Spin glass models with Kac interactions

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Abstract. In this paper I will review my work on disordered systems -spin glass model with two body and $p > 2$ body interactions- with long but finite interaction range $R$. I will describe the relation of these model with Mean Field Theory in the Kac limit and some attempts to go beyond mean field.

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1 Introduction

In statistical physics mean-field theories play the important role of offering a first rough approximate scheme to understand collective phenomena and phase transitions. However several well known pathologies plague this description, which can be traced in the fact that the finite range character of the interactions of physical systems is neglected. Many of the progresses in the comprehension of the physics of pure (i.e. non-disordered) equilibrium and dynamical systems in the statistical physics of 20th century can be viewed as amendment and extensions of mean-field schemes to account of more complex behavior. A remarkable example is given by nucleation theory of first order transitions and phase coexistence that can be viewed as a non-perturbative expansion in the inverse of the interaction range. The role of an infinite interaction range in the mean-field description of first-order transition and metastability phenomena, had been emphasized in classical papers by Kac, Uhlenbeck and Hemmer [2] in 1D and Lebowitz and Penrose [3] in arbitrary D, showing that when range of interaction $R$ is sent to infinity after the thermodynamic limit, mean-field description emerges, complemented by the Maxwell construction that eliminates possible unphysical thermodynamical instability of mean-field theory.

In this contribution I will discuss the relation between mean-field theory and finite D physics in spin glass models and generalized spin glass models with $p$-body interactions. Mean field theory for these systems predicts ergodicity breaking with the coexistence of many phases unre-
lated by physical symmetry. However for finite range spin glasses, the mere existence of an upper critical dimension above which there is a low temperature spin glass phase with the MF characteristics has been questioned and is still matter of debate [4]. Alternative theories describe the finite D spin glass phase in all dimensions as a coexistence of two phases related by spin reversal symmetry [5]. The case of generalized spin glasses, or \( p \)-spin models, which is interesting in connection with structural glass physics, also deserves some attention. Mean field theory in fact predicts the existence of an exponential multiplicity of metastable states, able to dynamically confine the system in high free-energy regions. For a long time it has been acknowledged the necessity to properly include finite interaction range effects in order to describe the barrier crossing between states. The understanding of this phenomenon in Kac models would pose the basis for a theory of activated processes in finite D glassy systems.

This is the organization of the present contribution: in Section 2 the basic models are defined. In section 3 I discuss the behavior of the systems in the Kac limit and physical implications for systems with large but finite interaction range. In Section 4 we discuss some numerical results for the 1D Kac spin glass. In Section 5 we discuss correlation lengths in \( p \)-spin models. Finally, we draw our conclusions.

2 Spin glasses with Kac-type interactions

Spin-glass with Kac interactions were first defined in [6]. Here we will refer to the models given in [7], where a \( p \)-spin model with Kac-like interaction is defined by the Hamiltonian:

\[
H^p(\sigma) = - \sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \tag{1}
\]

where \( \sigma_i \) (\( i = 1, \ldots, N \)) are Ising or spherical spins in the D dimensional cubic box \( \Lambda = \{1, 2, \ldots, L\}^D \) with periodic boundary conditions (with \( N = L^D \)) and the couplings \( J_{i_1 \ldots i_p} \) are Gaussian i.i.d. random variables with zero average and variance

\[
E J^2_{i_1 \ldots i_p} = \frac{p!}{R^{Dp}} \sum_{k \in \Lambda} \psi(|i_1 - k|/R) \cdots \psi(|i_p - k|/R) \tag{2}
\]

where \( \psi(x) \) is a range one positive function normalized in a way that \( \int_{R^d} d^d x \, \psi(|x|) = 1 \). The variable \( R \) is the interaction range: the form (2) is a convenient form to let only groups of variables within a distance of order \( R \) effectively interact.

Notice that the interaction is scaled in a way that if \( R = L \) one effectively recover the mean-field model where

\[
E J^2_{i_1 \ldots i_p} = \frac{p!}{N^{p-1}}.
\]

3 The Kac limit

3.1 Free-energy

In the previous section we have seen that the interaction range \( R \) is taken to be equal to \( L \), then one recovers the mean field model. The scope of this section is to discuss in an informal way the physical implication of two theorems concerning the behavior of the the disordered models (1) in the Kac limit. One would like then to understand for large \( R \), but \( R \ll L \). It is clear that a necessary condition
for applicability of mean-field theory to finite range systems is a smooth crossover from \( R \ll L \) to \( R \sim L \). The first quantity of interest is of course the free-energy. Consider the the finite volume - finite range quenched free-energy, with standard notations:

\[
f_{L,R}(T) = -\frac{T}{N}E \log Z_{L,R}(T; J).
\]  

(3)

One would like to compare the behavior of the infinite volume free-energy for finite \( R \)

\[
\lim_{R \to \infty} \lim_{L \to \infty} f_{L,R}(T)
\]

(4)
to the one of the mean field model, for which

\[
f^{MF}(T) = \lim_{L \to \infty} f_{L,L}(T).
\]

(5)

Let us consider the behavior of the infinite volume free-energy \( f_R(\beta) = \lim_{L \to \infty} f_{L,R}(\beta) \). This is an existing function, self-averaging with probability one with respect to the realization of the random couplings. A theorem first proved in [8], insures that for all temperatures in the Kac limit \( R \to \infty \) the Mean-Field free-energy is recovered. This is a continuity result: the numerical value of the free-energy for large \( R \) is close to the one of the MF function.

One may notice that in the context of spin glass models there are no non-convexities of the free-energy as a function of the temperature and it does not arise the need of Maxwell construction.\(^1\) Unfortunately, although there is no particular reason to doubt of the validity of the Maxwell construction, a formal prove is still to be provided.

Without entering into the details of the prove let us mention that this is based on interpolation inequalities [11] between the Kac models and their Mean Field correspondent.

### 3.2 Local Spin Glass order

Of course, as already emphasized by Kac Ulembeck and Hammer [2] in the non disordered context, the continuity of the free-energy has no implication for the phase diagram at finite \( R \), any analytic function (as e.g. the free-energy as a function of the temperature for large \( R \) in \( D=1 \)) can be well approximated by a non-analytic one as provided by the MF limit. However the result suggests that at the local level, on scales that diverge with the range of interaction, the physics of finite range models should be well described by mean-field theory. Indeed, this is the conclusion of the study of local order parameter in the Kac limit [7].

The spin-glass order parameter, capable to describe ergodicity braking in the mean-field models is the probability distribution of the overlap between identical copies of the system [14]. In the context of finite dimensional, extended systems it is natural to study the behavior of local overlaps. One can then define window overlaps on a box \( B_{\ell} \) of scale \( \ell \in (0, L] \) centered around an arbitrary point: given two spin configurations \( \sigma \) and \( \tau \),

\[
q_{\ell}(\sigma, \tau) = \frac{1}{|B_{\ell}|} \sum_{i \in B_{\ell}} \sigma_i \tau_i.
\]

(6)
The corresponding probability distribution induced by thermal and quenched disorder is:

\[ P_{L,\ell}(q) = E \left[ \frac{1}{Z} \sum_{\sigma, \tau} e^{-\beta (H(\sigma) + H(\tau))} \delta (q_{\ell}(\sigma, \tau) - q) \right]. \]  \hfill (7)

One would like to understand how the behavior of the order parameter depends on \( \ell \). Mean-Field long-range order would correspond to MF-like probability distribution functions (PDF) at the largest scale \( L \). A fundamental question about the nature of finite D spin glasses is whether this long-range order is possible. More modestly, one can investigate the possibility of local MF order and define a (possibly infinite) correlation length \( \xi \) marking the crossover from a non-trivial MF-like behavior of the box overlap to a trivial one.

A second result on the Kac limit concerns then box overlaps on the scale \( \ell = R \) of the interaction \[7\]. The overlap can be analyzed through linear response theory \[12\], considering a generalized model whose Hamiltonian is a sum of the original one plus small contributions of all \( p \)

\[ H = H^p + \sum_r C_r H^r \] \hfill (8)

with \( C_r \ll 1 \) and where each of the \( H^r \) is of the Kac kind for same range of interaction. For any \( L \) and \( R \), the derivative of the free-energy with respect to the couplings \( C_r \) generate the moments of the overlap distribution:

\[ \frac{\partial}{\partial C_r} f_{L,R}(T) = -\beta \left( 1 - \int P_{L,R}(q) q' \right). \] \hfill (9)

It was proved in \[7\] that for almost all choices of the parameters \( C_r \) in probabilistic sense, the function \( P_{L,R}(q) \) tends to the corresponding mean-field function in the Kac limit. The main implication of this result is that for large but finite \( R \) and infinite \( L \) at least on a local level on scales of order \( R \) mean field order holds. This puts a lower bounds to the overlap correlation length

\[ \xi \geq R. \] \hfill (10)

### 3.3 Coarse graining and replica field theory

In order to understand genuinely finite dimensional systems one needs to go beyond the Kac limit and study possibly large, but finite values of \( R \). Unfortunately, in this case mathematically rigorous analysis becomes prohibitively complicated. One has to resort then to the available non rigorous techniques of theoretical physics. In \[13\] the problem was addressed with the replica method. It was shown there that it is possible to coarse grain the space on scales \( 1 \ll \ell \ll R \ll L \) such that the \( n \)-times replicated partition function can be written as a functional integral of the exponential of an action for local overlap \( n \times n \) matrices \( Q_{ab}(x) \) on the coarse graining scales.

\[ E(Z^n) = \int \mathcal{D}Q_{ab}(x) \exp \left( -R^D \int d^D x S[Q, x] \right). \] \hfill (11)

The coarse grained action \( S \) has been computed explicitly in \[13\] and turns out to be independent of \( R \). Abstracting from the functional form of \( S \) an interesting aspect of formula (11) is the appearance of “the volume of interaction” \( R^D \) in front of the action. This suggests to treat finite dimensional effects through asymptotic expansions in \( 1/R \) based on instanton techniques where one seeks for non spatially homogeneous saddle points \( S \).
4 1D Spin glasses.

We consider in this section the spin glass model $p = 2$ in dimension $D=1$. In this case for any finite $R$ there is no phase transition: the equilibrium phase is paramagnetic and correspondingly the overlap correlation length $\xi_R$ stays finite at all temperatures. Studying the window overlap one should then observe at low temperature a cross-over from a spin-glass regime for to a paramagnetic one as the window size is increased. An interesting question concerns the behavior of the overlap correlation length as a function of $R$ below the mean-field critical temperature, specifically is its growth linear in $R$ as the analysis of the Kac limit suggests, or is it instead super-linear? This would appear as the simplest theoretical question to go beyond the Kac limit. Yet it has not been answered analytically in the low temperature phase so far. I would like to present here some results in this sense obtained through numerical simulations in [15].

The $p = 2$ model [1], where each spin interacts in 1D with order $R$ neighbors with a strength of order $1/\sqrt{R}$ is not very suitable for simulational purposes. A better choice of a model with a similar low temperature behavior consists in letting each spin $\sigma_i$ ($i = 1, \ldots, L$) to interact with a small number of other spins $j$ randomly chosen within the neighborhood $|i - j| \leq R$ which has been analyzed in the Kac limit in [16,17].

Figure 1 and 2 are an illustration of how the analysis of the previous section applies in a 1D spin glass (the details of the simulations are given in [15]). In figure 1 the window overlap probability distribution on the scale $R$ is plotted for a temperature below the mean-field $T_c$. One can see that increasing $R$ the shape of the curves tend to the characteristic form of mean field spin glasses, with two symmetric delta peaks at values $\pm q_{EA}$ and a continuous part for $|q| < q_{EA}$. Figure 2 on the other hand shows the window overlap distribution for the same temperature, fixed $R = 16$ as a function of the window size $\ell$. The curves, that for small $\ell$ are reminiscent of the mean-field shape, tend to a Gaussian shape for larger and larger $\ell$.

Quantifying numerically the growth of the overlap correlation length in the low temperature phase is a difficult task even in 1D. In [15] a theoretical and numerical estimate was given for $T = T_c$ where it is found that

$$\xi \sim R^{6/5}. \quad (12)$$

The value of the exponent is larger then the bound implied by the analysis of the Kac limit. The theoretical derivation [15], based on dimensional analysis, relies on replica theory, which predicts that the spin glass transition is governed by a cubic theory for $T > T_c$ [18].
5 \(p\)-spin models

The last years have seen a great interest of the statistical physics community in the structural glass transition. With at the basis a deep phenomenological analogy, disordered models of the family of the \(p\)-spin have been proposed as prototypical systems to understand the structural glass transition. Indeed they are capable to describe in a unified framework Mode-Coupling dynamics, the Kauzmann transition as well as the growth of dynamical correlations on approaching the glass transition \[19\].

The physics of \(p\)-spin mean-field models is dominated by metastable states. The dynamical Mode-Coupling-like transition at \(T_c\) and the static Kauzmann like transition at the lower transition \(T_K\) reflect the structure of these states. It is a fundamental problem to understand how the picture extends when non-mean-field effects are taken into account. In particular it is necessary to understand how the barriers, which are \(O(\text{volume})\) in mean field get modified in finite \(D\), and what are the mechanisms responsible for barrier crossing. The introduction of Kac models offer the possibility to study finite dimensional effect in an asymptotic expansion in \(R\) around mean field. The barriers in this case, remain finite in the thermodynamic limit, but diverge as \(O(R^D)\) for large \(R\) \[20\]. Thanks to this fact, for large \(R\) one can test in a controlled theoretical setting various phenomenological ideas that have been proposed to cure mean-field pathologies in finite dimension. In particular, it has been possible to study the behavior of two correlation lengths that have been proposed to be important for the glass transition \[21\].

The study of metastability and relaxation time in glassy systems for large \(R\) has been related in \[26\] to a free-energy difference between two systems subject to physical constraints, in analogy with the classical analysis of metastability of \[22\].

The idea of \[26\] is that dynamics proceeds as a passage from metastable state to metastable state and almost all relevant low temperature configurations can be taken as representative of a metastable configurations. Almost all the other configurations inside the metastable state, will have in all point of space a local overlap with the reference above a threshold value.

Barrier states, allowing the relaxation are at the border of a metastable state, consist then in configurations where the overlap with the reference is lower then threshold at least in one point in space. These barrier states can be computed considering considering a system in finite geometry (e.g. a sphere) constrained to have a high overlap with a reference on the boundary.
This analysis thus connects barrier to the “point-to-set” (PS) correlation functions recently proposed to identify correlation lengths in glassy systems in alternative to dynamical determinations [23].

The PS functions measure the correlation in a region of size \( \ell \) with a reference configuration fixed as boundary condition outside the region. The analysis of the PS correlations in p-spin in the Kac limit in [21] allowed to identify two different relevant lengths which, for temperatures above \( T_K \), scale linearly with the interaction range \( R \). A first length \( \xi_{MC} \), identified as the dominant length appearing in the dynamical four point function [24, 25] above \( T_c \) and diverging as \( R|T - T_c|^{1/4} \), represents the typical size of regions that can relax in time independent of \( R \) without needing activation. A second length \( \xi_{Mos} \), diverging at \( T_K \) as \( R(T - T_K)^{-1} \) represents the minimal size of regions that can relax through activation, on time scales exponentially divergent with \( R^d \). Below \( T_K \) it can be argued that the dominant relaxation mechanism operate on that scale.

The two lengths reflect then different relaxation mechanisms, with characteristic times displaying critical mode-coupling and activated scaling respectively:

\[
\tau_{MC} \sim \xi_{MC}^{z} \sim |T - T_c|^{2\nu} \\
\tau_{Mos} \sim \exp(R^d(\xi_{Mos}/R)^\psi C) \sim \exp(R^d(T - T_K)^{-\psi} C')
\]

where \( z \) is the dynamical exponent, and the value of the exponent \( \psi = D - 1 \) in mean field. We see that \( \tau_{Mos} \) has a form similar to the the Vogel-Fulcher form \( \tau \sim \exp(C/|T - T_K|) \) expected in 3 dimensions but with a different exponent [19].

Finally I would like to comment that while the Kac limit provides a limiting case where analytic progress is possible, the relevance of the large \( R \) analysis for systems with short range interaction is far from being obvious. This problem has been addressed in the context of a simple 1D model in [27], where it was found that it is only comparing the behavior of systems with different values of \( R \) that the large \( R \) scenario could be verified.

### 6 Conclusions

This paper presents a rapid excursus of my work, in various collaborations, on disordered models with Kac interactions. The main results presented are

- The rigorous analysis of the Kac limit. This shows that Mean Field theory provides an appropriate description of local properties of disordered systems. This is a necessary condition for Long Range order to hold, but it does not imply it.

- Numerical simulations and theoretical arguments show that even in \( D = 1 \) where for finite \( R \) the system is paramagnetic at all temperatures, the correlation length below and at mean field critical temperature \( T_c \) grows with \( R \) faster than the linear bound implied by the analysis of the Kac limit.

- Kac models can be investigate to test in a controlled setting phenomenological extension of mean-field theory to deal with metastable states in disordered finite dimensional systems. In that framework, two relevant lengths with different temperature dependence emerge,
that describe the size of cooperatively rearranging regions in different domains.

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