Solar Nebula Magnetohydrodynamics

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Abstract.

The dynamical state of the solar nebula depends critically upon whether or not the gas is magnetically coupled. The presence of a subthermal field will cause laminar flow to break down into turbulence. Magnetic coupling, in turn, depends upon the ionization fraction of the gas. The innermost region of the nebula ($\lesssim 0.1$ AU) is magnetically well-coupled, as is the outermost region ($\gtrsim 10$ AU). The magnetic status of intermediate scales ($\sim 1$ AU) is less certain. It is plausible that there is a zone adjacent to the inner disk in which turbulent heating self-consistently maintains the requisite ionization levels. But the region adjacent to the active outer disk is likely to be magnetically “dead.” Hall currents play a significant role in nebular magnetohydrodynamics.

Though still occasionally argued in the literature, there is simply no evidence to support the once standard claim that differential rotation in a Keplerian disk is prone to break down into shear turbulence by nonlinear instabilities. There is abundant evidence—numerical, experimental, and analytic—in support of the stabilizing role of Coriolis forces. Hydrodynamical turbulence is almost certainly not a source of enhanced turbulence in the solar nebula, or in any other astrophysical accretion disk.

Keywords: accretion, accretion disks – instabilities – MHD – solar system: formation

1. Introduction

Despite decades of activity, a reliable model for the dynamical state of the primitive solar nebula remains elusive. It is of course essential to have such an understanding if we are to make sense of nebular chemistry, transport processes, planet formation and migration, and evolutionary history. The nebular temperature profile is of particular importance, since much of the meteoritic data (e.g. condensation properties of refractory elements) bear directly upon this.

A significant source of heat for the solar nebula will be turbulent dissipation, at least in those regions of the disk prone to instability. This turbulence also results in angular momentum and outward energy transport at rates far in excess of those associated with molecular vis-
cosity; one speaks therefore of an enhanced turbulent, or “anomalous” viscosity. These properties are the essence of an accretion disk. They allow the central star and disk to evolve, and in some circumstances (e.g. FU Orionis outbursts) the evolution may be eruptive. In this contribution, we review developments in our understanding of accretion disk turbulence as it applies to the solar nebula, and by extension, to protostellar and protoplanetary disks more generally. Recent general disk reviews may be found in Papaloizou & Lin (1995), Lin & Papaloizou (1996), and Balbus & Hawley (1998).

2. Accretion Disk Turbulence

2.1. A Preliminary Comment

For many years, it has been the custom for disk articles to identify the source of anomalous viscosity as the principal difficulty of accretion disk theory, use words like “mysterious” to describe its origin, and pay homage to nonlinear instabilities, convection, and possibly magnetic fields. The author would then move on to a simple viscous model, which would form the basis of the discussion.

Substantial progress has been made in the last decade. The origin of accretion disk turbulence is not mysterious, it is in fact very simple. The presence of any kind of weak magnetic field (in a sense described below) in a conducting, differentially rotating fluid renders it violently unstable. Remarkably, the instability was known at some level (though its consequences were certainly not appreciated) forty years ago (Velikhov 1959, Chandrasekhar 1960). Its importance to accretion disks was noted by Balbus & Hawley (1991, 1992a,b), and its subsequent elucidation and development have been carried through by many investigators (e.g., Goodman and Xu 1994; Hawley, Gammie, and Balbus 1995; Brandenburg et al. 1995; Matsumoto et al. 1996; Stone et al. 1996; Gammie 1996; Terquem and Papaloizou 1996; Ogilvie and Pringle 1996; Balbus and Hawley 1998 for review).

If earlier work has inspired confidence in magnetohydrodynamic (MHD) turbulence, then the most recent findings compel outright chauvinism. Not only is it established that weakly magnetized disks become unstable, developing enhanced turbulent transport, it also clear that hydrodynamical turbulence is fundamentally unsuitable to serve in this role. One may as well try to excite convection in a top-heated fluid as excite shear turbulence in a Keplerian disk.\footnote{This pithy observation was apparently noted by Sir Harold Jeffreys in conversation with L. Mestel (Papaloizou, private communication).} Neither ven-
ture will work, “nonlinear instabilities” notwithstanding. This rather
equivocal position will be justified later in this paper.

2.2. THE MAGNETOROTATIONAL INSTABILITY

We begin with a brief presentation of the linear magnetic (“magne-
torotational”) instability, almost certainly the basis for any model of
turbulent transport in the nebula. Set up a standard cylindrical coor-
dinate system $(R, \phi, z)$ centered upon a central gravitating body. The
equations of motion for an element of fluid orbiting in a plane in the
field of central potential $\Phi$ are

\[
\frac{d^2 R}{dt^2} - R \left( \frac{d\phi}{dt} \right)^2 = -\frac{\partial \Phi}{\partial R} + f_R, \tag{1}
\]

\[
R \frac{d^2 \phi}{dt^2} + 2 \frac{dR}{dt} \frac{d\phi}{dt} = f_\phi, \tag{2}
\]

where $f_R$ and $f_\phi$ represent local forces in the $R$ and $\phi$ directions. We
allow the particle to make small excursions from a circular orbit $R = R_0, \phi = \Omega_0 t$ by introducing small quasicartesian $x$ and $y$ variables

\[
R = R_0 + x \quad \phi = \Omega_0 t + y/R_0. \tag{3}
\]

Substituting equations (3) into equations (1) and (2), canceling lead-
ing order terms, and retaining those linear in $x$ and $y$ leads to the Hill
equations. These were first used to study the Sun-Earth-Moon triplet
system:

\[
\ddot{x} - 2 \Omega \dot{y} = -x \frac{d\Omega^2}{d\ln R} + f_x, \tag{4}
\]

\[
\ddot{y} + 2 \Omega \dot{x} = f_y. \tag{5}
\]

We follow the convention of writing dots for time derivatives and have
resolved the local force into its $x$ and $y$ components. The angular ve-
clocity gradient $d\Omega^2/d\ln R$ refers to the circular velocity profile $\Omega(R)$,
and we have suppressed the “0” subscript for clarity.

In what follows, fluid pressure forces are unimportant, and we need
consider only magnetic stresses in the $f$-forces. Consider an equilibrium
vertical magnetic field. If we restrict our attention to fluid displace-
ments in the plane of the disk, varying only with $z$ and $t$ as $e^{i(\omega t - k z)}$,
these stresses take a very simple form: the magnetic tension force is

\[-(k \cdot u_A)^2 \xi \] (e.g., Balbus and Hawley 1992a), where $u_A$ is the Alfvén
velocity

\[
u_A = \frac{B}{\sqrt{4\pi \rho}}, \tag{6}
\]
and $\xi$ is the two-dimensional displacement vector. The equations of motion take the form

$$\ddot{x} - 2\Omega \dot{y} = -x \frac{d\Omega^2}{d\ln R} - (k \cdot u_A)^2 x,$$

$$\ddot{y} + 2\Omega \dot{x} = -(k \cdot u_A)^2 y. \tag{7}$$

These are equations for classical Alfvén waves (more precisely an Alfvén and a slow wave, which is degenerate with the former), modified by Coriolis forces and a radial tidal field. The magnetic tension terms proportional to $(k \cdot u_A)^2$ act like spring forces, with a tunable spring constant depending upon the magnitude of $k$.

Equations (7) and (8) may be satisfied provided that $\omega$ satisfies

$$\omega^4 - \omega^2[\kappa^2 + (k \cdot u_A)^2] + (k \cdot u_A)^2 \left[ (k \cdot u_A)^2 + \frac{d\Omega^2}{d\ln R} \right] = 0, \tag{9}$$

a simple quadratic in $\omega^2$. Here, $\kappa^2$ is the epicyclic frequency,

$$\kappa^2 = \frac{1}{R^3} \frac{dR^4\Omega^2}{dR}. \tag{10}$$

If

$$(k \cdot u_A)^2 + \frac{d\Omega^2}{d\ln R} < 0, \tag{11}$$

then the system will be unstable. Physically, the left side equation (11) is (with a minus sign) the net centrifugal force on a displaced element, the magnetic field here appearing as a restoring, stabilizing agent. The inequality is a statement that this force should have a radial outward component for destabilization.

Assuming that large enough wavelengths are allowable (a weak spring), when the angular velocity (not the angular momentum) decreases outwards, the disk will be unstable. But of course that is the behavior of a Keplerian (or any other astrophysical disk) rotation law. Only slightly more work is needed to reveal that there is a maximum growth rate

$$|\omega_{\max}| = \frac{1}{2} \frac{d\Omega}{d\ln R}, \tag{12}$$

which, for a Keplerian rotation law $\Omega \sim r^{-3/2}$, is obtained when

$$(k \cdot u_A)^2 = \frac{15}{16} \Omega^2. \tag{13}$$

The growth rate (12) is quite large, amounting to an amplification factor of about 100 per orbit.
In what sense must the field be “weak” for the instability to run? Equation (11) provides the key. The field must be sufficiently weak that the Alfvén tension frequency is less than \( \sqrt{3}\Omega \) in a Keplerian disk. Wavenumbers in excess of this limitation are stabilized by the tension forces. If the field is too strong, all wavelengths able to fit inside the thin disk will become stabilized. Generally, this restricts the unperturbed Alfvén speed to be less than order the disk sound speed (Balbus and Hawley 1991).

To understand the instability more fully, go back to our connecting spring analogy. When the spring is stretched so that one mass is orbiting slightly farther out than the other, the inner mass rotates more rapidly on the lower orbit. The tug of the spring pulls back on this mass, causing it to lose angular momentum. (This is the torque on the right hand side of equation [8]). This loss compels the mass to descend to a yet lower orbit, while the gain of angular momentum in the outer mass sends it to a higher orbit. The spring stretches more, and the process runs away. Note that this is an instability only if the spring is relatively weak; if the spring is too strong the mass points will simply oscillate as they orbit. The instability is strongest if the spring constant is comparable to \( \Omega^2 \), as indicated by equation (13). Note as well that angular momentum transport is not some nonlinear outcome of the instability, it is the essence of the instability.

Second, the instability is even more robust than our simplified presentation indicates (Balbus and Hawley 1998). The maximum growth rate (12) is independent, not only of the strength of the magnetic field, but also of the geometry of the magnetic field. Even a purely toroidal field is unstable (Balbus and Hawley 1992b; Terquem & Papaloizou 1996), with the same maximum growth rate (Balbus & Hawley 1998). The presence of buoyancy or other pressure effects does not affect the stability criterion, since the most unstable displacements lie in the plane of the disk. Furthermore, it has been conjectured that equation (12) represents the maximum possible growth rate that any instability tapping into the free energy of differential rotation can achieve (Balbus & Hawley 1992a).

Finally, three-dimensional numerical simulations of the nonlinear phase of the instability (Hawley et al. 1995), combined with analytic arguments (Balbus and Papaloizou 1999) eliminate any doubt that the instability leads to turbulence, that the turbulence is fully developed, and that significant angular momentum transport results. Values of the “\( \alpha \) parameter” (cf. next section) obtained in local simulations extend over the range \( 5 \times 10^{-3} \lesssim \alpha \lesssim 5 \times 10^{-1} \) (Brandenburg et al. 1995; Hawley et al. 1995), with lower values corresponding to initially uniform toroidal fields or fields with vanishing mean values, and the higher
values corresponding to the presence of a mean axial field. If the weak field MHD turbulence is present, it will act like a greatly enhanced disk viscosity, both diffusively and dissipatively. It will dictate the local thermal structure of the solar nebula. The question of importance becomes, where in the disk will the ionization levels be sufficiently high to permit good magnetic coupling?

2.3. **Turbulent Heating in the Nebula**

Viscous, turbulence-enhanced accretion has been a popular and stimulating model for the post-infall global evolution of the solar nebula (Lin and Papaloizou 1980). Classical viscous accretion disk models allow one to determine macroscopic disk properties in terms of two constants: the accretion rate $\dot{M}$, and a dimensionless disk viscosity parameter, $\alpha$. The latter is related to the (turbulent) viscosity $\nu_T$ by

$$\nu_T = \alpha c_S H \quad (14)$$

where $c_S$ is the isothermal sound speed, and $H$ is the disk scale height (Pringle 1981). (The quantity $c_S$ is computed with a mean mass per particle appropriate to a molecular hydrogen gas of cosmic abundances, $\sim 2.33 m_p$, where $m_p$ is a proton mass.) If, as we believe, MHD turbulence is the physics behind $\alpha$, then it should be possible to go beyond simple parameterizations.

We know, for example, that the disk must be hot enough to maintain an ion population capable of coupling the field to the molecular nebular gas. Where is this requirement self-consistently met? Using an $\alpha$ model as our starting point, let us parameterize our results in terms of the column density $\Sigma$ and midplane optical depth $\tau$. The midplane nebula temperature $T$ may be found from energy conservation:

$$\frac{3}{4} \tau T^4 = \frac{3GM\dot{M}}{8\pi R^3\sigma}, \quad (15)$$

and angular momentum conservation,

$$\dot{M} = 3\pi \Sigma \alpha c_S H. \quad (16)$$

(Shakura and Sunyaev 1973; Frank, King, and Raine 1992). Here $G$ is the gravitational constant, $M$ is the central mass, and $\sigma$ is the Stefan-Boltzmann constant. Equation (15) expresses the condition that surface radiation losses are balanced by internal mechanical energy dissipation;

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2 Note that this accretion rate refers to radial drift through the disk, not an external infall rate.
equation (16) employs a vanishing stress boundary condition at the inner edge of the disk.

These two equations lead to a temperature of

\[ T = 1140 \, \text{K} \left( \frac{\alpha}{10^{-2}} \right)^{1/3} \left( \frac{\Sigma T}{10^6} \right)^{1/3} R_{AU}^{-1/2}, \]  

(17)

where \( R_{AU} \) is the disk radial location in astronomical units. From the point of view of magnetic coupling, this is an interesting temperature regime, neither prohibitively cool, nor dominated by ions. Clearly, different regions of the nebula will have very different magnetic properties, very different internal temperatures, and very different dynamical states. The solar nebula is likely to have been highly inhomogeneous. Among other consequences, the ease with which a planet migrates through the nebula depends very much upon whether the gas is turbulent or not (Terquem, Papaloizou, and Nelson, this volume).

With the database of new planets and even new planetary systems rapidly swelling, understanding the extent over which protoplanetary disks maintain turbulence is a key problem.

3. Resistivity and Hall Currents

Outside of the first few 0.1 AU, the ionization fraction in the solar nebula is likely to be very small. Of course very little ionization is needed to ensure good coupling in astrophysical plasmas, but even this may not always be attainable. There are two important effects in a low ionization plasma which bear discussion.

3.1. Resistivity

The resistivity of partially ionized plasma is (e.g., Blaes & Balbus 1994)

\[ \eta = 230 \left( \frac{n_n}{n_e} \right) T^{1/2} \, \text{cm} \, \text{s}^{-2}, \]  

(18)

where \( n_n \) is the neutral number density, \( n_e \) is the electron number density, and \( T \) is the temperature. A measure of the relative importance of the resistivity is given by the magnetic Reynolds number

\[ Re_M = \frac{H c_s}{\eta} \sim \frac{c_s^2}{\Omega \eta}, \]  

(19)

where the final expression is our operational definition. When \( Re_M \) is smaller than \( c_s^2 / u_A^2 \), resistivity affects the most rapidly growing linear
wavelengths, and when $Re_M$ falls below $4\pi^2$, the linear instability is completely quenched (Stone et al. 1999). (Note that $c_s^2/\alpha_A^2$ is generally, though not inevitably, larger than $4\pi^2$.) On scales of 1 AU in the solar nebula, this corresponds to ionization fraction of about $10^{-13}$. [This is a small number, but it is large enough to be in a regime where we expect dust grains to be playing a relatively minor role (Umebayashi & Nakano 1988)]. The dominant ions will be Na$^+$ and K$^+$, abundant alkalis with low ionization potentials.

The ionization in the inner nebula will be thermal [tenuous outer layers may be regulated by cosmic or X-rays, (Gammie 1996, Glassgold et al. 1997)], and an ionization fraction of $10^{-13}$ is associated with temperatures of 900–1100 K over a broad range of densities. Taking this result together with equation (17), we find that formally, an accretion disk with $\alpha = 10^{-2}$ can maintain this minimal ionization out to radii of order 1 AU.

We need to be careful. Conditions for linear instability to be present and conditions for maintaining turbulence need not be the same. (One example of this is the linear two-dimensional magnetorotational instability, which cannot sustain turbulence because of the anti-dynamo theorem. Quite generally, shear turbulence requires three dimensions.) A local study by Fleming, Stone, & Hawley (1999), has shown a marked decrease in the level of MHD turbulence for $Re_M < 5 \times 10^4$, with turbulence decaying once $Re_M < 2 \times 10^4$. These values are far too high to affect the linear behavior of the instability, and indeed the simulations are indistinguishable early in the linear regime. It is possible that the upper limit represents the discreteness of the grid, since the behavior of $Re_M = 5 \times 10^4$ and “$Re_M = \infty$” (i.e., no explicit resistivity) are likewise indistinguishable. Thus, finite magnetic Reynolds number effects may be present at higher values as well.

$Re_M$ is extremely sensitive to the temperature. If the ionization fraction is much less than the abundance of K ($\sim 10^{-7}$), the Saha equation gives

$$\frac{n_e}{n_n} = 6.47 \times 10^{-13} a_{-7}^{1/2} T_3^{3/4} \left(\frac{2.4 \times 10^{15}}{n_n}\right)^{1/2} \frac{\exp(-25.188/T)}{1.15 \times 10^{-11}}$$

(20)

where $a_{-7}$ is the K abundance in unit of $10^{-7}$, $T_3$ is the temperature in units of $10^3$ K; the Boltzmann factor is normalized to its value at 1000 K, and the neutral density normalization is given by fundamental constants of nature which happen to lead to a value within an order of magnitude (or two) of that expected for the solar nebula. This leads to
a magnetic Reynolds number of

\[ Re_M = 15.8 T_3^{5/4} a_{-7}^{1/2} R_{AU}^{3/2} \left( \frac{2.4 \times 10^{15}}{n_n} \right)^{1/2} \frac{\exp(-25,188/T)}{1.15 \times 10^{-11}} \]  

At face value this is well below what the Fleming et al. simulations suggest is needed for unencumbered turbulence, but the exponential Boltzmann acts like a switch. One need not venture more than a few tenths of an AU inward for conditions to change dramatically, say, at \( T = 1400 \) K. We are then in the regime of full MHD turbulence.

This behavior is the most salient difference between a simple \( \alpha \) model for the nebula, and one based upon MHD turbulence: a very sharp change in the dynamical state of the inner 1 AU. Furthermore, it is also self-consistent to have a quiescent disk as we move inward of 1 AU, with much lower temperatures, and much lower ionization fractions. This would mean a smaller value for the turbulent stress, which is the basis of the self-consistency. More turbulence means more ions to help maintain the magnetic coupling, less turbulence means the converse. Clearly, the transition between quiescence and turbulence may be explosive: a slight increase in temperature turns into an increase in ionization, an increase in the alpha parameter, and yet higher temperatures. Gammie and Menou (1998) have argued that dwarf novae outbursts are related to just such a transition to MHD turbulence; it is very possible that FU Orionis outbursts are a similar phenomenon.

3.2. The Hall Effect

The second effect may be understood as follows. In the absence of resistivity, the standard induction equation for the magnetic field \( B \) is

\[ \frac{\partial B}{\partial t} = \nabla \times (v \times B), \]  

where \( v \) is the fluid velocity. The question is, which fluid? Under most circumstances, the question is moot because the difference between ion, electron, and neutral velocities is too small to matter. But when the ionization fraction becomes sufficiently small (and \( 10^{-13} \) qualifies), we must be careful with the distinction. A more accurate induction equation is

\[ \frac{\partial B}{\partial t} = \nabla \times (v_e \times B), \]  

that is, the electron velocity replaces the “fluid” velocity. This is because the magnetic field lines are tied most effectively to the most mobile charge carriers. In the case of interest, the inertia of the charge
carriers is negligible, and the fluid velocity is to a very good approximation that of the neutrals. Since

\[ v_e = v + (v_e - v_i) + (v_i - v), \]  

(24)

the electron velocity may be represented as the sum of the neutral velocity, a second term proportional to the current \( v_i \) is the ion velocity), plus a final term determined by the strength of ambipolar diffusion coupling. In interstellar molecular gas, this final term generally turns out to be more important than the second. But in protostellar disks, the second term, which gives rise to the Hall effect in the laboratory, is dominant. This point is made, using the formalism of conductivity tensors, in an important recent paper by Wardle (1999). We will adopt a more dynamical approach here.

Equation (24) may be written in terms of the current density \( j \),

\[ v_e = v - \frac{j}{n_i e}, \]  

(25)

where \( n_i \) is the ion number density (taken equal to the electron’s), and the final ion-neutral drift term is ignored. It is clear now why the Hall term on the right is important, with \( n_i \) small and the currents large enough to generate fields of interest. The induction equation (23) takes the form

\[ \frac{\partial B}{\partial t} = \nabla \times (v \times B - \frac{j \times B}{n_i e}), \]  

(26)

which introduces Coriolis-like terms in the behavior of the field lines. This makes the behavior of the fluid and the field symmetric: “field-freezing” tries to couple the magnetic field and fluid velocities, while Coriolis forces and Hall effect terms cause internal epicyclic motions in the matter and field respectively. Relative to the disk rotation, the fluid epicyclic motion is retrograde, while the sense of the field line motion depends on the magnetic helicity \( \Omega \cdot B \).

A more clear picture of the effect of the Hall term emerges, if we consider its behavior in a uniformly rotating disk, ignoring for the moment any effects of finite electrical resistance. If we once again consider perturbations to a vertical field \( B = B e_z \), and denote linear perturbations by \( \delta \), plane wave disturbances of the form \( \exp(ikz - i\omega t) \) satisfy the dynamical equations,

\[ -i\omega \delta v_R - 2\Omega \delta v_\phi - \frac{ikB}{4\pi \rho} \delta B_R = 0, \]  

(27)

\[ -i\omega \delta v_\phi + 2\Omega \delta v_R - \frac{ikB}{4\pi \rho} \delta B_\phi = 0, \]  

(28)
and the induction equations,

\[ -i \omega \delta B_R + \frac{k^2 Bc}{4\pi en_i} \delta B_\phi - ik B \delta v_R = 0 \quad (29) \]

\[ -i \omega \delta B_\phi - \frac{k^2 Bc}{4\pi en_i} \delta B_R - ikb \delta v_\phi = 0. \quad (30) \]

Note the symmetrical appearance of \( \delta B \) and \( \delta v \) in the above, and that the Hall terms enter for the magnetic field precisely as the Coriolis terms do for the velocity field, except that they may induce a different sense of rotation.

The dispersion relation for the above system is

\[ \omega^4 - \omega^2 \left[ 2k^2 u_A^2 + 4\Omega^2 + \frac{k^4 u_H^4}{4\Omega^2} \right] + k^4 (u_A^2 + u_H^2)^2 = 0 \quad (31) \]

where we have defined the Hall velocity \( u_H \) by

\[ u_H^2 = \frac{\Omega Bc}{2\pi en_i}; \quad (32) \]

where \( c \) is the speed of light. Note that \( u_H^2 \) may be positive or negative depending upon the sign of helicity factor \( \Omega \cdot B \). The Hall term alters both the effective epicyclic frequency of the coupled field-fluid system as well as the radial centrifugal force on a displaced fluid element, discussed in §2.2. The contribution of the magnetic field to the return force on a displaced element goes from \( k^2 u_A^2 \) to \( k^2 u_A^2 + k^2 u_H^2 \), and a simple measure of the importance of the Hall currents is the ratio \( x \equiv u_H^2 / u_A^2 \).

The dispersion relation in the presence of differential rotation and resistivity is more complicated, and seems best left to the literature for discussion (e.g., Wardle 1999). But the stability requirements are easily stated. In the absence of resistivity one simply replaces \( u_A^2 \) with \( u_A^2 + u_H^2 \),

\[ k^2 u_A^2 (1 + x) + \frac{d\Omega^2}{d\ln R} > 0 \quad \text{STABILITY} \quad (33) \]

In the presence of a resistivity \( \eta \) (units: cm sec\(^{-2}\)), matters are a bit more complicated:

\[ k^2 u_A^2 [1 + x + \zeta/(4 + x)] + \frac{d\Omega^2}{d\ln R} > 0 \quad \text{STABILITY} \quad (34) \]

where

\[ \zeta \equiv \frac{4k^2 \eta^2}{u_A^4}. \quad (35) \]
The effect of resistivity is to force the instability to longer wavelengths, lessening the perturbation currents. But once the wavelength exceeds the available linear dimensions, the system is stabilized.

Overall, however, the effect of Hall currents on the stability of accretion disks are not simple. They both stabilize and destabilize, depending upon whether $\Omega$ is aligned or counter aligned with $B$. One relatively straightforward effect which may be seen either in equation (33) or (34) is that when $B \to 0$, $x \gg 1$ and the Hall term dominates. The same magnetic tension force is provided by much longer wavelengths than would have been possible without the Hall currents. This extends the range of unstable field strengths before dissipation processes kill the instability.

We have seen that Hall effect currents affect the stability of a low ionization protostellar disks. How might they affect the nonlinear development of the turbulence? We are some ways from answering this, but one potential complication can be addressed. The Poynting contribution to the turbulent energy flux is

$$B \times (v_e \times B).$$

A key piece of the alpha disk formulation is that the energy flux be a linear function of the same turbulent flow quantities responsible for angular momentum transport: the radial drift velocity and the $R\phi$ component of the stress tensor (Balbus & Papaloizou 1999). The Poynting contribution is one of the dominant components of the energy flux, and the fact that $v_e$ appears above instead of $v$ looks like a complication—we have emphasized the importance of the difference $v - v_e$. But here it is the azimuthal component of the unperturbed rotational velocity $v_e$ which is responsible for the radial contribution to the energy flux. (What is important for a perturbation may not be important in the unperturbed state.) For the Hall term to be negligible in this context, we find

$$u_H^2 \ll (\ell \Omega)(R\Omega)$$

where $\ell$ is defined by $|\nabla \times B| \sim B/\ell$, a sort of Taylor microscale (e.g., Tennekes & Lumley 1972) for MHD turbulence. If $\ell$ is not much smaller than $H^2/R$, this inequality will be amply satisfied, and the alpha formulation of MHD turbulence should remain intact.
4. The Role of Self-Gravity and Hydrodynamical Processes

4.1. The Intermediate Disk

Beyond 1 AU, the body of the disk is likely to be magnetically active. $Re_M$ drops rapidly until one reaches beyond 10 AU or so, where cosmic rays or X-rays can penetrate and leave a sufficient residual ionization to ensure good coupling once again. Although the focus of this meeting has been the zone of terrestrial planets, some mention of how the inner and outer disk are related is needed.

It is now clear that many of the usual hydrodynamical mechanisms invoked in the past—turbulence generated by infall, convection, etc.—will not transport angular momentum outward. One that certainly does work, at least in principle, is self-gravity (e.g., Nelson et al. 1998). If the mass of the disk within a radius $R$ is in excess of $(H/R)M_\odot$, it will become Jeans unstable, developing local (possibly global) spiral structure, which in turn is an effective outward transport mechanism. At 5 AU, the required interior disk mass is $\sim 0.05 M_\odot$. This is much in excess of Jupiter’s $10^{-3} M_\odot$, but it is not unphysical. It is what one might guess if planet formation is efficient at the level of a few per cent, and consistent with the figure of $10^{-2} M_\odot$ inferred for the accreted mass of the inner disk in FU Orionis outbursts (Hartmann, Kenyon, and Hartigan 1993).

While this behavior may be common, gravitational replenishment is not likely to be universal, and certainly cannot go on indefinitely! What happens when the disk mass is too small? In some cases, we would expect to find passive disks with a hole on scales less than AU. At later stages in a disk’s evolution, this will be unavoidable, as mass is lost one way or another. Strom, Edwards, and Skrutskie (1993) have found evidence for disks with inner holes on precisely these scales. They argue that these probably represent later stage transition disks when accretion begins to cease, that the inner parts of the disk need to be optically thin, and that the inner disk region is “isolated” from the outer. Masses of the optically thick inner disks were estimated at between 0.01 and 0.1 $M_\odot$, consistent with self-gravity requirements. At the time of the Strom et al. review, planet formation was thought to be the most likely explanation. But among other possibilities (e.g., Königl 1991), these disks may be candidates for magnetic depletion. An interesting prediction is that the size of the inner hole should correlate with disk metallicity in this picture: higher abundances should produce bigger holes, since the ionization levels needed for higher alphas could be maintained farther out. Detailed modeling is needed to test these striking general agreements of scale and predicted morphology.
4.2. Why Hydrodynamical Turbulence Fails

The central feature of the picture of the Solar Nebula we have discussed here is that turbulent transport is regulated by magnetohydrodynamics, and that hydrodynamical transport is via self-gravity. If differential rotation by itself were for some reason unstable, we would be forced to a completely different picture, one in which the turbulence properties were much more uniform. This point of view retains prominent advocates (e.g., Dubrulle 1993, Richard & Zahn 1999), and should be addressed.

The classical laboratory set-up for studying hydrodynamical turbulence in differential rotation is to examine flow between two rotating cylinders, the Taylor-Couette experiment. According to the classic text of Landau & Lifshitz (1959, p. 110), flow predicted to be stable by the Rayleigh criterion is shown by experiments to break down into fully developed turbulence at sufficiently high Reynolds numbers. The textbook by Zel’dovich, Rumaikin, and Sokoloff (1983, p. 321) is even more explicit on this point, arguing that the nonlinear breakdown of Keplerian flow is supported by experiments.

These are extraordinary claims. We have found no such experiments in the literature; almost certainly none exist. The closest approximation to a nonlinear breakdown of Keplerian flow is seen in experiments dominated by the rotation of the outer cylinder, or with a very small but positive angular momentum gradient. The first is characterized by $d\Omega/\ln dR \gg 2\Omega$ (shear dominating Coriolis forces), the second by a nearly vanishing value of $\kappa^2$ (small vorticity). Though decidedly nonlinear in their stability properties, each of these flows differs significantly from Keplerian, which has comparable values of $\kappa = \Omega$, $d\Omega/d\ln R = -1.5\Omega$, and $2\Omega$. No Couette flow experiment has show any sign of instability when the inner and outer cylinders follow anything close to a Keplerian profile.

To be fair, the Russian texts are somewhat misrepresentative; most proponents of hydrodynamical turbulence would concede that the experiments performed to date do not in fact show that Keplerian profiles are unstable. But, it is asserted, this is just a matter of going to higher Reynolds number, $Re$. Keplerian flow will be unstable at high $Re$, because that is the way shear flows behave.

This intuition seems born of the classical bounded planar Couette and Poiseuille flow, and it presumes that the only dimensionless number characterizing the flow is $Re$. It also presumes an exquisite sensitivity to $Re$, since proponents are forced to argue that high resolution numerical simulations, which can easily recover the onset of laboratory Couette instabilities, once again simply have not achieved high enough $Re$. 
To begin with, there are, of course, two nondimensional numbers characterizing Couette flow, both of which influence stability. One is the $Re$, the other is the Rossby number $Ro$,

$$Ro = \left| \frac{1}{2} \frac{d \ln \Omega}{d \ln R} \right|$$

which measures the importance of inertial forces relative to Coriolis forces. Obviously, $Ro$ is a critical parameter in determining flow stability. As noted explicitly in the review of Bayly, Orszag, & Herbert (1988), the effect of viscosity on unbounded flows (this includes shear layers and disks) is that of a regular perturbation, and the instabilities affecting such systems are essentially inviscid. The viscosity, when introduced, perturbs the flow at a level proportional to $Re^{-1} \ll 1$, not, as in boundary layer theory, at the order unity level. Numerical Reynolds number effects are very small. The Rossby number is the only inviscid flow quantity of interest; to within a factor of 2 it is just the power law index of $\Omega$.

The nonlinear breakdown of shear flow has been studied in some detail (Bayly et al. 1988), both quasi-analytically and fully numerically. In all cases, it involves a two-stage process: a neutrally stable nonlinear two-dimensional equilibrium is first perturbed on top of the fundamental flow. This is a new “equilibrium.” In the next stage, this new solution is itself linearly perturbed in three dimensions. The new equilibrium is exponentially unstable. The existence of neutral behavior in the intermediate two-dimension solution is critical. It is a feature of systems whose linear restoring forces effectively vanish, as is the case for simple shear layers and constant angular momentum disks. It accounts for why nonlinear disturbances quickly become dominant. Systems similar to Keplerian disks, with comparable epicyclic and shear rates, do not allow the formation of neutrally stable nonlinear equilibria on top of the differential rotation.

Finally, we note that turbulence maintains itself in shear layers by vortex stretching (Tennekes and Lumley 1972). Stretching is possible in neutrally stable flows for which initially neighboring elements drift apart when perturbed. This includes disks with nearly constant angular momentum profiles; these are indeed nonlinearly unstable. But in a Keplerian (or similar) disk, epicyclic motions keep elements localized, and vortex stretching is extremely inefficient (Hawley, Balbus, and Winters 1999).

Formal analytic arguments of the moment equations for fluctuations can be advanced to support these arguments (Balbus & Hawley 1998). The basic point is that in each of the unstable laboratory flows a dominant source term is present in their equation, which acts obviously
differently for flows in the stable regime. Numerical PPM simulations on high resolution grids \((128^3)\) have been performed recently (Hawley et al. 1999) on local Keplerian flow. The effective Reynolds number at this resolution is very high, probably in excess of \(10^5\). The analytic arguments were confirmed in detail, and not a trace of instability was observed. By way of contrast, our group has found that nonlinear instabilities in constant angular momentum flows can be recovered on grids as small as \(8^3\)! When turbulence is directly imposed on a Keplerian disk in the form of driven convection, the angular momentum gradient compels \textit{inward} angular momentum transport (Stone & Balbus 1996); when imposed by random dynamical forces, no transport at all occurs (Balbus & Hawley 1998). It is difficult to understand how supposedly diffusive numerics could be at once large enough to damp Keplerian instability on a very fine grid, small enough to allow nonlinear instabilities on an extremely crude grid, and small enough to allow imposed thermal fluctuations to drive \textit{inward} angular momentum transport! Each of these behaviors is, however, a completely straightforward consequence of inviscid flow dynamics. It is simply untenable to cling to the notion that this is all somehow a gross misinterpretation of numerical artifacts.

An angular momentum gradient in the flow is no less stabilizing than an entropy gradient in a stratified fluid, to which it is exactly analogous (Jeffreys 1928). One would not argue that the radiative interior of the sun is convectively unstable because the temperature gradient is a source of free energy that nonlinear disturbances can tap into. One would not argue that the Richardson criterion for stability, which is due to the presence of gravitational stratification across a shear layer, is something that disappears at large Reynolds number for nonlinear perturbations. Angular momentum stratification stabilizes shear flows as effectively as gravity stabilizes a bottom-heavy fluid. A century of investigation has produced no evidence to the contrary.

5. An Overview of Solar Nebula Dynamics

We are at an interesting and exciting stage in our developing understanding of solar nebula dynamics. Nothing like a complete model is at hand of course, but even understanding how an isolated magnetized gas disk around a nascent sun behaves would represent important progress. And we have made progress.

We have at hand an understanding of the physical basis for generating and maintaining turbulence, and of enhancing transport in magnetized disks. The magnetorotational instability has broad analytic and numerical support.
We have not just an overall understanding of the way turbulence could work, we also have a handle on many of the details: ionization balance, Hall currents, nonlinear dynamics, for example. We know that metallicity is important, since the alkalis regulate magnetic coupling. There is, therefore, a direct connection between metallicity and gas dynamics. It is more usual, of course, for the metals to regulate thermodynamical properties.

Models in which accretion, mediated by an external ionizing agent, proceeds through the the exposed layers of the disk have been proposed by Gammie (1996) and Glassgold et al. (1997). However, a detailed study of how the magnetorotational instability actually behaves for this geometry and resistivity has not yet been done.

To summarize: the MHD picture of the (isolated) solar nebula is one in which turbulence is present within $\sim 1$ AU and beyond $\sim 10$ AU. Gravitational instability may be present in the early stages on intermediate scales, maintaining a mass influx and replenishing the inner disk. When the disk mass falls to the point where this is no longer possible, the inner regions of the disk will be depleted, an effect that should manifest itself as a deficit at near IR wavelengths. This spectral feature has been observed, but an inner planet, or disruptive stellar magnetosphere (Königl 1991) are also possible interpretations.

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