Vortices as Instantons in Noncommutative Discrete Space: Use of $\mathbb{Z}_2$ Coordinates

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Abstract

We show that vortices of Yang-Mills-Higgs model in $\mathbb{R}^2$ space can be regarded as instantons of Yang-Mills model in $\mathbb{R}^2 \times \mathbb{Z}_2$ space. For this, we construct the noncommutative $\mathbb{Z}_2$ space by explicitly fixing the $\mathbb{Z}_2$ coordinates and then show, by using the $\mathbb{Z}_2$ coordinates, that BPS equation for the vortices can be considered as a self-dual equation. We also propose the possibility to rewrite the BPS equations for vortices as ADHM equations through the use of self-dual equation.
1 Introduction

Topological solitons play an important role in various field theories. These are kink, vortex, baby-skyrmion, monopole, skyrmion, instanton and so on [1]. Some of the soliton equations are solved analytically, others are solved only numerically. It is interesting to look for the relations among the topological solitons. We consider a static soliton in Yang-Mills-Higgs (YMH) model in 2 + 1 dimensions. The static soliton is a vortex in 2-dimensional $R^2$ space. Some properties of Abelian vortex and non-Abelian vortex in YMH model have been studied [2, 3, 4]. The vortex configurations are solved numerically. The BPS (Bogomol’nyi-Prasad-Sommerfield) equations [5] for the vortex can be rewritten in terms of master equation plus half-ADHM (Atiyah-Drinfeld-Hitchin-Manin) equation [6]. The solution of half-ADHM equation contains information on the moduli space of the vortex, while instanton in 4-dimensional space are solved analytically by the ADHM method [7].

On the other hand, Higgs fields can be treated as gauge fields [8]. Note that, in these works discrete spaces are treated in terms of differential forms without the explicit use of the coordinates. We have been investigating the possibility of describing a vortex in 2-dimensional space as an instanton in 4-dimensional space, which is $R^2 \times Z_2$ space in this paper. In the previous paper [9], from a viewpoint of the noncommutative differential geometry and gauge theory in discrete space, we have shown that the instanton in $R^2 \times Z_2$ space is nothing but the vortex in $R^2$ space. This means that difference of vortex and instanton can be considered as that of the spaces $R^2 \times Z_2$ and $R^4$. In ref. [9], we did not explicitly discuss the relations between the Yang eq. and the master eq., due to lack of representation of the $Z_2$ coordinates. The ADHM method for vortices also requires the coordinate representation. By introducing the explicit form of the $Z_2$ coordinates, we can approach the problem from the new point of view. An attempt of this paper is the analysis using the explicit form of the noncommutative $Z_2$ coordinates. On the other hand, the arguments with the differential forms can not be cast straightforwardly into the coordinate picture. The purpose of this paper is to clarify the relation between instanton and vortex using the noncommutative $Z_2$ coordinates. We first define the coordinates for noncommutative $Z_2$ space and then investigate the relation between the instantons in $R^2 \times Z_2$ space and the vortices in $R^2$ space. In addition, we consider the relations among different descriptions of the vortices.

In section 2, we summarize properties of YMH model and fix the notations. In section 3, we construct a noncommutative $Z_2$ space. In section 4, we discuss relation between the instanton in $R^2 \times Z_2$ space and the vortex in YMH model in $R^2$ space. In section 5, we discuss relations among BPS, master and half-ADHM equations. The final section is devoted to summary and discussion.

2 Some properties of Yang Mills Higgs model

Let us summarize here some properties of the YMH model which has non-Abelian gauge symmetry [9]. The model contains a Higgs field, represented by $N_L \times N_R$ matrix, and two gauge fields corresponding to $U(N_L) \times U(N_R)$ gauge group. In this paper, we consider the models with $N_L = N_R = N$, where the solitons are local vortices. The Lagrangian in $2 + 1$
\[ \mathcal{L} = \text{Tr} \left( \frac{1}{2g^2} (F_L)^{\mu\nu} (F_L)^{\mu\nu} + \frac{1}{2g^2} (F_R)^{\mu\nu} (F_R)^{\mu\nu} \right) \]
\[ + \text{Tr} \left( (D_\mu H)^\dagger D^\mu H - \frac{g^2}{2} (H H^\dagger - c 1_N)^2 \right). \]  

(1)

Where, we define a covariant derivative \( D_\mu \) and field strength \( F_\mu^{\nu} \), \( F_\mu^{\nu} \) as
\[ D_\mu H = \partial_\mu H + L_\mu H - H R_\mu, \]  

(2)

\[ (F_L)^{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu], \]  

(3)

\[ (F_R)^{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu], \]  

(4)

and \( \text{Tr} \) is a trace over the adjoint representation of \( U(N) \). Two \( U(N) \) gauge fields \( L_\mu, R_\mu \) and the Higgs field \( H \) are represented by \( N \times N \) matrices. In the following we take \( g^2 = 2 \) and \( c = 1 \) for simplicity. The energy integral is of the form
\[ E = \int dx_1 dx_2 \text{Tr} \left( \frac{1}{2} |F_{12}^L|^2 + \frac{1}{2} |F_{12}^R|^2 + |D_1 H|^2 + |D_2 H|^2 + (H H^\dagger - 1_N)^2 \right). \]  

(5)

The BPS equations minimizing the energy are
\[ (D_1 \pm i D_2) H = 0, \]  

(6)

\[ i F_{12}^L \pm (H H^\dagger - 1_N) = 0, \]  

(7)

\[ i F_{12}^R \mp (H H^\dagger - 1_N) = 0, \]  

(8)

where we use the anti-Hermitian gauge fields \( L_\mu^\dagger = -L_\mu \) and \( R_\mu^\dagger = -R_\mu \) [4]. The solutions of the equations (6), (7), (8) are topologically stable solitons, called non-Abelian vortices. Where, 2 sets of equations are those for vortex and for anti-vortex. It is obvious that only pure gauge configurations are allowed at the spacial infinity \( |x| \to \infty \). This means that the topological property of the non-Abelian vortices is classified by the mapping index for \( S^1 \to U(N) \times U(N) \). On account of the fact that \( U(N) \) is equal to \( U(1) \times SU(N) \), the corresponding homotopy group is
\[ \pi_1 (U(N) \times U(N)) = \pi_1 (U(1) \times U(1)) = \mathbb{Z} \times \mathbb{Z}. \]  

(9)

We can take the topological charges corresponding to (9) as
\[ Q_{L-R} \equiv \frac{i}{2\pi} \int dx_1 dx_2 \text{Tr} (F_{12}^L - F_{12}^R) = 0, \pm 1, \pm 2, \cdots \]  

(10)

and
\[ Q_{L+R} \equiv \frac{i}{2\pi} \int dx_1 dx_2 \text{Tr} (F_{12}^L + F_{12}^R) = 0, \pm 1, \pm 2, \cdots. \]  

(11)

Here, \( Q_{L-R} \) is identified with the vortex number. On the other hand, topological charge \( Q_{L+R} \) is irrelevant to the vortex configuration, since gauge field \( \text{Tr}(L_\mu + R_\mu) \) does not interact with
other fields. Although the general configurations are classified by two topological charges \(Q_{L-R}\) and \(Q_{L+R}\), the vortex configurations are essentially classified by \(Q_{L-R}\). Because the BPS equations (7) and (8) mean that the \(U(1)\) part of \(F_{12}^L + F_{12}^R = 0\), and thus \(Q_{L+R} = 0\) for the vortex solutions.

Note that, although our YMH models have the \(U(N) \times U(N)\) gauge group with \(1 \leq N\), vortex solutions have some relations to those of the model with \(U(N)\) gauge group. Particularly, the model of \(U(1)_L \times U(1)_R\) gauge group is equivalent to the model of \(U(1)_{L-R}\) gauge group, since one of the combinations of gauge field, i.e. \(L + R\), decouples from other fields. For \(2 \leq N\), the relations among vortex solutions of the models with \(U(N)\) and \(U(N)_L \times U(N)_R\) gauge groups are shown in section 5.

Let us describe the notations for 4-dimensional space, since we construct the vortex in 2-dimensional space from a model in 4-dimensional space. The relation between Cartesian coordinates \((x_1, x_2, x_3, x_4)\) and complex coordinates \((z, \bar{z}, w, \bar{w})\) in 4-dimensional space are

\[
\begin{align*}
z &= \frac{1}{\sqrt{2}} (x_1 + ix_2), \quad \bar{z} = \frac{1}{\sqrt{2}} (x_1 - ix_2), \\
w &= \frac{1}{\sqrt{2}} (x_3 + ix_4), \quad \bar{w} = \frac{1}{\sqrt{2}} (x_3 - ix_4), \\
\partial_z &= \frac{1}{\sqrt{2}} (\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2}} (\partial_1 + i\partial_2), \\
\partial_w &= \frac{1}{\sqrt{2}} (\partial_3 - i\partial_4), \quad \partial_{\bar{w}} = \frac{1}{\sqrt{2}} (\partial_3 + i\partial_4).
\end{align*}
\] (12)

For \(R^4\) space, \((z, \bar{z}, w, \bar{w})\) are usual complex coordinates. While, for \(R^2 \times Z_2\) space used in this paper, \(w\) and \(\bar{w}\) are noncommutative coordinates to be defined in the next section. Gauge fields are defined by

\[
\begin{align*}
a_z &= \frac{1}{\sqrt{2}} (a_1 - ia_2), \quad a_{\bar{z}} = -\frac{1}{\sqrt{2}} (a_1 + ia_2), \\
a_w &= \frac{1}{\sqrt{2}} (a_3 - ia_4), \quad a_{\bar{w}} = -\frac{1}{\sqrt{2}} (a_3 + ia_4).
\end{align*}
\] (13)

Finally, relations between the gauge field strength in the complex and Cartesian coordinates are

\[
\begin{align*}
F_{zz} &= -iF_{12}, \quad F_{\bar{z}\bar{z}} = -iF_{34}, \\
F_{zw} &= -\frac{1}{2} (F_{13} + iF_{14} + F_{24} - iF_{23}), \\
F_{\bar{z}w} &= -\frac{1}{2} (F_{13} - iF_{14} + F_{24} + iF_{23}), \\
F_{zw} &= \frac{1}{2} (F_{13} - iF_{14} - F_{24} - iF_{23}), \\
F_{\bar{z}\bar{w}} &= \frac{1}{2} (F_{13} + iF_{14} - F_{24} + iF_{23}).
\end{align*}
\] (15)
3 Noncommutative $Z_2$ space

In this section, we construct a 2-dimensional noncommutative discrete $Z_2$ space, referring to the construction of noncommutative $R^2_{NC}$ space. In the case of $R^2_{NC}$ space, the complex coordinates are represented by the creation and annihilation operators on the Fock space $\{|n\rangle\}$ with $n = 0, 1, 2, 3, \cdots$ \[^{10}\] Then the commutation relation of the complex coordinates is proportional to the noncommutative parameter.

Now, we consider the coordinates $w$ and $\bar{w}$ of noncommutative discrete $Z_2$ space as the operators on the Fock space with 2-states $|0\rangle$ and $|1\rangle$. Our definition of $Z_2$ space is

$$w |0\rangle = 0, \quad \bar{w} |0\rangle = \sqrt{\theta} |1\rangle, \quad w |1\rangle = \sqrt{\theta} |0\rangle, \quad \bar{w} |1\rangle = 0,$$  

where $\theta$ is the noncommutative parameter. Then $Z_2$ coordinates can be represented by $2 \times 2$ matrices as

$$w = \sqrt{\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{w} = \sqrt{\theta} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where the Fock space is described by the vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

From (18), coordinates $w$ and $\bar{w}$ are characterized by anti-commutation relations

$$\{ w, w \} = \{ \bar{w}, \bar{w} \} = 0$$

and

$$\{ w, \bar{w} \} = \theta,$$  

where $\{ A, B \} \equiv AB + BA$. Note that, the noncommutative coordinates of $Z_2$ space satisfy the anti-commutation relations (20) and (21), in contrast to the case of $R^2_{NC}$ space, where the commutation relations $[w, w] = [\bar{w}, \bar{w}] = 0$ and $[w, \bar{w}] = \theta$ are satisfied. It means that the coordinates of $Z_2$ space are fermionic, while those of $R^2_{NC}$ space are bosonic.

Next, we define the differentiation by $w$ and $\bar{w}$ as “right-differential”, namely differentiation of a function $f(w, \bar{w})$ by $w$ (or $\bar{w}$) is defined by the following procedure. Move $w$ (or $\bar{w}$) to the right for each term in $f(w, \bar{w})$ with the help of (20) (21), and then differentiate by $w$ (or $\bar{w}$) on the right-hand side. This definition of the differentiation can also be described by use of the commutator as

$$\partial_w = -\theta^{-1} [\bar{w}, ] \sigma_3, \quad \partial_{\bar{w}} = \theta^{-1} [w, ] \sigma_3.$$  

Because of the nilpotency of $w$ and $\bar{w}$ (20), arbitrary function of $Z_2$ space can be expanded in five terms,

$$1, \quad w, \quad \bar{w}, \quad w\bar{w}, \quad \bar{w}w.$$  

Here, four terms are linearly independent under the relation (21). Explicit form of the differentials are given by

$$\partial_w 1 = \partial_w \bar{w} = 0, \quad \partial_w w = 1,$$

$$\partial_w \bar{w}w = -\bar{w}, \quad \partial_w w\bar{w} = \bar{w}.$$  

(24)
and
\[
\partial \bar{w} 1 = \partial \bar{w} w = 0, \quad \partial \bar{w} \bar{w} = 1, \\
\partial \bar{w} w = w, \quad \partial \bar{w} \bar{w} = -w.
\] (25)

These can also be represented by matrix form, corresponding to (13), as
\[
\partial \bar{w} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \frac{1}{\sqrt{\theta}} \left( \begin{array}{cc} B & 0 \\ -A + D & B \end{array} \right)
\] (26)
and
\[
\partial \bar{w} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \frac{1}{\sqrt{\theta}} \left( \begin{array}{cc} C & A - D \\ 0 & C \end{array} \right),
\] (27)
where we used the fact that
\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = A \frac{w \bar{w}}{\theta} + B \frac{w}{\sqrt{\theta}} + C \frac{\bar{w}}{\sqrt{\theta}} + D \bar{w} w \frac{1}{\theta}.
\] (28)

Furthermore, the integral in $w$ space is defined by the super trace on the Fock space \{\ket{0}, \ket{1}\} as
\[
\int_{Z_2} O d^2 w = \text{str} O \{ \langle 0 | O | 0 \rangle - \langle 1 | O | 1 \rangle \},
\] (29)
because of the anti-commutation relations (20) and (21). In the following, we take $\theta = 1$ for simplicity.

4 Vortices in $R^2$ space as instantons in $R^2 \times Z_2$ space

In this section, we discuss the YMH model in $R^2$ space which descends from the Yang-Mills (YM) model in $R^2 \times Z_2$ space, where $Z_2$ is the noncommutative discrete space. The following is the discussion on the self-dual equations in 4-dimensional $R^2 \times Z_2$ space and BPS equations for the vortex in 2-dimensional $R^2$ space. First, we sketch the argument in ref. [13] on the self-dual equations in pure $U(N)$ YM model in commutative $R^4$ space. As we shall see later, applying this discussion to the $R^2 \times Z_2$ space, BPS equations in YMH model can be obtained.

Their argument goes as follows. They consider the self-dual equation for pure $U(N)$ YM model in commutative $R^4$ space. From the relation (13), the self-dual equation
\[
F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}
\] (30)
can be rewritten as
\[
F_{\bar{z} w} = 0, \quad F_{\bar{z} \bar{w}} = 0, \quad F_{z \bar{z}} = F_{w \bar{w}}.
\] (31)

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can be rewritten as
\[
F_{\bar{z} w} = 0, \quad F_{\bar{z} \bar{w}} = 0, \quad F_{z \bar{z}} = F_{w \bar{w}}.
\] (31)
in commutative $z$ and $w$ coordinates. In this model, two $U(N)$ matrix functions $h$ and $\bar{h}$ are introduced with the definition of gauge field as

$$a_z = h^{-1} \partial_z h, \quad a_{\bar{z}} = \bar{h}^{-1} \partial_{\bar{z}} \bar{h},$$
$$a_w = h^{-1} \partial_w h, \quad a_{\bar{w}} = \bar{h}^{-1} \partial_{\bar{w}} \bar{h}.$$  \tag{34}

Then, a part of self-dual equations (31) and (32) are satisfied automatically. And from

$$F_{z\bar{z}} = h^{-1} \partial_z (g^{-1} \partial_{\bar{z}} g) h, \quad F_{w\bar{w}} = - (h^{-1} \partial_w (g^{-1} \partial_{\bar{w}} g) h),$$  \tag{35}

where

$$g = \bar{h} h^{-1},$$  \tag{36}

another equation (33) takes the form

$$\partial_z (g^{-1} \partial_{\bar{z}} g) + \partial_w (g^{-1} \partial_{\bar{w}} g) = 0.$$  \tag{37}

Equation (37) is called Yang equation [12].

To apply the above argument to the case of $R^2 \times Z_2$ space, where $Z_2$ space is noncommutative defined by (16), (17), we have to replace the coordinates $(w, \bar{w})$ in the previous argument by noncommutative discrete ones for the equations from (12) to (15) and from (30) to (37). Especially, the self-dual equations are

$$F_{z\bar{w}} = 0,$$
$$F_{z\bar{w}} = 0,$$
$$F_{z\bar{z}} = F_{w\bar{w}},$$  \tag{38}

where $(z, \bar{z})$ are the commutative $R^2$ coordinates and $(w, \bar{w})$ are the noncommutative $Z_2$ ones. Furthermore, $h$ and $\bar{h}$ are expressed by the $(N \times N) \otimes (2 \times 2)$ matrices

$$h = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad h^{-1} = \begin{pmatrix} 0 & c^{-1} \\ b^{-1} & 0 \end{pmatrix},$$  \tag{39}
$$\bar{h} = \begin{pmatrix} 0 & \bar{b} \\ \bar{c} & 0 \end{pmatrix}, \quad \bar{h}^{-1} = \begin{pmatrix} 0 & \bar{c}^{-1} \\ \bar{b}^{-1} & 0 \end{pmatrix}.$$

Namely, $h$ and $\bar{h}$ are expressed as $2 \times 2$ matrices (39), (40), and each matrix elements $b, c, \bar{b}, \bar{c}$ are $U(N)$ matrices. Replacing $R^4$ space by $R^2 \times Z_2$ space, the gauge field corresponding to $Z_2$ space can be considered as the Higgs field. In the following, we show the equivalence between the instanton of YM model in $R^2 \times Z_2$ space and the vortex of YMH model in $R^2$ space.

Now, we shall consider the gauge fields and strengths. We use the differential rules (22) or (26) (27) for the $Z_2$ coordinates. The gauge fields are given by

$$a_z = h^{-1} \partial_z h = \begin{pmatrix} c^{-1} \partial_z c & 0 \\ 0 & b^{-1} \partial_z b \end{pmatrix},$$  \tag{41}
$$a_{\bar{z}} = \bar{h}^{-1} \partial_{\bar{z}} \bar{h} = \begin{pmatrix} \bar{c}^{-1} \partial_{\bar{z}} \bar{c} & 0 \\ 0 & \bar{b}^{-1} \partial_{\bar{z}} \bar{b} \end{pmatrix}.$$  \tag{42}

7
\[ a_w = \bar{h}^{-1}\partial_w \bar{h} \]
\[ = \left( \begin{array}{cc} 0 & \bar{c}^{-1} \\ \bar{b}^{-1} & 0 \end{array} \right) \partial_w \left( \begin{array}{cc} 0 & \bar{b} \\ \bar{c} & 0 \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0 & \bar{c}^{-1}\bar{b} \\ 1 & 0 \end{array} \right) \]  
(43)

and

\[ a_{\bar{w}} = h^{-1}\partial_{\bar{w}} \bar{h} \]
\[ = \left( \begin{array}{cc} 0 & c^{-1} \\ b^{-1} & 0 \end{array} \right) \partial_{\bar{w}} \left( \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0 & 1 \\ b^{-1}c & 0 \end{array} \right). \]  
(44)

Then we define the gauge fields \( L, R \) and the Higgs field \( H \) as

\[ L_z = c^{-1}\partial_z c, \quad L_{\bar{z}} = \bar{c}^{-1}\partial_{\bar{z}} \bar{c}, \]
\[ R_z = b^{-1}\partial_z b, \quad R_{\bar{z}} = \bar{b}^{-1}\partial_{\bar{z}} \bar{b} \]  
(45)

and

\[ H = \bar{c}^{-1}\bar{b}, \quad H^\dagger = b^{-1}c, \]  
(46)

respectively. Here, \( h \) and \( \bar{h} \) are related as

\[ \bar{h}^\dagger = \bar{h}^{-1} \text{ or } (b^\dagger = b^{-1}, \quad c^\dagger = c^{-1}), \]  
(47)

and the gauge fields are anti-Hermitan

\[ L_z^\dagger = -L_{\bar{z}}, \quad R_z^\dagger = -R_{\bar{z}}. \]  
(48)

The field strengths are calculated as follows. First, \( F_{zw} \) and \( F_{\bar{z}\bar{w}} \) are calculated as

\[
F_{zw} = \partial_z a_w - \partial_w a_z + [a_z, a_w]
\]
\[ = \partial_z \left( \begin{array}{cc} 0 & \bar{c}^{-1}\bar{b} \\ 1 & 0 \end{array} \right) - \partial_w \left( \begin{array}{cc} \bar{c}^{-1}\partial_z \bar{c} & 0 \\ 0 & \bar{b}^{-1}\partial_z \bar{b} \end{array} \right) \]
\[ + \left[ \left( \begin{array}{cc} \bar{c}^{-1}\partial_z \bar{c} & 0 \\ 0 & \bar{b}^{-1}\partial_z \bar{b} \end{array} \right), \left( \begin{array}{cc} 0 & \bar{c}^{-1}\bar{b} \\ 1 & 0 \end{array} \right) \right] \]
\[ = \left( \begin{array}{cc} 0 & \partial_z (\bar{c}^{-1}\bar{b}) + (\bar{c}^{-1}\partial_z \bar{c}) \bar{c}^{-1}\bar{b} - \bar{c}^{-1}\bar{b} (\bar{b}^{-1}\partial_z \bar{b}) \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0 & D_z H \end{array} \right) \]  
(49)
and
\begin{align*}
F_{z\bar{w}} &= \partial_z a_{\bar{w}} - \partial_{\bar{w}} a_z + [a_z, a_{\bar{w}}] \\
&= \partial_z \begin{pmatrix} 0 & 1 \\
-1 & 0 
\end{pmatrix} - \partial_{\bar{w}} \begin{pmatrix} c^{-1} \partial_z c & 0 \\
0 & b^{-1} \partial_z b 
\end{pmatrix} \\
&+ \left[ \begin{pmatrix} c^{-1} \partial_z c & 0 \\
0 & b^{-1} \partial_z b 
\end{pmatrix}, \begin{pmatrix} 0 & 1 \\
0 & b^{-1} c 
\end{pmatrix} \right] \\
&= \begin{pmatrix} \partial_z (b^{-1} c) + (b^{-1} \partial_z b) b^{-1} c - b^{-1} c (c^{-1} \partial_z c) & 0 \\
0 & D_z H^\dagger 0 
\end{pmatrix},
\end{align*}
(50)
where
\begin{equation}
D_z H = \partial_z H + L_z H - H R_z
\end{equation}
(51)
and
\begin{equation}
D_z H^\dagger = (D_z H)^\dagger = \partial_{\bar{z}} H^\dagger - H^\dagger L_z + R_z H^\dagger.
\end{equation}
(52)

Note that the commutator term \([a, a]\) is needed even for the \(U(1)\) case because of the non-commutativity of \(Z_2\) space. As in the case of \(R^4\),
\begin{equation}
F_{z\bar{w}} = 0
\end{equation}
(53)
and
\begin{equation}
F_{z\bar{w}} = 0
\end{equation}
(54)
are satisfied automatically with the definition of gauge fields by \(h\) and \(\bar{h}\). Equations (49) (53) and (50) (54) mean
\begin{equation}
D_z H = 0
\end{equation}
(55)
and
\begin{equation}
D_z H^\dagger = (D_z H)^\dagger = 0
\end{equation}
(56)
respectively, and are nothing but a part of BPS equations for YMH model in \(R^2\).

Similarly,
\begin{align*}
F_{z\bar{w}} &= \begin{pmatrix} 0 & D_z H \\
0 & 0 
\end{pmatrix}, \quad F_{z\bar{w}} = \begin{pmatrix} 0 & 0 \\
D_z H^\dagger & 0 
\end{pmatrix}
\end{align*}
(57)
are derived, where
\begin{equation}
D_z H = \partial_z H + L_z H - H R_z
\end{equation}
(58)
and
\begin{equation}
D_z H^\dagger = (D_z H)^\dagger = \partial_{\bar{z}} H^\dagger - H^\dagger L_z + R_z H^\dagger.
\end{equation}
(59)
Finally, $F_{w\bar{w}}$ becomes
\[
F_{w\bar{w}} = \partial_w a_{\bar{w}} - \partial_{\bar{w}} a_w + [a_w, a_{\bar{w}}] \\
= \partial_w \left( \begin{array}{cc} 0 & 1 \\ b^{-1}c & 0 \end{array} \right) - \partial_{\bar{w}} \left( \begin{array}{cc} 0 & \bar{c}^{-1}\bar{b} \\ 1 & 0 \end{array} \right) \\
+ \left[ \left( \begin{array}{cc} 0 & \bar{c}^{-1}\bar{b} \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ b^{-1}c & 0 \end{array} \right) \right] \\
= \left( \begin{array}{cc} \partial_w (H H^\dagger - 1) & - \partial_{\bar{w}} (H^\dagger H - 1) \\ 0 & 0 \end{array} \right),
\] (60)

using (43) and (44). $F_{z\bar{z}}$ is calculated as
\[
F_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] \\
= \partial_z \left( \begin{array}{cc} c^{-1}\partial_z \bar{c} & 0 \\ 0 & \bar{b}^{-1}\partial_z \bar{b} \end{array} \right) - \partial_{\bar{z}} \left( \begin{array}{cc} c^{-1}\partial_{\bar{z}} c & 0 \\ 0 & b^{-1}\partial_{\bar{z}} b \end{array} \right) \\
+ \left[ \left( \begin{array}{cc} c^{-1}\partial_z c, \bar{c}^{-1}\partial_{\bar{z}} \bar{c} \\ 0 & \bar{b}^{-1}\partial_z b, \bar{b}^{-1}\partial_{\bar{z}} \bar{b} \end{array} \right) \right] \\
= -i \left( \begin{array}{cc} F_{12}^L & 0 \\ 0 & F_{12}^R \end{array} \right),
\] (61)

From (60) and (61), the self-dual equation (33)
\[
F_{z\bar{z}} = F_{w\bar{w}}
\] (62)

reduces to the BPS equations
\[
iF_{12}^L = 1 - HH^\dagger,
\] (63)
\[
iF_{12}^R = H^\dagger H - 1.
\] (64)

These are also expressed by Yang equation
\[
\partial_z (g^{-1}\partial_z g) + \partial_w (g^{-1}\partial_w g) = 0,
\] (65)

where
\[
g = \bar{h}h^{-1},
\] (66)

and $h$, $\bar{h}$ are given by (39) and (40).

The above argument shows that the vortex in $R^2$ space can be regarded as an instanton in $R^2 \times Z_2$ space, since the self-dual equations (38) of YM model in $R^2 \times Z_2$ space is equivalent to the BPS equations of YMH model in $R^2$ space.

Furthermore, we can see that the YM model in $R^2 \times Z_2$ space also reduces to the YMH model in $R^2$ space at the level of static part of the Lagrangian. For the static configurations, square of field strength becomes
\[
\frac{1}{4} |F_{w\bar{w}}|^2 = \frac{1}{2} |F_{zz}|^2 + \frac{1}{2} |F_{w\bar{w}}|^2 + \frac{1}{2} F_{zw}F_{z\bar{w}} + \frac{1}{2} F_{\bar{z}w}F_{z\bar{w}} \\
\equiv \left( \begin{array}{cc} L_1 & 0 \\ 0 & L_2 \end{array} \right),
\] (67)
where 
\[
L_1 = \frac{1}{2} (F_{i12}^L)^2 + \frac{1}{2} D_z HD_z H^\dagger + \frac{1}{2} (1 - HH^\dagger)^2, \\
L_2 = \frac{1}{2} (F_{i12}^R)^2 + \frac{1}{2} D_z H^\dagger D_z H + \frac{1}{2} (1 - H^\dagger H)^2.
\]

(68)

Then, in the case of YM model for the \(U(N) \times U(N)\) gauge fields and 1-Higgs field, the Lagrangian density of YM model in \(R^2 \times Z_2\) space is given by
\[
\mathcal{L} = \begin{pmatrix} \text{Tr} L_1 & 0 \\ 0 & \text{Tr} L_2 \end{pmatrix} \sigma_3,
\]

(69)

where \(\text{Tr}\) means the trace of \(U(N)\) matrix and \(\sigma_3\) comes from the volume element derived from the metric of the \(Z_2\) space. Then the action \(S\) is obtained as
\[
S = \int_{R^2 \times Z_2} \text{str} \mathcal{L} d^2 xd^2 w \\
= \int \text{str} \mathcal{L} d^2 x \\
= \int \text{str} \left\{ \begin{pmatrix} \text{Tr} L_1 & 0 \\ 0 & \text{Tr} L_2 \end{pmatrix} \sigma_3 \right\} d^2 x \\
= \int \text{Tr} \left\{ \frac{1}{2} (F_{i12}^L)^2 + \frac{1}{2} (F_{i12}^R)^2 + \frac{1}{2} D_z HD_z H^\dagger + \frac{1}{2} D_z H^\dagger H + (1 - HH^\dagger)^2 \right\} d^2 x.
\]

(70)

This gives the action of the YMH model in \(R^2\) space \(^{11}\) with \(g^2 = 2\) and \(c = 1\) for the static configurations. It can also be verified that the instanton number, denoted as \(Q_I\), in \(R^2 \times Z_2\) space is just the vortex number in \(R^2\) space as follows.
\[
Q_I \equiv -\frac{1}{8\pi} \int \text{str} \left\{ \left( \text{Tr} F \bar{F} \right) \sigma_3 \right\} d^2 x \\
= -\frac{1}{8\pi} \int \text{str} \left\{ \left( \text{Tr} \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} F_{\mu \nu} F_{\alpha \beta} \right) \sigma_3 \right\} d^2 x \\
= -\frac{1}{2\pi} \int \text{str} \left\{ \text{Tr} \left( -F_{zz} \bar{F}_{w \bar{w}} - F_{z \bar{w}} \bar{F}_{z \bar{w}} + F_{z \bar{w}} \bar{F}_{z \bar{w}} \right) \sigma_3 \right\} d^2 x \\
= -\frac{1}{2\pi} \int \left[ \text{Tr} \left\{ iF_{i12}^L \left( HH^\dagger - 1 \right) + iF_{i12}^R \left( 1 - H^\dagger H \right) \right\} \\
+ \text{Tr} \left\{ -D_z HD_z H^\dagger + D_z H^\dagger H \right\} \right] d^2 x \\
= \frac{i}{2\pi} \int \text{Tr} \left( F_{i12}^L - F_{i12}^R \right) d^2 x \\
= Q_{L-R}.
\]

(71)

5 BPS, master and half-ADHM equations

In the first part of this section, we study the BPS equation, master equation and half-ADHM equation for YMH models with \(U(N)\) and \(U(N) \times U(N)\) gauge groups. In the latter part
of this section, we study the relation between formulations for soliton equation discussed in section 4 and that of master equation plus half-ADHM equation. We show that, for these two models, the BPS equation for the vortex with certain topological number can be expressed by the master equation plus half-ADHM equation. Furthermore, we see that the vortex solution in two models satisfies the common half-ADHM equation. In addition, we comment on some Abelian and non-Abelian vortices in both YMH models. Finally, we obtain the relation between the variables in the two formulations.

First, we summarize the YMH model with $U(N)$ gauge group \[6\]. The Lagrangian is

$$\mathcal{L} = \text{Tr} \left( \frac{1}{2g^2} (F^L)^{\mu\nu} (F^L)^{\mu\nu} + (D_\mu H)^\dagger D^\mu H - \frac{g^2}{4} (HH^\dagger - c1_N)^2 \right).$$

(72)

Where, we define a covariant derivative $D_\mu$ and field strength $F^{L}_{\mu\nu}$ as

$$D_\mu H = \partial_\mu H + L_\mu H$$

(73)

and

$$F^{L}_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu].$$

(74)

We take $g^2 = 2$ and $c = 1$ in the following. BPS equations are

$$D_\bar{z} H = \partial_\bar{z} H + L_\bar{z} H = 0$$

(75)

and

$$iF_{12} = 1 - HH^\dagger.$$  

(76)

Let us introduce a $N \times N$ invertible matrix $S(z, \bar{z}) \in GL(N, \mathbb{C})$ and consider a gauge invariant quantity defined by

$$\Omega(z, \bar{z}) \equiv S(z, \bar{z}) S^\dagger(z, \bar{z}).$$

(77)

Then the Higgs field and gauge field can be written as

$$H = S^{-1} H_0,$$

(78)

$$L_\bar{z} = S^{-1} \partial_\bar{z} S.$$  

(79)

Here, $H_0(z)$ is the $N \times N$ matrix and has elements consisting of holomorphic functions of $z$. The first BPS equation (75) could be solved for arbitrary $S$ on account of these relations. And the second BPS equation (76) is written in the form of

$$\partial_\bar{z} (\Omega^{-1} \partial_\bar{z} \Omega) = 1 - \Omega^{-1} H_0 H_0^\dagger.$$  

(80)

This equation is called master equation \[6\] for the vortices. The vortex number is given by

$$Q \equiv \frac{i}{2\pi} \int dx_1 dx_2 \text{Tr} F_{12} = 0, \pm 1, \pm 2, \cdots.$$  

(81)

From the master equation, at $|z| \to \infty$

$$\Omega = H_0 H_0^\dagger$$  

(82)
for vortex configurations, since the left side of (80) is
\[ \partial_z (\Omega^{-1} \partial \Omega) = (S^*)^{-1} (F^{L}_{12}) S^* \to 0 \text{ at } |z| \to \infty. \] (83)
Then, the vortex number (81) can be rewritten as
\[ Q = k = \frac{1}{4\pi} \text{Im} \oint dz \partial_z \log \left( \det H_0 H_0^\dagger \right) = \frac{1}{2\pi} \text{Im} \oint dz \partial_z \log \left( \det H_0 \right). \] (84)
This representation for the topological charge makes it clear that \( H_0 \) behaves like \( \det H_0 \sim z^k \) at the spatial infinity \( |z| \to \infty \). Moreover, \( H_0(z) \) can be considered as a solution of the half-ADHM equation
\[ \nabla^\dagger L = 0. \] (85)
Here,
\[ L^\dagger \equiv (H_0(z), J(z)), \]
\[ \nabla \equiv \left( \begin{array}{c} -\Psi \\ z - Z \end{array} \right), \] (86)
and \( H_0, J, \Psi \) and \( Z \) are \( N \times N, k \times N, k \times N \) and \( k \times k \) matrices, respectively. \( \Psi \) and \( Z \) are constant matrices and have a meaning of moduli parameters. As a result, BPS equations reduce to the master equation plus half-ADHM equation by introducing variables \( S \) and \( H_0 \).
Here, \( H_0 \) is given as a solution of the half-ADHM equation. And, for given \( H_0 \), \( S \) is solved as a solution of the master equation.
Next, we extend the above argument to the case of \( U(N) \times U(N) \) gauge fields (\( L_\mu \) and \( R_\mu \)) \( 11 \). The BPS equations are
\[ D_z H = \partial_z H + L_z H - HR_z = 0, \] (87)
\[ iF^L_{12} = 1 - HH^\dagger, \] (88)
\[ iF^R_{12} = H^\dagger H - 1. \] (89)
Expressing the Higgs field and gauge field as
\[ H = S^{-1}(z, \bar{z}) H_0(z) T(z, \bar{z}), \] (90)
\[ L_z = S^{-1} \partial \bar{z} S, \] (91)
\[ R_z = T^{-1} \partial \bar{z} T, \] (92)
BPS equation (87) is satisfied automatically and BPS equations (88) (89) are reduced to the two master equations
\[ \partial_z (\Omega^{-1}_S \partial_z \Omega_S) = 1 - \Omega^{-1}_S H_0 \Omega_T H_0^\dagger, \]
\[ \partial_z (\Omega^{-1}_T \partial_z \Omega_T) = -1 + H_0^\dagger \Omega^{-1}_S H_0 \Omega_T, \] (93)
where

\[ \Omega_S \equiv S(z, \bar{z}) S(z, \bar{z})^\dagger, \]
\[ \Omega_T \equiv T(z, \bar{z}) T(z, \bar{z})^\dagger. \]  \hfill (94)

At \( |z| \to \infty \), we can see the following. Finite energy of the static energy (5) means that \( U(N) \times U(N) \) gauge fields \( L_\mu \) and \( R_\mu \) go to pure gauge configurations. It is possible to send the \( SU(N) \times SU(N) \) part of gauge fields to zero, because of the homotopy

\[ \pi_1(SU(N)) = 0. \]  \hfill (95)

Then \( S \) and \( T \) can be expressed by elements of \( U(1) \) as

\[ S(z, \bar{z}) = s(z, \bar{z}) \cdot 1_N, \quad T(z, \bar{z}) = t(z, \bar{z}) \cdot 1_N, \]  \hfill (96)

where \( s(z, \bar{z}) \) and \( t(z, \bar{z}) \) are scalar functions. Defining

\[ S' \equiv s(z, \bar{z}) t(z, \bar{z})^{-1}, \]  \hfill (97)

Higgs field \( H_0 \) and \( U(1) \) part of the gauge field are expressed as

\[ H = S'H_0 \]  \hfill (98)

and

\[ \text{Tr}(L_{\bar{z}} - R_\bar{z}) = (S')^{-1} \partial_{\bar{z}} S'. \]  \hfill (99)

Then, by the replacement \( S \to S' \), the topological charge \( Q_{L-R} \) in \( U(N) \) YMH model reduces to that in \( U(N) \times U(N) \) model.

As a result, vortex number \( Q_{L-R} \), given by (10), can be expressed by

\[ Q_{L-R} = \frac{1}{4\pi} \Im \oint dz \partial_z \log \left( \det H_0 H_0^\dagger \right), \]  \hfill (100)

which is same as (84). Therefore, \( H_0 \) satisfies the common half-ADHM equation in each case of YMH model with \( U(N) \) and \( U(N) \times U(N) \) gauge groups. On the other hand, the master equation turns to the coupled equations for \( \Omega_S \) and \( \Omega_T \) in the YMH model with \( U(N) \times U(N) \) gauge group.

Here, we comment on the vortex solutions of \( U(1) \times U(1) \) and \( U(N) \times U(N) \) YMH models. It is known that when \( F_{12}^* \) and \( H^* \) are a numerical vortex solution of \( U(1) \) YMH model, a vortex solution of \( U(N) \) YMH model can be constructed by embedding this vortex solution as

\[ F_{12} = U \text{diag}(F_{12}^*, 0, \cdots, 0) U^{-1}, \quad H = U \text{diag}(H^*, 1, \cdots, 1) U^{-1}, \]  \hfill (101)

where \( U \) takes a value in \( CP^{N-1} \). On the other hand, we can show that a vortex solution of \( U(1) \times U(1) \) YMH model is expressed by that of \( U(1) \) model by comparing both BPS equations. That is, denoting a vortex configuration with topological number \( m \) of \( U(1) \) model
with $g^2 = 4$ and $c = 1$ as $\tilde{F}^*_{12}$ and $\tilde{H}^*$, a vortex of the $U(1) \times U(1)$ YMH model (1) (with $g^2 = 2$ and $c = 1$) is given by

$$F^L_{12} = -F^R_{12} = \frac{1}{2} \tilde{F}^*_{12}, \quad H = \tilde{H}^*. \quad (102)$$

And a non-Abelian vortex of the $U(N) \times U(N)$ YMH model is constructed as

$$F^L_{12} = U \text{diag}(\frac{1}{2} \tilde{F}^*_{12}, 0, \ldots, 0) U^{-1}, \quad F^R_{12} = U \text{diag}(-\frac{1}{2} \tilde{F}^*_{12}, 0, \ldots, 0) U^{-1}, \quad H = U \text{diag}(\tilde{H}^*, 1, \ldots, 1) U^{-1}. \quad (103)$$

As mentioned in section 2, it is obvious that the topological charge $Q_{L+R} = 0$ for the Abelian vortex (102) and non-Abelian vortex (103), since $\text{Tr} (F^L_{12} + F^R_{12}) = 0$. And charge $Q_{L-R} \equiv \int \frac{i}{2\pi} dx_1 dx_2 \text{Tr} (F^L_{12} - F^R_{12}) = \int \frac{i}{2\pi} dx_1 dx_2 \tilde{F}^*_{12} = m$ counts the vortex number.

Finally, we consider the relation between variables $(h$ and $\tilde{h})$ and variables $(S, T$ and $H_0)$ in YMH model with $U(N) \times U(N)$ gauge group. A relation for the variables can take the form

$$\tilde{h} = \begin{pmatrix}
0 & \bar{b} \\
\bar{c} & 0
\end{pmatrix} = \begin{pmatrix}
0 & \tilde{H}^T_0 (z) S \\
H^S_0 (z) & 0
\end{pmatrix}. \quad (104)$$

We can check that two formulations lead the same Higgs field and gauge fields. As the formulation given above in this section, taking the variables $S, T, H_0$, Higgs field and gauge fields are given by

$$H = S^{-1} (z, \bar{z}) H_0 (z) T (z, \bar{z}) , \quad (105)$$

$$L_{\bar{z}} = S^{-1} \partial_{\bar{z}} S, \quad R_{\bar{z}} = T^{-1} \partial_{\bar{z}} T, \quad (106)$$

respectively. On the other hand, for the formulation discussed in section 4, taking the variable $\tilde{h}$ as (104), Higgs field and gauge fields are given by

$$H = \tilde{c}^{-1} \bar{b} = S^{-1} (H^S_0)^{-1} H^T_0 T, \quad (107)$$

$$L_{\bar{z}} = \tilde{c}^{-1} \partial_{\bar{z}} \bar{c} = S^{-1} \partial_{\bar{z}} S, \quad (108)$$

$$R_{\bar{z}} = \bar{b}^{-1} \partial_{\bar{z}} \bar{b} = T^{-1} \partial_{\bar{z}} T. \quad (109)$$

Then, the condition that the two formulations give the same fields is

$$(H^S_0 (z))^{-1} H^T_0 (z) = H_0 (z) . \quad (110)$$

There exists some ambiguity in the relations of variables. A simple relation is given by

$$\tilde{h} = \begin{pmatrix}
0 & \bar{b} \\
\bar{c} & 0
\end{pmatrix} = \begin{pmatrix}
0 & H_0 (z) T (z, \bar{z}) \\
S (z, \bar{z}) & 0
\end{pmatrix}. \quad (111)$$
6 Summary and Discussion

In this paper, we have defined the coordinates for noncommutative $Z_2$ space and have investigated the relation between the instantons in $R^2 \times Z_2$ space and the vortices in $R^2$ space. We have shown that the vortices of YMH model in $R^2$ space can be regarded as the instantons of YM model in $R^2 \times Z_2$ space. The BPS equation for the vortices can be considered as a self-dual Yang-Mills equation and is related to the ADHM equation. We also have obtained the relations between the master equation for the vortices and the Yang equation for the instantons.

It may be expected that the ADHM method can also be applied to the construction of the vortex solutions. However, extension of ADHM equation into the $R^2 \times Z_2$ space is not straightforward. The reason can be traced to noncommutativity of $\partial_3$ and $\partial_4$ (or $\partial_3$ and $\partial_4$).

Writing the Dirac operators as

$$D_x \equiv e^\mu \otimes D_\mu = e^\mu \otimes (\partial_\mu + A_\mu),$$
$$\bar{D}_x \equiv \bar{e}_\mu \otimes D_\mu = -D_\mu^1,$$ (112)

where

$$e_\mu = (-i\sigma_1, 1), \bar{e}_\mu = (i\sigma_1, 1)$$ (113)

are quaternions. Square of Dirac operators are written as

$$\bar{D}_x D_x = 1_2 \otimes D_\mu D_\mu + in_{\mu\nu}^{i(\pm)} \sigma_i \otimes D_\mu D_\nu,$$ (114)

where

$$n_{\mu\nu}^{i(\pm)} = \epsilon_{i\mu\nu4} \pm \delta_{i\mu} \delta_{i\nu} \mp \delta_{i\nu} \delta_{i\mu}$$ (115)

are 't Hooft symbols. The last term of equation (114) can be written for $R^4$ space as

$$in_{\mu\nu}^{i(\pm)} \sigma_i \otimes D_\mu D_\nu = i\sigma_1 \otimes \{F_{23} + F_{14}\} + i\sigma_2 \otimes \{-F_{13} + F_{24}\} + i\sigma_3 \otimes \{F_{12} + F_{34}\},$$ (116)

and the condition

$$[\bar{D}_x D_x, \sigma_i] = 0$$ (117)

leads to the (anti-)self-dual equation

$$F_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}. $$ (118)

For $R^2 \times Z_2$ space, however, we have

$$in_{\mu\nu}^{i(\pm)} \sigma_i \otimes D_\mu D_\nu = i\sigma_1 \otimes \{F_{23} + F_{14}\} + i\sigma_2 \otimes \{-F_{13} + F_{24}\} + i\sigma_3 \otimes \{F_{12} + F_{34} + [\partial_3, \partial_4]\},$$ (119)

and because of noncommutativity of $Z_2$ space (117) does not lead to the self-duality equation and we have to find a different constraint. Furthermore, unlike the case of noncommutative ADHM, $[\partial_3, \partial_4]$ is not a constant, thus we have to find a different modification. Consequently,
it is possible that ADHM equations are not pure algebraic equations but include differential equations in $R^2$ space. And this could be related to the fact that it is impossible to obtain the vortex solutions analytically.

We have compared our YMH model that contains two gauge fields with YMH model with only one gauge field. In the latter model, we can rewrite the BPS equations into the master equation plus half-ADHM equation. We can do the same in the former model, the BPS equations also reduce to the master plus half-ADHM equations and the half-ADHM equations in both models coincide exactly with each other. Furthermore we have studied both Abelian and non-Abelian vortices and the interrelations among them.

Although we have defined our $Z_2$ through equations (16), (17), there exist other possibilities and they are probably worthwhile to be considered. Furthermore, it has been proposed that there exists similar relation in the case of the model on compact Riemann surface [13]. It would be interesting to examine the relations between our work and their approach.

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