'_1...A'_n \) given by the expression

\[
\Delta(\varrho) = I(A_1A'_2...A'_n|A'_1|_{\varrho}) + \sum_{k=2}^{n} I(A_k:A_1...A_{k-1}A'_1...A'_{k-1}A'_{k+1}...A'_n|A'_k|_{\varrho}), \quad (68)
\]

in which all the summands are the extended tripartite QCMI described in Section 2.1 [27 Proposition 8]. This expression and upper bound (7) imply that

\[
\Delta(\varrho) \leq 2 \sum_{k=1}^{n} H(\varrho_{A_k}) \quad \forall \varrho \in \mathcal{S}(\mathcal{H}_{A_1...A_nA'_1...A'_n}). \quad (69)
\]

---

19In finite dimensions expression [18] was obtained in [30].
If \( \rho \) is a state in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n A'_1 \ldots A'_{n'}}) \) with finite marginal entropies then

\[
\Delta(\rho) = \sum_{k=1}^{n} H(A_k | A'_k)_{\rho} - H(A_1 \ldots A_n | A'_1 \ldots A'_{n'})_{\rho}.
\]

By using this representation, concavity of the conditional entropy and inequality (8) it is easy to show that the function \( f = \Delta \) satisfies inequality (20) with \( a_f = 1 \) and \( b_f = n \) for any states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n A'_1 \ldots A'_{n'}}) \) with finite marginal entropies. Using this, representation (68) and Corollary 9 in [27] one can prove that the function \( f = \Delta \) satisfies inequality (20) with \( a_f = 1 \) and \( b_f = n \) for arbitrary states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n A'_1 \ldots A'_{n'}}) \). Inequality (69) and nonnegativity of \( \Delta(\rho) \) mean that the function \( f = \Delta \) satisfies inequality (21) with \( c_f^* = 0 \) and \( c_f = 2 \).

These observations show that the function \( \Delta \) belongs to the class \( L_{2n}^n(2, n+1) \). It follows that the function \( 2E_I \) belongs to the class \( N_{2n}^{n+1}(2, n+1) \).

Thus, if the subsystems \( A_1, \ldots, A_n \) are finite-dimensional then Proposition 2 implies that

\[
2|E_I(\rho) - E_I(\sigma)| \leq 2\delta \ln \dim \mathcal{H}_{A_1 \ldots A_n} + (n + 1)g(\delta)
\]

for any states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) s.t. \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1 \), where \( \delta = \sqrt{\varepsilon(2 - \varepsilon)} \).

In the infinite-dimensional case continuity bounds for the function \( E_I \) can be obtained by using Theorems 1 and 2 (along with Corollary 2). They are presented in the following proposition, in which \( \mathcal{V}\mathcal{B}_H^n(\bar{E}, \varepsilon | C, D) \) denotes the expression in the r.h.s. of (15) defined by means of any continuous function \( \mathcal{F}_{HA} \) on \( \mathbb{R}_+ \) satisfying conditions (16) and (17).

**Proposition 6.** Let \( n \geq 2 \) be arbitrary and \( H_{A_1}, \ldots, H_{A_n} \) the Hamiltonians of quantum systems \( A_1, \ldots, A_n \) satisfying condition (15). Let \( \rho \) and \( \sigma \) be states in \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \) such that \( \sum_{k=1}^{n} \text{Tr} H_{A_k} \rho_{A_k}, \sum_{k=1}^{n} \text{Tr} H_{A_k} \sigma_{A_k} \leq nE \) and \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1 \). Let \( A^n = A_1 \ldots A_n \) and \( \delta = \sqrt{\varepsilon(2 - \varepsilon)} \). Then

\[
2|E_I(\rho) - E_I(\sigma)| \leq 2\sqrt{2\varepsilon} \mathcal{F}_{HA,n} \left[ \frac{n\bar{E}}{\delta} \right] + (n + 1)g(\sqrt{2\varepsilon}) \tag{70}
\]

where \( \mathcal{F}_{HA,n} \) is the function defined in (15) with \( A = A^n \) and \( \bar{E} = E - E_0^{A^n} / n \).

If \( A_k \cong A \) for \( k = 1, \ldots, n \) then

\[
2|E_I(\rho) - E_I(\sigma)| \leq \mathcal{V}\mathcal{B}_H^n(\bar{E}, \delta | 2, n + 1) \tag{71}
\]

for any \( t \in (0, 1/\delta) \), where \( \bar{E} = E - E_0^A \).

The right hand sides of (70) and (71) tend to zero as \( \varepsilon \to 0 \) for given \( \bar{E} \) and \( t \).

If \( A \) is the \( \ell \)-mode quantum oscillator then inequality (71) holds with the r.h.s. replaced by the r.h.s. of (72) with \( \delta \) instead of \( \varepsilon \), \( C = 2 \) and \( D = n + 1 \) for any \( t \in (0, 1/\delta) \).

Continuity bound (70) imply the following
Corollary 5. Let $A_1, \ldots, A_n$ be arbitrary quantum systems. If the Hamiltonians $H_{A_1}, \ldots, H_{A_n}$ satisfy condition (13) then

A) the function $E_I$ is uniformly continuous on the set of states $\rho$ in $\mathcal{S}(H_{A_1} \ldots A_n)$ such that $\sum_{k=1}^n \text{Tr} H_{A_k} \rho_{A_k} \leq E$ for any $E > E_0^{A_1} = E_{A_1}^0 + \ldots + E_{A_n}^0$;

B) the function $E_I$ is asymptotically continuous in the following sense (cf. [9]): if $\{\rho_d\}$ and $\{\sigma_d\}$ are any sequences of states such that $\rho_d, \sigma_d \in \mathcal{S}(H_{B_1} \ldots B_n)$, $\text{Tr} H_{B_d} \rho_d, \text{Tr} H_{B_d} \sigma_d \leq dE$, $\forall d$, and $\lim_{d \to +\infty} \|\rho_d - \sigma_d\|_1 = 0$,

where $X_d$ denotes $d$ copies of a system $X$, $B = A_1 \ldots A_n$ and $H_{B_d}$ is the Hamiltonian of the system $B_d$, then

$$\lim_{d \to +\infty} \frac{|E_I(\rho_d) - E_I(\sigma_d)|}{d} = 0.$$ 

Proof. The first assertion of the corollary directly follows from continuity bound (70), since the r.h.s. of (70) vanishes as $\varepsilon \to 0$.

The second assertion is derived from continuity bound (70) by repeating the arguments from the proof of Corollary 4. \qed

5 On preserving continuity bounds under local channels

Many characteristics of a multipartite quantum system $A_1 \ldots A_n$ are nonnegative and do not increase under actions of local channels, i.e. channels of the form

$$\Lambda = \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n,$$

(72)

where $\Phi_k$ is a channel from the system $A_k$ to any system $A'_k$, $k = 1, n$.

Assume now that $f$ is any function on $\mathcal{S}(H_{A_1} \ldots A_n)$ possessing the above properties and satisfying inequalities (20) and (21). Since a quantum channel is a linear map, it follows that for any local channel $\Lambda : A_1 \ldots A_n \rightarrow A'_1 \ldots A'_n$ the function $f \circ \Lambda$ also satisfies inequalities (20) and (21) with the same parameters. In terms of the classes introduced in Section 3 this means that

$$f \in \tilde{L}_n^m(C, D) \quad \Rightarrow \quad f \circ \Lambda \in \tilde{L}_n^m(C, D).$$

So, by applying Proposition [2], Theorem [1] and Theorem [2] to the function $f \circ \Lambda$ we obtain the same continuity bound for $f \circ \Lambda$ as for the function $f$.

For example, the nonnegativity and monotonicity of the quantum mutual information under local channels implies the following...
Proposition 7. Let $\Lambda : \mathcal{H}_{A_1..A_n} \rightarrow \mathcal{H}_{A'_1..A'_{n'}}$ be a channel having form (72).

A) If all the subsystems $A_1, ..., A_n$ are finite-dimensional then

$$|I(A'_1: \ldots : A'_{n'})_{\Lambda(\rho)} - I(A'_1: \ldots : A'_{n'})_{\Lambda(\sigma)}| \leq \varepsilon \ln \dim \mathcal{H}_{A_1..A_n} + n \varepsilon$$

(73)

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1..A_n})$ such that $\frac{1}{2}||\rho - \sigma||_1 \leq \varepsilon$.

B) If the assumptions of Proposition 3 hold then inequalities (56) and (57) remain valid with the left hand side replaced by

$$|I(A'_1: \ldots : A'_{n'})_{\Lambda(\rho)} - I(A'_1: \ldots : A'_{n'})_{\Lambda(\sigma)}|.$$

Note that inequality (73) holds regardless of the dimensions of the subsystems $A'_1, ..., A'_{n'}$ (which may be infinite).

Note also that the assertion of Proposition 7 remains valid for any positive trace preserving linear map $\Lambda : \mathcal{S}(\mathcal{H}_{A_1..A_n}) \rightarrow \mathcal{S}(\mathcal{H}_{A'_1..A'_{n'}})$ such that

$$I(A'_1: \ldots : A'_{n'})_{\Lambda(\rho)} \leq I(A_1: \ldots : A_{n'})_{\rho} \text{ for any } \rho \in \mathcal{S}(\mathcal{H}_{A_1..A_n}).$$

Proposition 7 states, roughly speaking, that the continuity bound for the quantum mutual information given by Proposition 3 is preserved by local channels. Similar assertion holds for both continuity bounds for the QCMI given by Proposition 4.

Concluding remarks. We have proposed universal methods for quantitative continuity analysis of characteristics of multipartite quantum systems. The limited size of the article allowed us to consider only several applications of these methods. In fact, they can be applied to many other characteristics of multipartite quantum systems, including the relative entropy of entanglement, conditional and unconditional dual total correlation [12] (also called secrecy monotones [5, 37]), the interaction information of an $n$-partite quantum system (the topological entanglement entropy in the case $n = 3$) [17, 18], etc.

I am grateful to A.S.Holevo and G.G.Amosov for the discussion that motivated this research. I am also grateful to S.N.Filippov and K.Zyczkowski for useful references.

References

[1] R.Alicki, M.Fannes, “Continuity of quantum conditional information”, Journal of Physics A: Mathematical and General, V.37, N.5, L55-L57 (2004); arXiv: quant-ph/0312081.

[2] K.M.R.Audenaert, “A sharp continuity estimate for the von Neumann entropy”, J. Math. Phys. A: Math. Theor. 40(28), 8127-8136 (2007).

[3] D.Avis, P.Hayden, I.Savov, “Distributed compression and multiparty squashed entanglement”, Journal of Physics A: Mathematical and General, 41(11):115301 (2008); arXiv:0707.2792.
[4] S.Becker, N.Datta, "Convergence rates for quantum evolution and entropic continuity bounds in infinite dimensions", Commun. Math. Phys. 374, 823–871 (2020); arXiv:1810.00863.

[5] N.J.Cerf, S.Massar, S.Schneider, "Multipartite classical and quantum secrecy monotones". Physical Review A, 66(4):042309, (2002); arXiv:quant-ph/0202103.

[6] M.Christandl, A.Winter, "Squashed entanglements - an additive entanglement measure", J. Math. Phys., V.45, 829-840 (2003).

[7] N.Davis, M.E.Shirokov, M.M.Wilde, "Energy-constrained two-way assisted private and quantum capacities of quantum channels", Phys. Rev. A, 97:6 (2018), 62310, 31 pp.; arXiv:1801.08102.

[8] I.Devetak, J.Yard, "The operational meaning of quantum conditional information", Phys. Rev. Lett. 100, 230501 (2008).

[9] J.Eisert, Ch.Simon, M.B.Plenio, "On the quantification of entanglement in infinite-dimensional quantum systems", J. Phys. A 2002. V.35, N.17. P.3911-3923.

[10] M.Fannes, "A continuity property of the entropy density for spin lattice systems", Commun. Math. Phys. V.31, 291-294 (1973).

[11] N.Gisin, "Quantum Measurements and Stochastic Processes". Physical Review Letters. 52 (19): 1657–1660 (1984).

[12] Te Sun Han, "Nonnegative entropy measures of multivariate symmetric correlations", Information and Control, 36(2):133–156, (1978).

[13] F.Herbut "On Mutual Information in Multipartite Quantum States and Equality in Strong Subadditivity of Entropy", J. Phys. A: Math. Gen. 37 (2004) 3535-3542; arXiv:quant-ph/0311193

[14] A.S.Holevo, "Quantum systems, channels, information. A mathematical introduction", Berlin, DeGruyter, 2012.

[15] A.S.Holevo, "Classical capacities of quantum channels with constrained inputs", Probability Theory and Applications. V.48, N.2, 359-374 (2003); arXiv:quant-ph/0211170

[16] L.P.Hughston, R.Jozsa, W.K.Wootters, "A complete classification of quantum ensembles having a given density matrix", Physics Letters A. 183 (1): 14–18 (1993).

[17] A.Jakulin, I.Bratko "Quantifying and Visualizing Attribute Interactions", arXiv:cs/0308002
[18] A.Kitaev, J.Preskill “Topological Entanglement Entropy”, Phys. Rev. Lett. 96, 110404.

[19] A.A.Kuznetsova, ”Quantum conditional entropy for infinite-dimensional systems”, Theory of Probability and its Applications, V.55, N.4, 709-717 (2011).

[20] G.Lindblad, ”Expectation and Entropy Inequalities for Finite Quantum Systems”, Comm. Math. Phys., V.39, N.2, 111-119 (1974).

[21] G.Lindblad, ”Entropy, information and quantum measurements”, Comm. Math. Phys., V.33, 305-322 (1973).

[22] E.H.Lieb, M.B.Ruskai, ”Proof of the strong suadditivity of quantum mechanical entropy”, J.Math.Phys. V.14. 1938 (1973).

[23] M.Mosonyi, F.Hiai, ”On the quantum Renyi relative entropies and related capacity formulas”, IEEE Trans. Inf. Theory 57(4) (2011), 2474-2487.

[24] O.Nagel, G.Raggio, ”Another state entanglement measure”, arXiv: quant-ph/0306024.

[25] M.A.Nielsen, I.L.Chuang, ”Quantum Computation and Quantum Information”, Cambridge University Press, 2000.

[26] M.E.Shirokov, ”Entropy characteristics of subsets of states. I”, Izv. Math., 70:6 (2006), 1265–1292.

[27] M.E.Shirokov, ”Measures of correlations in infinite-dimensional quantum systems”, Sbornik: Mathematics, 207:5, 724-768 (2016); arXiv:1506.06377.

[28] M.E.Shirokov, ”Squashed entanglement in infinite dimensions”, J. Math. Phys., 57:3 (2016), 32203, 22 pp; arXiv: 1507.08964.

[29] M.E.Shirokov, ”Adaptation of the Alicki-Fannes-Winter method for the set of states with bounded energy and its use”, Rep. Math. Phys., 81:1 (2018), 81–104; arXiv:1609.07044.

[30] M.E.Shirokov, ”Advanced Alicki–Fannes–Winter method for energy-constrained quantum systems and its use”, Quantum Inf. Process., 19 (2020), 164, 33 pp., arXiv:1907.02458.

[31] E.Schrodinger, ”Probability relations between separated systems”, Proceedings of the Cambridge Philosophical Society, 32 (3): 446–452 (1936).

[32] B.Synak-Radtke, M. Horodecki ”On asymptotic continuity of functions of quantum states”, arXiv:quant-ph/0506126
[33] R. Tucci, "Entanglement of distillation and conditional mutual information", arXiv: quant-ph/0202144.

[34] A. Wehrl, "General properties of entropy", Rev. Mod. Phys. V.50, 221-250 (1978).

[35] M. M. Wilde, "From Classical to Quantum Shannon Theory", arXiv:1106.1445.

[36] M. M. Wilde "Multipartite quantum correlations and local recoverability", Proceedings of the Royal Society A, 471, N.2177, (2015); arXiv:1412.0333.

[37] D. Yang, K. Horodecki, P. Horodecki, J. Oppenheim, W. Song, "Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof", IEEE Trans. Inf. Theory 55, 3375 (2009), arXiv:0704.2236.

[38] D. Yang, M. Horodecki, Z. D. Wang, "Conditional Entanglement", arXiv:quant-ph/0701149.

[39] D. Yang, M. Horodecki, Z. D. Wang "An additive and operational entanglement measure: conditional entanglement of mutual information", Phys. Rev. Lett. V.101, 140501 (2008); arXiv:0804.3683.

[40] A. Winter, "Coding theorem and strong converse for quantum channels", IEEE Transactions on Information Theory, V.45, N.7, 2481-2485 (1999).

[41] A. Winter, "Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints", Comm. Math. Phys., V.347 N.1, 291-313 (2016); arXiv:1507.07775 (v.6).

[42] Z. Xi, X.-M. Lu, X. Wang, Y. Li, "Necessary and sufficient condition for saturating the upper bound of quantum discord", Phys. Rev. A, 85, 032109 (2012), arXiv:1111.3837.

[43] A continuity bound \( \sup_{x,y \in S_a} |f(x) - f(y)| \leq B_a(x, y) \) depending on a parameter \( a \)
is called asymptotically tight for large \( a \) if \( \lim_{a \to +\infty} \sup_{x,y \in S_a} \frac{|f(x) - f(y)|}{B_a(x, y)} = 1 \).