POSETTED TREES AND BAKER-CAMPBELL-HAUSDORFF PRODUCT

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Abstract. We introduce the combinatorial notion of possetted trees and we use it in order to write an explicit expression of the Baker-Campbell-Hausdorff formula.

1. Introduction

If $a, b$ are continuous operators on a Hilbert space, we may write

$$e^a e^b = e^{a \bullet b}, \quad a \bullet b = a + b + \sum_{n=2}^{\infty} w_n(a, b),$$

where $w_n$ is a universal, non commutative, homogeneous polynomial of degree $n$ with rational coefficients. The product $\bullet$ is called, after [1, 2, 8], Baker-Campbell-Hausdorff (BCH) product: it is associative and the BCH theorem asserts that every polynomial $w_n$ is a Lie element, i.e., is a linear combination of nested commutators. However, the proof of the BCH theorem does not give directly an explicit description of $w_n$ as a Lie element; moreover, such description is not unique in view of the Jacobi identity.

The most famous explicit expression of $a \bullet b$, in terms of nested commutators, is probably the one due to E. Dynkin (see [4, Equation 1] or [3, Equation 1.7.3]):

\begin{equation}
  a \bullet b = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{p_1,q_1,\ldots,p_n,q_n} \frac{1}{p_1! q_1! \cdots p_n! q_n!} ad(a)^{p_1} ad(b)^{q_1} \cdots ad(a)^{p_n} ad(b)^{q_n-1} b,
\end{equation}

where $ad(x) = [x, -]$ is the adjoint operator, the second sum is over all possible combinations of $p_1, q_1, \ldots, p_k, q_k \in \mathbb{N}$ such that $p_i + q_i > 0$, for $i = 1, \ldots, k$, and $\sum_{i=1}^{k} (p_i + q_i) = n$.

The literature about Baker-Campbell-Hausdorff formula is huge. For instance: in 1998, V. Kathotia [9] derived a trees summation expression for the BCH product over the real numbers using M. Kontsevich’s universal formula for deformation quantization of Poisson manifolds; the coefficient of this formula are certain integrals on configuration spaces and it is still unknown if they are rational numbers. In the papers [5] and [6], the authors recognize the equation $a \bullet b \bullet c = 0$ as the Maurer-Cartan equation of the canonical $L_{\infty}$ structure on the conormalized complex of singular cochains, on the standard two dimensional simplex with values in a Lie algebra. Therefore, the possibility of an explicit description of $a \bullet b$, again as a trees summation formula, by using the standard tools of homological perturbation and homotopy transfer theory [11]. The reader may also consult [16] for a list of explicit and recursive formulas.

The aim of this paper is to give a simple and elementary combinatorial description of the polynomial $w_n$ that uses some notions about planar rooted trees. The necessary combinatorial background is summarized in Sections 2 and 3. In particular, for every finite planar rooted tree $\Gamma$, the set of its leaves admits a total ordering (from left to right) and also a partial ordering $\preceq$, which takes care of the position of leaves with respect to the subroots. Then, we define a possetted tree as a finite planar rooted tree, whose leaves are labelled by elements on a partially ordered set (a poset), monotonically with respect to $\preceq$.

Our main result (Theorem [11]) gives an explicit description of every $w_n$ as a linear combination with rational coefficients of nested commutators, indexed by a certain set of possetted trees with $n$ leaves. The

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formula of the coefficients involves the Bernoulli numbers and is completely described in terms of the combinatorial data of posseted trees.

2. Subroots of planar rooted trees

This section is devoted to introduce the notion, already known in the parallel logic programming community [14], of subroots of a planar rooted trees.

Recall that a tree is called a rooted tree if one vertex has been designated the root. Every rooted tree has a natural structure of directed tree such that, for every vertex $u$, there exists a unique directed path from $u$ to the root. We shall write $u \rightarrow v$ if the vertex $v$ belongs to the directed path from $u$ to the root. A leaf is a vertex without incoming edges: equivalently, a vertex $u$ is a leaf if the relation $v \rightarrow u$ implies $u = v$. A vertex is called internal if it is not a leaf; notice that, if a rooted tree has at least two vertices, then the root is an internal vertex.

Figure 1. A rooted tree, with $v \rightarrow u$.

From now on, we consider only planar rooted trees; following [12], we denote by $\mathcal{T}$ the set of finite planar rooted trees with the root at the top and the leaves at the bottom (i.e., every directed path moves upward), and such that every internal vertex has at least two incoming edges.

We also write

$$\mathcal{T} = \bigcup_{n>0} T_n,$$

where $T_n$ is the set of planar rooted trees with $n$ leaves and, for every $\Gamma \in \mathcal{T}$, we denote by $L(\Gamma)$ the set of leaves of $\Gamma$. The planarity of the tree gives, for every internal vertex $v$, a total ordering of the edges ending in $v$, from the leftmost to the rightmost (see Figure 2).

Figure 2. An element of $T_{12}$.

Definition 1. A rightmost branch of a planar rooted tree $\Gamma \in \mathcal{T}$ is a maximal connected subgraph $\Omega \subseteq \Gamma$, with the property that every edge of $\Omega$ is a rightmost edge of $\Gamma$. A rightmost branch is called non trivial if it has at least two vertices.
Definition 2. A **local rightmost leaf** is a leaf lying on a non trivial rightmost branch. Given an internal vertex \( v \), we call \( m(v) \) the leaf lying on the rightmost branch containing \( v \). We also denote by \( d(v) \) the distance between \( v \) and \( m(v) \), as defined in [13].

Definition 3. A **subroot** is the vertex of a non trivial rightmost branch which is nearest to the root. The set of subroots of a finite planar rooted tree \( \Gamma \) will be denoted by \( R(\Gamma) \).

Therefore, we have the natural bijections

\[
\{ \text{subroots} \} \cong \{ \text{non trivial rightmost branches} \} \cong \{ \text{local rightmost leaves} \}.
\]

Example 4. In the tree of Figure 4 the subroots are the vertices \( r, a, c \) and \( e \); the rightmost leaves are the leaves 2, 3, 5 and 7. Moreover, \( m(a) = 3, m(c) = 2, m(e) = 5 \) and \( m(r) = m(b) = m(f) = 7 \); and \( d(r) = 3, d(b) = 2 \) and \( d(a) = d(c) = d(e) = d(f) = 1 \).

A planar rooted tree \( \Gamma \in T \) is a **binary tree** if every internal vertex has exactly two incoming edges. We use the notation

\[
B = \bigcup_{n>0} B_n \subset T,
\]

where \( B_n \) is the set of planar binary rooted trees with \( n \) leaves. Using the notion introduced above, it is very easy to see that a tree \( \Gamma \in T_n \) is a binary tree if and only if it satisfies the equality:

\[
\sum_{v \in R(\Gamma)} d(v) = n - 1.
\]

Let \( R \) be a (non associative) algebra over a field \( \mathbb{K} \) and \( \Gamma \in B \) a planar rooted tree. Labelling the leaves of \( \Gamma \) with elements of \( R \), we can associate the product element in \( R \) obtained by the usual operadic rules [11, 12], i.e., we perform the product of \( R \) at every internal vertex in the order arising from the planar structure of the directed tree. For instance, the following labelled tree
gives the product \(((r_1r_2)r_3)((r_4r_5)(r_6r_7))\) \(\in R\).

Given any map \(f : L(\Gamma) \to R\) (the labelling), we denote by \(Z_\Gamma(f) \in R\) the corresponding product element.

If \(S \subset R\), then the elements \(Z_\Gamma(f)\), with \(\Gamma \in B\) and \(f : L(\Gamma) \to S\), are a set of generators of the subalgebra generated by \(S\). If \(R\) is either symmetric or skewsymmetric (e.g., a Lie algebra), then we may reduce the set of generators by a suitable choice of the labelling. Keeping in mind our main application (the BCH product), a possible way of doing that is by introducing the combinatorial notion of posetted trees.

3. Posetted trees

Using the notion of subroot, we can define a partial order \(\preceq\) on the set of leaves \(L(\Gamma)\).

**Definition 5.** Given two leaves \(l_1\) and \(l_2\) in a tree \(\Gamma \in T\), we say \(l_1 \preceq l_2\) if \(l_1 = l_2\) or there exists a subroot \(v \in R(\Gamma)\) such that \(l_2 = m(v)\) and \(l_1 \to v\).

**Figure 5.** Here, we have \(l_1 \preceq l_6\), \(l_2 \preceq l_3 \preceq l_5 \preceq l_6\) and \(l_4 \preceq l_5\).

**Definition 6.** For every poset \((A, \leq)\), we denote
\[
T(A) = \{(\Gamma, f) | \Gamma \in T, f : (L(\Gamma), \preceq) \to (A, \leq), f \text{ monotone}\}
\]
In a similar way, we define \(B(A)\), and, for every \(n > 0\), \(T_n(A)\) and \(B_n(A)\).

We call **posetted trees** the elements of \(T(A)\).

**Example 7.** The sets \(B_1(b \leq a)\), \(B_2(b \leq a)\) and \(B_3(b \leq a)\) contain 2, 3 and 8 posetted trees, respectively (see Figures 6 and 7).

**Figure 6.** The 5 posetted trees of \(B_i(b \leq a)\), \(i = 1, 2\).
Remark 8. If \( A = \{1, \ldots, m\} \) with the usual order, then there exists a natural inclusion of \( T_n(A) \) into the set of admissible graphs with \( n \) vertices of the first kind and \( m \) vertices of the second kind considered in [9, 10].

Assume that \( A \) is a subset of a (skew)commutative algebra \( R \) and choose a total ordering on \( A \). Then, it is easy to see that the elements \( Z(f) \), with \( (\Gamma, f) \in B(A) \) generate, as a \( \mathbb{K} \) vector space, the subalgebra generated by \( A \).

4. An expression of the Baker-Campbell-Hausdorff Product in terms of posetted trees

Let \( L \) be a Lie algebra over a field \( \mathbb{K} \) of characteristic 0, which is complete with respect to its lower descending series \( L^1 = L, L^{n+1} = [L^n, L] \). Denote by \( \bullet : L \times L \to L \) the Baker-Campbell-Hausdorff (BCH) product, obtained formally by the formula \( a \bullet b = \log(e^a e^b) \). It is well known that

\[
a \bullet b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] - \frac{1}{12}[b, [b, a]] + \cdots,
\]

is an element of the Lie subalgebra generated by \( a \) and \( b \) and, then, it can be expressed as an infinite sum

\[
a \bullet b = \sum_{(\Gamma, f) \in B(b \leq a)} s_{(\Gamma, f)} Z(f),
\]

for a sequence \( s_{(\Gamma, f)} \in \mathbb{K} \). Clearly, in view of the alternating properties of the product and Jacobi identity, such a sequence is not unique. The Dykin Formula [11] provides a sequence as above where \( s_{(\Gamma, f)} = 0 \), whenever \( \Gamma \) has at least 2 subroots: on the other hand, the explicit expression of the nonvanishing \( s_{(\Gamma, f)} \) is rather complicated.

Here, we describe another sequence \( b_{(\Gamma, f)} \) of rational numbers with the above properties. First of all, define the sequence of rational numbers \( \{b_n\} \), for every \( n \geq 0 \), by their ordinary generating function

\[
\sum_{n \geq 0} b_n x^n = \frac{x}{e^x - 1}.
\]

Notice that \( b_n = B_n n! \) where the \( B_n \) are the Bernoulli numbers. In particular, the only non trivial odd term of the sequence is \( b_1 = -\frac{1}{2} \) and we have:

\[
b_0 = 1, \quad b_2 = \frac{1}{12}, \quad b_4 = -\frac{1}{720}, \quad \ldots.
\]

Definition 9. Given a poset \( A \) and a posetted tree \( (\Gamma, f) \in T(A) \), let us define

\[
b_{(\Gamma, f)} := \prod_{v \in R(\Gamma)} \frac{b_{d(v)}}{t(v)},
\]

where the \( b_n \)'s are the rational numbers above and, for every subroot \( v \in R(\Gamma) \), we have

\( t(v) = \) number of leaves \( u \in L(\Gamma) \) such that \( u \to v \) and \( f(u) = f(m(v)) \).

We remind that \( m(v) \) is the leaf lying on the rightmost branch containing \( v \) (Definition 2).

\[\text{Figure 7. The 8 posetted trees of } B_3(b \leq a).\]
Example 10. Let $A = \{ b \leq a \}$ and consider the posetted tree

$$(\Gamma, f): \begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\bullet
\end{array}$$

Here, we have $d(u) = d(v) = 1; t(u) = 1; t(v) = 2$; therefore, $b_{(\Gamma, f)} = \frac{b_1}{1} \cdot \frac{b_1}{2} = \frac{1}{8}$.

Theorem 11. Let $L$ be a Lie algebra as above; then, for every positive integer $k$ and every $a_1, \ldots, a_k \in L$, we have

$$(1) \quad a_k \cdot a_{k-1} \cdot \cdots \cdot a_1 = \sum_{(\Gamma, f) \in B_1(a_1 \leq a_2 \leq \cdots \leq a_k)} b_{(\Gamma, f)} Z_{\Gamma}(f),$$

$$(2) \quad a_1 \cdot a_2 \cdot \cdots \cdot a_k = \sum_{n=1}^{\infty} \sum_{(\Gamma, f) \in B_n(a_1 \leq a_2 \leq \cdots \leq a_k)} (-1)^{n-1} b_{(\Gamma, f)} Z_{\Gamma}(f).$$

In particular, for $a, b \in L$, we have

$$(3) \quad a \cdot b = \sum_{(\Gamma, f) \in \mathcal{B}(b \leq a)} b_{(\Gamma, f)} Z_{\Gamma}(f).$$

Proof. Let us first prove Formula (4). Let $C'_{b \leq a} \subset B_{b \leq a}$ be the subset of posetted trees having every local rightmost leaf labelled with $a$ and denote by $C_{b \leq a} = C'_{b \leq a} \cup B_1(b)$.

Since the bracket is skew-symmetric, we have that $Z_{\Gamma}(f) = 0$, for every $(\Gamma, f) \notin C_{b \leq a}$; therefore,

$$\sum_{(\Gamma, f) \in B(b \leq a)} b_{(\Gamma, f)} Z_{\Gamma}(f) = \sum_{(\Gamma, f) \in C(b \leq a)} b_{(\Gamma, f)} Z_{\Gamma}(f).$$

In [3, Theorem. 1.6.1] and [7], the following recursive formula for the Baker-Campbell-Hausdorff product is proved:

$$a \cdot b = \sum_{r \geq 0} Z_r,$$

where

$$Z_0 = b, \quad Z_{r+1} = \frac{1}{r+1} \sum_{m \geq 0} b_m \sum_{i_1 + \cdots + i_m = r} (\text{ad } Z_{i_1})(\text{ad } Z_{i_2}) \cdots (\text{ad } Z_{i_m})a, \quad \text{for } r \geq 0.$$

For every $r > 0$, let $C_r \subset C(b \leq a)$ be the subset of posetted trees with exactly $r$ leaves labelled with $a$; we prove that, for every $r \geq 0$, we have

$$(5) \quad Z_r = \sum_{(\Gamma, f) \in C_r} b_{(\Gamma, f)} Z_{\Gamma}(f).$$

This is clear for $r = 0$; for $r = 1$, we have

$$Z_1 = \sum_{m \geq 0} b_m (\text{ad } Z_0)^m a = \sum_{m \geq 0} b_m (\text{ad } b)^m a,$$

whereas $C_1 = \{ \Omega_m \}, m \geq 0$, is the set of posetted trees of Bernoulli type [15], i.e.,

$$\Omega_m : \begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\bullet
\end{array}$$
where \( m \) is the number of leaves labelled with \( b \). Therefore, the coefficient \( b_{(\Omega_n)} \) is exactly \( b_m \) and so

\[
Z_1 = \sum_{(\Gamma, f) \in C,} b_{(\Gamma, f)} Z_{\Gamma}(f).
\]

Moreover, every element of \( C_{r+1} \) is obtained in a unique way starting from a tree \( \Omega_{m} \) and grafting, at each of the \( m \) leaves labelled with \( b \), the roots of elements of \( C_{i_1}, \ldots, C_{i_m} \), with \( i_1 + \cdots + i_m = r \) (for the definition of the grafting see [12, Definition 1.37]). Therefore, the proof of (5) follows easily by induction on \( r \).

Next, let us compute \( b \). Comparing the coefficients, we have

\[
a_k \cdot a_{k-1} \cdots \cdot a_1 = (-a_k) \cdots (-a_1),
\]

and Formula (4) follows immediately from (2). Finally, setting \( b = a_{k-1} \cdots \cdot a_1 \), we have that every posetted tree of \( B(a_1 \leq a_2 \leq \cdots \leq a_k) \) can be described in a unique way as a posetted tree in \( C(b \leq a_k) \), where at every leaf labelled with \( b \) is grafted the root of a posetted tree of \( B(a_1 \leq a_2 \leq \cdots \leq a_{k-1}) \). In view of the associativity relation

\[
a_k \cdot a_{k-1} \cdots \cdot a_1 = a_k \cdot b,
\]

we obtain that (2) is a consequence of \( a_k \cdot b = \sum_{(\Gamma, f) \in C(b \leq a_k)} b_{(\Gamma, f)} Z_{\Gamma}(f) \).

\[\square\]

Remark 12. Choose \( a_1 = b \) and \( a_2 = a \) in Equation (2), and \( a_1 = a \) and \( a_2 = b \) in Equation (3). Comparing the coefficient of the product \( ad(b)^n(a) \) in both equations, we obtain the following relations

\[
(1 + n(-1)^n) b_n = - \sum_{i=1}^{n-1} (-1)^i b_i b_{n-i}, \quad n > 0.
\]

Indeed, the coefficient of \( ad(b)^n(a) \) in Equation (3) comes from the Bernoulli tree \( \Omega_n \) and so it is exactly \( b_n \). On the other side, we need to consider the subset \( S(n) \) of trees \( (\Gamma, f) \in B(a \leq b) \) with only one subroot, \( n \) leaves labelled \( b \) and one leaf labelled \( a \). For any \( (\Gamma, f) \in S(n) \), we have \( Z_{\Gamma}(f) = \pm ad(b)^n(a) \), and we can define \( C_n \) as

\[
\sum_{(\Gamma, f) \in S(n)} b_{(\Gamma, f)} Z_{\Gamma}(f) = C_n ad(b)^n(a).
\]

Comparing the coefficients, we have

\[
C_n = (-1)^n b_n.
\]

Next, let us compute \( C_n \) recursively. There are two different types of contributions to \( C_n \) due to the following graphs. The first contribution is due to the graph with only one subroot; in this case, the coefficient is \( \frac{b}{n} ad(b)^{n-1}([a, b]) = - \frac{b}{n} ad(b)^n(a) \). The other contribution is due to the graphs obtained from a graph in \( S(i) \), for every \( i = 1, \ldots, n-1 \), and grafting, at the leaves labelled with \( a \), a graph of \( S(n-i) \). Therefore, for every fixed \( i \), the coefficient is

\[
\frac{b}{n} C_{n-i} ad(b)^{i-1}([ad(b)^{n-i}(a), b]) = - \frac{b}{n} C_{n-i} ad(b)^n(a).
\]

Summing up, we have

\[
C_n = - \frac{b}{n} - \sum_{i=1}^{n-1} \frac{b}{n} C_{n-i};
\]

and, since \( C_n = (-1)^n b_n \), we get the relation

\[
b_n (1 + n(-1)^n) = - \sum_{i=1}^{n-1} (-1)^i b_i b_{n-i}.
\]
Note that, in the previous computation, we have just used the fact that the product is associative and therefore apply for every associative product defined by Equation (2). More precisely, let \( a_n \) be any sequence in \( \mathbb{K} \), and for any \((\Gamma, f) \in B(b \leq a)\), define
\[
a_{(\Gamma, f)} := \prod_{v \in R(\Gamma)} \frac{a_{d(v)}^d(v)}{t(v)},
\]
and the product
\[
a \ast b = \sum_{(\Gamma, f) \in B(b \leq a)} a_{(\Gamma, f)} Z_{\Gamma}(f).
\] (7)

**Proposition 13.** In the notation above, the product \( \ast \) is associative if and only if there exists an \( h \in \mathbb{K} \) such that \( a_n = h^n b_n \), for every \( n > 0 \).

**Proof.** One implication is clear, if \( a_n = h^n b_n \), then
\[
a \ast b = \sum_{(\Gamma, f) \in B(b \leq a)} a_{(\Gamma, f)} Z_{\Gamma}(f) = \sum_{(\Gamma, f) \in B(b \leq a)} \prod_{v \in R(\Gamma)} h^{d(v)} b_{(\Gamma, f)} Z_{\Gamma}(f) = h^{-1}(h a) \bullet (h b);
\]
this implies that the product \( \ast \) is associative (in the last equality we use that \( \sum_{v \in R(\Gamma)} d(v) = n - 1 \)). As regards the other implication, assume that the product \( \ast \) is associative; then, Equation (3) holds for the product \( \ast \) instead of \( \bullet \). Arguing as in the above remark, we conclude that the numbers \( a_n \) must satisfy Equation (6), and this easily implies that \( a_n = (-2a)h^n b_n \), for every \( n > 0 \).

\[\square\]

**References**

[1] H. Baker: *Alternants and continuous groups*. Proc. London Math. Soc., (2) 3, (1905), 24-47.
[2] J. Campbell: *On a law of combination of operators*. Proc. London Math. Soc., (1) 28, (1897), 381-390; 29, (1898), 14-32.
[3] J.J. Duistermaat and J.A.C. Kolk: *Lie Groups*. Springer Universitext (2000).
[4] E. Dynkin: *Calculation of the coefficients of the Campbell-Hausdorff formula*. Dokl. Akad. Nauk., 57, (1947), 323-326. An English translation may be found in: E.B. Dynkin, A.A. Yushkevich, G.M. Seitz, A.L. Onishchik (Eds.), Selected Papers of E.B. Dynkin with Commentary, American Mathematical Society/International Press, Providence, R.I./Cambridge, Mass, (2000).
[5] D. Fiorenza and M. Manetti: *L∞-structures on mapping cones*. Algebra & Number Theory, 1, (2007), 301-330.
[6] D. Fiorenza, M. Manetti and E. Martinengo: *Semicosimplicial DGLAs in deformation theory*. Preprint arXiv:0803.0399v1.
[7] B.C. Hall: *Lie Groups, Lie Algebras, and representations*. An elementary introduction. Graduate Texts in Mathematics, 222, Springer-Verlag, New York Berlin, (2003).
[8] F. Hausdorff: *Die symbolische Exponentialformel in der Gruppentheorie*. Ber. Verh. Sachs. Akad. Wiss., Leipzig, Math. Phys. Kl., 58, (1906), 19-48.
[9] V. Khoroshkin: *Kontsevich universal formula for quantization and the Campbell-Baker-Hausdorff formula*. Internat. J. Math. 11, (2000), 523-551; arXiv:math/0403296.
[10] M. Kontsevich: *Deformation quantization of Poisson manifolds, I*. Letters in Mathematical Physics, 66, (2003) 157-216; arXiv:math/9812174.
[11] J.L. Loday and B. Vallette: *Algebraic Operads*. Draft version 0.99 (2010); available at the authors web pages.
[12] M. Markl, S. Schnider and J. Stasheff: *Operads in algebra, topology and physics*. Mathematical Surveys and Monographs, 96, American Mathematical Society, Providence, RI, (2002).
[13] O. Ore: *Theory of graphs*. Colloquium publications 38, American Mathematical Society, Providence, RI, (1962).
[14] D. Ranjan, E. Pontelli and G. Gupta: *Data structures for order-sensitive predicates in parallel nondeterministic system*. Acta Informatica, 37, (2000), 21-43.
[15] C. Torossian: *Sur la formule combinatoire de Kashiwara-Vergne*. J. Lie Theory, 12, (2002), 597-616.
[16] M. Weyrauch and D. Scholz: *Computing the Baker-Campbell-Hausdorff series and the Zassenhaus product*. Computer Physics Communications, 180, (2009), 1558-1565.
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