THE GENUS OF A CURVE OF FERMAT TYPE

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Abstract. In this paper we begin to study curves on a weighted projective plane with one trivial weight, \( \mathbb{P}(1, m, n) \), by determining the genus of curves of Fermat type. These are curves, \( C \), defined by the “homogeneous” polynomial \( x_0^{amn} + x_1^{an} = x_2^{am} \). We begin by finding local coordinates for the standard affine cover of \( \mathbb{P}(1, m, n) \), and then prove that the curve is smooth. This is done by pulling the curve up to the surface’s desingularization, \( D(1, m, n) \). Then a map, \( \phi : C \to \mathbb{P}^1 \) is constructed, and it’s ramification divisor is determined. We conclude by applying Hurwitz’s theorem to \( \phi \) to obtain \( C \)’s genus.

We begin by making some simplifications. First, a result of Dolgachev [D, 1.3.1] tells us that \( \mathbb{P}(1, am, an) \cong \mathbb{P}(1, m, n) \). Thus, we may assume that \( m \) and \( n \) are relatively prime. Next, note that \( [x_0, x_1, x_2] \mapsto [x_0, x_2, x_1] \) gives an isomorphism of the coordinate rings for \( \mathbb{P}(1, m, n) \) and \( \mathbb{P}(1, n, m) \). Subsequently, we may assume that \( m < n \).

In section 1 an affine cover for \( \mathbb{P}(1, m, n) \) is constructed using the classical description of a weighted projective space as \( \text{Proj} \) of a graded ring. The local (affine) forms of the equations defining a Fermat-type curve are then constructed. Section 2 gives the construction of \( \mathbb{P}(1, m, n) \) as a toric variety and describes the isomorphism with the classical construction by relating the generators of the rings for the affine cover. The goal of section 3 is to use the machinery of toric varieties write to down some of the polynomials in the monomial ideal \( \mathcal{I}(U_1) \) (resp. \( \mathcal{I}(U_2) \) ) of functions vanishing on the affine surface \( U_1 \) (resp. \( U_2 \) ). In section 4 the desingularization algorithm in \([K]\) and \([O]\) is used to show that a Fermat-type curve is smooth, while section 5 constructs a map from the curve to \( \mathbb{P}^1 \) and the ramification divisor of this map. Finally, Hurwitz’s theorem is used to determine the genus of a Fermat-type curve. Throughout this paper \( k \) is an algebraically closed field of characteristic 0.

1. The Classical Description

The standard affine cover of a weighted projective plane consists of three affine varieties, \( U_i = \{ [x_0, x_1, x_2] \in \mathbb{P}(a_0, a_1, a_2) | x_i \neq 0 \} \). This is the space \( \text{Spec}(k[U_i]) \) where \( k[U_i] \) is the degree zero part of the graded ring \( k[x_0, x_1, x_2][x_i^{-1}] \) where the grading is given by \( \text{deg}(x_i) = a_i \).

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In the case of \( \mathbb{P}(1,m,n) \) we can use the fact that \( \text{deg}(x_0) = 1 \) to see that \( U_0 \cong \mathbb{A}^2 \). This is because the degree zero part of \( k[x_0, x_1, x_2][x_0^{-1}] \) is generated by \( \frac{x_0}{x_1} \) and \( \frac{x_2}{x_0} \). Since there are no relations on these forms we see \( U_0 = \text{Spec} \left( k \left[ \frac{x_0}{x_1}, \frac{x_2}{x_0} \right] \right) = \text{Spec} (k [X, Y]) = \mathbb{A}^2 \).

The other two affine surfaces, however, are singular. Begin by finding generators for the \( k \)-algebras, \( k[U_1] \) and \( k[U_2] \). On \( U_1 \), the polynomial generator \( x_1 \) becomes a unit. Thus, \( k[x_0, x_1, x_2][x_1^{-1}] \) is generated over its quotient field by \( x_0 \) and \( x_2 \).

**Lemma 1.1.**

\( k[U_1] \) is generated as an affine \( k \)-algebra by

\[
(1.1) \quad z_j = \frac{x_0^{\left\lceil \frac{jm}{m} \right\rceil} - jn}{x_1^{\left\lceil \frac{jm}{m} \right\rceil} x_2^{j}}
\]

where \( 0 \leq j \leq m \) and \( \left\lceil x \right\rceil \) is the next largest integer than \( x \), with the exception that \( \left\lceil 0 \right\rceil = 1 \).

**Proof.** The goal is to generate the forms of degree 0 where \( x_1 \) is invertible. Begin with the obvious form, \( z_0 = \frac{x_0^m}{x_1^m} \), which is the reason for taking \( \left\lceil 0 \right\rceil = 1 \) to avoid a useless generator of 1.

The technique to generate the remaining \( z_j \)'s is to take successively higher powers of \( x_2 \). Then put just enough \( x_1 \) terms in the denominator to make the degree of the whole form negative, which is to say \( \left\lceil \frac{jm}{m} \right\rceil \) of them. Then to give a form of degree zero, take advantage of the fact that \( \text{deg}(x_0) = 1 \) and put just enough \( x_0 \) terms in the numerator to give the form a total degree of zero. The number of \( x_0 \)'s necessary is then

\[
\left\lceil \frac{jm}{m} \right\rceil \cdot \text{deg}(x_1) - j \cdot \text{deg}(x_2) = m \left\lceil \frac{jm}{m} \right\rceil - jn.
\]

With this method one keeps generating possibly distinct forms until the final form of \( z_m = \frac{x_0^m}{x_1^m} \) is reached. \( \square \)

In fact the exact same technique with the roles of \( x_1 \) and \( x_2 \) reversed will yield the corresponding result for the other singular affine surface, \( U_2 \). By continuing with the convention that \( \left\lceil 0 \right\rceil = 1 \) we have the following Lemma.

**Lemma 1.2.**

\( k[U_2] \) is generated as an affine \( k \)-algebra by

\[
(1.2) \quad w_j = \frac{x_0^{\left\lceil \frac{jm}{m} \right\rceil} - jm}{x_1^{\left\lceil \frac{jm}{m} \right\rceil} x_2^{j}}
\]

where \( 0 \leq j \leq n \).
Now that we have the rings for an affine cover of \( \mathbb{P}(1, m, n) \) we can write down the equation for a Fermat-type curve locally. Recall that such a curve is written in terms of it homogeneous coordinate ring as

\[
x_0^{amn} + x_1^{am} - x_2^{am}.
\]

On \( U_0 \) the element \( x_0 \) is a unit so that this equation becomes \( 1 + X^{am} - Y^{am} \).

By a similar process on \( U_1 \) using the local coordinates \( (z_0, \ldots, z_m) \) we see that the curve is defined by

\[
0 = \left( \frac{x_0^n}{x_1^m} \right)^a + 1 - \left( \frac{x_0^n}{x_1^m} \right)^a = z_0^{an} + 1 - z_m^a.
\]

The corresponding result for \( U_2 \) is then \( w_0^{am} + w_n^a - 1 \).

2. **The Toric Approach**

In this paper we use the description and notation for toric varieties found in [F].

A weighted projective space \( \mathbb{P}(a_0, \ldots, a_d) \), is the complete toric variety whose fan is given by \( \Delta(1) = \{v_0, \ldots, v_d\} \) where \( \text{Span}_{\mathbb{Z}}(v_0, \ldots, v_d) = \mathbb{Z}^d = \mathbb{N} \) is the lattice, and \( \sum_{\Delta(1)} a_j v_j = 0 \). In the case of \( \mathbb{P}(1, m, n) \) such a fan can be given by

\[
\Delta(1) = \left\{ \left[ \begin{array}{c} -m \\ n \\ 0 \\ 1 \\ 1 \end{array} \right] \right\}
\]

where each maximal cone \( \sigma_i \) is the cone generated over \( \mathbb{R}_+ \) by \( \Delta(1) \setminus \{v_i\} \).

A quick check of the determinants of the edges of each maximal cone will show that the surfaces \( U_{\sigma_1} \) and \( U_{\sigma_2} \) will be singular, while \( U_{\sigma_3} \) is smooth. To describe \( k[U_0] \) note that \( \sigma_0 = \langle e_1, e_2 \rangle \) where \( e_1, e_2 \) is the standard basis for \( \mathbb{Z}^2 \). Then the dual cone is \( \sigma_0^\vee = \langle e_2^\vee, e_1^\vee \rangle \) where \( e_1^\vee, e_2^\vee \) is dual to the standard basis. Then taking \( X = \chi e_1^\vee \) and \( Y = \chi e_2^\vee \) gives \( k[U_0] \cong k[Y, X] \).

Since the remaining surfaces are singular, it will be more difficult to construct their affine rings. To begin, note that their dual cones are

\[
\sigma_1^\vee = \left\langle \begin{array}{c} -1 \\ n \\ -m \end{array} \right\rangle \quad \text{and} \quad \sigma_2^\vee = \left\langle \begin{array}{c} n \\ -m \\ -1 \end{array} \right\rangle
\]

respectively.

Proceed with finding generators of the semi-group \( S_{\sigma_1} = \sigma_1^\vee \cap \mathbb{N}^* \) by finding lattice points in the parallelogram formed by \( -e_1^\vee \) and \( -ne_1^\vee + me_2^\vee \). Do this by considering rational numbers \( 0 \leq s, t \leq 1 \) where \( s = 0 \) if and only if \( t = 1 \) and vice-versa. Then the generators of \( S_{\sigma_1} \) are lattice points of the form \( s(-e_1^\vee) + t(-ne_1^\vee + me_2^\vee) = -(s + tn)e_1^\vee + (tm)e_2^\vee \). For the second coordinate to be integral we must have \( t = \frac{j}{m} \) for \( 0 \leq j \leq m \). Since \( 0 \leq s \leq 1 \) the first coefficient must then be \( \left\lceil \frac{am}{m} \right\rceil \) where \( \lceil \cdot \rceil \) denotes the next largest integer. It should be noted that in order to match the condition \( t = 0 \Rightarrow s = 1 \) we must again use the convention that \( \lceil 0 \rceil = 1 \).
Adopting this convention we see that generators for \( S_{\sigma_1} \) are the lattice points \( u_j = -\left\lceil \frac{nj}{m} \right\rceil e_1^\vee + je_2^\vee \) for \( 0 \leq j \leq m \). The generators for the \( k \)-algebra, \( k[\sigma_1] \), are then
\[
(2.1) \ ˜z_j = X^{-\left\lceil \frac{nj}{m} \right\rceil} Y^j.
\]
By a similar argument one sees that \( k[\sigma_2] \) is generated by
\[
(2.2) \ ˜w_j = X^j Y^{-\left\lceil \frac{mj}{n} \right\rceil}
\]
for \( 0 \leq j \leq n \).

**Theorem 2.1.** The isomorphism between classical and toric constructions of \( \mathbb{P}(1,m,n) \) is given by
\[
X \sim x_1^{x_0^m} x_2^{x_0^m} \quad \text{and} \quad Y \sim x_1^{x_0^m} x_2^{x_0^m},
\]
where \( X \) and \( Y \) are defined as in equation 2.1.

**Proof.** We prove this by showing that this correspondence gives an isomorphism of the three \( k \)-algebras giving the affine cover of the surface in each construction. In particular, it gives an isomorphism on the generators of these rings such that \( U_i \cong U_{\sigma_i} \).

Checking this on \( U_0 \) is trivial since \( k[Y, X] \cong k[\frac{x_2}{x_0}, \frac{x_1}{x_0}] \). To see the isomorphism on \( U_1 \) note that the correspondence means
\[
\tilde{z}_j = X^{-\left\lceil \frac{nj}{m} \right\rceil} Y^j = \left( \frac{x_1}{x_0^m} \right)^{-\left\lceil \frac{nj}{m} \right\rceil} \left( \frac{x_2}{x_0} \right)^j = \frac{x_0^m \left\lceil \frac{mj}{n} \right\rceil - nj}{x_1 \left\lceil \frac{mj}{n} \right\rceil} = z_j
\]
while a similar calculation gives \( \tilde{w}_j = w_j \) to complete the isomorphism on \( U_2 \).

\[
\square
\]

### 3. The Ideals for the Affine Cover

The homomorphism \( k[z_0, \ldots, z_m] \rightarrow k[\sigma_1] \) gives an embedding \( U_{\sigma_1} \hookrightarrow \mathbb{A}^{m+1} \). The kernel of this homomorphism, \( \mathcal{I}(U_1) \), is the ideal of functions on \( \mathbb{A}^{m+1} \) vanishing on \( U_1 \). In order to determine some of the equations in \( \mathcal{I}(U_1) \) note that \( k[\sigma_1] \) is generated by \( z_j = \chi^{u_j} \) for \( 0 \leq j \leq m \) where the \( u_j \)'s are lattice points in the cone \( \sigma_1^\vee \). By regarding a linearly independent pair \( u_{i_1}, u_{i_2} \) as a basis for the vector space \( \mathbb{N}^* \otimes \mathbb{Q} \), one can write the remaining \( u_j \in S_{\sigma_1} \) as a rational linear combination of them. Multiplication by the common denominator of these fractions yields an equation with integral coefficients \( a_j u_j = b_j u_{i_1} + c_j u_{i_2} \) that the map \( u_j \mapsto \chi^{u_j} \) turns into
\[
z_j^{a_j} = z_{i_1}^{b_j} z_{i_2}^{c_j}.
\]
Any negative exponents can be multiplied out to convert this to a polynomial in \( \mathcal{I}(U_1) \). For convenience we refer to a set of polynomials obtained in this fashion as having type \((i_1, i_2)\).

For example, the type \((0, m)\) equations are nearly already done for us since \( u_0 \) and \( u_m \) are the edges of \( \sigma_1^\vee \) that were used to find the other generators of \( S_{\sigma_1} \). Recall from the derivation of equation 2.1 that \( u_j = su_0 + tu_m \) where
t = \frac{j}{m} and s = \lceil \frac{mn}{m} \rceil - \frac{ni}{m}$. Multiplying both sides by $m$ and applying $\chi$ yields the $(0, m)$ polynomials of $I(U_1)$ as
\begin{equation}
(3.1) \quad z_j^m - z_0^{\frac{mn}{m}} - nj z_j^m
\end{equation}
for $1 \leq j \leq m - 1$. Using the same technique shows the $(0, n)$ equations of $I(U_2)$ to be
\begin{equation}
(3.2) \quad w_j^n = w_0^{\frac{mn}{n}} - mj w_j^n
\end{equation}
for $1 \leq j \leq n - 1$.

It is not difficult to use this same technique to generate the type $(0, 1)$ equations as well. Simply note that $u_0 = -e_1^y$ and $u_1 = -\frac{m}{n} e_1^y + e_2^y$. The result in this case is that $I(U_1)$ will contain
\begin{equation}
(3.3) \quad z_j^1 - z_0^{\frac{mn}{m}} - \frac{ni}{m} z_j
\end{equation}
for $2 \leq j \leq m$. Similarly, since $0 < m < n$, $I(U_2)$ contains
\begin{equation}
(3.4) \quad w_j^1 - w_0^{\frac{mn}{n}} w_j
\end{equation}
for $2 \leq j \leq n$.

### 4. An Aside on the Smoothness of Fermat-Type Curves

Showing that a Fermat-type curve is smooth will allow certain machinery to be applied to the study of these curves. Begin by noting that on $U_0$, the curve $C$ is defined by
\[
1 + \left(\frac{x_1}{x_0}\right)^{an} - \left(\frac{x_2}{x_0}\right)^{am}.
\]
In terms of the toric coordinates, $X = \chi e_1^y$ and $Y = \chi e_2^y$, this is $1 + X^{an} - Y^{am}$. Plugging this into the Jacobi criterion will show that $C$ is smooth where $x_0 \neq 0$.

All that remains now is to check that $C$ is smooth where it intersects the line $x_0 = 0$. Since the curve must satisfy $x_0^{an} + x_1^{an} - x_2^{am} = 0$, any points on this line will satisfy $x_1^{an} = x_2^{am}$. Consequently $x_1 = 0 \iff x_2 = 0$. This means that any points of $C$ with $x_0 = 0$ are contained in $U_1 \cap U_2$. Thus we need only verify the smoothness of $C$ on $U_1$.

To do this one could try combining Equations (3.1) and (3.3) with Equation (1.4) and using the Jacobi criterion. This attempt, however, would fail to do anything more than show that possible singular points lie on the line $x_0 = 0$, which we already know. This is because Equations (3.1) and (3.3) fail to generate the whole ideal $I(U_1)$.

On the other hand, because $\mathbb{P}(1, m, n)$ is a complete toric variety, it is normal so that any singularities have codimension at least 2. Since this is a surface the singularities will be isolated to the fixed points of the toric
action on $U_1$ and $U_2$, which are $[0, 0, 1]$ and $[0, 1, 0]$ in terms of homogeneous coordinates. But $C$ contains neither of these points. Thus, if we consider this surface’s desingularization $\pi : \mathbb{D}(1, m, n) \to \mathbb{P}(1, m, n)$ as in [K], we obtain an isomorphism $\pi^{-1}(C)$.

In fact the concern here is not the entirety of $\mathbb{D}(1, m, n)$, but rather $\pi^{-1}(U_\sigma)$ since we merely have to demonstrate the smoothness of $C$ on $U_1$. It is known that the desingularization of an affine toric surface corresponding to a cone, $\sigma$ is the toric surface obtained by subdividing $\sigma$ through the rays $\{l_0, \ldots, l_{s+1}\}$ given by [O] Lemma 1.20 where $l_0$ and $l_{s+1}$ are the edges of $\sigma$. The maximal cones for $\pi^{-1}(U_\sigma)$ are then $\tau_j = \langle l_{j-1}, l_j \rangle$.

An additional part of the algorithm of great importance is a collection of integers $\{b_1, \ldots, b_s\}$ with each $b_j \geq 2$. Geometrically, these numbers correspond to the self-intersection number of the $T$-equivariant divisors on $\pi^{-1}(U_\sigma)$ by $D(l_j) = -b_j$. It is also important to note from [O] Prop. 1.19 that they satisfy

$$l_{j-1} + l_{j+1} = b_j l_j.$$  (4.1)

We define elements of the dual lattice $\mathbb{M} = \text{Hom}(\mathbb{N}, \mathbb{Z})$ by letting $l^+_j$ be the unique element with $l^+_j(l_j) = 0$ and $l^+_j(l_{j-1}) = 1$ (or equivalently $l^+_j(l_{j+1}) = -1$). This is well defined since $\tau_j$ is non-singular, so $\det [l_{j-1}, l_j] = \pm 1$.

Using this notation each of the $k$-algebras $k[\tau_j]$ is simply $k[x_j, y_j]$ where $x_j = \chi^{-l^+_j}$ and $y_j = \chi^{l^+_j}$. Next up, we need to know how to change coordinates between $U_{\tau_j}$ and $U_{\tau_{j+1}}$.

**Lemma 4.1.** The $k$-algebra isomorphism between $k[\tau_j][y_j^{-1}]$ and $k[\tau_{j+1}][x_{j+1}^{-1}]$ is given by $x_j \mapsto x_{j+1}^{b_j} y_{j+1}$ and $y_j \mapsto x_{j+1}^{-1}$.

**Proof.** Both of these algebras are simply $k[\tau_j \cap \tau_{j+1}]$. In this region we may invert the element corresponding to their common edge, $l_j$. This leads one to observe that $y_j = \chi^{l_j} = (\chi^{-l^+_j})^{-1} = x_{j+1}^{-1}$.

To prove the $x_j$ piece of the isomorphism note that it is equivalent to the statement $-l^+_j = -b_j l^+_j + l^+_j$. This will be proven by showing that $l^+_j - l^+_j - b_j l^+_j$ vanishes on a basis for $\mathbb{N} = \mathbb{Z}^2$ (and hence, on all of $\mathbb{N}$). Since $U_{\tau_j}$ is smooth the vectors $l_{j-1}$ and $l_j$ constitute a suitable basis.

Begin by recalling that $l^+_j(l_j) = 0$, $l^+_j(l_{j+1}) = 1$, and $l^+_j(l_{j-1}) = -1$. Subsequently we have $l^+_j(l_j) + l^+_j(l_{j+1}) - b_j l^+_j(l_{j-1}) = -1 + 1 - b_j$ to show that the form vanishes on $l_j$. Using the same process for $l_{j-1}$ gives $l^+_j(l_{j-1}) - b_j$. In order to determine $l^+_j(l_{j-1})$ solve equation (4.1) for $l_{j-1}$ and use the linearity of $l^+_{j+1}$ to obtain

$$l^+_{j+1}(l_{j-1}) = b_j l^+_{j+1}(l_j) - l^+_{j+1}(l_{j+1}) = b_j$$
which can be plugged back in to find $t_{j-1}^j(l_{j-1}) + t_{j+1}^j(l_{j-1}) - b_jt_j^j(l_{j-1}) = b_j - b_j = 0$, concluding the proof.

This isomorphism will allow us to write the polynomial defining a Fermat-type curve on each of the open affine neighborhoods $U_{r_j} = \text{Spec}(k[x_j, y_j])$. In order to complete this process, a couple of auxiliary sequences will need to be obtained.

The first of these sequences, $\{r_{-1}, \ldots, r_{s-1}\}$, was constructed in \cite[Theorem 6.1]{K}. In the case of $\sigma_1$ for $\mathbb{P}(1, m, n)$, by expressing $n$ as $mk + r$ with $m, r$ relatively prime, the initial values in this sequence are $r_{-1} = m$ and $r_0 = r$. It was also shown that this is a sequence of positive integers satisfying

$$r_j = b_jr_{j-1} - r_{j-2}.$$ 

The other sequence, $\{t_0, \ldots, t_s\}$ is given by $t_0 = 0$, $t_1 = 1$ and $t_{j+1} = b_jt_j - t_{j-1}$. Using the fact that every $b_j \geq 2$, it is a simple matter to prove inductively that this sequence is increasing, which means each $t_j$ is non-negative.

**Lemma 4.2.** On the region $U_{r_j}$ a Fermat-type curve of degree $amn$ is determined by the polynomial

$$F_j = x_j^{an}y_j^{an} - x_j^{ar_j-1}y_j^{ar_j-2} + 1$$

for $1 \leq j \leq s + 1$.

**Proof.** The proof is by induction on $j$. When $j = 1$, \cite[Lemma 4.1]{K} shows that $\tau_1 = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$. This means that in terms of $X = \chi^{e_1}$ and $Y = \chi^{e_2}$ the local coordinates for $U_{\tau_1}$ are $x_1 = X^{-1}$ and $y_1 = X^{-k}Y$. Recall from equation \cite{A} that the curve on $U_{\tau_1}$ is defined by $z_m^a - z_m^a + 1$. Theorem 2.1 allows us to write this in toric coordinates as $X^{-am} - X^{-an}Y + 1$. A little algebra gives the corresponding polynomial on $U_{\tau_1}$ to be $x_1^{an} - x_1^{ar} + 1$ proving the case $j = 1$.

Now suppose the Lemma holds for $F_j$. Then $F_{j+1}$ will be the image of $F_j$ under the map from Lemma 4.1. This is turns out to be

$$\left(\frac{x_j}{y_j+1}x_j^{an}t_j\right)^{an} \left(\frac{y_j}{x_j+1}y_j^{an}t_j-1\right)^{an} - \left(\frac{b_j}{x_j+1}b_j^{an}t_j\right)^{ar_j-1} \left(\frac{b_j}{y_j+1}y_j^{an}t_j-1\right)^{ar_j-2} + 1.$$ 

Collecting terms and using the recursive definition of $r_j$ and $t_{j+1}$ then gives

$$x_j^{an}t_j+1 \frac{x_j^{an}t_j}{y_j+1} - \frac{b_j^{an}t_j}{x_j+1}b_j^{an}t_j+1 + 1$$

concluding the proof.

With the various local formulations of $\pi^{-1}(C \cap U_1)$ in hand it is not difficult to prove the following.

**Theorem 4.3.** A Fermat-type curve is smooth.
Proof. The only part of the proof that remains unfinished is checking the points of \( C \) with \( x_0 = 0 \), which all lie in \( U_1 \). Since \( C \cong \pi^{-1}(C) \) this means we just need to check for smoothness on \( \pi^{-1}(U_1) = \bigcup_{j=1}^{s+1} U_{\tau_j} \).

Begin by noting that for the interior cones \( 2 \leq j \leq s \) any point of \( \pi^{-1}(C) \cap U_{\tau_j} \) satisfies \( x_j \neq 0 \) and \( y_j \neq 0 \). Thus each such point is also contained in \( U_{\tau_{j-1}} \), so we really only need to check the cases \( j = 1 \) and \( j = s + 1 \).

On \( U_{\tau_1} \) note that \( C \) is defined by \( x_1^{an} - x_1 y_1^{am} + 1 \) which contains no points with \( x_1 = 0 \). Also, the differential is

\[
dF_1 = \left( an \cdot x_1^{an-1} - ar \cdot x_1^{ar-1} y_1^{am} \right) dx_1 - \left( am \cdot x_1 y_1^{am-1} \right) dy_1.
\]

Since \( x_1 \neq 0 \) the only way for the \( dy_1 \) coefficient to vanish is to have \( y_1 = 0 \). However, this results in a non-zero \( dx_1 \) coefficient, so that the curve is smooth on \( U_{\tau_1} \).

To handle the case \( j = s + 1 \) one must recall the sequence of rational numbers \( \{ \beta_0, \ldots, \beta_{s-1} \} \) in [K] Eq. 2.1 which are related to the \( r_j \)'s by \( \beta_j = \frac{r_j}{r_{j-1}} \). In particular, [K] Lem 2.1 tells us that \( \frac{r_j}{r_{j-1}} = b_j - \beta_{j-1} \). Since the last \( \beta_j \) occurs when \( \beta_{s-1} = b_s \in \mathbb{Z} \) this means that \( \frac{r_{s-1}}{r_{s-1}} = b_s - \beta_{s-1} = 0 \). Consequently, \( r_s = 0 \) (and \( r_{s-1} > 0 \)). Since the \( t_j \)'s are increasing this leaves the polynomial

\[
F_{s+1} = x_{s+1}^{an-t_{s+1}} y_{s+1}^{an-t_s} - x_{s+1}^{ar_{s-1}} + 1
\]

whose resulting curve contains no points with \( x_{s+1} = 0 \). Applying the Jacobi criterion as was done in the \( j = 1 \) case will complete the proof. \( \square \)

5. Ramification of a Map to the Projective Line

Now that we know a Fermat-type curve is smooth we can proceed with determining its genus. The approach will be to construct a map to \( \mathbb{P}^1 \), determine the degree and ramification divisor of this map, and at last use Hurwitz’s Theorem [H] Cor. IV.2.4] to determine \( C \)'s genus.

The map used will be the rational map \( \phi : \mathbb{P}(1, m, n) \to \mathbb{P}(1, m) \) sending \([x_0, x_1, x_2]\) to \([x_0, x_1]\) in terms of homogeneous coordinates. The only point at which this is undefined, \([0, 0, 1]\) is not on the curve, so it restricts to a morphism on \( C \). The following Lemma reveals that \( \mathbb{P}(1, m) \) is a rather simple space.

Proposition 5.1. If \( s, t \) are relatively prime, then \( \mathbb{P}(s, t) \cong \mathbb{P}^1 \).

Proof. Since \( \mathbb{P}(s, t) = \text{Proj} \left( k[x_0, x_1] \right) \) with the grading \( \deg(x_0) = s, \deg(x_1) = t \), it is covered by the two regions \( V_i = \{ [x_0, x_1] \text{ s.t. } x_i \neq 0 \} \). Specifically,

\[
V_0 = \text{Spec} \left( k \left[ \frac{x_1}{x_0} \right] \right) \quad V_1 = \text{Spec} \left( k \left[ \frac{x_0}{x_1} \right] \right).
\]

The resulting space is two affine lines with a coordinate change \( x \mapsto x^{-1} \), i.e. \( \mathbb{P}^1 \). \( \square \)
With this map in hand, and the local information obtained in section 3, we can determine both the the degree of $\phi$ and its ramification divisor, $R$.

**Lemma 5.2.** Let $C$ be a Fermat-type curve on $\mathbb{P}(1, m, n)$ of degree $amn$. The degree of the map $\phi : C \to \mathbb{P}^1 \cong \mathbb{P}^1$ given by $[x_0, x_1, x_2] \mapsto [x_0, x_1]$ is $am$.

**Proof.** The degree will be determined by finding the number of distinct points in a generic fiber of $\phi$. This may be done on the dense, open subset, $U_0 = \text{Spec}(k[X, Y])$ where $X = \frac{a}{x_0}$ and $Y = \frac{b}{x_0}$. Using this notation, $k[V_0] = k[X]$, and $\phi|_{U_0}$ corresponds to the inclusion of $k$-algebras, $k[X] \hookrightarrow k[X, Y]$. Consequently, the restriction of $\phi$ to $C \cap U_0$ is obtained by composing this with the natural projection to $k[C \cap U_0] \cong k[X, Y]/(1 + X^an - Y^{am})$.

Now note that unless $1 + X^an = 0$ (which only happens for finitely many points on $V_0$), this quantity will have $am$ distinct $am$th roots. Each of these corresponds to a distinct $Y$-value, yielding an equal number of distinct points in the fiber of $X$ and proving the Lemma.

In fact, the points on $V_0$ where $1 + X^an = 0$ are more than just the points where $\phi : C \to \mathbb{P}^1$ is not $am$-to-one. These are some of the branch points whose fibers will consist of ramification points of $\phi$. They are not, as we shall see, all of the branch points. Thus we begin our determination of the ramification divisor by splitting it into two pieces. The first, $\bar{R}$, will consist of those ramification points contained in $U_0$. The other piece, $R_0$ is merely those ramification points on the line $x_0 = 0$.

**Lemma 5.3.** The ramification divisor for the map $\phi : C \cap U_0 \to \mathbb{P}^1$ is

$$\bar{R} = \sum_{j=1}^{an} (am - 1)[1, \alpha_j, 0]$$

where each $\alpha_j$ is a distinct an$^{th}$ root of $-1$.

**Proof.** Note that since $x_0 \neq 0$ on both regions, that the image of $U_0$ is wholly contained in $V_0$. Then the proof of Lemma 5.2 revealed the branch points to be those with $1 + X^an = 0$, which are the $an$ distinct ideals $\langle X - \alpha_j \rangle$. A consequence of this is that the fibers of these branch points must satisfy $Y^{am} = 1 + \alpha_j^{an} = 0$, so the branch points, $X = \alpha_j$, are in one to one correspondence with the ramification points $X = \alpha_j$, $Y = 0$ (or in terms of homogeneous coordinates the branch points are $[1, \alpha_j]$ and the ramification points are $[1, \alpha_j, 0]$).

Next it remains to find the coefficient of each ramification point $P \in \bar{R}$. By [H] Prop. IV.2.2 this is merely $v_P(t) - 1$ where $t$ is the generator for the one-dimensional maximal ideal $\phi(P) \subset k[X]$ and $v_P$ is the standard valuation at the ramification point. Now $P$ is the ideal $\langle X - \alpha_j, Y \rangle$ and $\phi(P)$ is $\langle t \rangle = $
\[ X - \alpha_j \). In particular, the valuation, \( v_P \), is taking place in the ring \( k[C] = k[X,Y]/(1 + X^{an} - Y^{am}) \) localized at the ideal \( P = \langle X - \alpha_j, Y \rangle \). Since \( C \) is a smooth curve, this point must be a principal ideal generated by either \( X - \alpha_j \) or \( Y \).

Since \( P \in C \), in \( O_P \), we have \( Y^{an} = 1 + X^{an} = (X - \alpha_j) \cdot p(X) \) where \( p(\alpha_j) \neq 0 \), so \( p(X) \) is a unit. Subsequently, \( X - \alpha_j \in \langle Y \rangle \) so that \( Y \) generates the principal ideal, \( P \). Furthermore \( v_P(X - \alpha_j) = \alpha m \) so that the coefficient of \( P = [1, \alpha_j, 0] \) is \( (\alpha m - 1) \).

Now we turn our attention to the ramification points of \( C \) that lie on the line \( x_0 = 0 \). We saw in the beginning of section 4 that all such points lie in \( U_1 \cap U_2 \), so that we may work in the affine region \( U_1 \). This will mean using the local coordinates of Lemma 1.1 \( (z_0, \ldots, z_m) \). Furthermore, these points are all in the fiber of \( \phi \) sitting over the origin of \( V_1 \). As it turns out this is a branch point of \( \phi \) and every point of \( C \) with \( x_0 = 0 \) is a ramification point.

**Lemma 5.4.** In terms of local coordinates on \( U_1 \), the ramification occurring on the line \( x_0 = 0 \) is

\[ R_0 = \sum_{j=1}^{\alpha} (m - 1) \cdot (0, \ldots, 0, \gamma_j) \]

where each \( \gamma_j \) is a distinct \( j^{th} \) root of 1.

**Proof.** Begin by finding all of the points on \( C \) with \( x_0 = 0 \). To do this, note that Lemma 1.1 implies that they are of the form \( (0, \ldots, 0, z_m) \). Furthermore equation 1.4 says that any such points on \( a \) by the distinct \( p \)

Since \( U \) and every point of \( U_C = \langle 1 \rangle \cap \{ x_0 = 0 \} \) given by the distinct \( j^{th} \) roots of unity, \( z_m = \gamma_j \).

Also note that \( k[V_1] = k \left\{ \frac{z_0}{z_1} \right\} = k \left\{ \frac{X^{-1}}{1} \right\} = k[z_0] \), and the \( k \)-algebra homomorphism corresponding to \( \phi \) is the map \( k[z_0] \to k[z_0, \ldots, z_m]/I(U_1) \) sending \( z_0 \) to itself. Since all of the points \( P = \langle z_0, \ldots, z_m \rangle \) project to the origin of \( V_1 \), \( \langle z_0 \rangle \), all that remains is to show these are ramification points with \( v_P(z_0) = m \).

To do this note that \( z_m(P) = \gamma_j \neq 0 \) means \( z_m \notin P \) is a unit in the local ring \( O_P \), yielding \( v_P(z_m) = 0 \). Now consider equation 3.1 with \( j = 1 \), i.e. \( z_1^m = z_m^{-r} \). Taking valuations of both sides gives

\[ m v_P(z_1) = (m - r) v_P(z_0). \]

Since \( m \) and \( m - r \) are relatively prime this means that \( m | v_P(z_0) \). Now if we can show that \( v_P(z_0) | m \), we’ll be done.

Since \( C \) is a smooth curve at \( P \), the ideal \( P = \langle z_0, \ldots, z_m - \gamma_j \rangle \) must be principal. On the curve, however, \( z_0^m = z_m^{-r} - 1 \) so that \( z_m - \gamma_j \) fails to generate \( P \). Since \( P \) is principal, this means that for some \( 0 \leq i \leq m - 1 \) we have \( P = \langle z_i \rangle \) (i.e. \( v_P(z_i) = 1 \)). Now consider
the $i$th copy of equation \[3.3\] \( z_i^1 = z_i z_0^{[i]} \) and take valuations of both sides. Using equation \[5.2\] to substitute for \( v_P(z_1) \) one can solve for \( v_P(z_i) \):

\[
v_P(z_i) = \left( \left\lceil \frac{ri}{m} \right\rceil - \frac{ri}{m} \right) v_P(z_0).
\]

Since the term in parentheses is a rational number whose denominator is a factor of \( m \), the only way for \( v_P(z_i) \) to be 1 is for \( v_P(z_0) \) to divide \( m \).

Consequently, \( v_P(z_0) = m \), and \( R_0 = \sum_{j=1}^a (m-1) \cdot P \) concluding the proof. \( \square \)

Now that we have the degree and ramification divisor of \( \phi: C \to \mathbb{P}^1 \) in hand, determining \( C \)'s genus is a simple matter of plugging the results into Hurwitz's theorem.

**Theorem 5.5.** The genus of a Fermat-type curve on \( \mathbb{P}(1,m,n) \) of degree \( amn \) is

\[
g(C) = \frac{(am-1)(an-2) + a(m-1)}{2}.
\]

**Proof.** Hurwitz’s theorem [H, Cor. IV.2.4] states that given a finite map of curves, \( f: X \to Y \) with ramification divisor \( R \), the genus of each curve is related by

\[
2g(X) - 2 = \deg(f) (2g(Y) - 2) + \deg(R).
\]

Apply this result to the map \( \phi: C \to \mathbb{P}^1 \). Lemma \[5.2\] gives \( \deg(\phi) = am \). In order to compute the degree of the ramification divisor, appeal to Lemmas \[5.3\] and \[5.4\] and the fact that \( R = \bar{R} + R_0 \) to see that

\[
\deg(R) = \left( \sum_{j=1}^a am - 1 \right) + \left( \sum_{j=1}^a m - 1 \right) = an(am-1) + a(m-1).
\]

Since \( g(\mathbb{P}^1) = 0 \), equation \[5.3\] leaves

\[
2g(C) - 2 = am(-2) + an(am-1) + a(m-1)
\]

which may be solved for \( g(C) \). \( \square \)

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