FACTORIALS \((\text{mod } p)\) AND THE AVERAGE OF MODULAR MAPPINGS

CRISTIAN COBELL, ALEXANDRU ZAHARESCU

ABSTRACT. We have known that most sequences in \(M = \{1, 2, \ldots, M\}\) with length \(n\) will miss \(M e^{-\lambda}\) of the total numbers of \(\{1, 2, \ldots, M\}\) as the ratio \(n/M\) tends to \(\lambda\). Now we consider a more general case where the numbers in \(\{1, 2, \ldots, M\}\) are achieved exactly \(k\) times by a ‘random’ sequence \(f(1), f(2), \ldots, f(n)\). We show that if \(n/M \to \lambda\), then the limit has a Poisson distribution, that is, the proportion of sequences for which some number in \(M\) is achieved exactly \(k\) times has the limit \(\frac{\lambda^k}{k!} e^{-\lambda}\). We conjecture that this is the behavior of the factorial mapping modulo a prime and present a few supporting arguments.

1. INTRODUCTION

In contrast with many of their real siblings, which are continuous, modular functions are known to be chaotic in nature. This is true for polynomials, monomials in particular, and this behavior accentuates as the power increases from 2 to the value of the modulus minus 2, that is, from squares to inverses. The matter was studied extensively in the works of the authors et al. [CZ’00, CZ’01, CVZ’00a, CVZ’03, CGZ’03], P. Kurlberg et al. [KR’99, Kur’00, Kur’09], and I. E. Shparlinski [Shp’12].

Since the factorial mapping \(x \mapsto x!\) modulo a prime \(p\) acts like a ‘diagonal of the monomials’, one expects it to induce an even higher degree of randomness. Many visual representations, such as those in Figure 1, confirm the expectation. One of the main issues is raised by the following problem.

Conjecture 1 (R. Stauduhar). Let \(p\) be a prime and let \(h(p)\) be the number of distinct residues of \(1!, 2!, \ldots, (p−1)!\) (mod \(p\)). Then

\[
\lim_{p \to \infty} \frac{h(p)}{p} = 1 - \frac{1}{e}.
\]

Stauduhar’s Conjecture is stated as Problem 7 in a list of 114 problems, which, according to a nice tradition of the time, appeared in the Proceedings of the 1963 Number Theory Conference [NTC’63]. In several later articles, various authors referred to the same problem, but in the formulation of R. K. Guy [Guy’81, Problem F11]. There, Guy draws attention to the phenomenon that the sequence of factorials \(1!, 2!, \ldots, p!\) misses about \(p/e\) residue classes modulo \(p\). The conjecture remains unsolved, but it was proved [CVZ’00b] that it holds in average for all modular mappings or, in other words, this is a characteristic of randomly chosen sequences of \(p\) classes of residues modulo \(p\). The link with the ‘randomness’ was made earlier, as we have found in the last stage of the preparation of this manuscript. Thus, Brillhart [NTC’63, Problem 7, page 90] adds a comment to the statement of the conjecture, saying in parentheses that “from extensive numerical calculation the statement appears to be true” and ‘this is the “random result” for any set of numbers’.

Key words and phrases: Factorials; modular mappings; random sequences; Poisson distribution; fixed points; Stauduhar’s Conjecture.

2010 Mathematics Subject Classification: primary 11N69, secondary 49J55, 65C10.
In this article we show that in average, the expected number of residue classes that are reached exactly \( k \geq 0 \) times by a random modular function defined from \( \mathbb{F}_p \) with values in \( \mathbb{F}_p \) equals \( \frac{1}{p}e^{-\lambda} \).

The limit asymptotic values for the mean and variation are obtained in Theorems and the size of the error term are evaluated in Theorem 2. Accordingly, for factorials, this allows us to extend Stauduhar’s Conjecture, since wide-ranging numerical verifications confirm this trend.

**Conjecture 2.** For any integer \( k \geq 0 \), the proportion of elements \( y \in \mathbb{F}_p \) for which there are exactly \( k \) positive integers \( n \) for which \( n! \equiv y \pmod{p} \) tends to the limit \( \frac{1}{e^{\lambda}} \), as \( p \to \infty \).

Notice that Stauduhar’s Conjecture is just the \( k = 0 \) case of Conjecture 2 and the general statement says that if \( p \) is sufficiently large, then the elements of the sequence \( \{ n! (\pmod{p}) \}_{1 \leq n \leq p} \) behave like a Poisson process with mean \( \lambda = 1 \). More background data are presented in Section 2.

Our main result is a uniform estimation over all modular mappings of the average number of elements attained exactly a certain fixed number of times. Let \( \mathcal{M} \) be a finite set of \( M \) elements and let \( \mathcal{N} \subset \mathcal{M} \) be a subset of \( n \) elements. Denote by \( \mathcal{T} \) the set of \( n \)-tuples with components in \( \mathcal{M} \), that is, \( \mathcal{T} = \mathcal{M}^n \) or \( \mathcal{T} = \{ \mathbf{x} = (x_1, x_2, \ldots, x_n): x_1, \ldots, x_n \in \mathcal{M} \} \). We say that an element \( y \in \mathcal{N} \) is represented \( k \) times in the tuple \( \mathbf{x} \in \mathcal{T} \), if exactly \( k \) components of \( \mathbf{x} \) coincide with \( y \). Let \( m_k(x) \) be the number of elements of \( \mathcal{N} \) that are represented exactly \( k \) times by \( \mathbf{x} \), that is,

\[
m_k(\mathbf{x}) := \left| \{ y \in \mathcal{N}: y \text{ is represented exactly } k \text{ times in } \mathbf{x} \} \right|.
\]

It turns out that if the sets \( \mathcal{N} \) and \( \mathcal{M} \) have sufficiently many elements, than the proportions \( m_k(\mathbf{x})/n \) cluster around the values of a Poisson distribution. A controlled bound of the maximum deviation is obtain in the following theorem.

**Theorem 1.** Let \( \lambda \in (0, 1], \gamma \in [0, 1), \delta \in (0, (1 - \gamma)/2) \) and let \( k \geq 0 \) be integer. Suppose the integer variables \( M \) and \( n \) satisfy the inequalities \( k \leq n \leq M \) and \( M = n/\lambda + O(n^\gamma) \), uniformly on \( \lambda \) and \( k \) as \( n \) tends to infinity while \( \gamma \) is fixed. Then

\[
\frac{1}{|\mathcal{T}|} \left| \left\{ \mathbf{x}: \frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!} e^{-\lambda} < n^{-\delta} \right\} \right| = 1 - O\left( \frac{\lambda^k}{k!} \cdot \frac{1}{n^{1-2\delta}} \left( 1 + \frac{\lambda^{k+1}(\lambda + k)}{k!} n^\gamma \right) \right),
\]

and the constant involved in the big Oh term does not dependent on \( \lambda, \gamma, \delta \) and \( k \).
Let us notice that both summands in the error term have their particular type of contribution. The distinction can be seen whether or not $\lambda$ is very small, or whether $k = 0$, or else, whether $k$ is small or it becomes large enough to make one or the other of the terms of the sum dominate on one side or the other of the balance point, which is attained if $n \approx \left(k!/(\lambda^{k+1}(\lambda + k))^{1/\gamma}\right)$.

2. Ramification Points and Factorials ($\mod p$)

Referring to the distribution of the sequence of factorials modulo a prime $p$, an earlier simpler problem was proposed by P. Erdős [Guy’81, problem F11]. He asks whether there exists $p > 5$ for which the numbers $2!, 3!, \ldots, (p-1)!$ are all distinct modulo $p$. If there were such a prime, then T. Trudgian [Tru’14] verified that it must be greater than $10^9$. The problem of Erdős is still unsolved, although B. Rokowska, A. Schinzel [RS’60] and Trudgian [Tru’14] showed that for large classes of primes the sequence of factorials modulo $p$ can not be so close to a permutation, while Klurman and Munsch [KM’17] obtained non-trivial bounds for the average deviation.

In reality, for any given $p$, one can check that there are many residue classes modulo $p$ that are hit more than once by the sequence of factorials. Iterating the factorial function modulo $p$, two types of points distinguish in the complex created tree. They are the fixed points, or the roots of the tree, and the ramification points, which are the residue classes reached several times by the sequence of factorials. Iterating the factorial function modulo $p$ that are hit more than once by the sequence of factorials modulo $p$, two types of points distinguish in the complex created tree. They are the fixed points, or the roots of the tree, and the ramification points, which are the residue classes reached several times by the factorials.

In precise terms, we say that $1 \leq x \leq p - 1$ is a fixed point or, shortly, $x$ is an FP of the factorial function modulo $p$, if $x! \equiv x \pmod{p}$. The residue classes modulo $p$, viewed as stacks that are reached a different number of times by the sequence $1!, 2!, 3!, \ldots, (p - 1)! \pmod{p}$, are also called ramification points of the factorial function modulo $p$ or, shortly, RPs. Thus, rigorously, we say that $y \in \{1, \ldots, p - 1\}$ is a $k$-RP, if it is hit exactly $k$ times by the sequence $1!, 2!, 3!, \ldots, (p - 1)!$, that is, there exist exactly $k$ distinct integers $x_1, x_2, \ldots, x_k \in \{1, 2, \ldots, p - 1\}$ such that $x_1! \equiv \cdots \equiv x_k! \equiv y \pmod{p}$. For example, in Table 1, one sees that for $p = 23$, the points $a = 1, 2, 5, 9, 12, 22$ are FPs; $10, 15, 16, 17, 19, 20$ are 0-RPs (they are missed by the factorial function modulo 23); $2, 3, 4, \ldots, 9, 11, 12, 13, 14, 18, 21$ are 1-RPs (they are hit exactly once by the factorial function); 22 is a 3-RP (it is attained three times) and 1 is a 5-RP (it appears five times on the second row of the table).

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $x! \pmod{p}$ | 1 | 2 | 6 | 1 | 5 | 7 | 3 | 1 | 9 | 21 | 1 | 12 | 18 | 22 | 8 | 13 | 14 | 24 | 4 | 11 | 1 | 22 |

Remark that 1, 2 and $p - 1$ are always FPs (by Wilson’s Theorem), but congruences such as

\[ 4244208! \equiv 4244208 \pmod{9991769}, \quad 5112195! \equiv 5112195 \pmod{5444407} \]

are sporadic. Quite often, the median $m = (p + 1)/2$ is an almost trivial fixed point, too, since $m! \equiv m \pmod{p}$ or $m! \equiv m - 1 \pmod{p}$ for any prime $p \equiv 3 \pmod{4}$.

Also, notice this intrinsic connection between FPs and the particular ramification point 1: for any prime $p \geq 3$, the number of FPs is $r$ if and only if 1 is an $(r - 1)$-RP.

Let $m(p)$ be the number of 0-RPs, that is, the number of residual classes $1, 2, \ldots, p - 1$ that are missed by the sequence of factorials modulo $p$. Table 2 shows evidences in favor of Stauduhar’s Conjecture[1].

With so many residue classes missed by the factorials, we expect that the sequence of factorials is always very far from a permutation, but a proof of this fact is still needed. A comparison approach is taken by Lev [Lev’06] who tests the size of the partial sums of the elements of permutations in abelian groups. Upper and lower bounds for the number of distinct residue classes of $n! \pmod{p}$ under different conditions were obtained by Banks et al. [BLSS’05]. Garaev
et al. [GLS’04, GH’17], Klurman and Munsch [KM’17]. The interesting properties of \( n! \pmod{p} \) have been studied from different perspectives by us [CVZ’00b], Shub and Smale [SS’95], Markström [Mar’15], Luca et al. [LS’03, LS’05a, LS’05b], Cheng [Che’04], Broughan et al. [BB’09], Garaev et al. [GLS’04, GLS’05], Banks et al. [BLSS’05], Dai et al. [CD’06, Dai’08], Garcia [Gar’07], [Gar’08].

Table 2. The proportion of residue classes missed by factorials for different primes. Compare the values from the third column with \( 1/e \approx 0.3678794. \)

| \( n \) | \( p_n \) | \( m(p_n)/p_n \) | \( 1/e - m(p_n)/p_n \) |
|---|---|---|---|
| 5 | 11 | 0.5454545 | -0.1775751 |
| 7 | 17 | 0.3529412 | 0.0149382 |
| 8 | 19 | 0.4210526 | -0.051731 |
| 26 | 101 | 0.3663366 | 0.0015428 |
| 100 | 541 | 0.3792797 | -0.0110484 |
| 1000 | 7919 | 0.3725218 | -0.0046423 |
| 10000 | 104729 | 0.3679662 | -0.000867 |
| 100000 | 1299709 | 0.3679662 | -0.0000867 |
| 1000000 | 15485863 | 0.3676930 | 0.0001864 |

Together with our colleague M. Vâjâitu [CVZ’00b], we showed that when \( p \) is sufficiently large the number of 0-RPs is about \( p/e \) for almost all sequences. Since most sequences have no particular simple defining rule, we may say that the non-representation of \( p/e \) residue classes modulo \( p \) is a general feature that characterizes randomness.

Figure 2. The proportions of \( k \)-RPs of the 30th prime (the figures on the left) and of the 10000th prime (the figures on the right). In the figures at the bottom, only the RPs obtained from factorials from the first half, that is, only the frequencies of \( 1!, 2!, \ldots, \left( \frac{p-1}{2} \right)! \pmod{p} \) are counted. The proportions are compared with the Poisson distribution with means \( \lambda = 1 \) and \( \lambda = 1/2 \), respectively.
The more general statement [CVZ’00b, Theorem 1] says that if \( \lambda \in (0, 1) \) is fixed, \( p \) is a large prime number, and \( n \sim \lambda p \), then almost all sequences of length \( n \) chosen from a subset \( \mathcal{N} \subset \mathbb{F}_p \) having \( n \) elements omit about \( n/e^\lambda \) classes of \( \mathcal{N} \).

We remark that the proportions of \( k \)-RP's of the factorials are close to their conjectured limits even for small primes \( p \), as can be seen from the two example shown in Figure 2 for \( p = 113 \) and \( p = 104729 \) (the 30th and the 10000th primes). For \( p = 113 \), there are 40 points that are missed by factorials, 45 points that are hit exactly once, 19 points that are hit exactly twice and so on. The record values for the two primes are the residue classes 57, which is hit exactly 5 times, since

\[
20! \equiv 40! \equiv 50! \equiv 89! \equiv 101! \equiv 57 \pmod{113},
\]

and 78919, which is hit exactly 9 times, since

\[
2470! \equiv 2742! \equiv 29734! \equiv 36188! \equiv 39370! \equiv 39865! \equiv 57 \pmod{104729}.
\]

The fast decay of the number of \( k \)-RP's as \( k \) increases is a widespread phenomenon in different contexts and it can be investigated under the following generic query.

**Problem.** Let \( M \) be a set of positive integers and suppose \( \mathcal{N} \subset M \) is a 'large enough' subset. What is the expected proportion of numbers that are represented exactly \( k \) times by the sequence \( f(1), f(2), \ldots, f(n) \), where \( \mathcal{N} := \{ f(1), f(2), \ldots, f(n) \} \)?

The most expected answer to the question in this problem is given in the following section.

### 3. The Limits of the Average and of the Variation

Denote by \( A(k; M, n) \) the average of the proportions of elements of \( \mathcal{N} \subset M \) that are represented \( k \) times by vectors \( x \in T \), that is,

\[
A(k; M, n) := \frac{1}{|T|} \sum_{x \in T} \frac{m_k(x)}{n},
\]

where the counter \( m_k(x) \) is defined by \([1]\). For any pair of elements \( x, y \in M \), we define

\[
\delta(x, y) = \begin{cases} 
1, & \text{if } x = y \\
0, & \text{if } x \neq y
\end{cases}
\]

Similarly, for any vector \( x = (x_1, x_2, \ldots, x_n) \in T \) and any \( y \in M \), we denote

\[
\delta_k(x, y) = \begin{cases} 
1, & \text{if exactly } k \text{ components of } x \text{ coincide with } y, \\
0, & \text{else}.
\end{cases}
\]

Then, by the inclusion-exclusion principle, the counting function \( m_k(x) \) can be expressed as

\[
m_k(x) = \left| \{ y \in \mathcal{N} : \delta_k(x, y) = 1 \} \right|
= \sum_{y \in \mathcal{N}} \sum_{|\mathcal{L}| = k} \prod_{j \in \mathcal{L}} \delta(x_j, y) \left( 1 - \delta(x_i, y) \right).
\]

Since for each \( y \in \mathcal{N} \) there are \( \binom{n}{k} \) choices for the positions of the components of \( x \) occupied by \( y \), while the remaining \( M - k \) positions can take any of the remaining \( M - 1 \) values, by \([3]\) we obtain
the following closed form expression of the average introduced by \(2\):

\[
A(k; M, n) = \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}} \frac{m_k(x)}{n}
= \frac{1}{nM^n} \sum_{x \in \mathcal{T}} \sum_{y \in \mathcal{N}} \frac{\sum_{|\mathcal{L}| = k} \prod_{j \in \mathcal{L}} \delta(x_j, y) \left(1 - \delta(x_i, y)\right)}{\prod_{j \in \mathcal{L}} \delta(x_j, y)}
= \frac{1}{M^n} \binom{n}{k} (M-1)^{n-k}.
\]

This implies that

\[
A(k; M, n) = \frac{1}{k!} \cdot \frac{n(n-1)\cdots(n-k+1)}{M^k} \left(1 - \frac{1}{M}\right)^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda},
\]

provided that \(k\) is fixed and both \(M, n \to \infty\) such that the ratio \(n/M\) tends to \(\lambda\).

Next we find the limit of the variance of \(A(k; M, n)\) about its limit mean determined by the asymptotic estimate \(5\)

Let

\[
M_2(k; M, n) := \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}} \left(\frac{m_k(x)}{n} - \frac{\lambda^k}{k!} e^{-\lambda}\right)^2
\]

be the second moment about the mean. Expanding the binomial, we find that it can be written as

\[
M_2(k; M, n) = \frac{\lambda^{2k}}{k!^2} e^{-2\lambda} - \frac{2\lambda^k}{k!} e^{-\lambda} A(k; M, n) + S_2(k; M, n),
\]

where we denoted

\[
S_2(k; M, n) = \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}} \left(\frac{m_k(x)}{n}\right)^2 = \frac{1}{M^n n^2} \sum_{x \in \mathcal{T}} m_k^2(x).
\]

By \(3\) and the same argument used to derive the expression \(4\), we see that

\[
S_2(k; M, n) = \frac{1}{M^n n^2} \sum_{x \in \mathcal{T}} \left(\sum_{y \in \mathcal{N}} \delta_k(x, y)\right)^2
= \frac{1}{M^n n^2} \sum_{x \in \mathcal{T}} \sum_{y \in \mathcal{N}} \delta_k(x, y) \delta_k(x, y')
= \frac{1}{M^n n^2} \left(\sum_{x \in \mathcal{T}} \sum_{y \in \mathcal{N}} \delta_k(x, y) \delta_k(x, y') + \sum_{x \in \mathcal{T}} \sum_{y \in \mathcal{N}} \delta_k(x, y) \delta_k(x, y')\right)
= \frac{1}{M^n n^2} \left(n(n-1) \binom{n}{k} \binom{n-k}{k} (M-2)^{n-2k} + n \binom{n}{k} (M-1)^{n-k}\right).
\]

We denote and rewrite the two terms on the last line of \(8\) as

\[
S_I(k; M, n) = \left(1 - \frac{1}{n}\right) \cdot \frac{1}{k!^2} \cdot \frac{n(n-1)\cdots(n-k+1)}{M^k} \times \frac{(n-k)(n-k-1)\cdots(n-2k+1)}{M^k} \left(1 - \frac{2}{M}\right)^{n-2k},
\]

\[
S_{II}(k; M, n) = \frac{1}{n} \cdot \frac{1}{k!} \cdot \frac{n(n-1)\cdots(n-k+1)}{M^k} \left(1 - \frac{1}{M}\right)^{n-k}.
\]
Provided that $k$ is fixed and both $M, n \to \infty$ such that the ratio $n/ M$ tends to some fixed constant $\lambda \in (0, 1]$, it follows that

$$S_I(k; M, n) \to \frac{\lambda^{2k}}{k!^2} e^{-2\lambda} \quad \text{and} \quad S_{II}(k; M, n) \to 0,$$

which means that

$$\lim_{n, M \to \infty \atop n/M \to \lambda} S_2(k; M, n) = \left( \frac{\lambda^k}{k!} e^{-\lambda} \right)^2.$$

On inserting this limit and the asymptotic estimate (5) on the right side of relation (7), we find that the limit of the second moment about the mean is zero.

**Theorem 2.** Let $\lambda \in (0, 1]$ and let $k$ be a fixed integer. Suppose that $n$ and $M$ are integer variables and both increase tending to infinity while their ratio $n/ M$ tends to $\lambda$. Then, the average defined by relation (2) and the second square moment defined by (5) have the following limits:

$$\lim_{n, M \to \infty \atop n/M \to \lambda} A(k; M, n) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{and} \quad \lim_{n, M \to \infty \atop n/M \to \lambda} M_2(k; M, n) = 0.$$

4. Sharp uniform estimate of the average and of the second moment

**Theorem 3.** Let $\lambda \in (0, 1], \gamma \in [0, 1)$ and let $k \geq 0$ be integer. Suppose the integer variables $M$ and $n$ satisfy the inequalities $k \leq n \leq M$ and $M = n/ \lambda + O(n^{\gamma})$, uniformly on $\lambda$ and $k$ as $n$ tends to infinity while $\gamma$ is fixed. Then

$$A(k; M, n) = \frac{\lambda^k}{k!} e^{-\lambda} \left( 1 + O\left( \lambda (\lambda + k) n^{\gamma-1} \right) \right)$$

and

$$M_2(k; M, n) = O\left( \frac{\lambda^k}{k!} \cdot \frac{1}{n} \left( 1 + \frac{\lambda^{k+1} (\lambda + k)}{k!} n^{\gamma} \right) \right),$$

where the constants implied in the estimates are independent of $\lambda, \gamma$ and $k$.

**Proof.** By the hypothesis of the theorem, we see that $n/M = 1/(\lambda^{-1} + O(n^{\gamma-1}))$, so that $1/M = \lambda(n^{-1} + O(\lambda n^{\gamma-2}))$ and $n/M = \lambda (1 + O(\lambda n^{\gamma-1}))$. Then the asymptotic approximations of the main exp-log functions involved are:

$$\left( \frac{n}{M} \right)^k = \lambda^k \left( 1 + O\left( \lambda n^{\gamma-1} \right) \right)^k = \lambda^k \left( 1 + O\left( \lambda kn^{\gamma-1} \right) \right)$$

and, if $a = 1$ or 2, then

$$\left( 1 - \frac{a}{M} \right)^{n-k} = e^{(n-k) \log(1-a/M)} = e^{\frac{ak}{M} + O\left( \frac{1}{M} \right)} = e^{-a \lambda} \left( 1 + O\left( \lambda^2 n^{\gamma-1} + \lambda kn^{\gamma-1} \right) \right).$$

On combining (4), (12) and (13), we find that

$$A(k; M, n) = \frac{1}{k!} \cdot \lambda^k \left( 1 + O\left( \lambda kn^{\gamma-1} \right) \right) e^{-\lambda} \left( 1 + O\left( \lambda^2 n^{\gamma-1} + \lambda kn^{\gamma-1} \right) \right)$$

$$= \frac{\lambda^k}{k!} e^{-\lambda} \left( 1 + O\left( \lambda (\lambda + k) n^{\gamma-1} \right) \right).$$

The estimation of $M_2(k; M, n)$ is obtained similarly on combining (7), (9), (9), (12) and (13), which completes the proof of the theorem.
Now we can prove the uniform result in Theorem\textsuperscript{11}. For any \( \eta > 0 \), let us split \( \mathcal{T} \) in two disjoint parts, that is, \( \mathcal{T} = \mathcal{T}^<(\eta) \cup \mathcal{T}^\geq(\eta) \) and \( \mathcal{T}^<(\eta) \cap \mathcal{T}^\geq(\eta) = \emptyset \), where

\[
\mathcal{T}^<(\eta) = \left\{ x \in \mathcal{T} : \left| \frac{m_k(x)}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right| < \eta \right\} \quad \text{and} \quad \mathcal{T}^\geq(\eta) = \left\{ x \in \mathcal{T} : \left| \frac{m_k(x)}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \geq \eta \right\}.
\]

Then, accordingly,

\[
M_2(k; M, n) = \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}^<(\eta)} \left( \frac{m_k(x)}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right)^2 + \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}^\geq(\eta)} \left( \frac{m_k(x)}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right)^2.
\]

Ignoring the contribution of the first sum, we obtain for \( M_2(k; M, n) \) a bound from below:

\[
M_2(k; M, n) \geq \frac{1}{|\mathcal{T}|} \sum_{x \in \mathcal{T}^\geq(\eta)} \eta^2 = \frac{\eta^2}{|\mathcal{T}|} \left( |\mathcal{T}| - |\mathcal{T}^<(\eta)| \right).
\]

This can be rewritten conveniently as the following inequality:

\[
\frac{|\mathcal{T}^<(\eta)|}{|\mathcal{T}|} \geq 1 - \eta^{-2} M_2(k; M, n).
\]

On using the estimate (11) on the right side of (14) and choosing \( \eta = n^{-\delta} \), we find that

\[
\frac{|\mathcal{T}^<(n^{-\delta})|}{|\mathcal{T}|} \geq 1 - O \left( \frac{\lambda^k}{k!} \frac{1}{n^{1-2\delta}} \left( 1 + \frac{\lambda^{k+1}(\lambda + k)}{k!} n^{\delta} \right) \right),
\]

which concludes the proof of the theorem.

\textbf{Acknowledgements:} Calculations and plots created using the free open-source mathematics software system \textsc{Sage} \footnote{http://www.sagemath.org}.

\begin{thebibliography}{99}

\bibitem[NTC’63]{NTC63} \textit{***, Proposed problems of the 1963 Number Theory Conference}, Proc. Number Theory Conf., Boulder, 1963, 89–116.

\bibitem[BLLS’05]{BLLS05} Banks, William, Luca, Florian, Shparlinski, Igor E., Stichtenoth, Henning, \textit{On the value set of \( n! \) modulo a prime}, Turkish J. Math. \textbf{29} (2005), no. 2, 169–174.

\bibitem[BB’09]{BB09} Broughan, Kevin A., Barnett, A. Ross, \textit{On the missing values of \( n! \) modulo \( p \)}, J. Ramanujan Math. Soc. \textbf{24} (2009), no. 3, 277–284.

\bibitem[CD’06]{CD06} Chen, Yong-Gao, Dai, Li-Xia, \textit{Congruences with factorials modulo \( p \)}, Integers \textbf{6} (2006), A21, 3 pp.

\bibitem[Che’04]{Che04} Cheng, Qi, \textit{On the ultimate complexity of factorials}, Theoret. Comput. Sci. \textbf{326} (2004), no. 1-3, 419–429.

\bibitem[CGZ’03]{CGZ03} Cobeli, C. I., Gonek, S. M., Zaharescu, A., \textit{The distribution of patterns of inverses modulo a prime}, J. Number Theory \textbf{101} (2003), no. 2, 209–222.

\bibitem[CVZ’00a]{CVZ00a} Cobeli, Cristian, Văjâitu, Marius, Zaharescu, Alexandru, \textit{Average estimates for the number of tuples of inverses modulo \( p \) in short intervals}, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) \textbf{43(91)} (2000), no. 2, 155–164.

\bibitem[CVZ’00b]{CVZ00b} Cobeli, C., Văjâitu, M., Zaharescu, A., \textit{The sequence \( n! (\text{mod } p) \)}, J. Ramanujan Math. Soc. \textbf{15} (2000), no. 2, 153–154.

\bibitem[CVZ’03]{CVZ03} Cobeli, C., Văjâitu, M., Zaharescu, A., \textit{Distribution of gaps between the inverses modulo \( q \)}, Proc. Edinb. Math. Soc. (2) \textbf{46} (2003), no. 1, 185–203.

\bibitem[CZ’00]{CZ00} Cobeli, Cristian, Zaharescu, Alexandru, \textit{The order of inverses modulo \( q \)}, Mathematika \textbf{47} (2000), no. 1-2, 87–108 (2002).

\bibitem[CZ’01]{CZ01} Cobeli, Cristian, Zaharescu, Alexandru, \textit{Generalization of a problem of Lehmer}, Manuscripta Math. \textbf{104} (2001), no. 3, 301–307.

\bibitem[Dai’08]{Dai08} Dai, Lixia, \textit{Congruences with factorials modulo \( p, II \)}, J. Nanjing Norm. Univ. Nat. Sci. Ed. \textbf{31} (2008), no. 4, 33–36.

\bibitem[GH’17]{GH17} Garaev, M. Z., Hernández, J., \textit{A note on \( n! \) modulo \( p \)}, Monatsh. Math. \textbf{182} (2017), no. 1, 23–31.

\end{thebibliography}
FACTORIALS $\pmod{p}$ AND THE AVERAGE OF MODULAR MAPPINGS

[GLS'04] Garaev, Moubariz Z., Luca, Florian, Shparlinski, Igor E., *Character sums and congruences with $n!$*, Trans. Amer. Math. Soc. 356 (2004), no. 12, 5089–5102. 4

[GLS'05] Garaev, Moubariz Z., Luca, Florian, Shparlinski, Igor E., *Exponential sums and congruences with factorials*, J. Reine Angew. Math. 584 (2005), 29–44. 4

[Gar'07] García, Víctor C., *On the value set of $n!m!$ modulo a large prime*, Bol. Soc. Mat. Mexicana (3) 13 (2007), no. 1, 1–6. 4

[Gar'08] García, Víctor C., *Representations of residue classes by product of factorials, binomial coefficients and sum of harmonic sums modulo a prime*, Bol. Soc. Mat. Mexicana (3) 14 (2008), no. 2, 165–175. 4

[Guy'81] Guy, Richard K., *Unsolved problems in number theory*. Third edition. Problem Books in Mathematics. Springer-Verlag, New York, 2004 (first edition 1981). xviii+437 pp. 1, 3

[KM'17] Klurman, Oleksiy, Munsch, Marc, *Distribution of factorials modulo $p$*, J. Théor. Nombres Bordeaux 29 (2017), no. 1, 169–177. 3, 4

[Kur'00] Kurlberg, Pär, *The distribution of spacings between quadratic residues. II*, Israel J. Math. 120 (2000), part A, 205–224. 4

[Kur'09] Kurlberg, Pär, *Poisson spacing statistics for value sets of polynomials*, Int. J. Number Theory 5 (2009), no. 3, 489–513. 4

[KR'99] Kurlberg, Pär, Rudnick, Zeév, *The distribution of spacings between quadratic residues*, Duke Math. J. 100 (1999), no. 2, 211–242. 1

[Lev'06] Lev, Vsevolod F., *Permutations in abelian groups and the sequence $n! \pmod{p}$*, European J. Combin. 27 (2006), no. 5, 635–643. 3

[LS'05a] Luca, Florian, Shparlinski, Igor E., *Prime divisors of shifted factorials*, Bull. London Math. Soc. 37 (2005), no. 6, 809–817. 4

[LS'05b] Luca, Florian, Shparlinski, Igor E., *On the largest prime factor of $n! + 2^n - 1$*, J. Théor. Nombres Bordeaux 17 (2005), no. 3, 859–870. 4

[LS'03] Luca, Florian, Stanica, Pantelimon, *Products of factorials modulo $p$*, Colloq. Math. 96 (2003), no. 2, 191–205. 4

[Mar'15] Markström, Klas, *The straight line complexity of small factorials and primorials*, Integers 15 (2015), Paper No. A6, 8 pp. 4

[RS'60] Bokowska, B., Schinzel, A., *Sur un problème de M. Erdìs*, Elem. Math. 15 (1960), 84–85. 3

[Shp'12] Shparlinski, Igor E., *Modular hyperbolas*, Jpn. J. Math. 7 (2012), no. 2, 235–294. 4

[SS'95] Shub, Michael, Smale, Steve, *On the intractability of Hilbert’s Nullstellensatz and an algebraic version of “NP ≠ P”*, A celebration of John F. Nash, Jr. Duke Math. J. 81 (1995), no. 1, 47–54 (1996). 4

[Tru'14] Trudgian, Tim, *There are no socialist primes less than 109*, Integers 14 (2014), Paper No. A63, 4 pp. 3

CC: Simion Stoilow Institute of Mathematics of the Romanian Academy, 21 Calea Griviţei Street, P. O. Box 1-764, RO-014700, Bucharest, Romania

Email address: cristian.cobeli@imar.ro

AZ: Department of Mathematics, University of Illinois at Urbana-Champaign, Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA and Simion Stoilow Institute of Mathematics of the Romanian Academy, 21 Calea Griviţei Street, P. O. Box 1-764, RO-014700, Bucharest, Romania

Email address: zaharesc@illinois.edu