Comment on the paper ”The third-order perturbed Korteweg-de Vries equation for shallow water waves with a non-flat bottom” by M. Fokou, T.C. Kofané, A. Mohamadou and E. Yomba, Eur. Phys. J. Plus, 132, 410 (2017)

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Abstract The authors of the paper ”The third-order perturbed Korteweg-de Vries equation for shallow water waves with a non-flat bottom” [1] claim that they have derived the full third order perturbed KdV equation for the case of uneven bottom. We show that the authors’ derivation is not consistent due to the fact that they took into account only some of the third order corrections but not all of them. Moreover, we show that a consistent third order perturbed Korteweg-de Vries equation for shallow water waves with a non-flat bottom cannot be derived for a general form of bottom function.

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1 The extended KdV equation for uneven bottom

It is widely known that the ubiquitous Korteweg - de Vries equation, in the case of shallow water waves, is obtained by a perturbation approach which is first order in small parameters and assumes a flat bottom for the fluid container. The second order equation, also for this even bottom case, was derived first by Marchant and Smyth in 1990 [2] and named the extended KdV.

In the papers [3,4] we have derived, for the first time, the nonlinear equation describing shallow water gravity waves for uneven bottom

\[
0 = \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{1}{6} \beta \eta_{xx} - \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta \left( \frac{23}{24} \eta_x \eta_{xx} + \frac{5}{12} \eta_{xxx} \right) + \beta \frac{19}{360} \eta_{xxx} + \beta \delta \left( h_{3x} \eta + h_{2x} \eta - h_x \eta_{xx} + h \eta_{xxx} - \frac{2}{\beta} (h_x \eta + h \eta_x) \right).
\]

(1)

The equation (1) has been obtained by a perturbation method up to the second order in small parameters $\alpha$, $\beta$, $\delta$.

The authors of the discussed paper cite our equation as [1, Eq. (2)], which in their paper does not have the correct form. In [1, Eq. (2)] the coefficient of the linear term $\eta_{xx}$ is $\frac{1}{3} \beta$ instead of the correct $\frac{1}{6} \beta$. Moreover, all coefficients in the bracket $\beta \delta (\cdots)$ slightly differ from the correct ones. The authors write these terms as

\[
\beta \delta \left( \frac{13}{72} h_{3x} \eta + \frac{1}{8} h_{2x} \eta - \frac{7}{24} h_x \eta_{xx} - \frac{17}{72} h \eta_{xxx} - \frac{2}{3 \beta} (h_x \eta + h \eta_x) \right).
\]

(2)

whereas the correct form is

\[
\beta \delta \left( \frac{1}{4} (h_{3x} \eta + h_{2x} \eta) - \frac{1}{4} (h_x \eta_{xx} + h \eta_{xxx}) - \frac{1}{2 \beta} (h_x \eta + h \eta_x) \right).
\]

(3)

[Compare with [3, Eq. (35)] or [4, Eq. (18)].]

In our opinion most parts of the paper [1] are incorrect since the derivation presented there is inconsistent, which we show below.

The inconsistency has arisen since the authors have taken the boundary condition at the variable bottom in the form [1, Eq. (14)], that is,

\[
\phi_z = \beta \delta (h_x \phi_x) \quad \text{for} \quad z = \delta h(x).
\]

(4)

This is correct up to the second order in small parameters and it was enough in our papers [3,4] for consistent derivation of second order wave equation for a non-flat bottom. The equation (4) together with the Laplace equation [1, Eq. (10)] allows us to express all odd order functions $\phi^{(m)}$ through $\phi^{(0)}$, $h$ and their derivatives according to

\[
\phi^{(1)} = \beta \delta \left( h_x \phi_x^{(0)} + h \phi_{xx}^{(0)} \right)
\]

(5)
and 
\[ \phi^{(2k+1)} = \left( -\beta \right)^{k+1} \left( 2k + 1 \right)! \phi^{(1)}_x. \] (6)

A consistent perturbation approach of the third order requires, however, to take into account the bottom boundary condition to the same third order. As we pointed out in Eq. (14) this condition takes the form of a complicated differential equation
\[ \phi^{(1)} - \beta \delta \left( h_x \phi_x^{(0)} + h \phi_x^{(2)} \right) - \beta^2 \left( h h_x \phi_x^{(1)} + \frac{1}{2} h^2 \phi_x^{(1)} \right) = 0 \] (7)

which does not supply a simple expression for \( \phi^{(1)} \) such as [5]. For arbitrary bottom function \( h(x) \) it is not possible to find an explicit form of \( \phi^{(1)} \) as a function of \( \phi^{(0)}, h \) and their derivatives. Therefore consistent derivation of a KdV-type equation, third order in all three small parameters \( \alpha, \beta, \delta \) is not possible. This was the reason why in papers [3,4] we limited our study to the second order perturbation approach.

The authors of the critiqued paper [1] are not aware of the latter fact. They take bottom boundary condition [5] for granted and insert it into the equation for velocity potential [1, Eq. (15)]. Subsequently they proceed as recommended by Burde and Sergyeyev [3], taking into account terms up to third order in small parameters. This procedure is in some part third order and in another part second order and therefore totally inconsistent.

Moreover, the authors repeat several technical errors. The coefficient in front of the linear term \( \eta \beta_x \) is correctly written as \( \frac{1}{2} \beta \) in the KdV equation [1, Eq. (1)] (when surface tension is neglected), but written wrongly as \( \frac{1}{4} \beta \) in equations [1, Eq. (2)] and [1, Eq. (40)] and then again correctly in the equation [1, Eq. (42)] of the paper. The terms of the order \( \beta \delta \) shown here in [2] appear with an incorrect coefficient not only in [1, Eq. (2)] but also in both [1, Eq. (40)] and [1, Eq. (42)].

The most astonishing error consists in the term \( \frac{1}{2} \eta \beta_x \) in equations [1, Eq. (2)] and [1, Eq. (40)]. This term is well known in KdV, where it appears correctly as \( \frac{1}{4} \eta \beta_x \). None of the perturbative approaches of higher order can change it. It is to be questioned how this term was obtained in the discussed paper. We can not explain it, since in one of the previous papers [3] the same team of authors present the third order KdV equation where this term is correct. This term is correct in [1, Eq. (42)], as well.

### 2.1 Case of ascendant bottom

We focus on the simplest cases presented in Figs. 2 and 3 of the paper [1]. In [1, Fig. 2] the authors show the time evolution of the initial KdV soliton, given by [1, Eq. (43)] with \( \tau = 0 \). Time evolution is calculated numerically according to [1, Eq. (44)] with parameters \( \alpha = \beta = 0.1 \) and \( \delta = 0.5 \). The bottom function is taken as \( h(x) = \pm \frac{1}{6} (\tanh(0.055(x - 55)) + 1) \).

In Fig. 4 we display the time evolution of the same initial KdV soliton, the same values of \( \alpha, \beta, \delta \), the same interval \( x \in [0, 100] \) and the same space and time steps \( \Delta x, \Delta t \) as in [1, Fig. 2] but obtained according to a second order equation derived by us in [3,4]. Only time instances corresponding to particular profiles may be slightly different since this information is not supplied in the paper [1]. However, since solitons cover similar distances the comparison of both numerical time evolutions is possible.

In [1, Fig. 2] the amplitude of the wave increases from 1 to \( 1.8 \) and profiles are distorted (by secondary wave) behind the main part only. In our calculations presented in Fig. 4 we observe soliton radiation in front of the main wave and no distortions behind it. This radiation which occurs when a soliton enters a shallower region is known in shallow water theory, see, e.g. [7,8,9]. In our case the amplitude increases only from 1 to \( 1.3 \) at \( x \approx 70 \). The next decrease of the main wave amplitude is just the effect of the radiation mentioned above (which is relatively big since \( \delta = 0.5 \) parameter, contrary to \( \alpha = \beta = 0.1 \) is not small).

![Figure 1](image-url)  
**Figure 1.** Time evolution of the initial KdV soliton obtained with second order equation derived in [3,4]. The bottom function \( h_1(x) \) and the values of parameters \( \alpha, \beta, \delta \) are the same as in the Fig. 2 of [1].

One wonders why the authors do not obtain the soliton radiation in [1, Fig. 2]. In order to find the answer we made the following test. We inserted the coefficients of the equation [1, Eq. (40)] into our code and ran the time
evolution according to the equation
\[
\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{3}{8} \beta \eta_{3x}
\]
\[
\eta_t + \eta_x + \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta \left( \frac{23}{24} \eta \eta_x + \frac{5}{12} \eta_{3x} \right) + \frac{19}{360} \eta_{5x} + \beta \delta \left( \frac{13}{72} \eta_{3x} \right) + \frac{7}{24} \eta_x \eta_{3x} - \frac{17}{72} \eta \eta_{3x} - \frac{2}{3 \beta} \eta_{x} - \frac{2}{3 \beta} \eta_x
\]
\[
= 0
\]
that is, the equation [1 Eq. (40)] limited to second order. The result of this numerical integration is displayed in Fig. 2. This shows that soliton radiation preceding the main wave has disappeared. Is it the effect of the wrong coefficient in front of \( \eta_{3x} \) or is it the effect of wrong coefficients in \( \beta \delta \) terms? To answer this question we restored the coefficient \( \frac{1}{3} \beta \) to \( \frac{1}{6} \beta \) in (8) and ran the code once more. The result is presented in Fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure2}
\caption{Time evolution of the initial KdV soliton obtained with evolution according to equation (8) in which coefficient \( \frac{1}{3} \beta \) is replaced by the correct one \( \frac{1}{6} \beta \). The bottom function is \( h_+(x) \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure3}
\caption{Time evolution of the initial KdV soliton obtained with evolution according to equation (8) in which coefficient \( \frac{1}{3} \beta \) is replaced by the correct one \( \frac{1}{6} \beta \). The bottom function is \( h_+(x) \).}
\end{figure}

\subsection*{2.2 Case of sloping bottom}
In the case of the decreasing bottom we repeat the calculations with the bottom function \( h_-(x) \). The result of our second order numerical time evolution is displayed in Fig. 3. In our case the soliton amplitude decreases slower than in [1 Fig. 3] and the depression behind the main wave is much smaller.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure4}
\caption{Time evolution of the initial KdV soliton obtained with second order equation derived in [3,4]. The bottom function \( h_-(x) \) and the values of parameters \( \alpha, \beta, \delta \) are the same as in the Fig. 3 of [1].}
\end{figure}

Next, we repeat the same steps as in the previous subsection. Limiting the equation [1 Eq. (40)] to second order, that is, to (8) we obtain with the bottom function \( h_-(x) \) the evolution displayed in Fig. 3 which looks incorrect.

Now, as in the case of the ascendant bottom, we replace the incorrect term \( \frac{1}{3} \beta \eta_{3x} \) by the correct one \( \frac{1}{6} \beta \eta_{3x} \).
Figure 5. The same as in Fig. 2 but with the bottom function $h_-(x)$.

The evolution according to (3) in which the term $\beta\eta_3x$ is the right one gives the result presented in Fig. 5. The wave profiles in Figs. 4 and 5 differ only very little. This means that, the main reason for the wrong results presented in Figs. 2 and 3 of [1] is the use of the incorrect first order term in the form $\frac{1}{3}\beta\eta_3x$. Using the incorrect form of $\beta\delta(\cdots)$ terms has a much smaller influence since these terms are small and the coefficients used in [1, Eq. (40)] are only slightly different from the correct ones (compare (2) and (3)).

Figure 6. The same as in Fig. 4 but with the bottom function $h_-(x)$.

2.3 Surface tension effects

In order to check the results presented in [1] for nonzero surface tension let us rewrite the equation [1, Eq. (42)] but limiting it to second order

$$0 = \eta_t + \eta_x + \frac{3}{2}\beta\eta_3x + \frac{1}{6}\beta(1-3\tau)\eta_{3x} - \frac{3}{8}\beta^2\eta_x$$

$$+ \alpha\beta\left(\frac{23+15\tau}{24}\eta_x\eta_{2x} - \frac{5-3\tau}{12}\eta_{4x}\right) + \beta\left(19 - 30\tau - 45\tau^2\right)\eta_{5x}$$

$$+ \beta\delta\left(\frac{13}{72}h_{3x}\eta_x + \frac{1}{8}h_{2x}\eta_x - \frac{7}{24}h_{x}\eta_{2x} - \frac{17}{72}h_x\eta_{3x} - \frac{2}{3\beta}(h_x\eta_x + h\eta_x)\right).$$

(9)

In Fig. 7 we present the time evolution of the initial KdV soliton according to the equation (9) for all conditions as those used by the authors of [1] in their Fig. 6, that is, for $\alpha = \beta = 0.1$, $\delta = 0.3$ and the same bottom function.

Figure 7. The result of time evolution according to second order equation (9) with $\tau = 0$ to compare with [1, Fig. 6].

Comparing this result with [1, Fig. 6] we see that in our case the changes of the amplitude are much smaller and second order effects are much 'cleaner' than those contained in Fig. 6 of [1].

Now, we check the influence of the surface tension. In Fig. 10 of [1] the authors present the time evolution of the initial KdV soliton according to their third-order equation (42) in which third order terms are inconsistent. The parameters used were: $\alpha = \beta = 0.1$, $\delta = 0.3$ and $\tau = 0.1$. The bottom was taken as the Gaussian well followed by the symmetric Gaussian hump.

In Fig. 8 we show the time evolution of the same soliton but according to the equation (42) of [1] limited to second order, that is, the equation (9). The result is obtained with the same parameters $\alpha, \beta, \delta, \tau$ and the same bottom function. For comparison with the Fig. 10 of [1] we set the same vertical scale.

From Fig. 8 it is clear that all rapid oscillations seen in Fig. 10 of [1] for $30 < x < 40$ are not present in the
evolution according to consistent second order KdV equation (9). We conclude that the strange behaviour of the soliton motion over bottom changes presented by the authors of [1] is the result of an inconsistent derivation of third order equation.

3 Conclusions

From this short study the following conclusions may be drawn.

– Consistent incorporation of small amplitude bottom changes into KdV equation is possible exclusively in the perturbation approach of second order with respect to small parameters. This is because the bottom kinetic boundary equation (7) can be resolved for φ(1) only when it is taken in second order.

– Derivation of the third order KdV equation for the case of uneven bottom presented in [1] is inconsistent since one of equations of the Euler set of equations is taken in second order and the other ones in third order. Therefore these equations are flawed.

– Numerical tests made by us with consistent second order KdV equation for the case of a non-flat bottom show the absence of any strange behaviour of soliton evolution obtained in [1] with inconsistent third order equations.

References

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