Investigation of the finite size core helical vortex filament moving in cylindrical channel

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Abstract. A mathematical analysis of the problem on finding the shape of the core of an equilibrium helical vortex moving in a cylindrical tube has been carried out. Within the core, the vorticity was considered constant. The vorticity distribution was approximated by the Heaviside function. Integral equation for the vortex core boundary and stream function is formulated. An approximate solution on the stream function is found for the case of a circular section of the vortex core. An algorithm developed for the vortex ring [2] is adopted for solution of the problem under consideration.

1. Introduction
Vortex flows occur in many technical devices and apparatus, as well as in nature. In particular, the appearance of such phenomena in the process of operation of hydraulic units is inevitable, and their influence on the operation of hydroelectric power plants is significant. When the turbine is operating in a non-optimal mode, it is often accompanied by the formation of a helical vortex in the wake of the impeller. Many studies are devoted to research in this area, most of which are experimental. This is due to the complexity of mathematical equations describing systems of screw vortices.

Most of the works devoted to the study of helical vortices are based on experiments. For a helical vortex in an infinite fluid, the problem of finding the shape of the nucleus was solved in [1] with using complicated mathematical apparatus was used. In [2], the shape of a vortex ring in infinite space was found in the ideal fluid on the basis of the expansion of an unknown function in a Fourier series and the derivation of equations to search for the coefficients of the series.

In the work [3], the problem on finding the stream function in a cylindrical channel with an infinitely thin vortex filament was solved. In the present work we consider helical vortex with finite size core, motion of which is assumed to be stationary with the presence of a helical symmetry in the flow. It means that the flow pattern is the same in any plane perpendicular to the helical vortex axis. The fluid in which the vortex exists is assumed to be non-viscous. Outside the vortex core, the vorticity value is considered to be zero; inside the core, it is considered constant ($\omega_z = k = \text{const}$)

The aim of the work is to find the shape of vortex core cross-section providing equilibrium (without change of the core boundaries) motion of a helical vortex in a cylindrical tube, as well as to obtain dependencies of the speed of vortex motion on the problem parameters. Both the algorithm adaptation for search of the vortex core form [2] and way for obtaining the stream function induced by the finite size core helical vortex in the cylindrical channel has been proposed.
2. Formulation of the problem

Let us consider a vortex with axis represented by the helical line:

\[ x = a \cos(\theta), \quad y = a \sin(\theta), \quad z = l \theta \]  

(1)

where \( a \) is a vortex line radius and \( 2\pi l = h \) is the vortex pitch.

When describing the model, a coordinate system denoted for flows with helical symmetry [3] was used. As a basic vectors the following vectors was used: \( r \) is the classical radius vector, \( B \) is the coordinate along the tangent vector to the helix (vortex axis), and \( \chi \) is the coordinate along the binormal to the helix.

The components of the vorticity vector in the coordinate system \((r, \chi, z)\) (where \( \chi = \theta - z/l \)) for a flow with helical symmetry can be written as [3]:

\[ \omega_z = \frac{1}{r} \frac{\partial}{\partial \chi} \left( r B \frac{\partial \psi}{\partial r} \right) - 2u_0 \frac{B^2}{l}, \]

\[ \omega_{\theta} = \omega_z r/l; \quad B^2 = \frac{l^2}{l^2 + r^2}. \]

(2)

Here \( \psi \) – stream function, \( u_0 \) – velocity value on the cylinder axis. We introduce new function

\[ \phi = \psi + \frac{u_0 r^2}{2l}. \]

(3)

to provide a homogeneous equation on \( \phi \) for the area outside the vortex core.

Now the vorticity equation looks like:

\[ -\omega_z = \frac{\partial^2 \phi}{\partial r^2} + \frac{l^2 - r^2}{l^2 + r^2} \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{r^2 + l^2}{r l^2} \frac{\partial^2 \phi}{\partial \chi^2}. \]

(4)

The impermeability conditions on the wall were used as boundary conditions.

\[ u_r \big|_{r=a} = \frac{1}{r} \frac{\partial \psi}{\partial \chi} \big|_{r=a} = 0. \]

(5)

Here \( u_r \) is radial velocity component.

3. Results

According to the conditions of the problem, the vorticity is different from 0 only inside the vortex core. With the Heaviside function, we describe the vorticity as:

\[ \omega_z = \Omega \left( \Theta(r - A_1(\chi)) - \Theta(r - A_2(\chi)) \right) \left( \Theta(\chi - \chi_1) - \Theta(\chi - \chi_2) \right). \]

(6)

Here \( \Omega \) is const; \( \Theta \) is the Heaviside function; \( \chi_1 \) and \( \chi_2 \) are the angular bound points of the core boundary; \( A_1 \) and \( A_2 \) are the functions describing the boundary of the vortex core.

The solution of homogeneous equation corresponded to (4) was looking for in the complex function form. Stream function is equal to the real part of complex function:

\[ \Phi(r, \chi) = r \sum_{m=0}^{\infty} \exp(im\chi) p_m(r), \]

\[ \phi = \text{Re}(\Phi). \]

(7)

Here \( p_m(r) \) are unknown functions on \( r \).
The left part of (4) described in more detail in (6) must be also arranged in the Fourier series:

\[ \omega_m = - \sum_{m=-\infty}^{\infty} \frac{\Omega \left( l^2 + r^2 \right)}{l^2} \exp(im\chi) G_m(r), \]

\[ G_m(r) = \frac{1}{2\pi^2} \int_0^\infty \left( \Theta(r - A_1(\zeta)) - \Theta(r - A_2(\zeta)) \right) e^{-im\zeta} d\zeta. \]

\[ G(r) \] can be presented with analytical function like:

\[ G_m(r) = \frac{1}{\pi} \left( \Theta(r-a-\varepsilon) - \Theta(r-a+\varepsilon) \right) \sin(mS(r)) \frac{\sin(mS(r))}{m}, \]

where \( \varepsilon \) is the core radius, \( S(r) \) is a function reverse to the functions \( A_1(\zeta) \) and \( A_2(\zeta) \). Thus, the left side of (4) looks as follows:

\[ \omega_m = \sum_{m=-\infty}^{\infty} \left( -\frac{\Omega \left( l^2 + r^2 \right)}{\pi l^2} \right) \left( \Theta(r-a-\varepsilon) - \Theta(r-a+\varepsilon) \right) \exp(im\chi) \frac{\sin(mS(r))}{m}. \]

For the case when the vortex core has a circular cross-section of radius \( \varepsilon \) the functions \( A_1(\zeta) \), \( A_2(\zeta) \) and \( S(r) \) can be written down explicitly:

\[ A_2(\zeta) = a \cos(\zeta) + \sqrt{\varepsilon^2 - a^2 \sin^2(\zeta)}, \quad S'(r) = \frac{r^2 + a^2 - \varepsilon^2}{2ar}. \]

For more realistic vortex with the circular form in cross-section perpendicular to the helical line (1) the analytical representation for function \( S_h(r) \) also exists [4]:

\[ S'(r) = f'(r) + \alpha \sin \left( \frac{l^2}{ar} f'(r) \right), \quad f'(r) = 2 \left[ 1 + \frac{r^2}{l^2} + \frac{a^2}{l^2} + \frac{r^2}{l^2} - \frac{\varepsilon^2}{l^2} \right]^{1/2} - 2 - \frac{r^2}{l^2} + \frac{a^2}{l^2} + \frac{\varepsilon^2}{l^2} \right]^{1/2}. \]

Substitution of (7) and (10) into (4), yields a system of equations:

\[ \frac{\partial^2 p_m}{\partial \varepsilon^2} + \left( \frac{l^2 - r^2}{l^2 + r^2} + 2 \right) \frac{1}{r} \frac{\partial p_m}{\partial r} + \left[ \frac{l^2 - r^2}{l^2 + r^2} \right] \frac{m^2 (l^2 + r^2)^3}{r^2 l^2} \left[ \frac{l^2 - r^2}{l^2 + r^2} \right] \frac{m^2 (l^2 + r^2)^3}{r^2 l^2} \left[ \frac{l^2 - r^2}{l^2 + r^2} \right] \frac{m^2 (l^2 + r^2)^3}{r^2 l^2} \frac{m^2 (l^2 + r^2)^3}{r^2 l^2} \frac{m^2 (l^2 + r^2)^3}{r^2 l^2} \right] = f(r), \]

\[ f(r) = -\frac{\Omega \left( l^2 + r^2 \right) \sin(mS(r))}{m\pi l^2} \left( \Theta(r-a-\varepsilon) - \Theta(r-a+\varepsilon) \right). \]

In [3], it was found that partial solutions of the homogeneous equation corresponding to (11) are derivatives of modified Bessel functions or their linear combinations. Here it is also proposed to use these functions

\[ \begin{align*}
  p_m^{(1)} &= I_m'(mr/l), \\
  p_m^{(2)} &= K_m'(mr/l) - \alpha_m I_m'(mr/l)
\end{align*} \]

as linearly independent particular solutions. Find the Wronskian for (12):

\[ W\left(p_m^{(1)}, p_m^{(2)}\right) = \frac{m^2 (l^2 + r^2)^3}{l^2 r^3}. \]
The solution of the inhomogeneous equation (11) was searched by the formula [3]:

\[ P_m(r) = p_m^{(0)} \int_{0}^{r} \frac{P_m^{(2)}(r)}{W(p_m^{(0)}, p_m^{(2)})} dr + p_m^{(2)} \int_{0}^{r} \frac{p_m^{(2)} f(r)}{W(p_m^{(0)}, p_m^{(2)})} dr. \]  

(14)

where \( f(r) \) is defined in (10).

Substitution of (12), (13) and \( f(r) \) from (11) into (14), gives an expression for unknown functions:

\[ P_m(r) = -I_m'(mr/l)K_m'(mp/l)H(p) dp - K_m'(mr/l)I_m'(mp/l)H(p) dp + \]

\[ + \alpha_m I_m'(mr/l) \int_{0}^{r} I_m'(mp/l)H(p) dp, \]

\[ H(r) = \frac{\Omega}{\pi m^2 (m^2 + r^2)^2} \left( \Theta(r - a - \varepsilon) - \Theta(r - a + \varepsilon) \right). \]

Here \( \alpha_m \) are the coefficients responsible for satisfaction the boundary conditions.

Taking into account (7), one can write down the expression for the stream function for a helical vortex of finite size core moving in a cylindrical pipe:

\[ \psi(r) = r \sum_{m=-\infty}^{\infty} \cos(m \chi) \left\{ -I_m'(mr/l)K_m'(mp/l)H(p) dp - K_m'(mr/l)I_m'(mp/l)H(p) dp + \right\} \]

\[ + \alpha_m I_m'(mr/l) \int_{0}^{r} I_m'(mp/l)H(p) dp \]  

(16)

A helical vortex in a cylindrical pipe moves both translationally and rotates around the axis of the cylinder. Therefore, in the expression for the stream function written in the coordinate system associated with the vortex, additional terms appear:

\[ \varphi_{\text{tr}}(r) = \varphi(r) - \frac{1}{2} \Omega_M r^2 - \frac{U_M r^2}{2l}. \]  

(17)

Here \( \varphi \) is described in (16), \( U_M \) and \( \Omega_M \) are connected to each other through the flow velocity on the axis:

\[ U_M = -\frac{\alpha^2}{l} \Omega_M = u_0. \]  

(18)

The equation on the form of the vortex core cross-section.

In [2], to find the shape of a vortex ring, the condition was posed that at the core boundary the stream function has a constant value. Using this assumption for our case, we obtain the equation for finding the boundary of the helical vortex core:

\[ r \sum_{m=-\infty}^{\infty} \cos(m \chi) \left\{ -I_m'(mr/l)K_m'(mp/l)H(p) dp - K_m'(mr/l)I_m'(mp/l)H(p) dp + \right\} \]

\[ + \alpha_m I_m'(mr/l) \int_{0}^{r} I_m'(mp/l)H(p) dp \]

\[ - \frac{1}{2} \Omega_M r^2 - \frac{U_M r^2}{2l} = k \]  

(19)

Here \( k \) is a constant value of the stream function at the boundary of the vortex core. The algorithm for solving the problem looks similar to that proposed by Norbury [2] and involves numerical evaluation of the integrals in equation (19). This will make it possible to obtain various forms of the
helical vortex core depending on the flow parameters and to find the corresponding dependencies for $U_M$ and $\Omega_M$.

**Conclusion**
This stage of the work assumes a continuation in which the shape of the core will be parameterized with using Fourier series, and the dependencies for the coefficients of the series as well as speed of the vortex motion on the problem parameters will also be derived. This will allow us to find a family of equilibrium helical vortices moving in a cylindrical channel permanently and without changing the shape.

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**References**
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