The Coulomb Branch of $N = 2$ Supersymmetric Product Group Theories from Branes

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Abstract

We determine the low energy description of $N = 2$ supersymmetric $\prod_i SU(k_i)$ gauge theories with bifundamental and fundamental matter based on $M$-theory fivebrane configurations. The dependence on moduli and scales of the coefficients in the non-hyperelliptic Seiberg-Witten curves for these theories are determined by considering various field theory and brane limits. A peculiarity in the interpretation of these curves for the vanishing $\beta$-function case is noted.
1 Introduction

In recent months it has been shown that many field theory results for strongly interacting
gauge theories can be derived from string theory [1, 2]. Further results have been derived
from M-theory [3]. M-theory is particularly useful in identifying the form of the Seiberg-
Witten curve describing the Coulomb branch in cases where it is not hyperelliptic, since it
gives the order of the polynomial in $y$ and $x$ as shown in Ref. [3] (or $t$ and $v$ in the new
language). However, simply knowing the form of the polynomials is not the whole answer if
one actually were to extract the physics associated with the curve. One would also need to
know the dependence of the coefficients on the moduli and the dynamical scales of the theory.
Of course there are ambiguities in defining the moduli quantum mechanically. Nonetheless,
even with these ambiguities, one can derive physical quantities from the curve. Many of
the curves which have been derived so far reproduce old results or do not make manifest
the dependence of the coefficients of the curve on physical parameters; however there have
been several detailed new results which have been obtained from the M-theory picture (e.g.
[4, 5, 6]).

In this paper, from the M-theory starting point, we fill in the moduli and scale dependence
of the coefficients of the $N=2$ theory curves for arbitrary products of SU($k$) factors using
both field theoretic and brane considerations. These theories have also been studied in
the context of geometric engineering [7], but detailed comparison with the standard field
theoretic moduli is difficult from this perspective.

In the following section, we review the M-theory construction. We then discuss in detail
$N=2$ SU(2)$\times$SU(2) with a bifundamental hypermultiplet. This will allow us to introduce
the constraints which we found necessary to pin down the form of the coefficients, and to
verify our results by comparing the singularities to those of the known SO(4) theory. We
then generalize to SU($N)\times$SU($M$) (finding results in agreement with Giveon and Pelc [4])
and also to an arbitrary product of SU($k$) groups for asymptotically free theories. Our result
is that the curve for the $\prod_{i=1}^{M}$ SU($k_i$) theory with $k_0$ flavors of SU($k_1$) and $k_{M+1}$ flavors of
SU($k_M$) hypermultiplets is

$$t^{M+1} P_{k_0}(v) - t^M P_{k_1}(v) + \sum_{j=0}^{M-1} (-1)^{M-j+1} \prod_{n=1}^{M-j-1} \Lambda_n^{(M-n-j+1)\beta_n} t^j P_{k_{M-j+1}}(v) = 0.$$  (1)

The polynomials, which are explicitly given in the text, are the polynomials which reproduce
the classical singularities for the individual SU($n$) groups. This is in fact the simplest answer
one might have guessed; the paper shows that this is in fact the correct curve.

Finally, we discuss some aspects of the theories with vanishing beta function. We find
the intriguing result that in the classical limit the distance between the fivebranes in the
SU(2) theory with four flavors corresponds to the SU(2) coupling if we do not identify the
coupling $\tau$ appearing in the curve as the SU(2) coupling of the massless theory in the SW
renormalization scheme. In fact, our result seems to substantiate the claim in [8] regarding the interpretation of Seiberg-Witten curves in conformal theories.

2 M-Theory Construction

In this section, we review the basic elements of Witten’s M-theory construction in order to establish notation – the details can be found in [3]. We will first discuss the brane configuration in Type IIA string theory [2] and then review Witten’s interpretation of the configuration in M-theory.

The Type IIA picture involves the Neveu-Schwarz solitonic fivebranes and Dirichlet fourbranes in flat ten-dimensional Minkowski space. There are $N$ fourbranes located at $x^7 = x^8 = x^9 = 0$ and some fixed value of $v = x^4 + ix^5$ with world volume coordinates $x^0, x^1, x^2, x^3$ and $x^6$. When the open superstrings which end on these fourbranes are quantized, the massless excitations give a U($N$) gauge theory in ten dimensions with $N = 1$ supersymmetry (presence of the fourbranes breaks half the supersymmetries so 16 supercharges are left unbroken). The strings stretching between the fourbranes represent the $N^2$ gauge bosons. Dimensional reduction of this theory to the world volume of the fourbranes gives a U($N$) gauge theory in five dimensions. The theory on the world volume of the fivebrane has a U($N$) gauge field $A_i$ ($i = 0, 1, 2, 3, 6$) and five real scalar fields in the adjoint representation of U($N$) corresponding to the five transverse directions. The scalar fields can be interpreted geometrically as specifying the location of the fourbranes in the transverse space.

Now consider the configuration with the $N$ fourbranes stretched between two NS fivebranes located at $x^7 = x^8 = x^9 = 0$ and some fixed values of $x^6$ with world volume coordinates $x^0, x^1, x^2, x^3$ and $x^6$. Due to the compactness of the fourbrane in the $x^6$ direction, at low energies (i.e. at length scales much larger than the $x^6$ separation of the fivebranes), the world volume theory on the fourbranes is effectively four dimensional. Ignoring the dependence on the $x^6$ coordinate, there is a four dimensional theory with a U($N$) gauge field and one scalar field corresponding to $A_6$. This $A_6$, along with the scalar fields corresponding to the $x^7, x^8$ and $x^9$ directions is projected out of the low energy four dimensional theory on the world volume of the fourbranes. Since the fourbranes are free to move in the $v = x^4 + ix^5$ direction, the complex scalar field $\phi$ which corresponds to this motion remains in the low energy theory and combines with the gauge field to give an $N = 2$ vector multiplet. That the theory has $N = 2$ supersymmetry can easily be seen by the fact that the fivebranes break another half of the supersymmetries, leaving 8 unbroken supercharges corresponding to $N = 2$ in four dimensions.

At weak coupling, the coupling of the four dimensional gauge theory is $1/g^2 = \Delta x^6/\lambda$ where $\lambda$ is the string coupling constant. (In M-theory units the string coupling is replaced by the $M$-theory radius $R_{11}$.) The fact that the coupling of the gauge theory runs with scale is nicely reflected in the bending of the fivebranes due to the force exerted by the fourbranes.
Figure 1: The brane configuration corresponding to the $\text{SU}(k_1) \times \text{SU}(k_2) \times \cdots \times \text{SU}(k_M)$ theory with bifundamentals and $k_0$ flavors of $\text{SU}(k_1)$ and $k_{M+1}$ flavors of $\text{SU}(k_M)$. The thick vertical lines represent the NS-fivebranes, and the horizontal lines are the D-fourbranes.

as explained in [3]. We will give a more precise formula for the conformal case in Section 4.

The kinetic term of the ten dimensional $U(N)$ gauge theory produces a scalar potential of the form $V = \text{Tr} [\phi^*, \phi]^2$. This potential has flat directions corresponding to diagonal $\phi$ matrices. In each of these vacua, the $U(N)$ gauge symmetry is broken to $U(1)^N$—the diagonal entries of $\phi$ correspond to the distance between the fourbranes. As discussed in [3], the motion of the fourbranes results in the motion of the disturbance they produce on the fivebranes. The requirement of finite energy configurations imposes the condition that the average position in $v$ of the fourbranes is constant. Hence a $U(1)$ subgroup of $U(N)$ is non-dynamical and the configuration describes an $\text{SU}(N)$ gauge theory in its Coulomb phase.

An obvious extension of this setup is shown in Figure 1. There are $M + 1$ fivebranes labeled by $\alpha = 1, \ldots, M + 1$ with $k_\alpha$ fourbranes stretched between the $\alpha$th and $(\alpha + 1)$th fivebranes. The gauge group of the four-dimensional theory will be $\prod_{\alpha=1}^{M} \text{SU}(k_\alpha)$. The hypermultiplet spectrum of the theory will correspond to strings ending on fourbranes of adjacent groups. They will transform as $(k_1, \bar{k}_2) \oplus (k_2, \bar{k}_3) \oplus \cdots (k_{M-1}, \bar{k}_M)$. The bare mass of a hypermultiplet, $m_\alpha$ is the difference between the average position in the $v$ plane of the fourbranes to the left and right of the $\alpha$th fivebrane.

In the strong coupling limit of the Type IIA string theory, the low energy dynamics is described by eleven-dimensional supergravity, which is the semiclassical limit of an eleven-dimensional $M$-theory. This theory lives on $R^{1,9} \times S^1$ where $R^{1,9}$ is the ten dimensional Minkowski space and $S^1$ is a circle of radius $R$ in the tenth spatial direction $x^{10}$. The fourbranes and fivebranes of Type IIA string theory come from the same fivebrane of $M$-theory—the fivebrane is an $M$-theory fivebrane at a point in $x^{10}$, whereas the IIA fourbrane is the $M$-theory fivebrane wrapped once around the circle $S^1$ in $x^{10}$. By lifting the brane
configuration of Type IIA string theory discussed above to M-theory, the configuration is described by a single fivebrane which captures the nonperturbative physics of the gauge theory (as discussed in [3]). The world volume of this M-theory fivebrane is a continuous six dimensional surface embedded in an eight dimensional space \(-R^{1,3}\) which is the four dimensional Minkowski space \((x^0, x^1, x^2, x^3)\) and \(v = x^4 + ix^5\) and \(t = e^{-(x^6+ix^{10})/R}\). Since the construction in the Type IIA picture is translationally invariant in \(R^{1,3}\), the world volume of the M-theory fivebrane will factor as \(R^{1,3} \times \Sigma\) where \(\Sigma\) is a two-dimensional Riemann surface embedded in \(v, t\) space described by a single complex equation in \(t\) and \(v\). This surface is the Seiberg-Witten surface from which the gauge couplings of the various U(1)’s in the low energy theory can be derived. As discussed in [3], the surface describing the product group configuration in Figure 1 corresponds to the following curve:

\[p_{k_0}(v)t^{M+1} + p_{k_1}(v)t^M + p_{k_2}(v)t^{M-1} + \ldots + p_{k_M}(v)t + p_{k_{M+1}}(v) = 0\]

where \(p_{k_i}(v)\) are polynomials of order \(k_i\) in \(v\). In this paper, our objective is to find the explicit dependence of the polynomials \(p_{k_i}(v)\) on the moduli and scales of the gauge theory.

3 \(SU(2)_1 \times SU(2)_2\)

As a preliminary step in obtaining the full curve discussed above, we first derive in detail the exact curve for the simplest product group theory in this class, \(SU(2) \times SU(2)\).

In four dimensional \(N = 1\) language, the theory has vector multiplets associated with the \(SU(2)_1 \times SU(2)_2\) gauge group and the following chiral multiplets.

|       | \(SU(2)_1\) | \(SU(2)_2\) |
|-------|-------------|-------------|
| \(\Phi_1\) | \(\square\) | 1           |
| \(\Phi_2\) | 1           | \(\square\) |
| \(Q\)     | \(\square\) | \(\square\) |
| \(\tilde{Q}\) | \(\square\) | \(\square\) |

For other gauge groups the symmetric tensor generalizes to the adjoint of the gauge group. By adjusting the average position of the fourbranes for the two groups to be the same, we can set the bare mass of the hypermultiplet \((Q, \tilde{Q})\) to zero. We will also find it useful to scale \(t\) such that the middle brane is at \(x^6 = 0\), \(i.e.\ t = 1\) in the dimensionless convention for \(t\) which we will find most convenient). As discussed in the previous section, the Seiberg-Witten surface is described by the curve

\[t^3 + P_2(v, u_1, u_2, \Lambda_1, \Lambda_2)t^2 + \tilde{P}_2(v, u_1, u_2, \Lambda_1, \Lambda_2)t + k(\Lambda_1, \Lambda_2) = 0,\]

where \(P_2\) and \(\tilde{P}_2\) are polynomials quadratic in \(v\) and depend on scales \(\Lambda_1\) and \(\Lambda_2\) and moduli \(u_1 = \text{Tr } \Phi^2_1\), \(u_2 = \text{Tr } \Phi^2_2\). The constant \(k\) depends on \(\Lambda_1\) and \(\Lambda_2\). Note that we are considering
only the Coulomb branch and have taken the $Q$ field to vanish; the Higgs branch will be commented on later.

We will fix and check the dependence of the polynomials $P_2$ and $\tilde{P}_2$ on $u_1, u_2, \Lambda_1$ and $\Lambda_2$ using the following:

1. U(1) Symmetries
2. SU(2)$_1 \leftrightarrow$ SU(2)$_2$ and $t \leftrightarrow 1/t$ symmetry
3. Classical limits
4. Assuming dependence on the strong interaction scale $\Lambda$ is through instanton corrections and arises only through positive integer powers of $\Lambda^{b_0}$
5. Removing the middle brane
6. Comparison with SO(4) with a vector hypermultiplet

We will determine the curve up to an arbitrary function $h(\Lambda_1^2, \Lambda_2^2)$ and an integer $p$ by using the first four constraints and fix the curve uniquely with the fifth. Since SO(4) with a vector hypermultiplet is isomorphic to SU(2)$\times$SU(2) with a bifundamental hypermultiplet, we can use the known SO(4) curve as a check for the SU(2)$\times$SU(2) curve.

**U(1) Symmetries**

Unlike many $N = 1$ curves, the U(1) symmetries here are not very restrictive. Because of the hypermultiplet which couples to both adjoints, the only independent U(1) which helps restrict the curve (in the Coulomb phase where the VEVs of the hypermultiplets are set to zero) is the U(1) $R$-symmetry. However, this is not restrictive in that it is equivalent to requiring all terms in the curve to have the same dimension.

**SU(2)$_1 \leftrightarrow$ SU(2)$_2$ and $t \leftrightarrow 1/t$ symmetry**

From the brane picture, it is clear that the curve should be equivalent to the curve in which the role of SU(2)$_1$ and SU(2)$_2$ are interchanged if we also take $x_6 \rightarrow -x_6$, or equivalently $t \rightarrow 1/t$ if the middle brane is at $x_6 = 0$. When we rewrite the curve in terms of $t' = 1/t$, we get

$$
t'^2 + \frac{\tilde{P}_2(v, u_1, u_2, \Lambda_1, \Lambda_2)}{k(\Lambda_1, \Lambda_2)} t'^2 + \frac{P_2(v, u_1, u_2, \Lambda_1, \Lambda_2)}{k(\Lambda_1, \Lambda_2)} t'^2 + \frac{1}{k(\Lambda_1, \Lambda_2)} = 0.
$$

Since the middle brane has an equal number of fourbranes attached to it from the left and the right, it should be at a fixed value of $t$ for large $v$. We can scale $t$ such that the middle
brane is at $t = 1$ ($x^6 = 0$). Then $t \leftrightarrow 1/t$ corresponds to $SU(2)_1 \leftrightarrow SU(2)_2$. Hence we should have

$$
\frac{\tilde{P}_2(v, u_1, u_2, \Lambda_1, \Lambda_2)}{k(\Lambda_1, \Lambda_2)} = P_2(v, u_2, u_1, \Lambda_2, \Lambda_1)
$$

$$
\frac{1}{k(\Lambda_1, \Lambda_2)} = k(\Lambda_2, \Lambda_1)
$$

Classical Limits

The curve is a function of $\Lambda_1$, $\Lambda_2$, $u_1$, and $u_2$, or effectively three ratios. We can take the classical limits $\Lambda_1 \to 0$, $\Lambda_2 \to 0$, $1/u_1 \to 0$, where it is understood that this means $\Lambda_1 \to 0$ relative to the three other dimensionful parameters given above, etc.

$\Lambda_2 \to 0$

Pulling the rightmost brane to $x_6 = \infty$, i.e. taking $\Lambda_2 = 0$, we expect to get the curve for an $SU(2)$ gauge theory with two flavors. The curve should factorize as

$$
t \left( t^2 - \frac{4}{\Lambda_1^2} (v^2 - u_1 + \frac{\Lambda_2^2}{8}) t + \frac{4}{\Lambda_1^2} (v^2 - u_2) \right) = 0,
$$

where the factor $t$ corresponds to the rightmost brane at $x_6 = \infty$ ($t = 0$) and the rest is the Seiberg-Witten curve for an $SU(2)$ gauge theory with two flavors with bare masses $m$ and $-m$ such that $u_2 = m^2$. Note that the brane at $t = 0$ is infinitely far away and not relevant.

$\Lambda_1 \to 0$

Pulling the leftmost brane to $x_6 = -\infty$, the curve should factorize as

$$
t' \left( t'^2 - \frac{4}{\Lambda_2^2} (v^2 - u_2 + \frac{\Lambda_1^2}{8}) t' + \frac{4}{\Lambda_2^2} (v^2 - u_1) \right) = 0,
$$

which is again the brane at $x_6 = -\infty$ times the Seiberg-Witten curve with two flavors.

At this point, the most general curve we can write down consistent with the above conditions is

$$
t^3 - \frac{4}{\Lambda_1^2} \left( (v^2 - u_1) + \frac{\Lambda_1^2}{8} + O(u_1^2, u_2^2) \right) \frac{h(\Lambda_1^2, \Lambda_1^2)}{(\Lambda_1^2)^p} t^2
$$

$$
+ \frac{4}{\Lambda_2^2} \left( (v^2 - u_2) + \frac{\Lambda_2^2}{8} + O(u_1^2, u_2^2) \right) \frac{h(\Lambda_1^2, \Lambda_1^2)}{(\Lambda_2^2)^p} t - \left( \frac{\Lambda_2^2}{\Lambda_1^2} \right)^{p+1} = 0,
$$
where $p$ is a positive integer. $h(\Lambda_1^2, \Lambda_2^2)$ is a function such that

$$
\begin{align*}
h(\Lambda_1^2, 0) &= (\Lambda_1^2)^p, \\
h(\Lambda_1^2, \Lambda_2^2) &= h(\Lambda_2^2, \Lambda_1^2).
\end{align*}
$$

For example, $h(\Lambda_1^2, \Lambda_2^2)$ could be $(\Lambda_1^2 + \Lambda_2^2)^p$. In fact, there could in principle be independent functions $h_1, h_2, h_3$ multiplying $v^2, u, \Lambda^2$. However, there is freedom to redefine $u$ and $v$ (where the redefinition must agree with their semiclassical definitions in the $\Lambda_1, \Lambda_2 \to 0$ limit) which permits the curve to be written with the single function $h$ (which will in any case be shown to be trivial in the subsequent section). Notice that the function $h$ cannot be absorbed in $u$ because the classical limit of the function would not be correct. The function $h$ can exist because in the presence of two scales $\Lambda_1$ and $\Lambda_2$, one can construct dimensionless ratios which are consistent with the classical limits.

$O(u_1^2, u_2^2)$ are terms of higher order in $u_1$ and $u_2$ which vanish in the $\Lambda_2 \to 0$ limit and respect the SU(2)$_1 \leftrightarrow$ SU(2)$_2$ and $t \leftrightarrow 1/t$ symmetries. The terms should also be taken to respect the $u \to \infty$ limit not yet discussed, which in practice means $u_1$ must be multiplied by a sufficiently high power of $\Lambda_2$. Again because we can take dimensionless ratios with good classical limits, there are many such terms permitted at this point. Although these terms appear strange, one can of course multiply through so that all the instanton powers appear in the numerator.

By explicitly examining the $\Lambda_1 \to 0$ and $\Lambda_2 \to 0$ limits, it is clear that the curve must have the correct classical singularities (namely where either of the $u_i$ vanish and when $u_1 = u_2$). We demonstrate this explicitly for $p = 1$.

We will first consider the $p = 0, h(\Lambda_1^2, \Lambda_2^2) = 1$ case, which should manifestly have the correct classical singularities. As we will show, the singularities of the curve are at $u_1 = u_2$ and when the discriminant of the following polynomial vanishes:

$$
\Lambda_1^2 t^3 + (4u_1 - \frac{\Lambda_1^2}{2})t^2 - (4u_2 - \frac{\Lambda_2^2}{2})t - \Lambda_2^2.
$$

The discriminant of this polynomial is

$$
\Delta = 256\tilde{u}_1^2\tilde{u}_2^2 + 288\Lambda_1^2\Lambda_2^2\tilde{u}_1\tilde{u}_2 + 256\Lambda_2^2\tilde{u}_1^3 - 256\Lambda_1^2\tilde{u}_2^3 - 27\Lambda_1^4\Lambda_2^4,
$$

where $\tilde{u}_i = u_i - \frac{\Lambda_i^2}{8}$ for $i = 1, 2$. Now if one includes a nontrivial function $h$, one in fact obtains the same classical singularities. For example, consider explicitly the case where $p = 1$ and $h(\Lambda_1^2, \Lambda_2^2) = \Lambda_1^2 + \Lambda_2^2$. There are singularities when the discriminant of the following polynomial is zero:

$$
\frac{\Lambda_1^4}{\Lambda_1^2 + \Lambda_2^2} t^3 + (4u_1 - \frac{\Lambda_1^2}{2})t^2 - (4u_2 - \frac{\Lambda_2^2}{2})t - \frac{\Lambda_2^4}{\Lambda_1^2 + \Lambda_2^2}.
$$
The discriminant is
\[ \tilde{\Delta} = 256\tilde{u}_1^2\tilde{u}_2^2 - 288\frac{\Lambda_1^4\Lambda_2^4}{(\Lambda_1^2 + \Lambda_2^2)^4} \tilde{u}_1\tilde{u}_2 - 256\frac{\Lambda_1^4}{\Lambda_1^2 + \Lambda_2^2} \tilde{u}_2^3 + 256\frac{\Lambda_2^4}{\Lambda_1^2 + \Lambda_2^2} \tilde{u}_1^3 - 27\frac{\Lambda_1^8\Lambda_2^8}{(\Lambda_1^4 + \Lambda_2^4)^2}. \]

It is clear that this discriminant also gives the correct classical singularities, independent of the ratio \( \Lambda_1/\Lambda_2 \). If the leading term in the discriminant is defined without factors of \( \Lambda \), it is clear that the extra terms do involve instanton powers in the denominator; it is not obvious that such terms should be ruled out as they have a good classical limit and one can write the polynomial with instanton powers in the numerator. However, we will see in the next section that these other terms are otherwise excluded.

\[ u_1 \to \infty \]

Another way to obtain the classical limit is to take \( u_1 \to \infty \). In order to do this consistently, we need a finite \( \tilde{\Lambda}_2 \), where \( \tilde{\Lambda}_2^4 = \Lambda_2^2u_1 \). Because this amounts to taking \( \Lambda_2 \to 0 \), it does not provide an additional constraint on the curve. That one obtains the correct classical limit, zero flavor SU(2) can be readily seen by scaling \( t \) according to \( t' = t\sqrt{u_1/\Lambda_2} \). This limit does constrain the higher order terms in \( u_1 \) and \( u_2 \), but one can still construct functions which survive this limit.

**Removing the middle brane**

A further constraint on the curve can be obtained by examining the subspace of the moduli space where the Higgs branch which has SU(2) \(_1\times\)SU(2) \(_2\) broken to diagonal SU(2) joins the Coulomb branch. The Higgs branch arises when \( Q \) and \( \tilde{Q} \) fields become massless. In an SU(\( N \)) theory, this would correspond to the baryonic Higgs branch. The hypermultiplets are massless when the four branes on either side of the middle brane align. The meaning of this condition is clear semiclassically; the quantum mechanical condition is derived from the curve (with a given convention for the moduli).

In the brane picture, removing the middle brane corresponds to Higgsing the SU(2) \(_1\)×SU(2) \(_2\) group to the diagonal SU(2) subgroup. We can remove the middle brane if it is straight—this is the case when the fourbranes of both SU(2)'s are attached to the middle brane at the same point. A straight brane corresponds to the point where the Higgs branch joins the Coulomb branch. This should also correspond to a singularity of the curve since this is the point where the quark hypermultiplet becomes massless. The condition for factoring out a \((t - 1)\) from the curve is

\[ u_1 - \frac{\Lambda_1^2}{8} + \frac{(\Lambda_1^2)^{p+1}}{8h(\Lambda_1^2, \Lambda_2^2)} + O(u_1, u_2) = u_2 - \frac{\Lambda_2^2}{8} + \frac{(\Lambda_2^2)^{p+1}}{8h(\Lambda_1^2, \Lambda_2^2)} + O(u_1, u_2), \quad (2) \]
and the curve factorizes as
\[(t - 1) \left( t^2 + \left( 1 - \frac{4h(\Lambda_1^2, \Lambda_2^2)}{(\Lambda_1^2)^{p+1}} \left( v^2 - u_1 + \frac{\Lambda_1^2}{8} + O(u_1^2, u_2^2) \right) \right) t + \left( \frac{\Lambda_2^2}{\Lambda_1^2} \right)^{p+1} \right) = 0. \]

It is clear that if we pull the middle brane to infinity in the \(x^7, x^8, x^9\) direction, the brane configuration in the \(vt\) plane describes the diagonal SU(2) theory with no flavors. Due to the decoupling of the Higgs and Coulomb branches, we expect that even as we bring the middle brane to the same \(x^7, x^8, x^9\) values as the other fivebranes, this should still be the case, i.e. the factor of the curve multiplying \((t - 1)\) should describe an SU(2) gauge theory with dynamical scale \(\Lambda\) such that \(\Lambda_2^2 = \Lambda_1^2 \Lambda_2^2\). The factor of the curve
\[ t^2 + \left( 1 - \frac{4h(\Lambda_1^2, \Lambda_2^2)}{(\Lambda_1^2)^{p+1}} \left( v^2 - u_1 + \frac{\Lambda_1^2}{2} + O(u_1^2, u_2^2) \right) \right) t + \left( \frac{\Lambda_2^2}{\Lambda_1^2} \right)^{p+1} = 0 \]
can be written as
\[ \bar{t}^2 + \left( \frac{\Lambda_2^2}{\Lambda_1^2} \right)^{p+1} \left( 1 - \frac{4h(\Lambda_1^2, \Lambda_2^2)}{(\Lambda_1^2)^{p+1}} \left( v^2 - u_1 + \frac{\Lambda_1^2}{2} + O(u_1^2, u_2^2) \right) \right) \bar{t} + 1 = 0, \]

where \(\bar{t} = \left( \frac{\Lambda_2^2}{\Lambda_1^2} \right)^{p+1} t\). To get the right Seiberg-Witten curve, we need \(p = 0\), \(h(\Lambda_1^2, \Lambda_2^2) = 1\) and no terms of higher order in \(u_1\) and \(u_2\). We then get
\[ \bar{t}^2 - \frac{4}{\Lambda_2^2} (v^2 - U) \bar{t} + 1 = 0, \]

by using \(\Lambda_2^2 = \Lambda_1^2 \Lambda_2^2\), \(U = u_1 + \frac{\Lambda_1^2}{8}\) (the moduli have to agree in the semi-classical limit only) and \(\bar{t} = \frac{\Lambda_2^2}{\Lambda_1^2} t\). This is indeed the Seiberg-Witten curve for an SU(2) theory with no flavors.

The curve for the SU(2)\(_1\)×SU(2)\(_2\) theory is then uniquely determined to be
\[ t^3 - \frac{4}{\Lambda_1^2} \left( (v^2 - u_1) + \frac{\Lambda_1^2}{8} \right) t^2 + 4 \left( (v^2 - u_2) + \frac{\Lambda_2^2}{8} \right) t - \frac{\Lambda_2^2}{\Lambda_1^2} = 0. \quad (3) \]

The condition on the moduli for factoring out a middle brane (or where the Higgs branch joins the Coulomb branch) becomes
\[ u_1 + \frac{\Lambda_1^2}{8} = u_2 + \frac{\Lambda_2^2}{8}. \quad (4) \]

We can see that the curve has a singularity for moduli satisfying (4). The curve \(F(t, v) = 0\) is singular when
\[ F(t, v) = 0, \quad (5) \]
\[ \frac{\partial F}{\partial t}(t, v) = 0, \quad (6) \]
\[ \frac{\partial F}{\partial v}(t, v) = 0. \quad (7) \]
This is satisfied for \( t = 1, \quad v = \frac{u_1 + u_2}{2} + \frac{5\Lambda^2}{16} + \frac{5\Lambda^2}{16} \) and \( u_1 + \frac{\Lambda^2}{8} = u_2 + \frac{\Lambda^2}{8} \).

**Comparison with the \( \text{SO}(4) \) theory with a vector hypermultiplet**

We notice that for \( \Lambda_1 = \Lambda_2 = \Lambda \), the \( N = 2 \text{SU}(2)_1 \times \text{SU}(2)_2 \) theory with a bifundamental hypermultiplet is the same as an \( \text{SO}(4) \) theory with a vector hypermultiplet. This curve was given by Argyres, Plesser, and Shapere [9] and is

\[
y^2 = x(x - \phi_1^2)^2(x - \phi_2^2)^2 - 4\Lambda^2x^4
\]

(8)

Here \( \phi_1 \) and \( \phi_2 \) are the semiclassical eigenvalues appearing in the skew-diagonalized adjoint matrix. Notice that the curve for the \( \text{SO}(4) \) theory was given as a polynomial whose highest order term is \( t^2 \). Nonetheless, we will show that the singularities occur at the same locations for the \( \text{SU}(2)_1 \times \text{SU}(2)_2 \) curve (though we do not find an explicit transformation of coordinates).

By identifying the generators of the commuting \( \text{SO}(3) \) subgroups of \( \text{SO}(4) \), it is easy to check that \( u_1' = (\phi_1 + \phi_2)^2 \text{Tr} T_3^2 \) and \( u_2' = (\phi_1 - \phi_2)^2 \text{Tr} T_3^2 \). Including a minus sign associated with the trace, one derives \( \phi_1^2 + \phi_2^2 = -\frac{1}{2}(u_1' + u_2') \) and \( \phi_1\phi_2 = \frac{1}{4}(u_1' - u_2') \). The curve for this theory in terms of moduli \( u_1' \) and \( u_2' \) (\( u_1' \) and \( u_2' \) should be the same as \( u_1 \) and \( u_2 \) in the semi-classical limit) is

\[
y^2 = P(x) = x\left(x^2 + \frac{1}{2}(u_1' + u_2')x + \frac{1}{4}(u_1' - u_2')^2\right)^2 - 4\Lambda^2x^4.
\]

If the two theories describe the same physics, the singularities should coincide. The singularities of the curve occur when the discriminant of \( P(x) \) vanishes. The discriminant of \( P(x) \) is

\[
-256\Lambda^4(u_1' - u_2')^4(64u_1^2\Lambda^2 + 27u_1^2\Lambda^4 - 256u_1^2u_2^2 + 96u_1^2u_2^2\Lambda^2 - 54u_1^2\Lambda^4u_2' - 96u_1u_2\Lambda^2 + 27\Lambda^4u_2^2 + 64u_2^3\Lambda^2)
\]

For \( \Lambda_1 = \Lambda_2 = \Lambda \), the curve for the \( \text{SU}(2)_1 \times \text{SU}(2)_2 \) theory is

\[
F(t, v) = t^3 - \frac{4}{\Lambda^2}(v^2 - u_1 + \frac{\Lambda^2}{8})t^2 + \frac{4}{\Lambda^2}(v^2 - u_2 + \frac{\Lambda^2}{8})t - 1 = 0
\]

The singularities of a curve \( F(t, v) = 0 \) are given by solutions of equations (8,9). \( \frac{\partial F}{\partial v}(t, v) = 0 \Rightarrow vt(t - 1) = 0 \Rightarrow v = 0, t = 0, \text{ or } t = 1 \). Since \( F(0, v) = \Lambda^2 \neq 0, t \neq 0 \). For \( t = 1 \), we get a singularity of the curve at \( u_1 = u_2 \). For \( v = 0 \) we get the two equations

\[
F(t, 0) = t^3 - \frac{4}{\Lambda^2}(-u_1 + \frac{\Lambda^2}{8})t^2 + \frac{4}{\Lambda^2}(-u_2 + \frac{\Lambda^2}{8})t - 1 = 0
\]

(9)

\[
\frac{\partial F(t, 0)}{\partial t} = 0
\]

(10)
These two conditions are equivalent to the discriminant of $F(t, 0) = 0$. The discriminant of $F(t, 0)$ is

$$
\Delta_{F(t,0)} = -64u_1u_2^2\Lambda^2 - 64u_1^2u_2\Lambda^2 + 304u_1\Lambda^4u_2 + 256u_1^2u_2^2 + 256u_1^3\Lambda^2 \\
- 92\Lambda^4u_2^2 - 25\Lambda^6u_1 - 25\Lambda^6u_2 - 92u_1^2\Lambda^4 - \frac{375}{16}\Lambda^8 + 256u_2^3\Lambda^2.
$$

If we take $u'_1 = u_1 - \frac{5\Lambda^2}{8}$ and $u'_2 = u_2 - \frac{5\Lambda^2}{8}$, the two curves indeed have the same singularities.

Notice that the singularity where $u_1 = u_2$ corresponds to $\phi_1$ or $\phi_2$ vanishing. For the general $SO(N)$ theory, this singularity is not physical, as can be seen from the fact that the monodromy associated with this singularity is trivial [9]. That the singularity here is meaningful should be expected on physical grounds as $\phi_{[1,2]} = 0$ corresponds to the restoration of the nonabelian $SO(3)$ in this case.

We can now write the final result for this curve in a more symmetric way as

$$
\Lambda_1^2t^3 - 4(v^2 - u_1 + \frac{\Lambda_2^2}{8})t^2 + 4(v^2 - u_2 + \frac{\Lambda_2^2}{8})t - \Lambda_2^2 = 0.
$$

The extension to non-vanishing bare mass $m_0$ for the hypermultiplet is trivial; one makes the substitution $u_2 \rightarrow u_2 + m_0^2$.

## 4 Generalizations

In this section we generalize these results to arbitrary products of $N = 2$ SU($n$) gauge theories. For each gauge group there is an adjoint scalar, in addition to which there are bifundamental hypermultiplets for all neighboring pairs of gauge group factors. For the first and last gauge groups in the chain we also include fundamental flavor hypermultiplets via semi-infinite fourbranes.

Consider SU($k_1$) $\times$ SU($k_2$) gauge theory with a bifundamental hypermultiplet, $k_0$ flavor hypermultiplets of SU($k_1$) and $k_3$ flavor hypermultiplets of SU($k_2$). The brane configuration is shown in Figure [4] with $M=2$. The simplest guess for a curve which would reduce to our SU(2) $\times$ SU(2) curve is

$$
P_{k_0}t^3 - \frac{1}{\Lambda_1^{2k_1-k_2-k_0}}P_{k_1}t^2 + \frac{1}{\Lambda_1^{2k_1-k_2-k_0}}P_{k_2}t - \frac{\Lambda_2^{2k_2-k_1-k_3}}{\Lambda_1^{2k_1-k_2-k_0}}P_{k_3} = 0,
$$

which is in the conventions used by [4] who proposed this curve based on classical limits. In the remainder of this paper we use the notation

$$
P_k(v) = \det(v - a_i) = \sum_{i=0}^{k} s_i v^{k-i},
$$

11
\[ s_k = (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} \prod_j a_{i_j} . \] (13)

The \( s_k \) above are only defined semiclassically, and \( s_0 = 1 \). Because of the quantum mechanical ambiguity in the definition of these operators, the curves can have different forms corresponding to \( O(\Lambda) \) shifts. In cases where there are symmetries, the moduli can be uniquely defined, for example using the \( u \rightarrow -u \) symmetry which motivated our choice of the \( SU(2) \times SU(2) \) curve.

In principle, the classical limits permit additional terms of the form discussed earlier but we now argue that these terms are not present by Higgsing a general \( SU(k_1) \times SU(k_2) \) theory to \( SU(2) \times SU(2) \).

We can Higgs \( SU(k_1) \) to \( SU(k_1 - 2) \) by giving the adjoint \( SU(k_1) \) a large VEV of the form

\[
\Phi = \begin{pmatrix} m & -m \\ -m & 0 \\ \ddots & \ddots \end{pmatrix}.
\]

This also gives masses \( m \) and \( -m \) to two flavors of the \( SU(k_2) \) gauge group. The matching of scales is

\[
m^4 \tilde{\Lambda}_1^{\tilde{\beta}_1} = \Lambda_1^{\beta_1}, \\
\tilde{\Lambda}_2^{\tilde{\beta}_2} = -m^2 \Lambda_2^{\beta_2},
\] (14)

where \( \Lambda_i \) is the scale of the \( SU(k_i) \) theory, \( \tilde{\Lambda}_i \) is the scale in the \( SU(k_i - 2) \) theory, and the \( \beta \)-function coefficients are

\[
\beta_1 = 2k_1 - k_2 - k_0, \\
\tilde{\beta}_1 = 2(k_1 - 2) - k_2 - k_0, \\
\beta_2 = 2k_2 - k_1 - k_3, \\
\tilde{\beta}_2 = 2k_2 - (k_1 - 2) - k_3.
\] (15)

The curve (11) can be written in terms of the parameters of the Higgsed theory,

\[
P_{k_0}t^3 - \frac{1}{m^4 \tilde{\Lambda}_1^{\tilde{\beta}_1}} (v^2 - m^2) P_{k_1 - 2}t^2 + \frac{1}{m^4 \tilde{\Lambda}_1^{\tilde{\beta}_1}} P_{k_2}t + \frac{\tilde{\Lambda}_2^{\tilde{\beta}_2}}{m^6 \tilde{\Lambda}_1^{\tilde{\beta}_1}} P_{k_3} = 0.
\] (16)

Rescaling \( \tilde{t} = -m^2 t \),

\[
P_{k_0} \tilde{t}^3 + \frac{(v^2 - m^2)}{m^2} P_{k_1 - 2} \tilde{t}^2 + P_{k_2} \tilde{t} - \frac{\tilde{\Lambda}_2^{\tilde{\beta}_2}}{\tilde{\Lambda}_1^{\beta_1}} P_{k_3} = 0,
\] (17)
which reduces to
\[ P_{k_0} \tilde{t}^3 - P_{k_1-2} \tilde{t}^2 + P_{k_2} \tilde{t} - \frac{\tilde{\Lambda}_2^{\beta_2}}{\tilde{\Lambda}_1^{\beta_1}} = 0. \] (18)

This is consistent with (11) for SU(\(k_1-2\))×SU(\(k_2\)) with \(k_0\) flavors of SU(\(k_1-2\)) and \(k_3\) flavors of SU(\(k_2\)).

We can similarly Higgs to SU(\(k_1-3\))×SU(\(k_2\)). We give the adjoint of SU(\(k_1\)) a large VEV of the form
\[
\Phi = \begin{pmatrix}
\frac{2}{3}m & \frac{1}{3}m & -m & \cdots & 0 \\
\frac{1}{3}m & -m & \cdots & 0 \\
-m & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.
\]

The matching of scales is given by
\[
\left(\frac{2m^3}{9}\right)^2 \frac{\tilde{\Lambda}_1^{\beta_1}}{\tilde{\Lambda}_1^{\beta_1}} = \Lambda_1^{\beta_1},
\]
\[
\frac{\tilde{\Lambda}_2^{\beta_2}}{\tilde{\Lambda}_1^{\beta_1}} = \frac{2m^3}{9} \Lambda_2^{\beta_2},
\]
where the \(\beta\)-functions of the Higgsed theory in this case are,
\[
\beta_1 = 2(k_1-3) - k_2 - k_0,
\]
\[
\beta_2 = 2k_2 - (k_1-3) - k_3,
\] (19)
and the curve in terms of \(\tilde{t} = \frac{2m^3}{9} t\) becomes
\[ \tilde{t}^3 - P_{k_1-3} \tilde{t}^2 + P_{k_2} \tilde{t} - \frac{\tilde{\Lambda}_2^{\beta_2}}{\tilde{\Lambda}_1^{\beta_1}} = 0. \] (20)

Higgsing in these ways we can flow from any SU(\(k_1\))×SU(\(k_2\)) theory to SU(2), except the theory SU(3)×SU(3), for which a different Higgsing is necessary. Any SU(\(k\))×SU(\(k\)) theory can be Higgsed to SU(2)×SU(2) by Higgsing in succession, as in [4], via an adjoint VEV of the form
\[
\Phi_L = \begin{pmatrix}
m & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 \\
0 & 0 & m & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & -(k-1)m & 0 \\
\end{pmatrix},
\]
the bare masses in the resulting curves by $m$ and $-m$ on the right and left, respectively. With this shift and the substitution of scales with the effective scales (21), the curve (11) flow smoothly among theories with arbitrary $k_1$ and $k_2$, whereas theories with additional terms that are potentially allowed would not have this property.

The extension to more general products of SU($k$) gauge groups can be determined by induction and dimensions. We only consider theories which are asymptotically free or conformal in each SU($k$) factor. We first focus on the case where each of the gauge groups is asymptotically free. The curve will then contain the appropriate polynomial $t^M$ factor. We first focus on the case where each of the gauge groups is

$$
\Phi_R = \begin{pmatrix}
-m & 0 & 0 & 0 & 0 \\
0 & -m & 0 & 0 & 0 \\
0 & 0 & -m & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & (k-1)m
\end{pmatrix},
$$

breaking SU($k$)$\times$SU($k$) to SU($k-1$)$\times$SU($k-1$). Matching of scales in this case is given by

$$
km\Lambda^{N-1}_{L,R} = \Lambda^N_{L,R}.
$$

In this case, Higgsing also gives masses to the bifundamentals, which we cancel by shifting the bare masses in the resulting curves by $m$ and $-m$ on the right and left, respectively. With this shift and the substitution of scales with the effective scales (22), the curve (21) with $k_1 = k_2 = k$ reduces to one of the same form, with $k \to (k-1)$.

So indeed the most obvious generalization of the SU(2)$\times$SU(2) curves are correct, as they flow smoothly among theories with arbitrary $k_1$ and $k_2$, whereas theories with additional terms that are potentially allowed would not have this property.

The extension to more general products of SU($k$) gauge groups can be determined by induction and dimensions. We only consider theories which are asymptotically free or conformal in each SU($k$) factor. We first focus on the case where each of the gauge groups is asymptotically free. The curve will then contain the appropriate polynomial $P_{k_j}(v)$, given above, multiplying $t^{M-j+1}$ for the $\prod_{i=1}^M$ SU($k_i$) theory. If the gauge groups have non-vanishing $\beta$-functions, the dimensionful $\Lambda_j$ must appear in the curve so as to make the curve dimensionally consistent. As we will show shortly, the curve is

$$
t^{M+1}P_{k_0}(v) - t^M P_{k_1}(v) + \sum_{j=0}^{M-1} (-1)^{M-j+1} \left( \prod_{n=1}^{M-j-1} \Lambda_n^{(M-n-j+1)\beta_n} \right) t^j P_{k_{M-j+1}}(v) = 0, \quad (22)
$$

where $k_n$ is the number of fourbranes in the $n$th gauge group factor ($n = 1, \cdots, M$), and as before $P_k(v) = \det(v-a_i)$ semiclassically. In $P_{k_0}$ and $P_{k_{M+1}}$ the adjoint VEV’s $a_i$ are replaced by the bare masses of the flavor hypermultiplets in SU($k_1$) and SU($k_M$), respectively. For the product of three SU($k$) gauge groups ($M = 3$) the curve is

$$
P_{k_0}(v)t^4 - P_{k_1}(v)t^3 + \Lambda_1^{\beta_1} P_{k_2}(v)t^2 - \Lambda_1^{\beta_1} \Lambda_2^{\beta_2} P_{k_3}(v)t + \Lambda_1^{\beta_1} \Lambda_2^{\beta_2} \Lambda_3^{\beta_3} P_{k_3}(v) = 0. \quad (23)
$$

It is straightforward to check that this curve has the correct limits upon flowing to other curves. Pulling the rightmost fivebrane to $x_6 = \infty$, i.e. taking $\Lambda_3 \to 0$, we are left with the curve (21) for the SU($k_1$)$\times$SU($k_2$) theory with $k_3$ flavors of SU($k_2$) hypermultiplets and $k_0$ flavors of SU($k_1$) hypermultiplets (corresponding to the semi-infinite fourbranes in this limit). This is seen as follows: The polynomials $P_k(v)$ are functions of the adjoint VEV’s $a_i$ in the semiclassical limit. By the Higgs mechanism, as the scale $\Lambda_3 \to 0$ the $k_3$
flavors of SU($k_2$) hypermultiplets become massive with masses $m_i = a_i$. Hence in this limit $P_{k_3}(v, s_i(a_i)) \rightarrow P_{k_3}(v, s_i(m_i))$, where $s_i$ are the moduli which appear as coefficients in the curve. Setting $\Lambda_3 = 0$, the curve \([23]\) factorizes as

$$ t \left( P_{k_0}(v) t^3 - P_{k_1}(v) t^2 + \Lambda_1^{\beta_1} P_{k_2}(v) t - \Lambda_1^{\beta_2} \Lambda_2^{\beta_3} P_{k_3}(v) \right) = 0. \tag{24} $$

The factor $t$ corresponds to the fivebrane at $x_6 = \infty$, i.e. $t = 0$. The other factor is the curve \([14]\) (up to rescaling of $t$) of the SU($k_1$)$\times$SU($k_2$) theory with $k_3$ flavors of SU($k_2$) and $k_0$ flavors of SU($k_1$), as claimed.

Also as expected, the curve is normalized such that only instanton powers appear. This follows from dimensional considerations. We choose the coefficients of $t^{M+1}$ and $P_{k_1}(v)t^M$ in the curve to be one. This fixes the dimension of $t$ to be $[t] = k_1 - k_0$, and each term in the curve then has dimension $(M+1)k_1 - Mk_0$. The $\beta$-function coefficient for each group is $\beta_n = 2k_n - k_{n-1} - k_{n+1}$. In anticipation of the classical limits we require terms in the curve to be proportional to $P_{k_{M-j+1}}(v)t^j$. The coefficient $c_{M-1}$ of the term $P_{k_2}(v)t^{M-1}$ has dimension $[c_{M-1}] = 2k_1 - k_0 - k_2 = \beta_1$. The dimensions of the coefficients $c_M$ and $c_{M+1}$ are chosen to be zero. The claim is that the dimension of the coefficient $c_j$ of the term $P_{k_{M-j+1}}(v)t^j$ is

$$ [c_j] = \sum_{n=1}^{M-j-1} (M - n - j + 1)\beta_n \tag{25} $$

leading to the choice of powers of the scales $\Lambda_i$ in \([22]\). That \([25]\) is valid can be seen most easily by recursion. The dimension of $c_j$ is $[c_j] = (M+1-j)k_1 - (M-j)k_0 - k_{M-j+1}$, so

$$ [c_j] - [c_{j+1}] = \sum_{n=1}^{M-j-1} \beta_n = k_1 - k_0 + k_{M-j} - k_{M-j+1}. \tag{26} $$

With $[c_M] = 0$, \([23]\) follows.

It is convenient to redefine $t$ in order to test other limits of the curve. For an even number of SU($k$) factors we can write the curve in a symmetric way by rescaling $t \rightarrow t' = t \prod_{i=1}^{N/2} \Lambda_i^{-\beta_i}$. For example, we can write the curve for the product of four SU($k$) factors as

$$ P_{k_0}^{\beta_1} \Lambda_1^{2\beta_2} t^5 - \Lambda_2^{\beta_2} P_{k_1} t^4 + P_{k_2} t^3 - P_{k_3} t^2 + \Lambda_3^{\beta_3} P_{k_4} t' - \Lambda_3^{\beta_3} \Lambda_4^{\beta_4} P_{k_5} = 0. \tag{27} $$

If we make the further rescaling $\tilde{t} = \Lambda_2^{\beta_2} t'$ and take $\Lambda_3 \rightarrow 0$ we are again left with the SU($k_1$)$\times$SU($k_2$) curve with $k_3$ flavors of SU($k_2$) and $k_0$ flavors of SU($k_1$), as expected. In this form we also see that as $\Lambda_2 \rightarrow 0$ the curve reduces to that of the SU($k_3$)$\times$SU($k_4$) theory with $k_2$ flavors of SU($k_3$) and $k_5$ flavors of SU($k_4$) hypermultiplets, as expected. Other classical limits follow from further rescalings of $t$ by powers of the scales, and the passage to arbitrary number of SU($k$) factors follows by induction.

It is also interesting to explore the case where some or all of the $\beta$ functions vanish. When this happens, one expects the branes to be parallel asymptotically, and that the asymptotic
separation between the branes $\Delta x^6$ will correspond to $1/g^2$, where $g$ is the SU$(n)$ coupling. We study the SU$(2)$ theory, however, and find that the separation of the branes at weak coupling agrees with expectations only up to a constant which does not vanish at weak coupling. However, if we interpret $\tau$ appearing in the curve as the effective U$(1)$ coupling of the massless version of the theory, as opposed to the SU$(2)$ coupling, as proposed by Dorey, Khoze, and Mattis [8], we find that this numerical constant is absent and a consistent leading order result is obtained.

Let us first construct the extension of our curve to the case with vanishing $\beta$-function at weak coupling for some of the SU$(k)$ factors, which follows by the replacement of the $P_k(v)$ and scales in (22) with certain modular forms. A proposed curve for the $N = 2$ SU$(n)$ theory with $2n$ flavors was written down in [11]. In the present language, the curve can be written

$$t^2 - 2 \left[ P_n(v, l(q)s_i(a_i)) + \frac{L(q)}{4} \sum_{i=0}^n v^{n-i}s_i(m_i) \right] t + L(q) \prod_{i=1}^{2n} (v + l(q)m_i) = 0,$$

where $s_i$ are, as before, the symmetric polynomials

$$s_k = (-1)^k \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k},$$

$q = e^{i\pi \tau}$, $l(q)$ is a modular form which approaches 1 as $q \to 0$, and $L(q)$ is a modular form of weight zero which approaches 64 for small $q$. According to [11], $\tau$ is the SU$(n)$ coupling $\tau = 8\pi i/g^2 + \theta/\pi$; however there is freedom to redefine $\tau$ so long as it agrees at weak coupling. For SU$(2)$, $L(q)$ and $l(q)$ were given in [11] to be

$$L(q) = \frac{4\theta_{10}^4}{\theta_{00}^4},$$
$$l(q) = \frac{\theta_{01}^8}{\theta_{00}^8},$$

where the theta functions are

$$\theta_{00}(q) = \sum_{n \in \mathbb{Z}} q^{n^2},$$
$$\theta_{01}(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2},$$
$$\theta_{10}(q) = \sum_{n \in \mathbb{Z}} q^{(n + \frac{1}{2})^2}.\quad (31)$$

\[One should note a typographical error in [11], which states that $L(q) \to q$ as opposed to $64q$ in this limit.]
It was shown in [11] that (28) flows to the right limits as flavors are integrated out. It should be noted that this flow determines the leading term of the function $L(q)$, independent of the full functional form.

In fact, we have checked that the discriminant for this curve agrees with the discriminant of the curve in [12] to subleading order after a redefinition $q = q_{sw}(1 - 42q_{sw})$, where $q_{sw}$ appears in the curve of [12].

If the curve (28) is correct, the generalization of our curve including SU($k$) factors with vanishing $\beta$-function, follows by the replacement

$$
\Lambda_i \rightarrow L(q_i)
$$

$$
P_k(v, s_i(a_i)) \rightarrow P_k(v, t(q)s_i(a_i) + \frac{L(q)}{4} \sum_{i=0}^{N_c} v^{N_c-i} s_i(m_i)) .
$$

One can check that in the weak coupling limit, the curve (22) with the above replacements reduces appropriately just as for the case of nonvanishing $\beta$-function, and similarly for the Higgs limit. Integrating out flavors works just as in [11].

We now consider the distance between the branes according to the above curve for an SU($n$) factor with vanishing beta function. For large $v$, corresponding to the region far from the positions of the fourbranes, the curve (28) for the SU($n$) theory with $2n$ flavors factorizes (after the replacement $t \rightarrow t/v^{N_c}$) as

$$
v^{N_c} \left( t^2 - 2(1 + L(q)/4)t + L(q) \right) = 0 .
$$

The solutions for $t$ are the asymptotic positions of the fivebranes,

$$
t_{\pm} = (1 + L(q)/4) \pm (1 - L(q)/4) ,
$$

with the ratio

$$
\frac{t_-}{t_+} = \frac{L(q)}{4} .
$$

For SU(2) in the weak coupling limit, $L(q) \rightarrow 64q$ and the distance between the branes,

$$
\Delta x^6 = \log(t_-/t_+) \rightarrow \log q + \log 16 ,
$$

is proportional to the SU(2) coupling constant $\tau = (1/2\pi i) \log q$ up to a shift by $\log 16/2\pi i$.

Now, the relation $\Lambda = 64mq$ defined the renormalization scheme, but there is still freedom in the interpretation of $q$. Although it was implicitly identified with the SU(2) coupling, the scheme in which this is true was not explicit. The discrepancy found above indicates that in the renormalization scheme used for the Seiberg-Witten curve, the parameter $q$ which appears differs by a constant factor from $q_{SU(2)}$, where $q_{SU(2)} = e^{i\pi SU(2)}$ and $\tau_{SU(2)}$ is the SU(2) gauge coupling in the SW scheme. In other words, $g_{sw}^2$ is a power series in $g_{SU(2)}$. The
SW coupling can also be interpreted as the SU(2) coupling, but in a different renormalization scheme.

Dorey, Khoze, and Mattis find a similar discrepancy, in their case between the SW curve and direct instanton calculations. They suggest that the parameter $\tau$ should be identified with the U(1) coupling of the massless theory, as opposed to the SU(2) coupling. In the SW renormalization scheme, the matching to the three flavor theory was given by $\Lambda = 64mq_{sw}$. They argue that $q_{sw} = q_{SU(2)}/16$. This is precisely the numerical discrepancy we find in the distance between branes, and seems to support the interpretation of [8]. However, the redefinition of coupling given in [8] does not appear to work at higher order; we find that the redefinition of $\tau$ involves a single instanton correction, which does not appear in [8], who argue that only even instanton corrections should be present.

It seems that these discrepancies can only be resolved with a clear identification of the physical predictions of the curves, an identification of the parameters appearing in the curves, and a better understanding of the implications of modular invariance. We do not have a resolution of the discrepancy found above, but find the leading order result suggestive.

5 Conclusions

We have determined the coefficients in the Seiberg-Witten curve for $N = 2$ supersymmetric SU($n$) product group theories with bifundamental and fundamental hypermultiplets from a brane construction. The curves are non-hyperelliptic, and the result is one that would have been difficult to guess solely from field theoretic considerations. These curves are the obvious generalization of some of the results of [4], and we demonstrate from comparison of field theoretic and brane limits that the most natural ansatz is the correct one. Presumably the moduli dependence of other curves can be derived similarly; furthermore our result could be useful for constructing new $N = 1$ curves.

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