Properties of Periodic Fibonacci-like Sequences

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Abstract
Generalized Fibonacci-like sequences appear in finite difference approximations of the Partial Differential Equations based upon replacing partial differential equations by finite difference equations. This paper studies properties of the generalized Fibonacci-like sequence $F_{n+2} = A + BF_{n+1} + CF_n$. It is shown that this sequence is periodic with the period $T \in (2, \infty)$ if $C = -1$, $|B| < 2$.

Keywords: Generalized Fibonacci-like sequence, Periodicity, Discrete Dynamical System, PDE, Graph, Network, Digital Space

1. Preliminaries
Generalized Fibonacci sequences are widely used in many areas of science including applied physics, chemistry and biology. Over the past few decades, there has been a growth of interest in studying properties periodic Fibonacci sequences modulo m, Pell sequence and Lucas sequence. Many interesting results have been received in this area by researchers (see, e.g., [1, 4, 8, 10, 14]). Fibonacci like sequences are basic elements in computational solutions of partial differential equations using finite difference approximations of the PDE [6, 11], and in partial differential equations on networks and graphs [2, 7, 9]. The goals for this study is to investigate properties of the Fibonacci-like sequences, which are structural elements in discrete dynamic processes that can be encoded into graphs. The property we are interested in this study is the periodicity of a the Fibonacci-like sequence.

In Section 2, we study a periodic continuous function $f(x)$, where $f(x+1) = A + Bf(x) + Cf(x-1)$. We show that if $f(x)$ is periodic with period $T > 2$ then $|B| < 2$, $C = -1$.

We apply the obtained results to the sequence, which is defined by recurrence relation $F_{n+2} = A + BF_{n+1} + CF_n$ with $F_0 = a$, $F_1 = b$, $n \geq 0$, where A, B, C, a and b are real numbers. We show that if $|B| < 2$, $C = -1$ then $\{F_n\}$ is periodic with the period $T > 2$.

Then we present numerical examples of periodic Fibonacci-like sequences.

2. Periodic Fibonacci-like continuous functions and sequences
Consider properties of periodic continuous function $f(x)$ on domain R provided that $f(x+1) = A + Bf(x) + Cf(x-1)$.

Proposition 1
Let $f(x)$ be a continuous function on domain R, and let

$$f(x+1) = A + Bf(x) + Cf(x-1) \quad (1)$$

for any $x \in R$. If $f(x)$ is a periodic function with period $T > 2$ then $|B| < 2$, $C = -1$, $T = \frac{2\pi}{\omega}$, $\omega = \arccos \frac{B}{2}$.

Proof.
Let $f(x)$ is periodic with period $T = 2\pi/\omega$. Consider the Fourier series representing $f(x)$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad (2)$$

Using (2), it is easy to see that

$$f(x+1) - A - Bf(x) - Cf(x-1) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + d_n \sin nx, \quad (3)$$

$$c_n = a_n (\cos n - C \cos n - B) + b_n (1 + C) \sin n \quad (4)$$
Since $f(x+1) - A - Bf(x) - Cf(x-1) = 0$ for all $x$ then for all $n$, coefficients $c_0$, $c_n$ and $d_n$ are equal to zero.

$$c_0 = c_n = d_n \equiv 0. \quad (6)$$

Let $n=1$.

$$c_1 = a_1(\cos \omega - C \cos \omega - B) + b_1(1 + C)\sin \omega = 0$$
$$d_1 = -a_1(1 + C)\sin \omega + b_1(\cos \omega - C \cos \omega - B) = 0.$$  

Since $f(x)$ is periodic with basic period $T = \frac{2\pi}{\omega}$ then $a_1 \neq 0$ and/or $b_1 \neq 0$, which means that

$$\begin{align*}
(\cos \omega - C \cos \omega - B &= 0 \quad (7) \\
(1 + C)\sin \omega &= 0
\end{align*}$$

Suppose that $\sin \omega \neq 0$, i.e., $\omega \neq k\pi$, $T \neq \frac{2}{k}$. It follows from (7) that $1 + C = 0$, $\cos \omega = \frac{B}{1-C}$, $|\cos \omega| < 1$. Therefore, $C = -1$, $B = 2\cos \omega$, $|B| < 2$. Hence, $\omega = \arccos \frac{B}{2}$ and $T = \frac{2\pi}{\omega}$.

Let $n > 1$.

$$c_n = a_n(\cos \omega - C \cos \omega - B) + b_n(1 + C)\sin \omega = 0 \quad (8)$$
$$d_n = -a_n(1 + C)\sin \omega + b_n(\cos \omega - C \cos \omega - B) = 0.$$  

For the same reason as above, if $\cos \omega - C \cos \omega - B \neq 0$ then $a_n = b_n \equiv 0$, if $\cos \omega - C \cos \omega - B = 0$ then $a_n \neq 0$ and/or $b_n \neq 0$. Note that $\cos \omega - C \cos \omega - B = 0$ if $(n \pm 1)\omega = 2\pi k$, $n, k \in N$.

Finally, find $a_0$ from equation (3). Notice that $C = -1$, $|B| < 2$.

$$a_0 = \frac{A}{1 - C - B} = \frac{A}{2 - B}.$$  

Thus, if $|B| < 2$ and $C = 1$ then $f(x)$ is periodic with basic period $T = \frac{2\pi}{\omega}$ and $\omega = \arccos \frac{B}{2}$.

$$f(x) = a_0 + a_1 \cos \omega x + b_1 \sin \omega x + \sum_{n \in G} a_n \cos nx + b_n \sin nx, \text{ where } n \in G \text{ if } (n \pm 1)\omega = 2\pi k, \text{ } n, k \in N.$$  

This completes the proof. □

It is easy to check directly the following result.

**Proposition 2**

Let $f(x)$ be a continuous periodic function on domain $\mathbb{R}$ with period $T = \frac{2}{k}$, $k \in N, k \neq 0$, and for any $x \in R$, $f(x+1) = A + Bf(x) + Cf(x-1)$.

Then

$$f(x+1) = f(x) \text{ if } k = 2n,$$
$$f(x+1) = D - f(x) \text{ if } k = 2n + 1.$$  

**Remark 1**

If $f(x)$ is a continuous periodic function on domain $\mathbb{R}$ with period $T \leq 2$, $T \neq \frac{2}{k}$, $k \in N, k \neq 0$, then $f(x+1) \neq A + Bf(x) + Cf(x-1)$, $x \in R$.

We now outline a correspondence between the number $B$ and the period $T$ of function $f(x+1) = A + Bf(x) - f(x-1)$, where $|B| < 2$. Let’s investigate this question graphically. The graph $T = T(B) = 2\pi \left(\arccos \frac{B}{2}\right)^{-1}$ is shown in fig. 1. Obviously, $\lim_{B \to 0} T = \infty$, $\lim_{B \to -2} T = 2$.

In a numerical solution of the one-dimensional wave equation by the finite difference method, the spatial domain and the temporal domain are replaced by a set of mesh points,
the second-order derivatives can be replaced by central differences [2, 6, 7, 9]. The solution of the wave equation is replaced by the mesh function, which approximates the exact solution at the mesh points. The obtained discretization scheme has the form

$$u_{i}^{n+1} = 2u_{i}^{n} + c^2(u_{i-1}^{n} - 2u_{i}^{n} + u_{i+1}^{n}) - u_{i}^{n-1}, \text{ where } c = k(\Delta t/\Delta x).$$

By construction, this is a periodic in time sequence of real numbers. Periodic generalized Fibonacci sequences modulo m have been studied in a number of papers (see e.g. [3, 5, 12, 13]). Let us consider the following sequence

$$F_{n+2} = A + BF_{n+1} + CF_{n}, \ n \geq 0, \ F_0 = a, \ F_1 = b \quad (11)$$

In sequence (11), $A, B, C, a$ and $b$ are real numbers. In particular, $A, B, C, a$ and $b$ can be integer. The following propositions are a direct consequence of propositions 1 and 2.

**Proposition 3**

Let $F_{n+2} = A + BF_{n+1} + CF_{n}, \ n \geq 0, \ F_0 = a, \ F_1 = b$, be a sequence of real numbers, $(A, B, C, a, b) \in R$. If $|B| < 2, \ C = -1$ then $\{F_n\}$ is a periodic sequence with period $T = \frac{2\pi}{\omega}, \ \omega = arccos \frac{B}{2} \ T > 2$.

Proof. Consider the function $f(x + 1) = A + Bf(x) + Cf(x - 1)$ where $f(0) = a, \ f(1) = b$. Then $f(n+2) = F_{n+2}, \ f(n+1) = F_{n+1}, \ f(n) = F_{n}$ for $n \geq 0$.

According to proposition 1, if $|B| < 2, \ C = -1$, then f(x) is periodic with period $T = \frac{2\pi}{\omega}$, $\omega = arccos \frac{B}{2} \ T > 1, \ T \neq 2$. Therefore, sequence $\{F_n\}$ is also periodic with period $T = \frac{2\pi}{\omega}$.

This completes the proof.

Consider a special case when the period T=1, 2. The following proposition is a direct consequence of proposition 2.

**Proposition 4**

Let $F_{n+2} = A + BF_{n+1} + CF_{n}, \ n \geq 0, \ F_0 = a, \ F_1 = b$, be a sequence of real numbers, $(A, B, C, a, b) \in R$.

If $A = 0, \ B = 1, \ C = 0$ then $\{F_n\}$ is a periodic sequence with period $T = 1$, $F_{n+1} = F_n, \ n \geq 0, \ F_0 = F_1 = a$.

If $C \neq 1, \ B = C - 1$ then $\{F_n\}$ is a periodic sequence with period $T = 2$.
Consider examples of the periodic generalized Fibonacci sequence \( F_{n+2} = A + BF_{n+1} + CF_n, \) \( n \geq 0, \) \( F_0 = a, \) \( F_1 = b \) for different \( (A, B, C, a, b) \in \mathbb{R}. \)

**Example 1**

Let \( A = 3, \) \( B = 1.8, \) \( C = -1, \) \( F_0 = 1, \) \( F_1 = 5. \) Then \( F_{n+2} = 3 + 1.8F_{n+1} - F_n, \) \( n \geq 0. \)

![Figure 2. \( F_n \) profile at \( n=0, 1, \ldots, 50. \) \( F_{n+2} = 3 + 1.8F_{n+1} - F_n, \) \( n \geq 0, \) \( F_0 = 1, \) \( F_1 = 5. \) \( T = 13.93. \)](image)

For \( n=0, 1, \ldots, 14, \) the terms of \( F_n \) are

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( F_n \) | 1.0 | 5.0 | 11.0 | 17.8 | 24.0 | 28.5 | 30.2 | 28.9 | 24.8 | 18.8 | 12.0 | 5.8 | 1.4 | -0.2 | 1.2 |

The profile of \( F_n \) for \( n=1, \ldots, 50 \) is shown in fig.2. \( F_n \) is periodic with period \( T = 13.93, \) as it follows from formulas \( T = \frac{2\pi}{\omega}, \) \( \omega = a \cos \frac{B}{2}. \)

![Figure 3. \( F_n \) profile at \( n=0, 1, \ldots, 20. \) \( F_{n+2} = 3 + F_{n+1} - F_n, \) \( n \geq 0, \) \( F_0 = 1, \) \( F_1 = 5. \) \( T = 6. \)](image)

**Example 2**

Let \( A = 3, \) \( B = 1, \) \( C = -1, \) \( F_0 = 1, \) \( F_1 = 5. \) Then \( F_{n+2} = 3 + F_{n+1} - F_n, \) \( n \geq 0. \) For \( n=0, 1, \ldots, 14, \) the terms of \( F_n \) are

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( F_n \) | 1.0 | 5.0 | 7.0 | 5.0 | 1.0 | -1.0 | 1.0 | 5.0 | 7.0 | 5.0 | 1.0 | -1.0 | 1.0 | 5.0 | 7.0 |
Fig. 3 shows the profile of $F_n$ for $n=1,...,20$. $F_n$ is periodic with period $T = 6$, as it follows from formulas $T = \frac{2\pi}{\omega}$, $\omega = a \cos \frac{\theta}{2}$.

**Example 3**

Let $A = 3$, $B = 0$, $C = -1$, $F_0 = 1$, $F_1 = 5$. Then $F_{n+2} = 3 - F_n$, $n \geq 0$. For $n=0,1,...,14$, the terms of $F_n$ are

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| $F_n$ | 1.0 | 5.0 | -2.0 | 1.0 | 5.0 | 2.0 | -2.0 | 1.0 | 5.0 | 2.0 | -2.0 | 1.0 | 5.0 | 2.0 |

Fig. 4 shows the profile of $F_n$ for $n=1,...,20$. $F_n$ is periodic with period $T = 4$, as it follows from formulas $T = \frac{2\pi}{\omega}$, $\omega = a \cos \frac{\theta}{2}$.

**Example 4**

Let $A = 3$, $B = -1$, $C = -1$, $F_0 = 1$, $F_1 = 5$. Then $F_{n+2} = 3 - F_{n+1} - F_n$, $n \geq 0$. For $n=0,1,...,14$, the sequence $F_n$ is

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| $F_n$ | 1.0 | 5.0 | -3.0 | 1.0 | 5.0 | -3.0 | 1.0 | 5.0 | -3.0 | 1.0 | 5.0 | -3.0 | 1.0 | 5.0 | -3.0 |

**Fig. 5.** $F_n$ profile at $n=0,1,...,20$. $F_{n+2} = 3 - F_{n+1} - F_n$, $n \geq 0$. $F_0 = 1$, $F_1 = 5$. $T = 3$. 

**Fig. 4.** $F_n$ profile at $n=0,1,...,20$. $F_{n+2} = 3 - F_n$, $n \geq 0$. $F_0 = 1$, $F_1 = 5$. $T = 4$. 

**Fig. 5.** $F_n$ profile at $n=0,1,...,20$. $F_{n+2} = 3 - F_{n+1} - F_n$, $n \geq 0$. $F_0 = 1$, $F_1 = 5$. $T = 3$. 

**Fig. 4.** $F_n$ profile at $n=0,1,...,20$. $F_{n+2} = 3 - F_n$, $n \geq 0$. $F_0 = 1$, $F_1 = 5$. $T = 4$.
The profile of $F_n$ for $n=1,...,20$ is shown in fig. 5. $F_n$ is periodic with period $T = 3$, as it follows from formulas $T = \frac{2\pi}{\omega}$, $\omega = a \cos \frac{\beta}{2}$.

**Example 5**
Let $A = 3$, $B = -1.5$, $C = -1$, $F_0 = 1$, $F_1 = 5$. Then $F_{n+2} = 3 - 1.5F_{n+1} - F_n$, $n \geq 0$. For $n=0,1,...,14$, the sequence $F_n$ is

| $F_n$  | 1.0 | 5.0 | -5.5 | 6.3 | -0.9 | -1.9 | 6.8 | -5.2 | 4.1 | 2.1 | -4.3 | 7.3 | -3.6 | 1.2 | 4.9 |
|--------|-----|-----|------|-----|------|------|-----|------|-----|-----|------|-----|------|-----|-----|
| $n$    | 0   | 1   | 2    | 3   | 4    | 5    | 6   | 7    | 8   | 9   | 10   | 11  | 12   | 13  | 14  |

The profile of $F_n$ for $n=1,...,20$ is shown in fig. 6. $F_n$ is periodic with period $T = 2.51$, as it follows from formulas $T = \frac{2\pi}{\omega}$, $\omega = a \cos \frac{\beta}{2}$.

6. Conclusion
In connection with computational methods for approximating the solutions of partial differential equations, the generalized Fibonacci-like sequence $F_{n+2} = A + BF_{n+1} + CF_n$ with $F_0 = a$, $F_1 = b$, $n \geq 0$, where $A$, $B$, $C$, $a$ and $b$ are real numbers is investigated. It is shown that if $|B| < 2$, $C = -1$ then $\{F_n\}$ is periodic with the period $T > 2$. If period $T = 1$, then $A = C = 0$, $B = 1$, $F_{n+1} = F_n$, $n \geq 0$, $F_0 = F_1 = a$. If period $T = 2$ and $C \neq 1$ then $B = C - 1$, $D = \frac{A}{1-C}$, $F_{n+1} = D - F_n$, $n \geq 0$, $F_0 = a$, $F_1 = D - a$.

Numerical examples of periodic Fibonacci-like sequences are presented.

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