ROBUST ATTRACTORS FOR A PERTURBED NON-AUTONOMOUS EXTENSIBLE BEAM EQUATION WITH NONLINEAR NONLOCAL DAMPING

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ABSTRACT. The paper investigates the attractors and their robustness for a perturbed non-autonomous extensible beam equation with nonlinear nonlocal damping. We prove that the related evolution process has a finite-dimensional pullback attractor $\mathcal{A}_\kappa$ and a pullback exponential attractor $\mathcal{M}_{exp}$ for each extensibility parameter $\kappa \in [0,1]$, respectively, and both of them are stable on the perturbation $\kappa$. In particular, these stability holds for the global and exponential attractors when the non-autonomous dynamical system degenerates to an autonomous one, so the results of the paper deepen and extend those in recent literatures [22, 33].

1. Introduction. In this paper, we investigate the existence of pullback attractors and pullback exponential attractors and their robustness on the perturbation to the following non-autonomous extensible beam equation with nonlinear nonlocal damping

$$u_{tt} + \Delta^2 u - \kappa \phi(\|\nabla u\|^2)\Delta u + \sigma(\|\nabla u\|^2)g(u_t) + f(u) = h, \quad (x,t) \in \Omega \times (\tau, \infty),$$

(1)

where $\kappa \in [0,1]$ is an extensibility parameter, which is related to the extensibility of the beam, $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with the smooth boundary $\partial \Omega$, $h = h(x,t)$ is a time-dependent external load, $\| \cdot \|$ stands for the norm in $L^2(\Omega)$, and the assumptions on $\phi(s)$, $\sigma(s)$, $g(u_t)$, $f(u)$ and $h$ will be specified later, on which we consider the hinged boundary condition

$$u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial \Omega \times (\tau, \infty),$$

(2)

and the initial condition

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \quad x \in \Omega.$$  

(3)

Eq. (1) can be seen as a special case of more general model

$$u_{tt} + \Delta^2 u - \phi(\|\nabla u\|^2)\Delta u = F(x,t,u,u_t)$$

(4)

which was proposed by Berger [2] as a simplification of the von Karman plate equation which describes large deflection of plate (cf. [10]). When $\phi(s) = a +
bs, with $a, b > 0$, and $F = 0$, the corresponding equation had been introduced by Woinowsky-Krieger [32] in one-dimensional case in 1950 as a model for the transverse motion of an extensible beam.

Eq. (1) covers a few kinds of essential dissipations usually appearing in the literatures, say, the linear one $u_t$, the nonlinear one $g(u_t)$ and nonlocal Kirchhoff one $\sigma(\|\nabla u\|^2)u_t$.

For the linear damping case, the first result on the attractors was obtained by Hale [19] and then by Eden and Milani [14]. In those papers, they investigated the existence of global and exponential attractors and showed the regularity of the attractors. Replacing the linear damping $u_t$ by a structural one $(-\Delta)^\theta u_t$, $\theta \in [0, 1]$, Zelati [35] established the existence of global and exponential attractors with optimal regularity for the corresponding model equation. For the related study on this topic, one can see [35] and references therein.

For the nonlocal Kirchhoff damping $\sigma(\|\nabla u\|^2)u_t$, to our best knowledge it was first introduced by Lange and Perla Menzala [24] in the research of plate models and subsequently studied by Cavalcanti et al [4] in a viscoelastic context. Replacing this kind of dissipation by more general nonlocal structural one $\sigma(\|\nabla u\|^2)(-\Delta)^\theta u_t$, $\theta \in [0, 1]$, Silva and Narciso [21] showed the well-posedness and the existence of finite-dimensional global and exponential attractors for the corresponding model equation. For the related work on this topic, one can see [12].

For the natural nonlinear damping $g(u_t)$, Patcheu [27] investigated the existence and the decay property of the global solutions to the corresponding model equation, with $f(u) = 0$. Chueshov and Lasiecka [10] further established the existence of finite-dimensional global attractor to Eq. (1), with $\Omega \subset \mathbb{R}^2, \phi(s)$ being a linear function and without $f(u)$. Under the assumptions that the nonlinearities $f(u)$ and $g(u_t)$ are of subcritical growth, Cavalcanti et al [5] obtained the well-posedness of corresponding Eq. (1), with $h(x, t) \equiv h(x)$ and clamped boundary condition. Based on the works of [5], Ma and Narciso [25] further established the existence of global attractor. Recently, under the relaxed conditions that the growth exponent $p$ and $q$ of the nonlinearities $f(u)$ and $g(u_t)$ are permitted up to critical range: $2 \leq p \leq q \leq \frac{N+4}{(N-4)}$, Yang [33] proved the existence of finite-dimensional global and exponential attractor of Eq. (1), with $\sigma(s) \equiv 1$ and $h(x, t) \equiv h(x)$.

More recently, Silva and Narciso [22] presented a first analysis on the well-posedness and longtime dynamics of the autonomous model (1) (i.e., $h(x, t) \equiv h(x)$), which contains a nonlocal nonlinear damping $\sigma(\|\nabla u\|^2)g(u_t)$. They established the existence of finite-dimensional global attractors and exponential attractors for the related autonomous dynamical system. Their results cover the above mentioned a few classes of dissipations, perfect and generalize those related results existed in the literatures.

However, there still exist some questions. For example, extensibility parameter $\kappa$ is an important index of this kinds of models, say, when $\kappa = 0$, the extensible model (1) degenerates to usual beam equation, and $\kappa$ is usually an approximation of reality. What about the stability of the global and exponential attractors on the perturbation $\kappa \in [0, 1]$? Instead global and exponential attractor, pullback attractor and pullback exponential attractor are two basic concepts in the study of the longtime dynamics of non-autonomous dynamical system. For more complicated non-autonomous problem (1)-(3), what about the existence of the pullback and pullback exponential attractors? what about their stability on the perturbation $\kappa \in [0, 1]$? These questions are still unsolved.
To be more precise, a process acting on the Banach space $E$ is a two-parametrical family of operators $\{U(t, \tau) : E \to E | t, \tau \in \mathbb{R}, t \geq \tau\}$ satisfying
$$U(t, s)U(s, \tau) = U(t, \tau), \quad U(\tau, \tau) = I \text{ (identity operator)}, \quad t, s, \tau \in \mathbb{R}, \quad t \geq s \geq \tau.$$ 

**Definition 1.1.** A family of nonempty compact subsets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ in $E$ is said to be a pullback attractor of the process $\{U(t, \tau)\}$ if it is invariant, i.e., $U(t, s)\mathcal{A}(s) = \mathcal{A}(t)$, $t \geq s$, and it pullback attracts all bounded subsets of $E$, i.e., for every bounded subset $D \subset E$ and $t \in \mathbb{R},$
$$\lim_{s \to +\infty} \text{dist}_E\{U(t, t-s)D, \mathcal{A}(t)\} = 0.$$ 

**Definition 1.2.** A family of nonempty compact subsets $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ is said to be a pullback exponential attractor of the process $\{U(t, \tau)\}$ acting on the Banach space $E$ if
(i) it is semi-invariant, i.e., $U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$;
(ii) the fractal dimension in $E$ of the sections $\mathcal{M}(t)$, $t \in \mathbb{R}$ is uniformly bounded, i.e.,
$$\sup_{t \in \mathbb{R}} \dim f(\mathcal{M}(t), E) < +\infty;$$
(iii) it pullback attracts every bounded subset $B$ in $E$ at an exponential rate, i.e.,
$$\text{dist}_E\{U(t, t-s)B, \mathcal{M}(t)\} \leq C(\|B\|_E)e^{-\beta s},$$
for some $\beta > 0$, where $t, s \in \mathbb{R}, s \geq 0, \|B\|_E = \sup_{x \in B} \|x\|_E$.

There have been extensive researches on the theory of pullback attractors, pullback exponential attractors and their applications to various types of evolution equations arising from mathematical physics (cf. [3, 6, 23, 29] and references therein). The question on the stability of attractors with respect to perturbations of a dynamical system has also been widely concerned (cf. [1, 7, 8, 15, 17, 18, 31, 34] and references therein).

Generally speaking, the existence of a pullback exponential attractor implies the finite dimensionality of the sections of the pullback attractor. However, when the assumption is weaker and not enough to guarantee the existence of the former, how to prove the finite dimensionality of the latter? Based on recently developed quasi-stability estimate method, Chueshov [13] proposed an abstract criterion on the finite dimensionality of an invariant compact set for autonomous discrete dynamical system. Motivated by his idea in [13], we give a criterion on this topic for non-autonomous dynamical system and use it to show that under more relaxed conditions, the fractal dimensions of the sections of the pullback attractors are uniformly bounded.

We emphasize that the notion and method of quasi-stability estimate, which was originally introduced by Chueshov and Lasiecka in [9], have had a lot of consequences, including the existence, finite-dimensionality and smoothness of global attractors, and an existence of exponential attractor and so on (cf. [10, 11]). This method and framework and the idea behind this notion have been extensively used to study the longtime dynamics of various evolution equations such as Von Karman models (cf. [11]), Kirchhoff wave models (cf. [34]), extensible beam models (cf. [21, 22, 25]) and the systems with delay/memory terms (cf. [16]) and so on.

The main results of the present paper are that
(i) under the conditions that either Assumption 2.1 or Assumptions 2.2 (see below) hold, we show that the evolution process related to problem (1)-(3) has
a pullback attractor $\mathcal{A}_\kappa$ for each $\kappa$, which is upper semicontinuous on $\kappa \in [0, 1]$, especially when $h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{2}{a}}(\mathbb{R}; L^{1+\frac{2}{a}})$, $\mathcal{A}_\kappa$ is continuous on $\kappa \in I \subset [0, 1]$, where $I$ is a dense subset of $[0, 1]$; (see Theorem 2.3 and Theorem 2.5)

(ii) under Assumption 2.2, by using a criterion recently established in [34] we show that when $h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{2}{a}}(\mathbb{R}; L^{1+\frac{2}{a}})$, the process has a pullback exponential attractor $\mathcal{M}^e_{\kappa}$ for each $\kappa$ and it is Hölder continuous on the perturbation $\kappa \in [0, 1]$; (see Theorem 2.6)

(iii) under Assumption 2.2, with $h \in L^2(-\infty, t; L^2) \cap L^{1+\frac{2}{a}}(-\infty, t; L^{1+\frac{2}{a}})$, which is not enough to ensure the existence of the pullback exponential attractor, we give a criterion and use it to prove the uniformly finite dimensionality of the sections of the pullback attractor $\mathcal{A}_\kappa$, (see Theorem 2.7)

In particular, when $h(x, t) \equiv h(x)$, the above mentioned results hold for the related autonomous dynamical system (see Corollary 2.8), so the results of the paper deepen and extend those in [22, 33].

The paper is organized as follows. In Section 2, we state some assumptions used throughout the paper and the main results. In Section 3, we show some estimates of the weak solutions which will play key roles for us to discuss the longtime dynamics of the system later. In Section 4, we discuss the existence of the pullback attractors and their continuity on the extensibility parameter $\kappa \in [0, 1]$. In Section 5, we investigate the existence of pullback exponential attractors and their Hölder continuity on the perturbation $\kappa$. In Section 6, we give a criterion on the fractal dimension of the pullback attractors and use it to prove the finite dimensionality of the sections of the pullback attractors in more relaxed assumptions. In particular, when $h(x, t) \equiv h(x)$, the non-autonomous dynamical system degenerates to an autonomous one, the operator $S(t) = U(t + \tau, \tau)$ becomes a semigroup acting on the phase space, and as a consequence, the family of robust pullback and pullback exponential attractors become a robust global attractor and a robust exponential attractor, respectively.

2. Assumptions and main results. We begin with the following abbreviations

$L^p = L^p(\Omega)$, $H^k = W^{k, 2}(\Omega)$, $H_0^k = W_0^{k, 2}(\Omega)$, $\| \cdot \|_p = \| \cdot \|_{L^p}$, $\| \cdot \| = \| \cdot \|_{L^2}$, with $p \geq 1$. The notation $(\cdot, \cdot)$ for the $L^2$ inner product will also be used for the notation of paring between dual spaces. The sign $H_1 \hookrightarrow H_2$ denotes that the functional space $H_1$ continuously embeds into $H_2$ and $H_1 \hookrightarrow \hookrightarrow H_2$ denotes that $H_1$ compactly embeds into $H_2$. We denote the space $V_2 = H^2 \cap H_0^1$ and the operator $A : V_2 \to V_2'$

$$(Au, v) = (\Delta u, \Delta v), \quad \forall u, v \in V_2,$$

with $D(A) = \{ u \in H^4\|u, \Delta u \in H_0^1 \}$. Thus, we can define the powers $A^s$ of $A$ ($s \in \mathbb{R}$), and the Hilbert spaces $V_s = D(A^{s/2})$ with the scalar products and norms

$$(u, v)_s = (A^\frac{s}{2} u, A^\frac{s}{2} v), \quad \| u \|_{V_s} = \| A^\frac{s}{2} u \|,$$

respectively. Obviously,

$$\| u \|_{V_2} = \| A^{\frac{1}{2}} u \| = \| \Delta u \|, \quad \| u \|_{V_1} = \| A^{\frac{3}{2}} u \| = \| \nabla u \|;$$

$$\lambda_1 \| u \|^2 \leq \| u \|_{V_2}^2, \quad \sqrt{\lambda_1} \| u \|_{V_1}^2 \leq \| u \|_{V_2}^2, \quad \forall u \in V_2,$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $A$. We denote the phase space

$${\mathcal{H}} = V_2 \times L^2$$
equipped with the norm 
\[ \| (u, v) \|_{\mathcal{H}}^2 = \| u \|_{L^2}^2 + \| v \|_{L^2}^2, \quad \forall (u, v) \in \mathcal{H}. \]

Rewriting Eq. (1) at an abstract level, we get the Cauchy problem equivalent to problem (1)-(3):
\[ u_{tt} + Au + \alpha \phi(\| A^t u \|_2) A^t u + \sigma(\| A^t u \|_2) g(u) + f(u) = h, \quad (5) \]
\[ u(\tau) = u_0, \quad u_t(\tau) = u_1. \quad (6) \]

**Assumption 2.1.** (H1) \( g \in C^1(\mathbb{R}) \) satisfying \( g(0) = 0 \) and \( K_0 |s|^{q-1} \leq g'(s) \leq C(1 + |s|^{q-1}), \quad \forall s \in \mathbb{R}, \)
where \( K_0 > 0, C > 0, \) and \( q \geq 1 \) if \( 1 \leq N \leq 4; 1 \leq q \leq \frac{N+4}{N-4} \) if \( N \geq 5; \)

(H2) \( f \in C^1(\mathbb{R}) \) satisfying \( f(0) = 0 \) and
\[ |f'(s)| \leq C(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}, \]
\[ -C_1 - \frac{\alpha_1}{2} s^2 \leq F(s) \leq f(s) s + \frac{\alpha_1}{2} s^2, \quad \forall s \in \mathbb{R}, \]
where \( F(s) := \int_0^s f(r) dr, \) \( C_1 \geq 0, 0 \leq \alpha_1 < \lambda_1, \) and \( p \geq 1 \) if \( 1 \leq N \leq 4; 1 \leq p < \frac{N}{N-4} \) if \( N \geq 5; \)

(H3) \( \phi, \sigma \in C^1(\mathbb{R}^+) \) with
\[ \sigma(s) \geq \sigma_0, \quad \phi(s) \geq -\alpha_2, \quad \int_0^s \phi(r) dr \leq 2\phi(s)s + \alpha_2 s, \quad \forall s \in \mathbb{R}^+, \]
where \( \sigma_0 > 0, 0 \leq \alpha_2 < \sqrt{\lambda_1} \) and \( \alpha := 1 - \left( \frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\sqrt{\lambda_1}} \right) > 0; \)

(H4) either
\[ h \in L^2(-\infty, t; L^2) \cap L^{1+\frac{1}{2}}(-\infty, t; L^{1+\frac{1}{2}}), \quad \forall t \in \mathbb{R}, \quad (7) \]
or
\[ h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{1}{2}}(\mathbb{R}; L^{1+\frac{1}{2}}). \quad (8) \]

**Assumption 2.2.** Let Assumption 2.1 be valid, with \( g'(0) > 0 \) and replace condition (H2) there by

(H2') \( f \in C^2(\mathbb{R}) \) satisfying \( f(0) = 0 \) and
\[ f'(s) > -\alpha_1, \quad |f''(s)| \leq C(1 + |s|^{p-2}), \]
\[ F(s) \leq f(s)s + \frac{\alpha_1}{2} s^2, \quad \forall s \in \mathbb{R}, \]
where \( 0 \leq \alpha_1 < \lambda_1, \) and \( p \geq 2 \) if \( 1 \leq N \leq 4; 2 \leq p \leq q \leq \frac{N+4}{N-4} \) if \( 5 \leq N \leq 12. \)

**Remark 1.** (i) Condition (H1) implies that
\[ (g(s) - g(r))(s - r) \geq \frac{K_0}{2q} (|s|^{q-1} + |r|^{q-1})|s - r|^2 \geq \frac{K_0}{q} |s - r|^{q+1}, \quad \forall s, r \in \mathbb{R}. \quad (9) \]

(ii) Condition (H2') implies
\[ F(s) \geq -\frac{\alpha_1}{2} s^2, \quad \forall s \in \mathbb{R}. \]

(iii) Assumption 2.1 implies that \( V_2 \hookrightarrow L^{q+1}, V_2 \hookrightarrow L^{2p}, \) and there is no relations between \( q \) and \( p. \) While Assumption 2.2 implies that \( V_2 \hookrightarrow L^{q+1} \hookrightarrow L^{p+1}. \)
Theorem 2.1. (Well-posedness) Let either Assumption 2.1 or Assumption 2.2 be valid. Then for any \((u_0, u_1) \in \mathcal{H}, \kappa \in [0, 1]\), problem (5)-(6) possesses a unique weak solution \(u^\kappa\) with
\[
(u^\kappa, u^\kappa_t) \in L^\infty(\tau, T; \mathcal{H}) \cap C([\tau, T]; \mathcal{H}), u^\kappa_t \in L^{1+q}(\tau, T; L^{1+q}), u^\kappa_{tt} \in L^{1+\frac{1}{2}}(\tau, T; L^{1+\frac{1}{2}})
\]
for any \(T > \tau\). Moreover, the following Lipschitz stability holds:
\[
\|z(t) - z(t)\|_\mathcal{H} \leq C_l \|v(\tau) - v(\tau)\|_\mathcal{H}, \quad \forall t \in [\tau, T],
\]
where \(z(t) = u^\kappa(t) - v(\tau), u^\kappa\) and \(v(\tau)\) are two weak solutions of problem (5)-(6) corresponding to parameter \(\kappa \in [0, 1]\), and
\[
C_l = C_l(\|u^\kappa(\tau), u^\kappa_t(\tau)\|_\mathcal{H}, \|v(\tau), v_t(\tau)\|_\mathcal{H}, T)
\]
is a positive constant.

Repeating the similar arguments as in [22, 33] except for the treatment for \(h(x, t)\) (\(h(x, t) \equiv h(x)\) there), one easily proves Theorem 2.1. We omit proving process here.

Under the assumptions of Theorem 2.1, for each \(\kappa \in [0, 1]\), we define the solution operator \(U_\kappa(t, \tau) : \mathcal{H} \to \mathcal{H},\)
\[
U_\kappa(t, \tau)(u_0, u_1) = (u^\kappa(t), u^\kappa_t(t)), \quad \forall t \geq \tau, \quad \forall \tau \in \mathbb{R}, \quad (u_0, u_1) \in \mathcal{H},
\]
where \(u^\kappa\) is the weak solution of problem (5)-(6). By Theorem 2.1, the family of solution operators \(\{U_\kappa(t, \tau)\}_{t \geq \tau, \tau \in \mathbb{R}}\) constitutes a continuous evolution process acting on \(\mathcal{H}\) for each \(\kappa \in [0, 1]\). Moreover, the process \(U_\kappa(t, \tau)\) is also continuous on both \(t\) and \(\tau\).

Now, we state the main results of the paper.

Theorem 2.2. (Existence and upper semi-continuity of the pullback attractors) Let either Assumption 2.1 or Assumption 2.2 be valid. Then the process \(U_\kappa(t, \tau)\) has in \(\mathcal{H}\) a pullback attractor \(\mathcal{A}_\kappa = \{A_\kappa(t)\}_{t \in \mathbb{R}}\) for each \(\kappa \in [0, 1]\). Moreover, for any \([a, b] \subset \mathbb{R}\), the union
\[
\bigcup_{\kappa \in [0, 1]} \bigcup_{t \in [a, b]} A_\kappa(t) \text{ is precompact in } \mathcal{H},
\]
and the mapping \(\kappa \mapsto A_\kappa(t)\) is upper semi-continuous in the following sense:
\[
\lim_{\kappa \to \kappa_0} \sup_{t \in [a, b]} \text{dist}_\mathcal{H} (A_\kappa(t), A_{\kappa_0}(t)) = 0, \quad \forall \kappa_0 \in [0, 1].
\]

Definition 2.3. Let \(X\) be a complete metric space. The subset \(A\) of \(X\) is said to be nowhere dense if its closure contains no non-empty open sets. The set \(A\) is said to be residual if its complement is the countable union of nowhere dense sets.

Obviously, if \(A\) is a residual subset of \(X\), then \(A\) is dense in \(X\).

Theorem 2.4. (Continuity of the pullback attractors) Let either Assumption 2.1 or Assumption 2.2 be valid, with \(h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{1}{2}}(\mathbb{R}; L^{1+\frac{1}{2}})\). Then there exists a residual subset \(I \subset [0, 1]\), which is dense in \([0, 1]\), such that
\[
\lim_{\kappa \to \kappa_0} \text{dist}^\text{symm}_\mathcal{H} (A_\kappa(t), A_{\kappa_0}(t)) = 0, \quad \forall t \in \mathbb{R}, \quad \forall \kappa_0 \in I.
\]

Theorem 2.5. (Hölder continuity of the pullback exponential attractors) Let Assumption 2.2 be valid, with \(h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{1}{2}}(\mathbb{R}; L^{1+\frac{1}{2}})\). Then the process \(U_\kappa(t, \tau)\) has in \(\mathcal{H}\) a pullback exponential attractor \(\mathcal{M}_\kappa^{\exp} = \{M_\kappa(t)\}_{t \in \mathbb{R}}\) for each \(\kappa \in [0, 1]\). Moreover, there exist positive constants \(C\) and \(\gamma: 0 < \gamma < 1\) such that
\[
\sup_{t \in \mathbb{R}} \text{dist}^\text{symm}_\mathcal{H} (M_\kappa(t), M_{\kappa_0}(t)) \leq C|\kappa - \kappa_0|^\gamma, \quad \forall \kappa, \kappa_0 \in [0, 1].
\]
Theorem 2.6. (Finite dimensionality of the pullback attractors) Let Assumption 2.2 be valid. Then the fractal dimensions of the sections of the pullback attractors $\mathcal{A}_\kappa$ given by Theorem 2.2 are uniformly bounded, i.e.,

$$\sup_{\kappa \in [0,1]} \sup_{t \in \mathbb{R}} \dim(\mathcal{A}_\kappa(t), \mathcal{H}) < +\infty.$$ 

When $h(x,t) \equiv h(x)$, the nonautonomous evolution equation (5) degenerates to an autonomous one, the related evolution process $\{U_\kappa(t,\tau)\}_{t \geq \tau}$ degenerates a semigroup $\{S_\kappa(t)\}_{t \geq 0}$, with $S_\kappa(t) = U_\kappa(t + \tau, \tau)$, for each $\kappa \in [0,1]$, the pullback attractor $\mathcal{A}_\kappa$ and the pullback exponential attractor $\mathcal{M}_{ex}^\kappa$ become a global one $\mathcal{A}_\kappa$ and an exponential one $\mathcal{M}_{ex}$, respectively. Moreover, the above mentioned continuity on perturbation $\kappa$ still holds for the global and exponential attractors. In the concrete, we have

Corollary 1. (i) Let either Assumption 2.1 or Assumption 2.2 be valid, replacing condition $(H_4)$ there by $h(x,t) \equiv h(x) \in L^2(\Omega)$. Then the solution semigroup $S_\kappa(t)$ associated with problem (5)-(6) has in $\mathcal{H}$ a global attractor $\mathcal{A}_\kappa$ for each $\kappa \in [0,1]$, and the following upper semicontinuity and continuity on perturbation $\kappa$ hold:

$$\lim_{\kappa \to \kappa_0} \text{dist}_{\mathcal{H}}(\mathcal{A}_\kappa, \mathcal{A}_{\kappa_0}) = 0, \quad \forall \kappa_0 \in [0,1],$$

$$\lim_{\kappa \to \kappa_0} \text{dist}_{\mathcal{H}}^{symm}(\mathcal{A}_\kappa, \mathcal{A}_{\kappa_0}) = 0, \quad \forall \kappa_0 \in I,$$

where the subset $I$ is as shown in Theorem 2.4.

(ii) Let Assumption 2.2 be valid, with $h(x,t) \equiv h(x) \in L^2(\Omega)$. Then the solution semigroup $S_\kappa(t)$ has in $\mathcal{H}$ an exponential attractor $\mathcal{M}_\kappa$ for each $\kappa \in [0,1]$, and the following Hölder continuity on perturbation $\kappa$ holds:

$$\text{dist}_{\mathcal{H}}^{symm}(\mathcal{M}_\kappa, \mathcal{M}_{\kappa_0}) \leq C|\kappa - \kappa_0|^\gamma, \quad \forall \kappa, \kappa_0 \in [0,1]$$

for some $\gamma \in (0,1)$.

The proofs of Theorems 2.2 and 2.4 are given in Section 4. The proofs of Theorem 2.5 and Theorem 2.6 are given in Section 5 and Section 6, respectively.

3. Some key estimates. Our goal in this section is to provide some estimates to the solutions of problem (5)-(6) which will play key roles for us to discuss the longtime dynamics of the system later.

Proposition 1. (Dissipative estimate) Let the assumptions of Theorem 2.1 be valid. Then for any bounded subset $B$ of $\mathcal{H}$ and $t \in \mathbb{R}$, there exists a positive constant $\epsilon = \epsilon(B, t) > 0$ such that

$$\|U_\kappa(t, t - \tau)\xi_0\|^2_{\mathcal{H}} \leq C(B)e^{-\epsilon\tau} + C|\Omega| + C\int_{t-\tau}^t e^{-\epsilon(r)} \left(\|h(r)\|^2 + \|h(r)\|^{1 + \frac{4}{q}}\right) dr$$

for any $\tau \geq 0$, $\xi_0 \in B$ and $\kappa \in [0,1]$, where $C(B)$ is a positive constant depending only on $B$.

Proof. For the given $t \in \mathbb{R}$, $\tau \geq 0$ and $\xi_0 \in B$, let

$$(v(s), u_\kappa(s)) = (u(t - \tau + s), u_\kappa(t - \tau + s)) = U_\kappa(t - \tau + s, s)\xi_0, \quad s \in [0, \tau].$$

Then $v$ solves

$$v_{ss} + Av + \kappa \phi (\|A^{\frac{1}{2}} v\|^2) A^{\frac{1}{2}} v + \sigma (\|A^{\frac{1}{2}} v\|^2) g(v_s) + f(v) = h_1, \quad s \in (0, \tau],$$

$$v(0), v_1(0) = \xi_0, \quad (14)$$

$$v(t, 0) + (u(t, \tau)), \quad (15)$$
with \( h_1(s) = h(t - \tau + s) \). Using the multiplier \( v_s \) in Eq. (14) yields

\[
\frac{d}{ds} E_\kappa(v, v_s) + \sigma(\|A^\frac{1}{2}v\|^2) \int_\Omega g(v_s)v_sdx = \int_\Omega h_1v_sdx, \tag{16}
\]

where

\[
E_\kappa(v, v_s) = \frac{1}{2}\|v_s\|^2 + \|A^\frac{1}{2}v\|^2 + \frac{\kappa}{2} \int_0^\|A^\frac{1}{2}v\|^2 \phi(r)dr + \int_\Omega F(v)dx.
\]

By Assumption either 2.1 or 2.2 and Remark 1,

\[
\frac{\kappa}{2} \int_0^\|A^\frac{1}{2}v\|^2 \phi(r)dr + \int_\Omega F(v)dx \geq -\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{2\lambda_1}\right)\|A^\frac{1}{2}v\|^2 - C_1|\Omega|,
\]

\[
E_\kappa(v, v_s) \geq \frac{1}{2} \left(\|v_s\|^2 + \alpha\|A^\frac{1}{2}v\|^2\right) - C_1|\Omega|,
\]

\[
\sigma(\|A^\frac{1}{2}v\|^2) \int_\Omega g(v_s)v_sdx \geq \frac{\sigma_0K_0}{q} \|v_s\|_{q+1}^{g+1},
\]

\[
\int h_1v_sdx \leq \frac{\sigma_0K_0}{2q} \|v_s\|_{g+1}^{q+1} + C\|h_1\|_{1+\frac{1}{q}}^{1+\frac{1}{q}}.
\]

Therefore,

\[
\frac{d}{dt} E_\kappa(v, v_s) + \frac{\sigma_0K_0}{2q} \|v_s\|_{g+1}^{q+1} \leq C\|h_1\|_{1+\frac{1}{q}}^{1+\frac{1}{q}},
\]

\[
E_\kappa(v, v_s) + \frac{\sigma_0K_0}{2q} \int_0^s \|v_s(r)\|_{g+1}^{q+1}dr \leq E_\kappa(\xi_0) + C \int_0^s \|h_1(r)\|_{1+\frac{1}{q}}^{1+\frac{1}{q}}dr, \quad s \in [0, \tau],
\]

which means

\[
\|(v(s), v_s(s))\|_\mathcal{H} + \int_0^s \|v_s(r)\|_{g+1}^{q+1}dr \leq \mu(B, t), \quad s \in [0, \tau],
\]

where

\[
\mu(B, t) := \begin{cases} C(B) \left[1 + \int_{-\infty}^t \left(\|h(r)\|^2 + \|h(r)\|_{1+\frac{1}{q}}^{1+\frac{1}{q}}\right)dr \right], & \text{if (7) holds,} \\ C(B) \left[1 + \int_{\tau}^t \left(\|h(r)\|^2 + \|h(r)\|_{1+\frac{1}{q}}^{1+\frac{1}{q}}\right)dr \right] = \mu(B), & \text{if (8) holds.} \end{cases}
\]

Taking the multiplier \( v \) in Eq. (14) and letting

\[
F_\kappa(v, v_s) = E_\kappa(v, v_s) + \epsilon(v_s, v) \quad \text{with} \quad \epsilon \geq 0,
\]

we infer that

\[
\frac{d}{ds} F_\kappa(v, v_s) + H_\kappa(v, v_s) = 0, \tag{18}
\]

where

\[
H_\kappa(v, v_s) = \sigma(\|A^\frac{1}{2}v\|^2) \int_\Omega g(v_s)(cv + v_s)dx + \epsilon\|A^\frac{1}{2}v\|^2 + \epsilon \int_\Omega f(v)vdx
\]

\[+ \epsilon\kappa\phi(\|A^\frac{1}{2}v\|^2)\|A^\frac{1}{2}v\|^2 - \epsilon\|v_s\|^2 - \int_\Omega h_1(cv + v_s)dx.
\]

Obviously,

\[
F_\kappa(v, v_s) \geq \frac{\alpha}{4}\|v_s\|^2 + \|A^\frac{1}{2}v\|^2 - C_1|\Omega| \tag{19}
\]

for \( \epsilon > 0 \) suitably small. Let

\[
\Omega_{1s} = \{x \in \Omega \mid |v_s(x, s)| > 2\}, \quad \Omega_{2s} = \{x \in \Omega \mid |v_s(x, s)| \leq 2\}.
\]
By condition \((H_1)\), when \(x \in \Omega_{1s}\),

\[
g(v_s)v_s = \int_0^1 g'(rv_s)drv_s^2 \geq \int_\frac{1}{2}^1 g'(rv_s)drv_s^2 \geq \frac{K_0}{2}v_s^2.
\]

So,

\[
\|v_s\|^2 \leq \frac{2}{K_0} \int_{\Omega_{1s}} g(v_s)v_sdx + \int_{\Omega_{2s}} |v_s|^2dx \leq \frac{2}{K_0} \int\Omega g(v_s)v_sdx + C|\Omega|.
\]

By estimate \((17)\),

\[
\sup_{s \in [0, r]} \left[ \sigma(\|A^\frac{1}{4} v(s)\|^2) + \phi(\|A^\frac{1}{2} v(s)\|^2) \right] \leq \mu_1(B, t),
\]

where and in the following \(\mu_1(B, t)\) denotes a positive constant depending only on \(\mu(B, t)\). Taking account of the Sobolev embedding \(V_2 \rightarrow L^{q+1}\) and condition \((H_1)\), we have

\[
\left| \int_{\Omega_{1s}} g(v_s)v_xdx \right| \leq \left( \int_{\Omega_{1s}} g(v_s)v_xdx \right)^\frac{q}{q+1} \|v\|_{q+1}
\]

\[
\leq \left\{ \begin{array}{ll}
C\|A^\frac{1}{4} v\|_{\Omega_{1s}} g(v_s)v_xdx, & \text{if } \int_{\Omega_{1s}} g(v_s)v_xdx \geq 1, \\
\delta\|A^\frac{1}{2} v\|^2 + C(\delta) \left( \int_{\Omega_{1s}} g(v_s)v_xdx \right)^\frac{q}{q+1}, & \text{if } \int_{\Omega_{1s}} g(v_s)v_xdx < 1,
\end{array} \right.
\]

\[
\leq \delta\|A^\frac{1}{2} v\|^2 + C(\delta) \left( 1 + \|A^\frac{1}{2} v\| \right) \int_{\Omega_{1s}} g(v_s)v_xdx,
\]

and

\[
\left| \int_{\Omega_{2s}} g(v_s)v_xdx \right| \leq \lambda_1 \delta\|v\|^2 + C(\delta) \int_{\Omega_{2s}} g(v_s)v_s \int_0^1 g'(rv_s)drdx
\]

\[
\leq \delta\|A^\frac{1}{2} v\|^2 + C(\delta) \int_{\Omega_{2s}} g(v_s)v_xdx.
\]

Therefore, by estimate \((17)\) we obtain

\[
\epsilon \sigma(\|A^\frac{1}{2} v\|^2) \int_\Omega g(v_s)v_xdx \leq \mu_1(B, t) \left( 2\epsilon\delta\|A^\frac{1}{2} v\|^2 + \epsilon C(\delta) \int_\Omega g(v_s)v_xdx \right).
\]

(22)

Obviously,

\[
(h_1, v_s) \leq C\|h_1\|_1^{\frac{1}{4}} + \frac{\sigma_0 K_0}{8q} \|v_s\|_{q+1}^{\frac{q+1}{q}} \leq C\|h_1\|_1^{\frac{1}{4}} + \frac{\sigma_0}{8} \int_\Omega g(v_s)v_xdx,
\]

\[
(h_1, \epsilon v) \leq \frac{\epsilon \lambda_1}{8} \|A^\frac{1}{2} v\|^2 + C\epsilon\|h_1\|^2.
\]
By condition either \((H_2)\) or \((H'_2)\), condition \((H_3)\), estimates \((20)-(22)\), we have
\[
H_\alpha(v,v_s) - \epsilon F_\alpha (v,v_s)
\geq \sigma_0 \int_{\Omega} g(v_s) v_x dx + \epsilon \sigma (||A^{\frac{3}{4}} v||^2) \int_{\Omega} g(v_s) v dx + \frac{\epsilon}{2} ||A^{\frac{3}{4}} v||^2 + \int_{\Omega} \left( f(v) v - F(v) \right) dx
+ \epsilon k \left( \phi(||A^{\frac{3}{4}} v||^2) ||A^{\frac{3}{4}} v||^2 - \frac{1}{2} \int_{0}^{||A^{\frac{3}{4}} v||^2} \phi(r) dr \right)
- \frac{3\epsilon}{2} ||v_x||^2 - (H_1, \epsilon v + v_s) - \epsilon^2 (v_s, v)
\geq \left( \frac{\sigma_0}{3} - \frac{4\epsilon}{K_0} \right) \int_{\Omega} g(v_s) v_x dx + \epsilon \left( \frac{\alpha}{4} - 2\delta \mu_1(B,t) \right) ||A^{\frac{3}{4}} v||^2
- \epsilon C_\Omega - C \left( \left\| h_1 \right\|_{1+\frac{3}{8}}^{1+\frac{3}{8}} + \left\| h_1 \right\|^2 \right)
\geq - \epsilon C_\Omega - C \left( \left\| h_1 \right\|_{1+\frac{3}{8}}^{1+\frac{3}{8}} + \left\| h_1 \right\|^2 \right)
\tag{23}
\]
for \(\delta > 0\) suitably small and \(\epsilon \in [0, \epsilon_0]\), where \(\epsilon_0 = \epsilon_0(B,t) < 1\) is a small positive constant. Inserting \((23)\) into \((18)\) gives
\[
\frac{d}{ds} F_\alpha(v,v_s) + \epsilon F_\alpha(v,v_s) \leq \epsilon C_\Omega + C \left( \left\| h_1 \right\|_{1+\frac{3}{8}}^{1+\frac{3}{8}} + \left\| h_1 \right\|^2 \right), \quad s \in (0, \tau].
\]
By using the Gronwall inequality and estimate \((19)\), we obtain \((13)\). \qed

**Lemma 3.1.** [26] *(The Nakao inequality)* Let \(\varphi(t)\) be a nonnegative continuous function on \([0, T]\), with \(T > 1\), satisfying
\[
\sup_{t \leq s \leq t+1} \varphi(s)^{1+\gamma} \leq C_0 (\varphi(t) - \varphi(t+1)) + K, \quad 0 \leq t \leq T-1,
\tag{24}
\]
with some \(C_0 > 0, K > 0\) and \(\gamma > 0\). Then
\[
\varphi(t) \leq \left( C_0^{-1} \gamma (t-1)^+ + \left[ \sup_{0 \leq s \leq 1} \varphi(s) \right]^{-\gamma} \right)^{-1/\gamma} + K^{1/(\gamma+1)}, \quad 0 \leq t \leq T.
\tag{25}
\]
If \((24)\) holds with \(\gamma = 0\), then we have, instead of \((25)\),
\[
\varphi(t) \leq \sup_{0 \leq s \leq 1} \varphi(s) \left( \frac{C_0}{1+C_0} \right)^{[t]} + K, \quad 0 \leq t \leq T.
\tag{26}
\]

**Proposition 2.** Let Assumption 2.1 be valid. Then for any bounded subset \(B\) of \(\mathcal{H}, t \in \mathbb{R}\) and \(\xi_1, \xi_2 \in B\), we have
\[
\left\| U_{\alpha}(t, t-\tau) \xi_1 - U_{\alpha}(t, t-\tau) \xi_2 \right\|_{\mathcal{H}}^2 \leq \mu_1(B,t) \sup_{0 \leq s \leq \tau} \left[ \left\| A^{\frac{3}{4}} z(s) \right\|^2 + \left\| z(s) \right\|^2_{2p} \right]^{\frac{2}{1+\frac{1}{q}}}
+ \frac{1}{\alpha} \left[ q - \frac{1}{2\mu_1(B,t)} (\tau - 1) + \mu_1(B,t) \right]^{-\frac{2}{1+\frac{1}{q}}}, \quad \text{if } q > 1,
\]
and
\[
\left\| U_{\alpha}(t, t-\tau) \xi_1 - U_{\alpha}(t, t-\tau) \xi_2 \right\|_{\mathcal{H}}^2 \leq \mu_1(B,t) \sup_{0 \leq s \leq \tau} \left[ \left\| A^{\frac{3}{4}} z(s) \right\|^2 + \left\| z(s) \right\|^2_{2p} \right]
+ \frac{1}{\alpha} \mu_1(B,t) \left[ \frac{\mu_1(B,t)}{1+\mu_1(B,t)} \right]^{[\tau]}, \quad \text{if } q = 1.
\]
for all $\tau \geq 1$ and $\kappa \in [0,1]$, where

$$(z(s), z_s(s)) = (v^1(s), v_s^1(s)) - (v^2(s), v_s^2(s)) = U_e(t - \tau + s, t - \tau)\xi_1 - U_e(t - \tau + s, t - \tau)\xi_2, \quad s \in [0, \tau].$$

Proof. For the given $\tau \geq 1$ and $\kappa \in [0,1]$, it follows from estimate (17) that

$$\sum_{i=1,2} \left[ \sup_{s \in [0,\tau]} \| (v^i(s), v_s^i(s)) \|_H^2 + \int_0^\tau \| v_s^i(s) \|_{q+1}^2 \, ds \right] \leq \mu_1(B, t), \quad (27)$$

and $z(s)$ solves

$$z_{ss} + Az + \kappa \phi_1(s) A^{\frac{1}{2}} z + \sigma_1(s) (g(v_s^1) - g(v_s^2)) = -J_\sigma(s) g(v_s^2) - \kappa \phi_2(s) A^{\frac{1}{2}} v^2 - (f(v^1) - f(v^2)), \quad s \in (0, \tau], \quad (28)$$

$$(z(0), z_s(0)) = \xi_1 - \xi_2,$

where

$$\sigma_i(s) = \sigma(\| A^{\frac{1}{2}} v^i(s) \|), \quad \phi_i(s) = \phi(\| A^{\frac{1}{2}} v^i(s) \|), \quad i = 1, 2,$$

$$J_\sigma(s) = \sigma_1(s) - \sigma_2(s), \quad J_\phi(s) = \phi_1(s) - \phi_2(s), \quad s \in [0, \tau].$$

It follows from (27) that

$$\sup_{s \in [0,\tau]} \sum_{i=1,2} \left[ \sigma_i(s) + |\phi_i(s)| + |\phi'(\| A^{\frac{1}{2}} v^i(s) \|)| \right] \leq \mu_1(B, t),$$

$$|J_\sigma(s)| + |J_\phi(s)| \leq \mu_1(B, t) \| A^{\frac{1}{2}} z(s) \|, \quad s \in [0, \tau]. \quad (29)$$

Taking the multiplier $z_t$ in Eq. (28) we get

$$\frac{1}{2} \frac{d}{ds} E_1(z, z_s) + \sigma_1(s) \int_\Omega (g(v_s^1) - g(v_s^2)) z_s \, dx = \sum_{j=1}^4 I_j, \quad s \in (0, \tau], \quad (30)$$

where

$$E_1(z, z_s) = \| z_s \|^2 + \| A^{\frac{1}{2}} z \|^2 + \kappa \phi_1(s) \| A^{\frac{1}{2}} z \|^2,$$

$$I_1 = - \kappa J_\phi(s) (A^{\frac{1}{2}} v^2, z_s),$$

$$I_2 = \kappa \phi'(\| A^{\frac{1}{2}} v^1 \|) (A^{\frac{1}{2}} v^1, A^{\frac{1}{2}} v^1) \| A^{\frac{1}{2}} z \|^2,$$

$$I_3 = - J_\sigma(s) \int_\Omega g(v_s^2) z_s \, dx,$$

$$I_4 = - \int_\Omega (f(v^1) - f(v^2)) z_s \, dx.$$

It follows from conditions $(H_1), (H_3)$, Remark 2.1 and (29) that

$$\alpha \| (z, z_s) \|_H^2 \leq E_1(z, z_s) \leq \mu_1(B, t) \| (z, z_s) \|_H^2, \quad (31)$$

$$\sigma_1(s) \int_\Omega (g(v_s^1) - g(v_s^2)) z_s \, dx \geq \frac{\sigma_0 K_0}{2q} \int_\Omega \left[ |v_s^1|^{q-1} + |v_s^2|^{q-1} \right] |z_s|^2 \, dx \geq \frac{\sigma_0 K_0}{q} \| z_s \|_{q+1}^{q+1}. $$
By estimates (27) and (29),

$$|I_1| \leq \mu_1(B,t)\|A^{\frac{4}{5}}z\|\|z\|_s \leq \frac{\sigma_0 K_0}{6q} \|z\|_s^{q+1} + \mu_1(B,t)\|A^{\frac{4}{5}}z\|^{1+\frac{1}{q}},$$

$$|I_2| \leq \mu_1(B,t)\|A^{\frac{4}{5}}z\|^2 = \mu_1(B,t)\|A^{\frac{4}{5}}z\|^{\frac{q+1}{q}} \|A^{\frac{4}{5}}z\|^{1+\frac{1}{q}} \leq \mu_1(B,t)\|A^{\frac{4}{5}}z\|^{1+\frac{1}{q}},$$

$$|I_3| \leq \mu_1(B,t)\|A^{\frac{4}{5}}z\| \int_\Omega \left[|v_s^2| + |v^2|^q\right]|z|dx$$

$$\leq \mu_1(B,t)\left[\|A^{\frac{4}{5}}z\|\|z\|_s + \|A^{\frac{4}{5}}z\|\|v_s^2\|_{q+1}\|z\|_{q+1}\right]$$

$$\leq \frac{\sigma_0 K_0}{6q}\|z\|_{q+1} + \mu_1(B,t)(1 + \|v_s^2\|_{q+1})\|A^{\frac{4}{5}}z\|^{1+\frac{1}{q}}.$$

By the Sobolev embedding $V_2 \hookrightarrow L^{2p}$, we have

$$|I_4| \leq C\left[1 + \|v^1\|_{2p}^{-1} + \|v^2\|_{2p}^{-1}\right]\|z\|_{2p}\|z\|_s \leq \frac{\sigma_0 K_0}{6q}\|z\|_{q+1} + \mu_1(B,t)\|z\|_{2p}^{-\frac{1}{q}}.$$

Inserting above estimates into (30) yields

$$\frac{d}{ds}E_1(z, z_s) + \frac{\sigma_0 K_0}{2q}\|z\|_{q+1}^{q+1}$$

$$\leq \mu_1(B,t)(1 + \|v_s^2\|_{q+1}^{-1})\left[\|A^{\frac{4}{5}}z\|^{1+\frac{1}{q}} + \|z\|_{2p}^{-\frac{1}{q}}\right], \quad s \in (0, \tau].$$

For any $s \in [0, \tau - 1]$, integrating (32) over $[s, s + 1] \subset [0, \tau]$ and making use of (17), we have

$$\frac{\sigma_0 K_0}{2q} \int_s^{s+1} \|z_s(r)\|_{q+1}^{q+1}dr$$

$$\leq E_1(z(s), z_s(s)) - E_1(z(s + 1), z_s(s + 1))$$

$$+ \mu_1(B,t) \int_s^{s+1} (1 + \|v_s^2(r)\|_{q+1}^{-1})\left[\|A^{\frac{4}{5}}z(r)\|^{1+\frac{1}{q}} + \|z(r)\|_{2p}^{-\frac{1}{q}}\right]dr$$

$$\leq E_1(z(s), z_s(s)) - E_1(z(s + 1), z_s(s + 1))$$

$$+ \mu_1(B,t) \sup_{0 \leq r \leq \tau} \left[\|A^{\frac{4}{5}}z(r)\|^{1+\frac{1}{q}} + \|z(r)\|_{2p}^{-\frac{1}{q}}\right] := [W(s)]^2.$$

Obviously,

$$\int_s^{s+1} \|z_s(r)\|^2dr \leq \Omega^\frac{q+1}{q+4}\left(\int_s^{s+1} \|z_s(r)\|_{q+1}^{q+1}dr\right)^{\frac{q+1}{q+4}} \leq C_0[W(s)]^\frac{q+1}{q+4},$$

with $C_0 = \frac{2q^2}{\sigma_0 K_0}\Omega^\frac{q+1}{q+4}$, so there must be $s_1 \in [s, s + \frac{1}{4}]$ and $s_2 \in [s + \frac{3}{4}, s + 1]$ such that

$$\|z_s(s_i)\|^2 \leq 4C_0[W(s)]^\frac{q+1}{q+4}, \quad i = 1, 2.$$  \hspace{1cm} (35)

Multiplying Eq. (28) by $z$ and integrating over $[s_1, s_2] \subset [s, s + 1]$, we get

$$\int_{s_1}^{s_2} \left[\|A^{\frac{4}{5}}z(r)\|^2 + \kappa \phi_1(r)\|A^{\frac{4}{5}}z(r)\|^2\right]dr = \int_{s_1}^{s_2} \|z_s(r)\|^2dr + \sum_{j=1}^{5} \chi_j,$$  \hspace{1cm} (36)
where
\[ \chi_1 = \int_{\Omega} [z_x(s_1)z(s_1) - z_x(s_2)z(s_2)]dx, \]
\[ \chi_2 = -\int_{s_1}^{s_2} \sigma_1(r) \int_{\Omega} (g(v_1^2(r)) - g(v_x^2(r)))z(r)dxdr, \]
\[ \chi_3 = -\int_{s_1}^{s_2} \int_{\Omega} (f(v^3(r)) - f(v^2(r)))z(r)dxdr, \]
\[ \chi_4 = -\int_{s_1}^{s_2} J_x(r) \int_{\Omega} g(v_x^2(r))z(r)dxdr, \]
\[ \chi_5 = -\kappa \int_{s_1}^{s_2} J_o(r) \int_{\Omega} A^{\frac{1}{2}}v^2(r)z(r)dxdr. \]

By the Young inequality and (35),
\[ |\chi_1| \leq \left( \sum_{j=1,2} \|z_x(s_j)\| \right) \sup_{s \leq r \leq s + 1} \|z(r)\| \leq \frac{1}{12} \sup_{s \leq r \leq s + 1} E_1(z(r), z_4(r)) + \beta|W(s)|^{\frac{1}{12}}, \]
hereafter \( \beta = \beta(C_0) \) is a positive constant. Taking account of the Sobolev embedding \( V_2 \hookrightarrow L^{p+1} \) and making use of (27) we get
\[ |\chi_2| \leq \mu_1(B, t) \int_{s_1}^{s_2} \int_{\Omega} (1 + |v_1^1|^{|q-1|} + |v_2^1|^{|q-1|})z_x|z|dxdr \]
\[ \leq \mu_1(B, t) \left( \int_{s_1}^{s_2} \left( 1 + \|v^1_x\|_{q+1}^{q+1} + \|v^2_x\|_{q+1}^{q+1} \right)dr \right)^{\frac{1}{q+1}} \]
\[ \cdot \left( \int_{s_1}^{s_2} \|z_x\|_{q+1}^{q+1}dr \right)^{\frac{1}{q+1}} \]
\[ \leq \mu_1(B, t) \sup_{s \leq r \leq s + 1} \|A^{\frac{1}{2}}z(r)\| \left( \int_{s_1}^{s_2} \|z_x\|_{q+1}^{q+1}dr \right)^{\frac{1}{q+1}} \]
\[ \leq \frac{1}{12} \sup_{s \leq r \leq s + 1} E_1(z(r), z_4(r)) + \mu_1(B, t)|W(s)|^{\frac{1}{12}}, \]
\[ |\chi_3| \leq C \int_{s_1}^{s_2} (1 + |v_1^2|^{p-1} + |v_2^2|^{p-1})\|z\|_{p+1}^{2}dr \leq \mu_1(B, t) \sup_{0 \leq r \leq \tau} \|z(r)\|_{2p}^{2}. \]

By (29) and (27),
\[ |\chi_4| \leq \mu_1(B, t) \int_{s_1}^{s_2} \|A^{\frac{1}{2}}z\| \left( 1 + |v^1_x|^{|q|} \right)z_x|z|dxdr \]
\[ \leq \mu_1(B, t) \sup_{s \leq r \leq s + 1} \|A^{\frac{1}{2}}z(r)\| \left( \int_{s_1}^{s_2} \left( 1 + |v^1_x|^{|q+1|} \right)dxdr \right)^{\frac{1}{q+1}} \]
\[ \cdot \left( \int_{s_1}^{s_2} \|z_x\|_{q+1}^{q+1}dr \right)^{\frac{1}{q+1}} \]
\[ \leq \mu_1(B, t) \sup_{s \leq r \leq s + 1} \|A^{\frac{1}{2}}z(r)\| \sup_{s \leq r \leq s + 1} \|A^{\frac{1}{2}}z(r)\| \]
\[ \leq \frac{1}{12} \sup_{s \leq r \leq s + 1} E_1(z(r), z_4(r)) + \mu_1(B, t) \sup_{0 \leq r \leq \tau} \|A^{\frac{1}{2}}z(r)\|^{2}, \]
\[ |\chi_5| \leq \mu_1(B, t) \int_{s_1}^{s_2} \|A^{\frac{1}{2}}z\|^{2}dr \leq \mu_1(B, t) \sup_{0 \leq r \leq \tau} \|A^{\frac{1}{2}}z(r)\|^{2}, \]
and
\[
\sup_{0 \leq r \leq \tau} \left[ \| A^{\frac{t}{2}} z(r) \|^2 + \| z(r) \|_{2p}^2 \right] \leq \mu_1(B, t) \sup_{0 \leq r \leq \tau} \left[ \| A^{\frac{t}{2}} z(r) \|^{1 + \frac{t}{q}} + \| z(r) \|_{2p}^{1 + \frac{t}{q}} \right].
\]

Inserting above estimates into (36), letting
\[
K = \sup_{0 \leq r \leq \tau} \left[ \| A^{\frac{t}{2}} z(r) \|^{1 + \frac{t}{q}} + \| z(r) \|_{2p}^{1 + \frac{t}{q}} \right],
\]
and making use of (34) we obtain
\[
\int_{s_1}^{s_2} \left[ \| A^{\frac{t}{2}} z(r) \|^2 + \kappa \phi_1(R) \| A^{\frac{t}{2}} z(r) \|^2 \right] dr \\
\leq \frac{1}{4} \sup_{s \leq r \leq s + 1} E_1(z(r), z_s(r)) + (\beta + \mu_1(B, t)) |W(s)|^{\frac{2}{1 + t}} + \mu_1(B, t) K.
\]

The combination of the inequality with (34) yields
\[
\int_{s_1}^{s_2} E_1(z(r), z_s(r)) dr \\
\leq \frac{1}{4} \sup_{s \leq r \leq s + 1} E_1(z(r), z_s(r)) + (\beta + \mu_1(B, t)) |W(s)|^{\frac{2}{1 + t}} + \mu_1(B, t) K.
\]  

(37)

By the integral mean value theorem, there exists a point \( s_3 \in [s_1, s_2] \) such that
\[
\int_{s_1}^{s_2} E_1(z(r), z_s(r)) dr = E_1(z(s_3), z_s(s_3))(s_2 - s_1) \geq \frac{1}{2} E_1(z(s_3), z_s(s_3)),
\]
and there exists a point \( s_4 \in [s, s + 1] \) such that
\[
E_1(z(s_4), z_s(s_4)) = \sup_{s \leq r \leq s + 1} E_1(z(r), z_s(r)).
\]

Integrating (32) over \([s_3, s + 1]\) and making use of (37), we get
\[
E_1(z(s + 1), z_s(s + 1)) \\
\leq E_1(z(s_3), z_s(s_3)) + \mu_1(B, t) \int_{s_3}^{s + 1} \left( 1 + \| v_s^2 \|_{q + 1}^2 \right) \left[ \| A^{\frac{t}{2}} z \|^{1 + \frac{t}{q}} + \| z \|_{2p}^{1 + \frac{t}{q}} \right] \frac{1}{2} E_1(z(r), z_s(r)) dr + \mu_1(B, t) K \\
\leq 2 \int_{s_1}^{s_2} E_1(z(r), z_s(r)) dr + \mu_1(B, t) K \\
\leq \frac{1}{2} \sup_{s \leq r \leq s + 1} E_1(z(r), z_s(r)) + (\beta + \mu_1(B, t)) |W(s)|^{\frac{2}{1 + t}} + \mu_1(B, t) K,
\]
and integrating (32) over \([s, s_4]\) and making use of (33), we have
\[
E_1(z(s_4), z_s(s_4)) \\
\leq E_1(z(s), z_s(s)) + \mu_1(B, t) \int_{s}^{s_4} \left( 1 + \| v_s^2 \|_{q + 1}^2 \right) \left[ \| A^{\frac{t}{2}} z \|^{1 + \frac{t}{q}} + \| z \|_{2p}^{1 + \frac{t}{q}} \right] dr \\
\leq |W(s)|^2 + E_1(z(s + 1), z_s(s + 1)) + \mu_1(B, t) K \\
\leq \frac{1}{2} \sup_{s \leq r \leq s + 1} E_1(z(r), z_s(r)) + |W(s)|^2 + (\beta + \mu_1(B, t)) |W(s)|^{\frac{2}{1 + t}} + \mu_1(B, t) K,
\]
that is
\[
\sup_{s \leq r \leq s+1} E_1(z(r), z_s(r)) \\
\leq 2[W(s)]^{\frac{2q}{q+4}} \left[ \beta + \mu_1(B, t) + [W(s)]^{\frac{2q}{q+4}} \right] + \mu_1(B, t)K
\]
(38)
\[
\leq \mu_1(B, t)[W(s)]^{\frac{2q}{q+4}} + \mu_1(B, t)K,
\]
where we have used the estimate $|W(s)|^2 \leq \mu_1(B, t)$. It follows from (33) that
\[
\sup_{s \leq r \leq s+1} E_1(z(r), z_s(r))^{1+\frac{2q}{q+4}} \\
\leq \mu_1(B, t)[W(s)]^2 + \mu_1(B, t)K^{\frac{2q-4}{q+4}}
\]
(39)
\[
\leq \mu_1(B, t) \left( [E_1(z(s), z_s(s)) - E_1(z(s+1), z_s(s+1))] + K + K^{\frac{2q}{q+4}} \right)
\]
\[
\leq \mu_1(B, t) \left( [E_1(z(s), z_s(s)) - E_1(z(s+1), z_s(s+1))] + K \right), \quad s \in [0, \tau].
\]
Applying Lemma 3.1 to (39) and taking $s = \tau$ gives that when $q > 1$,
\[
E_1(z(\tau), z_s(\tau)) \\
\leq \left( \frac{q-1}{2\mu_1(B, t)}(\tau - 1) + \left( \sup_{0 \leq s \leq 1} E_1(z(s), z_s(s)) \right)^{\frac{2q-4}{q+4}} \right)^{-\frac{q+4}{q-2}} + (\mu_1(B, t)K)^{\frac{q}{q-2}}
\]
\[
\leq \left( \frac{q-1}{2\mu_1(B, t)}(\tau - 1) + \mu_1(B, t) \right)^{-\frac{q+4}{q-2}} + (\mu_1(B, t)K)^{\frac{q}{q-2}},
\]
and when $q = 1$,
\[
E_1(z(\tau), z_s(\tau)) \leq \sup_{0 \leq s \leq 1} E_1(z(s), z_s(s)) \left( \frac{\mu_1(B, t)}{1 + \mu_1(B, t)} \right)^{\tau} + \mu_1(B, t)K
\]
\[
\leq \mu_1(B, t) \left( \frac{\mu_1(B, t)}{1 + \mu_1(B, t)} \right)^{\tau} + \mu_1(B, t)K
\]
for all $\tau \geq 1$, which combined with (31) yields the conclusion of Proposition 2. \hfill \Box

**Proposition 3.** (Quasi-stability estimate) Let Assumption 2.2 be valid. Then for any bounded subset $B$ of $\mathcal{H}$, $t \in \mathbb{R}$ and $\xi_1, \xi_2 \in B$, there exists a positive constant $\sigma$ such that
\[
\|U_\kappa(t, t-\tau)\xi_1 - U_\kappa(t, t-\tau)\xi_2\|_{\mathcal{H}}^2 \leq \mu_1(B, t)e^{-\sigma \tau}\|\xi_1 - \xi_2\|_{\mathcal{H}}^2 + \mu_1(B, t)\int_0^\tau e^{-\sigma(\tau-s)}\|A^{\frac{1}{2}}z(s)\|^2 ds
\]
for all $\tau \geq 0$ and $\kappa \in [0, 1]$, where $z$ is as shown in Proposition 2.

*Proof.* Repeating the similar arguments as in [22], one can easily obtain the conclusion of Proposition 3. We omit precise details here. \hfill \Box

**Proposition 4.** (Uniform Lipschitz stability on the parameter $\kappa$) Let the assumptions of Theorem 2.1 be valid. Then for any bounded subset $B$ of $\mathcal{H}$ and $t \in \mathbb{R}$,
\[
\sup_{\xi \in B} \sup_{s \in [0, \tau]} \|U_{\kappa_1}(t-\tau + s, t-\tau)\xi - U_{\kappa_2}(t-\tau + s, t-\tau)\xi\|_{\mathcal{H}}^2 \leq \mu_1(B, t)|\kappa_1 - \kappa_2|, \quad (40)
\]
for all $\kappa_i \in [0, 1], i = 1, 2$ and $\tau \geq 0$.\hfill \Box
Eq. (41) gives

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Inserting (43)-(45) into (42) yields

\[ \text{By Remark 2.1, } \]

\[ v \]

\[ \text{Obviously, estimates (27) and (29) hold for } \]

\[ (\text{In the case that Assumption 2.2 holds, we obtain} \]

\[ \text{where } J_\sigma(s), J_\phi(s), \phi_i(s), \sigma_i(s) \text{ are as shown in (28). Taking the multiplier } w_s \text{ in Eq. (41) gives} \]

\[ \frac{1}{2} \frac{d}{ds} \left[ \| A^2 w \|^2 + \| w_s \|^2 \right] + \sigma_2(s) \int_\Omega (g(v_s^1) - g(v_s^2)) w_s dx + (f(v^1) - f(v^2), w_s) \]

\[ = - (\kappa_1 - \kappa_2) \phi_1(s) (A^2 v^1, w_s) - \kappa_2 J_\phi(s) (A^2 v^1, w_s) - J_\sigma(s) (g(v_s^1), w_s) - \kappa_2 \phi_2(s) (A^2 v, w_s). \]

By Remark 2.1,

\[ \sigma_2(s) \int_\Omega (g(v_s^1) - g(v_s^2)) w_s dx \geq \frac{\sigma_0 K_0}{2q} \int_\Omega \left[ |v_s^1|^q - 1 + |v_s^2|^q - 1 \right] |w_s|^2 dx. \]

It follows from estimates (27) and (29) that

\[ \left| - (\kappa_1 - \kappa_2) \phi_1(s) (A^2 v^1, w_s) \right| \leq \mu_1(B, t) |\kappa_1 - \kappa_2|, \]

\[ | - \kappa_2 J_\phi(s) (A^2 v^1, w_s) - \kappa_2 \phi_2(s) (A^2 w, w_s) | \leq \mu_1(B, t) \left[ \| A^2 w \|^2 + \| w_s \|^2 \right], \]

\[ | - J_\sigma(s) (g(v_s^1), w_s) | \leq \mu_1(B, t) \| A^2 w \| \int_\Omega \left( |v_s^1| + |v_s^1|^{q+1} + |v_s^2|^{q+1} \right) |w_s| dx \]

\[ \leq \mu_1(B, t) \left( 1 + \| v_s^1 \|_{2q}^{q+1} \right) \left[ \| A^2 w \|^2 + \| w_s \|^2 \right] + \frac{\sigma_0 K_0}{4q} \int_\Omega |v_s^1|^{q-1} |w_s|^2 dx. \]

In the case that Assumption 2.1 is valid, we have

\[ \left| (f(v^1) - f(v^2), w_s) \right| \leq C \left( 1 + \| v_s^1 \|_{2p}^{q-1} + \| v_s^2 \|_{2p}^{q-1} \right) \| w_s \|_{2p} \| w_s \| \leq \mu_1(B, t) \left[ \| A^2 w \|^2 + \| w_s \|^2 \right]. \]

Inserting (43)-(45) into (42) yields

\[ \frac{d}{ds} \left[ \| A^2 w \|^2 + \| w_s \|^2 \right] \leq \mu_1(B, t) \left( 1 + \| v_s^1 \|_{2q}^{q+1} \right) \left[ \| A^2 w \|^2 + \| w_s \|^2 \right] + \mu_1(B, t) |\kappa_1 - \kappa_2|. \]

Applying the Gronwall inequality to this inequality gives (40).

In the case that Assumption 2.2 holds, we obtain

\[ (f(v^1) - f(v^2), w_s) = \frac{1}{2} \frac{d}{ds} \int_\Omega \int_0^1 f'_{\eta \theta} |w| dx d\theta d\Omega - \frac{1}{2} \int_\Omega \int_0^1 f''_{\eta \theta} |w|^2 dx d\theta d\Omega, \]

where \( \eta \theta = \theta v^1 + (1 - \theta) v^2, \eta_{\theta s} = \theta v_s^1 + (1 - \theta) v_s^2. \) Let

\[ H_w(w, w_s) = \| A^2 w \|^2 + \| w_s \|^2 + \int_\Omega \int_0^1 f'_{\eta \theta} |w|^2 d\theta d\Omega \sim \| A^2 w \|^2 + \| w_s \|^2, \]
where we have used the fact:

\[-\frac{\alpha_1}{\lambda_1} \|A^2w\|^2 \leq -\alpha_1 \|w\|^2 \leq \int_0^1 \int_0^1 f'(\eta_\theta)|w|^2 d\theta dx \leq \mu_1(B,t) \|w\|^2, \quad s \in [0,\tau].\]

By condition \((H'_2)\),

\[-\frac{\alpha_1}{\lambda_1} \|A^2w\|^2 \leq -\alpha_1 \|w\|^2 \leq \int_0^1 \int_0^1 f''(\eta_\theta)\eta_\theta s|w|^2 d\theta dx \leq C(1 + \|v_1\|_{p+1}^2 + \|v_2\|_{p+1}^2)(\|v_1\|_{p+1} + \|v_2\|_{p+1}) \|w\|^2_{p+1}
\]

Inserting \((43)-(44)\) and \((46)\) into \((42)\) yields

\[-\frac{d}{ds}H_w(w,w_s) \leq \mu_1(B,t)(1 + \|v_1\|_{q+1}^2 + \|v_2\|_{q+1}^2)H_w(w,w_s) + \mu_1(B,t)|\kappa_1 - \kappa_2|.\]

Applying the Gronwall inequality to the above formula gives \((40)\). \(\square\)

**Remark 2.** If condition \((8)\) is valid, then \(\mu_1(B,t) \equiv \mu_1(B)\) (see \((17)\) and \((21)\)).

4. **Pullback attractors and their continuity.** For the readers’ convenience, we first state some preliminaries. In what follows, we suppose that \(E\) is a Banach space and \(U(t,\tau)\) is a process acting on \(E\).

**Definition 4.1.** A family of sets \(\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}\) is said to be a pullback absorbing family of the process \(U(t,\tau)\), if for any \(t \in \mathbb{R}\) and every bounded subset \(B \subset E\), there exists a \(T = T(t,B) > 0\) such that \(U(t,t-\tau)B \subset D(t)\) for \(\tau \geq T\). In addition, the family \(\mathcal{D}\) is said to be pullback \(\mathcal{D}\)-absorbing, if for any \(t \in \mathbb{R}\), there exists \(T_t > 0\) such that \(U(t,t-\tau)D(t-\tau) \subset D(t)\) for \(\tau \geq T_t\).

**Lemma 4.2.** [3] Let the family \(\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}\) be a pullback absorbing family of the process \(U(t,\tau)\), and \(U(t,\tau)\) be continuous in \(E\) and pullback \(\mathcal{D}\)-asymptotically compact, i.e., for any \(t \in \mathbb{R}\), any sequence \(\tau_n \to \infty\) and \(x_n \in D(t-\tau_n)\), the sequence \(\{U(t,t-\tau_n)x_n\}_{n \in \mathbb{N}}\) is precompact in \(E\). Then the family \(\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}\) defined by

\[A(t) = \bigcap_{s \geq 0} \left[ \bigcup_{\tau \geq s} U(t,t-\tau)D(t-\tau) \right]_E\]

for each \(t \in \mathbb{R}\) (47) is a pullback attractor of \(U(t,\tau)\). Moreover, if \(\mathcal{D}\) is pullback \(\mathcal{D}\)-absorbing, then \(A(t) \subset D(t)\) for all \(t \in \mathbb{R}\).

**Definition 4.3.** Let \(D\) be a bounded set of \(E\). A functional \(\Psi(\cdot,\cdot)\) defined on \(D \times D\) is said to be contractive, if for every sequence \(\{x_n\} \subset D\), there exists a subsequence \(\{x_{n_k}\}\) such that

\[\lim_{k \to \infty} \lim_{l \to \infty} \Psi(x_{n_k},x_{n_l}) = 0.\]

**Lemma 4.4.** [30] Let the family \(\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}\) be a pullback \(\mathcal{D}\)-absorbing family of the process \(U(t,\tau)\). Assume that for any \(\delta > 0\) and \(t \in \mathbb{R}\), there exist a \(\tau = \tau(t,\delta,\mathcal{D}) > 0\) and a contractive functional \(\Psi_{t,\tau}(\cdot,\cdot)\) defined on \(D(t-\tau) \times D(t-\tau)\) such that

\[\|U(t,t-\tau)x - U(t,t-\tau)y\|_E \leq \delta + \Psi_{t,\tau}(x,y), \quad \forall x, y \in D(t-\tau).\]

Then the process \(U(t,\tau)\) is pullback \(\mathcal{D}\)-asymptotically compact in \(E\).
Lemma 4.5. [31] Assume that for every $\lambda \in [0, 1]$, the process $U_\lambda(t, \tau)$ has a pullback absorbing family $\mathcal{D}_\lambda = \{D_\lambda(t)\}_{t \in \mathbb{R}}$ which is pullback $\mathcal{D}_\lambda$-absorbing, and $U_\lambda(t, \tau)$ is pullback $\mathcal{D}_\lambda$-asymptotically compact in $E$, $\mathcal{A}_\lambda = \{A_\lambda(t)\}_{t \in \mathbb{R}}$ is the pullback attractor in $E$ given by formula (47). Suppose that

$(L_1)$ the pullback absorbing family $\mathcal{D}_\lambda = \{D_\lambda(t)\}_{t \in \mathbb{R}}$ is independent of the choice of $\lambda \in [0, 1]$, i.e.,

$$D_\lambda(t) = D(t) \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad \lambda \in [0, 1];$$

$(L_2)$ for any $[a, b] \subset \mathbb{R}$, $\tau > 0$ and any bounded subset $B$ of $E$,

$$\lim_{\lambda \to \lambda_0} \sup_{\xi \in B} \sup_{t \in [a, b]} \|U_\lambda(t, a - \tau)\xi - U_{\lambda_0}(t, a - \tau)\xi\|_E = 0, \quad \forall \lambda_0 \in [0, 1];$$

$(L_3)$ for any $\tau > 0$, any sequence $t_n \in [a, b]$ with $t_n \to t_0$ and $\xi_n \to \xi_0$ in $E$,

$$\lim_{n \to \infty} \|U_\lambda(t_n, a - \tau)\xi_n - U_\lambda(t_0, a - \tau)\xi_0\|_E = 0, \quad \forall \lambda \in [0, 1].$$

Then for any $[a, b] \subset \mathbb{R}$,

$$\lim_{\lambda \to \lambda_0} \sup_{t \in [a, b]} \text{dist}_E\{A_\lambda(t), A_{\lambda_0}(t)\} = 0, \quad \forall \lambda_0 \in [0, 1],$$

and the union

$$\bigcup_{t \in [a, b]} \bigcup_{\lambda \in [0, 1]} A_\lambda(t) \quad \text{is precompact in} \quad E.$$

Lemma 4.6. [20] Let the process $U_\lambda(t, \tau)$ have in $E$ a pullback attractor $\mathcal{A}_\lambda = \{A_\lambda(t)\}_{t \in \mathbb{R}}$ as shown in (47) for every $\lambda \in [0, 1]$, and conditions $(L_1)$-$(L_3)$ hold. If

there exists a bounded subset $B_0$ of $E$ such that

$$\bigcup_{t \in \mathbb{R}} \bigcup_{\lambda \in [0, 1]} A_\lambda(t) \subset B_0; \quad (48)$$

and

$$\left[ \bigcup_{\lambda \in [0, 1]} A_\lambda(t) \right]_E \quad \text{is a compact set in} \quad E \quad \text{for each} \quad t \in \mathbb{R}. \quad (49)$$

Then there exists a residual subset $I$ of $[0, 1]$, which is dense in $[0, 1]$, such that

$$\lim_{\lambda \to \lambda_0} \text{dist}_{\mathcal{H}}\text{symm}\{A_\lambda(t), A_{\lambda_0}(t)\} = 0, \quad \forall t \in \mathbb{R}, \quad \lambda_0 \in I.$$

Proof of Theorem 2.2. For any $\kappa \in [0, 1]$ and $t \in \mathbb{R}$, let

$$D(t) = \{\xi \in \mathcal{H} \|\xi\|_\mathcal{H} \leq R(t)\} \quad (50)$$

with

$$R^2(t) = \begin{cases} 
1 + C|\Omega| + \int_0^t \left(\|h(s)\|^2 + \|h(s)\|_1^{1+\frac{2}{q}}\right)ds, & \text{if condition (7) holds,} \\
1 + C|\Omega| + \int_\mathbb{R} \left(\|h(s)\|^2 + \|h(s)\|_1^{1+\frac{2}{q}}\right)ds = R^2, & \text{if condition (8) holds,}
\end{cases}$$

where $C|\Omega|$ is as shown in (13). It follows from Proposition 1 that for any bounded $B$ of $\mathcal{H}$, there exists a constant $T_B > 0$ which is independent of $t$ such that

$$\bigcup_{\kappa \in [0, 1]} U_\kappa(t, t - \tau)B \subset D(t), \quad \forall \tau \geq T_B.$$
that is, the family \( \mathcal{D} = \{ D(t) \}_{t \in \mathbb{R}} \) is a common pullback absorbing family of the processes \( U_\kappa(t, \tau), \kappa \in [0, 1] \). Due to \( R(t) \leq R(t_0) \) for all \( t \leq t_0 \), there exists a \( T_t > 0 \) for any given \( t \in \mathbb{R} \) such that

\[
\bigcup_{\kappa \in [0, 1]} U_\kappa(t, t - \tau)D(t - \tau) \subset \bigcup_{\kappa \in [0, 1]} U_\kappa(t, t - \tau)D(t) \subset D(t), \quad \forall \tau \geq T_t, \tag{51}
\]

which means that the family \( \mathcal{D} \) is also pullback \( \mathcal{D} \)-absorbing for every \( \kappa \in [0, 1] \), i.e., the condition \((L_1)\) is valid.

If Assumption 2.1 is valid, then we infer from Proposition 2 that for any \( t \in \mathbb{R} \) and \( \delta > 0 \), there exists a \( \tau = \tau(\delta, t) > 0 \) such that

\[
\| U_\kappa(t, t - \tau)\xi_1 - U_\kappa(t, t - \tau)\xi_2 \|_\mathcal{H} \leq \delta + \Psi_{t, \tau}(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in D(t - \tau), \tag{52}
\]

where the mapping \( \Psi_{t, \tau} : D(t - \tau) \times D(t - \tau) \to \mathbb{R} \) is given by

\[
\Psi_{t, \tau}^2(\xi_1, \xi_2) = \begin{cases} 
\mu_1(D(t), t) \sup_{0 \leq s \leq \tau} \left[ \left\| A^{\frac{1}{2}} z(s) \right\|^{1+\frac{1}{p}} + \left\| z(s) \right\|_{2p}^{1+\frac{1}{p}} \right]^{\frac{2}{p}}, & \text{if } q > 1, \\
\mu_1(D(t), t) \sup_{0 \leq s \leq \tau} \left[ \left\| A^{\frac{1}{2}} z(s) \right\|^2 + \left\| z(s) \right\|_{2p}^2 \right], & \text{if } q = 1.
\end{cases}
\]

For any sequence \( \{ \xi_n \}_{n \in \mathbb{N}} \subset D(t - \tau) \), let

\[
(u^n(s), u^n_\tau(s)) = U_\kappa(t - \tau + s, t - \tau)\xi_n, \quad s \in [0, \tau].
\]

By estimate \((17)\) (replacing \( v(s) \) there by \( u^n(s) \)) and the Simon lemma \([28]\), \( \{ (u^n, u^n_\tau) \}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(0, \tau; \mathcal{H}) \) and

\[
\{ u^n \}_{n \in \mathbb{N}} \text{ is precompact in } C([0, \tau]; V_1 \cap L^{2p}) \tag{53}
\]

for \( V_2 \leftrightarrow V_1 \cap L^{2p} \). Thus there exists a subsequence \( \{ n_k \} \subset \{ n \} \) such that \( u^{n_k} \) is a Cauchy sequence in \( C([0, \tau]; V_1 \cap L^{2p}) \) and

\[
\lim_{k \to \infty} \lim_{l \to \infty} \Psi_{t, \tau}(\xi_{n_k}, \xi_{n_l}) = 0,
\]

that is, \( \Psi_{t, \tau} \) is a contractive functional on \( D(t - \tau) \times D(t - \tau) \). So by Lemma 4.4, the process \( U_\kappa(t, \tau) \) is pullback \( \mathcal{D} \)-asymptotically compact in \( \mathcal{H} \).

If Assumption 2.2 is valid, then we infer from Proposition 3 that for any \( t \in \mathbb{R} \) and \( \delta > 0 \), there exists a \( \tau = \tau(\delta, t) > 0 \) such that

\[
\bigcup_{\kappa \in [0, 1]} \| U_\kappa(t, t - \tau)\xi_1 - U_\kappa(t, t - \tau)\xi_2 \|_\mathcal{H} \leq \delta + \Psi'_{t, \tau}(\xi_1, \xi_2), \quad \forall \xi_1, \xi_2 \in D(t - \tau), \tag{54}
\]

where the mapping \( \Psi'_{t, \tau} : D(t - \tau) \times D(t - \tau) \to \mathbb{R} \) is given by

\[
\Psi'_{t, \tau}^2(\xi_1, \xi_2) = \mu_1(D(t), t) \int_0^\tau \| A^{\frac{1}{2}} z(s) \|^2 ds.
\]

Repeating the same arguments as above (replacing \((53)\) by \( \{ u^n \}_{n \in \mathbb{N}} \) is precompact in \( L^2(0, \tau; V_1) \)) for \( V_2 \leftrightarrow V_1 \), one sees that \( \Psi'_{t, \tau} \) is also a contractive functional on \( D(t - \tau) \times D(t - \tau) \) and the process \( U_\kappa(t, \tau) \) is pullback \( \mathcal{D} \)-asymptotically compact in \( \mathcal{H} \).

By the continuity of the process \( U_\kappa(t, \tau) \) in \( \mathcal{H} \) (see \((10)\)) and Lemma 4.2, we obtain that for each \( \kappa \in [0, 1] \), the process \( U_\kappa(t, \tau) \) has in \( \mathcal{H} \) a pullback attractor \( \mathcal{A}_\kappa = \{ A_\kappa(t) \}_{t \in \mathbb{R}} \) which is given by

\[
A_\kappa(t) = \bigcap_{s \geq 0} \left[ \bigcup_{\tau \geq s} U_\kappa(t, t - \tau)D(t - \tau) \right]_{\mathcal{H}},
\]

and \( A_\kappa(t) \subset D(t) \) for all \( t \in \mathbb{R} \).
For any bounded subset $B$ of $\mathcal{H}$, $\tau > 0$ and $[a, b] \subset \mathbb{R}$, we infer from Proposition 4 that
\[
\sup_{\xi \in B} \sup_{x \in [a, b]} \|U_\kappa(s, a - \tau)\xi - U_{\kappa_0}(s, a - \tau)\xi\|_\mathcal{H}^2 \\
\leq \sup_{\xi \in B} \sup_{x \in [0, \kappa - a + \tau]} \|U_\kappa(a - \tau + s, a - \tau)\xi - U_{\kappa_0}(a - \tau + s, a - \tau)\xi\|_\mathcal{H}^2 \\
\leq \mu_1(B, b)|\kappa - \kappa_0| \to 0 \quad \text{as} \quad \kappa \to \kappa_0
\]
for all $\kappa_0 \in [0, 1]$, i.e., condition $(L_2)$ is valid.

For any given $\kappa \in [0, 1]$, $\tau > 0$, $t_n \in [a, b]$ with $t_n \to t_0$ and $\xi_n \in \mathcal{H}$ with $\xi_n \to \xi_0$ in $\mathcal{H}$,
\[
\|U_\kappa(t_n, a - \tau)\xi_n - U_\kappa(t_0, a - \tau)\xi_0\|_\mathcal{H} \\
\leq \|U_\kappa(t_n, a - \tau)\xi_n - U_\kappa(t_n, a - \tau)\xi_0\|_\mathcal{H} + \|U_\kappa(t_n, a - \tau)\xi_0 - U_\kappa(t_0, a - \tau)\xi_0\|_\mathcal{H} \\
\leq C_1 \|\xi_n - \xi_0\|_\mathcal{H} + \|U_\kappa(t_n, a - \tau)\xi_0 - U_\kappa(t_0, a - \tau)\xi_0\|_\mathcal{H} \to 0 \quad \text{as} \quad n \to \infty,
\]
where we have used estimate (10) and the fact that $(\nu^\kappa, u^\kappa) \in C([\tau, T]; \mathcal{H})$ for any $T > \tau$. That is, condition $(L_3)$ holds. Therefore, we get the desired conclusions (12) and (11) by using Lemma 4.5.

**Proof of Theorem 2.4.** When $h \in L^2(\mathbb{R}; L^2) \cap L^{1+\frac{1}{2}}(\mathbb{R}; L^{1+\frac{1}{2}})$, it follows from (50) that $R(t) \equiv R$ and $D(t) \equiv D$. Therefore, the pullback attractors given by Theorem 2.2 are of the property:
\[
\bigcup_{\kappa \in [0, 1]} A_\kappa(t) \subset D, \quad \forall t \in \mathbb{R}.
\]
(55)

We infer from (11) that formula (49) holds. Therefore, By Lemma 4.6, we obtain the desired conclusion.

5. **Robust pullback exponential attractors.** In order to prove Theorem 2.5, we first quote a recently established criterion on the stability of pullback exponential attractors.

**Lemma 5.1.** [34] Let $M$ be a bounded closed subset of $E$ and $(U_\kappa(t, \tau), E)$ be a non-autonomous dynamical system for each $\kappa \in [0, 1]$. And assume that

(i) $M$ is a uniformly (w.r.t. $\kappa \in [0, 1]$ and $t \in \mathbb{R}$) absorbing set of the family of processes $(U_\kappa(t, \tau), \kappa \in [0, 1])$, i.e., for any bounded subset $B$ of $E$, there exists a constant $T_B > 0$ such that
\[
\bigcup_{\kappa \in [0, 1]} \bigcup_{t \in \mathbb{R}} U_\kappa(t + \tau, t)B \subset M, \quad \forall \tau \geq T_B;
\]

(ii) there exist some constants $T > 0$ and $L_T > 0$ such that, for any $t \in \mathbb{R}$,
\[
\bigcup_{\kappa \in [0, 1]} U_\kappa(t + \tau, t)M \subset M, \quad \tau \geq T,
\]
\[
\sup_{\tau \in [0, T]} \sup_{\kappa \in [0, 1]} \|U_\kappa(t + \tau, t)x - U_\kappa(t + \tau, t)y\|_E \leq L_T \|x - y\|_E, \quad \forall x, y \in M; \quad (56)
\]

(iii) there exist a Banach space $Z$ and a compact seminorm $n_Z(\cdot)$ on $Z$, and there exists a mapping $K_\kappa^*: M \to Z$ for each $\kappa \in [0, 1]$ and $n \in \mathbb{Z}$ such that for any $x, y \in M$,
\[
\sup_{\kappa \in [0, 1]} \sup_{n \in \mathbb{Z}} \|K_\kappa^n x - K_\kappa^n y\|_Z \leq L \|x - y\|_E. \quad (57)
\]
Then for each \( \kappa \in [0, 1] \), the dynamical system \( (U_n(t, \tau), E) \) possesses a pullback exponential attractor \( \{ \mathcal{M}^\kappa(t) \}_{t \in \mathbb{R}} \). Moreover, the mapping \( \kappa \mapsto \mathcal{M}^\kappa(t) \) is stable in the following sense: for any \( \kappa_0 \in [0, 1] \), if

\[
\Gamma(\kappa, \kappa_0) := \sup_{s \in [0, T]} \sup_{t} \sup_{\tau} \sup_{\omega} \| U_n(s + t, \omega) - U_n(s + t, \omega) \|_E < 1,
\]

then

\[
\sup_{t} \text{dist}^{\text{sym}}_E \{ \mathcal{M}^\kappa(t), \mathcal{M}^\kappa_0(t) \} \leq C[\Gamma(\kappa, \kappa_0)]^\lambda,
\]

where \( C > 0 \) and \( 0 < \lambda < 1 \) are constants independent of \( \kappa \).

**Proof of Theorem 2.5.** Under the assumptions of Theorem 2.5, it follows from (50) that

\[
\mu_1(B, t) \equiv \mu_1(B) \quad \text{and} \quad R(t) \equiv R, \quad \forall t \in \mathbb{R}.
\]

By Proposition 1, the set

\[
M \equiv D = \{ \xi \in \mathcal{H} \| \xi \|_\mathcal{H} \leq R \}
\]

is a uniformly \( (w.r.t. \kappa \in [0, 1] \) and \( t \in \mathbb{R} \) absorbing set of the family of processes \( \{ U_n(t, \tau) \}, \kappa \in [0, 1] \). Thus, there exists a \( T > 0 \) such that for each \( t \in \mathbb{R} \),

\[
\bigcup_{\tau \geq T} \bigcup_{\kappa \in [0, 1]} U_n(t + \tau, \kappa) M \subset M, \quad \eta^2 = \mu_1(M)e^{-\sigma T} < 1,
\]

and by (10)

\[
\sup_{t \in [0, T]} \sup_{\kappa \in [0, 1]} \| U_n(t + \tau, \kappa) \xi_1 - U_n(t + \tau, \kappa) \xi_2 \|_\mathcal{H} \leq C(M)\| \xi_1 - \xi_2 \|_\mathcal{H}, \quad \forall \xi_1, \xi_2 \in M.
\]

Let the space

\[
Z = \{ u \in L^2(0, T; V_2) \mid u_t \in L^2(0, T; L^2) \}
\]

equipped with the norm

\[
\| u \|_Z^2 = \int_0^T \| (u(t), u_t(t)) \|_H^2 \, dt, \quad \forall u \in Z.
\]

Obviously, \( Z \) is a Banach space, and

\[
n_Z(u) = \mu_1(M)\| u \|_{L^2(0, T; V_1)}
\]

is a compact seminorm on \( Z \) for \( V_2 \hookrightarrow V_1 \) (cf. [28]). For each \( \kappa \in [0, 1] \) and \( n \in \mathbb{N}^+ \), we define the mapping

\[
K_\kappa^n : M \to Z, \quad K_\kappa^n \xi = u(\cdot + nT), \quad \xi \in M,
\]

where \( (u(\cdot + nT), u_t(\cdot + nT)) = U_n(\cdot + nT, nT) \xi \) and \( u(\cdot + nT) \) means \( u(s + nT), s \in [0, T] \). For any \( \xi \in M, i = 1, 2 \), it follows from (59) that

\[
\sup_{\kappa \in [0, 1]} \| K_\kappa^n \xi_1 - K_\kappa^n \xi_2 \|_Z^2 = \sup_{\kappa \in [0, 1]} \int_0^T \| U_n(s + nT, nT) \xi_1 - U_n(s + nT, nT) \xi_2 \|_H^2 \, ds
\]

\[
\leq L^2 \| \xi_1 - \xi_2 \|_Z^2,
\]

where \( L^2 = C(M)/T \). By Proposition 3 and formula (61),

\[
\| U_n((n + 1)T, nT) \xi_1 - U_n((n + 1)T, nT) \xi_2 \|_\mathcal{H} \leq \eta\| \xi_1 - \xi_2 \|_\mathcal{H} + n_Z(K_\kappa^n \xi_1 - K_\kappa^n \xi_2)
\]

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Lemma 6.1. to guarantee the existence of the pullback exponential attractor. In order to prove ever, the relaxed assumption boundedness of the fractal dimensions of the sections of pullback attractor. How-

6. Finite dimensionality of pullback attractors. Under the assumptions of Theorem 2.5, the existence of pullback exponential attractor implies the uniform boundedness of the fractal dimensions of the sections of pullback attractor. However, the relaxed assumption \( h \in L^2(-\infty, t; L^2) \cap L^{1+\frac{1}{2}}(-\infty, t; L^{1+\frac{1}{2}}) \) is not enough to guarantee the existence of the pullback exponential attractor. In order to prove Theorem 2.6 in this case, we first give a lemma.

**Lemma 6.1.** Let \( U(t, \tau) \) be a process acting on \( E \) and have a pullback attractor \( A = \{ A(t) \}, \) Assume that

(i) there exist a \( t_0 \in \mathbb{R} \) and a bounded subset \( B \) of \( E \) such that \( \cup_{t \leq t_0} A(t) \subset B; \)

(ii) there exist some positive constants \( T \) and \( L_{t,T} \) such that for any \( t \in \mathbb{R}, \)

\[
\sup_{\tau \in [0, T]} \| U(t + \tau, t) x - U(t + \tau, t) y \|_E \leq L_{t,T} \| x - y \|_E; \tag{64}
\]

(iii) there exist a Banach space \( Z, \) a compact seminorm \( n_Z(\cdot) \) on \( Z, \) and operators \( K_t : B \to Z \) such that for any \( t \leq t_0, \)

\[
\sup_{t \leq t_0} \| K_t x - K_t y \|_Z \leq L \| x - y \|_E, \tag{65}
\]

\[
\| U(t + T, t) x - U(t + T, t) y \|_E \leq \eta \| x - y \|_E + n_Z(K_t x - K_t y), \tag{66}
\]

where \( \eta \in (0, 1/2), \)

Then the fractal dimensions of the sections \( A(t), t \in \mathbb{R} \) are uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R} \leq t_0} \dim_f(A(t), E) \leq \left[ \ln \frac{1}{2\eta} \right]^{-1} \ln m_Z \left( \frac{2L}{\eta} \right) < +\infty, \tag{67}
\]

where \( m_Z(R) \) is the maximal number of elements \( z_i \) in the ball \( \{ z \in Z \| z \|_Z \leq R \} \) such that \( n_Z(z_i - z_j) > 1. \)

**Proof.** Assumption (i) implies that there exists a \( R_0 > 0 \) such that \( N(B, R_0) < +\infty, \)

here \( N(B, \epsilon) \) denotes the cardinality of the minimal covering of the set \( B(\subset E) \) by the closed subsets of diameter \( \leq 2\epsilon. \)

For every \( t \leq t_0, \) due to

\[
N(A(t), R_0) \leq N(B, R_0) < +\infty,
\]

there exists the minimal covering \( \{ F_i^t \}_{i=1}^{N(A(t), R_0)} \) of \( A(t) \) by its closed subsets of diameter \( \leq 2R_0, \) that is,

\[
A(t) = \bigcup_{i=1}^{N(A(t), R_0)} F_i^t.
\]
Let
\[ n_i(t) := \mathbb{N}\{x_k \in F^i_t | n_Z(K_i x_k - K_i x_j) > \eta R_0, k \neq j\}, \forall i = 1, \cdots, N(A(t), R_0), t \leq t_0, \]
where \( \mathbb{N}\{\cdots\} \) denotes the maximal number of elements with the given properties. We claim that
\[ n_i(t) \leq m_Z \left( \frac{2L}{\eta} \right) < +\infty, \quad \forall i = 1, \cdots, N(A(t), R_0), t \leq t_0. \quad (68) \]
Indeed, for every \( t \leq t_0, i = 1, \cdots, N(A(t), R_0) \), let \( F^i_t = \{ K_i x | x \in F^j_t \} \). By (65),
\[ \text{diam} F^i_t \leq L \text{diam} F^j_t \leq 2R_0L. \]
Thus, \( F^i_t \subset B_{2R_0L}(y^i_t) = \{ z \in Z ||z - y^i_t||_Z \leq 2R_0L \} \)
for some \( y^i_t \in F^i_t \). By the linearity and compactness of the seminorm,
\[ n_i(t) = \mathbb{N}\{z_k \in F^i_t | n_Z(z_k - z_j) > \eta R_0, k \neq j\} \]
\[ \leq \mathbb{N}\{z_i \in B_{2R_0L}(y^i_t) | n_Z(z_k - z_j) > \eta R_0, k \neq j\} \]
\[ = \mathbb{N}\{z_k \in B_{2\eta L}(0) | n_Z(z_k - z_j) > 1, k \neq j\} = m_Z \left( \frac{2L}{\eta} \right) < +\infty. \]
Thus, there exists a maximal subset \( \{ x^{i,t}_{j} \}_{j=1}^{n_i(t)} \) of \( F^i_t \) such that \( n_Z(K_i x^{i,t}_{j} - K_i x^{i,t}_{j}) > \eta R_0 \)
and
\[ F^i_t = \bigcup_{j=1}^{n_i(t)} B^{i,t}_{j}, \quad B^{i,t}_{j} = \{ x \in F^i_t | n_Z(K_i x - K_i x^{i,t}_{j}) \leq \eta R_0 \}. \]
Consequently,
\[ A(t) = U(t, t - T)A(t - T) = \bigcup_{i=1}^{N(A(t - T), R_0)} A(t - T) = \bigcup_{j=1}^{N(A(t - T), R_0)} U(t, t - T) B^{i,t-T}_{j}. \]
By (66),
\[ \text{diam}(U(t, t - T) B^{i,t-T}_{j}) \leq \eta \text{diam} F^{i-t}_t + 2\eta R_0 \leq 4\eta R_0, \]
which means
\[ N(A(t), 2\eta R_0) \leq \sum_{i=1}^{N(A(t - T), R_0)} n_i(t - T) \leq N(A(t - T), R_0) m_Z \left( \frac{2L}{\eta} \right), \quad \forall t \leq t_0, \]
where we have used (68). Hence, for any \( t \leq t_0 \) and \( k \in \mathbb{N}^+ \), we obtain
\[ N(A(t), (2\eta)^k R_0) \leq N(A(t - kT), R_0) \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^k \leq N(B, R_0) \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^k. \]
For any \( \epsilon \in (0, 1) \), there exists a \( k(\epsilon) \in \mathbb{N}^+ \) such that \( \epsilon \in [(2\eta)^{k(\epsilon)} R_0, (2\eta)^{k(\epsilon)-1} R_0) \),
which means that
\[ N(A(t), \epsilon) \leq N(A(t), (2\eta)^{k(\epsilon)} R_0), \quad k(\epsilon) \leq \ln(R_0/\epsilon) \ln^{-1}(1/2\eta) + 1. \]
A simple calculation shows that
\[ \dim_f(A(t), E) = \limsup_{\epsilon \to 0} \frac{\ln N(A(t), \epsilon)}{\ln(1/\epsilon)} \leq \left[ \ln \left( \frac{1}{2\eta} \right)^{-1} \ln m_Z \left( \frac{2L}{\eta} \right) \right], \quad \forall t \leq t_0. \quad (69) \]
For any \( t > t_0 \), formula (64) shows that the mapping \( U(t, t_0) : A(t_0) \to A(t) \) is Lipschitz continuous in \( E \). Therefore,
\[
\dim_f(A(t), E) = \dim_f(U(t, t_0)A(t_0), E)
\]
\[
\leq \dim_f(A(t_0), E) \leq \left[ \ln \frac{1}{2\eta} \right]^{-1} \ln m_Z\left(\frac{2L}{\eta}\right). \tag{70}
\]
The combination of (69) and (70) gives (67).

\[\square\]

**Proof of Theorem 2.6.** Due to \( R(t) \leq R(0), \forall t \leq 0 \), we have
\[
\bigcup_{t \leq 0} A_\kappa(t) \subset \bigcup_{t \leq 0} D(t) \subset D(0) \equiv D.
\]
Proposition 3 implies that for all \( t \leq 0 \) and \( \xi_1, \xi_2 \in D \),
\[
\|U_\kappa(t, t-T)\xi_1 - U_\kappa(t, t-T)\xi_2\|_\mathcal{H}^2 \leq \eta^2\|\xi_1 - \xi_2\|_\mathcal{H}^2 + \mu_1(D, 0) \int_0^T \|A^\frac{1}{2}z(s)\|^2 ds. \tag{71}
\]
where \( T : \eta^2 \equiv \mu_1(D, 0)e^{-\sigma T} < 1/4 \). Let the Banach space \( Z \) and norm \( \| \cdot \|_Z \) be as shown in (60). Then
\[
n_Z(u) = \mu_1(D, 0)\|u\|_{L^2(0,T; V_1)} \quad \text{is a compact seminorm on} \quad Z \tag{72}
\]
for \( V_2 \hookrightarrow V_1 \) (cf. [28]). For any \( t \leq t_0 \equiv -T \), we define the mapping
\[
K_t : D \to Z, \quad K_t \xi = u(\cdot + t), \quad \xi \in D,
\]
where \( (u(\cdot + t), u(\cdot + t)) = U_\kappa(\cdot + t, t)\xi \) and \( u(\cdot + t) \) means \( u(s + t), s \in [0, T] \). It follows from (10) that for any \( t \leq t_0, \xi_1, \xi_2 \in D \),
\[
\sup_{s \in [0,T]} \|U_\kappa(s + t, t)\xi_1 - U_\kappa(s + t, t)\xi_2\|_\mathcal{H}^2 \leq C_T\|\xi_1 - \xi_2\|_\mathcal{H}^2, \tag{73}
\]
and
\[
\|K_t\xi_1 - K_t\xi_2\|_Z^2 = \int_0^T \|U_\kappa(s + t, t)\xi_1 - U_\kappa(s + t, t)\xi_2\|_\mathcal{H}^2 ds \leq L^2\|\xi_1 - \xi_2\|_\mathcal{H}^2, \tag{74}
\]
where \( L^2 = C_T \). And formula (71) means that
\[
\|U_\kappa(t, t-T)\xi_1 - U_\kappa(t, t-T)\xi_2\|_\mathcal{H} \leq \eta\|\xi_1 - \xi_2\|_\mathcal{H} + n_Z(K_{t-T}\xi_1 - K_{t-T}\xi_2) \tag{75}
\]
for all \( t \leq t_0, \xi_1, \xi_2 \in D \). Therefore, by Lemma 6.1,
\[
\sup_{t \in \mathbb{R}} \dim_f(A_\kappa(t), \mathcal{H}) \leq \left[ \ln \frac{1}{2\eta} \right]^{-1} \ln m_Z\left(\frac{2L}{\eta}\right) < +\infty.
\]
\[\square\]

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