ON THE MASS CENTER OF THE TENT MAP

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Abstract. It is well known that the time average or the center of mass for generic orbits of the standard tent map is 0.5. In this paper we show some interesting properties of the exceptional orbits, including periodic orbits, orbits without mass center, and orbits with mass centers different from 0.5. We prove that for any positive integer n, there exist n distinct periodic orbits for the standard tent map with the same center of mass, and the set of mass centers of periodic orbits is a dense subset of [0, 2/3]. Considering all possible orbits, then the set of mass centers is the interval [0, 2/3]. Moreover, for every x in [0, 2/3], there are uncountably many orbits with mass center x. We also show that there are uncountably many orbits without mass center.

1. Introduction

Let $T : [0, 1] \rightarrow [0, 1]$ be the standard tent map defined by $T(x) = 1 - |2x - 1|$ or

$$T(x) = \begin{cases} 2x & \text{when } x \in [0, \frac{1}{2}] \\ 2(1 - x) & \text{when } x \in [\frac{1}{2}, 1]. \end{cases}$$

This map is often introduced as the first example of chaotic maps in typical textbooks for dynamical systems. Its dynamics exhibit various features that are commonly used to identify chaotic systems. Some of these well-known features are sensitive dependence, topological transitivity (for almost all orbits in the Lebesgue sense), existence of infinitely many orbits with positive Lyapunov exponent, and being a topological factor of every unimodal map with topological entropy log 2. Details and further references can be found in [2], for instance.

The n-th iterate of T is given by

$$T^n(x) = \begin{cases} 2^n(x - \frac{2m}{2^n}) & \text{when } x \in [\frac{2m}{2^n}, \frac{2m+1}{2^n}] \\ 2^n(\frac{2(m+1)}{2^n} - x) & \text{when } x \in [\frac{2m+1}{2^n}, \frac{2(m+1)}{2^n}] \end{cases}$$

for $m \in \{0, 1, \ldots, 2^n-1\}$. Fixed points of $T^n$, or n-periodic points of T, are clearly of the form $\frac{2m}{2^n-1}$ or $\frac{2m}{2^n+1}$. Given an n-periodic point $x \in (0, 1)$, the average

$$\bar{x} = \frac{1}{n} \left(x + T(x) + \cdots + T^{n-1}(x)\right)$$

over its orbit is called the center of mass of the orbit.

In [3] Misiurewicz noted that two cycles with mirror itineraries have the same center of mass. He then raised an interesting question: Can there be three different cycles with the same center of mass? The answer is affirmative as can be easily verified from the distinct orbits of

$$\begin{array}{cccc}
166 & 202 & 278 & 418 \\
4095 & 4095 & 4095 & 4095.
\end{array}$$

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or the distinct orbits of
\[
\frac{42}{4097}, \frac{54}{4097}, \frac{134}{4097}, \frac{190}{4097}
\]
The orbits of
\[
\frac{2250}{16385}, \frac{2266}{16385}, \frac{2446}{16385}, \frac{2490}{16385}, \frac{2510}{16385}
\]
are also distinct but with the same mass center, and so are the orbits of
\[
\frac{1510}{65535}, \frac{1658}{65535}, \frac{2270}{65535}, \frac{3566}{65535}, \frac{3830}{65535}, \frac{3938}{65535}
\]
As the denominator increases there are more and more examples of distinct cycles with the same mass center. The search for \(n\) distinct cycles with the same mass center becomes a challenging task when \(n\) is large. In this paper we prove that

**Theorem 1.** For any positive integer \(n\), there exist \(n\) distinct cycles for the standard tent map with the same center of mass.

The definition for center of mass can be extended to non-periodic orbits. For any \(x \in [0, 1]\), if the partial time average \(\frac{1}{n} \sum_{k=0}^{n-1} T^k x\) converges as \(n \to \infty\), the limit is called the *center of mass* or the *time average* of the orbit. This definition clearly coincides with the definition on periodic orbits. Since the Lebesgue measure is an ergodic invariant measure, by the Birkhoff ergodic theorem, we know that for almost every \(x\) (in the sense of Lebesgue) the mass center of its orbit is \(\frac{1}{2}\), which is the space average of \(T\). Also, for almost every \(x\) the frequency of appearance of its forward orbit in a measurable set \(A\) is equal to the Lebesgue measure of \(A\).

For interval maps with an absolutely continuous ergodic invariant measure, in literature there is little study on the collection of exceptional orbits, such as orbits without mass center, and orbits with mass centers different from the space average. These orbits are also of our interest because they exhibit dynamics of the map with different degrees of randomness. What can we say about the set of orbits without mass center, except for being a zero measure set? We will show that

**Theorem 2.** There are uncountably many orbits of the standard tent map without mass center.

An interesting comparison with the tent map is the irrational rotation which maps \(x \in [0, 1]\) to \(x + \alpha \mod 1\) for some fixed \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). Irrational rotations also has the Lebesgue measure as an ergodic invariant measure but every orbit is equidistributed and thus has \(\frac{1}{2}\) as its mass center (see [1]); in other words, there is no exceptional orbit for irrational rotations.

Counting all possible orbits, periodic and non-periodic ones, how do their mass centers distribute over \([0, 1]\)? Consider the set of all possible mass centers. It might appear that this set is a meagre set, but it is indeed not the case. We will show in this paper that

**Theorem 3.** The set of all mass centers for the standard tent map is the interval \([0, \frac{2}{3}]\). The set of mass centers for cycles is a dense subset of this interval. Moreover, for every \(x\) in \([0, \frac{2}{3}]\), there are uncountably many orbits with mass center \(x\).

In section 2 we provide a sufficient condition for two periodic orbits to have the same mass center. Section 3 is devoted to the proof of Theorem 1. Theorem 2 and Theorem 3 are both proved in section 4. Several examples and related problems can be found in section 5 and 6.
2. Binary representations

In this section we introduce the concept of trace vector and establish a sufficient condition for periodic orbits of certain type to have the same mass center.

Throughout this and the next section we consider $n$-periodic points of the form $x = \frac{2m}{2^n - 1}$. The numerator $2m$ of $x$ can be uniquely written as

$$a_1 2^{n-1} + a_2 2^{n-2} + \cdots + a_{n-1} 2^1$$

with $a_i \in \{0, 1\}$ for all $i$. For convenience we add the term $a_n 2^0$ with coefficient $a_n = 0$ and write the numerator as a vector in $(\mathbb{Z}/2\mathbb{Z})^n$:

$$2m = [a_1, a_2, \cdots, a_n].$$

The tent map $T$ acts on the vector as a cyclic left-shift when $a_1 = 0$, while 0’s and 1’s are swapped after a cyclic left-shift when $a_1 = 1$. More precisely, the tent map induces a map $T_*$ from $\{0, 1, 2, \cdots, 2^n - 1\}$ to $\{0, 2, 4, \cdots, 2^{n-1} - 2\}$ given by

$$T_*[a_1, a_2, \cdots, a_n] = \begin{cases} [a_2, a_3, \cdots, a_n, a_1] & \text{if } a_1 = 0 \\ [1 - a_2, 1 - a_3, \cdots, 1 - a_n, 1 - a_1] & \text{if } a_1 = 1. \end{cases}$$

$$= [a_2, a_3, \cdots, a_n, a_1] + [a_1, a_1, \cdots, a_1, a_1]$$

$$= [a_1, a_2, \cdots, a_{n-1}, a_n]E F,$$

where $E, F \in M_{n \times n}(\mathbb{Z}/2\mathbb{Z})$:

$$E = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad F = I_n + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Note that the last entry of $T_*[a_1, a_2, \cdots, a_n]$ is always zero. Therefore,

$$T_*^k[a_1, a_2, \cdots, a_n] = [a_1, a_2, \cdots, a_n](EF)^k$$

$$= [a_1, a_2, \cdots, a_n]E^k (E^{1-k} F E^{k-1})(E^{2-k} F E^{k-2}) \cdots (E^{-1} F E^1) F$$

$$= [a_{k+1}, \cdots, a_n, a_1, \cdots, a_k] + [a_k, a_k, \cdots, a_k].$$

The last identity holds because

$$[a_1, a_2, \cdots, a_n]E^k = [a_{k+1}, \cdots, a_n, a_1, \cdots, a_k]$$

and the collected effect of

$$(E^{1-k} F E^{k-1})(E^{2-k} F E^{k-2}) \cdots (E^{-1} F E^1) F$$

is adding a vector $[c, c, \cdots, c]$ to $[a_{k+1}, \cdots, a_n, a_1, \cdots, a_k]$. Since the last entry of $T_*^k[a_1, a_2, \cdots, a_n]$ is zero, $c$ must be equal to $a_k$. 


The orbit of \( x \) is encoded in the \( n \times n \) matrix \( A(x) \in M_{n \times n}(\mathbb{Z}/2\mathbb{Z}) \):

\[
A(x) = \begin{bmatrix}
    a_1, a_2, \cdots, a_n \\
    T_r[a_1, a_2, \cdots, a_n] \\
    T_s[a_1, a_2, \cdots, a_n] \\
    \vdots \\
    T^{n-1}_s[a_1, a_2, \cdots, a_n]
\end{bmatrix}
\]

where \( \alpha = [a_1, a_2, \cdots, a_n]^T \). Let \( \beta = \alpha + E\alpha \) and

\[
C_k = I + E^{-1} + \cdots + E^{-k}.
\]

Then

\[
E^{-k}\alpha + E\alpha = (I + E^{-1} + \cdots + E^{-k})(\alpha + E\alpha) = C_k\beta.
\]

Therefore

\[
A(x) = [\beta, C_1\beta, \cdots, C_{n-1}\beta].
\]

The last column of \( A(x) \) is zero. Since all entries of the matrix \( C_{n-1} \) are 1, we conclude that the number of 1’s in \( \beta \) must be even. Also note that the first column \( \beta = \alpha + E\alpha \) of \( A(x) \) is exactly the itinerary of \( x = \frac{2m}{2^n-1} \); that is, \( \beta_1 = \chi_{[\frac{1}{2},1]}(T^{i-1}(x)) \) for each \( i \).

Note that \( \frac{2m}{2^n-1} \) and \( \frac{2k}{2^n-1} \) can not have the same itinerary unless \( m = k \). Since there are \( 2^{n-1} \) vectors of the form \( [a_1, \cdots, a_{n-1}, 0] \) and \( 2^{n-1} \) possible itineraries \( \beta \) (which must have evenly many 1’s), the correspondence between \( x = \frac{2m}{2^n-1} \) and \( \beta \) is necessarily bijective.

Let \( s_j(\beta) = s_j \) be the number of 1’s in the \( j \)-th column \( C_{j-1}\beta \) of \( A(x) \). We call

\[
s(\beta) = (s_1, s_2, \cdots, s_n) \in \mathbb{Z}^n
\]

the trace vector of \( x \). If follows easily from (1) that \( s_n = 0 \) and each \( s_j \) is even. Since

\[
C_{n-k-1}\beta = E^{1-(n-k)}\alpha + E\alpha = E^{1+k}\alpha + E\alpha = E^k(E\alpha + E^{1-k}\alpha) = E^kC_{k-1}\beta
\]

we also have

(2) \[ s_{n-k} = s_k \text{ for } k = 1, \cdots, n-1 \].

Write \( A(x) = [h_{ij}] \) with indices \( i \) and \( j \) ranging from 1 to \( n \), then \( h_{ij} = \sum_{r=0}^{j-1} \beta_{i+r} \). It is understood that the subindex \( i+r \) of each \( \beta_{i+r} \) is in \( \mathbb{Z}/n\mathbb{Z} \). The orbit of \( x \) has mass center

\[
\bar{x} = \frac{1}{(2^n-1)n} \sum_{i,j=1}^n 2^{-j} h_{ij} = \frac{1}{(2^n-1)n} \sum_{j=1}^n 2^{-j} s_j.
\]

Thus for two \( n \)-periodic points \( x = \frac{2m}{2^n-1} \) and \( y = \frac{2k}{2^n-1} \), a sufficient condition for \( \bar{x} = \bar{y} \) is that they have the same trace vector.
Let $\beta$ and $\gamma$ denote respectively the itineraries of $n$-periodic points $x$ and $y$. As explained earlier, $x = y$ if and only if $\beta = \gamma$. It follows that $x$ and $y$ are in the same cycle if and only if $\gamma = E^k\beta$ for some $k$, in which case we will say that $\beta$ and $\gamma$ are equivalent.

Let
\[
D = \begin{bmatrix}
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 
\end{bmatrix}.
\]

Then $D$ and $E$ generate a dihedral group of order $2n$ with relations
\[
E^n = D^2 = (ED)^2 = I_n.
\]

Observe that
\[
C_k E = E C_k, \quad C_k D = D E^k C_k
\]
for all $k$.

If $\gamma = E\beta$, then $C_k \gamma = C_k E \beta = E C_k \beta$ for all $k$. Thus $A(x)$ and $A(y)$ have the same trace vector as expected.

If $\gamma = D\beta$, then $C_k \gamma = C_k D \beta = D E^k C_k \beta$ for all $k$. Thus $A(x)$ and $A(y)$ have the same trace vector. Note that $\gamma$ is the mirror image of $\beta$. In this case $x$ and $y$ belong to different cycles except when $\beta = D E^k \beta$ for some $k$. For instance, the point $x = \frac{26}{127}$ has itinerary
\[
\beta = [0, 0, 1, 0, 1, 1]^T.
\]
The itinerary of $y = \frac{88}{127}$ is $\beta$ reversed. These two points $x$ and $y$ belong to two different cycles with the same center of mass $\frac{72}{127}$.

One might attempt to look for other permutation matrices $P$ with the nice property that $s(\beta) = s(P\beta)$ for all itineraries $\beta$. Unfortunately there are no such permutation matrices other than those in the dihedral group generated by $D$ and $E$. In order to find three or more different cycles with the same mass center, we need more explicit constructions, as to be shown in the next section.

### 3. A Family of Cycles with the Same Mass Center

In order to construct $n$ distinct cycles with the same mass center, we will find $n$ ($8n$)-periodic points such that their itineraries $\beta^{(1)}, \ldots, \beta^{(n)}$ have the same trace vector, yet $\beta^{(i)}$ and $\beta^{(j)}$ are non-equivalent when $i \neq j$. The “$n$” in this section is not to be confused with the “$n$” in the previous section, which is replaced by “$8n$” here for convenience.

For each $1 \leq m \leq n$, let
\[
\beta^{(m)} = [\beta_1^{(m)}, \beta_2^{(m)}, \ldots, \beta_{8n}^{(m)}]^T
\]
be defined by the following conditions
\[
\beta_i^{(m)} + \beta_{4n+i}^{(m)} = 1 \quad \text{for } i = 1, 2, \ldots, 4n,
\]
\[
\beta_i^{(m)} = \begin{cases} 
1 & \text{if } i \in \{2m, 2m + 1\} \\
0 & \text{if } i \in \{1, 2, \ldots, 4n\} \setminus \{2m, 2m + 1\}.
\end{cases}
\]
It is easy to see that these conditions uniquely determine \( \beta^{(m)} \). It is understood that the subindex \( i \) of each \( \beta_{i}^{(m)} \) is in \( \mathbb{Z}/8n\mathbb{Z} \). For instance, when \( n = 3 \), the \( \beta^{(m)} \)'s are

\[
\beta^{(1)} = [0, 1, 1, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 1, 1, 1, 1, 1, 1, 1, 1, 1]^{T}
\]

\[
\beta^{(2)} = [0, 0, 0, 1, 1, 1, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 1, 1, 1, 1, 1, 1, 1, 1, 1]^{T}
\]

\[
\beta^{(3)} = [0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^{T}
\]

The vertical bars are inserted to the vectors above to improve readability.

By (2), we only need to show that \( s_{k}(\beta^{(m)}) \) is independent of \( m \) for \( 1 \leq k \leq 4n \) because of the symmetry of the trace vector.

If \( k \) is odd, then for each \( i \in \mathbb{Z}/8n\mathbb{Z} \) we have

\[
b^{(m)}_{i,k} + b^{(m)}_{4n+i,k} = \sum_{j=0}^{k-1} \beta_{i+j}^{(m)} + \sum_{j=0}^{k-1} \beta_{4n+i+j}^{(m)} = k \equiv 1 \mod 2.
\]

Hence there are as many 1's as 0's in \( [b^{(m)}_{1,k} b^{(m)}_{2,k} \cdots b^{(m)}_{8n,k}]^{T} \), and therefore \( s_{k}(\beta^{(m)}) = 4n \) when \( k \) is odd.

Next we note that

\[
b^{(m)}_{i,4n} = \sum_{j=0}^{4n-1} \beta_{i+j}^{(m)} = b^{(m)}_{i+1,4n} + \beta_{i}^{(m)} - \beta_{4n+i}^{(m)} = b^{(m)}_{i+1,4n} + 1.
\]

Hence there are as many 1's as 0's in \( [b^{(m)}_{1,4n} b^{(m)}_{2,4n} \cdots b^{(m)}_{8n,4n}]^{T} \), and therefore \( s_{4n}(\beta^{(m)}) = 4n \).

We now assume that \( k = 2r \) where \( 1 \leq r < 2n \). Note that

\[
\beta^{(m)}_{2i-1} + \beta^{(m)}_{2i} = \begin{cases} 1 & \text{when } i \in \{m, m+1, 2n+m, 2n+m+1\}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\beta^{(m)}_{2i} + \beta^{(m)}_{2i+1} = \begin{cases} 1 & \text{when } i \in \{0, 2n\}, \\ 0 & \text{otherwise}. \end{cases}
\]

It follows that

\[
b^{(m)}_{2i-1,2r} = \sum_{j=0}^{r-1} \left( \beta^{(m)}_{2i+2j-1} + \beta^{(m)}_{2i+2j} \right) = \begin{cases} 1 & \text{if } i \in \{m+1, m-r+1, 2n+m+1, 2n+m+1\}, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
b^{(m)}_{2i,2r} = \sum_{j=0}^{r-1} \left( \beta^{(m)}_{2i+2j} + \beta^{(m)}_{2i+2j+1} \right) = \begin{cases} 1 & \text{if either } \{0, 1\} \text{ or } \{4n, 4n+1\} \subset \{2i, 2i+1, \cdots, 2i+2r-1\}, \\ 0 & \text{otherwise}. \end{cases}
\]

There are four 1's in the first case and 2r 1's in the second case. Hence \( s_{k}(\beta^{(m)}) = 2r + 4 = k + 4 \).

It remains to show that if \( i \neq j \) then \( \beta^{(i)} \) and \( \beta^{(j)} \) are not equivalent; that is, \( \beta^{(i)} \) cannot be obtained from \( \beta^{(j)} \) by a cyclic rotation. This follows from the fact that if we write \( \beta^{(m)} \) around a circle, then the length of the longest substring of consecutive 0's is \( 4n - 2m - 1 \). Thus the \( \beta^{(m)} \)'s
are itineraries of points in \( n \) distinct cycles with the same center of mass. The proof of Theorem 1 is now concluded.

**Remark 1.** It is easy to see that the definition of \( \beta^{(m)} \) can be extended to all \( 1 \leq m < 2n \). In fact, all these \( 2n - 1 \) itineraries are pairwise non-equivalent. Moreover there is another itinerary with the same trace vector, making a total of \( 2n \) distinct \( (8n) \)-cycles with the same mass center.

Replacing \( 2n \) by \( n \), we may construct \( n \) distinct \( (4n) \)-cycles with the same mass center as follows. For each \( 0 \leq m < n \), let

\[
\gamma^{(m)} = [\gamma_1^{(m)}, \gamma_2^{(m)}, \cdots, \gamma_{4n}^{(m)}] \in (\mathbb{Z}/2\mathbb{Z})^{4n}
\]

where \( \gamma_i^{(m)} + \gamma_{2n+i}^{(m)} = 1 \) for \( i = 1, 2, \cdots, 2n \). For \( m = 0 \) define

\[
\gamma_i^{(0)} = \begin{cases} 
1 & \text{if } i \in \{1,3\} \\
0 & \text{if } i \in \{1,2,\cdots,2n\} \setminus \{1,3\}
\end{cases}
\]

and for \( 1 \leq m < n \) define

\[
\gamma_i^{(m)} = \begin{cases} 
1 & \text{if } i \in \{2m,2m+1\} \\
0 & \text{if } i \in \{1,2,\cdots,2n\} \setminus \{2m,2m+1\}.
\end{cases}
\]

Then a similar calculation shows that \( s(\gamma^{(m)}) \) is independent of \( m \), with \( s_{2r-1} = s_{2n} = 2n \) and \( s_{2r} = 2r + 4 \) for \( r = 1, \cdots, n \). It is also not hard to verify that if \( i \neq j \) then \( \gamma^{(i)} \) and \( \gamma^{(j)} \) are not equivalent. Therefore we obtain \( n \) distinct \((4n)\)-cycles with the same mass center.

**Remark 2.** Let \( x^{(m)} \) denote the \((4n)\)-periodic point with itinerary \( \gamma^{(m)} \) for \( 0 \leq m < n \). Then it can be calculated that

\[
x^{(0)} = \frac{2}{3(2^{2n} + 1)} + \frac{2^{4n-2} - 2^{2n-2}}{2^{4n} - 1};
\]

\[
x^{(m)} = \frac{2}{3(2^{2n} + 1)} + \frac{2^{2n-2m} - 2^{2n} - 1}{2^{2n} - 1} \text{ for } m = 1, \cdots, n - 1.
\]

Using equations (2), (3) and the common trace vector of \( x^{(m)} \)'s, one can calculate the common mass center of their orbits:

\[
\bar{x} = \frac{1}{4n(2^{2n} - 1)} \left[ 2^{2n}s_{2n} + \sum_{j=1}^{4n-1} 2^{4n-j}s_j + \sum_{j=2}^{2n-2} 2^{4n-j}s_j + \sum_{j=2n+2}^{4n-2} 2^{4n-j}s_j \right]
\]

\[
= \frac{1}{4n(2^{2n} - 1)} \left[ 2n \left( 2^{2n} + \sum_{r=0}^{2n-1} 2^{2r+1} \right) + \sum_{r=0}^{n-1} \left( (2r+4)2^{4n-2r} + (2n-2r+4)2^{2n-2r} \right) \right]
\]

\[
= \frac{1}{3} + \frac{5}{9n} - \frac{13}{9n(2^{2n} + 1)} - \frac{2}{n(2^{4n} - 1)}.
\]

If we replace \( n \) by \( 2n \) in these formulas, then we obtain the \((8n)\)-periodic points with itineraries \( \beta^{(m)} \) and the common mass center of their orbits.

4. **Orbits without mass center and the set of mass centers**

In this section we construct uncountably many orbits without mass center. We also show that the set of all mass centers is the interval \([0, \frac{2}{3}]\) and the set of mass centers of cycles is dense in \([0, \frac{2}{3}]\). Moreover, for every \( x \) in \([0, \frac{2}{3}]\), there are uncountably many orbits with mass center \( x \).
We start with some simple calculations. Given a positive integer $k$, for any $0 \leq i \leq k$ we have

$$ T^i \left( \frac{2}{3} \left( \frac{1}{2^{k-i}} + 2^i \varepsilon \right) \right) = \frac{2}{3} \left( \frac{1}{2^{k-i}} + 2^i \varepsilon \right) \quad \text{if} \quad \frac{-1}{2^k} \leq \varepsilon \leq \frac{1}{2^{k+1}}, $$

and

$$ T^i \left( \frac{2}{3} (1 + \varepsilon) \right) = \frac{2}{3} \left( 1 + (-1)^i 2^i \varepsilon \right) \quad \text{if} \quad \frac{-1}{2^{k+1}} \leq \varepsilon \leq \frac{1}{2^k}. $$

In particular, both identities are valid for $|\varepsilon| \leq \frac{1}{2^{k+1}}$.

Consider an infinite sequence of even integers $\{a_n\}_{n=0}^{\infty}$, where $a_0 = 0$ and $a_n < a_{n+1}$ for all $n$.

Let

$$ c = \frac{2}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{a_k}}. $$

(8)

Then

$$ T^1(c) = \frac{2}{3} \left( 1 + \sum_{k \geq 1} (-1)^{k-1} \frac{1}{2^{a_k-1}} \right) $$

and

$$ T^{a_1}(c) = \frac{2}{3} \left( 1 + \sum_{k \geq 1} (-1)^{k-a_1} \frac{1}{2^{a_k-a_1}} \right) = \frac{2}{3} \sum_{k \geq 2} \frac{(-1)^k}{2^{a_k-a_1}} $$

$$ T^{a_1+1}(c) = \frac{2}{3} \sum_{k \geq 2} \frac{(-1)^k}{2^{a_k-a_1-1}} $$

and

$$ T^{a_2}(c) = \frac{2}{3} \sum_{k \geq 2} \frac{(-1)^k}{2^{a_k-a_2}} = \frac{2}{3} \left( 1 + \sum_{k \geq 3} (-1)^{k-a_2} \frac{1}{2^{a_k-a_2}} \right). $$

The $a_k$’s are assumed to be even so the signs come out just right. Proceed inductively, then for any $m \geq 0$ we have

$$ T^n(c) = \frac{2}{3} \left( 1 + \sum_{k \geq 2m+1} (-1)^{k-n} \frac{1}{2^{a_k-n}} \right) \quad \text{if} \quad a_{2m} \leq n \leq a_{2m+1} $$

and

$$ T^n(c) = \frac{2}{3} \sum_{k \geq 2m+2} \frac{(-1)^k}{2^{a_k-n}} \quad \text{if} \quad a_{2m+1} \leq n \leq a_{2m+2}. $$
Therefore,
\[
\sum_{n=a_{2m}}^{a_{2m+1}-1} T^n(c) = \frac{2}{3} \left[ (a_{2m+1} - a_{2m}) + \left( \sum_{n=a_{2m}}^{a_{2m+1}-1} \frac{(-1)^n}{2^{a_{2m+1}-n}} \right) \left( \sum_{k=2m+1}^{\infty} \frac{(-1)^k}{2^{a_k-a_{2m+1}}} \right) \right]
\]
\[
= \frac{2}{3} \left[ (a_{2m+1} - a_{2m}) + \frac{1}{3} \left( 1 - \frac{1}{2^{a_{2m+1}-a_{2m}}} \right) \left( \sum_{k=2m+1}^{\infty} \frac{(-1)^k}{2^{a_k-a_{2m+1}}} \right) \right]
\]
\[
\sum_{n=a_{2m+1}}^{a_{2m+2}-1} T^n(c) = \frac{2}{3} \left( \sum_{n=a_{2m+1}}^{a_{2m+2}-1} \frac{1}{2^{a_{2m+2}-n}} \right) \left( \sum_{k=2m+2}^{\infty} \frac{(-1)^k}{2^{a_k-a_{2m+2}}} \right)
\]
\[
= \frac{2}{3} \left( 1 - \frac{1}{2^{a_{2m+2}-a_{2m+1}}} \right) \left( \sum_{k=2m+2}^{\infty} \frac{(-1)^k}{2^{a_k-a_{2m+2}}} \right).
\]

It follows that
\[
\frac{2}{3} (a_{2m+1} - a_{2m}) < \sum_{n=a_{2m}}^{a_{2m+1}-1} T^n(c) < \frac{2}{3} (a_{2m+1} - a_{2m}) + 1
\]
and
\[
\frac{2}{3} (a_{2m+1} - a_{2m}) < \sum_{n=a_{2m+1}}^{a_{2m+2}-1} T^n(c) < \frac{2}{3} (a_{2m+1} - a_{2m}) + 1
\]

We can now make use of these calculations to prove Theorem 2. Let \( \{a_n\}_{n=0}^{\infty} \) be a strictly increasing sequence of even integers with \( a_0 = 0 \) and \( a_{2k} = 10^{2k}, a_{2k+1} = 10^{2k+1} \) for infinitely many \( k \)'s. For such a \( k \),
\[
\frac{1}{10^{2k}} \sum_{n=0}^{10^{2k}-1} T^n(c) < \frac{1}{10^{2k}} \left( 10^{2k-1} + \sum_{n=10^{2k-1}}^{10^{2k}-1} T^n(c) \right) < \frac{10^{2k-1} + 1}{10^{2k}} \leq \frac{1}{5}.
\]

On the other hand
\[
\frac{1}{10^{2k+1}} \sum_{n=0}^{10^{2k+1}-1} T^n(c) > \frac{1}{10^{2k+1}} \sum_{n=10^{2k}}^{10^{2k+1}-1} T^n(c) > \frac{2}{3} \left( \frac{10^{2k+1} - 10^{2k}}{10^{2k+1}} \right) = \frac{3}{5}.
\]

Thus the orbit of \( c \) has no mass center. It can be easily seen that there are uncountably many sequences \( \{a_n\}_{n=0}^{\infty} \) with the prescribed properties, where hence their corresponding points in (8) generate uncountably many distinct orbits (since every orbit is countable). This proves Theorem 2.

If \( \frac{2}{3} \leq x \leq 1 \), then the average
\[
\frac{x + T(x)}{2} = 1 - \frac{x}{2} \leq \frac{2}{3}.
\]

From this it follows easily that the mass center of an orbit can be at most \( \frac{2}{3} \), if it exists. We are now ready to show that mass centers of cycles are dense in the interval \([0, \frac{2}{3}]\). It suffices to show the following: for any rational number \( r \in [0, \frac{2}{3}] \) and any \( \epsilon > 0 \), there exists a periodic-point \( c \) such that \( |c - r| < \epsilon \) where \( c \) is the mass center of the periodic orbit of \( c \). We may assume that
Let \( r \neq \frac{2}{3} \). Let \( r = \frac{p}{q} \) where \( p \) and \( q \) are positive integers such that \( 3p < 2q \). Fix a positive even integer \( t \), for \( m \geq 0 \) let
\[
a_{2m} = m(2qt) \quad \text{and} \quad a_{2m+1} = 3pt + m(2qt).
\]
Then \( \{a_n\} \) is a strictly increasing sequence of even integers with \( a_0 = 0 \), and
\[
c = \frac{2}{3} \sum_{i=0}^{\infty} \frac{(-1)^i}{2^{a_i}} = \frac{2}{3} \left( 1 - \frac{1}{2^{a_1}} \right) \sum_{k=0}^{\infty} \frac{1}{2^{ka_2}} = \frac{2^{a_2-2} + (2^{a_1} - 1)}{3(2^{a_2} - 1)}.
\]
Since \( a_1 = 3pt \) is even, \( (2^{a_1} - 1) \) is divisible by \( 3 \). Therefore \( c \) is an \( a_2 \)-periodic point, and
\[
\bar{c} = \frac{1}{a_2} \sum_{n=0}^{a_2-1} T^n(c).
\]
Now we use the inequalities (9) and (10), then
\[
\frac{2}{3} (a_1 - 0) < \sum_{n=0}^{a_1-1} T^n(c) < \frac{2}{3} (a_1 - 0) + 1
\]
and
\[
0 < \sum_{n=a_1}^{a_2-1} T^n(c) < 1.
\]
Combining them we obtain
\[
\frac{p}{q} = \frac{\frac{2}{3} a_1}{a_2} < \frac{1}{a_2} \sum_{n=0}^{a_2-1} T^n(c) < \frac{(\frac{2}{3} a_1 + 1) + 1}{a_2} = \frac{p}{q} + \frac{1}{qt}.
\]
Choose \( t \) so that \( \frac{1}{qt} < \epsilon \), then
\[
|\bar{c} - r| = \left| \frac{1}{a_2} \sum_{n=0}^{a_2-1} T^n(c) - \frac{p}{q} \right| < \epsilon.
\]
We now show that for every \( x \) in the interval \([0, \frac{2}{3}]\), there are uncountably many orbits with mass center \( x \). First we deal with the case of \( x < \frac{2}{3} \). Let \( x_n = \lfloor nx \rfloor + e_n \), where \( e_n = 1 \) or \( 2 \); that is, \( x_n \) is one of the two possible integers such that
\[
x_n + 1 \leq x_n < nx + 3.
\]
Then there exists a positive integer \( n_0 \) such that \( 0 < 3x_n < 2n \) if \( n \geq n_0 \). For \( m \geq 0 \) let
\[
a_{2m} = 4 \sum_{k=0}^{m-1} (k + n_0) = 2m(m + 2n_0 - 1) \quad \text{and} \quad a_{2m+1} = a_{2m} + 6x_{m+n_0}.
\]
Then \( \{a_n\} \) is a strictly increasing sequence of even integers with \( a_0 = 0 \), so we may define \( c \) by (8). Combining inequalities (9) and (10) we obtain
\[
4x_{m+n_0} < \sum_{n=a_{2m}}^{a_{2m+2}-1} T^n(c) < 4x_{m+n_0} + 2.
\]
For any positive integer $N$, there exists $M$ such that $a_{2M} \leq N < a_{2M+2}$. Thus we have
\[
\sum_{n=0}^{N-1} T^n(c) \geq \sum_{m=0}^{M-1} \sum_{n=a_{2m}}^{a_{2m+2}-1} T^n(c) > 4 \sum_{m=0}^{M-1} x_{m+n_0} \geq 4 \sum_{m=0}^{M-1} [(m + n_0)x + 1] = 4x \sum_{m=0}^{M-1} (m + n_0) + 4M = xa_{2M} + 4M
\]
and
\[
\sum_{n=0}^{N-1} T^n(c) \leq \sum_{m=0}^{M} \sum_{n=a_{2m}}^{a_{2m+2}-1} T^n(c) < 4 \sum_{m=0}^{M} x_{m+n_0} + 2(M + 1) \leq 4 \sum_{m=0}^{M} [(m + n_0)x + 3] + 2(M + 1) = xa_{2M+2} + 14(M + 1).
\]
Hence
\[
x a_{2M} - \frac{4M}{N} < \frac{1}{N} \sum_{n=0}^{N-1} T^n(c) < x a_{2M+2} - \frac{14(M + 1)}{N}.
\]
Note that $a_{2M}$, $N$ and $a_{2M+2}$ are all of order $2M^2(1 + o(1))$. Let $N \to \infty$ and we obtain $\bar{c} = x$ as desired. Since we have two choices for each $x_n$, this produces uncountably many different $c$.

It remains to deal with the case $x = \frac{2}{3}$. In this case let $x_n = \lceil nx \rceil - e_n$, where $e_n = 1$ or 2. Once again there exists a positive integer $n_0$ such that $0 < 3x_n < 2n$ if $n \geq n_0$. The rest is pretty much the same as before. This concludes the proof of Theorem 3.

5. Further Examples

There are several patterns of itineraries that can be suitably permuted without altering their trace vectors. We will discuss one of them to which $\beta^{(1)}$ and $\gamma^{(1)}$ in section 3 belong.

To simplify our notations, we will denote itineraries by row vectors instead of column vectors in this section. For any string $u = [u_1, \cdots, u_k]$ consisting of 0’s and 1’s, $u^{-1}$ denotes the string obtained by reversing the order of $u_i$’s in $u$, and $|u|$ denotes the sum of $u_i$’s over $\mathbb{Z}/2\mathbb{Z}$. All itineraries are treated as cyclic vectors.

**Proposition 4.** Let $\beta$ be an itinerary of the form
\[
\beta = [0, p, 0, q, 1, p^{-1}, 1, r]
\]
where $p = [p_1, \cdots, p_j]$, $q = [q_1, \cdots, q_k]$ and $r = [r_1, \cdots, r_l]$ are subject to conditions
\[
q = q^{-1}, \quad r = r^{-1}, \quad |q| = |r|.
\]
Then the trace vector of $\beta$ does not change if $p$ and $p^{-1}$ are switched; that is, $s(\beta) = s(\gamma)$, where
\[
\gamma = [0, p^{-1}, 0, q, 1, p, 1, r].
\]

Note that the presence of the two 0’s and 1’s implies that $\gamma$ is in general not in the orbit of $\beta$ under the action of the dihedral group (4). We will call the two 0’s and 1’s the *four corners* of $\beta$ and $\gamma$. 
Proof. Let \( n = 2j + 2k + 4 \) denote the length of \( \beta \). By (2), we only need to show that \( s_l(\beta) = s_l(\gamma) \) for \( 1 \leq l \leq n/2 \). Let us fix such an \( l \). Define

\[
\beta' = [0, p^{-1}, 0, r, 1, p, 1, q], \quad \beta'' = [1, p, 1, q, 0, p^{-1}, 0, r], \quad \beta''' = [1, p^{-1}, 1, r, 0, p, 0, q];
\gamma' = [0, p, 0, r, 1, p^{-1}, 1, q], \quad \gamma'' = [1, p^{-1}, 1, q, 0, p, 0, r], \quad \gamma''' = [1, p, 1, r, 0, p^{-1}, 0, q].
\]

Then \( \beta', \beta'' \) and \( \beta''' \) are in the same orbit of \( \beta \) under the action of \( \text{dihedral group (4)} \), so they all have the same trace vector. Similarly \( \gamma', \gamma'' \) and \( \gamma''' \) are in the same orbit of \( \gamma \), so they all have the same trace vector as well. For a cyclic vector \( x = [x_1, \ldots, x_n] \), let \( \tilde{x} = [x_i, \ldots, x_{i+l-1}] \) where it is understood that the subindices are in \( \mathbb{Z}/n\mathbb{Z} \). Since \( l \leq n/2 \), the substring \( \tilde{\beta} \) covers at most two corners of \( \beta \).

Case 1: \( \tilde{\beta} \) covers zero corners. If \( \tilde{\beta} \) covers nothing of \( p \) and \( p^{-1} \), then note that \( \gamma, \gamma', \gamma'' \) and \( \gamma''' \) are obtained from \( \beta, \beta', \beta'' \) and \( \beta''' \) by switching \( p \) and \( p^{-1} \). If \( \tilde{\beta} \) covers nothing of \( q \) and \( r \), then note that \( \gamma, \gamma', \gamma'' \) and \( \gamma''' \) are obtained from \( \beta, \beta', \beta'' \) and \( \beta''' \) by switching \( q \) and \( r \). In either case we have \( \{[\beta], [\beta'], [\beta''], [\beta''']\} \) and \( \{[\gamma], [\gamma'], [\gamma''], [\gamma''']\} \) are the same as multisets.

Case 2: \( \tilde{\beta} \) covers one corner. Notice that if we switch 0 and 1 in the four corners, then \( \beta \) becomes \( \beta'' \) and \( \beta' \) becomes \( \beta''' \). Therefore as multisets \( \{[\beta], [\beta'], [\beta''], [\beta''']\} = \{0, 0, 1, 1\} \). For similar reason we also have \( \{[\gamma], [\gamma'], [\gamma''], [\gamma''']\} = \{0, 0, 1, 1\} \).

Case 3: \( \tilde{\beta} \) covers two corners. The argument here is similar to Case 1. If \( \tilde{\beta} \) covers a complete \( p \) (or \( p^{-1} \)), then it covers nothing of \( p^{-1} \) (or \( p \)) since \( l \leq n/2 \). In this case switch \( p \) and \( p^{-1} \). If \( \tilde{\beta} \) covers a complete \( q \) (or \( r \)), then it covers nothing of \( r \) (or \( q \)). In this case switch \( q \) and \( r \). Since \( |p^{-1}| = |p| \) and \( |q| = |r| \), in either case we have \( \{[\beta], [\beta'], [\beta''], [\beta''']\} = \{[\gamma], [\gamma'], [\gamma''], [\gamma''']\} \).

Now let \( i \) run through \( \{1, \ldots, n\} \), then we have \( 4s_l(\tilde{\beta}) = 4s_l(\gamma) \).

Remark 3. Same arguments can be applied to itineraries of the form

\[
\beta = [0, p, 1, q, 0, p^{-1}, 1, r]
\]

where \( p = [p_1, \ldots, p_j], q = [q_1, \ldots, q_k] \) and \( r = [r_1, \ldots, r_k] \) are subject to conditions (11).

Example 1. Both the \( \beta^{(1)} \) in (5) and \( \gamma^{(1)} \) in (7) have the structure described in Proposition 4. For instance, the \( \beta^{(1)} \) in (6) as a cyclic vector can be written

\[
\beta^{(1)} = [0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0].
\]

By Proposition 4, the itinerary

\[
\beta^{(0)} = [0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].
\]

has the same trace vector as \( \beta^{(1)} \). Their trace vector is

\[
(12, 6, 12, 8, 12, 10, 12, 12, 12, 14, 12, 12, 12, 12, 12, 12, 10, 12, 8, 12, 6, 12, 0).
\]

Proposition 4 also gives another reason why the \( \gamma^{(0)} \) and \( \gamma^{(1)} \) in (7) have the same trace vector.

Example 2. Ideally, one would hope to have itineraries where Proposition 4 could be applied multiple times to produce many distinct cycles. Here we present such an interesting example for the case \( n = 20 \). Unfortunately it is unclear how to generalize it to general \( n \). Let

\[
\beta^{(0)} = [0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].
\]
After reversing $p_i$'s and shifting the cyclic vector, the new itinerary $\beta^{(1)}$ also has the pattern studied in Proposition 4. By repeating this process, we obtain

$$
\beta^{(1)} = [0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] \\
\beta^{(2)} = [0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1] \\
\beta^{(3)} = [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0].
$$

We resume $\beta^{(0)}$ by reversing the $p_i$'s in $\beta^{(3)}$. It is easy to see that these itineraries and their mirror itineraries are all different, yielding eight distinct cycles with the same mass center. In the order of $\beta^{(0)}, \ldots, \beta^{(3)}$ followed by their mirror itineraries, they correspond respectively to

$$
\begin{align*}
33658 & \quad 504896 & \quad 523412 & \quad 39664 & \quad 776224 & \quad 17852 & \quad 337916 & \quad 125728 \\
1048575 & \quad 1048575 & \quad 1048575 & \quad 1048575 & \quad 1048575 & \quad 1048575 & \quad 1048575 & \quad 1048575
\end{align*}
$$

Their common trace vector is

$$(8, 8, 8, 10, 10, 12, 12, 12, 12, 12, 12, 12, 10, 10, 10, 8, 8, 8, 0),$$

and their common center of mass is $\frac{2170984}{00212575} \approx 0.414048$.

**Example 3.** There are patterns of itineraries that allow much more distinct cycles with the same mass centers than the one constructed in section 3. However it is unclear how to generalize these patterns. The following itineraries along with their mirror itineraries correspond 16 cycles of period 24 with the same center of mass.

$$
\begin{align*}
\beta^{(0)} &= [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1] \\
\beta^{(1)} &= [0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1] \\
\beta^{(2)} &= [0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0] \\
\beta^{(3)} &= [0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0] \\
\beta^{(4)} &= [0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0] \\
\beta^{(5)} &= [0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0] \\
\beta^{(6)} &= [0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1] \\
\beta^{(7)} &= [0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1].
\end{align*}
$$

From $\beta^{(0)}$ to $\beta^{(7)}$, they are respectively itineraries of

$$
\begin{align*}
183958 & \quad 300446 & \quad 309014 & \quad 343334 & \quad 398774 & \quad 547438 & \quad 596602 & \quad 900526 \\
16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215
\end{align*}
$$

The mirror itineraries of $\beta^{(0)}, \ldots, \beta^{(7)}$ correspond respectively to

$$
\begin{align*}
13821568 & \quad 15946304 & \quad 13752896 & \quad 13203776 & \quad 14373056 & \quad 15512608 & \quad 12366112 & \quad 15432544 \\
16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215 & \quad 16777215
\end{align*}
$$

Their common trace vector is

$$(12, 12, 10, 14, 12, 12, 12, 12, 10, 14, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12),$$

and their common center of mass is $\frac{2170984}{00212575} \approx 0.494671$. 
Example 4. The criterion in section 2 for two distinct cycles to have the same mass center is sufficient but not necessary. For example, the fractions $\frac{5414}{131071}$ and $\frac{10090}{131071}$ are points of period 17 with itineraries
\[
\beta = [0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1]
\]
\[
\gamma = [0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1].
\]
Their trace vectors are respectively
\[
s(\beta) = (10, 8, 8, 8, 10, 8, 6, 8, 6, 8, 10, 8, 8, 8, 10, 0)
\]
\[
s(\gamma) = (10, 8, 8, 8, 14, 14, 10, 10, 8, 14, 6, 8, 8, 8, 10, 0).
\]
One can easily check that $\frac{1185588}{2228207} \approx 0.532082$ is their common center of mass.

Example 5. Cycles of different lengths may have the same center of mass. For instance, the orbit of $\frac{6}{275}$ has length 20 and the orbit of $\frac{1158}{1735}$ has length 24. One can easily verify that their mass centers are both equal to $\frac{3}{5}$.

We have mentioned in section 1 that, by the Birkhoff ergodic theorem, the mass center of generic orbits is $\frac{1}{2}$, the space average or integral of the tent map. Periodic orbits are exceptional but there are actually examples of cycles where the mass center is also $\frac{1}{2}$. For instance,
\[
\begin{align*}
19 & \quad 17 & \quad 13 & \quad 151 \\
18564^* & \quad 14460^* & \quad 4820^* & \quad 55692
\end{align*}
\]
are all 24-periodic points and $\frac{1}{2}$ is the common mass center of their orbits.

6. Final Remarks and Related Problems

The discussions presented so far are focused on periodic points of the form $\frac{2n-1}{2^{n+1}}$. They are also applicable to periodic points of the form $\frac{2m}{2^n+1}$ because $\frac{2m}{2^n+1} = \frac{2m(2^n-1)}{2^{n+1}}$. We say an $n$-cycle is of the first type if it is generated by points of the form $\frac{2m}{2^n+1}$, and it is of the second type if otherwise. An $n$-periodic point of the second type is a $2n$-periodic point of the first type. Two of the examples in the introduction are of the second type.

Let $c(n)$ denote the maximum number of distinct first type $n$-cycles with the same center of mass, and $d(n)$ denote the same number for second type $n$-cycles. Exhaustive search by computer shows that $c(n)$ and $d(n)$ are

| $n$  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|
| $c(n)$ | 1  | 2  | 4  | 4  | 4  | 4  | 4  | 6  | 8  | 8  |
| $d(n)$ | 1  | 2  | 4  | 4  | 5  | 4  | 4  | 4  | 8  | 8  |

Let $e(n)$ be the maximum number of distinct $n$-cycles of either type with the same center of mass. It turns out that $e(n) = \max\{c(n), d(n)\}$ for $n \leq 31$. Also note that in our notation $n$-cycles include all cycles whose lengths are divisors of $n$. If we only consider cycles whose lengths are exactly $n$, then the values of $e(n)$, $c(n)$, and $d(n)$ remain the same for $n \leq 31$. Are these observations true for arbitrary $n$?

Another interesting question about cycles with the same mass center is how fast $e(n)$ grows. From section 3 we know that $e(4n) \geq n$. Is $e(n)$ of order $O(n)$ as $n$ goes to infinity? If so, what are the values of $\liminf_{n \to \infty} \frac{e(n)}{n}$ and $\limsup_{n \to \infty} \frac{e(n)}{n}$? Intuitively $e(n)$ grows faster on composite numbers because itineraries of $n$-cycles with prime $n$ seems to admit fewer possible trace-invariant permutations. Note that the values of $e(29)$ and $e(31)$ are respectively less than half of $e(28)$ and $e(30)$. Each of $e(17)$ and $e(19)$ is attained at only one example where not all of the distinct cycles have the same trace vector. How fast does $e(n)$ grow on prime numbers?
Let $C_n$ be the set of reals with the property that each point of $C_n$ is the mass center of at least $n$ distinct cycles. It would be interesting to understand the intersection of the nested sequence $\{C_n\}$. Is this intersection nonempty? Equivalently, is there a point which is the mass center of infinitely many cycles?

Our work and the problems raised above are related to the class of exceptional orbits to which the Birkhoff ergodic theorem does not apply. Exceptional orbits are rare but their dynamics can be used to characterize different chaotic maps. We have seen that, for the standard tent map, the dynamics exhibited by the exceptional orbits are rich and maybe surprising. Similar dynamical phenomena may occur for many other chaotic interval maps or even higher dimensional maps.

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