Monochromatic and rainbow 4-connectivity of moon graphs

M A Shulhany\textsuperscript{1*}, D K N Rachim\textsuperscript{2} and Y Rukmayadi\textsuperscript{3}

\textsuperscript{1} Department of Civil Engineering, Faculty of Engineering, Universitas Sultan Ageng Tirtayasa, Jln. Jend Sudirman KM. 3, Cilegon 42435, Banten, Indonesia
\textsuperscript{2} Mathematics lecturer, UIN Sultan Maulana Hasanuddin, Jln. Jenderal Sudirman No. 30, Penancangan, Serang 42118, Banten, Indonesia
\textsuperscript{3} Department of Mechanical Engineering, Faculty of Engineering, Universitas Sultan Ageng Tirtayasa, Jln. Jend Sudirman KM. 3, Cilegon 42435, Banten, Indonesia

*Email: ahmad.s@untirta.ac.id

Abstract. Monochromatic and rainbow connectivity are known as the concepts that can calculate the minimum number of passwords and security systems. In this paper, a security system is represented by a special mathematics object, namely a graph. The study focuses on the moon graphs or Mon. Moon graphs are chosen because it is one of the 4-connected graphs that has a minimum number of edges. The research method used is mathematical modeling equipped with direct proves and contradictions. The conclusion is that there is an efficiency of 75% when using the monochromatic connected number, and there is an efficiency between 62.5% - 75% when using the rainbow 4-connected number.

1. Introduction
Monochromatic connectivity has been introduced by Caro and Yuster in [2]. Whereas, rainbow connectivity has been introduced by Chartrand et al. [3]. Both of these concepts are used to calculate the minimum number of passwords needed on a network security system by taking care of their security level and efficiency. For example, a graph \( G = (V, E) \) with \( n \) vertices and \( m \) edges is a simple, non-trivial, undirected, and 4-connected graph. Edge coloring on \( G \) with colors \( s \in \mathbb{N} \) is called as edge \( s \)-coloring. A \( u - v \) path is the path on \( G \) with the vertices on its edges, \( u \) and \( v \), and has been subjected to edge \( s \)-coloring. The rainbow \( u-v \) path is a \( u-v \) path on \( G \) that has different colors, whereas the monochromatic \( u-v \) path on \( G \) is the \( u-v \) path that has similar colors.

The rainbow \( k \)-connection number, \( r_{c_k}(G) \) is the smallest number so that each pair of vertices on \( G \) has \( k \) pieces of rainbow \( u - v \) paths which are internally disjoint of each. Generally, the rainbow \( k \)-connection number has a lower bound, namely \( \left\lceil \frac{k+1}{2} \right\rceil \), see [4], and the upper bound is \( m \), just like the number of edges.

For example \( s \in \mathbb{N} \) and \( G \) is \( k \)-connected, \( P_{s}(u, v) \) is a family of \( k \) disjoint path between vertices \( u \) and \( v \), the \( k \) distance from vertices \( u \) and \( v \), \( d_k(u, v) \), is \( \min\{|P_{s}|\} \) and \( k \) diameter of \( G \), \( diam_k(G) \), is \( \max\{d_k(u, v)|(u, v) \in V(G)\} \) [6]. Next, the rainbow \( k \)-connection numbers depend on \( diam_k(G) \). Because

\[
diam(G) \leq diam_2(G) \leq \ldots \leq diam_k(G)
\]

and

\[
\left\lceil \frac{k+1}{2} \right\rceil \leq diam_k(G)
\]
so
\[ \text{diam}(G) \leq rc(G) \leq rc_2(G) \leq \ldots \leq rc_k(G) \leq m \] (3)
and
\[ \left\lceil \frac{k + 1}{2} \right\rceil \leq \text{diam}_k(G) \leq rc_k(G) \leq m \] (4)

For \( k = 1 \), the graphs which show their rainbow connection numbers are Diamond Graph [7], Stellar [8] and Oleander Graph [1]. For \( k = 2 \), the study conducted by Susanti, et.al [9] has found the upper bound for the Cartesian product of a path and a cycle. For \( n \geq 2 \), Chartrand, et.al [4] show the proof for:
\[ rc_{n-1}(K_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \] (5)

Furthermore, the monochromatic connection number, \( cr(G) \), is the maximum number so that each pair of vertices \( u \) and \( v \) on \( G \) has a monochromatic \( u - v \) path. Caro and Yuster [2] has found some bounds for the rainbow connection number. Those bounds are related to its minimum spanning tree and connectivity. The minimum spanning subgraph of \( G \), labeled \( G' \), is a connected subgraph which has the minimum edges with \( V' = V \), and the minimum spanning tree named \( T' \) is a minimum spanning subgraph in the form of a tree. The monochromatic connection number is related to the minimum spanning tree which has \( n - 1 \) vertices, coloring all edges \( T' \) with 1, therefore
\[ cr(G) \leq m - n + 2 \] (6)

And for \( G \) \( k \)-connected, at the vertex with degree \( k - 1 \) only two edges have the same color or at one vertex there are \( k - 1 \) colors so that the upper bound is obtained
\[ cr(G) \leq m - n + k + 1. \] (7)

Research related to monochromatic connection number, research by Gu, et al. [5] concerns on monochromatic coloring on random graphs. Unlike the previous research, the main objective of this study is determining \( rc_4(G) \) and \( cr(G) \) with \( G \) as new class graphs named Moon Graph, denoted by \( Mo_n \).

2. Method

The method used in this study is mathematical modeling accompanied by proof of the theorem using direct methods and contradictions. The research is carried out through three stages. First, study literature to find theorems about graph coloring. Second, formulate a hypothesis according to the found theorem. Finally, make the proof. The proof is divided into two parts, proof of the upper bound and proof of the lower bound. Verification of the lower limit is done by showing that the number found is the smallest number that satisfies, while the upper bound is done with two ways, namely making the coloring function and showing that every pair of vertices on graph colored contains a monochromatic path or 4 internally disjoint rainbow paths.

3. Main Result

The Moon Graph on \( n \) vertices and \( n \geq 5 \), named by \( Mo_n \), is the graph with set of vertices \( V(G) = \{v_i\} | 1 \leq i \leq n \} \) and set of edges \( E(G) = \{v_i v_{i+1} | 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{v_i v_{i+2} | 1 \leq i \leq n, v_{n+i} = v_i\} \). For \( 1 \leq n \leq 4 \), the graph formed is trivial. For example, figure 1 is \( Mo_3 \) Graph.

**Theorem 1.** Let \( n \) be a natural number with \( n \geq 5 \). If \( Mo_n \) is moon graphs on \( n \) vertices, then its monochromatic number is
\[ cr(Mo_n) = n + 2, \text{for } n \geq 5. \]

**Proof.** Based on (6), the proof of lower bound is obtained, then the upper bound of \( cr(Mo_n) \leq m - n + 2 = 2n - n + 2 = n + 2 \) can be shown by defining a monochromatic coloring.

Define a coloring \( a: E(Mo_n) \rightarrow [1, n + 2] \) as follows:
For example, See Monochromatic 11-coloring graph in figure 2. Then, every each pair of vertices \(v_i\) and \(v_j\) on colored \(Mo_n\) with \(i < j\) has monochromatic path \(v_i, v_{i+1}, ..., v_{j-1}, v_j\). See figure 2, vertex \(v_1\) and \(v_7\) has a monochromatic path, namely \(v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7\). It can be concluded that \(cr(Mo_n) = n + 2\).

**Theorem 2.** Let \(n\) be a natural number with \(n \geq 5\). If \(Mo_n\) is moon graph on \(n\) vertices, then

\[
rc_4(Mo_n) = \begin{cases} 
\frac{5}{2} + 2 & \text{for } n = 5; \\
\frac{3n}{4} & \text{for } n \text{ odd and } n \geq 6; \\
\end{cases}
\]

**Proof.**

**Case 1.** \(rc_4(Mo_5) = 5\). See that \(Mo_5 \equiv K_5\), according to (5), it is obtained that \(rc_4(Mo_5) = rc_4(K_5) \geq 2\left\lfloor \frac{5}{2} \right\rfloor - 1 = 5\). See monochromatic 5-coloring \((Mo_5)\) in figure 3, every pair of vertices on colored \(Mo_5\) has 4 internally disjoint rainbow paths, so that \(rc_4(Mo_n) \leq 5\). The proof for this case is complete.

**Case 2.** \(rc_4(Mo_n) = \frac{n}{2} + 2\) for \(n\) even and \(n \geq 8\). The proof for this case uses contradiction. If \(rc_4(Mo_n) \geq diam_4(Mo_n) = \frac{n}{2} + 1\). Build any \(Mo_n\), see vertices \(v_1\) dan \(v_3\) which have 4 internally disjoint rainbow paths, they are: \(v_1v_3; v_1v_2, v_2v_3; v_1v_4, v_4v_5, v_5v_6, v_6v_7; \) and \(v_2v_n, v_nv_{n-2}, v_{n-2}v_{n-4} ..., v_4v_3\). Without loss of generality, choose the longest \(v_1 - v_3\) path and assume that \(a(v_1v_{i+2}) = \frac{i}{2}\) for \(i\) even and \(4 \leq i \leq n - 2\), \(a(v_3v_4) = 1\), and \(a(v_nv_1) = \frac{n}{2}\). See \(v_nv_2\), we divide into two subcases.

**Subcase 2.1.1.** If \(a(v_nv_2) = \frac{n}{2}\), the contradiction happens because \(v_1\) and \(v_2\) can’t have 4 rainbow paths that internally disjoint each other.

**Subcase 2.1.2.** If \(a(v_nv_2) = i\), with \(i\) natural number \(2 \leq i \leq \frac{n}{2} - 1\), the contradiction happens because \(v_2\) and \(v_4\) can’t have 4 rainbow paths that internally disjoint each other.

So that \(a(v_nv_2) = 1\). Next, see \(v_2v_4\), we divide into two subcases.

**Subcase 2.2.1.** If \(a(v_2v_4) = 1\), the contradiction happens because \(v_2\) and \(v_3\) can’t have 4 rainbow paths that internally disjoint each other.

**Subcase 2.2.2.** If \(a(v_2v_4) = i\), with \(i\) natural number \(2 \leq i \leq \frac{n}{2} - 1\), the contradiction happens because \(v_2\) and \(v_4\) can’t have 4 rainbow paths that internally disjoint each other.

So that \(a(v_2v_4) = \frac{n}{2}\). Next, see \(v_1v_2\), we divide into two subcases.

**Subcase 2.3.1.** If \(a(v_1v_2) = 1\), the contradiction happens because \(v_1\) and \(v_2\) can’t have 4 rainbow paths that internally disjoint each other.
Subcase 2.3.2. If \( a(v_1 v_2) = \frac{n}{2} \), the contradiction happens because \( v_1 \) and \( v_2 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.3.3. If \( a(v_1 v_2) = i \), with \( i \) natural number \( 2 \leq i \leq \frac{n}{2} - 2 \), the contradiction happens because \( v_1 \) and \( v_{n-1} \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_1 v_2) = \frac{n}{2} - 1 \). Next, see \( v_2 v_3 \), we divide into two subcases.

Subcase 2.4. For \( n = 6 \).

Subcase 2.4.1. If \( a(v_2 v_3) = 1 \), the contradiction happens because \( v_3 \) and \( v_6 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.4.2. If \( a(v_2 v_3) = 2 \), the contradiction happens because \( v_3 \) and \( v_1 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.4.3. If \( a(v_2 v_3) = 3 \), the contradiction happens because \( v_3 \) and \( v_4 \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_2 v_3) = 4 \). Next, see \( v_3 v_6 \), we divide into 4 subcases.

Subcase 2.4.4. If \( a(v_3 v_6) = 1 \), the contradiction happens because \( v_2 \) and \( v_5 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.4.5. If \( a(v_3 v_6) = 2 \), the contradiction happens because \( v_3 \) and \( v_5 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.4.6. If \( a(v_3 v_6) = 3 \), the contradiction happens because \( v_1 \) and \( v_3 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.4.7. If \( a(v_3 v_6) = 4 \), the contradiction happens because \( v_1 \) and \( v_5 \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_3 v_6) = 5 \). The proof for this subcase is complete.

Subcase 2.5. For even \( n \geq 8 \).

Subcase 2.5.1. If \( a(v_2 v_3) = 1 \) or \( a(v_2 v_3) = i \), with \( i \) natural number \( 3 \leq i \leq \frac{n}{2} - 1 \), the contradiction happens because \( v_3 \) and \( v_5 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.5.2. If \( a(v_2 v_3) = \frac{n}{2} \), the contradiction happens because \( v_3 \) and \( v_4 \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_2 v_3) = 2 \). Next, see \( v_5 v_6 \), we divide into two subcases.

Subcase 2.6.1. If or \( a(v_2 v_3) = i \), with \( i \) natural number \( 1 \leq i \leq \frac{n}{2} - 1 \), the contradiction happens because \( v_3 \) and \( v_5 \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.6.2. If \( a(v_5 v_6) = \frac{n}{2} \), the contradiction happens because \( v_{n-1} \) and \( v_6 \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_5 v_6) = \frac{n}{2} + 1 \). Next, see \( v_{n-2} v_{n-1} \), we divide into two subcases.

Subcase 2.7.1. If \( a(v_2 v_3) = i \), with \( i \) natural number \( 2 \leq i \leq \frac{n}{2} \), the contradiction happens because \( v_1 \) and \( v_{n-1} \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.7.2. If \( a(v_{n-2} v_{n-1}) = 1 \), the contradiction happens because \( v_3 \) and \( v_{n-1} \) can’t have 4 rainbow paths that internally disjoint each other.

Subcase 2.7.3. If \( a(v_{n-2} v_{n-1}) = \frac{n}{2} + 1 \), the contradiction happens because \( v_5 \) and \( v_{n-1} \) can’t have 4 rainbow paths that internally disjoint each other.

So that \( a(v_{n-2} v_{n-1}) = \frac{n}{2} + 2 \). Therefore, \( rc_4(Mo_n) \geq \frac{n}{2} + 2 \) for \( n \) even number and \( n \geq 6 \).

Then, The proof for the upper bound \( rc_4(Mo_n) \geq \frac{n}{2} + 2 \) is using a coloring definition. Define an edge coloring \( a: E(Mo_n) \rightarrow [1, n + 2] \) as follows:
\[ a(v_i v_{i+1}) = \left\lceil \frac{n}{2} \right\rceil + 1, \text{ for } i \text{ odd number and } 1 \leq i \leq n - 1; \]
\[ a(v_i v_{i+1}) = \left\lceil \frac{n}{2} \right\rceil + 2, \text{ for } i \text{ even number } 2 \leq i \leq n - 2; \]
\[ a(v_i v_{i+2}) = \left\lceil \frac{n}{2} \right\rceil, \text{ for } 1 \leq i \leq n - 2. \]

For each pair of vertices \( u \) and \( v \), there are four (4) rainbow paths which are presented in Table 1. Based on Table 1 above, it can be concluded that \( r_c(M_{\alpha_0}) \geq \left\lceil \frac{n}{2} \right\rceil + 2 \), for \( n \) even number and \( n \geq 6 \). As the example, see the rainbow 6-coloring graph \( M_{\alpha_0} \) in Figure 4.

**Table 1.** The rainbow paths for Case 2 and Case 3.

| \( u \) | \( v \) | Conditions | Rainbow Paths |
|---|---|---|---|
| \( v_i \) | \( v_j \) | \( n \equiv 0 \mod 2, i < j, \text{ and } |i-j| = 1 \) | \( v_i v_j \) |
| \( v_i \) | \( v_j \) | \( n \equiv 1 \mod 2, i < j, \text{ and } |i-j| = 1 \) | \( v_i v_j \) |
| \( v_i \) | \( v_j \) | \( n \equiv 0 \mod 2, |i-j| \geq 2, \text{ and } i \text{ and } j \text{ have similar parity} \) | \( v_i v_{i+2}, \ldots, v_j v_{j-4} v_{j-2} v_i, v_j v_{j-2} v_{j-4} \)
| \( v_i \) | \( v_j \) | \( n \equiv 1 \mod 2, |i-j| \geq 2, \text{ and } i \text{ and } j \text{ have different parity} \) | \( v_i v_{i+2}, \ldots, v_j v_{j-3} v_{j-1} v_{j-5}, v_{j-5} v_{j-3} v_{j-1} v_i \)

**Case 3.** \( r_c(M_{\alpha_0}) \leq \left\lfloor \frac{3n}{4} \right\rfloor \), for \( n \) odd and \( n \geq 7 \). This case is divided into two subcases.

**Subcase 3.1.** \( n \equiv 1 \mod 4 \). Define a coloring \( a : E(M_{\alpha_0}) \to \left\lfloor \frac{3n}{4} \right\rfloor \) as follows:

\[ a(v_i v_3) = \left\lceil \frac{n}{2} \right\rceil; \]
\[ a(v_i v_{i+1}) = \left\lceil \frac{n}{4} \right\rceil, \text{ for } i \text{ odd and } 1 \leq i \leq n; \]
\[ a(v_i v_{i+1}) = \left\lceil \frac{n+i+2}{4} \right\rceil, \text{ for } i \text{ even and } 2 \leq i \leq n - 1; \]
\[ a(v_i v_{i+2}) = \left\lceil \frac{n+i}{4} \right\rceil, \text{ for } i \equiv 0 \mod 4 \text{ and } 4 \leq i \leq n - 1; \]
\[ a(v_i v_{i+2}) = \left\lceil \frac{i}{4} \right\rceil, \text{ for } i \equiv 1 \mod 4 \text{ and } 5 \leq i \leq n; \]
\[ a(v_i v_{i+2}) = \left\lceil \frac{n+i}{2} \right\rceil, \text{ for } i \equiv 2 \text{ mod } 4 \text{ or } i \equiv 3 \text{ mod } 4 \text{ with } 2 \leq i \leq n - 1. \]

**Subcase 3.2.** \( n \equiv 3 \text{ mod } 4. \) Define a coloring \( a: E(M_{on}) \to \left[ 1, \frac{3n}{4} \right] \) as follows:

\[
a(v_1 v_3) = \left\lfloor \frac{n}{2} \right\rfloor; a(v_2 v_n) = \left\lceil \frac{n+2}{2} \right\rceil; \\
a(v_i v_{i+1}) = \left\lfloor \frac{i}{4} \right\rfloor, \text{ for } i \text{ odd and } 1 \leq i \leq n; \\
a(v_i v_{i+1}) = \left\lceil \frac{n+i}{4} \right\rceil, \text{ for } i \text{ even and } 2 \leq i \leq n - 1; \\
a(v_i v_{i+2}) = \left\lfloor \frac{n+i}{4} \right\rfloor, \text{ for } i \equiv 2 \text{ mod } 4 \text{ and } 6 \leq i \leq n - 1; \\
a(v_i v_{i+2}) = \left\lceil \frac{i}{4} \right\rceil, \text{ for } i \equiv 1 \text{ mod } 4 \text{ and } 5 \leq i \leq n - 2; \\
a(v_i v_{i+2}) = \left\lfloor \frac{n+i+1}{2} \right\rfloor, \text{ for } i \equiv 3 \text{ mod } 4 \text{ or } i \equiv 0 \text{ mod } 4 \text{ with } 3 \leq i \leq n - 3; \]

Next, it will show each pair of vertices \( u \) and \( v \) in \( V(G) \) that has 4 rainbow paths which are internally disjoint each other. Based on table 1 and a coloring function on **Case 3**, we get \( r_{c4}(M_{on}) \leq \left\lceil \frac{3n}{4} \right\rceil. \) For example of this coloring can be seen at figure 5. Type equation here.

### 4. Conclusion

Based on the two theorems that we discussed earlier, the application of monochromatic connection numbers in the network in the form of moon graphs results in an efficiency of around 50%, while the use of rainbow 4-connection numbers results in an efficiency of around 62.5% to 75%.

**Discussion.** The number of sides of the moon graph is 2n. if the calculation of the number of passwords uses the monochromatic connection number which is \( n+2 \), then the ratio is around 50%. if it uses the rainbow 4-connection number used as much or to calculate the number of passwords, the ratio with the number of sides is around 37.5% or 25%.

**Open Problem:** For further research, we can research to determine the exact results of the rainbow 4-connection numbers for \( M_{on} \) with \( n \) odd number and \( n \geq 7 \).

![Figure 5](image-url) A 3-rainbow 9-coloring of \( M_{o11} \) The rainbow 10-coloring of \( M_{o13} \).
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