YET ANOTHER WAY OF CALCULATING MOMENTS OF THE KESTEN’S DISTRIBUTION AND ITS CONSEQUENCES FOR CATALAN NUMBERS AND CATALAN TRIANGLES

PAWEL J. SZABŁOWSKI

Abstract. We calculate moments of the so-called Kesten distribution by means of the expansion of the denominator of the density of this distribution and then integrate all summands with respect to the semicircle distribution. By comparing this expression with the formulae for the moments of Kesten’s distribution obtained by other means, we find identities involving polynomials whose power coefficients are closely related to Catalan numbers, Catalan triangles, binomial coefficients. Finally, as applications of these identities we obtain various interesting relations between the aforementioned numbers, also concerning Lucas, Fibonacci and Fine numbers.

1. Introduction

The purpose of this note is to calculate a sequence of moments of the Kesten’s distribution and thus, by comparison with the existing formulae, to obtain some polynomial type identities involving Catalan and some other sequences of numbers related to them (see Proposition 1). In 2015 in [7] and in 2020 in [8] Szabłowski calculated in two different ways the moments of Kesten distribution. Later T. Hasegawa and S. Saito in [2] evaluated these moments in some other ways and, equating the results, they found interesting identities involving Catalan and related numbers. So in this note, we will calculate these moments in yet another way and obtain some other identities, involving, surprisingly, other important numbers sequences like Fibonacci, Lucas and Fine numbers.

2. Basic Ingredients

We start with the modified semicircle distribution, i.e., distribution with the density

\[ f_S(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2} & \text{if } |x| \leq 2 \\ 0 & \text{if } |x| > 2 \end{cases} \]
It is well known that the famous Catalan numbers are the moment sequence of \( f_S \).
More precisely, we have
\[
\int_{-2}^{2} x^{2n} f_S(x) dx = C_n = \frac{1}{n+1} \binom{2n}{n}.
\]
We also know (see, e.g., [3] (4.8 p. 107)) that, after inserting proper values of parameters, the moment generating function of this distribution is equal to
\[
g_S(z) = \frac{2}{\sqrt{1 - 4z^2 + 1}},
\]
for \(|z| \leq 1/2\). One can also easily notice, that
\[
\sum_{k \geq 0} t^k C_k = \frac{2}{\sqrt{1 - 4t + 1}},
\]
which is valid for \(|t| \leq 1/4\), as the evaluation of the convergence radius and the study of the convergence interval show, thanks to the relation:
\[
\binom{2n}{n}/4^n \nrightarrow +\infty \quad n \rightarrow +\infty \quad O(n^{-1/2}).
\]

The other ingredient is the definition of the Kesten distribution. It was considered in many papers including [4], [7], [8], and recently in [2] with different parametrization. Let us consider the Kesten distribution parametrized basically as in [2] with parameter \( q \) replaced by \( r \), i.e., with the following density:
\[
f_K(x|p, r) = \begin{cases} 
0 & \text{if } |x| > 2\sqrt{r} \\
\frac{p \cdot \sqrt{4r - x^2}}{2\pi \left( p^2 - (p-r)x^2 \right)} & \text{if } |x| \leq 2\sqrt{r} 
\end{cases},
\]
for \( 0 < p \leq 2r \).

We point out that, if \( p = r \), then
\[
f_K(x|r, r) = \frac{1}{\sqrt{r}} f_S(x/\sqrt{r}).
\]

3. Main results

Thus, we have
\[
\int_{-2\sqrt{r}}^{2\sqrt{r}} x^{2n} f_K(x|p, r) dx = r^n C_n = \frac{r^n}{n+1} \binom{2n}{n}.
\]
Now notice that for
\[
\left| \frac{p - r}{p^2} \right| x^2 \leq \left| \frac{p - r}{p^2} \right| 4r < 1,
\]
we have the following expansion:
\[
f_K(x|p, r) = \frac{r \cdot \sqrt{4r - x^2}}{2\pi r^2} \sum_{k \geq 0} \left( \frac{p - r}{p^2} \right)^k x^{2k}.
\]
When we consider \( |x| \leq 2\sqrt{r} \) and parameters satisfying \( (3.1) \), the series for \( f_K(x|p, r) \) is uniformly convergent and we can integrate term by term obtaining
\[
M_{2m}(p, r) = \int_{-2\sqrt{r}}^{2\sqrt{r}} x^{2m} f_K(x|p, r) dx = \frac{r}{p} \sum_{k \geq 0} \left( \frac{p - r}{p^2} \right)^k r^{k+m} C_{k+m}.
\]
Let us introduce a new auxiliary variable

\[(3.3) \quad t = \frac{r}{p}.\]

General conditions on \(p\) and \(r\) require that \(t \geq 1/2\). From inequalities (3.1), we must also have \(|t(1-t)| < 1/4\), however taking into account (2.3), we can notice that (3.2) converges also for \(|t(1-t)| = 1/4\). This leads to the following condition

\[(3.4) \quad 1/2 \leq t \leq (1 + \sqrt{2})/2.\]

Summarizing, we get the following result.

**Theorem 1.** Let the Kesten distribution be defined by the density \(f_K\) given by (2.4). Let the real positive parameters \(p\) and \(r\) satisfy the following relationship

\[2r \geq p \geq 2(\sqrt{2} - 1)r,\]

then we have

\[(3.5) \quad M_{2m}(p, r) = \frac{p^m}{(1-t)^m} \left(1 - \sum_{k=0}^{m-1} t^{k+1} (1-t)^k C_k\right),\]

with \(t\) given by (3.3), with an obvious condition \(|t(1-t)| \leq 1/4\) and \(t \neq 1\). For \(t = 1\) we have \(M_{2m}(r, r) = r^m C_m\), because of (2.1) and (2.5).

**Proof.** Keeping in mind that \(r/p = t\), let us analyze first when the series (3.2) is convergent. Namely, it is absolutely convergent if

\[|t(1-t)| \leq \frac{1}{4},\]

because of (2.3), that is, when

\[-\frac{1}{4} \leq t - t^2 \leq \frac{1}{4}.\]

The inequality \(\frac{1}{p} - \left(\frac{1}{p}\right)^2 \leq \frac{1}{4}\) is equivalent to the following \((\frac{1}{2} - \frac{1}{p})^2 \geq 0\) which is always true when \(p \neq 0\), in particular for all \(p\) and \(r\) such that \(2r \geq p > 0\). The second one leads to inequality

\[t^2 - t - \frac{1}{4} \leq 0,\]

which gives \(\frac{1}{2} - \sqrt{2} \leq t \leq \frac{1}{2} + \sqrt{2}\), a condition that is always satisfied when \(2r \geq p \geq 2(\sqrt{2} - 1)r > 0\).

From the ordinary generating function (2.2) of Catalan numbers, since \(|t(1-t)| \leq 1/4\), we also have the following identity:

\[\sum_{n \geq 0} t^n (1-t)^n C_n = \frac{1}{t},\]
Using this identity, since $|t(1-t)| \leq 1/4$, from (3.2) when $t \neq 1$ we find

$$M_{2m}(p, r) = \frac{r}{p} \frac{p^{2m}}{(p-r)^m} \sum_{k \geq 0} t^{k+m}(1-t)^{k+m} C_{k+m} =$$

$$= \frac{r}{p} \frac{p^m}{(1-t/p)^m} \left( \sum_{k \geq 0} t^k(1-t)^k C_k - \sum_{k=0}^{m-1} t^k(1-t)^k C_k \right) =$$

$$= p^m \frac{t}{(1-t)^m} \left( \frac{1}{t} - \sum_{k=0}^{m-1} t^k(1-t)^k C_k \right).$$

Now it suffices to multiply the expression in round brackets by $t$. □

**Remark 1.** Formula (3.3) can be derived from the unnumbered formula

$$M_{2m} = \frac{p}{p-q} \left( \frac{p^{2k-1}}{(p-q)^k} - \sum_{m=1}^{k} \frac{2m-1}{m} q^m \frac{p^{2(k-m)}}{(p-q)^{k-m}} \right)$$

placed in Comment 1 of [2] thanks to equality

$$C_m = \frac{1}{m} \left( \frac{2m-2}{m-1} \right) = \frac{1}{2m-1} \left( \frac{2m-1}{m} \right)$$

and keeping in mind that in our notation $q = r$, $p \neq r$ and $t = r/p$. Indeed we have

$$M_{2k} = \frac{p}{p-r} \left( \frac{p^{2k-1}}{(p-r)^k} - \sum_{m=1}^{k} \frac{2m-1}{m} p^m \frac{p^{2(k-m)}}{(p-r)^{k-m}} \right) =$$

$$= \frac{p}{p-r} \left( \frac{p^{2k-1}}{(p-r)^k} - \sum_{j=0}^{k-1} C_j p^{j+1} \frac{p^{2(k-j-1)}}{(p-r)^{k-j-1}} \right) =$$

$$= \frac{p^{2k}}{(p-r)^k} \left( 1 - \sum_{j=0}^{k-1} p p^j \left( \frac{p-r}{p^2} \right)^j \right).$$

This calculation was done by the referee in his report.

Let us underline the important property of the sequences that we are considering, namely, that the sequences :

$$\{M_{2m}\}_{m \geq 0}, \left\{ \frac{1}{(1-t)^m} \left( 1 - \sum_{k=0}^{m-1} t^{k+1}(1-t)^k C_k \right) \right\}_{m \geq 0}$$

and also the sequences for $d \in [0, 1]$

$$\left\{ 1 - d \sum_{k=0}^{m-1} t^{k+1}(1-t)^k C_k \right\}_{m \geq 0},$$

are moment sequences. The last statements follow from the fact that the product of two moment sequences is another moment sequence and that the convex combination of two moment sequences is another moment sequence. For a brief recollection of facts concerning the moment sequences, see e. g., [10] or Appendix in [9].
Remark 2. As a matter of honesty, the author was able to see the first version of the paper of Hasegawa and Saito. In this version, there were not present Comment 1 and Comment 2. It was in May and June 2021. Their paper has inspired the author to write this note. To clarify everything, the final form of the paper appeared while in Comment 2 the expansion appeared. In the third comment, the authors stated that these formulae are promising and that they will research further on these formulae. Anyway, the Remark indicates that to get the crucial formula one did not need to exploit the new way of calculating even moments of Kesten distribution.

Let us now compare this result with known formulae for the moments of Kesten distribution.

Let us notice that we have the following equality
\[ f_{CN}(x|0, \rho, 0) = f_K(x\sqrt{r}, p, r)\sqrt{r} \]

involving the distribution \( f_{CN}(x|y, \rho, q) \) considered in [7] with \( y = q = 0 \) and \( \rho^2 = 1 - p/r \) and the distribution defined in (2.4). Thanks to this equality and to Proposition 3 part (i) in [7], we find
\[ M_{2m}(p, r) = r^m \sum_{k=0}^{m} \left( \frac{p}{r} - 1 \right)^{m-k} S_{m,k}, \]

where
\[ S_{m,k} = \binom{2m}{k} - \binom{2m}{k-1}, \]

with the understanding that \( \binom{2m}{-1} = 0 \). Hence in terms of \( t = r/p \) and \( t \neq 1 \), we have:

\[ M_{2m}(p, r) = p^m \sum_{k=0}^{m} \left( -r + 1 \right)^{m-k} \left( \frac{r}{p} \right)^k S_{m,k} = p^m \sum_{k=0}^{m} t^k (1 - t)^{m-k} S_{m,k}. \]

In [2] two other expressions for the moments of Kesten distribution can be found. Namely the following formulae:

\[ M_{2m}(p, r) = p \sum_{j=0}^{m-1} p^{m-1-j} r^j T_{m-1,j}, \]

\[ M_{2m}(p, r) = p \sum_{j=0}^{m-1} (p - r)^j r^{m-1-j} B_{m,j+1}, \]

where numbers \( T_{m,j} \) and \( B_{m,j} \) are called Catalan triangles, depending on the author. The numbers \( T_{m,j} \) are defined as
\[ T_{m,j} = \frac{m - j + 1}{m+1} \binom{m+j}{m}, \]

for integers \( m, j \) such that \( m \geq j \geq 0 \) and the related sequence is A009766 in Sloane’s Encyclopedia [5], the numbers \( B_{k,j} \), introduced by L.W. Shapiro in [6] and with related sequence A039598 in OEIS [5], are given by:
\[ B_{k,j} = \frac{j}{k} \binom{2k}{k-j}, \]
where the integers \( k, j \) satisfy \( k \geq j \geq 1 \). Comparing formulae (3.6), (3.7) and (3.8) we find the following result.

**Proposition 1.** i) For all \( m \geq 1 \) and \( t \in \mathbb{C} \) we obtain:

\[
(1 - t)^m \sum_{k=0}^{m} S_{m,k} t^k (1 - t)^{m-k} = (1 - t)^m \sum_{k=0}^{m-1} T_{m-1,k} t^k = (1 - t)^{m-1} \sum_{k=0}^{m-1} B_{m,k} t^k.
\]

\[
(3.9)
\]

\[
= (1 - t)^m \sum_{k=0}^{m-1} B_{m,k+1} (1 - t)^k t^{m-1-k} = 1 - \sum_{k=0}^{m-1} C_k t^{k+1} (1 - t)^k.
\]

\[
(3.10)
\]

ii) For all \( m \geq 1 \) and \( x \in \mathbb{C} \) we get:

\[
\sum_{k=0}^{m-1} S_{m,k} x^k = (x + 1) \sum_{k=0}^{m-1} B_{m,k+1} x^{m-1-k} =
\]

\[
(3.11)
\]

\[
= \sum_{k=0}^{m-1} T_{m-1,k} x^k (x+1)^{m-k} = (1+x)^{2m} - \sum_{k=0}^{m-1} C_k x^{k+1} (x+1)^{2m-2k-1}.
\]

\[
(3.12)
\]

**Proof.** i) Firstly, for \( t \) satisfying (3.4) we have from (3.5) and (3.6) the following equalities, which are true also when \( t = 1 \):

\[
\frac{(1-t)^m}{p^m} M_{2m} = (1 - t)^m \sum_{k=0}^{m} S_{m,k} t^k (1 - t)^{m-k} = 1 - \sum_{k=0}^{m-1} t^{k+1} (1 - t)^k C_k.
\]

Now with parametrization \( t = r/p \) formulae (3.7) and (3.8) become:

\[
\frac{(1-t)^m}{p^m} M_{2m} = (1 - t)^m \sum_{k=0}^{m-1} T_{m-1,k} t^k = (1 - t)^m \sum_{k=0}^{m-1} B_{m,k+1} t^{m-1-k} (1 - t)^k.
\]

Therefore, we obtain the chain of equalities given by (3.9) and (3.10). Finally, we observe that all these equalities involve polynomials in \( t \), so we extend their domain from any segment to all complex numbers and conclude that they hold for all \( t \in \mathbb{C} \).

ii) Having proved i) we consider \( x = t/(1-t) \), with \( t \neq 1 \). Then \( t = x/(x+1) \) and we consider the identities (3.9) and (3.11) for \( x \neq -1 \). Now we multiply both sides of each of them by \((1+x)^{2m}\). We get immediately forms (3.11) and (3.12). Now again we deal with polynomials hence we can drop assumption that \( x \neq -1 \). □

### 4. Applications

The equalities proved in Proposition 1 could have many useful applications. Thanks to them, we can find relationships between Catalan numbers, Catalan triangles, binomial coefficients and other special numbers like, e.g., Fibonacci and Lucas or Fine numbers. Indeed, in the next two examples we show some interesting identities obtained respectively from formulae (3.9), (3.10) and formulae (3.11), (3.12).

**Example 1.** Let us consider formulae (3.9) and (3.10), evaluating them for some special values of \( t \in \mathbb{C} \) we find other interesting identities.
a) For $t = 2$, $1 - t = -1$ and $t/(1-t) = -2$ and finally we get for all $m \geq 1$:

$$1 - 2 \sum_{k=0}^{m-1} (-2)^k C_k = \sum_{k=0}^{m} (-2)^k S_{m,k} =$$

$$= - \sum_{k=0}^{m-1} (-2)^{m-1-k} B_{m,k+1} = (-1)^m \sum_{k=0}^{m-1} 2^k T_{m-1,k}.$$ 

b) For $t = e^{i\pi/3}$, $1 - t = e^{-i\pi/3}$ and $t/(1-t) = e^{2i\pi/3}$ and finally we get for all $m \geq 1$:

$$1 - e^{i\pi/3} \sum_{k=0}^{m-1} C_k = e^{-im\pi/3} \sum_{k=0}^{m-1} e^{ik\pi/3} T_{m-1,k} =$$

$$= e^{-2im\pi/3} \sum_{k=0}^{m} e^{2ik\pi/3} S_{m,k} = e^{-i(2m-1)\pi/3} \sum_{k=0}^{m-1} e^{2(1-k)\pi/3} B_{m,k+1}.$$ 

From these formulae considering real parts we find:

$$1 - \frac{1}{2} \sum_{k=0}^{m-1} C_k = \sum_{k=0}^{m-1} \cos((m-k)\pi/3) T_{m-1,k} =$$

$$= \sum_{k=0}^{m} \cos(2(m-k)\pi/3) S_{m,k} = \sum_{k=0}^{m-1} \cos((2k+1)\pi/3) B_{m,k+1},$$

while, when considering imaginary parts, we get:

$$\sum_{k=0}^{m-1} C_k = \frac{2\sqrt{3}}{3} \sum_{k=0}^{m-1} \sin((m-k)\pi/3) T_{m-1,k} =$$

$$= \frac{2\sqrt{3}}{3} \sum_{k=0}^{m} \sin(2(m-k)\pi/3) S_{m,k} = \frac{2\sqrt{3}}{3} \sum_{k=0}^{m-1} \sin((2k+1)\pi/3) B_{m,k+1}.$$ 

We observe that the following functions of $m \cos(m\pi/3)$, $\cos(2m\pi/3)$, $\frac{2\sqrt{3}}{3} \sin(m\pi/3)$ and $\frac{2\sqrt{3}}{3} \sin(2m\pi/3)$ are periodic with periods equal to 6. Moreover, we have: $\cos(m\pi/3) \in \{\pm1/2, \pm1\}$ and $\frac{2\sqrt{3}}{3} \sin(m\pi/3) \in \{0, \pm1\}$.

c) For $t = (1 + \sqrt{5})/2$ and $1 - t = (1 - \sqrt{5})/2$ and thanks to the following relations involving Fibonacci numbers $F_n$ and Lucas numbers $L_n$ (respectively in OEIS A000045 and A000032):

$$\left(\frac{1 + \sqrt{5}}{2}\right)^k = \frac{L_n}{2} + \frac{\sqrt{5}F_n}{2},$$

$$\left(\frac{1 - \sqrt{5}}{2}\right)^k = \frac{L_n}{2} - \frac{\sqrt{5}F_n}{2}.$$
we get for all $m \geq 1$:

$$\frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m} S_{m,k} \frac{L_k + F_k \sqrt{5} L_{m-k} - F_{m-k} \sqrt{5}}{2} =$$

$$= \frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m-1} T_{m-1,k} \frac{L_k + F_k \sqrt{5}}{2} =$$

$$= \frac{L_m - F_m \sqrt{5}}{2} \sum_{k=0}^{m-1} B_{m,k+1} \frac{L_k - F_k \sqrt{5} L_{m-1-k} + F_{m-1-k} \sqrt{5}}{2} =$$

$$= 1 - \frac{L_1 + F_1 \sqrt{5}}{2} \sum_{k=0}^{m-1} C_k (-1)^k.$$

Now we divide all sides by $\frac{L_m - F_m \sqrt{5}}{2}$ getting $\frac{1}{(\frac{L_m - F_m \sqrt{5}}{2})} = (-1)^m (\frac{L_m + F_m \sqrt{5}}{2})$ and we perform the calculations inside the sums using also the following identities:

$$L_n F_k = F_{n+k} + (-1)^k F_{k-n}, \quad F_{-k} = (-1)^{k-1} F_k,$$

$$L_n L_k - 5 F_n F_k = (-1)^k 2 L_{n-k}, \quad L_{-n} = (-1)^n L_n.$$

Finally, we equate firstly the terms free from the factor $\sqrt{5}$ obtaining

$$\sum_{k=0}^{m} (-1)^k S_{m,k} L_{m-2k} = \sum_{k=0}^{m-1} T_{m-1,k} L_k = \sum_{k=0}^{m-1} (-1)^k B_{m,k+1} L_{m-2k-1} =$$

$$= (-1)^m L_m + (-1)^{m+1} L_{m+1} \sum_{k=0}^{m-1} (-1)^k C_k \quad (4.1)$$

and then the ones multiplied by $\sqrt{5}$ finding

$$\sum_{k=0}^{m} (-1)^k S_{m,k} F_{m-2k} = \sum_{k=0}^{m-1} T_{m-1,k} F_k = \sum_{k=0}^{m-1} (-1)^k B_{m,k+1} F_{m-1-2k} =$$

$$= (-1)^m F_m + (-1)^{m+1} F_{m+1} \sum_{k=0}^{m-1} (-1)^k C_k \quad (4.3)$$

$$= (-1)^m F_m + (-1)^{m+1} F_{m+1} \sum_{k=0}^{m-1} (-1)^k C_k =$$

$$= (-1)^m F_m + (-1)^{m+1} F_{m+1} \sum_{k=0}^{m-1} (-1)^k C_k \quad (4.4)$$

Example 2. Let us consider identities (3.11) and (3.12), we find some interesting identities using different values of $x$.

a) When $x = 1$ we get for $m \geq 1$:

$$\sum_{k=0}^{m} S_{m,k} = 2 \sum_{k=0}^{m-1} B_{m,k+1} = \sum_{k=0}^{m-1} T_{m-1,k} 2^{m-k} =$$

$$= 4^m - \sum_{k=0}^{m-1} C_k 2^{2m-2k-1},$$
b) When \( x = -1/2 \) after multiplying all sums by \( 2^m \), we find for \( m \geq 1 \):

\[
\begin{align*}
(4.5) \quad & (-1)^m \sum_{k=0}^{m} S_{m,k}(-2)^{m-k} = (-1)^{m-1} \sum_{k=0}^{m-1} B_{m,k+1}(-2)^k = \\
(4.6) \quad & = \sum_{k=0}^{m-1} (-1)^k T_{m-1,k} = \frac{1}{2^m} \sum_{k=0}^{m-1} (-1)^{k+1} C_k \frac{1}{2^{m-k}}.
\end{align*}
\]

c) Let us divide both sides of (3.11) and (3.12) by \( (x+1) \), if we pass to the limit \( x \to -1 \), taking into account that

\[
\lim_{x \to -1} \frac{x^n - (-1)^n}{x+1} = (-1)^{n-1} n,
\]

and that for \( m \geq 1 \)

\[
\sum_{j=0}^{m} S_{m,k}(-1)^k = 0,
\]

we obtain for all \( m \geq 1 \)

\[
\lim_{x \to -1} \frac{\sum_{k=0}^{m} S_{m,k}x^k}{x+1} = \sum_{k=0}^{m-1} (-1)^{k+1} S_{m,k} = \sum_{k=0}^{m-1} (-1)^{m-1-k} B_{m,k+1} = (-1)^{m-1} T_{m-1,m-1} = (-1)^{m-1} C_{m-1}.
\]

Remark 3. From Grimaldi’s monograph [1] (p. 285) we find, that Fine numbers \( \Phi_n \) are given by the relationship:

\[
\Phi_n = -\frac{1}{2} \Phi_{n-1} + \frac{1}{2} C_n,
\]

for \( n \geq 1 \) with \( \Phi_0 = 1 \). Using well-known formulae for the solutions of discrete, non-homogeneous, linear difference equations we immediately find that

\[
\Phi_n = \frac{1}{2} \left( \frac{-1}{2} \right)^n + \frac{1}{2} \sum_{j=0}^{n} C_j \left( \frac{-1}{2} \right)^{n-j}.
\]

Now, notice, that the previous formula is in direct relation with formulae (4.5) and (4.6). Indeed we have

\[
\frac{1}{2^n} \sum_{k=0}^{m-1} (-1)^{k+1} C_k \frac{1}{2^{m-k}} = (-1)^{m-1} \Phi_{m-1}.
\]

Hence, dividing every part of the equation (4.5) by \( (-1)^{m-1} \) and renaming the indexes, we obtain the following expressions for the Fine numbers:

\[
\Phi_n = \sum_{j=0}^{n} B_{n+1,j+1}(-2)^j = \sum_{j=0}^{n} T_{n,j}(-1)^{n-j} = -\sum_{j=0}^{n+1} S_{n+1,j}(-2)^{n+1-j}.
\]
References

[1] Grimaldi, Ralph P. Fibonacci and Catalan numbers. An introduction. *John Wiley & Sons, Inc., Hoboken, NJ*, 2012. xiv+366 pp. ISBN: 978-0-470-63157-7 MR2963306

[2] Hasegawa, Takehiro; Saito, Seiken, A note on the moments of the Kesten distribution. *Discrete Math.* 344 (2021), no. 10, Paper No. 112524, 10 pp. MR4283208

[3] Hora, Akihito; Obata, Nobuaki. Quantum probability and spectral analysis of graphs. With a foreword by Luigi Accardi. *Theoretical and Mathematical Physics*. Springer, Berlin, 2007. xviii+371 pp. ISBN: 978-3-540-48862-0 MR2316893

[4] Kesten, Harry. Symmetric random walks on groups. *Trans. Amer. Math. Soc.* 92 (1959), 336–354. MR0109367

[5] Sloane, N., J., A. The on-line Encyclopedia of Integer Sequences, https://oeis.org.

[6] Shapiro, L. W. A Catalan triangle. *Discrete Math.* 14 (1976), no. 1, 83–90. MR0387069

[7] Szabowski, Paweł J. Moments of $q$-normal and conditional $q$-normal distributions. *Statist. Probab. Lett.* 106 (2015), 65–72. MR3389972

[8] Szabowski, Paweł J. On the generalized Kesten–McKay distributions. *ESAIM Probab. Stat.* 24 (2020), 56–68. MR4069296, ArXiv: [http://arxiv.org/abs/1507.03191](http://arxiv.org/abs/1507.03191)

[9] Paweł J. Szabowski, On positivity of orthogonal series and its applications in probability, in print in Positivity, https://arxiv.org/abs/2011.02710.

[10] Sokal, Alan D. The Euler and Springer numbers as moment sequences. *Expo. Math.* 38 (2020), no. 1, 1–26. MR40982303

Emeritus in Department of Mathematics and Information Sciences, Warsaw University of Technology, ul Koszykowa 75, 00-662 Warsaw, Poland