Scale-free trees: the skeletons of complex networks

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We investigate the properties of the spanning trees of various real-world and model networks. The spanning tree representing the communication kernel of the original network is determined by maximizing total weight of edges, whose weights are given by the edge betweenness centralities. We find that a scale-free tree and shortcuts organize a complex network. The spanning tree shows robust betweenness centrality distribution that was observed in scale-free tree models. It turns out that the shortcut distribution characterizes the properties of original network, such as the clustering coefficient and the classification of networks by the betweenness centrality distribution.

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Complex network theories have attracted much attention in last few years with the advance in the understanding of the highly interconnected nature of various social, biological and communication systems [11]. The inhomogeneity of network structures is conveniently characterized by the degree distribution \( P_d(k) \), the probability for a vertex to have \( k \) edges toward other vertices. The emergence of scale-free distribution \( P_d(k) \sim k^{-\gamma} \) has been reported in many real-world networks, such as the coauthorship networks in social systems [2], the metabolic networks and the protein interaction networks in biological systems [2, 3], and Internet and World Wide Web in technological systems [2, 4].

It is important to study the dynamics on networks as well as to study the structural properties of networks since its application to the real-world. However, the dynamical phenomena on networks such as traffic and information flow are very difficult to predict from local information due to rich microstructures and corresponding complex dynamics. Thus, to understand the dynamical phenomena on networks, one must know the global properties of networks as well as the local properties such as degree distribution. It is the reason why the dynamics on complex networks has not been studied systematically so far.

Due to their inhomogeneous structure, traffic or information flow on complex networks would be also very inhomogeneous. As a simplified quantity to measure the traffic of networks, it is natural to use the betweenness centrality (BC) [2, 3, 10]. The BC of \( G \), either a vertex or an edge, is defined as

\[
b(G) = \sum_{i \neq j \neq k} b(i, j; G) = \sum_{i \neq j} c(i, j; G) / c(i, j),
\]

where \( c(i, j; G) \) denotes the number of shortest paths from a vertex \( i \) to \( j \) through \( G \), and \( c(i, j) \) is the total number of shortest paths from \( i \) to \( j \). In terms of the packet in the Internet, assuming every vertex sends a unit packet to each of other vertices, \( BC \) is the average amount of packets passing though a vertex or an edge.

In scale-free networks, the distribution of the vertex BC is known to follow the power-law with the exponents of either 2.2 or 2.0 [11]. Though the edge BC distribution does not follow power-law exactly, the distribution of the edge BC is also very inhomogeneous in scale-free networks [12]. This indicates that there exist extremely essential edges having large edge BCs which are used for communication very frequently. Thus, one can imagine a sub-network constructed only by the essential edges with global connectivity retained. We regard this network as a communication kernel, which handles most of the traffic on a network.

For simplicity, we define the communication kernel of a network as the spanning tree with a set of edges maximizing the summation of their edge BCs on the original networks. The constructing procedure is very similar to the minimum spanning tree algorithm [13]. We repeatedly select an edge according to the priority of the edge BC, and add the edge to the tree if it does not make any loop until the tree includes all vertices [14]. Note that the residual edges can be regarded as the shortcuts since they shorten paths on the spanning tree. This concept of the spanning tree and shortcuts corresponds to that of 1-D regular lattice and shortcuts, respectively, in the small-world networks [15].

In this paper, we investigate the structural and dynamical properties of the spanning tree of complex networks and the role of shortcuts in the networks. For various scale-free real-world and model networks, we find that the spanning trees show the scale-free behavior. We also find that the vertex and edge BC distributions follow the power-law with the robust exponent \( \eta = 2.0 \), regardless of the exponent value \( \eta = 2.0 \) or 2.2 of original networks. On the other hand, it turns out that the shortcut length distribution shows either Gaussian-like or monotonically decaying behavior depending on the BC distribution exponent \( \eta \) of original networks.

Firstly, we confirm the spanning tree to be a communication kernel by estimating the relative importance of selected edges in the obtained spanning tree and those...
from the random selection. If we select the edges randomly, the fraction $f$ of the edge BC summation over the selected edges and that over total edge would be approximately $f_0$, the ratio of the number of edges in the tree and that of network. However, it turns out that the real set of selected edges from the spanning tree possesses over 50% of the total edge BC of the network (see Table I), therefore $f \gg f_0$. For instance, the coauthorship network shows that $f$ is nearly three times larger than $f_0$ even though the number of edges in the spanning tree is only 16% of that in the original network. Thus we can call this spanning tree the communication kernel.

To find out more about this kernel, we measure the degree distribution of the spanning trees. It turns out that the degree distribution always follows the power-law [16], which is tested for various networks including the Barabási-Albert (BA) model [17], coauthorship network in neuroscience (NEURO) [18], protein interaction networks of yeast (PIN) [19], Internet at the autonomous systems (AS) level [20], and so on (see Fig. 1 and Table I). However, the details of the degree distribution depend on each of the networks. The exponents of the power-law degree distributions of the spanning trees do not always agree with those of the original networks (see Table 1). This indicates that the spanning trees are far from the random sampling of edges.

To confirm the scale-free behavior of the spanning tree, we investigate the time evolution of the degree in a growing network. Assuming that the fixed number of new vertices are introduced at each time step in growing networks, it is well known that the degree following the power-law $k_1(t) \sim t^{\beta}$ leads to the scale-free degree distribution $P_d(k) \sim k^{-\gamma}$ [21], where $k_1(t)$ is the degree of the vertex $i$ at time $t$ and $\gamma = 1/\beta + 1$. This argument can be naturally applied for the spanning tree of the BA model since it grows constantly. At each time step of the growth in the BA model, we obtain the spanning tree and measure the degree of every vertex. In Fig. 2(a), we show the time evolution of the degrees of several vertices. The degrees evolve with $\beta = 0.58$ that leads $\gamma_s = 2.7$ of the spanning tree, which agrees with our measurement from the actual degree distribution.

The high correlation between the degrees from spanning trees and the original networks also guarantees the preserved scale-free behavior of the spanning trees. The correlation coefficient between the degree of the original network $k$ and the degree of its spanning trees $k_s$ is defined as the Pearson’s correlation coefficient between $k$ and $k_s$, $r_p = \frac{(\langle kk_s \rangle - \langle k \rangle \langle k_s \rangle)}{\sqrt{(\langle k^2 \rangle - \langle k \rangle^2)(\langle k_s^2 \rangle - \langle k_s \rangle^2)}}$. Most networks exhibit strong degree correlation between the spanning
The assortativity is another interesting feature of the spanning trees. The assortativity $\mathcal{r}$ that measures the degree correlation of vertices directly connected by an edge, is defined by $\mathcal{r} = \frac{\left\langle (k_i - \langle k_i \rangle)(k_j - \langle k_j \rangle) \right\rangle}{\sigma_k^2}$, where $j$ and $k$ are the remaining degrees at the end of an edge and the angular brackets indicate the average over all edges. We find that all spanning trees show dissortative or neutral behavior regardless of the assortativity of original networks (see Table I). Thus, we can propose that it is general characteristic of the spanning trees of scale-free networks. We need further study to prove our conjecture.

We find that the BC distribution of the spanning tree is robust regardless of its original networks. For both of vertices and edges, the BC distribution follow the power-law with the robust exponent $\eta = 2.0$ in all spanning trees we studied (see Fig. 3 and Table I). This is consistent with the numerical results for the known scale-free tree models [11]. The same BC distribution for vertices and edges is the general feature of trees. In the mean field picture, the largest BC of edges belonging to a vertex gives dominant contribution to the BC of the vertex [22]. For our obtained spanning trees, we verify numerically that the largest edge BC of a vertex almost equals to the vertex BC for most of vertices (See Fig. 3(c)).

The spanning trees show the robust features, such as scale-free degree distribution, robust BC distribution, and dissortative or neutral degree correlation. Here one can ask what is the role of shortcuts which are not included in the spanning tree. To answer this question, we focus on the length of the shortcuts on the spanning trees. The length of a shortcut between vertices $i$ and $j$ is defined as the minimum number of hops from $i$ to $j$ on the spanning tree. The non-zero clustering coefficient of the original networks can now be explained by short-length shortcuts. Obviously, shortcuts with the length 2 build triangles of vertices, hence increase the clustering coefficient. All networks with non-vanishing clustering coefficient have the significant amount of the shortcuts with the length 2 (see Fig. 4).

Interestingly, we find that there are two types in the shortcut length distribution (see Fig. 4). In one distribution (Type I), most shortcuts distribute near a large mean value, similar to the Gaussian distribution, which shows that the network is the longer-loop dominant structure. In the other distribution (Type II), the number of shortcuts monotonically decreases as the length increases, which indicates that the network is tree-like. Most of networks including the BA model, coauthorship networks, and PIN belong to the type I. On the other hand, Internet AS and the adaptation model are type II. We find that our classification exactly agrees with the grouping by the exponent of the BC distribution [11]. The networks belonging to type I or type II show vertex BC distributions with the exponents of 2 or 2.2, respectively. The lines in (b) are linear fits with slope 2.0. The data points are shifted vertically to enhance the visibility. (c) The ratio of the largest value of edge BC ($b_{\text{max}}$) to vertex BC ($b$) of a vertex with degree $k_s$ for the BA tree (+) and the spanning trees of the BA model (×), NEURO (□), Internet AS (○), and PIN (∆) networks.
type II networks are as same as those of the scale-free trees. Because there exists mostly short length shortcuts in the type II networks with monotonically fast-decaying shortcut length distribution, the structure of the original networks are not significantly different from their spanning trees. Therefore, The BC exponents of the type II networks are unchanged at 2.0 of their spanning trees.

In summary, we study the properties of the spanning trees with maximum total edge betweenness centrality, which is regarded as the communication kernel on networks. We find that a complex network can be decomposed into a scale-free tree and additional shortcuts on it. The scale-free trees show robust characteristics in the betweenness centrality distribution and the degree correlation. The remaining shortcuts are responsible for the detailed characteristics of the networks such as the clustering property and the BC distribution. The distribution of the shortcut length clearly distinguishes the network into the two types, which coincides with the classes determined from the BC exponents [11].

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