The Mini-Superambitwistor Space

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We present the construction of the mini-superambitwistor space, which is suited for establishing a Penrose-Ward transform between certain bundles over this space and solutions to the $\mathcal{N} = 6$ super Yang-Mills equations in three dimensions.

The essential point underlying twistor string theory [1] is the marriage of Calabi-Yau and twistor geometry in the space $\mathbb{CP}^3/\mathbb{Z}_4$. This complex projective space is a Calabi-Yau supermanifold and simultaneously the supertwistor space of the complexified, compactified Minkowski space. The interest in a twistorial string theory is related to the fact that twistor geometry allows for a very convenient description of the solution spaces of classical gauge theories [2, 3]. In such a description, spacetime is associated with a certain complex manifold, its twistor space. Subsequently, the space of holomorphic vector bundles over this twistor space is mapped to the solution space of the gauge theory via a so-called Penrose-Ward transform.

Suitable twistor spaces are well-known for four-dimensional self-dual Yang-Mills theory and its supersymmetric extensions, as well as for four-dimensional Yang-Mills theory and its $\mathcal{N} = 3$ supersymmetric extension, see [4, 5, 6, 7]. Via dimensional reduction, one obtains so-called mini-twistor spaces upon which a description of the solution space to the Bogomolny monopole equation [7] and its supersymmetric extensions can be constructed, see e.g. [8, 9]. In this collection, the mini-twistor space suited for a Penrose-Ward transform yielding solutions to the three-dimensional Yang-Mills-Higgs equations and their $\mathcal{N} = 6$ supersymmetric extension is evidently missing. This gap was filled in [10], and here we will concisely report on the results and thus review the construction of the mini-superambitwistor space.

Let us first recall that twistors were invented by Penrose to give a unified description of General Relativity and quantum mechanics. Consider a light ray, which is given by the set of points $x^{\alpha\dot{\alpha}}$ satisfying the equation $x^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}}_0 + t\rho^{\alpha\dot{\alpha}}$. Here, $x^{\alpha\dot{\gamma}}_0$ is an arbitrary point on the light ray and $t \in \mathbb{R}$ is a parameter. Taking a light ray which does not pass through the origin, one can obviously choose $x^{\alpha\dot{\alpha}}_0$ to be null. Since one can decompose every null vector into a pair of commuting two-spinors, we can rewrite the equation defining our light ray as $x^{\alpha\dot{\alpha}} = c\omega^{\alpha\dot{\gamma}}\tilde{\omega}^{\dot{\gamma}\dot{\alpha}} + t\lambda^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}$. Multiplication of this equation by $\lambda_{\dot{\alpha}}$ together with the right choice of normalization $c = (\tilde{\omega}^{\dot{\gamma}}\lambda_{\dot{\alpha}})^{-1}$ gives rise to the incidence relation

$$\omega^{\alpha} = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}. \quad \text{(1)}$$

A twistor $Z^i$ is now a projective pair of two-spinors $Z^i = (\omega^{\alpha}, \lambda_{\dot{\alpha}}) \in \mathbb{CP}^3$, which transforms under coordinate shifts $x^{\alpha\dot{\alpha}} \rightarrow x^{\alpha\dot{\alpha}} + r^{\alpha\dot{\alpha}}$ according to the incidence relation.

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2the incidence relation is invariant under scaling
The space $\mathbb{C}P^3$ is the twistor space of the complexified, compactified Minkowski space $S^3$. Taking out the sphere $S^2 \cong \mathbb{C}P^1 = \{ (\omega^\alpha \neq 0, \lambda_\dot{\alpha} = 0) \}$ which is incident to $x^{\alpha\dot{\alpha}} = \infty$, we can consider $\lambda_\dot{\alpha}$ as the homogeneous coordinates on the Riemann sphere $\mathbb{C}P^1$. Due to the incidence relation, the $\omega^\alpha$’s are homogeneous polynomials of degree one in $\lambda_\dot{\alpha}$ and thus describe a section of the rank two complex vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$. This bundle’s total space, which we denote by $\mathcal{P}^3$, is the twistor space of $\mathbb{C}^4$.

The incidence relation allows us furthermore to establish the double fibration

$$
\begin{array}{ccc}
\mathbb{C}^4 \times \mathbb{C}P^1 & \xrightarrow{\pi_2} & \mathbb{C}P^3 \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
\mathbb{C}^4 & & \mathbb{C}^4
\end{array}
$$

where $\pi_2(x^{\alpha\dot{\alpha}}, \lambda_\dot{\alpha}) = (x^{\alpha\dot{\alpha}} \lambda_\dot{\alpha}, \lambda_\dot{\alpha})$ and $\pi_1(x^{\alpha\dot{\alpha}}, \lambda_\dot{\alpha}) = (x^{\alpha\dot{\alpha}})$, from which one can easily read off the following twistor correspondence: A point $x^{\alpha\dot{\alpha}} \in \mathbb{C}^4$ defines a sphere $S^2 \cong \mathbb{C}P^1$ embedded in $\mathcal{P}^3$, while a point $p \in \mathcal{P}^3$ is incident to a null two-plane $\mathbb{C}^2 \times \mathbb{C}^2$ in $\mathbb{C}^4$.

To obtain an $\mathcal{N}$-extended supertwistor space, one can simply start from the projective superspace $\mathbb{C}P^3|\mathcal{N}$, take out the $\mathbb{C}P^1|\mathcal{N}$ corresponding to the point at infinity and arrive at the supervector bundle

$$
\mathcal{P}^3|\mathcal{N} := \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^N \otimes \Pi \mathcal{O}(1)
$$

over the Riemann sphere $\mathbb{C}P^1$. The operator $\Pi$ simply inverts the parity of the fibre coordinates of a vector bundle, and one has therefore $\mathcal{N}$ additional homogeneous Grassmann coordinates $\eta_1, \ldots, \eta_N$. The incidence relation (1) is extended to

$$
\omega^\alpha = x^{\alpha\dot{\alpha}} \lambda_\dot{\alpha} \quad \text{and} \quad \eta_\alpha = \eta^{\dot{\alpha}} \lambda_\dot{\alpha},
$$

which naturally gives rise to the double fibration

$$
\begin{array}{ccc}
\mathbb{C}^4|2\mathcal{N} \times \mathbb{C}P^1 & \xrightarrow{\pi_2} & \mathcal{P}^3|\mathcal{N} \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
\mathbb{C}^4|2\mathcal{N} & & \mathbb{C}^4|2\mathcal{N}
\end{array}
$$

Here, $\pi_2$ is given by the extended incidence relations (4) and $\pi_1$ is the obvious projection.

In the special case $\mathcal{N} = 4$, the first Chern number of (the tangent bundle of) the supertwistor space vanishes. This is due to the fact that Berezin integration is equivalent to a differentiation and therefore the contribution of $\Pi \mathcal{O}(1)$ to the total first Chern number is $-1$. Altogether, we have a contribution of 2 from the tangent bundle to the Riemann sphere $TS^2 \cong \mathcal{O}(2)$ and 1 from each bosonic $\mathcal{O}(1)$, which is cancelled by the $-4$ of the fermionic line bundles $\Pi \mathcal{O}(1)$. Thus, $\mathcal{P}^3|4$ comes with a holomorphic measure $\Omega^{3,0|4,0}$ and this supertwistor space is a Calabi-Yau supermanifold.

One can now establish a relation between a topological string theory having $\mathcal{P}^3|4$ as its target space and $\mathcal{N} = 4$ self-dual Yang-Mills theory in four dimensions: The open topological B-model on $\mathcal{P}^3|4$ with a stack of $n$ space-filling D5-branes is equivalent to holomorphic Chern-Simons theory on the same space, which describes the dynamics of a $\mathfrak{gl}(n, \mathbb{C})$-valued connection $(0,1)$-form $\mathcal{A}^{0,1}$ on a rank $n$ complex vector bundle $\mathcal{E} \to \mathcal{P}^3|4$. The action of this holomorphic Chern-Simons theory reads as

$$
S = \int \Omega^{3,0|4,0} \wedge \text{tr} (\mathcal{A}^{0,1} \wedge \bar{\partial} \mathcal{A}^{0,1} + \frac{2}{3} \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1})
$$

(6)
where $\Omega^{3,0,4,0}$ is the holomorphic measure on $\mathcal{P}^{3|4}$. (Some minor assumptions about the explicit form of $\mathcal{A}^{0,1}$ have to be made at this point, see [1] ) The corresponding equations of motion are given by $\partial \mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0$ and their solutions describe holomorphic structures which promote the complex vector bundle $\mathcal{E}$ to holomorphic vector bundles $(\mathcal{E}, \mathcal{A}^{0,1})$. Via a generalized Penrose-Ward transform using supertwistor spaces [1, 6], one can map these holomorphic vector bundles to solutions to the $\mathcal{N} = 4$ extended SDYM equations on $\mathbb{C}^4$. These equations are the supersymmetric extensions of the self-dual Yang-Mills equations $F_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}$, which read in spinorial notation $F_{\mu \nu} \to F_{\dot{a} \dot{a} \beta \dot{\beta}} = \varepsilon_{\dot{a} \dot{b}} f_{\dot{a} \dot{b}} + \varepsilon_{\dot{a} \dot{b}} f_{\dot{a} \dot{b}}$ as

\[
\begin{align*}
\alpha &= 0 , \\
\nabla_{\dot{a}} \chi^{\dot{i}} &= 0 , \\
\square \phi^{[ij]} &= \frac{1}{2} \{ \chi^{\dot{i}}, \chi^{\dot{j}} \} , \\
\varepsilon^{\dot{i} \dot{j}} \nabla_{\dot{a}} \chi^{\dot{i} \dot{j} [kl]} &= -2 [\phi^{[ij]}, \chi^{\dot{k} \dot{l}]} , \\
\varepsilon^{\dot{i} \dot{j}} \nabla_{\dot{a}} \chi^{\dot{i} \dot{j} [kl]} &= -\{ \chi^{\dot{i}}, \chi^{\dot{j}} \} + [\phi^{ij}, \nabla_{\dot{a}} \phi^{kl}] ,
\end{align*}
\]

where the nontrivial fields $(f_{\dot{a} \dot{b}}, \chi^{\dot{i}}, \phi^{[ij]}, \chi^{\dot{i} \dot{j} [kl]}, \chi^{\dot{i} \dot{j} [kl]}, G^{[ij]kl])$ have helicities (+1, +1, 0, -1/2, -1). Neglecting the trivial field $f_{\dot{a} \dot{b}}$, the field content of $\mathcal{N} = 4$ self-dual Yang-Mills theory is identical to the one of $\mathcal{N} = 4$ super Yang-Mills theory, but the interactions in the two theories are different.

Let us now turn our attention to another twistor space, the so-called superambitwistor space, upon which a Penrose-Ward transform for the full $\mathcal{N} = 3$ super Yang-Mills equations can be constructed. Important here is the observation that the $\mathcal{N} = 3$ supermultiplet is reducible and splits into a self-dual and an anti-self-dual part. One is thus naturally led to glue together the twistor space $\mathcal{P}^{3|3}$ for $\mathcal{N} = 3$ self-dual Yang-Mills theory with a dual copy $\mathcal{P}^{3|3}$ for the anti-self-dual part. Denoting the homogeneous coordinates on these two spaces by $(\omega^\alpha, \lambda_\dot{a}, \eta_i)$ and $(\omega^*_\alpha, \lambda^*_\dot{a}, \eta^*_i)$, we can write the appropriate gluing condition as the quadric equation

\[
\kappa := \omega^\alpha \lambda^*_\alpha - \omega^*_\alpha \lambda_\dot{a} + 2 \eta^*_i \eta_i = 0 ,
\]

which defines the superambitwistor space $\mathcal{L}^{5|6}$ as a subset of $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}$. To examine the geometry of $\mathcal{L}^{5|6}$ more closely, note that the appropriate incidence relations for the space $\mathcal{P}^{3|3} \times \mathcal{P}^{3|3}$ read as

\[
\begin{align*}
\omega^\alpha &= x^\alpha R \lambda_\dot{a} , \quad \eta_i = \eta^i R \eta_i , \quad \omega^*_\alpha = x^\alpha L \lambda^*_\alpha , \quad \eta^*_i = \eta^i L \eta^*_i .
\end{align*}
\]

The quadric condition (8) is automatically and most generally solved, if we choose

\[
\begin{align*}
x^\alpha R &= x^\alpha \dot{a} - \eta^i \eta_i \dot{a} , \quad x^\alpha L &= x^\alpha \dot{a} + \eta^i \eta_i \dot{a} ,
\end{align*}
\]

and thus $x^\alpha R$ and $x^\alpha L$ are indeed right- and left-handed chiral coordinates on the chiral superspaces $\mathbb{C}^{4|12}_R$ and $\mathbb{C}^{4|12}_L$. These incidence relations, too, define a double fibration:

\[
\mathbb{C}^{4|12} \times \mathbb{C} P^1 \times \mathbb{C} P^1 \xrightarrow{\pi_2} \mathcal{L}^{5|6} \xrightarrow{\pi_1} \mathbb{C}^{4|12}
\]

\[\text{The word dual here refers to the spinor indices and not to the line bundles underlying } \mathcal{P}^{3|3}.\]
Over the superambitwistor space, one can then establish a Penrose-Ward transform which is a map between solutions to the $N = 3$ super Yang-Mills equations and certain holomorphic vector bundles over $\mathcal{L}^{5|6}$, see e.g. [6].

One can also find twistor spaces which describe self-dual Yang-Mills theory after a dimensional reduction $\mathbb{C}^4 \to \mathbb{C}^3$. In our conventions, one can make the following identification of vector fields

$$\mathcal{T}_3 := \frac{\partial}{\partial x^3} = -\frac{\partial}{\partial x^{12}} + \frac{\partial}{\partial x^{21}} \sim \frac{\partial}{\partial x^{[21]}}.$$  

(12)

Dimensional reduction of the $x^3$-direction thus implies eliminating the modulus $x^{[a\dot{a}]}$. On the twistor space side, this can be done by changing the incidence relation on $\mathcal{P}^3 = \mathcal{O}(1) \oplus \mathcal{O}(1)$ from $\omega^a = x^{a\dot{a}} \lambda_\dot{a}$ to $v = x^{a\dot{b}} \lambda_\dot{a} \lambda_{\dot{b}}$. The latter equation defines sections of the line bundle $\mathcal{P}^2 := \mathcal{O}(2)$ over the Riemann sphere $\mathbb{C}P^1$. More formally, one has $(\mathcal{O}(1) \oplus \mathcal{O}(1))/\mathcal{G} = \mathcal{O}(2)$, where $\mathcal{G}$ is the abelian group generated by the holomorphic vector field on $\mathcal{P}^3$ which corresponds to $\mathcal{T}_3$ [9]. The space $\mathcal{P}^2$ is called the mini-twistor space [7], and the corresponding double fibration reads as

$$\begin{array}{ccc}
\mathbb{C}^3 & \to & \mathbb{C}P^1 \\
\nu_2 & \downarrow & \nu_1 \\
\mathcal{P}^2 & \to & \mathbb{C}^3
\end{array}$$

(13)

After applying this reduction to the space $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$, we arrive at the space $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3}$ together with the incidence relations

$$v = y^{a\dot{b}} \lambda_{\dot{a}} \lambda_{\dot{b}}, \quad \eta_* = \eta_*^i \lambda_{\dot{a}}, \quad v_* = y_*^{a\dot{b}} \lambda_*^i \lambda_{\dot{b}}^i, \quad \eta_*^i = \eta_*^{a\dot{a}} \lambda_*^i.$$

(14)

Here, we adjusted the spinor indices anticipating that there is no distinction between left- and right-handed spinors on $\mathbb{C}^3$. The quadric equation (5) is correspondingly reduced to the equation

$$(v - v_* + 2\eta_*^i \eta_i^j)|_{\lambda = \lambda_*} = 0,$$

(15)

and this condition defines the mini-superambitwistor space as a subset of $\mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3}$ [10]. Altogether, we have the dimensional reductions

$$\begin{array}{ccc}
\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3} & \to & \mathcal{P}^{2|3} \times \mathcal{P}_*^{2|3} \\
\downarrow & \downarrow & \downarrow \\
\mathcal{L}^{5|6} & \to & \mathcal{L}^{4|6} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{C}P^1 \times \mathbb{C}P^1 & \to & \mathbb{C}P^1 \times \mathbb{C}P^1
\end{array}$$

(16)

The reduced quadric equation (15) is solved, if we choose the “chiral coordinates”

$$y^{a\dot{b}} = y_0^{a\dot{b}} - \eta_*^i (\dot{a} \eta_i^\dot{b}) \quad \text{and} \quad y_*^{a\dot{b}} = y_0^{a\dot{b}} + \eta_*^i (\dot{a} \eta_i^\dot{b})$$

(17)

in the incidence relations.

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4See also [9], where the explicit identification is slightly different.
The incidence relations (14) determine together with the reduced quadric equation (15) yielding (17) the dimensional reduction of the double fibration (11) to be

\[ \pi_2 : \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}^4, \]

\[ \pi_1 : \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}^4, \]

\[ \nu_2 : \mathbb{C}^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}^3, \]

\[ \nu_1 : \mathbb{C}^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}^3. \]

Although all the constructions seem to go through without difficulties, the geometry of \( \mathcal{L}^{46} \) contains some surprises. First of all, note that the reduced quadric condition (15) is not imposed over the whole base \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) of the supervector bundle \( \mathcal{P}^3 \times \mathcal{P}^3 \), but only over its diagonal \( \Delta := \text{diag}(\mathbb{C}P^1 \times \mathbb{C}P^1) \), which is the subspace of the base for which \( \lambda = \lambda^* \). Considering the projection \( \pi : \mathcal{L}^{46} \to \mathbb{C}P^1 \times \mathbb{C}P^1 \), we see that \( \pi^{-1}(\lambda, \lambda^*) \cong \mathbb{C}^2 \) for \( \lambda \neq \lambda^* \), but \( \pi^{-1}(\lambda, \lambda^*) \cong \mathbb{C} \) on the diagonal \( \Delta \). One can in fact show that \( \mathcal{L}^{46} \) is a fibration (10), but since its fibre dimension varies, it is evidently not a vector bundle. However, we will see in the following, that this seemingly unpleasant property does not impose any relevant obstructions.

First, let us find an interpretation of the geometries involved in the double fibration for the mini-superambitwistor space which is contained in (18). Recall that for the well-known double fibrations (2), (5) and (11), there is a nice interpretation in terms of flag manifolds (3). In the case of the double fibrations for the mini-supertwistor and mini-superambitwistor spaces, we find a quite similar description. For simplicity, we restrict our discussion to the bosonic subspaces, i.e. to the bodies of the considered superspaces.

After imposing reality conditions (9) on the spaces involved in the double fibration (13), we obtain

\[ \mathbb{R}^3 \times S^2 \]

\[ \mathcal{P}^2 \]

The space \( \mathbb{R}^3 \times S^2 \) on the top is the space of oriented lines in \( \mathbb{R}^3 \) with one marked point. Keeping the point and dropping the line evidently leads to an element of the space \( \mathbb{R}^3 \), while keeping the line and dropping the point – or, alternatively, moving the point as close as possible to the origin – leads to an element of \( \mathcal{P}^2 = \mathcal{O}(2) \cong TS^2 \). The projections \( \nu_1 \) and \( \nu_2 \) in (19) have therefore a clear geometric meaning.

The real double fibration for the bosonic part of the mini-superambitwistor space,

\[ \mathbb{R}^3 \times S^2 \times S^2 \]

\[ \mathcal{L}^4 \]

has a similar interpretation. The space \( \mathbb{R}^3 \times S^2 \times S^2 \) is the space of two oriented lines in \( \mathbb{R}^3 \) with a common marked point. Dropping the lines leads again to elements of \( \mathbb{R}^3 \), while moving the point on one of the lines (together with the attached second line) to its shortest distance to the origin yields an element of \( \mathcal{L}^4 \).

Ultimately, one is certainly interested in extending the discussion to the level of topological strings. Recall that the superambitwistor space is in fact a (local) Calabi-Yau
supermanifold \([1]\), and one can therefore use \(L^{5|6}\) as a target space for the topological B-model. Although the mini-superambitwistor space \(L^{4|6}\) is not a manifold, one nevertheless finds that a certain Calabi-Yau property still persists.

A Calabi-Yau manifold can be defined as a manifold whose tangent bundle has vanishing first Chern class. Chern classes of vector bundles in turn are related to certain degeneracy loci of a set of generic sections: On a rank \(n\) vector bundle, the first Chern class is Poincaré dual to the degeneracy loci of \(n\) generic sections. Straightforward calculations show that the appropriate degeneracy loci for \(L^{5|6}\) and \(L^{4|6}\) are rationally equivalent \([10]\). This is a strong hint that \(L^{4|6}\) comes with the necessary properties for using this space as target space for a topological B-model. Furthermore, if the conjecture \([11]\) by Aganagic, Neitzke and Vafa is correct and \(L^{5|6}\) is indeed the mirror symmetry partner of \(P^{3|4}\) then by applying dimensional reduction, it is only natural to conjecture that the mini-superambitwistor space \(L^{4|6}\) is the mirror of the mini-supertwistor space \(P^{2|4}\).

As far as the Penrose-Ward transform is concerned, the discussion over \(L^{4|6}\) follows essentially the lines of the discussion over \(L^{5|6}\). Since the mini-superambitwistor space is only a fibration and not a manifold, we have to slightly extend the notion of a vector bundle. We define an \(L^{4|6}\)-bundle of rank \(n\) by a Čech 1-cocycle \(\{f_{ab}\} \in H^1(L^{4|6}, \mathcal{S})\), where \(\mathcal{S}\) is the sheaf of smooth \(GL(n, \mathbb{C})\)-valued functions on \(L^{4|6}\). This 1-cocycle takes over the rôle of a transition function in an ordinary vector bundle. A holomorphic \(L^{4|6}\)-bundle is correspondingly defined by a holomorphic such Čech 1-cocycle. We call two \(L^{4|6}\)-bundles given by two 1-cocycles \(\{f_{ab}\}\) and \(\{f'_{ab}\}\) topologically equivalent, if there is a Čech 0-cochain \(\{\psi_a\} \in C^0(L^{4|6}, \mathcal{S})\) such that \(f_{ab} = \psi_a^{-1} f'_{ab} \psi_b\). In particular, an \(L^{4|6}\)-bundle is topologically trivial (topologically equivalent to the trivial bundle), if its defining 1-cocycle can be decomposed by a Čech 0-cochain according to \(f_{ab} = \psi_a^{-1} \psi_b\).

With these definitions, we can state that topologically trivial, holomorphic \(L^{4|6}\)-bundles, which become holomorphically trivial vector bundles upon restriction to any \(\mathbb{C}P^1 \times \mathbb{C}P^1_s \subset L^{4|6}\) are equivalent to solutions of the \(N = 6\) super Yang-Mills equations on \(\mathbb{C}^3\). The number of supersymmetries doubled in the dimensional reduction process, as the complex supercharges in four dimensions are converted into two real supercharges in three dimensions.

Recall at this point that the \(N = 3\) and \(N = 4\) super Yang-Mills equations are the same and only the field content differs by an additional reality condition \([4]\) in the case \(N = 4\); this condition renders the fourth supersymmetry linear. The \(N = 6\) and \(N = 8\) super Yang-Mills equations in three dimensions are identical in the same sense.

There is a further Penrose-Ward transform for ordinary Yang-Mills theory in four dimensions, which can also be translated to a Yang-Mills-Higgs theory in three dimensions. Here, one considers holomorphic vector bundles over a third-order thickening of \(L^5 \subset P^5 \times P^3_s\). That is, instead of demanding that \(\kappa\) in \([8]\) vanishes, we only impose the condition that \(\kappa^3 \sim 0\) and arrive at an infinitesimal neighborhood of \(L^5 \subset P^3 \times P^3_s\). For a recent review on such complex manifolds which have additional even nilpotent directions, see e.g. \([12]\). The Penrose-Ward transform, which can then be established \([4, 5]\) maps holomorphic vector bundles over the third-order thickening of \(L^5 \subset P^3 \times P^3_s\) to solutions to the ordinary Yang-Mills equations on \(\mathbb{C}^4\).

The corresponding mini-ambitwistor space \(L^4\) is obtained by simply dropping all fermionic coordinates of \(L^{4|6}\). For the Penrose-Ward transform, one has in fact to consider

\[\text{All the spaces in the following are derived from the corresponding superspaces by putting their fermionic coordinates to zero.}\]
a “subthickening”, i.e. the space $\mathcal{L}^4 \subset \mathcal{P}^2 \times \mathcal{P}^2$ with the formal third-order neighborhood of the diagonal $\Delta$ in the base $\mathbb{C}P^1 \times \mathbb{C}P^1$ of the fibration $\mathcal{L}^4$. We can then establish a Penrose-Ward transform, which states that holomorphic $\mathcal{L}^4$-bundles over a third order subthickening of $\mathcal{L}^4$ which become holomorphically trivial vector bundles upon restriction to any $\mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathcal{L}^4$ are in a one-to-one correspondence with solutions to the Yang-Mills-Higgs equations on $\mathbb{C}^3$ up to $\mathcal{L}^4$-bundle equivalence and gauge equivalence relations.

Summing up, we can state that there are twistor spaces for both Yang-Mills-Higgs theory and $\mathcal{N} = 6$ super Yang-Mills theory in three dimensions. Although these spaces are fibrations but no manifolds, they come with nice properties, and the mini-superambitwistor space $\mathcal{L}^4|6$ possibly plays an important rôle as the mirror manifold of the mini-supertwistor space $\mathcal{P}^2|4$. For future work, it remains to define a topological B-model on the mini-superambitwistor space and to substantiate the above pronounced mirror conjecture.

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