A note on identities in two variables for a class of monoids

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Received: 30 August 2019    Revised: 24 December 2019    Accepted: 12 January 2020

Abstract: In this note we consider identities in the alphabet $X = \{x, y\}$. This note is self-contained and the aim is to describe gradually the identities partition (with three parameters) of the free semigroup $X^+$ for the class of monoids $B_n = \langle a, b \mid ba = b^n \rangle$ $(n > 0)$.

Keywords: Semigroup identities, Checking identities, Identities partition.

2010 Mathematics Subject Classification: 68R15, 08A50.

1 Introduction and preliminaries

Section 4 of the recent paper by Geroldinger and Schwab [2] is devoted to the study of non-unique factorizations in a class of non-commutative monoids $\{B_n\}_{n>1}$. The monoids $B_n$, $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of nonnegative integers, are defined by the monoid presentation:

$$B_n = \langle a, b \mid ba = b^n \rangle.$$  

The elements of $B_n$ are words of the form $a^k b^m$ for $k, m \in \mathbb{N}$ with the understanding that $a^0 = b^0 = 1$. The monoid $B_0$ is the bicyclic monoid which plays a very important role in the structural theory of semigroups. The multiplication in $B_0$ is given by the rule
If there are three distinct ways, and Shleifer [4] studied also (only if \( X = \{ \} \)) identities using computer assistance.

In preparation we used only a few identities (Section 2).
2 The identities \((A_{i,j})\)

Since \(B_n\) (for all \(n \in \mathbb{N}\)) contains a copy of the infinite cyclic semigroup, any identity \(v \approx w\) for \(B_n\) is balanced. From the multiplication defined in \(B_n\) it follows that if \(v \approx w\) is an identity satisfied in \(B_n\) then the first letter of \(v\) and \(w\) coincide. We will consider identities with the first letter \(x\); changing the two letters \(x\) and \(y\) between them in each of the words of the identity \(v \approx w\) does not lead to a new identity in our convention. If \(n > 0\) then the right cancellation law holds in the set of all identities for \(B_n\) because the monoid \(B_n\) \((n > 0)\) is right cancellative.

**Proposition 2.1.** For any positive integers \(i \geq j\) and \(n > 0\),

\[
(A_{i,j}) \quad xy^{i+1}x \approx xy^{j}xy^{i-j+1} \quad (A'_{i,j}) \quad yx^{i+1}y \approx yx^{j}yx^{i-j+1}
\]

is an identity satisfied in \(B_n\).

**Proof.** To prove that \((A_{i,j})\) is an identity for \(B_n\) \((n > 0)\) we consider the substitution \(y = a^kb^m\), \(x = a^rb^s\). Then

\[
(a) \quad y^2x = \begin{cases} a^{2k+rm} & \text{if } m = 0 \\ a^{k+2m+(n-1)k+(n-1)r+s} & \text{if } m > 0, \end{cases}
\]

and

\[
(b) \quad yxy = \begin{cases} a^{2k+r} & \text{if } m = 0 \text{ and } s = 0 \\ a^{k+2m+(n-1)k+(n-1)r+s} & \text{if } m = 0 \text{ and } s > 0 \\ a^{k+2m+(n-1)k+(n-1)r+s} & \text{if } m > 0. \end{cases}
\]

Since \(y^2x = yxy\) in the cases \(m > 0\) and \(m = 0 = s\), we will consider hereinafter \(m = 0, s > 0\). Then

\[
xy^{i+1}x = xy^{i-1}(y^2x) = a^r b^s a^{(i-1)k} a^{2k+rm} b^s = a^r b^s a^{(i+1)k+r} b^s = a^r b^{2s+(n-1)(i+1)k},
\]

and

\[
xy^jy = xy^{i-1}(yxy) = a^r b^s a^{(i-1)k} a^{k+rm} b^{s+(n-1)k} = a^r b^s a^{ik+rm} b^{s+(n-1)k} = a^r b^{2s+(n-1)(i+1)k}.\]

So, \((A_{i,j})\) is an identity satisfied in \(B_n\) \((n > 0)\) if \(i = j\).

Now, the following sequence of identities satisfied in \(B_n, n > 0,\)

\[
xy^{i+1}x \approx xy^i xy \approx xy^{i-1}xy^2 \approx xy^{i-2}xy^3 \approx \ldots \approx xy^ixy^{i-j+1}
\]

finishes the proof of the proposition. \(\square\)

It is easy to check that nontrivial identities for \(B_n\) of length 2 and 3 do not exist. It follows that:

**Corollary 2.1.** For any \(n > 0\), the identity

\[
(A_{1,1}) \quad xy^2x \approx xyxy \quad (A'_{1,1}) \quad yx^2y \approx yxyx
\]

is the shortest nontrivial identity satisfied in the monoid \(B_n\).
Remark 2.1. The Adjan identity (I) and the identity (II) are both satisfied in $B_n$ since they are simple consequences of $(A'_{1,1})$ if $n > 0$:

$$(A'_{1,1}) \Rightarrow xy y^2 x y^2 x \approx xy y x y x y^2 x$$

that is (I);

$$(A'_{1,1}) \Rightarrow xy y^2 x y x^2 y \approx xy y x y x y^2 y$$

that is (II).

Remark 2.2. Example 4.4 of [3] sets that

$$xyx^i y x^{j-i} y^k x \approx xy x^j y x^{j-i} y^k x$$

$(0 \leq i < j < \ell$ and $k \geq 1)$

is an identity for $B_0$ if and only if $(k+1)(i+1) \geq \ell + 1 \geq 2(j+1)$. The problem gets a new look in the case $n > 0$. Using $(A'_{i,1})$ and $(A'_{j,1})$ we obtain the following two identities satisfied in $B_n$, $n > 0$:

$$xy x^i y x^{j-i} y^k x \approx xy x^j y x^{j-i} y^k x$$

and

$$xy x^i y x^{j-i} y^k x \approx xy x^j y x^{j-i} y^k x.$$

So, $xy x^i y x^{j-i} y^k x \approx xy x^j y x^{j-i} y^k x$ is an identity for $B_n$ (if $n > 0$) for any $i, j, k, \ell \in \mathbb{N}$ with $i, j \leq \ell$.

3 Main results

Unless otherwise indicated, we consider words $v$ (and identities) with $x$ the first letter and with $n_y(v) > 0$ (that is, words $v$ of the form $v = x^k u$, where $u$ is non-empty and $y$ is the first letter of $u$). We say that a word of the form

$$(*) \quad x^{\ell_1} (yx)^{\ell_2} z^{\ell_3}$$

(where $z \in \{x, y\}$, $\ell_1 > 0$ and $\ell_2, \ell_3 \geq 0$)

is a canonical form of the word $v$ (the words $(yx)^{\ell_2}$ and $z^{\ell_3}$ are the empty word if $\ell_2 = 0$ and $\ell_3 = 0$, respectively) if

$$v \approx x^{\ell_1} (yx)^{\ell_2} z^{\ell_3}$$

is an identity satisfied in $B_n$, $n > 0$ ($\ell_2$ can be 0 only if $\ell_3 > 0$ and $z = y$ since $n_y(v) > 0$).

Lemma 3.1. A canonical form of the word $v = x^k u$ ($y$ being the first letter of $u$), is given by

$$v \approx \begin{cases} 
  x^k (yx)^{n_y(u)} y^{n_y(u) - n_x(u)} & \text{if } n_y(u) \geq n_x(u) \\
  x^k (yx)^{n_y(u) - n_x(u)} & \text{if } n_y(u) < n_x(u)
\end{cases}.$$

Proof. A sequence of identities obtained by using (from left to right) only the identities $(A_{i,1})$ and $(A'_{i,1})$ (i.e., $xy^{i+1} x \approx xy x^i$ and $yx^{i+1} y \approx yxy^{i}$) leads us in the end to an identity for $B_n$ of the form

$$v \approx x^k y x y x \cdots y x z^m$$

$(m \geq 0)$,

where $z \in \{x, y\}$. It is clear that if $n_y(u) > n_x(u)$ then $z = y$ and the number of occurrences of $(yx)$ is $n_x(u)$. If $n_y(u) = n_x(u)$ then the number of occurrences of $(yx)$ is also $n_x(u)$. Since any identity for $B_n$ is balanced, it follows that $m = n_y(u) - n_x(u)$. Now, if $n_y(u) < n_x(u)$ then $z = x$ and the number of occurrences of $(yx)$ is $n_y(u)$. Obviously in this case $m = n_x(u) - n_y(u).$
Theorem 3.1. Let \( v \) and \( w \) be two words in the alphabet \( \{x, y\} \), \( v = x^k u \) and \( w = x^{k'} u' \) (\( y \) being the first letter of both words \( u \) and \( u' \)). Then the following statements are equivalent:

1. \( v \approx w \) is an identity satisfied in \( B_n, n > 0 \);
2. \( v \) and \( w \) have the same canonical form;
3. \( n_x(u) = n_x(u') \), \( n_y(u) = n_y(u') \) and \( k = k' \);
4. \( (v, w) \) is balanced and \( k = k' \).

Proof. (i) \( \iff \) (ii) If \( v \approx w \) have the same canonical form then obviously \( v \approx w \) is an identity satisfied in \( B_n \) if \( n > 0 \).

Conversely, if \( v \approx w \) is an identity for \( B_n, n > 0 \), and \( v \approx x^{\ell_1}(yx)^{\ell_2} z^{\ell_3} \), \( w \approx x^{\ell_1}(yx)^{\ell_2} z^{\ell_3'} \), are two canonical forms of \( v \) and \( w \) respectively, then we will prove that the two canonical forms are the same, that is: (1) \( \ell_1 = \ell_1', \ell_2 = \ell_2', \ell_3 = \ell_3' \), and (2) \( z = z' \) if \( \ell_3 = \ell_3' \neq 0 \).

Using the substitution \( \sigma_{1,1} \) by elements of \( B_n \) \( (n > 0) \) defined by \( x = a, y = b \),

\[
\sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3}) = a^{\ell_1} b^{\ell_2} z^{\ell_3} = a^{\ell_1} b^{\ell_2 + \ell_3} \quad \text{if } z = y
\]

and

\[
\sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3}) = a^{\ell_1} b^{\ell_2} z^{\ell_3} = a^{\ell_1} b^{\ell_2 + (n-1)\ell_3} \quad \text{if } z = x.
\]

Analogously,

\[
\sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3'}) = a^{\ell_1} b^{\ell_2 + \ell_3'} \quad \text{if } z' = y
\]

and

\[
\sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3'}) = a^{\ell_1} b^{\ell_2 + (n-1)\ell_3} \quad \text{if } z' = x.
\]

It is clear that \( \sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3}) = \sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3'}) \) implies

\[\ell_1 = \ell_1'.\]

Since any identity for \( B_n \) is balanced, it follows that

\[2\ell_2 + \ell_3 = 2\ell_2' + \ell_3'.\]

The equality \( \sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3}) = \sigma_{1,1}(x^{\ell_1}(yx)\ell_2 z^{\ell_3'}) \) implies also:

**Case 1.** \( z = z' = y \): \( n\ell_2 + \ell_3 = n\ell_2' + \ell_3' \), that is \( (n-2)(\ell_2' - \ell_2) = 0 \).

**Case 2.** \( z = z' = x \): \( n\ell_2 + (n-1)\ell_3 = n\ell_2' + (n-1)\ell_3' \), that is \( (n-2)(\ell_2' - \ell_2) = 0 \).

**Case 3.** \( z \neq z' \): if \( z = y \) and \( z' = x \) then \( n\ell_2 + \ell_3 = n\ell_2' + (n-1)\ell_3' \) implies \( 2n(\ell_2' - \ell_2) = 2\ell_3 - 2(n-1)\ell_3' \) and so, \( (n-2)(\ell_3 + \ell_3') = 0 \); analogously if \( z = x \) and \( z' = y \). Thus the hypothesis \( z \neq z' \) implies \( \ell_3 = \ell_3' = 0 \) if \( n \neq 2 \), and therefore \( z^{\ell_3} \) and \( z^{\ell_3'} \) are the empty word.
Taking into account that \( \sigma \) and \( \ell_3 \) are two words that \( \ell_2 = \ell_2, \ell_3 = \ell_3 \) if \( n \neq 2 \), and \( z = z' \) if \( \ell_3 = \ell_3 \neq 0 \) and \( n \neq 2 \) \((n > 0)\). The case \( n = 2 \) will be discussed below.

Let \( \sigma_{1,2} \) be the substitution by elements of \( B_2 \) defined by \( x = a, y = b^2 \). Then,

\[
\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2}a^{\ell_3} = a^{\ell_1}b^{3\ell_2+\ell_3} \quad \text{if } z = x
\]

and

\[
\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2}b^{2\ell_3} = a^{\ell_1}b^{3\ell_2+2\ell_3} \quad \text{if } z = y.
\]

Analogously,

\[
\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2+\ell_3} \quad \text{if } z' = x
\]

and

\[
\sigma_{1,2}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = a^{\ell_1}b^{3\ell_2+2\ell_3} \quad \text{if } z' = y.
\]

Taking into account that \( \ell_1 = \ell_1 \) and \( 2\ell_2 + \ell_3 = 2\ell_2 + \ell_3' \), the equality \( \sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) = \sigma_{1,1}(x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}) \) implies:

**Case 1.** \((z = z') = y)\: 3\ell_2 + \ell_3 = 3\ell_2 + \ell_3' \Rightarrow \ell_2 = \ell_2' \) (and therefore \( \ell_3 = \ell_3' \)).

**Case 2.** \((z = z') = x)\: 3\ell_2 + 2\ell_3 = 3\ell_2 + 2\ell_3' \Rightarrow \ell_2 = \ell_2' \) (and therefore \( \ell_3 = \ell_3' \)).

**Case 3.** \((z \neq z')\): if \( z = y \) and \( z' = x \) then \( 3\ell_2 + \ell_3 = 3\ell_2' + 2\ell_3' \Rightarrow \ell_2 = \ell_2' + \ell_3' \) and so

\[
2\ell_2' + \ell_3' = 2(\ell_2' + \ell_3') + \ell_3, \quad \text{that is } \ell_3' + \ell_3 = 0 \quad \text{and therefore } \ell_3' = \ell_3 = 0 \quad \text{(analogously if } z = x \text{ and } z' = y).
\]

Thus, if \( v \approx w \) is an identity for \( B_n, n > 0 \), and \( v \approx x^{\ell_1}(yx)^{\ell_2}z^{\ell_3}, w \approx x^{\ell_1}(yx)^{\ell_2}z^{\ell_3} \), are two canonical forms of \( v \) and \( w \) respectively, then the two canonical forms coincide.

\((ii) \Leftrightarrow (iii) \) follows from Lemma 3.1.

\((iii) \Leftrightarrow (iv) \) holds obviously.

**Remark 3.1.** Given two different words \( v \) and \( w \), if \( x^k \) \((k > 0)\) is the leftmost subword of the maximal length of both words \( v \) and \( w \) consisting of repetitions of \( x \), \( n_y(v) = n_y(w) = \ell > 1 \) and \( n_x(v) - k = n_x(w) - k = m > 0 \) then, and only then, \( v \approx w \) is a nontrivial identity for \( B_n \) \((n > 0)\). So, a triple of positive integers \((k, l, m)\), \( l > 1 \), determine a set of words and thus a set of nontrivial identities. For example, the triple of positive integers \((4, 2, 2)\) determine the set of words

\[
\{x^4y^2x^2, x^4yx^2y, x^4xyyx\}
\]

and the set of nontrivial identities

\[
\{x^4y^2x^2 \approx x^4yx^2y, x^4yx^2y \approx x^4xyyx, x^4y^2x^2 \approx x^4yx^2\}.
\]

Taking into account all possible cases, we conclude that
Theorem 3.2. The identities partition \( \mathcal{P}_{B_n} \) \((n > 0)\) is given by

\[
\mathcal{P}_{B_n} = \{P_{k,l,m}\}_{k,l>0,m\geq 0} \cup \{P_{k,0,0}\}_{k>0},
\]

where

\[
P_{k,l,m} = \{x^k u \mid \text{the first letter of } u \text{ is } y, \ n_y(u) = l \text{ and } n_x(u) = m\}
\]

if \(k, l > 0, m \geq 0\), and \(P_{k,0,0} \ (k > 0)\) are the singletons \(\{x^k\}\). The elements of this partition are finite sets and if \(k, l > 0, m \geq 0\), then

\[
|P_{k,l,m}| = \binom{l + m - 1}{l - 1}.
\]

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