Harmonic oscillators in a Snyder geometry

P. Valtancoli

Dipartimento di Fisica, Polo Scientifico Università di Firenze and INFN, Sezione di Firenze (Italy), Via G. Sansone 1, 50019 Sesto Fiorentino, Italy
E-mail: valtancoli@fi.infn.it

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We find that, in the presence of the Snyder geometry, the quantization of $d$ isotropic harmonic oscillators can be solved exactly.

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1. Introduction

Recent studies have suggested that a natural cutoff for the ultraviolet behavior of the physical theories can be obtained by modifying the Heisenberg algebra of the canonical commutation rules [1–5]. This implies a finite minimal uncertainty $\Delta x_0$ in the position measurement, modifying the structure of space-time at short distances.

Such a cutoff can already be applied in the non-relativistic approximation, i.e. in ordinary quantum mechanics. The most elementary example is the harmonic oscillator with a finite minimal uncertainty $\Delta x_0$, a problem solved in Ref. [1], using the momentum representation.

The aim of this article is the generalization of this result to the case of $d$ isotropic harmonic oscillators quantized with the Snyder algebra. We will show that this problem can be exactly solved, without any approximation, allowing us to discuss how the state degeneracy of $d$ independent harmonic oscillators is removed by the non-commutative deformation.

To achieve such an aim, we have studied the eigenvalue equation in a new representation, the variables $\rho_i$, which resolve the Snyder algebra. As an outcome of our research, we have been able to solve, at a mathematic level, a $d$-dimensional generalization of the well known Gegenbauer equation.

2. Harmonic oscillator revisited

In Ref. [1] the harmonic oscillator was quantized using the following modified quantization rule:

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2)$$

(2.1)

in the momentum representation. The corresponding eigenvalue equation is

$$(\hat{p}^2 + m^2 \omega^2 \hat{x}^2)\psi(p) = 2mE\psi(p).$$

(2.2)

The idea behind the present paper is to solve this problem in a new representation [2]:

$$\hat{x} = i\hbar \sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho} \quad 0 < \rho < \frac{1}{\sqrt{\beta}}$$

$$\hat{p} = \frac{\rho}{\sqrt{1 - \beta \rho^2}}.$$ 

(2.3)
This effort will be helpful in the next section to solve exactly the quantization of $d$ harmonic oscillators in the Snyder geometry.

The eigenvalue equation in the variables $\rho$ turns out to be

$$
\left( \frac{\rho^2}{1 - \beta \rho^2} - \frac{1}{\beta^2 \mu (\mu - 1)} \sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho} \sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho} \right) \psi(\rho) = 2mE \psi(\rho)
$$

(2.4)

where $m \omega \hbar \beta = [\mu (\mu - 1)]^{\frac{1}{2}}$.

For the ground state it is well known that the solution is

$$
\psi_0(\rho) = c_0 (1 - \beta \rho^2)^{\frac{\mu}{2}}
$$

(2.5)

with an eigenvalue

$$
E_0 = \frac{\hbar \omega}{2} \sqrt{\frac{\mu}{\mu - 1}}.
$$

(2.6)

To study the excited states we look for a solution of this type:

$$
\psi(\rho) \sim \chi(\rho)(1 - \beta \rho^2)^{\frac{\mu}{2}},
$$

(2.7)

from which we obtain the following differential equation for $\chi(\rho)$:

$$
(1 - \beta \rho^2) \frac{\partial^2}{\partial \rho^2} \chi(\rho) - \beta (2 \mu + 1) \rho \frac{\partial}{\partial \rho} \chi(\rho) + \beta \mu[2mE\beta(\mu - 1) - 1] \chi(\rho) = 0.
$$

(2.8)

Let us make the substitution

$$
E = \frac{\hbar \omega v^2 + (2v + 1)\mu}{2 \sqrt{\mu (\mu - 1)}}
$$

(2.9)

and change the variable $\rho \rightarrow z = \sqrt{\beta} \rho$, obtaining

$$
(1 - z^2) \frac{\partial^2}{\partial z^2} \chi(z) - (2 \mu + 1)z \frac{\partial}{\partial z} \chi(z) + v(v + 2 \mu) \chi(z) = 0.
$$

(2.10)

We can easily recognize the Gegenbauer equation, whose polynomial solutions are obtained for

$$
v = n \Rightarrow E_n = \frac{\hbar \omega n^2 + (2n + 1)\mu}{2 \sqrt{\mu (\mu - 1)}}.
$$

(2.11)

The corresponding eigenfunctions are proportional to the Gegenbauer polynomials, satisfying the recurrence equation

$$
\partial_z P_n^\mu(z) = 2\mu P_{n-1}^{\mu+1}(z).
$$

(2.12)

Similarly to the Hermite polynomials, there is a generating function for the Gegenbauer polynomials:

$$
\frac{1}{(1 - 2zt + t^2)^{\mu}} = \sum_{n=0}^{\infty} P_n^\mu(z)t^n.
$$

(2.13)

The first polynomials are

$$
P_0^\mu(z) = 1,
$$

$$
P_1^\mu(z) = 2\mu z,
$$

$$
P_2^\mu(z) = -\mu + 2\mu(1 + \mu)z^2,
$$

$$
P_3^\mu(z) = -2\mu(1 + \mu)z + \frac{4}{3}\mu(1 + \mu)(2 + \mu)z^3.
$$

(2.14)
3. Harmonic oscillators in \(d\) dimensions

The theory of the harmonic oscillator described in the previous section can be generalized in \(d\) dimensions. The Hamiltonian of isotropic oscillators of equal mass \(m\) and frequency \(\omega\) in \(d\) dimensions and Cartesian coordinates can be written as

\[
H = \sum_{i=1}^{d} \left( \frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i^2 \right). \tag{3.1}
\]

For the generic \(d\) case, we extend the commutation rule (2.1) to the Snyder algebra [3]:

\[
[x_i, p_j] = i\hbar(\delta_{ij} + \beta p_i p_j)
\]

\[
[p_i, p_j] = 0
\]

\[
[x_i, x_j] = i\hbar\beta(p_j x_i - p_i x_j). \tag{3.2}
\]

This algebra is resolved by the \(\rho\) representation [2]:

\[
x_i = i\hbar\sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho_i}
\]

\[
p_i = \frac{\rho_i}{\sqrt{1 - \beta \rho^2}} \quad 0 < \rho^2 < \frac{1}{\beta}. \tag{3.3}
\]

The eigenvalue equation for \(d\) oscillators with the same frequency \(\omega\) and mass \(m\) is as follows:

\[
\sum_{i=1}^{d}(\hat{p}_i^2 + m^2\omega^2\hat{x}_i^2)\psi(\rho) = 2mE\psi(\rho), \tag{3.4}
\]

which, rewritten in the \(\rho\) variables, reads

\[
\sum_{i=1}^{d}\left( \frac{\rho_i^2}{1 - \beta \rho^2} - \frac{1}{\beta^2 \mu(\mu - 1)} \sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho_i} \sqrt{1 - \beta \rho^2} \frac{\partial}{\partial \rho_i} \right)\psi(\rho) = 2mE\psi(\rho). \tag{3.5}
\]

The ground state is simply

\[
\psi_0(\rho) = c_0(1 - \beta \rho^2)^{\frac{\mu}{2}}. \tag{3.6}
\]

with an eigenvalue

\[
E_0 = d\hbar\omega\sqrt{\frac{\mu}{\mu - 1}}. \tag{3.7}
\]

To study the excited states, we introduce, as in the case \(d = 1\), the following ansatz:

\[
\psi(\rho) = \chi(\rho)(1 - \beta \rho^2)^{\frac{\mu}{2}}, \tag{3.8}
\]

from which we obtain

\[
(1 - \beta \rho^2)\sum_{i=1}^{d} \frac{\partial^2}{\partial \rho_i \partial \rho_i} \chi(\rho) - \beta(1 + 2\mu)\sum_{i=1}^{d} \rho_i \frac{\partial}{\partial \rho_i} \chi(\rho) + \beta \mu[2mE\beta(\mu - 1) - d]\chi(\rho) = 0, \tag{3.9}
\]

a \(d\)-dimensional generalization of the Gegenbauer equation.
Let us introduce the notation $\epsilon_{[n_i]} = \mu[2mE\beta(\mu - 1) - d]$ and $z_i = \sqrt{\beta}\rho_i$, where the parameter $\epsilon$ depends on some quantum numbers $n_i$, from which the equation to be solved is

$$
\left[1 - \left(\sum_{i=1}^{d} z_i^2\right)\right] \sum_{i=1}^{d} \frac{\partial^2}{\partial z_i \partial z_i} - (1 + 2\mu) \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i} + \epsilon_{[n_i]} \right] P_{[n_i]}(z_i) = 0, \quad (3.10)
$$

with the energy parameter given by

$$
E_{[n_i]} = \frac{\hbar \omega (\epsilon_{[n_i]} + d\mu)}{2 \sqrt{\mu(\mu - 1)}}. \quad (3.11)
$$

The symmetry of this equation suggests the introduction of the following ansatz, i.e. that the polynomial solution space is composed of two parts:

i) The solutions $P_{kk}$ of the free differential equation

$$
\left(\sum_{i=1}^{d} \frac{\partial^2}{\partial z_i \partial z_i} \right) P_{kk}(z_i) = 0, \quad (3.12)
$$

where $P_{kk}(z_i)$ is an homogeneous polynomial in the variables $z_i$ of degree $k$.

In this case it is easy to compute the corresponding energy eigenvalue

$$
\epsilon^\mu_{kk} = k(1 + 2\mu). \quad (3.13)
$$

ii) The solutions $P_{Nk}$ with $N = k + 2n$, where

$$
P_{Nk}(z_i) = P_{kk}(z_i) \left(1 + \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{d} z_j^2\right)^i\right). \quad (3.14)
$$

To compute the eigenvalue corresponding to $P_{Nk}(z_i)$ we apply the following differential operator $\sum_{i=1}^{d} \frac{\partial^2}{\partial z_i \partial z_i}$ to the eigenvalue equation (3.10). With simple algebraic steps we deduce that

$$
\left(\sum_{i=1}^{d} \frac{\partial^2}{\partial z_i \partial z_i} \right) P^\mu_{Nk}(z_i) \propto P^\mu_{(N-2)k}(z_i), \quad (3.15)
$$

completed with the energy constraint

$$
\epsilon^\mu_{Nk} = \epsilon^\mu_{(N-2)k} + 2(1 + 2\mu) + 2d
$$

$$
\epsilon^\mu_{kk} = k(1 + 2\mu). \quad (3.16)
$$

This recurrence equation (3.16) is solved by

$$
\epsilon^\mu_{Nk} = N(1 + 2\mu) + (N - k)(N + k + d - 2), \quad (3.17)
$$

from which we deduce that the energy eigenvalues depends only on two quantum numbers $N$ and $k$ (whose difference $N - k$ must be an even positive integer number):

$$
E_{Nk} = \frac{\hbar \omega [N(1 + 2\mu) + (N - k)(N + k + d - 2) + d\mu]}{\sqrt{\mu(\mu - 1)}}. \quad (3.18)
$$

We conclude that, in comparison with the case $\beta = 0$, the non-commutative deformation (3.2) reduces the states degeneracy from $d - 1$ degrees of freedom to $d - 2$.

Now it remains to show that the ansatz with which we have solved the differential equation (3.10) gives all the polynomial solutions. We know that for $\beta \to 0$ our problem reduces to $d$ independent
oscillators; in this case, fixing the level of \( N \), the number of independent eigenfunctions is given by the formula
\[
s_d(N) = \frac{(N + d - 1)!}{N!(d - 1)!}. \tag{3.19}
\]

For \( \beta \neq 0 \) we must simply count how many independent solutions \( s_d(k) \) exist of the free differential equation (3.12) defining \( P_{kk}(z_i) \). We have computed them up to \( d = 5 \) oscillators:
\[
\begin{align*}
s_2(k) &= 2 \quad k > 0, \quad s_2(0) = 1 \\
s_3(k) &= 2k + 1 \\
s_4(k) &= (k + 1)^2 \\
s_5(k) &= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2. \tag{3.20}
\end{align*}
\]

In all cases we can check that the following identities hold:
\[
\begin{align*}
s_d(N) &= \sum_{k=\text{even}}^{N} s_d(k) \quad N \text{ even} \\
&= \sum_{k=\text{odd}}^{N} s_d(k) \quad N \text{ odd}. \tag{3.21}
\end{align*}
\]

This concludes our proof that we have described a complete basis of the Hilbert space.

4. Conclusion

In this article we have shown that, even modifying the quantization rule, many problems of quantum mechanics can be exactly solved. In particular, we have found that the natural \( d \)-dimensional extension of the modified Heisenberg algebra (2.1) is surely the Snyder algebra (3.2). We have discussed in detail that the quantization of \( d \) isotropic oscillators in non-commutative geometry gives rise to two relevant quantum numbers, from which we can deduce that the residual degeneracy of the states is reduced to \( d - 2 \) degrees of freedom. The spectrum contains, besides a linear term in the main quantum number \( N \) (that in the commutative limit is the sum of the single particle quantum numbers \( n_i \)), a quadratic term also dependent on a secondary quantum number \( k \), such that \( N - k \) is an even positive integer number. In the limit \( d \to 1 \), our general formula reduces to the single harmonic oscillator spectrum studied in Ref. [1].

We therefore expect that the solvability of these examples can be extended to more complex cases. Work is in progress in this direction.

References

[1] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D 52, 1108 (1995) [arXiv:9412167 [hep-th]].
[2] P. Valtancoli, Int. J. Mod. Phys. A 27, 1250183 (2012) [arXiv:1209.4012].
[3] H. S. Snyder, Phys. Rev. 71, 38 (1947).
[4] A. Kempf, J. Phys. A 30, 2093 (1997) [arXiv:9604045 [hep-th]].
[5] F. Brau, J. Phys. A 32, 7691 (1999) [arXiv:9905033 [quant-ph]].