SOLVING TOPOLOGICAL FIELD THEORIES ON MAPPING TORI

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Abstract

Using gauge theory and functional integral methods, we derive con-
crete expressions for the partition functions of $BF$ theory and the $U(1|1)$ model of Rozansky and Saleur on $\Sigma \times S^1$, both directly and using equivalent two-dimensional theories. We also derive the partition function of a certain non-abelian generalization of the $U(1|1)$ model on mapping tori and hence obtain explicit expressions for the Ray-Singer torsion on these manifolds. Extensions of these results to $BF$ and Chern-Simons theories on mapping tori are also discussed. The topological field theory actions of the equivalent two-dimensional theories we find have the interesting property of depending explicitly on the diffeomorphism defining the mapping torus while the quantum field theory is sensitive only to its isomorphism class defining the mapping torus as a smooth manifold.
1 Introduction

Topological field theories in three dimensions have been approached in essentially three different ways. The first is through Witten’s original observation \cite{Witten} that one can get a handle on Chern-Simons theory by using surgery and known results from conformal field theory (essentially from the results of E. Verlinde \cite{Verlinde}). There is also a combinatorial approach as, for example, in defining the Turaev-Viro and Reshetikhin-Turaev invariants \cite{Turaev}. This approach makes use of quantum groups and so is intimately related to the first. The third method for computing is perturbation theory \cite{Brezin,DiFrancesco}. In this letter we will pursue a fourth approach, initiated in \cite{Dijkgraaf}, and examine and solve various topological field theories in three dimensions using certain functional integral and gauge theoretic methods which (unlike the usual perturbative approach) allow us to make maximal use of the symmetries of the problem. We derive concrete, computable expressions for the Ray-Singer torsion on the manifolds and the partition functions of the theories considered, and not merely formal expressions. In particular, we will calculate the partition function of $BF$ theory on three-manifolds of the form $\Sigma \times S^1$ by two different methods, analogous to those employed in \cite{Dijkgraaf} to solve Chern-Simons theory on $\Sigma \times S^1$: directly in three dimensions (via Abelianisation) and using an equivalent two dimensional theory derived from $BF$ theory on $\Sigma \times I$ (the $BF$ analog of $G/G$ gauged Wess-Zumino-Witten models). As a further application of these techniques we also provide a simple derivation of the partition function of the $U(1|1)$ model of Rozansky and Saleur \cite{Rozansky} on $\Sigma \times S^1$.

In \cite{Dijkgraaf}, we had also claimed that, in principle, these techniques are applicable to (topological) gauge theories on mapping tori $\Sigma_\beta$ associated to a diffeomorphism $\beta$ of $\Sigma$. Here we illustrate this in the context of a certain natural non-Abelian generalization of the $U(1|1)$ model (a “$G/T$” model). In particular, we obtain in this way topological field theories in two dimensions which have the novel property of depending on (the equivalence classes under isotopy and conjugation of) a diffeomorphism $\beta$. This thus provides us with a $\beta$-twisted analog of the 2d/3d correspondence familiar from Chern-Simons theory. A detailed investigation of these theories, however, as well as of those arising in a similar fashion from Chern-Simons and $BF$ theories will be left to a forthcoming publication \cite{Dijkgraaf2}.

The partition functions of the theories calculated here (integrals of the Ray-Singer torsion over the moduli space of flat connections) are topological
invariants of the manifolds considered. They may be thought of as general-
izations of the Johnson invariant to manifolds other than homology three-
spheres once we have dealt with the infinities that arise along the way.

2 The torus $\Sigma \times S^1$ and the cylinder $\Sigma \times I$

Here we discuss $BF$ theory on $\Sigma \times S^1$ and (as part of a second method of
solving the theory on $\Sigma \times S^1$) on $\Sigma \times I$, $I = [0,1]$. We also use these methods
to give a quick rederivation of the partition function of the $U(1|1)$ model of
Rozansky and Saleur \[7\] on $\Sigma \times S^1$.

2.1 BF theory

The general action for $BF$ theory (with a “cosmological constant” $\lambda$) on a
three manifold $M$ is

$$S(A,B) = \int_M B \wedge F(A) + \frac{\lambda}{3} B \wedge B \wedge B$$

(2.1)

Here $A$ is a connection on a principal $G$-bundle over $M$, $B$ is a corresponding
$L(G)$ (Lie algebra of $G$) valued 1-form and the integral is understood to
include a trace. We note that for $\lambda > 0$ the theory can be transformed into
a Chern-Simons theory of $G \times G$, whereas for $\lambda < 0$ it is the imaginary part
of a Chern-Simons theory for $G_C$. Here we will discuss the $\lambda = 0$ case which
can alternatively be regarded as a Chern-Simons theory for a non-compact
group commonly denoted $IG$ (the tangent bundle group $TG$) \[9, 10\]. The
field equations tell us what the classical phase space is. The equation from
the $B$ variation tells us immediately that we are dealing with the space of
flat connections on $M$. Fortunately the space of solutions to the $A$ variation
equation is equally easily characterized. This equation is just the linearized
form of the $A$ equation, telling us that we are dealing with the cotangent
bundle to the space of flat connections as a phase space. We have also to mod
out by the gauge group but this preserves the cotangent bundle structure
and we are left with $T^*\mathcal{M}(M)$ as the reduced phase space, where $\mathcal{M}(M)$ is
the moduli space of flat connections on $M$. General results \[11\] tell us that
we are calculating the volume of $T^*\mathcal{M}(M)$ with the Ray-Singer torsion as
the measure.

We restrict the group in question to be compact, semi-simple and simply-
connected, unless we specify $G = U(1)$. In either case we are dealing only
with trivial bundles, in the first case because that is all there is, and in the second because of the flatness condition. We also make use of an orthogonal decomposition of the Lie algebra:

\[ g = t \oplus \mathfrak{k} \]

(2.2)

where \( t \) is the Cartan subalgebra. For manifolds \( M \) of the form \( \Sigma \times S^1 \) there is a particularly useful gauge choice using this decomposition, namely

\[ \dot{A}_0^1 = 0, \quad A_0^k = 0 \]

(2.3)

plus the condition that \( A_0 \) be compact. The reason that one cannot fix to the gauge \( A_0 = 0 \) is that on the circle the holonomy is gauge invariant and cannot be set to zero. The compactness condition arises as follows. Even after we have imposed the conditions set out in (2.3) there are still ‘large’ periodic gauge transformations available which shift \( A_0^1 \) by elements of the integer lattice \( I \) of \( t \). These have the form

\[ g(t) = g(0) \exp t \gamma, \quad g(1) = g(0) \leftrightarrow \gamma \in I, \]

(2.4)

and shift \( A_0^1 \) by

\[ g^{-1}(t) A_0^1 g(t) + g(t)^{-1} \partial_0 g(t) = A_0^1 + \gamma. \]

(2.5)

In these formula the group element \( g(t) \in T \) may have \( \Sigma \) dependence (\( T \) is the maximal torus of \( G \) corresponding to the decomposition (2.2)). We note that the conjugation of \( A_0 \) into the torus can in general not be achieved globally and enforcing it introduces a sum over all torus bundles, as described in [12].

Using the above gauge conditions we can calculate an expression for the partition function for \( BF \) theory on \( \Sigma \times S^1 \).

### 2.1.1 Solution

We gauge fix \( B \) by imposing \( D_0 B_0 = 0 \). On the \( t \) and \( \mathfrak{k} \) components this condition becomes

\[ \partial_0 B_0^1 = 0, \quad D_0 B_0^k = 0. \]

(2.6)

We append to the action the ghost and gauge fixing terms

\[ \int_{\Sigma \times S^3} \Lambda D_0 B_0 + \bar{c} D_0 c + \bar{\rho} D_0^2 \rho. \]

(2.7)
These give us determinants and delta functions

$$\delta(\partial_0 B_0^\mu) \delta(B_0^k) \text{Det}_t(D_0) \big|_0^{-1} \text{Det}_t'(\partial_0) \big|_0 \text{Det}_t(D_0) \big|_0 \text{Det}(D_0^2) \big|_0$$

(2.8)

where the $|_0$ indicates evaluation on zero-forms. The prime on the Dets indicates that the zero modes are omitted in the evaluation. We integrate out the non-constant $B$ modes to be left with an action

$$\int_{\Sigma \times S^1} B_0(d\alpha + \frac{1}{2}[\alpha, \alpha]) + b(d\alpha + \text{ad}(\alpha))\alpha_0$$

(2.9)

plus the gauge fixing and ghosts and a new determinant $\text{Det}'(D_0) \big|_1^{-1}$ where the 1 indicates it is to be evaluated on the space of one-forms on $\Sigma$. The lower case letters in the above equation indicate constant modes of the fields considered. The $b$ integral now contributes delta functions

$$\delta(d\alpha_0)\delta([\alpha_0, \alpha^1]) = \delta(d\alpha_0)\delta(\alpha^1) \text{Det}_t(\text{ad}(\alpha_0))^{-1}.$$  

(2.10)

Using all the delta functions at our disposal the action then reduces to

$$\int_{\Sigma \times S^1} b_0^I d\alpha^I.$$  

(2.11)

This tells us that $b_0^I$ is constant and integral and thus the $b_0^I$ integral contributes an overall factor of $\zeta(0)$ to the partition function. Collecting the determinants from the ghosts and gauge fixing term and those from above leaves us with an integral over $\alpha_0$ (constant in space and time):

$$\int d\alpha_0 \frac{\text{Det}_t(\partial_0 + \alpha_0) \big|_0^2}{\text{Det}_t(\partial_0 + \alpha_0) \big|_1}$$

(2.12)

which gives, after the usual regularization described in [6],

$$Z = \int_T \text{det}_t(1 - \text{Ad}(t))^\chi$$

(2.13)

where $t$ is an element in the maximal torus $T$ and $\chi$ is the Euler character of $\Sigma$ and we define $\text{Ad}(g)\phi = g^{-1}\phi g$. Thus we obtain the Ray-Singer torsion on $S^1$ raised to the power of the Euler character, i.e. the Ray-Singer torsion of $\Sigma \times S^1$. Recall that the Chern-Simons partition function is the square root of this result and note that the integral here is not cut off by the level as it is in the Chern-Simons case. We have been assuming no $B_0$ zero modes. If
they do exist the determinants must be taken in the orthocomplement to the kernel of the operators involved. We further note that we have discarded an infinity that arises from the integral over $b^i$. The harmonic modes did not enter into the delta function on $\frac{d}{da}$, and thus we are left with an integral of the form $\prod_{i=1}^{2g} \int db^i$ which plainly contributes an infinity. We compare this with a formal result of Witten [13], that the partition function should take the form

$$Z = \int_{TAM} DB \int_{M} T_{M}^{RS}(A)$$ (2.14)

where $T_{M}^{RS}(A)$ denotes the Ray-Singer torsion of $M$ with respect to the flat connection $A$. The infinity that we find is that coming from the integral over the tangent space, whereas our finite results correspond to the integral over the moduli space of flat connections $\mathcal{M}$. For homology three-spheres this is the Johnson invariant and, as mentioned in the introduction, it is a topological invariant of the manifold. This is the justification for our remark that the quantities we are calculating are generalizations of this invariant to manifolds other than homology three-spheres.

2.2 BF theory on $\Sigma \times I$

As in [3] we will use the manifold $\Sigma \times I$ to construct a two-dimensional theory equivalent to $BF$ theory on $\Sigma \times S^1$ by imposing boundary conditions on $\Sigma \times I$ and then reconstructing the partition function on $\Sigma \times S^1$ by integrating over these boundary values while imposing a periodicity condition. The boundary values are interpreted as fields in two dimensions. We also gauge fix the $\Sigma \times S^1$ fields $A_0$ and $B_0$ by $A_0(x,t) = A_0(x)$ and $B_0(x,t) = B_0(x)$. Our choice of boundary conditions is

$$A_q(0) = A, \quad B_q(1) = B$$ (2.15)

where the subscript $q$ now differentiates the dynamical fields from the boundary conditions. This necessitates adding boundary terms to the action to have a well-defined variational principle. It is easily seen that this term has the form

$$+ \int_{\Sigma \times \{1\}} B_q \wedge A_q.$$ (2.16)

Thus we are evaluating the following partition function:

$$Z_{\Sigma \times S^1} = \int DA_0 DB_0 D\overline{A} D\overline{B} \delta(\partial_0 A_0) \delta(\partial_0 B_0) \text{Det}(D_0) \left|_{0}^{2} \right.$$
\[ e^{-i \int_{\Sigma} B A} Z_{\Sigma \times I}[A, B, A_0, B_0] \]  

(2.17)

where \( Z_{\Sigma \times I} \) is the partition function on \( \Sigma \times I \) with the given boundary conditions and with \( A_0 \) and \( B_0 \) as background fields. (For notational purposes we will initially denote these fields in the \( \Sigma \times I \) partition function by \( A_{q0} = A_0 \) and \( B_{q0} = B_0 \).) One gets to this formula by thinking of the path integral on \( \Sigma \times S^1 \) as

\[ \int DA(A, t=1 | A, t=0) = \int DADB(A, t=1 | B, t=1) \langle B, t=1 | A, t=0 \rangle, \]  

(2.18)

together with

\[ \langle A, t=1 | B, t=1 \rangle = e^{-i \int_{\Sigma} B A}. \]  

(2.19)

### 2.2.1 Reduction to a 2-dimensional theory

We will evaluate the model on \( \Sigma \times I \). The action in this case is

\[ \int_{\Sigma \times I} B_q F_{A_q} + \int_{\Sigma \times [1]} B_q A_q. \]  

(2.20)

Now write the action as

\[ \int_{\Sigma \times I} B_{q}^g F_{A_q^g} + \int_{\Sigma \times [1]} B_q A_q \]  

(2.21)

and choose \( g \) to solve \( A_{q0}^g = 0 \) with \( g(1) = 1 \). Now send \( B_q \to B_q^{g^{-1}} \) and \( A_q \to A_q^{g^{-1}} \). This has the effect of turning off \( A_{q0} \) in the action, which becomes

\[ \int_{\Sigma \times I} B_q F_{(A_q, 0)} + \int_{\Sigma \times [1]} B_q A_q. \]  

(2.22)

where now \( B_{q0} = g^{-1}B_0g \). Of course we see the change of variables in the boundary data which now reads

\[ \delta(A_q(0) - A_q^{g}) \delta(B_q(1) - B). \]  

(2.23)

The aim now is to trivialise the dependence on \( B_{q0} \). We rewrite the action as

\[ \int_{\Sigma \times I} \left( B_q + d(\Lambda_{q, 0}) \Lambda \right) F_{(A_q, 0)} + \int_{\Sigma \times [1]} B_q A_q - \int_{\Sigma \times [1]} \Lambda F_{A_q} - \int_{\Sigma \times [0]} \Lambda F_{A_q}. \]  

(2.24)
One picks $\Lambda$ to solve $Bq' + \partial_0 \Lambda = 0$ with $\Lambda(1) = 0$. That is, one sets $\Lambda = \int_1^t Bq'$. We now change variables, according to $\mathcal{B}_q \rightarrow \mathcal{B}_q - d\mathcal{A}_q \Lambda$. The net effect in the action is to set $Bq'$ to zero up to a boundary term,

$$
\int_{\Sigma \times [0]} g^{-1} \phi F_{\mathcal{A}_q} + \int_{\Sigma \times I} \mathcal{B}_q \partial_0 \mathcal{A}_q + \int_{\Sigma \times [1]} B\mathcal{A}_q. 
$$

(2.25)

where

$$
\phi = \int_0^1 Bq' = \frac{1}{\text{ad}(A_0)} (\text{Ad}(g^{-1}) - 1) B_0.
$$

(2.26)

The boundary conditions on $A_q$ and $B_q$ are not changed. The delta function constraints on $A_q$ imply that it is equal to $A^g$ while those on $B_q$ imply that this is $B$.

We have thus established that

$$
Z[A, B, A_0, B_0] = e^{i \int_{\Sigma \times [0]} g^{-1} \phi F_{A^g} \times \int_{\mathcal{A}_q} D\mathcal{A}_q \int_{\mathcal{B}_q} D\mathcal{B}_q e^{i \int_{\Sigma \times I} \mathcal{B}_q \partial_0 \mathcal{A}_q + i \int_{\Sigma \times [1]} B\mathcal{A}_q}. 
$$

(2.27)

Notice that the path integral

$$
\int_{B: A^g} D\mathcal{A}_q \int_{\mathcal{B}_q} D\mathcal{B}_q e^{i \int_{\Sigma \times I} \mathcal{B}_q \partial_0 \mathcal{A}_q + i \int_{\Sigma \times [1]} B\mathcal{A}_q} = e^{i \int_{\Sigma} B\mathcal{A}_g}, 
$$

(2.28)

so that finally we have

$$
Z[A, B, A_0, B_0] = e^{i \int_{\Sigma \times [0]} \phi F_{A^g} + i \int_{\Sigma} B\mathcal{A}_g}. 
$$

(2.29)

### 2.2.2 Rederivation of solution on $\Sigma \times S^1$

In order to calculate the path integral on $\Sigma \times S^1$ we put (2.29) into (2.17). Putting the pieces together (and dropping the underlining) one arrives at the partition function

$$
Z_{\Sigma \times S^1} = \int DA Dg DB D\phi e^{iS} 
$$

(2.30)

where the action is

$$
S = \int_{\Sigma} \phi F_A + B(A^g - A). 
$$

(2.31)
Notice that we have exchanged the measure $DA_0 \det'(D_0) |_0$ for $Dg$. This is in fact correct as $\det'(D_0) |_0 = \det((1 - \text{Ad}(g)) / \text{ad}(A_0))$ which is the required Jacobian. Notice also that the change of variables from $B_0$ to $\phi$ produces exactly the right Jacobian to cancel the remaining $\det'(D_0) |_0$.

The action still has a great deal of symmetry. Conventional gauge invariance is there as well as invariance under

$$\delta B = (d + \frac{1}{2} \text{ad}(A + A^\theta)) \Lambda, \quad \delta \phi = g(\Lambda g^{-1} - \Lambda). \quad (2.32)$$

We can conjugate the group field $g$ into the torus of the group $t \in T$. Once we have done this, as usual, all the non-trivial torus bundles are liberated. We must sum over all of the possible first Chern classes associated with these. As a next step of gauge fixing we also wish to fix $\phi$ to lie in the Cartan subalgebra $t$, the Lie algebra of $T$. This is done by making use of the symmetry (2.32). It is important to notice that this gauge is not achieved by conjugation but, rather, by a shift

$$\phi^t \to \phi^t + (1 - \text{Ad}(t)) \Lambda^t \quad (2.33)$$

so that one is not ‘mixing’ torus bundles. Also note that the $t$ part of $\Lambda$ does not appear in this transformation rule.

With $g$ in the torus, $B^t$ appears in the action solely in the term $B^t(1 - \text{Ad}(t))A^t$ so that on integrating it out we obtain

$$\det_t(1 - \text{Ad}(t)) |_0^{-1} \delta(A^t). \quad (2.34)$$

The $B^t$ integral sets $t$ to be position independent (modulo infinities from the harmonic modes, whose existence depends on whether we allow $t^{-1}dt$ to be non-exact). Furthermore, the integral over the torus component of the gauge field, including a sum over all possible Chern classes, gives a delta function constraint onto constant and integral $\phi$. Thus from the gauge fixing for $g$, the gauge fixing for $\phi$ and the final part of the functional integral we pick up exactly the right determinants to reproduce the result (2.13). Note that if we had chosen the gauge fixing $D_0 B_0 = 0$ instead of $\partial_0 B_0 = 0$, the ghost determinant would now be $\det(D_0) |_0^2$. Solving the gauge constraint would force $\phi$ to lie in the torus spanned by $A_0$ (when coupled with periodicity) and changing variables to eliminate this constraint would produce the determinants $(\det(\text{ad}(A_0)) |_0 \det(D_0) |_0)^{-1}$ which are just sufficient to leave an overall $\det'(D_0) |_0$ the determinant we obtain from (2.33) after gauge fixing $\phi$ to lie in the torus.
2.3 $U(1|1)$ model

Rozansky and Saleur introduced a cohomological field theory in [7] which they related to the Alexander polynomial. In this way they were able to give a field theoretic proof of the relationship between the Alexander polynomial and the Ray-Singer Torsion. The model they consider is a cousin to the topological field theory used to describe the Casson invariant. Indeed it is a type of $U(1)$ version of the Casson model (also known as three-dimensional super $BF$ theory [14]), the conventional $U(1)$ Casson model being trivial. In our previous examples the bundles were, from the outset, trivial. In principle we should now specify which $U(1)$ bundle we are talking about. However, the path integral has delta function support on flat connections and so we may as well fix our attention on the trivial $U(1)$ bundle.

The action is
\[
\int_M B dA + \bar{\psi}(d + A)\psi, \tag{2.35}
\]
and has an $N = 2$ topological supersymmetry as well as conventional gauge invariance,
\[
\begin{align*}
\delta A &= d\omega, \\
\delta B &= d\rho - \bar{\psi}\sigma + \bar{\sigma}\psi, \\
\delta \psi &= (d + A)\sigma, \\
\delta \bar{\psi} &= (d - A)\bar{\sigma}.
\end{align*} \tag{2.36}
\]

Rozansky and Saleur employ conformal field theory techniques to compute the partition function of this theory. We will reproduce their results by a direct path integral calculation.

Our choice of gauge on a three manifold $\Sigma \times S^1$ is
\[
\dot{A}_0 = 0, \quad \dot{B}_0 = 0 \quad (\partial_0 - A_0)\psi_0 = 0, \quad (\partial_0 + A_0)\bar{\psi}_0 = 0. \tag{2.37}
\]

The $B$ zero modes do not enter into the theory at all, and correspond to symmetries which may be gauge fixed to zero. This is understood to have been done. Here we have no problems with zero modes as the shift symmetry $\delta B = d\rho$ is manifest in the original action.

In order to implement these gauge choices one needs to append to the action (2.35)
\[
\int_{\Sigma \times S^1} E\partial_0 A_0 + E\partial_0 B_0 + \eta(\partial_0 - A_0)\psi_0 + \eta(\partial_0 + A_0)\bar{\psi}_0, \tag{2.38}
\]
as well as the following Faddeev-Popov ghost terms

\[
\int_{\Sigma \times S^1} \bar{\omega} \partial_0 \omega + \bar{\rho}(\partial_0 + \bar{\sigma} \psi_0 - \bar{\psi}_0 \sigma) + \bar{\phi}(\partial_0 - A_0)(\partial_0 + A_0) \sigma \\
+ \bar{\phi}(\partial_0 + A_0)(\partial_0 - A_0) \bar{\sigma}.
\]

(2.39)

Integrating over the \( B_i(t) \) field we obtain a delta function constraint

\[
\delta(\partial_0 A_i(t) - \partial_i A_0),
\]

(2.40)

which again, together with periodicity in the \( S^1 \) variables, tells us that \( A_i \) is time independent and that \( A_0 \) is constant (and will henceforth be renamed \( a_0 \) to make this explicit). As usual, we have the freedom to fix \( a_0 \) to lie on the circle and we do so. Now as \( a_0 \) is constant, and as long as it is not zero, one deduces from the gauge fixing conditions (2.37) that

\[
\psi_0 = \bar{\psi}_0 = 0.
\]

(2.41)

The path integral thus reduces to a product of the determinants

\[
\frac{\text{Det}'^{2}(\partial_0) \mid_0}{\text{Det}'(\partial_0) \mid_1} \cdot \frac{\text{Det}(\partial_0 + a_0) \mid_1}{\text{Det}(\partial_0 + a_0)(\partial_0 - a_0) \mid_0}
\]

(2.42)

times

\[
\int DB_0 DA \exp i \left( \int_{\Sigma} B_0 dA \right).
\]

(2.43)

The ratio of determinants (2.42) is essentially

\[
\int_{0+\epsilon}^{2\pi-\epsilon} da_0 (2 \sin a_0/2)^{-\chi(\Sigma)}.
\]

(2.44)

For \( g \geq 1 \) one may dispense with the regularisation and in this case we obtain

\[
V_M(M) = 2^{2g-2} \begin{pmatrix} 2g - 2 \\ g - 1 \end{pmatrix}.
\]

(2.45)

where \( V_M(M) \) is the volume of the moduli space of flat connections with the Ray-Singer torsion as the measure. In fact we can absorb overall factors \( a.b^{g-1} \) into the normalization [6], and it is the combinatorial factor that is interesting. This result is in agreement with Rozansky and Saleur [7]. One can also derive an equivalent two-dimensional description of the \( U(1|1) \) model along the lines of the calculation leading to (2.31).
In passing, we want to point out that the combinatorial factor \((2g-2)\) can be interpreted as the Euler character of the \((g-1)\)th symmetric power of \(\Sigma\). This result agrees with a calculation of the Seiberg-Witten-Casson invariant \(^{15}\) when the spin-c bundle is the canonical line bundle of \(\Sigma\).

3 The mapping torus \(\Sigma_\beta\)

Thus far our attention has been on manifolds of the type \(\Sigma \times S^1\). We may generalize somewhat and nevertheless keep solvability by passing to mapping tori.

Let \(\beta : \Sigma \to \Sigma\) be a diffeomorphism of \(\Sigma\). Associated with \(\beta\) and \(\Sigma\) is a three-manifold \(\Sigma_\beta\) known as the mapping torus of \(\beta\). It is obtained from the manifold \(\Sigma \times I\) by making the following identification: \((\beta x, 0) \sim (x, 1)\). This is clearly a fibration over \(S^1\) with the obvious projection. Fields on \(\Sigma_\beta\) may be regarded as fields on \(\Sigma \times I\) subject to the twisted periodicity condition

\[
\phi(t + 1) = \beta^* \phi(t). \tag{3.1}
\]

We will need to know one basic fact about the cohomology of \(\Sigma_\beta\). First of all, \(\beta\) induces an action on the cohomology of \(\Sigma\) (denoted \(\beta^*\) as it is essentially a pullback). The degree one cohomology is then generated by the pullback of the generator of \(H^1(S^1)\) and by the fixed points of the action of \(\beta^*\) on \(H^1(\Sigma, \mathbb{R})\). If there are no such fixed points, in other words when \((\beta^* - 1)\) is invertible, then \(H^1(\Sigma_\beta, \mathbb{R})\) is one-dimensional, and this is the case we will consider henceforth. In genus one this will be the case whenever the induced \(SL(2, \mathbb{Z})\) matrix \(U_\beta\) satisfies \(\text{Tr}(U_\beta) \neq 2\). It should be noted however that the reduction to two dimensions does not rely on this assumption, and the actions we obtain are therefore valid for general \(\beta\).

3.1 \(G/T\) model

The model we want to consider is very closely related to the \(U(1|1)\) model above but with a slightly different fermionic sector. The matter fields (which in fact we will take to be commuting or anti-commuting) take values in \(\mathfrak{k}\) and transform under the adjoint action of \(T\). For simplicity we will take the fields as having values in the \(\mathfrak{k}\) directions in \(SU(2)\) where the gauge group is \(U(1)\) although everything goes through in exactly the same way in the more general case. The reason for the choice of the adjoint action is that
the determinant is always unity and thus there is no problem in defining
the phase of the path integral, a subtlety that also arises in Chern-Simons
theory, reflecting the problem of choosing a framing.

The theory is then given by

\[ S(A,B,\psi,\bar{\psi}) = \int_M B dA + \bar{\psi} (d + \text{ad}(A)) \psi. \]  

(3.2)

This has the following infinitesimal symmetries depending on the statistics
of \( \psi \) and \( \bar{\psi} \):

\[
\begin{align*}
\delta A &= d \omega \\
\delta B &= d \rho - (-1)^\psi (\text{ad}(\bar{\sigma}) \psi + \text{ad}(\sigma) \bar{\psi}) \\
\delta \psi &= - \text{ad}(\omega) \psi + d_A \sigma \\
\delta \bar{\psi} &= - \text{ad}(\omega) \bar{\psi} + d_A \bar{\sigma}.
\end{align*}
\]  

(3.3)

The partition function for (3.2) is given by

\[ Z = \int_{T^{b_1}} T_M^{RS} (g)^{\pm 1} \]  

(3.4)

where \( b_1 \) denotes the first Betti number of \( M \) and the parameter \( g \) is given
by the holonomy along the non-trivial cycles of the manifold. We note that
a straightforward generalization of the argument used for the \( U(1|1) \) model
would enable us to solve this model for \( M = \Sigma \times S^1 \) but now we concentrate
on the mapping torus.

### 3.1.1 Gauge fixing

The first issue we have to deal with on \( \Sigma_\beta \) is that of gauge fixing. We want
to “project out” the two dimensional theory by gauge fixing away the third
dimension. This was achieved for \( \Sigma \times S^1 \) by gauge fixing \( A_0 \) to the class
of functions constant in time. The situation is more complicated for \( \Sigma_\beta \) as
we are not now dealing with a product manifold, and a global definition of
a “time” direction cannot be given. To make things clearer we shall work
on \( \Sigma \times \mathbb{R} \) with functions that satisfy the condition \( \beta^* \phi(x,t) = \phi(x,t+1) \),
or alternatively on the interval with functions satisfying \( \beta^* \phi(x,0) = \phi(x,1) \).
It is immediately obvious that we cannot simply gauge fix to the class of
time-constant functions, as they will not necessarily satisfy this condition.
Instead we must look for a class of functions that do satisfy the condition,
but which is no “larger” than the time-constant class. One possibility is
to multiply a constant function by a time dependent part that vanishes at
the boundary of the interval (or at every integer if we are thinking of the real line). We need to achieve this gauge via a gauge transformation that is the identity at the boundary of the interval (we have boundary conditions to preserve). In the present (abelian) case we find that the requisite gauge parameter is

$$\omega(x,t) = \int_0^t ds \left( f(s) \dot{A}_0(x) - A_0(x,s) \right).$$  \hspace{1cm} (3.5)

The requirement that it vanish also at \( t = 1 \) imposes the following condition:

$$\dot{A}_0(x) \int_0^1 dt f(t) = \int_0^1 dt A_0(x,t).$$  \hspace{1cm} (3.6)

The simplest thing is to normalize \( f \) to integrate to unity over the interval. Thus we can achieve our gauge choice, still parametrized by a function on \( \Sigma \), using suitable gauge transformations. The gauge fixing introduces two determinants of \( D_0 \). Given that the gauge group is abelian we have no need of these to provide Jacobians for a change of measure to the group. In fact they are cancelled, as in the case of \( BF \) theory on \( \Sigma \times S^1 \), by a field redefinition similar to that in equation (2.26).

### 3.1.2 Reduction to 2 dimensions

We proceed as we did for the reduction to two dimensions of \( BF \) theory, starting on \( \Sigma \times I \) specifying \( \psi \) and \( A \) at \( t = 0 \) and \( \bar{\psi} \) and \( B \) at \( t = 1 \). Using the results about gauge fixing above we then find the following two-dimensional action (the derivation is not completely trivial and will be explained in [8]):

$$S = \int \Sigma B_0 dA + \bar{\psi}(\beta^* - \text{Ad}(g))\psi + \psi_0 d_{\beta^*}A \bar{\psi} + \bar{\psi}_0 d_A \psi + B(\beta^*A - A - g^{-1}dg)$$  \hspace{1cm} (3.7)

where we omit the underlining on the now spatial one-forms \( A, B, \psi \) and \( \bar{\psi} \). \( g \) here is \( e^{A_0} \). While this action has an explicit dependence on \( \beta \in \text{Diff}(\Sigma) \), it can be shown [8] that the theory depends only on the smooth manifold \( \Sigma_\beta \) and is, in particular, invariant under isotopies of \( \beta \). We note that there is a symmetry of the form \( \delta B = d\sigma \) with compensating terms in \( \delta B_0 \), but that this is independent of the fields and will henceforth be ignored (assumed gauge fixed).

This action has the following local symmetry:

$$\psi \mapsto h^{-1} \psi h \quad \bar{\psi} \mapsto \beta^* h^{-1} \psi \beta^* h \quad g \mapsto h^{-1} g \beta^* h$$
$$\psi_0 \mapsto \beta^* h^{-1} \psi_0 \beta^* h \quad \bar{\psi}_0 \mapsto h^{-1} \bar{\psi}_0 h \quad A \mapsto A + h^{-1}dh.$$  \hspace{1cm} (3.8)
The somewhat unusual action of the gauge group on the group valued field $g$, $g \mapsto h^{-1} g \beta^* h$, will also appear in the analogous two-dimensional models for $BF$ and Chern-Simons theory to be discussed below. To solve we make the following steps:

1. We gauge fix the symmetry using $d * A = 0$. This condition together with the flatness condition imposed by the $B_0$ integral tells us that $A$ is harmonic. Note that this statement has not required the integral over the $B_0$ zero mode.

2. The $B$ integral is telling us that $(\beta^* - 1) A = g^{-1} d g$. Because of the gauge fixing we know that $A$ is an harmonic form, call it $\omega$. The exterior derivative commutes with the pullback and thus, by the Hodge decomposition we know that $\beta^* A$ can be written as

$$\beta^* A = \gamma + d \alpha$$

(3.9)

where $\gamma$ is harmonic (in the original metric) and $\alpha$ will of course depend on $\omega$. The right hand side $g^{-1} d g$ (as $\int_0^1 A_0$ is globally well-defined) is exact and so from orthogonality we are led to the two equations $\gamma = \omega$ and $d \alpha = g^{-1} d g$. Now making the assumption that $b_1 = 1$, so that $(\beta^* - 1)$ on the harmonic forms is invertible, we can conclude that $\omega = 0$. Thus $A$ and its pullback are zero and hence $g^{-1} d g$ is zero as well.

3. There is an extra symmetry to be gauge fixed viz:

$$\delta \psi = d_A \rho$$

$$\delta \psi_0 = -(-1)^\psi (\beta^* - Ad(g)) \rho$$

$$\delta B_0 = [\bar{\psi}_0, \rho]$$

$$\delta B = -(-1)^\psi [Ad(g) \rho, \bar{\psi}]$$

(3.10)

that depends on the statistics of $\psi$, and an analogous one for $\bar{\psi}$. These we fix by the conditions $d_A * \psi = 0$ and $d_A * \bar{\psi} = 0$, although we already know that $A = 0$ in fact. Then the $\psi_0$ and $\bar{\psi}_0$ integrals tell us that $\psi$ and $\bar{\psi}$ are harmonic also.

4. During the gauge fixing of $\psi$ and $\bar{\psi}$ we have not used the $\rho$ zero mode as $\delta \psi = d \rho$, so there are still constant $\rho$ gauge transformations which leave the gauge fixed action invariant. Notice also that $\psi_0$ constant modes do not appear in the action (3.7). This happy situation means that providing $\text{Det}(1 - Ad(g)) \neq 0$ we can use the constant $\rho$ symmetry by (3.10) to gauge fix the $\psi_0$ constant modes to zero and we do so. A similar story holds for $\bar{\psi}_0$. 

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5. From the gauge fixing of $\psi_0$ and $\bar{\psi}_0$ we pick up ghost determinants
\[
\det(1 - \text{Ad}(g))_{H^0(\Sigma)}^{\pm 2}.
\]
(3.11)

6. Thus all except the $g$ integral have been performed and we are left with the following expression for the partition function of the theory on $\Sigma_\beta$:
\[
Z[\beta] = \int dg \left\{ \det(\beta^* - \text{Ad}(g))_{H^1(\Sigma)} \cdot \det(1 - \text{Ad}(g))_{H^0(\Sigma)}^{-2} \right\}^{\pm 1}
\]
(3.12)
where the sign of the exponent depends on the statistics of $\psi$. Comparing this equation to (3.4) allows us to claim that on a mapping torus, the Ray-Singer torsion is
\[
T_{\Sigma_\beta}(g) = \det(\beta^* - \text{Ad}(g))_{H^1(\Sigma)} \cdot \det(1 - \text{Ad}(g))_{H^0(\Sigma)}^{-2}
\]
(3.13)
and this agrees with the result of Fried [16].

4 Conclusions and future work

We briefly discuss the theories one obtains on reducing $BF$ and Chern-Simons theories on the mapping torus to equivalent two dimensional theories.

Starting from the action for $BF$ theory we can follow a similar procedure to that in the $\Sigma \times S^1$ case to reduce the theory on $\Sigma_\beta$ to two dimensions. The result is that the two dimensional action becomes
\[
S = \int_{\Sigma} \phi F_A + B(A^g - \beta^* A)
\]
(4.1)
with gauge symmetries
\[
A \mapsto h^{-1} A h + h^{-1} d h \quad g \mapsto h^{-1} g \beta^* h
\]
\[
\phi \mapsto h^{-1} \phi h \quad B \mapsto \beta^* h^{-1} B \beta^* h
\]
(4.2)
and extra symmetries
\[
\delta \phi = g \Lambda g^{-1} - (\beta^*)^{-1} \Lambda \quad \delta B = [d + \frac{1}{2} \text{ad}(A^g + \beta^* A)] \Lambda
\]
(4.3)
In order to evaluate the partition function of (4.1) we cannot employ abelianization directly. Instead one makes use of a localization argument to proceed.
For Chern-Simons theory on a mapping torus one can also obtain an equivalent two-dimensional theory. One finds a $\beta$-twisted $G/G$ model, i.e. an anomaly-free $G_L \times G_R$ gauged WZW model (see [17]) with $A_R = \beta^* A_L$ and local gauge invariance given by the first line of (4.2). These two-dimensional theories derived from BF theory and Chern-Simons theory again explicitly involve a diffeomorphism $\beta \in \text{Diff}(\Sigma)$, and again only depend on the conjugacy class of the isotopy class of this diffeomorphism.

We have seen that for various admittedly simple three manifolds a direct gauge-theoretic evaluation of partition functions of topological field theories is possible. The same techniques allow one to calculate observables in these theories as well.

For the theories considered here we have at no time explicitly needed to introduce a metric on $\Sigma$. However for Chern-Simons theory on $\Sigma \times I$ the boundary data was specified relative to a metric $\tilde{g}$. At the end one proves that nothing depends on this choice. One can view the BF theories as special cases of Chern-Simons theory so that for example when $\lambda = 0$ one would find be the $IG/IG$ gauged WZW model as the two-dimensional equivalent. The relationship between such two-dimensional actions and the manifestly metric-independent actions obtained here is akin to the relationship between the $G/G$ model and the gauged WZ term described in [18] and [19]. The details of all these observations, as well as a comparison with the work on Chern-Simons theory on a mapping torus in [20], will be described in [8].

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