Independent transversal domination number of a graph

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Abstract: Let $G = (V, E)$ be a graph. A set $S \subseteq V(G)$ is a dominating set of $G$ if every vertex in $V \setminus S$ is adjacent to a vertex of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of $G$. Furthermore, a dominating set $S$ is an independent transversal dominating set of $G$ if it intersects every maximum independent set of $G$. The independent transversal domination number of $G$, denoted by $\gamma_{it}(G)$, is the cardinality of a minimum independent transversal dominating set of $G$. In 2012, Hamid initiated the study of the independent transversal domination of graphs, and posed the following two conjectures:

Conjecture 1. If $G$ is a non-complete connected graph on $n$ vertices, then $\gamma_{it}(G) \leq \lceil \frac{n}{2} \rceil$.

Conjecture 2. If $G$ is a connected bipartite graph, then $\gamma_{it}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$.

We show that Conjecture 1 is not true in general. Very recently, Conjecture 2 is partially verified to be true by Ahangar, Samodivkin, Yero. Here, we prove the full statement of Conjecture 2. In addition, we give a correct version of a theorem of Hamid. Finally, we answer a problem posed by Martínez, Almira, and Yero on the independent transversal total domination of a graph.

Keywords: Domination number; Independence number; Covering number; Independent transversal domination number

1 Introduction

We consider undirected finite simple graphs only, and follow the notations and terminology in [3]. Let $G = (V(G), E(G))$ be a graph. The order of $G$ is $|V(G)|$. For a vertex $v \in V(G)$, the set of neighbors is denoted by $N(v)$. The degree of $v$, denoted by $d(v)$, is number of edges incident with $v$ in $G$. Since $G$ is simple, $d(v) = |N(v)|$. The minimum degree of $G$, denoted by $\delta(G)$, is $\min\{d(v) : v \in V(G)\}$. For a set $S \subseteq V(G)$, let $N(S) = \cup_{v \in S} N(v)$. A set $S$ is

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said to be an *independent set* of $G$, if no pair of vertices of $S$ are adjacent in $G$. The *independence number* of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of $G$. We denote by $\Omega(G)$ the set of all maximum independent sets of $G$.

On the other hand, $S$ is said to be a *vertex covering* of $G$ if every edge of $G$ is incident with some vertex of $S$ in $G$. It is easy to see that $S$ is an independent set of $G$ if and only if $V(G) \setminus S$ is a vertex covering of $G$. The vertex covering number, denoted by $\beta(G)$, is the cardinality of a minimum vertex covering of $G$. So, for any graph $G$ of order $n$,

$$\alpha(G) + \beta(G) = n. \tag{1}$$

A set $M \subseteq E(G)$ is said to be a matching of $G$ if no pair of edges of $M$ have a common end vertex. The matching number of $G$, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of $G$. It is obvious that for a graph $G$ of order $n$,

$$\alpha'(G) \leq \min\{\frac{n}{2}, \beta(G)\}. \tag{2}$$

The well-known Kőnig-Egerváry theorem states that for a bipartite graph $G$,

$$\alpha'(G) = \beta(G). \tag{3}$$

A set $S \subseteq V(G)$ is a *dominating set* of $G$ if every vertex in $V \setminus S$ is adjacent to a vertex of $S$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A minimum dominating set of a graph $G$ is called a $\gamma(G)$-set of $G$.

It is clear that for a graph $G$,

$$\gamma(G) \leq \alpha(G), \tag{4}$$

and if $G$ has no isolated vertices,

$$\gamma(G) \leq \beta(G). \tag{5}$$

A set $S \subseteq V(G)$ is said to be an *independent transversal* of $G$ if $S \cap I \neq \emptyset$ for any $I \in \Omega(G)$. The *independent transversal number* of $G$, denoted by $\tau_t(G)$, is the cardinality of a minimum independent transversal of $G$. A dominating set $S$ of a graph $G$ is said to be an *independent transversal dominating set* if $S \cap I \neq \emptyset$ for any $I \in \Omega(G)$. By the definitions above, for any graph $G$,

$$\max\{\gamma(G), \tau_t(G)\} \leq \gamma_{it}(G). \tag{6}$$

The notion of independent transversal domination was first introduced by Hamid [8] in 2012. He proved that for a graph $G$ of order $n$, $\gamma_{it}(G) \leq n$, with equality if and only if $G \cong K_n$. 

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Theorem 1.1. (Hamid [8]) For a graph $G$ without isolated vertices, $\gamma_{it}(G) \leq \beta(G) + 1$.

Theorem 1.2. (Hamid [8]) If $G$ is a non-complete connected graph of order $n$ with $\alpha(G) \geq \frac{n}{2}$, then $\gamma_{it}(G) \leq \frac{n}{2}$.

In view of the above theorem, Hamid posed the following conjecture in [8].

Conjecture 1.1. (Hamid [8]) If $G$ is a non-complete connected graph on $n$ vertices, then $\gamma_{it}(G) \leq \lceil \frac{n}{2} \rceil$.

We show that the above conjecture is not true in general. Hamid [8] showed that $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$. In particular, $\gamma_{it}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$ for a tree $T$. So, he proposed the following conjecture.

Conjecture 1.2. (Hamid [8]) If $G$ is a connected bipartite graph, then $\gamma_{it}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$.

Recently, Ahangar, Samodivkin, Yero [1] proved that Conjecture 2 is valid for all unbalanced bipartite graphs.

Theorem 1.3. (Ahangar, Samodivkin, Yero [1]) Let $G$ be a bipartite graph with bipartition $(X, Y)$ such that $|X| \neq |Y|$. Then, $\gamma_{it}(G) \leq \gamma(G) + 1$. In particular, this is true when $G$ has odd order.

We show that the full statement of Conjecture 2 in the next section.

Complexity of independent transversal domination problem can be in $\Omega(n^2)$. Disproof of Conjecture 1.1 and Proof of Conjecture 1.2

First we begin with a useful observation.

Lemma 2.1. Let $G$ be non-complete graph of order $n$. If $G$ is the complement of a triangle-free graph, then $\tau_i(G) = \beta(G)$, and thus $\gamma_{it}(G) \geq n - \alpha(G)$.

Proof. Since $G$ is the complement of a triangle-free graph, $\alpha(G) = 2$. Thus, $\Omega(G) = \{\{u, v\} : uv \in E(G)\}$ and an independent transversal of $G$ is a vertex covering of $\overline{G}$, $\tau_i(G) = \beta(G)$, and thus $\gamma_{it}(G) \geq n - \alpha(G)$.

To disprove Conjecture 1.1, we recall a celebrated result on the Ramsey theory, due to Kim [10].

Theorem 2.2. (Kim [10]) For sufficiently large $n$, there exists a triangle-free graph $G$ of order $n$ with $\alpha(G) \leq 9\sqrt{n\log n}$.
Note that the symbol "≤" in the inequality above was misprinted as "≥" in [10]. The following result is an immediate consequence of Lemma 2.1 and Theorem 2.2.

**Corollary 2.3.** For sufficiently large $n$, there exists a non-complete graph $G$ (the complement of a triangle-free graph) of order $n$, $\gamma_{id}(G) \geq n - \alpha(G) \geq n - 9\sqrt{n \log n} > \lceil \frac{n}{2} \rceil$.

This disproves Conjecture 1. It is straightforward to check that the complement of the Petersen graph $P_{10}$ is also a counterexample to Conjecture 1. Alon [2] gave some explicit construction of triangle-free graphs with relatively small independence numbers contrast to their orders. The complements of these graphs are also counterexamples to Conjecture 1.

Let $\text{core}(G) = \cap \{S : S \in \Omega(G)\}$ be the set of vertices belonging to all maximum independent sets, and let $\xi(G) = |\text{core}(G)|$.

**Theorem 2.4.** (Boros, Golumbic, and Levit [4]) If $G$ is a connected graph with $\alpha(G) > \alpha'(G)$, then $\xi(G) \geq \alpha(G) - \alpha'(G) + 1$.

It can be deduced from the theorem above that $\tau_i(G) = 1$ for any connected graph $G$ with $\alpha(G) > \alpha'(G)$. It is an interesting problem for characterizing graphs with $\tau_i(G) = 1$.

**Problem 2.5.** What is the best upper bound of $\tau_i(G)$ for graphs $G$ in terms of their order $n$?

Now, we are ready to show Conjecture 2.

**Theorem 2.6.** If $G$ is a connected bipartite graph, then $\gamma_{id}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$.

**Proof.** Let $S$ be a $\gamma(G)$-set. If $\alpha(G) > \alpha'(G)$, then by Theorem 2.4, $\xi(G) \geq \alpha(G) - \alpha'(G) + 1 > 0$. So, $S \cup \{v\}$ is an independent transversal dominating set for a vertex $v \in \text{core}(G)$, and hence $\gamma_{id}(G) \leq |S \cup \{v\}| = \gamma(G) + 1$.

Next, we consider the case $\alpha(G) \leq \alpha'(G)$. Since $G$ is bipartite,

$$\alpha(G) \geq \frac{n}{2} \geq \alpha'(G). \tag{7}$$

Combining (7) with (1), (2), (3), we have

$$\alpha(G) = \alpha'(G) = \beta(G) = \frac{n}{2}. \tag{8}$$

If $\gamma(G) = \beta(G)$, then the result follows from Theorem 1.1. So, we assume that $\gamma(G) < \beta(G)$. Let $(X, Y)$ be the bipartition of $G$. By the equation (8), we have $|X| = |Y| = \frac{n}{2}$. Thus, $S \cap X \neq \emptyset$ and $S \cap Y \neq \emptyset$. Take a vertex $x \in S \cap X$, and let
Ωₙ be the set of all maximum independent set of G containing x. In particular, X ∈ Ωₓ. We consider G − x. It is clear that Ω \ Ωₓ = Ω(G − x) and Y ∈ Ω \ Ωₓ. Note that
\[ \alpha(G − x) = |Y| = \frac{n}{2} > \frac{n}{2} − 1 = |X \setminus \{x\}| \geq \alpha'(G − x). \]
By Theorem 2.4, ξ(G − x) > 0, and let y ∈ core(G − x). So, S ∪ {y} is an independent transversal dominating set of G. This shows γᵢ𝑡(G) ≤ γ(G) + 1.

3 Bipartite graphs G with γᵢ𝑡(G) = \frac{n}{2}

Hamid [8] obtained the following theorem.

**Theorem 3.1.** (Hamid [8]) For a bipartite graph G with bipartition (X, Y) such that |X| ≤ |Y| and γ(G) = |X|, γᵢ𝑡(G) = γ(G) + 1 if and only if every vertex in X is adjacent to at least two pendant vertices.

Actually, the theorem above is not complete, see for the counterexample in Figure 1. It is easy to see that for the graph G, γ(G) = 2 = |X| and γᵢ𝑡(G) = 3. However, there is vertex in X, which has no two pendent neighbors.

![Figure 1. A counterexample G](image)

Next we present the correct version of the theorem.

**Theorem 3.2.** Let G be a bipartite graph with bipartition (X, Y) such that |X| ≤ |Y| and γ(G) = |X|, then γᵢ𝑡(G) = γ(G) + 1 if and only if every vertex in X is either a pendent vertex, or is adjacent to at least two pendant vertices.

**Proof.** First, we show its necessity. By the assumption that |X| + 1 = γ(G) + 1 = γᵢ𝑡(G) and by Theorem 1.1, we have |X| ≤ β(G). On the other hand, since X is a vertex covering of G, β(G) ≤ |X|. So, X is a minimum covering of G, implying that Y is a maximum independent set of G.

Next, we show that δ(G) = 1. Suppose δ(G) ≥ 2. Since G is bipartite and γ(G) = |X|, X is a γ(G)-set. Let u ∈ X and v ∈ N(u). Since δ(G) ≥ 2, it follows that S = (X \ {u}) ∪ {v} is a dominating set of G.

**Claim 1.** S is an independent transversal dominating set of G.

Suppose it is not, and let I be a maximum independent set of G such that I ∩ S = ∅. Since |I| = α(G) = |Y| and S = (X \ {u}) ∪ {v}, we have I =
the graphs and independently, in 1985, Fink, Jacobson, Kinch and Roberts [7] characterized joining a new vertex \( v \) of order 2. Hamid [8] posed the following problem: characterizing all bipartite graphs \( G \) of order at most 2. By Theorem 1.2, we have \( \gamma_{it}(G) \leq |S| = |X| = \gamma(G) \), contradicting the assumption that \( \gamma_{it}(G) = |X| + 1 \). This proves \( \delta(G) = 1 \).

To complete the proof of the necessity, it suffices to show that if \( u \in X \) is not a pendent vertex, then \( u \) has at least two pendent neighbors. Suppose, on the contrary, that \( u \) has at most one pendent neighbor. Since \( d(u) \geq 2 \), we choose a neighbor \( v \) of \( u \) with degree as small as possible. Let \( S = (X \setminus \{u\}) \cup \{v\} \). We show that \( S \) is a dominating set of \( G \). If it is not, there exists a vertex \( y \in Y \setminus \{v\} \) not dominated by \( S \). Then \( y \) must be a pendent vertex and be adjacent to \( u \). Now, \( u \) has two pendent neighbors \( v \) and \( y \), a contradiction. So, \( S \) is a (minimum) dominating set of \( G \). Furthermore, by the similar argument as in the proof of Claim 1, we can show that \( S \) is an independent transversal dominating set of \( G \). Again, we have \( \gamma_{it}(G) \leq |S| = |X| = \gamma(G) \), contradicting the assumption that \( \gamma_{it}(G) = \gamma(G) + 1 \). This shows that \( u \) has at least two pendent neighbors.

To show its sufficiency, assume that every vertex in \( X \) is either a pendent vertex, or is adjacent to at least two pendent vertices. Let \( \{x_1, \ldots, x_k\} \) be the set of all pendent vertices in \( X \) for integer \( k \geq 0 \). Let \( y_i \) be the neighbor of \( x_i \) in \( Y \) for each \( i \). Then \( \Omega(G) = \{I : I = (Y \setminus Y') \cup X'\} \), where \( X' \subseteq \{x_1, \ldots, x_k\} \) and \( Y' = N(X') \cap \{y_1, \ldots, y_k\} \). In particular, if \( X' = \emptyset \), then \( Y' = \emptyset \), and \( I = Y \).

Note that if \( S \) is a minimum dominating set of \( G \), then there exists \( X' \subseteq \{x_1, \ldots, x_k\} \) such that \( S = (X \setminus X') \cup Y' \), where \( Y' \) is defined as above. However, \( I_S = (Y \setminus Y') \cup X' \) is a maximum independent set such that \( S \cap I_S = \emptyset \). It means that no minimum dominating set of \( G \) is an independent transversal dominating set, implying that \( \gamma_{it}(G) \geq \gamma(G) + 1 \). On the other hand, by Theorem 2.1, \( \gamma_{it}(G) \leq \beta(G) + 1 \leq |X| + 1 = \gamma(G) + 1 \). We conclude that \( \gamma_{it}(G) = \gamma(G) + 1 \).

The proof is completed.

By Theorem 1.2, \( \gamma_{it}(G) \leq \frac{n}{2} \) for any bipartite graph \( G \) without a component of order at most 2. Hamid [8] posed the following problem: characterizing all bipartite graphs \( G \) for which \( \gamma_{it}(G) = \frac{n}{2} \). In what follows, we partially answer the problem. If \( G \) is a connected bipartite graph with \( \gamma_{it}(G) = \frac{n}{2} \), then by Theorem 2.6,

\[
\frac{n}{2} - 1 \leq \gamma(G) \leq \frac{n}{2}.
\]

(9)

The corona of a graph \( G \), denoted by \( G \circ K_1 \), is the graph obtained from \( G \) by joining a new vertex \( v' \) to each vertex \( v \in V(G) \). In 1982, Payan and Xuong [12] and independently, in 1985, Fink, Jacobson, Kinch and Roberts [7] characterized the graphs \( G \) of order \( n \) with \( \gamma(G) = \frac{n}{2} \).

**Theorem 3.3.** (Fink et al. [7], Payan et al. [12]) For any graph \( G \) with even order \( n \) having no isolated vertices \( \gamma(G) = \frac{n}{2} \) if and only if the components of \( G \)
are $C_4$ or $H \circ K_1$ for any connected graph $H$.

**Corollary 3.4.** Let $G$ be a connected graph of even order $n \geq 4$. If $\gamma(G) = \frac{n}{2}$, then $\gamma_{it}(G) = \frac{n}{2}$.

Proof. Since $\gamma(G) = \frac{n}{2}$, by Theorem 3.3, either $G \cong C_4$ or $H \circ K_1$, where $H$ is a connected graph of order $\frac{n}{2} \geq 2$. It is straightforward to check that $\gamma_{it}(C_4) = 2$. If $G \cong H \circ K_1$ for a connected graph of order $\frac{n}{2}$, $\gamma_{it}(G) = \frac{n}{2}$, because the set of pendant vertices is the unique minimum independent transversal dominating set of $G$. So, the result follows.

**Theorem 3.5.** For a bipartite graph $G$ of even order $n$ with bipartition $(X, Y)$ and without a component of order at most 2, $\gamma_{it}(G) = \frac{n}{2}$, if one of the following cases occurs:

1. $\gamma(G) = \frac{n}{2}$.
2. $\gamma(G) = \frac{n}{2} - 1$, and $|X| = \frac{n}{2} - 1$, $|Y| = \frac{n}{2} + 1$, and every vertex in $X$ is either a pendant vertex, or is adjacent to at least two pendant vertices.

Proof. If $\gamma(G) = \frac{n}{2}$, by Corollary 3.4, $\gamma_{it}(G) = \frac{n}{2}$. If (2) holds, the result follows by Theorem 3.2.

By a long and difficult proof, Hansberg and Volkmann [9] were able to characterize even order trees $T$ with $\gamma(T) = \frac{n}{2} - 1$. By Theorems 3.2, 3.3, and 3.5, the bipartite graphs $G$ with $\gamma_{it}(G) = |X| = |Y| = \frac{n}{2}$ and $\gamma(G) = \frac{n}{2} - 1$ remain to be characterized. Some more effort must be used for completing this task.

## 4 Independent transversal total domination

A new variant of transversal in graphs was introduced very recently by Martínez, Almira, and Yero [11], called independent transversal total domination. A dominating set $S$ of a graph $G$ is called a total dominating set if $G[S]$ has no isolated vertices. The total dominating number of $G$, denoted by $\gamma_t(G)$, is the cardinality of a minimum total dominating set of $G$. Further, a total dominating set $S$ is called an independent transversal total dominating set if $S \cap I \neq \emptyset$ for each $I \in \Omega(G)$. The independent transversal total dominating number of $G$, denoted by $\gamma_{tt}(G)$, is the cardinality of a minimum independent transversal total dominating set of $G$. So, for any graph $G$, $\gamma_{tt}(G) \geq \gamma_{it}(G)$.

**Theorem 4.1.** (Cockayne, Dawes and Hedetneimi [3]) If $G$ is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2n}{3}$.

Brigham, Carrington, and Yitary [5] characterized the connected graphs of order at least 3 with total domination number exactly two-thirds their order.
Among other things, Martínez, Almira, and Yero showed that $\gamma_{tt}(G) \leq \frac{2n}{3}$ for some classes of graphs of order $n$. Based on these results, they further asked the following open problem:

- Is it true that $\gamma_{tt}(G) \leq \frac{2n}{3}$ for any graph of order $n$? If yes, then: Is it true that $\gamma_{tt}(G) = \frac{2n}{3}$ if and only if $\gamma_t(G) = \frac{2n}{3}$?

The answer for the question above is no. By Corollary 2.3, for sufficiently large $n$, there exists a graph $G$ (the complement of a triangle-free graph) of order $n$ with $\gamma_{tt}(G) \geq \gamma_t(G) \geq n - 9\sqrt{n\log n}$.

If $\gamma_{tt}(G) \leq \frac{2n}{3}$ holds for any graph $G$ of order $n$, we have $n - 9\sqrt{n\log n} \leq \frac{2n}{3}$, which is impossible for any sufficiently large $n$.

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