De-noising procedures for frame operators

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Abstract

The present paper provides a comprehensive study of de-noising properties of frames and, in
particular, tight frames, which constitute one of the most popular tools in contemporary signal
processing. The objective of the paper is to bridge the existing gap between mathematical and
statistical theories on one hand and engineering practice on the other and explore how one
can take advantage of a specific structure of a frame in contrast to an arbitrary collection of
vectors or an orthonormal basis. For both the general and the tight frames, the paper presents
a set of practically implementable de-noising techniques which take frame induced correlation
structures into account. These results are supplemented by an examination of the case when
the frame is constructed as a collection of orthonormal bases. In particular, recommendations
are given for aggregation of the estimators at the stage of frame coefficients. The paper is
concluded by a finite sample simulation study which confirms that taking frame structure and
frame induced correlations into account indeed improves de-noising precision.

Keywords: frames, tight frames, shrinkage, thresholding, aggregation

AMS (2000) Subject Classification: Primary 62G08; Secondary 42C15

1 Introduction

In the recent years, there has been resurgence of interest in de-noising by frames spanning different
communities. The effort was undertaken by mathematicians working in the area of approximation
theory, by statistics and computer science communities (the “large-p, small-n” problem and model
selection), and by engineering community (regularization theory and sparse coding of signals and
images).

The need for overcomplete representations stems from the fact that, though a single orthon-
ogonal basis allows very fast computations, it very often fails to efficiently represent a function of
interest, f, so that one needs a large number of coefficients to transmit or store. In fact, if f is
expanded over a much more exhaustive dictionary with p elements, it very often can be represented
with a very few nonzero coefficients. Moreover, one can reduce the error of representation beyond
what is possible when one orthonormal basis is used.

Mathematicians and statisticians dealt with this problem for years. However, the methods
which they designed were intended for an arbitrary dictionary and did not take advantage of their
particular structure. For this reason, those methods work very well in a regression-type set up when one does not need to obtain results instantaneously.

One of the most popular groups of methods relies on minimizing the difference between the function $f$ and its representation under some set of constraints. From the point of view of optimization theory this problem can be re-formulated as the problem of minimization of penalized risk of the representation of $f$. Various choices of penalties and risk functions were suggested leading to RIDGE regression (see, e.g., Brown and Zidek (1980), BRIDGE regression (Frank and Friedman (1993)), LASSO (Tibshirani (1996)), Dantzig selector (Candes and Tao (2007)), the least angle regression (Efron et al. (2004) and Support Vector regression (Smola and Schölkopf (2004)) among others.

Such methods neither assume nor exploit any specific structure of the dictionary and, as a result, are very computationally expensive. For this reason, those methods cannot be used for real-time problems and, as a result, are not very popular in practical engineering applications.

For many years, engineers have been using frames, especially, tight frames due to their simple reconstruction properties. However, when it comes to de-noising, engineers routinely treat frames, especially tight frames, as if they were orthonormal bases completely ignoring correlations between frame vectors and using thresholding methodologies developed for the case of orthonormal bases. This sentiment is well expressed in a recent paper of Yu, Mallat and Bacry (2008) which states that “a tight frame behaves like a union of $\alpha$ orthogonal bases.” For this reason, there is a multitude of engineering papers where methodologies designed for orthonormal basis are applied to frames without any consideration of frame structure.

There have been a growing sentiment in the statistics community that discounting correlations associated with frames reduce de-noising precision. Few authors focus their attention on universal threshold which is frequently used by engineers in the context of frames as if all frame functions are independent which leads to the threshold which is too large. Among them, Downie and Silverman (1998) considered multiwavelets which constitute a particular type of a frame. Walker and Chen (2010) studied universal thresholding in the case of Gabor frames with a Blackman window. Recently, Haltmeier and Munk (2012) derived the universal threshold for a general frame satisfying rather stringent conditions which ensure that the threshold depends on the number of frame functions but not on frame structure.

However, to the best of our knowledge, there have never been a comprehensive study of a de-noising properties of frames and, in particular, tight frames, which constitute one of the most popular tools in contemporary signal processing. Note that general statistical methods designed for correlated noise do not lead to fast computations and are impractical in this set up since the covariance matrix is too big. The objective of the present paper is to bridge the existing gap between mathematical and statistical theories on one hand and engineering practice on the other and explore how one can take advantage of a specific structure of a frame in contrast to an arbitrary collection of vectors or an orthonormal basis. In particular, the purpose of this paper is to provide a set of practically implementable de-noising techniques which take correlation structure of the frame coefficients into account.

In pursuing this goal, we start with derivation of the oracle (best in the mean square sense) linear diagonal shrinkage estimator. It turns out that, by construction, the i-th element of the diagonal shrinkage matrix depends not only on the i-th frame coefficient but also on coefficients related to it (not necessary in its vicinity). In this sense, the oracle can be regarded as a block shrinkage procedure, where the length and the constitution of the block is automatically determined by the correlation structure of frame coefficients induced by the frame transform. The oracle is followed by derivation of the Stein Unbiased Risk Estimator (SURE) in the case of a general frame and an arbitrary de-noising strategy. The SURE formulation is very similar to the one obtained
by Blu and Luisier (2007) for the case of interscale image de-noising. Next, we use this result for designing particular types of de-noising algorithms (linear shrinkage, soft or hard thresholding, etc.) The SURE provides a good assessment tool for de-noising with any kind of a frame and can, in fact, be used for construction of frames with specific properties. It also leads to fast computational procedures and, at the same time, better de-noising precision since it exploits both the sparsity and the correlation structure of the frame. Subsequently, we explore the case of a tight frame and show how the techniques suggested for general frames are naturally simplified and speeded up in this situation.

Finally, we consider the case when a tight frame is formed as a collection of orthonormal bases. In this situation, hypothetically, one can obtain estimators for each of the orthonormal bases separately and then combine them with weights which sum up to unity. Normally, in engineering practice, the estimators are just combined with equal weights as it is done, for example, in cycle spinning. However, one can use different weights with the objective of obtaining an estimator with better risk properties. The set of methods which exploit this idea is called aggregation and was studied extensively by statistics community in the last decade (see, for instance, Bunea and Nobel (2008), Bunea, Tsybakov and Wegkamp (2007), Gribonval (2003), Guleryuz (2007), Juditsky and Nemirovski (2000), Juditsky, Rigollet, and Tsybakov (2008), Leung and Barron (2006), Wegkamp (2003) and Yang (2001)).

Nevertheless, aggregation techniques have various limitations which make them unsuitable for engineering practice. The existing techniques treat estimators as constant (the risk is conditioned on those estimators) or require sequential constructions of regression estimators and, in both cases, lead to expensive computational procedures. In particular, the algorithm of Bunea, Tsybakov and Wegkamp (2007) involves high-dimensional optimization which is impossible to carry out in real-time computations. Leung and Barron (2006), on the other hand, treat each of the regression estimators as variable and work out an oracle expression for the risk which allows them to offer an explicit choice of weights. However, due to the fact that the estimators are combined at the final stage, the authors cannot take full advantage of their approach and are able to combine only one type of estimators, the least squares estimators, in particular, the least squares estimators based on one basis function each.

In what follows, we take a more general and flexible approach to the aggregation problem. We study the situation when both the risk and the estimators are variable and they are combined before reconstruction, at the stage of frame coefficients. In particular, we assume that a tight frame is constructed as a collection of orthonormal bases. The frame coefficients are subsequently de-noised and, finally, the function is re-constructed using variable weights for each of the bases. Using results of the earlier parts of the paper, we derive an oracle expression for the risk which is not conditioned on a particular estimation strategy and can take into account any explicit de-noising technique. Moreover, unlike in Leung and Barron (2006) and other aggregation papers, we derive an expression which contains unknown weights in explicit form, making it easier to carry out necessary optimization. Furthermore, our approach allows one to explore both the situation of data independent weights (or fixed estimators) and data dependent weights. In the former case, we validate one of the main reasons for popularity of frames in engineering. Indeed, we show that, if the frame is constructed as a combination of orthonormal bases, then the risk of any frame estimator obtained as a linear combination of the estimators in each basis is smaller than the linear combination of the risks.

The rest of the paper is organized as follows. Section 2 presents oracle expressions for the mean squared risks of the diagonal shrinkage and thresholding estimators in the case of general or tight frames. Section 3 provides SURE rules for those estimators. Results obtained in Sections 2
and are used in Section 4 for designing optimal thresholding or shrinkage algorithms. Section 5 treats the case when the frame is constructed as a collection of orthonormal bases, in particular, it gives recommendation how the estimators can be aggregated at the frame coefficient stage, before reconstruction. Section 6 studies performances of the methodologies developed in the paper via numerical simulations carried out on test and real signals. Section 7 concludes the paper with the discussion. Finally Section 8 contains the proofs of the statements presented in the paper.

2 Oracle expression for the risk for general or tight frames

A collection of functions \(\{w_i\}\) form a frame in a separable Hilbert space \(H\) if there exist two positive frame bounds \(C_l\) and \(C_u > 0\) such that, for any \(f \in H\),

\[
C_l \|f\|^2 \leq \sum_i |(f, w_i)|^2 \leq C_u \|f\|^2.
\]

(2.1)

As particular cases of frames one can list Gabor frames, in which set \(\{w_i\}\) comprises translated and modulated versions of the same function, short time (or windowed) Fourier transform and wavelet frames.

In the space of discrete signals of length \(n\), one usually considers \(N\) vectors \(w_i \in \mathbb{C}^n\), \(i = 1, \cdot \cdot \cdot, N\), which together form matrix \(W \in \mathbb{C}^{N \times n}\). In these notations, (2.1) implies that \(W\) is a matrix of a frame operator if for any \(f \in L_2(\mathbb{R}^n)\) one has

\[
C_l \|f\|^2 \leq \langle f, W^* W f \rangle \leq C_u \|f\|^2
\]

(2.2)

where \(W^*\) is a transpose conjugate of \(W\). The latter guarantees that eigenvalues of matrix \(V = W^* W\) are bounded above and below and, therefore, \(V\) is invertible.

If frame bounds are equal to each other, \(C_l = C_u = \alpha\), then the frame is called tight and \(\alpha\) is referred to as a frame constant. In the case of a tight frame the generalized Parseval’s identity holds and \(W^* W\) is proportional to the identity matrix. In what follows, we shall assume that if the frame is tight, then \(W^* W = \alpha I_n\). However, a tight frame can also be normalized so that \(\alpha = 1\), as it is done for the Gabor frame which is used for simulations in Section 6.

Consider a problem of recovering vector \(f \in \mathbb{R}^n\) from its noisy observation

\[
x = f + \delta, \ \delta \sim N(0, \sigma^2 I_n).
\]

(2.3)

Applying frame transform \(W\) to both sides of equation (2.3), obtain

\[
y = \theta + \varepsilon, \ \varepsilon \sim N(0, \sigma^2 U)
\]

(2.4)

where \(y = W x, \ \theta = W f, \ \varepsilon = W \delta\) and \(U = W W^* \in \mathbb{C}^{N \times N}\). The goal of the analysis is to reduce noise in the vector of frame coefficients \(y\) by shrinking or thresholding its components, thus, obtaining vector \(\hat{\theta}\) and, subsequently, to estimate \(f\) by

\[
\hat{f} = V^{-1} W^* \hat{\theta} = W^+ \hat{\theta},
\]

(2.5)

where \(W^+ = (W^* W)^{-1} W^*\) is the Moore-Penrose inverse of matrix \(W\).

We assume that the vector of frame coefficients \(\theta\) is estimated by \(\hat{\theta} = \Gamma y\) where \(\Gamma = \text{diag}(\gamma_1, \cdot \cdot \cdot, \gamma_N)\) is a fixed diagonal matrix in \([0, 1]^{N \times N}\). The next statement provides an oracle expression for the risk of this estimator.
Theorem 1 If \( \hat{\theta} = \Gamma y \) where \( \Gamma \) is a fixed diagonal matrix, then

\[
E\|\hat{f} - f\|^2 = \text{Tr}[U^{-}(I_N - \Gamma)\theta\theta^*(I_N - \Gamma) + \sigma^2\Gamma U U^{-}].
\] (2.6)

If the frame is tight, the previous expression takes the form

\[
E\|\hat{f} - f\|^2 = \alpha^{-2} \text{Tr}[U(I_N - \Gamma)\theta\theta^*(I_N - \Gamma) + \sigma^2\Gamma U U].
\] (2.7)

Here, \( U = WW^* \) and \( U^{-} = (W^+)^*W^+ \).

Proofs of this and later statements are given in Section 8.

Note that expressions (2.6) and (2.7) require simple minimization of quadratic forms due to the following identity

\[
\text{argmin}_{\Gamma = \text{diag}(\gamma)} \{ \text{Tr}[U^{-}(I_N - \Gamma)\theta\theta^*(I_N - \Gamma) + \sigma^2\Gamma U U^{-}] \} = \text{argmin}_\gamma \{ \gamma^* A \gamma - 2\gamma^* \hat{b} \} \] (2.8)

where \( A = (\theta\theta^*) \circ U^{-} + \sigma^2(U \circ U^{-}), \hat{b} = ((\theta\theta^*) \circ U^{-}) e_N, e_N \) is the vertical vector with all components equal to one and \( \circ \) denotes the Hadamard (element-wise) matrix product. According to identity (2.8), the optimal gain vector \( \gamma = \text{diag}(\Gamma) \) can be presented as

\[
\gamma = (\theta\theta^* \circ U^{-} + \sigma^2 U \circ U^{-})^{-1} (\theta\theta^* \circ U^{-}) e_N
\]

and, in the case of tight frame, it takes the form

\[
\gamma = (\theta\theta^* \circ U + \sigma^2 U \circ U)^{-1} (\theta\theta^* \circ U) e_N. \] (2.9)

It is worth noting that, by construction, the weights \( \gamma_i \) in the best linear diagonal estimator are functions not only of \( \theta_i \) but also of other coefficients in its neighborhood. In this sense, the best linear diagonal estimator is no longer diagonal and represents an overlapping block shrinkage procedure where the length of the block is automatically determined by the correlations induced by the frame operator. Moreover, observe that matrix \( A \) is invertible since the Hadamard product of two positive-definite matrices is positive-definite. Moreover, matrix \( U^{-} \) usually has a block structure, so that the inversion of \( A \) could be carried out by fast algorithms specifically designed for this case.

According to Theorem 1 for hard thresholding one needs to minimize risk (2.6) or (2.7) over the set of arbitrary diagonal matrices with zero or unit values. Observe that in the case of an orthonormal basis, the oracle (2.7) takes a familiar form

\[
E\|\hat{f} - f\|_{\text{hard}}^2 = \sum_{i=1}^n [\theta_i^2 I(\gamma_i = 0) + \sigma^2 I(\gamma_i = 1)]
\]

and motivates one to keep larger coefficients and discard smaller ones irrespective of the particular value of matrix \( W \). The situation changes when matrix \( W \) ceases to be unitary. Indeed, Theorem 1 implies that the choice of coefficients to “keep” or “kill” depends not only on their values but also on the entries of matrix \( U \).
3 SURE rules for general or tight frames

The advantage of the oracle expressions is that they allow to construct unbiased estimators for the risk. Indeed, matrix $\Theta = \theta^* \theta^*$ can be written as $\Theta = \mathbb{E}(yy^*) - \sigma^2 U$ and estimated by $\hat{\Theta} = yy^* - \sigma^2 U$. The latter leads to the following unbiased estimator for the risk:

**Corollary 1** If $\hat{\theta} = \Gamma y$ where $\Gamma$ is a fixed diagonal matrix, then

$$\mathbb{E}\|\hat{f} - f\|^2 = \sigma^2 n + \mathbb{E}\Delta$$

(3.1)

where

$$\Delta = y^*(I_N - \Gamma)U^-(I_N - \Gamma)y - 2\sigma^2 \text{Tr}[U^- U (I_N - \Gamma)].$$

(3.2)

In particular, if $\Gamma$ induces a hard thresholding rule, i.e. $\gamma_i = 1$ or $0$, then

$$\Delta = \sum_{i,j=1}^N \left[ y_i y_j U_{ij}^- - 2\sigma^2 (U^- U)_{ii} \mathbb{I}(i = j) \right] \mathbb{I}(\gamma_i = 0) \mathbb{I}(\gamma_j = 0).$$

(3.3)

Since matrix $\Theta$ is non-negative definite, all its diagonal elements should be non-negative which leads to the relations

$$\hat{\Theta}_{ii} = y_i^2 - \sigma^2 U_{ii} \geq 0.$$

These inequalities themselves enforce hard thresholds $\sigma \sqrt{U_{ii}}$ on the values of $y_i$. The oracle expression (2.7) allows for further reduction of the risk.

The oracles (2.6) and (2.7), though, are of limited value since they do not allow one to access risk of more sophisticated rules where matrix $\Gamma$ itself depends on $y$. In this case, one can write $\hat{\theta}$ as

$$\hat{\theta} = y + g(y).$$

(3.4)

Then, using modification of SURE, one obtains the following result:

**Theorem 2** Let the data follow model (2.3) and $y$ be of the form (2.4). Let $\hat{f}$ be given by formula (2.5) with $\hat{\theta}$ of the form (3.4) where $g(y) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuous and piecewise differentiable column vector function. Let $Z = \nabla_y g^*(y)$ be an $N \times N$-dimensional matrix with components

$$Z_{ij} = \frac{\partial}{\partial y_i} [g_j(y)].$$

(3.5)

Then, the mean quadratic risk is given by expression (3.1) with

$$\Delta = g^*(y) U^- g(y) + 2\sigma^2 \text{Tr}[U^- U Z].$$

(3.6)

In the frame is tight, then $U^- = \alpha^{-2} U$ and

$$\Delta = \alpha^{-2} g^*(y) U g(y) + 2\sigma^2 \alpha^{-1} \text{Tr}[U Z].$$

(3.7)

Note that Theorem 2 allows one to obtain explicit expressions for various type of thresholding or shrinkage procedures, as well as to construct unbiased estimators of the risk of those procedures. If one uses linear shrinkage $\Gamma$, then $g(y) = (\Gamma - I_N)y$ and $Z = \Gamma - I_N$, so that Theorem 2 recovers expression (3.2) for $\Delta$.

In the case of soft thresholding with variable threshold $t_i$, one has $\hat{\theta}_i = (y_i - \text{sgn}(y_i)t_i)\mathbb{I}(|y_i| > t_i)$, so that $g_i(y)$ is of the form

$$g_i(y) = -\text{sgn}(y_i) \min(|y_i|, t_i), \quad i = 1, \ldots, N.$$

(3.8)
Hence, $Z$ is a diagonal matrix with elements

$$Z_{ii} = -\mathbb{I}(|y_i| < t_i) \quad (3.9)$$

and the following corollary is valid.

**Corollary 2** If $g(y)$ is defined by (3.8), then the risk is of the form (3.1) with

$$\Delta = N \sum_{i,j=1}^{N} \left[ \text{sgn}(y_i y_j) \min(|y_i|, t_i) \min(|y_j|, t_j) U_{ij} - 2\sigma^2(U^- U)_{ii} \mathbb{I}(i=j) \mathbb{I}(|y_i| < t_i) \right]. \quad (3.10)$$

If the frame is tight, the previous expression simplifies to

$$\Delta = N \sum_{i,j=1}^{N} \left[ \alpha^{-2} \text{sgn}(y_i y_j) \min(|y_i|, t_i) \min(|y_j|, t_j) U_{ij} - 2\sigma^2\alpha^{-1} U_{ii} \mathbb{I}(i=j) \mathbb{I}(|y_i| < t_i) \right]. \quad (3.11)$$

It is easy to check that, in the case of an orthonormal basis, familiar expressions for the risks can be easily recovered from formula (3.11). Indeed, setting $n = N$, $\alpha = 1$ and $U = I_n$, as before, one obtains:

$$\mathbb{E} \|\hat{f} - f\|_{\text{soft}}^2 = \sigma^2 n + \mathbb{E} \left[ \sum_{i=1}^{n} \min(y_i^2, t_i^2) - 2\sigma^2 \sum_{i=1}^{n} \mathbb{I}(|y_i| < t_i) \right].$$

4 Designing optimal thresholding or shrinkage algorithms

One can use expressions derived in the previous section to design an optimal shrinkage or thresholding strategy. In what follows, just to be specific, we consider the case of linear shrinkage and hard thresholding only. Other shrinkage or thresholding techniques including soft thresholding can be analyzed in a similar manner. Note that since matrices $U$, $U^-$ and $U^- U$ are not data-dependent, they may be calculated in advance and, thus, the main computational complexity lies in solving the resulting optimization problems.

4.1 Linear shrinkage

Recall that the risk is of the form (3.1) where $\Delta = \text{Tr}[U^- (I_N - \Gamma) yy^* (I_N - \Gamma) + 2\sigma^2 U^- U \Gamma - 2\sigma^2 U^- U]$. Since the last term of the expression for $\Delta$ is independent of $\Gamma$, one needs to minimize

$$F(\Gamma) = \text{Tr}[U^- (I_N - \Gamma) yy^* (I_N - \Gamma) + 2\sigma^2 U^- U \Gamma].$$

Direct calculations show that this minimization takes the form of a quadratic programming problem. Indeed, if one define matrix $A = (yy^*) \circ U^-$ and vectors $\gamma = \text{diag}(\Gamma)$ and $b = (A - \sigma^2 U \circ U^-) e_n$, then

$$F(\Gamma) \equiv F(\gamma) = \gamma^* A \gamma - 2\gamma^* b, \quad (4.1)$$

and the optimal $\gamma \in [0, 1]^N$ which minimizes $F(\gamma)$ takes the form

$$\gamma = ((yy^*) \circ U^-)^{-1} ((yy^*) \circ U^- - \sigma^2 U \circ U^-) e_N. \quad (4.2)$$

Since matrices $U$, $U^-$ and $yy^*$ are nonnegative definite and Hermitian, matrix $A$ is also nonnegative definite and Hermitian, and, thus, the quadratic programming problem is convex. Furthermore,
note that matrix $A$ and vector $b$ are sparse. For example, in the case of a tight frame, expressions for $A$ and $b$ take the forms $A = \alpha^{-2}(yy^*) \circ U$ and $b = (A - \sigma^2 U \circ U)e_N$. Since the majority of entries of matrix $U$ are equal to zero, respective entries of matrix $A$ also vanish.

The optimization problem (4.1) can also be modified by adding a penalty term $\text{pen}(\gamma)$ to $F(\gamma)$. In particular, one can use a quadratic penalty term $\gamma^*P\gamma$ with the positive definite matrix $P$ or an $\ell_p$ penalty of the form $\text{pen}(\gamma) = \beta \|\gamma\|_{\ell_p}$ where $\| \cdot \|_{\ell_p}$ is a vector norm in $\ell_p$ space, which induces sparsity whenever $0 \leq p \leq 1$.

4.2 Hard thresholding

In the case of hard thresholding, the SURE is of the form (3.1) with $\Delta$ given by expression (3.3). In order to minimize expression for $\Delta$ in (3.3), introduce matrix $H$ with components

$$H_{ij} = \begin{cases} \frac{y_i y_j}{y_i^2} U_{ij} & \text{if } i \neq j, \\ \frac{y_i^2}{2(U-U)_{ii}} & \text{if } i = j \end{cases}$$

Consider a set of indices $J$ such that $j \in J$ if $\gamma_j = 0$ and $j \notin J$ otherwise. Then $\Delta$ can be re-written as

$$\Delta = \sum_{i,j \in J} H_{ij}$$

and the goal is to find a set of indices $J$ such that the sum of respective row and column elements of matrix $H$ is minimal. This minimizations can be accomplished by a kind of a greedy algorithm which can be carried out as follows.

Greedy algorithm

1. Since diagonal values of matrix $H$ are counted once while all other elements are counted twice, introduce modified matrix $\tilde{H}$ with elements

$$\tilde{H}_{ij} = \begin{cases} H_{ij}, & \text{if } i \neq j, \\ H_{ij}/2 & \text{if } i = j \end{cases}$$

Set $J = \{1, \cdots, N\}$.

2. Find a column $l$ of $\tilde{H}$ with the maximum sum of elements.

3. If the sum of elements of column $l$ is positive, then eliminate column $l$ and row $l$ from $\tilde{H}$ and index $l$ from set $J$, and RETURN TO STEP 2. If the sum of elements of column $l$ is zero or negative, then FINISH.

4. Set $\gamma_j = 0$ if $j \in J$ and $\gamma_j = 1$ if $j \notin J$.

5 Frame constructed as a collection of orthonormal bases

Consider the case when a frame is constructed as a collection of $m$ orthonormal bases. In this case, $N = nm$ and matrix $W$ has a block structure with $m$ vertical blocks $W^{(i)} \in C^{n \times n}$, $i = 1, \cdots, m$, such that $W^{(i)}(W^{(i)})^* = (W^{(i)})^*W^{(i)} = I_n$. Denote $U^{(i,j)} = W^{(i)}(W^{(j)})^*$. Then, matrix $U$ is a block matrix with blocks $U^{(i,j)}$, $i, j = 1, \cdots, m$, and $U^{(i,i)} = I_n$. It is easy to see that $W$ constitutes a tight frame with $\alpha = m$.

An interesting phenomenon for the frame of this type is that, since each of matrices $W^{(i)}$ allows complete reconstruction of $f$, one can combine those reconstructions with non-equal weights.
Let $\Lambda$ be a block-diagonal matrix with blocks $\Lambda^{(i,i)} = \lambda_i I_n$, $i = 1, \cdots, m$, where weights $\lambda_i$ sum to unity:

$$\sum_{i=1}^{m} \lambda_i = 1. \quad (5.1)$$

Note that, under condition $(5.1)$, one has

$$W^* \Lambda W = \sum_{i=1}^{m} (W^{(i)})^* \Lambda W^{(i)} = \sum_{i=1}^{m} \lambda_i (W^{(i)})^* W^{(i)} = I_n.$$ 

Therefore, if $\theta = W f$, then $f$ can be reconstructed as $f = W^* \Lambda \theta$ and estimated by

$$\hat{f} = W^* \Lambda \hat{\theta}. \quad (5.2)$$

Usually, in the current engineering practices, weights are chosen to be equal (as in, e.g., cycle spinning), however, this choice does not allow one to reduce the total risk by assigning a smaller weight to an estimator with a higher risk.

It is easy to see that the problem of choosing weights in this setup is ultimately related with aggregation problem studied in, for instance, Bunea and Nobel (2008), Bunea, Tsybakov and Wegkamp (2007), Gribonval (2003), Guleryuz (2007), Juditsky and Nemirovski (2000), Juditsky, Rigollet, and Tsybakov (2008), Leung and Barron (2006), Wegkamp (2003) and Yang (2001) among others. Indeed, note that one can consider an estimator of $f$ of the form $\hat{f}^{(i)} = (W^{(i)})^* \hat{\theta}^{(i)}$ where $\hat{\theta}^{(i)} = y^{(i)} + g^{(i)}(y^{(i)})$ is an estimator of $\theta^{(i)} = W^{(i)} f$, the coefficients of representation of $f$ in the $i$-th basis. Then, estimator $\hat{f}$ in $(5.2)$ can be re-written as

$$\hat{f} = \sum_{i=1}^{m} \lambda_i \hat{f}^{(i)}. \quad (5.3)$$

The difference between our approach and aggregation techniques, however, is that we carry out aggregation at the level of frame coefficients, not estimators of $f$ themselves. This will allow us to avoid, if desired, both conditioning on estimation technique (“constant estimators”) and treating weights as data-independent.

Nevertheless, we shall start with the case of data-independent weights and then investigate a more elaborate case where weights are data-dependent.

### 5.1 Data independent weights: better than the best basis

If the weights are data independent, then direct calculations show that

$$\mathbb{E} \| \hat{f} - f \|^2 = \sum_{i,j=1}^{m} \lambda_i \lambda_j \rho_{ij} \quad \text{with} \quad \rho_{ij} = \mathbb{E}[(\hat{\theta}^{(i)} - \theta^{(i)})^* U^{(i,j)}(\hat{\theta}^{(j)} - \theta^{(j)})]. \quad (5.4)$$

The statement below shows that the error of $\hat{f}$ is always smaller than the weighted sum of the errors of estimators $\hat{f}^{(i)}$.

**Theorem 3** If $\hat{f}$ is defined in $(5.2)$ and weights $\lambda_i$ are data independent and satisfy condition $(5.1)$, then

$$\mathbb{E} \| \hat{f} - f \|^2 = \sum_{i=1}^{m} \lambda_i \mathbb{E} \left[ \| \hat{f}^{(i)} - f \|^2 - \| \hat{f}^{(i)} - \hat{f} \|^2 \right], \quad (5.5)$$
so that if, furthermore, the weights are nonnegative,

$$\mathbb{E}\|\hat{f} - f\|^2 \leq \sum_{i=1}^{m} \lambda_i \mathbb{E}\|\hat{f}^{(i)} - f\|^2.$$  

Theorem 3 does not allow one to choose optimal weights, since weights enter expression (5.5) implicitly in the form of $\hat{f}$. In order to evaluate expression for the risk in the case of constant weights, one needs to combine expression (5.4) and formula (8.2) with $g^{(i)}(y^{(i)})$, $U^{(i,j)}$ and $Z^{(j,i)}$ instead of $g(y)$, $U$ and $Z$, respectively:

$$\mathbb{E}\|\hat{f} - f\|^2 = \sum_{i,j=1}^{m} \lambda_i \lambda_j \left\{ \sigma^2 \text{Tr}[U^{(i,j)} U^{(j,i)}] + \mathbb{E}[\{ (g^{(j)})^*U^{(j,i)}g^{(i)} \}^2 + 2\sigma^2 \text{Tr}[U^{(i,j)} Z^{(j,i)} U^{(j,i)}]\] \right\},$$

where for the sake of brevity, we denoted $g^{(i)}(y^{(i)}) = g^{(i)}$. Taking into account that $U^{(i,i)} = I_n$ and $U^{(i,j)} U^{(j,i)} = I_n$, one arrives at the risk of the form (3.1) with

$$\Delta = \sum_{i,j=1}^{m} \lambda_i \lambda_j (g^{(i)})^* U^{(i,j)} g^{(j)} + 2\sigma^2 \sum_{i=1}^{m} \lambda_i \text{Tr}[Z^{(i,i)}].$$  

(5.6)

The above expression contains weights in explicit form and allows to choose optimal weights for any kind of a shrinkage or thresholding technique. Observe that the choice of a “best” basis corresponds to one of the coefficients $\lambda_i$ being one and the others being zero. This would be an optimal choice if matrix $\rho$ with entries $\rho_{ij}$ defined in (5.4) were diagonal. However, since this is not the case, the choice of only one estimator versus a mixture may not be the best strategy, both, from the point of view of risk and even sparsity (see, e.g., Elad and Yavneh (2009)).

### 5.2 Data dependent weights

In order to study the case of data-dependent weights, recall that $y \in \mathbb{R}^N$ is a vector with $m$ block-components $y^{(i)} = \theta^{(i)} + \varepsilon^{(i)}$, where $\varepsilon^{(i)} \sim N(0, \sigma^2 I_n)$ and $\theta^{(i)}$ is estimated by $\hat{\theta}^{(i)} = y^{(i)} + g^{(i)}(y^{(i)})$. Introduce data dependent weights $\lambda_i(y)$ such that relation (5.1) is valid for any value of $y$.

Here, we ought to point out two essential features of our choice of weights. First, we explicitly choose weights depending on frame coefficients $y$ rather than raw data $x$. Second, weights for each basis depend on all frame coefficients. Re-writing (5.2), we obtain

$$\hat{f} = \sum_{i=1}^{m} \lambda_i(y) (W^{(i)})^* \hat{\theta}^{(i)} = \sum_{i=1}^{m} (W^{(i)})^* \left\{ \lambda_i(y) [y^{(i)} + g^{(i)}(y^{(i)})] \right\}.$$  

(5.7)

Note that $\hat{f}$ in the last expression can be presented as $\hat{f} = m^{-1}W^* \hat{\theta}$, the tight frame reconstruction of the estimator $\hat{\theta} = \hat{\theta}(y) = y + \tilde{g}(y)$ of frame coefficients. Here $\tilde{g}(y)$ is a block vector with blocks $g^{(i)}(y) = m \lambda_i(y) [y^{(i)} + g^{(i)}(y^{(i)})] - y^{(i)}$.

Hence, we can use expression (3.7) in Theorem 2 with $\alpha = m$ and $\tilde{g}$ and $\tilde{Z}$ instead of $g$ and $Z$, respectively.
\textbf{Theorem 4} If \( \hat{f} \) is defined in (5.2) and weights \( \lambda_i = \lambda_i(y) \) are data–dependent and satisfy condition (5.1) for every \( y \), then the risk is of the form (3.7) with
\[
\Delta = \sum_{i,j=1}^{m} \lambda_i(y)\lambda_j(y)(g^{(i)}y^{(j)})^*U^{(i,j)}y^{(j)} + 2\sigma^2 \sum_{i=1}^{m} \lambda_i(y) \text{Tr}(Z^{(i,i)}) + \Delta_0, \tag{5.8}
\]
\[
\Delta_0 = 2\sigma^2 \sum_{i,j=1}^{m} (\hat{\theta}^{(i)}y^{(i)})^*U^{(i,j)} \left[ \nabla y^{(j)} \lambda_i(y) \right].
\]

Here, same as before, \( \hat{\theta}^{(i)}y^{(i)} = y^{(i)} + g^{(i)}(y^{(i)}) \), and, for the sake of brevity, we omitted \( (y^{(i)}) \) in the expressions \( g^{(i)}(y^{(i)}) \) and \( (y^{(i)}) \).

Straightforward comparison shows that the first two terms in (5.8) coincide with the respective terms in (5.6) while the last term vanishes when the weights are data independent. Expression (5.8) contains weights explicitly, so, hypothetically, it can be used for choosing data dependent weights.

The difficulty with using formula (5.8), however, lies in the fact that one would like to choose weights depending not on frame coefficients \( y \) but rather on the risk of the \( i \)-th estimator \( \hat{\theta}^{(i)}y^{(i)} \), or, more precisely, on the Stein unbiased estimator of this risk. For this reason, one needs to learn how to find partial derivatives of the unbiased estimator of the risk, which is accomplished by the following statement.

\textbf{Lemma 1} Let the data follow model \( y = \theta + \varepsilon \) where \( \theta, \varepsilon \in \mathbb{R}^n \) and \( \varepsilon \sim N(0, \sigma^2 I_n) \). Let \( \hat{\theta} \) be an estimator of \( \theta \) of the form \( \hat{\theta}(y) = y + g(y) \). Then for the Stein unbiased estimator
\[
r(y) = \sigma^2 n + g^*(y)g(y) + 2\sigma^2 \text{Tr} \left[ \nabla g^*(y) \right]
\]
of the risk \( \mathbb{E} \| \hat{\theta}(y) - \theta \|^2 \) one has
\[
\nabla y r(y) = 2[\nabla g^*(y)]g(y) + 2\sigma^2 d(y) \tag{5.9}
\]
where \( d(y) \) is a column vector with components
\[
d_k(y) = \sum_{l=1}^{n} \frac{\partial^2 g_l(y)}{\partial y_l \partial y_k} \quad k = 1, \ldots, n. \tag{5.10}
\]

Recall that in the case of fixed linear shrinkage \( g_l(y) = (\Gamma_l - 1)y_l \) and for soft thresholding \( g_l(y) = -\text{sgn}(y_l) \text{min}(y_l, t) \), where \( t \) is the threshold, one has \( d(y) = 0 \). Therefore, in those two cases, \( \nabla y r(y) = 2[\nabla g^*(y)]g(y) \).

Expression (5.8) can be potentially used in order to minimize \( \Delta \) with respect to weights. However, this expression is too general to use. Hence, following Leung and Barron (2006), we consider weights in the exponential form.

\section*{5.3 Weights in the exponential form}

Let the weights be of the form
\[
\lambda_i(y) = \frac{\pi_i \exp \left[ -\beta \eta_i(y^{(i)}) \right]}{\sum_{l=1}^{m} \pi_l \exp \left[ -\beta \eta_l(y^{(i)}) \right]} \tag{5.11}
\]
where \( \pi_i \geq 0, i = 1, \ldots, m \). Presentation (5.11) guarantees that the weights \( \lambda_i(y) \) sum to unity. Usually, the most intuitive choice is \( \eta_i = r_i(y^{(i)}) \), the SURE of the \( i \)-th estimator \( \hat{\theta}^{(i)}y^{(i)} \).

The following corollary of Theorem 4 provides an explicit expression for the unbiased estimator of the risk for the weights in the form (5.11).
Corollary 3 If \( \hat{f} \) is defined in (3.2) and weights are in the form (7.11), then the risk is of the form (3.1) with \( \Delta \) given by formula (5.8) and

\[
\Delta_0 = 2\sigma^2 \beta \sum_{i=1}^{m} \lambda_i(y) \left[ \nabla_{g^{(i)}} \eta_i(y^{(i)}) \right]^* \left( W^{(i)} \hat{f} - \hat{g}^{(i)} \right)
\]

\[
= 2\sigma^2 \beta \left\{ \sum_{i,j=1}^{m} \lambda_i(y) \lambda_j(y) \left[ \nabla_{g^{(i)}} \eta_j(y^{(j)}) \right]^* U^{(j,i)} \hat{g}^{(i)} - \sum_{i=1}^{m} \lambda_i(y) \left[ \nabla_{g^{(i)}} \eta_i(y^{(i)}) \right]^* \hat{g}^{(i)} \right\}.
\]

Here, as before, \( \hat{g}^{(i)} = y^{(i)} + g^{(i)} \).

Note that representation (5.12) of the risk is more compact but does not contain the weights explicitly, while formula (5.13) is more convenient if one wants to minimize the risk with respect to \( \pi, \eta_i, i = 1, \ldots, m \), or \( \beta \).

If \( \eta_i(y^{(i)}) = r_i(y^{(i)}), i = 1, \ldots, m \), where \( r_i(y^{(i)}) \) is the unbiased estimator of the risk

\[
r_i(y^{(i)}) = \sigma^2 n + \left[ g^{(i)}(y^{(i)}) \right]^* g^{(i)}(y^{(i)}) + 2\sigma^2 \text{Tr} \left[ \nabla_{g^{(i)}} \left[ g^{(i)}(y^{(i)}) \right]^* \right],
\]

of the estimator \( \hat{f}^{(i)} \), then, by Lemma [1] one has

\[
\nabla_{y^{(i)}} r_i(y^{(i)}) = 2 \left[ \nabla_{y^{(i)}} \left[ g^{(i)}(y^{(i)}) \right]^* \right] g^{(i)}(y^{(i)}) + 2\sigma^2 d_i(y^{(i)})
\]

where \( d_i(y^{(i)}) \) is a column vector with components

\[
d_{ik} = \sum_{l=1}^{n} \frac{\partial^2 g_i^{(i)}(y^{(i)})}{\partial y_l^{(i)} \partial y_k^{(i)}}, \quad k = 1, \ldots, n.
\]

In particular, if one uses linear shrinkage or thresholding, soft or hard, then \( d_i(y^{(i)}) = 0 \).

6 Simulation Study

In this section, we carry out some numerical experiments to study the finite sample performances of the proposed estimators. It is well known that the choice of a frame is linked to the underlying function \( f \) to be de-noised. The advantage of using a frame compared to an orthogonal basis is that it can provide an efficient representation of a broad class of signals as well as better adaptivity for their parsimonious representation. In our simulation study, we use the classical Gabor frame with Hamming window. This is a tight frame which is particularly suitable for representation of fast oscillating signals such as audio signals. For that reason, we consider two fast oscillating standard test signals, *WernerSorrows* and *Mishmash*, reproducible by MakeSignal of the toolbox Wavelab, and two pieces of real audio signals *sp2-5k.wav* and *Glock.wav*. The test signals listed above are displayed in Figure [1]

The objective of this simulation study is to illustrate the gain in de-noising precision obtained by taking into the account the frame structure rather than be an exhaustive study of signal de-noising by frames. Results of all comparisons are represented in terms of the means and the standard deviations of the \( L_2 \) errors. In order to show the advantage attained by accounting for the frame structure, we compare the ideal best diagonal estimator obtained minimizing the true risk in (2.9) versus the ideal best diagonal estimator obtained by minimizing the true risk.
without taking into account the frame structure i.e., considering $U = I$. We denote these two estimators $\text{IDEAL}_U$ and $\text{IDEAL}_I$, respectively. Note that estimators $\text{IDEAL}_U$ and $\text{IDEAL}_I$ are not available in practice, but their comparison can give an idea of the best possible gain obtained by taking into account the frame structure. The empirical versions of these estimators are derived by substituting $\theta \theta^*$ with its unbiased estimator $y y^* - \sigma^2 U$ and $y y^* - \sigma^2 I_N$, respectively. In the first case, we obtain

$$
\gamma = (y y^T \circ U)^{-1} ((y y^T - \sigma^2 U) \circ U) e_N
$$

which coincides with solution (4.2) in the case $U^- = U$, while, in the second case, we obtain

$$
\gamma = (y y^T \circ I_N)^{-1} ((y y^T - \sigma^2 I_N) \circ I_N) e_N
$$

which component-wise reduces to the well known empirical Wiener filter $\gamma_i = (y_i^2 - \sigma^2)/y_i^2$. In what follows, we refer to these estimators as $\text{EMP}_U$ and $\text{EMP}_I$, respectively. Since matrices $(y y^T \circ U)$ and especially $(y y^T \circ I_N)$ sometimes have high condition numbers, in order to stabilize their inversion in our simulation study, we add a quadratic penalization term $\gamma^T P \gamma$ to the functional with matrix $P = \zeta I_N$.

Results for $\zeta = 10^{-4.5}$ are reported in Table 1 and are based on 100 simulational runs with signal-to-noise ratios (SNR) 1, 3 and 5, which represent, respectively, severe, moderate and low noise levels. As it is standard in the statistical literature, the signal-to-noise ratio (SNR) is defined here as the ratio of the standard deviations of the signal and the noise. The empirical estimators $\text{EMP}_U$ and $\text{EMP}_I$ approximate the corresponding ideal estimators $\text{IDEAL}_U$ and $\text{IDEAL}_I$ when the noise level is low (SNR=5) and may be quite far from them when the noise level is high.
(SNR=1). However, for all the test signals, the ideal gain (the difference between the first and the second columns) and the empirical gain (the difference between the third and the fourth columns) obtained by accounting for the frame structure is quite significant, especially, in the case of severe noise.

In a similar manner, we carry out comparisons between soft thresholding procedures obtained with and without consideration of the specific frame structure. In particular, we construct estimators $SOFT_U$ and $SOFT_I$ which are obtained by the formula $\hat{\theta}_i = (y_i - \text{sgn}(y_i)t)\mathbb{I}(|y_i| > t)$ with the global threshold $t$ obtained by minimizing, respectively, expression (3.11) when $t_i = t$ for all $i = 1, \cdots, N$, and

$$\arg\min_{t} \left\{ \sum_{i=1}^{n} \min(y_i^2, t^2) - 2\sigma^2 \sum_{i=1}^{n} \mathbb{I}(|y_i| < t) \right\}, \quad (6.1)$$

which is the classical expression of the SURE reported in Donoho and Johnstone (1995). Similarly, we compare estimators $VISU_U$ and $VISU_I$ obtained using hard thresholding procedure $\hat{\theta}_i = y_i\mathbb{I}(|y_i| > t)$ where, in the first case, the expression for the universal threshold is provided in Haltmeier and Munk (2012)

$$t = \sigma \sqrt{2\log N} + \sigma \left( \frac{2z - \log(\log N) - \log \pi}{2\sqrt{2\log N}} \right)$$

with $z = \pi/\sqrt{6}$, and, in the second case, $t$ is the classical universal threshold for the orthonormal bases $t = \sigma \sqrt{2\log N}$.

Results of comparisons are reported in Table 2 and are based on 100 simulation runs. It is easy to notice that the gain obtained by taking frame structure into account is much more significant in the case of SURE than for both $VISU_U$ and $VISU_I$ universal thresholding procedures. This is due to the fact that universal threshold is known to be too large for de-noising applications, as it has been already noted in statistical literature (see, e.g., Donoho and Johnstone (1995)). In fact, SURE-based soft thresholding procedures outperform the universal thresholding procedures even if the former does not take the fame structure into account: it follows from Table 2 that $SOFT_I$ has better precision than $VISU_U$ for every test signal and every noise level.

In order to examine the performance of the estimator proposed in Section 5, we study the simple case of data independent weights. In particular, we consider two classical orthonormal bases, COSINE and HAAR, and three test functions, WINDOW, LoSINE and a combination of the two (see Figure 2). The WINDOW and the LoSINE are classical test signals which are very well represented, respectively, by the HAAR and the COSINE bases. We evaluate estimator (5.3), where $\lambda$ is derived by minimizing expression (5.6) and $f^{(i)}$’s are obtained as soft thresholding estimators with the universal data-independent threshold. The risks of the estimators are presented in the forth column of Table 3 and the mean values of the estimated weights $\lambda$ are displayed in the fifth column. Table 3 also reports the risks of the single estimators (columns one and two) and of the average of the estimators obtained with the weights $\lambda_1 = \lambda_2 = 0.5$. Note that the aggregation estimator is always better or at least as good as the best basis estimator and it is always better then the estimator obtained by simple average (i.e., by the default frame reconstruction). Moreover, it is instructional to observe that the choice of weights $\lambda_i$ supplied by criterion (5.6) follows an intuitive preference. Indeed, one would favor COSINE basis for LoSINE signal, HAAR basis for WINDOW signal as well as a balanced combination of the two bases for the sum of these two signals: computations confirm those intuitive assessments.
Table 1: Results obtained over 100 runs and with parameter choices, \( n = 1280 \) and 64-sampled Hamming window.

| Method       | SNR=1       | SNR=3       | SNR=5       |
|--------------|-------------|-------------|-------------|
| WernerSorrows| 0.1327 (0.0096) | 0.2274 (0.0116) | 0.4964 (0.0240) | 5.7420 (0.1847) |
| SNR=3        | 0.0284 (0.0019) | 0.0404 (0.0022) | 0.0777 (0.0034) | 0.1343 (0.0049) |
| SNR=5        | 0.0126 (0.0006) | 0.0167 (0.0007) | 0.0321 (0.0011) | 0.0412 (0.0015) |
| MeshMash     | 0.1026 (0.0106) | 0.1837 (0.0139) | 0.4881 (0.0254) | 6.2411 (0.2290) |
| SNR=3        | 0.0211 (0.0017) | 0.0284 (0.0021) | 0.0752 (0.0036) | 0.1113 (0.0044) |
| SNR=5        | 0.0094 (0.0007) | 0.0122 (0.0008) | 0.0324 (0.0013) | 0.0286 (0.0015) |
| sp2-5k       | 0.1533 (0.0112) | 0.2474 (0.0127) | 0.5201 (0.0254) | 6.2648 (0.1745) |
| SNR=3        | 0.0363 (0.0022) | 0.0548 (0.0024) | 0.0849 (0.0039) | 0.1771 (0.0058) |
| SNR=5        | 0.0168 (0.0009) | 0.0244 (0.0011) | 0.0349 (0.0014) | 0.0614 (0.0022) |
| Glock        | 0.0845 (0.0075) | 0.1305 (0.0093) | 0.4529 (0.0245) | 6.4889 (0.2079) |
| SNR=3        | 0.0192 (0.0014) | 0.0278 (0.0016) | 0.0737 (0.0037) | 0.1232 (0.0043) |
| SNR=5        | 0.0089 (0.0006) | 0.0123 (0.0007) | 0.0322 (0.0013) | 0.0326 (0.0015) |

Figure 2: Normalized test signals of length 1024.
Table 2: Results obtained over 100 runs and with parameter choices, \( n = 1280 \) and 64-sampled Hamming window.

| WernerSorrows | SOFT\_\_ | SOFT\_L | VISU\_\_ | VISU\_L |
|---------------|---------|---------|----------|---------|
| SNR=1         | 0.3748 (0.0188) | 0.8511 (0.0414) | 0.8987 (0.0199) | 0.9024 (0.0200) |
| SNR=3         | 0.0763 (0.0041) | 0.1342 (0.0114) | 0.3748 (0.3748) | 0.3965 (0.3965) |
| SNR=5         | 0.0327 (0.0016) | 0.0481 (0.0039) | 0.1230 (0.0041) | 0.1275 (0.0041) |

| MishMash      | SOFT\_\_ | SOFT\_L | VISU\_\_ | VISU\_L |
|---------------|---------|---------|----------|---------|
| SNR=1         | 0.3519 (0.0216) | 0.8970 (0.0695) | 0.9733 (0.0173) | 0.9756 (0.0158) |
| SNR=3         | 0.0602 (0.0040) | 0.1063 (0.0095) | 0.2434 (0.0148) | 0.2573 (0.0160) |
| SNR=5         | 0.0251 (0.0013) | 0.0414 (0.0035) | 0.0749 (0.0039) | 0.0786 (0.0039) |

| SP2-5k        | SOFT\_\_ | SOFT\_L | VISU\_\_ | VISU\_L |
|---------------|---------|---------|----------|---------|
| SNR=1         | 0.3893 (0.0197) | 0.8555 (0.0693) | 0.9934 (0.0120) | 0.9952 (0.0105) |
| SNR=3         | 0.0917 (0.0047) | 0.1689 (0.0164) | 0.3881 (0.0143) | 0.4023 (0.0142) |
| SNR=5         | 0.0457 (0.0026) | 0.0626 (0.0059) | 0.1740 (0.0052) | 0.1799 (0.0050) |

| Glock         | SOFT\_\_ | SOFT\_L | VISU\_\_ | VISU\_L |
|---------------|---------|---------|----------|---------|
| SNR=1         | 0.2853 (0.0186) | 0.5064 (0.0462) | 0.9181 (0.0453) | 0.9350 (0.0462) |
| SNR=3         | 0.0516 (0.0029) | 0.0981 (0.0083) | 0.1591 (0.0076) | 0.1628 (0.0078) |
| SNR=5         | 0.0228 (0.0012) | 0.0406 (0.0034) | 0.0898 (0.0026) | 0.0919 (0.0026) |

Table 3: Results obtained over 100 runs by COSINE and HAAR bases. Parameter choices are \( n = 2^{10} \) and \( J = 3 \) for the Haar basis.

| function       | SNR | COSINE | HAAR | averaging | aggregation | \((\lambda_1, \lambda_2)\) |
|----------------|-----|--------|------|-----------|-------------|-----------------|
| WINDOW         | 1   | 0.2291 | 0.1719 | 0.1719    | 0.1648      | (0.2200 0.7800) |
|                | 3   | 0.0742 | 0.0214 | 0.0364    | 0.0214      | (0.0105 0.9955) |
|                | 5   | 0.0444 | 0.0076 | 0.0182    | 0.0077      | (0.0045 0.9955) |
| LOSINE         | 1   | 0.1284 | 0.9440 | 0.4118    | 0.1284      | (1.0000 0.0000) |
|                | 3   | 0.0427 | 0.6682 | 0.2221    | 0.0444      | (0.9836 0.0164) |
|                | 5   | 0.0259 | 0.3617 | 0.1177    | 0.0305      | (0.9181 0.0819) |
| WINDOW + LOSINE| 1   | 0.2644 | 0.3496 | 0.2673    | 0.2563      | (0.7693 0.2307) |
|                | 3   | 0.0855 | 0.2011 | 0.1037    | 0.0827      | (0.8650 0.1350) |
|                | 5   | 0.0509 | 0.1666 | 0.0720    | 0.0496      | (0.8448 0.1552) |
7 Discussion

The present paper provides a comprehensive study of de-noising properties of frames and, in particular, tight frames, which constitute one of the most popular tools in contemporary signal processing. The objective of the paper is to bridge the existing gap between mathematical and statistical theories on one hand and engineering practice on the other and explore how one can take advantage of a specific structure of a frame in contrast to an arbitrary collection of vectors or an orthonormal basis.

For both the general and the tight frames, the paper presents a set of practically implementable de-noising techniques which take frame induced correlation structures into account. These results are supplemented by an examination of the case when the frame is constructed as a collection of orthonormal bases. In particular, recommendations are given for aggregation of the estimators at the stage of frame coefficients. The paper is concluded by a finite sample simulation study which confirms that taking frame structure and frame induced correlations into account indeed improves de-noising precision.

Acknowledgements

Marianna Pensky was supported in part by National Science Foundation (NSF), grant DMS-1106564.

8 Appendix

Proof of Theorem 1. To verify expression (2.6), note that
\[E\|\hat{f} - f\|^2 = \text{Tr}[W^+(\Gamma y - \theta)(\Gamma y - \theta)^*(W^+)^*] = \Delta_1 + \Delta_2\]
where
\[\Delta_1 = \text{Tr}[W^+\Gamma\mathbb{E}(\epsilon\epsilon^*)\Gamma^*(W^+)^*] = \sigma^2\text{Tr}[\Gamma U\Gamma^*]\]
and
\[\Delta_2 = \text{Tr}[W^+(I_N - \Gamma)\theta\theta^*(I_N - \Gamma)(W^+)^*] = \text{Tr}[U^-(I_N - \Gamma)\theta\theta^*(I_N - \Gamma)],\]
which completes the proof of (2.6). To prove (2.7), note that in the case of a tight frame, one has \(U^{-1}U = \alpha^{-2}U^2 = \alpha^{-2}WW^*WW^* = \alpha^{-1}U\) since \(W^*W = \alpha I\).

Proof of Corollary 1. Note that
\[\text{Tr}[\Gamma U\Gamma^*] = \text{Tr}[UU^- + (I_N - \Gamma)U(I_N - \Gamma)U^- - 2(I_N - \Gamma)UU^-]\]
Now, to prove (3.2), replace \(\theta\theta^*\) by \(yy^* - \sigma^2U\) in (2.6) and observe that
\[\text{Tr}[UU^-] = \text{Tr}[WW^*(W^+)^*W^+] = \text{Tr}[(W^+W)^*W^+W] = n\]
(8.1)
since \(W^+W = I_n\). In order to obtain (3.2), replace \(U^-\) with \(\alpha^{-2}U\).

Proof of Theorem 2. First, let us show that under conditions of Theorem 2 one has
\[E[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^*] = \sigma^2U + \mathbb{E}[g(y)g^*(y)] + 2\sigma^2U\mathbb{E}[Z].\]
(8.2)
To this end, note that
\[ \mathbb{E}[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^*) = \mathbb{E}[(y - \theta)(y - \theta)^* + g(y)g^*(y)] + 2\mathbb{E}[(y - \theta)g^*(y)] \equiv \Omega_1 + 2\Omega_2. \]

Here \( \Omega_1 = \sigma^2U + \mathbb{E}[g(y)g^*(y)] \) and, due to representations \( y = Wx \) and \( \theta = Wf \), the expression for \( \Omega_2 \) may be written as \( \Omega_2 = W\mathbb{E}[(x - f)g^*(Wx)]. \)

Denote \( C_\sigma = (2\pi\sigma^2)^{-n/2} \) and observe that \( Q = \mathbb{E}[(x - f)g^*(Wx)] \) is the \( n \times N \) matrix with components
\[ Q_{ij} = \mathbb{E}[(x_i - f_i)g_j(Wx)] = C_\sigma \int \cdots \int (x_i - f_i)g_j(Wx) \exp\left(-\|x - f\|^2/2\sigma^2\right) dx \]
\[ = -C_\sigma \sigma^2 \int \cdots \int g_j(Wx) \, dt (\exp(-0.5 \sigma^{-2} \|x - f\|^2)) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \]
\[ = C_\sigma \sigma^2 \int \cdots \int \frac{\partial}{\partial x_i} g_j(Wx) \exp(-\|x - f\|^2/2\sigma^2) \, dx = \sigma^2 \mathbb{E} \left[ \frac{\partial}{\partial x_i} g_j(Wx) \right]. \]

In the expression above, we denoted differential with respect to \( x_i \) by \( dt \) and used integration by parts.

Applying the chain rule, derive that
\[ \frac{\partial}{\partial x_i} [g_j(Wx)] = \sum_{i=1}^N \frac{\partial}{\partial y_i} [g_j(y)] W_{il} = \sum_{i=1}^N Z_{ij} W_{il} = (W^* Z)_{ij}. \]

Therefore,
\[ \Omega_2 = \sigma^2 \mathbb{E}(WW^* Z) = \sigma^2 U \mathbb{E}(Z), \]
which yield expression (8.2).

Now, to complete the proof of (3.6), observe that
\[ \mathbb{E}\|\hat{f} - f\|^2 = \mathbb{E} \text{ Tr} \left[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^*(W^+)^*W^+\right] \]
\[ = \mathbb{E} \text{ Tr} \left[\sigma^2UU^* + g(y)g^*(y)U^* + 2\sigma^2UUZU^* \right] \]
and recall that, by formula (8.1), one has \( \text{Tr} [UU^*] = n \). Validity of formula (3.7) follows from the fact that, in the case of a tight frame, one has \( U^*U = \alpha^2U^2 \).

**Proof of Corollary 2** Validity follows directly from Theorem 2 and relations (3.8) and (3.9).

**Proof of Theorem 3** Note that
\[ \mathbb{E}\|\hat{f} - f\|^2 = \left(\sum_{i=1}^m \lambda_i \right) \mathbb{E}\|\hat{f} - f\|^2 = \sum_{i=1}^m \lambda_i \mathbb{E}\|\hat{f} - f^{(i)}\|^2 \]
\[ = \sum_{i=1}^m \lambda_i \left[ \mathbb{E}\|\hat{f} - f^{(i)}\|^2 + \mathbb{E}\|\hat{f}^{(i)} - f\|^2 + 2\mathbb{E}\left(\hat{f} - f^{(i)}\right)^* \left(f^{(i)} - f\right) \right]. \]

By direct calculations, it is easy to check that
\[ \sum_{i=1}^m \lambda_i \left(\hat{f} - f^{(i)}\right)^* \left(f^{(i)} - f\right) = \|\hat{f}\|^2 - \sum_{i=1}^m \lambda_i \|\hat{f}^{(i)}\|^2 = -\sum_{i=1}^m \lambda_i \|\hat{f}^{(i)} - f\|^2. \]
Therefore, changing the order of expectation and summation (due to the fact that the weights are data-independent) and using identity above, we derive

$$
E\|\hat{f} - f\|^2 = E\sum_{i=1}^{m} \lambda_i \left[ \|\hat{f}^{(i)} - \hat{f}\|^2 + \|\hat{f}^{(i)} - f\|^2 - 2\|\hat{f}^{(i)} - \hat{f}\|^2 \right],
$$

which completes the proof.

**Proof of Theorem 4.** Applying Theorem 2 with \(\alpha = m\) and a block vector \(\tilde{g}\) with blocks given by formula (5.7), one can write \(\Delta\) as

$$
\Delta = m^{-2} \tilde{g}(y) U \tilde{g}(y) + 2\sigma^2 m^{-1} \text{Tr}[U \tilde{Z}] = \Delta_1 + \Delta_2. \quad (8.3)
$$

Here, re-arranging \(\tilde{g}^{(i)}(y)\), we obtain

$$
\Delta_1 = \sum_{i,j=1}^{m} \lambda_i(y) \lambda_j(y) (g^{(i)})^* U^{(i,j)} g^{(j)} + m^{-2} \sum_{i,j=1}^{m} [1 - m \lambda_i(y)][1 - m \lambda_j(y)](y^{(i)})^* U^{(i,j)} y^{(j)}
$$

$$
- 2m^{-1} \sum_{i,j=1}^{m} \lambda_i(y)[1 - m \lambda_j(y)](g^{(i)})^* U^{(i,j)} y^{(j)}.
$$

Since \(U^{(i,j)} y^{(j)} = W^{(i)} (W^{(j)})^* y^{(j)} = W^{(i)} x = y^{(i)}\) and \(\sum_{j=1}^{m} (1 - \lambda_j) = 0\), the second and the third terms in the last expression are equal to zero and

$$
\Delta_1 = \sum_{i,j=1}^{m} \lambda_i(y) \lambda_j(y) (g^{(i)})^* U^{(i,j)} g^{(j)}. \quad (8.4)
$$

Now, consider \(\Delta_2\). Note that \(\tilde{Z}\) is a block matrix with blocks \(\tilde{Z}^{(i,j)}\) which, with the help of the product rule, can be presented in the form

$$
\tilde{Z}^{(i,j)} = \nabla_{y^{(i)}} (\tilde{g}^{(j)})^* (y) = m[\nabla_{g^{(i)}} \lambda_j(y)](\tilde{\theta}^{(j)})^* + m \lambda_i(y) [I_n + Z^{(i,i)}] I_n - I_n I_i = I_j.
$$

Substituting the last expression into \(\Delta_2\) in (8.3) and recalling that \(U^{(i,i)} = I_n\), we arrive at

$$
\Delta_2 = 2\sigma^2 m^{-1} \sum_{i,j=1}^{m} \text{Tr}[U^{(j,i)} \tilde{Z}^{(i,j)}] = 2\sigma^2 \sum_{i,j=1}^{m} \text{Tr} \left[ U^{(j,i)} \nabla_{g^{(i)}} \lambda_j(y) \right] (\tilde{\theta}^{(j)})^* \quad (8.5)
$$

$$
+ 2\sigma^2 \sum_{i=1}^{m} \lambda_i(y) \text{Tr}[I_n + Z^{(i,i)}] - 2\sigma^2 n.
$$

Now, interchange \(i\) and \(j\) in the first term of \(\Delta_2\), and also note that

$$
\sum_{i=1}^{m} \lambda_i(y) \text{Tr}[I_n + Z^{(i,i)}] = n + \sum_{i=1}^{m} \lambda_i(y) \text{Tr}[Z^{(i,i)}].
$$

To complete the proof, combine (8.3), (8.4) and (8.5).

**Proof of Lemma 1.** Observe that \(\nabla_{y} [g^*(y) g(y)]\) is a column vector with components

$$
\frac{\partial}{\partial y_i} [g^*(y) g(y)] = 2 \sum_{k=1}^{n} \frac{\partial g_k(y)}{\partial y_i} g_k(y).
$$
Hence, $\nabla_y [g^*(y)g(y)] = 2[\nabla_y g^*(y)]g(y)$. Similarly, $\nabla_y \text{Tr} [\nabla_y g^*(y)]$ is a column vector with components

$$d_k(y) = \frac{\partial}{\partial y_k} \left[ \sum_{l=1}^{n} \frac{\partial g_l(y)}{\partial y_l} \right],$$

which coincides with (5.10).

**Proof of Corollary 3** Denote

$$\Psi(y) = \sum_{l=1}^{m} \pi_l \exp(-\beta \eta_l),$$

so that, $\log(\lambda_i(y)) = \log(\pi_i) - \beta \eta_i(y^{(i)}) - \log(\Psi(y))$. Then, $\nabla_{y^{(j)}} \lambda_i(y) = \lambda_i(y) \nabla_{y^{(j)}} [\log(\lambda_i(y))]$ where

$$\nabla_{y^{(j)}} [\log(\lambda_i(y))] = -\beta \nabla_{y^{(j)}} [\eta_i(y^{(i)})] + \beta \sum_{l=1}^{m} \lambda_l(y) \nabla_{y^{(j)}} [\eta_l(y^{(l)})].$$

Taking into account that $\nabla_{y^{(j)}} [\eta_i(y^{(i)})] = 0$ if $i \neq j$, we derive

$$\nabla_{y^{(j)}} \lambda_i(y) = \beta \lambda_i(y) \left[ \lambda_j(y) \nabla_{y^{(j)}} [\eta_j(y^{(j)})] - \nabla_{y^{(j)}} [\eta_i(y^{(i)})]I(i = j) \right].$$

Now, to complete the proof of (5.13), recall from Theorem 4 that

$$\Delta_0 = 2\sigma^2 \sum_{i,j=1}^{m} \left[ \nabla_{y^{(j)}} \lambda_i(y) \right]^* U^{(j,i)} \hat{\theta}^{(i)}$$

and insert the expression for $\nabla_{y^{(j)}} [\lambda_i(y)]$ into $\Delta_0$. To show validity of (5.12), note that

$$\sum_{i,j=1}^{m} \lambda_i(y) \lambda_j(y) \left[ \nabla_{y^{(j)}} \eta_j(y^{(j)}) \right]^* U^{(j,i)} \hat{\theta}^{(i)} =$$

$$\sum_{j=1}^{m} \lambda_j(y) \left[ \nabla_{y^{(j)}} \eta_j(y^{(j)}) \right]^* W^{(j)} \sum_{i=1}^{m} \lambda_i(y) \left( W^{(i)} \right)^* \hat{\theta}^{(i)} =$$

$$\sum_{j=1}^{m} \lambda_j(y) \left[ \nabla_{y^{(j)}} \eta_j(y^{(j)}) \right]^* W^{(j)} \hat{f}.$$  

Acknowledgements

Marianna Pensky was supported in part by National Science Foundation (NSF), grant DMS-1106564.

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