Renormalization of chiral two-pion exchange NN interactions with Δ-excitations: Central Phases and the Deuteron.

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The renormalization of the chiral np interaction with Δ-excitations in intermediate states is considered at next-to-leading order (NLO) and next-to-next-to-leading order (N2LO) for central waves. The inclusion of the Δ-excitation as an explicit degree of freedom improves the convergence properties of the effective field theory results for np scattering with respect to Δ-less theory, and allows the existence of a deuteron bound state in the infinite cut-off limit. The \(^1\)S\(_0\) singlet and \(^3\)S\(_1\) triplet phase shifts reproduce data for \(p \sim m_n\). The role of spectral function regularization is also discussed.

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I. INTRODUCTION

The nucleon-nucleon (NN) interaction problem has a well deserved reputation of being a difficult one (for a review see e.g. \cite{1}). One obvious reason may be found in the theoretical and experimental inaccessibility of the shortest distance interaction region relevant to Nuclear Physics. This lack of fundamental knowledge might be remedied in the future by \textit{ab initio} lattice calculations, for which some incipient results exist already \cite{2,3}. However, the current situation does not prevent from addressing many facets of NN interactions, particularly long distance features, provided short distance insensitivity is guaranteed. Despite the great efforts during the years based on successful phenomenology \cite{4,5,6,7,8,9}, only during the last decade has the parentage to the underlying theory been made more explicit after the proposal \cite{10,11,12}. EFT starts out with intensive use of Chiral Symmetry (for comprehensive reviews see e.g. Ref. \cite{10,11,12}). EFT starts out with the most general Lagrangean in terms of the relevant degrees of freedom compatible with chiral symmetry, dimensional analysis and perturbative renormalization, organized by a prescribed power counting based on assuming a large scale suppression on the parameters \(4\pi f_\pi \sim M_N \sim 1\)GeV. Thus, by construction, EFT complies to the expectation of short distance insensitivity, provided that enough counter-terms encoding the unknown short distance physics are added in perturbation theory to a finite order \cite{13,14,15,16,17,18,19,20}. However, the physics of bound states, such as the deuteron, is genuinely non-perturbative and an infinite resummation of diagrams proves necessary (see however \cite{21} for a renewed perturbative set up). There arises the problem of non-perturbative renormalization and/or modifications of the original power counting for which no universally accepted scheme has been found yet, mainly because the requirements may vary. Thus, although the EFT method does comply to model independence it is far from trivial to achieve regulator independence and a converging pattern dictated by a reasonable power counting within a non-perturbative set up.

The NN renormalization problem may be cast efficiently in the traditional language of potentials and the corresponding Schrödinger equation. Besides phrasing the NN interaction into the familiar and more intuitive non-relativistic quantum mechanical framework, this procedure has also the advantage that the perturbative determination of the potential complies to the expected short distance insensitivity for the potential itself \cite{2,8,13,14,15,16,22}. The unconventional feature is that chiral expansions necessarily involve singular potentials at short distances, i.e. \(r^2 |V(r)| \to \infty\) for \(r \to 0\). Indeed, in the limit \(r \ll 1/m_N\) (or equivalently large momenta) pion mass effects are irrelevant in loop integrals and hence at some fixed order of the expansion one has for the coupled channel potential \(^1\)

\[ V(r) \to \frac{M_N c_{2n+m+r}}{(4\pi f_\pi)^{2n} M_N^2 \Delta^r r^{2n+m+r}}, \]

\textsuperscript{1}where \(n, m, r\) are nonnegative (≥ 0) integers, \(\Delta\) the N\(\Delta\) splitting, and \(c_k\), with \(k = 2n + m + r\), is a dimensionless matrix of van der Waals coefficients in coupled channel space. The dimensional argument is reproduced by loop calculations in the Weinberg dimensional power counting \cite{8,13,14,15,16,22,23,24} and is scheme

\(1\) The only exception is the singlet channel-One-Pion-Exchange (OPE) case which behaves as \(\sim m_N^2/\pi^2 r\), see below.
independent. It is thus conceivable that much of our understanding on the physics deduced from chiral potentials might be related to a proper interpretation of these highly singular potentials, which degree of singularity increases with the order of the expansion. This of course raises the question about in what sense higher order potentials are smaller. One obvious way to ensure this desired smallness is by keeping a finite and sufficiently moderate cut-off [22, 23, 24, 25, 26, 27, 28], so the singularity of the potential is not probed, effectively recovering the power counting expectations for the size of higher order contributions. The disadvantage is that results are strongly cut-off dependent for scales about \( r_c \sim 0.5 - 1.0 \text{ fm} \), similar to the ones probed in NN scattering [29, 30, 31]. However, this is not the only possibility. Explicit computations [29, 30, 31] show that, when the cut-off is removed, the higher order contributions will continue to generate only small changes in the amplitudes under specific circumstances which can be determined \emph{a priori}. As emphasized in previous works [24, 30, 31] renormalization is the most natural tool provided 1) we expect the potential to be realistic at long distances and 2) we want short distance details to be inessential in the description. As discussed above, this is precisely the situation we face most often in Nuclear Physics. A surprising and intriguing feature is that knowledge on the attractive or repulsive character of the singularity, i.e. the sign of the eigenvalues of \( c_k \) in Eq. (1), turns out to be crucial to successfully achieve this program and ultimately depends on the particular scheme or power counting used to compute the potential.

The singularities of chiral potentials may be disconcerting \(^2\), but they can be handled in a way that do not differ much from the standard treatment of well-behaved regular potentials which one usually encounters in nuclear physics \([33]\). Renormalization is the mathematical implementation of the appealing physical requirement of short distance insensitivity and hence a convenient tool to search for typical long distance model- and regulator independent results. In a non-perturbative setup such as the NN problem, renormalization imposes rather tight constraints on the interplay between the unknown short distance physics and the perturbatively computable long distance interactions \([29, 30, 31]\). This viewpoint provides useful insights and it is within such a framework that we envisage a systematic and model independent description of the NN force based on chiral interactions. In this regard it is amazing to note how the sophisticated machinery of perturbative renormalization in Quantum Field Theory used to compute the chiral potentials has not been so extensively developed when the inevitable non-perturbative physics must be incorporated; quite often the renormalization process is implemented by trial and error methods by adding counter-terms suggested by the a priori power counting in momentum space. However, detailed analyses in coordinate and momentum space \([29, 30, 31, 36, 37, 38]\) show that the allowed structure of counter-terms can be anticipated on purely analytical grounds and cannot be chosen independently on the long distance potential (see also Refs. \([39, 40]\) for numerical work). Specifically, for a channel-subspace with good total angular momentum the number of counter-terms is \( n(n+1)/2 \) with \( n \) the number of negative eigenvalues of the van der Waals matrix \( c_k \) in Eq. (1). In fact, the much simpler coordinate space renormalization has been shown to be fully equivalent to the popular momentum space treatment for purely contact theories \([41]\) and theories containing additional long range physics \([42, 43]\).

On the light of these latter studies the smallness of increasingly singular potentials such as Eq. (1) is triggered quite naturally by choosing the regular solution as \( u_p(r) \sim (4\pi f_\pi r)^{2m+1} \) (modulo prefactors depending on the attractive or repulsive character of the potential). Thus, as shown in Ref. \([43]\), when the short distance cut-off \( r_c \) approaches a fixed scale, \( \sim 1/(4\pi f_\pi) \), an increasing \( O(r_c^{m+1}/r^{2+1}) \) insensitivity of phase shifts and deuteron properties is guaranteed as the power of the singularity of the potential increases. Indeed, calculations with chiral Two Pion Exchange (TPE) potentials reproducing low energy NN data display this insensitivity for reasonable scales of \( r_c \sim 0.5 \text{ fm} \) \([29, 30]\) which correspond to the shortest wavelength probed by NN scattering in the elastic region. Within such a scenario, the discussion on whether one should remove the cut-off or not \([24, 30, 32, 44, 45]\) would become less relevant as the order of the chiral expansion is increased in the computations. This requires of course that the same renormalization conditions are implemented, in order for the computations to represent an specific physical situation, and that the cut-off lies in the stability region \( r_c \sim 0.5 \text{ fm} \), which in turn means that renormalization has effectively been achieved.

However, despite all these findings, the question on how a sensible hierarchy for NN interactions should be organized has been left open. Although we know \( \text{whether and, in positive case, how} \) this can be made compatible with the desired short distance insensitivity \([24, 30, 31]\), not all chiral potentials based on any given power counting are necessarily eligible. The Weinberg counting based in a heavy baryon approach at leading order (LO) \([11]\) for \( \hat{1}S_0 \) and \( \hat{3}S_1 - \hat{3}D_1 \) states turns out to be renormalizable \([29]\) due to an attractive-repulsive character of the coupled channel eigen potentials at short distances. There is at present no logical need why this ought to be so, for the simple reason that power counting does not anticipate the sign of the interaction at short distances. In fact when one goes to next-to-leading order (NLO) the short distance \( 1/r^5 \) singular repulsive character of the potential makes the deuteron unbound \([33]\) because the interaction becomes singular and repulsive.

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\(^2\) Early treatments can be found in \([33]\) (see Ref. \([34]\) for an early review).
Finally, next-to-next-to-leading order (NNLO) potentials diverge as \(-1/r^6\) and are, again, compatible with Weinberg counting in the deuteron \[30\], in this case because the interaction is singular and attractive. Further inconsistencies between Weinberg’s power counting and renormalization have been reported in Refs. \[32,42\]. The previous examples show clearly that the requirement of renormalizability can be in open conflict with the idea of a convergent pattern based on a preconceived chiral expansion and the mere dimensional power counting of the original proposal \[8\]. In particular, the assumption that the NN potential cannot be completely determined suggests further investigation since it appears as an obvious candidate where all necessary requirements for a convergent and short distance insensitive scheme might be met. In the present paper we check that the naive expectations based on the simple Eq. \[2\] are indeed verified for the long distance chiral potentials including \(\Delta\) degrees of freedom \[9,13,14,52\].

The delta-isobar has played a crucial role in the development of nuclear and particle physics (see e.g. \[53,54\] and references therein). Besides fixing the number of colours \(N_c = 3\) in QCD to comply with Pauli principle both at the quark as well as at the hadron level, this state can be clearly seen in \(\pi N\) scattering as a resonance, and it ubiquitously appears whenever the nucleon excitation energy is about the Delta-Nucleon splitting, \(\Delta = M_\Delta - M_N = 293\text{MeV}\). Already in the earliest implementations of Weinberg’s ideas the influence of Delta degrees of freedom was considered \[3,55,56\] in old-fashioned perturbation theory where energy dependent potentials are generated. Energy independent potentials have been obtained using the Feynman diagram technique a decade ago \[13,14\] and only recently the N2LO contributions have also been worked out in Ref. \[52\], where peripheral np phase shifts are computed in perturbation theory for these \(\Delta\) contributions. Finite cut-off calculations involving \(\Delta\)-degrees of freedom to NLO have been analyzed in momentum space \[23\]. It has been shown that the inclusion of the \(\Delta\)-excitation improves the convergence of the chiral expansion of the NN interaction \[53,58\]. The renormalization of the \(\Delta\)-NLO chiral potential has also been discussed recently in momentum space \[42\]. Short distance insensitivity has also been discussed in other contexts different from NN collisions such as \(\pi d\) scattering at threshold \[54\] where a rationale for the multiple scattering expansion was indeed supported by the chiral singular potentials.

Unfortunately, boost corrections being proportional to the average relative pn kinetic energy in the deuteron, \(\langle p^2 \rangle_M / M\), turned out to diverge due to the infinitely many short oscillations of the wave function. Recently, the benefits of including the \(\Delta\) in such a process are discussed in \[60\] where it is shown how a proper reorganization of the boost corrections not only makes them finite but also numerically small after renormalization in harmony with phenomenological expectations. The drawback is a proliferation of low energy constants when including the \(\Delta\), which can render the \(\Delta\)-full theory impractical at higher orders. The implications of a \(\Delta\)-based power counting for the three nucleon problem are analyzed in Refs. \[53,58\].

In the present paper we focus our interest in the crucial role played by the Delta-isobar in NN scattering in the elastic region from the point of view of renormalization of chiral nuclear forces, and how it might solve a long-standing problem. Besides an acceptable phenomenology in the s-wave phases we also show that precisely because of the built-in short distance insensitivity and unlike previous calculations where the \(\Delta\)-degrees of freedom were absent, the deuteron is always bound up to N2LO-\(\Delta\), the

\[3\] In the coupled channel case that means all eigen-channels being repulsive at short distances.
highest order computed at present \[52\].

The paper is organized as follows. In Section II we review the formalism as applied to the energy independent potentials in a ∆-full theory \[13, 14, 52\]. In Section III we discuss the simpler \(^1S_0\) channel including either one or two counter-terms by means of an energy dependent boundary condition. We analyze the deuteron bound state and its properties in Section IV. Scattering states in the \(^3S_1\) channel as well as the corresponding phase shifts are constructed by orthogonality to the deuteron and compared to the results obtained with the Nijmegen II potential \[6\], which has a \(\chi^2\) per datum near one and therefore can be considered as an alternative partial wave analysis, compatible with the original Nijmegen PWA \[3\]. The final expressions for the potential can be written as

\[
V_{NN}(\vec{x}) = V_C(r) + \tau W_C(r) + \sigma (V_S(r) + \tau W_S(r)) + S_{12} (V_T(r) + \tau W_T(r)),
\]

where spin-orbit and quadratic spin-orbit terms have been ignored as they are not present in the NLO-∆ and N2LO-∆ potentials from Ref. \[52\]. The operators \(\tau, \sigma\) and \(S_{12}\) are given by

\[
\tau = \vec{r}_1 \cdot \vec{r}_2 = 2t(t + 1) - 3, \\
\sigma = \vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2s(s + 1) - 3, \\
S_{12} = 3 \vec{\sigma}_1 \cdot \vec{r} \vec{\sigma}_2 \cdot \vec{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2,
\]

where \(\vec{r}_1(2)\) and \(\vec{\sigma}_1(2)\) represent the proton(neutron) isospin and spin operators; \(\vec{r}_1 \cdot \vec{r}_2\) and \(\vec{\sigma}_1 \cdot \vec{\sigma}_2\) are evaluated for total isospin \(t = 0, 1\) and total spin \(s = 0, 1\). Note that in Ref \[52\] the potential is local and there are no relativistic mass corrections at N2LO-∆. For states with good total angular momentum \(j\) and total spin \(s = 1\), the tensor operator reads

\[
S_{12}^j = \begin{pmatrix}
-\frac{2(j-1)}{2j+1} & 0 & 6\sqrt{j(j+1)} \\
0 & 2 & 0 \\
6\sqrt{j(j+1)} & 0 & -\frac{2(j+2)}{2j+1}
\end{pmatrix},
\]

where the matrix indices represent the orbital angular momentum \(l = j - 1, j, j + 1\). For \(s = 0\) states the tensor operator vanishes \(S_{12}^0 = 0\). We remind that Fermi-Dirac statistics implies \((-1)^{j+1+s} = -1\).

A remarkable feature which happens at LO, NLO-∆ and N2LO-∆ \[9, 28, 52\] is that the spin-orbit coupling vanishes, as well as any relativistic corrections. As a consequence, for a given isospin the potential can be diagonalized. The corresponding eigen-potentials depend just on the isospin of the channel and not on the total angular momentum \(j\). All the \(j\) dependence goes into the matrix which diagonalizes the potential. This can be seen in the following formulas:

\[
V^{ij} = (V_C + \tau W_C) - 3(V_S + W_S \tau) + S_{12}^i (V_T + \tau W_T),
\]

where \(V^{ij}\) is the triplet-channel potential written in matrix form, with

\[
A = (V_C + \tau W_C) - 3(V_S + W_S \tau), \\
B = (V_T + \tau W_T), \\
M_j = \begin{pmatrix}
\cos \theta_j & 0 & -\sin \theta_j \\
0 & 1 & 0 \\
\sin \theta_j & 0 & \cos \theta_j
\end{pmatrix},
\]

where the spin and isospin indices have not been explicitly written; \(M = 2M_pM_n/(M_p + M_n)\) is twice the reduced proton-neutron mass. In coordinate space the potential can be written as

\[
V_{NN}(\vec{x}) = V_C(r) + \tau W_C(r) + \sigma (V_S(r) + \tau W_S(r)) + S_{12} (V_T(r) + \tau W_T(r)),
\]


\footnote{Although the effect of keeping or ignoring the mass corrections is indeed small (see end of this section).}
TABLE I: van der Waals $M_C$ coefficients (in fm$^3$) for the different spin-isospin components of the NLO-$\Delta$ and N2LO-$\Delta$ potentials. We use the $\pi N$ motivated Fits 1 and 2 of Ref. [52]. Fit 1 involves the SU(4) quark-model relation $h_A = 3g_A/(2\sqrt{2}) = 1.34$ for $g_A = 1.26$.

| $M_C^{\text{NLO-\Delta}}$ | $M_C^{\text{NLO-\Delta}}$ | $M_C^{\text{NLO-\Delta}}$ | $M_C^{\text{NLO-\Delta}}$ | $M_C^{\text{NLO-\Delta}}$ |
|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| Fit-1                     | -7.233                    | -1.201                    | 0.601                     | 0.301                     |
| Fit-2                     | -3.867                    | -0.833                    | 0.417                     | 0.177                     |

| $M_C^{\text{N2LO-\Delta}}$ | $M_C^{\text{N2LO-\Delta}}$ | $M_C^{\text{N2LO-\Delta}}$ | $M_C^{\text{N2LO-\Delta}}$ | $M_C^{\text{N2LO-\Delta}}$ |
|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| Fit-1                     | -1.241                    | -0.489                    | 0.244                     | 0.351                     |
| Fit-2                     | -4.381                    | -1.121                    | 0.561                     | 0.439                     |

where

$$\cos \theta_j = \sqrt{\frac{j}{2j+1}}. \quad (9)$$

The transformation $M_f$ diagonalizes the full potential but not the Schrödinger equation, since it contains in addition to the potential the centrifugal barrier, which is a diagonal operator in the $j\ell$s basis, but does not remain diagonal in the rotated basis. Specific knowledge of the short distance behaviour is needed to carry on with the renormalization program. Generally, on purely dimensional grounds, we have for $r \to 0$

$$V_i(r) \to \frac{C_{V,i}^k}{r^k}, \quad W_i(r) \to \frac{C_{W,i}^k}{r^k}, \quad (10)$$

where

$$C_{k=2n+m+r+1}^i \sim \frac{1}{f_{2n}^2 M_N^m N^{r+1}}. \quad (11)$$

with $\Delta$ the $N\Delta$ splitting, and $n$, $m$ and $r$ nonnegative integers. At short distances, the angular momentum dependence may be neglected when the index $k > 2$. The relevant issue to carry out the renormalization procedure and to generate finite results is to know whether the interaction is attractive or repulsive. It turns out that both NLO-$\Delta$ and N2LO-$\Delta$ potentials have a leading singularity behaviour of $1/r^6$. In Appendix A we list the analytical expressions for the van der Waals coefficients $C_k^i$ for $k = 6$ (i.e. the leading singularity of the potential).

In our numerical calculations we take $f_{\pi} = 92.4$ MeV, $m_{\pi} = 138.03$ MeV, $2\mu_{\pi} = M_N = 2M_p M_n/(M_p + M_n) = 938.918$ MeV, $g_A = 1.29$ in the OPE piece to account for the Goldberger-Treiman discrepancy and $g_A = 1.26$ in the TPE piece of the potential. The corresponding pion nucleon coupling constant takes then the value $g_{\pi NN} = 13.083$ for the OPE piece of the potential. We use the $\Delta N$ splitting $\Delta = 293$ MeV. For $h_A$ and the low energy constants $c_1, c_2, c_3, c_4, b_4$ and $b_5$ we take the values from Fit 1 and 2 of Ref. [52] (Table I within that reference), where they are deduced from a fit to $\pi N$ threshold parameters in S- and P-waves to the data of Ref. [62]. These values are compatible with all $\pi N$ threshold parameters except $b_{0+}$ which is about twice its recommended value [62] at this level of approximation. Fit 1 uses the SU(4) quark-model relation $h_A = 3g_A/2\sqrt{2}$ (= 1.34 for $g_A = 1.26$), while Fit 2 uses $h_A = 1.05$.

The numerical values of the van der Waals coefficients $M_C^i$ are summarized in Table I for the different components of the NLO-$\Delta$ and N2LO-$\Delta$ potentials. The van der Waals coefficients are additive: therefore one can obtain the coefficient corresponding to some given partial wave by adding the individual contributions, i.e.

$$M_C = M_C^{6,V,C}(r) + \tau M_C^{6,W,C} + \sigma (M_C^{6,V,S} + M_C^{6,W,S}) + S_{12} (M_C^{6,V,T} + \tau M_C^{6,W,T}). \quad (12)$$

This is done for the singlet $^1S_0$ and the triplet $^3S_1 - ^3D_1$ channels in Table I.

We also present N2LO-$\Delta$ results for comparison purposes. For them we use the same parameters as in the N2LO-$\Delta$ computation, except for the $c_1, c_3$ and $c_4$ LECs, for which we use the following two different determinations: the so-called set IV of Ref. [34], which was obtained in Ref. [24] by fitting to NN scattering data, and the values from Ref. [52] for N2LO-$\Delta$ which we refer to as set $\pi N$ and allow a better comparison with the N2LO-$\Delta$ computations presented here, as they are also fitted to reproduce $\pi N$ S- and P-wave threshold parameters. Due to the different fitting procedures, any direct comparison between the results of Ref. [34], i.e. set IV, and the N2LO-$\Delta$ results should be done with care. It should be mentioned too that the results of Ref. [34] contain 1/M corrections to the potential. These corrections are nevertheless small and, if excluded, would only induce small differences in the results of Ref. [34].

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5 Strictly speaking $g_A = 1.26$ both in the OPE and TPE pieces of the potential, but it happens that at NLO and higher orders the OPE piece receives a contribution from the $d_{18}$ LEC, related to the Goldberger-Treiman discrepancy, see, for example, the expressions of Ref. [24] for details. This is equivalent to consider the original expression for the OPE potential, but taking $g_A = 1.29$ instead of $g_A = 1.26$.

6 As an illustration, ignoring the 1/M corrections will yield to
### III. THE SINGLET CHANNEL

#### A. Equations and boundary conditions

The $^1S_0$ wave function in the pn center-of-mass (c.m.) system can be written as

$$\Psi(\vec{x}) = \frac{1}{\sqrt{4\pi r}} u(r) \chi_{pm}^{m_s} \, ,$$  \hspace{1cm} (13)

with the total spin $s = 0$ and $m_s = 0$. The function $u(r)$ is the reduced $S$-wave function, and satisfies the following reduced Schrödinger equation

$$-u''(r) + U_{1S_0}(r) u(r) = k^2 u(r),$$  \hspace{1cm} (14)

where $k$ is the center-of-mass momentum, and $U_{1S_0}$ the reduced potential defined as

$$U_{1S_0}(r) = M \left( V_C(r) + W_C(r) - 3V_S(r) - 3W_S(r) \right).$$  \hspace{1cm} (15)

For a finite energy scattering state we solve for the chiral potential with the asymptotic normalization

$$u_k(r) \to \frac{\sin(kr + \delta_0(k))}{\sin\delta_0(k)},$$  \hspace{1cm} (16)

with $\delta_0(k)$ the phase shift. For a potential falling off exponentially $\sim e^{-m_s r}$ at large distances, we have the effective range expansion, valid for momenta $|k| < m_s/2$,

$$k \cot \delta_0(k) = \frac{1}{\alpha_0} + \frac{1}{2} r_0 k^2 + v_2 k^4 + \ldots$$  \hspace{1cm} (17)

with $\alpha_0$ the scattering length and $r_0$ the effective range. At short distances the NN chiral potential behaves as

$$U_{1S_0}(r) \to MC_{6,1}^{\text{NLO}-\Delta} \frac{r^6}{r^6} = -R^4_{\text{c}},$$  \hspace{1cm} (18)

where

$$MC_{6,1}^{\text{NLO}-\Delta} = MC_{6,1}^{\text{NLO}-\Delta} + MC_{6,1}^{\text{N2LO}-\Delta},$$  \hspace{1cm} (19)

which is a van der Waals type interaction. The numerical values for $MC_{6,1}^{\text{NLO}-\Delta}$ and $MC_{6,1}^{\text{N2LO}-\Delta}$ are listed in Tab. II. The value of the coefficient is negative, with the typical length scale $R = (MC_6)^{1/4}$. The solution at short distances is of oscillatory type:

$$u_k(r) \to A \left( \frac{r}{R} \right)^{3/2} \sin \left[ \frac{1}{2} \left( \frac{R}{r} \right)^2 + \varphi \right],$$  \hspace{1cm} (20)

where $A$ is a normalization constant and $\varphi$ an undetermined phase, which may in principle depend on energy.

We will present below two calculations. In the first one the renormalization is carried out with one counter-term, which in turn means one renormalization condition. In such a case the short distance phase becomes energy independent and orthogonality conditions are satisfied. In the second calculation we proceed with two counter-terms, i.e. two renormalization conditions, for which the short distance phase acquires a very specific energy dependence.

#### B. Renormalization with one counter-term

As mentioned, the phase shift is determined from Eq. (16), but to fix the undetermined phase $\varphi$ we impose orthogonality for $r > r_c$ between the zero energy state and the state with momentum $p$. As shown in [30], orthogonality turns out to be equivalent to the following condition between 0- and k-momentum reduced wave functions at $r = r_c$

$$u_k'(r_c) u_0(r_c) - u_0'(r_c) u_k(r_c) = 0.$$  \hspace{1cm} (21)

Taking the limit $r_c \to 0$ implies that the short distance phase $\varphi$ is energy independent [31]. Thus, for the zero energy state we solve

$$-u_0''(r) + U_{1S_0}(r) u_0(r) = 0,$$  \hspace{1cm} (22)

the following modifications for the results of Ref. [31] (original results in parentheses): $A_2 = 0.887(0.884) \text{ fm}^{-1/2}$, $r_m = 1.972(1.967) \text{ fm}$, $Q_d = 0.278(0.276) \text{ fm}^2$, $P_D = 8(8) \%$, $(r^{-1}) = 0.442(0.447) \text{ fm}^{-1}$, $(r^{-2}) = 0.276(0.284) \text{ fm}^{-2}$.

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7 It should be noted that the NLO-\Delta potential is less singular than the NLO-\Delta one at short distances, diverging as $1/r^3$ only.
with the asymptotic normalization at large distances
\[ u_0(r) \rightarrow 1 - \frac{r}{\alpha_0}, \quad (23) \]
where \( \alpha_0 \) is the scattering length. In this equation \( \alpha_0 \) is an input, so one needs to integrate Eq. (22) from infinity to the origin (in contrast with the usual procedure of integrating from the origin to infinity). The effective range, defined as
\[ r_0 = 2 \int_0^\infty dr \left[ \left(1 - \frac{r}{\alpha_0}\right)^2 - u_0(r)^2 \right], \quad (24) \]
can be computed. Due to the superposition principle, we can write the zero momentum wave function as the following linear combination
\[ u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r), \quad (25) \]
where \( u_{0,c}(r) \to 1 \) and \( u_{0,s}(r) \to r \) correspond to cases where the scattering length is either infinity or zero respectively. Using this decomposition, we get
\[ r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2}, \quad (26) \]
where \( A, B \) and \( C \), defined as:
\[ A = 2 \int_0^\infty dr (1 - u_{0,c}^2), \quad (27) \]
\[ B = -4 \int_0^\infty dr (r - u_{0,c} u_{0,s}), \quad (28) \]
\[ C = 2 \int_0^\infty dr (r^2 - u_{0,s}^2), \quad (29) \]
depend on the potential parameters only. The interesting thing is that all dependence on the scattering length \( \alpha_0 \)

FIG. 1: Renormalized phase shifts for the LO, NLO-\( \Delta \) and N2LO-\( \Delta \) potentials as a function of the c.m. np momentum \( k_{\text{c.m.}} \) in the \( 1S_0 \) singlet channel, compared to the Nijmegen II potential results [6]. The computations are done both with \( h_A = 1.34 \) (upper panels) and with \( h_A = 1.05 \) (lower panels). For the LO result, the calculation is always done with one counterterm, while for the NLO and N2LO results, computations are done with one counter-term (left panels) and two counter-terms (right panels), corresponding this last case to the standard Weinberg counting for the \( 1S_0 \) channel.
TABLE III: Predicted threshold parameters in the singlet $^1S_0$ channel with one counter-term for Fits 1 and 2 of Ref. [32]. We compare our renormalized results given by the cut-off independent universal formula (29) for $r_0$ and its extension for $v_2$ to finite cut-off NN calculations using their scattering length as an input. The experimental values for the scattering length and effective range are taken from Ref. [3].

|            | Calculation | $\alpha_0$ (fm) | $r_0$ (fm) | $v_2$ (fm$^3$) |
|------------|-------------|-----------------|------------|----------------|
| LO        | Ref. [30]   | Input           | 1.44       | -2.11          |
| NLO-Δ     | Ref. [30]   | This work       | 2.29       | -1.02          |
| NLO-Δ ($h_A = 1.34$) | This work | Input           | 2.91       | -0.32          |
| NLO-Δ ($h_A = 1.05$) | This work | Input           | 2.76       | -0.53          |
| N2LO-Δ (Set IV) | Ref. [30] | This work       | 2.87       | -0.38          |
| N2LO-Δ (πN)  | This work   | Input           | 3.05       | -0.12          |
| N2LO-Δ (Fit 1) | This work | Input           | 2.92       | -0.31          |
| N2LO-Δ (Fit 2) | This work | Input           | 3.04       | -0.13          |
| Nijm H     | Ref. [5, 6] | -23.73          | 2.67       | -0.48          |
| Reid 93    | Ref. [5, 6] | -23.74          | 2.75       | -0.49          |
| Exp.       | -           | -23.74(2)       | 2.77(5)    | -              |

is displayed explicitly by Eq. (20). Numerically we get

\[
\begin{align*}
 r_0 &= 2.661 - \frac{5.707}{\alpha_0} + \frac{5.988}{\alpha_0^2} \quad (\text{NLO}-\Delta, h_A = 1.34), \\
 r_0 &= 2.517 - \frac{5.430}{\alpha_0} + \frac{5.811}{\alpha_0^2} \quad (\text{NLO}-\Delta, h_A = 1.05), \\
 r_0 &= 2.789 - \frac{5.992}{\alpha_0} + \frac{6.187}{\alpha_0^2} \quad (\text{N2LO-\Delta}, \text{Fit 1}), \\
 r_0 &= 2.780 - \frac{5.969}{\alpha_0} + \frac{6.171}{\alpha_0^2} \quad (\text{N2LO-\Delta}, \text{Fit 2}).
\end{align*}
\]

The corresponding numerical values when the experimental $\alpha_0 = -23.74$ fm is taken, as well as the $v_2$ parameter can be looked up in Table III. As can be seen, the value of the effective range has a clear convergence pattern when going from LO to NLO-Δ. The contribution coming from N2LO-Δ is very small compared to the NLO-Δ contribution, in agreement with the findings of Ref. [32]. This is in contrast with the Δ-less theory, for which N2LO generates a great correction over the NLO results. Unfortunately the N2LO-Δ results converge to wrong values, about 3 fm for both Fit 1 and 2. The same happens for the N2LO-Δ results, although with a weaker convergence pattern. The reason for this discrepancy is that the NLO and N2LO potentials, both in Δ-less and Δ-full theories, are too attractive at intermediate distances, thus yielding a bigger value for $r_0$ than the one obtained with phenomenological potentials like Nijmegen II or Reid93 [31, 42].

Using the orthogonality condition, Eq. (21), the phase-shift can be determined from the scattering length and the potential as independent parameters when the limit $r_c \to 0$ is taken. The renormalized phase shift is presented in Fig. 11 (left). The phase shifts are compared with the ones obtained from the Nijmegen II potential [6], which are compatible with the Nijmegen PWA [5]. As we see the trend in the effective range $r_0$ and the $v_2$ parameter is reflected in the behavior of the phase shift. In Ref. [42] a similar calculation was carried out in momentum space with inclusion of Δ degrees of freedom at NLO and one counter-term. The present NLO-Δ coordinate space results agree with that calculation when the SU(4) relation, $h_A = 1.34$, is used. As discussed in the previous paragraph for the effective range parameters, at N2LO-Δ the phase shifts have already converged, although to the wrong value, due to the excessive strength of the intermediate range of the potential. In the next section we will see how the situation changes when an extra counter-term is included in the computations. This counter-term will be fitted to reproduce the effective range.

C. Renormalization with two counter-terms

In the standard Weinberg counting both the long distance potential as well as the short distance potential are dimensionally expanded in momentum space. For the short range piece, one writes $V_S(p', p) = C_0 + C_2(p^2 + p'^2) + \ldots$, where $C_0$, $C_2$, etc, are referred to as counter-terms. In Ref. [42] the interrelation between the counter-terms and short distance boundary conditions in coordinate space has been discussed at length. At LO in the Weinberg counting one has only one counter term. This corresponds to the situation described in the previous section. At NLO and N2LO in the Weinberg counting (both in the Δ-less and Δ-full cases), one adds two counter-terms $C_0$ and $C_2$ for a fixed momentum space cut-off $\Lambda$, which may be fixed by reproducing the scattering length $\alpha_0$ and the effective range $r_0$. The interesting finding was that such a procedure does not generate a converging $^1S_0$ phase shift in the limit $\Lambda \to \infty$ [42]. Thus, the momentum space polynomial parameterization of Weinberg counting is not compatible with renormalization. However, this does not necessarily mean that one cannot impose two renormalization conditions to fix the scattering length and the effective range. Actually, on a wider perspective one may pose on the one hand the problem of obtaining a finite phase shift embodying the chiral potential and on the other the problem of fixing both $\alpha_0$ and $r_0$ as independent parameters. Fortunately, there exists a unique procedure in coordinate space meeting the two previous conditions and yielding converging phase shifts, which we apply below to discuss the $^1S_0$ channel when two counter-terms are considered for NLO-Δ and N2LO-Δ chiral potentials. We will refer to this scheme as Weinberg counting with Δ.

The equivalent coordinate space procedure [37, 42] consists of expanding the wave function in powers of the energy

\[
u_k(r) = u_0(r) + k^2 u_2(r) + \ldots
\]
where $u_0(r)$ and $u_2(r)$ satisfy the following equations,
\begin{align}
-u_0''(r) + U_{1S_0}(r)u_0(r) &= 0, \quad u_0(r) \xrightarrow{r \to \infty} 1 - \frac{r}{\alpha_0}, \tag{32}
-u_2''(r) + U_{1S_0}(r)u_2(r) &= u_0(r), \quad u_2(r) \xrightarrow{r \to \infty} \left(\frac{r^3 - 3\alpha_0 r^2 + 3\alpha_0 r_0 r}{6\alpha_0}\right). \tag{33}
\end{align}

The asymptotic conditions correspond to fixing $\alpha_0$ and $r_0$ as independent parameters (two counter-terms). The matching condition at the boundary $r = r_c$ becomes energy dependent \cite{36}
\begin{align}
\frac{u_k'(r_c)}{u_k(r_c)} &= \frac{u_0'(r_c) + k^2u_2'(r_c) + \ldots}{u_0(r_c) + k^2u_2(r_c) + \ldots}, \tag{34}
\end{align}

whence the corresponding phase shift may be deduced by integrating in Eq. (32) and Eq. (33) and integrating out Eq. (14) with Eq. (16). It is worth mentioning that the energy dependent matching condition, Eq. (34), is quite unique since this is the only representation for the boundary condition guaranteeing the existence of the limit $r_c \to 0$ for singular potentials \cite{36}. As pointed out in Refs. \cite{36,42}, polynomial expansions in $k^2$ such as suggested e.g. in the Nijmegen PW A \cite{5}, are implemented later on for chiral TPE potentials \cite{16}, do not work for $r_c \to 0$, and in fact generate undesired oscillations for $r_c \ll 1.4$ fm in the phase-shifts. The validity of these features can be deduced analytically in coordinate space as a consequence of the RG-invariance of a Moebius bilinear transformation \cite{36}. Equivalent parameterizations in momentum space may likely exist, but are so far unknown. As already mentioned, the widely used polynomial representation of short distance interactions in momentum space $V_S(p', p) = C_0 + C_2(p^2 + p'^2) + \ldots$ of standard NLO and NNLO Weinberg counting implies that for $\Lambda \to \infty$ either $C_2$ is irrelevant when only $\alpha_0$ is kept fixed or the phase shift does not converge when both $\alpha_0$ and $r_0$ are simultaneously fixed \cite{36,42}.

In Fig. 1 (right panel) we show the results for the two counter-term renormalized phase shift at NLO-$\Delta$ and N2LO-$\Delta$. As we see, the second counter-term is responsible for a certain improvement over the elastic region, in particular in the region $k < m_\pi$ where TPE effects are expected to dominate in the singlet $^1S_0$-channel \footnote{In principle for $k < m_\pi$ OPE (instead of TPE) effects should dominate, but in the case of the $^1S_0$ singlet channel this is not the case due to the weakness of OPE in this channel. This only happens in the singlet channel; in the case of the triplet channel we recover the naive expectations, and OPE clearly dominates for $k < m_\pi$.}.

At higher momenta, however, the discrepancy with the Nijmegen II potential results \cite{6} and therefore with the Nijmegen PW A \cite{5} persists. The second counter-term is therefore unable to provide the necessary repulsion to compensate for the excessive intermediate range attraction present in the NLO-$\Delta$ and N2LO-$\Delta$ potentials. This result agrees with the findings of Ref. \cite{42} and extends them from NLO-$\Delta$ to N2LO-$\Delta$. Possible solutions to this disturbing situation include finite cut-off computations, which help to increase the effect of the second counter-term, and the inclusion of spectral function regularization, proposed in Ref. \cite{29} precisely to treat the problem of the intermediate strength of the N2LO-$\Delta$ potential in peripheral partial waves. The case of spectral function regularization will be discussed in Section VI.

IV. THE TRIPLET CHANNEL

A. Equations and boundary conditions

The $^3S_1 - ^3D_1$ wave function in the pn c.m. system can be written as
\begin{align}
\Psi(\vec{x}) &= \frac{1}{\sqrt{4\pi r}} \left[u(r)\sigma_p \cdot \sigma_n + \frac{w(r)}{\sqrt{8}} (3\sigma_p \cdot \vec{x}\sigma_n \cdot \vec{x} - \sigma_p \cdot \sigma_n)\right] \chi_{pn}^{s_m}, \tag{35}
\end{align}

with total spin $s = 1$ and $m_s = 0, \pm 1$; $\sigma_p$ and $\sigma_n$ are the Pauli matrices for the proton and the neutron respectively. The functions $u(r)$ and $w(r)$ are the reduced S- and D-wave components of the relative wave function respectively. They satisfy the coupled set of equations in the $^3S_1 - ^3D_1$ channel
\begin{align}
-u''(r) + U_{3S_1}(r)u(r) + U_E(r)w(r) &= k^2u(r),
-w''(r) + U_{3D_1}(r)u(r) + \left[U_{3D_1}(r) + \frac{6}{r^2}\right]w(r) &= k^2w(r), \tag{36}
\end{align}

with $U_{3S_1}(r)$, $U_E(r)$ and $U_{3D_1}(r)$ the corresponding matrix elements of the coupled channel potential, which are
\begin{align}
U_{3S_1} &= V_C - 3W_C + V_S - 3W_S, \\
U_E &= 2\sqrt{2}(V_T - 3W_T), \\
U_{3D_1} &= V_C - 3W_C + V_S - 3W_S - 2V_T + 6W_T. \tag{37}
\end{align}

At short distances one has
\begin{align}
U_{3S_1} &\xrightarrow{r \to 0} \frac{MC_{6,3,S_1}}{r^6}, \\
U_E &\xrightarrow{r \to 0} \frac{MC_{6,3}}{r^6}, \\
U_{3D_1} &\xrightarrow{r \to 0} \frac{MC_{6,3,D_1}}{r^6}, \tag{38}
\end{align}

where the van der Waals coefficients are given by
\begin{align}
C_{6,3,S_1} &= C_{6,3}^{NLO} + C_{6,3}^{N2LO}, \\
C_{6,3,E} &= C_{6,3}^{NLO} + C_{6,3}^{N2LO}, \\
C_{6,3,D_1} &= C_{6,3}^{NLO} + C_{6,3}^{N2LO}. \tag{39}
\end{align}
Their numerical values are listed in Table II for Fits 1 and 2 of Ref. [52]. One can diagonalize the corresponding matrix of van der Waals coefficients

\[
\begin{pmatrix}
MC_{6,3S_{1}} & MC_{6,E_{1}} \\
MC_{6,E_{1}} & MC_{6,3D_{1}}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
-R_{+}^{1} & 0 \\
0 & -R_{-}^{1}
\end{pmatrix} \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
\]

(40)

where, according to Eq. (11) the angle is

\[\cos \theta = \frac{1}{\sqrt{3}},\]

(41)
i.e. \(\theta = 54.7^\circ\) and common to both the NLO-\(\Delta\) and N2LO-\(\Delta\) potentials. That means that the eigenvalues of the NLO-\(\Delta\) and N2LO-\(\Delta\) van der Waals matrices are additive, and therefore can be summed up directly from Table I

\[-R_{\pm}^{1} = -R_{\pm,\text{NLO}}^{1} - R_{\pm,\text{N2LO}}^{1}.
\]

As we see from Table II the NLO-\(\Delta\) matrix is negative definite, but N2LO-\(\Delta\) matrix is not so for Fit 1. However, what counts is the NLO-\(\Delta +\) N2LO-\(\Delta\) contribution which, as can be checked, is negative definite. In the diagonal basis one has at short distances

\[
\begin{pmatrix}
u
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
v_{+} \\
v_{-}
\end{pmatrix},
\]

(43)

where the short distance eigen functions are

\[
v_{+}(r) = \left(\frac{r}{R_{+}}\right)^{\frac{3}{2}} C_{+} \sin \left[\frac{1}{2} \frac{R_{+}^{2}}{r^{2}} + \varphi_{+}\right]
\]

\[
v_{-}(r) = \left(\frac{r}{R_{-}}\right)^{\frac{3}{2}} C_{-} \sin \left[\frac{1}{2} \frac{R_{-}^{2}}{r^{2}} + \varphi_{-}\right],
\]

(44)

and \(\varphi_{\pm}\) are short distance phases which must be fixed independently on the potential and \(C_{\pm}\) suitable normalization constants. Orthogonality of solutions of different energy requires these phases to be energy independent. Following the procedure of Ref. [30] we fix them from deuteron physical properties, namely the binding energy and asymptotic D/S ratio (see below). Once these two quantities are fixed, scattering states can be completely determined by fixing in addition the scattering length of the \(3S_{1}\) phase, and then imposing orthogonality to the deuteron state. This is equivalent to renormalizing with three counter-terms in momentum space [13]. Of course, more counter-terms could be considered if orthogonality is given up by an energy dependent boundary condition, as done above for the \(1S_{0}\) channel.

In the following two subsections we will consider the description of the deuteron bound state and the scattering states and discuss our results.

B. The Deuteron

In the case of negative energy we consider Eq. (36) with

\[k^{2} = -\gamma^{2} = -MB_{d},\]

(45)

with \(\gamma\) the deuteron wave number and \(B_{d}\) the deuteron binding energy. We solve Eq. (36) together with the asymptotic condition at infinity

\[u(r) \rightarrow A_{S}e^{-\gamma r},\]

\[w(r) \rightarrow A_{D}e^{-\gamma r}\left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^{2}}\right),\]

(46)

where \(A_{S}\) and \(A_{D}\) are the s- and d-wave normalization factors. The asymptotic D/S ratio parameter \(\eta\) is defined as \(\eta = A_{D}/A_{S}\).
\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Set        & $\gamma$ (fm$^{-1}$) & $\eta$ & $A_S$ (fm$^{-1/2}$) & $r_m$ (fm) & $Q_d$(fm$^2$) & $P_D$ (%) & $\langle r^{-1}\rangle$ & $\langle r^{-2}\rangle$ \\
\hline
LO         & Input               & 0.2633 & 0.8681(1) & 1.9351(5) & 0.2762(1) & 7.31(1) & 0.486 (1) & 0.434(3) \\
NLO-$\Delta$ & Unbound            & –      & –        & –        & –        & –        & –        & –        \\
NLO-$\Delta$ ($h_A = 1.34)$ & Input & Input & 0.884(3) & 1.963(7) & 0.274(9) & 5.9(4) & 0.446(10) & 0.29(2) \\
NLO-$\Delta$ ($h_A = 1.05)$ & Input & Input & 0.84(4) & 1.86(8) & 0.24(3) & 12(5) & 0.62(15) & 0.84(4) \\
N2LO-$\Delta$ (Set IV) & Input & Input & 0.884(4) & 1.967(6) & 0.276(3) & 8(1) & 0.447(5) & 0.284(8) \\
N2LO-$\Delta$ ($\pi N$) & Input & Input & 0.896(2) & 1.990(3) & 0.282(5) & 6.1(8) & 0.4287(13) & 0.253(2) \\
N2LO-$\Delta$ (Fit 1) & Input & Input & 0.892(2) & 1.980(4) & 0.279(5) & 5.9(9) & 0.4336(15) & 0.262(5) \\
N2LO-$\Delta$ (Fit 2) & Input & Input & 0.890(2) & 1.975(3) & 0.278(5) & 5.8(9) & 0.4470(15) & 0.268(2) \\
NijmII      & 0.231605             & 0.02521 & 0.8845 & 1.9675 & 0.2707 & 5.635 & 0.4502 & 0.2868 \\
Reid93      & 0.231605             & 0.02514 & 0.8845 & 1.9686 & 0.2703 & 5.699 & 0.4515 & 0.2924 \\
Exp.        & 0.231605             & 0.0256(4) & 0.8838(4) & 1.971(5) & 0.2860(15) & –        & –        & –        \\
\hline
\end{tabular}
\caption{Deuteron properties for the OPE and TPE potentials. The computation is made by fixing $\gamma$ and $\eta$ to their experimental values. The errors quoted in both TPE computations reflect the uncertainty in the non-potential parameters $\gamma$, $\eta$, and $\alpha_0$ only. For the OPE (LO) we take $g_{NN} = 13.1(1)$. For the LEC’s in the TPE calculation, by set IV we refer to the determination of Ref.\cite{24} (see main text). For the $\Delta$ case we use Fits 1 and 2 of Ref.\cite{52}. Fit 1 involves the SU(4) quark-model relation $h_A = 3g_A/(2\sqrt{2}) = 1.34$ for $g_A = 1.26$. Fit 2 takes $h_A = 1.05$. The experimental values are taken from the following references: $\eta$ from \cite{63}, $A_S$ from \cite{64}, $r_m$ from \cite{65} and $Q_d$ from \cite{66} (see also Ref. \cite{67} for a brief review).}
\end{table}

In order to obtain the regularized wave functions, we fix $\gamma$ and $\eta$ to their experimental values (see Table IV), and obtain $u(r)$ and $w(r)$ by integrating Eq. (50) from $r \to \infty$ to $r = 0$ with Eq. (19) as boundary conditions. $A_S$ can be later determined from

$$
\int_0^\infty dr \left[u(r)^2 + w(r)^2\right] = 1, \quad (47)
$$
i.e., from demanding the deuteron normalization to be equal to unity. The renormalized deuteron wave functions are depicted in Fig. 2 at LO, NLO-$\Delta$ and N2LO-$\Delta$, and compared to the Nijmegen II wave functions \cite{67}. The asymptotic normalization $u \to e^{-\gamma r}$ has been adopted and the asymptotic D/S ratio is taken to be $\eta = 0.0256$. One can see in Fig. 2 the appearance of an increasing number of oscillations in the wave function when the radius approaches zero. They have no appreciable effect on the physics of the deuteron, as they happen at very short distances. Therefore they have a very small effect on the computation of deuteron observables and cannot be resolved by external probes at the energies for which the effective theory description is valid. This latter point is shown explicitly in Ref. \cite{43} for elastic electron-deuteron scattering. Perhaps the most remarkable aspect of the present calculation is the convergence of the proposed scheme when the $\Delta$-resonance is considered, which of course implies that the deuteron is bound at all computed orders of the potential. This is in contrast with the delta-less theory, where at NLO-$\Delta$ the deuteron became unbound (as discussed extensively in Ref. \cite{51}).

Here, we also compute the matter radius, which reads,

$$
r_m^2 = \frac{\langle r^2 \rangle}{4} = \frac{1}{4} \int_0^\infty r^2(u(r)^2 + w(r)^2)dr, \quad (48)
$$
the quadrupole moment (without meson exchange currents)

$$
Q_d = \frac{1}{20} \int_0^\infty r^2 w(r)(2\sqrt{2}u(r) - w(r))dr, \quad (49)
$$
the D-state probability

$$
P_D = \int_0^\infty w(r)^2 dr, \quad (50)
$$
and the inverse moments of the radius

$$
\langle r^{-n}\rangle = \int_0^\infty r^{-n}(u(r)^2 + w(r)^2)dr, \quad (51)
$$
which, as is well known, appear in the multiple scattering expansion of the $\pi$-deuteron scattering length. Some results for the inverse moments in $\Delta$-less effective field theory can be found in Refs. \cite{53, 55}. In Tab. IV we show our results in a variety of situations. In general, the results for NLO-$\Delta$ and N2LO-$\Delta$ are in agreement with the experimental value of the deuteron observables, with the exception of the quadrupole moment, which presents a discrepancy of 0.01 fm$^2$. The reason for it lies in meson exchange current contributions to the quadrupole moment, which were estimated for ChPT in Ref. \cite{69}, and are of the order of the difference of our results with respect to the experimental value. When comparing the $\Delta$-full computations with the $\Delta$-less ones, one can notice, in the first place, that a renormalized result exists for NLO-$\Delta$, unlike the NLO-$\Delta$ case (as was explained in more detail in Ref. \cite{51}). This supports the better convergence properties of effective theory when including the $\Delta$ degree of freedom. It can also be noticed that the D-wave probability $P_D$, although not an observable, is better described in N2LO-$\Delta$ than in N2LO-$\Delta$, when compared with the results coming from the phenomenological potentials \cite{65}. The consequence is that the magnetic moment of the deuteron would be better described.
in the $\Delta$-full theory than in the $\Delta$-less theory, although no actual computation has been made on that respect in the present paper. The reason for that is that in a non-relativistic framework, the deuteron magnetic moment depends solely on $P_D$ \cite{70}. Before comparing the results for other observables, one important comment must be made: one should only directly compare the N2LO-$\Delta$ results with the N2LO- $\Delta$ ($\pi N$) ones. Comparison with N2LO-$\Delta$ set IV should be done with care, as for this case the LECs are fitted to reproduce NN data \cite{24}. This is why they look slightly better than the other results of Table [IV]. Therefore, the comparison of N2LO- $\Delta$ with N2LO- $\Delta$ ($\pi N$) results implies in particular that the inclusion of $\Delta$ enhances the compatibility between $\pi N$ and $NN$ scattering, as one would expect within the EFT philosophy.

C. Phase Shifts

Finally, in the case of positive energy we consider Eq. \cite{36} with

$$E_{c.m.} = \frac{k^2}{M}$$ \hspace{1cm} (52)

where $k$ is the corresponding c.m. momentum. We solve Eq. \cite{36} for the two linear independent scattering states, which are usually labelled as the $\alpha$ and $\beta$ states. They are defined by the asymptotic normalization

$$u_{k,\alpha}(r) \rightarrow \cos \left( j_0 (kr) \cot \delta_1 - \hat{y}_0 (kr) \right),$$

$$w_{k,\alpha}(r) \rightarrow \sin \left( j_2 (kr) \cot \delta_1 - \hat{y}_2 (kr) \right),$$

$$u_{k,\beta}(r) \rightarrow -\sin \left( j_0 (kr) \cot \delta_2 - \hat{y}_0 (kr) \right),$$

$$w_{k,\beta}(r) \rightarrow \cos \left( j_2 (kr) \cot \delta_2 - \hat{y}_2 (kr) \right),$$ \hspace{1cm} (53)

where $\hat{j}_1(x) = xj_1(x)$ and $\hat{y}_1(x) = xy_1(x)$ are the reduced spherical Bessel functions and $\delta_1$ and $\delta_2$ are the eigenphases in the $^3S_1$ and $^3D_1$ channels; $\epsilon$ is the mixing angle $E_1$. The orthogonality constraints between the deuteron and scattering states generate the following boundary conditions

$$u_{\gamma} u'_{k,\alpha} - u'_{\gamma} u_{k,\alpha} + w_{\gamma} w'_{k,\alpha} - w'_{\gamma} w_{k,\alpha} \bigg|_{r=r_c} = 0,$$

$$u_{\gamma} u'_{k,\beta} - u'_{\gamma} u_{k,\beta} + w_{\gamma} w'_{k,\beta} - w'_{\gamma} w_{k,\beta} \bigg|_{r=r_c} = 0.$$ \hspace{1cm} (55)

The use of the superposition principle for the $\alpha$ and $\beta$ scattering states, plus the deuteron wave functions, allows to deduce the corresponding $^3S_1 - ^3D_1$ eigen phase-shifts. The results are depicted in Fig. [3] at LO, NLO-$\Delta$ and N2LO-$\Delta$ compared to the Nijmegen II potential results \cite{6}. We observe a clear improvement in the threshold region, and quite remarkably for the $E_1$ mixing phase. However, there is no improvement in the $^3D_1$ phase-shift.

V. COMPARISON WITH THE DELTA-LESS THEORY

In our previous work \cite{30}, the renormalization of Delta-less theory was analyzed. As discussed above, it was found that at NLO-$\Delta$ there was no deuteron bound state if the cut-off was removed. The reason was due to the short distance $\sim g_A^4/(f^2\rho^2)$ repulsive singular character of the potential in the $^3S_1 - ^3D_1$ channel. However, at N2LO-$\Delta$ three counter-terms where needed due to the short distance attractive singular character of the potential. In fact the agreement with more sophisticated calculations \cite{24} was remarkable. More recently, the quality of these chiral wave functions has been tested in electron-deuteron scattering \cite{43} using LO currents, with an amazingly good agreement up to momentum transfer of $q = 1$GeV. It is interesting to compare the results of the Delta-less theory \cite{30} with the ones found here after inclusion of the $\Delta$ in the potential. The first aspect to note is that the NLO-$\Delta$ potential has an attractive $\sim g_A^4/(\Delta f^4\rho^6)$ short distance singularity, a feature kept at the N2LO-$\Delta$, however with different scales.

In Fig. [4] the np $^1S_0$ singlet phase shifts are depicted as a function of the c.m. momentum for the N2LO-$\Delta$ and N2LO-$\Delta$ compared to the Nijmegen II potential phase shifts [6]. As we see, with only one counter-term, i.e. fixing the scattering length $a_0$, the result is slightly worsened when the $\Delta$ is included. This is consistent with the change in the effective range reported in Table [III]. Of course, if $r_0$ is fixed as an additional renormalization condition, there is an improvement in the low energy region but the difference between including or not $\Delta$ degrees of freedom is hardly visible.

The situation for the deuteron wave functions is slightly different. As we see in Fig. [5] the present N2LO-$\Delta$ deuteron wave functions resemble slightly better the Nijmegen II ones \cite{5,6}. In particular, the D-wave becomes smaller when one goes from our previous TPE ones \cite{30}. Finally, in Fig. [6] we show the np eigen phase shifts in the $^3S_1 - ^3D_1$ channel as a function of the c.m. momentum for the N2LO-$\Delta$ and N2LO-$\Delta$ compared to the Nijmegen II potential results \cite{6}. The noticeable improvement in the $E_1$ phase is in agreement with the smaller $D$-wave deuteron wave function.
VI. SCHEME DEPENDENCE AND CUT-OFFS

A. The relevant scales

As we see our scheme including the ∆ does not reproduce the $^1S_0$ phase shift for momenta larger than the pion mass even if the effective range is adjusted to its experimental value. This trend has also been observed in other renormalized calculations including TPE effects and $1/M$ corrections in a Heavy Baryon formalism [30], using a relativistic potential [71] or including N3LO contributions [42]. Within the chiral approach to NN interactions the main candidates for explaining such a disagreement would be $3\pi$-exchange or relativistic corrections.

It is interesting at this point to ask which are the relevant scales which build the full strength of the phase shifts. As already found in [29, 30, 31, 42, 71], this is about 0.4 – 0.5 fm for Delta-less calculations. For illustration purposes, the short distance cut-off dependence is shown in Fig. 7 for the effective range in the case of renormalization with one counter-term in the Heavy Baryon Delta-less theory to LO, NLO-∆ and N2LO-∆ where also the Nijmegen II potential [6] is considered. As we see, and regardless on the full renormalized effective range value, the Nijmegen II saturating scale is in between NLO-∆ and N2LO-∆, a not unreasonable result. Likewise the N2LO-∆ effective range displays a similar approach to the renormalized result.

These scales are comparable to the nucleon size but also to the range where $3\pi$ exchange starts contributing since it behaves as $e^{-3m_{\pi}r}$ at long distances. According to the results of Kaiser [72, 73, 174] (and in the absence
FIG. 5: Deuteron wave functions, $u$ (left panel) and $w$ (right panel), as a function of the radius (in fm) for the N2LO-$\Delta$ potential and the NLO and N2LO-$\Delta$ potentials, compared to the Nijmegen II wave functions. The asymptotic normalization $u \to e^{-\gamma r}$ has been adopted and the value $\eta = 0.0256(4)$ is taken for the asymptotic D/S ratio.

FIG. 6: The np spin triplet (eigen) phase shifts for the total angular momentum $j = 1$ as a function of the c.m. momentum for the N2LO-$\Delta$ and N2LO-$\Delta$ potentials compared to the Nijmegen II potential results. In this context, one can identify two kind of finite cut-off sources of errors: the first one is related with the explicit form of the regulator employed in the computations, and the second one with the actual size of the finite cut-off. This second source of errors is the only one which is usually assessed in most effective field theory works, while the role played by the regulator is commonly ignored, probably resulting in an underestimation of the errors. While in our present renormalization scheme the most sensible thing to do in the $^1S_0$ singlet channel is to completely remove the cut-off, this may not be the case in other regularization schemes. Although this could be in fact considered as a good motivation for keeping a finite cut-off or introducing form factors, we think that this kind of procedure is difficult to justify from the EFT depend on the previous choice.

of $\Delta$) the potential is attractive, and using dimensional estimates for the $3\pi$-exchange potential, it behaves as $g_\Delta^6/(f_\pi^6 r^7)$ at short distances indicating a stronger singularity and therefore a stronger short distance suppression as well. Thus, we expect that including these effects would only slightly worsen the results by reducing the phase-shift.

Another interesting observation is that by taking larger coordinate space cut-offs the phase shift is not improved, as in our regularization procedure it converges from below, so in this case the best possible cut-off is $r_\text{c} \to 0$.

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9 Improving or not the phase shifts by using a finite cut-off while keeping the same renormalization conditions is, in fact, a regularization scheme dependent feature, and hence a further good reason to remove the cut-off. This choice however does not resolve the representation dependence of the potential on the choice of the pion field. Therefore, the cut-off less solution will obviously depend on the previous choice.
viewpoint.

The influence of relativistic effects on the results is less obvious, since they generate either energy dependence or non-localities, but experience in renormalizing relativistic potentials indicates that they are not crucial, at least in the $^1S_0$ channel [71], and are saturated by scales larger than the nucleon Compton wavelength, i.e. well above a possible influence from the $NN$ channel.

Thus, we see that in all these calculations including TPE effects the saturating scales are of the order of 0.5 fm, regardless on detailed issues, in particular including or not the $\Delta$. This is comparable to other scales, $2/m_\rho \sim 0.51$ fm, which may equally represent vector meson exchange or nucleon size effects. Therefore the discrepancy between renormalized phase shifts and phenomenological ones should be considered a real one, in the sense that finite size effects may be important physically. Moreover, it should be kept in mind that once the short distance cut-off becomes smaller than the nucleon size, the effect of the counter-terms is not enough to reproduce intermediate energy phase shifts, meaning that they cannot mimic the finite nucleon size or other short-distance physical effects which may be responsible of the observed discrepancy.

B. Remarks on Spectral Regularization

The calculation of the NN potentials can advantageously be carried out by using the method of dispersion relations [13, 14, 52]. This has motivated the use of the so-called spectral regularization [26, 27, 28, 52] in NN calculations where finite momentum space cut-offs have been implemented. Remarkably, a good fit to the $^1S_0$ phase was achieved. This is in contrast to the N3LO computation carried out in Ref. [42] with one counter-term, where a systematic underestimation of the data was found. In our view, the relevant issue is to disentangle the cut-off artifacts from clearly attributable physical effects. In this section we analyze the issue of cutting-off the potential.

The two pion exchange potential satisfies a dispersion relation based on the $NN \rightarrow 2\pi$ amplitude. One has the representations

$$V_C(r, \tilde{\Lambda}) = \frac{1}{2\pi^2 r^6} \int_{2m}^{\tilde{\Lambda}} d\mu \rho_C(\mu) e^{-\mu r},$$

$$V_S(r, \tilde{\Lambda}) = -\frac{1}{6\pi^2 r^7} \int_{2m}^{\tilde{\Lambda}} d\mu \left( \mu^2 \rho_T(\mu) - 3 \rho_S(\mu) \right) e^{-\mu r},$$

$$V_T(r, \tilde{\Lambda}) = -\frac{1}{6\pi^2 r^8} \int_{2m}^{\tilde{\Lambda}} d\mu \mu^3 (3 + 3 \mu r + \mu^2 r^2) \rho_T(\mu) e^{-\mu r},$$

(56)

with $\rho_i(\mu) = \text{Im} V_i(\mu)$ and similar relations for the $W_i$ potentials. It is straightforward to see that for small $\tilde{\Lambda}$ and $r \ll \tilde{\Lambda}^{-1}$, the potentials behave as

$$V_C(r, \tilde{\Lambda}) \rightarrow \frac{1}{2\pi^2 r^6} \int_{2m}^{\tilde{\Lambda}} d\mu \rho_C(\mu),$$

$$V_S(r, \tilde{\Lambda}) \rightarrow -\frac{1}{6\pi^2 r^7} \int_{2m}^{\tilde{\Lambda}} d\mu \left( \mu^2 \rho_T(\mu) - 3 \rho_S(\mu) \right),$$

$$V_T(r, \tilde{\Lambda}) \rightarrow -\frac{1}{6\pi^2 r^8} \int_{2m}^{\tilde{\Lambda}} d\mu \mu^3 3 \rho_T(\mu).$$

(57)

Thus, although $V_C$, $W_C$, $V_S$ and $W_S$ become regular, the tensor contributions $V_T$ and $W_T$ remain singular, despite higher energy states being cut off. That means that regularization of the scattering problem in triplet channels is mandatory to obtain well defined results.

Note that this short distance behaviour is quite different from the one obtained for $r > 0$ when $\tilde{\Lambda} \rightarrow \infty$ (see appendix A). In Fig. 8 the ratio, $V(r, \tilde{\Lambda})/V(r)$, of $^1S_0$ potentials as they enter in the phase shift calculation with and without spectral regularization is depicted. We note that there is a sizeable distortion at 2-3 fm’s with $\tilde{\Lambda} = 700$ MeV, decreasing the strength of the interaction and hence providing effectively a repulsion in the singlet potential. The effects of the spectral regularization on the potential can be seen in Fig. 9 for NLO-Δ and N2LO-Δ both with one counter-term as well as with two counter-terms. As we see, it is possible to describe the data successfully for suitable values of the spectral cut-off $\tilde{\Lambda} \sim 700$ MeV, in agreement with the findings of Ref. [52]. It is noteworthy that this happens regardless on the usage of one or two counter-terms, reinforcing the conclusion that the agreement is mainly due to the distortion in the potential. In regard to the behaviour with respect to spectral regularization, the trends presented for the $\Delta$-full theory are also reproduced in the $\Delta$-less scheme. We remind that an accurate description of the data was achieved in Ref. [28] at N3LO-Δ with a spectral regularized potential with finite momentum space cut-offs and 4 counter-terms in the $^1S_0$ channel (the full N3LO computation has 24 counter-terms). On the other hand the data were not described for large values of $\rho$ in the N3LO-Δ and (spectrally) unregularized calculation of Ref. [42].

As before, it is interesting to analyze the relevant scales building up the effective range in the case of renormalization with one counter-term. In Fig. 10 we show the effective range $r_0$ as a function of the short distance cut-off $\tilde{\Lambda}$ for several values of the spectral cut-off $\tilde{\Lambda}$. Clearly, the stability region in $r_0$ is shifted towards lower values as the spectral cut-off is decreased. This is consistent with the large distortion of the potential at intermediate scales.

We have also analyzed the impact of the spectral regularization on the already successful description of the deuteron presented in previous papers and the present one. The results do not change noticeably after (spectrally) regularizing the potential: they lie between the
results obtained with the OPE potential and with NLO- and N2LO-\(\Delta\). This is compatible with the weakening of these contributions to the chiral potential. Although the final results are rather simple to summarize, some remarks should be added on the renormalization procedure for the deuteron channel when spectral regularization is applied. The modified NLO- and N2LO-\(\Delta\) chiral potentials have now an attractive and repulsive eigen-channel, instead two attractive ones, which lead us to the following alternative: either we remove the cutoff, in which case we can only fix the binding energy and \(\eta\). Provided that the finite cut-off is sensible, about \(r_c \sim 0.5\) fm (i.e. the saturation scale) for a spectral cutoff \(\Lambda \sim 600 - 800\) MeV, both procedures give equivalent results. The previous discussion on the effects of spectral cut-off agree with the remarks presented in Ref. \[43\] regarding the effects of this regularization of the potential for the deuteron form factors. For completeness we show the results in Table V.

In summary, reducing the strength of the potential at short distances by means of the spectral regularization is phenomenologically preferred in the singlet channel and innocuous in the triplet channel, but distorts largely the chiral potential in a region much larger than the nucleon size. Moreover, we find that within our scheme the agreement with the data is achieved regardless of the additional counter-terms invoked by Weinberg’s counting, being in fact superfluous after renormalization. This said, although the spectral regularization improves over the standard finite cut-off approaches and seems phenomenologically favoured, the inclusion of finite size effects in a model independent manner would certainly be very useful.

VII. CONCLUSIONS

Since more than fifteen years there has been a growing interest in pursuing an EFT description of NN scattering below pion production threshold where the spontaneous breakdown of chiral symmetry in QCD is manifestly exploited. One of the reasons which have greatly hindered the EFT developments within the NN context has been the lack of a credible power counting for the potential
TABLE V: Deuteron properties for the OPE and TPE potentials with spectral regularization. The computation is made by fixing $\gamma$ and $\eta$ to their experimental values or by fixing $\gamma$ and predicting $\eta$ depending on the singularity structure of the potential and the coordinate space cut-off. For each case we present three computations: (i) the complete computation, i.e. the results obtained when the cut-off is completely removed and there is no spectral cut-off, (ii) the spectrally regularized TPE computations and in the spectrally regularized potential with a finite cut-off $r_c = 0.5$ fm reflect the uncertainty in the non-potential parameters $\gamma$ and $\eta$. In the spectrally regularized potentials, the errors represent the cut-off dependence of the results for $r_c$ ranging from 0.1 to 0.2 fm. For the OPE (LO) we take $g_{nN} = 13.1(1)$. We take set IV \cite{24} for the LEC's in the TPE calculation. For the $\Delta$ case we use Fits 1 and 2 of Ref. \cite{52}. Fit 1 involves the SU(4) quark-model relation $h_A = 3g_A/(2\sqrt{2}) = 1.34$ for $g_A = 1.26$. Fit 2 takes $h_A = 1.05$.

| Set        | $\gamma$ (fm$^{-1}$) | $\eta$ | $A_S$ (fm$^{-1/2}$) | $r_m$ (fm) | $Q_d$(fm$^2$) | $P_D$ (%) | $(r^{-1})$ | $(r^{-2})$ |
|------------|----------------------|-------|--------------------|------------|--------------|----------|-----------|-----------|
| LO        | Input                | 0.02633 | 0.8681(1) | 1.9351(5) | 0.2762(1) | 7.31(1) | 0.486 (1) | 0.434(3) |
| NLO-$\Delta$ | Unbound              | -     | -                  | -          | -            | -        | -         | -         |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.2669 | 0.851(7) | 1.894(15) | 0.270(4) | 8.9(1.0) | 0.6(2) | 1.8(1.5) |
| NLO-$\Delta$ (h$_A = 1.34$) | $r_c = 0.5$ fm       | Input | 0.884(3) | 1.963(7) | 0.274(9) | 5.9(4) | 0.446(10) | 0.29(2) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.2637 | 0.8720(10) | 1.938(2) | 0.2781(6) | 7.31(12) | 0.48(2) | 0.5(2) |
| NLO-$\Delta$ (h$_A = 1.05$) | $r_c = 0.5$ fm       | Input | 0.867(13) | 1.93(3) | 0.263(14) | 6.6(12) | 0.48(4) | 0.35(8) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.854(17) | 1.90(4) | 0.26(2) | 7(2) | 0.51(5) | 0.41(11) |
| N2LO-$\Delta$ (Set IV) | $r_c = 0.5$ fm       | Input | 0.884(3) | 1.967(6) | 0.276(3) | 8(1) | 0.447(5) | 0.284(8) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.877(2) | 1.946(5) | 0.264(2) | 5.6(7) | 0.48(2) | 0.42(6) |
| N2LO-$\Delta$ ($\pi N$) | $r_c = 0.5$ fm       | Input | 0.875(8) | 1.942(15) | 0.270(2) | 8(2) | 0.46(2) | 0.27(6) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.80(12) | 1.954(5) | 0.2756(14) | 6.58(6) | 0.463(4) | 0.35(3) |
| N2LO-$\Delta$(Fit 1) | $r_c = 0.5$ fm       | Input | 0.883(4) | 1.95(6) | 0.274(9) | 5.9(8) | 0.447(13) | 0.28(2) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.892(2) | 1.980(4) | 0.279(5) | 5.9(9) | 0.4336(15) | 0.263(2) |
| N2LO-$\Delta$(Fit 2) | $r_c = 0.5$ fm       | Input | 0.878(3) | 1.950(7) | 0.2774(2) | 6.77(3) | 0.465(4) | 0.36(4) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.880(6) | 1.954(14) | 0.272(10) | 5.9(6) | 0.45(2) | 0.29(4) |
| N2LO-$\Delta$(Fit 2) | $r_c = 0.5$ fm       | Input | 0.890(2) | 1.975(3) | 0.278(5) | 5.8(9) | 0.447(15) | 0.268(2) |
| $\Lambda = 700$ MeV | $r_c = 0.5$ fm       | Input | 0.8769(4) | 1.947(9) | 0.2770(2) | 6.83(3) | 0.467(5) | 0.37(4) |
| N2LO-$\Delta$(Set IV) | $r_c = 0.5$ fm       | Input | 0.878(6) | 1.950(15) | 0.271(10) | 5.9(6) | 0.46(2) | 0.30(4) |
| NijmII     | 0.231605             | 0.02521 | 0.8845 | 1.9675 | 5.635 | 0.2707 | 0.4502 | 0.2868 |
| Reid93     | 0.231605             | 0.02514 | 0.8845 | 1.9686 | 0.2703 | 5.699 | 0.4515 | 0.2924 |
| Exp.       | 0.231605             | 0.0256(4) | 0.8838(4) | 1.971(5) | 0.2860(15) | - | - | - |

which at the same time complies to short distance insensitivity. Given the tight constraints under which this might actually happen, it has not been obvious which particular form of the chiral expansion indeed embodies these desirable properties. In the present work we have shown how the inclusion of $\Delta$ degrees of freedom in the potential not only complies to a phenomenologically well founded motivation, but also provides the requested short distance insensitivity of the central phases and the deuteron properties after the necessary renormalization is carried out. This improves the previous situation without explicit $\Delta$ degrees of freedom: while at LO-$\Delta$ and N2LO-$\Delta$ the existence of a deuteron was compatible with renormalizability, at NLO-$\Delta$ that was not the case. This has raised reasonable doubts on the suitability and usefulness of non-perturbative renormalization per se to chiral potentials. A very rewarding aspect of the present investigation is the existence of a deuteron bound state at LO, NLO-$\Delta$ and N2LO-$\Delta$. Of course, a proof of consistency to all orders, including relativistic, spin-orbit, three pion exchange corrections, etc., remains to be done. This is so because, although the power law behaviour of the chiral potential at short distances can be trivially fixed by dimensional arguments and power counting, the determination of the attractive-repulsive character of the potential can so far only be fixed by actual calculations. The found deuteron properties seem to obey a converging pattern and in the $E_1$ phase a clear improvement is observed at N2LO-$\Delta$. Moreover, our N2LO-$\Delta$ results resemble much those of N2LO-$\Delta$ after renormalization for
FIG. 9: np $^1S_0$ renormalized phase shifts for the spectral-regularized N2LO-$\Delta$ potential for different values of the spectral cut-off $\tilde{\Lambda}$ as a function of the c.m. momentum (in MeV) compared to the Nijmegen II potential results \cite{Nijmegen} with one counter-term (left panel) and with two counter-terms (right panel), corresponding this last choice to the standard Weinberg counting.

reasonable parameter values describing the $\pi N$ reaction close to threshold. This suggests that despite the previous inconsistency in NLO-$\Delta$, the N2LO-$\Delta$ deuteron wave functions can be used for practical purposes, despite the theoretically unpleasant and disturbing “jumping” of the NLO-$\Delta$ calculation. In addition, it should be reminded that within these approximations the $\pi N$ threshold properties can be fitted, with the sole exception of $b_{0,+}$, and thus the overall consistency between the $\pi N$ and $NN$ sectors is almost satisfied. Therefore, the inclusion of the $\Delta$ resonance complies with the original EFT motivation of describing simultaneously $\pi N$ and $NN$ scattering at low energies.

We have also found that at the level of approximations involved in the present and previous renormalized calculations, the $^1S_0$ phase shift is not entirely reproduced for c.m. momenta larger than the pion mass if we insist on a reasonable $\pi N$ physics. Several effects might be responsible for this persistent discrepancy. We have discussed those on the light of the relevant scales, $r \geq r_c = 0.5$ fm, which practically provide the total contribution to observables. We have further discussed parameterizations of the NN force based on a spectral regularization of the NN potential with a cut-off of $\tilde{\Lambda} = 700$ MeV. We have shown that agreement to data is achieved mainly due to a large distortion of the potential at 2-3 fm and regardless on additional inclusion of counter-terms. These length scales are much larger than the expected finite nucleon size or vector meson exchange effects, casting doubts on the usefulness of spectral regularization and suggesting the need for a more controllable and better founded description of the missing short distance physics in the $^1S_0$ channel.

In the present paper we have restricted to central phases and the deuteron, but the calculation of higher partial waves is also of interest, as well as the inclusion of Coulomb effects in pp scattering. In all cases, the negative definite character of the potential at short distances guarantees the existence of convergent results. Finally, the present results can have some impact on calculations including deuteron properties such as deuteron form factors, pion-deuteron scattering, and further low energy matrix elements of electroweak deuteron reactions where short distance insensitivity and chiral symmetry are both expected to play a significant role.

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APPENDIX A: SHORT DISTANCE EXPANSION OF THE POTENTIALS

Using the spectral representation the short distance expansion of the potentials can be done by expanding spectral functions for $\mu \gg m$, with $\mu r$ fixed. This way
one obtains

\[ V_C(r) = \frac{C_{V,C}^{W}}{r^6} + \ldots \]
\[ W_C(r) = \frac{C_{W,C}^{V}}{r^6} + \ldots \]
\[ V_S(r) = \frac{C_{V,S}^{W}}{r^6} + \ldots \]
\[ W_S(r) = \frac{C_{W,S}^{V}}{r^6} + \ldots \]
\[ V_T(r) = \frac{C_{V,T}^{W}}{r^6} + \ldots \]
\[ W_T(r) = \frac{C_{W,T}^{V}}{r^6} + \ldots \] (A1)

where the van der Waals coefficients at NLO-\( \Delta \) are given by

\[ C_{6,V,C}^{\text{NLO-}\Delta} = -\frac{(9g_A^2 + 4h_A^2)h_A^2}{36f_\pi^2\Delta} \] (A2)
\[ C_{6,W,C}^{\text{NLO-}\Delta} = -\frac{(9g_A^2 - 2h_A^2)h_A^2}{108f_\pi^2\Delta} \] (A3)
\[ C_{6,V,S}^{\text{NLO-}\Delta} = \frac{(9g_A^2 - 2h_A^2)h_A^2}{216f_\pi^2\Delta} \] (A4)
\[ C_{6,W,S}^{\text{NLO-}\Delta} = \frac{(9g_A^2 + h_A^2)h_A^2}{648f_\pi^2\Delta} \] (A5)
\[ C_{6,V,T}^{\text{NLO-}\Delta} = -\frac{(9g_A^2 - 2h_A^2)h_A^2}{216f_\pi^2\Delta} \] (A6)
\[ C_{6,W,T}^{\text{NLO-}\Delta} = -\frac{(9g_A^2 + h_A^2)h_A^2}{648f_\pi^2\Delta} \] (A7)

and at N2LO-\( \Delta \) by

\[ C_{6,V,C}^{\text{N2LO-}\Delta} = \frac{4b h_A^2}{9f_\pi^2} + \frac{c_3 h_A^2}{6 f_\pi^2} + \frac{9c_9 g_A^2}{16 f_\pi^2} \] (A8)
\[ C_{6,W,C}^{\text{N2LO-}\Delta} = \frac{2b h_A^2}{27 f_\pi^2} - \frac{b g_A^2 h_A}{6 f_\pi^2} \] (A9)
\[ C_{6,V,S}^{\text{N2LO-}\Delta} = \frac{b g_A^2 h_A}{12 f_\pi^2} - \frac{b h_A^3}{27 f_\pi^2} \] (A10)
\[ C_{6,W,S}^{\text{N2LO-}\Delta} = -\frac{b h_A^3}{162 f_\pi^2} + \frac{c_4 h_A^2}{36 f_\pi^2} + \frac{c_4 g_A^2}{16 f_\pi^2} \] (A11)
\[ C_{6,V,T}^{\text{N2LO-}\Delta} = \frac{b h_A^3}{27 f_\pi^2} - \frac{b g_A^2 h_A}{12 f_\pi^2} \] (A12)
\[ C_{6,W,T}^{\text{N2LO-}\Delta} = -\frac{b h_A^3}{162 f_\pi^2} + \frac{c_4 h_A^2}{36 f_\pi^2} + \frac{c_4 g_A^2}{16 f_\pi^2} \] (A13)

where \( \hat{b} = b_3 + b_8 \).

[1] R. Machleidt and I. Slaus, J. Phys. G27, R69 (2001), nucl-th/0101056.
[2] S. R. Beane, P. F. Bedaque, K. Orginos, and M. J. Savage, Phys. Rev. Lett. 97, 012001 (2006), hep-lat/0602010.
[3] N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007), nucl-th/0611096.
[4] R. Machleidt, K. Holinde, and C. Elster, Phys. Rept. 149, 1 (1987).
[5] V. G. J. Stoks, R. A. M. Kompl, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. C48, 792 (1993).
[6] V. G. J. Stoks, R. A. M. Kompl, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C49, 2950 (1994), nucl-th/9406039.
[7] R. Machleidt, Phys. Rev. C63, 024001 (2001), nucl-th/0006014.
[8] S. Weinberg, Phys. Lett. B251, 288 (1990).
[9] C. Ordóñez, L. Ray, and U. van Kolck, Phys. Rev. C53, 2086 (1996), hep-ph/9511380.
[10] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, 339 (2002), nucl-th/0203055.
[11] E. Epelbaum, Prog. Part. Nucl. Phys. 57, 654 (2006), nucl-th/0509032.
[12] R. Machleidt and D. R. Entem, J. Phys. G31, S1235 (2005), nucl-th/0503025.
[13] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. A625, 758 (1997), nucl-th/9706045.
[14] N. Kaiser, S. Gerstendorfer, and W. Weise, Nucl. Phys. A637, 395 (1998), nucl-th/9802071.
[15] J. L. Friar, Phys. Rev. C60, 034002 (1999), nucl-th/9901082.
[16] M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar, and J. J. de Swart, Phys. Rev. Lett. 82, 4992 (1999), nucl-th/9901054.
[17] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998), nucl-th/9801034.
[18] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B534, 329 (1998), nucl-th/9802075.
[19] S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. A700, 377 (2002), nucl-th/0104030.
[20] T. Frederico, V. S. Timoteo, and L. Tomio, Nucl. Phys. A653, 209 (1999), nucl-th/9902052.
[21] S. R. Beane, D. B. Kaplan, and A. Vuorinen (2008), 0812.3938.
[22] D. R. Entem and R. Machleidt, Phys. Rev. C66, 014002 (2002), nucl-th/0202039.
20

[23] D. R. Entem and R. Machleidt, Phys. Lett. B524, 93 (2002), nucl-th/0108057.

[24] D. R. Entem and R. Machleidt, Phys. Rev. C68, 041001 (2003), nucl-th/0304018.

[25] E. Epelbaum, W. Glöckle, and U.-G. Meißen, Nucl. Phys. A671, 295 (2000), nucl-th/9910064.

[26] E. Epelbaum, W. Glöckle, and U.-G. Meißen, Eur. Phys. J. A19, 125 (2004), nucl-th/0304037.

[27] E. Epelbaum, W. Glöckle, and U.-G. Meißen, Eur. Phys. J. A19, 401 (2004), nucl-th/0308010.

[28] E. Epelbaum, W. Glöckle, and U.-G. Meißen, Nucl. Phys. A747, 362 (2005), nucl-th/0405048.

[29] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C72, 054002 (2005), nucl-th/0504067.

[30] M. P. Valderrama and E. Ruiz Arriola, Phys. Rev. C74, 054001 (2006), nucl-th/0506047.

[31] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C74, 064004 (2006), nucl-th/0507075.

[32] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C72, 054006 (2005), nucl-th/0506005.

[33] K. M. Case, Phys. Rev. 80, 797 (1950).

[34] W. Frank, D. J. Land, and R. M. Spector, Rev. Mod. Phys. 43, 36 (1971).

[35] E. Ruiz Arriola, A. Calle Cordon, and M. Pavon Valderrama (2007), arXiv:0710.2770 [nucl-th].

[36] M. P. Valderrama and E. Ruiz Arriola, Annals Phys. 323, 1037 (2008), 0705.2952.

[37] C. J. Yang, C. Elster, and D. R. Phillips, Phys. Rev. C77, 014002 (2008), 0706.1242.

[38] C. J. Yang, C. Elster, and D. R. Phillips (2009), nucl-th/0901.2663.

[39] C. J. Yang, C. Elster, and D. R. Phillips (2009), nucl-th/0901.2663.

[40] U. van Kolck, Nucl. Phys. A645, 273 (1999), nucl-th/9808007.

[41] D. R. Entem, E. Ruiz Arriola, M. Pavon Valderrama, and R. Machleidt, Phys. Rev. C77, 044006 (2008), 0709.2770.

[42] M. P. Valderrama, A. Nogga, E. Ruiz Arriola, and D. R. Phillips, Eur. Phys. J. A36, 315 (2008), 0711.4785.

[43] E. Epelbaum and U. G. Meißen (2006), nucl-th/0609037.

[44] M. C. Birse, Phys. Rev. C74, 014003 (2006), nucl-th/0507077.

[45] D. R. Entem, F. Fernandez, and A. Valcarce, Phys. Rev. C62, 034002 (2000).

[46] D. Bartz and F. Stancu, Phys. Rev. C63, 034001 (2001), nucl-th/0009010.

[47] A. Valcarce, H. Garcilazo, F. Fernandez, and P. Gonzalez, Rept. Prog. Phys. 68, 965 (2005), hep-ph/0502173.

[48] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Lett. 22, 83 (1969).

[49] E. Jenkins and A. V. Manohar, Phys. Lett. B255, 558 (1991).

[50] T. R. Hemmert, B. R. Holstein, and J. Kambor, J. Phys. J. A32, 127 (2007), nucl-th/0703087.

[51] T. E. O. Ericson and W. Weise (1988), Oxford, UK: Clarendon 479 p. (The International Series of Monographs on Physics, 74).

[52] E. Epelbaum and U. G. Meißen, Eur. Phys. J. A32, 127 (2007), nucl-th/0703087.

[53] M. C. Birse, Phys. Rev. C74, 014003 (2006), nucl-th/0507077.

[54] D. R. Entem, F. Fernandez, and A. Valcarce, Phys. Rev. C62, 034002 (2000).

[55] D. Bartz and F. Stancu, Phys. Rev. C63, 034001 (2001), nucl-th/0009010.

[56] A. Valcarce, H. Garcilazo, F. Fernandez, and P. Gonzalez, Rept. Prog. Phys. 68, 965 (2005), hep-ph/0502173.