A universality class for RNA-like polymers and double polymers

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October 18, 2024

Abstract

We examine the statistics of conformations of a linear polymer in a solvent. The polymer is allowed to form double polymers. We closely follow a classical technique to derive a field theory for the problem from an \(O(n)\) symmetric spin model. The field theory is a model for RNA or DNA with constant binding energy per monomer.

It is shown that there is a stable renormalization group fixed point, at which the double polymer decouples from the single-strand polymer and becomes a branched polymer of the conventional type with a three-point interaction. To reach this fixed point, at least one parameter must be adjusted. The critical dimension is eight. Fisher-renormalization, equation of state and critical exponents are reproduced in this limit. The single-strand polymer depends on the double-strand polymer and disappears at the critical point, but has its own critical exponents.

Introduction

The statistical physics of linear polymers, branched polymers and RNA-like polymer networks in a solvent has a long history. Classical results are a method to calculate the statistics of polymer configurations with path integrals [1] and the mapping of the configurations of a linear polymer to the spin configurations of \(O(n)\)-symmetric magnetic systems in the \(n \to 0\) limit [2].

According to this \(O(n)\)-correspondence, the radius of gyration of a linear polymer of length \(s\) grows like \(s^\nu\) for long polymers, where the exponent \(\nu\) is universal and greater than \(1/2\) when polymer-polymer repulsion (excluded volume effect) is taken into account. The excluded volume effect is irrelevant for linear polymers above four dimensions, where \(\nu = 1/2\).

The limit \(n \to 0\) of a special \(O(n)\)-symmetric spin system also was used by Lubensky and Isaacson [3, 4] to examine branched polymers, that is, polymer configurations containing two-valent and multi-valent (star like) monomers.

These authors examine connections to percolation, lattice animals, gelation, and the Potts model. There also exists a mapping of the critical properties of such polymer networks to the Lee-Yang edge singularity of a spin system in \(d - 2\) dimensions [5]. A related result is a mapping to a hard-core gas in \(d - 2\) dimensions [6, 7].

At a more formal level, the authors find a critical dimension 8 when excluded volume effects are taken into account and a critical dimension 6 in a theta solvent (without polymer-polymer repulsion).

More complicated branched polymers, and in particular RNA-like conformations, have been examined with many different methods [8, 9, 10, 11, 12, 13], only to cite a few. The irregular and not completely random base sequence of RNA plays a significant role in most realistic cases.

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RNA molecules with a periodic base sequence like ATAT... constitute a more tractable problem, and it suggests itself to start with this case [3]. It turns out that the techniques used by Lubensky and Isaacson for simple branched polymers are directly applicable to this problem.

We follow this path and start with a rather explicit derivation of a spin model analogous to Ref. [4]. The examination of the resulting field theory then in principle is standard, but as for one-component branched polymers, the theory is very rich [4].

**Partition sum**

The starting point is an expression for the partition sum $Z$ of the system, essentially the number of polymer configurations in a lattice model. The expression makes use of a probability distribution $P_N (\sigma_1, \sigma_2, ..., \sigma_N) d^N \sigma$ for $N$ spin variables $\sigma$ with the property $\langle \sigma_\alpha \sigma_\beta \rangle = \delta_{\alpha, \beta}$ and $\langle \sigma_\alpha_1 \ldots \sigma_\alpha_k \rangle = O (N^{k^2/2 - 1})$ for $k > 2$, see appendix A [14]. Expectation values with an odd power of any $\sigma_\alpha$ vanish. Examples are

$$
\langle \sigma_i^2 \rangle = 1,
$$

$$
\langle \sigma_i^4 \rangle = \frac{3N}{N+2},
$$

$$
\langle \sigma_i^6 \rangle = \frac{15N^2}{(N+2)(N+4)},
$$

$$
\langle \sigma_1^2 \sigma_2^2 \sigma_3^2 \rangle = \frac{N^2}{(N+2)(N+4)}.
$$

The essential point is that in the limit $N \to 0$ there only remains $\langle 1 \rangle = 1$ and $\langle \sigma_\alpha \sigma_\beta \rangle = \delta_{\alpha, \beta}$.

We now consider a rectangular lattice with $n$ spins $s^\mu$ and $\frac{1}{2} n (n + 1)$ spins $S^{\mu \nu} = S^{\nu \mu}$ with $1 \leq \mu, \nu \leq n$ at each lattice point. Spins $s^\mu$ can be identified with the single-polymer, spins $S^{\mu \nu}$ with the double-polymer. For each lattice point we use the probability distribution $P_N \left( s^\mu, S^{\mu \nu} / \sqrt{2}, (S^{\mu \nu})_{\mu < \nu} \right)$ for $N = n + \frac{1}{2} n (n + 1)$ spins. This leads to

$$
\langle s^\mu s^{\nu} \rangle = \delta_{\mu \nu},
$$

$$
\langle S^{\mu \nu} S^{\rho \tau} \rangle = \delta_{\mu \rho} \delta_{\nu \tau} + \delta_{\mu \tau} \delta_{\nu \rho},
$$

and all other correlation functions vanish in the limit $n \to 0$. The partition sum is given by

$$
Z = \left( \prod_{i,j;\mu} (1 + \frac{1}{2} s_i^\mu v_{i,j} s_j^\mu) \prod_{i,j;\mu,\nu} (1 + \frac{1}{2} S^{\mu \nu}_{i,j} v_{i,j} S^{\mu \nu}_{j,i}) \prod_i (1 + f (s_i, S_i)) \right),
$$

where $i$ and $j$ denote lattice points and $v_{i,j} = 1$ for next neighbors and $v_{i,j} = 0$ otherwise. The function $f$ is the interaction. To understand the connection between $Z$ and polymer configurations consider next neighbor pair $i, j$ of the first product. To this pair there corresponds the factor

$$
z_{i,j} = \prod_{\mu} (1 + X_{i,j}^\mu) = 1 + \sum_{\mu} X_{i,j}^\mu + \frac{1}{2} \sum_{\mu,\nu} X_{i,j}^\mu X_{i,j}^\nu - \frac{1}{2} \sum_{\mu} X_{i,j}^\mu X_{i,j}^\mu + O ((X_{i,j}^\mu)^2) \to \left( 1 + \sum_{\mu} s_i^\mu s_j^\mu \right),
$$

where $X_{i,j}^\mu = s_i^\mu s_j^\mu + \frac{1}{4} s_i^\mu s_j^\mu s_i^\mu s_j^\mu$, and in the last step all terms not contributing to the $\langle \ldots \rangle$ average in the $n \to 0$ limit have been omitted. The remaining „links“ $\sum_{\mu} s_i^\mu s_j^\mu$ must be paired in $Z$, and generate all self-avoiding paths with constant $\mu$ along a path (a real path also requires an external source and sink).

The double polymer spin $S$ in the second product of $Z$ has two indices, but can be decoded in the same way. It counts all self-avoiding double polymer paths twice, with index pair $\mu \nu$ along a path conserved or exchanged. The alternatives are caused by $S^{\mu \nu} = S^{\nu \mu}$ and can be interpreted as direct and twisted connections. In fact, because of $\langle s_i^2 s_j^2 \rangle = 0$ single and double polymer paths avoid each other mutually.
The action integral can be written in the form:

\[
S = \int d^d x \left\{ -\frac{1}{2} \sum_\mu \varphi_\mu \left( r_0 - \nabla^2 \right) \varphi_\mu - \frac{1}{2} \sum_{\mu\nu} \chi_{\mu\nu} \left( \tau_0 - \nabla^2 \right) \chi_{\mu\nu} + g \sum_{\mu\nu} \chi_{\mu\nu}\varphi_\mu\varphi_\nu + g' \bar{\chi} \sum_\mu \varphi_\mu\varphi_\mu \right\}
\]

(1)

The interaction term \( f \) resembles the interaction in the lattice model of one-component branched polymers [4] and is described in appendix B. The transition from \( Z \) to a field theory is standard and can be performed as in ref. [4].

**Action integral**

The action integral can be written in the form

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\]

(1)

The field \( \varphi_\mu \) denotes the single-strand polymer, the field \( \chi_{\mu\nu} = \chi_{\nu\mu} \) the double-strand polymer, \( \bar{\chi} \) is an abbreviation for \( \bar{\chi} = \sum_\mu \chi_{\mu\mu} \). The indices run from 1 to \( n \), at the end one takes the limit \( n \to 0 \). The limit eliminates all types of closed loops. What graphically looks like a \( \chi \)-loop may still give a contribution because of \( \chi \)-twists, an example is graph \( F' \) in fig. [2].

The lattice model originally generates the interactions with coupling constants \( g, u_i \) and \( w_j \). The \( g \)-interaction, however, generates other interactions, see fig. [1]. It will turn out that \( g', \lambda' \) and \( \lambda'' \) decouple and can be ignored, but the linear term with coupling constant \( H \) and the bilinear term with coupling constant \( \alpha \) play a special role. The diagrams \( A \) and \( H \) in fig. [1] generate such contributions with originally \( \alpha < 0 \).

The \( \chi \)-Propagator can be found by inverting the matrix \( A_{\mu\nu,\rho\tau} \) in the \( \chi \)-\( \chi \) two-form of \( S \). For wave vectors \( k \) and \( p \) and with the abbreviation \( v_k = \tau_0 + k^2 \) one finds

\[
\langle \chi_k^{\mu\nu} \chi_p^{\rho\tau} \rangle = \left( \frac{\delta_{\mu\rho}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\rho}}{2 v_k} - \frac{\alpha \delta_{\mu\nu}\delta_{\rho\tau}}{v_k (v_k + n \alpha)} \right) (2\pi)^d \delta^d (k + p),
\]

\[
\langle \chi_k^{\mu\nu} \bar{\chi}_p \rangle = \left( \frac{1}{v_k} - \frac{\alpha n}{v_k (v_k + n \alpha)} \right) \delta_{\mu\nu} (2\pi)^d \delta^d (k + p),
\]

\[
\langle \bar{\chi}_k \bar{\chi}_p \rangle = \left( \frac{n}{v_k} - \frac{\alpha n^2}{v_k (v_k + n \alpha)} \right) (2\pi)^d \delta^d (k + p).
\]

The second and third lines are contractions of the first line over \( \rho\tau \) and \( \mu\nu \).

The \( \langle \chi\chi \rangle \)-propagator is analogous to the propagator of simple polymer networks [4, 5], and it is expedient to think of two propagators - a less singular normal propagator \( (\sim v_k^{-1}) \) and more singular propagator with \( \alpha \)-insertions \( (\sim -\alpha v_k^{-2} \text{ for } n = 0) \). The \( \alpha \)-propagator is graphically denoted with
Fisher-renormalization of critical exponents

Hairpin diagrams like $H$ in fig.(1) and $H'$ in fig.(2) generate a linear contribution to $S$ like $-H \int d^d x \bar{\chi}$ with a negative or positive (when $\alpha$ is positive) $H$. This term is strongly relevant and must be included in the action integral. As usual one shifts the field $\chi_{\mu \nu} \to \delta_{\mu \nu} Q + \chi_{\mu \nu}$, $\bar{\chi} \to n Q + \bar{\chi}$ to eliminate the linear term \[4, 5\]. For $n \to 0$ this leads to

$$S (H, \tau_0, r_0) \to S (0, \tau_0 - 6 \lambda Q, r_0 - 2 g Q)$$

$$+ \int d^d x \left( [-H - \tau_0 Q + 3 \lambda Q^2] \bar{\chi} - 4 u_2 Q^2 \bar{\chi}^2 - 4 u_2 Q \bar{\chi} \sum \lambda_{\mu \nu}^2 + \ldots \right) .$$

One could imagine that $H = H(\tau_0)$ depends on $\tau_0$ in such a way that $[\ldots] \bar{\chi}$ vanishes for some constant $Q = Q_c$ in all orders of the loop expansion. One then has a simplified version of the field theory with shifted temperature variables $\tau_0$ and $r_0$ and constant $Q$. Linear terms can be ignored, the contributions proportional to $u_2 Q$ and $u_2 Q^2$ do not change the structure of the field theory (except when $\alpha$ originally is negative and now becomes positive). The $\chi$-sector of this field theory, as will be seen below, is related to the Lee-Yang edge singularity in $d - 2$ dimensions.

Actually, however, $H$ is constant, and the linear term only vanishes if $Q$ depends on $\tau_0$. A rather generic Ansatz is

$$Q = Q (\tau_0) \cong Q_c + a \tau - A \tau^\beta H,$$

where $\tau = \tau_0 - \tau_c$ and $\beta_H$ is some exponent. For this $Q (\tau_0)$ the field theory has the same structure as for $Q = \text{const}$, but with a temperature variable $Q (\tau) + 6 \lambda A \tau^\beta H$ instead of $Q_0$ (the variable $r_0 - 2 g Q$ still is linear in $\tau_0$).

If $\beta_H < 1$, then $\tau^\beta H$ dominates $\tau$. The critical exponents $\nu_H$ and $\gamma_H$ of the $\chi$-field thus are Fisher-renormalized \[13\] versions of the $Q = \text{const}$ exponents $\nu$ and $\gamma$ ($\eta$ is not renormalized)

$$\nu_H = \beta_H \nu, \quad \gamma_H = \beta_H \gamma.$$

The correlation lenghth, for instance is $\xi \propto \tau^{-\nu_H}$ instead of $\xi \propto \tau^{-\nu}$. All this is analogous to simple polymer networks \[4\]. In tree-approximation the linear term vanishes for

$$Q (\tau_0) = \frac{1}{6 \lambda} \left( \tau_0 - \sqrt{\tau_0^2 - 12 \lambda H} \right) = \frac{1}{6 \lambda} \left( \tau_0 - \sqrt{(\tau_0 - \tau_c) (\tau_0 + \tau_c)} \right) ,$$

$$\tau' = \tau_0 - 6 \lambda Q = \sqrt{(\tau_0 - \tau_c) (\tau_0 + \tau_c)} .$$

There remains the model without the linear term, but with a temperature variable $\tau' \sim \sqrt{(\tau_0 - \tau_c)}$ instead of $\tau_0$. In tree approximation the critical exponents for the $\chi$ fields thus are $\nu_H = \frac{1}{4}, \gamma_H = \frac{1}{4}$ and $\beta_H = \frac{1}{2} \[16\].

Renormalization group fixed points

A renormalization group calculation can be performed for $Q = \text{const}$. A one-loop calculation leads to the coupling constant flow

$$d \bar{\lambda} / dl = \bar{\lambda} \left( \frac{\xi}{2} - 81 \bar{\lambda}^2 \right) ,$$

$$d \bar{g} / dl = \bar{g} \left( \frac{\xi}{2} - 2 \bar{g}^2 - 24 \bar{g} \bar{\lambda} + 9 \bar{\lambda}^2 \right) ,$$

(4)
Figure 2: One-loop diagrams. The dot denotes $\bar{\chi}$, the bar denotes the $\alpha$-propagator (center right). Diagrams marked with an asterisk (with an external dot or two internal dots or bars) only renormalize $\lambda'$ and $\lambda''$.

Figure 3: The coupling constant flow for $\alpha > 0, \epsilon > 0$ (schematic).

where $\epsilon = 8 - d$ and $\bar{\lambda} = \lambda \sqrt{\alpha K_d}$ and $\bar{g} = g \sqrt{\alpha K_d}$ with $K_d = 2^{1-d} \pi^{-d/2}/\Gamma(d/2)$. The contributing diagrams are depicted in fig. [2], the coupling constant flow is shown graphically in fig. [3]. The flow is inversion-symmetric in the $(\bar{g}, \bar{\lambda})$ plane, only the region $\bar{g} \geq 0, \bar{\lambda} \geq 0$ is of interest. The trivial fixed point $\bar{g} = \bar{\lambda} = 0$ ist unstable below the critical dimension 8.

At the fixed point $\bar{g} = \frac{1}{2} \sqrt{\epsilon}, \bar{\lambda} = 0$ at the bottom right of fig. [3] the finite $\bar{g}$ couples $\chi$ and $\varphi$, but there is no $\bar{\lambda}\chi^3$ interaction. However, $\bar{\lambda}$ is generated from $\bar{g}$, and the fixed point is unstable in $\bar{\lambda}$-direction.

The two fixed points with a finite $\bar{\lambda} \approx \frac{1}{18} \sqrt{2 \epsilon}$ are identical as far as the double-strand field $\chi$ is concerned. The flow equation (4) for $\bar{\lambda}$ and the critical exponents for $\chi$ are independent of $\bar{g}$, also in higher orders of the loop expansion. The fixed point with finite $\bar{g} = \frac{1}{6} \sqrt{2 \epsilon}$ in some way thus only describes a decoration of $\chi$ with $\varphi$.

The critical exponents $\eta = -\frac{\epsilon}{3} + O(\epsilon^2)$ and $\nu = \frac{1}{2} + \frac{5}{36} \epsilon + O(\epsilon^2)$ for $\chi$ agree with the exponents of one-component branched polymers [4]. And in fact, the critical behavior of $\chi_{\mu\nu}$ is contained in the correlation function $\langle \chi_{11}\chi_{11} \rangle$, in which there only occurs the $\chi_{11}$-component of $\chi_{\mu\nu}$ - the tensor nature of $\chi$ (and double polymer twisting) plays no role.

Branched polymers of this type have been examined extensively [4], the most interesting aspect is the equivalence to the Lee-Yang edge-singularity in $d - 2$ dimensions [5, 6]. The Lee-Yang exponents have been calculated in 5-loop order [17]. The physically relevant exponents are the Fisher-renormalized versions $\nu_H = \beta_H \nu$ and $\gamma_H = \beta_H \gamma$ of $\nu$ and $\gamma$, see table [1]. For example, the gyration radius of $\chi$ grows with polymer length $s_\chi$ like $s_\chi^{\nu_H}$. In three dimensions the exact value is $\nu_H = \frac{1}{2}$ [5]. The field theory [1] and the RG calculation thus have shown that the double-strand $\chi$ belongs to this universality class, the three-point interaction originates from the $\chi\varphi\varphi$ vertex.

In contrast, the flow equation of the single-strand field $\varphi$ depends on the double-strand field $\chi$. The critical exponents for the $\varphi$-field at the stable fixed point in one-loop order are

$$\eta_\varphi = -\frac{\epsilon}{18}, \quad \nu_\varphi = \frac{1}{2} + \frac{\epsilon}{36}, \quad \gamma_\varphi = \nu_\varphi (2 - 2\eta_\varphi) = 1 + \frac{\epsilon}{9}.$$ 

The role of the $\varphi$-polymer is examined below.
The effective coupling constants of the action integral (1) are strongly interact, but this cannot be described in detail with the field theory.

Table 1: Exponents for $\lambda$-field at stable fixed point in two-loop order $[13]$, $\epsilon = 8 - d$, and exact values for $d = 3$. Exponents with a suffix $H$ are Fisher-renormalized.

| $d$ | $\eta$ | $\nu = \left(2 + \frac{n-1}{2}\right)^{-1}$ | $\gamma = \nu(2 - \eta)$ | $\beta_H = \frac{1}{1 + \gamma}$ | $\nu_H = \nu/\beta_H$ | $\gamma_H = \gamma/\beta_H$ |
|-----|--------|---------------------------------|----------------------------|-------------------------------|------------------|------------------|
| 3   | -1     | -1                              | -3                         | $-\frac{1}{2}$                | $\frac{1}{2}$   | $\frac{3}{2}$   |

The parameter flow (4) is based on the assumption of a positive $\alpha$. Diagram $A$ in fig. (1) generates a negative contribution to $\alpha$. This diagram involves the $\varphi$-field. Only a sufficiently strong excluded-volume effect ($u_2Q^2$ large in in eq.(2)) generates a positive $\alpha$.

If $\alpha$ is negative (which might occur in theta solvents), then in the flow equation (4) the signs of $\bar{\lambda}^2$, $\bar{g}^2$ and $\bar{g}\bar{\lambda}$ in the brackets change, and there is no stable fixed point any more. The polymers strongly interact, but this cannot be described in detail with the field theory.

### Scale invariance

The effective coupling constants of the action integral (1) are $\bar{\lambda} = \sqrt{\alpha K_d}\lambda$ and $\bar{g} = \sqrt{\alpha K_d}g$. In addition to these coupling constants the action integral contains the dimensionful parameter $\alpha$, which is known to exactly scale like $\alpha \sim k^2$. Our conventions for the critical exponents are contained in the scaling equivalences

$$
\chi \sim k^{d/2-1+\eta_x}, \quad \varphi \sim k^{d/2-1+\eta_\varphi}, \quad k \sim (\tau_0 - \tau_c)^{\nu_x} \sim (r_0 - r_c)^{\nu_\varphi},
$$

Critical exponents without a suffix $H$ are not Fisher-renormalized.

The scale invariance implied by the renormalization group consists of a scaling of length, the fields and the parameter $\alpha$. Vertex functions $\Gamma_{\text{loop}}$ calculated directly in a loop expansion (for instance with graph matching) normally scale as predicted by the RG. This correspondence is violated if $\Gamma$ contains an $\alpha$-factor - which is constant in a direct calculation.

An example is the vertex function $\Gamma_{\chi\chi\chi} \sim k^{\epsilon/2-1-3\eta_x}$ with three amputated external $\chi$-lines, which contains a factor $\lambda = \alpha^{-1/2}K_d^{-1/2}\bar{\lambda}$. A direct calculation thus gives $\Gamma_{\chi\chi\chi}^{\text{loop}} \sim \alpha^{1/2}\Gamma_{\chi\chi\chi} \sim k^{\epsilon/2-1-3\eta_x}$.

Other examples are

$$
\begin{align*}
\Gamma_{\chi\chi\chi} & \sim k^{\epsilon/2-1-3\eta_x}, & \Gamma_{\chi\chi\chi}^{\text{loop}} & \sim k^{\epsilon/2-3\eta_x}, \\
\Gamma_{\chi\varphi\varphi} & \sim k^{\epsilon/2-1-\eta_x-2\eta_\varphi}, & \Gamma_{\chi\varphi\varphi}^{\text{loop}} & \sim k^{\epsilon/2-\eta_x-2\eta_\varphi}, \\
\Gamma_{\chi} & \sim k^{5-\epsilon/2-\eta_x}, & \Gamma_{\chi}^{\text{loop}} & \sim k^{4-\epsilon/2-\eta_x}.
\end{align*}
$$

This allows to determine the exponent $\beta_H$ of the equation of state (3), which also determines the Fisher-renormalization of the critical exponents.

### Equation of state

If the $\chi$ field is shifted like $\chi_{\mu\nu} \rightarrow \delta_{\mu\nu}Q(\tau_0) + \chi_{\mu\nu}$, then the action integral takes the form (2). One requires that for $Q(\tau_0) \cong Q_c + a\tau - A\tau^{1+\beta_H}$ the leading coefficient of $\dot{\chi}$ in the effective potential $\Gamma$ vanishes in any order of the loop expansion. Inserting $Q(\tau_0)$ into eq.(2) leads to

$$
[...] + \Gamma_{\chi}^{\text{loop}} = (-H - \tau_c Q_c + 3\lambda Q_c^2) + A(\tau_c - 6\lambda Q_c)\tau^{1+\beta_H} \\
- (Q_c + a\tau_c - 6\lambda a Q_c)\tau + (A - 6\lambda a A)\tau^{1+\beta_H} + 3\lambda A^2 \tau^{2\beta_H} + \Gamma_{\chi}^{\text{loop}} = 0.
$$
The 1-point vertex function $\Gamma^\text{loop}_\chi = a_0 + a_1 \tau^\beta + a_2 \tau^{2\beta} + a_3 \tau^{(d/2-\eta_\beta)} + \ldots$ contains regular and singular contributions in the Fisher-renormalized variable $\tau^\beta$, see eq. (5). The $O(\tau^0)$ and $O(\tau^\beta)$ parts of eq. (6) vanish with an appropriate choice of $\tau_c$ and $Q_c$.

The $O(\tau)$-term of eq. (6) only can vanish for $\beta_H = 1/(1 + \gamma)$ \[18\]. The critical exponents of the $\chi$-field are listed in table (1). The condition $\beta_H < 1$ for Fisher-renormalization is fulfilled. The value $\beta_H^{d=3} = -\frac{1}{2}$ for $d = 3$ is known exactly, and thus $Q^{d=3} \sim \tau^{-1/2}$ in the vicinity of the fixed points with $\lambda > 0$.

**Polymer lengths**

The critical point of the double-strand polymer $\chi$, as shown above, is independent of the single-strand $\varphi$. The critical exponents are the exponents of the Lee-Yang edge singularity in $d-2$ dimensions, with an additional Fisher-renormalization.

In contrast, the single-strand polymer $\varphi$ depends on the double-strand polymer $\chi$. A realistic physical situation is one single-strand polymer with (originally) length $s$. Within the model \[11\] this can be achieved with a weak spatially constant external field in $\mu = 1$-direction. The physics is contained in the correlation function $\langle \varphi_1 \varphi_1 \rangle_{k=0} = \langle \varphi_1(0) \int d^d x \varphi_1(x) \rangle$. The polymer lengths $s_\varphi$ and $s_\chi$ of the polymer types are determined by $\tau_0$ and $\tau_0$, which in $e^{-S}$ act like negative chemical potentials. In a real experiment, however, the total length $s = s_\varphi + 2s_\chi$ is given, and effectively only one parameter (temperature, composition of solution, ...) can be varied. We now write $r = r_0 - \tau_c$ and $\tau = \tau_0 - \tau_c$.

The amounts (or lengths) of the polymer types are (fig.(4))

$$s_\varphi = \left( \varphi_1 \varphi_1 \right)_{k=0} / \langle \varphi_1 \varphi_1 \rangle = \langle \varphi_1 \varphi_1 \rangle \Gamma^\varphi_\varphi \sim -2 \frac{d}{dr} \ln \langle e^{S} \varphi_1 \varphi_1 \rangle \sim \frac{1}{\tau},$$

$$s_\chi = \left( \varphi_1 \varphi_1 \right)_{k=0} / \langle \varphi_1 \varphi_1 \rangle = 2 \langle \varphi_1 \varphi_1 \rangle \langle \chi_{11} \chi_{11} \rangle Q \Gamma^\text{loop}_\chi \varphi \varphi.$$

The length $s_\varphi$ contains $\Gamma^\varphi_\varphi$, a two-point vertex function with a $\varphi^2$ insertion. The length $s_\chi$ contains a contribution proportional to $Q$ because of redefinition \[2\]. $\chi^2_{\mu \nu}$ is negligible. The vertex function $\Gamma^\text{loop}_\chi \varphi \varphi$ is defined in eq. (5). In Fisher-renormalized tree-approximation

$$s_\varphi = 1/r, \quad s_\chi = aQ / (r \tau^{1/2}),$$

where $a$ is a constant. The ratio is $s_\varphi / s_\chi = \tau^{1/2} / (aQ)$. With the constraint $s = s_\varphi + 2s_\chi$ it follows

$$s_\varphi = \frac{s}{1 + 2s_\chi / s_\varphi} \approx \frac{s}{2aQ} \tau^{1/2},$$

$$s_\chi = \frac{s}{2 + s_\varphi / s_\chi} \approx \frac{1}{2} s \left( 1 - \frac{1}{2aQ} \tau^{1/2} \right).$$

In the final stage of the condensation $s_\varphi$ thus decreases like $\tau^{1/2}$. 

Figure 4: Polymer lengths, up to one loop. Self energies of external lines are not shown.
In general, if $\tau$ and $r$ are small then near a fixed point the vertex functions scale like $\Gamma_{\varphi^2\varphi^2} \sim k^{2-1/\nu_\varphi-2\eta_\varphi}$ and $\Gamma^\text{loop}_{\chi\varphi\varphi} \sim k^{\xi/2-\eta_\chi-2\eta_\varphi}$ where $k \sim \tau^\nu_H \sim r^{\nu_\varphi}$. The variable $\tau$ drops out from the vertex functions and from $\langle \varphi \varphi \rangle$ for $\tau \to 0$. From scaling arguments $\langle \varphi \varphi \rangle \sim r^{-\gamma_\varphi}$. The $\chi$-propagator $\langle \chi \chi \rangle \sim \tau^{-\gamma_H}$ only depends on $\tau$. This gives for $\tau \to 0$

$$s_\varphi \sim r^{-\gamma_\varphi \nu_\varphi (2-1/\nu_\varphi-2\eta_\varphi)} \sim r^{-1},$$

$$s_\chi \sim (Q_c + A_H \tau^\beta_H) r^{-\gamma_H \nu_\varphi - \beta_H \nu_\varphi / \nu_H},$$

where the second line the hyperscaling equation $\beta_H / \nu_H = 2 - \frac{\xi}{2} + \eta_\chi$ has been used \[18\].

We now assume $\beta_H > 0$. In the final stage of the condensation $s_\chi$ is constant, and thus $r^{-\beta_H \nu_\varphi / \nu_H} \sim \tau^{\gamma_H}$. It follows

$$s_\varphi \sim r^{-1} \sim \tau^z,$$

with $z = \gamma_H \nu_H / \beta_H \nu_\varphi = \frac{1}{2} + \frac{7}{36} \epsilon + ...$

This agrees with the mean-field result for $\epsilon = 0$.

If $\beta_H < 0$, which is the case in $d = 3$ where $\nu_H = \frac{1}{2}$, $\gamma_H = \frac{3}{2}$ and $\beta_H = -\frac{1}{2}$, then there is no reasonable solution. The exponent $z$ diverges if $\beta_H$ approaches zero from above, and one might conclude that $s_\varphi$ suddenly drops to zero at some $\tau$-value when $\tau$ is diminished. But the details of the condensation of a single strand to a double strand in three dimensions are not understood.

**Conclusions**

Our original starting point was a similar lattice model and a similar field theory with length variables instead of $O(n)$ indices \[19\]. This model contains directed polymers and can describe the case where aligned and anti-aligned polymer strands form double polymers with different binding energies. It also would allow to examine the case where the binding energy of aligned and anti-aligned polymer strands form double polymers with different binding energies. It would be of interest to compare the predictions with experiment. For instance, the radius of gyration for the single-strand-component of the model.

Within the simpler $O(n)$-model examined here it becomes clear, that there is a close relationship to branched polymers consisting of one type of polymer. The double-strand polymer acquires a three-point interaction, decouples from the single-strand polymer near the critical point, and thus becomes a branched polymer of the conventional type.

We have reproduced Fisher-renormalization of the critical exponents and the equation of state in a more direct way. The critical exponents in three dimensions are known exactly, and it would be of interest to compare the predictions with experiment. For instance, the radius of gyration for the $\chi$-field should grow like $s_\chi^{1/2}$ with polymer length $s_\chi$. Under most conditions, RNA strands in a solvent are far from ideal, and one needs long polymers to reach a scaling limit. But in principle this could be tested with RNA with a base sequence like ATATA...

On the other hand, the single-strand polymer is somewhat ephemeral near the critical point. It is affected by the double-strand polymer, but disappears at the critical point, when the condensation to a double-strand is complete. Critical exponents can be calculated in an expansion around $d = 8$, but an extrapolation to three dimensions is difficult. And it appears that the connection between branched polymers and the Lee-Yang edge singularity in $d = 2$ dimensions \[5\] \[6\] cannot be extended to the single-strand-component of the model.

**Appendix**

**Appendix A: Average over spin variables**

The analytic expression for the partition sum makes use of an average over components $\sigma_i$ of an $N$-dimensional vector $(\sigma_1, \sigma_2, ..., \sigma_N)$ with weight $P_N(\sigma) d^N \sigma = \pi^{-N/2} 2^{N/2} \Gamma(\frac{N}{2}) \delta \left( |\sigma| / \sqrt{N} - 1 \right) d^N \sigma$. One finds
\[ \langle \sigma_{m_1} \ldots \sigma_{m_k} \rangle_N = \int P_N(\sigma) \sigma_{m_1} \ldots \sigma_{m_k} dN \sigma = \frac{\Gamma \left( \frac{N}{2} \right) N^{(m_1+\ldots+m_k)/2}}{\Gamma \left( \frac{N+m_1+\ldots+m_k}{2} \right) 2^{(m_1+\ldots+m_k)/2}} \langle \sigma_{m_1} \ldots \sigma_{m_k} \rangle_G, \]

where \( \langle \ldots \rangle_G \) is the normalized average over the gaussian function \( \exp \left( -\frac{1}{2} \sigma^2 \right) \). The gaussian averages can be calculated by pairwise contraction of the \( \sigma \)-factors, and produce a value of order 1. This gives \( \langle 1 \rangle_N = 1 \) and \( \langle x_i x_j \rangle = \delta_{ij} \). Averages are analytic in \( N \), and expectation values of \( m > 2 \) \( \sigma \)-factors vanish like \( N^{(m-2)/2} \) in the limit \( N \to 0 \).

**Appendix B: Interaction at the level of the lattice model**

The ansatz for the interaction at a given lattice point is

\[ f = f_1 + bf_2 + cf_3, \]

where

\[ f_1 = \frac{(N+2)(N+4)}{2N^2} \sum_{\mu \nu} S^{\mu \nu} s^\mu s^\nu, \quad f_2 = \sum_{\mu \nu} S^{\mu \mu} s^\nu s^\nu, \quad f_3 = \sum_{\mu} S^{\mu \mu}. \]

One requires \( \langle S^{\alpha \beta} s^{\rho} s^{\tau} f \rangle = \delta_{\alpha \rho} \delta_{\beta \tau} + \delta_{\alpha \tau} \delta_{\beta \rho} \) and \( \langle S^{\alpha \beta} f \rangle = 0 \). This is possible only with factors \( 1/N \). It follows

\[
\begin{align*}
\langle S^{\alpha \beta} s^{\rho} s^{\tau} f_1 \rangle &= \delta_{\alpha \rho} \delta_{\beta \tau} + \delta_{\alpha \tau} \delta_{\beta \rho} + \delta_{\alpha \beta} \delta_{\rho \tau}, \\
\langle S^{\alpha \beta} s^{\rho} s^{\tau} f_2 \rangle &= \frac{(N+2)(N+4)}{2N^2} \delta_{\alpha \beta} \delta_{\rho \tau}, \\
\langle S^{\alpha \beta} s^{\rho} s^{\tau} f_3 \rangle &= \frac{2N^2}{N+2} \delta_{\alpha \beta} \delta_{\rho \tau}.
\end{align*}
\]

The solution is

\[ b = \frac{(N+2)(N+4)}{2N^2(2+N-n)}, \quad c = -Nb. \]

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