The Brown measure of a family of free multiplicative Brownian motions

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Abstract

We consider a family of free multiplicative Brownian motions $b_{s,\tau}$ parametrized by a real variance parameter $s$ and a complex covariance parameter $\tau$. We compute the Brown measure $\mu_{s,\tau}$ of $ub_{s,\tau}$, where $u$ is a unitary element freely independent of $b_{s,\tau}$. We find that $\mu_{s,\tau}$ has a simple structure, with a density in logarithmic coordinates that is constant in the $\tau$-direction. These results generalize those of Driver–Hall–Kemp and Ho–Zhong for the case $\tau = s$. We also establish a remarkable “model deformation phenomenon,” stating that all the Brown measures with $s$ fixed and $\tau$ varying are related by push-forward under a natural family of maps. Our proofs use a first-order nonlinear PDE of Hamilton–Jacobi type satisfied by the regularized log potential of the Brown measures. Although this approach is inspired by the PDE method introduced by Driver–Hall–Kemp, our methods are substantially different at both the technical and conceptual level.

Mathematics Subject Classification 46L54 Free probability and free operator algebras · 60B20 Random matrices (probabilistic aspects) · 35F21 Hamilton-Jacobi equations · 58J65 Diffusion processes and stochastic analysis on manifolds

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1 Introduction

1.1 Additive and multiplicative models and their large-$N$ limits

The most basic random matrix models, such as the Gaussian unitary ensemble and the Ginibre ensemble, are described by Gaussian measures. Thus, in light of the central limit theorem, such a model can be thought of as arising as the sum of a large number of small independent, identically distributed random matrices. More explicitly, a Gaussian random matrix model can be described in terms of Brownian motion in the appropriate vector space of matrices.
It is then natural to consider “multiplicative” random matrix models, which are constructed as products of independent random matrices that are close to the identity. More explicitly, we can consider random matrix models arising from a left-invariant Brownian motion in some group of invertible matrices, such as the unitary group \( U(N) \) or the general linear group \( GL(N; \mathbb{C}) \). For such a Brownian motion \( g_t \), the multiplicative increments, computed as \( g_s^{-1} g_t \) for \( s < t \), are independent.

In [3], Biane constructed the large-\( N \) limit of Brownian motions \( U^N_t \) in \( U(N) \), the so-called free unitary Brownian motion \( u_t \), as an element of an operator algebra with a trace. Biane then computed the law of \( v_t \), which is a probability measure \( \nu_t \) on the unit circle. Zhong [36] then extended Biane’s result by computing the law of \( uu_t \), where \( u \) is an arbitrary unitary element that is freely independent of \( u_t \). Meanwhile, in [4], Biane introduced a free multiplicative Brownian motion \( b_t \) (denoted \( \Lambda_t \) in [4]). As conjectured by Biane and proved by Kemp [25], \( b_t \) is the large-\( N \) limit, in the sense of \( * \)-distribution, of a Brownian motion \( B^N_t \) in \( GL(N; \mathbb{C}) \).

The paper [13] computed the Brown measure of \( b_t \). This Brown measure is supported in a domain \( \Sigma_t \) introduced by Biane and has a special structure. It is also related to the law \( \nu_t \) of the free unitary Brownian motion. The results of [13] were then extended by Ho and Zhong [24] to compute the Brown measure of \( ub_t \), where \( u \) is an arbitrary unitary element freely independent of \( b_t \).

It is possible to study also a two-parameter family of Brownian motions in \( GL(N; \mathbb{C}) \), where the two parameters control the diffusion rates in the unitary and positive directions. This family was introduced in the context of the Segal–Bargmann transform in [9, 18] and then used also in the large-\( N \) limit of the Segal–Bargmann transform in [11, 22, 25]. This family of Brownian motions was also studied from the point of view of random matrix theory (that is, looking at the eigenvalue distribution) in [27] and [21].

In [12], it was then noted that one can naturally include a third parameter that controls the correlation between the unitary and positive diffusions. According to [12, Theorem 3.2], this three-parameter family of Brownian motions is (up to multiplying by a scalar process) the most general family of Brownian motions that is invariant under the left action of \( GL(N; \mathbb{C}) \) and the right action of \( U(N) \). See also the work of Chan [8] on the large-\( N \) limit of the Segal–Bargmann transform associated to this three-parameter family.

Following [12], it is convenient to describe these Brownian motions by one real parameter \( s \) and one complex parameter \( \tau \), where \( s \) controls the overall diffusion rate, while the real and imaginary parts of \( \tau \) control the diffusion rate in the unitary direction and the correlations between the unitary and positive diffusions. The complex parameter \( \tau \) is nonzero and satisfies

\[
|\tau - s| \leq s.
\]

See Sect. 2.1 for details of the parametrization. We therefore obtain a family \( B^N_{s,\tau}(r) \) of Brownian motions in \( GL(N; \mathbb{C}) \), where \( s \) and \( \tau \) are parameters and \( r \) is the time variable. In this paper, we will consider a “free” version \( b_{s,\tau}(r) \) of these Brownian motions. For any \( r > 0 \), the \( * \)-distribution of \( b_{s,\tau}(r) \) is the same as that of \( b_{rs,\tau}(1) \). It is therefore sufficient to consider the case \( r = 1 \) and we use the nota

\[
b_{s,\tau} := b_{s,\tau}(1).
\]
The case $\tau = s$ gives the “standard” free multiplicative Brownian motion $b_s$, as defined by Biane and denoted by $A_s$ in [4, Section 4]. When $\tau$ is real, results of Kemp [25] show that $b_{s,\tau}$ is the large-$N$ limit of $B^{N}_s,\tau$ in the sense of $\ast$-distribution. We expect that this result extends in a straightforward way to general values of $\tau$, but this has not been proved.

We then consider an arbitrary unitary element $u$ freely independent of $b_{s,\tau}(r)$ and we consider the element

$$ub_{s,\tau}.$$ (1.2)

We let $\mu_{s,\tau}$ denote the “Brown measure” of $ub_{s,\tau}$, as defined in Sect. 1.2. We believe that the Brown measure of $ub_{s,\tau}$ coincides with the limiting eigenvalue distribution of any random matrix model of the form $U^NB^{N}_s,\tau$, where $U^N$ is unitary and independent of $B^{N}_s,\tau$, and where and limiting eigenvalue distribution of $U^N$ is the law of the unitary element $u$ in (1.2). Our belief is supported by numerical simulations of $U^NB^{N}_s,\tau$, as in Fig. 1.

In this paper, we will compute the Brown measure $\mu_{s,\tau}$ and find that it has a remarkably simple structure, with a density in logarithmic coordinates that is constant in the $\tau$-direction. (See Sect. 1.3.) We will also establish a “model deformation phenomenon” as follows. Suppose we deform the free random matrix model $ub_{s,\tau}$ by varying $\tau$ with $s$ and $u$ fixed. Then the Brown measures $\mu_{s,\tau}$ vary in a simple way: the Brown measures are all related by push-forward under a natural family of maps. (See Sect. 1.4.) Assuming that the Brown measures are the same as the limiting eigenvalue distributions of the associated random matrix models, this result represents a new phenomenon in random matrix theory: the limiting eigenvalue distribution of one model can be obtained by applying an explicitly computable plane map to the eigenvalues of a different random matrix model.

In [23], the second author obtains the additive counterpart of the results of the present paper, in the special case that $\tau$ is real. That is to say, when $\tau$ is real, [23] gives the Brown measure of $x_0 + w_{s,\tau}$, where $w_{s,\tau}$ is the free (additive) Brownian motion $w_{s,\tau}(r)$ in Sect. 2.1 evaluated at $r = 1$ and where $x_0$ is a self-adjoint element freely independent of $w_{s,\tau}$. The results of [23] are based on [20], which uses the PDE method.

For general complex values of $\tau$ (with $|\tau - s| \leq s$), the results of the present paper should have straightforward analogs for the “additive case,” that is, the Brown measure of $x_0 + w_{s,\tau}$, where $x_0$ is self-adjoint and freely independent of $w_{s,\tau}$. Indeed, all the proofs given here should adapt to the additive case in a straightforward way, except that certain parts of the argument become simpler. We plan to provide the details of this argument in a later publication.

Meanwhile, after the first version of this paper was posted on the arXiv, Zhong [37] posted a preprint that computes the Brown measure of $x_0 + w_{s,\tau}$, where $x_0$ is assumed to be freely independent of $w_{s,\tau}$ but is otherwise arbitrary. Zhong’s results therefore go well beyond the additive counterpart of our results here, which rely heavily on the assumption that the element $u$ in (1.2) is unitary. Zhong uses free probability in place of the PDE method.
1.2 Random matrices and the Brown measure

Girko’s method for determining the eigenvalues of a non-normal random matrix $Z$ consists of looking at the function

$$s(\lambda) := \frac{1}{N} \log(|\det(Z - \lambda)|^2) = \frac{2}{N} \sum_{j=1}^{N} \log |\lambda - \lambda_j|,$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $Z$. Since $\frac{1}{2\pi} \log |\lambda - \lambda_j|$ is the Green’s function of the Laplacian in the plane, we find that $\frac{1}{4\pi}$ time the Laplacian of $s$, computed in the distribution sense, is just the empirical eigenvalue distribution of $Z$:

$$\frac{1}{4\pi} \Delta s(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}(\lambda).$$

It is then an elementary exercise to show that $s$ may also be computed as

$$s(\lambda) = \frac{1}{N} \text{trace} \left[ \log((Z - \lambda)^*(Z - \lambda)) \right],$$

(1.3)

where “log” denotes the self-adjoint logarithm of a self-adjoint, positive matrix.

Meanwhile, pioneering work of Voiculescu developed an operator-algebra formalism for understanding the large-$N$ limits of random matrix models. Specifically, if $Z^N$ is a sequence of $N \times N$ random matrices (not necessarily Hermitian or even normal) one looks for a von Neumann algebra $\mathcal{A}$ with a “trace” $\text{tr} : \mathcal{A} \to \mathbb{C}$ and an element $z$
of $\mathcal{A}$ such that

$$\lim_{N \to \infty} \mathbb{E} \left\{ \frac{1}{N} \text{trace} \left[ p(Z^N, (Z^N)^*) \right] \right\} = \text{tr} \left[ p(z, z^*) \right]$$

(1.4)

for every polynomial $p$ in two noncommuting variables. (Here $\text{tr}$ is not the trace in the sense of trace-class operators, but is a linear functional having properties similar to the normalized trace of matrices.) One indication of the power of this approach is the work of Voiculescu [34, 35], which showed that, under quite general circumstances, independent random matrices become freely independent in such limits. (See also Chapter 23 of [30] and Section 4.2 of [28].)

If $z$ is an element of a tracial von Neumann algebra $(\mathcal{A}, \text{tr})$, one can define the Brown measure of $z$ [6] denoted $\mu^z$, by imitating the preceding construction for matrices. (See also Chapter 11 of [28].) One first introduces a regularized analog of (1.3),

$$S(\lambda, \varepsilon) = \text{tr} \left[ \log((z - \lambda)^* (z - \lambda) + \varepsilon^2) \right], \quad \varepsilon > 0,$$

(1.5)

and then defines

$$S_0(\lambda) = \lim_{\varepsilon \to 0^+} S(\lambda, \varepsilon).$$

The Brown measure $\mu^z$ is then $\frac{1}{4\pi}$ times the distributional Laplacian of $S_0$:

$$\mu^z = \frac{1}{4\pi} \Delta S_0.$$

An important technical issue with this approach is that if even if $Z^N$ converges to $z$ in the sense described in (1.4), the empirical eigenvalue distribution of $Z^N$ need not converge almost surely to the Brown measure of $z$. On the other hand, tools have been developed to show that, in many interesting cases, $\mu^z$ is indeed the limiting eigenvalue distribution. We mention here work of Girko [15], Bai [1], Tao–Vu [33], and Guionnet–Krishnapur–Zeitouni [17], among others. (Although these works do not explicitly use the Brown measure terminology, they all use Girko’s approach to computing the eigenvalue distribution in the large-$N$ limit).

1.3 Formula for the Brown measure

The first main result of this paper is a computation of the following measure:

$$\mu_{s, \tau} = \text{Brown measure of } ub_{s, \tau},$$

where the answer will depend on the choice of the unitary element $u$. The case $\tau = s$ corresponds to the Brown measure computed in [13, 24].
We now briefly indicate the structure of the Brown measure, leaving a precise statement of the results to Sect. 2. We first identify a certain open set \( \Sigma_{s, \tau} \) containing all the mass of the Brown measure \( \mu_{s, \tau} \) and then give a formula for the density of \( \mu_{s, \tau} \) on \( \Sigma_{s, \tau} \). We use the notation

\[
\tau = \tau_1 + i \tau_2, \quad \tau_1, \tau_2 \in \mathbb{R},
\]

and introduce twisted logarithmic coordinates of a nonzero complex number \( \lambda \), given as

\[
v = \frac{1}{\tau_1} \log |\lambda|; \quad \delta = \arg \lambda - \frac{\tau_2}{\tau_1} \log |\lambda|.
\]

These coordinates are the unique real numbers \( v \) and \( \delta \) such that

\[
\lambda = e^{v\tau} e^{i\delta}.
\]

We are now ready to state our first main result.

**Theorem 1.1** There is a smooth, increasing function \( \phi^{s, \tau} \) such that the Brown measure in \( \Sigma_{s, \tau} \) is given by the formula

\[
d\mu_{s, \tau}(\lambda) = \frac{1}{2\pi \tau_1} \frac{1}{|\lambda|^2} \frac{d\phi^{s, \tau}(\delta)}{d\delta} d\lambda d\mu,
\]

where \( \lambda = x + iy \). We can also write the result in logarithmic coordinates, \( \rho = \log |\lambda| \) and \( \theta = \arg \lambda \), as

\[
d\mu_{s, \tau} = \frac{1}{2\pi \tau_1} \frac{d\phi^{s, \tau}(\delta)}{d\delta} d\rho d\theta.
\]

Since the map \( (\rho, \theta) \mapsto (\rho + \tau_1, \theta + \tau_2) \) does not change the value of the \( \delta \)-coordinate, we see that the density of \( \mu_{s, \tau} \) in logarithmic coordinates is constant in the \( \tau \)-direction.

The geometric meaning of the function \( \phi^{s, \tau} \) is explained in Sect. 2.2; see Figs. 8 and 9 and Notation 2.7. We could also write the Brown measure in the twisted logarithmic coordinates \( v \) and \( \delta \), but this change only affects the density by a constant, since \( v \) and \( \delta \) are obtained from \( \rho \) and \( \theta \) by a linear change of variables.

Figure 2 gives a density plots of an example of the Brown measure, in both rectangular and logarithmic coordinates. See also Fig. 3 for additional examples of Brown measures, showing how the Brown measure changes as \( s \) changes with \( \tau \) fixed. See Sect. 7 for details about Theorem 1.1.

### 1.4 Relating different values of \( \tau \)

Our second main result is that all the Brown measures of \( uh_{s, \tau} \) with \( s \) and \( u \) fixed and \( \tau \) varying, are related.
Fig. 2 The Brown measure $\mu_{s,\tau}$ in rectangular coordinates (left) and in logarithmic coordinates (right). Shown with $u = 1$, $s = 3$, and $\tau = 1 + i$. The density in logarithmic coordinates is constant in the $\tau$-direction.

Fig. 3 The Brown measure $\mu_{s,\tau}$ with $\tau = 1 + i/2$ and the law of $u$ putting equal mass at the four points $\pm 1$ and $\pm i$. Shown for $s = 0.9$, $s = 1$, and $s = 1.1$.

**Theorem 1.2** Fix a positive real number $s$ and the unitary element $u$ and consider a nonzero complex number $\tau$ satisfying $|s - \tau| \leq s$. Then there is an invertible map $\Phi_{s,\tau} : \Sigma_s \to \Sigma_{s,\tau}$ with the following properties. First, $\Phi_{s,\tau}$ maps each radial segment in $\Sigma_s$ to a curve with a constant value of the coordinate $\delta$ in $\Sigma_{s,\tau}$; this curve is a portion of an exponential spiral. Second, the associated Brown measures $\mu_{s,s}$ and $\mu_{s,\tau}$ are related by push-forward:

$$ (\Phi_{s,\tau})_* (\mu_{s,s}) = \mu_{s,\tau}. $$
Furthermore, let $\Phi_s$ denote the limit of $\Phi_{s, \tau}$ as $\tau$ tends to zero. Then $\Phi_s$ maps $\Sigma_s$ into the unit circle and the push-forward of $\mu_{s, \tau}$ by $\Phi_s$ is the law of $uu_s$, where $u_s$ is the free unitary Brownian motion and $u$ is as in (1.2).

See Sect. 8 for details. The map $\Phi_{s, \tau}$ is illustrated in Fig. 4. Figure 5 then shows the domains and Brown measures for a fixed value of $s$ and several different values of $\tau$. Although the different Brown measures in the figure look quite different from one another, they are actually all connected by maps of the form $\Phi_{s, \tau} \circ \Phi_{s, \tau}'$. In particular, the topology of the closed support of $\mu_{s, \tau}$ is independent of $\tau$ with $s$ fixed.

Theorem 1.2 demonstrates what we call a “model deformation phenomenon.” As we noted in Sect. 1.1, we believe that the Brown measure $\mu_{s, \tau}$ is the limiting eigenvalue distribution of the random matrix model $U_N B_{s, \tau}^N$. Assuming this is the case, Theorem 1.2 tells us that the limiting eigenvalue distribution of one random matrix model $U_N B_{s, \tau}^N$ can be converted into the limiting eigenvalue distribution of another random matrix model $U_N B_{s, \tau}'^N$ by applying the map $\Phi_{s, \tau} \circ \Phi_{s, \tau}'^{-1}$ to the eigenvalues of the first model. As we will explain in the next paragraph, the limiting case of this result as $\tau \to 0$ was established in [13] and [24]. Otherwise, it seems to be a new phenomenon in random matrix theory. (After the first version of this paper was posted on the arXiv, Zhong posted a preprint [37] obtaining similar results in the additive case.)

The paper [13] showed that the push-forward of the Brown measure of $b_{s, s}$ under a certain map $\Phi_s$ is equal to the law of the free unitary Brownian motion $u_s$. This result was extended in [24] to relate the Brown measure of $ub_s$ to the law of $uu_s$, where $u$ is an arbitrary unitary element freely independent of $b_s$ and of $u_s$. The map $\Phi_s$ in [13, 24] is the same as the one in the last part of Theorem 1.2, showing that the push-forward result in those papers is simply a limiting case ($\tau \to 0$) of a much more general family of results.
Fig. 5 The domains $\Sigma_{s, \tau}$ and associated Brown measures for $s = 1$ and several different values of $\tau$, with the law of $u$ giving equal mass to the four points $\pm 1$ and $\pm i$. The real part of $\tau$ increases from left to right and the imaginary part of $\tau$ is zero on the bottom row and increases from bottom to top. The case $\tau = s$ is in the middle of the bottom row.

1.5 The method of proof

The proofs use a variant of the PDE method introduced in [13], which studied the “standard” free multiplicative Brownian motion with trivial initial condition. (See also [19] for a gentle introduction to the PDE method.) In the nota of the present paper, [13] corresponds to taking $\tau = s$ and $u = 1$. The method in [13] was then extended by the second author and Zhong [24] to analyze the standard free multiplicative Brownian motion with arbitrary unitary initial condition ($\tau = s$ and $u$ is arbitrary). The method has also been used in [10] and [20] and discussed in the physics literature in [16].

Although the results of the present paper include the results of [13] and [24] as special cases, the PDE we use here does not include the PDE in those papers as a special case. The PDEs differ because in our PDE we vary $\tau$ with $s$ fixed, whereas in
[13, 24], \( \tau \) is always equal to \( s \). This point is discussed in detail in Sect. 4.2. Although the approach we are using has substantial advantages (described in Sect. 4.2) relative to the approach in [13, 24], it also requires new techniques. Specifically, the technical details associated to analyzing the Brown measure in the region where it is not zero are substantially different here than in previous papers. See the beginning of Sect. 7 for a discussion of the relevant differences. To derive the relevant PDE, we use a factorization result (Theorem 4.3) for elements of the form \( b_{s, \tau} \) in (1.1) that is of independent interest. The proof of this result is surprisingly subtle; see Appendix A.

1.6 Table of notation

We collect the following notations.

**Table of Notation**

| \( u_s \) | free unitary Brownian motion starting at 1 |
| \( b_{s, \tau} \) | free multiplicative Brownian motion in Notation 2.1 |
| \( u \) | unitary element freely independent of \( b_{s, \tau} \) |
| \( \mu_0 \) | law of \( u \) |
| \( \mu_s \) | law of \( uu_s \) |
| \( m_s \) | density of \( \mu_s \) w.r.t. normalized Lebesgue measure on \( S^1 \) |
| \( \mu_{s, \tau} \) | Brown measure of \( ub_{s, \tau} \) |
| \( f_\beta \) | holomorphic function in Definition 2.2 |
| \( \Sigma_s \) and \( \Sigma_{s, \tau} \) | domains in Definitions 2.3 and 2.5 |
| \( S(s, \tau, \lambda, \varepsilon) \) | regularized log potential of \( ub_{s, \tau} \) in (4.1) |
| \( S_0(s, \tau, \lambda) \) | limiting value of \( S(s, \tau, \lambda, \varepsilon) \) as \( \varepsilon \) tends to zero from above |
| \( v \) and \( \delta \) | twisted logarithmic coordinates defined in (1.7) |
| \( \tau_1 \) and \( \tau_2 \) | real and imaginary parts of \( \tau \) |

2 Set-up and statement of results

2.1 The free Brownian motions

We now consider a tracial von Neumann algebra \( \mathcal{A} \), that is, a von Neumann algebra together with a faithful, normal, tracial state. To avoid a conflict of notation with the complex parameter \( \tau \), we denote the trace on \( \mathcal{A} \) by

\[
\text{tr} : \mathcal{A} \rightarrow \mathbb{C},
\]

while emphasizing that \( \text{tr} [\cdot] \) is not the trace in the sense of trace-class operators. Rather, \( \text{tr} [\cdot] \) is a linear functional that behaves like the normalized trace of matrices.

In free probability, a semicircular element is the large-\( N \) limit of a GUE. More precisely, an element \( X \) of a tracial von Neumann algebra \((\mathcal{A}, \text{tr})\) is **semicircular with variance** \( t \) if its law is the semicircular measure supported on the interval \([-2\sqrt{t}, 2\sqrt{t}]\). There then exists a **semicircular Brownian motion** \( x_t \), characterized as a continuous
process with \( x_0 = 0 \) and having freely independent increments such that \( x_t - x_s \) is semicircular with variance \( t - s \) for all \( t > s \). See Section 1.1 of [3].

We then define a **rotated elliptic element** to be one of the form

\[
Z = e^{i\theta}(aX + ibY),
\]

(2.1)

where \( X \) and \( Y \) are freely independent semicircular elements, \( a, b, \) and \( \theta \) are real numbers, and we assume \( a \) and \( b \) are not both zero. The \( * \)-distribution of \( Z \) is unchanged if we switch the sign of \( a \) or \( b \). The \( * \)-distribution is also unchanged if we interchange \( a \) and \( b \) and then replace \( \theta \) by \( \theta + \pi \). Finally, if \( a = b \), then \( Z \) is a circular element with \( * \)-distribution independent of \( \theta \).

We parametrize rotated elliptic elements by two parameters: a positive variance parameter \( s \) and a complex covariance parameter \( \tau \) defined by

\[
s = \text{tr} \left[ Z^*Z \right],
\]

(2.2)

\[
\tau = \text{tr} \left[ Z^*Z \right] - \text{tr} \left[ Z^2 \right].
\]

(2.3)

By the Cauchy–Schwarz inequality for the inner product \( \langle A, B \rangle := \text{tr}(A^*B) \), any rotated elliptic element must satisfy

\[
|\tau - s| \leq s.
\]

We can recover the parameters \( a, b, \) and \( \theta \)—up to the transformations in the previous paragraph—by plugging (2.1) into (2.2) and (2.3), giving

\[
\text{tr} \left[ Z^*Z \right] = a^2 + b^2 = s
\]

\[
\text{tr} \left[ Z^2 \right] = e^{2i\theta}(a^2 - b^2) = s - \tau
\]

where we have used that \( \text{tr} \left[ X^2 \right] = \text{tr} \left[ Y^2 \right] = 1 \) and \( \text{tr} [XY] = \text{tr} [X] \text{tr} [Y] = 0 \). If we assume \( a \geq b \geq 0 \), then we easily find

\[
\theta = \frac{1}{2} \arg(s - \tau)
\]

\[
a = \sqrt{\frac{1}{2}(s + |\tau - s|)}
\]

\[
b = \sqrt{\frac{1}{2}(s - |\tau - s|)}.
\]

(2.4)

In the special case \( \tau = s \), we may take \( a = b = \sqrt{s/2} \) and choose \( \theta \) is arbitrarily. If \( |\tau - s| = s \), we have \( a = \sqrt{s} \) and \( b = 0 \), so that \( Z = e^{i\theta} \sqrt{s} X \). In particular, if \( \tau = 0 \), then \( Z \) is just a semicircular element of variance \( s \).

We then define a **free additive \((s, \tau)\)-Brownian motion** as a continuous process \( w_{s, \tau}(r) \) with \( w_{s, \tau}(0) = 0 \) having freely independent increments such that for all
\[ r_2 > r_1, \]
\[
\frac{w_{s,\tau}(r_2) - w_{s,\tau}(r_1)}{\sqrt{r_2 - r_1}} \tag{2.5}
\]
is a rotated elliptic element with parameters \( s \) and \( \tau \). Explicitly, we can construct \( w_{s,\tau}(r) \) as
\[
w_{s,\tau}(r) = e^{i\theta}(aX_r + ibY_r)
\]
where \( a \) and \( b \) are chosen as above and where \( X_r \) and \( Y_r \) are freely independent semicircular Brownian motions.

We then construct a **free multiplicative** \((s, \tau)\)-Brownian motion \( b_{s,\tau}(r) \) as the solution to the free stochastic differential equation
\[
db_{s,\tau}(r) = b_{s,\tau}(r) \left( i \, dw_{s,\tau}(r) - \frac{1}{2} (s - \tau) \, dr \right), \tag{2.6}
\]
with
\[
b_{s,\tau}(0) = 1.
\]
The \( dr \) term on the right-hand side of (2.6) is an “Itô term” equal to half the square of \( i \, dw_{s,\tau}(r) \), computed using the Itô rules in Sect. 4.3. Existence and uniqueness of the solution to (2.6) follows from a standard Picard iteration argument, as in Proposition A.1 of [7]. (The key making the Picard iteration work is the Burkholder–Davis–Gundy estimate for the norm of a free stochastic integral, as in [5, Theorem 3.2.1] or [31, Theorem 3.1.12].) It follows from (2.5) that \( w_{s,\tau}(r) \cong w_{rs,\tau r}(1) \), where \( \cong \) indicates that the two elements have the same \( * \)-distribution. It is then not hard to see that
\[
b_{s,\tau}(r) \cong b_{rs,\tau r}(1),
\]
We will therefore usually assume that \( r = 1 \), without loss of generality.

We also fix a unitary element \( u \) that is freely independent of \( b_{s,\tau}(r) \).

**Notation 2.1** For \( s > 0 \) and \( |\tau - s| \leq s \), we use \( b_{s,\tau} \) to denote the value of \( b_{s,\tau}(r) \) at \( r = 1 \):
\[
b_{s,\tau} := b_{s,\tau}(1).
\]
Then we define
\[
\mu_{s,\tau} = \text{Brown measure of } ub_{s,\tau}.
\]
When \( \tau = 0 \), we may take \( \theta = 0 \), \( a = \sqrt{s} \), and \( b = 0 \) in (2.4) so that \( w_{s,\tau}(r) = \sqrt{s} X_r \). In that case, the SDE (2.6) becomes

\[
\frac{db_{s,\tau}(r)}{dr} = b_{s,\tau}(r) \left( i \sqrt{s} \frac{dX_r}{dr} - \frac{1}{2} s \right),
\]

and the solution is a free unitary Brownian motion, with time-parameter scaled by \( s \). We express this result as

\[
b_{s,\tau}(r) \bigg|_{\tau=0} \cong u_{s r}, \tag{2.7}
\]

where \( u_s \) is a free unitary Brownian motion with initial condition 1, as introduced by Biane in [3].

### 2.2 Main results

Throughout the paper, we will assume that \( \tau \) is a nonzero complex number satisfying \( |\tau - s| \leq s \). We always write \( \tau \) in the form

\[
\tau = \tau_1 + i \tau_2, \quad \tau_1, \tau_2 \in \mathbb{R}.
\]

We also assume that the unitary element \( u \) is fixed and we define a probability measure \( \mu_0 \) on the unit circle by

\[
\mu_0 = \text{law of } u. \tag{2.8}
\]

That is to say, \( \mu_0 \) is the unique probability measure on \( S^1 \) such that

\[
\int_{S^1} \xi^k d\mu_0(\xi) = \text{tr} \left[ u^k \right]
\]

for every integer \( k \). Although most of the objects defined below depend on the measure \( \mu_0 \), we do not indicated this dependence in the nota. We also consider the free unitary Brownian motion \( u_s \) introduced in [3] and let

\[
\mu_s = \text{law of } uu_s.
\]

The measure \( \mu_s \) was computed by Zhong in [36]. In particular, Proposition 3.6 of Zhong states that \( \mu_s \) is absolutely continuous with respect to the Lebesgue measure on the circle.

**Definition 2.2** For any complex number \( \beta \), we define a function \( f_\beta \) by

\[
f_\beta(z) = z \exp \left\{ \frac{\beta}{2} \int_{S^1} \frac{\xi + z}{\xi - z} d\mu_0(\xi) \right\}
\]

for any \( z \) for which the integral is absolutely convergent.
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In the case that \( \mu_0 \) is the \( \delta \)-measure at 1 and \( \beta = s > 0 \), the function \( f_s \) reduces to the one defined by Biane in [4] and used in [13] to compute the Brown measure of \( b_{s,s} \). The general definition of \( f_s \) was given by Zhong in [36] and was used in [24] to compute the Brown measure of \( u b_{s,s} \). In general, the function \( f_{\beta} \) is certainly defined and holomorphic on the open set \( \mathbb{C} \setminus \text{supp}(\mu_0) \), where \( \text{supp}(\mu_0) \) is the closed support (in \( S^1 \)) of the measure \( \mu_0 \). In this paper, the motivation for Definition 2.2 is that the characteristic curves of a certain family of PDEs will be expressed in terms of functions of the form \( f_{\beta} \); see Remark 2.6.

It is easily verified that for any positive real number \( s \), the function \( f_s \) satisfies

\[
 f_s(z) = \left( f(1/\bar{z}) \right)^{-1}.
\]  

By Proposition 2.3 of [36] (drawing on results of [2]), \( f_s \) has a right inverse function \( \chi_s \) defined on the unit disk. We may then extend \( \chi_s \) holomorphically to the set of \( z \) such that \( |z| \neq 1 \) by defining

\[
 \chi_s(z) = \left( \chi_s(1/\bar{z}) \right)^{-1},
\]  

for \( |z| > 1 \). By (2.9), this extended \( \chi_s \) is still a right inverse to \( f_s \). Then by [36, Theorem 3.8], \( \chi_s \) extends by continuity to a neighborhood of any point \( \xi \) in the unit circle outside the closed support of the measure \( \mu_s \). Then \( \chi_s \) is holomorphic on \( \mathbb{C} \setminus \text{supp}(\mu_s) \). By applying the Schwarz reflection principle to the function \( w \mapsto \frac{1}{i} \log \chi_s(e^{iW}) \), we see that \( \chi_s \) is holomorphic on \( \mathbb{C} \setminus \text{supp}(\mu_s) \). It also satisfies

\[
 f_s(\chi_s(z)) = z, \quad z \in \mathbb{C} \setminus \text{supp}(\mu_s).
\]

We then define a domain \( \Sigma_s \) as the interior of the complement of the image of \( \chi_s \), as follows.

**Definition 2.3** For any \( s > 0 \), we define sets \( \tilde{\Sigma}_s \) and \( \Sigma_s \) by

\[
 \tilde{\Sigma}_s = \left[ \chi_s(\mathbb{C} \setminus \text{supp}(\mu_s)) \right]^c, \quad \Sigma_s = \text{interior of } \tilde{\Sigma}_s.
\]

Since Proposition 2.4 will show that \( \tilde{\Sigma}_s \) is the closure of \( \Sigma_s \), we will subsequently use the nota \( \Sigma_s \) in place of \( \tilde{\Sigma}_s \). See Fig. 6.

In Sect. 3.1, we will give the following characterization of the domain \( \Sigma_s \).

**Proposition 2.4** There is a unique continuous function \( r_s \) on the unit circle satisfying \( 0 < r_s(\theta) \leq 1 \) such that

\[
 \left| f_s(r_s(\theta)e^{i\theta}) \right| = 1
\]

and such that the domain \( \Sigma_s \) may be computed as

\[
 \Sigma_s = \{ r e^{i\theta} \mid r_s(\theta) < r < 1/r_s(\theta) \}.
\]
Fig. 6 The map \( \chi_s = f_s^{-1} \) maps the complement of the support of \( \mu_s \) (left) to the complement of the closure of \( \Sigma_s \) (right). Shown for \( s = 2 \) and \( \mu_0 = \delta_1 \).

Furthermore, we have

\[ \tilde{\Sigma}_s = \Sigma_s. \]

We also have that

\[ f_s(r_s(\theta)e^{i\theta}) = f_s(r_s(\theta)^{-1}e^{i\theta}). \]

We now introduce a domain \( \Sigma_{s,\tau} \), which we define by specifying its complement.

**Definition 2.5** Fix a positive real number \( s \) and a nonzero complex number \( \tau \) with \( |s - \tau| \leq s \). Let \( \Sigma_{s,\tau} \) be the domain defined by

\[
(\tilde{\Sigma}_{s,\tau})^c = f_{s-\tau}(\tilde{\Sigma}_s^c) = f_{s-\tau}(\chi_s(\mathbb{C}\setminus \text{supp}(\mu_s)))) \tag{2.11}
\]

\[
\Sigma_{s,\tau} = \text{interior of } \tilde{\Sigma}_{s,\tau}. \tag{2.12}
\]

We will see in Sect. 3 that \( \tilde{\Sigma}_{s,\tau} \) is the closure of \( \Sigma_{s,\tau} \), so we will subsequently use the nota \( \tilde{\Sigma}_{s,\tau} \) in place of \( \Sigma_{s,\tau} \). See Fig. 7. In the case that \( \tau \) is real and \( u = 1 \), the domain \( \Sigma_{s,\tau} \) and the map \( f_{s-\tau} \circ \chi_s \) were studied in the paper [22] of the second author.

**Remark 2.6** The way (the complement of) the domain \( \Sigma_{s,\tau} \) arises in this paper is the following. To compute the Brown measure \( \mu_{s,\tau} \) of \( ub_{s,\tau} \), we consider the regularized logarithmic potential \( S \) of \( ub_{s,\tau} \) (as in (1.5)). We will derive (Sect. 4) a partial differential equation for \( S \) and then analyze this equation by the Hamilton–Jacobi method. In this method (Sect. 5), we compute \( S \) along certain “characteristic curves” \( \lambda(\tau) \) and \( \varepsilon(\tau) \), which depend on a choice of \( \lambda_0 \) and \( \varepsilon_0 \). Since the definition of the Brown measure involves letting \( \varepsilon \) tend to zero, we will try to choose \( \lambda_0 \) and \( \varepsilon_0 \) so that \( \varepsilon(\tau) = 0 \). The condition \( \varepsilon(\tau) = 0 \) can be achieved by taking \( \varepsilon_0 = 0 \)—but with the restriction that \( \lambda_0 \) cannot be in the closed support \( \text{supp}(\mu_s) \) of the measure \( \mu_s \) in (2.8). (This restriction arises because the initial momenta in the problem become undefined if \( \lambda_0 \in \text{supp}(\mu_s) \).

In Sect. 6, we will see that the Brown
measure \( \mu_{s,\tau} \) is zero in a neighborhood of any point of the form \( \lambda(\tau) \) computed with \( \varepsilon_0 = 0 \) and \( \lambda_0 \in \mathbb{C} \setminus \text{supp}(\mu_s) \).

Now, if we compute \( \lambda(\tau) \) with \( \varepsilon_0 = 0 \) and \( \lambda_0 \) outside \( \text{supp}(\mu_s) \), we will find (Proposition 5.6) that

\[
\lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0)).
\]

Thus, by (2.11), we find that \( \Sigma_{s,\tau} \) is the interior of the complement of the set of points of the form \( \lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0)) \). Since the Brown measure is zero near each point of the form \( \lambda(\tau) \), we see that the closed support of \( \mu_{s,\tau} \) is contained in \( \tilde{\Sigma}_{s,\tau} = \tilde{\Sigma}_{s,\tau} \).

We now introduce “twisted logarithmic coordinates” \( \nu \) and \( \delta \) for a nonzero complex number \( \lambda \), which are real numbers satisfying

\[
\lambda = e^{\nu_1}e^{i\delta}.
\]

We can compute \( \nu \) and \( \delta \) explicitly as

\[
\nu = \frac{1}{\tau_1} \log |\lambda|; \quad \delta = \arg \lambda - \frac{\tau_2}{\tau_1} \log |\lambda|, \quad (2.13)
\]

where \( \arg \lambda \) is the argument of \( \lambda \) and where \( \tau = \tau_1 + i\tau_2 \). If we fix \( \delta \) and let \( \nu \) vary, \( \log \lambda \) varies along a straight line in the \( \tau \)-direction in the plane, while \( \lambda \) itself varies along an exponential spiral that intersects the unit circle at angle \( \delta \).

We will show that for each \( \delta \), if an exponential spiral \( \{ e^{\nu_1} e^{i\delta} \}_{\nu \in \mathbb{R}} \) intersects \( \Sigma_{s,\tau} \), this spiral intersects the boundary of \( \Sigma_{s,\tau} \) in exactly two points, corresponding to \( \nu = \nu_1(\delta) \) and \( \nu = \nu_2(\delta) \), with \( \nu_1(\delta) < \nu_2(\delta) \). For any two such boundary points, we will show that there is a unique pair \( r_s(\theta)e^{i\theta} \) and \( r_s(\theta)^{-1}e^{i\theta} \) of points on the boundary of \( \Sigma_s \) such that

\[
f_{s-\tau}(r_s(\theta)e^{i\theta}) = e^{\nu_1(\delta)\tau}e^{i\delta}, \quad f_{s-\tau}(r_s(\theta)^{-1}e^{i\theta}) = e^{\nu_2(\delta)\tau}e^{i\delta}. \quad (2.14)
\]
Fig. 8 The dependence of $\phi$ on $\theta$. Shown for $s = 2$ and $\mu_0 = \delta_1$

Fig. 9 The dependence of $\delta$ on $\theta$. Shown for $s = 2$, $\tau = \frac{3}{2} + \frac{1}{2}i$, and $\mu_0 = \delta_1$

Proposition 2.4 then tells us that $f_s$ maps the points $r_s(\theta)e^{i\theta}$ and $r_s(\theta)^{-1}e^{i\theta}$ to the same point $e^{i\phi}$ on the unit circle. Thus, to each such $\delta$, we associate first an angle $\theta$ and then an angle $\phi$. The relationship among $\phi$, $\theta$, and $\delta$ is shown in Figs. 8 and 9.

Notation 2.7 Fix a $\delta$ such that the exponential spiral $e^{\nu r}e^{i\delta}$ intersects $\Sigma_{s,\tau}$. Let $\theta$ be the angle such that (2.14) holds. Then define $\phi^{s,\tau}(\delta)$ by

$$\phi^{s,\tau}(\delta) = \text{arg } f_s(r_s(\theta)e^{i\theta})$$

$$= \text{arg } f_s(r_s(\theta)^{-1}e^{i\theta}).$$

We are now ready to state our main result.

Theorem 2.8 All the mass of the Brown measure $\mu_{s,\tau}$ is concentrated on the open set $\Sigma_{s,\tau}$. The measure $\mu_{s,\tau}$ has a density on $\Sigma_{s,\tau}$ given by

$$d\mu_{s,\tau}(\lambda) = \frac{1}{2\pi \tau_1 |\lambda|^2} \frac{d\phi^{s,\tau}(\delta)}{d\delta} \ dx \ dy,$$

(2.15)
where \( \lambda = x + iy \). In logarithmic coordinates \( \rho = \log |\lambda| \) and \( \theta = \arg \lambda \), (2.15) becomes

\[
d\mu_{s,\tau}(\lambda) = \frac{1}{2\pi \tau_1} \frac{d\phi^{s,\tau}(\delta)}{d\delta} d\rho \, d\theta,
\]
so that the density in logarithmic coordinates is constant in the \( \tau \)-direction.

Here, when we say “constant in the \( \tau \)-direction,” we are thinking of the complex number \( \tau \) as a vector in \( \mathbb{C} \cong \mathbb{R}^2 \). See the right-hand side of Fig. 2. The density of \( \mu_{s,\tau} \) in the twisted logarithmic coordinates \( v \) and \( \delta \) differs from the density in ordinary logarithmic coordinates by a constant, since \( v \) and \( \delta \) are linear functions of \( \rho \) and \( \theta \).

Let us consider what happens to (2.15) in the case \( \tau = s \). In that case, \( f_{s-\tau}(z) = f_0(z) = z \), so that \( \Sigma_{s,\tau} = \Sigma_s \) and in Fig. 9, \( \delta \) is equal to \( \theta \). Thus, writing \( \theta \) in place of \( \delta \) and \( s \) in place of \( \tau \), we have

\[
d\mu_{s,s}(\lambda) = \frac{1}{2\pi s} \frac{1}{|\lambda|^2} \frac{d\phi^{s,s}(\theta)}{d\theta} dx \, dy.
\]

This formula agrees with the result previously obtained in Theorem 2.2 and Proposition 8.5 of [13] (in the case \( \mu_0 = \delta_1 \)) and Proposition 4.24 and Theorem 4.28 of [24] (in general).

### 3 The domains \( \Sigma_s \) and \( \Sigma_{s,\tau} \)

#### 3.1 The domain \( \Sigma_s \)

In this section, we study the domain \( \Sigma_s \) in Definition 2.3.

As in [13, Equation (4.1)] and [24, Equation (4.32)], we define a function \( T : \mathbb{C} \to \mathbb{R} \) by

\[
T(\lambda) = \frac{\log(|\lambda|^2)}{|\lambda|^2 - 1} \left( \int_{S_1} \frac{1}{|\lambda - \xi|^2} d\mu_0(\xi) \right)^{-1},
\]

where we give \( T \) the value 0 if the integral in the definition has the value \( +\infty \). The function \( T \) is positive for \( \lambda \) outside the closed support of the measure \( \mu_0 \). If \( |\lambda| = 1 \), we interpret \( \log(|\lambda|^2)/(|\lambda|^2 - 1) \) as having the value 1 in accordance with the limit \( \lim_{x \to 1} \log(x)/(x - 1) = 1 \). The motivation for this definition is this: If \( |z| \neq 1 \), then for any real number \( s \), we can easily compute that

\[
|f_s(z)| = 1 \iff T(z) = s.
\]

(In [13, 24], the function \( T \) also gives the small-\( \varepsilon_0 \) lifetime to a solution of a certain system of ODEs.) By [24, Lemma 4.15], \( T \) satisfies \( T(re^{i\theta}) = T(r^{-1}e^{i\theta}) \) and \( T(re^{i\theta}) \) is a decreasing function of \( r \) for \( 0 < r < 1 \) with \( \theta \) fixed.
As in \[24, \text{Equation (2.16)}\], we then define a function \(r_s\) by
\[
rs(\theta) = \begin{cases} 
\text{the unique } r < 1 \text{ such that } T(re^{i\theta}) = s \text{ if such an } r \text{ exists} \\
1 \text{ if no such } r \text{ exists}
\end{cases}
\]
In light of (3.2), we may equivalently say that \(rs(\theta)\) is the unique \(r < 1\) for which
\[
|f_s(rs e^{i\theta})| = 1
\]
or 1 if no such \(r\) exists. Since \(|f_s(z)| = 1\) when \(|z| = 1\), we see that
\[
|f_s(rs e^{i\theta})| = 1 \quad (3.3)
\]
for all \(\theta\), whether \(rs(\theta) < 1\) or \(rs(\theta) = 1\). The function \(rs\) is continuous everywhere and analytic when \(rs(\theta) < 1\) \[24, \text{Remark 2.8}\]. As a consequence of (2.9) and (3.3), we also have the relation
\[
f_s(rs(\theta) e^{i\theta}) = f_s(rs(\theta)^{-1} e^{i\theta}). \quad (3.4)
\]

The following result is a strengthening of Proposition 2.4.

**Proposition 3.1** The domain \(\Sigma_s\) in Definition 2.3 can be computed as
\[
\Sigma_s = \left\{ \{re^{i\theta} \mid rs(\theta) < r < \frac{1}{rs(\theta)} \right\} \quad (3.5)
\]
and also as
\[
\Sigma_s = \{ z \in \mathbb{C} \mid T(z) < s \}. \quad (3.6)
\]
Finally, if \(z\) is outside of \(\Sigma_s\) then \(T(z) > s\) and \(z\) is outside the closed support of \(\mu_0\).

**Proof** The set \(\mathbb{C} \setminus \text{supp}(\mu_s)\) consists of three parts: the set \(\{|z| < 1\}\), the set \(\{|z| > 1\}\), and \(S^1 \setminus \text{supp}(\mu_s)\). According to \[36, \text{Theorem 3.2}\], we have
\[
\chi_s(\{|z| < 1\}) = \{re^{i\theta} \mid r < rs(\theta) \}.
\]
Then by (2.10),
\[
\chi_s(\{|z| > 1\}) = \{re^{i\theta} \mid r > rs(\theta)^{-1} \}.
\]
Then by Proposition 3.7 and Theorem 3.8 of \[36\], the image of \(S^1 \setminus \text{supp}(\mu_s)\) is the set of points \(e^{i\theta}\) in the unit circle for which \(rs \equiv 1\) near \(\theta\). Thus, in light of Definition 2.3, \(\Sigma_s\) is the complement of the union of these three images. It is easy to see that \(\Sigma_s\) is the closure of the set on the right-hand side of (3.5). Then \(\Sigma_s\), defined as the interior of \(\Sigma_s\), is simply the set on the right-hand side of (3.5), so that \(\Sigma_s = \Sigma_s\). Theorem 4.10 of \[24\] then gives the alternative characterization of \(\Sigma_s\) in (3.6) and shows that \(T > s\) outside the closure of \(\Sigma_s\). That points outside the closure of \(\Sigma_s\) are outside the
closed support of $\mu_0$ follows by the same argument as in the proof of Lemma 6.3 of [20].

We will use the following nota.

**Definition 3.2** We define a continuous function $\phi^s$ from $S^1$ to $S^1$ by the relation

$$e^{i\phi^s(\theta)} = f_s(r(s(\theta))e^{i\theta}) = f_s(r(s(\theta))^{-1}e^{i\theta}).$$

Here we have used the identity (3.4). The nota is illustrated in Fig. 8. According to [36, Proposition 3.7], the map $\phi_s$ defines a homeomorphism of the unit circle to itself.

### 3.2 The domain $\Sigma_{s,\tau}$

We study the domain $\Sigma_{s,\tau}$ in Definition 2.5. We will use the twisted logarithmic coordinates introduced in (1.7). Recall that the $(v, \delta)$-coordinates of a nonzero complex number $z$ may be obtained by writing $z$ as

$$z = e^{v\tau} e^{i\delta},$$

with $v$ and $\delta$ in $\mathbb{R}$.

We let $J$ denote the Herglotz integral of $\mu_0$, given by

$$J(z) = \int_{S^1} \frac{\xi + z}{\xi - z} d\mu_0(\xi),$$

which appears in Definition 2.2 defining the function $f_\beta$. Using the Cauchy–Schwarz inequality, we compute that

$$|J(z)| = \left| 1 + 2z \int_{S^1} \frac{1}{\xi - z} d\mu_0(\xi) \right| \leq 1 + 2|z| \left( \int_{S^1} \frac{1}{|\xi - z|^2} d\mu_0(\xi) \right)^{1/2}.$$  

Thus, $J(z)$ is defined and finite whenever the integral in the definition (3.1) of the function $T$ is finite.

If there is some $s > 0$ for which $z$ is outside $\Sigma_s$, then by (3.6), $T(z) \neq 0$ and $J(z)$ will be defined and finite. Furthermore,

$$|J(z_1) - J(z_2)| = 2|z_1 - z_2| \left| \int_{S^1} \frac{\xi}{(\xi - z_1)(\xi - z_2)} d\mu_0(\xi) \right| \leq 2|z_1 - z_2| \left( \int_{S^1} \frac{1}{|\xi - z_1|^2} d\mu_0(\xi) \int_{S^1} \frac{1}{|\xi - z_2|^2} d\mu_0(\xi) \right)^{1/2}.$$  

(3.8)
From this and (3.6), we can easily verify that \( J \) is continuous on the complement of \( \Sigma_s \), for every \( s > 0 \).

We will prove the following results. One of the results concerns the \( \delta \)-coordinate of \( f_{s-\tau}(r_s(\theta)e^{i\theta}) \); the other gives a characterization of \( \Sigma_{s,\tau} \).

**Theorem 3.3** Fix a positive real number \( s \) and a nonzero complex number \( \tau \) satisfying \( |\tau - s| \leq s \).

(1) Define a function \( \delta^{s,\tau} : \mathbb{R} \to \mathbb{R} \) by

\[
\delta^{s,\tau}(\theta) = \theta + \frac{s - |\tau|^2}{2 |\tau|^1} \text{Im} \left[ \int_{S^1} \left( \frac{\xi + r_s(\theta)e^{i\theta}}{\xi - r_s(\theta)e^{i\theta}} \right) d\mu_0(\xi) \right].
\]

Then \( \delta^{s,\tau} \) is continuous and strictly increasing and satisfies

\[
\delta^{s,\tau}(\theta + 2\pi) = \delta^{s,\tau}(\theta) + 2\pi.
\]

Furthermore, the \( \delta \)-coordinate of \( f_{s-\tau}(r_s(\theta)e^{i\theta}) \) may be computed as

\[
\delta(f_{s-\tau}(r_s(\theta)e^{i\theta})) = \delta^{s,\tau}(\theta).
\]

(2) For each \( \delta \in \mathbb{R} \), choose \( \theta \in \mathbb{R} \) so that \( \delta^{s,\tau}(\theta) = \delta \), which is possible by Point 1. Then define \( v_1^{s,\tau}(\delta) \) and \( v_2^{s,\tau}(\delta) \) to be the \( v \)-coordinates of \( f_{s-\tau}(r_s(\theta)e^{i\theta}) \) and \( f_{s-\tau}(r_s(\theta)^{-1}e^{i\theta}) \), respectively, so that

\[
f_{s-\tau}(r_s(\theta)e^{i\theta}) = e^{\tau v_1^{s,\tau}(\delta)} e^{i\delta}; \quad f_{s-\tau}(r_s(\theta)^{-1}e^{i\theta}) = e^{\tau v_2^{s,\tau}(\delta)} e^{i\delta}.
\]

Then \( v_1^{s,\tau}(\delta) \leq v_2^{s,\tau}(\delta) \) for all \( \delta \) and the domain \( \Sigma_{s,\tau} \) may be computed as

\[
\{ e^{\tau v} e^{i\delta} \mid v < v_2^{s,\tau}(\delta) \} \quad (3.11)
\]

and \( \Sigma_{s,\tau} = \Sigma^{s,\tau} \).

Note that if \( v_1^{s,\tau}(\delta) = v_2^{s,\tau}(\delta) \), then there are no points \( \Sigma_{s,\tau} \) of the form \( e^{\tau v} e^{i\delta} \).

Point 1 of the theorem is proved in Sect. 3.3. Section 3.4 then shows that \( f_{s-\tau} \) is injective on the complement of \( \Sigma_s \). The characterization of the domain in (3.11) is then given in Sect. 3.5. We briefly outline the argument for this characterization. The complement of \( \Sigma_s \) in the Riemann sphere can be computed as the union of two closed topological disks, \( \{ re^{i\theta} \mid r \leq r_s(\theta) \} \) and \( \{ re^{i\theta} \mid r \geq r_s(\theta)^{-1} \} \cup \{ \infty \} \), which intersect at the points (if any) where \( r_s(\theta) = 1 \). These two disks are shown in dark gray and light gray, respectively, on the left-hand side of Fig. 10. We will show that \( f_{s-\tau} \) maps these disks to two closed topological disks, namely

\[
\{ e^{\tau v} e^{i\delta} \mid v \leq v_1^{s,\tau}(\delta) \} \cup \{ 0 \} \quad (3.12)
\]
Fig. 10  The domains $\Sigma_s$ (left) and $\Sigma_{s,\tau}$ (right). The complement of each domain in the Riemann sphere consists of two topological disks (dark and light gray), which may overlap on the boundary. The map $f_{s-\tau}$ takes each disk on the left homeomorphically to the corresponding disk on the right. Shown for $\mu_0 = \delta_1$, $s = 2$, and $\tau = \frac{3}{2} + \frac{1}{2}i$.

and

$$\{e^{x+iy} | v \geq v_2^{s,\tau}(\delta) \} \cup \{\infty\}. \tag{3.13}$$

These disks are shown in dark gray and light gray, respectively, on the right-hand side of Fig. 10. Since the disks (3.12) and (3.13) cover $v \leq v_1^{s,\tau}(\delta)$ and $v \geq v_2^{s,\tau}(\delta)$, the complement of their union is the set in (3.11) where $v_1^{s,\tau}(\delta) < v < v_2^{s,\tau}(\delta)$. Once this result is established, we can easily compute $\tilde{\Sigma}_{s,\tau} := (\Sigma_{s,\tau})^c$ and verify that $\Sigma_{s,\tau} := \text{int}(\tilde{\Sigma}_{s,\tau})$ is the set in (3.11).

3.3 Behavior of $f_{s-\tau}$ on the boundary of $\Sigma_s$

Throughout the section, we assume that $s$ is a positive real number and $\tau$ is a nonzero complex number satisfying $|\tau - s| \leq s$. As usual, we let $\tau_1$ and $\tau_2$ denote the real and imaginary parts of $\tau$, respectively.

3.3.1 Proof of Point 1 of Theorem 3.3.

We now work our way toward the proof of the first part of Theorem 3.3. We define $R_s(\theta)$ and $I_s(\theta)$ as the real and imaginary parts of the Herglotz function $J$ in (3.7), on the curve $r_s(\theta)e^{i\theta}$:

$$R_s(\theta) = \text{Re}[J(r_s(\theta)e^{i\theta})]; \quad I_s(\theta) = \text{Im}[J(r_s(\theta)e^{i\theta})]. \tag{3.14}$$
Lemma 3.4 The $\delta$-coordinate of the point $f_{s-\tau}(r_s(\theta)e^{i\theta})$ may be computed as in (3.9) and we have the inequality

$$0 < \frac{|\tau|^2}{s\tau_1} \leq 2. \quad (3.15)$$

Proof We may easily compute that $f_{s-\tau}(z) = f_s(z)e^{-\frac{\tau}{2}J(z)}$. Since $|f_s(r_s(\theta)e^{i\theta})|$ equals 1, we must have

$$f_s(r_s(\theta)e^{i\theta}) = e^{i\theta + \frac{s}{2}I_s(\theta)}. \quad (3.16)$$

Thus,

$$f_{s-\tau}(r_s(\theta)e^{i\theta}) = \exp\left\{i\theta + \frac{s}{2}I_s(\theta)\right\} \exp\left\{-\frac{\tau}{2}R_s(\theta)\right\} \exp\left\{-\frac{i\tau}{2}I_s(\theta)\right\}.$$  

Now, multiplying a complex number $z$ by $e^{i\alpha}$, $\alpha \in \mathbb{R}$, just adds $\alpha$ to the $\delta$-coordinate of $z$, while multiplying $z$ by $e^{v\tau}$, $v \in \mathbb{R}$, does not change the $\delta$-coordinate of $z$. Thus,

$$\delta\left(f_{s-\tau}(r_s(\theta)e^{i\theta})\right) = \theta + \frac{s}{2}I_s(\theta) + \delta\left(\exp\left\{-\frac{i\tau}{2}I_s(\theta)\right\}\right)$$

$$= \theta + \frac{s}{2}I_s(\theta) - \frac{|\tau|^2}{2\tau_1}I_s(\theta),$$

where the second equality is a direct computation using the definition (1.7) of $\delta$. This expression is the claimed result (3.9).

Since $\tau$ is nonzero and satisfies $|\tau - s| \leq s$, both $|\tau|^2$ and $\tau_1$ are positive, so that $0 < |\tau|^2/\tau_1$. The inequality $|\tau|^2/(2\tau_1) \leq s$ follows from the assumption $|\tau - s|^2 \leq s^2$, after writing $|\tau - s|^2$ as $|\tau|^2 - 2s\tau_1 + s^2$. \hfill \Box

Next, we estimate the derivative of $\delta^{s,\tau}(\theta)$ at points $\theta$ where $r_s(\theta) < 1$.

Proposition 3.5 The function $\delta^{s,\tau}$ in (3.9) is differentiable at every point $\theta$ for which $r_s(\theta) < 1$, and at every such point, we have

$$0 < \frac{d\delta^{s,\tau}}{d\theta} < 2.$$  

Proof If $I_s$ is as in (3.14), then (3.9) says that

$$\delta^{s,\tau}(\theta) = \theta + \frac{s - |\tau|^2/\tau_1}{2}I_s(\theta). \quad (3.17)$$

Meanwhile, the relation (3.16) shows that the function $\phi^s$ in (3.2) can be computed as

$$\phi^s(\theta) = \theta + \frac{s}{2}I_s(\theta). \quad (3.18)$$
If we solve for \( I_s \) in (3.18), substitute into (3.17), and differentiate, we get

\[
\frac{d\delta^{s,\tau}(\theta)}{d\theta} = \frac{|\tau|^2}{s \tau_1} + \left(1 - \frac{|\tau|^2}{s \tau_1}\right) \frac{d\phi^s}{d\theta}.
\]

Now, [24, Lemma 4.20] shows that for every \( \theta \) with \( r_s(\theta) < 1 \), the function \( \phi^s \) is differentiable at \( \theta \) with the estimate \( 0 < \frac{d\phi^s}{d\theta} < 2 \). Then if \( |\tau|^2 / (s \tau_1) < 1 \), we get

\[
\frac{|\tau|^2}{s \tau_1} < \frac{d\delta^{s,\tau}}{d\theta} < 2 - \frac{|\tau|^2}{s \tau_1},
\]

and if \( |\tau|^2 / (s \tau_1) > 1 \), we get

\[
2 - \frac{|\tau|^2}{s \tau_1} < \frac{d\delta^{s,\tau}}{d\theta} < \frac{|\tau|^2}{s \tau_1},
\]

while if \( |\tau|^2 / (s \tau_1) = 1 \), we get

\[
\frac{d\delta^{s,\tau}(\theta)}{d\theta} = \frac{|\tau|^2}{s \tau_1} = 1.
\]

In all cases, the inequality (3.15) shows that the lower bound on \( \frac{d\delta^{s,\tau}}{d\theta} \) is greater than or equal to zero and the upper bound is less than or equal to 2. \( \Box \)

**Proposition 3.6** The function \( \delta^{s,\tau} \) is strictly increasing on \( \{ \theta \in \mathbb{R} | r_s(\theta) = 1 \} \). That is, if \( \theta_2 > \theta_1 \) and \( r_s(\theta_1) = r_s(\theta_2) = 1 \), then \( \delta^{s,\tau}(\theta_2) > \delta^{s,\tau}(\theta_1) \).

**Proof** Denote \( |\tau|^2 / \tau_1 \) by \( t \) and let \( \theta_2 > \theta_1 \). Note that \( (\xi + e^{i\theta_j})/(\xi - e^{i\theta_j}) \), \( j = 1, 2 \), are purely imaginary. If \( r_s(\theta_j) = 1 \), we compute that

\[
\delta^{s,\tau}(\theta_2) - \delta^{s,\tau}(\theta_1) = \left(\theta_2 - \frac{s - t}{2} I_s(\theta)\right) - \left(\theta_1 - \frac{s - t}{2} I_s(\theta)\right)
= (\theta_2 - \theta_1) \left(1 - (s - t) \frac{e^{i\theta_2} - e^{i\theta_1}}{\theta_2 - \theta_1} \int_{S^1} \frac{\xi \ d\mu_0(\xi)}{(\xi - e^{i\theta_2})(\xi - e^{i\theta_1})}\right). \tag{3.19}
\]

Since \( \theta_1 \neq \theta_2 \), we have

\[
\frac{|e^{i\theta_2} - e^{i\theta_1}|}{\theta_2 - \theta_1} = \frac{\sin[(\theta_2 - \theta_1)/2]}{(\theta_2 - \theta_1)/2} < 1. \tag{3.20}
\]

\( \Box \) Springer
Now, since \( r_s(\theta_j) = 1, j = 1, 2 \), Proposition 2.4 tells us that the point \( e^{i\theta_j} \) is not in \( \Sigma_s \). Thus, by Proposition 3.1, \( T(e^{i\theta_j}) \geq s \), or
\[
\int_{S^1} \frac{1}{|\xi - e^{i\theta_j}|^2} \, d\mu_0(\xi) \leq \frac{1}{s} \log r_s(\theta_j)^2 - 1 = \frac{1}{s}.
\]
We thus obtain, using the Cauchy–Schwarz inequality, that
\[
\left| \int_{S^1} \frac{\xi \, d\mu_0(\xi)}{(\xi - e^{i\theta_2})(\xi - e^{i\theta_1})} \right| \leq \frac{1}{s}.
\] (3.21)
By (3.15), we have \( 0 < t \leq 2s \) or \(-s < t - s < s\). It then follows from (3.20) and (3.21) that
\[
\left| (s - t)e^{i\theta_2} - e^{i\theta_1} \int_{S^1} \frac{\xi \, d\mu_0(\xi)}{(\xi - e^{i\theta_2})(\xi - e^{i\theta_1})} \right| < 1.
\]
Plugging this estimate into (3.19) shows that \( \delta^{s, \tau}(\theta_2) - \delta^{s, \tau}(\theta_1) > 0 \). \( \square \)

**Proof of Point 1 of Theorem 3.3** We have already shown (Lemma 3.4) that \( \delta^{s, \tau} \) is a local continuous version of the \( \delta \)-coordinate of \( f_{s-\tau}(r_s(\theta)e^{i\theta}) \) for \( \theta \) in a neighborhood of \( \theta_0 \). By (3.8), \( f_{s-\tau} \) is continuous on \( \Sigma^*_s \), so \( \delta^{s, \tau} \) is continuous.

Next, we show that \( \delta^{s, \tau} \) is strictly increasing on \( \mathbb{R} \). Let \( \theta_1 < \theta_2 \). We consider four cases, corresponding to whether \( r_s(\theta_1) \) and \( r_s(\theta_2) \) are 1 or less than 1. If both \( r_s(\theta_1) \) and \( r_s(\theta_2) \) are both 1, Proposition 3.6 shows that \( \delta^{s, \tau}(\theta_1) < \delta^{s, \tau}(\theta_2) \). If \( r_s(\theta_1) = 1 \) but \( r_s(\theta_2) < 1 \), then let \( \alpha \) be the infimum of the interval \( I \) around \( \theta_2 \) on which \( r_s \) is positive, so that \( r_s(\alpha) = 1 \) and \( \theta_1 \leq \alpha \). Then \( \delta^{s, \tau}(\theta_1) < \delta^{s, \tau}(\theta_2) \) by the positivity of \( d\delta/d\theta \) on \( I \) in Proposition 3.5. The remaining cases are similar; the case where both \( r_s(\theta_1) \) and \( r_s(\theta_2) \) are less than 1 can be subdivided into two cases depending on whether or not \( \theta_1 \) and \( \theta_2 \) are in the same interval where \( r_s \) is less than 1.

Meanwhile, from (3.9), we can see that \( \delta^{s, \tau}(\theta) - \theta \) is 2\( \pi \)-periodic and continuous on \( \mathbb{R} \). Since also \( \delta^{s, \tau} \) is strictly increasing, it defines a homeomorphism of \( S^1 \) to \( S^1 \). \( \square \)

### 3.3.2 The functions \( v_1 \) and \( v_2 \)

In the following lemma, we show that the function \( f_{s-\tau} \) maps the points \( r_s(\theta)e^{i\theta} \) and \( r_s(\theta)^{-1}e^{i\theta} \) to a pair of points lying on the same exponential spiral; the point at which this spiral crosses the unit circle is \( e^{i\delta^{s, \tau}(\theta)} \). Recall the definition of the functions \( v_1^{s, \tau} \) and \( v_2^{s, \tau} \) in (3.10).

**Lemma 3.7** The points \( f_{s-\tau}(r_s(\theta)e^{i\theta}) \) and \( f_{s-\tau}(r_s(\theta)^{-1}e^{i\theta}) \) have the same \( \delta \)-coordinate, which we denote as \( \delta^{s, \tau}(\theta) \). Furthermore, we have that \( v_1^{s, \tau}(\delta^{s, \tau}(\theta)) < v_2^{s, \tau}(\delta^{s, \tau}(\theta)) \) if and only if \( r_s(\theta) < 1 \). In fact,
\[
v_2^{s, \tau}(\delta^{s, \tau}(\theta)) - v_1^{s, \tau}(\delta^{s, \tau}(\theta)) = -\frac{2}{s} \log r_s(\theta).
\] (3.22)
Proof The Herglotz integral $J(z)$ in (3.7) is easily seen to satisfy
\[ J(1/\bar{z}) = -J(z), \] (3.23)
so that
\[
fs - \tau (z) = fs(z)e^{-\frac{\tau}{2}J(z)} \\
fs - \tau (1/\bar{z}) = fs(1/\bar{z})e^{-\frac{\tau}{2}J(1/\bar{z})} = \left(\frac{fs(z)}{fs(1/\bar{z})}\right)^{-1} e^\tau J(z).
\]

If we evaluate at $z = r_s(\theta)e^{i\theta}$ where $|fs(z)| = 1$ and use the nota $R_s(\theta)$ and $I_s(\theta)$ defined in (3.14), we obtain
\[
fs - \tau (r_s(\theta)e^{i\theta}) = fs(r_s(\theta)e^{i\theta}) \exp\left\{-\frac{\tau}{2} R_s(\theta)\right\} \exp\left\{-i \frac{\tau}{2} I_s(\theta)\right\} \\
fs - \tau (r_s(\theta)^{-1}e^{i\theta}) = fs(r_s(\theta)e^{i\theta}) \exp\left\{\frac{\tau}{2} R_s(\theta)\right\} \exp\left\{-i \frac{\tau}{2} I_s(\theta)\right\}.
\]

Since multiplying a complex number by $e^{iv}$ for $v \in \mathbb{R}$ does not change the $\delta$-coordinate, $fs - \tau (r_s(\theta)e^{i\theta})$ and $fs - \tau (r_s(\theta)^{-1}e^{i\theta})$ have the same $\delta$-coordinates.

Meanwhile, recalling that the $v$-coordinate of a point $\lambda$ is computed as $\log|\lambda|/\tau_1$, we find that the difference of the $v$-coordinates is $R_s(\theta)$. But using the identity $|fs(r_s(\theta)e^{i\theta}| = 1$, we may compute that
\[
R_s(\theta) = -\frac{2}{s} \log r_s(\theta),
\]
giving the claimed formula (3.22).

\[\square\]

3.4 Injectivity of $fs - \tau$ on the complement of $\Sigma_s$

In this section, we prove the following result. We continue to assume $s$ is a positive real number and $\tau$ is a nonzero complex number with $|\tau - s| \leq s$.

Theorem 3.8 The map $fs - \tau$ is injective on the complement of $\Sigma_s$, including on the boundary.

We begin with the following lemma, whose proof is deferred until the end of this section.

Lemma 3.9 For all complex numbers $w_1$ and $w_2$, we have
\[
\left|\frac{e^{w_1} - e^{w_2}}{w_1 - w_2}\right|^2 \leq \frac{(e^{2\Re w_1} - 1)(e^{2\Re w_2} - 1)}{(2\Re w_1)(2\Re w_2)}. \tag{3.24}
\]

Here all fractions are interpreted as having their limiting values when the denominator is zero. Equality holds in (3.24) only if $w_2 = -\bar{w}_1$.
Proof of Theorem 3.8 Suppose, towards a contradiction, that \( f_{s - \tau}(z_1) = f_{s - \tau}(z_2) \) for two distinct points \( z_1 \) and \( z_2 \) outside of \( \Sigma_s \). Since \( f_{s - \tau}(z) = 0 \) if and only if \( z = 0 \), we may assume \( z_1 \) and \( z_2 \) are nonzero. Using the definition of \( f_{s - \tau} \) and a bit of algebraic manipulation, we obtain

\[
\frac{z_1}{z_2} = \exp \left\{ (s - \tau) (z_2 - z_1) \int_{S^1} \frac{\xi}{(\xi - z_1)(\xi - z_2)} \, d\mu_0(\xi) \right\}.
\]

We can then find some choice of a logarithm of \( z_1 \) and a logarithm of \( z_2 \) such that taking the log of both sides gives

\[
\frac{\log z_1 - \log z_2}{z_1 - z_2} = -(s - \tau) \int_{S^1} \frac{\xi}{(\xi - z_1)(\xi - z_2)} \, d\mu_0(\xi).
\]

(3.25)

Applying the Cauchy–Schwarz inequality, we find that, since \( \xi \in S^1 \), we have

\[
\left| \int_{S^1} \frac{\xi}{(\xi - z_1)(\xi - z_2)} \, d\mu_0(\xi) \right| \leq \left( \int_{S^1} \frac{1}{|\xi - z_j|^2} \, d\mu_0(\xi) \right)^{1/2} \left( \int_{S^1} \frac{1}{|\xi - z_1|^2} \, d\mu_0(\xi) \right)^{1/2}.
\]

(3.26)

Now, since \( z_1 \) and \( z_2 \) are outside the domain \( \Sigma_s \), Proposition 3.1 tells us that \( T(z_j) \geq s \), \( j = 1, 2 \), and \( T(z_j) > s \) unless \( z_j \) is in the boundary of \( \Sigma_s \). Thus,

\[
\int_{S^1} \frac{1}{|\xi - z_j|^2} \, d\mu_0(\xi) \leq \frac{1}{s} \log \left( \frac{|z_j|^2}{|z_j|^2 - 1} \right),
\]

(3.27)

and the inequality is strict unless \( z_j \) is in the boundary of \( \Sigma_s \).

Assume at least one of \( z_1 \) and \( z_2 \) is not in the boundary of \( \Sigma_s \) and use (3.26) and (3.27) to get

\[
\left| \int_{S^1} \frac{\xi}{(\xi - z_1)(\xi - z_2)} \, d\mu_0(\xi) \right| < \frac{2}{s} \left( \frac{\log |z_1| \log |z_2|}{(|z_1|^2 - 1)(|z_2|^2 - 1)} \right)^{1/2}.
\]

(3.28)

By taking the absolute value of (3.25) and using (3.28) and the assumption \(|\tau - s| \leq s\), we get

\[
\left| \frac{\log z_1 - \log z_2}{z_1 - z_2} \right| < 2 \left( \frac{\log |z_1| \log |z_2|}{(|z_1|^2 - 1)(|z_2|^2 - 1)} \right)^{1/2}.
\]

(3.29)

We now let \( w_1 = \log z_1 \) and \( w_2 = \log z_2 \), square both sides of (3.29), and take reciprocals, giving

\[
\left| \frac{e^{w_1} - e^{w_2}}{w_1 - w_2} \right|^2 > \frac{(e^{2 \Re w_1} - 1)(e^{2 \Re w_2} - 1)}{(2 \Re w_1)(2 \Re w_2)}.
\]

(3.30)
What we have shown is that if \( f_{s-\tau}(z_1) = f_{s-\tau}(z_2) \) for two distinct points \( z_1 \) and \( z_2 \) outside \( \Sigma_s \) with at least one of \( z_1 \) and \( z_2 \) not in \( \partial \Sigma_s \), then there exist some choices \( w_1 \) and \( w_2 \) of the logarithms of \( z_1 \) and \( z_2 \), respectively, for which (3.30) holds. But (3.30) contradicts Lemma 3.9.

It remains, then, to address the case in which both \( z_1 \) and \( z_2 \) are on the boundary of \( \Sigma_s \). By Proposition 3.1, any point \( z \) on the boundary of \( \Sigma_s \) must have the form \( z = r_s(\theta) e^{i\theta} \) or \( z = r_s(\theta) e^{-i\theta} \). If \( z_1 \) and \( z_2 \) have the same value of \( \theta \), then they must have the form \( z = r_s(\theta) e^{i\theta} \) and \( r_s(\theta) e^{-i\theta} \) with \( r_s(\theta) < 1 \), in which case Lemma 3.7 tells us that the \( v \)-coordinates of \( f_{s-\tau}(z_1) \) and \( f_{s-\tau}(z_2) \) are different. If \( z_1 \) and \( z_2 \) have different values of \( \theta \), then Point 1 of Theorem 3.3 tells us that the \( \delta \)-coordinates of \( f_{s-\tau}(z_1) \) and \( f_{s-\tau}(z_2) \) are different. \( \square \)

We note that if \( \tau = 0 \), Lemma 3.7 does not make sense as written, because the coordinate \( v \) is defined as \( \log |\lambda| / \tau_1 \). But if we rewrite the lemma using the coordinate \( \rho = \log |\lambda| \), we find that \( \rho_{s,\tau}^2(\delta_{s,\tau}(\theta)) = \rho_{1,\tau}^2(\delta_{s,\tau}(\theta)) \) when \( \tau = 0 \). Thus, when \( \tau = 0 \), injectivity of \( f_{s-\tau} = f_s \) fails on the boundary of \( \Sigma_s \), as we have already observed in (3.4).

We conclude this section by supplying the proof of Lemma 3.9.

**Proof** If we let \( a = (w_1 + w_2)/2 \) and \( b = (w_1 - w_2)/2 \), we can compute that

\[
\left| \frac{e^{w_1} - e^{w_2}}{w_1 - w_2} \right|^2 = e^{2 \text{Re}(a)} \left| \frac{\sinh(b)}{b} \right|^2. \tag{3.31}
\]

We now claim that

\[
\left| \frac{\sinh(b)}{b} \right|^2 \leq \left( \frac{\sinh(\text{Re}(b))}{\text{Re}(b)} \right)^2, \tag{3.32}
\]

with equality only if \( b \) is real. To verify (3.32), we first compute that with \( b = x + iy \), we have

\[
|\sinh(b)|^2 = \sinh^2(x) + \sin^2(y).
\]

We then use the elementary inequalities

\[
\sinh^2(x) \geq x^2; \quad \sin^2(y) \leq y^2,
\]

which hold with equality only at \( x = 0 \) and \( y = 0 \), respectively. These inequalities give

\[
\left| \frac{\sinh(b)}{b} \right|^2 \leq \frac{\sinh^2(x) + y^2}{x^2 + y^2} \leq \frac{\sinh^2(x)}{x^2}, \tag{3.33}
\]

with equality only if \( x = 0 \) or \( y = 0 \). This result establishes the inequality (3.32). If equality holds in (3.32), the equality holds in (3.33), so that either \( y = 0 \)—meaning
that \( b \) is real—or \( x = 0 \). But if \( x = 0 \), (3.32) says that \( |\sin y/y| \leq 1 \), and equality holds there only if \( y = 0 \).

Combining (3.31) and (3.32), we see that

\[
\left| \frac{e^{w_1} - e^{w_2}}{w_1 - w_2} \right|^2 \leq e^{2 \Re a} \left( \frac{\sinh(\Re b)}{\Re b} \right)^2 \\
= e^{\Re w_1 + \Re w_2} \left( \frac{\sinh((\Re w_1 - \Re w_2)/2)}{(\Re w_1 - \Re w_2)/2} \right)^2, \tag{3.34}
\]

with equality only if \( \Im b = 0 \), that is, if \( \Im w_1 = \Im w_2 \). Now, the function \( \log(\sinh(x)/x) \) is strictly convex on the real line, which we can prove by computing its second derivative as \( 1/x^2 - 1/\sinh^2 x \), which is positive for all \( x \). Thus,

\[
\frac{\sinh^2 \left( \frac{u+v}{2} \right)}{\left( \frac{u+v}{2} \right)^2} \leq \frac{\sinh u \sinh v}{u/v}
\]

with equality only when \( u = v \). Applying this with \( u = \Re w_1 \) and \( v = -\Re w_2 \) gives

\[
\left( \frac{\sinh((\Re w_1 - \Re w_2)/2)}{(\Re w_1 - \Re w_2)/2} \right)^2 \leq \frac{\sinh(\Re w_1)}{\Re w_1} \frac{\sinh(\Re w_2)}{\Re w_2}, \tag{3.35}
\]

with equality only if \( \Re w_2 = -\Re w_1 \). Plugging (3.35) back into (3.34) and simplifying give the claimed formula.

\[\square\]

### 3.5 The characterization of \( \Sigma_{s,\tau} \)

Finally, we are ready to prove Point 2 of Theorem 3.3, using the strategy indicated in Fig. 10. We use the following elementary topological result.

**Lemma 3.10** Suppose \( D_1 \) and \( D_2 \) are closed topological disks in \( \mathbb{C} \) and \( f \) is an injective, continuous map of \( D_1 \) into \( \mathbb{C} \). Suppose \( f \) maps at least one point in the interior of \( D_1 \) into the interior of \( D_2 \) and that \( f \) maps the boundary of \( D_1 \) homeomorphically onto the boundary of \( D_2 \). Then the image of \( D_1 \) is contained in \( D_2 \) and \( f \) is a homeomorphism of \( D_1 \) onto \( D_2 \).

**Proof** Pick \( z_1 \) in the interior of \( D_1 \) mapping to the interior of \( D_2 \). If some \( z_2 \) in the interior of \( D_1 \) maps outside of \( D_2 \), connect \( z_1 \) to \( z_2 \) by a continuous path in the interior of \( D_1 \). The image of this path then travels from the interior of \( D_2 \) to the complement of \( D_2 \) and must cross \( \partial D_2 \). Thus, there is some \( z_3 \) in the interior of \( D_1 \) mapping to \( \partial D_2 \). But \( f \) maps \( \partial D_1 \) onto \( \partial D_2 \), so there is \( z_4 \) in \( \partial D_1 \) with \( f(z_4) = f(z_3) \), contradicting the injectivity of \( f \).

Meanwhile, \( f \) maps \( \partial D_1 \) to a curve that winds exactly once around each point in the interior of \( D_2 \). It then follows easily that \( f \) must map \( D_1 \) onto \( D_2 \). \[\square\]
Proof of Point 2 of Theorem 3.3 The claim that \( v_1(\delta) \leq v_2(\delta) \) follows from Lemma 3.7. To establish the characterization of \( \Sigma_{s, \tau} \), we apply the lemma with \( f = f_{s-\tau} \), with \( D_1 = \{ re^{i\theta} \mid 0 \leq r < r_s(\theta) \} \), and with \( D_2 = \{ e^{\tau v} e^{i\delta} \mid -\infty \leq v \leq v_1^{s, \tau}(\delta) \} \). (We define the \( v \)-coordinate of 0 to be \(-\infty\).) Theorem 3.8 tells us that \( f_{s-\tau} \) is injective on \( D_1 \). Furthermore, \( f_{s-\tau} \) maps the point 0 in \( D_1 \) to the point 0 in \( D_2 \). Finally, Point 1 of Theorem 3.3, together with the definition (3.10), tells us that \( f_{s-\tau} \) maps \( \partial D_1 \) homeomorphically onto \( \partial D_2 \). Thus, Lemma 3.10 tells us that \( f_{s-\tau} \) maps \( D_1 \) into \( D_2 \) and is a homeomorphism. A similar argument shows that \( f_{s-\tau} \) maps the disk \( \{ re^{i\theta} \mid r \geq r_s(\theta)^{-1} \} \cup \{ \infty \} \) in the Riemann sphere onto the disk \( \{ e^{\tau v} e^{i\delta} \mid v \geq v_2^{s, \tau}(\delta) \} \cup \{ \infty \} \).

We therefore conclude that

\[
 f_{s-\tau}(\Sigma^c_s) = \{ e^{\tau v} e^{i\delta} \mid -\infty \leq v \leq v_1^{s, \tau}(\delta) \text{ or } v \geq v_2^{s, \tau}(\delta) \}.
\]

Now, the boundary of \( \Sigma_s \) consists of points of the form \( r_s(\theta) e^{i\theta} \) and \( r_s(\theta)^{-1} e^{i\theta} \) where \( r_s(\theta) < 1 \), together with limits of such points. By (3.10) and Lemma 3.7, \( f_{s-\tau} \) will map \( \partial \Sigma_s \) to points with \( v \)-coordinates \( v_1^{s, \tau}(\delta) \) or \( v_2^{s, \tau}(\delta) \) where \( v_1^{s, \tau}(\delta) < v_2^{s, \tau}(\delta) \), together with limits of such points. Then as in the proof of Proposition 3.1, we can easily see that \( \Sigma_s \) is the closure of the set in (3.11), so that \( \Sigma_s \) is the set in (3.11) and \( \Sigma_s = \overline{\Sigma_s} \).

4 The PDE for \( \Sigma_s \)

4.1 The main result

Recall the definition of \( b_{s, \tau} \) in (1.1) and recall that \( u \) is a unitary element freely independent of \( b_{s, \tau} \).

Definition 4.1 Define a function \( S \) by

\[
 S(s, \tau, \lambda, \varepsilon) = \text{tr} \left[ \log((ub_{s, \tau} - \lambda)^s(ub_{s, \tau} - \lambda) + \varepsilon^2) \right] \tag{4.1}
\]

for all \( s > 0 \), \( \tau \in \mathbb{C} \) with \( |\tau - s| \leq s \), \( \lambda \in \mathbb{C} \), and \( \varepsilon > 0 \).

We call \( S \) the regularized log potential of \( ub_{s, \tau} \). Note that we regularize the logarithm on the right-hand side of (4.1) by \( \varepsilon^2 \). By contrast, [13] and [24] use \( \varepsilon \).

Theorem 4.2 The function \( S \) in Definition 4.1 satisfies the PDE

\[
 \frac{\partial S}{\partial \tau} = \frac{1}{8} \left[ 1 - \left( 1 - \varepsilon \frac{\partial S}{\partial \varepsilon} - 2\lambda \frac{\partial S}{\partial \lambda} \right)^2 \right]. \tag{4.2}
\]
for all $\lambda \in \mathbb{C}$, all $\varepsilon > 0$, and all $\tau > 0$ satisfying $|\tau - s| < s$, with the initial condition at $\tau = 0$ given by

$$S(s, 0, \lambda, \varepsilon) = \text{tr} \left[ \log((uu_s - \lambda)^*(uu_s - \lambda) + \varepsilon^2) \right],$$

where $u_s$ is the free unitary Brownian motion. More explicitly, we have

$$S(s, 0, \lambda, \varepsilon) = \int_{S^1} \log(|\xi - \lambda|^2 + \varepsilon^2) \, d\mu_s(\xi),$$

where $\mu_s$ is the law of the unitary element $uu_s$.

This result is proved in Sect. 4.5. Note that although in Definition 4.1 we allow $|\tau - s| \leq s$, Theorem 4.2 only asserts that the PDE (4.2) holds for $\tau$ in the open set $|\tau - s| < s$.

Note that $S$ takes positive real values and therefore cannot be holomorphic in $\tau$ or $\lambda$ (unless it is independent of that variable). Thus, the derivative with respect to $\tau$ in (4.2) should be interpreted as the usual complex partial derivative

$$\frac{\partial}{\partial \tau} = \frac{1}{2} \left( \frac{\partial}{\partial \tau_1} - i \frac{\partial}{\partial \tau_2} \right),$$

where $\tau = \tau_1 + i\tau_2$, and similarly for $\partial/\partial \lambda$.

### 4.2 A discussion of our approach

Recall that the three-parameter free multiplicative Brownian motion $b_{s, \tau}$ is defined as the solution $b_{s, \tau}(r)$ to a the free SDE (2.6) evaluated at $r = 1$ (Notation 2.1). The reader might then naturally expect that we would analyze the Brown measure of $ub_{s, \tau}$ by deriving a PDE for the regularized logarithmic potential $S$ of $ub_{s, \tau}(r)$, with $r$ playing the role of the time variable. Indeed, this is the approach taken in [13] and [24] in the case $\tau = s$. Although we will, in fact, derive such a PDE—see (4.4) below—it is not the PDE we will actually use to compute the Brown measure. Instead, to compute the Brown measure, we will use the PDE (4.2) in Theorem 4.2, in which we differentiate the regularized log potential $S$ of $ub_{s, \tau}$ with respect to $\tau$ while keeping $s$ fixed.

This approach can be motivated by the work of the first author with Driver and Kemp on the complex-time Segal–Bargmann transform [12]. In that paper, the space of $L^2$ functions on $U(N)$ with respect to a heat kernel measure $\rho_s$ are transformed unitarily into holomorphic $L^2$ functions on $GL(N; \mathbb{C})$ with respect to a heat kernel measure $\nu_{s, \tau}$ by means of the heat operator with complex time $\tau$. The measure $\nu_{s, \tau}$ is just the law of the finite-$N$ Brownian motion $B_{s, \tau}^N(1)$ and the appearance of the heat operator with time $\tau$ makes it natural to differentiate in $\tau$ with $s$ fixed.

Regardless of that motivation, there are several advantages to differentiating with respect to $\tau$ with $s$ fixed. First, the PDE in (4.2) is simpler than the one obtained by
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differentiating in \( r \), which reads

\[
\frac{\partial S}{\partial r} = \text{Re} \left[ \frac{\tau}{4} \left( 1 - \left( 1 - \epsilon \frac{\partial S}{\partial \epsilon} - 2\lambda \frac{\partial S}{\partial \lambda} \right)^2 \right) \right] \\
+ s \text{ Re} \left[ \lambda^2 \left( \frac{\partial S}{\partial \lambda} \right)^2 - \lambda \frac{\partial S}{\partial \lambda} + \frac{|\lambda|^2}{4} \left( \frac{\partial S}{\partial \epsilon} \right)^2 \right].
\]

(Apply Theorem 4.4 below with \((s, \tau)\) replaced by \((0, 0)\) and \((s', \tau')\) replaced by \((s, \tau)\).) Although the first term on the right-hand side of (4.4) is closely related to (4.2), there is a complicated second term. Second, in the PDE (4.2), it is possible (at least formally) to set \( \epsilon = 0 \) and obtain a self-contained PDE for

\[
S_0(s, \tau, \lambda) := S(s, \tau, \lambda).
\]

That is to say, if we set \( \epsilon = 0 \) in (4.2), all derivatives with respect to \( \epsilon \) also disappear, which is not the case for the second term in the PDE (4.4). Since we are ultimately interested in setting \( \epsilon = 0 \), this property of the PDE will prove useful. Third, related to the second, differentiating with respect to \( \tau \) with \( s \) fixed will help us understand a crucial property of the Brown measures \( \mu_{s, \tau} \), namely that there is close relationship between two Brown measures with the same value of \( s \) but different values of \( \tau \). To understand this property from the PDE perspective, we first set \( \epsilon = 0 \), after justifying that this is allowed. As we have noted, putting \( \epsilon = 0 \) gives a self-contained PDE for the un-regularized log potential \( S_0 \) of \( ub_{s, \tau} \). We will then apply the Laplacian to both sides of this equation, obtaining a formula for how the Brown measure \( \mu_{s, \tau} \) changes as \( \tau \) changes with \( s \) fixed. See Sect. 8.

One disadvantage of differentiating in \( \tau \) with \( s \) fixed is that the initial condition—computed from \( ub_{s, \tau} \) at \( \tau = 0 \)—is not “trivial.” Specifically, when \( \tau = 0 \), we have

\[
ub_{s,0} = uu_s,
\]

where \( u_s \) is Biane’s free unitary Brownian motion and \( u \) is our unitary element, assumed to be freely independent of \( u_s \). By contrast, if we differentiated \( b_{s, \tau}(r) \) with respect to \( r \), then the initial condition would be computed from the value of \( ub_{s, \tau}(r) \) at \( r = 0 \), which is simply \( u \). Fortunately, Zhong [36] has computed the law of \( uu_s \) in terms of the law of \( u \) using Biane’s computation of the law of \( u_s \) and the technique of free multiplicative convolution. We will use Zhong’s results in Sect. 5.4 (and then indirectly in Sects. 6 and 7).

Although our PDE is in many ways simpler than the one in [13] and [24], the analysis of it requires new techniques. Thus, the technical details involved in computing the Brown measure are completely different here as compared to those earlier papers. See the beginning of Sect. 7 for more information.
4.3 Itô rules

The theory of stochastic integration with respect to a semicircular Brownian motion was developed by Biane and Speicher [5]. This paper is foundational to the whole theory of free stochastic calculus and gave the first version of a free Itô formula [5, Proposition 4.3.4]. We will need an extension of that result to a pair of freely independent semicircular Brownian motions. If \( x_r \) and \( \tilde{x}_r \) are two freely independent semicircular Brownian motions, they satisfy the following free Itô rules for a continuous adapted process \( A_r \):

\[
\begin{align*}
\text{tr}[A_r dx_r] &= \text{tr}[A_r d\tilde{x}_r] = 0 \\
 dx_r A_r dx_r &= d\tilde{x}_r A_r d\tilde{x}_r = \text{tr}[A_r] dr \\
 dx_r A_r d\tilde{x}_r &= d\tilde{x}_r A_r dx_r = 0 \\
 dx_r A_r dr &= dr A_r dx_r = 0, \\
\end{align*}
\]

(4.5)

together with a stochastic product rule for processes \( m^1_r \) and \( m^2_r \) satisfying SDEs involving \( x_r \) and \( \tilde{x}_r \):

\[
d(m^1 m^2)_r = dm^1_r m^2_r + m^1_r dm^2_r + (dm^1_r)(dm^2_r).
\]

(4.6)

These rules are widely used in the literature and were established by Kümmerer and Speicher in the setting of the Cuntz algebra [26]. We will use a general form of these rules developed by Nikitopoulos [31, Theorem 3.2.5], stated in an equivalent form for a circular Brownian motion and its adjoint.

The following Itô rules for the Brownian motions \( w_{s,\tau} \) introduced in Sect. 2.1 can be obtained easily from (4.5):

\[
\text{tr}[A_r dw_{s,\tau}(r)] = \text{tr}[A_r d\tilde{w}_{s,\tau}(r)] = 0,
\]

(4.7)

and

\[
\begin{align*}
 dw_{s,\tau}(r) A_r dw^*_{s,\tau}(r) &= dw^*_{s,\tau}(r) A_r dw_{s,\tau}(r) = s \text{tr}[A_r] dr \\
 dw_{s,\tau}(r) A_r dw_{s,\tau}(r) &= (s - \tau) \text{tr}[A_r] dr \\
 dw^*_{s,\tau}(r) A_r dw^*_{s,\tau}(r) &= (s - \bar{\tau}) \text{tr}[A_r] dr \\
\end{align*}
\]

(4.8)

and

\[
\begin{align*}
 dw_{s,\tau}(r) A_r dr &= dr A_r dw_{s,\tau}(r) = 0 \\
 dw^*_{s,\tau}(r) A_r dr &= dr A_r dw^*_{s,\tau}(r) = 0. \\
\end{align*}
\]

(4.9)

4.4 The PDE with respect to \( r \)

We will make use of the following result.
Theorem 4.3 (Factorization Theorem) Choose \( s, s' \geq 0 \) and \( \tau, \tau' \in \mathbb{C} \) so that \( |\tau - s| \leq s \) and \( |\tau' - s'| \leq s' \). Let \( b_{s, \tau} \) and \( b'_{s', \tau'} \) be two elements constructed as in Notation 2.1 but chosen to be freely independent. Then \( b_{s+s', \tau+\tau'} \) and \( b_{s, \tau}b'_{s', \tau'} \) have the same \( * \)-distribution.

The proof of Theorem 4.3 is subtle and is deferred to Appendix A.

Define

\[
g = ub_{s, \tau}b'_{s', \tau'}(r)
\]

and then set

\[
P(r, \lambda, \varepsilon) = \text{tr} \left[ \log( (g - \lambda)^*(g - \lambda) + \varepsilon^2 ) \right], \tag{4.10}
\]

where \( b'_{s', \tau'}(r) \) is freely independent of \( b_{s, \tau} \) and \( u \) and where \( s, s', \tau, \) and \( \tau' \) are fixed. By Theorem 4.3 and the fact that \( b'_{s', \tau'}(r) \) has the same \( * \)-distribution as \( br_{s, \tau'}(1) \), we see that

\[
P(r, \lambda, \varepsilon) = S(s + rs', \tau + r\tau', \lambda, \varepsilon). \tag{4.11}
\]

We note that the process \( ub_{s, \tau}b'_{s', \tau'}(r) \) satisfies the same SDE as \( b'_{s', \tau'}(r) \), but with a different initial condition.

Theorem 4.4 The function \( P \) satisfies the PDE

\[
\frac{\partial P}{\partial r} = \text{Re} \left[ \frac{\tau'}{4} \left( 1 - \left( 1 - \varepsilon \frac{\partial P}{\partial \varepsilon} - 2\lambda \frac{\partial P}{\partial \lambda} \right)^2 \right) \right] + s' \text{Re} \left[ \frac{\lambda}{4} \left( \frac{\partial P}{\partial \lambda} \right)^2 - \frac{\partial P}{\partial \lambda} + \frac{|\lambda|^2}{4} \left( \frac{\partial P}{\partial \varepsilon} \right)^2 \right]. \tag{4.12}
\]

Remark 4.5 If we take \( s = \tau = 0 \) and \( s' = \tau' = 1 \), then the function \( P \) in (4.10) becomes \( S(r, r, \lambda, \varepsilon) \), which is the function considered in [13] and [24]—except that those papers regularize with \( \varepsilon \) instead of \( \varepsilon^2 \). In this case, the PDE (4.12) simplifies to

\[
\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \varepsilon} \left( \varepsilon + \frac{1}{4} \left( |\lambda|^2 - \varepsilon^2 \right) \frac{\partial P}{\partial \varepsilon} - \frac{\varepsilon}{2} \left( x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y} \right) \right)
\]

where \( \lambda = x + iy \). This is just the PDE in [13] and [24], after making the change of variable \( \varepsilon \leftrightarrow \varepsilon^2 \) to account for the difference in regularization (\( \varepsilon \) in [13] and [24] and \( \varepsilon^2 \) here).

We will prove Theorem 4.4 after establishing some preliminary results. We use the nota

\[
g = ub_{s, \tau}b'_{s', \tau'}(r)
\]
\[ g_\lambda = g - \lambda \]
\[ m_r = g^*_\lambda g_\lambda \]
\[ R = (m_r + \varepsilon^2)^{-1}. \]

Note that
\[ P(r, \lambda, \varepsilon) = \text{tr}[\log(m_r + \varepsilon^2)]. \quad (4.13) \]

**Proposition 4.6** We have
\[ \frac{\partial}{\partial r} P(r, \lambda, \varepsilon) = \frac{\text{tr}[R dm_r]}{dr} - \frac{1}{2} \frac{\text{tr}[R dm_r R dm_r]}{dr} \quad (4.14) \]

where
\[ \frac{\text{tr}[R dm_r]}{dr} = -\frac{s' - \bar{\tau}'}{2} \text{tr}[R g^*_\lambda g_\lambda] - \frac{s' - \tau'}{2} \text{tr}[R g^*_\lambda g_\lambda] + s' \text{tr}[g^* g] \text{tr}[R] \quad (4.15) \]

and
\[ \frac{\text{tr}[R dm_r R dm_r]}{dr} = -2 \text{Re} \left\{ (s' - \tau') \text{tr}[R g^*_\lambda g_\lambda] \text{tr}[R g^*_\lambda g_\lambda] \right\} 
+ 2s' \text{tr}[R] \text{tr}[R g^*_\lambda g g^*_\lambda g_\lambda]. \quad (4.16) \]

Observe that the right-hand sides of (4.15) and (4.16) involve \( g \) and \( g^* \) in addition to \( g_\lambda = g - \lambda \) and \( g^*_\lambda = (g - \lambda)^* \). Since the derivatives of \( P \) with respect to \( \lambda \) and \( \varepsilon \) (Lemma 4.7) involve only \( g_\lambda \) and \( g^*_\lambda \), we will eventually need to rewrite \( g \) as \( g_\lambda + \lambda \) and simplify in (4.15) and (4.16).

**Proof** Using (4.13), we can see that the expression (4.14) is a consequence of Eq. (33) in Example 3.5.5 of [31]. To compute more explicitly, we use the stochastic product rule (4.6) to obtain
\[ dm_r = (dg^*) g_\lambda + g^*_\lambda (dg) + (dg^*)(dg), \quad (4.17) \]

and then compute \( dg \) and \( dg^* \) (thinking of \( r \) as the time variable) using the SDE (2.6) for \( b_{s', \bar{\tau}'}(r) \). When computing \( \text{tr}[R dm_r]/dr \), we use (4.7) and (4.9), which tell us that we keep only the \( dr \) terms in the formulas for \( dg \) and \( dg^* \) in the first two terms on the right-hand side of (4.17), but that we can drop the \( dr \) terms in the last term on the right-hand side of (4.17). Three terms will remain, which correspond to the three terms in (4.15).

We then compute that
\[ \text{tr}[R dm_r R dm_r] = \text{tr} \left\{ R \left[ (dg^*) g_\lambda + g^*_\lambda (dg) \right] R \left[ (dg^*) g_\lambda + g^*_\lambda (dg) \right] \right\}, \quad (4.18) \]
where we note that the \((dg^*)(dg)\) term in (4.17) does not contribute here. Furthermore, the \(dr\) terms in the formula for \(dg\) and \(dg^*\) do not contribute. We are then left with four terms, which we compute using (4.8) to obtain (4.16).

\[ \frac{\partial P}{\partial \varepsilon} = 2\varepsilon \text{tr}[R]; \quad \frac{\partial P}{\partial \lambda} = -\text{tr}[Rg^*_\lambda]; \quad \frac{\partial P}{\partial \bar{\lambda}} = -\text{tr}[Rg_\lambda]. \]

**Proof** Direct calculation using Lemma 1.1 in Brown’s paper [6], which states that
\[ \frac{d}{du} \text{tr} \left[ \log(f(u)) \right] = \text{tr} \left[ f(u)^{-1} \frac{df}{du} \right] \]
for any smooth function \(f(u)\) taking values in the set of strictly positive elements of \(A\).

We now provide the proof of Theorem 4.4.

**Proof of Theorem 4.4** Our strategy is (1) to simplify the results in (4.15) and (4.16) using the relations \(g = g_\lambda + \lambda, g^* = g^*_\lambda + \bar{\lambda}\), and
\[ Rg^*_\lambda g_\lambda = R(g^*_\lambda g_\lambda + \varepsilon - \varepsilon) = 1 - \varepsilon R, \]
and (2) to express the resulting expressions in terms of derivatives of \(P\) with respect to \(\lambda\) and \(\varepsilon\) using Lemma 4.7. There is, however, a difficulty with this approach, namely that the last term in (4.15) and the last term in (4.16) will give rise to “bad” terms (such as \(\text{tr}[g^*_\lambda g_\lambda]\)) that do not show up in Lemma 4.7. Fortunately, the bad terms arising from (4.15) cancel with the bad terms arising from (4.16). We omit the details of the argument, which is similar to the proof of Theorem 2.8 in Section 5 of [13].

4.5 The PDE with respect to \(\tau\)

We now prove Theorem 4.2, establishing the PDE (4.2).

**Proof of Theorem 4.2** Recall the definition of \(P\) in (4.10) that
\[ P(r, \lambda, \varepsilon) = S(s + rs', \tau + r\tau', \lambda, \varepsilon). \]
If we differentiate both sides of this relation with respect to \(r\) at \(r = 0\), using the chain rule, we obtain
\[ \left. \frac{\partial P}{\partial r} \right|_{r=0} = s' \frac{\partial S}{\partial s}(s, \tau, \lambda, \varepsilon) + \tau \frac{\partial S}{\partial \tau}(s, \tau, \lambda, \varepsilon) + \bar{\varepsilon} \frac{\partial S}{\partial \bar{\tau}'}(s, \tau, \lambda, \varepsilon). \]

We can then compute \(\partial S/\partial \tau\) as
\[ \frac{\partial S}{\partial \tau} = \left. \frac{\partial P}{\partial \tau'} \right|_{r=0}. \]
If we evaluate the PDE (4.12) at $r = 0$ and then differentiate the right-hand side with respect to $\tau'$, we obtain (4.2).

\[\square\]

5 The Hamilton–Jacobi analysis

5.1 The complex-time Hamilton–Jacobi formulas

We now analyze solutions to the PDE (4.2). We introduce the complex-valued Hamiltonian function by replacing $\partial S/\partial \lambda$ by $p_\lambda$ and $\partial S/\partial \varepsilon$ by $p_\varepsilon$ on the right-hand side of (4.2), with an overall minus sign:

\[H(\lambda, \varepsilon, p_\lambda, p_\varepsilon) = -\frac{1}{8} \left[ 1 - (1 - \varepsilon p_\varepsilon - 2\lambda p_\lambda)^2 \right].\]  \hspace{1cm} (5.1)

Here the variables $\lambda$ and $p_\lambda$ are complex valued and the variables $\varepsilon$ and $p_\varepsilon$ are real valued. Define

\[p_{\lambda,0} = \frac{\partial}{\partial \lambda_0} S(s, 0, \lambda_0, \varepsilon_0) \]  \hspace{1cm} (5.2)

\[p_{\varepsilon,0} = \frac{\partial}{\partial \varepsilon_0} S(s, 0, \lambda_0, \varepsilon_0). \]  \hspace{1cm} (5.3)

Writing $S(s, 0, \lambda_0, \varepsilon_0)$ as in (4.3), we obtain

\[p_{\lambda,0} = -\int_{S^1} \frac{\bar{\xi} - \bar{\lambda}}{|\xi - \lambda_0|^2 + \varepsilon_0^2} d\mu_s(\xi) \]  \hspace{1cm} (5.4)

and

\[p_{\varepsilon,0} = \int_{S^1} \frac{2\varepsilon_0}{|\xi - \lambda_0|^2 + \varepsilon_0^2} d\mu_s(\xi). \]  \hspace{1cm} (5.5)

Then define curves $\lambda(\tau)$, $\varepsilon(\tau)$, $p_\lambda(\tau)$, and $p_\varepsilon(\tau)$ by

\[\lambda(\tau) = \lambda_0 \exp \left\{ \frac{\tau}{2} (\varepsilon_0 p_\varepsilon,0 + 2\lambda_0 p_\lambda,0 - 1) \right\} \]  \hspace{1cm} (5.6)

\[\varepsilon(\tau) = \varepsilon_0 \exp \left\{ \text{Re} \left[ \frac{\tau}{2} (\varepsilon_0 p_\varepsilon,0 + 2\lambda_0 p_\lambda,0 - 1) \right] \right\} \]  \hspace{1cm} (5.7)

\[p_\lambda(\tau) = p_{\lambda,0} \exp \left\{ -\frac{\tau}{2} (\varepsilon_0 p_\varepsilon,0 + 2\lambda_0 p_\lambda,0 - 1) \right\} \]  \hspace{1cm} (5.8)

\[p_\varepsilon(\tau) = p_{\varepsilon,0} \exp \left\{ -\text{Re} \left[ \frac{\tau}{2} (\varepsilon_0 p_\varepsilon,0 + 2\lambda_0 p_\lambda,0 - 1) \right] \right\} \]  \hspace{1cm} (5.9)

We now derive Hamilton–Jacobi formulas for solutions to the PDE (4.2).
Theorem 5.1 For all $\tau$ with $|\tau - s| \leq s$, we have the first Hamilton–Jacobi formula

$$S(s, \tau, \lambda(\tau), \varepsilon(\tau)) = S(s, 0, \lambda_0, \varepsilon_0) + 2 \Re[\tau H_0] + \frac{1}{2} \Re \left[ \tau (\varepsilon_0 p_{\varepsilon,0} + 2\lambda_0 p_{\lambda,0}) \right],$$

(5.10)

where $H_0 = H(\lambda_0, \varepsilon_0, p_{\lambda,0}, p_{\varepsilon,0})$, and the second Hamilton–Jacobi formulas

$$\frac{\partial S}{\partial \lambda}(s, \tau, \lambda(\tau), \varepsilon(\tau)) = p_\lambda(\tau) \tag{5.11}$$

$$\frac{\partial S}{\partial \varepsilon}(s, \tau, \lambda(\tau), \varepsilon(\tau)) = p_\varepsilon(\tau). \tag{5.12}$$

Note that the ordinary Hamilton–Jacobi method is not directly applicable to the PDE (4.2), because the “time” variable $\tau$ in that equation is complex. Our method for proving Theorem 5.1 will then be to reduce the problem to an ordinary Hamilton–Jacobi PDE with real time variable.

5.2 Ordinary Hamilton–Jacobi method

Our strategy for proving Theorem 5.1 is to consider the function

$$S^\tau(s, t, \lambda, \varepsilon) := S(s, t\tau, \lambda, \varepsilon), \tag{5.13}$$

where $t$ is a positive real number. Note that on the right-hand side of (5.13), we are scaling the complex number $\tau$ by the positive real number $t$. With $\tau$ fixed, $S^\tau(s, t, \lambda, \varepsilon)$ is defined for all $t$ such that $|t\tau - s| \leq s$.

Proposition 5.2 The function $S^\tau$ satisfies the PDE

$$\frac{\partial S^\tau}{\partial t} = \Re \left[ \frac{\tau}{4} \left( 1 - \left( 1 - \varepsilon \frac{\partial S^\tau}{\partial \varepsilon} - 2\lambda \frac{\partial S^\tau}{\partial \lambda} \right)^2 \right) \right] \tag{5.14}$$

for all $t$ such that $|t\tau - s| < s$, with the initial condition at $t = 0$ given by

$$S^\tau(s, 0, \lambda, \varepsilon) = \text{tr}\left[ \log((uu_s - \lambda)^* (uu_s - \lambda) + \varepsilon^2) \right] \tag{5.15}$$

The PDE for $S^\tau$ is of ordinary Hamilton–Jacobi type with respect to the real time variable $t$. Note that the coefficients of the PDE in (5.14) do not depend on $s$; the $s$-dependence in the problem is only in the initial condition (5.15).

Proof Using the chain rule in the definition (5.13) of $S^\tau$, we find that

$$\frac{\partial S^\tau}{\partial t} = \tau \frac{\partial S}{\partial \tau} + \bar{\tau} \frac{\partial S}{\partial \bar{\tau}}.$$

The result now follows from the PDE (4.2) for $S$. \qed

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We will analyze the function $S^\tau$ using the Hamilton–Jacobi method. The following proposition recalls the general Hamilton–Jacobi method.

**Proposition 5.3** Fix an open set $U \subset \mathbb{R}^n$, a time-interval $[0, T]$, and a function $H(x, p) : U \times \mathbb{R}^n \to \mathbb{R}$. Consider a function $S(t, x)$ satisfying

$$\frac{\partial S}{\partial t} = -H(x, \nabla_x S), \quad x \in U, \ t \in [0, T].$$

Consider a curve $(x(t), p(t))$ with $x(t) \in U$, $p(t) \in \mathbb{R}^n$, and $t$ ranging over an interval $[0, T_1]$ with $T_1 \leq T$. Assume this curve satisfies Hamilton’s equations:

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}(x(t), p(t)); \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}(x(t), p(t))$$  \hspace{1cm} (5.16)

with initial conditions of the special form

$$x(0) = x_0; \quad p(0) = (\nabla_x S)(0, x_0).$$  \hspace{1cm} (5.17)

Then we have the first Hamilton–Jacobi formula

$$S(t, x(t)) = S(0, x_0) - tH(x_0, p_0) + \int_0^t p(s) \cdot \frac{dx}{ds} ds$$  \hspace{1cm} (5.18)

and the second Hamilton–Jacobi formula

$$(\nabla_x S)(t, x(t)) = p(t).$$  \hspace{1cm} (5.19)

See the proof of Proposition 5.3 in [13] for a self-contained proof and see also Section 3.3 in the book [14] of Evans.

We now record how this general result looks in the case of the function $S^\tau(s, t, \lambda, \varepsilon)$, taking the open set $U$ to be the set of $(s, \lambda, \varepsilon)$ with $\varepsilon > 0$. Recall that we write $\lambda = x + i y$. We use the nota

$$p_\lambda = \frac{1}{2}(p_x - ip_y)$$

$$p_\bar{\lambda} = \frac{1}{2}(p_x + ip_y).$$  \hspace{1cm} (5.20)

This is a convenient nota, so that $p_\lambda$ and $p_\bar{\lambda}$ correspond to $\partial S^\tau / \partial \lambda$ and $\partial S^\tau / \partial \bar{\lambda}$, respectively, in the PDE and in the second Hamilton–Jacobi formula. We then compute that

$$\left( \frac{\partial}{\partial p_x} + i \frac{\partial}{\partial p_y} \right) p_\lambda = 1; \quad \left( \frac{\partial}{\partial p_x} - i \frac{\partial}{\partial p_y} \right) p_\lambda = 0;$$

$$\left( \frac{\partial}{\partial p_x} + i \frac{\partial}{\partial p_y} \right) p_\bar{\lambda} = 0; \quad \left( \frac{\partial}{\partial p_x} - i \frac{\partial}{\partial p_y} \right) p_\bar{\lambda} = 1.$$

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The Hamiltonian, read off (with a minus sign) from the PDE (5.14), is

\[ H_{\tau} = -\text{Re} \left[ \frac{\tau}{4} \left( 1 - (1 - \varepsilon p_\varepsilon - 2\lambda p_\lambda)^2 \right) \right]. \] (5.22)

We now use the notation \( \lambda^\tau, \varepsilon^\tau, p^\tau_\lambda, \) and \( p^\tau_\varepsilon \) for solutions to Hamilton’s equations with Hamiltonian \( H_{\tau} \).

**Proposition 5.4** Suppose \((\lambda^\tau (t), \varepsilon^\tau (t), p^\tau_\lambda (t), p^\tau_\varepsilon (t))\) satisfies the Hamilton’s equations (5.16) with Hamiltonian \( H_{\tau} \), with \( \varepsilon^\tau (t) > 0 \) and initial conditions computed as in (5.17). Then for all \( t > 0 \) such that \( |t\tau - s| < s \), we have

\[ S^\tau (s, t, \lambda^\tau (t), \varepsilon^\tau (t)) = S^\tau (s, 0, \lambda_0, \varepsilon_0) + t H^\tau_0 + t \text{Re} \left[ \frac{\tau}{2} (\varepsilon_0 p_\varepsilon, 0 + 2\lambda_0 p_\lambda, 0) \right], \] (5.23)

where \( H^\tau_0 \) is the value of the Hamiltonian at \( t = 0 \). Furthermore,

\[
\begin{align*}
\frac{\partial S^\tau}{\partial \lambda}(s, t, \lambda^\tau (t), \varepsilon^\tau (t)) &= p^\tau_\lambda (t) \\
\frac{\partial S^\tau}{\partial \lambda}(s, t, \lambda^\tau (t), \varepsilon^\tau (t)) &= p^\tau_\lambda (t) \\
\frac{\partial S^\tau}{\partial \varepsilon}(s, t, \lambda^\tau (t), \varepsilon^\tau (t)) &= p^\tau_\varepsilon (t).
\end{align*}
\] (5.24)

**Proof** In the general Hamilton–Jacobi method, we note that \( dx_j / ds = \partial H / \partial p_j \), so that \( \mathbf{p}(s) \cdot d\mathbf{x}/ds = \mathbf{p}(s) \cdot \nabla_p H (\mathbf{x}(s), \mathbf{p}(s)) \). We then note that the operator \( \mathbf{p} \cdot \nabla_p \) is the homogeneity operator in the \( \mathbf{p} \) variables, acting as \( k \) times the identity on a function that is homogeneous of degree \( k \) in \( \mathbf{p} \). We then write the Hamiltonian (5.22) as \( H^\tau = H_2 + H_1 \), with

\[
\begin{align*}
H_2 &= \text{Re} \left[ \frac{\tau}{4} (\varepsilon p_\varepsilon + 2\lambda p_\lambda)^2 \right]; \\
H_1 &= -\text{Re} \left[ \frac{\tau}{2} (\varepsilon p_\varepsilon + 2\lambda p_\lambda) \right],
\end{align*}
\]

where \( H_2 \) and \( H_1 \) are, respectively, homogeneous of degrees 2 and 1 in \( \mathbf{p} \). (The terms of degree zero in \( \mathbf{p} \) cancel.) Thus,

\[ \mathbf{p} \cdot \nabla_p H^\tau = 2H_2 + H_1 = 2H^\tau - H_1. \]

It is then easy to check that \( H^\tau \) and \( H_1 \) are both constants of motion, so that \( \int_0^t \mathbf{p}(s) \cdot d\mathbf{x}/ds \, ds \) equals \( t \) times the value of \( 2H^\tau - H_1 \) at \( t = 0 \). The claimed formula (5.23) then follows easily. Meanwhile, the formulas in (5.24) follow immediately from the general formula (5.19), together with the definitions (5.20) of \( p_\lambda \) and \( p_\perp \). \( \square \)

**Proposition 5.5** Suppose \( \tau \) is a nonzero complex number such that \( |\tau - s| = s \). Then the Hamilton–Jacobi formulas (5.23) and (5.24) continue to hold at \( t = 1 \).
Note that in the borderline case $|\tau - s| = s$, we have only established that the PDE (5.14) holds for $t < 1$. (See Theorem 5.2.) The Proposition says that the Hamilton–Jacobi formulas nevertheless remain valid at $t = 1$.

**Proof** We will show in Appendix B that the element $b_{s,\tau}$ in Notation 2.1 depends continuously (in the operator norm topology) on the parameter $\tau$, for all $\tau$ satisfying $|\tau - s| \leq s$. (See Theorem B.2.) Let us then fix a nonzero $\tau$ with $|\tau - s| = s$. We will use the continuity of the operator logarithm on strictly positive self-adjoint operators, which may be established by expanding the function $\log x$ in a Taylor series based at $a$, for some large positive $a$. It is then easily verified that the function $S^\tau (s, t, \lambda, \varepsilon)$ in and its derivatives with respect to $\lambda$ and $\varepsilon$ (computed in Lemma 4.7) depend continuously on the operator $b_{s,\tau}$ and therefore also on $\tau$, even up to the boundary. We may therefore let $t$ tend to 1 on both sides of the Hamilton–Jacobi formulas (5.23) and (5.24).

Following the general Hamilton–Jacobi method (as in (5.17)), the initial momentum $p_{\lambda,0}$ is computed as

$$p_{\lambda,0} = \frac{\partial}{\partial \lambda_0} S^\tau (s, 0, \lambda_0, \varepsilon_0).$$

Since $S^\tau (s, 0, \lambda_0, \varepsilon_0) = S(s, 0, \lambda_0, \varepsilon_0)$, we see that the initial momentum $p_{\lambda,0}$ associated to $S^\tau$ agrees with the initial momentum $p_{\lambda,0}$ associated to $S$, as in (5.2). A similar statement applies to the initial momentum $p_{\varepsilon,0}$.

**5.3 Proof of the main theorem**

We now prove Theorem 5.1.

**Proof of Theorem 5.1** From the definition (5.13) of $S^\tau$, we have

$$S(s, \tau, \lambda, \varepsilon) = S^\tau (s, t, \lambda, \varepsilon)\big|_{t=1}.$$  

(5.25)

Thus, to compute $S(s, \tau, \lambda, \varepsilon)$, we will apply the Hamilton–Jacobi analysis in the previous subsection and then evaluate at $t = 1$. It is therefore natural to define

$$\lambda(\tau) = \lambda^\tau (t)\big|_{t=1} ; \quad \varepsilon(\tau) = \varepsilon^\tau (t)\big|_{t=1} ;$$

$$p_{\lambda}(\tau) = p_{\lambda}^\tau (t)\big|_{t=1} ; \quad p_{\varepsilon}(\tau) = p_{\varepsilon}^\tau (t)\big|_{t=1}.$$  

(5.26)

As noted at the end of the previous subsection, the initial momenta $p_{\lambda,0}$ and $p_{\varepsilon,0}$ associated to $p^\tau_{\lambda}$ and $p^\tau_{\varepsilon}$ agree with the initial momenta in the statement of the complex-time Hamilton–Jacobi formulas, as in (5.2) and (5.3), because $S^\tau (s, 0, \lambda, \varepsilon)$ coincides with $S(s, 0, \lambda, \varepsilon)$.

We now show that the curves in (5.26) are given by the formulas in (5.6)–(5.9). We have, for example,

$$\frac{d\lambda^\tau}{dt} = \frac{dx^\tau}{dt} + i \frac{dy^\tau}{dt} = \left( \frac{\partial}{\partial \lambda_{\tau}^x} + i \frac{\partial}{\partial \lambda_{\tau}^y} \right) H^\tau.$$ 

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Then in (5.22), we write the real part as half the sum of the indicated expression and its conjugate. Using (5.21), this gives

$$\frac{d\lambda^\tau}{dt} = \tau (\epsilon^\tau p_{\dot{\epsilon}}^\tau + 2\lambda^\tau p_{\dot{\lambda}}^\tau - 1) \lambda^\tau.$$ 

Since $\lambda^\tau p_{\dot{\lambda}}^\tau$ and $\epsilon^\tau p_{\dot{\epsilon}}^\tau$ are constants of motion, as the reader may easily verify, we find

$$\lambda^\tau(t) = \lambda_0 \exp \left\{ \frac{t \tau}{2} (\epsilon_0 p_{\dot{\epsilon},0} + 2\lambda_0 p_{\dot{\lambda},0} - 1) \right\}.$$ 

Setting $t = 1$ gives the curve in (5.6). The verifications for the remaining curves are extremely similar.

Using (5.25), we can easily see that the Hamilton–Jacobi formulas (5.10), (5.11), and (5.12) are just restatements of (5.23) and (5.24) using the curves defined in (5.26). By Proposition 5.5, the formulas continue to hold in the borderline case $|\tau - s| = s$. $\square$

### 5.4 The $\epsilon_0 = 0$ case

We now wish to look at what happens to the curves $\lambda(\tau)$ and $\epsilon(\tau)$ in the limit as $\epsilon_0$ approaches zero from above. (We are not currently claiming that the Hamilton–Jacobi formulas remain valid up to $\epsilon_0 = 0$; we are merely looking at the limiting behavior of the curves.) In the formulas (5.6) and (5.7), the limit is obtained by simply setting $\epsilon_0 = 0$, provided that the initial momenta $p_{\dot{\epsilon},0}$ and $p_{\dot{\lambda},0}$ remain defined in this limit. But looking at the formulas in (5.4) and (5.5), we see that if $\epsilon_0 = 0$ and $\lambda_0$ belongs to the closed support $\text{supp}(\mu_s)$ of $\mu_s$, then the integrals defining $p_{\dot{\epsilon},0}$ and $p_{\dot{\lambda},0}$ may be divergent. We therefore assume $\lambda_0 \notin \text{supp}(\mu_s)$ in the following result.

**Proposition 5.6** Suppose $\lambda_0 \notin \text{supp}(\mu_s)$. Then with $\epsilon_0 = 0$, we have

$$2\lambda_0 p_{\dot{\lambda},0} - 1 = -\int_{S^1} \frac{\xi + \lambda_0}{\xi - \lambda_0} d\mu_s(\xi), \quad (\epsilon_0 = 0). \quad (5.27)$$

We also have the alternative formula

$$2\lambda_0 p_{\dot{\lambda},0} - 1 = -\int_{S^1} \frac{\xi + \chi_s(\lambda_0)}{\xi - \chi_s(\lambda_0)} d\mu_0(\xi), \quad (\epsilon_0 = 0), \quad (5.28)$$

where $\chi_s$ is the inverse function to $f_s$. Note that the integral in (5.27) is with respect to $\mu_s$ (the law of uu$S$) while the integral in (5.28) is with respect to $\mu_0$ (the law of u).

When $\epsilon_0 = 0$, the formula for $\lambda(\tau)$ in (5.6) becomes

$$\lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0)), \quad (5.29)$$
while the formula for \( \varepsilon(\tau) \) in (5.7) becomes
\[
\varepsilon(\tau) \equiv 0.
\]

**Corollary 5.7** For each fixed \( \tau \), consider the map
\[
\lambda_0 \mapsto \lambda(\tau)
\]
where \( \lambda(\tau) \) is computed with \( \lambda(0) = \lambda_0 \) and \( \varepsilon_0 = 0 \). Then this map is a biholomorphism of \( \mathbb{C} \setminus \text{supp}(\mu_s) \) onto \((\tilde{\Sigma}_{s,\tau})^c\).

**Proof** By Definition 2.3, \( \chi_s \) maps \( \mathbb{C} \setminus \text{supp}(\mu_s) \) to the complement of \( \tilde{\Sigma}_s = \Sigma_s \) and by the last point in Proposition 3.1, the complement of \( \Sigma_s \) does not intersect \( \text{supp}(\mu_s) \). Thus, the Herglotz integral \( J \) in (3.7) is defined and holomorphic on the complement of \( \text{supp}(\mu_s) \) and also on the complement of \( \Sigma_s \). Thus, the right-hand side of (5.29) is a holomorphic function of \( \lambda_0 \). The claimed bijectivity of the map then follows from the definitions of the domains in Definitions 2.3 and 2.5, the injectivity of \( f_{s-t} \) in Theorem 3.8, and the fact that \( \tilde{\Sigma}_{s,\tau} \) is the closure of its interior. (This last point follows from the proof of Theorem 3.3 in Sect. 3.5.) \( \square \)

**Proof of Proposition 5.6** From the definition (5.20) of \( p_\lambda \), we have
\[
p_{\lambda,0} = \frac{\partial}{\partial \lambda_0} S(s, 0, \lambda_0, \varepsilon_0).
\]
If we then specialize to the case \( \varepsilon_0 = 0 \), we have
\[
2\lambda_0 p_{\lambda,0} - 1 = 2\lambda_0 \frac{\partial}{\partial \lambda_0} \int S^1 \log(|\lambda_0 - \xi|^2) \, d\mu_s(\xi) - 1
\]
\[
= 2 \int S^1 \frac{\lambda_0}{\lambda_0 - \xi} \, d\mu_s(\xi) - 1
\]
\[
= - \int S^1 \frac{\xi + \lambda_0}{\xi - \lambda_0} \, d\mu_s(\xi),
\]
provided that \( \lambda_0 \) is outside the closed support of \( \mu_s \), establishing (5.27).

We now apply results from Zhong’s paper [36], specifically, the relation \( \eta_{\mu_t}(z) = \eta_\mu(\eta_t(z)) \) on p. 1357, after Theorem 2.1, along with Eq. (2.1) and the relation \( \Phi_{t,\mu}(\eta_{t_0}(z)) = z \) following Lemma 2.2. The relation \( \eta_{\mu_t}(z) = \eta_\mu(\eta_t(z)) \) implies also that \( \psi_{\mu_t}(z) = \psi_\mu(\eta_t(z)) \), where \( \psi_\mu \) is defined on p. 1356 of [36]. If we apply this last result with Zhong’s \( \mu \) corresponding to our \( \mu_0 \) (the push-forward of \( \mu_0 \) under the complex-conjugation map) and Zhong’s \( \mu_t \) corresponding to our \( \mu_t \), we find that Zhong’s function \( \eta_t \) corresponds to our \( \chi_s \) and we get the relation
\[
\int S^1 \frac{\xi + \lambda_0}{\xi - \lambda_0} \, d\mu_s(\xi) = \int S^1 \frac{\xi + \chi_s(\lambda_0)}{\xi - \chi_s(\lambda_0)} \, d\mu_0(\xi), \tag{5.30}
\]
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establishing (5.28). The identity (5.30) initially holds for $|\lambda_0| < 1$, but extends to $|\lambda_0| > 1$ by the identities (3.23) and (2.9) and then to $\lambda_0 \notin \text{supp}(\mu_s)$ by continuity.

Meanwhile, when $\varepsilon_0 = 0$, we have $\varepsilon_0 p_{\varepsilon,0} = 0$ and (5.28) applies, so that the formula for $\lambda(\tau)$ in (5.6) becomes

$$
\lambda(\tau) = \lambda_0 \exp \left\{ -\frac{\tau}{2} \int_{S^1} \frac{\xi + \chi_s(\lambda_0)}{\xi - \chi_s(\lambda_0)} d\mu_0(\xi) \right\}.
$$

Now, we can compute that

$$
fs_{-\tau}(z) = fs(z) \exp \left\{ -\frac{\tau}{2} \int_{S^1} \frac{\xi + z}{\xi - z} d\mu_0(\xi) \right\}.
$$

Applying this result with $z = \chi_s(\lambda_0)$ and recalling that $\chi_s$ is a right inverse of $f_s$ gives the claimed formula (5.29). \qed

## 6 Outside the domain

### 6.1 Outline

Our goal is to compute the function

$$
S_0(s, \tau, \lambda) := \lim_{\varepsilon \to 0^+} S(s, \tau, \lambda, \varepsilon), \quad (6.1)
$$

and its Laplacian with respect to $\lambda$, using the Hamilton–Jacobi formulas (5.10), (5.11), and (5.12). We wish to choose the initial conditions $\lambda_0$ and $\varepsilon_0$—with the initial momenta given by (5.4) and (5.5)—so that $\lambda(\tau)$ is very close to $\lambda$ and $\varepsilon(\tau)$ is very close to zero. From the formula (5.7), we see that if $\varepsilon_0$ approaches zero, then $\varepsilon(\tau)$ will also approach zero.

The obstruction to letting $\varepsilon_0$ approach 0 is that if $\varepsilon_0 = 0$ and $\lambda_0$ is in the closed support of the measure $\mu_s$ (inside the unit circle), then the integrals (5.4) and (5.5) defining the initial momenta may be singular. Thus, in this section, we always assume that $\lambda_0$ is outside $\text{supp}(\mu_s)$.

For $\lambda_0 \in \text{supp}(\mu_s)^c$, we have already analyzed in Sect. 5.4 the behavior the solution $\lambda(\tau)$ in the limit as $\varepsilon_0$ approaches zero. Let us then assume that we can simply set $\varepsilon_0 = 0$ in the second Hamilton–Jacobi formulas (5.11) and (5.12). Note that we are not really allowed to do this, because $S(s, \tau, \lambda, \varepsilon)$ is not known ahead of time to be smooth up to $\varepsilon = 0$; this is a point we will need to return to in Sect. 6.2. If we multiply (5.11) by $\lambda(\tau)$ and set $\varepsilon_0 = 0$, then $\varepsilon(\tau) = 0$ and we obtain

$$
\lambda(\tau) \frac{\partial S}{\partial \lambda}(s, \tau, \lambda(\tau), 0) = \lambda(\tau) p_{\lambda}(\tau) = \lambda_0 p_{\lambda,0},
$$

where the second equality is evident from the formulas (5.6) and (5.8) for $\lambda(\tau)$ and $p_{\lambda}(\tau)$.
Now, since \( \varepsilon_0 = 0 \), the formula (5.29) in Proposition 5.6 applies:

\[
\lambda(\tau) = f_{s-\tau}(\chi_{s}(\lambda_0)).
\]

Furthermore, Corollary 5.7 tell us that \( f_{s-\tau} \circ \chi_s \) maps \( \text{supp}(\mu_s) \) injectively onto \((\overline{\Sigma}_{s,\tau})^c\). Thus, for \( \lambda \in (\overline{\Sigma}_{s,\tau})^c \), we may choose

\[
\lambda_0 = (f_{s-\tau} \circ \chi_{s})^{-1}(\lambda), \quad (6.2)
\]

so that \( \lambda_0 \) depends holomorphically on \( \lambda \). Thus, we obtain

\[
\lambda \frac{\partial S_0}{\partial \lambda} = \lambda_0 p_{\lambda,0} \big|_{\lambda_0 = (f_{s-\tau} \circ \chi_{s})^{-1}(\lambda)}, \quad \lambda \in (\overline{\Sigma}_{s,\tau})^c. \quad (6.3)
\]

Finally, we note from (5.27) in Proposition 5.6 that \( \lambda_0 p_{\lambda,0} \) depends holomorphically on \( \lambda_0 \), for \( \lambda_0 \) outside \( \text{supp}(\mu_s) \). Thus, we conclude from (6.3) that \( \lambda \partial S_0/\partial \lambda \) is a holomorphic function of \( \lambda \), for \( \lambda \) outside \( \overline{\Sigma}_{s,\tau} \). Thus, \( \partial S_0/\partial \lambda \) is also holomorphic outside \( \overline{\Sigma}_{s,\tau} \), except possibly at the origin. But we will see, once we compute \( \lambda \partial S_0/\partial \lambda \) more explicitly in Theorem 6.1, that the singularity at the origin is removable, so \( \partial S/\partial \lambda \) is holomorphic on all of \( (\overline{\Sigma}_{s,\tau})^c \). Thus,

\[
\Delta_{\lambda} S_0(s, \tau, \lambda) = 4 \frac{\partial}{\partial \lambda} \frac{\partial S_0}{\partial \lambda} = 0 \quad (6.4)
\]

for \( \lambda \) outside of \( \overline{\Sigma}_{s,\tau} \), showing that the Brown measure is zero there.

In the next section, we will make the argument sketched here rigorous.

### 6.2 Details

We use Definitions 2.3 and 2.5 along with Theorem 3.3 and the injectivity of \( f_{s-\tau} \) (Theorem 3.8) and the injectivity of \( \chi_s \) (from the relation \( f_{s}(\chi_{s}(z)) = z \)). These results tell us that \( f_{s-\tau} \circ \chi_s \) maps \( \text{supp}(\mu_s) \) injectively onto \((\overline{\Sigma}_{s,\tau})^c\). Our main objective in this section is to rigorize the argument presented in Sect. 6.1, showing that the Brown measure \( \mu_{s,\tau} \) is zero outside \( \overline{\Sigma}_{s,\tau} \).

Our first main result is a refinement of (6.3). Recall the definition (6.1) of \( S_0 \).

**Theorem 6.1** For all \( \lambda \) outside \( \overline{\Sigma}_{s,\tau} \), we have

\[
\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) = \chi_{s-\tau}(\lambda) \int_{S_1} \frac{1}{\chi_{s-\tau}(\lambda) - \xi} d\mu_0(\xi)
\]

and the Brown measure \( \mu_{s,\tau} \) is zero outside \( \overline{\Sigma}_{s,\tau} \). Here, \( \chi_{s-\tau} \) is the inverse function to \( f_{s-\tau} \), which exists by Definition 2.5 and Theorem 3.8.

The reason that the argument in Sect. 6.1 is not rigorous is that we are not allowed to put \( \varepsilon_0 = 0 \)—which results in \( \varepsilon(\tau) = 0 \) for all \( \tau \)—in the Hamilton–Jacobi formulas.
After all, $S(s, \tau, \lambda, \epsilon)$ is initially defined only for $\epsilon > 0$ and is not known ahead of time to have a smooth extension up to $\epsilon = 0$.

Our strategy will then be to prove that $S(s, \tau, \lambda, \epsilon)$ does have a smooth extension to $\epsilon = 0$, for $\lambda \in (\Sigma_{s, \tau})^c$. To do this, we look at the map $(\lambda_0, \epsilon_0) \mapsto (\lambda(\tau), \epsilon(\tau))$ for a fixed $\tau$. This map is defined by solving the Hamilton’s equations with Hamiltonian (5.22) and, as we will see, it makes perfect sense to put $\epsilon_0 = 0$, provided $\lambda_0$ is outside the closed support of the measure $\mu_s$. We will show that this map has a smooth inverse at $(\lambda_0, 0)$ and then use the inverse function theorem and the first Hamilton–Jacobi formula to construct the smooth extension of $S$.

**Proposition 6.2** Consider the map $\Psi_\tau$ from $\mathbb{C} \times (0, \infty)$ into $\mathbb{C} \times (0, \infty)$ given by

$$\Psi_\tau(\lambda_0, \epsilon_0) = (\lambda(\tau), \epsilon(\tau)),$$

where $\lambda(\tau)$ and $\epsilon(\tau)$ are computed with $\lambda(0) = \lambda_0$, $\epsilon(0) = \epsilon_0$, and the initial momenta given by (5.4) and (5.5). Then for all $\lambda_0 \in \text{supp}(\mu_s)^c$, the map $\Psi_\tau$ extends analytically to a neighborhood of $\epsilon_0 = 0$, and the Jacobian of $\Psi_\tau$ at $(\lambda_0, 0)$ is invertible. Furthermore, the $\epsilon$-component of the extended map is an odd function of $\epsilon_0$.

**Proof** As long as $\lambda_0$ is outside the closed support of $\mu_s$, the formulas (5.4) and (5.5) defining the initial momenta $p_{\lambda, 0}$ and $p_{\epsilon, 0}$ remain well defined and analytic, even in a neighborhood of $\epsilon_0 = 0$. The formulas (5.6) and (5.7) for $\lambda(\tau)$ and $\epsilon(\tau)$ then depend analytically on $\lambda_0$ and $\epsilon_0$. It is then easily checked that all quantities involved are even functions of $\epsilon_0$—including $\epsilon_0 p_{\epsilon, 0}$—except for the leading factor of $\epsilon_0$ in the formula for $\epsilon(\tau)$, making the $\epsilon$-component an odd function of $\epsilon_0$.

We then compute the Jacobian of $\Psi_\tau$ at $(\lambda_0, 0)$, writing $\lambda_0 = x_0 + iy_0$. If we differentiate (5.7) and then evaluate at $\epsilon_0 = 0$, the leading factor of $\epsilon_0$ in the formula gives a simple result:

$$\frac{\partial \epsilon}{\partial x_0} = 0; \quad \frac{\partial \epsilon}{\partial y_0} = 0; \quad \frac{\partial \epsilon}{\partial \epsilon_0} = \exp \left\{ \text{Re} \left( \frac{\tau}{2} (2\lambda_0 p_{\lambda, 0} - 1) \right) \right\}. \quad (6.6)$$

Meanwhile, when $\epsilon_0 = 0$, we may use the formula (5.29): $\lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0))$. Thus, the Jacobian of $\Psi_\tau$ at $(\lambda_0, 0)$ has the form

$$\begin{pmatrix} J & \frac{\partial \epsilon}{\partial \epsilon_0} \\ 0 & \frac{\partial \epsilon}{\partial \epsilon_0} \end{pmatrix},$$

where $J$ is the $2 \times 2$ Jacobian of the map $f_{s-\tau} \circ \chi_s$ and the value of $*$ is irrelevant. Since, by the identity $f_s(\chi_s(z)) = z$ and Theorem 3.8, $f_{s-\tau} \circ \chi_s$ is an injective holomorphic map, its complex derivative cannot be zero and therefore $J$ must be invertible. Since, also, $\partial \epsilon / \partial \epsilon_0$ is positive by (6.6), we find that the Jacobian of $\Psi_\tau$ is invertible. ☐

**Proposition 6.3** Fix $\tau$ with $|\tau - s| \leq s$ and a point $\tilde{\lambda} \in (\Sigma_{s, \tau})^c$. Then the map

$$(\lambda, \epsilon) \mapsto S(s, \tau, \lambda, \epsilon),$$
initially defined for \( \varepsilon > 0 \), has an analytic extension defined for \((\lambda, \varepsilon)\) in a neighborhood of \((\tilde{\lambda}, 0)\) and this extension is even in \(\varepsilon\).

**Proof** We define a function \(HJ(\tau, \lambda_0, \varepsilon_0)\) by the right-hand side of the first Hamilton–Jacobi formula (5.10), namely

\[
HJ(s, \tau, \lambda_0, \varepsilon_0) = S(s, 0, \lambda_0, \varepsilon_0) + 2 \text{Re}[\tau H_0] + \frac{1}{2} \text{Re} \left[ \tau (\varepsilon_0 p_{\varepsilon, 0} + 2\lambda_0 p_{\lambda, 0}) \right],
\]

where it is understood that the initial momenta \(p_{\lambda, 0}\) and \(p_{\varepsilon, 0}\) are always computed as functions of \(\lambda_0\) and \(\varepsilon_0\) as in (5.4) and (5.5). Fix \(\tilde{\lambda}\) in \((\Sigma_{s, \tau})^c\). By Corollary 5.7, we can find \(\lambda_0\) in \(\text{supp}(\mu_s)\) so that with \(\varepsilon_0 = 0\), we have \(\lambda(\tau) = \tilde{\lambda}\). By Proposition 6.2 and the inverse function theorem, \(\Psi_{\tau}\) has an analytic inverse near \((\lambda_0, 0)\). Since, by (5.7), \(\varepsilon(\tau)\) always has the same sign as \(\varepsilon_0\), we see that the \(\varepsilon_0\)-component of \(\Psi_{\tau}^{-1}(\lambda, \varepsilon)\) is positive whenever \(\varepsilon\) is positive. We then consider the function

\[
\tilde{S}(s, \tau, \lambda, \varepsilon) = HJ(s, \tau, \Psi_{\tau}^{-1}(\lambda, \varepsilon)),
\]

which is an analytic function. Since the first Hamilton–Jacobi formula tells us that \(\tilde{S}\) agrees with \(S\) when \(\varepsilon > 0\), we see that \(\tilde{S}\) is the desired extension. Since the \(\varepsilon\)-component of \(\Psi_{\tau}\) is an odd function of \(\varepsilon_0\), the \(\varepsilon_0\)-component of \(\Psi_{\tau}^{-1}\) is an odd function of \(\varepsilon\). Since, as is easily checked, \(HJ\) is an even function of \(\varepsilon_0\), we see that the extended \(S\) is even in \(\varepsilon\). \(\square\)

**Proof of Theorem 6.1** We fix \(\lambda\) in \((\Sigma_{s, \tau})^c\) and we choose \(\lambda_0 \in \text{supp}(\mu_s)^c\) so that with \(\varepsilon_0 = 0\), we get \(\lambda(\tau) = \lambda\) and \(\varepsilon(\tau) = 0\). (Such a \(\lambda_0\) exists by Corollary 5.7.) We now multiply the second Hamilton–Jacobi formula (5.11) by \(\lambda(\tau),\) using this value of \(\lambda_0\) and, initially, \(\varepsilon_0 > 0:\)

\[
\lambda(\tau) \frac{\partial S}{\partial \lambda}(s, \tau, \lambda(\tau), \varepsilon(\tau)) = \lambda(\tau) p_{\lambda}(\tau) = \lambda_0 p_{\lambda, 0},
\]

where the second equality is evident from the formulas (5.6) and (5.8) for \(\lambda(\tau)\) and \(p_{\lambda}(\tau).\) As \(\varepsilon_0\) tends to zero, \(\lambda(\tau)\) tends to \(\lambda\) and \(\varepsilon(\tau)\) tends to 0. Thus, by the regularity of \(S\) established in Proposition 6.3, we may let \(\varepsilon_0\) tend to zero and obtain

\[
\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) = \lambda_0 p_{\lambda, 0}.
\]

We now apply the formula (5.28), from which we can easily solve for \(\lambda_0 p_{\lambda, 0},\) to get

\[
\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) = \chi_s(\lambda_0) \int_{S^1} \frac{1}{\chi_s(\lambda_0) - \xi} d\mu_0(\xi). \tag{6.7}
\]

Finally, we recall that \(\lambda = f_s^\tau(\chi_s(\lambda_0)),\) so that (recalling that \(\chi_s\) is the inverse function to \(f_s^\tau),\) \(\lambda_0 = f_s^\tau(\chi_{s-\tau}(\lambda)).\) Plugging this value for \(\lambda_0\) into (6.7) gives the claimed formula (6.5) for \(\lambda \partial S / \partial \lambda.\) If we divide (6.5) by \(\lambda\) and note that \(\chi_s(\lambda_0) = \chi_{s-\tau}(\lambda_0) \neq 0,\) we obtain

\[
\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) = \chi_s(\lambda_0) \int_{S^1} \frac{1}{\chi_s(\lambda_0) - \xi} d\mu_0(\xi).
\]
0, we see that $\partial S/\partial \lambda$ has a removable singularity at the origin. Thus, $\partial S/\partial \lambda$ is a holomorphic function of $\lambda \in (\Sigma_{s,\tau})^c$, so that as in (6.4), the Brown measure $\mu_{s,\tau}$ is zero on $(\Sigma_{s,\tau})^c$.

7 Inside the domain

We now briefly summarize the way the computation of the Brown measure “in the domain” (that is, where it is not zero) works in the present paper, since this is the main way our paper differs from earlier ones. In the present paper, as in earlier works such as [10, 13, 24], and [20], one initially attempts to achieve the desired condition $\varepsilon(\tau) = 0$ by taking $\varepsilon_0 = 0$, leading to a determination of the region where the Brown measure is zero. (Sect. 6 of the present paper.) But the way the idea of letting $\varepsilon_0$ tend to zero fails to work is different here than in the previous works. In the just-cited papers, letting $\varepsilon_0$ tend to 0 fails at points $\lambda_0$ where, with $\varepsilon_0 = 0$, the solution ceases to exist before one gets to the $\tau$-value one is interested in. For each such $\lambda_0$, one must look for a positive value of $\varepsilon_0$ that gives $\varepsilon(\tau) = 0$, and then compute the associated $\lambda(\tau)$.

The situation in the present paper is different in that the lifetime of the paths is always infinite, even if $\varepsilon_0$ tends to zero. (As a result, we are able to avoid the involved analysis of the lifetime of solutions in Sections 6.3 and 6.4 of [13] and Section 4.3 of [24].) In this paper, letting $\varepsilon_0$ tend to 0 fails at points $\lambda_0$ for which the initial momenta $p_{\varepsilon,0}$ and $p_{\lambda,0}$ become ill defined in this limit. The set of such points is one dimensional, consisting of the support of the measure $\mu_s$ inside the unit circle.

For points $e^{i\phi}$ in supp($\mu_s$), we will let $\lambda_0$ approach $e^{i\phi}$ while simultaneously letting $\varepsilon_0$ tend to zero. A key point is that approaching $(e^{i\phi}, 0)$ along different paths in $(\lambda_0, \varepsilon_0)$-space give different limiting values of $\lambda(\tau)$. Thus, although supp($\mu_s$) is one dimensional, the set of $\lambda(\tau)$’s obtained will be two dimensional, consisting of the entire domain $\Sigma_{s,\tau}$.

7.1 Outline

Recall the function $\phi^s$ introduced in Definition 3.2 and the function $\delta^{s,\tau}$ introduced in Theorem 3.3. For a point $\lambda$ in $\Sigma_{s,\tau}$ with $\delta$-coordinate $\delta$, we let $\phi^{s,\tau}(\delta)$ be as in Notation 2.7, which means that

$$\phi^{s,\tau}(\delta) = \phi^s(\theta^{s,\tau}(\delta)),$$

where $\theta^{s,\tau}$ is the inverse function to $\delta^{s,\tau}$. This definition means simply that $\phi = \phi^{s,\tau}(\delta)$ is related to $\delta$ as in Figs. 8 and 9.

In this section, we provide an outline of the computation of the Brown measure $\mu_{s,\tau}$ inside the domain $\Sigma_{s,\tau}$, including a brief (but not entirely rigorous) derivation of the formula (1.8) for the Brown measure.
7.1.1 Mapping into the domain

Suppose we take $\varepsilon_0 = 0$ and $\lambda_0 = e^{i\phi}$, where the density of the measure $\mu_s$ at $e^{i\phi}$ is positive. Then the initial momenta $p_{\lambda,0}$ and $p_{\varepsilon,0}$ become ill defined, because the integrals (5.4) and (5.5) defining them are not absolutely convergent. Now, if we take $\varepsilon_0 = 0$ and we let $\lambda_0$ approach such a point $e^{i\phi}$, either from inside or from outside the unit circle, we get two different limiting values of $\lambda(\tau)$, both lying on the boundary of the domain $\Sigma_{s,\tau}$. Specifically, the limiting values of $\chi_s(\lambda_0)$ will be $r_s(\theta)e^{i\theta}$ and $r_s(\theta)^{-1}e^{i\theta}$, which are the two circled points on the left-hand side of Fig. 8; see [36, Proposition 2.3 and Proposition 3.7]. The limiting values of $\lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0))$ will then be as on the right-hand side of Fig. 9. To get limiting values of $\lambda(\tau)$ in the interior of $\Sigma_{s,\tau}$, we will choose a family of paths in $(\lambda_0, \varepsilon_0)$-space approaching $(e^{i\phi}, 0)$ so that the limiting value of $\varepsilon(\tau)$ is always zero, but limiting value of $\lambda(\tau)$ depends on the path.

We compute the initial momenta in (untwisted) logarithmic coordinates $\rho = \log |\lambda|$ and $\theta = \arg \lambda$, so that the initial momenta are $p_{\rho,0}$, $p_{\theta,0}$, and $p_{\varepsilon,0}$, that is, the derivatives of $S(s, 0, \lambda_0, \varepsilon_0)$ with respect to $\rho$, $\theta$, and $\varepsilon_0$. We will find paths in $(\lambda_0, \varepsilon_0)$-space approaching $(e^{i\phi}, 0)$ so that: (1) $\varepsilon_0 p_{\varepsilon,0}$ always approaches zero, (2) $p_{\theta,0}$ approaches a single number independent of the choice of path, and (3) $p_{\rho,0}$ can approach a whole range of different values, depending on the choice of path. Along all paths, the value of $\varepsilon(\tau)$ will approach 0. We then compute the value of $\lambda(\tau)$ in twisted logarithmic coordinates and we find that the range of possible values for $p_{\rho,0}$ gives a range of possible values for the $v$-coordinate of $\lambda(\tau)$, so that the values of $\lambda(\tau)$ trace out a segment of an exponential spiral in $\Sigma_{s,\tau}$. As $\phi$ varies over all points where the density of $\phi$ is positive, these spiral segments will trace out the whole domain $\Sigma_{s,\tau}$. See Sect. 7.2 for details of this argument.

7.1.2 Computing the Brown measure

In this section, we use the twisted logarithmic coordinates $v$ and $\delta$ introduced previously:

$$v = \frac{1}{\tau_1} \log |\lambda|; \quad \delta = \arg \lambda - \frac{\tau_2}{\tau_1} \log |\lambda|,$$

where $\tau = \tau_1 + i\tau_2$. We may easily verify the following formulas for the derivatives of $S$ in those coordinates:

$$\frac{\partial S}{\partial v} = 2 \, \text{Re} \left[ \tau \lambda \frac{\partial S}{\partial \lambda} \right],$$

$$\frac{\partial S}{\partial \delta} = -2 \, \text{Im} \left[ \lambda \frac{\partial S}{\partial \lambda} \right].$$

(7.1)

We recall the nota $S_0$ for the function

$$S_0(s, \tau, \lambda) = \lim_{\varepsilon \to 0^+} S(s, \tau, \lambda, \varepsilon).$$
We now use the second Hamilton–Jacobi formula (5.11), which gives a formula for \( \partial S / \partial \lambda \) at the point \((s, \tau, \lambda, \epsilon(\tau))\). As \((\lambda_0, \epsilon_0)\) approaches \((e^{i\phi}, 0)\) along our specially chosen paths, \(\epsilon(\tau)\) tends to zero and \(\lambda(\tau)\) tends to a point \(\lambda\) in \(\Sigma_{s, \tau}\). Assuming that \(S\) has a \(C^1\) extension to a neighborhood of \((s, \tau, \lambda, 0)\), we can multiply by \(\lambda\) and let \(\epsilon_0\) tend to zero in the second Hamilton–Jacobi formula to obtain

\[
\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) = \lambda(\tau)p_\lambda(\tau) = \lambda_0 p_{\lambda, 0}.
\]  

(7.2)

where the second equality follows directly from the formulas (5.6) and (5.8).

Meanwhile, let us take the formula (5.6) for \(\lambda(\tau)\) and evaluate the \(v\)-coordinates of this point in our limiting case, where \(\epsilon_0 = 0\) and \(|\lambda_0| = 1\). We obtain

\[
v(\tau) = \frac{1}{\tau_1} \text{Re} \left[ \frac{\tau}{2} \left( 2\lambda_0 p_{\lambda, 0} - 1 \right) \right] = \frac{1}{\tau_1} \text{Re} \left[ \frac{\tau}{2} \left( 2\lambda \frac{\partial S_0}{\partial \lambda} - 1 \right) \right] = \frac{1}{2\tau_1} \frac{\partial S_0}{\partial v} - \frac{1}{2},
\]  

(7.3)

where we have used (7.2) and (7.1). A key point here is that \(v(\tau)\) does not depend on the value of \(\lambda_0\), provided \(\lambda_0\) is in the unit circle.

We can solve (7.3) for \(\partial S_0 / \partial v\) as

\[
\frac{\partial S_0}{\partial v} = 2\tau_1 v + \tau_1.
\]  

(7.4)

There are two remarkable features of this formula: first, that it is very simple and explicit, and second that (inside the domain) \(\partial S_0 / \partial v\) depends only on \(v\) and not on \(\delta\). We can then take another derivative and obtain

\[
\frac{\partial^2 S_0}{\partial v^2} = 2\tau_1.
\]

We now compute the \(\delta\)-coordinate of \(\lambda(\tau)\) (with \(\epsilon_0 = 0\) and \(|\lambda_0| = 1\)). After computing that the \(\delta\)-coordinate of \(e^{i\theta} e^{\tau z}\) is \(\theta + \frac{|\tau|^2}{\tau_1} \text{Im}(z)\), we obtain

\[
\delta(\tau) = \arg \lambda_0 + \frac{|\tau|^2}{\tau_1} \text{Im} \left[ \lambda_0 p_{\lambda, 0} \right] = \arg \lambda_0 - \frac{1}{2} \frac{|\tau|^2}{\tau_1} \frac{\partial S_0}{\partial \delta},
\]

from which we obtain

\[
\frac{\partial S_0}{\partial \delta} = \frac{2\tau_1}{|\tau|^2} (\arg \lambda_0 - \delta).
\]  

(7.5)
This formula is not as explicit as the formula (7.4) for $\partial S_0/\partial v$ because the result depends on the argument of the point $\lambda_0$ in the unit circle.

Now, as discussed in Sect. 7.1.1, the point $\lambda_0$ on the unit circle is related to the point $\lambda$ in $\Sigma_{s,\tau}$ as in Figs. 8 and 9, so that $\arg \lambda_0 = \phi_{s,\tau}(\delta)$. Thus, taking a second derivative, we obtain

$$\frac{\partial^2 S_0}{\partial \delta^2} = \frac{2\tau_1}{|\tau|^2} \left( \frac{d\phi_{s,\tau}(\delta)}{d\delta} - 1 \right).$$

We then observe from (7.4) that $\partial S_0/\partial v$ is independent of $\delta$, so that the mixed derivative $\partial^2 S_0/\partial \delta \partial v$ is zero. Using the easily computed formula for the Laplacian in $(v, \delta)$ coordinates, we obtain

$$\Delta_{\lambda} S_0 = \frac{1}{|\lambda|^2} \left( \frac{1}{\tau_1^2} \frac{\partial^2 S}{\partial v^2} - 2 \frac{\tau_2}{\tau_1^2} \frac{\partial^2 S}{\partial \delta \partial v} + \frac{|\tau|^2}{\tau_1^2} \frac{\partial^2 S}{\partial \delta^2} \right)$$

$$= \frac{2}{\tau_1} \frac{1}{|\lambda|^2} \frac{d\phi}{d\delta}.$$

**Conclusion 7.1** If the preceding argument can be made rigorous, the Brown measure $\mu_{s,\tau}$ in $\Sigma_{s,\tau}$ is given by

$$d\mu_{s,\tau} = \frac{1}{2\pi \tau_1 |\lambda|^2} \frac{d\phi}{d\delta} \ dx \ dy$$

where $\lambda = x + iy$. Here $\phi$ and $\delta$ are related as in Notation 2.7 or Figs. 8 and 9. We therefore recover the claimed formula (1.8) for the Brown measure.

### 7.2 Surjectivity

In this section, we provide the details of the strategy outlined in Sect. 7.1.1, consisting of letting $\epsilon_0$ approach zero while simultaneously letting $\lambda_0$ approach a point $e^{i\phi}$ in the support of $\mu_s$. By letting $(\lambda_0, \epsilon_0)$ approach $(e^{i\phi}, 0)$ along different paths, we will get different limiting values of $\lambda(\tau)$, while still ensuring that $\epsilon(\tau)$ approaches 0. The following theorem shows that for any $\lambda$ in $\Sigma_{s,\tau}$, we can find a point $e^{i\phi}$ and a path in $(\lambda_0, \epsilon_0)$-space approaching $(e^{i\phi}, 0)$ so that the limiting value of $\lambda(\tau)$ is $\lambda$.

We use the notation

$$\lambda(\tau; \lambda_0, \epsilon_0); \ \epsilon(\tau; \lambda_0, \epsilon_0)$$

to denote the curves $\lambda(\tau)$ and $\epsilon(\tau)$ in (5.6) and (5.7) with the indicated initial conditions and with the initial momenta given by (5.4) and (5.5).

**Theorem 7.2** Pick a point $e^{i\phi}$ in $S^1$ where the density of the measure $\mu_s$ is positive and choose $\lambda_0$ to have the form

$$\lambda_0 = (1 + c\epsilon_0)e^{i\phi}, \ c \in \mathbb{R}.$$  

(7.6)
The Brown measure of a family…

Fig. 11 A point $e^{i\phi}$ is “blown up” into an interval by approaching $e^{i\phi}$ along curves of the form (7.6) with different values of $c$ (left). This interval then maps to a portion of an exponential spiral in $\Sigma_{s,\tau}$ (right). As $\phi$ varies over points where the density of $m_s$ is positive and $c$ varies over $\mathbb{R}$, the entire domain $\Sigma_{s,\tau}$ is swept out.

Then for all $c \in \mathbb{R}$, we have

$$\lim_{\varepsilon_0 \to 0^+} \varepsilon(\tau; (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0) = 0.$$ 

Furthermore, as $c$ varies over $\mathbb{R}$, the value of

$$\lambda(\tau) = \lim_{\varepsilon_0 \to 0^+} \lambda(\tau; (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0)$$

traces out a curve in $\Sigma_{s,\tau}$ with $\delta$-coordinate fixed and $v$-coordinate varying between $v_1^{s,\tau}(\delta)$ and $v_2^{s,\tau}(\delta)$. As $e^{i\phi}$ varies over all points where the density of $\mu_s$ is positive and $c$ varies over $\mathbb{R}$, the value of $\lambda(\tau)$ fills out the entire domain $\Sigma_{s,\tau}$.

Theorem 7.2 is illustrated in Fig. 11.

The key to proving Theorem 7.2 is to understand how the initial momenta—that is, the derivatives of $S(s, \tau, \lambda_0, \varepsilon_0)$ at $\tau = 0$—behave as $(\lambda_0, \varepsilon_0)$ approaches $(e^{i\phi}, 0)$ along a curve of the form (7.6). If we set $\tau = \varepsilon_0 = 0$, the resulting function $S_0(s, 0, \lambda_0)$ is the logarithmic potential of the measure $\mu_s$. This function has a fold-type singularity along the portion of the unit circle where the density of $\mu_s$ is positive.

The radial derivative of $S_0(s, 0, \lambda_0)$ therefore has a jump discontinuity on portions of the unit circle, while the angular derivative of $S_0(s, 0, \lambda_0)$ is continuous. But as soon as $\varepsilon_0$ becomes positive, $S(s, 0, \lambda_0, \varepsilon_0)$ becomes smooth in $\lambda_0$, so that the radial derivative takes on a continuous range of values, as shown in Fig. 12. It is then plausible that for any number $p$ between the “inner” and “outer” values of $\frac{\partial}{\partial \rho} S_0(s, 0, e^{i\phi})$, we can find a $\rho$ close to 0 and an $\varepsilon_0$ close to zero for which $\frac{\partial}{\partial \rho} S(s, 0, e^{\rho + i\phi}, \varepsilon_0)$ is close to $p$; this claim is verified in Proposition 7.4.

We start by recording what happens when $\varepsilon_0 = 0$ and $\lambda_0$ approaches the unit circle.
**Proposition 7.3** Fix a point $e^{i\phi}$ in the unit circle where the density of $\mu_s$ is positive. Then the limits of $\partial S_0 / \partial \rho(s, 0, \lambda_0)$ as $\lambda_0$ approaches $e^{i\phi}$ from inside and outside the unit circle, respectively, exist and are finite. We denote these limits as

$$\frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})_{\text{in}} \quad \text{and} \quad \frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})_{\text{out}}.$$

Furthermore the limit of $\partial S_0 / \partial \theta(s, 0, \lambda_0)$ as $\lambda_0$ approaches $e^{i\phi}$ from points not in the unit circle exists. We denote this limit as

$$\frac{\partial S_0}{\partial \theta}(s, 0, e^{i\phi}).$$

We will compute these limits and show that, in general, $\partial S_0 / \partial \rho(s, 0, e^{i\phi})_{\text{in}}$ and $\partial S_0 / \partial \rho(s, 0, e^{i\phi})_{\text{out}}$ are not equal. We now turn to the limiting values of the momenta along the curves of the form (7.6).

**Proposition 7.4** Fix a point $e^{i\phi}$ in the unit circle where the density of $\mu_s$ is positive. First, we have

$$\lim_{\varepsilon_0 \to 0^+} \frac{\partial S}{\partial \theta}(s, 0, (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0) = \frac{\partial S_0}{\partial \theta}(s, 0, e^{i\phi}),$$

independent of $c$. Second, we have

$$\lim_{\varepsilon_0 \to 0} \varepsilon_0 \frac{\partial S}{\partial \varepsilon_0}(s, 0, (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0) = 0$$

independent of $c$. Third, we have

$$\lim_{\varepsilon_0 \to 0^+} \frac{\partial S}{\partial \rho}(s, 0, (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0) = \frac{1}{2} \left( 1 - \frac{c}{\sqrt{1 + c^2}} \right) \frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})_{\text{in}} + \frac{1}{2} \left( 1 + \frac{c}{\sqrt{1 + c^2}} \right) \frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})_{\text{out}}.$$
Thus, as $c$ varies from $-\infty$ to $\infty$, the limiting value of $\partial S/\partial \rho$ varies continuously from $\partial S_0/\partial \rho(s, 0, e^{i\phi})^{\text{in}}$ to $\partial S_0/\partial \rho(s, 0, e^{i\phi})^{\text{out}}$.

It is convenient to relate the initial momenta in the preceding propositions to the Herglotz integral of the measure $\mu_s$, namely the integral

$$\int_{S^1} \frac{e^{i\phi'} + qe^{i\phi}}{e^{i\phi'} - qe^{i\phi}} \, d\mu_s(e^{i\phi'}),$$

(7.10)

for $q \neq 1$. We will first use this idea to verify the statements in Propositions 7.3 and 7.4 about $\partial S/\partial \theta$.

Proof of results about $\partial S/\partial \theta$. We try to connect $\partial S/\partial \theta$ to the imaginary part of the integral in (7.10) by seeking $q \neq 1$ such that

$$\frac{\partial S}{\partial \theta}(s, 0, r_0 e^{i\phi}, \varepsilon_0) = \text{Im} \left[ \int_{S^1} \frac{e^{i\phi'} + qe^{i\phi}}{e^{i\phi'} - qe^{i\phi}} \, d\mu_s(e^{i\phi'}) \right].$$

(7.11)

The left-hand side of (7.11) is computed in terms of the measure $\mu_s$ using (4.3). After computing both sides, (7.11) becomes

$$\int_{S^1} \frac{2\sin(\phi - \phi')}{(r_0^2 + 1 + \varepsilon_0^2)/r_0 - 2\cos(\phi - \phi')} \, d\mu_s(e^{i\phi'})$$

$$= \int_{S^1} \frac{2\sin(\phi - \phi')}{(q + 1/q) - 2\cos(\phi - \phi')} \, d\mu_s(e^{i\phi'}),$$

(7.12)

which gives

$$q + \frac{1}{q} = \frac{r_0^2 + 1 + \varepsilon_0^2}{r_0}.$$  

(7.13)

We first consider the case $\varepsilon_0 = 0$, which gives $q = r_0$ or $q = 1/r_0$, so that as $r_0$ tends to 1, $q$ also tends to 1. Still assuming $\varepsilon_0 = 0$, suppose $qe^{i\phi}$ approaches a point on the unit circle where the density of $\mu_s$ is positive, from inside the circle. Then since the density is analytic where it is positive [36, Proposition 3.6], we can rewrite the integral on the right-hand side of (7.11) as a contour integral and deform the contour to see that the limit exists. The limit from outside the circle similarly exists and the two limits are equal because the right-hand side of (7.11), as computed on the right-hand side of (7.12), is invariant under $q \mapsto 1/q$. This establishes the claim about $\partial S_0/\partial \theta$ in Proposition 7.3.

We next consider the case $r_0 = 1 + ce\varepsilon_0$. Since the left-hand side of (7.13) is invariant under $q \mapsto 1/q$, the two solutions are reciprocals of one another, with the solution with $q < 1$ simplifying to

$$q = 1 + \frac{1}{2} \frac{(1 + c^2)\varepsilon_0^2}{1 + ce\varepsilon_0} - \frac{1}{2} \frac{\sqrt{(1 + c^2)(4 + \varepsilon_0^2 + 4c\varepsilon_0 + c^2\varepsilon_0^2)}}{1 + c\varepsilon_0},$$

(7.14)
provided $1 + c\varepsilon_0 > 0$. As $\varepsilon_0 \to 0^+$ with $c$ fixed, $q$ tends to 1. Thus,

$$\lim_{\varepsilon_0 \to 0^+} \frac{\partial S}{\partial \theta}(s, 0, (1 + c\varepsilon_0)e^{i\phi}, \varepsilon_0) = \lim_{q \to 1^-} \text{Im} \left[ \int_{S^1} \frac{e^{i\phi'} + qe^{i\phi}}{e^{i\phi'} - qe^{i\phi}} \, d\mu_s(e^{i\phi'}) \right],$$

which is the same limit as we obtained with $\varepsilon_0 = 0$ and $r_0$ approaching 1, verifying (7.7).

We now turn to $\partial S/\partial \rho$.

Proof of results about $\partial S/\partial \rho$. Since $\partial / \partial \rho = r \partial / \partial r$, we may compute that

$$\frac{\partial S}{\partial \rho}(s, 0, r_0e^{i\phi}, \varepsilon_0) = 1 + \frac{r_0^2 - 1 - \varepsilon_0^2}{r_0} \int_{S^1} \frac{1}{(r_0^2 + 1 + \varepsilon_0^2)/r_0 - 2 \cos(\phi - \phi')} \, d\mu_s(e^{i\phi'}).$$

Meanwhile, we compute that

$$\text{Re} \left[ \int_{S^1} \frac{e^{i\phi'} + qe^{i\phi}}{e^{i\phi'} - qe^{i\phi}} \, d\mu_s(e^{i\phi'}) \right] = -\left( q - \frac{1}{q} \right) \int_{S^1} \frac{1}{q + 1/q - 2 \cos(\phi - \phi')} \, d\mu_s(e^{i\phi'}). \quad (7.15)$$

We therefore find that if $q$ satisfies (7.13), then

$$\frac{\partial S}{\partial \rho}(s, 0, r_0e^{i\phi}, \varepsilon_0) = 1 - \frac{1}{q - 1/q} \frac{r_0^2 - 1 - \varepsilon_0^2}{r_0} \text{Re} \left[ \int_{S^1} \frac{e^{i\phi'} + qe^{i\phi}}{e^{i\phi'} - qe^{i\phi}} \, d\mu_s(e^{i\phi'}) \right]. \quad (7.16)$$

We start with the case $\varepsilon_0 = 0$, in which case, we may take $q = r_0$ in (7.13). Then (7.16) reduces to

$$\frac{\partial S}{\partial \rho}(s, 0, r_0e^{i\phi}, 0) = 1 - \text{Re} \left[ \int_{S^1} \frac{e^{i\phi'} + r_0e^{i\phi}}{e^{i\phi'} - r_0e^{i\phi}} \, d\mu_s(e^{i\phi'}) \right]. \quad (7.17)$$

If we let $r_0e^{i\phi}$ approach a point on the unit circle where the density of $\mu_s$ is positive, the limit from inside the disk exists, giving

$$\frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})^{\text{in}} = 1 - \lim_{r_0 \to 1^-} \text{Re} \left[ \int_{S^1} \frac{e^{i\phi'} + r_0e^{i\phi}}{e^{i\phi'} - r_0e^{i\phi}} \, d\mu_s(e^{i\phi'}) \right] \quad \square
where $m_s$ is the density of the measure $\mu_s$ with respect to the normalized Lebesgue measure on $S^1$. (The second equality is obtained by recognizing the right-hand side of (7.15) as the Poisson integral of $\mu_s$, evaluated at $qe^{i\phi}$.) The limit from outside the disk also exists but is not equal to the limit from inside, because the right-hand side of (7.15) changes sign if $q$ is replaced by $1/q$, giving

$$
\frac{\partial S_0}{\partial \rho}(s, 0, e^{i\phi})_{\text{out}} = 1 + \lim_{r_0 \to 1^-} \Re \left[ \int_{S^1} \frac{e^{i\phi'} + r_0 e^{i\phi}}{e^{i\phi'} - r_0 e^{i\phi}} \, d\mu_s(e^{i\phi'}) \right]
= 1 + m_s(\phi). \tag{7.19}
$$

Meanwhile, if we take $r_0 = 1 + c \varepsilon_0$ in (7.16) and choose $q$ as in (7.14), we may compute that

$$
\frac{1}{q - 1/q} \frac{r_0^2 - 1 - \varepsilon_0^2}{\varepsilon_0 - c(2 + c \varepsilon_0)} = \frac{\varepsilon_0 - c(2 + c \varepsilon_0)}{\sqrt{(1 + c^2)(4 + \varepsilon_0^2 + 4c \varepsilon_0 + c^2 \varepsilon_0^2)}}. \tag{7.20}
$$

The right-hand side of this expression tends to $-c/\sqrt{1 + c^2}$ as $\varepsilon_0$ tends to zero. Meanwhile, as $\varepsilon_0$ tends to zero, the quantity $q$ in (7.14) tends to 1 from below. Thus,

$$
\lim_{\varepsilon_0 \to 0^+} \frac{\partial S}{\partial \varepsilon_0}(s, 0, (1 + c \varepsilon_0)e^{i\phi}, \varepsilon_0) = 1 + \frac{c}{\sqrt{1 + c^2}} m_s(\phi). \tag{7.21}
$$

Using this result and (7.18) and (7.19), we can easily check that the left-hand side of (7.9) agrees with the right-hand side.

We finally verify (7.8).

**Proof** We compute that

$$
\frac{\partial S}{\partial \varepsilon_0}(s, 0, r_0 e^{i\phi}, \varepsilon_0) = 2 \varepsilon_0 \int_{S^1} \frac{1}{(r_0^2 + 1 + \varepsilon_0^2)/r_0 - 2 \cos(\phi - \phi')} \, d\mu_s(e^{i\phi'}),
$$

so that, if $q$ satisfies (7.13), we get

$$
\frac{\partial S}{\partial \varepsilon_0}(s, 0, r_0 e^{i\phi}, \varepsilon_0) = -\frac{\varepsilon_0}{q - 1/q} \Re \left[ \int_{S^1} \frac{e^{i\phi'} + q e^{i\phi}}{e^{i\phi'} - q e^{i\phi}} \, d\mu_s(e^{i\phi'}) \right]. \tag{7.22}
$$

After checking that $q - 1/q = \sqrt{\alpha^2 + 4\alpha}$, we can easily check that the leading coefficient on the right-hand side of (7.22) remains finite as $\varepsilon_0$ tends to zero. Thus, after multiplying by $\varepsilon_0$, the limit becomes zero.  \qed
**Proof of Theorem 7.2** According to Proposition 7.4, the quantity \( \varepsilon_0 p_{\varepsilon, 0} \) tends to zero along the path \( \lambda_0 = (1 + c \varepsilon_0) e^{i \phi} \). Meanwhile, we can easily compute from (5.2) that

\[
2\lambda_0 p_{\lambda, 0} - 1 = \left( \frac{\partial S}{\partial \rho} - i \frac{\partial S}{\partial \theta} \right) \bigg|_{\tau = 0} - 1, \quad (7.23)
\]

where the limiting behavior of \( \partial S/\partial \rho \) and \( \partial S/\partial \theta \) was worked out in Proposition 7.4. We then use the formulas (5.7) and (5.6) for \( \varepsilon(\tau) \) and \( \lambda(\tau) \). From (5.7), we see that in the limit, since \( \varepsilon_0 \) is tending to zero and the initial momenta remain finite, \( \varepsilon(\tau) \) tends to zero. We then apply (5.6), using (7.23) and recalling that \( \varepsilon_0 p_{\varepsilon, 0} \) goes to zero, giving

\[
\lambda(\tau) = \lambda_0 e^{-\tau/2} \exp \left\{ -i \frac{\tau}{2} \left. \frac{\partial S}{\partial \theta} \right|_{\tau = 0} \right\} \exp \left\{ \frac{\tau}{2} \left. \frac{\partial S}{\partial \rho} \right|_{\tau = 0} \right\}. \quad (7.24)
\]

According to Proposition 7.4, the limiting value of \( \partial S/\partial \theta \) is independent of the value of \( c \), while the value of \( \partial S/\partial \rho \) varies between \( \partial S/\partial \rho_{\text{in}} \) and \( \partial S/\partial \rho_{\text{out}} \).

If we substitute the formulas (7.7) and (7.9) from Proposition 7.4 into (7.24), we see that

\[
\lim_{c \to -\infty} \lambda(\tau) = \lambda_0 \exp \left\{ \frac{\tau}{2} (2\lambda_0 p_{\lambda, 0} - 1) \right\} = f_{\text{s} - \tau} (\chi_{S}^{\text{in}}(e^{i \phi}));
\]

\[
\lim_{c \to +\infty} \lambda(\tau) = \lambda_0 \exp \left\{ \frac{\tau}{2} (2\lambda_0 p_{\lambda, 0} - 1) \right\} = f_{\text{s} - \tau} (\chi_{S}^{\text{out}}(e^{i \phi})), \quad (7.25)
\]

where \( \chi_{S}(e^{i \phi})^{\text{in}} \) and \( \chi_{S}(e^{i \phi})^{\text{out}} \) denote the limiting value of \( \chi_{S}(\lambda_0) \) as \( \lambda_0 \) approaches \( e^{i \phi} \) from inside and outside the unit disk, respectively. These two limiting values of \( \lambda(\tau) \) then lie on the boundary of \( \Sigma_{S, \tau} \) and their \( \delta \)-coordinate is related to \( \phi \) as in Figs. 8 and 9. Meanwhile, a curve of the form \( z e^{\tau x} \), where \( z \) is fixed and \( x \) varies over \( \mathbb{R} \), is an exponential spiral along which the \( \delta \)-coordinate is constant and the \( v \)-coordinate is varying. (Recall the definition (1.7).) Thus, as \( c \) varies with \( \phi \) fixed, the value of \( \lambda(\tau) \) in (7.24) moves along a segment of an exponential spiral connecting the two limiting values in (7.25). (See Fig. 11.) By [36, Theorem 3.8], as \( \phi \) varies over all points where the density of \( \mu_{s} \) is positive, \( \theta \) will vary over all angles where \( r_{3}(\theta) < 1 \), so that (Lemma 3.7), \( \delta \) will vary over all angles for which \( \nu_{1}^{s, \tau}(\delta) < \nu_{2}^{s, \tau}(\delta) \). Thus, as \( \phi \) varies over all points where the density of \( \mu_{s} \) is positive, the spiral segments will fill up the whole of \( \Sigma_{S, \tau} \). \( \square \)

### 7.3 Computation of the Brown measure inside the domain

In this section, we justify the computation of the Brown measure in \( \Sigma_{S, \tau} \) outlined in Sect. 7.1.2. The computation given there is mostly rigorous, except that we need to show \( S(s, \tau, \lambda, \varepsilon) \) extends smoothly up to \( \varepsilon = 0 \).
7.3.1 Regularity of $S$ at $\varepsilon = 0$

In this section, we establish the regularity for $S(\varepsilon, \tau, \lambda, \varepsilon)$ near $\varepsilon = 0$ that is needed to rigorize the computation of the Brown measure. We will show that the function

$$(\lambda, \varepsilon) \mapsto S(s, \tau, \lambda, \varepsilon),$$

initially defined for $\varepsilon > 0$, has a $C^1$ extension to a neighborhood of $(\tilde{\lambda}, 0)$, for any $\tilde{\lambda}$ in $\Sigma_{s, \tau}$. (See Theorem 7.6 below.)

We consider points $(\lambda_0, \varepsilon_0)$, where at first we require $\varepsilon_0 > 0$. We label such points using the coordinates

$$\varepsilon_0; \quad \phi = \arg(\lambda_0); \quad c = \frac{|\lambda_0| - 1}{\varepsilon_0}.$$

Now, if we let $\varepsilon_0$ tend to zero with a fixed value of $c$, the value $|\lambda_0|$ will approach 1 and $\lambda_0$ itself will approach $e^{i\phi}$. In this limit, $\lambda_0$ is approaching a single point $e^{i\phi}$, independent of $c$—and yet the limiting value of $\lambda(\tau)$, computed in the previous subsection, does depend on $c$ (Theorem 7.2). It is therefore natural to consider a “blown up” domain in which we consider all non-negative values of $\varepsilon_0$ and all real values of $c$. The point here is that even when $\varepsilon_0 = 0$, we count points with the same value of $\phi$ but different values of $c$ as being different.

To establish the desired $C^1$ extension of $S$, we need to allow the variable $\varepsilon_0$ to be negative. Our blown up domain then consists of triples $(\varepsilon_0, c, \phi)$, where $\phi$ is an angle and $\varepsilon_0$ and $c$ are real numbers with $1 + c\varepsilon_0 > 0$. We may then define $\lambda_0$ in terms of $(\varepsilon_0, c, \phi)$ as $\lambda_0 = (1 + c\varepsilon_0)e^{i\phi}$. Note that if $\varepsilon_0 = 0$, then $\lambda_0 = e^{i\phi}$; hence, $|\lambda_0|$ must equal 1 when $\varepsilon_0 = 0$. The blown up domain can be understood geometrically as in Fig. 13. The variable $c$ is the slope of a line in the $(\varepsilon_0, |\lambda_0|)$ plane passing through the point $(0, 1)$ and we decree that we obtain different points if we approach $(0, 1)$ along lines lines of different slope.

The first main result of this section is that the map sending $\lambda_0$ and $\varepsilon_0$ to $\lambda(\tau)$ and $\varepsilon(\tau)$, viewed as a map on the blown up domain, has a $C^1$ extension with nonsingular Jacobian up to (and a little beyond) $\varepsilon_0 = 0$. Once this result is established, we will use the first Hamilton–Jacobi formula to obtain the desired $C^1$ extension of $S$.

**Theorem 7.5** Fix $\tau$ and consider the map

$$(\lambda_0, \varepsilon_0) \mapsto (\lambda(\tau), \varepsilon(\tau)),$$

initially defined for $\lambda_0 \in \mathbb{C}$ and $\varepsilon_0 > 0$. We compute using the coordinates $(\varepsilon_0, \phi, c)$ on the domain side and $(\varepsilon, \delta, v)$ on the range side, giving a map

$$\Psi_\tau(\varepsilon_0, \phi, c) = (\varepsilon, \delta, v),$$

initially defined for $\varepsilon_0 > 0$, $\phi \in [-\pi, \pi)$, and $c \in \mathbb{R}$. Fix a real number $\tilde{c}$ and a point $\tilde{\phi}$ where the density $m_s$ of the measure $\nu_s$ is positive. Then (1) $\Psi_\tau$ extends to a $C^1$
function defined for \((\varepsilon_0, \phi, c)\) in a neighborhood of \((0, \tilde{\phi}, \tilde{c})\) inside \(\mathbb{R}^3\), and (2) the Jacobian of \(\Psi_\tau\) at \((0, \tilde{\phi}, \tilde{c})\) is invertible.

We separate the proof into two parts, the proof of smoothness and the computation of the Jacobian.

**Proof (smoothness)** The formulas for \(\lambda(\tau)\) and \(\varepsilon(\tau)\) in (5.6) and (5.7) are smooth functions of \(\lambda_0, \varepsilon_0\), and the quantities \(\varepsilon_0 p_{\varepsilon, 0}\) and \(\lambda_0 p_{\lambda, 0}\). Since \(\lambda_0 = (1 + c \varepsilon_0)e^{i\phi}\) is a smooth function of \((\varepsilon_0, \phi, c)\), it suffices to verify that \(\varepsilon_0 p_{\varepsilon, 0}\) and \(\lambda_0 p_{\lambda, 0}\) are \(C^1\) functions of \((\varepsilon_0, \phi, c)\). In light of (7.23), it then suffices to verify smoothness of \(\varepsilon_0 p_{\varepsilon, 0}\), \(\partial S(s, 0, \lambda_0, \varepsilon_0)/\partial \rho\), and \(\partial S(s, 0, \lambda_0, 0)/\partial \theta\) as functions of \(\varepsilon_0, \phi,\) and \(c\). To obtain the desired smoothness, we use the expressions (7.22), (7.16) (together with (7.20)), and (7.11). We then use the previously noted “deform the contour” argument to show that all these expressions extend smoothly as functions of \(q\) past \(q = 1\). Finally, \(q\) itself, given as a function of \(c\) and \(\varepsilon_0\) by (7.14), has a smooth extension past \(\varepsilon_0 = 0\).

**Proof (Jacobian)** From (5.6) and (5.7), we have

\[
\varepsilon = \varepsilon_0 \frac{|\lambda|}{|\lambda_0|} = \varepsilon_0 \frac{|\lambda|}{1 + c \varepsilon_0}.
\]

(7.26)

Since we have shown that \(|\lambda|\) is a smooth function of \((\varepsilon_0, \phi, c)\), the leading factor of \(\varepsilon_0\) in (7.26) easily gives that

\[
\left. \frac{\partial \varepsilon}{\partial \varepsilon_0} \right|_{\varepsilon_0=0} = |\lambda|; \quad \left. \frac{\partial \varepsilon}{\partial \phi} \right|_{\varepsilon_0=0} = \frac{\partial \varepsilon}{\partial c} \left|_{\varepsilon_0=0} \right. = 0.
\]

Let us now evaluate \(\lambda(\tau)\) at \(\varepsilon_0 = 0\)—that is, using fixed values of \(c\) and \(\phi\) and letting \(\varepsilon_0\) tend to zero as in Proposition 7.4. Let us use the superscript “in” to mean that we evaluate the relevant quantity with \(\varepsilon_0 = 0\) in the limit as \(\lambda_0\) approaches \(e^{i\phi}\)
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from *inside* the unit circle. Then, using (7.21), we obtain
\[
\lambda(\tau)|_{\varepsilon=0} = \lambda_0 \exp\left\{ \frac{\tau}{2} (2\lambda_0 p_{\lambda_0} - 1)^{in} \right\} \exp\left\{ \frac{\tau}{2} \left( 1 + \frac{c}{\sqrt{1 + c^2}} \right) \right\} m_s(\phi) .
\]

where \( m_s \) is the density of \( \phi_s \), or, by Proposition 5.6,
\[
\lambda(\tau) = f_{s-\tau}(\chi_s(\lambda_0))^{in} \exp\left\{ \frac{\tau}{2} \left( 1 + \frac{c}{\sqrt{1 + c^2}} \right) \right\} m_s(\phi) .
\]

(7.27)

We see, then, that the factor of \( f_{s-\tau}(\chi_s(\lambda_0))^{in} \) depends only on \( \phi \) and not on \( c \). Furthermore, because the final exponential factor in (7.27) has the form \( e^{a\tau} \) for some real number \( a \), this factor only affects the value of \( v \) and not the value of \( \delta \). Thus, the value of \( \delta \) comes from \( f_{s-\tau}(\chi_s(\lambda_0))^{in} \) which does not depend on \( c \), and we see
\[
\frac{\partial \delta}{\partial c} = 0
\]
and we find, using (7.26) that the Jacobian at \( \varepsilon_0 = 0 \) has the form
\[
\begin{pmatrix}
|\lambda| & 0 & 0 \\
* & \frac{\partial \delta}{\partial \phi} & 0 \\
* & * & \frac{\partial v}{\partial \varepsilon} \\
\end{pmatrix}.
\]

Now, since the value of \( \delta \) comes from \( f_{s-\tau}(\chi_s(\lambda_0))^{in} \), we see that \( \delta \) depends on \( \phi \) as in Figs. 8 and 9. Thus, by Proposition 3.5 and the estimate [24, Lemma 4.20] for \( d\phi/d\theta \), we see that \( \partial \delta / \partial \phi \) is positive. Furthermore, \( \partial v / \partial c \) is easily computed (from the final exponential factor in (7.27)) to be positive, provided we evaluate at \( \phi = \tilde{\phi} \), with \( m_s(\tilde{\phi}) > 0 \). Finally, we recall that the point \( \lambda = 0 \) is never in \( \Sigma_{s,\tau} \) and we can see that the determinant of the Jacobian is positive. \( \square \)

**Theorem 7.6** Pick a point \( \tilde{\lambda} \in \Sigma_{s,\tau} \). Then the map
\[(\lambda, \varepsilon) \mapsto S(s, \tau, \lambda, \varepsilon),
\]
initially defined for \( \varepsilon > 0 \), has a \( C^1 \) extension defined for \( (\lambda, \varepsilon) \) in a neighborhood of \( (\tilde{\lambda}, 0) \).

We emphasize that for \( \varepsilon < 0 \), the \( C^1 \) extension of \( S(s, \tau, \lambda, \varepsilon) \) is not given by the formula (4.1). Indeed, since \( \varepsilon(\tau)p_{\varepsilon}(\tau) \) is independent of \( \tau \) (from the formulas (5.7) and (5.9)), we can compute \( p_{\varepsilon}(\tau) \) as \( p_{\varepsilon,0}\varepsilon_0/\varepsilon(\tau) \). If \( \lambda_0 = (1 + c\varepsilon_0)e^{i\phi} \), with \( m_s(\phi) > 0 \), then as \( \varepsilon_0 \) tends to zero, both \( p_{\varepsilon,0} \) and \( \varepsilon_0/\varepsilon(\tau) \) tend to nonzero limits. (For \( p_{\varepsilon,0} \), use (7.22) and for \( \varepsilon_0/\varepsilon(\tau) \), use (5.7).) Using the second Hamilton–Jacobi formula (5.12), we find that \( \partial S / \partial \varepsilon \) evaluated at \( \varepsilon = 0 \) is nonzero, so that a \( C^1 \) extension of \( S \) cannot be an even function of \( \varepsilon \). To put it differently, if we did define \( S \) by the same formula (4.1) even for \( \varepsilon < 0 \), then \( S \) would not be \( C^1 \) but would have a singularity of absolute-value type as a function of \( \varepsilon \) near \( \varepsilon = 0 \).

**Corollary 7.7** Consider the function \( S_0 \) given by \( S_0(s, \tau, \lambda) = \lim_{\varepsilon \to 0^+} S(s, \tau, \lambda, \varepsilon) \). Then for \( \tau \) satisfying \|\tau - s\| < s and \( \lambda \) in \( \Sigma_{s,\tau} \), the function \( S_0 \) satisfies the PDE
obtained by formally setting $\varepsilon = 0$ in (4.2), namely

$$\frac{\partial S_0}{\partial \tau} = -\frac{1}{2} \left( \lambda^2 \left( \frac{\partial S_0}{\partial \lambda} \right)^2 - \lambda \frac{\partial S_0}{\partial \lambda} \right).$$  

(7.28)

Furthermore, suppose we define curves $\lambda(\tau)$ and $p_{\lambda}(\tau)$ by taking the curves in (5.6) and (5.8) with $\lambda_0 = (1 + c\varepsilon_0)e^{i\phi}$ as in (7.6) and letting $\varepsilon_0$ tend to zero. Then $S_0$ satisfies the first Hamilton–Jacobi formula

$$S_0(s, \tau, \lambda(\tau)) = S_0(s, 0, \lambda_0) + 2 \text{Re}[\tau H_0] + \text{Re} \left[ \tau \lambda_0 p_{\lambda, 0} \right]$$  

(7.29)

and the second Hamilton–Jacobi formula

$$\frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda(\tau)) = p_{\lambda}(\tau),$$  

(7.30)

where $H_0$ is the value of the Hamiltonian $H$ in (5.1) with $\varepsilon p_\varepsilon = 0$ and $\lambda p_\lambda$ replaced by $\lambda_0 p_{\lambda, 0}$.

Proof Theorem 7.6 justifies formally setting $\varepsilon = 0$ in the PDE. Using Theorem 7.6 again along with Proposition 7.4 allow us to let $\varepsilon_0$ tend to zero in the Hamilton–Jacobi formulas (5.10), (5.11), and (5.12).

Proof of Theorem 7.6 We define a function $\text{HJ}(s, \tau, \lambda_0, \varepsilon_0)$ by the right-hand side of the first Hamilton–Jacobi formula, namely

$$\text{HJ}(s, \tau, \lambda_0, \varepsilon_0) = S(s, 0, \lambda_0, \varepsilon_0) + 2 \text{Re}[\tau H_0] + \frac{1}{2} \text{Re} \left[ \tau (\varepsilon_0 p_{\varepsilon, 0} + 2\lambda_0 p_{\lambda, 0}) \right]$$

where it is understood that the initial momenta $p_{\varepsilon, 0}$ and $p_{\lambda, 0}$ are always computed as functions of $\lambda_0$ and $\varepsilon_0$ as in (5.4) and (5.5). We express $(\lambda_0, \varepsilon_0)$ in terms of the coordinates $(\varepsilon_0, c, \phi)$ and work in the blown up domain. Then the proof of Theorem 7.5 shows that $\varepsilon_0 p_{\varepsilon, 0}$ and $\lambda_0 p_{\lambda, 0}$ extend to $C^1$ functions of these variables defined near $(0, \tilde{c}, \tilde{\phi})$. A similar argument shows that $S(s, 0, \lambda_0, \varepsilon_0)$, which is the regularized log potential of the measure $\mu_s$, also has a $C^1$ extension.

We now appeal to Theorem 7.5 and the inverse function theorem to construct an inverse to the map $\Psi_{\tau}$ near $(0, \delta, v)$. Since the extended map $\Psi_{\tau}$ uses the same formula (5.7) for $\varepsilon(\tau)$, but with $C^1$ extensions of the momenta, we see that the sign of $\varepsilon$ is always the same as the sign of $\varepsilon_0$. Thus, the $\varepsilon_0$-component of $\Psi_{\tau}^{-1}(\varepsilon, \delta, v)$ is positive when $\varepsilon$ is positive. We then consider

$$\tilde{S}(s, \tau, \lambda, \varepsilon) = \text{HJ}(s, \tau, \Psi_{\tau}^{-1}(\varepsilon, \delta, v)),$$

which is a $C^1$ function. The first Hamilton–Jacobi formula tells us that $\tilde{S}$ agrees with $S$ when $\varepsilon > 0$. Thus, $\tilde{S}$ is the desired extension.  

\[ \Box \]
7.3.2 The Brown measure

We now rigorize the argument outlined in Sect. 7.1.2.

Proof of Conclusion 7.1 Choose an arbitrary point $\lambda$ in $\Sigma_{s, \tau}$. By Theorem 7.2, we can find $\phi$ and $c$ so that if $\lambda_0 = (1 + c\varepsilon_0)e^{i\phi}$, then as $\varepsilon_0$ tends to zero, $\varepsilon(\tau)$ tends to zero and $\lambda(\tau)$ tends to $\lambda$. As shown in the proof of Theorem 7.2, the $\delta$-coordinate of $\lambda$ is then related to $\phi$ as in Figs. 8 and 9. Thus, as in Sect. 7.1.2, the limiting value of arg $\lambda_0$ is $\phi_{s, \tau}(\delta)$.

We then apply the second Hamilton–Jacobi formula (5.11) with this choice of $\lambda_0$, where we initially require $\varepsilon_0 > 0$. After multiplying by $\lambda$, we have

$$\lambda \frac{\partial S}{\partial \lambda}(s, \tau, \lambda(\tau), \varepsilon(\tau)) = \lambda(\tau)p_{\lambda}(\tau) = \lambda_0p_{\lambda,0},$$

(7.31)

where the second equality follows from the formulas (5.6) and (5.8) for $\lambda(\tau)$ and $p_{\lambda}(\tau)$. Theorem 7.6 allows us to let $\varepsilon_0$ tend to zero on the left-hand side of (7.31) to obtain

$$\frac{\partial}{\partial \lambda}S_0(s, \tau, \lambda),$$

$$\lambda \frac{\partial}{\partial \lambda}S_0(s, \tau, \lambda, \varepsilon(\tau)) = \lambda(\tau)p_{\lambda}(\tau) = \lambda_0p_{\lambda,0},$$

where $\lambda_0$ and $p_{\lambda,0}$ can be computed as continuous functions of $\lambda$ over all of $\mathbb{C}$, even when $\lambda$ is in the boundary of $\Sigma_{s, \tau}$. (Check that the “inside” values of $\lambda_0$ and $p_{\lambda,0}$ in Sect. 7.2 agree with the “outside” values of $\lambda_0$ and $p_{\lambda,0}$ in Sect. 6 when $\lambda$ approaches a point on the boundary of $\Sigma_{s, \tau}$.)

If we then integrate $S_0(s, \tau, \lambda)$ against $\Delta\chi(\lambda)$ for some test function $\lambda$, we may split the integral into integrals over $\Sigma_{s, \tau}$ and its complement. We may then hope to integrate by parts using Green’s second identity. Since $S_0$ and its first derivatives are continuous up to the boundary, the boundary terms from inside and from outside should cancel, giving the integral of $(\Delta S_0)\chi$ over $\Sigma_{s, \tau}$ and the integral of $(\Delta S_0)\chi$ over $\Sigma^{c}_{s, \tau}$. Since the “outside” integral is zero by Theorem 6.1, we would be left with
the integral of \((\Delta S_0)\chi\) over \(\Sigma_{s,\tau}\), showing that the Brown measure is supported on \(\Sigma_{s,\tau}\). This argument is carried out the case \(\tau = s\) and \(\mu_0 = \delta_1\) in [13, Proposition 7.13].

The difficulty with carrying this argument out in general is that we would have to justify the use of Green’s second identity, using, say, Federer’s theorem [29, Section 12.2]. But to use Federer’s theorem, we would need to know that the boundary \(\Sigma_{s,\tau}\) is a rectifiable curve, which we do not know if \(\mu_0\) is a completely arbitrary probability measure on the unit circle. (By contrast, if \(\mu_0 = \delta_1\), as in [13], then the boundary of \(\Sigma_{s,\tau}\) is smooth, except possibly at a single point.)

We will therefore use an indirect approach, similar to the one developed in [24, Lemma 4.25] and used also in [20, Theorem 7.9]. In [24, Proposition 4.31], it is shown that when \(\tau = s\), the Brown measure \(\mu_{s,s}\) gives mass one to the domain \(\Sigma_{s,s} = \Sigma_s\).

In Sect. 8, we will use a push-forward result to prove that \(\mu_{s,\tau}\) assigns mass 1 to \(\Sigma_{s,\tau}\), for all nonzero \(\tau\) with \(|\tau - s| \leq s\). See Sect. 8.5 for the details of this argument.

8 Relating different values of \(\tau\)

The goal of this section is to prove Theorem 1.2, which states that the Brown measure \(\mu_{s,\tau}\) is the push-forward of \(\mu_{s,s}\) under a certain map \(\Phi_{s,\tau} : \Sigma_s \rightarrow \Sigma_{s,\tau}\). Recall the nota

\[
S_0(s, \tau, \lambda) = \lim_{\varepsilon \to 0^+} S(s, \tau, \lambda, \varepsilon),
\]

so that the Brown measure \(\mu_{s,\tau}\) is the distributional Laplacian of \(S_0(s, \tau, \lambda)\) with respect to \(\lambda\).

8.1 The maps

Throughout this section, we assume that \(\mu_0\) (the law of the unitary element \(u\) in the expression \(ub_{s,\tau}\)) is fixed. Recall the “characteristic curves” \(\lambda(\tau)\) defined in (5.6). We will define the map \(\Phi_{s,\tau}\) in terms of these curves and then compute the map more explicitly in Proposition 8.3.

Definition 8.1  Fix a positive real number \(s\) and a nonzero complex number \(\tau\) such that \(|\tau - s| \leq s\). For each \(\lambda \in \Sigma_s\), define \(\lambda_0\) in the unit circle by

\[
\lambda_0 = \exp \left\{ i \phi^s(\theta) \right\},
\]

where \(\theta = \arg \lambda\) and \(\phi^s\) is as in Definition 3.2. Let \(p_{\theta,0} = \partial S_0/\partial \theta(0, 0, \lambda_0)\) be computed from \(\lambda_0\) as in Proposition 7.3. By Theorem 7.2, we can choose \(p_{\rho,0}\) between \(\partial S_0/\partial \rho^\text{in}\) and \(\partial S_0/\partial \rho^\text{out}\) so that with initial conditions \((\lambda_0, p_{\theta,0}, p_{\rho,0})\)—and with \(\varepsilon_0 p_{\varepsilon,0} = 0\)—we have

\[
\lambda(s) = \lambda. \quad (8.1)
\]
To compute $\Phi_{s, \tau} (\lambda)$, we choose initial conditions $(\lambda_0, p_{\theta, 0}, p_{\rho, 0})$ so that $\lambda(s) = \lambda$ and then set $\Phi_{s, \tau} (\lambda)$ equal to $\lambda(\tau)$, computed with the same initial conditions. Then define a map $\Phi_{s, \tau} : \Sigma_s \to \Sigma_{s, \tau}$ by setting

$$\Phi_{s, \tau} (\lambda) = \lambda(\tau),$$

where $\lambda(\tau)$ is computed using the same initial conditions $(\lambda_0, p_{\theta, 0}, p_{\rho, 0})$ (with $\varepsilon_0 p_{\varepsilon, 0} = 0$) that give (8.1).

See Fig. 14. The following is the main result about the maps $\Phi_{s, \tau}$.

**Theorem 8.2** Fix a positive real number $s$ and a nonzero complex number $\tau$ such that $|\tau - s| \leq s$. Then $\Phi_{s, \tau}$ is a homeomorphism of $\Sigma_s$ onto $\Sigma_{s, \tau}$ and a diffeomorphism of $\Sigma_s$ onto $\Sigma_{s, \tau}$. Furthermore, the Brown measure $\mu_{s, \tau}$ is the push-forward of the Brown measure $\mu_{s,s}$ under the map $\Phi_{s, \tau}$.

In light of Definition 8.1, the theorem can be interpreted as follows: as $\tau$ varies, the Brown measure $\mu_{s, \tau}$ pushes forward along the characteristic curves of the PDE for its log potential.

Note that we have already established (Theorem 6.1) that the Brown measure $\mu_{s,s}$ is zero outside $\Sigma_s = \Sigma_{s,s}$. It is therefore meaningful to speak of pushing forward $\mu_{s,s}$ by a map that is only defined on $\Sigma_s$. The homeomorphism and diffeomorphism properties of $\Phi_{s, \tau}$ are established in Proposition 8.3 in this section. The proof of the push-forward property is given in Sect. 8.4.

In [13, Proposition 2.5] and [24, Corollary 4.30], a map $\Phi_s$ from $\Sigma_s$ into the unit circle is constructed with the property that the push-forward of the Brown measure $\mu_{s,s}$ is the measure $\mu_s$, the law of $u u_s$, where $u_s$ is the free unitary Brownian motion and $u$ is the unitary element in the expression $u b_{s, \tau}$, taken to be freely independent of $u_s$. Now, $u u_s$ is just the limit in $*$-distribution of $u b_{s, \tau}$ as $\tau$ tends to zero (see (2.7) and Appendix B), and we will show in Proposition 8.3 that $\Phi_s$ is the the limit of $\Phi_{s, \tau}$ as $\tau$ tends to zero. Theorem 8.2 thus shows that the relationship between $\mu_{s,s}$ and $\mu_s$ in [13, 24] is a limiting case of a whole family of relationships among the measures $\mu_{s, \tau}$ with $s$ fixed and $\tau$ varying.
Of course, it follows from Theorem 8.2 that the push-forward of \( \mu_{s, \tau_1} \) under the map \( \Phi_{s, \tau_2} \circ \Phi_{s, \tau_1}^{-1} \) is \( \mu_{s, \tau_2} \), for any two nonzero complex numbers \( \tau_1 \) and \( \tau_2 \) with \( |\tau_1 - s| \leq s \) and \( |\tau_2 - s| \leq s \).

We now compute the map \( \Phi_{s, \tau} \) more explicitly.

**Proposition 8.3** Fix a positive number \( s \) and a nonzero complex number \( \tau \) satisfying \( |\tau - s| \leq s \). Then for all \( \lambda = re^{i\theta} \) in \( \Sigma_s \), we may compute \( \Phi_{s, \tau}(\lambda) \) as

\[
\Phi_{s, \tau}(re^{i\theta}) = \left( \frac{r}{r_s(\theta)} \right)^{\frac{\tau}{s}} f_{s-\tau}(r_s(\theta)e^{i\theta}). \tag{8.2}
\]

In particular, \( \Phi_{s, \tau} \) agrees with \( f_{s-\tau} \) on the boundary of \( \Sigma_s \) and maps each radial segment to a portion of an exponential spiral (a curve of the form \( u \mapsto ce^{ut} \) for some \( c \in \mathbb{C} \)) in \( \Sigma_{s, \tau} \). From these formulas, we can verify that \( \Phi_{s, \tau} \) is a homeomorphism of \( \Sigma_s \) onto \( \Sigma_{s, \tau} \) and a diffeomorphism of \( \Sigma_s \) onto \( \Sigma_{s, \tau} \). Furthermore, we have

\[
\lim_{\tau \to 0} \Phi_{s, \tau}(\lambda) = \Phi_s(\lambda), \tag{8.3}
\]

where, as in [13, 24], \( \Phi_s : \Sigma_s \to S^1 \) is given by

\[
\Phi_s(re^{i\theta}) = f_s(r_s(\theta)e^{i\theta}). \tag{8.4}
\]

See Fig. 4. We note that \( \Phi_{s, \tau}(\lambda) \) depends holomorphically on \( \tau \) with \( \lambda \) and \( s \) fixed; this property of \( \Phi_{s, \tau} \) will be used in Sect. 8.3. On the other hand, \( \Phi_{s, \tau}(\lambda) \) does not depend holomorphically on \( \lambda \) with \( s \) and \( \tau \) fixed.

**Proof** Suppose we take \( \varepsilon_0 \rho_{e, 0} = 0 \), fix a point \( \lambda_0 = e^{i\phi} \) in the support of \( \mu_s \) in the unit circle, and take \( p_{\theta, 0} \) as in Proposition 7.3. Then the proof of Theorem 7.2 shows that the point \( \lambda(s) \) will have the form \( \lambda_0 e^{i\alpha r} \), where \( r > 0 \) and the value of \( \alpha \) is independent of the limiting value of \( p_{\rho, 0} \). As \( p_{\rho, 0} \) varies between \( \delta S/\partial \rho^{\text{in}} \) and \( \partial S/\partial \rho^{\text{out}} \), the value of \( r \) will vary, so that \( \lambda(s) \) will range over a radial segment inside \( \Sigma_s \) connecting the inner and outer boundaries. Then by (5.6), \( \lambda(\tau) \) will have the form \( \lambda_0 e^{i\tau \alpha s r \tau/s} \). Thus, \( \Phi_{s, \tau} \) will map the radial segment in \( \Sigma_s \) in a log-linear fashion to a spiral segment in \( \Sigma_{s, \tau} \).

We then check that the formula in (8.2) captures this behavior. We can verify that the right-hand side of (8.2) agrees with \( f_{s-\tau}(re^{i\theta}) \) both on the inner boundary at \( r = r_s(\theta) \) (trivially) and on the outer boundary at \( r = r_s(\theta)^{-1} \) (using Lemma 3.7). It follows that the endpoints of the radial segment map to the endpoints of the spiral segment. As \( r \) varies, the right-hand side of (8.2) traces out an exponential spiral of the form \( cr^{\tau/s} \), along which \( \delta \) is constant. Thus, (8.2) agrees with the map in the previous paragraph.

That \( \Phi_{s, \tau} \) is a homeomorphism of \( \Sigma_s \) onto \( \Sigma_{s, \tau} \) follows from the bijective nature of the correspondence between \( \theta \) and \( \delta \) (Point 1 of Theorem 3.3), together with Lemma 3.7, which tells us that a radial segment in \( \Sigma_s \) cannot collapse to a single point in \( \Sigma_{s, \tau} \). That \( \Phi_{s, \tau} \) is a diffeomorphism on \( \Sigma_s \) follows by using polar coordinates \((r, \theta)\) in \( \Sigma_s \) and twisted logarithmic coordinates \((v, \delta)\) in \( \Sigma_{s, \tau} \). Then \( \delta \) depends only on \( \theta \) and, by
Proposition 3.5, \(d\delta/d\theta > 0\). Furthermore, \(\partial v/\partial r > 0\) from the explicit formula in (8.2). Thus, the Jacobian of \(\Phi_{s,\tau}\) in these coordinates is upper triangular with positive entries on the diagonal.

Finally, the expression on the right-hand side of (8.2) is continuous up to \(\tau = 0\) and

\[
\lim_{\tau \to 0} \Phi_{s,\tau}(r e^{i\theta}) = f_s(r_s(\theta)e^{i\theta}),
\]

as claimed in (8.3) and (8.4).

We now give formulas for the composite map \(\Phi_{s,\tau'} \circ \Phi_{s,\tau}^{-1}\) and the \(\bar{\lambda}\)-derivative of its logarithm. Equation (8.5) shows that \(\Phi_{s,\tau'} \circ \Phi_{s,\tau}^{-1}\) can be computed from the Brown measure \(\mu_{s,\tau}\) (and its log potential). Equation (8.6), meanwhile, shows precisely how \(\Phi_{s,\tau'} \circ \Phi_{s,\tau}^{-1}\) (or, equivalently, its logarithm) fails to be conformal on \(\Sigma_{s,\tau}\).

**Proposition 8.4** Let \(\tau\) and \(\tau'\) be two nonzero complex numbers such that \(|s - \tau| \leq s\) and \(|s - \tau'| \leq s\). Then the map \(\Phi_{s,\tau'} \circ \Phi_{s,\tau}^{-1}\) can be computed as follows:

\[
\Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda)) = \lambda \exp \left\{ \frac{(\tau' - \tau)}{2} \left( 2\lambda \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) - 1 \right) \right\}. \tag{8.5}
\]

Therefore, the \(\bar{\lambda}\)-derivative of \(\log \Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda))\) has a direct relation to the density \(W(s, \tau, \lambda)\) of \(\mu_{s,\tau}\), as follows:

\[
\bar{\lambda} \frac{\partial}{\partial \lambda} \log \Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda)) = \pi(\tau' - \tau) |\lambda|^2 W(s, \tau, \lambda), \quad \lambda \in \Sigma_{s,\tau}. \tag{8.6}
\]

**Proof** Fix \(\lambda \in \Sigma_{s,\tau}\) and choose initial conditions \(\lambda_0\) and—through the limiting process in Definition 8.1—\(p_{\lambda,0}\) such that, with \(\varepsilon_0, p_{\varepsilon_0} = 0\), we have

\[
\lambda(\tau) = \lambda.
\]

Then by Definition 8.1, we will have \(\Phi_{s,\tau}^{-1}(\lambda) = \lambda(s)\), where \(\lambda(s)\) is computed with the same initial conditions as \(\lambda(\tau)\). Then since \(\lambda(s) = \Phi_{s,\tau}^{-1}(\lambda)\), we may continue to use the same initial conditions and compute

\[
\Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda)) = \lambda(\tau').
\]

By Corollary 7.7, \(\lambda(\tau)\) and \(\lambda(\tau')\) can be computed using (5.6) with \(\varepsilon_0, p_{\varepsilon_0} = 0\). Using initial conditions \(\lambda_0\) and \(p_{\lambda,0}\) as in the previous paragraph, we then have

\[
\lambda = \lambda(\tau) = \lambda_0 \exp \left( \frac{\tau}{2} (2\lambda_0 p_{\lambda,0} - 1) \right)
\]

and

\[
\Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda)) = \lambda(\tau') = \lambda_0 \exp \left( \frac{\tau'}{2} (2\lambda_0 p_{\lambda,0} - 1) \right).
\]
Thus,

\[
\Phi_{s',\tau}(\Phi_{s,\tau}(\lambda)) = \lambda \exp\left(\frac{(\tau' - \tau)}{2} (2\lambda_0 p_{\lambda,0} - 1)\right) = \lambda \exp\left(\frac{(\tau' - \tau)}{2} (2\lambda(\tau) p_{\lambda}(\tau) - 1)\right),
\]

where, in the second equality, we have used that \(\lambda(\tau) p_{\lambda}(\tau)\) is independent of \(\tau\).

Since \(\lambda(\tau) = \lambda\), we may apply the second Hamilton–Jacobi formula (Corollary 7.7) to obtain (8.5). Finally, (8.6) follows by applying \(\partial/\partial \bar{\lambda}\) to the logarithm of (8.5), recalling that \(W = \frac{1}{4\pi} \Delta_{\lambda} S_0\).

We now begin working toward the proof of Theorem 8.2. In Sect. 8.2, we will compute how the Brown measure \(\mu_{s,\tau}\) varies as \(\tau\) varies. Then in Sect. 8.3, we will compute how the push-forward of \(\mu_{s,s}\) by \(\Phi_{s,\tau}\) varies as \(\tau\) varies. The proof (Sect. 8.4) will be based on a similarity between these two computations. The proof does not rely on the formula for the Brown measure obtained in Sect. 7 but instead uses only that the log potential of the Brown measure satisfies the \(\varepsilon = 0\) PDE in Corollary 7.7.

8.2 The derivative of the density of the Brown measure

We let \(W(s, \tau, \lambda)\) denote the density (in \(\Sigma_{s,\tau}\)) of the Brown measure \(\mu_{s,\tau}\). We begin by computing how \(W(s, \tau, \lambda)\) varies as \(\tau\) varies with \(s\) fixed. Recall that \(S_0(s, \tau, \lambda)\) is the limit of \(S(s, \tau, \lambda, \varepsilon)\) as \(\varepsilon\) tends to zero from above and that the density \(W\) of the Brown measure is \(\frac{1}{4\pi}\) times the Laplacian of \(S_0\) with respect to \(\lambda\).

**Theorem 8.5** For all \(s > 0\), the function \(W\) satisfies the equation

\[
\frac{\partial W}{\partial \tau} = -\frac{1}{\partial \lambda} \left[ \lambda \left( \frac{\partial S_0}{\partial \lambda} - \frac{1}{2} \right) W \right]
\]

for all \(\tau\) with \(|\tau - s| < s\) and all \(\lambda\) in \(\Sigma_{s,\tau}\).

Note that (8.7) is not a self-contained PDE for \(W\), since the right-hand side involves \(S_0\) as well as \(W\).

**Proof** According to Corollary 7.7, \(S_0\) satisfies the \(\varepsilon = 0\) version of the PDE (4.2) inside \(\Sigma_{s,\tau}\), as given in (7.28). We now apply the operator

\[
\frac{1}{4\pi} \Delta_{\lambda} = \frac{1}{\pi} \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}}
\]

to both sides of (7.28), leaving the \(\lambda\)-derivative on the right-hand side unevaluated. This gives

\[
\frac{1}{\pi} \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}} \frac{\partial S_0}{\partial \tau} = -\frac{1}{2 \pi} \frac{\partial}{\partial \lambda} \left( 2\lambda^2 \frac{\partial S_0}{\partial \lambda} \frac{\partial^2 S_0}{\partial \lambda \partial \bar{\lambda}} - \lambda \frac{\partial^2 S_0}{\partial \lambda \partial \bar{\lambda}} \right).
\]

\[\square\] Springer
After interchanging the $\tau$-derivative with the $\lambda$- and $\bar{\lambda}$-derivatives on the left-hand side of (8.8), we get the claimed result. 

8.3 Pushing forward the Brown measure from $\tau = s$

It is shown in Lemma 4.25 and Theorem 4.28 of [24] that the Brown measure $\mu_{s,s}$ has full mass on the open set $\Sigma_s$. (This statement is stronger than what we have proved in Theorem 6.1, which tells us only that $\mu_{s,s}$ has full mass on $\Sigma_s$.) In this section, we start with the Brown measure $\mu_{s,s}$ on $\Sigma_s$ and push it forward to each domain of the form $\Sigma_{s,\tau}$, with $\tau \neq 0$ and $|\tau - s| \leq s$. We then compute how the density of the pushed-forward measure varies with $\tau$. Thus, we define

$$\hat{\mu}_{s,\tau} = \text{push-forward of } \mu_{s,s} \text{ by } \Phi_{s,\tau}$$

and we let $\hat{W}(s, \tau, \lambda)$ denote the density (in $\Sigma_{s,\tau}$) of $\hat{\mu}_{s,\tau}$.

**Theorem 8.6** For each fixed $s > 0$, the function $\hat{W}$ satisfies the same equation as the function $W$ in Theorem 8.5, namely

$$\frac{\partial \hat{W}}{\partial \tau} = -\frac{\partial}{\partial \lambda} \left[ \lambda \left( \bar{\lambda} \frac{\partial S_0}{\partial \lambda} - \frac{1}{2} \right) \hat{W} \right],$$

for all $\tau$ with $|\tau - s| < s$.

We begin by computing $\partial \hat{W}/\partial \tau$ in a general way. The following result does not rely on the specific form of the map $\Phi_{s,\tau}$ but only on one crucial property of it—that $\Phi_{s,\tau}(\lambda)$ depends holomorphically on $\tau$ with $\lambda$ fixed.

**Proposition 8.7** For each fixed $s > 0$, and all $\tau$ with $|\tau - s| < s$, we have

$$\frac{\partial \hat{W}}{\partial \tau} = -\frac{\partial}{\partial \lambda} \left[ Q \hat{W} \right],$$

where

$$Q(s, \tau, \lambda) = \left. \frac{\partial}{\partial \tau'} \Phi_{s,\tau'}(\Phi_{s,\tau}^{-1}(\lambda)) \right|_{\tau' = \tau}. \quad (8.9)$$

Before proving the proposition, we establish a useful lemma.

**Lemma 8.8** Suppose $f_r$ is a family of diffeomorphisms such that $f_r$ and $f_r^{-1}$ depend smoothly on $r$, $\alpha$ is a fixed differential form, and we define $\beta_r$ as the pull-back of $\alpha$ by $f_r^{-1}$. Then

$$\frac{d \beta_r}{dr} = -\mathcal{L}_X \beta_r, \quad (8.10)$$
where \( X_r \) is the vector field given by

\[
X_r(m) = \left. \frac{d}{dr} f_r'(f_r^{-1}(m)) \right|_{r'=r}.
\]

**Proof** We write \( \alpha \) locally as

\[
\alpha = g \, dx_1 \wedge \cdots \wedge dx_n,
\]

so that

\[
\beta_r = (g \circ f_r^{-1}) d[x_1 \circ f_r^{-1}] \wedge \cdots \wedge d[x_n \circ f_r^{-1}],
\]

from which an elementary calculation shows

\[
\frac{d\beta_r}{dr} = Y_r(g \circ f_r^{-1}) d[x_1 \circ f_r^{-1}] \wedge \cdots \wedge d[x_n \circ f_r^{-1}]
\]

\[
+ \sum_j (g \circ f_r^{-1}) d[x_1 \circ f_r^{-1}] \wedge \cdots \wedge d[Y_r(x_j \circ f_r^{-1})] \wedge \cdots \wedge d[x_n \circ f_r^{-1}]
\]

\[
= L_{Y_r} \beta_r,
\]

where

\[
Y_r(m) = (f_r)_* \left( \frac{d}{dr} f_r^{-1}(m) \right).
\]

We then differentiate the identity \( f_r(f_r^{-1}(m)) = m \) with respect to \( r \) to show that \( Y_r = -X_r \).

**Proof of Proposition 8.7** It will be convenient to use the symbol “vect” to convert a complex number to a vector in the plane; that is,

\[
\text{vect}(\alpha + i\beta) = (\alpha, \beta).
\]

Consider a curve of the form

\[
\tau(r) = \tau + r\tau'
\]

and define a vector field \( V_r \) by

\[
V_r(\lambda) = \text{vect}\left( \frac{d}{d\tau'} \Phi_{s,\tau}(\lambda) \left( \Phi_{s,\tau}^{-1}(\lambda) \right) \bigg|_{r'=r} \right).
\]

Since \( \Phi_{s,\tau}(\lambda) \) depends holomorphically on \( \tau \) with \( \lambda \) fixed, we may compute

\[
\frac{d}{d\tau'} \Phi_{s,\tau}(\lambda) \left( \Phi_{s,\tau}^{-1}(\lambda) \right) \bigg|_{r'=r} = \left. \frac{\partial \Phi_{s,\tau}}{\partial \tau} (\Phi_{s,\tau}(\lambda)) \frac{d\tau}{dr} \right|_{\tau=\tau(r)}
\]

\[
= \tau' \left. \frac{\partial \Phi_{s,\tau}}{\partial \tau} (\Phi_{s,\tau}(\lambda)) \right|_{\tau=\tau(r)}.
\]
Now, since $\hat{W}(s, \tau(r), \cdot)$ is the density of the push-forward of $\mu_{s, s}$ by $\Phi_{s, \tau(r)}$, the associated differential form $\hat{W}(s, \tau(r), \lambda) \, dx \wedge dy$ is the pull-back of $W(s, s, \lambda) \, dx \wedge dy$ by the inverse map $\Phi_{s, \tau(r)}^{-1}$. By Lemma 8.8, we obtain

$$\frac{d}{dr} \hat{W}(s, \tau(r), \lambda) \, dx \wedge dy = -\mathcal{L}_{V_r}[\hat{W}(s, \tau(r), \lambda) \, dx \wedge dy]. \quad (8.15)$$

We now compute $\mathcal{L}_{V_r}$ using Cartan’s formula, one term of which is zero because $\hat{W} \, dx \wedge dy$ is a top-degree form. Now, using (8.14), we find that the vector field $V_r$ in (8.13) is equal to $\text{vect}(\tau' Q(s, \tau(r), \lambda))$, where $Q$ is as in (8.9). We may then easily compute that

$$\mathcal{L}_{V_r}[\hat{W} \, dx \wedge dy] = \left( \tau' \frac{\partial}{\partial \lambda}(Q \hat{W}) + \overline{\tau'} \frac{\partial}{\partial \lambda}(\overline{Q} \hat{W}) \right) \, dx \wedge dy. \quad (8.16)$$

We now argue as in the proof of Theorem 4.2. Recalling that $\tau(r) = \tau + r \tau'$, we find that

$$\left. \frac{\partial}{\partial r} \hat{W}(s, \tau(r), \lambda) \right|_{r=0} = \frac{\partial \hat{W}}{\partial \tau'}(s, \tau, \lambda) \tau' + \frac{\partial \hat{W}}{\partial \overline{\tau}}(s, \tau, \lambda) \overline{\tau'}.$$

Thus, using (8.15) and (8.16), we obtain

$$\frac{\partial \hat{W}}{\partial \tau}(s, \tau, \lambda) = \frac{\partial}{\partial \tau'} \left( \frac{\partial}{\partial r} \hat{W}(s, \tau(r), \lambda) \bigg|_{r=0} \right) = -\left. \frac{\partial}{\partial \lambda}(Q(s, \tau(r), \lambda) \hat{W}) \right|_{r=0},$$

as claimed. \(\square\)

To prove Theorem 8.6, we simply need to compute the function $Q$ in Proposition 8.7.

**Proof of Theorem 8.6** To compute $\Phi_{s, \tau}^{-1}(\lambda)$, using Definition 8.1, we choose our initial conditions so that $\lambda(\tau) = \lambda$, and then $\Phi_{s, \tau}^{-1}(\lambda)$ is equal to $\lambda(s)$. Then to compute $\Phi_{s, \tau'}(\Phi_{s, \tau}^{-1}(\lambda))$, we continue to use the same initial conditions—since they give $\lambda(s) = \Phi_{s, \tau}^{-1}(\lambda)$—and then $\Phi_{s, \tau'}(\Phi_{s, \tau}^{-1}(\lambda))$ is equal to $\lambda(\tau')$. Thus, the values $\Phi_{s, \tau'}(\Phi_{s, \tau}^{-1}(\lambda))$ have the form $\lambda(\tau')$, where all curves are computed using a fixed set of initial conditions. We then differentiate the formula (5.6) for $\lambda(\tau)$ with $\varepsilon_0 p_{\varepsilon, 0} = 0$, and find that

$$\frac{d\lambda(\tau)}{d\tau} = \lambda(\tau) \left( \lambda_0 p_{\lambda, 0} - \frac{1}{2} \right).$$
Since $\lambda(\tau) p_\lambda(\tau)$ is independent of $\tau$ (from the formulas (5.6) and (5.7)) and using the second Hamilton–Jacobi formula (7.30) for $S_0$, we get

$$\frac{d\lambda(\tau)}{d\tau} = \lambda(\tau) \left( \lambda(\tau) \left( \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda(\tau)) - \frac{1}{2} \right) \right).$$

(8.17)

Thus,

$$Q(s, \tau, \lambda) = \lambda \left( \lambda \left( \frac{\partial S_0}{\partial \lambda}(s, \tau, \lambda) - \frac{1}{2} \right) \right)$$

and Theorem 8.6 follows from Proposition 8.7.

8.4 Proof of the push-forward property

We now prove our main push-forward result, Theorem 8.2. The proof is based on the fact that $W$ (the density of the measure $\mu_{s, \tau}$) and $\hat{W}$ (the density of the push-forward of $\mu_{s, \tau}$ by $\Phi_{s, \tau}$) satisfy the same PDE with the same initial condition at $r = 0$. (In both equations, recorded in Theorems 8.5 and 8.6, we regard $S_0$ as a known quantity and try to solve for $W$ or $\hat{W}$.) The agreement between the PDEs, in turn, relies on an agreement between the expression for $d\lambda/d\tau$ in (8.17) and the coefficient of $W$ inside the square brackets on the right-hand side of (8.7).

Now, the quantity in square brackets in (8.7), comes from applying $\frac{1}{\pi} \partial/\partial \bar{\lambda}$ to the right-hand side of the PDE (7.28), while the formula for $\partial \lambda/\partial \tau$ comes from the Hamilton–Jacobi analysis of the same PDE. While we will refrain from stating a precise result, one could develop a general theory whenever a real-valued function $S_0$ satisfies a PDE of the form

$$\frac{\partial S_0}{\partial \tau} = F\left( \lambda, \frac{\partial S_0}{\partial \lambda} \right),$$

where $F$ is a holomorphic of two complex variables.

Proof of Theorem 8.2 We will prove that the function $W$ in Sect. 8.2 and the function $\hat{W}$ in Sect. 8.3 are equal, showing that the push-forward of $\mu_{s, \tau}$ by $\Phi_{s, \tau}$ agrees with $\mu_{s, \tau(r)}$ on the open set $\Sigma_{s, \tau(r)}$. Then since the set $\Sigma_s$ has measure 1 with respect to $\mu_{s, \tau}$ [24, Lemma 4.25 and Theorem 4.28], the set $\Sigma_{s, \tau(r)}$ must have measure 1 with respect to $\mu_{s, \tau(r)}$. Thus, the push-forward of $\mu_{s, \tau}$ by $\Phi_{s, \tau}$ is actually equal to $\mu_{s, \tau}$.

We treat the function $S_0$ as a known quantity, and note that this function occurs both in the PDE for $W$ (Theorem 8.5) and in the identical PDE for $\hat{W}$ (Theorem 8.6). We can then solve this PDE for $W$ (or $\hat{W}$) by differentiating along the characteristic curves $\lambda(\tau)$. We compute that

$$\frac{\partial}{\partial \tau} W(s, \tau, \lambda(\tau)) = \frac{\partial W}{\partial \tau}(s, \tau, \lambda(\tau)) + \frac{\partial W}{\partial \lambda}(s, \tau, \lambda(\tau)) \frac{\partial \lambda}{\partial \tau}.$$
Then using the PDE (8.7) and the formula (8.17) for $\partial \lambda / \partial \tau$, we obtain, after noting a cancellation and dividing both sides by $W$:

$$\frac{\partial}{\partial \tau} \log W(s, \tau, \lambda(\tau)) = - \frac{\partial}{\partial \lambda} \left[ \lambda \left( \lambda \frac{\partial S_0}{\partial \lambda}(\tau, \lambda) - \frac{1}{2} \right) \right].$$

We now consider a nonzero $\tau$ with $|\tau - s| \leq s$ and consider the curve $\tau(r) = s + r(\tau - s)$, which is simply the curve in (8.12) with $\tau = s$ and $\tau' = (\tau - s)$. We then wish to show that

$$W(s, \tau(r), \lambda) = \hat{W}(s, \tau(r), \lambda) \quad (8.18)$$

for all $0 \leq r \leq 1$, where equality at $r = 1$ shows that $W = \hat{W}$. We then compute that for $0 \leq r < 1$,

$$\frac{d}{dr} \log W(s, \tau(r), \lambda(\tau(r))) = \frac{\partial}{\partial \tau} \log W(s, \tau(r), \lambda(\tau(r))) \frac{d\tau}{dr}$$

$$+ \frac{\partial}{\partial \bar{\tau}} \log W(s, \tau(r), \lambda(\tau(r))) \frac{d\bar{\tau}}{dr}$$

$$= -2 \Re \left[ (\tau - s)H_r(\lambda(\tau(r))) \right] \quad (8.19)$$

where

$$H_r(\lambda) = - \frac{\partial}{\partial \lambda} \left[ \lambda \left( \lambda \frac{\partial S_0}{\partial \lambda}(\tau, \lambda) - \frac{1}{2} \right) \right].$$

Of course, $\hat{W}$ satisfies the same equation with $W$ changed to $\hat{W}$ everywhere. We impose the restriction $r < 1$ because in the borderline case $|\tau - s| = s$, the PDEs for $W$ and $\hat{W}$ are not known to hold at $\tau$.

At $r = 0$, both functions equal the density of the Brown measure $\mu_{s,s}$. We may then integrate in (8.19) to obtain

$$\log W(s, \tau, \lambda(\tau(r))) = \log \hat{W}(s, \tau, \lambda(\tau(r)))$$

$$= \log W(s, s, 0) - 2 \int_0^r \Re \left[ (\tau - s)H_r'(\lambda(\tau(r'))) \right] dr',$$

for $r < 1$. We now wish to let $r$ tend to 1. This is only a problem if $|\tau - s| = s$. (If $|\tau - s| < s$, the PDEs for $\hat{W}$ and $W$ are applicable even a little beyond $r = 1$.) As long as $\tau \neq 0$, all the estimates in Sect. 3.3 concerning the boundary behavior of $f_{s-\tau}$ remain applicable even when $|\tau - s| = s$, from which good behavior of the map $\Phi_{s,\tau}$ follows. Thus, we can see that $\hat{W}(s, \tau, \lambda(\tau(r)))$ is continuous up to $r = 1$. The analysis in Sect. 7.3.1 of $S$ at $\varepsilon = 0$ is similarly applicable for $\tau \neq 0$ and $|\tau - s| = s$, from which we can see that $W(s, \tau, \lambda(\tau(r)))$ is continuous up to $r = 1$. Thus, (8.18) holds at $r = 1$, showing that $W = \hat{W}$. \qed
8.5 The proof of Theorem 7.8

The theorem in question says that the Brown measure $\mu_{s,\tau}$ gives full mass to the open set $\Sigma_{s,\tau}$. This is a strengthening of Theorem 6.1, which says that $\mu_{s,\tau}$ gives full mass to the closed set $\overline{\Sigma}_{s,\tau}$. As discussed after the statement of Theorem 7.8 in Sect. 7.3.2, if the boundary of $\Sigma_{s,\tau}$ is a rectifiable curve, Theorem 7.8 follows from the agreement between the first derivatives of $S_0$ as we approach the boundary from inside and from outside the domain—but we do not know that rectifiability of the boundary holds in general.

We actually have already proved Theorem 7.8 in general as part of the proof of Theorem 8.2 in the previous subsection. The proof relies on a result of Ho and Zhong [24, Lemma 4.25 and Theorem 4.28], that the Brown measure of $\mu_{s,s}$ has full mass on $\Sigma_s$. Then the push-forward result on the open domains then tells us that

$$
\mu_{s,\tau}(\Sigma_{s,\tau}) = \mu_{s,s}(\Phi_{s,\tau}^{-1}(\Sigma_{s,\tau})) = \mu_{s,s}(\Sigma_s) = 1.
$$

Now, the proof of the just-cited result of Ho and Zhong also relies on a push-forward result. Recall that $\mu_s$ denotes the law of the unitary element $uu_s$, which is known [36, Proposition 3.6] to have a continuous density with respect to Lebesgue measure on the unit circle. The push-forward result of Ho and Zhong (implicit in their Lemma 4.25 and stated explicitly in Proposition 4.31) asserts that the push-forward of the restriction of $\mu_{s,s}$ to $\Sigma_s$ coincides with the restriction of $\mu_s$ to a certain open set $V_s$ in the unit circle. By Theorems 2.11 and 4.10 in [24], the set $V_s$ is precisely the set where the density of $\mu_s$ is positive, so that $\mu_s(V_s)$ equals 1. It then follows that $\mu_{s,s}(\Sigma_s)$ equals 1 as well.

Intuitively, the push-forward result in [24] can be understood as the $\tau \to 0$ limit of Theorem 8.2 in the present paper, but their result does not actually follow from our result, without additional effort.

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Appendix A. A factorization of $b_{s,\tau}$

In this section, we establish the factorization result in Theorem 4.3 for elements of the form $b_{s,\tau}$ (Notation 2.1). This result is used in the derivation of the PDE for $S$ in Sect. 4.

One could prove a similar result in the finite-$N$ setting of [12], replacing, say, $b_{s,\tau}(r)$ by the corresponding Brownian motion in $GL(N; \mathbb{C})$. According to the equation between Eq. (1.13) and Eq. (1.14) in [12], the generator of the $(s, \tau)$-Brownian motion in $GL(N; \mathbb{C})$ is the operator $s\Delta - \tau \partial^2 - \overline{\tau} \overline{\partial}^2$, where $\Delta$, $\partial^2$, and $\overline{\partial}^2$ are...
certain left-invariant differential operators on $GL(N; \mathbb{C})$. In this setting, the factorization result holds because all three of these operators commute with one another \[12, Corollary 5.7\].

One obvious strategy for trying to prove Theorem 4.3 in the free setting is to try to prove that $b_{s,\tau}(r)b'_{s',\tau}(r)$ and $b_{s+s',\tau+\tau'}(r)$ have the same $\ast$-distribution for all $r$. We may then try to show that both sets of $\ast$-moments satisfy the same differential equation with respect to $r$, with the same trivial initial condition at $r = 0$. This approach does not, however, work in general. Actually, we will implement precisely this strategy in Sect. A.1 in the special case where $\tau' = 0$. In the $\tau' = 0$ case, we get $b'_{s',0}(r) = u_{rs'}$, where $u.$ is the free unitary Brownian motion, and the unitarity of $u_{rs'}$ is used in an essential way in the analysis. (See, specifically, the derivation of (A.9) in the proof of Proposition A.1.)

To show that the $\ast$-moments of $b_{s,\tau}b'_{s',\tau'}$ and $b_{s+s',\tau+\tau'}$ are the same, we follow a three-step process. In Step 1, we will prove this result (as we have just discussed) in the case $\tau' = 0$. In Step 2, we will determine how the $\ast$-moments of $b_{s,\tau}$ (multiplied on the left and right by freely independent elements) change as we vary $\tau$ with $s$ fixed. To determine this, we will need to use the factorization result in Step 1. Finally, in Step 3, we will use Step 2 to show that the $\ast$-moments of

$$b_{s,t}\ b'_{s',t\tau'}, \text{ and } b_{s+s',t(\tau+\tau')}$$

satisfy the same differential equation in $t$, with the same value at $t = 0$. We will then conclude the $\ast$-moments are equal for all $t$; setting $t = 1$ gives the factorization theorem.

A.1 Factoring with a free unitary Brownian motion

In this section, we carry out Step 1 in the proof of Theorem 4.3, corresponding to the special case $\tau' = 0$. In this case, the free multiplicative Brownian motion $b'_{s',0}(r)$ is computable as

$$b'_{s',0}(r) = u_{rs'},$$

where $u.$ is a free unitary Brownian motion freely independent of $b_{s,\tau}(r)$. The unitarity of $u.$ gives us the identities

$$b_{s,\tau}(r)b_{s,\tau}(r)^* = (b_{s,\tau}(r)u_{rs'})(b_{s,\tau}(r)u_{rs'})^*$$

and

$$u_{rs'}^*u_{rs'} = 1,$$

which of course do not hold if $u_{rs'}$ is replaced by a generic element of the form $b'_{s',\tau'}(r)$. These identities will play an essential role in the proof of the following result.

Proposition A.1 Choose $s > 0$, $s' > 0$, and $\tau \in \mathbb{C}$ so that $|\tau - s| \leq s$. Let $u.$ be a free unitary Brownian motion that is freely independent of $b_{s,\tau}(r)$. Then for any $r > 0$, the
random variables $b_{s+s', \tau}(r)$ and

$$a_{s, s', \tau}(r) := b_{s, \tau}(r)u_{rs'}$$  \hspace{1cm} (A.2)

have the same $*$-distribution.

We will prove Proposition A.1 using the free Itô rules. If we think of $u_{rs'}$ as a free stochastic process with time-variable $r$, we obtain the scaled version of the free SDE for $u_r$, namely

$$du_{rs'} = u_{rs'}\left(i\sqrt{s'}\,dx_s - \frac{s'}{2}\,dr\right).$$

Using this result, the SDE (2.6) for $b_{s, \tau}(r)$, and the stochastic product rule (4.6), we find that the process $a_{s, s', \tau}(r)$ satisfies the following free SDE:

$$da_{s, s', \tau}(r) = i\sqrt{s'}a_{s, s', \tau}(r)\,dx_r + ib_{s, \tau}(r)\,dw_{s, \tau}(r)u_{rs'} - \frac{1}{2}(s + s' - \tau)a_{s, s', \tau}(r)\,dr,$$  \hspace{1cm} (A.3)

where $x_r$ is a semicircular Brownian motion that is freely independent of $w_{s, \tau}(r)$.

**Proof of Proposition A.1** In this proof, we assume $s_0$, $s$, and $\tau$ are fixed and use the notations

$$b_r = b_{s+s', \tau}(r); \quad w_r = w_{s+s', \tau}(r);$$

$$\tilde{b}_r = b_{s, \tau}(r); \quad a_r = a_{s, s', \tau}(r); \quad \tilde{w}_r = w_{s, \tau}(r). \quad .$$

We prove the result by induction on the length of the $*$-moment. It is clear that $b_r$ and $a_r$ have the same zeroth-order $*$-moments. Suppose that all the $*$-moments of order less than $n$ of $b_r$ and $a_r$ are equal and consider a $*$-moment of length $n$, which we write as $\text{tr} \left[b_r^{e_1} \cdots b_r^{e_n}\right]$ or $\text{tr} \left[a_r^{e_1} \cdots a_r^{e_n}\right]$, with $e_j \in \{1, *\}$.

In the case of the $*$-moment of $b_r$, the stochastic product rule tells us that

$$\frac{d}{dr} \text{tr} \left[b_r^{e_1} \cdots b_r^{e_n}\right] = \sum_j \text{tr} \left[b_r^{e_1} \cdots db_r^{e_j} \cdots b_r^{e_n}\right] + \sum_{j<k} \text{tr} \left[b_r^{e_1} \cdots db_r^{e_j} \cdots db_r^{e_k} \cdots b_r^{e_n}\right],$$

where the sum over $j < k$ is empty (having a value of 0) if $n = 1$. Using the SDE (2.6) of $b_r$ and the Itô rules (4.8), we may easily compute that

$$\frac{d}{dr} \text{tr} \left[b_r^{e_1} \cdots b_r^{e_n}\right] = -\frac{1}{2} \text{tr} \left[b_r^{e_1} \cdots b_r^{e_n}\right] \sum_j (s + s' - \tau e_j).$$

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− \sum_{\varepsilon_j=\varepsilon_k=1} (s+s'-\tau) \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_j} b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_n} \right] \text{tr} \left[ b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_k} \right] \\
− \sum_{\varepsilon_j=\varepsilon_k=*} (s+s'-\bar{\tau}) \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_j} b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_n} \right] \text{tr} \left[ b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_k} \right] \\
+ \sum_{\varepsilon_j=1, \varepsilon_k=*} (s+s') \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_j} b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_n} \right] \text{tr} \left[ b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_k-1} \right] \\
+ \sum_{\varepsilon_j=*, \varepsilon_k=1} (s+s') \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_{j-1}} b_r^{\varepsilon_{j+1}} \cdots b_r^{\varepsilon_n} \right] \text{tr} \left[ b_r^{\varepsilon_{j}} \cdots b_r^{\varepsilon_k} \right], \quad (A.4)

where in the last four lines, each sum is over all pairs \((j, k)\) with \(j < k\) and \(\varepsilon_j\) and \(\varepsilon_k\) satisfying the indicated conditions. In the first line, we take \(\tau^{\varepsilon_j} = \bar{\tau}\) if \(\varepsilon_j = *\).

The first term on the right-hand side of (A.4) is a multiple of the \(*\)-moment on the left-hand side of the equation. All the remaining terms on the right-hand side of (A.4) are either multiples of the left-hand side (e.g., the term \(j = k - 1\) in the second sum on the right-hand side) or products of \(*\)-moments of \(b\) of lower degree. Thus, (A.4) has the form

\[
\frac{d}{dr} \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_n} \right] = C \text{tr} \left[ b_r^{\varepsilon_1} \cdots b_r^{\varepsilon_n} \right] + f(r), \quad (A.5)
\]

where \(f(r)\) may be considered, inductively, as a known quantity. The equation (A.5) may then be solved by the method of integrating factors.

We now show that \(\text{tr} \left[ a_r^{\varepsilon_1} \cdots a_r^{\varepsilon_n} \right]\) satisfies the same ODE in (A.5). By the stochastic product rule,

\[
d \text{tr} \left[ a_r^{\varepsilon_1} \cdots a_r^{\varepsilon_n} \right] = \sum_j \text{tr} \left[ a_r^{\varepsilon_1} \cdots da_r^{\varepsilon_j} \cdots a_r^{\varepsilon_n} \right] \\
+ \sum_{j<k} \text{tr} \left[ a_r^{\varepsilon_1} \cdots da_r^{\varepsilon_j} \cdots da_r^{\varepsilon_k} \cdots a_r^{\varepsilon_n} \right]. \quad (A.6)
\]

By the SDE (A.3) of \(a\) and the free Itô rules (4.5) and (4.8), the first term on the right-hand side of (A.6) is simply

\[
− \frac{1}{2} \text{tr} \left[ a_r^{\varepsilon_1} \cdots a_r^{\varepsilon_n} \right] \sum_j (s+s'−\tau^{\varepsilon_j}). \quad (A.7)
\]

This term is the same as the first term in the right-hand side of (A.4), after replacing the \(b_r\) by \(a_r\).

For the second sum on the right-hand side of (A.6), we separate the computation into four cases of \((\varepsilon_j, \varepsilon_k) \in \{1, *\}^2\). If \((\varepsilon_j, \varepsilon_k) = (1, 1)\), then, after dropping terms that are equal to zero, we get

\[
\text{tr} \left[ a_r^{\varepsilon_1} \cdots da_r^{\varepsilon_j} \cdots da_r^{\varepsilon_k} \cdots a_r^{\varepsilon_n} \right]
\]
\[
= -s' \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_j} dx_r \cdots a_r^{e_k} dx_r \cdots a_r^{e_n} \right] \\
+ \text{tr} \left[ a_r^{e_1} \cdots (i \tilde{b}_r d \tilde{w}_r u_{rs'}) \cdots (i \tilde{b}_r d \tilde{w}_r u_{rs'}) \cdots a_r^{e_n} \right].
\]

We then apply the Itô rules (4.5) and (4.8). The second term on the right-hand side of (A.8) is the crucial one: after using (4.8), the \( \tilde{b}_r \) to the left of the first \( d \tilde{w}_r \) and the \( u_{rs'} \) to the right of the second \( d \tilde{w}_r \) combine to form \( a_r = a_r^{e_j} \). Meanwhile, after a cyclic permutation inside the trace, the \( u_{rs'} \) to the right of first \( d \tilde{w}_r \) and the \( \tilde{b}_r \) to the left of second \( d \tilde{w}_r \) also combine to form \( a_r = a_r^{e_k} \). After combining terms, we end up with

\[
\text{tr} \left[ a_r^{e_1} \cdots da_r^{e_j} \cdots da_r^{e_k} \cdots a_r^{e_n} \right] \\
= -(s + s' - \tau) \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_j} a_r^{e_{k+1}} \cdots a_r^{e_n} \right] \text{tr} \left[ a_r^{e_{j+1}} \cdots a_r^{e_k} \right].
\]

This result matches the \( \varepsilon_j = \varepsilon_k = 1 \) term in (A.4). The case where \( (\varepsilon_j, \varepsilon_k) = (\ast, \ast) \) similarly matches the corresponding term in (A.4).

We now turn to the case \( (\varepsilon_j, \varepsilon_k) \in (1, \ast) \). We compute

\[
\text{tr} \left[ a_r^{e_1} \cdots da_r^{e_j} \cdots da_r^{e_k} \cdots a_r^{e_n} \right] = s' \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_j} dx' \cdots dx' a_r^{e_k} \cdots a_r^{e_n} \right] \\
+ \text{tr} \left[ a_r^{e_1} \cdots \tilde{b}_r d \tilde{w}_r u_{rs'} \cdots u_{rs'}^{*} d \tilde{w}_r^{*} \tilde{b}_r^{*} \cdots a_r^{e_n} \right].
\]

Using the Itô rules (4.5) and (4.8), we obtain

\[
\text{tr} \left[ a_r^{e_1} \cdots da_r^{e_j} \cdots da_r^{e_k} \cdots a_r^{e_n} \right] \\
= s' \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_j} a_r^{e_{k+1}} \cdots a_r^{e_n} \right] \text{tr} \left[ a_r^{e_{j+1}} \cdots a_r^{e_{k-1}} \right] \\
+ s \text{tr} \left[ a_r^{e_1} \cdots \tilde{b}_r \tilde{b}_r^{*} \cdots a_r^{e_n} \right] \text{tr} \left[ u_{rs'} a_r^{e_{j+1}} \cdots a_r^{e_{k-1}} u_{rs'}^{*} \right] \\
= (s + s') \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_j} a_r^{e_{k+1}} \cdots a_r^{e_n} \right] \text{tr} \left[ a_r^{e_{j+1}} \cdots a_r^{e_{k-1}} \right],
\]

where, in the last equality, we have used both lines of (A.1). Note that the unitarity of \( u_{rs'} \) is used in an essential way here. The result of (A.9) matches the \( \varepsilon_j = 1, \varepsilon_k = \ast \) term in (A.4). The case \( (\varepsilon_j, \varepsilon_k) = (\ast, 1) \) is similar. Summing up all the four cases and (A.7), we see that \( \text{tr} \left[ a_r^{e_1} \cdots da_r^{e_j} \cdots da_r^{e_k} \cdots a_r^{e_n} \right] \) satisfies the same equation as in (A.4), after changing \( b_r \) to \( a_r \).

By the induction hypothesis, all the lower-order \( \ast \)-moments of \( a_r \) and \( b_r \) are equal. Thus, \( \text{tr} \left[ a_r^{e_1} \cdots a_r^{e_n} \right] \) satisfies the same ODE as \( \text{tr} \left[ b_r^{e_1} \cdots b_r^{e_n} \right] \) as in (A.5), with the same constant \( C \) and the same function \( f \). Since, also, both functions equal 1 at \( r = 0 \), they are equal for all \( r \). \( \square \)
A.2 Varying $\tau$ with $s$ fixed

In this section, we carry out Step 2 in the proof of Theorem 4.3, by looking at how the $\ast$-moments of $b_{s,\tau}$—multiplied on the left and right by freely independent elements—vary as $\tau$ varies with $s$ fixed. When doing the calculation, we will obtain a convenient cancellation that eliminates certain problematic terms, a cancellation that would not occur if we were varying both $s$ and $\tau$ simultaneously. This computation supports the overall philosophy of this paper, that we obtain the nicest results when $\tau$ is varied while keeping $s$ fixed.

Lemma A.2 Fix $s$ and $\tau$ with $|\tau - s| \leq s$ and fix a sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ with values in $\{1, \ast\}$. Suppose $a_1, a_2 \in A$ and $b_{s,\tau}(r)$ are all freely independent. Define

$$B = a_1 b_{s,\tau} a_2$$

and

$$f(s, t) = \text{tr}[B^{\varepsilon_1} \cdots B^{\varepsilon_k}].$$

Then for all $t < 1$, we have

$$\frac{\partial f}{\partial t} = \frac{f(s, t)}{2} \sum_j \tau^{\varepsilon_j}$$

$$- \sum_{\varepsilon_j = \varepsilon_k} \tau^{\varepsilon_j} \text{tr}[B^{\varepsilon_1} \cdots B^{\varepsilon_j} B^{\varepsilon_{k+1}} \cdots B^{\varepsilon_n}] \text{tr}[B^{\varepsilon_{j+1}} \cdots B^{\varepsilon_k}],$$

(A.10)

where $\tau^{\varepsilon_j} = \bar{\tau}$ if $\varepsilon_j = \ast$ and where the second sum on the right-hand side of (A.10) is over all pairs $j < k$ with $\varepsilon_j = \varepsilon_k$.

We emphasize that the second sum on the right-hand side of (A.10) contains only terms where $\varepsilon_j = \varepsilon_k$; the terms with $\varepsilon_j \neq \varepsilon_k$ cancel out in the process of computing the derivative with respect to $t$.

**Proof** Let $B_r = a_1 b_{s,\tau}(r) a_2$, so that $B = B_1$. Since $b_{s,\tau}(r)$ has the same $\ast$-distribution as $b_{rs,\tau}(r)$, we have

$$f(rs, rt) = \text{tr}[B_r^{\varepsilon_1} \cdots B_r^{\varepsilon_k}].$$

(A.11)

Then by the chain rule, we have

$$\frac{d}{dr} f(rs, rt) = s \frac{\partial f}{\partial s}(rs, rt) + t \frac{\partial f}{\partial t}(rs, rt).$$

Solving this for the $\partial f/\partial t$ term and setting $r = 1$ gives

$$\frac{\partial f}{\partial t}(s, t) = \frac{1}{t} \left. \frac{d}{dr} f(rs, rt) \right|_{r=1} - \frac{s}{t} \frac{\partial f}{\partial s}(s, t).$$

(A.12)
We now proceed to compute each of the two terms on the right-hand side of (A.12). We first differentiate $f(rs, rt)$ with respect to $r$, using (A.11). Using the stochastic product rule and the Itô formulas, we easily obtain

$$
\frac{d}{dr} f(rs, rt) = \sum_j \frac{\text{tr}[B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_j} \cdots B_r^{\varepsilon_n}]}{dr} + \sum_{j<k} \frac{\text{tr}[B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_k} \cdots B_r^{\varepsilon_n}]}{dr}
$$

$$
= -f(rs, rt) \sum_j s - t \tau^\varepsilon_j \frac{1}{2} + \sum_{j<k} \frac{\text{tr}[B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_k} \cdots B_r^{\varepsilon_n}]}{dr}.
$$

(A.13)

We will analyze the last sum in (A.13) in a moment.

We next differentiate $f(s, t)$ with respect to $s$. Since $|t\tau - s| < s$ for any $t < 1$, we can choose $s_0 < s$ such that $|t\tau - s_0| < s$. Then by Proposition A.1, the element

$$
U_s := a_1 b_{s_0, t\tau} u_{s_0} a_2
$$

has the same $*$-distribution as $a_1 b_{s, t\tau} a_2$. We then calculate

$$
\frac{\partial f}{\partial s}(s, t) = \sum_j \frac{\text{tr}[U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_j} \cdots U_s^{\varepsilon_n}]}{ds} + \sum_{j<k} \frac{\text{tr}[U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_k} \cdots U_s^{\varepsilon_n}]}{ds}
$$

$$
= -\frac{n}{2} f(s, t) + \sum_{j<k} \frac{\text{tr}[U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_k} \cdots U_s^{\varepsilon_n}]}{ds},
$$

(A.14)

where we have computed the first sum as

$$
\sum_{j=1}^n \frac{\text{tr}[U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_j} \cdots U_s^{\varepsilon_n}]}{ds} = -\frac{1}{2} \sum_{j=1}^n \text{tr}[U_s^{\varepsilon_j} \cdots U_s^{\varepsilon_n}] = -\frac{n}{2} f(s, t).
$$

We will analyze the last sum in (A.14) in a moment.

Using (A.12) together with (A.13) and (A.14), we obtain

$$
\frac{\partial f}{\partial t} = \frac{f(s, t)}{2} \sum_j \tau^\varepsilon_j
$$

$$
= \sum_{j<k} \left( \frac{1}{t} \frac{\text{tr}[B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_j} \cdots d B_r^{\varepsilon_k} \cdots B_r^{\varepsilon_n}]}{dr} \bigg|_{r=1} - \frac{s}{t} \frac{\text{tr}[U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_j} \cdots d U_s^{\varepsilon_k} \cdots B_r^{\varepsilon_n}]}{ds} \right).
$$

(A.15)
We then analyze the sum on the right-hand side of (A.15) case by case, starting with the critical cases where \((\varepsilon_j, \varepsilon_k) = (1, \ast)\) or \((\ast, 1)\). If \((\varepsilon_j, \varepsilon_k) = (1, \ast)\), then

\[
\begin{align*}
\text{tr}[B^{\varepsilon_1}_r \cdots d B^{\varepsilon_j}_r \cdots d B^{\varepsilon_k}_r \cdots B^{\varepsilon_n}_r] \\
= \text{tr}[B^{\varepsilon_1}_r \cdots (a_1 b_{s,t\tau}(r) dw_r a_2) \cdots (a_2^* dw_r^* b_{s,t\tau}(r) a_1^*) \cdots B^{\varepsilon_n}_r] \\
= s \text{tr}[B^{\varepsilon_1}_r \cdots a_1 b_{s,t\tau}(r) b_{s,t\tau}(r) a_1^* \cdots B^{\varepsilon_n}_r] \text{tr}[a_2 B^{\varepsilon_j+1}_r \cdots B^{\varepsilon_k-1}_r a_2^*] \, dr
\end{align*}
\]

where \(\text{tr}[a_2 B^{\varepsilon_j+1}_r \cdots B^{\varepsilon_k-1}_r a_2^*]\) means \(\text{tr}[a_2 a_2^*]\) if \(k = j + 1\). Similarly,

\[
\begin{align*}
\text{tr}[U^{\varepsilon_1}_s \cdots d U^{\varepsilon_j}_s \cdots d U^{\varepsilon_k}_s \cdots U^{\varepsilon_n}_s] \\
= \text{tr}[U^{\varepsilon_1}_s \cdots (a_1 b_{s_0,t\tau} u_{s_0-s_0} d x_s a_2) \cdots (a_2^* d x_s^* b_{s_0,t\tau} a_1^*) \cdots U^{\varepsilon_n}_s] \\
= \text{tr}[U^{\varepsilon_1}_s \cdots (a_1 b_{s_0,t\tau} b_{s_0,t\tau} a_1^*) \cdots U^{\varepsilon_n}_s] \text{tr}[a_2 U^{\varepsilon_j+1}_s \cdots U^{\varepsilon_k-1}_s a_2^*] \, ds.
\end{align*}
\]

In the last line of (A.16), we note that \(b_{s_0,t\tau} b_{s_0,t\tau}^* = (b_{s_0,t\tau} u_{s_0-s_0})(b_{s_0,t\tau} u_{s_0-s_0})^*\). Then after recalling the factorization result in Proposition A.1, we get a cancellation:

\[
\frac{1}{t} \text{tr}[B^{\varepsilon_1}_r \cdots d B^{\varepsilon_j}_r \cdots d B^{\varepsilon_k}_r \cdots B^{\varepsilon_n}_r] \bigg|_{r=1} - \frac{s}{t} \text{tr}[U^{\varepsilon_1}_s \cdots d U^{\varepsilon_j}_s \cdots d U^{\varepsilon_k}_s \cdots U^{\varepsilon_n}_s] \bigg|_{r=1} = 0.
\]

The above equality also holds for the case \((\varepsilon_j, \varepsilon_l) = (\ast, 1)\) by a similar computation that again uses the unitarity of \(u_{s_0-s_0}\).

We now consider the case \(\varepsilon_j = \varepsilon_l = 1\). In this case, we can directly verify that

\[
\frac{1}{t} \text{tr}[B^{\varepsilon_1}_r \cdots d B^{\varepsilon_j}_r \cdots d B^{\varepsilon_k}_r \cdots B^{\varepsilon_n}_r] \bigg|_{r=1} = \frac{s - t\tau}{t} \text{tr}[B^{\varepsilon_1}_1 \cdots B^{\varepsilon_j}_1 B^{\varepsilon_{j+1}} \cdots B^{\varepsilon_k}_r \cdots B^{\varepsilon_n}_r] \text{tr}[B^{\varepsilon_{j+1}} \cdots B^{\varepsilon_k}]
\]

and

\[
\frac{s}{t} \text{tr}[U^{\varepsilon_1}_s \cdots d U^{\varepsilon_j}_s \cdots d U^{\varepsilon_k}_s \cdots U^{\varepsilon_n}_s] \bigg|_{r=1} = \frac{s - t\tau}{t} \text{tr}[U^{\varepsilon_1}_s \cdots U^{\varepsilon_j}_s U^{\varepsilon_{j+1}} \cdots U^{\varepsilon_n}_s] \text{tr}[U^{\varepsilon_{j+1}} \cdots U^{\varepsilon_k}]
\]

Subtracting these equations gives a cancellation of the terms involving \(s/t\). The above equations hold in the case \(\varepsilon_j = \varepsilon_l = \ast\), except with \(s - t\tau\) replaced by \(s - t\bar{\tau}\) in the first line. Combining the four cases in (A.15), we obtain the result claimed in the proposition. \(\square\)
A.3 The proof of the main result

We are now ready for the proof of our main factorization result.

Proof of Theorem 4.3 Write

\[ A_{t,t'} = b_{s,t} b_{s',t',t'} \]
\[ B_t = b_{s,s',t(t+t')} \]

We want to show that \( A_{t,t} \) and \( B_t \) have the same \( \ast \)-moments by mathematical induction on the order \( n \) of \( \ast \)-moments, starting from the trivial case \( n = 0 \). In the induction step, we will obtain differential equations for the \( \ast \)-moments of \( A_{t,t} \) and \( B_t \) with respect to \( t \).

Assume all the \( \ast \)-moments of length less than \( n \) are equal, consider a sequence \( (\epsilon_1, \ldots, \epsilon_n) \) taking values in \( \{1, \ast\} \), and consider the functions

\[ f(t) = \text{tr}[A_{t,t}^{\epsilon_1} \cdots A_{t,t}^{\epsilon_k}] \]

and

\[ g(t) = \text{tr}[B_t^{\epsilon_1} \cdots B_t^{\epsilon_k}] \]

Since \( u_s u_{s'} \) has the same \( \ast \)-distribution as \( u_s + u' \) where \( u \) and \( u' \) are freely independent free unitary Brownian motions, \( f(0) = g(0) \). Our goal is to show that \( f(t) = g(t) \) for \( 0 \leq t \leq 1 \).

Let

\[ h(t,t') = \text{tr}[A_{t,t'}^{\epsilon_1} \cdots A_{t,t'}^{\epsilon_k}] \]

Applying Lemma A.2 with \( a_1 = 1, a_2 = b_{s,t} \), we have

\[
\frac{\partial h}{\partial t}(t,t') = \frac{h(t,t')}{2} \sum_j (\tau^{\epsilon_j}) \quad - \quad \sum_{\epsilon_j = \epsilon_k} (\tau^{\epsilon_j}) \text{tr}[A_{t,t'}^{\epsilon_1} \cdots A_{t,t'}^{\epsilon_j} A_{t,t'}^{\epsilon_{k+1}} \cdots A_{t,t'}^{\epsilon_n}] \text{tr}[A_{t,t'}^{\epsilon_j+1} \cdots A_{t,t'}^{\epsilon_k}] .
\]

Similarly, by applying Lemma A.2 with \( a_1 = b_{s,t} \), \( a_2 = 1 \), and with \( s', \tau' \) in place of \( s, \tau \), we have

\[
\frac{\partial h}{\partial t'} = \frac{h(t,t')}{2} \sum_j (\tau'^{\epsilon_j}) \quad - \quad \sum_{\epsilon_j = \epsilon_k} (\tau'^{\epsilon_j}) \text{tr}[A_{t,t'}^{\epsilon_1} \cdots A_{t,t'}^{\epsilon_j} A_{t,t'}^{\epsilon_{k+1}} \cdots A_{t,t'}^{\epsilon_n}] \text{tr}[A_{t,t'}^{\epsilon_j+1} \cdots A_{t,t'}^{\epsilon_k}] .
\]
Since \( f(t) = h(t, t) \), the chain rule tells us that
\[
\frac{df}{dt} = \frac{\partial h}{\partial t}(t, t) + \frac{\partial h}{\partial t'}(t, t) = \frac{f(t)}{2} \sum_j (\tau + \tau')^{\varepsilon_j} - \sum_{\varepsilon_j = \varepsilon_k} (\tau + \tau')^{\varepsilon_j} \text{tr}[A_{t,t}^{\varepsilon_1} \cdots A_{t,t}^{\varepsilon_j} A_{t,t}^{\varepsilon_{j+1}} \cdots A_{t,t}^{\varepsilon_n}] \text{tr}[A_{t,t}^{\varepsilon_1} \cdots A_{t,t}^{\varepsilon_k}].
\] (A.17)

Meanwhile, by directly applying Lemma A.2, we find that the function \( g \) satisfies the same formula as in (A.17), but with \( A_{t,t} \) replaced by \( B_t \) everywhere. Now, by the induction hypothesis, all the lower moments of \( A_{t,t} \) and \( B_t \) are equal. We therefore conclude that \( f \) and \( g \) satisfy the same first-order linear ODE. Since we also have \( f(0) = g(0) \), as we have noted, we conclude that \( f(t) = g(t) \) for all \( t < 1 \). Equality also holds when \( t = 1 \) by the continuous dependence of \( b_s, \tau \) on \( \tau \) (Appendix B). This completes the proof using mathematical induction, and we conclude that \( A_{t,t} \) and \( B_t \) have the same \(*\)-distribution.

\( \square \)

**Appendix B. Continuous dependence of \( b_{s, \tau} \) on \( \tau \)**

We consider a free SDE of the form
\[
\text{d} b_r = b_r (\alpha \, d x_r + \beta \, d \tilde{x}_r + \gamma \, d r)
\]
with initial condition \( b_0 \), where \( \alpha, \beta, \) and \( \gamma \) are fixed complex numbers. In integral form, this reads as
\[
b_r = b_0 + \alpha \int_0^r b_u \, d x_u + \beta \int_0^r b_u \, d \tilde{x}_u + \gamma \int_0^r b_u \, d u.
\] (B.1)

We will show that the solution depends continuously on the constants \( \alpha, \beta, \) and \( \gamma \). We first give an a priori bound on the size of the solutions.

**Proposition B.1** Given any \( r_0 > 0 \), we have the inequality
\[
\| b_r \|^2 \leq \| b_0 \|^2 \left( 1 + 2 \sqrt{2} (|\alpha| + |\beta|) + \sqrt{r_0} |\gamma| \right)^2 e^{(1 + (2 \sqrt{2} (|\alpha| + |\beta|) + \sqrt{r_0} |\gamma|))^2 r}
\]

for all \( r \leq r_0 \).

**Proof** We take the norm of both sides of (B.1) and apply the free Burkholder–Davis–Gundy inequality [5, Theorem 3.2.1], giving
\[
\| b_r \| \leq \| b_0 \| + 2 \sqrt{2} (|\alpha| + |\beta|) \left( \int_0^r \| b_u \|^2 \, d u \right)^{1/2} + |\gamma| \int_0^r \| b_u \| \, d u.
\] (B.2)
Using the Cauchy–Schwarz inequality twice (once for the last integral in (B.2) and once for numbers) we have

\[ \|b_r\| \leq \|b_0\| + (2\sqrt{2}(|\alpha| + |\beta|) + |\gamma|\sqrt{r_0}) \left( \int_0^r \|b_u\|^2 \, du \right)^{1/2} \leq \left( \|b_0\|^2 + \int_0^r \|b_u\|^2 \, du \right)^{1/2} \left( 1 + (2\sqrt{2}(|\alpha| + |\beta|) + \sqrt{r_0}|\gamma|)^2 \right)^{1/2}. \] (B.3)

If we square both sides of (B.3) and apply Gronwall’s inequality [32, Theorem 1.2.2], we obtain

\[ \|b_r\|^2 \leq \|b_0\|^2 \left( 1 + (2\sqrt{2}(|\alpha| + |\beta|) + \sqrt{r_0}|\gamma|)^2 \right)^r e^{\left( 1 + (2\sqrt{2}(|\alpha| + |\beta|) + \sqrt{r_0}|\gamma|)^2 \right)^{1/2}} \]

for all \( r \leq r_0 \), as desired. \( \square \)

Now consider \( b_r \) and \( b'_r \) satisfying (B.1) with constants \( \alpha, \beta, \gamma \) and \( \alpha', \beta', \gamma' \), respectively, and define

\[ v(r) = \|b_r - b'_r\|. \]

Then we have the following estimate, showing that \( b_r \) is close to \( b'_r \) when \( (\alpha, \beta, \gamma) \) is close to \( (\alpha', \beta', \gamma') \).

**Theorem B.2** Fix \( r_0 \) and assume \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \) lie in a disk of some fixed radius. Then there exists a constant \( C > 0 \) such that

\[ v(r)^2 \leq C \left( 2\sqrt{2}(|\alpha - \alpha'| + |\beta - \beta'| + \sqrt{r_0}|\gamma - \gamma'|)^2 \right) e^{Cr}, \quad r \leq r_0. \]

In particular, \( v(r) \to 0 \) uniformly in \( r \in [0, r_0] \) as \( (\alpha, \beta) \to (\alpha', \beta') \).

**Proof** We compute

\[
\begin{aligned}
b_t - b'_t &= \int_0^t [(\alpha - \alpha')b_r + \alpha'(b_r - b'_r)] \, dx_r + \int_0^t [(\beta - \beta')b_r + \beta'(b_r - b'_r)] \, d\tilde{x}_r \\
&\quad + \int_0^t [(\gamma - \gamma')b_r + \gamma'(b_r - b'_r)] \, dr.
\end{aligned}
\]

By the free Burkholder–Davis–Gundy inequality,

\[
\begin{aligned}
v(r) &\leq 2\sqrt{2} \left( \int_0^r [|\alpha - \alpha'| + |\beta - \beta'|]\|b_u\| + (|\alpha'| + |\beta'|)u(u)\|^2 \, du \right)^{1/2} \\
&\quad + \int_0^r (|\gamma - \gamma'| \|b_u\| + |\gamma'| v(u)) \, du.
\end{aligned}
\]
We now apply the Minkowski inequality to the first integral and Cauchy–Schwarz inequalities to the second integral and collect terms to obtain

\[
v(r) \leq \left(2\sqrt{2}(|\alpha - \alpha'| + |\beta - \beta'|) + \sqrt{r_0} |\gamma - \gamma'|\right) \left(\int_0^r \|b_u\|^2 du\right)^{1/2} \\
+ \left(2\sqrt{2}(|\alpha'| + |\beta'|) + \sqrt{r_0} |\gamma'|\right) \left(\int_0^r v(u)^2 du\right)^{1/2}.
\]

Applying the two-dimensional Cauchy–Schwarz inequality to the above estimate, we have

\[
v(r)^2 \leq \left[\left(2\sqrt{2}(|\alpha - \alpha'| + |\beta - \beta'|) + \sqrt{r_0} |\gamma - \gamma'|\right)^2 + \int_0^r v(u)^2 du\right]^2 \\
\times \left[\int_0^r \|b_u\|^2 du + \left(2\sqrt{2}(|\alpha'| + |\beta'|) + \sqrt{r_0} |\gamma'|\right)^2\right]. \tag{B.4}
\]

We may now bound the second factor on the right-hand side of (B.4) by a constant $C$ using Proposition B.1. Applying Gronwall’s inequality gives the claimed estimate. □

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