ON GRADED $E_\infty$-RINGS AND PROJECTIVE SCHEMES IN SPECTRAL ALGEBRAIC GEOMETRY

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Abstract We define $\mathbb{N}$-graded and $\mathbb{Z}$-graded $E_\infty$-rings, by using an $\infty$-operad constructed from the symmetric monoidal categories $\{0\} \subset \mathbb{N} \subset \mathbb{Z}$ and the Day convolution. We consider the universality of one-element localization, and obtain projective schemes. We define a certain finiteness assumption on an $\mathbb{N}$-graded $E_\infty$-ring $A$ and $\mathbb{Z}$-graded $A$-modules and show that, under the finiteness assumption, the $\infty$-category of perfect quasi-coherent sheaves over Proj $A$ is equivalent to the localization of a certain $\infty$-category of $\mathbb{Z}$-graded $A$-modules with respect to an $\infty$-subcategory of locally compact $\mathbb{Z}$-compact graded $A$-modules which are gradedly bounded above.

1. INTRODUCTION

Let $k$ be a field, and $A$ a commutative ring of finite type over $k$. As is well-known, the category of $A$-modules is corresponding to the category of quasi-coherent sheaves over Spec $A$. For a finitely generated commutative graded algebra $A = \oplus_{n \geq 0} A_n$ over $k$, where $A$ is generated by $A_1$ over $A_0$, the category of quasi-coherent sheaves over Proj $A$ is equivalent to the quotient category of the category of graded $A$-modules with the subcategory of graded $A$-modules gradedly bounded above [18, Corollary-Definition 0.3]. The construction in the algebraic geometry goes back to Serre [16], so that the correspondence is called the Serre theorem, and has been studied by Gabriel, Manin, and so on.

In noncommutative geometry and differential geometry, there are many approaches to define projective variety via abelian category (cf.[1]) and derived category, which already have been considered in 1990s after Serre. However, the graded module categories still provide fruitful observations of invariants via the Serre theorem. For example calculations of cyclic homology and algebraic $K$-theory of polynomials or $\mathbb{F}_1$-algebras, even when the coefficient ring is not commutative.

Mandell, May, Schwede and Shipley defined the symmetric monoidal structure of the category of $\mathcal{D}$-spectra in [12], which is given by the Day convolution (cf. [12, Definition 1.9,
Definition 21.4) which is introduced at first in [2]. The theory of \( \mathcal{D} \)-spectra can be applied to the category of \( \mathbb{Z} \)-indexed spectra. Recall that the symmetric monoidal structure on the category is the coend

\[
\int_{(c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}} \text{Hom}_\mathbb{Z}(c_1 + c_2, -) \otimes X_{c_1} \otimes X_{c_2},
\]

and we also have the inner hom \([X, Y]_{\hat{\otimes}}\) by the end

\[
\int_{(c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}} \text{Map(Hom}_\mathbb{Z}(c_1 + c, c_2), \text{Map}(X_{c_1}, Y_{c_2})).
\]

There already have been the several generalization of \( \mathcal{D} \)-spectra, especially \( \mathbb{Z} \)-index, to the \( \infty \)-category theory. One of the main points is to define suitable \( \mathbb{Z} \)-grading on spectra and the monoidal structure on them in the setting of \( \infty \)-category, which is mainly introduced by Lurie in his papers [7], [6].

In [7], the constant simplicial set \( N_\Delta(\mathbb{Z}) \) plays a role of degree zero part on \( \mathbb{Z} \)-grading in the setting of \( E_2 \)-rings.

Another spectra with \( \mathbb{Z} \)-grading together with the Day convolution appears in [6, Section 2.2.6, Proposition 6.3.1.12], which is given by the generalized \( \infty \)-operad \( \mathbb{Z} \times N_\Delta(\mathbb{F}^{fin}) \) in the setting of \( E_1 \)-rings.

Our construction is slightly different from these constructions since we require the \( E_\infty \)-ring structure to consider a spectral projective scheme by glueing covering sieves. We use \( \infty \)-operads obtained by the symmetric monoidal categories \( \{0\} \subset \mathbb{N} \subset \mathbb{Z} \) whose morphisms are just isomorphisms.

We define \( \mathbb{Z} \)-graded and \( \mathbb{N} \)-graded \( E_\infty \)-rings by lax monoidal functors from \( N_\Delta(\mathbb{O}_Z^\otimes) \) and \( N_\Delta(\mathbb{O}_N^\otimes) \) to \( \text{Sp}^\otimes \) respectively, where \( \mathbb{O}_Z^\otimes \) and \( \mathbb{O}_N^\otimes \) are categories given by colored operad obtained by the monoidal structure of \( \mathbb{Z} \) and \( \mathbb{N} \) with respect to +. The \( \infty \)-category of those functors admits the Day convolution [6] [15], which we write as \( \hat{\otimes} \). It makes \( \text{Fun}(N_\Delta(\mathbb{O}_Z^\otimes), \text{Sp}^\otimes)\) the right closed symmetric monoidal \( \infty \)-category, so that \( \text{Fun}(N_\Delta(\mathbb{O}_Z^\otimes), \text{Sp}^\otimes)\) naturally inherits the structure of enriched \( \infty \)-category [4]. By using this fact, we define the localization of graded \( E_\infty \)-rings with respect to one element.

We remark that our construction also makes sense for ordinary graded rings if we replace \( \infty \)-operads \( N_\Delta(\mathbb{O}^\otimes) \) with a category \( \mathbb{O}^\otimes \) obtained from colored operads as in Example 3.14.

We define projective spectral schemes as a subsheaf of a certain affine spectral scheme by specifying certain collections of morphisms in the setting of Lurie [5]. Our notion of projective spectral schemes includes the notion of projective spaces in [11, Section 5.4.1]. Gabriel-Zisman [3] showed that the localization of category corresponds to a certain thick subcategory. The Serre theorem said that the localization with respect to sheafification of finitely presented modules specifies the thick subcategory consisting of gradedly bounded above modules. We consider an assumption in spectral algebraic geometry which corresponds to
the “finiteness” of ordinary graded rings and modules. We generalize the Serre theorem as follows.

**Theorem 1.1.** Let $R$ be an $E_{\infty}$-ring and $A$ an $\mathbb{N}$-graded $E_{\infty}$-ring. Assume the condition in Definition 4.1 holds. Let $\text{Mod}^{\text{f c p t},\text{Agr}}_{R}$ be the $\infty$-category locally compact $\mathbb{Z}$-compact graded $A$-modules over $R$ defined in Definition 4.1. Let $N_{i} l_{R}$ be the full $\infty$-subcategory of $\text{Mod}^{\text{f c p t},\text{Agr}}_{R}$ spanned by those objects whose essential images via the Serre twist defined in Definition 3.9 is zero for sufficiently large positive $n \in \mathbb{N}$. Let $\text{Mod}^{\text{f c p t},\text{Agr}}_{R}/N_{i} l_{R}$ be the Dwyer-Kan localization with respect to the morphisms $M \to N$ in $\text{Mod}^{\text{f c p t},\text{Agr}}_{R}$ such that it induces $M_{i} \simeq N_{i}$ for sufficiently large positive $i \in \mathbb{N}$. Then, a functor

$$\text{QCoh}(\text{Proj} A)^{\text{perf}} \to \text{Mod}^{\text{f c p t},\text{Agr}}_{R}/N_{i} l_{R}$$

which sends a quasi-coherent perfect sheaf $\mathcal{F}$ on $\text{Proj} A$ to the coproduct of global sections $\coprod_{n \in \mathbb{Z}} \Gamma(\text{Proj} A, \mathcal{F}[n])$ gives an equivalence of $\infty$-categories.

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2. **Day Convolution on $\text{Fun}_{N_{\Delta}(\text{Fin}_{+})}(N_{\Delta}(\mathcal{O}^{\circ}), \text{Sp}^{\circ})$**

Set $\langle n \rangle = \{*, 1, 2, \ldots, n\}$. Let $\text{Fin}_{+}$ be the category of finite pointed sets $\langle n \rangle$ whose base point is $\ast$. Morphisms in $\text{Fin}_{+}$ are maps of pointed finite sets [6, Notation 2.0.0.2, Remark 4.1.1.4]. A map $f : \langle m \rangle \to \langle n \rangle$ in $\text{Fin}_{+}$ is called inert if $f^{-1}(i)$ has exactly one element for each $1 \leq i \leq n$. For every pair of integers $1 \leq i \leq n$, let $\rho^{i} : \langle n \rangle \to \langle 1 \rangle$ denote the morphism given by the formula

$$\rho^{i}(j) = \begin{cases} 1 & (i = j) \\ \ast & (i \neq j). \end{cases}$$

Let $p : X \to S$ be an inner fibration of simplicial sets. An edge $f : x \to y$ of $X$ is $p$-Cartesian if the induced map on overcategories (cf.[5] Proposition 1.2.9.2)

$$X_{l f} \to X_{l y} \times_{S_{l p(y)}} S_{l p(f)}$$

is a trivial fibration of simplicial sets. An edge $f : x \to y$ of $X$ is $p$-coCartesian [5, Definition 2.4.1.3] if it is Cartesian with respect to the inner fibration $p^{\text{op}} : X^{\text{op}} \to S^{\text{op}}$.

A map $p : X \to S$ of simplicial sets is a Cartesian fibration if $p$ is an inner fibration and for every edge $f : x \to y$ in $S$ and every vertex $\tilde{y}$ of $X$ with $p(\tilde{y}) = y$, there exists a $p$-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$. The map $p$ is called coCartesian fibration if $p : X^{\text{op}} \to S^{\text{op}}$ is a Cartesian fibration [6, Definition 2.4.2.1].
Recall that a symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : C^\otimes \to N_\Delta(\text{Fin}_*)$ with the following property: the functors $\rho_i^j : C^\otimes_{(n)} \to C^\otimes_{(1)}$, induced from maps $\rho_i^j : (n) \to (1)_i \leq i \leq n$, determine an equivalence $C^\otimes_{(n)} \to (C^\otimes_{(1)})^n$ [6, Definition 2.0.0.7].

**Example 2.1** ([6], Remark 2.0.0.5, Construction 2.1.1.7). Let $(C, \otimes)$ be the symmetric monoidal category. We define a category $\mathcal{C}^\otimes$ by setting [6, 2.0.0.1, Example 2.1.1.5]

$$\text{Mul}_C([X_i]_{i \in I}, Y) = \text{Hom}_C(\otimes_{i \in I} X_i, Y).$$

From Construction [6, Construction 2.1.1.7], we obtain a category $\mathcal{O}^\otimes$ from a colored operad $\mathcal{O}$.

A morphism $\partial$ in an $\infty$-operad is said to be inert if it is $p$-coCartesian, its image in $N_\Delta(\text{Fin}_*)$ is inert and its image in $\mathcal{C}$ is an equivalence. For a symmetric monoidal $\infty$-category $p : C^\otimes \to N_\Delta(\text{Fin}_*)$, a commutative algebra object of $C^\otimes$ is a section $N_\Delta(\text{Fin}_*) \to C^\otimes$ of $p$ such that it carries an inert morphism in $N_\Delta(\text{Fin}_*)$ to an inert morphism in $C^\otimes$ [6, Definition 2.1.2.7].

We recall the Day convolution on $\infty$-category.

**Definition 2.2.** Let $C^\otimes \to N_\Delta(\text{Fin}_*)$ be a symmetric monoidal $\infty$-category. Let $\text{Fun}(N_\Delta(C^\otimes_{Z}), C^\otimes)$ be the $\infty$-category of functors from $N_\Delta(C^\otimes_{Z})$ to $C^\otimes$. By the construction [15, Proposition 2.11] and [15, Lemma 2.13], we obtain a symmetric monoidal $\infty$-category $\text{Fun}_{N_\Delta(\text{Fin}_*)}(N_\Delta(C^\otimes_{Z}), C^\otimes)^\hat{} \to N_\Delta(\text{Fin}_*)$ by the Day convolution which is denoted by $\hat{}$.

Let $X$ be an object in $\text{Fun}_{N_\Delta(\text{Fin}_*)}(N_\Delta(C^\otimes_{Z}), C^\otimes)^\hat{}$. We denote by $X_i$ the value at $i \in Z$ of the underlying functor of $X$.

**Proposition 2.3** ([15], Lemma 2.13). *If the tensor product on the symmetric monoidal $\infty$-categories $C^\otimes$ and $D^\otimes$ commutes with colimits in each variable separately, the tensor product on $\text{Fun}_{N_\Delta(\text{Fin}_*)}(C^\otimes, D^\otimes)^\otimes$ also commutes with each variable separately.*

Now we introduce $\infty$-operads which we mainly use in this paper.

**Definition 2.4.** We can regard $\mathbb{N}$ and $\mathbb{Z}$ as a symmetric monoidal categories whose morphisms are just isomorphisms and the monoidal structure is obtained by $\cdot$. We can make a colored operad $\mathcal{O}_{\mathbb{N}}$ from $\mathbb{N}$ and $\mathbb{Z}$ respectively by setting $\text{Mul}([X_i]_{i \in I}, Y) = \text{Hom}(\otimes_{i \in I} X_i, Y)$. Then, we obtain a symmetric monoidal $\infty$-categories $N_\Delta(C^\otimes_{\mathbb{N}}) \to N_\Delta(\text{Fin}_*)$ and $N_\Delta(C^\otimes_{\mathbb{Z}}) \to N_\Delta(\text{Fin}_*)$ by the construction as in [6, Construction 2.1.2.21] whose underlying $\infty$-category is $N_\Delta(\mathbb{N})$ $N_\Delta(\mathbb{Z})$ respectively.

**Remark 2.5.** Let us consider a category $[0]$ consisting of one element $[0] \subset \mathbb{Z}$ regarded as the full subcategory of the symmetric monoidal category $\mathbb{Z}$ with respect to $\cdot$. The inclusions $[0] \subset \mathbb{N} \subset \mathbb{Z}$ are the symmetric monoidal functors. Then, we have a forgetful functor from $[0]$ to the Segal category $\text{Fin}_*$, and it is an isomorphism of categories.
Proposition 2.6. Let \( \mathcal{C} \) be a small symmetric monoidal \( \infty \)-category such that \( \text{Ind}(\mathcal{C}^{cpt}) \cong \mathcal{C} \), where \( \mathcal{C}^{cpt} \) is the full \( \infty \)-subcategory of \( \mathcal{C} \) consisting of \( \omega \)-compact objects in \( \mathcal{C} \) and assumed to be closed under the tensor product. We have a morphism \( j' : \text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O} \otimes \mathbb{Z}), \text{Ind}(\mathcal{C}^{cpt})\right) \rightarrow \text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O} \otimes \mathbb{Z}), \text{Ind}(\mathcal{C})\right) \) induced by Yoneda embedding. Here, for an \( \infty \)-category \( X \) with a symmetric monoidal structure, \( \text{Ind}(X) \) inherits the tensor product induced by the Day convolution. Then, the following morphism induced by the Yoneda embedding is an equivalence

\[
\text{Ind}(\text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O} \otimes \mathbb{Z}), \mathcal{C}^{cpt}\right)) \rightarrow \text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O} \otimes \mathbb{Z}), \text{Ind}(\mathcal{C})\right).
\]

Proof. By [15, Theorem 3.2], Yoneda embedding is symmetric monoidal and fully faithful. By [5, Proposition 5.3.4.13], the functor in right hand side is \( \omega \)-compact if and only if it values in \( \mathcal{C}^{cpt} \). □

3. \( \mathbb{Z} \)-Graded (\( \mathbb{N} \)-Graded) \( \mathbb{E}_\infty \)-Rings and \( \mathbb{Z} \)-Graded Modules

We use the notation \( \text{Alg}_{\mathcal{O}'/\mathcal{O}}(M) \) for the fibration \( M \rightarrow \mathcal{O} \) of \( \infty \)-operads and given \( \infty \)-operads \( \mathcal{O}' \rightarrow \mathcal{O} \) as in [6, Definition 2.1.3.1].

We will define a \( \mathbb{Z} \)-graded (\( \mathbb{N} \)-graded) \( \mathbb{E}_\infty \)-ring \( A \) and graded \( A \)-module to be a monoid objects and \( A \)-module objects with respect to the graded tensor as follows.

Definition 3.1. (i) We define the \( \infty \)-category of \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-rings by

\[
\text{Alg}_{N\Delta}(\mathcal{F}_{\text{fin}})/N\Delta(\mathcal{F}_{\text{fin}})(\text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O} \otimes \mathbb{Z}), \text{Sp}^{\otimes}\right)),
\]

and call its objects \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-rings.

(ii) We say that a \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-ring \( A \) is connective if each \( A_i \) for \( i \in \mathbb{Z} \) (as in Definition 2.2) is a connective spectrum.

(iii) The inclusion map \( \mathbb{N} \rightarrow \mathbb{Z} \) is symmetric monoidal, and induces a map \( N\Delta(\mathcal{O}^{\otimes}) \rightarrow N\Delta(\mathcal{O}^{\otimes}) \) of \( \infty \)-operads. We say that a \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-ring is \( \mathbb{N} \)-graded if the underlying functor values 0 at any negative integer \( n \in \mathbb{Z} \).

(iv) Let \( R \) be an \( \mathbb{E}_\infty \)-ring. We define the \( \infty \)-category of \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-rings over \( R \) by

\[
\text{Alg}_{N\Delta}(\mathcal{F}_{\text{fin}})/N\Delta(\mathcal{F}_{\text{fin}})(\text{Fun}_{N\Delta}(\mathcal{F}_{\text{fin}})\left(\mathcal{N}(\mathcal{O}^{\otimes}), \text{Mod}^{\otimes}_R\right)),
\]

and call its objects \( \mathbb{Z} \)-graded \( \mathbb{E}_\infty \)-rings over \( R \).

Let us denote the \( \infty \)-category of \( \mathbb{Z} \)-graded and \( \mathbb{N} \)-graded \( \mathbb{E}_\infty \)-rings by \( \text{CAlg}^{gr\mathbb{Z}} \) and \( \text{CAlg}^{gr\mathbb{N}} \) respectively. We usually identify objects of \( \text{CAlg}^{gr\mathbb{N}} \) with that of \( \text{CAlg}^{gr\mathbb{Z}} \).

We denote by \( \text{CAlg}_{R}^{gr\mathbb{Z}} \) and \( \text{CAlg}_{R}^{gr\mathbb{N}} \) the \( \infty \)-category of \( \mathbb{Z} \)-graded and \( \mathbb{N} \)-graded \( \mathbb{E}_\infty \)-rings over \( R \).
Definition 3.2. Consider the following diagram

\[
\begin{array}{ccc}
N_\Delta(O^\otimes_Z)^{\oplus} & \xrightarrow{i} & N_\Delta(O^\otimes_Z) \\
\wedge & A & \wedge \\
N_\Delta(F^\otimes_{\text{Fin}_+}) & \xrightarrow{0} & N_\Delta(F^\otimes_{\text{Fin}_+})
\end{array}
\]

where 0 is obtained by the symmetric monoidal inclusion map \(\{0\} \to \mathbb{Z}\) and \(i\) is induced by the inclusion. Here we denote by \(\hat{A}\) the induced morphisms obtained by the left Kan extensions of \(A\) along \(i\) and by \(A_0\) the induced morphism by composing 0 respectively. The extension \(\hat{A}\) of \(A\) is a colimit diagram; we denote \(A(\infty)\) the colimit of \(A\), where \(\infty\) denotes the cone point. We call \((-)_0\) the truncation functor.

Proposition 3.3. Under the notation of Definition 3.2, assume that \(A\) is a \(\mathbb{Z}\)-graded \(\mathbb{E}_\infty\)-ring. Then,

(i) \(A_0\) is an \(\mathbb{E}_\infty\)-ring, i.e., \(A_0\) is an object in \(\text{Alg}_{N_\Delta(F^\otimes_{\text{Fin}_+})}/N_\Delta(F^\otimes_{\text{Fin}_+})\) \((\text{Sp}^\otimes, \hat{\otimes})\),

(ii) \(A(\infty)\) is an \(\mathbb{E}_\infty\)-ring, i.e., an object in \(\text{Alg}_{N_\Delta(F^\otimes_{\text{Fin}_+})}/N_\Delta(F^\otimes_{\text{Fin}_+})\) \((\text{Sp}^\otimes, \hat{\otimes})\).

Especially, the truncation functor \((-)_0\) commutes with the graded tensor.

Proof. Since \(\{0\} \subset \mathbb{Z}\) is a fully faithful symmetric monoidal functor, the composed functor \(A_0\) also has a \(\mathbb{E}_\infty\)-ring structure induced from \(A\). So (i) holds. Since the Day convolution preserves the colimits separably in each variable, it commutes with the left Kan extension. So (ii) holds. \(\square\)

Definition 3.4. We call \(A(\infty)\) the underlying \(\mathbb{E}_\infty\)-ring of \(A\). By virtue of Proposition 3.3, we sometimes regard \(A(\infty)\) as \(A\).

Since we take isomorphisms as morphisms in \(\mathbb{Z}\), we can write \(A(\infty)\) as \(\bigsqcup A_i\), where \(A_i\) is a spectrum valued at \(i \in \mathbb{Z}\) in Definition 2.2.

Remark 3.5. The condition of Definition 3.1(ii) and (iii) now can be written as the conditions that \(A(\infty)\) is connective and \(A(\infty) \simeq A'(\infty)\), where \(A'\) is the image of \(A\) under the morphism \(\text{Alg}_{N_\Delta(F^\otimes_{\text{Fin}_+})}/N_\Delta(F^\otimes_{\text{Fin}_+})\) \((\text{Fun}_{N_\Delta(F^\otimes_{\text{Fin}_+})}(O^\otimes_N, \text{Sp}^\otimes)) \to \text{Alg}_{N_\Delta(F^\otimes_{\text{Fin}_+})}/N_\Delta(F^\otimes_{\text{Fin}_+})\) \((\text{Fun}_{N_\Delta(F^\otimes_{\text{Fin}_+})}(O^\otimes_N, \text{Sp}^\otimes))\) induced from \(N_\Delta(O^\otimes_N) \to N_\Delta(O^\otimes_N)\).

Next, we will denote by \(\text{Mod}^{gr\mathbb{Z}}\) the \(\infty\)-category of \(\mathbb{Z}\)-graded modules.

Definition 3.6. For a \(\mathbb{Z}\)-graded \(\mathbb{E}_\infty\)-ring \(A\) and an \(\mathbb{E}_\infty\)-ring \(R\),

(i) we define the \(\infty\)-category of \(\mathbb{Z}\)-graded \(A\)-modules by

\[
\text{Mod}_A(\text{Fun}_{N_\Delta(F^\otimes_{\text{Fin}_+})}(N_\Delta(O^\otimes_Z)^{\otimes}, \text{Sp}^\otimes), \hat{\otimes}),
\]

where the notation \(\text{Mod}_A(\cdot)\) is in the sense of Lurie. Let us denote the \(\infty\)-category of \(\mathbb{Z}\)-graded \(A\)-modules by \(\text{Mod}^{gr\mathbb{Z}}\).
We define the $\infty$-category of $\mathbb{Z}$-graded $A$-modules over $R$ by
\[ \text{Mod}_A(\text{Fun}_{N_A}(\text{Fin}_n)(N_A(\mathbb{O}_Z)^0, \text{Mod}_R^\otimes)^\hat{\otimes}). \]

We will denote by $\text{Mod}_{Ag_r}^\mathbb{Z}$.

(iii) We say that a $\mathbb{Z}$-graded $A$-module $M$ is connective if each value $M_i$ of the underlying functor is a connective spectrum for $i \in \mathbb{Z}$.

We call a morphism in $\text{CAlg}_{Ag_r}^\mathbb{Z}$ and $\text{Mod}_{Ag_r}^\mathbb{Z}$ a morphism of degree 0 or a morphism of graded $E_\infty$-rings and of graded $A$-modules.

Remark 3.7. Since the graded tensor commutes with filtered colimits in each variable, we obtain a lax symmetric monoidal functor $\text{Mod}_{Ag_r}^\mathbb{Z} \to \text{Mod}_A(\mathbb{Z})$ by the diagram in Definition 3.2, i.e., a $\mathbb{Z}$-graded $A$-module for an $\mathbb{N}$-graded $E_\infty$-ring $A$ gives an $A(\mathbb{Z})$-module $M(\mathbb{Z})$.

3.1. Enrichment and localization of $\mathbb{Z}$-graded spectra with one element. Recall that we have the symmetric monoidal $\infty$-categories $\text{Sp}^\otimes \to N_A(\text{Fin}_n)$ and $\text{Mod}_R^\otimes \to N_A(\text{Fin}_n)$, where $R$ is an $E_\infty$-ring. Since the symmetric monoidal structure $\text{Sp}$ (resp. $\text{Mod}_R$) is right closed, it is $\text{Sp}$-enriched (resp. $\text{Mod}_R$-enriched) (cf. [6, Example 4.2.1.32, Proposition 4.8.2.18]). From the point of view, we can regard the mapping spaces as objects in $\text{Sp}$ (resp. $\text{Mod}_R$).

Now, we take the symmetric monoidal $\infty$-categories $\text{Sp}^\otimes$ and $\text{Mod}_R^\otimes$, where $R$ is an $E_\infty$-ring, and consider the graded tensor $\hat{\otimes}$. Since $\text{Sp}$ and $\text{Mod}_R$ are presentable, the underlying $\infty$-category of $\text{Fun}_{N_A}(\text{Fin}_n)(N_A(\mathbb{O}_Z)^0, \text{Sp}^\otimes)^\hat{\otimes}$ and $\text{Fun}_{N_A}(\text{Fin}_n)(N_A(\mathbb{O}_Z)^0, \text{Mod}_R^\otimes)^\hat{\otimes}$ are also presentable by [5, Proposition 5.5.3.6].

We regard a mapping space $\text{Map}_{\text{Mod}_R}(M, N)$ (resp. $\text{Map}_{\text{Sp}}(M, N)$) for $R$-modules $M$ and $N$ (resp. for spectra $M$ and $N$) as an $R$-module in $\text{Mod}_R$ (resp. as a spectrum in $\text{Sp}$).

Lemma 3.8. The graded tensor on the $\infty$-category $\text{Fun}(N_A(\mathbb{Z}), \text{Sp}^\otimes)$, which is the underlying $\infty$-category of $\text{Fun}(N_A(\mathbb{O}_Z)^0, \text{Sp}^\otimes)$, admits the right adjoint. Therefore, $\text{Fun}(N_A(\mathbb{O}_Z)^0, \text{Sp}^\otimes)$ is the $\text{Fun}(N_A(\mathbb{O}_Z)^0, \text{Sp}^\otimes)$-enriched $\infty$-category.

Proof. Since the Day convolution preserves colimits in each variable by Proposition 2.3, it admits the right adjoint, so that it is right closed. Therefore, $\text{Fun}(N_A(\mathbb{O}_Z)^0, \text{Sp}^\otimes)$ admits self-enrichment. □

Definition 3.9. Let $A$ be a $\mathbb{Z}$-graded $E_\infty$-ring, and $A(\mathbb{Z})$ be its underlying $E_\infty$-ring.

(i) Let us take an element $a \in \pi_0 A(\infty)$ which is homogeneous in usual sense. As in Definition 3.4, it is equivalent to take an element $a$ of $\pi_0 (A_d)$ for a certain $d \in \mathbb{Z}$ and the class of elements $a^k$ in each $\pi_0 (A_{dk})$ for $k \in \mathbb{Z}$. We say that an element $a \in \pi_0 (A(\infty))$ is degree $d$ if $a$ lies in the essential image of $A_i \to A(\infty)$.
(ii) Let \( (k) : N_\Delta(\mathcal{O}_Z^\otimes) \to N_\Delta(\mathcal{O}_Z^\otimes) \) be a functor given by the assignment of the underlying functor \( Z \to Z; n \mapsto n + k \). We take the left Kan extension of a functor \( X \) along \( (k) \):

\[
\begin{array}{ccc}
N_\Delta(\mathcal{O}_Z^\otimes) & \xrightarrow{X} & \text{Sp}^\otimes \\
(k) \downarrow & & \downarrow \\
N_\Delta(\mathcal{O}_Z^\otimes) & \xrightarrow{X'} & \\
\end{array}
\]

We write the left Kan extension \( X' \) of \( X \) as \( X(k) \). By the diagram, we have \( X(k)_n = X_{n+k} \).

Remark 3.11. Since the Day convolution on \( \text{Fun}(N_\Delta(\mathcal{O}_Z^\otimes), \text{Sp}^\otimes) \) is right closed by Lemma 3.8, if \( f \) is degree 1, this universality is just for the universality with respect to the inner mapping spectrum, so that the localization is compatible with the graded tensor, i.e., we have the equivalence \( \text{Map}_{\text{Calg}^{gr 2}}(Z[\text{Map}(A[f^{-1}], B)]) \simeq \text{Map}_{\text{Calg}^{gr 2}}(Z \hat{\otimes} A[f^{-1}], B) \), so that we easily see \( B[f^{-1}] \hat{\otimes} A[f^{-1}] \to ((B \hat{\otimes} A) C)[f^{-1}] \) for \( Z \)-graded \( E_\infty \)-rings \( Z, A, B \) and \( C \) by the associativity of the Day convolution. By the universality, we also have \( A[f^{-1}][g^{-1}] \simeq A[(fg)^{-1}], \) where the second \( g \) in the left hand side is the image of \( g \) in \( \pi_0(A[f^{-1}]|\infty) \) and the \((fg)^{-1}\) in the right hand side means \( g^{-1}f^{-1}\).

Remark 3.12. Note that \( A \simeq \coprod A_i \) is an omega spectrum, so that \( f \in \pi_0(\coprod A_i) \simeq \coprod \pi_0(A_i) \) gives an evaluation \( * \to \coprod A_i \). We can see the existence of \( A[f^{-1}] \) by iteratedly using the map given by the ring structure of \( A \) together with the evaluation.

Proposition 3.13. Let \( A \) be a \( \mathbb{N} \)-graded \( E_\infty \)-ring, \( M \) and \( N \) \( \mathbb{Z} \)-graded \( A \)-modules. If we take degree 1 elements \( a, b \in \pi_0(A(\infty)) \), then

(i) We have an equivalence \( A[a^{-1}]_0 \otimes (b/a)^{-1} \to A[(ab)^{-1}]_0 \), where we denote by the second \([b/a]^{-1}\] the localization of (non-graded) \( E_\infty \)-ring.

(ii) We have an equivalence \( M[a^{-1}]_0 \otimes (a^{-1})_0 N[a^{-1}]_0 \to ((M \hat{\otimes} A N)[a^{-1}]_0 \).
(iii) For an \( \mathbb{N} \)-graded \( \mathbb{E}_\infty \)-ring \( B \), let \( f, f', g, g' \) be homogenous elements in \( \pi_0(A(\infty)) \) which have degree 1. We have a diagram:

\[
\begin{array}{ccc}
A[f^{-1}]_0 & \rightarrow & A[(fg)^{-1}]_0 \\
\downarrow & & \downarrow \\
B[f'^{-1}]_0 & \rightarrow & B[(f'g')^{-1}]_0
\end{array}
\]

and we also have the diagram of graded modules

\[
\begin{array}{ccc}
M[f^{-1}]_0 \otimes_{A[f^{-1}]_0} N[f^{-1}]_0 & \rightarrow & M[(fg)^{-1}]_0 \otimes_{A[(fg)^{-1}]_0} N[(fg)^{-1}]_0 \\
\downarrow & & \downarrow \\
(M \hat{\otimes}_A N)[f^{-1}]_0 & \rightarrow & (M \hat{\otimes}_A N)[(fg)^{-1}]_0
\end{array}
\]

**Proof.** Note that the graded tensor commutes with the truncation \((-)_0\). So, by the observation of Remark 3.11, the first part of (iii) holds. By the associativity of the relative tensor product, we obtain (ii). The second part of (iii) follows from (ii) and Remark 3.11. For (i), by the universality of the localization, we have a morphism \( A[a^{-1}] \rightarrow A[a^{-1}][b^{-1}] = A[(ab)^{-1}] \), which corresponds to morphisms \( A[a^{-1}] \rightarrow B \) which send the image of \( b \) in \( \pi_0(A[a^{-1}](\infty)) \) to an invertible element of \( \pi_0(B(\infty)) \). Note that the image of \( b \) in \( \pi_0(A[a^{-1}](\infty)) \) corresponds to a class \( \{ba^k\} \) as in Definition 3.9, in degree 0 part \( \pi_0(A[a^{-1}]_0) \), the element \( b/a \). So, by the universality of localization of (non-graded) \( \mathbb{E}_\infty \)-ring and the fact that the localization is determined up to equivalence, we obtain an equivalence \( A[a^{-1}]_0[(b/a)^{-1}] \rightarrow A[(ab)^{-1}]_0 \). 

\[\square\]

**Example 3.14 (Ordinary grading).** Let \( A \) be a discrete \( \mathbb{N} \)-graded ring and \( (\text{AMod}^{gr}) \) be the ordinary category of \( (\mathbb{Z}) \)-graded \( A \)-modules. We can apply our procedure together with \( O_\mathbb{Z} \) to the ordinary case; a graded algebra and a graded \( A \)-module are obtained by an algebra object and an \( A \)-module object with respect to the Day convolution on the functor category whose underlying category is just the category of symmetric monoidal functors \( \mathbb{Z} \rightarrow (\text{ZMod}) \). The unit object is \( I \), where the degree 0 part of \( I \) is \( \mathbb{Z} \) and the other degree part is 0. We can construct a graded category \( (\text{AMod}^{gr})^{gr} \) whose associated category \( (\text{AMod}^{gr})^0 \) is the category \( (\text{AMod}^{gr}) \) since the Day convolution has the right adjoint in the category \( \text{AMod}^{gr} \). As objects of \( (\text{AMod}^{gr})^{gr} \) we take \( \mathbb{Z} \)-graded modules. Given \( \mathbb{Z} \)-graded modules \( L \) and \( M \). The set of homomorphisms are given by

\[
\text{Hom}_{(\text{AMod}^{gr})^{gr}}(L, M) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{(\text{AMod}^{gr})}(L, M[n]),
\]

where \( M[n] \) is a graded \( A \)-module given by \( M[n]_i = M_{i+n} \) and \( \text{Hom}_{(\text{AMod}^{gr})}(L, M[n]) \) is the degree 0 of \( A \)-module morphisms. This Hom-set can be regarded as a graded \( A \)-module, so
that \((\text{AMod}^g)\) can be enriched by \((\text{AMod}^g)\) itself. Here, the \(\text{Hom}_{(\text{AMod}^g)^g}(M, -)\) is the right adjoint of the Day convolution.

4. PROJECTIVE SCHEMES AND ITS QUASI-COHERENT SHEAVES

(i) In this section and Section 5, for a \(Z\)-graded \(E_\infty\)-ring \(A\) and \(Z\)-graded \(A\)-module \(M\), let us denote \(A(\infty)\) and \(M(\infty)\) by \(A\) and \(M\) for simplicity. We will also denote by \(\text{Spec} A\) the Zariski spectrum of the underlying \(E_\infty\)-ring \(A(\infty)\) and the \(\tilde{M}\) for the sheafification of \(M(\infty)\) as in Definition 3.4.

(ii) We also denote by \(\text{Spec} A[\![a^{-1}]\!]_0\) the Zariski spectrum of the truncation at degree 0 part of \(A[\![a^{-1}]\!]\). It makes sense by the following diagram; by Definition 3.2, we have the diagram

\[
\begin{array}{ccc}
\text{Mod}_{\text{Agr}Z} & \xrightarrow{(-)_{\infty}} & \text{Mod}_{\text{Agr}Z} \\
\text{Mod}_A & \xrightarrow{(-)_{0}} & \text{Mod}_A \\
\end{array}
\]

where the bottom morphism is obtained by the whole diagram in Definition 3.2, which we also denote by \((-)_{0}\).

For a graded \(Z\)-graded \(E_\infty\)-ring \(A\), we have an equivalence \((\coprod_n A(n))[\![a^{-1}]\!]_0 \simeq A[\![a^{-1}]\!]_0\) by the above diagram. One may wonder if \((\coprod_n A(n)[\![a^{-1}]\!]_0 \simeq A[\![a^{-1}]\!]_0\). It is true since all the morphisms in the above diagram preserve monoidalicity.

Let \(\mathcal{C}\) be an \(\infty\)-category. A sieve on \(\mathcal{C}\) is a full \(\infty\)-subcategory \(\mathcal{C}^{(0)} \subset \mathcal{C}\) having the property that if \(f : C \rightarrow D\) is a morphism in \(\mathcal{C}\) and \(D\) belongs to \(\mathcal{C}^{(0)}\), then \(C\) also belongs to \(\mathcal{C}^{(0)}\) [5, Definition 6.2.2.1]. Let \(\mathcal{C}\) be a small \(\infty\)-category, and \(j : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, S)\) the Yoneda embedding. Let \(C \in \mathcal{C}\) be an object, and \(i : U \rightarrow j(C)\) be a monomorphism in \(\text{Fun}(\mathcal{C}^{\text{op}}, S)\). We denote by \(\mathcal{C}/C(U)\) the full subcategory of \(\mathcal{C}\) spanned by those objects \(f : D \rightarrow C\) of \(\mathcal{C}/C\) such that there exists a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i} & j(C) \\
\downarrow{j(D)} & \nearrow{j(f)} & \\
\end{array}
\]

Then, \(\mathcal{C}/C(U)\) becomes a sieve on \(C\), and there is a bijection between the subobjects of \(j(C)\) and the sieves on \(C\) given by sending \(i : U \rightarrow j(C)\) to \(\mathcal{C}/C(U)\) [5, Proposition 6.2.2.5]. Let \(S\) be a sphere spectrum. Recall that \(\mathcal{G}^{Sp}_{\text{Zar}}(S)\) is an \(\infty\)-category with Grothendieck topology determined as follows [10, Definition 2.10].

(i) On the level of \(\infty\)-categories, \(\mathcal{G}^{Sp}_{\text{Zar}}(S)\) is the opposite of the \(\infty\)-category of compact algebras. If \(A\) is a compact algebra, let \(\text{Spec} A\) denote the corresponding object of \(\mathcal{G}^{Sp}_{\text{Zar}}(S)\).

(ii) A morphism \(f : \text{Spec} A \rightarrow \text{Spec} B\) in \(\mathcal{G}^{Sp}_{\text{Zar}}(S)\) is admissible if and only if there exists an element \(b \in \pi_0 B\) such that \(f\) sends \(b\) an invertible element in \(\pi_0 A\) and the \(B[\!b^{-1}] \rightarrow A\) is an equivalence.
A Grothendieck topology on Ind$(\mathcal{S}^{Sp}_{\text{Zar}})_{op}$ is determined by $\mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S})$ as follows, called Zariski topology [9, Notation 2.2.6]. Let $j : \mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S}) \to \text{Ind}(\mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S}))_{op}$ be the Yoneda embedding. Take an object $X$ in Ind$(\mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S}))_{op}$. A sieve $S$ on an object $U \to X$ in Ind$(\mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S}))_{op}$ is covering if there exists an object $U'$ of $\mathcal{S}^{Sp}_{\text{Zar}}(\mathcal{S})$, a finite collection of admissible morphisms $\{V_i \to U'\}_{1 \leq i \leq n}$ which generate a covering sieve on $U'$, and a morphism $U \to j(U')$ such that each pullback $U \times_{j(U')} j(V_i) \to U$ belongs to $S$.

In classical sense, coherent modules are finitely presented modules. It is different from the coherentness for an $R$-module over an $\mathbb{E}_\infty$-ring. Also, in classical sheaf theory, we obtained the tensor product of sheaves on a projective scheme of a discrete $\mathbb{N}$-graded ring $A$ by glueing up the affine cover together if the ideal $A_+$ of $A$ consisting of positive degree elements is spanned by the set $A_1$ of degree 1 elements of $A$, and the projective scheme associated to $A$ is quasi-compact when we assume that $A_+$ is generated by finite number of elements in $A_1$.

We will consider the several conditions to realize the similar situation. We define notations and the condition as follows, which we call the finite presentation condition.

4.1. Finite presentation condition.

(i) An $\mathbb{E}_\infty$-ring $R$ is coherent if any ideal of $\pi_0 R$ is finitely presented and $\pi_n R$ is finitely presented over $\pi_0 R$.

(ii) Let $R$ be a coherent $\mathbb{E}_\infty$-ring. An $R$-module $M$ is almost perfect if and only if $\pi_i M \simeq 0$ for sufficiently small $i$ and $\pi_n M$ is finitely presented over $\pi_0 R$.

(iii) For a module $M$ over a coherent $\mathbb{E}_\infty$-ring $A$, $M$ is perfect if and only if it is almost perfect and finite Tor-amplitude.

We say that an $\mathbb{E}_\infty$-ring $R$ is coherent if any ideal of $\pi_0 R$ is finitely presented and $\pi_n R$ is finitely presented over $\pi_0 R$. (This condition may be now said Noetherian coherent).

Let $A$ be an $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring. We say that a $\mathbb{Z}$-graded $A$-module $M$ is locally perfect if every $M[a^{-1} \cdot 0]$ is perfect $A[a^{-1} \cdot 0]$-module for a homogeneous element $a$ of $A$.

**Definition 4.1** (Finite presentation condition). By [8, Proposition 2.2.2] and Proposition 2.6, we take a $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring $A$ and a $\mathbb{Z}$-graded $A$-module whose underlying functors valued in compact objects respectively, and say that such a pair $(A, M)$ is $\mathbb{Z}$-compact graded. One may wonder if $A_0$-module $A_1$ is compact. This is compact also by [8, Proposition 2.2.2] and the fact that the sphere spectrum $\mathcal{S}$ is compact. Let us denote the $\infty$-category of $\mathbb{Z}$-compact graded $A$-modules over $R$ for a compact graded $\mathbb{E}_\infty$-ring $A$ by $\text{Mod}_{\mathbb{E}_\infty^{\text{cptGr}}(\mathbb{Z})}$. Note that if $A$ is $\mathbb{Z}$-graded compact, a localization $A[a^{-1}]$ is $\mathbb{Z}$-graded compact, so is $A[a^{-1} \cdot 0]$. We also note that, if $B$ is a $\mathbb{Z}$-graded compact over a coherent $\mathbb{E}_\infty$-ring $R$, the homotopy group $\pi_*(B(\infty))$ is bounded below.
Recall the notation from Definition 2.2 and Definition 3.4. We say that, for an $\mathcal{E}_\infty$-ring $R$, an $\mathbb{N}$-graded $\mathcal{E}_\infty$-ring $A$ over $R$ and $\mathbb{Z}$-graded $A$-modules over $R$ satisfy the finite presentation condition if

(i) $A$ is an $\mathbb{N}$-graded compact as $R$-module, locally perfect, and connective $\mathcal{E}_\infty$-ring. Here $R$ is a regular $\mathcal{E}_\infty$-ring.

(ii) There exists $x_1, \cdots, x_n$ such that they give a Serre fibration $A_0[x_1, \cdots, x_n] \to \text{Sym}_{A_0}(A_1)$.

We may identify each $x_i$ with a vertex in $A_0[x_1, \cdots, x_n]$ which gives an indeterminant $x_i$ in $\pi_0(A_0[x_1, \cdots, x_n]) \simeq \pi_0(A_0)[x_1, x_2, \cdots, x_n]$. 

(iii) There is a Serre fibration $\text{Sym}_{A_0}(A_1) \to A(\infty)$ (which means $A \simeq A_0(A_1)$).

(iv) A $\mathbb{Z}$-graded $A$-module $M$ is locally compact if there exists $n_0 \in \mathbb{Z}$ such that the restricted colimit $N_\Delta(\mathcal{O}_Z^\otimes)^{\geq n_0} \to \text{Sp}^\otimes$ is locally perfect. Here, $N_\Delta(\mathcal{O}_Z^\otimes)_{\geq n_0}$ is full $\infty$-subcategory of $N_\Delta(\mathcal{O}_Z^\otimes)$ spanned by the vertices $\mathbb{Z} \geq n_0$. Let us denote the $\infty$-category of locally compact $\mathbb{Z}$-compact graded $A$-modules over $R$ for a compact graded $\mathcal{E}_\infty$-ring $A$ by $\text{Mod}_{\text{fct A gr}^\mathbb{Z}}$.

Let $k$ be a field. The Eilenberg-MacLane spectrum $HK$ is corent $\mathcal{E}_\infty$-ring. Moreover, it is regular, so that an almost perfect module over $HK$ is automatically perfect. This is one reason we often take a field as a base ring in classical case. So, we replace $HK$ by a regular $\mathcal{E}_\infty$-ring $R$ to proceed the same argument. However, note that the sphere spectrum $\mathbb{S}$ is not regular, since the $K$-theory is different from the $G$-theory on the sphere $\mathbb{S}$.

4.2. Projective schemes. We will explain a glueing of graded $\mathcal{E}_\infty$-rings to define a projective spectral scheme.

Let $A$ and $B$ be $\mathbb{N}$-graded $\mathcal{E}_\infty$-rings. Consider a morphism $\phi : A \to B$ of degree 0. Take $a \in \pi_0 A(\infty)$ and $b = \phi(a) \in \pi_0 B(\infty)$. Then, we have a morphism $A[a^{-1}] \to B[b^{-1}]$ of degree 0 between $\mathbb{Z}$-graded $\mathcal{E}_\infty$-rings. Then, we have a morphism $A[a^{-1}]_0 \to B[b^{-1}]_0$, so that $\text{Spec } B[b^{-1}]_0 \to \text{Spec } A[a^{-1}]_0$.

**Definition 4.2.** We say that a functor $X : \text{Ind}(\mathcal{O}_{\mathbb{S}^{\mathcal{Z}}_{\mathcal{Z}ar}}^{\mathbb{S}^p}(\mathbb{S})^{op}) \to \mathbb{S}$ is a projective spectral scheme if there exists a collection $\{U_a\}$ such that $U_a$ covers $X$ and there exists $A \in \mathcal{C} \mathcal{A} \mathcal{G}^{gr Z}$ such that $(U_a, \mathcal{O}_X) \simeq (\text{Spec } (A[a^{-1}]_0), \mathcal{O}_{\text{Spec } A[a^{-1}]_0})$ for each $U_a$ and for degree more than 1 elements $a$ in $\pi_0 A(\infty)$.

Assume Definition 4.1(iii). By the truncation $A \to A_0$, we have a morphism $\text{Spec } A_0 \to \text{Spec } A$. Since the truncation is functorial, by Proposition 3.13(i), the class $\{\text{Spec } A[a^{-1}]_0 \to \text{Spec } A[a^{-1}] \to \text{Spec } A\}$ is a sieve, which is corresponding to the monomorphism $\text{Proj } A \to \text{Spec } A(\infty)$ [5, Proposition 6.2.2.5].

Since the $\infty$-category of quasi-coherent sheaves on $X = \text{Proj } A$ can be obtained as the limit $\lim_{\text{Spec } A[a^{-1}]_0 \to X} \text{Mod}_{\text{Spec } A[a^{-1}]_0}$, we modify the sheafification on $X$ as follows.
**Definition 4.3.** Let $A$ be an $\mathbb{N}$-graded $E_\infty$-ring and $M$ a $\mathbb{Z}$-graded $A$-module. Set $X = \text{Proj} \ A$. We define the sheafification of $M$ on $X$ to be an image of sheaves $\tilde{M}[a^{-1}]_0$ on each affine $\text{Spec} \ A[a^{-1}]_0$ in $\lim_{\text{Spec} \ A[a^{-1}]_0 \to \text{X Mod Spec} \ A[a^{-1}]_0}$. Here we denote by $(\tilde{\_})$ the sheafification on $X$.

Note that the $\infty$-category of $\mathcal{O}_X$-modules becomes a symmetric monoidal $\infty$-category [8, Section 1.5]. The following Corollary follows from Proposition 3.13.

**Corollary 4.4.** Let $A$ be an $\mathbb{N}$-graded $E_\infty$-ring $A$ satisfies Definition 4.1 (iii), $M$ and $N$ be $\mathbb{Z}$-graded $A$-modules. We obtain an equivalence between $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ and the sheafification of $M \otimes_A N$.

**Proposition 4.5.** If an $\mathbb{N}$-graded $E_\infty$-ring $A$ satisfies Definition 4.1 (iii), $\text{A}(\ell)$ is a locally free sheaf of rank 1 over $\tilde{A}$.

*Proof.* Take $\{\text{Spec} (A[a^{-1}]_0) \to \text{Spec} (A_0)\}_{a \in A_1}$, and it suffices to see that $\text{A}(\ell)$ is equivalent to $\tilde{A}$ when they are restricted on each $\text{Spec} (A[a^{-1}]_0)$. By taking the restrictions on $\text{Spec} (A[a^{-1}]_0)$, $\text{A}(\ell)$ is $A(\ell)[a^{-1}]_0$ and $\tilde{A}$ is $A[a^{-1}]_0$. We have the following diagram for a $\mathbb{Z}$-graded $E_\infty$-ring $B$

$$
\begin{array}{ccc}
A & \rightarrow & A[a^{-1}] \\
\downarrow & & \downarrow \\
A(\ell) & \rightarrow & A(\ell)[a^{-1}] \\
\downarrow & & \downarrow \\
A = A(\ell)(-\ell) & \rightarrow & A(\ell)[a^{-1}](-\ell) \\
\downarrow & & \downarrow \\
 & & B(n)
\end{array}
$$

where the horizontal morphisms are assumed to be obtained by those morphisms which send $a$ to an invertible element and the vertical morphisms are associated with the extensions $(-)(\ell)$ and $(-)(-\ell)$. Note that we write the middle term $A(\ell)[a^{-1}]$ since the upper left square is obtained by the graded tensor $A(\ell) \otimes_A A[a^{-1}]$. It suffices to see that the lower middle term $A(\ell)[a^{-1}](-\ell)$ is equivalent to $A[a^{-1}] = A(\ell)(-\ell)[a^{-1}]$. Since $n$ runs over the set $\mathbb{Z}$ of all integers, by the characterization of localization, the lower horizontal morphism $A \rightarrow B(n)$ factors through $A[a^{-1}]$. On the other hand, by the construction and the extension is determined up to equivalence, $A \rightarrow B(n)$ factors through $A(\ell)[a^{-1}](-\ell)$, so that, by using Proposition 3.13, the degree 0 part of $A(\ell)[a^{-1}]$ can be identified with $A[a^{-1}]_0$ via the morphism induced by the middle vertical morphism, which induces $a^\ell$-multiplication on $\pi_0$. \hfill \Box

**Definition 4.6.** $\text{A}(n)$ is denoted by $\mathcal{O}_X(n)$ for $n \in \mathbb{Z}$, and we define $\mathcal{F}(n)$ to be $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

**Definition 4.7.** For an $E_\infty$-ring $R$, let $A$ be an $\mathbb{N}$-graded $E_\infty$-ring over $R$. Let $X = \text{Proj} \ A$ be a projective spectral scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Note that we have a morphism $\text{Proj} \ A \rightarrow \text{Spec} R$. Assume Definition 4.1 (iii). Let us take a class of morphisms
\[ \text{Spec } A[a_i^{-1}]_0 \to \text{Spec } A[a_i^{-1}]_0. \] Since the restriction \( \mathcal{F}|_{\text{Spec } A[a_i^{-1}]_0} \) to each affine is a quasi-coherent sheaf over an affine spectral scheme, we can take an \( A[a_i^{-1}]_0 \)-module \( M_i \) whose sheafification on \( \text{Spec } A[a_i^{-1}]_0 \) is equivalent to \( \mathcal{F}|_{\text{Spec } A[a_i^{-1}]_0} \). We define the section functor \( \Gamma(X, -) : \text{Qcoh}(X) \to \text{Mod}_R \) to be the module obtained by the equalizer
\[ (4.1) \quad \Gamma(X, \mathcal{F}) \longrightarrow \prod_i \Gamma(\text{Spec } A[a_i^{-1}]_0, \mathcal{F}|_{\text{Spec } A[a_i^{-1}]_0}) \longrightarrow \prod_{i,j} M_{ij} \longrightarrow \cdots \]

This definition is equivalent to the definition of the global section in the sense of Lurie as in [8] if we take an affine spectral scheme.

Now, assume Definition 4.1(i) (ii) (iii). For a \( \mathbb{N} \)-graded \( \mathbb{E}_{\infty} \)-ring \( A \) over \( R \), let \( X = \text{Proj}(A) \) and \( \prod_{n \in \mathbb{Z}} \Gamma(\text{Proj } A, \mathcal{F}(n)) \) a \( \mathbb{Z} \)-graded \( A \)-module whose underlying functor assigns each \( n \in \mathbb{Z} \) to \( \Gamma(X, \mathcal{F}(n)) \). Let
\[ \Gamma_*(X, -) : \text{Qcoh}(\text{Proj } A) \to \text{Mod}_R^{\text{Grd } \mathbb{Z}} \]
be a functor given by \( \mathcal{F} \mapsto \prod_{n \in \mathbb{Z}} \Gamma(\text{Proj } A, \mathcal{F}(n)) \).

Now, we will explain why \( \Gamma_*(X, -) \) takes the values in \( \text{Mod}_R^{\text{Grd } \mathbb{Z}} \) as the following lemma. We compare \( A \) with \( \Gamma_*(X, \mathcal{O}_X) \).

Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. We can see \( \mathcal{O}_X(\ell) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(\ell + m) \).

**Lemma 4.8.** Assume Definition 4.1(i) (ii) (iii). For a \( \mathbb{N} \)-graded \( \mathbb{E}_{\infty} \)-ring \( A \) over \( R \). Let \( X = \text{Proj}(A) \), then

(i) \( \Gamma_*(X, \mathcal{O}_X) \) is a \( \mathbb{Z} \)-graded \( \mathbb{E}_{\infty} \)-ring

(ii) \( \Gamma_*(X, \mathcal{F}) \) is a \( \mathbb{Z} \)-graded \( \Gamma_*(X, \mathcal{O}_X) \)-module

(iii) we have a natural equivalence \( \mathcal{O}_X \cong \widehat{A} \cong \Gamma_*(X, \mathcal{O}_X) \). Moreover, we have a morphism \( A \to \Gamma_*(X, \mathcal{O}_X) \) of \( \mathbb{Z} \)-graded \( \mathbb{E}_{\infty} \)-rings.

**Proof.** We show (i) and (iii). We have a morphism \( A(\ell) \to A(\ell)[a^{-1}] \) of \( \mathbb{Z} \)-graded \( A \)-modules for a degree 1 element \( a \in \pi_0 A(\infty) \) as in Proposition 4.5. By virtue of Definition 4.1(i), (ii) and (iii), especially the condition Definition 4.1 (i), \( A \) itself is compact in usual sense and \( \mathbb{Z} \)-graded compact, so that we obtain that \( A \) is locally perfect, and \( \Gamma_*(X, \mathcal{O}_X) = \Gamma_*(X, \prod_n \mathcal{O}_X(n)) \cong \Gamma_*(X, \prod_n \mathcal{O}_X(n)) \) by [10, Lemma 3.21].

We have \( \prod_n \Gamma(\text{Spec } A[a^{-1}]_0, A(\ell)) \cong \Gamma(\text{Spec } A[a^{-1}]_0, (\prod_\ell A(\ell))) \cong (\prod_\ell A(\ell)) \otimes _{\mathbb{E}_\infty} A[a^{-1}]_0 \cong A[a^{-1}]_0 \) by the diagram (4). Thus we obtain (i) and the equivalence \( \mathcal{O}_X \cong \widehat{A} \cong \Gamma_*(X, \mathcal{O}_X) \).

Since \( M(\ell)[a^{-1}]_0 \cong M(\ell) \), we have \( \Gamma_*(X, \mathcal{F}) \cong \mathcal{F} \) for a quasi-coherent \( \mathcal{O}_X \)-module \( F \) under this assumption. So, \( \Gamma_* \) is the right adjoint of the sheafification (\( \sim \)) in this case. Thus, we also obtain a morphism \( A \to \Gamma_*(X, \mathcal{O}_X) \) of \( \mathbb{Z} \)-graded \( \mathbb{E}_{\infty} \)-rings since \( \Gamma_* \) commutes with tensoring affine covers.

We have the \( \Gamma(X, \mathcal{O}_X) \)-module \( \Gamma(X, \prod_n \mathcal{F}(n)) \) if \( X \) is quasi-compact. Thus (ii) is proved.

Thus, in Definition 4.7, \( \Gamma_* \) values in \( \text{Mod}_R^{\text{Grd } \mathbb{Z}} \). \( \square \).
For any quasi coherent $\mathcal{O}_X$-module $\mathcal{F}$, with a class of morphisms $\{\text{Spec } A[a_i^{-1}]_0 \to \text{Spec } A\}_{a_i \in A_1}$, there exists a $\mathbb{Z}$-graded $A$-module $M$ such that $\mathcal{F}|_{\text{Spec } A[a_i^{-1}]_0} \cong M[a_i^{-1}]_0$.

**Proposition 4.9.** Let $A$ be an $\mathbb{N}$-graded $\mathbb{E}_\infty$-ring over $R$ and $X = \text{Proj } (A)$ a projective spectral scheme. Let $\mathcal{F}$ be an quasi-coherent perfect $\mathcal{O}_X$-module. Assume the condition Definition 4.1 (i), (ii) and (iii) holds. Then,

(i) the sheafification $\tilde{A}$ is perfect as an $\mathcal{O}_X$-module,

(ii) and

\[
\Gamma_*(X, \mathcal{F}) = \lim_{\text{Spec } A[a_i^{-1}]_0 \to X} \Gamma(\text{Spec } A[a_i^{-1}]_0, \bigsqcup_n \mathcal{F}(n))
\]

itself is $\infty$-quasi compact.

**Proof.** We will show this proposition like the previous lemma. By Definition 4.1 (ii) and (iii), we can take a finite covering $\{\text{Spec } A[a_i^{-1}]_0 \to X\}$ of $X$ and by Definition 4.1 (i) $\text{Spec } A[a_i^{-1}]_0$ is $\infty$-quasi compact. Since $i$ runs through the finite index set, $X$ is also $\infty$-quasi compact. Since $\tilde{A}(\mathcal{F}) = \tilde{A}$ is perfect by the proof of Lemma 4.8, $\mathcal{F}(n)$ is also perfect by [10, Lemma 3.21] together with Definition 4.1 (i). So, the assertion follows since the limit in the right hand side is reduced to the finite limit.

□

**Proposition 4.10.** Let $\mathcal{F}$ be a quasi-coherent perfect $\mathcal{O}_X$-module, and assume Definition 4.1 (i), (ii) and (iii). Then $\Gamma_*(X, \mathcal{F})[a^{-1}]_0 \to \Gamma(\text{Spec } A[a^{-1}]_0, \mathcal{F})$ is induced and it is an equivalence.

**Proof.** Since $X$ becomes $\infty$-quasi compact by Definition 4.1 (i)(ii)(iii) and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, we identify $\Gamma_*(X, \mathcal{F})$, as in Definition 4.7, with the equalizer

\[
\Gamma_*(X, \mathcal{F}) \to \bigsqcup_i \Gamma(\text{Spec } A[a_i^{-1}]_0, \bigsqcup_n \mathcal{F}(n)) \to \bigsqcup_{i,j} \bigsqcup_n M^n_{ij} \to \cdots,
\]

where we denote by $M^n_{ij}$ an $A[a_{ij}^{-1}]_0$-module whose sheafification on $\text{Spec } A[a_{ij}^{-1}]_0$ is equivalent to $\mathcal{F}(n)|_{\text{Spec } A[a_{ij}^{-1}]_0}$. We take $M^n_{ij} as M_{ij}$ in the equation (4.1).

We have a natural morphism $\Gamma_*(X, \mathcal{F})[a^{-1}] \to \Gamma(\text{Spec } A[a^{-1}]_0, \mathcal{F})$ in Lemma 4.8. Then, we reduce the assertion to the case $(\bigsqcup_i M[a^{-1}]_0) \cong (\bigsqcup_i M[a^{-1}]_0)$ which is true as in the second (*) of Section 4.

□

**Definition 4.11** (Gradedly bounded above modules). We define the $\infty$-category $\text{Nil}^A_R$ by the full $\infty$-subcategory of $\text{Mod}_R^{\text{fpt, Agr}}$ spanned by those objects whose essential images via the Serre twist $(n)$ is zero for sufficiently large positive $n \in \mathbb{N}$. Here, $\text{Mod}_R^{\text{fpt, Agr}}$ is defined in Definition 4.1. We define the $\infty$-category $\text{Mod}_R^{\text{fpt Agr}}/\text{Nil}^A_R$ as the Dwyer-Kan localization with respect to the class of morphisms $M \to N$ in $\text{Mod}_R^{\text{fpt Agr}}$ such that it induces $M_i \simeq N_i$ for sufficiently large positive $n \in \mathbb{N}$.
5. The proof of Theorem 1.1

From now on, we assume that all condition of Definition 4.1.

Proposition 5.1. If a \( \mathbb{Z} \)-graded \( A \)-module \( M \) is in \( \text{Nil}_R \), its sheafification is equivalent to zero.

Proof. Let us denote the truncation of \( M \) by \( M_{\geq n} \) via the inclusion \( N_{\Delta} (\mathcal{O}_X)^{\otimes n} \rightarrow N_{\Delta} (\mathcal{O}_X) \).

Since \( M \) is gradedly bounded above, there exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( M_{\geq k} \simeq 0 \). Take the cofiber sequence \( M_{\geq k} \rightarrow M \). Then, the cofiber is also gradedly bounded above. Since the sheafification and the cofiber sequence commute with the grade tensor, we have \( \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X (k+1) \simeq 0 \). By Definition 4.1(i), \( \mathcal{O}_X (k+1) \) is dualizable, so that it admits a dual \( \mathcal{S} \). We have the conclusion of the proposition. \( \Box \)

Since we take \( \mathbb{Z} \)-graded compact objects, we can write an \( A(\infty) \)-module \( M(\infty) \) by the form \( \coprod_{i} M_i \).

Since we already saw in Proposition 4.9 that, for a locally perfect \( \mathbb{Z} \)-graded \( A \)-module \( M \), its sheafification \( \tilde{M} \) is equivalent to a quasi coherent sheaf (note that the cofiber argument also shows that the locally compactness make its sheafification perfect) given by a \( \mathbb{Z} \)-graded \( A \)-module of the form \( \coprod_{i} M_i \) and that \( \Gamma(X, \tilde{M}) \simeq \tilde{M} \), we have the essentially surjective sheafification functor from \( \text{Mod}^{fcpt\text{Agr}}_R / \text{Nil}_R \) to \( \text{QCoh}(X)^{perf} \).

Proposition 5.2. Let \( X = \text{Proj}(A) \). Assume that all of the finite presentation condition Definition 4.1 holds. Then, the functor \( \coprod_{n} \Gamma(X, -(n)) \) gives an equivalence of \( \infty \)-categories from \( \text{QCoh}(X)^{perf} \) to \( \text{Mod}^{fcpt\text{Agr}}_R / \text{Nil}_R \).

Proof. It suffices to show that, by the universality of the Dwyer-Kan localization, we have \( \tilde{M} \simeq 0 \) if and only if \( M \) is gradedly bounded above, for a \( \mathbb{Z} \)-graded \( A \)-module \( M \) with the finite presentation condition.

So, assume that \( \tilde{M} \simeq 0 \). Since \( M \) is locally compact by Definition 4.1, there exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( M_{\geq k} \) is locally perfect. Take the cofiber sequence \( M_{\geq k} \rightarrow M \). Then, the cofiber is gradedly bounded above, so its sheafification is zero. Thus, we have \( M_{\geq k} \simeq 0 \). By the assumption \( \tilde{M} \simeq 0 \), we obtain \( \tilde{M}_{\geq k} = 0 \). It shows that the sheafification of \( \coprod_{\ell \geq k} \Gamma(X, \tilde{M}(\ell)) \), which is equivalent to \( \tilde{M} \), is gradedly bounded above. Therefore, by restricting to affines \( M_{| \text{Spec} \mathbb{A}^{-1}_{\mathbb{Z}}}_0 \simeq M| \text{A}^{-1}_{\mathbb{Z}}}_0 \), the fully faithfulness is reduced to the affine case together with Proposition 4.9. \( \Box \)

Remark 5.3. The stabilization of the \( \infty \)-category \( \text{Mod}^{\text{Agr}}_R \) via the Serre twist (1), an \( \infty \)-category \( \text{Stab}_1(\text{Mod}^{\text{Agr}}_R) \) by replacing the mapping space \( \text{Map}(M, N) \) of the corresponding simplicial category of \( \text{Mod}^{\text{Agr}}_R \) as \( \text{colim}_{m \rightarrow \infty} \text{Map}(M[m], N[m]) \), could be defined. Then, we obtain a certain Verdier quotient \( \text{Mod}^{\text{Agr}}_R / \text{Nil}_R \) of stable presentable \( \infty \)-category \( \text{Mod}^{\text{Agr}}_R \).
with the full $\infty$-subcategory $Nil_R^A$ spanned by the gradedly bounded above $A$-modules similarly defined as in Definition 5.1. The following proof as in [14, p.18, Section I.3] provides an equivalence $\text{Stab}_{\mathbb{I}}\text{Mod}_R^{AgrZ} \simeq \text{Mod}_R^{AgrZ}/\text{Nil}_R^A$ of $\infty$-categories.

Let $I_{X,Y} = \{(X', Y') \in P(X) \times P(Y) | \text{ cofib } X' \to X, Y' \in \text{Nil}_R^A\}$, where $P(Z)$ is the partially ordered set of equivalence classes of submodules of $Z$ and the ordering $(X', Y') \leq (X'', Y'')$ holds if and only if $X' \to X''$ and $Y' \to Y''$ are submodules. By the definition of Verdier quotient in [14, p.18, Section I.3], we have

$$\text{Map}_{\text{Mod}_R^{AgrZ}/\text{Nil}_R^A}(X, Y) = \text{colim}_{(X', Y') \in I_{X,Y}} \text{Map}_{\text{Mod}_R^{AgrZ}}(X', \text{ cofib } Y' \to Y).$$

Let us take $(X', Y') \in I_{X,Y}$, where the cofiber $X' \to X$ and $Y'$ are in $\text{Nil}_R^A$. There exists $m$ such that $(\text{ cofib } X' \to X)[m] \simeq 0$ and $Y'[m] \simeq 0$. By definition, we have $(X', Y') \leq (X[m][−m], Y')$ and

$$\text{Map}_{\text{Mod}_R^{AgrZ}}(X[m][−m], \text{ cofib } Y' \to Y) \simeq \text{Map}_{\text{Mod}_R^{AgrZ}}(X[m][−m], Y[m][−m]) \
\simeq \text{Map}_{\text{Mod}_R^{AgrZ}}(X[m], Y[m]).$$

This equivalence shows that the main theorem in this paper is a generalization of Serre theorem which specifies the thick subcategory consisting of gradedly bounded above modules.

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