Pair production in classical Stueckelberg-Horwitz-Piron electrodynamics

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Abstract.
We calculate pair production from bremsstrahlung as a classical effect in Stueckelberg-Horwitz electrodynamics. In this framework, worldlines are traced out dynamically through the evolution of events $x^\mu(\tau)$ parameterized by a chronological time $\tau$ that is independent of the spacetime coordinates. These events, defined in an unconstrained 8D phase space, interact through five $\tau$-dependent gauge fields induced by the event evolution. The resulting theory differs in its underlying mechanics from conventional electromagnetism, but coincides with Maxwell theory in an equilibrium limit. In particular, the total mass-energy-momentum of particles and fields is conserved, but the mass-shell constraint is lifted from individual interacting events, so that the Feynman-Stueckelberg interpretation of pair creation/annihilation is implemented in classical mechanics.

We consider a three-stage interaction which when parameterized by the laboratory clock $x^0$ appears as (1) particle-1 scatters on a heavy nucleus to produce bremsstrahlung, (2) the radiation field produces a particle/antiparticle pair, (3) the antiparticle is annihilated with particle-2 in the presence of a second heavy nucleus. When parameterized in chronological time $\tau$, the underlying process develops as (1) particle-2 scatters on the second nucleus and begins evolving backward in time with negative energy, (2) particle-1 scatters on the first nucleus and releases bremsstrahlung, (3) particle-2 absorbs radiation which returns it to forward time evolution with positive energy.

1. Introduction
In the historical introduction to his book on quantum field theory [1], Weinberg devotes a paragraph to deprecation of Dirac’s hole theory of antiparticles, observing that QFT had made the theory “unnecessary, even though it lingers on in many textbooks.” It might also have been mentioned that the essential idea of hole theory also lingers on productively as the quasiparticle formalism in condensed matter physics and many-particle theory[2]. Still, as an interpretation of particle/antiparticle processes, the Feynman-Stueckelberg time reversal formalism, expressed in QFT through the Feynman propagator, has many conceptual advantages. One advantage not mentioned by Weinberg is that besides not requiring the Dirac sea, it does not actually require quantum field theory. The description of an antiparticle as a particle propagating

1 More generally, Dirac’s fundamental insight that the absence of a physical object can behave like the presence of an inverse object has been influential in many fields, especially psychology, and can be compared to the remark attributed to Miles Davis that, “Music is the space between the notes. It’s not the notes you play; it’s the notes you don’t play.”
backward in time was first proposed by Stueckelberg [2] in the context of classical relativity, without resort to quantum ideas or phenomena. In this model, a pair process is represented by a single worldline, generated dynamically by a classical event whose time coordinate advances or retreats with respect to the laboratory clock, as its instantaneous energy changes sign under interaction with a field.

In order to generate worldlines of this type, Stueckelberg constructed a covariant Hamiltonian theory of interacting spacetime events, in which the events evolve dynamically, as functions of a Poincaré invariant parameter $\tau$. As shown in Figure 1, the particle worldline is traced out in terms of the values taken on by the four-vector $x^\mu(\tau)$ as the parameter proceeds monotonically from $\tau = -\infty$ to $\tau = \infty$.

![Figure 1: World Lines][2]

A: Usual type, with a unique solution to $t(\tau) = x^0$ for each $x^0$
B: Annihilation type, with two solutions to $t(\tau) = x^0$ for $x^0 \ll 0$ and no solution for $x^0 \gg 0$
C: Creation type, with two solutions to $t(\tau) = x^0$ for $x^0 \gg 0$ and no solution for $x^0 \ll 0$

By explicitly distinguishing the Einstein coordinate time $x^0 = t$ from the temporal order [3], the parameter time $\tau$ becomes formally similar to the Galilean invariant time in Newtonian theory, serving Stueckelberg’s broader goal of generalizing the techniques of nonrelativistic classical and quantum mechanics to covariant form. Stueckelberg identified pair creation in worldlines of type C in Figure 1, because there are two solutions to $t(\tau) = t_2$, but no solution to $t(\tau) = t_1$. The observer will therefore first encounter no particle trajectories and then encounter two. It seems clear that the intrinsic electric charge should not change along the worldline, but to identify one part of the worldline as an antiparticle trajectory, requires that the measured charge reverse sign. Charge reflection may be grasped intuitively as in Dirac’s hole model: carrying positive charge in one time direction should be equivalent to carrying negative charge in the opposite time direction. In standard QFT, charge conjugation is accomplished through the action of an operator with no classical analog. In Stueckelberg’s classical formalism, the 0-component of the current includes the electric charge multiplied by $d x^0/d \tau$, which becomes negative when the event $x^\mu(\tau)$ evolves toward earlier values of $t = x^0$. Thus, particles and antiparticles do not appear as distinct classes of solutions to a defining equation, but as a
single event whose qualitative behavior depends instantaneously on the dynamical value of its velocity.

A standard technique for pair creation in the laboratory is the two-step process by which Anderson [4] first observed positrons in 1932: high energy electrons are first scattered by heavy nuclei to produce bremsstrahlung radiation, and electron/positron pairs are then created from the radiation field. The Bethe-Heitler mechanism [5] describes this technique as the quantum process,

$$e^- + Z \rightarrow e^- + Z + \gamma$$

$$Z + \gamma \rightarrow Z + e^- + e^+$$

involving a quantized radiation field and the external Coulomb field of the nuclei. The Feynman diagrams describing the second step in QED are shown in Figure 2 [6].

![Figure 2: Bethe-Heitler mechanism in QED](image)

The incoming real photon carries 4-momentum $k\gamma$ and the real outgoing electron and positron carry 4-momentum $q_e$ and $q_p$. The intermediate virtual state carries 4-momentum $\tilde{p}_e$ or $\tilde{p}_p$, and the virtual photon exchanged with the nucleus is represented by the dashed line.

A modern experimental setup is shown in Figure 3 [7].

![Figure 3: Bethe-Heitler mechanism in the laboratory](image)
Electrons are accelerated to high energy by focusing an intense laser pulse on a thin gold disk. The electrons are strongly deflected in the Coulomb field of the nuclei and emit bremsstrahlung represented in the illustration as $\gamma$-rays. In the Coulomb field of a nucleus, the photon can decay into an electron/positron pair.

Stueckelberg was not able to provide a classical account of pair processes, because the mass-shell constraint $p^2 = (M\dot{x})^2 = -M^2$ prevents continuous evolution of the event trajectory from the timelike region into the spacelike region on its way to time-reversed timelike motion. He considered adding a vector component to his Lorentz force that would overcome the constraint, but dropped the idea, finding no justification from first principles. Just such a vector field appears naturally in a gauge-invariant approach to Stueckelberg’s theory [8, 9].

In this paper, we give a brief overview of Stueckelberg-Horwitz electrodynamics and use the formalism to provide a classical description of the pair creation process described by the Bethe-Heitler mechanism. Our goal is to calculate the classical trajectories that produce the two-step process shown in Figure 4.

![Figure 4: Bethe-Heitler mechanism in classical electrodynamics](image)

At chronological time $\tau_1$ positive energy particle-1 arrives at the laboratory time coordinate $t = t_3$ and scatters in the Coulomb field of nucleus $Z$. Particle-1 emerges with negative energy moving backward in $t$. At a subsequent time $\tau_2 > \tau_1$ positive energy particle-2 arrives at $t = t_1$ and scatters to positive energy in the field of another nucleus, emitting classical bremsstrahlung. This radiation impinges on particle-1 at $t = t_2$, providing sufficient energy to divert it back to positive energy evolution.

In the laboratory, where events are recorded in the order determined by clock $t$, the process appears as particle-2 scattering at $t = t_1$ and emitting bremsstrahlung, followed by the appearance at $t = t_2$ of a particle/antiparticle pair. Then at $t = t_3$, the antiparticle encounters another particle causing their mutual annihilation.
In section 2 we present those features of Stueckelberg-Horwitz electrodynamics required to describe this scattering process. In section 3 we provide a description of Coulomb scattering required for the events at \( t = t_3 \) and \( t = t_1 \). In section 4 we calculate the acceleration of a particle in the radiation field of a scattering event, providing a description of the pair creation event. In both the pair creation event at \( t = t_2 \) and the pair annihilation event at \( t = t_3 \), the basic requirement is that the classical interaction energy be greater than the masses of the created particles. Section 5 presents a discussion of the results and directions for further study.

2. Overview of Horwitz-Stueckelberg electrodynamics

The generalized Stueckelberg-Schrödinger equation

\[
(i\partial_\tau + e_0\phi)\psi(x, \tau) = \frac{1}{2M} (p^\mu - e_0a^\mu)(p_\mu - e_0a_\mu)\psi(x, \tau)
\]

describes the interaction of an event characterized by the wavefunction \( \psi(x, \tau) \) with five gauge fields \( a_\mu(x, \tau) \) and \( \phi(x, \tau) \). Equation (1) is invariant under local gauge transformations

\[
\psi(x, \tau) \rightarrow e^{i\mu(x, \tau)}\psi(x, \tau)
\]

Vector potential

\[
a_\mu(x, \tau) \rightarrow a_\mu(x, \tau) + \partial_\mu \Lambda(x, \tau)
\]

Scalar potential

\[
\phi(x, \tau) \rightarrow \phi(x, \tau) + \partial_\tau \Lambda(x, \tau)
\]

whose \( \tau \)-dependence is the essential departure from Stueckelberg’s work, and determines the structure of the resulting theory \[8\] \[9\]. The corresponding global gauge invariance leads to the conserved Noether current

\[
\partial_\mu j^\mu + \partial_\tau \rho = 0
\]

where

\[
j^\mu = -\frac{i}{2M} \left\{ \psi^* (\partial^\mu - ie_0a^\mu)\psi - \psi (\partial^\mu + ie_0a^\mu)\psi^* \right\}
\]

\[
\rho = \left| \psi(x, \tau) \right|^2.
\]

Adopting the formal designations

\[
x^3 = \tau \quad \partial_3 = \partial_\tau \quad \partial^3 = \rho \quad a_5 = \phi
\]

and the index convention

\[
\lambda, \mu, \nu = 0, 1, 2, 3 \quad \alpha, \beta, \gamma = 0, 1, 2, 3, 5
\]

the gauge and current conditions can be written

\[
a_\alpha \rightarrow a_\alpha + \partial_\alpha \Lambda \quad \partial_\alpha j^\alpha = 0.
\]

The classical mechanics of a relativistic event is found by rewriting the Stueckelberg-Schrödinger equation in the form

\[
i\partial_\tau \psi(x, \tau) = \left[ \frac{1}{2M} (p^\mu - e_0a^\mu)^2 - e_0a_5 \right] \psi(x, \tau) = K\psi(x, \tau)
\]

and transforming the classical Hamiltonian to Lagrangian as

\[
L = \dot{x}^\mu p_\mu - K = \frac{1}{2} M\dot{x}^\mu \dot{x}_\mu + e_0\dot{x}^\alpha a_\alpha
\]
from which the Euler-Lagrange equations

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0
\]  

are

\[
\frac{d}{d\tau} \left[ M \ddot{x}^\mu + e_0 a^\mu(x, \tau) \right] = e_0 \dot{x}^\alpha \partial^\mu a_\alpha(x, \tau)
\]

leading to the Lorentz force

\[
M \dddot{x}^\mu = e_0 \left[ \dot{x}^\alpha \partial^\mu a_\alpha - (\dot{x}^\nu \partial_\nu + \partial_\tau) a^\mu \right] = e_0 f^\mu_\alpha(x, \tau) \dot{x}^\alpha
\]

where

\[
f^\mu_\alpha = \partial^\mu a_\alpha - \partial_\alpha a^\mu \quad \dot{x}^\alpha = \tau = 1
\]

As required for time reversal, particles may exchange mass with fields

\[
\frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = -M \ddot{x}_\mu \dot{x}_\mu = -e_0 \dot{x}^\mu (f_{\mu 5} + f_{\mu \nu} \dot{x}^\nu) = -e_0 \dot{x}^\mu f_{\mu 5}
\]

and in this formalism, the mass shell is demoted from the status of constraint to that of conservation law for interactions in which \(f_{\mu 5} = 0\). Analysis of the mass-energy-momentum tensor shows that the total mass, energy, and momentum of the particles and fields are conserved. To write an electromagnetic action requires the choice of a kinetic term for the gauge field, which must be both gauge and \(O(3,1)\) invariant. We write

\[
S_{em} = \int d^4 x d\tau \left\{ e_0 j^\alpha(x, \tau) a_\alpha(x, \tau) - \int ds \frac{\lambda}{4} \left[ f^{\alpha \beta}(x, \tau) \Phi(\tau - s) f_{\beta \gamma}(x, s) \right] \right\}
\]

where the local event current

\[
j^\alpha(x, \tau) = \dot{X}^\alpha(\tau) \delta^4(x - X(\tau))
\]

has support at the instantaneous location \(X(\tau)\) of the event. The \(\tau\)-integral of (16) along the worldline concatenates the event current into the Maxwell particle current in the usual form. Taking the field interaction kernel to be \(\Phi(\tau) = \delta(\tau) - \lambda^2 \delta^{(\mu)}(\tau) = \int \frac{d\kappa}{2\pi} \left[ 1 + (\lambda \kappa)^2 \right] e^{-i\kappa \tau} \),

the inverse function becomes

\[
\Phi^{-1}(\tau) = \frac{1}{2\lambda} e^{-|\tau|/\lambda}
\]

which will be seen to spread the current along the worldline. The classical action can be written using (17) as

\[
S = \int d\tau \left( \frac{1}{2} M \ddot{x}_\mu + \int d^4 x d\tau \left\{ e_0 a_\alpha f^{\alpha \beta} - \frac{\lambda}{4} f_{\alpha \beta} f^{\alpha \beta} - \frac{\lambda^3}{4} \left( \partial_\gamma f^{\alpha \beta} \right) \left( \partial_\beta f_{\alpha \gamma} \right) \right\} \right)
\]
in which the gauge and O(3,1) invariance are manifest. The $\tau$ derivatives in the last term explicitly break any formal higher symmetry in the electromagnetic terms. Varying the action in the form (15) with respect to the fields and applying (18) leads to the field equations

$$\partial_\beta f^{\alpha\beta}(x, \tau) = \frac{e_0}{\lambda} \int ds \varphi(\tau - s) j^\alpha(x, s) = e j^\alpha_p(x, \tau)$$

(20)

$$\partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0$$

(21)

which are formally similar to 5D Maxwell equations with $e = e_0/\lambda$. The source $j^\alpha_p(x, \tau)$ of the field in (21) is the instantaneous current $j^\alpha(x, \tau)$ defined in (16) with its support along the worldline smoothed by the kernel function $\varphi(\tau)$. For $\lambda$ very small, $\varphi$ becomes a delta function which narrows the source to a small neighborhood around the event inducing the current. The parameter $\lambda$ plays the role of a correlation length, characterizing the range of the electromagnetic interaction.

Rewriting the field equations in vector and scalar components, they take the form

$$\partial_\nu f^{\mu\nu} - \partial_\tau f^{5\mu} = e j^\mu_p$$
$$\partial_\mu f^{5\nu} = e j^5_p$$
$$\partial_\nu f_{\rho\mu} + \partial_\rho f_{\nu\mu} + \partial_\mu f_{\nu\rho} = 0$$
$$\partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \partial_\tau f_{\nu\mu} = 0$$

(22)

which may be compared with the 3-vector form of Maxwell equations

$$\nabla \times \mathbf{B} - \partial_0 \mathbf{E} = e \mathbf{J}$$
$$\nabla \cdot \mathbf{E} = \varepsilon_0 J^0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0$$

(23)

with $f_{5\mu}$ playing the role of the vector electric field and $f^{\mu\nu}$ playing the role of the magnetic field. It is sometimes notionally convenient to further expand the field into 3-vector components as

$$(\mathbf{e})^i = f^{0i}_p$$
$$(\mathbf{h})_{ij} = \varepsilon_{ijk} f^{jk}_p$$

(24)

The connection with Maxwell theory is found, as seen for the instantaneous event current, by concatenation — integration over $\tau$ along the worldline,

$$\partial_\beta f^{\alpha\beta}(x, \tau) = e j^\alpha_p(x, \tau)$$
$$\partial_\alpha f_{\beta\gamma} = 0$$
$$\partial_\gamma f_{\alpha\beta} = 0$$

(25)

$$\partial_\nu F^{\mu\nu}(x) = e J^\mu(x)$$
$$\partial_\mu F^{5\nu}(x) = 0$$
$$\partial_\nu f_{5\mu}(x) = 0$$

where

$$A^\mu(x) = \int d\tau a^\mu(x, \tau)$$
$$F^{\mu\nu}(x) = \int d\tau f^{\mu\nu}(x, \tau)$$
$$J^\mu(x) = \int d\tau j^\mu(x, \tau)$$

(26)

The field equations (22) are called pre-Maxwell equations, and together with the Lorentz force (12) describe a microscopic event dynamics for which Maxwell theory can be understood as an equilibrium limit. Since $e_0 a^\mu$ must have the dimensions of $e A^\mu$, it follows that $e_0$ and $\lambda$ have the dimension of time and $e = e_0/\lambda$ is dimensionless. The pre-Maxwell equations lead to the wave equation

$$\partial_\alpha \partial^\alpha a^\beta(x, \tau) = (\partial_\mu \partial^\mu - \partial_\tau^2) a^\beta(x, \tau) = -e j^\beta_p(x, \tau)$$

(27)
whose solutions may respect 5D symmetries broken by the \( \text{O}(3,1) \) symmetry of the event dynamics. The principal part Green’s function is

\[
G(x, \tau) = -\frac{1}{2\pi} \delta(x^2)\delta(\tau) - \frac{1}{2\pi^2} \frac{\partial}{\partial x^2} \frac{\theta(x^2 - \tau^2)}{\sqrt{x^2 - \tau^2}}, \quad x^2 = x^\mu x_\mu
\]

where \( D(x) \) is the 4D Maxwell Green’s function and \( G_{\text{correlation}} \) vanishes under concatenation. In this paper we neglect the correlation term. The ‘static’ Coulomb potential in this framework is induced by an isolated event moving uniformly along the \( t \) axis. Writing the event as

\[
x(\tau) = (\tau, 0, 0, 0)
\]

produces the currents

\[
j^0(x, \tau) = j^5(x, \tau) = \delta(t - \tau) \delta^4(x) \quad j(x, \tau) = 0
\]

\[
j^0_\phi(x, \tau) = j^5_\phi(x, \tau) = \phi(t - \tau) \delta^4(x) \quad j_\phi(x, \tau) = 0
\]

inducing the Yukawa-type potential

\[
a^0(x, \tau) = a^5(x, \tau) = \frac{e}{4\pi |x|} \phi(t - |x|)
\]

and recovering the standard Coulomb potential

\[
A^0(x) = \int d\tau \ a^0(x, \tau) = \frac{e}{4\pi |x|}
\]

under concatenation. A test event on the lightcone of this event will experience the force

\[
M\ddot{x} = e^2 \nabla \left[ \frac{e^{-|x|/\lambda}}{4\pi |x|} \right]
\]

in which \( 1/\lambda \) represents the mass spectrum of the pre-Maxwell field. If \( \lambda \) is small (so that \( \phi \) approaches a delta function and the current narrows to around the event) the mass spectrum becomes wide. If \( \lambda \) is large, the support of the current spreads along the worldline and the potential becomes Coulomb-like.

An arbitrary event \( X^\mu(\tau) \) induces the current

\[
j^\mu_\phi(x, \tau) = \int ds \ \phi(\tau - s) \dot{X}^a(s) \delta^4(x - X(s))
\]

leading to the Liénard-Wiechert potential

\[
a^\hat{\beta}(x, \tau) = -e \int d^4x' d\tau' D(x - x') \delta(\tau - \tau') j^\mu_\phi(x', \tau')
\]

\[
= \frac{e}{2\pi} \int ds \ \phi(\tau - s) \dot{r}^a(s) \delta\left((x - X(s))^2\right) \theta^{\tau_0}
\]

Using identity

\[
\int d\tau f(\tau) \delta[g(\tau)] = \frac{f(\tau_R)}{|g'(\tau_R)|}
\]
where \( \tau_R \) is the retarded time found from
\[
g (\tau) = (x - X(\tau_R))^2 = 0 \quad \text{and} \quad \theta^{\text{ret}} = \theta (x^0 - X^0 (\tau_R)) ,
\]
provides
\[
a^\beta (x, \tau) = \frac{e}{4\pi} \phi (\tau - \tau_R) \frac{\dot{X}^\beta (\tau_R)}{(x^\mu - X^\mu (\tau_R)) X_\mu (\tau_R)} .
\]
Notice that the \( \tau \)-dependence is limited to the smoothing kernel \( \phi (\tau - \tau_R) \) and again \( \lambda \) plays the role of a correlation length that localizes the interaction to the neighborhood \( \tau_R \pm \lambda \). Using this potential and writing
\[
u^\mu = \dot{X}^\mu (\tau) \quad z^\mu = x^\mu - X^\mu (\tau)
\]
we find the field strengths, separated into the retarded and radiation parts, as
\[
f^{\mu\nu}_{\text{rel}} (x, \tau) = -e \phi (\tau - \tau_R) \frac{z^\mu u^\nu - z^\nu u^\mu}{4\pi (u \cdot z)^3} u^2 \sim \frac{1}{z^2}
\]
\[
f^{\mu\nu}_{\text{rad}} (x, \tau) = e \phi (\tau - \tau_R) \frac{z^\mu u^2 - u^\mu (u \cdot z)}{4\pi (u \cdot z)^3} \sim \frac{1}{z^2}
\]
\[
f^{\mu\nu}_{\text{rad}} (x, \tau) = -e \phi (\tau - \tau_R) \left( \frac{z^\mu u^\nu - z^\nu u^\mu}{(u \cdot z)^2} \right) \frac{d}{d\tau_R} \phi (\tau - \tau_R) \sim \frac{1}{|z|}
\]
\[
f^{\mu\nu}_{\text{rad}} (x, \tau) = e \phi (\tau - \tau_R) \left( \frac{z^\mu u^\nu - z^\nu u^\mu}{(u \cdot z)^2} \right) - e \frac{z^\mu u^\nu - z^\nu u^\mu}{4\pi (u \cdot z)^2} \frac{d}{d\tau_R} \phi (\tau - \tau_R) \sim \frac{1}{|z|}
\]

3. Coulomb scattering

We begin by analyzing the scattering at \( \tau_1 \) in Figure 4. Initially (at time \( \tau \to -\infty \)) the target nucleus \( Z \) and incoming particle are widely separated. The nucleus is at rest in the laboratory frame,
\[
X_Z (\tau) = (t_Z, x_Z) = (1, 0) \tau
\]
and the incoming particle approaches on the trajectory
\[
X_{in} (\tau) = (t, x, y, z) = u \tau + s = i_{in} (1, v, 0, 0) \tau + (s_t, 0, s_y, 0)
\]
where
\[
u = \frac{d}{d\tau} (t, x, y, z) \quad \frac{dx}{dt} = \frac{dx}{dt} i_{in} = v i_{in} \quad i_{in} = \frac{dt}{d\tau} = \sqrt{1 - v^2}
\]
The scattering takes place in the plane \( z = 0 \) so that the spatial distance between the incoming particle and the target is
\[
R (\tau) = |x| = \sqrt{x^2 + y^2} = \sqrt{(vi_{in} \tau)^2 + s_y^2} .
\]
It is convenient take the correlation length $\lambda$ to be small so that the support of the fields is narrowly centered around the retarded time $\tau$. Taking $\lambda \approx R(\tau_1)$ allows us to approximate $\phi(\tau - \tau_1) \approx \delta(\tau - \tau_1)$, so that $\tau_1$ is determined from the causality conditions for the initial trajectories,

$$[X_{in}(\tau_1) - X_Z(\tau_1)]^2 = 0 \quad X_{in}^0(\tau_1) - X_Z^0(\tau_1) > 0$$ (50)

These equations have the solution

$$\tau_1 = \frac{1}{v t_{in}(1 - \eta_v^2)} \left( \eta_v s_t + \sqrt{s_t^2 - s_y^2 (1 - \eta_v^2)} \right) \quad \underset{v \ll 1}{\longrightarrow} \frac{\sqrt{s_t^2 - s_y^2}}{v}$$ (51)

where it is convenient to introduce the smooth parameter

$$\eta_v = \frac{1}{v} \left( 1 - \frac{1}{t_{in}} \right) \rightarrow \begin{cases} 0, & v = 0 \\ 1, & v = 1 \end{cases}.$$ (52)

Notice that the 0-component $s_t$ of the impact parameter must be positive in order for the interaction to take place. The location of the incoming particle at the time of interaction is found to be

$$x(\tau_1) = R \hat{R} \quad t(\tau_1) = t_{in} \tau_1 + s_t$$ (53)

where

$$R = \frac{1}{1 - \eta_v^2} \left( \eta_v \sqrt{s_t^2 - s_y^2 (1 - \eta_v^2)} + s_t \right) \quad \underset{v \ll 1}{\longrightarrow} s_t$$ (54)

$$\hat{R} = \frac{\left( \eta_v s_t + \sqrt{s_t^2 - s_y^2 (1 - \eta_v^2)} s_y, 0 \right)}{s_t + \eta_v \sqrt{s_t^2 - s_y^2 (1 - \eta_v^2)}} \quad \underset{v \ll 1}{\longrightarrow} \left( \sqrt{1 - \frac{s_y^2}{s_t^2}}, \frac{s_y}{s_t} \right).$$ (55)

From equation (52) the potential induced by the target nucleus is

$$a^0(x, \tau) = a^5(x, \tau) = \frac{Ze}{4\pi R} \delta(\tau - \tau_1) \quad a^i = 0$$ (56)

so that the nonzero field strengths can be written

$$e^i = \partial^0 a^i - \partial^0 a^i \quad f^{5i} = \partial^5 a^i - \partial^5 a^i \quad f^{30} = \partial^3 a^0 - \partial^0 a^5$$ (57)

$$e = -\nabla a^0 \quad f^5 = -\nabla a^5 = e \quad f^{50} = -\left( 1 + \frac{1}{t_{in}} \right) \partial_\tau a^0.$$ (58)

Using these expressions in (12) provides the Lorentz force on the incoming particle in the form

$$\ddot{t} = -\frac{e_0}{M} e \cdot \dot{x} - \frac{e_0}{M} f^{50} = \frac{\lambda e}{M} \left( \dot{x} \cdot \nabla + \left( 1 + \frac{1}{t_{in}} \right) \partial_\tau \right) a^0(x, \tau)$$ (59)

$$\ddot{x} = -\frac{e_0}{M} e \ddot{t} + \frac{e_0}{M} f^5 = \frac{\lambda e}{M} (t + 1) \nabla a^0(x, \tau).$$ (60)
The delta function in (56) enables immediate integration of the force equations as

\[
\begin{align*}
\dot{t}_f - \dot{t}_\text{in} &= \frac{\lambda}{M} \int_{\tau_1 - \lambda/2}^{\tau_1 + \lambda/2} d\tau \left( \dot{x} \cdot \nabla \right) \left( 1 + \frac{1}{\dot{t}_\text{in}} \right) \frac{Ze^2}{4\pi R} \delta (\tau - \tau_1) \\
&= \frac{\lambda}{M} \dot{x} (\tau_1) \cdot \nabla \frac{Ze^2}{4\pi R} \\
&= -\frac{\lambda}{M} \frac{Ze^2}{4\pi R^2} \dot{x} (\tau_1) \cdot \hat{R} \quad (61)
\end{align*}
\]

\[
\dot{x}_f - \dot{x}_\text{in} = \frac{\lambda}{M} \int_{\tau_1 - \lambda/2}^{\tau_1 + \lambda/2} d\tau \left( \dot{t} + 1 \right) \nabla \frac{Ze^2}{4\pi R} \delta (\tau - \tau_1) \\
= -\frac{\lambda}{M} \frac{Ze^2}{4\pi R^2} (\dot{t} (\tau_1) + 1) \hat{R} \quad (62)
\]

where the velocities are evaluated at the interaction point as

\[
(t, \dot{t}) (\tau_1) = \frac{1}{2} \left[ (t, \dot{t})_f + (t, \dot{t})_\text{in} \right] .
\]

We introduce the dimensionless parameter

\[
g_e = \frac{\lambda}{M} \frac{Ze^2}{4\pi R^2} = \frac{\lambda}{R} \times \frac{Ze^2}{4\pi R} \frac{1}{M} = \frac{\text{correlation length}}{\text{impact parameter}} \times \frac{\text{interaction energy}}{\text{mass energy}}
\]

which appears in (61) and (62) as the factor controlling the strength of the interaction. Writing

\[
\alpha_x = \frac{1}{2} g_e \hat{R}_x \quad \alpha_y = \frac{1}{2} g_e \hat{R}_y
\]

we can expand the Lorentz force as components in the form

\[
\begin{bmatrix}
1 & \alpha_x & \alpha_y \\
\alpha_x & 1 & 0 \\
\alpha_y & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{t}_f \\
\dot{x}_f \\
\dot{y}_f
\end{bmatrix}
= \begin{bmatrix}
1 & -\alpha_x & 0 \\
-\alpha_x & 1 & 0 \\
-\alpha_y & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{t}_\text{in} \\
\dot{v}_\text{in} \\
0
\end{bmatrix}
- 2 \begin{bmatrix}
0 \\
\alpha_x \\
\alpha_y
\end{bmatrix}
\]

(66)

and solve for the final velocity

\[
\begin{bmatrix}
\dot{t}_f \\
\dot{x}_f \\
\dot{y}_f
\end{bmatrix}
= \frac{1}{1 - \frac{1}{4} g_e^2}
\begin{bmatrix}
\dot{t}_\text{in} \\
\dot{v}_\text{in} \\
0
\end{bmatrix}
- g_e \begin{bmatrix}
\dot{t}_\text{in} \hat{R}_x \\
(\dot{t}_\text{in} + 1) \hat{R}_y
\end{bmatrix}
+ \frac{1}{4} g_e^2 \begin{bmatrix}
\dot{t}_\text{in} + 2 \\
(2 \hat{R}_x \dot{R}_y - \hat{R}_y \dot{R}_x) \dot{v}_\text{in}
\end{bmatrix}
\]

(67)

Before considering pair annihilation, we examine the low velocity and low interaction energy limit of this result. Taking

\[
|\dot{t}| = v \ll 1 \quad \dot{t}_\text{in} \to 1 \quad \eta_v \to 0 \quad g_e \ll 1
\]

the initial velocity reduces to

\[
\dot{X}_\text{in} (\tau) \to (1, v, 0, 0)
\]

(69)
the final velocity becomes

\[ t_f \approx t_{in}, \quad \dot{x}_f \approx \dot{x} - 2g_e \dot{R}, \quad \dot{R} = \left( \sqrt{1 - \frac{s_y^2}{s_t}}, \frac{s_y}{s_t}, 0 \right) \equiv R = s_t \]  \hspace{1cm} (70)

and the scattering angle can be found as

\[ \cos \theta = \frac{\dot{x}_f \cdot \dot{\dot{x}}}{|\dot{x}_f| |\dot{\dot{x}}|} = \frac{\dot{x}_f^2 - 2g_e \dot{R} \cdot \dot{\dot{x}}}{|\dot{x}_f| |\dot{\dot{x}}|} = \frac{v - 2g_e \dot{R}_x}{|\dot{x}_f|}. \]  \hspace{1cm} (71)

If we impose the nonrelativistic condition for conservation of energy, we obtain a new constraint in the form

\[ \dot{x}_f^2 = v^2 = \dot{x}_f^2 = [\dot{x} - 2g_e \dot{R}]^2 \Rightarrow v \dot{R}_x = g_e \]  \hspace{1cm} (72)

in which case

\[ \cos \theta = \frac{1}{|\dot{x}_f|} [v - g_e \dot{R}_x] = 1 - 2 \dot{R}_x^2 \]  \hspace{1cm} (73)

Now, using (64) we find

\[ \cot \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} = \frac{\dot{R}_y}{\dot{R}_x} = \frac{s_y}{s_t} \frac{v}{g_e} = \frac{s_t}{\lambda v} \times \frac{4\pi M v^2 s_y}{Z e^2} \]  \hspace{1cm} (74)

which recovers the Rutherford scattering formula if

\[ \frac{s_t}{\lambda v} = 1. \]  \hspace{1cm} (75)

But from (68) we have \( s_t = R (\tau_1) \) which we assumed to be comparable to \( \lambda \). Since \( \lambda v \ll \lambda \) in this low velocity case, (75) cannot be maintained. This result is unsurprising because the short-range potential cannot provide an adequate model of nonrelativistic Rutherford scattering.

Removing these restrictions and returning to the relativistic case, the condition for pair annihilation is that particle-1 scatters to negative energy, that is \( t_f < 0 \) for some value of \( g_e \). From (61)

\[ t_f = \frac{t_{in} (1 - g_e v \dot{R}_x) + \frac{1}{4} g_e^2 (t_{in} + 2)}{1 - \frac{1}{4} g_e^2} \]  \hspace{1cm} (76)

and we see that for small values of \( g_e \),

\[ t_f \rightarrow t_{in} \geq 1. \]  \hspace{1cm} (77)

Since \( v < 1 \) and \( R_x < 1 \), the discriminant of the numerator satisfies

\[ (v R_x)^2 - \left[ 1 + \frac{2}{t_{in}} \right] < 0 \]  \hspace{1cm} (78)

so that the numerator is positive definite. The denominator becomes negative when

\[ 1 - \frac{1}{4} g_e^2 < 0 \Rightarrow g_e = \frac{\text{correlation length}}{\text{impact parameter}} \times \frac{\text{interaction energy}}{\text{mass energy}} > 2 \]  \hspace{1cm} (79)
and since we take the correlation length $\lambda$ approximately equal to the impact parameter $R$, the requirement for pair annihilation is

$$\frac{Ze^2}{4\pi R} > 2M$$

meaning that the interaction energy is greater than the mass energy of the annihilated particles. As $g_e$ approaches 2 from below $\dot{t}_f$ becomes very large. After $g_e$ passes this critical value, $\dot{t}_f$ decreases from large negative values, taking the limiting value

$$\dot{t}_f \rightarrow -\frac{(E_{in} + 2M)}{E_{in}}$$

so that the outgoing trajectory is timelike for all values of $g_e$.

4. Bremsstrahlung

Having found the condition for pair annihilation at time $\tau_1$ we now apply the general expression (67) to describe the scattering at time $\tau_2$. Particle-2 approaches a second nucleus along some trajectory $x_\mu(\tau)$ and emerges from the interaction along trajectory $x_\mu(\tau)$ with positive energy. To find the radiation field emitted by the scattering and acceleration of particle-2, at a point $y^\mu$ along a line of observation

$$z = y - x(\tau_2)$$

we write the initial and final 4-velocities as

$$u_{in} = \dot{x}_{in} \quad u_f = \dot{x}_f$$

so that

$$\Delta u = u_f - u_{in}$$

$$u(\tau) = u_{in} + \Delta u \delta (\tau - \tau_2)$$

$$\dot{u}(\tau) = \Delta u \delta (\tau - \tau_2)$$

$$u(\tau_2) = \bar{u} = \frac{1}{2} [u_f + u_{in}]$$

From (44) and (45) we rewrite the radiation fields produced by an arbitrary trajectory in the form

$$f_{\mu\nu} = -e\varphi(\tau - \tau_2) F_{\mu\nu}(z, u, \bar{u}) - e\varphi'(\tau - \tau_2) G_{\mu\nu}$$

$$f^{5\mu} = e\varphi(\tau - \tau_2) F^{5\mu}(z, u, \bar{u}) - e\varphi'(\tau - \tau_2) G^{5\mu}$$

where

$$F_{\mu\nu} = \left[ \frac{(z \cdot \bar{u}) (u \cdot z) - (z \cdot u) (\bar{u} \cdot z)}{4\pi (u \cdot z)^3} \right]_{\mu\nu} \quad G_{\mu\nu} = \frac{1}{4\pi} \left[ \frac{z \cdot u}{(u \cdot z)^2} \right]_{\mu\nu}$$

$$F^{5\mu} = \left[ \frac{(\bar{u} \cdot z) z}{4\pi (u \cdot z)^3} \right]_\mu \quad G^{5\mu} = \left[ \frac{z - u (u \cdot z)}{4\pi (u \cdot z)^2} \right]_\mu$$
As pictured in Figure 4, the radiation emitted by the scattering of particle-2 is absorbed by the negative energy particle-1 arriving at \( y^\mu \). Using the Lorentz force equations \cite{12} we calculate the change in velocity to particle-1 caused by the incoming radiation. Since we take \( \lambda \) to be small, we may approximate the smoothing kernel as

\[
\varphi (\tau - \tau_2) = \frac{1}{2\lambda} \left[ \theta (\tau - (\tau_2 - \lambda)) - \theta (\tau - (\tau_2 + \lambda)) \right].
\]  

(92)

The \( \tau \) integrations over \( \varphi' (\tau - \tau_2) \) vanish, leaving the change in velocity \( \dot{y}^\mu (\tau_2) \) of particle-1 in the form

\[
\Delta \dot{y}^\mu = \frac{\lambda e}{M} \int_{-\infty}^{\infty} d\tau \left[ f^{\mu\nu}_{\text{rad}} \dot{y}_\nu + f^{\mu5}_{\text{rad}} \dot{y}_5 \right] = -\frac{e^2}{2M} \int_{\tau_2 - \lambda}^{\tau_2 + \lambda} d\tau \left[ F^{\mu\nu} \dot{y}_\nu + F^{\mu5} \dot{y}_5 \right]
\]

\[
= -\frac{e^2}{2M} \left[ F^{\mu\nu} (z, \bar{u}, \Delta u) \dot{y}_\nu + F^{\mu5} (z, \bar{u}, \Delta u) \right]
\]  

(93)

expressed in terms of the velocity change \( \Delta u \) and average velocity \( \bar{u} \), which are found from \cite{67} to be

\[
\Delta u = -\frac{g_e}{1 - \frac{4}{4}g_e^2} \left[ \frac{v t_{\text{in}} \hat{R}_x}{(t_{\text{in}} + 1) \hat{R}_x} \right] + \frac{1}{4}g_e \left[ \frac{v t_{\text{in}} \hat{R}_x}{(t_{\text{in}} + 1) \hat{R}_y} \right]
\]

\[
\bar{u} = \frac{1}{1 - \frac{4}{4}g_e^2} \left[ \frac{t_{\text{in}}}{v t_{\text{in}}} \hat{R}_x \right] - \frac{1}{1 - \frac{4}{4}g_e^2} \left[ \frac{v t_{\text{in}} \hat{R}_x}{(t_{\text{in}} + 1) \hat{R}_x} \right] + \frac{1}{4}g_e \left[ \frac{v t_{\text{in}} \hat{R}_x}{(t_{\text{in}} + 1) \hat{R}_y} \right]
\]

(94)

(95)

Since the support of \( \varphi(\tau - \tau_2) \) is narrowly centered on \( \tau_2 \), the line of observation \( z^\mu \) must be a lightlike vector, which we write as

\[
z^\mu = y^\mu - x^\mu (\tau_2) = \rho \hat{n}^\mu \quad \hat{n} = (1, \hat{n}) \quad \hat{n}^2 = 1.
\]

(96)

From \cite{93} the Lorentz force acting on particle-1 at \( \tau_2 \) can be written

\[
\dot{y}^\mu_f = \frac{e^2}{4M} \left( \frac{z^\mu \Delta u_v - z_v \Delta u^\mu}{4\pi (\bar{u} \cdot z)^3} \right) \dot{y}^\mu_f
\]

\[
= \dot{y}^\mu_i - \frac{e^2}{4M} \left( \frac{z^\mu \Delta u_v - z_v \Delta u^\mu}{4\pi (\bar{u} \cdot z)^3} \right) \dot{y}^\mu_i
\]

\[
+ \frac{e^2}{2M} \left( \Delta u \cdot z \right) \frac{\rho \gamma}{4\pi (\bar{u} \cdot z)^3} \hat{R}_x
\]

(97)

Making the simplifying choice \( \hat{R} \cdot \hat{n} = 0 \), we find

\[
\bar{u} \cdot z = -\rho \gamma \left[ 1 - v \left( \hat{n}_x + \frac{1}{2}g_e \hat{R}_x \right) \right] \quad \Delta u \cdot z = g_c \gamma v \rho \hat{R}_x
\]

(98)

so that taking \( v \ll 1 \) and neglecting \( g_e^2 \), the Lorentz force equations become

\[
\dot{y}^\mu_f = g_c \hat{R} \cdot \dot{y}_f = \dot{y}^\mu_{\text{in}} + g_c \hat{R} \cdot \dot{y}_i
\]

(99)
\[ \dot{y}_f + g_e g_R \left[ (\hat{n} \cdot \dot{y}_f - \dot{y}_f^0) \hat{R} - \hat{n} (\hat{R} \cdot \dot{y}_f) \right] = \dot{y}_{in} - g_e g_R \left[ (\hat{n} \cdot \dot{y}_{in} - \dot{y}_{in}^0) \hat{R} - \hat{n} (\hat{R} \cdot \dot{y}_{in}) \right] \] (100)

where

\[ g_R = \frac{e^2}{2M 4\pi \rho} . \] (101)

We write the velocity of incoming negative energy particle-1 as

\[ \dot{y}_{in}^0 < -1 \quad \text{and} \quad \dot{y}_{in} \cdot \hat{n} = 0 \Rightarrow \dot{y}_{in} = |\dot{y}_{in}| \hat{R} \] (102)

and write the Lorentz force in components, with \( g = g_e g_R \), as

\[
\begin{bmatrix}
1 & -g\hat{R}_x & -g\hat{R}_y \\
-g\hat{R}_x & 1 - g\hat{n}_x\hat{R}_x & -g\hat{n}_x\hat{R}_y \\
-g\hat{R}_y & -g\hat{n}_y\hat{R}_x & 1 - g\hat{n}_y\hat{R}_y
\end{bmatrix}
\begin{bmatrix}
\dot{y}_{in}^0 \\
\dot{y}_{sf} \\
\dot{y}_{gf}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
g\hat{R}_x & 1 & 0 \\
g\hat{R}_y & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{y}_f^0 \\
\dot{y}_x \\
\dot{y}_y
\end{bmatrix} + g |\dot{y}_f| \begin{bmatrix}
1 \\
\hat{n}_x \\
\hat{n}_y
\end{bmatrix} \tag{103}
\]

so that the final velocity of particle-1 after absorbing the radiation is

\[
\begin{bmatrix}
\dot{y}_f^0 \\
\dot{y}_f
\end{bmatrix} = \frac{1}{1 - \alpha^2} \begin{bmatrix}
\dot{y}_{in}^0 \\
|\dot{y}_{in}| \hat{R}
\end{bmatrix} + \frac{2g}{1 - \alpha^2} \begin{bmatrix}
|\dot{y}_{in}| \\
|\dot{y}_{in}| \hat{R} + |\dot{y}_{in}| \hat{n}
\end{bmatrix} + \frac{g^2}{1 - \alpha^2} \begin{bmatrix}
|\dot{y}_{in}| \\
2|\dot{y}_{in}| \hat{n} + 2 |\dot{y}_{in}| \hat{R}
\end{bmatrix} + \frac{g^3}{1 - \alpha^2} \begin{bmatrix}
|\dot{y}_{in}| \\
|\dot{y}_{in}| \hat{n}
\end{bmatrix} . \tag{104}
\]

The 0-component is

\[ \dot{y}_f^0 = \frac{1 + g^2}{1 - \alpha^2} \dot{y}_{in}^0 + g \frac{2 + g + \alpha^2}{1 - \alpha^2} |\dot{y}_{in}| \] (105)

approximated at low velocity as

\[ \dot{y}_f^0 \approx \frac{1 + g^2}{1 - \alpha^2} \dot{y}_{in}^0 = -\alpha \dot{y}_{in}^0 \] (106)

where

\[ \alpha^2 = \frac{\alpha + 1}{\alpha - 1} \Rightarrow \alpha = -\frac{1 + g^2}{1 - g^2} \] (107)

is written so that \( \alpha > 1 \) for a positive energy timelike particle. The exact final velocity of the scattered particle is

\[
\begin{bmatrix}
\dot{y}_f^0 \\
\dot{y}_f
\end{bmatrix} = \frac{\alpha - 1}{2} \begin{bmatrix}
\dot{y}_{in}^0 \\
|\dot{y}_{in}| \hat{R}
\end{bmatrix} + \sqrt{\alpha^2 - 1} \begin{bmatrix}
|\dot{y}_{in}| \\
|\dot{y}_{in}| \hat{R} + |\dot{y}_{in}| \hat{n}
\end{bmatrix}
\]

\[ -\frac{\alpha + 1}{2} \begin{bmatrix}
\dot{y}_{in}^0 \\
|\dot{y}_{in}| \hat{R}
\end{bmatrix} - \frac{\alpha + 1}{2} \sqrt{\frac{\alpha + 1}{\alpha - 1}} \begin{bmatrix}
|\dot{y}_{in}| \\
|\dot{y}_{in}| \hat{n}
\end{bmatrix} \] (108)

with 0-component

\[ \dot{y}_f^0 = -\alpha \dot{y}_{in}^0 - \frac{\alpha + 1}{2} \left[ 1 + \frac{3 - \alpha}{\sqrt{\alpha^2 - 1}} \right] |\dot{y}_{in}| . \] (109)

A pair creation event is observed at \( \tau_2 \) for \( \alpha > 1 \) which requires that \( g^2 = g_e^2 g_R^2 > 1 \). In this case, \( g_e \) is the interaction strength for the scattering of particle-2 to positive energy, so \( g_e < 2 \). From (101) the requirement for pair creation is then

\[ \frac{e^2}{4\pi \rho} > \frac{2M}{g_e} . \] (110)
5. Conclusions

In this paper we have shown that a classical equivalent of the Bethe-Heitler mechanism is permitted in Stueckelberg-Horwitz electrodynamics. Although Stueckelberg proposed his model with the goal of providing such a description, the calculation has not been previously carried out in detail. The process begins at $\tau_1$ with a pair annihilation event produced by the scattering of an incoming particle in the Coulomb field of a target nucleus. The general solution for the change in velocity leads to a requirement for this pair process

$$ g_e = \lambda \frac{Z e^2}{4\pi R M} > 2 \Rightarrow \lambda \frac{Z e^2}{4\pi R} > 2M \quad (111) $$

which is reasonable on relativistic grounds and can be seen as consistent with the QED requirement of interaction energy greater than the total mass energy of the particle pair. The next stage in the mechanism is the scattering to positive energy ($g_e < 2$ for this interaction) of a second particle in the Coulomb field of another nucleus. The outgoing 4-velocity is found from the general solution for Coulomb scattering, and this allows us to calculate the radiation emitted by the accelerating particle. Using the Lorentz force on the first particle produced by absorption of bremsstrahlung we find the requirement on pair creation to be

$$ g_e g_R > 1 \Rightarrow \frac{e^2}{4\pi \rho} > \frac{2M}{g_e} \quad (112) $$

which is similarly reasonable on relativistic grounds.

Further work is required to obtain a realistic description of the classical Bethe-Heitler mechanism. To describe the interaction for the long range Coulomb force, it is necessary to solve the nonlinear differential equations that arise from the smoothing function

$$ \varphi(\tau) = \frac{1}{2\lambda} e^{-|\tau|/\lambda} \quad (113) $$

with a large enough $\lambda$ to accurately model Rutherford scattering. It is also necessary to examine the mass transfer in the pair processes and check mass conservation among particles and fields.

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