NEW PROOFS OF BASIC THEOREMS IN CALCULUS

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Abstract. New proofs of three basic calculus theorems are presented. We first present a quantitative necessarily and sufficient condition for a function to be uniformly continuous, and as a by-product we obtain explicitly the optimal delta for the given epsilon. The uniform continuity of a continuous function defined on a compact metric space follows as a simple consequence. We proceed with the extreme value theorem and present two proofs of this theorem: the “envelope proof” in which the largest possible maximal point is found using an envelope function, and the “programmer’s proof”, which does not use the costume argument of proving boundedness first. We finish with the intermediate value theorem, which is generalized to a class of discontinuous functions and in which the meaning of the intermediate value property is re-examined. At the end we discuss briefly in which sense the proofs are constructive.

1. Introduction

In this note we present new proofs of three classical calculus theorems. Although these theorems are well-known, in each proof we obtain something which seems to be unknown.

We first present a quantitative necessarily and sufficient condition for a function between two metric spaces to be uniformly continuous, and as a by-product we obtain explicitly the optimal delta for the given epsilon. The uniform continuity of a continuous function defined on a compact metric space follows as a simple consequence. We proceed with the extreme value theorem and present two proofs of this theorem: the “envelope proof” in which the largest possible maximal point is found using an envelope function, and the “programmer’s proof”, which does not use the costume argument of proving boundedness first. We finish with the intermediate value theorem, which is generalized to a class of discontinuous functions and in which the meaning of the intermediate value property is re-examined. At the end we discuss briefly in which sense the proofs are constructive.

The proofs presented here have an elementary character. However, in contrast to the well-known proofs of these theorems (see e.g., [4, 5, 7]), these proofs are probably not suitable for a first semester course in calculus.

2. Uniform continuity

A well-known fact is that any real continuous function defined on a compact interval \( I \) is uniformly continuous, i.e., for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y \in I \), if \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). Known proofs of this fact, e.g., the ones taken from [4, p. 193],[5, p. 168-169], [6, p. 91], and [7, p. 143-144]

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show the existence of such positive \( \delta \), but do not give any single clue for finding it explicitly. In particular, they do not give any information on how to find the largest possible such \( \delta \). Is it possible to find explicitly this optimal \( \delta \)? The following lemma shows that the answer is positive. A key step is simply to reformulate the condition of uniform continuity. The uniform continuity of a continuous function defined on a compact space is a simple consequence of this lemma.

**Lemma 2.1.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces, and let \( f : X \rightarrow Y \).

(a) \( f \) is uniformly continuous if and only if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y \in X \), if \( d_Y(f(x), f(y)) \geq \epsilon \), then \( d_X(x, y) \geq \delta \).

(b) Suppose that \( f \) is non-constant and let \( M \in (0, \infty] \) be defined by \( M := \sup \{d_Y(f(x), f(y)) : x, y \in X \} \). Then \( f \) is uniformly continuous if and only if the function \( \delta : [0, M) \rightarrow [0, \infty) \) defined by

\[
\delta(\epsilon) = \inf \{d_X(x, y) : (x, y) \in X^2, d_Y(f(x), f(y)) \geq \epsilon \}
\]

satisfies \( \delta(\epsilon) > 0 \) for each \( \epsilon \in (0, M) \). Moreover, the function \( \delta \) defined above assigns to each \( \epsilon \in (0, M) \) the largest possible delta from the definition of uniform continuity.

**Proof.** (a) Simple.

(b) Since the set \( A_\epsilon := \{(x, y) \in X^2 : d_Y(f(x), f(y)) \geq \epsilon \} \) is nonempty by the choice of \( \epsilon \), it follows that the function \( \delta \) is well defined. Now, if \( \delta(\epsilon) > 0 \) for each \( \epsilon \in (0, M) \), then from (a small modification of) the previous part \( f \) is uniformly continuous. On the other hand, suppose that \( f \) is uniformly continuous, and let \( \delta \) be any positive number with the property that \( d_X(x, y) < \delta \) implies \( d_Y(f(x), f(y)) < \epsilon \). Then for each \( (x, y) \in A_\epsilon \) we must have \( \delta \leq d_X(x, y) \), i.e., \( \delta \) is a lower bound of the set \( \{d_X(x, y) : (x, y) \in A_\epsilon \} \). Hence \( \delta \leq \delta(\epsilon) \), and in particular \( \delta(\epsilon) \) is positive and it is the largest possible delta from the definition of uniform continuity.

\( \square \)

**Theorem 2.2.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces, and let \( f : X \rightarrow Y \) be continuous. If \( X \) is compact, then \( f \) is uniformly continuous.

**Proof.** The assertion is obvious if \( f \) is constant, so from now on assume it is not. By Lemma 2.1 it is sufficient to show that the function \( \delta \) defined in (1) satisfies \( \delta(\epsilon) > 0 \) for each \( \epsilon \in (0, M) \). Assume to the contrary that there exists some \( \epsilon \in (0, M) \) for which \( \delta(\epsilon) = 0 \). Then \( 0 = \delta(\epsilon) = \lim_{n \rightarrow \infty} d_X(x_n, y_n) \) for some sequence \( ((x_n, y_n)) \) contained in \( A_\epsilon = \{(x, y) \in X^2 : d_Y(f(x), f(y)) \geq \epsilon \} \), and by passing to a convergent subsequence we find that \( 0 = d_X(x, y) \) for some \( (x, y) \in X^2 \). But \( f \) is continuous, so \( d_Y(f(x), f(y)) \geq \epsilon \), a contradiction.

\( \square \)

To the best of our knowledge, the issue of the “optimal delta” is not treated in the literature/calculus courses, although it is sometimes raised by curious teachers/students/readers. We note that \( \delta(\epsilon) \) from (1) is somewhat dual, but definitely different, from the modulus of (uniform) continuity

\[
w(\delta) = \sup \{d_Y(f(x), f(y)) : x, y \in X, d_X(x, y) \leq \delta \},
\]

which assigns to a given \( \delta \geq 0 \) the smallest possible \( \epsilon \geq 0 \) from the definition of uniform continuity, when one allows weak inequalities. However, \( \delta(\epsilon) \) can be
regarded as a modulus of its own. Local versions of this modulus can be defined similarly.

It is of some interest to compute explicitly $\delta(\epsilon)$ in some simple cases. For example, let $X = [0, b], Y = \mathbb{R}$ and let $f : X \to Y$ be defined by $f(x) = x^\alpha$, where $\alpha, b \in (0, \infty)$. Then

$$
\delta(\epsilon) = \begin{cases}
  b - (b^\alpha - \epsilon)^{1/\alpha} & 1 \leq \alpha, \\
  \epsilon^{1/\alpha} & 0 < \alpha \leq 1.
\end{cases}
$$

To derive this, one simply finds the minimum of $g(x, y) = |x - y|$, or equivalently of $h(x, y) = (x - y)^2$, on the set $A_\epsilon = \{(x, y) \in X^2 : |f(x) - f(y)| \geq \epsilon\}$. This minimum is always attained on the boundary of $A_\epsilon$. It is interesting whether one can compute $\delta(\epsilon)$ using the standard definition of uniform continuity. Note that here $\epsilon < M = b^\alpha$, so $\delta(\epsilon)$ is indeed well defined when $\alpha \geq 1$. In fact, the above formula shows that $f$ is uniformly continuous on $[0, \infty)$ for $0 < \alpha \leq 1$, but cannot be uniformly continuous there if $\alpha > 1$.

It is tempting to conjecture, and the above example supports this, that the optimal delta is a continuous function of $\epsilon$. Unfortunately, in general this is not true: the “decreasing chainsaw” function $f : [0, 1] \to \mathbb{R}$ defined by $f(0) = 0$ and (here $n \in \mathbb{N}$) by

$$
f(t) = \begin{cases}
  \frac{1}{n+1} - (2n+1) \left( t - \frac{1}{n+1} \right) & t \in [1/(n+1), 2/(2n+1)], \\
  (2n+1) \left( t - \frac{2}{2n+1} \right) & t \in [2/(2n+1), 1/n],
\end{cases}
$$

shows that in general $\delta$ may be discontinuous at infinitely many points (here $\delta(1/n) = 1/(n(2n+1)) < 1/(n(2n-1)) \leq \delta(\epsilon)$ whenever $1/n < \epsilon$). However, it can be easily verified that $\delta$ is always lower semicontinuous. In addition, it is increasing, so it is continuous, and actually differentiable, almost everywhere. See [1, 3] for a related discussion about the latter issue.

3. The Extreme Value Theorem

**Theorem 3.1.** Let $(I, d)$ be a compact metric space and let $f : I \to \mathbb{R}$ be continuous. Then $f$ has both a minimum and a maximum on $X$.

We present two different proofs of Theorem 3.1. The first one is only for the special case where $I$ is a compact interval. It is based on a certain “envelope” function, and uses the costume argument of proving first that $\sup_{x \in I} f(x)$ and $\inf_{x \in I} f(x)$ are finite, and then proving that they are attained; this argument appears in almost all the proofs we know, including the topological one [6, p. 89]. The only exception one is the proof of Fort [2]. The second proof is for the general case, and it does not use the above argument. The proof is significantly different than that of Fort, but can be thought as a dual to his one, in the sense that in his proof one proves the existence of the extreme value by starting from “above” (the whole space) and going “downward” (to smaller subsets), and in our case we start from “below” (a finite subset) and go “upward” (to a dense subset).

**Proof 1: the “envelope proof”**. The case where $I$ is a singleton is obvious, so from now on assume that $I$ contains at least two points. The proof consists of two
steps.

**Step 1:** We show that $f$ is bounded using the “real induction” argument. This part of the proof is not really new (it is a modification of [7, p. 135]), but it is included for the sake of completeness. Let

$$A = \{ x \in I : f \text{ is bounded on } [a, x] \}.$$  

$A$ is nonempty because $a \in A$. Since $f$ is continuous, each $x \in X$ has a neighborhood in which $f$ is bounded. Hence $A$ has the property that if $x \in A$, then also $[x, x + \delta] \cap I \subseteq A$ for some $\delta > 0$, and in particular $[a, a + \delta] \subset A$ for some $\delta > 0$. Let $s = \sup A$. Because $f$ is continuous at $s$, there are $M_1, \delta \in (0, \infty)$ such that $|f(t)| \leq M_1$ for all $t \in (s - \delta, s + \delta) \cap I$. By the definition of $s$ there exists $x \in A \cap (s - \delta, s)$, and by the definition of $A$ we know that $\sup_{t \in [a, x]} |f(t)| \leq M_2$ for some $M_2 \in (0, \infty)$. Hence $f$ is bounded on $[a, s + \delta) \cap I$ by $M_1 + M_2$, so in particular $s \in A$. But now, by the property of $A$ described above, it must be that $s = b$, otherwise $s$ is not the supremum of $A$. Thus $A = I$, and this establishes the first step.

**Step 2:** We now find explicitly a point $x_0 \in I$ at which $f$ attains a maximal value. $x_0$ is in fact the largest possible such point; a similar argument can be used for showing that $f$ attains a minimal value. Let

$$g(x) = \sup\{ f(t) : t \in [a, x] \}, \quad \forall x \in I.$$  

By the first step the “envelope” function $g$ is well defined (and actually continuous, but we will not use this fact), and it is obviously increasing, so it has a maximum at $b$. Let

$$C = \{ x \in I : g(x) = g(b) \}, \quad x_0 = \inf C.$$  

We finish by showing that $f(x_0) = g(b) = \sup_{x \in I} f(x)$. Let $\epsilon > 0$. By the continuity of $f$ there exists $\delta > 0$ such that $f(x) < f(x_0) + \epsilon$ for each $x$ in the intersection $I \cap [x_0 - \delta, x_0 + \delta]$. By the definition of $x_0$ there exists $x \in C \cap [x_0, x_0 + \delta]$. If $x_0 = a$, then $g(b) = g(x) \leq f(x_0) + \epsilon$. Otherwise, assume $\delta < \min(x_0 - a, b - a)$. We have

$$g(b) = g(x) \leq \max(\sup\{ f(t) : t \in [a, x_0 - \delta] \}, \sup\{ f(t) : t \in [x_0 - \delta, x_0 + \delta] \})$$

$$\leq \max(g(x_0 - \delta), f(x_0) + \epsilon).$$

If $g(b) \leq g(x_0 - \delta)$, then there is equality because $g$ is increasing, so we obtain a contradiction to the definition of $x_0$. Hence also $a < x_0$ implies $g(b) \leq f(x_0) + \epsilon$. Since $\epsilon$ was arbitrary, we conclude that $f(x_0) = g(b)$.

**Proof 2:** the “programmer proof”. Consider an increasing sequence $(E_n)_n$ of finite subsets of $I$ such that $\bigcup_{n=1}^{\infty} E_n = I$. If $I$ is a compact interval $[a, b]$, then we can take

$$P_k = a + \frac{(b - a)k}{2^n}, \quad E_n = \{ P_k : k = 0, 1, \ldots, 2^n \}$$

for each $n \in \mathbb{N} \cup \{0\}$ and each $k = 0, 1, \ldots, 2^n$. In general, we can define $E_{n+1} = E_n \cup F_n$, $E_0 = F_0$, where $F_n$ is the set of centers of the balls of a given $2^{-n}$-net of $I$. Let

$$M_n = \max\{ f(x) : x \in E_n \},$$

$$m_n = \min\{ f(x) : x \in E_n \}.$$
The function \( f \) attains a maximum on the finite set \( E_n \), i.e., there exists \( x_n \in E_n \) such that \( f(x_n) = M_n \). Let \((x_n)_j\) be any convergent subsequence of \((x_n)_n\) and let \( x_\infty = \lim_{j \to \infty} x_{n_j} \). Then \( x_\infty \in I \), and because \( f \) is continuous,

\[
f(x_\infty) = \lim_{j \to \infty} f(x_{n_j}) = \lim_{j \to \infty} M_{n_j}.
\]

Actually, the whole sequence \((M_n)_{n=0}^\infty\) converges to \( f(x_\infty) \), since it is an increasing sequence with a convergent subsequence. We now show that \( f \) attains its maximal value at \( x_\infty \), i.e., \( f(x) \leq f(x_\infty) \) for all \( x \in I \). Let \( x \in I \) and let \( \epsilon > 0 \). Since \( f \) is continuous on \( I \), it is continuous at \( x \), so there exists \( \delta > 0 \) such that if \( y \in I \) satisfies \( d(y, x) < \delta \), then \( |f(x) - f(y)| < \epsilon \). By the construction of the sequence \((E_n)_n\), for \( n \) large enough there exists \( t_n \in E_n \) such that \( d(t_n, x) < \delta \). Therefore

\[
f(x) \leq f(t_n) + \epsilon \leq M_n + \epsilon \lim_{n \to \infty} f(x_\infty) + \epsilon.
\]

But \( \epsilon \) was arbitrary, so \( f(x) \leq f(x_\infty) \), and since \( x \) was arbitrary this means that \( f \) has a maximum at \( x_\infty \). By the same way \( f \) has a minimum on \( I \). \( \square \)

4. The intermediate value theorem

**Theorem 4.1.** Let \( I = [a, b] \subset \mathbb{R} \). If \( f : I \to \mathbb{R} \) is continuous and if \( c \in \mathbb{R} \) is between \( f(a) \) and \( f(b) \), then there exists \( x \in I \) such that \( f(x) = c \).

Theorem 4.1 is a consequence of the following more general theorem which generalizes the intermediate value theorem to a class of discontinuous functions, and also re-examines the meaning of the intermediate value. Before stating it, recall that a topological space \( X \) is called connected if it cannot be represented as \( X = A \cup B \), where \( A \) and \( B \) are two nonempty, disjoint and open sets in \( X \). A simple consequence of this definition and the completeness axiom is that every interval in \( \mathbb{R} \) is a connected space. For a subset \( D \) of \( X \) we denote by \( \text{Int}(D) \), \( \partial D \) and \( \text{Ext}(D) = X \setminus (D \cup \partial D) = \text{Int}(X \setminus D) \) its interior, boundary and exterior respectively. We will use the following terminology.

**Definition 4.2.** Let \( X, Y \) be two topological spaces. A function \( f : X \to Y \) is said to be continuous with respect to \( D \subseteq Y \) if \( f^{-1}(D) \) is an open set in \( X \).

For instance, \( f : X \to Y \) is continuous if and only if it is continuous with respect to all open subsets of \( Y \), and \( f : X \to \mathbb{R} \) is lower semicontinuous if and only if it is continuous with respect to all the intervals of the form \((a, \infty)\).

**Theorem 4.3.** Let \( X \) be a connected topological space and let \( Y \) be a topological space. Let \( D \subseteq Y \). If \( f : X \to Y \) is continuous with respect to both \( \text{Int}(D) \) and \( \text{Ext}(D) \), and if there are \( a, b \in X \) such that \( f(a) \in D \) and \( f(b) \notin D \), then there exists \( x \in X \) such that \( f(x) \in \partial D \). In particular this is true if \( f \) is continuous.

**Proof.** If \( f(a) \in \partial D \) or \( f(b) \in \partial D \), then the proof is complete. Otherwise, since \( f(a) \in D \) and \( f(b) \notin D \), it follows that \( f(a) \in \text{Int}(D) \) and \( f(b) \in \text{Ext}(D) \), so \( f^{-1}(\text{Int}(D)) \) and \( f^{-1}(\text{Ext}(D)) \) are nonempty sets and they are open by our assumption. Now, since

\[
X = f^{-1}(Y) = f^{-1}(\text{Int}(D) \cup \partial D \cup \text{Ext}(D)) = f^{-1}(\text{Int}(D)) \cup f^{-1}(\partial D) \cup f^{-1}(\text{Ext}(D)),
\]
it follows that if $f^{-1}(\partial D)$ is empty, then $X$ is a union of two open, disjoint and nonempty sets and this contradicts the assumption that $X$ is connected. Hence $f^{-1}(\partial D)$ is nonempty, i.e., there exists $x \in X$ such that $f(x) \in \partial D$. □

Proof of Theorem 4.1: Denote $D = (-\infty, c)$. Without loss of generality we can assume that $f(a) < c < f(b)$. Hence $f(a) \in \text{Int}(D) = D$, $f(b) \in \text{Ext}(D) = (c, \infty)$ and $f^{-1}(\text{Int}(D)), f^{-1}(\text{Ext}(D))$ are open because $f$ is continuous. Since $I$ is connected, by Theorem 4.3 there is $x \in I$ such that $f(x) \in \partial D = \{c\}$, i.e., $f(x) = c$.

Example 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ when $x$ is irrational, $f(x) = 2x$ when $x \in \mathbb{Q}\{1/n : n \in \mathbb{N}\}$ and $f(1/n) = 1, n \in \mathbb{N}$. Obviously $f^{-1}((0, \infty))$ and $f^{-1}((-, 0))$ are open sets, so the conditions of Theorem 4.3 are satisfied and indeed $f(x) \in \partial (0, \infty)$ for $x = 0$. But $f$ is discontinuous at every point. This shows that the type of continuity expressed in Definition 4.2 is a very weak one. It would be interesting to find more useful examples.

Example 4.5. Let $I = [a, b]$ and suppose that $f : I \to I$ has the property that both $f^{-1}((0, \infty))$ and $f^{-1}((-, 0))$ are open sets in $I$. Then $f$ has a fixed point in $I$. Indeed, if $f(a) \neq a$ and $f(b) \neq b$, then the usual trick of defining $g(x) = f(x) - x$ works here too, because $g(a) \in (0, \infty), g(b) \in (-\infty, 0)$ and both $g^{-1}((0, \infty))$ and $g^{-1}((-, 0))$ are open sets in $I$ as a simple check shows. Thus Theorem 4.3 implies that $g(t) = 0$ for some $t \in I$, i.e., $t$ is a fixed point of $f$.

Remark. There is another theorem which generalizes the intermediate value theorem [6, p. 93]. This theorem says that the image of a connected topological space by a continuous function is a connected topological space.

Both Theorem 4.3 and the above theorem generalize the classical intermediate value theorem. However, there are two main differences between them. First, in Theorem 4.3 the function $f$ is not necessarily continuous, but rather satisfies a mild condition of continuity. Second, the intermediate value property is expressed differently in both cases: in the theorem mentioned above it is expressed in the connectivity of $f(X)$, while in Theorem 4.3 it is expressed in the fact that if $f$ passes through both $D$ and its complement $Y \setminus D$, then it also passes through the boundary $\partial D$, which can be thought of as being an intermediate set between $D$ and $Y \setminus D$ (or between $\text{Int}(D)$ and $\text{Ext}(D)$).

5. Concluding remarks

The proofs given above raise some questions regarding the sense in which they are constructive.

In the proof of Theorem 2.2, for any $\epsilon > 0$ the corresponding $\delta(\epsilon)$ was found explicitly. However, the proof that $\delta(\epsilon) > 0$ for each $\epsilon > 0$ is based on nonconstructive arguments.

The first proof of Theorem 3.1 gives a representation for the largest possible maximal point $x_0$, but this representation is too vague to be considered as constructive. The second proof is somewhat constructive in the practical sense, since the monotone sequence $(M_n)_n$, which converges to $\max_{x \in I} f(x)$ can be computed easily and explicitly (but slowly), at least when $I$ is a compact interval. The sequence $(x_n)_n$ of points for which $f(x_n) = M_n$ can also be easily computed.
On the other hand, one can argue against the sense in which this proof is constructive. First, no error estimates were given for the convergence of \((M_n)_n\). Second, the point \(x_\infty\) at which \(f\) attains its maximal value usually cannot be found in a constructive manner, because it is a limit of a convergent subsequence, which usually cannot be computed explicitly in advance (recall that the existence of such subsequence follows from an infinite version of the Dirichlet pigeonhole principle, so it is highly nonconstructive).

Nevertheless, if some additional information is given about \(f\), and hence about \((x_n)_n\), then we can say more about \(x_\infty\). For instance, if it is known that \(x_\infty\) is unique, as it is the case when \(f\) is strictly concave, then the proof shows that the whole sequence \((x_n)_n\) converges to \(x_\infty\).

The proof of Theorem 4.3 can be regarded as a pure existence proof, i.e., a proof without any single constructive clue. However, if some additional information is known about the connected space \(X\), then by a repeated application of Theorem 4.3 one can compute explicitly with error estimates an “intermediate” point \(x \in X\) for which \(f(x) \in \partial D\).

For example, Let \(X, Y, D \subseteq Y\) and \(f : X \to Y\) satisfy the conditions of Theorem 4.3, where \(X = [a, b] \subseteq \mathbb{R}, f(a) \in D\) and \(f(b) \notin D\). Theorem 4.3 ensures that there exists \(x_0 \in X\) such that \(f(x_0) \in \partial D\). Now, for the point \(P_1 = a + (b - a)/2\), either \(f(P_1) \in D\) or \(f(P_1) \notin D\). Hence there are \(a_1, b_1 \in \{a, P_1, b\}\) such that \(f(a_1) \in D\), \(f(b_1) \notin D\) and \([a_1, b_1] \subseteq X\). Since \(f^{-1}(\text{Int}(D)) \cap [a_1, b_1]\) and \(f^{-1}(\text{Ext}(D)) \cap [a_1, b_1]\) are open sets in the connected space \([a_1, b_1]\), by Theorem 4.3 there exists \(x_1 \in [a_1, b_1]\) such that \(x_1 \in \partial D\), so one has a better estimate for an intermediate point. Continuing in this way, one essentially gets the bisection method and finds an intermediate point to within an error of \((b - a) \cdot 2^{-n}\) in the \(n\)-th step.

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