Scalable Planning in Multi-Agent MDPs

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Abstract—Multi-agent Markov Decision Processes (MMDPs) arise in a variety of applications including target tracking, control of multi-robot swarms, and multiplayer games. A key challenge in MMDPs occurs when the state and action spaces grow exponentially in the number of agents, making computation of an optimal policy computationally intractable for medium- to large-scale problems. One property that has been exploited to mitigate this complexity is transition independence, in which each agent’s transition probabilities are independent of the states and actions of other agents. Transition independence enables factorization of the MMDP and computation of local agent policies but does not hold for arbitrary MMDPs. In this paper, we propose an approximate transition dependence property, called δ-transition dependence and develop a metric for quantifying how far an MMDP deviates from transition independence. Our definition of δ-transition dependence recovers transition independence as a special case when δ is zero. We develop a polynomial time algorithm in the number of agents that achieves a provable bound on the global optimum when the reward functions are monotone increasing and submodular in the agent actions. We evaluate our approach on two case studies, namely, multi-robot control and multi-agent patrolling example.

I. INTRODUCTION

A variety of distributed planning and decision-making problems, including multiplayer games, search and rescue, and infrastructure monitoring, can be modeled as Multi-agent Markov Decision Processes (MMDPs). In such processes, the state transitions and rewards are determined by the joint actions of all of the agents. While there is a substantial body of work on computing such optimal joint policies [1], [2], [3], a key challenge is that the total number of states and actions grows exponentially in the number of agents. This increases the complexity of computing an optimal policy, as well as storing and implementing the policy on the agents.

One approach to mitigating this complexity is to identify additional problem structures. One such structure is transition-independence (TI) [4]. In a TI-MDP, the state transitions of an agent are independent of the states and actions of the other agents. Such MDPs may arise, for example, in multi-robot scenarios where the motion of each robot is independent of the others. TI-MDPs can be approximately solved by factoring into multiple MDPs, one for each agent, and then obtaining a local policy, in which each agent’s next action depends only on that agent’s current state. When the TI property holds and the MDP possesses additional structure, such as submodularity, this approach may yield scalable algorithms for computing near-optimal policies [5].

The TI property, however, does not hold for general MMDPs when there is coupling between the agents. Coupling occurs when two agents must cooperate to reach a particular state, or when the actions of agents may interfere with each other. In this case, the existing results providing near-optimality do not hold, and at present there are no scalable algorithms for local policy selection in non-TI-MMDPs.

In this paper, we investigate the problem of computing approximately optimal local policies for non-TI MMDPs. We propose δ-transition dependence, which captures the deviation of the MDP from transition independence. We make the following contributions:

• We define the δ-transition dependence property, in which the parameter δ characterizes the maximum change in the probability distribution of any agent due to changes in the states and actions of the other agents.
• We propose a local search algorithm for computing local policies of the agents, in which each agent computes an optimal policy assuming that the remaining agents follow given, fixed policies.
• We prove that, when the reward functions are monotone and submodular in the agent actions, the proposed algorithm achieves a provable optimality bound as a function of the dependence parameter δ and the ergodicity of the MMDP.
• We evaluate our approach on two numerical case studies, namely, a patrolling example and a multi-robot target tracking scenario. On average, our approach achieves 0.95-optimality in the multi-robot scenario and 0.99-optimality in the multi-agent patrolling example, while requiring 10-20% of the runtime of an exact optimal algorithm.

The paper is organized as follows. Section II presents the related work. Section III contains preliminary results. Section IV presents our problem formulation and algorithm. Section V presents optimality and complexity analyses. Section VI presents simulation results. Section VII concludes the paper.

II. RELATED WORK

MDPs have been extensively studied as a framework for multi-agent planning and decision-making [6], [7]. Most existing works focus on selecting an optimal joint strategy for the agents, which maps each global system state to an action for each agent [1], [2], [3]. These methods can be shown to converge to a locally optimal policy, in which no agent...
can improve the overall reward by unilaterally changing its policy. These joint decision-making problems can be viewed as special cases of multi-agent games in which all agents have a shared reward [8]. These approaches, however, suffer from a “curse of dimensionality,” in which the state space grows exponentially in the number of agents, and hence do not scale well to large numbers of agents.

Transition-independent MDPs (TI-MDPs) provide problem structure that can be exploited to speed up the computation [4]. In a TI-MDP, each agent’s transitions probabilities are independent of the actions and states of the other agents, allowing the MDP to be factored and approximately solved [9], [10], [11], [12]. Extensions of the TI-MDP approach to POMDPs were presented in [13], [14]. A greedy algorithm for TI-MDPs with submodular rewards was proposed in [5]. The goal of the present paper is to extend these works to non-TI MDPs by relaxing transition independence, enabling optimality bounds for a broader class of MDPs. A local policy algorithm was proposed in [15] that leverages a fast-decaying property that is distinct from the approximate transition independence that we consider.

Our optimality bounds rely on submodularity of the reward functions. Submodularity is a diminishing-returns property of discrete functions that has been studied in discrete multi-agent planning [5], sensor scheduling [20], and solving POMDPs [21]. Submodularity for transition-dependent MDPs, however, has not been explored.

III. BACKGROUND AND PRELIMINARIES

This section gives background on perturbations of Markov chains, as well as definition and relevant properties of submodularity.

A. Perturbations of Markov Chains

A finite-state, discrete-time Markov chain is a stochastic process defined over a finite set $S$, in which the next state is chosen according to a probability distribution $P(s, \cdot)$, where $s \in S$ is the current state. A Markov chain over $S$ is defined by its transition matrix $P$, in which $P(s, s')$ represents the probability of a transition from state $s$ to state $s'$. The following theorem describes the steady-state behavior of a class of Markov chains.

Theorem 1 (Ergodic Theorem [22]): Consider a Markov chain with transition matrix $P$. Suppose there exists $T_0 > 0$ such that $(P^t)_{ij} > 0$ for all $t > T_0$ and $i, j \in S$. Then there is a probability distribution $\pi$ over $S$ such that, for any distribution over the initial state,

$$\lim_{t \to \infty} \frac{\eta(s, t)}{t} = \pi(s),$$

where $\eta(s, t)$ is the number of times the Markov chain reaches state $s$ in the first $t$ time steps. Moreover, $\pi$ is the unique left eigenvector of $P$ with eigenvalue 1.

A Markov chain satisfying the conditions of Theorem 1 is ergodic. The probability distribution defined in Theorem 1 is the stationary distribution of the chain. Intuitively, a Markov chain is ergodic if the relative frequency of reaching each state is independent of the initial state. The ergodicity coefficient of a matrix $P$ is defined by

$$\Lambda_1(P) = \frac{1}{2} \max_{i,j} \sum_k |P_{ik} - P_{jk}|.$$

We next state preliminary results on perturbations of ergodic Markov chains. First, we define the total variation distance between two probability distributions as follows. For two probability distributions $\mu$ and $\nu$ over a finite space $\Omega$, the total variation distance is defined by

$$||\mu - \nu||_{TV} \triangleq \max_{\Theta \subseteq \Omega} |\mu(\Theta) - \nu(\Theta)|.$$

The total variation distance satisfies [23]

$$||\mu - \nu||_{TV} \leq \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

Let $P$ and $P'$ denote the transition matrices of two ergodic Markov chains on the same state space with stationary distributions $\mu$ and $\nu$, and define $\Delta = P - P'$. The 1-norm of the matrix $\Delta$ is defined by

$$||\Delta||_1 = \max_{i} \left\{ \sum_{j} |\Delta_{ij}| \right\},$$

where $\Delta_{ij}$ is the $(i, j)$-th entry of $\Delta$. The group inverse of $P$, denoted $P^\#$, is the unique square matrix satisfying

$$PP^\#P = P, \quad P^\#P P^\# = P^\#, \quad PP^\# = PP^\#.$$

Let $Z = I - P$, where $I$ denotes the identity. It is known [24] that $Z^\# = (I - P + \mu 1^T)^{-1} - \mu 1^T$, where $1$ denotes the vector with all 1’s.

The following result gives a bound on the distance between $\mu$ and $\nu$ as a function of the perturbation $\Delta$.

Lemma 1 ([25]): The total variation distance between the stationary distributions $\mu$ and $\nu$ of Markov chains with transition matrices $P$ and $P'$, respectively, satisfies

$$||\mu - \nu||_{TV} \leq \frac{1}{2} \Lambda_1(Z^\#)||P - P'||_1,$$

where $Z^\#$ is the group inverse of $(I - P)$.

B. Background on Submodularity and Matroids

A function $f : 2^V \to \mathbb{R}$ is submodular [26] if, for any sets $S \subseteq T \subseteq V$ and any element $v \notin T$, we have

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T).$$

The function $f$ is monotone if $f(S) \leq f(T)$ for $S \subseteq T$. A matroid is defined as follows.

Definition 1: Let $V$ denote a finite set and let $\mathcal{I}$ be a collection of subsets of $V$. Then $\mathcal{N} = (V, \mathcal{I})$ is a matroid if

(i) $\emptyset \in \mathcal{I}$, (ii) $S \subseteq T$ and $T \in \mathcal{I}$ implies that $S \in \mathcal{I}$, and (iii) for any $S, T \in \mathcal{I}$ with $|S| < |T|$, there exists $v \in T \setminus S$ such that $(S \cup \{v\}) \in \mathcal{I}$.

The rank of a matroid $\mathcal{N}$ is equal to the cardinality of the maximal independent set in $\mathcal{I}$. A matroid basis is a maximal
independent set in \( \mathcal{I} \), i.e., a set \( S \) such that \( S \in \mathcal{I} \) and \((S \cup \{v\}) \notin \mathcal{I}\) for all \( v \notin S \). A partition matroid is defined by a partition of the set \( V \) into \( V_1 \cup \cdots \cup V_k \), where \( V_i \cap V_j = \emptyset \) for \( i \neq j \). A set \( S \) is independent in the partition matroid if, for all \( i \), \(|S \cap V_i| \leq 1 \).

The following result leads to optimality bounds on local search algorithms for submodular maximization.

**Lemma 2 ([16]):** Suppose that \( S \) is a basis of matroid \( \mathcal{N} = (V, \mathcal{I}) \), \( f \) is a monotone submodular function, and there exists \( \epsilon > 0 \) such that, for any \( u \in S \) and \( v \notin S \) with \((S \cup \{v\}) \notin \mathcal{I} \),

\[
    f(S) \geq \frac{1}{1+\epsilon} f(S \cup \{v\} \setminus \{u\}).
\]

Then we have \( f(S) \geq \frac{1}{1+\epsilon} f(T) \) for any \( T \in \mathcal{I} \), where \( k \) is the rank of \( \mathcal{N} \).

**IV. PROBLEM FORMULATION AND PROPOSED ALGORITHM**

In this section, we first present our problem formulation, followed by the proposed algorithm.

**A. System Model and Problem Formulation**

We consider a Markov Decision Process (MDP)\(^1\) defined by a tuple \( \mathcal{M} = (S, A, P, R) \), where \( S \) and \( A \) denote the state and action spaces, respectively. The transition probability function \( P(s, a, s') \) denotes the probability of transitioning from state \( s \in S \) to state \( s' \in S \) after taking action \( a \in A \). The reward function \( R(s, a) \) defines the reward from taking action \( a \) in state \( s \). The goal is to maximize the average reward per step, denoted by \( \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} R(s_t, a_t) \), where \( s_t \) and \( a_t \) denote the state and action at time \( t \).

The state and action spaces of \( \mathcal{M} \) can be decomposed between a set of \( m \) agents and an underlying environment. We write \( S_0 \) to denote the state space of the environment, \( S_i \) the state space of agent \( i \), and \( A_i \) to denote the action space of agent \( i \). We then have \( S = S_0 \times S_1 \times \cdots \times S_m \) and \( A = A_1 \times \cdots \times A_m \). Throughout the paper, we use \( s_i \) to denote a state in \( S_i \) and \( s_{-i} \) to denote a tuple of state values for the agents excluding agent \( i \). Similarly, we denote an action in \( A_i \) as \( a_i \) and let \( a_{-i} \) denote a tuple of actions for the agents excluding agent \( i \).

We assume that the reward function is a monotone and submodular function of the agent actions for any fixed state value. Define \( R_{\text{max}} = \max \{ R(s, a) : s \in S, a \in A \} \) and \( R_{\text{min}} = \min \{ R(s, a) : s \in S, a \in A \} \). We observe that the size of the state space may grow exponentially in the number of agents, increasing the complexity of computing the transition probabilities and optimal policy. A problem structure that is known to simplify these computations is transition independence, defined as follows.

**Definition 2 ([4]):** An MDP is transition independent (TI) if there exist transition functions \( P_0 : S_0 \to S_0 \) and \( P_i : S_i \times A_i \to S_i \), \( i = 1, \ldots, m \), such that

\[
    P_i(s_i, a_i, s_i') = \sum_{s_{-i}} P_i(s_{-i}, a_{-i}, s_{-i}') \quad \text{for} \quad i = 1, \ldots, m.
\]

Transition independence implies that the state transitions of each agent depend only on that agent’s states and actions, thus enabling factorization of the MDP and reducing the complexity of simulating and solving the MDP. We observe, however, that the TI property does not hold for general MDPs, and introduce the following relaxation.

**Definition 3:** Let

\[
    \mu_i(s_i, a_i, s_{-i}, a_{-i}) = \sum_{s_i'} P(s_i', a_i, s_{-i}') \quad \text{for} \quad i = 1, \ldots, m.
\]

An MDP is \( \delta \)-transition dependent (or \( \delta \)-dependent) if

\[
    \max_{i \neq i', a_i, a_i', s_{-i}, s_{-i}', a_{-i}, a_{-i}'} ||\mu_i(a_i, s_{-i}, s_{-i}', a_{-i}) - \mu_i(a_i, s_{-i}, s_{-i}', a_{-i}')||_{TV} \leq \delta
\]

Intuitively, the \( \delta \)-dependent property implies that the impact of the other agents on agent \( i \)’s transition probabilities is bounded by \( \delta \). When \( \delta = 0 \), our definition of \( \delta \)-dependent MDP reduces to TI-MDP defined in [4].

The agents choose their actions at each time step by following a policy, which maps the current and previous state values to the action at time \( t \). We focus on stationary policies of the form \( \pi : S \to A \), which only incorporate the current state value when choosing the next action. We assume that, for any stationary policy, the resulting induced Markov chain is ergodic. We let \( \lambda \) denote the maximum value of the ergodic number \( A_1(Z^\lambda) \) over all stationary policies. Furthermore, to reduce the complexity of storing the policy at the agents, each agent follows a local policy \( \pi_i : S_0 \times S_i \to A_i \). Hence, each agent’s actions only depend on the environment and the agent’s internal state. Any policy \( \pi \) with this structure can be expressed as \( \{ \pi_1, \ldots, \pi_m \} \), where \( \pi_i \) denotes the policy of agent \( i \). We let \( \pi_{-i} \) denote the set of policies of the agents excluding \( i \).

The problem is formulated as follows. Define the value function for policies \( \{ \pi_i : i = 1, \ldots, m \} \) by

\[
    J(\pi) = \lim_{T \to \infty} \mathbb{E} \left\{ \frac{1}{T} \sum_{t=0}^{T-1} R(s_t, \pi(s_t)) \right\}.
\]

When it is not clear from the context, we let \( J_M(\pi) \) denote the average reward from policy \( \pi \) on MDP \( \mathcal{M} \). The goal is then to select \( \pi \) that maximizes \( J(\pi) \). As a preliminary, we say that a policy \( \pi \) is locally optimal if, for all \( i \), \( J(\pi) \geq J(\tilde{\pi}_i, \pi_{-i}) \) for any agent \( i \) policy \( \tilde{\pi}_i \). We say that \( \pi \) is \( \theta \)-locally optimal if \((1 + \theta)J(\pi) \geq J(\tilde{\pi}_i, \pi_{-i}) \) for all \( i \) and all policies \( \tilde{\pi}_i \) for agent \( i \).

\(^1\)In this paper, we use MDP and MMDP interchangeably.
B. Proposed Algorithm

To motivate our approach, we first map the problem to a combinatorial optimization problem as in [5]. Consider the finite set of agent policies, which we write as $\Pi = \Pi_1 \cup \cdots \cup \Pi_m$, where $\Pi_i$ denotes the set of possible local policies for agent $i$. The collection $\Pi_i$ is formally defined as the set of functions of the form $\{\pi_i : S_0 \times S_i \to A_i\}$. The problem of selecting an optimal collection of local policies can therefore be mapped to the combinatorial optimization problem

$$\max J(\pi)$$

s.t. $\pi \in \Pi, |\pi \cap \Pi_i| = 1 \ \forall i = 1, \ldots, m$ (2)

In (2), the policy $\pi$ is interpreted as a set, in which each element represents the policy of a single agent. Since there is exactly one policy per agent, the constraint $|\pi \cap \Pi_i| = 1$ is a partition matroid constraint. The following proposition provides additional structure for a special case of (2).

Proposition 1 ([5]): If the MDP $\mathcal{M}$ is transition-independent and the rewards $R(s, a)$ are monotone and submodular in $a$ for any fixed state $s$, then the function $J^*_\mathcal{M} : \Pi \to \mathbb{R}$ is monotone and submodular in $\pi$.

Proposition 1 implies that, when the MDP is TI and reward function is submodular, efficient heuristic algorithms will lead to provable optimality guarantees. One such algorithm is local search, which attempts to improve the current set of policies $\{\pi_1, \ldots, \pi_m\}$ by searching for policies $\pi'_i$ satisfying $J(\pi'_i, \pi_{-i}) > J(\pi)$. If no such policy can be found, then the policy $\pi$ is a local optimum of (2), and hence Lemma 2 can be used to obtain a $\frac{1}{2}$-optimality guarantee.

The difficulty in the above approach arises from the fact that the number of possible policies $\Pi_i$ for each agent $i$ grows exponentially in the number of states $S_i$. Hence, instead of brute force search, the approach of [5] leverages the fact that, in a TI-MDP in which all other agents adopt stationary policies, the optimal policy for agent $i$ can be obtained as the solution to an MDP. This MDP has reward function and transition matrix, respectively, given by

$$R_i(s, a_i) = \sum_{s \sim s} q(s \sim s) R(\{s, s \sim s\}, \{a_i, \pi_{-i}(s \sim s)\})$$

and $P_i(s, a, s') = P_i$, where $q$ denotes the stationary distribution of the joint states under the chosen policies. Using this property, an optimal policy for agent $i$, conditioned on the policies $\{\pi_{-i}\}$ of the other agents, can be obtained by solving this equivalent MDP.

We now present our proposed approach, which generalizes this idea from TI to non-TI MDPs. Our algorithm is initialized as follows. Choose a parameter $\epsilon > 0$. First, for each agent $i$, choose a probability distribution $\mu_i$ over the states in $S_{-i}$ and a policy $\pi_{-i}(s_{-i})$. Next, define a local transition function for each agent $i$ as

$$P_i(s, a, s') = \mathbb{E}_{\mu_i}(P(\{s, s_{-i}\}, \{a_i, \pi_i(s_{-i})\}, \{s', s'_{-i}\})),$$ (3)

where the expectation is over $s_{-i}$ from distribution $\mu_{-i}$. We then choose policies $\pi_1^0, \ldots, \pi_m^0$ arbitrarily, and set $q_i^0$ as the stationary distribution on the state $s_i$ induced by the policy $\pi_i^0$ under transition function $P_i^0$.

At the $k$-th iteration of the algorithm, each agent $i$ updates its policy $\pi_i$ while the other agent policies are held constant. The optimal policy of agent $i$ is approximated by the solution to a local MDP denoted $\mathcal{M}_i^k = (S_i, A_i, P_i, R_i^k)$, where

$$R_i^k(s, a_i) = \sum_{s \sim s} \left( \prod_{j \neq i} \hat{q}_j(s_j) \right) R(\{s, s_{-i}\}, \{a_i, \pi_{-i}(s_{-i})\}).$$ (4)

A policy $\pi_i$ is then obtained as the optimal policy for $\mathcal{M}_i^k$.

Pseudocode for this algorithm is given in Algorithm 1.

Algorithm 1 Approximate algorithm for selecting local policies

1: Input: MDP $(S, A, P, R)$
2: Output: Policies $\{\pi_i : i = 1, \ldots, m\}$
3: Initialization: $\pi_i^0 : i = 1, \ldots, m, s_i \in S_i, k \rightarrow 1$
4: Compute $P_i : i = 1, \ldots, m$ according to (3)
5: Compute $q_i^0 : i = 1, \ldots, m$ as the stationary distribution of $P_i$ under policy $\pi_i^0$
6: while $1$ do
7: $\pi_i^k \leftarrow \pi_i^{k-1}, i = 1, \ldots, m, found \leftarrow 0, \hat{q}_i^k \leftarrow \hat{q}_i^{k-1}$
8: for $i = 1, \ldots, m$ do
9: Compute $R_i^k$ according to (4)
10: Solve local MDP $\mathcal{M}_i^k = (S_i, A_i, P_i, R_i^k)$ to obtain new policy $\pi_i$
11: if $J_{\mathcal{M}_i^k}(\pi_i) > (1 + \epsilon)J_{\mathcal{M}_i^k}(\pi_i^{k-1})$ then
12: $\pi_i^k \leftarrow \pi_i$
13: $\hat{q}_i^k \leftarrow \text{stationary distribution of } P_i \text{ under policy } \pi_i$
14: $\text{found} \leftarrow 1; \text{Break}$
15: end if
16: end for
17: if $\text{found} == 0$ then
18: $\text{Break}$
19: else
20: $k \leftarrow k + 1$
21: end if
22: end while

V. Optimality Analysis

We analyze the optimality in three stages. First, we define a TI-MDP, and prove that the policies returned by our algorithm are within a provable bound of a local optimum of the TI-MDP. We then use submodularity of the reward function to prove that the local optimal policies provide a constant-factor approximation to the global optimum on
the TI-MDP. Finally, we prove that the approximate global optimum on the TI-MDP is also an approximate global optimum for the original MDP.

We define $\hat{\pi} = \{\hat{\pi}_1, \ldots, \hat{\pi}_m\}$ to be the policy returned by our algorithm. Let $\hat{q}$ be the joint stationary distribution of the agents in the MDP $M$ arising from these policies. We construct a TI-MDP $\hat{M} = (S, A, \hat{P}, \hat{R})$. The transition function $\hat{P}$ is defined by

$$\hat{P}(s, a, s') = \sum_{i=1}^{m} \hat{P}_i(s_i, a_i, s'_i).$$

The reward function $\hat{R} = R(s, a)$. We observe that, by construction, if $M_i$ is the local MDP obtained at the last iteration of Algorithm 1, then $J_{M_i}(\hat{\pi}_i) = J_{\hat{M}_i}(\hat{\pi})$ for all $i$.

Lemma 3: The policy $\hat{\pi}$ returned by Algorithm 1 is a $1/(1+\epsilon)$-local optimum for MDP $\hat{M}$.

Proof: By construction, the algorithm terminates if, for all $i$, there is no policy $\pi_i$ such that

$$J_{\hat{M}_i}(\pi_i) = J_{\hat{M}_i}(\hat{\pi}_i) + (1+\epsilon)J_{\hat{M}_i}(\hat{\pi}),$$

implying that $\hat{\pi}$ is a $1/(1+\epsilon)$-local optimum of $M$.

Based on the local optimality, we can derive the following optimality bound for $\hat{\pi}$.

Lemma 4: Let $\pi^*$ denote the optimal local policies for MDP $M$. Then

$$J_{\hat{M}}(\hat{\pi}) \geq \frac{1}{2 + \epsilon m} J_{\hat{M}}(\pi^*).$$

Proof: The proof follows from the submodularity of $J_M$ (Proposition 1) and Lemma 2.

Lemma 2 provides an optimality bound with respect to the TI-MDP $\hat{M}$. Next, we leverage the $\delta$-dependent property to derive an optimality bound with respect to the given MDP $M$. We start with the following preliminary results.

Lemma 5: For any state $s$ and policy $\pi$, $|P(s, \pi, s') - \bar{P}(s, \pi, s')|_{TV} \leq m\delta$, where $P$ and $\bar{P}$ are the transition matrices corresponding to $M$ and $\hat{M}$, respectively.

A proof can be found in the technical appendix. We next exploit the bound in Lemma 2 to approximate the gap between the $J_M$ and $J_{\hat{M}}$.

Lemma 6: Suppose that $M$ is $\delta$-dependent MDP. For any policy $\pi$, $|J_M(\pi) - J_{\hat{M}}(\pi)| \leq (R_{\text{max}} - R_{\text{min}})(2\lambda m\delta)$.

Proof: Let $q_M$ and $q_{\hat{M}}$ denote the stationary distributions induced by policy $\pi$ on the MDPs $M$ and $\hat{M}$. We have

$$|J_M(\pi) - J_{\hat{M}}(\pi)| \leq (R_{\text{max}} - R_{\text{min}})||q_M - q_{\hat{M}}||_1.$$

By Lemma 2

$$||q_M - q_{\hat{M}}||_1 \leq 2\lambda m\delta,$$

giving the desired result.

Combining these derivations yields the following.

Theorem 2: Let $M$ be an $\delta$-dependent MDP and $\pi$ and $\pi^*$ denote the output of Algorithm 1 and the optimal policies, respectively. Then

$$J_M(\pi^*) - J_M(\hat{\pi}) \leq 4R_{\text{max}}\lambda m\delta + (1 + \epsilon)J_M(\hat{\pi}) + J_{\hat{M}}(\hat{\pi}).$$

Proof: We have

$$J_M(\pi^*) - J_M(\hat{\pi}) \leq |J_M(\pi) - J_{\hat{M}}(\pi) + J_{\hat{M}}(\hat{\pi}) - J_M(\pi^*)| + |J_{\hat{M}}(\pi^*) - J_{\hat{M}}(\hat{\pi})|$$

$$\leq 4R_{\text{max}}\lambda m\delta + (1 + \epsilon)|\hat{M}_i(\hat{\pi})|$$

by Lemmas 2 and 6.

VI. SIMULATION

In this section, we present our simulation results. We consider two scenarios, namely, multi-robot control and a multi-agent patrolling example. Both simulations are implemented using Python 3.8.5 on a workstation with Intel(R) Xeon(R) W-2145 CPU @ 3.70GHz processor and 128 GB memory. Given the transition and reward matrices, an MDP is solved using Python MDP Toolbox [27].

A. Multi-robot Control

1) Simulation Settings: We consider a set of $N > 1$ robots whose goal is to cover maximum number of targets from a set of fixed targets $B$ positioned in a $L \times L$ grid environment. Robots initially start from a fixed set of grid locations. At each time $t = 1, 2, \ldots$, each robot $i$ can move one grid position horizontally or vertically from the current grid position by taking some action $a_i \in A_i := \{\text{left}, \text{down}, \text{right}, \text{up}\}$. For each robot $i$, let $D_i := \{d(a_i)\}_{a_i \in A_i}$ denotes the set of grid positions that can be reached from the current grid position $s_i$ under each action $a_i \in A_i$, $d(a_i)$ is the grid position corresponds to $(s_i, a_i)$. Note that $d(a_i) = \emptyset$, if $a_i$ is not a valid action $e.g., a_i \in \{\text{left}, \text{down}\}$ at the bottom left corner of the grid are not valid actions). Let $P_i$ be the transition probability function associated with robot $i$ and $P_i(s_i, a_i, s'_i)$ be the probability of robot $i$ transitions from a grid position $s_i$ to $s'_i$ under action $a_i$. Let $n(s'_i)$ be the number of robots at $s'_i$ after taking actions $(a_i, a_i \ldots)$. Then,

$$P_i(s_i, a_i, s'_i) = \begin{cases} \frac{1}{|D_i|}, & \text{if } s'_i \in D_i \setminus d(a_i) \text{ and } n(s'_i) < K \\ \frac{1}{|D_i| - 1}, & \text{if } s'_i = d(a_i) \text{ and } n(s'_i) < K \\ 0, & \text{otherwise} \end{cases}$$

where $K \geq 1$. Uncertainty in the environment is modeled by the parameter $0 < c \leq 1$ and the transition dependencies between the robots are modeled by the parameter $0 \leq \delta \leq 1$.

We model the multi-robot control problem as an MDP $M = (S, A, P, R)$. The state space $S = S_1 \times \ldots \times S_N$, where $S_i = \{0, \ldots, L^2 - 1\}$ for all $i = 1, \ldots, N$. The action space $A = A_1 \times \ldots \times A_N$. The transition probability matrix is denoted as $P$. The probability of transitioning from a state $s = (s_1, \ldots, s_N) \in S$ to some target state $s' = (s'_1, \ldots, s'_N) \in S$ by taking action $a = (a_1, \ldots, a_m) \in A$ is given as $P(s, a, s') = \prod_{i=1}^{m} P_i(s_i, a_i, s'_i)$. The submodular reward $R$ of $M$ is given by $R(s_i, \pi(s_i)) = \sum_{b \in B} (1 - (1 - \eta)N_b)$, where $N_b$ is the number of robots visiting target $b$ following a joint policy $\pi(s_i)$ at a state state $s_i$. The
parameter $\eta$ captures the effectiveness of having $N_b$ agents at target $b$. Similar submodular reward has been used in [5].

2) Simulation Results: We use Algorithm 1 to find a set of policies for the robots that maximizes their average reward. Parameters $\epsilon$, $K$, $\eta$, and $\beta$ are set to 0, 1, 0.75 and 0.9, respectively. The transition probability for each agent is calculated by evaluating (3) over $\lfloor N_t \times L \rfloor$ samples of actions $\{a_i, a_{-i}\}$ and states $\{s'_i, s'_{-i}\}$. We test Algorithm 1 under different sizes of grids $L$, number of agents $N$, number of targets $T$, and initial locations of the agents. For each setting, we execute Algorithm 1 for 100 trials, and take the average over the trials as the performance of Algorithm 1. We compare Algorithm 1 with the global MDP approach, while our proposed approach only requires 1.45 Mb memory to calculate the policy. Therefore, the global MDP approach is not computationally efficient for larger example sizes.

Table II shows the simulation results obtained using Algorithm 1 and the global MDP approach. We observe that our proposed approach provides more than 90% optimality with respect to the average reward achieved by the agents for all settings, while incurring comparable run time when the example size is small and much less run time when the example sizes increase. Particularly, as the number of agents and the grid size increase, e.g., two agents, two targets, and $10 \times 10$ grid, our proposed approach maintains more than 100% optimality with only 12.62% run time, compared with the global MDP approach. Hence, our proposed approach shows scalability to multi-agent scenarios with $\delta$-dependent property.

B. Multi-Agent Patrolling Example

1) Simulation Settings: We implement our proposed approach on a patrolling example with multiple patrol units capturing multiple adversaries among a finite set of locations $L$ as an evaluation. At each time, each patrol unit can be deployed at some location $l \in L$.

| Number of Agents and Targets $(N, B)$ | Grid Size $L$ | Target Location | Agents’ Initial Locations | State Space Size | Action Space Size | Algorithm 1 | Global MDP Approach | Ratio | Algorithm 1 | Global MDP Approach | Ratio |
|---------------------------------------|---------------|------------------|---------------------------|-----------------|-----------------|-------------|---------------------|-------|-------------|---------------------|-------|
| 2, 1                                  | 3 × 3         | 6                | (0,2)                     | 81              | 16              | 0.312       | 0.333              | 93.69% | 0.246       | 0.142              | 73.7% |
| 2, 2                                  | 5 × 4         | (20,24)          | (3,5)                     | 625             | 16              | 0.288       | 0.269              | 99.63% | 3.08        | 2.01               | 102.7%|
| 3, 1                                  | 3 × 3         | 8                | (1,1,2)                   | 729             | 64              | 0.460       | 0.304              | 91.21% | 2.51        | 24.1               | 27.07%|
| 3, 2                                  | 2 × 4         | 11               | (0,0,1,1)                 | 499             | 64              | 0.313       | 0.335              | 94.93% | 4.19        | 44.1               | 9.395%|
| 4, 3                                  | 5 × 4         | 2                | (0,0,1,1)                 | 256             | 64              | 0.378       | 0.476              | 98.96% | 4.76        | 58.7               | 13.35%|
| 3, 2                                  | 3 × 10        | 51               | (90,99)                   | 10,000          | 16              | 0.125       | 0.125              | 100%   | 56.9        | 451                | 12.62%|
| 4, 3                                  | 10 × 10       | 55               | (55,77)                   | 15,992          | 16,000          | 0.244       | 0.244              | 100%   | 56.8        | 429                | 13.24%|

TABLE I COMPARISON OF AVERAGE REWARD AND RUN TIME WHEN MULTI-ROBOT CONTROL PROBLEM IS SOLVED USING ALGORITHM 1 VERSUS USING RELATIVE VALUE ITERATION ALGORITHM IN PYTHON MDP Toolbox [27] ON CORRESPONDING GLOBAL MDP ($\mathcal{M}$). BOTH APPROACHES ARE COMPARED UNDER DIFFERENT SIZES OF GRIDS $L$, NUMBER OF AGENTS $N$, NUMBER OF TARGETS $B$, AND INITIAL LOCATIONS OF THE AGENTS. THE ‘RATIO’ OF AVERAGE REWARD IN TABLE I IS OBTAINED AS THE AVERAGE REWARD OF ALGORITHM 1 DIVIDED BY THAT OF THE GLOBAL MDP APPROACH. THE ‘RATIO’ OF RUN TIME IN TABLE I IS OBTAINED AS THE RUN TIME INCURRED BY ALGORITHM 1 DIVIDED BY THAT OF THE GLOBAL MDP APPROACH.

| Patroll Unit, Adversary Number | Location Number | State Space Size | Patroll Units’ Action Space Size | Algorithm 1 | Global MDP Approach | Ratio | Algorithm 1 | Global MDP Approach | Ratio |
|--------------------------------|-----------------|-----------------|---------------------------------|-------------|---------------------|-------|-------------|---------------------|-------|
| 2, 1                           | 3               | 27              | 9                               | 0.774       | 0.775               | 99.87%| 0.133       | 0.435               | 30.57%|
| 3, 1                           | 3               | 81              | 27                              | 0.685       | 0.866               | 99.88%| 1.93        | 6.49                | 29.73%|
| 3, 2                           | 3               | 243             | 27                              | 1.73        | 1.73                | 100%  | 22.84       | 157                 | 14.54%|
| 2, 1                           | 3               | 125             | 23                              | 0.768       | 0.768               | 100%  | 5.36        | 18.13               | 30.61%|
| 3, 1                           | 3               | 625             | 123                             | 0.856       | 0.856               | 100%  | 2.39        | 28.23               | 8.46% |
| 3, 2                           | 3               | 345             | 49                              | 0.766       | 0.766               | 100%  | 68.7        | 358                 | 19.19%|
| 2, 1                           | 8               | 512             | 64                              | 0.766       | 0.766               | 100%  | 188         | 1295                | 14.52%|

TABLE II COMPARISON OF AVERAGE REWARD AND RUN TIME WHEN MULTI-AGENT PATROLLING EXAMPLE IS SOLVED USING ALGORITHM 1 VERSUS USING RELATIVE VALUE ITERATION ALGORITHM IN PYTHON MDP Toolbox [27] ON CORRESPONDING GLOBAL MDP ($\mathcal{M}$). BOTH APPROACHES ARE COMPARED UNDER DIFFERENT NUMBER OF PATROL UNITS, ADVERSARIES, AND LOCATIONS. THE ‘RATIO’ OF AVERAGE REWARD IN TABLE II IS OBTAINED AS THE AVERAGE REward OF ALGORITHM 1 DIVIDED BY THAT OF THE GLOBAL MDP APPROACH. THE ‘RATIO’ OF RUN TIME IN TABLE II IS OBTAINED AS THE RUN TIME INCURRED BY ALGORITHM 1 DIVIDED BY THAT OF THE GLOBAL MDP APPROACH.
The objective of the patrol units is to compute a policy to patrol the locations to capture the adversaries. Each adversary is assumed to follow a heuristic policy as follows. If there exists no patrol unit that is deployed at the adversary’s target location \( l \), then with probability \( d \) the adversary transitions to location \( l \) and with probability \( (1-d)/(L-1) \) the adversary transitions to some other location \( l' \neq l \). If the adversary’s target location \( l \) is being patrolled by some unit, then with probability \( \hat{d} \) the adversary transitions to location \( l \), and with probability \( (1-\hat{d})/(L-1) \) the adversary transitions to some other location \( l' \neq l \). The adversaries’ policies are assumed to be known to the patrol units.

The patrolling example is modeled by an MDP \( \mathcal{M} = (S, A, P, R) \), where \( S = (x_iS_i) \times (x_jS_j) \) is the set of joint locations of the patrol units and adversaries, with \( S_i = L \) is the set of locations at which patrol unit \( i \) is deployed and \( S_j = L \) is the set of locations where adversary \( j \) can be located. The action set of each patrol unit and adversary is \( A_i = A_j = L \). Thus the joint action space \( A = (x_iA_i) \times (x_jA_j) \). We shall remark that the joint action space is defined as the Cartesian product of the action spaces of all patrol units and adversaries so that we can accurately capture the transition probabilities of all patrol units and adversaries.

We solve the problem by optimizing over the joint action space of all the patrol units, since the adversaries’ policies are known to the patrol units. For each patrol unit \( i \) and adversary \( j \), we let

\[
P_i(s_i,a_i,s'_i) = \begin{cases} 
\frac{c}{|L|} & \text{if } a_i = s'_i, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{\delta c}{|L|} & \text{if } a_i = s'_i, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{d}{|L|} & \text{if } a_i = s'_i, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{1-\beta d}{|L|} & \text{if } a_i \neq s'_i, \beta l' \neq i \text{ s.t. } a_i = a_i
\end{cases}
\]

\[
P_j(s_j,a_j,s'_j) = \begin{cases} 
\frac{c}{|L|} & \text{if } a_j = s'_j, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{\delta c}{|L|} & \text{if } a_j = s'_j, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{d}{|L|} & \text{if } a_j = s'_j, \beta l' \neq i \text{ s.t. } a_i = a_i \\
\frac{1-\beta d}{|L|} & \text{if } a_j \neq s'_j, \beta l' \neq i \text{ s.t. } a_i = a_i
\end{cases}
\]

Here parameters \( c, d \in [0,1] \) capture the transition uncertainties, parameter \( \delta \in [0,1] \) captures the transition dependency among the patrol units, and \( \beta \in [0,1] \) captures the adversaries’ reactions to the patrol units’ actions. Let \( s = (x_iS_i) \times (x_jS_j) \) and \( s' = (x_iS'_i) \times (x_jS'_j) \) be two joint locations. Then \( P(s,a,s') = \prod_{i,j} P_i(s_i,a_i,s'_i)P_j(s_j,a_j,s'_j) \), where \( a = (x_iS_i) \times (x_jS_j) \). We define the reward function \( R(s,a) \) for each \( s \in S \) and \( a \in A \) as \( R(s,a) = \sum_{s' \in S} P'(s,a,s')P(s,a,s') \), where \( r'(s,a,s') = \sum_{l \in L} (1-(1-\eta)^k)x_l \), where \( \eta \in [0,1] \) is the effectiveness parameter, \( k \) and \( x \) are the number of patrol units and adversaries that are in location \( l \) corresponding to \( s' \), respectively.

2) Simulation Results: We use Algorithm 1 to compute the policies for the patrol units, given the adversaries’ policies. Parameters \( c, d, \delta, \beta, \eta \) are set as \( 0.9, 1, 0.9, 0.9 \), and \( 0.75 \), respectively. We calculate the transition probability of each patrol unit \( i \) by evaluating (3) over all possible actions \((x_i\{a_i\}) \times (x_j\{a_j\})\) and all possible states \((x_iS_i) \times (x_jS_j)\) of all adversaries and all the other patrol units except \( i \). We implement our proposed approach under various settings by varying the number of patrol units, adversaries, and locations. For each setting, we run Algorithm 1 for 100 trials and take the average over the trials as its performance. We compare Algorithm 1 with the global MDP approach that implements relative value iteration algorithm on MDP \( \mathcal{M} \).

Table II shows the simulation results obtained using Algorithm 1 and the global MDP approach. We observe that our proposed approach achieves more than 99% of optimality with respect to the average reward, while incurring at most 30.57% of run time over all settings. By comparing the first row, 4-th row, 6-th row, and 7-th row in Table II we have that the run time advantage provided by our proposed approach remains when we increase the number of locations. By comparing the first three rows in Table II we observe that our proposed approach remains close to optimal average reward (more than 99%), but scales better when the number of agents including patrol units and adversaries increases.

VII. CONCLUSIONS

This paper presented an approach for selecting decentralized policies for transition dependent MMDPs. We proposed a property of \( \delta \)-transition dependence, which we defined based on the maximum total variation distances for each agent’s state transitions conditioned on the actions of the other agents. In the special case of \( \delta = 0 \), the MMDP is transition-independent. We derived optimality bounds on the policies obtained from our algorithm as a function of \( \delta \). Our results were verified through numerical studies on a patrolling example and a multi-robot control scenario.

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to an element of $T$. We define $T_i(s_{1:(i-1)})$ by

$$T_i(s_{1:(i-1)}) = \{ s'_{i} \in S_i : \{ s'_i, s_{1:(i-1)} \} \subseteq Q \in T \},$$

i.e., $\{ s'_1, \ldots, s'_i \}$ can be completed to an element of $T$.

We can then write the probability $P(s' \in T| s, \pi(s))$ as

$$\begin{align*}
\prod_{i=1}^{m} P(s'_i \in T_i(s_{1:(i-1)})) &\leq \prod_{i=1}^{m} P(s'_i \in T_i(s_{1:(i-1)}))|s_{1:(i-1)} \in T_{1:(i-1)} | s, \pi(s))
\end{align*}$$

Hence, the total variation distance is equivalent to

$$\begin{align*}
\max_T \prod_{i=1}^{m} P(s'_i \in T_i(s_{1:(i-1)})) | s_{1:(i-1)} &\in T_{1:(i-1)} | s, a \rangle
- \prod_{i=1}^{m} \overline{P}(s'_i \in T_i(s_{1:(i-1)})) | s_{1:(i-1)} \in T_{1:(i-1)}
\leq \max_T \sum_{i=1}^{m} | P(s'_i \in T_i(s_{1:(i-1)}) | s_{1:(i-1)} \in T_{1:(i-1)})
- \overline{P}(s'_i \in T_i(s_{1:(i-1)})) | s_{1:(i-1)} \in T_{1:(i-1)} | s, a \rangle
\end{align*}$$

The total variation distance is equivalent to

$$\begin{align*}
\sum_{i=1}^{m} \max_{T, U} \left| \sum_{s'} P(s' \in T | s'_{-i} \in U, s, a) - P(s' \in T | s'_{-i} \in U, s, a) \right|
\end{align*}$$

We therefore have

$$\begin{align*}
\sum_{i=1}^{m} \max_{T, U} \left| \sum_{s'} \sum_{s'_{-i} \in U} \left( P(s'_i | s_{i-1}, \pi_{-i}, a_i, \pi_{-i}) - P(s'_i | s_{i-1}, a_i, \pi_{-i}) \right) \right|
\end{align*}$$

which is equal to

$$\begin{align*}
\sum_{i=1}^{m} \max_{T, U} \left| \sum_{s'_{-i} \in U} \left( P(s'_i | s_{i-1}, \pi_{-i}, a_i, \pi_{-i}) - P(s'_i | s_{i-1}, a_i, \pi_{-i}) \right) \right|
\end{align*}$$

We can then rearrange the order of summation to obtain

$$\begin{align*}
\sum_{i=1}^{m} \max_{T, U} \left| \sum_{s'_{-i} \in U} \left( P(s'_i | s_{i-1}, \pi_{-i}, a_i, \pi_{-i}) - P(s'_i | s_{i-1}, a_i, \pi_{-i}) \right) \right|
\end{align*}$$

This summation can be bounded above by

$$\begin{align*}
\sum_{i=1}^{m} \max_{s'_{-i}} \sum_{\pi_{-i}} \left( P(s'_i | s_{i-1}, \pi_{-i}, a_i, \pi_{-i}) - P(s'_i | s_{i-1}, a_i, \pi_{-i}) \right)
\end{align*}$$

The inner maximum is equal to the total variation distance, which is bounded above by $\delta$ by the definition of $\delta$-dependent property.