ON THE SPEED OF CONVERGENCE OF DISCRETE PICKANDS CONSTANTS TO CONTINUOUS ONES

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Abstract: In this manuscript, we address open questions raised by Dieker & Yakir (2014), who proposed a novel method of estimation of (discrete) Pickands constants \( H_\alpha \) using a family of estimators \( \xi_\alpha(T) \), \( T > 0 \), where \( \alpha \in (0, 2] \) is the Hurst parameter, and \( \delta \geq 0 \) is the step-size of the regular discretization grid. We derive an upper bound for the discretization error \( H_\alpha - \xi_\alpha(T) \), whose rate of convergence agrees with Conjecture 1 of Dieker & Yakir (2014) in case \( \alpha \in (0, 1] \) and agrees up to logarithmic terms for \( \alpha \in (1, 2) \). Moreover, we show that all moments of \( \xi_\alpha(T) \) are uniformly bounded and the bias of the estimator decays no slower than \( \exp\{ -CT^{\alpha} \} \), as \( T \) becomes large.

Key Words: fractional Brownian motion; Pickands constants; Monte Carlo simulation; discretization error

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1. Introduction

For any \( \alpha \in (0, 2] \) let \( \{B_\alpha(t), t \in \mathbb{R}\} \) be a fractional Brownian motion (later on, fBm) with Hurst parameter \( H = \alpha/2 \), that is, \( B_\alpha(t) \) is a centered Gaussian process with covariance function given by

\[
\text{Cov}(B_\alpha(t), B_\alpha(s)) = \frac{|t|^\alpha + |s|^\alpha - |t-s|^\alpha}{2}, \quad t, s \in \mathbb{R}, \quad \alpha \in (0, 2].
\]

In this manuscript we consider the classical Pickands constant defined by

\[
H_\alpha := \lim_{S \to \infty} \frac{1}{S} \mathbb{E} \left\{ \sup_{t \in [0,S]} e^{\sqrt{2}B_\alpha(t) - t^\alpha} \right\} \in (0, \infty), \quad \alpha \in (0, 2].
\]

The constant \( H_\alpha \) was first defined by Pickands [30, 31] to describe the asymptotic behavior of the maximum of stationary Gaussian processes. Since then, Pickands constants played an important role in the theory of Gaussian processes, appearing in various asymptotic results related to the supremum; see monographs [32, 33]. In [21], it was recognized that discrete Pickands constant can be interpreted as an extremal index of a Brown-Resnick process. This new realization motivated generalization of Pickands constants beyond the realm of Gaussian processes. We refer to [12, 13] for further references, who give an excellent account of the history of Pickands constants, their connection to the theory of max-stable processes and most recent advances in the theory.

While being omnipresent in the asymptotic theory of stochastic processes, to this date, the value of \( H_\alpha \) is known only in two very special cases: \( \alpha = 1 \) and \( \alpha = 2 \). In these cases, the distribution of the supremum of process \( B_\alpha \) is well-known — \( B_1 \) is a standard Brownian motion, while \( B_2 \) is a straight line with random, normally distributed slope. When \( \alpha \notin \{1, 2\} \), one may attempt to estimate the numerical value of \( H_\alpha \).
from the definition (1) using Monte-Carlo methods. However, there is a number of problems associated with this approach:

(i) Firstly, Pickands constant $\mathcal{H}_\alpha$ in (1) is defined as a limit $S \to \infty$, so one must approximate it by choosing some (large) $S$. This results in bias in estimation, which we call the \textit{truncation error}. The truncation error was shown to decay faster than $S^{-p}$ for any $p < 1$, [11, Corollary 3.1].

(ii) Secondly, for every $\alpha \in (0, 2)$, the variance of the truncated estimator blows up, as $S \to \infty$, i.e.

$$
\lim_{S \to \infty} \text{Var} \left\{ \frac{1}{S} \sup_{t \in [0, S]} \exp \{ \sqrt{2} B_\alpha(t) - t^\alpha \} \right\} = \infty.
$$

This can be easily seen by considering the second moment of $\frac{1}{S} \exp \{ \sqrt{2} B_\alpha(S) - S^\alpha \}$. This directly affects the \textit{sampling error} (standard deviation) of the Crude Monte Carlo estimator. As $S \to \infty$, one needs more and more samples to prevent its variance from blowing up.

(iii) Finally, there are no methods available for the exact simulation of $\sup_{t \in [0, S]} \exp \{ \sqrt{2} B_\alpha(t) - t^\alpha \}$ for $\alpha \not\in \{1, 2\}$. One must therefore resort to some method of approximation. Typically, one would simulate fBm on a regular $\delta$-grid, i.e. on the set $\delta \mathbb{Z}$ for $\delta > 0$; cf. Eq. (2) below. This approximation leads to a bias, which we call the \textit{discretization error}.

In the following, for any fixed $\delta > 0$ we define the discrete Pickands constant

$$
\mathcal{H}^\delta_\alpha := \lim_{S \to \infty} \frac{1}{S} \mathbb{E} \left\{ \sup_{t \in [0, S]_\delta} e^{\sqrt{2} B_\alpha(t) - t^\alpha} \right\}, \quad \alpha \in (0, 2],
$$

where for $a, b \in \mathbb{R}$ and $\delta > 0$ $[a, b)_\delta = [a, b] \cap \delta \mathbb{Z}$. Additionally, we set $0 \mathbb{Z} = \mathbb{R}$, so that $\mathcal{H}^0_\alpha = \mathcal{H}_\alpha$. In light of the discussion in item (iii) above, the discretization error equals to $\mathcal{H}_\alpha - \mathcal{H}^\delta_\alpha$. We should note that the quantity $\mathcal{H}^\delta_\alpha$ is well-defined and $\mathcal{H}^\delta_\alpha \in (0, \infty)$ for $\delta \geq 0$. Moreover, $\mathcal{H}^\delta_\alpha \to \mathcal{H}_\alpha$, as $\delta \to 0$, which means that the discretization error diminishes, as the size of the gap of the grid goes to 0. We refer to [12] for the proofs of these properties.

In recent years, [22] proposed a new representation of $\mathcal{H}^\delta_\alpha$, which does not involve the limit operation. They show [22, Proposition 3], that for all $\delta \geq 0$, and $\alpha \in (0, 2]$:

$$
\mathcal{H}^\delta_\alpha = \mathbb{E} \left\{ \xi^\delta_\alpha \right\}, \quad \text{where} \quad \xi^\delta_\alpha := \sup_{t \in \delta \mathbb{Z}} \frac{e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}}}{\delta \sum_{t \in \delta \mathbb{Z}} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}}},
$$

where for $\delta = 0$ the denominator in the fraction above is substituted by $\int_{\mathbb{R}} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}} \, dt$. In fact, the denominator can be substituted with $\eta \sum_{t \in \delta \mathbb{Z}} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}}$ for any $\eta$, which is an integer multiple of $\delta$, see [12, Theorem 2]. While one would ideally estimate $\mathcal{H}_\alpha$ using $\xi^0_\alpha$, it is unfortunately unfeasible due to the lack of exact simulation methods of $\xi^\delta_\alpha$ (see also item (iii) above). For that reason, the authors define the ‘truncated’ version of random variable $\xi^\delta_\alpha$, namely

$$
\xi^\delta_\alpha(T) := \frac{\sup_{t \in [-T, T]_\delta} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}}}{\delta \sum_{t \in [-T, T]_\delta} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}}},
$$

where for $\delta = 0$ the denominator in the fraction above is substituted by $\int_{-T}^{T} e^{\sqrt{2} B_\alpha(t) - |t|^{\alpha}} \, dt$. For any $\delta, T \in (0, \infty)$, the estimator $\xi^\delta_\alpha(T)$ is a functional of a fractional Brownian motion on a finite grid and, as such, it can be simulated exactly; see e.g. [23] for the survey of methods of simulation of fBm. The side effect of this approach is that the new estimator induces both the truncation and discretization errors described in items (i) and (iii) above.
In this manuscript we rigorously show that the estimator $\xi_\delta(T)$ is well-suited for simulation. In Theorem 2.1, we address the conjecture about the asymptotic behavior of the discretization error between the continuous and discrete Pickands constant for a fixed $\alpha \in (0,2]$, which was stated by the inventors of the estimator $\xi_\alpha$, namely:

[22. Conjecture 1] For all $\alpha \in (0,2]$ it holds, that $\lim_{\delta \to 0} \frac{\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta}{\delta^{\alpha/2}} \in (0,\infty)$.

We establish that the conjecture is true when $\alpha = 1$ and it is not true when $\alpha = 2$; see Theorem 2.1(iii-iv) below, where the exact asymptotics of the discretization error are derived in these two special cases. Furthermore, in Theorem 2.1(i), we show that $\limsup_{\delta \to 0} \delta^{-\alpha/2}(\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta) \in (0,\infty)$ for $\alpha \in (0,1)$ and in Theorem 2.1(ii) we show that $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ is upper-bounded by $\delta^{\alpha/2}$ up to logarithmic terms for $\alpha \in (1,2)$ and all $\delta > 0$ small enough. These results support the claim of the conjecture for all $\alpha \in (0,2)$.

Secondly, we consider the truncation and sampling errors induced by $\xi_\delta(T)$. In Theorem 2.3 we derive a uniform upper bound for the tail of $\xi_\delta^\alpha$ which implies that all moments of $\xi_\delta^\alpha$ exist and are uniformly bounded in $\delta \in [0,1]$. In Theorem 2.4 we establish that for any $\alpha \in (0,2)$ and $p \geq 1$, the difference $|\mathbb{E}(\xi_\delta^\alpha(T))^p - \mathbb{E}(\xi_\alpha^\delta)^p|$ decays no slower than $\exp\{-C T^{\alpha}\}$, as $T \to \infty$, uniformly for all $\delta \in [0,1]$. This implies that the truncation error of the Dieker-Yakir estimator decays no slower than $\exp\{-C T^{\alpha}\}$ and together with Theorem 2.3 they imply that $\xi_\delta^\alpha(T)$ has a uniformly bounded sampling error, i.e.

$$\sup_{(\delta,T) \in [0,1] \times [1,\infty)} \text{Var}\left\{\xi_\delta^\alpha(T)\right\} < \infty.$$

The manuscript is organized as follows. In Section 2 we present our main results and discuss their extensions and relation to other problems. All the proofs are provided in Section 3.

2. Main Results

In the following, we give an upper bound for $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ for all $\alpha \in (0,1) \cup (1,2)$ for small $\delta > 0$. When $\alpha \in \{1,2\}$ we provide the exact asymptotics of the discretization error, as $\delta \to 0$. Below, $\zeta$ is the Euler-Riemann zeta function.

**Theorem 2.1.** It holds, that

(i) for all $\alpha \in (0,1)$ there exists $C > 0$ such that for all $\delta > 0$ sufficiently small,

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq C \delta^{\alpha/2};$$

(ii) for all $\alpha \in (1,2)$ there exists $C > 0$ such that for all $\delta > 0$ sufficiently small,

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq C \delta^{\alpha/2} |\log \delta|^{1/2};$$

(iii) $\lim_{\delta \to 0} \frac{\mathcal{H}_1 - \mathcal{H}_1^\delta}{\sqrt{\delta}} = -\frac{\zeta(1/2)}{\sqrt{\pi}}$;

(iv) $\lim_{\delta \to 0} \frac{\mathcal{H}_2 - \mathcal{H}_2^\delta}{\delta^2} = \frac{1}{12\sqrt{\pi}}$.

The exact asymptotic behavior of the discretization error, as $\delta \to 0$ could be derived for $\alpha \in \{1,2\}$ because, in these cases, the explicit formulas for $\mathcal{H}_\alpha^\delta$ are known. For convenience, we collect these results in Proposition 2.2 below. In the following, $\Phi$ is the cumulative distribution function of a standard Gaussian random variable.

**Proposition 2.2.** It holds, that
implies that all moments of \( \xi_\alpha^\delta \) are finite and uniformly bounded in \( \delta \in [0, 1] \) for any fixed \( \alpha \in (0, 2) \).

**Theorem 2.3.** For any \( \alpha \in (0, 2) \) there exist positive constants \( C_1, C_2 \) such that
\[
P\left\{ \xi_\alpha^\delta > x \right\} \leq C_1 e^{-C_2 \log^2 x}
\]
for all \( x \geq 1, \delta \in [0, 1] \).

Evidently, Theorem 2.3 implies that all moments of \( \xi_\alpha^\delta \) are finite and uniformly bounded in \( \delta \in [0, 1] \) for any fixed \( \alpha \in (0, 2) \).

**Theorem 2.4.** For any \( \alpha \in (0, 2) \) and \( p > 0 \) there exist positive constants \( C_1, C_2 \) such that
\[
\left| \mathbb{E} \left\{ (\xi_\alpha^\delta(T))^p \right\} - \mathbb{E} \left\{ (\xi_\alpha^\delta)^p \right\} \right| \leq C_1 e^{-C_2 T^\alpha}
\]
for all \( (\delta, T) \in [0, 1] \times [1, \infty) \).

2.1. **Discussion.** We believe that finding the exact asymptotics of the speed of the discretization error \( \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \) is closely related to the behavior of fractional Brownian motion around the time of its supremum. We motivate this by the following heuristic:
\[
\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta = \mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} - \sup_{t \in \mathbb{Z}} \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} \right\}
\approx \mathbb{E} \left\{ \Delta(\delta) \cdot \sup_{t \in \mathbb{Z}} \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} \right\}
\approx \mathbb{E} \{ \Delta(\delta) \} \cdot \mathcal{H}_\alpha^\delta,
\]
where \( \Delta(\delta) \) is the difference between the supremum on the continuous and discrete grid, i.e. \( \Delta(\delta) := \sup_{t \in \mathbb{R}} \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} - \sup_{t \in \mathbb{Z}} \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} \). The first approximation above is due to the mean value theorem and the second approximation is based on the assumption that \( \Delta(\delta) \) and \( \xi_\alpha^\delta \) are asymptotically independent, as \( \delta \to 0 \). Now, we believe that \( \Delta(\delta) \sim C \delta^{\alpha/2} \) due to self-similarity, where \( C > 0 \) is some constant, which would imply that \( \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \sim C \mathcal{H}_\alpha \delta^{\alpha/2} \). This heuristic reasoning can be made rigorous in case \( \alpha = 1 \), when \( \sqrt{2} B_\alpha(t) - |t|^{1/\alpha} \) is a Lévy process (Brownian motion with drift). In this case, the asymptotic behavior of functionals such as \( \mathbb{E} \{ \Delta(\delta) \} \), as \( \delta \to 0 \) can be explained by the weak convergence of trajectories around the time of supremum to the so-called Lévy process conditioned to be positive, see [25] for more information on the topic. In fact, Theorem 2.1(iii) can be proven using the tools developed in [5]. To the best of the authors’ knowledge, there are no such results available for a general fractional Brownian motion. Although, it is worth mentioning that recently [2] considered a related problem of penalizing fractional Brownian motion for being negative.

A problem related to the asymptotic behavior of \( \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \) was considered in [6, 7], who have shown that \( \mathbb{E} \sup_{t \in [0, 1]} B_\alpha(t) - \mathbb{E} \sup_{t \in [0, 1]} B_\alpha(t) \), decays like \( \delta^{\alpha/2} \) up to logarithmic terms. We should emphasize
that in Theorem 2.1, case \( \alpha \in (0,1) \) we were able to establish that the upper bound for the discretization error decays exactly like \( \delta^{\alpha/2} \). In light of the discussion above, we believe that the result and the proving methodology of Theorem 2.1(i), could be useful in further research related to the discretization error for fractional Brownian motion.

Discretization error for asymptotic constants in the ruin theory for Gaussian processes. Although arguably most celebrated, Pickands constants are not the only constants appearing in the asymptotic theory of Gaussian processes. Depending on the setting, other constants might appear. Among others, we distinguish: Parisian Pickands constants [14, 15, 26], sojourn Pickands constants [17, 19], Piterbarg-type constants [3, 27, 32, 33], and generalized Pickands constants [10, 20]. Just like the classical Pickands constants, their numerical values are known typically only in case \( \alpha \in \{1,2\} \). In other cases, they need to be estimated and one encounters problems described in items (i-iii) in Section 1. In particular, possibility is an approximation by discretization. We believe, that under appropriate assumptions, using the technique from the proof of Theorem 2.1(ii), one could derive upper bounds for the discretization error, which are exact up to logarithmic terms.

Monotonicity of Pickands constants. Based on the definition (2), it is clear that for any \( \alpha \in (0,2) \), the sequence \( \{H_{\alpha}, H_{2 \alpha}, H_{4 \alpha}, \ldots\} \) is decreasing for any fixed \( \delta > 0 \). It is therefore natural to speculate that \( \delta \mapsto H_{\alpha}^\delta \) is a decreasing function. The explicit formulas for \( H_1^\delta \) and \( H_2^\delta \) given in Proposition 2.2 allow us to give the positive answer to this question in these cases. Namely,

**Corollary 2.5.** \( H_1^\delta \) and \( H_2^\delta \) are strictly decreasing functions with respect to \( \delta \) for all \( \delta \geq 0 \).

### 3. Proofs

In this section we give proofs. Define for \( \alpha \in (0,2) \)

\[
Z_\alpha(t) = \sqrt{2} B_\alpha(t) - |t|^\alpha, \quad t \in \mathbb{R}.
\]

Assume that all considered random processes and variables are defined on a complete general probability space \( \Omega \) equipped with a probability measure \( \mathbb{P} \). Let \( C, C_1, C_2, \ldots \) be some positive constants that may differ from line to line.

#### 3.1. Proof of Theorem 2.1 case \( \alpha \in (0,1) \).

The proof of Theorem 2.1 in case \( \alpha \in (0,1) \) is based on the following three results. In what follows, \( \eta \) is independent of \( \{Z_\alpha(t), t \in \mathbb{R}\} \) and follows a standard exponential distribution.

**Lemma 3.1.** For all \( \alpha \in (0,2) \)

\[
H_\alpha^\delta/2 - H_\alpha^\delta = \delta^{-1} \mathbb{P} \left\{ \sup_{t \in \mathbb{R} \setminus \{0\}} Z_\alpha(t) < 0, \sup_{t \in \mathbb{R} \setminus \{0\}} Z_\alpha(t - \frac{\delta}{2} \cdot \text{sgn}(t)) + \eta < 0 \right\}.
\]

As a side note, we remark that the representation in Lemma 3.1 yields a straightforward lower bound

\[
H_\alpha^{\delta/2} - H_\alpha^\delta \geq \delta^{-1} \mathbb{P} \left\{ \sup_{t \in (\delta/2) \mathbb{Z} \setminus \{0\}} Z_\alpha(t) + \eta < 0 \right\}
\]

for all \( \alpha \in (0,2), \delta > 0 \).

**Lemma 3.2.** For all \( \alpha \in (0,1) \) and \( \delta > 0 \)

\[
H_\alpha^{\delta/2} - H_\alpha^\delta \leq \delta^{-1} \mathbb{P} \left\{ \sup_{t \in \mathbb{Z} \setminus \{0\}} Z_\alpha(t) + \eta < 0 \right\}.
\]


Proposition 3.3. For any $\alpha \in (0, 2)$, there exists $C > 0$ such that
\[
\mathbb{P} \left\{ \sup_{t \in \mathbb{Z} \setminus \{0\}} Z_\alpha(t) + \eta < 0 \right\} \leq C \delta^{1+\alpha/2}
\]
for all $\delta > 0$ small enough.

Proof of Theorem 2.1, $\alpha \in (0, 1)$. Using the fact that $\mathcal{H}_\alpha^\delta \to \mathcal{H}_\alpha$, as $\delta \to 0$, we may represent the discretization error $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$, as a telescopic series, that is
\[
\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta = \sum_{k=0}^{\infty} \mathcal{H}_\alpha^{2^{-(k+1)}\delta} - \mathcal{H}_\alpha^{2^{-k}\delta}.
\]
Combining Lemma 3.2 and Proposition 3.3 we find that there exists $C_0 > 0$ such that
\[
\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq \sum_{k=0}^{\infty} C_0 (2^{-k}\delta)^{\alpha/2}
\]
for all $\delta$ small enough. This completes the proof with $C = \frac{C_0}{1-2^{-\alpha/2}}$. \qed

We remark that if the upper bound in Lemma 3.2 holds also for $\alpha \in (1, 2)$, then the upper bound in Theorem 2.1(i) would hold for all $\alpha \in (0, 2)$. The remainder of this section is devoted to proving Lemma 3.1, Lemma 3.2, and Proposition 3.3.

In what follows, for any $\alpha \in (0, 2)$, let $\{X_\alpha(t), t \in \mathbb{R}\}$ be a centered, stationary Gaussian process with $\text{Var} \{X_\alpha(t)\} = 1$, whose covariance function satisfies
\[
\mathbb{Cov}(X_\alpha(t), X_\alpha(0)) = 1 - |t|^{\alpha} + o(|t|^{\alpha}), \quad t \to 0.
\]
Before we give the proof of Lemma 3.1, we introduce the following result.

Lemma 3.4. The finite-dimensional distributions of $\{u(X_\alpha(u^{-2/\alpha}t) - u) \mid X_\alpha(0) > u, t \in \mathbb{R}\}$ converge weakly to the finite-dimensional distributions of $\{Z_\alpha(t) + \eta, t \in \mathbb{R}\}$, where $\eta$ is a random variable independent of $\{Z_\alpha(t), t \in \mathbb{R}\}$ following a standard exponential distribution.

The result in Lemma 3.4 is well-known; see, e.g., [1, Lemma 2], where the convergence of finite-dimensional distributions is established on $t \in \mathbb{R}_+$. The extension to $t \in \mathbb{R}$ is straightforward.

Proof of Lemma 3.1. The following proof is very similar in its flavor to the proof of [4, Lemma 3.2]. From [33, Lemma 9.2.2] and the classical definition of Pickands constant it follows that for any $\alpha \in (0, 2)$ and $\delta \geq 0$
\[
\mathcal{H}_\alpha^\delta = \lim_{T \to \infty} \lim_{u \to \infty} \mathbb{P} \left\{ \sup_{t \in [0, T]_\delta} X_\alpha(u^{-2/\alpha}t) > u \right\},
\]
where $\Psi(u)$ is the complementary cdf (tail) of the standard normal distribution and $\{X_\alpha, t \in \mathbb{R}\}$ is the process introduced above Eq. (5). Therefore,
\[
\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^\delta = \lim_{T \to \infty} \lim_{u \to \infty} \mathbb{P} \left\{ \max_{t \in [0, T]_\delta/2} X_\alpha(u^{-2/\alpha}t) > u, \max_{t \in [0, T]_\delta} X_\alpha(u^{-2/\alpha}t) < u \right\}.
\]
Now, notice that we can decompose the event in the numerator above into a sum of disjoint events
\[
\frac{1}{\Psi(u)} \mathbb{P} \left\{ \max_{t \in [0, T]_{\delta/2}} X_\alpha(u^{-2/\alpha}t) > u, \max_{t \in [0, T]_{\delta}} X_\alpha(u^{-2/\alpha}t) < u \right\}
\]
Using the stationarity of the process $X_\alpha$, the above is equal to

$$
\sum_{\tau \in [0,T]_{\delta/2}} \mathbb{P}\left\{ \max_{t \in [0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}t) \leq X_\alpha(u^{-2/\alpha} \tau), \ max_{t \in [0,T]_{\delta}} X_\alpha(u^{-2/\alpha}t) \leq u \mid X_\alpha(u^{-2/\alpha} \tau) > u \right\}.
$$

Applying Lemma 3.4 to each element of the sum above, we find that the sum above converges to

$$
\sum_{\tau \in [0,T]_{\delta/2}} \mathbb{P}\left\{ \max_{t \in [0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}(t - \tau)) \leq X_\alpha(0), \ max_{t \in [0,T]_{\delta}} X_\alpha(u^{-2/\alpha}(t - \tau)) \leq u \mid X_\alpha(0) > u \right\}.
$$

We have now established that $\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} = \lim_{T \to \infty} \frac{1}{T} \sum_{\tau \in [0,T]_{\delta/2}} U(\tau, T)$. Clearly,

$$
U(\tau, \infty) = U(0, \infty) = \mathbb{P}\left\{ \max_{t \in (\delta/2)\mathbb{Z} \setminus \{0\}} Z_\alpha(t) < 0, \ max_{t \in \delta \mathbb{Z} \setminus \{0\}} Z_\alpha(t - \frac{\delta}{2} \cdot \text{sgn}(t)) + \eta < 0 \right\}.
$$

We will now show that $\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta}$ is lower bounded and upper bounded by $\delta^{-1} U(0, \infty)$, which will complete the proof. For the lower bound see that

$$
\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} \geq \lim_{T \to \infty} \frac{1}{T} \sum_{\tau \in [0,T]_{\delta/2}} U(\tau, \infty),
$$

where the limit is equal to $\delta^{-1} U(0, \infty)$ because the sum above has $[T(\delta/2)^{-1}]$ elements, half of which is equal to 0 and the other half is equal to $U(0, \infty)$. In order to show the upper bound consider $\varepsilon > 0$. For any $\tau \in (\varepsilon T, (1 - \varepsilon)T)_{\delta/2}$ we have

$$
U(\tau, T) \leq U(T, \varepsilon) := \mathbb{P}\left\{ \max_{t \in (\varepsilon T, (1 - \varepsilon)T)_{\delta}} Z_\alpha(t) \leq 0, \ max_{t \in (\varepsilon T, (1 - \varepsilon)T)_{\delta}} Z_\alpha(t) + \eta \leq 0 \right\}.
$$

Furthermore, we have the following decomposition

$$
\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} = \lim_{T \to \infty} \frac{1}{T} \left( \sum_{\tau \in (\delta/2)\mathbb{Z} \setminus I_-} U(\tau, T) + \sum_{\tau \in (\delta/2)\mathbb{Z} \setminus I_0} U(\tau, T) + \sum_{\tau \in (\delta/2)\mathbb{Z} \setminus I_+} U(\tau, T) \right),
$$

where $I_- := [0, \varepsilon T]$, $I_0 := (\varepsilon T, (1 - \varepsilon)T)$, $I_+ := [(1 - \varepsilon), T]$. The first and the last sum can be bounded by their number of elements $[\varepsilon T(\delta/2)^{-1}]$ because $U(\tau, T) \leq 1$. The middle sum can be bounded by $\frac{1}{2} \cdot [(1 - 2\varepsilon)T(\delta/2)^{-1}] U(T, \varepsilon)$ because half of its elements are equal to 0 and the other half can be upper bounded by $U(T, \varepsilon)$. After passing with $T \to \infty$ this gives us

$$
\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} \leq 4\varepsilon \delta^{-1} + (1 - 2\varepsilon)\delta^{-1} U(0, \infty)
$$

because $U(T, \varepsilon) \to U(0, \infty)$, as $T \to \infty$. Finally, after passing $\varepsilon \to 0$ we obtain the desired result.

**Proof of Lemma 3.2.** In the light of Lemma 3.1, it suffices to show that

$$
\mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} Z_\alpha(t - \frac{\delta}{2} \cdot \text{sgn}(t)) + \eta < 0 \right\} \leq \mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} Z_\alpha(t) + \eta < 0 \right\}.
$$

The left-hand side of the above equals to

$$
\mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} \sqrt{2} B_\alpha(t - \frac{\delta}{2} \cdot \text{sgn}(t)) - |t - \frac{\delta}{2} \cdot \text{sgn}(t)|^\alpha + \eta < 0 \right\}
$$
\begin{align*}
&= \mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} \frac{\sqrt{2}B_{\alpha}(t - \frac{\delta}{2} \cdot \text{sgn}(t))}{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2}} - \frac{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2} + \frac{\eta}{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2}} < 0 \right\} \\
&\leq \mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} \frac{\sqrt{2}B_{\alpha}(t - \frac{\delta}{2} \cdot \text{sgn}(t))}{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0 \right\}.
\end{align*}

Now, for all \(t, s \in \delta \mathbb{Z} \setminus \{0\}\) it holds that

\begin{equation}
\text{Cov}\left( \frac{\sqrt{2}B_{\alpha}(t - \frac{\delta}{2} \cdot \text{sgn}(t))}{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2}}, \frac{\sqrt{2}B_{\alpha}(s - \frac{\delta}{2} \cdot \text{sgn}(s))}{|s - \frac{\delta}{2} \cdot \text{sgn}(s)|^{\alpha/2}} \right) \leq \text{Cov}\left( \frac{\sqrt{2}B_{\alpha}(t)}{|t|^{\alpha/2}}, \frac{\sqrt{2}B_{\alpha}(s)}{|s|^{\alpha/2}} \right),
\end{equation}

which is shown below. Since for \(t = s\), the covariances in Eq. (6) are equal, we may use Slepian lemma \[33, \text{Lemma 2.1.1}\] and obtain

\begin{align*}
\mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} \frac{\sqrt{2}B_{\alpha}(t - \frac{\delta}{2} \cdot \text{sgn}(t))}{|t - \frac{\delta}{2} \cdot \text{sgn}(t)|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0 \right\} &\leq \mathbb{P}\left\{ \sup_{t \in \delta \mathbb{Z} \setminus \{0\}} \frac{\sqrt{2}B_{\alpha}(t)}{|t|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0 \right\}
\end{align*}

and the claim follows. It is left to show Eq. (6). Let \(t, s \in \mathbb{R}\) be fixed and let

\[c(\delta; t, s) := \text{Cov}\left( \frac{B_{\alpha}(t + \delta \cdot \text{sgn}(t))}{|t + \delta \cdot \text{sgn}(t)|^{\alpha/2}}, \frac{B_{\alpha}(s + \delta \cdot \text{sgn}(s))}{|s + \delta \cdot \text{sgn}(s)|^{\alpha/2}} \right).\]

We will show that \(\delta \mapsto c(\delta; t, s)\) is a non-decreasing function, which will conclude the proof. We have

\[c(\delta; t, s) = \frac{|t + \delta \cdot \text{sgn}(t)|^{\alpha} + |s + \delta \cdot \text{sgn}(s)|^{\alpha} - |t - s + \delta \cdot (\text{sgn}(t) - \text{sgn}(s))|^{\alpha}}{2|s + \delta \cdot \text{sgn}(s)|^{\alpha/2} \cdot |t + \delta \cdot \text{sgn}(t)|^{\alpha/2}}.\]

We consider two cases: (i) \(t, s > 0\), and (ii) \(s < 0 < t\). Consider case (i) first. Without loss of generality we assume that \(t \geq s\), then

\[c(\delta; t, s) = \frac{(t + \delta)^{\alpha} + (s + \delta)^{\alpha} - (t - s)^{\alpha}}{2((s + \delta)(t + \delta))^{\alpha/2}}.\]

It suffices to show that the first derivative of \(\delta \mapsto c(\delta; t, s)\) is nonnegative. We have

\[
\frac{\partial}{\partial \delta} c(\delta; t, s) = \frac{\alpha(1 - x)(t + \delta)^{\alpha/2}/4}{(s + \delta)^{1+\alpha/2}} \cdot \left( (1 - x)^{\alpha-1}(1 + x) - x^{\alpha} - 1 \right),
\]

where \(x := \frac{s + \delta}{t + \delta} \in (0, 1]\). The derivative above is non-negative iff \(G_1(x, \alpha) := (1 - x)^{\alpha-1}(1 + x) + x^{\alpha} - 1 \geq 0\) for all \(x \in (0, 1]\). It is easy to see that, for any fixed \(x \in (0, 1]\), \(\alpha \mapsto G_1(x, \alpha)\) is a non-decreasing function; this observation combined with the fact that \(G(x, 2) = 0\) completes the proof of case (i). In case (ii) we need to show that

\[c(\delta; t, -s) = \frac{(t + \delta)^{\alpha} + (s + \delta)^{\alpha} - (t + s + 2\delta)^{\alpha}}{2((s + \delta)(t + \delta))^{\alpha/2}}\]

is a non-decreasing function of \(x\) for any \(s, t > 0\). Without the loss of generality let \(0 < s \leq t\). Again, we take the first derivative of the above and see that

\[
\frac{\partial}{\partial \delta} c(\delta; t, s) = \frac{\alpha(1 - x)(t + \delta)^{\alpha/2}/4}{(s + \delta)^{1+\alpha/2}} \cdot \left( (1 - x)(1 + x)^{\alpha-1} + x^{\alpha} - 1 \right),
\]

where \(x := \frac{s + \delta}{t + \delta} \in (0, 1]\). The derivative above is non-negative iff \(G_2(x, \alpha) := (1 - x)(1 + x)^{\alpha-1} + x^{\alpha} - 1 \geq 0\). Notice that \(G_2(x, 1) = 0\). We will now show that \(\frac{\partial}{\partial \alpha} G_2(x, \alpha) \leq 0\) for all \(\alpha \in [0, 1]\) and \(x \in (0, 1]\), which will conclude the proof. We have

\[
\frac{\partial}{\partial \alpha} G_2(x, \alpha) = (1 - x)(1 + x)^{\alpha-1} \log(x + 1) + x^{\alpha} \log x
\]

\[
\leq (1 - x)x^{\alpha-1} \log(x + 1) + x^{\alpha} \log x
\]
so that, with
\[(11) \quad \Sigma(\lambda) = \begin{pmatrix} \text{Cov}(Z_\alpha(-\delta), Z_\alpha(-\delta)) & \text{Cov}(Z_\alpha(-\delta), Z_\alpha(1)) \\ \text{Cov}(Z_\alpha(1), Z_\alpha(-\delta)) & \text{Cov}(Z_\alpha(1), Z_\alpha(1)) \end{pmatrix} = \begin{pmatrix} 2 & 2 - 2\alpha \\ 2 - 2\alpha & 2 \end{pmatrix}. \]

By the self-similarity property of fBm, the covariance matrix \(\Sigma(\delta)\) of \((Z_\alpha(-\delta), Z_\alpha(\delta))\) equals \(\Sigma(\delta) = \delta^\alpha \Sigma\).

With \(\mathbf{1}_2 = (1, 1)^\top\) we define:
\[(11) \quad a(x) := x^\top \Sigma^{-1} x, \quad b(x) := x^\top \Sigma^{-1} \mathbf{1}_2, \quad c := 1_2^\top \Sigma^{-1} 1_2, \]
so that, with \(|\Sigma|\) denoting the determinant of matrix \(\Sigma\) we have
\[
f(x; \delta) = \frac{1}{2\pi |\Sigma|^{1/2} \delta^\alpha} \exp \left\{ - (x + 1_2 \delta^\alpha)^\top \Sigma(\delta)^{-1} (x + 1_2 \delta^\alpha) \right\} \\
= \frac{1}{2\pi |\Sigma|^{1/2} \delta^\alpha} \exp \left\{ - a(x) + 2b(x)\delta^\alpha + c\delta^{2\alpha} \right\}.
\]

**Lemma 3.5.** For any \(\lambda > 0\) there exist \(C_0, C_1 > 0\) such that
\[C_0 \delta^{\alpha/2} \leq q(\delta, \lambda) \leq C_1 \delta^{\alpha/2}\]
for all \(\delta > 0\) sufficiently small.

**Proof.** For the lower bound see that
\[
q(\delta, \lambda) = \mathbb{P} \left\{ \sqrt{2} B_\alpha(-\delta) - \delta^\alpha + \lambda^{-1} \eta < 0, \sqrt{2} B_\alpha(\delta) - \delta^\alpha + \lambda^{-1} \eta < 0 \right\} \\
\geq \mathbb{P} \left\{ \sqrt{2} B_\alpha(-\delta) + \lambda^{-1} \eta < 0, \sqrt{2} B_\alpha(\delta) + \lambda^{-1} \eta < 0, \lambda^{-1} \eta < \delta^{\alpha/2} \right\}
\]
In the following lemma, we establish the formulas for \( f^- \) and \( g^- \); most notably, we show that \( g^- \) can be upper bounded by \( f^- \) uniformly in \( \delta \), up to a positive constant.

**Lemma 3.6.** For any \( \lambda > 0 \),

(i) \( f^-(x; \delta) = p(\delta)^{-1} f(x; \delta) \);

(ii) \( g^-(x; \delta, \lambda) = q(\delta, \lambda)^{-1} f(x; \delta) \int_0^\infty \lambda \exp \left\{ -cz^2 + 2z((\lambda - c)\delta^{-2} - b(x)) \right\} dz \);

(iii) there exists \( C > 0 \) depending only on \( \lambda \), such that for all \( \delta \) small enough:

\[
g^-(x; \delta, \lambda) \leq Cf^-(x; \delta), \text{ for all } x \leq 0;
\]

Proof. Part (i) follows directly from the definition. For part (ii), for \( x \leq 0 \) we have

\[
g^-(x; \delta, \lambda) = q(\delta, \lambda)^{-1} \int_0^\infty f(x - 1_2 z; \delta) \cdot \lambda e^{-\lambda z} dz
\]

\[
= \int_0^\infty \frac{q(\delta, \lambda)^{-1}}{2\pi |\Sigma|^{1_2}} \exp \left\{ -\frac{\left( x + 1_2(\delta^{-2} - z) \right)^\top \Sigma^{-1}(x + 1_2(\delta^{-2} - z))}{2\delta^2} \right\} \cdot \lambda e^{-\lambda z} dz
\]

\[
= \int_0^\infty \frac{q(\delta, \lambda)^{-1}}{2\pi |\Sigma|^{1_2}} \exp \left\{ -\frac{a(x) + 2b(x)(\delta^{-2} - z) + c(\delta^2 - 2\delta^2 z + z^2) + 2\lambda \delta^2 z}{2\delta^2} \right\} dz
\]

\[
= q(\delta, \lambda)^{-1} f(x; \delta) \int_0^\infty \lambda \exp \left\{ -\frac{cz^2 + 2z((\lambda - c)\delta^{-2} - b(x))}{2\delta^2} \right\} dz.
\]

For part (iii), we have \( \Sigma^{-1} = (2^2(4 - 2^2))^{-1} \cdot (2^2 - 2^2 2^2) \), thus \( \Sigma^{-1} 1_2 \geq 0 \) element-wise. It then follows that \( b(x) \leq 0 \) when \( x \leq 0 \), which yields the following upper bound

\[
g^-(x; \delta, \lambda) \leq q(\delta, \lambda)^{-1} f(x; \delta) \int_0^\infty \lambda \exp \left\{ -\frac{cz^2 + 2z(\lambda - c)\delta^{-2}}{2\delta^2} \right\} dz.
\]

Now, it is easy to see that for all \( \delta > 0 \) small enough

\[
\int_0^\infty \lambda \exp \left\{ -\frac{cz^2 + 2z(\lambda - c)\delta^{-2}}{2\delta^2} \right\} dz \leq \int_0^\infty \lambda \exp \left\{ -cz^2 + 2z(\lambda - c) \right\} dz < \infty,
\]

hence, using part (i), there exists \( C > 0 \) such that

\[
g^-(x; \delta, \lambda) \leq C\delta^{\alpha/2} q(\delta, \lambda)^{-1} p(\delta)^{-1} f^-(x; \delta).
\]

The proof is concluded by noting that \( p(\delta) \to \mathbb{P} \{ B_{\alpha}(-1) < 0, B_{\alpha}(1) < 0 \} > 0 \) and \( \delta^{\alpha/2} q(\delta, \lambda)^{-1} = O(1) \), due to Lemma 3.5.
Recall the definition of $\Sigma$ in Eq. (10). In what follows, we put, for $k \in \mathbb{Z}$

$$
\begin{bmatrix}
    c^-(k) \\
    c^+(k)
\end{bmatrix}
:= \Sigma^{-1} \cdot 
\begin{bmatrix}
    Cov(Z_{\alpha}(k), Z_{\alpha}(-1)) \\
    Cov(Z_{\alpha}(k), Z_{\alpha}(1))
\end{bmatrix}
$$

(12)

$$
\frac{1}{2^\alpha(4 - 2^\alpha)} \cdot \begin{bmatrix}
    2 & 2^\alpha - 2 \\
    2^\alpha - 2 & 2
\end{bmatrix} \cdot 
\begin{bmatrix}
    k^\alpha + 1 - (k + \text{sgn}(k))\alpha \\
    k^\alpha + 1 - (k - \text{sgn}(k))\alpha
\end{bmatrix}.
$$

**Lemma 3.7.** For $k \in \mathbb{Z} \setminus \{0\}$,

(i) $(2 - 2^{\alpha-1})^{-1} < c^-(k) + c^+(k) \leq 1$ when $\alpha \in (0, 1)$;

(ii) $1 \leq c^-(k) + c^+(k) \leq (2 - 2^{\alpha-1})^{-1}$ when $\alpha \in (1, 2)$;

**Proof.** After some algebraic transformations, from (12), we find that

$$
c^-(k) + c^+(k) = \frac{2 + 2|k|^\alpha - (|k| - 1)^\alpha - (|k| + 1)^\alpha}{4 - 2^\alpha}, \quad k \in \mathbb{Z} \setminus \{0\}
$$

Let $f(x) = 2x^\alpha - (x - 1)^\alpha - (x + 1)^\alpha$. We have for $x > 1$

$$
f'(x) = \alpha x^{\alpha-1} (2 - (1 - 1/x)^{\alpha-1} - (1 + 1/x)^{\alpha-1})
$$

$$
= \alpha x^{\alpha-1} \left[ 2 - \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha - 1) \cdots (\alpha - n)}{n!} x^{-n} - \sum_{n=0}^{\infty} \frac{(\alpha - 1) \cdots (\alpha - n)}{n!} x^{-n} \right]
$$

$$
= \alpha x^{\alpha-3}(\alpha - 1)(2 - \alpha) \left[ \frac{1}{2!} + \sum_{n=1}^{\infty} \frac{\alpha - 3 \cdots (\alpha - 2(n + 1))}{(2(n + 1))!} x^{-2n} \right].
$$

We see that each of the terms of the sum above is positive, so $\text{sgn}(f'(x)) = \text{sgn}(\alpha - 1)$. Thus, $f'(x)$ is negative for $\alpha \in (0, 1)$ and positive for $\alpha \in (1, 2)$. Finally, since

$$
\lim_{x \to \infty} \frac{2 + 2x^\alpha - (x - 1)^\alpha - (x + 1)^\alpha}{4 - 2^\alpha} = \frac{2}{4 - 2^\alpha} = (2 - 2^{\alpha-1})^{-1}
$$

and $c^-(1) + c^+(1) = 1$, the claim follows. \(\square\)

We are now ready to prove Proposition 3.3. In what follows, for any $\delta > 0$ and $t \in \mathbb{R}$ let

$$
Y_\alpha^\delta(t) := Z_\alpha(t) - \mathbb{E} \{Z_\alpha(t) \mid (Z_\alpha(-\delta), Z_\alpha(\delta))\}
$$

$$
= Z_\alpha(t) - (c^-(k)Z_\alpha(-\delta) + c^+(k)Z_\alpha(\delta)).
$$

It is a well known fact that $\{Y_\alpha^\delta(t), t \in \mathbb{R}\}$ is independent of $(Z_\alpha(-\delta), Z_\alpha(\delta))$.

**Proof of Proposition 3.3.** Recall the definition of the events $A(\delta, \lambda)$ and $A(\delta)$ in (7). We have

$$
P \left\{ \sup_{t \in \mathbb{R} \setminus \{0\}} Z_\alpha(t) + \eta \leq 0 \right\}
$$

$$
= \mathbb{P} \left\{ \sup_{k \in \mathbb{Z} \setminus \{-1, 0, 1\}} Y_\alpha^\delta(\delta k) + \left( c^-(k)Z_\alpha(-\delta) + c^+(k)Z_\alpha(\delta) \right) + \eta < 0, A(\delta, 1) \right\},
$$

with $Y_\alpha(\delta)$ defined in (13). Let $\lambda^* := 1$ when $\alpha \in (0, 1]$, and $\lambda^* := (2 - 2^{\alpha-1})^{-1}$ when $\alpha \in [1, 2)$. Due to Lemma 3.7 we have $\lambda^* \geq 1$ and $c^-(k) + c^+(k) \leq \lambda^*$, thus $A(\delta, 1) \subseteq A(\delta, \lambda^*)$ and the display above is upper bounded by

$$
P \left\{ A(\delta, \lambda^*) < 0, \sup_{k \in \mathbb{Z} \setminus \{-1, 0, 1\}} Y_\alpha^\delta(\delta k) + \left( c^-(k)(Z_\alpha(-\delta) + \frac{\eta}{\lambda^*}) + c^+(k)(Z_\alpha(\delta) + \frac{\eta}{\lambda^*}) \right) < 0 \right\}
Finally, we have
\[ q(\delta, \lambda^*) \cdot \mathbb{P} \left\{ \sup_{k \in \mathbb{Z} \setminus \{ -1, 0, 1 \}} Y_0^\delta(\delta k) + \left( c_0^-(k)(Z_0(-\delta) + \frac{p}{k}) + c_0^+(k)(Z_0(\delta) + \frac{q}{k}) \right) < 0 \middle| A(\delta, \lambda^*) \right\} \]
\[ = q(\delta, \lambda^*) \cdot \int \mathbb{P} \left\{ \sup_{k \in \mathbb{Z} \setminus \{ -1, 0, 1 \}} Y_0^\delta(\delta k) + \left( c_0^-(k)x_1 + c_0^+(k)x_2 \right) < 0 \right\} g^-(x; \delta, \lambda^*) \mathrm{d}x, \]
with \( g^- \) defined in (9). Now, using Lemma 3.6(iii) we know that there exists \( C > 0 \) such that for all \( \delta > 0 \) small enough, the expression above is upper bounded by
\[ q(\delta, \lambda^*) \cdot \int \mathbb{P} \left\{ \sup_{k \in \mathbb{Z} \setminus \{ -1, 0, 1 \}} Y_0^\delta(\delta k) + \left( c_0^-(k)x_1 + c_0^+(k)x_2 \right) < 0 \right\} C f^-(x; \delta) \mathrm{d}x, \]
\[ = C q(\delta, \lambda^*) \mathbb{P} \left\{ \sup_{k \in \mathbb{Z} \setminus \{ -1, 0, 1 \}} Y_0^\delta(\delta k) + \left( c_0^-(k)Z_0(-\delta) + c_0^+(k)Z_0(\delta) \right) < 0 \right\} A(\delta) \]
\[ = C q(\delta, \lambda^*) p^\delta \mathbb{P} \left\{ A(\delta), \sup_{k \in \mathbb{Z} \setminus \{ -1, 0, 1 \}} Y_0^\delta(\delta k) + \left( c_0^-(k)Z_0(-\delta) + c_0^+(k)Z_0(\delta) \right) < 0 \right\} \]
\[ = C q(\delta, \lambda^*) p^\delta \mathbb{P} \left\{ \sup_{t \in \mathbb{Z} \setminus \{ 0 \}} Z_0(t) \leq 0 \right\} \]
Finally, we have \( \mathbb{P} \left\{ \sup_{t \in \mathbb{Z} \setminus \{ 0 \}} Z_0(t) \leq 0 \right\} \sim \delta^{-1} \mathcal{H}_\alpha \), see [22, Proposition 4]. This completes the proof due to Lemma 3.5 and the fact that \( p^\delta \rightarrow \mathbb{P} \{ B_\alpha(-1) < 0, B_\alpha(1) < 0 \} > 0. \]

3.2. Proof of Theorem 2.1, case \( \alpha \in (1, 2) \). The following lemma provides a crucial bound for \( \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \).

Lemma 3.8. For sufficiently small \( \delta > 0 \) it holds, that
\[ \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq 2 \mathbb{E} \left\{ \sup_{t \in [0, 1]} e^{Z_0(t)} - \sup_{t \in [0, 1], \delta} e^{Z_0(t)} \right\}. \]

Proof of Lemma 3.8. As follows from the proof of [16, Theorem 1], the first equation on p. 12 with \( c_\delta := \lfloor 1/\delta \rfloor \), where \( \lfloor \cdot \rfloor \) is the integer part of a real number, it holds, that
\[ \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq c_\delta^{-1} \mathbb{E} \left\{ \sup_{t \in [0, c_\delta]} e^{Z_0(t)} - \sup_{t \in [0, c_\delta], \delta} e^{Z_0(t)} \right\} \]
\[ \leq 2 \mathbb{E} \left\{ \sup_{t \in [0, c_\delta]} e^{Z_0(t)} - \sup_{t \in [0, c_\delta], \delta} e^{Z_0(t)} \right\} \leq 2 \mathbb{E} \left\{ \sup_{t \in [0, 1]} e^{Z_0(t)} - \sup_{t \in [0, 1], \delta} e^{Z_0(t)} \right\}, \]
which completes the proof. \( \square \)

Now we are ready to prove Theorem 2.1(ii).

Proof of Theorem 2.1, \( \alpha \in (1, 2) \). Note that for any \( y \leq x \) it holds, that \( e^x - e^y \leq (x-y)e^x \). Implementing this inequality we find that, for \( s, t \in [0, 1], \)
\[ \left| e^{Z_0(t)} - e^{Z_0(s)} \right| \leq e^{\max(Z_0(t), Z_0(s))} |Z_0(t) - Z_0(s)| \]
\[ \leq e^{\sqrt{2} \max_{w \in [0, 1]} B_\alpha(w)} \left| \sqrt{2}(B_\alpha(t) - B_\alpha(s)) - (t^\alpha - s^\alpha) \right|. \]
Next, by Lemma 3.8 we have
\[ \mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq 2 \mathbb{E} \left\{ \sup_{t, s \in [0, 1], \left| t-s \right| \leq \delta} \left| e^{Z_0(t)} - e^{Z_0(s)} \right| \right\} \]
ON THE SPEED OF CONVERGENCE OF DISCRETE PICKANDS CONSTANTS TO CONTINUOUS ONES

\[ \leq 2\sqrt{2}E \left\{ e^{\sqrt{2} \max_{w \in [0,1]} B_\alpha(w)} \sup_{t,s \in [0,1],|t-s| \leq \delta} |B_\alpha(t) - B_\alpha(s)| \right\} + 2\sqrt{2}E \left\{ e^{\sqrt{2} \max_{w \in [0,1]} B_\alpha(w)} \sup_{t,s \in [0,1],|t-s| \leq \delta} |t^\alpha - s^\alpha| \right\}. \]

Clearly, the second term is upper-bounded by \( C_1 \delta \) for all \( \delta \) small enough. Using Hölder inequality, the first term can be bounded by

\[ 2\sqrt{2}E \left\{ e^{\sqrt{2} \max_{w \in [0,1]} B_\alpha(w)} \right\}^{1/2} E \left\{ \left( \sup_{t,s \in [0,1],|t-s| \leq \delta} (B_\alpha(t) - B_\alpha(s)) \right)^2 \right\}^{1/2}. \]

The first expectation is finite. The random variable inside the second expectation is called the uniform modulus of continuity. From [9, Theorem 4.2, p. 164] it follows that there exists \( C > 0 \) such that

\[ E \left\{ \left( \sup_{t,s \in [0,1],|t-s| \leq \delta} (B_\alpha(t) - B_\alpha(s)) \right)^2 \right\}^{1/2} \leq C\delta^{\alpha/2} |\log(\delta)|^{1/2}. \]

This concludes the proof.

\[ \square \]

**Remark 3.9.** Note that the proofs of Lemma 3.8 and Theorem 2.1 work also in case \( \alpha \in (0,1] \).

### 3.3. Proof of Theorem 2.1, case \( \alpha \in \{1,2\} \).

**Proof of Theorem 2.1, \( \alpha = 1 \).** In the following \( \phi, \Psi \) stand for the pdf and the survival function of a standard Gaussian random variable and

\[ v(\eta) := \eta \exp\left(2 \sum_{k=1}^{\infty} \Psi\left(\sqrt{\eta k/2}/k\right)\right), \quad \eta > 0. \]

Before giving the proof we formulate and prove the following auxiliary lemma

**Lemma 3.10.** It holds, that for any \( \eta > 0 \)

\[ v'(\eta) = \exp\left(2 \sum_{k=1}^{\infty} \Psi\left(\sqrt{\eta k/2}/k\right)\right) \left(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{e^{-\eta k}}{\sqrt{k}}\right). \]

**Proof of Lemma 3.10.** To prove the lemma it is sufficient to show that for any \( \eta > 0 \)

\[ \frac{\partial}{\partial \eta} \left( \sum_{k=1}^{\infty} \Psi\left(\sqrt{\eta k/2}/k\right) \right) = \sum_{k=1}^{\infty} \frac{\partial}{\partial \eta} \left(\Psi\left(\sqrt{\eta k/2}/k\right)\right). \]

Take \( a, b > 0 \) such that \( \eta \in [a,b] \), \( f(\eta) = \sum_{k=1}^{\infty} \Psi(\sqrt{\eta k/2}/k) \) and \( f_n(\eta) = \sum_{k=1}^{n} \Psi(\sqrt{\eta k/2}/k), n \in \mathbb{N}. \)

According to paragraph 3.1 p. 385 in [29] to claim the line above it is enough to show that

1) there exists \( \eta_0 \in [a,b] \) such that the sequence \( \{f_n(\eta_0)\}_{n \in \mathbb{N}} \) converges to a finite limit,
2) \( f_n'(\eta), \eta \in [a,b] \) converge uniformly to some function.
The first condition holds since $\Psi(x) < e^{-x^2/2}$ for $x > 0$. For the second condition we need to prove that uniformly for all $\eta \in [a, b]$ it holds, that $\sum_{k=n+1}^{\infty} f_k^j(\eta) \to 0$ as $n \to \infty$. We have
\[
\sum_{k=n+1}^{\infty} f_k^j(\eta) = \sum_{k=n+1}^{\infty} \frac{\phi(\sqrt{\eta}k/2)}{2\sqrt{2k\eta}} = \sum_{k=n+1}^{\infty} \frac{e^{-\eta k/4}}{4\sqrt{\pi k\eta}} \leq Ce^{-C_1n} \to 0, \quad n \to \infty
\]
and the claim holds. \qed

Now we are ready to continue proof of Theorem 2.1 for $\alpha = 1$. By Proposition 2.2(i), we find that
\[
\mathcal{A} := \lim_{\eta \to 0} \frac{\mathcal{H}_0 - \mathcal{H}_\eta}{\sqrt{\eta}} = \lim_{\eta \to 0} \frac{1 - 1/v(\eta)}{\sqrt{\eta}}.
\]
Since $\mathcal{H}_\eta = v(\eta)^{-1} \to 1$ as $\eta \to 0$ (see, e.g., [22]) we conclude that $\lim_{\eta \to 0} v(\eta) = 1$ and hence $\mathcal{A} = \lim_{\eta \to 0} \frac{v(\eta)-1}{\sqrt{\eta}}$. Implementing the L'Hôpital's rule we obtain by Lemma 3.10
\[
\mathcal{A} = \lim_{\eta \to 0} \frac{v'(\eta)}{1/(2\sqrt{\eta})} = 2 \lim_{\eta \to 0} \sqrt{\eta} \exp \left( 2 \sum_{k=1}^{\infty} \frac{\Psi(\sqrt{\eta}k/2)}{k} \right) \left( 1 - \sqrt{\eta} \sum_{k=1}^{\infty} e^{-\eta k/4} \right).
\]
Note that by the definition of $v(\eta)$ observation that $\lim_{\eta \to 0} v(\eta) = 1$ implies
\[
\sqrt{\eta} \exp \left( 2 \sum_{k=1}^{\infty} \frac{\Psi(\sqrt{\eta}k/2)}{k} \right) \sim \frac{1}{\sqrt{\eta}}, \quad \eta \to 0
\]
and hence
\[
\mathcal{A} = \lim_{\eta \to 0} \frac{2}{\sqrt{\eta}} \left( 1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} e^{-\eta k/4} \right).
\]
Let $x = \sqrt{\eta}/2$, thus
\[
\mathcal{A} = \lim_{x \to 0} \frac{1}{x} \left( 1 - \frac{x}{\sqrt{\pi}} \sum_{k=1}^{\infty} e^{-x^2 k} \right) = \lim_{x \to 0} \frac{1}{x} \left( 1 - \frac{x}{\sqrt{\pi}} \log \frac{1}{\sqrt{\pi}} \right),
\]
where $\log$ is the polylogarithm function, see, e.g., [8]. As follows from [34, Eq. (9.3)]
\[
\lim_{x \to 0} \frac{1}{x} \left( 1 - \frac{x}{\sqrt{\pi}} \log \frac{1}{\sqrt{\pi}} \right) = \lim_{x \to 0} \frac{1}{x} \left( 1 - \frac{x}{\sqrt{\pi}} \Gamma(1/2)(x^2)^{-1/2} + \zeta(1/2) + \sum_{k=1}^{\infty} \zeta(1/2 - k)(-x^2)^k/k! \right)
\]
\[
= \frac{\zeta(1/2)}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \lim_{x \to 0} \left( \sum_{k=1}^{\infty} \zeta(1/2 - k)x^{2k}(-1)^k/k! \right).
\]
Thus, to prove the claim is it enough to show that
\[
\lim_{x \to 0} \sum_{k=1}^{\infty} \zeta(1/2 - k)x^{2k}(-1)^k/k! = 0.
\]
By the Riemann functional equation (equation (2.3) in [24]) and observation that $\zeta(s)$ is strictly decreasing for real $s > 1$ we have for any natural number $k$
\[
|\zeta(1/2 - k)| \leq 2^{1/2-k} \pi^{-1/2-k} \Gamma(1/2 + k) \zeta(1/2 + k) \leq 2^{-k} \Gamma(k + 1) \zeta(3/2) = \frac{\zeta(3/2)k!}{2^k}.
\]
Thus, for $|x| < 1$ we have
\[
\left| \sum_{k=1}^{\infty} \zeta(1/2 - k)x^{2k}(-1)^k/k! \right| \leq x^2 \sum_{k=1}^{\infty} \frac{\zeta(1/2 - k)}{k!} \leq x^2 \zeta(3/2) \sum_{k=1}^{\infty} 2^{-k} = \zeta(3/2)x^2
\]
and (14) follows, this completes the proof of case \( \alpha = 1 \).

\[ \square \]

**Proof of Theorem 2.1, \( \alpha = 2 \).** By Proposition 2.2(ii) we have, as \( \delta \to 0 \),

\[
\mathcal{H}_2 - \mathcal{H}_2^\delta = \frac{1}{\sqrt{2\pi}} - \frac{1}{\delta} \left( \Phi\left( \frac{\delta}{\sqrt{2}} \right) - \frac{1}{2} \right) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\sqrt{2}}{\delta} \int_0^{\delta/\sqrt{2}} e^{-x^2/2} \, dx \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\sqrt{2}}{\delta} \int_0^{\delta/\sqrt{2}} \left( 1 - \frac{x^2}{2} \right) \, dx + O(\delta^0) \right) = \frac{1}{\sqrt{2\pi}} \frac{2}{12\sqrt{2}}
\]

and the claim follows. \[ \square \]

3.4. **Proofs of other Results.**

**Proof of Proposition 2.2.** For (i) consult, e.g., [18, 28] and for (ii) see [13, Eq. (2.9)]. \[ \square \]

Let \( \delta \geq 0 \). Define measure \( \mu_\delta \) such that for real numbers \( a \leq b \)

\[
\mu_\delta([a, b]) = \begin{cases} 
\delta \cdot \#(a, b)_\delta, & \delta > 0 \\
b - a, & \delta = 0.
\end{cases}
\]

**Proof of Theorem 2.3.** We have for any \( T > 0, x \geq 1 \) and \( \delta \geq 0 \)

\[
\mathbb{P} \left\{ \xi_\alpha^\delta > x \right\} \leq \mathbb{P} \left\{ \frac{\sup_{t \in \mathbb{R}} Z_\alpha(t)}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} > x \right\}
\]

\[
= \mathbb{P} \left\{ \frac{\sup_{t \in [-T, T]} Z_\alpha(t)}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} > x \text{ and } Z_\alpha(t) \text{ achieves its maximum at } t \in [-T, T] \right\}
\]

\[
+ \mathbb{P} \left\{ \frac{\sup_{t \in \mathbb{R} \setminus [-T, T]} Z_\alpha(t)}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} > x \text{ and } Z_\alpha(t) \text{ achieves its maximum at } t \in \mathbb{R} \setminus [-T, T] \right\}
\]

\[
\leq \mathbb{P} \left\{ \frac{\sup_{t \in [-T, T]} Z_\alpha(t)}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} > x \right\} + \mathbb{P} \left\{ \exists t \in \mathbb{R} \setminus [-T, T] : Z_\alpha(t) > 0 \right\}
\]

\[
\leq \mathbb{P} \left\{ \frac{\sup_{t \in [-T, T]} Z_\alpha(t)}{\int_{[-T, T]} e^{Z_\alpha(t)} \, d\mu_\delta} > x \right\} + \mathbb{P} \left\{ \exists t \geq T : Z_\alpha(t) > 0 \right\}
\]

\[ (15) \quad =: p_1(T, x) + 2p_2(T). \]

**Estimation of \( p_2(T) \).** By the self-similarity of fBm we have

\[
p_2(T) \leq \sum_{k=1}^{\infty} \mathbb{P} \left\{ \exists t \in [kT, (k+1)T] : \sqrt{2}B_\alpha(t) - t^{\alpha} > 0 \right\}
\]

\[
= \sum_{k=1}^{\infty} \mathbb{P} \left\{ \exists t \in [1, 1 + \frac{1}{k}] : \sqrt{2}B_\alpha(t)(kT)^{\alpha/2} > (kT)^{\alpha}t^{\alpha} \right\}
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{P} \left\{ \exists t \in [1, 2] : \sqrt{2}B_\alpha(t) > (kT)^{\alpha/2} \right\}
\]

and thus using Borel-TIS inequality, we find that for all \( T \geq 1 \)

\[ (16) \quad p_2(T) \leq \sum_{k=1}^{\infty} C e^{-\frac{(kT)^{\alpha}}{10}} \leq C e^{-\frac{T^{\alpha}}{10}}. \]
Estimation of \(p_1(T, x)\). Observe that for \(T, x \geq 1\) and \(\delta \in [0, 1]\),

\[
p_1(T, x) \leq \mathbb{P}\left\{ \frac{\sum_{k=-T}^{T-1} a_k(\omega)}{\sum_{k=-T}^{T-1} b_k(\omega)} > x \right\} =: \mathbb{P}\left\{ \frac{\sum_{k=-T}^{T-1} a_k(\omega)}{\sum_{k=-T}^{T-1} b_k(\omega)} > x \right\}.
\]

Since the event \(\{\sum_{k=-T}^{T-1} a_k(\omega)/\sum_{k=-T}^{T-1} b_k(\omega) > x\}\) implies \(\{a_k(\omega)/b_k(\omega) > x, \text{ for some } k \in [-T, T-1]\}\), thus

\[
\mathbb{P}\left\{ \frac{\sum_{k=-T}^{T-1} a_k(\omega)}{\sum_{k=-T}^{T-1} b_k(\omega)} > x \right\} \leq \mathbb{P}\left\{ \sum_{k=-T}^{T-1} a_k(\omega) > x \right\} \leq 2T \sup_{k \in [-T, T]} \mathbb{P}\left\{ \sup_{t \in [k, k+1]} e^{Z_\alpha(t)} \int_{[k, k+1]} e^{Z_\alpha(t)} d\mu_\delta > x \right\}.
\]

Therefore, we obtain that for \(x, T \geq 1\)

\[
p_1(T, x) \leq 2T \sup_{k \in [-T, T]} \mathbb{P}\left\{ \sup_{t \in [k, k+1]} e^{Z_\alpha(t)} \int_{[k, k+1]} e^{Z_\alpha(t)} d\mu_\delta > x \right\}.
\]

Next we have by the stationarity of the increments of fBm for \(x, T \geq 1\)

\[
\mathbb{P}\left\{ \sup_{t \in [k, k+1]} e^{Z_\alpha(t)} \int_{[k, k+1]} e^{Z_\alpha(t)} d\mu_\delta > x \right\} \leq \mathbb{P}\left\{ \sup_{t \in [k, k+1]} e^{Z_\alpha(t)} > x \right\} \leq \mathbb{P}\left\{ \exists t, s \in [k, k+1]: B_\alpha(t) - B_\alpha(s) > \frac{\log(\frac{\delta}{T})}{\sqrt{2}} \right\}
\]

\[
\leq \mathbb{P}\left\{ \exists t, s \in [k, k+1]: B_\alpha(t) - B_\alpha(s) > \frac{\log(\frac{\delta}{T}) - \sup_{t, s \in [k, k+1]} \left( |t|^\alpha - |s|^\alpha \right)}{\sqrt{2}} \right\},
\]

\[
\leq \mathbb{P}\left\{ \exists t \in [0, 1]: B_\alpha(t) > \frac{\log x - C \max(1, T^{\alpha-1})}{\sqrt{2}} \right\},
\]

where in the second line we used that \(\mu_\delta[k, k+1] \geq 1/2\) for \(\delta \in [0, 1]\). Thus, for \(x, T \geq 1\)

\[
(17) \quad p_1(T, x) \leq 2T \mathbb{P}\left\{ \exists t \in [0, 1]: B_\alpha(t) > \frac{\log x - C \max(1, T^{\alpha-1})}{\sqrt{2}} \right\}
\]

and combining the statement above with (15) and (16) we have for \(x, T \geq 1\)

\[
\mathbb{P}\left\{ \sup_{t \in \mathbb{R}} e^{Z_\alpha(t)} \int_{\mathbb{R}} e^{Z_\alpha(t)} d\mu_\delta > x \right\} \leq \tilde{C} e^{-\frac{x^{\alpha}}{10}} + 2T \mathbb{P}\left\{ \exists t \in [0, 1]: B_\alpha(t) > \frac{\log x - C \max(1, T^{\alpha-1})}{\sqrt{2}} \right\}.
\]

Assume that \(\alpha \leq 1\). Then choosing \(T = x\) in the line above we have by Borell-TIS inequality for all \(x \geq 1\)

\[
\mathbb{P}\left\{ \sup_{t \in \mathbb{R}} e^{Z_\alpha(t)} \int_{\mathbb{R}} e^{Z_\alpha(t)} d\mu_\delta > x \right\} \leq C_1 e^{-C_2 \log^2 x}.
\]

Assume that \(\alpha > 1\). Taking \(T = C' (\log x)^{\frac{1}{\alpha-1}}\) with sufficiently small \(C' > 0\) we obtain for \(x \geq 1\)

\[
\mathbb{P}\left\{ \sup_{t \in \mathbb{R}} e^{Z_\alpha(t)} \int_{\mathbb{R}} e^{Z_\alpha(t)} d\mu_\delta > x \right\} \leq \tilde{C} e^{-C''(\log x)^{\frac{\alpha}{\alpha-1}}} + C_1 e^{-C_2 \log^2 x} \leq C_3 e^{-C \log^2 x}
\]

and the claim follows. \(\square\)

**Proof of Theorem 2.4.** Observe that \(|x^p - y^p| \leq p |x - y|(x^{p-1} + y^{p-1})\) for all \(x, y \geq 0\) and \(p \geq 1\), this can be shown straightforwardly by the differentiation. Hence we have

\[
\left| \mathbb{E}\left\{ (\xi_\alpha(T))^p \right\} - \mathbb{E}\left\{ (\xi_\alpha^\delta)^p \right\} \right| \leq \mathbb{E}\left\{ (\xi_\alpha(T))^p - (\xi_\alpha^\delta)^p \right\} \leq p \mathbb{E}\left\{ |\xi_\alpha(T) - \xi_\alpha^\delta| \cdot (|\xi_\alpha(T)|^{p-1} + |\xi_\alpha^\delta|^{p-1}) \right\}
\]
We have
\[
\beta := \frac{\sup_{t \in \mathbb{Z}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} \cdot \frac{\sup_{t \in [-T,T]} e^{Z_\alpha(t)}}{\int_{[-T,T]} e^{Z_\alpha(t)} \, d\mu_\delta} = \beta_1 - \beta_2 \beta_3,
\]
where
\[
\beta_1 = \frac{\sup_{t \in \mathbb{Z}} e^{Z_\alpha(t)} - \sup_{t \in [-T,T]} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \, d\mu_\delta} \geq 0,
\beta_2 = \frac{\sup_{t \in [-T,T]} e^{Z_\alpha(t)}}{\int_{[-T,T]} e^{Z_\alpha(t)} \, d\mu_\delta} > 0,
\beta_3 = \frac{\sup_{t \in [-T,T]} e^{Z_\alpha(t)}}{\int_{[-T,T]} e^{Z_\alpha(t)} \, d\mu_\delta} > 0.
\]
After applying Hölder inequality we obtain
\[
E \{ \beta^2 \} \leq 2E \{ \beta_1^2 \} + 2\sqrt{E \{ \beta_2^4 \} E \{ \beta_3^4 \}}.
\]
We have by (17) that for \( x, T \geq 1 \)
\[
P \{ \beta_3 > x \} \leq 2T P \left\{ \exists t \in [0,1]: B_\alpha(t) > \frac{\log x - \max(1,T^{\alpha-1})}{\sqrt{2}} \right\}
\]
implies that for \( T \geq 1 \)
\[
E \{ \beta_3^4 \} = \int_0^\infty P \{ \beta_2 > x^{1/4} \} \, dx
\]
\[
\leq 2T \int_0^\infty P \left\{ \exists t \in [0,1]: B_\alpha(t) > \frac{\log x - \max(1,T^{\alpha-1})}{\sqrt{2}} \right\} \, dx
\]
\[
\leq 2T \left( \int_0^{\exp(5\max(1,T^{\alpha-1}) - 1)} 1 \, dx + \int_{\exp(5\max(1,T^{\alpha-1}) - 1)}^\infty P \{ \exists t \in [0,1]: B_\alpha(t) > C_3 \log x \} \, dx \right)
\]
\[
\leq C_1 e^{C \max(1,T^{\alpha-1})}.
\]
Finally for \( \alpha \in (0,2) \) and \( T \geq 1 \) we have
\[
E \{ \beta_3^4 \} \leq C_1 e^{C \max(1,T^{\alpha-1})}.
\]
Next, we focus on properties of \( \beta_2 \). We have for \( k > 0 \) and sufficiently large \( T \)
\[
P \left\{ \int_{[kT,(k+1)T]} e^{Z_\alpha(t)} \, d\mu_\delta > e^{-1/2} T^{\alpha k^2} \right\} \leq P \left\{ (T + 1) \sup_{t \in [kT,(k+1)T]} e^{Z_\alpha(t)} > e^{-1/2} T^{\alpha k^2} \right\}
\]
\[
= P \left\{ \log(T + 1) + \sup_{t \in [kT,(k+1)T]} Z_\alpha(t) > \frac{1}{2} T^{\alpha k^2} \right\}
\]
\[
= P \left\{ \exists t \in [kT,(k+1)T]: \frac{B_\alpha(t)}{t^{\alpha/2}} > \frac{t^{\alpha/2} - \frac{1}{2} T^{\alpha k^2} - \log(T + 1)}{\sqrt{2t^{\alpha/2}}} \right\}
\]
\[
\leq P \left\{ \exists t \in [kT,(k+1)T]: \frac{B_\alpha(t)}{t^{\alpha/2}} > (Tk)^{\alpha/2}/3 \right\},
\]
which, according to Borell-TIS inequality, is upper-bounded by $e^{-e^{T\alpha}/19}$. By the lines above we obtain that with probability at least $1 - \sum_{k \in \mathbb{Z}\setminus\{0\}} e^{-|k|^{2}\alpha T^{\alpha}/19} \geq 1 - e^{-T^{\alpha}/20}$ for large $T$

$$\int_{\mathbb{R}\setminus[-T,T]} e^{Z_{\alpha}(t)} \, d\mu_{\delta} \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} e^{-\frac{\alpha}{2} T^{\alpha}|k|^{\alpha}} \leq e^{-T^{\alpha}/3}.$$ Combining everything together we obtain that for sufficiently large $T$

$$\Pr \left\{ \int_{\mathbb{R}\setminus[-T,T]} e^{Z_{\alpha}(t)} \, d\mu_{\delta} > e^{-T^{\alpha}/3} \right\} \leq e^{-T^{\alpha}/20}. \tag{21}$$

Next we notice that for $\beta_{2}$

$$\Pr \left\{ \int_{[-T,T]} e^{Z_{\alpha}(t)} \, d\mu_{\delta} < e^{-\frac{T^{\alpha}}{4}} \right\} \leq \Pr \left\{ \int_{[0,1]} e^{Z_{\alpha}(t)} \, dt \mu_{\delta} < e^{-\frac{T^{\alpha}}{4}} \right\} \leq \Pr \left\{ \sup_{t \in [0,1]} Z_{\alpha}(t) < -\frac{T^{\alpha}}{4} \right\}$$

so by Borell-TIS inequality, the above is bounded by $e^{\frac{2T^{\alpha}}{65}}$ for all sufficiently large $T$. This result, in combination with (21) gives us $\Pr \{ \beta_{2} > e^{-T^{\alpha}/12} \} \leq e^{-T^{\alpha}/21}$ for all sufficiently large $T$. Thus, since $\beta_{2} \in [0,1]$ from the line above we immediately obtain that

$$\mathbb{E} \{ \beta_{2}^{2} \} \leq C_{1}e^{-CT^{\alpha}}. \tag{22}$$

for $T \geq 1$. By (16) we observe that

$$\Pr \{ \beta_{1} > 0 \} \leq \Pr \{ \exists t \notin [-T,T] : Z_{\alpha}(t) > 0 \} \leq 2p_{2}(T) \leq e^{-CT^{\alpha}}.$$

Next by Theorem 2.3 we have for $x \geq 1$ that

$$\Pr \{ \beta_{1} > x \} \leq \Pr \left\{ \sup_{t \in \mathbb{Z}} e^{Z_{\alpha}(t)} \int_{\mathbb{R}} e^{Z_{\alpha}(t)} \, d\mu_{\delta} > x \right\} \leq C_{1}e^{-C_{2}\log^{2}x}$$

and thus $\mathbb{E} \{ \beta_{1}^{2} \} < C$ with positive constant $C$ that does not depend on $T$. With $A_{T} := \{ \beta_{1}(\omega) > 0 \}$, by Hölder inequality for large $T$ we have

$$\mathbb{E} \{ \beta_{1}^{2} \} = \mathbb{E} \{ \beta_{1}^{2} \cdot \mathbb{I}(A_{T}) \} \leq \sqrt{\mathbb{E} \{ \beta_{1}^{4} \}} \sqrt{\mathbb{E} \{ \mathbb{I}(A_{T}) \}} \leq C_{1}e^{-C_{2}T^{\alpha}}.$$

By the line above and (20), (22) and (18) we obtain for $T \geq 1$

$$\mathbb{E} \{ \beta^{2} \} \leq C_{2}e^{-C_{1}T^{\alpha}}. \tag{23}$$

Our next aim is estimation of $\mathbb{E} \{ \kappa_{p} \}$. We have for $T \geq 1$ that

$$\mathbb{E} \left\{ (\xi_{\alpha}(T))^{2p-2} \right\} = \int_{0}^{\infty} \Pr \{ \xi_{\alpha}(T) > x^{\frac{1}{2p-2}} \} \, dx = \int_{0}^{\infty} \Pr \{ p_{1}(T, x^{\frac{1}{2p-2}}) \} \, dx \leq \int_{0}^{\infty} \Pr \{ \exists t \in [0,1] : B_{\alpha}(t) > \frac{\log x^{\frac{1}{2p-2}} - C \max(1, T^{\alpha-1})}{\sqrt{2}} \} \, dx.$$ By the same arguments, as in Eq. (19), the last integral above does not exceed $C_{1}e^{\max \{C_{2}(T^{\alpha-1}, 1)\}$ and since by Theorem 2.3 $\xi_{\alpha}^{\delta}$ has all finite moments uniformly bounded for all $\delta \geq 0$ we obtain that for $T \geq 1$

$$\kappa_{p} \leq C_{1}e^{\max \{C_{2}(T^{\alpha-1}, 1)\}}.$$ Combining the bound above with (23) we have $\sqrt{2p} \mathbb{E} \{ \beta^{2} \} \mathbb{E} \{ \kappa_{p} \} \leq e^{-C_{1}T^{\alpha}}$ for sufficiently large $T$ and the claim follows. \hfill \Box
Proof of Corollary 2.5. Case $\alpha = 1$. First we show that $v(\eta)$ is an increasing function for $\eta > 0$, that is equivalent with fact that $v'(\eta) > 0$ for $\eta > 0$. In the light of Lemma 3.10 it is sufficient to show 

$$\frac{\sqrt{\eta}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} e^{-\frac{\eta}{4k}} < 1, \quad \eta > 0.$$ 

We have

$$\sqrt{\eta} \sum_{k=1}^{\infty} \frac{\sqrt{k}}{2\sqrt{\pi}} e^{-\frac{\eta k}{4}} < \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{\eta k}{4}} z^{-1/2} dz = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{\eta x}{4}} \left(\frac{\eta z}{4}\right)^{-1/2} d\left(\frac{\eta z}{4}\right) = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = 1$$

and hence $H^0_1 = 1/v(\eta)$ is decreasing for $\eta > 0$. Since by the classical definition $H^0_1 > H^0_\eta$ for any $\eta > 0$ we obtain the claim.

Case $\alpha = 2$. We have by Proposition 2.2(ii)

$$H^\delta_2 = \frac{2}{\delta} \left( \Phi(\delta/\sqrt{2}) - \frac{1}{2} \right) = \frac{2}{\delta \sqrt{2\pi}} \int_{0}^{\delta/\sqrt{2}} e^{-x^2/2} dx = \frac{1}{\eta \sqrt{\pi}} \int_{0}^{\eta} e^{-x^2/2} dx,$$

where $\eta = \delta/\sqrt{2}$. The derivative of the last integral above with respect to $\eta$ equals

$$\frac{1}{\sqrt{\pi}} \left( \frac{1}{\eta^2} \int_{0}^{\eta} e^{-x^2/2} dx + \frac{1}{\eta} e^{-\eta^2/2} \right) = \frac{1}{\sqrt{\pi} \eta^2} \left( \int_{0}^{\eta} (e^{-\eta^2/2} - e^{-x^2/2}) dx \right) < 0$$

and the claim follows.

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