Drinfeld twists and symmetric Bethe vectors of supersymmetric fermion models

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Abstract. We construct the Drinfeld twists (factorizing $F$-matrices) of the $gl(m|n)$-invariant fermion model. Completely symmetric representation of the pseudo-particle creation operators of the model are obtained in the basis provided by the $F$-matrix (the $F$-basis). We resolve the hierarchy of the nested Bethe vectors in the $F$-basis for the $gl(m|n)$ supersymmetric model.

Keywords: algebraic structures of integrable models, quantum integrability (Bethe ansatz), solvable lattice models
1. Introduction

In [1], it was realized that the R-matrices for the one-dimensional integrable XXX and XXZ spin chain systems are factorized in terms of certain non-degenerate lower-triangular F-matrices (Drinfeld twists [2])

\[ R_{12}(u_1, u_2) = F_{21}^{-1}(u_2, u_1) F_{12}(u_1, u_2). \]

(1.1)

This leads to the natural F-basis for the analysis of these models. Working in the F-basis, the pseudo-particle creation operators of the systems take the completely symmetric form. Compared to the original Bethe vectors of the models, Bethe vectors in the F-basis are dramatically simplified and can be written down explicitly. These results allow form factors, correlation functions and spontaneous magnetizations of the systems to be represented in exact and compact form [3, 4].

The results of [1] were generalized to other models including the models associated with any finite-dimensional irreducible representations of the Yangian \( Y[gl(2)] \) [5], the gl(n) rational Heisenberg model [6], the elliptic XYZ and Belavin models [7, 8].

There are integrable models which do not come from a bosonic algebra. Examples include the Perk–Schultz model [9], whose exact solution and the Bethe ansatz equations were obtained in [10]–[12]. The actual algebra underlying the Perk-Schultz model is the Lie superalgebra \( gl(m|n) \): using the transformation introduced in [13], the R-matrix of the Perk-Schultz model is related to the \( gl(m|n) \)-invariant R-matrix. Integrable models with \( gl(m|n) \) supersymmetry are physically important because they give strongly
correlated fermion models of superconductivity. Interesting examples include the \( gl(2|1) \)-invariant supersymmetric \( t-J \) model, the \( gl(2|2) \)-invariant electronic model [14] and the supersymmetric \( U \) model [15], proposed in an attempt to understanding high-\( T_c \) superconductivities. The application of the hierarchy of the algebraic Bethe ansatz to spin systems related to Lie superalgebras was given in [16].

Recently [17, 18], we have successfully constructed the Drinfeld twists for the \( t-J \) models with \( gl(2|1) \) and \( U_\ell(gl(2|1)) \) supersymmetries, and resolved the hierarchies of the nested Bethe vectors of the two models. In this paper, we construct the factorizing \( F \)-matrices of the \( gl(m|n) \)-invariant fermion model. Working in the \( F \)-basis, we obtain the symmetric representations of the monodromy matrix and the creation operators. Moreover we resolve the hierarchy of the nested Bethe vectors in the \( F \)-basis for the \( gl(m|n) \) model.

The present paper is organized as follows. In section 2, we introduce some basic notation on the \( gl(m|n) \)-invariant fermion model. In section 3, we construct the \( F \)-matrix and its inverse. In section 4, we give the representation of the monodromy matrix and creation operators in the \( F \)-basis. In section 5, specializing to the \( gl(2|2) \) invariant superconductive electronic model, we resolve its nested Bethe vectors in the \( F \)-basis. In section 6, we resolve the hierarchy of the Bethe vectors of the \( gl(m|n) \) model in the \( F \)-basis. We conclude the paper by offering some discussion in section 7.

2. Basic definitions and notation

Let \( V \) be a \( (m + n) \)-dimensional \( gl(m|n) \)-module and \( R \in \text{End}(V \otimes V) \) be the \( R \)-matrix associated with this module. \( V \) is \( Z_2 \)-graded, and in the following we choose the following grading for \( V \): \([1] = \cdots = [m] = 1, [m + 1] = \cdots = [m + n] = 0\). The graded permutation operator \( P \) is defined by

\[
P(a \otimes b) = (-1)^{|a||b|}(b \otimes a)
\]

or in matrix form

\[
(P)_{ab} = (-1)^{|c||d|}\delta_{ad}\delta_{bc}.
\]

The \( R \)-matrix depends on the difference of two spectral parameters \( u_1 \) and \( u_2 \) associated with the two copies of \( V \), and is given by

\[
R_{12}(u_1, u_2) = R_{12}(u_1 - u_2) = a_{12}I + b_{12}P,
\]

where \( I \) is the identity operator, and

\[
a_{12} = a(u_1, u_2) \equiv \frac{u_1 - u_2}{u_1 - u_2 + \eta}, \quad b_{12} = b(u_1, u_2) \equiv \frac{\eta}{u_1 - u_2 + \eta}
\]

with \( \eta \in \mathbb{C} \) being the crossing parameter. One can easily check that the \( R \)-matrix satisfies the unitary relation

\[
R_{21}R_{12} = 1.
\]

Here and throughout, \( R_{12} \equiv R_{12}(u_1, u_2) \) and \( R_{21} \equiv R_{21}(u_2, u_1) \). The \( R \)-matrix satisfies the graded Yang–Baxter equation (GYBE) [13]

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

In terms of the matrix elements defined by

\[
R(u)(v^i \otimes v^j) = \sum_{i,j} R(u)^{ij'}_{ij} (v^i \otimes v^{j'}),
\]

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It follows that
\[ R_{i\ell k}^{j\ell k'} R_{i\ell k}^{j'\ell k'} \left( -1 \right)^{\left| j' \right| + \left| j'' \right|} \]
where the index 0 refers to the auxiliary space and
\[ T \]
permutation group
where
\[ T \]
\[ \sigma \]
may generalize (2.14) to an \( N \)-fold tensor product of spaces:
\[ \text{STR} \] denotes the supertrace over the auxiliary space. Then the Hamiltonian of the
\[ R \]
\[ \equiv \]
\[ T_{0,23} \]
\[ T_{0,03} R_{02} \]
It follows that \( R_{1...N}^{\sigma} \) is a product of elementary \( R \)-matrices, corresponding to a decomposition of \( \sigma \) into elementary transpositions. With the help of the GYBE, one may generalize (2.14) to an \( N \)-fold tensor product of spaces:
\[ R_{1...N}^{\sigma} T_{0,1...N} = T_{0,\sigma(1...)N} R_{1...N}^{\sigma} \]
where \( T_{0,1...N} \equiv R_{0N} \ldots R_{01} \). This implies the ‘decomposition’ law
\[ R_{1...N}^{\sigma,\sigma'} = R_{\sigma'(1...)N}^{\sigma} R_{1...N}^{\sigma'} \]
...N is a product of elementary
\[ T \]
\[ \equiv \]
\[ \sigma_0 \]
\[ A \]
\[ B \]
\[ C \]
\[ D \]
This model is integrable thanks to the commutativity of the transfer matrix for different parameters,
\[ [t(u), t(v)] = 0 \]
which can be verified by using the GYBE.
Following \[1\], we now introduce the notation \( R_{1...N}^{\sigma} \), where \( \sigma \) is any element of the permutation group \( S_N \). We note that we may rewrite the GYBE as
\[ R_{23}^{\sigma_{23}} T_{0,23} = T_{0,32} R_{23}^{\sigma_{23}} \]
...N \equiv \]
...N \equiv \]
...N \equiv \]
...N \equiv \]
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for a product of two elements in \( S_N \). We remark here that for the elementary transposition \( \sigma_{i, i+1} \), \( R_{1 \ldots N}^{\sigma_{i, i+1}} = R_{i, i+1} \), and for any \( \sigma \in S_N \), \( R_{1 \ldots N}^{\sigma} \) can be obtained with the help of (2.16).

Note that \( R_{1 \ldots N}^{\sigma(1 \ldots N)} \) satisfies the relation

\[
R_{1 \ldots N}^{\sigma(1 \ldots N)} T_{0, \sigma(1 \ldots N)} = T_{0, \sigma(1 \ldots N)} R_{1 \ldots N}^{\sigma(1 \ldots N)}. \tag{2.17}
\]

As in [6], we write the elements of \( R_{1 \ldots N}^{\sigma} \) as

\[
(R_{1 \ldots N}^{\sigma})_{\beta N \ldots \beta_1}^{\alpha (N) \ldots \alpha (1) \ldots \alpha_1}, \tag{2.18}
\]

where the labels in the upper indices are permuted relative to the lower indices according to \( \sigma \).

### 3. \( F \)-matrices for the \( gl(m|n) \) supersymmetric model

In this section, we construct the factoring \( F \)-matrix and its inverse for the supersymmetric \( gl(m|n) \) model.

#### 3.1. The \( F \)-matrix

For the \( R \)-matrix (2.2), we define the \( F \)-matrix

\[
F_{12} = \sum_{m+n \geq \alpha_1 > \alpha_2} P_{1}^{\alpha_1} P_{2}^{\alpha_2} + c_{12} \sum_{\gamma=1}^{m} P_{1}^{\gamma} P_{2}^{\gamma} + \sum_{m+n \geq \alpha_1 > \alpha_2} P_{1}^{\alpha_1} P_{2}^{\alpha_2} R_{12}, \tag{3.1}
\]

where \( (P_{i})_{k}^{l} = \delta_{k, \alpha} \delta_{l, \alpha} \) is the projector acting on the \( i \)th space, and \( c_{12} = a_{12} - b_{12} \). Then from the \( R \)-matrix (2.2) and \( F \)-matrix (3.1), we have

\[
F_{12} = \left( \sum_{m+n \geq \alpha_1 > \alpha_2} P_{2}^{\alpha_2} P_{1}^{\alpha_1} + c_{21} \sum_{\gamma=1}^{m} P_{2}^{\gamma} P_{1}^{\gamma} + \sum_{m+n \geq \alpha_2 > \alpha_1} P_{2}^{\alpha_2} P_{1}^{\alpha_1} R_{21} \right) R_{12}
\]

\[
= \sum_{m+n \geq \alpha_1 > \alpha_2} P_{2}^{\alpha_2} P_{1}^{\alpha_1} R_{12} + (1 + c_{21}) \sum_{\gamma=1}^{m} P_{2}^{\gamma} P_{1}^{\gamma} + \sum_{m+n \geq \alpha_2 > \alpha_1} P_{2}^{\alpha_2} P_{1}^{\alpha_1} R_{21}
\]

\[
= \sum_{m+n \geq \alpha_1 > \alpha_2} P_{2}^{\alpha_2} P_{1}^{\alpha_1} R_{12} + (1 + c_{21}) \sum_{\gamma=1}^{m} P_{2}^{\gamma} P_{1}^{\gamma} + \sum_{\gamma=m+1}^{m+n} P_{2}^{\gamma} P_{1}^{\gamma}
\]

\[
= \sum_{m+n \geq \alpha_1 > \alpha_2} P_{2}^{\alpha_2} P_{1}^{\alpha_1} R_{12} + c_{12} \sum_{\gamma=1}^{m} P_{2}^{\gamma} P_{1}^{\gamma} + \sum_{m+n \geq \alpha_2 > \alpha_1} P_{2}^{\alpha_2} P_{1}^{\alpha_1}
\]

\[
= F_{12}. \tag{3.2}
\]

Here we have used \( R_{12} R_{21} = 1 \) and \( c_{12} c_{21} = 1 \). Some remarks are in order. The solutions to (1.1), i.e. the \( F \)-matrices satisfying (1.1), are not unique [1, 6]. In this paper, we only consider a particular solution of the form (3.1), which is lower-triangular.

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We now generalize the $F$-matrix to the $N$-site problem. As is pointed out in [6], the generalized $F$-matrix should satisfy three properties: (i) lower-triangularity; (ii) non-degeneracy; and

\[ (iii) \ F_{\sigma(1...N)}(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) R_{\sigma(1...N)}^\sigma(z_1, \ldots, z_N) = F_{1...N}(z_1, \ldots, z_N), \]

where $\sigma \in S_N$ and $z_i$, $i = 1, \ldots, N$, are generic inhomogeneous parameters.

Define the $N$-site $F$-matrix:

\[ F_{1...N} = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(i)} \ldots \alpha_{\sigma(N)}} \prod_{j=1}^N P^i_{\sigma(j)} S(c, \sigma, \alpha) R_{1...N}^\sigma, \]

where the sum $\sum^*$ is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

\[ \alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} \quad \text{if} \ \sigma(i+1) > \sigma(i) \]

\[ \alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} \quad \text{if} \ \sigma(i+1) < \sigma(i) \]

and the $c$-number function $S(c, \sigma, \alpha)$ is given by

\[ S(c, \sigma, \alpha) \equiv \exp \left\{ \sum_{i<k=1}^N \delta^{\gamma}_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)} \sigma(l)) \right\} \]

with $\gamma = 1, \ldots, m$, $\delta^{\gamma}_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} = 1$ for $\alpha_{\sigma(k)} = \alpha_{\sigma(l)} = \gamma$, and $\delta^{\gamma}_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} = 0$ otherwise.

The definition of $F_{1...N}$, (3.4), and the summation condition (3.5) imply that $F_{1...N}$ is a lower-triangular matrix. Moreover, one can easily check that the $F$-matrix is non-degenerate because all diagonal elements are non-zero.

We now prove that the $F$-matrix (3.4) satisfies the property (iii). Any given permutation $\sigma \in S_N$ can be decomposed into elementary transpositions of the group $S_N$ as $\sigma = \sigma_1 \ldots \sigma_k$ with $\sigma_i$ denoting the elementary permutation $(i, i+1)$. By (2.16), we have, if the property (iii) holds for elementary transposition $\sigma_1$,

\[ F_{\sigma(1...N)} R_{1...N}^\sigma = F_{\sigma_1...\sigma_k(1...N)} R_{\sigma_1...\sigma_k-1(1...N)}^\sigma R_{\sigma_1...\sigma_k-2(1...N)}^\sigma \ldots R_{1...N}^\sigma \]

\[ = F_{\sigma_1...\sigma_k-1(1...N)} R_{\sigma_1...\sigma_k-2(1...N)}^\sigma \ldots R_{1...N}^\sigma \]

\[ = \ldots = F_{\sigma_1(1...N)} R_{1...N}^\sigma = F_{1...N}. \]

For the elementary transposition $\sigma_i$, we have

\[ F_{\sigma_1(1...N)} R_{1...N}^\sigma = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma_1(1...N)}} \prod_{j=1}^N P^i_{\sigma_1\sigma(j)} S(c, \sigma_1\sigma, \alpha_{\sigma_1\sigma}) R_{\sigma_1(1...N)}^\sigma R_{1...N}^\sigma \]

\[ = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma_1(1...N)}} \prod_{j=1}^N P^i_{\sigma_1\sigma(j)} S(c, \sigma_1\sigma, \alpha_{\sigma_1\sigma}) R_{1...N}^\sigma \]

\[ = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma_1(1...N)}} \prod_{j=1}^N P^i_{\sigma(j)} S(c, \tilde{\sigma}, \alpha_{\tilde{\sigma}}) R_{1...N}^\sigma, \]

where $\tilde{\sigma}$ is a function of $\sigma$. This completes the proof of property (iii).
where $\tilde{\sigma} = \sigma_i \sigma$, and the summation sequence of $\alpha_{\tilde{\sigma}}$ in $\sum'^{(i)}$ now has the form
\[
\begin{align*}
\alpha_{\tilde{\sigma}(j+1)} &\geq \alpha_{\tilde{\sigma}(j)} & \text{if } \sigma_i \tilde{\sigma}(j+1) > \sigma_i \tilde{\sigma}(j), \\
\alpha_{\tilde{\sigma}(j+1)} &> \alpha_{\tilde{\sigma}(j)} & \text{if } \sigma_i \tilde{\sigma}(j+1) < \sigma_i \tilde{\sigma}(j).
\end{align*}
\]

Comparing (3.9) with (3.5), we find that the only difference between them is the transposition $\sigma_i$ factor in the ‘if’ conditions. For a given $\tilde{\sigma} \in S_N$ with $\tilde{\sigma}(j) = i$ and $\tilde{\sigma}(k) = i+1$, one finds that if $|j-k| > 1$, then $\sigma_i$ does not affect the sequence of $\alpha_{\tilde{\sigma}}$ at all, that is, the sign of inequality ‘$>$’ or ‘$\geq$’ between two neighbouring root indexes is unchanged with the action of $\sigma_i$; and if $|j-k| = 1$, then in the summation sequences of $\alpha_{\tilde{\sigma}}$, when $\tilde{\sigma}(j+1) = i+1$ and $\tilde{\sigma}(j) = i$, the sign ‘$\geq$’ changes to ‘$>$’, while when $\tilde{\sigma}(j+1) = i$ and $\tilde{\sigma}(j) = i+1$, ‘$>$’ changes to ‘$\geq$’. Thus (3.5) and (3.8) differ only when equal labels $\alpha_{\tilde{\sigma}}$ appear. With the help of the $c$-number $S$ in the definition of $F_{1,...,N}$ (3.4) and the relation $c_{12}c_{21} = 1$, one easily shows that for equal labels, $F_{\sigma_1(1)...N} R_{\sigma(1)...N}^{\sigma_1} - F_{1,...,N} = 0$. (For a detailed proof, please see [17].)

Therefore, we have proved that the Drinfeld twist factorizes the $R$-matrix of the $N$-site $gl(m|n)$ model:
\[
R_{\sigma_1(1)...N}^\sigma(z_1, \ldots, z_N) = F_{\sigma_1(1)...N}^{-1}(z_\sigma(1), \ldots, z_\sigma(N)) F_{1,...,N}(z_1, \ldots, z_N).
\]

In summary, the factorizing $F$-matrix $F_{1,...,N}$ of the $gl(m|n)$ model is proved to satisfy all three properties.

3.2. Inverse $F_{1,...,N}^{-1}$ of the $F$-matrix

The non-degenerate property of the $F$-matrix implies that we can find the inverse matrix $F_{1,...,N}^{-1}$. To do so, we first define
\[
F_{1,...,N}^{-1} = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}} \sum_{*} S(c, \sigma, \alpha_{\sigma}) R_{\sigma(1)...N}^{\sigma_1} \prod_{j=1}^{N} P_{\sigma(j)}^{\alpha_{\sigma(j)}},
\]
where the sum $\sum^{**}$ is taken over all possible $\alpha_i$ which satisfy the following non-increasing constraints:
\[
\begin{align*}
\alpha_{\sigma(i+1)} &\leq \alpha_{\sigma(i)} & \text{if } \sigma(i+1) < \sigma(i), \\
\alpha_{\sigma(i+1)} &< \alpha_{\sigma(i)} & \text{if } \sigma(i+1) > \sigma(i).
\end{align*}
\]

Now we compute the product of $F_{1,...,N}$ and $F_{1,...,N}^{-1}$. Substituting (3.4) and (3.11) into the product, we have
\[
F_{1,...,N} F_{1,...,N}^{-1} = \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}} \sum_{*} S(c, \sigma, \alpha_{\sigma}) S(c', \sigma', \beta_{\sigma'})
\]
\[
\times \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma(1)...N}^\sigma R_{\sigma'(1)...N}^{\sigma'} \prod_{i=1}^{N} P_{\sigma'(i)}^{\beta_{\sigma'(i)}}
\]
\[
= \sum_{\sigma \in S_N} \sum_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(N)}} \sum_{*} \sum_{**} S(c, \sigma, \alpha_{\sigma}) S(c', \sigma', \beta_{\sigma'})
\]
\[
\times \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma(1)...N}^{\sigma-1} \prod_{i=1}^{N} P_{\sigma'(i)}^{\beta_{\sigma'(i)}}.
\]

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To evaluate the rhs, we examine the matrix element of the $R$-matrix
\[
\left( R_{\sigma' (1...N)}^{\sigma (1...N)} \right)^{\alpha_{\sigma (1)...\alpha_{\sigma (1)}}}_{\beta_{\sigma'(1)}...\beta_{\sigma'(N)}} .
\] (3.14)

Note that the sequence $\{ \alpha_{\sigma} \}$ is non-decreasing and $\{ \beta_{\sigma'} \}$ is non-increasing. Thus the non-vanishing condition of the matrix element (3.14) requires that $\alpha_{\sigma}$ and $\beta_{\sigma'}$ satisfy
\[
\beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)} .
\] (3.15)

One can verify [6] that (3.15) is fulfilled only if
\[
\sigma'(N) = \sigma(1), \ldots, \sigma'(1) = \sigma(N).
\] (3.16)

Let $\bar{\sigma}$ be the maximal element of the $S_N$ which reverses the site labels
\[
\bar{\sigma}(1, \ldots, N) = (N, \ldots, 1).
\] (3.17)

Then from (3.16), we have
\[
\sigma' = \sigma \bar{\sigma}.
\] (3.18)

Substituting (3.15) and (3.18) into (3.13), we have
\[
F_{1...N}F^*_{1...N} = \sum_{\sigma \in S_N} \sum_{\alpha_1 \ldots \alpha_N} S(c, \sigma, \alpha_\sigma) S(c, \sigma, \alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma(i)} R_{\sigma(N)...1}^{\bar{\sigma}} \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma(i)} .
\] (3.19)

The decomposition of $R^{\bar{\sigma}}$ in terms of elementary $R$-matrices is unique modulo GYBE. One deduces from (3.19) that $FF^*$ is a diagonal matrix:
\[
F_{1...N}F^*_{1...N} = \prod_{i<j} \Delta_{ij}.
\] (3.20)

where
\[
[\Delta_{ij}]^{\alpha, \beta}_{\alpha_i, \alpha_j} = \delta_{\alpha_i, \beta_i} \delta_{\alpha_j, \beta_j} \begin{cases} a_{ij} & \text{if } \alpha_i > \alpha_j \\ a_{ji} & \text{if } \alpha_i < \alpha_j \\ 4a_{ij}a_{ji} & \text{if } \alpha_i = \alpha_j = 1, 2, \ldots, m \\ 1 & \text{if } \alpha_i = \alpha_j = m + 1, \ldots, m + n. \end{cases}
\] (3.21)

Therefore, the inverse of the $F$-matrix is given by
\[
F^{-1}_{1...N} = F^*_{1...N} \prod_{i<j} \Delta^{-1}_{ij} .
\] (3.22)

4. The monodromy matrix in the $F$-basis

In the previous section, we see that the $gl(m|n)$ $R$-matrix factorizes in terms of the $F$-matrix and its inverse which we constructed explicitly. The column vectors of the inverse of the $F$-matrix form a basis set on which $gl(m|n)$ acts. In this section, we study the generators of $gl(m|n)$ and the elements of the monodromy matrix in the $F$-basis.

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4.1. $gl(m|n)$ generators in the $F$-basis

Denote by $E^\gamma \gamma^{\pm 1}$ the simple generators of the $N$-site $gl(m|n)$ supersymmetric system. Then $E^\gamma \gamma^{\pm 1} = E^\gamma_{(1)} \gamma^{\pm 1} + \cdots + E^\gamma_{(N)} \gamma^{\pm 1}$, where $E^\gamma_{(k)}$ acts on the $k$th component of the tensor product space. Let $\tilde{E}^\gamma \gamma^{\pm 1}$ denote the corresponding simple generators in the $F$-basis: $\tilde{E}^\gamma \gamma^{\pm 1} = F_{1\ldots N} E^\gamma \gamma^{\pm 1} F_{1\ldots N}^{-1}$. For later use, we derive $\tilde{E}^\gamma \gamma^{+1}$. From the expressions for $F$ and its inverse, we have

\[
\tilde{E}^\gamma \gamma^{+1} = F_{1\ldots N} E^\gamma \gamma^{+1} F_{1\ldots N}^{-1}
\]

\[
= \sum_{\sigma, \sigma' \in S_N} \sum_{\alpha(1), \ldots, \alpha(N)} \sum_{\beta(1), \ldots, \beta(N)} S(c, \sigma, \alpha) S(c', \sigma', \beta)
\]

\[
\times \prod_{i=1}^N P^{\sigma(i)} R_{\sigma(1)\ldots N}^\sigma E^\gamma \gamma^{+1} R_{\sigma'(1)\ldots N}^{\sigma'} \prod_{i=1}^N P^{\sigma'(i)} \prod_{i<j} \Delta_{ij}^{-1}
\]

\[
= \sum_{\sigma, \sigma' \in S_N} \sum_{k=1}^N E^\gamma \gamma^{+1}_{(\sigma(a)\ldots k)} \sum_{\alpha(1), \ldots, \alpha(N)} \sum_{\beta(1), \ldots, \beta(N)} S(c, \sigma, \alpha) S(c', \sigma', \beta)
\]

\[
\times \prod_{i=1}^N P^{\sigma(i)} \prod_{i=1}^N P^{\sigma(i)}_{\sigma(a)\ldots k} E^\gamma \gamma^{+1}_{(\sigma(a)\ldots k)} \prod_{i=1}^N P^{\sigma'(i)} \prod_{i<j} \Delta_{ij}^{-1}
\]

where in (4.1), we have used $[E^\gamma \gamma^{\pm 1}, R^\sigma_{1\ldots N}] = 0$. The element of $R_{\sigma'(1)\ldots N}^{\sigma'}$ connecting $P^{\sigma(1)} \ldots P^{\sigma(a)+1} \ldots P^{\sigma(N)}$ and $P^{\beta(1)} \ldots P^{\beta(1)}$ is denoted as

\[
\left( R_{\sigma'(1)\ldots N}^{\sigma'} \right)^{\alpha(1)}_{\sigma(1)} \ldots \alpha(a)_{\sigma(a)} \sigma(a)+1_{\sigma(a)+1} \ldots \sigma(N)_{\sigma(N)}
\]

(4.3)

We call the sequence $\{\alpha(1)\}$ normal if it is arranged according to the rules in (3.5); otherwise, we call it abnormal.

It is now convenient for us to discuss the non-vanishing condition of the $R$-matrix element (4.3). Comparing (4.3) with (3.14), we find that the difference between them lies in the $k$th site. Because the group label in the $k$th space has been changed, the sequence $\{\alpha_{\sigma}\}$ is now an abnormal sequence. However, it can be permuted to the normal sequence by some permutation $\hat{\sigma}_k$. That is, $\alpha_{\gamma_{\gamma+1}}$ in the abnormal sequence can be moved to a suitable position by using the permutation $\hat{\sigma}_k$ according to rules in (3.5). (It is easy to verify that $\hat{\sigma}_k$ is unique by using (3.5).) Thus, by a procedure similar to that in the

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previous section, we find that when
\[ \sigma' = \hat{\sigma}_k \sigma \hat{\sigma} \quad \text{and} \quad \beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}, \] (4.4)
the \( R \)-matrix element (4.3) is non-vanishing.

Because the non-zero condition of the elementary \( R \)-matrix element \( R_{ij}^{i'j'} \) is \( i + j = i' + j' \), the following \( R \)-matrix elements:
\[ (R_{\sigma'(1)}^{\sigma(1)} \cdots \sigma_{\sigma'(N)}^{\sigma(N)} \cdots)_{\beta_{\sigma'(N)}^{\beta_{\sigma(1)}}} \] (4.5)
with \( a + 1 \leq n < a + l \) are also non-vanishing.

Therefore, (4.2) becomes
\[ \tilde{E}^{\gamma+1}_{\gamma} = \sum_{\sigma \in \mathcal{S}_N} \sum_{k=1}^{N} \sum_{\alpha_1, \ldots, \alpha_N}^{*} S(c, \sigma, \alpha_{\sigma}) S(c, \hat{\sigma}_k \sigma, \alpha_{\hat{\sigma}_k \sigma}) \times [E^{\gamma+1}_{\sigma(a+l)} P^{\alpha_{\sigma(1)}}_{\sigma(1)} \cdots P^{\alpha_{\gamma+1}}_{\sigma(\gamma+1)} \cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)} + \cdots + E^{\gamma+1}_{\sigma(a+n)} P^{\alpha_{\sigma(1)}}_{\sigma(1)} \cdots P^{\alpha_{\gamma+1}}_{\sigma(\gamma+1)} \cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)} + \cdots \times \prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} \prod_{i<j}^{N} \Delta_{ij}^{-1} \] (4.6)
\[ = \sum_{k=1}^{N} E^{\gamma+1}_{(k)} \otimes_{j \neq k} G^{\gamma+1}_{(j)}(k, j). \] (4.7)
Here in (4.6) \( \hat{\sigma}_k \) is the element of \( \mathcal{S}_N \) which permutes the first abnormal sequence in the square bracket of (4.6) to the normal sequence and \( G^{\gamma+1}(i, j) \) in (4.7) has the following elements: for \( 1 < \gamma + 1 \leq m \),
\[ G^{\gamma+1}(i, j)_{kl} = \delta_{k,l} \left\{ \begin{array}{ll}
2 & k = \gamma \\
(2a_{ij})^{-1} & k = \gamma + 1 \\
1 & \text{otherwise};
\end{array} \right. \] (4.8)
for \( \gamma = m \)
\[ G^{\gamma+1}(i, j)_{kl} = \delta_{k,l} \left\{ \begin{array}{ll}
2 & k = \gamma \\
1 & \text{otherwise};
\end{array} \right. \] (4.9)
and for \( m + 1 \leq \gamma < m + n \)
\[ G^{\gamma+1}(i, j)_{kl} = \delta_{k,l} \left\{ \begin{array}{ll}
(a_{ij})^{-1} & k = \gamma \\
1 & \text{otherwise}.\end{array} \right. \] (4.10)

The non-simple generators are generated by the simple generators through the relation
\[ \tilde{E}^{m+n-\alpha \ m+n} = [\tilde{E}^{m+n-\alpha \ m+n+1}, [\tilde{E}^{m+n-\alpha+1 \ m+n+2}], \ldots, [\tilde{E}^{m+n-2 \ m+n-1}, \tilde{E}^{m+n \ m+n}] \ldots]. \] (4.11)

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We have:

1. For \( m = 0 \), the \( gl(m|n) \) supersymmetric model reduces to the bosonic \( gl(n) \) model, which has been discussed in [6] by Albert et al.

2. For \( n = 0, \alpha < m \),

\[
\hat{E}^{m-\alpha} m = \sum_{k=1}^{\alpha} \sum_{i_1 \neq \cdots \neq i_k} \prod_{\gamma=1}^{k-1} \frac{\eta}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \sum_{\alpha = \beta_0 > \cdots > \beta_k = 0} \otimes_{i=1}^{k} \hat{E}_{(i_1)}^{m-\beta_{i-1}} \begin{pmatrix} m-\beta_i \\ \beta_0 - \beta_1 \\ \vdots \\ \beta_{k-1} - \beta_k \end{pmatrix} \quad (4.12)
\]

3. For \( m, n \geq 1 \) and \( \alpha < n \),

\[
\hat{E}^{m+n-\alpha} m+n = \sum_{k=1}^{\alpha} \sum_{i_1 \neq \cdots \neq i_k} \prod_{\gamma=1}^{k-1} \frac{\eta}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \sum_{\alpha = \beta_0 > \cdots > \beta_k = 0} \otimes_{i=1}^{k} \hat{E}_{(i_1)}^{m+n-\beta_{i-1}} \begin{pmatrix} m+n-\beta_i \\ \beta_0 - \beta_1 \\ \vdots \\ \beta_{k-1} - \beta_k \end{pmatrix} \quad (4.13)
\]

4. For \( m, n \geq 1 \) and \( \alpha \geq n \),

\[
\hat{E}^{m+n-\alpha} m+n = \sum_{k=1}^{\alpha} \sum_{i_1 \neq \cdots \neq i_k} \prod_{\gamma=1}^{\min(k-1,n-1)} \frac{\eta}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \prod_{\gamma=n+1}^{k-1} \frac{\eta}{z_{i_{\gamma}} - z_{i_{\gamma+1}}} \times \sum_{\alpha = \beta_0 > \cdots > \beta_k = 0} \otimes_{i=1}^{k} \hat{E}_{(i_1)}^{m+n-\beta_{i-1}} \begin{pmatrix} m+n-\beta_i \\ \beta_0 - \beta_1 \\ \vdots \\ \beta_{k-1} - \beta_k \end{pmatrix} \quad (4.14)
\]
4.2. Elements of the monodromy matrix in the $F$-basis

In the $F$-basis, the monodromy matrix $T(u)$, (2.9), becomes

$$
\tilde{T}(u) = \begin{pmatrix}
\hat{A}_{11}(u) & \cdots & \hat{A}_{1 \, m+n-1}(u) & \tilde{B}_1(u) \\
\vdots & \ddots & \vdots & \vdots \\
\hat{A}_{m+n-1 \, 1}(u) & \cdots & \hat{A}_{m+n-1 \, m+n-1}(u) & \tilde{B}_{m+n-1}(u) \\
\tilde{C}_1(u) & \cdots & \tilde{C}_{m+n-1}(u) & \tilde{D}(u)
\end{pmatrix}.
$$

(4.15)

We first study the diagonal element $\tilde{D}(u)$. Acting the $F$-matrix on $D(u)$, we have

$$
F_{1\ldots N} D = \sum_{\sigma \in S_N} \sum_{\alpha_{(1)}, \ldots, \alpha_{(N)}}^{*} S(c, \sigma, \alpha_{(N)}) \prod_{i=1}^{N} P_{\sigma(i)}^{m+n} R_{1\ldots N}^{m+n} T_{0,1\ldots N}^{m+n} P_{0}^{m+n}
$$

$$
= \sum_{\sigma \in S_N} \sum_{\alpha_{(1)}, \ldots, \alpha_{(N)}}^{*} S(c, \sigma, \alpha_{(N)}) \prod_{i=1}^{N} P_{\sigma(i)}^{m+n} T_{0,\sigma(1\ldots N)}^{m+n} P_{0}^{m+n} R_{1\ldots N}^{m+n}.
$$

(4.16)

Following [6], we can split the sum $\sum^{*}$ according to the number of occurrences of the index $m+n$:

$$
F_{1\ldots N} T^{m+n} = \sum_{\sigma \in S_N} \sum_{k=0}^{N} \sum_{\alpha_{(1)}, \ldots, \alpha_{(N)}}^{*} S(c, \sigma, \alpha_{(N)}) \prod_{j=N-k+1}^{N} \delta_{\alpha(j), m+n} P_{\sigma(j)}^{m+n}
$$

$$
\times \prod_{j=1}^{N-k} P_{\sigma(j)}^{m+n} T_{0,\sigma(1\ldots N)}^{m+n} P_{0}^{m+n} R_{1\ldots N}^{m+n}.
$$

(4.17)

Consider the prefactor of $R_{1\ldots N}^{m+n}$. We have

$$
\prod_{j=1}^{N-k} P_{\sigma(j)}^{m+n} T_{0,\sigma(1\ldots N)}^{m+n} P_{0}^{m+n}
$$

$$
= \prod_{j=1}^{N-k} P_{\sigma(j)}^{m+n} T_{0,\sigma(1\ldots N)}^{m+n} P_{0}^{m+n}
$$

$$
= \prod_{j=1}^{N-k} P_{\sigma(j)}^{m+n} T_{0,\sigma(1\ldots N)}^{m+n} P_{0}^{m+n}
$$

$$
= \prod_{i=1}^{N-k} (R_{0,\sigma(i)})^{m+n} \prod_{j=1}^{N-k+1} P_{\sigma(i)}^{m+n} P_{0}^{m+n}
$$

$$
= \prod_{i=1}^{N-k} \prod_{j=1}^{N-k} P_{\sigma(i)}^{m+n} P_{0}^{m+n}.
$$

(4.18)

where $c_{0i} = c(u, z_i)$, $a_{0i} = a(u, z_i)$. Substituting (4.18) into (4.17), we have

$$
F_{1\ldots N} T^{m+m+n} = \otimes_{i=1}^{N} \text{diag}(a_{0i}, \ldots, a_{0i}, 1)_{(i)} F_{1\ldots N}.
$$

(4.19)

Therefore,

$$
\tilde{D}(u) = \otimes_{i=1}^{N} \text{diag}(a_{0}, \ldots, a_{0}, 1)_{(i)}.
$$

(4.20)
The creation operators in the monodromy matrix can then be obtained as follows:
\[
\hat{C}_\alpha(u) = [\hat{E}^{m+n}, \hat{D}(u)], \quad (1 \leq \alpha < m + n), \tag{4.21}
\]
which follows from the \( gl(m|n) \) invariance of the \( R \)-matrix, i.e. in terms of the monodromy matrix,
\[
[\hat{T}(u), \hat{E}^\alpha_{\alpha}(0) + \hat{E}^\beta_{\beta}(0)] = 0. \tag{4.22}
\]
Substituting \( \hat{E}^{m+n}, \hat{E}^{m+n}_{\alpha} \) and \( \hat{T}^{m+n} m+n \) into the above relations yields: for \( n = 0 \),
\[
\hat{C}_{m-\alpha}(u) = -\sum \sum b_{i_k} \prod_{\gamma=1}^{k-1} \frac{a_{0,\eta}}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \sum \otimes_{l=1}^{k} E_{(i_l)}^{m-\beta_{l-1} m-\beta_l} \otimes \text{diag} \left( a_{0j}, \ldots, a_{0j}, 2a_{0j}, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1} \right) \right); \tag{4.23}
\]
for \( m, n \geq 1 \) and \( \alpha < n \),
\[
\hat{C}_{m+n-\alpha}(u) = \sum \sum b_{i_k} \prod_{\gamma=1}^{k-1} \frac{a_{0,\eta}}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \sum \otimes_{l=1}^{k} E_{(i_l)}^{m+n-\beta_{l-1} m+n-\beta_l} \otimes \text{diag} \left( a_{0j}, \ldots, a_{0j}, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1}, 1 \right) \right); \tag{4.24}
\]
and for \( m, n \geq 1 \) and \( \alpha \geq n \),
\[
\hat{C}_{m+n-\alpha}(u) = \sum \sum b_{i_k} \prod_{\gamma=1}^{\min(k-1,n-1)} \frac{a_{0,\eta}}{z_{i_{\gamma+1}} - z_{i_{\gamma}}} \prod_{\gamma=n+1}^{k-1} \frac{a_{0,\eta}}{z_{i_{\gamma}} - z_{i_{\gamma+1}}} \sum \otimes_{l=1}^{k} E_{(i_l)}^{m+n-\beta_{l-1} m+n-\beta_l} \otimes \text{diag} \left( a_{0j}, \ldots, a_{0j}, 2a_{0j}, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1}, \ldots, a_{0j}a_{i_{1j}}^{-1}, 1 \right) \right). \tag{4.25}
\]
Here, $a_{0j}$ and $b_{0j}$ stand for $a(u, z_j)$ and $b(u, z_j)$, respectively.

5. $gl(2|2)$ Bethe vectors in the $F$-basis

In this section, specializing to the superconductive $gl(2|2)$ electronic system, we will resolve its nested Bethe vectors in the $F$-basis.

In the framework of the algebraic Bethe ansatz, the pseudo-vacuum state is

$$|0\rangle = \bigotimes_{k=1}^{N} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{(k)}. \quad (5.1)$$

The Bethe vector of the model is then defined by

$$\Phi_N(v_1, \ldots, v_{n_1}) = \sum_{d_1 \ldots d_{n_1}} (\phi_{n_1}^{(1)})^{d_1 \ldots d_{n_1}} C_{d_1}(v_1) \ldots C_{d_{n_1}}(v_{n_1}) |0\rangle, \quad (5.2)$$

where $d_i = 1, 2, 3, (\phi_{n_1}^{(1)})^{d_1 \ldots d_{n_1}}$ is a function of the spectral parameter $v_j$. In the algebraic Bethe ansatz, $(\phi_{n_1}^{(1)})^{d_1 \ldots d_{n_1}}$ is also associated with the three-dimensional nested Bethe vector

$$\phi_{n_1}^{(1)}(v_1^{(1)}, \ldots, v_{n_2}^{(1)}) = \sum_{d_1 \ldots d_{n_2}} (\phi_{n_2}^{(2)})^{d_1 \ldots d_{n_2}} C_{d_1}^{(1)}(v_1^{(1)}) \ldots C_{d_{n_2}}^{(1)}(v_{n_2}^{(1)}) |0\rangle^{(1)} \quad (5.3)$$

where $d_i = 1, 2$, $|0\rangle^{(1)}$ is the nested pseudo-vacuum state

$$|0\rangle^{(1)} = \bigotimes_{k=1}^{n_2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{(k)}. \quad (5.4)$$

$(\phi_{n_2}^{(2)})^{d_1 \ldots d_{n_2}}$ is a function of the spectral parameter $v_j^{(1)}$ and $C^{(1)}$ are the creation operators of the nested $gl(2|1)$ system. Here $\phi_{n_2}^{(2)}$ is the second-level nested Bethe vector associated with the $gl(2)$ model. As usual, the second-level nested Bethe vector is defined by

$$\phi_{n_2}^{(2)}(v_1^{(2)}, \ldots, v_{n_3}^{(2)}) = C^{(2)}(v_1^{(2)}) \ldots C^{(2)}(v_{n_3}^{(2)}) |0\rangle^{(2)}, \quad (5.5)$$

with

$$|0\rangle^{(2)} = \bigotimes_{k=1}^{n_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(k)}. \quad (5.6)$$

Applying the $gl(2|2)$ $F$-matrix, i.e. $m = 2, n = 2$ in (3.4), to the pseudo-vacuum state (5.1), we find that it is invariant. This is due to the fact that only terms whose roots all equal 4, in the definition expression of $F^{(1)}$, produce non-zero results. Therefore the Bethe vector (5.2) in the $F$-basis becomes

$$\tilde{\Phi}_N(v_1, \ldots, v_{n_1}) = F_{1 \ldots N} \Phi_N(v_1, \ldots, v_{n_1})$$

$$= \sum_{d_1 \ldots d_{n_1}} (\phi_{n_1}^{(1)})^{d_1 \ldots d_{n_1}} \tilde{C}_{d_1}(v_1) \ldots \tilde{C}_{d_{n_1}}(v_{n_1}) |0\rangle. \quad (5.7)$$

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For the Bethe vector (5.7), one checks that the spectral parameters of the system preserve the exchange symmetry
\[ \tilde{\Phi}_N(v_{\sigma(1)}, \ldots, v_{\sigma(n_1)}) = \tilde{\Phi}_N(v_1, \ldots, v_{n_1}), \]  
while the creation operators satisfy the following commutation relation [14]:
\[ \tilde{C}_i(u)\tilde{C}_j(v) = (-1)^{|i||j|} \frac{1}{a(v, u)} \tilde{C}_j(v)\tilde{C}_i(u) - (-1)^{|i|} \frac{b(v, u)}{a(v, u)} \tilde{C}_j(u)\tilde{C}_i(v). \]  
(5.9)
These two properties mean that we may propose the following special sequence of the creation operators with the \( p_1 \) number \( d_i = 1 \), the \( p_2 - p_1 \) number \( d_i = 2 \) and the \( n_1 - p_2 \) number \( d_i = 3 \):
\[ \tilde{C}_1(v_1) \ldots \tilde{C}_1(v_{p_1}) \tilde{C}_2(v_{p_1+1}) \ldots \tilde{C}_2(v_{p_2}) \tilde{C}_3(v_{p_2+1}) \ldots \tilde{C}_3(v_{n_1}). \]  
(5.10)
With the help of the commutation relation (5.9), we may rewrite the sequence as
\[ \tilde{C}_1(v_1) \ldots \tilde{C}_1(v_{p_1}) \tilde{C}_2(v_{p_1+1}) \ldots \tilde{C}_2(v_{p_2}) \tilde{C}_3(v_{p_2+1}) \ldots \tilde{C}_3(v_{n_1}) = h(v_1, \ldots, v_{n_1}) \tilde{C}_3(v_{p_2+1}) \ldots \tilde{C}_3(v_{n_1}) \tilde{C}_2(v_{p_1+1}) \ldots \tilde{C}_2(v_{p_2}) \]  
\[ \times \tilde{C}_1(v_1) \ldots \tilde{C}_1(v_{p_1}) + \ldots \]  
(5.11)
with
\[ h(v_1, \ldots, v_{n_1}) = \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{p_2} \left( -\frac{1}{a(v_l, v_k)} \right) \prod_{k=1}^{p_2} \prod_{l=p_2+1}^{n_1} \left( \frac{1}{a(v_l, v_k)} \right). \]
Here the prefactor \( h \) comes from the first term of (5.9), and ‘...’ stands for the other terms contributed by the second term. Considering the exchange symmetry (5.8), one easily checks that the other terms ‘...’ can be represented by
\[ C_3(v_{\sigma(p_2+1)}) \ldots \tilde{C}_3(v_{\sigma(n_1)}) \tilde{C}_2(v_{\sigma(p_1+1)}) \ldots \tilde{C}_2(v_{\sigma(p_2)}) \tilde{C}_1(v_{\sigma(1)}) \ldots \tilde{C}_1(v_{\sigma(p_1)}), \]  
(5.12)
where \( \sigma \in S_{n_1} \). Substituting (5.10) into the Bethe vector (5.7), we then propose the following Bethe vector \( \tilde{\Phi}_N^{(p_1, p_2)} \) corresponding to the quantum numbers \( p_1 \) and \( p_2 \):
\[ \tilde{\Phi}_N^{(p_1, p_2)}(v_1, \ldots, v_{n_1}) = \frac{1}{p_1!(p_2 - p_1)!(n_1 - p_2)!} \sum_{\sigma \in S_{n_1}} (\phi_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3} \tilde{C}_1(v_1) \ldots \tilde{C}_1(v_{p_1}) \tilde{C}_2(v_{p_1+1}) \ldots \tilde{C}_2(v_{p_2}) \tilde{C}_3(v_{p_2+1}) \ldots \tilde{C}_3(v_{n_1}) |0 \rangle \]  
\[ \times \tilde{C}_2(v_{\sigma(p_1+1)}) \ldots \tilde{C}_2(v_{\sigma(p_2)}) \tilde{C}_1(v_{\sigma(1)}) \ldots \tilde{C}_1(v_{\sigma(p_1)}) |0 \rangle, \]  
(5.13)
where \( \sigma \) in \( \phi_{n_1}^{(1, \sigma)} \) implies the permutation of the inhomogeneous parameters of the original nested Bethe vector.
Applying \( \tilde{\Phi} \) between where '\( B_0 \) Therefore, substituting (5.14)–(5.16) into (5.13), we obtain

\[
\hat{C}_3 = \sum_{i=1}^{N} b_i E_{(i)}^{34} \otimes \sum_{j \neq i} \text{diag}(a_{0j}, a_{0j}, a_{0j} a_{ij}^{-1}, 1)_{(j)},
\]

(5.14)

\[
\hat{C}_2 = \sum_{i=1}^{N} b_i E_{(i)}^{24} \otimes \sum_{j \neq i} \text{diag}(a_{0j}, 2a_{0j}, a_{0j} a_{ij}^{-1}, 1)_{(j)} + \cdots,
\]

(5.15)

\[
\hat{C}_1 = \sum_{i=1}^{N} b_i E_{(i)}^{14} \otimes \sum_{j \neq i} \text{diag}(2a_{0j}, a_{0j} a_{ij}^{-1}, a_{0j} a_{ij}^{-1}, 1)_{(j)} + \cdots,
\]

(5.16)

where '\( \ldots \) stands for terms which contain more than one generator, e.g. \( E_i^{12} \otimes E_j^{24} \) etc.

Applying \( \hat{C}_2 \) and \( \hat{C}_1 \) to the pseudo-vacuum state (5.1), one finds that all other terms equal zero. Therefore, substituting (5.14)–(5.16) into (5.13), we obtain

\[
\tilde{\Phi}_N^{(p_1, p_2)}(v_1, \ldots, v_{n_1})
\]

\[
= \frac{1}{p_1!(p_2 - p_1)!(n_1 - p_2)!} \times \sum_{i_1 < \cdots < i_{n_1}} B^{(0)}_{n_1, (p_1, p_2)}(v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_1}^{(1)}, \ldots, v_{p_2}^{(1)}; z_{i_1}, \ldots, z_{i_{n_1}})
\]

\[
\times \prod_{j=p_{2}+1}^{i_{p_1}} E_{(j)}^{34} \prod_{j=p_{1}+1}^{i_{p_2}} E_{(j)}^{24} \prod_{j=1}^{i_{p_1}} E_{(j)}^{14} |0\rangle,
\]

(5.17)

where

\[
B^{(0)}_{n_1, (p_1, p_2)}(v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_1}^{(1)}, \ldots, v_{p_2}^{(1)}; z_{i_1}, \ldots, z_{i_{n_1}})
\]

\[
= \sum_{\sigma \in S_{n_1}} \prod_{k=1}^{p_2} b(\sigma(1)_k, z_{i_k}) \prod_{l=p_{2}+1}^{n_1} a(\sigma(1)_l, z_{i_l}) \prod_{k=1}^{p_2} \prod_{l=p_{2}+1}^{n_1} (a(\sigma(1)_k, z_{i_k}) - a(\sigma(1)_l, z_{i_l}))
\]

\[
\times (\phi_{p_1}^{(1)}|1^{12 \cdots 23 \cdots 3} B_{n-p_1}(v_{\sigma(p_2+1)}, \ldots, v_{\sigma(n_1)}|z_{i_{p_2}+1}, \ldots, z_{i_{n_1}})
\]

\[
\times B^*_{p_2-p_1}(v_{\sigma(p_1+1)}, \ldots, v_{\sigma(p_2)}|z_{i_{p_1}+1}, \ldots, z_{i_{p_2}}) B^*_p(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)}|z_{i_1}, \ldots, z_{i_{p_1}})
\]

(5.18)

with

\[
B_p(v_1, \ldots, v_p|z_1, \ldots, z_p) = \sum_{\sigma \in S_p} \prod_{m=1}^{p} b(v_m, z_{\sigma(m)}) \prod_{l=m+1}^{p} a(v_m, z_{\sigma(l)})
\]

(5.19)

\[
B^*_p(v_1, \ldots, v_p|z_1, \ldots, z_p) = \sum_{\sigma \in S_p} \text{sgn}(\sigma) \prod_{m=1}^{p} b(v_m, z_{\sigma(m)}) \prod_{l=m+1}^{p} 2a(v_m, z_{\sigma(l)}).
\]

In (5.18), we still need to determine the form of \( (\phi_{n_1}^{(1)})^{1 \cdots 12 \cdots 23 \cdots 3} \), which should be evaluated in the original basis. Define \( \tilde{\phi}_{n_1}^{(1)} \equiv F_{1 \cdots n_1} \phi_{n_1}^{(1)} \). We now examine the relation between \( \phi_{n_1}^{(1)} \) and \( \tilde{\phi}_{n_1}^{(1)} \).
Write the nested pseudo-vacuum vector (5.4) as
\[
|0(1)\rangle \equiv |3\cdots 3\rangle(1),
\]
where the number of 3 in the above state is \(n_1\). Then the nested Bethe vector (5.3) can be rewritten as
\[
\phi_{n_1}(v_1^{(1)} \ldots v_{d_1}^{(1)}) \equiv |\phi_{n_1}^{(1)}\rangle = \sum_{d_1 \ldots d_{n_1}} (\phi_{n_1}^{(1)})^{d_1 \ldots d_{n_1}} |d_1 \ldots d_{n_1}\rangle^{(1)}.
\]

Acting with the \(gl(2|1)\) \(F\)-matrix \(F(1)\) from the left on the above equation, we have
\[
\tilde{\phi}_{n_1}(v_1^{(1)} \ldots v_{d_1}^{(1)}) \equiv |\tilde{\phi}_{n_1}^{(1)}\rangle = F(1)|\phi_{n_1}^{(1)}\rangle = \sum_{d_1 \ldots d_{n_1}} (\tilde{\phi}_{n_1}^{(1)})^{d_1 \ldots d_{n_1}} |d_1 \ldots d_{n_1}\rangle^{(1)}.
\]

It follows that
\[
(\phi_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3} = \langle 1 \ldots 12 \ldots 23 \ldots 3 | \tilde{\phi}_{n_1}^{(1)} \rangle = \langle 1 \ldots 12 \ldots 23 \ldots 3 | F(1) |\phi_{n_1}^{(1)}\rangle
= \langle 1 \ldots 12 \ldots 23 \ldots 3 | \sum_{\sigma \in S_{n_1}} \sum_{\alpha_{\sigma(1)} \cdots \alpha_{\sigma(n_1)}} \prod_{j=1}^{n_1} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \times S(c, \sigma, \alpha_{\sigma}) R_{1 \ldots n_1}^{\sigma} |\phi_{n_1}^{(1)}\rangle
= \langle 1 \ldots 12 \ldots 23 \ldots 3 | \sum_{\alpha_{\sigma(1)} \cdots \alpha_{\sigma(n_1)}} \prod_{j=1}^{n_1} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \right|_{\sigma=\text{id}} \times S(c, \sigma, \alpha_{\sigma}) |\phi_{n_1}^{(1)}\rangle
= t(c) \langle 1 \ldots 12 \ldots 23 \ldots 3 | \phi_{n_1}^{(1)} \rangle = t(c) (\phi_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3}
\]
with \(t(c) = \prod_{j>i=1}^{p_1} (1 + c_{ij}) \prod_{j>i=p_1+1}^{p_2} (1 + c_{ij})\). It follows that \((\phi_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3}\) may be computed by using the \(F\)-transformed version \((\tilde{\phi}_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3}\).

In the following, we compute \((\phi_{n_1}^{(1)})^{1 \ldots 12 \ldots 23 \ldots 3}\) with the help of the nested \(gl(2|1)\) \(F\)-basis.

5.1. The first-level nested \(gl(2|1)\) Bethe vector
Denote by \(F_{1 \ldots n_1}^{(1)}\) the \(n_1\)-site \(gl(2|1)\) \(F\)-matrix. Acting with the \(F\)-matrix on (5.4), one finds that the nested \(gl(2|1)\) pseudo-vacuum state is also invariant. Thus in the \(F\)-basis, the nested Bethe vector (5.3) becomes
\[
\tilde{\phi}_{n_1}^{(1)}(v_1^{(1)} \ldots v_{d_1}^{(1)}) = \sum_{d_1 \ldots d_{n_2}} (\tilde{\phi}_{n_2}^{(2)})^{d_1 \ldots d_{n_2}} C_{d_1}^{\sigma(1)}(v_1^{(1)}) \ldots C_{d_2}^{\sigma(n_2)}(v_{d_2}^{(1)}) |0(1)\rangle.
\]

We may prove that the nested Bethe vector satisfies the following exchange symmetry:
\[
\tilde{\phi}_{n_1}^{(1)}(v_{\sigma(1)}^{(1)}, \ldots, v_{\sigma(n_2)}^{(1)}) = \frac{1}{c_{1 \ldots n_2}^{\sigma}} \tilde{\phi}_{n_1}^{(1)}(v_1^{(1)}, \ldots, v_{n_2}^{(1)}),
\]
where \(c_{1 \ldots n}^{\sigma}\) has the decomposition law
\[
c_{1 \ldots n}^{\sigma \alpha} = c_{\sigma(1 \ldots n)}^{\sigma} c_{1 \ldots n}^{\alpha}
\]
with \(c_{1 \ldots n}^{\sigma} = c_{i \ i+1}^{\sigma} \equiv c(v_i, v_{i+1})\) for an elementary permutation \(\sigma_i\).
This enables one to concentrate on a particularly simple term in the sum (5.24) of the form with the $p_1$ number $d_1 = 1$ and the $n - p_1$ number $d_j = 2$:

$$
\tilde{C}_1^{(1)}(v^{(1)}) \cdots \tilde{C}_1^{(1)}(v_{p_1}^{(1)}) \tilde{C}_2^{(1)}(v_{p_1+1}^{(1)}) \cdots \tilde{C}_2^{(1)}(v_{n_2}^{(1)}).
$$

(5.27)

The commutation relation between $C_i(v)$ and $C_j(u)$ [14] in the $F$-basis becomes

$$
\tilde{C}_i^{(1)}(v) \tilde{C}_j^{(1)}(u) = -\frac{1}{a(u, v)} \tilde{C}_j^{(1)}(u) \tilde{C}_i^{(1)}(v) + \frac{b(u, v)}{a(u, v)} \tilde{C}_j^{(1)}(v) \tilde{C}_i^{(1)}(u).
$$

(5.28)

Then using (5.28), all $\tilde{C}_1^{(1)}$s in (5.27) can be moved to the right of all $\tilde{C}_2^{(1)}$s, yielding

$$
\tilde{C}_1^{(1)}(v^{(1)}) \cdots \tilde{C}_1^{(1)}(v_{p_1}^{(1)}) \tilde{C}_2^{(1)}(v_{p_1+1}^{(1)}) \cdots \tilde{C}_2^{(1)}(v_{n_2}^{(1)})
= g(v^{(1)}, \ldots, v_{n_2}^{(1)}) \tilde{C}_2^{(1)}(v_{p_1+1}^{(1)}) \cdots \tilde{C}_2^{(1)}(v_{n_2}^{(1)}) \tilde{C}_1^{(1)}(v_{p_1}^{(1)}) \cdots \tilde{C}_1^{(1)}(v_{n_1}^{(1)}) + \cdots,
$$

(5.29)

where $g(v^{(1)}, \ldots, v_{n_2}^{(1)}) = \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n_2} (-1/a(v_l^{(1)}, v_k^{(1)}))$ is the contribution from the first term of (5.28). As before, the other terms $\ldots$ can be given with the help of the permutation operator $\sigma \in S_{n_2}$. Thus, we rewrite the nested Bethe vector as

$$
\phi_{n_1}^{(1), p_1}(v^{(1)}, \ldots, v_{n_2}^{(1)})
= \frac{1}{p_1!(n_2 - p_1)!} \sum_{\sigma \in S_{n_2}} \epsilon^{\sigma} \cdots \epsilon^{n_2} (\phi_{n_2}^{(2), \sigma}) \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n_2} \left( -\frac{1}{a(v_k^{(1)}, v_l^{(1)})} \right)
\times \tilde{C}_2^{(1)}(v^{(1)}_{\sigma(p_1+1)}) \cdots \tilde{C}_2^{(1)}(v^{(1)}_{\sigma(n_2)}) \tilde{C}_1^{(1)}(v^{(1)}_{\sigma(1)}) \cdots \tilde{C}_1^{(1)}(v^{(1)}_{\sigma(n_1)}) |0\rangle.
$$

(5.30)

From (4.24) and (4.25), the $gl(2|1)$ creation operators in the $F$-basis are given by

$$
\tilde{C}_2 = \sum_{i=1}^{N} b_{i1} E_{(i)}^{23} \otimes j \neq i \text{ diag}(a_{0j}, 2a_{0j}, 1_{(j)}),
$$

(5.31)

$$
\tilde{C}_1 = \sum_{i=1}^{N} b_{i1} E_{(i)}^{13} \otimes j \neq i \text{ diag}(2a_{0j}, a_{0j}a_{ij}^{-1}, 1_{(j)} + \cdots.
$$

(5.32)

Substituting these expressions for $\tilde{C}_i$ into (5.30), we have

$$
\phi_{n_1}^{(1), p_1}(v^{(1)}, \ldots, v_{n_2}^{(1)})
= \frac{1}{p_1!(n_2 - p_1)!} \sum_{i_1 < \cdots < i_{n_2}} B_{n_2, p_1}^{(1)} (v_{i_1}^{(1)}, \ldots, v_{i_2}^{(1)}, v_{i_1}^{(2)}, \ldots, v_{i_1}^{(1)} | v_{i_2}^{(1)}, \ldots, v_{i_{n_2}}^{(1)})
\times \prod_{j=i_{p_1+1}}^{i_{n_2}} E_{(j)}^{23} \prod_{j=i_1}^{i_{p_1}} E_{(j)}^{13} |0\rangle,
$$

(5.33)
where

\[ B_{n_2,p_1}^{(1)}(v_1^{(1)}, \ldots, v_n^{(1)}; v_1^{(2)}, \ldots, v_p^{(2)} | v_{i_1}, \ldots, v_{i_n}) = \sum_{\sigma \in S_{n_2}} \epsilon^*_{\sigma} \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n_2} \left( \frac{-\alpha(v_{\sigma(l)}^{(1)}, v_{\sigma(k)}^{(1)})}{\alpha(v_{\sigma(l)}^{(1)} - v_{\sigma(k)}^{(1)})} \right) \times (\phi_{n_2}^{(2),\sigma})^{11\ldots12\ldots2} B_{n_2-p_1}^{(2)}(v_1^{(1)}; \ldots, v_{n_2}^{(1)} | v_{i_1}, \ldots, v_{i_p}) \times B_{p_1}^{(1)}(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)} | v_{i_1}, \ldots, v_{i_p}). \]  

(5.34)

Denote by \( F^{(2)} \) the second-level nested \( gl(2) \) \( F \)-matrix. Applying the \( F \)-matrix to \( \phi_{n_2}^{(2)} \), one obtains

\[ (\phi_{n_2}^{(2)})^{11\ldots12\ldots2} = (t'(c))^{-1}(\phi_{n_2}^{(2)})^{11\ldots12\ldots2} \]  

(5.35)

with \( t'(c) = \prod_{j>i}^{p_1}(1+c_{ij}) \prod_{j>i=p_1+1}^{n_2}(1+c_{ij}) \). Therefore as before, the \( (\phi_{n_2}^{(2),\sigma})^{11\ldots12\ldots2} \) in (5.34) can be determined in the \( gl(2) \) \( F \)-basis.

### 5.2. The second-level nested \( gl(2) \) Bethe vector

The \( m=2 \), and \( n=0 \) limit of (3.4) gives the \( gl(2) \) \( F \)-matrix \( F^{(2)} \). In this \( F \)-basis, the \( n_2 \)-site simple generator \( E^{(12)} \) is given by

\[ \tilde{E}^{(12)} = \sum_{i=1}^{n_2} E_{(i)}^{(12)} \otimes_{j \neq i} \text{diag}(2, (2a_{ij})^{-1})_{(j)}, \]  

(5.36)

the diagonal element \( D^{(2)}(u) \) of the \( gl(2) \) monodromy matrix is

\[ \tilde{D}^{(2)}(u) = \otimes_{i=1}^{n_2} \text{diag}(a_{0i}, c_{0i})_{(0)} \]  

(5.37)

and the creation operator \( C^{(2)}(u) \) becomes

\[ \tilde{C}^{(2)}(u) = -\sum_{i=1}^{n_2} b_{0i} E_{(i)}^{(12)} \otimes_{j \neq i} \text{diag}(2a_{0j}, (2a_{ij})^{-1}c_{0j})_{(j)}. \]  

(5.38)

Applying \( F^{(2)} \) to the nested Bethe vector \( \phi_{n_2}^{(2)} \), we obtain

\[ \tilde{\phi}_{n_2}^{(2)}(v_1^{(2)}, \ldots, v_{n_2}^{(2)}) \equiv F^{(1)}_{1\ldots n_2} \phi^{(2)}(v_1^{(2)}, \ldots, v_{n_2}^{(2)}) = s_{n_3}(c) \tilde{C}^{(2)}(v_1^{(2)}) \tilde{C}^{(2)}(v_2^{(2)}) \ldots \tilde{C}^{(2)}(v_{n_3}^{(2)}) |0\rangle^{(2)}, \]  

(5.39)

where \( s_{n}(c) = \prod_{i=1}^{n}(1+c_{ij}) \) is from the action of \( F^{(2)} \) on the nested pseudo-vacuum state \( |0\rangle^{(1)} \). With the help of (5.38), the \( F \)-transformed nested Bethe vector is given by

\[ \tilde{\phi}_{n_2}^{(2)}(v_1^{(2)}, \ldots, v_{n_2}^{(2)}) = s_{n_3}(c) \tilde{C}^{(2)}(v_1^{(2)}) \tilde{C}^{(2)}(v_2^{(2)}) \ldots \tilde{C}^{(2)}(v_{n_3}^{(2)}) |0\rangle^{(2)} \]

\[ = s_{n_3}(c) \sum_{i_1 < \ldots < i_{n_3}} B_{n_3}^{(2)}(v_1^{(2)}, \ldots, v_{n_3}^{(2)} | v_{i_1}^{(1)}, \ldots, v_{i_{n_3}}^{(1)}) E_{(i_1)}^{(12)} \ldots E_{(i_{n_3})}^{(12)} |0\rangle^{(2)}, \]  

(5.40)
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where

\[
B_{n_3}^{(2)}(v_1^{(2)}, \ldots, v_{n_3}^{(2)}|v_1^{(1)}, \ldots, v_{n_3}^{(1)})
= \sum_{\sigma \in S_{n_3}} \prod_{k=1}^{n_3} (-b(v_k^{(2)}, v_{\sigma(k)}^{(1)})) \prod_{j \neq \sigma(1), \ldots, \sigma(k)}^{n_2} \frac{c(v_k^{(2)}, v_j^{(1)})}{2a(v_{\sigma(k)}^{(1)}, v_j^{(1)})} \prod_{l=k+1}^{m} 2a(v_k^{(2)}, v_{\sigma(l)}) \quad (5.41)
\]

Here we note that if we exchange the spectral parameter in (5.39), the \(gl(2)\) Bethe vector is invariant. Therefore substituting the representation of \(\tilde{\phi}^{(2)}\), i.e. \(B^{(2)}\), into (5.34), we may rewrite the nested \(gl(2|1)\) Bethe vector as

\[
\tilde{\phi}^{(1)}_{p_1}(v_1^{(1)}, \ldots, v_{n_2}^{(1)})
= \frac{s_{p_1}(c)}{p_1!(n_2 - p_1)!} \sum_{i_1 < \ldots < i_{n_2}} \prod_{j=1}^{i_1} E_{(j)}^{23} \prod_{j=i_1}^{i_{n_2}} E_{(j)}^{13} |0\rangle \quad (5.42)
\]

with

\[
B_{n_2,p_1}^{(1)}(v_1^{(1)}, \ldots, v_{n_2}^{(1)}, v_1^{(2)}, \ldots, v_{p_1}^{(2)}|v_1, \ldots, v_{n_2})
= \sum_{\sigma \in S_{n_2}} \prod_{k=1}^{n_2} \prod_{l=p_1+1}^{p_{k-1}+1} \left( \prod_{i=0}^{k} \frac{a(v_{\sigma(i)}^{(1)}, v_i^{(1)})}{1 + c_{\sigma(i)\sigma(j)}} \right)^{-1} \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n_2} \left( -a(v_{\sigma(i)}^{(1)}, v_k^{(1)}) \right) \prod_{k=1}^{n_2} B_{n_2,p_1}^{(2)}(v_1^{(2)}, \ldots, v_{p_1}^{(2)}|v_{\sigma(1)}^{(1)}, \ldots, v_{\sigma(p_1)}^{(1)})
\times B_{n_2-p_1}^{(2)}(v_{\sigma(p_1+1)}, \ldots, v_{\sigma(n_2)}^{(1)}|v_{p_1+1}, \ldots, v_{n_2})
\times B_{p_1}^{(2)}(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)}^{(1)}|v_1, \ldots, v_{p_1}) \quad (5.43)
\]

Having resolved the nested \(gl(2|1)\) Bethe vector, we now go back to the Bethe vectors of the \(gl(2|2)\) electronic model. By the exchange symmetry of the \(gl(2|1)\) Bethe vector (5.25), we represent the Bethe vector (5.44) in the \(F\)-basis by

\[
\tilde{\Phi}^{(p_1,p_2)}_{F}(v_1, \ldots, v_{n_1})
= \frac{s_{p_1}(c)}{(p_1)!^2((p_2 - p_1)!)^2(n_1 - p_2)!} \prod_{i_1 < \ldots < i_{n_1}} \prod_{j=1}^{i_{n_1}} E_{(j)}^{34} \prod_{j=i_{n_1}+1}^{i_{p_1}} E_{(j)}^{14} \prod_{j=i_1}^{i_{p_2}} E_{(j)}^{24} |0\rangle \quad (5.44)
\]
with
\[
B_{n_1,(p_1,p_2)}^{(0)}(v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_2}^{(1)}|z_{i_1}, \ldots, z_{i_{n_1}}) \\
= \sum_{\sigma \in S_{n_1}} \left( \prod_{i=0}^{p_1} \prod_{j=0}^{p_2} \left( 1 + c_{\sigma(i)\sigma(j)} \right) \right)^{-1} \times \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{p_2} \frac{a(v_{\sigma(l)}, z_{i_l})}{a(v_{\sigma(l)}, v_{\sigma(k)})} \times \prod_{k=1}^{p_2} \prod_{l=p_2+1}^{p_1} \frac{a(v_{\sigma(l)}, z_{i_l})}{a(v_{\sigma(l)}, v_{\sigma(k)})} \\
\times B_{p_2-p_1}^{(1)}(v_1^{(1)}, \ldots, v_{p_2}^{(1)}; v_1^{(2)}, \ldots, v_{p_1}^{(2)}|v_{\sigma(1)}, \ldots, v_{\sigma(i_{p_2})}) \\
\times B_{n-p_2}^{(1)}(v_{\sigma(p_2+1)}, \ldots, v_{\sigma(n_1)}|z_{i_{p_1+1}}, \ldots, z_{i_{n_1}}) \\
\times B_{p_1}^{(1)}(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)}|z_{i_1}, \ldots, z_{i_{p_1}}). \quad (5.45)
\]

6. The resolution of the $gl(m|n)$ nested Bethe vectors in the $F$-basis

In this section, we generalize the results in the previous section to the $gl(m|n)$ supersymmetric model. The procedure is similar to that of the $gl(2|2)$ case. Here we only give the final results.

Associated with the $(m + n)$-dimensional $gl(m|n)$ representation space, we have the orthogonal states $|j\rangle$ ($j = 1, 2, \ldots, m + n$) defined by
\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix}, \quad |m + n\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \end{pmatrix}. \quad (6.1)
\]

Consider the $gl(m|n)$ Bethe state vectors with quantum numbers $p_1, p_2 - p_1, \ldots, p_{m+n-2} - p_{m+n-3}, n_1 - p_{m+n-2}$, and $N - n_1$, which label the numbers of states $|1\rangle$, $|2\rangle, \ldots, |m + n\rangle$ in the Bethe state, respectively. Define the pseudo-vacuum state of the $N$-site system:
\[
|0\rangle = \otimes_{k=1}^{N} |m + n\rangle(k).
\]

Then the $gl(m|n)$ Bethe vector in the $F$-basis corresponding to the special quantum numbers, $\Phi_N^{(p_1, \ldots, p_{m+n-2})}(v_1, \ldots, v_{n_1})$, is given by
\[
\Phi_N^{(p_1, \ldots, p_{m+n-2})}(v_1, \ldots, v_{n_1}) \\
= \frac{\prod_{\alpha=1}^{m+n-1} s_{\alpha}(c)p_1!}{\prod_{\alpha=1}^{m+n-1}((p_{\alpha} - p_{\alpha-1})!)^{m+n-\alpha}} \times \sum_{i_1 < \cdots < i_{n_1}} B_{n_1,(p_1, \ldots, p_{m+n-2})}^{(0)} \\
\times (v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_1}^{(1)}, \ldots, v_{p_{m+n-2}}^{(1)}|z_{i_1}, \ldots, z_{i_{n_1}}) \\
\times \prod_{\alpha=1}^{m+n-1} \prod_{j=i_{\alpha-n-\alpha+1}}^{i_{\alpha-n-\alpha}} E^{m+n-\alpha}_{(j)} |0\rangle, \quad (6.2)
\]
where for $m = 1$

$$
B^{(0)}_{n_1,(p_1,\ldots,p_{m+n-2})}(v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_1}^{(1)}, \ldots, v_{p_{m+n-2}}^{(1)} | z_{i_1}, \ldots, z_{i_{n_1}}) = B_{p_1}^* \left( v_1^{(n-1)}, \ldots, v_{p_1}^{(n-1)} | v_{\sigma^2(1)}^{(n-2)}, \ldots, v_{\sigma^2(p_1)}^{(n-2)} \right)
$$

$$\times \prod_{\beta=2}^{n-1} \sum_{\sigma^3 \in S_{\beta}} \left( \prod_{\sigma^3(j) > \sigma^3(i)=1} (1 + c_{\sigma^3(i)\sigma^3(j)}) \right)^{-1}
$$

$$\times \prod_{\alpha=1}^{p} \prod_{k=1}^{p_{\alpha+1}} a \left( v_{\sigma^3(l)}^{(n-\beta)}, v_{\sigma^3(k)}^{(n-1-\beta)} \right)
$$

$$\times \prod_{\gamma=2}^{\beta} B_{p_{\gamma-1}}^* \left( v^{(n-\theta)}_{\sigma^3(p_{\gamma-1}+1)}, \ldots, v^{(n-\theta)}_{\sigma^3(p_{\gamma})} | v^{(n-1-\theta)}_{\sigma^3(p_{\gamma}+1)}, \ldots, v^{(n-1-\theta)}_{\sigma^3(p_{\gamma}+1)} \right)
$$

(6.3)

and for $m > 1$

$$
B^{(0)}_{n_1,(p_1,\ldots,p_{m+n-2})}(v_1, \ldots, v_{n_1}; v_1^{(1)}, \ldots, v_{p_1}^{(1)}, \ldots, v_{p_{m+n-2}}^{(1)} | z_{i_1}, \ldots, z_{i_{n_1}}) = B_{p_1}^{**} \left( v_1^{(m+n-2)}, \ldots, v_{p_1}^{(m+n-2)} | v^{(m+n-3)}_{\sigma^2(1)}, \ldots, v^{(m+n-3)}_{\sigma^2(p_1)} \right)
$$

$$\times \prod_{\beta=2}^{m-1} \sum_{\sigma^3 \in S_{\beta}} \left( \prod_{\epsilon=0}^{\beta-1} \prod_{\sigma^3(j) > \sigma^3(i)=p_{\alpha+1}} (1 + c_{\sigma^3(i)\sigma^3(j)}) \right)^{-1}
$$

$$\times \prod_{\alpha=1}^{p} \prod_{k=1}^{p_{\alpha+1}} a \left( v_{\sigma^3(l)}^{(m+n-1-\beta)}, v_{\sigma^3(k)}^{(m+n-1-\beta)} \right)
$$

$$\times \prod_{\theta=2}^{m+n-1} \prod_{\alpha=1}^{p_{\alpha-1}} B_{p_{\alpha-1}}^{**} \left( v^{(m+n-1-\theta)}_{\sigma^3(p_{\alpha-1}+1)}, \ldots, v^{(m+n-1-\theta)}_{\sigma^3(p_{\alpha})} | v^{(m+n-2-\theta)}_{\sigma^3(p_{\alpha})}, \ldots, v^{(m+n-2-\theta)}_{\sigma^3(p_{\alpha})} \right)
$$

$$\times \prod_{\beta=m}^{m+n-1} \left[ \sum_{\sigma^3 \in S_{\beta}} \exp \left\{ \delta_{\beta,m} \ln c_{1,p_{\beta}}^{\sigma^3} \right\} \right]
$$

$$\times \left( \prod_{\epsilon=0}^{m-1} \prod_{\sigma^3(j) > \sigma^3(i)=p_{\alpha+1}} (1 + c_{\sigma^3(i)\sigma^3(j)}) \right)^{-1}
$$

$$\times \prod_{\alpha=1}^{p} \prod_{k=1}^{p_{\alpha+1}} a \left( v_{\sigma^3(l)}^{(m+n-1-\beta)}, v_{\sigma^3(k)}^{(m+n-1-\beta)} \right)
$$

$$\times \prod_{\gamma=2}^{m+n-1} \prod_{\alpha=1}^{p_{\gamma-1}} a \left( v_{\sigma^3(l)}^{(m+n-1-\beta)}, v_{\sigma^3(k)}^{(m+n-1-\beta)} \right)
$$

(6.3)
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\[ \times \prod_{\alpha=m}^{m+n-2} \prod_{k=1}^{p_{\alpha+1}} a \left( v^{(m+n-1-\beta)}_{\alpha\gamma(l)}, v^{(m+n-2-\beta)}_{\sigma^{\alpha+1}(l)} \right) \]

\[ \times \prod_{\theta=m}^{\beta} \left( \prod_{\alpha=1}^{m} B_{p_{\alpha} - p_{\alpha-1}} \left( v^{(m+n-1-\theta)}_{\sigma^{\alpha}(p_{\alpha}+1)} \right) \right) \]

\[ \times \prod_{\gamma=m+1}^{\theta} B_{p_{\gamma} - p_{\gamma-1}} \left( v^{(m+n-1-\theta)}_{\sigma^{\gamma}(p_{\gamma}+1)} \right) \]

\[ \times \prod_{j=m}^{\theta} (v^{(m+n-1-\theta)}_{\sigma^{j}(p_{j}+1)}, \ldots, v^{(m+n-2-\theta)}_{\sigma^{j+1}(p_{j})}) \]  

(6.4)

with the conventions \( p_0 = 0, p_{m+n-1} = n+1 \), \( v^{(0)} = v, v^{(-1)} = z_i, \sigma^{m+n} = 1 \) and \( B_n (v^{(l)}, \ldots, v^{(l-1)}) = B_n \left( v^{(l)}_1, \ldots, v^{(l)}_n \right) \).

7. Discussion

In this paper, we have constructed the factorizing \( F \)-matrices for the \( gl(m|n) \)-invariant fermion model. In the basis provided by the \( F \)-matrix (the \( F \)-basis), the monodromy matrix and the creation operators take completely symmetric forms. Moreover, we have obtained a symmetric representation of the Bethe vector of the system.

The authors of [19] derived a formula that expresses the local spin and field operators of fundamental graded models in terms of the elements of the monodromy matrix. In particular they reconstructed the local operators \( (E^{ij}) \) in terms of operators figuring in the \( gl(m|n) \) monodromy matrix. This together with the results of the present paper in the \( F \)-basis should enable one to get the exact representations of form factors and correlation functions of the supersymmetric fermion models. These are under investigation and results will be reported elsewhere.

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