AN EFFICIENT ITERATION FOR THE EXTREMAL SOLUTIONS OF DISCRETE-TIME ALGEBRAIC RICCATI EQUATIONS

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Abstract. Algebraic Riccati equations (AREs) have been extensively applicable in linear optimal control problems and many efficient numerical methods were developed. The most attention of numerical solutions is the (almost) stabilizing solution in the past works. Nevertheless, it is an interesting and challenging issue in finding the extremal solutions of AREs which play a vital role in the applications. In this paper, based on the semigroup property, an accelerated fixed-point iteration (AFPI) is developed for solving the extremal solutions of the discrete-time algebraic Riccati equation. In addition, we prove that the convergence of the AFPI is at least R-suplinear with order \( r > 1 \) under some mild assumptions. Numerical examples are shown to illustrate the feasibility and efficiency of the proposed algorithm.

Key words. discrete-time algebraic Riccati equation, extremal solution, stabilizing solution, antistabilizing solution, accelerated fixed-point iteration, semigroup property

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1. Introduction. In this paper we are mainly concerned with the extremal solutions of the discrete-time algebraic Riccati equation (DARE)

\[
X = A^H X A - A^H X B (R + B^H X B)^{-1} B^H X A + C^H C,
\]

or its equivalent expression

\[
X = A^H X (I + GX)^{-1} A + H,
\]

where \( A \in \mathbb{C}^{n \times n} \), \( B \in \mathbb{C}^{n \times m} \), \( R \in \mathbb{C}^{m \times m} \) is positive definite, \( C \in \mathbb{C}^{l \times n} \) with \( m, l \leq n \), \( I \) is the identity matrix of compatible size, \( G := BR^{-1}B^H \) and \( H = C^H C \), respectively, and the \( n \)-square matrix \( X \) is the unknown Hermitian matrix that is to be determined. For the sake of simplicity, the matrix operator \( \mathcal{R} : \text{dom}(\mathcal{R}) \to \mathbb{H}_n \) is defined by \( \mathcal{R}(X) := A^H X (I + GX)^{-1} A + H \), which is used to rewrite the equation (1.1b) into the compact expression \( X = \mathcal{R}(X) \), where \( \mathbb{H}_n \) is the set of all \( n \times n \) Hermitian matrices and \( \text{dom}(\mathcal{R}) := \{ X \in \mathbb{H}_n \mid \det(R + B^H X B) \neq 0 \} \). The following sets

\[
\mathcal{R}_* := \{ X \in \text{dom}(\mathcal{R}) \mid X = \mathcal{R}(X) \}, \quad \mathcal{R}_\geq := \{ X \in \text{dom}(\mathcal{R}) \mid X \geq \mathcal{R}(X) \},
\]

will play an important role in our main results given below.

The nonlinear matrix equation of the form (1.1a) arises from the linear-quadratic (LQ) optimal control problem that minimizes the cost functional

\[
\mathcal{J}(u) := \sum_{k=0}^{\infty} (y_k^H y_k + u_k^H Ru_k)
\]

subject to the linear discrete-time system

\[
x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k, \quad k \geq 0,
\]

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where \( x_k \in \mathbb{C}^n \) is the state variable, \( u_k \in \mathbb{C}^m \) is the control variable and \( y_k \) is the output variable, respectively. Let \( F_X = (R + B^HXB)^{-1}B^HXA \) and \( T_X = A - BF_X \) for any Hermitian solution \( X \) of (1.1). Note that \( T_X \equiv (I + GX)^{-1}A \). If \( (A,B) \) is stabilizable and \( (A,C) \) is detectable, it is well-known that there exists an optimal state feedback control

\[
u_k^* := -F_Xx_k, \quad k \geq 0\]

such that the cost functional \( J(u_k^*) = x_0^Hx_kx_0 \) is minimized and the corresponding closed-loop system

\[
x_{k+1} = T_Xx_k, \quad k \geq 0\]

is asymptotically stable, i.e., all eigenvalues of the close-loop matrix \( T_X \) are inside the open unit disk \( \mathbb{D} \) in the complex plane, where \( X_s \geq 0 \) is the unique stabilizing solution of the DARE (1.1a), see, e.g., [1, 13]. Here, we denote \( M \geq 0 \) (resp. \( M > 0 \)) if \( M \in \mathbb{C}^{n \times n} \) is a Hermitian positive semidefinite (resp. positive definite) matrix. Analogously, we denote \( M \leq 0 \) (resp. \( M < 0 \)) if \( M \) is a Hermitian negative semidefinite (resp. negative definite) matrix. A matrix operator \( f : \mathbb{H}_n \to \mathbb{H}_n \) is order preserving (resp. reversing) on \( \mathbb{H}_n \) if \( f(A) \geq f(B) \) (resp. \( f(A) \leq f(B) \)) when \( A \geq B \) and \( A, B \in \mathbb{H}_n \). It is proved that the operator \( \mathcal{R}(\cdot) \) is order preserving. Throughout the paper the set of all \( n \times n \) Hermitian positive semidefinite (resp. negative semidefinite) matrices is denoted by \( \mathbb{H}_n \) (resp. \( -\mathbb{H}_n \)). For any \( M, N \in \mathbb{H}_n \), we write \( M \geq N \) (resp. \( M \leq N \)) if \( M - N \geq 0 \) (resp. \( M - N \leq 0 \)). Moreover, \( \sigma(M) \) and \( \rho(M) \) are the spectrum and the spectral radius of the square matrix \( M \), respectively, and define the quantities

\[
\mu(M) := \min\{ |\lambda| \mid \lambda \in \sigma(M) \}, \quad \rho_\mathbb{D}(M) := \max\{ |\lambda| \mid \lambda \in \sigma(M) \cap \mathbb{D} \},
\]

which will be used below. We also denote the closed unit disk by \( \mathbb{D} \), the boundary of \( \mathbb{D} \) by \( \partial \mathbb{D} \), the region outside the open unit disk by \( \mathbb{D}^c \), and the spectral or Euclidean norm for matrices and vectors in the context by \( \| \cdot \|_2 \), respectively.

If \( (A,B) \) is stabilizable and \( (A,C) \) is detectable, which are standard assumptions in the LQ optimal control problem mentioned previously, then \( \mathcal{R}_\mathbb{R}_+ \cap \mathbb{N}_n = \{ X_s \} \). In this case, there are dozens of numerical methods for solving the DAREs with small to medium sizes in the literature, see, e.g., [5, 7, 11, 14, 15, 20, 21] and the references therein. The standard way for computing the stabilizing solution is to utilize the Schur method [7, 15], which solves the stable deflating subspace \( V = \text{span}(X_1, X_2) \) of the associated symplectic pencil

\[
M - \lambda L := \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix} - \lambda \begin{bmatrix} I & G \\ 0 & A^H \end{bmatrix}
\]

using the ordered QZ algorithm, and thus \( X = X_2X_1^{-1} \geq 0 \) is the unique stabilizing solution of the DARE (1.1b).

On the other hand, regarding the DARE (1.1a) or (1.1b) as a nonlinear matrix equation, it is natural to apply Newton’s method for finding the unique stabilizing solution \( X_s \) under the stabilizability and detectability conditions, see, e.g., [14, 18]. Moreover, since \( H = C^H C \geq 0 \), it follows from Theorem 1.1 of [8] that \( X_s \) is also the maximal solution to the DARE (1.1b) satisfying \( X_s \geq S \) for all \( S \in \mathcal{R}_+ \), see
also [14, Theorem 13.1.1]. In addition, stating with some initial \( X_0 \in \mathbb{H}_n \), the author proposed the theoretical characterizations for the linearly convergent behavior of Newton’s method when solving the DARE (1.1b) with all unimodular eigenvalues of \( T_X \) being semisimple. Inspired by the doubling algorithm [2, 11], starting from the coefficient matrices \( A, G \) and \( H \), Chu et. al. proposed structure-preserving doubling algorithms (SDAs) [5] for solving the DARE (1.1b) and periodic DAREs, respectively. When the control weighting matrix \( R \in \mathbb{H}_n \) is singular, we developed a variant of the structured doubling algorithm for computing the stabilizing solution to the DARE (1.1a), please refer to [4].

It is well known that the set \( \mathcal{R}_{\geq} \cap \mathbb{N}_n \) associated with the DARE (1.1b), equipped with the partial ordering “\( \geq \)”, forms a complete lattice if \((A, B)\) is stabilizable and \((A, C)\) is not detectable, see, e.g., [12, 22]. Under the same assumptions the DARE (1.1b) might have the almost stabilizing solution \( X_s \geq 0 \) with \( \rho(T_{X_s}) \leq 1 \), which is the maximal element of the solution set \( \mathcal{R}_\approx \), and the optimizing solution \( Y \geq 0 \) with the cost functional \( J(u^*_0) = u^*_0^T Y x_0 \) being minimized, which is the minimal element of the solution set \( \mathcal{R}_{\geq} \cap \mathbb{N}_n \). On the other hand, it follows from [10, 23] that the DARE (1.1b) has a unique almost antistabilizing solution \( \hat{X} \in -\mathbb{N}_n \) with \( \mu(T_{\hat{X}}) \geq 1 \), which is the minimal element of the solution set \( \mathcal{R}_\approx \) satisfying \( \hat{X} \leq S \) for all \( S \in \mathcal{R}_\approx \), if \( A \) is nonsingular and rank \([A - \lambda I, B]\) = \( n \) for all \( \lambda \in \bar{D} \\setminus \{0\} \). Under the same assumptions, it is shown in Theorem 3.1 of [24] that \( \hat{Y} = -\bar{Y} \) is the maximal element of the solution set \( \mathcal{R}_\approx \cap -\mathbb{N}_n \) if and only if \( \bar{Y} \) is the minimal positive semidefinite solution to the dual DARE of first kind

\[
X = \hat{A}^H \hat{X} \hat{A} - (\hat{C}^H + \hat{A}^H X \hat{B})(\hat{R} + \hat{B}^H X \hat{B})^{-1}(\hat{C} + \hat{B}^H X \hat{A}) + \hat{H},
\]

where the coefficient matrices are given by

\[
\hat{A} = A^{-1}, \quad \hat{B} = A^{-1}B, \quad \hat{H} = (A^{-1})^H HA^{-1}, \quad \hat{C} = B^H \hat{H}, \quad \hat{R} = R + \hat{C}B,
\]

respectively. The Riccati equation (1.2) is also called the reverse discrete-time algebraic Riccati equation presented in [9], and a numerically stable algorithm was proposed in [19] for computing the minimal and antistabilizing solution \( \hat{X} \in -\mathbb{N}_n \) to the DARE (1.1a) with nonsingular \( A \).

Recently, it is shown in Theorem 5.1 of [24] that the existence and uniqueness of the minimal solution to the DARE (1.1a) can be constructed iteratively, through a Newton-type iteration for computing the maximal solution to the dual DARE (1.2), when rank \([A - \lambda I, B]\) = \( n \) for all \( \lambda \in \bar{D} \\setminus \{0\} \) and \( A \) is nonsingular. In this paper, applying the methodology of the fixed-point iteration (FPI), we will give constructive proofs for the existence (or uniqueness) theorems of these four extremal solutions to the DARE under some wild and reasonable assumptions. More restrictive assumptions have been addressed in [6] for the uniqueness of the maximal solution to the DARE (1.1b) under the framework of the FPI. Starting with some appropriate initial matrices, we will develop the accelerated fixed-point iteration (AFPI), based on the semigroup property of the DARE with an integer parameter \( r \geq 2 \) [17], for computing the external elements \((X, Y)\) of the set \( \mathcal{R}_{\geq} \cap \mathbb{N}_n \) simultaneously. Specifically, the AFPI with the initial \( X_0 = 0 \) is just the SDA presented in [5] when \( r = 2 \). In analogous way, a variant of the AFPI will be derived from the dual DARE (1.2) for solving the extremal elements \((\hat{X}, \hat{Y})\) of the set \( \mathcal{R}_\approx \cap -\mathbb{N}_n \) simultaneously, if \((A, B)\) is antistabilizable and \( A \) is nonsingular.

The paper is organized as follows. In Section 2 we present some preliminary lemmas that will be used in our main theorems. Under some wild assumptions, the
existence (or uniqueness) of the extremal solutions to the DARE will be constructed iteratively via the standard FPI in Section 3. Moreover, based on the framework of the semigroup property for the DARE (1.1b), two accelerated variants of the FPI will be proposed for computing the extremal elements of $R_+ \cap \mathbb{N}$ and $R_- \cap -\mathbb{N}_n$, respectively, and will discuss the rate of convergence for the AFPI presented in this section. In Section 4 some illustrate examples are presented for demonstrating the feasibility and efficiency of the proposed AFPI$(r)$ with different values of $r$. Finally, we conclude this paper in Section 5.

2. Preliminaries. In this section we introduce some definitions and auxiliary results that will be used below. Firstly, the concepts of R-linear and R-superlinear convergence of a sequence will be used in the following sections, see, e.g., Definition 1.1 of [17].

**Definition 2.1.** Let $\{X_k\}_{k=0}^\infty$ be a sequence of $n \times n$ complex matrices and $\| \cdot \|$ be an induced matrix norm.

(a) $\{X_k\}_{k=0}^\infty$ converges at least R-linearly to $X_* \in \mathbb{C}^{n\times n}$, if there exists a scalar $\sigma \in (0,1)$ such that

$$\lim sup_{k \to \infty} \sqrt[k]{\|X_k - X_*\|} \leq \sigma.$$

(b) $\{X_k\}_{k=0}^\infty$ converges at least R-superlinearly to $X_* \in \mathbb{C}^{n\times n}$ with order $r > 1$, if there exists a scalar $\sigma \in (0,1)$ such that

$$\lim sup_{k \to \infty} r^k \sqrt[k]{\|X_k - X_*\|} \leq \sigma.$$

Let the Stein operator $S_A : \mathbb{H}_n \to \mathbb{H}_n$ associated with a matrix $A \in \mathbb{C}^{n\times n}$ be defined by

$$S_A(X) := X - A^HXA,$$

for all $X \in \mathbb{H}_n$. In general, the operator $S_A$ is neither order-preserving nor order-reversing. However, under the assumption that $\rho(A) < 1$, the inverse operator $S_A^{-1}$ exists and it is order-preserving, since $S_A^{-1}(X) = \sum_{k=0}^\infty (A^k)^HXA^k \geq \sum_{k=0}^\infty (A^k)^HYA^k = S_A^{-1}(Y)$ for all $X \geq Y$. For the sake of clarity the results in this section can be separated into three major parts.

2.1. Some useful identities. The following lemma provides some identities with respect to the Stein matrix operator defined by (2.1), which will play an important role in our main results below.

**Lemma 2.2.** Let $X, \bar{X} \in \text{dom}(\mathcal{R})$. The following identities hold, which state the relationship between DARE and a specific Stein matrix equation:

(i) If $A_F := A - BF$ for any $F \in \mathbb{C}^{m\times n}$ and $H_F := H + F^HRF$, then

$$X - \mathcal{R}(X) = S_{A_F}(X) - H_F + K_F(X),$$

where $K_F(X) := (F - F_X)(R + B^HXB)(F - F_X)$.

(ii) If $K(\bar{X}, X) := K_{F_X}(X)$ and $H_{\bar{X}} := H + F_{\bar{X}}^HRF_{\bar{X}}$, then (2.2a) can be rewritten as

$$X - \mathcal{R}(X) = S_{T_{\bar{X}}}(X) - H_{\bar{X}} + K(\bar{X}, X).$$
Furthermore, we also have

\[(2.2c) \quad X - \mathcal{R}(X) = S_{T_{\tilde{X}}}(X) - H + [K(\tilde{X}, X) - K(\tilde{X}, 0)].\]

(iii) If \( U_{\tilde{X}} := \hat{X}T_{\tilde{X}}, \) then \( B^H U_{\tilde{X}} = RF_{\tilde{X}}, \) \( H_{\tilde{X}} = H + U^H_{\tilde{X}}GU_{\tilde{X}} \) and thus \( (2.2b) \) can be rewritten as

\[(2.2d) \quad X - \mathcal{R}(X) = S_{T_{\tilde{X}}}(X) - H_{\tilde{X}} + (U_{\tilde{X}} - U_X)H(G + GXG)(U_{\tilde{X}} - U_X).\]

**Proof.**

(i) Observe that

\[X - \mathcal{R}(X) = \Gamma_1(X) + \Gamma_2(X),\]

where \( \Gamma_1(X) := S_{A}(X) - H \) and \( \Gamma_2(X) := A^HXB(R + B^HXB)^{-1}B^HXA. \) A direct computation yields

\[
\begin{align*}
\Gamma_2(X) &= F^H_X(R + B^HXB)F_X = K_F(X) + [F^H(R + B^HXB)F_X \\
&\quad + F^H_X(R + B^HXB)F] - F^H(R + B^HXB)F \\
&= K_F(X) - F^HRF + [F^HB^HXA + A^HB^HBF - F^HBB^HBF] \\
&= K_F(X) - F^HRF + S_{A^H}(X) - S_{A}(X).
\end{align*}
\]

We conclude that

\[X - \mathcal{R}(X) = (S_A(X) - H) + (K_F(X) - F^HRF + S_{A^H}(X) - S_A(X)) = S_{A^H}(X) - H_F + K_F(X).\]

(ii) The first result is clearly true from \( (2.2a) \). For the proof of the remaining part, an easy computation shows that

\[
\begin{align*}
RF_{\tilde{X}} &= B^H\hat{X}(I + BR^{-1}B^H\hat{X})^{-1}A = B^H\hat{X}T_{\tilde{X}}, \\
F^H_{\tilde{X}}RF_{\tilde{X}} &= (RF_{\tilde{X}})^HR^{-1}(RF_{\tilde{X}}) = (\hat{X}T_{\tilde{X}})^HG(\hat{X}T_{\tilde{X}}).
\end{align*}
\]

We immediately obtain this formula due to \( K(\tilde{X}, 0) = F^H_{\tilde{X}}RF_{\tilde{X}}. \)

(iii) A trivial verification shows that

\[
\begin{align*}
B^H U_{\tilde{X}} &= B^H\hat{X}T_{\tilde{X}} = (R + B^H\hat{X}B)^{-1}B^H\hat{X}A = RF_{\tilde{X}}, \\
H_{\tilde{X}} &= H + F^H_{\tilde{X}}RF_{\tilde{X}} = H + U^H_{\tilde{X}}BRB^H U_{\tilde{X}} = H + U^H_{\tilde{X}}GU_{\tilde{X}}.
\end{align*}
\]

The result \( (2.2d) \) follows from \( (2.2b) \) immediately.

In addition, under different setting, Eq. \( (2.2c) \) can be reformulated as follows.

1. If \( X = \tilde{X} \in \mathbb{H}_n, \) then
   \[
   (2.3a) \quad S_{T_{\tilde{X}}}(\tilde{X}) = \tilde{X} - \mathcal{R}(\tilde{X}) + H + K(\tilde{X}, 0).
   \]

2. If \( X \in \mathbb{R}_+ \) and \( \tilde{X} \in \mathbb{H}_n, \) then
   \[
   (2.3b) \quad S_{T_{\tilde{X}}}(X) = H + [K(\tilde{X}, 0) - K(\tilde{X}, X)].
   \]

3. If \( X = \tilde{X} \in \mathbb{R}_+, \) then
   \[
   (2.3c) \quad S_{T_{\tilde{X}}}(X) = H + K(X, 0).
   \]
For any $X, \tilde{X} \in \mathbb{H}_n$, consider a subset of $\mathbb{H}_n$ defined by

$$\mathcal{S}_\pm := \{ Y \in \mathbb{H}_n \mid \mathcal{S}_{TX}(Y) = K(\tilde{X}, X) \}.$$  

Clearly, from (2.3a), the solution set $\mathcal{R}_\pm$ has the equivalent expression given by

$$\mathcal{R}_\pm = \{ Y \in \mathbb{H}_n \mid \mathcal{S}_{TY}(Y) = H + K(Y, 0) \}.$$  

The relationship between these two sets $\mathcal{R}_\pm$ and $\mathcal{S}_\pm$ is characterized in the following lemma, which will be used in the proof of Theorem 3.3 later on.

**Lemma 2.3.** For $\mathcal{S}_\pm$ and $\mathcal{R}_\pm$ defined by (2.4)–(2.5), the following statements hold:

(i) If $X, \tilde{X} \in \mathcal{R}_\pm$, then $\tilde{X} - X \in \mathcal{S}_\pm$.

(ii) If $X \in \mathcal{R}_\pm$ and $\tilde{X} - X \in \mathcal{S}_\pm$, then $\tilde{X} \in \mathcal{R}_\pm$.

(iii) If $\tilde{X} \in \mathcal{R}_\pm$ and $\tilde{X} - X \in \mathcal{S}_\pm$, then $X \in \mathcal{R}_\pm$.

**Proof.** With the aid of (2.3), the results are proven as follows.

(i) From (2.3b) we obtain

$$\mathcal{S}_{TX}(\tilde{X} - X) = \mathcal{S}_{TX}(\tilde{X}) - \mathcal{S}_{TX}(X) = H + K(\tilde{X}, 0) - [H + K(\tilde{X}, 0) - K(\tilde{X}, X)] = K(\tilde{X}, X).$$

That is, $\tilde{X} - X \in \mathcal{S}_\pm$.

(ii) From (2.3b) we obtain

$$\mathcal{S}_{TX}(\tilde{X}) = \mathcal{S}_{TX}(X) + K(\tilde{X}, X) = H + K(\tilde{X}, 0) - K(\tilde{X}, X) + K(\tilde{X}, X) = H + K(\tilde{X}, 0).$$

That is, $\tilde{X} \in \mathcal{R}_\pm$.

(iii) From (2.3c) and (2.2c) we obtain

$$\mathcal{S}_{TX}(X) = \mathcal{S}_{TX}(\tilde{X}) - K(\tilde{X}, X) = H + K(\tilde{X}, 0) - K(\tilde{X}, X) = \mathcal{S}_{TX}(X) - [X - \mathcal{R}(X)].$$

That is, $X = \mathcal{R}(X)$ or $X \in \mathcal{R}_\pm$. \hfill \Box

### 2.2. Some useful facts and properties.

For any $M \in \mathbb{C}^{n \times n}$, the generalized eigenspace of $M$ corresponding an eigenvalue $\lambda$ is defined by $E_\lambda(M) = \text{Ker}(M - \lambda I)^n$. The following lemma characterizes the inheritance of (almost) stability property under the setting of a Stein inequality.

**Lemma 2.4.** Let $B \in \mathbb{C}^{n \times n}$ and $Q \geq 0$. If $X_0$ is a positive semidefinite solution of the Stein inequality $\mathcal{S}_B(X) \geq Q$, and $\text{Ker}(Q) \subseteq \text{Ker}(B - A)$ for some $A \in \mathbb{C}^{n \times n}$, then $\rho(B) \leq \max\{1, \rho(A)\}$. Furthermore, we have

(i) $\rho(B) \leq 1$ if $\rho(A) \leq 1$.

(ii) $\rho(B) < 1$ if $\rho(A) < 1$ or $\text{Ker}(Q) \cap \sigma(B) = \{0\}$ for some $\lambda \in \sigma(B)$.

**Proof.** Let $\lambda \in \sigma(B)$ and a nonzero vector $x \in E_\lambda(B)$. Then $x^H \mathcal{S}_B(X_0)x = (1 - |\lambda|^2)x^H X_0 x \geq x^H Q x \geq 0$. If $x \not\in \text{Ker}(Q)$, then $x \not\in \text{Ker}(X_0)$ and thus $|\lambda| < 1$. Otherwise, $x \in \text{Ker}(Q)$ implies that $Ax = Bx + (A - B)x = \lambda x$ and thus $|\lambda| \leq \rho(A)$. We complete the proof. \hfill \Box
According to the following theorem, the positive semidefiniteness of the operator $K(\cdot, \cdot)$ defined by (2.2b) might depend on the positive definiteness of the matrix $R + B^H XB$ for any $X \in \mathcal{R}_\geq$.

**Theorem 2.5.** Let $X \in \mathcal{R}_\geq$. Then $R + B^H XB > 0$ and $K(\hat{X}, X) \geq 0$ for any $\hat{X} \in \mathbb{H}_n$.

**Proof.** We first notice that $X \in \mathcal{R}_\geq$ is equivalent to $X \geq H + T_X^H (X + XG)TX$.

Let $R = \hat{R}\hat{R}^H$ for some $\hat{R} > 0$, e.g., the Cholesky decomposition of $R$, and $\hat{B} := \hat{R}^{-H}B$. Then $G = \hat{B}\hat{B}^H$ and a direct computation yields

\[
\begin{bmatrix}
R & -B^HXT_X \\
-T_X^HXB^H & X - T_X^HXT_X
\end{bmatrix} \geq \begin{bmatrix}
R & -B^HXT_X \\
-T_X^HXB^H & T_X^HXT_X
\end{bmatrix} = \begin{bmatrix}
\hat{R} & -\hat{B}^HXT_X \\
-T_X^HXB & T_X^HXT_X
\end{bmatrix} \geq 0.
\]

Therefore, we see that

\[
(2.6) \quad \begin{bmatrix} R + B^HXB & 0 \\ 0 & X \end{bmatrix} \succeq \begin{bmatrix} B^HXB & B^HXT_X \\ T_X^HXB & T_X^HXT_X \end{bmatrix} = U^HUX,
\]

where $U := \begin{bmatrix} B & T_X \end{bmatrix} \in \mathbb{C}^{n \times (m+n)}$. Next, we shall deduce that $R + B^HXB > 0$. If not, there is a nonzero $x_1 \in \mathbb{C}^m$ such that $x_1^H(R + B^HXB)x_1 < 0$. Choose a scalar $\lambda \not\in \sigma(T_X)$ with $|\lambda| = 1$, and let $x_2 := (\lambda I - T_X)^{-1}Bx_1$ and $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively. It is easily seen that $Ux = \lambda x_2$ and hence, from (2.6), we immediately obtain

\[
x_1^H(R + B^HXB)x_1 + x_2^HXX_2 \geq |\lambda|^2(x_2^HXX_2) = x_2^HXX_2.
\]

Then $x_1^H(R + B^HXB)x_1 \geq 0$, contradicting the assumption of $x_1$. Thus, we conclude that $R + B^HXB \succeq 0$ is nonsingular or $R + B^HXB > 0$ for any $X \in \mathcal{R}_\geq$.

The remaining part immediately follows from the definition of the operator $K(\cdot, \cdot)$ in Lemma 2.2. 

**Remark 2.6.** From Theorem 2.5, we see that $I + GX = I + B\hat{R}^{-1}B^H X$ is nonsingular if $X \in \mathcal{R}_\geq$ (since $I + R^{-1}B^HXB$ is nonsingular).

Based on the following lemma, the uniqueness of the almost stabilizing solution to the DARE (1.1) will be established in Theorem 3.3 below.

**Lemma 2.7.** Let $\rho(A) \leq 1$ and $Q_1 \succeq 0$. If $X_0 \in \mathbb{H}_n$ is a solution of the matrix inequality $S_A(X) \succeq (XA)^HQ_1(XA)$, and the generalized eigenspaces of $A^H$ satisfy

\[
(2.7) \quad E_\lambda(A^H) \cap \text{Ker}(Q_1) = \{0\}
\]

for all $\lambda \in \sigma(A)$ with $|\lambda| = 1$, then $X_0 \succeq 0$.

**Proof.** For the sake of convenience, we factorize $Q_1$ as the Cholesky decomposition $Q_1 = Q_2Q_2^H$. Note that $\text{Ker}(Q_1) = \text{Ker}(Q_2^H)$. Let $\lambda \in \sigma(A)$ with $|\lambda| = 1$ and $Av_1 = \lambda v_1$ for some nonzero $v_1 \in \mathbb{C}^n$. Then we see that

\[
0 = v_1^H S_A(X_0)v_1 = (Q_2^H X_0^H v_1) (Q_2^H X_0^H v_1) = (v_1^H X_0 Q_2)(v_1^H X_0 Q_2)^H,
\]

and this implies that

\[
0 = v_1^H A^H X_0 Q_2 = \lambda v_1^H X_0 Q_2, \quad 0 = v_1^H S_A(X_0) = \lambda v_1^H X_0 (\lambda I - A).
\]
Thus, it follows from the assumption (2.7) that \( X_0v_1 = 0 \). Let \((A - \lambda I)v_2 = v_1\). Observe that the quadratic form \( v_2^H S_A(X_0)v_2 \) again satisfies
\[
v_2^H (X_0 - A^H X_0 A)v_2 = - (\lambda v_1^H X_0 v_2 + \bar{\lambda} v_2^H X_0 v_1 + v_2^H X_0 v_1) = 0,
\]
and we then have
\[
0 = v_2^H AX_0 Q_2 = \bar{\lambda} v_2^H X_0 Q_2, \quad 0 = v_2^H S_A(X_0) = \bar{\lambda} v_2^H X_0 (\lambda I - A).
\]
Therefore, \( X_0v_2 = 0 \) by using the assumption (2.7). Inductively, we can show that \( X_0v_j = 0 \) for a Jordan chain \( \{v_j\} \) corresponding to the unimodular eigenvalue \( \lambda \) of \( A \).

Let \( J_A = P^{-1}AP \) be the Jordan canonical form of \( A \). Suppose that \( J_A = J_1 \oplus J_s \), where \( J_1 \in \mathbb{C}^{m_1 \times m_1} \) with \( \sigma(J_1) \subseteq \partial \mathbb{D} \) and \( J_s \in \mathbb{C}^{m_2 \times m_2} \) with \( \sigma(J_s) \subseteq \mathbb{D} \). If we let \( Y_0 := S_A(X_0) \), then \( Y_0 \geq (X_0A)^H Q_1(X_0A) \geq 0 \) and one deduces that
\[
S_A(\tilde{X}_0) = \tilde{X}_0 - J_A^{-1} \tilde{X}_0 J_A = \tilde{Y}_0,
\]
where \( \tilde{X}_0 := P^H X_0 P \) and \( \tilde{Y}_0 := P^H Y_0 P \geq 0 \). From the above discussion we have \( \tilde{X}_0 = 0_{m_1} \oplus \tilde{X}_{0,s} \), \( \tilde{Y}_0 = 0_{m_1} \oplus \tilde{Y}_{0,s} \) and \( \tilde{X}_{0,s} \in \mathbb{H}^{m_2} \) satisfies the Stein equation \( S_{J_s}(\tilde{X}_{0,s}) = \tilde{Y}_{0,s} \geq 0 \). Therefore, \( X_0 = P^{-H}(0_{m_1} \oplus \tilde{X}_{0,s})P^{-1} \geq 0 \), since the matrix \( \tilde{X}_{0,s} = S_{J_s}^{-1}(\tilde{Y}_{0,s}) \) is positive semidefinite if \( \rho(J_s) < 1 \).

In Lemma 3.5, the dual DARE of second kind will possess a positive definite stabilizing solution if the sufficient conditions of the following lemma are fulfilled, in which \( \mathbb{D}^c = \{ z \in \mathbb{C} \mid |z| \geq 1 \} \) is the complement of the open unit disk \( \mathbb{D} \).

**Lemma 2.8.** Assume that \( X_0 \) is a positive semidefinite solution of the Stein inequality \( S_A(X) \geq BB^H \) with \( B \in \mathbb{C}^{n \times m} \). Then

(i) \( \rho(A) < 1 \) if \( \text{Ker}(A - \lambda I) \cap \text{Ker}(B^H) = \{ 0 \} \) for \( \lambda \in \sigma(A) \cap \mathbb{D}^c \) or, equivalently,
\[
\text{rank} \begin{bmatrix} B^H \\ A - \lambda I \end{bmatrix} = n \text{ for } \lambda \in \sigma(A) \cap \mathbb{D}^c.
\]

(ii) \( X_0 > 0 \) if \( \text{Ker}(A - \lambda I) \cap \text{Ker}(B^H) = \{ 0 \} \) for \( \lambda \in \sigma(A) \) or, equivalently,
\[
\text{rank} \begin{bmatrix} B^H \\ A - \lambda I \end{bmatrix} = n \text{ for } \lambda \in \sigma(A).
\]

**Proof.** Under slightly different conditions, the results are derived as follows.

(i) Let \( u \in \text{Ker}(A - \lambda I) \) with \( u \neq 0 \) for \( \lambda \in \sigma(A) \cap \mathbb{D}^c \). Then
\[
0 \geq (1 - |\lambda|^2) u^H X_0 u \geq (B^H u)^H (B^H u) \geq 0.
\]

From the assumption we conclude that \( u \in \text{Ker}(A - \lambda I) \cap \text{Ker}(B^H) = \{ 0 \} \), which leads to a contradiction with \( u \neq 0 \). Thus, \(|\lambda| < 1\) for all \( \lambda \in \sigma(A) \) or, equivalently, \( \rho(A) < 1 \).

(ii) Let \( \lambda \in \sigma(A) \). For any \( v_1 \in \text{Ker}(A - \lambda I) \cap \text{Ker}(X_0) \), we see that
\[
0 = (1 - |\lambda|^2) v_1^H X_0 v_1 \geq (B^H v_1)^H (B^H v_1) \geq 0.
\]

Then \( v_1 \in \text{Ker}(A - \lambda I) \cap \text{Ker}(B^H) = \{ 0 \} \) and thus \( v_1 = 0 \). This implies that \( \text{Ker}(A - \lambda I) \cap \text{Ker}(X_0) = \{ 0 \} \) for \( 1 \leq i \leq k \) and \( k \in \mathbb{N} \). For any \( v_{k+1} \in \text{Ker}(A - \lambda I)^k \cap \text{Ker}(X_0) \), let \( v_k := (A - \lambda I)v_{k+1} \) or \( Av_{k+1} = v_k + \lambda v_{k+1} \). Thus, we further obtain
\[
0 \geq - v_k^H X_0 v_k = v_{k+1}^H X_0 v_{k+1} - (v_k^H + \bar{\lambda} v_k^H) X_0 (v_k + \lambda v_{k+1})
\]
\[
= v_{k+1}^H S_A(X_0) v_{k+1} \geq (B^H v_{k+1})^H (B^H v_{k+1}) \geq 0.
\]
This leads to $X_0v_k = 0$ and hence $v_k \in \text{Ker}(A - \lambda I)^k \cap \text{Ker}(X_0) = \{0\}$ or $v_k = 0$. Then $v_{k+1} \in \text{Ker}(A - \lambda I) \cap \text{Ker}(B^H) = \{0\}$ or $v_{k+1} = 0$. That is, $\text{Ker}(A - \lambda I)^{k+1} \cap \text{Ker}(X_0) = \{0\}$. It follows from mathematical induction that $\text{Ker}(A - \lambda I)^\ell \cap \text{Ker}(X_0) = \{0\}$ for each $\ell \in \mathbb{N}$. Therefore, $\text{Ker}(X_0) = \text{Ker}(X_0) \cap \mathbb{C}^n = \text{Ker}(X_0) \cap \left( \bigoplus_{\lambda \in \sigma(A)} E_\lambda(A) \right) = \{0\}$ or, equivalently, $X_0 \geq 0$ is nonsingular.

2.3. The formulation of the dual DARE. Let $A$ be nonsingular through this subsection. The positive semidefinite solutions of dual DARE play a central role in the set of negative semidefinite solutions of the original DARE. For the DARE (1.1b) of the compact form, we shall provide the construction of two kinds of dual DARE (2.8) and (2.14), in terms of the coefficient matrices $A$, $G$ and $H$, in this subsection.

2.3.1. The first kind of dual DARE. For the sake of simplicity the matrix products $A^{-H}XA^{-1}$ is denoted by $X^{(A)}$ for any $X \in \mathbb{C}^{n \times n}$. Note that $I + GH^{(A)}$ is nonsingular since $G$, $H^{(A)} \geq 0$. Then Eq. (1.1b) is equivalent to

\[(I + GX)^{-1} = I - G(X - H^{(A)}) = I + GH^{(A)} - GX^{(A)} = \left( I + GH^{(A)} \right) A \left( I - A^{-1} (I + GH^{(A)})^{-1} GA^{-H} X \right) A^{-1} = \tilde{A}^{-1} (I - \tilde{G}X) \tilde{A}^{-1}. \]

We immediately obtain $I - \tilde{G}X$ is nonsingular and

\[(2.8) \quad [(I + GX)^{-1} A] \times [(I + \tilde{G}Y)^{-1} \tilde{A}] = I, \]

where $Y = -X$, $\tilde{A} = A^{-1} (I + H^{(A)})^{-1}$ and $\tilde{G} = ((I + GH^{(A)})^{-1} G)^{H^H} = \tilde{A}GA^H$. Eq. (2.8) provides the formulation of the first kind of dual DARE

\[(2.9) \quad Y = D_1(Y) := \tilde{H} + \tilde{A}^H Y (I + \tilde{G}Y)^{-1} \tilde{A}, \]

where

\[
\tilde{H} = -X + \tilde{A}^H X (I - \tilde{G}X)^{-1} \tilde{A} = -X + \tilde{A}^H X A^{-1} (I + GX)
\]
\[
= -X + \tilde{A}^H HA^{-1} (I + GX) + (I + H^{(A)} G)^{-1} X
\]
\[
= \tilde{A}^H HA^{-1} + \left[ - I + \tilde{A}^H HA^{-1} G + (I + H^{(A)} G)^{-1} \right] X
\]
\[
= \tilde{A}^H HA^{-1} + \left[ - I + (I + H^{(A)} G)^{-1} H^{(A)} G + (I + H^{(A)} G)^{-1} \right] X
\]
\[
= \tilde{A}^H HA^{-1}.
\]

Notice that if we write $G = BR^{-1}B^H$, then it is easily seen that

\[(2.10a) \quad \tilde{A} = A^{-1} - \tilde{B} \tilde{R}^{-1} B^H H^{(A)}, \]
\[(2.10b) \quad \tilde{G} = \tilde{B} \tilde{R}^{-1} \tilde{B}^H \geq 0, \]
\[(2.10c) \quad \tilde{H} = H^{(A)} - B^H H^{(A)} \tilde{R}^{-1} H^{(A)} B, \]

where $\tilde{B} = A^{-1} B$ and $\tilde{R} = R + B^H H^{(A)} B > 0$. (2.10) are the same as the coefficient matrices defined in [9, Theorem 3.1]. More precisely, we have

\[
\tilde{A} = \tilde{A} - \tilde{B} \tilde{R}^{-1} \tilde{C}, \quad \tilde{H} = \tilde{H} - \tilde{C}^H \tilde{R}^{-1} \tilde{C}.
\]
Finally, it is straightforward to verify

\[(2.11) \quad (\mathcal{R}(X) - X)^{(A)} = (\mathcal{D}_1(Y) - Y)(I + GH^{(A)} ),\]

i.e., \(Y = -X\) is a solution of dual DARE \((2.9)\) if and only if \(X\) is a solution of DARE \((1.1b)\). Notice that from \((2.8)\) we have

\[\sigma((I + \hat{G}Y)^{-1}\hat{A}) = \sigma(T_X^{-1}).\]

### 2.3.2. The second kind of dual DARE

Let \(X\) be a Hermitian solution of the DARE \((1.1b)\), then \(X - H\) is nonsingular and

\[(2.12) \quad [X(I + GX)^{-1}A] \times [(X - H)^{-1}A^H] = I.\]

Furthermore, if \(X\) is nonsingular, it follows from \((2.12)\) that \(Y := -X^{-1}\) satisfies

\[(2.13) \quad [X(I + GX)^{-1}A] \times [Y(I + HY)^{-1}A^H] = -I\]

or, equivalently, \([Y(I + HY)^{-1}A^H] \times [X(I + GX)^{-1}A] = -I\). It thus implies that \(Y\) is a Hermitian solution to the second kind of dual DARE

\[(2.14) \quad Y = D_2(Y) := AY(I + HY)^{-1}A^H + G.\]

In addition, it is easily seen that the Hermitian matrices \(X\) and \(Y\) also fulfill

\[(2.15a) \quad Y - D_2(Y) = A[(X - H)^{-1} - (\mathcal{R}(X) - H)^{-1}]A^H,\]

\[(2.15b) \quad X - \mathcal{R}(X) = A^H[(Y - G)^{-1} - (D_2(Y) - G)^{-1}]A,\]

i.e., \(Y = -X^{-1}\) is a nonsingular solution of dual DARE \((2.14)\) if and only if \(X = -Y^{-1}\) is a nonsingular solution of DARE \((1.1b)\). Notice that from \((2.13)\) we have

\[(2.16) \quad \sigma((I + HY)^{-1}A^H)) = \sigma(XT_X^{-1}X^{-1}) = \sigma(T_X^{-1}).\]

Analogously, let \(\text{dom}(\mathcal{D}_1) := \{X \in \mathbb{H}_n \mid \det(I + \hat{G}X) \neq 0\}\), \(\text{dom}(\mathcal{D}_2) := \{X \in \mathbb{H}_n \mid \det(I + HX) \neq 0\}\) and \(\mathcal{D}_i^{(2)} := \{X \in \text{dom}(\mathcal{D}_i) \mid X \geq \mathcal{D}_i(X)\}\), with \(i = 1, 2\) for the sake of explanation.

### 3. Extremal solutions of the DARE

In this section, the existence of extremal solutions to the DARE \((1.1)\) will be established iteratively through the FPI given by

\[(3.1) \quad X_{k+1} = \mathcal{R}(X_k), \quad k \geq 0,\]

with a suitable initial matrix \(X_0 \in \mathbb{N}_n\).

According to the formulations \((2.11)\) and \((2.15)\), when \(A\) is nonsingular, it is shown that \(X \in \mathbb{H}_n\) is a solution of the DARE \((1.1)\) if and only if \(-X \in \mathbb{H}_n\) is a solution of its dual DARE of first kind \((2.11)\), and \(X \in \mathbb{H}_n\) is a nonsingular solution of the DARE \((1.1)\) if and only if \(-X^{-1} \in \mathbb{H}_n\) is a solution of its dual DARE \((2.14)\) of second kind. With this reason, we thus consider two FPIs defined by

\[(3.2a) \quad Y_{k+1} = \mathcal{D}_1(Y_k), \quad k \geq 0,\]

\[(3.2b) \quad Z_{k+1} = \mathcal{D}_2(Z_k), \quad k \geq 0,\]
with $Y_0, Z_0 \in \mathbb{N}_n$ being the initial guesses.

For the sake of explanation, $X_{+, M}, X_{+, m}, X_{-, M}$ and $X_{-, m}$ denote the maximal positive semidefinite solution, minimal positive semidefinite solution, maximal negative semidefinite solution and minimal negative semidefinite solution of DARE, respectively, if they exist. For the sake of clarity, positive semidefinite and negative semidefinite extremal solutions of the DARE (1.1) will be discussed separately in the following subsections.

### 3.1. Positive semidefinite extremal solutions

The following theorem, quoted from [3, Theorem 3.2], guarantees the existence of the minimal solution $X_{+, m} \in \mathbb{N}_n$ to the DARE (1.1b), i.e., $X_{+, m} \leq S$ for all $S \in R_\geq \cap \mathbb{N}_n$, under reasonable assumptions.

**Theorem 3.1.** [3] If $R_\geq \cap \mathbb{N}_n \neq \emptyset$, then the FPI (3.1) generates a nonincreasing sequence of positive semidefinite matrices $\{X_k\}_{k=0}^\infty$ with $0 \leq X_0 \leq H$, which converges at least R-linearly to the minimal element $X_{+, m}$ of the solution set $R_\geq \cap \mathbb{N}_n$ with the rate of convergence

$$\limsup_{k \to \infty} \sqrt[k]{\|X_k - X_{+, m}\|} \leq \rho_D(T_{X_{+, m}})^2 < 1.$$

As a special case of Theorem 1.1 in [8], if $(A, B)$ is stabilizable and $H \geq 0$, then the DARE (1.1a) has a unique almost stabilizing $X_* \in \mathbb{H}_n$, which is the maximal element of the solution set $R_\geq$ satisfying $R + B^H X_* B > 0$. In the following theorem the existence and uniqueness of the almost stabilizing solution to the DARE (1.1) can be established under the framework of the fixed-point iteration.

**Theorem 3.2.** If there exists $X_* \in \mathbb{H}_n$ satisfying $\rho(T_{X_*}) < 1$, then the following statements hold:

1. $S_\geq = \{X \in \mathbb{H}_n | S_{T_{X_*}}(X) \geq H_{X_*}\}$ is a nonempty subset of $R_\geq \cap \mathbb{N}_n$, where $H_{X_*} = H + F_{X_*}^H R F_{X_*}$.
2. The FPI (3.1) generates a nonincreasing sequence of positive semidefinite matrices $\{X_k\}_{k=0}^\infty$ with $X_0 = S_{T_{X_*}}^{-1}(H_{X_*})$, which converges at least R-linearly to the maximal element $X_{+, M}$ of $R_\geq \cap \mathbb{N}_n$ with the rate of convergence

$$\limsup_{k \to \infty} \sqrt[k]{\|X_k - X_{+, M}\|} \leq \rho(T_{+, M})^2,$$

provided that $\rho(T_{+, M}) < 1$.

1. For each $k \geq 0$, $\rho(T_{X_k}) < 1$ and thus $\rho(T_{+, M}) \leq 1$.

**Proof.** Notice that the operator $S_{T_{X_*}}$ is invertible because $\rho(T_{X_*}) < 1$.

1. Since $H_{X_*} \geq 0$, it follows from Lemma 2.2 that $X_0 = S_{T_{X_*}}^{-1}(H_{X_*}) \in S_\geq \cap \mathbb{N}_n$, and we further deduce that

$$X - R(X) = S_{T_{X_*}}(X) - H_{X_*} + K(X_*, X) \geq 0$$

and $X \geq 0$ for all $X \in S_\geq$, since $K(X_*, X) \geq 0$ in (2.2b) for $X \geq 0$.

1. Because $X_0 = S_{T_{X_*}}^{-1}(H_{X_*}) \geq 0$ is an element of the set $R_\geq$ and $H \geq 0$, we see that $X_0 \geq R(X_0) = X_1 \geq 0$ and hence it is easily seen that $X_k \geq X_{k+1} \geq 0$ for $k \geq 1$ by induction, since the operator $R(\cdot)$ is order preserving. Thus, $\{X_k\}_{k=0}^\infty$ generated by the FPI (3.1) is a nonincreasing sequence of positive semidefinite matrices, which converges to a limit $X_{+, M} \in \mathbb{N}_n$ eventually. On the other hand, for any $X_+ \in R_\geq$, it follows from Theorem 2.5 and (2.2b) that $K(X_*, X_+) \geq 0$ and $H_{X_*} = S_{T_{X_*}}(X_+) + K(X_*, X_+) \geq S_{T_{X_*}}(X_+)$. Since
the inverse of $S_{T_X}$ is order preserving. We see that $X_0 = S_{T_X}^{-1}(H_{X_0}) \geq X_+$. Suppose that $X_i \geq X_+$ for $i = 0, 1, 2, \ldots, k$. Then $X_{k+1} - X_k = R(X_k) - R(X_+) \geq 0$ and we thus deduce that $X_k \geq X_+$ for all $k$. This yields $X_{\infty} := \lim_{k \to \infty} X_k \geq X_+$ for all $X_+ \in R_+$, i.e., the limit of $\{X_k\}_{k=0}^\infty$ must be the maximal element of the solution set $R_+$ or $X_{+, M} = X_{\infty}$. Moreover, the proof of the rate of convergence for $\{X_k\}_{k=0}^\infty$ follows from the Appendix of [16].

(iii) For each $k \geq 0$, from (2.2b) we see that

\begin{align}
(3.3a) \\
X_k - X_{k+1} &= S_{T_{X_k}}(X_k) - H_{X_k}, \\
(3.3b) \\
S_{T_{X_{k-1}}}(X_k) - H_{X_{k-1}} &\geq X_k - T_{H_{X_{k-1}}}^\dagger X_{k-1} - H_{X_{k-1}} = 0
\end{align}

if we let $X_- := X_+ \in R_n$. According to the Eq. (3.3a), one deduces that $X_{k+1} = T_{H_{X_k}}^\dagger X_k T_{X_k} + H_{X_k}$ for $k \geq 0$, and it follows from (3.3b) that

$S_{T_{X_{k-1}}}(X_k) - H_{X_{k-1}} \geq X_k - T_{H_{X_{k-1}}}^\dagger X_{k-1} - H_{X_{k-1}} = 0$

for all $k \geq 1$. Therefore, this yields the following Stein inequality

$S_{T_{X_k}}(X_k) \geq H_{X_k} + K(X_{k-1}, X_k) \geq 0$

for each $k \geq 0$, where the equality holds for $k = 0$. Furthermore, we also have

$\text{Ker}(K(X_{k-1}, X_k)) = \text{Ker}(B^H(U_{X_{k-1}} - U_{X_k})) = \text{Ker}(F_{X_{k-1}} - F_{X_k}) \subseteq \text{Ker}(BF_{X_{k-1}} - BF_{X_k}) = \text{Ker}(T_{X_{k-1}} - T_{X_k})$.

Applying Lemma 2.4, we conclude that $\rho(T_{X_k}) < 1$, if $\rho(T_{X_{k-1}}) < 1$, for each $k \geq 0$. \hfill \Box

Now, in the following theorem, we will give some equivalent conditions for the stabilizability of the pair $(A, B)$ from a different point of view.

**Theorem 3.3.** The following statements are equivalent:

(i) The pair $(A, B)$ is stabilizable.

(ii) There exists a matrix $X_+ \in R_n$ satisfying $\rho(T_{X_+}) < 1$.

(iii) The DARE (1.1) has a unique almost stabilizing solution $X \in R_n$.

(iv) The DARE (1.1) has a maximal and almost stabilizing solution $X \in R_n$.

**Proof.** The equivalence of these statements is given as follows.

(i)⇒(ii): If $(A, B)$ is stabilizable, then $A_F = A - BF$ is d-stable, i.e., $\rho(A_F) < 1$, for some $F \in C^{m \times n}$. Recall that $H_F = H + F^HRF \geq 0$, then $X_+ = S_{T_F}^{-1}(H_F) \geq 0$ and we further have

$S_{T_{X_+}}(X_+) = (F - F_{X_+})^H(I + B^H X_+ B)(F - F_{X_+}) + H + (F - F_{X_+})^H F_{X_+} \geq 0$

Since $\text{Ker}(F - F_{X_+}) \subseteq \text{Ker}(A_F - T_{X_+})$, it follows from Lemma 2.4 and $\rho(A_F) < 1$ that the closed-loop matrix associated with $X_+ \in R_n$ is also d-stable, i.e., $\rho(T_{X_+}) < 1$.

(i)⇔(ii): Suppose that the statement (ii) holds. If we let $F = (R + B^H X_+ B)^{-1}B^H X_+ A$, then $\rho(A - BF) = \rho(T_{X_+}) < 1$ and hence $(A, B)$ is stabilizable.
(i)⇒(iii): Suppose that \((A, B)\) is stabilizable. Firstly, it follows from Theorem 3.2 that the DARE (1.1) has an almost stabilizing solution \(X \geq 0\), which is the maximal element of the solution set \(\mathcal{R}_{\infty}\), if either the statement (i) or (ii) holds. For the uniqueness of the solution \(X\), we assume that the DARE (1.1) has another solution \(\bar{X} \in \mathbb{H}_n\) with \(\rho(T_{\bar{X}}) \leq 1\). It is clear that \(\bar{X} \leq X\) because of the maximality of \(X\). On the other hand, from Lemma 2.7 and let \(\Delta := \bar{X} - X\), we have \(S_{\bar{T}}(\Delta) = K(\bar{X}, X)\). From \(F_{\bar{X}} - F_X = (R + B^HXB)^{-1}B^H\Delta T_X\) we deduce that
\[
S_{\bar{T}}(\Delta) = K(\bar{X}, X) = T_{\bar{X}}^H\Delta Q_1\Delta T_{\bar{X}},
\]
with \(Q_1 := B(R + B^HXB)^{-1}B^H \geq 0\). Since \((A, B)\) is stabilizable and \(T_{\bar{X}} = A - BF_{\bar{X}}\), it follows that \(\text{rank}[T_{\bar{X}} - \lambda I B] = n\) for all \(\lambda \in \mathbb{D}^c\), and hence \(\Delta \geq 0\) or \(\bar{X} \geq X\) follows from Lemma 2.7. Therefore, the DARE (1.1) must have a unique almost stabilizing solution \(X = \bar{X}\).

(i)⇐(iii): Suppose that the statement (iii) holds and that \((A, B)\) is not stabilizable. Then \(\text{rank}[A - \lambda I, B] < n\) for some \(\lambda \in \mathbb{C}\) with \(|\lambda| \geq 1\). If \(|\lambda| > 1\), then \(\text{rank}(A - BF_X - \lambda I) < n\) or \(\det(T_X - \lambda I) = 0\), and we thus have \(\lambda \in \sigma(T_X)\), which leads to a contradiction with \(\rho(T_X) \leq 1\). On the other hand, if \(\text{rank}[A - \lambda I, B] < n\) for some \(\lambda \in \mathbb{C}\) with \(|\lambda| = 1\), there is a nonzero \(y \in \mathbb{C}^n\) such that \(y^H[A - \lambda I B] = 0\). Hence this implies that \(\Delta := yy^H\) satisfies the following equation
\[
S_{\bar{T}}(\Delta) = T_{\bar{X}}^H\Delta B(R + B^HXB)^{-1}B^H\Delta T_{\bar{X}} = K(\Delta, 0).
\]
Therefore, it follows from Lemma 2.3 that \(\bar{X} := X + \Delta \in \mathbb{H}_n\) is also a solution of the DARE (1.1) with \(\sigma(T_{\bar{X}}) = \sigma(T_X) \subseteq \mathbb{D}\), which leads to a contradiction with the uniqueness of the almost stabilizing solution \(X\) to the DARE (1.1). Consequently, the pair \((A, B)\) must be stabilizable.

(i)⇒(iv): This follows from Theorem 3.2 directly, since the statements (i) and (ii) are equivalent.

(i)⇐(iv): Assume that the DARE (1.1) has an almost stabilizing and maximal solution \(X \in \mathbb{H}_n\). If \((A, B)\) is not stabilizable, applying the similar arguments described previously, there exist a nonzero \(y \in \mathbb{C}\) and a matrix \(\Delta = yy^H \geq 0\) such that \(X + \Delta\) is also an almost stabilizing solution of the DARE (1.1a). Because of the maximality of \(X\), we obtain \(X + \Delta \leq X\) and hence \(\Delta = 0\), which leads to a contradiction with \(y \neq 0\). Thus \((A, B)\) is stabilizable.

3.2. Negative semidefinite extremal solutions. In this section, when \(A \in \mathbb{R}^{n \times n}\) is assumed to be nonsingular, the existence of the negative semidefinite extremal solutions to the DARE (1.1) will be addressed under the framework of two FPIs (3.2).

3.2.1. The first approach. Recall that the matrices \(\hat{A}\) and \(\hat{B}\) are defined by (2.10). For a nonzero \(\lambda\), the block row matrix \(\begin{bmatrix} \hat{A} - \lambda I & \hat{B} \end{bmatrix}\) can be decomposed into
\[
\begin{bmatrix} \hat{A} - \lambda I & \hat{B} \end{bmatrix} = A^{-1} \begin{bmatrix} A - \frac{1}{\lambda} I & B \end{bmatrix} \begin{bmatrix} -\lambda I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ -\hat{R}^{-1}B^H(H^{(A)} \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix}.
\]
Thus, we see that the pair \((\hat{A}, \hat{B})\) is stabilizable if and only if \(\text{rank}[A - \lambda I B] = n\) for \(\lambda \in \mathbb{D}\{0\}\). Suppose that \(\text{rank}[A - \lambda I B] = n\) for all \(\lambda \in \mathbb{D}\{0\}\). Due to the
stabilizability of \((\hat{A}, \hat{B})\) there exists a \(\hat{F} \in \mathbb{C}^{m \times n}\) such that the matrix \(\hat{A}_p : \hat{A} - \hat{B} \hat{F}\) satisfies \(\sigma(\hat{A}_p) \subseteq \mathbb{D}\).

The following results can be proven by applying the similar arguments described in the previous sections. Therefore, we state two theorems without proof as follows.

**Theorem 3.4.** Assume that \(\hat{H} \geq 0\). Then the following statements hold:

(i) If \(D^{(1)}_{\geq} \cap N_n \neq \emptyset\), then FPI (3.2a) generates a nondecreasing sequence of positive semidefinite matrices \(\{Y_k\}_{k=0}^\infty\) with \(0 \leq Y_0 \leq \hat{H}\), which converges at least \(R\)-linearly to \(-X_{-M}\) with the rate of convergence

\[
\lim_{k \to \infty} \frac{\|Y_k + X_{-M}\|}{\rho_D(T_{X_{-M}}^{-1})^2} < 1.
\]

(ii) If \(\text{rank}[A - \lambda I \ B] = n\) for all \(\lambda \in \hat{\mathbb{D}} \setminus \{0\}\), then the FPI (3.2) generates a nonincreasing sequence of positive semidefinite matrices \(\{Y_k\}_{k=0}^\infty\) with \(Y_0 = S_{\hat{A}_p}(\hat{H}) \geq 0\), which converges to \(-X_{-m}\) with the rate of convergence

\[
\lim_{k \to \infty} \frac{\|Y_k + X_{-m}\|}{\rho(T_{X_{-m}}^{-1})^2} = \mu(T_{X_{-m}})^2 \leq 1,
\]

where \(\hat{H} = \hat{H} + \hat{F}^H R \hat{F}\).

**3.2.2. The second approach.** For solving the dual DARE (2.14) of second kind, we will apply FPI (3.2b) given by

\[
Z_{k+1} = D_2(Z_k) = AZ_k(I + HZ_k)^{-1}A^H + G, \quad k \geq 0,
\]

where \(Z_0 \geq 0\) is some initial matrix, \(G = BR^{-1}B^H\) and \(H \geq 0\), respectively. The following result guarantees that the limit of the sequence \(\{Z_k\}_{k=0}^\infty\), if it exists, will be a unique (almost) stabilizing solution of the dual DARE (2.14) under wild assumptions.

**Lemma 3.5.** Assume that \(D^{(2)}_{\geq} \cap N_n \neq \emptyset\). Let \(\{Z_k\}_{k=0}^\infty\) be a sequence generated by the FPI (3.2b). Then the following statements hold:

(i) If \(0 \leq Z_0 \leq G\), then \(\{Z_k\}_{k=0}^\infty\) is a nondecreasing sequence of positive semidefinite matrices, which converges to the minimal element \(Z_{\infty}\) of the set \(D^{(2)}_{\geq} \cap N_n\).

(ii) Assume that \((A, B)\) is controllable, where \(A \in \mathbb{C}^{n \times n}\) and \(B \in \mathbb{C}^{n \times m}\). Then \(Z_k > 0\) for \(k \geq n\) if \(Z_0 = 0\), and \(Z_k > 0\) for \(k > n - 1\) if \(Z_0 = G\). Moreover, the dual DARE (2.14) has a unique almost stabilizing solution \(Z_{\infty} > 0\).

(iii) If \(0 \leq Z_0 \leq G\) and \((A(I + Z_{\infty}H)^{-1}, B)\) is controllable, where \(Z_{\infty}\) is the minimal element of \(D^{(2)}_{\geq} \cap N_n\), then \(Z_{\infty}\) is a positive definite stabilizing solution to the dual DARE (2.14).

**Proof.**

(i) This statement follows from Theorem 3.1 directly.

(ii) Suppose that the pair \((A, B)\) is controllable. It is well known that the matrix \(B_n := [B, AB, \ldots, A^{n-1}B]\) has full rank. Now, if \(Z_0 = 0\), then \(Z_1 = BB^H := B_1R_1^{-1}B_1^H\), where \(B_1 = B\) and \(R_1 = I_m\) is the \(m \times m\) identity matrix. Thus we further see that

\[
Z_2 = D_2(Z_1) = AB_1R_1^{-1}B_1^H(I + HB_1R_1^{-1}B_1^H)^{-1}A^H + BB^H
= BB^H + AB_1(R_1 + B_1^H H B_1)^{-1}B_1^H A^H
= [B_1 \ AB_1] \begin{bmatrix} R_1 & 0 \\ 0 & R_1 + B_1^H H B_1 \end{bmatrix}^{-1} \begin{bmatrix} B_1^H \\ B_1^H A^H \end{bmatrix} := B_2 R_2^{-1} B_2.
\]
with \( B_2 := [B_1\ AB_1] = [B\ AB] \) and \( R_2 := R_1 \oplus (R_1 + B^H_kHB_1) > 0 \).

Inductively, we obtain \( Z_{k+1} = B_{k+1}R_{k+1}^{-1}R_{k+1}^H \), where \( B_{k+1} := [B_1\ AB_k] = [B\ AB\ A^2B\ \cdots\ A^kB] \) and \( R_{k+1} := R_1 \oplus (R_k + B^H_kHB_k) > 0 \) for \( k \geq 1 \). In particular, when \( k = n - 1 \), we see that \( Z_n = B_nR_n^{-1}B_n^H \) with rank\( (B_n) = n \).

Let \( B_n^H = Q \begin{bmatrix} L^H \\ 0 \end{bmatrix} \) be the QR-decomposition with \( Q \in \mathbb{C}^{mn \times mn} \) and \( L \in \mathbb{C}^{n \times n} \) being unitary and nonsingular lower-triangular matrices, respectively.

This implies that \( Z_n = [L\ 0](Q^H\ R_n^{-1}Q) \begin{bmatrix} L^H \\ 0 \end{bmatrix} > 0 \). Furthermore, continuing this process, it is still valid that \( Z_k > 0 \) for each \( k \geq n \). For the case that \( Z_0 = G \), the result can be shown similarly. Since \( \{Z_k\}_{k=0}^{\infty} \) is a nondecreasing sequence, we see that \( Z_k \geq Z_0 > 0 \) for some positive integer \( k_0 \) and thus \( Z_\infty = \lim_{k \to \infty} Z_k \geq Z_0 > 0 \).

Let \( W_{Z_\infty} := (I + HZ_\infty)^{-1}A^H \) be the closed-loop matrix associated with \( Z_\infty \) and we shall prove that \( \rho(W_{Z_\infty}) \leq 1 \). If not, there is a \( \lambda \in \mathbb{C} \) and a nonzero \( y \in \mathbb{C}^n \) such that \( |\lambda| = \rho(W_{Z_\infty}) > 1 \) and \( W_{Z_\infty}y = \lambda y \). Note that the matrix \( Z_\infty > 0 \) also satisfies

\[
Sw_{Z_\infty}(Z_\infty) = W_{Z_\infty}^H(Z_\infty + Z_\infty HZ_\infty)W_{Z_\infty} + BR_1^{-1}B^H \geq BB^H.
\]

Thus this implies \( 0 > (1 - |\lambda|^2)(y^HZ_\infty y) \geq y^H(BR_1^{-1}B^H)y \), which leads to a contradiction with the positive semidefiniteness of \( G = BR_1^{-1}B^H \). Consequently, \( Z_\infty > 0 \) is the unique almost stabilizing solution of the dual DARE (2.14).

(iii) Let \( W_{Z_\infty} \) be the closed-loop matrix defined previously. Since \( (W_{Z_\infty}^H, B) \) is controllable by the assumption, it follows that rank \( \begin{bmatrix} W_{Z_\infty} - \lambda I \\ B^H \end{bmatrix} = n \) for all \( \lambda \in \sigma(W_{Z_\infty}) \). Thus, from Lemma 2.8 and (3.4), we obtain \( \rho(W_{Z_\infty}) < 1 \) and \( Z_\infty > 0 \) immediately.

Based on (i) and (ii) of Lemma 3.5 and (2.16), the minimal negative semidefinite solution of the DARE (1.1) can be obtained from the limit of \( \{Z_k\}_{k=0}^{\infty} \), which is summarized in the following theorem.

**Theorem 3.6.** Assume that \( D_2^{(2)} \cap \mathbb{N}_n \neq \emptyset \). If \((A, B)\) is controllable, then FPI (3.2b) generates a nondecreasing sequence of positive semidefinite matrices \( \{Z_k\}_{k=0}^{\infty} \) with \( Z_0 = 0 \), which converges at least \( R \)-linearly to a positive definite matrix \( Z_\infty \) with the rate of convergence

\[
\limsup_{k \to \infty} \sqrt[k]{\|Z_k - Z_\infty\|} \leq \rho_0(T_{Z_\infty}^{-1})^2 < 1.
\]

Furthermore, the minimal negative semidefinite solution of DARE (1.1) can be obtained by \( X_{-,m} = -Z_\infty^{-1} \).

Finally, the relationship between the almost stabilizing solution and positive semidefinite extremal solutions of DARE (1.1b) is summarized in the following proposition without proof.

**Proposition 3.7.** If \( X_s \) is the unique almost stabilizing solution of the DARE (1.1b), then

(i) \( X_{+,M} \) exists and \( X_s = X_{+,M} \geq 0 \).

(ii) \( X_{+,M} \) exists. Moreover, \( X_{+,M} = X_{+,m} \) if \( \rho(T_{X_{+,m}}) \leq 1 \), and \( X_{+,m} \neq X_{+,M} \) if \( \rho(T_{X_{+,m}}) > 1 \).
4. Acceleration of fixed-point iteration. In previous sections we have addressed the convergence of fixed-point iteration \( X_{k+1} = R(X_k) \), \( k \geq 0 \), for the existence of the extremal solutions to DARE (1.1b), in which \( X_0 \in \mathbb{H}_n \) is some appropriate initial guess. Since the FPI is usually linearly convergent, a numerical method of higher order of convergence is always required in the practical computation and many real-life applications. In this section, for any positive integer \( r > 1 \), we will revisit an accelerated FPI (AFPI) of the form

\[
\hat{X}_{k+1} = R^{(r^k-r^k)}(\hat{X}_k), \quad k \geq 1, \\
\hat{X}_1 = R^{(r)}(\hat{X}_0), \quad k = 1
\]

with \( \hat{X}_0 = X_0 \), for solving the numerical solutions of DARE (1.1b). Here \( R^{(\ell)}(\cdot) \) denotes the composition of the operator \( R(\cdot) \) itself for \( \ell \) times, where \( \ell \geq 1 \) is a positive integer. Theoretically, the iteration of the form (4.1) is equivalent to the formula

\[
\hat{X}_k = R^{(r^k)}(\hat{X}_0), \quad k \geq 1,
\]

with \( \hat{X}_0 = X_0 \), and we see that

\[
\hat{X}_k = X_{r^k}
\]

for each \( k \geq 1 \). Therefore, it is an interesting and challenging issue to make sure that the explicit expression of \( \hat{X}_k \) is of the form (4.3) for each \( k \geq 1 \). In the following sections we will show that the AFPI algorithm is more efficient than the iteration (4.2), or even some Newton-type methods [8, 24], for solving the extremal solutions of the DARE (1.1b).

4.1. Equivalent formulation of the fixed-point iteration. The following definition modifies the semigroup property of the iteration associated a binary operator.

**Definition 4.1.** [17] Let \( \mathbb{K}_n \subseteq \mathbb{C}^{p \times q} \) and \( F : \mathbb{K}_n \times \mathbb{K}_n \rightarrow \mathbb{K}_n \) be a binary matrix operator, where \( p \) and \( q \) are positive integers. We call that an iteration

\[
X_{k+1} = F(X_k, X_0), \quad k \geq 0,
\]

has the semigroup property if the operator \( F \) satisfies the following associative rule:

\[
F(F(Y,Z),W) = F(Y,F(Z,W))
\]

for any \( Y, Z \) and \( W \) in \( \mathbb{K}_n \).

It is interesting to point out that the sequence \( \{X_k\} \) satisfies the so-called discrete flow property [17], that is,

\[
X_{i+j+1} = F(X_i, X_j)
\]

for any nonnegative integers \( i \) and \( j \). Here the subscript of (4.4) is an equivalent adjustment to the original formula [17, Theorem 3.2].

For the FPI defined by \( X_{k+1} = R(X_k) \), in which \( X_0 \in \mathbb{H}_n \) is an initial matrix and the operator \( R(\cdot) \) is defined by (1.1b), it is shown in [16] that the fixed-point iteration can be rewritten as the following formulation

\[
X_{k+1} = R^{(k)}(R(X_0)) = R^{(k+1)}(X_0) = H_k + A_k^H X_0 (I + G_k X_0)^{-1} A_k,
\]
where the sequence of matrices \( \{ (A_k, G_k, H_k) \}_{k=0}^{\infty} \) is generated by the following iteration

\[
X_{k+1} = F(X_k, X_0) := \begin{bmatrix}
A_0 \Delta G_k H_0 A_k \\
G_0 + A_0 \Delta G_k H_0 G_k A_0^H \\
H_k + A_k^H H_0 \Delta G_k H_0 A_k
\end{bmatrix},
\]

with \( X_k := [A_k^H \ G_k \ H_k]^H \) and \( X_0 := [A^H \ G \ H]^H \) for each \( k \geq 0 \), provided that the matrices \( \Delta G_i, H_i := (I + G_i H_i)^{-1} \) exists for all \( i, j \geq 0 \). Notice that \( F : \mathbb{K}_n \times \mathbb{K}_n \rightarrow \mathbb{K}_n \) is a binary operator with \( \Delta G_0, H_0 = 0 \) when \( F \) is a binary operator with \( \Delta G_0, H_0 = G \). Moreover, it has been shown in Theorem 4.2 of [17] that the iteration (4.6) has the semigroup property and thus satisfies (4.4). Based on the equivalent expression of the FPI (4.5), we shall develop an accelerated FPI for computing the extremal solutions to the DARE (1.1b).

**Remark 4.2.** From the FPI (3.2b) and (4.6), it is worth mentioning that \( Z_k = G_k \) for each \( k \geq 0 \) when \( Z_0 = G_0 = G \). Analogously, it follows from the discrete flow property (4.4) and iteration (4.6) that \( X_k = H_k \) for each \( k \geq 0 \) if \( X_0 = H_0 = H \).

### 4.2. The accelerated fixed-point iteration

In this section, we will mainly aim at some efficient ways for generating the sequence of matrices \( \{ (A_k, G_k, H_k) \}_{k=0}^{\infty} \) presented in the FPI (4.5), where it may converge rapidly. For the sake of simplicity, the operator \( F : \mathbb{K}_n \rightarrow \mathbb{K}_n \) is defined recursively by

\[
F_{\ell+1}(X) = F(X, F_\ell(X)), \quad \ell \geq 1,
\]

with \( F_1(X) = X \) for all \( X \in \mathbb{K}_n \) and \( F(\cdot, \cdot) \) being defined by (4.6). Furthermore, if we let \( X_k := [A_k^H \ G_k \ H_k]^H \in \mathbb{K}_n \) for \( k \geq 0 \) and \( X_0 := X_0 \) be defined as in (4.6), then the sequence \( \{ A_k, G_k, H_k \} \) generated by the iteration

\[
X_{k+1} = F(X_k, X_k) = F_2(X_k), \quad k \geq 0,
\]

which is equivalent to the doubling or structured doubling algorithms [2, 5]. Indeed, according the semigroup and discrete flow property (4.4), we have \( A_k = A_{2k-1} \), \( G_k = G_{2k-1} \) and \( H_k = H_{2k-1} \) for each \( k \geq 0 \). That is, under the iteration (4.8), the sequence of matrices \( \{ (A_k, G_k, H_k) \}_{k=0}^{\infty} \) proceeds rapidly with their subscripts being the exponential numbers of base number \( r = 2 \). From the theoretical point of view, starting with a suitable matrix \( X_0 \in \mathbb{K}_n \), the sequence \( \{ X_k \}_{k=0}^{\infty} \) generated by (4.5) might also converge rapidly to its limit, if the limit exists.

Analogously, when the base number is \( r = 3 \), it is suggested in [17] to consider an accelerated iteration of the form

\[
X_{k+1} = F(X_k, F(X_k, X_k)) = F_3(X_k), \quad k \geq 0,
\]

with \( X_0 = X_0 \), and we have \( \{ A_k, G_k, H_k \} = \{ A_{3^k-1}, G_{3^k-1}, H_{3^k-1} \} \) for each \( k \geq 0 \).

Recently, for any positive integer \( r > 1 \), Lin and Chiang proposed an efficient iterative method for generating the sequence \( \{ (A_k, G_k, H_k) \}_{k=0}^{\infty} \) with order \( r \) of R-convergence, provided that the operator \( F(\cdot, \cdot) \) in (4.6) is well-defined, see, e.g., Algorithm 3.1 in [17]. Theoretically, this algorithm utilizes the following accelerated iteration

\[
X_{k+1} = F_r(X_k), \quad k \geq 0,
\]
Algorithm 4.1 The Accelerated Fixed-Point Iteration with \( r \) (AFPI(\( r \))) for solving
\[
X = H_0 + A_0^H X (I_n + G_0 X)^{-1} A_0.
\]

Require: \( A_0 = A_0 \in \mathbb{C}^{n \times n}, \ G_0 = B_0 R_0^{-1} B_0^H \geq 0, \ H_0 = H_0 \geq 0, \ (A_0, B_0) \) is stabilizable, initial matrix \( \hat{X}_0 \in \mathbb{H}_n \) and \( r > 1 \).

Ensure: the maximal positive semidefinite solution \( \hat{X}_{k_1} \) and the minimal positive semidefinite solution \( H_{k_2} \).

for \( k = 0, 1, 2, \ldots \) do
  \[
  A_k^{(1)} = A_k, \ G_k^{(1)} = G_k, \ H_k^{(1)} = H_k;
  \]
  for \( l = 1, 2, \ldots, r - 2 \) do
    \[
    A_k^{(l+1)} = A_k^{(l)} (I + G_k H_k^{(l)})^{-1} A_k;
    \]
    \[
    G_k^{(l+1)} = G_k^{(l)} (I + G_k H_k^{(l)})^{-1} G_k (A_k^{(l)})^H;
    \]
    \[
    H_k^{(l+1)} = H_k + A_k^H H_k^{(l)} (I + G_k H_k^{(l)})^{-1} A_k;
    \]
  end for
  \[
  A_{k+1} = A_k^{(r-1)} (I + G_k H_k^{(r-1)})^{-1} A_k;
  \]
  \[
  G_{k+1} = G_k^{(r-1)} (I + G_k H_k^{(r-1)})^{-1} G_k (A_k^{(r-1)})^H;
  \]
  \[
  H_{k+1} = H_k + A_k^H H_k^{(r-1)} (I + G_k H_k^{(r-1)})^{-1} A_k;
  \]
  \[
  \hat{X}_{k+1} = A_{k+1}^H \hat{X}_0 (I + G_{k+1} \hat{X}_0)^{-1} A_{k+1} + H_{k+1};
  \]
  if \( \hat{X}_{k_1} \) and \( H_{k_2} \) satisfy the stoping criterion for some positive \( k_1, k_2 \) then
    return the positive semidefinite matrices \( \hat{X}_{k_1} \) and \( H_{k_2} \);
  end if
end for

with \( X_0 := [A^H \ G \ H]^H \) and \( F_r(\cdot) \) being the operator defined by \( (4.7) \), for constructing \( A_k = A_{k-1} \), \( G_k = G_{k-1} \) and \( H_k = H_{k-1} \), respectively. Therefore, combining \( (4.5) \) and \( (4.9) \), we obtain the pseudocode of AFPI summarized in Algorithm 4.1.

Due to the property presented in \( (4.3) \), it seems that the superlinear convergence of the AFPI algorithm would be expected in practical computation. In our numerical experiments, we adopted \( A_0 = A, \ B_0 = B, \ R_0 = R \) and \( H_0 = H \) in Algorithm 4.1 for computing the positive semidefinite extremal solutions of the DARE \( (1.1) \). On the other hand, the positive semidefinite extremal solutions of the dual DARE \( (2.9) \) were computed if we adopted \( A_0 = \hat{A}, \ B_0 = \hat{B}, \ R_0 = \hat{R} \) and \( H_0 = \hat{H} \) defined by \( (2.10) \), respectively, as the initial data in Algorithm 4.1.

4.3. Convergence analysis of the AFPI. As for Algorithm 4.1, worth mentioning is that it can be reduced to SDA iteration when \( r = 2 \) \([5]\). The following theorem guarantees, under the same sufficient conditions as in Theorem 3.2 and the equivalent conditions presented in Theorem 3.3, the R-superlinear convergence of \( \{\hat{X}_k\}_{k=0}^{\infty} \) and \( \{H_k\}_{k=0}^{\infty} \) generated by Algorithm 4.1.

Theorem 4.3. Let the sequences \( \{H_k\}_{k=0}^{\infty} \) and \( \{\hat{X}_k\}_{k=0}^{\infty} \) be generated by Algorithm 4.1 with \( A_0 = A, \ G_0 = G \) and \( H_0 = H \), respectively. If the hypotheses of Theorem 3.2 are satisfied, then

(i) \( \{H_k\}_{k=0}^{\infty} \) converges at least \( R \)-superlinearly to \( X_{+,m} \) with the rate of conver-
Remark 4.2

Theorem 4.3

Theorem 3.4

under the stabilizability assumption. As mentioned in

1.1b

and

Lemma 3.5

1.1b.

4.4

, with

Theorem 3.1

has the semigroup property and thus satisfies the discrete flow pr-

Algorithm 4.1

for computing the negative semidefinite extremal solutions of the DARE

by

controllability of the pair

The hypotheses of

Algorithm 4.1

will be stated without proof as follows.

Corollary 4.4. Let

Algorithm 4.1

with

A_0 = A, G_0 = G

and

H_0 = H

respectively. If

(A, B)

is controllable and

D_r^{(2)} \cap \mathbb{N}_n \neq \emptyset

then the following statements hold:

(i) \{X_k\}^\infty_{k=0}

is generated by (3.2b)

with

G_0 = Z_0 = G.

Moreover,

\{X_k\}^\infty_{k=0}

converges at least R-superlinearly to the unique almost stabilizing solution

G_\infty > 0

of the dual DARE (2.14) with the rate of convergence

\limsup_{k \to \infty} \frac{\rho(T_{G,\infty})^2}{\rho(T_{X,\infty})^2} < 1.

If, in addition, \( \hat{A} \) is nonsingular, then \( X_{-m} = -G_\infty^{-1} \leq 0 \) is the unique

almost antistabilizing solution to the DARE (1.1b).

(ii) \{X_k\}^\infty_{k=0}

have the same results as Theorem 4.3.

When \( A \) is nonsingular, according to Theorem 3.4, it is possible to characterize the

R-superlinear convergence of Algorithm 4.1, with

A_0 = \hat{A}, G_0 = \hat{G}

and

H_0 = \hat{H},

for computing the negative semidefinite extremal solutions of the DARE (1.1b). It

will be stated without proof as follows.

Theorem 4.5. Let the sequences \{H_k\}^\infty_{k=0}

and \{\hat{X}_k\}^\infty_{k=0}

be generated by Algorithm 4.1 with

A_0 = \hat{A}, G_0 = \hat{G}

and

H_0 = \hat{H}

being defined by (2.10), respectively. If the hypotheses of Theorem 3.4 are satisfied, then

(i) \{H_k\}^\infty_{k=0}

converges at least R-superlinearly to \(-X_{-M}\) with the rate of convergence

\limsup_{k \to \infty} \frac{\rho(T_{X,\infty})^2}{\rho(T_{H,\infty})^2} < 1.

(ii) \{\hat{X}_k\}^\infty_{k=0}

converges at least R-superlinearly to \(-X_{-m}\) with the rate of con-
Corollary 4.4

Theorem 4.3

or

5.1 is inconclusive when

might have different speed of convergence, even though the minimal solution $X_{-,m}$ can be computed from their limits. It can be seen that the convergence of $\{G_k\}_{k=0}^\infty$ is at least R-superlinear, but the convergence of $\{\hat{X}_k\}_{k=0}^\infty$ is unclear if $\mu(T_{X_{-,m}}) = 1$.

Remark 4.6.

(i) Under the hypotheses of Corollary 4.4, the sequences $\{G_k\}_{k=0}^\infty$ and $\{\hat{X}_k\}_{k=0}^\infty$ obtained in Theorem 4.5 might have different speed of convergence, even though the minimal solution $X_{-,m}$ can be computed from their limits. It can be seen that the convergence of $\{G_k\}_{k=0}^\infty$ is at least R-superlinear, but the convergence of $\{\hat{X}_k\}_{k=0}^\infty$ is unclear if $\mu(T_{X_{-,m}}) = 1$.

(ii) Although the statement (ii) in either Theorem 4.3 or Theorem 4.5 is inconclusive when $\rho(T_{X_{+,m}}) = 1$ or $\mu(T_{X_{-,m}}) = 1$, it is observed from our numerical experiments that the sequence $\{\hat{X}_k\}_{k=0}^\infty$ may converge R-linearly with at least the rate $1/r$, which will be investigated in the future and report the theoretical results elsewhere.

5. Numerical examples. In this section, we present four examples to illustrate the accuracy and efficiency of the AFPI algorithm for solving the extremal solutions of the DARE (1.1b). In the first three examples we compared the AFPI algorithm, through the sequence $\{\hat{X}_k\}_{k=0}^\infty$ starting with some suitable initial $\hat{X}_0$, with Newton’s method (NTM) [8] for solving the maximal and (almost) stabilizing solution $X_{+,M} \geq 0$ of DARE (1.1b). Here $\hat{X}_0 \geq 0$ is the unique solution of Stein matrix equation

$$S_{Af}(X) := X - A^H A = H + F^H RF,$$

which can be computed by MATLAB command dlyap directly, if $A_F = A - BF$ is d-stable for some $F \in \mathbb{C}^{m\times n}$. Moreover, we assume $R = I$ in all numerical examples.

As a byproduct, the sequence $\{H_k\}_{k=0}^\infty$ generated by Algorithm 4.1 converges at least R-superlinearly to the minimal positive semidefinite solution $X_{+,m}$ of the DAREs in Example 5.1 and Example 5.2, which is shown theoretically in Theorem 4.3. Moreover, we also demonstrate the ability of the AFPI algorithm for computing the negative semidefinite extremal solutions to the DARE in Example 5.4. Starting with $A, \hat{G}, \hat{H}$ and $\hat{X}_0$ defined as in (2.10), it is seen in Theorem 4.5 that the AFPI produces two sequences of matrices $\{-\hat{X}_k\}_{k=0}^\infty$ and $\{-H_k\}_{k=0}^\infty$ converging to the minimal negative semidefinite solution $X_{-,m}$ and the maximal negative semidefinite solution $X_{-,M}$ of the DARE (1.1b), respectively.

For an approximate solution $Z$ to the DARE (1.1), we will report its normalized residual

$$NRes(Z) := \frac{\|Z - R(Z)\|}{\|Z\| + \|A^H Z (I + GZ)^{-1} A\| + \|H\|},$$

and one of the following two quantities associated with $T_Z := (I + GZ)^{-1} A$

$$\rho(T_Z) := \max\{\lambda | \lambda \in \sigma(T_Z)\}, \quad \mu(T_Z) := \min\{\lambda | \lambda \in \sigma(T_Z)\}$$

in the tables below. We terminated the numerical methods AFPI and NTM when $NRes \leq 1.0 \times 10^{-15}$ in Examples 5.1–5.3, and the AFPI algorithm terminated when $NRes \leq 1.0 \times 10^{-12}$ in Example 5.4, respectively. All numerical experiments were
performed on ASUS laptop (ROG GL502VS-0111E7700HQ), using Microsoft Windows 10 operating system and MATLAB Version R2019b, with Intel Core i7-7700HQ CPU and 32 GB RAM.

Example 5.1. Let the coefficient matrices of DARE (1.1b) be given by

\[ A = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \]

Then it is easily seen that the pair \((A, B)\) is stabilizable, but \((A, C)\) is not detectable. Moreover, this DARE (1.1b) has only two positive semidefinite solutions, namely,

\[ X_{+,M} = \begin{bmatrix} 8 & 0 \\ 0 & 4/3 \end{bmatrix}, \quad X_{+,m} = \begin{bmatrix} 0 & 0 \\ 0 & 4/3 \end{bmatrix}, \]

with \( H = C^H C \). The matrix \( X_{+,M} \) is the maximal and stabilizing solution of the DARE such that the eigenvalues of \( T_{X_{+,M}} = (I + GX_{+,M})^{-1} A \) are 1/3 and 1/2, and \( \sigma(T_{X_{+,m}}) = \sigma(A) \), respectively. Thus, \( X_{+,m} \) is the minimal positive semidefinite solution of the DARE (1.1b) with the property \( \rho(T_{X_{+,m}}) = 3 > 1 \).

If we choose \( F = [3, 0] \) so that \( A_F = A - BF \) is d-stable and \( \hat{X}_0 \) is determined by (5.1), then the numerical results of AFPI(2) and NTM are presented in Table 1 and Table 2, respectively, where \( A_0 = A, \ G_0 = BB^H \) and \( H_0 = H \) are required matrices for Algorithm 4.1. Notice that our algorithm AFPI(2) generates two highly accurate approximations to the extremal solutions \( X_{+,M} \) and \( X_{+,m} \) simultaneously, with the relative errors being \( 2.3 \times 10^{-16} \) and \( 0.0 \times 10^0 \) respectively, even though \( \|A_k\| \) increases rapidly as \( k \) proceeds. Nevertheless, starting from the same initial matrix \( X_0 \), the NTM only produces an accurate approximation to the solution \( X_{+,M} \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
k & NRes(X_k) & NRes(H_k) & \rho(T_{X_k}) & \rho(T_{H_k}) & \|A_k\| \\
\hline
1 & 5.7 \times 10^{-4} & 2.4 \times 10^{-2} & 5.0 \times 10^{-1} & 3.0 \times 10^0 & 9.0 \times 10^0 \\
2 & 7.1 \times 10^{-6} & 1.5 \times 10^{-3} & 5.0 \times 10^{-1} & 3.0 \times 10^0 & 8.1 \times 10^1 \\
3 & 1.1 \times 10^{-9} & 5.7 \times 10^{-6} & 5.0 \times 10^{-1} & 3.0 \times 10^0 & 6.6 \times 10^3 \\
4 & 1.0 \times 10^{-16} & 8.7 \times 10^{-11} & 5.0 \times 10^{-1} & 3.0 \times 10^0 & 4.3 \times 10^7 \\
5 & 0.0 \times 10^0 & 3.0 \times 10^0 & 3.0 \times 10^0 & 1.9 \times 10^{15} \\
\hline
\end{array}
\]

**Table 1**
Numerical results of AFPI(2) for Example 5.1.

\[
\begin{array}{|c|c|}
\hline
k & NRes(X_k) \\
\hline
1 & 5.7 \times 10^{-4} \\
2 & 8.7 \times 10^{-8} \\
3 & 2.1 \times 10^{-15} \\
4 & 0.0 \times 10^0 \\
\hline
\end{array}
\]

**Table 2**
Numerical results of NTM for Example 5.1.

Example 5.2. In this example we consider the DARE (1.1b) with its \( 5 \times 5 \) coeffi-
cient matrices being defined by

\[
A = \begin{bmatrix}
2.9 & 1 & 0 & 0 & 0 \\
0 & 2.9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad H = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 200 & -0.5 & 0 \\
0 & 0 & -0.5 & 200 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and \( B = \text{diag}(\sqrt{2}, 1, 0, 0, 1) \), respectively. It can be shown that the minimal positive semidefinite solution \( X_{+,m} \) of the DARE (1.1b) is

\[
X_{+,m} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 200 & -0.5 & 0 \\
0 & 0 & -0.5 & 200 & 0 \\
0 & 0 & 0 & 0 & (1 + \sqrt{5})/2
\end{bmatrix},
\]

which is almost the same as \( H \) except the \((5,5)\)-entry. If we let \( F = \text{diag}(2, 3, 0, 0, 0.5) \) so that \( \rho(A_F) < 1 \) and \( X_0 \) is determined by (5.1), the numerical results of AFPI(2) and NTM are reported in Table 3 and Table 4, respectively, for solving the DARE (1.1b) from the same initial \( X_0 \geq 0 \).

| \( k \) | \( \text{NRes}(X_k) \) | \( \text{NRes}(H_k) \) | \( \rho(T_{X_k}) \) | \( \rho(T_{H_k}) \) | \( \|A_k\| \) |
|-----|------------------|------------------|------------------|------------------|---------------|
| 1   | \( 9.0 \times 10^{-9} \) | \( 2.5 \times 10^{-4} \) | \( 3.8 \times 10^{-1} \) | \( 2.9 \times 10^{0} \) | \( 1.2 \times 10^{4} \) |
| 2   | \( 1.7 \times 10^{-6} \) | \( 5.6 \times 10^{-6} \) | \( 3.8 \times 10^{-1} \) | \( 2.9 \times 10^{0} \) | \( 1.3 \times 10^{2} \) |
| 3   | \( 7.1 \times 10^{-10} \) | \( 2.6 \times 10^{-9} \) | \( 3.8 \times 10^{-1} \) | \( 2.9 \times 10^{0} \) | \( 1.5 \times 10^{4} \) |
| 4   | \( 7.2 \times 10^{-17} \) | \( 5.2 \times 10^{-16} \) | \( 3.8 \times 10^{-1} \) | \( 2.9 \times 10^{0} \) | \( 1.4 \times 10^{8} \) |

Table 3
Numerical results of AFPI(2) for Example 5.2.

| \( k \) | \( \text{NRes}(X_k) \) | \( \rho(T_{X_k}) \) |
|-----|------------------|------------------|
| 1   | \( 3.4 \times 10^{-4} \) | \( 3.8 \times 10^{-1} \) |
| 2   | \( 1.3 \times 10^{-6} \) | \( 3.8 \times 10^{-1} \) |
| 3   | \( 2.6 \times 10^{-11} \) | \( 3.8 \times 10^{-1} \) |
| 4   | \( 2.3 \times 10^{-18} \) | \( 3.8 \times 10^{-1} \) |

Table 4
Numerical results of NTM for Example 5.2.

Notice that, after 4 iterations, the AFPI(2) generates accurate approximations to the maximal and minimal positive semidefinite solutions of the DARE (1.1b) simultaneously, but NTM merely computes an approximation to the solution \( X_{+,M} \). Since the exact maximal solution is unknown, we may apply the MATLAB command \texttt{dare} to produce the accurate approximation of \( X_{+,M} \). The relative errors of the 4th iterates computed by AFPI(2) are

\[
\frac{\|\hat{X}_4 - X_{+,M}\|}{\|X_{+,M}\|} \approx 1.6 \times 10^{-16}, \quad \frac{\|H_4 - X_{+,M}\|}{\|X_{+,M}\|} \approx 1.2 \times 10^{-15},
\]

respectively, which shows the feasibility of our proposed algorithm.
Example 5.3. This example is modified from Example 6.2 of [8]. For \( \varepsilon \geq 0 \), the coefficient matrices of DARE (1.1b) are defined by

\[
A = \text{diag}\left(\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & \varepsilon \\
0 & 0 & 1
\end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & \sqrt{2} & 0 \\
0 & 1 & \varepsilon \\
0 & 0 & 1
\end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{\sqrt{2}} \\
0 & 0 & 1
\end{bmatrix}\right),
\]

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix},
\]

\[
H = C^H C = 0 \in \mathbb{R}^{8 \times 8}.
\]

Then this DARE has a unique positive semidefinite solution \( X_{+M} = X_{+m} = 0 \), where \( X_{+M} \) is the almost stabilizing solution with \( \sigma(T_{+M}) = \sigma(A) \) for all \( \varepsilon \geq 0 \). Note that this DARE (1.1b) is just the same as the one appeared in Example 6.2 of [8] when \( \varepsilon = 0 \), in which all unimodular eigenvalues of \( A \) are semisimple.

If we choose \( F = \text{diag}(-1, 1, 1, 1, 0.1, 0.1, 0.1) \) for any \( \varepsilon \geq 0 \), then \( A_F = A - BF \) is d-stable so that the initial matrix \( \hat{X}_0 \) is the unique positive semidefinite solution to the Stein equation (5.1). With \( H_0 = H = 0 \), the AFPI\((r)\) generates \( H_k = 0 \) for all \( k \geq 0 \) and \( r \geq 2 \). When \( \varepsilon = 0 \), the convergence histories of NTM and AFPI\((r)\) are presented in Figure 1 for \( r = 2, 4, 8 \) and 100, respectively. Moreover, the CPU times of these numerical methods are reported in Table 5.

| Method    | Iter. No. | CPU Time (sec.) |
|-----------|-----------|-----------------|
| AFPI(2)   | 50        | \( 8.78 \times 10^{-3} \) |
| AFPI(4)   | 25        | \( 1.45 \times 10^{-2} \) |
| AFPI(8)   | 17        | \( 1.35 \times 10^{-2} \) |
| AFPI(100) | 8         | \( 1.51 \times 10^{-2} \) |
| NTM       | 50        | \( 3.08 \times 10^{-2} \) |

Table 5: The CPU times of numerical methods for Example 5.3 with \( \varepsilon = 0 \).

As for \( \varepsilon = 1 \), the matrix \( A \) contains an \( 2 \times 2 \) Jordan block corresponding to the eigenvalue \( \lambda = 1 \), and the other unimodular eigenvalues are semisimple. In this case, the NTM encountered a breakdown after 54 iterations, due to the stagnation of dlyap for solving the corresponding Stein equation. It requires about \( 3.16 \times 10^{-2} \) second to achieve the absolute error \( \| X_k - X_{+M} \| \approx 5.4 \times 10^{-9} \). On the other hand, the AFPI(100) produced a highly accurate approximation to the solution \( X_{+M} = 0 \) with \( \| \hat{X}_k - X_{+M} \| \approx 2.0 \times 10^{-14} \) after 7 iterations, starting from the same initial matrix \( \hat{X}_0 \) determined as NTM by (5.1). Moreover, the numerical results of AFPI(100) are reported in Table 6. From the third column of Table 6 it seems that the AFPI(100) converges linearly with rate of convergence being \( \frac{1}{r} = 1.00 \times 10^{-2} \) approximately, but this phenomenon will be further investigated in the future and reported elsewhere.

Example 5.4. This example will demonstrate the feasibility of our AFPI algorithm for solving the negative semidefinite extremal solutions of the DARE (1.1b). As quoted
from Example 6.2 of [24], the coefficient matrices of DARE (1.1b) are given by

\[ A = \begin{bmatrix} 4 & 3 \\ -9 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad H = \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}. \]

Moreover, the authors in [24] suggested that the given DARE (1.1b) should be transformed into its dual DARE (1.2) for computing the negative semidefinite extremal solutions of the DARE (1.1b), where

\[ \tilde{A} = \begin{bmatrix} 7 & 6 \\ -9 & -8 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 12 \\ -14 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 24 & 16 \end{bmatrix}, \quad \tilde{R} = 65, \quad \tilde{H} = H. \]

It is shown in [24] that the given DARE (1.1b) has three extremal solutions, namely,

\[ X_{+,M} = X_{+,m} = \begin{bmatrix} \frac{9}{2} + \frac{9}{8}\sqrt{17} & 3 + \frac{3}{4}\sqrt{17} \\ 3 + \frac{3}{4}\sqrt{17} & 2 + \frac{\sqrt{17}}{2} \end{bmatrix}. \]
Corollary 4.4

Algorithm 4.1

1.1b

\| \end{equation}

Table 7

and

\[
X_{+,M} = \begin{bmatrix}
\frac{9}{2} - \frac{9}{2} \sqrt{17} \\
3 - \frac{3}{2} \sqrt{17} \\
- \frac{3}{2} \sqrt{17}
\end{bmatrix}, \quad X_{-,m} = \begin{bmatrix}
- \frac{103}{12} - \frac{\sqrt{17}}{8} \\
\frac{39}{4} - \frac{\sqrt{17}}{4} \\
- \frac{13}{4} - \frac{\sqrt{17}}{4}
\end{bmatrix},
\]

where \( X_{+,M} \geq 0 \) is the maximal and stabilizing solution, \( X_{-,M} \) is the maximal negative semidefinite solution and \( X_{-,m} \leq 0 \) is the minimal solution, respectively.

Since it can be checked that \( 90I \in D^2_{\geq} \cap N_\rho \) and \((A,B)\) is controllable, the sufficient conditions of Corollary 4.4 are satisfied. If we let \( F = [-0.58, -0.68] \) so that \( A_F \) is d-stable, then the AFPI(2), with \( A_0 = A, G_0 = G, H_0 = H \) and \( X_0 \) being determined by (5.1), produced highly accurate approximations to \( X_{+,M} \) and \( X_{-,m} \) with relative errors being \( 1.8 \times 10^{-16} \) and \( 5.9 \times 10^{-14} \), respectively, after 5 iterations.

In addition, for computing the negative semidefinite extremal solutions to the DARE (1.1b), we consider the dual DARE (2.9) with \( \hat{B} = \hat{B}, R = \hat{R}, \hat{A} = \hat{A} - BR^{-1}C, \hat{G} = BR^{-1}B^H \) and \( \hat{H} = H - C^H R^{-1}C, \) respectively. Since the pair \((\hat{A}, \hat{B})\) is stabilizable, we may choose \( \hat{F} = [0.62, 0.52] \) so that the matrix \( \hat{A}_\hat{F} := \hat{A} - \hat{B} \hat{F} \) is d-stable. Let \( \hat{X}_0 \geq 0 \) be the unique solution to the Stein equation

\[
S_{\hat{A}_\hat{F}}(X) = \hat{H} + \hat{F}^H \hat{R} \hat{F} \geq 0.
\]

Applying Algorithm 4.1 with \( A_0 = \hat{A}, G_0 = \hat{G}, H_0 = \hat{H} \) and \( X_0 \), the AFPI(4) generates highly accurate approximations \( H_3 \approx -X_{-,M} \) and \( \hat{X}_3 \approx -X_{-,m} \) with relative errors being \( 4.1 \times 10^{-14} \) and \( 7.4 \times 10^{-16} \), respectively, after 3 iterations. Moreover, the numerical results of AFPI(4) are reported in Table 7. From the 5th column of Table 7, we see that our AFPI can provide a good approximation to the minimal and antistabilizing solution \( X_{+,M} \) of the DARE (1.1).

| \( k \) | \( NRes(-H_k) \) | \( NRes(-X_k) \) | \( \mu(T_{-H_k}) \) | \( \mu(T_{-X_k}) \) | \( \|A_k\| \) |
|---|---|---|---|---|---|
| 1 | \( 2.1 \times 10^{-13} \) | \( 9.5 \times 10^{-1} \) | \( 5.0 \times 10^{-1} \) | \( 2.0 \times 10^{1} \) | \( 5.7 \times 10^{1} \) |
| 2 | \( 2.1 \times 10^{-13} \) | \( 5.1 \times 10^{-8} \) | \( 5.0 \times 10^{-1} \) | \( 2.0 \times 10^{0} \) | \( 2.3 \times 10^{5} \) |
| 3 | \( 2.1 \times 10^{-13} \) | \( 7.4 \times 10^{-13} \) | \( 5.0 \times 10^{-1} \) | \( 2.0 \times 10^{0} \) | \( 1.3 \times 10^{-1} \) |

Table 7

Numerical results of AFPI(4) for Example 5.4.

6. Concluding remarks. In most of the past works, it is always assumed that the DARE has a unique maximal positive semidefinite solution \( X \) with \( \rho(T_X) \leq 1 \) and another meaningful solutions are lacking in brief discussion. Our contribution fills in the existing gap in finding four extremal solutions of the DARE. More precisely, we have studied the existence of minimal and maximum solutions under mild assumptions. It is important to note that we no longer need the traditional assumptions on the distribution of the spectrum of the symplectic matrix pencil or Popov matrix function, but give mathematical conditions from different point of view.

This paper concerns comprehensive convergence analysis of the most recent advanced algorithms AFPI, including its variant SDA, for solving the extremal solutions of DARE. We verify that the AFPI works efficiently and the convergence speed is R-superlinear under the mild assumptions. As compared to the previous works, the theoretical results presented here are merely deduced from the framework of the fixed-point iteration, using basic assumptions and elementary matrix theory. We believe
the results we obtain are novel on this topic and could provide considerable insights into the study of unmixed solutions of DARE.

REFERENCES

[1] J. Ackermann, Sampled-Data Control Systems: Analysis and Synthesis, Robust System Design, Springer Berlin Heidelberg, Berlin, Heidelberg, 1985.
[2] B. D. O. Anderson, Second-order convergent algorithms for the steady-state Riccati equation, in 1977 IEEE Conference on Decision and Control Including the 16th Symposium on Adaptive Processes and A Special Symposium on Fuzzy Set Theory and Applications, New Orleans, LA, USA, Dec. 1977, IEEE, pp. 948–953.
[3] C.-Y. Chiang, The convergence analysis of an accelerated iteration for solving algebraic Riccati equations, J. Franklin Inst., In Press (2021).
[4] C.-Y. Chiang, H.-Y. Fan, and W.-W. Lin, A structured doubling algorithm for discrete-time algebraic Riccati equations with singular control weighting matrices, Taiwanese J. Math., 14 (2010), pp. 933–954.
[5] E. K.-W. Chu, H.-Y. Fan, W.-W. Lin, and C.-S. Wang, Structure-preserving algorithms for periodic discrete-time algebraic Riccati equations, Int. J. Control, 77 (2004), pp. 767–788.
[6] S. M. El-Sayed and A. C. M. Ran, On an iteration method for solving a class of nonlinear matrix equations, SIAM J. Matrix Anal. & Appl., 23 (2002), pp. 632–645.
[7] T. Gudmundsson, C. Kenney, and A. Laub, Scaling of the discrete-time algebraic Riccati equation to enhance stability of the Schur solution method, IEEE Trans. Automat. Contr., 37 (1992), pp. 513–518.
[8] C.-H. Guo, Newton’s method for discrete algebraic Riccati equations when the closed-loop matrix has eigenvalues on the unit circle, SIAM J. Matrix Anal. & Appl., 20 (1998), pp. 279–294.
[9] V. Ignescu, Reverse discrete-time Riccati equation and extended Nehari’s problem, Linear Algebra Appl., 236 (1996), pp. 59–94.
[10] E. Jonckheere, On the existence of a negative semidefinite, antistabilizing solution to the discrete-time algebraic Riccati equation, IEEE Trans. Automat. Control, 26 (1981), pp. 707–712.
[11] M. Kimura, Convergence of the doubling algorithm for the discrete-time algebraic Riccati equation, Int. J. Syst. Sci., 19 (1988), pp. 701–711.
[12] V. Kučera, The discrete Riccati equation of optimal control, Kybernetika, 8 (1972), pp. 430–447.
[13] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, Wiley Interscience, New York, 1972.
[14] P. Lancaster and L. Rodman, Algebraic Riccati Equations, Oxford Science Publications, Clarendon Press ; Oxford University Press, Oxford : New York, 1995.
[15] A. Laub, A Schur method for solving algebraic Riccati equations, IEEE Trans. Automat. Contr., 24 (1979), pp. 913–921.
[16] M. Lin and C.-Y. Chiang, An accelerated technique for solving one type of discrete-time algebraic Riccati equations, J. Comput. Appl. Math., 338 (2018), pp. 91–110.
[17] ———, On the semigroup property for some structured iterations, J. Comput. Appl. Math., 374 (2020), p. 112768.
[18] V. Mehrmann, The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution, no. 163 in Lecture Notes in Control and Information Sciences, Springer, Berlin, 1991.
[19] C. Oară, Stabilizing solution to the reverse discrete-time Riccati equation: A matrix-pencil-based approach, Linear Algebra Appl., 246 (1996), pp. 113–130.
[20] T. Pappas, A. Laub, and N. Sandell, On the numerical solution of the discrete-time algebraic Riccati equation, IEEE Trans. Automat. Contr., 25 (1980), pp. 631–641.
[21] P. Van Dooren, A generalized eigenvalue approach for solving Riccati equations, SIAM J. Sci. and Stat. Comput., 2 (1981), pp. 121–135.
[22] H. Wimmer, The set of positive semidefinite solutions of the algebraic Riccati equation of discrete-time optimal control, IEEE Trans. Automat. Contr., 41 (1996), pp. 660–671.
[23] H. K. Wimmer, On the existence of a least and negative-semidefinite solution of the discrete-time algebraic Riccati equation, J. Math. Systems Estim. Control, 5 (1995), pp. 445–457.
[24] L. Zhang, M. Z. Q. Chen, and C. Li, The dual algebraic Riccati equations and the set of all solutions of the discrete-time Riccati equation, Int. J. Control, 90 (2017), pp. 1371–1388.