On “Upper error bounds for quadrature formulas on function classes” by K. K. Frolov

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Abstract

This is a tutorial paper that gives the complete proof of a result of Frolov [2] that shows the optimal order of convergence for numerical integration of functions with bounded mixed derivatives. The presentation follows Temlyakov [5], see also [7].

1 Introduction

We study cubature formulas for the approximation of the \(d\)-dimensional integral

\[
I(f) = \int_{[0,1]^d} f(x) \, dx
\]

for functions \(f\) with bounded mixed derivatives. For this, let \(D^\alpha f\), \(\alpha \in \mathbb{N}_0^d\), be the usual partial derivative of a function \(f\) and define the norm

\[
\|f\|_{s,\text{mix}}^2 := \sum_{\alpha \in \mathbb{N}_0^d: \|\alpha\|_\infty \leq s} \|D^\alpha f\|_{L_2}^2,
\]

where \(s \in \mathbb{N}\). In the following we will study the class (or in fact the unit ball)

\[
H_{d,s,\text{mix}} := \{ f \in C^{sd}([0,1]^d): \|f\|_{s,\text{mix}} \leq 1 \},
\]

i.e. the closure in \(C([0,1]^d)\) (with respect to \(\| \cdot \|_{s,\text{mix}}\)) of the set of \(sd\)-times continuously differentiable functions \(f\) with \(\|f\|_{s,\text{mix}} \leq 1\). Additionally, we will study the class

\[
\hat{H}_{d,s,\text{mix}} := \{ f \in H_{d,s,\text{mix}}: \text{supp}(f) \subset (0,1)^d \}.
\]
The algorithms under consideration are of the form

\[ Q_n(f) = \sum_{j=1}^{n} a_j f(x^j) \]  

(4)

for a given set of nodes \( \{x^j\}_{j=1}^{n} \), \( x^j = (x^j_1, \ldots, x^j_d) \in [0,1]^d \), and weights \( (a_j)_{j=1}^{n} \), \( a_j \in \mathbb{R} \), i.e. the algorithm \( A_n \) uses at most \( n \) function evaluations of the input function.

The worst case error of \( Q_n \) in the function class \( H \) is defined as

\[ e(Q_n, H) = \sup_{f \in H} |I(f) - Q_n(f)|. \]

We will prove the following theorem.

**Theorem 1.** Let \( s, d \in \mathbb{N} \). Then there exists a sequence of algorithms \( (Q_n)_{n \in \mathbb{N}} \) such that

\[ e(Q_n, \tilde{H}_d^{s,\text{mix}}) \leq C_{s,d} n^{-s} (\log n)^{d-1}, \]

where \( C_{s,d} \) may depend on \( s \) and \( d \).

Using standard techniques, see e.g. [6, Sec. 2.12] or [8, Thm. 1.1], one can deduce (constructively) from the algorithm that is used to prove Theorem 1 a quadrature rule for \( \tilde{H}_d^{s,\text{mix}} \) that has the same order of convergence. This results in the following corollary.

**Corollary 2.** Let \( s, d \in \mathbb{N} \). Then there exists a sequence of algorithms \( (Q_n)_{n \in \mathbb{N}} \) such that

\[ e(Q_n, H_d^{s,\text{mix}}) \leq \tilde{C}_{s,d} n^{-s} (\log n)^{d-1}, \]

where \( \tilde{C}_{s,d} \) may depend on \( s \) and \( d \).

The proof of Theorem 1 and hence also of Corollary 2 is constructive, i.e. we will show how to construct the nodes and weights of the used algorithms.

**Remark 3.** The upper bounds of Theorem 1 and Corollary 2 that will be proven in the next section for a specific algorithm, see (10), are best possible in the sense of the order of convergence. That is, there are matching lower bounds that hold for arbitrary quadrature rules that use only function values, see e.g. [8, Theorem 3.2].

**Remark 4.** There is a natural generalization of the spaces \( \tilde{H}_d^{s,\text{mix}} \), say \( \tilde{H}_d^{s,\text{mix},p} \), where the \( L_2 \)-norm in (11) is replaced by an \( L_p \)-norm, \( 1 < p < \infty \). The same lower bounds as mentioned in Remark 3 are valid also in this case.

Obviously, the upper bounds from Theorem 2 hold for these spaces if \( p \geq 2 \), since the spaces get smaller for larger \( p \). That the same algorithm satisfies the optimal order if \( 1 < p < 2 \) was proven by Skriganov [5, Theorem 2.1]. We refer to [8] and references therein for more details on this, the more delicate case \( p = 1 \), and the generalization to non-integer smoothness.
2 Proof of Theorem 1

2.1 The algorithm for $H_d^{s,\text{mix}}$

We start with the construction of the nodes of our quadrature rule. See Sloan and Joe \[6\] for a more comprehensive introduction to this topic. In the setting of Theorem 1 the set $X \subset [0,1)^d$ of nodes will be a subset of a lattice $X \subset \mathbb{R}^d$, i.e. $x, y \in X$ implies $x \pm y \in X$. In fact, we take $X = \mathbb{X} \cap [0,1)^d$.

The lattice $\mathbb{X}$ will be “$d$-dimensional”\footnote{It is well known that every lattice in $\mathbb{R}^d$ can be written as $T(\mathbb{Z}^m)$ for some $m \leq d$ and some $d \times m$-matrix $T$ with linearly independent columns. The number $m$ is called the dimension of the lattice.}, i.e. there exists a non-singular $d \times d$-matrix $T$ such that

$$X := T(\mathbb{Z}^d) = \{Tx : x \in \mathbb{Z}^d\}.$$  \hspace{1cm} (5)

The matrix $T$ is called the generator of the lattice $X$. Obviously, every multiple of $X$, i.e. $cX$ for some $c \in \mathbb{R}$, is again a lattice and note that while $X$ is a lattice, it is not necessarily an integration lattice, i.e. in general we do not have $X \supset \mathbb{Z}^d$.

The nodes for our quadrature rule for functions from $H_d^{s,\text{mix}}$ will be all points inside the cube $[0,1)^d$ of the shrinked lattice $a^{-1}X$, $a > 1$. That is, we will use the set of points

$$X_a^d := (a^{-1}X) \cap [0,1)^d, \quad a > 1.$$  \hspace{1cm} (6)

For the construction of the nodes it remains to present a specific generator matrix $T$ that is suitable for our purposes. For this, define the polynomials

$$P_d(t) := \prod_{j=1}^d (t - 2j + 1) - 1, \quad t \in \mathbb{R}.$$  \hspace{1cm} (7)

Obviously, the polynomial $P_d$ has only integer coefficients, and it is easy to check that it is irreducible\footnote{A polynomial $P$ is called irreducible over $\mathbb{Q}$ if $P = GH$ for two polynomials $G, H$ with rational coefficients implies that one of them has degree zero. In fact, every polynomial of the form $\prod_{j=1}^d (x - b_j) - 1$ with different $b_j \in \mathbb{Z}$ is irreducible, but has not necessarily $d$ different real roots.} (over $\mathbb{Q}$) and has $d$ different real roots. Let $\xi_1, \ldots, \xi_d \in \mathbb{R}$ be the roots of $P_d$. Using these roots we define the $d \times d$-matrix $B$ by

$$B = (B_{i,j})_{i,j=1}^d := \left(\xi_i^{j-1}\right)_{i,j=1}^d.$$  \hspace{1cm} (8)

This matrix is a Vandermonde matrix and hence invertible and we define the generator matrix of our lattice by

$$T = (B^\top)^{-1},$$  \hspace{1cm} (9)
where $B^\top$ is the transpose of $B$. It is well known that $X^* := B(Z^d)$ is the dual lattice associated with $X = T(Z^d)$, i.e. $y \in X^*$ if and only if $\langle x, y \rangle \in \mathbb{Z}$ for all $x \in X$.

We define the quadrature rule for functions $f$ from $\hat{H}^{s,\text{mix}}_d$ by

$$\hat{Q}_a(f) := a^{-d} \det(T) \sum_{x \in X^d_a} f(x), \quad a > 1.$$  \hfill (10)

In the next subsection we will prove that $\hat{Q}_a$ has the optimal order of convergence for $\hat{H}^{s,\text{mix}}_d$.

Note that $\hat{Q}_a(f)$ uses $|X^d_a|$ function values of $f$ and that the weights of this algorithm are equal, but do not (in general) sum up to one. While the number $|X^d_a|$ of points can be estimated in terms of the determinant of the corresponding generator matrix, it is in general not equal. In fact, if $a^{-1}X$ would be an integration lattice, then it is well known that $|X^d_a| = a^d \det(T^{-1})$, see e.g. [6]. For the general lattices that we consider, we know, however, that these numbers are of the same order, see Skriganov [5, Theorem 1.1].

**Lemma 5.** Let $X = T(Z^d) \subset \mathbb{R}^d$ be a lattice with generator $T$ of the form (9), and let $X^d_a$ be given by (6). Then there exists a constant $C_T$ that is independent of $a$ such that

$$\left| |X^d_a| - a^d \det(T^{-1}) \right| \leq C_T \ln^{d-1}(1 + a^d)$$

for all $a > 1$. In particular, we have

$$\lim_{a \to \infty} \frac{|X^d_a|}{a^d \det(T^{-1})} = 1.$$ 

**Remark 6.** It is still not clear if the corresponding QMC algorithm, i.e. the quadrature rule (10) with $a^{-d} \det(T)$ replaced by $|X^d_a|^{-1}$, has the same order of convergence. In fact, if true, this would imply the optimal order of the $L_2$-discrepancy of a modification of the set $X^d_a$, see [3]. We leave this as an open problem.

In the remaining subsection we prove the crucial property of these nodes. For this we need the following corollary of the Fundamental Theorem of Symmetric Polynomials, see, [4, Theorem 6.4.2].

**Lemma 7.** Let $P(x) = \prod_{j=1}^d (x - \xi_j)$ and $G(x_1, \ldots, x_d)$ be polynomials with integer coefficients. Additionally, assume that $G(x_1, \ldots, x_d)$ is symmetric in $x_1, \ldots, x_d$, i.e. invariant under permutations of $x_1, \ldots, x_d$. Then, $G(\xi_1, \ldots, \xi_d) \in \mathbb{Z}.$

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3Skriganov proved this result for admissible lattices. The required property will be proven in Lemma 8 see also [5, Lemma 3.1(2)].
We obtain that the elements of the dual lattice $B(\mathbb{Z}^d)$ satisfy the following

**Lemma 8.** Let $0 \neq z = (z_1, \ldots, z_d) \in B(\mathbb{Z}^d)$ with $B$ from (8). Then, $\prod_{j=1}^d z_i \in \mathbb{Z} \setminus 0$.

**Proof.** Fix $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ such that $Bm = z$. Hence,

$$z_i = \sum_{j=1}^d m_j \xi_i^{j-1}$$

depends only on $\xi_i$. This implies that $\prod_{j=1}^d z_i$ is a symmetric polynomial in $\xi_1, \ldots, \xi_d$ with integer coefficients. By Lemma 7 we have $\prod_{j=1}^d z_i \in \mathbb{Z}$.

It remains to prove $z_i \neq 0$ for $i = 1, \ldots, d$. Define the polynomial $R_1(x) := \sum_{j=1}^d m_j x_j^{j-1}$ and assume that $z_\ell = R_1(\xi_\ell) = 0$ for some $\ell = 1, \ldots, d$. Then there exist unique polynomials $G$ and $R_2$ with rational coefficients such that

$$P_d(x) = G(x)R_1(x) + R_2(x),$$

where $\text{deg}(R_2) < \text{deg}(R_1)$. By assumption, $R_2(\xi_\ell) = 0$. If $R_2 \equiv 0$ this is a contradiction to the irreducibility of $P_d$. If not, divide $P_d$ by $R_2$ (instead of $R_1$). Iterating this procedure, we will eventually find a polynomial $R^*$ with $\text{deg}(R^*) > 0$ (since it has a root) and rational coefficients that divides $P_d$: a contradiction to the irreducibility. This completes the proof of the lemma. \qed

We finish the subsection with a result on the maximal number of nodes in the dual lattice that lie in a axis-parallel box of fixed volume.

**Corollary 9.** Let $B$ be the matrix from (8) and $a > 0$. Then, for each axis-parallel box $\Omega \subset \mathbb{R}^d$ we have

$$\left|aB(\mathbb{Z}^d) \cap \Omega\right| \leq a^{-d} \text{vol}_d(\Omega) + 1.$$

**Proof.** Assume first that $\text{vol}_d(\Omega) < a^d$. If $\Omega$ contains 2 different points $z, z' \in aB(\mathbb{Z}^d)$, then, using that this implies $z'' = z - z' \in aB(\mathbb{Z}^d)$, we obtain

$$\text{vol}_d(\Omega) \geq \prod_{i=1}^d |z_i - z_i'| \geq \prod_{i=1}^d |z''_i| \geq a^d$$

from Lemma 8, a contradiction. For $\text{vol}_d(\Omega) \geq a^d$ we divide $\Omega$ along one coordinate in $\lfloor a^{-d} \text{vol}_d(\Omega) + 1 \rfloor$ equal pieces, i.e. pieces with volume less than $a^d$, and use the same argument as above. \qed
Remark 10. Although we focus in the following on the construction of nodes that is based on the polynomial $P_d$ from (7), the same construction works with any irreducible polynomial of degree $d$ with $d$ different real roots and leading coefficient 1. For example, if the dimension is a power of 2, i.e. $d = 2^k$ for some $k \in \mathbb{N}$, we can be even more specific. In this case we can choose the polynomial
$$P_d^*(x) = 2 \cos \left( d \cdot \arccos \left( \frac{x}{2} \right) \right),$$
cf. the Chebyshev polynomials. The roots of this polynomial are given by
$$\xi_i = 2 \cos \left( \frac{\pi (2i - 1)}{2d} \right), \quad i = 1, \ldots, d.$$
Hence, the construction of the lattice $X_d$ that is based on this polynomial is completely explicit. For a suitable polynomial if $2d + 1$ is prime, see [4]. We didn’t try to find a completely explicit construction in the intermediate cases.

2.2 The error bound

In this subsection we prove that the algorithm $\hat{Q}_a$ from (10) has the optimal order of convergence for functions from $H^{s,\text{mix}}_d$, i.e. that
$$e(\hat{Q}_a, \hat{H}^{s,\text{mix}}_d) \leq C_{s,d} n^{-s} (\log n)^{d-1},$$
where $n = n(a, T) := |X_d^a|$ is the number of nodes used by $\hat{Q}_a$ and $C_{s,d}$ is independent of $n$.

For this we need the following two lemmas. Recall that the Fourier transform of an integrable function $f \in L_1(\mathbb{R}^d)$ is given by
$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle y, x \rangle} \, dx, \quad y \in \mathbb{R}^d,$$
with $\langle y, x \rangle := \sum_{j=1}^d y_j x_j$. Furthermore, let
$$\nu_s(y) = \prod_{j=1}^d \left( \sum_{\ell=0}^s |2\pi y_j|^{2\ell} \right) = \sum_{\alpha \in \mathbb{N}_0^d : \|\alpha\|_\infty \leq s} \prod_{j=1}^d |2\pi y_j|^{2\alpha_j}, \quad y \in \mathbb{R}^d. \quad (11)$$
Clearly,
$$\nu_s(y) |\hat{f}(y)|^2 = \sum_{\alpha \in \mathbb{N}_0^d : \|\alpha\|_\infty \leq s} \left| \int_{\mathbb{R}^d} \prod_{j=1}^d (-2\pi i y_j)^{\alpha_j} f(x) e^{-2\pi i \langle y, x \rangle} \, dx \right|^2$$
$$= \sum_{\alpha \in \mathbb{N}_0^d : \|\alpha\|_\infty \leq s} \left| \overline{D^\alpha f(y)} \right|^2.$$
for all \( f \in H_d^{s,\text{mix}} \) with compact support and \( y \in \mathbb{R}^d \).

We begin with the following result on the sum of values of the Fourier transform.

**Lemma 11.** Let \( f \in H_d^{s,\text{mix}}(\mathbb{R}^d) \) with \( \text{supp}(f) \subset A \) for some compact \( A \subset \mathbb{R}^d \). Additionally, define \( M_A := \# \{ m \in \mathbb{Z}^d : A \cap (m + (0,1)^d) \neq \emptyset \} \). Then,

\[
\sum_{y \in \mathbb{Z}^d} \nu_s(y)|\hat{f}(y)|^2 \leq M_A \| f \|^2_{s,\text{mix}}.
\]

**Proof.** Let \( \Gamma := \{ \alpha \in \mathbb{N}_0^d : \| \alpha \|_\infty \leq s \} \). Define the function \( g(x) := \sum_{m \in \mathbb{Z}^d} f(m + x), \) \( x \in [0,1]^d \), and note that at most \( M_A \) of the summands are not zero. Obviously, \( g \) is 1-periodic. Hence, we obtain by Parseval’s identity and Jensen’s inequality that

\[
\sum_{y \in \mathbb{Z}^d} \nu_s(y)|\hat{f}(y)|^2 = \sum_{\alpha \in \Gamma} \sum_{y \in \mathbb{Z}^d} \left| \hat{D}_\alpha f(y) \right|^2 = \sum_{\alpha \in \Gamma} \sum_{y \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^d} D_\alpha f(x) e^{-2\pi i \langle y,x \rangle} \, dx \right|^2
\]

\[
= \sum_{\alpha \in \Gamma} \sum_{y \in \mathbb{Z}^d} \left| \int_{[0,1]^d} D_\alpha f(m + x) e^{-2\pi i \langle y,x \rangle} \, dx \right|^2
\]

\[
= \sum_{\alpha \in \Gamma} \sum_{y \in \mathbb{Z}^d} \left| \int_{[0,1]^d} D_\alpha g(x) e^{-2\pi i \langle y,x \rangle} \, dx \right|^2 = \sum_{\alpha \in \Gamma} \int_{[0,1]^d} |D_\alpha g(x)|^2 \, dx
\]

\[
= M_A^2 \sum_{\alpha \in \Gamma} \int_{[0,1]^d} \left| \frac{1}{M_A} \sum_{m \in \mathbb{Z}^d} D_\alpha f(m + x) \right|^2 \, dx
\]

\[
\leq M_A^2 \sum_{\alpha \in \Gamma} \int_{[0,1]^d} \frac{1}{M_A} \sum_{m \in \mathbb{Z}^d} |D_\alpha f(m + x)|^2 \, dx
\]

\[
= M_A \sum_{\alpha \in \Gamma} \int_{\mathbb{R}^d} |D_\alpha f(x)|^2 \, dx = M_A \| f \|^2_{s,\text{mix}}
\]

as claimed. \( \Box \)

Additionally, we need the following version of the **Poisson summation formula** for lattices.

**Lemma 12.** Let \( \mathbb{X} = T(\mathbb{Z}^d) \subset \mathbb{R}^d \) be a full-dimensional lattice and \( \mathbb{X}^* \subset \mathbb{R}^d \) be the associated dual lattice. Additionally, let \( f \in H_d^{s,\text{mix}}, \ s \geq 1 \). Then,

\[
\det(T) \sum_{x \in \mathbb{X} \cap [0,1]^d} f(x) = \sum_{y \in \mathbb{X}^*} \hat{f}(y).
\]

In particular, the right-hand-side is convergent.
Proof. To ease the notation we identify each \( f \in \tilde{H}_d^{s,\text{mix}} \) with the continuation to the whole space by zero, i.e. \( f(x) = 0 \) for \( x \notin [0,1]^d \). Let \( g(x) = f(Tx) \), \( x \in \mathbb{R}^d \). Then, by the definition of the lattice, we have

\[
\sum_{x \in X \cap [0,1]^d} f(x) = \sum_{x \in X} f(x) = \sum_{x \in \mathbb{Z}^d} f(Tx) = \sum_{x \in \mathbb{Z}^d} g(x).
\]

Additionally, note that \( B = (T^\top)^{-1} \) is the generator of \( X^* \) and hence

\[
\sum_{y \in X^*} \hat{f}(y) = \sum_{y \in \mathbb{Z}^d} \hat{f}(By) = \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle By,x \rangle} \, dx = \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle y,B^\top x \rangle} \, dx
\]
\[
= \det(T) \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(Tz) e^{-2\pi i \langle y,z \rangle} \, dz = \det(T) \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{R}^d} g(z) e^{-2\pi i \langle y,z \rangle} \, dz
\]
\[
= \det(T) \sum_{y \in \mathbb{Z}^d} \hat{g}(y),
\]

where we performed the substitution \( x = Tz \). (Here, we need that the lattice is full-dimensional.) In particular, the series on the left hand side converge if and only if the right hand side does. For the proof of this convergence note that \( f \in \tilde{H}_d^{s,\text{mix}} \), \( s \geq 1 \), implies \( \|g\|_{1,\text{mix}} \leq \|g\|_{s,\text{mix}} < \infty \). We obtain by Lemma 11 that

\[
\sum_{y \in \mathbb{Z}^d} \nu_1(y) |\hat{g}(y)|^2 \leq M_{T^{-1}([0,1]^d)} \|g\|_{1,\text{mix}}^2 < \infty
\]

with \( M_{T^{-1}([0,1]^d)} \) from Lemma 11 since \( \text{supp}(g) \subset T^{-1}([0,1]^d) \). Hence,

\[
\sum_{y \in \mathbb{Z}^d} |\hat{g}(y)| \leq \left( \sum_{y \neq 0} \nu_1(y) \right)^{1/2} \left( \sum_{y \neq 0} \nu_1(y) |\hat{g}(y)|^2 \right)^{1/2} < \infty,
\]

which proves the convergence. We finish the proof of Lemma 12 by

\[
\sum_{y \in \mathbb{Z}^d} \hat{g}(y) = \sum_{y \in \mathbb{Z}^d} \int_{\mathbb{R}^d} g(z) e^{-2\pi i \langle y,z \rangle} \, dz = \sum_{y \in \mathbb{Z}^d} \int_{[0,1]^d} \sum_{m \in \mathbb{Z}^d} g(m + z) e^{-2\pi i \langle y,z \rangle} \, dz = \sum_{m \in \mathbb{Z}^d} g(m).
\]

The last equality is simply the evaluation of the Fourier series of the function \( \sum_{m \in \mathbb{Z}^d} g(m + x) \), \( x \in [0,1]^d \), at the point \( x = 0 \). It follows from the absolute convergence of the left hand side that this Fourier series is pointwise convergent. \( \square \)
By Lemma 12 we can write the algorithm $\hat{Q}_a$, $a > 1$, as

$$\hat{Q}_a(f) = a^{-d} \det(T) \sum_{x \in X^d_a} f(x) = \sum_{z \in aB(\mathbb{Z}^d)} \hat{f}(z), \quad f \in \tilde{H}^{s,\text{mix}}_d,$$

where $aB$ (see (8)) is the generator of the dual lattice of $a^{-1}T(\mathbb{Z}^d)$ (see (9)) and $X^d_a = (a^{-1}X) \cap [0,1]^d$. Since $I(f) = \hat{f}(0)$ we obtain

$$|I(f) - \hat{Q}_a(f)| = \left| \sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} \hat{f}(z) \right| \leq \sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} |\nu_s(z)|^{-1/2} \left| \nu_s(z)^{1/2} \hat{f}(z) \right|$$

$$\leq \left( \sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} |\nu_s(z)|^{-1} \right)^{1/2} \left( \sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} |\nu_s(z)| |\hat{f}(z)|^2 \right)^{1/2}.$$ 

with $\nu_s$ from (11). We bound both sums separately. First, note that Lemma 11 implies that

$$\sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} |\nu_s(z)| |\hat{f}(z)|^2 = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \nu_s(aBm) |\hat{f}(aBm)|^2 \leq \sum_{\alpha: \|\alpha\|_{\infty} \leq s} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \left| \int_{\mathbb{R}^d} D^\alpha f(x) e^{-2\pi i \langle aBm, x \rangle} dx \right|^2$$

$$= (a^{-d} \det(T))^2 \sum_{\alpha: \|\alpha\|_{\infty} \leq s} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \left| \int_{\mathbb{R}^d} D^\alpha f(a^{-1}Ty) e^{-2\pi i \langle m, y \rangle} dy \right|^2$$

$$\leq (a^{-d} \det(T))^2 M_{aT^{-1}([0,1]^d)} \sum_{\alpha: \|\alpha\|_{\infty} \leq s} \int_{\mathbb{R}^d} |D^\alpha f(a^{-1}Ty)|^2 dy$$

$$= C(a, T) \|f\|_{s,\text{mix}}^2$$

with $C(a, T) := a^{-d} \det(T) M_{aT^{-1}([0,1]^d)}$. Using that $T^{-1}([0,1]^d)$ is Jordan measurable, we obtain $\lim_{a \to \infty} C(a, T) = 1$ and, hence, for $a > 1$ large enough,

$$\left( \sum_{z \in aB(\mathbb{Z}^d) \setminus \{0\}} \nu_s(z) |\hat{f}(z)|^2 \right)^{1/2} \leq 2 \|f\|_{s,\text{mix}}.$$

$$\text{(12)}$$

Now we treat the first sum. Recall from Lemma 8 that $\prod_{j=1}^d z_j \in \mathbb{Z} \setminus \{0\}$ for all $z \in B(\mathbb{Z}^d) \setminus \{0\}$ and define, for $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, the sets

$$\rho(m) := \{x \in \mathbb{R}^d: |2^{m_j-1}| \leq |x_j| < 2^{m_j} \text{ for } j = 1, \ldots, d\}.$$
Note that $\prod_{j=1}^d |x_j| < 2^{||m||_1}$ for all $x \in \rho(m)$. This shows that $|(aB(\mathbb{Z}^d) \setminus 0) \cap \rho(m)| = 0$ for all $m \in \mathbb{N}_0^d$ with $||m||_1 \leq |d \log_2(a)| =: r$. Hence, with $|\bar{z}| := \prod_{j=1}^d \max\{1, 2\pi |z_j|\}$, we obtain

$$
\sum_{z \in aB(\mathbb{Z}^d) \setminus 0} |\nu_s(z)|^{-1} \leq \sum_{z \in aB(\mathbb{Z}^d) \setminus 0} |\bar{z}|^{-2s} = \sum_{\ell=r+1}^{\infty} \sum_{m: ||m||_1 = \ell} \sum_{z \in (aB(\mathbb{Z}^d) \cap \rho(m))} |\bar{z}|^{-2s}.
$$

Note that for $z \in \rho(m)$ we have $|\bar{z}| \geq \prod_{j=1}^d \max\{1, 2\pi [2^{m_j-1}]\} \geq 2^{||m||_1}$. Since $\rho(m)$ is a union of $2^d$ axis-parallel boxes each with volume less than $2^{||m||_1}$, Corollary 9 implies that $|(aB(\mathbb{Z}^d) \setminus 0) \cap \rho(m)| \leq 2^d (a^{-d}||m||_1 + 1) \leq 2^{d+1} a^{-d} 2^{||m||_1} \leq 2^{d+2} 2^{||m||_1-r}$ for $m$ with $||m||_1 \geq r$. Additionally, note that $|\{m \in \mathbb{N}_0^d: ||m||_1 = \ell\}| = (d+\ell-1) < (\ell + 1)^{d-1}$. We obtain

$$
\sum_{z \in aB(\mathbb{Z}^d) \setminus 0} |\nu_s(z)|^{-1} \leq \sum_{\ell=r+1}^{\infty} \sum_{m: ||m||_1 = \ell} (|B(\mathbb{Z}^d) \setminus 0) \cap \rho(m)| 2^{-2s} ||m||_1
$$

$$
\leq 2^{d+2} \sum_{\ell=r+1}^{\infty} \sum_{m: ||m||_1 = \ell} 2^{||m||_1-r} 2^{-2s} ||m||_1
$$

$$
\leq 2^{d+2} \sum_{\ell=r+1}^{\infty} (\ell + 1)^{d-1} 2^{\ell-r} 2^{-2s \ell} = 2^{d+2} \sum_{t=1}^{\infty} (t + r + 1)^{d-1} 2^t 2^{-2s(t+r)}
$$

$$
\leq 2^{2d+2} 2^{-2sr} \sum_{t=1}^{\infty} (t + 1)^{d-1} 2^{(1-2s)t}
$$

$$
\leq 2^{2d+2s+2} a^{-2sd} \log_2(a^d)\sum_{t=1}^{\infty} (t + 1)^{d-1} 2^{(1-2s)t},
$$

where we’ve used that $d \log_2(a) - 1 \leq r \leq d \log_2(a)$. Note that the last sum equals $2^{2s-1} \text{Li}_{1-d}(2^{1-2s}) - 1$, where $\text{Li}$ is the polylogarithm (also known as Jonqui`ere’s function), i.e.

$$
\text{Li}_s(z) := \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^s}.
$$

So, all together

$$
e(\hat{Q}_a, H_d^{s, \text{mix}}) \leq c_{s,d} a^{-sd} \left( \log_2(a^d) \right)^{d-1} 2^{\frac{d-1}{2}} \tag{13}
$$

where $c_{s,d} = 2^{d+2s+1} \text{Li}_{1-d}(2^{1-2s})^{1/2}$.

From Lemma 5 we know that the number of nodes used by $\hat{Q}_a$ is proportional to $a^d$. This proves Theorem 1.
Remark 13. It is interesting to note that the proof of Theorem 1 is to a large extent independent of the domain of integration. For an arbitrary Jordan measurable set $\Omega \subset \mathbb{R}^d$ we can consider the algorithm $\tilde{Q}_a$ from (10) with the set of nodes $X^d_a$ replaced by $X^d_a(\Omega) = (a^{-1}T(\mathbb{Z}^d)) \cap \Omega$. The only difference in the estimates would be that $C(a, T)$, cf. (12), converges to $\text{vol}_d(\Omega)$ instead of 1. Thus, we have

$$e(\tilde{Q}_a, \tilde{H}^s_{d, \text{mix}}(\Omega)) \leq c_{s,d,\Omega} a^{-sd} \left(\log_2(a^d)\right)^{\frac{d-1}{2}}$$

for large enough $a > 1$ with $c_{s,d,\Omega} = c_{s,d,\text{vol}_d(\Omega)^{1/2}}$ as in (13). Since $a^d$ is in this case proportional to $|X^d_a(\Omega)|/(\det(T^{-1})\text{vol}_d(\Omega))$, there would be some additional volume dependent terms in $C_{s,d}$ from Theorem 1.

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