NON-LANDING PARAMETER RAYS OF THE MULTICORNS

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Abstract. It is well known that every rational parameter ray of the Mandelbrot set lands at a single parameter. We study the rational parameter rays of the multicorns, the connectedness loci of unicritical antiholomorphic polynomials, and give a complete description of their accumulation properties. One of the principal results is that the parameter rays accumulating on the boundaries of odd period (except period 1) hyperbolic components of the multicorns do not land, but accumulate on arcs of positive length consisting of parabolic parameters.

We also show the existence of undecorated real-analytic arcs on the boundaries of the multicorns, which implies that the centers of hyperbolic components are not equidistributed with respect to the harmonic measures of the multicorns.

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1. Introduction

We consider unicritical anti-polynomials \( f_c(z) = \bar{z}^d + c \) for any degree \( d \geq 2 \) and \( c \in \mathbb{C} \). In analogy to the holomorphic case, the set of all points which remain bounded under all iterations of \( f_c \) is called the Filled-in Julia set \( K(f_c) \). The boundary of the Filled-in Julia set is defined to be the Julia set \( J(f_c) \) and the complement of the Julia set is defined to be its Fatou set \( F(f_c) \). This leads, as in the holomorphic case, to the notion of Connectedness Locus of degree \( d \) unicritical anti-polynomials:

Definition. The multicorn of degree \( d \) is defined as \( \mathcal{M}_d = \{ c \in \mathbb{C} : K(f_c) \text{ is connected} \} \)

The dynamics of anti-quadratic maps and its connectedness locus, \( \mathcal{M}_2 \) (also known as the tricorn), was first studied in [CHRC90] and their numerical experiments showed differences between the Mandelbrot set and the tricorn in that there are bifurcations from the period 1 hyperbolic component to period 2 hyperbolic...
components along arcs in the latter. However, it was Milnor who first observed the importance of the multicorns; he found little tricorn and multicorn-like sets as prototypical objects in the parameter space of real cubic polynomials [Mil92] and in the real slices of rational maps with two critical points [Mil00]. Nakane [Nak93] proved that the tricorn is connected, in analogy to Douady and Hubbard’s classical proof on the Mandelbrot set. This generalizes naturally to multicorns of any degree. Later, Nakane and Schleicher, in [NS03], studied the structure of hyperbolic components of $M_d^*$ via the multiplier map (even period case) and the critical value map (odd period case). These maps are branched coverings over the unit disk of degree $d - 1$ and $d + 1$ respectively, branched only over the origin. Hubbard and Schleicher [HS14] proved that the multicorns are not pathwise connected, confirming a conjecture of Milnor. Recently, in an attempt to explore the topological aspects of the parameter spaces of unicritical anti-polynomials, the combinatorics of external dynamical rays of such maps were studied in [Muk14] in terms of orbit portraits and this was used in [MNS14] where the bifurcation phenomena, boundaries of odd period hyperbolic components and the combinatorics of parameter rays were described.

The combinatorics and topology of the multicorns differ in many ways from those of their holomorphic counterparts, the multibrot sets, which are the connectedness loci of degree $d$ unicritical polynomials. At the level of combinatorics, this is manifested in the structure of orbit portraits [Muk14] Theorem 2.6, Theorem 3.1]. The topological features of the multicorns have quite a few properties in common with the connectedness locus of real cubic polynomials, e.g. discontinuity of landing points of dynamical rays, bifurcation along arcs, existence of real-analytic curves containing q.c.-conjugate parabolic parameters, lack of local connectedness of the connectedness loci etc. [Lav89], [KN04], [HS14 Corollary 3.7], [MNS14 Theorem 3.2, Theorem 6.2]. These are in stark contrast with the multibrot sets.

![Figure 1](image1.png)

**Figure 1.** Left: Landing of the parameter rays at fixed angles on parabolic arcs containing undecorated sub-arcs. Right: Non-trivial accumulation of a parameter ray at an odd-periodic angle.

One of the main purposes of this paper is to give a complete description of the landing/accumulation properties of the rational parameter rays of $M_d^*$ (see Theorem
The landing of rational parameter rays of the Mandelbrot set is related to the fact that the parabolic parameters with given combinatorics are isolated in the Mandelbrot set [GM93, Theorem C.7], [Sch00, Proposition 3.1]. This remains true for even period parabolics of the multicorns. Hence, if the accumulation set of a parameter ray $\mathcal{R}_d^t$ of $\mathcal{M}_d^*$ contains a parameter $c$ having an even-periodic parabolic cycle, then it lands at a single point. This statement was proved in [MNS14, Lemma 7.2]. But the odd periodic parabolic parameters of the multicorns are far from being isolated, there are real-analytic arcs of combinatorially equivalent parabolic parameters of any given odd period. This, at least heuristically, tells that there is no good reason for a parameter ray to land at a single point of a parabolic arc, unless it does so for some symmetry reasons. The following theorem confirms this heuristics (compare Figure 1).

**Theorem 1.1 (Non-Landing Parameter Rays).** The accumulation set of every parameter ray accumulating on the boundary of a hyperbolic component of odd period (except period one) of $\mathcal{M}_d^*$ contains an arc of positive length.

The wiggling behavior is stated precisely and proved in Section 3. Section 4 gives a complete description of which rays of the multicorns land and which ones have this wiggling property, in terms of the combinatorics of the angle.

It is worth noting that non-trivial accumulation of some stretching rays in the parameter space of real cubic polynomials was proved by Nakane and Komori in [KN04] by different methods. It has been empirically observed that there are infinitely many small tricorn-like sets in the parameter space of real cubic polynomials (and possibly elsewhere). Our techniques can be naturally generalized to these small tricorn-like sets (of course, these need to be defined rigorously) yielding the non-trivial accumulation of the stretching rays that approach the boundaries of these small tricorn-like sets.

In the last section, we will study another topological property of the multicorns. In [HS14], it was asked whether the parabolic arcs of the multicorns can contain undecorated sub-arcs. In section 5 we answer this question affirmatively by showing that:

**Theorem 1.2 (Undecorated Arcs on The Boundary).** For $d \geq 2$, every period 1 parabolic arc of $\mathcal{M}_d^*$ contains an undecorated sub-arc.

There are some interesting consequences of the previous theorem. One of them is that the centers of hyperbolic components as well as the Misiurewicz parameters (with strictly pre-periodic critical points) are not dense on the boundary of $\mathcal{M}_d^*$ (Corollary 5.1). This is another item in the list of the topological differences between the multicorns and the multibrot sets. The fact that the centers of hyperbolic components (or the Misiurewicz parameters) are dense on the boundaries of the multibrot sets follows by an easy application of Montel’s theorem.

Equidistribution problems are of great interest in the study of the parameter spaces of polynomials (or rational maps) and a good deal of work has been done in this direction in the recent years (see [DF07, Duj14, Duj09, FG13, Gau14] and the references therein). In this spirit, we deduce in Corollary 5.2 that the centers of the hyperbolic components of the multicorns are not equidistributed with respect to their harmonic measures. The situation is, once again, opposite to that for the multibrot sets (see [Lev90]). This essentially tells that the harmonic measures of the multicorns are rather uninteresting from a dynamical point of view. However,
the question of finding a more dynamically meaningful measure (an analogue of the bifurcation measure) for the multicorns remains open.

We would like to thank Adam Epstein for many helpful discussions. Special thanks go to Dierk Schleicher for his useful suggestions to improve the original manuscript and for allowing us to reproduce one of the figures from [HS14]. The second author gratefully acknowledges the support of Deutsche Forschungsgemeinschaft DFG during this work.

2. PARABOLIC POINTS IN ANTIHOLONOMIC DYNAMICS

In this section, we briefly recall some known results on antiholomorphic dynamics and their parameter spaces, which we will need for in the rest of the paper. The next result was proved by Nakane (see [Nak93]).

**Theorem 2.1 (Real-Analytic Uniformization).** The map \( \Phi : \mathbb{C} \setminus M^*_d \to \mathbb{C} \setminus \mathbb{D} \), defined by \( c \mapsto \phi_c(c) \) (where \( \phi_c \) is the Böttcher coordinate near \( \infty \) for \( f_c \)) is a real-analytic diffeomorphism. In particular, the multicorns are connected.

The previous theorem also allows us to define parameter rays of the multicorns.

**Definition (Parameter Ray).** The parameter ray at angle \( \theta \) of the multicorn \( M^*_d \), denoted by \( R^d_{\theta} \), is defined as \( \{ \Phi^{-1}(re^{2\pi i \theta}) : r > 1 \} \), where \( \Phi \) is the real-analytic diffeomorphism from the exterior of \( M^*_d \) to the exterior of the closed unit disc in the complex plane constructed in Theorem 2.1.

**Remark.** Some comments should be made on the definition of the parameter rays. Observe that unlike the multibrot sets, the parameter rays of the multicorns are not defined in terms of the Riemann map of the exterior. In fact, the Riemann map of the exterior of \( M^*_d \) has no obvious dynamical meaning. We have defined the parameter rays via a dynamically defined diffeomorphism of the exterior of \( M^*_d \) and it is easy to check that this definition of parameter rays agree with the notion of stretching rays in the family of polynomials \( (z^d + a)^d + b \).

One of the main features of the antiholomorphic parameter spaces is the existence of abundant parabolics. In particular, the boundaries of odd period hyperbolic components of the multicorns consist only of parabolic parameters.

**Lemma 2.2 (Indifferent Dynamics of Odd Period).** The boundary of a hyperbolic component of odd period \( k \) consists entirely of parameters having a parabolic orbit of exact period \( k \). In local conformal coordinates, the \( 2k \)-th iterate of such a map has the form \( z \mapsto z + z^{q+1} + \ldots \) with \( q \in \{1, 2\} \).

**Proof.** See [MNS14, Lemma 2.5]. \( \square \)

This leads to the following classification of odd periodic parabolic points.

**Definition (Parabolic Cusps).** A parameter \( c \) will be called a cusp point if it has a parabolic periodic point of odd period such that \( q = 2 \) in the previous lemma. Otherwise, it is called a simple parabolic parameter.

In holomorphic dynamics, the local dynamics in attracting petals of parabolic periodic points is well-understood: there is a local coordinate \( \zeta \) which conjugates the first-return dynamics to the form \( \zeta \mapsto \zeta + 1 \) in a right half place (see Milnor [Mil06, Section 10]). Such a coordinate \( \zeta \) is called a Fatou coordinate. Thus the quotient of the petal by the dynamics is isomorphic to a bi-infinite cylinder, called the Ecalle
Note that Fatou coordinates are uniquely determined up to addition by a complex constant.

In antiholomorphic dynamics, the situation is at the same time restricted and richer. Indifferent dynamics of odd period is always parabolic because for an indifferent periodic point of odd period \( k \), the \( 2k \)-th iterate is holomorphic with positive real multiplier, hence parabolic as described above. On the other hand, additional structure is given by the antiholomorphic intermediate iterate.

**Lemma 2.3 (Fatou Coordinates).** Suppose \( z_0 \) is a parabolic periodic point of odd period \( k \) of \( f_c \) with only one petal (i.e. \( c \) is not a cusp) and \( U \) is a periodic Fatou component with \( z_0 \in \partial U \). Then there is an open subset \( V \subset U \) so that for every \( z \in U \), there is an \( n \in \mathbb{N} \) with \( f_c^{2nk}(z) \in V \). Moreover, there is a univalent map \( \psi: V \to \mathbb{C} \) with \( \psi(f_c^{2nk}(z)) = \psi(z) + 1/2 \), and \( \psi(V) \) contains a right half plane. This map \( \psi \) is unique up to horizontal translation.

**Proof.** See [HS14, Lemma 2.3]. \( \square \)

The map \( \psi \) will be called an antiholomorphic Fatou coordinate for the petal \( V \). The antiholomorphic iterate interchanges both ends of the Ecalle cylinder, so it must fix one horizontal line around this cylinder (the equator). The change of coordinate has been so chosen that the equator is the projection of the real axis. We will call the vertical Fatou coordinate the Ecalle height. Its origin is the equator. Of course, the same can be done in the repelling petal as well. The existence of this distinguished real line, or equivalently an intrinsic meaning to Ecalle height, is specific to antiholomorphic maps.

The Ecalle height of the critical value plays a special role in antiholomorphic dynamics. The next theorem proves the existence of real-analytic arcs of non-cusp parabolic parameters on the boundaries of odd period hyperbolic components of the multicorns.

**Theorem 2.4 (Parabolic arcs).** Let \( c_0 \) be a parameter such that \( f_{c_0} \) has a parabolic orbit of odd period and suppose that \( c_0 \) is not a cusp. Then \( c_0 \) is on a parabolic arc in the following sense: there exists a real-analytic arc of non-cusp parabolic parameters \( c(t) \) (for \( t \in \mathbb{R} \)) with quasiconformally equivalent but conformally distinct dynamics of which \( c_0 \) is an interior point and the Ecalle height of the critical value of \( f_{c(t)} \) is \( t \).

**Proof.** See [MNS14, Theorem 3.2]. \( \square \)

Following [MNS14], we classify parabolic arcs into two types.

**Definition (Root Arcs and Co-Root Arcs).** We call a parabolic arc a root arc if, in the dynamics of any parameter on this arc, the parabolic orbit disconnects the Julia set. Otherwise, we call it a co-root arc.

The structure of the hyperbolic components of odd period plays an important role in the global topology of the parameter spaces. Let \( H \) be a hyperbolic component of odd period \( k \neq 1 \) (with center \( \tilde{c} \)) of the multicorn \( \mathcal{M}_d^* \). The first return map of the closure of the characteristic Fatou component of \( \tilde{c} \) fixes exactly \( d + 1 \) points on its boundary. Only one of these fixed points disconnect the Julia set and is the landing point of two distinct dynamical rays at \( 2k \) periodic angles. Let the set of the angles of these two rays be \( S' = \{ \alpha_1, \alpha_2 \} \). Each of the remaining \( d \) fixed points is the landing point of precisely one dynamical ray at a \( k \) periodic angle; let the
collection of the angles of these rays be $S = \{\theta_1, \theta_2, \ldots, \theta_d\}$. We can, possibly after renumbering, assume that $0 < \alpha_1 < \theta_1 < \theta_2 < \cdots < \theta_d < \alpha_2$ and $\alpha_2 - \alpha_1 < \frac{1}{H}$.

By [MNS14, Theorem 1.2], $\partial H$ is a simple closed curve consisting of $d + 1$ parabolic arcs and the same number of cusp points such that every arc has two cusp points at its ends. Exactly 1 of these $d + 1$ parabolic arcs is a root arc and the parameter rays at angles $\alpha_1$ and $\alpha_2$ accumulate on this arc. The rest of the $d$ parabolic arcs are co-root arcs and each of them contains the accumulation set of exactly one $R_d^\theta$. Furthermore, the rational lamination remains constant throughout the closure of the hyperbolic component $H$ except at the cusp points.

The main technical tool used in the proof of the non-trivial accumulation of parameter rays is the perturbation of antiholomorphic parabolic points. For the rest of this section, we will assume that $c$ is a non-cusp parabolic parameter of odd period $k$ lying on the parabolic arc $C$ on the boundary of the hyperbolic component $H$.

We briefly recall the concepts of near-parabolic antiholomorphic Fatou coordinates and the transit map following [HS14, §4]. There exists an open neighborhood $U$ of $c$ such that in $U^- := U \setminus \overline{H}$, the characteristic parabolic point splits into two simple periodic points and the Fatou coordinates persist throughout $U^-$. More precisely, there exists an incoming domain $V_{c'}^{\text{in}}$ and an outgoing domain $V_{c'}^{\text{out}}$ separated by a curve joining the two simple periodic points (called the “gate”) such that the points in the incoming domain eventually transit through the gate and escape to the outgoing domain (as shown in Figure 2). Furthermore, for every $c' \in U^-$, the quotients $C_{c'}^{\text{in}} := V_{c'}^{\text{in}}/f^{2k}$ and $C_{c'}^{\text{out}} := V_{c'}^{\text{out}}/f^{2k}$ (the quotients of $V_{c'}^{\text{in}}$ and $V_{c'}^{\text{out}}$ by the dynamics, identifying points that are on the same finite orbits entirely in $V_{c'}^{\text{in}}$ or in $V_{c'}^{\text{out}}$) are complex annuli isomorphic to $\mathbb{C}/\mathbb{Z}$. The isomorphisms are given by Fatou coordinates which depend continuously on the parameter throughout $U^-$. 

![Figure 2. The typical dynamical picture after perturbation of the parabolic point. (Figure provided by Dierk Schleicher)](image)

Since the map $f_{c'}^{2k}$ commutes with $f_{c'}^{2k}$, it induces antiholomorphic self-maps from $C_{c'}^{\text{in}}$ (respectively $C_{c'}^{\text{out}}$) to itself. As $f_{c'}^{2k}$ interchanges the two periodic points at the ends of the gate, it interchanges the ends of the cylinders, so it must fix a closed geodesic in the cylinders $\mathbb{C}/\mathbb{Z}$. This is similar to the situation at the
parabolic parameter $c$, so we will call this invariant geodesic the equator. One can choose complex coordinates in these cylinders for which the equator has imaginary part 0 and thus we can again define Ecalle height as the imaginary part in these coordinates. We will denote the Ecalle height of a point $z \in C_{c'}^\text{in/out}$ by $E(z)$.

The incoming and the outgoing cylinders are isomorphic to each other by a natural biholomorphism, namely $f \circ 2k_{c'}$. This isomorphism is called the “Transit map” and denoted by $T_{c'}$. The transit map clearly depends continuously on the parameter $c \in U$. It maps the fixed geodesic of the incoming cylinder to the fixed geodesic of the outgoing cylinder and preserves the upper (respectively lower) ends of the cylinders. Thus it must preserve Ecalle heights. The existence of this special isomorphism allows us to relate the Ecalle heights of points in the incoming and the outgoing cylinders and is one of the principal tools in our study.

Finally, we are in a position to state the key technical lemma that helps us to transfer the dynamical information at a parabolic parameter to the parameter plane. One can define the disjoint unions:

\[
C^\text{in} = \bigsqcup_{c' \in U^-} C_{c'}^\text{in} \quad \text{and} \quad C^\text{out} = \bigsqcup_{c' \in U^-} C_{c'}^\text{out}
\]

which are topologically trivial bundles with fibers isomorphic to $\mathbb{C}/\mathbb{Z}$. Choose a smooth real curve $s \mapsto c(s)$ in $U$ (in parameter space), parametrized by $s \in [0, \delta]$ for some $\delta > 0$, with $c(0) = c$ and $c(s) \in U^-$ for $s > 0$. Choose a smooth curve $s \mapsto \zeta(s)$ (in the dynamical planes, typically the critical value), also defined for $s \in [0, \delta]$ such that $\zeta(s) \in V_{c(s)}^\text{in}$ for all $s \in [0, \delta]$. Then $s \mapsto \zeta(s)$ induces a map $\sigma : [0, \delta] \to C^\text{in}$ with $\sigma(s) \in C_{c(s)}^\text{in}$. The following was proved in [HS14, Proposition 4.8].

**Lemma 2.5 (Limit of Perturbed Fatou Coordinates).** The curve $\gamma := s \mapsto T_{c(s)}(\sigma(s))$ in $C^\text{out}$, parametrized by $s \in (0, \delta)$, spirals as $s \downarrow 0$ towards the circle in $C^\text{out}$ at Ecalle height $E(\sigma(0))$.

The other technical ingredient in our proof is the asymptotic behavior of the horn maps as one approaches to the ends of a parabolic arc $C$. Once again, we exploit the symmetry between the upper and lower ends of the Ecalle cylinders provided by the antiholomorphic return map. For the sake of completeness, we include the basic definitions and properties of the horn maps. More comprehensive accounts on these ideas can be found in [BE02, §2].

The characteristic parabolic point $z_c$ (say) of $f_c$ has exactly two petals, one attracting and one repelling (denoted by $P_{\text{att}}$ and $P_{\text{rep}}$ respectively). The intersection of the two petals has two connected components. We denote by $U^+$ the connected component of $P_{\text{att}} \cap P_{\text{rep}}$ whose image under the Fatou coordinates is contained in the upper half-plane and by $U^-$ the one whose image under the Fatou coordinates is contained in the lower half-plane. We define the “sepal” $S^\pm$ by

\[
S^\pm = \bigcup_n f_c^o 2nk(U^\pm)
\]

Note that each sepal contains a connected component of the intersection of the attracting and the repelling petals and they are invariant under the first holomorphic return map of the parabolic point. The attracting Fatou coordinate $\psi_{\text{att}}$ (respectively the repelling Fatou coordinate $\psi_{\text{rep}}$) can be extended to $P_{\text{att}} \cup S^+ \cup S^-$. 


Proof. It follows from the symmetry of the two lifted horn maps that they conjugate the first holomorphic return map to the translation $\zeta \mapsto \zeta + 1$.

**Definition (Lifted horn maps).** Let us define $V^- = \psi^-_\text{rep}(\mathcal{S}^-)$, $V^+ = \psi^+_\text{rep}(\mathcal{S}^+)$, $W^- = \psi^-_\text{att}(\mathcal{S}^-)$ and $W^+ = \psi^+_\text{att}(\mathcal{S}^+)$. Then, denote by $H^-_c : V^- \to W^-$ the restriction of $\psi^-_\text{att} \circ \psi^-_\text{rep}^{-1}$ to $V^-$ and by $H^+_c : V^+ \to W^+$ the restriction of $\psi^+_\text{att} \circ \psi^+_\text{rep}^{-1}$ to $V^+$. We refer to $H_c^\pm$ as lifted horn maps for $f_c$ at $z_c$.

The regions $V^\pm$ and $W^\pm$ are invariant under translation by 1. Moreover, the asymptotic development of the Fatou coordinates implies that the regions $V^+$ and $W^+$ contain an upper half-plane, whereas the regions $V^-$ and $W^-$ contain a lower half-plane. Consequently, under the projection $\pi : \zeta \mapsto w = \exp(2it\pi\zeta)$, the regions $V^+$ and $W^+$ project to punctured neighborhoods $V^+$ and $W^+$ of 0, whereas $V^-$ and $W^-$ project to punctured neighborhoods $V^-$ and $W^-$ of $\infty$.

The lifted horn maps $H^\pm_c$ satisfy $H^\pm_c(\zeta + 1) = H^\pm_c(\zeta) + 1$ on $V^\pm$. Thus, they project to mappings $h^\pm_c : V^\pm \to W^\pm$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V^\pm & \xrightarrow{H^\pm_c} & W^\pm \\
\downarrow{\pi} & & \downarrow{\pi} \\
V^\pm & \xrightarrow{h^\pm_c} & W^\pm
\end{array}
\]

**Definition (Horn Maps).** The maps $h^\pm_c$ are called horn maps for $f_c$ at $z_c$.

It is well-known that $\exists \eta_c, \eta'_c \in \mathbb{C}$ such that $H^+_c(\zeta) \approx \zeta + \eta_c$ when $\Im(\zeta) \to +\infty$ and $H^-_c(\zeta) \approx \zeta + \eta'_c$ when $\Im(\zeta) \to -\infty$. This proves that $h^+_c(w) \to 0$ as $w \to 0$. Thus, the horn map $h^+_c$ extends analytically to 0 by $h^+_c(0) = 0$. One can show similarly that the horn map $h^-_c$ extends analytically to $\infty$ by $h^-_c(\infty) = \infty$. Observe that the constants $\eta_c$ and $\eta'_c$ are, in general, not well-defined as they depend on particular normalizations of the Fatou coordinates. However, in the antiholomorphic situation, we can and will choose the normalizations of Fatou coordinates described in Lemma 2.3 and these Fatou coordinates conjugate the first (antiholomorphic) return map in both petals to $\zeta \mapsto \zeta + 1/2$. This choice involves an adjustment of the vertical degree of freedom of the Fatou coordinates. Consequently, the two lifted horn maps $H^+_c$ and $H^-_c$ are conjugated to each other by $\zeta \mapsto \zeta + \frac{1}{2}$. It follows that $\eta'_c = \overline{\eta_c}$. Note that with the chosen normalizations of the Fatou coordinates, the imaginary parts of $\eta_c$ and $\eta'_c$ (which are the asymptotic vertical translation constants of the lifted horn maps) become well-defined real numbers. We need to study the asymptotic behavior of $\eta_c$ as $c$ tends to the ends of the parabolic arc $\mathcal{C}$.

**Lemma 2.6.** $\Im(\eta_c) \to +\infty$ as $c$ tends to the ends of the parabolic arc $\mathcal{C}$.

**Proof.** It follows from the symmetry of the two lifted horn maps that the two horn maps $h^+_c$ and $h^-_c$ which are defined respectively in neighborhoods of 0 and of $\infty$ are conjugated by $w \mapsto -1/w$ and they asymptotically look like $w \mapsto \exp(2\pi in_c)w$ and $w \mapsto \exp(2\pi i/\eta_c)w$ respectively. Clearly, $(h^+_c)'(0) = \exp(2\pi in_c)$ and $(h^-_c)'(\infty) = \exp(-2\pi i/\eta_c)$. By [HLS12 Proposition 1], $\exp(-4\pi \Im(\eta_c)) = (h^+_c)'(0)/(h^-_c)'(\infty) = \exp(4\pi^2(1 - \kappa_c))$, where $\kappa_c$ is the holomorphic fixed-point index of $f_c^{2k}$ at the parabolic fixed point $z_c$. Towards the ends of a parabolic arc, the fixed-point index $\kappa_c$ at the characteristic parabolic point tends to $+\infty$ in $\mathbb{R}$ ([HLS14 Proposition 3.7]). Hence, $\Im(\eta_c) \to +\infty$ towards the ends of a parabolic arc. $\square$
3. Wigglers of Parameter Rays

The goal of this section is to prove Theorem 1.1. We begin with a couple of definitions.

**Definition (Accumulation Set of a Ray).** The accumulation set of a parameter ray \( R^d_t \) of \( \mathcal{M}_d^* \) is defined as \( L_{\mathcal{M}_d^*}(t) := \overline{R^d_t} \cap \mathcal{M}_d^* \).

**Definition (Rational Lamination).** The rational lamination of an anti-polynomial \( f_c \) is defined as an equivalence relation on \( \mathbb{Q}/\mathbb{Z} \) such that \( \theta_1 \sim \theta_2 \) if and only if the dynamical rays \( R_c(\theta_1) \) and \( R_c(\theta_1) \) land at the same point of \( J(f_c) \). It is denoted by \( \mathcal{R}(f_c) \).

**Lemma 3.1.** The parameter rays \( \{ R^d_t : t = 0, \frac{1}{d+1}, \frac{2}{d+1}, \ldots, \frac{d}{d+1} \} \) of \( \mathcal{M}_d^* \) land.

**Proof.** Let \( \omega = \exp(\frac{2\pi i}{d+1}) \). The anti-polynomials \( f_c \) and \( f_{\omega c} \) are conformally conjugate via the linear map \( z \mapsto \omega z \). It follows that \( \mathcal{M}_d^* \) has a \((d+1)-fold rotational symmetry and \( f_c \sim f_{\omega c} \sim f_{\omega^2 c} \sim \cdots \sim f_{\omega^{d-1} c} \). Also, \( K(f_{\omega c}) = \omega^j K(f_c) \) and \( J(f_{\omega^j c}) = \omega^j J(f_c) \). The Böttcher maps are related by \( \omega^j \phi_c(z) = \phi_{\omega^j c}(\omega^j z) \). This reads, in terms of external rays, as \( \omega^j R_c(\theta) = R_{\omega^j c}(\theta + \frac{j}{d+1}) \).

The map \( \Phi : \mathbb{C} \setminus \mathcal{M}_d^* \to \mathbb{C} \setminus \mathbb{D} \), defined by \( c \mapsto \phi_c(c) \) (where \( \phi_c \) is the Böttcher coordinate near \( \infty \)) is a real-analytic diffeomorphism. This map defines the parameter rays of the multicorns. It follows that \( \Phi(\omega^j c) = \omega^j \Phi(c) \).

Now,

\[
R^d_0 = \{ c \in \mathbb{C} : \Phi(c) = r, r > 1 \}
= \{ c \in \mathbb{C} : \frac{1}{\omega^j} \Phi(\omega^j c) = r, r > 1 \}
= \{ c \in \mathbb{C} : \Phi(\omega^j c) = \exp\left(\frac{2\pi i j}{d+1}\right), r > 1 \}
= \{ c \in \mathbb{C} : \omega^j c \in R^d_{\frac{j}{d+1}} \}
= \frac{1}{\omega^j} R^d_{\frac{j}{d+1}}.
\]

Hence, \( \omega^j R^d_0 = R^d_{\frac{j}{d+1}} \). Since \( R^d_0 \subset \mathbb{R} \), it lands. It follows that \( R^d_{\frac{j}{d+1}} \), being the image of \( R^d_0 \) under a rotation, must land as well. \( \square \)

Having taken care of the parameter rays at fixed angles, we now turn our attention to the parameter rays that accumulate on the boundaries of hyperbolic components of odd period greater than 1. The proof of Theorem 1.1 is carried out in various steps. Let us sketch the key ideas of the proof to stop the readers from getting lost in the technicalities. We will stick to the terminologies of Section 2.

Let \( t \in S \cup S' \) and \( C \) be the parabolic arc where the parameter ray \( R^d_t \) accumulates. For every parameter on \( C \), the dynamical ray at angle \( t \) lands at the characteristic parabolic point through the unique repelling petal. We first show that if for some \( c \in C \), the dynamical ray \( R_c(t) \) projects to a horizontal line under the repelling Fatou coordinate, then the rational lamination of \( f_c \) must be invariant under a certain affine transformation. This is achieved by considering a pair of dynamically meaningful involutions in the repelling petal. A simple combinatorial exercise then shows that such invariant laminations can never exist when the period
of $H$ is greater than 1. This proves that for every parameter on $C$, the projection of the dynamical ray $R_c(t)$ under the repelling Fatou coordinate must traverse a non-degenerate interval of Ecalle heights. The final part of the proof involves a careful parabolic perturbation argument which allows us to transfer the variation of Ecalle heights of the dynamical rays at angle $t$ to the wiggling of the corresponding ray in the parameter plane.

We denote the repelling Fatou coordinate at the characteristic parabolic point of $f_c$ by $\psi_{\text{rep}} : \mathcal{P}_{\text{rep}} \to \mathbb{H}_{\text{Left}}$ and the Böttcher coordinate by $\phi_c : \mathcal{A}_\infty(f_c) \to \mathbb{C} \setminus \mathbb{D}$.

**Lemma 3.2** (Invariance of Lamination). Let $c \in C$ where $C$ is a parabolic arc on the boundary of $H$. Suppose that the dynamical ray $R_c(t)$ projects to a horizontal line under the repelling Fatou coordinate. Then the rational lamination $R L (f_c)$ is invariant under the transformation $s \mapsto 2t - s$.

**Remark.** The projection of the basin of infinity onto the repelling Ecalle cylinder is a conformal annulus bounded by fractal structures (the Julia set and the decorations thereof) from above and below and the projection of the dynamical ray is the unique simple closed geodesic (the core curve) of this conformal annulus. The core curve is a round circle if and only if the annulus is symmetric with respect to this round circle and heuristically speaking, this is an extremely unlikely situation for a polynomial. This lemma essentially tells that such a miracle could happen only if the polynomial had some strong global symmetry.

**Proof.** Let $L$ be the the horizontal line which is the image of the dynamical ray $R_c(t)$ under the repelling Fatou coordinate. We denote the reflection in $\mathbb{H}_{\text{Left}}$ w.r.t. the line at angle $t$ (denoted by $\iota_1$). This gives a local antiholomorphic diffeomorphism $(\psi_{\text{rep}}^{-1} \circ \iota_1 \circ \psi_{\text{rep}})$ in the domain of definition of the repelling Fatou coordinate. On the other hand, consider the reflection in the Böttcher coordinate with respect to the radial line at angle $t$ (denoted by $\iota_2$). This gives an antiholomorphic diffeomorphism $(\phi_c^{-1} \circ \iota_2 \circ \phi_c)$ in the basin of infinity $\mathcal{A}_\infty(f_c)$, preserving $R_c(t)$ and mapping a dynamical ray $R_c(s)$ to $R_c(2t - s)$ (see Figure 3). We will first show that these two diffeomorphisms agree on $\mathcal{P}_{\text{rep}} \cap \mathcal{A}_\infty(f_c)$.

Let $\psi_{\text{rep}} \circ \phi_c^{-1}$ be defined on a domain $D$ which we can assume to be symmetric w.r.t. the radial line at angle $t$ (under the map $\iota_2$). Since $\psi_{\text{rep}} \circ \phi_c^{-1}$ maps the radial line at angle $t$ to a horizontal line in the right half-plane, the Schwarz reflection principle implies that $\psi_{\text{rep}} \circ \phi_c^{-1}(w) = \iota_1 \circ \psi_{\text{rep}} \circ \phi_c^{-1} \circ \iota_2(w)$ for all $w \in U$. This implies that $\phi_c^{-1} \circ \iota_2 \circ \phi_c = \psi_{\text{rep}}^{-1} \circ \iota_1 \circ \psi_{\text{rep}}$ on $\mathcal{P}_{\text{rep}} \cap \mathcal{A}_\infty(f_c)$. Hence, the local antiholomorphic diffeomorphism $(\psi_{\text{rep}}^{-1} \circ \iota_1 \circ \psi_{\text{rep}})$ in the repelling petal maps a co-landing ray pair to another co-landing ray pair. It follows that for rational angles $s_1$ and $s_2$, if $R_c(t + s_1)$ and $R_c(t + s_2)$ land at the same point close to the characteristic parabolic point, then so do $R_c(t - s_1)$ and $R_c(t - s_2)$. This proves the local invariance of the rational lamination under the map $s \mapsto 2t - s$.

We now spread this local invariance to the entire rational lamination. Let the rational dynamical rays $R_c(s_1)$ and $R_c(s_2)$ co-land. By the density of iterated pre-images in the Julia set, there exists a co-landing rational ray pair $R_c(s_1')$ and $R_c(s_2')$ such that their common landing point lies in $\mathcal{P}_{\text{rep}}$ and $(-d)^{2m} s_1' = s_1$, $(-d)^{2m} s_2' = s_2$, for some $m \in \mathbb{N}$. By the local invariance, $R_c(2t - s_1')$ and $R_c(2t - s_2')$ co-land. By continuity, $f_c^{2m} (R_c(2t - s_1')) = R_c(2t - s_1)$ and $f_c^{2m} (R_c(2t - s_2')) = R_c(2t - s_2)$ co-land as well. This completes the proof of the lemma. \(\Box\)
Figure 3. Top: The reflections with respect to the straight line $L$ (respectively the radial line at angle $t$) defined in the left-half plane (respectively in the exterior of the closed unit disk). Bottom: These reflections transported to the dynamical plane via the Fatou (respectively the Böttcher) coordinates agree on their common domain of definitions, namely on $P_{\text{rep}} \cap A_{\infty}(f_c)$.

The proof of the above lemma does not use any fact specific to antiholomorphic dynamics, and hence, the conclusion of the lemma holds for any polynomial of degree $d \geq 2$ with connected Julia set. In general, one does not expect the rational lamination of a polynomial to be invariant under such an affine transformation. This can be easily seen in our case, which is the content of the following:

**Lemma 3.3 (No Invariant Lamination).** Let $c \in \mathcal{C} \subset \partial \mathcal{H}$. Then the rational lamination $\mathcal{RL}(f_c)$ cannot be invariant under the transformation $s \mapsto 2t - s$.

**Proof.** We will assume that the rational lamination $\mathcal{RL}(f_c)$ is invariant under the given transformation and arrive at a contradiction.

**Case 1. $t \in S'$**

Without loss of generality, we assume that $t = \alpha_1$. In the dynamical plane of $c$, $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land at a common point (namely, at the characteristic parabolic point). By the invariance, the dynamical rays at angles $\alpha_1$ and $2\alpha_1 - \alpha_2$ must land at a common point as well. This is clearly impossible since exactly two rays land at the characteristic parabolic point.
Case 2. \( t \in S, \, 2t \neq \alpha_1 + \alpha_2 \)

Note that in the dynamical plane of \( f_c \), the dynamical rays \( R_c(\alpha_1) \) and \( R_c(\alpha_2) \) land at a common point. By the invariance, the dynamical rays at angles \( 2t - \alpha_1 \) and \( 2t - \alpha_2 \) must land at a common point as well. By our assumption, the rays \( R_c(2t - \alpha_1) \) and \( R_c(2t - \alpha_2) \) lie in different connected components of \( \mathbb{C} \setminus \left( R_c(\alpha_1) \cup R_c(\alpha_2) \right) \). This forces four different rays \( R_c(\alpha_1), \, R_c(\alpha_2), \, R_c(2t - \alpha_1) \) and \( R_c(2t - \alpha_2) \) to land at a common point (we have used the fact that \( 0 < \alpha_1 < t < \alpha_2 \) and \( \alpha_2 - \alpha_1 < 1/2 \)), which contradicts [Muk14, Theorem 2.6].

![Figure 4](image)

**Figure 4.** Left: Parameter rays accumulating on the boundary of a hyperbolic component of period 5 of the tricorn. Right: The corresponding dynamical rays landing on the boundary of the characteristic Fatou component in the dynamical plane of a parameter on the boundary of the same hyperbolic component.

Case 3. \( t \in S, \, 2t = \alpha_1 + \alpha_2 \)

We have to work a little harder in this case. Note that for any \( i \in \{0, 1, \cdots, d\} \), the parameter ray \( R^d_{\frac{i}{d+1}} \) lands on the parabolic arc \( C_i \) on the boundary of the period 1 hyperbolic component. Define the wake \( W_i \) to be the connected component of \( \mathbb{C} \setminus \{ R^d_{\frac{i}{d+1}} \cup R^d_{\frac{i+1}{d+1}} \cup C_i \cup C_{i+1} \} \) not containing 0. Then each parameter in \( W_i \) has a repelling periodic orbit admitting the orbit portrait \( P_i = \{ \frac{i}{d+1}, \frac{i+1}{d+1} \} \) such that the dynamical rays at angles \( \frac{i}{d+1} \) and \( \frac{i+1}{d+1} \) together with their common landing point separate the critical value from the critical point.

It is easy to see that \( H \) must be contained in some \( W_j \). In particular, \( R_c(\frac{j}{d+1}) \) and \( R_c(\frac{j+1}{d+1}) \) land at the same point (and no other ray lands there) and \( \frac{j}{d+1} < t < \frac{j+1}{d+1} \). By the invariance property, the rays \( R_c(2t - \frac{j}{d+1}) \) and \( R_c(2t - \frac{j+1}{d+1}) \) must land at the same point as well. By arguing as in Case 2, we can conclude that \( t = \frac{2j+1}{2(d+1)} \).

Since \( t \) is a periodic angle under multiplication by \(-d\), it follows that \( d \) must be odd.
A simple computation now shows that \((-d)^2t = t\), which contradicts the fact that the period of \(t\) is an odd integer \(k \neq 1\). This completes the proof of the lemma. □

**Remark.** Case 3 of the previous lemma never occurs for the tricorn. For any hyperbolic component \(H\) of odd period \(k(\neq 1)\) of the tricorn, we have \(S = \{\theta_1, \theta_2\}\), \(S' = \{\alpha_1, \alpha_2\}\) where \((1 + 2^k) \cdot (\theta_1 - \alpha_1) = (\alpha_2 - \alpha_1) = (1 + 2^k) \cdot (\alpha_2 - \theta_2)\) (Compare [MNS14] Corollary 5.12 and Figure [4]).

**Proof of Theorem 1.1.** Let \(\mathcal{R}_t^d\) accumulates on \(\mathcal{C} \subset \partial H\).

**Case 1.** \(t \in S\).

Let \(c_0\) be the parameter on \(\mathcal{C}\) whose critical value has (incoming) Ecalle height 0. Note that \(\mathcal{C}\) must be a co-root arc of \(H\). By Lemma 3.2 and Lemma 3.3, the projection of the dynamical ray \(R_{c_0}(t)\) under the repelling Fatou coordinate must traverse a non-degenerate interval of (outgoing) Ecalle heights. Since this dynamical ray is fixed by the first antiholomorphic return map, the interval of (outgoing) Ecalle heights traversed by it must be of the form \([-h, h]\) for some \(h > 0\). Since the rays \(R_{c}(t)\) depend uniformly continuously on the parameter \(c\), and since the projection into Ecalle cylinders is also continuous, we can choose a small neighborhood \(U\) of \(c_0\) such that for all \(c \in \overline{U} \setminus H\), the projection of the rays \(R_{c}(t)\) into the Ecalle cylinders traverse (outgoing) heights at least \([-h + \varepsilon, h - \varepsilon]\).

(Note that in the outgoing cylinder of \(c_0\), \(R_{c_0}(t)\) traverses Ecalle heights \([-h, h]\). To transfer the variation of Ecalle height of \(R_{c}(t)\) to wiggling of the parameter ray \(\mathcal{R}^d_t\), we employ [HS14] Proposition 4.8).

Let \(c_h \in \mathcal{C}\) be the parameter on \(\mathcal{C}\) whose critical value has (incoming) Ecalle height \(h\). We pick a \(c_h \in U\) with \(h' \in [-h + 2\varepsilon, h - 2\varepsilon]\) and choose any smooth path \(\gamma : [0, \delta] \to \overline{U}\) with \(\gamma(0) = c_{h'}\) but so that, except for \(\gamma(0)\), the path avoids closures of hyperbolic components of period \(k\) and so that the path is transverse to \(\mathcal{C}\) at \(c_{h'}\).

For \(s \in [0, \delta]\), let \(z(s)\) be the critical value. For \(s > 0\), the critical orbit “transits” from the incoming Ecalle cylinder to the outgoing cylinder; as \(s \downarrow 0\), the image of the critical orbit in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to \(h' \in [-h + 2\varepsilon, h - 2\varepsilon]\), while the phase tends to infinity. Therefore, there is \(s \in (0, \delta)\) arbitrarily close to 0 at which the critical value, projected into the incoming cylinder and sent by the transfer map to the outgoing cylinder, lands on the projection of the rays \(R_{\gamma(s)}(t)\). But in the dynamics of \(f_{\gamma(s)}\), this means that the critical value is on one of the dynamical rays \(\mathcal{R}^d_{\gamma(s)}(t)\), so \(\gamma(s)\) is on the parameter ray \(\mathcal{R}_t^d\). Hence, any smooth path starting at \(c_{h'} \in \mathcal{C} \cap U\) (with \(h' \in [-h + 2\varepsilon, h - 2\varepsilon]\)) and living inside \(\overline{U} \setminus \overline{H}\) thereafter, intersects the parameter ray \(\mathcal{R}_t^d\) infinitely often. This proves that \(\mathcal{R}_t^d\) cannot land.

**Case 2.** \(t \in S'\).

Let \(c_h \in \mathcal{C}\) be the parameter on \(\mathcal{C}\) whose critical value has (incoming) Ecalle height \(h\) and the interval of (outgoing) Ecalle heights traversed by \(R_{c_h}(t)\) be \([l_1(c_h), u_t(c_h)]\). By Lemma 3.2 and Lemma 3.3, \(u_t(c_h) > l_1(c_h)\) for every parameter \(c_h\). If we knew that there was a parameter \(c_h \in \mathcal{C}\) with \(h \in (l_t(c_h), u_t(c_h))\) (observe that this was automatic in Case 1), then the proof of the wiggling of the parameter ray \(\mathcal{R}_t^d\) would proceed exactly as in the previous case. Hence, it suffices to prove the existence of such a parameter \(c_h\).
Consider parameters $c_h$ on the parabolic arc $\mathcal{C}$ where $h$ is sufficiently close to $+\infty$. For such parameters, the critical value lies in the intersection of the attracting and the repelling petals. Let $h'$ be the outgoing Ecalle height of such a critical value (or its forward iterate). It follows from Lemma 2.6 that $h \approx h' + \Im(\eta_{c_h})$ for some large positive $\Im(\eta_{c_h})$. Since the critical value lies in the Fatou component and the Fatou component lies above the dynamical ray in the outgoing Ecalle cylinder for $f_c$ with $c \not\in \mathcal{P}$ close to $c_h$, it follows that $h > h' > l_t(c_h)$ for parameters $c_h$ with $h$ is sufficiently close to $+\infty$. Similarly, for parameters $c_h \in \mathcal{C}$ with $h$ is sufficiently close to $-\infty$, $h < h' < u_t(c_h)$. The previous two inequalities together with the fact that the (incoming) Ecalle height of the critical value as well as the interval of (outgoing) Ecalle heights traversed by the dynamical ray at angle $t$ depend continuously on $h$ imply that there is some parameter $c_h$ on $\mathcal{C}$ for which $h \in (l_t(c_h), u_t(c_h))$. This completes the proof of the theorem. \qed

4. A Combinatorial Classification

In this section, we will give an algorithm to find whether a rational parameter ray $\mathcal{R}_t^d$ lands or oscillates based only on the combinatorics of $t$. The following lemma will be useful for this purpose.

Recall that finite collection $\mathcal{P} = \{A_1, A_2, \cdots, A_k\}$ of subsets of $\mathbb{Q}/\mathbb{Z}$ satisfying the five properties of [Muk14, Theorem 2.6] is called a formal orbit portrait.

Lemma 4.1. Let $t \in \mathbb{Q}/\mathbb{Z}$ has period $2k$ under multiplication by $-d$, where $k$ is an odd integer. Consider the collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ given by $\mathcal{P} = \{A_1, A_2, \cdots, A_k\}$, where $A_1 = \{t, (-d)^k t\}$ and $A_{i+1} = (-d)A_i, i \pmod{k}$. Then the parameter ray $\mathcal{R}_t^d$ accumulates on the parabolic root arc of a hyperbolic component of period $k$ if and only if $\mathcal{P}$ satisfies the properties of a formal orbit portrait with characteristic angles $t$ and $(-d)^k t$.

Proof. If $\mathcal{R}_t^d$ accumulates on a sub-arc of the parabolic root arc of an odd period hyperbolic component, then the period of the hyperbolic component must be $k$ and the dynamical rays $R_{(-d)^k t}$ and $R_{(-d)^k t}$ co-land at the dynamical root of the characteristic Fatou component of the center $\hat{c}$ of $H$. In fact, these are the only rays landing there. It is easy to see that $t$ and $(-d)^k t$ generate the orbit portrait $\mathcal{P}$ and they are also the characteristic angles of the orbit portrait.

The proof the converse is similar to [MNS14 Lemma 5.3]. For completeness, we work out the details here. Without loss of generality, we can assume that the characteristic arc of $\mathcal{P}$ is $(t, (-d)^k t)$. Note that any accumulation point of a parameter ray at a $2k$-periodic angle is either a parabolic parameter of odd period $k$ or a parabolic parameter of even period $r$ with $r|2k$ such that the corresponding dynamical ray of period $2k$ lands at the characteristic parabolic point in the dynamical plane of that parameter. Thus the set of accumulation points of $\mathcal{R}_t^d$, for $\theta \in A_1 \cup \cdots \cup A_k$, is contained in $F = \{\text{The closure of the finitely many root arcs of period } k\} \cup \{\text{The finitely many parabolic parameters of even period and of ray period } 2k\}$. Consider the connected components $U_i$ of $\mathbb{C} \setminus \left( \bigcup_{\theta \in A_1 \cup \cdots \cup A_k} \mathcal{R}_t^d \cup F \right)$. There are only finitely many components $U_i$ and they are open. The same rays with angles in $A_1 \cup \cdots \cup A_k$ land at common points throughout every component $U_i$.

Let $U_1$ be the component which contains all parameters $c$ outside $\mathcal{M}_c^\nu$ with external angle $t(c) \in (t, (-d)^k t)$ (there is such a component as $(t, (-d)^k t)$ does
not contain any other angle of $\mathcal{P}$). $U_1$ must have the two parameter rays $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)t}^d$ on its boundary. By the proof of [Muk14 Theorem 3.1], each $c \in U_1 \setminus \mathcal{M}_d^*$ has a repelling periodic orbit admitting the portrait $\mathcal{P}$. If the two parameter rays at angles $t$ and $(-d)^k t$ do not land at a common point or accumulate on a common root arc, then $U_1$ would contain parameters $c$ outside $\mathcal{M}_d^*$ with $t(c) \notin \{t, (-d)^k t\}$. It follows from the remark at the end of [Muk14 §3] that such a parameter can never admit the orbit portrait $\mathcal{P}$, a contradiction. Hence, the parameter rays $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)t}^d$ must land at a common even period parabolic parameter of ray period $2k$ or accumulate on a common root arc of period $k$ of $\mathcal{M}_d^*$. But it is easy to see that if $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)t}^d$ co-land at a parabolic parameter, then the period of its parabolic orbit must be odd, ruling out the first possibility.

\textbf{Theorem 4.2} (Combinatorial Classification). Let $t \in \mathbb{Q}/\mathbb{Z}$.

1) If the period of $t$ under multiplication by $-d$ is $4k$ for some $k \in \mathbb{N}$, then $\mathcal{R}_t^d$ lands at a parabolic parameter on the boundary of a hyperbolic component of period $4k$.

2) If the period of $t$ under multiplication by $-d$ is an odd integer $k$, then it lands if $k = 1$ and accumulates on a sub-arc (of positive length) of a parabolic co-root arc on the boundary of a hyperbolic component of period $k$ otherwise.

3) If the period of $t$ under multiplication by $-d$ is $2k$ for some odd integer $k$, then it accumulates on a sub-arc (of positive length) of the parabolic root arc of a hyperbolic component of period $k$ if and only if the collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ given by $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$, where $A_1 = \{t, (-d)^k t\}$ and $A_{i+1} = (-d)A_i$, $i \pmod{k}$, satisfies the properties of a formal orbit portrait with characteristic angles $t$ and $(-d)^k t$. Otherwise, it lands at a parabolic parameter on the boundary of a hyperbolic component of period $2k$.

4) If $t$ is strictly pre-periodic under multiplication by $-d$, then $\mathcal{R}_t^d$ lands at a Misiurewicz parameter.

\textbf{Proof.} 1) See [MNS14 Lemma 7.2].

2) By [MNS14 Corollary 5.9], every rational parameter ray at an angle $t$ of odd period $k$ lands/accumulates on a sub-arc of a parabolic co-root arc of period $k$. By Theorem [1.1] only the rays at fixed angles land at a single point of a parabolic arc, so the others must accumulate on a sub-arc of positive length.

3) This directly follows from [MNS14 Lemma 7.2], Lemma [4.1] and Theorem [1.1].

4) Arguing as in [Sch00 Theorem 1.1 (3)], one sees that for any limit point $c$ of $\mathcal{R}_t^d$, the critical value $c$ is pre-periodic under $f_c$ with fixed period and pre-period. This implies that all the critical points are strictly pre-periodic (with fixed pre-periods and periods) for the holomorphic polynomial $f_c^{d^2}$. Since the accumulation set of a parameter ray is connected, it now suffices to prove that there are only finitely many parameters with these algebraic data.

In fact, it is not hard to see that there are only finitely pairs of complex numbers $(a, b)$ such that the polynomial $(z^d + a)^d + b$ has strictly pre-periodic critical points with fixed pre-periods and periods. The conditions on the critical points determine a pair of distinct algebraic curves in $\mathbb{C}^2$ and their intersection is contained in the connectedness locus of the family of polynomials of degree $d^2$ (recall that the Julia set of a polynomial is connected if and only if all the critical orbits are bounded). Since the connectedness locus is compact [BHSS], it follows from Bézout’s theorem that the two algebraic curves under consideration must intersect at a finite set of
points. This shows that there are only finitely many Misiurewicz parameters with fixed period and pre-period in the space of degree $d$ unicritical anti-polynomials. □

5. **Undecorated Arcs on The Boundaries of The Multicorns**

Recall that every parabolic arc has, at both ends, an interval of positive length at which a bifurcation from a hyperbolic component of odd period $k$ to a hyperbolic component of period $2k$ occurs. The decorations attached to these period $2k$ components accumulate on sub-arcs of positive length of the parabolic arc [HSI14 Theorem 7.3]. In this section, we will prove that for the parabolic arcs of period 1, the accumulation sets of these decorations do not overlap; they stay at a positive distance away from the Ecalle height 0 parameters.

**Proof of Theorem 1.2.** Every multicorn $\mathcal{M}_d^*$ contains a parabolic arc $C$ of period 1 intersecting the positive real axis at a unique (non-cusp) parameter $c_d = d^{-\frac{d-1}{d}}$. Note that $f_{c_d}$ has a unique parabolic fixed point $d^{-\frac{d}{d-1}}$ on the real line. Due to the rotational symmetries of the multicorns, it suffices to prove the result for this arc. In the dynamical plane of $f_{c_d} = z^d + c_d$, the parabolic fixed point has a unique access through the unique repelling petal and the critical value $c_d$ has incoming Ecalle height 0 (in fact, the incoming and outgoing equators are both contained in the real line and so is the critical value). The projection of the Julia set in the repelling cylinder is a pair of disjoint simple closed curves (the Julia set in this case is simply the boundary of the immediate basin of attraction of the parabolic fixed point) and together they bound a cylinder $C$ of finite modulus. This finite modulus cylinder $C$ is the projection of the basin of infinity in the repelling Ecalle cylinder. We will first show that the cylinder $C$ contains a round cylinder containing the equator.

We choose the repelling Fatou coordinate at the parabolic fixed point so that the equator is mapped to the real line. Since $f_{c_d}$ commutes with complex conjugation, our Fatou coordinates also have the same property. This implies that the upper and the lower components (disjoint simple closed curves) of the projection of the Julia set in the repelling Ecalle cylinder are symmetric with respect to the real line. It follows that both these curves stay at a bounded distance away from the real line; in other words, the projection of the basin of infinity in the repelling Ecalle cylinder contains a round cylinder $S^1 \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Alternatively, it is easy to see that the dynamical ray $R_{c_d}(0)$ (and its image under the repelling Fatou coordinate) is contained in the real line and hence coincides with the equator in the repelling petal. This shows that the equator is contained in the basin of infinity. Hence, the projection of the basin of infinity in the repelling Ecalle cylinder contains a horizontal round circle and thus also contains a round cylinder $S^1 \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

The final step is to transfer this round cylinder to an undecorated sub-arc in the parameter plane. It is known that the basin of infinity can not get too small when $c_d$ is perturbed a little bit (compare [Dou94 Theorem 5.1(a)]). Since the critical value and the Fatou coordinates depend continuously on the parameter, we can choose a small neighborhood $U$ of $c_d$ such that for all $c \in U \setminus H$, the round cylinder $S^1 \times [-\varepsilon/2, \varepsilon/2]$ is contained in projection of the basin of infinity into the repelling Ecalle cylinder (note that in the outgoing cylinder of $c_d$, the round cylinder
$S^1 \times [-\varepsilon, \varepsilon]$ is contained in the projection of the basin of infinity) and such that the critical value of $f_c$ has incoming Ecalle height in $[-\varepsilon/4, \varepsilon/4]$.

We claim that $\overline{U} \setminus \overline{H}$ is contained in the exterior of the multicorns. Indeed, let $c \in \overline{U} \setminus \overline{H}$. In the dynamical plane of $f_c$, the critical orbit of $f_c$ “transits” from the incoming Ecalle cylinder to the outgoing cylinder and the Ecalle height is preserved in the process. By our construction, this would provide with a point of the critical orbit with outgoing Ecalle height in $[-\varepsilon/4, \varepsilon/4]$ in the repelling Ecalle cylinder. But since $c \in \overline{U} \setminus \overline{H}$, any point in the repelling cylinder with (outgoing) Ecalle height in $[-\varepsilon/2, \varepsilon/2]$ is contained in the projection of the basin of infinity. Therefore, the critical orbit is contained in the basin of infinity; i.e. $c \notin M^*$. This implies that $\overline{U} \cap M^*_d \subset \overline{H}$. Hence, $C$ contains a sub-arc containing the Ecalle height 0 parameter, no point of which is a limit point of further decorations; i.e. $C$ has an undecorated sub-arc. □

Remark. a) One can prove the following slightly stronger statement for the tricorn: the parabolic arcs of period 1 and 3 contain undecorated sub-arcs. Indeed, in the dynamical plane of a parameter on a parabolic arc of odd period $k$, the projection of the basin of infinity into the repelling Ecalle cylinder is either an annulus of modulus $\frac{\pi k \ln 2}{2k \ln 2}$ or two disjoint annuli, each of modulus $\frac{\pi k \ln 2}{2k \ln 2}$ (depending on whether the parameter is on a co-root or root arc). For $k = 1$ and 3, this modulus is greater than 1/2; i.e. the corresponding annuli are not too thin. It is well-known (see [BDH04, Theorem I], for instance) that such a conformal annulus contains a round annulus centered at the origin. In other words, there is an interval $I$ of outgoing Ecalle heights such that in the repelling Ecalle cylinder, the round cylinder $S^1 \times I$ is contained in the projection of the basin of infinity. One can now prove the existence of undecorated sub-arcs by using the same technique as in Theorem 1.2.

b) Numerical experiments show that away from the real line, the parabolic arcs of sufficiently high periods of the multicorns do not contain undecorated sub-arcs, rather the accumulation sets of the decorations attached to the two bifurcating hyperbolic components at the ends of such an arc overlap. This overlapping phenomenon would automatically make the corresponding parameter rays wiggle on such arcs. However, we do not know how to prove this statement.

We finish with a few interesting consequences of the previous theorem.

**Corollary 5.1.** The centers of hyperbolic components as well as the Misiurewicz parameters are not dense on the boundary of $M^*_d$.

**Corollary 5.2** (Centers Are Not Equidistributed with Respect to The Harmonic Measure). Let, $A_n = \{ c \in \mathbb{C} : f^{\circ n}_c(0) = 0 \}$. Then, no subsequence of the sequence of measures $\mu_n := \frac{1}{d^{n-1}} \sum_{x \in A_n} \delta_x$ weakly converges to the harmonic measure of $M^*_d$.

Proof. It follows from Theorem 1.2 (and its proof) that each period 1 parabolic arc $C$ contains a sub-arc $C'$ with the following properties:

1. No point of $C'$ is a limit point of centers of hyperbolic components and hence, $C'$ is not charged by any subsequential limit of $\{ \mu_n \}_{n \geq 1}$.
2. Each point of $C'$ is accessible from the exterior of $M^*_d$. It is a consequence of a theorem of Lindelöf (see [GM05, Lemma VI.3.1] for a proof) that every point on the boundary of a full compact set in $\mathbb{C}$ that is accessible from its complement, is necessarily the landing point of an external ray (inverse
image of a radial line under the Riemann map of the complement of the full compact). Consequently, one can pick any two distinct points on the undecorated arc $C'$ and these would be the landing points of two distinct external rays (coming from the Riemann map of the complement of $M^*_d$).

It is now easy to see that there exists a non-degenerate interval of angles in $\mathbb{R}/\mathbb{Z}$ such that all external rays at angles in this interval land on $C'$.

Therefore, the harmonic measure of $M^*_d$ assigns a positive mass to $C'$.

This completes the proof of the lemma. □

In the last corollary, we show that there are no bifurcations near the Ecalle height 0 parameters of the parabolic arcs. This was first proved in [HST14, Theorem 7.1], the present proof is somewhat simpler and almost readily follows from the previous results.

**Corollary 5.3 (No Bifurcation near Ecalle Height Zero).** On every parabolic arc of period $k$, the point with Ecalle height zero has a neighborhood (along the arc) that does not intersect the boundary of a hyperbolic component of period $2k$.

**Proof.** For the parabolic arcs of period one, the statement readily follows from Theorem 1.2. By the proof of Theorem 1.1, the Ecalle height 0 parameter on any co-root arc has an open neighborhood (along the arc) which lies in the accumulation set of a parameter ray; hence this neighborhood does not intersect the bifurcating period $2k$ components.

To finish the proof, assume that $C$ is a root arc. Let $\alpha_1$ and $\alpha_2$ be the angles of the parameter rays accumulating on $C$. In the dynamical plane of any $c \in C$, the corresponding dynamical rays land at the characteristic parabolic point through two different accesses in the repelling petal. These two accesses are separated by a parabolic Hubbard tree, which is invariant under the first anti-holomorphic return map. Clearly, the tree either projects to the equator in the repelling cylinder or its projection traverses an interval of Ecalle heights $[-a, a]$ for some $a > 0$. We can, without loss of generality, assume that the dynamical $\alpha_1$ (respectively $\alpha_2$)-ray lies ‘above’ (respectively ‘below’) the hubbard tree (more precisely, this means that the image of the $\alpha_1$-ray under the repelling Fatou coordinate lies in a complementary component of the image of the Hubbard tree containing an upper half plane). We denote the interval of Ecalle heights traversed by $R_c(\alpha_1)$ (respectively, $R_c(\alpha_2)$) by $[l_1(c), u_1(c)]$ (respectively, $[l_2(c), u_2(c)]$). It now follows that $u_1(c) > 0$ and $l_2(c) < 0$ $\forall c \in C$. Arguing as in case 2 of Theorem 1.1 we can find a parameter $c_h \in C$ (respectively $c_{h'}$) with critical Ecalle height $h > 0$ (respectively $h' < 0$) so that $h \in (l_1(c_h), u_1(c_h))$ (respectively $h' \in (l_2(c_{h'}), u_2(c_{h'}))$). This implies that $c_h$ and $c_{h'}$ are in the accumulation sets of the parameter rays at angles $\alpha_1$ and $\alpha_2$ respectively. Hence, the convex hull (along $C$) of the accumulation sets of these two parameter rays contain the Ecalle height 0 parameter on $C$. This shows that the accumulation sets of two parameter rays bound the Ecalle height 0 parameter on every root arc away from the bifurcating period $2k$ components and completes the proof of the corollary. □
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