Markov semi-groups generated by elliptic operators with divergence-free drift

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Abstract

In this paper we construct a conservative Markov semi-group with generator \( L = \Delta + b \cdot \nabla \) on \( \mathbb{R}^n \), where \( b \) is a divergence-free vector field which belongs to \( L^2 \cap L^p \) with \( p < n \). The research is motivated by the question of understanding the blow-up solutions of the fluid dynamic equations, which attracts a lot of attention in recent years.

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1 Introduction

In fluid dynamics, the velocity \( u(t,x) \) of fluid particles is described by, in the case of incompressible fluids, the Navier-Stokes equations

\[
\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p, \quad \nabla \cdot u = 0,
\]

in a domain of Euclidean space \( \mathbb{R}^3 \), subject to certain initial and boundary conditions. Here \( p(t,x) \) is the pressure which is uniquely determined by \( u(t,x) \) up to a constant at every \( t \), and it solves the Poisson equation \( \Delta p = -\nabla u \otimes \nabla u \). Hence \( p(t,x) \) is a non-linear and non-local term in the Navier-Stokes equations.

The first equation in (1.1) can be written as a parabolic type equation:

\[
(\partial_t - \nu \Delta + u \cdot \nabla) u = -\nabla p,
\]

which however possesses no much common features as (local) parabolic equations, but nevertheless the theory of parabolic equations is helpful in the analysis of the Navier-Stokes equations. It is a matter of fact that many quantities related to fluid flows such as the vorticity, the rate-of-stress tensor fields also satisfy the same kind of parabolic evolution equations with the principal parabolic operator \( \partial_t - L \), where \( L = \nu \Delta - u \cdot \nabla \). The operator \( L \) is time non-homogeneous since \( u \) depends on \( t \), and its formal adjoint \( L^* = \nu \Delta + u \cdot \nabla \) is the infinitesimal generator of a diffusion process, called Taylor’s diffusion, which models the fluid flows in terms of Brownian particles. Taylor’s diffusion solves formally the following stochastic differential equation

\[
dX_t = u(t,X_t) + \sqrt{2\nu}dW_t,
\]
where $W_t$ is the Brownian motion. Taylor's diffusion has been an important tool in the study of
turbulent flows and in the development of numerical simulations to the solutions of the Navier-
Stokes equations (such as vortex methods). For the Navier-Stokes equations, only global weak
solutions have been constructed in general, and knowledge of weak solutions is still limited. There
is a vast literature addressing the regularity of weak solutions, see e.g. [18, 21]. Leray’s weak
solution $u(t, x)$ satisfies the energy balance equation, which implies that

$$u(t, x) \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2).$$

For the most interesting case where dimension is three, this regularity only implies that $u(t, x) \in
L^2(0, T; L^6(\mathbb{R}^3)) \cap L^\infty(0, T; L^2(\mathbb{R}^3))$ and the classical parabolic regularity theory fails to apply.

Consider the following parabolic equation of second order with singular divergence-free drift

$$\partial_t u(t, x) - \sum_{i,j=1}^n \partial_i (a_{ij}(t, x) \partial_j u(t, x)) + \sum_{i=1}^n b_i(t, x) \partial_i u(t, x) = 0 \quad (1.4)$$
on $\mathbb{R}^n$, where $(a_{ij}(t, x))$ is symmetric. In this equation, we denote $b$ as the drift velocity vector
field and $u$ as a solution. When $a_{ij} = v \delta_{ij}$, equation (1.4) corresponds to Taylor’s diffusion (1.3).
A classical monograph on such parabolic equations is [9] by Ladyzhenskaya et al., in which the
existence of a unique Hölder continuous weak solution $u$ is proved under the conditions that $a$ is
uniform elliptic and $b \in L^1(0, T; L^2(\mathbb{R}^n))$ with $\frac{2}{q} + \frac{n}{q} \leq 1$, $l \neq \infty$. It is not known whether a Leray’s
weak solution has this regularity or not, which motivates us to consider the cases that $\frac{2}{q} + \frac{n}{q} > 1$
together with the assumption that $b$ is divergence-free. Throughout this article, we will always
assume that there exists a constant $\lambda > 0$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2 \quad \text{E}$$

and that $b$ is divergence-free:

$$\sum_{i=1}^n \partial_i b_i(t, x) = 0 \quad \text{S}$$

for every $t \geq 0$.

If $(a_{ij})$ and $(b_i)$ are smooth, then there exists a unique fundamental solution $\Gamma^{(a, b)}(t, x; \tau, \xi)$
(or simply by $\Gamma(t, x; \tau, \xi)$ if we only work with one pair $(a, b)$) associated with the Cauchy initial
problem (1.4). In [17], the following Aronson type estimate in the time-inhomogeneous case with
a super-critical drift $b$ has been established, see also the related estimates in [1, 16, 23].

**Theorem 1.** Suppose $a = (a_{ij})$ and $b = (b_i)$ are smooth which satisfy conditions (E), (S) and
assume that $b \in L^q(0, T; L^p(\mathbb{R}^n))$ for some $n \geq 3, l > 1, q > \frac{2}{l}$ such that $1 \leq \frac{2}{l} + \frac{n}{q} < 2$. If $\mu \equiv \frac{2}{2 - \gamma + \frac{n}{q} + \frac{2}{q}} > 1$ with $\gamma = \frac{2}{l} + \frac{n}{q}$, then the fundamental solution has upper bound

$$\Gamma^{(a, b)}(t, x; \tau, \xi) \leq \begin{cases} \frac{C_1}{(t - \tau)^{n/2}} \exp \left(-\frac{1}{C_2} \frac{(|x - \xi|)^2}{(t - \tau)^{1/2}} \right), & \text{if } |\mu|^{p-2} < 1 \\ \frac{C_1}{(t - \tau)^{n/2}} \exp \left(-\frac{1}{C_2} \frac{(|x - \xi|)^2}{(t - \tau)^{1/2}} \right), & \text{if } |\mu|^{p-2} \geq 1 \end{cases} \quad (1.5)$$

where $\nu = \frac{2 - \gamma}{2 - \gamma + \frac{n}{q} + \frac{2}{q}}$. $\Lambda = \|b\|_{L^q(0, T; L^p(\mathbb{R}^n))}$. $C_1 = C_1(l, q, n, \lambda)$, and $C_2 = C_2(l, q, n, \lambda, \Lambda)$. If $\mu = 1$, so that $q = \infty$, then

$$\Gamma^{(a, b)}(t, x; \tau, \xi) \leq \frac{C_1}{(t - \tau)^{n/2}} \exp \left(-\frac{(C_1 \Lambda (t - \tau)^{\nu} - |x - \xi|^2)}{4C_1 (t - \tau)} \right). \quad (1.6)$$
The upper bound in the theorem implies that \( \Gamma^{(a,b)}(t,x;\tau,\xi) \) decays exponentially in space variables, which yields the pre-compactness of the family of the probability measures defined by \( \Gamma^{(a,b)} \), in the sense that, the family of finite dimensional distributions
\[
\prod_{i=1}^{n} \Gamma^{(a,b)}(t_i,x_i;\tau_{i-1},\xi_{i-1}) \, dx_1 \cdots dx_n
\]
for fixed \( s \leq t_0 < t_1 < \cdots < t_n \), is pre-compact under the topology of weak convergence for measures. The pre-compactness allows us to construct \( \Gamma^{(a,b)}(t,x;\tau,\xi) \) for Borel measurable \( a \) and \( b \) which satisfy \( (E), (S) \) and \( b \in L^2(0,T;L^q(\mathbb{R}^n)) \) with \( \frac{2}{q} + \frac{q}{2} \in [1,2) \). The weak convergence for measures is too week to ensure the Chapman–Kolmogorov equation:
\[
\Gamma^{(a,b)}(t,x;\tau,\xi) = \int_{\mathbb{R}^n} \Gamma^{(a,b)}(t,x;y)\Gamma^{(a,b)}(y;\tau,\xi) \, dy.
\]

We leave it as unsolved problem, i.e. to construct a Markov process for the time non-homogeneous case. However, for the time-homogeneous case, we will prove the Chapman–Kolmogorov equation and construct the corresponding Markov semi-group.

Therefore, we consider the time-homogeneous parabolic equation
\[
\partial_t u(t,x) - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(t,x)) + \sum_{i=1}^{n} b_i(x)\partial_{x_i}u(t,x) = 0 \tag{1.7}
\]
where \((a,b)\) satisfies conditions \((E)\) and \((S)\). We study the Markov semi-group associated with \( \Gamma^{(a,b)} \) for \( b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \), \( q > \frac{n}{2} \). The corresponding bi-linear form
\[
\mathcal{E}(u,v) = \int_{\mathbb{R}^n} [\langle \nabla u,a \cdot \nabla v \rangle + (b \cdot \nabla u)v] \, dx \tag{1.8}
\]
is not sectorial in general in the sense defined in [13] and the theory of non-symmetric Dirichlet forms does not apply in this case. On the other hand, due to the divergence-free condition \((S)\), the symmetric part of the bi-linear form is given by
\[
\mathcal{E}_s(u,v) = \int_{\mathbb{R}^n} \langle \nabla u,a \cdot \nabla v \rangle \, dx,
\]
which is however sectorial, and \((\mathcal{E}_s, D(\mathcal{E}_s))\) is a Dirichlet form. See for example [6, 13].

We are now in a position to state the main result of this paper.

**Theorem 2.** Suppose conditions \((E), (S)\) hold and \( b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \) for \( q > \frac{n}{2} \). There is a unique Markov semi-group \((P_t)_{t \geq 0}\) on \( L^2(\mathbb{R}^n) \) associated with the bi-linear form \((1.8)\) which has transition probability kernel \( \Gamma(t,x,y) \) for \( t > 0, x,y \in \mathbb{R}^n \). Moreover, the uniqueness of weak solutions holds for the Cauchy initial problem to \((1.7)\) and is given by the representation
\[
u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,y)u_0(y) \, dy \tag{1.9}
\]
for any initial data \( u_0 \in L^2(\mathbb{R}^n) \).

When the dimension \( n = 3 \), the condition of the theorem above is satisfied if \( b \in L^2(\mathbb{R}^3) \). As a consequence of Theorem 2, we have the following result which is interesting by its own.

**Corollary 3.** Let \( b \) be a \( C^1\)-vector field in \( \mathbb{R}^n \) with \( n \geq 3 \) such that \( \nabla \cdot b = 0 \). If \( b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) where \( q \geq \frac{n}{2} \), then the diffusion process defined by solving
\[
dX_t = dB_t + b(X_t)dt, \quad X_0 = x
\]
is conservative, i.e. its transition density function \( \Gamma(t,x,y) \) (with respect to the Lebesgue measure) satisfies that

\[
\int_{\mathbb{R}^n} \Gamma(t,x,y) \, dy = 1, \tag{1.10}
\]

where \( \Gamma(t,x,y) = \mathbb{P}[X_t = y | X_0 = x] \) formally.

The closest conditions in literature to ensure the stochastic completeness (1.10) for unbounded \( b \) are those on the symmetric tensor \( \nabla^2 b \) (Ricci curvature or Bakry-Émery condition) and \( \nabla \cdot b \) is the trace of \( \nabla^2 b \). Hence our condition impose a constrain on the “scalar curvature” of the operator \( L = \Delta - b \cdot \nabla \).

There have been many works on the construction of Markov semi-groups from non-sectorial bi-linear forms, which is an important topic in stochastic analysis. In [8], Kovalenko and Semenov proved the existence of a semi-group on \( L^p \) for \( p \) larger than a certain number under an entropy condition on \( b \). Their entropy condition is still a critical condition on \( b \). Using ideas from Dirichlet form, it is proved in [11] that there exists a \( C_0 \)-semigroup if the drift \( b \) is form bounded. Later, Sobol and Vogt [19] proved the existence of a strong continuous semi-group on \( L^p \) for any \( p \in [1, \infty) \) if the space \( Q(b^2) \cap D(\mathcal{E}) \) is core for \( \mathcal{E} \), where \( Q(b^2) = \{ u \in L^2 : u^2 b^2 \in L^1 \} \), and \( \mathcal{E} \) is accretive. Their idea is to use the continuity argument. They first add a potential \( V \) to the bi-linear form in order to remove the singularity appearing from the drift, and then send the potential to zero. So the association of the semi-group \( e^{tL} \) and bi-linear form \( \mathcal{E} \) is established through the correspondence of \( e^{t(L-V)} \) and \( \mathcal{E} + V \). Our approach is to directly approximate \( b \) by smooth \( b_k \), which gives the existence and conservative of the kernel. Later in [12], Liskevich and Sobol further proved the heat kernel estimate of these semi-groups under additional functional conditions on the bi-linear form, by using the idea developed in [3], which is similar to proving upper bound in time-inhomogeneous cases in [16, 17].

In [25], Zhikov considered the following type of parabolic equations

\[
\partial_t u - \text{div}((A + B) \cdot \nabla u) = 0, \tag{1.11}
\]

and constructed the unique approximation semi-group for periodic \( B \in L^\infty(\mathbb{R}^n) \), \( \text{div} B \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( \sup_{r \geq 1} \frac{1}{r} ||B||_{L^q(B(0,r))}^q < \infty \). Here \( A \) is a symmetric matrix-valued and \( B \) is a anti-symmetric matrix-valued. It is easy to see that such problems are equivalent to (1.4) with divergence-free \( b \) if we set \( a = A \) and \( b = -\text{div} B \).

In order to establish the existence and uniqueness of a Markov semi-group associated with parabolic equation (1.7), which also defines the unique weak solution, we use an idea from [25]. For \( b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \) with \( q > \frac{n}{d} \), there are divergence-free vector fields \( b_k \in C^\infty_0(\mathbb{R}^n) \) for \( k = 1, 2, \cdots \) such that \( b_k \to b \) in \( L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \). For the existence of such approximation sequence to divergence-free vector fields, see Section 1.5 in [18]. Throughout the paper, \( L \) denotes the elliptic operator \( \text{div}(a \cdot \nabla) - b \cdot \nabla \), and its adjoint operator is

\[
L^* = \text{div}(a \cdot \nabla) + b \cdot \nabla
\]

as \( b \) is divergence-free. The fact that the dual operator has the same form will be of great importance to our arguments in what follows.

This paper is organized as follows. In section 2, we proved the existence of weak solution for \( b \in L^2(\mathbb{R}^n) \). In section 3, we give the proof to Theorem 2 with the stronger assumption that \( b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \). A similar result was proved in [25] for (1.11) using the same idea.
We use $\Gamma^{(a,b)}(t,x,\xi)$ ($t > 0$) to denote the fundamental solution (recall that $a, b$ are independent of $t$) which is defined by $\Gamma^{(a,b)}(t - \tau, x, \xi) = \Gamma^{(a,b)}(t, x; \tau, \xi)$.

**Definition 4.** A function $u \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n))$ is a weak solution to (1.7) corresponding to $(a, b)$ and initial data $u_0$ if

$$\int_0^T \int_{\mathbb{R}^n} u(t,x) \partial_t \varphi(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla \varphi(t,x) \rangle \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{R}^n} \langle b(x), \nabla u(t,x) \rangle \varphi(t,x) \, dx \, dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) \, dx$$

for any $\varphi \in C_0^\infty(0, T) \times \mathbb{R}^n$.

When $b \in L^q(\mathbb{R}^n)$ with $q \geq n$, for any initial data $u_0 \in L^2(\mathbb{R}^n)$, there exists a unique weak solution $u$ satisfying that $\partial_t u \in L^2(0, T; H^{-1}(\mathbb{R}^n))$ and $u \in C([0,T], L^2(\mathbb{R}^n))$. Moreover, it satisfies the energy identity

$$\frac{1}{2} \| u(T) \|^2 + \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla u(t,x) \rangle \, dx \, dt = \frac{1}{2} \| u_0 \|^2_2. \tag{2.1}$$

For more details, we refer to [9].

**Proposition 5.** Suppose conditions (E) and (S) are satisfied and $b \in L^2(\mathbb{R}^n)$, there exists a weak solution to (1.7) with initial data $u_0 \in L^2(\mathbb{R}^n)$.

**Proof.** Denote $u_k$ the weak solution corresponding to $(a, b_k)$, where $b_k \in C^0_0(\mathbb{R}^n)$ are divergence-free and $b_k \rightarrow b$ in $L^2(\mathbb{R}^n)$. Then $\{u_k\}$ is uniformly bounded in $L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n))$ and hence has a sub-sequence which converges weakly to some $u$. This weak convergence allows us to take limit as $k \rightarrow \infty$ in the equation:

$$\int_0^T \int_{\mathbb{R}^n} u_k(t,x) \partial_t \varphi(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u_k(t,x), \nabla \varphi(t,x) \rangle \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{R}^n} \langle b_k(x), \nabla u_k(t,x) \rangle \varphi(t,x) \, dx \, dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) \, dx$$

to obtain that

$$\int_0^T \int_{\mathbb{R}^n} u(t,x) \partial_t \varphi(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla \varphi(t,x) \rangle \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{R}^n} \langle b(x), \nabla u(t,x) \rangle \varphi(t,x) \, dx \, dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) \, dx.$$

We call the weak solution constructed in this way an approximation solution. Next, we show that every weak solution is an approximation solution in a weaker sense. This result follows from a similar argument in [23].

**Proposition 6.** Suppose $b \in L^2(\mathbb{R}^n)$ and $b_k \in C^0_0(\mathbb{R}^n)$ are divergence-free such that $b_k \rightarrow b$ in $L^2(\mathbb{R}^n)$. Let $u$ and $\{u_k\}$ be the weak solution to (1.7) on $[0,T] \times \mathbb{R}^n$ with initial data $u_0$, and drifts $b$ and $\{b_k\}$ respectively. Then $u$ is the $L^\infty(0,T; L^1(\mathbb{R}^n))$ limit of functions $\{u_k\}$.
Proof. Choose a sequence $b_k \to b$ in $L^2(\mathbb{R}^n)$. Consider the Cauchy problem

$$\partial_t u_k - \text{div}(a \cdot \nabla u_k) + b_k \cdot \nabla u_k = 0$$

with initial data $u_k(x,0) = u(x,0) = u_0(x)$. Clearly $u_k - u$ is a weak solution to

$$\partial_t (u_k - u) - \text{div}(a \cdot \nabla (u_k - u)) + b_k \cdot \nabla (u_k - u) = (b - b_k) \cdot \nabla u$$

with 0 as the initial value. By assumption, $\| (b - b_k) \cdot \nabla u \|_{L^2(0,T;L^1(\mathbb{R}^n))} \to 0$ as $k \to \infty$. Since $b_k \in C^\infty_0(\mathbb{R}^n)$, we have a representation given by

$$(u_k - u)(t,x) = \int_0^t \int_{\mathbb{R}^n} \Gamma_k(t-t',x,\xi)(b-b_k) \cdot \nabla u(\xi,\tau) \, d\xi \, d\tau,$$

where $\Gamma_k$ is the fundamental solution corresponding to $b_k$ on $\mathbb{R}^n$. Then $\Gamma_k^\ast(t,\xi,x) := \Gamma_k(t,x,\xi)$ is the fundamental solution to $(\partial_t - L_k^\ast)u = 0$, which is of the same form as the original equation (1.7) up to a sign on the drift. Hence

$$\int_{\mathbb{R}^n} \Gamma_k(t-t',x,\xi) \, dx = 1$$

(2.2)

for any fixed $(t,\tau,\xi)$. This implies that

$$\int_{\mathbb{R}^n} |u_k - u|(t,x) \, dx \leq \int_0^t \int_{\mathbb{R}^n} |b-b_k| |\nabla u| \, d\xi \, d\tau \to 0$$

and the proof is done. \qed

The proposition above implies that any week solution is an approximation solution. Here the divergence-free condition is the key to have the dual operator being conservative to obtain (2.2).

3 Uniqueness of the approximation semi-group and its kernel

In this section we prove our main result Theorem 2. The idea is to construct a unique approximation Markov semi-group corresponding to generator $L = \text{div}(a \cdot \nabla) - b \cdot \nabla$. Since $a$ is only Borel measurable, the generator $L$ is not well defined as a differential operator. Hence we will construct $L$ in the following, while we still use formal expression $L = \text{div}(a \cdot \nabla) - b \cdot \nabla$, if no confusion may arise, for simplicity of notations. We start with the bi-linear form

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla v \rangle + (b \cdot \nabla u) v \, dx.$$

Naturally we consider the elliptic problem and its weak solutions. The approach is standard in literature.

**Definition 7.** Let $(a,b)$ satisfies (E), (S) and $b \in L^2(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n)$, if there exists a $u \in H^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla \phi \rangle + (b \cdot \nabla u) \phi + \alpha u \phi \, dx = \int_{\mathbb{R}^n} f \phi \, dx$$

for all $\phi \in C^\infty_0(\mathbb{R}^n)$, we call $u$ a weak solution to the elliptic problem $(\alpha - L, f)$, where $\alpha \geq 0$.

For $b \in C^\infty_0(\mathbb{R}^n)$, the bi-linear form is actually a Dirichlet form. We recall the following result on Dirichlet forms in [13, Chapter 1].
Theorem 8. Let \((a,b)\) satisfies (E), (S) and \(b \in C_0^\infty(\mathbb{R}^n)\). Then \(\mathcal{E},H^1(\mathbb{R}^n)\), where
\[
\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot a \cdot \nabla v + (b \cdot \nabla u)v \, dx
\]
for \(u,v \in H^1(\mathbb{R}^n)\), is a (non-symmetric) Dirichlet form. We still use \(L\) together with its domain \(D(L)\) to denote the generator associated with the Dirichlet form \(\mathcal{E},H^1(\mathbb{R}^n)\). The resolvent \(R_\alpha = (\alpha - L)^{-1}\) for \(\alpha > 0\) is a bounded linear operator from \(L^2(\mathbb{R}^n)\) to \(L^2(\mathbb{R}^n)\) with \(||(\alpha - L)^{-1}||_{L^2 \to L^2} \leq \alpha^{-1}\), and it satisfies
\[
\mathcal{E}(R_\alpha f, v) + \alpha(R_\alpha f, v) = (f, v).
\]
(3.1)
Thus for \(b \in C_0^\infty(\mathbb{R}^n)\), \(\textrm{div}(a \cdot \nabla) - b \cdot \nabla\) is understood as the generator \(L\) defined as in Theorem 8 above. Clearly, for any \(f \in L^2(\mathbb{R}^n)\), \((\alpha - L)^{-1} f\) is the unique weak solution to \((\alpha - L, f)\). We can take \(v = (\alpha - L)^{-1} f\) and derive that
\[
||f||_{L^2} \leq \frac{1}{\min\{\lambda, \alpha\}} ||f||_{L^2} \quad \text{and} \quad ||f||_{L^2} \leq \frac{1}{\alpha} ||f||_{L^2}
\]
for all \(\alpha > 0\) and \(f \in L^2(\mathbb{R}^n)\). The following estimate on \(R_\alpha\), which follows from [25], plays an important role in proving our main result.

Lemma 9. Suppose \(b \in C_0^\infty(\mathbb{R}^n)\) and \(L\) as in Theorem 8, set \(u = (1 - L)^{-1} f\) for \(f \in C_0^\infty(\mathbb{R}^n)\). Then for \(n \geq 3\), we have
\[
\int_{\mathbb{R}^n} [\ln(|x|^2 + e)]^{2^\gamma} u^2(x) \, dx \leq C_0 \int_{\mathbb{R}^n} [\ln(|x|^2 + e)]^{2^\gamma} f^2(x) \, dx
\]
with sufficiently small positive \(\gamma\) and constant \(C_0\) depending only on \(n, \lambda, \gamma\) and \(||b||_{L^q(\mathbb{R}^n)}\) with \(q > \frac{n}{2}\).

Proof. Let \(\psi = \gamma \psi_0, \psi_0 = \ln(|x|^2 + e), \) for \(\gamma > 0\), and consider the operator \(L_\psi = e^\psi Le^{-\psi}\). For \(v = e^\psi u\), we have \(L_\psi v - v = g = e^\psi f\) and
\[
\int_{\mathbb{R}^n} -\langle \nabla (e^\psi v), a \cdot \nabla (e^{-\psi} v) \rangle - b \cdot \nabla (e^{-\psi} v)e^\psi v - v^2 \, dx = \int_{\mathbb{R}^n} gv \, dx.
\]
It follows, together with (E) and (S), that
\[
\int_{\mathbb{R}^n} \lambda |\nabla v|^2 - \frac{1}{\lambda} \gamma^2 |\nabla \psi_0|^2 v^2 - \gamma (b \cdot \nabla \psi_0) v^2 + v^2 \, dx \leq -\int_{\mathbb{R}^n} gv \, dx.
\]
Notice that
\[
|\nabla \psi_0| \leq \frac{2|x|}{(|x|^2 + 1)\ln(|x|^2 + e)},
\]
which is bounded. Hence we have
\[
\int_{\mathbb{R}^n} (b \cdot \nabla \psi_0)v^2 \, dx \leq C ||b||_{L^q} ||\nabla \psi_0||_{L^q} ||v||_{L^2}^{1-\theta} ||\nabla v||_{L^2}^{1+\theta} \leq C ||b||_{L^q} ||\nabla \psi_0||_{L^q} C(\theta) (||v||_{L^2}^{1+\theta} + ||\nabla v||_{L^2}^{1+\theta})
\]
where \(\theta = \frac{n}{q} - 1\) and \(C\) depends on \(n,q\). Now we can take \(\gamma\) small enough such that \(||v||_{L^2} \leq C_0 ||g||_{L^2}\) and the proof is complete. \(\square\)

7
Given divergence-free \( b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and a sequence of smooth functions \( b_k \to b \) in \( L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), this lemma implies that for each fixed \( f \in C_0^\infty(\mathbb{R}^n) \),

\[
\lim_{k \to \infty} \int_{|x|>r} |(1 - L_k)^{-1} f|^2 = 0
\]

uniformly in \( k \). Using them, we will prove the compactness of resolvent operators \( \{ (\alpha - L_k)^{-1} \} \).

**Lemma 10.** Given divergence-free \( b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), smooth approximations \( b_k \to b \) in \( L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), and \( f \in L^2(\mathbb{R}^n) \), the sequence \( \{ (1 - L_k)^{-1} f \} \) is strongly compact in \( L^2(\mathbb{R}^n) \) and weakly compact in \( H^1(\mathbb{R}^n) \).

**Proof.** Since

\[
\|(1 - L_k)^{-1} f\|_{H^1} \leq \frac{1}{\min\{\lambda, 1\}} \|f\|_{L^2}
\]

(3.3)

the sequence \( \{ (1 - L_k)^{-1} f \} \) is weakly compact in \( H^1(\mathbb{R}^n) \). To prove the strong compactness in \( L^2(\mathbb{R}^n) \), recall that we have proved \( \|(1 - L_k)^{-1}\|_{L^2 \to L^2} \leq \alpha^{-1} \) for all \( \alpha > 0 \) and

\[
(\alpha - L_k)^{-1} f \to (\alpha - L)^{-1} f \quad \text{in} \quad L^2(\mathbb{R}^n)
\]

as \( k \to \infty \), for all \( \alpha > 0 \) and \( f \in L^2(\mathbb{R}^n) \).

The previous lemma allows us to take limit as \( k \to \infty \) and to define the generator \( L \) for singular \( b \).

**Lemma 11.** Given \( L_k \) defined as in Theorem 8 corresponding to \( b_k \) which converges to \( b \) in \( L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), after a possible selection of a sub-sequence (denoted as \( L_k \) again), there exists a closed operator \( L \) defined on a dense subset of \( L^2(\mathbb{R}^n) \) such that \( \|(\alpha - L)^{-1}\|_{L^2 \to L^2} \leq \alpha^{-1} \) for all \( \alpha > 0 \) and

\[
(\alpha - L_k)^{-1} f \to (\alpha - L)^{-1} f \quad \text{in} \quad L^2(\mathbb{R}^n)
\]

as \( k \to \infty \), for all \( \alpha > 0 \) and \( f \in L^2(\mathbb{R}^n) \).

**Proof.** We first consider the case when \( \alpha = 1 \). We apply Lemma 10 to \( f \) in a countable dense subset of \( L^2(\mathbb{R}^n) \), by Theorem 6 in [10, Ch15] and Cantor’s diagonal argument, we can find a sub-sequence of \( (1 - L_k)^{-1} \) that converges strongly. We still denote the sub-sequence as \( (1 - L_k)^{-1} \) and denote its limit as \( S \), i.e.

\[
(1 - L_k)^{-1} f \to Sf
\]

strongly in \( L^2(\mathbb{R}^n) \) for \( f \in L^2(\mathbb{R}^n) \). Since \( (1 - L_k)^{-1} f \) is weakly compact in \( H^1 \), it also converges to \( Sf \) weakly in \( H^1 \). It is easy to see that the limit \( Sf \) is a weak solution to \( (1 - L, f) \). Since \( S \) is bounded linear operator from \( L^2(\mathbb{R}^n) \) to itself, we can define its adjoint operator \( S^* \) by \( \langle Sf, g \rangle = \langle f, S^* g \rangle \) for all \( f, g \in L^2(\mathbb{R}^n) \). We already know that

\[
\lim_{k \to \infty} \langle (1 - L_k)^{-1} f, g \rangle = \langle Sf, g \rangle
\]

for all \( f, g \in L^2(\mathbb{R}^n) \) and

\[
\langle (1 - L_k)^{-1} f, g \rangle = \langle f, (1 - L_k^*)^{-1} g \rangle.
\]

Hence we can see that \( S^* g \) is a weak solution to \( (1 - L^*, g) \). Proposition 12 implies that both \( S \) and \( S^* \) have kernels \( K(S) = K(S^*) = 0 \) and hence they have dense range in \( L^2(\mathbb{R}^n) \) due to the equality that \( K(S^*) = R(S)_1^\perp \). Now we can define \( L = 1 - S^{-1} \), which has dense domain \( D(L) \) and \( D(L) \subset H^1 \). Since \( S = R_1 = (1 - L)^{-1} \) is the resolvent, we also have that \( L \) is a closed
operator. Clearly, for each \( u \in D(L) \), it is the weak solution to \((-L, -Lu)\). Hence \((\alpha - L)^{-1} f\) is weak solution to \((\alpha - L, f)\) for \( f \) in the range of \((\alpha - L)\), i.e. \( f \in R(\alpha - L) \). From last theorem, we already know that \( R_\alpha = (\alpha - L)^{-1} \) is bounded linear operator. We therefore need to show that \( R(\alpha - L) = L^2(\mathbb{R}^n) \). We can show that for each \( f \in L^2(\mathbb{R}^n) \), there is a unique weak solution \( u \in D(L) \) to \((\alpha - L, f)\). This is because for each \( u \in D(L), f = (1 - L)u + (\alpha - 1)u \in L^2 \) and \( u \) is the weak solution to \((\alpha - L, f)\). Finally we can apply Theorem 1.3 in [7, Ch.8] to the approximation sequence \( L_k \) to obtain that

\[
(\alpha - L_k)^{-1} f \to (\alpha - L)^{-1} f \quad \text{in } L^2(\mathbb{R}^n)
\]
as \( k \to \infty \), for all \( \alpha > 0 \) and \( f \in L^2(\mathbb{R}^n) \).

Let \( u \) be the limit of \((1 - L_k)^{-1} f \) weakly in \( H^1(\mathbb{R}^n) \). Then it is easy to check that \( u \) is a weak solution to \((1 - L, f)\). Next we show that for \( b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), there is a unique \( S \) defined as in last Theorem. The uniqueness of \( S \) implies that the definition of \( L \) is independent of the choice of the convergent sub-sequence.

**Proposition 12.** Suppose \((a, b)\) satisfies conditions \((E)\) and \((S)\). For any \( f \in L^2 \), there exists a unique weak solution \( u \in H^1 \) to the elliptic problem \((\alpha - L, f)\) for \( n \geq 3, b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and \( \alpha > 0 \).

**Proof.** We already showed the existence of weak solution by an approximation approach. Given a weak solution \( u \) where \( f = 0 \), actually we can take a test function as \( h = \bar{u} \phi \) with \( \bar{u} = u \wedge N \lor (-N) \) and \( \phi \in C_0^\infty \), because \( b \cdot \bar{u} \in L^2 \). We let

\[
\phi_r = \begin{cases} 
1 & |x| \leq \frac{r}{2}, \\
0 & |x| \geq r
\end{cases}
\]

for any \( r > 0 \) and \( 0 \leq \phi_r \leq 1 \). Then we have

\[
\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla (\bar{u} \phi_r) \rangle + b \cdot \nabla u (\bar{u} \phi_r) + \alpha u (\bar{u} \phi_r) \ dx = 0.
\]

Because \( \bar{u} \phi_r \to \bar{u} \) in \( H^1(\mathbb{R}^n) \) and almost everywhere, by taking \( r \to \infty \), we obtain that

\[
\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla (\bar{u}) \rangle + b \cdot \nabla u (\bar{u}) + \alpha u (\bar{u}) \ dx = 0.
\]

Next we consider the second term in the equation above. Since \( \int_{\mathbb{R}^n} b \cdot \nabla \bar{u} \bar{u} \ dx = 0 \), we have

\[
\int_{\mathbb{R}^n} b \cdot \nabla \bar{u} \ dx = \int_{\mathbb{R}^n} b \cdot (\nabla u - \nabla \bar{u}) \bar{u} \ dx = N \int_{\{|u| > N\}} b \cdot (\nabla u - \nabla \bar{u}) \ dx - N \int_{\{|u| < -N\}} b \cdot (\nabla u - \nabla \bar{u}) \ dx = 0,
\]

and therefore

\[
\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla u \rangle + \alpha u^2 \ dx = 0
\]

by taking limit as \( N \to \infty \). Hence \( u = 0 \).

Finally, to prove the representation (1.9), we also need the convergence of the fundamental solution.
Definition 13. (Tightness) Given a family of probability measures \( \{P_i\}_{i \in I} \) on a metric space. If for every \( \varepsilon > 0 \), there is a compact set \( K \) such that \( \sup_{i \in I} P_i(K) > 1 - \varepsilon \). Then we call this family of measures tight.

Proposition 14. Given a sequence of probability measures \( \{P_n\} \) on \( \mathbb{R}^n \) which have densities \( \{f_n\} \) uniformly bounded from above by a continuous function \( h \). Suppose \( h \) satisfies
\[
\lim_{R \to \infty} \int_{B(0,R)} h(x) \, dx = 0,
\]
where \( B(0,R) \) is the open ball in \( \mathbb{R}^n \) centered at 0 with radius \( R \). Then \( \{P_n\} \) is weakly compact in the space of probability measure. Suppose we take a convergent sub-sequence, then its limit \( P \) has density \( f \) which is also bounded from above by \( h \).

Proof. It is easy to see that \( \{P_n\} \) is tight, which implies that it is weakly compact by Prohorov’s theorem. So we just need to show that \( P \) has density \( f \) which is bounded by \( h \). Firstly, we show that \( P \) is absolutely continuous with respect to the Lebesgue measure \( m \). Suppose \( A \subset \mathbb{R}^n \) such that \( m(A) = 0 \), then there is a decreasing sequence of open sets \( \{O_j\} \) containing \( A \) such that \( \lim_{j \to \infty} m(O_j) = 0 \). Therefore \( \lim_{j \to \infty} P_n(O_j) = 0 \) uniformly for all \( P_n \). By Portmanteau theorem [20, Theorem 1.1.1], we have \( P(O_i) \leq \limsup_{n \to \infty} P_n(O_i) \), which implies that \( \lim_{i \to \infty} P(O_i) = 0 \) and hence \( P(A) = 0 \). So \( P \) has a density \( f \) by Radon–Nikodym’s theorem.

Next we show that this \( f \) is bounded by \( h \). If not, we can find a bounded set \( A \) such that \( m(A) > 0 \) and \( f > h \) a.e. on \( A \). Since \( h \) is continuous, we can find an open set \( O \) small enough such that it contains \( A \) and \( P(O) > \int_O h > P_n(O) \) for all \( n \). Clearly this contradicts to that \( P_n \to P \) weakly in measure.

Now we are in a position to complete the proof of Theorem 2.

Proof. By the fundamental approximation theorem of semi-groups in [7, Cp 9, Theorem 2.16], the convergence of resolvents in Theorem 11 implies that \( e^{tL_k} \to e^{tL} \) as bounded linear operators from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) and are uniform for \( t \) in any finite interval \([0,T]\). Further, Proposition 6 yields that \( e^{tL} \) is the unique semi-group which generates the unique weak solution. Let \( \Gamma_k(t,x,y) \) be the fundamental solution to \( (\partial_t - L_k)u = 0 \).

Then
\[
\int_{\mathbb{R}^n} \Gamma_k(t,x,y) u_0(y) \, dy = e^{tL_k} u_0
\]
for any \( u_0 \in L^2(\mathbb{R}^n) \) and \( k = 1,2,\ldots \). By Theorem 1 and Proposition 14, we have that for each fixed \( (t,x) \) (and \( (t,y) \)), the family of transition probabilities \( \{\Gamma_k(t,x,y) \, dy\} \) (and also the family \( \{\Gamma_k(t,x,y) \, dx\} \)) is tight and hence converges weakly in measure to some \( \Gamma(t,x,y) \) which has the same upper bound as that of \( \Gamma_k(t,x,y) \). Define
\[
u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,y) u_0(y) \, dy
\]
for \( u_0 \in C_0^{\infty}(\mathbb{R}^n) \), then \( u_k(t,x) \to u(t,x) \) by the weak convergence of measure. As we have proved above that \( u_k \to e^{tL}u_0 \) in \( L^2(\mathbb{R}^n) \), so \( u = e^{tL}u_0 \) in \( L^2(\mathbb{R}^n) \). Since \( C_0^{\infty}(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), we can extend it to conclude that operator \( e^{tL} \) has a kernel \( \Gamma(t,x,y) \).

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