CLASSIFICATION OF SINGULARITIES IN THE COMPLETE CONFORMALLY FLAT YAMABE FLOW

PANAGIOTA DASKALOPOULOS* AND NATASA SESUM**

ABSTRACT. We show that an eternal solution to a complete locally conformally flat Yamabe flow, \( \frac{\partial}{\partial t} g = -R g \), with uniformly bounded scalar curvature and positive Ricci curvature at \( t = 0 \), where the scalar curvature assumes its maximum, is a gradient steady soliton. As an application of this result, we study the blow up behavior of \( g(t) \) at the maximal time of existence \( T < \infty \). We assume that \((M, g(\cdot, t))\) satisfies (i) the injectivity radius bound such as in (1.2) and that its sectional curvature is bounded at each time, for \( t \in [0, T) \) or (ii) the Schouten tensor is positive at time \( t = 0 \) and the Ricci curvature is bounded at each time-slice. We show that the singularity that the flow develops at time \( T \) is always of type I.

1. Introduction

We consider the complete Yamabe flow

\[
\frac{\partial}{\partial t} g_{ij} = -R g_{ij},
\]

on a simply connected, locally conformally flat manifold \( M \). It is well known that in many cases this flow develops a singularity at some finite time \( T < \infty \).

Definition 1.1. We say that the singularity at \( T < \infty \) is of type I if

\[
\sup_{t \in [0, T)} m(t)(T - t) \leq C, \quad \text{where } m(t) = \sup_{M} |Rm(g(t))|.
\]

Otherwise we say that the singularity is of type II.

Examples of type I singularities of the Yamabe flow forming at finite time \( T \) are the shrinking spheres, in the compact case, and the Barenblatt self-similar solutions, in the conformally flat complete non-compact case. We can view the spheres as the...
conformally flat metrics on $\mathbb{R}^n$ given by $g = U_\lambda^{\frac{4}{n+2}} \, ds^2$, where

$$U_\lambda(x, t) = (T-t)^{\frac{4}{n+2}} \left( \frac{C_n \lambda}{|x|^2 + \lambda^2} \right)^{\frac{4}{n+2}}$$

for a fixed constant $C_n > 0$, depending on dimension, and any $\lambda > 0$.

The Barenblatt solutions correspond to conformally flat metrics $g = B_k^{\frac{4}{n+2}} \, ds^2$, with

$$B_k(x, t) = (T-t)^{\frac{4}{n+2}} \left( \frac{C^*_n}{|x|^2 + k(T-t)^{n-2}} \right)^{\frac{4}{n+2}}$$

for a fixed constant $C^*_n > 0$, depending on dimension, and any $k > 0$.

It is well known (c.f. [5], [3], [1]) that if the manifold $M$ is compact, then locally conformally flat metrics evolving by (1.1) develop only type I singularities.

In [4] we showed that the vanishing profile of a solution $u(x, t)$ to the fast-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^{\frac{n-2}{2}}$$

when $u(x, 0)$ starts trapped in between two Barenblatt solutions with the same vanishing time, is given by a Barenblatt solution. This means that the metric $g = u^{\frac{4}{n+2}} \, ds^2$, when evolves by (1.1), develops a type I singularity in finite time.

The goal of this paper is to show that, under certain conditions, locally conformally flat solutions to the complete Yamabe flow develop always a type I singularity.

**Theorem 1.2.** Let $(M, g(t))$ be a complete locally conformally flat Yamabe flow that satisfies an injectivity radius bound (defined below) with its sectional curvature bounded at each time slice. Assume that the Ricci curvature is positive at time $t = 0$. Then, either the flow exists forever or it develops a type I singularity at time $T < \infty$.

The proof of Theorem 1.2 goes by contradiction. If the statement of the theorem were not true, that is, if the singularity were of type II, then by taking the limit of an appropriate sequence of blow up solutions we would produce an eternal solution of the Yamabe flow with positive Ricci curvature and bounded curvature, whose scalar curvature attains its maximum at an interior space-time point. Using the Harnack expression for the Yamabe flow, Theorem 2.1 below, we will show, by similar methods to the ones introduced by Hamilton in [8], that such a limit has to be a gradient steady soliton. Finally we will argue that this is not possible, under the assumptions of Theorem 1.2.
Let $m(t) = \sup_M |\text{Rm}(g(t))|$ and let $\rho(t)$ denote the infimum of the injectivity radii at all points at time $t$. We will say that the manifold satisfies an injectivity radius bound if there exists a constant $c > 0$ so that

$$\rho(t) \geq \frac{c}{\sqrt{m(t)}}.$$  

By the Topogonov injectivity radius theorem ([2], page 17), if $(M, g)$ is an orientable manifold with positive bounded sectional curvature $0 < K < \infty$, then

$$\text{injrad}(M, g) \geq \frac{\pi}{\sqrt{\sup_M K}}.$$  

If

$$S_{ij} = \frac{1}{N-2}(R_{ij} - \frac{R}{2(N-1)}g_{ij})$$

is the Schouten tensor, since in a locally conformally flat case the Weyl tensor is zero, in geodesic coordinates at a fixed point, the sectional curvature is given in terms of the Schouten tensor by

$$R_{jiji} = S_{ii}g_{jj} + S_{jj}g_{ii}.$$  

This shows that the sectional curvature is positive, if the Schouten tensor $S_{ij}$ is a positive definite tensor. In [3] Chow proved that if $R > 0$ and $R_{ij} \geq c R g_{ij}$, for some $c \geq 0$, then the inequality $R_{ij} \geq c R g_{ij}$ is preserved under the Yamabe flow. His proof also works in the complete setting since all curvature quantities are uniformly bounded in space, at each time-slice, and the evolution equation for $R_{ij} - c R g_{ij}$ has a form to which we can apply the maximum principle for complete manifolds. This together with the evolution of $R$ ([2.1] tells us that if we start with $R > 0$, it remains so) imply that the positivity of the Ricci curvature is preserved along the Yamabe flow. Moreover, the positivity of the Schouten tensor is preserved under the flow. Using this observation the following result follows from Theorem [1.2]

**Corollary 1.3.** Let $(M, g(t))$ be a complete locally conformally flat Yamabe flow, with $S_{ij} > 0$ at time $t = 0$ and the Ricci curvature bounded at each time-slice. Then, either the flow exists forever or it develops a type I singularity at time $T < \infty$.

The key point in proving Theorem [1.2] is the following classification result for eternal solutions to (1.1). This is the analogue of the classification of eternal solutions to the Ricci flow equation which was proved by Hamilton in [8] and turned out to be very useful in understanding the singularity forming along the flow.
Theorem 1.4. Let \( g(x,t) \) be a complete eternal solution to the locally conformally flat Yamabe flow on a simply connected manifold \( M \), with uniformly bounded sectional curvature and strictly positive Ricci curvature. If the scalar curvature \( R \) assumes its maximum at an interior space-time point \( P_0 \), then \( g(x,t) \) is necessarily a gradient steady soliton.

The organization of the paper is as follows: In section 2 we give some preliminaries. In section 3 we prove Theorem 1.4. The proof of Theorem 1.2 is given in the final section 4.

Acknowledgment: We are grateful to our colleague R. Hamilton for many useful discussions. We would also like to thank the referee for many useful comments.

2. Preliminaries

It is well known that if \( g(t) \) evolves by (1.1), then the scalar curvature evolves by

\[
\frac{\partial}{\partial t} R = (n - 1) \Delta R + R^2.
\]

Since \((M, g)\) is locally conformally flat, the following identities hold (see [6])

\[
\nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(\nabla_j R g_{ik} - \nabla_k R g_{ij}) = 0
\]

and

\[
R_{kijl} = \frac{1}{n-2}(R_{kl} g_{ij} + R_{ij} g_{kl} - R_{kij} g_{kl} - R_{ij} g_{kl}) - \frac{R(g_{kl} g_{ij} - g_{kj} g_{il})}{(n-1)(n-2)}.
\]

Using the identities (2.2) and (2.3), Chow ([3]) obtained that the evolution equation for \( R_{ij} \) can be written as a heat type equation, namely

\[
\frac{\partial}{\partial t} R_{ij} = (n - 1)(\Delta R_{ij} + R_{kij} R_{kl} - R_{ij}^2).
\]

The proof of Theorem 1.2 relies on a Harnack estimate, for the curvature \( R_{ij} \), proven by Chow in [3]. He found that the right Harnack expression for the Yamabe flow is given by

\[
Z(g, X) = (n - 1) \Delta R + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij} X^i X^j + R^2 + \frac{R}{t}
\]

for an arbitrary vector field \( X \), and he showed the following result.
Theorem 2.1 (B. Chow). Assume that at time $t = 0$, $(M, g_0)$ be a compact locally conformally flat manifold with positive Ricci curvature. Then, under the Yamabe flow, we have

$$Z(g, X) \geq 0$$

for any 1-form $X$.

Remark 2.2. One can follow the proof of Theorem 2.1 in [3] to show that the theorem holds for any manifold satisfying the identities (2.2) and (2.3).

We will also need the following maximum principle for complete manifolds (see [13] and [14]).

Lemma 2.3 (Shi). Let $g_{ij}(t)$ be a family of complete Riemannian metrics on a noncompact complete manifold $M$ such that

1. $g_{ij}(x, t)$ varies smoothly in $t$,
2. for all $t \in [0, T)$, $g_{ij}(x, t)$ is equivalent to $g_{ij}(x, 0)$, and
3. for all $t$, $g_{ij}(x, t)$ has bounded curvature.

Suppose $f(x, t)$ is a smooth bounded function on $M \times [0, T)$ such that

1. $f(x, 0) \geq 0$,
2. $\frac{\partial}{\partial t} f = \Delta_t f + Q(f, x, t)$, and
3. $Q(f, x, t) \geq 0$ whenever $f \leq 0$,

where $\Delta_t$ denotes the Laplacian of the metric $g_{ij}(x, t)$. Then, we have $f(x, t) \geq 0$ on $M \times [0, T)$.

Combining the previous lemma and Hamilton’s maximum principle for tensors (Theorem 9.1 in [11]) yield to the maximum principle for tensors on a complete manifold, if their norms are uniformly bounded in space at each time slice.

3. Eternal solutions to the Yamabe flow

In this section we will give the proof of Theorem 1.4. Choose a vector field $X$ to satisfy

$$\nabla_i R + \frac{1}{n-1} R_{ij} X^j = 0.$$  \hspace{1cm} (3.1)

The vector field $X$ is well defined since $\text{Ric} > 0$ (and therefore defines an invertible matrix). Following Chow [3] we define the Harnack expression for the eternal
Yamabe flow, namely
\[(3.2) \quad Z(g, X) = (n - 1)\Delta R + \frac{1}{2}(\nabla R, X) + \frac{1}{2(n-1)} \cdot R_{ij} X^i X^j + R^2.\]

Note that (2.4) gives (3.2) if we use (3.1) and drop the term $R/t$, due to the fact we have an eternal solution.

Also, from (3.1) we have that $X_j = -(n-1)R_{ij} \nabla R_i$. Since $R_t = (n-1)\Delta R + R^2$ and since at the point $P_0 = (x_0, t_0)$ where $R$ assumes its maximum, we have $\partial R/\partial t = 0$ and $\nabla_i R = 0$, we conclude that $Z(g, X) = 0$, at $P_0$.

The idea is to apply the strong maximum principle to get that $Z \equiv 0$, which implies that $\nabla_i X_j = R g_{ij}$ (this will follow from the evolution equation for $Z$). To simplify the notation, we define $\Box = \partial_t - (n-1)\Delta$.

**Lemma 3.1.** The quantity $Z$ defined by (3.2) evolves by
\[(3.3) \quad \Box Z = 3RZ - R^3 + \frac{1}{2}(R_{kij} - R_{ij}^2)X_i X_j + \frac{1}{2(n-1)} R R_{ij} X_i X_j - \frac{(n-1)(n-2)}{2} |\nabla R|^2 \]
where $A_{ij}$ is the same matrix that Chow defines by (3.13) in [3].

**Proof.** We have the following equation due to Chow (3) after dropping all terms with $1/t$:
\[(3.4) \quad \Box Z = 3RZ - R^3 + \frac{1}{2}(R_{kij} - R_{ij}^2)X_i X_j - \frac{1}{2(n-1)} R R_{ij} X_i X_j - \frac{(n-1)(n-2)}{2} |\nabla R|^2 \]
\[- (n-1)R_{ij} \nabla_i RX_j + \langle \nabla R, \Box X \rangle + \frac{R_{ij} X_i \Box X_j}{n-1} \]
\[- 2R_{ij} \nabla_k X_i \nabla_k X_j - 2 \nabla_k R_{ij} \nabla_k X_i X_j - 2(n-1) \langle \nabla \nabla R, \nabla X \rangle \]
\[+ R_{ij} \nabla_k X_i \nabla_k X_j.\]

Since the evolution equation for $Z$ is independent of the choice of coordinates, we may choose coordinates such that $g_{ij} = \delta_{ij}$ at a given point, and the Ricci tensor is diagonal at that point. Then, we have
\[(3.5) \quad \langle \nabla R, \Box X \rangle + \frac{R_{ij}}{n-1} \Box X_i X_j = [\nabla_j R + \frac{R_{ij} X_i}{n-1}] X_j = 0\]
We also have
\[
\frac{2R_{ij}}{n} \nabla_k X_i \nabla_k X_j + 2 \nabla_k R_{ij} \nabla_k X_i X_j + 2(n-1) \nabla \nabla R \nabla X
\]
(3.6)

\[
= 2(\nabla_k X_i \cdot \nabla_k X_j (R_{ij}) + (n-1) \nabla_k X_i \nabla_k \nabla_i R)
\]

\[
= 2 \nabla_k X_i \nabla_k (R_{ij}) X_j + (n-1) \nabla_i R) = 0
\]
by the definition (3.1) of our vector field X. Combining (3.2), (3.3) and (3.6) yield to the equation

\[
\Box Z = 3RZ - R^3 + \frac{1}{2} (R_{ki} R_{ki} - R_{ij}^2) X_i X_j - \frac{1}{2(n-1)} R R_{ij} X_i X_j
\]
(3.7)

\[
- \frac{(n-1)(n-2)}{2} \nabla |R| - (n-1) R_{ij} \nabla X_j + R_{ij} \nabla_k X_i \nabla_k X_j
\]

If we contract identity (2.3) by \( R_{kl} \) we get

\[
R_{ki} R_{kl} = \frac{1}{n-2} (R_{kl}^2 g_{ij} + R_{ij} R_{kl} g_{kl} - R_{ij} R_{kl} g_{kl} - R_{kl} R_{kl} g_{ij} - R_{kl} (g_{kl} g_{ij} - g_{kl} g_{ij}))
\]

= \frac{1}{n-2} \left( |\text{Ric}|^2 \delta_{ij} + R_{ij} R \delta_{ij} - R_{ij}^2 \delta_{ij} - R_{ij}^2 \delta_{ij} - \frac{R}{n-1} (R \delta_{ij} - R_{ij}) \right)

= \frac{1}{n-2} \left( |\text{Ric}|^2 + \frac{n}{n-1} R R_{ij} - 2 R_{ij}^2 - \frac{R^2}{n-1} \right) \delta_{ij}

and therefore

\[
\frac{1}{2} (R_{ki} R_{kl} - R_{ij}^2) = \frac{1}{2(n-2)} \left[ |\text{Ric}|^2 + \frac{n}{n-1} R R_{ij} - n R_{ij}^2 - \frac{R^2}{n-1} \right] \delta_{ij}
\]
(3.8)

We also have \( \nabla_i R = -\frac{1}{n-2} R_{ij} X^j \), hence

\[
|\nabla R|^2 = g^{ij} \nabla_i R \nabla_j R = \frac{1}{(n-1)^2} R_{ik} X_k R_{ij} X^i
\]
(3.9)

\[
= \frac{1}{(n-1)^2} R_{ik} R_{kl} X_k X_l \delta_{ij}
\]

\[
= \frac{1}{(n-1)^2} R_{ij}^2 X_i X_j \delta_{ij}
\]

and

\[
(n-1) R_{ij} \nabla_i X_j = R_{ik} X^k R_{ij} X_j = R_{ij}^2 X_i X_j \delta_{ij}
\]
(3.10)

Combining (3.7), (3.8) and (3.10) yield to the equation

\[
\Box Z = 3RZ - R^3 + \frac{X_i X_j \delta_{ij}}{n-2} \left[ \frac{|\text{Ric}|^2}{2} + \frac{R R_{ij}}{n-1} - \frac{R^2}{2(n-1)} - \frac{n}{2(n-1)} R_{ij}^2 \right] + R_{ij} \nabla_k X_i \nabla_k X_j
\]

Denoting by

\[
A_{ij} = \frac{1}{n-2} \left[ \frac{|\text{Ric}|^2}{2} + \frac{R R_{ij}}{n-1} - \frac{R^2}{2(n-1)} - \frac{n}{2(n-1)} R_{ij}^2 \right]
\]
we have

\begin{equation}
\Box Z = 3RZ - R^3 + A_{ij}X_i X_j \delta_{ij} + R_{ij} \nabla_k X_i \nabla_k X_j.
\end{equation}

Direct computation gives

\begin{equation}
2RZ - R^3 = 2R((n-1)\Delta R + \langle \nabla R, X \rangle) + \frac{1}{2(n-1)}R_{ij}X^i X^j + R^2 - R^3
\end{equation}

\begin{equation}
= 2(n-1)R\Delta R + 2R\langle \nabla R, X \rangle + \frac{RR_{ij}}{n-1}X^i X^j + R^3.
\end{equation}

Also, after taking the covariant derivative \( \nabla_i \) of (3.1), we find that

\begin{equation}
\nabla_i \nabla_i R + \frac{1}{n-1} \nabla_i R_{ij} X^j + \frac{1}{n-1} R_{ij} \nabla_i X^j = 0.
\end{equation}

We can see now that in the chosen coordinates at the considered point

\( \nabla_i X_j = -(n-1)R_{ij}^{-1}(\nabla_i \nabla_i R + \frac{\nabla_i R_{il} X^l}{n-1}) \).

Since the matrix Ric\(^{-1}\) is diagonal at that point, we get immediately that

\begin{equation}
\nabla_i X_j = 0, \quad \text{for } i \neq j.
\end{equation}

If we sum (3.13) over \( i \), by the contracted Bianchi identity \( \nabla_i R_{ij} = \frac{1}{2} \nabla_j R \), we get

\begin{equation}
\Delta R + \frac{1}{2(n-1)} \nabla_j RX^j + \frac{1}{n-1} R_{ij} \nabla_i X^j = 0.
\end{equation}

By (3.1) and the previous identity we have

\begin{equation}
2(n-1)R\Delta R = \frac{RR_{ij}}{n-1}X_i X_j - 2R R_{ij} \nabla_i X_j
\end{equation}

which combined with (3.12) yields to

\begin{equation}
2RZ - R^3 = \frac{RR_{ij}}{n-1}X_i X_j - 2R R_{ij} \nabla_i X_j + \frac{RR_{ij}}{n-1}X_i X_j + R^3
\end{equation}

\begin{equation}
= R^3 - 2R R_{ij} \nabla_i X_j.
\end{equation}

It is easy to see that in our chosen coordinates around a point, we have

\begin{equation}
R^3 - 2R R_{ii} \nabla_i X_i + R_{ii} \nabla_i X_i \nabla_i X_i = \sum_i R_{ii}(R^2 g_{ii} - 2R \nabla_i X_i + (\nabla_i X_i)^2)
\end{equation}

\begin{equation}
= \sum_i R_{ii}(R g_{ii} - \nabla_i X_i)^2.
\end{equation}

Equations (3.11), (3.15) and (3.14) now yield to

\begin{equation}
\Box Z = RZ + A_{ij}X_i X_j + g^{kl} R_{ij}(R g_{ik} - \nabla_i X_k) \cdot (R g_{jl} - \nabla_j X_l).
\end{equation}
The matrix $A_{ij}$ is the same one that Chow defines by (3.13) in [3]. In local coordinates $\{x_i\}$, where $g_{ij} = \delta_{ij}$ and the Ricci tensor is diagonal at a point, we have

$$R_{ij} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

hence

(3.16) $$A_{ij} = \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_n \end{pmatrix}$$

where

$$\nu_i = \frac{1}{2(n-1)(n-2)} \sum_{k,l \neq i, k > l} (\lambda_k - \lambda_l)^2.$$ 

This finishes the proof of the proposition.

To finish the proof of Theorem 1.4 we need the following version of the strong maximum principle.

**Lemma 3.2.** If $Z(g, X) = 0$ at some point at $t = t_0$, then $Z(g, X) \equiv 0$ for all $t < t_0$.

**Proof.** The proof is similar to the proof of Lemma 4.1 in [8]. Our Lemma will be a consequence of the usual strong maximum principle, which assures that if we have a function $h \geq 0$ which solves

$$h_t = \Delta h$$

for $t \geq 0$ and if we have $h > 0$ at some point when $t = 0$, then $h > 0$ everywhere for $t > 0$.

Assume there is a $t_1 < t_0$ such that $Z(g, X) \neq 0$ at some point, at time $t_1$. We may assume, without the loss of generality, that $t_1 = 0$. Define $F_0 := Z(0)$ and allow $F_0$ to evolve by the equation

$$F_t = (n-1) \Delta F.$$ 

From the result of Chow we know that $F(0) \geq 0$ and therefore it will remain so for $t \geq 0$, by the maximum principle. Since, by our assumption, there is a point at $t = 0$ at which $F(0) > 0$, we conclude by the strong maximum principle that $F > 0$ everywhere as soon as $t > 0$. 

Take $\phi = \delta e^{At} f(x)$, where $f(x)$ is the function constructed in [9] with $f(x) \to \infty$ as $x \to \infty$, $f(x) \geq 1$ everywhere, with all the covariant derivatives bounded, and $A$ is big enough (depending on $\delta$) so that

$$\phi_t > (n-1) \Delta \phi.$$ 

Observe next that since $R, Z \geq 0, A_{ij}X_iX_j \geq 0$ and $Ric \geq 0$, all terms on the right hand side of (3.3) are nonnegative, therefore

$$Z_t \geq (n-1) \Delta Z.$$ 

Hence, $\tilde{Z} := Z - F + \phi$ satisfies the differential inequality

$$\tilde{Z}_t \geq (n-1) \Delta \tilde{Z} - F_t + (n-1) \Delta F + \phi_t - (n-1) \Delta \phi$$

and from the choice of $\phi$ and $f$

$$\tilde{Z}_t > (n-1) \Delta \tilde{Z}.$$ 

Since $\phi(x) \to \infty$ as $x \to \infty$, $\tilde{Z}$ attains the minimum inside a bounded set and by the maximum principle, we have

$$(\tilde{Z}_{\text{min}})_t > 0$$

which implies that

$$\tilde{Z}_{\text{min}}(t) \geq \tilde{Z}_{\text{min}}(0) = \phi(0) > 0.$$ 

We conclude that $Z \geq F - \phi$ everywhere, for $t \geq 0$. We now let $\delta \to 0$ in the choice of $\phi$. This yields to the lower bound

$$Z \geq F > 0, \quad \text{as soon as } t > 0.$$ 

On the other hand, $Z(g, X) = 0$ at time $t_0 > 0$, at the point where $R$ attains its maximum, which contradicts (3.17). This implies $Z(g, X) \equiv 0$ everywhere, for $t < t_0$, and finishes the proof of Lemma 3.2. □

We are now ready to conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. The theorem readily follows from Lemma 3.1 and Lemma 3.2. Since $Z \equiv 0$ and since all terms on the right hand side of (3.3) are nonnegative, we obtain from (3.3) the identity

$$\nabla_i X_j = R_{ij}.$$
that is, $g$ is a steady soliton. Since $\nabla_i X_j = \nabla_j X_i$ and since our manifold is simply connected, the vector field $X$ is the gradient of a function, which means that the metric $g$ is a gradient steady soliton. 

4. Type I singularity in the Yamabe flow

In this section we will prove Theorem 1.2 and Corollary 1.3 using a contradiction argument. Since the proofs of both results are almost the same, we will focus on the former one. We will first mention the following continuation result.

Claim 4.1. If $T < \infty$, then $\limsup_{t \to T} \sup_M |R| (\cdot, t) = \infty$.

Proof. Assume $\sup_M |R| (\cdot, t) \leq C$, uniformly in time $t \in [0, T)$. Since the Ricci curvature is assumed to be positive (Chow showed in [3] that this condition is preserved along the flow), we have $\sup_M |\text{Ric}| \leq C$, for all $t \in [0, T)$ and by (2.3) the uniform curvature bounds as well. Recall that the evolution equation for Ric is:

$$\frac{\partial}{\partial t} R_{ij} = (n-1)\Delta R_{ij} + \frac{1}{n-2} B_{ij},$$

where

$$B_{ij} = (n-1)|\text{Ric}|^2 g_{ij} + nRR_{ij} - n(n-1)R_{ij}^2 - R^2 g_{ij}.$$  

We can rewrite the equation for Ric as

$$\frac{\partial}{\partial t} \text{Ric} = (n-1)\Delta \text{Ric} + \text{Ric} \ast \text{Ric},$$

where $\text{Ric} \ast \text{Ric}$ stands for any linear combination of tensors formed by contraction on $R_{ij} \cdot R_{kl}$. Notice that the evolution for Ric along the Yamabe flow has the same form as the evolution for $R_m$ along the Ricci flow. Techniques similar to Shi’s in [14] apply to our case as well, and we can show that all the covariant derivatives of Ric are uniformly bounded on $[0, T)$. Differentiating (2.3) this yields to uniform bounds in space and time on all the covariant derivatives of $R_m$, and therefore we can extend the flow $g(t)$ smoothly, past time $T < \infty$, which contradicts the maximality of $T$. 

Proof of Theorem 1.2. There are two possibilities for $g(t)$, either the flow exists forever, in which case we are done, or the maximal time of existence $T < \infty$. Assume the singularity at $T$ is of type II, namely that

$$\limsup_{t \to T} m(t) (T - t) = \infty,$$
where \( m(t) = \sup_M |\text{Rm}(g(t))| \). By our assumption \((M, g(t))\) satisfies the injectivity radius bound, that is,

\[
\rho(t) \geq \frac{1}{\sqrt{m(t)}}.
\]

Following R. Hamilton’s approach for the Ricci flow ([10]), we define a sequence of blow up solutions, carefully choosing sequences of points \( p_i \in M \) and times \( t_i \to T \) around which we will perform the blow up (we choose them as in [10], keeping in mind that the max \( R \) controls the curvature since \( \text{Ric} > 0 \) and we are in the locally conformally flat case). Let

\[
g_i = Q_i g(t_i + tQ_i^{-1}), \quad \text{with} \quad Q_i = R(P_i, t_i).
\]

After taking the limit of \((M, g_i(t_i), P_i)\) (as in [10]) we obtain an eternal complete non-flat solution to the Yamabe flow \( \bar{g}(t) \) on \( M_\infty \) with strictly positive Ricci curvature and uniformly bounded curvature operator, where the scalar curvature attains its maximum at the interior point. By Theorem 1.4 the solution \( \bar{g}(t) \) has to be a gradient steady soliton, that is,

\[
R \bar{g}_{ij} = \nabla_i \nabla_j f
\]

for some function \( f \), with \( R > 0 \) everywhere on \( M \). The following claim will contradict the nonflatness of \( \bar{g}(t) \).

**Claim 4.2.** \( \bar{g}(t) \) has to define a flat metric, unless \( (M_\infty, \bar{g}(t)) \) is a compact manifold.

**Proof of Claim.** After passing to the limit \( i \to \infty \), \( Z(\bar{g}, X) \) satisfies (3.3) and \( \bar{g} \) is a gradient steady soliton with strictly positive Ricci curvature and uniformly bounded scalar curvature, such that the scalar curvature attains its maximum at an interior space-time point. The proof of Theorem 1.4 implies that \( Z(\bar{g}, X) \equiv 0 \) and since the matrix \( A_{ij} \) in (3.3) is a positive semi-definite quadratic form, we must have

\[
A_{ij} \nabla_i f \cdot \nabla_j f = 0.
\]

Recall that \( \bar{g} \) also satisfies

\[
R \bar{g}_{ij} = \nabla_i \nabla_j f
\]

which immediately tells us that since \( R > 0 \) everywhere on \( M_\infty \), \( \text{Hess}(f) \) is a strictly positive definite quadratic form. Define

(i) \( U_{jk} := \{ x \in M_\infty \mid \nabla_j f(x) \neq 0, \nabla_k f(x) \neq 0 \} \), for \( j \neq k \)
(ii) $S := \{ x \in M_\infty \mid \nabla_j f(x) = 0, \text{ for all } j \}$

(iii) $V_i := \{ x \in M_\infty \mid \nabla_j f(x) = 0, \text{ for all } j \neq i \}$

so that $M_\infty = (\cup U_{jk}) \cup (\cup V_i) \cup S$. The sets $U_{jk}$ are open in $M_\infty$ and if $x \in U_{jk}$, since $n \geq 3$, (4.2) and (3.16) yield to the equality

$$\lambda_1 = \cdots = \lambda_n$$

which implies that

$$R_{ij} = \frac{R}{n} \bar{g}_{ij}.$$ 

As a corollary of the second Bianchi identity (see [7]), we get that $R_{ij} = \lambda \bar{g}_{ij}$, for a constant $\lambda$, on $U_{jk}$. Hence, we have

$$\cup_{j \neq k, 0 \leq j, k \leq n} U_{jk} \subset \{ x \in M_\infty \mid \text{Ric} = \frac{R}{n} \bar{g} \}. \quad (4.4)$$

The set $S$ of critical points is closed in $M_\infty$ and since the map $F := \nabla f : M_\infty \to \mathbb{R}^n$ has the property that $DF|_x$ is an invertible matrix for every $x \in M_\infty$, we easily obtain that $S$ is of measure zero in $M_\infty$, that is, $\mu(S) = 0$.

Denote by

$$B := \{ x \in M_\infty \mid \text{Ric} = \frac{R}{n} \bar{g} \} \cup S.$$ 

The relation (4.3) yields to

$$B^c = \cup_{1 \leq i \leq n} (B^c \cap V_i).$$

Since $B^c$ is an open set, if it is not empty, at least one of $B^c \cap V_i$ has to be open (note that $(B^c \cap V_i) \cap (B^c \cap V_j) = \emptyset$ since otherwise if $x \in (B^c \cap V_i) \cap (B^c \cap V_j)$, then $x \in S \subset B$). Assume $B^c \cap V_1$ is open and nonempty. Take any $i \neq 1$. Since $\nabla_i f = 0$ on $B^c \cap V_1$, we have

$$R\bar{g}_{ji} = \nabla_j \nabla_i f = 0$$

which implies $R = 0$ on $B^c \cap V_1$ which is not possible. Hence $B^c = \emptyset$ and

$$M_\infty = \{ x \in M_\infty \mid \text{Ric} = \frac{R}{n} \bar{g} \} \cup S$$

where $\nu(S) = 0$. By continuity, we obtain

$$\text{Ric} = \frac{R}{n} \bar{g}, \quad \text{on } M_\infty$$

and therefore $\text{Ric} = \lambda \bar{g}$, for a positive constant $\lambda$. By Mayer’s theorem, if $\lambda > 0$, $M_\infty$ has to be compact, otherwise $\lambda = 0$ and $(M_\infty, \bar{g}(t))$ is flat. This finishes the proof of the claim. $\square$
To finish the proof of Theorem 1.2, notice that if we start with a complete non-compact manifold \((M, g_0)\), its blow up limit cannot be compact. Then by Claim 4.2 the metric \(\bar{g}(t)\) has to be a flat metric, which is not possible as a matter of rescaling of our original solution. If we start with something compact, the previous claim shows that we get the limit \(\bar{g}(t)\) of the rescaled sequence of blow ups to be either flat or the Einstein metric of positive curvature and therefore of constant positive sectional curvature (since \(\bar{g}(t)\) satisfies (2.3)). The former case is excluded because of the way we rescale (we keep curvature to be 1 at the points around which we rescale). Having a metric of constant positive sectional curvature would imply that \(M_\infty\) had to be a sphere \(S^n\), and therefore \((M, g(t))\) would have a type I blow up again. The proof of Theorem 1.2 is now complete. Notice that the compact Yamabe flow has already been completely understood (II).

Proof of Corollary 1.3. Let \(T\) be the maximal time of existence of a solution \(g(t)\) to (1.1). The evolution equation for the Schouten tensor \(S_{ij}\) is

\[
\frac{\partial}{\partial t} S_{ij} = (n - 1) \Delta S_{ij} + \frac{1}{N - 2} B_{ij}
\]

where \(B_{ij}\) is the tensor given by (4.1). Since all curvature quantities are uniformly bounded in space at each time-slice, by the maximum principle, similarly as in the proof of Lemma 2.9 in [3], we have that if \(S_{ij} > 0\) at time \(t = 0\) then \(S_{ij} > 0\) along the flow. From the discussion in the introduction, \(S_{ij}\) being positive definite means the sectional curvature \(K\) is positive. Since \(\sup_M K(t) \leq C(t)\), we have

\[
iinjrad(M, g(t)) \geq \frac{\pi}{\sqrt{C(t)}}
\]

We can now apply Theorem 1.2 to finish the proof.

References

[1] Brendle, S., *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. 69 (2005), no. 2, 217–278.
[2] Burago, Yu.D., Zalgaller, V.A., *Convex sets in Riemannian spaces of nonnegative curvature*; Russian Math Surveys 32:3 (1977), 1–57.
[3] Chow, B., *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. 65 (1992), 1003–1014.
[4] Daskalopoulos, P., Sesum, N., *On the extinction profile of solutions to fast-diffusion*, J. Reine Angew. Math. 622 (2008), 95–119.
[5] del Pino, M.; Sáez, M., On the extinction profile for solutions of \(u_t = \Delta u^{(N-2)/(N+2)}\). Indiana Univ. Math. J. 50 (2001), no. 1, 611–628.
[6] Eisenhart, L.P., *Riemannian geometry*, Princeton University Press, 1927.
[7] Gallot, S., Hullin, D., Lafontaine, J. *Riemannian geometry*, Berlin-Heidelberg: Springer-Verlag, 1987.
[8] Hamilton, R., *Eternal solutions to the Ricci flow*, J.Diff.Geom. 38 (1993), 1–11.
[9] Hamilton, R., *The Harnack estimate for the Ricci flow*, J.Diff.Geom. 37 (1993), 225–243.
[10] Hamilton, R., *Formation of the singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
[11] Hamilton, R., *Three manifolds with positive Ricci curvature*, J.Diff.Geom. 17 (1982), 255-306.
[12] Peterson, P., *Riemannian geometry*, Springer-Verlag, ISBN 0-387-98212-4.
[13] Shi, W.X., *Ricci flow deformation of the metric on complete Riemannian manifolds*, J.Diff.Geom. 30 (1989), 303-394.
[14] Shi, W.X., *Ricci flow and the uniformization of the metric on complete noncompact Kähler manifolds*, J.Diff.Geom. 45 (1997), 94-220.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, USA

E-mail address: pdaskalo@math.columbia.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, USA

E-mail address: natasas@math.columbia.edu