STATIC RESPONSE FUNCTION FOR LONGITUDINAL AND TRANSVERSE EXCITATIONS IN SUPERFLUID HELIUM

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Abstract. The sum rule formalism is used to evaluate rigorous bounds for the density and current static response functions in superfluid $^4$He at zero temperature. Both lower and upper bounds are considered. The bounds are expressed in terms of ground state properties (density and current correlation functions) and of the inter-atomic potential. The results for the density static response significantly improve the Feynman approximation and turn out to be close to the experimental (neutron scattering) data. A quantitative prediction for the transverse current response is given. The role of one-phonon and multi-particle excitations in the longitudinal and transverse channels is discussed.

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I. INTRODUCTION

It is well known that the linear response of a many body system to an external probe (density, current, . . . ) contains relevant information on the dynamic correlations among particles [1]. If the probe is static and coupled to the density of the system, one has to deal with the static density response function $\chi(q)$, related to the dynamic structure function $S(q, \omega)$ through the equation

$$\chi(q) = -2 \int_0^\infty \frac{S(q, \omega)}{\hbar \omega} d\omega .$$

In the low-$q$ limit this quantity is fixed by the well known compressibility sum rule

$$-\lim_{q \to 0} \chi(q) = \frac{N}{Mc^2} ,$$

where $N$, $M$ and $c$ are the particle number, particle mass and sound velocity respectively. In the opposite case, $q \to \infty$, one finds the free-particle limit

$$-\lim_{q \to \infty} \chi(q) = \frac{4NM}{\hbar^2 q^2} .$$

In superfluid $^4$He the integral of Eq.(1) can be extracted, with rather good accuracy, from neutron scattering experiments [2]. An important characteristic of $\chi(q)$ is a pronounced peak in the region of the roton wave vectors. It reflects the strong interaction between particles which tends to produce solid-like correlations in the system. This feature of the static response function plays an important role in the context of density functional theories for inhomogeneous Bose systems [3,4].

The calculation of $\chi(q)$ represents a challenging theoretical problem. In this work we use the sum rule formalism to provide rigorous lower and upper bounds.
to the static response function at zero temperature. The basic idea was already introduced by Hall and Feenberg [5], who obtained bounds to $\chi(q)$ using the so-called Feynman approximation. The difference between the lower and the upper bounds was, however, too large to make their approach of practical use. More recently an improved lower bound has been proposed [6] with the help of additional sum rules. In the first part of this work we will derive and evaluate explicitly new lower and upper bounds for the static density response function. The ground state properties which enter the relevant sum rules, needed to calculate the bounds, are taken from Monte Carlo calculations [7,8]. The resulting difference between the new lower and upper bounds turns out to be relatively small and, thus, they allow for a rather precise estimate of the true static response in very good agreement with the available experimental results at zero pressure.

In the second part of this work we apply the same formalism to investigate the response to current excitations. We give a quantitative estimate of the static response in the transverse channel, through the use of a new lower bound, and we compare the results with the longitudinal case.

II. DENSITY-DENSITY RESPONSE FUNCTION

2.1 General formalism

The linear response function characterizes the behavior of a quantum many-body system subject to a small external perturbation. For static density excitations,
at zero temperature, one can write the total Hamiltonian as

\[ H(\lambda) = H + \lambda \rho_q \dagger, \]

where

\[ H = \sum_j \frac{p_j^2}{2M} + \sum_{i \neq j} V(|r_i - r_j|) \]

is the unperturbed Hamiltonian, \( \lambda \) is the strength of the perturbing field, and

\[ \rho_q \dagger = \sum_{j=1}^N e^{i\mathbf{q} \cdot \mathbf{r}_j} \]

is the usual density operator. The static response function is defined as

\[ \chi(q) = \lim_{\lambda \to 0} \frac{\langle \lambda | \rho_q | \lambda \rangle}{\lambda} , \]

where \( |\lambda\rangle \) is the ground state of \( H(\lambda) \). At zero temperature, standard perturbation theory gives \( \chi(q) \) in terms of the eigenstates and eigenvalues of \( H \) as follows:

\[ \chi(q) = -2 \sum_n \frac{|\langle n | \rho_q \dagger | 0 \rangle|^2}{\hbar \omega_n} . \]

With the usual definition of the dynamic structure function

\[ S(q, \omega) = \sum_n |\langle n | \rho_q \dagger | 0 \rangle|^2 \delta(\omega - \omega_n) \]

and its moments

\[ m_p(q) = \int_0^\infty (\hbar \omega)^p S(q, \omega) d\omega = \sum_n |\langle n | \rho_q \dagger | 0 \rangle|^2 (\hbar \omega_n)^p , \]

Eq.(6) reads

\[ \chi(q) = -2 \int_0^\infty \frac{S(q, \omega)}{\hbar \omega} d\omega = -2m_{-1}(q) , \]
The dynamic structure function $S(q, \omega)$ contains the detailed structure of the elementary excitations of the system. It is directly accessible to inelastic neutron scattering experiments on a relevant range of wave vectors. In particular, the integral in Eq.(9) can be estimated rather precisely from the observed spectrum; the factor $1/\omega$ makes the integral rapidly convergent at high $\omega$, reducing the effects of the complex structure of multiphonon excitations. An accurate measurement of $\chi(q)$ is available, at present, only at zero pressure [2]. The theoretical determination of $\chi(q)$ is a much harder problem; a direct approach, in fact, implies the calculation of non-uniform perturbed states with high enough accuracy to extract the linear limit; first results on this line are now becoming available [9]. In this work we choose an alternative approach based on the use of the moments $m_p$ with $p \geq 0$, which, differently from $m_{-1}$, can be expressed in terms of known ground state properties with the help of sum rules.

By using the completeness relationship $\sum_n |n\rangle\langle n| = 1$ in Eq.(8), one finds

$$m_0(q) = \int d\omega \ S(q, \omega) = \langle \rho_q^\dagger \rho_q \rangle$$ \hspace{1cm} (10a)

$$m_1(q) = \int d\omega \ h\omega \ S(q, \omega) = \frac{1}{2} \langle [\rho_q^\dagger, [H, \rho_q]] \rangle$$ \hspace{1cm} (10b)

$$m_2(q) = \int d\omega \ (h\omega)^2 \ S(q, \omega) = \langle [\rho_q^\dagger, H][H, \rho_q] \rangle$$ \hspace{1cm} (10c)

$$m_3(q) = \int d\omega \ (h\omega)^3 \ S(q, \omega) = \frac{1}{2} \langle [[[\rho_q^\dagger, H], H], [H, \rho_q]] \rangle.$$ \hspace{1cm} (10d)

The above equations relate properties of the excitation spectrum to mean values on the ground state. They are known as sum rules and have been extensively used in the theory of Bose liquids [1,5,6]. The moment $m_0$ coincides with the static form
factor

\[ NS(q) = m_0(q) \quad , \tag{11} \]

where \( N \) is the number of particles. \( S(q) \) is related to the radial distribution function \( g(r) \) by means of

\[ S(q) - 1 = \int d\mathbf{r} \ (g(r) - 1) \ e^{i\mathbf{q} \cdot \mathbf{r}} \quad . \tag{12} \]

Several \textit{ab initio} calculations are available for the pair correlation function \( g(r) \) (see for example Refs. 7 and 8). The corresponding \( S(q) \) well agrees with the experimental static form factor [10].

The energy weighted sum rule \( m_1 \) is the model independent f-sum rule

\[ m_1(q) = N \frac{\hbar^2 q^2}{2M} \quad , \tag{13} \]

which follows from the particle number conservation [1]. It can be easily derived from Eq.(10b) taking into account that the interatomic potential commutes with the density operator.

The \( m_2 \) sum rule can be expressed in terms of the current correlation function using the continuity equation

\[ [\rho_{\mathbf{q}}, H] = \hbar \mathbf{q} \cdot \mathbf{J}_{\mathbf{q}} \quad , \tag{14} \]

with the current density operator given by

\[ \mathbf{J}_{\mathbf{q}} = \frac{1}{2} \sum_{j=1}^{N} \left( \frac{\mathbf{p}_j}{M} e^{-i\mathbf{q} \cdot \mathbf{r}_j} + e^{-i\mathbf{q} \cdot \mathbf{r}_j} \frac{\mathbf{p}_j}{M} \right) \quad . \tag{15} \]

If \( \mathbf{q} \) is taken along \( z \), one finds

\[ m_2(q) = \hbar^2 q^2 \langle J_{z\mathbf{q}}^\dagger J_{z\mathbf{q}} \rangle \quad . \tag{16} \]
Eq.(16) shows that \( m_2(q) \) is proportional to the longitudinal component of the current correlation function in \( q \)-space. The \( m_2 \) moment can be also written in the following form [5]:

\[
m_2(q) = N \left[ (2 - S(q)) \left( \frac{\hbar^2 q^2}{2M} \right)^2 + \frac{\hbar^4 q^2}{M^2} D(q) \right],
\]

(17)

where \( D(q) \) is the so called kinetic structure function

\[
D(q) = \int dr_1 dr_2 \cos(q(z_1 - z_2)) \nabla_1^z \nabla_2^z \rho^{(2)}(r_1, r_2; r_1', r_2') \mid_{r_1 = r_1', r_2 = r_2'}.
\]

(18)

While \( S(q) \) is fixed by the diagonal components of the two-body density matrix, the kinetic structure function \( D(q) \) requires the knowledge of the non-diagonal components. At \( P = 0 \) we can use the Path Integral Monte Carlo calculations of Pollock and Ceperley [8,11] for the current density correlations in order to evaluate \( D(q) \).

In Fig. 1 we plot the resulting curve. At \( q \) smaller than about 1 Å\(^{-1}\) the accuracy is poor, due to the finite-size box of the PIMC calculations. At large \( q \) the numerical results are consistent with the asymptotic limit [5]

\[
\lim_{q \to \infty} D(q) = \frac{2}{3} \frac{M}{\hbar^2} \langle KE \rangle,
\]

(19)

where \( \langle KE \rangle \) is the mean kinetic energy per particle in the ground state.

Finally we note that the cubic energy weighted moment \( m_3 \) can be rather easily evaluated carrying out the commutators in Eq.(10d). One finds [12]

\[
m_3(q) = N \left[ \left( \frac{\hbar^2 q^2}{2M} \right)^3 + \frac{\hbar^4 q^4}{M^2} \langle KE \rangle + \frac{\rho_o \hbar^4}{2M^2} \int dr \ g(r) \left( 1 - \cos(q \cdot r) \right) (q \cdot \nabla)^2 V(r) \right],
\]

(20)

where \( \rho_o \) and \( V(r) \) are the particle density and the interatomic potential respectively.
2.2 The Feynman approximation

So far we have shown how the sum rules \( m_0, m_1, m_2 \) and \( m_3 \) can be determined from known properties of the ground state. Now we use them to fix rigorous bounds to \( m_{-1} \) and, consequently, to the static response function \( \chi(q) \). We notice that \( S(q, \omega) \) is a positive function and the inequality

\[
\int_0^\infty d\omega \frac{S(q, \omega)}{\hbar \omega} (1 + \alpha \hbar \omega)^2 \geq 0
\]

holds for any real \( \alpha \). Using the definition of the moments \( m_p \), one has

\[
m_{-1} \geq -(2 \alpha m_0 + \alpha^2 m_1).
\]

One can vary the parameter \( \alpha \) to make the r.h.s. of eq.(22) maximum. This yields \( \alpha = -m_0/m_1 \) and

\[
m_{-1} \geq \frac{(m_0)^2}{m_1} = \frac{NS(q)}{\hbar \omega_F(q)},
\]

where

\[
\hbar \omega_F(q) = \frac{m_1(q)}{m_0(q)}
\]

is the energy of the phonon-roton excitation branch in the so called Feynman approximation. Equation (23) provides a first rigorous bound to the static response function at \( T = 0 \). Using Eqs. (11) and (13) the same inequality can be written in the form [5]

\[
m_{-1}(q) \geq m_{-1}^F(q) = \frac{2NMS^2(q)}{\hbar^2 q^2}.
\]

The quantity \( m_{-1}^F(q) \) corresponds to the Feynman approximation to \( m_{-1} \).

In a similar way one can find an upper bound to \( \chi(q) \). The crucial point is that in superfluid \(^4\)He there are no excitations with energy lower than the energy \( \hbar \omega_0 \) of
the phonon-roton branch. Thus, it is possible to write

\[ \int_0^\infty \frac{S(q, \omega)}{\hbar \omega} d\omega \leq \int_0^\infty \frac{S(q, \omega)}{\hbar \omega_o} d\omega , \]  

(26)

which implies [5]

\[ m_{-1} \leq \frac{m_0}{\hbar \omega_o} . \]  

(27)

Precise measurements of \( \hbar \omega_o \) are available [2,13], so that the upper bound (27) can be accurately estimated at several pressures. One notices that the two bounds (25) and (27) would collapse in the exact \( m_{-1} \) if the excitation spectrum were exhausted by a single collective phonon-roton mode. In that case also the Feynman energy \( \hbar \omega_F \) would coincide with the true phonon-roton energy \( \hbar \omega_o \).

In Fig. 2 the lower (25) and upper (27) bounds, calculated at zero pressure, are plotted as dashed lines, together with the experimental data. Experiments are consistent with the theoretical bounds which, however, turn out to be quite far each other. The sizable difference between the two bounds is a measure of the role of multiphonon excitations and is a signature of the inadequacy of the Feynman approximation.

### 2.3 New bounds for \( m_{-1} \)

Equations (21) and (26) can be generalized in a natural way through the proper inclusion of additional sum rules. Let’s begin with the inequality

\[ \int_0^\infty d\omega \frac{S(q, \omega)}{\hbar \omega}(1 + \alpha \hbar \omega + \beta \hbar^2 \omega^2)^2 \geq 0 , \]  

(28)
valid for any real $\alpha$ and $\beta$. As before, we can write Eq.(28) as a lower bound for $m_{-1}$ and vary both $\alpha$ and $\beta$. After a straightforward calculation one gets [6]

$$m_{-1}(q) \geq \frac{m_{-1}^F(q)}{1 - \Delta(q)/\epsilon(q)},$$

(29)

where

$$\epsilon(q) = m_3 \frac{m_1^2}{m_0^2} - 2 \frac{m_2}{m_0} \left( \frac{m_2}{m_1} - \frac{m_1}{m_0} \right)^{-1},$$

(30)

and

$$\Delta(q) = m_2 \frac{m_1}{m_0} - m_1 \frac{m_0}{m_0}.$$  

(31)

The ratio $\Delta/\epsilon$ takes important contributions from multiphonon excitations, through the moments $m_2$ and $m_3$. As a consequence we expect a significant improvement with respect to the Feynman approximation (25).

As concerns the upper bound we generalize Eq.(26) in the following way [14]:

$$\int_{0}^{\infty} d\omega \frac{S(q,\omega)}{\hbar \omega} (1 + \gamma \hbar \omega)^2 \leq \int_{0}^{\infty} d\omega \frac{S(q,\omega)}{\hbar \omega_0} (1 + \gamma \hbar \omega)^2$$

(32)

or, equivalently,

$$m_{-1} \leq \frac{m_0}{\hbar \omega_0} + 2 \gamma \left( \frac{m_1}{\hbar \omega_0} - m_0 \right) + \gamma^2 \left( \frac{m_2}{\hbar \omega_0} - m_1 \right) .$$

(33)

Minimization with respect to $\gamma$ yields

$$m_{-1}(q) \leq \frac{m_0}{\hbar \omega_0} \left[ 1 - \frac{m_0}{m_1} \left( \frac{m_1}{m_0} - \hbar \omega_0 \right)^2 \frac{m_2}{m_1} \left( m_2 - \hbar \omega_0 \right)^{-1} \right] .$$

(34)

Note that since both $m_1/m_0$ and $m_2/m_1$ differ from $\hbar \omega_0$, due to the important role of multiphonon excitations, Eq.(34) yields a significant lowering with respect to the Feynman upper bound (27). In Fig. 2 the new bounds (29) and (34) at zero
pressure are plotted as solid lines. At small $q$ the microscopic ingredients used in the present sum rule analysis (density and current correlation functions of the ground state) suffer from a lack of accuracy due to the finite-size of the cell for Monte Carlo simulations. For this reason we have not shown the curves for the upper and lower bounds below $q \simeq 1\text{Å}^{-1}$. As in the case of the Feynman approximation the experimental data fulfil the theoretical bounds. But now the two bounds are much closer each other over all the relevant range of $q$‘s. This means that Eqs. (29) and (34) provide an estimate of the static response function close to the exact value. We stress again that the evaluation of the two bounds (29) and (34) involves only ground state properties and, consequently, is much simpler than the explicit \textit{ab initio} calculation of the static response function.

\section{III. THE CURRENT-CURRENT RESPONSE}

\subsection{3.1 Longitudinal current excitations}

In this section we rewrite the formalism of Section 2.1 for the current response function. The current operator has been already defined in Eq.(15). As before, one adds a small perturbation to the Hamiltonian of the system. The perturbation is now a vector field proportional to the current density operator. The response of the system is given by the current response tensor

\begin{equation}
\chi_{\mu\nu}(q, \omega) = \sum_n \left[ \frac{\langle 0|J_{\mu q}|n\rangle \langle n|J_{\nu q}^\dagger|0\rangle}{\hbar \omega - \hbar \omega_n + i\eta} - \frac{\langle 0|J_{\nu q}^\dagger|n\rangle \langle n|J_{\mu q}|0\rangle}{\hbar \omega + \hbar \omega_n + i\eta} \right].
\end{equation}

The transverse and longitudinal components of the response tensor can be studied
separately. Let’s begin with the longitudinal one. We take \( \mathbf{q} \) along \( z \) and define the longitudinal response function

\[
\chi^L(q, \omega) = \sum_n \frac{2\hbar\omega_n}{(\hbar\omega + i\eta)^2 - (\hbar\omega_n)^2} |\langle n|J^\dagger_{z\mathbf{q}}|0\rangle|^2 .
\]  

(36)

Then we define the quantity

\[
\Upsilon^L(q, \omega) = \sum_n |\langle n|J^\dagger_{z\mathbf{q}}|0\rangle|^2 \delta(\omega - \omega_n)
\]  

(37)

and its moments

\[
m^L_p(q) = \int_0^\infty (\hbar\omega)^p \Upsilon^L(q, \omega) d\omega = \sum_n |\langle n|J^\dagger_{z\mathbf{q}}|0\rangle|^2 (\hbar\omega_n)^p .
\]  

(38)

The longitudinal static response function is the \( \omega \to 0 \) limit of Eq.(36). Using the definition of \( \Upsilon^L \) one has

\[
-\chi^L(q) = 2 \int d\omega \frac{\Upsilon^L(q, \omega)}{\hbar\omega} .
\]  

(39)

Indeed the determination of the static response function (39) is trivial. The key point is the continuity equation (14) which connects the matrix elements of the longitudinal current with the ones of the density operator:

\[
\langle n|J^\dagger_{z\mathbf{q}}|0\rangle = \frac{\omega_n}{q} \langle n|\rho^\dagger_{\mathbf{q}}|0\rangle .
\]  

(40)

One easily obtains [1,15]

\[
-\chi^L(q) = 2m^L_{-1}(q) = 2\frac{m_1(q)}{\hbar^2 q^2} = \frac{N}{M} ,
\]  

(41)

where \( m_1 \) is the density \( f \)-sum rule (13). In the same way one finds

\[
m^L_0(q) = \langle J^\dagger_{z\mathbf{q}} J_{z\mathbf{q}} \rangle = \frac{m_2(q)}{\hbar^2 q^2}
\]  

(42)
and
\[ m_T^L(q) = \frac{1}{2} \langle [[J_{zq}^T, H], J_{zq}] \rangle = \frac{m_3(q)}{\hbar^2 q^2} , \tag{43} \]
where \( m_2 \) and \( m_3 \) are the sum rules (16) and (20). The simplicity of results (41) reflects the fact that the response to a static longitudinal probe is, actually, a fictitious problem, related to gauge invariance properties [1].

### 3.2 Transverse current excitations

The response to transverse probes plays a crucial role in the theory of superfluidity. Actually the \( q \to 0 \) limit of the transverse response function defines the normal (non superfluid) density [1,15] of the system. This limit was extracted in Ref. 8 at several temperatures through a Path Integral Monte Carlo calculation of the transverse current correlation function and the use of the fluctuation-dissipation theorem. In this section we provide a first estimate of \( \chi^T \) at finite \( q \) by calculating a rigorous lower bound at zero temperature.

Let’s rewrite Eqs. (36-39) by replacing the superscript \( L \) with \( T \) and the \( z \)-component of the current with an arbitrary component orthogonal to \( q \). Similarly to the longitudinal case one has
\[ m_0^T(q) = \langle J_{xq}^T J_{xq} \rangle \tag{44} \]
and
\[ m_1^T(q) = \frac{1}{2} \langle [[J_{xq}^T, H], J_{xq}] \rangle . \tag{45} \]
The calculation of these two sum rules follows exactly the procedure used for the \( m_2 \) and \( m_3 \) sum rules in the case of density excitations (see Eqs. (16) and (20)).
In particular, the moment $m^T_0$ is the Fourier transform of the transverse current correlation function. The $m^T_1$ moment can be evaluated by carrying out explicitly the commutators in Eq.(45). One finds

$$m^T_1(q) = N \left[ \frac{\hbar^2 q^2}{3M^2} \langle KE \rangle + \frac{\rho_0 \hbar^2}{2M^2} \int d\mathbf{r} g(r)(1 - \cos(qz))\nabla^2_x V(r) \right]. \quad (46)$$

The transverse static response function is given by

$$-\chi^T(q) = 2m^T_1 = 2 \int d\omega \frac{\Upsilon^T(q, \omega)}{\hbar \omega}. \quad (47)$$

The quantity $\Upsilon^T(q, \omega)$ is positive, so that the inequality (22) holds even for the transverse moments and, thus,

$$-\chi^T(q) \geq 2\frac{(m^T_0(q))^2}{m^T_1(q)}. \quad (48)$$

This is a rigorous lower bound, valid at zero temperature. We have explicitly evaluated the r.h.s. of Eq.(48) at zero temperature, taking $m^T_0$ from the transverse current correlation function calculated in Ref. 8. The resulting lower bound is shown in Fig. 3. Since the PIMC calculations of Ref. 8 are carried out at low but finite temperature (the lowest value is $T = 1.18$ K), the use of the corresponding $m^T_0$ in Eq.(48) is meaningful only for $q$ much greater than $kT/c$; the curve in Fig. 3 corresponds to values of $q$ well above this limit.

At zero temperature the function $\chi^T(q)$ should vanish at $q = 0$ because the transverse current operator cannot excite phonons, which, in this limit, are the dominant excitations (the system is entirely superfluid). At higher $q$ multiphonon processes take place and $\chi^T(q)$ no longer vanishes. The first contribution to the static response is expected to be in $q^2$. In the opposite case $q \to \infty$ one approaches
the free particle limit and all the ratios $m_{p+1}/m_p$ tend to the same energy $\hbar^2 q^2/2M$.

This implies

$$-\lim_{q \to \infty} \chi^T(q) = 2 \lim_{q \to \infty} \frac{(m^T_0(q))^2}{m^T_1} = N \frac{8 \langle KE \rangle}{3 \hbar^2 q^2}, \quad (49)$$

where we have used the asymptotic values

$$\lim_{q \to \infty} m^T_0 = \frac{2N}{3M} \langle KE \rangle \quad (50)$$

and

$$\lim_{q \to \infty} m^T_1 = \frac{N\hbar^2 q^2}{3M^2} \langle KE \rangle. \quad (51)$$

The asymptotic behavior of $\chi^T(q)$ is shown in Fig. 3 as a dot-dashed line. The position of the maximum of the solid curve provides a characteristic coherence length for superfluidity [1]. As expected, it is of the same order as the roton wave vector. The height at the maximum measures the strength of the interaction between particles. In fact in a free Bose gas at zero temperature the function $\chi^T(q)$ would be identically zero.

Clearly Eq.(48) provides only a lower bound for $-\chi^T(q)$. A rough estimate of the difference between this lower bound and the exact value of the transverse static response can be made using the following arguments. First we note that only multiparticle excitations affect the transverse response function, since no transverse current is carried by the elementary excitations (phonons, rotons) of the system. If one assumes that the average energy and the spreading of multiparticle excitations are the same in the longitudinal as in the transverse channels, one can estimate the relative difference between $m^T_{-1}$ and $(m^T_0)^2/m^T_1$ by analysing the multiparticle contribution to the corresponding longitudinal sum rules. Such a contribution can
be explicitly extracted from the experimental data on \( S(q, \omega) \) [2]. With this procedure we conclude that the lower bound (48) should underestimate the exact value of \(-\chi^T(q)\) by about 30% in the roton region.

IV. CONCLUSIONS

In this work we have investigated lower and upper bounds for the static response function of superfluid \(^4\text{He}\) at zero temperature. In the case of the density response \(\chi(q)\) the new bounds improve significantly the Feynman approximation, yielding estimates in agreement with the experimental data. In the case of the current response function we have given a first rigorous bound to the static response \(\chi^T(q)\) and made a direct comparison with the longitudinal response, stressing the different role of one-phonon and multiparticle excitations.

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per particle is overestimated by about 1K. We checked that this problem has negligible quantitative effects on our results.

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**FIGURE CAPTIONS**

Fig. 1  Kinetic structure function extracted from Eq.(17) with $m_2(q)$ and $S(q)$ taken from Monte Carlo calculations [7,8]. In the limit $q \to \infty$ one finds the asymptotic value $0.8 \text{ Å}^{-2}$, as in Eq.(19).

Fig. 2  Static response function for density excitations. Open circles: experimental values [2]; dashed lines: upper and lower bounds in Feynman approximation; solid lines: upper and lower bounds given in Section 2.3.

Fig. 3  Static response function for transverse current excitations. Solid line: lower bound (48); dot-dashed line: $q \to \infty$ asymptotic curve for the exact response function (see Eq.(49)).