The internet differential equation and fractal networks

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Abstract. The Internet is an example of a general physical problem dealing with motion near the speed of light relative to different time frames of reference. The second order differential equation (DE) takes the form of ‘time diffusion’ near the speed of light or alternatively, considered as a complex variable with real time and imaginary longitudinal components. Congestion waves are generated by peak global traffic from different time zones following the Earth’s revolution defined by spherical harmonics and a day/night bias. The DE is essentially divided into space and time operators constrained by the speed of light c, band capacity w and a fractal dimension Z (Hausdorff dimension). This paper explores the relationship between the dynamics and the network including the addition of fractal derivatives to the DE for regional networks for \(0 < Z < 1\).

1. Introduction

The Internet forms the physical network of connectivity (such as, optical cables and phone wires), where there are nodes or ‘routers’ that navigate packets of data from one computer to another. The location of these computers around the Earth means that pings of information packets can be transmitted to and from different time zones relative to the arbitrary International Date Line. There will never be instantaneous transfers of packets because of the fundamental limit of the speed of light. The time taken to transmit this data between computers is termed ‘latency’. Packet loss is a proxy for peak demand, since in heavy traffic periods, too many packets arrive and the routers then hold them in buffers until the traffic decreases. If these buffers fill during these times of congestion, the routers drop packets and this is measured as ‘packet loss’. Sites that are not well-connected in the network, suffer high packet loss and this can be a proxy for congestion. However, with the significant increase in band capacity over the last decade, such congestion has dissipated within the global network. Nevertheless, packet loss still remains a very useful proxy for modelling and such data compiled during the Stanford Linear Accelerator (SLAC) internet experiments for 2000 and 2004 [1] are a valuable resource to test physical modelling.

There are a number of approaches employed to study internet traffic, such as, network analysis [2] [3] [4], the fluid dynamics of congestion flows through the system [5] [6], stochastic evolution [7] [8] and the measurement of ping times and geographical distance relative to capacity [9]. An internet differential equation has been developed to describe the dynamics of pings around a global network where information can move either forwards or backwards between relative time zones [7] [10].
This paper summarises and develops the modeling dynamics by looking at two further aspects raised briefly by [10]. Firstly, since we are dealing with relative time, where pings directions can be either forwards or backwards relative to the Earth’s rotation, the concept of time for the internet can be framed as a complex variable with real and imaginary components. The meridians of longitude can be visualized as imaginary time lines relative to a real time rotation and can be described as such mathematically. Secondly, the link between the dynamics of global traffic movement and regional network contributions to congestion is critical to any physical description of internet traffic. There is a possibility of a connection between the Gaussian nature of internet traffic movement and the fractal nature of a bound Gaussian network, when the Hausdorff dimension is $Z = 2$ [10]. This has not proved to be very fruitful because there no easy way to mathematically link congestion patterns and the network fractal. This paper looks briefly at introducing a fractal calculus $F^Z$ derivative in the internet DE for the condition $0 < Z < 1$ where the congestion measurement is proportional to the Hausdorff dimension $Z$.

2. Background

Random walk modeling has been used to develop a set of internet equations between the Earth’s time zones for pings between monitoring machines and remote hosts located within different time zones [7]. Traffic can move forwards and backwards relative to the arbitrary International Date Line with the novelty that one can experience their birthday twice traveling from Sydney to Los Angeles. There is real time movement from the Earth’s revolution, but pings traveling near the speed of light can access imaginary time lines elsewhere on the revolving Earth designated by meridians of longitude. This is the difference between information transfer by the internet in the 21st century and telegraphic transfers by cables in the late 19th century, since spatial diffusion then was a major problem for trans-Atlantic cabling.

Internet traffic within and between time zones can be defined by a set of equations for the exchange rates $E(x,t)$, continuity conditions for origin-destination exchanges and the congestion transfers as [7]:

$$E(x,t) = -(w/c^2)\left(\partial \phi / \partial t + \nu \phi \right) \quad (1)$$

$$\partial \phi / \partial x = -\partial E / \partial t \quad (2)$$

$$\partial \phi / \partial t = (w/c^2)\left(\partial^2 \phi / \partial t^2 - \nu \partial \phi / \partial t \right) \quad (3)$$

where $\phi_0 (x, t)$ is the initial site density for a time distance $p$ between pairs assumed to be equal for remote hosts and the $i^{th}$ monitoring site (located at $x$ and having meridians of longitude as time bands between $t_i = ip$); $w$ the transfer bandwidth; $c$ the speed of light and $\nu$ as bias in traffic from the Earth’s 24-hour revolution. The ping latency $\Delta t$ and the partitioning of distance $\Delta x$ are constrained by the inequality [7]:

$$\left(2c^2/w\right) \Delta x \leq \Delta t^2 \quad (4)$$

This inequality states that there will never be an instantaneous transfer of information because of the speed of light but improvement in capacity will see pings approach the event horizon (figure 1). For the time wave on a sphere, Eq.3 can be expressed by spherical polar coordinates for the day/night
rotation \((l = 1, m = 1)\). This means that packet loss, as a density proxy at any of the monitoring sites, can be modeled with longitudinal change on the sphere.

Figure 1. Latency shift towards the limit of transmission for global traffic, SLAC 2000 & 2004

Figure 2. The wave nature of global demand congestion for hepnrce.hep.net, Chicago US

The novelty for the physics is that information can come to a computer located at \(x\) from positive and/or negative time directions, because the time zone band \(\Delta t\) is squared, making travel time Gaussian in nature. Also, the return ping to the monitoring site requires a convergence without any fluctuations. This will occur if all the coefficients in the exchange have the same sign and are positive in a Taylor expansion at location \(x\) for a modulus \(\lambda\) stopped at the first order \(\Delta x\). Therefore, for positive coefficients only, the ratio \(\lambda\) of space to time in any network must obey the inequality \(0 \leq \lambda \leq \frac{1}{2}\) [7].

Equations (1)-(4) have been applied to the Stanford Linear Accelerator Centre internet experimental data set [1]. The SLAC experiments in 2000 used 27 monitoring sites on five continents and up to 216 remote hosts in a global network. In 2004 there were 25 monitoring sites and up to 269 remote hosts. The monitoring sites pinged the network every 30 minutes and SLAC recorded the average ping latency and packet loss between each origin-destination pairs. The raw hourly ping latency data was collated and averaged for each hour of the week. The data begins Monday 00:00 local time of the remote host and is truncated to extract the days (Monday to Sunday 00:00 to 24:00 local time). This is then standardised to Greenwich Mean Time. Using the actual ping time and the distance at which the interaction between the monitoring site and remote host occurs (using latitude and longitude references), plots are generated for all yearly ping time averages.

The key aspects of the differential equation have been compared for each operator subset of the equation to the SLAC global network [10]. The limit of internet information transfer for the whole network \((kd \leq \Delta t^2)\) is shown for 2000 and 2004 (figure 1). The wave metric of congestion \((\partial^2 / \partial r^2)\) is demonstrated for the hepnrce.hep.net monitoring machine in Chicago US (figure 2). The distance metric \((\partial / \partial x)\) yields a negative exponential distance metric. Regression of the residuals of the phase metric \(\chi\) of the network gives a proxy for global connectivity of the monitoring site for \(0 < \chi < 1\). It can be calculated for each monitoring site, where, for \(\chi \rightarrow 0\), the site is increasingly well-connected globally. For example, for the well connected hepnrce.hep.net site, \(\chi = 0.184\), whereas for the isolated caia.swin.edu.au site at Melbourne Australia, \(\chi = 0.764\) (figures 7 and 8). The slope of the regression line is calibrated as unity from traffic in phases with the rotation of the Earth, so there is no
significance testing. The day-night divide can also be described through spherical polar coordinates for \( l = 1 \) and \( m = 1 \) (figure 4) and this can be used to animate the SLAC data [10].

2. Imaginary Time Zones and Internet Pings

The model in Eq.3 has been applied to the Internet where pings can travel near the speed of light to and from different time zones relative to the Earth’s rotation. Whilst the rotation is real, the longitudes are imaginary in the sense that motion can be forwards in time or backwards in time, relative to the arbitrary International Date Line. In recognition of this internal time relativity between longitudes of imaginary time, Eq.1 can be rewritten including the imaginary time lines as:

\[
\left( \frac{iD}{2\pi} \right) \frac{\partial^2 \phi}{\partial t^2} - \nu \frac{\partial \phi}{\partial t} = \mu \frac{\partial \phi}{\partial x}
\]

This can be derived simply by using the complex wave form of \( \Phi(t,x) = \exp(2\pi i (kt - \beta x)) \), differentiating time twice and space once, recognising that traffic can travel with or against the Earth’s rotation. This is the Schrödinger equation in one spatial dimension with the second order operator in time, where the time quantums are just the time zones defined by meridians of longitude. Time is framed as complex with real (physical time) and imaginary components (time relative to the arbitrary International Date Line). As in the case of quantum mechanics, probability densities can be constructed from wave functions between 0 and T of the Earth’s 24-hour rotation \( \int_0^{24} |\Phi|^2 dt \) and the expectation value of packet loss \( \Psi(t) \):

\[
\bar{\Psi}(t) = \int_0^{24} \Phi^* \Psi(t) \Phi \, dt
\]

If \( \Psi(t) \) varies with location, then there is still an expectation value, since \( \Phi(t) \) must be evaluated at a particular time as \( \Phi \) is also a function of location. Therefore, averages in packet loss can be assigned to any monitoring machine measuring internet traffic supporting the data smoothing in averaging the ping data for each hour per week.

The solution of a time Schrödinger equation in spherical co-ordinates is interesting since the separation constant is the rotational period of the Earth. The distribution of global internet traffic...
suggests (figure 5) three principle axes of demand (Europe, US and Asia) in three time zones. An interesting idea is whether the solutions for time zones according to quantum numbers \( l \) and \( m \) in the imaginary time Schrödinger equation are self-similar (figure 6) in that the demand loops for an individual computer in the inner circle have the same fractal dimension of global traffic as the outer circle. If this is the case, then the oscillation with a real rotation between three imaginary time zones would constitute a Julia Set (figure 10).

Figure 5. Internet traffic 2000 (source CADA)

Figure 6. Self-similar oscillation in a Julia Set

4. Fractal Dimensions and Network Analysis

The contribution of the regional network configuration to congestion is an important element of understanding the character of internet traffic. This has been recognized in the origin-destination exchange equation [10] with the addition of the regional network contribution \( -Z\Phi \) to congestion resulting in the differential equation, namely:

\[
E(x,t) = -(w_\chi / c^2) \left( \partial \phi / \partial t + v\phi - Z\phi \right)
\]

(7)

\[
\partial \phi / \partial x = (w_\chi / c^2) \phi^2 \partial \phi^2 / \partial t^2 - (1-Z) \partial \phi / \partial t
\]

(8)

The measure of phase connectivity in the network involves two elements, firstly a global phase measurement and secondly a regional network contribution to congestion. The global index \( \chi \) uses a linear regression plot of the residuals of the wave against the standardised global line for the range \((0,1)\). For well-connected sites, \( \chi \) approaches 0 and the amplitude of the wave diminishes accordingly (figure 7). This phase line was constructed for points distributed relative to a 360° manifold with a boundary. Conversely, a regional \( \kappa \)-statistic is defined by a slope of zero and is set for residuals of packet loss that are in phase with local congestion. The residuals are distributed along a standardised horizontal line relative to the local time zone of the monitoring machine (figure 8). As \( \kappa \) approaches 0, there is substantial phase congestion from local traffic dropping packets in what could be inferred as poorly connected sites. The relationship between the \( \chi \)- and \( \kappa \)-statistics are summarised below:

\[
\begin{array}{c|c|c}
Z = 0 & Z = 1 (\text{wave only}) & Z = 2 (\text{bound Gaussian}) \\
\hline
\chi = 1 & \chi = 2 \chi & \kappa = 1
\end{array}
\]

Regional Phase
If Z is a fractal dimension, for $Z = 2$, the probability of network jumps for a time-space similarity $\lambda$ of the ping distribution can be defined through a Lévy distribution as [11]:

$$P(\eta) = 2 \exp \left( - b \left| \lambda \right|^Z \right) \quad 0 < Z \leq 2$$  \hspace{1cm} (9)

When $Z = 2$, the network cluster of computers is a bound Gaussian of Brownian motion from the pings from the monitoring machine. In this case, equation 9 simplifies to equation 3 and so there is an implicit Gaussian network within the time Gaussian. For $Z = 1$, there is zero congestion from the regional networks and no bias from the Earth’s rotation. There is only a pure wave of congestion and this is when the global index $\chi$ is calculated for the monitoring computer network.

For the global phase regression for a well-connected hepnc.hep.net monitoring machine in Chicago US (figure 7) in 2000 $\chi = 0.184$ and the regional phase $\kappa = 0.564$ whilst for a less well-connected machine in Melbourne Australia (caia.swin.edu.au) in 2000 $\chi = 0.764$ and $\kappa = 0.115$. For well-connected regional networks for the US in 2000, the mean value for the 14 monitoring sites was 0.488 and for the 11 monitoring machines in 2004 it was 0.485 (figure 9). However, the regional median for US monitoring sites 2004 was 0.395, possibly due to the substantial reduction in periodic congestion measurements used in the regression, due to improvements in capacity. Why would a regional network that is well-connected approach $\kappa \approx 0.5$? It could simply be the relationship between the ratio of the time-space network to global phase is limited to the ratio of space to time in any network mesh which must obey the inequality $0 \leq \lambda \leq \frac{1}{2}$; otherwise it fluctuates and is chaotic [7].

![Figure 7](image1.png)  \hspace{1cm} ![Figure 8](image2.png)

**Figure 7.** Global phase regression for the well-connected hepnc.hep.net site Chicago US  
**Figure 8** Global phase regression for the poorly connected caia.swin.edu.au site Melbourne Australia 2004

![Figure 9](image3.png)  \hspace{1cm} ![Figure 10](image4.png)

**Figure 9.** Box and whisker plot of regional phase congestion $\kappa$ in the US sites  
**Figure 10.** Generalised Cantor set for $\lambda = 0.48$ (source: Wolfram Demonstrator)
US sites are in phase with global traffic and well-connected regionally, suggesting only positive coefficients in a Taylor expansion around $\Delta x$. If this ratio is self-similar for every computer within this regional network (figure 11), then $\kappa$ may be proportional to a fractal dimension and be non-chaotic for $0 \leq \kappa \leq \frac{1}{2}$.

If the well-connected US and European sites do indeed have a fractal $Z$ dimension of $\sim 0.5$, then the exchange of traffic between the US and Europe may cascade through a doubling of the time zones as a Feigenbaum attractor where the traffic across the Atlantic is purely the wave ($Z = 1$), whilst the network in the US and Europe may be chaotic for $Z > 0.5$ (figure 11). The Hausdorff dimension for the Feigenbaum attractor is calculated at 0.538, which is close to the $Z$ boundary for stability. If $Z \sim 0.5$ for regional phase (and $\chi \rightarrow 0$), equation 9 reduces to the continuity condition (equation 2) for a return ping across the Atlantic.

![Figure 11](image.png)

**Figure 11.** The US and European networks and cable link as a Feigenbaum attractor of a logistic mapping function with a period doubling cascade for deterministic $Z = 1$ and chaotic for $Z = 0.538$

An alternative idea to model the network contribution to congestion is to introduce a fractal derivative in equation 3. The use of fractional derivatives as a fractal subset of ordinary calculus is an area of growing interest (see [12] for a review). The $\Phi^z$-derivative is best suited for functions like the Cantor staircase which change only on the fractal boundary. The cumulative distance demand graphs for each monitoring computer in the SLAC 2000 network (such as the hepnre.hep.net monitoring machine in Chicago, figure 3) can be normalised into a Cantor staircase distribution with the $\Phi^z$-integral and $\Phi^z$-derivative characteristics [12]. There is a mass function $E(\Phi, x, t)$ that is proportional to the Hausdorff dimension and because there are finite sub-divisions in regional networks, $\kappa$ may be analogous to a separation constant for a $\Phi^z$-derivative of congestion from the network. It is fortuitous that the phase of the time derivative from the regression plot of residuals from pings in a regional network is $(0, 1)$ and this in the range for the $\Phi^z$-derivative of $0 < Z < 1$. When $Z = 1$ we have to only consider the global wave contribution. Equation 8 can therefore be defined for a constant network congestion threshold $\Phi_o$ as:

$$E(x, t) + E^Z(x, t) = -\frac{w}{c^2} \left( \frac{\partial \phi}{\partial t} + v\phi - Z\phi \right) + \Phi_o$$

(10)

We assume there is a fractal continuity condition between the fractional derivatives in space and time as:

$$\partial^Z x^Z = \partial^Z E^Z t^Z$$

(11)
Applying this assumption to equation 11 and noting that the \( \Phi_z \)-derivative of a constant function is zero, equation 10 simplifies for \( \nu = 1 \) to:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} = (w / c^2) \frac{\partial^2 \phi}{\partial t^2} - (1 - Z) \frac{\partial \phi}{\partial t}
\]

Equation 12 is stating that changes in distance constraint on congestion and the configuration of the spatial network to transmit this congestion is equivalent to the periodic congestion wave of demand with bias relative to whether the local network is in phase with the global network.

We may define the regional network as a generalised Cantor set \(-\log(2)/\log(1-k/2)\) (figure 10). For example, the stability condition \( 0 \leq \kappa \leq 1/2 \), the hep.re.hep.net monitoring site in Chicago with a ratio for network demand of \( k = 0.48 \) yields a Cantor set in figure 11 for pings coming in and out of the network for a standardised distance.

This paper has summarised the work on dynamic modelling of internet pings as time-based random walks where time lines can be imaginary relative to the International Date Line. There are various ideas presented on how to account the contribution of the network to traffic congestion on Earth. The definition of a regional network as a Cantor set allows for the possibility of introducing \( \Phi_z \)-derivatives into the dynamics of the internet differential equation and a description of the network from the cumulative patronage as a standardised function of distance.

5. References

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