Sharpness of Seeger-Sogge-Stein orders for the weak (1,1) boundedness of Fourier integral operators

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Abstract. Let $X$ and $Y$ be two smooth manifolds of the same dimension. It was proved by Seeger et al. in (Ann Math 134(2): 231–251, 1991) that the Fourier integral operators with real non-degenerate phase functions in the class $I^\mu_1(X,Y;\Lambda)$, $\mu \leq -\frac{n-1}{2}$, are bounded from $H^1$ to $L^1$. The sharpness of the order $-(n-1)/2$, for any elliptic operator was also proved in (Seeger et al. Ann Math 134(2): 231–251, 1991) and extended to other types of canonical relations in (Ruzhansky Hokkaido Math J 28(2): 357–362, 1992). That the operators in the class $I^\mu_1(X,Y;\Lambda)$, $\mu \leq -\frac{n-1}{2}$, satisfy the weak (1,1) inequality was proved by Tao (J Aust Math Soc 76(1):1–21, 2004). In this note, we prove that the weak (1,1) inequality for the order $-(n-1)/2$ is sharp for any elliptic Fourier integral operator, as well as its versions for canonical relations satisfying additional rank conditions.

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1. Introduction. Let $X$ and $Y$ be smooth manifolds of dimension $n$. In this work, we analyse the sharpness of the order $-(n-1)/2$ for the weak (1,1) inequality of elliptic Fourier integral operators $T \in I^\mu_1(X,Y;\Lambda)$ with order $\mu \leq -(n-1)/2$. For the general aspects of the theory of Fourier integral operators, we refer the reader to Hörmander [5], Duistermaat and Hörmander [1], and Melin and Sjöstrand [8,9].

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First, let us review the mapping properties of Fourier integral operators. By following [1,5], these classes are denoted by $I^\mu_p(X,Y;\Lambda)$, with $\Lambda$ being considered locally a graph of a symplectomorphism from $T^*X \setminus 0$ to $T^*Y \setminus 0$, which are equipped with the canonical symplectic forms $d\sigma_X$ and $d\sigma_Y$, respectively. Such Fourier integral operators are called non-degenerate. The symplectic structure of $\Lambda$ is determined by the symplectic 2-form $\omega = \sigma_X \oplus -\sigma_Y$. Let $\pi_{X \times Y}$ be the canonical projection from $T^*X \times T^*Y$ into $X \times Y$. As in the case of pseudo-differential operators, non-degenerate Fourier integral operators of order zero are bounded on $L^2$. The fundamental work of Segger, Sogge, and Stein [12] establishes the boundedness of $T \in I^\mu_p(X,Y;\Lambda)$ from $L^p_{\text{comp}}(Y)$ into $L^p_{\text{loc}}(X)$ with the order

$$\mu \leq -(n-1)|1/2 - 1/p| \quad (1.1)$$

if $1 < p < \infty$, and from $H^1_{\text{comp}}(Y)$ into $L^1_{\text{loc}}(X)$ if $p = 1$. Also, for $p = 1$ in (1.1) and as a consequence of the weak (1,1) estimate in Tao [14], an operator $T$ of order $-(n-1)/2$ is locally of weak (1,1) type.

The critical Seeger–Sogge–Stein order (1.1) is sharp if $d\pi_{X \times Y}|_\Lambda$ has full rank equal to $2n-1$ somewhere and if $T$ is an elliptic operator. When $d\pi_{X \times Y}|_\Lambda$ does not attain the maximal rank $2n-1$, the upper bound for the order (1.1) is not sharp and may depend on the singularities of $d\pi_{X \times Y}|_\Lambda$. In conclusion, as it was observed in [12], the mapping properties of the classes $I^\mu_p(X,Y;\Lambda)$ of Fourier integral operators depend on the singularities and on the maximal rank of the canonical projection. So, an additional condition on the canonical relation $\Lambda$ was introduced in [12], the so called factorisation condition for $\pi_{X \times Y}$. Roughly speaking, it can be introduced as follows. Assume that there exists $k \in \mathbb{N}$, with $0 \leq k \leq n-1$, such that for any $\lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in \Lambda$, there is a conic neighborhood $U_{\lambda_0} \subset \Lambda$ of $\lambda_0$, and a smooth homogeneous of order zero map $\pi_{\lambda_0} : U_{\lambda_0} \to \Lambda$, such that

$$(\text{RFC}): \text{rank}(d\pi_{\lambda_0}) = n + k,$$ 

and $\pi_{X \times Y}|_{U_{\lambda_0}} = \pi_{X \times Y}|_\Lambda \circ \pi_{\lambda_0}$. \quad (1.2)

Under the real factorisation condition (RFC) in (1.2), Seeger, Sogge, and Stein in [12] proved that the order

$$\mu \leq -(k + (n-k)(1-\rho))(1/2 - 1/p) \quad (1.3)$$

guarantees that any Fourier integral operator $T \in I^\mu_p(X,Y;\Lambda)$, with $\rho \in [1/2, 1]$, is bounded from $L^p_{\text{comp}}(Y)$ into $L^p_{\text{loc}}(X)$, and for $p = 1$, from $H^1_{\text{comp}}(Y)$ into $L^1_{\text{loc}}(X)$. For a set $A$, $1_A$ denotes its characteristic function. We recall that $T$ is locally of weak $(1,1)$ type if, for any pair of compact subsets $K \subset Y$, and $K' \subset X$, the localised operator $1_K \cdot T1_{K'} : L^1(Y) \to L^{1,\infty}(X)$ is bounded.

If $\Lambda$ satisfies the factorisation condition (RFC) in (1.2), because rank$(d\pi_{\lambda_0}) = n + k$, we have that rank$(d\pi_{X \times Y}|_{U_{\lambda_0}}) \leq n + k$, in an open neighborhood $U_{\lambda_0}$ of $\lambda_0$. The following Theorem 1.1 proves that if the rank $n + k$ is attained somewhere, the order in (1.3) for $\rho = 1$, that is $\mu \leq -k/2$, is sharp for $0 \leq k \leq \mu$. The sharpness of the order $\mu \leq -k/2$ for the $H^1_{\text{loc}}(Y) - L^1_{\text{loc}}(X)$-boundedness of elliptic Fourier integral operators has been proved in [10], with the case $k = n - 1$ known from [12]. Here, we observe that the geometric construction in [10] implies also the following sharp Theorem 1.1, proving for
$k = n - 1$, the converse of the weak (1,1)-inequality in Tao [14] for elliptic Fourier integral operators. More specifically, we have:

**Theorem 1.1.** Let the real canonical relation $\Lambda$ be a local canonical graph such that the inequality $\text{rank}(d\pi_{X \times Y}|_{\Lambda}) \leq n + k$ holds with $0 \leq k \leq n - 1$, and the rank $n + k$ is attained at some point. Then elliptic operators $T \in I^k_1(X, Y; \Lambda)$ are not locally of weak (1,1) type provided that $\mu > -k/2$.

**Remark 1.2.** In the endpoint case $k = n - 1$, $\text{rank}(d\pi_{X \times Y}|_{\Lambda}) \leq 2n - 1$, the elliptic operators $T \in I^k_1(X, Y; \Lambda)$ are not locally of weak (1,1) type provided that $\mu > -(n - 1)/2$. In view of the weak (1,1) estimate in Tao [14] for the class $I^{-(n-1)/2}_1(X, Y; \Lambda)$, when the real canonical relation has full rank $\text{rank}(d\pi_{X \times Y}|_{\Lambda}) \leq 2n - 1$, the main [14, Theorem 1.1] together with Theorem 1.1 imply the following result.

**Corollary 1.3.** Let the real canonical relation $\Lambda$ be a local canonical graph such that (RFC) in (1.2) is satisfied for $k = n - 1$. Let $T \in I^k_1(X, Y; \Lambda)$ be an elliptic Fourier integral operator. Then $T$ is locally of weak (1,1) type if and only if $\mu \leq -(n - 1)/2$.

2. Preliminaries.

2.1. Basics on symplectic geometry. Let $M, X,$ and $Y$ be (paracompact) smooth real manifolds of dimension $n$. So, using partitions of the unity, the spaces $L^1(Y)$ and $L^{1,\infty}(X)$ are defined by the set of functions that under any changes of coordinates belong to $L^1(\mathbb{R}^n)$ and $L^{1,\infty}(\mathbb{R}^n)$, respectively. For instance, we can take $M = X \times Y$ or $M = X \times Y$. In this section, we will follow [11, Chapter I].

A 2-form $\omega$ is called symplectic on $M$ if $d\omega = 0$, and for all $x \in M$, the bilinear form $\omega_x$ is antisymmetric and non-degenerate on $T_x M$. The canonical symplectic form $\sigma_M$ on $M$ is defined as follows. Let $\pi := \pi_M : T^* M \to M$ be the canonical projection. For any $(x, \xi) \in T^* M$, let us consider the linear mappings

$$d\pi_{(x, \xi)} : T_{(x, \xi)}(T^* M) \to T_x M \text{ and } \xi : T_x M \to \mathbb{R}.$$ 

The composition $\alpha_{(x, \xi)} := \xi \circ d\pi_{(x, \xi)} \in T^*_{(x, \xi)}(T^* M)$, that is $\xi \circ d\pi_{(x, \xi)} : T_{(x, \xi)}(T^* M) \to \mathbb{R}$, defines a 1-form $\alpha$ on $T^* M$. Then the canonical symplectic form $\sigma_M$ on $M$ is defined by

$$\sigma_M := d\alpha. \quad (2.1)$$

Because $\sigma_M$ is an exact form, it follows that $d\sigma_M = 0$ and then that $\sigma_M$ is symplectic. If $M = X \times Y$, it follows that $\sigma_{X \times Y} = \sigma_X \oplus -\sigma_Y$. Now we record the kind of submanifolds that are necessary when one defines the canonical relations.

- A submanifold $\Lambda \subset T^* M$ of dimension $n$ is called Lagrangian if $T_{(x, \xi)} \Lambda = \{ y \in T_{(x, \xi)}(T^* M) : \sigma_M(y, y') = 0 \ \forall y' \in T_{(x, \xi)} \Lambda \}$.
- We say that $\Lambda \subset T^* M \setminus 0$ is conic if $(x, \xi) \in \Gamma$ implies that $(x, t\xi) \in \Gamma$ for all $t > 0$. 

Let $\Sigma \subset X$ be a smooth submanifold of $X$ of dimension $k$. Its conormal bundle in $T^*X$ is defined by
\[
N^*\Sigma := \{(x, \xi) \in T^*X : x \in \Sigma, \xi(\delta) = 0 \forall \delta \in T_x\Sigma\}. \tag{2.2}
\]
The following facts characterise the Lagrangian submanifolds of $T^*M$.

- Let $\Lambda \subset T^*M \setminus 0$ be a closed submanifold of dimension $n$. Then $\Lambda$ is a conic Lagrangian manifold if and only if the 1-form $\alpha$ in (2.1) vanishes on $\Lambda$.
- Let $\Sigma \subset X$ be a submanifold of dimension $k$. Then its conormal bundle $N^*\Sigma$ is a conic Lagrangian manifold.

The Lagrangian manifolds have the following property.

- Let $\Lambda \subset T^*M \setminus 0$ be a conic Lagrangian manifold and let
\[
d\pi_{(x, \xi)} : T_{(x, \xi)}\Lambda \to T_xM \tag{2.3}
\]
have constant rank equal to $k$ for all $(x, \xi) \in \Lambda$. Then each $(x, \xi) \in \Gamma$ has a conic neighborhood $\Gamma$ such that
1. $\Sigma = \pi(\Gamma \cap \Lambda)$ is a smooth manifold of dimension $k$.
2. $\Gamma \cap \Lambda$ is an open subset of $N^*\Sigma$.

The Lagrangian manifolds have a local representation defined in terms of phase functions that can be defined as follows. For this, let us consider a local trivialisation $M \times (\mathbb{R}^N \setminus 0)$, where we can assume that $M$ is an open subset of $\mathbb{R}^n$.

**Definition 2.1 (Real-valued phase functions).** Let $\Gamma$ be a cone in $M \times (\mathbb{R}^N \setminus 0)$. A smooth function $\phi : M \times (\mathbb{R}^N \setminus 0) \to \mathbb{R}$, $(x, \theta) \mapsto \phi(x, \theta)$, is a real phase function if it is homogeneous of degree one in $\theta$ and has no critical points as a function of $(x, \theta)$, that is
\[
\forall t > 0, \phi(x, t\theta) = t\phi(x, \theta), \text{ and } d(x, \theta)\phi(x, \theta) \neq 0 \ \forall (x, \theta) \in M \times (\mathbb{R}^N \setminus 0). \tag{2.4}
\]
Additionally, we say that $\phi$ is non-degenerate in $\Gamma$ if for $(x, \theta) \in \Gamma$ such that $d_\theta\phi(x, \theta) = 0$, one has that
\[
d(x, \theta)\frac{\partial \phi}{\partial \theta_j}(x, \theta), \ 1 \leq j \leq N, \tag{2.5}
\]
is a system of linearly independent vectors.

The following facts describe locally a Lagrangian manifold in terms of a phase function.

- Let $\Gamma$ be a cone in $M \times (\mathbb{R}^N \setminus 0)$, and let $\phi$ be a non-degenerate phase function in $\Gamma$. Then there exists an open cone $\tilde{\Gamma}$ containing $\Gamma$ such that the set
\[
U_\phi = \{(x, \theta) \in \tilde{\Gamma} : d_\theta\phi(x, \theta) = 0\} \tag{2.6}
\]
is a smooth conic submanifold of $M \times (\mathbb{R}^N \setminus 0)$ of dimension $n$. The mapping
\[
L_\phi : U_\phi \to T^*M \setminus 0, \ L_\phi(x, \theta) = (x, d_x\phi(x, \theta)), \tag{2.7}
\]
is an immersion. Let us denote $\Lambda_\phi = L_\phi(U_\phi)$.

- Let $\Lambda \subset T^* M \setminus 0$ be a submanifold of dimension $n$. Then $\Lambda$ is a conical Lagrangian manifold if and only if every $(x, \xi) \in \Lambda$ has a conic neighborhood $\Gamma$ such that $\Gamma \cap \Lambda = \Lambda_\phi$ for some non-degenerate phase function $\phi$.

**Remark 2.2.** The cone condition on $\Lambda$ corresponds to the homogeneity of the phase function.

**Remark 2.3.** Although we have given a definition when a real phase function of $(x, \theta)$ is non-degenerate, the same can be defined if one considers functions of $(x, y, \theta)$. Indeed, a real valued phase function $\phi(x, y, \theta)$ homogeneous of order $1$ at $\theta \neq 0$ that satisfies the following two conditions

$$
\det \partial_x \partial_\theta (\phi(x, y, \theta)) \neq 0, \quad \det \partial_y \partial_\theta (\phi(x, y, \theta)) \neq 0, \quad \theta \neq 0,
$$

is called non-degenerate.

### 2.2. Smooth factorisation condition for real phases.

We can assume that $X, Y$ are open sets in $\mathbb{R}^n$. One defines the class of Fourier integral operators $T \in I^\mu_\rho(X, Y; \Lambda)$ by the (microlocal) formula

$$
T f(x) = \int_Y \int_{\mathbb{R}^N} e^{i \Psi(x, y, \theta)} a(x, y, \theta) f(y) d\theta dy,
$$

where the symbol $a$ is a smooth function locally in the class $S^\mu_{\rho, 1-\rho}(X \times Y \times (\mathbb{R}^n \setminus 0))$, with $1/2 \leq \rho \leq 1$. This means that $a$ satisfies the symbol inequalities

$$
|\partial_x^\alpha \partial_y^\beta a(x, y, \theta)| \leq C_{\alpha, \beta} (1 + |\theta|)^{\mu-\rho|\alpha|+(1-\rho)|\beta|}
$$

for $(x, y)$ in any compact subset $K$ of $X \times Y$, and $\theta \in \mathbb{R}^N \setminus 0$, while the real-valued phase function $\Psi$ satisfies the following properties:

1. $\Psi(x, y, \lambda \theta) = \lambda \Psi(x, y, \theta)$ for all $\lambda > 0$;
2. $d\Psi \neq 0$;
3. $\{d_\theta \Psi = 0\}$ is smooth (i.e. $d_\theta \Psi = 0$ implies $d_{(x, y, \theta)} \partial \Psi / \partial \theta_j$ are linearly independent).

Here $\Lambda \subset T^*(X \times Y) \setminus 0$ is a Lagrangian manifold locally parametrised by the phase function $\Psi$,

$$
\Lambda = \Lambda_\Psi = \{(x, d_x \Psi, y, d_y \Psi) : d_\theta \Psi = 0\}.
$$

The canonical relation associated with $T$ is the conic Lagrangian manifold in $T^*(X \times Y) \setminus 0$, defined by $\Lambda' = \{(x, \xi, y, -\eta) : (x, \xi, y, \eta) \in \Lambda\}$. In view of the equivalence-of-phase-functions theorem (see e.g. [11, Theorem 1.1.3, Page 9]), the notion of Fourier integral operator becomes independent of the choice of a particular phase function associated to a Lagrangian manifold $\Lambda$. Because of the diffeomorphism $\Lambda \cong \Lambda'$, we do not distinguish between $\Lambda$ and $\Lambda'$ by saying also that $\Lambda$ is the canonical relation associated with $T$. 

Let us consider the canonical projections:
\[
\begin{array}{cccc}
T^*X & \xrightarrow{\pi_X} & \Lambda & \subset T^*X \times T^*Y \xrightarrow{\pi_Y} T^*Y. \\
& & \downarrow & \\
& & X \times Y
\end{array}
\]

The smooth factorisation condition for \( \Psi \) can be formulated as follows (see [12] or e.g. [11, Page 45]). Suppose that there exists a number \( k \), with \( 0 \leq k \leq n - 1 \), such that for any \( \lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in \Lambda \Psi \), there exists a conic neighborhood \( U_{\lambda_0} \subset \Lambda \Psi \) of \( \lambda_0 \), and a smooth homogeneous of order zero map
\[
\pi_{\lambda_0} : U_{\lambda_0} \to \Lambda \Psi,
\]
with constant rank \( \text{rank}(d\pi_{\lambda_0}) = n + k \), for which one has
\[
(\text{RFC}): \pi_{X \times Y|U_{\lambda_0}} = \pi_{X \times Y|\Lambda \Psi \circ \pi_{\lambda_0}}.
\]
In this case, we say that the canonical relation \( \Lambda := \Lambda \Psi \) satisfies the factorisation condition (RFC).

3. Sharpness of Seeger–Sogge–Stein orders. Now, we will prove the sharpness of the Seeger–Sogge–Stein order for \( \rho = 1 \) in the case of elliptic Fourier integral operators.

Proof of Theorem 1.1. Let \( \mu > -k/2 \). Let us follow the argument in [11, Page 42] (see also [10]) and to analyse the weak (1,1) estimate at the end of the proof. Using again the equivalence-of-phase-function theorem, it is enough to consider elliptic operators \( T \) on \( \mathbb{R}^n \) with kernels, locally defined by
\[
K(x, y) = \int_{\mathbb{R}^n} e^{i\Phi(x, y, \xi)} b(x, y, \xi) d\xi, \quad \Phi(x, y, \xi) := x \cdot \xi - \phi(y, \xi),
\]
with symbols \( b(x, y, \xi) \) compactly supported in \( (x, y) \). That \( \Lambda \) satisfies the local graph condition means that the real-value phase function \( \phi \) satisfies
\[
\det \partial_y \partial_\xi \phi(y, \xi) \neq 0
\]
on the support of the symbol \( b \), and \( \xi \neq 0 \). Let us observe that
\[
\Lambda_0 := \{ \lambda \in \Lambda : \text{rank } d\pi_{X \times Y|\Lambda}(\lambda) = n + k \}
\]
is not empty and is open in \( \Lambda \). Fix a point \( \lambda_0 \in \Lambda_0 \). Let \( \Delta := \sum_{j=1}^n \partial_{x_j}^2 \) be the standard Laplacian on \( \mathbb{R}^n \), and define the distribution
\[
x_0(y) := (1 - \Delta)^{-s/2} \delta_{y_0}
\]
for a fixed point \( y_0 \in Y \), where \( s > 0 \) is small enough in such a way that
\[
-\frac{k}{2} + s < \mu.
\]
Since \((1 - \Delta)^{-s/2}\) is an elliptic pseudo-differential operator of order \(-s\), its Schwartz kernel \(K_s\) is of Calderón-Zygmund type, and it satisfies the inequality 
\[ |K_s(y, y_0)| \leq C|y - y_0|^{-n+s}, \]

in some local coordinate system. So, the fact that \(s > 0\) implies that
\[
\varphi(y) = \int_{\mathbb{R}^n} K_s(y, z) \delta_{y_0}(z) dz = K_s(y, y_0) \in L^1_{\text{loc}}.
\] (3.6)

Now, we are going to introduce the geometric construction in [10] (see also [11, Page 42]) in order to obtain a useful parametrisation of the phase function \(\Phi\). Let \(\Sigma = \pi_{X \times Y}(C \cap U)\), where \(U \subset \Lambda_0\) is a neighborhood of \(\lambda_0\). Taking into account that rank \(d\pi_{X \times Y}|_U = n + k\), that is the rank of \(d\pi_{X \times Y}\) is constant in \(U\), \(\Sigma\) is a \(k\)-dimensional submanifold defined by the equations
\[
h_j(x, y) = 0, \ 1 \leq j \leq n - k,
\] (3.7)
in a neighborhood of \(y_0\), with the system \(\{\nabla h_j : 1 \leq j \leq n - k\}\) being linearly independent on \(\Sigma\). Then \(\Lambda\) is the conormal bundle of \(\Sigma\), and the phase function of \(T\) takes the representation
\[
\Phi(x, y, \lambda) = \sum_{j=1}^{n-k} \lambda_j h_j(x, y).
\] (3.8)

Because compositions of Fourier integral operators with pseudo-differential operators leaves invariant the canonical relation \(\Lambda\), we have that \(T \circ (1 - \Delta)^{-s/2} \in I^{n-s} (X, Y, \Lambda)\). So, in local coordinates, we have
\[
T \varphi(x) = T \circ (1 - \Delta)^{-s/2} \delta_{y_0} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-k}} e^{i\sum_{j=1}^{n-k} \lambda_j h_j(x, y)} a(x, \lambda) \delta_{y_0}(y) d\lambda dy
\]
\[
= \int_{\mathbb{R}^{n-k}} e^{i\overline{\lambda} \cdot \overline{h}(x, y_0)} a(x, \lambda) d\lambda = (2\pi)^{n-k} \mathcal{F}^{-1} a(x, \overline{h}(x, y_0)),
\]
where \(\overline{\lambda}\) and \(\overline{h}\) are vectors with components \(\lambda_j\) and \(h_j\), respectively, and \(\mathcal{F}\) denotes the Fourier transform. The symbol \(a \in S^{\mu-s+\frac{k}{2}} (\mathbb{R}^{n-k})\) is obtained from the symbol of the operator \(T \circ (1 - \Delta)^{-s/2}\) by using the stationary method phase, where we have eliminated \(k\)-variables. Computing the second argument of \((2\pi)^{n-k} \mathcal{F}^{-1} a\), one has
\[
(2\pi)^{n-k} \mathcal{F}^{-1} a(x, \zeta) = \int_{\mathbb{R}^{n-k}} e^{i\lambda \cdot \zeta} a(x, \lambda) \mathcal{F}(\delta_0)(\lambda) d\lambda = P \delta_0(\zeta),
\] (3.9)
where \(P\) is a pseudo-differential operator in \(\mathbb{R}^{n-k}\) of order \(m = n - s + \frac{k}{2}\). Denoting the Schwartz kernel of \(P\) by \(K_P\), we have that \(P \delta_0(\zeta) = K_P(\zeta, 0)\), and then one has
\[
|K_P(\zeta, 0)| \sim |\zeta|^{-(n-k)-m}, \ m = n - s + \frac{k}{2}.
\] (3.10)

Define the set
\[
\Sigma_{y_0} := \{x : (x, y_0) \in \Sigma\}.
\]
We have that distance\( (x, \Sigma_{y_0}) \preceq |\overline{h}(x, y_0)| \). Consequently, 
\[
|(2\pi)^{n-k} \mathcal{F}^{-1} a(x, \zeta)| \preceq \text{distance}(x, \Sigma_{y_0})^{-(n-k)-(\mu-s+\frac{k}{2})},
\]
locally uniformly in \( x \). The identity (3.9) implies that \( T\varphi \) is smooth on \( \Sigma_{y_0} \). Now, we apply the geometric construction above to test the operator \( T \) on the distribution \( \varphi \) to see that 
\[
|T\varphi(x)| = |K_P(\overline{h}(x, y_0), 0)| \preceq |\overline{h}(x, y_0)| \preceq \text{distance}(x, \Sigma_{y_0})^{-(n-k)-(\mu-s+\frac{k}{2})},
\]
and that the singularities of \( T\varphi \) can appear only in transversal directions to \( \Sigma \). Now, to finish the proof, let \( \Omega = \overline{B(y_0, r)} \) be the compact neighborhood of \( y_0 \), with radius \( r > 0 \). Observing that for any \( x \in \Omega \), \( \overline{h}(x, y_0) \in \mathbb{R}^{n-k} \), \( T\varphi \in L^1(\Omega) \), if and only if 
\[
(n-k) + (\mu-s + \frac{k}{2}) < n-k,
\]
and that \( T\varphi \in L^{1,\infty}(\Omega) \setminus L^1(\Omega) \) if and only if 
\[
(n-k) + (\mu-s + \frac{k}{2}) = n-k,
\]
it follows that \( T\varphi \notin L^{1,\infty}(\Omega) \) if and only if 
\[
(n-k) + (\mu-s + \frac{k}{2}) > n-k, \tag{3.11}
\]
or equivalently \( \mu > s - \frac{k}{2} \) as in (3.5). In conclusion, we have that \( \varphi \) is locally in \( L^1 \) and that \( T\varphi \notin L^{1,\infty}(\Omega) \) showing that \( T \) is not locally of weak \((1,1)\) type. \( \square \)

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