FREE BOUNDARY MINIMAL SURFACES OF ANY TOPOLOGICAL TYPE IN EUCLIDEAN BALLS VIA SHAPE OPTIMIZATION

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ABSTRACT. For any compact surface $\Sigma$ with smooth, non-empty boundary, we construct a free boundary minimal immersion into a Euclidean ball $\mathbb{B}^N$ where $N$ is controlled in terms of the topology of $\Sigma$. We obtain these as maximizing metrics for the isoperimetric problem for the first non-trivial Steklov eigenvalue. Our main technical result concerns asymptotic control on eigenvalues in a delicate glueing construction which allows us to prove the remaining spectral gap conditions to complete the program by Fraser–Schoen and the second named author to obtain such maximizing metrics. Our construction draws motivation from earlier work by the first named author with Siffert on the corresponding problem in the closed case.

1. Introduction

Minimal surfaces naturally appear considering soap films: for instance, in the classical Plateau problem asking for area-minimizing disks whose boundary is a closed curve in $\mathbb{R}^3$. After this problem was independently solved by Douglas and Radó, Courant [Cou40] generalized this question, looking for disks minimizing the area, letting the boundary lie in a constraint surface of $\mathbb{R}^3$. This created a lot of activity around so-called free boundary minimal surfaces (see the surveys by Hildebrandt [Hil85] and M. C. Li [Li20]). In the current paper, we focus on free boundary minimal surfaces in Euclidean unit balls.

In their celebrated, pioneering paper [FS16], Fraser and Schoen made a one to one link between free boundary minimal immersions of a surface with boundary into a Euclidean unit ball and critical metrics of Steklov eigenvalues on this surface among metrics with boundary of unit length. They were inspired by the seminal work by Nadirashvili [Nad96] and then El-Soufi and Ilias [ESI00], who notably gave the one to one link between critical metrics for Laplace eigenvalues on closed surfaces among metrics with unit area and minimal immersions into a round sphere. Since then, the topic of free boundary minimal surface has gained more attention again. In particular, uniqueness questions and construction of examples for a large variety of topologies have been studied extensively in recent years. The work by Fraser and Schoen [FS16] and then extended by the second author [Pet14, Pet18, Pet19] gave a natural program for the construction of minimal surfaces by solving isoperimetric optimization problems for eigenvalues. This has been a permanent source of inspiration for the research of the authors, and also for the current paper. Our main result completes the existence question of free boundary minimal immersion into a Euclidean unit ball, for any topology of the surface.

Theorem 1.1. Let $\Sigma$ be a compact surface with non-empty boundary. Then there is $N \geq 3$ depending on the topology of $\Sigma$ and a branched free boundary minimal immersion $\Phi: \Sigma \to \mathbb{B}^N$. Let $S \subset \mathbb{B}^N$ be a free boundary minimal surface. For a fixed genus but
varying number of boundary components at most finitely many of the above immersions can be branched covers over $S$. Similarly, for a fixed number of boundary components but varying genus, infinitely many of the above immersions cannot be branched covers over $S$ provided $S \neq \mathbb{D}^2$.

Notice that in our result, the dimension of the target ball is controlled by the multiplicity of the first Steklov eigenvalue associated to the pull-back of the Euclidean metric along $\Phi$. This multiplicity is controlled in terms of the topology of $\Sigma$ (see [KKP14]). Beyond the work initiated by Fraser and Schoen [FS16], there are by now plenty of other constructions of free boundary minimal surfaces. Using perturbation techniques, Folha, Pacard, and Zolotareva [FPZ17] obtained the existence of examples in $\mathbb{B}^3$ with genus 0 and 1 and $k$ boundary components for $k$ large. Using an equivariant version of min-max theory, Ketover obtained the existence of free boundary minimal surfaces in $\mathbb{B}^3$ of unbounded genus and three boundary components [Ke17, Ke17a]. Examples of the same topological type using desingularization techniques were found by Kapouleas and Li [KL17]. Examples with high genus and connected boundary were constructed by Kapouleas and Wiygul [KW17]. Another recent result by Carlotto, Franz and Schulz [CFS20] gives existence of free boundary minimal surfaces with arbitrary genus, connected boundary, and dihedral symmetry.

We obtain Theorem 1.1 by completely resolving the existence question in the isoperimetric problem for the first non-trivial Steklov eigenvalue on compact surfaces with non-empty boundary.

Recall that for a compact Riemannian surface $(\Sigma, g)$ with non-empty boundary the first non-trivial Steklov eigenvalue $\sigma_1(\Sigma, g)$ is the smallest non-zero eigenvalue of the Dirichlet-to-Neumann operator given by $Tu = \partial_\nu \hat{u}$, where $\hat{u} \in C^\infty(\Sigma)$ denotes the harmonic extension of $u \in C^\infty(\partial \Sigma)$, and $\nu$ is the outward pointing normal field along $\partial \Sigma$. We also write $L_g(\partial \Sigma)$ for the length of the boundary.

**Theorem 1.2.** Let $\Sigma$ be a compact surface with non-empty boundary. Then there is a smooth metric $g$ on $\Sigma$ such that

$$\sigma_1(\Sigma, g) L_g(\partial \Sigma) \geq \sigma_1(\Sigma, h) L_h(\partial \Sigma)$$

for any smooth metric $h$ on $\Sigma$.

This generalizes and also reproves a result due to Fraser and Schoen if $\Sigma$ is orientable and has genus 0, [FS16]. In fact, the proof of [FS16 Proposition 4.3] (a special case of Theorem 1.3 below) appears not to be complete, cf. [GL20 Remark 1.5 and Appendix A].

The analogous result for the first eigenvalue of the Laplace operator on closed surfaces is known by work of the second named author [Pet14] and the first named author with Siffert [MS19a]. Very recently, in their interesting paper [KS20] Karpukhin and Stern have obtained some related results. They show that for fixed genus $\gamma$ there is an infinite number of $b \in \mathbb{N}$ such that Theorem 1.2 holds if $\Sigma$ has genus $\gamma$ and $b$ boundary components. Their argument relies on a comparison result between Steklov and Laplace eigenvalues combined with the main result of [MS19a].

The main result of the second named author in [Pet19] applied to the first non-zero Steklov eigenvalue $\sigma_1$ states that if we set

$$\sigma_1(\gamma, k) = \sup_g \sigma_1(\Sigma, g) L_g(\partial \Sigma)$$
for an orientable surface $\Sigma$ of genus $\gamma$ with $k$ boundary components, and if one has that
$$\sigma_1(\gamma, k) > \sigma_1(\gamma - 1, k + 1) \text{ for } \gamma \geq 1 \text{ and } k \geq 1$$

and
$$\sigma_1(\gamma, k) > \sigma_1(\gamma, k - 1) \text{ for } \gamma \geq 0 \text{ and } k \geq 2,$$

then $\sigma_1(\gamma, k)$ is achieved by a smooth metric. Theorem 1.2 then follows by induction from the following gluing theorem which is our main technical result.

**Theorem 1.3.** Let $(\Sigma, g)$ be a compact surface with smooth, non-empty boundary. Suppose that $\Sigma'$ is topologically obtained from $\Sigma$ by attaching a strip along two opposite sides of its boundary along two disjoint portions of the boundary of $\Sigma$. Then there is a smooth metric $g'$ on $\Sigma'$ such that
$$\sigma_1(\Sigma', g') L_g(\partial \Sigma') > \sigma_1(\Sigma, g) L_g(\partial \Sigma).$$

This result gives the required gaps as soon as $\sigma_1(\gamma - 1, k + 1)$ and $\sigma_1(\gamma, k - 1)$ are achieved by a smooth metric. We explain this in more detail in Section 2.2. As mentioned above, by induction and a combination of [Pet19] and Theorem 1.3 also using that the flat disk achieves $\sigma_1(0, 1)$, we obtain Theorem 1.2. While it is not written in [Pet19], the non-orientable case follows along the very same lines. We refer to [MS17] for more details on the non-orientable closed case. Theorem 1.3 is the analogue of [MS19a, Theorem 1.3] in the Steklov case.

**Ruling out branched covers.** The assertion from Theorem 1.1 that only few of these minimal surfaces can arise as branched covers over a fixed surface is an easy consequence of Theorem 1.3 coarse eigenvalue bounds, and the lower bound on the area for free boundary minimal surfaces proved by Fraser–Schoen (see also Brendle for the higher dimensional case). Let us recall the latter two results. For a 2-dimensional free boundary minimal surface $S$ in $\mathbb{B}^N$ we have that

$$L(\partial S) = 2|S| \geq 2\pi,$$

as proved by Fraser and Schoen in [FST11]. This has also been obtained by Brendle including the higher dimensional analogue. In addition he obtained the corresponding rigidity assertion that we will also use below, [Bre12]. The coarse eigenvalue bounds by Fraser–Schoen and Kokarev state that

$$\sigma_1(\gamma, k) \leq \min \left\{ 8\pi \left[ \frac{\gamma + 3}{2} \right], 2\pi (\gamma + k) \right\},$$

[FST11, Kok11].

Now suppose that a maximizing metric for the normalized first Steklov eigenvalue on $\Sigma_{\gamma, k}$ is given as a branched cover over $S$. First of all notice that this implies that the coordinate functions of $S$ are necessarily first Steklov eigenfunctions and hence $\sigma_1(\Sigma_{\gamma, k}, g) = 1$ for the maximizing metric $g$. Then we find from the area formula that

$$\sigma_1(\gamma, k) = l L(\partial S) \geq 2l\pi$$

for some positive integer $l$ thanks to (1.4). Now assume that there are positive integers $k_1 < k_2 < \cdots < k_j$ such that $\Sigma_{\gamma, k_i}, i = 1, \ldots, j$, all admit a maximizing metric that arises
as a branched cover over $S$. Then the above remark combined with Theorem 1.3 implies that

$$\sigma_1(\gamma, k_j) \geq 2\pi(l + j) \geq 2\pi(j + 1).$$

We immediately find from (1.5) that we need to have

$$j \leq 4 \left\lfloor \frac{\gamma + 3}{2} \right\rfloor - 1.$$ 

The argument in the genus is similar but slightly more subtle. Assume that there is $\gamma_0 \geq 0$ such that for $\gamma \geq \gamma_0$ all of the above minimal surfaces are covers over some $S \neq \mathbb{D}^2$. Then, thanks to rigidity in the area bound (1.4) we have that $L(\partial S) \geq 2\pi + \delta$ for some $\delta > 0$.

The same arguments as above imply in this case that

$$\liminf_{\gamma \to \infty} \frac{\sigma_1(\gamma, k)}{\gamma} \geq 2\pi + \delta$$

which contradicts the coarse eigenvalue bound $\sigma_1(\gamma, k) \leq 2\pi(\gamma + k)$ for $\gamma$ sufficiently large.

**Eigenvalues in Glueing Constructions.** We now describe some of the key difficulties we have to overcome to prove Theorem 1.3. Some of those difficulties are also present in the closed case and discussed in some detail in [MS19, MS19a]. These are generally related to some serious obstructions on glueing constructions for which one can hope to obtain the monotonicity result from Theorem 1.3.

Let us start by recalling the first naive approach one would like to follow in the closed case. The most natural idea was to attach a thin flat cylinder of length $L$ and radius $\varepsilon$ to a surface $\Sigma$ of genus $\gamma$ along the boundary of two removed disks of radius $\varepsilon$ to obtain a family of surfaces $\Sigma_\varepsilon$. In fact, once the limit of the spectrum as $\varepsilon \to 0$ is computed (this was done in [Ann87]), it is not hard to use this constructing to obtain the non-strict spectral gap condition, see [CES03].

It is then natural but much harder to compute the first non-zero term in the asymptotic expansion of the first non-zero eigenvalue on the perturbed surface $\Sigma_\varepsilon$ of genus $\gamma + 1$, as $\varepsilon \to 0$. Of course, the first eigenvalue on $\Sigma_\varepsilon$ might very well be smaller than the first eigenvalue on $\Sigma$. But one can hope that the positive extra-term of size $\varepsilon$ given by the asymptotic expansion of the area compensates this loss as $\varepsilon \to 0$. For deep reasons, this is not always possible. One expects the range of parameters $L$ for which there is hope to get sufficiently strong asymptotic control on the eigenvalue to be such that the perturbed surface $\Sigma_\varepsilon$ enjoys some interaction between the spectra on the thick and the thin part, respectively. In other words, this is the range of parameters at which one can observe the change of topology on a spectral level. Of course, one crucially has to use the fact that the topology changes in order to obtain the strict monotonicity.

More precisely, one should adjust the length of the cylinder $L$ (potentially depending on $\varepsilon$) so that the first eigenvalue of the interval of length $L$ is close to the first eigenvalue of the thick part to observe this interaction phenomenon. However, there is a fatal obstruction term in the asymptotic expansion of the eigenvalue containing $u_*(p_0) + u_*(p_1)$ where $u_*$ is a specific first eigenfunction of $\Sigma$ and the handle is attached near the points $p_0$ and $p_1$, [MS19]. If this term does not vanish, one can never obtain the strict inequality from Theorem 1.3 by this technique for such parameters $L$. (Remember that we do not expect other parameter to work either.) This can occur when the first eigenvalue has multiplicity on $\Sigma$, which is exactly the situation we have for a maximal metric.
To be slightly more precise, the eigenvalues near $\lambda_1(\Sigma)$ of the Laplacian in this case have the same asymptotic expansion as the eigenvalues of the matrix

$$
\begin{pmatrix}
\lambda_1(\Sigma) & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_1(\Sigma) & 0 \\
0 & \cdots & 0 & \lambda_1(\Sigma) & c_0(u_*(p_0) + u_*(p_1))\varepsilon^{1/2} \\
0 & \cdots & 0 & c_0(u_*(p_0) + u_*(p_1))\varepsilon^{1/2} & \frac{\pi}{L^2}
\end{pmatrix}
$$

up to the scale $O(\varepsilon \log(1/\varepsilon))$, for some constant $c_0 > 0$, [MS19]. In particular, for $L$ such that $\pi L^2$ is close to $\lambda_1(\Sigma)$ we have that

$$
\lambda_1(\Sigma_\varepsilon) = \lambda_1(\Sigma) - f_\varepsilon(L)\varepsilon^{1/2},
$$

for a function $f$ that is locally uniformly and positively bounded below within $L \pm O(\varepsilon^{1/2})$. It is important to understand the origin of the scaling of the off-diagonal terms. These come from the scaling of the first Dirichlet-eigenfunction on the cylindrical part. If $\phi_\varepsilon$ denotes an $L^2$-normalized $\lambda_0(S^1(\varepsilon) \times [0, L])$-eigenfunction, then

$$
-\int_{\partial(S^1(\varepsilon) \times [0, L])} \partial_\nu \phi_\varepsilon dH^1 = \lambda_0(S^1(\varepsilon) \times [0, L]) \int_{S^1(\varepsilon) \times [0, L]} \phi_\varepsilon dH^2 \sim c_1 \varepsilon^{1/2}
$$

for another positive constant $c_1 > 0$ closely related to $c_0$. This is propagated to the thick part of $\Sigma_\varepsilon$ through the integral kernel of $(\Delta - \lambda)^{-1}$ with poles at $p_0$ and $p_1$ resulting precisely in the obstruction term on scale $\varepsilon^{1/2}$. Also note that this scaling is not very particular to the flat collapsing cylinder $S^1(\varepsilon) \times [0, L]$. As long as the collapsing part resembles a model with an isolated eigenvalue at the bottom of the spectrum, we get the very same behaviour.

Even though not contained in [MS19] this discussion applies completely analogously to the Steklov spectrum. The natural strip that one would like to use is a flat rectangle of size $L\varepsilon \times \varepsilon^2$ that we attach along intervals of length $\varepsilon^2$ on the boundary of $\Sigma$. See also [FS19] for a related construction, in fact corresponding to the case $L \to 0$, where we do not expect to detect the change in topology on a spectral level. An analogous analysis gives the obstruction term $u_*(p_0) + u_*(p_1)$ where $u_*$ is a first Steklov eigenfunction of $\Sigma$. Notice that in the special case of the sphere or the disk, respectively, attaching the thin part along antipodal points gives that the obstruction term from above vanishes. But in general there might be no way to obtain this. In the closed case this happens for instance on the flat equilateral torus (the unique maximizer among tori [Nad96]).

Therefore, in order to prove these type of glueing results one has to drastically change the geometry of the attached thin part. The guiding principle here, originating from (1.6) and (1.7), cf. [MS19], is that the convergence rate can not be any better than the $L^1$-norm of an $L^2$-normalized eigenfunction on the thin part. (For the flat rectangle this is of size $\varepsilon^{1/2}$ loosing against the additional boundary length on scale $\varepsilon$.) In particular, this forces us to work with a geometry that observes the formation of continuous spectrum in the limit $\varepsilon \to 0$. This clearly introduces a number of serious technicalities, which any proof of a strict spectral gap result has to overcome. Also, we notably introduce a big asymmetry between the attaching boundaries, while we try to have a thin part with computable spectra.
The glueing construction for the closed case in [MS19a] by the first author jointly with Siffert uses a thin, hyperbolic cusp of area $\varepsilon$ truncated at $R = R(\varepsilon)$, whose Dirichlet and Neumann spectrum is perfectly computable by separation of variables, and has an infinite number of eigenvalues converging to $\frac{1}{4}$ as $\varepsilon \to 0$. Again, one can play with a parameter of dilatation $t$, such that $\frac{t^2}{4}$ is the first eigenvalue of the thin part and is close to the first eigenvalue of the thick part, in order to capture the interaction between both spectra, and cleverly choose the parameters $t$ and $R$ so that the expected strict inequality occurs. Already because of the infinite number of eigenvalues accumulating at $\frac{t^2}{4}$ it is technically much more challenging to understand the behaviour of the eigenfunctions, but in addition one has to compute the asymptotic expansion up to a much smaller scale.

In principal, we would like to follow a similar line of thought to prove Theorem 1.3. However, compared to the approach in [MS19a] we have to face a number of serious new difficulties. These are most significantly related to the fact that the Steklov problem does not enjoy as strong separation of variables properties as available for the Laplace spectrum. The argument in [MS19a] heavily relies on an explicit understanding and very specific properties of all the eigenvalues and eigenfunctions on the collapsing part in a fixed interval $\left[\frac{t^2}{4}, \frac{t^2}{4} + \delta\right]$. Note that this interval contains an unbounded number of eigenvalues as $\varepsilon \to 0$. For the Steklov problem we are not aware of any domain known to resemble this with the required accuracy. However, we expect the cuspidal domain that we use to have all of these properties, but our current methods only apply to a finite number of eigenvalues, compare also the discussion in Section 2.3.2. Therefore, we are not able to follow the approach from [MS19a] directly, even though we borrow a lot of motivation from there. On the other hand, we strongly believe that the more robust techniques developed here - whose necessity grew out of the aforementioned reasons - could be used to shorten the argument in [MS19a].

**Strategy of the proof of Theorem 1.3.** We now try to describe some of the key ideas that we introduce in order to overcome the problems described above.

The domain analogous to hyperbolic cylinders in the Steklov case is a cuspidal domain introduced by Nazarov-Taskinen [NT08], as the region on the plane such that $y > 0$ and $-\frac{y^2}{2} \leq x \leq \frac{y^2}{2}$, since it is the simplest example having a continuous spectrum bounded away from zero. We discovered that there is a strong link between the structure of the spectrum and associated eigenfunctions on a truncated hyperbolic cusp and on these cuspidal domains truncated at $r \leq y \leq 1$, and this is the reason why the approach in the closed case and Steklov case are so related. Our glueing construction makes use of an appropriately collapsing domain modelled on this.

One of the key ingredients in our analysis is a good ansatz for the eigenvalue equation in these degenerating domains strongly motivated by the eigenvalue equation in a hyperbolic cusp. Thanks to our specific choice of the change of variables we are even able to handle eigenfunctions with only very crude control on the boundary values at the original scale. Such control in turn can be obtained for eigenfunctions on the glued surface by exploiting the compactness of the thick part. Moreover, thanks to a good choice of test functions we can obtain energy bounds which roughly speaking imply that the first few eigenfunctions become symmetric on the collapsing part much fast as $\varepsilon \to 0$ than functions with merely bounded energy.
These three ingredients then are the key input in our technical main tool: A precise asymptotic expansion (in energy norm) of eigenfunctions in the thin part. This expansion gives precise information when the corresponding eigenfunction has some of its $L^2$-norm concentrated on the thin part. We achieve this by carefully adjusting the metric by a dilation on the thin part. At this stage, we can already prove a bound on scale $O(\varepsilon)$, where $\varepsilon$ denotes the scale of gain in length of the boundary.

Next, we proceed to find a second low energy eigenfunction for which we can also obtain this good asymptotic expansion on the thin part. The presence of at least two eigenfunctions with this behaviour is only possible in the range of dilation parameters that observe interaction of the two parts of the spectrum.

Given the improved understanding of those eigenfunctions on the thin part, we can also prove improved pointwise estimates in the attaching region of the thick part. The key difficulty in obtaining strong pointwise estimates is the lack of good gradient estimates in the attaching region. This is really at the heart of the matter. If we had sufficiently strong gradient estimates available (even in a space as weak as $L^2$) the glueing theorem would follow immediately via integration by parts - to be precise it is integration by parts twice, to control the mean value and the energy. At the same time the potential lack of these gradient estimates prevents us from comparing the $L^2$-norm of an eigenfunction and its gradient on the thick part to one another. Such an estimate is a key ingredient to obtain quantitative control on the limiting behaviour of eigenfunctions via rescaling and elliptic estimates\footnote{This is precisely the step circumvented in [MS19a] that heavily makes use of the precise knowledge of eigenvalues and eigenfunctions on the thin part.}. We only manage to obtain such an estimate in Section 6 by integrating by parts on the thin part exploiting the precise asymptotic expansion available there. This estimate has an error term that falls exactly short of allowing us to conclude. We circumvent this by building a test function out of the two eigenfunctions mentioned above by arranging for cancelation in the asymptotic expansion to remove precisely the aforementioned error term in the integration by parts estimate. However, at the same time this also leads to cancelation of almost all the $L^2$-norm on the thick part and we need to invoke our precise estimates to work out the precise scales of the energy and $L^2$-norm for this test function getting us eventually to the desired $o(\varepsilon)$-bound.

**Organization of the paper.** In Section 2 we explain the glueing construction and the ansatz for the behaviour of solutions to the eigenvalue equation of the Dirichlet-to-Neumann operator on cuspidal domains.

In Section 3 we give a good upper bound on the first few eigenvalues.

We then give a first pointwise estimate on eigenfunctions in the attaching region in Section 4.

In Section 5 we perform our fine analysis on the eigenvalue equation on the thin part.

Finally, in Section 6 we use the results from Section 2 and the refined energy estimates from Section 5 to choose an improved test function for $\sigma_*$ and complete the proof of Theorem 1.3.

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2. The glueing construction

In this section we introduce the two parameter family of competitors for the problem
that we use to obtain Theorem 1.3. We also discuss some of the properties of the cus-
pidal domain that we use. In particular, we give several analogies regarding its spectral
properties with the family of truncated hyperbolic cusps used in the closed case.

2.1. The construction. Let \((\Sigma, g)\) be a compact connected Riemannian surface with a
non-empty smooth boundary \(\partial \Sigma\) of length \(L_g(\partial \Sigma) = 1\).

Let \(\varepsilon > 0\). We set 
\[
\Omega_\varepsilon = \left\{ (x, y) \in \mathbb{R}^2; r_\varepsilon \leq y \leq 1, -\frac{y^2}{2} \leq x \leq \frac{y^2}{2} \right\}
\]
for a parameter \(0 < r_\varepsilon < 1\) endowed with the metric
\[
g_\varepsilon = \frac{e^4 dx^2 + e^2 dy^2}{t_\varepsilon^2}
\]
for some \(t_\varepsilon > 0\). Note that \((\Omega_\varepsilon, g_\varepsilon)\) is isometric to 
\[
\left\{ (x, y) \in \mathbb{R}^2; \varepsilon r_\varepsilon \leq y \leq \varepsilon, -\frac{y^2}{2} \leq x \leq \frac{y^2}{2} \right\}
\]
endowed with the metric \(dx^2 + dy^2\). We also write 
\[
I_\varepsilon^\pm = \left\{ (x, y) \in \mathbb{R}^2; r_\varepsilon \leq y \leq 1, y = \pm \frac{x^2}{2} \right\}.
\]
We glue the cuspidal domain \(\Omega_\varepsilon\) at the neighborhood of two points \(p_0, p_1 \in \partial \Sigma\) in the
following way. For \(i = 0, 1\), let \(\varphi_i : B_i \to \mathbb{D}_2^+\) be a conformal chart at the neighborhood of 
\(p_i \in B_i\) such that \(\varphi_i(p_i) = 0, \varphi_i(B_i) = \mathbb{D}_2^+, \varphi_i(B_i \cap \partial \Sigma) = \mathbb{R}_{\pm}\times \{0\}\) and 
\[
g = \varphi_\varepsilon^* (e^{2\omega_i} (dx^2 + dy^2))
\]
for a smooth function \(\omega_i : \mathbb{D}_2^+ \to \mathbb{R}\). We denote by \(\Sigma_\varepsilon\) the surface (which also depends on
our choice of \(r\) and \(t\) but those in turn will depend on \(\varepsilon\)) that we obtain by gluing \(\Sigma\) and 
\(\Omega_\varepsilon\) along the intervals \([-1, 1] \times \{0\}\) in the charts 
\[
f_0 : B_0 \to \mathbb{D}_2^+_{\varepsilon} \text{ and } g_0 : \Omega_\varepsilon \to E_0,
\]
at the neighborhood of \(p_0\), where
\[
E_0 = \left\{ (x, z); \frac{r - 1}{\varepsilon} \leq z \leq 0 \text{ and } -\frac{(1 + \varepsilon z)^2}{2} \leq x \leq \frac{(1 + \varepsilon z)^2}{2} \right\}
\]
and in the charts 
\[
f_1 : B_1 \to \mathbb{D}_2^+_{\varepsilon} \text{ and } g_1 : \Omega_\varepsilon \to E_1
\]
at the neighborhood of \(p_1\), where
\[
E_1 = \left\{ (x, z); \frac{r - 1}{\varepsilon} \leq z \leq 0 \text{ and } -\frac{(1 - \varepsilon z)^2}{2} \leq x \leq \frac{(1 - \varepsilon z)^2}{2} \right\}
\]
and where \(f_i\) and \(g_i\) are defined by the formulæ 
\[
f_0^{-1}(x, z) = \varphi_0^{-1}(\varepsilon^2 (x, z)) \text{ and } g_0^{-1}(x, z) = (x, 1 + \varepsilon z),
\]

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so that in the chart at the neighborhood of \( p_0 \) in \( \Sigma_\varepsilon \) the metric is given by

\[
\begin{cases}
\varepsilon^4 e^{2\omega(\varepsilon^2(x,z))} (dx^2 + dz^2) & \text{if } 0 \leq z < \frac{2}{\varepsilon^2} \text{ and } -\frac{1}{2} \leq x \leq \frac{1}{2} \\
\varepsilon^4 (dx^2 + dz^2) & \text{if } \frac{r-1}{\varepsilon} \leq z \leq 0 \text{ and } -\frac{(1+\varepsilon^2)^2}{2} \leq x \leq \frac{(1+\varepsilon^2)^2}{2}
\end{cases}
\]

and in the chart at the neighborhood of \( p_1 \) in \( \Sigma_\varepsilon \) the metric is given by

\[
\begin{cases}
r^4 \varepsilon^4 e^{2\omega(r^2\varepsilon^2(x,z))} (dx^2 + dz^2) & \text{if } 0 \leq z < \frac{2}{\varepsilon^2} \text{ and } -\frac{1}{2} \leq x \leq \frac{1}{2} \\
r^4 \varepsilon^4 \frac{1}{r^2} (dx^2 + dz^2) & \text{if } \frac{r-1}{r^2} \leq z \leq 0 \text{ and } -\frac{(1-r^2\varepsilon^2)^2}{2} \leq x \leq \frac{(1-r^2\varepsilon^2)^2}{2}.
\end{cases}
\]

We denote by \( \sigma_* \) the first non-zero Steklov eigenvalue of the surface \( \Sigma \). Moreover, \( \sigma^1_\varepsilon \) denotes the first non-zero Steklov eigenvalue on \( \Sigma_\varepsilon \). We aim at proving that for suitable choices of the parameters \( t_\varepsilon \) and \( r_\varepsilon \) we have that

\[
\sigma^1_\varepsilon = \sigma_* + o(\varepsilon),
\]

where \( \varepsilon \) is the scale of the extra length of the boundary when we glue \( \Omega_\varepsilon \) to \( \Sigma \).

We also remark that it is easy to approximate the metric on \( \Sigma_\varepsilon \) by smooth metrics e.g. using [Kok14] Lemma 4.1 combined with the observation that \( \Sigma_\varepsilon \) carries a smooth conformal structure.

### 2.2. The topological change.

Note that the new surface \( \Sigma_\varepsilon \) will differ topologically from the initial surface \( \Sigma \). Through different choices of the points \( p_i \) and orientations of the charts \( f_i, g_i \) this gives rise to various options for the topological type of \( \Sigma_\varepsilon \). None of these choices will affect our analytic arguments at all. However, it is of fundamental importance for our application towards Theorem 1.2 that we do not miss any topological changes required to apply the main result from [Pet19]. Let us briefly explain how to achieve this.

We first discuss the case that \( \Sigma \) is orientable with genus \( \gamma \geq 0 \) and \( k \geq 1 \) boundary components. If we take \( p_0 \) and \( p_1 \) to lie in the same component of the boundary we can obtain two types of surfaces by attaching the cuspidal domain \( \Omega_\varepsilon \). If we choose compatible orientations for the charts \( f_i \) and \( g_i \), \( \Sigma_\varepsilon \) is orientable has genus \( \gamma \) and \( k+1 \) boundary components. If we reverse the orientation of one of the charts, the resulting surface will be non-orientable, have \( k \) boundary components and non-orientable genus \( 2\gamma+1 \).

If we assume that \( k \leq 2 \) there is also the option of taking \( p_0 \) and \( p_1 \) to lie in different components of \( \partial \Sigma \). In this case, if we attach \( \Omega_\varepsilon \) such that we obtain an orientable surface (i.e. we choose compatible orientations for the charts), we find that \( \Sigma_\varepsilon \) has genus \( \gamma + 1 \) and \( k-1 \) boundary components. If we reverse the orientation of one of the charts in which we glue, the new surface will have non-orientable genus \( 2\gamma+2 \) and \( k-1 \) boundary components.

If we start with a non-orientable surface \( \Sigma \) of non-orientable genus \( \gamma \) and \( k \) boundary components, the topological type depends only on the location (in the same or in different components of the boundary) of the points \( p_0 \) and \( p_1 \). It will either have non-orientable genus \( \gamma \) and \( k+1 \) boundary components, or non-orientable genus \( \gamma + 1 \) and \( k \) boundary components.
2.3. Spectral properties of the cuspidal domain. We briefly discuss some spectral properties of the cuspidal domains $\Omega_\varepsilon$. While we do not give any proofs here, we give some motivation behind our ansatz for the asymptotic expansion. Later we deal with eigenfunctions on the cuspidal domain that are restrictions of eigenfunctions on $\Sigma_\varepsilon$ and hence obey weaker control on the boundary values.

2.3.1. The energy on one dimensional functions and the main ansatz. For simplicity we consider $t = 1$ for the moment. First of all, notice that for fixed $y \in [r, 1]$ the standard Poincaré inequality applied along line segments $\{(x, y) : -\frac{y^2}{2} \leq x \leq \frac{y^2}{2}\}$ implies that functions with bounded energy on $\Omega_\varepsilon$ endowed with $g_\varepsilon$ become more and more constant in the $x$-direction for $\varepsilon$ small. Therefore, as a first ansatz, we would like to consider functions $\phi : \Omega_\varepsilon \to \mathbb{R}$ with $\phi(x, y) = \phi(y)$. Of course, these will never be exact eigenfunctions. The energy of these functions is given by

$$
\int_{\Omega_\varepsilon} |\nabla \phi|^2 dA_\varepsilon = \varepsilon \int_r^1 y^2 \phi_y^2 dy,
$$

while the boundary mass is

$$
\int_{I^+ \cup I^-} |\phi|^2 dl_\varepsilon = 2\varepsilon(1 + O(\varepsilon^2)) \int_r^1 |\phi|^2 dy.
$$

We want to point out that this resembles (up to the error term on the scale $\varepsilon^2$, a factor of 2, and changing $y \to 1/y$) exactly the Dirichlet energy and the $L^2$-norm, respectively, on rotationally symmetric functions on a truncated hyperbolic cusp.

This suggest to use the following change of variables that is the starting point of our asymptotic analysis. For a function $\phi \in W^{1,2}(\Omega_\varepsilon)$ we write

$$
(2.3) \quad \phi(x, y) = \sqrt{\varepsilon} y^{-\frac{1}{2}} \theta \left( x, \frac{\ln(y)}{\ln(r)} \right),
$$

where $\theta$ is defined on

$$
\tilde{\Omega} = \left\{(x, v) \in \mathbb{R}^2; 0 \leq v \leq 1, -\frac{v^2}{2} \leq x \leq \frac{v^2}{2}\right\}.
$$

We introduce new coordinates for $\Omega$ by the change of variables $(x, y) = (x, r^v)$. It is convenient to write, for a function $\theta$ defined on $\tilde{\Omega}$, for the mean value on horizontal lines

$$
\bar{\theta} = r^{-2v} \int_{-\frac{v^2}{2}}^{\frac{v^2}{2}} \theta dx.
$$

We can then write in the new coordinates

$$
(2.4) \quad \int_{I^\pm} \phi^2 dl_\varepsilon = \int_0^1 \theta^2 \left( \pm \frac{r^2 v}{2}, v \right) \sqrt{1 + \varepsilon^2 r^{2v}} dv,
$$

$$
(2.5) \quad \int_{I^\pm} \phi dl_\varepsilon = \sqrt{\frac{\ln \frac{1}{r}}{\varepsilon r}} \int_0^1 r^{\frac{v}{2}} \theta \left( \pm \frac{r^{2v}}{2}, v \right) \sqrt{1 + \varepsilon^2 r^{2v}} dv,
$$

$$
\text{where } \theta \text{ is defined on } \tilde{\Omega} = \left\{(x, v) \in \mathbb{R}^2; 0 \leq v \leq 1, -\frac{v^2}{2} \leq x \leq \frac{v^2}{2}\right\}.
$$
and

\[\int_\Omega |\nabla \phi|^2 dA_\varepsilon = \varepsilon \left( \frac{1}{\varepsilon^2} \int_0^1 \left( \int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}} \theta_x^2 dx \right) dv + \int_0^1 \left( \frac{\theta}{2} + \frac{\theta_v}{\ln \frac{1}{r}} \right)^2 dv \right)\]

where \(dA_\varepsilon = \frac{\varepsilon^2}{\varepsilon^2} dxdy\), \(|\nabla \phi|^2_{g_\varepsilon} = t_\varepsilon^2 (\varepsilon^{-4} \phi^2_x + \varepsilon^{-2} \phi^2_y)\) and \(dl_\varepsilon = \frac{\varepsilon}{\varepsilon^2} \sqrt{1 + \varepsilon^2 y^2} dy\) on \(I^\pm\), noticing that

\[\phi_x = \frac{\sqrt{\varepsilon} y^{-\frac{3}{2}}}{\sqrt{\varepsilon/ln \frac{1}{r}}} \theta_x \text{ and } \phi_y = -\frac{\sqrt{\varepsilon} y^{-\frac{3}{2}}}{\sqrt{\varepsilon/ln \frac{1}{r}}} \left( \frac{\theta}{2} + \frac{\theta_v}{\ln \frac{1}{r}} \right)\].

Most of our arguments will take place on the level of \(\theta\) since this is the scale on which we can hope to get good control on eigenfunctions. Broadly speaking we aim at proving that if \(\phi\) is an eigenfunction on \(\Omega_\varepsilon\), then \(\theta\) converges to a solution of \(f'' + \nu f = 0\) for \(\varepsilon \to 0\) in a sufficiently strong sense.

### 2.3.2. The asymptotic behaviour of the spectrum for the cuspidal domains.

The argument in [MS19a] relies on the following properties of the spectrum of the truncated hyperbolic cusp (with parameter \(\alpha \in (1/3, 1/2)\)) with area on scale \(\varepsilon\):

- The first non-trivial Dirichlet eigenvalue is bounded away from zero.
- The \(L^1\)-norm of the normal derivative along the boundary of the first non-trivial \(L^2\)-normalized eigenfunction is of size \(o(\varepsilon)\). (It is on scale \(\varepsilon^{(3\alpha+1)/2}\).)
- The separation of two consecutive Dirichlet eigenvalues is much larger than the \(L^1\)-norm of the normal derivative along the boundary of the normalized eigenfunctions. (It is \(\ell \varepsilon^2\alpha\) versus \(\varepsilon^{(3\alpha+1)/2}\) for the \(l\)-th eigenvalue.)
- The first non-trivial Neumann and Dirichlet eigenvalues are related by \(\lambda_0 \leq \mu_1\).

The starting point of our construction was to find an analogue for the Steklov problem. It turns out that this is given by the cuspidal domains that we use. In fact, one can prove similar assertions on their spectrum. However, the more robust asymptotic techniques developed here to attack this problem turn out to apply more generally directly to restrictions of eigenfunctions from \(\Sigma_\varepsilon\) using that one can propagate some control on the boundary values from the compactness of \(\Sigma\).

### 3. Upper bounds for eigenvalues of the glued surface

Recall that we denote by \(\sigma_*\) the first non-trivial Steklov eigenvalue of \(\Sigma\). Moreover, we write \(K = \text{mult} \sigma_*\) for the multiplicity of \(\sigma_*\).

Using appropriate extensions of \(\sigma_*\)-eigenfunctions and an eigenfunction of the limiting quadratic form on \(\Omega_\varepsilon\) we can obtain some upper bounds on \(\sigma_{\varepsilon}^1, \ldots, \sigma_{\varepsilon}^{K+1}\) through a classical test function argument on the variational characterization of eigenvalues.

**Claim 3.1.** The first eigenvalue on \(\Sigma_\varepsilon\) satisfies

\[\sigma_{\varepsilon}^1 \leq \min \left\{ \sigma_* + O\left( \frac{\varepsilon}{(\ln \frac{1}{r})^2} + \varepsilon^2 \right), \frac{t_\varepsilon}{8} + \frac{t_\varepsilon \pi^2}{2(\ln r)^2} + O\left( \frac{\varepsilon}{(\ln \frac{1}{r})^3} + \varepsilon^2 \right) \right\},\]

as \(\varepsilon \to 0\) and \(r \to 0\). Moreover, we have that

\[\sigma_{\varepsilon}^{K+1} \leq \max \left\{ \sigma_* + \frac{t_\varepsilon}{8} + \frac{t_\varepsilon \pi^2}{2(\ln r)^2} \right\} + O\left( \frac{\varepsilon}{\ln \frac{r}{2}} + \frac{\varepsilon^3}{(\ln \frac{1}{r})^3} + \varepsilon^2 \right)\]
Indeed, we test the variational characterization of $\sigma$ and prove.

\[ u_\varepsilon = \begin{cases} (u_\varepsilon(p_1) - u_\varepsilon(p_0)) \ln y + u_\varepsilon(p_0) & \text{in } \Omega_\varepsilon \\ u_\varepsilon + \eta(\varepsilon^2 \varphi_0)(u_\varepsilon(p_0) - u_\varepsilon) + \eta(r^2 \varepsilon^2 \varphi_1)(u_\varepsilon(p_1) - u_\varepsilon) & \text{in } \Sigma \end{cases} \]

where $\eta$ is a smooth cut-off function such that $\eta = 1$ on $\mathbb{D}$, $\eta = 0$ in $\mathbb{R}^2 \setminus \mathbb{D}_2$ and $\nabla \eta$ is a bounded function, so that $u_\varepsilon$ is a Lipschitz function on $\Sigma \varepsilon$ which extends $u_\varepsilon$ well to $\Omega_\varepsilon$ up to slightly modifying $u_\varepsilon$ at the neighbourhood of $p_0$ and $p_1$.

By the variational characterization of the first eigenvalue we have that

\[ \sigma_\varepsilon^1 \leq \frac{\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2_g dA_\varepsilon}{\int_{\partial \Sigma_\varepsilon} (u_\varepsilon)^2 dl_\varepsilon - \left( \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2_g dA_\varepsilon + \int_{\Omega_\varepsilon} \nabla u_\varepsilon)^2_g dA_g}{\int_{\Omega_\varepsilon} u_\varepsilon^2 d\varepsilon - O(\varepsilon^2)} \right)^2 \]

as $\varepsilon \to 0$, where we noticed in the denominator that $|u_\varepsilon - u_\varepsilon(p_0)| = O(\varepsilon^2)$ and $|u_\varepsilon - u_\varepsilon(p_1)| = O(\varepsilon^2)$ in the neighbourhood of $p_0$ and $p_1$ on which $u_\varepsilon$ and $u_\varepsilon$ do not agree. We also noticed that $u_\varepsilon$ is uniformly bounded on $\Omega_\varepsilon$ so that the mean value on the boundary is controlled by the length of order $\varepsilon$ of the boundary. For the gradient, we have

\[ \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2_g dA_\varepsilon = \varepsilon \int_r^1 y^2 |\partial_y u_\varepsilon|^2 dy \leq \frac{\varepsilon}{(\ln \varepsilon)^2} \]

and

\[ \int_{\Sigma} |\nabla u_\varepsilon|^2_g dA_g \leq \int_{\Sigma} |\nabla u_\varepsilon|^2_g dA_g + 2 \int_{\Sigma} |\nabla u_\varepsilon| |(u_\varepsilon - u_\varepsilon)| dA_g + \int_{\Sigma} |\nabla (u_\varepsilon - u_\varepsilon)|^2 dA_g \]

\[ \leq \int_{\Sigma} |\nabla u_\varepsilon|^2_g dA_g + 2C \varepsilon^2 \|\nabla (u_\varepsilon - u_\varepsilon)\|_{L^2} + \|\nabla (u_\varepsilon - u_\varepsilon)\|^2_{L^2} \]

as $\varepsilon \to 0$, where we used conformality of $\varepsilon^2 \varphi_0$ and $\varepsilon^2 \varphi_1$ and again that $|u_\varepsilon - u_\varepsilon(p_0)| = O(\varepsilon^2)$ and $|u_\varepsilon - u_\varepsilon(p_1)| = O(\varepsilon^2)$ on $\text{supp} \nabla(\varepsilon^2 \varphi_0)$ and $\text{supp} \nabla(\varepsilon^2 \varphi_1)$, respectively. This gives the first term in the right-hand side of the inequality \(3.3\).

Now, we aim at proving the inequality

\[ \sigma_\varepsilon^1 \leq \frac{t_\varepsilon}{8} + \frac{t_\varepsilon \pi^2}{2(\ln r)^2} + O \left( \varepsilon \left( \ln \frac{1}{r} \right)^{-3} + \varepsilon^2 \right) \]

Indeed, we test the variational characterization of $\sigma_\varepsilon$ with the functions

\[ \phi(y) = \frac{\sqrt{t_\varepsilon} \sqrt{\ln y \frac{1}{\varepsilon}}}{\sqrt{\varepsilon}} \sin \left( \pi \left( \ln \frac{y}{\ln(r)} \right) \right) \]

That is, thanks to \(2.3\), we compute the quantities on the right hand sides in \(2.4\) and \(2.6\) for $f(v) = \sin(\pi v)$. We have that

\[ \int_0^1 \left( \frac{f}{2} + \frac{f}{\ln \frac{1}{r}} \right)^2 dv = \int_0^1 \frac{f^2}{4} dv + \int_0^1 \frac{f}{\ln \frac{1}{r}}^2 dv + \int_0^1 \frac{f f'}{\ln \frac{1}{r}} dv \]
and that
\[ \int_0^1 f f' \, dv = \int_0^1 \frac{(f^2)'}{2 \ln \frac{1}{r}} \, dv = \frac{f(1)^2 - f(0)^2}{2 \ln \frac{1}{r}} = 0. \]

We denote by \( I = \int_0^1 r^2 \sin(\pi v) \, dv \). Integrating by parts twice, we get
\[ I \left( 1 + \frac{4\pi^2}{(\ln r)^2} \right) = \frac{4\pi}{(\ln r)^2} \left( 1 + r^2 \right) \]
so that
\[ \varepsilon^{\frac{1}{2}} \sqrt{\ln \frac{1}{r}} \int_0^1 r^2 f(v) \sqrt{1 + \varepsilon^2 r^2} \, dv = \varepsilon^{\frac{1}{2}} \sqrt{\ln \frac{1}{r}} I(1 + O(\varepsilon^2)) = O \left( \varepsilon^{\frac{1}{2}} \left( \ln \frac{1}{r} \right)^{-\frac{3}{2}} \right) \]
as \( \varepsilon \to 0 \). Thanks to (2.4), (2.5) and (2.6) we get from the variational characterization that
\[ \sigma^*_\varepsilon \leq \frac{\int_{\Omega_\varepsilon} |\nabla \phi|_{g_\varepsilon}^2 \, dA_{g_\varepsilon}}{\int_{I^+ \cup I^-} \phi^2 \, dl_{g_\varepsilon} - \left( \frac{\int_{I^+ \cup I^-} \phi \, dl_{g_\varepsilon}}{1 - 2\pi^2 + \frac{2}{\varepsilon^2} + O(\varepsilon^2)} \right)^2} \leq t_\varepsilon \frac{\int_0^1 \frac{f^2}{r} \, dv + \int_0^1 \frac{(f')^2}{(\ln \frac{1}{r})^2} \, dv}{2 \int_0^1 f^2 \, dv - O \left( \varepsilon \left( \ln \frac{1}{r} \right)^{-3} + \varepsilon^2 \right)} \]
as \( \varepsilon \to 0 \), which gives the inequality (3.4).

Now, in order to prove the inequality (3.3) on \( \sigma^*_\varepsilon^{K+1} \), it suffices to take the previous test functions (note that we have \( K \) linearly independent functions of the first type) and notice that they are orthogonal up to a small error term we shall compute:
\[ \int_{\Omega_\varepsilon} \langle \nabla u_\varepsilon, \nabla \phi \rangle_{g_\varepsilon} \, dA_{g_\varepsilon} = \varepsilon \int_0^1 y^2 u_\varepsilon, y \phi_y \sqrt{1 + \varepsilon^2 y^2} \, dy \]
\[ = \frac{t_\varepsilon^2 \varepsilon^{\frac{1}{2}}}{(\ln \frac{1}{r})^{\frac{1}{2}}} \int_0^1 r^2 \left( \varepsilon^2 f' + f \varepsilon^2 \right) \sqrt{1 + \varepsilon^2 r^2} \, dv \]
\[ = O \left( \varepsilon^{\frac{1}{2}} \left( \ln \frac{1}{r} \right)^{-\frac{5}{2}} \right) + O(\varepsilon^2) \]
where we use the computation of \( I = \int_0^1 r^2 \sin(\pi v) \, dv \) and we also compute \( \int_0^1 r^2 \cos(\pi v) \, dv \) by integration by parts. We also have that
\[ \int_{I^+ \cup I^-} u_\varepsilon \phi \, dl_{g_\varepsilon} = \frac{\varepsilon}{t_\varepsilon} \int_0^1 u_\varepsilon \phi \sqrt{1 + \varepsilon^2 y^2} \, dy \]
\[ = \frac{\varepsilon^{\frac{1}{2}} (\ln \frac{1}{r})^{\frac{1}{2}}}{t_\varepsilon^{\frac{1}{2}}} \int_0^1 r^2 \left( (u_\varepsilon(p_0) - u_\varepsilon(p_1))v + u_\varepsilon(p_0) \right) f(v) \sqrt{1 + \varepsilon^2 r^2} \, dv \]
\[ = O \left( u_\varepsilon(p_0) \varepsilon^{\frac{1}{2}} \left( \ln \frac{1}{r} \right)^{-\frac{3}{2}} + \varepsilon \varepsilon^{\frac{1}{2}} \left( \ln \frac{1}{r} \right)^{-\frac{5}{2}} + \varepsilon^2 \right), \]
where we can prove by several integrations by parts that \( \int_0^1 r^2 \varepsilon^2 v \sin(\pi v) \, dv = O \left( (\ln \frac{1}{r})^{-3} \right) \).
Moreover, if \( u_{\star} \) and \( v_{\star} \) are two orthonormal \( \sigma_{\star} \)-eigenfunctions on \( \Sigma \), we have that

\[
\int_{\Sigma} |(\nabla u_{\varepsilon}, \nabla v_{\varepsilon})| dA_{\varepsilon} \leq \int_{\Sigma} |\nabla u_{\star}| |\nabla (v_{\varepsilon} - v_{\star})| + \int_{\Sigma} |\nabla v_{\star}| |\nabla (u_{\varepsilon} - u_{\star})| \\
+ \int_{\Sigma} |(\nabla u_{\varepsilon} - u_{\star})| |\nabla (v_{\varepsilon} - v_{\star})| + \varepsilon \int_{r}^{1} y^2 u_{\varepsilon,y} v_{\varepsilon,y} dy \\
\leq C_{\varepsilon}^{4} + C_{\varepsilon} \frac{(u_{\star}(p_0) - u_{\star}(p_1))(v_{\star}(p_0) - v_{\star}(p_1))}{(\ln \frac{1}{r})^2} \\
\leq C_{\varepsilon}^{4} + \frac{C_{\varepsilon}}{(\ln \frac{1}{r})^2}.
\]

Similarly, assuming that \( u_{\star}(p_0) = 0 \), we also have that

\[
\int_{\partial \Sigma} |u_{\varepsilon} v_{\varepsilon}| dl_{\varepsilon} \leq C_{\varepsilon}^{2} + C_{\varepsilon} \frac{\int_{r}^{1} y^2 u_{\varepsilon,y} v_{\varepsilon,y} dy}{\ln r} \\
\leq C_{\varepsilon}^{4} + \frac{C_{\varepsilon}}{\ln \left(\frac{1}{r}\right)}.
\]

Finally let us choose \( u_{1,\star}^{\varepsilon}, \ldots, u_{K,\star}^{\varepsilon} \) an orthonormal basis of \( \sigma_{\star} \)-eigenfunctions such that \( u_{i,\star}^{\varepsilon}(p_0) = 0 \) for \( i \geq 2 \) and denote by \( u_{1,\varepsilon}^{\star}, \ldots, u_{K,\varepsilon}^{\star} \) the corresponding extensions constructed above. The estimate (3.3) now easily follows from the estimates above and the variational characterization of eigenvalues applied to the space spanned by \( u_{1,\varepsilon}^{\star}, \ldots, u_{K,\varepsilon}^{\star}, \phi \) from above. \( \Box \)

4. Pointwise estimates on eigenfunctions

We first aim at giving pointwise estimates on eigenfunctions at the neighbourhood of \( p_0 \) and \( p_1 \) on \( \Sigma \) in order to get some control on the boundary values of the eigenvalue equation on \( \Omega_{\varepsilon} \). More precisely, we want to get estimates on \( \bar{\theta}(0) \) and \( \bar{\theta}(1) \), where we use the change of variables to the function \( \theta \) as described in Section 2.3.1. It is natural to compare them to \( u_{\star}(p_0) \) and \( u_{\star}(p_1) \), where \( u_{\star} \) is the weak limit in \( W^{1,2}(\Sigma) \) (and strong limit in \( L^2(\partial \Sigma) \)) of \( u_{\varepsilon} \). Notice that by the elliptic estimates of [Rob11] on domains with corners, the functions \( u_{\varepsilon} \) are \( C^{0,\alpha} \) up to the boundary. We have the following compatibility conditions:

\[
\begin{align*}
(4.1) \quad & \frac{1}{\varepsilon^2} \int_{-\frac{\varepsilon^2}{2}}^{\frac{\varepsilon^2}{2}} (u_{\varepsilon} \circ \varphi_{0}^{-1})(x,0) dx = \bar{\varphi}(0) = \frac{\sqrt{\varepsilon} \bar{\theta}(0)}{\sqrt{\ln \frac{1}{r}}} \\
(4.2) \quad & \frac{1}{r^2 \varepsilon^2} \int_{-\frac{r^2 \varepsilon^2}{2}}^{\frac{r^2 \varepsilon^2}{2}} (u_{\varepsilon} \circ \varphi_{1}^{-1})(x,0) dx = \bar{\varphi}(r) = \frac{\sqrt{\varepsilon} r^{-\frac{1}{2}} \bar{\theta}(1)}{\sqrt{\ln \frac{1}{r}}}.
\end{align*}
\]

Because of the factor \( r^{-\frac{1}{2}} \) in (4.2) that is not present in (4.1), we see that \( \bar{\theta}(0) \) and \( \bar{\theta}(1) \) do not play the same role. More precisely, \( \bar{\theta}(1) \) will be much smaller. Therefore, we will never need a very precise estimate in the neighborhood of \( p_1 \).
Claim 4.3. Let \( u_\varepsilon \) be an \( L^2(\partial \Sigma_\varepsilon) \)-normalized \( \sigma_i^l \)-eigenfunction for \( l \in \{1, \ldots, K+1\} \). We have a constant \( C > 0 \) independent of \( \varepsilon \) and \( r \) such that
\[
|u_\varepsilon \circ \varphi_i^{-1}(x)| \leq C \ln \frac{1}{r \varepsilon}
\]
for any \( x \in D_{r \varepsilon}^{+} \) up to the boundary, and
\[
|u_\varepsilon \circ \varphi_0^{-1}(x)| \leq C \ln \frac{1}{\varepsilon}
\]
for any \( x \in D_{\varepsilon}^{+} \) up to the boundary. More precisely,
\[
|u_\varepsilon \circ \varphi_0^{-1}(x) - u_\varepsilon(p_0)| \leq C \left( \|u_\varepsilon - u_\varepsilon\|_{W^{1,2}(\Sigma)} + \|\nabla \phi_\varepsilon\|_{L^2(F)} + |\sigma_\varepsilon - \sigma_*| + \varepsilon + b_\varepsilon \ln \frac{1}{\varepsilon} \right)
\]
for any \( x \in \partial D_{\varepsilon}^{+} \) up the boundary, where \( F_\varepsilon = \{(x,y) \in \Omega; 1 - \varepsilon \leq y \leq 1\} \) and
\[
b_\varepsilon = \frac{\varepsilon}{\pi} \varphi_y(1).
\]
Proof. We set
\[
m_i^\varepsilon(\rho) = \frac{1}{\pi} \int_0^\pi (u_\varepsilon \circ \varphi_i^{-1})(\rho \cos \theta, \sin \theta)) d\theta
\]
for \( i = 0, 1 \), the mean value of \( u_\varepsilon \) on the arc of radius \( \rho \) in the neighbourhood of \( p_i \) in the conformal chart \( \varphi_i \). Since the charts \( \varphi_i \) are conformal and thanks to the eigenvalue equation on \( \partial \Sigma \setminus (A_0 \cup A_1) \), we know that
\[
\begin{aligned}
\Delta (u_\varepsilon \circ \varphi_i^{-1}) &= 0 & \text{in } D_2^+ \\
-\partial_y (u_\varepsilon \circ \varphi_i^{-1}) &= \varepsilon^{10} \sigma_\varepsilon (u_\varepsilon \circ \varphi_0^{-1}) & \text{on } [-2,2] \times \{0\} \setminus \left( [-\frac{\rho^2}{2}, \frac{\rho^2}{2}] \times \{0\} \right) \text{ if } i = 0 \\
-\partial_y (u_\varepsilon \circ \varphi_0^{-1}) &= \varepsilon^{10} \sigma_\varepsilon (u_\varepsilon \circ \varphi_1^{-1}) & \text{on } [-2,2] \times \{0\} \setminus \left( [-\frac{\rho^2}{2}, \frac{\rho^2}{2}] \times \{0\} \right) \text{ if } i = 1
\end{aligned}
\]
so that
\[
-\frac{1}{\rho} \partial_\rho \left( \rho (m^\varepsilon_i)'(\rho) \right) = \frac{\varepsilon^{10} \sigma_\varepsilon}{\pi \rho} (u_\varepsilon \circ \varphi_i^{-1}(\rho,0) + u_\varepsilon \circ \varphi_i^{-1}(-\rho,0))
\]
for any \( \frac{\rho^2}{2} < \rho \leq 1 \) if \( i = 0 \) and for any \( \frac{\rho^2}{2} < \rho \leq 1 \) if \( i = 1 \). We integrate this equation to obtain
\[
\rho (m^\varepsilon_i)'(\rho) = (m^\varepsilon_i)'(1) - \frac{\sigma_\varepsilon}{\pi} \int_{J_1 \setminus J_\rho} u_\varepsilon d\sigma
\]
where \( J_s = \varphi_i^{-1}([-s, s] \times \{0\}) \). Integrating again, we get that
\[
m^\varepsilon_i(\rho) = m^\varepsilon_i(1) + \ln \rho (m^\varepsilon_i)'(1) - \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon d\sigma \right) ds.
\]
Moreover, we have by Hölder’s inequality that
\[
\left| \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon d\sigma \right) ds \right| \leq \int_1^\rho \frac{1}{s} \int_{J_1 \setminus J_\rho} |u_\varepsilon| d\sigma ds \leq \left( \ln \frac{1}{\rho} \right) \sqrt{N} \sqrt{L_g(J_1 \setminus J_\rho)}.
\]
which then implies by the estimates above that

\[ \left| m^i_\varepsilon(\rho) \right| \leq \left| m^i_\varepsilon(1) \right| + \left| (m^i_\varepsilon)'(1) \right| \left( \ln \frac{1}{\rho} \right) + \frac{\sigma_\varepsilon}{\pi} \sqrt{N} \sqrt{L_g(J_1 \setminus J_\rho)} \left( \ln \frac{1}{\rho} \right). \]

By standard elliptic theory on (4.9), we know that

\[ \left| m^i_\varepsilon(1) \right| + \left| (m^i_\varepsilon)'(1) \right| \leq C \sqrt{N} \]

as \( \varepsilon \to 0 \) so that we obtain

(4.11) \[ \left| m^i_\varepsilon(\rho) \right| \leq C \sqrt{N} \left( 1 + \ln \frac{1}{\rho} \right) \]

for \( \rho \geq \varepsilon^2 \) if \( i = 0 \) and for \( \rho \geq r\varepsilon^2 \) if \( i = 1 \).

Let us be more precise for \( i = 0 \). We set

\[ b_\varepsilon = \frac{\varepsilon^2}{2} \left( m^0_\varepsilon \right)' \left( \frac{\varepsilon^2}{2} \right). \]

Notice that by the computation above (more precisely (4.10)) and a similar application of Green’s formula, we have that

(4.12) \[ b_\varepsilon = \left( m^0_\varepsilon \right)'(1) - \frac{\sigma_\varepsilon}{\pi} \int_{J_1 \setminus J_\frac{1}{2}^\varepsilon} \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon dl_g \right) ds = \frac{\varepsilon}{\pi} \varphi_0(1), \]

so that in particular our notation for \( b_\varepsilon \) is consistent with (4.7). By integrating (4.12) as before we then deduce the following formula

\[ m^0_\varepsilon(\rho) = m^0_\varepsilon(1) + \ln \rho \left( m^0_\varepsilon \right)'(1) - \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon dl_g \right) ds \]

\[ = m^0_\varepsilon(1) + b_\varepsilon \ln \rho + \ln \rho \frac{\sigma_\varepsilon}{\pi} \int_{J_1 \setminus J_\frac{1}{2}^\varepsilon} u_\varepsilon dl_g - \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon dl_g \right) ds \]

\[ = m^0_\varepsilon(1) + b_\varepsilon \ln \rho + \ln \rho \frac{\sigma_\varepsilon}{\pi} \int_{J_1 \setminus J_\rho} u_\varepsilon dl_g + \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon dl_g \right) ds \]

\[ = m^0_\varepsilon(1) + b_\varepsilon \ln \rho + \ln \rho \frac{\sigma_\varepsilon}{\pi} \int_{J_1 \setminus J_\rho} u_\varepsilon dl_g + \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_s} u_\varepsilon dl_g \right) ds. \]

We can do the very same computation for \( m^0_\star \), the mean value of \( u_\star \) on the arc of radius \( \rho \) in the neighbourhood of \( p_0 \) in the conformal chart \( \varphi_0 \), where \( u_\star \circ \varphi_0^{-1} \) satisfies the same equation with eigenvalue \( \sigma_\star \) but now also along \( J_{\varepsilon^2} \) (which does not hold for \( u_\varepsilon \)). This gives that

\[ m^0_\star(\rho) = m^0_\star(1) + \ln \rho \frac{\sigma_\star}{\pi} \int_{J_\rho} u_\star dl_g + \frac{\sigma_\star}{\pi} \int_1^\rho \frac{1}{s} \left( \int_{J_1 \setminus J_\rho} u_\star dl_g \right) ds. \]
The difference between \( m_\varepsilon^0 \) and \( m_\varepsilon^0 \) gives
\[
m_\varepsilon^0(\rho) = m_\varepsilon^0(\rho) + b_\varepsilon \ln \rho + (m_\varepsilon^0 - m_\varepsilon^0)
\]
(1)
\[
+ \ln \rho \left( \frac{\sigma_\varepsilon}{\pi} \int J_\rho \Delta v_\varepsilon \, dg + \frac{\sigma_\varepsilon - \sigma_*}{\pi} \int J_\rho \frac{u_\varepsilon - u_*}{\rho} \, dg \right)
\]
\[
+ \frac{\sigma_\varepsilon}{\pi} \int_1^\rho \frac{1}{s} \left( \int J_\rho \frac{u_\varepsilon - u_*}{\rho} \, dg \right) ds + \frac{\sigma_\varepsilon - \sigma_*}{\pi} \int_1^\rho \frac{1}{s} \left( \int J_\rho u_\varepsilon \, dg \right) ds.
\]
Moreover, we can estimate
\[
\left| \int_1^\rho \frac{1}{s} \left( \int J_\rho (u_\varepsilon - u_*) \, dg \right) \right| ds \leq \int_1^\rho \frac{1}{s} \sqrt{L_g(J_\rho \setminus J_\rho)} |u_\varepsilon - u_*| L^2(J_\rho \setminus J_\rho) ds
\]
\[
\leq C |u_\varepsilon - u_*| L^2(\partial \Omega) \int_1^\rho \frac{(s - \rho)^{1/2}}{s} ds
\]
\[
\leq C |u_\varepsilon - u_*| L^2(\partial \Omega)
\]
and similarly for the last term in the difference of the mean values above. Therefore, we get by standard elliptic estimates on a compact subset of \( \Omega \setminus \{p_0, p_1\} \) and Hölder’s inequality that there is a constant \( C \) independent of \( \varepsilon \) and \( \rho \) such that
\[
|m_\varepsilon^0(\rho) - m_\varepsilon^0(\rho) - b_\varepsilon \ln \rho| \leq C \left( |u_\varepsilon - u_*| L^2(\partial \Omega) + |\sigma_\varepsilon - \sigma_*| + \varepsilon^2 \ln \frac{1}{\rho} \right)
\]
for any \( \rho \in D \setminus D_\varepsilon \).

Now we look at \( u_\varepsilon \) in the charts \( f_i \) and \( g_i \) that we used to define the gluing between \( \Omega_\varepsilon \) and \( \Omega \) at the neighborhood of \( p_i \) for \( i = 0, 1 \). We denote by \( v_\varepsilon^i \) the functions in these charts and the equation (4.9) becomes
\[
\begin{align*}
\Delta v_\varepsilon^0 &= 0 & \text{in } D_\varepsilon^+ \cup E_0 \\
-\partial_y v_\varepsilon^0 &= \varepsilon^2 e^{\omega_0(\varepsilon^2(x,z))} \sigma_\varepsilon v_\varepsilon^0 & \text{in } [-\frac{2}{r_\varepsilon^2}, \frac{2}{r_\varepsilon^2}] \times \{0\} \setminus \left( \left[-\frac{1}{r_\varepsilon^2}, \frac{1}{r_\varepsilon^2}\right] \times \{0\} \right) \\
\partial_{\nu_0} v_\varepsilon^0 &= \frac{\varepsilon^2}{r_\varepsilon^2} \sigma_\varepsilon v_\varepsilon^0 & \text{if } r_\varepsilon^2 \leq z \leq 0 \text{ and } x = \pm \left(1 + \varepsilon^2\right)^{\frac{1}{2}} \\
\Delta v_\varepsilon^1 &= 0 & \text{in } D_\varepsilon^+ \cup E_1 \\
-\partial_y v_\varepsilon^1 &= \varepsilon^2 e^{\omega_1(\varepsilon^2(x,z))} \sigma_\varepsilon v_\varepsilon^1 & \text{in } [-\frac{2}{r_\varepsilon^2}, \frac{2}{r_\varepsilon^2}] \times \{0\} \setminus \left( \left[-\frac{1}{r_\varepsilon^2}, \frac{1}{r_\varepsilon^2}\right] \times \{0\} \right) \\
\partial_{\nu_0} v_\varepsilon^1 &= \frac{\varepsilon^2}{r_\varepsilon^2} \sigma_\varepsilon v_\varepsilon^1 & \text{if } \frac{r_\varepsilon^2}{1 + \varepsilon^2} \leq z \leq 0 \text{ and } x = \pm \left(1 - \varepsilon^2\right)^{\frac{1}{2}}
\end{align*}
\]
where \( E_i \) for \( i = 0, 1 \), defined by (2.1) and (2.2), are the image of \( \Omega_\varepsilon \) under the charts \( g_i \), and \( \nu_\varepsilon^\pm \) is the outpointing normal on the boundaries of \( E_i \) endowed with the flat metric. By elliptic regularity on domains with corners [Rob11], we know that \( v_\varepsilon^i \in C^{0,\alpha} \) and we have the estimates
\[
\|v_\varepsilon^0 - m_\varepsilon^0(\varepsilon^2)\|_{C^k(F_0)} \leq C(\varepsilon + 1)
\]
\[
\|v_\varepsilon^1 - m_\varepsilon^1(\varepsilon^2)\|_{C^k(F_1)} \leq C(\varepsilon + 1)
\]
for some constant \( C \) as soon as \( F_i \) is a bounded set at the neighborhood of \((0,0)\). Therefore, (4.5) and (4.4) hold true. The energy which appears in the right-hand term of the pointwise estimate for \( i = 0 \) gives the precise estimate (4.6). \( \Box \)
5. **Asymptotic Expansion on the First Eigenvalue and First Eigenfunction**

In this section we prove our main technical tool, a precise asymptotic expansion of an eigenfunction on the cuspidal domain with control on boundary values given by Claim 4.3.

5.1. **Preliminary computations.** Let \( u_\varepsilon \) be an eigenfunction associated to \( \sigma_1^\varepsilon \) with unit norm, that is \( \int_{\partial \Sigma_\varepsilon} u_\varepsilon^2 d\varepsilon = 1 \). Integrating the equation satisfied by \( u_\varepsilon \), we get that

\[
\int_{\Sigma} |\nabla u_\varepsilon|^2 g_\varepsilon dA_\varepsilon - \sigma_\varepsilon N + \int_{\Omega_\varepsilon} |\nabla \phi_\varepsilon|^2 g_\varepsilon dA_\varepsilon = \sigma_\varepsilon M ,
\]

where we write \( \phi_\varepsilon = u_\varepsilon \big|_{\Omega_\varepsilon} \) for the eigenfunction \( u_\varepsilon \) in the chart \( \Omega_\varepsilon \) of \( \Sigma_\varepsilon \),

\[
M = \int_{I^+ \cup I^-} \phi_\varepsilon^2 d\varepsilon \quad \text{and} \quad N = \int_{\partial \Sigma \setminus (A_0 \cup A_1)} u_\varepsilon^2 d\varepsilon
\]

are the boundary masses of the eigenfunctions on the cuspidal domain and on the surface where

\[
A_0 = \varphi_0^{-1} \left( \mathbb{D}_r^+ \right) \cap \partial \Sigma \quad \text{and} \quad A_1 = \varphi_1^{-1} \left( \mathbb{D}_r^2 \right) \cap \partial \Sigma.
\]

Notice that by assumption \( M + N = 1 \). We define a new function \( \theta \) by the change of variables (2.3), and thanks to (2.6) we can rewrite the gradient term over \( \Omega_\varepsilon \) so that (5.1) becomes

\[
\delta_\varepsilon + \frac{1}{2 \varepsilon^2} \| \theta \|_{L^2(\tilde{\Omega})}^2 + \int_0^1 \left( \frac{\theta^2 + \theta_v}{2 + \ln \frac{1}{r}} \right)^2 dv = \sigma_\varepsilon M ,
\]

where

\[
\delta_\varepsilon = \int_{\Sigma} |\nabla u_\varepsilon|^2 g_\varepsilon dA_\varepsilon - \sigma_\varepsilon N
\]

and we have that

\[
\int_0^1 \left( \frac{\theta + \theta_v}{2 + \ln \frac{1}{r}} \right)^2 dv = \int_0^1 \left( \frac{\theta^2}{4} + \frac{\theta \theta_v}{\ln \frac{1}{r} + \frac{\theta^2}{\ln \frac{1}{r}}} \right) dv ,
\]

so that (5.2) becomes

\[
M \left( \frac{\sigma_\varepsilon}{\delta_\varepsilon} - \frac{1}{8} \right) (1 + O(\varepsilon^2)) = \frac{\delta_\varepsilon}{\varepsilon^2} + \frac{\| \theta \|_{L^2(\tilde{\Omega})}^2}{\varepsilon^2} + \frac{1}{(\ln r)^2} \int_0^1 \theta_v^2 dv + I_1 + I_2 ,
\]

where

\[
I_1 = \frac{1}{8} \int_0^1 \left( 2 \theta^2 - \theta^2 \left( \frac{\theta^2}{2}, v \right) - \theta^2 \left( \frac{-\theta^2}{2}, v \right) \right) dv
\]

and

\[
I_2 = \frac{1}{\ln \frac{1}{r}} \int_0^1 \theta_v dv .
\]

We have that

\[
\begin{align*}
\theta \theta_v &= \left( \theta - \bar{\theta} \right) \theta_v + \theta \theta_v = \left( \theta - \bar{\theta} \right) \theta_v + \bar{\theta} \cdot \theta_v .
\end{align*}
\]

But since \( \left( \bar{\theta} \right)' = \bar{\theta} + \ln \left( \frac{1}{r} \right) \left( 2 \bar{\theta} - \theta \left( \frac{\theta^2}{2}, v \right) - \theta \left( \frac{-\theta^2}{2}, v \right) \right) \) and

\[
2 \int_0^1 \bar{\theta} \left( \bar{\theta} \right)' dv = \bar{\theta}^2 (1) - \bar{\theta}^2 (0) ,
\]
we get
\[ I_2 = \frac{1}{2 \ln \frac{r}{\eta}} \left( \tilde{\theta}'(1) - \tilde{\theta}'(0) \right) + I_3, \]
where
\[ I_3 = \frac{1}{\ln \frac{r}{\eta}} \int_0^1 \left( \frac{\theta - \bar{\theta}}{\eta} \right) \theta_v dv - \int_0^1 \tilde{\theta} \left( 2\tilde{\theta} - \theta \left( \frac{r^2v}{2}, v \right) - \theta \left( -\frac{r^2v}{2}, v \right) \right) dv. \]

Let \(-\frac{r^2v}{2} \leq x \leq \frac{r^2v}{2}\), then
\begin{equation}
|\theta(x, v) - \bar{\theta}(v)| = r^{-2v} \left| \int_{-\frac{r^2v}{2}}^{\frac{r^2v}{2}} \left( \int_x^s \theta_v(t, v) dt \right) ds \right| \leq \left( r^{4v} \tilde{\theta}^2(v) \right)^{1/2}.
\end{equation}

Note that since \(r^{2v} \leq 1\) this implies that we also have that
\begin{equation}
||\tilde{\theta}||_{L^2(0, 1)} \leq \left| \tilde{\theta} - \theta \left( \frac{r^{2v}}{2}, v \right) \right|_{L^2(0, 1)} + \sqrt{M} \leq ||\theta_x||_{L^2(\tilde{\Omega})} + \sqrt{M}.
\end{equation}

Similarly, discarding \(r^{2v} \leq 1\) again, thanks to Hölder’s inequality,
\[ |I_1| \leq \frac{1}{8} \left( \left| \theta + \theta \left( \frac{r^{2v}}{2}, v \right) \right|_{L^2(0, 1)} + \left| \tilde{\theta} + \theta \left( \frac{-r^{2v}}{2}, v \right) \right|_{L^2(0, 1)} \right) \left| \theta_x \right|_{L^2(\tilde{\Omega})} \]
\[ \leq \frac{1}{2} \left( ||\theta_x||_{L^2(\tilde{\Omega})} + \sqrt{M} \right) \left| \theta_x \right|_{L^2(\tilde{\Omega})}, \]
and
\[ \int_0^1 \left| (\theta - \bar{\theta}) \theta_v \right| dv \leq \int_0^{\frac{r^2v}{2}} \int_{-\frac{r^2v}{2}}^{\frac{r^2v}{2}} \tilde{\theta}_v^2 \theta_v dx dv \]
\[ \leq \left( \int_0^{\frac{r^2v}{2}} \int_{-\frac{r^2v}{2}}^{\frac{r^2v}{2}} \tilde{\theta}_v^2 dx dv \right)^{1/2} \left( \int_0^{\frac{r^2v}{2}} \int_{-\frac{r^2v}{2}}^{\frac{r^2v}{2}} \theta_v^2 dx dv \right)^{1/2} \]
\[ \leq ||\theta_x||_{L^2(\tilde{\Omega})} \left( \int_0^{\frac{r^2v}{2}} \tilde{\theta}_v^2 dv \right)^{1/2}. \]

After discarding \(r^{2v} \leq 1\) in the line above we then find from the above estimates that
\[ |I_3| \leq \frac{1}{\ln \frac{r}{\eta}} \left( \int_0^{\frac{r^2v}{2}} \tilde{\theta}_v^2 dv \right)^{1/2} ||\theta_x||_{L^2(\tilde{\Omega})} + 2 ||\tilde{\theta}||_{L^2(0, 1)} \left| \theta_x \right|_{L^2(\tilde{\Omega})}. \]

Gathering our estimates, using again that \(||\theta_x||_{L^2(\tilde{\Omega})} \leq \varepsilon\) by \([5,4]\), we then have thanks to Young’s inequality that
\begin{equation}
e = |I_1| + |I_3| = O \left( \frac{||\theta_x||_{L^2(\tilde{\Omega})}^2}{\eta} \right) \left( \frac{\left| \tilde{\theta}_v^2 \right|_{L^2(0, 1)}^2}{\eta^2} \right) + \frac{\eta}{(\ln r)^2} + \eta M. \end{equation}
for any $\eta \in (0, 1]$. Now, we can rewrite (5.4) as follows (5.8)

$$M \left( \sigma_\varepsilon - \frac{1}{8} \right) = \frac{\delta_\varepsilon}{t_\varepsilon} + \frac{\|\theta_\varepsilon\|_{L^2(\tilde{\Omega})}^2}{\varepsilon^2} + \frac{1}{(\ln r)^2} \int_0^1 \frac{\bar{\theta}_\varepsilon^2 dv}{\varepsilon} + \frac{1}{2 \ln \frac{1}{r}} \left( \bar{\theta}_\varepsilon^2(1) - \bar{\theta}_\varepsilon^2(0) \right) + O(\varepsilon + \varepsilon^2 M) .$$

The formula (5.8) gives the main connection between the behaviour of the eigenfunction $u_\varepsilon$ on the thick part $\Sigma$ and $\Omega_\varepsilon$ the thin part.

We now incorporate our pointwise estimate Claim 4.3 into (5.8). Notice that by (4.4) at the neighbourhood of $p_1$ and by (4.2), we have that

$$\frac{\bar{\theta}_\varepsilon^2(1)}{2 \ln \frac{1}{r}} = \frac{r_\varepsilon}{t_\varepsilon} \frac{\bar{\phi}(1)^2}{2} = O \left( \frac{r_\varepsilon}{t_\varepsilon} \left( \ln \frac{1}{r_\varepsilon} \right)^2 \right)$$

and (5.8) becomes

(5.9)

$$M \left( \sigma_\varepsilon - \frac{1}{8} \right) = \frac{\delta_\varepsilon}{t_\varepsilon} + \frac{\|\theta_\varepsilon\|_{L^2(\tilde{\Omega})}^2}{\varepsilon^2} + \frac{1}{(\ln r)^2} \int_0^1 \frac{\bar{\theta}_\varepsilon^2 dv - \bar{\theta}_\varepsilon^2(0)}{2 \ln \frac{1}{r}} + O \left( \varepsilon + \varepsilon^2 M + r_\varepsilon \left( \ln \frac{1}{r_\varepsilon} \right)^2 \right) .$$

Since $L_g(\partial \Sigma) = 1$ by the Poincaré trace inequality, we have that

$$\sigma_\ast \left( N - \left( \int_{\partial \Sigma} u_\varepsilon dl_g \right)^2 \right) \leq \sigma_\ast \left( \int_{\partial \Sigma} u_\varepsilon^2 dl_g - \left( \int_{\partial \Sigma} u_\varepsilon dl_g \right)^2 \right) \leq \int_{\Sigma} |\nabla u_\varepsilon|^2 dA_g,$$

where we recall that $\sigma_\ast$ is the first non-zero Steklov eigenvalue on $\Sigma$. This immediately implies that $\delta_\varepsilon$ defined by (5.3) can be estimated according to

(5.10)

$$N \left( \sigma_\ast - \sigma_\varepsilon \right) \leq \delta_\varepsilon + \sigma_\ast \left( \int_{\partial \Sigma} u_\varepsilon dA_g \right)^2 .$$

Since $u_\varepsilon$ has zero mean value on $\partial \Sigma_\varepsilon$, we have that

$$\left( \int_{\partial \Sigma \setminus (A_0 \cup A_1)} u_\varepsilon dl_g \right)^2 = \left( \int_{I^+ \cup I^-} \phi_\varepsilon dl_\varepsilon \right)^2 .$$

Moreover, we have thanks to Claim 4.3 that

$$\int_{A_0 \cup A_1} u_\varepsilon dl_g = O \left( \varepsilon^2 \ln \varepsilon \right)$$

so that we find

(5.11)

$$\left( \sigma_\ast - \sigma_\varepsilon \right) \leq \frac{\delta_\varepsilon}{N} + \frac{2 \sigma_\ast}{N} \left( \left( \int_{I^+ \cup I^-} \phi_\varepsilon dl_\varepsilon \right)^2 + O \left( \varepsilon^4 \left( \ln \varepsilon \right)^2 \right) \right),$$

where we also remark that

$$\left( \int_{I^+ \cup I^-} \phi_\varepsilon dl_\varepsilon \right)^2 = O(M\varepsilon)$$

thanks to Hölder’s inequality.

The first idea is then to make $\delta_\varepsilon$ as small as possible in order to have the expected inequality.
5.2. Choice of the parameter $t_\varepsilon$. We now aim at choosing the adapted parameters $t_\varepsilon$ (near $t_*$) and $r_\varepsilon$ for which we we have a chance to minimize $\delta_\varepsilon$. We choose

$$r_\varepsilon = \exp\left(-\frac{1}{\varepsilon^\alpha}\right)$$

for $0 < \alpha < \frac{1}{2}$ and. Let us also introduce the parameter

$$t_* = 8\sigma_*.$$

The following claim is our tool to make a good choice for $t_\varepsilon$.

**Claim 5.12.** Let $\alpha$ and $r_\varepsilon$ be as above and $2\alpha < \tau < 1$. For any $\eta > 0$, there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $\varepsilon^{1-\tau} < \xi_\varepsilon < 1 - \eta$, there is $t_\varepsilon > 0$ (converging to $t_*$ as $\varepsilon \to 0$) such that $M = \xi_\varepsilon$ for some first eigenfunction $u_\varepsilon$.

**Remark 5.13.** We have not ruled out the possibility that the first eigenvalue of $\Sigma_\varepsilon$ has multiplicity. In particular note that the assignment $t \mapsto M$ might not be a well-defined map but depends on the choice of a normalized first eigenfunction.

**Proof.** We define a function $\tilde{M} : [t_0, t_1] \to [0, 1]$ as follows.

$$\tilde{M} = \inf_{\{u \in E_1 : \|u\|_{L^2(\partial\Sigma_\varepsilon)} = 1\}} \int_{t_1^+ \cup t_1^-} |u|^2 \, dl_\varepsilon$$

where $E_1$ denotes the space of $\sigma_1^\varepsilon$-eigenfunctions (note that all of this depends on the parameter $t$ as well).

In a first step we argue that

$$\tilde{M}(t_1) \leq \varepsilon^{1-\tau}$$

for $t_1 > t_*$ fixed and $\varepsilon \leq \varepsilon_0$.

If for $t_1 > t_*$, we have $\tilde{M}(t) > \varepsilon^{1-\tau}$ along a sequence $\varepsilon_n \to 0$, then

$$\frac{(\ln r)^2 \varepsilon^2 (1)^2}{2M} \leq C\varepsilon^{2-2\alpha} (\ln \varepsilon)^2 \to 0$$

for some constant $C$ as $\varepsilon \to 0$ by (4.5) and (4.1). Using also the inequality (5.11), where $\alpha < \frac{1}{2}$, combined with (3.2), we find from (5.9) that $(\ln r)^2 \left(\sigma_1^\varepsilon - \frac{1}{8}\right)$ is bounded from below. It is clear by (3.2) that it is also bounded from above by $\pi^2$. Therefore, it converges up to a subsequence to a constant $\lambda \in [0, \pi^2]$.

Therefore, we have that for any $t$ in a compact neighbourhood of $t_*$,

$$\sigma_1^\varepsilon = t \left(\frac{1}{8} + \frac{\lambda}{2 (\ln r)^2} + o \left(\frac{1}{(\ln r)^2}\right)\right),$$

which is impossible by the eigenvalue bound from (3.2) for $\varepsilon$ sufficiently small.

Next, we claim that

$$\lim_{\varepsilon \to 0} \tilde{M}(t_0) = 1.$$  

(5.14)

For $t_0 < t_*$ fixed we take $u_\varepsilon$ a normalized first eigenfunction. We have that $u_\varepsilon$ is bounded in $W^{1,2}(\Sigma)$ since

$$\int_{\Sigma} |\nabla u_\varepsilon| \leq \sigma_1^\varepsilon \leq \sigma_* + o(1)$$
thanks to (3.2), and
\[ \int_{\partial\Sigma} |u_\varepsilon|^2 \leq N + C\varepsilon^2 (\ln \varepsilon)^2 \]
by (4.5). Therefore, it follows from standard Sobolev trace theory that \((u_\varepsilon)\) is bounded \(W^{1,2}(\Sigma)\). Hence, after potentially taking a subsequence, \(u_\varepsilon\) converges weakly in \(W^{1,2}(\Sigma)\) and strongly in \(L^2(\partial\Sigma)\) to a function \(u_0\) on \(\Sigma\). Moreover, by ellipticity of the eigenvalue equation, \(u_\varepsilon\) converges in \(C^2_{\text{loc}}(\Sigma \setminus \{p_0, p_1\})\) to \(u_0\). Therefore, \(u_0\) satisfies a Steklov eigenvalue equation on \(\Sigma\) with eigenvalue \(\lim_{\varepsilon \to 0} \sigma_\varepsilon^1\). Thanks to the inequality (3.4), we know that \(\lim_{\varepsilon \to 0} \sigma_\varepsilon^1 \leq \frac{t}{8}\). Moreover, using that \(\frac{\sigma(0)^2}{\ln \frac{1}{\varepsilon}} = O(\varepsilon (\ln \varepsilon)^2)\) as \(\varepsilon \to 0\), that \(M + N = 1\), and the definition of \(\delta_\varepsilon\) in formula (5.9), we get that \(\lim_{\varepsilon \to 0} \sigma_\varepsilon^1 \geq \frac{t}{8}\). Therefore, we arrive at
\[ \lim_{\varepsilon \to 0} \sigma_\varepsilon^1 = \frac{t}{8} \in (0, \sigma_*) \]
where \(\sigma_*\) is the first non-zero Steklov eigenvalue on \(\Sigma\) and we conclude that \(u_0 = 0\). By strong convergence in \(L^2(\partial\Sigma)\) we then have that \(\lim_{\varepsilon \to 0} N = 0\) so that \(\lim_{\varepsilon \to 0} M = 1\). Since the argument applies to any sequence of eigenfunctions we conclude (5.14).

We can now decrease \(\varepsilon_0 > 0\) if necessary such that we have
\[ \tilde{M}(t_1) \leq \varepsilon^{1-\tau} < \xi < 1 - \eta \leq \tilde{M}(t_0) \]
for any \(\varepsilon \in (0, \varepsilon_0]\). In particular,
\[ t_\varepsilon = \sup\{t \in [t_0, t_1] : \tilde{M} > \xi\} \in (t_0, t_1) \]
is well-defined for \(\varepsilon \in (0, \varepsilon_0]\). We claim that for \(t_\varepsilon\) we can find a first eigenfunction on \(\Sigma_\varepsilon\) as desired.

In fact, by construction, we find sequences \(r_n \nearrow t_\varepsilon\) and \(s_n \searrow t_\varepsilon\) and associated normalized first eigenfunctions \(v_n\) and \(w_n\) respectively with
\[ \int_{I_\varepsilon^1 \cup I_\varepsilon^-} |v_n|^2 d\varepsilon \leq \xi \leq \int_{I_\varepsilon^1 \cup I_\varepsilon^-} |v_n|^2 d\varepsilon. \]
Thanks to (3.2) and the compact embedding \(W^{1,2}(\Sigma_\varepsilon) \to L^2(\partial\Sigma_\varepsilon)\) (note that this holds also along a sequence of parameters \(t\) since all the norms are uniformly equivalent for different choices of \(t \in [t_0, t_1]\) and \(\varepsilon > 0\) fixed) we may assume \(v_n \rightharpoonup v\) and \(w_n \rightharpoonup w\) weakly in \(W^{1,2}(\Sigma_\varepsilon)\) and strongly in \(L^2(\partial\Sigma_\varepsilon)\). We now have two options: If \(v = \pm w\) we are done immediately. In any other case we can easily find a linear combination of \(v\) and \(w\) with the desired property.  \(\diamondsuit\)

5.3. The asymptotic expansion. We now provide the precise asymptotic expansion of the first eigenfunction on the cuspidal domain.

Recall our choice of \(2\alpha < \tau < 1\) and \(r_\varepsilon = \exp\left(-\frac{1}{\varepsilon^\alpha}\right)\). Moreover, thanks to Claim 5.12 we may assume that \(t_\varepsilon\) is such that \(M \geq \varepsilon^{1-\tau}\) for some first eigenfunction. All the results in this section refer to this specific eigenfunction.

We first focus on the function \(\theta\) defined on \(\tilde{\Omega}\). We know that \(\phi\) satisfies
\[
\begin{cases}
\varepsilon^4 \Delta_{y^2}\phi := \phi_{xx} + \varepsilon^2 \phi_{yy} = 0 & \text{in } \Omega_\varepsilon \\
\frac{\partial \phi}{\partial t} := \frac{\pm \phi_x \phi_x - \varepsilon^2 y \phi_y}{\varepsilon^2 \sqrt{1 + \varepsilon^2 y^2}} = \frac{\sigma_x}{t_\varepsilon} \phi & \text{on } T_\varepsilon^\perp
\end{cases}
\]
(5.15)
where \( I^\pm_\varepsilon = \{(x, y) \in \mathbb{R}^2; r \leq y \leq 1, y = \pm \varepsilon^2 \frac{r}{2}\} \) and \( \nu^\pm_\varepsilon = \frac{t_\varepsilon (\pm 1, -\varepsilon^2 y)}{\varepsilon^2 \sqrt{1 + \varepsilon^2 y^2}} \) is the outward pointing normal along \( I^\pm_\varepsilon \) with respect to \( g_\varepsilon \). Therefore, \( \theta \) satisfies the following equation

\[
\begin{cases}
\frac{r^2}{2} v_{xx} + \varepsilon^2 \left( \frac{3\phi}{4} + \frac{\phi v}{\ln \frac{1}{\varepsilon^2 r^2}} \right) = 0 & \text{in } \tilde{\Omega} \\
\pm \theta_x + \varepsilon^2 \left( \frac{\theta}{2} + \frac{\theta v}{\ln \frac{1}{\varepsilon^2 r^2}} \right) = \varepsilon^2 \sqrt{1 + \varepsilon^2 r^2 \frac{\sigma_\varepsilon}{t_\varepsilon} \theta} & \text{if } x = \pm \frac{r^2}{2} \varepsilon^2 \sqrt{1 + \varepsilon^2 r^2} \varepsilon^2 \theta
\end{cases}
\]

Thanks to (5.16), \( \bar{\theta}_v = r^{-2v} \int_{-\frac{r^2}{2}}^{\frac{r^2}{2}} \theta_v(x, v) dx \) satisfies the equation

\[
- \bar{\theta}_v' = (\ln r)^2 \left( \theta \left( \frac{r^2}{2}, v \right) + \theta \left( -\frac{r^2}{2}, v \right) \right) \left( \frac{\sigma_\varepsilon}{t_\varepsilon} \sqrt{1 + \varepsilon^2 r^2} - \frac{1}{2} \right) + \frac{3}{4} \theta.
\]

That is, if we denote

\[
\mu(v) = \left( 2\bar{\theta}(v) - \theta \left( \frac{r^2}{2}, v \right) - \theta \left( -\frac{r^2}{2}, v \right) \right)
\]

and

\[
\eta = \sqrt{1 + \varepsilon^2 r^2 v - 1}
\]

we have the equation

\[
- \bar{\theta}_v' = \nu \bar{\theta} + (\ln r)^2 \left( 2 \frac{\sigma_\varepsilon}{t_\varepsilon} \eta \bar{\theta} + \left( \frac{1}{2} - \frac{\sigma_\varepsilon}{t_\varepsilon} (\eta + 1) \right) \mu \right)
\]

We also have that

\[
\bar{\theta}' = \left( \ln \frac{1}{r} \right) \mu + \bar{\theta}_v
\]

where we set

\[
\nu_\varepsilon = (\ln r)^2 \left( 2 \frac{\sigma_\varepsilon}{t_\varepsilon} - \frac{1}{4} \right)
\]

Thanks to (5.5), \( \|\mu\|_{L^2(0,1)} \) is controlled by \( \|\theta_x\|_{L^2(\Omega)} \), which has to be very small as \( \varepsilon \to 0 \). \( \|\eta\|_{L^\infty(0,1)} \) is controlled by \( \varepsilon^2 \) and is also very small. Therefore, the equation on \( \bar{\theta} \) resembles to \( f'' + \nu f = 0 \) as \( \varepsilon \to 0 \).

Let’s first prove that \( \nu_\varepsilon \) is positive for the choice of parameter \( t_\varepsilon \) we did in Claim 5.12, using the compatibility conditions (4.1) and (4.2).

Claim 5.21.

\[
\lim_{\varepsilon \to 0} \left( (\ln r)^2 \left( 2 \frac{\sigma_\varepsilon}{t_\varepsilon} - \frac{1}{4} \right) \right) = \pi^2
\]

Proof. Using that \( M \geq \varepsilon^{1-\tau}, 2\alpha < \tau < 1 \), and the pointwise estimates (4.5) in combination with the compatibility condition (4.1), we get that

\[
\frac{(\ln r)^2 \varepsilon^2 \theta(1)^2}{2M} \leq C \varepsilon^{-2\alpha} (\ln \varepsilon)^2 \to 0 \quad \text{as } \varepsilon \to 0.
\]
From this combined with (3.2), (5.9) and (5.11) we then deduce that \((\ln r)^2 \left(\frac{2\sigma_{\varepsilon}}{t_{\varepsilon}} - \frac{1}{4}\right)\) is bounded from below. Notice also that by (3.2), it is also bounded from above by \(\pi^2\). Therefore, up to taking a non-relabeled subsequence\(^2\), we may assume that

\[
(\ln r)^2 \left(\frac{2\sigma_{\varepsilon}}{t_{\varepsilon}} - \frac{1}{4}\right) \to \lambda \in [0, \pi^2].
\]

We then deduce from (5.5) and (5.9) that

\[
(5.23) \quad \left\| \frac{\mu}{\sqrt{M}} \right\|_{L^2(0,1)} \leq \left\| \frac{\theta_x}{c_0} \right\|_{L^2(0,1)} = O \left( \varepsilon \left( \ln \frac{1}{r} \right)^{-1} \right) \text{ as } \varepsilon \to 0.
\]

Moreover, by (5.6) we also have that \(\frac{\theta}{\sqrt{M}}\) is bounded in \(L^2(0,1)\). By (5.9) and Jensen’s inequality we find that \(\frac{\bar{\theta}_\nu}{\sqrt{M}}\) is bounded in \(L^2(0,1)\). Thanks to (5.18), we can then deduce that \(\frac{\bar{\theta}_\nu}{\sqrt{M}}\) is bounded in \(W^{1,2}(0,1)\). We may therefore take a subsequence such that

\[
\frac{\bar{\theta}_\nu}{\sqrt{M}} \to g \quad \text{weakly in } W^{1,2}(0,1) \text{ and strongly in } C_0(0,1).
\]

By (5.19) and (5.23), \(\frac{\theta}{\sqrt{M}}\) is bounded in \(W^{1,2}(0,1)\) and we may thus assume that

\[
\frac{\theta}{\sqrt{M}} \to f \quad \text{weakly in } W^{1,2}(0,1). \quad \text{Again, by (5.19) combined with (5.23), } \frac{\theta}{\sqrt{M}} \text{ converges strongly in } W^{1,2}(0,1) \text{ to } f \text{ and } f' = g. \quad \text{Thanks to the pointwise bounds (4.4) and (4.5) combined with the compatibility conditions (4.1) and (4.2), we have that } f(0) = 0 \text{ and } f(1) = 0. \quad \text{Therefore, passing to the limit in (5.18) we find that}
\]

\[
(5.24) \quad \begin{cases} f'' + \lambda f = 0 & \text{in } (0,1) \\ f = 0 & \text{if } v = 0 \text{ or } v = 1. \end{cases}
\]

Since \(\int_0^1 f^2(v)dv = \frac{1}{2}\), we have that \(f \neq 0\) and since \(\lambda \leq \pi^2\), we must have \(\lambda = \pi^2\) and \(f(v) = \pm \sin(\pi v)\), i.e. (5.22) holds true.

In this section, we first aim at proving the following Claim:

**Claim 5.25.** the \(C^1(0,1)\) asymptotic expansion of \(\theta\) and \(\bar{\theta}_\nu\)

\[
(5.26) \quad \begin{cases} \bar{\theta}(v) = d \cos(\sqrt{\nu} v) + e \sin(\sqrt{\nu} v) + O \left( (\ln r)^2 \left( \left\| \theta_x \right\|_{L^2(\Omega)} + \varepsilon^2 \sqrt{M} \right) \right) \\ \bar{\theta}_\nu(v) = \sqrt{\nu} \left( -d \sin(\sqrt{\nu} v) + e \cos(\sqrt{\nu} v) \right) + O \left( (\ln r)^2 \left( \left\| \theta_x \right\|_{L^2(\Omega)} + \varepsilon^2 \sqrt{M} \right) \right) \end{cases}
\]

\(^2\)We ignore the issue of taking subsequence from here on, since we only work with precompact sequences with limit independent of the subsequence.
as \( \varepsilon \to 0 \), where
\[
(5.27) \quad d = \bar{d}(0),
\]
\[
(5.28) \quad e = \bar{e}(0) \sqrt{\nu}.
\]
and the following asymptotic expansion on the eigenvalue
\[
(5.29) \quad \sqrt{\nu} = \pi - \arctan \left( \frac{d}{e} \right) + O \left( (\ln r)^2 \left( \frac{\|\theta_x\|_{L^2(\overline{\omega})}}{\sqrt{M}} + \varepsilon^2 \right) + r \frac{\varepsilon^2}{\sqrt{M}} \left( \ln \frac{1}{r} \right)^{\frac{1}{2}} \right)
\]
as \( \varepsilon \to 0 \) and on the mass
\[
(5.30) \quad M = d^2 + e^2 + \frac{de}{\sqrt{\nu}} + O \left( (\ln r)^2 \left( \frac{\|\theta_x\|_{L^2(\overline{\omega})}}{\sqrt{M}} + \varepsilon^2 M \right) + \frac{r \varepsilon^2}{\sqrt{M}} \left( \ln \frac{1}{r} \right)^{\frac{1}{2}} M \right)
\]
as \( \varepsilon \to 0 \).

Proof. The system of equations (5.18) and (5.19) resembles to the linear equation on \([0,1]\) on \((f, g)\) such that \(-g' = \nu f\) and \(f' = g\). We have the following ansatz by a variation of constants method as soon as \( \nu \) is positive: let \( A, B \) be functions on \([0,1]\) such that
\[
(5.31) \quad \begin{cases}
\bar{\theta} = A \cos \left( \sqrt{\nu} v \right) + B \sin \left( \sqrt{\nu} v \right) \\
\bar{\theta}_v = \sqrt{\nu} \left( -A \sin \left( \sqrt{\nu} v \right) + B \cos \left( \sqrt{\nu} v \right) \right)
\end{cases}
\]
Writing the equations (5.18) and (5.19) on \( \bar{\theta} \) and \( \bar{\theta}_v \) with respect to \( A \) and \( B \) thanks to (5.31), we obtain the following
\[
(5.32) \quad \begin{cases}
A' = \frac{(\ln r)^2}{\sqrt{\nu}} \kappa_1 \\
B' = \frac{(\ln r)^2}{\sqrt{\nu}} \kappa_2
\end{cases}
\]
where
\[
\kappa_1 = \sin \left( \sqrt{\nu} v \right) \left( 2 \frac{\sigma_x}{t_{\varepsilon}} \eta \bar{\theta} + \left( \frac{1}{2} - \frac{\sigma_x}{t_{\varepsilon}} (\eta + 1) \right) \mu \right) + \nu \ln r \cos \left( \sqrt{\nu} v \right) \mu
\]
and
\[
\kappa_2 = -\cos \left( \sqrt{\nu} v \right) \left( 2 \frac{\sigma_x}{t_{\varepsilon}} \eta \bar{\theta} + \left( \frac{1}{2} - \frac{\sigma_x}{t_{\varepsilon}} (\eta + 1) \right) \mu \right) + \frac{\nu}{\ln r} \sin \left( \sqrt{\nu} v \right) \mu
\]
so that
\[
(5.33) \quad \begin{cases}
\bar{\theta}(v) = d \cos \left( \sqrt{\nu} v \right) + e \sin \left( \sqrt{\nu} v \right) + R(v) \\
\bar{\theta}_v(v) = \sqrt{\nu} \left( -d \sin \left( \sqrt{\nu} v \right) + e \cos \left( \sqrt{\nu} v \right) \right) + S(v)
\end{cases}
\]
where we recall that \( d \) and \( e \) are defined by \( d = \bar{d}(0) \) and \( e = \frac{\bar{e}(0)}{\sqrt{\nu}} \) and we define
\[
(5.34) \quad \begin{cases}
R(v) = (\ln r)^2 \left( (\int_0^v \kappa_1(s) ds) \cos \left( \sqrt{\nu} v \right) + (\int_0^v \kappa_2(s) ds) \sin \left( \sqrt{\nu} v \right) \right) \\
S(v) = (\ln r)^2 \left( - (\int_0^v \kappa_1(s) ds) \sin \left( \sqrt{\nu} v \right) + (\int_0^v \kappa_2(s) ds) \cos \left( \sqrt{\nu} v \right) \right)
\end{cases}
\]
By definition of \( R, S, \kappa_1 \) and \( \kappa_2 \), it is clear that
\[
\|R\|_{C^1} + ||S||_{C^1} = O \left( (\ln r)^2 \left( \frac{\|\theta_x\|_{L^2(\overline{\omega})}}{\sqrt{M}} + \varepsilon^2 \sqrt{M} \right) \right)
\]
as \( \varepsilon \to 0 \) so that we proved (5.26).
Moreover, by the compatibility condition (4.2) for \( v = 1 \) given by
\[
\frac{1}{r^2 \varepsilon^2} \int_{r^2 \varepsilon^2}^{2, 2} (u_\varepsilon \circ \varphi^{-1}) (x, 0) dx = \bar{\vartheta}(r) = \frac{\sqrt{T_\varepsilon r^{-\frac{3}{2}} \bar{\vartheta}(1)}}{\sqrt{\varepsilon} \sqrt{\ln \frac{1}{r}}}.
\]
we obtain that
\[
\bar{\vartheta}(1) = O \left( r^\frac{1}{2} \varepsilon^\frac{1}{2} \left( \ln \frac{1}{r} \right)^{\frac{1}{2}} \right).
\]

Thanks to (5.26), the choice of the parameter \( t_\varepsilon \) and (4.1), we notice that
\[
M \sim e^2 \text{ as } \varepsilon \to 0
\]
and that
\[
|R(1)| + |S(1)| = O \left( \ln \frac{1}{r} \left( \| \theta_x \|_{L^2(\tilde{\Omega})} + \varepsilon^2 \sqrt{M} \right) \right)
\]
as \( \varepsilon \to 0 \). Then computing \( \bar{\vartheta}(1) \) with (4.2) and (5.26) and the previous estimate on \( R(1) \) and \( S(1) \), we obtain (5.29). The expansion (5.30) of the mass is then obtained through a straightforward computation by integrating \((d \cos(\sqrt{nuv}) + e \sin(\sqrt{nuv}))^2\):
\[
M = d^2 + e^2 + \frac{d^2 - e^2}{\sqrt{\nu}} \sin(2\sqrt{\nu}) - 2de \cos(2\sqrt{\nu}) + O \left( \ln \frac{1}{r} \left( \| \theta_x \|_{L^2(\tilde{\Omega})} + \varepsilon^2 \sqrt{M} \right) \right)
\]
taking into account from (5.29) that
\[
\tan(\sqrt{\nu}) = \frac{d}{e} + O \left( (\ln r)^2 \left( \| \theta_x \|_{L^2(\tilde{\Omega})} \sqrt{M} + \varepsilon^2 M \right) + r^\frac{1}{2} \varepsilon^\frac{1}{2} \left( \ln \frac{1}{r} \right)^{\frac{1}{2}} M \right)
\]
leading to the additional cancelations.

\[\diamondsuit\]

Now, as already said, we aim at obtaining a good estimate for \( \delta_\varepsilon \) in order to apply (5.11). We have by (4.1) and (5.9) that
\[
\frac{\| \theta_x \|_{L^2(0, 1)}^2}{\varepsilon^2} + \int_0^1 \left( \frac{\theta_v}{\ln \frac{1}{r}} + \frac{\theta_v}{2} - \frac{\theta_v}{\ln \frac{1}{r}} + \frac{\theta_v}{2} \right)^2 + \frac{\delta_\varepsilon}{\varepsilon^2} \varepsilon^2 = 2 \int_0^1 \bar{\vartheta}^2 \left( \frac{\sigma_\varepsilon}{\ln \frac{1}{r}} - \frac{1}{4} \right)
\]
(5.35)
\[
- \int_0^1 \left( \frac{\theta_v}{\ln \frac{1}{r}} + \frac{\theta_v}{2} \right)^2 + O \left( \sqrt{M} \| \theta_x \|_{L^2(\tilde{\Omega})} + \varepsilon^2 M \right),
\]
so that setting \( \zeta = \frac{\theta_v}{\ln \frac{1}{r}} + \frac{\theta_v}{2} \), \( f = d \cos(\sqrt{nuv}) + e \sin(\sqrt{nuv}) \), where \( d \) and \( e \) are given in Claim 5.25, so that \( f \) is a solution of \( f'' + \nu f = 0 \), and recalling the definition (5.20) of \( \nu_\varepsilon \) we have
\[
\frac{\| \theta_x \|_{L^2(\tilde{\Omega})}^2}{\varepsilon^2} + \int_0^1 (\zeta - \bar{\theta}(0))^2 + \frac{\delta_\varepsilon}{\varepsilon^2} \varepsilon^2 = \frac{\bar{\theta}(0)^2}{2 \ln \frac{1}{r}} - \frac{f(0) f'(0)}{(\ln r)^2}
\]
(5.36)
\[
+ O \left( \sqrt{M} \| \theta_x \|_{L^2} + M \varepsilon^2 + r \varepsilon \left( \ln \frac{1}{r} \right)^2 \right)
\]
thanks to Claim 5.25 and then, the inequality (5.11) becomes

\[ \sigma_* - \sigma_\varepsilon \leq \varepsilon \bar{\phi}(1)^2 + O\left( \frac{\varepsilon^{1+\alpha}}{N} \right), \]

where we have that \( t_\varepsilon \varepsilon^{2\alpha} d^2 = \varepsilon^{1+\alpha} \bar{\phi}(1)^2 \). We now choose for the rest of the argument

\[ M = N = \frac{1}{2}, \]

which is possible thanks to Claim 5.12. If we can prove that \( \bar{\phi}(1) = o(1) \) as \( \varepsilon \to 0 \), we can conclude the proof of Theorem 1.3. Indeed, (5.37) would give that \( \sigma_* - \sigma_\varepsilon = o(\varepsilon) \) and the extra-length of \( \Sigma_\varepsilon \) with respect to \( \Sigma \) is of order \( \varepsilon \).

However, there is in general no reason that \( \bar{\phi}(1) = o(1) \) as \( \varepsilon \to 0 \). Indeed, using (5.36) we can prove that the \( W^{1,2} \) and energy norms in the right-hand term of (4.6) converge to 0 as fast as \( \varepsilon^{3} \) (see formula (6.12) in Claim 6.11 in the next section). Therefore, \( \bar{\phi}(1) \) converges to \( u_\varepsilon(p_0) \), where \( u_\varepsilon \) is the weak limit of \( u_\varepsilon \) in \( W^{1,2}(\Sigma) \). As a \( \sigma_* \)-eigenfunction on \( \Sigma \), \( u_\varepsilon \) does not necessarily vanish at some point \( p_0 \). In the case when \( \sigma_* \) is simple, one can choose an attaching point \( p_0 \) such that \( u_\varepsilon(p_0) = 0 \). More generally, if any eigenfunction associated to \( \sigma_* \) vanishes at some point \( p_0 \), we get the theorem. This is not necessarily true.

Therefore, the Poincaré inequality we invoked to prove (5.10) and then (5.11) is not sufficient to get \( \sigma_* - \sigma_\varepsilon = o(\varepsilon) \). The key idea to improve this estimate is to use a better perturbation of a first eigenfunction to get good control on the mean value. (Before, we simply did this by a constant function.) The way we do this is by using another eigenfunction with eigenvalue close to \( \sigma_*^1 \) which perturbs the eigenvalue equation on a much smaller scale.

### 6. The improved test function

Recall all the choices up to this point. We have \( r_\varepsilon = \exp\left(-\frac{1}{\varepsilon^\alpha}\right) \), with \( \alpha < \frac{1}{2} \) and \( t_\varepsilon \) chosen with the help of Claim 5.12 such that we have \( u_\varepsilon^1 \) a normalized \( \sigma_*^1 \)-eigenfunction with \( M = N = \frac{1}{2} \). From here on we assume in addition that also \( \alpha > \frac{1}{3} \) such that \( \varepsilon^{\frac{3\alpha}{2} + \frac{1}{3}} = o(\varepsilon) \) for the scale in the estimates above.

In order to conclude our main result we would like to choose a better test function by finding a good linear combination of the first eigenfunction and another eigenfunction with eigenvalue close to \( \sigma_*^1 \). We would like to have a similar asymptotic expansion on the cuspidal domain available for the corresponding eigenfunction in order to arrange for cancelations in our asymptotic estimates. In order to do so we first need to locate another eigenfunction with some concentration of mass on the cuspidal domain.

#### 6.1. Locating another good eigenfunction.

Let us consider an eigenvalue \( \sigma_*^l \) on \( \Sigma_\varepsilon \) with \( 2 \leq l \leq K + 1 \) where we recall that \( K = \text{mult} \sigma_* \) denotes the multiplicity of the first non-trivial eigenvalue on \( \Sigma \). In practice, \( l \) could depend on \( \varepsilon \) but up to taking a subsequence we may assume that it is fixed.

By the estimates from the previous sections, we can prove that

\[ \sigma_*^l = \frac{t_\varepsilon}{8} + \frac{t_\varepsilon \pi^2}{2} \varepsilon^{2\alpha} + O\left( \varepsilon (\ln \varepsilon)^2 \right) \]

as \( \varepsilon \to 0 \). Indeed, by (3.3), if \( \sigma_*^l \) is controlled by the second term in the maximum, (6.1) holds true (as we know the lower bound even for \( \sigma_*^1 \)). If not, we have that \( \sigma_*^l - \sigma_* = \ldots \).
$O\left(\varepsilon^{\frac{33}{2}+\frac{1}{2}}\right)$ as $\varepsilon \to 0$. Therefore, using (4.5) and (5.37), we have

$$\sigma_{0}^{l} = \sigma_{*}^{l} - \sigma_{*} + \sigma_{*}^{1} + \sigma_{*}^{2} = \sigma_{*} + O\left(\varepsilon (\ln \varepsilon)^{2}\right) \quad \text{as} \quad \varepsilon \to 0,$$

and using (5.29), we get the conclusion (6.1).

Therefore since

$$\nu_{\varepsilon}^{l} := (\ln r)^{2}\left(\frac{2\pi}{t_{\varepsilon}} - \frac{1}{4}\right) \quad \text{(6.2)}$$

satisfies $\pi^{2} + o(1) = \nu_{\varepsilon}^{l} \leq \nu_{\varepsilon}^{l} \leq \pi^{2} + o(1)$ by (5.29) and (4.5), we have the following claim (6.4):

**Claim 6.3.** the $C^{1}(0, 1)$ asymptotic expansion of $\bar{\theta}_{l}$ and $(\bar{\theta}_{l})_{v}$

$$\begin{align*}
\bar{\theta}_{l}(v) &= d_{l} \cos\left(\sqrt{\nu_{l}^{2}}v\right) + e_{l} \sin\left(\sqrt{\nu_{l}^{2}}v\right) + O\left((\ln r)^{2}\left(\|\bar{\theta}_{l}\|_{L^{2}(\tilde{\Omega})} + \varepsilon^{2}\sqrt{M_{l}}\right)\right), \\
(\bar{\theta}_{l})_{v}(v) &= \sqrt{\nu_{l}}\left(-d_{l} \sin\left(\sqrt{\nu_{l}^{2}}v\right) + e_{l} \cos\left(\sqrt{\nu_{l}^{2}}v\right)\right) + O\left((\ln r)^{2}\left(\|\bar{\theta}_{l}\|_{L^{2}(\tilde{\Omega})} + \varepsilon^{2}\sqrt{M_{l}}\right)\right),
\end{align*} \quad \text{as} \quad \varepsilon \to 0,$$

where

$$d_{l} = \bar{\theta}_{l}(0), \quad e = \frac{(\bar{\theta}_{l})_{v}(0)}{\sqrt{\nu_{l}}} \quad \text{(6.6)}$$

and the following asymptotic expansion on the eigenvalue

$$\sqrt{\nu_{l}^{2}} = \pi - \arctan\left(\frac{d_{l}}{e_{l}} + O\left((\ln r)^{2}\left(\frac{\|\bar{\theta}_{l}\|_{L^{2}(\tilde{\Omega})}}{\sqrt{M_{l}}} + \varepsilon^{2}\right) + \frac{r^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}(\ln \frac{1}{r})^{\frac{1}{2}}}{\sqrt{M_{l}}}\right)\right) \quad \text{(6.7)}$$

as $\varepsilon \to 0$ and on the mass

$$M_{l} = (d_{l})^{2} + (e_{l})^{2} + \frac{d_{l}e_{l}}{\sqrt{\nu_{l}}} + O\left((\ln r)^{2}\left(\frac{\|\bar{\theta}_{l}\|_{L^{2}(\tilde{\Omega})}}{\sqrt{M_{l}}} + \varepsilon^{2}M_{l}\right) + \frac{r^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}(\ln \frac{1}{r})^{\frac{1}{2}}}{\sqrt{M_{l}}}\right) \quad \text{(6.8)}$$

as $\varepsilon \to 0$.

We need to make sure that the eigenfunction has some of its mass concentrated on the cuspidal domain. We achieve this by choosing $l$ appropriately in the next claim.

**Claim 6.9.** For $\varepsilon > 0$ sufficiently small we can find $l \in \{2, \ldots, K+1\}$ such that there is a normalized $\sigma_{\varepsilon}^{l}$-eigenfunction orthogonal to $u_{\varepsilon}^{1}$ with

$$\int_{I_{\varepsilon}^{1} \cup I_{\varepsilon}^{2}} |u_{\varepsilon}^{l}|^{2} dl_{\varepsilon} \geq \frac{1}{4K}. \quad \text{Proof.}$$

We argue by contradiction and may thus take a collection $u_{\varepsilon}^{2}, \ldots, u_{\varepsilon}^{K+1}$ of orthonormal eigenfunctions all orthogonal to $u_{\varepsilon}^{1}$, such that $u_{\varepsilon}^{l}$ is an $\sigma_{\varepsilon}^{l}$-eigenfunction and

$$\int_{I_{\varepsilon}^{1} \cup I_{\varepsilon}^{2}} |u_{\varepsilon}^{l}|^{2} dl_{\varepsilon} \leq \frac{1}{4K}.$$
for any $l = 2, \ldots, K + 1$. In particular, also using the corresponding bound for the first eigenfunction, we find by Cauchy–Schwarz for $w_\varepsilon = \sum_{i=1}^{K+1} t_i u_i^\varepsilon$ that
\[
\int_{I_\varepsilon^i \cup U_\varepsilon^i} |w_\varepsilon|^2 \sigma_\varepsilon \, dl_\varepsilon \leq \int_{I_\varepsilon^i \cup U_\varepsilon^i} \left( \sum_{i=1}^{K+1} t_i \right) \left( \sum_{i=1}^{K+1} |u_i^\varepsilon|^2 \right) \sigma_\varepsilon \, dl_\varepsilon \leq \frac{3}{4} \|w_\varepsilon\|^2_{L^2(\partial \Sigma_\varepsilon)}.
\]
After taking appropriate subsequences we can assume that $u_i^\varepsilon \rightharpoonup u_i^*$ weakly in $W^{1,2}(\Sigma)$ and strongly in $L^2(\partial \Sigma)$ for $l = 1, \ldots, K + 1$. As in the proof of Claim 5.12 we find that all of these are $\sigma_\varepsilon$-eigenfunctions $u_i^\varepsilon$. But this can be seen to be impossible as follows. We can choose $w_\varepsilon = \sum_{i=1}^{K+1} t_i u_i^\varepsilon$ with $\|w_\varepsilon\|_{L^2(\partial \Sigma_\varepsilon)} = 1$ such that $\int_{\partial \Sigma} w_\varepsilon u_i^\varepsilon = 0$ for any $l = 1, \ldots, K + 1$ since the multiplicity of $\sigma_\varepsilon$ is only $K$, so these are in fact only $K$ linear conditions. By strong convergence in $L^2(\partial \Sigma)$ we find that $\|w_\varepsilon\|_{L^2(\partial \Sigma_\varepsilon)} \to 0$ as $\varepsilon \to 0$.

When combined with (6.10) this gives
\[
\|w_\varepsilon\|^2_{L^2(\partial \Sigma_\varepsilon)} \leq \int_{I_\varepsilon^i \cup U_\varepsilon^i} |w_\varepsilon|^2 \sigma_\varepsilon \, dl_\varepsilon + o(1) \leq \frac{3}{4} \|w_\varepsilon\|^2_{L^2(\partial \Sigma_\varepsilon)} + o(1),
\]
which is a contradiction for $\varepsilon$ sufficiently small.

6.2. Improved pointwise estimates. We define by $u_i^*$ the strong limit in $L^2(\partial \Sigma)$ of $\frac{u_i^\varepsilon}{\sqrt{N_i}}$ for $i = 1, l$, Note that this differs from our previous convention, where we did not rescale by the mass. Recall that $N_1 = \frac{1}{2}$. Similarly, we also have that $N_l$ is bounded away from zero. This can be seen as follows. Thanks to Claim 5.25 and (5.5) we have that
\[
\int_{I_\varepsilon^i \cup U_\varepsilon^i} \frac{\sqrt{L_\varepsilon y}}{\sqrt{\varepsilon} \ln \frac{1}{\varepsilon}} \sin \left( \frac{\pi \ln y}{\ln r} \right) u_i^\varepsilon \, dl_\varepsilon = \sqrt{M_i} + o(1)
\]
as $\varepsilon \to 0$ for $i = 1, l$. From this we conclude that $M_1 + M_l \leq 1 + o(1)$, which gives that $N_i \geq \frac{1}{2} - o(1)$.

Claim 6.11. Then we have that the following estimates for $i = 1, l$:
\[
\|u_i^\varepsilon - \sqrt{N_i} u_i^*\|_{W^{1,2}(\partial \Sigma)} = O \left( \varepsilon^{\frac{1}{2}} + \frac{\varepsilon^2}{|\sigma_\varepsilon - \sigma_i^*|} \right)
\]
and
\[
\varepsilon^{\frac{q+1}{2}} \sqrt{L_\varepsilon} = \sqrt{N_i} u_i^*(p_0) + O \left( \varepsilon^{\frac{1}{2}} + \frac{\varepsilon^2}{|\sigma_\varepsilon - \sigma_i^*|} \right)
\]
as $\varepsilon \to 0$.

Proof. We go back to (4.6) in Claim 4.3 which holds for $i = 1, l$ and gives, after incorporating the estimates from the end of Section 5 that
\[
\left| (u_i^\varepsilon \circ \varphi_0^{-1}) (x) - \sqrt{N_i} u_i^*(p_0) \right| \leq C \left( \|u_i^\varepsilon - \sqrt{N_i} u_i^*\|_{W^{1,2}(\Sigma)} + \|\nabla \phi_i\|_{L^2(F_\varepsilon)} + \varepsilon \ln \frac{1}{\varepsilon} \right).
\]
for any \( x \in \varphi^{-1}_0(\mathbb{D}_{\varepsilon^2}) \) where \( F_\varepsilon = \{(x,y) \in \Omega; 1 - \varepsilon \leq y \leq 1\} \). We also have the estimate (6.36), which is also true for \( i = 1, l \),

\[
\frac{\|(\theta_i)_x\|^2}{\varepsilon^2} + \int_0^1 \frac{(\zeta_i - \zeta_i')^2}{t_\varepsilon} + \frac{(\delta_i)_x}{t_\varepsilon} = \frac{\varepsilon \phi_i(1)^2}{2} + O \left( \sqrt{M_i} \|(\theta_i)_x\|_{L^2} + M_i \varepsilon^2 + r \varepsilon \left( \ln \frac{1}{r} \right)^2 \right)
\]

as \( \varepsilon \to 0 \). By the same computations as in Section 5 on \( \theta_i \), we know that

\[
\frac{1}{t_\varepsilon} \|\nabla \phi_i\|^2_{L^2(F_\varepsilon)} = \left( \frac{1}{\varepsilon^2} \int_0^{\ln(1-\varepsilon)} \int_0^{\frac{\ln(1-\varepsilon)}{ln r}} (\theta_i)_x^2 \, dx \, dv + \int_0^{\ln(1-\varepsilon)} \left( \frac{\theta_i}{2} + \frac{(\theta_i)_x}{\ln \frac{1}{r}} \right)^2 \, dv \right)
\]

\[
\leq \frac{\|(\theta_i)_x\|^2}{\varepsilon^2} + \int_0^1 (\zeta_i - \zeta_i')^2 \, dv
\]

\[
+ \frac{1}{(\ln r)^2} \int_0^{\ln(1-\varepsilon)} \left( (\theta_i)_x \right)^2 \, dv + \frac{(\theta_i^2 \left( \frac{\ln(1-\varepsilon)}{\ln r} \right) - \theta_i^2 (0))}{2 \ln \frac{1}{r}}
\]

\[
+ \frac{1}{4} \int_0^{\ln(1-\varepsilon)} \theta_i^2 \, dv + O \left( \|(\theta_i)_x\|_{L^2(G)} \right)
\]

as \( \varepsilon \to 0 \) for \( i = 1, l \). Remember that with (3.2) and that \( \sigma^1 - \sigma^1 = o(\varepsilon) \), we obtain that \( \sigma^1 - \sigma_\ast \leq o(\varepsilon) \). Then by (6.15), the asymptotic analysis on \( \theta_i \) and \( (\theta_i)_x \) and (3.4) we have

(6.16) \[
\|\nabla \phi_i\|_{L^2(F_\varepsilon)} = O \left( \varepsilon^2 \right)
\]

as \( \varepsilon \to 0 \). Let \( v \) be a first eigenfunction associated to \( \sigma_\ast \), bounded in \( W^{1,2}(\Sigma) \). We have

\[
\int_\Sigma |\nabla (u_\varepsilon^\ast - v)|^2 = \int_\Sigma |\nabla u_\varepsilon^\ast|^2 + \int_\Sigma |v|^2 - 2 \int_\Sigma \langle \nabla u_\varepsilon^\ast, \nabla v \rangle
\]

\[
= (\delta_i)_x + \sigma_\ast \int_{\partial \Sigma} (u_\varepsilon^\ast)^2 + \sigma_\ast \int_{\partial \Sigma} v^2 - 2 \sigma_\ast \int_{\partial \Sigma} v u_\varepsilon^\ast + O \left( \varepsilon^2 \ln \frac{1}{\varepsilon} \right)
\]

(6.17)

\[
= (\delta_i)_x + (\sigma_\varepsilon^\ast - \sigma_\ast) \int_{\partial \Sigma} (u_\varepsilon^\ast)^2 + \sigma_\ast \int_{\partial \Sigma} (v - u_\varepsilon^\ast)^2 + O \left( \varepsilon^2 \ln \frac{1}{\varepsilon} \right)
\]

\[
\leq C \frac{\varepsilon \phi_i(1)^2}{2} + \sigma_\ast \int_{\partial \Sigma} (v - u_\varepsilon^\ast)^2 + o(\varepsilon)
\]

where the second equality comes from integrating by parts and the pointwise estimate (4.5) on \( u_\varepsilon^\ast \) and the fourth inequality uses (5.37), (6.15) and that \( \sigma_\varepsilon^\ast - \sigma_\ast \leq o(\varepsilon) \). In particular, for \( v = \sqrt{N_i} u_\varepsilon^\ast \), this means that the right-hand term of (6.14) converges to 0 and that the \( W^{1,2} \)-norm of \( u_\varepsilon^\ast - \sqrt{N_i} u_\varepsilon^\ast \) is controled by its \( L^2 \)-norm up to a term of order \( \varepsilon^2 \).

Let us now prove (6.12). This will complete the proof of Claim 6.11.

For any closed set \( E \) in \( L^2(\partial \Sigma) \), we denote by \( P_E \) the orthogonal projection in \( L^2 \) on \( E \). Let \( E_\ast \) be the space generated by the eigenfunctions associated to \( \sigma_\ast \). We denote \( F_\ast \) and \( G_\ast \) the spaces such that

(6.18) \[
F_\ast = \{ v \in E_\ast : v(p_0) = 0 \} \text{ and } G_\ast = F_\ast^\bot \cap E_\ast.
\]
Notice that $F_\ast$ has codimension at most 1 in $E_\ast$. We then have by the Pythagorean theorem
\begin{equation}
\|u_\varepsilon^i\|_{L^2(\partial\Sigma)}^2 = \|u_\varepsilon^i - P_{E_\ast}(u_\varepsilon^i)\|_{L^2(\partial\Sigma)}^2 + \|P_{E_\ast}(u_\varepsilon^i)\|_{L^2(\partial\Sigma)}^2 + \|P_{\ast}(u_\varepsilon^i)\|_{L^2(\partial\Sigma)}^2
\end{equation}
We first prove that
\begin{equation}
\|u_\varepsilon^i - P_{E_\ast}(u_\varepsilon^i)\| = o\left(\varepsilon^\frac{1}{2}\right) \quad \text{as} \quad \varepsilon \to 0.
\end{equation}
We set $R_\varepsilon^i = \|u_\varepsilon^i - P_{E_\ast}(u_\varepsilon^i)\|$ and assume by contradiction that $\varepsilon^{\frac{1}{2}} = O\left(\|R_\varepsilon^i\|_{L^2(\partial\Sigma)}\right)$ as $\varepsilon \to 0$.

We have the following equation on $R_\varepsilon^i = u_\varepsilon^i - P_{E_\ast}(u_\varepsilon^i)$
\begin{equation}
\begin{cases}
\Delta_g R_\varepsilon^i = 0 & \text{in} \Sigma \\
\partial_\nu R_\varepsilon^i = \sigma_\ast R_\varepsilon^i - (\sigma_\ast - \sigma_\varepsilon^i) P_{E_\ast}(u_\varepsilon^i) & \text{on} \partial\Sigma \setminus (A_0 \cup A_1).
\end{cases}
\end{equation}
We have from (6.17) applied to $v = P_{E_\ast}(u_\varepsilon^i)$ combined with $\|R_\varepsilon^i\|_{L^2(\partial\Sigma)} \gtrsim \varepsilon^{\frac{1}{2}}$ that
\[\frac{R_\varepsilon^i}{\|R_\varepsilon^i\|_{L^2(\partial\Sigma)}} \text{ is uniformly bounded in } W^{1,2}(\Sigma)\]
Therefore, we may take
\[\frac{R_\varepsilon^i}{\|R_\varepsilon^i\|_{L^2(\partial\Sigma)}} \to R_\ast^i
\]
weakly in $W^{1,2}(\Sigma)$ and strongly in $L^2(\partial\Sigma)$. By the strong convergence in $L^2(\partial\Sigma)$ we have that $\|R_\ast^i\|_{L^2(\partial\Sigma)} = 1$. Since $\sigma_\ast - \sigma_\varepsilon^i = O(\varepsilon)$ as $\varepsilon \to 0$, by standard elliptic theory on any compact subset of $\Sigma \setminus \{p_0, p_1\}$, we get the following equation at the limit
\begin{equation}
\begin{cases}
\Delta_g R_\ast^i = 0 & \text{in} \Sigma \setminus \{p_0, p_1\}
\partial_\nu R_\ast^i = \sigma_\ast R_\ast^i & \text{on} \partial\Sigma \setminus \{p_0, p_1\}.
\end{cases}
\end{equation}
Since $R_\ast^i \in W^{1,2}(\Sigma)$, the equation (6.22) holds on all of $\Sigma$. Then, since we have that $R_\ast^i$ is orthogonal to the eigenspace associated to $\sigma_\ast$ by construction, we must in fact have that $R_\ast^i = 0$. This contradicts that $\|R_\ast^i\|_{L^2(\partial\Sigma)} = 1$. We finally get (6.20).

We now prove that
\begin{equation}
\|P_{E_\ast}(u_\varepsilon^i)\| = O\left(\varepsilon^2|\sigma_\ast - \sigma_\varepsilon^i|\right) \quad \text{as} \quad \varepsilon \to 0.
\end{equation}
We integrate the equation satisfied by $u_\varepsilon^i$ on $\Sigma \setminus \left(\varphi_0^{-1}(D^+_2) \cup \varphi_1^{-1}(D^+_2)\right)$, against a first eigenfunction $v$ associated to $\sigma_\ast$ and we get thanks to (6.14) combined with the estimates above that
\[ (\sigma_\ast - \sigma_\varepsilon^i) \int_{\partial\Sigma} vu_\varepsilon^i \, d\sigma + O\left(\varepsilon^2\sqrt{N_i}\right) = \int_{\varphi_0^{-1}(S^+_2) \cup \varphi_1^{-1}(S^+_2)} (v\partial_\nu u_\varepsilon^i - u_\varepsilon^i\partial_\nu v).
\]
By elliptic theory at the scale $\varepsilon^2$ at the neighborhood of $p_0$, we have the following gradient estimate
\begin{equation}
\varepsilon^4 \|\nabla u_\varepsilon^i\|^2(\varphi_0^{-1}(x)) \leq C\left(\varepsilon + \int_{\partial\Sigma} (u_\varepsilon^i)^2\right)
\end{equation}
for any $x \in \mathbb{D}_{2\varepsilon}^+ \setminus \mathbb{D}_{\frac{2\varepsilon}{3}}^+$ and at the scale $r^2\varepsilon^2$ at the neighbourhood of $p_1$

$$
(6.25) \quad \varepsilon^4 r^4 \left| \nabla \left( u^i_\varepsilon \right) \right|^2 (\varphi^{-1}_1(x)) \leq C \left( \varepsilon + \int_{\partial \Sigma} \left( u^i_\varepsilon \right)^2 \right)
$$

for any $x \in \mathbb{D}_{2\varepsilon}^+ \setminus \mathbb{D}_{\frac{2\varepsilon}{3}}^+$. We simply have that

$$
(\sigma_* - \sigma^i_\varepsilon) \int_{\partial \Sigma} vu^i_\varepsilon dl_g = \int_{\varphi^{-1}_1(\mathbb{S}_{r_\varepsilon}^2)} v \partial_r u^i_\varepsilon + \int_{\varphi^{-1}_1(\mathbb{S}_{r_\varepsilon/2}^2)} (v - v(p_1)) \partial_r u^i_\varepsilon + v(p_1) \int_{\varphi^{-1}_1(\mathbb{S}_{r_\varepsilon/2}^2)} \partial_r u^i_\varepsilon + O \left( \varepsilon^2 \sqrt{N_i} \right).
$$

By integration by parts combined with the pointwise estimates from Claim 4.3 we also have that

$$
v(p_1) \int_{\varphi^{-1}_1(\mathbb{S}_{r_\varepsilon}^2)} \partial_r u^i_\varepsilon = -v(p_1)\varepsilon r^2 \varphi_y + O \left( r^2 \varepsilon^2 \log \left( \frac{1}{\varepsilon r} \right) \right) = O(r) \text{ as } \varepsilon \to 0,
$$

We also have that $v - v(p_1) = O \left( r^2 \varepsilon^2 \right)$, uniformly on $\mathbb{S}_{r_\varepsilon}^2$. If we assume in addition that $v$ satisfies $v(p_0) = 0$, we have that $|v| = O(\varepsilon^2)$ uniformly on $\mathbb{S}_{r_\varepsilon}^2$. Therefore, by the uniform estimates (6.24) and (6.25) on the gradient, we have for such $v$ that

$$
(6.26) \quad (\sigma_* - \sigma^i_\varepsilon) \int_{\partial \Sigma} vu^i_\varepsilon dl_g = O \left( \varepsilon^2 \sqrt{N_i} \right)
$$

as $\varepsilon \to 0$. We easily obtain (6.23) from (6.26).

Now thanks to (6.19), (6.20) and (6.23), we get that

$$
(6.27) \quad \left\| P_{G_*} (u^i_\varepsilon) \right\|^2 = \left| u^i_\varepsilon \right|^2 - O \left( \varepsilon^4 + \frac{\varepsilon^4}{\left| \sigma_* - \sigma^i_\varepsilon \right|^2} \right)
$$

as $\varepsilon \to 0$. We assume that $\varepsilon^2 = o \left( \left| \sigma_* - \sigma^i_\varepsilon \right| \right)$ up to the end of the Claim. (If not, (6.12) and (6.13) are obvious and moreover Theorem (1.3) is proved). Then passing to the limit, we have $u^i_\varepsilon \in G^*$. Then

$$
(6.28) \quad \left\| P_{G_*} (u^i_\varepsilon) - \sqrt{N_i} u^i_* \right\| = \sqrt{N_i} - \left\| P_{G_*} (u^i_\varepsilon) \right\| = \frac{\left\| u^i_\varepsilon \right\|^2 - \left\| P_{G_*} (u^i_\varepsilon) \right\|^2}{\left\| P_{G_*} (u^i_\varepsilon) \right\| + \sqrt{N_i}} = \frac{\left\| u^i_\varepsilon - P_{G_*} (u^i_\varepsilon) \right\|^2}{\left\| P_{G_*} (u^i_\varepsilon) \right\| + \sqrt{N_i}} \
\leq \frac{\left\| u^i_\varepsilon - \sqrt{N_i} u^i_* \right\|^2}{\left\| P_{G_*} (u^i_\varepsilon) \right\| + \sqrt{N_i}}.
$$

Using a Pythagorean theorem again, we get that

$$
\left\| u^i_\varepsilon - \sqrt{N_i} u^i_* \right\|^2 = O \left( \varepsilon^4 + \frac{\varepsilon^4}{\left| \sigma_* - \sigma^i_\varepsilon \right|^2} + \frac{\left\| u^i_\varepsilon - \sqrt{N_i} u^i_* \right\|^4}{N_i} \right)
$$

so that Claim 6.11 is proved. \qed
6.3. Integration by parts estimate on the thin part. Now we aim at capturing the interactions between \( u_\varepsilon^1 \) and \( u_\varepsilon^l \). Before the final arguments, we aim at using some linear combinations \( \Psi = \gamma_1 u_1 + \gamma_l u_l \) of the two eigenfunctions as test functions on \( \Sigma \) in order to improve the estimates on the eigenvalue \( \sigma_\star \). We would like to compare \( \sigma_\star \) to \( \frac{\sigma_1 + \sigma_l}{2} \).

Let us start with some preliminary computations on \( \Omega_\varepsilon \) of \( L^2 \) norms of \( \Psi \). We define \( \Delta \) as

\[
\Delta = \frac{\sigma_1 + \sigma_l}{2} \int_{I^+ \cup I^-} \Psi^2 d\varepsilon - \int_{\Omega_\varepsilon} |\nabla \Psi|^2 dA_\varepsilon.
\]

Let \( \Theta \) be defined by

\[
\Psi(x, y) = \frac{\sqrt{t_\varepsilon y^{-\frac{1}{2}}}}{\sqrt{\varepsilon \ln \frac{1}{r}}} \Theta \left( x, \frac{\ln(y)}{\ln(r)} \right).
\]

Notice that \( \Theta = \gamma_1 \theta_1 + \gamma_l \theta_l \). By our computations in the preceding section (see (5.2), (5.4), (5.8) and (5.9)), we have that

\[
\int_{I^+ \cup I^-} \Psi^2 d\varepsilon = \int_0^1 \left( \Theta^2 \left( \frac{r^2 v}{2}, v \right) + \Theta^2 \left( \frac{r^2 v}{2}, -v \right) \right) = 2 \int_0^1 \Theta^2 dv + O \left( \|\Theta_x\|_{L^2(\tilde{\Omega})} + \varepsilon^2 \right)
\]

and that

\[
\int_{\Omega} |\nabla \Psi|^2 dA_\varepsilon = t_\varepsilon \left( \frac{1}{4} \int_0^1 \Theta^2 dv + \frac{1}{(\ln r)^2} \int_0^1 \Theta^2 dv + \frac{1}{\ln r} \int_0^1 \Theta \Theta_v^2 dv \right)
\]

\[
+ t_\varepsilon \left( \int_0^1 \frac{\Theta_v^2}{\varepsilon^2} dv + \int_0^1 \sqrt{\xi^2 - \frac{\Theta_v^2}{\varepsilon^2}} dv + O \left( \|\Theta_x\|_{L^2(\tilde{\Omega})} + \varepsilon^2 \right) \right),
\]

where \( \xi = \frac{\Theta_v^2}{\ln r} + \frac{\Theta^2}{2} \). Therefore, we have that

\[
\Delta = \frac{\sigma_1 + \sigma_l}{2} \int_0^1 \frac{1}{4} \Theta^2 dv + \frac{1}{(\ln r)^2} \int_0^1 \Theta^2 dv + \frac{1}{\ln r} \int_0^1 \Theta \Theta_v^2 dv + E_\varepsilon,
\]

where

\[
E_\varepsilon = -\frac{\|\Theta_x\|_{L^2(\tilde{\Omega})}^2}{\varepsilon^2} - \int_0^1 \left( \xi - \frac{\Theta_v^2}{\varepsilon^2} \right)^2 dv + O \left( \|\Theta_x\|_{L^2(\tilde{\Omega})} + \varepsilon^2 \right).
\]

Now, we set \( f = \gamma_1 f_1 + \gamma_l f_l \) where for \( i = 1, l \), \( f_i(v) = d_i \cos(\sqrt{\nu_i}v) + e_i \sin(\sqrt{\nu_i}v) \) are defined in (6.4), and are solutions of

\[
f_i'' + \nu_i f_i = 0.
\]

and we set

\[
\tilde{M}_i = \int_0^1 (f_i)^2 \text{ and } \tilde{J} = \frac{\int_0^1 f_1 f_l}{\sqrt{\tilde{M}_1 \tilde{M}_l}}.
\]
Thanks to the convergence properties of \( \bar{\theta}_I \) and \( \bar{\theta}_t \) in Claim 6.3 and because \( \Theta_v = \frac{\mu}{\ln(\frac{1}{r})} + \Theta' \), where \( \|\mu\|_{L^2(0,1)} = O\left(\|\Theta_x\|_{L^2(\bar{\Omega})}\right) \), we have that

\[
\Delta = \frac{\nu_1 + \nu_t}{(\ln r)^2} \int_0^1 f^2 - \frac{1}{(\ln r)^2} \int_0^1 \left( f' \right)^2 - \frac{1}{\ln \frac{1}{r}} \int_0^1 f f' + F_\varepsilon \\
= \frac{1}{(\ln r)^2} \left( \frac{\nu_1 + \nu_t}{2} \int_0^1 f^2 - \int_0^1 \left( -f'' \right) f - f'(0) f(0) \right) + \int_0^1 \frac{f(0)^2}{2 \ln \frac{1}{r}} + F_\varepsilon \\
= \frac{1}{(\ln r)^2} \left( \frac{\nu_1 + \nu_t}{2} \left( \int_0^1 f^2 - \int_0^1 \left( -f'' \right) f - f'(0) f(0) \right) \right) + \int_0^1 \frac{f(0)^2}{2 \ln \frac{1}{r}} + F_\varepsilon \\
= \frac{1}{(\ln r)^2} \left( \frac{\nu_1 - \nu_t}{2} \left( \int_0^1 f^2 - \int_0^1 \left( -f'' \right) f - f'(0) f(0) \right) \right) + \int_0^1 \frac{f(0)^2}{2 \ln \frac{1}{r}} + F_\varepsilon \\
(6.33)
\]

where we know that

\[ F_\varepsilon = E_\varepsilon + O\left(\|\Theta_x\|_{L^2(\bar{\Omega})} + \varepsilon^2\right). \]

Now since

\[
(\gamma_1)^2 M_1 - (\gamma_t)^2 M_t = (\gamma_1)^2 \int_{I^+ \cup I^-} (\phi_1)^2 \, d\varepsilon - (\gamma_t)^2 \int_{I^+ \cup I^-} (\phi_t)^2 \, d\varepsilon \\
= 2 (\gamma_1) \int_0^1 \left( \bar{\theta}_1 \right)^2 - 2 (\gamma_t) \int_0^1 \left( \bar{\theta}_t \right)^2 \\
+ O\left(\|\theta_1\|_x \|\theta_1\|_{L^2(\bar{\Omega})} + \|\theta_t\|_x \|\theta_t\|_{L^2(\bar{\Omega})}\right) + o(\varepsilon^2) \\
= 2 (\gamma_1) \int_0^1 \left( f_1 \right)^2 - 2 (\gamma_t) \int_0^1 \left( f_t \right)^2 \\
+ O\left(\varepsilon^{-2\alpha} \left(\|\theta_1\|_x \|\theta_1\|_{L^2(\bar{\Omega})} + \|\theta_t\|_x \|\theta_t\|_{L^2(\bar{\Omega})}\right)\right) + O(\varepsilon^{2-2\alpha}) \\
(6.34)
\]

and since \( t_\varepsilon (\nu_t - \nu_1) = 2 (\ln r)^2 (\sigma_t - \sigma_1) \) that

\[
\frac{\Delta}{t} = \frac{\sigma_t - \sigma_1}{2t} \left( (\gamma_1)^2 M_1 - (\gamma_t)^2 M_t \right) - \frac{f'(0) f(0)}{(\ln r)^2} + \int_0^1 \frac{f(0)^2}{2 \ln \frac{1}{r}} + G_\varepsilon \\
(6.35)
\]

where

\[
G_\varepsilon = -\frac{\|\Theta_x\|_{L^2(\bar{\Omega})}}{\varepsilon^2} + O\left(\|\Theta_x\|_{L^2(\bar{\Omega})} + \varepsilon^2\right) + O\left(\varepsilon^{\frac{1}{2} + \frac{3\alpha}{2}} - 2\alpha \left(\|\theta_1\|_x + \|\theta_t\|_x\|\right)\right) \\
\leq O(\varepsilon^2) \text{ as } \varepsilon \to 0, \\
(6.36)
\]

since we know that \( \sigma_t - \sigma_1 = O\left(\varepsilon^{\frac{1}{2} + \frac{3\alpha}{2}}\right) \) and that \( \frac{\|\theta_i\|_x^2}{\varepsilon^2} = O(\varepsilon) \text{ for } i = 1, l. \)
6.4. Quantitative (non-)orthogonality. Guided by Claim 6.11 in order to have a $L^2$ convergence of $u^1_\epsilon$ and $u^2_\epsilon$ at the rate $\epsilon^{\frac{3}{2}}$, we assume now that $\epsilon^{\frac{3}{2}} = O((\sigma_\star - \sigma_\epsilon))$ (if not, then Theorem (1.3) is obviously proved). Since $u^i_\star \in G_\star$, where $G_\star$, defined by (6.18) is a set of dimension 1, we have that

$$u_\star := u^1_\star = \pm u^l_\star.$$  

Now, to capture the interactions between $u^1_\epsilon$ and $u^l_\epsilon$ we use that $u^1_\epsilon$ and $u^l_\epsilon$ are orthogonal in $L^2(\partial \Sigma_\epsilon)$. We set

$$I = \int_{\partial \Sigma} \frac{u^1_\epsilon}{\sqrt{N_1}} \frac{u^l_\epsilon}{\sqrt{N_l}} \quad \text{and} \quad J = \int_{\Omega_\epsilon} \frac{\phi^1_\epsilon}{\sqrt{M_1}} \frac{\phi^l_\epsilon}{\sqrt{M_l}}$$

It is clear that $|I| < 1$ and $|J| < 1$ and that

$$\sqrt{N_1 N_l} I = -\sqrt{M_1 M_l} J.$$  

By a direct computation of $J$ with Claim 6.3, we have that

$$J = 2 \frac{\int_0^1 \theta_1 \theta_l}{\sqrt{M_1 M_l}} + o(1) = \frac{e_1 e_l}{\sqrt{M_1 M_l}} + o(1)$$

as $\epsilon \to 0$. For $i = 1, l$ we assume up to change $u^i_\epsilon$ into $-u^i_\epsilon$ we have that $e_1 > 0$ and $e_l > 0$. By Claim 6.3 we have that $J \to 1$ as $\epsilon \to 0$. In particular $I < -\sqrt{M_1 M_l} + o(1)$. Therefore (6.37) becomes

$$\left\| \frac{u^1_\epsilon}{\sqrt{N_1}} - u_\star \right\| + \left\| \frac{u^2_\epsilon}{\sqrt{N_2}} + u_\star \right\| = O(\epsilon^{\frac{1}{2}})$$

in $L^2(\partial \Sigma)$ as $\epsilon \to 0$.

In the following Claim, we compute the asymptotic expansion of $I$ by a very precise analysis in the thick part on $\left\| \frac{u^1_\epsilon}{\sqrt{N_1}} + \frac{u^l_\epsilon}{\sqrt{N_l}} \right\|$. We also compute the asymptotic expansion on $J$ in the thin part thanks to Claim 6.3.

At that stage, we also know that $\left\| \frac{d_1}{\sqrt{N_1}} + \frac{d_l}{\sqrt{N_l}} \right\| = O(\epsilon^{1-\frac{2}{2}})$ as $\epsilon \to 0$ but we will need a better estimate. We deduce it from better estimates on $\left\| \frac{u^1_\epsilon}{\sqrt{N_1}} + \frac{u^l_\epsilon}{\sqrt{N_l}} \right\|$

Claim 6.40.

$$I = -1 + O \left( \frac{\epsilon^{2+\alpha}}{(\sigma_\star - \frac{\sigma_1 + \sigma_2}{2})^2} \right)$$

and

$$J = 1 - C \left| u_\star(p_0) \right|^2 \epsilon^{1-\alpha} + o(\epsilon^{1-\alpha})$$

as $\epsilon \to 0$, where $C > 0$ is a positive constant. Moreover, we have the following estimate

$$\left| \frac{d_1}{\sqrt{N_1}} + \frac{d_l}{\sqrt{N_l}} \right| = O(\epsilon^{\frac{3}{2} - \frac{\alpha}{2}})$$

as $\epsilon \to 0$. 
Proof. We first prove (6.42). We have using Claim 6.3

\[ J \sqrt{M_1} \sqrt{M_l} = -2 \int_0^1 \frac{\partial \Psi_i}{\partial \Psi_i} + O \left( \left( \| (\theta_1)_x \|_{L^2(0,1)} + \| (\theta_l)_x \|_{L^2(0,1)} \right) + \epsilon^2 \right) \]

(6.44)

\[ = -2 \int_0^1 f_i f_i + O \left( \| (\theta_1)_x \|_{L^2(0,1)} \epsilon^{-2\alpha} + \| (\theta_l)_x \|_{L^2(0,1)} \epsilon^{-2\alpha} + \epsilon^{2-2\alpha} \right) \]

where we set \( \omega_i = \sqrt{\beta_i} \) and \( f_i = d_i \cos (\omega_i v) + e_i \sin (\omega_i v) \). We set for \( i = 1, l \),

\[ x_i = \frac{d_i}{e_i} \]

so that \( x_i = O \left( \frac{\epsilon^{1/2}}{\epsilon} \right) \) as \( \epsilon \to 0 \). We deduce from (6.7) that

(6.45)

\[ \omega_i = \pi - x_i + O \left( |x_i|^3 \right) \]

as \( \epsilon \to 0 \). Then, from

\[ \frac{2 \int_0^1 f_i f_i}{e_i e_i} = (x_1 x_l + 1) \frac{\sin (\omega_1 + \omega_l)}{\omega_1 + \omega_l} + (x_1 x_l - 1) \frac{\sin (\omega_1 - \omega_l)}{\omega_1 - \omega_l} + (x_1 + x_l) \frac{1 - \cos (\omega_1 + \omega_l)}{\omega_1 + \omega_l} + (x_l - x_1) \frac{1 - \cos (\omega_1 - \omega_l)}{\omega_1 - \omega_l} \]

(6.46)

we deduce from (6.45) and (6.47) that

\[ \frac{2 \int_0^1 f_i f_i}{e_i e_i} = 1 + \frac{x_1 + x_l}{2\pi} + x_1 x_l + \frac{(x_1 + x_l)^2}{4 \pi^2} + \frac{(x_1 - x_l)^2}{3} + O \left( (|x_1| + |x_l|)^3 \right) \]

(6.47)

Thanks to (6.8), we have that for \( i = 1, l \),

\[ \frac{M_i}{(e_i)^2} = 1 + \frac{x_i}{\pi} + (x_i)^2 \left( 1 + \frac{1}{\pi^2} \right) + O \left( |x_i|^3 \right) \]

(6.48)

so that since we assumed \( e_i > 0 \),

\[ \frac{e_i}{\sqrt{M_i}} = 1 - \frac{x_i}{2\pi} - (x_i)^2 \left( 1 + \frac{1}{4 \pi^2} \right) + O \left( |x_i|^3 \right) \]

(6.49)

and we deduce from (6.48)

\[ \frac{2 \int_0^1 f_i f_i}{\sqrt{M_i} \sqrt{M_l}} = 1 - \left( \frac{1}{6} + \frac{1}{4 \pi^2} \right) (x_1 - x_l)^2 + \left( \frac{1}{3} + \frac{1}{4 \pi^2} \right) x_1 x_l + O \left( (|x_1| + |x_l|)^3 \right) \]

(6.50)

\[ = 1 - C |u_*(p_0)|^2 \epsilon^{1-\alpha} + o(\epsilon^{1-\alpha}) \]

for a positive constant \( C \) since the product \( x_1 x_l \) has to be negative by (6.13), (6.39) and (6.45). We then proved (6.42).

Now let’s prove (6.41). We set \( u = \frac{u_1}{\sqrt{N_1}} + \frac{u_l}{\sqrt{N_l}} \) and we have that

\[ I = -1 + \frac{|u|^2}{L^2(\partial \Sigma)} \]

(6.51)

Now let’s look for estimates on \( |u|_{L^2(\partial \Sigma)} \). We recall that

\[ F_* = \{ u \in E_*; u(p_0) = 0 \} \]

and \( G_* = E_* \cap F_*^\perp \).
where \( E_* \) is the space of eigenfunctions associated to the first eigenvalue \( \sigma_* \). We have by the Pythagorean theorem that

\[
\|u\|_{L^2(\Omega)}^2 = \|P_{E_*}(u)\|_2^2 + \|P_{G_*}(u)\|_2^2 + \|u - P_{E_*}(u)\|_2^2.
\]

(6.53)

We know from (6.11) and since \( u_* := u_1^* = -u_2^2 \) that

\[
\|u_1 - \sqrt{N_1}u_*\|_{L^2(\partial\Omega)}^2 + \|u_2 + \sqrt{N_2}u_*\|_{L^2(\partial\Omega)}^2 = O(\varepsilon^2)
\]

as \( \varepsilon \to 0 \). Therefore, we have the following estimate

\[
\|P_{G_*}(u)\| = \int_{\partial\Omega} uu_* = \frac{1}{2} \left( \|u_2 + u_*\|_{L^2(\partial\Omega)}^2 - \|u_1 - u_*\|_{L^2(\partial\Omega)}^2 \right)
\]

\[
= \frac{1}{2} \left( \|u_2 + u_*\|_{L^2(\partial\Omega)}^2 - \frac{1}{2} \|u_1 - u_*\|_{L^2(\partial\Omega)}^2 \right).
\]

Then

\[
\|P_{G_*}(u)\| = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\]

(6.54)

Let’s now estimate \( \|u - P_{E_*}(u)\|^2 \). Integrating the equation satisfied by \( u \) against \( u \),

\[
\begin{cases}
\Delta_g u = 0 & \text{in } \Sigma_* \\
\partial_{\nu} u = \frac{\sigma_1 + \sigma_l}{2} u + \frac{\sigma_l - \sigma_1}{2} \left( \frac{u_1}{\sqrt{N_1}} - \frac{u_2}{\sqrt{N_2}} \right) & \text{on } \partial\Sigma_*
\end{cases}
\]

we get that

\[
\int_{\Sigma} |\nabla u|^2 = \frac{\sigma_1 + \sigma_l}{2} \int_{\Sigma} u^2 + \frac{\sigma_l - \sigma_1}{2} \int_{\partial\Sigma} \left( \frac{\phi_1}{\sqrt{N_1}} \right)^2 - \left( \frac{\phi_l}{\sqrt{N_l}} \right)^2
\]

\[
+ \frac{\sigma_1 + \sigma_l}{2} \int_{\partial\Sigma} \left( \frac{\phi_1}{\sqrt{N_1}} + \frac{\phi_l}{\sqrt{N_l}} \right)^2 - \int_{\Omega_*} \left| \nabla \left( \frac{\phi_1}{\sqrt{N_1}} + \frac{\phi_l}{\sqrt{N_l}} \right) \right|^2.
\]

(6.55)

Therefore,

\[
\int_{\Sigma} |\nabla u|^2 \leq \frac{\sigma_1 + \sigma_l}{2} \int_{\partial\Sigma} u^2 + \frac{\sigma_1 - \sigma_l}{2} \left( \frac{M_1}{N_1} - \frac{M_l}{N_l} \right) + \Delta
\]

(6.56)

where we define \( \Delta \) as

\[
\Delta = \frac{\sigma_1 + \sigma_l}{2} \int_{\Sigma} \left( \frac{\phi_1}{\sqrt{N_1}} + \frac{\phi_l}{\sqrt{N_l}} \right)^2 - \int_{\Omega_*} \left| \nabla \left( \frac{\phi_1}{\sqrt{N_1}} + \frac{\phi_l}{\sqrt{N_l}} \right) \right|^2.
\]

(6.57)

Now, denoting \( f = \frac{f_1}{\sqrt{N_1}} + \frac{f_l}{\sqrt{N_l}} \) where \( f_i \) are defined in (6.4), we know from (6.35) and (6.36) with \( \gamma_i = \frac{1}{\sqrt{N_i}} \) that

\[
\Delta = \frac{\sigma_l - \sigma_1}{2t} \left( \frac{M_1}{N_1} - \frac{M_l}{N_l} \right) - \frac{f'(0)f(0)}{(\ln r)^2} - \frac{f(0)^2}{2\ln \frac{1}{r}} + O(\varepsilon^2). \]

(6.58)

We obtain

\[
\int_{\Sigma} |\nabla u|^2 \leq \frac{\sigma_1 + \sigma_l}{2} \int_{\partial\Sigma} u^2 - \frac{f'(0)f(0)}{(\ln r)^2} + \frac{f(0)^2}{2\ln \frac{1}{r}} + O(\varepsilon^2). \]
We know from Claim 6.11 that \( f(0) = d_1 + d_1 = O\left(\varepsilon^{1-\frac{\alpha}{2}}\right) \) so that
\[
(6.59) \quad \int_{\Sigma} |\nabla u|^2 \leq \frac{\sigma_1 + \sigma_l}{2} \int_{\partial\Sigma} u^2 + O\left(\varepsilon^{1+\frac{3\alpha}{2}}\right).
\]
Therefore, similarly to the proof of Claim 6.11, we have for any \( v \in E_* \) first eigenfunction:
\[
(6.60) \quad \int_{\Sigma} |\nabla (u - v)|^2 = \left(\frac{\sigma_1 + \sigma_l}{2} - \sigma_*\right) \int_{\partial\Sigma} u^2 + \sigma_* \int_{\partial\Sigma} (u - v)^2 + O\left(\varepsilon^{1+\frac{3\alpha}{2}}\right).
\]
We then obtain
\[
(6.61) \quad \int_{\Sigma} |\nabla (u - v)|^2 = \sigma_* \int_{\partial\Sigma} (u - v)^2 + O\left(\varepsilon^{1+\frac{3\alpha}{2}} + \left|\frac{\sigma_1 + \sigma_l}{2} - \sigma_*\right| |u|^2\right).
\]
Therefore, applying the same argument as in Claim 6.11 with \( v = P_{E_*}(u) \), we get that
\[
(6.62) \quad \|u - P_{F_*}(u)\|^2 = O\left(\varepsilon^{1+\frac{3\alpha}{2}} + \left|\frac{\sigma_1 + \sigma_l}{2} - \sigma_*\right| |u|^2\right)
\]
as \( \varepsilon \to 0 \).

Now we integrate the equation satisfied by \( u \) against a first eigenfunction \( v \in F_* \) of unit \( L^2 \) norm. We argue similarly to the proof of Claim 6.11 using that \( v = O\left(\varepsilon^2\right) \) on \( S_{\varepsilon^2}(p_0) \) and \( v = O\left(\varepsilon^2r^2\right) \) on \( S_{r^2\varepsilon^2}(p_1) \) uniformly and the weak estimates on the gradient, we obtain
\[
\left(\sigma_* - \frac{\sigma_1 + \sigma_l}{2}\right) \int_{\partial\Sigma} vu = \sigma_1 - \frac{\sigma_1 + \sigma_l}{2} \int_{\partial\Sigma} v \left(\frac{u_1}{\sqrt{N_1}} - \frac{u_l}{\sqrt{N_l}}\right) + O\left(\varepsilon^2\right).
\]
Since \( v \) is orthogonal to \( u_* \),
\[
\int_{\partial\Sigma} v \left(\frac{u_1}{\sqrt{N_1}} - \frac{u_l}{\sqrt{N_l}}\right) = \int_{\partial\Sigma} v \left(\frac{u_1}{\sqrt{N_1}} - u_*\right) - \int_{\partial\Sigma} v \left(\frac{u_l}{\sqrt{N_l}} + u_*\right) = O\left(\varepsilon^{\frac{1}{2}}\right)
\]
as \( \varepsilon \to 0 \). We then get:
\[
(6.63) \quad \left|\sigma_* - \frac{\sigma_1 + \sigma_l}{2}\right| \|P_{F_*}(u)\| = O\left(\varepsilon^{1+\frac{3\alpha}{2}}\right)
\]
Gathering (6.54), (6.62) and (6.63) in (6.53), we get
\[
(6.64) \quad \|u\|^2_{L^2(\partial\Sigma)} = O\left(\varepsilon^{1+\frac{3\alpha}{2}}\right) + O\left(\frac{\varepsilon^{2+3\alpha}}{|\sigma_* - \frac{\sigma_1 + \sigma_l}{2}|^2}\right)
\]
as \( \varepsilon \to 0 \).

and knowing that \( |\sigma_* - \frac{\sigma_1 + \sigma_l}{2}| = O(\varepsilon) \), we get a constant \( C_2 > 0 \) such that
\[
(6.65) \quad \|u\|^2_{L^2(\partial\Sigma)} \leq C_2 \frac{\varepsilon^{2+3\alpha}}{|\sigma_* - \frac{\sigma_1 + \sigma_l}{2}|^2}
\]
Then (6.52) and (6.65) complete the proof of the Claim.

Finally, we prove (6.43). We keep the notation \( u = \frac{u_1}{\sqrt{N_1}} + \frac{u_l}{\sqrt{N_l}} \). We work similarly to the proof of (6.13), but on \( u - P_{F_*}(u) \) instead of \( u_i \pm \sqrt{N_i} u_* \). We set \( \delta = \left|\frac{d_1}{\sqrt{N_1}} + \frac{d_l}{\sqrt{N_l}}\right| \). It is clear that we have on the following mean value of \( u - P_{F_*}(u) \)
\[
(6.66) \quad \left|\frac{1}{\varepsilon^2} \int_{-\varepsilon^2}^{\varepsilon^2} (u - P_{F_*}(u)) o \varphi_0 \, dx\right| = \sqrt{\varepsilon} \varepsilon^{\frac{\alpha-1}{2}} \delta + O\left(\varepsilon^2\right)
\]
as $\varepsilon \to 0$. From (6.54), (6.62) in (6.53), we know that
\begin{equation}
\|u - P_{\epsilon_2}(u)\|_{L^2(\partial \Sigma)}^2 = O(\varepsilon^{2\alpha} + \varepsilon^2)
\end{equation}
as $\varepsilon \to 0$.

We go back to (4.6) in Claim 4.3. A similar formula holds for $|u - P_{\epsilon_2}(u)|$. Then we obtain a formula similar to (6.14):
\begin{equation}
\sum l \leq \varepsilon^2 \phi (\varepsilon_{\alpha} \sigma_2) = \varepsilon^{2\alpha} \delta + \varepsilon^2
\end{equation}
as $\varepsilon \to 0$.

We are now in position to prove the theorem. We assume that $\varepsilon^{2\alpha} = o\left(\sigma_\ast - \frac{\sigma_1 + \sigma_l}{2}\right)$. If not, since $\sigma_l - \sigma_1 = O\left(\varepsilon^{2\alpha} \delta + \varepsilon^2\right)$, the Theorem (1.3) would be proved.

Since $N_l = 1 - M_l$ and $M_l = N_1 = \frac{1}{2}$, we obtain from (6.38) and (6.42) that
\begin{equation}
\sqrt{M_l} - \sqrt{1 - M_l} + (1 + I)\sqrt{1 - M_1} = \sqrt{M_l} C |u_\ast(p_0)|^2 \varepsilon^{1-\alpha} + o(\varepsilon^{1-\alpha}),
\end{equation}
and from (6.41) that
\begin{equation}
2M_l - 1 \sim C |u_\ast(p_0)|^2 \varepsilon^{1-\alpha},
\end{equation}
since by our assumption $\frac{\varepsilon^{2\alpha} \delta}{\sigma_\ast - \frac{\sigma_1 + \sigma_l}{2}} = o\left(\varepsilon^{1-\alpha}\right)$ as $\varepsilon \to 0$.

As a final argument, we test the function $u = \frac{u_1}{\sqrt{N_1}} + \frac{u_2}{\sqrt{N_2}}$ for the first eigenvalue $\sigma_\ast$ of $\Sigma$:
\begin{equation}
\sigma_\ast \leq \frac{\int_{\Sigma} |\nabla u|^2}{\int_{\Sigma} u^2} = \frac{\sigma_1 + \sigma_l - \int_{\Omega_2} \nabla u^2}{2 - \int_{I + \cup I^-} u^2 d\epsilon + O(\varepsilon^2) - (\int_{I + \cup I^-} u^2)^2}
\end{equation}
\begin{equation}
\leq \frac{\sigma_1 + \sigma_l}{2} + \frac{\sigma_1 + \sigma_l}{2} \frac{\int_{I + \cup I^-} u^2 - \int_{\Omega_2} |\nabla u|^2}{2 - \int_{I + \cup I^-} u^2 d\epsilon + O(\varepsilon^2)}
\end{equation}
where the mean value satisfies
\begin{equation}
\int_{\partial \Sigma} u d\gamma = -\int_{I + \cup I^-} u d\epsilon + O(\varepsilon^2) = O\left(\varepsilon^{2\alpha} \delta + \varepsilon^2\right)
\end{equation}
as $\varepsilon \to 0$.

We set
\begin{equation}
A = \frac{\sigma_1 + \sigma_l}{2} \int_{I + \cup I^-} u^2 - \int_{\Omega_2} |\nabla u|^2.
\end{equation}
Now, denoting $f = \frac{f_1}{\sqrt{N_1}} + \frac{f_l}{\sqrt{N_l}}$ where $f_i$ are defined in (6.4), we know from (6.35) and (6.36) with $\gamma_i = \frac{1}{\sqrt{N_i}}$ for $i = 1, l$ that
\[
A t = \frac{\sigma_l - \sigma_1}{2t} (M_1 - M_l) - \frac{f'(0)f(0)}{(\ln r)^2} + \frac{f(0)^2}{2 \ln \frac{r}{r_0}} + O(\varepsilon^2).
\]
We have that $f(0) = \frac{d_1}{\sqrt{N_1}} + \frac{d_l}{\sqrt{N_l}} = O(\varepsilon^3)$ by (6.43) and by (6.70) we obtain that $A$ is negative:
\[
A t \leq \frac{\sigma_l - \sigma_1}{2t} \left( \frac{M_1}{N_1} - \frac{M_l}{N_l} \right) + O(\varepsilon^2)
\]
for a positive constant $D > 0$. Then $\sigma_* \leq \frac{\sigma_1 + \sigma_l}{2}$. Since $\sigma_1 - \sigma_l = O \left( \varepsilon^{\frac{1}{2} + \frac{3\alpha}{2}} \right)$, we can conclude Theorem 1.3.

Remark 6.75. We can prove from Claim 6.40 that
\[
\|u\|_{L^2(\partial \Sigma)}^2 = 2 - \int_{I_+ \cup I_-} u^2 d\xi \sim \varepsilon^{2 + 3\alpha} \left| \sigma_* - \frac{\sigma_1 + \sigma_l}{2} \right|^2
\]
as $\varepsilon \to 0$. Then, with (6.71) and (6.74), we obtain that $\frac{\sigma_1 + \sigma_l}{2} - \sigma_* = O \left( \varepsilon^{\frac{1}{2} + \frac{3\alpha}{2}} \right)$ as $\varepsilon \to 0$, which contradicts the assumption above so that
\[
\left| \sigma_* - \frac{\sigma_1 + \sigma_l}{2} \right| = O \left( \varepsilon^{\frac{1}{2} + 2\alpha} \right)
\]
as $\varepsilon \to 0$ and in this case notice that $\sigma_1 - \sigma_* < 0$ and $\sigma_l - \sigma_* > 0$ for $\varepsilon$ small.

References

[Ann87] C. Anné, Spectre du laplacien et écrasement d’anses, Ann. Sci. École Norm. Sup. (4) 20, 1987, 271–280.
[Bre12] S. Brendle, A sharp bound for the area of minimal surfaces in the unit ball, Geom. Funct. Anal. 22, 2012, 621–625.
[CFS20] A. Carlotto, G. Franz, M. B. Schulz Free boundary minimal surfaces with connected boundary and arbitrary genus, arXiv preprint 2020, arXiv:2001.04920, 16pp.
[CES03] B. Colbois, A. El Soufi, Extremal eigenvalues of the Laplacian in a conformal class of metrics: the ‘conformal spectrum’, Ann. Global Anal. Geom. 24, 2003, no.4, 337–349.
[Con40] R. Courant, The existence of minimal surfaces of given topological structure under prescribed boundary conditions, Acta Math. 72, 1940, 51-98.
[ESI00] A. El Soufi, S. Ilias, Riemannian manifolds admitting isometric immersions by their first eigen-functions, Pacific J. Math. 195, 2000, 91-99.
[FS11] A. Fraser, R. Schoen, The first Steklov eigenvalue, conformal geometry, and minimal surfaces, Adv. Math. 226, 2011, 4011–4030.
[FS16] A. Fraser, R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball, Invent. Math. 203, 2016, 823–890.
[FS19] A. Fraser, R. Schoen, Some results on higher eigenvalue optimization, arXiv preprint 2019, arXiv:1910.03547, 25pp.
[FPZ17] A. Folha, F. Pacard, T. Zolotareva, Free boundary minimal surfaces in the unit 3-ball, Manuscripta Math. 154, 2017, 359–409.
[GL20] A. Girouard, J. Lagacé, Large Steklov eigenvalues via homogenisation on manifolds, arXiv preprint 2020, arXiv:2004.04044, 30pp.
[Hil85] S. Hildebrandt, *Free boundary problems for minimal surfaces and related questions*, Comm. Pure Appl. Math. 39 1986, no. S, suppl., S11-S138, Frontiers of the mathematical sciences: 1985 (New York, 1985).

[KL17] N. Kapouleas, M.M-C. Li, *Free Boundary Minimal Surfaces in the Unit Three-Ball via Desingularization of the Critical Catenoid and the Equatorial Disk*, arXiv preprint 2017, arXiv:1709.08556, 45 pp.

[KW17] N. Kapouleas, D. Wiygul, *Free-boundary minimal surfaces with connected boundary in the 3-ball by tripling the equatorial disc*, arXiv preprint 2017, arXiv:1711.00818, 33 pp.

[KKP14] M.A. Karpukhin, G. Kokarev, I. Polterovich, *Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces*, Ann. Inst. Fourier (Grenoble), Université de Grenoble. Annales de l’Institut Fourier, 64, 2014, 6, 2481–2502.

[KS20] M. Karpukhin, D. L. Stern *Min-max harmonic maps and a new characterization of conformal eigenvalues*, arXiv preprint 2020, arXiv:2004.04086, 59 pp.

[Ke17] D. Ketover, *Free boundary minimal surfaces of unbounded genus*, arXiv preprint 2016, arXiv:1612.08691, 32 pp.

[Ke17a] D. Ketover, *Equivariant min-max theory*, arXiv preprint 2016, arXiv:1612.08692, 42 pp.

[Kok14] G. Kokarev, *Variational aspects of Laplace eigenvalues on Riemannian surfaces*, Adv. Math. 258, 2014, 191–239.

[Li20] M. M-C. Li, *Free boundary minimal surfaces in the unit ball: recent advances and open questions*, to appear in Proceedings of the first annual meeting of the ICCM, 2020.

[MS17] H. Matthiesen, A. Siffert, *Existence of metrics maximizing the first eigenvalue on non-orientable surfaces*, to appear in Journal of Spectral Theory, 2017, 14pp.

[MS19] H. Matthiesen, A. Siffert, *Sharp asymptotics for the first eigenvalue on some degenerating surfaces*, to appear in Trans. Amer. Math. Soc, 2019, 35pp.

[MS19a] H. Matthiesen, A. Siffert, *Handle attachment and the normalized first eigenvalue*, arXiv preprints 2019, arXiv:1909.03105v2, 65pp.

[Nad96] N. Nadirashvili, *Berger’s isoperimetric problem and minimal immersions of surfaces*, Geom. Funct. Anal. 6, 1996, 877–897.

[NT08] S. A. Nazarov, J. Taskinen *On the spectrum of the Steklov problem in a domain with a peak*, Vestnik St. Petersburg University: Mathematics volume 41, 2008, 45-52.

[Pet14] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*, Geom. Funct. Anal. 24, 2014, 1336–1376.

[Pet18] R. Petrides, *On the existence of metrics which maximize Laplace eigenvalues on surfaces*, Int. Math. Res. Not. 14, 2018, 4261–4355.

[Pet19] R. Petrides, *Maximizing Steklov eigenvalues on surfaces*, J. Differential Geom. Volume 113, 2019, no.1, 95–188.

[Rob11] N. Robin, *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains*, J. Differential Equations 251, 2011, 860–880.

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