ON THE RELIABILITY OF POLARIZATION ESTIMATION USING ROTATION MEASURE SYNTHESIS

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ABSTRACT

We benchmark the reliability of the rotation measure (RM) synthesis algorithm using the 1005 Centaurus A field sources of Feain et al. The RM synthesis solutions are compared with estimates of the polarization parameters using traditional methods. This analysis provides verification of the reliability of RM synthesis estimates. We show that estimates of the polarization parameters can be made at lower signal-to-noise ratio (S/N) if the range of RMs is bounded, but reliable estimates of individual sources with unusual RMs require unconstrained solutions and higher S/N. We derive from first principles the statistical properties of the polarization amplitude associated with RM synthesis in the presence of noise. The amplitude distribution depends explicitly on the amplitude of the underlying (intrinsic) polarization signal. Hence, it is necessary to model the underlying polarization signal distribution in order to estimate the reliability and errors in polarization parameter estimates. We introduce a Bayesian method to handle the distribution of intrinsic amplitudes based on the distribution of measured amplitudes. The theoretically derived distribution is compared with the empirical data to provide quantitative estimates of the probability that an RM synthesis solution is correct as a function of S/N. We provide quantitative estimates of the probability that any given RM synthesis solution is correct as a function of measured polarized amplitude and the intrinsic polarization amplitude compared to the noise.

Key words: galaxies: individual (Centaurus A, NGC 5128) – techniques: polarimetric

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1. INTRODUCTION

The birefringence of the magnetized plasma that pervades interstellar and intergalactic space causes the observed linear polarization properties of most astrophysical radio sources to be strongly frequency dependent. In cold plasmas the effect of Faraday rotation causes the position angle of the linear polarization vector to increase by an amount $\lambda$RM. The magnitude of the rotation measure (RM) ranges from $\sim 10$ rad m$^{-2}$ for typical sources observed through the interstellar medium of our Galaxy (Johnston-Hollitt et al. 2004; Schnitzeler 2010) to $\sim 10^3$ rad m$^{-2}$ for the cores of some radio sources to as high as $\sim 5 \times 10^5$ rad m$^{-2}$ for the compact radio source at the center of the Milky Way, Sgr A* (Macquart et al. 2006; Marrone et al. 2007).

It is often desirable to derive the RM along a particular line of sight and recover the linear polarization vector intrinsic to the radio source at the point of emission. This requires observations over a sufficiently large frequency range that the effect of Faraday rotation is measurable but with fine enough spectral resolution that Faraday rotation does not cause appreciable rotation of the polarization vector across individual spectral channels (bandwidth smearing). However, in many situations of astrophysical interest this trade-off results in a situation where the signal-to-noise ratio (S/N) of the polarization measurement in each spectral channel is too low to compute either the RM or intrinsic polarization vector without employing a method that simultaneously utilizes the measurements across the entire spectral band. This is particularly pertinent to observations made with next-generation backends, such as that of the Expanded Very Large Array (Perley et al. 2011) and the Compact Array Broadband Backend (Wilson et al. 2011).

The technique of RM synthesis was developed by N.E.B. Killeen et al. in 1999 (unpublished preprint submitted to MNRAS) to search for polarization from Sgr A* in situations where the large expected RM required such high frequency resolution that the S/N per channel was too low. This technique, which was already implicit in the treatment of Burn (1966), was independently discovered by Brentjens & de Bruyn (2005) to handle the large Galactic rotations seen at low radio frequency. Over the past few years RM synthesis has emerged as the method of choice for the derivation of Faraday RMs from radio polarimetric data. RM synthesis utilizes both the amplitude and position angle of the polarization vector at all observed frequencies and is capable of disentangling emission at multiple RMs (or Faraday depths) combined within the telescope beam. Because all the observed frequency-dependent information is used without bandwidth smearing and without the nonlinear operations needed to calculate the variation in polarization position angle with frequency, it should also be the optimum method to extract other polarization information from noisy data. This technique is fundamentally different from previous methods that derived RMs by fitting position angle estimates at a usually small number of often widely spaced frequencies.

Despite its extensive current use, a number of questions remain concerning the practical application of RM synthesis to real data sets. This includes debate on the confidence of the results of the technique as a function of S/N (e.g., Law et al. 2011) and on the reliability of the technique when there are two emission regions with closely spaced RM and polarization vectors within the same telescope beam (Farnsworth et al. 2011). There is also confusion regarding the appropriate means of treating the effects of polarization bias, associated with the fact that the mean magnitude of the polarization vector is nonzero even if no polarized signal is present (Simmons & Stewart 1985).

The objective of this paper is to resolve these issues. The technique and nomenclature of RM synthesis are summarized...
in Section 2. In Section 3, we proceed to examine several specific facets of the performance of RM synthesis in detail. We utilize the 1005 source data set of a field surrounding Centaurus A (Feain et al. 2009) to examine its performance in estimating polarization parameters.

In Section 4, we derive the distribution of polarization amplitudes expected from RM synthesis data as a function of the amplitude of the input polarization signal. We use this to demonstrate the derivation of the intrinsic distribution of polarization amplitudes within a sample. This is applied to the Cen A data set to examine the polarization flux density distribution down to the sub-mJy level; this presents a means of measuring the polarization distribution to well below the detection threshold of any given source, in a manner somewhat akin to that of “stacking” used in other surveys to achieve statistical detections of populations at detection thresholds below that of an individual source.

In Section 5, we examine the Cen A data set for sources with unusual polarization behavior, including sources with extreme or multiple RMs. Analysis of the changes in the goodness of fit (as quantified by the reduced-\( \chi^2 \) statistic) is used to identify the small number of sources with multiple RMs. The conclusions are presented in Section 6.

2. RM SYNTHESIS: TERMINOLOGY AND SUMMARY

For the purposes of the following sections, it is useful to briefly recapitulate how RM synthesis determines the RM and polarization of a source subject to Faraday rotation. The input data for RM synthesis are a stream of \( Q \) and \( U \) Stokes parameters, \( Q_1, Q_2, \ldots, Q_N \) and \( U_1, U_2, \ldots, U_N \), where there are \( N \) spectral channels corresponding to wavelengths \( \lambda_1, \ldots, \lambda_N \). The polarization data can be equivalently represented as a series of complex numbers where \( P_j = Q_j + iU_j = P_i e^{i\phi_j} \). (Throughout the text we use \( P \) to denote the complex polarization and \( P = |P| \) to denote its amplitude.) RM synthesis works by winding up the measurements of \( Q_j \) and \( U_j \) for a range of different trial RMs, \( \phi \). If the weights corresponding to the spectral channels are denoted \( w_j \), then the approximation to the Faraday dispersion function is

\[
\tilde{F}(\phi) = K \sum_{j=1}^{N} \tilde{P}_j e^{-2i\phi(\lambda_j^2 - \lambda_0^2)},
\]

where

\[
K = \left( \sum_{j=1}^{N} w_j \right)^{-1},
\]

and

\[
\tilde{P}_k = \frac{w_k P_k}{w_k^2 + P_k^2}
\]

It is often necessary to recover the Faraday dispersion function, \( F \), by deconvolving \( \tilde{F} \) from the RMTF, particularly if the source contains emission at multiple Faraday depths and if the S/N of the observations is high. To this end, we implemented a variation of the CLEAN algorithm (Högborn 1974; Heald et al. 2009) to handle the fact that both \( \tilde{F} \) and the RMTF are complex valued. It searches for the highest peak in \(|\tilde{F}|\) over a specified range of \( \phi \) and removes a suitably scaled copy of the RMTF centered on its corresponding Faraday depth. The algorithm iterated over successive peaks until either a maximum number of allowed iterations was reached or the rms of the fluctuations in \( \tilde{F} \) in the search region matched the rms of \( \tilde{F} \) exterior to the search region. We used a loop gain of \( \gamma = 1.0 \) in the CLEAN algorithm. Lower loop gain values, as normally used in two-dimensional image deconvolution, were trialed, but it was found that since these stop recovering the polarization signal once it reaches the noise floor, solutions in which \( \gamma < 1 \) are used systematically underestimate the polarization amplitude by at least 1σ.

The RM synthesis code was supplemented by several additional features. Because the functions \( \tilde{F} \) and \( R \) were computed on a grid of Faraday depths much finer than the width of \( R \), it is sometimes desirable to group together CLEAN components that are very tightly clustered together and identify them as members of a single component. By grouping components in this way, we could obtain the \( \chi^2 \) estimate for each independent component. We introduced a Faraday depth bunching tolerance, \( \Delta \phi \), such that any two components with depths \( \phi_j \) and \( \phi_k \) were merged if \( |\phi_j - \phi_k| \leq \Delta \phi \). A tolerance \( \Delta \phi = 5 \) rad m\(^{-2}\) was found to be suitable for the data set examined in the present paper.

The code outputs a list of clean components ranked in order of amplitude, together with their corresponding Faraday depth and

\[ R(\phi) = K \sum_{j=1}^{N} w_j e^{-2i\phi(\lambda_j^2 - \lambda_0^2)}. \]

In the simple case in which all channel weights are equal, the above relations simplify to

\[
\tilde{F}(\phi) = \frac{1}{N} \sum_{j=1}^{N} P_j e^{-2i\phi(\lambda_j^2 - \lambda_0^2)},
\]

\[
R(\phi) = \frac{1}{N} \sum_{j=1}^{N} e^{-2i\phi(\lambda_j^2 - \lambda_0^2)}.
\]

Now suppose a source contains polarization emission at one or more Faraday depths, \( \phi_k \), and associated with each depth is a polarization vector \( P_{\text{src.k}} \). It is clear from Equation (5) that for a data set of polarization measurements for a source with Faraday depths \( \phi_0, \phi_1, \ldots \) the sum over channels results in a coherent summation of the polarization vectors, and the amplitude of the function \( F(\phi) \) will be large. The tendency for this sum to add coherently for values of \( \phi_k \) that match (i.e., correctly de-rotate) the polarization data means that \(|F(\phi)|\) peaks at the locations of the RM(s) in the source, with the quantity \(|F(\phi_k)| = |P_{\text{src.k}}|\) giving the amplitude of the polarization vector that is rotating at each Faraday depth, \( \phi_k \). Thus, the problem of measuring the Faraday depths and rotation polarization vectors that are significant in a given data set is one of locating those peaks in the function \(|\tilde{F}(\phi)|\) that exceed some given threshold.

2.1. RM Synthesis Deconvolution Implementation (CLEAN)

When the S/N is sufficient, this enables us to “super-resolve” \( \tilde{F} \) and search for multiple peaks in \( \phi \) a scale smaller than the width of \( R \).
polarization vector $\mathbf{P}$, evaluated at the mean-square wavelength $\lambda_0$. The code also computed the reduced $\chi^2$ associated with the polarization solution as a function of the number of clean components used. To be specific, for measurements with standard deviations $\sigma_Q$ and $\sigma_U$ in $Q$ and $U$, respectively, and clean components with Faraday depths $\phi_1$, $\phi_2$, ... and associated polarization vectors $\mathcal{P}_c_1$, $\mathcal{P}_c_2$, ..., the reduced $\chi^2$ associated with a fit to the data by including the first $n$ clean components is

$$\chi^2 = \frac{1}{2N - 3n} \sum_{i}^{N} (Q_i - Q(\phi_1, ..., \phi_n; \mathcal{P}_c_1, ..., \mathcal{P}_c_n; \lambda_i))^2$$

$$+ \frac{[U_i - U(\phi_1, ..., \phi_n; \mathcal{P}_c_1, ..., \mathcal{P}_c_n; \lambda_i)]^2}{\sigma_U^2},$$

(7)

where

$$Q(\phi_1, ..., \phi_n; \mathcal{P}_c_1, ..., \mathcal{P}_c_n; \lambda_k)$$

$$+ iU(\phi_1, ..., \phi_n; \mathcal{P}_c_1, ..., \mathcal{P}_c_n; \lambda_k)$$

$$= \sum_{j=1}^{n} \mathcal{P}_c j e^{2i\lambda_k\phi_j}.$$  

(8)

This $\chi^2$ statistic was found to be an excellent estimator of the number of significant clean components required to model the polarization behavior of each source. It was also useful to identify cases in which WM synthesis solution differed significantly from the polarization behavior of the source; this highlighted cases in which the polarization is not noise-like but where its behavior is chiefly governed by a mechanism other than Faraday rotation (Section 5 discusses specific examples of this).

3. RM SYNTHESIS OF CENTAURUS A DATA

In this section, we compare the performance of RM synthesis against other techniques of determining the polarization and RMs of radio sources. As an initial step, we investigate the accuracy of the RMs reported by RM synthesis as a function of $S/N$. We then compare the RMs derived by RM synthesis against a technique that mimics the technique used to derive the RMs extracted from the NRAO VLA Sky Survey (Condon et al. 1998; Taylor et al. 2009) based on measurements of the polarization in two closely separated bands at 1.4 GHz. We then compare the performance of RM synthesis against a more sophisticated technique that performs a least-squares fit to the polarization data.

This analysis is based on the ATCA observations of 1005 background sources contained in a ~34 deg$^2$ field centered on the radio galaxy Centaurus A (Feain et al. 2009). The data have a bandwidth of 192 MHz divided into 24 × 8 MHz channels between 1296 and 1480 MHz. Because of the effect of the reweight option in Miriad, a 5% correlation was introduced between channels, resulting in 22.75 independent channels. The total bandwidth and spectral resolution of these data mean that the FWHM in Faraday rotation for a single polarization component is 280 rad m$^{-2}$, and that it is sensitive to RMs with magnitudes less than $\approx 3500$ rad m$^{-2}$. The relatively low RM resolution in these data avoids the potential complexity in the interpretation of any sources with closely spaced multiple RMs, as discussed by Farnsworth et al. (2011). We allowed our RM synthesis code to search to RMs of $\pm 4000$ rad m$^{-2}$. The average rms error in $U$ and $Q$ is 0.089 mJy, and the corresponding average sigma in $P$ (which is the measure of noise used throughout this paper) is 0.12 mJy.

3.1. Confidence versus $S/N$ Based on RM Synthesis-derived Parameters

We performed RM synthesis on the 1005 sources in the Cen A field to investigate the reliability of the derived Faraday depths as a function of $S/N$. Several recent papers adopt an $S/N$ of 7 as the threshold below which the results of RM synthesis are regarded as unreliable and are not reported (Feain et al. 2009; Law et al. 2011). It is therefore propitious to investigate whether this threshold represents a sharp barrier to the believability of RM synthesis results or whether results below this level only decline gradually in accuracy. The large size of our sample enables us to empirically derive the level of confidence associated with RM synthesis solutions over a large range of $S/N$.

Figures 1 and 2 show the distribution of derived RMs for varying $S/N$ thresholds. For the purposes of this analysis, we have used only the single most significant Faraday depth derived by RM synthesis; the small number of sources in which there are multiple significant RMs has a negligible influence on the statistics (see Section 5). The signal was derived from the value of $|F|$ at the most significant Faraday depth. Some RFI had to be flagged in the time-based data, resulting in a small variation in $S/N$ between channels. This also varied from source to source. The noise, $\sigma$, is taken to be the average rms noise per channel for each source divided by the square root of the weighted number of channels used. To simplify the theoretical analysis in the following sections, we have assumed equal noise per channel, but it was then essential to correct for the effect of this non-uniform noise in the data by including the non-uniform weighting in the $S/N$ estimates and in the estimates of the
Figure 2. Histograms showing the distribution of Faraday depths as a function of the S/N of the polarization detection. The logarithmic scaling on the y-axis accentuates bins populated by only a single source.
number of independent channels. We estimate this average rms noise per channel from the observations using the fact that the rms errors add in quadrature. Feain et al. (2009) provide specifics on the estimation of errors for the 8 MHz $Q$ and $U$ channels, which were interpolated from the correlator output. The effect of RFI flagging, which resulted in non-uniform noise varying from source to source, had to be included in the analysis. Even though these effects are small and subtle, it is essential to account for them when analyzing the polarization statistics.

It is evident that all the observed RMs for high-S/N detections are in the range $-150$ to $+50$ rad m$^{-2}$, and we assume that the lower S/N detections are drawn from the same distribution. Polarization detections with RMs outside this range are most likely to be due to noise. At the lower S/N regime of interest here, the expected distribution is broadened by noise, so we increase the false RM range by $2\sigma$ and use the range $-210$ to $+110$ rad m$^{-2}$, which is appropriate down to signal strengths of $5\sigma$. If Faraday depths associated with false detections are evenly distributed over the depth search range—as appears to be the case based on Figures 1 and 2—there is only a 4% chance that any given Faraday depth found by RM synthesis will fall in the range of expected real detections by accident. Table 1 gives the distribution of "true" and "false" polarization detections as a function of S/N.

Some RMs outside this range could be real (see Section 4.5), and while this does not significantly affect the statistical analysis, individual sources with high intrinsic RM are of interest (see Section 5). This population is rare in the flux density range of our sample since only one of the 309 sources detected above $6\sigma$ possesses an intrinsic RM magnitude larger than $200$ rad m$^{-2}$.

The one high RM source in the $6 < S/N < 7$ bin, 132446–460218, merits comment. This source has a clear double peak in the RM synthesis spectrum. The two peaks have nearly equal polarization: one is at $297$ rad m$^{-2}$ but the second at $-59$ rad m$^{-2}$ does fall within the expected RM range. The S/N difference in amplitude is $<10\%$. As discussed at the end of Section 5, the statistical treatment of a second RM component in the presence of a significant polarized signal is well beyond the analysis in this paper, and this may be a correct but rare detection.

3.2. Comparison with Traditional RM Methods

The traditional method to determine the RM has been to determine the position angle of polarization in each frequency channel and then to make a scalar least-squares fit to the position angles. Figure 3 (top) shows the result of this analysis when applied to sources in the Cen A polarization catalog. When compared to the RM synthesis solution for the same sources (bottom panel in Figure 3), it can be seen that a small number of spurious high RM values are obtained. Only those sources with integrated polarization signals S/N > 7, as identified by RM synthesis, were used in the analysis. It is evident that there is a tail in the distribution of the Faraday depths derived by fitting polarization position angles. Of the 251 sources with S/N > 7 used in the analysis, 16 of these fall outside the range $[-210, 110]$ rad m$^{-2}$. This can be explained by noting that with the lower S/N in each channel the least-squares fit for weak sources can find solutions that are randomly distributed over the RM space included in the least-squares solution. This high RM tail is reminiscent of the tail of high RMs sometimes seen in existing RM catalogs, e.g., see the comparison of the NVSS catalogs and the Kronberg & Newton–McGee compilation in Pshirkov et al. (2011). The small number of genuine high RMs seen in our data set (<1%) indicates that this class of radio source is rare in surveys at 1.4 GHz, presumably because sources with high intrinsic RM are more likely to be strongly depolarized.

Taylor et al. (2009) report the RMs of 37,543 sources from the NVSS based on Very Large Array (VLA) snapshot images in two 42 MHz wide bands centered on 1364.9 MHz and 1435.1 MHz. It is opportune to investigate the accuracy of this catalog given that it currently represents one of the largest resources of RMs and is rapidly being adopted as a de facto standard in analyses of the Galactic and extragalactic magnetic fields (Harvey-Smith et al. 2011; Law et al. 2011; Oppermann et al. 2011; Govoni et al. 2010; McClure-Griffiths et al. 2010; Schnitzeler 2010; Stasyszyn et al. 2010).

We used the Cen A polarization catalog to investigate the accuracy of this technique used to derive NVSS RMs. The data were obtained at a comparable frequency, and we have collapsed them into two spectral channels with similar center frequencies of 1364 and 1436 MHz. As for the NVSS sample, the range of Faraday depths in our sample is sufficiently small that phase wrap ambiguities will not occur. The RM results for the two-band measurements are in full agreement with the RM synthesis results. The agreement with the two-band RM synthesis results is unsurprising given that the bands are contiguous.

3.3. Comparison of RM Synthesis and Least-squares Fits

Although RM synthesis outperforms traditional frequency-channel-based position-angle fitting algorithms such as the ones discussed above, it remains to be seen whether it makes the best use of the polarization information available. We address this by

| S/N | >7σ | >6σ | >5σ | >4σ | >3σ | >2σ | All |
|-----|-----|-----|-----|-----|-----|-----|-----|
| Approx P(mJy) | >0.63 | >0.54 | >0.45 | >0.36 | >0.27 | >0.18 |
| Total number | 239 | 288 | 352 | 488 | 775 | 1003 | 1005 |
| Number with good RM | 239 | 287 | 346 | 423 | 499 | 527 | 528 |
| Number with false RM | 0 | 1 | 6 | 65 | 276 | 476 | 477 |
| Differential number | 49 | 64 | 136 | 287 | 228 |
| Differential number with good RM | 48 | 59 | 77 | 76 | 28 |
| Differential number with false RM | 0 | 1 | 5 | 59 | 211 | 200 |

In the NVSS, the actual rotation measure of a source is related to the deduced value assuming no phase wraps, $RM_0$, by the relation $RM = RM_0 + (681 \, \text{rad m}^{-2})n$, where $m$ represents the number of integral phase wraps that occurred between the two observing frequencies. Taylor et al. (2009) assume $|m| \leq 1$, given that RMs of the magnitude required to effect a phase wrap are rare when the source is located off the Galactic plane and away from the inner Galaxy.
considering the output of a brute-force minimum least-squares search for detected polarization at any RM on all the sources, subject to the assumption that the source contains only a single polarized component that is assumed to have a flat spectrum. To be specific, we implemented a code that seeks the parameters $p_0, \chi_0$, and $\phi$ (polarization amplitude, position angle, and Faraday depth) that minimize the penalty function,

$$\chi^2 = \frac{1}{2N - 3} \sum_j \frac{|Q_j - \text{Re}[p_j(P, \chi_0, \phi)]|^2}{\sigma_Q^2} + \frac{|U_j - \text{Im}[p_j(P, \chi_0, \phi)]|^2}{\sigma_U^2},$$

(9)

where the model for the polarization of the $j$th channel is

$$p_j(P, \chi_0, \phi) = P \exp \left[ 2i \left( \chi_0 + \lambda^2 \phi \right) \right].$$

(10)

Figure 4 compares the solutions from the RM synthesis with the brute-force least-squares fit approach. The two methods agree very well in both RM and polarization amplitude. At high S/N the solutions are identical. The increased scatter at lower S/N is not surprising as the two methods find different solutions, but the gap (along the Faraday depth axis) surrounding the identical solution indicates that both methods converge to the identical solution if the solutions are close. There is no systematic change in the ratio of signal amplitude between the two methods with S/N, and the average ratio for the two methods is 1.0.

The behavior of the solutions for the sources with a “false” detection based on the RM has a number of undesirable properties. The brute-force solution has a much narrower distribution of RMs at low S/N ($\pm 500 \text{ rad m}^{-2}$). The amplitudes of the bad solutions are lower and the reduced $\chi^2$ worse, indicating that it does not find the global minimum for noise-dominated signals.

In Section 4.2, it is demonstrated that the noise does fall as $N^{-1/2}$ for RM synthesis. The naive expectation would be that it falls similarly for traditional methods, since the estimated standard deviation in the slope of a regression line scales as $N^{-1/2}$. However, in practice this dependence is more complicated for traditional methods in a way that is hard to analyze; $n\pi$ phase ambiguities in the position angle introduce additional degrees of freedom into the fitting problem, and the fit error depends in detail on how the channels are spaced.

4. THE STATISTICS OF POLARIZED SOURCES

Determination of the distribution of polarization amplitudes intrinsic to a set of sources involves disentangling the contribution of noise from the polarization amplitudes deduced by RM synthesis. This is most important for sources whose intrinsic polarization is near the noise level.

We are motivated to examine the noise statistics in detail with a view to accounting for its effects in the distribution of polarization amplitudes measured in a set of sources. Specifically, the objective is to evaluate the quantity $p(P|P_0)$, namely, the probability of obtaining an observed polarization amplitude $P$ conditioned on the amplitude of the true intrinsic polarization, $P_0$. This distribution function is the basis for deriving the probability distribution of intrinsic (i.e., “noise-corrected”) polarization amplitudes based on a set of measured polarization amplitudes determined by RM synthesis. We conclude this section by applying the formalism to investigate the distribution of polarization amplitudes in the Cen A data set at low polarization amplitude levels.

4.1. The Polarization Distribution of Noise

Here, we derive the distribution of polarization amplitudes recovered by RM synthesis from pure noise. The input measurements $Q_j$ and $U_j$ are therefore assumed to possess the properties of thermal noise and are independent and normally distributed with zero mean and noise per spectral channel $\sigma$.

Recall that RM synthesis de-rotates the channelized measurements of $Q$ and $U$ for a range of different trial RMs, $\phi$. The resultant summed (complex) number is

$$\mathcal{P} = \frac{1}{N} \sum_j P_j e^{i\phi_j} e^{2i\lambda_j \phi}. $$

(11)
The polarization vector derived by RM synthesis is the value of $P$ for which the choice of $\phi$ gives a maximum in the value of $P$. The task here is to determine the distribution of the amplitude of $P$. If $\phi$ is given, it is straightforward to recover the polarization distribution. However, an additional complication arises because the RM synthesis algorithm is allowed the extra degree of freedom to choose any value of $\phi$ that maximizes $P$. The freedom to rotate the measured polarization vectors over a range of trial values of $\phi$ ensures that the polarization vector amplitude returned by RM synthesis is at least as large as that found in the $\phi = 0$ case. This is because the algorithm may de-rotate the measured polarization vectors with any value of $\phi$ that improves the coherence of the sum in Equation (11).

There is a direct analogy between the noise characteristics of RM synthesis and those encountered in interferometric radio imaging, which involves deriving the distribution of noise amplitudes from (complex) visibilities (see Section 9.3 of Thompson et al. 2001). In both cases one is attempting to “wind up” complex valued quantities whose real and imaginary components are both normally distributed. In the case of RM synthesis, they are wound up for various values of Faraday depth. In the interferometric imaging case the complex-valued interferometric visibilities are searched across a range of fringe rotation values, corresponding to the possible position of the source. Thus, the two treatments are mathematically identical. We elaborate on this connection in the analysis below.

### 4.1.1. Constant $\phi$

First consider the simple, well-known case in which we explicitly hold the RM at a fixed value. Since the input data are purely noise-like, the actual value of $\phi$ is unimportant: any noise-like polarization vector that is transformed by de-rotation to any nonzero value of $\phi$ still possesses the same statistical noise-like properties. Here, for the purpose of simplicity, we set $\phi = 0$. The polarization found by RM synthesis is

$$P_{\phi=0} = \frac{1}{N} \left( \sum_{j} Q_{j} + i \sum_{j} U_{j} \right).$$

The statistics of the sum of $N$ independent normally distributed random variates is also normally distributed, with standard deviation $\sigma_{Q,U}/\sqrt{N}$. Thus, $S_{Q} = N^{-1} \sum_{j} Q_{j}$ also follows a normal distribution with zero mean and standard deviation $\sigma_{Q,U}/\sqrt{N}$. An identical relation holds for $S_{U} = N^{-1} \sum_{j} U_{j}$. It is then straightforward to show that the probability distribution for the polarization amplitude when there is no signal present and $\phi = 0$ is just a Rayleigh distribution,

$$p_{P}(P)dP = \frac{P}{\Sigma^{2}} \exp\left(-\frac{P^{2}}{2\Sigma^{2}}\right)dP,$$

where we define $\Sigma^{2} = \sigma^{2}/N$. This result is well known in the analogous interferometric imaging case pertaining to noise in very long baseline interferometry (VLBI) observations, where it corresponds to the statistics of the fringe amplitude for a specific fringe rotation when no signal is present (Thompson et al. 2001, p. 318).

### 4.1.2. Unconstrained $\phi$

Now consider the general case in which RM synthesis performs a fit to the polarization data that is unconstrained in

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Figure 4. Comparison of the RM and polarization amplitude derived by RM synthesis compared to a brute-force least-squares fit to the polarization data, as described in Section 3.3.
Faraday depth, $\phi$. This consists of two separate steps. The first is to find the specific value of $\phi$ that maximizes the value of $P$ for a given data set. The second step is to find the distribution of $P$ corresponding to these maxima.

The square of the polarization amplitude is

$$\mathcal{P} P^* = \sum_{k,m}^N P_k P_m e^{i(\psi_k - \psi_m) e^{2i\phi(\lambda_k^2 - \lambda_m^2)}} = \sum_{k,m}^N P_k P_m \cos \left[ \psi_k - \psi_m + 2\phi \left( \lambda_k^2 - \lambda_m^2 \right) \right].$$  \hspace{1cm} (14)

The second line follows because the sine function has odd symmetry, so the imaginary part of every pair $k, m$ in the sum is canceled by every pair $m, k$.

The RM $\phi$ should obey the following condition at the maximal value of $P$:

$$\frac{\partial (P P^*)}{\partial \phi} = 0 = \sum_{k,m}^N 2i(\lambda_k^2 - \lambda_m^2) P_k P_m e^{i(\psi_k - \psi_m) e^{2i\phi(\lambda_k^2 - \lambda_m^2)}}.$$  \hspace{1cm} (15)

Since the imaginary terms are antisymmetric with respect to exchange of the indices $k$ and $m$, we see that the imaginary part of this sum is zero for any value of $\phi$. So only the real part of the sum yields a nontrivial constraint on $\phi$:

$$0 = \sum_{k,m}^N \left( \lambda_k^2 - \lambda_m^2 \right) P_k P_m \sin \left[ \psi_k - \psi_m + 2\phi \left( \lambda_k^2 - \lambda_m^2 \right) \right].$$  \hspace{1cm} (16)

The foregoing relations show that the distribution of $\mathcal{P}$ exhibits a detailed dependence on the spectral coverage (i.e., the values of $\lambda_k^2$). This indicates that the exact location and amplitude of the noise peak depend on the $\lambda_k^2$, and that the noise distribution for any given spectral setup is best found empirically.

It is possible to construct a simple approximate analytic formulation independent of the detailed spectral coverage. This result has already been derived in the analogous case of the noise properties in VLBI images, where one is concerned with the statistics of the fringe amplitude across (Thompson et al. 2001, p. 211–319). We briefly summarize the argument here and place it in the context of RM synthesis.

The statistics of the polarization amplitude at any one value of $\phi$ obey a Rayleigh distribution. At any given value of $\phi$ the probability that we detect a polarization amplitude lower than some prescribed threshold amplitude, say, $Z$, is the cumulative probability distribution corresponding to Equation (13), namely, $p(Z < P) = 1 - \exp(-Z^2/2\Sigma^2)$. If the RM synthesis search is conducted over a large range of possible Faraday depths, $-\phi_0 < \phi < \phi_0$, and the width of the amplitude of the RMTF is approximately $\Delta\phi$, there are $N \approx 2\phi_0/\Delta\phi$ independent Faraday depth “pixels,” or trials, over which the RM search is being performed. The probability that we do not detect a polarization amplitude above some amplitude $Z$ after these $N$ trials is $p^N$. The probability of finding $Z \leq |P|$ is just the cumulative probability distribution function of the amplitude of the polarization vector,

$$CDF_n(Z) = \left[ 1 - \exp \left( -\frac{Z^2}{2\Sigma^2} \right) \right]^N,$$  \hspace{1cm} (17)

where the subscript $n$ emphasizes that the distribution is associated with a purely noise-like signal. The corresponding probability density function is

$$p_n(Z) = \frac{NZ}{\Sigma^2} \left[ 1 - \exp \left( -\frac{Z^2}{2\Sigma^2} \right) \right]^{N-1} \exp \left( -\frac{Z^2}{2\Sigma^2} \right).$$  \hspace{1cm} (18)

This expression describes the expected distribution of the polarization amplitude when the input data are purely noise-like and in which the RM is searched over approximately $N$-independent Faraday depth pixels. It is identical to the characteristics of the noise in a search for fringes from $N$-independent samples (Thompson et al. 2001, Equation (9.57)). The exact noise distribution pertaining to the Cen A data set is discussed at the conclusion of this section. We pre-empt this by noting that we find empirically that the noise distribution for the Cen A observations (shown in Figure 6) differs slightly from the analytic formula derived here.

The behavior of the noise distribution has a number of important implications for the statistics of polarization amplitudes determined by RM synthesis. Equation (18) shows that the distribution of polarization amplitudes derived from noise narrows as $N$ increases (for constant $\Sigma$), and the peak of the distribution shifts to increasing values of $P/\Sigma$. The location of the peak is determined by the solution to the transcendental equation,

$$P^2[N - \exp(P^2/2\Sigma^2)] = \Sigma^2[1 - \exp(P^2/2\Sigma^2)].$$  \hspace{1cm} (19)

This has solutions $P/\Sigma = 1.93, 2.24, 2.84, 3.07,$ and $3.54$, respectively, for $N = 5, 10, 50, 100,$ and 500.

Thus, the threshold $S/N$, $P/\Sigma$, for the believability of a polarization signal extracted using RM synthesis exhibits an important, albeit weak, dependence on the number of independent trials, $N$. The distribution peaks sharply on the low $P/\Sigma$ side of the distribution, implying that one expects very few polarization amplitudes with $S/N$ values less than 2.0 once $N$ exceeds 10. For the analysis in this paper $N = 30$.

4.2. The Polarization Distribution of a Nonzero Polarization Signal Containing Noise

There is an important distinction between the polarization distribution obtained from a purely noise-like signal from RM synthesis and the polarization distribution derived when an intrinsic (nonzero) polarization signal is present. The distinction arises because, in the absence of an intrinsic polarization signal, the RM synthesis algorithm picks out the polarization vector associated with the Faraday depth that maximizes the length of the “Faraday-de-rotated” noise vector. However, this freedom to optimize the length of the polarization noise vector is removed provided that the intrinsic signal is sufficiently large that the RM synthesis code finds the highest amplitude at the correct Faraday depth of the intrinsic signal (i.e., there is no other Faraday depth at which a purely noise-like signal yields a higher polarization amplitude). In this instance the RM synthesis code is constrained to de-rotate the signal at the Faraday depth of the intrinsic polarization signal.

We compute the probability distribution of the polarization signal subject to the assumption that the intrinsic polarization is nonzero and that the RM synthesis code de-rotates the signal at
the Faraday depth associated with the intrinsic signal. Suppose we have a nonzero intrinsic signal whose Faraday de-rotated channel measurements of the linear polarization are \( P_0 = (Q_0, U_0) \). Associated with each spectral channel there is also a noise contribution. The noise follows a Gaussian distribution. It is also uncorrelated with the intrinsic signal and thus its statistical properties remain identical after de-rotation at the Faraday depth of the intrinsic signal. Then the polarization vector derived by RM synthesis is
\[
P = (Q_0 + S_Q, U_0 + S_U).
\]

where \( S_Q = N^{-1} \sum Q_i \) and \( S_U = N^{-1} \sum U_i \) (as defined previously). The polarization vector, \( P \), therefore obeys the statistics of a random walk with a constant offset \( P_0 = (Q_0, U_0) \).

Thus, the polarization amplitude in the presence of an intrinsic signal follows the Rician distribution:
\[
p_{\delta}(Z = P|P_0) = \frac{Z}{\Sigma} \exp \left[ -\frac{Z^2 + P_0^2}{2\Sigma^2} \right] I_0 \left( \frac{Z|P_0}{\Sigma} \right),
\]
where \( I_0(x) \) is the modified Bessel function of the first kind of order zero and the subscript \( s \) denotes that the distribution is valid only when a nonzero polarization signal is present so that the signal is de-rotated at the Faraday depth corresponding to the intrinsic signal, \( P_0 \). The associated cumulative probability distribution function (i.e., the probability of \( Z \leq P \)) is
\[
C_{DF}(Z|P_0) = 1 - Q_1 \left( \frac{P_0}{\Sigma}, \frac{Z}{\Sigma} \right).
\]

where \( Q_1(a, b) \) is Marcum’s \( Q \) function. This result is analogous to the probability distribution of the interferometric fringe amplitude in which one measures a nonzero signal in the presence of noise (Thompson et al. 2001, Equation (9.37)).

Some remarks about the treatment of polarization noise bias are in order. The polarization amplitude distribution in Equation (21) demonstrates that the measured amplitude is, on average, biased by noise. In particular, one has
\[
\langle P^2 \rangle = P_0^2 + 2\Sigma^2.
\]

Thus, the appropriate noise bias to subtract from a single measurement of \( P^2 \) derived on the basis of RM synthesis is twice the channel-integrated noise variance, \( \Sigma^2 \). While correct on average, bias subtraction is clearly inexact for any given source; in as much as the noise can assume a range of values, so is there a range of values of intrinsic \( P_0 \) that are consistent with a given measured \( P \). Our analysis shows that it is entirely inappropriate to simply remove a noise bias from the data and treat the resulting polarization amplitudes as if they were distributed with a Gaussian error distribution.

### 4.3. Likelihood of Detection

In the two foregoing subsections, we have derived the probability distributions of the polarization amplitude both when a signal is absent and when one is present and its amplitude is sufficiently large that it dominates the noise (i.e., provided that \( P_0 \) exceeds some threshold).

Here, we deduce the probability distribution of the polarization amplitude that is valid for all \( P_0 \). The key to this derivation is that there is a nonzero probability that the value of \( P \) drawn from the signal distribution \( p_s \) is lower than the value that would be drawn from the noise distribution \( p_n \). Under this circumstance, the value of \( P \) picked by the RM synthesis algorithm is the greater of the two values, namely, that drawn from the noise distribution.

The probability of obtaining a polarization amplitude of \( Z = P \) is the sum of (1) the probability of drawing \( Z \) from the noise distribution and requiring that its value exceeds the amplitude drawn from the signal distribution and (2) the probability that \( Z \) was instead drawn from the signal distribution and requiring its value exceeds the amplitude drawn from the noise distribution. Thus, one obtains the combined probability distribution,
\[
p_{\delta}(Z = P|P_0) = p_s(Z|P_0)C_{DF}(Z|P_0) + p_n(Z|P_0)C_{DF}(Z).
\]

This distribution is valid for any value of \( P_0 \). It obeys the expected property that at low \( Z \) and \( P_0 \) the distribution is dominated by the noise distribution, \( p_n(Z) \), because \( C_{DF}(Z) \) is almost zero in this region, while it is dominated by \( p_s(Z|P_0) \) at high \( Z \) because \( p_s(Z) \) is almost zero in this domain.

By way of illustration, Figure 5 compares the combined probability distribution for some specific choices of \( P_0 \) against \( p_s \) and \( p_n \).

One can extend the foregoing logic to determine the likelihood that a signal found by RM synthesis at a given amplitude, \( Z \), is a genuine signal. The probability that the signal has been de-rotated to the correct value of \( \phi \) is
\[
p_{\phi \text{ correct}}(Z|P_0) = \frac{p_s(Z|P_0)C_{DF}(Z)}{p_s(Z|P_0)}.
\]

It is apparent that, even for small values of \( Z \) and \( P_0 \), there is a nonzero probability that the RM synthesis code deduces the correct Faraday depth of the signal.

This is borne out by the Cen A data: close inspection of the lower panel in Figure 1 indicates that even at \( S/N \sim 3.5 \) the RM synthesis algorithm preferentially finds signals within the range of physically acceptable values of \( \phi \).

### 4.4. Deduction of Unbiased Polarization Amplitude Source Counts

The foregoing formalism provides an obvious prescription to determine the distribution of intrinsic polarization amplitudes, \( N(P_0) \), based on the measured distribution of polarization amplitudes as deduced by RM synthesis, namely, \( N(P) \):
\[
N(P) = \int dP_0 \, p(P|P_0)N(P_0).
\]

The solution of Equation (26) to find \( N(P_0) \) is formally an ill-posed problem, but it is of a form that is often encountered in numerical optimization theory.

---

6 The contribution of noise, at the Faraday depth of the intrinsic signal, may, in principle, slightly alter the depth at which the peak amplitude occurs and hence the Faraday depth reported by the RM synthesis algorithm. This may, in turn, influence the statistics of \( P \). However, this is a small effect in most practical situations. If the Faraday depth derived by RM synthesis is sufficiently close to the correct depth, it will not alter the statistics of \( P \); this is the case provided that the \( S/N \) is sufficient that the error in Faraday depth is a small fraction of the width of the RM transfer function, \( R(\phi) \) (i.e., the \( S/N \gtrsim 2 \)). Winding up an intrinsic polarization signal \( P_0 \) at a depth that is wrong by an amount \( \Delta \phi \) yields an intrinsic signal of amplitude \( \sim P_0 \cos \lambda_0^2 \Delta \phi \), so the error in the amplitude of the intrinsic polarization vector, \( \approx P_0(\lambda_0^2 \Delta \phi^2/2) \), is quadratic in \( \lambda_0^2 \Delta \phi \). This is a small quantity since, for \( S/N \gtrsim 2 \), the peak is localized to within the width of \( R(\phi) \), and one has \( \lambda_0^2 \Delta \phi < 1 \), so that \( p(Z|P_0) \approx p(Z|P_0 \cos \lambda_0^2 \phi) \). We see in the following subsection that the condition \( S/N \gtrsim 2 \) is always satisfied.
It is instructive to demonstrate how the solution of Equation (26) is readily amenable to numerical solution. The distribution of intrinsic polarization amplitudes, $N(P_0)$, must be solved by sampling onto a grid with resolution, say, $\Delta P$. We represent this distribution by the vector $y_j$. The corresponding distribution of measured polarization amplitudes is discretized onto a grid in $P$ with identical resolution, and this is represented by the vector $x_i$. If one similarly discretizes $p_c(Z | P_0)$ into a matrix whose $i,j$th element corresponds to the parameters $Z_i$ and $P_{0j}$, with resolution $\Delta P$ in both dimensions, the integral in Equation (26) can be cast in the form

$$x_i = \sum_j A_{ij} y_j \Delta P.$$  \tag{27}

This class of problems is amenable to solution by iterative deconvolution and expectation-maximization algorithms.

A simpler but more robust approach is to forward model the expected distribution $N(|P_0|)$ to determine whether it is consistent with the observed amplitude distribution.

4.5. Application to Centaurus A Data

As an illustration of the foregoing formalism, we apply it to determine the distribution of intrinsic polarization amplitudes in the Cen A data set.

In order to model the effect of noise in this data set precisely, we determined the noise distribution $p_n$ empirically based on the specific spectral coverage of the ATCA observations. The same RM synthesis code used to process the Cen A data was employed. We synthesized 50,000 purely noise-like sources where, for each source, synthetic $Q$ and $U$ data for each spectral channel were drawn from a normal distribution of zero mean and unit standard deviation and assigned to spectral channels with frequencies identical to those used in the Cen A observations.

For each source the “signal” was extracted at the value of $\phi$ at which the amplitude in the Faraday dispersion spectrum was a maximum. The statistics of the polarization amplitude at a fixed Faraday depth, $\phi = 0$, were also recorded. The resulting probability distributions are shown in Figure 6. There is excellent agreement between the distribution for the $\phi = 0$ case and the Rayleigh distribution discussed in Section 4.1.1. The lower panel of Figure 6 shows the amplitude distribution determined from the synthetic noise data set when the RM synthesis algorithm is allowed to find a source anywhere over
the Faraday depth range, [−4000, +4000] rad m⁻². One expects naively, on the basis of the width of the RMTF and the range of the search in Faraday depth, that there are \( N \approx 29 \) independent Faraday depth trials over the search space. There is good agreement in the shape of the theoretically and empirically derived distributions for this value of \( N \). However, it is evident that the theoretical distribution peaks at a lower \( S/N \) than the empirically derived distribution. This cannot be accounted for in terms of a change in \( N \) since, although this would shift the peak of the distribution to higher \( S/N \), it would also narrow the distribution to a value lower than is observed. We note that, from a purely empirical viewpoint, the two distributions agree closely if the \( S/N \) axis is scaled such that the recovered \( S/N \) values from the simulated sources are systematically higher than those expected in the theoretical distribution by 8.5%. The distribution corresponding to this is also shown in Figure 6. However, the reason for such an \( S/N \) discrepancy is not obvious; the lower right panel of Figure 4 shows that there is no systematic bias in the \( S/N \) of sources recovered by the RM synthesis technique and a brute-force fit to the polarization data.

One can directly compare the polarization amplitude distribution for the sources in the Cen A field with “false” RMs (see Section 3.1) against the distribution, \( p_0 \), as determined by the synthetic pure-noise sources. This comparison is shown in Figure 7. In this plot we see excellent agreement between the polarization amplitude distribution of sources with false RMs and the expected noise distribution. However, there is an excess of sources in the region \( 4 < S/N < 5 \) that deserves further investigation. A likely explanation for this excess is that a number of these sources are mistakenly identified as false detections. It is possible that several contain real signal, but their \( S/N \) is sufficiently low that their Faraday depth is poorly localized by RM synthesis (i.e., they are only localized to about the \( \sim 280 \) rad m⁻² width of the RM transfer function), thus placing the detection RM outside the nominal range [−210, 110] rad m⁻² considered to constitute a “good” detection. The excess of sources can be seen in Figure 2 as the concentration of sources in the \( 4 < S/N < 5 \) bin with RMs in the range [0, 500] rad m⁻². The results of Section 4.2 (and Figure 5) indicate that when some small signal is present the polarization amplitude distribution shifts to higher \( S/N \). This suggests that some sources with supposedly “false” RMs do indeed contain intrinsic signal.

A related statistic is \( P_0 \), the expected fraction of sources whose Faraday depths are identified correctly as a function of \( P \) for a given \( P_0 \). We compute this quantity using the numerically derived distribution, \( p_0 \). Figure 8 displays this likelihood. This gives a direct estimate that the probability is correct at any given RM. At \( S/N \approx 3 \), the probability of retrieving the correct Faraday depth is \( \sim 50\% \), and it approaches 100% for \( S/N \) values between 5 and 6, depending on the value of the intrinsic polarization value, \( P_0 \).

It is interesting to compare the differential source counts from the Cen A data in Table 1 against this curve, despite uncertainty in the intrinsic polarization amplitudes that contribute to the Cen A data. At \( S/N \) values of 3, 4, and 5, the measured fractions of good RMs are 26%, 57%, and 92%, respectively, while the predicted fractions at these \( S/N \) values are in the ranges 25%−57%, 61%−93%, and 94%−100%, respectively. In estimating the range of fractions, we have assumed that the \( S/N \) of the intrinsic polarization, \( P_0 \), is between 2.5 and 5.

4.5.1. Forward Modeling of Polarization Source Counts

It is beyond the scope of this paper to implement a complete iterative deconvolution scheme to recover the intrinsic polarization source counts, \( N(P_0) \), using Equation (26) in conjunction with \( p_0(Z \mid P_0) \).

We have instead developed a simple iterative forward-modeling scheme in which the intrinsic polarized source count distribution is taken to be of the form

\[
N(P_0) = A \begin{cases} (C/|\Sigma|)^{\alpha_1} (P_0/|\Sigma|)^{\alpha_2}, & P_0/\Sigma < C \chi^2, \\ (P_0/\Sigma)^{\alpha_1}, & P_0/\Sigma < C, \end{cases}
\]

which is a broken power-law distribution with index \( \alpha_1 < 0 \) that turns over at an \( S/N \) of \( C \) and scales with an index \( -4 < \alpha_2 < 4 \) below this \( S/N \) break, and where \( A \) is an overall normalization.\(^7\) This distribution was gridded onto the vector \( y \) with a resolution

\(^7\) Since the distribution \( p(P \mid P_0) \) is zero for \( P_0/\Sigma \leq 2 \), the fact that \( N(P_0) \) diverges for small \( P_0 \) when \( \alpha_2 < 0 \) does not pose a problem for our modeling. We are effectively insensitive to the behavior of the distribution at \( P_0/\Sigma \leq 2 \).
Figure 9. Observed source counts in P (cyan) and the best-fitting model of it (black).

(A color version of this figure is available in the online journal.)

Table 2
The Fit Quality and Associated Fit Parameters as a Function of Location in the Spectral Break, C

| C       | $\chi^2$ | $\sigma_1$ | $\sigma_2$ | $\alpha$ |
|---------|----------|------------|------------|----------|
| 0.0     | 1.12     | -0.96      | -           | 119      |
| 1.0     | 0.93     | -1.53      | 4          | 1.56 x 10^3 |
| 2.0     | 0.88     | -1.90      | 1.01       | 8.09 x 10^3 |
| 3.0     | 0.86     | -2.04      | -0.05      | 1.48 x 10^4 |
| 3.72    | 0.85     | -2.17      | -0.30      | 2.60 x 10^4 |
| 4.0     | 0.85     | -2.22      | -0.37      | 3.27 x 10^4 |
| 5.0     | 0.87     | -2.40      | -0.54      | 7.39 x 10^4 |
| 6.0     | 0.91     | -2.54      | -0.66      | 1.45 x 10^5 |
| 7.0     | 0.95     | -2.64      | -0.74      | 2.50 x 10^5 |

Notes. This table illustrates the covariance between the break S/N and the spectral indices of the polarization amplitude count distribution.

in $\Delta P/\Sigma = 0.1$ and substituted into Equation (27) to find the predicted distribution of measured $P$ values, which we label $x_{\text{pred}}$. This is compared against the actual distribution vector $x_{\text{meas}}$ by computing the reduced $\chi^2$ of the fit,

$$\chi^2 = \frac{1}{N_{\text{bins}}} \sum_{i=1}^{N_{\text{bins}}} \frac{|x_{\text{pred},i} - x_{\text{meas},i}|^2}{\sigma_i^2},$$

(29)

where we take the errors to be $\sigma_i^2 = x_{\text{meas},i}$, (and set $\sigma_i = 1$ when $x_{\text{meas},i} = 0$). The quantity $\chi^2$ was searched for a minimum in terms of the parameters $\alpha$, $C$, $\sigma_1$, and $\sigma_2$. The minimum occurs at $\chi^2 = 0.85$ for $(C, \sigma_1, \sigma_2, A) = (3.72, -2.17, -0.30, 2.60 \times 10^4)$, and the best-fitting solution is plotted over the measured polarization distribution in Figure 9. However, the minimum formally supports a range of indices in the range $-2.9 \lesssim \alpha_1 \lesssim -1.0$, and we note that the values of $\alpha_1$ and $\alpha_2$ and $C$ are all highly correlated. To illustrate this, Table 2 below lists the derived fit parameters for a range in the break $S/N$, $C$.

The best-fitting index of the slope of the differential counts in $P$ at high $S/N$, $\alpha_1 = -2.17$, is very close to the value of $-2.2$ obtained by fitting just the stronger well-detected sources in the sample with $P$ in the range 0.6–6 mJy. This indicates that the source counts in $P$ continue with the same slope down to at least 0.35 mJy, and possibly lower. Note that this forward-modeling method does not require any assumptions about polarization amplitude bias at low $S/N$.

Following the technique used by Tucci et al. (2004) we can now estimate the change in fractional polarization as a function of flux density by comparing the source counts in $I$ and $P$ at the same source density. For both FIRST (White et al. 1997) and these Centaurus A field sources the differential counts in $I$ have a slope of $-1.8$ in this flux range. The fractional polarization is $3.0\%$ for the strongest sources ($> 200$ mJy), and this increases to $3.5\%$ for sources below 20 mJy. This can be compared with the mean fractional polarization of ATLSB sources ($S_{\nu} > 0.4$ mJy) of $4.3\%$. Taylor et al. (2007) make a detailed analysis of polarization in the Elais N1 field. They have 786 sources with 83 polarization detections. We have 1005 sources and at least 346 polarization detections ($5\Sigma$) in the same flux density range. We agree well in the source density and polarized fraction at the higher flux density, but our fractional polarization for sources below 20 mJy (3.7\%) is less than the increase to 4.8% seen by Taylor et al. (2007).

5. SOURCES WITH COMPLEX POLARIZATION

To evaluate the reality of sources that have multiple RMs, we evaluated the reduced $\chi^2$ after RM synthesis identified each new CLEAN component$^8$ and used this to help determine the significance of each new component. To evaluate the success of this process and to see what type of source has high reduced $\chi^2$, we looked at the 21 sources with reduced $\chi^2 > 2.4$ and $S/N > 5$. We can divide these sources into four categories.

1. Some residual bad data points were readily identified in this process, and these (six) sources do not appear in the list when the bad data are flagged out.

2. High $S/N$ (three sources). The high $\chi^2$ associated with some high-$S/N$ sources could be due to real but weak components with different RM that are only visible at high $S/N$. It may also result from instrumental dynamic range errors.

3. Multiple RM (eight sources). These are well fit by multiple components. Note that for sources with multiple RMs we are unable to uniquely separate components within our RM resolution of $\sim 280$ rad m$^{-2}$.

4. Significant spectral change in $P$ across the band (four sources). This is likely to be caused by regions with closely spaced RM and polarization vectors as discussed by Farnsworth et al. (2011), although it may in principle be caused by depolarization effects.

We illustrate these effects with two sources from this list: 131943$-$445904 has three significant and comparable amplitude RM components that are just separated at our RM resolution. The three components are a good fit and reduce $\chi^2$ from 4.5 to 1.6. See Figure 10 and Table 3.

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$^8$ Recall that CLEAN recognized all components that were within 5 rad m$^{-2}$ of each other as being part of the same component.
Figure 10. Three-component polarization solution for the source identified in Feain et al. (2009) as 131943−445904. The fit parameters for each component are shown in Table 3, including the reduced $\chi^2$ value after each successive component is added to the model.

(A color version of this figure is available in the online journal.)

| Component No. | RM (rad m$^{-2}$) | Reduced $\chi^2$ after Inclusion | $P_0$ (mJy) | S/N |
|---------------|-------------------|----------------------------------|-------------|-----|
| 1             | −74.8             | 2.7                              | 13.1        | 66.8|
| 2             | 118.4             | 1.7                              | 0.56        | 2.9 |
| 3             | −686.8            | 1.1                              | 0.45        | 2.3 |

Table 4
The Fit Parameters for the Source 131713−410934, as Shown in Figure 11

131713−410934 is a high-S/N case with a dominant RM components at −75 rad m$^{-2}$. It also has a barely resolved component at 118 rad m$^{-2}$ that causes the slope in $P$ and a much weaker (3%) well-resolved component at RM = −687 rad m$^{-2}$. See Figure 11 and Table 4.

It is beyond the scope of this paper to present a detailed discussion of the sources with complex RM structure, but we can make a few useful observations. Sources with RM structure less than our resolution (280 rad m$^{-2}$) can still cause strong variations in polarization amplitude across the band, and the RM clean process can fit this change in amplitude with components separated by approximately the half-power beam width. While this is still a valid indication of significant RM structure, the “super resolution” of our CLEAN process (i.e., the ability to resolve RM components within the width of the RMTF) is unlikely to find a unique solution for the actual RM components.

Although we do see clear evidence of multiple RMs in 12 of the 359 sources with S/N in polarization amplitude greater than 5, this is only 3% of our sample and the majority of sources are well fit by a single RM component.

To better quantify this fraction, we can ask down to what S/N one can detect a second RM component by examining to what degree the addition of the second component improves the fit to the spectropolarimetric data. We start with a model consisting of one polarized component $M_1$ and compare it against the measurement vector $\mathbf{m}$. Let us assume that the measurements in each spectral channel have the same error, $\sigma$. Then, if the vector...
\[
\chi^2_1 = \frac{1}{2N_{\text{chan}} - 3}\sigma^2 (M_1 - m)^2. \tag{30}
\]

Now let us introduce a second polarized component, \(M_2\), that improves the fit and produces a new reduced \(\chi^2\) of
\[
\chi^2_2 = \frac{1}{2N_{\text{chan}} - 6}\sigma^2 (M_1 + M_2 - m)^2. \tag{31}
\]

Subtracting the two foregoing expressions and if the data are well explained by two polarized components, one has \(M_1 - m \approx M_2\), and we find
\[
\Sigma^2 [\chi^2_2 (2N_{\text{chan}} - 6) - \chi^2_1 (2N_{\text{chan}} - 3)] \approx 3|M_2|^2 / N_{\text{chan}}. \tag{32}
\]

where we use the fact that \(\Sigma = \sigma / \sqrt{N_{\text{chan}}}\). Recognizing that \(|M_2|^2 / N_{\text{chan}}\) is the average squared amplitude of the second polarized component, say, \(P_2^2\), we find
\[
\frac{P_2^2}{\Sigma^2} \approx \frac{1}{3} [\chi^2_2 (2N_{\text{chan}} - 6) - \chi^2_1 (2N_{\text{chan}} - 3)]. \tag{33}
\]

For our measurements, one has \(N_{\text{chan}} = 22\) and we can see that, to improve the fit from \(\chi^2 = 2.4\) to 1.3, one requires a component with an \(S/N\) of \(P_2^2/\Sigma \approx 3.6\).

We note that the statistical treatment for the detection of a second RM component in the presence of another strongly polarized signal is very different from the analysis of the detection of a polarized signal in the presence of noise. The perturbation that an additional polarized signal generates will be closer to the normal detection statistics, so the weak multiple components we find at the \(S/N \gtrsim 3\) level are already significant.

6. CONCLUSIONS

Our major conclusions are as follows:

1. We have assessed the distribution of false and correct RM detections in the 1005-source sample of Feain et al. (2009). This yields a quantitative estimate of the likelihood that a given RM, as found by RM synthesis, is correct, as a function of \(S/N\). If \(4 < S/N < 5\), we find that the likelihood of finding a correct RM is about 52%. Restricting the range of possible RM solutions to the range
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—210 to 110 rad m$^{-2}$ (the range found for the high-S/N sources) further decreases the likelihood of a false RM detection; there is only a 4% chance that any given Faraday depth found by RM synthesis in a search over the range $[-4000, 4000]$ rad m$^{-2}$ will fall in the range of expected real detections by accident.

2. There is no systematic difference in the polarization amplitude recovered by RM synthesis compared to a least-squares fitting approach as a function of S/N, and the average ratio of the amplitudes found by the two methods is 1.0.

3. We have investigated the effect of noise in RM synthesis and examined the probability distribution of the polarization amplitude as determined by this algorithm. From this, we have developed a formalism to recover the distribution of intrinsic polarization amplitudes from a measured distribution that is affected by noise. This enables us to recover the polarization distribution at S/N levels well below the confidence level for any single detection. We have applied this to examine the polarization counts of the sources in the Feain et al. (2009) Cen A catalog; the best-fitting polarization source count distribution follows $N \propto P^{-2.35}$, which is consistent with $N \propto P^{-2.16}$ found by fitting only to S/N $> 7$ detections, and with no turnover at an amplitude above 0.2 mJy. A related point is that polarization “bias” should not be subtracted from the derived polarization amplitudes when analyzing the distribution of $P$. Subtraction of this bias is often at best misleading, since the distribution of $P$ is highly non-Gaussian.

4. Pre-existing analyses have used an S/N cutoff of $\approx 7$ as a limit to the believability of results from RM synthesis. We derive here the likelihood that a given RM detection is correct as a function of S/N and the intrinsic polarization amplitude ($P_0$) of the polarization signal. We also derive the criteria for the likelihood of the detection of a second RM component. For instance, a detection at an S/N $\sim 3$ is sufficient to believe the existence of a second component if the addition of the second component reduces the $\chi^2$ value of the fit to the polarization data from $\approx 2.4$ to $\approx 1.3$.

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Note added in proof. Since the paper has been in press, it has come to our attention that George et al. (2011) obtain a number of results pertaining to the statistics of the polarization amplitude derived from RM synthesis similar to those derived in the present paper.

REFERENCES

Brentjens, M. A., & de Bruyn, A. G. 2005, A&A, 441, 1217
Burn, B. J. 1966, MNRAS, 133, 67
Condon, J. J., Cotton, W. D., Greisen, E. W., et al. 1998, AJ, 115, 1693
Farnsworth, D., Rudnick, L., & Brown, S. 2011, AJ, 41, 191
Feain, I. J., Ekers, R. D., Murphy, T., et al. 2009, ApJ, 707, 114
George, S. J., Stil, J. M., & Keller, B. W. 2011, PASA, in press (arXiv:1106.5362)
Govoni, F., Dolag, K., Murigia, M., et al. 2010, A&A, 522, A105
Harvey-Smith, L., Madsen, G. J., & Gaensler, B. M. 2011, ApJ, 736, 83
Heald, G., Braun, R., & Edmonds, R. 2009, A&A, 503, 409
Högbom, J. A. 1974, A&AS, 15, 417
Johnston-Hollitt, M., Hollitt, C. P., & Ekers, R. D. 2004, in The Magnetized Interstellar Medium, ed. B. Uyaniker, W. Reich, & W. Wielebinski (Katlenburg-Lindau: Copernicus GmbH), 13
Law, C. J., Gaensler, B. M., Bower, G. C., et al. 2011, ApJ, 728, 57
Macquart, J.-P., Bower, G. C., Wright, M. C. H., Backer, D. C., & Falcke, H. 2006, ApJ, 646, L111
Marrone, D. P., Moran, J. M., Zhao, J.-H., & Rao, R. 2007, ApJ, 654, L57
McClure-Griffiths, N. M., Madsen, G. J., Gaensler, B. M., McConnell, D., & Schnitzeler, D. H. F. M. 2010, ApJ, 725, 275
Oppermann, N., Junklewitz, H., Robbers, G., & Enßlin, T. A. 2011, A&A, 530, A89
Perley, R. A., Chandler, C. J., Butler, B. J., & Wrobel, J. M. 2011, ApJ, 739, L1
Pshirkov, M. S., Tinyakov, P. G., Kronberg, P. P., & Newton-McGee, K. J. 2011, ApJ, 738, 192
Schnitzeler, D. H. F. M. 2010, MNRAS, 409, L99
Simmons, J. F. L., & Stewart, B. G. 1985, A&A, 142, 100
Stasyszyn, F., Nuza, S. E., Dolag, K., Beck, R., & Donnert, J. 2010, MNRAS, 408, 684
Taylor, A. R., Stil, J. M., Grant, J. K., et al. 2007, ApJ, 666, 201
Taylor, A. R., Stil, J. M., & Sunstrum, C. 2009, ApJ, 702, 1230
Thompson, A. R., Moran, J. M., & Swenson, G. W., Jr. 2001, Interferometry and Synthesis in Radio Astronomy, ed. A. R. Thompson, J. M. Moran, & G. W. Swenson, Jr. (2nd ed.; New York: Wiley)
Tucci, M., Martinez-Gonzalez, E., Toffolatti, L., Gonzalez-Nuevo, J., & De Zotti, G. 2004, MNRAS, 349, 1267
White, R. L., Becker, R. H., Helfand, D. J., & Gregg, M. D. 1997, ApJ, 475, 479
Wilson, W. E., Ferris, R. H., Axtens, P., et al. 2011, MNRAS, 416, 832