HARDY AND BERGMAN SPACES ON
HYPERCONVEX DOMAINS AND THEIR
COMPOSITION OPERATORS

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1. Introduction

The theory of Hardy and Bergman spaces of analytic functions on the unit disk is one of the most developed and useful branch of function theory. In several variables such spaces were studied on the unit ball ([R1]), strongly pseudoconvex domains ([St], [BFG]) and polydisks ([R2]). However, in several variables none of the listed classes of domains can serve as a “model” domain similar to the unit disk in one variable.

The goal of this paper is to develop a technique leading to a meaningful and uniform function theory of spaces of analytic functions on a broad class of domains in $\mathbb{C}^n$. We introduce Hardy and weighted Bergman spaces on hyperconvex domains and prove their basic properties. Our prime focus is on geometric aspects rather than reproducing kernels and duality. The geometry of the domain is hidden in the exhaustion functions, but the Nevanlinna counting functions determined by the exhaustions reveals the geometric nature of the norm and, therefore, spaces. Since the introduced spaces are expected to behave under holomorphic transformations of domains similar to the classical ones, in the last part of the paper we prove some estimates for composition operators induced by holomorphic mappings. These estimates might be viewed as generalized embedding results. They lead to embedding theorems for holomorphic isomorphisms.

Let us briefly present the content of the paper. In the classical theory, methods of potential theory play an important role. In several variables, where most of other methods either disappear or become technically difficult, we still can rely on methods of pluripotential theory (see [Kl] and [D3]). In this theory the role of the Laplacian is played

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by the Monge–Ampere operator. For this operator Demailly in ([D1]) proved on hyperconvex domains a fundamental Lelong-Jensen formula, which may be viewed as an analogue of the classical Littlewood-Paley identity. Recall that a domain $D \subset \mathbb{C}^n$ is called hyperconvex, if there is a continuous negative plurisubharmonic function $u$ on $D$, called an exhaustion function, such that $\lim_{z \to \partial D} u(z) = 0$. To make our presentation self-contained we stated Lelong-Jensen formula along with other background materials in Section 2.

Lelong-Jensen formula suggests hyperconvex domains as a natural class of domains where one might have a rich function theory. This class is very wide: by the theorem of Demailly ([D1]) every bounded pseudoconvex domain with Lipschitz boundary is hyperconvex. At the same time as we show the presence of Lelong-Jensen formula suffices to prove many interesting results.

Using this formula in Section 3 we introduce Hardy and Bergman norms on the cone of non-negative plurisubharmonic functions on a hyperconvex domain $D \subset \mathbb{C}^n$. In general, such a norm depends on the choice of an exhausting function $u$, but we prove that if two exhausting functions have the same rate of decay near the boundary of the domain, then the norms are equivalent. Moreover the smallest norms are obtained when $u$ belongs to the compact class $\mathcal{E}_0(D)$, i.e., $(dd^c u)^n$ has a compact support in $D$. Pluricomplex Green functions are in this class.

In Section 4 we define Hardy and weighted Bergman norms of a holomorphic function $f$ as the norms from Section 3 of $|f|^p$. Both norms depend on the choice of the exhaustion function $u$ and, consequently, Hardy and Bergman spaces depend on $u$. The largest spaces are obtained when $u \in \mathcal{E}_0(D)$. We show that the latter spaces coincide with spaces that were studied on the unit ball ([Ru1]), strongly pseudoconvex domains ([St]) and polydisks (in the case of Hardy spaces) ([Ru2]).

The choice of a pluricomplex Green function as an exhausting function not only frequently leads to a simplification of estimates and resulting formulas but also gives bounded point evaluations in all cases. Using this we prove that the introduced spaces are Banach. In the same section we also prove a version of the Littlewood’s subordination principle for holomorphic mappings of a hyperconvex domain into the unit disk.

The formulas for the norms are more complicated comparatively to the plane case and this might be viewed as an obstruction to the new theory. Fortunately, Lelong-Jensen formula provides a splitting for
\[ \|f\|_{A_{u,\alpha}}^p, \text{ namely,} \]
\[ \|f\|_{A_{u,\alpha}}^p = \int_D \sigma_\alpha(u)|f|^p(dd^c u)^n + \int_D \gamma_\alpha(u)|f|^p(dd^c u)^{n-1}, \]

where \( \sigma_\alpha \) and \( \gamma_\alpha \) are some auxiliary functions (see Section 6). The first integral when, for example, \( u \in \mathcal{E}_0(D) \), is over a compact part of \( D \) and can be easily estimated.

We define and use the Nevanlinna counting function \( N_{\alpha,f}(w) \) of a holomorphic function \( f \) on \( D \) to understand the geometry of the second integral. Earlier, Nevanlinna counting functions in several variables were constructed and used in multidimensional Nevanlinna theory by Griffiths and King [GK]. Our construction differs from Griffiths and King’s in two aspects. First, we consider the hyperbolic case instead of parabolic. The second and the main difference is that we construct Nevanlinna functions for an arbitrary current of a type described below, while in [GK] such functions are defined only for special exhausting functions.

The geometric meaning of the Nevanlinna counting functions is similar to the classical one. If in the classical theory they, basically, count the number of points where \( f(z) = w \) in several variables they count areas of the sets \( \{ f = w \} \) in the pseudo-metric \( dd^c u \).

To define the Nevanlinna counting functions in Section 5 we consider the current \( T = u_0 dd^c u_1 \wedge ... \wedge dd^c u_{n-1} \), where \( u_0, ..., u_{n-1} \) are continuous plurisubharmonic functions on \( D \) satisfying some mild conditions. Given a compactly supported \( C^\infty \)-function \( \phi \) in \( D \), a holomorphic function \( f \) in \( D \) and \( w \in \mathbb{C} \) we define
\[ N_{f,T}(w,\phi) = N(w,\phi) = \int_D \phi dd^c \log |f - w| \wedge T. \]

The main result of this most technical section is Theorem 5.4, which establishes a change of variables formula. It states that for every holomorphic function \( f \) mapping \( D \) into \( \Omega \subset \mathbb{C} \), every \( C^\infty \)-function \( \phi \) compactly supported in \( D \), a \( C^\infty \)-function \( \psi \) compactly supported in \( \Omega \) and every subharmonic function \( v \) on \( \Omega \), whose infinite locus does not intersect the image of infinite locus of \( u_j \), \( j = 0, ..., n-1 \), we have
\[ \int_D \phi \psi \circ f dd^c (v \circ f) \wedge T = \int_{\Omega} \psi N_{f,T}(w,\phi \psi \circ f) dd^c v. \]

In Section 6 we set the functions \( u_0, ..., u_{n-1} \) equal to an exhaustion function \( u \) on \( D \) and define the Nevanlinna counting functions of order
\[ N_{f,\alpha}(w) = \int_D \gamma\alpha(u)(dd^*u)^{n-1} \wedge dd^* \log |f - w|. \]

The change of variables formula implies that

\[ \|f\|^p_{A^p_{u,\alpha}} = \int_D \sigma\alpha(u)|f|^p(dd^*u)^n + \int_C N_{\alpha,f}(w)|dd\log|w| |^p. \]

The latter formula generalizes classical Littlewood-Paley identity. When a pluricomplex Green function is chosen as the exhaustion, the formula takes exactly the form of this identity.

Section 7 is devoted to properties of counting functions. First, we show their equivalence when exhausting functions are equivalent. We also prove that, like in the one-dimensional case, the Nevanlinna counting function satisfies Shapiro’s mean value inequality. Finally, we prove that for every holomorphic function which takes \( D \) into the unit disk \( \mathbb{D} \), the classical logarithmic estimate holds for the Nevanlinna counting function, i.e., \( N_{u,f}(w) \leq c \log |w| \) for every \( w \in \mathbb{D} \) sufficiently close to the boundary.

It is well known that the projection \((z_1, z_2) \to z_1\) induces an isometry of the classical Bergman space in the unit disk into the Hardy space in the unit ball in \( \mathbb{C}^2 \). This and many other embedding theorems for spaces of analytic functions have numerable applications to operator theory. In the rest of the paper we investigate more general problem: how our spaces on domains \( D_1 \subset \mathbb{C}^n \) and \( D_2 \subset \mathbb{C}^m \) are transformed by a holomorphic mapping \( F : D_1 \to D_2 \), which induces the operator \( C_F f = f \circ F \) called the composition operator.

The case when \( D_1 = D_2 = \mathbb{D} \) has been intensively investigated since 1960-s and is well understood. A good exposition of main results of one-dimensional theory can be found in monographs ([Sha1]) and ([CoM]) and references there.

Contrary to this, results in the multivariable case are sparse. It was understood quite a while ago that the situation in several variables is considerably harder than in the classical setting. For instance, Littlewood subordination principle implies that every holomorphic self-mapping of the unit disk induces a composition operator which acts boundedly from every weighted Bergman space \( A^p_{\alpha}(\mathbb{D}) \) into itself \((p > 0, \alpha \geq -1, \text{and, as usually, if } \alpha = -1, \text{the corresponding space is the Hardy space})\). For domains in \( \mathbb{C}^n \) this is not the case. Even a quadratic polynomial self-mapping of the unit ball in \( \mathbb{C}^n \) need not
to induce a bounded operator acting on the Hardy space. Counterexamples were constructed by Shapiro, Cima and Wogen ([CW2]) and others.

In the classical case conditions for a composition operator to act continuously or compactly from $A^p_\alpha(D)$ to $A^p_\beta(D)$ are naturally expressed in terms of the Nevanlinna counting functions. This was discovered by Shapiro in his paper [Sha2] when $\alpha = \beta = -1$ and then by Smith ([Sm]) for any $\alpha$ and $\beta$. Roughly speaking, the function $N_{F,\beta}(w)$ must decay as $\gamma_\alpha(w)$ at the boundary for the continuity of $C_F$ and faster than that for compactness.

In section 8 we prove two theorems which respectively give sufficient and necessary conditions of boundedness and compactness of a composition operator induced by a holomorphic mapping of arbitrary hyperconvex domains as an operator acting from one weighted Bergman space into another. These theorems might be considered as multidimensional analogs of the results by Shapiro and Smith, though our necessary and sufficient conditions in general case seem to be different.

However, in the case when $D_2 = \mathbb{D}$ as we show in Section 9 the gap between our necessary and sufficient conditions disappears and they become identical in the form to the conditions of Shapiro and Smith.

In section 10 we prove that if the domain $D_2$ is strongly pseudoconvex then $C_F$ maps $A^p_{\rho,\alpha}(D_2)$ continuously into $A^p_{u,n+\alpha-1}(D_1)$, $\alpha \geq -1$. In such a generality this result cannot be improved. Previously, MacCluer and Mercer ([MM]) proved this for self-mappings of a strongly convex domain in $\mathbb{C}^n$ and $\alpha = -1$. Cima and Mercer ([CM]) extended this result to Bergman spaces on a strongly convex domain and any $\alpha \geq -1$.

A similar result for mappings between balls of arbitrary dimensions (not necessarily equal) was obtained recently by Koo and Smith ([KS]). Mappings of polydisks were considered in ([SZ1], [SZ2]).

The proof exploits the fact that boundedness and compactness of composition operators are naturally expressed in terms of Carleson measures. This approach was first explicitly stated by MacCluer ([Mc]) (earlier Carleson measures were used by Cima and Wogen ([CW1]) to give a compactness criterion for Toeplitz operators).

The general result stated above might be significantly improved under some additional assumptions. We describe two such situations. The first is when $D_1$ is strongly pseudoconvex, $D_2 = \mathbb{D}$ and $F$ is holomorphic in a neighborhood of $D_1$. In this case we use our sufficient conditions from Section 9 to show that $C_F$ acts boundedly from $A^p_\alpha(\mathbb{D})$ into $A^p_{u,\beta}(D)$ when $\alpha \leq \beta + (n-1)/2$ and compactly when $\alpha < \beta + (n-1)/2$. 
An example in this section of a quadratic polynomial $F$ shows that the result cannot be improved.

The second is when $F$ is a proper holomorphic mappings of a hyperconvex domains $D_1$ into a hyperconvex domain $D_2$ of the same dimension, then for every pair of exhausting function $u_1 \in \mathcal{E}_i(D_1)$ and $u_2 \in \mathcal{E}_i(D_2)$ the composition operator $C_F$ acts boundedly from $A^p_a(D_2)$ into $A^p_a(D_1)$ and has a left inverse with the same properties. In particular, every automorphism of a strongly pseudoconvex domain induces a bounded composition operators acting on Hardy and Bergman spaces.

2. BACKGROUND RESULTS

2.1. Differential forms and currents. Let $D$ be a domain in $\mathbb{C}^n$ and let $C^\infty_0(D)$ be the space of all smooth functions on $D$ with compact supports. A sequence $\{\phi_j\} \subset C^\infty_0(D)$ converges to 0 if the supports of all $\phi_j$ belong to a compact set $K \subset D$ and the functions $\phi_j$ with all derivatives converge uniformly to 0.

We denote by $\mathcal{D}^{p,q}(D)$ the space of all differential form

$$\omega = \sum_{|I|=p, |J|=q} \omega_{I,J} dz_I \wedge d\overline{z}_J$$

of bidegree $(p,q)$, where $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ are subsets of $\{1, \ldots, n\}$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ and $\omega_{I,J} \in C^\infty_0(D)$. Equipped with the topology of uniform convergence on compacta with all derivatives, $\mathcal{D}^{p,q}$ has a structure of a linear topological space.

The space $\mathcal{D}'_{p,q}(D)$ of continuous linear functionals on $\mathcal{D}^{p,q}(D)$ is called the space of currents of bidimension $(p,q)$ or of bidegree $(n-p, n-q)$. If $\phi \in \mathcal{D}'_{p,q}(D)$ then

$$\phi = \sum_{|I|=n-p, |J|=n-q} \phi_{I,J} dz_I \wedge d\overline{z}_J,$$

where $\phi_{I,J}$ are distributions and the pairing $\langle \phi, \omega \rangle$ is given by

$$\langle \phi, \omega \rangle = \sum_{|I|=n-p, |J|=n-q} \langle \phi_{I,J}, \omega_{I,J} \rangle.$$

A current $\phi \in \mathcal{D}'_{p,p}$ is positive if $\langle \phi, \omega \rangle \geq 0$ for every test form

$$\omega = i \omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge i \omega_p \wedge \overline{\omega}_p, \quad \omega_j \in \mathcal{D}^{1,0}(D).$$

In this case the coefficients $\phi_{I,J}$ are positive measures.

The differential of $\omega$ is defined by $d\omega = \partial \omega + \bar{\partial} \omega$, where

$$\partial \omega = \sum \frac{\partial \omega_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J, \quad \bar{\partial} \omega = \sum \frac{\partial \omega_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$
The operator \(d^c\) is defined by \(d^c = i(\bar{\partial} - \partial)\). For \(\phi \in C^2(D)\) we have
\[
 dd^c \phi = 2i \sum \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.
\]

Given a current \(T\) we define \(dT\) by the formula: \(\langle dT, \omega \rangle = \langle T, d\omega \rangle\) and \(dd^c T\) by the formula: \(\langle dd^c T, \omega \rangle = \langle T, dd^c \omega \rangle\). A current \(T\) is closed if \(dT = 0\).

Every plurisubharmonic function generates a closed positive \((1,1)\)-current. The following result can be found in [Kl, Prop. 3.3.5].

**Theorem 2.1.** If \(u\) is a plurisubharmonic function on \(\Omega\), then \(dd^c u\) is a closed, positive \((1,1)\)-current with measure coefficients.

The wedge product of a \((p,q)\)-form \(\omega\) and \((n-r,n-s)\)-current \(T\) is a \((n-r+p,n-s+q)\)-current \(T \wedge \omega\) defined by
\[
\langle T \wedge \omega, \psi \rangle = \langle T, \psi \wedge \omega \rangle.
\]

In some cases it is possible to make sense of the wedge product even when \(\omega\) is not smooth (and is considered as a current). We will use three results of this kind. To formulate them, we introduce the infinite locus \(L(u)\) of a plurisubharmonic function \(u\) on \(D\) as the set of points \(z \in D\) such that \(u\) is not bounded on any neighborhood of \(z\).

The first result is due to Demailly [D2, Cor. 2.3].

**Theorem 2.2.** Suppose that \(u\) is a plurisubharmonic function, the set \(L(u)\) is compact and \(T\) is a closed positive current of bidegree \((1,1)\), then the current \((dd^c u)^{n-1} \wedge T\) is well defined and has locally finite mass on \(D\). Moreover, if \(\{u_k\}\) is a sequence of decreasing plurisubharmonic functions converging to \(u\), then the currents \((dd^c u_k)^{n-1} \wedge T\) converge weak-* to \((dd^c u)^{n-1} \wedge T\).

Let \(\mathcal{H}_k\) be the Hausdorff measure of dimension \(k\). The second result was proved by Demailly [D2] and then improved by Fornæss and Sibony [FS, Cor. 3.6].

**Theorem 2.3.** Suppose that \(u_0, \ldots, u_n\) are plurisubharmonic functions on \(D\) and for any choice of indexes \(0 \leq j_1 < \cdots < j_m \leq n\)
\[
\mathcal{H}_{2(n-m+1)}(L(u_{j_1}) \cap \ldots \cap L(u_{j_m})) = 0.
\]  \(1\)

Then the function \(u_0\) is locally integrable with respect to the measure \(dd^c u_1 \wedge \ldots dd^c u_n\).

Moreover, if for every \(0 \leq j \leq n\) the sequences of plurisubharmonic function \(\{u_{j_k}\}\) converge to \(u_j\) in \(L^1_{\text{loc}}(D)\) and \(u_{jk} \geq u_j\), then
\[
u_{0k} dd^c u_{1k} \wedge \cdots \wedge dd^c u_{nk} \to u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_n
\]
in the sense of currents.
The third theorem we will need is due to Coman [C, Theorem 3.3].

**Theorem 2.4.** Let \( \Omega \) be a hyperconvex domain in \( \mathbb{C}^n \), \( v \) be a continuous plurisubharmonic exhaustion function on \( \Omega \), \( T \) be a closed positive \((1,1)\)-current on \( \Omega \), and \( u \) be a negative plurisubharmonic function on \( \Omega \) such that

\[
\int_{\Omega} dd^c u \wedge T < \infty.
\]

Then

\[
\int_{\Omega} v dd^c u \wedge T \geq \int_{\Omega} u dd^c v \wedge T.
\]

2.2. **Maximal plurisubharmonic functions.** Let \( \Omega \subset \mathbb{C}^n \) be an open set and \( u : \Omega \to \mathbb{R} \) be a plurisubharmonic function. Recall that \( u \) is called maximal, if for every relatively compact open subset \( G \) in \( \Omega \), and for every upper semicontinuous function \( v \) on \( \overline{G} \) such that \( v \) is plurisubharmonic in \( G \) and \( v \leq u \) on \( \partial G \), we have \( v \leq u \). The following result of Bedford and Taylor ([Kl], p. 131) gives a necessary and sufficient condition of maximality.

**Theorem 2.5.** Let \( \Omega \subset \mathbb{C}^n \) be an open set and \( u \) be a plurisubharmonic locally bounded function in \( \Omega \). Then \( u \) is maximal if and only if

\[
(dd^c u)^n = 0.
\]

If \( D \) is hyperconvex and \( w \in D \), then pluricomplex Green function with pole at \( w \) is a unique plurisubharmonic in \( z \) continuous exhaustion function \( g_D(z, w) \) on \( D \times D \) such that \((dd^c g_D(z, w))^n = (2\pi)^n \delta_w \) and \(|g_D(z, w) - \log |z - w|\) is bounded on \( D \). It is possible to show that

\[
g_D(z, w) = \sup\{u(z) : u \text{ is negative and plurisubharmonic in } D \text{ and for some constant } C, u(z) \leq \log |z - w| + C \text{ near } w\}.
\]

(2)

(see ([Kl], p. 221 or [D1] for details). By Theorem 2.5 \( g_D(z, w) \) is maximal in \( D \setminus \{w\} \).

2.3. **Lelong-Jensen formula.** Let \( D \) be a hyperconvex domain in \( \mathbb{C}^n \) and \( u \) be a continuous negative plurisubharmonic exhausting function on \( D \).

We define \( B_u(r) = \{z \in D : u(z) < r\} \) and \( S_u(r) = \{z \in D : u(z) = r\} \). Following [D1] we let

\[
\mu_{u,r} = (dd^c u_r)^n - \chi_{D \setminus B_u(r)}(dd^c u)^n,
\]

where \( u_r = \max\{u, r\} \). The measure \( \mu_{u,r} \) is nonnegative and supported by \( S_u(r) \). In [D1, Theorem 1.7] Demailly had proved the following fundamental Lelong–Jensen formula.
Theorem 2.6. For all \( r < 0 \) and every plurisubharmonic function \( \phi \) on \( D \)

\[
\mu_{u,r}(\phi) = \int_D \phi \mu_{u,r}
\]

is finite and

\[
\mu_{u,r}(\phi) - \int_{B_u(r)} \phi (dd^c u)^n = \int_{B_u(r)} (r - u) dd^c \phi \wedge (dd^c u)^{n-1}
\]

\[
= \int_{-\infty}^r dt \int_{B_u(t)} dd^c \phi \wedge (dd^c u)^{n-1}. \tag{3}
\]

The last integral in this formula can be equal to \( \infty \). Then the integral in the left side is equal to \( -\infty \). This cannot happen if \( \phi \geq 0 \).

The function

\[
\Phi(t) = \int_{B_u(t)} dd^c \phi \wedge (dd^c u)^{n-1} < \infty
\]

is, evidently, increasing and by Theorem 2.2 \( \Phi(t) < \infty \) for all \( t < 0 \).

Thus the function

\[
\Psi(r) = \int_{-\infty}^r dt \int_{B_u(t)} dd^c \phi \wedge (dd^c u)^{n-1}
\]

is either identically equal to \( \infty \) or is a continuous function. It follows that the function \( \mu_{u,r}(\phi) \) is increasing and continuous from the left.

2.4. Poincaré-Lelong formula. Let \( f \) be a holomorphic function on a domain \( D \subset \mathbb{C}^n \) and let \( X = \{ f = 0 \} \). It is well-known that the set \( X \) consists of countably many connected components. The critical set \( \{ \nabla f = 0 \} \) also consists of countably many components and the function \( f \) is constant on each of components. Since only finitely many of them intersect a compact set \( K \subset D \), \( f \) has only finitely many critical values on \( K \). If 0 is in the range of \( f \) and is not a critical value of \( f \) (and, therefore, \( f \) is not a constant), then the analytic set \( X \) is a complex manifold of codimension 1. If 0 is a critical value of \( f \), and \( f \) is not a constant function, then we denote by \( X^{\text{reg}} \) the regular part of \( X \). The singular part \( X^{\text{sing}} \) of \( X \) has codimension at least 2. These and other basic facts about analytic sets could be found, for example, in [Chi].

Let \( A \) be an irreducible component of \( X \). The multiplicity of \( f \) on \( A \) is defined as follows (cf. [Chi]). Let \( z_0 \in A \cap X^{\text{reg}} \). Since \( A \)
is a smooth complex manifold, it is possible to find local coordinates
\( \zeta = (\zeta_1, \ldots, \zeta_n) \) near \( z_0 \) which map \( z_0 \) to the point of origin and the set \( A \) into the set \( \{ \zeta_1 = 0 \} \). Then in \( \zeta \)-coordinates we have \( f(\zeta) = \zeta_1^m g(\zeta) \), where \( m \) is a natural number and \( g(0) \neq 0 \).

Let \( h \) be a holomorphic branch of \( g^{1/m} \) defined in a neighborhood of \( 0 \). In coordinates \( \xi_1 = h(\zeta)\zeta_1, \xi_j = \zeta_j, j \geq 2 \), we have \( f(\xi) = \zeta_1^m \).

The number \( m = m_{A,f} \) is a continuous integer value function on \( X_{\text{reg}} \) and, consequently, is a constant on \( A_{\text{reg}} \). It is called the \textit{multiplicity} of \( f \) on \( A \).

The next theorem contains Poincaré–Lelong formula (4). Its proof can be found in [D3, Proposition 3.2.15].

**Theorem 2.7.** If \( f \) is a holomorphic function on a domain \( D \subset \mathbb{C}^n \) and \( \phi \in \mathcal{D}^{n-1,n-1}(D) \), then

\[
\int_D \phi \wedge dd^c \log |f| = \sum_{A} m_{A,f} \int_{A_{\text{reg}}} \phi, \tag{4}
\]

where the summation runs over all irreducible components \( A \) of \( X = \{ f = 0 \} \). Both sides of this formula define a closed positive current \( T \) on \( D \) of bidegree \( (1,1) \).

### 3. Spaces of plurisubharmonic functions

Let \( D \) be a hyperconvex domain in \( \mathbb{C}^n \). We define the space \( \mathcal{P} \mathcal{S}_u(D) \) as the set of all nonnegative plurisubharmonic functions \( \phi \) on \( D \) such that

\[
\limsup_{r \to 0^-} \mu_{u,r}(\phi) < \infty.
\]

(A similar definition for strongly pseudo-convex domains was given by Hörmander [H]).

Since \( \mu_{u,r}(\phi) \) is an increasing function of \( r \) for all \( r < 0 \), we can replace \( \limsup \) in the definition of the space \( \mathcal{P} \mathcal{S}_u(D) \) by \( \lim \). So we can introduce the norm on \( \mathcal{P} \mathcal{S}_u(D) \) as

\[
\| \phi \|_u = \lim_{r \to 0^-} \mu_{u,r}(\phi).
\]

If \( \mathbb{D} \) is the unit disk in \( \mathbb{C} \) then the weighted Bergman space \( \mathcal{A}^p_u \) is defined as the set of all holomorphic function \( f \) on \( \mathbb{D} \) such that

\[
\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p dV = \int_{-\infty}^0 \int_{0}^{2\pi} (1 - e^{2\pi i r})^\alpha e^{i r} |f(e^{r+i\theta})|^p d\theta dr < \infty.
\]
Similar to these spaces we introduce the weighted spaces $PS_{u,\alpha}(D)$, $\alpha > -1$, as the set of all nonnegative plurisubharmonic functions $\phi$ on $D$ such that

$$\|\phi\|_{u,\alpha} = \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(\phi) \, dr < \infty.$$  

Clearly, $PS_u(D) \subset PS_{u,\alpha}(D) \subset PS_{u,\beta}(D)$ when $\beta \geq \alpha$. In what follows, to facilitate the system of notation we let $PS_u(D) = PS_{u,-1}(D)$ and $\|\phi\|_u = \|\phi\|_{u,-1}$.

The following theorem shows that faster decaying near the boundary of $D$ exhausting functions determine dominating norms.

**Theorem 3.1.** Let $u$ and $v$ be continuous plurisubharmonic exhaustion functions on $D$ and let $F$ be a compact set in $D$ such that $F \subset B_u(r_0)$ for some $r_0 < 0$ and $v(z) \leq u(z)$ for all $z \in D \setminus F$. Then for any $c > 1$ and any $a < 1 - c^{-1}$ we have

$$\mu_{u,r}(\phi) \leq c^n \mu_{v,ar}(\phi)$$

when $r \geq r_0$. Moreover, $PS_v(D) \subset PS_u(D)$ and $\|\phi\|_u \leq \|\phi\|_v$, $PS_{v,a}(D) \subset PS_{u,\alpha}(D)$ and there is a constant $C$ depending only on $r_0$ and $\alpha$ such that $\|\phi\|_{u,\alpha} \leq C\|\phi\|_{v,a}$.

**Proof.** Take a $b > 1$ and let $v_1 = bv$, $\beta = 1 - c^{-1}$ and $\alpha = c^{-1}$. Take any $r \geq r_0$ and consider the function $w = \max\{v_1, \alpha u + \beta r\}$. If $z \in S_u(r)$ then $\alpha u(z) + \beta r = r > v_1$ and $w \equiv \alpha u + \beta r$ on a neighborhood of $S_u(r)$. Moreover, if $z \in S_w(r)$ then $z \in S_u(r)$. Hence $S_u(r) = S_w(r)$ and $\mu_{u,r} = \alpha^n \mu_{v,ar}$.

Let $r_1 = \beta r > r$. If $z \in D \setminus B_{v_1}(r_1)$, i.e. $v_1(z) \geq r_1$, then $v_1 \geq \beta r > \alpha u + \beta r$. Hence $w \equiv v_1$ on a neighborhood of $D \setminus B_{v_1}(r_1)$ and $w$ is an exhaustion function.

It follows that

$$b^n \mu_{v,b^{-1}r_1}(\phi) = \mu_{v_1,r_1}(\phi) = \mu_{w,r_1}(\phi) \geq \mu_{w,r}(\phi) = \alpha^n \mu_{u,r}(\phi).$$

Thus

$$\lim_{b \to 1^+} \mu_{v,b^{-1}r}(\phi) \geq \alpha^n \mu_{u,r}(\phi)$$

and we see that $\mu_{u,r}(\phi) \leq c^n \mu_{v,ar}(\phi)$ when $r \geq r_0$.

Hence if $\phi \in PS_v(D)$ then $\phi \in PS_u(D)$ and $\|\phi\|_u \leq \|\phi\|_v$.

If $\phi \in PS_{v,a}(D)$ then we let $c = 2$ and $a = 1/4$. We write

$$\int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(\phi) \, dr = \int_{-\infty}^{r_0} |r|^\alpha e^r \mu_{u,r}(\phi) \, dr + \int_{r_0}^{0} |r|^\alpha e^r \mu_{u,r}(\phi) \, dr.$$
The first integral does not exceed $C_1(r_0, \alpha) \mu_{u, r_0}(\phi)$, where

$$C_1(r_0, \alpha) = \int_{-\infty}^{r_0} |r|^\alpha e^r \, dr.$$

As we proved above $\mu_{u, r_0}(\phi) \leq 2^n \mu_{v, r_0/4}(\phi)$, and

$$\|\phi\|_{v, \alpha} \geq \int_{r_0/4}^{0} |r|^\alpha e^r \mu_{u, r}(\phi) \, dr \geq \mu_{v, r_0/4}(\phi) \int_{r_0/4}^{0} |r|^\alpha e^r \, dr.$$

Hence,

$$\int_{-\infty}^{r_0} |r|^\alpha e^r \mu_{u, r}(\phi) \, dr \leq \frac{2^n C_1(r_0, \alpha)}{C_2(r_0, \alpha, c)} \|\phi\|_{v, \alpha},$$

where

$$C_2(r_0, \alpha, c) = \int_{r_0/4}^{0} |r|^\alpha e^r \, dr.$$

The second integral does not exceed

$$2^n \int_{r_0}^{0} |t|^\alpha e^t \mu_{v, r/4}(\phi) \, dt = 2^{2\alpha+n+2} \int_{r_0/4}^{0} |t|^\alpha e^t \mu_{v, t}(\phi) \, dt,$$

which, in turn, does not exceed

$$2^{2\alpha+n+2} \int_{r_0/4}^{0} |t|^\alpha e^t \mu_{v, t}(\phi) \, dt \leq 2^{2\alpha+n+2} \|\phi\|_{v, \alpha}.$$

Combining these estimates we get that

$$\|\phi\|_{u, \alpha} \leq \left(\frac{2^n C_1(r_0, \alpha)}{C_2(r_0, \alpha, c)} + 2^{2\alpha+n+2}\right) \|\phi\|_{v, \alpha}.$$  

We are done. \qed

This theorem has a couple of useful corollaries.

**Corollary 3.2.** Let $u$ and $v$ be continuous plurisubharmonic exhaustion functions on $D$ and let $F$ be a compact set in $D$ such that $bv(z) \leq u(z)$ for some constant $b > 0$ and all $z \in D \setminus F$. Then $PS_v(D) \subset PS_u(D)$ and $\|\phi\|_u \leq b^n \|\phi\|_v$, $PS_{v, \alpha}(D) \subset PS_{u, \alpha}(D)$ and there is a constant $C$ depending only on $r_0$, $b$ and $\alpha$ such that $\|\phi\|_{u, \alpha} \leq C \|\phi\|_{v, \alpha}$.  

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Proof. Remark that $b v_r = \max \{b v, b r\}$, and, thus, $b^n \mu_{v, r} = \mu_{b v, b r}$. Now Corollary follows immediately from Theorem 3.1. □

**Corollary 3.3.** Let $u$ and $v$ be continuous plurisubharmonic exhaustion functions on $D$ and let $F$ be a compact set in $D$ such that

$$ b v \leq u \leq b^{-1} v $$

(5)

for some constant $b > 0$ and all $z \in D \setminus F$. Then $PS_v(D) = PS_u(D)$, $PS_{v, \alpha}(D) = PS_{u, \alpha}(D)$ and the identity mappings are continuous.

Let us describe a class of functions for which the inequality (5) holds automatically. We denote by $E_0(D)$ the set of all plurisubharmonic exhaustion functions $u$ on $D$ with compactly supported $(dd^c u)^n$. The set $E_0(D)$ is not empty: pluricomplex Green functions are in $E_0(D)$. Moreover, by (3) if $u \in E_0(D)$ then the space $A^p_{u, \alpha}(D)$ contains constants and, consequently, all bounded holomorphic functions.

**Lemma 3.4.** If $u, v \in E_0(D)$ then there is a constant $b > 0$ such that $b u(z) \leq v(z) \leq b^{-1} u(z)$ near $\partial D$.

Proof. Let us take a compact set $F \subset D$ containing the supports of $(dd^c u)^n$ and $(dd^c v)^n$ such that both $u$ and $v$ are bounded on $\partial F$ and (5) holds on $\partial F$ for some number $b > 0$. By the maximality of $u$ and $v$ on $D \setminus F$ this inequality holds on $D \setminus F$ also. □

Thus functions in $E_0(D)$ generate the same spaces $PS_{u, \alpha}(D)$ and $PS_u(D)$ with equivalent norms. As the following proposition shows these spaces are the largest in our class.

**Proposition 3.5.** Let $u \in E_0(D)$ and let $v$ be a continuous plurisubharmonic exhaustion function on $D$. Then $PS_{v, \alpha}(D) \subset PS_{u, \alpha}(D)$ for each $\alpha \geq -1$ and there is a constant $C(\alpha)$ such that $\|\phi\|_{u, \alpha} \leq C(\alpha)\|\phi\|_{v, \alpha}$.

Proof. Take a number $r < 0$ such that $\text{supp}(dd^c u)^n \subset B_u(r)$. There is a constant $b > 0$ such that $b v \leq u$ on $S_u(r)$. By the maximality of $u$ the same inequality holds on $D \setminus B_u(r)$. Now the proposition follows from Corollary 3.2. □

Using pluricomplex Green functions we can get estimates for point evaluations. If $u(z) = g_D(z, w)$, $w \in D$, then by (3) (see also [D1, Thm. 5.1]) we get

$$ (2\pi)^n \phi(w) = \mu_{u, r}(\phi) - \int_{B_u(r)} (r - u) dd^c \phi \wedge (dd^c u)^{n-1} \leq \mu_{u, r}(\phi). $$

(6)
**Theorem 3.6.** Let \( v \) be a continuous plurisubharmonic exhaustion function on \( D \). Then for any compact set \( K \subset D \) and any \( \alpha \geq -1 \) there is a constant \( c \) such that for all \( w \in K \) and all nonnegative plurisubharmonic functions on \( D \) we have \( \phi(w) \leq c \| \phi \|_{v,\alpha} \).

**Proof.** By [D1, Th. 4.14] the function \( u_w(z) = g_D(z, w) \) is continuous on \( D \times D \). If \( V \subset D \) is an open set containing \( K \), then there is a negative constant \( a \) such that \( a \leq u_w(z) \) for all \( w \in K \) and \( z \in \partial V \). Consequently, there is a constant \( c > 0 \) such that \( v(z) \leq cu_w(z) \) for all \( w \in K \) and \( z \in \partial V \). The maximality of \( g_D \) outside \( F \) implies that that \( v(z) \leq cu_w(z) \) for all \( w \in K \) and \( z \in D \setminus V \).

There is a number \( r_0 < 0 \) such that \( V \subset B_{u_w}(r_0) \) for all \( w \in K \). Hence by (6) and Theorem 3.1

\[
\phi(w) \leq \frac{c^n}{(2\pi)^n} \mu_{u_w,r}(\phi) \leq \frac{c^n}{(2\pi)^n} \| \phi \|_{v,-1}
\]

and

\[
\phi(w) \int_{r_0}^{0} |r|^\alpha e^r \, dr \leq \frac{c^n}{(2\pi)^n} \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u_w,r}(\phi) \, dr \leq C \| \phi \|_{v,\alpha}.
\]

\[\square\]

We denote by \( \mathcal{E}(D) \) the class of continuous plurisubharmonic exhaustion functions \( v \) on \( D \) such that the inequality \( bv \leq u \leq b^{-1}v \) holds near \( \partial D \) for some function \( u \in \mathcal{E}_0(D) \) and constant \( b > 0 \). By Corollary 3.3 these exhaustion functions generate the same space as functions from \( \mathcal{E}_0(D) \) with equivalent norms. The class \( \mathcal{E}(D) \) contains many important functions.

**Proposition 3.7.** A plurisubharmonic exhaustion function \( v \in C^1(\overline{D}) \) on a domain \( D \) belongs to \( \mathcal{E}(D) \).

**Proof.** Fix \( w \in D \) and let \( u(z) = g_D(z, w) \). As it was explained in the proof of Proposition 3.5 the inequality \( bv \leq u \) holds for some constant \( b \) near \( \partial D \). On the other hand by Hopf’s lemma \( u(z) \leq -cd(z, \partial D) \), where \( c \) is some positive constant and \( d(z, \partial D) \) is the distance from \( z \) to the boundary of \( D \). Since for some positive constant \( a \) the function \( \rho(z) > -ad(z, \partial D) \) near \( \partial D \), the proposition follows. \[\square\]

The classical approach to the definition of spaces \( PS_{u,\alpha}(D) \) and \( PS_u(D) \) would be to restrict the studies to domains \( D \) for which there is a smooth function \( \rho \) defined on a neighborhood of \( \overline{D} \) and such that
Let $D = \{ \rho < 0 \}$ and $\nabla \rho \neq 0$ on $\partial D$. Then the definitions of spaces will go through as above with the replacement of $\mu_{u,r}(\phi)$ by

$$\nu_{\rho,r}(\phi) = \int_{\{\rho=r\}} \phi \, d\sigma,$$

where $d\sigma$ is the surface measure. We will denote the new norms by $\|\phi\|_{\rho}$ and $\|\phi\|_{\rho,\alpha}$.

The following theorem shows that both approaches coincide in the case of strongly pseudoconvex domains.

**Theorem 3.8.** Let $D$ be a strongly pseudoconvex domain and let $\rho$ be a strictly plurisubharmonic function defined on a neighborhood of $\bar{D}$ such that $D = \{ \rho < 0 \}$ and $\nabla \rho \neq 0$ on $\partial D$. If $u \in \mathcal{E}(D)$ then for each $\alpha \geq -1$ there is a constant $C > 1$ such that

$$C^{-1}\|\phi\|_{u,\alpha} \leq \|\phi\|_{\rho,\alpha} \leq C\|\phi\|_{u,\alpha}.$$

**Proof.** We just note that by [D1, (1.5)] $\mu_{\rho,r} = (dd^c \rho)^{n-1} \wedge d^c \rho$. Since this form is continuous and strictly positive on $D$, there are positive constants $a_1$ and $a_2$ such that $a_1 \mu_{\rho,r} \leq d\sigma \leq a_2 \mu_{\rho,r}$. The rest follows from Proposition 3.7 and Corollary 3.3. \(\square\)

4. **Hardy and Bergman spaces**

Let $u$ be a continuous plurisubharmonic exhaustion function on a hyperconvex domain $D$. We define the Hardy space $H^p_u(D)$, $p > 0$, as the space of all holomorphic functions $f$ on $D$ such that $|f|^p \in PS_u(D)$. The Hardy norm $\|f\|_{H^p_u} = \|\|f|^p\|_{u}^{1/p}$.

We define the Bergman space $A^p_{u,\alpha}(D)$, $p > 0$, as the space of all holomorphic functions $f$ on $D$ such that $|f|^p \in PS_{u,\alpha}(D)$. The Bergman norm $\|f\|_{A^p_{u,\alpha}} = \|\|f|^p\|_{u,\alpha}^{1/p}$.

Similar to the case of nonnegative plurisubharmonic functions we denote $A^p_{u,-1}(D) = H^p_u(D)$ and $\|f\|_{A^p_{u,-1}} = \|\|f|^p\|_u$.

The classical definition of Hardy and Bergman spaces when $D$ is the unit ball $B \subset \mathbb{C}^n$ instead of the measures $\mu_{u,t}$ uses the measures $\nu_r = dS_r$, where $dS$ is the normalized surface area, on spheres of radius $r$. If we take $u(z) = \log |z|$ as an exhaustion function, then $dd^c u_t$ is a rotationally invariant measure which is supported by the sphere $\{|z| = r = e^t\}$ and is a constant multiple of $dS$. By Riesz representation formula or by the Lelong–Jensen formula

$$\int_D dd^c u_t = (2\pi)^n.$$

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So $\mu_{u,t} = (2\pi)^n dS$ and we see that our definition coincides with the classical one.

If $D = \mathbb{D}^2$ is the unit bidisk in $\mathbb{C}^2$ with coordinates $z_1$ and $z_2$ and $u(z) = \log \max\{|z_1|, |z_2|\}$, then $(dd^c u)^2 = (2\pi)^2 \delta_0$ and the measure $(dd^c u_r)^2$ is supported by the set $\{|z_1| = |z_2| = r\}$. The latter measure is also rotationally invariant, so $\mu_{u,r}$ is a constant multiple of the normalized surface area on the torus $\{|z_1| = |z_2| = r\}$ and again our definition coincides with the classical definition of Hardy (but not Bergman) spaces.

Let us show that Hardy and Bergman spaces are Banach.

**Theorem 4.1.** Let $u$ be a continuous plurisubharmonic exhaustion function on $D$. Then the spaces $A_{u,\alpha}^p(D)$, $p \geq 1$, are Banach.

**Proof.** Clearly $\|u\|_{A_{u,\alpha}^p}$ is a norm on $A_{u,\alpha}^p(D)$.

Suppose that $\{f_j\}$ is a Cauchy sequence in $H_u^p(D)$. By Theorem 3.6 $\{f_j\}$ is a Cauchy sequence in the uniform metric on any compact set in $D$. Hence this sequence converges to a holomorphic function $f$ on $D$ uniformly on compacta. In particular, for every fixed $r < 0$

$\mu_{u,r}(|f - f_j|^p) \to 0$ as $j \to \infty$.

There is $A > 0$ such that $\|f_j\|_{H_u^p} \leq A$ for all $j$. Hence

$$(\mu_{u,r}(|f|^p))^{1/p} \leq (\mu_{u,r}(|f - f_j|^p))^{1/p} + A$$

and we see that $f \in H_u^p(D)$ and $\|f\|_{H_u^p} \leq A$.

Suppose that $\limsup \|f - f_j\|_{H_u^p} \neq 0$. Switching to a subsequence we may assume that $\|f - f_j\|_{H_u^p} \geq a > 0$ for all $j$. Take $i$ such that $\|f_i - f_j\|_{H_u^p} < a/4$ when $j > i$ and then find $r < 0$ such that $\mu_{u,r}(|f - f_i|^p) > (a/2)^p$. Then

$$(\mu_{u,r}(|f - f_j|^p))^{1/p} \geq (\mu_{u,r}(|f - f_i|^p))^{1/p} - \|f_i - f_j\|_{H_u^p} \geq \frac{a}{4},$$

But the left side converges to 0 as $j \to \infty$ and this proves that $\lim \|f - f_j\|_{H_u^p} = 0$.

The proof for the Bergman spaces is basically the same. $\square$

Our next goal is to prove a multidimensional analog of Littlewood’s subordination principle.

**Theorem 4.2.** Let $D$ be a hyperconvex domain, $z_0 \in D$, $u(z) = g_D(z, z_0)$ and let $f : D \to \mathbb{D}$ be a holomorphic mapping with $f(z_0) = w_0$. If $v(w) = \log |(w-w_0)/(1-\overline{w_0} w)|$ then $\mu_{u,r}(\phi \circ f) \leq (2\pi)^{n-1} \mu_{v,r}(\phi)$ for every subharmonic function $\phi$ on $\mathbb{D}$.

**Proof.** The relation (2) implies that $v^*(z) = v(f(z)) \leq u(z)$ on $D$. Hence $f(B_u(r)) \subset B_{v}(r)$.
Let \( r < t < 0 \) and let \( h_t \) be the harmonic function on \( B_v(t) \) equal to \( \phi \) on \( S_v(t) \). Then \( h_t(f(z)) \geq \phi(f(z)) \) on \( S_u(r) \) and therefore
\[
\mu_{u,r}(\phi \circ f) \leq \mu_{u,r}(h_t \circ f).
\]
Since the function \( h_t \circ f \) is pluriharmonic, \( dd^c(h_t \circ f) \equiv 0 \). Since \( h_t \circ f \) is defined on \( B_u(t) \) we can use (6) to get
\[
\mu_{u,r}(h_t \circ f) = (2\pi)^n h_t(w_0).
\]
Since \( \mu_{v,t}(\phi) = \mu_{v,t}(h_t) = 2\pi h_t(w_0) \), we see that \( \mu_{u,r}(\phi \circ f) \leq \mu_{v,t}(\phi) \).

The following corollary follows immediately from the previous theorem.

**Corollary 4.3.** Let \( v(w) = \log|w|, \ w \in \mathbb{D} \), and let \( D \) be a hyperconvex domain, \( u \in E(D) \), and \( f : D \to \mathbb{D} \) be a holomorphic mapping. Then the composition operator \( C_f \) maps \( A_{v,\alpha}(\mathbb{D}) \) into \( A_{u,\alpha}(D) \). Moreover, there is a constant \( A \) depending only on \( |w_0| \), \( w_0 = f(z_0) \), and \( u \) such that \( \|C_f\|_{u,\alpha} \leq A\|\phi\|_{v,\alpha} \).

Consequently, \( C_f \) maps continuously \( A^p_{v,\alpha}(\mathbb{D}) \) into \( A^p_{u,\alpha}(D) \).

For a holomorphic mapping of a domain \( D \) into another domain \( F \) we introduce the measure \( \nu_{f,\alpha} \) on \( F \) defined as
\[
\nu_{f,\alpha}(E) = \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(\chi_E(f(z))) \, dr,
\]
where \( \chi_E \) is the characteristic function of \( E \). The proof of the following lemma is similar to the proof of Theorem 2.6 in [CoM].

**Lemma 4.4.** Let \( E(w,s) = \{|z - w| < s\} \cap \mathbb{D} \). In the assumptions of Corollary 4.3 there is a constant \( a \geq 1 \) depending only on \( |w_0| \) such that \( \nu_{f,\alpha}(E(w,s)) \leq as^{\alpha+2} \) when \( s \leq 1 \) for all \( w \in S \).

**Proof.** It suffices to prove this lemma for \( w = 1 \). Let us take \( t = 1/(1 + 2s) \), \( 0 < s < 1 \), and consider the function
\[
\phi(z) = \frac{1}{1 - tz}^{2\alpha+4}.
\]
Then by Corollary 4.3
\[
\int_D \phi \, d\nu_{f,\alpha} = \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(\phi(f(z))) \, dr
\]
\[
= \|\phi \circ f\|_{A^1_{u,\alpha}(D)} \leq c\|\phi\|_{A^1_{v,\alpha}(\mathbb{D})} = c_1 \frac{1}{(1 - t^2)^\alpha + 2}.
\]
If $|1 - z| < s$, then
\[
\frac{1}{|1 - tz|} = \frac{1 + 2s}{|1 - z + 2s|} \geq \frac{1 + 2s}{3s} \geq \frac{1}{3s}
\]
and
\[
\frac{1}{1 - t^2} \leq \frac{3}{s}.
\]
Thus
\[
c_1 \left( \frac{3}{s} \right)^{\alpha+2} \geq c_v \frac{1}{(1 - t^2)^{\alpha+2}} \geq \int_{\mathbb{D}} \phi \, d\nu_{f,\alpha} \geq \left( \frac{1}{3s} \right)^{2\alpha+4} \nu_{f,\alpha}(E(w, s)).
\]
Hence $\nu_{f,\alpha}(E(w, s)) \leq as^{\alpha+2}$ for all $w \in S$. \hfill \Box

5. The change of variables formula

Throughout this section $D$ is a domain in $\mathbb{C}^n$ and $u_0, \ldots, u_{n-1}$ are continuous plurisubharmonic functions on $D$. We set
\[
T = u_0 dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1}.
\]
We will assume that for any choice of indexes $0 \leq j_1 < \cdots < j_m \leq n-1$
\[
\mathcal{H}_{2(n-m+1)}(L(u_{j_1}) \cap \cdots \cap L(u_{j_m})) = 0. \tag{8}
\]
We fix a holomorphic function $f$ on $D$ mapping $D$ into a domain $\Omega \subset \mathbb{C}$. For a subharmonic function $v$ on $\Omega$ we define a functional on $D^0(D) \times D^0(\Omega)$
\[
\beta_{f,v,T}(\phi, \psi) = \beta_f(\phi, \psi) = \int_{D} \phi \psi^* dd^c v^* \wedge T,
\]
where $\psi^*$ and $v^*$ are $\psi \circ f$ and $v \circ f$ respectively.

Lemma 5.1. If the functions $u_0, \ldots, u_{n-1}$ satisfy (8) and for any choice of indexes $0 \leq j_1 < \cdots < j_m \leq n-1$
\[
\mathcal{H}_{2(n-m)}(L(v^*) \cap L(u_{j_1}) \cap \cdots \cap L(u_{j_m})) = 0,
\]
then for a fixed $\phi$ the functional $\beta_{f,v,T}(\phi, \psi)$ is a current of bidegree $(0, 0)$ on $\Omega$ and for a fixed $\psi$ the functional $\beta_{f,v,T}(\phi, \psi)$ is a current of bidegree $(0, 0)$ on $D$.

Proof. It follows from Theorem 2.3 that the current $dd^c v^* \wedge T$ has a locally finite mass. Since the function $\psi^*$ is bounded on $D$, we see that $\beta_{f,v,T}(\phi, \psi)$ is a current of bidegree $(0, 0)$ on $D$.

Let $\phi \in D^0(D)$ be fixed. Then the functional $\beta_{f,v,T}(\phi, \psi)$ is defined for all $\psi \in D^0(\Omega)$. To show that the functional $\beta_{f,v,T}(\phi, \psi)$ is continuous we note that if a sequence $\{\psi_j\}$ converges to 0, then the functions
{φψj} converge to 0 uniformly on D and, therefore, βf,v,T(φ,ψj) → 0 as j → ∞.

The main goal of this section is to get a formula for the measure on Ω defining βf,v,T. For this we take a function φ ∈ D0(D) and for w ∈ C introduce the function

\[ N_{f,T}(w, φ) = N(w, φ) = \int_D φ dd^c \log |f - w| ∧ T. \]

This function need not be finite. For example if \( D = D, T = \log |z| \), \( f(z) = z \) and \( w = 0 \), then \( N(0, φ) = -\infty \) when \( φ(0) > 0 \). The following proposition gives sufficient conditions for the function \( N(w, φ) \) to be finite and states some of its properties.

Let \( X_w = \{ f = w \} \). Write

\[ N_0(w, φ) = \sum_A m_{A,f} \int_{A^{\text{reg}}} φT, \]

where the summation runs over all irreducible components \( A \) of \( X_w \) and \( m_{A,f} \) is the multiplicity of \( f \) on \( A^{\text{reg}} \).

**Proposition 5.2.** If the functions \( u_0, \ldots, u_{n-1} \) satisfy (8) and \( \mathcal{L}(u_j) ∩ X_w = \emptyset \) for any \( 0 \leq j \leq n - 1 \), then \( N(w, φ) < \infty \) and \( N(w, φ) = N_0(w, φ) \). Moreover, if, additionally, \( φ \geq 0 \) on \( D \) and the function \( u_0 < 0 \) on \( \text{supp} \ φ \), then the function \( N(τ, φ) \) is upper semicontinuous (as a function of \( τ \)) at \( w \). Moreover, if the set \( X_w \) is smooth then \( N(τ, φ) \) is continuous at \( w \).

**Proof.** The fact that \( N(w, φ) \) is bounded follows immediately from Theorem 2.3.

To show that \( N(w, φ) = N_0(w, φ) \) we observe that the function \( u_0 \) does not exceed some number \( a > 0 \) on a neighborhood \( V \subset D \) of \( \text{supp} \ φ \). Thus

\[ T = (u_0 - a)dd^c u_1 ∧ ⋯ ∧ dd^c u_{n-1} + add^c u_1 ∧ ⋯ ∧ dd^c u_{n-1} \]

is the sum of a negative and a positive current. Hence, without loss of generality we may assume that \( u_0 < 0 \) on \( V \).

Then we take a decreasing sequence of smooth plurisubharmonic functions \( u_{jk} \) defined on \( V \) and converging to \( u_j \) for every \( 0 \leq j \leq n-1 \). By Theorem 2.3

\[ \int_D φT ∧ dd^c \log |f - w| = \lim_{k→∞} \int_V φT_k ∧ dd^c \log |f - w|, \]

where \( T_k = u_{0k}dd^c u_{1k} ∧ ⋯ ∧ dd^c u_{n-1,k} \).
By Poincaré–Lelong formula (4)

\[ \int_V \phi T_k \wedge dd^c \log |f - w| = \sum_A m_{A,f} \int_{A^{\text{reg}}} \phi T_k. \]

Now we choose an increasing sequence of nonnegative functions \( \psi_j \in D^{0,0}(V) \) equal to 0 on \( X_w^{\text{sing}} \) and converging to 1 on \( \text{supp} \phi \setminus X_w^{\text{sing}} \). Since the functions \( u_{jk} \) are bounded on \( X_w \) and converge to \( u_j \),

\[ \lim_{k \to \infty} \int_{A^{\text{reg}}} \phi \psi_j T_k = \int_{A^{\text{reg}}} \phi \psi_j T. \]

Thus,

\[ \int_V \phi \psi_j T \wedge dd^c \log |f - w| = \sum_A m_{A,f} \int_{A^{\text{reg}}} \phi \psi_j T. \]

If \( \phi \geq 0 \) on \( V \), then letting \( j \to \infty \) by the monotone convergence theorem we get that

\[ \int_{V \setminus X_w^{\text{sing}}} \phi T \wedge dd^c \log |f - w| = \sum_A m_{A,f} \int_{A^{\text{reg}}} \phi T. \] (9)

If \( \phi \) is not positive on \( V \) then we take a function \( \tilde{\phi} \in D^{0,0}(V) \) such that \( \tilde{\phi} \geq |\phi| \). Since the function \( \psi = \tilde{\phi} - \phi \geq 0 \) on \( D \) and \( \phi = \tilde{\phi} - \psi \), the equality (9) holds in this case also.

Now let us show that

\[ \int_{X_w^{\text{sing}}} \phi T \wedge dd^c \log |f - w| = 0. \]

Using a partition of unity we can reduce it to the case when \( D = B(z_0, r) \), \( \phi \geq 0 \), \( z_0 \in X_w^{\text{sing}} \) and the analytic set \( X_w^{\text{sing}} \) is the set of common zeros of holomorphic function \( f_1, \ldots, f_k \) on \( D \). Then for the plurisubharmonic function

\[ \tilde{u}_0 = u_0 + \log \sum_{i=1}^k |f_i|^2 \]

the set \( L(\tilde{u}_0) = L(u_0) \cup X_w^{\text{sing}} \). If \( \tilde{T} = \tilde{u}_0 dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1} \) and \( u_n = \log |f - w| \), then the conditions (8) hold when at least one of the indexes \( j_i \) is between 1 and \( n-1 \) because \( L(u_j) \cap X_w = \emptyset \) for all \( j \). If \( m = 1 \) then \( \mathcal{H}_{2n}(L(u_0)) = \mathcal{H}_{2n}(X_w) = 0 \), and, therefore, \( \mathcal{H}_{2n}(L(\tilde{u}_0)) = 0 \). If \( m = 2 \), \( j_1 = 0 \) and \( j_2 = n \), then the set \( L(\tilde{u}_0) \cap L(u_n) = X_w^{\text{sing}} \) is an analytic set of codimension at least 2 and, therefore, \( \mathcal{H}_{2n-2}(L(\tilde{u}_0) \cap L(u_n)) = 0 \). Hence, by Theorem 2.3, the current \( \tilde{T} \wedge \log |f - w| \) is locally integrable.
But $\tilde{u}_0 \equiv -\infty$ on $X_w^{\text{sing}}$. Therefore, the current $dd^c u_1 \wedge \ldots dd^c u_n$ has no mass on $X_w^{\text{sing}}$, and

$$\int_{X_w^{\text{sing}}} \phi T \wedge dd^c \log |f - w| = 0.$$ 

Let us show that the function $N_0(w, \phi)$ is upper semicontinuous at $w$ which we assume to be equal to 0. Take a compact set $K_A$ in a component $A_0^{\text{reg}}$, cover $K_A$ by open balls $V_k \subset V$, $1 \leq k \leq k_0$, such that $f(\zeta) = \zeta^m$, $m = m_{A,f}$, in some new coordinates $(\zeta_1, \ldots, \zeta_n)$ on $V_k$ and form a partition of unity $\psi_k \in D^{0,0}(V_k)$ such that $\psi = \sum_j \psi_k = 1$ on a neighborhood $W_A$ of $K_A$.

There is an $\varepsilon > 0$ such that for every $k$ and every $w \neq 0$, $|w| < \varepsilon$, the set $X_w \cap V_k$ is a complex manifold consisting of $m$ connected components $X_{kj}^w$, $1 \leq j \leq m$, given by the equations $\zeta_1 = w^{1/m}$.

If $\zeta = (\zeta_2, \ldots, \zeta_n)$ then by continuity of the functions $u_i$ the functions $u_{ikj}(\zeta) = u_i(w^{1/m}, \zeta)$ uniformly converge to $u_i(0, \zeta)$ as $w \to 0$ and, therefore,

$$\lim_{w \to 0} \int_{X_{kj}^w} \phi \psi_k T = \int_{X_0 \cap W_k} \phi \psi_k T.$$ 

Since $\phi \geq 0$ and $u_0 \leq 0$, so $\phi T$ is a negative current, summation over $j$ and $k$ results in

$$\int_{K_A} \phi T \geq \limsup_{w \to 0} \int_{X_w \cap W_A} \phi T,$$

and our statement follows.

To finish the proof of the proposition let us show that $N_0(0, r)$ is continuous at 0 when $X_0$ is smooth. In this case we can take a compact set $K = X_0 \cap \text{supp} \phi$ and the balls $V_k$, covering $K$, can be chosen so that each of them contains only one component of $X_0$. Now the previous argument yields the proof. \qed

**Remark.** This result does not hold when $L(u_j) \cap X_w \neq \emptyset$. For example, if $D$ is the unit ball in $\mathbb{C}^2$ with coordinates $z_1$ and $z_2$, $u_0(z) \equiv 1$, $u_1(z) = \log |z|$ and $f(z) = z_1 z_2$, then

$$N(0, \phi) = \int_D \phi dd^c u_1 \wedge dd^c \log |f| = 2 \phi(0).$$

However, $u_1(z)$ is equal to either $\log |z_1|$ or $\log |z_2|$ on $X_0^{\text{reg}}$. Therefore, $dd^c u_1 = 0$ on $X_0^{\text{reg}}$ and $N_0(0, \phi) = 0$.

We will need the following version of Fubini’s theorem.
Lemma 5.3. Let $X$ be a complex manifold of dimension $n - 1$, $U$ be a domain in $\mathbb{C}$, $v$ be a subharmonic function on $U$ and $u_0, \ldots, u_{n-1}$ be bounded pluriharmonic functions on $X \times U$. If $\phi \in D^{0,0}(X \times U)$ then
\[
\int_{X \times U} \phi T \wedge dd^c v = \int_{U} \left( \int_{X \times \{w\}} \phi T \right) dd^c v.
\]

Proof. Let us show that the lemma holds when the functions $u_j$ and $v$ are smooth. Introducing a partition of unity we can reduce it to the case when $X$ is a domain in $\mathbb{C}^{n-1}$ with coordinates $z = (z_1, \ldots, z_{n-1})$. Let $w$ be a coordinate on $U$ and
\[
dd^c z u_j = 2i \sum_{i,k=1}^{n-1} \frac{\partial^2 u_j}{\partial z_i \partial \bar{z}_k}(z,w) dz_i \wedge d\bar{z}_k.
\]
Note that $dd^c z u_j$ is the restriction of $dd^c u_j$ to $X \times \{w\}$. Hence
\[
\int_{X \times \{w\}} \phi T = \int_{X \times \{w\}} \phi T_w,
\]
where $T_w = u_0 dd^c z u_1 \wedge \cdots \wedge dd^c z u_{n-1}$. Since $\phi T \wedge dd^c v = \phi T_w \wedge dd^c v$, the lemma follows immediately from Fubini’s theorem.

In the general case let us fix open sets $U_1 \subset \subset U$ and $X_1 \subset \subset X$ such that $\text{supp} \phi \subset X_1 \times U_1$. We take decreasing sequences $\{u_{jk}\}$, $0 \leq j \leq n-1$, and $\{v_k\}$ of smooth pluriharmonic functions on $X_1 \times U_1$ and $U_1$ converging to $u_j$ and $v$ respectively.

Let $T_k = u_{0k} dd^c z u_{1k} \wedge \cdots \wedge uu_{n-1,k}$. Note that the functions
\[
\Phi_k(w) = \int_{X \times \{w\}} \phi T_k
\]
are smooth and have compact support on $U$. Since $dd^c v_m$ converge weak-$*$ to $dd^c v$ on $U$ we see that
\[
\int_{X \times U} \phi T_k \wedge dd^c v = \int_{U} \Phi_k(w) \, dd^c v.
\]
As $k \to \infty$ the monotonic convergence of $u_{jk}$ implies that
\[
\lim_{k \to \infty} \int_{X \times U} \phi T_k \wedge dd^c v = \int_{X \times U} \phi T \wedge dd^c v.
\]
and for each \( w \) the functions \( \Phi_k(w) \) converge to

\[
\Phi(w) = \int_{X \times \{w\}} \phi T.
\]

Since \( u_{jk} \geq u_{j,k+1} \geq u_j \) and the functions \( u_j \) are bounded, there is a constant \( a \) such that the \( L^{\infty} \)-norms of \( u_{jk} \) on \( X_1 \times U_1 \) do not exceed \( a \). By Chern-Levine-Nirenberg inequality (see [Kl], p.111-112) there is another constant \( C \) such that \( |\Phi_k(w)| \leq C \). The support of \( \Phi_k(w) \) lies in \( U_1 \subset U \) and \( dd^c v \) has a finite mass on \( U_1 \). Thus, by the dominated convergence theorem

\[
\lim_{k \to \infty} \int_U \Phi_k(w) dd^c v = \int_U \Phi(w) dd^c v.
\]

Hence,

\[
\int_{X \times U} \phi T \wedge dd^c v = \int_U \Phi(w) dd^c v.
\]

\( \square \)

The following theorem, which is the main goal of this section, gives us a change of variables formula.

**Theorem 5.4.** Let \( f \) be a holomorphic function on a domain \( D \subset \mathbb{C}^n \) and \( u_0, \ldots, u_{n-1} \) be continuous plurisubharmonic functions on \( D \). Let \( \Omega \) be a domain in \( \mathbb{C} \) such that \( f(D) \subset \Omega \) and let \( v \) be a subharmonic function on \( \Omega \) and \( v^* = v \circ f \). Suppose that the sets \( L(u_j) \) are finite and \( f(L(u_j)) \cap L(v) = \emptyset \) for \( 0 \leq j \leq n-1 \). If \( \phi \in D^{0,0}(D) \) and \( \psi \in D^{0,0}(\Omega) \), then

\[
\beta_{f,v,T}(\phi, \psi) = \int_{\Omega} N(w, \phi \psi^*) dd^c v.
\]

**Proof.** Let us assume for a while that the functions \( \phi \) and \( \psi \) are non-negative and \( u_0 \leq 0 \) on \( \text{supp} \phi \). For a set \( E \subset D \) we define the measure

\[
\nu(E) = -\int_E \phi \psi^* T \wedge dd^c v^*.
\]

Let \( W_1 \) be the union of the sets \( f(L(u_j)) \), \( 0 \leq j \leq n - 1 \). This is a finite set consisting of \( p \) points \( \{w_j\} \). Let \( X = \bigcup_{j=1}^{p} X_{w_j} \) and \( \tilde{u}_0 = u_0 + \sum_{j=1}^{p} \log |f - w_j| \). Let \( u_n = v^* \). If \( j_m = n \) and \( 0 \leq j_k \leq n - 1 \) at least once for a choice of indexes \( 0 \leq j_1 < \cdots < j_m \), then \( L(u_{j_1}) \cap \cdots \cap L(u_{j_m}) = \emptyset \) and

\[
\mathcal{H}_0(L(u_{j_1}) \cap \cdots \cap L(u_{j_m})) = 0.
\]
If $m \geq 2$ and $1 \leq j_m < n$ then the sets $L(u_{j_1}) \cap \cdots \cap L(u_{j_m})$ are finite and $H_1(L(u_{j_1}) \cap \cdots \cap L(u_{j_m})) = 0$. If $m = 1$ and $j_1 = 0$ or $j_1 = n$ then $H_2n(X) = 0$. So we see that (1) holds and the current $\tilde{u}_0 dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1} \wedge dd^c v^*$ has locally finite mass. Since $\tilde{u}_0 = -\infty$ in $X$, 

$$\int_X dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1} \wedge dd^c v^* = 0$$

and, consequently, $\nu(X) = 0$.

Now, let $W_2 = \{w_1, \ldots, w_q\}$ be the set of those critical values of $f$ on $\text{supp} \phi$ which do not belong to any $f(L(u_j))$. We denote by $Y$ the union of $X_{w_k}^{\text{sing}}$ for all $k$. Since the functions $u_j$ are bounded on $f^{-1}(W_2)$, the argument of Proposition 5.2 shows that 

$$\int_Y dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1} \wedge dd^c v^* = 0.$$ 

Again $\nu(Y) = 0$.

Write $Z = X \cup Y$. It follows that

$$\nu(E) = -\int_{E \setminus Z} \phi \psi^* T \wedge dd^c v^*.$$ 

We fix a point $w_0 \in C \setminus W_1$ that we assume to be equal to 0. Since $X_0 \setminus Z$ is a complex manifold, for every its point there is a neighborhood and new coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$ on it such that $f(\zeta) = \zeta^m$. A preferred neighborhood $U_\zeta(\varepsilon)$ of a point $z \in X_0 \setminus Z$ is an open set that in new $\zeta$-coordinates has the form $\{|\zeta_1| < \varepsilon^{1/m}, \zeta = (\zeta_2, \ldots, \zeta_n) \in \mathbb{D}^{n-1}(0, \varepsilon)\}$.

Now, let us take an increasing sequence of nonnegative functions $\phi_k \in \mathcal{D}^{0,0}(D \setminus Z)$ converging to 1 on $D \setminus Z$. There are finitely many components $A_{kl}$, $1 \leq l \leq l_k$, of $X_0$ intersecting the set $F_k = \text{supp} \phi_k$.

Let us fix $k$ and choose a finite set of preferred neighborhoods

$$U_{kl} = \{|\zeta_1| < \varepsilon^{1/m_{kl}}, \zeta \in \mathbb{D}^{n-1}(0, \varepsilon_{kl})\},$$

where $m_{kl}$ is the multiplicity of $f$ on $A_{kl}$, covering each set $A_{kl} \cap F_k$, and nonnegative functions $\psi_{kl} \in \mathcal{D}^{0,0}(U_{kl})$ such that $\sum \psi_{kl} \equiv 1$ on a neighborhood $W_k$ of $X_0 \cap F_k$. Clearly there is an $\varepsilon_0 > 0$ such that in the preferred coordinates the set

$$U_{kl}(\varepsilon) = \{\zeta \in U_{kl}, |\zeta_1| < \varepsilon^{1/m_{kl}}, \zeta \in \mathbb{D}^{n-1}(0, \varepsilon_{kl})\} \subset W_k$$

when $\varepsilon \leq \varepsilon_0$. Note that $U_{kl}(\varepsilon) = \{\zeta \in U_{kl} : |f(\zeta)| < \varepsilon\}$. 

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Let \( V(\varepsilon) = f^{-1}(\mathbb{D}(0, \varepsilon)) \). There is a positive \( \varepsilon_1 < \varepsilon_0 \) such that

\[
V(\varepsilon) \cap F_k \subset U_k(\varepsilon) = \bigcup_{l,i} U_{kli}(\varepsilon)
\]

when \( \varepsilon < \varepsilon_1 \). Hence, if the measure \( \nu_k \) is defined by

\[
\nu_k(E) = -\int_E \phi \phi_k \psi^* T \wedge dd^c v^*,
\]

then

\[
\nu_k(V(\varepsilon)) = -\sum_{l,i} \int_{U_{kli}(\varepsilon)} \phi \phi_k \psi_{kli} \psi^* T \wedge dd^c v^* \tag{10}
\]

when \( \varepsilon < \varepsilon_1 \).

Since every preferred neighborhood \( U_{kli} \) has a structure of a direct product \( \mathbb{D}(0, \varepsilon_{kli}^{1/m_{kl}}) \times \mathbb{D}^{n-1}(0, \varepsilon_{kli}) \) and \( v^*(\zeta) = v(\zeta_{kli}^{m_{kl}}) = v_1(\zeta) \) on \( U_{kli} \), by Lemma 5.3

\[
\Psi_{kli}(\varepsilon) = -\int_{U_{kli}(\varepsilon)} \phi \phi_k \psi_{kli} \psi^* T \wedge dd^c v^* = -\int_{\mathbb{D}(0, \varepsilon^{1/m_{kl}})} \psi \psi_{kli} \int_{A_\xi} \phi \phi_k \psi_{kli} T,
\]

where \( A_\xi = \{ \zeta \in U_{kli} : \zeta_1 = \xi \} \).

Since the functions \( u_j \) are continuous and not equal to \(-\infty\) near \( X_0 \), the functions

\[
\Phi_{kli}(\xi) = \int_{A_\xi} \phi \phi_k \psi_{kli} T
\]

are also continuous.

If

\[
\eta(E) = \int_E dd^c v,
\]

then

\[
\int_{\mathbb{D}(0, \varepsilon^{1/m_{kl}})} dd^c v_1 = m_{kl} \eta(\mathbb{D}(0, \varepsilon)).
\]

Hence,

\[
\lim_{\varepsilon \to 0} \frac{\Psi_{kli}(\varepsilon)}{\eta(\mathbb{D}(0, \varepsilon))} = m_{kl} \psi(0) \Phi_{kli}(0).
\]

By (10)

\[
\lim_{\varepsilon \to 0} \frac{\nu_k(V(\varepsilon))}{\eta(\mathbb{D}(0, \varepsilon))} = -\sum_{l,i} m_{kl} \psi(0) \Phi_{kli}(0).
\]
But the last sum is equal to
\[- \sum_A m_A \int_{X_0} \phi_k \psi^* T = N_0(0, \phi_k \psi^*) = N(0, \phi_k \psi^*).\]

Thus, we see that the density of the measure \(\tilde{\nu}_k(E) = \nu_k(f^{-1}(E))\) with respect to \(\eta\) is equal to \(N(0, \phi_k \psi^*)\) at the points that do not belong to \(f(W_1)\). Since \(v(w)\) is bounded near a point \(w_0 \in f(W_1)\), its Laplacian has no mass at \(w_0\). Hence,
\[\nu_k(D) = - \int_{\Omega} N(w, \phi_k \psi^*) dd^c v.\]

The functions \(\phi_k \psi^*\) form an increasing sequence and, therefore,
\[\lim_{k \to \infty} \nu_k(D) = - \int_{\Omega} \phi^* T \wedge dd^c v^*,\]
while
\[\lim_{k \to \infty} N_0(0, \phi_k \psi^*) = N_0(0, \psi^*) = N(0, \psi^*).\]
Hence
\[\int_{\Omega} \phi^* T \wedge dd^c v^* = \int_{\Omega} N(w, \psi^*) dd^c v\]
and we proved the theorem in the case of \(\phi, \psi \geq 0\) and \(u_0 \leq 0\) on \(\text{supp} \phi\).

For the proof of the general case we just note that the function \(u_0\) does not exceed some constant \(a > 0\) and, therefore, is the difference of two plurisubharmonic functions negative on \(\text{supp} \phi\). The functions \(\phi\) and \(\psi\) also can be written as differences of nonnegative test functions. Hence, the general case can be reduced to the case we have just considered. \(\square\)

6. NEVANLINNA COUNTING FUNCTIONS

Let \(D \subset \mathbb{C}^n\) be a hyperconvex domain with a continuous exhaustion function \(u\) and let \(f\) be a holomorphic function on \(D\). For \(r < 0\) we set \(T_r = (u - r)(dd^c u)^{n-1}\). Let us introduce the Nevanlinna counting functions as
\[N_{u,f}(w, r) = N(w, r) = - \int_{B_u(r)} T_r \wedge dd^c \log |f - w|\]
and

\[ N_0(w, r) = -\sum_A m_{A,f} \int_{A_{w,r}^{\text{reg}}} T_r, \]

where the summation runs over all irreducible components \( A_w \) of \( X_w \) and \( A_{w,r}^{\text{reg}} = A_w^{\text{reg}} \cap B_u(r) \).

Theorem 5.4 leads to the following result.

**Theorem 6.1.** If \( w_0 \notin f(L(u)) \) then \( N(w_0, r) = N_0(w_0, r) \) and the function \( N(w, r) \) is lower semicontinuous at \( w_0 \).

Let \( \Omega \) be a domain in \( \mathbb{C} \) such that \( f(D) \subset \Omega \) and let \( v \) be a subharmonic function on \( \Omega \) and \( v^* = v \circ f \). If \( L(u) \) is a finite set and \( f(L(u)) \cap L(v) = \emptyset \), then

\[ \int_{B_u(r)} (r - u)(dd^c u)^{n-1} \wedge dd^c v^* = \int_{\Omega} N(w, r) dd^c v. \]

**Proof.** In Theorem 5.4 let \( u_0 = u - r \) and \( u_1 = \cdots = u_{n-1} = u \). Take increasing sequences of nonnegative functions \( \{\phi_k\} \subset D^{0,0}(D) \) with \( \text{supp} \phi_k \subset B_u(r) \) and \( \{\psi_k\} \subset D^{0,0}(\mathbb{D}) \) converging to 1 on \( B_u(r) \) and \( \Omega \) respectively. Now Theorem 5.4 and the monotone convergence theorem yield the result. \( \square \)

Let us explain why we call these functions the Nevanlinna counting functions. Our first observation is that by Fubini’s theorem

\[ N(w, r) = \int_{B_u(r)} (r - u)(dd^c u)^{n-1} \wedge dd^c |f - w| \]

\[ = \int_{-\infty}^r dt \int_{B_u(t)} (dd^c u)^{n-1} \wedge dd^c \log |f - w|. \]

Therefore, if

\[ n_{u,f}(w, r) = n(w, r) = \int_{B_u(r)} (dd^c u)^{n-1} \wedge dd^c |f - w|, \]

then

\[ N(w, r) = \int_{-\infty}^r n(w, t) dt. \]

The function \( n(w, t) \) has a nice geometric interpretation. First of all, if \( n = 1 \) then \( n(w, r) \) is equal to the number of points in \( B_u(r) \), where \( f(z) = w \), counted with their multiplicities. Hence, it coincides
with the function \( n(w, r) \) from Nevanlinna’s theory. If \( D = \mathbb{D} \) and \( u(z) = \log |z| \), then

\[
N(w, \log \rho) = \int_{0}^{\rho} \frac{n(w, \log t)}{t} dt
\]

and we see again a classical formula.

If \( n > 1 \) and \( w \notin f(L(u)) \), then by Proposition 5.2

\[
n(w, r) = \int_{X \cap B_u(r)} (dd^c u)^{n-1}.
\]

The form \( (dd^c u)^{n-1} \) can be viewed as \( 2(n-1) \)-dimensional volume form on complex manifolds in the pseudo-metric \( dd^c u \). For example, if \( D \) is the unit ball and \( u(z) = |z|^2 - 1 \), then \( (dd^c u)^{n-1} \) is exactly a scalar multiple of this volume form. So in the multidimensional case, like in the classical parabolic setting, we count not the number of preimages of \( w \) in \( B_u(r) \) but their pseudo-areas.

However, contrary to the classical parabolic case we are more interested in the final form of the counting function, i.e., the function

\[
N_{u,f}(w) = N(w) = -\int_{D} u(dd^c u)^{n-1} \wedge dd^c \log |f - w|,
\]

which is the limit of \( N(w, r) \) as \( r \to 0^- \).

For \( \alpha > -1 \) and \( u < 0 \) we introduce the following auxiliary functions

\[
\sigma_{\alpha}(u) = \int_{u}^{0} |r|^\alpha e^r dr = \frac{1}{\alpha + 1} |u|^{\alpha + 1} + o(|u|^{\alpha + 1})
\]

and

\[
\gamma_{\alpha}(u) = \int_{u}^{0} |r|^\alpha e^r (r - u) dr = -\sigma_{\alpha+1}(u) - u \sigma_{\alpha}(u).
\]

Note that both functions are positive, \( \sigma_{\alpha}(u) \leq \Gamma(\alpha + 1) \) and

\[
\gamma_{\alpha}(u) = \frac{1}{(\alpha + 1)(\alpha + 2)} |u|^{\alpha + 2} + o(|u|^{\alpha + 2}).
\]

We also set \( \sigma_{-1}(u) = 1 \) and \( \gamma_{-1}(u) = -u \).

We define the Nevanlinna counting function of order \( \alpha \) as

\[
N_{u,f,\alpha}(w) = N_{\alpha}(w) = \int_{D} \gamma_{\alpha}(u)(dd^c u)^{n-1} \wedge dd^c \log |f - w|.
\]
By Fubini’s theorem

\[ N_\alpha(w) = \int_{-\infty}^{0} |r|^\alpha e^r N(w, r) \, dr. \] (11)

If \( n = 1 \) then \( N_\alpha(w) = \sum_{f(z)=w} \gamma_\alpha(u(z)). \)

The importance of the Nevanlinna counting function is explained by the fact that the Hardy and weighted Bergman norms of functions can be expressed in terms of their Nevanlinna counting functions. The following result may be viewed as a generalization of the classical Littlewood-Paley formula.

**Theorem 6.2.** Let \( D \) be a hyperconvex domain in \( \mathbb{C}^n \) with an exhaustion function \( u \) such that the set \( L(u) \) is finite. If \( f \) is a holomorphic function on \( D \), then

\[
\|f\|_{A^p_{\mu,\alpha}}^p - \int_D \sigma_\alpha(u)|f|^p(\dd\bar{u}u)^n = \int_D \gamma_\alpha(u) \dd\bar{u} |f|^p \wedge (\dd\bar{u}u)^{n-1} \\
= \int N_\alpha(w) \dd\bar{u} |w|^p.
\]

**Proof.** The case \( \alpha = -1 \) follows immediately from the definitions of the Hardy norm and the function \( N(w) \) and from Theorem 5.4 where \( v(w) = |w|^p \).

When \( \alpha > -1 \) we recall that

\[
\|f\|_{A^p_{\mu,\alpha}}^p = \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u,r}(|f|^p) \, dr = \int_{-\infty}^{0} |r|^\alpha e^r \left( \int_{B_u(r)} |f|^p \dd\bar{u}u)^n \right) \, dr \\
+ \int_{-\infty}^{0} |r|^\alpha e^r \left( \int_{B_u(r)} (r - u) \dd\bar{u} |f|^p \wedge (\dd\bar{u}u)^{n-1} \right) \, dr.
\]

By Fubini’s theorem the first double integral is equal to

\[
\int_D \sigma_\alpha(u)|f|^p(\dd\bar{u}u)^n
\]

and the second integral is equal to

\[
\int_D \gamma_\alpha(u) \dd\bar{u} |f|^p \wedge (\dd\bar{u}u)^{n-1}.
\]
By Theorem 5.4 the second integral is equal to
\[
\int_{-\infty}^{0} |r|^\alpha e^r \left( \int_{C} N(w, r) dd^c |w|^p \right) dr = \int_{C} \left( \int_{-\infty}^{0} |r|^\alpha e^r N(w, r) dr \right) dd^c |w|^p
\]
\[= \int_{C} N_\alpha(w) dd^c |w|^p. \]

□

In the case when \( u = g_D(z, z_0) \) is a pluricomplex Green function the theorem above takes exactly the form of the Littlewood–Paley identity.

**Corollary 6.3.** If in the assumptions of Theorem 6.2 the function \( u = g_D(z, z_0) \), then
\[
\|f\|_{A^{p}_{\alpha, \alpha}}^p = \Gamma(\alpha + 1)|f(z_0)|^p + \int_{C} N_\alpha(w) dd^c |w|^p
\]

7. **Properties of the Nevanlinna counting functions**

By their definition the Nevanlinna counting functions depend on the choice of the exhaustion. The next result shows that this dependence can be estimated.

**Theorem 7.1.** Let \( u \) and \( v \) be two continuous exhausting functions on \( D \) such that \( c v \leq u \leq c^{-1}v \) for all \( z \in D \). If \( \phi \) is a plurisubharmonic function on \( D \) and
\[
\int_{B_u(r)} (r - u) dd^c \phi \wedge (dd^c(u + v))^{n-1} < \infty
\]
for all \( r < 0 \), then for all \( r < 0 \)
\[
\int_{B_u(r)} (r - u) dd^c \phi \wedge (dd^c u)^{n-1} \leq c \int_{B_u(r)} (r_n - v) dd^c \phi \wedge (dd^c v)^{n-1},
\]
where \( r_n = c^{-2n+1}r \).

**Proof.** The inequality \( cv \leq u \leq c^{-1}v \) on \( D \) implies two facts: first, \( B_v(cr) \subset B_u(r) \subset B_v(r/c) \) and, secondly, \( c(c^{-1}r - v) \geq r - u \). Thus
\[
\int_{B_u(r)} (r - u) dd^c \phi \wedge (dd^c u)^{n-1} \leq c \int_{B_u(c^{-1}r)} (c^{-1}r - v) dd^c \phi \wedge (dd^c u)^{n-1}. \quad (12)
\]
Since \((dd^c(u + v))^{n-1} \geq (dd^c v)^k \wedge (dd^c u)^{n-k-1}, 0 \leq k \leq n - 1\), we conclude from the assumptions of the theorem that
\[
\int_{B_u(r)} dd^c \phi \wedge (dd^c v)^k \wedge (dd^c u)^{n-k-1} < \infty \tag{13}
\]
for all \(r < 0\).

If \(T = dd^c \phi \wedge (dd^c v)^k \wedge (dd^c u)^{n-k-2}\), then \(T\) is a closed positive current of bidimension \((1, 1)\) and by (13)
\[
\int_{B_u(r)} dd^c u \wedge T < \infty.
\]

Since the function \(c^{-1}r - u \geq 0\) on \(B_v(r)\), we have by Theorem 2.4
\[
\int_{B_u(r)} (r - v)(dd^c u) \wedge T \leq \int_{B_u(r)} (c^{-1}r - u)(dd^c v) \wedge T.
\]

But \(c^{-1}r - u \leq c(c^{-2}r - v)\) and, therefore, the latter integral does not exceed
\[
c \int_{B_u(c^{-2}r)} (c^{-2}r - v) dd^c \phi \wedge (dd^c v)^{k+1} \wedge (dd^c u)^{n-k-2}.
\]

Thus,
\[
\int_{B_u(r)} (r - v) \wedge dd^c \phi \wedge (dd^c v)^k \wedge (dd^c u)^{n-k-1}
\]
\[
\leq c \int_{B_u(r_1)} (r_1 - v) dd^c \phi \wedge (dd^c v)^{k+1} \wedge (dd^c u)^{n-k-2}, \tag{14}
\]

where \(r_1 = c^{-2}r\).

Starting with (12) and applying (14) \(n - 1\) times to the right-hand side of (12) to eliminate \(dd^c u\) we get the theorem. \(\square\)

Since by Theorem 5.2 \(N_{u+v,f}(w, r) < \infty\) when \(L(u) \cap X_w = \emptyset\) and \(L(v) \cap X_w = \emptyset\), in the case when \(\phi(z) = \log |f(z) - w|\), the previous result and (11) imply the following corollary.

**Corollary 7.2.** When \(L(u) \cap X_w = \emptyset\) and \(L(v) \cap X_w = \emptyset\), in the assumptions of Theorem 7.1 \(c^{-n}N_u(w, r_n) \leq N_u(w, r) \leq c^n N_v(w, r_n)\), where \(r_n = c^{-2n+1}r\). Consequently, \(c^{-n}N_{v,\alpha}(w) \leq N_{u,\alpha}(w) \leq c^n N_{v,\alpha}(w)\)
It is well-known that for a self-mapping of the unit disk \( f \) which fixes the origin, the classical Nevanlinna counting function satisfies the estimate \( N(w) \leq -\log |w| \). Our next result generalizes this statement to the multivariable case.

**Theorem 7.3.** Let \( u \in \mathcal{E}(D) \) and let \( f : D \to \mathbb{D} \) be a holomorphic function. Then there is a number \( r < 1 \) and a constant \( c \) depending only on \( u \) and \( f \) such that for every \( w \in \mathbb{D}, |w| > r \), \( N_u(w) \leq -c \log |w| \).

Moreover if \( w(z) = g_D(z, z_0) \) and \( f(z_0) = w_0 \), then

\[
N_u(w) \leq -(2\pi)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right|.
\]

**Proof.** We fix a point \( z_0 \) in \( D \) and let \( v(z) = g_D(z, z_0) \). If \( w_0 = f(z_0) \) we set \( v_1(\zeta) = \log |\zeta - w_0|/|1 - \overline{w_0}\zeta| \) and \( \phi(\zeta) = \log |\zeta - w|/|1 - \overline{w}\zeta| \), \( \zeta \in \mathbb{D} \).

By Theorem 4.2 \( \mu_{v,r}(\phi \circ f) \leq (2\pi)^{n-1}\mu_{v_1,v}(\phi) \). If \( w \neq w_0 \), then

\[
\mu_{v,r}(\phi \circ f) = (2\pi)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right| + N_v(w, r)
\]

and

\[
\mu_{v_1,v}(\phi) = 2\pi \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right| + \int_{B_{v_1}(r)} (r - v_1) \, dd^c \phi = 2\pi r.
\]

Hence,

\[
N_v(w, r) \leq (2\pi)^n r - (2\pi)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right|
\]

and

\[
N_v(w) \leq -(2\pi)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right|.
\]

For an arbitrary function \( u \in \mathcal{E}(D) \) we take the compact set \( K = \overline{B_u(-1)} \cup \{ u < -1 \} \). Let \( u_1 = \max\{ u, -1 \} \) and \( v_1 = \max\{ v, -1 \} \). If \( f(K) \subseteq \mathbb{D}(0, r) \) and \( |w| > r \), then \( N_u(w, r) = N_{u_1}(w, r) \) and \( N_v(w, r) = N_{v_1}(w, r) \). Since there is a constant \( c > 1 \) such that \( cv_1 \leq u_1 \leq c^{-1} v_1 \) for all \( z \in D \), by Corollary 7.2

\[
N_u(w, r) \leq c^n N_{v}(w, r_n) \leq (2\pi c)^n r_n - (2\pi c)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right|,
\]

where \( r_n = c^{-2n+1} r \). Hence,

\[
N_u(w) \leq -(2\pi c)^n \log \left| \frac{w_0 - w}{1 - \overline{w_0}w} \right| \leq -(2\pi c)^n \log |w|
\]

when \( |w| > r \). \( \square \)
Our next result establishes a multidimensional analog of Shapiro’s mean value inequality for counting functions.

**Theorem 7.4.** Let $f$ be a analytic function on a hyperconvex domain $D \subset \mathbb{C}^n$ with an exhausting function $u$, and let $K = \text{supp}(dd^c u)^n$ be a compact set. If $w_0 \in \mathbb{C}$ and $0 < \rho < r_0 = \text{dist}(w_0, f(K))$, then

$$N_{u,\alpha}(w_0) \leq \frac{1}{2\pi i \rho^2} \int_{\mathbb{D}(w_0, \rho)} N_{u,\alpha}(w) \, dw \wedge d\overline{w}.$$  

**Proof.** Note that the function $\Phi(w, r) = \mu_{u,r}(\log |f(z) - w|)$ is subharmonic and

$$\Phi(w_0, r) \leq \frac{1}{2\pi i t^2} \int_{\mathbb{D}(w_0, t)} \Phi(w, r) \, dw \wedge d\overline{w}.$$  

The function

$$\Psi(w) = \int_D \log |f(z) - w|(dd^c u)^n$$

is harmonic outside of $f(K)$. By (3) $N_u(w, r) = \Phi(w, r) - \Psi(w)$ when $K \subset B_u(r)$. Hence, the function $N_u(w, r)$ is subharmonic outside of $f(K)$. Since this family of functions is increasing in $r$ and $N_u(w) = \lim_{r \to 0} -N_u(w, r)$, the theorem follows from (11).  

□

This theorem has an important corollary.

**Corollary 7.5.** Let $f$ be an analytic function on a hyperconvex domain $D \subset \mathbb{C}^n$ with an exhausting function $u \in \mathcal{E}(D)$, and let $K \subset D$ be a compact set such that $L(u)$ lies in the interior of $K$. There is a constant $c > 0$ depending only on $u$ and $F$ such that if $w_0 \in \mathbb{C}$ and $0 < \rho < r_0 = \text{dist}(w_0, f(K))$, then

$$N_{u,\alpha}(w) \leq \frac{c}{2\pi i \rho^2} \int_{\mathbb{D}(w_0, \rho)} N_{u,\alpha}(w) \, dw \wedge d\overline{w}.$$  

**Proof.** Let $z_0$ be an interior point of $K$. We take the function $g_D(z, z_0)$, let $a$ be the maximum of $g_D(z, z_0)$ on a closed ball $B$ lying in the interior of $F$ and set $v(z) = \max\{g_D(z, z_0), a\}$. Then supp$(dd^c v)^n$ lies in the interior of $K$.

If $b$ is the maximum of $g_D(z, z_0)$ on $B$ and $u_1(z) = \max\{u(z), b\}$, then there is a constant $k \geq 1$ such that $k^{-1}v \leq u_1 \leq kv$ on $D$. By Corollary 7.2 $k^{-n}N_{u_1,\alpha} \leq N_{v,\alpha} \leq k^n N_{u_1,\alpha}$. Since $N_{u_1,\alpha}(w) = N_{u,\alpha}(w)$ when $w \not\in f(K)$ and

$$N_{v,\alpha}(w) \leq \frac{c}{2\pi i \rho^2} \int_{\mathbb{D}(w_0, \rho)} N_{v,\alpha}(w) \, dw \wedge d\overline{w},$$

we have

$$N_{u,\alpha}(w) \leq \frac{c}{2\pi i \rho^2} \int_{\mathbb{D}(w_0, \rho)} N_{u,\alpha}(w) \, dw \wedge d\overline{w}.$$
the corollary follows.

8. Application to composition operators: boundedness and compactness

Now, let \( F : D_1 \rightarrow D_2 \) be a holomorphic mapping between hyperconvex domains \( D_1 \subset \mathbb{C}^n \) and \( D_2 \subset \mathbb{C}^m \) with exhausting functions \( u_1 \in \mathcal{E}(D_1) \) and \( u_2 \in \mathcal{E}(D_2) \) respectively. If \( f \) is a holomorphic function on \( D_2 \) then we denote by \( f^* \) the function \( f \circ F \).

Let us introduce the “tail” part of the Nevanlinna function

\[
N^*_{u_1,F,f,\beta}(w,r) = \int_{T(r)} \gamma_\beta(u_1)(dd^c u_1)^{n-1} \wedge dd^c \log |f^* - w|,
\]

where \( T(r) = D_1 \setminus \overline{B}_{u_2}(r) = \{ z \in D_1 : u_2(F(z)) > r \} \). We define the \((\beta, \alpha)\)-deficiency of \( F \) as

\[
\delta_{u_1,u_2,F,\beta,\alpha}(r) = \delta_{F,\beta,\alpha}(r) = \sup \frac{N^*_{u_1,F,f,\beta}(w,r)}{N_{u_2,f,\alpha}(w)},
\]

where the supremum is taken over all \( f \in A_{\alpha}^p(D_2) \) and all \( w \in \mathbb{C} \). We assume that the ratio \( N^*_{u_1,F,f,\beta}(w,r)/N_{u_2,f,\alpha}(w) = 0 \) when \( N_{u_2,f,\alpha}(w) = 0 \). Clearly, the function \( \delta_{F,\beta,\alpha}(r) \) is decreasing in \( r \).

In what follows, given a holomorphic function \( h \) on a domain \( D \) and coordinates \( \zeta_1, \ldots, \zeta_n \) in \( \mathbb{C}^n \) we denote by \( \nabla h \) the complex gradient of \( h \),

\[
\nabla h = \left( \frac{\partial h}{\partial \zeta_1}, \ldots, \frac{\partial h}{\partial \zeta_n} \right),
\]

and for vectors \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) in \( \mathbb{C}^n \) we let \( \langle a, b \rangle = \sum a_i b_i \).

**Lemma 8.1.** Let \( F \) be a holomorphic mapping from a domain \( D_1 \subset \mathbb{C}^n \) into domain \( D_2 \subset \mathbb{C}^m \), and let \( h \) be a holomorphic function on \( D_2 \). If \( h^* = h \circ F \) and \( F = (F_1, \ldots, F_m) \), then at every point \( \zeta \in D_1 \) the following estimate holds

\[
dd^c |h^*|^p \leq \frac{p^2}{4} |h^*|^{p-2}(\zeta) |\nabla h(F(\zeta))|^2 \sum_{i=1}^m dd^c |F_i(\zeta)|^2.
\]

**Proof.** A direct calculation for a holomorphic function \( h \) shows that

\[
dd^c |h|^p = \frac{p^2}{4} |h|^{p-2} dd^c |h|^2
\]
and $dd^c|h^*|^2 = 2i\partial h^* \wedge \overline{\partial h^*}$. The quadratic form corresponding to $dd^c|h^*|^2$ is
\[ \sum_{i,j=1}^{n} \frac{\partial^2 |h^*|^2}{\partial \zeta_i \partial \overline{\zeta}_j} (\zeta) \xi_i \xi_j = \sum_{i,j=1}^{n} \frac{\partial h^*}{\partial \zeta_i} (\zeta) \frac{\partial \overline{h^*}}{\partial \overline{\zeta}_j} (\zeta) \xi_i \xi_j = |\langle \nabla h^*(\zeta), \xi \rangle|^2, \]
where $\xi = (\xi_1, \ldots, \xi_n)$. Now
\[ |\langle \nabla h^*(\zeta), \xi \rangle| = |\langle \nabla h, F'(\zeta) \xi \rangle| \leq |\nabla h(F(\zeta))||F'(\zeta)|. \]
But
\[ |F'(\zeta)|^2 = \sum_{i=1}^{m} |\langle \nabla F_i(\zeta), \xi \rangle|^2. \]
Thus,
\[ |\langle \nabla h^*(\zeta), \xi \rangle|^2 \leq |\nabla h(F(\zeta))|^2 \sum_{i=1}^{m} |\langle \nabla F_i(\zeta), \xi \rangle|^2. \]
Since
\[ dd^c h^* = 2i \sum_{i,j=1}^{n} \frac{\partial h^*}{\partial \zeta_i} \frac{\partial h^*}{\partial \overline{\zeta}_j} d\zeta_i \wedge d\overline{\zeta}_j \]
and
\[ dd^c |F_i|^2 = 2i \sum_{i,j=1}^{n} \frac{\partial F_i}{\partial \zeta_i} \frac{\partial F_i}{\partial \overline{\zeta}_j} d\zeta_i \wedge d\overline{\zeta}_j, \]
by Corollary 3.2.5 from ([Kl], p.102) we conclude that the differential form
\[ |\nabla h(F(\zeta))|^2 \sum_{i=1}^{m} dd^c |F_i(\zeta)|^2 - dd^c |h^*(\zeta)|^2 \]
is positive. \qed

**Lemma 8.2.** Let $F : D_1 \to D_2$ be a holomorphic mapping from a hyperconvex domain $D_1 \subset \mathbb{C}^n$ with an exhausting functions $u$ such that the set $L(u)$ is finite, into a domain $D_2 \subset \mathbb{C}^m$. If $V$ is an open set in $D_1$, $\phi$ is a nonnegative continuous function on $D_1$, $W$ is an open set in $D_2$, $F(V) \subset W$ and $f$ is a holomorphic function on $W$ such that $0 < c_1 < |f(z)| < c_2$ and $|\nabla f| < c_3$ on $W$, then
\[ \int_V \gamma_\beta(u) dd^c |f^*|^p \wedge (dd^c u)^{n-1} \leq c_1^{p-2} c_3^2 \Gamma(\beta + 1) \sum_{i=1}^{m} \|F_i\|^2_{H^2_0(D_1)}, \]
where the constant $c$ is equal to $c_2$ when $p - 2 \geq 0$ and $c_1$ when $1 \leq p \leq 2$. \hfill \[35\]}
Proof. The integral in question is equal to
\[ \frac{p^2}{4} \int_{V} \gamma_{\beta}(u)|f^*|p-2d\bar{c}|f^*|^2 \wedge (d\bar{c}u)^{n-1}. \]
Since \(0 < c_1 < |f(z)| < c_2\) and \(|\nabla f| < c_3\) on \(V\), by Lemma 8.1 this integral does not exceed
\[ c^{p-2}c_3^2 \sum_{i=1}^{m} \int_{D_1} \gamma_{\beta}(u)d\bar{c}|F_i|^2 \wedge (d\bar{c}u)^{n-1}, \]
By Theorem 6.2
\[ \int_{D_1} \gamma_{\beta}(u)d\bar{c}|F_i|^2 \wedge (d\bar{c}u)^{n-1} = \int_{-\infty}^{0} |r|^\beta e^r \int_{B_u(r)} (r-u)d\bar{c}|F_i|^2 \wedge (d\bar{c}u)^{n-1}. \]
Since
\[ \int_{B_u(r)} (r-u)d\bar{c}|F_i|^2 \wedge (d\bar{c}u)^{n-1} \leq \|F_i\|^2_{H_{\beta}^2(D_1)}, \]
we see that
\[ \int_{V} \gamma_{\beta}(u)d\bar{c}|f^*|^p \wedge (d\bar{c}u)^{n-1} \leq c^{p-2}c_3^2\Gamma(\beta+1) \sum_{i=1}^{m} \|F_i\|^2_{H_{\beta}^2(D_1)}. \]
\[ \square \]

The following result gives sufficient conditions of boundedness and compactness of the composition operator \(C_F\) as an operator acting from \(A^p_{\alpha}(D_2)\) into \(A^p_{\beta}(D_1)\). It could be viewed as an extension of the appropriate results of Shapiro [Sha2] and Smith [Sm] to the multi-variable case.

**Theorem 8.3.** Let \(F : D_1 \to D_2\) be a holomorphic mapping between hyperconvex domains \(D_1 \subset \mathbb{C}^n\) and \(D_2 \subset \mathbb{C}^m\) with exhausting functions \(u_1\) and \(u_2\) respectively such that the sets \(L(u_1)\) and \(L(u_2)\) are finite.

1. If there exists \(r_0 < 0\) such that \(\delta_{F,\alpha,\beta}(r_0) < \infty\), then \(C_F\) is a bounded operator from \(A^p_{\alpha_2,\alpha}(D_2)\) into \(A^p_{\alpha_1,\beta}(D_1)\).

2. If the function \(\delta_{F,\alpha,\beta}(r)\) converges to 0 as \(r \to 0\), then \(C_F\) is a compact operator from \(A^p_{\alpha_2,\alpha}(D_2)\) into \(A^p_{\alpha_1,\beta}(D_1)\).

**Proof.** To prove the first part of the theorem we take a function \(f \in A^p_{\alpha}(D_2)\) with \(\|f\|_{A^p_{\alpha}(D_2)} = 1\). If \(K\) is a compact set in \(D_1\) such that \((d\bar{c}u_1)^n \equiv 0\) outside of \(K\) and \(K' = F(K)\), then by Theorem 3.6 there
is a constant $C_1$ not depending on $f$ such that $|f(z)| \leq C_1$ on $K'$. Consequently,

$$\int_{D_1} \sigma_{\beta}(u_1)|f^*|^p(\dd^c u_1)^n \leq C_1^p \Gamma(\beta + 1) \int_{D_1} (\dd^c u_1)^n. \quad (15)$$

By Theorem 3.6 there are constants $C_2, C_3 > 0$ not depending on $f$ such that $|f(z)| \leq C_2$ and $|\nabla f(z)|^2 \leq C_3$ when $z \in B_{u_2}(r_0/2)$. Let $g(z) = f(z) + 2C_2$. Then $C_2 < |g(z)| < 3C_2$ on $B_{u_2}(r_0/2)$ and

$$\|g\|_{A^p_\alpha(D_2)} \leq 1 + (2C_2) \left( \int_{D_2} \sigma_\alpha(u_2)(\dd^c u_2)^n \right)^{1/p} = C_4.$$ 

By Lemma 8.2 we have

$$\int_{B_{u_2}^*(r_0/2)} \gamma_\beta(u_1) \dd^c|g|^p \wedge (\dd^c u_1)^{n-1} \leq c^{p-2} C_3^2 \Gamma(\beta + 1) \sum_{i=1}^m \|F_i\|^2_{H_{\alpha_2}^p(D_1)}.$$ 

But the functions $F_i$ are bounded and, therefore, $\|F_i\|^2_{H_{\alpha_2}^p(D_1)} < \infty$. Hence

$$\int_{B_{u_2}^*(r_0/2)} \gamma_\beta(u_1) \dd^c|g|^p \wedge (\dd^c u_1)^{n-1} < C_5, \quad (16)$$

where the constant $C_5$ depends only on $c, C_3, p$ and $F$.

By Theorem 5.4

$$\int_{T(r_0)} \gamma_\beta(u_1) \dd^c|g|^p \wedge (\dd^c u_1)^{n-1} = \int_{\mathbb{C}} N_{u_1,F,g,\beta}(w,r_0) \dd^c|w|^p.$$ 

Since $N_{u_1,F,g,\beta}(w,r_0) \leq \delta_{u_1,u_2,F,\beta,\alpha}(r_0) N_{u_2,g,\alpha}(w)$,

$$\int_{\mathbb{C}} N_{u_1,F,g,\beta}(w,r_0) \dd^c|w|^p \leq \delta_{u_1,u_2,F,\beta,\alpha}(r_0) \int_{\mathbb{C}} N_{u_2,g,\alpha}(w) \dd^c|w|^p \leq \delta_{u_1,u_2,F,\beta,\alpha}(r_0) \|g\|^p_{A^p_{\alpha_2,\alpha}(D_2)}. \quad (17)$$

Combining together (15) and (16) we see that

$$\|g^*\|^p_{A^p_{\alpha_1,\beta}(D_1)} \leq \delta_{u_1,u_2,F,\beta,\alpha}(r_0) \|g\|^p_{A^p_{\alpha_2,\alpha}(D_2)}.$$ 

Since

$$\|f^*\|^p_{A^p_{\alpha_1,\beta}(D_1)} \leq \|g^*\|^p_{A^p_{\alpha_1,\beta}(D_1)} + 2C_2 \|1\|^p_{A^p_{\alpha_1,\beta}(D_1)},$$

this implies that the composition operator is bounded.
Suppose that the second condition is satisfied. To show the compactness of $C_f$ we need to show that if holomorphic functions $g_k$ converge to 0 uniformly on compacta in $D_2$ and $\|g_k\|_{A^p_\beta(D_2)} \leq 1$, then $\|g^*_k\|_{A^p_\beta(D_1)} \to 0$ as $k \to \infty$.

For $\varepsilon > 0$ we take $r_0 < 0$ such that $\delta_{F,\alpha,\beta}(r_0) < \varepsilon$ and let $r = r_0/2$. We also take $r_1 > r$ such that the set $B_{u_1}(r)$ compactly belongs to $B_{u_2}(r)$. Let $\varepsilon_k(r_1)$ be the supremum of $|g_k|$ on $B_{u_2}(r_1)$. Then by Cauchy inequalities there is a constant $C_1$ such that $|\nabla g_k| < C_1 \varepsilon_k(r_1)$ on $B_{u_2}(r)$. If $h_k = 2\varepsilon_k(r_1) + g_k$ then $0 < \varepsilon_k(r_1) < |h_k| < 3\varepsilon_k(r_1)$ and $|\nabla h_k| < C_1 \varepsilon_k(r_1)$ on $B_{u_2}(r)$. By Lemma 8.2

$$\int_{B_{u_2}(r)} \gamma_\beta(u_1) d\bar{z}|h_k^*|^p \wedge (d\bar{z}u_1)^{n-1} \leq C e^{p-2} C_1 \varepsilon_k^2(r_1),$$

where $C = \Gamma(\beta + 1) \sum_{i=1}^m \|F_i\|^2_{H^2_i(D_1)}$ and the constant $c$ is equal to $3\varepsilon_k(r_1)$ when $p - 2 \geq 0$ and $\varepsilon_k(r_1)$ when $1 \leq p \leq 2$. In any case the integral does not exceed $9C C_1^2 \varepsilon_k^p(r_1)$. Thus

$$\lim_{k \to \infty} \int_{B_{u_2}(r)} \gamma_\beta(u_1) d\bar{z}|h_k^*|^p \wedge (d\bar{z}u_1)^{n-1} = 0. \quad (18)$$

By (17)

$$\int_{T(r_0)} \gamma_\beta(u_1) d\bar{z}|h_k^*|^p \wedge (d\bar{z}u_1)^{n-1} \leq \delta_{F,\alpha,\beta}(r_0) \|h_k\|^p_{A^p_{u_2,\alpha}(D_2)} < a_k \varepsilon,$$

where $a_k = \|\varepsilon_k(r_1)\|_{A^p_{u_2,\alpha}(D_2)}$. Combining (18) with the latter estimate we get that

$$\limsup_{k \to \infty} \int_{D_1} \gamma_\beta(u_1) d\bar{z}|h_k^*|^p \wedge (d\bar{z}u_1)^{n-1} \leq \varepsilon.$$

Evidently,

$$\lim_{k \to \infty} \int_{D_1} \sigma_\beta(u_1)|g_k^*|^p (d\bar{z}u_1)^n = 0,$$

and, therefore, the functions $h_k$ and, consequently, by Theorem 6.2 $g_k^*$, converge to 0 in $A^p_{u_1,\beta}(D_1)$.

To provide necessary conditions we fix a compact set $K \subset D_1$ whose interior contains $L(u_1)$ and for a holomorphic function $f \in A^p_{u_2,\alpha}(D_2)$
introduce the function

\[
\nu_{F,\alpha,\beta}(w, f) = \frac{|w|^p N_{u_1, f^*, \beta}(w)}{\|f\|_{A_{u_2, \alpha}^p}^p}
\]

(here and below we use the same notation: for a function \(h\) on \(D_2\) we denote by \(h^*\) the composition \(h\) and \(F,\ h^* = h \circ F\)). For \(a > 1\) we set

\[
\rho_{u_1, u_2, F, \alpha, \beta}(a) = \rho(a) = \sup \nu_{F,\alpha,\beta}(w, f),
\]

where the supremum is taken over all \(f \in A_{u_2, \alpha}^p(D_2)\) and all \(w \in \mathbb{C}\), \(|w| > a \max_{\zeta \in K} |f^*(\zeta)|\). Note that \(\nu_{\alpha, \beta}(cw, cf) = \nu_{\alpha, \beta}(w, f)\). Thus, we may assume that \(\|f\|_{A_{u_2, \alpha}^p(D_2)} = 1\) in the definition of \(\rho_{F,\alpha,\beta}\).

We also will need another characteristic of the mapping \(F\). For its definition we fix an open ball \(B \subset D_2\) and for \(t > 0\) and \(a > 1\) we set

\[
\tilde{\rho}_{u_1, u_2, F, \alpha, \beta}(t, a) = \sup \nu_{\alpha, \beta}(w, f),
\]

where the supremum is taken over all \(w \in \mathbb{C}\), \(|w| > a \max_{\zeta \in K} |f^*(\zeta)|\), and all \(f \in A_{u_2, \alpha}^p(D_2)\) such that \(\|f\|_{A_{u_2, \alpha}^p(D_2)} = 1\) and \(|f| < t\) on \(B\).

**Theorem 8.4.** Let \(F : D_1 \to D_2\) be a holomorphic mapping between hyperconvex domains \(D_1 \subset \mathbb{C}^n\) and \(D_2 \subset \mathbb{C}^m\) with exhausting functions \(u_1(\zeta) \in E(D_1)\) and \(u_2 \in E(D_2)\) respectively and such that the sets \(L(u_1)\) and \(L(u_2)\) are finite.

1. If \(C_F\) is a bounded operator from \(A_{u_2, \alpha}^p(D_2)\) into \(A_{u_1, \beta}^p(D_1)\), then \(\rho_{F,\alpha,\beta}(a) < \infty\) for all \(a > 1\).

2. If \(C_F\) is a compact operator from \(A_{u_2, \alpha}^p(D_2)\) into \(A_{u_1, \beta}^p(D_1)\), then the function \(\tilde{\rho}_{F,\alpha,\beta}(t)\) converges to \(0\) as \(t \to 0^+\) for all \(a > 1\).

**Proof.** If the first part of the theorem does not hold, then there are functions \(f_j \in A_{u_2, \alpha}^p(D_2)\), \(\|f_j\|_{A_{u_2, \alpha}^p} = 1\), and \(w_j \in \mathbb{C}\) such that \(|w_j| > a \max_{\zeta \in K} |f_j^*(\zeta)|\) and

\[
|w_j|^p N_{u_1, f_j^*, \beta}(w_j) \geq j.
\]

Let \(r_j = C_0|w_j|\), where \(C_0 = (1 - a^{-1})/2\) and \(V_j = \mathbb{D}(w_j, r_j) \subset \mathbb{C}\). Since \(|w| \geq C_1|w_j|\) on \(V_j\), where \(C_1 = (a^{-1} + 1)/2\), we get

\[
\|f_j^*\|_{A_{u_1, \beta}^p} \geq \int_{V_j} N_{u_1, f_j^*, \beta}(w) \, dd^c|w|^p = \frac{p^2}{4} \int_{V_j} |w|^{p-2} N_{u_1, f_j^*, \beta}(w) \, dd^c|w|^2 \\
\geq \frac{p^2}{4} C_1^p |w_j|^p \int_{V_j} |w|^{-2} N_{u_1, f_j^*, \beta}(w) \, dd^c|w|^2.
\]

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But \(|w| \leq C_2|w_j|\) on \(V_j\), where \(C_2 = (3 + a^{-1})/2\). Hence,
\[
\|f_j^*\|^p_{A^p_{u_1, \beta}} \geq \frac{p^2 \pi C_1^p C_0^p |w_j|^p}{2C_2^2} \int_{V_j} N_{u_1, f_j^*, \beta}(w) \, dd^c |w|^2.
\]
But the disk \(V_j\) lies outside of \(f_j^*(K)\). Thus, by Corollary 7.5
\[
\|f_j^*\|^p_{A^p_{u_1, \beta}} \geq \frac{\pi p^2 C_2^p C_0^p}{2C_2^2} |w_j|^p N_{u_1, f_j^*, \beta}(w_j) \geq C_j,
\]
where the constant \(C = \frac{\pi p^2 C_2^p C_0^p}{2C_2^2}\) depends only on \(a, K, \lambda\) and \(p\). Therefore, the norms \(\|f_j^*\|_{A^p_{u_1, \beta}} \to \infty\) as \(j \to \infty\) and we get a contradiction.

For the proof of the second part we assume that the statement does not hold and take sequences of complex numbers \(\{w_j\}\) and functions \(f_j \in A^p_{u_2, \alpha}\), \(\|f\|^p_{A^p_{u_2, \alpha}} = 1\), such that \(|w_j| > a \max_{\zeta \in K} |f_j^*(\zeta)|\), \(|f_j| < 1/j\) on \(B\) and \(\nu_{\alpha, \beta}(w_j, f_j) \geq c > 0\). Clearly, the sequence \(f_j\) converges to 0 uniformly on compacta in \(D_2\). But the same estimates as in the first part show that \(\|f_j^*\|_{A^p_{u_1, \beta}} \geq C > 0\) and this contradicts the compactness of \(C_F\).

\[\square\]

9. Composition operators induced by mappings into the unit disk

It turns out that if \(D_2\) is the unit disk, the necessary and sufficient conditions, given by Theorems 8.4 and 8.3 respectively, agree. Of course, this is caused by the simple structure of divisors in \(\mathbb{C}\). In this case the results of the previous section allow us to state and prove a theorem which gives necessary and sufficient conditions for boundedness and compactness of a composition operator in the form of results of [Sha2] and [Sm].

**Theorem 9.1.** Let \(F : D \to \mathbb{D}\) be a holomorphic mapping from a hyperconvex domain \(D \subset \mathbb{C}^n\) into the unit disk \(\mathbb{D}\) with exhausting functions \(u \in \mathcal{E}(D)\) such that the set \(L(u)\) is finite and \(v(z) = \log |z|\) respectively. Then the condition
\[
N_{F, u, \beta}(z) = O(\gamma_\alpha(\log |z|)) \quad \text{as} \quad |z| \to 1
\]
is necessary and sufficient for the operator \(C_F\) to map continuously \(A^p_{u, \alpha}(\mathbb{D})\) into \(A^p_{u, \beta}(D)\) and the condition
\[
N_{F, u, \beta}(z) = o(\gamma_\alpha(\log |z|)) \quad \text{as} \quad |z| \to 1
\]
is necessary and sufficient for the operator \(C_F : A^p_{u, \alpha}(\mathbb{D}) \to A^p_{u, \beta}(D)\) to be compact.
Proof. Let us evaluate \( N^*_{u,F,f,\beta}(w,r) \). We take the set of all points \( \{z_i\} \) in \( \mathbb{D} \) such that \( f(z_i) = w \) and \( v(z_i) > r \). Let \( \cup_j A_{ij} = F^{-1}(z_i) \) be the decomposition of the preimage of \( z_i \) under \( F \) into irreducible components. The multiplicity of \( f^* \) on \( A_{ij} \) is equal to \( m_im_{ij} \), where \( m_{ij} \) is the multiplicity of \( F \) on \( A_{ij} \) and \( m_i \) is the multiplicity of \( f \) at \( z_i \). If \( r \) is so small that \( L(u_1) \cap T(r) = \emptyset \), where \( T(r) = \{ z \in D : v(F(z)) > r \} \), then by Proposition 5.2

\[
N^*_{u,F,f,\beta}(w,r) = \sum_i m_i \sum_j m_{ij} \int_{A_{ij} \cap T(r)} \gamma_\beta(u)(dd^c u)^{n-1}.
\]

Considering \( F \) as a holomorphic function on \( D_1 \), we remark that

\[
\sum_j m_{ij} \int_{A_{ij} \cap T(r)} \gamma_\beta(u)(dd^c u)^{n-1} = N_{F,\beta}(z_i),
\]

so

\[
N^*_{u,F,f,\beta}(w,r) = \sum_i m_i N_{F,\beta}(z_i).
\]

We also observe that

\[
N_{v,f,\alpha}(w) \geq \sum_i m_i \gamma_\alpha(v(z_i)).
\]

Hence,

\[
\frac{N^*_{u,F,f,\beta}(w,r)}{N_{v,f,\alpha}(w)} \leq \frac{\sum_i m_i N_{F,\beta}(z_i)}{\sum_i m_i \gamma_\alpha(v(z_i))} \leq \max_i \left\{ \frac{N_{F,\beta}(z_i)}{\gamma_\alpha(v(z_i))} \right\}.
\]

Thus,

\[
\delta_{u,v,F,\beta,\alpha}(r) \leq \sup_{|z| > r} \frac{N_{F,\beta}(z)}{\gamma_\alpha(v(z))}.
\]

On the other hand if \( f(z) = z \), then

\[
\frac{N^*_{u,F,f,\beta}(w,r)}{N_{v,f,\alpha}(w)} = \frac{N_{F,\beta}(z)}{\gamma_\alpha(v(z))}.
\]

Hence,

\[
\delta_{u,v,F,\beta,\alpha}(r) = \sup_{|z| > r} \frac{N_{F,\beta}(z)}{\gamma_\alpha(v(z))}.
\]

Now the sufficiency of our conditions for \( C_F \) to be bounded or compact follows from Theorem 8.3.

To show the necessity of the conditions we, firstly, note that if \( |F| < a < 1 \) on \( D \), then the operator \( C_F \) is both bounded and compact and \( N_{F,\beta}(z) = 0 \) when \( |z| > a \). So this case is trivial.
Now we assume that there is a sequence \( \{ \zeta_j \} \subset D \) such that the points \( z_j = F(\zeta_j) \) converge to the unit circle. Following the standard argument (cf. [Sm], [CoM]) we consider the functions

\[
k_j(z) = \frac{(1 - |z_j|^2)^{(\alpha+2)/p}}{(1 - \overline{z_j}z)^{2(\alpha+2)/p}}
\]
on \( D \). It can be shown (see [Sm]) that the norms \( \|k_j\|_{A^\alpha_p} \approx 1 \) and the functions \( k_j \) converge to 0 uniformly on compacta.

Note that \( k_j(z_j) = w_j = \frac{1}{(1 - |z_j|^2)^{(\alpha+2)/p}} \).

Thus,

\[
w_j^p \approx \frac{1}{\gamma_\alpha(\log |z_j|)}.
\]

Since \( N_{u,k_j^*,\beta}(w_j) \geq N_{F,\beta}(z_j) \) this implies

\[
\nu_{F,\alpha,\beta}(w_j, k_j) \geq C_1 w_j^p N_{u,k_j^*,\beta}(w_j) \geq C_2 \frac{N_{F,\beta}(z_j)}{\gamma_\alpha(\log |z_j|)}.
\]

Since \( k_j^* \) converges to 0 on any compact set \( K \subset D \), for a fixed \( a > 1 \) the condition \( w_j > a \sup_{\zeta \in K} |k_j^*(\zeta)| \) is satisfied and, therefore,

\[
\rho_{u_1,u_2,F,\alpha,\beta}(a) \geq C_2 \frac{N_{F,\beta}(z_j)}{\gamma_\alpha(\log |z_j|)}.
\]

For the same reason \( \tilde{\rho}_{u_1,u_2,F,\alpha,\beta}(t,a) \) approaches 0 as \( t \to 0 \). Now the result follows from Theorem 8.4.

We want to conclude this section with a useful formula for the norms of composition operators when a hyperconvex domain \( D \subset \mathbb{C}^n \) is mapped by a holomorphic function \( F \) into the unit disk.

**Theorem 9.2.** Let \( D \subset \mathbb{C}^n \) be a hyperconvex domain with an exhausting function \( u \) such that the set \( L(u) \) is finite. If \( F \) is a holomorphic function mapping \( D \) into the unit disk, \( f \) is a holomorphic function on \( D \) and \( f^* = C_F f \), then

\[
\|f^*\|_{A^\alpha_p}^p = \int_D \sigma_\alpha(u)|f|^p(dd^c u)^n + \int_D N_{F,\alpha}(w)dd^c|f|^p.
\]

**Proof.** By Theorem 5.4

\[
-\int_D u d\overline{d}^c |f^*|^p \wedge (dd^c u)^{n-1} = \int_D N_{F,u}(w) d\overline{d}^c|f|^p.
\]

Thus, the case \( \alpha = -1 \) follows immediately from Theorem 6.2.
In the case $\alpha > -1$ we note that
\[
0 \int_{-\infty}^{0} |r|^\alpha e^{r} \left( \int_{B_u(r)} (r - u)^{f^*} \wedge (dd^c u)^{n-1} \right) dr
= \int_{-\infty}^{0} |r|^\alpha e^{r} \int_{\mathbb{D}} N_{F,u}(w, r) dd^c f^p dr = \int_{\mathbb{D}} N_{F,\alpha}(w) dd^c |f|^p.
\]
Again, the identity follows from Theorem 6.2. \hfill \Box

10. COMPOSITION OPERATORS INDUCED BY MAPPINGS INTO STRONGLY PSEUDOCONVEX DOMAINS

If $B$ is the unit ball in $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ and $u(z) = \log |z|$, then (see [Ru1, Prop. 1.4.10]) the function $\phi(z) = (1 - z_1)^{-1} \in H^p_u(B)$ if and only if $p < n$. So if $f : \mathbb{D} \to B$ is defined as $f(z) = (z,0,\ldots,0)$, then $C_f \phi \not\in H^p(\mathbb{D})$ when $p \geq 1$. On the other hand, the function $(1 - z)^{-1}$ is in $A^p(\mathbb{D})$ if and only if $p < \alpha + 2$. So whatever is $p < n$, the function $C_f \phi \in A^p_{\alpha}(\mathbb{D})$.

This calculation motivates the main result of this section. Before proving it we mention that some special cases were considered in [MM], [CM], [KS] and [SZ1]. A similar theorem for mappings of polydisks was obtained in [SZ2].

**Theorem 10.1.** Let $D_1 \subset \mathbb{C}^n$ be a hyperconvex domain and $u \in \mathcal{E}(D_1)$. Let $D_2 \subset \mathbb{C}^m$ be a strongly pseudoconvex domain with a $C^3$ exhausting strongly plurisubharmonic function $\rho$ such that $\nabla \rho \not= 0$ on $\partial D$. If $F : D_1 \to D_2$ is a holomorphic mapping then $C_F$ acts boundedly from $A^p_{\rho,\alpha}(D_2)$ into $A^p_{\rho,n+\alpha-1}(D_1)$, $\alpha \geq -1$.

**Proof.** Let $T_\zeta$ be the complex tangent plane to $\partial D_2$ at $\zeta \in \partial D_2$ and $U(\zeta, t)$ be the ball in $T_\zeta$ of radius $\sqrt{t}$ centered at $\zeta$. Let
\[
A(\zeta, t) = \{ z \in D_2 : \text{dist}(z, U(\zeta, t)) < t \}.
\]
A theorem of Hörmander [H] states that if $\mu$ is a positive measure on $D_2$ such that there is a constant $C$ so that for every $\zeta \in \partial D$ and $t > 0$,
\[
\mu(\zeta, t) < C t^n,
\]
then there is a constant $C_1$ such that for every function $f \in H^p(D_2)$
\[
\int_{D_2} |f|^p d\mu \leq C_1 \| f \|^p_{H^p(D_2)}.
\]
The corresponding result for weighted Bergman spaces was proved by Cima and Mercer ([CM]). Namely, if
\[
\mu(A(\zeta, t)) < C t^{\alpha+n+1},
\]
then there is a constant $C_1$ such that for every function $f \in A^{p,\alpha}_t(D_2)$
\[
\int_{D_2} |f|^p d\mu \leq C_1 \|f\|^p_{A^{p,\alpha}_t(D_2)}.
\]

By Corollary 3.2 from [Ra, Ch. VII] there are $\delta > 0$ and a $C^2$ function $H(\zeta, w)$ on $\partial D \times D_\delta$, where $D_\delta = \{\rho < \delta\}$, such that $H$ is holomorphic in $w$, $|H(\zeta, w)| < 1$ when $z \in \overline{D} \setminus \{\zeta\}$ and $H(\zeta, \zeta) = 1$. We may assume that the first and second derivatives in $w$ of $H(\zeta, w)$ on $\partial D \times D_\delta$ do not exceed some constant $C_1$. There is $t_0 > 0$ such that the sets $U(\zeta, t) \subset D_\delta$ when $0 < t \leq t_0$. Since the derivatives of $H(\zeta, w)$ along the complex tangent plane $T_\zeta$ are equal to 0 at $\zeta$, there is a constant $C_2$ such that $|H(\zeta, w) - 1| \leq C_2 t^{3/2}$ when $t \leq t_0$ and $w \in U(\zeta, t)$. Therefore, $|H(\zeta, w) - 1| \leq C_3 t$ for some constant $C_3$ when $t \leq t_0$ and $w \in A(\zeta, t)$. It follows that there is a constant $C_4$ such that the sets $A(\zeta, t)$ are contained in the sets $F(\zeta, C_4 t) = \{w \in D_2 : |H(\zeta, w) - 1| \leq C_4 t\}$.

Fix a point $z_0 \in D_1$ and let $v(z) = g_{D_1}(z, z_0)$. Write $h_\zeta(z) = H(\zeta, F(z))$ and consider the following measure

\[
\lambda_\zeta(E) = \int_{-\infty}^{0} |r|^{n+\alpha-1} e^r \mu_{v,r}(\chi_E(h_\zeta(z))) \, dr.
\]
on the unit disk $\mathbb{D}$. If $\nu_F$ is a measure on $D_2$ defined by

\[
\nu_F(E) = \int_{-\infty}^{0} |r|^{n+\alpha-1} e^r \mu_{v,r}(\chi_E(F(z))) \, dr,
\]
then $\nu_F(A(\zeta, t)) \leq \lambda_\zeta(E(1, C_4 t))$ for all $t$. Since there is $b < 1$ such that $|h_\zeta(z_0)| \leq b$ for all $\zeta \in \partial D_2$, by Lemma 4.4 there is a constant $a$ such that $\lambda_\zeta(E(1, C_4 t)) \leq a t^{\alpha+n+1}$. Thus, $\nu_F(A(\zeta, t)) \leq a_1 t^{\alpha+n+1}$ for all $\zeta \in \partial D$ and all $t$.

Therefore, by theorems of Hörmander and Cima and Mercer we have for $\phi \in A^{p,\alpha}_t(D_2)$
\[
\|C_\nu \phi\|^{p}_{A^{p,\alpha+n+1}_t(D_1)} \leq \int_{D_2} |\phi|^p d\nu \leq C \|\phi\|^p_{A^{p,\alpha}_t(D_2)}.
\]
\[
\square
\]
Under some additional assumptions about the mapping $F$ one might get a considerable improvement of the result of the previous Theorem. Below we consider two such cases.

In the first case $D \subset \mathbb{C}^n$ is a strongly pseudoconvex domain with a strictly plurisubharmonic exhaustion function $\rho \in C^3(V)$, where $V$ is a neighborhood of $\overline{D}$, and a holomorphic mapping $F$ is defined on $V$ and takes $D$ into the unit disk $\mathbb{D}$.

We will need a lemma (see [Sh, Ch. 3.13.37]).

**Lemma 10.2.** There is $r_0 > 0$ and $C > 0$ such that for every point $z_0$ on the boundary of $D$ there are holomorphic coordinates $(z_1, \ldots, z_n)$ on $B(z_0, r_0)$ with the following properties: $z_0 = 0$ and in these coordinates

$$\rho(z) = \text{Re} z_n + \frac{1}{2} H(z) + \phi(z), \quad (21)$$

where

$$H(z) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_i}(0) z_i \overline{z}_j$$

and $|\phi(z)| \leq C|z|^3$.

We will call such coordinates preferred. In these coordinates the real tangent hyperplane to $\partial D$ at 0 is $\{\text{Re} z_n = 0\}$ and the complex tangent hyperplane to $\partial D$ at 0 is $\{z_n = 0\}$.

The next lemma gives us a local estimate for the area of the sets $\{F = w\}$.

**Lemma 10.3.** Suppose that $F$ is a holomorphic function on a neighborhood $V$ of $\overline{D}$ mapping $D$ into $\mathbb{D}$. Then there are positive numbers $C$, $c$ and $r_1$ such that for every $z_0 \in \partial D$ with $F(z_0) = w_0$ and $|w_0| = 1$ and for every $w \in \mathbb{C}$ and $r < 0$ either the set $\{F(z) = w\} \cap B(z_0, r_1) \cap \{\rho(z) < r\}$ is empty or $r + c|w - w_0| \geq 0$ and the area of this set does not exceed $C(r + c|w - w_0|)^{(n-1)/2}$ when $r + c|w - w_0| < 1$.

**Proof.** We may assume that $w_0 = 1$, otherwise we consider the function $\overline{w_0} f(z)$. Since $\rho(z) > 0$ when $|F(z)| > 1$ it is easy to see that in preferred coordinates $\nabla F(0) = (0, \ldots, 0, \lambda)$, $\text{Im} \lambda = 0$ and $\lambda > 0$.

Let us write the Taylor expansion of $F$ as

$$F(z) = 1 + \lambda z_n + a_0(z) + z_n b_0'(z) + c_0 z_n^2 + F_3(z),$$

where $'z = (z_1, \ldots, z_{n-1})$,

$$a_0(z) = \sum_{i,j=1}^{n-1} \frac{\partial^2 F}{\partial z_j \partial \overline{z}_i}(0) z_i z_j,$$
\[ b_0(z') = \sum_{j=1}^{n-1} \frac{\partial^2 F}{\partial z_j \partial z_n}(0) z_j, \]

\[ c_0 = \frac{\partial^2 F}{\partial^2 z_n}(0) \text{ and } |F_3(z)| < C_1|z|^3 \text{ for some constant } C_1. \]

By the Implicit Function Theorem there is a positive \( r_1 < r_0 \) such that if \( F(z) = 1 + w \) at some point \( z \in B(0, r_1) \), then the solution of the equation \( F(z) = 1 + w \) in \( B(0, r_1) \) can be represented as \( z_n = g(z', w) \), where \( g \) is a holomorphic function. We write \( g \) as

\[ g(z', w) = \lambda^{-1} w + a(z') + wb(z') + cw^2 + g_3(z', w'), \tag{22} \]

where

\[ a(z') = \sum_{i,j=1}^{n-1} \frac{\partial^2 g}{\partial z_j \partial z_i}(0) z_i z_j, \]

\[ b(z') = \sum_{j=1}^{n-1} \frac{\partial^2 g}{\partial z_j \partial w}(0) z_j, \]

\[ c = \frac{\partial^2 g}{\partial^2 w}(0) \text{ and } |g_3(z', w')| < C_2|z'|^3 \text{ for some constant } C_2. \]

Substituting (22) into the identity \( F(z', g(z', w')) = w \) we see that \( a(z') = -\lambda^{-1} a_0(z') \), \( b(z') = -\lambda^{-2} b_0(z') \) and \( c = -\lambda^{-3} c_0 \). Thus

\[ \rho_1(z', w) = \rho(z', g(z', w)) = \lambda^{-1} \text{Re} w + L(z', w) + \psi(z', w), \]

where

\[ L(z', w) = -\text{Re} \left( \lambda^{-1} a_0(z') + \lambda^{-2} w b_0(z') + \lambda^{-3} c_0 w^2 \right) + \frac{1}{2} H(z', \lambda^{-1} w) \]

and \( |\psi(z', w)| \leq C_3|z'|^3 \) for some constant \( C_3 \).

Since \( \rho_1(z', w) > 0 \) when \( |1 + w|^2 > 1 \) or \( 2 \text{Re} w > -|w|^2 \), we see that

\[ L(z', w) \geq \frac{1}{2\lambda} |w|^2. \]

Hence, the quadratic form \( L \) is nonnegative and, therefore, the set of vectors \( (z', w) \), where \( L(z', w) = 0 \) is a real linear subspace. The quadratic form

\[ L_1(z') = L(z', 0) = -\frac{1}{\lambda} \text{Re} a_0(z') + \frac{1}{2} H(z', 0) \]

is also nonnegative. Let us show that the real linear space \( N \) where \( L_1(z') = 0 \) does not contain complex lines. Observe that if \( L_1(\zeta' z) = 0 \) for all \( \zeta \in \mathbb{C} \), then

\[ L_1(z') = -\frac{1}{\lambda} \text{Re} \zeta a_0(z') + \frac{1}{2} H(\zeta' z, 0). \]
Since $H$ is strictly positive we see that $L_1(\zeta') > 0$ when $\zeta = -\overline{w}_0(\zeta)$. This contradiction shows that $N$ does not contain complex lines and, therefore, its real dimension is at most $n - 1$.

The quadratic form $L'(\z, w) = L_1(\z') + \Re w b_1(\z') + c_1 |w|^2$, where $b_1$ is a linear function and $c_1 > 0$. Note that the coefficients of $b_1$ and $c_1$ depend only on the second derivatives of $F$ and $\rho$ and, therefore, are uniformly bounded. Hence if $\rho_1(\z, w) < r$, then there is a constant $c > 0$ such that $L_1(\z') \leq r + c|w|$. Hence, if the set $\{F(z) = w\} \cap B(z_0, r_1) \cap \{\rho(z) < r\}$ is non-empty, then $r + c|w| \geq 0$.

We can introduce orthonormal coordinates $x = (x_1, \ldots, x_{2n-2})$ on the ball in $\mathbb{C}^{n-1}$ such that

$$L_1(x) = \sum_{j=1}^{k} d_j |x_j|^2,$$

where all $d_j > 0$ and $k \geq n - 1$. The volume of the set $\{x \in B : L_1(x) < r + c|w|\}$, where $B \subset \mathbb{C}^{n-1}$ is the ball of radius $r_1$ centered at the origin, does not exceed $C_4(r + c|w|)^{k/2}$. Since the orthogonal projection of the set $\{F = w\}$ in $B(0, r_1)$ has the Jacobian close to 1 we see that the area of the set $\{F(z) = w\} \cap B(z_0, r_1) \cap \{\rho(z) < r\}$ does not exceed $C_4(r + c|w|)^{k/2} \leq C_4(r + c|w|)^{(n-1)/2}$ when $r + c|w| < 1$. □

This lemma allows us to estimate the counting functions.

**Lemma 10.4.** In the assumptions of Lemma 10.3 there are positive numbers $\delta, c > 0$ and $C$ such that $n_{F, \rho}(w, r) \leq C(r + c|w|)^{(n-1)/2}$, $N_{F, \rho}(w, r) \leq C(r + c|w|)^{(n+1)/2}$ and $N_{F, \rho, \beta}(w) \leq C|w|^\beta(n+3)/2$ when $|w| \geq 1 - \varepsilon$.

**Proof.** Let us chose finitely many points $z_1, \ldots, z_m$ in the set $G = \{z \in \partial D : |F(z)| = 1\}$ such that $G \subset W = \bigcup_{j=1}^{m} B(z_j, r_1)$. Clearly, there is $\varepsilon > 0$ such that $z \in W$ when $w = f(z)$ and $|w| > 1 - \varepsilon$. Since $\nabla F \neq 0$ on $G$ we may assume that $\varepsilon$ is so small that the set $\{f(z) = w\}$ is smooth when $|w| > 1 - \varepsilon$. By Theorem 6.1

$$n_{\rho, F}(w, r) = \int_{\{f(w)\} \cap B_\rho(r)} (dd^c \rho)^{n-1}.$$

But $dd^c \rho$ is equivalent to the Euclidean metric and, therefore, $n_{\rho, F}(w, r)$ does not exceed the area of the set $\{f(z) = w\} \cap B_\rho(r)$ times some constant. In every ball $B(z_j, r_1)$ this area does not exceed $C(r + c|w|)^{(n-1)/2}$. Hence $n_{\rho, F}(w, r) \leq mC(r + c|w|)^{(n-1)/2}$. □
The function

\[ N_{\rho,F}(w,r) = \int_{-c|w|}^{r} n_{\rho,F}(w,t) \, dt \leq \frac{2mC}{n+1} (r + c|w|)^{(n+1)/2}. \]

Finally,

\[ N_{\rho,F,\beta}(w) = \int_{-c|w|}^{0} |r|^\beta e^{r} N_{\rho,F}(w,t) \, dt \leq mC \int_{-c|w|}^{0} |r|^\beta (r + c|w|)^{(n+1)/2} \, dt. \]

To estimate the last integral we notice that

\[ \int_{-c|w|/2}^{-c|w|/2} |r|^\beta (r + c|w|)^{(n+1)/2} \, dt \leq C_1 |w|^{\beta+(n+3)/2} \]

and

\[ \int_{-c|w|/2}^{0} |r|^\beta (r + c|w|)^{(n+1)/2} \, dt \leq C_2 |w|^{\beta+(n+3)/2}. \]

Thus \( N_{\rho,F,\beta}(w) \leq C_3 |w|^{\beta+(n+3)/2} \).

The following theorem is an immediate consequence of these lemmas and Theorem 9.1.

**Theorem 10.5.** Suppose that \( F \) is a holomorphic function on a neighborhood \( V \) of \( \overline{D} \) mapping a strongly pseudoconvex domain \( D \) into \( \mathbb{D} \). Then the composition operator \( C_F \) maps \( A^p_\alpha(D) \) into \( A^p_\beta(D) \) when \( \alpha \leq \beta + (n-1)/2 \). This operator is compact when \( \alpha < \beta + (n-1)/2 \).

It is interesting to note that this theorem cannot be improved even for quadratic polynomials. Let \( B \) be the unit ball in \( \mathbb{C}^n \) centered at the origin and \( F(z) = z_1^2 + \cdots + z_n^2 \). If \( z_j = r_j e^{i\phi_j} \) then \( |F(z)| = 1 \) if and only if \( \sum r_j^2 = 1 \) and \( e^{2i\phi_j} = 1 \). Let us estimate the area of the set \( \{ F = w \} \) near \( z = (0, \ldots, 0, 1) \). The equation \( F(z) = 1 + w \) has a holomorphic solution \( g'(z,w) \) with the following Taylor expansion of order 2

\[ g'(z,w) = 1 + \frac{w}{2} - \frac{r z^2}{2} - \frac{w^2}{4}, \]

where \( r z^2 = z_1^2 + \cdots + z_{n-1}^2 \).

If \( |r z|^2 + |g'(z,w)|^2 < 1 + r, \, r < 0 \), then \( \text{Re} \, w - \text{Re} \, r z^2 + |r z|^2 < cr \) for some constant \( c > 0 \). If \( z_j = x_j + iy_j \) this inequality is equivalent to \( y_1^2 + \cdots + y_{n-1}^2 \leq cr - \text{Re} \, w \). Now the estimates similar to what was used
in the proof of Lemma 10.3 tell us that \( n_{\rho,F}(w,r) \geq C (cr + |w|)^{(n-1)/2} \). Consequently, \( N_{\rho,F,\beta}(w) \geq C|w|^\beta + (n+3)/2 \).

Our next example shows that Hardy and Bergman spaces are preserved by proper mappings of domains with equal dimension.

Suppose that domains \( D_1 \) and \( D_2 \) are in \( \mathbb{C}^n \) and \( f : D_1 \to D_2 \) is a proper holomorphic mapping, i.e., \( f^{-1}(K) \) is compact when \( K \subset D_2 \) is compact. In particular, for every \( w \in D_2 \) the set \( \{f^{-1}(w)\} \) is compact and analytic, so it is finite. The branch set \( B_f \) of \( f \) is the set of points in \( D_1 \), where \( f \) is not locally homeomorphic. It is contained in the analytic set \( \{J_f = 0\} \), where \( J_f \) is the Jacobian of \( f \) and, consequently, has the dimension less than \( n \). Thus, if points \( w_1, w_2 \in D_2 \setminus f(B_f) \) then we can connect them by a continuous curve \( \gamma \subset D_2 \setminus f(B_f) \) and considering its preimage to see that the sets \( \{f^{-1}(w_1)\} \) and \( \{f^{-1}(w_2)\} \) have the same number of points \( m \). This number is called the multiplicity of \( f \).

If \( z_0 \in D_1 \) and \( w_0 = f(z_0) \) we consider a ball \( B_r \subset \subset D_2 \) centered at \( w_0 \) and of radius \( r \). If \( W \) is a connected component of \( f^{-1}(B_r) \) containing \( z_0 \), then the restriction of \( f \) to \( W \) maps \( W \) properly on \( B_r \) with the multiplicity \( m_r \). The limit of \( m_r \) as \( r \) goes to 0 is called the multiplicity of \( f \) at \( z_0 \) and denoted by \( m(z_0,f) \).

If \( \phi \) is a function on \( D_2 \), then as above we set \( \phi^*(z) = \phi(f(z)) \). If \( \phi \) is a function on \( D_1 \) then we let

\[
\phi_*(w) = \frac{1}{m} \sum_{f(z) = w} \phi(z),
\]

where the summation counts multiplicities. Clearly, \( \phi_* \) is plurisubharmonic when \( \phi \) is plurisubharmonic and holomorphic when \( \phi \) is holomorphic.

The following theorem is well-known. Unfortunately, we could not find a reference so we give a proof.

**Theorem 10.6.** Suppose that domains \( D_1 \) and \( D_2 \) are in \( \mathbb{C}^n \) and \( f : D_1 \to D_2 \) is a proper holomorphic mapping of multiplicity \( m \).

If \( v_1, \ldots, v_n \) are plurisubharmonic functions on \( D_2 \) and \( \phi \) is a non-negative Borel function on \( D_2 \), then

\[
m \int_{D_2} \phi d\bar{\phi} v_1 \wedge \cdots \wedge d\bar{\phi} v_n = \int_{D_1} \phi^* d\bar{\phi}^* v_1^* \wedge \cdots \wedge d\bar{\phi}^* v_n^*.
\]

If \( \phi \) is a non-negative Borel function on \( D_1 \), then

\[
m \int_{D_2} \phi_* d\bar{\phi}_* v_1 \wedge \cdots \wedge d\bar{\phi}_* v_n = \int_{D_1} \phi^* d\bar{\phi}^* v_1^* \wedge \cdots \wedge d\bar{\phi}^* v_n^*.
\]
Proof. To prove the first part we take an exhaustion of $D_2$ by domains $V_n$ such that $V_k \subset \subset V_{k+1} \subset \subset D_2$. It is known that for each $k$ there are smooth plurisubharmonic functions $v_{jk}$, $1 \leq j \leq n$, defined on $V_{k+1}$ and such that $v_{jk} \geq v_{j,k+1}$ on $V_k$ and $\lim v_{jk} = v_j$. We also take continuous non-negative functions $h_k \leq 1$ on $D_2$ such that $h_k \equiv 1$ on $V_k$ and $h_k \equiv 0$ on $D \setminus V_{k+1}$.

By the result of Bedford and Taylor ([Kl], p.114)

$$\lim_{k \to \infty} \int_{D_2} h_t \phi dd^c v_{1k} \wedge \cdots \wedge dd^c v_{nk} = \int_{D_2} h_t \phi dd^c v_1 \wedge \cdots \wedge dd^c v_n.$$  

Since the sets $f(B_f)$ and $f^{-1}(f(B_f))$ have a zero measure,

$$m \int_{D_2} h_t \phi dd^c v_{1k} \wedge \cdots \wedge dd^c v_{nk} = \int_{D_1} h_t \phi dd^c v^*_{1k} \wedge \cdots \wedge dd^c v^*_{nk}.$$  

Taking first the limit as $k \to \infty$ and then the limit as $l \to \infty$ we get our statement.

The second statement has a similar proof. \qed

Now we can prove a theorem about the composition operators of proper holomorphic mappings.

**Theorem 10.7.** Let $D_1$ and $D_2$ be hyperconvex domains with exhaustion functions $u_j \in E(D_j)$, $j = 1, 2$, and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Then the composition operator $C_f$ maps $PS_{u_2}(D_2)$ into $PS_{u_1}(D_1)$ and $A_{u_2,\alpha}(D_2)$ into $A_{u_1,\alpha}(D_1)$. Moreover, there is a constant $A \geq 1$ such that $A^{-1} \|\phi\|_{u_2} \leq \|C_f \phi\|_{u_1} \leq A \|\phi\|_{u_2}$ and $A^{-1} \|\phi\|_{u_2,\alpha} \leq \|C_f \phi\|_{u_1,\alpha} \leq A \|\phi\|_{u_2,\alpha}$.

Consequently, $C_f$ maps continuously Banach spaces $H^p_{u_2}(D_2)$ into $H^p_{u_1}(D_1)$ and $A^p_{u_2,\alpha}(D_2)$ into $A^p_{u_1,\alpha}(D_1)$.

Proof. We fix a point $w_0 \in D_2$. Then we consider the exhaustion functions $v(z) = g_{D_2}(w, w_0)$ on $D_2$ and $v^*(z)$ on $D_1$. Both functions belong to $E(D_2)$ and $E(D_1)$ respectively.

Note that $(v^*)_r = (v_r)^*$ and $(dd^c v)^n \equiv 0$ outside of $\overline{D}_v(r)$ while $(dd^c v^*)^n \equiv 0$ outside of $\overline{D}_{v^*}(r)$. Thus

$$\mu_{v,r}(\phi) = \int_{D_2} \phi(dd^c v_r)^n \quad \text{and} \quad \mu_{v^*,r}(\phi^*) = \int_{D_1} \phi^*(dd^c v^*_{r})^n.$$  

By Theorem 10.6 $m \mu_{v,r}(\phi) = \mu_{v^*,r}(\phi^*)$, where $m$ is the multiplicity of $f$. Hence, if $\phi$ is in $PS_v(D_2)$ or in $A_{v,\alpha}(D_2)$, then $\phi^*$ is in $PS_{v^*}(D_1)$ or in $A_{v^*,\alpha}(D_1)$ and $\|\phi^*\|_{v^*} = m\|\phi\|_v$, and $\|\phi^*\|_{v^*,\alpha} = m\|\phi\|_{v,\alpha}$ respectively.

Now our theorem follows immediately from Theorem 3.1. \qed
The operator \( L_f \) mapping the functions \( \phi \) on \( D_1 \) into the functions \( \phi \) on \( D_2 \) is linear and \( L_f \circ C_f \) is the identity operator. As the next theorem shows this operator has the same properties as \( C_f \) and the proof is also the same.

**Theorem 10.8.** In the assumptions of Theorem 10.7 the operator \( L_f \) maps \( PS_{u_1}(D_1) \) into \( PS_{u_2}(D_2) \) and \( A_{u_1,\alpha}(D_1) \) into \( A_{u_2,\alpha}(D_2) \). Moreover, there is a constant \( A \geq 1 \) such that \( A^{-1} \|\phi\|_{u_1} \leq \|L_f\phi\|_{u_2} \leq A \|\phi\|_{u_1} \) and \( A^{-1} \|\phi\|_{u_1,\alpha} \leq \|L_f\phi\|_{u_2,\alpha} \leq A \|\phi\|_{u_1,\alpha} \).

Consequently, \( L_f \) maps continuously Banach spaces \( H^p_{u_1}(D_1) \) into \( H^p_{u_2}(D_2) \) and \( A^p_{u_1,\alpha}(D_1) \) into \( A^p_{u_2,\alpha}(D_2) \).

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