Gauge/string correspondence in curved space

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**Abstract**

We discuss Gubser-Klebanov-Polyakov proposal for the gauge/string theory correspondence for gauge theories in curved space. Specifically, we consider Klebanov-Tseytlin cascading gauge theory compactified on $S^3$. We explain regime when this gauge theory is a small deformation of the superconformal $\mathcal{N} = 1$ gauge theory on the world volume of regular D3-branes at the tip of the conifold. We study closed string states on the leading Regge trajectory in this background, and attempt to identify the dual gauge theory twist two operators.

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1 Introduction

Probably the most intriguing aspect of the gauge theory/string theory duality [1–3] (see [4] for a review) is the fact that it provides a dynamical principle for the nonperturbative definition of string theory in the asymptotically Anti de Sitter spacetime where there is no notion of an $S$-matrix\(^1\). It thus appear promising that this dual “definition” of string theory (in terms of certain non-gravitational gauge theory) will be useful for formulating string theory in cosmologically relevant backgrounds\(^2\). Following on ideas of constructing the supergravity dual to gauge theories in curved space-time [8], it was proposed in [9, 10] that certain gravitating de Sitter backgrounds of string theory are dual to gauge theories formulated in classical, non-gravitational de Sitter space-time.

Most work on the Maldacena proposal\(^3\), including the specific computations in [8–10], dealt with the supergravity limit (corresponding to the one-loop approximation of the sigma model) of the gauge/string theory correspondence. It has been recently explained in [2, 3] how to identify string states in the dual gauge theory. Specifically, it has been argued in [3] (GKP) that certain $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory states with large quantum numbers are described by solitons of the nonlinear sigma model in $AdS_5 \times S^5$. Motivated mainly by the potential application of the deformations of gauge/string theory correspondence, where the gauge theory is formulated in curved space-time toward observable cosmology, in this paper we attempt to extend analysis of [3] for (static) deformations of [8]. More precisely, we study\(^4\) closed string states on the leading Regge trajectory in the supergravity background [8], dual to the Klebanov-Tseytlin (KT) cascading gauge theory [12], formulated in $R \times S^3$. Somewhat surprisingly, in the regime where we can trust both the gauge theory and the sigma model analysis, we find that natural candidate dual twist two operators of the gauge theory have subleading correction to the anomalous dimensions different from the corresponding correction to the energy of the highly excited closed string states.

The paper is organized as follows. In the next section we briefly review the correspondence of [3] and explain, from the gauge theory perspective, why the KT gauge theory in $R \times S^3$ is a natural (and computationally controllable) deformation of the

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\(^1\)This is emphasized in particular in [5].

\(^2\)Problems of defining $S$-matrix in backgrounds with observer dependent horizons are discussed in [5–7].

\(^3\)See citation to [1].

\(^4\)GKP proposal for nonconformal gauge/string theory correspondence has been discussed in [11].
GKP computation where one breaks the conformal invariance. We perform the computation of the anomalous dimension of certain twist two operators in this gauge theory near the infrared conformal fixed point. These are the operators analogous to the ones identified in [3] as the gauge theory dual to highly excited (“long”) strings on the leading Regge trajectory in $AdS_5$. In section 3 we review and clarify the dual supergravity background originally constructed in [8]. We then study closed strings spinning in this background, and find a discrepancy with the gauge theory computation of section 2. The disagreement (at the technical level) can be traced back to the fact (mentioned in [8]) that the leading correction to the global $AdS_5 \times T^{1,1}$ background due to the 3-form flux does not have the precise asymptotics of the extremal Klebanov-Tseytlin background. We do not have a physical understanding of the discrepancy at this stage. We comment on the difficulties of establishing gauge/string correspondence for the KT gauge theory in the far ultraviolet, where it is also believed to be almost conformal. Since the bulk of the paper is rather technical, in section 4 we summarize the main logical steps, avoiding formulas, leading to the puzzle.

2 The gauge theory story

The original Maldacena correspondence [1] relates $\mathcal{N} = 4$ $SU(N)$ superconformal Yang-Mills theory in four dimensional Minkowski space-time and type IIB string theory in $AdS_5 \times S^5$. The gauge theory description is valid for small ’t Hooft coupling $\lambda \equiv g_Y^2 N \ll 1$, while the dual supergravity has small curvatures in the opposite regime $\lambda \gg 1$. The main challenge in extending the correspondence beyond the supergravity approximation comes from the fact that anomalous dimensions of the gauge theory operators dual to excited string states (rather than to the supergravity modes) are generically expected to grow as $\lambda^{1/4}$ [13,14], and thus appear to be beyond the grasp of the perturbative gauge theory analysis in the regime of the validity of the supergravity approximation.

2.1 GKP proposal

In [3], Gubser, Klebanov and Polyakov considered twist two operators in $\mathcal{N} = 4$ SYM. In the free field theory these are the operators with the lowest conformal dimension for a given spin $n$, for example,

$$O_{(\mu_1 \cdots \mu_n)} = \text{Tr} \Phi' \nabla_{(\mu_1} \cdots \nabla_{\mu_n)} \Phi'^{\prime}, \quad (1)$$
where $\Phi^I$ are the $N=4$ scalars, and the symmetrization $(\cdots)$ denotes also a removal of traces. The gauge covariant derivative is $\nabla_\mu \equiv \partial_\mu + igYM A_\mu$. Classically, the operator $O_{(\mu_1\cdots\mu_n)}$ has dimension $\Delta_{O_n} = n + 2$, hence twist 2. In is expected [15], that the leading term in the anomalous dimensions of operators such as (1) grows as $\ln n$ (exactly as a one-loop perturbative correction!) to all orders in perturbation theory and also non-perturbatively. Thus, one expects that in the full interacting gauge theory

$$\Delta_{O_n} - (n + 2) = f(\lambda) \ln n + o(\ln n),$$

(2)

where $f(\lambda)$ is a certain function of the 't Hooft coupling, which has perturbative SYM expansion

$$f(\lambda) = a_1 \lambda + a_2 \lambda^2 + \cdots, \quad (\lambda \ll 1).$$

(3)

Notice that, provided (2) is correct, the anomalous dimension of operators $O_n$ is small compare to the classical one, in the limit $n \to \infty$

$$\frac{\Delta_{O_n} - (n + 2)}{n} \to 0.$$  

(4)

GKP proposed [3] that such operators in the dual supergravity picture are described by folded macroscopic strings rotating in $AdS_5$. Remarkably, closed strings on the leading Regge trajectory with large spin $S \gg 1$ in the $AdS_5$ have energy

$$E = S + \frac{\sqrt{\lambda}}{\pi} \ln S + o(\ln S),$$

(5)

which agrees (up to the functional dependence on $\lambda$) with the gauge theory result (2), once we employ the standard gauge/gravity dictionary

$$E \leftrightarrow \Delta, \quad S \leftrightarrow n.$$  

(6)

Notice that GKP proposal predicts that the strong coupling expansion of $f(\lambda)$ in (2) is

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \tilde{a}_1 + \frac{\tilde{a}_2}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right).$$

(7)

2.2 KT gauge theory on $S^3$

We would like to extend GKP analysis to KT cascading gauge theory compactified on $S^3$. We explain the regime in which this theory is a “small deformation” of the

\(^5\)The 1-loop string correction $\tilde{a}_1$ has been computed in [16].
Klebanov-Witten (KW) $\mathcal{N} = 1$ superconformal gauge theory [17], and estimate leading correction to the anomalous dimensions of operators analogous to (1).

Consider $N$ D3-branes at the tip of the singular conifold in type IIB string theory [17]. In the limit of small 't Hooft coupling $g_s N \ll 1$, the gauge theory on the world volume of the D-branes is weakly coupled. We find $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N)$ gauge theory with two chiral superfields $A_1, A_2$ in the $(N, \overline{N})$ representation, and two fields $B_1, B_2$ in the $(\overline{N}, N)$ representation. Additionally, there is a superpotential

$$W \sim \text{tr} (A_i B_j A_k B_l) \epsilon^{ik} \epsilon^{jl} .$$

The theory has $SU(2) \times SU(2) \times U(1)$ global symmetry, with the first (second) $SU(2)$ factor rotating the flavor index of the $A_i$ ($B_i$), while the “baryon” $U(1)$ acts as $A_i \rightarrow A_i e^{i \alpha}$, $B_i \rightarrow B_i e^{-i \alpha}$. There is also anomaly free $U(1)_R$ symmetry under which $A_i, B_j$ superfields have R-charge $\frac{7}{2}$. As argued in [18], this theory flows in the IR to a superconformal fixed point with the same $U(1)_R$ symmetry, and hence exactly marginal superpotential (8). At the IR fixed point, the theory has two exactly marginal deformations, parameterized by the gauge couplings $g_1, g_2$ of the $SU(N) \times SU(N)$.

The dual supergravity description of the IR fixed point of the above gauge theory is represented by the backreaction of the D3-branes on the conifold geometry in the limit $g_s N \gg 1$. In this case one finds [17] $AdS_5 \times T^{1,1}$ ($T^{1,1} = (SU(2) \times SU(2))/U(1)$) background with $N$ units of the five form flux through the $T^{1,1}$. The $AdS_5$ factor reflects the conformal invariance of the dual gauge theory. The string coupling is constant, and is related to the sum of the two gauge couplings

$$\frac{1}{g_s} = \frac{4\pi}{g_1^2} + \frac{4\pi}{g_2^2} .$$

The other modulus, the difference of the gauge couplings, is related to a (constant) NSNS 2-form flux $B_2$ through the $S^2$ of the cone base. Finally, the symmetries of the $T^{1,1}$ coset are realized as global symmetries of the gauge theory.

One can literally repeat the GKP analysis in this case. The energy-spin relation for the macroscopic rotating strings with $S \gg 1$ is still given by (5), while the dual gauge theory twist two operators are

$$O_{(\mu_1 \cdots \mu_n)}^{ij} = \text{tr} A_i \nabla_{(\mu_1} \cdots \nabla_{\mu_n)} B_{j)} ,$$

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$^6$ $g_s$ is the string coupling.

$^7$ We assign R-charge to gauginos to be 1.

$^8$ Topologically $T^{1,1}$ is $S^2 \times S^3$. 
with $\nabla_\mu \equiv \partial_\mu + ig_1 A_\mu^{(1)} + ig_2 A_\mu^{(2)}$. It is easy to see that the one-loop perturbative contribution to the anomalous dimension scales as

$$\triangle_{\phi_i^{(1)}} - (n + 2) \sim N(g_1^2 + g_2^2) \ln n.$$  \hfill (11)

Interestingly, while the perturbative gauge theory computation (11) indicates the dependence on both parameters along exactly marginal deformations $g_1, g_2$, the corresponding classical Regge trajectory depends only on the 't Hooft parameter

$$\lambda \equiv g_s N = \frac{g_1^2 g_2^2 N}{4\pi(g_1^2 + g_2^2)}.$$  \hfill (12)

The obvious reason is that the classical string rotating in $AdS_5$ does not “know” about the NSNS 2-form flux through the two-cycle of the $T^{1,1}$, parameterizing the other marginal direction

$$\int_{S^2} B_2 \sim \frac{1}{g_1^2} - \frac{1}{g_2^2}.$$  \hfill (13)

We expect however, that the dependence on (13) would arise in 1-loop sigma model correction to the Regge trajectory. It would be very interesting to verify this explicitly by extending the computation of [16]. The changes in the $\lambda$ scaling from the weak to strong coupling regime is familiar from other gauge/gravity computations (like the coefficient of Wilson loops), or the GKP analysis. The coupling dependence we are finding here is much more involved and deserves a better understanding. In what follows, we assume that in the strong coupling regime of the superconformal KW gauge theory, the anomalous dimension of twist two operators (10) indeed scales as $\sqrt{\lambda}$, as predicted by the analysis of the sigma model solitons (5).

In [19] it was shown that adding $M$ fractional D3-branes (D5-branes wrapping the 2-cycle of the conifold) breaks the conformal invariance. The resulting $\mathcal{N} = 1$ supersymmetric gauge theory has been studied in details in\(^9\) [21]. Here, the gauge theory on the world volume of the branes is $SU(N + M) \times SU(N)$ with the same matter content as in the KW gauge theory: chiral superfields $A_i$ in the $(N + M, \overline{N})$ representation, and chiral superfields $B_j$ in $(\overline{N} + M, N)$. We also have the same superpotential (8). The $M$-deformed theory still has $SU(2) \times SU(2) \times U(1)$ global symmetry. As argued in [21], the sum of the gauge couplings (9) still remains the exactly marginal direction, while the difference of the couplings runs

$$\frac{4\pi}{g_2^2} - \frac{4\pi}{g_1^2} \sim M \ln(\mu/\Lambda)[3 + 2(1 - \gamma)],$$  \hfill (14)

\(^9\)For a nice review see [20].
where $\gamma$ is the anomalous dimension of operators $\text{tr} A_i B_j$, and $\Lambda$ is the strong coupling scale. As a result of (14), the $M$-deformed KW theory undergoes a series of self-similarity transformations (a cascade [21] of Seiberg dualities [22]) which can be succinctly characterized as if the rank $N$ (the number of regular D3-branes at the tip of the conifold) develops an “anomalous dimension”, so that [23]

$$N \rightarrow N_{\text{eff}}(\mu) \sim g_s M^2 \ln(\mu/\Lambda).$$

(15)

The support for the interpretation (15) comes from studies of the high temperature thermodynamics of this gauge theory [23, 24], and the computation of the correlation functions [25]. The above description of the deformed KW gauge theory is clearly physically inadequate in the IR, where the effective rank (15) becomes negative. In [21], Klebanov and Strassler analyzed in details the case when

$$N = 1, \quad \text{mod } M.$$

(16)

They showed that in this case the cascade actually stops before reaching $N_{\text{eff}} < 0$. The physical reason is rather simple. It turns out, for (16), the gauge theory develops a mass gap in the IR, so that in the effective low energy description the gauge couplings stop running (because of the mass gap there are no charged zero modes), and as a result

$$N_{\text{eff}}(\mu) \equiv 1, \quad \mu < \Lambda.$$

(17)

With this physical mechanism of stopping the duality cascade in mind, it is easy to understand now how to terminate Klebanov-Strassler duality cascade in the IR at the conformal fixed point. Indeed, consider conformal compactification of the KW gauge theory on the $S^3$. Such a theory has a mass gap, and as a result its $M$-deformation (changing the rank of one of the gauge groups $N \rightarrow N + M$) would stop cascading at energy scale $\mu_0$ set by the size of the compactification $S^3$. Obviously, it is possible to arrange “initial” value $N_{\text{eff}}(\mu_0) = N_0 \gg 1$, so that the duality cascade would stop with $SU(N_0 + M) \times SU(N_0)$ gauge theory, which can be made arbitrarily close of being conformally invariant if $\frac{M}{N_0} \ll 1$. Notice that according to (15) (see [20] for a more precise statement for $N_{\text{eff}}(\mu)$), there is a window of energy scales

$$\mu_0 \ll \mu_i < \mu < \mu_f,$$

(18)

such that

$$M \ll g_s M^2 \ln(\mu/\Lambda) \ll N_0.$$

(19)
Additionally, we can take
\[ \ln \frac{\mu}{\Lambda} \gg 1, \tag{20} \]
provided \( \frac{M^2}{N_0} \) is small enough. Physically, we need conditions (18)—(20) so that the cascading gauge theory is (a) perturbative in the appropriate description along the Seiberg duality cascade, (b) being probed at scales much shorter than the \( S^3 \) compactification scale, so that we can use the flat-space renormalization group flow equations, and (c), though at these energy scales the theory underwent many steps of the Seiberg duality cascade, it is still a small deformation of the \( SU(N_0) \times SU(N_0) \mathcal{N} = 1 \) superconformal gauge theory. The dual supergravity background to the \( M \)-deformed KW gauge theory in the regime (18)—(20) has been constructed analytically to the leading order in the deformation in [8].

We now turn to the leading correction to the anomalous dimensions of operators (10) due to the \( M \)-deformation, with constraints (18)—(20). The perturbative computation (11) would go through with the only change \( N \to N_{\text{eff}}(\mu) \). The dual supergravity computation predicts that at large ’t Hooft coupling \( \lambda \gg 1 \), in the conformally invariant KW theory the anomalous dimensions of these operators scale as
\[ \triangle \sigma^{\mu}_{ij} - (n + 2) \sim \sqrt{\lambda} \ln n. \tag{21} \]
Since in the regime (18)—(20) the deformed theory is almost conformally invariant, we expect that the leading correction to (21) would be due\(^{10}\)
\[ \lambda \to \lambda_{\text{eff}}(\mu) \equiv g_s N_{\text{eff}}(\mu) \sim g_s N_0 \left( 1 + \frac{M^2}{N_0} \ln \frac{\mu}{\Lambda} + M^2 \frac{\ln(\ln(\mu/\Lambda)) + o(M^2)}{\ln(\ln(\mu/\Lambda)) + o(M^2)} \right), \tag{22} \]
thus we expect the leading correction to be
\[ \triangle \sigma^{\mu}_{ij} - (n + 2) \sim \sqrt{\lambda_0} \ln n + \frac{\sqrt{\lambda_0} M^2}{N_0} \ln n \ln \frac{\mu}{\Lambda}, \tag{23} \]
where \( \lambda_0 \equiv g_s N_0 \). In the next section we compare (23) with the energy of the highly excited string states on the leading Regge trajectory in the dual supergravity background [8]. We argue that these string states should be thought of as being dual to gauge theory operators at energy scales \( \ln \mu/\Lambda \sim \ln n \). In this case the subleading correction to the anomalous dimensions of operators (10) is predicted from (23) to scale
\[^{10}\text{From the supergravity analysis [8], in the KT gauge theory on the } S^3 \text{ the string coupling is not constant, but rather } g_s = g_s(\mu). \text{ The scale dependence is actually very mild and comes with a factor of } M^2/N_0.\]
as $M^2/N_0 \ln^2 n$. We rather find the leading in $M^2/N_0$ correction to the energy of these string states to scale as $M^2/N_0 \ln^3 S$, where $S$ is a spin of the state.

We would like to conclude this section with some conjectures about the properties of operators (10) in the ultraviolet following from the discussion above. Clearly, in the UV, whether we compactify KT gauge theory or not, should be irrelevant. Since in the UV ($\mu \to \infty$), $M/N_{\text{eff}}(\mu) \to 0$, it is reasonable to assume that KT gauge theory approaches conformal fixed point, which has the properties of the standard $\mathcal{N} = 4$ superconformal gauge theory with scale dependent number of colors, determined by (15). This statement is definitely not new, and is implicit in many studies of the KT gauge theory. In this case, motivated by (5), we would expect anomalous dimension of operators (10) to be\footnote{Note that in the UV the t’ Hooft coupling is always large, that’s why we should use the sigma model result (5).}

$$\Delta_{O^{ij}} - (n + 2) \sim \sqrt{\lambda_{\text{eff}}(\mu) \ln n} \sim g_s^{1/2} M \sqrt{\ln \frac{\mu}{\Lambda} \ln n}. \quad (24)$$

Obviously, it makes sense to talk about anomalous dimension only when

$$\sqrt{\lambda_{\text{eff}}(\mu) \ln n} \ll n. \quad (25)$$

Unfortunately, as we explain in details in the following section, the lack of the full nonlinear solution for the dual supergravity does not allow us to test (24).

## 3 The supergravity story

In this section we would like to compare (23) with the dual sigma model computation in the supergravity background [8]. After reviewing the construction of the dual supergravity to the deformed KW gauge theory, we extract the leading Regge trajectory of closed strings. In the regime dual to (18)-(20) we find subleading correction to the energy of highly excited string states to differ from that implied by (23). We also comment on the difficulty studying “very long strings”, which probe the anomalous dimension of twist two operators (10) far in the ultraviolet, expected to be given by (24). A related work appeared in [26].

### 3.1 SUGRA dual to the KT gauge theory on $S^3$

We begin with reviewing the supergravity solution of [8] realizing supergravity dual to the KT gauge theory compactified on $S^3$. 
The deformed 10-d Einstein frame metric takes the form

\[ ds_E^2 = - f_1^{-1/2} dX_0^2 + \rho f_2^{-1/2} (dS^3)^2 + \frac{d\rho^2}{4\rho(1-\rho)^2} \]

\[ + f_3^{1/2} e_\psi^2 + f_4^{1/2} (e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2) , \]

with

\[ (dS^3)^2 = d\beta_1^2 + \cos^2 \beta_1 (d\beta_2^2 + \cos^2 \beta_2 d\phi^2) , \]

and \( e_\psi, e_{\theta_1}, e_{\phi_1} \) are the standard \( T^{1,1} \) vielbeins (see for example (2.4) of [8]). The \( p \)-form fields are as in the extremal KT solution

\[ F_3 = P e_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) , \quad B_2 = P k(\rho) (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) , \]

\[ F_5 = \mathcal{F} + * \mathcal{F} , \quad \mathcal{F} = K(\rho) e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} , \quad K(\rho) = 4 + 2P^2 k(\rho) , \]

where the \( p = 0 \) normalization of the five form flux \( K \) is such that the background \( AdS_5 \) radius is \( L = 1 \). Also we take the bare string coupling to be \( g_s = 1 \). With this normalization, \( P \) is related to the gauge theory parameters \( M, N_0 \) of the previous section as follows

\[ P^2 \equiv \frac{M^2}{N_0} . \]

In (26) the radial coordinate \( \rho \in [0,1) \) and the warp factors \( f_i \) differ by \( O(P^2) \) terms from the \( AdS_5 \times T^{1,1} \) geometry\(^\text{12}\)

\[ f_1(\rho) = (1 - \rho)^2 + P^2 \phi_1(\rho) , \quad f_2(\rho) = (1 - \rho)^2 + P^2 \phi_2(\rho) , \]

\[ f_3(\rho) = 1 + P^2 \phi_3(\rho) , \quad f_4(\rho) = 1 + P^2 \phi_4(\rho) , \quad \Phi(\rho) = P^2 \phi(\rho) , \]

where \( \Phi \) is the dilaton\(^\text{13}\).

The analytical solution for the warp factors \( \phi_i \) and \( k(\rho) \) to leading order in \( P^2 \) was

\(^\text{12}\)The standard \( AdS_5 \) metric in global coordinates \( ds_{AdS} = - \cosh^2 r \; dX_0^2 + \sinh^2 r \; (dS^3)^2 + dr^2 \) is obtained with the identification \( \rho = \tanh^2 r \).

\(^\text{13}\)The complete nonlinear system of differential equations for the warp factors \( f_i(\rho) \), the dilaton \( \Phi(\rho) \equiv \ln g_s(\rho) \), and the 3-form flux \( k(\rho) \) obtained from the type IIB supergravity equations of motion is given in Appendix.
found in [8]. Here we reproduce only the IR/UV ($\rho \to 0/\rho \to 1_-$) asymptotics.

$$k(\rho) = \frac{\rho}{2} + \frac{\rho^2}{4} + O(\rho^3), \quad \phi(\rho) = \frac{\rho}{8} + \frac{\rho^2}{48} + O(\rho^3),$$

$$\phi_1(\rho) = \left(15\delta - \frac{57}{8} + \frac{2}{3}\pi^2\right)\rho + \left(\frac{13}{48} - \frac{5}{3}\delta - \frac{2}{27}\pi^2\right)\rho^2 + O(\rho^3),$$

$$\phi_2(\rho) = \left(\frac{35}{3}\delta - \frac{67}{12} + \frac{14}{27}\pi^2\right)\rho + \left(-\frac{107}{120} + \delta + \frac{2}{45}\pi^2\right)\rho^2 + O(\rho^3),$$

$$\phi_3(\rho) = 3\delta + \left(-\frac{31}{8} + 12\delta + \frac{1}{3}\pi^2\right)\rho + O(\rho^2),$$

$$\phi_4(\rho) = \left(3\delta - \frac{7}{4} + \frac{1}{6}\pi^2\right) + \left(-\frac{25}{4} + 12\delta + \frac{7}{12}\pi^2\right)\rho + O(\rho^2),$$

as $\rho \to 0_+$ and

$$k(x) = -\frac{1}{2}\ln x, \quad \phi(x) = \left(-\frac{1}{4} + \frac{1}{24}\pi^2\right) - \frac{1}{4}x + O(x^2\ln x),$$

$$\phi_1(x) = \left(\frac{1}{16}\ln^2 x - \frac{5}{16}\ln x\right)x^2 + O(x^2),$$

$$\phi_2(x) = \left(\frac{1}{16}\ln^2 x - \frac{5}{16}\ln x\right)x^2 + O(x^2),$$

$$\phi_3(x) = \left(-\frac{1}{4}\ln x + \frac{1}{8}\right) + \frac{1}{12}x + O(x^2),$$

$$\phi_4(x) = \left(-\frac{1}{4}\ln x + \frac{1}{8}\right) - \frac{1}{24}x + O(x^2),$$

as $x \equiv (1 - \rho) \to 0_+$. In (31) $\delta$ (referred to as $\alpha_1$ in (5.21) of [8]) has been computed numerically\(^{14}\). Clearly we can trust “ultraviolet” asymptotics (32) as long as

$$P^2\ln^2 x \ll 1, \quad (33)$$

which is a gravity dual to the gauge theory requirement of staying close to the IR conformal fixed point (the second inequality in (19)). Notice that the supergravity constraint (33) appears to be stronger than the corresponding gauge theory statement, once we identify\(^{15}\)

$$\ln \frac{\mu}{\Lambda} \sim -\ln x, \quad (34)$$

\(^{14}\)From (5.23) and the footnote (22) of [8], $\delta \approx 0.0646108$.

\(^{15}\)This identification comes from comparing the anomalous dimension of the rank (15) with the dual supergravity statement of the leading $P^2$ radial dependence of the 3-form flux: $k(x) = -\frac{1}{2}\ln x$. 

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A related point (mentioned in [8]), is that the UV asymptotics for $\phi_1, \phi_2$ differ from the corresponding asymptotics of the extremal KT solution [12]. It is this difference that will be responsible for the disagreement of the leading Regge trajectory of the long strings with the gauge theory result (23).

Before we move on to study string solitons in the deformed KT background, we would like to explain one subtlety\(^\text{16}\) associated with the leading $P^2$ solution of [8], giving rise to asymptotics (31), (32). This (regular) supergravity solution does not have a free parameter apart from the bare string coupling (which we set equal to one in the infrared), once $P$ is set, and we choose $L = 1$. Physically, we expect a one parameter family of nonsingular KT deformations, represented by

$$\xi \equiv \frac{\Lambda}{\mu_0},$$

(35)

where $\Lambda$ is the strong coupling scale of KT gauge theory, and $\mu_0$ is the gauge theory compactification scale set by the size of the $S^3$. We expect that this parameter appears as nonlinear effect in $P^2$ in the “infrared” of the supergravity solution [8]. Presumably this is due to the fact that, in the infrared, the deformation is about the conformal background, which is insensitive to a specific value of scale $\mu_0$, as long as it is nonzero. While it is possible to identify a candidate parameter in the “ultraviolet” (where the supergravity background is a small deformation of the extremal KT geometry [12]), to conclusively settle this issue one needs the full nonlinear solution for the deformed background. Consider the most general regular in the IR (small $\rho$) solution to (74). We find\(^\text{17}\)

\[
\begin{align*}
k(\rho) &= \delta_1 \rho + O(\rho^2), \\
e^{\Phi(\rho)} &= 1 + \frac{1}{4} P^2 \delta_1 \rho + O(\rho^2), \\
f_1 &= 1 - \frac{1}{4} \delta_1 \left( P^2 + 32 \delta_1 \delta_2^{1/2} \right) \rho + O(\rho^2), \\
f_2 &= 1 + \frac{1}{18} \left( 24 \sqrt{2} \delta_1^{1/2} \delta_2^{1/4} - 128 \delta_1^{3/2} \delta_2^{1/2} - 5 \delta_1 P^2 - 8 \delta_1 \delta_2 - 24 \right) \rho + O(\rho^2), \\
f_3 &= \delta_2 - \frac{1}{4} \delta_1 \delta_2 \left( 32 \delta_1 \delta_2^{1/2} - 16 \delta_2 + 3 P^2 \right) \rho + O(\rho^2), \\
f_4 &= \frac{1}{2} \delta_1^{-1/2} \delta_2^{-1/2} - \frac{1}{8} \delta_1^{-1/2} \delta_2^{-1/2} \left( 8 \delta_1^{1/2} \delta_2 - 12 \sqrt{2} \delta_2^{1/4} + 32 \delta_1^{3/2} \delta_2^{1/2} + P^2 \delta_1^{1/2} \right) \rho + O(\rho^2), \\
\end{align*}
\]

\(^{16}\)I would like to thank Leo Pando Zayas for raising this issue and a very useful discussion.

\(^{17}\)As in [8], we set $f_1(\rho = 0) = 1$. This initial value is related to the freedom of rescaling the time coordinate, and is also present in KT background [12].
where $\delta_1, \delta_2$ are free parameters. Matching (36) with (31), necessary to reproduce the leading KT asymptotics in the UV, gives

$$\delta_1 = \frac{1}{2} + \left( \frac{7}{8} - \frac{9}{4}\delta - \frac{1}{12}\pi^2 \right) P^2 + O(P^4),$$

$$\delta_2 = 1 + 3\delta P^2 + O(P^4).$$

(37)

Indeed, both parameters are fixed at order $P^2$. As we show now, small deformation of the extremal KT solution in the UV has a single free parameter. To study UV asymptotics ($\rho \to 1_-$) of (74), it is convenient to use the 3-form flux function $k(\rho)$ as a new radial coordinate $r \equiv k(\rho)$. The UV asymptotics are then simply $r \to \infty$. Rewriting (74) for the new radial coordinate, we find the following asymptotics as $r \to \infty$

$$g_s = g_0 \left( 1 + e^{-2r/g_0} G(r) + O(e^{-4r/g_0}) \right),$$

$$f_1 = \gamma \left( 1 + \frac{1}{2} P^2 r + \frac{1}{8} P^2 g_0 \right) e^{-4r/g_0} \left( 1 + e^{-2r/g_0} F_1(r) + O(e^{-4r/g_0}) \right),$$

$$\rho^{-2} f_2 = \gamma \left( 1 + \frac{1}{2} P^2 r + \frac{1}{8} P^2 g_0 \right) e^{-4r/g_0} \left( 1 + e^{-2r/g_0} F_2(r) + O(e^{-4r/g_0}) \right),$$

$$f_3 = \left( 1 + \frac{1}{2} P^2 r + \frac{1}{8} P^2 g_0 \right) \left( 1 + e^{-2r/g_0} F_3(r) + O(e^{-4r/g_0}) \right),$$

$$f_4 = \left( 1 + \frac{1}{2} P^2 r + \frac{1}{8} P^2 g_0 \right) \left( 1 + e^{-2r/g_0} F_4(r) + O(e^{-4r/g_0}) \right),$$

(38)

and (compare with (26))

$$\frac{(d\rho)^2}{4\rho(1-\rho)^2} \equiv G_{rr}(dr)^2,$$

(39)

with

$$G(r) = -\frac{1}{4} P^2 g_0 \gamma^{1/2},$$

$$F_1(r) = \frac{1}{36} P^2 g_0 \gamma^{1/2} \left( \frac{120 P^2 r + 23 P^2 g_0 + 240}{4P^2 r + P^2 g_0 + 8} \right),$$

$$F_2(r) = P^2 \gamma^{1/2} + \frac{1}{12} \gamma^{1/2} \left( 13P^2 g_0 + 24 \right) r^{-1} + O(r^{-2}),$$

$$F_3(r) = F_1(r) - \frac{3}{4} P^2 g_0 \gamma^{1/2},$$

$$F_4(r) = F_1(r) - \frac{7}{8} P^2 g_0 \gamma^{1/2},$$

(40)

where $\gamma$ is the parameter we conjecture is related to $\xi$ in (35), and $g_0$ is the asymptotic value of the string coupling. In (40) all functions except $F_2$ have been determined.
analytically. The function $F_2$ is given by

$$F_2(r) = \frac{8^{1/2} e^{2r/g_0}}{g_0 r} \left( \alpha + \int_1^{1/r} \frac{e^{-2/(g_0 x)} (P^2 + 2x)}{x^3 (x(P^2 g_0 + 8) + 4P^2)^2} \right) \left( x^2(11P^4 g_0^2 + 78P^2 g_0 + 144) + 3P^2 x(13P^2 g_0 + 48) + 36P^4 \right),$$

with the integration constant $\alpha$ determined from the regularity requirement as $r \to \infty$, (40). We could not evaluate (41) in elementary functions. Finally,

$$G_{rr} = g_0^{-4} \left( 1 + \frac{1}{2} P^2 r + \frac{1}{8} P^2 g_0 \right) \left( 1 + e^{-2r/g_0} g_{rr}(r) + O(e^{-4r/g_0}) \right),$$

with

$$g_{rr} = \frac{\gamma^{1/2}}{36} \frac{144P^4 r^2 + 12P^2 (48 + 19P^2 g_0) r + 41P^4 g_0^2 + 456P^2 g_0 + 576}{4P^2 r + P^2 g_0 + 8}.$$  

Curiously, even though $g_s$ approaches constant both in the IR and in the UV, these constants are not the same: from (31), (32) we find

$$\ln g_0 = \left( -\frac{1}{4} + \frac{1}{24} \pi^2 \right) P^2 + O(P^4).$$

### 3.2 Rotating string in deformed KT geometry

We now discuss classical string solutions in the background geometry (26).

We take sigma model action in the conformal gauge\(^a\)

$$S = \frac{1}{4 \pi \alpha'} \int d^2 \sigma G_{ij} \partial_\sigma X^i \partial^\sigma X^j.$$  

This action is supplemented by the constraints

$$G_{ij} \left( \partial_r X^i \partial_r X^j + \partial_\sigma X^i \partial_\sigma X^j \right) = 0,$$

$$G_{ij} \partial_r X^i \partial_\sigma X^j = 0.$$  

Consider a closed string rotating in the $(\rho, \phi)$ plane in the deformed KT background (26)

$$\rho = \rho(\sigma), \quad \phi = \omega \tau, \quad X_0 = k \tau,$$

$$\beta_i = 0, \quad \theta_i = 0, \quad \phi_i = 0, \quad \psi = 0,$$

\(^a\)Note that $G_{ij}$ is the string frame metric.
for constants \((k, \omega)\). Equations of motion and the constraints are satisfied provided

\[
0 = \partial_\sigma (G_{pp} \partial_\rho \rho) - \frac{1}{2} \partial_\rho (G_{pp}) (\partial_\sigma \rho)^2 + \frac{1}{2} \partial_\rho \left( G_{\phi\phi} \omega^2 - G_{X_0 X_0} k^2 \right),
\]

\[
0 = G_{pp} (\partial_\sigma \rho)^2 + \left( G_{\phi\phi} \omega^2 - G_{X_0 X_0} k^2 \right),
\]

where

\[
G_{X_0 X_0} = e^{\Phi/2} f_1^{-1/2},
G_{\phi\phi} = e^{\Phi/2} \rho f_2^{-1/2},
G_{\rho\rho} = e^{\Phi/2} \frac{1}{4\rho(1-\rho)^2}.
\]

The space-time energy is given by

\[
E = \frac{k}{2\pi \alpha'} \int_0^{2\pi} d\sigma \ G_{X_0 X_0},
\]

and the spin is

\[
S = \frac{\omega}{2\pi \alpha'} \int_0^{2\pi} d\sigma \ G_{\phi\phi}.
\]

We consider the simplest “one-fold” string configuration where the interval \(0 \leq \sigma < 2\pi\) is split into 4 segments: for \(0 < \sigma < \pi/2\) the function \(\rho(\sigma)\) increases from 0 to its maximum value \(\rho_0\) such that \(\rho'(\pi/2) = 0\)

\[
0 = G_{\phi\phi}(\rho_0) \omega^2 - G_{X_0 X_0}(\rho_0) k^2,
\]

then for \(\pi/2 < \sigma < \pi\) decreases to zero, etc. The periodicity of \(\sigma\) implies additional condition on the parameters

\[
2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\rho_0} d\rho \sqrt{\frac{G_{pp}}{k^2 G_{X_0 X_0} - \omega^2 G_{\phi\phi}}},
\]

In what follows we introduce

\[
\frac{\omega^2}{k^2} = 1 + \eta.
\]

3.2.1 Short strings

A short string limit corresponds to

\[
\eta \gg 1.
\]

\(^{19}\)Primes denote derivatives with respect to \(\sigma\).
From (52) we find
\[ \rho_0 \approx \frac{1}{\eta} \ll 1. \] (56)

In this case the rotating string is hardly stretched, so we can replace the complicated deformed KT geometry (26) with almost flat space. We expect the leading Regge trajectory to be a small deformation of that in flat space. Indeed, a somewhat tedious but straightforward computation gives

\[ 2\pi\alpha' E = \frac{1}{\eta^{1/2}} \left( \frac{1}{4} \right)^{1/2} \left( 1 + \frac{1}{\eta} \left( \frac{44\delta - 831/40 + 88\pi^2/45}{40\delta - 37/2 + 16\pi^2/9} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ - \frac{|P|}{\eta} \left( \frac{20}{3} \delta - \frac{37}{12} + \frac{8}{27} \pi^2 \right)^{1/2} \left( 1 + \frac{1}{\eta} \left( \frac{22\pi^2/9 - 267/10 + 55\delta}{16\pi^2/9 - 96/5 + 40\delta} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ - \frac{\pi P^2}{\eta^{3/2}} \left( \frac{5}{18} \pi^2 - 3 + \frac{25}{4} \delta \right) \left( 1 + \frac{1}{\eta} \left( \frac{86\pi^2/45 - 1047/50 + 43\delta}{8\pi^2/5 - 432/25 + 36\delta} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ + O \left( P^3 \right), \] (57)

for the energy, and

\[ 2\pi\alpha' S = \frac{\sqrt{1 + \eta}}{2\eta^{3/2}} \left( \frac{1}{2} \right)^{1/2} \left( 1 + \frac{1}{\eta} \left( \frac{24\delta - 461/40 + 16\pi^2/15}{40\delta - 37/2 + 16\pi^2/9} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ - \frac{|P|}{\eta^{3/2}} \left( \frac{20}{3} \delta - \frac{37}{12} + \frac{8}{27} \pi^2 \right)^{1/2} \left( 1 + \frac{1}{\eta} \left( \frac{24\pi^2/9 - 267/10 + 55\delta}{16\pi^2/9 - 96/5 + 40\delta} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ - \frac{\pi P^2}{\eta^2} \left( \frac{5}{24} \pi^2 - \frac{9}{4} + \frac{75}{16} \delta \right) \left( 1 + \frac{1}{\eta} \left( \frac{86\pi^2/45 - 1047/50 + 43\delta}{8\pi^2/5 - 432/25 + 36\delta} \right) + O \left( \frac{1}{\eta^2} \right) \right) \]

\[ + O \left( P^3 \right), \] (58)

for the spin. Furthermore, we find

\[ \frac{E^2}{2\alpha' S} = \left( 1 + O \left( \frac{1}{\eta} \right) \right) + \frac{|P|}{\eta^{3/2}} \left( \alpha_1 + O \left( \frac{1}{\eta} \right) \right) + \frac{P^2}{\eta^2} \left( \alpha_2 + O \left( \frac{1}{\eta} \right) \right), \] (59)

where \( \alpha_i \) are some constants easily computable from (57),(58). As in [3], reintroducing the scale \( L \), (59) reproduces the standard operator-state correspondence of the AdS/CFT duality.

### 3.2.2 Long strings

The long strings correspond to
\[ \eta \ll 1. \] (60)
Here a priori we have to consider two cases

\[ P^2 (\ln \eta)^2 \ll 1, \quad (61) \]

which corresponds to the small deformation of the conformal infrared fixed point, or

\[ P^2 (\ln \eta)^2 \gg 1, \quad (62) \]

for the “very long strings” corresponding to the UV of the KT gauge theory. It is the first regime, (61) that is of most interest. Using the asymptotics (32), from (52) we find

\[ \rho_0 \approx \frac{1}{1 + \eta} + O(P^4). \quad (63) \]

From (50), (51) we find

\[ 2\pi \alpha' (E - S) = 2 \int_0^{\rho_0} d\rho \frac{(1 - (1 + \eta)^{1/2}\rho)}{\rho^{1/2}(1 - \rho)^{3/2}(1 - (1 + \eta)\rho)^{1/2}} \left(1 - \frac{P^2}{64} \ln^2(1 - \rho) + \cdots\right) \quad (64) \]

where \( \cdots \) denotes subleading in \( P^2 \) correction to the one indicated. Notice that in the limit \( \eta \to 0 \) (we still have to satisfy (61)! the integral diverges. The divergence comes solely from the upper limit, and thus justifies the use of (32). We interpret this “localization” of the divergence as the statement that the stretched rotating string “probes” anomalous dimension of the dual gauge theory operators at energy scale dual to its radial extent. Using gauge/gravity renormalization group relation (34) this scale is

\[ \ln \frac{\mu}{\Lambda} \sim - \ln(1 - \rho_0) \sim - \ln \eta. \quad (65) \]

Carefully extracting the divergent as \( \eta \to 0 \) part of (64) we find

\[ 2\pi (\epsilon - s) \equiv 2\pi \alpha' (E - S) = (-2 \ln \eta + o(\ln \eta)) + P^2 \left(\frac{1}{96}(\ln \eta)^3 + o((\ln \eta)^3)\right). \quad (66) \]

We also need to relate \( \eta \) and \( s \). From (51),

\[ 2\pi \equiv 2\pi \alpha' S = \left(\frac{4}{\eta} + \ln \eta + O(\eta \ln \eta)\right) - \frac{P^2}{16\eta} (\ln \eta)^2 + O(\ln \eta) + O(P^3), \quad (67) \]

so that

\[ \eta = \frac{2}{\pi s} \left(1 - \frac{\ln s}{2\pi s} + \cdots\right) \left(1 - P^2 \left\{\frac{(\ln s)^2}{64} + \cdots\right\}\right), \quad (68) \]
where again we kept only the leading terms. With (68), we arrive at our final expression for the energy of closed string states on the leading Regge trajectory in the deformed KT geometry

$$\epsilon - s = \left( \frac{1}{\pi} \ln \frac{s \pi}{2} + O \left( \frac{\ln s}{s} \right) \right) + \frac{P^2}{2\pi} \left( \left( \ln \frac{s \pi}{2} \right)^3 + O \left( (\ln s)^2 \right) \right).$$  \hfill (69)

Two comments are in order. First, (69) does not reproduce the leading correction to the anomalous dimension of operators (10) at energy scales specified in (18)-(20): with the standard gauge/gravity dictionary (6) the agreement would imply

$$\epsilon - s = \left( \frac{1}{\pi} \ln \frac{s \pi}{2} + O \left( \frac{\ln s}{s} \right) \right) + \frac{P^2}{2\pi} \left( A (\ln s)^2 + O \left( (\ln s)^2 \right) \right),$$  \hfill (70)

for some constant $A$. This is rather puzzling, as the twist two operators (10) are the natural candidate dual, motivated by the translation of the GKP analysis to the deformed KW conformal gauge theory. Second, we would have found the agreement for the leading $P^2$ scaling, had the UV asymptotics of the deformed KT geometry (32) contained single logarithms, rather than $\ln^2 x$. As claimed in [8], it is the single logarithm asymptotics for $\phi_1$ and $\phi_2$ of (32), that is required for the precise agreement with the UV of the extremal KT geometry. In (32) we extended the asymptotic analysis of [8] and confirmed the $x^2 \ln^2 x$ leading asymptotic. We did not manage to find a different from [8] regular deformation of the KW background by the 3-form fluxes of the KT type.

### 3.2.3 Very long strings

In the previous section, using only asymptotic linear in $P^2$ geometry of the deformed KT background we determined the leading Regge trajectory of the long strings (61). The question we would like to address here is whether using the far ultraviolet asymptotics of the deformed KT geometry (38)-(43), we can still identify very long rotating strings, (62), with operators (10) in the far ultraviolet. The latter are expected to have anomalous dimensions given by (24).

Unfortunately, we can not do so in a computationally controllable way. The main difficulty stems from the fact that while the energy/spin of the rotating string in case (61) was dominated by the contribution from the “stretched end” (so that the use of (32) for the background geometry was indeed justified), in case (62) we find energy/spin to be very sensitive to the infrared region of the geometry. Furthermore,
while the infrared geometry is completely regular for the asymptotics (32), it is unphysical (singular) for (38).

To give some more details let’s compare the expression for the energy of the “long” (61), and the “very long” (62) string. To illustrate the point, in the case of long strings it is sufficient to set \( P = 0 \). The gravity background is then simply \( AdS_5 \), and the energy (50) of a rotating string (we use \( \rho \) as a radial coordinate) is

\[
2\pi\alpha' E_{\text{long}} = 2 \int_0^{\rho_0=1/(1+\eta)} \frac{d\rho}{\rho^{1/2}(1-\rho)^{3/2}(1-(1+\eta)\rho)^{1/2}}. \tag{71}
\]

Indeed, as \( \eta \to 0 \) the integral (71) is dominated at the upper bound, and diverges as \( \eta^{-1} \). In other words, in this case the main contribution does come from the ends of the folded stretched string. As the latter stretches almost to the boundary of the global \( AdS_5 \), we are justified to use asymptotics (32) for the \( P^2 \) deformation. Note also that these leading UV asymptotics are regular for small \( \rho \) (we have to set \( x \to 1_- \) in (32)), thus the contribution from the lower bound in (71) is still small.

Consider now the energy of the very long string. Here, using the asymptotics (38) and keeping the leading terms in the integrand we find

\[
2\pi\alpha' E_{\text{very long}} \approx 4\sqrt{2\gamma} e^{-2r_0/g_0} \int_{r_{ir}/g_0}^{r_0/g_0} \frac{e^t}{(g_0 P^2 \gamma^{1/2} t e^{-2t} - 2\eta)^{1/2}} dt, \tag{72}
\]

where we had to introduce the infrared cutoff \( r_{ir} \) to make sense of (72), also

\[
\frac{1}{2} T_0 e^{-2r_0/g_0} P^2 \gamma^{1/2} \equiv \eta. \tag{73}
\]

Notice that unlike (71), the integral (72) crucially depends on the infrared cutoff \( r_{ir} \). It is possible that the problems with the long strings are due to a bad choice of radial coordinate. Recall that here we used the 3-form flux dependence \( k(\rho) \) as a radial coordinate of the deformed KT geometry. It is also possible that these very long strings are unsuitable probes for the operators (10) in the far UV. This is an interesting open question to pursue further.

4 Summary

In this paper we discussed the gauge/string correspondence when gauge theory is formulated in curved space. We argued that in an appropriate regime, the cascading
gauge theory on the world-volume of regular and fractional D3-branes is a small deformation about the infrared conformal fixed point. Following the GKP prescription we attempted to reproduce the anomalous dimension of certain twist two operators from the dual nonlinear sigma model computation in the supergravity background of [8]. We did not succeed in doing this.

We will now summarize the main steps of the gauge/string correspondence studied in this paper.

- We assumed the validity of the GKP identification of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory operators (1) with the long strings spinning in $AdS_5$. The leading Regge trajectory of these highly excited string states then provides a prediction for the anomalous dimensions of twist two operators (1) at large 't Hooft coupling.

- It follows then that the GKP proposal would go through for the Klebanov-Witten $\mathcal{N} = 1$ $SU(N) \times SU(N)$ superconformal gauge theory [17], with twist two operators now being (10). This is intuitively clear once we recall that the KW gauge theory is an infrared fixed point of the $\mathbb{Z}_2$ orbifold of the $\mathcal{N} = 4$ gauge theory, deformed by the mass term for the adjoint chiral superfields in the $\mathcal{N} = 2$ vector multiplets. By (literally) repeating GKP analysis, we then have a prediction for the anomalous dimension of operators (10) at large 't Hooft coupling.

- Next, we considered deformation of the KW gauge theory, when one of the ranks of the two gauge groups is shifted $N \rightarrow N + M$. We assumed that the resulting gauge theory would still be almost conformally invariant if $M/N \ll 1$. As shown in [21], while classically we can think of $N$ and $M$ as being some constants, this is inconsistent quantum mechanically: the deformation $N \rightarrow N + M$ breaks conformal invariance, and under the renormalization group flow this theory very fast becomes strongly coupled. The proper (perturbative) quantum-mechanical description of this deformed theory is in terms of “cascading” gauge theory $SU(N(\mu)) \times SU(N(\mu) + M)$ where the rank $N$ becomes scale dependent. In the ultraviolet, the cascade goes forever as $N(\mu) \rightarrow \infty$ as $\mu/\Lambda \rightarrow \infty$, while in the infrared it stops dynamically with $N(\mu/\Lambda \rightarrow 0) < M$. Thus in the original KT model, the cascading gauge theory is far from being conformally invariant in the infrared. While, it is tempting to say that KT gauge theory is almost “conformal” in the ultraviolet because $M/N(\mu) \rightarrow 0$, the precise meaning of this statement is not clear.

- A way to terminate the duality cascade in the IR with a conformal gauge theory was proposed in [8]. The idea is simply to conformally compactify KW gauge theory,
and then deform it by the shift \( N \to N + M \). While there is still the Seiberg duality cascade (induced by the RG flow) in this compactified gauge theory, which still continues forever in the UV, the termination of the cascade in the IR is rather different: because of the kinematic mass gap in the theory (due to the compactification) the RG flow stops below the scale of the lightest charged states, in turn stopping the duality cascade. We called this energy scale \( \mu_0 \).

- We then explained the constraints on the parameters of the model, namely \( N_0 \equiv N(\mu_0), M \), and the range of energy scales such that the deformed cascading gauge theory is close to the superconformal \( \mathcal{N} = 1 \) \( SU(N_0) \times SU(N_0) \) gauge theory. First, since the cascading KT gauge theory always has gauge group ranks differing by the multiple of \( M \), we should choose \( N_0 \gg M \). Second, in order to use the flat-space cascading picture of Klebanov and Strassler, [21], we should be able to study the theory at energy scales \( \mu \) much higher than the compactification scale \( \mu \gg \mu_0 \). One the other hand, the energy scales of interest should not be too high, if we still want this cascading theory to be close to the infrared superconformal theory: \( N(\mu) - N_0 \ll N_0 \). From the gauge theory perspective all these conditions are mutually compatible for small enough \( M^2/N_0 \), and the appropriate range of energy scales.

- Accepting above steps, it is natural to assume (and we did this in a paper) that the dominant correction to the anomalous dimension of operators (10) for large \('t\) Hooft coupling \( \lambda = g_s N_0 \gg 1 \) due to the deformation \( N_0 \to N_0 + M \) for one of the gauge groups rank, would come from simply replacing \( \sqrt{\lambda_0} \) in the (dual gravitational) expression for the leading Regge trajectory of closed string in \( AdS_5 \) with \( \sqrt{\lambda_{eff}(\mu)} \), properly accounting for the Seiberg duality cascade in the theory.

- We proposed to test the latter prediction by studying closed strings spinning in the supergravity background [8]. This background was constructed as a leading (in \( P^2 \equiv M^2/N_0 \)) regular deformation of the global \( AdS_5 \times T^{1,1} \) geometry (dual to the superconformal \( SU(N_0) \times SU(N_0) \)) gauge theory by turning on certain 3-form fluxes on \( T^{1,1} \), dual to the gauge theory shift deformation \( N_0 \to N_0 + M \).

- As in the \( AdS_5 \) space-time, we found that the closed string states on the leading Regge trajectory with very high spin, were realized by strings stretched almost to the boundary of the (\( P^2 \) deformed) global \( AdS_5 \). Dominant contribution for the energy/spin of such states was coming from the region close to the boundary. We proposed to identify these stretched strings as gravity dual to operators (10), probed at energy scales dual to the radial extent of these stretched stings. This energy scale
was determined unambiguously by identifying the radial dependence of the 3-form flux in the dual supergravity with the gauge theory equation for the scale dependence of the effective rank of the cascading gauge theory $N(\mu)$.

- We found different order $P^2$ correction to the Regge trajectory of long strings in the background [8], compare with the $M^2/N_0$ corrected anomalous dimension of KW operators (10). We traced the difference to the fact that the asymptotics of the (linear in $P^2$) supergravity solution of [8] does not reproduce precisely the asymptotics of the extremal Klebanov-Tseytlin background [12].

We did not find a physically satisfactory explanation for the discrepancy between the long strings Regge trajectory in the background [8] and the (supposedly gauge theory dual) anomalous dimension of operators (10). One possibility is that there is a different from [8] deformation of the KW supergravity background (i.e. $AdS_5 \times T^{1,1}$), also regular in the limit of vanishing 3-form flux. We searched for such solution, still within the ansatz of only radial deformation as in [8], and did not find one. Another possible explanation is related to the subtlety of the background [8], discussed in section 3. The point is that the leading $P^2$ deformation of [8] does not contain parameter $\mu_0$. It is thus not clear whether the gauge theory requirement $\mu \gg \mu_0$ is actually satisfied before $P^2 \ln^2 \mu$ in Eq. (32) becomes much larger then one, invalidating the use of these asymptotics for the study of long spinning strings relevant to operators (10) with anomalous dimensions given by (23). To resolve the latter issue it is necessary either to construct the next order correction in $P^2$ to the background [8] (to see the appearance of $\mu_0$, and verify that $P^2 \ln^2 \mu_0 \ll 1$ is indeed possible to arrange), or to understand the renormalization group flow of the compactified cascading gauge theory at energy scales $\mu \sim \mu_0$. We hope to report on this in the future.

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Appendix

Type IIB supergravity equations of motion for the background (26), (28) reduce to the following system

\[
0 = \left( \frac{(k')^4 f_3 \rho^8 (1 - \rho)^4}{f_1 f_3^2 g_s^4} \right)' - \frac{2(k')^3 f_3^{1/2} (2 + P^2 k) \rho^7 (1 - \rho)^2}{f_1 f_3^2 f_4 g_s^4},
\]

\[
0 = \left( \frac{(g'_s)^4 f_3 f_4^4 \rho^8 (1 - \rho)^4}{f_1 f_2^3 g_s^4} \right)' + \frac{P^2 (g'_s)^3 f_3^4 \rho^7 (1 - \rho)^2}{f_1 f_3^2 g_s^4} \left( 4(k')^2 f_3 \rho (1 - \rho)^2 - f_3^{1/2} g_s^2 \right),
\]

\[
0 = \left( \frac{(f'_4)^4 f_3}{f_1 f_3^2} \rho^8 (1 - \rho)^4 \right)' + \frac{(f'_4)^3 f_3^2 \rho^7 (1 - \rho)^2}{f_1 f_3^2 f_4 g_s} \left( 4g_s f_3^{1/2} (2 + P^2 k)^2 + P^2 g_s f_3^{1/2} f_4 + 4P^2 (k')^2 f_3 f_4 \rho (1 - \rho)^2 \right),
\]

\[
0 = \left( \frac{(f'_3)^4 f_4^4 \rho^8 (1 - \rho)^4}{f_1 f_2^3 f_3^3} \right)' + \frac{(f'_3)^3 f_4^2 \rho^7 (1 - \rho)^2}{f_1 f_3^2 f_4^2 g_s} \left( 4g_s (2 + P^2 k)^2 + g_s f_4 (3P^2 g_s - 16 f_3) - 4P^2 (k')^2 f_3 f_4 \rho (1 - \rho)^2 \right),
\]

\[
0 = \left( \frac{(f'_1)^4 f_3 f_4^4 \rho^8 (1 - \rho)^4}{f_1 f_3^2} \right)' + \frac{(f'_1)^3 f_3^2 \rho^7 (1 - \rho)^2}{f_1 f_2^2 g_s} \left( 4g_s f_3^{1/2} (2 + P^2 k)^2 + P^2 g_s f_3^{1/2} f_4 + 4P^2 (k')^2 f_3 f_4 \rho (1 - \rho)^2 \right),
\]

\[
0 = \left( \frac{(f'_2)^4 f_3 f_4^4 \rho^{14} (1 - \rho)^4}{f_1 f_2^2} \right)' - \frac{2(f'_2)^3 f_4}{f_1 f_2^2} \rho^{13} (1 - \rho)^4
\]

\[
+ \frac{(f'_2)^3 f_4^2 \rho^{12} (1 - \rho)^2}{f_1 f_2^{13/2} g_s} \left( 4P^2 (k')^2 f_2^{1/2} f_3 f_4 \rho^2 (1 - \rho)^2 + 4g_s f_2^{1/2} f_3^{1/2} (2 + P^2 k)^2 \rho
\]

\[
+ g_s f_4 \left( P^2 g_s f_2^{1/2} f_3^{1/2} \rho + 8f_2^{1/2} f_3 f_4 (2r - 1) (1 - \rho) + 8f_2 f_3 f_4 \right) \right),
\]

(74)

where primes denote derivatives with respect to $\rho$. There is also a first order constraint (consistent with (74)) coming from gauge fixing the radial coordinate in (26).
References

[1] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from $N = 4$ super Yang Mills,” JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [arXiv:hep-th/0204051].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large $N$ field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[5] E. Witten, “Quantum gravity in de Sitter space,” arXiv:hep-th/0106109.

[6] S. Hellerman, N. Kaloper and L. Susskind, “String theory and quintessence,” JHEP 0106, 003 (2001) [arXiv:hep-th/0104180].

[7] W. Fischler, A. Kashani-Poor, R. McNees and S. Paban, “The acceleration of the universe, a challenge for string theory,” JHEP 0107, 003 (2001) [arXiv:hep-th/0104181].

[8] A. Buchel and A. A. Tseytlin, “Curved space resolution of singularity of fractional D3-branes on conifold,” Phys. Rev. D 65, 085019 (2002) [arXiv:hep-th/0111017].

[9] A. Buchel, “Gauge / gravity correspondence in accelerating universe,” Phys. Rev. D 65, 125015 (2002) [arXiv:hep-th/0203041].

[10] A. Buchel, P. Langfelder and J. Walcher, “On time-dependent backgrounds in supergravity and string theory,” arXiv:hep-th/0207214.

[11] A. Armoni, J. L. Barbon and A. C. Petkou, “Orbiting strings in AdS black holes and $N = 4$ SYM at finite temperature,” JHEP 0206, 058 (2002) [arXiv:hep-th/0205280]; A. Armoni, J. L. Barbon and A. C. Petkou, “Rotating strings in confining AdS/CFT backgrounds,” JHEP 0210, 069 (2002) [arXiv:hep-th/0209224].
[12] I. R. Klebanov and A. A. Tseytlin, “Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories,” Nucl. Phys. B 578, 123 (2000) [arXiv:hep-th/0002159].

[13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[14] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[15] G. P. Korchemsky, “Asymptotics Of The Altarelli-Parisi-Lipatov Evolution Kernels Of Parton Distributions,” Mod. Phys. Lett. A 4, 1257 (1989); G. P. Korchemsky and G. Marchesini, “Structure function for large x and renormalization of Wilson loop,” Nucl. Phys. B 406, 225 (1993) [arXiv:hep-ph/9210281].

[16] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S**5,” JHEP 0206, 007 (2002) [arXiv:hep-th/0204226].

[17] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [arXiv:hep-th/9807080].

[18] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory,” Nucl. Phys. B 447, 95 (1995) [arXiv:hep-th/9503121].

[19] I. R. Klebanov and N. A. Nekrasov, “Gravity duals of fractional branes and logarithmic RG flow,” Nucl. Phys. B 574, 263 (2000) [arXiv:hep-th/9911096].

[20] C. P. Herzog, I. R. Klebanov and P. Ouyang, “D-branes on the conifold and N = 1 gauge / gravity dualities,” arXiv:hep-th/0205100.

[21] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].

[22] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) [arXiv:hep-th/9411149].

[23] A. Buchel, “Finite temperature resolution of the Klebanov-Tseytlin singularity,” Nucl. Phys. B 600, 219 (2001) [arXiv:hep-th/0011146].
[24] S. S. Gubser, C. P. Herzog, I. R. Klebanov and A. A. Tseytlin, “Restoration of chiral symmetry: A supergravity perspective,” JHEP 0105, 028 (2001) [arXiv:hep-th/0102172].

[25] M. Krasnitz, “Correlation functions in a cascading $N = 1$ gauge theory from supergravity,” arXiv:hep-th/0209163.

[26] A. A. Tseytlin, “Semiclassical quantization of superstrings: AdS(5) x S**5 and beyond,” arXiv:hep-th/0209116.