Multiplier ideals of plane curve singularities via Newton polygons

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**ABSTRACT**

We give a description of the multiplier ideals and jumping numbers associated with a plane curve singularity in a smooth surface in terms of Newton polygons. Our approach is inspired by a theorem of Howald about multiplier ideals of Newton non-degenerate hypersurfaces and our results provide a generalization of it to the case of plane curve singularities. We use toroidal embedded resolutions, which can be applied to the case of quasi-ordinary hypersurface singularities.

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**Introduction**

Let $S$ be a smooth complex algebraic variety, $C$ be an integral effective divisor on $S$, and $\xi > 0$ be a rational number. The multiplier ideal associated with $C$ and $\xi$ is defined by

$$J(\xi C) := \pi_* \mathcal{O}_Y(K_\pi - \lfloor \xi \pi^*(C) \rfloor),$$

where $\pi : Y \to S$ is a log-resolution of $C$ and $K_\pi$ is the relative canonical divisor. For any point $o \in S$ there exists an increasing sequence $(\xi_i)_i$ of positive rational numbers, called the jumping numbers of $C$ at $o$, such that if $\xi_i \leq \xi < \xi_{i+1}$ then $J(\xi C) = J(\xi_i C) \supseteq J(\xi_{i+1} C)$. More generally, the multiplier ideals and their jumping numbers can be associated with an ideal sheaf on $S$, even if $S$ has some mild singularities. The study of multiplier ideals have become a central aspect of birational geometry, thanks to the vanishing theorems of Kawamata, Viehweg and Nadel, which were inspired by Kodaira vanishing theorem. We refer to Lazarsfeld’s book [26] for the historical aspects and more information on the subject. Many properties of the jumping numbers were studied by Ein, Lazarsfeld, Smith and Varolin [11]. The multiplier ideals provide a subtle measure of the singularities of the pair $(S, C)$ and enjoy a wealth of relations with other notions like the Hodge modules, mixed Hodge modules, $V$-filtrations of $D$-modules and poles of Igusa zeta functions which are of interest in singularity theory (see Budur’s survey [5]). The multiplier ideals can be defined in complex analytic terms without using any resolution of singularities. Favre and Jonsson studied multiplier ideals associated with an ideal of the ring of germs of holomorphic functions at a point of a smooth surface by using tree potentials on the valuative tree (see [13, 14] and Jonsson’s survey [25, Section 7]). Multiplier ideals are also connected with the study...
of singularities over fields of positive characteristic through test ideals (see for instance [35, Section 4]). Algorithms to compute multiplier ideals and jumping numbers were given Shibuta [32], and Berkesch and Leykin [4].

If $S$ is a complex surface there are many results about multiplier ideals and their associated jumping numbers. Järvilehto described the jumping numbers of a simple complete ideal in a two-dimensional regular local ring, and deduced a formula for the jumping numbers of a branch $(C, o)$ on a smooth surface $S$ (see [24]). Smith and Thompson introduced the notion of contribution of an exceptional prime divisor of the minimal embedded resolution $\pi$ of a plane curve singularity $C$ on a smooth surface $S$, and they related this notion with the rupture components of the divisor $\pi^*(C)$ (see [34]). Tucker [36] and Naie [27] gave also descriptions of the jumping numbers in this case. Tucker studied multiplier ideals on a surface $S$ with rational singularities by using the notion of critical contribution of a reduced exceptional divisor to a jumping number and applied it to give an algorithm to compute the jumping numbers (see [36, 37]). Moreover, he established an iterative relation between the jumping numbers of a branch and those of its approximate roots (see [36, Chapter VI]). Alberich-Carramiñana, Álvarez Montaner, and Dachs-Cadedu gave a different algorithm to compute the jumping numbers and the associated multiplier ideals (see [1]). Later on Alberich-Carramiñana, Álvarez Montaner and Blanco gave an algorithm to compute the integral closure of an ideal of $\mathbb{C}(x, y)$ in terms of monomials in a set of maximal contact elements of its minimal log-resolution. Combining this algorithm with the one in [1], they obtained an algorithm to compute the multiplier ideals and jumping numbers of an ideal of $\mathbb{C}(x, y)$ in terms of a system of generators of it (see [2]). In the case of an analytically irreducible plane curve singularity, a similar result describing the associated multiplier ideals was obtained with different methods in [16] and also in Zhang’s preprint [38]. Other recent works about jumping numbers of multiplier ideals in this context are [20, 21].

The aim of this paper is to develop an algorithmic and conceptually new description of the multiplier ideals associated with a plane curve singularity on a smooth surface. We give a combinatorial algorithm to provide the generators of the multiplier ideals and the jumping numbers associated with a plane curve singularity on a smooth surface in terms of a finite set of Newton polygons appearing in a toroidal embedded resolution process of the plane curve singularity. Our approach is inspired by Howald’s result about multiplier ideals of hypersurface singularities which are Newton non-degenerate (see [23]), but we do not require any Newton non-degeneracy hypothesis. Our motivation was to develop a method which can be extended to the study of higher-dimensional singularities, as irreducible quasi-ordinary hypersurface singularities (see [31, chapter 6]). In contrast to previous results in the literature, our techniques do not pass through the correspondence between antinef divisors and complete ideals, used in [2] for instance. This paper is a development of the PhD Theses of the third and fourth named authors [16, 31].

In order to state our results, we outline briefly the construction of a toroidal embedded resolution of a plane curve singularity $(C, o)$ embedded in a complex smooth surface $(S, o)$. These kinds of resolutions are described in Section 3.4, following the presentation given by García Barroso, Popescu-Pampu and the first named author in [18]. The process of toroidal embedded resolution that we recall here is a slight generalization of the toroidal resolution processes of plane curves by Oka [30], Lê and Oka [10], A’Campo and Oka [3], the first named author [19] and, Cassou-Noguès and Libgober [8].

Let us denote by $\mathcal{O} \cong \mathbb{C}(x, y)$ the local ring of germs of holomorphic functions of $S$ at $o$. We start by fixing a cross $(R, L)$ at $o$, which is an ordered pair of smooth transversal branches defined by the vanishing locus of the entries of a local coordinate system $(x, y)$ of $S$ at $o$. We assume that $R$ is not a component of $C$. We denote by $\mathcal{N}_{R,L}(C)$ the Newton polygon of any defining function $f_C \in \mathcal{O}$ of $C$. Then, we consider the regularized Newton modification of $C$ with respect to the cross $(R, L)$, which is a toric modification defined in terms of the Newton polygon $\mathcal{N}_{R,L}(C)$. Its exceptional curve $E$ intersects the strict transform of $C$ only at smooth points of $E$. If $o'$ is any of these points, we denote by $R'$ the germ of $E$ at $o'$, and we choose a cross $(R', L')$ at $o'$ by taking $L'$ as a curvettta of $E$ at $o'$. Then, we iterate this procedure until we get a toroidal embedded resolution $\pi : \Sigma \to S$ of $C$. In the process, we have a finite number of
crosses \((R_i, L_i)\) at some infinitely near points \(o_i\) of \(o\), for \(i \in I_\pi\). The completion \(\hat{C}_\pi\) of \(C\) relative to \(\pi\) is the reduced plane curve whose branches are \(R\) and the projections by \(\pi\) of the smooth branches \(L_i\), for \(i \in I_\pi\). By definition every branch of \(C\) is a branch of \(\hat{C}_\pi\).

It is possible to choose \(\pi\) in such a way that \(\pi\) is the minimal embedded resolution of \(C\) and the components of \(\hat{C}_\pi\) are components of \(C\) or maximal contact curves of \(\pi\) (see Definition 2.3 and Remark 4.22).

Let us fix any toroidal embedded resolution \(\pi\) of \(C\). If \(h \in O\) defines a plane curve singularity \(C_h \subset S\), we denote by \(\hat{N}_{R_i,L_i}(C_h)\) the Newton polygon of the total transform of \(C_h\) with respect to the cross \((R_i, L_i)\) and by \(\lambda_{R_i}\) the log-discrepancy of \(R_i\), for \(i \in I_\pi\). In Theorem 4.9 we prove that:

**Theorem 1.** The multiplier ideal \(J(\xi C)_0\) consists of the functions \(h \in O\) such that the inclusion
\[
\hat{N}_{R_i,L_i}(C_h) + (\lambda_{R_i}, 1) \subseteq \text{Int}(\xi, \hat{N}_{R_i,L_i}(C))
\]
is satisfied for all \(i \in I_\pi\).

The proof is based on a basic property of Newton polyhedra, which is combined with the description of the log-discrepancies of the exceptional divisors in the toroidal embedded resolution (see Proposition 4.3). If \(C\) is Newton non-degenerate with respect to the coordinate system \((x, y)\) then we recover a particular case of Howald’s Theorem [23].

By convenience, if \(D\) is an exceptional prime divisor (resp. \(D\) is a branch on \(S\)) we denote by \(v_D\) the associated divisorial valuation (resp. vanishing order valuation). We prove that a function \(h \in O\) belongs to the multiplier ideal \(J(\xi C)_0\) if and only if \(v_D(C_h) + \lambda_D > \xi v_D(C)\), for \(D\) running through the rupture components of the divisor \(\pi^*(C)\) and the components of \(C\) (see Corollary 4.17). Denote by \([x_0, \ldots, x_s]\) the set obtained by taking a defining function for every branch of \(\hat{C}_\pi\). Every monomial \(M\) in \(x_0, \ldots, x_s\) determines the rational number
\[
\xi_M := \min_D = \{(v_D(M) + \lambda_D)(v_D(C))^{-1}\},
\]
where \(D\) runs through the rupture components of the divisor \(\pi^*(C)\) and the branches of \(C\).

The following theorem is based upon previous works of Spivakovsky [33], Delgado, Galindo, and Nuñez [9] and Robredo [31] about generating sequences of valuations (see Theorem 4.20).

**Theorem 2.** Every jumping number of \(C\) is of the form \(\xi_M\) and the multiplier ideal \(J(\xi C)_0\) is generated by the finite set of monomials \(M\) in \(x_0, \ldots, x_s\) such that \(\xi < \xi_M \leq \xi + 1\).

As a consequence of our main results, the computation of the multiplier ideals and the associated jumping numbers of \(C\) boils down to an optimization problem in terms of log discrepancies of the rupture components of \(\pi^*(C)\) and the values of the corresponding exceptional divisors on the functions \(x_0, \ldots, x_s\).

We have formulated our results in the complex analytic category, but they also hold for algebroid curves on a smooth surface over algebraically closed fields of arbitrary characteristic (see Remark 4.25).

The structure of the paper is as follows. In Section 1 we recall the basic notions about multiplier ideals. In Section 2 we introduce some notations and well-known results about plane curve singularities and valuations. In Section 3 we describe an algorithm of toroidal embedded resolution following [18]. The main results of the paper are proven in Section 4, where we illustrate our results with a detailed example.

### 1. Multiplier ideals and jumping numbers

In this section we briefly review basic definitions and properties of the theory of multiplier ideals. For further details we refer to [26, Chapter 9].
Let \( X \) be a smooth complex algebraic variety and let \( a \subseteq \mathcal{O}_X \) be an ideal sheaf. A **log-resolution** of \( a \) is a modification (proper and birational map) \( \pi : Y \to X \), with \( Y \) smooth, exceptional locus \( E \), and such that \( \pi^*a = \mathcal{O}_Y(-F) \), where \( F \) is an effective divisor such that \( F + E \) has simple normal crossings.

For \( a \in \mathbb{Q} \), we denote by \( \lfloor a \rfloor \) the greatest integer lower than or equal to \( a \). For a \( \mathbb{Q} \)-divisor \( D = \sum a_jD_j \), supported on the prime divisors \( D_j \), we denote by \( \lfloor D \rfloor := \sum \lfloor a_j \rfloor D_j \).

Let \( \pi : Y \to X \) be a log-resolution of an ideal sheaf \( a \) of \( \mathcal{O}_X \). Denote by \( \mathcal{K}_\pi \) the relative canonical divisor, which is equal to the divisor associated with the Jacobian determinant of \( \pi \). The **multiplier ideal sheaf** \( \mathcal{J}(a^\xi) \) associated to \( \xi \in \mathbb{Q}_{>0} \) and \( a \) is defined as \( \mathcal{J}(a^\xi) := \pi_* \mathcal{O}_Y(K_\pi - \lfloor \xi F \rfloor) \).

The definition of the multiplier ideal \( \mathcal{J}(a^\xi) \) relies on the choice of a log-resolution of \( a \), but it is independent of it (see [26, Theorem 9.2.8]). The multiplier ideal \( \mathcal{J}(a^\xi) \) can be characterized in terms of valuations. If \( E_i \) is a prime divisor on \( Y \) we denote by \( v_{E_i} \) the vanishing order valuation along \( E_i \). A prime divisor \( E_i \) contained in the support of \( F + K_\pi \) is either the strict transform of a divisor on \( X \) or must be contained in the exceptional divisor of \( \pi \). Let us write \( F = \sum r_iE_i \), and \( K_\pi = \sum (\lambda_{E_i} - 1)E_i \), where the \( E_i \) are the prime divisors in the support of \( E + F \) on \( Y \). Then,

\[
\mathcal{J}(a^\xi) = \{ h \in \mathcal{O}_X | v_{E_i}(h) \geq \lfloor \xi r_i \rfloor - (\lambda_{E_i} - 1) \text{ for all } i \},
\]

or, equivalently,

\[
\mathcal{J}(a^\xi) = \{ h \in \mathcal{O}_X | v_{E_i}(h) + \lambda_{E_i} \geq \xi r_i \text{ for all } i \}.
\]

The equivalence follows since for any \( a \in \mathbb{Z} \) and \( b \in \mathbb{Q} \) it holds that \( a \geq \lfloor b \rfloor \) if and only if \( a > b - 1 \). The number \( \lambda_{E_i} \) is called the **log-discrepancy** of the exceptional prime divisor \( E_i \). If \( B \subseteq Y \) is the strict transform of a prime divisor in \( X \) we set \( \lambda_B := 1 \).

The next lemma introduces some numerical invariants of set of multiplier ideals of an ideal sheaf \( a \) of \( \mathcal{O}_X \) at a point \( x \in X \).

**Lemma 1.3** (See [26], Lemma 9.3.21). Let \( X \) be a smooth algebraic variety and let \( a \subseteq \mathcal{O}_X \) be an ideal sheaf and \( x \in X \). There exists a strictly increasing discrete sequence \( (\xi_i) \) of positive rational numbers such that if \( \xi \in \mathbb{Q} \cap [\xi_i, \xi_{i+1}) \), then \( \mathcal{J}(a^\xi)_x = \mathcal{J}(a^{\xi_i})_x \supseteq \mathcal{J}(a^{\xi_{i+1}})_x \).

The numbers \( \xi_i \) are called the **jumping numbers** associated with \( a \) at \( x \). The smallest jumping number \( \xi_1 \) is called the **log-canonical threshold** of \( a \) at \( x \).

If \( D \) is an effective integral divisor on \( X \), determining the line bundle \( \mathcal{O}_X(-D) = \{ h \in \mathcal{O}_X | \text{div}(h) - D \geq 0 \} \), we denote by \( \mathcal{J}(\mathcal{O}_X(-D)^\xi) \) the multiplier ideal \( \mathcal{J}(\mathcal{O}_X(-D)^\xi) \) associated with the ideal sheaf \( \mathcal{O}_X(-D) \) and the number \( \xi \). The following periodicity property of the multiplier ideals \( \mathcal{J}(\mathcal{O}_X(-D)^\xi) \) imply that their jumping numbers are determined by the finitely many of them lying in the unit interval \((0, 1]\) (see [11, Example 1.7 and Remark 1.15] or [26, Example 9.2.12 and Proposition 9.2.31]).

**Lemma 1.4**. Let \( D \) be an effective integral divisor on a smooth variety \( X \). Then, \( \mathcal{J}(D) = \mathcal{O}_X(-D) \), the number \( 1 \) is a jumping number of the multiplier ideals of \( D \) and \( \mathcal{J}(\mathcal{O}_X(-D)^{(\xi + 1)}D) = \mathcal{J}(\mathcal{O}_X(-D)^{\xi}D) \otimes \mathcal{O}_X(-D) \).

**Remark 1.5.** If \( X \) is a smooth variety and \( f \) is a germ of complex analytic function at \( x \in X \), then the above definitions of multiplier ideals and jumping numbers of \( (f) \) generalize to this local setting (see [11, Remark 1.26]).

### 2. Basic notions about plane curve singularities

Let \( S \) be a smooth complex algebraic or analytic surface and \( o \in S \) a fixed closed point. Denote by \( \mathcal{O}_s \), the local ring of germs of holomorphic functions on \( S \) at \( o \).

A germ of a complex analytic curve \( (C, o) \) is defined by an equation, \( f_C = 0 \), where \( f_C \in \mathcal{O} \) is a representative of \( (C, o) \) (defined up to multiplication by a unit of the ring \( \mathcal{O} \)). Similarly, if \( f \in \mathcal{O} \) we
denote \( C_f \) the curve germ defined by \( f = 0 \). If the base point \( o \) of the germ is clear from the context we denote \(( C, o)\) simply by \( C \). The curve \( C \) is a branch if \( f_C \) is irreducible. In general, if we decompose \( f_C = f_1^{a_1} \cdots f_r^{a_r} \) as a product of irreducible elements in \( O \) with \( f_i \) non-associated to \( f_j \) for \( i \neq j \), and we put \( C_i = C_{f_i} \), then we represent \( C = \sum_{i=1}^{r} a_i C_i \) as the effective divisor defined by \( f_C \).

A cross at the point \( o \) of \( S \) is an ordered pair \((R, L)\) of transversal smooth branches. A local coordinate system \((x, y)\) on \((S, o)\) is an ordered pair of elements generating the maximal ideal of \( O \). It defines a cross \((R, L)\) with \( R = C_x, L = C_y \).

A model of \((S, o)\) is a proper and birational map \( \pi : (S_{\pi}, E_{\pi}) \rightarrow (S, o) \) such that the restriction of \( \pi \) to \( S_{\pi} \setminus E_{\pi} \rightarrow S \setminus \{o\} \) is an isomorphism. If \( \pi \) is not the identity map, it is a composition of blow ups of points infinitely near to \( o \), hence the irreducible components the reduced divisor \( E_{\pi} := \pi^{-1}(o) \) are projective lines. We denote by \( E(\pi) \) the set of prime divisors of the exceptional divisor \( E_{\pi} \), that is, we can write \( E_{\pi} = \bigcup_{i \in E(\pi)} E_i \).

**Definition 2.1.** Let \( \pi \) be a model of \((S, o)\) and let \( C \) be a plane curve germ on \( S \) at a point \( o \). The total transform \( \pi^*(C) \) is the divisor of \( f_C \circ \pi \). The strict transform \( \overline{C}^\pi \) of \( C \) by \( \pi \) is sum of components of \( \pi^*(C) \) which are supported on the closure of \( \pi^{-1}(C \setminus \{o\}) \). The model \( \pi \) is an embedded resolution of \( C \) (also called log-resolution) if \( \pi^*(C) \) has simple normal crossings.

Any embedded resolution of \( C \) is a composition of blow ups of a finite set of infinitely near points of \( o \). There exists a unique minimal embedded resolution of \( C \), which is the one requiring the smallest number of blow ups.

We introduce several notions about the dual graph associated to certain divisor on a smooth surface. Let \( D = \sum_{i \in J} D_j \) be a reduced divisor with simple normal crossings on a smooth surface \( \Sigma \). The dual graph \( G(D) \) of \( D \) is the combinatorial graph with vertex set \( J \) and whose edges are in bijection with the singular points of \( D \). If \( p \) is a singular point of \( D \) then there are unique elements \( j, k \in J \) such that \( p \in E_j \cap E_k \). Then \( E_p = \{j, k\} \) is the corresponding edge of \( G(D) \). The valency of a vertex \( j \) of \( G(D) \) is the number of edges incident to \( j \). If \( D \) is an effective divisor we denote also by \( G(D) \) the dual graph associated to the reduction of \( D \). A prime divisor \( D_j \) whose corresponding vertex in the dual graph \( G(D) \) has valency \( \geq 3 \) is called a rupture component of \( D \). We denote by \( E(D) \) the set consisting of prime divisors of \( D \) defining end vertices of the dual graph \( G(D) \), that is, vertices of valency one.

**Definition 2.2.** Let \( \pi \) be a model of \((S, o)\). The dual graph \( G(\pi) \) of \( \pi \) is \( G(E_{\pi}) \). If \( \pi \) is an embedded resolution of a plane curve \( C \), we denote by \( G(\pi, C) \) the dual graph of \( \pi^*(C) \), by \( R(\pi, C) \) the set of rupture components of the divisor \( \pi^*(C) \).

The dual graphs \( G(\pi, C) \) and \( G(\pi) \) are finite trees, that is, connected graphs with a finite number of vertices and with no cycles.

Let \( \pi \) be a model of \((S, o)\). A branch \( K_j \) in \( S \) is a curvetta at a component \( E_i \) of the exceptional divisor \( E_{\pi} \) if \( K^2_j + E_{\pi} \) is a simple normal crossing divisor such that \( K^2_j \cap E_i \neq \emptyset \). In particular, in this case \( \pi \) is an embedded resolution of \( K_j \). We define now some classes of finite subsets whose elements are curvettas.

**Definition 2.3.** Let \( \pi \) be a model of \((S, o)\), different from the identity map of \( S \) or the blow up of \( o \) and let \( C \) be a plane curve germ on \( S \) at \( o \).

- A set of maximal contact curves of \( \pi \) contains exactly one curvetta \( K_i \) at \( E_i \), for every \( E_i \in E(\pi) \).
- If \( \pi \) is an embedded resolution of \( C \), a set of maximal contact curves of the pair \((\pi, C)\) consist of the components of \( C \) together with one curvetta \( K_i \) at \( E_i \), for \( E_i \) running through the components of \( E_{\pi} \) in the set \( E(\pi^*(C)) \).
Let us recall some facts about the valuations which we use in this paper.

**Definition 2.4.** A valuation of the local ring \( O \) is a function \( \nu : O \to [0, \infty] \) such that

(a) \( \nu(fg) = \nu(f) + \nu(g) \),
(b) \( \nu(f + g) \geq \min(\nu(f), \nu(g)) \) for all \( f, g \in O \);
(c) \( \nu(f) = \infty \iff f = 0 \),

where \( [0, \infty] := \mathbb{R}_{\geq 0} \cup \{ \infty \} \) is considered as a semigroup with addition in the usual sense.

If \( D \) is a germ of curve on \( S \) and \( \nu \) is a valuation of \( O \) then we denote \( \nu(D) := \nu(f_D) \).

Let us introduce two useful types of valuations associated to a branch \( C \) or to an exceptional prime divisor.

The vanishing order valuation along a branch \( C \), denoted by \( [\nu_C] \), is given by \( \nu_C(h) = a \), if \( 0 \neq h \in O \), and \( h = f_C^ah \) with \( \gcd(f_C, g) = 1 \). We set \( \nu_C(0) = \infty \). Notice that the vanishing order valuation along a branch is well defined because \( O \) is a unique factorization domain.

Let \( \pi \) be a model of \( S \) at \( o \) and \( E_i \) be an irreducible component of \( E_\pi \). The function \( [\nu_{E_i}] : O \to [0, \infty] \), which maps \( h \in O \) to the order of vanishing along \( E_i \) of \( h \circ \pi \), is a valuation of \( O \), called the divisorial valuation of \( E_i \).

Denote by \( \geq \) the poset relation in \( (\mathbb{R} \cup \{ \infty \})^s \) given by \( (a_1, \ldots, a_s) \geq (b_1, \ldots, b_s) \) if \( a_i \geq b_i \) for every index \( i \in \{1, \ldots, s\} \). Let \( \nu_1, \ldots, \nu_s \) be valuations of \( O \). The valuation ideal \( I_\xi^{(s)} \) associated with \( \xi = (c_1, \ldots, c_s) \in \mathbb{R}_{\geq 0}^s \) and \( \nu := (\nu_1, \ldots, \nu_s) \) is \( I_\xi^{(s)} := \{ f \in O \mid \nu(f) \geq \xi \} \).

The notion of generating sequence was considered in [33, Definition 1.1] for one valuation, and studied in [6, 9] for tuples of divisorial valuations.

**Definition 2.5.** Let \( \nu = (\nu_1, \ldots, \nu_s) \) be a tuple of valuations of \( O \). A set of elements \( \{z_j\}_{j \in J} \) in the maximal ideal of \( O \) is a generating sequence of \( \nu \) if for every \( \xi \in \mathbb{R}_{\geq 0}^s \) the valuation ideal \( I_\xi^{(s)} \) is generated by the set \( \big\{ \prod_{j \in J} z_j^{b_j} \mid b_j \in \mathbb{Z}_{\geq 0}, \sum b_j \nu(z_j) \geq \xi \big\} \). A generating sequence is minimal if every proper subset of it fails to be a generating sequence.

For instance, a minimal generating sequence of the divisorial valuation associated with the exceptional divisor of the blow up of \( o \) in \( S \) is \((x, y)\), where \((x, y)\) is a local system of coordinates of \( S \) at \( o \).

Let \( \nu = (\nu_1, \ldots, \nu_s) \) be a tuple of divisorial valuations. Campillo and Galindo proved that \( \nu \) has a finite generating sequence (see [6, Theorem 3]). If \( \{z_j\}_{j \in J} \) is a finite generating sequence of \( \nu_j \) then it is a generating sequence of \( \nu_1 \) (see [9]), in particular, it is a set of generators of the maximal ideal of \( O \) (see [33]). A model \( \pi : (\Sigma, E) \to (S, o) \) is a minimal embedded resolution of \( \nu \) if \( \pi \) is a composition of the minimal number of blowing ups of points such that there exists a component \( E_i \) of the exceptional divisor \( E \) such that \( \nu_i = \nu_{E_i} \), for \( i = 1, \ldots, s \). We have the following characterization of the minimal generating sequences associated with a tuple of divisorial valuations:

**Theorem 2.6.** [9, Theorem 5] Let \( \nu = (\nu_1, \ldots, \nu_s) \) be a tuple of divisorial valuations of \( O \). We assume that if \( s = 1 \) then \( \nu_1 \) is different from divisorial valuation associated with the blow up of \( o \) in \( S \). Let us denote by \( \pi : \Sigma \to S \) the minimal embedded resolution of \( \nu \). Take a defining function for every curvetta in a set of maximal contact curves of \( \pi \). Then, these functions define a minimal generating sequence of \( \nu \).
divisorial valuations associated with the rupture components of the divisor $\pi^*(C)$ and the vanishing order valuations of the branches of $C$.

Proof. We have that $\pi$ is also the minimal resolution of the divisorial valuations of the rupture components of the divisor $\pi^*(C)$. Let us denote by $\tilde{C}$ the sum of maximal contact curves of $(\pi, C)$, considered as a reduced effective divisor. We have that $\pi$ is also the minimal embedded resolution of $\tilde{C}$. The dual graph $G(\pi)$ is obtained from $G(\pi, \tilde{C})$ by deleting the arrows corresponding to the components of $\tilde{C}$. The first assertion follows by Theorem 2.6, since a set of maximal contact elements of $(\pi, C)$ contains a set of maximal contact elements of $\pi$. We refer to [31, Corollary 4.160] for a proof of the second statement.

3. Toroidal embedded resolutions of plane curves

In this section we recall the construction of a toroidal embedded resolution of a plane curve germ and its associated combinatorics which is encoded in the associated fan tree.

3.1. Fans, cones and toric varieties

We introduce some standard notations of toric geometry following [7, 12, 15, 29].

A lattice $[N]$ is a free group of finite rank $d$. We denote by $[\mathbb{Z}N]:= N \otimes \mathbb{Z} \mathbb{R}$ the real vector space spanned by $N$ and by $M_{\mathbb{R}}$ and $M$ its duals respectively. We denote by

$$\langle \cdot, \cdot \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}, \ (u, v) \mapsto \langle u, v \rangle = v(u)$$

the duality pairing between these two vector spaces. A cone $\mathbf{C} \subseteq N_{\mathbb{R}}$ is rational with respect to $N$ if it is of the form $\mathbf{C} = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s$ for $v_1, \ldots, v_s \in N$ and $s \in \mathbb{N}$. It is strictly convex if $\{0\}$ is the biggest subspace contained in it. A face of the cone $\mathbf{C}$ is the intersection of it with a supporting hyperplane, that is a subspace of codimension one such that one of its half-spaces contains $\mathbf{C}$. A cone is regular if it is spanned by a subset of an orbit of the torus action on $N_{\mathbb{R}}$. There is an action of the torus $\mathbb{G}_m^d$ on $N_{\mathbb{R}}$ by characterizing $\mathbf{C}$ as a rational cone.

We denote by $[\mathbf{F}]$ the set of primitive integral vectors of the lattice $N$ which spans the rays of the fan $\mathbf{F}$. The fan $\mathbf{F}$ is regular if all its cones are regular.

If $\mathbf{C} \subseteq N_{\mathbb{R}}$ is a strictly convex rational cone in $N_{\mathbb{R}}$, then $\mathbf{C}$ is rational with respect to $N$. A fan $\mathbf{F}$ of the lattice $N$ is a finite set of strictly convex rational cones in $N_{\mathbb{R}}$ such that it is closed under the operation of taking faces of its cones and the intersection of any two cones in the fan is a face of each of them. The support $[\mathbf{F}]$ of the fan $\mathbf{F}$ is the union of its faces. We denote by $[\mathbf{F}]_{\text{prim}}$ the set of primitive integral vectors of the lattice $N$ which spans the rays of the fan $\mathbf{F}$. The fan $\mathbf{F}$ is regular if all its cones are regular.

If $\mathbf{C} \subseteq N_{\mathbb{R}}$ is a strictly convex rational cone, then $\mathbf{C}$ is regular if all its cones are regular. A fan $\mathbf{F}$ of the lattice $N$ is a finite set of strictly convex rational cones $\mathbf{C}$ in $N_{\mathbb{R}}$ such that it is closed under the operation of taking faces of its cones and the intersection of any two cones in the fan is a face of each of them. The support $[\mathbf{F}]$ of the fan $\mathbf{F}$ is the union of its faces. We denote by $[\mathbf{F}]_{\text{prim}}$ the set of primitive integral vectors of the lattice $N$ which spans the rays of the fan $\mathbf{F}$. The fan $\mathbf{F}$ is regular if all its cones are regular.
Let $\mathcal{F}$ and $\mathcal{F}'$ be two fans of the lattice $N$. The fan $\mathcal{F}'$ is a subdivision of $\mathcal{F}$ if $|\mathcal{F}| = |\mathcal{F}'|$ and for every $\theta' \in \mathcal{F}'$ there exists $\theta \in \mathcal{F}$ such that $\theta' \subset \theta$. If $\mathcal{F}'$ is a subdivision of $\mathcal{F}$ we have a toric morphism $\psi_{\mathcal{F}'/\mathcal{F}} : X_{\mathcal{F}'} \to X_{\mathcal{F}}$, obtained by glueing the maps $\psi_\theta$ for every $\theta \in \mathcal{F}$, $\theta' \in \mathcal{F}'$ such that $\theta' \subset \theta$.

The morphism $\psi_{\mathcal{F}'/\mathcal{F}}$ is a modification that is, it is proper and birational. In addition, $\mathcal{F}'$ is a regular subdivision of $\mathcal{F}$ if $\mathcal{F}'$ is a regular fan containing every regular cone of $\mathcal{F}$.

We apply now these notions when the lattice $N$ has rank two and we endow it with a fixed basis $e_1, e_2$. We denote by $\sigma_0$ the regular cone spanned by the basis $e_1, e_2$ of $N$. Note that $X_{\sigma_0} = C^2$ and $\mathbb{C}[X_{\sigma_0}] = \mathbb{C}[x,y]$ where $x = \chi^\hat{e}_1$, $y = \chi^\hat{e}_2$ are the characters associated with the dual basis $\hat{e}_1, \hat{e}_2$ of $e_1, e_2$. If $\mathcal{F}$ is a fan of $N$ subdividing $\sigma_0$, then there exists a unique minimal regular subdivision $\mathcal{F}_{\text{reg}}$ of $\mathcal{F}$, if $\mathcal{F}'$ is any other regular subdivision of $\mathcal{F}$ then $\mathcal{F}'$ subdivides $\mathcal{F}_{\text{reg}}$ (see [7, Section 10.2]). If $\theta' = \mathbb{R}_{\geq 0} u + \mathbb{R}_{\geq 0} v \in \mathcal{F}_{\text{reg}}$ is a two dimensional cone, with $u = a_1 e_1 + a_2 e_2$ and $v = b_1 e_1 + b_2 e_2$, then $a_1 b_2 - a_2 b_1 = \pm 1$ and the chart $\psi_{\sigma_0} : X_\theta \to X_{\sigma_0}$ of the modification

$$
\psi_{\sigma_0} : X_{\mathcal{F}_{\text{reg}}} \to X_{\sigma_0}
$$

is the monomial map given by:

$$
x = x_\theta^{a_1} y_\theta^{b_1},
y = x_\theta^{a_2} y_\theta^{b_2}.
$$

If $\bar{u}, \bar{v} \in M$ denotes the dual basis of $u, v$ then, one has $x_\theta := \chi^{\bar{u}} y_\theta := \chi^{\bar{v}}$ and $\mathbb{C}[X_{\theta}] = \mathbb{C}[x_\theta, y_\theta]$. Notice that the closure of the orbit $O_\rho$ associated with a ray $\rho = \mathbb{R}_{\geq 0} u$, is defined on the chart (3.2) by $x_\theta = 0$.

### 3.2. Newton polytopes, Newton fans and support functions

Let $f = \sum a_{ij} x^i y^j \in \mathbb{C}[[x,y]]$ be a nonzero power series. The support $\text{supp}(f)$ of the power series $f$ consist of those vectors $(i,j) = i\hat{e}_1 + j\hat{e}_2 \in M$ with nonzero coefficient $a_{ij}$. Recall that $\sigma_0 = \mathbb{R}_{\geq 0} \hat{e}_1 + \mathbb{R}_{\geq 0} \hat{e}_2$ is the dual cone of $\sigma_0 = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2$ (see the notation of Section 3.1).

The Newton polygon $\mathcal{N}(f)$ is the convex hull of $\text{supp}(f) + \sigma_0$. The face $E_u$ of $\mathcal{N}(f)$ defined by a vector $u \in \sigma_0$ is the set of elements $v \in \mathcal{N}(f)$ such that $\langle u, v \rangle = \min \{ \langle u, v' \rangle \mid v' \in \mathcal{N}(f) \}$. The face $E_u$ is compact when $u$ belongs to the interior of $\sigma_0$. All the faces of $\mathcal{N}(f)$ are of this form. If $\mathcal{E}$ is a face of $\mathcal{N}(f)$ the closure of the set $\{ u \in \sigma_0 \mid E_u = \mathcal{E} \}$ is a cone $\theta_\mathcal{E} \subset \sigma_0$ which is rational for the lattice $N$. The set $\mathcal{F}(f)$ consisting of cones $\theta_\mathcal{E}$, for $\mathcal{E}$ running through the faces of $\mathcal{N}(f)$, is a fan of $N$ supported on $\sigma_0$, called the Newton fan of $f$. We denote by $\mathcal{F}_{\text{reg}}(f)$ the minimal regular subdivision of the fan $\mathcal{F}(f)$.

The support function $\Phi_{\mathcal{N}} : \sigma_0 \to \mathbb{R}_{\geq 0}$ of the polyhedron $\mathcal{N} := \mathcal{N}(f)$ is defined by $\Phi_{\mathcal{N}}(v) = \min_{u \in \sigma_0} \{ \langle v, u \rangle \}$. The support function $\Phi_{\mathcal{N}}$ is linear precisely on each cone of the fan $\mathcal{F}(f)$. If $\theta \in \mathcal{F}(f)$ is a two dimensional cone then there exists a unique vertex $u_\theta$ of the Newton polygon $\mathcal{N}(f)$ such that $\Phi_{\mathcal{N}}(v) = \langle v, u_\theta \rangle$ for all $v \in \theta$. In addition, for any real number $\xi > 0$ we have that

$$
\Phi_{\xi \mathcal{N}} = \xi \Phi_{\mathcal{N}}.
$$

Notice that the notion of support function can be defined for any convex polyhedra and determines it (see [12, Theorem 3.8, and Theorem 6.8]). In particular, we have

$$
\mathcal{N}(f) = \cap_{v \in \mathcal{F}(f)_{\text{prim}}} \{ u \in \mathbb{M} \mid \langle v, u \rangle \geq \Phi_{\mathcal{N}}(v) \} = \cap_{v \in \sigma_0} \{ u \in \mathbb{M} \mid \langle v, u \rangle \geq \Phi_{\mathcal{N}}(v) \}.
$$

A vector $v \in \sigma_0$ defines a monomial valuation $\text{ord}_v$ of the completion $\hat{\mathcal{O}} \cong \mathbb{C}[[x,y]]$ of the local ring $\mathcal{O}$, given by

$$
\text{ord}_v(h) := \min \{ \langle v, u \rangle \mid u \in \text{supp}(h) \}, \text{ for } h \in \mathbb{C}[[x,y]] \setminus \{0\}.
$$
It follows that for any $v \in \sigma_0$ and any plane curve $C$ we have
\[ \text{ord}_v(C) = \Phi_{N(c)}(v). \]  
(3.6)

The following lemma will be useful later.

**Lemma 3.7.** Let $A$ be a non-empty subset of $\tilde{\sigma}_0$, set $A := A + \tilde{\sigma}_0$, consider a rational number $\xi > 0$, and let $G$ be subdivision of $F(f)$. The following conditions are equivalent.

(a) $A$ is contained in the interior of the polygon $\xi N(f)$,
(b) $\Phi_A(v) > \xi \Phi_{N(f)}(v)$ for any $v \in G_{\text{prim}}$,
(c) $\Phi_A(v) > \xi \Phi_{N(f)}(v)$, for any $v \in F(f)_{\text{prim}}$.

**Proof.** By (3.3) and (3.4), the interior of $\xi N(f)$ is the intersection of open half-spaces
\[ \bigcap_{v \in F(f)_{\text{prim}}} \{ u \in M_{\mathbb{R}} \mid \langle v, u \rangle > \xi \Phi_{N(f)}(v) \} = \bigcap_{v \in \sigma_0} \{ u \in M_{\mathbb{R}} \mid \langle v, u \rangle > \xi \Phi_{N(f)}(v) \}. \]

This implies the equivalence between the conditions (a), (b), and (c). □

### 3.3. Newton modifications

We present the properties of the Newton modification associated with a germ of plane curve on a smooth surface relative to a cross following [18].

Let $(R, L)$ define a cross at a point $o$ of a smooth surface $S$. The set of divisors supported on $R + L$ is a rank two lattice $M_{R,L}$ with basis $R, L$. The map $M_{R,L} \to M$, which sends $aR + bL$ to $a\tilde{e}_1 + b\tilde{e}_2$ is an isomorphism of lattices, and it extends to an isomorphism of real vector spaces which maps the cone $\tilde{\sigma}_0^{R,L}$ of real effective divisors supported on $R + L$ onto the cone $\tilde{\sigma}_0$. We denote by $N_{R,L} \simeq N$ the dual lattice of $M_{R,L}$, by $[e_R, e_L] \in N_{R,L}$ the dual basis of $R =: \tilde{e}_R, L =: \tilde{e}_L$, and by $\sigma_0^{R,L}$ the cone $\mathbb{R}_{\geq 0} e_R + \mathbb{R}_{\geq 0} e_L$.

Let us consider the projectivization map:
\[ \phi : (N_{R,L})_{\mathbb{R}} \setminus \{0\} \to \mathbb{P}((N_{R,L})_{\mathbb{R}}), \]
\[ ae_R + be_L \mapsto (a : b), \]
where $(a : b)$ denote homogeneous coordinates. If $u = ae_R + be_L$ is nonzero the slope $S(u) := b/a \in \mathbb{R} \cup \{\infty\}$ is the affine coordinate of the point $\phi(u) \in \mathbb{P}((N_{R,L})_{\mathbb{R}}) \cong \mathbb{P}^1_{\mathbb{R}}$.

**Notation 3.8.** If $a, b \in \mathbb{N}$ are coprime we abuse of notation by denoting the image of the vector $u = ae_R + be_L$ by $\phi$ also by $u$.

**Definition 3.9.** Let $F$ be a fan of $N_{R,L}$ subdividing $\sigma_0^{R,L}$. The **trunk** $\theta(F)$ of $F$ is the segment $\phi(\sigma_0^{R,L} \setminus \{0\}) = [e_R, e_L]$ with the finite set of marked points defined by the image by $\phi$ of the rays of $F$. It is endowed with the slope coordinate function
\[ S : [e_R, e_L] \to [0, \infty) \subset \mathbb{P}^1_{\mathbb{R}}. \]

For example, with Notation 3.8 we have that if $F$ is the subdivision of $\sigma_0^{R,L}$ with rays spanned by the vectors $e_R$, $e_R + e_L$ and $e_L$ the trunk $\theta(F)$ is the segment $[e_R, e_L]$ with marked points $e_R$, $e_R + e_L$ and $e_L$ which have coordinate $0$, $1$ and $\infty$ respectively.

Any regular fan $F$ with respect to $N_{R,L}$ subdividing $\sigma_0^{R,L}$ defines a model of $(S, o)$:
\[ \psi_{F}^{\theta} : S_F \to S, \]
(3.10)
which is defined by gluing monomial maps of the form $\psi_{\sigma_0}^{\theta}$ for $\theta \in F$. 
Remark 3.11. There is a natural bijection between the set of marked points of the trunk $\theta(\mathcal{F})$ and the set of irreducible components of $(\psi_{R,L}^{\mathcal{F}})^{-1}(R + L)$ which sends a marked point $p$ to the irreducible component $D_p$ of $(\psi_{R,L}^{\mathcal{F}})^{-1}(R + L)$ which contains the orbit labeled by the ray of $\mathcal{F}$ of slope $S(p)$. In particular, if $p = e_R$ (resp. $p = e_L$) then one has that $D_p = R$ (resp. $D_p = L$).

Notation 3.12. Using this bijection, the marked points of $\theta(\mathcal{F})$ can be relabeled as $e_D$, for $D$ running through the irreducible components of $(\psi_{R,L}^{\mathcal{F}})^{-1}(R + L)$.

Remark 3.13. If $v \in \sigma_0^{R,L}$, then the monomial valuation $\text{ord}_v$ defined in (3.5) is independent of the choice of local coordinates $(x, y)$ defining the cross $(R, L)$. In the particular case of the vectors $e_R$ and $e_L$, we get that $\text{ord}_{e_R} = v_R$, and $\text{ord}_{e_L} = v_L$ are the vanishing order valuations along the branches $R$ and $L$ respectively. If $v = ne_R + me_L \in N_{R,L}$ for $n, m \in \mathbb{N}$ coprime, then the monomial valuation $\text{ord}_v$ is a divisorial valuation. In order to see this, take any regular fan $\mathcal{F}$ of $N_{R,L}$ subdividing $\sigma_0$ and containing the ray $\mathbb{R}_{\geq 0}v$. Then, if $p \in (e_R, e_L)$ has slope $m/n$ then we have that $\text{ord}_v = v_{D_p}$.

Let $(x, y)$ be a pair of local coordinates defining the cross $(R, L)$. Let $C$ be a plane curve singularity defined by a power series $f \in \mathbb{C}[[x, y]]$. The Newton polygon of $C$ with respect to the cross $(R, L)$ is $\mathcal{N}_{R,L}(C)$ just the polygon $\mathcal{N}(f)$ seen as a subset of $(M_{R,L})_{\mathbb{R}}$. The Newton fan of $C$ with respect to the cross $(R, L)$ is denoted by $\mathcal{F}_{R,L}(C)$ is just the fan $\mathcal{F}(f)$, whose support is seen as a subset of $(N_{R,L})_{\mathbb{R}}$. The polygon $\mathcal{N}_{R,L}(C)$ and the fan $\mathcal{F}_{R,L}(C)$ are independent of the choices of local coordinates $(x, y)$ defining the cross $(R, L)$ and the function $f_C \in \mathbb{C}[[x, y]]$ defining $C$ (see [18, Section 4.1]).

Definition 3.14. [18, Definition 4.14] We denote by $\mathcal{F}_{R,L}^{\text{reg}}(C)$ the minimal regular subdivision of the fan $\mathcal{F}_{R,L}(C)$. The map

$$\psi_{R,L}^{\text{reg}} : \psi_{R,L}^{\mathcal{F}_{R,L}(C)} : S_{\mathcal{F}_{R,L}(C)} \rightarrow S$$

is called the regularized Newton modification of $S$ defined by $C$ with respect to the cross $(R, L)$.

Notice that the map defined by (3.15) is a model of $(S, o)$, in particular, its exceptional divisor has simple normal crossings. Some concrete cases are discussed in Example 3.23.

Remark 3.16. In [18], a more general notion of Newton modification $\psi_{R,L}^{C} := \psi_{R,L}^{\mathcal{F}_{R,L}(C)} : S_{\mathcal{F}_{R,L}(C)} \rightarrow S$, is considered. Notice that the surface $S_{\mathcal{F}_{R,L}(C)}$ may be singular.

### 3.4. Toroidal resolutions of plane curves

In this section we summarize an algorithm of toroidal embedded resolution given in [18, Section 4].

Definition 3.17. [18, Definitions 3.29 and 4.15] A (smooth) toroidal surface is a smooth complex analytic surface $\Sigma$ endowed with a normal crossing divisor $\partial \Sigma$ called its boundary. A modification $\pi : (\Sigma_2, \partial \Sigma_2) \rightarrow (\Sigma_1, \partial \Sigma_1)$ between toroidal surfaces is a toroidal modification if $\pi^{-1}(\partial \Sigma_1) \subset \partial \Sigma_2$.

Let $C$ be a plane curve singularity on a germ of smooth surface $(S, o)$ endowed with a germ of normal crossing divisor $\partial S$, and let $(\Sigma, \partial \Sigma)$ be a smooth toroidal surface. A toroidal modification $\pi : \Sigma \rightarrow S$ is a toroidal embedded resolution of $C$ if the boundary $\partial \Sigma$ of $\Sigma$ contains the reduction of the total transform $\pi^*(C)$ of $C$ by $\pi$. The reduction of the image $\pi(\partial \Sigma)$ of $\partial \Sigma$ in $S$ is called the completion $\hat{C}_\pi$ of $C$ relative to $\pi$. 
Example 3.18. If \( \pi : S_\pi \to S \) is the minimal embedded resolution of \( C \) then \( \pi \) is a toroidal embedded resolution of \( C \), where \( \partial S := \emptyset \) and \( \partial S_\pi \) is the reduced divisor of \( \pi^*(C) \). In this case the completion \( \hat{C}_\pi \) is the reduction of \( C \).

We focus from now on toroidal modifications \( \pi : \Sigma \to S \), where \( (R, L) \) is a fixed cross on \( (S, o) \) and \( \partial S := R + L \).

Example 3.19. Denote by \( \psi : S_\psi \to S \) the regularized Newton modification (3.15) of \( S \) defined by \( C \) with respect to the cross \((R, L)\). Then, \( \psi \) is a modification of toroidal surfaces, when we take \( \partial S := R + L \) and \( \partial S_\psi \) is any reduced normal crossing divisor on \( S_\psi \) containing the reduction of \( \psi^*(R + L) \).

Proposition 3.20. [18, Proposition 4.18] Let \((C, o)\) be a plane curve singularity on a smooth surface \( S \), and let \((R, L)\) be a cross at \( o \). Assume that neither \( R \) nor \( L \) is a branch of \( C \). Denote by \( \psi : S_\psi \to S \) the regularized Newton modification (3.15) of \( S \) defined by \( C \) with respect to the cross \((R, L)\). Then, the strict transform \( C^\psi \) of \( C \) intersects the reduced divisor of \( \psi^*(R + L) \) only at smooth points of it.

By Proposition 3.20, one of the following two cases holds at each point of intersection \( o_i \) of the strict transform \( C^\psi \) with the exceptional divisor of \( \psi \).

1. The germ of the strict transform \( C^\psi \) at the point \( o_i \) is smooth and transversal to the exceptional divisor of \( \psi \). Then, only one branch of \( C^\psi \) passes through \( o_i \), and together with the germ \( R_i \) of the exceptional divisor define a canonical cross on \( S_\psi \).

2. Otherwise, we can choose a smooth germ \( L_i \) transversal to the germ \( R_i \) of exceptional divisor at \( o_i \) and then \((R_i, L_i)\) defines a cross at \( o_i \).

In the second case we have a plane curve singularity, the germ of strict transform \( C^\psi \) of \( C \) at \( o_i \), and a cross at \( o_i \), so that we can apply to it the associated regularized Newton modification defined by it with respect to this cross. This leads to the following algorithm.

Algorithm 3.21. [18, Algorithm 4.22 and Proposition 5.1] Let \((S, o)\) be a smooth germ of surface, \( R \) be a smooth branch on \((S, o)\), and \( C \) be a reduced germ of curve on \((S, o)\), which does not contain the branch \( R \) in its support.

**STEP 1.** If \((R, C)\) is a cross, then **STOP**.

**STEP 2.** Choose a smooth branch \( L \) on \((S, o)\), possibly included in \( C \), such that \((R, L)\) is a cross.

**STEP 3.** Consider the regularized Newton modification \( \psi := \psi_{R, L}^{C, \text{reg}} : (S_\psi, \partial S_\psi) \to (S, \partial S) \) of \( S \) defined by \( C \) with respect to the cross \((R, L)\), where \( \partial S := R + L \) and \( \partial S_\psi := \psi^{-1}(R + L) \), and the strict transform \( C^\psi \) of \( C \) by \( \psi \). **STEP 4.** For each point \( \tilde{o} \) belonging to \( C^\psi \cap \partial S_\psi \), denote:

- \( R := \text{the germ of } \partial S_\psi \text{ at } \tilde{o} \);
- \( C := \text{the germ of } C^\psi \text{ at } \tilde{o} \);
- \( o := \tilde{o} \);
- \( S := S_\psi \).

**STEP 5.** Go to step 1.

Proposition 3.22. [18, Proposition 5.1] The Algorithm 3.21 stops after finitely many iterations and provides a toroidal embedded resolution \( \pi : (\Sigma, \partial \Sigma) \to (S, \partial S) \) of \( C \) with the following boundaries.
• $\partial S := R + L$, where $L$ is the branch fixed at the first iteration of STEP 1 or STEP 2 of the Algorithm 3.21.
• $\partial \Sigma$ is the reduced normal crossing divisor which contains the reduction of $\pi^*(C)$ and the strict transforms of the components of the crosses considered when running the Algorithm 3.21.

Example 3.23. Let $(x,y)$ be a local coordinate system on the surface $(S,o)$ and consider $f_1 = (y^2 + x^3)^2 + x^2 y, f_2 = y^3 + x^3 \in O$. We describe a toroidal embedded resolution of the plane curve singularity $C = C_{f_1} + C_{f_2}$ following Algorithm 3.21. First, we fix the smooth branch $R_1 = C_x$. At the second step of the algorithm we choose $L_1 = C_y$, and the cross $(R_1, L_1)$ at $o_1$. The Newton polygon $N_{R_1,L_1}(C)$ has two compact edges which are orthogonal to the vectors $2eR_1 + 3eL_1$ and $3eR_1 + 5eL_1$. These vectors span rays of the fan $\mathcal{F}_1 := \mathcal{F}^{\text{reg}}_{R_1,L_1}(C)$, which is represented in the left part of Figure 1.

Let us take the regularized Newton modification $\psi := \psi_{R_1,L_1} : S_{\mathcal{F}_1} \to S$. By Proposition 3.20, the strict transform $C^{\psi_1}$ of $C$ intersects the exceptional divisor of $\psi_1$ at points $o_2 \in D_2$ and $o_3 \in D_3$, where the labels are those of Figure 1. We can check this on the chart $x = x_2^2 x_3^3, y = x_3^3 x_2^5$ of $\psi_1$, where $D_2 = C_{x_2}$ and $D_3 = C_{x_3}$, and the total transform of $C_1$ (resp. of $C_2$) is defined by $x_2^{12} x_3^{18} ((x_3 + 1)^2 + x_2^2 x_3^2) = 0$ (resp. by $x_2^9 x_3^{15}(1 + x_2) = 0$). Thus, the point $o_2$ (resp. $o_3$) has coordinates $(x_2, x_3) = (0, -1)$ (resp. $(x_2, x_3) = (-1, 0)$). Then, we iterate the algorithm at the points $o_3$ and $o_2$:
- At the point $o_3$ we get the cross $(R_3 := D_3, L_3 := C^{\psi_1} = C^{\psi_3})$, and we stop at step 1.
- At the point $o_2$, we choose the cross $(R_2 := D_2, L_2 := C_{x_3+1})$ at the second step of the algorithm. The fan $\mathcal{F}_2 := \mathcal{F}^{\text{reg}}_{R_2,L_2}(C)$, is represented in Figure 1. It defines the regularized Newton modification $\psi := \psi_{R_2,L_2} : S_{\mathcal{F}_2} \to S_{\mathcal{F}_1}$. By Proposition 3.20, the strict transform of $C$ intersects the exceptional divisor of $\psi_2$ at a point $o_4 \in D_6$ (the labels of the components of the exceptional divisor are indicated in the right part of Figure 1). We obtain that $\pi := \psi_1 \circ \psi_2$ is a toroidal embedded resolution of $C$, since at the point $o_4$ we get the cross $(R_4 := D_6, L_4 := C^{\pi} = C^{\pi_1})$, that is, the iteration of the algorithm at the point $o_4$ stops at step 1. The image of $L_2$ on the initial surface is the branch $C_{y^2+x^3}$.

3.5. The fan tree of the toroidal embedded resolutions of Algorithm 3.21

We explain now how a tree, called the fan tree, can be associated with a toroidal embedded resolution of the form given in Proposition 3.22, following [18]. The fan tree encodes the combinatorial structure of
the toroidal resolution process. The fan tree is a variant of the Newton tree considered by Cassou-Noguès and Libgober with equivalent decorations (see [8]). We introduce first some notations.

**Notation 3.24.** Assume that one executes Algorithm 3.21 on \((S, o)\), starting from the curve singularity \(C\) and the smooth branch \(R\), which is not a component of \(C\). We denote by \(\{I_\pi\}\) a finite set labeling the infinitely near points \(o_i\) of \(o\) at which one applies STEP 1 or STEP 2. We assume that \(1 \in I_\pi\) and then \(o_1 = o\) and \(R_1 = R\). If \(i \in I_\pi\) we denote by \((R_i, L_i)\) the corresponding cross at \(o_i\). Denote by \([e_{R_i}, e_{L_i}]\) the canonical basis of the weight lattice \(N_{R_i, L_i}\). If \(i \in I_\pi\) and \(i \neq 1\), the branch \(R_i\) is included in the exceptional divisor of the regularized Newton modification performed at the previous step. We denote by \(L_{i, \pi}\), or simply by \(L_i\), the projection on \(S\) of the curve \(L_i\) at \(o\). We denote by \(\psi_{ij} := \psi_{R_i, L_i}^{C, \text{reg}}\) the regularized Newton modification of \(C\) with respect to the cross \((R_j, L_j)\) at the point \(o_i\). We consider the trunk \(\theta_{ij}\) of the fan \(\mathcal{F}_{R_i, L_i}^{\text{reg}}(C)\), for \(i \in I_\pi\). The trunk \(\theta_{ij}\) is the segment \([e_{R_i}, e_{L_i}]\) endowed with the slope coordinate function \(S_i : [e_{R_i}, e_{L_i}] \rightarrow [0, +\infty)\), and with marked points defined by the edges of the fan \(\mathcal{F}_{R_i, L_i}^{\text{reg}}(C)\) (see Definition 3.9). Recall that we label the marked points of the trunk \(\theta_{ij}\) by \(e_D\), where \(D\) runs through the irreducible components of the reduction of the divisor \(\psi_{ij}^*(R_i + L_j)\) (see Notation 3.12). By definition, the strict transform of \(D\) on the final surface \(\Sigma\), which we denote also by \(D\), is a component of the boundary \(\partial \Sigma\). Notice that \(D\) is an exceptional prime divisor of \(\pi\) precisely when \(e_D \in (e_{R_i}, e_{L_i})\), or \(D = R_i\) for \(i \neq 1\). Otherwise \(D \subset \partial \Sigma\) is the strict transform of a component of \(\mathcal{C}_\pi = R + \sum_{j \in I_{\pi}} L_{j, \pi}\).

**Remark 3.25.** If \(C_j\) is a component of \(C\) then there exists a unique index \(i_j \in I_\pi\) such that \(C_j = L_{i_j, \pi}\). This means that the algorithm stops at the point \(o_{i_j}\), since it is the point of intersection of the strict transform of \(C_j\) with the exceptional divisor \(E_{\pi}\). In this case the morphism \(\psi_{i_j}\) is the identity map and the marked points of the fan tree \(\theta_{ij}\) are \(e_{R_{i_j}}\) and \(e_{C_j}\).

**Definition 3.26.** The fan tree \(\mathcal{F}_\pi(C)\) of the toroidal embedded resolution \(\pi : \Sigma \rightarrow S\) of \(C\) of Proposition 3.22 is a tree endowed with a finite set of marked points. The set \(\theta_{\pi}(C)\) is obtained from the disjoint union of the trunks \(\theta_{ij} = \theta_{(\mathcal{F}_{R_i, L_i}^{\text{reg}}(C))}\), for \(i \in I_\pi\), by identifying the marked points labeled by the same irreducible component of \(\partial \Sigma\).

We denote in the same way each interval \([e_{R_i}, e_{L_i}]\) and its image in \(\theta_{\pi}(C)\). By definition, if \(e \in \theta_{\pi}(C)\) and \(e \neq e_R\) there exists a unique index \(i \in I_\pi\) such that \(e \in (e_{R_i}, e_{L_i})\). This property allows to endow the fan tree with its slope function: \(S_\pi : \theta_{\pi}(C) \rightarrow [0, \infty]\), defined by

\[
S_\pi(e) = \begin{cases} 
0 & \text{if} \quad e = e_R, \\
S_i(e) & \text{if} \quad e \in (e_{R_i}, e_{L_i}) \text{ for some } i \in I_\pi.
\end{cases}
\]

This function is not continuous precisely on the set ramification points \(\{e_{R_i} \mid i \in I_\pi, i \neq 1\}\) of the tree \(\theta_{\pi}(C)\).

By construction, the toroidal embedded resolution \(\pi\) is also an embedded resolution of the completion \(\mathcal{C}_\pi\), and the boundary \(\partial \Sigma\) is the reduced divisor of the total transform of \(\mathcal{C}_\pi\). The associated dual graph is determined in terms of the fan tree.

**Proposition 3.27.** [18, Proposition 4.35] The fan tree \(\theta_{\pi}(C)\) is isomorphic to the dual graph \(G(\partial \Sigma)\) of the boundary \(\partial \Sigma\) of the source of the toroidal embedded resolution \(\pi : \Sigma \rightarrow S\) of \(C\) of Proposition 3.22 by an isomorphism which respects the labels.

**Remark 3.28.** By Proposition 3.27 the marked points of \(\theta_{\pi}(C)\) which are of valency \(\geq 2\) are labeled by the irreducible components of the exceptional divisor of \(\pi\). The ramification points of \(\theta_{\pi}(C)\) are labeled by the elements of \(\mathcal{R}_\pi(C)\), that is, by the rupture components of the divisor \((\pi^*)(\mathcal{C}_\pi))^{\text{red}} = \partial \Sigma\). The end points of the tree \(\theta_{\pi}(C)\), which correspond to the vertices of \(G(\partial \Sigma)\) of valency one, are labeled by the
irreducible components of the completion \( \hat{C}_\pi \). This implies that the set of components of \( \hat{C}_\pi \) contains a set of maximal contact curves of the pair \((\pi, C)\), see Definition 2.3.

**Remark 3.29.** The minimal embedded resolution of \( C \) can be obtained as a toroidal embedded resolution by choosing a suitable reference smooth branch \( R \) and suitable auxiliary branches \( L_i \) at the second step of the Algorithm 3.21. One may take a maximal contact toroidal embedded resolution, see [31, Section 4.2]. It is also the case of some toroidal resolutions described in [28].

**Example 3.30.** We describe the fan tree of the toroidal embedded resolution of Example 3.23. In the left side of Figure 2 we have represented the trunks \( \theta_i(\mathcal{F}_{R_i,L_i}(C)) \) for \( i = 1, \ldots, 4 \). The fan tree \( \theta_\pi(C) \), which is represented in right part of Figure 2, is obtained from these trunks by gluing the points with the same label. The completion of the toroidal embedded resolution \( \pi \) is \( \hat{C}_\pi = R_1 + L_1 + L_2 + C_1 + C_2 \). One may check that this is the minimal embedded resolution of \( C \).

The following notion of representing divisor is equivalent to the notion of representing divisor of a rational point of an Eggers-Wall tree in [17], see [18, Section 1.6].

**Definition 3.31.** A point \( p \in \theta_\pi(C) \) is **rational** if \( S_\pi(p) \in \mathbb{Q}^* \). The **representing divisor** \( D_p \) of a rational point \( p \in \theta_\pi(C) \) is defined as follows. By Definition 3.26 there exists a unique \( i \in I_\pi \) such that \( p \in (e_{R_i}, e_{L_i}) \) and then \( S_i(p) = \frac{m}{n} \), where \( n, m > 0 \) are two coprime integers. Take a regular subdivision \( \mathcal{F} \) of the fan \( \mathcal{F}_{R_i,L_i}(C) \) which contains the ray spanned by the vector

\[
e_{D_p} := ne_{R_i} + me_{L_i} \in N_{R_i,L_i}.
\]

Then, the prime exceptional divisor \( D_p \) is an irreducible component of on the source of the toric model \( \psi^\mathcal{F}_{R_i,L_i} \) (see Remark 3.11).

We can replace the model \( \psi_i \) by the map \( \psi^\mathcal{F}_{R_i,L_i} \) considered in Definition 3.31, in the running of Algorithm 3.21. Then, the output is a model \( \pi' : (\Sigma', \partial\Sigma') \to (S, \partial S) \) of \( C \) dominating \( \pi : (\Sigma, \partial\Sigma) \to (S, \partial S) \), such that the exceptional divisor \( D_p \) appears as a component of \( \partial\Sigma' \). In particular, the representing divisor \( D_p \) of a rational point \( p \) appears on \( \Sigma \) if and only if \( p \) is a marked point \( p \) of valency \( \geq 2 \) of \( \theta_\pi(C) \).

![Diagram](image-url)

**Figure 2.** The trunks and the fan tree \( \theta_\pi(C) \) of Example 3.23.
4. Multiplier ideals and Newton polygons

In this section we fix a plane curve singularity \(C\), a smooth branch \(R\) which is not a component of \(C\) and a toroidal embedded resolution \(\pi\) of \(C\) given by Proposition 3.22. Recall that the set \(I_\pi\) indexing the crosses appearing in the Algorithm 3.21 was introduced in Notation 3.24.

**Definition 4.1.** The log-discrepancy vector at \(o_i\), for \(i \in I_\pi\) is

\[
\lambda_i := \lambda_{R, R_i + L_i} \in M_{R, R_i + L_i}.
\]

Let \(D\) be a prime component of \(\hat{\pi}_i\). Recall that if \(D\) is exceptional then \(\lambda_D\) denotes the log-discrepancy \(D\). If \(D\) is not exceptional we set \(\lambda_D := 1\).

**Notation 4.2.** Let \(A\) be a curve on \(S\). If \(i \in I_\pi\) we denote by \(\mathcal{N}_{R_i, L_i}(A)\) the Newton polygon of the germ of the total transform of \(A\) at \(o_i\), relative to the cross \((R_i, L_i)\), and by \(\mathcal{F}_{R_i, L_i}(A)\) the corresponding Newton fan. Recall that \(\Phi_{\mathcal{N}_{R_i, L_i}(A)}\) denotes the support function of the Newton polygon \(\mathcal{N}_{R_i, L_i}(A)\) (see Section 3.2).

In the following proposition we apply the notion of representing divisor \(D_p\) of a rational point \(p\) in the fan tree (see Definition 3.31).

**Proposition 4.3.** Let \(p\) be a rational point of the fan tree \(\theta(C)\), let \(i \in I_\pi\) be the unique index such that \(p \in (R_i, L_i)\), and consider the vector \(e_{D_p} \in N_{R_i, L_i}\) defined by (3.32). Then, for any plane curve \(A\) on \(S\) we have

\[
\nu_{D_p}(A) = \Phi_{\mathcal{N}_{R_i, L_i}(A)}(e_{D_p}), \tag{4.4}
\]

\[
\nu_{L_i}(A) = \Phi_{\mathcal{N}_{R_i, L_i}(A)}(e_{L_i}). \tag{4.5}
\]

In addition, the log-discrepancy of the exceptional prime \(D_p\) is given by

\[
\lambda_{D_p} = \langle e_{D_p}, \lambda_i \rangle. \tag{4.6}
\]

**Proof.** Take local coordinates \((x_i, y_i)\) defining the cross \((R_i, L_i)\) at \(o_i\). Since \(p\) is rational, we have that \(\mathcal{N}_{R_i, L_i}(A)\) is a regular fan of the lattice \(F\), where \(n, m > 0\) are coprime. The monomial valuation \(\text{ord}_{e_{D_p}}\) is defined on the completion of the local ring at \(o_i\) (identified with \(\mathbb{C}[[x_i, y_i]]\)), by

\[
\text{ord}_{e_{D_p}}(x_i) = n, \quad \text{and} \quad \text{ord}_{e_{D_p}}(y_i) = m.
\]

By definition the valuation \(\nu_{D_p}\) is composed with the monomial valuation \(\text{ord}_{e_{D_p}}\). This means that if \(\psi_i : S_i \to S\) is the composition of modifications factoring \(\pi\) and appearing in the Algorithm 3.21 until the point \(o_i\) appears in \(S_i\), then

\[
\nu_{D_p}(h) = \text{ord}_{e_{D_p}}(h \circ \psi_i), \tag{4.7}
\]

for any \(h \in \mathcal{O}\). If \(A\) is the plane curve defined by \(h = 0\) on \(S\) we get that formula (4.4) follows from (3.6) and (4.7). Formula (4.5) follows from Remark 3.13 by the same argument.

Recall that \(\lambda_i \in M_{R, R_i + L_i}\) and \(e_{D_p} = ne_{R_i} + me_{L_i}\) belongs to the lattice \(N_{R_i, L_i}\) which is dual to \(M_{R_i, L_i}\). We get that

\[
\langle e_{D_p}, \lambda_i \rangle = \langle ne_{R_i} + me_{L_i}, \lambda_{R, R_i + L_i} \rangle = n\lambda_{R_i} + m. \tag{4.8}
\]

Let \(\mathcal{F}\) be a regular fan of the lattice \(N_{R_i, L_i}\) which subdivides the cone \(\sigma_0^{R_i, L_i}\) and contains the ray \(\mathbb{R}_{\geq 0} e_{D_p}\). Consider the modification \(\phi := \psi_{R_i, L_i} : S_i^F \to S_i\) (see (3.10)). There is a chart of \(\phi\) of the form

\[x_i = z^n t^a, \quad y_i = z^m t^b,\]

with \(nb - am = \pm 1\),
where the representing divisor $D_p$ is defined by $z = 0$. We deduce from this that the order of vanishing of the Jacobian of $\phi$ along $D_p$ is equal to $n + m - 1$.

By definition the order of vanishing of the Jacobian of $\psi_i$ along $R_i$ is equal to $\lambda_{R_i} - 1$. By the chain rule the order of vanishing of the Jacobian of $\psi_i \circ \phi$ along $D_p$ is equal to

$$(\lambda_{R_i} - 1) \cdot n + n + m - 1 = n\lambda_{R_i} + m - 1.$$ 

The log-discrepancy of $D_p$ is equal to the order of vanishing of the Jacobian of $\psi_i \circ \phi$ along $D_p$ plus one, that is, it is equal to $n\lambda_{R_i} + m$. Then, formula (4.6) follows by (4.8). \hfill \qed

4.1. Main results

The following theorem is a generalization of Howald’s description of multiplier ideals of functions which are nondegenerate with respect their Newton polyhedra (see [23]) or his description of the multiplier ideals of monomial ideals (see [22]).

**Theorem 4.9.** Let $C = \sum_{i=1}^{\infty} a_i C_i$, $a_i \in \mathbb{N}^*$ be a plane curve singularity at a point $o$ of a smooth surface $S$. Let $R$ be a smooth branch which is not a component of $C$ and $C : \Sigma \to S$ a toroidal embedded resolution of $C$ given by Proposition 3.22. Then, for any rational number $\xi > 0$, we have

$$\mathcal{J}(\xi C)_o = \{ h \in \mathcal{O} \mid v_D(C_h) + \lambda_D > \xi v_D(C) \text{ for } D \in E(\pi) \cup \{ C_1, \ldots, C_r \} \}. \tag{4.10}$$

**Proof.** We use (1.2) to obtain that:

$$\mathcal{J}(\xi C)_o = \{ h \in \mathcal{O} \mid v_D(C_h) + \lambda_D > \xi v_D(C) \text{ for } D \in E(\pi) \cup \{ C_1, \ldots, C_r \} \}. \tag{4.11}$$

We translate condition (4.11) in terms of the fan tree (see Remark 3.28). Let us take a segment $[e_{R_i}, e_{L_i}]$ for some $i \in I_{\pi}$ of the decomposition $D_{\pi}(C)$ of the fan tree $\theta_\pi(C)$ (see Definition 3.26). By (4.4), for any marked point $p$ of $\theta_\pi(C)$ lying on the segment $[e_{R_i}, e_{L_i}]$ we have

$$v_{D_p}(C_h) = \Phi_{N_{R_i,L_i}(C_h)}(e_{D_p}). \tag{4.12}$$

In particular, if $C = C_h$ we get that

$$\xi v_{D_p}(C) = \xi \Phi_{N_{R_i,L_i}(C)}(e_{D_p}) = \Phi_{\xi N_{R_i,L_i}(C)}(e_{D_p}). \tag{4.13}$$

By (4.6) and (4.12) we get

$$v_{D_p}(C_h) + \lambda_{D_p} = \Phi_{N_{R_i,L_i}(C_h)}(e_{D_p}) + (e_{D_p}, \lambda_{D_p}) = \Phi_{N_{R_i,L_i}(C_h)}(e_{D_p}). \tag{4.14}$$

It follows that the condition:

$$v_{D_p}(C_h) + \lambda_{D_p} > \xi v_{D_p}(C),$$

for any marked point $p$ in the segment $[e_{R_i}, e_{L_i}]$, is equivalent to:

$$\Phi_{N_{R_i,L_i}(C_h)}(e_{D_p}) > \Phi_{\xi N_{R_i,L_i}(C)}(e_{D_p}). \tag{4.15}$$

By Lemma 3.7, the expression (4.15) for $p \in [e_{R_i}, e_{L_i}]$, is equivalent to the inclusion:

$$N_{R_i,L_i}(C_h) + \lambda_{D_p} \subset \text{Int}(\xi N_{R_i,L_i}(C)). \tag{4.16}$$

Taking this into account for every $i \in I_{\pi}$ ends the proof of (4.10). \hfill \qed

**Corollary 4.17.** With the hypothesis and notation of Theorem 4.9 we have that:

$$\mathcal{J}(\xi C)_o = \{ h \in \mathcal{O} \mid v_D(C_h) + \lambda_D > \xi v_D(C) \text{ for } D \in \mathcal{R}_\pi(C) \cup \{ C_1, \ldots, C_r \} \}. \tag{4.18}$$

In addition, if $C$ is reduced and $0 < \xi < 1$ we have

$$\mathcal{J}(\xi C)_o = \{ h \in \mathcal{O} \mid v_D(C_h) + \lambda_D > \xi v_D(C) \text{ for } D \in \mathcal{R}_\pi(C) \}. \tag{4.19}$$
Proof. We keep the notation of the proof of Theorem 4.9 assuming that \( \pi : \Sigma \to S \) is a maximal contact toroidal embedded resolution of \( C \).

Notice that if \( p = e_{L_i} \) and \( L_i \) is not a branch of \( C \) then condition (4.15) is always satisfied since \( \lambda_{L_i} = 1 \) and \( v_{L_i}(C) = 0 \). The same happens if \( p = e_R \) since \( R \) is not a component of \( C \) by hypothesis. By Lemma 3.7, the inclusion (4.16) is equivalent the inequality (4.15) for \( p \) running through the ramification points of the tree \( \theta_\pi(C) \) which belong to the segment \([e_R, e_{L_i}]\). This proves formula (4.18).

Assume now that \( C \) is reduced, that is, \( a_j = 1 \) for \( j = 1, \ldots, r \). If \( 0 < \xi < 1 \) and if \( p = e_{L_i} \) is a branch of \( C \) then condition (4.15) always holds since \( \lambda_{L_i} = 1 \) and \( v_{L_i}(C) = 1 \). This implies (4.19).

We describe now the jumping numbers and the generators of the multiplier ideals:

**Theorem 4.20.** Let \( C = \sum_{j=1}^{r} a_i C_i, a_i \in \mathbb{N}^* \) be a plane curve singularity at a point \( o \) of a smooth surface \( S \). Let \( R \) be a smooth branch which is not a component of \( C \) and \( \pi : \Sigma \to S \) a toroidal embedded resolution of \( C \) given by Proposition 3.22. Denote by \( \{x_0, \ldots, x_s\} \) the set obtained by taking a defining function for every irreducible component of the completion \( \hat{C}_\pi \). We associate with a monomial \( M \) in \( x_0, \ldots, x_s \), the number

\[
\xi_M := \min\left\{ \frac{v_D(M) + \lambda_D}{v_D(C)} \mid D \in R_\pi(C) \cup \{C_1, \ldots, C_r\} \right\}.
\]

Then:

(1) For any rational number \( \xi > 0 \), the multiplier ideal \( J(\xi C)_o \) is generated by the finite set of monomials \( M \) in \( x_0, \ldots, x_s \) such that \( \xi < \xi_M \leq \xi + 1 \).

(2) The jumping numbers of the multiplier ideals of \( C \) is the set of rational numbers \( \xi_M \) for \( M \) running through the monomials \( x_0, \ldots, x_s \).

**Proof.** By Corollary 4.17 we have that the multiplier ideal \( J(\xi C)_o \) is a valuation ideal with respect to the divisorial valuations \( v_D \) for \( D \in R_\pi(C) \) and the vanishing order valuations \( v_{C_j} \), for \( j = 1, \ldots, r \). By Remark 3.28, the set of components of \( \hat{C}_\pi \) contains a set of maximal contact curves of the pair \( (\pi, C) \). Corollary 2.7 implies that \( J(\xi C)_o \) is generated by monomials in \( x_0, \ldots, x_s \). By (4.11) a monomial \( M \) in \( x_0, \ldots, x_s \) belongs to the multiplier ideal \( J(\xi C)_o \) if and only if \( \xi < \xi_M \).

Assume that \( \xi + 1 < \xi_M \) and let us prove that \( M \) is not a generator of the ideal \( J(\xi C)_o \). Our assumption implies that \( 1 < \xi_M \) therefore \( M \) belongs to \( J(1 - C) = (f_C) \), where \( f_C \) is a defining function of \( C \) (see Lemma 1.4). By Definition 2.3, we can assume that \( f_C \) is a monomial in \( \{x_0, \ldots, x_s\} \). It follows that \( M = M' \cdot f_C \) where \( M' \) is a monomial in \( x_0, \ldots, x_s \). Then, we check that \( \xi_M = 1 + \xi_M' \). We iterate this argument to obtain a monomial \( M'' \) in \( \{x_0, \ldots, x_s\} \) dividing \( M \) and such that \( M'' \in J(\xi C)_o \), and \( \xi < \xi_M'' \leq \xi + 1 \). This ends the proof of the first assertion.

Let us prove the second assertion. If \( \xi > 0 \) is a jumping number of the multiplier ideals of \( C \) there exists \( 0 < \epsilon \) small enough, such that we have

\[
J(\xi C)_o \subseteq J(\xi' C)_o \text{ and } J(\xi' C)_o = J(\xi'' C)_o,
\]

for any \( \xi', \xi'' \in (\xi - \epsilon, \xi) \). There exists a monomial \( M \) in \( x_0, \ldots, x_s \) such that \( M \in J(\xi' C)_o \) for \( \xi' \in (\xi - \epsilon, \xi) \) and \( M \notin J(\xi C)_o \). By (4.18) for any \( D \in R_\pi(C) \cup \{C_1, \ldots, C_r\} \) we have

\[
v_D(M) + \lambda_D > \xi' v_D(C), \text{ for } \xi' \in (\xi - \epsilon, \xi),
\]

while there exists \( D_0 \in R_\pi(C) \cup \{C_1, \ldots, C_r\} \) such that the condition \( v_{D_0}(M) + \lambda_{D_0} > \xi v_{D_0}(C) \) is not satisfied. This shows that \( \xi = \xi_M \) by continuity. Conversely, for every monomial \( M \) the number \( \xi_M \) is a jumping number since \( M \in J(\xi' C)_o \) for \( \xi' \in (\xi_M - \epsilon, \xi_M) \) for \( 0 < \epsilon \) small enough, while \( M \notin J(\xi_M C)_o \).

**Remark 4.22.** The toroidal embedded resolution \( \pi \) is the composition of \( \ell_\pi \) regularized Newton modifications. Assume that \( \pi \) is such that the integer \( \ell_\pi \) is the smallest one. Then, by [28, Theorem...
one has that \( \pi \) is the minimal embedded resolution of \( C \) and the set \( \{x_0, \ldots, x_s\} \) in Theorem 4.20 is obtained by taking a defining function for every element in a sequence of maximal contact curves of the pair \((\pi, C)\). Up to relabeling, let us denote by \( \{x_0, \ldots, x_{s'}\} \) a subset \( \{x_0, \ldots, x_s\} \) consisting of the defining functions of a set of maximal contact curves of \( \pi \). If \( C \) is reduced and \( 0 < \xi < 1 \) then the multiplier ideal \( \mathcal{J}(\xi C) \) is generated by monomials in the sequence \( \{x_0, \ldots, x_{s'}\} \). This is consequence of (4.19) and Corollary 2.7. Then, reasoning as in the proof of Theorem 4.20 we obtain that if \( 0 < \xi_0 < 1 \) is a jumping number there exists a monomial \( M \) in \( x_0, \ldots, x_{s'} \) such that \( \xi_0 \) equals

\[
\xi_0^{\text{red}} := \min\{(v_D(M) + \lambda_D)(v_D(C))^{-1} \mid D \in \mathcal{R}_{\pi}(C)\}. \tag{4.23}
\]

Conversely, if \( M \) is a monomial in \( x_0, \ldots, x_{s'} \) such that \( 0 < \xi_0^{\text{red}} < 1 \) then \( \xi_0^{\text{red}} \) is a jumping number of \( C \). Notice that if \( \xi' > 1 \) is not an integer then \( 0 < \xi := \xi' - \lfloor \xi' \rfloor < 1 \) and by Lemma 1.4, we have that \( \mathcal{J}(\xi C)_o = f[\xi'] : \mathcal{J}(\xi C)_o \).

**Remark 4.24.** A different proof of the monomiality in Theorem 4.20 was given in [2]. Their proof holds more generally for integrally closed ideals of \( \mathcal{O} \) and uses the correspondence between antinef divisors and complete ideals. The approach of this paper may be also generalized for the study of the multiplier ideals of ideals of \( \mathcal{O} \) (see [31, section 5.3]).

**Remark 4.25.** The results of this paper also hold for an algebraic curve \((C, o)\) on a smooth surface \( S \) over algebraically closed field of arbitrary characteristic. Notice that in this case we have a unique minimal embedded resolution of the plane curve \( C \) and any other embedded resolution factors through it. The argument given in [26, Theorem 9.2.18] implies that the multiplier ideals \( \mathcal{J}(\xi C)_o \) are independent of the choice of embedded resolution. Then, we use that the toroidal embedded resolutions can be built in this setting independently of the characteristic of the base field. In particular, there is no need to pass through Newton-Puiseux series in order to connect the fan tree \( \theta_{\pi}(C) \) of a toroidal embedded resolution with the functions in the valuative tree (see [18, Section 1.6.6] and [17] for details).

**Remark 4.26.** The slope function of \( S_\pi \) of the fan tree \( \theta_{\pi}(C) \) was introduced in [18], see Definition 3.26. This function determines explicitly the values of the log-discrepancies of a rupture component \( D \) of \( \pi^+(C) \) (see [17, Proposition 8.16 (2)]) and the values of the divisorial valuation \( v_D \) at the branches of the completion \( \hat{C}_\pi \) (see [17, Corollary 3.26, Propositions 7.18 and 8.16 (1)]). That is all the data required to compute the multiplier ideals \( \mathcal{J}(\xi C)_o \) as in Example 4.27 below. In [18, Section 6.5] it is shown how to identify the fan tree \( \theta_{\pi}(C) \) with the Eggers-Wall tree of \( \hat{C}_\pi \). This identification is related to the embedding of the fan tree \( \theta_{\pi}(C) \) in the valuative tree \( V_R \) of normalized semivaluations with respect to the smooth branch \( R \) (see [17]), which sends a rational point \( p \in \theta_{\pi}(C) \) to the normalized valuation \( \frac{v_D}{v_D(p)(R)} \). The valuative tree \( V_R \) has been intensively studied by Favre and Jonsson in [13], see also [25, Section 7].

**Example 4.27.** Let us consider the toroidal embedded resolution \( \pi \) of the plane curve \( C \) of Example 3.23. The set of rupture components of \( \pi^+(C) \) is \( \mathcal{R}_{\pi}(C) = \{R_2, R_3, R_4\} \). The log-discrepancies of the exceptional divisors \( R_2, R_3 \) and \( R_4 \) are

\[
\lambda_{R_2} = 5, \lambda_{R_3} = 8, \text{ and } \lambda_{R_4} = 13. \tag{4.28}
\]

Table 1 provides the the values of the divisorial valuations \( v_{R_i} \) of the rupture components \( R_i \), for \( i = 2, 3, 4 \), at the branches of \( \hat{C}_\pi \).

**Table 1.** List of values for the divisorial valuations in Example 4.27.

| \( v_{R_2} \) | 2 | 3 | 6 | 12 | 9 | 21 |
| \( v_{R_3} \) | 3 | 5 | 9 | 18 | 15 | 33 |
| \( v_{R_4} \) | 4 | 6 | 15 | 30 | 18 | 48 |
We illustrate in this example how the computation of the set of jumping numbers smaller than one and of a system of generators of the corresponding multiplier ideals reduces to an optimization problem in terms of the data of values of Table 1 and the log-discrepancies (4.28).

Recall from Example 3.30 that \( R_1 = C_6 \), \( L_1 = C_6 \), and \( L_2 = C_2 \), where \( z = y^2 + x^3 \). By Theorem 2.6, the functions \( x \), \( y \) and \( z \) define a minimal generating sequence of the divisorial valuations \( v_{R_2} \), \( v_{R_1} \) and \( v_{R_1} \). By Corollary 4.20 and Remark 4.22, if \( 0 < \xi \leq 1 \), if \( \xi \) is a jumping number then there exists a monomial \( M = x^a y^b z^c \) such that \( \xi = \frac{\xi_{\text{red}}}{M} \) where

\[
\xi_{\text{red}} := \min\{(2a + 3b + 6c + 5)/21, (3a + 5b + 9c + 8)/33, (4a + 6b + 15c + 13)/48\}.
\]

The multiplier ideal \( J(\xi C)_o \) is generated by monomials in \( x \), \( y \) and \( z \) such that \( \xi < \xi_{\text{red}} \leq \xi + 1 \). We indicate below the multiplier ideals ideals \( J(\xi C)_o \) for \( \xi \) running through the 30 jumping numbers in the interval \((0, 1)\), where we have underlined a monomial \( M = x^a y^b z^c \in J(\xi C)_o \) if \( \xi_{\text{red}} > \xi \) is the next jumping number after \( \xi \). The output is not a minimal set of generators of the ideals, for instance, we get \( J(\frac{5}{21} C)_o = (x, y, z) \), which is equal \((x, y)\).

\[
\begin{align*}
J(\frac{5}{21} C)_o & = (x, y, z) \\
J(\frac{1}{3} C)_o & = (x^2, y, z) \\
J(\frac{8}{21} C)_o & = (x^2, y, x^2, y^2, z) \\
J(\frac{3}{11} C)_o & = (x^3, y, y^2, z) \\
J(\frac{10}{21} C)_o & = (x^3, x^2 y, y^2, z) \\
J(\frac{13}{21} C)_o & = (x^4, x^2 y, x^2 y^2, x^3 y^2, y^3, x z, y z, z^2) \\
J(\frac{17}{21} C)_o & = (x^4, x^2 y, x^2 y^2, y^3, x z, y z, z^2) \\
J(\frac{19}{21} C)_o & = (x^5, x^3 y, x^2 y^2, y^3, x^2 z, x z, y z, z^2) \\
J(\frac{22}{21} C)_o & = (x^5, x^3 y, x^2 y^2, y^3, x^2 z, x z, y z, z^2) \\
J(\frac{24}{21} C)_o & = (x^6, x^4 y, x^2 y^2, y^3, x^2 y z, x y z, y z, z^2) \\
J(\frac{26}{21} C)_o & = (x^6, x^4 y, x^2 y^2, y^3, x^2 y z, x y z, y z, z^2) \\
J(\frac{29}{21} C)_o & = (x^6, x^5 y, x^3 y^2, x^2 y z, x^2 y z, y z, z^2) \\
J(\frac{32}{21} C)_o & = (x^7, x^6 y, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2) \\
J(\frac{34}{21} C)_o & = (x^7, x^6 y, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2) \\
J(\frac{36}{21} C)_o & = (x^7, x^6 y, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2) \\
J(\frac{38}{21} C)_o & = (x^8, x^7 y, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2) \\
J(\frac{40}{21} C)_o & = (x^8, x^7 y, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2) \\
J(\frac{42}{21} C)_o & = (x^9, x^8 y, x^5 y^2, x^4 y^2, x^3 y^2, x^2 y z, x y z, y z, z^2)
\end{align*}
\]
Remark 4.29. The fourth author wrote a program in Python to compute the jumping numbers and finite presentations for the multiplier ideals of a plane curve singularity in terms the log-discrepancies of the exceptional divisors $D \in \mathcal{R}_\pi(C)$, and the values of $\nu_D$ on the components of $\hat{C}_\pi$. The code is available at https://github.com/tdimitch/jumping-numbers.

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