FINITE ELEMENT ANALYSIS FOR IDENTIFYING THE REACTION COEFFICIENT FROM BOUNDARY OBSERVATIONS

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Abstract. This work is devoted to the nonlinear inverse problem of identifying the reaction coefficient in an elliptic boundary value problem from single Cauchy data. We then examine simultaneously two elliptic boundary value problems generated from the available Cauchy data. The output least squares method with the Tikhonov regularization is applied to find approximations of the sought coefficient. We discretize the PDEs with piecewise linear finite elements. The stability and convergence of this technique are then established. A numerical experiment is presented to illustrate our theoretical findings.

Key words. Reaction coefficient, finite element method, Tikhonov regularization, Neumann problem, mixed problem, ill-posed problem

AMS subject classifications. 35R25, 47A52, 35R30, 65J20, 65J22

1. Introduction. Let Ω be an open, bounded and connected domain of \( \mathbb{R}^d \), \( d \geq 2 \) with the boundary \( \partial \Omega \) and \( \Gamma \subset \partial \Omega \) be an accessible part of the boundary which is relatively open. In this paper we are related with the following elliptic system

\[
\begin{align*}
-\nabla \cdot (\alpha \nabla \Phi) + \beta \Phi &= f & \text{in } \Omega, \\
\alpha \nabla \Phi \cdot \mathbf{n} + \sigma \Phi &= j^\dagger & \text{on } \Gamma, \\
\alpha \nabla \Phi \cdot \mathbf{n} + \sigma \Phi &= j_0 & \text{on } \partial \Omega \setminus \Gamma, \\
\Phi &= g^\dagger & \text{on } \Gamma,
\end{align*}
\]

where \( \mathbf{n} \) is the unit outward normal on \( \partial \Omega \), the boundary conditions \( j^\dagger \in H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^* \), \( j_0 \in H^{-1/2}(\partial \Omega \setminus \Gamma) \), \( g^\dagger \in H^{1/2}(\Gamma) \), the source term \( f \in H^{-1}(\Omega) := H^1(\Omega)^* \) and the functions \( \alpha \), \( \sigma \) are assumed to be known. Here, \( \sigma \in L^\infty(\Omega) \) with \( \sigma(x) \geq 0 \) a.e. on \( \partial \Omega \) and \( \alpha := (\alpha_{rs})_{1 \leq r,s \leq d} \in L^\infty(\Omega)^{d \times d} \) is a symmetric diffusion matrix satisfying the uniformly elliptic condition

\[
\alpha(x)\xi \cdot \xi = \sum_{1 \leq r,s \leq d} \alpha_{rs}(x)\xi_r \xi_s \geq \underline{\alpha}|\xi|^2
\]

a.e. in \( \Omega \) for all \( \xi = (\xi_r)_{1 \leq r \leq d} \in \mathbb{R}^d \) with some constant \( \underline{\alpha} > 0 \). In case \( \alpha = \alpha \cdot I_d \), the unit \( d \times d \)-matrix \( I_d \) and \( \alpha : \Omega \to \mathbb{R} \), then \( \alpha \) is called the scalar diffusion.

If the reaction coefficient

\[
\beta \in S_{ad} := \{ \beta \in L^\infty(\Omega) \mid 0 < \beta \leq \overline{\beta} \text{ a.e. in } \Omega \}
\]

is given also, there may be no \( \Phi \) satisfying the system (1.1)–(1.4), where the constants \( 0 < \beta \leq \overline{\beta} \) are known. Our aim in this paper is to reconstruct the coefficient \( \beta \in S_{ad} \) from several sets of observation data \( (j^\dagger_i, g^\dagger_i)_{i=1}^I \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \) of the exact \( (j^\dagger, g^\dagger) \) satisfying the deterministic noise model

\[
\frac{1}{I} \sum_{i=1}^I \left( \| j^\dagger_i - j^\dagger \|_{H^{-1/2}(\Gamma)} + \| g^\dagger_i - g^\dagger \|_{H^{1/2}(\Gamma)} \right) \leq \delta
\]

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with \( \delta > 0 \) denoting the error level of the observations.

The problem arises from different contexts of applied sciences, e.g., from aquifer analysis, optical tomography which attracted great attention of many scientists in the last 40 years or so. For surveys on the subject, we refer the reader to [3, 24, 25]. Although there have been many papers devoted to the subject, which were employed the assumption about distributed observations in the whole domain, see, e.g., Alt [2], Coloniuz and Kunisch [6], Engl et al. [7], Kaltenbacher and Hofmann [15], Kaltenbacher and Klassen [16], Neubauer [18], Resmerita and Scherzer [21] and [9, 10, 11, 12] and the references therein. We mention that the boundary measurement subject for this identification problem, which is more realistic from the practical point of view, has not yet been investigated so far.

For simplicity of exposition we below consider one observation pair \((j_\delta, g_\delta)\) being available, i.e. \( I = 1 \), while the approach described here can be naturally extended to multiple measurements. We from the available observation data \((j_\delta, g_\delta)\) simultaneously examine the Neumann boundary value problem

\[
\begin{align}
-\nabla \cdot (\alpha \nabla u) + \beta u &= f & \text{in } \Omega, \\
\alpha \nabla u \cdot \mathbf{n} + \sigma u &= j_\delta & \text{on } \Gamma,
\end{align}
\]

and the mixed boundary value problem

\[
\begin{align}
-\nabla \cdot (\alpha \nabla v) + \beta v &= f & \text{in } \Omega, \\
v &= g_\delta & \text{on } \Gamma,
\end{align}
\]

Let \( N_{j_\delta}(\beta) \) and \( M_{g_\delta}(\beta) \) be the unique weak solutions of (1.7)–(1.9) and (1.10)–(1.12), respectively. We then consider a minimizer \( \beta_{\delta, \rho} \) of the Tikhonov regularized minimization problem

\[
\min_{\beta \in S_{ad}} J_{\delta, \rho}(\beta), \quad J_{\delta, \rho}(\beta) := \|N_{j_\delta}(\beta) - M_{g_\delta}(\beta)\|^2_{L^2(\Omega)} + \rho\|\beta - \beta^*\|^2_{L^2(\Omega)} \quad (P_{\delta, \rho})
\]
as reconstruction, where \( \rho > 0 \) is the regularization parameter and \( \beta^* \) is an a priori estimate of the true coefficient. The motivation for using the above cost function is that \( \|N_{j_\delta}(\beta) - M_{g_\delta}(\beta)\|_{L^2(\Omega)} \geq 0 \) and at the sought coefficient \( \beta \) it holds the identity \( \|N_{j_\delta}(\beta) - M_{g_\delta}(\beta)\|_{L^2(\Omega)} = 0 \).

Let \( N_{j_\delta}^h(\beta) \) and \( M_{g_\delta}^h(\beta) \) be corresponding approximations of \( N_{j_\delta}(\beta) \) and \( M_{g_\delta}(\beta) \) in the finite dimensional space \( V_1^h \) of piecewise linear, continuous finite elements. Utilizing the variational discretization concept [13] of \((P_{\delta, \rho})\) that avoids explicit discretization of the control variable, we then consider the discrete regularized problem corresponding to \((P_{\delta, \rho})\), i.e. the following minimization problem

\[
\min_{\beta \in S_{ad}} J_{\delta, \rho}^h(\beta), \quad J_{\delta, \rho}^h(\beta) := \|N_{j_\delta}^h(\beta) - M_{g_\delta}^h(\beta)\|^2_{L^2(\Omega)} + \rho\|\beta - \beta^*\|^2_{L^2(\Omega)} \quad (P_{\delta, \rho}^h)
\]
which also attains a minimizer \( \beta_{\delta, \rho}^h \) satisfying the relation (cf. Section 2)

\[
\beta_{\delta, \rho}^h(x) = P_{[\delta, \rho]} \left( \frac{1}{\rho} (N_{j_\delta}^h(\beta_{\delta, \rho}^h)(x)A_N^h(\beta_{\delta, \rho}^h)(x) - M_{g_\delta}^h(\beta_{\delta, \rho}^h)(x)A_M^h(\beta_{\delta, \rho}^h)(x)) + \beta^*(x) \right)
\]
generates an inner product on the space \( \mathcal{V} \), i.e. there exist positive constants \( c_i \) and \( c_{ii} \) which are independent of both \( h \) and \( \rho > 0 \), such that

\[
\| \cdot \|_{\mathcal{V}} \leq \| \cdot \| \leq c_i \| \cdot \|_{\mathcal{V}}.
\]

In Section 3 we show that the proposed finite element method is stable, i.e. if the regularization parameter and the observation data are both fixed, then the sequence of minimizers \( (\beta^h)_{\rho>0} \) to \( (P_{\delta,\rho})_h \) converges to a solution of \( (P_{\delta,\rho})_h \) as the mesh size \( h \) of the triangulation \( T_h \) tends to zero. Furthermore as \( h, \delta \to 0 \) and with an appropriate priori regularization parameter choice \( \rho = \rho(h, \delta) \to 0 \), the whole sequence \( (\beta^h)_{\rho>0} \) converges in the \( L^2(\Omega) \)-norm to the \( \beta^* \)-minimum norm solution \( \beta^* \) of the identification problem defined by

\[
\beta^* = \arg \min_{\beta \in S_{\alpha,\delta}} \min_{N_{g_i} (\beta) = M_{g_i} (\beta)} \| \beta - \beta^* \|_{L^2(\Omega)}.
\]

The corresponding state sequences \( (N_{g_i}^h (\beta^h_{\delta,\rho}))_{\rho>0} \) and \( (M_{g_i}^h (\beta^h_{\delta,\rho}))_{\rho>0} \) then converge in the \( H^1(\Omega) \)-norm to the exact state \( \Phi^* = \Phi(\beta^*, g^1, \beta^*) \) of the problem (1.1)–(1.4).

Our numerical implementation will be presented in Section 4. First, for the numerical solution of the discrete regularized problem \( (P_{\delta,\rho}^h) \) we employ a gradient projection algorithm with Armijo steplength rule. In Example 4.1 we assume that observations are available on the bottom surface of the domain. Example 4.2 is a continuity of the first one, where we investigate the effect of the regularization parameter choice rule and previous iteration processes as well on the final computed numerical result. In case observations taking on the bottom and left surface the computation is given in Example 4.3, while Example 4.4 is devoted to multiple measurements.

To complete this introduction we wish to mention briefly about some coefficient identification problems in PDEs from boundary observations. The authors Xie and Zou [27], Xu and Zou [29] have used finite element methods to numerically recovered the fluxes on the inaccessible boundary \( \Gamma_i \) from measurement data of the state on the accessible boundary \( \Gamma_a \), while the problem of identifying the Robin coefficient on \( \Gamma_i \) is also investigated by Xu and Zou [28]. Recently, authors of [14] adopted the variational approach of Kohn and Vogelius combining with total variation regularization technique to the scalar diffusion coefficient identification using observations available on the whole boundary.

Throughout the paper the symbol \( A \preceq B \) refers to the inequality \( A \leq cB \) for some constant \( c \) independent of both \( A \) and \( B \). In the Lebesgue space \( L^2(Q) \), where \( Q \) is either \( \Omega, \partial \Omega \) or \( \Gamma \), we use for all \( y, \tilde{y} \in L^2(Q) \) the inner product and the corresponding norm as \( (y, \tilde{y})_Q := \int_Q y(x) \cdot \tilde{y}(x) dx \) and \( \|y\|_Q := ((y, y)_Q)^{1/2} \). We also use the standard notion of Sobolev spaces \( H^k(Q) := W^k_2(Q) \) from, e.g., [1] with notations of its inner product \( (\cdot, \cdot)_k, \), the norm \( \| \cdot \|_k, \) and the semi-norm \( | \cdot |_k, \). Note that \( \| \cdot \|_{0, Q} = | \cdot |_{0, Q} = \| \cdot \|_Q \).

2. Finite element discretization.

2.1. Preliminaries. We remark that the expression

\[
[u, v] := [u, v]_{(\alpha, \beta, \sigma)} := (\alpha \nabla u, \nabla v)_{\Omega} + (\beta u, v)_{\Omega} + (\sigma u, v)_{\partial \Omega}
\]

generates an inner product on the space \( H^1(\Omega) \) which is equivalent to the usual one, i.e. there exist positive constants \( c_1, c_2 \) such that

\[
c_1 \| u \|_{1, \Omega} \leq [u, u] \leq c_2 \| u \|_{1, \Omega}
\]
for all $u \in H^1(\Omega)$. Therefore, for each $\beta \in S_{ad}$ the Neumann boundary value problem (1.7)–(1.9) defines a unique weak solution $u = u(\beta) := N_{js}(\beta)$ in the sense that $N_{js}(\beta) \in H^1(\Omega)$ and the equation

$$
\langle N_{js}(\beta), \phi \rangle = \langle f, \phi \rangle + \langle j_s, \phi \rangle + \langle j_0, \phi \rangle_{\partial \Omega} \Gamma
$$

is satisfied for all $\phi \in H^1(\Omega)$, where $\langle \cdot, \cdot \rangle_{\Omega}$ and $\langle \cdot, \cdot \rangle_{\Gamma}$ stand for the dual pairs $\langle \cdot, \cdot \rangle_{(H^{-1}(\Omega), H^1(\Omega))}$ and $\langle \cdot, \cdot \rangle_{(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))}$, respectively. Furthermore, there holds the estimate

$$
\|N_{js}(\beta)\|_{1,\Omega} \leq \|j_s\|_{H^{-1/2}(\Gamma)} + \|j_0\|_{H^{-1/2}(\partial \Omega \setminus \Gamma)} + \|f\|_{H^{-1}(\Omega)}.
$$

A function $v = v(\beta) := M_{gs}(\beta)$ is said to be a unique weak solution of the mixed boundary value problem (1.10)–(1.12) if $M_{gs}(\beta) \in H^1(\Omega)$ with $M_{gs}(\beta)_{\partial \Omega} = g_s$ and the equation

$$
\langle M_{gs}(\beta), \phi \rangle = \langle f, \phi \rangle + \langle j_0, \phi \rangle_{\partial \Omega \setminus \Gamma}
$$

is satisfied for all $\phi \in H^1(\Omega)$, where $H^1(\Omega \cup \Gamma) := C^\infty(\Omega \cup \Gamma)^{H^1(\Omega)} = \{ \phi \in H^1(\Omega) \mid \phi|_{\Gamma} = 0 \}$, the bar denotes the closure in $H^1(\Omega)$ and $C^\infty(\Omega \cup \Gamma)$ is the set of all functions $\phi \in C^\infty(\Omega)$ with $\text{supp}\phi$ being a compact subset of $\Omega \cup \Gamma$ (see, e.g., [26, pp. 9, 67]). The above weak solution satisfies the estimate

$$
\|M_{gs}(\beta)\|_{1,\Omega} \leq \|g_s\|_{H^{1/2}(\Gamma)} + \|j_0\|_{H^{1/2}(\partial \Omega \setminus \Gamma)} + \|f\|_{H^{-1}(\Omega)}.
$$

**Remark 2.1.** We now state some properties of the coefficient-to-solution operators $N_{js}, M_{gs} : S_{ad} \rightarrow H^1(\Omega)$.

**Lemma 2.2.** Assume that the dimension $d \leq 4$. Then the operator $N_{js}$ and $M_{gs}$ are infinitely Fréchet differentiable on the set $S_{ad}$ with respect to the $L^2(\Omega)$-norm. For $\beta \in S_{ad}$ and $(\kappa_1, \ldots, \kappa_m) \in C^\infty(\Omega)^m$ the $m$-th order differentials $D^{(m)}_{N_{js}} := N_{js}^{(m)}(\beta)(\kappa_1, \ldots, \kappa_m) \in H^1(\Omega)$ and $D^{(m)}_{M_{gs}} := M_{gs}^{(m)}(\beta)(\kappa_1, \ldots, \kappa_m) \in H^1_0(\Omega \cup \Gamma)$ are the unique solutions to the variational equations

$$
\left[ D^{(m)}_{N_{js}}, \phi \right]_{(\alpha, \beta, \sigma)} = -\sum_{i=1}^{m} \left( \kappa_i N_{js}^{(m-1)}(\beta) \xi_i, \phi \right)_\Omega, \quad \forall \phi \in H^1(\Omega)
$$

and

$$
\left[ D^{(m)}_{M_{gs}}, \phi \right]_{(\alpha, \beta, \sigma)} = -\sum_{i=1}^{m} \left( \kappa_i M_{gs}^{(m-1)}(\beta) \xi_i, \phi \right)_\Omega, \quad \forall \phi \in H^1_0(\Omega \cup \Gamma)
$$

for all $m \in \mathbb{N}$.
with $\xi_i := (\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_m) \in L^\infty(\Omega)^{m-1}$, respectively. Furthermore,

$$\max \left( \|D_N^{(m)}\|_{H^1(\Omega)}, \|D_M^{(m)}\|_{H^1(\Omega)} \right) \leq \prod_{i=1}^m \|\kappa_i\|_{L^2(\Omega)}.$$

**Proof.** The proof is based on standard arguments, therefore omitted here. \qed

We mention that the restriction on the dimension $d \leq 4$ in the above Lemma 2.2 is removed if the $L^\infty(\Omega)$-norm is taken into account instead of the $L^2(\Omega)$-norm (see, e.g., [9, 11]).

**Lemma 2.3.** Assume that the sequence $(\beta_n)_n \subset S_{ad}$ converges weakly in $L^2(\Omega)$ to an element $\beta$. Then the sequences $(N_{j_t}(\beta_n))_n$ and $(M_{g_t}(\beta_n))_n$ converge respectively to $N_{j_t}(\beta)$ and $M_{g_t}(\beta)$ weakly in $H^1(\Omega)$ and strongly in the $L^2(\Omega)$-norm.

**Proof.** We first note that since $S_{ad}$ is a convex and closed subset of $L^2(\Omega)$, it is weakly closed in $L^2(\Omega)$ which implies that $\beta \in S_{ad}$. Furthermore, it is a weakly* compact subset of $L^\infty(\Omega)$ (see, e.g., [20, Remark 2.1]). Therefore, by the inequality (2.3) and the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ being compact, the sequence $(N_n)_n := (N_{j_t}(\beta_n))_n$ has a subsequence denoted by the same symbol such that

$$\beta_n \rightharpoonup \beta \text{ weakly* in } L^\infty(\Omega), \text{ i.e. } (\beta_n, \xi)_\Omega \to (\beta, \xi)_\Omega \text{ for all } \xi \in L^1(\Omega),$$

$$N_n \to \theta \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega)$$

as $n \to \infty$, where $\theta$ is an element of $H^1(\Omega)$. Since $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is continuous as $d \leq 4$, one can show that $\theta = N_{j_4}(\beta)$. With similar arguments we also obtain that $(M_{g_t}(\beta_n))_n$ converges to $M_{g_t}(\beta)$ weakly in $H^1(\Omega)$, which finishes the proof. \qed

Together with (1.7)–(1.12), we consider two adjoint problems

\begin{align*}
(2.8) & \quad -\nabla \cdot (\alpha \nabla A_N) + \beta A_N = N_{j_t}(\beta) - M_{g_t}(\beta) \text{ in } \Omega \\
(2.9) & \quad \alpha \nabla A_N \cdot \vec{n} + \sigma A_N = 0 \text{ on } \partial \Omega
\end{align*}

and

\begin{align*}
(2.10) & \quad -\nabla \cdot (\alpha \nabla A_M) + \beta A_M = N_{j_t}(\beta) - M_{g_t}(\beta) \text{ in } \Omega \\
(2.11) & \quad A_M = 0 \text{ on } \Gamma \\
(2.12) & \quad \alpha \nabla A_M \cdot \vec{n} + \sigma A_M = 0 \text{ on } \partial \Omega \setminus \Gamma
\end{align*}

that attain unique weak solutions $A_N = A_N(\beta)$ and $A_M = A_M(\beta)$ in the sense that $A_N \in H^1(\Omega)$ and $A_M \in H^1_0(\Omega \cup \Gamma)$ satisfy the variational equations

\begin{align*}
(2.13) & \quad [A_N, \phi] = (N_{j_t}(\beta) - M_{g_t}(\beta), \phi)_{\Omega}, \quad \forall \phi \in H^1(\Omega) \\
(2.14) & \quad [A_M, \phi] = (N_{j_t}(\beta) - M_{g_t}(\beta), \phi)_{\Omega}, \quad \forall \phi \in H^1_0(\Omega \cup \Gamma).
\end{align*}

Furthermore,

$$\max (\|A_N\|_{1, \Omega}, \|A_M\|_{1, \Omega}) \leq \|j_0\|_{H^{-1/2}(\Gamma)} + \|g_0\|_{H^{1/2}(\Gamma)} + \|j_0\|_{H^{-1/2}(\partial \Omega \setminus \Gamma)} + \|f\|_{H^{-1}(\Omega)}.$$

**Theorem 2.4.** The minimization problem

$$\min_{\beta \in S_{ad}} J_{\delta, \rho}(\beta), \quad J_{\delta, \rho}(\beta) := \|N_{j_t}(\beta) - M_{g_t}(\beta)\|^2_{\Omega} + \rho \|\beta - \beta^*\|^2_{\Omega} \quad (P_{\delta, \rho})$$
attains a minimizer $\beta_{\delta,\rho}$ which satisfies the identity
\[ \beta_{\delta,\rho}(x) = \mathcal{P}_{[\beta,\bar{\beta}]} \left( \frac{1}{\rho} (N_{js}(\beta_{\delta,\rho})(x)A_N(\beta_{\delta,\rho})(x) - M_{gs}(\beta_{\delta,\rho})(x)A_M(\beta_{\delta,\rho})(x)) + \beta^*(x) \right) \]
a.e. in $\Omega$, where $A_N$, $A_M$ come from (2.8)–(2.12).

Proof. The existence of a minimizer $\beta$ follows directly from Lemma 2.3. It remains to show the above identity. Due to the first order optimality condition for the minimizer $\beta$, we get for all $\gamma \in \mathcal{S}_{ad}$ that $J'_{\delta,\rho}(\beta)(\gamma - \beta) \geq 0$. Setting $\kappa := \gamma - \beta$, we then obtain
\[ (N_{js}(\beta) - M_{gs}(\beta), N'_{js}(\beta)\kappa - M'_{gs}(\beta)\kappa)_\Omega + \rho(\beta - \beta^*, \kappa)_\Omega \geq 0. \]
It follows from Lemma 2.2 that
\[ (N_{js}(\beta) - M_{gs}(\beta), N'_{js}(\beta)\kappa - M'_{gs}(\beta)\kappa)_\Omega = [A_N(\beta), N'_{js}(\beta)\kappa] - [A_M(\beta), M'_{gs}(\beta)\kappa] = -(A_N(\beta)N_{js}(\beta), \kappa)_\Omega + (A_M(\beta)M_{gs}(\beta)\kappa)_\Omega \]
which yields
\[ \left( \frac{1}{\rho} (N_{js}(\beta)A_N(\beta) - M_{gs}(\beta)A_M(\beta)) + \beta^* - \beta, \gamma - \beta \right)_\Omega \leq 0 \]
for all $\gamma \in \mathcal{S}_{ad}$. This completes the proof. \qed

2.2. Finite element discretization. Let $(T_h)^{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size $h$. For the definition of the discretization space of the state functions let us denote
\[ \mathcal{V}_k^h := \{ v^h \in C(\overline{\Omega}) \mid v^h |_{\Gamma} \in P_k, \ \forall \Gamma \in T_h \} \quad \text{and} \quad \mathcal{V}_{1,0}^h := \mathcal{V}_1^h \cap H_0^1(\Omega \cup \Gamma), \]
where $P_k$ consists of all polynomial functions of degree less than or equal to $k$. For each $\beta \in \mathcal{S}_{ad}$ the variational equations
\begin{align*}
(2.15) \quad [u^h, \phi^h]_{[\alpha,\beta,\sigma]} &= \langle f, \phi^h \rangle_{\Omega} + \langle j_0, \phi^h \rangle_{\partial \Omega} + \langle j_h, \phi^h \rangle_{\{h\} \Omega} \quad \forall \phi^h \in \mathcal{V}_1^h \\
(2.16) \quad [u^h, \phi^h]_{[\alpha,\beta,\sigma]} &= \langle f, \phi^h \rangle_{\Omega} + \langle j_0, \phi^h \rangle_{\partial \Omega} \quad \forall \phi^h \in \mathcal{V}_{1,0}^h \quad \text{and} \quad v^h |_{\Gamma} = g_{\delta} 
\end{align*}
admit unique solutions $u^h := N_{js}^h(\beta) \in \mathcal{V}_1^h$ and $v^h := M_{gs}^h(\beta) \in \mathcal{V}_{1,0}^h$, respectively. Furthermore, the estimates
\begin{align*}
(2.17) \quad \|N_{js}^h(\beta)\|_{H^1(\Omega)} &\leq \|j_0\|_{H^{-1/2}(\Gamma)} + \|j_h\|_{H^{-1/2}((\partial \Omega) \backslash \Gamma)} + \|f\|_{H^{-1}(\Omega)} \\
(2.18) \quad \|M_{gs}^h(\beta)\|_{H^1(\Omega)} &\leq \|g_{\delta}\|_{H^{1/2}(\Gamma)} + \|j_0\|_{H^{-1/2}((\partial \Omega) \backslash \Gamma)} + \|f\|_{H^{-1}(\Omega)} 
\end{align*}
hold true.

The solutions $A_N = A_N(\beta)$ and $A_M = A_M(\beta)$ of the adjoint problems (2.8)–(2.12) are approximated by $A_N^h = A_N^h(\beta) \in \mathcal{V}_1^h$ and $A_M^h = A_M^h(\beta) \in \mathcal{V}_{1,0}^h$ satisfying
\begin{align*}
(2.19) \quad [A_N^h, \phi^h] &= \langle N_{js}^h(\beta) - M_{gs}^h(\beta), \phi^h \rangle_{\Omega} \quad \forall \phi^h \in \mathcal{V}_1^h \\
(2.20) \quad [A_M^h, \phi^h] &= \langle N_{js}^h(\beta) - M_{gs}^h(\beta), \phi^h \rangle_{\Omega} \quad \forall \phi^h \in \mathcal{V}_{1,0}^h.
\end{align*}
The Sobolev number of $W_q^m(\Omega)$ is defined by $\text{sob}(W_q^m(\Omega)) := m - d/q$. 

Lemma 2.5 (Quasi-interpolation operator, see, e.g., [5, 19, 23]). There exists an operator $\Pi^h : L^1(\Omega) \to V^1_h$ such that $\Pi^h \phi^h = \phi^h$ for all $\phi^h \in V^1_h$ and the limit
\begin{equation}
(2.21) \lim_{h \to 0} \| \phi - \Pi^h \phi \|_{W^q_q(\Omega)} = 0 \quad \text{for all} \quad 1 \le q \le \infty, \quad 0 \le m \le 1, \quad \phi \in W^m_q(\Omega).
\end{equation}
Furthermore, for all $T \in T^h$ we have the local estimate
\begin{equation}
(2.22) \| \phi - \Pi^h \phi \|_{W^k_k(T)} \le C h_T \text{sup}(W^m_q(\Omega) - \text{sup}(W^k_k(\Omega))) \| \phi \|_{W^m_q(\Omega)},
\end{equation}
where $0 \le k \le m \le 2$ and $1 \le p, q \le \infty$ such that $\text{sup}(W^m_q(\Omega)) > \text{sup}(W^k_k(\Omega))$. If $\phi \in W^1_1(\Omega)$ has a vanishing trace on a part $\Gamma \subset \partial \Omega$, then so does $\Pi^h \phi$.

With the above notations at hand, the continuous regularized problem $(P_{\delta, \rho})$ can be discretized by
\[ \min_{\beta \in S_{\text{ad}}} J^h_{\delta, \rho}(\beta), \quad J^h_{\delta, \rho}(\beta) := \| N^h_{j_1} (\beta) - M^h_{g_1}(\beta) \|^2_{\Omega} + \rho \| \beta - \beta^* \|_{\Omega}^2. \quad (P^h_{\delta, \rho}) \]

Theorem 2.6. The discrete problem $(P^h_{\delta, \rho})$ has a minimizer $\beta^h_{\delta, \rho}$ which satisfies for a.e. in $\Omega$ the relation
\[ \beta^h_{\delta, \rho}(x) = P_{[0, \infty)} \left( \frac{1}{h} (N^h_{j_1}(\beta^h_{\delta, \rho})(x)A^h_{g_1}(\beta^h_{\delta, \rho})(x) - M^h_{g_1}(\beta^h_{\delta, \rho})(x)A^h_{M}(\beta^h_{\delta, \rho})(x)) + \beta^*(x) \right). \]

Proof. The proof follows exactly as in the continuous case, we therefore omit here.

3. Stability and Convergence. We are in position to prove the stability of the proposed finite element method and the convergence of the regularized finite element approximations to the $\beta^*$-minimum norm solution of the identification problem.

Theorem 3.1 (Stability). Assume that the regularization parameter $\rho$ and the observation data $(j_1, g_1) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ are fixed. For each $n \in \mathbb{N}$ let $\beta_n := \beta^h_{\delta, \rho}$ be an arbitrary minimizer of $(P^h_{\delta, \rho})$. Then the sequence $(\beta_n)$ has a subsequence converging in the $L^2(\Omega)$-norm to an element $\beta_{\delta, \rho} \in S_{\text{ad}}$. Furthermore, $\beta_{\delta, \rho}$ is a minimizer of $(P^h_{\delta, \rho})$.

Proof. In view of the proof of Lemma 2.3 we deduce that a subsequence of $(\beta_n)$ which is not relabeled and an element $(\beta_{\infty}, \theta_N, \theta_M) \in S_{\text{ad}} \times H^1(\Omega) \times H^1(\Omega)$ exist such that
\[ \beta_n \to \beta_{\infty} \quad \text{weakly}^* \text{ in } L^\infty(\Omega), \]
\[ \left( N_{j_1}(\beta_n), \ M_{g_1}(\beta_n), \ N_{j_1}^h(\beta_n), \ M_{g_1}^h(\beta_n) \right) \to \left( N_{j_1}(\beta_{\infty}), \ M_{g_1}(\beta_{\infty}), \ \theta_N, \ \theta_M \right) \]
weakly in $H^1(\Omega)$ as $n \to \infty$. We first show that $\theta_N = N_{j_1}(\beta_{\infty})$, i.e. the limit
\[ \lim_{n \to \infty} \left[ N_{j_1}^h(\beta_n) - N_{j_1}(\beta_{\infty}), \varphi \right]_{\alpha, \beta_{\infty}, \sigma} = 0 \quad \text{holds true for all} \quad \varphi \in H^1(\Omega). \]
In fact, we can rewrite
\begin{equation}
(3.1) \quad \left[ N_{j_1}^h(\beta_n), \varphi \right]_{\alpha, \beta_{\infty}, \sigma} = \left[ N_{j_1}^h(\beta_n), \varphi \right]_{\alpha, \beta_{\infty}, \sigma} + \left( N_{j_1}^h(\beta_n), \beta_{\infty} \varphi \right)_{\Omega} - \left( \beta_n, N_{j_1}^h(\beta_n), \varphi \right)_{\Omega}.
\end{equation}
It follows from (2.15) and (2.2) that
\[ [N_{j_h}^n(\beta_n), \varphi]_{\alpha, \beta_n, \sigma} = [\beta_{j_h}^n(\beta_n), \Pi_{j_h}^n(\varphi)]_{\alpha, \beta_n, \sigma} + [N_{j_h}^n(\beta_n), \varphi - \Pi_{j_h}^n(\varphi)]_{\alpha, \beta_n, \sigma} \]
\[ = (f, \Pi_{j_h}^n(\varphi))_{\Omega} + \langle \beta, \Pi_{j_h}^n(\varphi) \rangle_{\Gamma} + \langle j_0, \Pi_{j_h}^n(\varphi) \rangle_{\partial \Omega \setminus \Gamma} + [N_{j_h}^n(\beta_n), \varphi - \Pi_{j_h}^n(\varphi)]_{\alpha, \beta_n, \sigma} \]
\[ = (f, \varphi)_{\Omega} + \langle j_h, \varphi \rangle_{\Gamma} + \langle j_0, \varphi \rangle_{\partial \Omega \setminus \Gamma} + [N_{j_h}^n(\beta_n), \varphi - \Pi_{j_h}^n(\varphi)]_{\alpha, \beta_n, \sigma} + \langle f, \Pi_{j_h}^n(\varphi - \varphi) \rangle_{\Omega} + \langle j_0, \Pi_{j_h}^n(\varphi - \varphi) \rangle_{\partial \Omega \setminus \Gamma} \]
and so
\[ \lim_{n \to \infty} [N_{j_h}^n(\beta_n), \varphi]_{\alpha, \beta_n, \sigma} = (f, \varphi)_{\Omega} + \langle j_h, \varphi \rangle_{\Gamma} + \langle j_0, \varphi \rangle_{\partial \Omega \setminus \Gamma} = [N_{j_h}(\beta_\infty), \varphi]_{\alpha, \beta_\infty, \sigma}, \]
where we used (2.17) and (2.21). Furthermore, we have that
\[ (N_{j_h}^n(\beta_n), \beta_\infty \varphi)_{\Omega} - (\beta_n, N_{j_h}^n(\beta_n), \varphi)_{\Omega} \]
\[ = (N_{j_h}^n(\beta_n), \beta_\infty \varphi)_{\Omega} - (\beta_n, \theta N(\varphi))_{\Omega} - (\beta_n \varphi, N_{j_h}^n(\beta_n) - \theta N(\varphi))_{\Omega} \]
\[ \to (\theta N, \beta_\infty \varphi)_{\Omega} - (\beta_\infty, \theta N(\varphi))_{\Omega} \]
\[ = 0. \]
We thus derive from (3.1)–(3.3) \( \lim_{n \to \infty} [N_{j_h}^n(\beta_n), \varphi]_{\alpha, \beta_\infty, \sigma} = [N_{j_h}(\beta_\infty), \varphi]_{\alpha, \beta_\infty, \sigma} \).
This also yields
\[ \lim_{n \to \infty} \| N_{j_h}^n(\beta_n) - N_{j_h}(\beta_\infty) \|_{\Omega} = 0. \]
Likewise, we can show \( \theta_M = M_{j_h}(\beta_\infty) \) and
\[ \lim_{n \to \infty} \| M_{j_h}^n(\beta_n) - M_{j_h}(\beta_\infty) \|_{\Omega} = 0. \]
Consequently, for all \( \beta \in S_{\alpha, \sigma} \) we arrive at
\[ \| N_{j_h}(\beta_\infty) - M_{j_h}(\beta_\infty) \|_{\Omega}^2 + \rho \| \beta_\infty - \beta^* \|_{\Omega}^2 \]
\[ \leq \lim_{n \to \infty} \| N_{j_h}^n(\beta_n) - M_{j_h}^n(\beta_n) \|_{\Omega}^2 + \lim \inf_{n \to \infty} \rho \| \beta_n - \beta^* \|_{\Omega}^2 \]
\[ = \lim \inf_{n \to \infty} \left( \| N_{j_h}^n(\beta_n) - M_{j_h}^n(\beta_n) \|_{\Omega}^2 + \rho \| \beta_n - \beta^* \|_{\Omega}^2 \right) \]
\[ \leq \lim \sup_{n \to \infty} \left( \| N_{j_h}^n(\beta_n) - M_{j_h}^n(\beta_n) \|_{\Omega}^2 + \rho \| \beta_n - \beta^* \|_{\Omega}^2 \right) \]
\[ \leq \lim \sup_{n \to \infty} \left( \| N_{j_h}^n(\beta_n) - M_{j_h}^n(\beta_n) \|_{\Omega}^2 + \rho \| \beta_n - \beta^* \|_{\Omega}^2 \right) \]
\[ = \| N_{j_h}(\beta_\infty) - M_{j_h}(\beta_\infty) \|_{\Omega}^2 + \rho \| \beta_\infty - \beta^* \|_{\Omega}^2. \]
This means that \( \beta_\infty \) is a minimizer of \( (P_{\beta, \varphi}) \). It remains to show that \( (\beta_n) \) converges to \( \beta_\infty \) in the \( L^2(\Omega) \)-norm. For this purpose we take \( \beta = \beta_\infty \) in the last equation to get
\[ \lim_{n \to \infty} \left( \| N_{j_h}^n(\beta_n) - M_{j_h}^n(\beta_n) \|_{\Omega}^2 + \rho \| \beta_n - \beta^* \|_{\Omega}^2 \right) \]
\[ = \| N_{j_h}(\beta_\infty) - M_{j_h}(\beta_\infty) \|_{\Omega}^2 + \rho \| \beta_\infty - \beta^* \|_{\Omega}^2 \]
and then write
\[
\rho \| \beta_n - \beta_{\infty} \|_{\Omega}^2 = \rho \| (\beta_n - \beta^*) - (\beta_{\infty} - \beta^*) \|_{\Omega}^2 \\
= \rho \| \beta_n - \beta^* \|_{\Omega}^2 + \rho \| \beta_{\infty} - \beta^* \|_{\Omega}^2 - 2 \rho (\beta_n - \beta^*, \beta_{\infty} - \beta^*)_{\Omega} \\
= \rho \| \beta_{\infty} - \beta^* \|_{\Omega}^2 - 2 \rho (\beta_n - \beta^*, \beta_{\infty} - \beta^*)_{\Omega} \\
- \| N_{j_4}^h (\beta_n) - M_{g_4}^h (\beta_n) \|_{\Omega}^2 + \| N_{j_4}^h (\beta_n) - M_{g_4}^h (\beta_n) \|_{\Omega}^2 + \rho \| \beta_{\infty} - \beta^* \|_{\Omega}^2.
\]
By the equations (3.4)–(3.6), we obtain
\[
\rho \lim_{n \to \infty} \| \beta_n - \beta_{\infty} \|_{\Omega}^2 = \rho \| \beta_{\infty} - \beta^* \|_{\Omega}^2 - 2 \rho (\beta_{\infty} - \beta^*, \beta_{\infty} - \beta^*)_{\Omega} \\
- \| N_{j_4} (\beta_{\infty}) - M_{g_4} (\beta_{\infty}) \|_{\Omega}^2 + \| N_{j_4} (\beta_{\infty}) - M_{g_4} (\beta_{\infty}) \|_{\Omega}^2 + \rho \| \beta_{\infty} - \beta^* \|_{\Omega}^2 = 0,
\]
which finishes the proof.

To go further, we remark that, due to the assumption on consistency of the system (1.1)–(1.4), the set
\[
\Pi_{\mathcal{S}_{ad}} (j^1, g^1) := \{ \beta \in \mathcal{S}_{ad} \mid N_{j_4} (\beta) = M_{g_4} (\beta) \}
\]
is nonempty, convex, bounded and closed in the \(L^2(\Omega)\)-norm. As a result, there exists a unique solution \(\beta^*\) of the problem
\[
\min_{\beta \in \Pi_{\mathcal{S}_{ad}} (j^1, g^1)} \| \beta - \beta^* \|_{L^2(\Omega)},
\]
which is called by \(\beta^*\)-minimum norm solution to the identification problem. Let
\[
\varrho_j^{h_{j_4}, g_4} (\beta) := \| N_{j_4} (\beta) - N_{j_4}^h (\beta) \|_{L^2(\Omega)} + \| M_{g_4} (\beta) - M_{g_4}^h (\beta) \|_{L^2(\Omega)}.
\]
Due to the standard theory of the finite element method for elliptic problems (see, e.g., [4]), we get
\[
\lim_{h \to 0} \varrho_j^{h_{j_4}, g_4} (\beta) = 0 \quad \text{and} \quad 0 \leq \varrho_j^{h_{j_4}, g_4} (\beta) \leq h^2
\]
in case \(N_{j_4} (\beta), M_{g_4} (\beta) \in H^2(\Omega)\).

**Theorem 3.2 (Convergence).** Let \(\lim_{n \to \infty} h_n = 0\). Assume that \((\delta_n)_{n}\) and \((\rho_n)_{n}\) be any positive sequences such that
\[
\rho_n \to 0, \quad \frac{\delta_n}{\sqrt{\rho_n}} \to 0, \quad \text{and} \quad \frac{\varrho_{j_n}^{h_n, g^1} (\beta^1)}{\sqrt{\rho_n}} \to 0 \quad \text{as} \quad n \to \infty.
\]
Furthermore, assume that \((j_{\delta_n}, g_{\delta_n}) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)\) is a sequence satisfying the inequality
\[
\| j_{\delta_n} - j^1 \|_{H^{-1/2}(\Gamma)} + \| g_{\delta_n} - g^1 \|_{H^{1/2}(\Gamma)} \leq \delta_n
\]
and \(\beta_n := \beta_{\delta_n, \rho_n}^h\) denotes an arbitrary minimizer of \((\varrho_{j_n}^{h_n, g^1, \rho_n})\) for each \(n \in \mathbb{N}\). Then:
- (i) The whole sequence \((\beta_n)_{n}\) converges in the \(L^2(\Omega)\)-norm to \(\beta^*\).
- (ii) The corresponding state sequences \((N_{j_{\delta_n}} (\beta_n))_{n}\) and \((M_{g_{\delta_n}}^h (\beta_n))_{n}\) converge in the \(H^1(\Omega)\)-norm to the exact state \(\Phi^1 = \Phi (j^1, g^1, \beta^1)\) of the problem (1.1)–(1.4).
Note that in case the exact solution \( \Phi^t \in H^2(\Omega) \), cf. Remark 2.1, then the convergences (i) and (ii) are obtained if the regularization parameter is chosen such that \( \rho_n \to 0 \), \( \delta_n/\sqrt{\rho_n} \to 0 \) and \( h_n^2/\sqrt{\rho_n} \to 0 \) as \( n \to \infty \).

**Proof of Theorem 3.2.** We have from the optimality of \( \beta_n \) for each \( n \) that

\[
\|N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n)\|_\Omega^2 + \rho_n\|\beta_n - \beta^*\|_\Omega^2 \\
\leq \|N_{j_n}^{h_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega^2 + \rho_n\|\beta^t - \beta^*\|_\Omega^2.
\]

(3.10)

Note that \( N_{j_n}(\beta^t) = M_{g_n}(\beta^t) \), thus we have

\[
\|N_{j_n}^{h_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega = \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t) + M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega \\
\leq \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega + \|M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega \\
+ \|M_{g_n}^{h_n}(\beta^t) - M_{g_n}(\beta^t)\|_\Omega + \|N_{j_n}^{h_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega \\
= \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega + \|M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega + \|M_{g_n}^{h_n}(\beta^t) - M_{g_n}(\beta^t)\|_\Omega.
\]

By the identities (2.15)–(2.16) and the inequality (3.9), we get

\[
\|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega \leq \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega + \|M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega \\
\leq \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega \leq \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega + \|M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega + \|M_{g_n}^{h_n}(\beta^t) - M_{g_n}(\beta^t)\|_\Omega.
\]

(3.11)

which yields

\[
\|N_{j_n}^{h_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega \leq \|N_{j_n}^{h_n}(\beta^t) - N_{j_n}(\beta^t)\|_\Omega + \|M_{g_n}(\beta^t) - M_{g_n}^{h_n}(\beta^t)\|_\Omega + \|M_{g_n}^{h_n}(\beta^t) - M_{g_n}(\beta^t)\|_\Omega.
\]

(3.12)

It follows from (3.10)–(3.12) that

\[
\lim_{n \to \infty} \|N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n)\|_\Omega = 0
\]

(3.13)

and

\[
\limsup_{n \to \infty} \|\beta_n - \beta^*\|_\Omega \leq \|\beta^t - \beta^*\|_\Omega.
\]

(3.14)

Next, we get that

\[
\|N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n)\|_\Omega \\
= \|N_{j_n}^{h_n}(\beta_n) - N_{j_n}^{h_n}(\beta_n) + N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n) + M_{g_n}^{h_n}(\beta_n) - M_{g_n}(\beta_n)\|_\Omega \\
\leq \|N_{j_n}^{h_n}(\beta_n) - N_{j_n}^{h_n}(\beta_n)\|_\Omega + \|M_{g_n}^{h_n}(\beta_n) - M_{g_n}(\beta_n)\|_\Omega + \|N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n)\|_\Omega \\
\leq \delta_n + \|N_{j_n}^{h_n}(\beta_n) - M_{g_n}^{h_n}(\beta_n)\|_\Omega 
\]

as \( n \to \infty \), by (3.13). Furthermore, since \( \beta_n \in S_{ad} \) for all \( n \in \mathbb{N} \), in view of the proof of Theorem 3.1, a subsequence of \( \beta_n \) not relabeled and an element \( \hat{\beta} \in S_{ad} \) exist such that

\[
\beta_n \to \hat{\beta} \quad \text{weakly* in} \quad L^\infty(\Omega),
\]

(3.15)

\[
\|\hat{\beta} - \beta^*\|_\Omega \leq \liminf_{n \to \infty} \|\beta_n - \beta^*\|_\Omega.
\]

\[
\left( N_{j_n}^{h_n}(\beta_n), \ M_{g_n}^{h_n}(\beta_n) \right) \to \left( N_{j_n}^{h_n}(\hat{\beta}), \ M_{g_n}(\hat{\beta}) \right) \quad \text{weakly in} \quad H^1(\Omega)
\]
which also implies that
\[
\left\| N_{x_n}^h(\beta) - M_{g^t}(\beta) \right\|_{L^4(\Omega)} = \lim_{n \to \infty} \left\| N_{x_n}^h(\beta_n) - M_{g^t}(\beta_n) \right\|_{L^4(\Omega)} = 0
\]
and so that \( \hat{\beta} \in \Pi_{S_{ad}}(g^t, g^t) \). Then, combining (3.15) with (3.14) and using the uniqueness of the \( \beta^* \)-minimum norm solution, we obtain
\[
(3.16) \\
\hat{\beta} = \beta^t \text{ and } \lim_{n \to \infty} \| \beta_n - \hat{\beta} \|_{L^4(\Omega)} = 0,
\]
the assertion (i) is thus proved. For (ii) we have for all \( \phi^h \in \mathcal{V}_1^h \) that
\[
[N_{x_n}^h(\beta^t) - N_{x_n}^h(\beta_n), \phi^h_n]_{\mathcal{A}, \beta^t, \sigma} = (\beta_n - \beta^t, N_{x_n}^h(\beta_n)\phi^h_n)
\]
which together with the inequality (2.1) and the continuous embedding \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) imply
\[
\left\| N_{x_n}^h(\beta^t) - N_{x_n}^h(\beta_n) \right\|^2_{L^4(\Omega)} \\
\leq \| \beta_n - \beta^t \|_{L^4(\Omega)} \left\| N_{x_n}^h(\beta_n) - N_{x_n}^h(\beta_n) \right\|_{L^4(\Omega)} \\
\leq \| \beta_n - \beta^t \|_{L^4(\Omega)} \| N_{x_n}^h(\beta_n) - N_{x_n}^h(\beta_n) \|_{L^4(\Omega)} \\
\leq \| \beta_n - \beta^t \|_{L^4(\Omega)} \| N_{x_n}^h(\beta^t) - N_{x_n}^h(\beta_n) \|_{L^4(\Omega)}.
\]
Therefore, we arrive at
\[
(3.17) \\
\lim_{n \to \infty} \| N_{x_n}^h(\beta^t) - N_{x_n}^h(\beta_n) \|_{L^4(\Omega)} \leq \lim_{n \to \infty} \| \beta_n - \beta^t \|_{L^4(\Omega)} = 0.
\]
by the aid of the limit (3.16). Consequently, we obtain from (3.11) and (3.17) that
\[
\left\| N_{x_n}^h(\beta_n) - N_{x_n}^h(\beta^t) \right\|_{L^4(\Omega)} \\
\leq \left\| N_{x_n}^h(\beta_n) - N_{x_n}^h(\beta_n) \right\|_{L^4(\Omega)} + \left\| N_{x_n}^h(\beta^t) - N_{x_n}^h(\beta^t) \right\|_{L^4(\Omega)} \\
\to 0
\]
as \( n \to \infty \). By the similar arguments, we also get \( \lim_{n \to \infty} \| M_{g^t}(\beta_n) - M_{g^t}(\beta^t) \|_{L^4(\Omega)} = 0 \), which finishes the proof. \( \square \)

4. Gradient projection algorithm with Armijo steplength rule and numerical implementation.

4.1. Algorithm. In this section we present the gradient projection algorithm with Armijo steplength rule (cf. [17, 22]) for numerical solution of the minimization problem \( (P_{\delta, \rho}) \).

In view of the proof of Theorem 2.4, we first note that for each \( \beta \in S_{ad} \) the \( L^2 \)-gradient of the cost function \( J_{\delta, \rho}^h \) of the problem \( (P_{\delta, \rho}) \) at \( \beta \) is given by
\[
\nabla J_{\delta, \rho}^h(\beta) = M_{g^t}(\beta)A_{M}^h(\beta) - N_{j_n}^h(\beta)A_{N}^h(\beta) + \rho(\beta - \beta^*).
\]
The algorithm is then read as: given a step size control \( \mu \in (0, 1) \), an initial approximation \( \beta_0 \), number of iteration \( N \) and setting \( k = 0 \).
1. Compute $N^h_j(\beta_k)$ and $M^h_g(\beta_k)$ from the equations

\[
\begin{align*}
[N^h_j(\beta_k), \phi^h](\alpha, \beta_k, \sigma) &= \langle f, \phi^h \rangle_\Omega + \langle j_\delta, \phi^h \rangle_\Gamma + \langle j_0, \phi^h \rangle_{\partial \Omega \setminus \Gamma} \quad \forall \phi^h \in V^h_1, \\
[M^h_g(\beta_k), \phi^h](\alpha, \beta_k, \sigma) &= \langle f, \phi^h \rangle_\Omega + \langle j_0, \phi^h \rangle_{\partial \Omega \setminus \Gamma} \quad \forall \phi^h \in V^h_{1,0},
\end{align*}
\]

and then the solutions $A^h_\Delta(\beta_k)$ and $A^h_M(\beta_k)$ of the adjoint problems

\[
\begin{align*}
[A^h_\Delta(\beta_k), \phi^h] &= (N^h_j(\beta_k) - M^h_g(\beta_k), \phi^h)_\Omega \quad \text{for all } \phi^h \in V^h_1, \\
[A^h_M(\beta_k), \phi^h] &= (N^h_j(\beta_k) - M^h_g(\beta_k), \phi^h)_\Omega \quad \text{for all } \phi^h \in V^h_{1,0}.
\end{align*}
\]

2. Compute the corresponding value of the cost functional

\[J^h_{k, \rho}(\beta) := \|N^h_j(\beta_k) - M^h_g(\beta_k)\|^2_\Omega + \rho \|\beta_k - \beta^*\|^2_\Omega\]

as well as the gradient

\[\nabla J^h_{k, \rho}(\beta_k) = M^h_g(\beta_k)A^h_M(\beta_k) - N^h_j(\beta_k)A^h_N(\beta_k) + \rho(\beta_k - \beta^*).\]

3. Compute

\[\widehat{\beta}_k := \max \left(\beta_k, \min \left(\widehat{\beta}_k - \mu \nabla J^h_{k, \rho}(\beta_k)\right)\right)\]

and then the corresponding states $N^h_j(\widehat{\beta}_k)$ and $M^h_g(\widehat{\beta}_k)$, the value of the cost functional $J^h_{k, \rho}(\widehat{\beta}_k)$ as well.

4. Compute the quantity

\[Q := J^h_{k, \rho}(\beta_k) - J^h_{k, \rho}(\widehat{\beta}_k) + \tau \mu \|\beta_k - \beta^*\|^2_\Omega\]

for a small positive constant $\tau = 10^{-4}$.

(a) If $Q \geq 0$

\[\text{go to the next step (b) below}\]

else

\[\mu := \frac{\mu}{2}\]

and then go back to the step 3.

(b) Update $\beta_k = \widehat{\beta}_k$, set $k = k + 1$.

5. Compute

\[\text{Tolerance} := \|\nabla J^h_{k, \rho}(\beta_k)\|_\Omega - \tau_1 - \tau_2 \|\nabla J^h_{k, \rho}(\beta_0)\|_\Omega\]

for $\tau_1 := 10^{-3} h$ and $\tau_2 := 10^{-2} h$. If Tolerance $\leq 0$ or $k > N$, then stop; otherwise go back to the step 1.

### 4.2. Numerical implementation.

For illustrating the theoretical result we consider the system (1.1)–(1.4) with $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1, x_2 < 1\}$. For discretization we divide the interval $(-1, 1)$ into $\ell$ equal segments, and so the domain $\Omega = (-1, 1)^2$ is divided into 2$\ell^2$ triangles, where the diameter of each triangle is $h_{\ell} = \frac{\sqrt{\ell}}{\ell}$.

The source functional $f$ is assumed to be also discontinuous and defined as

\[f := \frac{3}{2} \chi_D - \frac{1}{2} \chi_{\Omega \setminus D},\]

where $\chi_D$ is the characteristic functional of the Lebesgue measurable set

\[D := \{(x_1, x_2) \in \Omega \mid |x_1| \leq 1/2 \quad \text{and} \quad |x_2| \leq 1/2\}.\]
Furthermore, the special function \( \sigma \) is defined as
\[
\chi := \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \in \partial \Omega, \\ \frac{1}{2} & \text{otherwise}, \end{cases}
\]
where \( \chi_D \) is the characteristic functional of the Lebesgue measurable set \( D \) and
\[
\Omega_{11} := \{(x_1, x_2) \in \Omega \mid |x_1| \leq 3/4 \text{ and } |x_2| \leq 3/4\},
\]
\[
\Omega_{12} := \{(x_1, x_2) \in \Omega \mid |x_1| + |x_2| \leq 3/4\},
\]
\[
\Omega_{22} := \{(x_1, x_2) \in \Omega \mid x_1^2 + x_2^2 \leq 9/16\}.
\]

Furthermore, the special function \( \sigma \) is chosen to be zero while the constants appearing in the admissible set \( S_{ad} \) defined by (1.5) are chosen as \( \beta = 0.05 \) and \( \beta = 10 \).

The sought reaction coefficient \( \beta^\dagger \) is assumed to be discontinuous and given by
\[
\beta^\dagger := 3 \chi_{\Omega_0} + \chi_{\Omega_0^c}.
\]
with
\[
\Omega_0 := \{(x_1, x_2) \in \Omega \mid 4x_1^2 + 9x_2^2 \leq 1\}.
\]
The Neumann boundary condition on the bottom and left surface is given by
\[
j^\dagger := A \cdot \chi_{(0.1) \times \{1\}} + B \cdot \chi_{[-1,0) \times \{1\}} + C \cdot \chi_{[-1,0] \times \{-1\}} + D \cdot \chi_{\{1\} \times \{0,1\}}
\]
and on the right and top surface
\[
j_0 := 4 \chi_{\{1\} \times \{-1,0\}} - 3 \chi_{\{1\} \times \{0,1\}} + 2 \chi_{\{0,1\} \times \{1\}} - \chi_{[-1,0] \times \{1\}}
\]
with the constants \( A, B, C \) and \( D \) discussed in details later. The exact state \( \Phi^\dagger \) is then computed from the finite element equation \( K \Phi^\dagger = F \), where \( K \) and \( F \) are the stiffness matrix and the load vector associated with the problem (1.1)–(1.3), respectively. The Dirichlet boundary condition \( g^\dagger \) in (1.4) is then defined as \( g^\dagger = \gamma_D \Phi^\dagger \), the Dirichlet trace of \( \Phi^\dagger \) on the boundary \( \Gamma \).

We use the algorithm which is described in Subsection 4.1 for computing the numerical solution of the problem \((P_{\alpha, \beta})\). The step size control is chosen with \( \mu = 0.75 \) and the initial approximation is the constant function defined by \( \beta_0 = 1.5 \). Our computational process will be started with the coarsest level \( \ell = 4 \). In each iteration \( k \) we compute Tolerance defined by (4.1). Then the iteration is stopped if Tolerance \( \leq 0 \) or the number of iterations reaches the maximum iteration counted of 600. After obtaining the numerical solution of the first iteration process with respect to the coarsest level \( \ell = 4 \), we use its interpolation on the next finer mesh \( \ell = 8 \) as the initial approximation \( \beta_0 \) for the algorithm on this finer mesh, and so on for \( \ell = 16, 32, 64 \).

**Example 4.1.** In this example we assume \((A, B, C, D) = (1, -2, 3, -4)\) while observations are taken on the bottom surface \( \Gamma_{\text{observation}} := \Gamma_{\text{bottom}} := [-1, 1] \times \{1\} \) only. We assume that noisy observations are available in the form
\[
(j_{\delta}, g_{\delta}) = (j^\dagger + \theta \cdot R_j^\dagger, g^\dagger + \theta \cdot R_g^\dagger)
\]
for some \( \theta > 0 \) depending on \( \ell \), where \( R_j^\dagger \) and \( R_g^\dagger \) are \( \partial M^{\delta \epsilon} \times 1 \)-matrices of random numbers on the interval \((-1, 1)\) which are generated by the MATLAB function \( \text{rand} \) and \( \partial M^{\delta \epsilon} \) is the number of boundary nodes of the triangulation \( T^{\delta \epsilon} \) which belong to
\[ \delta_\ell = \| j_\ell - f \|_{L^2(\partial \Omega)} + \| g_\ell - g^\dagger \|_{L^2(\partial \Omega)}. \]

To satisfy the condition (3.8) in Theorem 3.2 we below take \( \theta_\ell = h_\ell \sqrt{\Omega} \cdot \rho_\ell \) and the regularization parameter \( \rho = \rho_\ell = 0.001 \sqrt{h_\ell}. \)

Let \( \beta_\ell \) denote the reaction obtained at the final iterate of the algorithm corresponding to the refinement level \( \ell \). We then use the following abbreviations for the errors

\[
L_\beta^2 = \| \beta_\ell - \beta^\dagger \|_{\Omega}, \quad L_N^2 = \| N_{j_\ell}^{h_\ell}(\beta_\ell) - N_{j_\ell}^{h_\ell}(\beta^\dagger) \|_{\Omega}, \quad L_D^2 = \| D_{g_\ell}^{h_\ell}(\beta_\ell) - D_{g_\ell}^{h_\ell}(\beta^\dagger) \|_{\Omega},
\]

where

\[
g_\ell^\dagger = \begin{cases} g_\ell & \text{on } \Gamma = \Gamma_{\text{bottom}}, \\ \gamma_\ell \Phi^\dagger & \text{on } \partial \Omega \setminus \Gamma. \end{cases}
\]

The numerical results are summarized in Table 1, where we present the refinement level \( \ell \), the mesh size \( h_\ell \) of the triangulation, the regularization parameter \( \rho_\ell \), the measurement noise \( \delta_\ell \), and the errors \( L_\beta^2, L_N^2, L_D^2 \).

| \( \ell \) | \( h_\ell \) | \( \rho_\ell \) | \( \delta_\ell \) | \( L_\beta^2 \) | \( L_N^2 \) | \( L_D^2 \) |
|---|---|---|---|---|---|---|
| 4  | 0.7071 | 8.4090e-4 | 0.1116 | 1.2185 | 0.3026 | 0.1289 |
| 8  | 0.3536 | 5.9460e-4 | 4.0042e-2 | 0.5989 | 0.1377 | 9.4845e-2 |
| 16 | 0.1767 | 1.9375e-4 | 7.9021e-3 | 5.9892e-2 | 0.1472 | 1.6441e-2 |
| 32 | 0.1767 | 2.9730e-4 | 7.9021e-3 | 7.4264e-2 | 2.0169e-2 | 6.9882e-3 |
| 64 | 0.1767 | 2.9730e-4 | 7.9021e-3 | 7.4264e-2 | 2.0169e-2 | 6.9882e-3 |

The convergence history given in Table 1 shows that the algorithm performs well for our identification problem. We also observe a decrease of all errors as the noise level gets smaller as expected from our convergence result in Theorem 3.2.

All figures presented hereafter correspond to the finest level \( \ell = 64 \). Figure 1 from left to right shows the numerical solution \( \beta_\ell \) computed by the algorithm at the final 523\textsuperscript{th}-iteration and the differences \( I_{1}^{h_\ell} \beta_\ell - \beta^\dagger, N_{j_\ell}^{h_\ell}(\beta_\ell) - N_{j_\ell}^{h_\ell}(\beta^\dagger), D_{g_\ell}^{h_\ell}(\beta_\ell) - D_{g_\ell}^{h_\ell}(\beta^\dagger) \),

where \( I_{1}^{h_\ell} \) is the usual Lagrange node value interpolation operator.

**Fig. 1.** Computed numerical solution \( \beta_\ell \) of the algorithm at the final 523\textsuperscript{th}-iteration, and the differences \( I_{1}^{h_\ell} \beta_\ell - \beta^\dagger, N_{j_\ell}^{h_\ell}(\beta_\ell) - N_{j_\ell}^{h_\ell}(\beta^\dagger), D_{g_\ell}^{h_\ell}(\beta_\ell) - D_{g_\ell}^{h_\ell}(\beta^\dagger) \) for \( \ell = 64, \rho = \rho_\ell = 0.001 \sqrt{h_\ell} \) and \( \delta_\ell = 3.2289e-3. \)
Example 4.2. In this example we consider regularization parameter in the forms

\[ \rho_\ell = h_\ell^2, \ 0.1 h_\ell^2, \ 0.01 h_\ell^2, \ \text{and} \ 0.001 h_\ell^2, \]

while smaller \( \theta_\ell = 10^{-2} h_\ell \sqrt{\rho_\ell}. \) Using the computational process which was described as in Example 4.1 starting with \( \ell = 4, \) in Table 2 we perform the numerical results for the finest grid \( \ell = 64 \) and with different values of \( \rho_\ell \) above.

| Regularization parameter | \( \delta_\ell \) | \( L_2^\delta \) | \( L_2^\beta \) | \( L_2^D \) |
|--------------------------|-----------------|-----------------|-----------------|-----------------|
| \( \rho_\ell = h_\ell^2 e - 0 = 1.9531 e - 3 \) | 3.4240e-5 | 5.0702 | 1.5779 | 0.1805 |
| \( \rho_\ell = h_\ell^2 e - 1 = 1.9531 e - 4 \) | 9.3133e-6 | 2.4302 | 1.1983 | 0.1439 |
| \( \rho_\ell = h_\ell^2 e - 2 = 1.9531 e - 5 \) | 3.3618e-6 | 0.1028 | 3.6218e-2 | 1.5681e-2 |
| \( \rho_\ell = h_\ell^2 e - 3 = 1.9531 e - 6 \) | 1.0207e-6 | 8.1651e-2 | 3.4587e-2 | 1.3142e-2 |

We note that the numerical results are not so good in first two cases, where \( \rho_\ell = h_\ell^2 \) and \( \rho_\ell = 0.1 h_\ell^2 \) even though the noise levels \( \delta_\ell \) are quite small. This indicates that previous iteration processes corresponding the coarse grid levels \( \ell = 4, 8, \ldots \) affects strongly on the final obtained numerical result with respect to the finest grid level \( \ell = 64. \)

With \( \rho_\ell = h_\ell^2, \) Figure 2 from left to right shows the computed numerical solution \( \beta_\ell \) at the final iteration, the differences \( N_{h\beta\ell} (\beta_\ell) - N_{h\beta} (\beta^\ell) \) and \( D_{h\beta\ell} (\beta_\ell) - D_{h\beta} (\beta^\ell); \)

while the differences \( I_{h\beta} \beta^\ell - \beta_\ell \) and \( I_{h\beta} \beta^\ell (x_1, 0) - \beta_\ell (x_1, 0) \) as the second variable \( x_2 = 0 \) are shown respectively in left two figures of Figure 3. The right figure of Figure 3 performs the graphs \( I_{h\beta} \beta^\ell (x_1, 0) \) and \( \beta_\ell (x_1, 0). \)

![Fig. 2. With \( \rho_\ell = h_\ell^2 \): computed numerical solution \( \beta_\ell \) at the final iteration, the differences \( N_{h\beta\ell} (\beta_\ell) - N_{h\beta} (\beta^\ell) \) and \( D_{h\beta\ell} (\beta_\ell) - D_{h\beta} (\beta^\ell). \)](image-url)
Fig. 3. With $\rho_\ell = h_\ell^2$: Differences $I^{h_\ell}_1 \beta^\dagger - \beta_\ell$, $I^{h_\ell}_1 \beta^\dagger(x_1, 0) - \beta_\ell(x_1, 0)$ and the graphs $I^{h_\ell}_1 \beta^\dagger(x_1, 0)$ as well as $\beta_\ell(x_1, 0)$.

Figure 4 and Figure 5 perform analogous numerical computations for $\rho = 0.1h_\ell^2$.

Fig. 4. With $\rho_\ell = 0.1h_\ell^2$: computed numerical solution $\beta_\ell$ at the final iteration, the differences $N^{h_\ell}_{\beta_\ell}(\beta_\ell) - N^{h_\ell}_{\beta^\dagger}(\beta^\dagger)$ and $D^{h_\ell}_{\beta_\ell}(\beta_\ell) - D^{h_\ell}_{\beta^\dagger}(\beta^\dagger)$.

Fig. 5. With $\rho_\ell = 0.1h_\ell^2$: Differences $I^{h_\ell}_1 \beta^\dagger - \beta_\ell$, $I^{h_\ell}_1 \beta^\dagger(x_1, 0) - \beta_\ell(x_1, 0)$ and the graphs $I^{h_\ell}_1 \beta^\dagger(x_1, 0)$ as well as $\beta_\ell(x_1, 0)$.

Example 4.3. We now consider the case $\Gamma_{\text{observation}} := \Gamma_{\text{bottom-left}}$ includes the bottom surface and the left surface of the domain $\Omega$, i.e. $\Gamma_{\text{bottom-left}} = \{ x = (x_1, x_2) \in \overline{\Omega} \mid x_2 = -1 \} \cup \{ x = (x_1, x_2) \in \overline{\Omega} \mid x_1 = -1 \}$. In this case $\theta_\ell$ in (4.3) is changed to $\theta_\ell = \frac{1}{2} h_\ell \sqrt{10} \cdot \rho_\ell$, while the regularization parameter $\rho = \rho_\ell = 0.001 \sqrt{h_\ell}$ as in Example 4.1. Computational results show in Table 3 below.
Convergence history for $\Gamma_{\text{observation}} := \Gamma_{\text{bottom-left}}$

| $\ell$ | $\delta_\ell$ | $L_2^\beta$ | $L_2^N$ | $L_2^D$ |
|--------|------------|-----------|-----------|-----------|
| 4      | 7.6402e-2 | 1.1379    | 0.3066    | 0.1911    |
| 8      | 3.2528e-2 | 0.5102    | 0.2215    | 0.1125    |
| 16     | 1.4637e-2 | 0.2056    | 0.1443    | 5.2079e-2 |
| 32     | 5.4159e-3 | 0.1192    | 5.6266e-2 | 2.0737e-2 |
| 64     | 2.3331e-3 | 6.6896e-2 | 1.4888e-2 | 6.17142e-3|

**Table 3**

Refinement level $\ell$, measurement noise $\delta_\ell$, and errors $L_2^\beta$, $L_2^N$, $L_2^D$.

Compared with Table 1, we do not see the difference clearly in obtained numerical results between $\Gamma_{\text{observation}} := \Gamma_{\text{bottom}}$ and $\Gamma_{\text{observation}} := \Gamma_{\text{bottom-left}}$. Figure 6 below performs the graphs of computations in this case.

![Computed numerical solution $\beta_\ell$ of the algorithm at the final 556th-iteration, and the differences $I_{\delta_\ell}^h \beta_\ell - \beta_\ell$, $N_{\delta_\ell}^h (\beta_\ell) - N_{\delta_\ell}^h (\beta_\ell^\dagger)$, $D_{\delta_\ell}^h (\beta_\ell) - D_{\delta_\ell}^h (\beta_\ell^\dagger)$ for $\ell = 64$, $\rho = \rho_\ell = 0.001 \sqrt{h_\ell}$ and $\delta_\ell = 2.3331e^{-3}$.

**Fig. 6.**

Example 4.4. In this example we assume that $I$ multiple measurements on the bottom surface and the left surface $(j_{\delta_\ell}, g_{\delta_\ell})_{i=1,...,I}$ are available. Then, the cost functional $J_{h,\rho,\delta}$ and the problem $(\mathcal{P}_{h,\rho,\delta})$ can be rewritten as

$$
\min_{\beta \in S_{ad}} J_{h,\rho,\delta}^\beta(\beta), \quad J_{h,\rho,\delta}^\beta(\beta) := \frac{1}{I} \sum_{i=1}^{I} \| N_{j_{\delta_\ell}}^h (\beta) - M_{g_{\delta_\ell}}^h (\beta) \|_\Omega^2 + \rho \| \beta - \beta^\star \|_\Omega^2.
$$

which also attains a solution $\beta_{h,\rho,\delta}^\star$. Let $J_{(A,B,C,D)}^\beta(\beta)$ be given by (4.2) which depends on the constants $A, B, C, D$ and $g_{(A,B,C,D)}^\dagger := \gamma \tau N_{j_{(A,B,C,D)}}^h (\beta^\dagger)$. The noisy observations are assumed to be given by

$$
\left( \beta_{(A,B,C,D)}, \mathcal{G}_{(A,B,C,D)}^\dagger \right) = \left( \beta_{(A,B,C,D)} + \theta \cdot R_{j_{(A,B,C,D)}}^\dagger, g_{(A,B,C,D)}^\dagger + \theta \cdot R_{g_{(A,B,C,D)}}^\dagger \right),
$$

where $\theta > 0$ is independent of the mesh size, regularization parameter and noise level. $R_{j_{(A,B,C,D)}}$ and $R_{g_{(A,B,C,D)}}$ denote $\partial M_{h_E} \times 1$-matrices of random numbers on the interval $(-1, 1)$, and $\partial M_{h_E}$ is the number of boundary nodes of the triangulation $\mathcal{T}_{h_E}$ which belong to $\Gamma_{\text{observation}} := \Gamma_{\text{bottom-left}}$.

The numerical results in Table 4 present for $\theta = 0.2$ and with respect to

- $I = 1$ measurement: $(A, B, C, D) = (1, -2, 3, -4)$,
- $I = 6$ measurements: fixing $D = -4$ and taking $(A, B, C)$ equals to all permutations of the set $\{1, -2, 3\}$,
- $I = 16$ measurements: taking $(A, B, C, D)$ equals to all permutations of the set $\{1, -2, 3, -4\}$.

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We observe that the use of multiple measurements improves the solution to yield an acceptable result even in the presence of relatively large noise as Table 4 below.

| Number of measurements $I$ | Iterate $L^2$ | $L^2_N$ | $L^2_D$ |
|----------------------------|---------------|---------|---------|
| 1  | 600          | 0.5889  | 0.3040  | 0.1338  |
| 6  | 600          | 0.3846  | 0.1597  | 6.2670e-2 |
| 16 | 600          | 0.2746  | 0.8772e-2 | 4.5228e-2 |

Table 4: Numerical results for $\ell = 64$, $\theta = 0.2$

Finally, in Figure 7–Figure 9 we perform the graphs of computations for multiple measurements $I = 1, 6, 16$, respectively, which include the computed numerical solution $\beta_\ell$ of the algorithm, and the differences $L^h_i(\beta^\dagger) - \beta_\ell$, $N^h_{\beta_\ell}(\beta_\ell) - N^h_{\beta^\dagger}(\beta^\dagger)$, $D^h_i(g^\dagger_{\beta_\ell}) - D^h_i(g^\dagger_{\beta^\dagger})$ for $\ell = 64$, $\rho = \rho_\ell = 0.001\sqrt{\ell}$ and $\delta_\ell = 0.4773$.

![Fig. 7. I = 1 measurement.](image)

![Fig. 8. I = 6 measurements.](image)

![Fig. 9. I = 16 measurements.](image)

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