Statistics of Two Kinds of Entangled Quantum Many-body Systems

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In this paper, we show two kinds of entangled many-body systems with special statistic properties. Firstly, an entangled fermions system with a pairwise entanglement between every two particles in the lowest energy level obeys the fractional statistics. As a check, for particle number $N=2, N=3$ and $N=4$, considering that any two fermions in the lowest Landau level are entangled in a proper way, the Laughlin wave function can be derived. The results reveals the explicit entanglement pattern of the Laughlin states. Secondly, we noticed that both Bose-Einstein statistics and Fermi-Dirac distributions are derived from computing the partial function of a free quantum many-body system in a certain ensemble without considering entanglement. We extend the computation of the partial function to an entangled quantum many-body system without interaction, in this system we assume that every particle in energy level $\epsilon_i$ is entangled with a particle in the energy level $\epsilon_{i+1}$ ($i = 1, 3, 5, ...$) and also every particle in energy level $\epsilon_i + 1$ is entangled with a particle in the energy level $\epsilon_i$ ($i = 1, 3, 5, ...$), which indicates that the two energy level have the same number of particles. In the entangled system, we find that the partial function will be changed. As a results, both the Bose-Einstein Statics and the Fermi-Dirac distributions will be modified at finite temperature.

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I. INTRODUCTION

Recently, entanglement has been used to get a novel perspective into the many-body quantum systems that appear in Nature\cite{1}. Important results have been obtained in understanding the entanglement properties in the vicinity of quantum critical points\cite{2}. The topological entanglement entropy is already used to study topological phases\cite{3, 4}. In this paper we try to understand the relationship between entanglement and statistics of a quantum entanglement identical particles system. People might suspect that an entangled system will have non-trivial statistics, in order to answer the question, we assume that we already have two entangled systems then we study the statistic properties of such a specially entangled system. We show that at least for two kinds of entangled many body systems, we will see non-trivial statistic properties like the fractional statistics. When studying a possible entangled many body system, usually we firstly have a Hamiltonian then we find the ground state of the system. However, the strategy we used is somehow ideally since we assumed that we already have an entangled many body state, which can be the ground state of a special Hamiltonian. The spirit of this paper is to show that there is a strong connection between entanglement and statistics.

II. ENTANGLEMENT BETWEEN FERMIONS IN THE SAME ENERGY LEVEL AND FRACTIONAL STATISTICS

We think about an entangled fermions system with a pairwise entanglement between every two particles in the lowest energy level $\epsilon_1$, with the state density $f(\epsilon_1) = M > 2$. We will see that the entangled fermions system obey the fractional statistics. At zero temperature, assuming that we have two fermions $A$ and $B$, the number of states in the lowest energy level $\epsilon_1$ is $M$, we label every states occupied by Fermion $A$ or $B$ as $\Psi_i(A)$ or $\Psi_i(B)$, $0 < i < j \leq M$ labels the $M$ states. The entanglement we consider can be expressed as the following form

$$\Psi(A, B) = \sum_{i<j\leq M} C_{ij} [\Psi_i(A)\Psi_j(B) - \Psi_i(B)\Psi_j(A)], \quad (1)$$

after permutation of the two fermions we see that $\Psi(B, A) = -\Psi(A, B)$. Since there are $M$ states while we have two fermions occupy $M$ states then we have the fractional statistics with filling factor $v = 2/M$. When $M = 2$ we recover the total filled case, then the Eq.(1) is just the Slater determinant.

In the following we see that the pairwise entanglement between every two particles in the lowest Landau level as Eq.(1) will give the Laughlin wave function of two fermions, three fermions and four fermions. In the lowest Landau level the single particle state is $\Psi_i(A) = (z_A)^i - 1 e^{-\frac{1}{2} B |z_A|^2}$, in which $z_A$ is the coordinate of particle $A$. $B$ is the magnetic field, $i_{\text{max}}$ is the degree of degeneration. When we choose proper coefficients in Eq.(1) as

$$\Psi(A, B) = [\Psi_1(A)\Psi_4(B) - \Psi_4(B)\Psi_1(A)] + 3[\Psi_2(A)\Psi_3(B) - \Psi_3(B)\Psi_2(A)], \quad (2)$$

in which $C_{14} = 1$, $C_{23} = 3$ and other $C_{ij} = 0$. With this particular coefficients above we recover exactly the Laughlin wave function for two Fermions $N = 2$ with filling factor $v = 1/3$ that $\Psi(A, B) = (z_A - z_B)^3 e^{-\frac{1}{2} B |z_A|^2 + |z_B|^2}$. Thinking of the case $v = 1/5$,
the Laughlin wave function reads

$$\Psi(A, B) = e^{-\frac{1}{k_B}(|z_A|^2 + |z_B|^2)} (z_A - z_B)^3$$

which is still a special case of Eq. (1). We can also check that the Laughlin functions for $N = 2$ with other filling factor $v = 1/7, 1/9$ are special case of Eq.(1).

We further check that the $N = 3$ Laughlin function can also comes from a pairwise entanglement between every two fermions. We label the three fermions as $A, B$ and $C$, we expand the Laughlin function for $N = 3$ with filling factor $v = 1/3$ as

$$\Psi(A, B, C) = (z_A - z_B)^3 (z_A - z_C)^3 (z_B - z_C)^3 e^{-\frac{1}{k_B}(|z_A|^2 + |z_B|^2 + |z_C|^2)}$$

$$= (z_A^5 - z_B^5 + 5z_A z_B z_C + 10z_A^3 z_B^2 - 10z_A^2 z_B z_C)$$

We collect the expansion above properly that the every line of wave function Eq.(4) can be expressed as

$$\Psi = \sum_{i<j<k\leq M} C_{ijk} \Psi_i(A) \Psi_j(B) \Psi_k(C) - \Psi_k(D) \Psi_l(I)$$

which still expresses an entanglement between every two fermions of the three fermions $A, B$ and $C$. As an example, we can see that the two terms with underline in Eq.(4)

$$\sum_{i<j<k\leq M} C_{ijkl} \Psi_i(A) \Psi_j(B) \Psi_k(C) - \Psi_l(I) = \Psi_1(C) \Psi_2(B) - \Psi_2(C) \Psi_1(B)$$

which is exactly a special case of Eq.(4) with $C_{147} = -1$ and other coefficients are zero. In fact we can also check that the Laughlin function for $N = 3$ with filling factor $v = 1/5, 1/7, 1/9$ can be also expressed as a proper case of Eq.(4).

$N = 4$ case is more complex, however, if we still expand the Laughlin wave function and regroup the polynomials carefully as Eq.(4) we can easily find that the Laughlin function $\prod_{j<i} (z_j - z_i)^{1/5} e^{-\frac{1}{k_B}(|z_A|^2 + |z_B|^2 + |z_C|^2 + |z_D|^2)}$ of four fermions $A, B, C, D$ can be also expressed as a special case that every two fermions of the four fermions are entangled as

$$\sum_{i<j<k<l} C_{ijkl} \Psi_i(A) \Psi_j(B) \Psi_k(C) \Psi_l(D) - \Psi_k(D) \Psi_l(I)$$

by choosing proper coefficients $C$. With the results of $N = 2, 3, 4$, we conjecture that for any number $N$, the Laughlin function can still be expressed as a linear combination of wave function that for every function there are two and only two fermions are entangled. To prove this is difficult and we leave it to future work. If the number of particle $N$ equals to the degree of degeneration $M$ we come back to the full filled case with $v = 1.$
III. ENTANGLEMENT BETWEEN PARTICLES IN DIFFERENT ENERGY LEVEL AND STATISTICS

We assume there is an entangled quantum many-body system that every particle in energy level $\epsilon_i$ is entangled with a particle in energy level $\epsilon_{i+1}$ ($i = 1, 3, 5, ...$) and also every particle in energy level $\epsilon_{i+1}$ is entangled with a particle in the energy level $\epsilon_i$ ($i = 1, 3, 5, ...$). With the simple pairwise correlation, it can be derived that if the mean number of particles in energy level $\epsilon_i$ is $n_i$, then the mean number of particles $n_{i+1}$ in energy level $\epsilon_i$ is also $n_i$. As a result the partition function will be modified. We want to emphasize that the system is totally ideal which may not exist in Nature, but the ideal system is helpful for us to understand the relationship between entanglement and statistics.

Firstly we study the bosonic system with conserved numbers of particles, which means we discuss the grand canonical ensemble that the temperature $T$ and chemical potential $\mu$ are fixed. The partition function of the system reads

$$Z = \sum_R e^{-\beta n_1(\epsilon_1 - \mu) + n_2(\epsilon_2 - \mu) + ...},$$

(8)

in which $R$ is the possible quantum states of the whole particles. Here the particles are to be considered as indistinguishable, so that the state of the gas can be specified by merely listing the number of particles in each energy level: $n_1, n_2, n_3, ...$. Since there is no limit to the number of particles that can occupy a state energy level, $n_i$ can equal 0, 1, 2, 3, ... for each state(energy level) $i$. If we do not consider the effect of entanglement between particles, the summation is over all values $n_i = 0, 1, 2, 3, ...$ for each $i$.

$$Z = \left( \sum_{n_1=0}^{\infty} e^{-\beta n_1(\epsilon_1 - \mu)} \right) \left( \sum_{n_2=0}^{\infty} e^{-\beta n_2(\epsilon_2 - \mu)} \right) ...$$

(9)

It is easy to derived that $\ln Z = -\sum_i \ln(1 - e^{-\beta \epsilon_i})$, then we get the Bose-Einstein distribution $\bar{n}_i = \frac{1}{e^{\beta \epsilon_i} - 1}$.

Assuming there is a kind of pairwise entanglement between any particle $A$ in energy level $\epsilon_i$ and a particle $B$ in energy level $\epsilon_{i+1}$ ($i = 1, 3, 5, ...$) as discussed above, the mean number of particles $n_{i+1}$ in energy level $\epsilon_{i+1}$ is equal to the number of particles $n_i$ in energy level $\epsilon_i$, the partition function $Z$ can be rewritten as

$$Z = \prod_{i=1,3,5,...} \left( \sum_{n_i=0}^{\infty} e^{-\beta [n_i(\epsilon_i - \mu) + n_i(\epsilon_{i+1} - \mu) + ...]} \right),$$

(10)

$i$ is the label of the energy level. Then we can easily have

$$\ln Z = \sum_{i=1,3,5,...} \ln(1 - e^{(\epsilon_i + \epsilon_{i+1} - 2\mu)}).$$

(11)

So the average particle number in energy level $\epsilon_i$ or $\epsilon_{i+1}$ reads

$$\bar{n}_i = \bar{n}_{i+1} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_i} = \frac{1}{e^{\beta(\epsilon_i + \epsilon_{i+1} - 2\mu)} - 1}$$

(12)

From the results we see that the Boson-Einstein statistics is modified when considering the effect of a very simple entanglement correlation. However when we choose the zero temperature limit we see that the results recovers the results without entanglement.

Now we move to Fermions, we discuss the grand canonical ensemble, still we also think there is a pairwise entanglement between any fermion $A$ in energy level $\epsilon_i$ and its pair $B$ in energy level $\epsilon_{i+1}$. Combine the fact fermions have to obey the Pauli exclusion principle and the mean number of particles $n_{i+1}$ in energy level $\epsilon_{i+1}$ is equal to $n_i$, the grand partition function is given by

$$Z = \prod_{i=1,3,5,...} \left( \sum_{n_i=0}^{1} e^{-\beta [n_i(\epsilon_i - \mu) + n_i(\epsilon_{i+1} - \mu) + ...]} \right).$$

(13)

We can easily get

$$Z = \prod_{i=1,3,5,...} \left( 1 + e^{-\beta(\epsilon_i + \epsilon_{i+1} - 2\mu)} \right)$$

(14)

Then the mean number of fermions in energy level $\epsilon_i$ or $\epsilon_{i+1}$ is given by

$$\bar{n}_i = \bar{n}_{i+1} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_i} = \frac{1}{e^{\beta(\epsilon_i + \epsilon_{i+1} - 2\mu)} + 1}$$

(15)

If we do not consider the effect of entanglement then the partition function reads

$$Z = \prod_{i=1,2,3,...} \left( 1 + e^{-\beta(\epsilon_i - \mu)} \right),$$

(16)

as a result we can easily come back to the Fermi-Dirac statistics $\bar{n}_i = \frac{1}{e^{\beta \epsilon_i} - 1}$.

Until now we assume an ideal many body system with $n_i = n_{i+1}$, $(i = 1, 3, 5, ...)$, the fact the two energy level has the same number of particles can be a result of that there is an entanglement correlation between every particle in one energy level and another particle in the other energy level. The $n_i = n_{i+1}$, $(i = 1, 3, 5, ...)$ will affect the computation of the partition function then the statistic at finite temperature will change, as a results, other properties of the boson or fermion gas, for example the heat capability, conductivity, the critical temperature of Bose-Einstein condensation will be changed. In the above discussion we assume that the state density $f(\epsilon_i)$ of every energy level $\epsilon_i$ equal to 1, if $f(\epsilon_i) > 1$ we have to multiply the state density $f(\epsilon_i)$ in front of the mean number $\bar{n}_i$ when calculate the particle numbers in energy level $\epsilon_i$. 
IV. CONCLUSION

In conclusion, we find that a pairwise entanglement between every two fermions in lowest Landau level will result in fractional statistics. The Laughlin wave function can be expressed as a simple combination of many body wave functions, in every wave function there are two fermions and only two fermions in the fermionic system are entangled. Usually, the entanglement between many body systems can not be reduced to the case of two particles, however we already show that at least for the Laughlin state, the entanglement can be expressed in the sum of states, in every state only two fermions are entangled. According to the finding we conjectured that the system with factional statistics is nothing but a specially entangled many body state. This picture shows the explicit entanglement pattern of the Laughlin states, the founding of the explicit entanglement pattern should be helpful to understand the entanglement entropy in the Laughlin state studied in literatures [6-8]. we also show that if we ideally consider there is an specially entangled quantum many-body system at the very beginning, the Bose-Einstein distributions and Fermi-Dirac distributions will be modified at finite temperature.

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