A STATIONARY CORE-SHELL ASSEMBLY IN A TERNARY INHIBITORY SYSTEM

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Abstract. A ternary inhibitory system motivated by the triblock copolymer theory is studied as a nonlocal geometric variational problem. The free energy of the system is the sum of two terms: the total size of the interfaces separating the three constituents, and a longer ranging interaction energy that inhibits micro-domains from unlimited growth. In a particular parameter range there is an assembly of many core-shells that exists as a stationary set of the free energy functional. The cores form regions occupied by the first constituent of the ternary system, the shells form regions occupied by the second constituent, and the background is taken by the third constituent. The constructive proof of the existence theorem reveals much information about the core-shell stationary assembly: asymptotically one can determine the sizes and locations of all the core-shells in the assembly. The proof also implies a kind of stability for the stationary assembly.

1. Introduction. A pattern forming, multi-constituent, inhibitory physical or biological system is characterized by two properties: growth and inhibition. A deviation from homogeneity has a strong positive feedback on its further increase. In the meantime a longer ranging confinement mechanism prevents unlimited spreading. Together they lead to a locally self-enhancing and self-organizing process.

Nakazawa and Ohta introduced a ternary inhibitory system in [15] to study triblock copolymers. A Gamma-limit of the system was derived by Ren and Wei in [20] for the sharp interface case.

This Gamma-limit is a nonlocal geometric variational problem. Let \( D \) be a bounded domain in \( \mathbb{R}^2 \). A functional is defined on non-overlapping pairs of subsets of \( D \) with the fixed size; namely on the admissible set

\[
\mathcal{A} = \left\{ (\Omega_1, \Omega_2) : \Omega_i \subset D \text{ is measurable, } |\Omega_i| = \omega_i |D|, \ i = 1, 2, \ |\Omega_1 \cap \Omega_2| = 0 \right\}.
\]

The numbers \( \omega_i \in (0,1) \) are given and \( \omega_1 + \omega_2 < 1 \). They are the first set of the parameters in this problem. One denotes by \( |\Omega_i| \) and \( |D| \) the Lebesgue measures of \( \Omega_i \) and \( D \) respectively.

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For each pair $(\Omega_1, \Omega_2) \in \mathcal{A}$, let $\Omega_3 = D \setminus (\Omega_1 \cup \Omega_2)$. The free energy of the system is given by

$$J(\Omega_1, \Omega_2) = \frac{1}{2} \sum_{i=1}^{3} P_D(\Omega_i)$$

$$+ \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_D \left( (-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i) \right) \left( (-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j) \right) dx.$$  

In the first term $P_D(\Omega_i)$ is the perimeter of $\Omega_i$ in $D$. If the boundary of $\Omega_i$ in $D$, $\partial \Omega_i \cap D$, is $C^1$, then $\partial \Omega_i \cap D$ is the interface of $\Omega_i$ and $P_D(\Omega_i)$ is simply the length of $\partial \Omega_i \cap D$. Note that $\partial \Omega_i$ is the boundary of $\Omega_i$ in $\mathbb{R}^2$. Part of $\partial \Omega_i$ may overlap with the domain boundary $\partial D$, but the interface of $\Omega_i$ does not include this part.

The interface is $\partial \Omega_i \cap D$, which is the part of the boundary of $\Omega_i$ that is inside $D$. For a general measurable set $\Omega_i$, perimeter is defined in (2.3).

There are possibly three types of interfaces: $\partial \Omega_1 \cap \partial \Omega_2 \cap D$, the interface separating $\Omega_1$ and $\Omega_2$, $\partial \Omega_2 \cap \partial \Omega_3 \cap D$, the interface separating $\Omega_2$ and $\Omega_3$, and $\partial \Omega_3 \cap \partial \Omega_1 \cap D$, the interface separating $\Omega_3$ and $\Omega_1$. The first term in (1.2) is the combined length of all three type interfaces in $D$. The half is put there to eliminate double counting since each type of interface is counted twice in the sum.

The numbers $\gamma_{ij}$ in the second term of (1.2) form a two by two symmetric and positive definite matrix. The operator $(-\Delta)^{-1/2}$ is defined as follows. Let $f \in L^2(D)$ such that $\int_D f(x) dx = 0$. One solves the Poisson’s equation with the Neumann boundary condition:

$$-\Delta v = f \text{ in } D, \quad \partial_D v = 0 \text{ on } \partial D, \quad \int_D v(x) dx = 0. \quad (1.3)$$

If the domain $D$ is sufficiently smooth, this equation has a unique solution in $H^2(D)$, and hence it defines an operator $(-\Delta)^{-1/2}$ from the space $\{ f \in L^2(D) : \int_D f(x) dx = 0 \}$ with the $L^2$ norm to the space $\{ v \in H^2(D) : \partial_D v = 0 \text{ on } \partial D, \int_D v(x) dx = 0 \}$ with the $H^2$ norm so that $v = (-\Delta)^{-1/2} f$. The operator $(-\Delta)^{-1}$ is positive definite and $(-\Delta)^{-1/2}$ in (1.2) is its positive square root. The latter acts on $\chi_{\Omega_i} - \omega_i$ where $\chi_{\Omega_i}$ is the characteristic function of $\Omega_i$ ($\chi_{\Omega_i}(x) = 1$ if $x \in \Omega_i$ and 0 otherwise).

A stationary point of $J$ is an element in $[\mathcal{A}]$ where the first variation of $J$ vanishes. In this paper we are interested in one type of stationary points: stationary core-shell assemblies. A core-shell assembly is a collection of many core-shells; each core-shell consists of a perturbed disc, called a core, surrounded by a perturbed ring, called a shell; see the second plot in Figure 1. The union of the cores is $\Omega_1$ and the union of the shells is $\Omega_2$. The many core-shells in the assembly do not intersect and they do not touch the domain boundary $\partial D$. In a core-shell assembly there are only two types of interfaces $\partial \Omega_1 \cap \partial \Omega_2$ and $\partial \Omega_2 \cap \partial \Omega_3$, namely the interface separating $\Omega_1$ from $\Omega_2$ and the interface separating $\Omega_2$ from $\Omega_3$; there is no interface between $\Omega_1$ and $\Omega_3$.

If a core-shell assembly has $C^2$ interfaces, then it is stationary if and only if it satisfies the following two Euler-Lagrange equations:

$$\kappa_1 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{12} - \gamma_{22}) I_{\Omega_2} = \lambda_1 \text{ on } \partial \Omega_1 \cap \partial \Omega_2, \quad (1.4)$$

$$\kappa_2 + \gamma_{12} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} = \lambda_2 \text{ on } \partial \Omega_2 \cap \partial \Omega_3. \quad (1.5)$$

In (1.4) and (1.5), $\kappa_1$ and $\kappa_2$ are the curvatures of the curves $\partial \Omega_1 \cap \partial \Omega_2$ and $\partial \Omega_2 \cap \partial \Omega_3$, respectively. These are signed curvatures defined with respect to a
choice of normal vectors. For instance, a circle has positive curvature if the normal vector is inward pointing. On $\partial \Omega_1 \cap \partial \Omega_2$ the normal vector points from $\Omega_2$ towards $\Omega_1$, i.e. inward with respect to $\Omega_1$ and outward with respect to $\Omega_2$. On $\partial \Omega_2 \cap \partial \Omega_3$, the normal vector points inward with respect to $\Omega_2$ and outward with respect to $\Omega_3$. Also in (1.4) and (1.5) are two functions $I_{\Omega_1}$ and $I_{\Omega_2}$ called inhibitors, which are obtained by solving (1.3) with $f = \chi_{\Omega_i} - \omega_i$; in other words,

$$I_{\Omega_i} = (\Delta)^{-1}(\chi_{\Omega_i} - \omega_i), \quad i = 1, 2.$$  

(1.6)

The constants $\lambda_1$ and $\lambda_2$ in (1.4) and (1.5) are Lagrange multipliers corresponding to the constraints $|\Omega_i| = \omega_i|D|$, $i = 1, 2$. They are unknown and are to be found with $\Omega_1$ and $\Omega_2$.

Introduce a fixed number $m \in (0, 1)$ and a small $\epsilon$ so that $\omega_1|D| = \epsilon^2m$ and $\omega_2|D| = \epsilon^2(1 - m)$. The area constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$ now take the form

$$|\Omega_1| = \epsilon^2m, \quad |\Omega_2| = \epsilon^2(1 - m).$$  

(1.7)

Instead of $\omega_1$ and $\omega_2$, $\epsilon$ becomes one parameter of our problem. The fixed number $m$ measures the relative size of $|\Omega_1|$ vs $|\Omega_2|$ since $|\Omega_1| = \frac{m}{1 - m}$. The other parameter is the matrix $\gamma$. It must be positive definite and satisfy a uniform positivity condition. Namely, there exists $\epsilon > 0$ so that $\overline{\lambda}(\gamma) \leq \overline{\lambda}(\gamma)$ where $\overline{\lambda}(\gamma)$ and $\overline{\lambda}(\gamma)$ are the two eigenvalues of $\gamma$ such that $0 < \overline{\lambda}(\gamma) \leq \overline{\lambda}(\gamma)$. The larger eigenvalue $\overline{\lambda}(\gamma)$ may be used as a norm of $\gamma$:

$$|\gamma| = \overline{\lambda}(\gamma),$$  

(1.8)

although any equivalent norm is acceptable.

Let $S^2$ be the set of 2 by 2 symmetric matrices. For each $\Gamma \in S^2$ and each positive integer $l$, define another 2 by 2 symmetric matrix

$$M_l(\Gamma) = \begin{bmatrix} \left(n \pi \frac{\Gamma_{12} - \Gamma_{11} \Gamma_{22}}{2l^2} - \frac{\Gamma_{11} - \Gamma_{12}}{2l^2} - \frac{\Gamma_{12} - \Gamma_{22}}{2l^2} \right) & \frac{(\Gamma_{12} - \Gamma_{22})m^{l/2}}{2l^2} \\ \frac{(\Gamma_{12} - \Gamma_{22})m^{l/2}}{2l^2} & \left(n \pi \frac{\Gamma_{12} - \Gamma_{11} \Gamma_{22}}{2l^2} - \frac{\Gamma_{11} - \Gamma_{12}}{2l^2} - \frac{\Gamma_{12} - \Gamma_{22}}{2l^2} \right) \end{bmatrix}.$$  

(1.9)

Figure 1. A double bubble assembly on the left and a core-shell assembly on the right.
We single out a special class of symmetric matrices:

$$S = \left\{ \Gamma \in S^2 : \Gamma_{22} > \Gamma_{12}, \Gamma > 0, M_l(\Gamma) > 0 \ \forall l \geq 2 \right\}. \quad (1.10)$$

Note that in (1.10), $l$ starts from 2, not 1. Also $\Gamma > 0$ (resp. $M_l(\Gamma) > 0$) means that the matrix $\Gamma$ (resp. $M_l(\Gamma)$) is positive definite. One can show that the interior of $S$, denoted $\text{int}(S)$, is non-empty; see Lemma 4.1.

The main result in this paper is the following existence theorem.

**Theorem 1.1.** Let $D$ be a bounded, sufficiently smooth domain in $\mathbb{R}^2$, $m \in (0, 1)$, $n \in \mathbb{N}$, and $\iota \in (0, 1 \setminus \mathbb{N})$. For each compact subset $K \subset \text{int}(S)$, there exist positive numbers $\delta, \sigma$ depending on $D, m, n, K, \iota$ only, such that if

1. $0 < \epsilon < \delta$,
2. $\frac{\sigma}{\epsilon^3 \log \frac{1}{\epsilon}} < \overline{\lambda}(\gamma)$ in the case $n \geq 2$,
3. $\epsilon^3 \gamma \in K$,
4. $\iota \overline{\lambda}(\gamma) \leq \overline{\lambda}(\gamma),$

then $J$ admits a stationary assembly of $n$ perturbed core-shells, satisfying the constraints (1.7).

A few remarks are in order. The smoothness assumption on $D$ is only needed to ensure that (1.3) is solvable, $(-\Delta)^{-1}$ is well defined, and a Green's function exists (see (1.13)).

The condition 3, $\epsilon^3 \gamma \in K \subset \text{int}(S)$, has several implications. First, $\gamma$ must be positive definite. Second,

$$\gamma_{22} > \gamma_{12}, \quad (1.11)$$

seen from the definition of $S$. This requirement is related to our assumption that cores are formed by the first constituent and shells formed by the second constituent. If the two constituents were reversed, we would have $\gamma_{11} > \gamma_{12}$. Third, the condition also implies, since $K$ is compact, that there exists $\overline{\sigma} > 0$ such that

$$\overline{\lambda}(\gamma) < \frac{\overline{\sigma}}{\epsilon^3}. \quad (1.12)$$

This upper bound for $\overline{\lambda}(\gamma)$ is by an order (albeit only $\log \frac{1}{\epsilon}$) greater than the lower bound for $\overline{\lambda}(\gamma)$ in condition 2.

Some of the techniques used in this paper were developed by Xie in [32], where she proved the existence of a single, perturbed core-shell as a stationary point of $J$, a special case of Theorem 1.1 with $n = 1$. There are significant differences between the $n = 1$ case and $n \geq 2$ case. The condition 2 is not needed if $n = 1$. If $n \geq 2$, the condition 2 prevents a coarsening phenomenon, where some core-shells grow bigger and others shrink and disappear. Much of our work is to ensure that under the condition 2, $n$ core-shells will keep their shapes and remain stable in an assembly.

Although the focus of this paper is on the existence of a stationary assembly, the construction process in the proof of Theorem 1.1 shows that the stationary assembly has a certain kind of stability.

The proof of Theorem 1.1 contains definitions for the centers of the perturbed core-shells in the stationary assembly and the radii of their interfaces; see the discussion after (4.7). The perturbed core-shells turn out to be of the similar size, so all the inner radii are approximately equal and all the outer radii are also approximately equal. The centers of the core-shells are determined asymptotically by the
Green’s function of $\{1.3\}$. Recall that the Green’s function of the $-\Delta$ operator on $D$ with the Neumann boundary condition is a function $G(x, y)$ that solves

$$-\Delta G(\cdot, y) = \delta(\cdot - y) - \frac{1}{|D|} \quad \text{in } D, \quad \partial_n G(\cdot, y) = 0 \quad \text{on } \partial D, \quad \int_D G(x, y) dx = 0,$$

(1.13)

as a function of $x$ for each $y \in D$. One can write $G$ as a sum of two terms:

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y|} + R(x, y).$$

(1.14)

The first term $\frac{1}{2\pi} \log \frac{1}{|x - y|}$ is the fundamental solution of the Laplace operator; the second term $R$ is the regular part of the Green’s function, a smooth function of $(x, y) \in D \times D$.

For $n$ distinct points $\xi^k, k = 1, 2, \ldots, n$, in $D$ let

$$F(\xi_1, \ldots, \xi_n) = \sum_{k=1}^n R(\xi^k, \xi^k) + \sum_{k=1}^n \sum_{l=1, l \neq k}^n G(\xi^k, \xi^l).$$

(1.15)

It is known that $R(x, x) \to \infty$ if $x \to \partial D$. Consequently $F(\xi^1, \ldots, \xi^n) \to \infty$, if one of the $\xi^k$’s approaches $\partial D$, or if the distance of two points $\xi^k$ and $\xi^l$ approaches 0. This ensures that $F$ is minimized by $n$ distinct points in $D$.

**Theorem 1.2.** Let $\xi^{*, k}$ be the center of the $k$-th perturbed core-shell of the stationary assembly in Theorem 1.1 and $r_1^{*, k}$ and $r_2^{*, k}$ be the radii of the inner and outer interfaces, respectively.

1. As $\epsilon \to 0$,

$$\frac{r_1^{*, k}}{\epsilon} \to \sqrt{\frac{m}{n\pi}}, \quad \frac{r_2^{*, k}}{\epsilon} \to \sqrt{\frac{1}{n\pi}}$$

for all $k = 1, 2, \ldots, n$.

2. If $\xi^{*, k} \to \xi^{\circ, k}$ for all $k = 1, 2, \ldots, n$, possibly along a subsequence, as $\epsilon \to 0$, then

$$F(\xi^{\circ, 1}, \xi^{\circ, 2}, \ldots, \xi^{\circ, n}) = \min \{ F(\xi^1, \ldots, \xi^n) : \xi^1, \ldots, \xi^n \in D, \xi^k \neq \xi^l \text{ if } k \neq l \}.$$

Although experimentally an almost unlimited number of architectures can be synthetically accessed in ternary systems like triblock copolymers [3 Figure 5 and the magazine’s cover], mathematical study of $\mathcal{F}$ is still in an early stage due to its complexity. The core-shell stationary assembly found in Theorem 1.1 is only the second stationary assembly discovered on a general two dimensional domain. The first is an assembly of perturbed double bubbles (the first plot in Figure 1) found by Ren and Wei in [29]. The special case of a single, perturbed double bubble stationary point was found earlier in [20, 27]. One dimensional stationary points were found in [20, 5].

The functional $\mathcal{F}$ has a simpler counterpart in a binary inhibitory system. Let $\omega \in (0, 1)$ and $\gamma > 0$. This time $\gamma$ is a number. For $\Omega \subset D$ with the fixed area: $|\Omega| = \omega|D|$, the binary free energy of $\Omega$ is

$$\mathcal{F}_B(\Omega) = \mathcal{P}_D(\Omega) + \frac{\gamma}{2} \int_D \left| (-\Delta)^{-1/2}(\chi_\Omega - \omega) \right|^2 dx.$$ 

(1.16)

A stationary set of this functional satisfies the equation

$$\kappa + \gamma I_{\Omega} = \lambda$$

(1.17)

on $\partial \Omega \cap D$. Equation (1.17) or functional (1.16) may be derived from the Ohta-Kawasaki theory [17] for diblock copolymers; see [16, 19]. The equation can also
be derived from the Gierer-Meinhardt system [25]. This binary problem has been studied intensively in recent years. All solutions to (1.17) in one dimension are known to be local minimizers of \( J_B \) [19]. There is even a dynamic counterpart of (1.17), and Fife and Hilhorst proved that any time dependent solution converges to one of the local minimizers [8]. Many solutions in two and three dimensions have been found that match the morphological phases in diblock copolymers [18, 22, 21, 23, 24, 11, 12, 25, 28, 31]. Global minimizers of \( J_B \) are studied in [2, 30, 14, 4, 13, 10] for various parameter ranges. Applications of the second variation of \( J_B \) and its connections to minimality and Gamma-convergence are found in [6, 1].

A relevant result in [21] states that when \( \omega \) and \( \gamma \) are in a proper range, (1.17) admits a solution that is an assembly of perturbed discs. The discs have approximately the same size, and the centers of the discs nearly minimize the same function \( F \) in Theorem 1.2.

2. Exact core-shell assembly. We reiterate that our data comprises the domain \( D \), the number \( m \in (0, 1) \), the number of core-shells \( n \) in assemblies, a compact subset \( K \) of the set \( S \) given in (1.10), and finally \( \epsilon \in (0, 1) \) which bounds the ratio of the eigenvalues of the matrix \( \gamma \). From \( D, m, n, K \), and \( \epsilon \), one proceeds to find \( \delta \) and \( \sigma \). These two numbers specify the range for \( \epsilon \) and \( \gamma \), where Theorem 1.1 will hold. Moreover Theorem 1.2 will be proved if \( \epsilon \to 0 \) within the range.

The two terms in \( \mathcal{J} \) are denoted by \( \mathcal{J}_s \) and \( \mathcal{J}_l \) for short and long range interactions respectively:

\[
\mathcal{J}_s(\Omega_1, \Omega_2) = \frac{1}{2} \sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) \\
\mathcal{J}_l(\Omega_1, \Omega_2) = \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_D \left( (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_i} - \omega_i) \right) \left( (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_j} - \omega_j) \right).
\]

For \( \Omega_1, \Omega_2 \) or \( \Omega_3 \) of a general member \((\Omega_1, \Omega_2)\) in \( \mathcal{A} \) defined by (1.1), each perimeter in \( \mathcal{J}_s \) is given by

\[
\mathcal{P}_D(\Omega_i) = \sup \left\{ \int_{\Omega_i} \text{div} \, g(x) \, dx : g \in C_0^1(D, \mathbb{R}^2), |g(x)| \leq 1 \text{ for all } x \in D \right\},
\]

where \( \text{div} \, g \) is the divergence of the \( C^1 \) vector field \( g \) on \( D \) with compact support and \( |g(x)| \) stands for the Euclidean norm of the vector \( g(x) \in \mathbb{R}^2 \); see [7] or [9] for more information on perimeter. Using the Green’s function \( G \) of (1.13) one can rewrite \( \mathcal{J}_l(\Omega_1, \Omega_2) \) as

\[
\mathcal{J}_l(\Omega_1, \Omega_2) = \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_{\Omega_1} \int_{\Omega_j} G(x, y) \, dx \, dy.
\]

Let \( \xi^1, \xi^2, ..., \xi^n \) be \( n \) points in \( D \) and \( r_1^1, r_2^1, r_1^2, r_2^2, ..., r_1^n, r_2^n \) be \( 2n \) numbers such that \( 0 < r_1^k < r_2^k, k = 1, 2, ..., n \). Denote by \( B^k \) an exact core-shell. More precisely \( B^k = (B_1^k \setminus B_2^k) \), where \( B_1^k = \{ x \in \mathbb{R}^2 : |x - \xi^k| < r_1^k \} \) and \( B_2^k = \{ x \in \mathbb{R}^2 : |x - \xi^k| < r_2^k \} \), so \( B_1^k \) is the core and \( B_2^k \setminus B_1^k \) is the shell. Also introduce \( w_i^k, i = 1, 2 \) and \( k = 1, 2, ..., n \), so that

\[
\pi(r_1^k)^2 = w_1^k \epsilon^2, \quad \pi((r_2^k)^2 - (r_1^k)^2) = w_2^k \epsilon^2.
\]

The \( w_i^k \)'s belong to the set \( W \) which is the closure of

\[
W = \left\{ (w_i^k) \in \mathbb{R}^{2n} : \left| w_1^k - \frac{m}{n} \right| < h, \left| w_2^k - \frac{1 - m}{n} \right| < h, \forall k \right\}.
\]
\[
\sum_{k=1}^{n} w_k = m, \quad \sum_{k=1}^{n} w_{k/2} = 1 - m \}.
\]

For now assume that
\[
0 < h < \min \left\{ \frac{m}{2n}, \frac{1 - m}{2n} \right\};
\]
later \( h \) will be restricted to a smaller range. Initially the \( w_k \)'s are fixed. Later they will vary in \( \overline{W} \). Of course \( w_k \) can vary only if \( n > 1 \). If \( n = 1 \), there is no need to introduce \( w_k \).

Recall the function \( F \) defined in (1.5). The domain of \( F \) is
\[
\Xi = \{ \xi = (\xi^1, \xi^2, \ldots, \xi^n) : \xi^k \in D \forall k, \xi^k \neq \xi^l \text{ if } k \neq l \}.
\]
Since \( F(\xi) \to \infty \) as \( \xi \to \partial \Xi \) where \( \Xi \) is viewed as an open subset of \( \mathbb{R}^{2n} \), one can find a small enough \( \delta > 0 \) such that
\[
\min_{\xi \in \Xi} F(\xi) < \min_{\xi \in \Xi \backslash \Xi_\delta} F(\xi).
\]
Here \( \Xi_\delta \) is a subset of \( \Xi \) defined as
\[
\Xi_\delta = \{ \xi \in \Xi : d(\xi^k, \partial D) > \delta \forall k, d(\xi^k, \xi^l) > 2\delta \forall k \neq l \}.
\]
In (2.10) “\( d \)” stands for the Euclidean distance in \( \mathbb{R}^2 \). The centers \( \xi^k \) of the core-shells \( B_k = (B^1_k, B^2_k \backslash B^1_k) \) will always be in the closure of \( \Xi_\delta \):
\[
\xi = (\xi^1, \xi^2, \ldots, \xi^n) \in \overline{\Xi_\delta}.
\]

The number \( \delta \) remains fixed throughout this paper. At this point we state our initial requirement on \( \delta \) of Theorem 1.1 which is the bound for \( \epsilon \). The number \( \delta \) must be small enough so that
\[
0 < r^k_\delta = \epsilon \sqrt{\frac{w^1_k + w^2_k}{\pi}} < \delta < \sqrt{\frac{w^1_k + w^2_k}{\pi}} < \frac{\delta}{2}
\]
holds for the radius \( r^k_\delta \) of any \( B^1_k \) as long as \( w = (w^1_k) \in \overline{W} \).

With this choice of \( \delta \) and with \( \epsilon < \delta \), let \( z^k \in B^1_2 \). Then for any \( x \in \partial D \),
\[
d(x, z^k) \geq d(x, \xi^k) - d(\xi^k, z^k) \geq \delta - r^k_\delta > \frac{\delta}{2}.
\]
For \( z^k \in B^1_2 \) and \( z^l \in B^1_2 \), where \( k \neq l \),
\[
d(z^k, z^l) \geq d(\xi^k, \xi^l) - d(\xi^k, z^k) - d(\xi^l, z^l) \geq 2\delta - r^k_\delta - r^l_\delta > \delta.
\]

Hence the core-shells are all inside \( D \) and they do not intersect. Moreover with \( z^k \in B^1_2 \), \( z = (z^1, z^2, \ldots, z^n) \) is in \( \Xi_\delta/2 \), where the set \( \Xi_\delta/2 \) is defined as in (2.10).

Let \( B_1 = \cup_{k=1}^{n} B^1_k \) and \( B_2 = \cup_{k=1}^{n} B^2_k \). Then \( (B_1, B_2 \backslash B_1) \) is in the admissible set \( A \) of (1.4). It is an assembly of exact core-shells, which is used as an approximate solution of our problem. For now the centers \( \xi^k \) and the radii \( r^k_\delta \) are taken arbitrarily from \( \overline{\Xi_\delta} \) and \( \overline{W} \) respectively. They will be determined near the end of the paper. Our first result gives \( J(B_1, B_2 \backslash B_1) \), the energy of the exact core-shell assembly.
Lemma 2.1.
\[ \mathcal{J}(B_1, B_2 \setminus B_1) \]
\[ = \sum_k 2\pi (r_k^1 + r_k^2) + \frac{\gamma_{11}\pi^2}{2} \left[ \sum_k \left( -\frac{(r_k^1)^4 \log r_k^1}{2\pi} + \frac{(r_k^1)^4}{8\pi} + (r_k^1)^4 R(\xi_k, \zeta_k) \right) \right. \]
\[ + \sum_{k \neq l} (r_k^1)^2 (r_l^1)^2 G(\xi_k, \zeta_l) + \sum_{k,l} \left( \frac{(r_k^1)^2 (r_l^1)^4}{8|D|} + \frac{(r_k^1)^4 (r_l^1)^2}{8|D|} \right) \right] \]
\[ + \gamma_{12}\pi^2 \left[ \sum_k \left( -\frac{(r_k^1)^2 (r_k^2)^2 \log r_k^2}{2\pi} - \frac{(r_k^1)^4}{2\pi} + \frac{(r_k^1)^2}{2\pi} \right) \right. \]
\[ \left. + \sum_{k \neq l} \frac{(r_k^1)^2 (r_k^2)^2 - (r_l^1)^4 + (r_k^1)^2((r_k^1)^2 - (r_l^1)^2)R(\xi_k, \zeta_k)}{4\pi} \right] \]
\[ + \gamma_{22}\pi^2 \left[ \sum_k \left( -\frac{(r_k^1)^4 \log r_k^2 + (r_k^1)^4 \log r_k^1}{2\pi} + \frac{(r_k^1)^2 (r_k^2)^2 \log r_k^2}{2\pi} \right) \right. \]
\[ \left. + \sum_{k \neq l} \frac{(r_k^1)^2 - (r_l^1)^2)((r_k^1)^2 - 3(r_l^1)^2) + ((r_k^1)^2 - (r_l^1)^2)^2 R(\xi_k, \zeta_k)}{8\pi} \right] \]
\[ + \sum_{k,l} \frac{(r_k^1)^4 - (r_l^1)^2 + (r_k^1)^4 - (r_l^1)^2)}{8|D|} + \sum_{k,l} \frac{(r_k^1)^4 - (r_l^1)^4 + (r_k^2)^2 - (r_l^2)^2}{8|D|} \].

Proof. The short range part \( \mathcal{J}_s(B_1, B_2 \setminus B_1) \) is the total length
\[ \mathcal{J}_s(B_1, B_2 \setminus B_1) = \sum_{k=1}^n 2\pi (r_k^1 + r_k^2). \] (2.15)

The long range part of \( \mathcal{J}(B_1, B_2 \setminus B_1) \) is
\[ \mathcal{J}_l(B_1, B_2 \setminus B_1) = I + II + III, \] (2.16)

where
\[ I = \frac{\gamma_{11}}{2} \int_{B_1} \int_{B_1} G(x, y) \, dx \, dy \]
\[ = \frac{\gamma_{11}}{2} \int_{B_1} v_1, \] (2.17)
\[ II = \gamma_{12} \int_{B_2 \setminus B_1} \int_{B_1} G(x, y) \, dx \, dy \]
\[ = \gamma_{12} \left[ \int_{B_2 \setminus B_1} v_1 - \int_{B_1} v_1 \right], \] (2.18)
\[ III = \frac{\gamma_{22}}{2} \int_{B_2 \setminus B_1} \int_{B_2 \setminus B_1} G(x, y) \, dx \, dy \]
\[
\int_{B_2} v_2 - \int_{B_1} v_1 - \int_{B_1} v_2 + \int_{B_1} v_1 = \frac{\gamma_{22}}{2} \left[ \int_{B_2} v_2 - \int_{B_1} v_1 - \int_{B_1} v_2 + \int_{B_1} v_1 \right].
\]

(2.19)

Here \(v_1\) and \(v_2\) are respectively the solutions of

\[- \Delta v_1 = \chi_{B_1} - \omega_1 \text{ in } D, \quad \partial_{\nu} v_1 = 0 \text{ on } \partial D, \quad \int_D v_1 = 0, \]

(2.20)

and

\[- \Delta v_2 = \chi_{B_2} - (\omega_1 + \omega_2) \text{ in } D, \quad \partial_{\nu} v_2 = 0 \text{ on } \partial D, \quad \int_D v_2 = 0. \]

(2.21)

One can write

\[v_1 = \sum_{l=1}^{n} v^l_1, \quad v_2 = \sum_{l=1}^{n} v^l_2,\]

(2.22)

where \(v^l_1\) solves

\[- \Delta v^l_1 = \chi_{B^l_1} - \frac{\pi (r^l_1)^2}{|D|} \text{ in } D, \quad \partial_{\nu} v^l_1 = 0 \text{ on } \partial D, \quad \int_D v^l_1 = 0.\]

As shown in [21],

\[v^l_1(x) = P^l_1(x - \xi^l) + \pi (r^l_1)^2 R(x, \xi^l) + \frac{\pi (r^l_1)^4}{8|D|},\]

(2.23)

where

\[P^l_1(x) = \begin{cases} 
-\frac{|x|^2}{4} + \frac{(r^l_1)^2}{2} \log r^l_1 & \text{if } |x| < r^l_1 \\
\frac{(r^l_1)^2}{2} \log |x| & \text{if } |x| \geq r^l_1 
\end{cases}\]

(2.24)

Similar to the calculations in [21] one notes that \(\int_{B_1} v_2 = \int_{B_2} v_1\) and derives from \(\frac{2.22}{2.23}\) and \(\frac{2.24}{2.24}\) that

\[
\int_{B_1} v_1 = \pi^2 \left[ \sum_k \left( -\frac{(r^k_1)^4}{2\pi} \log r^k_1 + \frac{(r^k_1)^4}{8\pi} + \frac{(r^k_1)^4}{8\pi} R(\xi^k, \xi^k) \right) 
+ \sum_{k \neq l} (r^k_1)^2(r^l_1)^2 G(\xi^k, \xi^l) + \sum_{k,l} \frac{(r^k_1)^2(r^l_1)^4}{8|D|} + \frac{(r^k_1)^4(r^l_1)^2}{8|D|} \right],
\]

(2.25)

\[
\int_{B_1} v_2 = \pi^2 \left[ \sum_k \left( -\frac{(r^k_2)^4}{2\pi} \log r^k_2 + \frac{(r^k_2)^4}{8\pi} + \frac{(r^k_2)^4}{8\pi} R(\xi^k, \xi^k) \right) 
+ \sum_{k \neq l} (r^k_2)^2(r^l_2)^2 G(\xi^k, \xi^l) + \sum_{k,l} \frac{(r^k_2)^2(r^l_2)^4}{8|D|} + \frac{(r^k_2)^4(r^l_2)^2}{8|D|} \right],
\]

(2.26)

\[
\int_{B_2} v_2 = \pi^2 \left[ \sum_k \left( -\frac{(r^k_1)^4}{2\pi} \log r^k_1 + \frac{(r^k_1)^4}{8\pi} + \frac{(r^k_1)^4}{8\pi} R(\xi^k, \xi^k) \right) 
+ \sum_{k \neq l} (r^k_1)^2(r^l_1)^2 G(\xi^k, \xi^l) + \sum_{k,l} \frac{(r^k_1)^2(r^l_1)^4}{8|D|} + \frac{(r^k_1)^4(r^l_1)^2}{8|D|} \right].
\]

(2.27)

The lemma then follows from \(\text{(2.20)}\) - \(\text{(2.24)}\).
3. Perturbed core-shells. We set up a framework to study perturbed core-shells in this section. Let \((\phi_1^k, \phi_2^k), k = 1, 2, ..., n, \) be \(n\) pairs of \(2\pi\) periodic functions, collectively denoted as

\[
\phi = \left( \begin{bmatrix} \phi_1^1 \\ \phi_2^1 \\ \phi_1^2 \\ \phi_2^2 \\ \vdots \\ \phi_1^n \\ \phi_2^n \end{bmatrix} \right).
\]

(3.1)

Using \(\phi\), we define \(2n\) sets

\[
\Omega_1^k = \left\{ \xi^k + t e^{i\theta} : \theta \in S^1, \ t \in \left[ 0, \sqrt{(r_1^k)^2 + 2\phi_1^k(\theta)} \right] \right\},
\]

(3.2)

\[
\Omega_2^k = \left\{ \xi^k + t e^{i\theta} : \theta \in S^1, \ t \in \left[ \sqrt{(r_1^k)^2 + 2\phi_1^k(\theta)}, \sqrt{(r_2^k)^2 + 2\phi_2^k(\theta)} \right] \right\}.
\]

(3.3)

Here \(S^1\) is the unit circle identified with \([0, 2\pi]\).

Since our domain is in \(\mathbb{R}^2\) we often use the complex notation for simplicity. In (3.2) and (3.3) \(e^{i\theta}\) is just \((\cos \theta, \sin \theta)\). The reader will see things like \(e^{i\theta} : x\) which is the inner product of two vectors \(e^{i\theta}\) and \(x\) in \(\mathbb{R}^2\).

Both \(\phi_1^k(\theta)\) and \(\phi_2^k(\theta)\) must be small compared to \((r_1^k)^2\) and \((r_2^k)^2\) respectively, so \(\Omega_1^k\) is a perturbation of \(B_1^k\) and \(\Omega_2^k\) is a perturbation of \(B_2^k\setminus B_1^k\). The areas of \(\Omega_1^k\) and \(\Omega_2^k\) are respectively

\[
|\Omega_1^k| = \int_0^{2\pi} \int_0^{\sqrt{(r_1^k)^2 + 2\phi_1^k(\theta)}} t \, dt \, d\theta,
\]

(3.4)

\[
|\Omega_2^k| = \int_0^{2\pi} \int_0^{\sqrt{(r_2^k)^2 + 2\phi_2^k(\theta)}} t \, dt \, d\theta
\]

(3.5)

The definitions (3.2) and (3.3) are motivated by the fact that if the \(\phi_i^k\)'s satisfy

\[
\sum_{k=1}^n \int_0^{2\pi} \phi_1^k \, d\theta = 0, \quad \sum_{k=1}^n \int_0^{2\pi} \phi_2^k \, d\theta = 0,
\]

(3.6)

then \(\Omega_1 = \bigcup_{k=1}^n \Omega_1^k\) and \(\Omega_2 = \bigcup_{k=1}^n \Omega_2^k\) satisfy the constraints

\[
|\Omega_1| = \epsilon^2 m, \ |\Omega_2| = \epsilon^2 (1 - m).
\]

(3.7)

Let us define some Hilbert spaces. First

\[
Z = \left\{ \phi = (\phi^1, \phi^2, ..., \phi^n) : \phi^k = \begin{bmatrix} \phi_1^k \\ \phi_2^k \end{bmatrix}, \ \phi_i^k \in L^2(S^1), \right\}
\]

\[
\sum_k \int_0^{2\pi} \phi_i^k \, d\theta = 0, \ \forall i = 1, 2
\]

(3.8)

The inner product on \(Z\) is

\[
\langle \phi, \psi \rangle = \sum_{i,k} \int_0^{2\pi} \phi_i^k \psi_i^k \, d\theta.
\]

(3.9)
When $\phi^k_1$ and $\phi^k_2$ are small compared to $(r^k_1)^2$ and $(r^k_2)^2$, they define a perturbed core-shell $(\Omega^k_1, \Omega^k_2)$. In order to define the energy $\mathcal{F}$ on an assembly of such perturbed core-shells, one needs some smoothness on $\phi^k_i$. Let

$$\mathcal{Y} = \{ \phi = (\phi^1, \phi^2, ..., \phi^n) : \phi^k = \begin{bmatrix} \phi^k_1 \\ \phi^k_2 \end{bmatrix}, \in H^1(S^1), \sum_k \int_0^{2\pi} \phi^k_i d\theta = 0, \forall i = 1, 2 \}$$

be a subspace of $\mathcal{Z}$. Here $H^1(S^1)$ is a usual Sobolev space on $S^1$. The norm of $\mathcal{Y}$ is given by

$$||\phi||_{\mathcal{Y}}^2 = \sum_{i,k} \int_0^{2\pi} ((\phi^k_i')^2 + (\phi^k_i)^2) d\theta. \quad (3.11)$$

If $\xi$ and $w$ are held fixed, $\mathcal{F}$ is viewed as a functional of $\phi$ with the domain

$$\text{Dom}(\mathcal{F}) = \{ \phi \in \mathcal{Y} : ||\phi||_{\mathcal{Y}} < \beta \epsilon^2 \}. \quad (3.12)$$

Recall that for all $\epsilon < \delta$, $\xi \in \mathbb{R}^n$ and $w \in \mathbb{W}$, the exact core-shell assembly $(B_1, B_2 \setminus B_1)$ determined by $\epsilon$, $\xi$, and $w$ has the property that $z = (z^1, z^2, ..., z^n) \in \Xi_{\epsilon/2}$ if $z^k \in B^2_2$ for $k = 1, 2, ..., n$. Choose $\beta$ in (3.12) sufficiently small so that for all $\epsilon < \delta$, all $(\xi, w) \in \Xi_{\epsilon/2} \times \mathbb{W}$, and all $\phi \in \text{Dom}(\mathcal{F})$, the perturbed core-shell assembly $(\Omega_1, \Omega_2)$ specified by $\epsilon$, $(\xi, w)$, and $\phi$ has the property that $z = (z^1, z^2, ..., z^n) \in \Xi_{\epsilon/4}$ if $z^k \in \Omega^k_1 \cup \Omega^k_2$. Hence the perturbed core-shells $\Omega^k$ in $(\Omega_1, \Omega_2)$ do not intersect, and they all stay inside $D$, away from $\partial D$.

One writes $\mathcal{F}(\phi)$ for $\mathcal{F}(\Omega_1, \Omega_2)$. Then $\mathcal{F}(\phi) = \mathcal{F}_s(\phi) + \mathcal{F}_l(\phi)$, where $\mathcal{F}_s(\phi)$ and $\mathcal{F}_l(\phi)$ are given in terms of $\phi$ as

$$\mathcal{F}_s(\phi) = \sum_{k,l} \int_0^{2\pi} \sqrt{(r^k_1)^2 + 2\phi^k_1(\theta)} + \frac{((\phi^k_1')^2(\theta))^2}{(r^k_1)^2 + 2\phi^k_1(\theta)} d\theta \quad (3.13)$$

$$\mathcal{F}_l(\phi) = \frac{\gamma_{11}}{2} \sum_{k,l} \int_0^{2\pi} \sqrt{(r^k_1)^2 + 2\phi^k_1(\theta)} dt \int_0^{2\pi} \sqrt{(r^l_1)^2 + 2\phi^l_1(\theta)} d\eta \int_0^{2\pi} \sqrt{(r^k_1)^2 + 2\phi^k_1(\theta)} d\tau$$

$$G(1 + te^{i\theta}, 1 + te^{i\eta}) \frac{dt}{\tau}$$

$$+ \frac{\gamma_2}{2} \sum_{k,l} \int_0^{2\pi} \sqrt{(r^k_1)^2 + 2\phi^k_1(\theta)} dt \int_0^{2\pi} \sqrt{(r^l_1)^2 + 2\phi^l_1(\theta)} d\eta \int_0^{2\pi} \sqrt{(r^k_1)^2 + 2\phi^k_1(\theta)} d\tau$$

$$G(1 + te^{i\theta}, 1 + te^{i\eta}) \frac{dt}{\tau} \quad (3.14)$$

Next consider the first variation of the functional $\mathcal{F}$ with respect to $\phi$. For convenience introduce functions $L^k_i$:

$$L^k_i((\phi^k_i)'', \phi^k_i) = \sqrt{(r^k_1)^2 + 2\phi^k_1} + \frac{((\phi^k_i)'')^2}{(r^k_1)^2 + 2\phi^k_1}. \quad (3.15)$$

A deformation in $\mathcal{Y}$ is simply

$$\phi \rightarrow \phi + \epsilon \psi \quad (3.16)$$
for $\phi \in \text{Dom}(J)$ and $\psi \in \mathcal{Y}$. The first variation of $J$ is the directional derivative
\[
\frac{dJ(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} = \frac{dJ_s(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} + \frac{dJ_l(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0},
\] (3.17)
where
\[
\frac{dJ_s(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} = \sum_{k,i} \int_0^{2\pi} \left( D_1 L^k_i (\phi^i_k)' , \psi^i_k \right) d\theta \tag{3.18}
\]
and
\[
\frac{dJ_l(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} = (\gamma_{11} - \gamma_{12}) \sum_k \int_0^{2\pi} I_{\Omega_1} (\xi^k + \sqrt{(r^k_1)^2 + 2\phi^k_1 e^{i\theta}}) \psi^k_1 (\theta) d\theta
\]
\[
+ (\gamma_{12} - \gamma_{22}) \sum_k \int_0^{2\pi} I_{\Omega_2} (\xi^k + \sqrt{(r^k_2)^2 + 2\phi^k_2 e^{i\theta}}) \psi^k_2 (\theta) d\theta
\]
\[
+ \gamma_{12} \sum_k \int_0^{2\pi} I_{\Omega_1} (\xi^k + \sqrt{(r^k_2)^2 + 2\phi^k_1 e^{i\theta}}) \psi^k_1 (\theta) d\theta
\]
\[
+ \gamma_{22} \sum_k \int_0^{2\pi} I_{\Omega_2} (\xi^k + \sqrt{(r^k_2)^2 + 2\phi^k_2 e^{i\theta}}) \psi^k_2 (\theta) d\theta. \tag{3.19}
\]
Here $D_1 L^k_i (\cdot, \cdot)$ is the derivative of $L^k_i$ with respect to its first argument and $D_2 L^k_i (\cdot, \cdot)$ the derivative with respect to its second argument. We would like to have operators $S_s$ and $S_l$ so that
\[
\frac{dJ_s(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} = \langle S_s(\phi), \psi \rangle, \quad \frac{dJ_l(\phi + \epsilon \psi)}{d\epsilon} |_{\epsilon=0} = \langle S_l(\phi), \psi \rangle. \tag{3.20}
\]
This is always possible for $J_l$. However for $J_s$, one must restrict $\phi$ to a smaller, more smooth space. Define
\[
\mathcal{X} = \left\{ \phi = (\phi^1, \phi^2, ..., \phi^n) : \phi^k = \begin{bmatrix} \phi^k_1 \\ \phi^k_2 \end{bmatrix}, \phi^k_i \in H^2(S^1), \sum_k \int_0^{2\pi} \phi^k_i d\theta = 0, \ \forall i = 1, 2 \right\}. \tag{3.21}
\]
with the norm
\[
\|\phi\|_{\mathcal{X}}^2 = \sum_{1,k} \int_0^{2\pi} \left( ((\phi^k_1)'')^2 + ((\phi^k_2)'')^2 + (\phi^k_1)'^2 \right) d\theta. \tag{3.22}
\]
Clearly $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$. One can define $S = S_s + S_l$ on
\[
\text{Dom}(S) = \{ \phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} < \beta \epsilon^2 \} \tag{3.23}
\]
where $\beta$ is the same as the number in (3.12). Therefore $\text{Dom}(S) \subset \text{Dom}(J)$. The nonlinear operators $S_s$ and $S_l$ map from $\text{Dom}(S)$ to $\mathcal{Z}$ as follows. The component $S^k_{s,i}(\phi)$ is given as
\[
S^k_{s,i}(\phi) = H^k_i (\phi^k_i) - \Lambda_{s,i}(\phi_i) \tag{3.24}
\]
Similarly, is the curvature operator, and \( \Lambda_{s,i}(\phi_i) \) is a number, depending on \( \phi_i = (\phi_{i1}, \phi_{i2}, ..., \phi_{in}) \), so chosen that
\[
\sum_k \int_0^{2\pi} S^k_{s,i}(\phi) = 0, \quad i = 1, 2.
\] (3.26)

Note that \( \Lambda_{s,i}(\phi_i) \) is the same for all \( k = 1, 2, ..., n \). The components of \( S_i(\phi) \) are
\[
S^k_{l,1}(\phi) = (\gamma_{i1} - \gamma_{i2}) I_{l1}(\zeta^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta}}) + (\gamma_{i2} - \gamma_{i2}) I_{l2}(\zeta^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta}}) - \Lambda_{l,1}(\phi)
\]
\[
S^k_{l,2}(\phi) = \gamma_{i1} I_{l1}(\zeta^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta}}) + \gamma_{i2} I_{l2}(\zeta^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta}}) - \Lambda_{l,2}(\phi)
\] (3.27)

where each \( \Lambda_{l,i}(\phi_i) \) is a number that ensures
\[
\sum_k \int_0^{2\pi} S^k_{l,i}(\phi) = 0, \quad i = 1, 2.
\] (3.28)

We need to write \( S^k_{l,i}(\phi) \) more explicitly. Let
\[
A^k_i(\phi^k) = -\frac{\gamma_{i1} - \gamma_{i2}}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log \left( \frac{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}}{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}} \right) ttd\eta
d\]
\[
B^k_i(\phi^k) = (\gamma_{i1} - \gamma_{i2}) \int_0^{2\pi} \int_0^{2\pi} \left( \frac{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}}{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}} \right) ttd\eta.
\]

When \( l \neq k \), let
\[
C^{kl}_{1}(\phi^k, \phi^l)
\]
\[
= (\gamma_{i1} - \gamma_{i2}) \int_0^{2\pi} \int_0^{2\pi} \left( \frac{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}}{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}} \right) ttd\eta.
\]

Similarly,
\[
A^k_2(\phi^k) = \frac{-\gamma_{i1}}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log \left( \frac{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}}{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}} \right) ttd\eta
d\]
\[
-\frac{-\gamma_{i2}}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log \left( \frac{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}}{r_1^k + \sqrt{(r_1^k)^2 + 2\phi_1^k e^{i\theta} - te^{i\eta}}} \right) ttd\eta.
\]
The next lemma estimates $S^k(0)$, up to the constants $\lambda_i(0)$.

Lemma 3.1.

$$S^k(0) + \lambda_1(0) = \frac{1}{r_1^2} + \frac{1}{r_2^2} \left[ -\frac{(r_1^k)^2}{2} \log r_1^k + \frac{(r_2^k)^2}{2} \log r_2^k + (r_1^k)^2 \right]$$

When $\xi$ and $\omega$ are fixed, the exact core-shell assembly is represented by $\phi = 0$. The next lemma estimates $S^k(0)$, up to the constants $\lambda_i(0)$.

$$S^k(0) + \lambda_2(0) = \frac{1}{r_2^2} \left[ -\frac{(r_2^k)^2}{2} \log r_2^k + \pi (r_2^k)^2 \left( R(\xi^k, \omega) + \nabla R(\xi^k, \omega) r_2^k e^{i\theta} \right) + O(|\gamma| \epsilon^4) \right] + O(1)$$
\[
\begin{align*}
&+ \sum_{l \neq k} \pi(r_1^l)^2 \left( G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_2^k e^{i\theta} \right) \\
&+ \gamma_{22} \left[ - \frac{(r_2^k)^2 - (r_1^k)^2}{2} \log r_2^k \\
&+ \pi((r_2^k)^2 - (r_1^k)^2) \left( R(\xi^k, \xi^k) + \nabla R(\xi^k, \xi^k) r_2^k e^{i\theta} \right) \\
&+ \sum_{l \neq k} \pi((r_2^l)^2 - (r_1^l)^2) \left( G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_2^k e^{i\theta} \right) \right] + O(\gamma |e|) \\
\end{align*}
\]

Proof. The interfaces of the exact assembly are circles, so

\[
\mathcal{H}_1^k(0) = \frac{1}{r_1^k}. \quad (3.32)
\]

Recall \( v_l^j \) from (2.23) and \( v_1 = \sum_{i=1}^n v_1^i, \ v_2 = \sum_{i=1}^n v_2^i \). Then,

\[
\begin{align*}
I_{B_1}(\xi^k + r_1^k e^{i\theta}) &= \ v_1(\xi^k + r_1^k e^{i\theta}) = \sum_{i=1}^n v_1^i(\xi^k + r_1^k e^{i\theta}) \\
&= - \frac{(r_1^k)^2}{2} \log r_1^k + \pi(r_1^k)^2 (R(\xi^k, \xi^k) + \nabla R(\xi^k, \xi^k) r_1^k e^{i\theta}) \\
&+ \sum_{l \neq k} \pi((r_1^l)^2 - (r_1^k)^2) (G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_1^k e^{i\theta}) \\
&+ O(e^4), \quad (3.33)
\end{align*}
\]

\[
\begin{align*}
I_{B_2 \setminus B_1}(\xi^k + r_1^k e^{i\theta}) &= \ v_2(\xi^k + r_1^k e^{i\theta}) - \ v_1(\xi^k + r_1^k e^{i\theta}) \\
&= - \frac{(r_2^k)^2}{2} \log r_2^k - \frac{(r_1^k)^2}{2} \log r_1^k + \frac{(r_2^k)^2 - (r_1^k)^2}{4} \\
&+ \pi((r_2^k)^2 - (r_1^k)^2) (R(\xi^k, \xi^k) + \nabla R(\xi^k, \xi^k) r_1^k e^{i\theta}) \\
&+ \sum_{l \neq k} \pi((r_2^l)^2 - (r_1^l)^2) (G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_1^k e^{i\theta}) \\
&+ O(e^4), \quad (3.34)
\end{align*}
\]

\[
\begin{align*}
I_{B_1}(\xi^k + r_2^k e^{i\theta}) &= \ v_1(\xi^k + r_2^k e^{i\theta}) \\
&= - \frac{(r_1^k)^2}{2} \log r_1^k + \pi(r_1^k)^2 (R(\xi^k, \xi^k) + \nabla R(\xi^k, \xi^k) r_2^k e^{i\theta}) \\
&+ \sum_{l \neq k} \pi((r_1^l)^2 - (r_1^k)^2) (G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_2^k e^{i\theta}) \\
&+ O(e^4), \quad (3.35)
\end{align*}
\]

\[
\begin{align*}
I_{B_2 \setminus B_1}(\xi^k + r_2^k e^{i\theta}) &= \ v_2(\xi^k + r_2^k e^{i\theta}) - \ v_1(\xi^k + r_2^k e^{i\theta}) \\
&= - \frac{(r_2^k)^2}{2} \log r_2^k + \frac{(r_2^k)^2 - (r_1^k)^2}{2} \\
&+ \pi((r_2^k)^2 - (r_1^k)^2) (R(\xi^k, \xi^k) + \nabla R(\xi^k, \xi^k) r_2^k e^{i\theta}) \\
&+ \sum_{l \neq k} \pi((r_2^l)^2 - (r_1^l)^2) (G(\xi^k, \xi^l) + \nabla G(\xi^k, \xi^l) r_2^k e^{i\theta}) \\
&+ O(e^4). \quad (3.36)
\end{align*}
\]

The lemma follows from (3.33) - (3.36). \( \square \)
4. Linear analysis. The Fréchet derivative of $S$ at $\phi \in \text{Dom}(S)$ is denoted by $S'(\phi)$. It can also be interpreted as the second variation of $J$ because

$$\left. \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2} J(\phi + \varepsilon_1 u + \varepsilon_2 v) \right|_{\varepsilon_1 = \varepsilon_2 = 0} = \langle S'(\phi)(u), v \rangle,$$

for every $\phi \in \text{Dom}(S)$, $u \in X$, and $v \in Y$. Note that the left side of (4.1) is meaningful provided $\phi \in \text{Dom}(J)$, $u \in Y$, and $v \in Y$, while the right side of (4.1) is defined if $\phi \in \text{Dom}(S)$, $u \in X$, and $v \in Z$.

In this section we study $S'(0)$, i.e. the linearized operator at the exact core-shell assembly. Because of (3.29), we consider the Fréchet derivative of each of the terms in $S$. Let $\alpha^k = \frac{\partial^2}{\partial \varepsilon \partial \gamma}$. Calculations show that

$$\left(\mathcal{H}^k_1\right)'(0)(u^k) = -\frac{1}{(r^k_1)^3}((u^k_1)'' + u^k_1)$$

$$(\mathcal{A}^k_1)'(0)(u^k)(\theta) = 2\gamma_{12} - \gamma_{11} - \gamma_{22} \int_0^{2\pi} u^k_1(\eta) \log |r^k_1 e^{i\theta} - r^k_2 e^{i\eta}| d\eta$$

$$-\frac{\gamma_{11} - \gamma_{22}}{2} u^k_1(\theta)$$

$$-\frac{\gamma_{12} - \gamma_{22}}{2} \int_0^{2\pi} u^k_2(\eta) \log |r^k_1 e^{i\theta} - r^k_2 e^{i\eta}| d\eta$$

$$(\mathcal{B}^k_1)'(0)(u^k)(\theta) = -2\gamma_{12} - \gamma_{11} - \gamma_{22} \int_0^{2\pi} u^k_1(\eta) R(\xi^k + r^k_1 e^{i\theta}, \xi^k + r^k_2 e^{i\eta}) d\eta$$

$$+(\gamma_{11} - \gamma_{22}) \frac{u^k_1(\theta)}{r^k_1} \int_{B^1_1} \nabla R(\xi^k + r^k_1 e^{i\theta}, y) \cdot e^{i\theta} dy$$

$$+(\gamma_{12} - \gamma_{22}) \frac{u^k_1(\theta)}{r^k_1} \int_{B^2_1 \setminus B^1_1} \nabla R(\xi^k + r^k_1 e^{i\theta}, y) \cdot e^{i\theta} dy$$

$$+(\gamma_{12} - \gamma_{22}) \int_0^{2\pi} u^k_2(\eta) R(\xi^k + r^k_1 e^{i\theta}, \xi^k + r^k_2 e^{i\eta}) d\eta$$

$$(\mathcal{C}^k_1)'(0)(u^k, u^l)(\theta) = -2\gamma_{12} - \gamma_{11} - \gamma_{22} \int_0^{2\pi} u^l_1(\eta) G(\xi^k + r^k_1 e^{i\theta}, \xi^l + r^l_1 e^{i\eta}) d\eta$$

$$+(\gamma_{11} - \gamma_{22}) \frac{u^l_1(\theta)}{r^k_1} \int_{B^1_1} \nabla G(\xi^k + r^k_1 e^{i\theta}, y) \cdot e^{i\theta} dy$$

$$+(\gamma_{12} - \gamma_{22}) \frac{u^l_1(\theta)}{r^k_1} \int_{B^2_1 \setminus B^1_1} \nabla G(\xi^k + r^k_1 e^{i\theta}, y) \cdot e^{i\theta} dy$$

$$+(\gamma_{12} - \gamma_{22}) \int_0^{2\pi} u^l_2(\eta) G(\xi^k + r^k_1 e^{i\theta}, \xi^l + r^l_2 e^{i\eta}) d\eta$$

$$(\mathcal{A}^k_2)'(0)(u^k)(\theta)$$
We define \( E \).

Let us separate \( E(k) = (\gamma_1 u^2 + \gamma_2 u_2)(\alpha^k)^2 \)

\[
\begin{align*}
E(k) &= \frac{\gamma_1 \gamma_2 - \gamma_1 \gamma_2}{2} \int_0^{2\pi} u_1^2(\eta) \log |r_1^k e^{i\theta} - r_1^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} \int_0^{2\pi} u_1^2(\eta) \log |r_1^k e^{i\theta} - r_1^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta
\end{align*}
\]

\[
\begin{align*}
&= \frac{\gamma_1 \gamma_2 - \gamma_1 \gamma_2}{2} \int_0^{2\pi} u_1^2(\eta) \log |r_1^k e^{i\theta} - r_1^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta
\end{align*}
\]

Let us separate \( S'(0) \) to a dominant part \( E \) and a minor part \( F \): \( S'(0) = E + F \).

We define \( E^k \), the \( k \)-th component of \( E \), to be \( E^k = (E_1^k, E_2^k) \) and

\[
\begin{align*}
E_1^k(u)(\theta) &= \frac{1}{(r_1^k)^3} ((u_1^k)''(\theta) + u_1^k(\theta))
+ \frac{2 \gamma_1 \gamma_2 - \gamma_1 \gamma_2}{2 \pi} \int_0^{2\pi} u_1^2(\eta) \log |r_1^k e^{i\theta} - r_1^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} u_1^k(\theta)
+ \frac{\gamma_1 \gamma_2}{2} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta + e_1(u),
\end{align*}
\]

\[
\begin{align*}
E_2^k(u)(\theta) &= \frac{1}{(r_2^k)^3} ((u_2^k)''(\theta) + u_2^k(\theta))
+ \frac{\gamma_2 \gamma_2 - \gamma_1 \gamma_2}{2 \pi} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta
- \frac{\gamma_1 \gamma_2}{2} u_2^k(\theta)(\alpha^k)^2
- \frac{\gamma_2 \gamma_2}{2 \pi} \int_0^{2\pi} u_2^2(\eta) \log |r_2^k e^{i\theta} - r_2^k e^{i\eta}| d\eta
- \frac{\gamma_2 \gamma_2}{2} u_2^k(\theta)(1 - (\alpha^k)^2) + e_2(u).
\end{align*}
\]
The real valued linear operator $e_i$ is independent of $k$. It is so chosen that $E$ maps from $\mathcal{X}$ to $\mathcal{Z}$. The rest of $S'(0)$ is denoted by $F$. Note that $E$ is determined by $H$ and $A$, and $F$ is determined by $B$ and $C$.

Identify $\cos l\theta \begin{bmatrix} v^k_1 \\ v^k_2 \end{bmatrix}$ with

$$\begin{bmatrix} 0 \\ 0 \\ \ldots \cos l\theta \begin{bmatrix} v^k_1 \\ v^k_2 \end{bmatrix}, \ldots, 0 \end{bmatrix}$$

in $\mathcal{X}$, where the nontrivial term appears in the $k$-th position. Same is done with $\sin l\theta \begin{bmatrix} v^k_1 \\ v^k_2 \end{bmatrix}$. Then

$$E^k \left( \cos l\theta \begin{bmatrix} v^k_1 \\ v^k_2 \end{bmatrix} \right) = \cos l\theta \begin{bmatrix} (l^2-1) & -212-222 \gamma_1-222 \gamma_2 \\ 212-222 & 212-212 \gamma_1 \end{bmatrix} \begin{bmatrix} (l^2-1) \gamma_1 & -212+222 \gamma_2 \\ 212-222 & 212 \gamma_1-212 \gamma_2 \end{bmatrix} \begin{bmatrix} v^k_1 \\ v^k_2 \end{bmatrix}$$

and the same holds if $\cos l\theta$ is replaced by $\sin l\theta$. Note that

$$E^k \left( \cos \theta \begin{bmatrix} \alpha^k_1 \\ 1 \end{bmatrix} \right) = E^k \left( \sin \theta \begin{bmatrix} \alpha^k_1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (4.3)$$

Hence 0 is an eigenvalue for $E$ on $\mathcal{X}$ and

$$\cos \theta \begin{bmatrix} \alpha^1_1 \\ 1 \end{bmatrix}, \ldots, \cos \theta \begin{bmatrix} \alpha^n_1 \\ 1 \end{bmatrix}, \sin \theta \begin{bmatrix} \alpha^1_1 \\ 1 \end{bmatrix}, \ldots, \sin \theta \begin{bmatrix} \alpha^n_1 \\ 1 \end{bmatrix} \quad (4.4)$$

are the associated eigenvectors.

Although an exact core-shell ($B^k_1, B^k_2 \setminus B^k_1$) has the well defined center $\xi^k$ and the radii $r^k_1$ and $r^k_2$. After perturbation $\phi^k$ to $(\Omega^k_1, \Omega^k_2)$, one cannot expect the perturbed core-shell always to have the same center and radii. Nevertheless, there is a special class of perturbations that preserve centers and radii.

Let $\Pi$ be the orthogonal projection operator from $\mathcal{Z}$ to a subspace $\mathcal{Z}_y$, where

$$\mathcal{Z}_y = \left\{ \phi \in \mathcal{Z} : \phi^k \perp \cos \theta \begin{bmatrix} \alpha^k_1 \\ 1 \end{bmatrix}, \phi^k \perp \sin \theta \begin{bmatrix} \alpha^k_1 \\ 1 \end{bmatrix}, \phi^k \perp \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \phi^k \perp \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \quad (4.5)$$

Here by $\phi^k \perp \cos \theta \begin{bmatrix} \alpha^k_1 \\ 1 \end{bmatrix}$ we mean

$$\int_0^{2\pi} (\phi^k_1 \phi^k_1 + \phi^k_2 \phi^k_2) \cos \theta d\theta = 0 \quad (4.6)$$

e etc. Also define

$$\mathcal{Y}_y = \mathcal{Y} \cap \mathcal{Z}_y, \quad \mathcal{X}_y = \mathcal{X} \cap \mathcal{Z}_y \quad (4.7)$$

When $\phi \in \mathcal{X}_y$ (or $\mathcal{Y}_y$, or $\mathcal{Z}_y$), the perturbed core-shells in the assembly $\phi$ have well defined centers and radii. The $k$-th perturbed core-shell described by $\phi^k$ is considered to be centered at $\xi^k$ of radii $r^k_1$ for the inner interface and $r^k_2$ for the outer interface. If $\phi \in \mathcal{Z} \setminus \mathcal{Z}_y$, then $\xi^k$ and $r^k_i$ cannot be interpreted as centers and radii of the core-shells in the assembly represented by $\phi$. 


We are more interested in $\Pi S'(0)$ and $\Pi E$ restricted on $X$. By the self-adjointness of $E$, $E$ maps $X$ into $\mathcal{Z}_0$, so $\Pi E = E$ on $X$.

The formula (4.2) diagonalizes $E$ on $X$ into blocks represented by matrices

$$
\tilde{M}^k_l = \begin{bmatrix}
\frac{\lambda^2-1}{(\lambda^2)^2} - \frac{21\lambda-71\lambda-22}{21\lambda} & \frac{11\lambda-71\lambda-22}{21\lambda} \\
\frac{21\lambda-71\lambda-22}{21\lambda} & \frac{-21\lambda-71\lambda-22}{21\lambda}
\end{bmatrix}
$$

or their scaled versions

$$
e^\lambda \tilde{M}^k_l = \begin{bmatrix}
\frac{(\lambda^2-1)}{(\lambda^2)^2} - \frac{21\lambda-71\lambda-22}{21\lambda} & \frac{11\lambda-71\lambda-22}{21\lambda} \\
\frac{21\lambda-71\lambda-22}{21\lambda} & \frac{-21\lambda-71\lambda-22}{21\lambda}
\end{bmatrix}
$$

Motivated by (4.10) and (4.12) we define

$$M_l(v, \Gamma)$$

Uniformly with respect to $\Gamma \in B(0, t)$, as $l \to \infty$. The right side of (4.12) is a positive definite matrix. On the other hand, since

$$M_l(0) = \begin{bmatrix}
\frac{(n\pi)^{3/2}}{m} & 0 \\
0 & (n\pi)^{3/2}
\end{bmatrix}
$$

where $0$ is the zero matrix, for each $l \geq 2$ one can find $B(0, t_l), t_l > 0$, such that $M_l(\Gamma) > 0$ if $\Gamma \in B(0, t_l)$. The lemma follows from these two properties.

**Lemma 4.1.** The set $S$ given in (1.10) has a non-empty interior.

**Proof.** Identify $S^2$ with $\mathbb{R}^3$; namely every $\Gamma \in S^2$ corresponds to the vector $(\Gamma_{11}, \Gamma_{12}, \Gamma_{22}) \in \mathbb{R}^3$. On each ball $B(0, t), t > 0$, in $\mathbb{R}^3$,

$$M_l(\Gamma) \to \begin{bmatrix}
\frac{(n\pi)^{3/2}}{m} & 0 \\
0 & (n\pi)^{3/2}
\end{bmatrix}
$$

uniformly with respect to $\Gamma \in B(0, t), as \ l \to \infty$. The right side of (4.12) is a positive definite matrix. On the other hand, since

$$M_l(0) = \begin{bmatrix}
\frac{(n\pi)^{3/2}}{m} & 0 \\
0 & (n\pi)^{3/2}
\end{bmatrix}
$$

where $0$ is the zero matrix, for each $l \geq 2$ one can find $B(0, t_l), t_l > 0$, such that $M_l(\Gamma) > 0$ if $\Gamma \in B(0, t_l)$. The lemma follows from these two properties.

Recall that $K$ is a fixed compact subset of the interior of $S$. Coming back to $M_l(v, \Gamma)$, we have

**Lemma 4.2.** Let $K$ be a compact subset of $\text{int}(S)$, the interior of $S$. Let $V(h) = V_1(h) \times V_2(h)$, where $V_1(h) = (\frac{m}{n} - h, \frac{m}{n} + h), V_2(h) = (\frac{1-m}{n} - h, \frac{1-m}{n} + h)$.

1. For any $0 < h_1 < \min\{\frac{m}{n}, \frac{1-m}{n}\}$, there exists $d_1 \in (0, 1)$ such that if $v \in V(h_1)$ and $\Gamma \in K$, then $\lambda_{l_1}(v, \Gamma) = 0, \lambda_{l_2}(v, \Gamma) \in [d_1, \frac{1}{d_1}]$.

2. There exist $h_2 > 0$ and $d_2 \in (0, 1)$ such that if $v \in V(h_2)$ and $\Gamma \in K$, then $\lambda_{l_1}(v, \Gamma) = \lambda_{l_2}(v, \Gamma)$ are both in $[d_2, \frac{1}{d_2}]$ for all $l = 2, 3, 4, \ldots$. 
Proof. When \( l = 1 \),

\[
M_1(v, \Gamma) = \begin{bmatrix}
\frac{\Gamma_{22} - \Gamma_{12}}{2} & \frac{\Gamma_{12} - \sqrt{Q_1Q_2}}{2} & \frac{\Gamma_{22} - \sqrt{Q_1Q_2}}{2}
\end{bmatrix},
\]

and

\[
\lambda_{1,1}(v, \Gamma) = 0, \quad \lambda_{1,2}(v, \Gamma) = \frac{\Gamma_{22} - \Gamma_{12}}{2} \left( 1 + \frac{Q_1}{Q_2} \right)^2
\] (4.14)

whose associated eigenvectors are respectively

\[
\begin{bmatrix}
\frac{\sqrt{Q_1}}{1}
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
-\frac{\sqrt{Q_2}}{Q_2}
\end{bmatrix}.
\] (4.15)

Part 1 follows from the fact that \( K \) is compact and \( Q_1, Q_2 \) are bounded. We also used the fact that \( \Gamma_{22} > \Gamma_{12} \) when \( \Gamma \) is in \( S \).

When \( l \geq 2 \), let \( h \) be any number satisfying \( 0 < h < \min\{\frac{m}{n}, \frac{1-m}{m} \} \). If \( l \to \infty \), then

\[
M_l(v, \Gamma) \to \begin{bmatrix}
\frac{(Q_1)^3}{Q_1} & 0 & 0
\end{bmatrix}
\]

uniformly on \( V(h) \times K \). Take \( d_3 > 0 \) such that

\[
d_3 < \inf \left\{ \frac{1}{Q_2^2} : Q_2 = \sqrt{\frac{v_1 + v_2}{\pi}}, \ v \in V(h) \right\}
\]

and

\[
\frac{1}{d_3} > \sup \left\{ \frac{1}{Q_1^2} : Q_1 = \sqrt{\frac{v_1}{\pi}}, \ v \in V_1(h) \right\}.
\] (4.16)

Thus, there exists \( L \) such that if \( l \geq L \), \( \lambda_{i,l}(v, \Gamma) \in [d_1, \frac{d_2}{r}] \), where \( i = 1, 2 \).

For any \( l < L \), first let \( v_1 = \frac{m}{n} \) and \( v_2 = \frac{1-m}{m} \). Then \( M_l(v, \Gamma) \) becomes \( M_l(\Gamma) \), defined in (1.9), which is positive definite for any \( \Gamma \in S \). Since \( K \) is a compact subset of the interior of \( S \), there exists \( d_4 > 0 \) such that \( \lambda_{i,l}(\Gamma) \in [d_4, \frac{d_4}{r}] \) for whenever \( l < L \) and \( \Gamma \in K \). Then take \( h_2 < h \) such that \( \lambda_{i,l}(v, \Gamma) \in [\frac{d_2}{r}, \frac{d_2}{r}] \) for all \( l < L, v \in V(h_2) \), and \( \Gamma \in K \). Since \( v \) is also in \( V(h) \), \( \lambda_{i,l}(v, \Gamma) \in [d_3, \frac{d_2}{r}] \) for all \( l \geq L \).

Finally set \( d_2 = \min\{d_3, \frac{d_2}{r}\} \).

The eigenvalues of \( \mathcal{E} \) on \( \mathcal{X}_0 \) are eigenvalues of the matrices \( \tilde{M}^k \) of (4.8) which we denote by \( \tilde{\lambda}_{k,i}^l, k = 1, 2, \ldots, n, l = 1, 2, \ldots, i = 1, 2 \). Here one must exclude \( \tilde{\lambda}_{1,1}^k, \tilde{\lambda}_{1,1}^2, \ldots, \tilde{\lambda}_{1,1}^n \), because their corresponding eigenvectors are perpendicular to \( \mathcal{X}_0 \). The eigenvalues \( \tilde{\lambda}_{k,i}^l \) are related to \( \lambda_{i,l}(v, \Gamma) \) of Lemma 4.2 if \( v_1 = w_1^k, v_2 = w_2^k \) and \( \Gamma = c_3^3 \Gamma \) (that is, \( Q_1 = R_1^k = \frac{r^k}{c}, Q_2 = R_2^k = \frac{r_k}{c} \)), by

\[
\tilde{\lambda}_{k,i}^l = c_3^{-3l} \lambda_{i,l}(w^k, c_3^3 \Gamma).
\] (4.17)

Lemma 4.2 (4.17), and a simple Fourier series argument yield the following estimates on \( \mathcal{E} \).

**Lemma 4.3.** There exist \( h > 0 \) and \( c_2 > 0 \) such that if \( W \) of (2.6) is defined by \( h \), then for all \( u \in \mathcal{X}_0 \)

1. \( \langle \Pi \mathcal{E}(u), u \rangle \geq 2c_2\varepsilon^{-3} \|u\|_{\mathcal{Y}}^2 \)

and
4.3. We use a bound of the second variation of $u$ at 0. The second part $F$ in $S'(0)$ is a minor part.

**Lemma 4.4.** There exists $C_2 > 0$ depending on $D$, $m$ and $n$ only such that for all $u \in X_0$,

$$
\|F(u)\|_2 \leq C_2 |\epsilon| \|u\|_2.
$$

**Proof.** Let $F^k = (F^k_1, F^k_2)$ be the $k$-th component of $F$. Then

$$
F^k_1(u)(\theta) = \left( 2\gamma_{12} - \gamma_{11} - \gamma_{22} \right) \int_0^{2\pi} u_1^k(\eta) R(\xi^k + r_1^k e^{i\theta}, \xi^k + r_1^k e^{i\eta}) d\eta \\
+ \left( 2\gamma_{11} - \gamma_{12} - \gamma_{22} \right) \int_0^{\pi} u_1^k(\theta) \int_{B_2^1 \setminus B_1^1} \nabla R(\xi^k + r_1^k e^{i\theta}, y) \cdot e^{i\theta} dy \\
+ \left( 2\gamma_{12} - \gamma_{22} \right) \int_0^{2\pi} u_2^k(\eta) R(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
- \sum_{l \neq k} (2\gamma_{12} - \gamma_{11} - \gamma_{22}) \int_0^{2\pi} u_1^l(\eta) G(\xi^k + r_1^k e^{i\theta}, \xi^k + r_1^k e^{i\eta}) d\eta \\
+ \sum_{l \neq k} (2\gamma_{11} - \gamma_{12} - \gamma_{22}) \int_0^{2\pi} u_1^l(\theta) \int_{B_2^1 \setminus B_1^1} \nabla G(\xi^k + r_1^k e^{i\theta}, \xi^k + r_1^k e^{i\eta}) d\eta \\
+ \sum_{l \neq k} (2\gamma_{12} - \gamma_{22}) \int_0^{2\pi} u_2^l(\eta) G(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ f_1(u),
$$

$$
F^k_2(u)(\theta) = \left( 2\gamma_{12} - \gamma_{11} - \gamma_{22} \right) \int_0^{2\pi} u_1^k(\eta) R(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \gamma_{11} \frac{u_1^k(\theta)}{r_2^k} \int_{B_1^2 \setminus B_1^1} \nabla R(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \gamma_{12} \frac{u_2^k(\theta)}{r_2^k} \int_{B_2^1 \setminus B_1^1} \nabla R(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \gamma_{12} \int_0^{2\pi} u_2^k(\eta) R(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \sum_{l \neq k} (2\gamma_{12} - \gamma_{11} - \gamma_{22}) \int_0^{2\pi} u_1^l(\eta) G(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \sum_{l \neq k} \gamma_{11} \frac{u_1^l(\theta)}{r_2^k} \int_{B_1^2 \setminus B_1^1} \nabla G(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta \\
+ \sum_{l \neq k} \gamma_{12} \frac{u_2^l(\theta)}{r_2^k} \int_{B_2^1 \setminus B_1^1} \nabla G(\xi^k + r_2^k e^{i\theta}, \xi^k + r_2^k e^{i\eta}) d\eta.
$$
implies that
\[
\int_{0}^{2\pi} u_2^i(\eta)G(\xi^k + r_{j}^k e^{i\theta}, \xi^l + r_{j}^l e^{i\eta}) d\eta + f_2(u)
\]
where \( f_1(u) \) and \( f_2(u) \) are real valued and independent of \( k \). They are included so that \( F(u) \) is in \( Z \).

Because
\[
R(\xi^k + r_{j}^k e^{i\theta}, \xi^l + r_{j}^l e^{i\eta}) - R(\xi^k, \xi^l) = O(\epsilon),
\]
and \( \int_{0}^{2\pi} u_2^i(\eta)d\eta = 0 \), where \( 1 \leq i, j \leq 2, 1 \leq k, l \leq n \), we obtain that for \( s, t = 1, 2 \)
\[
\left\| \frac{\gamma_{st} u_k^j(\eta)}{r_{j}^k} \int_{B_{r_k}^l} \nabla R(\xi^k + r_{j}^k e^{i\theta}, y) \cdot e^{i\theta} dy \right\|_Z \leq C|\gamma| \epsilon \left\| u_k^j \right\|_Z,
\]
\[
\left\| \frac{\gamma_{st} u_k^j(\eta)}{r_{j}^k} \int_{B_{r_k}^l \setminus B_{r_l}^l} \nabla R(\xi^k + r_{j}^k e^{i\theta}, y) \cdot e^{i\theta} dy \right\|_Z \leq C|\gamma| \epsilon \left\| u_k^j \right\|_Z,
\]
\[
\left\| \frac{\gamma_{st} u_k^j(\eta)}{r_{j}^k} \int_{B_{r_k}^l \setminus B_{r_l}^l} \nabla G(\xi^k + r_{j}^k e^{i\theta}, y) \cdot e^{i\theta} dy \right\|_Z \leq C|\gamma| \epsilon \left\| u_k^j \right\|_Z,
\]
\[
\left\| \frac{\gamma_{st} u_k^j(\eta)}{r_{j}^k} \int_{B_{r_k}^l \setminus B_{r_l}^l} \nabla G(\xi^k + r_{j}^k e^{i\theta}, y) \cdot e^{i\theta} dy \right\|_Z \leq C|\gamma| \epsilon \left\| u_k^j \right\|_Z.
\]
The condition
\[
\sum_{k=1}^{n} F_k^i(u)(\theta) = 0,
\]
implies that
\[
|f_i(u)| \leq C|\gamma| \epsilon \left\| u \right\|_Z,
\]
where \( i = 1, 2 \). The lemma then follows.

Lemma 4.5. There exists \( c_2 > 0 \) such that when \( \delta \) is small enough, for all \( u \in X_0 \)
1. \[
\langle \Pi S'(0)(u), u \rangle \geq c_2 \epsilon^{-3} \left\| u \right\|_X^2
\]

and
2. \[
\left\| \Pi S'(0)(u) \right\|_Z \geq c_2 \epsilon^{-3} \left\| u \right\|_X.
\]

3. Moreover the operator \( \Pi S'(0) \) is one-to-one from \( X_0 \) onto \( Z_\epsilon \).

Proof. By (1.12), there exists \( \tilde{\sigma} > 0 \) such that
\[
e^3|\gamma| \leq \tilde{\sigma}.
\]
According to Lemma 4.4 and condition 1 of Theorem 1.1 by letting \( \delta \) small enough,
\[
\left\| F(u) \right\|_Z \leq C|\gamma| \epsilon \left\| u \right\|_Z \leq C\tilde{\sigma} \epsilon^{-3} \left\| u \right\|_Z \leq c_2 \epsilon^{-3} \left\| u \right\|_Z
\]
for all \( u \in \mathcal{X}_\epsilon \). By Lemma 4.3 part 1 and (4.19)
\[
\langle \Pi S'(0)(u), u \rangle = \langle \Pi E(u), u \rangle + \langle \Pi F(u), u \rangle \\
\geq 2c_2\epsilon^{-3}\|u\|_Y^2 - c_2\epsilon^{-3}\|u\|_Z^2 \\
\geq c_2\epsilon^{-3}\|u\|_Z^2,
\]
for all \( u \in \mathcal{X}_\epsilon \). This proves part 1.

By Lemma 4.3 part 2 and (4.19)
\[
\|\Pi S'(0)(u)\|_Z \geq \|\Pi E(u)\|_Z - \|\Pi F(u)\|_Z \\
\geq 2c_2\epsilon^{-3}\|u\|_\mathcal{X} - c_2\epsilon^{-3}\|u\|_Z \\
\geq c_2\epsilon^{-3}\|u\|_\mathcal{X},
\]
for all \( u \in \mathcal{X}_\epsilon \). This proves part 2.

To prove the third part it suffices to show that \( \Pi S'(0) \) is from \( \mathcal{X}_\epsilon \) onto \( \mathcal{Z}_\epsilon \). Note that \( \Pi S'(0) \) is an unbounded self-adjoint operator on \( \mathcal{Z}_\epsilon \) with the domain \( \mathcal{X}_\epsilon \subset \mathcal{Z}_\epsilon \). If \( v \in \mathcal{Z}_\epsilon \) is perpendicular to the range of \( \Pi S'(0) \), i.e. \( \langle \Pi S'(0)(u), v \rangle = 0 \) for all \( u \in \mathcal{X}_\epsilon \), then the self-adjointness of \( \Pi S'(0) \) implies that \( v \in \mathcal{X}_\epsilon \) and \( \Pi S'(0)(v) = 0 \). By part 2, \( v = 0 \). Hence, the range of \( \Pi S'(0) \) is dense in \( \mathcal{Z}_\epsilon \). Part 2 also implies that the range of \( \Pi S'(0) \) is a closed subspace of \( \mathcal{Z}_\epsilon \). Therefore \( \Pi S'(0) \) is onto. \( \square \)

Finally one needs an estimate on the second Fréchet derivative of \( \mathcal{S} \), i.e. the third variation of \( J \).

**Lemma 4.6.** There exists \( C_3 > 0 \) such that for all \( \phi \in \text{Dom}(\mathcal{S}) \), the following estimates hold for \( u = (u^1, u^2, \ldots, u^n) \in \mathcal{X}, v = (v^1, v^2, \ldots, v^n) \in \mathcal{X} \), where \( u^k = (u^k_1, u^k_2) \) and \( v^k = (v^k_1, v^k_2) \).

1. \( \|(H^{(2)}_k)(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5}\|u^k\|_\mathcal{X}\|v^k\|_\mathcal{X} \).
2. \( \|(B^{(2)}_k)(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5}\|u^k\|_\mathcal{X}\|v^k\|_\mathcal{Y} \).
3. \( \|(C^{(2)}_k)(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5}\|u^k\|_\mathcal{Y}\|v^k\|_\mathcal{Y} \).
4. \( \|(A^{(2)}_k)(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5}\|u^k\|_\mathcal{Y}\|v^k\|_\mathcal{Y} + \|v^k\|_\mathcal{Y} \).
5. \( \|\mathcal{S}''(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5} + |\gamma|\epsilon^{-2}\|u^k\|_\mathcal{X}\|v^k\|_\mathcal{X} \).

In summary,
\[
\|\mathcal{S}''(\phi)(u, v)\|_Z \leq C_3\epsilon^{-5} + |\gamma|\epsilon^{-2}\|u^k\|_\mathcal{X}\|v^k\|_\mathcal{X}.
\]

We choose to denote the constants here by \( C_3 \) to remind the reader that \( C_3 \) is used in the estimate of the third variation of \( J \). The proof of Lemma 4.6 which is skipped, is straightforward estimation, similar to the proofs of [22] Lemma 3.2, and [21] Lemma 6.1.

5. **Reduction.** We view \( \mathcal{S} \) as a nonlinear operator from \( \mathcal{X} \) to \( \mathcal{Z} \). In this section it will be proved that, for each \( (\xi, w) \in \Xi_\epsilon \times \mathcal{W} \), there exists \( \phi^*(\xi, w) \in \mathcal{X}_\epsilon \) such that
\[
\Pi S(\phi^*(\xi, w)) = 0. \tag{5.1}
\]
In other words there exist \( A^{k,1}, A^{k,2}, A^{k,3}, A^{k,4} \in \mathbb{R} \) such that
\[
\mathcal{S}^{(k)}(\phi^*(\theta)) = A^{k,1}\cos \theta \left[ \alpha^k_1 \right] + A^{k,2}\sin \theta \left[ \alpha^k_1 \right] + A^{k,3}\left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + A^{k,4}\left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \tag{5.2}
\]
where \( k = 1, 2, \ldots, n \). Note that \( \phi^* \) is sought in \( \mathcal{X}_\epsilon \), so the core-shells in the assembly \( \phi^*(\xi, w) \) are centered at \( \xi^k \) of radii \( r^k \). In the next section we will find a particular \( (\xi, w) \), denoted \( (\xi^*, w^*) \), at which \( A^{k,1} = A^{k,2} = A^{k,3} = A^{k,4} = 0 \), i.e. \( \mathcal{S}(\phi^*(\xi^*, w^*)) = 0 \). This means that by finding \( \phi^*(\xi, w) \) we reduce the original problem \([124] \) and \([135] \) to a problem of finding a \((\xi^*, w^*)\) in the set \( \Xi_\epsilon \times \mathcal{W} \).
Lemma 5.1. When $\delta$ is small, there exists $\phi^* = \phi^*(\theta, \xi, w)$ such that for every $(\xi, w) \in \mathbb{R}^2 \times W, \phi^*(\cdot, \xi, w) \in X$, solves (5.1). Moreover $\|\phi^*(\cdot, \xi, w)\|_X \leq \frac{2C_1}{c_2} |\gamma| \epsilon^2$.

Proof. Expand $S(\phi)$ as

$$S(\phi) = S(0) + S'(0)(\phi) + R(\phi)$$

(5.3)

where $R$ is a higher order term defined by (5.3). Turn (5.1) to a fixed point form:

$$\phi = T(\phi)$$

(5.4)

where

$$T(\phi) = -(II S'(0))^{-1}(II S(0) + II R(\phi))$$

(5.5)

is an operator on $W = \{ \phi \in X : \|\phi\|_X \leq b \epsilon^2 \} \subset \text{Dom}(S)$. Here $b > 0$ is to be determined.

By Lemma 4.6, $S(0)$ is a sum of a $\theta$ independent part, an $r_k^i e^{i \theta}$ part, and a quantity of order $O(|\gamma| \epsilon^4)$. After one applies the projection operator $\Pi$, the $\theta$ independent part and $r_k^i e^{i \theta}$ part vanish due to the fact that $\left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \cos \theta \left[ \begin{array}{cc} \alpha_k & 0 \\ 0 & 1 \end{array} \right]$, and $\sin \theta \left[ \begin{array}{cc} \alpha_k & 0 \\ 0 & 1 \end{array} \right]$ are all perpendicular to $Z_\theta$, so we have $\|II S(0)\|_X = O(|\gamma| \epsilon^4)$.

Consequently, there exists $C_1 > 0$ such that

$$\|II S(0)\|_X \leq C_1 |\gamma| \epsilon^4.$$  (5.6)

We use $C_1$ to remind the reader that it is used in the estimate of the first variation of $J$.

By part 2 of Lemma 4.5

$$\|II S'(0)\|_X \leq \frac{C_1}{c_2} |\gamma| \epsilon^7.$$  (5.7)

Lemma 4.6 implies that

$$\|R(\phi)\|_X \leq C_3 (\epsilon^{-5} + |\gamma| \epsilon^{-2}) \|\phi\|_X^2.$$  (5.8)

By part 2 of Lemma 4.5

$$\|II S'(0)\|_X \leq \frac{C_3}{c_2} (\epsilon^{-2} + |\gamma| \epsilon) \|\phi\|_X^2.$$  (5.9)

For $\phi \in W$, by condition 1 of Theorem 1, (4.18), (5.5), (5.7), and (5.9) we deduce

$$\|T(\phi)\|_X \leq \frac{C_1}{c_2} |\gamma| \epsilon^7 + \frac{C_3}{c_2} (\epsilon^2 + |\gamma| \epsilon^5) b^2 \leq \left( \frac{C_1}{c_2} \epsilon \delta^2 + \frac{C_3}{c_2} b^2 + \frac{C_3}{c_2} \delta b^2 \right) \epsilon^2.$$

Take

$$b = \min \left\{ \frac{c_2}{4C_3(1 + \delta)}, \frac{C_1}{2} \right\}.$$  (5.10)

Let $\delta$ be small enough such that $\delta < \frac{b c_2}{2C_3}$ and Lemma 4.5 holds. Then

$$\|T(\phi)\|_X \leq b \epsilon^2.$$  

Therefore $T$ maps $W$ into itself.

Next we show that $T$ is a contraction. Let $\phi_1, \phi_2 \in W$. First note that

$$T(\phi_1) - T(\phi_2) = -(II S'(0))^{-1}(II(R(\phi_1) - R(\phi_2)))$$  (5.11)

Because

$$R(\phi_1) - R(\phi_2) = S(\phi_1) - S(\phi_2) - S'(0)(\phi_1 - \phi_2),$$  (5.12)
we deduce, with the help of Lemma 4.6 that
\[ \| R(\phi_1) - R(\phi_2) \|_Z \]
\[ \leq \| S'(\phi_2)(\phi_1 - \phi_2) - S'(0)(\phi_1 - \phi_2) \|_Z + \frac{C_3}{2} (\epsilon^{-5} + |\gamma|\epsilon^{-2})\|\phi_1 - \phi_2\|_Z^2 \]
\[ \leq C_3 (\epsilon^{-5} + |\gamma|\epsilon^{-2})\|\phi_2\|_X\|\phi_1 - \phi_2\|_X + \frac{C_3}{2} (\epsilon^{-5} + |\gamma|\epsilon^{-2})\|\phi_1 - \phi_2\|_X^2 \]
\[ \leq C_3 (\epsilon^{-5} + |\gamma|\epsilon^{-2}) (b + b)\epsilon^2\|\phi_1 - \phi_2\|_X \]
\[ \leq 2b C_3 (\epsilon^{-3} + |\gamma|)\|\phi_1 - \phi_2\|_X. \]

Then part 2 of Lemma 4.5, 4.18 and 5.10 imply that
\[ \| T(\phi_1) - T(\phi_2) \|_X \leq \frac{2b C_3}{c_2} (1 + \tilde{\sigma})\|\phi_1 - \phi_2\|_X \leq \frac{1}{2}\|\phi_1 - \phi_2\|_X, \]
i.e. \( T \) is a contraction map. A unique fixed point \( \phi^* \) is found in \( W \).

By the definition of \( W \), \( \| \phi^* \|_X = O(\epsilon^2) \). However this can be much improved.

Revisit the equation \( \phi = T(\phi) \), satisfied by \( \phi^* \), and derive from 4.5, 5.1 and 5.3 that
\[ \| \phi^* \|_X \leq \| (IL)^{-1}ILS(0) \|_X + \| (IL)^{-1}ILV(\phi) \|_X \]
\[ \leq \frac{C_1}{c_2} |\gamma|\epsilon^7 + \frac{C_3}{c_2} (\epsilon^{-2} + |\gamma|\epsilon)\|\phi^*\|_X^3. \]

Rewrite the above as
\[ (1 - \frac{C_3}{c_2} (\epsilon^{-2} + |\gamma|\epsilon)\|\phi^*\|_X)\|\phi^*\|_X \leq \frac{C_1}{c_2} |\gamma|\epsilon^7. \]

In 5.14 estimate
\[ \frac{C_3}{c_2} (\epsilon^{-2} + |\gamma|\epsilon)\|\phi^*\|_X \leq \frac{C_3}{c_2} (1 + |\gamma|\epsilon^3)b \leq \frac{C_3}{c_2} (1 + \tilde{\sigma})b \leq \frac{1}{4} \]
by 4.18 and 5.10. The estimate of \( \phi^* \) follows from 5.14 and 5.15.

We state a result regarding \( S' \) at \( \phi^*(\cdot, \xi, w) \). It implies that \( \phi^* \) is stable with respect to perturbations within \( X_0 \).

Lemma 5.2. When \( \delta \) is sufficiently small, there exists \( c_2' > 0 \) such that for all \( u \in X_0 \),

1. \[ \langle ILS'(\phi^*)(u), v \rangle \geq c_2' \epsilon^{-3} \| v \|_Y^2, \]

and 2. \[ \| ILS'(\phi^*)(u) \|_Z \geq c_2' \epsilon^{-3} \| v \|_X. \]

Proof. Part 1 of Lemma 4.5, Lemmas 4.6 and 5.1 imply that
\[ \langle ILS'(\phi^*)(u), v \rangle = \langle ILS'(0)(u), v \rangle + \langle (S'(\phi^*) - S'(0))u, v \rangle \]
\[ \geq c_2 \epsilon^{-3} \| u \|_2^2 - C_3 (\epsilon^{-3} + |\gamma|\epsilon^{-2})\|\phi^*\|_X \| u \|_2^2 \]
\[ \geq \left( c_2 - \frac{2C_1 C_3 \tilde{\sigma}}{c_2} (1 + \tilde{\sigma})\delta^2 \right) \epsilon^{-3} \| u \|_2^2. \]

Let \( \delta \) be small enough such that \( 2C_1 C_3 \tilde{\sigma} (1 + \tilde{\sigma})\delta^2 < c_2 \) and \( c_2' = c_2 - 2C_1 C_3 \tilde{\sigma} (1 + \tilde{\sigma})\delta^2. \) This proves the first part of this lemma.
By part 2 of Lemma 4.5, Lemmas 4.6 and 5.1, we deduce

$$\|\Pi S(\phi^*)(u)\|_z \geq \|\Pi S'(0)(u)\|_z - \|\Pi(S'(\phi^*) - S'(0))(u)\|_z \geq c_2\epsilon^{-3}\|u\|_X - C_3(\epsilon^{-5} + |\gamma|\epsilon^{-2})\|\phi^*\|_X \|u\|_X \geq \left(c_2 - \frac{2C_1C_3\tilde{\epsilon}}{c_2}(1 + \tilde{\epsilon})\epsilon^{-2}\right)\|u\|_X.$$ 

As before, the second part follows if $c_2' = c_2 - \frac{2C_1C_3\tilde{\epsilon}}{c_2}(1 + \tilde{\epsilon})\epsilon^{-2}$.

6. Existence.

**Lemma 6.1.** It holds uniformly for all $(\xi, u) \in \bar{E}_x \times W$,

$$|\mathcal{J}(\phi^*) - \mathcal{J}(0)| \leq |\gamma|^2\epsilon^{11}\left(\frac{C_2^2}{c_2} + \frac{10C_3C_1^3}{3c_2^3}(1 + |\gamma|\epsilon^3)|\epsilon|\right).$$

**Proof.** Expanding $\mathcal{J}(\phi^*)$ yields

$$\mathcal{J}(\phi^*) = \mathcal{J}(0) + \langle S(0), \phi^* \rangle + \frac{1}{2}\langle S'(0)(\phi^*), \phi^* \rangle + \frac{1}{6}\langle S''(\tau\phi^*),(\phi^*, \phi^*) \rangle \quad (6.1)$$

for some $\tau \in (0, 1)$. On the other hand expanding $S(\phi^*)$ and applying $\Pi$ on both sides give

$$\|\Pi S(\phi^*) - \Pi S(0) - \Pi S'(0)(\phi^*)\|_z \leq \sup_{\tau \in (0, 1)} \frac{1}{2}\|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z. \quad (6.2)$$

Since $\Pi S(\phi^*) = 0$, (6.2) shows that

$$\|\Pi S(0) + \Pi S'(0)(\phi^*)\|_z \leq \sup_{\tau \in (0, 1)} \frac{1}{2}\|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z,$$

which implies that

$$\|\langle \Pi S(0), \phi^* \rangle + \langle \Pi S'(0)(\phi^*), \phi^* \rangle\|_z \leq \left(\sup_{\tau \in (0, 1)} \frac{1}{2}\|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z\right)\|\phi^*\|_X. \quad (6.3)$$

Since $\phi^* \in X_x$,

$$\langle \Pi S(0), \phi^* \rangle = \langle S(0), \phi^* \rangle, \text{ and } \langle \Pi S'(0)(\phi^*), \phi^* \rangle = \langle S'(0)(\phi^*), \phi^* \rangle.$$  

Then (6.3) shows that

$$\|\langle S(0), \phi^* \rangle + \langle S'(0)(\phi^*), \phi^* \rangle\|_z \leq \left(\sup_{\tau \in (0, 1)} \frac{1}{2}\|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z\right)\|\phi^*\|_X. \quad (6.4)$$

By (6.3), (6.1) yields that

$$|\mathcal{J}(\phi^*) - \mathcal{J}(0) - \frac{1}{2}\langle S(0), \phi^* \rangle| \leq \frac{5}{12}\left(\sup_{\tau \in (0, 1)} \|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z\right)\|\phi^*\|_X.$$ 

Therefore (5.3), Lemmas 4.6 and 5.1, show that

$$|\mathcal{J}(\phi^*) - \mathcal{J}(0)| \leq \frac{1}{2}\langle S(0), \phi^* \rangle + \frac{5}{12}\left(\sup_{\tau \in (0, 1)} \|\Pi S''(\tau\phi^*),(\phi^*, \phi^*)\|_z\right)\|\phi^*\|_X \leq \frac{1}{2}\langle C_2|\gamma|\epsilon^4 \rangle \frac{2C_1}{c_2}|\gamma|\epsilon^7 + \frac{5}{12}C_3(\epsilon^{-5} + |\gamma|\epsilon^{-2})\left(\frac{2C_1}{c_2}|\gamma|\epsilon^7\right)^3 \leq \frac{1}{2}\langle C_2^2\rangle \frac{10C_3C_1^3}{3c_2^3}(1 + |\gamma|\epsilon^3)|\epsilon|$$

which proves the lemma. $\Box$
At this point, we resurrect the dependence on \((\xi, w)\) and write \(\phi^*\) as \(\phi^*(\cdot, \xi, w)\). Let \((\xi, w)\) vary in \(\Xi_\sigma \times \overline{W}\). The energy of \(\phi^*(\cdot, \xi, w)\), \(\mathcal{J}(\phi^*(\cdot, \xi, w))\), is now viewed as a function of \((\xi, w)\) on \(\Xi_\sigma \times \overline{W}\). We treat both \(\Xi_\sigma\) and \(\overline{W}\) as manifolds with boundary. The product of the two, \(\Xi_\sigma \times \overline{W}\), is also a manifold with boundary. It turns out that every critical point of \(\mathcal{J}(\phi^*(\cdot, \xi, w))\) in \(\Xi_\sigma \times \overline{W}\), the interior of \(\Xi_\sigma \times \overline{W}\), corresponds to an exact solution of (1.4) and (1.5).

**Lemma 6.2.** If \((\xi, w)\) is an (interior) critical point of the function \((\xi, w) \rightarrow \mathcal{J}(\phi^*(\cdot, \xi, w))\), then \(\phi^*(\cdot, \xi, w)\) solves \(\mathcal{S}(\phi^*(\cdot, \xi, w)) = 0\), i.e. the core-shell assembly represented by \(\phi^*(\cdot, \xi, w)\) satisfies (1.4) and (1.5).

The proof of this lemma is similar to that of [21, Lemmas 8.3 and 8.4], so we omit the details.

Since \(\Xi_\sigma \times \overline{W}\) is compact, there exists \((\xi^*, w^*) \in \Xi_\sigma \times \overline{W}\), that minimizes \(\mathcal{J}(\phi^*(\cdot, \xi, w))\). The next lemma asserts that \((\xi^*, w^*)\) is actually attained in the interior of \(\Xi_\sigma \times \overline{W}\), and hence is a critical point of \(\mathcal{J}(\phi^*(\cdot, \xi, w))\).

**Lemma 6.3.** Let \((\xi^*, w^*) \in \Xi_\sigma \times \overline{W}\) be a minimum of \(\mathcal{J}(\phi^*(\cdot, \xi, w))\). When \(\delta\) is sufficiently small and \(\sigma\) is sufficiently large, \((\xi^*, w^*)\) must be in \(\Xi_\sigma \times \overline{W}\), the interior of \(\Xi_\sigma \times \overline{W}\).

**Proof.** Let \((\xi^*, w^*) \rightarrow (\xi^0, w^0)\) as \(\epsilon \rightarrow 0\), possibly along a subsequence. First we show that \(w^0 = \tilde{w}\) where

\[
\tilde{w} = \left( \left( \frac{m}{n}, \frac{1 - m}{n} \right), \ldots, \left( \frac{m}{n}, \frac{1 - m}{n} \right) \right).
\]  

(6.5)

Lemmas 2.4, 6.1 and 2.5 imply that

\[
\mathcal{J}(\phi^*(\cdot, \xi, w)) = \sum_k 2\pi(r_1^k + r_2^k) + \frac{\gamma_{11}\pi^2}{2} \sum_k \left( -\frac{(r_1^k)^4 \log r_1^k}{2\pi} \right)
+ 2\pi^2 \sum_k \left( -\frac{(r_1^k)^2 (r_2^k)^2 \log r_1^k}{2\pi} - \frac{(r_1^k)^4 \log r_1^k}{2\pi} \right)
+ \gamma_{22}\pi^2 \sum_k \left( -\frac{(r_1^k)^4 \log r_2^k}{2\pi} + \frac{(r_1^k)^4 \log r_1^k}{2\pi} + \frac{(r_1^k)^2 (r_2^k)^2 \log r_2^k}{2\pi} \right) + O(\gamma \epsilon^4)
= 2\sqrt{\pi} \epsilon \sum_{k=1}^n \left( \sqrt{w_1^k} + \sqrt{w_2^k} \right)
+ \epsilon^4 \log \frac{1}{\epsilon} \sum_{k=1}^n \left( \frac{\gamma_{11}}{2\pi} w_1^k w_2^k + \frac{\gamma_{12}}{2\pi} w_1^k w_2^k + \frac{\gamma_{22}}{2\pi} w_1^k w_2^k \right) + O(\gamma \epsilon^4).
\]

Then

\[
\frac{\mathcal{J}(\phi^*(\cdot, \xi, w))}{\lambda(\gamma) \epsilon^4 \log \frac{1}{\epsilon}} = \frac{2\sqrt{\pi}}{\lambda(\gamma) \epsilon^3 \log \frac{1}{\epsilon}} \sum_{k=1}^n \left( \sqrt{w_1^k} + \sqrt{w_2^k} \right)
+ \epsilon^4 \log \frac{1}{\epsilon} \sum_{k=1}^n \frac{\gamma_{ij}}{2\pi} w_1^k w_2^k + O(\lambda(\gamma) \log \frac{1}{\epsilon}).
\]

(6.7)
Let \( \frac{1}{\lambda(\gamma) e^3 \log \frac{1}{\epsilon}} \to \Delta \) and \( \frac{2}{\lambda(\gamma)} \to M \) as \( \epsilon \to 0 \), possibly along a subsequence. By condition 4 of Theorem 1.1

\[
1 \leq \overline{\lambda}(M) \leq \overline{\lambda}(M) \leq \frac{1}{\epsilon},
\]

(6.8)

where \( \overline{\lambda}(M) \) and \( \overline{\lambda}(M) \) are the two eigenvalues of \( M \). Thus as \( \epsilon \to 0 \), \( \frac{|r_1|}{\lambda(\gamma) e^3 \log \frac{1}{\epsilon}} \to 0 \) and

\[
\frac{\mathcal{J}(\phi^*(\cdot, \xi, w))}{\lambda(\gamma) e^4 \log \frac{1}{\epsilon}} \to 2\sqrt{\pi} \Delta \sum_{k=1}^{n} \left( \sqrt{w_1^k} + \sqrt{w_1^k + w_2} \right) + \sum_{k=1}^{n} \sum_{i,j=1}^{\gamma} \frac{M_{ij} w_i^k w_j^k}{4\pi}
\]

(6.9)

uniformly for \( (\xi, w) \in \Xi \times \mathbb{W} \).

By condition 2 of Theorem 1.1 \( 0 \leq \Delta \leq \frac{1}{\gamma} \). If \( \sigma \) is sufficiently large, then \( \Delta \) is sufficiently small. The right side of (6.9) can be seen as a function of \( w \), which is studied in [26 Appendix C]. It is proved there that when \( \Delta \) is small, the right side of (6.9) is minimized at \( w = \bar{w} \). If \( w^0 \) were not \( \bar{w} \), then \( \mathcal{J}(\phi^*(\cdot, \xi^*, w^*)) > \mathcal{J}(\phi^*(\cdot, \xi^*, \bar{w})) \) when \( \epsilon \) is sufficiently small, a contradiction to the assumption that \( (\xi^*, w^*) \) is a minimum of \( \mathcal{J}(\phi^*(\cdot, \xi, w)) \).

Next we show that

\[
F(\xi^o) = \min_{\xi \in \Xi} F(\xi).
\]

(6.10)

Take \( w = w^* \) and denote the corresponding \( r \) by \( r^* \). Define

\[
H_\epsilon(\xi) = \frac{1}{\lambda(\gamma) e^3} \left\{ \mathcal{J}(\phi^*(\cdot, \xi, w^*)) - \left[ 2\pi \sum_{k=1}^{n} (r_1^{*,k} + r_2^{*,k}) + \frac{\gamma_1 e^2}{2} \sum_{k=1}^{n} \left( \frac{(r_1^{*,k})^4 \log r_1^{*,k}}{2\pi} + \frac{(r_1^{*,k})^4}{8\pi} \right) + \frac{\gamma_2 \pi}{2} \sum_{k=1}^{n} \left( \frac{(r_2^{*,k})^2 (r_2^{*,k})^2 \log r_2^{*,k} - (r_1^{*,k})^4 \log r_1^{*,k}}{2\pi} + \frac{(r_1^{*,k})^2 (r_2^{*,k})^2 - (r_1^{*,k})^4}{4\pi} \right) + \frac{(r_2^{*,k})^2 - (r_1^{*,k})^2}{2\pi} \right) \right\}
\]

By Lemmas 2.3, 6.1 and 6.4, we deduce that

\[
H_\epsilon(\xi) = \frac{1}{\lambda(\gamma)} \left\{ \sum_{k=1}^{n} \left( w_1^{*,k} \right)^2 R(\xi^k, \xi^k) + \sum_{k=1}^{n} \sum_{l \neq k} \left( w_1^{*,k} w_1^{*,l} G(\xi^k, \xi^l) \right) + \gamma_1 \sum_{k=1}^{n} \sum_{l \neq k} \left( w_1^{*,k} w_2^{*,k} R(\xi^k, \xi^k) + \sum_{k=1}^{n} \sum_{l \neq k} \left( w_1^{*,k} w_2^{*,k} G(\xi^k, \xi^l) \right) \right) + \gamma_2 \sum_{k=1}^{n} \sum_{l \neq k} \left( w_2^{*,k} w_2^{*,k} R(\xi^k, \xi^k) + \sum_{k=1}^{n} \sum_{l \neq k} \left( w_2^{*,k} w_2^{*,k} G(\xi^k, \xi^l) \right) \right) \right\}
\]

Note that the first \( O(\cdot) \) term in the last line comes from the smallest terms, of order \( O(|\gamma| e^3) \), in Lemma 2.7 the second \( O(\cdot) \) term comes from Lemma 6.3.


According to (4.18), (6.8), and the fact that \( w^* \to \bar{w} \) of (6.5), one obtains that as \( \epsilon \to 0 \),

\[
H_\epsilon(\xi) \to \frac{1}{2} \left( M_{11} \left( \frac{m}{n} \right)^2 + 2 M_{12} \left( \frac{m}{n} \right) \left( \frac{1-m}{n} \right) + M_{22} \left( \frac{1-m}{n} \right)^2 \right) F(\xi) \tag{6.11}
\]

uniformly for \( \xi \in \mathbb{R}^d \). In (6.11), the coefficient of \( F(\xi) \) is positive since \( M \) is positive definite. If \( \xi^0 \) were not a minimum of \( F \), then let \( \xi \) be a minimum of \( F \). This means \( F(\xi) < F(\xi^0) \). One finds that \( H_\epsilon(\xi^0) > H_\epsilon(\xi) \) when \( \epsilon \) is sufficiently small, a contradiction to the fact that \( (\xi^*, w^*) \) is a minimum of \( J(\phi^*(\cdot, \xi, w)) \).

Since \( \bar{w} \) is in \( W \) and any minimum of \( F \) is attained in \( \mathbb{R}^d \) by (229), \( (\xi^*, w^*) \) is in \( \mathbb{R}^d \times W \) when \( \epsilon \) is sufficiently small.

**Proof of Theorem 1.1** By Lemma 6.3 the minimum \( (\xi^*, w^*) \) of the function \( (\xi, w) \to J(\phi^*(\cdot, \xi, w)) \) is attained in the interior of \( \mathbb{R}^d \times \overline{W} \), so it is a critical point. Lemma 6.2 then asserts that \( \phi^*(\cdot, \xi^*, w^*) \) is a stationary point of \( J \).

**Proof of Theorem 1.2** In Lemma 6.3 it is proved that, as \( \epsilon \to 0 \),

\[
w^{*,k}_1 \to \frac{m}{n}, \quad w^{*,k}_2 \to \frac{1-m}{n}. \tag{6.12}
\]

In terms of \( r^* \), this means

\[
\frac{r^{*,k}_1}{\epsilon} \to \sqrt{\frac{m}{n\pi}}, \quad \frac{r^{*,k}_2}{\epsilon} \to \sqrt{\frac{1}{n\pi}}. \tag{6.13}
\]

Also proved in Lemma 6.3 is that as \( \epsilon \to 0 \), any limit point of \( \xi^* \) must be a minimum of \( F \); see (6.10).

A systematic study of the stability of solutions to (1.4) and (1.5) is beyond the scope of this paper. Our assertion that the solution \( \phi^*(\cdot, \xi^*, w^*) \) is in some sense stable is interpreted by its local minimization property. Recall that the solution \( \phi^*(\cdot, \xi^*, w^*) \) is found in two steps. First for each \( (\xi, w) \in \mathbb{R}^d \times \overline{W} \), a fixed point \( \phi^*(\cdot, \xi, w) \) is constructed in a class of assemblies whose core-shells are centered at \( \xi^k \) of radii \( r^k_i \) (\( r^k_i \) is related to \( w^k_i \)). This fixed point is shown to be locally minimizing \( J \) in this class; see Lemma 5.2 part 1. In the second step \( J \) is minimized among the \( \phi^*(\cdot, \xi, w)'s \) where \( (\xi, w) \) ranges over \( \mathbb{R}^d \times \overline{W} \), and \( \phi^*(\cdot, \xi^*, w^*) \) emerges as a minimum. As a minimum of locally minimizing assemblies from these classes of core-shell assemblies, \( \phi^*(\cdot, \xi^*, w^*) \) is stable in this sense.

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