A Comment on the Estimation of Angular Power Spectra in the Presence of Foregrounds

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It is common practice to estimate the errors on the angular power spectrum which could be obtained by an experiment with a given angular resolution and noise level. Several authors have also addressed the question of foreground subtraction using multi-frequency observations. In such observations the angular resolution of the different frequency channels is rarely the same. In this report we point out how the “effective” beam size and noise level change with \( \ell \) in this case, and give an expression for the error on the angular power spectrum as a function of \( \ell \).

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I. INTRODUCTION

There are many experiments planned which hope to reap the rich harvest of information available in the anisotropies of the Cosmic Microwave Background (CMB) [1]. For the purposes of estimating cosmological parameters or constraining models it is important to know how well an experiment can constrain the angular power spectrum of CMB anisotropies.

It is common practice to estimate the errors on the angular power spectrum which could be obtained by an experiment with a given angular resolution and noise level. Such an estimate can give us insight into what regions of the angular power spectrum one could constrain, and what are the limiting factors in an experiment designed e.g. to constrain cosmological parameters. In this report we point out some simple limiting cases which allow one to gain intuition about the effect of multiple observing frequencies with differing beam sizes in the presence of (somewhat idealized) foregrounds.

If we write the angular power spectrum of the anisotropy as \( C_\ell \) and the noise power spectrum as \( N_\ell \) (typically \( N_\ell = 4\pi f_{\text{sky}}\sigma^2/N_{\text{pix}} \)), where \( f_{\text{sky}} \) is the fraction of sky covered, \( \sigma \) is the rms pixel noise and \( N_{\text{pix}} \) is the number of pixels) then

\[
\delta C_\ell \sim \sqrt{\frac{2}{(2\ell + 1)f_{\text{sky}}}} \left( C_\ell + \frac{N_\ell}{W_\ell} \right)
\]

where \( W_\ell \) is the window function of the experiment and we have assumed that the noise is gaussian. For a gaussian beam of width \( \theta_b \), \( W_\ell = \exp(-\ell^2\theta_b^2) \). Eq. (1) assumes that the only sources of variance in an experiment are the anisotropy signal and the receiver noise, i.e. it neglects foregrounds. We have also implicitly assumed that \( \delta C_\ell \) is a gaussian error, thus we imagine we are working at reasonable high-\( \ell \) and will bin our power spectrum estimates into finite width bins in \( \ell \). Eq. (1) has been widely used to estimate how well upcoming satellite experiments could constrain cosmological parameters [2].

Several authors have also addressed the question of foreground subtraction using multi-frequency observations [3–6]. With multi-frequency observations it appears possible to separate the desired anisotropy signal from the foregrounds with encouraging precision. The authors of [3] have shown that one can regard foreground subtraction as an enhancement of the noise in a foreground free experiment. The noise enhancement factor has been called the foreground degradation factor (FDF) by Dodelson [3] who gave a simple expression for it. A method of foreground subtraction using both frequency and spatial information was proposed in Ref. [3]. Various methods of foreground subtraction have been compared recently by Tegmark [3].

Note that the “noise term” in Eq. (1) depends exponentially on \( \ell \) once \( \ell > \theta_b^{-1} \). In multi-frequency observations the angular resolution of the different frequency channels is rarely the same, leading to the question of which beam size to use. In this report we discuss a simple heuristic “effective” beam size and noise level. Since how well we can take out foregrounds depends on the angular structure of the foregrounds, the “noise” will be a function of the foreground angular power spectrum.

The formalism is directly comparable to that in [3] in that we use a minimum variance estimator of the CMB anisotropy power spectrum. An extension to “real world foregrounds” is straightforward, but the purpose of this report is to gain intuition through simple examples so we do not pursue this line of development.

II. FORMALISM

As noted by Tegmark [3], one can consider foregrounds as an additional noise component which is highly correlated among different frequency channels. If we label the
frequency channels by a greek subscript we can define a
noise correlation matrix

\[ N_{\alpha\beta} = \frac{4\pi}{N_{\text{pix}}} \sum_i W_{\ell\alpha}^{1/2} \langle f_{\ell\alpha} f_{\ell\beta}^* \rangle \frac{W_{\ell\beta}^{1/2}}{W_{\ell\alpha}^{1/2} F_{\ell\alpha}} + \frac{4\pi}{N_{\text{pix}}} \sigma_\alpha^2 \delta_{\alpha\beta} \]  

which has contributions from the pixel noise (assumed
uncorrelated between channels here for simplicity) and
foregrounds labelled by superscript \( i \). Here \( \langle f_{\ell\alpha} f_{\ell\beta}^* \rangle \) is
the correlation matrix of foreground \( i \) in channels \( \alpha \) and
\( \beta \), with the \( 4\pi/N_{\text{pix}} \) inserted for later convenience. If the
foregrounds are 100\% correlated between the channels
then \( \langle f_{\ell\alpha} f_{\ell\beta}^* \rangle = f_{\ell\alpha} f_{\ell\beta}^* \) where \( f_{\ell\alpha} \) is the rms intensity as
a function of frequency. We shall assume this case from
now on, but see Ref. [1].

It is straightforward to derive the minimum variance
estimator of \( C_\ell \), as a linear combination of measure-
ments at different frequencies. If we have measured mul-
tiple moments \( a_{\ell m}^\alpha \) at frequency \( \alpha \) then we write the
estimate of the CMB component as \( C_\ell = \sum_\alpha a_{\ell m}^\alpha \). Imagine that we can write the observed signal \( a_{\ell m}^\alpha \)
then

\[ W_{\ell\alpha}^{1/2} F_{\ell\alpha} W_{\ell\beta}^{1/2} F_{\ell\beta} \]

where \( \theta_\ell^2 \) is the average of \( \theta_{\ell m}^2 \) over \( m \),

\[ F_{\alpha} = \sum_\beta C_\ell W_{\ell\beta}^{1/2} \left( W_{\ell\beta}^{1/2} C_\ell W_{\ell\alpha}^{1/2} + N_{\alpha\beta} \right)^{-1} \]

and we have left the \( \ell \)-dependence of \( F_{\alpha} \) implicit for
notational convenience. The vector \( F_{\alpha} \) projects out the \( \ell \)th
CMB multipole moment from the signal in each channel in
a minimum variance sense.

If we assume that this estimator is Gaussian then we
can replace Eq. (3) by

\[ \delta C_\ell = \sqrt{\frac{2}{(2\ell + 1)}} \left( C_\ell + \sum_{\alpha\beta} \frac{(F_{\alpha} N_{\alpha\beta} F_{\beta})_\ell}{(W_{\ell\alpha}^{1/2} F_{\alpha})^2} \right) \]  

where we have set \( f_{\text{sky}} = 1 \) for simplicity (the scaling
with \( f_{\text{sky}} \) is given in Eq. (3)). Note that in the limit of
one frequency channel and no foregrounds, the sums over
\( \alpha \) and \( \beta \) are trivial, the noise term is independent of \( F \)
and we recover Eq. (1).

Eq. (4) is the general result, and we consider several
examples to gain intuition in the next section.

III. EXAMPLES

Let us consider various limits of Eq. (1). We will focus
on the noise term, since the part of the error proportional
to \( C_\ell \) simply reflects cosmic plus sample variance [3].
In the signal dominated limit \( (C_\ell \gg N_{\ell}) F_{\alpha} \) is just the
inverse square root of \( W_{\ell\alpha} \) and the noise term in Eq. (3)
reduces to

\[ \frac{N_{\ell}}{W_{\ell\alpha}} \left( \frac{N_{\ell}}{W_{\ell\alpha}} \right)^{-1} = \frac{N_{\text{pix}}}{4\pi} \sum_\alpha \frac{W_{\ell\alpha}}{\sigma_\alpha^2} \]  

where \( W_{\ell} = \sum_{\alpha} W_{\ell\alpha} \). Thus, in the absence of fore-
grounds the noise is the sum of the noises in those chan-
nels able to resolve features with multipole number \( \ell \),
divided by the number of such channels squared. The
addition of foregrounds increases the variance, but by
assumption both contributions to \( \delta C_\ell \) are sub-dominant.
We expect this regime to occur at low-\( \ell \), where we are
cosmic and sample variance limited.

Now let us consider the opposite limit. First imagine
there are no foregrounds. In this limit \( (N_{\ell} \gg C_\ell) \) the
noise term in Eq. (3) reduces to

\[ \frac{N_{\ell}}{W_{\ell\alpha}} \left( \frac{N_{\ell}}{W_{\ell\alpha}} \right)^{-1} = \frac{N_{\text{pix}}}{4\pi} \sum_\alpha \frac{W_{\ell\alpha}}{\sigma_\alpha^2} \]

which would be the obvious way to combine the chan-
nels to obtain the effective noise: recall the error on a
weighted mean is \( \sigma^2 = \sum \sigma_i^2 \).

Now we can enhance this last example by the addition
of foregrounds. For simplicity imagine a two channel ex-
periment with 1 foreground \( f_{\ell\alpha} \) in addition to uncor-
related noise \( \sigma_\alpha \), with \( \alpha = 1, 2 \). For definiteness imagine
that channel 1 is a low frequency channel with a “large”
beam, while channel 2 is a high frequency channel with a
“small” beam. Some trivial matrix algebra allows us to
write \( (N_{\ell}/W_{\ell})_{\text{eff}} \) as

\[ \frac{4\pi}{N_{\text{pix}}} \frac{\sigma_1^2 \sigma_2^2 + \sigma_1^2 f_{\ell1}^2 W_{\ell1} + \sigma_2^2 f_{\ell2}^2 W_{\ell2}}{\sigma_1^2 W_{\ell1} + \sigma_2^2 W_{\ell2} + (f_{\ell1} - f_{\ell2})^2 W_{\ell1} W_{\ell2}} \]

which reduces to Eq. (6) as \( f_{\ell\alpha} \rightarrow 0 \).

To bring out the essential details let us take \( \sigma_1 = \sigma_2 \)
and ignore the different resolutions by working at low
enough \( \ell \) that \( W_{\ell\alpha} \rightarrow 1 \),

\[ \frac{N_{\ell}}{W_{\ell\alpha}} \rightarrow \frac{4\pi \sigma^2}{N_{\text{pix}}} \left( \frac{\sigma^2 + f_{\ell1}^2 + f_{\ell2}^2}{2\sigma^2 + (f_{\ell1} - f_{\ell2})^2} \right) \]  

The term in parentheses is the increase in the noise
over the one channel result. As we decrease \( \sigma^2 \), this goes
from \( \frac{1}{2} \) (the case of no foregrounds: co-add the
channels) to \( (1 + x^2)(1 - x)^{-2} \) where \( x = f_{\ell2}/f_{\ell1} \). For this
case, the increase in the noise is just the FDF defined
by Dodelson [1], written in a slightly different notation.
The point $x = 1$ is when the frequency dependence of the foreground is the same as the CMB. Generically the minimum variance estimator Eq. (3) does better than the FDF would indicate, as has been pointed out in Ref. [6].

Now let us reinstate the window functions, but imagine we are at intermediate $\ell$, where $W_{\ell 1} \ll 1$. In this regime

$$\left( \frac{N_{\ell}}{W_{\ell}} \right)_{\text{eff}} = \frac{4\pi \sigma_{2}^{2}}{N_{\text{pix}} W_{\ell 2}} \left( 1 + \frac{f_{\ell 2}^{2} W_{\ell 2}}{\sigma_{2}^{2}} \right).$$

So the appropriate resolution is that of channel 2, and the noise is the channel 2 noise plus a contribution from the foreground at the higher frequency. Note that the properties of channel 1 do not enter the expression, as expected. The foreground contribution declines as one approaches the resolution of channel 2, so that asymptotically the error is just $4\pi \sigma_{2}^{2}/N_{\text{pix}} W_{\ell 2}$.

Finally note that if our foregrounds have a “steep” power spectrum, i.e. fall rapidly with $\ell$, then at high-$\ell$ we have the limit $f_{\alpha} \to 0$ and reproduce Eq. (7). At low-$\ell$ the foreground should dominate over the noise which is the case discussed below Eq. (3). This shows that for foregrounds without a lot of small-scale structure there is little effect at high-$\ell$ from removing the foregrounds (in this simplified example where the foreground properties are assumed known exactly).

### IV. SUMMARY

Let us summarize our results with $\ell$ increasing from the signal- to the noise-dominated limits. Under our assumptions, the error on the angular power spectrum starts dominated by cosmic and sample variance at low-$\ell$. Moving to higher $\ell$, if the noise or foregrounds start to dominate on scales which all channels can resolve, the effect of a foreground looks like an increase in the noise by a factor which is always less than Dodelson’s FDF $[4]$. If the noise or foregrounds dominate on scales smaller than the size of the widest beam, but larger than the size of the smallest beam, the noise is that of the channel with the smaller beam size, increased by the variance of the foreground at that frequency. At $\ell > \theta_{b}^{-1}$ for even the highest frequency channel the error is simply that obtained from the noise in the highest frequency channel, regardless of foregrounds. If the foregrounds are always negligible compared to the noise, the appropriate noise level is the weighted sum of the noises in each channel which can resolve the given $\ell$, as shown in Eq. (5).

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