Risk-sensitive Reinforcement Learning via Distortion Risk Measures

Nithia Vijayan and Prashanth L.A.

Abstract—We address the problem of control in a risk-sensitive reinforcement learning (RL) context via distortion risk measures (DRM). We propose policy gradient algorithms, which maximize the DRM of the cumulative reward in an episodic Markov decision process in on-policy as well as off-policy RL settings. We employ two different approaches in devising the policy gradient algorithms. In the first approach, we derive a variant of the policy gradient theorem that caters to the DRM objective, and use this theorem in conjunction with a likelihood ratio-based gradient estimation scheme. In the second approach, we estimate the DRM from the empirical distribution of cumulative rewards, and use this estimation scheme along with a smoothed functional-based gradient estimation scheme. For policy gradient algorithms using either approach, we derive non-asymptotic bounds that establish the convergence to an approximate stationary point of the DRM objective.

Index Terms—Distortion risk measure, risk-sensitive RL, non-asymptotic analysis, policy gradient.

I. INTRODUCTION

In a classical reinforcement learning (RL) problem, the objective is to learn a policy that maximizes the mean of the cumulative rewards. But, in many practical applications, we may end up with unsatisfactory policies if we only consider the mean of the rewards. Instead of focusing only on the mean, it is rational to consider other aspects of a cumulative reward distribution, viz., variance, shape, and tail probabilities. In literature, a statistical measure, called a risk measure is used to quantify these aspects.

While several risk measures are available in the literature, there is no consensus on an ideal risk measure. A risk measure is said to be coherent if it is translation invariant, sub-additive, positive homogeneous, and monotonic [1]. Coherent risk measures are very desirable as the aforementioned properties help to avoid risky decisions. Value-at-Risk (VaR) is a popular risk measure that lacks coherence as it is not sub-additive. Conditional Value-at-Risk (CVaR) [2] is a conditional expectation of outcomes not exceeding VaR, and is a coherent risk measure. However, as suggested in [3], CVaR is not preferable since it treats all outcomes below VaR equally, and ignores those beyond VaR. Cumulative prospect theory (CPT) [4] is a popular risk measure in human-centered decision making problems. However, CPT is a non-coherent risk measure. Instead of giving equal focus to all the outcomes, or treating only a fraction of the outcomes using a tail-based risk measure such as CVaR, it is preferable to consider all outcomes with the right emphasis, while retaining coherency. We describe such a risk measure next.

A family of risk measures called distortion risk measures (DRM) [5], [6] is widely used for optimization in finance and insurance. A DRM uses a distortion function to distort the original distribution, and calculate the mean of the rewards with respect to the distorted distribution. A distortion function allows one to vary the emphasis on each possible reward value. The choice of the distortion function governs the risk measure. A DRM with an identity distortion function is simply the mean of the rewards. Further, choosing a concave distortion function ensures that the DRM is coherent [7]. As an aside, the spectral risk functions are equivalent to distortion functions [8]. The popular risk measures like VaR and CVaR can be expressed as a DRM using appropriate distortion functions. But, the distortion function is discontinuous for VaR, and though continuous, it is not differentiable at every point for CVaR. Hence in [3], the author disfavors such distortion functions, and focuses on smooth distortion functions. In this paper, we consider smooth distortion functions. Some examples of such distortion functions are dual-power function, quadratic function, square-root function, exponential function, and logarithmic function (see [6], [9] for more examples).

Risk-sensitive RL has been studied widely in the literature, with focus on specific risk measures like expected exponential utility [10], variance related measures [11], CVaR [12], [13], and CPT [14]. In this paper, instead of deriving algorithms that cater to specific risk measures, we consider the whole family of DRMs with smooth distortion functions. The risk-neutral RL approach gives equal importance to all the events, and hence occasional high reward events gets equal priority as all other events. But using DRMs, since the distortion function is manipulating the reward distribution, we can give more emphasis to frequent events. Also, since the distortion function is not discarding any information, we still account for infrequent high severity events. As there is no universally accepted “ideal” risk measure, we may choose a risk measure which best fits our particular problem by picking an appropriate distortion function.

We consider a risk-sensitive RL problem, in which an optimal policy is learned by maximizing the DRM of cumulative rewards in an episodic Markov decision process (MDP). We consider this problem in on-policy as well as off-policy settings, and employ the policy gradient solution approach. The basis for a policy gradient algorithm is the expression for the gradient of the performance objective. In the risk-
neutral case, such an expression is derived using the likelihood ratio (LR) method \cite{15}. We derive a DRM analogue to the policy gradient theorem. In the case of DRM, policy gradient estimation is challenging since DRM of a given policy cannot be estimated using a sample mean. As in \cite{14}, we employ the empirical distribution function (EDF) to estimate DRM, and this leads to a biased estimate of the DRM policy gradient. In contrast, policy gradient estimation is considerably simpler in a risk-neutral setting as the task is to estimate the mean cumulative reward, and using a sample mean leads to an unbiased gradient estimate.

In this paper, we propose two methods to optimize DRM. In our first method, we derive the policy gradient expression for the DRM objective using the LR method. We characterize the mean squared error (MSE) in DRM policy gradient estimates. In particular, we establish that the MSE is of order $O(1/m)$, where $m$ is the batch size (or the number of episodes). Using the DRM policy gradient expression, we propose two policy gradient algorithms which cater to on-policy and off-policy RL settings, respectively. To the best of our knowledge, we are first to derive a policy gradient theorem under a DRM objective, and devise/analyze policy gradient algorithms to optimize DRM in a RL context.

In our second method, we estimate the DRM from sample episodes using the EDF as a proxy for the true distribution. We provide a non-asymptotic bound on the MSE of this estimator, and this may be of independent interest. In particular, we establish that the mean-square error is of order $O(1/m)$. With these DRM estimates, we estimate the DRM gradient, using the smoothed functional (SF) method \cite{16}, \cite{17}, and propose two algorithms which cater to on-policy and off-policy RL settings, respectively. We use a variant of SF which use two function measurements corresponding to two perturbed policies. An SF-based estimation scheme may be restrictive for some applications in an on-policy RL setting, since we need separate sets of episodes corresponding to two perturbed policies. An SF-based estimation scheme may be restrictive for some applications in an on-policy RL setting, since we need separate sets of episodes corresponding to two perturbed policies. But, in an off-policy RL context, we only need a single set of episodes corresponding to a behavior policy, and this makes our off-policy gradient algorithm practically amenable.

We provide bounds on the bias and variance of the aforementioned gradient estimates. Using these bounds, we establish that our algorithms converge to an approximate stationary point of the DRM objective at a rate of $O(1/\sqrt{N})$. Here $N$ denotes the total number of iterations of the DRM policy gradient algorithm. Our algorithms based on the LR method require $O(\sqrt{N})$ episodes per iteration for both on-policy and off-policy RL settings. On the other hand, our off-policy algorithm based on SF method requires only $O(1)$ episodes per iteration, while the on-policy variant requires $O(N)$ episodes per iteration.

**Related work.** In \cite{18}, the authors consider a policy gradient algorithm for an abstract coherent risk measure. They derive a policy gradient theorem using the dual representation of a coherent risk measure. Next, using the EDF of the cumulative reward distribution, they propose an estimate of the policy gradient, and this estimation scheme requires solving a convex optimization problem. Finally, they establish asymptotic consistency of their proposed gradient estimate. In \cite{14}, the authors consider a CPT-based objective in an RL setting, and they employ simultaneous perturbation stochastic approximation (SPSA) method for policy gradient estimation, and provide asymptotic convergence guarantees for their algorithm. In \cite{19}, the authors survey policy gradient algorithms for optimizing different risk measures in a constrained as well as an unconstrained RL setting. In a non-RL context, the authors in \cite{20} study the sensitivity of DRM using an estimator that is based on the generalized likelihood ratio method, and establish a central limit theorem for their gradient estimator.

In comparison to the aforementioned works, we would like to note the following aspects:

(i) For an abstract coherent risk measure, \cite{18} uses gradient estimation scheme which requires solving a convex optimization sub-problem, whereas our algorithms can directly estimate the gradient from the samples without solving any optimization sub-problem. Thus our gradient estimation schemes are computationally inexpensive compared to the one in \cite{18}.

(ii) Using the DRM gradient estimate, we analyze policy gradient algorithms, and provide a convergence rate result of order $O(1/\sqrt{N})$. But, the convergence guarantees in \cite{18} are asymptotic in nature.

(iii) In \cite{14}, the guarantees for a policy gradient algorithm based on SPSA are asymptotic in nature, and is for CPT in an on-policy RL setting. CPT is also based on a distortion function, but the distortion function underlying CPT is neither concave nor convex, and hence, it is non-coherent.

(iv) In \cite{19}, the authors derive a non-asymptotic bound of $O(1/N^{1/3})$ for an abstract smooth risk measure. They use abstract gradient oracles which satisfies certain bias-variance conditions. In contrast, we provide concrete gradient estimation schemes in RL settings, and our bounds feature an improved rate of $O(1/\sqrt{N})$.

The rest of the paper is organized as follows: Section \[II\] describes the DRM-sensitive episodic MDP. Section \[III\] introduces our proposed gradient estimation methods and corresponding algorithms. Section \[IV\] presents the non-asymptotic bounds for our algorithms. Sections \[V\], \[VI\] provide the detailed proofs of convergence. Finally, Section \[VIII\] provides the concluding remarks.

**II. PROBLEM FORMULATION**

**A. Distortion risk measure (DRM)**

Let $F_X$ denote the cumulative distribution function (CDF) of a r.v. $X$. The DRM of $X$ is defined using the Choquet integral of $X$ w.r.t. a distortion function $g$ as follows:

$$
\rho_g(X) = \int_{-\infty}^{0} (g(1-F_X(x))-1)dx + \int_{0}^{\infty} g(1-F_X(x))dx.
$$

The distortion function $g : [0, 1] \rightarrow [0, 1]$ is non-decreasing, with $g(0)=0$ and $g(1)=1$. Some examples of the distortion functions are given in Table \[I\] and their plots in Figure \[I\].

The DRMs are well studied from an ‘attitude towards risk’ perspective, and we refer the reader to \cite{21}, \cite{22} for details. In this paper, we focus on ‘risk-sensitive decision making under uncertainty’, with DRM as the chosen risk measure. We
TABLE I: Examples of distortion functions

| Function Type    | Formula                                                                 |
|------------------|-------------------------------------------------------------------------|
| Dual-power       | \( g(s) = 1 - (1 - s)^r, \quad r \geq 2 \)                              |
| Quadratic        | \( g(s) = (1 + r)s - rs^2, \quad 0 \leq r \leq 1 \)                      |
| Exponential      | \( g(s) = 1 - \exp(-rs^2) / 1 - \exp(-r), \quad r > 0 \)                  |
| Square-root      | \( g(s) = \sqrt{1 + rs^2} / \sqrt{1 + r}, \quad r > 0 \)                  |
| Logarithmic      | \( g(s) = \log(1 + rs) / \log(1 + r), \quad r > 0 \)                     |

Fig. 1: Examples of distortion functions

incorporate DRMs into a risk-sensitive RL framework, and the
following section describes our problem formulation.

**B. DRM-sensitive MDP**

We consider an MDP with a state space \( S \) and an action space \( A \). We assume that \( S \) and \( A \) are finite spaces. Let \( r : S \times A \times S \rightarrow [0,1], r_{max} \in \mathbb{R}^+ \) be the scalar reward function, and \( p \in \mathcal{P}([0,1]) \) be the transition probability function. The actions are selected using parameterized stochastic policies \( \{ \pi_\theta : \mathcal{P}(S \times A \times \mathbb{R}^d) \rightarrow [0,1], \theta \in \mathbb{R}^d \} \). We consider episodic problems, where each episode starts at a fixed state \( S_0 \), and terminates at a special absorbing state \( 0 \). We denote by \( S_t \) and \( A_t \) the state and action at time \( t \in \{0,1,\cdots\} \) respectively. We assume that the policies \( \{ \pi_\theta, \theta \in \mathbb{R}^d \} \) are proper, i.e., they satisfy the following assumption:

\((A1)\): \( \exists M > 0 : \max_{\theta \in \mathbb{R}^d} \mathbb{P} (S_M \neq 0 | S_0 = s, \pi_\theta) < 1, \forall \theta \in \mathbb{R}^d \).

The assumption \((A1)\) is frequently made in the analysis of episodic MDPs (cf. [23], Chapter 2).

The cumulative discounted reward \( R^\theta \) is defined by

\[
R^\theta = \sum_{t=0}^{T-1} \gamma^t r(S_t, A_t, S_{t+1}), \quad \forall \theta \in \mathbb{R}^d,
\]

where \( A_t \sim \pi_\theta(S_t, A_t, S_{t+1}), \quad \gamma \in (0,1), \quad T \) is the random length of an episode. Notice that \( \forall \theta \in \mathbb{R}^d \), \( |R^\theta| < \frac{M}{1-\gamma} \) a.s. From \((A1)\) we infer that \( \mathbb{E}[T] < \infty \). This fact in conjunction with \( \gamma \geq 0 \) implies the following bound:

\[
\exists M > 0 : T \leq M \quad \text{a.s.}
\]

The DRM \( \rho_\theta(\cdot) \) using a distortion function \( g(\cdot) \) is given by

\[
\rho_\theta(\cdot) = \int_{-M}^{\infty} g(1 - F_{R^\theta}(x)) dx + \int_{0}^{M} g(1 - F_{R^\theta}(x)) dx,
\]

where \( F_{R^\theta} \) is the CDF of \( R^\theta \), and \( M_r = \frac{M}{1-\gamma} \).

Our goal is to find a \( \theta^* \) which maximizes \( \rho_\theta(\cdot) \), i.e.,

\[
\theta^* \in \arg\max_{\theta \in \mathbb{R}^d} \rho_\theta(\theta).
\]

**III. DRM POLICY GRADIENT ALGORITHMS**

An iterative gradient-based algorithm can solve (4), using the following update iteration:

\[
\theta_{k+1} = \theta_k + \alpha \nabla \rho_\theta(\theta_k),
\]

where \( \theta_0 \) is set arbitrarily, and \( \alpha \) is the step-size.

But, in a typical RL setting, we do not have direct measurements of the gradient \( \nabla \rho_\theta(\cdot) \). In the following sections, we describe two different approaches in devising the policy gradient algorithms in on-policy as well as off-policy RL settings. In the first approach, we derive a variant of the policy gradient theorem that caters to the DRM objective, and use this theorem in conjunction with a LR-based gradient estimation scheme. In the second approach, we estimate the DRM from the empirical distribution of cumulative rewards, and use this estimation scheme along with a SF-based gradient estimation scheme.

**A. DRM policy gradient**

In this section, we present a DRM analogue to the policy gradient theorem under the following assumptions:

\((A2)\): \( \exists M > 0 : \forall \theta \in \mathbb{R}^d, \| \nabla \log \pi_\theta(a | s) \| \leq M_d, \forall a \in A, s \in S \), where \( \| \cdot \| \) is the \( d \)-dimensional Euclidean norm.

\((A3)\): \( \exists M_{g^\theta} > 0 : \forall t \in (0,1), |g'(t)| \leq M_{g^\theta} \).

The assumptions \((A2)\)|(A3) ensure the boundedness of the DRM policy gradient. An assumption like \((A2)\) is common to the analysis of policy gradient algorithms (cf. [24], [25]). A few examples of distortion functions, which satisfy \((A3)\) are given in Table [I]

For deriving a policy gradient theorem variant with DRM as the objective, we first express the CDF \( F_{R^\theta}(\cdot) \), and its gradient \( \nabla F_{R^\theta}(\cdot) \) as expectations w.r.t the episodes from the policy \( \pi_\theta \).

Starting with

\[
F_{R^\theta}(x) = \mathbb{E} \left[ \mathbb{1}\{ R^\theta \leq x \} \right],
\]

we obtain an expression for \( \nabla F_{R^\theta}(x) \) in the lemma below.

**Lemma 1**: \( \forall x \in (-M_r, M_r), \)

\[
\nabla F_{R^\theta}(x) = \mathbb{E} \left[ \mathbb{1}\{ R^\theta \leq x \} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t) \right].
\]

**Proof**: See Section [V-A]

We now state the DRM policy gradient theorem below.

**Theorem 1**: (DRM policy gradient) Assume \((A1)\)|(A3) Then the gradient of the DRM in (3) is given by

\[
\nabla \rho_\theta(\theta) = - \int_{-M}^{\infty} g'(1 - F_{R^\theta}(x)) \nabla F_{R^\theta}(x) dx.
\]

**Proof**: See Section [V-B]

We make the following additional assumptions to ensure the smoothness of the DRM \( \rho_\theta(\theta) \).

\((A4)\): \( \exists M_h > 0 : \forall \theta \in \mathbb{R}^d, \| \nabla^2 \log \pi_\theta(a | s) \| \leq M_h, \forall a \in A, s \in S \), where \( \| \cdot \| \) is the operator norm.

\((A5)\): \( \exists M_{g''} > 0 : \forall t \in (0,1), |g''(t)| \leq M_{g''} \).
An assumption like \([A4]\) is common in literature for the non-asymptotic analysis of the policy gradient algorithms (cf. \([24, 26]\)). A few examples of distortion functions, which satisfy \([A5]\) are given in Table I. Since \(g(\cdot)\) is bounded by definition, we can see that any \(g(\cdot)\) that satisfies \([A5]\) will satisfy \([A3]\) also.

The following results establish that the DRM \(\rho_g(\theta)\) is Lipschitz as well as smooth in the parameter \(\theta\).

**Lemma 2:** Assume \([A1]\)\(\leftarrow\)\([A3]\) Then, \(\forall \theta_1, \theta_2 \in \mathbb{R}^d\),

\[
|\rho_g(\theta_1) - \rho_g(\theta_2)| \leq L_\rho \|	heta_1 - \theta_2\|, \quad L_\rho = 2M_t M_g'M_g M_d.
\]

**Proof:** See Section V-C

**Lemma 3:** Assume \([A1]\)\(\leftarrow\)\([A5]\) Then \(\forall \theta_1, \theta_2 \in \mathbb{R}^d\),

\[
\|\nabla \rho_g(\theta_1) - \nabla \rho_g(\theta_2)\| \leq L_{\rho'} \|	heta_1 - \theta_2\|,
\]

where \(L_{\rho'} = 2M_t M_g' (M_t M_g' + M_c M_d^2 (M_g' + M_g''))\).

**Proof:** See Section V-C

**B. DRM optimization: LR-based gradient estimation**

In the following sections, we describe gradient algorithms that use (8) to derive LR-based DRM gradient estimates.

1) **DRM optimization in on-policy RL settings:** We generate \(m\) episodes using the policy \(\pi_\theta\), and estimate \(F_{\theta_x}(\cdot)\) and \(\nabla F_{\theta_x}(\cdot)\) using sample averages. We denote by \(R^0\) the cumulative reward, and \(T^a\) the length of the episode \(i\). Also, we denote by \(A^i_t\) and \(S^i_t\) the action and state at time \(t\) in episode \(i\), respectively. Let \(G^m_{\rho_x}(\cdot)\) denote the EDF of \(F_{\theta_x}(\cdot)\), and is defined by

\[
G^m_{\rho_x}(x) = \frac{1}{m} \sum_{i=1}^{m} \left\{ \mathbb{1} \{ R^0_i \leq x \} \right. \left. \right\}
\]

We form the estimate \(\hat{\nabla} G^m_{\rho_x}(\cdot)\) of \(\nabla F_{\theta_x}(\cdot)\) as follows:

\[
\hat{\nabla} G^m_{\rho_x}(x) = \frac{1}{m} \sum_{i=1}^{m} \left\{ \mathbb{1} \{ R^0_i \leq x \} \right. \left. \right\} \sum_{t=0}^{T^a_i-1} \nabla \log \pi_\theta(A^i_t | S^i_t) \]

(12)

Using the estimates from (9) and (10), we estimate the gradient \(\nabla \rho_g(\theta)\) in (8) as follows:

\[
\hat{\nabla} \rho_g(\theta) = - \int_{-\infty}^{M_r} g'(1 - G^m_{\rho_x}(x)) \hat{\nabla} G^m_{\rho_x}(x) dx.
\]

The gradient estimator in (11) is biased since \(\mathbb{E}[g'(1 - G^m_{\rho_x}(\cdot))] \neq g'(1 - F_{\theta_x}(\cdot))\), but the bias can be controlled by increasing the number of episodes \(m\). A bound for the MSE of this estimator is given below.

**Lemma 4:**

\[
\mathbb{E} \left[ \left\| \hat{\nabla} G_{\rho_g}(\theta) - \nabla \rho_g(\theta) \right\|^2 \right] \leq \frac{32 M_t^2 M_g^2 (c^2 M_g^2 + M_g'^2)}{m}
\]

**Proof:** See Section V-A

Using order statistics of \(m\) samples \(\{R^0_i\}_{i=1}^{m}\), we can compute the integral in (11) as given in the lemma below.

**Lemma 5:**

\[
\hat{\nabla} G_{\rho_g}(\theta) = \frac{1}{m} \sum_{i=1}^{m-1} \left( R^0_{i+1} - R^0_i \right) g' \left( 1 - \frac{i}{m} \right) \sum_{j=1}^{i} \nabla \rho_g(\theta_j) + \frac{1}{m} \left( R^0_m - M_r \right) g'_+(0) \sum_{j=1}^{m} \nabla \rho_g(\theta_j).
\]

**Proof:** See Appendix C

Algorithm 1: DRM-OnP-LR

1: **Input**: Parameterized form of the policy \(\pi_\theta\), iteration limit \(N\), step-size \(\alpha\), and batch size \(m\);
2: **Initialize**: Policy \(\theta_0 \in \mathbb{R}^d\), and discount factor \(\gamma \in (0, 1)\);
3: for \(k = 0, \ldots, N - 1\) do
4: Generate \(m\) episodes each using \(\pi_\theta_k\);
5: Use (12) to estimate \(\nabla C_{\rho_g}(\theta_k)\);
6: Use (13) to calculate \(\theta_{k+1}\);
7: end for
8: **Output**: Policy \(\theta_R\), where \(R \sim U\{1, N\}\).

In the above, \(R^0(i)\) is the \(i^{th}\) smallest order statistic from the samples \(\{R^0_i\}_{i=1}^{m}\), and \(\nabla \rho_g(\theta) = \sum_{i=0}^{T(i)-1} \nabla \log \pi_\theta(A^i_t | S^i_t)\), with \(T(i)\) denoting the length, and \(S^i_t\) and \(A^i_t\) the state and action at time \(t\) of the episode corresponding to \(R^0(i)\). Here \(g'_+(0)\) is the right derivative of the distortion function \(g\) at 0.

We solve (4) using the following update iteration:

\[
\theta_{k+1} = \theta_k + \alpha \hat{\nabla} C_{\rho_g}(\theta_k).
\]

Algorithm [1] presents the pseudocode of DRM-OnP-LR.

2) **DRM optimization in off-policy RL settings**: In an off-policy RL setting, we optimize the DRM of \(R^0\) from the episodes generated by a behavior policy \(b\), using the importance sampling ratio. We require the behavior policy \(b\) to be proper, i.e.,

\[(A6): \exists M > 0: \max_{s \in S} \mathbb{P}(S_M \neq 0 \mid S_0 = s, b) < 1\]

We also assume that the target policy \(\pi_\theta\) is absolutely continuous w.r.t. the behavior policy \(b\), i.e.,

\[(A7): \forall \theta \in \mathbb{R}^d, b(a|s) = 0 \Rightarrow \pi_\theta(a|s) = 0, \forall a \in A, \forall s \in S\]

An assumption like \([A7]\) is common for the analysis in an off-policy RL setting (cf. [27]).

The cumulative discounted reward \(R^b\) is defined by

\[
R^b = \sum_{t=0}^{T-1} \gamma^t r(S_t, A_t, S_{t+1}),
\]

where \(A_t \sim b(\cdot, S_t), S_{t+1} \sim p(\cdot, S_t, A_t), \gamma \in (0, 1),\) and \(T\) is the random length of an episode. From \([A6]\) we infer that \(\mathbb{E}[T] < \infty\). This fact in conjunction with \(T \geq 0\) implies the following bound:

\[
\exists M_e > 0: T \leq M_e, \ a.s.
\]

The importance sampling ratio \(\psi^b\) is defined by

\[
\psi^b = \prod_{t=0}^{T-1} \frac{\pi_\theta(A_t | S_t)}{b(A_t | S_t)}.
\]

From \([A2]\) and \([A7]\) we obtain \(\forall \theta \in \mathbb{R}^d, \pi_\theta(a|s) > 0\) and \(b(a|s) > 0, \forall a \in A, \forall s \in S\). This fact in conjunction with \(\psi^b\) implies the following bound for \(\psi^b\):

\[
\exists M_s > 0: \forall \theta \in \mathbb{R}^d, \psi^b \leq M_s, \ a.s.
\]

We express the CDF \(F_{\theta_x}(\cdot)\), and its gradient \(\nabla F_{\theta_x}(\cdot)\) as expectations w.r.t the episodes from the policy \(b\). Starting with

\[
F_{\theta_x}(x) = \mathbb{E} \left[ \mathbb{1} \{ R^b \leq x \} \psi^b \right],
\]

we obtain the following analogue of Lemma 1 for the off-policy case.
\[ \nabla F_{R^b}(x) = \mathbb{E} \left[ \{ R^b \leq x \} \psi^\theta \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t) \right]. \] (19)

**Proof:** See Section VII-B.

We generate \( m \) episodes using the policy \( b \) to estimate \( F_{R^b}(\cdot) \) and \( \nabla F_{R^b}(\cdot) \) using sample averages. We denote by \( R_i^b \) the cumulative reward, \( \psi_i^\theta \) the importance sampling ratio.

We form the estimate \( \hat{H}_{R^b}^m(x) \) of \( H_{R^b}(x) \) as follows:

\[ \hat{H}_{R^b}^m(x) = \min \{ \hat{H}_{R^b}^m(x), 1 \}, \] (20)

\[ \hat{H}_{R^b}^m(x) = \frac{1}{m} \sum_{i=1}^{m} \{ R_i^b \leq x \} \psi_i^\theta. \] (21)

In the above, because of the importance sampling ratio, \( \hat{H}_{R^b}^m(x) \) can get a value above 1. Since we are estimating a CDF, we restrict \( \hat{H}_{R^b}^m(x) \) to one in \( H_{R^b}(x) \).

We form the estimate \( \hat{\nabla} H_{R^b}^m(\cdot) \) of \( \nabla F_{R^b}(\cdot) \) as follows:

\[ \hat{\nabla} H_{R^b}^m(x) = \frac{1}{m} \sum_{i=1}^{m} \{ R_i^b \leq x \} \psi_i^\theta \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t). \] (22)

Using (20) and (22), we estimate \( \nabla \rho_g(\theta) \) by

\[ \hat{\nabla} H_{R^b}(\theta) = - \int_{-M_r}^{M_r} \nabla \hat{H}_{R^b}^m(x) dx. \] (23)

As in the on-policy case, the estimator in (23) is biased, but can be controlled by increasing the number of episodes \( m \). A bound on the MSE of our estimator is given below.

**Lemma 7:** \( \mathbb{E} \left[ \left\| \hat{\nabla} H_{R^b}(\theta) - \nabla \rho_g(\theta) \right\|^2 \right] \leq 32 M_r^2 M_2^2 M_2^2 \left( c^2 M_r^2 + M_g^2 M_3^2 \right). \)

**Proof:** See Section VII-B.

As in the on-policy case, the integral in (23) can be computed using order statistics of the samples \( \{ R_i^b \}_{i=1}^{m} \), as given in the lemma below.

**Lemma 8:** \( \hat{\nabla} H_{R^b}(\theta) = \frac{1}{m} \sum_{i=1}^{m-1} \left( \left( R_{(i)}^b - R_{(i+1)}^b \right) \right) \times g'(1 - \min \left( \left\{ \left( 1, \frac{1}{m} \sum_{j=1}^{i} \psi_j \right) \right\} \right) \sum_{j=1}^{i} \nabla \pi_\theta(A_t | S_t) \psi_j \]

\[ + \frac{1}{m} \left( R_{(m)}^b - M_r \right) g'_\theta(0) \sum_{j=1}^{m} \nabla \pi_\theta(A_t | S_t) \psi_j. \] (24)

**Proof:** See Appendix I-D.

In the above, \( R_{(i)}^b \) is the \( i \)th smallest order statistic from the samples \( \{ R_i^b \}_{i=1}^{m} \), and \( \psi_i \) is the importance sampling ratio corresponding to \( R_{(i)}^b \). Also, \( \nabla \pi_\theta(A_t | S_t) = \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t), \) with \( T \) denoting the length, and \( S_t \) and \( A_t \) are the state and action at time \( t \) of the episode corresponding to \( R_{(i)}^b \).

We solve (24) using the following update iteration:

\[ \theta_{k+1} = \theta_k + \alpha \hat{\nabla} H_{R^b}(\theta_k). \] (25)

The pseudo-code of DRM-OffIP-LR algorithm is similar to Algorithm 11 except that we generate episodes using the policy \( b \), and use (24) and (25) in place of (12) and (13), respectively.

### C. DRM optimization: SF-based gradient estimation

We now present our second method which uses an SF-based gradient estimation scheme [17], [28]. The SF-based methods form a smoothed version of the DRM \( \rho_g(\cdot) \) as \( \rho_{g,\mu}(\cdot) \), and use the gradient \( \nabla \rho_{g,\mu} \) as an approximation for \( \nabla \rho_g \). The smoothed functional \( \rho_{g,\mu}(\theta) \) is defined as

\[ \rho_{g,\mu}(\theta) = \mathbb{E}_{a \in \mathcal{A}} \left[ \rho_g(\theta + \mu a) \right], \] (26)

where \( u \) is sampled uniformly at random from the unit ball \( \mathbb{B} = \{ x \in \mathbb{R}^d \mid \| x \| \leq 1 \} \), and \( \mu \in (0, 1] \) is the smoothing parameter. From [29, Lemma 2.1], we obtain the gradient of \( \rho_{g,\mu}(\theta) \) as follows:

\[ \nabla \rho_{g,\mu}(\theta) = \mathbb{E}_{v \in \mathbb{B}^{-1}} \left[ \frac{d}{\mu} \rho_g(\theta + \mu v) v \right]. \] (27)

where \( v \) is sampled uniformly at random from the unit sphere \( \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d \mid \| x \| = 1 \} \). The gradient \( \nabla \rho_{g,\mu}(\theta) \) is estimated as follows:

\[ \hat{\nabla} \rho_{g,\mu}(\theta) = \frac{d}{n} \sum_{i=1}^{n} \frac{\rho_g(\theta + \mu v_i) - \rho_g(\theta - \mu v_i)}{2\mu} v_i, \] (28)

where \( v_i \) is sampled uniformly at random from \( \mathbb{S}^{d-1} \). The gradient estimate is averaged over \( n \) unit vectors to reduce the variance. Using the proof technique from [30], we show that \( \hat{\nabla} \rho_{g,\mu}(\theta) \) is an unbiased estimator of \( \nabla \rho_{g,\mu}(\theta) \).

**Lemma 9:** \( \mathbb{E}[\hat{\nabla} \rho_{g,\mu}(\theta) | \theta] = \nabla \rho_{g,\mu}(\theta). \)

**Proof:** See Section VII-A.

As we do not have a direct measurement of the DRM \( \rho_g(\cdot) \), we first estimate the DRM and then approximate the gradient \( \nabla \rho_{g,\mu}(\cdot) \). The following sections describe our algorithms.

1. **DRM optimization in on-policy RL settings:** In an on-policy RL setting, we generate \( m \) episodes using the policy \( \pi_\theta \). We consider the CDF estimate \( G_{R^b}(\cdot) \) of \( F_{R^b}(\cdot) \) as in (9), and we form an estimate \( \hat{\rho}_g(\cdot) \) of \( \rho_g(\cdot) \) as follows:

\[ \hat{\rho}_g^G(\theta) = \int_{-M_r}^{M_r} (g(1 - G_{R^b}(x)) - 1) dx + \int_{0}^{M_r} g(1 - G_{R^b}(x)) dx. \] (29)

Comparing (29) with (3), it is apparent that we have used the EDF \( G_{R^b} \) in place of the true cdf \( F_{R^b} \).

The DRM estimator in (29) is biased since \( \mathbb{E}[g(1 - G_{R^b}(x))] \neq g(1 - F_{R^b}(x)) \), but we can control the bias by increasing the number of episodes \( m \). A bound on the MSE of this estimator is given below.

**Lemma 10:** \( \mathbb{E} \left[ \left| \hat{\rho}_g(\theta) - \hat{\rho}_g^G(\theta) \right|^2 \right] \leq \frac{16 M_r^2 M_2^2}{m}. \)

**Proof:** See Section VII-B.

Using order statistics of \( m \) samples \( \{ R_i^b \}_{i=1}^{m} \), we can compute the integral in (29) as given below.

**Lemma 11:** \( \hat{\rho}_g(\theta) = \frac{1}{m} \sum_{i=1}^{m} R_i^b \left( 1 - i - 1 \right) - g \left( 1 - i \right) \). (30)

**Proof:** See Appendix I-A.

In the above, \( R_i^b \) is the \( i \)th smallest order statistic of the samples \( \{ R_i^b \}_{i=1}^{m} \). If we choose the distortion function as the identity function, then the estimator in (30) is merely the sample mean.
We estimate the gradient \( \nabla \rho_g(\theta) \) using two randomly perturbed policies, namely \( \theta + \mu v \) and \( \theta - \mu v \), where \( v \) is a random unit vector sampled uniformly from the surface of a unit sphere. The estimate \( \hat{\nabla}_{\mu,n} \rho_g^G(\theta) \) of \( \nabla \rho_g(\theta) \) is formed as follows:

\[
\hat{\nabla}_{\mu,n} \rho_g^G(\theta) = \frac{d}{n} \sum_{i=1}^{n} \frac{\hat{\rho}_g^G(\theta + \mu v_i) - \hat{\rho}_g^G(\theta - \mu v_i)}{2\mu}, \tag{31}
\]

where \( \forall i, v_i \) is sampled uniformly at random from \( \mathbb{S}^{d-1} \), and \( \hat{\rho}_g^G(\cdot) \) is as defined in (30).

The gradient estimator in (31) is biased, but we can control it by controlling the smoothing parameter \( \mu \), and the parameter \( n \) that is used for averaging. A bound on the MSE of this estimator is given below.

**Lemma 12:**

\[
\mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \rho_g^G(\theta) - \nabla \rho_g(\theta) \right\|^2 \right] 
\leq \frac{\mu^2 d^2 L_g^2 + 4d^2 L_{\mu}^2 + 16d^2 M_g^2 M_{\mu}^2}{\mu^2 mn}.
\]

**Proof:** See Section VII-B.

The update iteration in DRM-OnP-SF is as follows:

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + \alpha \hat{\nabla}_{\mu,n} \hat{\rho}_g^H(\hat{\theta}_k). \tag{32}
\]

Algorithm 2 presents the pseudocode of DRM-OnP-SF.

**2) DRM optimization in off-policy RL settings:** Each iteration of DRM-OnP-SF requires \( 2mn \) episodes corresponding to \( 2n \) perturbed policies (see Algorithm 2). In some practical applications, it may not be feasible to generate system trajectories corresponding to different perturbed policies. In our second algorithm DRM-OffP-SF, we overcome the aforementioned problem by performing off-policy evaluation, i.e., we collect episodes from a behavior policy \( b \), and estimate the values of the perturbed target policies. Using off-policy setting, the number of episodes needed in each iteration of our algorithm can be reduced to \( m \).

We generate \( m \) episodes using the policy \( b \) to estimate the cdf \( F_{R^b}(\cdot) \) using importance sampling. We define the cumulative discounted reward \( R^b \) and the importance sampling ratio \( \psi^b \) as in (14) and (16) respectively, and form the CDF estimate \( H_{R^b}^m(\cdot) \) of \( F_{R^b}(\cdot) \) as in (20).

Now we form an estimate \( \hat{\rho}_g^H(\theta) \) of \( \rho_g(\theta) \) as

\[
\hat{\rho}_g^H(\theta) = \int_{-M_r}^{0} (g(1-H_{R^b}^m(x))-1)dx + \int_{0}^{M_r} g(1-H_{R^b}^m(x))dx. \tag{33}
\]

Now we form an estimate \( \hat{\rho}_g^H(\theta) \) of \( \rho_g(\theta) \) as

**Lemma 13:**

\[\mathbb{E} \left[ \left| \rho_g(\theta) - \hat{\rho}_g^H(\theta) \right|^2 \right] \leq \frac{16M_g^2 M_{\mu}^2 M_{\mu}^2}{m}.
\]

**Proof:** See Section VII-C.

As in the on-policy case, we compute the integral in (33) using order statistics of the samples \( \{R_{i,b}^b\}_{i=1}^m \), as given below.

**Lemma 14:**

\[
\hat{\rho}_g^H(\theta) = R_{i,b}^b + \sum_{i=2}^{m} R_{i,b}^b \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{k=1}^{i-1} \psi^b(k) \right\} \right) - \sum_{i=1}^{m-1} R_{i,b}^b \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{k=1}^{i} \psi^b(k) \right\} \right), \tag{34}
\]

where \( R_{i,b}^b \) is the \( i \)-th smallest order statistic of the samples \( \{R_{i,b}^b\}_{i=1}^m \), and \( \psi^b \) is the importance sampling ratio of \( R_{i,b}^b \).

**Proof:** See Appendix I-B.

Now, we use the SF-based gradient estimation scheme as in on-policy RL settings to form \( \hat{\nabla}_{\mu,n} \hat{\rho}_g^H(\theta) \) as follows:

\[
\hat{\nabla}_{\mu,n} \hat{\rho}_g^H(\theta) = \frac{d}{n} \sum_{i=1}^{n} \hat{\rho}_g^H(\theta + \mu v_i) - \hat{\rho}_g^H(\theta - \mu v_i), \tag{35}
\]

As in on-policy case, the estimator in (35) is biased. A bound for the MSE of this estimator is given below.

**Lemma 15:**

\[\mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \hat{\rho}_g^H(\theta) - \nabla \rho_g(\theta) \right\|^2 \right] \leq \frac{4d^2 L_{\mu}^2 + 2d^2 L_{\mu}^2 + 16d^2 M_g^2 M_{\mu}^2}{\mu^2 mn}.
\]

**Proof:** See Section VII-C.

The update iteration in DRM-OffP-SF is as follows:

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + \alpha \hat{\nabla}_{\mu,n} \hat{\rho}_g^H(\hat{\theta}_k). \tag{36}
\]

The pseudocode of DRM-offP-SF is similar to Algorithm 2, but in each iteration, we get \( m \) episodes from policy \( b \). Then, we generate \( n \) DRM estimates \( \{\hat{\rho}_g^H(\theta_k + \mu v_i), i \in \{1, \ldots, n\}\} \) using (34). We estimate the gradient using (35), and use the policy parameter update rule (36).

**Remark 1:** Our DRM-offP-SF algorithm has a similar format to that of the algorithm OffP-SF in [30]. In [30], the objective function is the mean of the cumulative rewards, and hence the authors could derive an unbiased estimator for the same. However, for the DRM objective, we do not have an unbiased estimator, leading to significant deviations in the convergence analysis for DRM-offP-SF.

**IV. MAIN RESULTS**

Our non-asymptotic analysis establishes a bound on the number of iterations of our proposed algorithms to find an \( \epsilon \)-stationary point of the DRM, which is defined below.

**Definition 1:** (\( \epsilon \)-stationary point) Let \( \theta_R \) be the output of an algorithm. Then, \( \theta_R \) is called an \( \epsilon \)-stationary point of problem (4), if \( \mathbb{E} \left[ \left\| \nabla \rho_g(\theta_R) \right\|^2 \right] \leq \epsilon \).
In an RL setting, the objective need not be convex. Hence, it is common in the literature to establish convergence of policy gradient algorithms to an $\epsilon$-stationary point (cf. [25], [26]). We derive convergence rate of our algorithms for a random iterate $\theta_R$, that is chosen uniformly at random from the policy parameters $\{\theta_1, \cdots, \theta_N\}$.

### A. Non-asymptotic bounds (LR-based scheme)

We provide a convergence rate for the algorithm DRM-OnP-LR and DRM-OffP-LR below.

**Theorem 2:** (DRM-OnP-LR) Assume \[ (A1) \] \[ (A5) \] Let $\{\theta_i\}_{i=1}^N$ be the policy parameters generated by DRM-OnP-LR using (13), and let $\theta_R$ be chosen uniformly at random from this set. Then,

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4M^2 M_2^2 M_d^2}{\sqrt{N}} + 8\left( e^2 M_2^2 + M_2^2 \right) + \frac{8M_2^2}{m}.
$$

(37)

In the above, $\rho_0^g = \max_{\theta \in \mathbb{R}^d} \rho_g(\theta)$, $M_\rho = \frac{\sqrt{\mathbb{E} }\left[\rho_g^2\right]}{\max \left(\rho_g^2\right)}$, and $L_{\rho'}$ is as in Lemma 2. The constants $M_2, M_d, M_2'$, and $M_\rho$ are as defined in (A2)(A5) while $M_c$ is an upper bound on the episode length from (2).

**Proof:** See Section VII-A.

We specialize the bound in (37) to a particular choice of step-size $\alpha$, and batch size $m$ in the corollary below.

**Corollary 1:** Set $\alpha = \frac{1}{\sqrt{N}}$, and $m = \sqrt{N}$. Then, under the conditions of Theorem 2, we have

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4M^2 M_2^2 M_d^2}{\sqrt{N}} + 8\left( e^2 M_2^2 + M_2^2 \right).
$$

(38)

**Theorem 3:** (DRM-OffP-LR) Assume \[ (A1) \] \[ (A7) \] Let $\{\theta_i\}_{i=1}^N$ be the policy parameters generated by DRM-OffP-LR using (25), and let $\theta_R$ be chosen uniformly at random from this set. Then,

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4M^2 M_2^2 M_d^2}{\sqrt{N}} + 8\left( e^2 M_2^2 + M_2^2 \right) M_\rho\rho_0^g M_\rho^2 M_\rho^2 L_{\rho'} + \frac{8e^2 M_2^2}{m}.
$$

where $\rho_0^g$, $M_\rho$, $L_{\rho'}$, $M_\rho$, $M_\rho$, and $M_\rho'$ are as defined in Theorem 2. The constant $M_\rho$ is an upper bound on the importance sampling ratio from (17), and $M_c$ is an upper bound on the episode length from (15).

**Proof:** See Section VII-B.

We specialize the bound in (38) to a particular choice of step-size $\alpha$, and batch size $m$ in the corollary below.

**Corollary 2:** Set $\alpha = \frac{1}{\sqrt{N}}$, and $m = \sqrt{N}$. Then, under the conditions of Theorem 3, we have

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4M^2 M_2^2 M_d^2}{\sqrt{N}} + 8\left( e^2 M_2^2 + M_2^2 \right) M_\rho\rho_0^g M_\rho^2 M_\rho^2 L_{\rho'} + \frac{8e^2 M_2^2}{m}.
$$

(38)

### B. Non-asymptotic bounds (SF-based scheme)

We provide a convergence rate for the algorithm DRM-OnP-SF and DRM-OffP-SF below.

**Theorem 4:** (DRM-OnP-SF) Assume \[ (A1) \] \[ (A7) \] Let $\{\theta_i\}_{i=1}^N$ be the policy parameters generated by DRM-OnP-SF, and let $\theta_R$ be chosen uniformly at random from this set. Then,

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4d^2 L_\rho^2}{\sqrt{N}} + \frac{\mu^2 d^2 L_{\rho'}^2}{\mu^2 m n} + \frac{\mu^2 d^2 L_{\rho'}^2}{\mu^2 m n}.
$$

where $\rho_0^g$, $M_\rho$, $L_{\rho'}$, $L_{\rho}$, and $M_\rho'$ are as in Theorem 4, while $M_\rho$ is as in (17).

**Proof:** See Section VII-C.

We specialize the bound in (39) to a particular choice of step-size $\alpha$, smoothing parameter $\mu$, and batch sizes $m$ and $n$ in the corollary below.

**Corollary 3:** Set $\alpha = \frac{1}{\sqrt{N}}$, $\mu = \frac{1}{\sqrt{N}}$, $n = N$, and $m = C_1$. Then, under the conditions of Theorem 4, we have

$$
E \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \frac{2}{N\alpha} \left( \rho_0^g - \rho_g(\theta_0) \right) + \frac{4d^2 L_\rho^2}{\sqrt{N}} + \frac{\mu^2 d^2 L_{\rho'}^2}{\mu^2 m n} + \frac{\mu^2 d^2 L_{\rho'}^2}{\mu^2 m n}.
$$

where $\rho_0^g$, $M_\rho$, $L_{\rho'}$, $L_{\rho}$, and $M_\rho'$ are as in Theorem 4, while $M_\rho$ is as in (17).

**Proof:** See Section VII-C.
Also, let

$$R(\omega) = \sum_{t=0}^{T(\omega)-1} \gamma^t r(S_t(\omega), A_t(\omega), S_{t+1}(\omega)).$$

Also, let

$$\mathbb{P}_\theta(\omega) = \prod_{t=0}^{T(\omega)-1} \pi_\theta(A_t(\omega)|S_t(\omega)) p(S_{t+1}(\omega), S_t(\omega), A_t(\omega)).$$

From (41), we obtain

$$\frac{\nabla \mathbb{P}_\theta(\omega)}{\mathbb{P}_\theta(\omega)} = \sum_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega)|S_t(\omega)).$$

Now,

$$\nabla F_{\mathbb{P}_\theta}(x) = \nabla \mathbb{E}\{1 \{R^d \leq x\} \} = \sum_{\omega \in \Omega} \nabla \mathbb{E}\{1 \{R(\omega) \leq x\} \} \mathbb{P}_\theta(\omega)$$

$$= \sum_{\omega \in \Omega} \nabla \{1 \{R(\omega) \leq x\} \} \mathbb{P}_\theta(\omega)$$

$$= \sum_{\omega \in \Omega} \nabla \mathbb{P}_\theta(\omega) \mathbb{P}_\theta(\omega)$$

$$= \sum_{\omega \in \Omega} \nabla \sum_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega)|S_t(\omega)) \mathbb{P}_\theta(\omega)$$

$$= \mathbb{E}\left[1 \{R^d \leq x\} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t) \right].$$

In the above, (a) follows by an application of the dominated convergence theorem to interchange the differentiation and the expectation operation. The aforementioned application is allowed since (i) \(\Omega\) is finite and the underlying measure is bounded, as we consider an MDP where the state and actions spaces are finite, and the policies are proper, (ii) \(\nabla \log \pi_\theta(A_t|S_t)\) is bounded, as we consider an MDP where the state and actions spaces are finite, and the policies are proper, (iii) \(\nabla \log \pi_\theta(A_t|S_t)\) is bounded from (A2). The step (b) follows, since for a given episode \(\omega\), the cumulative reward \(R(\omega)\) does not depend on \(\theta\), and (c) follows from (A2).

Next, we derive an expression for the gradient of the CDF. We use this expression to prove the Lipschitz property of the CDF gradient (see Lemma 18) which in turn is used to prove that the DRM \(\rho_\theta(\theta)\) is a smooth function of \(\theta\), see Lemma 3.

Let \(\Omega\) denote the set of all sample episodes. For any episode \(\omega \in \Omega\), we denote by \(T(\omega)\), its length, and \(S_t(\omega)\) and \(A_t(\omega)\), the state and action at time \(t \in \{0, 1, \ldots\}\) respectively.

For any episode \(\omega\), let the cumulative discounted reward be

$$R(\omega) = \sum_{t=0}^{T(\omega)-1} \gamma^t r(S_t(\omega), A_t(\omega), S_{t+1}(\omega)).$$

The following lemma establishes an upper bound for the gradient and the Hessian of the CDF.

**Lemma 17:** \(\forall x \in (-M, M_r),\)

$$\|\nabla F_{\mathbb{P}_\theta}(x)\| \leq M_e M_d, \quad \text{and} \quad \|\nabla^2 F_{\mathbb{P}_\theta}(x)\| \leq M_e M_h + M_r^2 M_2^2.$$

**Proof:** From (A2) (A4) and (3), for any \(x \in (-M_r, M_r)\), we have

$$\|1 \{R^d \leq x\} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t)\| \leq M_e M_d \text{ a.s.}, \quad (44)$$

and

$$\|1 \{R^d \leq x\} \left(\sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t)\right) \left(\sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t)\right)^T\| \leq M_e M_h + M_r^2 M_2^2 \text{ a.s.}, \quad (45)$$

From Lemma 1 and 16, for any \(x \in (-M_r, M_r)\), we have

$$\|\nabla F_{\mathbb{P}_\theta}(x)\| \leq \mathbb{E}\left[1 \{R^d \leq x\} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t|S_t)\right].$$

Remark 3: From Corollary 3 (or 4), we conclude that after \(N\) iterations of (32) (or 36), algorithm DRM-OnP-SF (or DRM-OffP-SF) returns an iterate that satisfies

$$\mathbb{E}\left[\left\|\nabla \rho_\theta(\theta)\right\|^2\right] = O(1/\sqrt{N}).$$

So we conclude that both SF and LR algorithm variants have the same rate of convergence. However, LR method require \(O(\sqrt{N})\) episodes per iteration for both on-policy and off-policy RL settings, whereas the SF method requires only \(O(1)\) episodes per iteration for off-policy RL setting, though it requires \(O(N)\) episodes per iteration for on-policy RL setting.

Remark 4: Since a DRM with an identity distortion function is simply the mean of the rewards, we could compare our convergence rate with that of a classical policy gradient algorithm [31]. In [30], in an off-policy RL setting, the authors establish a non-asymptotic bound of the order \(O(1/\sqrt{N})\) for a policy gradient algorithm with a gradient estimation scheme or a LR-based gradient estimation scheme, in the spirit of REINFORCE. Specializing the bound in Theorem 3 to the case of identity distortion function, it is apparent that the rate we obtain is comparable to that of a classical policy gradient algorithm.
where these inequalities follow from (44), (45), and the assumption that the state and action spaces are finite.

The following lemma establishes the Lipschitzness of the CDF and its gradient.

**Lemma 18:** \( \forall x \in (-M_r, M_r), \\
\|F_{R_{b1}}(x) - F_{R_{b2}}(x)\| \leq M_e M_d \|\theta_1 - \theta_2\|, \) and
\[ \|\nabla F_{R_{b1}}(x) - \nabla F_{R_{b2}}(x)\| \leq (M_e M_h + M_e^2 M_d^2) \|\theta_1 - \theta_2\|. \]

**Proof:** The result follows from Lemma 17 and [32, Lemma 1.2.2].

**B. DRM policy gradient theorem**

**Proof of Theorem 7** Notice that
\[
\n\n\left(\n\right)
\n\n
\n\n\n\n\nIn the above, (a) follows by an application of the dominated convergence theorem to interchange the differentiation and the expectation operation. The aforementioned application is allowed since (i) \( \rho_\theta(\theta) \) is finite for any \( \theta \in \mathbb{R}^d; \)
(ii) \( |g'(\cdot)| \leq M_{g'} \) from (A3) and \( \nabla F_{R_b}(\cdot) \) is bounded from Lemma 17. The bounds on \( g' \) and \( \nabla F_{R_b} \) imply
\[
\int_{-M_r}^{M_r} \|g'(1 - F_{R_{b1}}(x))\| \|\nabla F_{R_{b1}}(x)\| \|\theta_1 - \theta_2\| \]
\[
\leq 2 M_e M_d M_e M_d \|\theta_1 - \theta_2\|,
\]
where (a) follows from (A3) and Lemmas 19 and 17 and (b) follows from Lemma 18.

**C. Lipschitzness and smoothness of the DRM \( \rho_\theta(\theta) \)**

We first establish that \( g \) and \( g' \) are Lipschitz in the lemma below. This result will be used for proving Lemmas 24 later.

**Lemma 19:** \( \forall t, t' \in (0, 1), \\
g(t) - g(t') \leq M_g |t - t'|; \quad g'(t) - g'(t') \leq M_{g'} |t - t'|. \)

**Proof:** Using the mean value theorem, we obtain
\[ g(t) - g(t') = g'(\tilde{t})(t - t'), \tilde{t} \in (t, t'), \]
and from (A3) we have \( |g'(\tilde{t})| \leq M_{g'}, \forall \tilde{t} \in (0, 1). \) Hence,
\[ |g(t) - g(t')| \leq M_{g'} |t - t'| \quad \forall t, t' \in (0, 1). \]

Similarly, we obtain
\[ g'(t) - g'(t') = g''(\tilde{t})(t - t'), \tilde{t} \in (t, t'), \]
and from (A3) we have \( |g''(\tilde{t})| \leq M_{g''}, \forall \tilde{t} \in (0, 1). \) Hence,
\[ |g'(t) - g'(t')| \leq M_{g''} |t - t'| \quad \forall t, t' \in (0, 1). \]

**Proof of Lemma 2** From (3), we obtain
\[ |\rho_\theta(\theta_1) - \rho_\theta(\theta_2)| \]
\[ \leq \int_{-M_r}^{M_r} |g(1 - F_{R_{b1}}(x)) - g(1 - F_{R_{b2}}(x))| \|\theta_1 - \theta_2\| \]
\[ \leq M_{g'} \int_{-M_r}^{M_r} |F_{R_{b1}}(x) - F_{R_{b2}}(x)| \|\theta_1 - \theta_2\| \]
\[ \leq 2 M_e M_{g'} M_e M_d \|\theta_1 - \theta_2\| = L_{\rho} \|\theta_1 - \theta_2\|,
\]
where (a) follows from Lemma 19 and (b) follows from Lemma 18.

**Proof of Lemma 3** From Theorem 1 we obtain
\[ \|\nabla \rho_\theta(\theta_1) - \nabla \rho_\theta(\theta_2)\| \]
\[ \leq \int_{-M_r}^{M_r} \|g'(1 - F_{R_{b1}}(x))\| \|\theta_1 - \theta_2\| \]
\[ \leq \int_{-M_r}^{M_r} \|g'(1 - F_{R_{b2}}(x))\| \|\theta_1 - \theta_2\| \]
\[ \leq \int_{-M_r}^{M_r} \|g'(1 - F_{R_{b1}}(x))\| \|\theta_1 - \theta_2\| \]
\[ \leq 2 M_e M_{g'} M_e M_d \|\theta_1 - \theta_2\|,
\]
where (a) follows from (A3) and Lemmas 19 and 17 and (b) follows from Lemma 18.

**VI. NON-ASYMPTOTIC CONVERGENCE ANALYSIS**

**(LR-BASED GRADIENT ESTIMATION SCHEME)**

**A. Analysis of DRM-OnP-LR**

In the lemma below, we establish an upper bound on the variance of the DRM gradient estimate \( \nabla G \rho_\theta(\theta) \) as defined in (11). Subsequently, we use this result to prove Lemma 4 and Theorem 2.

**Lemma 20:** \( \mathbb{E} \|\nabla G \rho_\theta(\theta)\|^2 \leq 4M_{\rho}^2 M_{\rho}^2 M_{\rho}^2. \)

**Proof:** From (10), using (44) from Lemma 17 we obtain
\[ \|\nabla G_{R_b}(x)\|^2 \leq M_{\rho}^2 M_{\rho}^2 \text{ a.s., } \forall x \in (-M_r, M_r). \]

From (11), and (A3) we obtain
\[ \mathbb{E} \|\nabla G \rho_\theta(\theta)\|^2 \leq M_{\rho}^2 \mathbb{E} \left[ \int_{-M_r}^{M_r} \|\nabla G_{R_b}(x)\|^2 dx \right]. \]
\[ a \leq 2M_r M_g^2 E \left[ \int_{-M_r}^{M_r} \left\| \nabla G_{R^e}^m(x) \right\|^2 dx \right] \]
\[ b \leq 2M_r M_g^2 \int_{-M_r}^{M_r} E \left[ \left\| \nabla G_{R^e}^m(x) \right\|^2 \right] dx \]
\[ c \leq 2M_r M_g^2 \int_{-M_r}^{M_r} M_e^2 M_f^2 dx = 4M_r^2 M_g^2 M_e^2 M_f^2, \]

where (a) follows from the Cauchy-Schwarz inequality, (b) follows from the Fubini’s theorem, and (c) follows from (48).

**Proof of Lemma 4** From the fact that \( \forall x \in (-M_r, M_r), \) \( 1 \{ R^e \leq x \} \leq 1 \) a.s., we observe that \( \left\{ \left( G_{R^e}^m(x) - F_{R^e}(x) \right) \right\}_{m=1}^\infty \) is a set of partial sums of bounded mean zero r.v.s, and hence they are martingales. Using Azuma-Hoeffding’s inequality, we obtain \( \forall x \in (-M_r, M_r), \)

\[ \mathbb{P} \left( \left| G_{R^e}^m(x) - F_{R^e}(x) \right| > \epsilon \right) \leq 2 \exp \left( -\frac{m\epsilon^2}{2} \right), \]

and

\[ \mathbb{E} \left[ \left| G_{R^e}^m(x) - F_{R^e}(x) \right|^2 \right] = \int_0^\infty \mathbb{P} \left( \left| G_{R^e}^m(x) - F_{R^e}(x) \right| > \sqrt{\epsilon} \right) d\epsilon \]
\[ \leq \int_0^\infty 2 \exp \left( -\frac{m\epsilon}{2} \right) d\epsilon = \frac{4}{m}. \]

(49)

Using (44) from Lemma 17, we observe that \( \left\{ \left( \nabla G_{R^e}^m(x) - \nabla F_{R^e}(x) \right) \right\}_{m=1}^\infty \) is a set of partial sums of bounded mean zero r.v.s, and hence they are martingales. Using vector version of Azuma-Hoeffding inequality from Theorem 1.8-1.9 in [33], for any \( x \in (-M_r, M_r), \) we have

\[ \mathbb{P} \left( \left| \nabla G_{R^e}^m(x) - \nabla F_{R^e}(x) \right| > \epsilon \right) \leq 2e^2 \exp \left( -\frac{m\epsilon^2}{2M_e^2 M_f^2} \right), \]

\[ \mathbb{E} \left[ \left| \nabla F_{R^e}(x) - \nabla G_{R^e}^m(x) \right|^2 \right] = \int_0^\infty \mathbb{P} \left( \left| \nabla F_{R^e}(x) - \nabla G_{R^e}^m(x) \right| > \sqrt{\epsilon} \right) d\epsilon \]
\[ \leq \int_0^\infty 2e^2 \exp \left( -\frac{m\epsilon}{2M_e^2 M_f^2} \right) d\epsilon = \frac{4e^2M_e^2 M_f^2}{m}. \]

(50)

From (48), for any \( x \in (-M_r, M_r), \) we have

\[ \mathbb{E} \left[ \left| g'(1 - F_{R^e}(x)) - g'(1 - G_{R^e}^m(x)) \right| \nabla G_{R^e}^m(x) \right)^2 \]
\[ \leq M_e^2 M_f^2 \mathbb{E} \left[ \left| g'(1 - F_{R^e}(x)) - g'(1 - G_{R^e}^m(x)) \right|^2 \right] \leq \frac{4M_e^2 M_g^2 M_f^2}{m}, \]

(51)

where the last two inequalities follow from Lemma 19 in Section 1 and (49). From (A.3) for any \( x \in (-M_r, M_r), \) we have

\[ \mathbb{E} \left[ \left| g'(1 - F_{R^e}(x)) \right| \nabla F_{R^e}(x) - \nabla G_{R^e}^m(x) \right|^2 \]
\[ \leq M_g^2 \mathbb{E} \left[ \left| \nabla F_{R^e}(x) - \nabla G_{R^e}^m(x) \right|^2 \right] \leq \frac{4e^2M_g^2 M_e^2 M_f^2}{m}, \]

(52)

where the last inequality follows from (50).

From (8), (11), and Cauchy-Schwarz inequality, we have

\[ \mathbb{E} \left[ \left\| \nabla G_{\rho^2}(\theta) - \nabla \rho^2(\theta) \right\|^2 \right] \leq 2M_r \mathbb{E} \left[ \left\| g'(1 - F_{R^e}(x)) \nabla F_{R^e}(x) - g'(1 - G_{R^e}^m(x)) \nabla G_{R^e}^m(x) \right\|^2 \right] \]
\[ \leq 2M_r \mathbb{E} \left[ \left\| g'(1 - F_{R^e}(x)) - g'(1 - G_{R^e}^m(x)) \nabla G_{R^e}^m(x) \right\|^2 \right] \]
\[ \leq 4M_r \mathbb{E} \left( \left\| g'(1 - F_{R^e}(x)) \nabla F_{R^e}(x) - \nabla G_{R^e}^m(x) \right\|^2 \right) \]
\[ + \mathbb{E} \left( \left\| g'(1 - F_{R^e}(x)) - g'(1 - G_{R^e}^m(x)) \nabla G_{R^e}^m(x) \right\|^2 \right) \]
\[ \leq 32M_e^2 M_g^2 M_f^2 (e^2M_e^2 M_f^2 + M_g^2), \]

(53)

where (a) follows from the Fubini’s theorem, (b) follows from the fact that \( \| x + y \|^2 \leq 2\| x \|^2 + 2\| y \|^2, \) and (c) follows from (51) and (52).

**Proof of Theorem 2** Using the fundamental theorem of calculus, we obtain

\[ \rho^2(\theta_k) - \rho^2(\theta_{k+1}) = \langle \nabla \rho^2(\theta_k), \theta_k - \theta_{k+1} \rangle \]
\[ + \int_0^1 \langle \nabla \rho^2(\theta_k + \tau(\theta_k - \theta_{k+1})), \theta_k - \theta_{k+1} \rangle d\tau \leq \langle \nabla \rho^2(\theta_k), \theta_k - \theta_{k+1} \rangle \]
\[ + \int_0^1 \| \nabla \rho^2(\theta_k + \tau(\theta_k - \theta_{k+1})) \| \| \theta_k - \theta_{k+1} \| d\tau \]
\[ \leq \langle \nabla \rho^2(\theta_k), \theta_k - \theta_{k+1} \rangle + \left( \frac{\Lambda}{2} \right) \| \theta_k - \theta_{k+1} \|^2 \int_0^1 (1 - \tau) d\tau \]
\[ = \langle \nabla \rho^2(\theta_k), \theta_k - \theta_{k+1} \rangle + \frac{\Lambda}{2} \| \theta_k - \theta_{k+1} \|^2 \]
\[ = \alpha \left( \nabla \rho^2(\theta_k), -\nabla G \rho^2(\theta_k) \right) + \frac{\alpha^2}{2} \left\| \nabla G \rho^2(\theta_k) \right\|^2 \]
\[ = \alpha \left( \nabla \rho^2(\theta_k), \nabla \rho^2(\theta_k) - \nabla G \rho^2(\theta_k) \right) \]
\[ - \frac{\alpha}{2} \| \nabla \rho^2(\theta_k) \|^2 + \frac{\alpha^2}{2} \left\| \nabla G \rho^2(\theta_k) \right\|^2 \]
\[ \leq \frac{\alpha}{2} \| \nabla \rho^2(\theta_k) \|^2 \]
\[ + \frac{\alpha^2}{2} \left\| \nabla G \rho^2(\theta_k) \right\|^2 \]
\[ - \frac{\alpha}{2} \| \nabla \rho^2(\theta_k) \|^2 + \frac{\alpha^2}{2} \left\| \nabla G \rho^2(\theta_k) \right\|^2 \]
\[ = \frac{\alpha}{2} \| \nabla \rho^2(\theta_k) \|^2 \]

(54)
where (a) follows from Lemma 3 and (b) follows from the fact that $2(x, y) \leq \|x\|^2 + \|y\|^2$.

Taking expectations on both sides of (54), we obtain

$$\alpha \mathbb{E} \left[ \|\nabla \rho_g(\theta_k)\|^2 \right] \leq 2\mathbb{E} \left[ \rho_g(\theta_{k+1}) - \rho_g(\theta_k) \right] + L_{\rho'}\alpha^2 \mathbb{E} \left[ \|\nabla G \rho_g(\theta_k)\|^2 \right] + \alpha \mathbb{E} \left[ \|\nabla \rho_g(\theta_k) - \nabla G \rho_g(\theta_k)\|^2 \right] \leq 2\mathbb{E} \left[ \rho_g(\theta_{k+1}) - \rho_g(\theta_k) \right] + \alpha 4M^2 \epsilon M_d^2 \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right), \quad (55)$$

where the last inequality follows from Lemmas 4 and 20.

Summing up (55) from $k = 0, \ldots, N-1$, we obtain

$$\alpha \sum_{k=0}^{N-1} \mathbb{E} \left[ \|\nabla \rho_g(\theta_k)\|^2 \right] \leq \frac{2}{N} \left( \rho_g^2(\theta_0) - \rho_g(\theta_N) \right) + N\alpha 4M^2 \epsilon M_d^2 \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right).$$

Since $\theta_R$ is chosen uniformly at random from the policy iterates $\{\theta_1, \ldots, \theta_N\}$, we obtain

$$\mathbb{E} \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} \left[ \|\nabla \rho_g(\theta_k)\|^2 \right] \leq \frac{2}{N} \left( \rho_g^2(\theta_0) - \rho_g(\theta_N) \right) + 4M^2 \epsilon M_d^2 \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right).$$

$\Box$

B. Analysis of DRM-offP-LR

Proof of Lemma 21. We use parallel arguments to the proof of Lemma 7. For any episode $\omega$, let the importance sampling ratio be

$$\psi^\theta(\omega) = \prod_{t=0}^{T(\omega)-1} \pi_\theta(A_t(\omega) | S_t(\omega)) \frac{b(A_t(\omega) | S_t(\omega))}{p(S_{t+1}(\omega), S_t(\omega), A_t(\omega))}. \quad (56)$$

Also, let

$$\mathbb{P}_b(\omega) = \prod_{t=0}^{T(\omega)-1} b(A_t(\omega) | S_t(\omega)) p(S_{t+1}(\omega), S_t(\omega), A_t(\omega)). \quad (57)$$

From (41), (42), (56), and (57), we obtain

$$\frac{\nabla \mathbb{P}_b(\omega)}{\mathbb{P}_b(\omega)} = \psi^\theta(\omega) \prod_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega) | S_t(\omega)). \quad (58)$$

From (43), we obtain

$$\nabla F^\theta_R(x) = \sum_{\omega \in \Omega} \mathbb{I} \{ R(\omega) \leq x \} \nabla \mathbb{P}_b(\omega) = \sum_{\omega \in \Omega} \mathbb{I} \{ R(\omega) \leq x \} \psi^\theta(\omega) \prod_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega) | S_t(\omega)) \mathbb{P}_b(\omega) \quad (59)$$

$$= \sum_{\omega \in \Omega} \mathbb{I} \{ R(\omega) \leq x \} \psi^\theta(\omega) \sum_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega) | S_t(\omega)) \mathbb{P}_b(\omega) \quad (60)$$

$$= \sum_{\omega \in \Omega} \mathbb{I} \{ R(\omega) \leq x \} \psi^\theta(\omega) \prod_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega) | S_t(\omega)) \mathbb{P}_b(\omega) \quad (61)$$

$$= \mathbb{E} \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq 4M^2 \epsilon M_d^2 M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right), \quad (62)$$

where (a) follows from (58).

In the Lemma below, we establish an upper bound on the variance of the gradient estimate $\nabla H^\theta R^\omega(\theta)$ as defined in (23). Subsequently, we use this result to prove Lemma 7 and Theorem 3.

Lemma 21: $\mathbb{E} \left[ \|\nabla H^\theta R^\omega(\theta)\|^2 \right] \leq 4M^2 \epsilon M_d^2 M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right).$ $\Box$

Proof: From (22), (A2) (15), and (17), we obtain

$$\mathbb{E} \left[ \|\nabla H^\theta R^\omega(\theta)\|^2 \right] \leq M^2 \epsilon M_d^2 M^2 \epsilon M_d^2 \frac{\alpha}{m}, \quad \forall x \in (-M_r, M_r). \quad (59)$$

The result follows by using similar arguments as in Lemma 20 along with (59). $\Box$

Proof of Lemma 7. We use parallel arguments to the proof of Lemma 4. From (17), we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{E} \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq \mathbb{E} \left[ \|\nabla \rho_g(\theta_R)\|^2 \right] \leq 4M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right)$. Hence, we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta)\|^2 > \epsilon \right) \\leq 2 \exp \left( -\frac{m \epsilon^2}{2M_d^2} \right). \quad (60)$

From (20) and (21), we observe that

$$\mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta)\|^2 > \epsilon \right) \leq \mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta)\|^2 > \epsilon \right).$$

Hence, we obtain $\forall \omega \in \{-M_r, M_r\}$,

$$\mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta)\|^2 > \epsilon \right) \leq 2 \exp \left( -\frac{m \epsilon^2}{2M_d^2} \right). \quad (61)$$

Using similar arguments as in (49) along with (61), we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta)\|^2 > \epsilon \right) \leq 4M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right). \quad (62)$

From (A2) (15), and (17), we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{E} \left[ \|\nabla H^\theta R^\omega(\theta)\|^2 \right] \leq 4M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right). \quad (63)$$

We observe that $\left\{ m \left( \nabla H^\theta R^\omega(\theta) - \nabla F^\theta_R(\theta) \right) \right\}_{\omega \in \Omega}$ is a set of partial sums of bounded mean zero r.v.s from (63), and hence they are martingales. Using Azuma-Hoeffding’s inequality from Theorem 1.8-1.9 in [33], we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{P} \left( \|\nabla H^\theta R^\omega(\theta) - \nabla F^\theta_R(\theta)\|^2 > \epsilon \right) \leq 2 \epsilon^2 \exp \left( -\frac{m \epsilon^2}{2M_d^2} \right). \quad (64)$

Using similar arguments as in (50) along with (64), we obtain $\forall \omega \in \{-M_r, M_r\}, \mathbb{E} \left[ \|\nabla F^\theta_R(\theta) - \nabla H^\theta R^\omega(\theta)\|^2 \right] \leq 4e^2 M^2 \epsilon M_d^2 M^2 \epsilon M_d^2 \frac{\alpha}{m} \left( \alpha M_d^2 L_{\rho'} + \frac{8(e^2 M_d^2 + M_d^2 \rho')}{m} \right). \quad (65)$
Using similar arguments as in (51), along with (62), and (59), we obtain \( \forall x \in (-M_r, M_r) \),
\[
\mathbb{E} \left[ \|g'(1 - F_{R^h}(x)) - g'(1 - H_{R^h}^m(x))\| \right] 
\leq \frac{4M_{g'}^2M_e^4M_{\hat{e}}^2M_d^2}{m}.
\] (66)

Using similar arguments as in (52) along with (65), we obtain \( \forall x \in (-M_r, M_r) \),
\[
\mathbb{E} \left[ \left\| \nabla H_{R^h}^m(x) \right\|^2 \right] 
\leq \frac{4e^2M_{g'}^2M_e^2M_{\hat{e}}^2M_d^2}{m}.
\] (67)

The result follows by using similar arguments as in (53) along with (66) and (67).

**Proof of Theorem 3.** By using a completely parallel argument to the initial passage in the proof of Theorem 2 leading up to (55), we obtain
\[
\mathbb{E} \left[ \nabla \rho_g(\theta_k) \right] 
\leq 2\mathbb{E} [\rho_g(\theta_{k+1}) - \rho_g(\theta_k)] + L_{\rho'}\alpha^2 \mathbb{E} \left[ \left\| \nabla H \rho_g(\theta_k) \right\|^2 \right]
+ \alpha \mathbb{E} \left[ \nabla \rho_g(\theta_k) \right] - \nabla H \rho_g(\theta_k) \right] 
\leq 2\mathbb{E} [\rho_g(\theta_{k+1}) - \rho_g(\theta_k)]
+ \alpha 4M_{g'}^2M_e^2M_{\hat{e}}^2M_d^2 \left( \alpha M_{g'}^2 L_{\rho'} + \frac{8(e^2M_{g'}^2 + M_{\hat{e}}^2,M_{\hat{e}}^2)}{m} \right)
\]

where the last inequality follows from Lemmas 7 and 21. Summing the above equation from \( k = 0, \ldots, N - 1 \), we obtain
\[
\alpha \sum_{k=0}^{N-1} \mathbb{E} \left[ \nabla \rho_g(\theta_k) \right] 
\leq 2\mathbb{E} [\rho_g(\theta_N) - \rho_g(\theta_0)]
+ N \alpha 4M_{g'}^2M_e^2M_{\hat{e}}^2M_d^2 \left( \alpha M_{g'}^2 L_{\rho'} + \frac{8(e^2M_{g'}^2 + M_{\hat{e}}^2,M_{\hat{e}}^2)}{m} \right).
\]

Since \( \theta_R \) is chosen uniformly at random from the policy iterates \( \{\theta_1, \ldots, \theta_N\} \), we obtain
\[
\mathbb{E} \left[ \left\| \nabla \rho_g(\theta_R) \right\|^2 \right] = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} \left[ \left\| \nabla \rho_g(\theta_k) \right\|^2 \right]
\leq 2 \left( \rho_g(\theta_0) - \rho_g(\theta_N) \right)
+ \frac{4M_{g'}^2M_e^2M_{\hat{e}}^2M_d^2}{m} \left( \alpha M_{g'}^2 L_{\rho'} + \frac{8(e^2M_{g'}^2 + M_{\hat{e}}^2,M_{\hat{e}}^2)}{m} \right).
\]

**VII. NON-ASYMPTOTIC CONVERGENCE ANALYSIS (SF-BASED GRADIENT ESTIMATION SCHEME)**

**A. SF-based gradient estimate**

**Proof of Lemma 3.** We follow the technique from [34]. Since \( v_{1:n} \) are i.i.d r.v.s, and have symmetric distribution around the origin, we obtain
\[
\mathbb{E} \left[ \nabla_{\mu,n} \rho_g(\theta) \mid \theta \right] = \mathbb{E}_{v_{1:n}} \left[ \nabla_{\mu,n} \rho_g(\theta) \right]
= \frac{d}{2\mu n} \sum_{i=1}^{n} \mathbb{E}_v [\rho_g(\theta + \mu v) - \rho_g(\theta - \mu v)]
= \frac{d}{4\mu} \left( \mathbb{E}_v [\rho_g(\theta + \mu v)] + \mathbb{E}_v [\rho_g(\theta + \mu(-v))(-v)] \right)
= \frac{d}{\mu} \mathbb{E}_v [\rho_g(\theta + \mu v)] = \nabla \rho_g, \mu(\theta),
\]
where last equality follows from [29, Lemma 2.1].

The following lemmas establish some results related to the SF-based gradient estimates \( \nabla \rho_{g,\mu}(\theta) \) and \( \nabla_{\mu,n} \rho_g(\theta) \) as defined in (27) and (28) respectively. Subsequently, we use these results to prove Lemma 12. The proof techniques are similar to that in [30].

**Lemma 22:** \( \left\| \nabla \rho_{g,\mu}(\theta) - \nabla \rho_{g}(\theta) \right\| \leq \frac{\mu dL_{\rho'}}{2} \).

**Proof:** The result follows from [35, Proposition 7.5] along with Lemma 3.

**Lemma 23:** \( \mathbb{E} \left[ \left\| \nabla_{\mu,n} \rho_g(\theta) \right\|^2 \right] \leq \frac{d^2L_{\rho}^2}{n} \).

**Proof:** Since \( \forall v \in \mathbb{R}^{d-1} \), \( \|v\| = 1 \), from (28), we have
\[
\mathbb{E}_{v_{1:n}} \left[ \left\| \nabla_{\mu,n} \rho_g(\theta) \right\|^2 \right] 
\leq \frac{d^2}{4\mu^2n^2} \sum_{i=1}^{n} \mathbb{E}_v \left[ \left( \rho_g(\theta + \mu v) - \rho_g(\theta - \mu v) \right) \right] 
\leq \frac{d^2}{4\mu^2n} \mathbb{E}_v \left[ \left( \rho_g(\theta + \mu v) - \rho_g(\theta - \mu v) \right) \right] 
\leq \frac{d^2L_{\rho}^2}{n} \mathbb{E}_{\|v\|^2} = \frac{d^2L_{\rho}^2}{n} \mathbb{E}_{\|v\|^2}.
\]

Finally,
\[
\mathbb{E} \left[ \left\| \nabla_{\mu,n} \rho_g(\theta) \right\|^2 \right] 
\leq \mathbb{E} \left[ \mathbb{E}_{v_{1:n}} \left[ \left\| \nabla_{\mu,n} \rho_g(\theta) \right\|^2 \right] \right] 
\leq \frac{d^2L_{\rho}^2}{n} \mathbb{E}_{\|v\|^2}.
\]

**B. Analysis of DRM-ONP-SF**

**Proof of Lemma 17.** Notice that
\[
\mathbb{E} \left[ \left\| \rho_g(\theta) - \rho_g^2(\theta) \right\|^2 \right] 
= \mathbb{E} \left[ \left( \int_{-M_r}^{M_r} (g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x))) dx \right)^2 \right]
\leq 2M_r \left( \int_{-M_r}^{M_r} \left| (g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x))) \right|^2 dx \right)
\leq 2M_r \left( \int_{-M_r}^{M_r} \left( g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x)) \right)^2 dx \right)
\leq 2M_r \left( \frac{M_r}{M_r} \int_{-M_r}^{M_r} \left( g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x)) \right)^2 dx \right)
\leq 2M_r^2 \left( \frac{M_r}{M_r} \int_{-M_r}^{M_r} \left( g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x)) \right)^2 dx \right)
\leq 2M_r^2 \left( \frac{M_r}{M_r} \int_{-M_r}^{M_r} \left( g(1 - F_{R^h}(x)) - g(1 - G_{R^h}^m(x)) \right)^2 dx \right)
\leq \frac{16M_r^2M_{\hat{e}}^2}{m}, \quad (68)
\]
where \( (a) \) follows from the Cauchy-Schwarz inequality, \( (b) \) follows from the Fubini’s theorem, \( (c) \) follows from Lemma 19 and \( (d) \) follows from (49) in Lemma 4.

The following result is used in the proofs of Lemmas 12 and 25 later.
Lemma 24:

\[ \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) \right\|^2 \right] \leq \frac{4d^2 M_g^2 M_g^2}{\mu^2 n} . \]

**Proof:** Notice that

\[
\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) \right\|^2 \right]
\leq \frac{d^2}{4n^2 \mu^2} \mathbb{E} \left[ \left\| \sum_{i=1}^{n} \left( \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) \right) v_i \right\|^2 \right]
\leq \frac{d^2}{4n^2 \mu^2} \mathbb{E} \left[ \left\| \sum_{i=1}^{n} \left( \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) \right) v_i \right\|^2 \right].
\]

where \((a)\) follows from the fact that \(v_i \) are i.i.d. mean zero r.v.s. and \(\hat{\rho}_{g}^C (\cdot)\) and \(\rho_g (\cdot)\) are bounded, \((b)\) follows from the fact that \( \left\| v_i \right\| = 1 \), and \((c)\) follows from Lemma 10.

**Proof of Lemma 13** Notice that

\[
\hat{\nabla}_{\mu,n} \hat{\rho}_{g}^C (\theta) = \frac{d}{n} \sum_{i=1}^{n} \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) v_i
\]
\[= \frac{d}{n} \sum_{i=1}^{n} \rho_g (\theta + \mu v_i) - \rho_g (\theta - \mu v_i) v_i + \frac{d}{n} \sum_{i=1}^{n} \hat{\rho}_{g}^C (\theta + \mu v_i) - \rho_g (\theta + \mu v_i) v_i
\]
\[= \frac{d}{n} \sum_{i=1}^{n} \rho_g (\theta - \mu v_i) - \rho_g (\theta - \mu v_i) v_i + \frac{d}{n} \sum_{i=1}^{n} \rho_g (\theta - \mu v_i) - \rho_g (\theta + \mu v_i) v_i.
\]

From \((69)\) and Lemma 24, we obtain

\[
\mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \hat{\rho}_{g}^C (\theta) - \nabla \rho_g (\theta) \right\|^2 \right]
\leq 2 \mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \rho_g (\theta) - \nabla \rho_g (\theta) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 n}
\]
\[\leq 4 \mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \rho_g (\theta) - \nabla \rho_g (\theta) \right\|^2 \right]
\leq 4 \mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \rho_g (\theta) - \nabla \rho_g (\theta) \right\|^2 \right]
\leq \frac{4d^2 L^2_{\mu}}{n} + \frac{\mu^2 d^2 L^2_{\rho'}}{\mu^2 mn} + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn},
\]

where \((a)\) follows from Lemma 9, \((b)\) follows from the fact that \(\mathbb{E} \left[ \left\| X - \mathbb{E}[X|Y] \right\|^2 \right] \leq \mathbb{E} \left[ \left\| X \right\|^2 \right] \) and Lemma 22, and \((c)\) follows from Lemma 23.

In the lemma below, we establish a bound on the variance of the DRM gradient estimate \( \hat{\nabla}_{\mu,n} \hat{\rho}_{g}^C (\theta) \). Subsequently, we use this result in the proof of Theorem 4.

Lemma 25:

\[ \mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \hat{\rho}_{g}^C (\theta) \right\|^2 \right] \leq \frac{2d^2 L^2_{\mu}}{n} + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn} . \]

**Proof:** Using \((69)\) and Lemma 24 we obtain

\[
\mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \hat{\rho}_{g}^C (\theta) \right\|^2 \right] \leq 2 \mathbb{E} \left[ \left\| \hat{\nabla}_{\mu,n} \rho_g (\theta) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}
\leq \frac{2d^2 L^2_{\mu}}{n} + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn},
\]

where the last inequality follows from Lemma 23.

**Proof of Theorem 4** By using a completely parallel argument to the initial passage in the proof of Theorem 2 leading up to \((55)\), we obtain

\[ \alpha \mathbb{E} \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right]
\leq 2 \mathbb{E} \left[ \left\| \rho_g (\theta_{k+1}) - \rho_g (\theta_k) \right\|^2 \right]
\leq 2 \mathbb{E} \left[ \left\| \rho_g (\theta_{k+1}) - \rho_g (\theta_k) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}
\leq 2 \mathbb{E} \left[ \left\| \rho_g (\theta_{k+1}) - \rho_g (\theta_k) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}.
\]

Summing up \((70)\) from \( k = 0, \ldots, N - 1 \), we obtain

\[ \alpha \sum_{k=0}^{N-1} \mathbb{E} \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right]
\leq 2 \mathbb{E} \left[ \left\| \rho_g (\theta_{N}) - \rho_g (\theta_0) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}
\leq 2 \mathbb{E} \left[ \left\| \rho_g (\theta_{N}) - \rho_g (\theta_0) \right\|^2 \right] + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}.
\]

Since \( \theta_R \) is chosen uniformly at random from the policy iterates \( \{ \theta_1, \ldots, \theta_N \} \), we obtain

\[ \mathbb{E} \left[ \left\| \nabla \rho_g (\theta_R) \right\|^2 \right] \leq \frac{2 \left( \rho_* - \rho_g (\theta_0) \right)}{N \alpha} + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}
\leq \frac{2 \left( \rho_* - \rho_g (\theta_0) \right)}{N \alpha} + \frac{16d^2 M_g^2 M_g^2}{\mu^2 mn}.
\]

C. Analysis of DRM-OFFP-SF

**Proof of Lemma 73** The result follows using similar arguments as in \((68)\) from Lemma 10 along with \((62)\) from Lemma 7.

The following result is used in the proofs of Lemmas 15 and 27 later.

**Lemma 26:**

\[ \mathbb{E} \frac{d}{n} \sum_{i=1}^{n} \hat{\rho}_{g}^H (\theta \pm \mu v_i) - \rho_g (\theta \pm \mu v_i) v_i \leq \frac{4d^2 M_g^2 M_g^2}{\mu^2 mn} . \]

**Proof:** The result follows using similar arguments as in Lemma 24 along with Lemma 13.
Proof of Lemma 25. The result follows using similar arguments as in Lemma 24 along with Lemma 26.

In the lemma below, we bound the variance of the DRM gradient estimate $\hat{\nabla}_{\mu,n}^H(\theta)$ as defined in (35). Subsequently, we use this result in the proof of Theorem 3.

**Lemma 27:** $E \left[ \left\| \hat{\nabla}_{\mu,n}^H(\theta) \right\|^2 \right] \leq \frac{2d^2L_n^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn}$.

**Proof:** The result follows using similar arguments as in Lemma 25 along with Lemma 26.

**Proof of Theorem 3:** By using a completely parallel argument to the initial passage in the proof of Theorem 2 leading up to (35), we obtain

\[
\alpha E \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \leq 2E \left[ \rho_g(\theta_{k+1}) - \rho_g(\theta_k) \right] + L_{\rho} \alpha^2 E \left[ \left\| \hat{\nabla}_{\mu,n}^H(\theta_k) \right\|^2 \right] + \alpha E \left[ \left\| \nabla \rho_g(\theta_k) - \hat{\nabla}_{\mu,n}^H(\theta_k) \right\|^2 \right] \leq 2E \left[ \rho_g(\theta_{k+1}) - \rho_g(\theta_k) \right] + L_{\rho} \alpha^2 \left( \frac{2d^2L_n^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn} \right) + \alpha \left( \frac{4d^2L_n^2}{n} + \frac{\mu^2d^2L_{\rho}^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn} \right),
\]

where the last inequality follows from lemmas 15 and 27. Summing up (72) from $k = 0, \ldots, N - 1$, we obtain

\[
\alpha \sum_{k=0}^{N-1} E \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \leq 2E [\rho_g(\theta_N) - \rho_g(\theta_0)] + NL_{\rho} \alpha^2 \left( \frac{2d^2L_n^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn} \right) + N \alpha \left( \frac{4d^2L_n^2}{n} + \frac{\mu^2d^2L_{\rho}^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn} \right).
\]

Since $\theta_R$ is chosen uniformly at random from the policy iterates $\{\theta_1, \ldots, \theta_N\}$, we obtain

\[
E \left[ \| \nabla \rho_g(\theta_R) \|^2 \right] = \frac{1}{N} \sum_{k=0}^{N-1} E \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \leq \frac{2}{N} \left( \rho^*_\theta - \rho_\theta(\theta_0) \right) + L_{\rho} \alpha \left( \frac{2d^2L_n^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn} \right) + \frac{4d^2L_n^2}{n} + \frac{\mu^2d^2L_{\rho}^2}{n} + \frac{16d^2N^2_m^2M_s^2M_2^2}{\mu^2mn}.
\]

**VIII. CONCLUSIONS**

We proposed DRM-based policy gradient algorithms for risk-sensitive RL control. We employed LR and SF gradient estimation schemes in on-policy as well as off-policy RL settings, and provided non-asymptotic bounds that establish convergence to an approximate stationary point of the DRM.

As a future work, it would be interesting to study DRM optimization in a risk-sensitive RL setting with feature-based representations, and function approximation. In this setting, one could consider an actor-critic algorithm for DRM optimization, and study its non-asymptotic performance.

**APPENDIX I**

**SIMPLIFYING THE ESTIMATE OF THE DRM AND ITS GRADIENT USING ORDER STATISTICS**

**A. Proof of Lemma 27**

Our proof follows the technique from [36]. Let $R^\theta_{(i)}$ be the $i^{th}$ smallest order statistic from the samples $\{R^\theta_{(i)}\}_{i=1}^m$. We rewrite (9) as given below.

\[
G^m_{Re}(x) = \begin{cases} 
0, & \text{if } x < R^\theta_{(1)} \\
\frac{i}{m}, & \text{if } R^\theta_{(i)} \leq x < R^\theta_{(i+1)} \\
1, & \text{if } x \geq R^\theta_{(m)}. 
\end{cases}
\]

We assume WLOG that $R^\theta_{(i)} < 0 < R^\theta_{(i+1)}$, and obtain,

\[
\hat{\rho}_g^G(\theta) = \int_{-\infty}^{0} \left( g(1 - G^m_{Re}(x)) - 1 \right) dx + \int_{0}^{\infty} \hat{G}^m_{Re}(x) dx = \int_{-\infty}^{R^\theta_{(1)}} (g(1 - G^m_{Re}(x)) - 1) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} (g(1 - G^m_{Re}(x)) - 1) dx + \int_{R^\theta_{(m)}}^{\infty} (g(1 - G^m_{Re}(x)) - 1) dx.
\]

\[
= \sum_{i=2}^{m} \int_{R^\theta_{(i-1)}}^{R^\theta_{(i)}} (g(1 - \frac{j}{m}) - 1) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx.
\]

\[
= \sum_{i=2}^{m} \int_{R^\theta_{(i-1)}}^{R^\theta_{(i)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx.
\]

\[
= \sum_{i=2}^{m} \int_{R^\theta_{(i-1)}}^{R^\theta_{(i)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(i)}}^{R^\theta_{(i+1)}} \left( g(1 - \frac{j}{m}) - 1 \right) dx + \int_{R^\theta_{(m)}}^{\infty} \left( g(1 - \frac{j}{m}) - 1 \right) dx.
\]
B. Proof of Lemma 14

We rewrite (20) as given below.

\[ H_{R}^{m}(x) = \begin{cases} 0, & \text{if } x < R_{(1)}^{b} \\ \min\{\frac{1}{m} \sum_{j=1}^{i} \psi_{(j)}^{\theta} \}, & \text{if } R_{(1)}^{b} \leq x < R_{(i+1)}^{b}, \\ 1, & \text{if } x \geq R_{(m)}^{b}, \end{cases} \tag{74} \]

where \( R_{(i)}^{b} \) is the \( i^{th} \) smallest order statistic from the samples \( \{R_{(i)}^{b}\}_{i=1}^{m} \), and \( \psi_{(i)}^{\theta} \) is the importance sampling ratio of \( R_{(i)}^{b} \).

We assume WLOG that \( R_{(j)}^{b} < 0 < R_{(j+1)}^{b} \), and obtain,

\[
\hat{\rho}_{\theta}^{g}(\theta) = \int_{-M_{R}}^{0} (g(1-H_{R}^{m}(x)) - 1)dx + \int_{0}^{M_{R}} g(1-H_{R}^{m}(x))dx \\
= \int_{-M_{R}}^{R_{(1)}^{b}} (g(1-H_{R}^{m}(x)) - 1)dx \\
+ \int_{R_{(1)}^{b}}^{R_{(i+1)}^{b}} (g(1-H_{R}^{m}(x)) - 1)dx \\
+ \int_{R_{(i+1)}^{b}}^{R_{(m)}^{b}} \int_{R_{(i)}^{b}}^{R_{(i+1)}^{b}} g(1-H_{R}^{m}(x))dx \\
+ \int_{R_{(m)}^{b}}^{M_{R}} g(1-H_{R}^{m}(x))dx \\
= \sum_{i=2}^{j} \int_{R_{(i-1)}^{b}}^{R_{(i)}^{b}} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) - 1 \right) dx \\
+ \int_{R_{(j)}^{b}}^{R_{(i+1)}^{b}} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) - 1 \right) dx \\
+ \int_{R_{(i+1)}^{b}}^{R_{(m)}^{b}} \int_{R_{(i)}^{b}}^{R_{(i+1)}^{b}} g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) dx \\
+ \int_{R_{(m)}^{b}}^{M_{R}} g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) dx \\
= \sum_{i=2}^{j} \left( R_{(i)}^{b} - R_{(i-1)}^{b} \right) \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) - 1 \right) \\
- R_{(j)}^{b} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) - 1 \right) \\
+ R_{(i+1)}^{b} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) - 1 \right) \\
+ \sum_{i=2}^{m} \left( R_{(i+1)}^{b} - R_{(i)}^{b} \right) \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) + R_{(i)}^{b} \right) \\
+ \sum_{i=2}^{m} \left( R_{(i+1)}^{b} - R_{(i)}^{b} \right) \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) \right) \\
= R_{(1)}^{b} + \sum_{i=2}^{m} R_{(i)}^{b} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) \right) \\
- \sum_{i=1}^{m} R_{(i)}^{b} \left( g(1-\min\left\{1, \frac{1}{m} \sum_{k=1}^{i} \psi_{(k)}^{\theta} \right\}) \right). \]

C. Proof of Lemma 5

Following the proof of Lemma 14, we rewrite \( G_{R}^{m}(x) \) from (20) as (23). Let \( \nabla l_{(i)}^{\theta} = \sum_{t=0}^{T_{(i)}} -1 \log \pi_{\theta}(A_{t}^{(i)} | S_{t}^{(i)}) \), where \( T_{(i)} \) is the length, and \( S_{t}^{(i)} \) and \( A_{t}^{(i)} \) are the state and action at time \( t \) of the episode corresponding to \( R_{(i)}^{b} \). We rewrite (10) as given below.

\[
\hat{\nabla} G_{R}^{m}(x) = \begin{cases} 0, & \text{if } x < R_{(1)}^{b} \\ \frac{1}{m} \sum_{j=1}^{i} \nabla l_{(j)}^{\theta}, & \text{if } R_{(i)}^{b} \leq x < R_{(i+1)}^{b}, \\ \frac{1}{m} \sum_{j=1}^{i} \nabla l_{(j)}^{\theta}, & \text{if } x \geq R_{(m)}^{b}. \end{cases} \tag{75} \]

Now,

\[
\hat{\nabla} G_{\rho_{g}}(\theta) = - \int_{-M_{R}}^{M_{R}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
= - \int_{-M_{R}}^{R_{(1)}^{b}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
- \sum_{i=1}^{m-1} \int_{R_{(i)}^{b}}^{R_{(i+1)}^{b}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
- \int_{R_{(m)}^{b}}^{M_{R}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
= - \frac{1}{m} \sum_{i=1}^{m-1} \int_{R_{(i)}^{b}}^{R_{(i+1)}^{b}} g'(1-\frac{i}{m}) \sum_{j=1}^{i} \nabla l_{(j)}^{\theta} dx \\
- \frac{1}{m} \sum_{i=1}^{m} g'_+(0) \sum_{j=1}^{m} \nabla l_{(j)}^{\theta} dx \\
= \frac{1}{m} \sum_{i=1}^{m-1} \left( R_{(i)}^{b} - R_{(i+1)}^{b} \right) g'_+(1-\frac{i}{m}) \sum_{j=1}^{i} \nabla l_{(j)}^{\theta} \\
+ \frac{1}{m} \left( R_{(m)}^{b} - M_{R} \right) g'_+(0) \sum_{j=1}^{m} \nabla l_{(j)}^{\theta}, \]

where \( g'_+(0) \) is the right derivative of the distortion function \( g \) at 0.

D. Proof of Lemma 3

Following the proof of Lemma 14, we rewrite \( H_{R}^{m}(x) \) from (20) as (23). Let \( \nabla l_{(i)}^{\theta} = \sum_{t=0}^{T_{(i)}} -1 \log \pi_{\theta}(A_{t}^{(i)} | S_{t}^{(i)}) \), where \( T_{(i)} \) is the length, and \( S_{t}^{(i)} \) and \( A_{t}^{(i)} \) are the state and action at time \( t \) of the episode corresponding to \( R_{(i)}^{b} \). We rewrite (22) as given below.

\[
\hat{\nabla} H_{R}^{m}(x) = \begin{cases} 0, & \text{if } x < R_{(1)}^{b} \\ \frac{1}{m} \sum_{j=1}^{i} \nabla l_{(j)}^{\theta}, & \text{if } R_{(i)}^{b} \leq x < R_{(i+1)}^{b}, \\ \frac{1}{m} \sum_{j=1}^{i} \nabla l_{(j)}^{\theta}, & \text{if } x \geq R_{(m)}^{b}. \end{cases} \tag{76} \]

Now,

\[
\hat{\nabla} \rho_{g}(\theta) = - \int_{-M_{R}}^{M_{R}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
= - \int_{-M_{R}}^{R_{(1)}^{b}} g'(1-H_{R}^{m}(x)) \hat{\nabla} H_{R}^{m}(x)dx \\
\]
\[ -\sum_{i=1}^{m-1} \int_{R_{(i+1)}^b}^R \left( 1 - H_{R_{(i+1)}^b}^m(x) \right) \nabla H_{R_{(i+1)}^b}^m(x) dx \]
\[ -\int_{R_{(1)}^b}^M \left( 1 - H_{R_{(1)}^b}^m(x) \right) \nabla H_{R_{(1)}^b}^m(x) dx \]
\[ = -\frac{1}{m} \sum_{i=1}^{m-1} \int_{R_{(i+1)}^b}^R \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{j=1}^{i} \psi^{(j)} \right\} \right) \nabla \psi^{(j)} dx 
- \frac{1}{m} \int_{R_{(1)}^b}^M \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{j=1}^{i} \psi^{(j)} \right\} \right) \nabla \psi^{(j)} dx \]
\[ \times \frac{1}{m} \sum_{j=1}^{m-1} \left( R_{(j)}^b - R_{(j+1)}^b \right) \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{j=1}^{i} \psi^{(j)} \right\} \right) \nabla \psi^{(j)} \psi^{(j)} dx 
+ \frac{1}{m} \sum_{j=1}^{m-1} \left( R_{(j+1)}^b - R_{(j)}^b \right) \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{j=1}^{i} \psi^{(j)} \right\} \right) \nabla \psi^{(j)} \psi^{(j)} dx \]
\[ \times \frac{1}{m} \sum_{j=1}^{m-1} \left( R_{(j+1)}^b - R_{(j)}^b \right) \left( 1 - \min \left\{ 1, \frac{1}{m} \sum_{j=1}^{i} \psi^{(j)} \right\} \right) \nabla \psi^{(j)} \psi^{(j)} dx. \]

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