BELL NUMBERS, LOG-CONCAVITY, AND LOG-CONVEXITY

NOBUHIRO ASAI, IZUMI KUBO, AND HUI-HSIUNG KUO

Abstract. Let \( \{b_k(n)\}_{n=0}^{\infty} \) be the Bell numbers of order \( k \). It is proved that the sequence \( \{b_k(n)/n!\}_{n=0}^{\infty} \) is log-concave and the sequence \( \{b_k(n)\}_{n=0}^{\infty} \) is log-convex, or equivalently, the following inequalities hold for all \( n \geq 0 \),

\[
1 \leq \frac{b_k(n+2)b_k(n)}{b_k(n+1)^2} \leq \frac{n+2}{n+1}.
\]

Let \( \{\alpha(n)\}_{n=0}^{\infty} \) be a sequence of positive numbers with \( \alpha(0) = 1 \). We show that if \( \{\alpha(n)/n!\}_{n=0}^{\infty} \) is log-convex, then

\[
\alpha(n)\alpha(m) \leq \alpha(n+m), \quad \forall n, m \geq 0.
\]

On the other hand, if \( \{\alpha(n)/n!\}_{n=0}^{\infty} \) is log-concave, then

\[
\alpha(n+m) \leq \left(\frac{n+m}{n}\right)\alpha(n)\alpha(m), \quad \forall n, m \geq 0.
\]

In particular, we have the following inequalities for the Bell numbers

\[
b_k(n)b_k(m) \leq b_k(n+m) \leq \left(\frac{n+m}{n}\right)b_k(n)b_k(m), \quad \forall n, m \geq 0.
\]

Then we apply these results to white noise distribution theory.

1. The main theorems

For an integer \( k \geq 2 \), let \( \exp_k(x) \) denote the \( k \)-times iterated exponential function

\[
\exp_k(x) = \exp\left(\exp\cdots\left(\exp(x)\right)\right)\text{, \( k \text{-times} \).}
\]

Let \( \{B_k(n)\}_{n=0}^{\infty} \) be the sequence of numbers given in the power series of \( \exp_k(x) \)

\[
\exp_k(x) = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} x^n.
\]

The Bell numbers \( \{b_k(n)\}_{n=0}^{\infty} \) of order \( k \) are defined by

\[
b_k(n) = \frac{B_k(n)}{\exp_k(0)}, \quad n \geq 0.
\]

The numbers \( b_2(n), n \geq 0 \), with \( k = 2 \) are usually known as the Bell numbers. The first few terms of these numbers are 1, 1, 2, 5, 15, 52, 203. Note that \( \exp_2(0) = e \) and so we have

\[
e^{e^x-1} = \sum_{n=0}^{\infty} \frac{b_2(n)}{n!} x^n.
\]

Research supported by the Daiko Foundation 1998 (N.A.), U.S. Army Research Office grant #DAAH04-94-G-0249, Academic Frontier in Science of Meijo University, and the National Science Council of Taiwan (H.-H.K.).
A sequence \( \{\delta(n)\}_{n=0}^{\infty} \) of nonnegative real numbers is called log-concave if
\[
\delta(n)\delta(n+2) \leq \delta(n+1)^2, \quad \forall n \geq 0.
\]
It is called log-convex if
\[
\delta(n)\delta(n+2) \geq \delta(n+1)^2, \quad \forall n \geq 0.
\]

The main purpose of this paper is to prove the following theorems.

**Theorem 1.** Let \( \{b_k(n)\}_{n=0}^{\infty} \) be the Bell numbers of order \( k \). Then the sequence \( \{b_k(n)/n!\}_{n=0}^{\infty} \) is log-concave and the sequence \( \{b_k(n)\}_{n=0}^{\infty} \) is log-convex.

Note that the conclusion of this theorem is equivalent the inequalities
\[
1 \leq \frac{b_k(n)b_k(n+2)}{b_k(n+1)^2} \leq \frac{n+2}{n+1}, \quad \forall n \geq 0.
\]

A different proof of the log-convexity of \( \{b_2(n)\}_{n=0}^{\infty} \) has been given earlier by Engel [3]. In [3] Canfield showed that the log-concavity of \( \{b_2(n)/n!\}_{n=0}^{\infty} \) holds asymptotically. In a recent paper [4], Cochran et al. used the log-convexity of certain sequences to study characterization theorems. However, they did not show whether the sequence \( \{b_k(n)/n!\}_{n=0}^{\infty} \) is log-concave. Thus our Theorem 1 fills up this gap (for details, see Section 3.)

**Theorem 2.** Let \( \{\alpha(n)\}_{n=0}^{\infty} \) be a sequence of positive numbers with \( \alpha(0) = 1 \).

(a) If \( \{\alpha(n)\}_{n=0}^{\infty} \) is log-convex, then
\[
\alpha(n)\alpha(m) \leq \alpha(n+m), \quad \forall n, m \geq 0.
\]

(b) If \( \{\alpha(n)/n!\}_{n=0}^{\infty} \) is log-concave, then
\[
\alpha(n+m) \leq \binom{n+m}{n}\alpha(n)\alpha(m), \quad \forall n, m \geq 0.
\]

We will prove Theorems 1 and 2 in Section 3. The next theorem is an immediate consequence of these two theorems.

**Theorem 3.** The Bell numbers \( \{b_k(n)\}_{n=0}^{\infty} \) of order \( k \) satisfy the inequalities
\[
b_k(n)b_k(m) \leq b_k(n+m) \leq \binom{n+m}{n}b_k(n)b_k(m), \quad \forall n, m \geq 0.
\]

In a recent paper [5] it is shown that for any \( k \geq 2 \) there exist constants \( c_2 \) and \( c_3 \), depending on \( k \), such that for all \( n, m \geq 0 \),
\[
b_k(n+m) \leq c_2^{n+m}b_k(n)b_k(m), \quad b_k(n)b_k(m) \leq c_3^{n+m}b_k(n+m).
\]

Observe that from Eq. (3) we get \( b_k(n)b_k(m) \leq b_k(n+m) \leq 2^{n+m}b_k(n)b_k(m) \). Thus in fact we can take \( c_2 = 2 \) and \( c_3 = 1 \) for the Bell numbers of any order \( k \).

## 2. Proofs of Theorems 1 and 2

For the proof of Theorem 1 we prepare two lemmas and state the Bender-Canfield theorem 2.

**Lemma 1.** If \( \{\beta(n)/n!\}_{n=0}^{\infty} \) is a log-concave sequence and \( r \) is a nonnegative real number such that \( \beta(2) \leq r\beta(1)^2 \), then the sequence \( 1, r\beta(n)/(n-1)! \), \( n \geq 1 \), is log-concave.
Proof. By assumption we have
\[ \frac{\beta(n) \beta(n+2)}{n! (n+2)!} \leq \left( \frac{\beta(n+1)}{n!} \right)^2. \]
When \( n \geq 1 \) this inequality is equivalent to
\[ \frac{(n+1)^2}{n+2} \left( \frac{\beta(n) \beta(n+2)}{(n-1)! (n+1)!} \right) \leq \left( \frac{\beta(n+1)}{n!} \right)^2. \]
Note that \((n+1)^2 \geq n(n+2)\). Hence for \( n \geq 1 \),
\[ \frac{\beta(n) \beta(n+2)}{(n-1)! (n+1)!} \leq \left( \frac{\beta(n+1)}{n!} \right)^2. \]
Thus for any constant \( r \) we have
\[ \frac{r \beta(n) \beta(n+2)}{(n-1)! (n+1)!} \leq \left( \frac{r \beta(n+1)}{n!} \right)^2, \quad \forall n \geq 1. \]
Moreover, the assumption \( \beta(2) \leq r \beta(1)^2 \) implies that \( 1 \cdot (r \beta(2)) \leq (r \beta(1))^2 \). Thus the sequence \( 1, r \beta(n)/n! \), \( n \geq 1 \), is log-concave.

Bender-Canfield Theorem \[2]: Let \( 1, Z_1, Z_2, \ldots \) be a log-concave sequence of nonnegative real numbers and define the sequence \( \{a(n)\}_{n=0}^\infty \) by
\[ \sum_{n=0}^{\infty} \frac{a(n)}{n!} x^n = \exp \left( \sum_{j=1}^{\infty} \frac{Z_j}{j} x^j \right). \]
Then the sequence \( \{a(n)/n!\}_{n=0}^\infty \) is log-concave and the sequence \( \{a(n)\}_{n=0}^\infty \) is log-convex.

Lemma 2. The sequence \( \{b_2(n)/n!\}_{n=0}^\infty \) is log-concave and the sequence \( \{b_2(n)\}_{n=0}^\infty \) is log-convex.

Proof. Note that \( e^{x-1} = \exp \left( \sum_{j=1}^{\infty} \frac{1}{j!} x^j \right) \). Hence by Eq. (2) we have
\[ \exp \left( \sum_{j=1}^{\infty} \frac{1}{j!} x^j \right) = \sum_{n=0}^{\infty} \frac{b_2(n)}{n!} x^n. \]
Let \( Z_j = \frac{1}{(j-1)!} \) for \( j \geq 1 \). It is easy to check that the sequence \( 1, Z_1, Z_2, \ldots \) is log-concave. Thus this lemma follows from the above Bender-Canfield theorem.

Proof of Theorem \[1\]
We prove the theorem by mathematical induction. By Lemma 2 the theorem is true for \( k = 2 \). Assume the theorem is true for \( k \). Note that
\[ \exp_{k+1}(x)/\exp_k(0) = \exp \left( \exp_k(x) - \exp_k(0) \right). \]
Hence
\[ \exp \left( \exp_k(x) - \exp_k(0) \right) = \sum_{n=0}^{\infty} \frac{b_{k+1}(n)}{n!} x^n. \]
But \( \exp_k(x) - \exp_k(0) = \sum_{j=1}^{\infty} \frac{B_k(j)}{j!} x^j \). Thus we get
\[ \exp \left( \sum_{j=1}^{\infty} \frac{B_k(j)}{j!} x^j \right) = \sum_{n=0}^{\infty} \frac{b_{k+1}(n)}{n!} x^n. \]
Let $Z_j = \frac{B_k(j)}{(j-1)!}$, $j \geq 1$. Then the above equation becomes
\[ \exp \left( \sum_{j=1}^{\infty} Z_j \frac{x^j}{j!} \right) = \sum_{n=0}^{\infty} \frac{b_{k+1}(n)}{n!} x^n. \] (6)

By the induction assumption, the sequence $\{b_k(n)/n!\}_{n=0}^{\infty}$ is log-concave. This implies in particular that $b_k(0)b_k(2)/2 \leq b_k(1)^2$. But $b_k(0) = 1$ and $\exp_k(0) > 2$. Hence
\[ b_k(2) \leq 2b_k(1)^2 < \exp_k(0) b_k(1)^2. \]
Thus we can apply Lemma 1 with $\alpha = 1$ to conclude that the sequence
\[ 1, \exp_k(0) \frac{b_k(n)}{(n-1)!}, n \geq 1, \]
is log-concave. Note that for $n \geq 1$,
\[ \exp_k(0) \frac{b_k(n)}{(n-1)!} = \frac{B_k(n)}{(n-1)!} = Z_n. \]

Hence the sequence $1, Z_1, Z_2, \ldots$ is log-concave. Upon applying the Bender-Canfield theorem, we see from Eq. (6) that the sequence $\{b_k+1(n)/n!\}_{n=0}^{\infty}$ is log-concave and the sequence $\{b_k+1(n)\}_{n=0}^{\infty}$ is log-convex.

Proof of Theorem 2

To prove (a), let $\{\alpha(n)\}_{n=0}^{\infty}$ be log-convex. Then $\alpha(n)\alpha(n+2) \geq \alpha(n+1)^2$. Hence $\alpha(n+1)/\alpha(n) \leq \alpha(n+2)/\alpha(n+1)$ and this implies that for any $n \geq 0$ and $m \geq 1$,
\[ \frac{\alpha(1)}{\alpha(0)} \leq \frac{\alpha(2)}{\alpha(1)} \leq \cdots \leq \frac{\alpha(n+m)}{\alpha(n+m-1)}. \]

Therefore, for any $n \geq 0$ and $m \geq 1$,
\[ \frac{\alpha(1)}{\alpha(0)} \frac{\alpha(2)}{\alpha(1)} \cdots \frac{\alpha(m)}{\alpha(m-1)} \leq \frac{\alpha(n+1)}{\alpha(n)} \frac{\alpha(n+2)}{\alpha(n+1)} \cdots \frac{\alpha(n+m)}{\alpha(n+m-1)}. \]
After the cancellation we get $\alpha(n)\alpha(m) \leq \alpha(0)\alpha(n+m)$. But $\alpha(0) = 1$ and so Eq. (6) is true when $n \geq 0$ and $m \geq 1$. When $m = 0$, Eq. (6) obviously holds for any $n \geq 0$. Hence we have proved assertion (a).

For the proof of (b), first note that $\{\alpha(n)/n!\}_{n=0}^{\infty}$ is log-concave if and only if for all $n \geq 0$,
\[ \frac{\alpha(n+1)}{\alpha(n)} \geq \frac{n+1}{n+2} \frac{\alpha(n+2)}{\alpha(n+1)}. \]
By using this inequality repeatedly, we get the following inequalities for any $n \geq 0$ and $m \geq 1$,
\[ \frac{\alpha(1)}{\alpha(0)} \geq \frac{1}{2} \frac{\alpha(2)}{\alpha(1)} \geq \frac{1}{3} \frac{\alpha(3)}{\alpha(2)} \geq \cdots \geq \frac{1}{n+m} \frac{\alpha(n+m)}{\alpha(n+m-1)}. \]
Hence for any $0 \leq j \leq m-1$,
\[ \frac{\alpha(j+1)}{\alpha(j)} \geq \frac{j+1}{n+m} \frac{\alpha(n+m)}{\alpha(n+m-1)}. \]
Therefore,
\[
\begin{align*}
\frac{\alpha(1) \alpha(2) \cdots \alpha(m)}{\alpha(0) \alpha(1) \cdots \alpha(m-1)} & \geq \left( \frac{1}{n+1} \frac{\alpha(n+1)}{\alpha(n)} \right) \left( \frac{2}{n+2} \frac{\alpha(n+2)}{\alpha(n+1)} \right) \cdots \left( \frac{m}{n+m} \frac{\alpha(n+m)}{\alpha(n+m-1)} \right).
\end{align*}
\]
After the cancellation we get
\[
\frac{\alpha(m)}{\alpha(0)} \geq \frac{n! m!}{(n+m)!} \frac{\alpha(n+m)}{\alpha(n)}.
\]
But \(\alpha(0) = 1\). Hence we have proved that for any \(n \geq 0, m \geq 1\),
\[
\alpha(n+m) \leq \binom{n+m}{n} \alpha(n) \alpha(m).
\]
Note that when \(m = 0\), Eq. (4) obviously holds for any \(n \geq 0\). Thus assertion (b) is proved.

3. Application to white noise analysis

• Characterization of test and generalized functions

The Bell numbers \(\{b_k(n)\}_{n=0}^{\infty}\) for \(k \geq 2\) provide important examples in white noise distribution theory \([8]\). In a recent paper \([4]\) Cochran et al. have constructed a space \([V]_\alpha\) of test functions and its dual space \([V]_\alpha^*\) of generalized functions from a nuclear space \(V\) and a sequence \(\{\alpha(n)\}_{n=0}^{\infty}\) of positive numbers satisfying the following conditions:

1. \(\alpha(0) = 1\).
2. \(\inf_{n \geq 0} \alpha(n) > 0\).
3. \(\lim_{n \to \infty} \left( \frac{\alpha(n)}{n!} \right)^{1/n} = 0\).

For the characterization of generalized functions in \([V]_\alpha^*\) (Theorem 6.4 in \([4]\)) they assume the following condition

\[
\limsup_{n \to \infty} \left( \frac{n!}{\alpha(n)} \inf_{x > 0} G_\alpha(x) x^n \right)^{1/n} < \infty.
\]

where \(G_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} x^n\) is the exponential generating function of the sequence \(\{\alpha(n)\}_{n=0}^{\infty}\). Furthermore, by Corollary 4.4 in \([4]\), if the sequence \(\{\alpha(n)/n!\}_{n=0}^{\infty}\) is log-concave, then the condition in Eq. (6) is satisfied.

For the case \(\alpha(n) = b_k(n)\), Cochran et al. showed in Proposition 7.4 in \([4]\) that the condition in Eq. (6) is satisfied. However, they did not show whether the sequence \(\{b_k(n)/n!\}_{n=0}^{\infty}\) is log-concave. Our Theorem 1 shows that this is indeed the case.

The other conclusion in Theorem 1, i.e., \(\{b_k(n)\}_{n=0}^{\infty}\) being log-convex, can be used to characterize the test functions. First we point out the following fact which can be easily checked.

**Fact.** If \(\{\beta(n)\}_{n=0}^{\infty}\) is log-convex, then \(\{\frac{1}{\beta(n)/n!}\}_{n=0}^{\infty}\) is log-concave.

Recall from Theorem 1 that the sequence \(\{b_k(n)\}_{n=0}^{\infty}\) is log-convex. Hence by the above fact the sequence \(\{\frac{1}{b_k(n)/n!}\}_{n=0}^{\infty}\) is log-concave.

In \([4]\) Cochran et al. did not study the characterization of test functions in \([V]_\alpha\). In a recent paper \([3]\) and our ongoing project initiated in \([1]\) several theorems on
the characterization of test functions and related results have been obtained. For test functions, we need to assume the following condition

$$\limsup_{n \to \infty} \left( \alpha(n)! \inf_{x > 0} \frac{G_{1/\alpha}(x)}{x^n} \right)^{1/n} < \infty,$$

where $G_{1/\alpha}(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha(n)!} x^n$ is the exponential generating function of the sequence $\{ \frac{1}{\alpha(n)!} \}_{n=0}^{\infty}$. The same argument as in the proof of Corollary 4.4 in [6] can be used to show that if $\{ \frac{1}{\alpha(n)!} \}_{n=0}^{\infty}$ is log-concave, then the condition in Eq. (8) is satisfied.

In particular, when $\alpha(n) = b_k(n)$, we know from Theorem [6] that the sequence $\{ \frac{1}{\alpha(n)!} \}_{n=0}^{\infty}$ is log-concave. Thus the condition in Eq. (8) is satisfied.

- *Inequality conditions on the sequence $\{ \alpha(n) \}_{n=0}^{\infty}$*

In order to carry out the white noise distribution theory for the spaces $|\mathcal{V}|_{\alpha}$ and $|\mathcal{V}|^*_{\alpha}$ the following three conditions have been imposed on $\{ \alpha(n) \}_{n=0}^{\infty}$ in [6]:

1. There exists a constant $c_1$ such that for any $n \leq m$,

   $\alpha(n) \leq c_1^n \alpha(m)$.

2. There exists a constant $c_2$ such that for any $n$ and $m$,

   $\alpha(n + m) \leq c_2^{n+m} \alpha(n) \alpha(m)$.

3. There exists a constant $c_3$ such that for any $n$ and $m$,

   $\alpha(n) \alpha(m) \leq c_3^{n+m} \alpha(n + m)$.

Note that $c_i \geq 1$ for all $i = 1, 2, 3$ since $\alpha(0) = 1$. As shown in Section 3 in [6], condition (c-3) implies condition (c-1). Moreover, it has been proved in Theorem 4.8 in [6] that the Bell numbers $\{ b_k(n) \}_{n=0}^{\infty}$ satisfy conditions (c-1), (c-2), and (c-3). Below we give further comments on the constants $c_1$, $c_2$, and $c_3$.

Obviously, if a sequence $\{ \alpha(n) \}_{n=0}^{\infty}$ is non-decreasing, then condition (c-1) is satisfied and $c_1 = 1$ is the best constant satisfying condition (c-1).

From Eq. (7.5) in [6] we have the formula for the sequence $\{ B_k(n) \}_{n=0}^{\infty}$ defined in Eq. (6) for $k \geq 2$:

$$B_k(n) = \sum_{j=0}^{\infty} \frac{B_{k-1}(j)}{j!} j^n,$$  \hspace{1cm} (9)

where $B_1(n) = 1$ for all $n$. On the other hand, we can differentiate both sides of Eq. (6) and then compare the coefficients of $x^n$ to get the formula:

$$B_k(n + 1) = \sum_{j_1 + \cdots + j_k = n} \frac{n!}{j_1! \cdots j_k!} B_1(j_1) \cdots B_k(j_k).$$  \hspace{1cm} (10)

We see from either Eq. (6) or (10) that the sequence $\{ B_k(n) \}_{n=0}^{\infty}$ is increasing. But $b_k(n) = B_k(n)/\exp_b(0)$ and so the sequence $\{ b_k(n) \}_{n=0}^{\infty}$ is also increasing. Hence the Bell numbers satisfy condition (c-1) and the best constant for $c_1$ is $c_1 = 1$.

As mentioned at the end of Section 6, the Bell numbers of any order $k \geq 2$ satisfy the inequalities:

$$b_k(n)b_k(m) \leq b_k(n + m) \leq 2^{n+m}b_k(n)b_k(m).$$
Hence the Bell numbers satisfy conditions (c-2) and (c-3) with $c_2 = 2$ and $c_3 = 1$. Obviously, $c_3 = 1$ is the best constant for condition (c-3). As for the best constant for $c_2$ we have the following

**Conjecture.** The best constant $c_2$ in the condition (c-2) for the Bell numbers \( \{b_k(n)\}_{n=0}^{\infty} \) of any order $k \geq 2$ is $c_2 = 2$.

Here we prove that the conjecture is true for $k = 2$. It follows from Theorem 3 that $b_2(n + m) \leq 2^{n+m}b_2(n)b_2(m)$. Hence the best constant $c_2$ must be $c_2 \leq 2$. On the other hand, by Theorem 4.3 in [7],

\[
\log b_2(n) = n \log n - n \log \log n - n + o(n).
\]

From this equality we obtain that

\[
\log b_2(2n) - 2 \log b_2(n) = 2n \log 2 - 2n (\log \log (2n) - \log \log n) + o(n).
\]

Then we get the following limit

\[
\lim_{n \to \infty} \frac{1}{2n} \log \frac{b_2(2n)}{b_2(n)^2} = \log 2.
\]  

(11)

Now, put $m = n$ in condition (c-2) to get $b_2(2n) \leq c_2^n b_2(n)^2$. This inequality implies that for all $n \geq 1$,

\[
\frac{1}{2n} \log \frac{b_2(2n)}{b_2(n)^2} \leq \log c_2.
\]  

(12)

Obviously, Eqs. (11) and (12) show that $\log 2 \leq \log c_2$. Hence $c_2 \geq 2$. But we already noted above that $c_2 \leq 2$. Therefore, $c_2 = 2$.

**REFERENCES**

[1] N. Asai, I. Kubo, and H.-H. Kuo, *Log-concavity and growth order in white noise analysis* (in preparation)

[2] E. A. Bender and E. R. Canfield, *Log-concavity and related properties of the cycle index polynomials*, J. Combinatorial Theory, Series A 74 (1996), 57–70.

[3] E. R. Canfield, *Engel’s inequality for Bell numbers*, J. Combinatorial Theory, Series A 72 (1995), 184–187.

[4] W. G. Cochran, H.-H. Kuo, and A. Sengupta, *A new class of white noise generalized functions*, Infinite Dimensional Analysis, Quantum Probability and Related Topics 1 (1998), 43–67.

[5] K. Engel, *On the average rank of an element in a filter of the partition lattice*, J. Combinatorial Theory, Series A 65 (1994), 67–78.

[6] I. Kubo, *On characterization theorems for CRK-spaces in white noise analysis*, Preprint (1998)

[7] I. Kubo, H.-H. Kuo, and A. Sengupta, *White noise analysis associated with sequences of numbers*, Preprint (1998)

[8] H.-H. Kuo, *White Noise Distribution Theory*, CRC Press, 1996.

Nobuhiro Asai: Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, JAPAN

Izumi Kubo: Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Hiroshima 739-8526, JAPAN

Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA