Semi-infinite $A$-variations of Hodge structure
over extended Kähler cone

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§1. Introduction

According to Kontsevich’s Homological Mirror Conjecture [K1], a mirror pair, $X$ and $\hat{X}$, of Calabi-Yau manifolds has two associated $A_\infty$-categories, the derived category of coherent sheaves on $X$ and the Fukaya category of $\hat{X}$, equivalent. In particular, the moduli spaces of $A_\infty$-deformations of these two categories must be isomorphic implying

$$H^*(X, \wedge^* T_X) = H^*(\hat{X}, \mathbb{C}).$$

Another expected corollary is the equivalence of two Frobenius manifold structures, the first one is generated on $H^*(X, \wedge^* T_X)$ by the periods of semi-infinite variations of Hodge structure on $X$ [B2, B3], and the second one is generated on $H^*(\hat{X}, \mathbb{C})$ by the Gromov-Witten invariants.

The l.h.s. in the above equality can be identified with the tangent space at $X$ to the extended moduli space, $\mathcal{M}_{\text{compl}}$, of complex structures [BK]. This moduli space is the base of semi-infinite $B$-variations, $\text{VHS}^B(X)$, of the standard Hodge structure in $H^*(X, \mathbb{C})$ [B2]. Moreover, it was shown in [B2] how to construct a family of Frobenius manifold structures, $\{\Phi^W_{\text{compl}}(X)\}$, on $\mathcal{M}_{\text{compl}}$ parameterized by isotropic increasing filtrations, $W$, in the de Rham cohomology $H^*(X, \mathbb{C})$ which are complementary to the standard decreasing Hodge filtration. Presumably, the compactification $\overline{\mathcal{M}}_{\text{compl}}$ contains a point with maximal unipotent monodromy, and the associated limiting weight filtration $W_0$ gives rise, via Barannikov’s semi-infinite variations of Hodge structure, to the solution, $\Phi^{W_0}_{\text{compl}}(X)$, of the WDVV equations which coincides precisely with the potential, $\Phi_{GW}(\hat{X})$, built out of the Gromov-Witten invariants on the mirror side. This has been checked for complete Calabi-Yau intersections in [B1].

It is widely believed that $\Phi_{GW}(\hat{X})$ can itself be reconstructed from $A$-model variations of Hodge structure (see [CF, CK, Mo] for the small quantum cohomology case). In this paper we propose a symplectic version of the Barannikov’s construction which, presumably, provides a correct framework for extending the results of [CF, CK, Mo] to the full quantum cohomology group. We study semi-infinite $A$-variations, $\text{VHS}^A(X)$, of Hodge structure over the extended moduli space, $\mathcal{M}_{\text{sympl}}$, of Kähler forms on the mirror partner $\hat{X}$, and then use Barannikov’s technique [B2] to build out of $\text{VHS}^A(\hat{X})$ a family of solutions of the WDVV equations, $\{\Phi^W_{\text{sympl}}(\hat{X})\}$, parameterized by isotropic increasing
filtrations, $W'$, in $H^*(X, \wedge^* T_X)$ which are complementary to the standard Hodge type filtration in $\oplus_{i,j} H^i(X, \wedge^j T_X)$. The tangent space to $\mathcal{M}_{\text{sympl}}$ is $H^*(\hat{X}, C)$ which is the r.h.s. in the above “mirror” equality of cohomology groups.

Thus, for any Calabi-Yau manifold $X$ there are two semi-infinite variations of Hodge structure, $\text{VHS}^A(X)$ and $\text{VHS}^B(X)$, and two families of solutions, $\{\Phi^W_{\text{compl}}(X)\}$ and $\{\Phi^{W'}_{\text{compl}}(X)\}$, to WDVV equations. In the idealized situation when $X$ and $\hat{X}$ are dual torus fibrations over the same Monge-Ampère manifold [KS, L, SYZ], one has

$$\text{VHS}^A(X) = \text{VHS}^B(\hat{X}), \quad \text{VHS}^B(X) = \text{VHS}^A(\hat{X}),$$

implying

$$\Phi^W_{\text{compl}}(X) = \Phi^W_{\text{sympl}}(\hat{X}), \quad \Phi^{W'}_{\text{sympl}}(X) = \Phi^{W'}_{\text{compl}}(\hat{X}),$$

for appropriately related filtrations $(W, \hat{W})$ and $(W', \hat{W}')$. To extend these equalities to an arbitrary mirror pair $X$ and $\hat{X}$, one has to find a conceptual way of incorporating instanton corrections (which vanish for dual torus fibrations) into the definition of $\text{VHS}^A$.

As a purely algebraic exercise, we show in this paper that semi-infinite variations of Hodge structure and the associated construction of solutions of WDVV equations make sense for any differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra satisfying Manin’s axioms [Ma].

The paper is organized as follows. Section 2 gives an outline of deformation theory and introduces the basic algebraic input. In Sections 3 and 4 we construct $\text{VHS}^A(\hat{X})$ and $\{\Phi^{W'}_{\text{sympl}}(X)\}$. In Section 5 we establish isomorphisms, $\text{VHS}^A(X) = \text{VHS}^B(\hat{X})$ and $\text{VHS}^B(X) = \text{VHS}^A(\hat{X})$, for dual torus fibrations. (Semi-infinite) Variations of a dGBV theme are collected in Section 6.

§2. An outline of deformation theory

2.1. Sign conventions. In the deformation theory context, it is more suitable to work with the odd version of the usual notion of differential Lie superalgebra. By definition, this is a $\mathbb{Z}_2$-graded vector space, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, equipped with two odd linear maps

$$d : \mathfrak{g} \to \mathfrak{g}, \quad \text{and} \quad [\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

such that

(a) $d^2 = 0$,

(b) $[a \cdot b] = -(\bar{a}+1)(\bar{b}+1)[b \cdot a]$,

(c) $d[a \cdot b] = [da \cdot b] - (-1)^{\bar{a}}[a \cdot db]$,
(d) \([a \bullet [b \bullet c]] = [[a \bullet b] \bullet c] + (-1)^{(\bar{a}+1)(\bar{b}+1)}[b \bullet [a \bullet c]],\)

for all \(a, b, c \in g_0 \cup g_1.\)

In many important examples, the \(\mathbb{Z}_2\)-grading in \(g\) comes from an underlying \(\mathbb{Z}\)-grading, i.e. \(g = \bigoplus_{i \in \mathbb{Z}} g^i, \ g_0 = \bigoplus_{i \ even} g^i, \ g_1 = \bigoplus_{i \ odd} g^i,\) and the basic operations satisfy \(d g^i \subset g^{i+1},\)

\([g^i \bullet g^j] \subset g^{i+j-1}.\)

Clearly, the parity change functor transforms this structure into the usual structure of differential Lie superalgebra on the vector superspace \(\Pi g.\) Thus the above notion is nothing new except slightly unusual sign conventions.

### 2.2. Deformation theory.

Here is a latest guide for constructing a versal moduli space, \(\mathcal{M},\) of deformations of a given mathematical structure:

**Step 1:** Associate to the mathematical structure a “controlling” differential graded (dg, for short) Lie algebra \((g = \bigoplus_{i \in \mathbb{Z}} g^i, d, [\bullet \bullet]).\)

**Step 2:** Find a mini-versal smooth formal pointed dg-manifold \((M, \delta, \ast)\) (i.e. a triple consisting of a smooth formal \(\mathbb{Z}\)-graded manifold \(M,\) a point \(\ast \in M,\) and an odd vector field \(\delta\) on \(M\) such that \([\delta, \delta] = 0)\) which represents the deformation functor, \(\text{Def}_g:\{\text{the category, } \mathcal{A}, \text{ of dg Artin algebras}\} \longrightarrow \{\text{the category of sets}\} \ (A, d_A) \longrightarrow \text{Def}_g^\ast(A, d_A),\)

\[\text{Def}_g(A, d_A) := \left\{ \Gamma \in (g \otimes m_A)^2 \mid d\Gamma + d_A\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0 \right\} \exp (g \otimes m_B)_{\bar{1}}.\]

Here \(m_A\) stands for the maximal ideal in \(A,\) and the quotient is taken with respect to the following action of the gauge group,

\[\Gamma \to \Gamma^g = e^{ad_g} \Gamma - \frac{e^{ad_g} - 1}{ad_g} (d + d_A)g, \quad \forall g \in (g \otimes m_A)^1.\]

It is proven in [Me2] that such a smooth dg-manifold \((M, \delta, \ast)\) always exists. Moreover, \((M, \ast)\) can be identified with a neighborhood of zero in the cohomology super-space, \(H(g) = \ker d/\text{Im} d,\) so that the main job in Step 2 is to find the vector field \(\delta.\)

**Step 3:** Try to make sense to the quotient space, \(\mathcal{M},\) of the subspace \(\text{zeros}(\delta) \subset M\) with respect to the foliation governed by the integrable distribution \(\text{Im} \delta := [\delta, TM], TM\) being the tangent sheaf. In the analytic category, the Kuranishi technique (that is, a cohomological splitting of \(g\) induced by a suitably chosen norm) should do the job. This \(\mathcal{M}\) is a desired mini-versal moduli space of deformations.

\[\text{In the sense that } \text{Def}_g(A, d_A) \simeq \text{Mor}((A, d_A, 0)^{op}, (M, \delta, \ast)), \text{ where } (A, d_A, 0)^{op} \text{ is the representative of } (A, d_A) \text{ in the opposite category } \mathcal{A}^{op}.\]
The functor $\text{Def}_g$ is called non-obstructed if the vector field $\partial$ vanishes. In this case $\mathcal{M} = M$.

2.3. Remarks. The classical deformation functor, $\text{Def}_g$, as defined in the works of Deligne, Goldman, Kontsevich, Millson and others (see [GM, K2] and references therein), is just a restriction of $\text{Def}$ to the subcategory, $\{(A, d_A = 0)\}$, of usual (non differential) Artin superalgebras. Here is an evidence in support of the Cyrillic version:

(i) The modification $\text{Def}_g \to \text{Def}_g$ does not break the Main Theorem of Deformation Theory: if dg Lie algebras $g_1$ and $g_2$ are quasi-isomorphic, then $\text{Def}_{g_1} \cong \text{Def}_{g_2}$.

(ii) Contrary to Def, the functor $\text{Def}$ is always representable by smooth geometric data $(M, \partial, *)$.

(iii) If $\text{Def}_g$ admits a mini-versal (usually, singular) moduli space $\mathcal{M}$, then the latter can be reconstructed from $(M, \partial, *)$ as in Step 3. In particular, $\text{Def}_g$ is non-obstructed in the usual sense if and only if $\text{Def}_g$ is non-obstructed.

Rather than working with a singular moduli space $\mathcal{M}$, it is more convenient to work with its smooth resolution $(M, \partial, *)$. This is the main, purely technical, advantage of $\text{Def}$ over $\text{Def}$.

Note that the tangent space to the functor $\text{Def}_g$ is the full cohomology group $H(g) = \oplus_i H^i(g)$ rather than its subgroup $H^2(g)$ as in many classical deformation problems (see examples below). This is because we allowed the solutions, $\Gamma$, of Maurer-Cartan equations to lie in $(g \otimes m_A)^2$ rather than in $g^2 \otimes m_A$. In this sense the moduli space $\mathcal{M}$ (or its resolution $(M, \partial, *)$) describes extended deformations of the mathematical structure under consideration. The classical moduli space, $\mathcal{M}_{cl}$, is a proper subspace of $\mathcal{M}$.

2.4. Example (Deformations of complex structures). The dg Lie algebra controlling deformations of a given complex structure on a $2n$-dimensional manifold $X$ is given by

$$g = \left( \bigoplus_{i=0}^{2n} g^i, g^i = \bigoplus_{p+q=i} \Gamma(M, \wedge^p T_X \otimes \Omega^{q,0}_X), \left[ \cdot, \cdot \right], \partial \right)$$

where $T_X$ stands for the sheaf of holomorphic vector fields, $\Omega^{s,q}_X$ for the sheaf of smooth differential forms of type $(s, q)$, and $\left[ \cdot, \cdot \right] = \text{Schouten brackets} \otimes \text{wedge product}$.

In general, the deformation theory is obstructed. However, if $X$ is a Calabi-Yau manifold, then $\text{Def}_g$ (or $\text{Def}_g$) is non-obstructed [BK], and the associated mini-versal moduli space $\mathcal{M}$ is isomorphic to an open neighbourhood of zero in $H(g) = H^*(X, \wedge^2 T_X)$. The embedding $\mathcal{M}_{cl} \subset \mathcal{M}$ corresponds to the inclusion $H^1(X, T_X) \subset H^*(X, \wedge^2 T_X)$.

2.5. Example (Deformations of Poisson and symplectic structures). The dg Lie algebra controlling deformations of a given Poisson structure, $\nu_0 \in \Gamma(X, \wedge^2 T_R)$, on a real smooth manifold $X$ is given by

$$\left( \bigoplus_{i=0}^{\dim X} \Gamma(X, \wedge^i T^*_R), \left[ \cdot, \cdot \right] = \text{Schouten brackets}, d = [\nu_0 \cdot \ldots] \right),$$
where \( T^\mathbb{R}_X \) stands for the sheaf of real tangent vectors.

If \( \nu_0 \) is non-degenerate, that is, \( \nu_0 = \omega^{-1} \) for some symplectic form \( \omega \) on \( X \), then the natural “lowering of indices map” \( \omega^\Lambda : \Lambda^r T^\mathbb{R}_X \to \Lambda^r T^\ast_\mathbb{R} \) sends \( \nu_0 \cdot \ldots \) into the usual de Rham differential. The image of the Schouten brackets under this isomorphism we denote by \( \omega \). In this way we make the de Rham complex of \( X \) into a dg Lie algebra,

\[
g = \left( \bigoplus_{i=0}^{\dim X} \Gamma(X, \Lambda^i T^\ast_\mathbb{R}), \ [ \cdot ]_\omega, \ d = \text{de Rham differential} \right),
\]
which controls the extended deformations of the symplectic structure \( \omega \). More explicitly,

\[
[k_1 \bullet k_2]_\omega := (-1)^{\tilde{k}_1} [i_{\omega^{-1}}, d](k_1 \wedge k_2) - (-1)^{\tilde{k}_1} ([i_{\omega^{-1}}, d]k_1) \wedge k_2 - k_1 \wedge [i_{\omega^{-1}}, d]k_2, \ \forall k_1, k_2 \in \Lambda^r T^\ast_\mathbb{R},
\]
with \( i_{\omega^{-1}} : \Lambda^r T^\mathbb{R}_X \to \Lambda^{r-2} T^\ast\mathbb{R}_X \) being the natural contraction with the 2-vector \( \omega^{-1} = \nu_0 \).

It is proven in [Me1] that the associated deformation functor is non-obstructed provided the symplectic manifold \((X, \omega)\) is Lefschetz (in particular, Kähler), that is, that the cup product

\[
[\omega^k] : H^{n-k}(X, \mathbb{R}) \longrightarrow H^{n+k}(X, \mathbb{R})
\]
is an isomorphism for any \( k \leq n =: 1/2 \dim X \). In this case the associated mini-versal moduli space, \( \mathcal{M} \), of extended symplectic structures is smooth, and locally isomorphic to \( H^\ast(X, \mathbb{R}) \). The embedding \( \mathcal{M}_{cl} \subset \mathcal{M} \) corresponds to the inclusion \( H^2(X, \mathbb{R}) \subset H^\ast(X, \mathbb{R}) \).

### 2.6. Example (extended deformations of Kähler structures)
Assume \( X \) is a complex manifold, and consider the Lie algebra,

\[
g' = \left( \bigoplus_{i=0}^{2n} g^i, \ g^i = \bigoplus_{p+q=i} \Gamma(X, \Lambda^p T^\ast_X \otimes \Lambda^q T_X), \ [ \cdot ] = \text{Schouten brackets} \otimes \text{wedge product} \right)
\]
If \( \omega \) is a Kahler form on \( X \) and \( \omega^{-1} \in \Gamma(X, T^\ast_X \otimes T_X) \) is its inverse, then \([\omega^{-1} \cdot \ldots] \) defines a differential on \( g' \). Clearly, the resulting dg Lie algebra controls deformations of the Kähler structure. It has a more convenient embodiment.

The Kähler form \( \omega \) induces the “lowering of indices” isomorphism,

\[
\Gamma(X, \Lambda^p T^\ast_X \otimes \Lambda^q T_X) \longrightarrow \Gamma(X, \Omega^{p,q}_X),
\]
which sends \([\omega^{-1} \cdot \ldots] \) into \( \bar{\partial} \), and hence makes the Doulbeaut complex into a dg Lie algebra,

\[
g = \left( \bigoplus_{i=0}^{2n} g^i, \ g^i = \bigoplus_{p+q=i} \Gamma(X, \Omega^{p,q}_X), \ [ \cdot ]_\omega, \ \bar{\partial} \right)
\]
The odd brackets can be written explicitly as in 2.5 but with \([i_{\omega^{-1}}, d] \) replaced by \([i_{\omega^{-1}}, \bar{\partial}] \). The associated deformation functor is non-obstructed, and the associated versal moduli space, \( \mathcal{M} \), of extended Kähler forms is locally isomorphic, as a formal pointed supermanifold, to \( H^\ast(X, \mathbb{C}) = H^\ast(X, \Omega^\ast_X) \), where \( \Omega^\ast_X \) stands for the sheaf of holomorphic differential
forms. We often call \((\mathcal{M}, \ast)\) the extended Kähler cone\(^2\) of \(X\). The embedding \(\mathcal{M}_{cl} \subset \mathcal{M}\) corresponds to the inclusion \(H^1(X, \Omega^1) \subset H^\ast(X, \Omega_X^\ast)\).

\[ \text{§3. A local system on the extended Kähler cone} \]

### 3.1. From \(\mathfrak{g}\)-modules to vector bundles.

Let \((\mathfrak{m}, \bullet, d)\) be a dg module over the dg Lie algebra \((\mathfrak{g}, [\bullet, \bullet], d)\), that is, a \(\mathbb{Z}\)-graded vector space \(\mathfrak{m}\) together with two odd linear maps, \(d: \mathfrak{m} \rightarrow \mathfrak{m}\) and \(\bullet: \mathfrak{m} \otimes \mathfrak{g} \rightarrow \mathfrak{m}\) such that

1. \(d^2 = 0\),
2. \(d(\kappa \bullet a) = (d\kappa) \bullet a - (-1)^{\hat{\kappa}} \kappa \bullet da\),
3. \(\kappa_1 \bullet \kappa_2 \bullet a - (-1)^{(\hat{\kappa}_1+1)(\hat{\kappa}_2+1)} \kappa_2 \bullet \kappa_1 \bullet a = [\kappa_1 \bullet \kappa_2] \bullet a\)

for all \(\kappa_1, \kappa_2 \in \mathfrak{g}\) and \(a \in \mathfrak{m}\).

Let \((\mathcal{M}, \ast, \partial)\) be the mini-versal dg moduli space associated with the functor \(\text{Def}_\mathfrak{g}\).

The dg Lie algebra structure on the vector superspace \(\mathfrak{g}\) can be geometrically represented as an odd homological vector field, \(Q_\mathfrak{g}\), on \(\mathfrak{g}\) viewed as a supermanifold. For any \(\alpha \in \mathfrak{g}^\ast\) (interpreted now as a function on the supermanifold \(\mathfrak{g}\)) and any point \(\gamma \in \mathfrak{g}\), one has

\[ Q_\mathfrak{g}\alpha \mid_\gamma = -(-1)^{\hat{\alpha}} \langle \alpha, d\gamma + \frac{1}{2}[\gamma \bullet \gamma] \rangle. \]

There always exists a map of pointed dg manifolds,

\[ \Gamma: (\mathcal{M}, \ast, \partial) \longrightarrow (\mathfrak{g}, 0, Q_\mathfrak{g}) \]

such that \(d\Gamma: T_\ast \mathcal{M} \rightarrow T_0 \mathfrak{g}\) is a monomorphism. Moreover, this map is unique up to a gauge transformation as in Step 2 of Sect. 2.2 (see, e.g., [Me2] for a proof).

Any such \(\Gamma\) gives rise to a flat \(\partial\)-connection,

\[ D_\partial^\Gamma: O_M \otimes \mathfrak{m} \longrightarrow O_M \otimes \mathfrak{m} \]

\[ fa \longrightarrow D_\partial^\Gamma(fa) := \partial fa + (-1)^{\bar{f}} f(da + \Gamma \bullet a), \]

in the trivial vector bundle \(M \times \mathfrak{m}\). Though this connection is not gauge invariant, the associated cohomology sheaf,

\[ E_m := O_M \otimes_{\text{Ker} \partial} \text{Ker} D_\partial^\Gamma/\text{Im} D_\partial^\Gamma, \]

together with its natural flat \(\partial\)-connection, \(D_\partial\), is well defined, i.e. the pair \((E_m, \partial)\) does not depend on the choice of a particular map \(\Gamma\).

\(^2\)This terminology could be misleading as we do not fix a particular isomorphism \((\mathcal{M}, \ast) \simeq (H^\ast(X, \mathbb{C}), 0)\). Eventually, however, we will employ a distinguished family of such isomorphism parameterized by isotropic filtrations of \(H^\ast(X, \wedge^n T_X)\) which are complementary to the standard Hodge one.
In summary, we have the following

3.1.1. Proposition. Let \((g, [\bullet], d)\) be a dg Lie algebra and let \((M, \ast, \partial)\) be the mini-versal dg moduli space representing the functor \(\text{Def}_g\). Any dg \(g\)-module \((m, \bullet, d)\) gives canonically rise

(i) to an \(O_M\)-module, \(\pi : E_m \to M\), such that \(\pi^{-1}(\ast) = H(m)\), the cohomology of the complex \(m\), and

(ii) to a flat \(\partial\)-connection \(D_\partial : E_m \to E_m\).

It is clear that the pair \((E_m, D_\partial)\) on \(M\) gives rise to a well defined \(O_M\)-module on the derived moduli space \(M = \text{Zeros}(\partial)/\text{Im} \partial\) (see [B2]).

3.1.2. Remark. The above Proposition can be strengthened as follows: the derived category of dg modules over a given dg Lie algebra \(g\) is equivalent to the purely geometric category of vector bundles over \((M, \partial, \ast)\) equipped with flat \(\partial\)-connections. We omit the proof.

In fact the base, \((M, \partial, \ast)\), can itself be identified with \(F(g)\), where \(F\) is the functor

\[
\left\{ \begin{array}{c}
\text{the category of} \\
\text{dg Lie algebras}
\end{array} \right\} \xrightarrow{F} \left\{ \begin{array}{c}
\text{the derived category of} \\
\text{dg Lie algebras}
\end{array} \right\}.
\]

This gives another meaning to the dg moduli space \((M, \partial, \ast)\) which we first encountered in the context of the \(\text{Def}\)ormation theory.

3.2. Flat structure. Let \((m, d, \bullet)\) be a dg \(g\)-module and assume that there is an even linear map,

\[
\circ : g \otimes m \to m
\]

\[
\kappa \otimes a \mapsto \kappa \circ a
\]

such that

\[
k_1 \circ k_2 \circ a = (-1)^{\tilde{k}_1 \tilde{k}_2} k_1 \circ a,
\]

\[
d(k \circ a) = (d\kappa) \circ a + (-1)^{\tilde{k}} k \circ da + (-1)^{\tilde{k}} k \bullet a,
\]

and

\[
k_1 \circ k_2 \bullet a - (-1)^{\tilde{k}_1 \tilde{k}_2} k_1 k_2 \bullet k_1 \circ a = -(-1)^{\tilde{k}_2} [k_1 \bullet k_2] \circ a.
\]

for any \(k, k_1, k_2 \in g\) and \(a \in m\).

Choosing a map \(\Gamma : (M, \ast, \partial) \to (g, 0, Q_g)\) as in Sect. 3.1, we may define a flat connection,

\[
\nabla^\Gamma : TM \otimes (\mathcal{O}_M \otimes m) \to \mathcal{O}_M \otimes m
\]

\[
v \otimes (fa) \mapsto \nabla^\Gamma_v(fa) := (vf)a + (-1)^{\tilde{f}} f((v\Gamma) \circ a),
\]

in the trivial vector bundle \(M \times m\). It is not hard to check that

\[
e^{-\Gamma_0}(d + \partial)e^{\Gamma_0} = D_\partial^\Gamma,
\]
and
\[ e^{-\Gamma^\circ(v)}e^{\Gamma^\circ} = \nabla^\Gamma_v, \]
for any vector field \( v \) on \( M \). Hence,
\[ [\nabla^\Gamma_v, \mathcal{D}^\Gamma] = \nabla^\Gamma_{[v, \partial]} \]
implying the following

### 3.2.1. Proposition \[\text{[B2]}\]. Let \((\mathfrak{g}, [ \cdot, \cdot ], d)\) be a dg Lie algebra, \((M, *, \overline{\partial})\) the associated mini-versal dg moduli space, and \(\mathcal{M} = \text{Zeros}(\overline{\partial})/\text{Im}(\overline{\partial})\) the associated derived moduli space. Any dg \(\mathfrak{g}\)-module \((\mathfrak{m}, \cdot, \circ, d)\) as above gives canonically rise to a pair, \((E_m, \nabla)\), where \(E_m\) is a vector bundle on \(M\) with typical fibre \(H^*\mathfrak{m}\), and \(\nabla\) is a flat connection.

### 3.2.2. Remark. The flat connection \(\nabla\) identifies the linear space of horizontal sections of \(\pi : E_m \to \mathcal{M}\) with the fibre \(\pi^{-1}(\ast)\). Explicitly, the identification goes as follows

\[ H^*(\mathfrak{m}) \xrightarrow{1:1} \text{Space of horizontal sections} \xrightarrow{e^{-\Gamma^\circ}} e^{-\Gamma^\circ}a. \]

In particular, the parallel transport establishes a canonical isomorphism of the fibres \(\pi^{-1}(t) \simeq \text{Ker}(d + \Gamma \circ)/\text{Im}(d + \Gamma \circ)\) with the fibre, \(\pi^{-1}(\ast) \simeq \text{Ker} d/\text{Im} d\), over the base point.

### 3.3. Example (deformations of complex structures). One of the key observations in \[\text{[B2]}\] is that, for any compact complex manifold \(X\), the pair consisting of the dg Lie algebra,

\[ \mathfrak{g} = \bigoplus_{i=0}^{2n} \mathfrak{g}^i, \quad \mathfrak{g}^i = \bigoplus_{p+q=i} \Gamma(M, \wedge^p T_X \otimes \Omega^0_X), \quad \cdot, \circ, d = \partial + \overline{\partial} \]

and the \(\mathfrak{g}\)-module,

\[ \mathfrak{m} = \bigoplus_{i=0}^{2n} \mathfrak{m}^i, \quad \mathfrak{m}^i = \bigoplus_{p+q=i} \Gamma(X, \Omega^{p, q}_X), \quad \cdot, \circ, d = \partial + \overline{\partial} \]

with \(\kappa \circ a := [i_k, \partial]\) and \(\kappa \circ a := i_k a\), do satisfy the conditions of Proposition 3.2.1. Thus the extended moduli space of complex structures (which, for Calabi-Yau \(X\), is smooth and locally isomorphic to \(H^*(X, \wedge^* T_X)\)) comes equipped with a flat vector bundle whose typical fibre is the de Rham cohomology \(H^*(X, \mathbb{C})\).

### 3.4. Example (deformations of Kähler forms). In this subsection we shall present one more example to which Proposition 3.2.1 is applicable. Curiously it inverses the roles of \(\mathfrak{g}\) and \(\mathfrak{m}\) in Barannikov’s example, and hence gives rise to a local system whose base is locally isomorphic to the de Rham cohomology \(H^*(X, \mathbb{C})\) and whose typical fibre is \(H^*(X, \wedge^* T_X)\).

We assume from now on that \((X, \omega)\) is a Kähler manifold, and denote by \(\mathfrak{g}\) the dg Lie algebra controlling extended deformations of the Kähler structure (see Sect. 2.6). We omit from now on the subscript \(\omega\) in the Lie brackets notation.

### 3.4.1. Auxiliary operators. We want to study the following morphisms of sheaves:
(i) The inverse Kähler form \( \omega^{-1} \) induces a natural “raising of indices” map
\[
\#: \Omega^{0,q}_X \longrightarrow T^*_X \otimes \Omega^{0,q-1}_X,
\]
which, combined with the antisymmetrisation, extends to the map
\[
\#: \wedge^* T^*_X \otimes \Omega^{0,*}_X \longrightarrow \wedge^{*+1} T^*_X \otimes \Omega^{0,*-1}_X.
\]

(ii) For any \( \kappa \in \Gamma(X, \Omega^{s,t}_X) \) there is a natural map,
\[
i_\kappa : \wedge^* T^*_X \otimes \Omega^{0,*}_X \longrightarrow \wedge^{*-s} T^*_X \otimes \Omega^{0,*+t}_X
\]
which is a combination of contraction and wedge product.

3.4.2. Lemma. For any Kähler manifold \( X \), one has

(a) The commutator,
\[
Q := [\#, \bar{\partial}] : \wedge^* T^*_X \otimes \Omega^{0,*}_X \longrightarrow \wedge^{*+1} T^*_X \otimes \Omega^{0,*},
\]
is a differential, i.e. \( Q^2 = 0 \).

(b) \([Q, \bar{\partial}] = 0\).

(c) \([Q, \bar{\epsilon}] = 0\).

(d) \([i_{\kappa_1}, [Q, i_{\kappa_2}]] = -i_{[\kappa_1, \kappa_2]} \quad \forall \kappa_1, \kappa_2 \in \mathfrak{g}\).

(e) \([i_{\kappa_1 \wedge \kappa_2}, Q] = i_{\kappa_1} [i_{\kappa_2}, Q] + (-1)^{\tilde{\kappa}_1 \tilde{\kappa}_2} i_{\kappa_2} [i_{\kappa_1}, Q] \quad \forall \kappa_1, \kappa_2 \in \mathfrak{g}\).

A comment on the proof. One can identify the sheaf \( \Omega^{*,*}_X \) with the structure sheaf on the supermanifold \( \Pi T^*_C \), where \( \Pi \) is the parity change functor and \( T^*_C = T_R \otimes \mathbb{C} \). Then, in a natural local coordinate system \((z^a, \psi^a := dz^a, \bar{\psi}^a := dz^a)\), one may explicitly represent the (odd) Lie brackets as Poisson ones,
\[
[i_{\kappa_1 \wedge \kappa_2}, Q] = (\kappa_1 \leftrightarrow \kappa_2)
\]
where the summation over repeated indices is assumed, and \( \omega^{ab} \) stand for the coordinate components of the inverse Kähler form \( \omega^{-1} \).

Analogously, one can identify the sheaf \( \wedge^* T^*_X \otimes \Omega^{0,*}_X \) with a sheaf of functions on the total superspace of the bundle \( \Pi T^*_X \oplus \Pi T^*_X \). In a natural local coordinate chart, \((z^a, \psi_a := \Pi \partial / \partial z^a, \bar{\psi}^a := d\bar{z}^a)\), one may explicitly represent the basic operators as follows
\[
\bar{\partial} = \bar{\psi}^a \frac{\partial}{\partial \bar{z}^a},
\]
\[
\# = \omega^{ab} \psi_a \frac{\partial}{\partial \psi^b},
\]
\[
Q = \omega^{ab} \psi_a \frac{\partial}{\partial \bar{z}^b} - \bar{\psi}^c \frac{\partial \omega^{ab}}{\partial \bar{z}^c} \psi_a \frac{\partial}{\partial \psi^b},
\]
\[
i_\kappa = (-1)^{pq} \kappa_{a_1 \ldots a_p} \bar{\psi}^{b_1} \ldots \bar{\psi}^{b_q} \frac{\partial^p}{\partial \psi_{a_1} \ldots \partial \psi_{a_p}}, \quad \forall \kappa \in \Omega^{p,q}_M.
\]
The main technical advantage of this point of view is that
- it is enough to check all the claims (a)-(e) at one arbitrary point $*$ ∈ $X$,
- due to the Kähler condition on $\omega$, one can always choose the coordinates at $*$ in such a way that

$$\frac{\partial \omega^{ab}}{\partial \bar{z}^c} |_* = 0.$$ 

With this observation all the above expressions can be dramatically simplified making the claims either transparent or requiring a minimal calculation. 

3.4.3. Proposition. The (odd) linear map,

$$\bullet : \ g \otimes m \longrightarrow m \quad \kappa \otimes a \longrightarrow -[i_\kappa, Q]a$$

makes the dg vector space,

$$m := \left( \bigoplus_{i=0}^{2n} m^i, \ m^i = \bigoplus_{p+q=i} \Gamma(X, \Lambda^p T_X \otimes \Omega^q_X), \ d = \bar{\partial} + Q \right).$$

into a dg module over the dg Lie algebra $g$.

Proof. For any $\kappa_1, \kappa_2 \in g$ and any $a \in m$, we have, by Lemma 2.1(d),

$$\kappa_1 \bullet \kappa_2 \bullet a + (-1)^{\bar{\kappa}_1 \bar{\kappa}_2 + \bar{\kappa}_1 + \bar{\kappa}_2} \kappa_2 \bullet \kappa_1 \bullet a = [i_{\kappa_1}, Q][i_{\kappa_2}, Q]a + (-1)^{\bar{\kappa}_1 + \bar{\kappa}_2 + \bar{\kappa}_1} [i_{\kappa_1}, Q][i_{\kappa_2}, Q]a$$

$$= -[i_{[\kappa_1, \kappa_2]}, Q]a$$

$$= [\kappa_1 \bullet \kappa_2] \bullet a.$$ 

which means that $(m, \bullet)$ is a $g$-module. Consistency of $\bullet$ with the differentials is also an easy check:

$$d(\kappa \bullet a) = (\bar{\partial} + Q)[i_\kappa, Q]a$$

$$= \bar{\partial} i_\kappa Qa - (-1)^{\bar{\kappa}} \bar{\partial} Q i_\kappa a + Q i_\kappa Qa$$

$$= [\bar{\partial}, i_\kappa] Qa - (-1)^{\bar{\kappa}} i_\kappa Q \bar{\partial} a + (-1)^{\bar{\kappa}} Q [\bar{\partial}, i_\kappa] a + Q i_\kappa \bar{\partial} a - (-1)^{\bar{\kappa}} [i_\kappa, Q] Qa$$

$$= [i_{\bar{\partial} \kappa}, Q] a - (-1)^{\bar{\kappa}} [i_\kappa, Q] Qa - [i_\kappa, Q] \bar{\partial} a$$

$$= (\bar{\partial} \kappa) \bullet a - (-1)^{\bar{\kappa}} \kappa \bullet da.$$ 

3.4.4. Lemma. An (even) linear map

$$\circ : \ g \otimes m \longrightarrow m \quad \kappa \otimes a \longrightarrow i_\kappa a$$

satisfies,

$$d(\kappa \circ a) = (\bar{\partial} \kappa) \circ a + (-1)^{\bar{\kappa}} \kappa \circ da + (-1)^{\bar{\kappa}} \kappa \bullet a,$$
and

\[ \kappa_1 \circ \kappa_2 \bullet a - (-1)^{\tilde{\kappa}_1}\tilde{\kappa}_2 + \tilde{\kappa}_1 \kappa_2 \bullet \kappa_1 \circ a = -(-1)^{\tilde{\kappa}_2}[\kappa_1 \bullet \kappa_2] \circ a. \]

for any \( \kappa, \kappa_1, \kappa_2 \in g \) and \( a \in m \).

**Proof.** We have

\[
\begin{align*}
\delta (\kappa \circ a) &= (\bar{\partial} + Q)i_\kappa a \\
&= [\bar{\partial}, i_\kappa]a + (-1)^{\tilde{\kappa}_1}i_\kappa \bar{\partial}a + [Q, i_\kappa]a + (-1)^{\tilde{\kappa}_2}i_\kappa Qa \\
&= i_{\bar{\partial} \kappa}a + (-1)^{\tilde{\kappa}_1}i_\kappa (\bar{\partial} + Q)a - (-1)^{\tilde{\kappa}_2}[Q, i_\kappa]a \\
&= (\bar{\partial} \kappa) \circ a + (-1)^{\tilde{\kappa}_1} \kappa \circ da + (-1)^{\tilde{\kappa}_2} \kappa \bullet a.
\end{align*}
\]

and, using Lemma 2.1(d),

\[
\begin{align*}
\kappa_1 \circ \kappa_2 \bullet a - (-1)^{\tilde{\kappa}_1}\tilde{\kappa}_2 + \tilde{\kappa}_1 \kappa_2 \bullet \kappa_1 \circ a &= -i_{\kappa_1}[i_{\kappa_2}, Q]a + (-1)^{\tilde{\kappa}_1}\tilde{\kappa}_2 + \tilde{\kappa}_1[i_{\kappa_2}, Q]i_{\kappa_1}a \\
&= (-1)^{\tilde{\kappa}_2}[i_{\kappa_1}, [Q, i_{\kappa_2}]]a \\
&= -(-1)^{\tilde{\kappa}_2}i_{[\kappa_1 \bullet \kappa_2]}a \\
&= -(-1)^{\tilde{\kappa}_2}[\kappa_1 \bullet \kappa_2] \circ a.
\end{align*}
\]

In conclusion, the pair consisting of the dg Lie algebra,

\[ g = \bigoplus_{i=0}^{2n} g^i, \quad g^i = \bigoplus_{p+q=i} \Gamma(X, \Omega^p_X \otimes \Omega^q_X), \quad \bullet, \circ, d = \bar{\partial} + Q \bigg), \]

and the dg \( g \)-module,

\[ m := \bigoplus_{i=0}^{2n} m^i, \quad m^i = \bigoplus_{p+q=i} \Gamma(X, \Lambda^p T_X \otimes \Omega^q_X), \quad \bullet, \circ, d = \bar{\partial} + Q \bigg), \]

satisfy the conditions of Proposition 3.2.1, and hence gives rise to a flat vector bundle, \((E_m, \nabla)\), over the moduli space of extended Kähler structures, \( M = M \simeq H^*(X, \mathbb{C}) \), with typical fibre \( H^*(X, \wedge^* T_X) \).

**3.5. Remark.** In Sect. 5 we shall give more examples of pairs \((g, m)\) to which Proposition 3.2.1 is applicable — one for each differential Gerstenhaber-Batalin-Vilkoviski (dGBV, for short) algebra. This will produce, in particular, one more local system, \((E'_m, \nabla)\), over the extended Kähler cone of \( X \), with typical fibre \( H^*(X, \mathbb{C}) \). If \( X \) is Ricci flat, then \((E'_m, \nabla)\) is isomorphic to the one constructed in subsect. 3.4, but in general it is different.
§4. Frobenius manifolds from semi-infinite variations of Hodge structure in $H^*(X, \wedge^*T_X)$

4.1. Semi-infinite $B$-variations of Hodge structure in $H^*(X, \mathbb{C})$. A complex structure $J_t$ on a compact manifold $X$ gives rise to the Hodge decomposition, $\bigoplus_{i,j} H^i(X, \Omega^j)$, of the de Rham cohomology group $H^*(X, \mathbb{C})$. When the complex structure is deformed, the associated Hodge filtration $F^r_0$ gets deformed into another one, $F^r_t$, $t \in \mathcal{M}_d$; a remarkable fact is that this deformation satisfies Griffiths transversality condition with respect to the Gauss-Manin connection on the bundle $E_m$ (see Examples 2.4 and 3.3) restricted to $\mathcal{M}_d \subset \mathcal{M}$.

What happens to the Hodge filtration when one moves from a “classical” point $J_t \in \mathcal{M}_d$ to a generic point in the extended moduli space, $\mathcal{M}$, of complex structures? An answer to this question was given in [B2] by a creative usage of Sato type Grassmanian, $Gr_{\infty}$, of semi-infinite subspaces in $H^*(X, \mathbb{C})[[h, h^{-1}]]$. Moreover, for Calabi-Yau $X$, the resulting datum was used as an input for producing a family of Frobenius manifold structures on $\mathcal{M} \simeq H^*(X, \wedge^*T_X)$ parametrized by isotropic (with respect to the Poincare metric on $H^*(X, \mathbb{C})$) increasing filtrations which are complementary to the Hodge one.

We present below a symplectic version of the Barannikov’s construction. The parallelism is so strong that we can afford being sketchy.

4.2. Semi-infinite $A$-variations of Hodge structure in $H^*(X, \wedge^*T_X)$. Let $(X, \omega)$ be a compact $n$-dimensional Kähler manifold, and let $(\mathcal{M}, *) \simeq (H^*(X, \mathbb{C}), 0)$ be the associated moduli space of extended Kähler structures. As before, we denote by $\mathfrak{g}$ the dg Lie algebra,

$$ \left( \bigoplus_{i=0}^{2n} \mathfrak{g}^i, \mathfrak{g}^i = \bigoplus_{p+q=i} \Gamma(X, \Omega^p_X), [\bullet, \omega], \bar{\partial} \right), $$

which controls deformations of the Kähler structure. However, instead of the $\mathfrak{g}$-module $m$ defined in Sect. 3.4, we need its slight modification involving a formal parameter $\hbar$,

$$ m := (\Gamma(X, \Lambda^*T_X \otimes \Omega^0_X)[[\hbar, \hbar^{-1}]])^{\bullet}, \circ, \partial = \bar{\partial} + hQ, $$

with $\kappa \circ a := -[i, Q]a$ and $\kappa \circ a := \frac{1}{\hbar}i_\kappa a$. All the formulae and claims of Sect. 3.4 remain true; in particular, the pair $(\mathfrak{g}, m)$ gives rise to a flat vector bundle, $(E_h, \nabla)$, over the moduli space $(\mathcal{M}, *)$. As

$$ \bar{\partial} + hQ = l_h^{-1}h^{\frac{1}{2}}(\partial + Q)l_h, $$

where $l_h$ is a linear automorphism of $\Gamma(X, \Lambda^*T_X \otimes \Omega_X^0)[[\hbar, \hbar^{-1}]]$ given by

$$ l_h a := \hbar^{\frac{n+2-p}{2}}a, \forall a \in \Gamma(X, \Lambda^pT_X \otimes \Omega^0_X), $$

the fibre of the bundle $E_h$ over the base point $*$ is isomorphic to $H^*(X, \wedge^*T_X)[[\hbar, \hbar^{-1}]]$.

\[\text{3The associated linear automorphism of } H^*(X, \wedge^*T_X)[[\hbar, \hbar^{-1}]] \text{ is denoted by the same letter } l_h.\]
The standard Hodge decreasing filtration on $H^*(X, \wedge^* T_X)$,

$$
0 = F^\geq_{\frac{n+2}{2}} \subset F^\geq_{\frac{n}{2}} \subset \ldots \subset F^\geq_{\frac{n}{2}} ,
0 = F^\geq_{\frac{n+1}{2}} \subset F^\geq_{\frac{n-1}{2}} \subset \ldots \subset F^\geq_{\frac{1-n}{2}}
$$

is given by

$$
F^\geq_r := \bigoplus_{\frac{p-q}{2} \geq r} H^q(X, \wedge^p T_X), \quad r \in \mathbb{Z}[\frac{1}{2}]
$$

(so that $F^\geq_{\frac{n}{2}} = H^0(X, \wedge^n T_X)$, $F^\geq_{\frac{n-1}{2}} = H^0(X, \wedge^{n-1} T_X) \oplus H^1(X, \wedge^n T_X)$, etc.). There is associated a linear subspace,

$$
L_0 := \text{span}_{r \in \mathbb{Z}[\frac{1}{2}]} F^\geq_r h^{-r+\frac{n}{2}}[[h]]
= \text{span}_{r \in \mathbb{Z}[\frac{1}{2}]} l_h H^*(X, \wedge^* T_X)[[h]] \subset H^*(X, \wedge^* T_X)[[h, h^{-1}]].
$$

Next one considers a relative Grassmanian, $Gr_{\frac{n}{2}}(E_h)$, whose fibre over a generic point $t \in \mathcal{M}$ is the Sato Grassmaninan of semi-infinite subspaces in the fibre of $\pi_h : E_h \rightarrow \mathcal{M}$ over $t$. The latter has a canonical global section, $\Phi$, which associates to any $t \in \mathcal{M}$ the vector space, $\Phi(t)$, of all elements in $\pi_h^{-1}(t)$ which are (formal) analytic in $h$. We can use the flat connection $\nabla$ to compare the values $\Phi(t)$ at different points $t \in \mathcal{M}$ via the parallel transport to the base point. In this way we get a well defined map to the projective limit of Sato Grassmanians,

$$
\mathcal{M} \rightarrow Gr_{\frac{n}{2}}(H^*(X, \wedge^* T_X)[[h, h^{-1}]])(\mathcal{O}_*)
$$

where $L_t$ is the $\mathcal{O}_*$-submodule of $H^*(X, \wedge^* T_X)[[h, h^{-1}]] \otimes \mathcal{O}_*$ generated by $l_h P^\nabla_t \Phi(t)$, where $P^\nabla_t$ stands for the parallel transport from $t$ to $*$. Taking into account Remark 3.2.2, we may write $L_t = l_h e^T(t) \otimes \mathcal{O}_*$. Note that $L_*$ is precisely the submodule $L_0 \otimes \mathcal{O}_*$ corresponding to the standard Hodge filtration on $H^*(X, \wedge^* T_X)$.

**4.3. Frobenius manifolds.** We assume from now on that $X$ is an $n$-dimensional Ricci flat Kähler manifold, i.e. a Calabi-Yau manifold. In this case the line bundle $\wedge^n T_X$ admits a global nowhere vanishing section which we denote by $\eta$. The associated nowhere vanishing global holomorphic $n$-form is denoted by $\Omega$. Note that $\eta \in L_0$.

Let

$$
(0 = W_{< \frac{n}{2}} \subset W_{< \frac{n-2}{2}} \subset \ldots \subset W_{< \frac{n-2}{2}} ,
0 = W_{< \frac{n-1}{2}} \subset W_{< \frac{n-3}{2}} \subset \ldots \subset W_{< \frac{n-3}{2}})
$$

be an increasing filtration on $H^*(X, \wedge^* T_X)$ which is complementary to the Hodge filtration in the sense that

$$
\bigoplus_{i+j=2r} H^i(X, \wedge^j T_X) = F^\geq_r \oplus W_{\leq r}.
$$

There is associated a linear subspace,

$$
L_W := \text{span}_{r \in \mathbb{Z}[\frac{1}{2}]} W_{\leq r} h^{-r+\frac{n}{2}}[[h^{-1}]] \subset H^*(X, \wedge^* T_X)[[h, h^{-1}]].
$$
Moreover,
\[ H^\ast(X, \wedge^* T_X)[[h, h^{-1}]] = L_0 \oplus L_W \]
and, for \( t \in \mathcal{M} \) “sufficiently close” to the base point \(*\), the intersection \( L_t \cap (\eta + L_W) \) consists of a single element. Hence there is a well defined composition
\[
\Psi^W : \mathcal{M} \rightarrow H^\ast(X, \wedge^* T_X)[[h, h^{-1}]] \rightarrow H^\ast(X, \Omega^{n-\ast})[[h, h^{-1}]]
\]
where \( \lrcorner \Omega \) stands for the natural contraction with the holomorphic volume form. Moreover its equivalence class,
\[
\hat{\Psi}^W := (\Psi^W - \eta) \mod h^{-1} L_W \lrcorner \Omega,
\]
gives rise to a composition
\[
\hat{\Psi}^W : \mathcal{M} \rightarrow (L_W \text{ mod } h^{-1} L_W) \lrcorner \Omega = \left( \bigoplus_r W_{\leq r}/W_{\leq r-1} \right) \lrcorner \Omega = H^\ast(X, \mathbb{C}),
\]
which is obviously a local diffeomorphism. If \( \{\Delta_a\} \) is a vector space basis in \( H^\ast(X, \mathbb{C}) \) and \( \{t^a\} \) the associated dual basis, then the functions \( t^a_W := (\hat{\Psi}^W)^{-1}(t^a) \) define a distinguished coordinate system, \( \{t^a_W\} \), on \( \mathcal{M} \). In this coordinate system the map \( \hat{\Psi}^W \) satisfies the equations (Proposition 6.5 in [B2]),
\[
\frac{\partial^2 \hat{\Psi}^W(t_W, h)}{\partial t^a_W \partial t^b_W} = h^{-1} \sum_c A^c_{ab}(t_W) \frac{\partial \hat{\Psi}^W(t_W, h)}{\partial t^c_W},
\]
and hence gives rise to a pencil, \( \partial / \partial t^a_W + \lambda A^c_{ab}(t_W) \), of flat connections on the moduli space of extended Kähler structures. That is, the structure functions \( A^c_{ab}(t_W) \) satisfy,
\[
\frac{\partial A^c_{ab}}{\partial t^d_W} = (-1)^{\tilde{a}\tilde{d}} \frac{\partial A^c_{ab}}{\partial t^d_W}, \quad \sum_c A^c_{ab} A^e_{cd} = (-1)^{\tilde{a}(\tilde{b}+\tilde{d})} \sum_c A^c_{bd} A^e_{ca}.
\]
Thus the product in the tangent sheaf \( \mathcal{T}_\mathcal{M} \) defined by
\[
\frac{\partial}{\partial t^a_W} \circ \frac{\partial}{\partial t^b_W} := \sum_c A^c_{ab} \frac{\partial}{\partial t^c_W}
\]
is associative.

The Poincare form on \( H^\ast(X, \mathbb{C}) \) together with the given holomorphic volume form \( \Omega \) induce a non-degenerate paring on \( H^\ast(X, \wedge^* T_X)[[h, h^{-1}]] \),
\[
(v, w) := \int_X (v \lrcorner \Omega) \wedge (w \lrcorner \Omega).
\]
If we assume that the complementary filtration \( W \) is isotropic in the sense that
\[
(W_{\leq r}, W_{\leq -r+1}) = 0, \quad \forall \ r \in \mathbb{Z}[\frac{1}{2}],
\]
\[\footnote{More precisely, the map \( \hat{\Psi}^W \) defines a distinguished flat structure, \( \nabla \), on \( \mathcal{M} \).} \]
then,
\[(v, w) \in \mathbb{h}^{n-2}[h^{-1}], \quad \forall v, w \in L_W.\]

On the other hand,
\[
\int_X \frac{\partial \Psi^W (t_W, h)}{\partial t_W^a} \frac{\partial \Psi^W (t_W, -h)}{\partial t_W^b} \in \mathbb{h}^{n-2}[h],
\]
so that the functions
\[
g_{ab}(t_W) := h^{2-n} \int_X \frac{\partial \Psi^W (t_W, h)}{\partial t_W^a} \frac{\partial \Psi^W (t_W, -h)}{\partial t_W^b}
\]
do not depend on \(h\), and hence define a non-degenerate metric, \(\langle \frac{\partial}{\partial t_W^a}, \frac{\partial}{\partial t_W^b} \rangle\), on the formal pointed manifold \((M, \ast)\). Moreover, differentiating the above equation with respect to \(t_W\), we immediately conclude that the metric \(g_{ab}\) is constant in these coordinates, and that the tensor,
\[
A_{abc} := A^d_{ab} g_{dc},
\]
is totally symmetric and hence potential,
\[
A_{abc} = \frac{\partial^3 \Phi^W}{\partial t_W^a \partial t_W^b \partial t_W^c},
\]
for scalar function \(\Phi^W(t_W)\) on \(M\). The above quadratic equations for \(A^c_{ab}\) mean that the latter must be a solution of the WDVV equations. It is not hard to check (cf. \([33]\)) that the equation,
\[
\frac{\partial \Psi^W (t_W, h)}{\partial h} = -h^{-1} \sum_c E^c (t_W) \frac{\partial \Psi^W (t_W, h)}{\partial t_W^c},
\]
holds for some vector field \(E\) on \(M\). Differentiating the defining equation of the metric with respect to \(h\), one immediately concludes that \(E\) is a conformal Killing vector with respect to the metric \(g\),
\[
\mathcal{L}_E g = (2 - n) g.
\]

Doing the same to the equation defining the structure functions, one proves the homogeneity,
\[
\mathcal{L}_E (\circ) = \circ,
\]
of the product in \(T_M\).

The basis \(\{\Delta_a\}\) may be chosen in such a way that \(\Delta_0\) is the unit of the supercommutative algebra \(H^*(X, \mathbb{C})\). We denote by 1 the canonical lift of \(\Delta_0\) into \(\mathfrak{g}\). The map of pointed dg manifolds, \(\Gamma : (M, \ast, 0) \to (\mathfrak{g}, 0, Q_\mathfrak{g})\), can always be normalized in such a way that \([32]\)
\[
\frac{\partial \Gamma}{\partial t_W^0} = 1.
\]

This immediately implies,
\[
\frac{\partial \Psi^W}{\partial t_W^0} = h^{-1} \Psi^W,
\]

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which in turn implies that the vector field \( e = \partial/\partial t^0_W \) is a unit with respect to the multiplication \( \circ \).

Thus applying the remarkable Barannikov’s construction \([B2, B3]\) to the deformation theory of Kähler forms, one gets a family, \((\Phi^W, g, E, e)\), of Frobenius manifold structures on \( \mathcal{M} \simeq H^*(X, \mathbb{C}) \) parameterized by isotropic filtrations in \( H^*(X, \wedge^* T_X) \) which are complementary to the standard Hodge one.

### 5. Dual torus fibrations

#### 5.1. Monge-Ampère manifolds \([H, KS]\)

Such a manifold is a triple, \((Y, g, \nabla)\), consisting of a smooth manifold \( Y \), a smooth Riemannian metric \( g \), and a flat torsion-free affine connection \( \nabla \) such that

(i) the metric is potential in the sense that its coefficients in a local affine coordinate system \( \{x^i, i = 1, \ldots, \dim Y\} \) are given by

\[
g_{ij} = \frac{\partial^2 K}{\partial x^i \partial x^j},
\]

for some smooth (convex) function \( K \);

(ii) The Monge-Ampère equation,

\[
\det \left( \frac{\partial^2 K}{\partial x^i \partial x^j} \right) = \text{const},
\]

is satisfied.

Due to the convexity of \( K \) the system of equations,

\[
\frac{\partial \hat{x}_i(x^j)}{\partial x^j} = g_{ij},
\]

can solved in any local affine coordinate system \( \{x^i\} \) on \((Y, g, \nabla)\). The resulting functions \( \hat{x}_i = \hat{x}_i(x^j) \) give rise to a new local coordinate system on \( Y \) and hence to a new torsion-free affine connection, \( \hat{\nabla} \), on \( Y \) which has \( \{y_i\} \) as local affine coordinates. Moreover, potentiality of the metric and the Monge-Ampère equation imply,

\[
g^{ij} = \frac{\partial^2 \hat{K}}{\partial y_i \partial y_j},
\]

for some smooth function \( \hat{K} \) satisfying

\[
\det \left( \frac{\partial^2 \hat{K}}{\partial y_i \partial y_j} \right) = \text{const}^{-1}.
\]
Thus we have the following

5.1.1. **Proposition** [H, KS, L]. For any Monge-Ampère manifold \((Y, g, \nabla)\) there is canonically associated dual Monge-Ampère manifold \((\hat{Y}, \hat{g}, \hat{\nabla})\) such that \((Y, g) = (\hat{Y}, \hat{g})\) as Riemannian manifolds, while the local systems \((T_Y, \nabla)\) and \((T_{\hat{Y}}, \hat{\nabla})\) are dual to each other.

5.2. **Dual torus fibrations** [H, KS, L, SYZ]. Let \((Y, g, \nabla)\) be a Monge-Ampère manifold. The flat connection produces a natural splitting, 

\[ T_T Y = \pi^\ast (T_Y) \oplus \pi^\ast (T_Y), \]

where \(TY\) is the total space of the tangent bundle to \(Y\). The almost complex structure defined with respect to this splitting as 

\[ J: (v_1, v_2) \rightarrow (-v_2, v_1) \]

is integrable and hence makes \(TY\) into a complex manifold. Using the same splitting, one introduces a Riemannian metric on \(TY\), 

\[ g_{TY} = \pi^\ast g \oplus \pi^\ast g, \]

which is Kähler with respect to \(J\). Its potential turns out to be \(\pi^\ast (K)\) so that the Monge-Ampère equation for \(K\) turns into the Ricci flatness of \(g_{TY}\). The net result is that the total space of the tangent bundle to a Monge-Ampère manifold is canonically a (non-compact) Calabi-Yau manifold.

If the holonomy of the flat connection \(\nabla\) is contained in \(SL(n, \mathbb{Z})\), then one can choose a \(\nabla\)-parallel lattice \(T^\mathbb{Z}_Y\) and take the quotient, \(X = TY/T^\mathbb{Z}_Y\). The result is a Calabi-Yau manifold \(X\) which is a torus fibration over \(Y\) with typical fiber \(T^n \simeq T_{Y,y}/T^\mathbb{Z}_{Y,y}, n = \dim Y\). Applying the same sequence of constructions to the dual Monge-Ampère manifold \((\hat{Y}, \hat{g}, \hat{\nabla})\), one gets another Calabi-Yau manifold \(\hat{X}\) which is also a torus fibration over \(Y\). Now the typical fiber is \(\hat{T}^n \simeq T_{Y,y}/\hat{T}^\mathbb{Z}_{Y,y}\), where \(\hat{T}^\mathbb{Z}_{Y,y}\) is the lattice in \(T_{Y,y}\) which is dual to \(T^\mathbb{Z}_{Y,y}\) with respect to the metric \(g\). If \(Y\) is compact, it was argued by many authors that the associated pair of Calabi-Yau manifolds, \(X\) and \(\hat{X}\), is a mirror pair [KS, L, SYZ].

5.3. **Mirror symmetry between semi-infinite variations of Hodge structure.**

With any compact Calabi-Yau manifold \(X\) one can associate two models:

- **semi-infinite \(B\)-variations of Hodge structure,**

\[ VHS^B(X) : (H^\ast (X, \wedge^\ast T_X), 0) \rightarrow Gr_{\mathbb{Z}}^2 (H^\ast (X, \mathbb{C})[[h, h^{-1}]])(\mathcal{O}_0) \]

over the extended moduli space of complex structures, and a family, \(\{\Phi^\ast_{\text{complex}}(X)\}\), of Frobenius manifold structures parameterized by isotropic increasing filtrations, \(W\), in \(H^\ast (X, \mathbb{C})\) which are complementary to the Hodge one (this is the original Barannikov construction, see [B2]);

- **semi-infinite \(A\)-variations of Hodge structure,**

\[ VHS^A(X) : (H^\ast (X, \mathbb{C}), 0) \rightarrow Gr_{\mathbb{Z}}^2 (H^\ast (X, \wedge^\ast T_X[[h, h^{-1}]])(\mathcal{O}_0) \]

over the extended moduli space of Kähler forms, and a family, \(\{\Phi^\ast_{\text{sympl}}(X)\}\), of Frobenius manifold structures parameterized by isotropic increasing filtrations, \(W'\), in \(H^\ast (X, \wedge^\ast T_X)\) which are complementary to the Hodge one (the symplectic version of the Barannikov construction, see Sect. 4).
5.3.1. **Theorem.** Let \( X \) and \( \hat{X} \) be dual torus fibrations over a compact Monge-Ampère manifold \( Y \). Then

\[
VHS^A(X) = VHS^B(\hat{X}), \quad VHS^B(X) = VHS^A(\hat{X}).
\]

In particular,

\[
\Phi^W_{\text{complex}}(X) = \Phi^W_{\text{sympl}}(\hat{X}), \quad \Phi^W_{\text{complex}}(\hat{X}) = \Phi^W_{\text{sympl}}(X)
\]

for appropriately related filtrations \((W, \hat{W})\) and \((W', \hat{W}')\).

**Proof.** It is enough to study \( T^n \)-invariant subsheaves of the sheaves of differential forms and polivector fields on both \( X \) and \( \hat{X} \). Which are related to each other via the following composition of isomorphisms of \( \mathcal{O}_Y \)-modules,

\[
\phi : (\wedge^*T_X \otimes \Omega^0_X)_T \overset{i_1}{\longrightarrow} \wedge^*T_Y \otimes \Omega^*_Y \otimes \mathbb{C} \overset{i_2}{\longrightarrow} \Omega^*_Y \otimes \Omega^*_Y \otimes \mathbb{C} \overset{i_3}{\longrightarrow} (\Omega^*_Y)^n_T,
\]

and a similar one with \( X \) and \( \hat{X} \) exchanged. Here we implicitly used the fact that \( Y \) can be canonically embedded into both \( X \) and \( \hat{X} \) as the zero section. The middle isomorphism,

\[
\wedge^*T_Y \otimes \Omega^*_Y \otimes \mathbb{C} \overset{i_2}{\longrightarrow} \Omega^*_Y \otimes \Omega^*_Y \otimes \mathbb{C}
\]

comes from the “lowering of indices” map induced by the metric \( g \) on the first tensor factor and the identity maps on the other two factors. In corresponding local affine coordinate systems this map is given on generators by,

\[
\phi : \frac{\partial}{\partial z^i} \overset{i_1}{\longrightarrow} \frac{\partial}{\partial x^i} \overset{i_2}{\longrightarrow} \sum_j g_{ij} dx^j = d\hat{x}_i \overset{i_3}{\longrightarrow} d\hat{z}_i,
\]

and

\[
\phi : dz^i \overset{i_1}{\longrightarrow} dx^i \overset{i_2}{\longrightarrow} dx^i = \sum_j g^{ij} d\hat{x}_j \overset{i_3}{\longrightarrow} \sum_j g^{ij} d\hat{z}_j.
\]

Then we have, for example,

\[
\phi(\bar{\partial}) = \phi(\sum_i dz^i \frac{\partial}{\partial z^i})
\]

\[
= \sum_{ij} g^{ij} d\hat{z}_j \frac{\partial}{\partial x^i}
\]

\[
= \sum_j d\hat{z}_j \frac{\partial}{\partial \hat{x}_j}
\]

\[
= \sum_j d\bar{z}_j \frac{\partial}{\partial \bar{z}_j}
\]

\[
= \bar{\partial}.
\]
and

\[
\phi(Q) = \phi \left( \sum_{ij} \psi_i \left( g^{ij} \frac{\partial}{\partial z^j} - \sum_{k} d\bar{z}^k \frac{\partial g^{ij}}{\partial \bar{z}^k} \frac{\partial}{\partial d\bar{z}^j} \right) \right)
\]

\[
= \sum_{ij} d\bar{z}_i \left( g^{ij} \frac{\partial}{\partial x^j} + \sum_{k} dx^k \frac{\partial g^{ij}}{\partial x_k} \frac{\partial}{\partial dx^j} \right)
\]

\[
= \sum_{ij} d\bar{z}_i \left( g^{ij} \sum_k \left( \frac{\partial x_k}{\partial x^j} \frac{\partial}{\partial x_k} + \frac{\partial x_k}{\partial x^j} \frac{\partial}{\partial dx_k} \right) \right) + \sum_k dx^k \frac{\partial g^{ij}}{\partial x_k} \frac{\partial}{\partial dx^j}
\]

\[
= \sum_{i} d\bar{z}_i \frac{\partial}{\partial x^i}
\]

\[
= \sum_{i} d\bar{z}_i \frac{\partial}{\partial z^i}
\]

\[
= \partial.
\]

Analogously one checks that the map \( \phi \) sends the Schouten brackets into the Lie brackets defined in Sect. 2.6. As a result, the map \( \phi \) canonically identifies the pair consisting of the dg Lie algebra,

\[
\left( \bigoplus_{i=0}^{2n} g^i, \ g^i = \bigoplus_{p+q=i} \Gamma(M, \wedge^p T_X \otimes \Omega_{X}^{0,q} T^*_n, [ \cdot \ ]_{\text{Sch}}, \bar{\partial}) \right),
\]

and its module,

\[
\left( \bigoplus_{i=0}^{2n} m^i, \ m^i = \bigoplus_{p+q=i} \Gamma(X, \Omega_{X}^{p,q} T^*_n, \cdot, \circ, d = \partial + \bar{\partial}) \right),
\]

with the pair consisting of the dg Lie algebra

\[
\left( \bigoplus_{i=0}^{2n} g^i, \ g^i = \bigoplus_{p+q=i} \Gamma(\hat{X}, \Omega_{\hat{X}}^{p,q} \hat{T}^*_n, [ \cdot \ ]_{\omega}, \bar{\partial}) \right),
\]

and its module

\[
\left( \bigoplus_{i=0}^{2n} m^i, \ m^i = \bigoplus_{p+q=i} \Gamma(\hat{X}, \Lambda^p T_{\hat{X}} \otimes \Omega_{\hat{X}}^{0,q} \hat{T}^*_n, \cdot, \circ, \bar{\partial} + Q) \right).
\]

The same statement, but with \( X \) and \( \hat{X} \) interchanged, is also true. All the above claims follow immediately. \( \square \)
6. Semi-infinite VHS in dGBV algebras

6.1. Differential Gerstenhaber-Batalin-Vilkovosky algebras. Such an algebra is a quadriple \((A, \circ, d, \Delta)\), where \((A, \circ)\) is a unital supercommutative algebra over a field \(k\), and \((d, \Delta)\) is a pair of supercommuting odd derivations of \((A, \circ)\) of order 1 and 2 respectively which satisfy \(d^2 = \Delta^2 = 0\).

Equivalently, a dGBV algebra is a differential supercommutative algebra with unit, \((A, \circ, d)\), plus an odd linear map \(\Delta : A \to A\) satisfying

(i) \(\Delta^2 = 0\),

(ii) \(d\Delta + \Delta d = 0\),

(iii) and, for any \(a, b, c \in A\),

\[
\Delta(a \circ b \circ c) = \Delta(a \circ b) \circ c + (-1)^{\hat{a}(\hat{b}+1)}b \circ \Delta(a \circ c) + (-1)^{\hat{a}}a \circ \Delta(b \circ c)
\]

\[ -\Delta(a) \circ b \circ c - (-1)^{\hat{a}}a \circ \Delta(b) \circ c - (-1)^{\hat{a}+\hat{b}}a \circ b \circ \Delta(c). \]

Note that \(\Delta(1) = 0\).

It is not hard to check (see, e.g. [Ma]) that the linear map

\[
[a \bullet] : A \otimes A \to A \\
a \otimes b \mapsto [a \bullet b] := (-1)^{\hat{a}}a \circ \Delta(b) - \Delta(a) \circ b - a \circ \Delta(b)
\]

makes \(A\) into a Lie superalgebra. Moreover, both the triples \((A, [ \bullet ], d)\) and \((A, [ \bullet ], \Delta)\) are differential Lie superalgebras, and the following odd Poisson identity,

\[
[a \bullet (b \circ c)] = [a \bullet b] \circ c + (-1)^{\hat{a}(\hat{b}+1)}b \circ [a \bullet b],
\]

holds for any \(a, b \in A\).

6.2. A dGBV algebra as a \(g\)-module. For any \(a \in A\), we denote

\[
l_a : A \to A \\
b \mapsto l_a(b) := a \circ b.
\]

The parity of this linear map is equal to the parity of \(a\).

6.2.1. Proposition. Let \((A, \circ, d, \Delta)\) be a dGBV algebra. Then the linear map,

\[
\bullet : A \otimes A \to A \\
a \otimes b \mapsto a \bullet b := -[l_a, \Delta]b,
\]

makes the triple \(m = (A, \bullet, d+\Delta)\) into a differential module over the differential Lie algebra \(g = (A, [ \bullet ], d)\).
Proof. Let us first show that \((A, \cdot)\) is a module over the Lie algebra \((A, [\cdot, \cdot])\), that is,
\[
a_1 \cdot a_2 \cdot b - (-1)^{(\tilde{a}_1 + 1)(\tilde{a}_2 + 1)} a_2 \cdot a_1 \cdot b = [a_1 \cdot a_2] \cdot b,
\]
holds for all \(a_1, a_2, b \in A\).

Using the definition and the nilpotency of \(\Delta\), we have
\[
\text{l.h.s.} = a_1 \circ \Delta(a_2 \circ \Delta b) - 2(-1)^{\tilde{a}_1} \Delta(a_1 \circ a_2 \circ \Delta b) + (-1)^{\tilde{a}_1 + \tilde{a}_2} \Delta(a_1 \circ \Delta(a_2 \circ b))
\]
\[\quad + (-1)^{\tilde{a}_1 \tilde{a}_2 + \tilde{a}_1 + \tilde{a}_2} a_2 \circ \Delta(a_1 \circ \Delta b) + (-1)^{\tilde{a}_1 + \tilde{a}_2} \Delta(a_2 \circ \Delta(a_1 \circ b)).
\]
On the other hand, using definitions only, we have
\[
\text{r.h.s.} = -[l_{[a_1, a_2]}, \Delta] b
\]
\[\quad = -(-1)^{\tilde{a}_1} \Delta(a_1 \circ a_2) \circ \Delta b + (-1)^{\tilde{a}_1} \Delta(a_1) \circ a_2 \circ \Delta b + a_1 \circ \Delta(a_2) \circ \Delta b
\]
\[\quad - (-1)^{\tilde{a}_2} \Delta((\Delta(a_1) \circ a_2) \circ b - \Delta(a_1) \circ a_2 \circ b - (-1)^{\tilde{a}_1} a_1 \circ \Delta(a_2) \circ b).
\]
As \(\Delta \Delta(a_1 \circ a_2 \circ b)\) vanishes identically,
\[
\Delta((\Delta(a_1 \circ a_2) \circ b - \Delta(a_1) \circ a_2 \circ b - (-1)^{\tilde{a}_1} a_1 \circ \Delta(a_2) \circ b) =
\]
\[= -\Delta((-1)^{\tilde{a}_2(\tilde{a}_1 + 1)} a_2 \circ \Delta(a_1 \circ b) + (-1)^{\tilde{a}_1} a_1 \circ \Delta(a_2 \circ b) - (-1)^{\tilde{a}_1 + \tilde{a}_2} a_1 \circ a_2 \circ \Delta b),
\]
so that using the decomposition formula for \(\Delta(a_1 \circ a_2 \circ b)\) one gets eventually the desired equality
\[
\text{r.h.s.} = \text{l.h.s.}
\]
Next,
\[
(d + \Delta)(a_1 \circ b) = (d + \Delta)(-a_1 \circ \Delta b + (-1)^{\tilde{a}_1} \Delta(a_1 \circ b))
\]
\[= -da_1 \circ \Delta b + (-1)^{\tilde{a}_1} a_1 \circ \Delta(d + \Delta) b - (-1)^{\tilde{a}_1} \Delta(da_1 \circ b)
\]
\[\quad - \Delta(a_1 \circ db) - \Delta(a_1 \circ \Delta b)
\]
\[= da_1 \circ b - (-1)^{\tilde{a}_1} a_1 \circ (d + \Delta) b,
\]
confirming the consistency with the differentials. \(\Box\)

6.3. Local systems from dGBV-algebras. In the notions of Proposition 6.2.1, we reinterpret the algebra structure in \(A\) as an even map,
\[
\circ : \quad g \otimes m \longrightarrow m
\]
\[\quad a \otimes b \longrightarrow a \circ b.
\]

6.3.1. Proposition. For any \(a_1, a_2, b \in A\), we have
\[
(d + \Delta)(a_1 \circ b) = (da_1) \circ b + (-1)^{\tilde{a}_1} a_1 \circ (d + \Delta) b + (-1)^{\tilde{a}_1} a_1 \circ b,
\]
and
\[
a_1 \circ a_2 \circ b - (-1)^{\tilde{a}_1(\tilde{a}_2 + 1)} a_2 \circ a_1 \circ b = -(-1)^{\tilde{a}_2}[a_1 \circ a_2] \circ b.
\]
Proof. The first equality is obvious. We check the second one,

\[
l.h.s. = -a_1 \circ a_2 \circ \Delta b + (-1)^{\tilde{a}_2} a_1 \circ \Delta (a_2 \circ b) + (-1)^{\tilde{a}_1(\tilde{a}_2+1)} a_2 \circ (a_1 \circ b)
\]

\[
-(-1)^{\tilde{a}_1+\tilde{a}_2} \Delta (a_1 \circ a_2 \circ b)
\]

\[
= -(-1)^{\tilde{a}_1+\tilde{a}_2} \Delta (a_1 \circ a_2) \circ b + (-1)^{\tilde{a}_1+\tilde{a}_2} \Delta (a_1 \circ a_2) \circ b + (-1)^{\tilde{a}_1} a_1 \circ \Delta (a_2) \circ b
\]

\[
= r.h.s.
\]

By Proposition 3.2.1, the dGBV algebra \((A, \circ, d, \Delta)\) gives rise to a flat vector bundle, \((E_m, \nabla)\), over the derived moduli space \(\mathcal{M}\) representing the deformation functor \(\text{Def}_{\mathfrak{g}}\). The typical fibre of \(\pi : E_m \rightarrow \mathcal{M}\) is isomorphic to the cohomology group \(H(A, d + \Delta)\).

6.4. Semi-infinite variations of Hodge structure in \(A\). There is an obvious formal parameter extension of all the results in Sections 6.2 and 6.3. More precisely, for any dGBV algebra \((A, \circ, d, \Delta)\), the pair, \((\mathfrak{g}, m_h)\), consisting of the differential Lie superalgebra

\[
\mathfrak{g} = (A, [\cdot, \cdot], d),
\]

and its differential module,

\[
m_h := (A[[h, h^{-1}]],[\cdot, \cdot], \circ := h^{-1} \circ, d + h\Delta),
\]

do satisfy the conditions of Proposition 3.2.1 and hence give rise to a local system, \((E_h, \nabla)\) over the derived moduli space \(\mathcal{M}\).

Proposition 6.4.1. Assume that the dGBV algebra \(A\) satisfies,

\[
\text{Im} d \cap \text{Ker} \Delta = \text{Im} \Delta \cap \text{Ker} d = \text{Im} d \cap \text{Im} \Delta,
\]

then

(i) the deformation functor \(\text{Def}_{\mathfrak{g}}\) is non-obstructed so that \((\mathcal{M}, *) \simeq (H(A, d), 0)\) as formal pointed supermanifolds;

(ii) the typical fibre of the associated bundle, \(\pi : E_h \rightarrow \mathcal{M}\), is isomorphic to \(H(A, d)[[h, h^{-1}]]\).

Proof. It follows from [Ma] that \(\delta = 0\) and that \(H(A, d) = H(A, \Delta) = \text{Ker} d \cap \Delta/\text{Im} d \Delta\).

Assuming further that the dGBV algebra is \(\mathbb{Z} \times \mathbb{Z}\)-graded (or \(\mathbb{Z}\)-graded), \(A = \oplus_{i,j} A^{i,j}\), with \(\circ\) of bidegree \((0, 0)\), \(d\) of bidegree \((0, 1)\), \(\Delta\) of bidegree \((-1, 0)\), and \(\dim H(A, d) < \infty\), we can use Barannikov’s technique [B2] to define semi-infinite variations of the Hodge structure in \(H(A, d)\), and then construct a family of pencils of flat connections on \(\mathcal{M}\),
\nabla^W$, depending on the choice of increasing filtrations, $W$, which are complementary to the Hodge one.

Moreover, if $(A, \circ, d, \nabla)$ admits a linear map (called an *integral* in [Ma]),

$$\int : A \rightarrow k$$

satisfying

$$\int (da) \circ b = -(-1)^{\tilde{a}+1} \int a \circ db, \quad \forall a, b \in A,$$

$$\int (\Delta a) \circ b = (-1)^{\tilde{a}} \int a \circ \Delta b, \quad \forall a, b \in A$$

and inducing a non-degenerate metric, $g([a], [b]) = \int a \circ b$, on $H(A, d)$, then one can make one step further and construct a family of solutions, $\{\Phi^W\}$, of WDVV equations on $M$ which are parameterized by the $g$-isotropic subclass of the above class of filtrations. This links Barannikov’s construction with the one studied in [Ma].

### 6.5. Examples.

(i) Let $(M, \omega)$ be a Lefschetz symplectic manifold, then the data

$$(A = \Gamma(M, \Omega^*_M), \circ = \text{wedge product}, d, \Delta = [d, \Lambda_\omega])$$

where $\Lambda_\omega : \Omega^*_M \rightarrow \Omega^{*-2}$ is the map of contraction with $\omega^{-1}$ is a dGBV algebra satisfying the conditions of Proposition 6.4.1 and admitting an integral.

(ii) If $(M, \omega)$ is a Kähler manifold, then the following variant of the above data,

$$(A = \Gamma(M, \Omega^*_M \circ \omega), \circ = \text{wedge product}, \bar{\partial}, \Delta = [\bar{\partial}, \Lambda_\omega])$$

is a dGBV algebra satisfying all the conditions of subsection 6.4 necessary for constructing solutions, $\{\Phi^W\}$, of WDVV equations from semi-infinite variations of the Hodge structure. If $M$ is Calabi-Yau, this construction is equivalent to the one discussed in Section 4, but in general it is different.

(iii) If $M$ is a Calabi-Yau manifold with a global holomorphic volume form $\Omega$, then

$$(A = \Gamma(M, \Lambda^*T_M \otimes \Omega^{0,*}_M), \circ = \text{wedge products}, \bar{\partial}, \Delta = i_{\Omega}^{-1} \bar{\partial}i_{\Omega})$$

with $i_{\Omega}$ being the isomorphism, $\Lambda^pT_M \otimes \Omega^{0,q}_M \rightarrow \Omega^{n-p,q}_M$, induced by the volume form, is a dGBV algebra satisfying all the conditions of subsection 6.4. It is easy to check that the resulting construction of Frobenius manifolds is equivalent to the original Barannikov’s one [B2].

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