A unitary “quantization commutes with reduction” map for the adjoint action of a compact Lie group

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Abstract

Let \( K \) be a simply connected compact Lie group and \( T^*(K) \) its cotangent bundle. We consider the problem of “quantization commutes with reduction” for the adjoint action of \( K \) on \( T^*(K) \). We quantize both \( T^*(K) \) and the reduced phase space using geometric quantization with half-forms. We then construct a geometrically natural map from the space of invariant elements in the quantization of \( T^*(K) \) to the quantization of the reduced phase space. We show that this map is a constant multiple of a unitary map.

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Many classical systems of interest in physics arise as the reduction of some larger system by the action of a group. Of particular importance are gauge field theories, in which reduction implements gauge symmetry. In quantizing such systems, one could plausibly attempt to perform the quantization either before or after reduction. It is then of interest to compare the results of these two procedures.

In this paper, we consider the holomorphic approach to quantization, with the Segal–Bargmann space over $T^\ast(R^n) \cong \mathbb{C}^n$ serving as a prototypical example. This approach to quantization allows one, for example, to construct a family of coherent states and often facilitates various aspects of the semiclassical limit. Specifically, we follow the approach of geometric quantization using a complex (i.e., Kähler) polarization. See [14] for background on Segal–Bargmann spaces and [39] or Chapters 22 and 23 of [17] for background on geometric quantization.

In the holomorphic approach to quantization, the first major result comparing quantization before reduction to quantization after reduction is the 1982 paper [10] of Guillemin and Sternberg. They work in the setting of compact Kähler manifolds, using geometric quantization without half-forms. Under certain regularity assumptions, they establish a geometrically natural invertible linear map between the “first quantize then reduce” space and the “first reduce and then quantize” space. There have been numerous extensions of this work,
many of which work with a notion of quantization based on the index of a certain operator. The reader is referred to the survey article of Sjamaar [34] for the state of the art in this area as of 1996.

1.2 Unitarity in “quantization commutes with reduction”

An issue not addressed in [10] is the matter of unitarity of the map. This issue is important because for physical applications, it is essential to have not just a vector space but a Hilbert space: The inner product matters! In [19], the first author and Kirwin analyzed the Guillemin–Sternberg map and found that in general it is not even asymptotically unitary as Planck’s constant tends to zero. The paper [19] then introduces a modified Guillemin–Sternberg-type map in the presence of half-forms and shows that this map is at least asymptotically unitary as Planck’s constant tends to zero. Much of the analysis in [19] can be applied even if the Kähler manifold in question is not compact.

A natural question remains whether there are interesting examples in which a map of Guillemin–Sternberg type is actually unitary (or unitary up to a constant), not just asymptotically unitary. Work of Kirwin [27] represents a step in the direction of answering this question, by analyzing the higher-order asymptotics of the map introduced in [19]. Nevertheless, there is no known general criterion for obtaining exact unitarity.

The first example of unitarity we are aware of is in an infinite-dimensional setting, that of the quantization of (1 + 1)-dimensional Yang–Mills theory on a space-time cylinder. Fix a connected compact Lie group $K$, called the structure group, with Lie algebra $\mathfrak{k}$. The unreduced configuration space for the problem is then the space $\mathcal{A}$ of connections (that is, $\mathfrak{k}$-valued 1-forms) on the spatial circle, and the unreduced phase space is the cotangent bundle $T^*\mathcal{A}$. We consider at first the based gauge group $G_0$ consisting of maps from the circle into $K$ that equal the identity at one fixed point. The symplectically reduced phase space $T^*(\mathcal{A})/G_0$ is then naturally identified with the finite-dimensional symplectic manifold $T^*(K)$. Furthermore, if we choose a natural complex structure on $T^*(\mathcal{A})$, the quotient inherits a complex structure, given by identifying $T^*(K)$ with $K_C$, the complexification of $K$. (This identification is described in Section 2.)

The quantization-versus-reduction problem for holomorphic quantization of $T^*(\mathcal{A})$ has been considered using two different methods. First, Landsman and Wren [29] and Wren [38] use the “generalized Rieffel induction” method of Landsman [28] to “project” the coherent states for $T^*(\mathcal{A})$ into the (nonexistent) gauge-invariant subspace. The article [38] shows that the projected coherent states are precisely the coherent states obtained in [13, 15] from the holomorphic quantization of $T^*(K) \cong K_C$.

Meanwhile, Driver–Hall [4] approach the problem using a Gaussian measure of large variance to approximate the nonexistent Lebesgue measure on $\mathcal{A}$. (See also [16].) The gauge invariant subspace is then identified as an $L^2$ space of holomorphic functions on $K_C$ that converges in the large variance limit to the one obtained in [15] from quantizing $T^*(K) \cong K_C$. Both [38] and [4] identify the
space obtained by first quantizing and then reducing as being the same space with the same inner product as the one obtained by first reducing and then quantizing.

The goal of the present paper is to provide a finite-dimensional example of a unitary “quantization commutes with reduction” map.

1.3 The adjoint action of $K$ on $T^*(K)$

We now let $K$ be a connected compact Lie group, which we assume for simplicity to be simply connected. The cotangent bundle $T^*(K)$ has a natural Kähler structure obtained by identification of $T^*(K)$ with the complexified group $K_C$. (See Section 2) Geometric quantization of $T^*(K) \cong K_C$, using a Kähler polarization and half-forms, is carried out in [13]. The quantum Hilbert space turns out to be naturally isomorphic to the generalized Segal–Bargmann space over $K_C$ introduced in [13]. Furthermore, the BKS pairing map between the Kähler-polarized and vertically polarized spaces turns out to coincide with the generalized Segal–Bargmann transform of [13]; in particular, the pairing map is unitary in this case. Related aspects of the geometric quantization of $T^*(K) \cong K_C$ have been studied by Florentino, Matias, Mourão, and Nunes [7, 8], by Lempert and Szőke [31, 32], and by Szőke [36]. All of these authors study families of complex structures on $T^*(K)$ and parallel transport in the resulting bundle of quantum Hilbert spaces.

Meanwhile, Florentino, Mourão, and Nunes [6, Theorems 2.2 and 2.3] give a remarkable explicit formula for norm of a holomorphic class function on $K_C$ (in the Hilbert space obtained by quantizing $T^*(K) \cong K_C$) as an integral over a complex maximal torus. The formula involves the Weyl denominator function and can be viewed as a complex version of the Weyl integral formula. The results of [6] are a vital ingredient in the present paper. Indeed, our main result can be summarized by saying that the just-cited result of [6] can be interpreted as a unitary “quantization commutes with reduction map.”

Now $K$ acts on itself by the adjoint action (i.e., by conjugation), and this action lifts to a symplectic action of $K$ on the cotangent bundle $T^*(K)$. There is then an equivariant momentum map

$$\phi : T^*(K) \rightarrow \mathfrak{k}^*.$$  

Although the action of $K$ on $\phi^{-1}(0)$ is not free (even generically), we may attempt to construct the symplectic quotient

$$T^*(K)/\text{Ad}_K := \phi^{-1}(0)/\text{Ad}_K.$$  

The quotient will not be a manifold, but will have singularities that must be dealt with in the analysis.

The reduction of $T^*(K)$ is a natural problem for various reasons. First, from the point of view of the Yang–Mills example described in the previous subsection, we have said that $T^*(K)$ is the symplectic quotient of $T^*(\mathcal{A})$ by the based gauge group $\mathcal{G}_0$. Let $\mathcal{G}$ denote the full gauge group, consisting of all maps of
into $K$. The adjoint action of $K$ on $T^*(K)$ is then the “residual” action of the full gauge group $G$ on $T^*(A)/G_0$. Second, the quotient $T^*(K)/\text{Ad}_K$ is a geometrically interesting example of a singular symplectic quotient. Quantization of this quotient has been studied by several researchers, including Wren [37], Huebschmann [23], Huebschmann, Rudolph and Schmidt [24], and Boeijink, Landsman, and van Suijlekom [3]. Third, the quotient $T^*(K)/\text{Ad}_K$ is a prototype—the case of a single plaquette—for the study of lattice gauge systems. See [9] for a study of more general cases.

The paper [3] of Boeijink, Landsman, and van Suijlekom, in particular, considers the quotient from the point of view of the quantization versus reduction problem. The authors consider “Dolbeault–Dirac quantization” of both $T^*(K)$ and of (the regular points in) $T^*(K)/\text{Ad}_K$. The Dolbeault–Dirac quantization ultimately turns out to give the same result as geometric quantization of these spaces without half-forms [3, Theorem 3.14]. (The authors also consider “spin quantization” of $T^*(K)$, which ultimately gives the same result [3, Theorem 3.15] as geometric quantization with half-forms, but their results on “quantization commutes with reduction” are for the Dolbeault–Dirac quantization.) The authors determine (1) the invariant subspace of the quantization of $T^*(K)$, and (2) the quantization of $T^*(K)/\text{Ad}_K$. They then show that both of these spaces can be identified unitarily with the space of Weyl-invariant elements in the Hilbert space $L^2(T)$ [3, Theorem 4.18]. Although this result constitutes a form of “quantization commutes with reduction,” the isomorphism between these spaces is constructed in an indirect way, making use of a very general Segal–Bargmann-type isomorphism from [13, Section 10]. Indeed, the authors say, “the quantization commutes with reduction theorem would get more body if there were a way to identify quantization after reduction with reduction after quantization differently from mere unitary isomorphism of Hilbert spaces” [3, p. 31].

By contrast, we will consider quantization of $T^*(K)$ and $T^*(K)/\text{Ad}_K$ with half-forms. We will consider a natural map of Guillemin–Sternberg type between the space of invariant elements in the quantization of $T^*(K)$ (“first quantize and then reduce”) and the quantization of $T^*(K)/\text{Ad}_K$ (“first reduce and then quantize”). This map will be similar to the one constructed in [19] and will include a mechanism for converting half-forms of one degree to half-forms of a smaller degree. Our main result will be that this geometrically natural map is a constant multiple of a unitary map.

### 1.4 The main results

We consider geometric quantization of $T^*(K)$ with half-forms, using a Kähler polarization obtained by identifying $T^*(K)$ with $K_{\mathbb{C}}$. We denote the resulting Hilbert space by $\text{Quant}(K_{\mathbb{C}})$. (Whenever we write $K_{\mathbb{C}}$, we always mean “$K_{\mathbb{C}}$ as identified with $T^*(K)$.”) The adjoint action of $K$ on $T^*(K) \cong K_{\mathbb{C}}$ then induces an unitary “adjoint” action of $K$ on $\text{Quant}(K_{\mathbb{C}})$. We let $\text{Quant}(K_{\mathbb{C}})^{\text{Ad}_K}$ denote the space of elements in $\text{Quant}(K_{\mathbb{C}})$ that are fixed by this action. We think of $\text{Quant}(K_{\mathbb{C}})^{\text{Ad}_K}$ as being the reduced quantum Hilbert space, that is, the space
obtained by first quantizing and then reducing.

Now let $T \subset K$ be a fixed maximal torus, let $T^*(T)$ be its cotangent bundle, and let $T_C \subset K_C$ be the complexification of $T$. Since $T$ is also a compact Lie group, we may similarly identify $T^*(T)$ with $T_C$ and perform geometric quantization with half-forms. We let $\text{Quant}(T_C)$ denote the quantization of the “Weyl-alternating” subspace of $\text{Quant}(T_C)$, which we denote as $\text{Quant}(T_C)^{W-}$.

The adjoint action of $K$ on $T^*(K)$ admits an equivariant momentum map $\phi$. We consider, finally, the reduced phase space

$$T^*(K)/\text{Ad}_K := \phi^{-1}(0)/\text{Ad}_K,$$

which we also write as $K_C/\text{Ad}_K$. Since the reduced phase space is not a manifold, we will quantize it by quantizing only the set of regular points—which is called the “principal stratum” in [21] and [6]—which is an open dense subset. (An argument for the reasonableness of this procedure is given in Section 5.2.) We denote the quantization of the reduced phase space by $\text{Quant}(K_C/\text{Ad}_K)$.

Now, in Section 3.1, we will see that $\text{Quant}(K_C)$ and $\text{Quant}(T_C)$ can be identified as $L^2$ spaces of holomorphic functions with respect to certain measures $\gamma_\hbar$ and $\gamma_\hbar'$, respectively. Then in Section 3.2, we will see that $\text{Quant}(K_C)^{\text{Ad}_K}$ corresponds to the space of holomorphic class functions in $L^2(K_C, \gamma_\hbar)$. Let $\sigma : T \to \mathbb{R}$ be the Weyl denominator function and let $\sigma_C : T_C \to \mathbb{C}$ be the analytic continuation of $\sigma$. (Since $K$ is assumed simply connected, $\sigma$ is a single-valued function on $T$.) We now record a crucial result of Florentino, Mourão, and Nunes [6].

**Theorem 1** There is a constant $c_\hbar$ such that for every holomorphic class function $F$ on $K_C$, we have

$$\int_{K_C} |F(g)|^2 \, d\gamma_\hbar(g) = c_\hbar \int_{T_C} |\sigma_C(z) F(z)|^2 \, d\gamma_\hbar'(z).$$

This result is Theorem 2.2 in [6]. The goal of the present paper is this: To interpret Theorem 1 as giving a unitary (up to a constant) “quantization commutes with reduction” map between $\text{Quant}(K_C)^{\text{Ad}_K}$ and $\text{Quant}(K_C/\text{Ad}_K)$. To accomplish this goal, we must perform two tasks. First, we must show that we can quantize the reduced phase space $K_C/\text{Ad}_K$ in such a way that $\text{Quant}(K_C/\text{Ad}_K)$ may be identified with $\text{Quant}(T_C)^{W-}$. Second, we must show that the map

$$F \mapsto (\sigma_C)(F|_{T_C})$$

implicit in Theorem 1 can be computed by means of a natural Guillemin–Sternberg-type map with half-forms.

In the second task, the key issue is to account for the appearance of the analytically continued Weyl denominator $\sigma_C$ in (1). We will argue that this factor is not something one simply needs to insert by hand in order to obtain the nice result in Theorem 1. Rather, we will show that this factor arises naturally in the
process of converting half-forms on $K_C$ to half-forms (of a different degree!) on $T_C$. Specifically, we will use a procedure—similar to that in [19]—of contracting half-forms with the vector fields representing the infinitesimal adjoint action of $K$ on $K_C$. The analytically continued Weyl denominator $\sigma_C$ will arise naturally in this process. (See Sections 6.1 and 6.2.)

We describe our results in schematic form.

**Theorem 2** It is possible to quantize the reduced phase space in such way that the elements of the quantum Hilbert space may be identified with the Weyl-alternating elements of the quantization of $T_C$:

$$\text{Quant}(K_C/\!/\text{Ad}_K) \cong \text{Quant}(T_C)^{W_-}.$$ 

In proving this result, we will exploit the freedom that is present in geometric quantization to choose the prequantum line bundle.

**Theorem 3** There is a geometrically natural “quantization commutes with reduction” map

$$B : \text{Quant}(K_C)^{\text{Ad}_K} \to \text{Quant}(K_C/\!/\text{Ad}_K) \cong \text{Quant}(T_C)^{W_-}$$

that corresponds, after suitable identifications, to the map $F \mapsto (\sigma_C)(F|_{T_C})$ in [17]. Thus, by Theorem 1 the map $B$ is a constant multiple of a unitary map.

Although the map $B$ is similar to the map $B_k$ in [19], modifications are needed in the present setting. First, because the adjoint action of $K$ on $\phi^{-1}(0)$ is not even generically free, we must contract at each point with a subset of the vector fields representing the infinitesimal adjoint action. Second, the contraction process requires a choice of orientation and it turns out to be impossible to choose the orientation consistently over all of $\phi^{-1}(0)$. It is therefore necessary to choose the prequantum line bundle in the quantization of the reduced phase space carefully in order to ensure that the “quantization commutes with reduction” map is globally defined.

## 2 Preliminaries

Let $K$ be a connected compact Lie group of dimension $n$, assumed for simplicity to be simply connected. This assumption ensures that the Weyl denominator function (Section 2.2) is single valued and that the centralizer of each regular semisimple element in the complexified group is a complex maximal torus (Section 4.2). We fix an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of $K$ that is invariant under the adjoint action of $K$. There is then a unique bi-invariant Riemannian metric on $K$ whose value at the identity is $\langle \cdot, \cdot \rangle$. We let $n$ denote the dimension of $K$. 

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2.1 The complex structure on $T^*(K)$

We let $K_\mathbb{C}$ be the complexification of $K$, which may be described as the unique simply connected Lie group whose Lie algebra is $\mathfrak{k}_\mathbb{C} := \mathfrak{k} \oplus i \mathfrak{k}$. The inclusion of $\mathfrak{k}$ into $\mathfrak{k}_\mathbb{C}$ induces a homomorphism of $K$ into $K_\mathbb{C}$, which is well known to be injective. (See [13, Section 3].) We will identify $K$ with its image inside $K_\mathbb{C}$, which is a compact and therefore closed subgroup of $K_\mathbb{C}$. As an example, we may take $K$ to be the special unitary group $SU(N)$ and $K_\mathbb{C}$ to be the special linear group $SL(N; \mathbb{C})$.

As our initial phase space (before reduction) we take the cotangent bundle $T^*(K)$, with the canonical 2-form $\omega$. Results of Lempert and Szöke [30, 35] and of Guillemin and Stenzel [11, 12] show that there is a natural globally defined "adapted complex structure" on $T^*(K)$ determined by the choice of bi-invariant metric on $K$. This complex structure can be described explicitly as follows. First, use left translation to identify $T^*(K)$ with $K \times \mathfrak{k}^\ast$. Then use the inner product on $\mathfrak{k}$ to identify $K \times \mathfrak{k}^\ast$ with $K \times \mathfrak{k}$. Finally, use the diffeomorphism $\Psi : K \times \mathfrak{k} \to K_\mathbb{C}$ given by the polar decomposition for $K_\mathbb{C}$:

$$\Psi(x, \xi) = xe^{-i\xi}, \quad x \in K, \ \xi \in \mathfrak{k}. \quad (2)$$

We then use $\Psi$ to pull back the complex structure on $K_\mathbb{C}$ to $T^*(K)$. The resulting complex structure on $T^*(K)$ fits together with the symplectic structure to make $T^*(K)$ into a Kähler manifold. This claim is a consequence of general results in the theory of adapted complex structures (e.g., the theorem on p. 568 of [11]), but can be verified directly by the calculations in the first appendix of [15].

In this paper, we follow the sign conventions in [17]. With these conventions, the minus sign in the exponent on the right-hand side of (2) is necessary in order to achieve the positivity condition in the definition of a Kähler manifold. Actually, a similar minus sign is needed even in the case of $T^*(\mathbb{R})$. If the canonical 2-form is defined as $\omega = dp \wedge dx$ (as in [17]), then the complex structure must be defined as $z = x - ip$ rather than $x + ip$ in order to for $\omega(X, JX)$ to be non-negative.

2.2 Maximal tori

We fix throughout the paper a maximal torus $T$ of $K$ and we denote its Lie algebra by $\mathfrak{t}$. We let $r$ denote the dimension of $T$. We let $T_\mathbb{C}$ denote the connected subgroup of $K_\mathbb{C}$ whose Lie algebra is $\mathfrak{t}_\mathbb{C} := \mathfrak{t} \oplus i \mathfrak{t}$. If $T$ is isomorphic to $(S^1)^r$, it follows from the polar decomposition for $K_\mathbb{C}$ that $T_\mathbb{C}$ is isomorphic to $(\mathbb{C}^*)^r$, so that $T_\mathbb{C}$ is the complexification of $T$ in the sense of Section 3 of [13].

The bi-invariant metric on $K$ restricts to an invariant metric on $T$. We may then regard the cotangent bundle $T^*(T)$ as a submanifold of $T^*(K)$ using the metrics:

$$T^*(T) \cong T(T) \subset T(K) \cong T^*(K). \quad (3)$$

We may also identify the cotangent bundle $T^*(T)$ with $T_\mathbb{C}$ identifying $T^*(T)$ with $T \times \mathfrak{t}^\ast$ and then with $T \times \mathfrak{t}$ and then applying the map $\Psi' : T \times \mathfrak{t} \to T_\mathbb{C}$.
given by the same formula as in (2):
\[ \Psi'(t, H) = te^{-iH}, \quad t \in T, \ H \in t. \] (4)

We say that a Lie subgroup \( S \) of \( K_C \) is a complex torus if it is isomorphic as a complex Lie group to a direct product of copies of \( \mathbb{C}^* \). We say that \( S \) is a complex maximal torus if it is a complex torus that is not properly contained in another complex torus. The group \( T_C \) is a complex maximal torus and all complex maximal tori are conjugate. (See Corollary A to Theorem 2.13 in [25].)

We let \( R \subset t \) denote the root system associated to the pair \((K,T)\). (Specifically, \( R \) is the set of “real roots,” in the sense of [18, Definition 11.34].) We fix once and for all a set \( R^+ \) of positive roots. We also let \( W := N(T)/T \) denote the Weyl group. By Theorem 11.36 in [18], \( W \) may be identified with the subgroup of the orthogonal group \( O(t) \) generated by the reflections about the hyperplanes perpendicular to the roots. The adjoint action of \( W \) on \( T \) extends to an action on \( T_C \).

We let \( \sigma : T \to \mathbb{R} \) denote the Weyl denominator, given by
\[
\sigma(e^H) = (2i)^m \prod_{\alpha \in R^+} \sin \left( \frac{\langle \alpha, H \rangle}{2} \right), \quad H \in t, \] (5)
where \( m \) is the number of positive roots, or by the alternative expression,
\[
\sigma(e^H) = \sum_{w \in W} \text{sign}(w) e^{i(w \cdot \delta, H)}, \] (6)
where \( \delta \) is half the sum of the positive roots. (See [18, Lemma 10.28] for the equality of these two expressions.) Since \( K \) is simply connected, \( \delta \) is an analytically integral element [18, Corollary 13.21] and \( \sigma \) is therefore a single-valued function on \( T \). We also let \( \sigma_C : T_C \to \mathbb{C} \) be the analytic continuation of the Weyl denominator, which is given by either of the expressions (5) or (6), but with \( H \) now belonging to \( t_C \).

We say that a function \( f \) on \( T \) or \( T_C \) is Weyl alternating if
\[ f(w \cdot z) = \text{sign}(w) f(z) \]
for all \( z \) in \( T \) or \( T_C \). Using either of the expressions for the Weyl denominator, one easily shows that \( \sigma \) and \( \sigma_C \) are Weyl alternating.

### 2.3 The momentum map

We refer to Section 4.2 in [1] for general information about momentum maps. We consider the adjoint action of \( K \) on itself, given by
\[ x \cdot y = xyx^{-1}, \]
and also the induced adjoint action of \( K \) on \( T^*(K) \). Since the action of \( K \) on \( T^*(K) \) is induced from an action on the base, there is an equivariant momentum map
\[ \phi : T^*(K) \to t^* \]
that is linear on each fiber of $T^*(K)$. (See Corollary 4.2.11 in [1].) To describe $\phi$, let us introduce the following notation. For each $\eta \in \mathfrak{k}$, we define the $\eta$-component of $\phi$ to be the function $\phi_\eta : T^*(K) \to \mathbb{R}$ be given by

$$
\phi_\eta(x, \xi) = \phi(x, \xi)(\eta).
$$

(These functions have the property that the Hamiltonian flow generated by $\phi_\eta$ is just the action adjoint action of the one-parameter subgroup of $K$ generated by $\eta$.) Then $\phi$ is determined by the following formula

$$
\phi_\eta(x, \xi) = \xi(Y^\eta(x)),
$$

where $Y^\eta$ is the vector field representing the infinitesimal adjoint action of $\eta$ on $K$.

Let us identify $T^*(K)$ with $K \times \mathfrak{k}$ using left translation and the inner product on $\mathfrak{k}$. Then we may easily compute that $Y^\eta(x) = \text{Ad}_x^{-1}(\eta) - \eta$, so that

$$
\phi_\eta(x, \xi) = \langle \xi, \text{Ad}_x^{-1}(\eta) - \eta \rangle = \langle \text{Ad}_x(\xi) - \xi, \eta \rangle.
$$

Thus, the momentum map, viewed as a map of $K \times \mathfrak{k}$ into $\mathfrak{k}^* \cong \mathfrak{k}$ is given explicitly as

$$
\phi(x, \xi) = \text{Ad}_x(\xi) - \xi. \quad (7)
$$

3 Reduction of the quantum Hilbert space

In this section we consider the Hilbert space obtained by first quantizing the phase space $T^*(K)$ and then reducing by the adjoint action of $K$, which we write as $\text{Quant}(T^*(K))^{\text{Ad}_K}$. We will identify $\text{Quant}(T^*(K))$ as an $L^2$ space of holomorphic functions on $K_\mathbb{C}$ and $\text{Quant}(T^*(K))^{\text{Ad}_K}$ as the corresponding $L^2$ space of holomorphic class functions on $K_\mathbb{C}$.

3.1 Quantization of the cotangent bundle

We briefly explain some of the results in [15], in which the phase space $T^*(K)$ is quantized using geometric quantization with half-forms. We follow the sign conventions in the book [17, Chapters 22 and 23], which differ from those in [15]. We let $\omega$ denote the canonical 2-form on $T^*(K)$, given in local coordinates as $\omega = \sum dp_j \wedge dx_j$. We let $\theta$ be the canonical 1-form on $T^*(K)$, satisfying $d\theta = \omega$. We let $L = T^*(K) \times \mathbb{C}$ be the trivial line bundle over $T^*(K)$, so that sections of $L$ are identified with complex-valued functions on $T^*(K)$. We use the trivial Hermitian structure on $L$, so that the magnitude of a section is just the absolute value of the corresponding function. We define a connection $\nabla$ on $L$ by setting

$$
\nabla_X f = Xf - \frac{i}{\hbar} \theta(X) f
$$

for each smooth section (i.e., function) $f$ and each vector field $X$. 

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We say that a smooth section $f$ of $L$ is a **holomorphic section** if
\[ \nabla_X f = 0 \]
for all vector fields $X$ of type $(0, 1)$ on $T^*(K) \cong K_C$. Although we identify sections with functions, the holomorphic sections do not correspond to holomorphic functions. Rather, the function $\kappa(x, \xi) = |\xi|^2$ is a Kähler potential for $T^*(K) \cong K_C$. This claim follows from a general result [11, p. 568] about adapted complex structures, and is verified by direct computation in the present case in the first appendix to [15]. It then follows easily that the holomorphic sections are precisely those of the form
\[ f = Fe^{-|\xi|^2/(2\hbar)}, \]
where $F$ is a holomorphic function on $K_C$. Here the expression “$\xi$” is defined as a function on $K_C$ by means of the diffeomorphism $\Psi$ in (2).

The **canonical bundle** $K$ for $T^*(K) \cong K_C$ is the holomorphic line bundle whose holomorphic sections are holomorphic $n$-forms, where $n$ is the complex dimension of $K_C$. The canonical bundle is holomorphically trivial, and we will choose a nowhere-vanishing, left-$K_C$-invariant holomorphic $n$-form $\beta$. (The form $\beta$ is unique up to a constant.) We then take a trivial square root $K_{1/2}$ to the canonical bundle, with a trivializing section $\sqrt{\beta}$ satisfying
\[ \sqrt{\beta} \otimes \sqrt{\beta} = \beta. \]

We define a Hermitian structure on $K_{1/2}$ by setting
\[ \left| \sqrt{\beta} \right|^2 = \left[ \frac{\beta \wedge \bar{\beta}}{b\varepsilon} \right]^{1/2}, \tag{9} \]
where $\varepsilon$ is the Liouville volume form on $T^*(K) \cong K_C$:
\[ \varepsilon := \frac{\omega^n}{n!}, \]
and where $b$ is chosen so that at each point $\beta \wedge \bar{\beta}$ is a positive multiple of $b\varepsilon$. We may take, for example,
\[ b = (2i)^n(-1)^n(n^{-1})/2. \]
The quotient $\beta \wedge \bar{\beta}/(b\varepsilon)$ should be interpreted as the unique function $j$ such that $\beta \wedge \bar{\beta} = j b\varepsilon$.

The elements of the **unreduced quantum Hilbert space** $\text{Quant}(K_C)$ are square integrable holomorphic sections of $L \otimes K_{1/2}$. Each section $\psi$ can be expressed uniquely as
\[ \psi = F e^{-|\xi|^2/(2\hbar)} \otimes \sqrt{\beta}, \]
where $F$ is a holomorphic function on $K_C$. The norm of such a section is computed as
\[ \| \psi \|^2 = \int_{K_C} |F(g)|^2 e^{-|\xi|^2/\hbar} \eta \varepsilon, \]
where
\[ \eta = \left[ \frac{\beta \wedge \bar{\beta}}{b \varepsilon} \right]^{1/2} \] (10)
is the function on the right-hand side of (9). An explicit formula for \( \eta \) is given in Eq. (2.10) of [15].

**Conclusion 4** We may quantize the phase space \( T^*(K) \cong K_C \) in such a way that each element \( \psi \) of the \( \text{Quant}(K_C) \) has the form
\[ \psi = Fe^{-|\xi|^2/(2\hbar)} \otimes \sqrt{\beta}, \]
where \( F \) is a holomorphic function on \( K_C \). The norm of \( \psi \) is the \( L^2 \) norm of \( F \) with respect to the measure
\[ \gamma_\hbar := e^{-|\xi|^2/\hbar \varepsilon}, \] (11)
where \( \varepsilon \) is the Liouville volume measure and \( \eta \) is as in (10).

The preceding result is a straightforward computation, first done in [15], using the methods of geometric quantization. What is remarkable about the result is that the measure \( \gamma_\hbar \) coincides up to a constant with a measure on \( K_C \) introduced from a very different point of view in [13].

**Proposition 5** For each \( \hbar > 0 \), there is a constant \( c_\hbar > 0 \) such that the measure \( \gamma_\hbar \) in (11) coincides with the “K-averaged heat kernel measure” \( \nu_\hbar(g) \ dg \) occurring in [13, Theorem 2].

The paper [15] also considers the “BKS pairing map” between the quantization of \( T^*(K) \cong K_C \) obtained using the Kähler polarization and the quantization obtained using the vertical polarization. The result is that the pairing map coincides up to a constant with the generalized Segal–Bargmann transform introduced in [13]. In particular, the pairing map is a constant multiple of a unitary map, something that is certainly not true for a typical pair of polarizations on a symplectic manifold.

For the purposes of the present paper, the importance of Proposition 5 is that it is used in the proof of a critical result—described in Section 3.3—of Florentino, Mourão, and Nunes.

### 3.2 The invariant subspace

For each smooth function \( \phi \) on \( T^*(K) \), we let \( X_\phi \) denote the associated Hamiltonian vector field, which satisfies \( \omega(X_\phi, \cdot) = \mathcal{d}\phi \). We then define the prequantum operator \( Q_{\text{pre}}(\phi) \), acting on the space of smooth sections of \( L \), by
\[ Q_{\text{pre}}(\phi) = i\hbar \nabla_{X_\phi} + \phi. \] (12)

If the Hamiltonian flow generated by \( \phi \) preserves the polarization on \( T^*(K) \)—that is, if the Hamiltonian flow is holomorphic on \( K_C \)—then we can define a
(typically unbounded) quantum operator $Q(\phi)$ on the quantum Hilbert space by the formula

$$Q(\phi)[f \otimes \sqrt{\beta}] = (Q_{\text{pre}}(\phi)f) \otimes \sqrt{\beta} + if \otimes \left[\frac{1}{2} \frac{L_{X_{\phi}}(\beta)}{\beta} - \sqrt{\beta}\right].$$

(There is a typographical error in Definition 23.52 of [17]; the sign on the right-hand side should be plus rather than minus.)

We now specialize to the case in which $\phi$ is $\phi_\eta$, one of the components of the momentum map. Under our identification of $T^*(K)$ with $K_C$, the adjoint action of $K$ on $T^*(K)$ corresponds to the conjugation action of $K$ on $K_C$, which is holomorphic. Thus, $Q(\phi_\eta)$ is a well-defined operator on the quantum Hilbert space.

**Definition 6** We say that an element $\psi$ of the quantization of $T^*(K)$ is invariant if

$$Q(\phi_\eta)\psi = 0$$

for all $\eta \in \mathfrak{k}$. The reduced quantum Hilbert space is the space of all invariant sections.

We now compute the space of invariant sections explicitly.

**Proposition 7** Suppose we write an element $\psi$ of the quantum Hilbert space as

$$\psi = Fe^{-|\xi|^2/(2\hbar)} \otimes \sqrt{\beta}$$

as in Conclusion 4. Then $\psi$ is invariant in the sense of Definition 6 if and only if the holomorphic function $F$ is a class function on $K_C$. Thus, the reduced quantum Hilbert space $\text{Quant}(K_C)^{\text{Ad}_K}$ is the space of holomorphic class functions on $K_C$ that are square integrable with respect to the measure $\gamma_\hbar$ in (11).

**Proof.** We first make an observation about the operator $Q_{\text{pre}}(\phi)$ in (12). By the definition of the covariant derivative, we have

$$Q_{\text{pre}}(\phi) = \mathcal{L}_{X_{\phi}} + \theta(X_{\phi}) + \phi,$$

where $\theta$ satisfies $d\theta = \omega$. Now, by Cartan’s formula for the Lie derivative

$$\mathcal{L}_{X_{\phi}} = d[i_{X_{\phi}}\theta] + i_{X_{\phi}}(d\theta)$$

$$= d[\theta(X_{\phi})] + \omega(X_{\phi}, \cdot)$$

$$= d[\theta(X_{\phi}) + \phi].$$

Thus, if $\mathcal{L}_{X_{\phi}} \theta = 0$, we have $d[\theta(X_{\phi}) + \phi] = 0$.

For any $\eta \in \mathfrak{k}$, the Hamiltonian vector field $X_{\phi_\eta}$ is the generator of the adjoint action of the one-parameter subgroup $e^{t\eta}$ on $T^*(K)$. We denote this vector field more compactly as $X^\eta$. Now, if we take $\phi = \phi_\eta$, the action of $X_{\phi_\eta} = X^\eta$ on $T^*(K)$ is induced from an action on the base. But any such action will
preserve the canonical 1-form $\theta$, showing that $\mathcal{L}_{X^\eta}\theta = 0$. We conclude, then, that $d[\theta(X^\eta) + \phi_\eta] = 0$, showing that $\theta(X^\eta) + \phi_\eta$ is a constant for each $\eta \in \mathfrak{t}$. But since $\theta(X^\eta)$ and $\phi_\eta$ are easily seen to be zero on the zero section inside $T^*(K)$, we actually have $\theta(X^\eta) + \phi_\eta = 0$. Thus, we have simply

$$Q_{pre}(\phi_\eta) = i\hbar X^\eta. \quad (13)$$

Finally, we claim that the form $\beta$ is invariant under the adjoint action of $K$. To see this, observe that if we transform $\beta$ by the adjoint action of $x \in K$, the resulting form $x \cdot \beta$ will still be a left-invariant holomorphic $n$-form, which must agree with $\beta$ up to a constant. It thus suffices to compare $x \cdot \beta$ to $\beta$ at the identity. But at the identity, $x \cdot \beta = \det(\text{Ad}_x)\beta = \beta$, because $\text{Ad}_x \in SO(\mathfrak{t}) \subset SO(\mathfrak{k}_C; \mathbb{C})$. Since the function $e^{-|\xi|^2/(2\hbar)}$ is also invariant under the adjoint action of $K$, we obtain

$$Q(\phi_\eta)\psi = (X^\eta F)e^{-|\xi|^2/(2\hbar)} \otimes \sqrt{\beta}. \quad (13)$$

Thus, the invariant elements are those for which $X^\eta F = 0$, i.e., those invariant under the adjoint action of $K$. But since $F$ is holomorphic, if $F$ is invariant under the adjoint action of $K$, it is also invariant under the adjoint action of $K_C$; that is, $F$ is a holomorphic class function on $K_C$. 

### 3.3 The theorem of Florentino, Mourão, and Nunes

The goal of this section is to describe a formula, obtained by Florentino, Mourão, and Nunes in [6], for computing the $L^2$ norm of a holomorphic class function $F$ with respect to the measure $\gamma_\hbar$ in (11). The formula expresses the square of the $L^2$ of $F$ as a certain integral of $|F|^2$ over $T_C$. Now, almost every point in $K_C$ is conjugate to a point—unique up to the action of $W$—in $T_C$. It is therefore easy to show that there is some $W$-invariant measure $\mu_\hbar$ on $T_C$ such that

$$\int_{K_C} |F(g)|^2 \, d\gamma_\hbar = \int_{T_C} |F(z)|^2 \, d\mu_\hbar(z),$$

for all functions $F$ (not necessarily holomorphic). What is not obvious is whether there is any way to compute $\mu_\hbar$ explicitly.

To describe the result of Florentino, Mourão, and Nunes, we make use of Proposition 5 which relates the measure $\gamma_\hbar$ in (11) to a heat kernel measure on $K_C$. We fix a Haar measure $dg$ on $K_C$ and consider the “$K$-averaged heat kernel” $\nu_\hbar$ on $K_C$ [13] Theorem 2], normalized so that $\nu_\hbar(g) \, dg$ is a probability measure. (This measure is just the heat kernel measure for the noncompact symmetric space $K_C/K$, viewed as a $K$-invariant measure on $K_C$.) We let $dz$ and $\nu'_h$ be the analogous objects on $T_C$.

**Theorem 8 (Florentino, Mourão, and Nunes)** If $F$ is a holomorphic class function on $K_C$, then

$$\int_{K_C} |F(g)|^2 \nu_\hbar(g) \, dg = \frac{e^{-\hbar||\xi||^2}}{|W|} \int_{T_C} |\sigma_C(z)F(z)|^2 \nu'_h(z) \, dz, \quad (14)$$

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where $\delta$ is half the sum of the positive roots. Furthermore, if $\Phi : T_C \to \mathbb{C}$ is a $W$-alternating holomorphic function for which

$$\int_{T_C} |\Phi(z)|^2 \nu_h(z) \, dz < \infty,$$

then there exists a unique holomorphic class function $F$ on $K_C$ that is square integrable with respect to $\nu_h$ and such that

$$(\sigma_C)(F|_{T_C}) = \Phi.$$

Recall from Proposition 5 that the measure $\nu_h(g) \, dg$ coincides up to a constant with the measure $\gamma_h$ on $T^*(K) \cong K_C$ that arises in geometric quantization. (See (11).) A similar statement applies to the measure $\nu'_h(z) \, dz$ on $T^*(T) \cong T_C$. Thus, Theorem 8 gives us an explicit way of computing the norm of an invariant element of the quantum Hilbert space as an integral over $T_C$. Note also the key role played by the analytically continued Weyl denominator $\sigma_C$: The integral on the right-hand side of (14) is computing the square of the $L^2$ norm of $(\sigma_C)(F|_{T_C})$—rather than the norm of $F|_{T_C}$—with respect to $\nu'_h(z) \, dz$.

As we have discussed in Section 1.4.1 the main goal of this paper is to interpret Theorem 8 as a unitary “quantization commutes with reduction” result. To achieve this goal, we must (1) show that the quantization of $K_C//\text{Ad}_K$ can be identified with a space of holomorphic functions on $T_C$, and (2) show that the map

$$F \mapsto (\sigma_C)(F|_{T_C})$$

arises from a Guillemin–Sternberg-type map with half-forms, similar to the one in [19].

Theorem 8 result is remarkably similar to the Weyl integral formula (e.g., [18, Proposition 12.24]), which states that if $f$ is a continuous class function on $K$, we have

$$\int_K |f(x)|^2 \, dx = \frac{1}{|W|} \int_T |\sigma(t)f(t)|^2 \, dt,$$

where $dx$ and $dt$ are the normalized Haar measures on $K$ and $T$, respectively. In passing from $K$ and $T$ to $K_C$ and $T_C$, we merely replace the Haar measures with heat kernel measures, change $\sigma$ to $\sigma_C$, and add a factor of $e^{-h\|b\|^2}$ on the right-hand side.

A result similar to Theorem 8 for holomorphic functions in the dual noncompact setting was given by Hall and Mitchell. (Compare the isometry theorem in [21] in the general case to the isometry theorem in [20] in the radial case.)

Florentino, Mourão, and Nunes give two proofs of Theorem 8, one of which (proof of Theorem 2.3 in [6]) actually applies to an arbitrary measurable (i.e., not necessarily holomorphic) class function $F$. Since the holomorphic case of Theorem 8 is a vital result for this paper, we outline the proof of this case, following [6]. Note that the statements in [6] differ by various factors of $2\pi$ from our statement of Theorem 8 because of differences in the scaling of the heat equation.
The ingredients of the proof are the generalized Segal–Bargmann transform for the group $K$ (see [13]), the analogous transform for $T$, and the Weyl integral formula. The Segal–Bargmann transform for $K$ is the map $C_\hbar : L^2(K) \to \mathcal{H}(K_C)$ given by

$$C_\hbar(f) = (e^{I\Delta_K/2} f)_C,$$

where $\Delta_K$ is the (negative) Laplacian for $K$, $e^{I\Delta_K/2}$ is the associated heat operator, and $\cdot)_C$ denotes analytic continuation from $K$ to $K_C$. The transform is a unitary map from $L^2(K, dx)$ onto $\mathcal{H}L^2(K_C, \nu_\hbar(g) \, dg)$, where $dx$ is the normalized Haar measure on $K$ and $\mathcal{H}L^2$ denotes the space of square integrable holomorphic functions [13, Theorem 2]. Since $T$ is also a connected compact Lie group, there is a similar unitary map from $L^2(T, dt)$ to $\mathcal{H}L^2(T_C, \nu_\hbar(z) \, dz)$.

**Proof.** For each dominant integral element $\mu$, let $\chi_\mu : K \to \mathbb{C}$ denote the character of the irreducible representation of $K$ with highest weight $\mu$. Then $\chi_\mu$ has a holomorphic extension to $K_C$, denoted $(\chi_\mu)_C$. We consider first the case that $F = (\chi_\mu)_C$. Now, the character $\chi_\mu$ satisfies

$$\Delta_K(\chi_\mu) = -\langle |\mu + \delta|^2 - |\delta|^2 \rangle \chi_\mu. \quad (15)$$

(This claim follows easily from Proposition 10.6 in [18].) Thus, if we take $f = e^{I\hbar(\langle |\mu + \delta|^2 - |\delta|^2 \rangle \chi_\mu}$, we will have $C_\hbar(f) = F$. By the isometricity of $C_\hbar$ and the Weyl integral formula, we then have

$$\int_{K_C} |F(g)|^2 \nu_\hbar(g) \, dg = \int_K |f(x)|^2 \, dx = \frac{e^{I\hbar(\langle |\mu + \delta|^2 - |\delta|^2 \rangle \chi_\mu}}{|W|} \int_T |\sigma_\chi_\mu(t)|^2 \, dt, \quad (16)$$

where $dx$ and $dt$ are the normalized Haar measures on $K$ and $T$, respectively.

Meanwhile, the function $\sigma_\chi_\mu$ on $T$ satisfies

$$\Delta_T(\sigma_\chi_\mu) = -\langle |\mu + \delta|^2 \rangle \chi_\mu; \quad (17)$$

note the shift in the eigenvalue between (15) and (17). This claim follows from the special form of the “radial part” of the Laplacian on a compact Lie group. (See Proposition 2.3 on p. 278 of [2]; the proof is essentially the same as the proof of Proposition 3.10 in Chapter II of [22] in the dual noncompact setting.) But (17) also follows easily from the Weyl character formula: The numerator in the character formula is easily seen to be an eigenfunction of $\Delta_T$ with the stated eigenvalue. The isometricity of the Segal–Bargmann transform for $T$ then tells us that

$$\frac{e^{I\hbar(\langle |\mu + \delta|^2 - |\delta|^2 \rangle \chi_\mu}}{|W|} \int_T |\sigma_\chi_\mu(t)|^2 \, dt$$

$$= \frac{e^{-I\hbar|\delta|^2}}{|W|} \int_{T_C} |\sigma_\chi_\mu(z)|^2 \langle (\chi_\mu)_C(z) \rangle^2 \nu_\hbar(z) \, dz. \quad (18)$$

Combining (16) and (18) establishes (14) when $F = (\chi_\mu)_C$.

Now, it follows from the “holomorphic Peter–Weyl theorem” of [13, Theorem 9] that the functions $(\chi_\mu)_C$ form an orthogonal basis for the space of holomorphic
class functions in \( L^2(K, \nu_\hbar(g) \, dg) \). Meanwhile, it is not hard to show that the functions \((\chi_\mu)_C\) are also orthogonal in \( L^2(T, \nu'_\hbar(z) \, dz) \). (Using the Segal–Bargmann transform for \( T \) along with \([17]\), the desired result reduces to the orthogonality of the functions \( \sigma_\chi_\mu \) in \( L^2(T, dt) \), which is a consequence of the Weyl character formula.) Thus, the general version of \([14]\) reduces to the already established case for characters.

Finally, suppose \( \Phi : T \to \mathbb{C} \) is as in the second part of the theorem. We can expand \( \Phi \) in a Fourier–Laurent series in terms of the exponential functions

\[
 f_\lambda(e^H) = e^{i\langle \lambda, H \rangle}, \quad H \in t_C,
\]

where \( \lambda \) ranges over all integral elements in \( t \). (If we identify \( T \) with \((C^*)^r\), these functions are just the monomials.) If \( \Phi \) is \( W \)-alternating, the coefficients in the expansion of \( \Phi \) must also be \( W \)-alternating. Thus, the coefficient of \( f_\lambda \) will be zero if \( \lambda \) belongs to any of the walls of the Weyl chambers. The coefficients where \( \lambda \) is not in the wall of any chamber, meanwhile, can be grouped into Weyl orbits. If \( \lambda \) is in the interior of the fundamental Weyl chamber, then \( \lambda = \mu + \delta \) for some \( \mu \) in the closed fundamental Weyl chamber \([18, \text{Proposition 8.38}]\). The group of exponentials coming from the Weyl-orbit of \( \lambda \) is then the numerator in the Weyl character formula for the representation with highest weight \( \mu \). The desired \( F \) can then be constructed as a linear combination of the analytically continued characters \((\chi_\mu)\), with the isometricity in \([14]\) guaranteeing convergence of the expansion.

\[ \blacksquare \]

### 4 Reduction of the classical phase space

Recall from \([4]\) that we think of \( T^*(T) \) as a submanifold of \( T^*(K) \). We are going to identify a “regular set” \( \phi^{-1}(0)^{\text{reg}} \) inside the zero set \( \phi^{-1}(0) \) of the momentum map. We are mainly interested in the regular part of the reduced phase space,

\[
 \phi^{-1}(0)^{\text{reg}}/\Ad_K,
\]

which is referred to as the “principal stratum” in \([24]\) and \([3]\). We will see that \( T^*(T) \) is contained in \( \phi^{-1}(0) \); we then define \( T^*(T)^{\text{reg}} \) as the intersection of \( T^*(T) \) with \( \phi^{-1}(0)^{\text{reg}} \). We will show that the regular part of the reduced phase space is a smooth symplectic manifold, which may be identified as

\[
 \phi^{-1}(0)^{\text{reg}}/\Ad_K = T^*(T)^{\text{reg}}/W.
\]

In addition, we will show that \( \phi^{-1}(0)^{\text{reg}}/\Ad_K \) inherits a Kähler structure from the Kähler structure on \( T^*(K) \cong K_C \). As a complex manifold, we have

\[
 \phi^{-1}(0)^{\text{reg}}/\Ad_K \cong T_C^{\text{reg}}/W,
\]

where \( T_C^{\text{reg}} \) denotes the set of “regular semisimple” points in \( T_C \). The reader who wishes to take these identifications on faith may look at the statements of Theorems \([14]\) and \([17]\) and then proceed to Section \([5]\).
Although some of the calculations in this section have appeared elsewhere (e.g., Section 1 of [23] or Section 4.1 of [3]), we give special emphasis to identifying the regular set and it is therefore simplest to give complete proofs.

### 4.1 The zero set of the momentum map

Recall the formula for the momentum map $\phi : T^*(K) \to \mathfrak{t}^*$ given in (7) in Section 2.3. From the formula, we immediately obtain that the zero-set of $\phi$ is as follows:

$$\phi^{-1}(0) = \{ (x, \xi) \in T^*(K) | \text{Ad}_x(\xi) = \xi \}.$$  \hspace{1cm} (19)

Recall also that we identify $T^*(T)$ as a subset of $T^*(K)$ as in (3).

**Proposition 9** Every point in $T^*(T)$ belongs to $\phi^{-1}(0)$ and each $\text{Ad}_K$-orbit in $\phi^{-1}(0)$ intersects $T^*(T)$ in exactly one $W$-orbit.

**Proof.** First, since $T$ is commutative, every point in $T^*(T)$ certainly satisfies the condition in (19). Second, suppose that $(x, \xi) \in \phi^{-1}(0)$. Then $x$ commutes with every element of the connected, commutative subgroup $S := \{ e^{t\xi} \}_{t \in \mathbb{R}}$ of $K$. Thus, by Lemma 11.37 of [18], there is a maximal torus $S'$ that contains both $x$ and $S$. By the torus theorem, $S'$ is conjugate to $T$. Thus, there is some $y \in K$ such that $h := yxy^{-1}$ belongs to $T$ and $H := y\xi y^{-1}$ belongs to $t$, showing that $(x, \xi)$ can be moved to a point in $T^*(T)$.

Last, suppose $(t, H)$ and $(t', H')$ in $T^*(T)$ belong to the same $\text{Ad}_K$-orbit. A standard result in the theory of compact groups says that if two elements of $T$ are conjugate in $K$, they belong to the same Weyl group orbit. The same proof applies without change here to show that $(t, H)$ and $(t', H')$ must be in the same Weyl group orbit. In the proof of Theorem 11.39 in [18], for example, we may simply replace the centralizer of $t$ by the stabilizer of $(t, H)$ and the argument goes through without change. \[\blacksquare\]

### 4.2 Regular points

The action of $K$ on $\phi^{-1}(0)$ is not even generically free. We can nevertheless identify a “regular set” in $\phi^{-1}(0)$ where the stabilizer is as small as possible.

Define, for each $(x, \xi) \in \phi^{-1}(0)$, the **stabilizer** $S_{(x, \xi)}$ as

$$S_{(x, \xi)} = \{ y \in K | yxy^{-1} = x, \text{ Ad}_y(\xi) = \xi \}.$$  

It follows from Proposition 9 that for all $(x, \xi) \in \phi^{-1}(0)$, the stabilizer of $(x, \xi)$ contains a maximal torus in $K$.

**Definition 10** A point $(x, \xi)$ in $\phi^{-1}(0)$ is called **regular** if $S_{(x, \xi)}$ is a maximal torus in $K$.

The set of regular points is referred to as the “principal stratum” in [24] and [3]. The other strata in those papers are defined by specifying the conjugacy class of the stabilizer.
We would like to understand when a point in $\phi^{-1}(0)$ is regular. In light of Proposition 9 it suffices to consider points in $T^*(T)$.

**Theorem 11** Consider a point $(e^{H_1}, H_2)$ in $T^*(T) \subset \phi^{-1}(0)$. Then $(e^{H_1}, H_2)$ is regular if and only if for each root $\alpha$, we have either

$$\langle \alpha, H_1 \rangle \not\in 2\pi \mathbb{Z}$$

or

$$\langle \alpha, H_2 \rangle \neq 0.$$  

It should be emphasized that this characterization of the regular set in $\phi^{-1}(0)$ is valid only because of our standing assumption that $K$ is simply connected. (This assumption is used in the proof of Proposition 15.) Note that the set of regular points in $T^*(T)$ is open and dense in $T^*(T)$.

**Corollary 12** The set of regular points in $\phi^{-1}(0)$ is open and dense.

**Proof.** Given $(x, \xi) \in \phi^{-1}(0)$, we can find (Proposition 9) some $y \in K$, some $t \in T$, and some $H \in \mathfrak{t}$ such that $(x, \xi) = y \cdot (t, H)$. We can then find some $(t', H')$ very near $(t, H)$ in $T^*(T)$ that satisfies the condition in Theorem 11. Thus, $y \cdot (t', H')$ is a regular point in $\phi^{-1}(0)$ very near to $(x, \xi)$, showing that the regular set is dense.

Suppose now that $(x, \xi) \in \phi^{-1}(0)$ is regular and that $(x_n, \xi_n)$ is a sequence in $\phi^{-1}(0)$ converging to $(x, \xi)$. Then we can write $(x_n, \xi_n) = y_n \cdot (t_n, H_n)$ for some $y_n \in K$, $t_n \in T$, and $H_n \in \mathfrak{t}$. Since $\xi_n$ is converging to $\xi$, there is some constant $C$ such that

$$\|H_n\| = \|\xi_n\| \leq C.$$  

Thus, using compactness, we can extract convergent sequences and assume that $y_n \to y$, $t_n \to t$, and $H_n \to H$. Then

$$(x, \xi) = \lim_{n \to \infty} (x_n, \xi_n) = \lim_{n \to \infty} y_n \cdot (t_n, H_n) = y \cdot (t, H).$$

Since $(x, \xi)$ is assumed regular, $(t, H)$ must be in the regular set in $T^*(T)$. But this set is open in $T^*(T)$, showing that $(t_n, H_n)$ and therefore also $y_n \cdot (t_n, H_n)$ are regular for all sufficiently large $n$. ■

We now give the proof of Theorem 11, which consists of a series of propositions.

**Proposition 13** For each $(x, \xi) \in \phi^{-1}(0)$, the stabilizer $S_{(x, \xi)}$ coincides with the intersection of the centralizer of $\Psi(x, \xi) := xe^{-i\xi} \in K_{\mathbb{C}}$ with $K$:

$$S_{(x, \xi)} = C_K(xe^{-i\xi}) = \{ y \in K | y(xe^{-i\xi})y^{-1} = xe^{-i\xi} \}.$$

**Proof.** Clearly, if $y$ commutes with both $x$ and $\xi$, then $y$ commutes with $xe^{-i\xi}$. Conversely, if $y \in K$ and $y$ commutes with $xe^{-i\xi}$, then $y$ must commute with both $x$ and $\xi$. After all,

$$y(xe^{-i\xi})y^{-1} = (yx) e^{-iAd_{\xi}}(x).$$
By the uniqueness of the polar decomposition, the above quantity equals \( xe^{-i\xi} \) only if \( yxy^{-1} = x \) and \( \text{Ad}_y(\xi) = \xi \). 

**Definition 14** An element \( g \) of \( K_C \) is called **regular semisimple** if the centralizer of \( g \) is a complex maximal torus in \( K_C \). We denote the set of regular semisimple elements in \( K_C \) by \( K^{rs}_C \) and the set of regular semisimple elements in \( T_C \) by \( T^{rs}_C \).

Since \( K \) and \( K_C \) are assumed simply connected, Steinberg’s theorem [26, Theorem 2.11] says that the centralizer of every semisimple element is connected. Thus, in our setting, Definition 14 is equivalent to the usual definition of a regular semisimple element (e.g., [26, Section 1.6]).

**Proposition 15** For each root \( \alpha \), let \( \phi_\alpha : T_C \to \mathbb{C}^* \) be the associated root homomorphism given by

\[
\phi_\alpha(e^H) = e^{i\langle \alpha, H \rangle}.
\]

(20)

Then \( z \in T_C \) is regular semisimple if and only if for all \( \alpha \in \mathbb{R} \), we have \( \phi_\alpha(z) \neq 1 \).

The factor of \( i \) in the exponent in the formula for \( \phi_\alpha \) is a result of our convention for using real roots [18, Definition 11.34].

**Proof.** As we have noted, the assumption that \( K \) is simply connected ensures that our notion of a regular semisimple element is equivalent to the one in [26]. The result then follows immediately from Proposition 2.3 in [26].

**Proposition 16** A point \( (x, \xi) \in \phi^{-1}(0) \) is regular if and only if the element \( \Psi(x, \xi) = xe^{-i\xi} \) in \( K_C \) is regular semisimple.

**Proof.** In light of Proposition 15, it is harmless to assume that \( x \in T \) and \( \xi \in t \), so that \( xe^{-i\xi} \in T_C \). We will prove that \( (x, \xi) \) fails to be regular if and only if \( xe^{-i\xi} \) fails to be regular semisimple. Suppose first that \( (x, \xi) \) fails to be regular. Then the stabilizer of \( (x, \xi) \) contains an element \( y \in K \) that is not in \( T \). Then \( y \) is not in the complexification \( T_C \) of \( T \), since \( T_C \cap K = T \). Then the centralizer of \( xe^{-i\xi} \) contains \( y \) and thus properly contains the complex maximal torus \( T_C \), showing that \( xe^{-i\xi} \) is not regular semisimple.

Suppose now that \( z := xe^{-i\xi} \in T_C \) fails to be regular semisimple. Then by Proposition 14, \( z \) belongs to the kernel of \( \phi_\alpha \) for some \( \alpha \), and therefore also to the kernel of \( \phi_{-\alpha} = 1/\phi_\alpha \). But then if \( X \) is in the root space \( (\mathfrak{t}_C)_\alpha \) we have

\[
\text{Ad}_z(X) = \phi_\alpha(z)X = X
\]

and similarly if \( X \in (\mathfrak{t}_C)_{-\alpha} \). Thus, the Lie algebra of the centralizer of \( z \) contains \( (\mathfrak{t}_C)_\alpha \oplus (\mathfrak{t}_C)_{-\alpha} \), which contains elements of \( \mathfrak{t} \) not in \( t \) [18, Corollary 7.20].

We are now in a position to complete the proof of Theorem 11. By Proposition 15 a point \((t, H)\) in \( T^*(T) \subset \phi^{-1}(0) \) is regular if and only if the corresponding point \( z = xe^{-i\xi} \) in \( T_C \) is regular semisimple, which holds (Proposition 14) if and only if \( \phi_\alpha(z) \) is different from 1 for all \( \alpha \). But

\[
\phi_\alpha(e^{H_1}e^{-iH_2}) = e^{i\langle \alpha, H_1 \rangle}e^{i\langle \alpha, H_2 \rangle} = 1
\]

if and only if \( \langle \alpha, H_1 \rangle \in 2\pi\mathbb{Z} \) and \( \langle \alpha, H_2 \rangle = 0 \).
4.3 The reduced phase space

Suppose \((M, \omega)\) is a symplectic manifold and \(K\) is a compact Lie group acting symplectically on \(M\). If the action of \(K\) admits an equivariant momentum map \(\phi\), the \textit{symplectic quotient} (or Marsden–Weinstein quotient) \(M//G\) is defined as the ordinary quotient of \(\phi^{-1}(0)\) by \(K\):

\[
M//K := \phi^{-1}(0)/K.
\]

Suppose, for example, that \(M = T^*(N)\) and the action of \(K\) on \(T^*(N)\) is induced from a regular action of \(K\) on \(N\). (The action is called regular if the stabilizers of any two points are conjugate, for example, if the action is free.) Then an equivariant momentum map may be constructed that is linear on each fiber in \(T^*(N)\), and we have

\[
T^*(N)//K \cong T^*(N/K).
\]  \hspace{1cm} (21)

(See Section 4.3 in [1] and especially the \(\mu = 0\) case of Theorem 4.3.3.)

In our case, \(M = T^*(K)\) and \(K\) acts on itself—and therefore also on \(T^*(K)\)—by the adjoint action. The adjoint action of \(K\) on itself is not regular, however, and the quotient \(K/\text{Ad}K\) is not a manifold. Rather, \(K/\text{Ad}K\) is identified with \(T/W\), which even when \(K = SU(2)\) is a closed interval rather than a smooth manifold. In light of (21), we expect that \(T^*(K)/\text{Ad}K\) should be something like \(T^*(T/W)\). Since \(T/W\) is not a manifold, however, the correct statement is that \(T^*(K)/\text{Ad}K\) is homeomorphic to \(T^*(T)/W\). (This claim follows from Proposition 9.)

In this paper, we will focus on the set \(\phi^{-1}(0)_{\text{reg}}\) of regular points in \(\phi^{-1}(0)\), and the quotient \(\phi^{-1}(0)_{\text{reg}}/\text{Ad}K\). To describe this quotient, recall that we think of \(T^*(T)\) as a submanifold of \(T^*(K)\) as in (3) and that \(T^*(T)\) is contained in \(\phi^{-1}(0)\). We define the regular set \(T^*(T)_{\text{reg}}\) in \(T^*(T)\) as

\[
T^*(T)_{\text{reg}} = T^*(T) \cap \phi^{-1}(0)_{\text{reg}}.
\]

Then \(T^*(T)_{\text{reg}}\) is an open dense subset of \(T^*(T)\) and the Weyl group acts freely on this set.

\textbf{Theorem 17} The quotient \(\phi^{-1}(0)_{\text{reg}}/\text{Ad}K\) is a smooth manifold, which may be identified as

\[
\phi^{-1}(0)_{\text{reg}}/\text{Ad}K \cong T^*(T)_{\text{reg}}/W.
\]

This manifold inherits a Kähler structure from the Kähler structure on \(T^*(K) \cong K_C\). The symplectic structure on the quotient comes from the canonical symplectic structure on \(T^*(T)\) and the complex structure on the quotient comes from the complex structure obtained by identifying \(T^*(T)\) with \(T_C\).

Recall also that (Proposition 10) under the identification of \(T^*(T)\) with \(T_C\), the set of regular points in \(T^*(T)\) corresponds to the set of regular semisimple points in \(T_C\). Thus, as a complex manifold, we may think of the regular reduced phase space as

\[
\phi^{-1}(0)_{\text{reg}}/\text{Ad}K \cong T_C^*/W.
\]  \hspace{1cm} (22)
We now give the proof of Theorem 17, which consists of a series of propositions.

Proposition 18  The set $\phi^{-1}(0)^{\text{reg}}$ is a smooth embedded submanifold of $T^*(K)$.

Proof. By Proposition 9, every element of $\phi^{-1}(0)^{\text{reg}}$ can be obtained from a point in $T^*(T)^{\text{reg}}$ by the action of $K$, and the point in $T^*(T)^{\text{reg}}$ is unique up to the action of $W$. Since the stabilizer of each point in $T^*(T)^{\text{reg}}$ is $T$, we have a smooth surjective map

$$f : T^*(T)^{\text{reg}} \times (K/T) \to \phi^{-1}(0)^{\text{reg}}$$

that is $|W|$ to one, given by

$$f((t,H), [y]) = y \cdot (t,H).$$

We claim that $f$ has a continuous local inverse. This claim amounts to saying that for $(x,\xi)$ in a small open set in $\phi^{-1}(0)^{\text{reg}}$, it is possible to choose the point in $T^*(T)^{\text{reg}}$ to depend continuously on $(x,\xi)$. This last point is a standard argument similar to the proof of Corollary 12 and is omitted.

It thus suffices to show that the differential of $f$ is injective at each point. By the $K$-equivariance of $f$, it suffices to check this at a point of the form $((t,H), [e])$ with $(t,H) \in T^*(T)^{\text{reg}}$. Let us identify the tangent space to $T^*(K)$ at any point with $\mathfrak{k} \oplus \mathfrak{k}$ using left translation and the inner product on $\mathfrak{k}$. Then the image of $f_*$ at the point $((t,H), [e])$ is easily computed to consist of vectors of the form

$$(H_1, H_2) + (\text{Ad}_{t^{-1}}(X) - X, [H,X]), \quad H_1, H_2 \in \mathfrak{t}, \quad X \in \mathfrak{k},$$

where the second term is easily seen to lie in $\mathfrak{t}^\perp \oplus \mathfrak{t}^\perp$. Now, if

$$(\text{Ad}_{t^{-1}}(X) - X, [H,X]) = (0,0),$$

then $X$ is in the Lie algebra of $S(x,\xi)$, i.e., $X \in \mathfrak{t}$. If follows that the dimension of the image of $f_*$ equals $2 \dim T + \dim(\mathfrak{k}) - \dim(\mathfrak{t})$, showing that $f_*$ is injective.

Proposition 19  The action of $K$ on $\phi^{-1}(0)^{\text{reg}}$ set is regular (i.e., all stabilizers are conjugate). Thus, the quotient is a manifold, which may be identified as

$$\phi^{-1}(0)^{\text{reg}}/\text{Ad}_K \cong T^*(T)^{\text{reg}}/W.$$

Proof. By definition, the stabilizer of every point in $\phi^{-1}(0)^{\text{reg}}$ is a maximal torus in $K$, and all such tori are conjugate. A general result then shows that the quotient is a manifold. (In general, the quotient of a manifold by a compact group action is “stratified” by manifolds associated to different strata, but if all stabilizers are conjugate, there is only one stratum. See, for example, Section 2.7 in [5].) By Proposition 9 every element of $\phi^{-1}(0)^{\text{reg}}$ can be moved by the action of $K$ to a point in $T^*(T)^{\text{reg}}$ that is unique up to the action of $W$, giving the claimed identification of the quotient.
Proposition 20  The quotient manifold $T^*(T)^{reg}/W$ inherits a symplectic structure, which comes from the canonical symplectic structure on $T^*(T)^{reg}$. 

Proof. For each point in the quotient, we choose a preimage in $\phi^{-1}(0)^{reg}$, which may be taken to be a point $(t, H)$ in $T^*(T)^{reg}$. Let $V$ denote the tangent space to $\phi^{-1}(0)^{reg}$ at $(t, H)$ and let $W \subset V$ denote the tangent space to the $Ad_K$-orbit through $(t, H)$. We then restrict the canonical 2-form $\omega$ on $T^*(K)$ to $V$. By an elementary general result, $\omega(w, v) = 0$ for all $w \in W$ and $v \in V$. (See [1, Lemma 4.3.2].) Thus, $\omega$ descends to the quotient space $V/W$, which is just the tangent space to the reduced manifold $\phi^{-1}(0)^{reg}/Ad_K$. 

We now compute this reduced form. The tangent space to $\phi^{-1}(0)^{reg}$ at $(t, H)$ is the direct sum of the tangent space to $T^*(T)$ and the tangent space to $K/T$, which is just the tangent space to the $Ad_K$-orbit of $(t, H)$. In light of the just-cited general result, the symplectic form on the reduced space will be just the restriction of the canonical 2-form on $T^*(K)$ to $T^*(T)$, which is the canonical 2-form on $T^*(T)$. ■ 

Proposition 21  The complex structure on $T^*(K) \cong K_C$ descends to the complex structure on $T^*(T)^{reg}/W$ given by the identification of $T^*(T)^{reg}$ with $T_C^{reg}$. 

Proof. For each point in the quotient, we choose a preimage in $\phi^{-1}(0)^{reg}$, which may be taken to be a point $(t, H)$ in $T^*(T)^{reg}$. Let $V$ denote the tangent space to $\phi^{-1}(0)^{reg}$ at $(t, H)$ and let $U$ denote the space of vectors $X$ in $V$ for which $JX$ is also in $V$. (Here $J$ is the complex structure on $T^*(K) \cong K_C$.) It is not hard to compute that $U$ is just the tangent space to $T^*(T)$ and thus that $V$ is the direct sum of $U$ and the tangent space to the $Ad_K$-orbit through $(t, H)$. Thus, the restriction of $J$ to $U$ descends to a map on the quotient, which is just the complex structure on $T^*(T) \cong T_C$. ■ 

Remark 22  Let us regard $\phi^{-1}(0)$ as a subset of $K_C$ by means of the identification of $T^*(K)$ with $K_C$. It is then not hard to show that if $g \in K_C^{reg}$, the conjugacy class of $g$ intersects $\phi^{-1}(0)^{reg}$ in exactly one $Ad_K$-orbit. We thus have an alternative characterization of the regular part of the reduced phase space as $\phi^{-1}(0)^{reg}/Ad_K \cong K_C^{reg}/Ad_K$. 

Although we will not use this result in what follows, it is an illuminating way of thinking about the complex structure on the reduced phase space.

5  Quantization of the reduced phase space 

5.1  Quantization of $T_C$ 

The reduced phase space is a quotient of an open, dense subset of $T_C$ by the action of the Weyl group. It is therefore natural to first consider the quantization of $T_C$. Since $T$ is a compact, connected Lie group, we can (and do) quantize
$T_C \cong T^*(K)$ the same way we quantized $K_C \cong T^*(K)$. Elements of $\text{Quant}(T_C)$ have the form

$$\psi = Fe^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'},$$

where $\beta'$ is a nowhere-vanishing, invariant holomorphic $r$-form and $F$ is a holomorphic function on $T_C$. The norm of $\psi$ is the $L^2$ norm of $F$ with respect to the measure

$$\gamma'_h := e^{-|H|^2/h}\eta' \varepsilon'$$

(24)

where $\varepsilon'$ is the symplectic volume measure on $T^*(T)$ and $\eta'$ is defined, analogously to (9), as

$$\eta' := \left[ \frac{\beta' \wedge \bar{\beta'}^{-1/2}}{b' \varepsilon'} \right].$$

(25)

Here the quantity “$H$” on $T_C$ is defined by means of the identification of $T_C$ with $T^*(T)$. Actually, since $T_C$ is commutative, the function $\eta'$ is easily seen to be constant. We follow the notational convention of using primes to distinguish constructs on $T_C$ from their counterparts on $K_C$.

Since we are going to quotient (an open dense subset of) $T_C$ by $W$, it is natural to look for subspaces of the above Hilbert space with particular transformation properties under $W$. The difficulty with this idea is that $\beta'$ is not invariant under the action of $W$, but rather transforms according to the sign of the Weyl-group element:

$$w \cdot \beta' = \text{sign}(w)\beta'.$$

Thus, the the most natural way for $W$ to act on the Hilbert space is by the following projective unitary action

$$U(w) \left( F(z)e^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'} \right) = \sqrt{\text{sign}(w)}F(w^{-1} \cdot z)e^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'}. \quad (26)$$

Here we allow both possible signs for the square root of $\sqrt{\text{sign}(w)}$, so that $U(w)$ is actually a pair of unitary operators differing by a sign. These operators satisfy (for any choice of the signs involved)

$$U(w_1w_2) = \pm U(w_1)U(w_2).$$

Now, if $\alpha$ is a root and $s_\alpha$ is the associated reflection, then for either choice of the sign in the definition, we have $U(s_\alpha)^2 = -I$. Thus, there are, strictly speaking, no nonzero “Weyl-invariant” elements in the Hilbert space! Nevertheless, we can make the following definition.

**Definition 23** For each root $\alpha$, let $s_\alpha$ be the associated reflection. Let us choose a sign for the operator $U(s_\alpha)$ by choosing the factor of $\sqrt{\text{sign}(s_\alpha)} = \sqrt{-1}$ to have the value

$$\sqrt{\text{sign}(s_\alpha)} = i. \quad (27)$$

For each $\psi$ in the Hilbert space, we say that $\psi$ is **Weyl-invariant** if

$$U(s_\alpha)\psi = i\psi, \quad \forall \alpha \in R,$$
and we say $\psi$ is **Weyl-alternating** if

$$U(s_\alpha)\psi = -i\psi, \quad \forall \alpha \in R.$$  

Of course, Definition 23 is just a fancy way of saying that if $\psi = F e^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'}$, then $\psi$ is Weyl invariant if $F$ is Weyl invariant and $\psi$ is Weyl alternating if $F$ is Weyl alternating. Note that the Weyl-invariant and Weyl-alternating elements have very different behavior as we approach the singular points.

**Proposition 24** Suppose $F$ is a holomorphic function on $T^{rss}_C$ that is square integrable with respect to the measure $\gamma'_h$ in (24). Then $F$ has a unique holomorphic extension to $T_C$.

There certainly exist holomorphic functions on $T^{rss}_C$ that do not extend holomorphically to $T_C$, such as the reciprocal of the analytically continued Weyl denominator $\sigma_C$. We are claiming, however, that such functions cannot be square integrable.

**Proof.** The set of irregular points is a complex analytic subvariety of $T_C$ defined by the vanishing of the holomorphic function

$$F(z) = \prod_{\alpha \in R^+} (\phi_\alpha(z) - 1),$$

where $\phi_\alpha : T_C \to \mathbb{C}^*$ is the root homomorphism in (20). The result then follows from a standard removable singularities theorem for square-integrable holomorphic functions, such as Theorem 1.13 and Proposition 1.14 in [33].

5.2 **Quantization of $T^{rss}_C/W$**

Recall that the full reduced phase space $\phi^{-1}(0)/\text{Ad}_K$ is not a manifold. We deal with this difficulty in a simple way, by quantizing only the set of regular points, $\phi^{-1}(0)^{\text{reg}}/\text{Ad}_K$, which we have identified with $T^{rss}_C/W$. Our justification for ignoring the singular points is that we will quantize $T^{rss}_C/W$ in such a way that elements of the quantization may be identified as $W$-alternating holomorphic functions on $T^{rss}_C$ that are square integrable with respect to the measure $\gamma'_h$. By Proposition 24, every such function extends holomorphically to all of $T_C$.

Suppose there were some quantization of the full reduced phase space. It is not clear what nicer properties an element of this quantization should have than those already possessed by elements of the quantization of the regular set.

As just mentioned, we are going to quantize $T^{rss}_C/W$ in such a way that the Hilbert space corresponds to the space of **Weyl-alternating elements of the quantization of $T_C$**. In practical terms, we “need” this to be the case, in the sense that the “quantization commutes with reduction” map only makes sense if the quantization of $T^{rss}_C/W$ is done in this way. (See Section 6.) We will show that for a suitable choice of the prequantum line bundle, the desired outcome can be obtained by exploiting the freedom in the standard procedure of geometric quantization with half-forms to choose the prequantum line bundle.
Proposition 25 The Weyl group acts freely on $T^\text{rss}_C$ and the analytically continued Weyl denominator $\sigma_C$ is nowhere vanishing on $T^\text{rss}_C$.

Proof. If $z \in T^\text{rss}_C$ were fixed by some nontrivial element of $W := N(T)/T$, then $z$ would commute with some element of $N(T)$ not in $T$, so that $z$ would not be regular semisimple. Meanwhile, from the formula for $\sigma_C$, we see that if $\sigma_C(e^H) = 0$, then $\langle \alpha, H \rangle \in 2\pi \mathbb{Z}$, from which it follows that $\phi_\alpha(z) = 1$, showing that $z$ is not regular semisimple.

We need to understand the canonical bundle $K'$ for the quotient $T^\text{rss}_C/W$. Since the volume form $\beta'$ on $T^*_C$ is not invariant under the action of the Weyl group, it does not descend to a form on the quotient. On the other hand, since the Weyl denominator function $\sigma_C$ is alternating, the form $\sigma_C \beta'$ is $W$-invariant and (by Proposition 25) nowhere vanishing on $T^\text{rss}_C$. Thus, we may regard this form a nowhere vanishing form on the quotient. In particular, we have established that the canonical bundle $K'$ for $T^\text{rss}_C/W$ is trivial. We may therefore take a trivial square root $K'_1/2$ of $K'$ with trivializing holomorphic section $\sqrt{\sigma_C \beta'}$.

We would like to quantize $T^\text{rss}_C/W$ in such a way that the sections have the form $Fe^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'}$, with $F$ being a Weyl-alternating holomorphic function on $T^\text{rss}_C$. We can formally rewrite such an object as

$$F \frac{e^{-|H|^2/(2\hbar)}}{\sqrt{\sigma_C \beta'}} \otimes \sqrt{\sigma_C \beta'},$$

where as above we may regard $\sqrt{\sigma_C \beta'}$ as a trivializing section of the canonical bundle $T^\text{rss}_C/W$. We now construct a line bundle $L'$ over $T^\text{rss}_C/W$ in such way that the expression $Fe^{-|H|^2/(2\hbar)}/\sqrt{\sigma_C}$ can be interpreted as a holomorphic section of $L'$. (The prime distinguishes $L'$ from the prequantum bundle $L$ over $T^*(K)$.)

We define $L'$ as the complex line bundle over $T^\text{rss}_C/W$ whose sections are $W$-alternating functions $f$ on $T^\text{rss}_C$. That is to say, the fiber of $L'$ over each $W$-orbit $O$ in $T^\text{rss}_C$ is the one-dimensional complex vector space of $W$-alternating functions from $O$ into $C$. To each section $f$ of $L'$ we associate the formal object

$$\frac{f}{\sqrt{\sigma_C}}.$$  

We emphasize that $\sqrt{\sigma_C}$ is not a single-valued function on $T^\text{rss}_C$; the expression (28) is simply a mnemonic device that will help us remember the definition of the Hermitian structure and connection on $L'$.

Motivated by (28) we define a Hermitian structure on $L'$ by setting

$$|f|_{L'}(z) = \frac{|f(z)|}{|\sigma_C(z)|^{1/2}}, \quad z \in T^\text{rss}_C,$$  

(29)
for each $W$-alternating function $f$. To define a connection on $L'$, we observe that if $\sqrt{\sigma_C}$ is any local square root of $\sigma_C$ and $X$ is a vector field, we have

$$X\left(\frac{f}{\sqrt{\sigma_C}}\right) = \frac{1}{\sqrt{\sigma_C}} \left(Xf - \frac{1}{2} \frac{X\sigma_C}{\sigma_C} f\right).$$

Thus, the formal expression (28) suggests to define a connection on $L'$ by setting

$$\nabla_X f = Xf - \frac{1}{2} \frac{X\sigma_C}{\sigma_C} f - i \hbar \theta(X)f,$$

whenever $X$ is a vector field on $T_{\operatorname{rss}}C/W$, viewed as a $W$-invariant vector field on $T_{\operatorname{rss}}C$, and $f$ is a $W$-alternating function. Note that since $X$ is $W$-invariant and $\sigma_C$ is $W$-alternating, $X\sigma_C/\sigma_C$ is $W$-invariant, so that $(X\sigma_C/\sigma_C)f$ is still $W$-alternating. Then, as usual in geometric quantization, we define a smooth section $f$ of $L'$ to be holomorphic if

$$\nabla_X f = 0$$

for all vectors of type $(0, 1)$.

**Proposition 26** The curvature of $L'$ with respect to the connection in (30) is $\omega/\hbar$. A section $f$ is holomorphic if and only if

$$\left(X - \frac{i}{\hbar} \theta(X)\right)f = 0$$

for each vector field of type $(0, 1)$, and this condition holds if and only if $f$ has the form

$$f = Fe^{-\mid H^{\prime} \mid^2/(2\hbar)}$$

for some Weyl-alternating holomorphic function $F$ on $T_{\operatorname{rss}}C$.

**Proof.** The connection (30) differs from the usual one in prequantization by the addition of the term involving $X\sigma_C/\sigma_C$. Locally, this change amounts to replacing $\theta$ by $\theta' = \theta + d\psi$, where $\psi$ is a multiple of the locally defined logarithm $\log(\sigma_C)$. Since the curvature is computed from $d\theta$, this change does not affect the curvature. Similarly, since $\sigma_C$ is holomorphic, the term involving $\sigma_C$ will vanish whenever $X$ is of type $(0, 1)$, so the condition for a holomorphic section is still (31). Finally, since $T$ is also a connected compact Lie group, the analysis we carried out in the quantization of $T^*(K)$ applies also here, showing that solutions to (31) have the form (32).

We summarize the preceding discussion in the following definition.

**Definition 27** (Quantization of the reduced phase space) Let $L'$ be the complex line bundle over $T_{\operatorname{rss}}C/W$ whose sections are $W$-alternating functions $f$ on $T_{\operatorname{rss}}C$, with Hermitian structure and connection on $L'$ as in (29) and (30). Take a trivial square root $K_{\gamma/2}'$ of the canonical bundle $K'$ over $T_{\operatorname{rss}}C/W$ with
trivializing section $\sqrt{\sigma C \beta'}$, with a Hermitian structure on $K'_{1/2}$ defined similarly to (9). We define our quantization of $T^\text{rss}_C / W$ as the space of square-integrable holomorphic sections of $L' \otimes K'_{1/2}$. In accordance with the formal expression (28), we write elements $\psi$ of the quantum Hilbert space as

$$\psi = \frac{Fe^{-|H|^2/(2\hbar)}}{\sqrt{\sigma C}} \otimes \sqrt{\sigma C \beta'},$$

or, suggestively, as

$$\psi = Fe^{-|H|^2/(2\hbar)} \otimes \sqrt{\beta'},$$

where $F$ is a $W$-alternating holomorphic function on $T^\text{rss}_C$.

The norm of such an element is computed as

$$||\psi||^2 = \frac{1}{|W|} \int_{T^*C} |F(z)|^2 e^{-|H|^2/(\hbar \eta') \varepsilon'},$$

where $\varepsilon'$ is the Liouville volume measure on $T^*(T) \cong T_C$ and where $\eta'$ is as in (25). In particular, we have identified

$$\text{Quant}(K_C / / \text{Ad}_K) = \text{Quant}(T^\text{rss}_C / W)$$

with the $W$-alternating subspace of $\text{Quant}(T_C)$, and this identification is unitary up to a constant. (The constant arises because of the factor of $1/|W|$ in (33).)

6 The “quantization commutes with reduction” map

In this section, we construct a “natural” map $B$ from the first-reduce-then-quantize Hilbert space $\text{Quant}(K_C)^{\text{Ad}_K}$ to the first-quantize-then-reduce Hilbert space $\text{Quant}(K_C / / \text{Ad}_K)$. The map includes a mechanism for converting half-forms of degree $n$ (over $K_C$) to half-forms of degree $r$ (over the regular part of the reduced phase). The main result will be that $B$ coincides, after suitable identifications, with the map of Florentino, Moura, and Nunes and therefore (Theorem 8) that $B$ is a constant multiple of a unitary map.

Recall that $L$ and $L'$ denote the prequantum line bundles over $T^*(K) \cong K_C$ and the reduced phase space, respectively, and that $K_{1/2}$ and $K'_{1/2}$ denote chosen square roots of the corresponding canonical bundles. We will introduce a contraction mechanism that will allow us to convert invariant holomorphic sections of $K_{1/2}$ to holomorphic sections of $K'_{1/2}$. A crucial factor of the Weyl denominator will arise in this process. The process depends, however, on a certain choice of orientations and it will not be possible to make this choice consistently over all of $\phi^{-1}(0)^\text{reg}$. Thus, the contraction procedure only makes sense locally.

We also introduce a “restriction” map for mapping invariant holomorphic sections of $L$ to sections of $L'$. This map is similarly defined only locally. When
we combine the two maps, however, we get a globally defined map $B$ of invariant holomorphic sections of $L \otimes K_{1/2}$ to holomorphic sections of $L' \otimes K'_{1/2}$. The map $B$ is our “quantization commutes with reduction map” from $\text{Quant}(K_C)^{\text{Ad}_K}$ to $\text{Quant}(K_C/\text{Ad}_K)$. Our main result is that $B$ after suitable identifications, $B$ is the map described by the theorem of Florentino, Mourão, and Nunes and therefore that $B$ is a constant multiple of a unitary map.

We now give a very brief summary of how $B$ is defined; details are given below. To each element $\psi = Fe^{-|\xi|^2/(2\hbar)} \otimes \sqrt{\beta}$ of $\text{Quant}(K_C)^{\text{Ad}_K}$, we formally associate the quantity

$$\psi' := F|_{T_C} e^{-|H|^2/(2\hbar)} \otimes (\sigma_C \sqrt{\beta'}),$$

where, as we shall see, the factor of $\sigma_C$ comes from the contraction process. We then formally rewrite $\psi'$ by moving the factor of $\sigma_C$ to the other side and multiplying and dividing by $\sqrt{\sigma_C}$ giving

$$\psi' = (\sigma_C)(F|_{T_C}) e^{-|H|^2/(2\hbar)} \otimes \sqrt{\sigma_C \beta'}.$$

We then note that $(\sigma_C)(F|_{T_C})$ is a Weyl-alternating holomorphic function, so that $\psi'$ is indeed an element of $\text{Quant}(K_C/\text{Ad}_K) \cong \text{Quant}(T_C)_{W^-}$. Note that the function $(\sigma_C)(F|_{T_C})$ occurs also on the right-hand side of Theorem 8.

6.1 Relating the canonical bundles

We begin with considering the relationship between the canonical bundles over $K_C$ and over the reduced phase space $T_C^{\text{reg}}/W$. Let $n$ be the complex dimension of $K_C$ and $r$ the complex dimension of $T_C$. Suppose $b$ is a holomorphic $n$-form on $K_C$ that is invariant under the adjoint action of $K$. We hope to associate to $b$ a holomorphic $r$-form $\tilde{b}$ on the regular part of the reduced phase space,

$$\phi^{-1}(0)_{\text{reg}}/\text{Ad}_K \cong T_C^{\text{reg}}/W.$$  

The only reasonable way to do this is to restrict $b$ to $\phi^{-1}(0)_{\text{reg}}$ and then contract with $n - r$ vector fields to convert $b$ from a $n$-form to an $r$-form. The only reasonable choice for the vector fields are the vector fields $X^\eta$, $\eta \in \mathfrak{k}$, describing the infinitesimal adjoint action of $K$ on $\phi^{-1}(0)_{\text{reg}}$.

We now investigate this contraction process in detail. For each $(x, \xi)$ in $\phi^{-1}(0)_{\text{reg}}$, let $S(x, \xi)$ be the stabilizer of $(x, \xi)$—which is a maximal torus in $K$ because $(x, \xi)$ is assumed regular—and let $\mathfrak{s}(x, \xi)$ be the Lie algebra of $S(x, \xi)$. Let $\eta_1, \ldots, \eta_{n-r}$ be an orthonormal basis for the orthogonal complement of $\mathfrak{s}_{(x, \xi)}$ of $\mathfrak{s}(x, \xi)$. We may then consider the contraction

$$\tilde{b} := i_{X^{\eta_1} \wedge \cdots \wedge X^{\eta_{n-r}}} (b).$$  

This contraction is easily seen to be unchanged if we replace $\eta_1, \ldots, \eta_{n-r}$ by another orthonormal basis for $\mathfrak{s}_{(x, \xi)}$ with the same orientation, but changes sign if we replace $\eta_1, \ldots, \eta_{n-r}$ by an orthonormal basis with the opposite orientation.
We now consider the issue of trying to choose the orientations on $s_{(x,\xi)}^\perp$ consistently over $\phi^{-1}(0)^\text{reg}$. 

**Proposition 28** For each $(x,\xi)$ in $\phi^{-1}(0)^\text{reg}$, let $s_{(x,\xi)}$ denote the Lie algebra of the stabilizer of $(x,\xi)$ and $s_{(x,\xi)}^\perp$ denote the orthogonal complement of $s_{(x,\xi)}$ in $\mathfrak{k}$.

1. For each $(x_0,\xi_0) \in \phi^{-1}(0)^\text{reg}$, we can find an open, $\text{Ad}_K$-invariant set $U \subset \phi^{-1}(0)^\text{reg}$ containing $(x_0,\xi_0)$ such that the orientation of $s_{(x,\xi)}^\perp$ can be chosen in a continuous, $\text{Ad}_K$-invariant fashion for all $(x,\xi) \in U$.

2. The orientation of $s_{(x,\xi)}^\perp$ cannot be chosen in a continuous, $\text{Ad}_K$-invariant fashion for all $(x,\xi) \in \phi^{-1}(0)^\text{reg}$.

Note that if $(x,\xi) \in \phi^{-1}(0)$ and $y \in K$, then

$$S_{y,(x,\xi)} = yS_{(x,\xi)}y^{-1},$$

so that $s_{y,(x,\xi)} = \text{Ad}_y(s_{(x,\xi)})$ and $s_{y,(x,\xi)}^\perp = \text{Ad}_y(s_{(x,\xi)}^\perp)$. Thus, $\text{Ad}_K$-invariance in the choice of orientation would mean, explicitly, that $\text{Ad}_y$, viewed as a map from $s^\perp_{(x,\xi)}$ to $s^\perp_{y,(x,\xi)}$, is orientation preserving.

**Proof.** For Point 1, it is harmless to assume that $(x_0,\xi_0)$ is equal to a point $(t_0, H_0)$ in $T^*(T)^\text{reg}$. Since $W$ acts freely on $T^*(T)^\text{reg}$, we can take a neighborhood $V$ of $(t_0, H_0)$ in $T^*(T)^\text{reg}$ such that $V \cap w \cdot V = \emptyset$ for all $w \neq 1$. Then $U := \text{Ad}_K \cdot V$ is an open set in $\phi^{-1}(0)^\text{reg}$ homeomorphic to $V \times (K/T)$. Each stabilizer of $(t, H) \in V$ is simply $T$, so we may choose orientations over $V$ by using one fixed orientation on $t^\perp$. Since the stabilizer (i.e., $T$) of each point $(t, H) \in V$ is connected, the action of the stabilizer on $s_{(t, H)}^\perp = t^\perp$ is orientation preserving. This fact guarantees that we can extend the choice of orientation from $V$ to $U$ in an unambiguous, invariant fashion.

For Point 2, note that we may make a continuous choice of orientation over $T^*(T)^\text{reg}$ simply by using one fixed orientation on $t^\perp$. Now, it is easily verified that $T^*(T)^\text{reg}$ is connected; it follows that using one fixed orientation on $t^\perp$ is the unique continuous choice of orientations over $T^*(T)^\text{reg}$.

Now fix a Weyl group element $w$ with $\text{sign}(w) = -1$ and pick a representative $y$ of $w$ in $N(T)$. Then $\text{Ad}_y$, viewed as a map from $t$ to itself, is orientation reversing. But by the connectedness of $K$, $\text{Ad}_y$, viewed as a map of $\mathfrak{k}$ to itself, is orientation preserving. Thus, $\text{Ad}_y$, viewed as a map of $t^\perp$ to itself must be orientation reversing. Thus, any continuous choice of orientation even over $T^*(T)^\text{reg}$ fails to be invariant under the adjoint action of $N(T) \subset K$.

We now come to a key computation that ultimately explains the geometric origin of the analytically continued Weyl denominator $\sigma_C$ in the “quantization commutes with reduction” map. To state our result, we now fix the normalization of the left-invariant holomorphic forms $\beta$ and $\beta'$ on $K_C$ and $T_C$. Let us fix an orientation of $t$ and an orientation of $\mathfrak{t}$. This then determines an orientation of $t^\perp$: If we take an oriented orthonormal basis $\eta_1, \ldots, \eta_r$ for $t$ and extend it to
an oriented orthonormal basis $\eta_1, \dotsc, \eta_n$ for $\mathfrak{t}$, then $\eta_{r+1}, \dotsc, \eta_n$ should be an oriented basis for $\mathfrak{t}^\perp$. Let us normalize $\beta$ and $\beta'$ so that at the identity, we have

$$\beta'(\eta_1, \dotsc, \eta_r) = 1; \quad \beta(\zeta_1, \dotsc, \zeta_n) = 1$$

whenever $\eta_1, \dotsc, \eta_r$ and $\zeta_1, \dotsc, \zeta_n$ are oriented orthonormal bases for $\mathfrak{t}$ and $\mathfrak{t}$, respectively. Note that $\beta$ and $\beta'$ are defined on the complex vector spaces $\mathfrak{t}_C$ and $\mathfrak{t}_C$, respectively, but that the normalizations are fixed on bases of the underlying real vector spaces $\mathfrak{t}$ and $\mathfrak{t}$.

**Proposition 29** Consider a point $(x, \xi) \in \phi^{-1}(0)^{\text{reg}}$. Let $U \subset \phi^{-1}(0)^{\text{reg}}$ be an open, $\text{Ad}_K$-invariant set containing $(x, \xi)$ as in Proposition 28, and let us fix a continuous, $\text{Ad}_K$-invariant choice of orientation on $z^*_{(x, \xi)}$, $(x, \xi) \in U$. Let $b$ be a holomorphic, $\text{Ad}_K$-invariant $n$-form on $T^*(K) \cong K_C$ and let $\tilde{b}$ be the $r$-form on $U$ defined by (34). Then $\tilde{b}$ descends to a holomorphic $r$-form on $U'$, the image of $U$ in $\phi^{-1}(0)^{\text{reg}}/\text{Ad}_K \cong T_C^{\text{reg}}/W$.

Suppose, specifically, that $b = \beta$. Let $[z_0] \in T_C^{\text{reg}}/W$, and let $z_0$ be a representative of $z$ in $T_C^{\text{reg}}$. If we then identify a neighborhood of $[z_0]$ in $T_C^{\text{reg}}/W$ with a neighborhood of $z_0$ in $T_C^{\text{reg}}$, we have

$$\tilde{\beta}(z) = \pm \sigma_C(z)^2 \beta'(z), \quad z \in T_C,$$

where the sign depends both on the choice of orientations on $U$ and on the choice of the representative $z$ of $[z]$.

Note that since the function $\sigma_C^2$ on $T_C$ is Weyl invariant and the form $\beta'$ is Weyl alternating, the form $\sigma_C^2 \beta'$ is Weyl alternating. We thus see very clearly the effect of the nonexistence of a global choice of orientation over $\phi^{-1}(0)^{\text{reg}}$. The form $\tilde{\beta}$ in (34) is not a Weyl-invariant form on $T_C^{\text{reg}}$ and it therefore does not descend to a form on $T_C^{\text{reg}}/W$. On the other hand, if $z_0 \in T_C^{\text{reg}}$, we can pick a small neighborhood $V$ of $z_0$ such that the sets $w \cdot V, w \in W$, are disjoint. Then there does exist a Weyl-invariant form on the union of the $w \cdot V$’s whose restriction to $V$ is $\sigma_C(z)^2 \beta'(z)$, namely the one whose restriction to $w \cdot V$ is $\text{sign}(w) \sigma_C(z)^2 \beta'(z)$.

**Proof.** Let $b$ be as in the first part of the proposition. Define a form $\tilde{b}$ on $U \subset \phi^{-1}(0)^{\text{reg}}$ by (34). We think of $\tilde{b}$ as a locally defined form on $\phi^{-1}(0)^{\text{reg}}$, meaning that we only plug into $\tilde{b}$ vectors that are tangent to $\phi^{-1}(0)^{\text{reg}}$ Since $b$ is assumed to be invariant under the adjoint action of $K$ and since the orientations over $U$ are chosen invariantly, it is easy to check that $\tilde{b}$ is invariant under the adjoint action of $K$ on $U$.

Fix some $(x, \xi)$ in $\phi^{-1}(0)^{\text{reg}}$, let $V$ denote the tangent space to $\phi^{-1}(0)^{\text{reg}}$ at $(x, \xi)$ and let $W \subset V$ denote the tangent space to the $\text{Ad}_K$-orbit through $(x, \xi)$. Then $b(Y_1, \dotsc, Y_r) = 0$ if even one of the $Y_j$’s is in $W$. (After all, $b$ is obtained from $b$ by contracting with a basis $X^{\eta_1}, \dotsc, X^{\eta_{n-r}}$ for $W$.) Thus, $\tilde{b}$ descends to a $r$-linear, alternating form on $V/W$, which is just the tangent space to the reduced phase space. Since $\tilde{b}$ is invariant under adjoint action of $K$, the value
of $\tilde{b}$ at a point in the reduced space is independent of the choice of point in the corresponding $K$-orbit in $\phi^{-1}(0)_{\text{reg}}$.

It is presumably possible to verify that $\tilde{b}$ is holomorphic by an argument similar to the one in [29 Section 3.2]. In this situation, however, we can work by direct computation. We first note that if $F$ is an $\text{Ad}_K$-invariant holomorphic function on $K_C$, then the restriction of $F$ to $\phi^{-1}(0)_{\text{reg}}$ is also $\text{Ad}_K$-invariant, so that this restriction descends to a function $\tilde{F}$ on $\phi^{-1}(0)_{\text{reg}}/\text{Ad}_K$. It should be clear from the way the complex structure on $\phi^{-1}(0)_{\text{reg}}/\text{Ad}_K$ is defined that $\tilde{F}$ is again holomorphic. (Explicitly, if we identify $F$ by direct computation. We first note that if $F$ is simply the function on $K$ which is again holomorphic.) Now, every $\text{Ad}_K$-invariant holomorphic $n$-form on $K_C$ is expressible as an $\text{Ad}_K$-invariant holomorphic function times the form $\beta$. It thus suffices to check holomorphicity in the case $b = \beta$, which we do in what follows.

Under our standing identification (2) of $T^*(K)$ with $K_C$, the adjoint action of $K$ on $T^*(K)$ corresponds to the adjoint action of $K$ on $K_C$. We identify the tangent space at each point in $K_C$ with $\mathfrak{t}_C$ by means of left translation. With this identification, the value of the vector field $X^\eta$ at a point $g \in K_C$ is easily computed to be

$$X^\eta = \text{Ad}_{g^{-1}}(\eta) - \eta.$$

Suppose now that $z \in T^*_z K_C$ and that $\eta \in \mathfrak{t}^\perp$. Then

$$X^\eta = \text{Ad}_{z^{-1}}(\eta) - \eta$$

is easily seen to lie in $\mathfrak{t}^\perp_C$.

Now let $\eta_1, \ldots, \eta_{n-r}$ be an oriented orthonormal basis for $\mathfrak{g}^\perp_C = \mathfrak{t}^\perp$. Then

$$\beta(X^{\eta_1}, \ldots, X^{\eta_{n-r}}, Y_1, \ldots, Y_r) = \det(A) \beta(\eta_1, \ldots, \eta_{n-r}, Y_1, \ldots, Y_r) = \det(A) \beta(Y_1, \ldots, Y_r, \eta_1, \ldots, \eta_{n-r})$$

for all $Y_1, \ldots, Y_r \in \mathfrak{t}_C$, where $A : \mathfrak{t}_C^\perp \to \mathfrak{t}_C^\perp$ is the unique linear transformation such that $A(\eta_j) = X^{\eta_j}$. (There is no minus sign in the second equality because $n - r$ is the number of roots, which is even.) From (37), we can identify $A$ as

$$A = \text{Ad}_z^\prime - I,$$

where $\text{Ad}_z^\prime$ is the restriction of $\text{Ad}_z$ to $\mathfrak{t}_C^\perp$.

Now, the eigenvalues of $\text{Ad}_z^\prime$ are the numbers of the form $\phi_\alpha(z^{-1})$, $\alpha \in R$, where $\phi_\alpha$ is the root homomorphism in (20). Thus,

$$\det(A) = \prod_{\alpha \in R} (\phi_\alpha(z^{-1}) - 1).$$

After group the roots into pairs, $\{\alpha, -\alpha\}$ with $\alpha \in R^+$, this result simplifies to

$$\det(A) = \prod_{\alpha \in R^+} [(e^{i\alpha.H}/2 - e^{-i\alpha.H}/2)(e^{-i\alpha.H}/2 - e^{i\alpha.H}/2)]$$

$$= (-1)^m \sigma_C(z)^2,$$
where $m$ is the number of positive roots. In particular, if $Y_1, \ldots, Y_r$ are in $t_C$, we have

$$\beta(X^{\eta_1}, \ldots, X^{\eta_{n-r}}, Y_1, \ldots, Y_r) = (-1)^m \sigma_C(z) \beta'(Y_1, \ldots, Y_r).$$

This equality certainly holds up to a constant because both sides are $r$-linear alternating functions of $Y_1, \ldots, Y_r$. The constant may then be checked from (38) when $Y_1, \ldots, Y_r$ form an oriented orthonormal basis for $t$. □

### 6.2 Relating the half-form bundles

We now observe that the locally defined contraction process on sections of the canonical bundle $K$ extends to a locally defined contraction process on sections of $K_{1/2}$. The idea is simple. We start with an invariant holomorphic section $c$ of $K_{1/2}$ and square it to an invariant holomorphic section $b := c \otimes c$ of $K$. Then we contract $b$ to a locally defined section $b'$ of $K'$. Finally, we look for a locally defined holomorphic section $c'$ of $K'_{1/2}$ with $c' \otimes c' = b'$. It is easy to see that the preceding procedure can be carried out locally, which is all that we hope for at the moment. (In the setting of [19], this contraction process on the half-form bundles can be done globally; see Theorem 3.1 there.)

Using the computations in the previous subsection, we can read off the results of the contraction process on the half-form bundles, as follows.

**Proposition 30** The contraction process on the canonical bundles induces a locally defined contraction process on the half-form bundles. For each $[z_0] \in T^\text{rss}_C/W$, we choose a representative $z_0$ of $[z_0]$ in $T^\text{rss}_C$, which we use to identify $T^\text{rss}_C/W$ locally with $T^\text{rss}_{\mathfrak{t}}$. Then the contraction process on half-forms is determined by its action on $\sqrt{\beta}$, which is given by

$$\sqrt{\beta} \mapsto \pm \sigma_C \sqrt{\beta}. \quad (39)$$

At the moment, the sign in (39) is undefined, since the section $c'$ in the above description is only unique up to a sign.

### 6.3 Relating the prequantum bundles

We now construct a natural local map from the space of invariant holomorphic sections of the prequantum bundle $L$ over $T^*(K) \cong K_C$ to the space of holomorphic sections of the prequantum bundle $L'$ over $T^*(T)^{reg}/W \cong T^\text{rss}_C/W$. Suppose $f$ is an invariant section of $L$, that is, that $Q_{pre}(\phi_\eta)f = 0$ for all $\eta \in \mathfrak{t}$. Then by (13), $f$ is a function that is invariant under the adjoint action of $K$. Invariant holomorphic sections of $L$ then have the form $Fe^{-(k^2/(2\hbar))}$, where $F$ is an $\text{Ad}_K$-invariant holomorphic function.

Pick a point $[z_0]$ in $T^\text{rss}_C/W$ and a representative $z_0$ of $[z_0]$ in $T^\text{rss}_C$. Let us identify a small neighborhood of $[z_0] \in T^\text{rss}_C/W$ with a neighborhood $V$ of $z \in T^\text{rss}_C$, where $V$ is chosen to be simply connected and so that the sets $wV$, $w \in W$, are disjoint. Pick a local holomorphic square root of $\sigma_C$ on $V$, denoted $[\sigma_C]^{1/2}$.
Then to each invariant holomorphic section \( f = Fe^{-|ξ|^2/(2ℏ)} \) of \( L \), we associate the function
\[
f'(z) = [σ_C(z)]^{1/2}F(z)e^{-|H|^2/(2ℏ)}, \quad z \in V,
\]
which we also write as the formal object
\[
\frac{[σ_C(z)]^{1/2}F(z)e^{-|H|^2/(2ℏ)}}{√σ_C}.
\]
Note that the \( √σ_C \) in the denominator is just a formal expression that reminds of the definition of the Hermitian structure and connection on \( L' \). The factor of \( [σ_C(z)]^{1/2} \), by contrast, is an actual holomorphic function on \( V \subset T_{\operatorname{rss}}^C \).

As in the case of the map between sections of the half-form bundle, the correspondence \( f \mapsto f' \) preserves the pointwise magnitude of sections, in the sense that if \([z] \in T_{\operatorname{rss}}^C/W\) comes from the \( \operatorname{Ad}_K \)-orbit of \((x,ξ) ∈ φ^{-1}(0)_{\operatorname{reg}}\), then
\[
|f'(z)|_{L'} = |f(x,ξ)|.
\]
(Recall from (29) that the pointwise magnitude of a section of \( L \) is simply the absolute value of the corresponding function.)

As in the case of the map between sections of the half-form bundle, the correspondence \( f \mapsto f' \) does not extend to a globally defined map of invariant holomorphic sections of \( L \) to holomorphic sections of \( L' \), because \( [σ_C]^{1/2} \) does not extend to a globally defined \( W \)-alternating holomorphic function on \( T_{\operatorname{rss}}^C \).

6.4 The map

Recall from Section 5.2 that elements of \( \operatorname{Quant}(K_C//\operatorname{Ad}_K) \) are holomorphic sections of the bundle \( L' ⊗ K'_{1/2} \) over \( T_{\operatorname{rss}}^C/W \). We write such a section in the form
\[
Fe^{-|H|^2/(2ℏ)} ⊗ √σ_Cβ',
\]
or in the suggestive alternative form
\[
Fe^{-|H|^2/(2ℏ)} ⊗ √β',
\]
where \( F \) is a \( W \)-alternating holomorphic function on \( T_{\operatorname{rss}}^C \). We now show that the local mappings described in the two previous sections combine into a global mapping \( B \) of invariant holomorphic sections of \( L ⊗ K_{1/2} \) to holomorphic sections \( L' ⊗ K'_{1/2} \). After suitable identifications, \( B \) will coincide with the correspondence in the theorem of Florentino, Mourão, and Nunes (Theorem 8), showing that \( B \) is a multiple of a unitary map.

If \( ψ \) is an invariant section of \( L ⊗ K_{1/2} \), we write
\[
ψ = Fe^{-|ξ|^2/(2ℏ)} ⊗ √β.
\]
If we apply the local mappings of the two previous subsections to $F e^{-|ξ|^2/(2ℏ)}$ and to $√β$, we obtain the local section

$$ψ' = \frac{[σC(z)]^{1/2} F(z) e^{-|H|^2/(2ℏ)}}{√σC} \otimes (σC √β').$$

Moving a factor of $[σC]^{1/2}$ from right to left in the tensor product allows us to rewrite this as

$$ψ' = \frac{σC(z) F(z) e^{-|H|^2/(2ℏ)}}{√σC} \otimes √σC β'$$

or, suggestively, as

$$ψ' = (σC(z) F(z)) e^{-|H|^2/(2ℏ)} \otimes √β'.$$

We now observe that $ψ'$ is actually a globally defined holomorphic section of $L' ⊗ K_{1/2}$. After all, if $F$ is a holomorphic class function on $K_C$, then the restriction of $F$ to $T_C^{un}$ is Weyl invariant, so that the function

$$(σC)(F|_{T_C^{un}})$$

is Weyl alternating.

**Theorem 31 (Quantization commutes with reduction)**  The locally defined maps in the previous subsections combine to give a globally defined map from the space of invariant holomorphic sections of $L ⊗ K_{1/2}$ to the space of holomorphic sections of $L' ⊗ K'_{1/2}$. Thus, we obtain a geometrically natural map

$$B : \text{Quant}(K_C)^{Ad_K} \rightarrow \text{Quant}(K_C/Ad_K) \cong \text{Quant}(T_C)^{W^{-}},$$

which may be computed explicitly as follows:

$$B(F e^{-|ξ|^2/(2ℏ)} ⊗ √β) = \frac{(σC F) e^{-|H|^2/(2ℏ)}}{√σC} \otimes √σC β'.$$

The map $B$ is a constant multiple of a unitary map.

**Proof.** We have already established that $B$ is well defined and computed as in the theorem. As a map of $\text{Quant}(K_C)^{Ad_K}$ to $\text{Quant}(T_C)^{W^{-}}$, $B$ is the map sending the holomorphic class function $F$ on $K_C$ to the function $(σC)(F|_{T_C})$ on $T_C$. The unitarity claim then follows from Proposition 5 and Theorem 8.

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