Characterization by detectability inequality for periodic stabilization of linear time-periodic evolution systems

Yashan Xu*

Abstract

Given a linear time-periodic control system in a Hilbert space with a bounded control operator, we present a characterization of periodic stabilization in terms of a detectability inequality. Similar characterization was built up in [8] for time-invariant systems.

Keywords: Periodic evolution systems, periodic stabilization, detectability inequality.

1 Introduction

Control system. Let $Y$ be a real Hilbert space (state space) with the norm and the inner product $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. Let $U$ be another real Hilbert space (control space) with the norm and the inner product $\| \cdot \|_U$ and $\langle \cdot, \cdot \rangle_U$ respectively. We identify $Y$ (resp., $U$) with its dual $Y'$ (resp., $U'$). Let $T > 0$ be arbitrarily given. In this paper, we will study the periodic stabilization for the linear control system:

$$y'(t) = A(t)y(t) + B(t)u(t), \quad t \in \mathbb{R}^+ \triangleq [0, \infty), \quad (1.1)$$

under the following hypotheses:

$(H_1)$ The family of operators $\{A(t)\}_{t \geq 0}$ satisfies that $A(t) = A + D(t)$ for a.e. $t \in (0, \infty)$, where the operator $A$, with its domain $D(A) \subset Y$, generates a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $Y$; and the operator-valued function $D(\cdot) \in L^1_{loc}(0, \infty; L(Y))$ is $T$-periodic in time, i.e., $D(t + T) = D(t)$ for a.e. $t \in \mathbb{R}^+$.

$(H_2)$ The operator-valued function $B(\cdot) \in L^\infty(0, \infty; L(U,Y))$ is $T$-periodic. (We denote its norm by $\| B \|_{L^\infty}$.)

$(H_3)$ Each control $u$ is taken from the space $L^2(0, \infty; U)$.

*School of Mathematical Sciences, Fudan University, KLMNS, Shanghai, 200433, China (yashanxu@fudan.edu.cn). This work is supported in part by NNSF Grant 11871166.
Given \( u \in L^2(t, \infty; U) \), \( z \in Y \) and \( t \geq 0 \), we write \( y(\cdot; t, z, u) \) for the solution of the equation \((1.1)\) over \([t, \infty)\) with the initial condition: \( y(t) = z \). (When \( z \in Y \) and \( u \in L^2(0, \hat{T}; U) \) for some \( \hat{T} > 0 \), we still use \( y(\cdot; 0, z, u) \) to denote the solution \( y(\cdot; 0, z, \hat{u}) \), where \( \hat{u} = u \) over \((0, \hat{T})\) and \( \hat{u} = 0 \) over \((\hat{T}, \infty)\).) Let

\[
E \triangleq \{(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 \leq s \leq t < \infty\}.
\]

Let \( \Phi(\cdot, \cdot) : E \to \mathcal{L}(Y) \) be the evolution system generated by \( A(\cdot) \). (When \( t \geq s \geq 0 \), we denote by \( \| \Phi(t, s) \|_{\mathcal{L}(Y)} \) the operator norm of \( \Phi(t, s) \)). Then we have that (see Lemma 5.6 on Page 68 in [4]) that \( \Phi(\cdot, \cdot) \) is strongly continuous over \( E \), and that

\[
\Phi(t, s)z = S(t-s)z + \int_s^t S(t-r)B(r)\Phi(r, s)zdr, \quad \text{when } 0 \leq s \leq t < \infty \quad \text{and} \quad z \in Y. \quad (1.2)
\]

Moreover, it follows by \((H_1)-(H_2)\) and \((1.2)\) that

\[
\Phi(t + T, s + T) = \Phi(t, s) \quad \text{for any } 0 \leq s \leq t < \infty. \quad (1.3)
\]

**Concepts on the stabilization.** Several concepts related to the periodic stabilization of the system \((1.1)\) are given in order.

- The system \((1.1)\) is said to be periodically exponentially stabilizable (periodically stabilizable, for short), if there is \( K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(Y, U)) \), with \( K(T + t) = K(t) \) for a.e. \( t \in \mathbb{R}^+ \), so that the system \( y'(t) = [A(t) + B(t)K(t)]y(t) \) \((t \geq 0)\) is stable, i.e., for some \( M > 0 \) and \( \omega > 0 \),

\[
\| \Phi_K(t, s)z \| \leq Me^{-\omega(t-s)}\|z\| \quad \text{for any } z \in Y \quad \text{and} \quad t \geq s \geq 0. \quad (1.4)
\]

Here and thereafter, \( \Phi_K(\cdot, \cdot) : E \to \mathcal{L}(Y) \) denotes the evolution system generated by \( A_K(\cdot) \triangleq A(\cdot) + B(\cdot)K(\cdot) \).

- Given \( n \in \mathbb{N}^+ \triangleq \{1, 2, \ldots\} \), the following system is called an adjoint equation of \((1.1)\) over \([0, nT]\):

\[
\begin{cases}
\varphi_n(t) = -A^*(t)\varphi_n(t) \quad \text{in } [0, nT], \\
\varphi_n(nT) = \psi.
\end{cases}
\quad (1.5)
\]

where \( \psi \in Y \). We write \( \varphi_n(\cdot; \psi) \) for the solution to the system \((1.5)\).

**Main result.** The main result of this paper is to present a characterization by an inequality for the periodic stabilization of the system \((1.1)\).

**Theorem 1.1.** Suppose that \((H_1)-(H_3)\) are true. Then the following statements are equivalent:

\((E_1)\) The system \((1.1)\) is periodically stabilizable.

\((E_2)\) For any \( \delta \in (0, 1) \), there is \( n_\delta \in \mathbb{N}^+ \) and \( C_\delta > 0 \) so that for any \( k \in \mathbb{N}^+ \),

\[
\| \varphi_{kn_\delta}(0; \psi) \| \leq \delta^k\| \psi \| + C_\delta\| B(\cdot)^*\varphi_{kn_\delta}(\cdot; \psi) \|_{L^2(0, kn_\delta T; U)}, \quad \text{when } \psi \in Y. \quad (1.6)
\]
There is $\delta \in (0,1)$, $n \in \mathbb{N}^+$ and $C > 0$ so that
\[ \|\varphi_n(0; \psi)\| \leq \delta\|\psi\| + C\|B(\cdot)\varphi_n(\cdot; \psi)\|_{L^2(0,nT;U)}, \quad \text{when } \psi \in Y. \] (1.7)

Several notes on Theorem 1.1 are given in order.

- The similar equivalence results in Theorem 1.1 were obtained in [8] for time-invariant systems. Different kinds of characterizations of the periodic stabilization for time-periodic systems have been studied in [5], [6], [9] and [10]. The characterization (for the system (1.1)), given in Theorem 1.1 seems to be new.

- We prefer to call (1.6) (or (1.7)) a detectability inequality rather than a weak observability inequality (which was used in [8]). The reason will be given in Subsection 3.1.

- It is well known that the null controllability of (1.1) is equivalent to the standard observability inequality; the null controllability of (1.1) implies the periodic stabilization of (1.1). (For the later, we refer readers to [10].) Comparing the standard observability inequality and the detectability inequality (1.7) (or (1.6)), we see that the gap between the null controllability and the periodic stabilization can be quantified by adding the term $\delta\|\psi\|$ on the right hand side of the standard observability inequality.

The rest of the paper is organized as follows: Section 2 proves Theorem 1.1. Section 3 gives some further discussions on the periodic stabilization.

## 2 Proof of Theorem 1.1

Several lemmas will be used in the proof of Theorem 1.1. The first one, i.e., Lemma 2.1, is a direct consequence of Theorem 1.4 in [10]. To state it, we define, for each $z \in Y$, the LQ problem:

\[ (LQ)_z \quad W(z) \triangleq \inf_{u \in L^2(0,\infty;U)} J(u; z), \] (2.1)

where
\[ J(u; z) \triangleq \int_0^\infty \left[ \|y(s; 0, z, u)\|^2 + \|u(s)\|_U^2 \right] ds, \quad u \in L^2(0,\infty;U). \] (2.2)

**Lemma 2.1.** The following assertions are equivalent:

(i) The system (1.1) is periodically stabilizable.

(ii) The functional $W(\cdot)$, given by (2.1), is finite valued, i.e., $W(z) < \infty$ for each $z \in Y$.

The second one, i.e., Lemma 2.2, takes some ideas from Proposition 6 in [8].

**Lemma 2.2.** Let $\delta \in (0,1)$, $n \in \mathbb{N}^+$, and $C > 0$. Then the following statements are equivalent:

(i) The system (1.1) has the property: for any $z \in Y$, there is $u_z \in L^2(0,nT;U)$ so that
\[ \|y(nT; 0, z, u_z)\| \leq \delta\|z\| \quad \text{and} \quad \|u_z\|_{L^2(0,nT;U)} \leq C\|z\|. \] (2.3)
(ii) The adjoint equation (1.5) has the property:

\[ \| \varphi_n(0; \psi) \| \leq \delta \| \psi \| + C \| B(\cdot)^* \varphi_n(\cdot; \psi) \|_{L^2(0,nT;U)} \quad \text{for any } \psi \in Y. \] (2.4)

**Proof.** (i) \(\implies\) (ii). Arbitrarily fix \(\psi \in Y\) and \(z \in Y\). Let \(u_z \in L^2(0,nT;U)\) be given by (i). Then by (1.1) and (1.5), we have that for any \(\tilde{z}\) by (1.1) and (1.5), we have that for any \(z \in Y\),

\[ \langle y(nT;0,z,u_z), \psi \rangle - \langle z, \varphi_n(0; \psi) \rangle = \int_0^{nT} \langle u_z(t), B(t)^* \varphi_n(t; \psi) \rangle_U dt, \]

where \(u_z \in L^2(0,nT;U)\) is given by (i). From the above, it follows that

\[ \| \varphi_n(0; \psi) \| = \sup_{z \neq 0} \frac{\langle z, \varphi_n(0; \psi) \rangle}{\| z \|} \]

\[ = \sup_{z \neq 0} \frac{\langle y(nT;0,z,u_z), \psi \rangle - \int_0^{nT} \langle u_z(t), B(t)^* \varphi_n(t; \psi) \rangle_U dt}{\| z \|} \]

\[ \leq \sup_{z \neq 0} \frac{1}{\| z \|} \left[ \| y(nT;0,z,u_z) \| \| \psi \| + \| u_z \|_{L^2(0,nT;U)} \| B(\cdot)^* \varphi_n(\cdot; \psi) \|_{L^2(0,nT;U)} \right] \]

\[ \leq \sup_{z \neq 0} \frac{\| y(nT;0,z,u_z) \| \| \psi \| + \sup_{z \neq 0} \| u_z \|_{L^2(0,nT;U)} \| B(\cdot)^* \varphi_n(\cdot; \psi) \|_{L^2(0,nT;U)}}{\| z \|}. \]

This, along with (2.3), yields (2.4). Hence, (ii) is true.

(ii) \(\implies\) (i). Arbitrarily fix \(z \in Y\). Define a space:

\[ \tilde{Y} \triangleq \left\{ \left( \psi, B(\cdot)^* \varphi_n(\cdot; \psi) \right) \mid \psi \in Y \right\} \subset Y \times L^2(0,nT;U), \]

with the norm:

\[ \left\| \left( \psi, B(\cdot)^* \varphi_n(\cdot; \psi) \right) \right\|_{\tilde{Y}} = \delta \| \psi \| + C \| B(\cdot)^* \varphi_n(\cdot; \psi) \|_{L^2(0,nT;U)}. \] (2.5)

We next define a functional \(\mathcal{F}\) on this space by

\[ \mathcal{F}\left( \psi, B(\cdot)^* \varphi_n(\cdot; \psi) \right) = \langle z, \varphi_n(0; \psi) \rangle \quad \text{for any } \psi \in Y. \] (2.6)

From (2.4)-(2.6), we see that \(\| \mathcal{F} \|_{Y^*} \leq \| z \|\). Then by the Hahn-Banach theorem, there is a functional \(\tilde{\mathcal{F}}\) defined on \(Y \times L^2(0,nT;U)\) so that

\[ \tilde{\mathcal{F}}\left( \psi, B(\cdot)^* \varphi_n(\cdot; \psi) \right) = \mathcal{F}\left( \psi, B(\cdot)^* \varphi_n(\cdot; \psi) \right) \quad \text{for any } \psi \in Y \] (2.7)

and

\[ \left| \tilde{\mathcal{F}}(\xi, \eta) \right| \leq \| z \|\left( \| \xi \| + C \| \eta \|_{L^2(0,nT;U)} \right) \quad \text{for any } (\xi, \eta) \in Y \times L^2(0,nT;U). \] (2.8)

From the Riesz representation theorem, we can find \((y_z, u_z) \in Y \times L^2(0,nT;U)\) so that

\[ \tilde{\mathcal{F}}(\xi, \eta) = \langle y_z, \xi \rangle + \int_0^{nT} \langle u_z(t), \eta(t) \rangle_U dt \quad \text{for any } (\xi, \eta) \in Y \times L^2(0,nT;U). \] (2.9)
By taking \((\xi, \eta) = (\psi, B(\cdot)^* \varphi_n(\cdot; \psi))\) (with \(\psi \in Y\) arbitrarily fixed) in (2.9), using (2.7) and (2.6), we find
\[
\langle z, \varphi_n(0; \psi) \rangle = \langle y_z, \psi \rangle + \int_0^{nT} \langle u_z(t), B(t)^* \varphi_n(t; \psi) \rangle_U \, dt \quad \text{for any } \psi \in Y.
\]

From this, as well as (1.1) (where \(u = u_z\)) and (1.5), one can easily verify
\[
y_z = y(nT; 0, z, -u_z). \tag{2.10}
\]
Meanwhile, it follows from (2.8) and (2.9) that
\[
\|y_z\| \leq \delta \|z\| \quad \text{and} \quad \|u_z\|_{L^2(0,nT; U)} \leq C \|z\|.
\]
These, together with (2.10), give (2.3). Hence, (i) is true.

Thus, we end the proof of Lemma 2.2.

The next Lemma 2.3 is on the connection between the periodic stabilization and the property (i) in Lemma 2.2.

**Lemma 2.3.** The following assertions are equivalent:

(i) The system (1.1) is periodically stabilizable.

(ii) Given \(\delta \in (0, 1)\), there is \(n_\delta \in \mathbb{N}^+\) and \(C_\delta > 0\) so that for each \(z \in Y\), there is \(u_z \in L^2(0,n_\delta T; U)\) satisfying
\[
\|y(n_\delta T; 0, z, u_z)\| \leq \delta \|z\| \quad \text{and} \quad \|u_z\|_{L^2(0,n_\delta T; U)} \leq C_\delta \|z\|. \tag{2.11}
\]

(iii) There is \(\delta \in (0, 1)\), \(n \in \mathbb{N}\), and \(C > 0\) so that for each \(z \in Y\), there is \(u_z \in L^2(0,nT; U)\) satisfying
\[
\|y(nT; 0, z, u_z)\| \leq \delta \|z\| \quad \text{and} \quad \|u_z\|_{L^2(0,nT; U)} \leq C \|z\|. \tag{2.12}
\]

**Proof.** (i) \(\implies\) (ii). Since (1.1) is periodically stabilizable, there is \(T\)-periodic \(K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(Y, U))\) satisfying (1.4). Note that \(\Phi_K(\cdot, 0)z\), with \(z \in Y\), is the solution to the equation:
\[
\begin{aligned}
y'(t) &= [A(t) + B(t)K(t)]y(t), \quad t \geq 0, \\
y(0) &= z.
\end{aligned}
\]
Define, for each \(z \in Y\), a control \(u_z : \mathbb{R}^+ \to U\) by
\[
u_z(t) = K(t)\Phi_K(t, 0)z \quad \text{for a.e. } t \in \mathbb{R}^+. \tag{2.13}
\]
Then we have that for each \(z \in Y\),
\[
\Phi_K(t, 0)z = y(t; 0, z; u_z) \quad \text{for each } t \geq 0. \tag{2.14}
\]
Arbitrarily fix $\delta \in (0,1)$. Let

$$n_\delta = \left\lceil \frac{1}{\omega} \ln \frac{M}{\delta} \right\rceil + 1,$$

$$C_\delta = \frac{M}{\sqrt{2\omega}} \|K\|_{L^\infty(\mathbb{R}^+;L(Y,U))}$$

where $M$ and $\omega$ are given by (1.4). Then by (2.14), (2.13) and (1.4), after some direct computations, we get (2.11).

(ii) $\implies$ (iii). It is clear.

(iii) $\implies$ (i). Suppose that (iii) holds for some $\delta \in (0,1)$, $n \in \mathbb{N}^+$ and $C > 0$. We claim that $W(\cdot)$, given by (2.1), is finite valued. When this is done, we can apply Lemma 2.1 to get (i) of this lemma.

Now we prove the above claim. Arbitrarily fix $z \in Y$. Set $z_0 \triangleq z$. By (iii), we find $u_1 \triangleq u_{z_0} \in L^2(0,nT;U)$ satisfying (2.12) with $z = z_0$. Following this way, we can inductively get $\{u_k\}_{k=1}^\infty \subset L^2(0,nT;U)$ and $\{z_k\}_{k=0}^\infty \subset Y$ so that

$$\|z_k\| \leq \delta^k\|z_0\| \text{ and } \|u_k\|_{L^2(0,nT;U)} \leq C\|z_k-1\| \leq C\delta^{k-1}\|z_0\| \text{ for all } k \in \mathbb{N}^+;$$

and so that

$$z_k = y(nT;0,z_k-1,u_k) \text{ and } u_{k+1} = u_{z_k} \text{ for all } k \in \mathbb{N}^+. \quad (2.16)$$

Define a control $\tilde{u}_z : \mathbb{R}^+ \to U$ by

$$\tilde{u}_z(knT+t) = u_{k+1}(t), \ t \in [0,nT), \ k \in \mathbb{N}. \quad (2.17)$$

Then by (2.17) and (2.15), we see

$$\|\tilde{u}_z\|_{L^2(0,\infty;U)}^2 = \sum_{k=0}^{+\infty} \|u_{k+1}\|_{L^2(0,nT;U)}^2 < \infty. \quad (2.18)$$

By (2.17), (1.3), (2.16), (2.15) and (2.18), we find

$$\|y(\cdot;0,z,\tilde{u}_z)\|_{L^2(0,\infty;Y)}^2 = \sum_{k=0}^{+\infty} \int_0^{nT} \left\| \Phi(t,0)z_k + \int_0^t \Phi(t,\tau)B(\tau)u_{k+1}(\tau)d\tau \right\|^2 dt\leq \sum_{k=0}^{+\infty} \int_0^{nT} \left[ 2 \max_{0 \leq \tau \leq nT} \|\Phi(t,0)\|_{L^2(Y)}^2 \|z_k\|^2 + \int_0^t \|\Phi(t,\tau)B(\tau)\|_{L^2(U,Y)}^2 d\tau \int_0^t \|u_{k+1}(\tau)\|_{L^2(U)}^2 d\tau \right] dt \leq 2nT \max_{0 \leq \tau \leq nT} \|\Phi(t,\tau)\|_{L^2(Y)}^2 (1 + nT\|B\|_{L^\infty}) \sum_{k=0}^{+\infty} \left[ \|z_k\|^2 + \|u_{k+1}\|_{L^2(0,nT;U)}^2 \right] \leq 2nT \max_{0 \leq \tau \leq nT} \|\Phi(t,\tau)\|_{L^2(Y)}^2 (1 + nT\|B\|_{L^\infty}) (1 + C^2) \sum_{k=0}^{+\infty} \|z_k\|^2 < \infty. \quad (2.19)$$

Now, from (2.2), (2.13) and (2.19), we see that $J(\tilde{u}_z,z) < +\infty$ for all $z \in Y$. Thus, we have $W(z) < \infty$ for all $z \in Y$, which leads to the desired claim.

Thus, we end the proof.
Now we begin to prove Theorem 1.1.

*Proof.* "(E1) $\implies$ (E2)". First of all, by the $T$-periodicity of $\Phi(\cdot, \cdot)$ and $B(\cdot)$ (see (1.3) and (H2)), we have that

$$\Phi(t, s)^* = \Phi(r, s)^* \Phi(t, r)^*$$

and that for any $n \in \mathbb{N}^+$ and any $\ell, k \in \mathbb{N}^+$, with $k \geq \ell$,

$$B(t)^* \Phi(\ell nT, t)^* = B((k - \ell)nT + t)^* \Phi(knT, (k - \ell)nT + t)^*$$

for any $t \in [0, nT]$.

Suppose that (E1) holds. Let $\delta \in (0, 1)$. CLAIM ONE: There is $n_\delta \in \mathbb{N}$ and $C_\delta > 0$ so that (1.6) (where $\delta = \hat{\delta}$) holds.

To show the above claim, we set

$$\delta \triangleq \hat{\delta}/\sqrt{2} \in (0, 1)$$

By Lemma 2.3 and (E1), We have (ii) of Lemma 2.3 in particular, for $\delta$ given by (2.22), we have

$$n_\delta \in \mathbb{N} \quad \text{and} \quad C_\delta > 0$$

so that (2.11) holds. This, along with Lemma 2.2 yields that

$$\| \varphi_{n_\delta}(0; \psi) \| \leq \delta \| \psi \| + C_\delta \| B(\cdot)^* \varphi_{n_\delta}(\cdot; \psi) \|_{L^2(0, n_\delta T; U)}$$

for any $\psi \in Y$, (2.24)

where $\delta$ is given by (2.22), $n_\delta$ and $C_\delta$ are given by (2.23). By (2.24) and the Cauchy-Schwarz inequality, we find

$$\| \varphi_{n_\delta}(0; \psi) \|^2 \leq 2\delta^2 \| \psi \|^2 + 2C_\delta^2 \| B(\cdot)^* \varphi_{n_\delta}(\cdot; \psi) \|^2_{L^2(0, n_\delta T; U)}$$

for any $\psi \in Y$. (2.25)

Arbitrarily fix $z \in Y$ and $k \in \mathbb{N}$. Taking $\psi = [\Phi(n_\delta T, 0)^*]^{\ell-1} z$, with $\ell = 1, \cdots, k$, in (2.25), then multiplying the both sides by $(2\delta^2)^{k-\ell}$, using (2.20), we obtain that when $\ell = 1, \cdots, k$,

$$\begin{align*}
(2\delta^2)^{k-\ell} \| \varphi_{n_\delta}(0; z) \|^2 &= (2\delta^2)^{k-\ell} \left\| \varphi_{n_\delta}(0; [\Phi(n_\delta T, 0)^*]^{\ell-1} z) \right\|^2 \\
&\leq (2\delta^2)^{k-\ell+1} \left\| [\Phi(n_\delta T, 0)^*]^{\ell-1} z \right\|^2 + (2\delta^2)^{k-\ell} 2C_\delta^2 \| B(\cdot)^* \varphi_{n_\delta}(\cdot; [\Phi(n_\delta T, 0)^*]^{\ell-1} z) \|^2_{L^2(0, n_\delta T; U)} \\
&\leq (2\delta^2)^{k-\ell+1} \| \varphi_{(\ell-1)n_\delta}(0; z) \|^2 + 2C_\delta^2 \| B(\cdot)^* \Phi(\ell n_\delta T, t)^* z \|^2_{L^2(0, n_\delta T; U)}.
\end{align*}$$

(In the last inequality of (2.26), we used the fact: $2\delta^2 < 1$.) Meanwhile, by (2.20) and (2.21), we see that when $\ell = 1, \cdots, k$,

$$\int_{0}^{n_\delta T} \| B(t)^* \Phi(\ell n_\delta T, t)^* z \|^2_U dt = \int_{(k-\ell-1)n_\delta T}^{(k-\ell)n_\delta T} \| B(\hat{t})^* \Phi(kn_\delta T, \hat{t})^* z \|^2_U d\hat{t}$$

$$= \| B(\cdot)^* \varphi_{kn_\delta}(\cdot; z) \|^2_{L^2((k-\ell)n_\delta T, (k-\ell+1)n_\delta T; U)}$$

(2.27)
Thus, by (2.27) and (2.26), we find that when \( \ell = 1, \cdots, k \),
\[
(2\delta^2)^{k-\ell}\|\varphi_{\ell n_d}(0;z)\|^2 \\
\leq (2\delta^2)^{k-\ell+1}\|\varphi_{(\ell-1)n_d}(0;z)\|^2 + 2C_\delta^2\|B(\cdot)^*\varphi_{kn_d}(\cdot;z)\|^2_{L^2((k-\ell)n_dT,(k-\ell+1)n_dT;U)}.
\]
(2.28)
Taking the sum from \( \ell = 1 \) to \( \ell = k \) in (2.28), we get
\[
\|\varphi_{kn_d}(0;\psi)\|^2 \\
\leq (2\delta^2)^k\|z\|^2 + 2C_\delta^2\|B(\cdot)^*\varphi_{kn_d}(\cdot;z)\|^2_{L^2(0, kn_dT;U)}
\]
\[
\leq \left[ (\sqrt{2}\delta)^k\|z\| + \sqrt{2C_\delta}\|B(\cdot)^*\varphi_{kn_d}(\cdot;z)\|^2_{L^2(0, kn_dT;U)} \right]^2.
\]
(2.29)
Now, by (2.29) and by letting \( n_\delta = n_d \) and \( C_\delta = \sqrt{2C_\delta} \) (where \( n_\delta \) and \( C_\delta \) are given by (2.23)), we get CLAIM ONE. Hence, (E2) is true.
\[
(E_2) \implies (E_3). \text{ It is trivial.}
\]
\[
(E_3) \implies (E_1). \text{ According to Lemma 2.2, (E3) is equivalent to (iii) of Lemma 2.3. Then by Lemma 2.3, we get (E1).}
\]
In summary, we complete the proof of Theorem 1.1.

3 Further discussions

3.1 On the detectability inequality

In this subsection, we will explain why we call (1.6) (or (1.7)) as a detectability inequality.

The concept of the detectability arises from the finite-dimensional system. Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). Consider the system:
\[
z'(t) = Az(t); \quad w(t) = Bz(t), \quad t \geq 0.
\]
(3.1)
We quote the concept of detectability from [1]: The system (3.1) is detectable, if
\[
w(\cdot) = 0 \text{ over } [0, \infty) \implies \lim_{t \to +\infty} z(t) = 0.
\]
(3.2)
For the pair of matrices \([A, B]\), the following statements are equivalent: (See [1] or [7].)
\[
(a_1) \text{ The system } y'(t) = Ay(t) + Bu(t), \quad t \geq 0 \text{ is stabilizable.}
\]
\[
(a_2) \text{ The system (3.2) is detectable.}
\]
\[
(a_3) \text{ There is } L \in \mathbb{R}^{n \times m} \text{ so that } A^* + LB^* \text{ is stable.}
\]
With the aid of the above equivalence, the concept of the detectability for infinite-dimensional time-invariant systems was extended in [11]. Motivated by the extension in [11], we define the detectability for \([A(\cdot), B(\cdot)]\) in the infinitely-dimensional periodic setting \((H_1)-(H_3)\) in the following manner:
\[
\text{The system}
\]
\[
z'(t) = A(t)^*z(t); \quad w(t) = B(t)^*z(t), \quad t \geq 0
\]
(3.3)

is detectable, if there is a $T$-periodic operator-valued function $L(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(Y,U))$ so that the system
\[ z'(t) = [A(t)^* + L(t)B(t)^*]z(t), \quad t \geq 0, \]
is exponentially stable.

From the above definition and Theorem 1.1 we can easily verify what follows:

(b1) The system (1.1) is periodically stabilizable if and only if the system (3.3) is detectable. (Here, we used the fact: $\varphi_n(nT - \cdot, \psi)$ solves the first equation of (3.3), over $[0, nT]$, with the initial condition $z(0) = \psi$.) Thus, the inequality (1.6) (or (1.7)) is equivalent to the detectability of (3.3). This is why we call (1.6) (or (1.7)) as a detectability inequality.

For $[A(\cdot), B(\cdot)]$ in the infinitely-dimensional periodic setting $(H_1)-(H_3)$, it deserves mentioning the following connection between ”detectability of (3.3)” and (3.2) (where $(z, w)$ solves (3.3)):

(c1) We have that ”detectability of (3.3)” $\implies$ (3.2). In fact, by (a3) and (b1), we see that the system (1.1) is periodically stabilizable. Then given $\delta \in (0,1)$, we can use Theorem 1 to find $n_\delta \in \mathbb{N}^+$ and $C_\delta > 0$ so that (1.6) holds for any $k \in \mathbb{N}$. Meanwhile, arbitrarily fix $k \in \mathbb{N}^+$ and $\psi \in Y$. Let
\[ z(t; k, \psi) = \varphi_{kn_\delta}(kn_\delta T - t; \psi), \quad t \in [0, kn_\delta T]. \]
Then $z(\cdot; k, \psi)$ solves the first equation of (3.3), over $[0, kn_\delta T]$, with the initial condition $z(0) = \psi$. This, along with (1.6), yields
\[ \|z(kn_\delta T)\| \leq \delta^k\|z(0)\| + C_\delta\|w(\cdot)\|_{L^2(0,kn_\delta T;U)}, \]
which leads to
\[ \lim_{k \to +\infty} z(kn_\delta T) = 0, \quad \text{when } w(\cdot) = 0. \]
So (3.2) is true.

(c2) It is not true that (3.2) $\implies$ ”detectability of (3.3)”. Here is an counterexample. Let $B(\cdot) = 0$ and $D(\cdot) = 0$. Take an operator $A$ so that the system $\dot{y} = Ay$ is polynomially stable but not exponentially stable. Then we clearly have (3.2), but have no the stabilization. Then by (b1), the system (3.3) is not detectable in this case.

### 3.2 Connection of detectability inequality and unique continuation

In this subsection, besides $(H_1)-(H_3)$, we further assume

(H4) The operator $A$ is compact.

We will present an equivalence between the detectability inequality (1.6) (or (1.7)) and a qualitative unique continuation property for the adjoint system (1.5), under the setting $(H_1)-(H_4)$. This qualitative unique continuation was introduced in [9] (see also [10]).
We start with introducing some notation: We write $Y^C$ for the complexification of $Y$. For each $L \in \mathcal{L}(Y)$, we denote by $L^C$ the complexification of $L$. We write $\mathbb{B}$ for the open unit ball in $\mathbb{C}^1$ and $\mathbb{B}(0, \delta)$ for the open ball in $\mathbb{C}^1$, centered at the origin and of radius $\delta > 0$.

We next introduce some concepts.

(I) **The Poincaré map.** We recall the following Poincaré map (see Page 197, [2]):

$$\mathcal{P}(t) \triangleq \Phi(t + T, t), \ t \in \mathbb{R}^+.$$ 

We have

$$\sigma(\mathcal{P}(t)^C) \setminus \{0\} = \{\lambda_j\}_{j=1}^\infty \text{ for each } t \geq 0,$$

where $\lambda_j$, $j = 1, 2, \ldots$, are all distinct non-zero eigenvalues of the compact operator $\mathcal{P}(0)^C$ so that $\lim_{j \to \infty} |\lambda_j| = 0$. Thus, there is a unique $n \in \mathbb{N}^+$ so that

$$|\lambda_j| \geq 1, \ j \in \{1, 2, \ldots, n\} \quad \text{and} \quad |\lambda_j| < 1, \ j \in \{n + 1, n + 2, \ldots\}.$$ 

Set

$$\bar{\delta} \triangleq \max\{|\lambda_j| \mid j > n\} < 1. \quad (3.4)$$

Let $l_j$ be the algebraic multiplicity of $\lambda_j$ for each $j \in \mathbb{N}^+$, and write

$$n_0 \triangleq l_1 + \cdots + l_n. \quad (3.5)$$

(II) **The Kato projection.** Arbitrarily fix $\delta \in (\bar{\delta}, 1)$, where $\bar{\delta}$ is given by (3.4). Let $\Gamma$ be the circle $\partial \mathbb{B}(0, \delta)$ with the anticlockwise direction in $\mathbb{C}^1$. The Kato projection is given by (see [2])

$$K(t) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda \text{id} - \mathcal{P}(t)^C)^{-1} d\lambda, \ t \geq 0.$$ 

Let

$$P(t) \triangleq (I_Y - K(t))|_Y \quad \text{(the restriction of } (I_Y - K(t)) \text{ on } Y), \ t \geq 0,$$

where $I_Y$ is the identity operator on $Y$. Let

$$P \triangleq P(0) \text{ and } Y_u \triangleq P(0)Y. \quad (3.6)$$

**Theorem 3.1.** Suppose that $(H_1)$-$\ (H_4)$ hold. Let $P$ and $Y_u$ be given by (3.6). Let $n_0$ be given by (3.5). Then the following statements are equivalent:

(i) There is $\delta \in (0, 1)$, $n \in \mathbb{N}^+$ and $C > 0$ so that the detectability inequality (1.6) holds.

(ii) If $\xi \in P^*Y_u$ and $B^*(\cdot)\Phi(n_0 T, \cdot)^* \xi = 0$ over $(0, n_0 T)$, then $\xi = 0$.

**Proof.** According to Theorem 1.1, the periodic stabilization of (1.1) is equivalent to (i) in Theorem 3.1. Meanwhile, according to Theorem 2.1 in [10], the periodic stabilization of (1.1) is equivalent to (ii) in Theorem 3.1. Hence, the statements (i) and (ii) in Theorem 3.1 are equivalent. This ends the proof.
3.3 Example

We will give a time periodic controlled heat equation which can be put into our framework \((H_1)-(H_3)\), and which is not null controllable and not stable with the null control, but satisfies the detectability inequality, consequently, is periodically stabilizable (by Theorem 1.1).

**Example 3.2.** Consider the following heat equation

\[
\begin{aligned}
y_t(x, t) - (\triangle + 3 \sin^2 t) y(x, t) &= u(t) \sin x, \quad (x, t) \in (0, \pi) \times (0, \infty), \\
y(0, t) &= y(\pi, t) = 0, \quad t \in (0, \infty), \\
y(\cdot, 0) &\in L^2(0, \pi).
\end{aligned}
\]  

(3.7)

The equation (3.7) can be put into the setting \((H_1)-(H_3)\) in the following manner: Let \(Y = L^2(0, \pi)\); \(U = \mathbb{R}\) and \(T = \pi\). Let \(A = \triangle\), with \(D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)\); \(D(t) = (3 \sin^2 t) I_Y\) for each \(t \geq 0\); \(B(t) \equiv B\) for all \(t \geq 0\), where \(B : U \to Y\) is defined by \(B(\alpha) = \alpha \sin x\), \(\alpha \in U\).

First, we show that (3.7), with \(u = 0\), is not stable. To this end, we let \(y\) be the solution to (3.7), where \(u = 0\) and \(y(x, 0) = \sin x\), \(x \in (0, \pi)\). Write

\[
a(t) \triangleq \int_0^\pi y(x, t) \sin x dx, \quad t \geq 0.
\]  

(3.8)

Then by (3.7) and (3.8), we find that

\[
\dot{a}(t) = (-1 + 3 \sin^2 t) a(t), \quad t \geq 0; \quad a(0) > 0.
\]  

(3.9)

Since

\[
\int_0^t (-1 + 3 \sin^2 s) ds = \frac{t}{2} - \frac{3}{4} \sin(2t), \quad t \geq 0,
\]

it follows from (3.9) that

\[
a(t) = e^{\frac{t}{2} - \frac{3}{4} \sin(2t)} a(0), \quad t \geq 0; \quad a(0) > 0.
\]

These yields that \(\lim_{t \to \infty} a(t) = \infty\). From this, we see that (3.7), where \(u = 0\), is unstable.

Next, we show that (3.7) is not null controllable. For this purpose, we arbitrarily fix a control \(u\). Let \(y(x, t; u)\) be the solution of (3.7) with the aforementioned \(u\) and with \(y(x, 0) = \sin(2x)\), \((x \in (0, \pi))\). Write

\[
z(t; u) \triangleq \int_0^\pi y(x, t; u) \sin(2x) dx, \quad t \geq 0.
\]

Then by (3.7), we have

\[
\begin{aligned}
\dot{z}(t; u) &= (3 \sin^2 t - 4) z(t; u), \quad t \geq 0, \\
z(0; u) &= \pi/2.
\end{aligned}
\]  

(3.10)

From (3.10), we see that \(z(t; u) \neq 0\) for any \(t \geq 0\). Thus, for any \(u\) and any \(t \geq 0\), we have \(y(\cdot, t; u) \neq 0\). So (3.7) is not null controllable.
Now we show the detectability inequality (1.7) holds for this example. To this end, we take
\begin{equation}
\delta = e^{-\pi}; \quad n = 1; \quad C = 2e^{2\pi}.
\end{equation}

Since \( T = \pi \) in this example, we see from (3.11) that \( nT = \pi \). So the adjoint equation (1.5) in the current case reads as:

\begin{equation}
\varphi_t(x,t) + (\Delta + 3\sin^2 t) \varphi(x,t) = 0 \quad (x,t) \in (0,\pi) \times [0,\pi]; \quad \varphi(x,\pi) = \psi,
\end{equation}

where \( \psi \in L^2(0,\pi) \). Write \( \varphi(x,t;\psi) \) \((x \in (0,\pi), t \in [0,\pi])\) for the solution to (3.12). Then it is exactly the solution \( \varphi_n(\cdot;\psi) \) to (1.5) with \( n = 1 \). Let

\begin{equation}
\varphi(x,t;\psi) = \sum_{k=1}^{\infty} a_k(t) \sin kx, \quad x \in (0,\pi), t \in [0,\pi].
\end{equation}

By (3.13) and (3.12), we have that for each \( k \in \mathbb{N}^+ \),

\begin{equation}
\dot{a}_k(t) = (k^2 - 3\sin^2 t)a_k(t), \quad t \in [0,\pi].
\end{equation}

From the above, one has

\begin{equation}
|a_k(0)| \leq e^{-(k^2-3)\pi}|a_k(\pi)|, \quad k \in \mathbb{N}; \quad |a_1(t)| \geq e^{t-\pi}|a_1(\pi)|, \quad t \in [0,\pi].
\end{equation}

By (3.14) and (3.13), we can easily check that

\begin{align*}
\|\varphi_n(0;\psi)\|^2 &= \int_0^\pi \varphi^2(x,0;\psi)dx \\
&= \pi^2 \sum_{k=1}^{\infty} a_k(0)^2 \\
&\leq \pi^2 \left[ e^{4\pi a_1(\pi)^2} + \sum_{k=2}^{\infty} e^{-(k^2-3)\pi} a_k(\pi)^2 \right] \\
&\leq \pi^2 \left[ e^{4\pi a_1(\pi)^2} + \sum_{k=1}^{\infty} e^{-2\pi} a_k(\pi)^2 \right] \\
&= \pi^2 e^{4\pi a_1(\pi)^2} + e^{-2\pi}\|\psi\|^2
\end{align*}

and

\begin{align*}
\|B(\cdot)^*\varphi_n(\cdot;\psi)\|^2_{L^2(0,\pi)} &= \int_0^\pi a_1(t)^2 \left( \frac{\pi}{2} \right)^2 dt \\
&\geq \frac{\pi^2}{8} (1 - e^{-2\pi}) a_1(\pi)^2 \\
&\geq \frac{\pi}{8} a_1(\pi)^2.
\end{align*}

From (3.15) and (3.16), we get (1.7).

Finally, by Theorem 1.1 and (1.7), we see that the system (3.7) is periodically stabilizable.
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