ON FRACTIONAL SCHRÖDINGER SYSTEMS OF CHOQUARD TYPE

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ABSTRACT. In this article, we first employ the concentration compactness techniques to prove existence and stability results of standing waves for nonlinear fractional Schrödinger-Choquard equation

\[ i\partial_t \Psi + (-\Delta)\alpha \Psi = a|\Psi|^{s-2}\Psi + \lambda \left( \frac{1}{|x|^{N-\beta}} \ast |\Psi|^p \right) |\Psi|^{p-2}\Psi \quad \text{in } \mathbb{R}^{N+1}, \]

where \( N \geq 2, \alpha \in (0,1), \beta \in (0,N), s \in (2, 2 + \frac{4}{N}), p \in [2, 1 + \frac{2\alpha + \beta}{N}], \) and the constants \( a, \lambda \) are nonnegative satisfying \( a + \lambda > 0 \). We then extend the arguments to establish similar results for coupled standing waves of nonlinear fractional Schrödinger systems of Choquard type. The same argument works for equations with an arbitrary number of combined nonlinearities and when \( |x|^{\beta-N} \) is replaced by a more general convolution potential \( K : \mathbb{R}^N \to [0, \infty) \) under certain assumptions. The arguments can be applied and the results are identical for the case \( \alpha = 1 \) as well.

1. Introduction

The fractional nonlinear Schrödinger (fNLS) equation

\[ i\Psi_t + (-\Delta)^\alpha \Psi = f(|\Psi|)\Psi \quad \text{in } \mathbb{R}^N \times (0, \infty) \quad (1.1) \]

is a fundamental equation of the space-fractional quantum mechanics (SFQM). Here the fractional Laplacian \((-\Delta)^\alpha\) of order \( \alpha \in (0,1) \) is defined as

\[ (-\Delta)^\alpha u(x) = C_{N,\alpha} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} dy, \quad \forall u \in \mathcal{S}(\mathbb{R}^N), \quad (1.2) \]

where P.V. stands for the principle value of the integral. The term SFQM provides a natural extension of the standard quantum mechanics when the Brownian trajectories in the well-known Feynman path integrals are replaced by the Lévy flights (see [12]). The fNLS equations with \( \alpha = 1/2 \) have been also used as models to describe

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Boson-stars. Recently, an optical realization of the fractional Schrödinger equation was proposed by Longhi [18].

An important issue concerning nonlinear evolution equations such as (1.1) is to study their standing wave solutions. A standing wave solution of (1.1) is a solution of the form \( \Psi(x, t) = e^{-i\omega t}u(x) \), where \( \omega \in \mathbb{R} \) and \( u \) satisfies the stationary equation

\[
(-\Delta)^\alpha u + \omega u = f(|u|)u \quad \text{in } \mathbb{R}^N. \tag{1.3}
\]

Equation (1.3) with the space derivative of order \( \alpha = 1 \) (the standard Schrödinger equation) and its variants have been extensively studied in the mathematical literature. Recently, there has been growing interest in extending similar results in the case \( 0 < \alpha < 1 \). One way to obtain solutions of (1.3) in \( H^\alpha(\mathbb{R}^N) \) is to look for the critical points of the functional

\[
J_\omega(u) = \frac{1}{2} \left\| (-\Delta)^{\alpha/2} u \right\|_{L^2}^2 + \frac{\omega}{2} \left\| u \right\|_{L^2}^2 - \int_{\mathbb{R}^N} F(u) \, dx,
\]

where \( F'(u) = f(|u|)u \). That is, ground state standing waves are characterized as solutions to the minimizing problem

\[
\inf_{u \in \mathcal{N}} J_\omega(u), \quad \mathcal{N} := \{ u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : J'_\omega(u)(u) = 0 \}.
\]

In this approach, since the parameter \( \omega \) is assumed to be fixed, we can not have a priori knowledge about the \( L^2 \)-norm of the standing wave profile. A natural question concerning (1.3) is thus the case of normalized solutions, i.e., solutions \( u \) satisfying \( \left\| u \right\|_{L^2}^2 = \sigma \) for given \( \sigma > 0 \). This paper is devoted to the study of such special solutions to fNLS equations and their coupled systems.

We first consider (1.3) with combined power and Hartree type nonlinearities

\[
(-\Delta)^\alpha u + \omega u = a |u|^{s-2} u + \lambda T_\beta^p(u) |u|^{p-2} u, \tag{1.4}
\]

where \( a, \lambda \) are nonnegative satisfying \( a + \lambda \neq 0 \) and for any \( r \), the Riesz potential \( T_\beta^p(u) \) of order \( \beta \in (0, N) \) for \( u \) is defined by

\[
T_\beta^r(u)(x) = \frac{1}{|x|^{N-\beta}} \ast |u|^r = \int_{\mathbb{R}^N} \frac{|u(y)|^r}{|x - y|^{N-\beta}} \, dy, \quad x \in \mathbb{R}^N. \tag{1.5}
\]

A natural way to obtain normalized solutions to (1.4) in \( H^\alpha(\mathbb{R}^N) \) is to describe them as solutions of the minimization problem constrained to the \( L^2 \) sphere of radius \( \sigma \),

\[
\minim \{ J(u) \} \text{ subject to } \int_{\mathbb{R}^N} |u|^2 \, dx = \sigma > 0, \quad u \in H^\alpha(\mathbb{R}^N),
\]

where the functional \( J \) represents the energy and is given by

\[
J(u) = \frac{1}{2} \left\| (-\Delta)^{\alpha/2} u \right\|_{L^2}^2 - a \frac{\| u \|_r^s}{s} \, dx - \frac{\lambda}{2p} \int_{\mathbb{R}^N} T_\beta^p(u) |u|^p \, dx. \tag{1.7}
\]
Parameter $\omega$ in this approach becomes the Lagrange multiplier of the constrained variational problem. The first part of this paper is devoted to proving the existence of minimizers for the problem (1.6).

In the specific case $N = 3, a = 0, \beta = 2, \alpha = p = 2$, equation (1.4) reduces to

$$-\Delta u + \omega u = \lambda \left( \frac{1}{|x|} \ast |u|^2 \right) u \quad \text{in} \quad \mathbb{R}^3,$$

(1.8)

and is also known as the Choquard-Pekar equation. Equation (1.8) is the form that appears as a model in quantum theory of a polaron at rest (see [20]). The time-dependent form of (1.8) also describes the self-gravitational collapse of a quantum mechanical wave-function (see [21]), in which context it is usually called the Schrödinger-Newton (SN) equation. In the plasma physics context, the stationary form of the SN equation is also known as Choquard equation (see [13] for details).

In pure mathematics, there is a huge literature concerning ground states for the Choquard and related equations. Among many others, we mention the paper by Ma Li and Zhao Lin [12] where the existence problem of ground states for (1.4) with $\alpha = 1$ and $a = 0$ was formulated, and the radial symmetry as well as the regularity of solutions have been proved (see also [19]). Recently, the ground state solution of (1.4) with $a = 0$ has been studied by D’Avenia et al. in [7]. Surprisingly, only a very few papers address normalized solutions, although they are the most relevant from the physical point of view. In [13], Lieb proved the existence and uniqueness of normalized solutions to the Choquard equation using the symmetrization techniques. In their papers [6, 16], Cazenave and Lions studied the existence and stability issues of normalized solutions for the Choquard and related equations.

Another interesting question is whether similar results can be proved for coupled systems of fNLS equations. In the past, systems of standard NLS equations (in the case $\alpha = 1$) have been widely studied and a fairly complete theory has been developed to study standing wave solutions to such systems (see [3], for instance, concerning the results on normalized solutions). No such theory yet exists, however, for the coupled systems of fNLS equations. In this paper, we are also interested in generalizing existence and stability results to the following coupled system of fNLS equations with Choquard type nonlinearities

$$\begin{cases}
(-\Delta)^\alpha u_1 + \omega_1 u_1 = \lambda_1 \mathcal{I}_\beta^{p_1}(u_1)|u_1|^{p_1-2}u_1 + c\mathcal{I}_\beta^q(u_2)|u_1|^{q-2}u_2, \\
(-\Delta)^\alpha u_2 + \omega_2 u_2 = \lambda_2 \mathcal{I}_\beta^{p_2}(u_2)|u_2|^{p_2-2}u_2 + c\mathcal{I}_\beta^q(u_1)|u_2|^{q-2}u_2,
\end{cases}$$

(1.9)

where $u_1, u_2 : \mathbb{R}^N \to \mathbb{C}$, $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, the constants $\lambda_1, \lambda_2$, and $c$ are positive, and $\mathcal{I}_\beta(f)$ is as defined in [15]. As in the scalar case, we search for normalized solutions to (1.9), i.e., solutions $(u_1, u_2)$ in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ satisfying $\|u_1\|_{L^2}^2 = \sigma_1$ and $\|u_2\|_{L^2}^2 = \sigma_2$ for given $\sigma_1 > 0$ and $\sigma_2 > 0$. To obtain normalized solutions to (1.9),
we look for minimizers of the problem
\[
\text{minimize } E(u_1, u_2) \text{ subject to } \int_{\mathbb{R}^N} |u_j|^2 \, dx = \sigma_j > 0, \quad j = 1, 2,
\]
where the associated energy functional \( E \) is given by
\[
E(u_1, u_2) = \frac{1}{2} \sum_{j=1}^{2} \| \nabla u_j \|_{L^2}^2 - \sum_{j=1}^{2} \frac{\lambda_j}{2p_j} \int_{\mathbb{R}^N} \frac{|u_j(x)|^{p_j}|u_j(y)|^{p_j}}{|x-y|^{N-\beta}} \, dx dy - \frac{c}{q} \int_{\mathbb{R}^N} \frac{|u_1(x)|^q|u_2(y)|^q}{|x-y|^{N-\beta}} \, dx dy.
\]
Parameters \( \omega_1 \) and \( \omega_2 \) in (1.9) appear as the Lagrange multipliers. Apart from the existence result for minimizers of (1.10), we also provide some results concerning structures of the set of minimizers.

All results established in Section 2 below are easily extendable to versions of (1.4) with an arbitrary number of combined nonlinearities and when \( |x|^\beta - N \) is replaced by a more general convolution potential \( K : \mathbb{R}^N \to [0, \infty) \), that is, the equation of the form
\[
(-\Delta)^\alpha u + \omega u = \sum_{j=1}^{m} a_j |u|^{s_j - 2} u + \sum_{k=1}^{d} \lambda_k (K * |u|^{p_k}) |u|^{p_k - 2} u,
\]
where \( N \geq 3, \ 0 < \alpha < 1, \) the constants \( a_j, \lambda_k \) are all nonnegative but not all zero, and the potential \( K \in L^r_w(\mathbb{R}^N) \) is radially symmetric satisfying some assumptions (see Theorem 2.3 below). Normalized solutions to (1.12) in \( H^\alpha(\mathbb{R}^N) \) are obtained as minimizers of the energy functional
\[
\tilde{J}(u) = \frac{1}{2} \| (-\Delta)^{\alpha/2} u \|_{L^2}^2 - \sum_{j=1}^{m} b_j \| u \|_{L^{s_j}}^{s_j} - \sum_{k=1}^{d} \gamma_k \int_{\mathbb{R}^N} (K * |u|^{p_k}) |u|^{p_k} \, dx
\]
satisfying the constraint \( \| u \|_{L^2}^2 = \sigma > 0 \), where \( b_j = a_j/s_j \) and \( \mu_k = \lambda_k/2p_k \).

Our approach here involves in studying the global minimization properties of stationary solutions via the concentration-compactness principle of P. L. Lions (see Lemma I.1 of [16]). The advantage of utilizing this technique in our context is that this not only gives the existence of stationary solutions but also addresses the important stability issue of associated standing waves, since the energy and the power(s) involved in variational problems are conservation laws for the flow of associated NLS-type evolution equations (see [6, 11] for the illustration of the method). To study the two-parameter problem (1.10), we follow the techniques developed in a series of papers [2, 3, 4] where the concentration compactness principle was used to study solitary waves for coupled Schrödinger and KdV systems. In order to establish relative compactness of energy minimizing sequences (and hence, existence and stability of minimizers) in the spirit of concentration compactness technique, one require to check certain strict inequalities
involving the infimum of the minimization problem. The proof of the strict inequality is a notable difficulty when one uses this technique and most proofs of these strict inequalities in the literature are based on some homogeneity-type property. The proof is much less understood for problems which violate homogeneity-type assumptions and for multi-constrained variational problems. This might be one reason why there are only a very few papers concerning the relative compactness of minimizing sequences, and hence existence and stability of normalized standing waves.

The paper is organized as follows. We analyze separately the scalar case and the coupled one. In Section 2, we analyze the fNLS equation with combined local and nonlocal nonlinearities via concentration compactness method. Theorem 2.1, Corollary 2.2, and Theorem 2.3 are the main results of Section 2. Section 3 provides the analysis of the two-constraint problem for the coupled fNLS system with nonlocal nonlinearities. Theorem 3.1 and its Corollary are the existence and stability results for coupled standing waves. To the author’s knowledge, this is the first paper which proves existence results of normalized solutions to coupled Schrödinger systems involving convolution type interaction terms. As far as we know, there is also no paper which employs the concentration compactness technique (in the presence of strict inequality) to obtain standing waves for nonlinear Schrödinger equations with an arbitrary number of combined nonlinearities.

Notation. The ball with radius $R$ and center $x \in \mathbb{R}^N$ will be denoted by $B_R(x)$. For any $r \geq 1$, we denote by $L^r(\mathbb{R}^N)$ the space of all complex-valued $r$-integrable functions $f$ with the norm $\|f\|_{L^r} = \int_{\mathbb{R}^N} |f|^r \, dx$. For any $1 \leq r < \infty$, the weak-$L^r$ space $L^r_w(\mathbb{R}^N)$ is the set of all measurable functions $f : \mathbb{R}^N \to \mathbb{C}$ such that $\|f\|_{L^r_w} = \sup_{M > 0} M \{ x : |f(x)| > M \}^{1/r} < \infty$.

We have the proper inclusion $L^r(\mathbb{R}^N) \hookrightarrow L^r_w(\mathbb{R}^N)$ (see [15] for details). The fractional Laplacian $(-\Delta)^\alpha, \alpha \in (0,1)$, defined in (1.2), can be equivalently defined via Fourier transform as $\hat{(-\Delta)^\alpha u}(\xi) = |\xi|^{2\alpha} \hat{u}(\xi), \forall u \in \mathcal{S}(\mathbb{R}^N)$. We denote by $H^\alpha(\mathbb{R}^N), 0 < \alpha < 1$, the fractional Sobolev space of all complex-valued functions $u \in L^2(\mathbb{R}^N)$ with the norm $\|u\|_{H^\alpha}^2 = \|(-\Delta)^{\alpha/2} u\|_{L^2}^2 + \|u\|_{L^2}^2$, where up to a multiplicative constant $\|(-\Delta)^{\alpha/2} u\|_{L^2}^2 = \iint_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy$, also known as Gagliardo seminorm. Throughout the paper, we denote by $Y^\alpha(\mathbb{R}^N)$ the product space $Y^\alpha(\mathbb{R}^N) = H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ and its norm by $\|(u,v)\|_{Y^\alpha}^2 = \|u\|_{H^\alpha}^2 + \|v\|_{H^\alpha}^2$. 


We do not develop the elements of the theory of fractional Sobolev spaces here, but use a number of fractional Sobolev type inequalities throughout the paper (for a detailed account of fractional Sobolev spaces, the reader may consult [8]). The same symbol \( C \) will frequently be used to denote different constants in the same string of inequalities whose exact value is not important for our analysis.

2. Fractional Schrödinger-Choquard equation

In this section we study existence and orbital stability issues of standing waves for fNLS equation with combined power and Choquard type nonlinearities

\[
i\Psi_t + (-\Delta)^\alpha \Psi = a|\Psi|^{s-2}\Psi + \lambda I^p_\beta(\Psi)|\Psi|^{p-2}\Psi, \quad (x,t) \in \mathbb{R}^N \times (0, \infty), \tag{2.1}
\]

where \( N \geq 2 \) and \( a, \lambda \) are nonnegative constants satisfying \( a + \lambda \neq 0 \). Throughout this section, we assume that the following conditions hold:

\[
N \geq 2, \beta \in (0, N), 0 < \alpha < 1, 2 < s < 2 + \frac{4\alpha}{N}, \quad 2 \leq p < \frac{N + 2\alpha + \beta}{N}. \tag{2.2}
\]

We first provide the statement of our main results. For any \( \sigma > 0 \), we denote by \( \Sigma_\sigma \) the set of fixed power \( \Sigma_\sigma = \{ u \in H^\alpha(\mathbb{R}^N) : \|u\|_{L^2}^2 = \sigma \} \) and define

\[ J(\sigma) = \inf_{f \in \Sigma_\sigma} J(f). \]

The following theorem and its corollary are the main results of this section:

**Theorem 2.1.** Suppose that (2.2) holds. For every \( \sigma > 0 \), let \( P_\sigma \) denotes the set of standing wave profiles of (2.1), that is,

\[ P_\sigma = \{ u \in H^\alpha(\mathbb{R}^N) : u \in \Sigma_\sigma \text{ and } J(u) = J(\sigma) \}. \]

Then, the set \( P_\sigma \) is nonempty. If \( u \in P_\sigma \), then \(|u| \in P_\sigma \) and \(|u| > 0 \) on \( \mathbb{R}^N \). Moreover, \(|u|^* \in P_\sigma \) whenever \( u \in P_\sigma \), where \( f^* \) represents the symmetric decreasing rearrangement of \( f \).

Furthermore, if \( f_n \in H^\alpha(\mathbb{R}^N) \), \( \|f_n\|_{L^2}^2 \to \sigma \), and \( J(f_n) \to J(\sigma) \), then

1. the sequence \( \{f_n\}_{n \geq 1} \) is relatively compact in \( H^\alpha(\mathbb{R}^N) \) up to a translation, i.e., there exists a subsequence which we still denote by the same, a sequence \( \{y_1, y_2, \ldots \} \subset \mathbb{R}^N \), and a function \( u \in H^\alpha(\mathbb{R}^N) \) such that \( f_n(\cdot + y_n) \to u \) strongly in \( H^\alpha(\mathbb{R}^N) \) and \( u \in P_\sigma \).
2. The following holds:

\[
\lim_{n \to \infty} \inf_{u \in P_\sigma} \inf_{y \in \mathbb{R}^N} \|f_n(\cdot + y) - u\|_{H^\alpha} = 0.
\]
3. \( f_n \to P_\sigma \) in the following sense,

\[
\lim_{n \to \infty} \inf_{u \in P_\sigma} \|f_n - u\|_{H^\alpha} = 0.
\]

A straightforward consequence is the following result.
Corollary 2.2. For every $\sigma > 0$, the solution set $P_\sigma$ is stable in the following sense: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^\alpha(\mathbb{R}^N)$, $u \in P_\sigma$, and $\|u_0 - u\|_{H^\alpha} < \delta$, then the solution $\Psi(x, t)$ of (2.1) with $\Psi(x, 0) = u_0(x)$ satisfies
$$\inf_{v \in P_\sigma} \|\Psi(\cdot, t) - v(x)\|_{H^\alpha} < \varepsilon, \forall t \geq 0.$$ 

In order to obtain analogue results for the equation (1.12) involving an arbitrary number of combined power and Hartree type nonlinearities, we require that the powers $s_j$ and $p_k$ satisfy
$$2 < s_j < 2 + \frac{4\alpha}{N}, \ 1 \leq j \leq m, \text{ and } 2 \leq p_k < \frac{2r(\alpha + N) - N}{N\tau}, \ 1 \leq k \leq d. \quad (2.3)$$
Furthermore, we require that the convolution potential $K(x)$ satisfies the following assumptions

(H1) $K : \mathbb{R}^N \to [0, \infty)$ is radially symmetric, i.e., $K(x) = K(|x|)$, satisfying
$$\lim_{r \to \infty} K(r) = 0 \quad \text{and} \quad K \in L_w^r(\mathbb{R}^N) \text{ for } r > \frac{N}{2N + 2\alpha - Np_k}, \ \forall 1 \leq k \leq d.$$

(H2) $K$ satisfies the following condition
$$\forall \theta > 1, \ K(\theta x) \geq \left(\frac{1}{\theta}\right)^\Gamma K(x) \text{ for some } \Gamma > 0.$$ 

(H3) The number $\Gamma$ satisfies $\Gamma < 2\alpha + N(2 - p_k)$ for all $1 \leq k \leq d$.

Analogue existence result for (1.12) is the following:

Theorem 2.3. Suppose that $K$ satisfy the assumptions (H1), (H2), (H3), and the powers $s_j$ and $p_k$ satisfy the conditions (2.3). For any $\sigma > 0$, define
$$\tilde{J}(\sigma) = \inf_{f \in \Sigma_\sigma} \tilde{J}(f) \quad \text{and} \quad \tilde{P}_\sigma = \left\{u \in H^\alpha(\mathbb{R}^N) : u \in \Sigma_\sigma, \tilde{J}(u) = \tilde{J}(\sigma)\right\}, \quad (2.4)$$
where $\Sigma_\sigma$ and $\tilde{J}(f)$ are as defined in Section 1. Then, the solution set $\tilde{P}_\sigma$ for the problem $\tilde{J}(\sigma)$ is nonempty and the compactness property is also enjoyed by its minimizing sequences. Each function $u \in \tilde{P}_\sigma$ solves the equation (1.12) for some $\omega > 0$. Furthermore, if $u \in \tilde{P}_\sigma$, then $|u| \in \tilde{P}_\sigma$ and $|u| > 0$.

The proof of Theorem 2.3 is quite similar to the proof of Theorem 2.1 and we do not provide all details of it. We only indicate the parts which require changing.

For the reader's convenience, we first recall some well-known results from the fractional Sobolev spaces. The following lemma is the fractional Sobolev embedding.

Lemma 2.4. Let $0 < \alpha < 1$ be such that $N > 2\alpha$. Then there exists a constant $C = C(N, \alpha) > 0$ such that
$$\|u\|_{L^{2N/(N-2\alpha)}}^2 \leq C\|(-\Delta)^{\alpha/2} u\|_{L^2}^2, \forall u \in H^\alpha(\mathbb{R}^N), \quad (2.5)$$
Thus, $H^\alpha(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for any $2 \leq r \leq \frac{2N}{N - 2\alpha}$ and compactly embedded into $L^r_{\text{loc}}(\mathbb{R}^N)$ for every $2 \leq r < \frac{2N}{N - 2\alpha}$.

**Proof.** See for example, Theorem 6.5 of [8].

We require the fractional Gagliardo-Nirenberg inequality.

**Lemma 2.5.** Let $1 \leq r < \infty, 0 < \alpha < 1$, and $N > 2\alpha$. Then, for any $u \in H^\alpha(\mathbb{R}^N)$, one has

$$
\|u\|_{L^r} \leq C\|(-\Delta)^{\alpha/2}u\|_{L^2}^{\rho}\|u\|_{L^1}^{1-\rho},
$$

where $t \geq 1, \rho \in [0,1], C = C(N, \alpha, \rho)$, and $\rho$ satisfies the identity

$$
\frac{N}{r} = \frac{\rho(N - 2\alpha)}{2} + \frac{N(1 - \rho)}{t}.
$$

**Proof.** It is a consequence of the the Hölder and fractional Sobolev-type inequalities. If $r = 1$, it is obvious. For $r > 1$, the Hölder inequality yields

$$
\|u\|_{L^r} \leq \|u\|_{L^{2N/(N-2\alpha)}}^{\rho}\|u\|_{L^1}^{1-\rho},
$$

where $\frac{1}{r} = \frac{1 - \rho}{t} + \frac{\rho(N - 2\alpha)}{2N}$. Using the Sobolev inequality (2.5), we obtain that

$$
\|u\|_{L^r} \leq (C(N, \alpha))^{\rho/2}\|(-\Delta)^{\alpha/2}u\|_{L^2}^{\rho}\|u\|_{L^1}^{1-\rho},
$$

which gives the desired inequality. \(\square\)

We will make frequent use of the Hardy-Littlewood-Sobolev inequality:

**Lemma 2.6.** Suppose $1 < r, t < +\infty$ and $0 < \gamma < N$ with $\frac{1}{r} + \frac{1}{t} + \frac{\gamma}{N} = 2$. If $u \in L^r(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$, then there exists $C(r, t, \alpha, N) > 0$ such that

$$
\iint_{\mathbb{R}^N} |u(x)||x - y|^{-\gamma}v(y)\,dxdy \leq C(r, t, \alpha, N)\|u\|_{L^r}\|v\|_{L^t}.
$$

**Proof.** See Theorem 4.3 of [15]. \(\square\)

We also need the weak version of Young’s inequality for convolutions which states that for any three measurable functions $f \in L^q(\mathbb{R}^N), g \in L^r_{w}(\mathbb{R}^N)$, and $h \in L^t(\mathbb{R}^N)$,

$$
\left|\iint_{\mathbb{R}^N} f(x)g(x - y)h(x)\,dxdy\right| \leq C(q, N, r)\|f\|_{L^q}\|g\|_{L^r_{w}}\|h\|_{L^t},
$$

where $1 < q, r, t < \infty$ and

$$
\frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 2.
$$

The weak Young’s inequality (2.6) was proved by Lieb [14] as a corollary of the Hardy-Littlewood-Sobolev inequality.
Remark 2.7. In view of the weak version of Young’s inequality, we see that the integral
\[ \iint_{\mathbb{R}^N} \mathcal{K}(x - y)|u(x)||^p|u(y)||^p \, dxdy \]
is well-defined if \( |u|^p \in L^t(\mathbb{R}^N) \) for all \( t > 1 \) satisfying the condition
\[ \frac{2}{t} + \frac{1}{r} = 2, \quad \text{or} \quad t = \frac{2r}{2r - 1}. \]
In the present context, since \( u \in H^\alpha(\mathbb{R}^N) \), we must require \( tp \in [2, \frac{2N}{N-2\alpha}] \). By our assumption on \( p = p_k \), it follows that
\[ \frac{1}{tp} = \frac{2r - 1}{2pr} = \frac{1}{p} - \frac{1}{2pr} > \frac{1}{p} - \frac{2N + 2\alpha - pN}{2pN} = \frac{1}{2} - \frac{\alpha}{pN} > \frac{1}{2} - \frac{\alpha}{N}, \quad (2.7) \]
and so, \( |u|^p \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) \) for every \( u \in H^\alpha(\mathbb{R}^N) \). When the convolution potential is \( \mathcal{K}(x) = \mathcal{K}_\beta(x) = |x|^{-(N-\beta)} \), one has \( \mathcal{K}_\beta \in L^p_\sigma(\mathbb{R}^N) \) if and only if \( N-\beta = N/r \) and (2.7) reduces to
\[ \frac{N + \beta}{2Np} = \frac{1}{2p} + \frac{\beta}{2Np} > \frac{1}{2p} + \frac{pN - N - 2\alpha}{2Np} = \frac{1}{2} - \frac{\alpha}{Np} > \frac{N - 2\alpha}{2N}, \]
and so, \( |u|^p \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) \) for every \( u \in H^\alpha(\mathbb{R}^N) \). One can also see that the condition (H3), namely \( \Gamma < 2\alpha + N(2-p) \), is equivalent to \( p < (N+2\alpha+\beta)/N \). These observations illustrate that the assumptions on the powers \( p \) and \( p_k \) as given in (2.2) and (2.3) are quite natural in the present setting.

We now establish some important properties of the function \( J(\sigma) \). We have broken the proof into several lemmas so that later, in the case of coupled system of fNLS equations, it will be easy to identify the parts which require changing.

In what follows, we denote by \( b = a/s \) and \( \mu = \lambda/2p \) (see definition (1.7) of the energy functional). We denote for any \( r > 0 \),
\[ D_r(|f|, |g|) = \frac{|f(x)g(y)|^r}{|x - y|^{N-\beta}}, \quad x, y \in \mathbb{R}^N. \]
With this notation, the Coulomb energy functional has the following form
\[ \mathbb{D}_r(f, g) = \int_{\mathbb{R}^N} \mathcal{T}_\beta(g)x \, dx = \int_{\mathbb{R}^N} D_r(|f|, |g|) \, dxdy. \]
In particular, we simply write \( \mathbb{D}_r(f) \) for the functional \( \mathbb{D}_r(f, f) \).

Lemma 2.8. Suppose that (2.2) holds. For any \( \sigma > 0 \), the following statements hold.

(i) If \( f_n \in H^\alpha(\mathbb{R}^N) \), \( \|f_n\|_{L^2}^2 \rightarrow \sigma \), and \( J(f_n) \rightarrow J(\sigma) \), then there exists \( B > 0 \) such that \( \|f_n\|_{H^\alpha} \leq B \) for all \( n \).

(ii) \( J(\sigma) \) is bounded from below on \( \Sigma_\alpha \). Moreover, \( J(\sigma) < 0 \).
Proof. Applying the fractional Gagliardo-Nirenberg inequality and using the boundedness of \( \|f_n\|_{L^2} \), we obtain

\[
\|f_n\|_{L^2}^s \leq C\|(-\Delta)^{\alpha/2}f_n\|_{L^2}^{\lambda'} \leq C\|(-\Delta)^{\alpha/2}f_n\|_{L^2}^{\lambda'} \leq C\|f_n\|_{H^\alpha}^{\lambda'},
\]

where \( \lambda' = N(s - 2)/2\alpha \). Next, by our assumption on \( p \), we have \( |f|^p \in L_{2N/(N+\beta)}^\infty(\mathbb{R}^N) \) for every \( f \in H^\alpha(\mathbb{R}^N) \). Applying the Hardy-Littlewood-Sobolev inequality with \( r = t = 2N/(N + \beta) \) and the fractional Gagliardo-Nirenberg inequality, we get

\[
\int_{\mathbb{R}^N} D_p(|f_n|,|f_n|) \, dx \, dy \leq C\|f_n\|^2_{L^{2N/(N+\beta)}} = C\|f_n\|_{L^{2N/(N+\beta)}}^{2p}.
\]

where \( \mu' = (Np - N - \beta)/2\alpha \). We now write

\[
\frac{1}{2}\|f_n\|_{H^\alpha}^2 = J(f_n) + b\|f_n\|_{L^2} + \mu \int_{\mathbb{R}^N} D_p(|f_n|,|f_n|) \, dx \, dy + \frac{1}{2}\|f_n\|_{L^2}^2.
\]

Since \( \{J(f_n)\}_{n \geq 1} \) is a convergent sequence of numbers, so it is bounded. Utilizing the estimates (2.8) and (2.9), the above identity implies that

\[
\frac{1}{2}\|f_n\|_{H^\alpha}^2 \leq C \left(1 + \|f_n\|_{H^\alpha}^\lambda + \|f_n\|_{H^\alpha}^{2\mu}\right).
\]

By our assumption on \( p \), we have that \( pN - N - \beta < N \) and so

\[
2\mu' = \frac{pN - N - \beta}{\alpha} < \frac{N}{\alpha} < \frac{2\alpha}{\alpha} = 2.
\]

Since \( \lambda' \in (0,2) \) and \( 0 < \frac{N - \beta}{\alpha} \leq 2\mu' < 2 \), it follows from (2.10) that that there exists a constant \( B > 0 \) such that \( \|f_n\|_{H^\alpha} \leq B \) for each \( n \). The statement that \( J(\sigma) > -\infty \) easily follows from the estimates (2.8) and (2.9).

To show \( J(\sigma) < 0 \), first we observe that if \( u_\theta(\cdot) = \theta^Au(\theta^{B} \cdot) \), for \( A, B \in \mathbb{R} \) and \( \theta > 0 \), then we have that for any \( p \),

\[
I_\beta^p(u_\theta)(x) = \theta^{pA+B(N-\beta)} \int_{\mathbb{R}^N} \frac{|u(\theta^By)|^p}{|\theta^{BX - \theta^{B}y|^{N-\beta}dy} = \theta^{pA-B\beta}I_\beta^p(u)(\theta^{B}x).
\]

Now let \( f \in \Sigma_\sigma \) be fixed. For any \( \theta > 0 \), define the scaled function \( f_\theta(x) = \theta^{1/2}f(\theta^{1/N}x) \). Then \( f_\theta \in \Sigma_\sigma \) as well. If both \( a > 0 \) and \( \lambda > 0 \); or \( a = 0 \) and \( \lambda > 0 \), then we have that

\[
J(f_\theta) \leq \frac{\theta^{2\alpha/N}}{2}\|(-\Delta)^{\alpha/2}f\|_{L^2}^2 - \mu\theta^\kappa \int_{\mathbb{R}^N} D_p(|f|,|f|) \, dx \, dy, \tag{2.11}
\]

where \( \kappa = (Np - \beta - N)/N \). Since \( \kappa \) satisfies \( 0 < \frac{N-\beta}{N} \leq \kappa < \frac{2\alpha}{N} \), we obtain that \( J(f_\theta) < 0 \) for sufficiently small \( \theta \). This proves that \( J(\sigma) \leq J(f_\theta) < 0 \). When \( \lambda = 0 \), we
have that

\[ J(f_0) \leq \frac{\theta^{2\alpha/N}}{2} \|(-\Delta)^{\alpha/2} f\|_{L^2}^2 - b\theta^{(s-2)/2}\|f\|_{L^s}^s. \]

Since the condition \(2 < s < 2 + \frac{2\alpha}{N}\) implies \((s-2)/2 \in (0, 2\alpha/N)\), we again obtain that \(J(f_0) < 0\) for sufficiently small \(\theta\).

\[ \square \]

**Remark 2.9.** Analogue of Lemma 2.8 for \(\tilde{J}(\sigma)\) can be proved by applying the weak version of Young’s inequality and the fractional Gagliardo-Nirenberg inequality. To see this, observe first that our assumption on \(p_k\) guarantees \(|f|^{p_k} \in L^{\frac{2}{r}}\) for every \(f \in H^\alpha(\mathbb{R}^N)\) and all \(1 \leq k \leq d\). Then, analogue of estimate (2.9) for any minimizing sequence \(\{f_n\}_{n \geq 1}\) of \(\tilde{J}(\sigma)\) takes the form

\[
\int_{\mathbb{R}^N} (K \ast |f_n|^{p_k}) |f_n|^{p_k} \, dx \leq \|K\|_{L^\infty} \|f_n\|_{L^{2p_k}}^{2p_k} \leq C \|f_n\|_{L^2}^{2(1-\mu_k')p_k} \|\nabla f_n\|_{L^2}^{2\mu_k'p_k}
\]

\[
\leq C \|\nabla f_n\|_{L^2}^{2\mu_k'p_k}, \quad \forall \ 1 \leq k \leq d,
\]

where the numbers \(\mu_k'\) are given by

\[ \mu_k' = \frac{N(rp_k - 2r + 1)}{2r\alpha p_k}, \quad 1 \leq k \leq d, \]

and by our assumption on \(p_k\), the powers \(2\mu_k'p_k\) satisfy

\[ 2\mu_k'p_k = \frac{Np_k}{\alpha} - \frac{2N}{\alpha} + \frac{N}{r\alpha} < \frac{N}{\alpha} \left( \frac{2\alpha}{N} + 2 - \frac{1}{r} \right) - \frac{2N}{\alpha} + \frac{N}{r\alpha} = 2. \]

An analogue of the estimate (2.8) remains true for each \(s = s_j, 1 \leq j \leq m\). Then, an analogue of estimate (2.10) proves that any minimizing sequence \(\{f_n\}_{n \geq 1}\) of \(\tilde{J}(\sigma)\) is bounded in \(H^\alpha(\mathbb{R}^N)\). The proof that \(\tilde{J}(\sigma) \in (-\infty, 0)\) will go through unchanged.

**Lemma 2.10.** Suppose that \(\lambda \neq 0\). If \(f_n \in H^\alpha(\mathbb{R}^N), \|f_n\|_{L^2}^2 \to \sigma, \) and \(J(f_n) \to J(\sigma)\), then there exists \(\delta > 0\) and \(n_0 \in \mathbb{N}\), depending on \(\delta\), such that for every \(n \geq n_0\),

\[ D_p(f_n) = \int_{\mathbb{R}^N} D_p(|f_n|, |f_n|) \, dxdy = \int_{\mathbb{R}^N} T_p^\sigma(f_n)||f_n||^p \, dx \geq \delta. \]

Moreover, for any \(T > 1\), there exists \(n_0 \in \mathbb{N}\), depending on \(T\), such that for all \(n \geq n_0\),

\[ J(T^{1/2} f_n)|_{a=0} < T J(f_n)|_{a=0}.\]

If \(a > 0 \) and \(\lambda \geq 0\), the same conclusions hold with \(D_p(f_n)\) replaced by \(\|f_n\|_{L^s}^s\) and \(J(g)|_{a=0}\) replaced by \(J(g)|_{\lambda=0}\).

**Proof.** To prove the first statement, suppose to the contrary that \(\liminf_{n \to \infty} D_p(f_n) = 0\). Then, it is obvious that

\[ J(\sigma) = \lim_{n \to \infty} J(f_n) \geq \liminf_{n \to \infty} \|(-\Delta)^{\alpha/2} f_n\|_{L^2}^2 \geq 0, \quad (2.12) \]
which contradicts the fact $J(\sigma) < 0$ and hence, the first statement follows. To prove the second statement, an easy calculation gives

$$J(T^{1/2}f_n)|_{a=0} = T\left(\frac{1}{2}\|
abla\Delta^{\alpha/2}f_n\|_{L^2}^2 - \mu \int_{\mathbb{R}^N} D_p(|f_n|, |f_n|) \, dx \, dy\right)$$

$$+ (T - T^p) \mu \int_{\mathbb{R}^N} D_p(|f_n|, |f_n|) \, dx \, dy. \tag{2.13}$$

By the first statement, we have $D_p(f_n) \geq \delta$. Since $p > 1$ and $\mu > 0$, the desired result follows from (2.13). The proof in the case $a > 0$ is similar. □

**Lemma 2.11.** Assume that the sequences $\{z_n\}_{n \geq 1}$ and $\{\nabla\Delta^{\alpha/2}z_n\}_{n \geq 1}$ are bounded in $L^2(\mathbb{R}^N)$. If there is some $R > 0$ satisfying

$$\lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |z_n(x)|^2 \, dx \right) = 0, \tag{2.14}$$

then $\lim_{n \to \infty} \|z_n\|_{L^q} = 0$ for every $2 < q < 2N/(N-2\alpha)$.

**Proof.** The result is a version of Lemma I.1 of [16]. See [3] for a proof in the case $\alpha = 1$ (the same argument works for the case $0 < \alpha < 1$ with obvious modification). □

The idea behind the proof of relative compactness of minimizing sequence $\{f_n\}_{n \geq 1}$ of $J(\sigma)$ is that, we can employ the concentration compactness principle to the sequence of non-negative functions $\rho_n$ defined by $\rho_n = |f_n|^2$. To do this, for $n = 1, 2, \ldots$ and $R > 0$, consider the associated concentration function $M_n(R)$ defined by

$$M_n(R) = \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \rho_n \, dx,$$

where $B_R(x)$ stands for the $n$-ball of radius $R$ and center $x \in \mathbb{R}^N$. Suppose that evanescence of the energy minimizing $\{f_n\}_{n \geq 1}$ occurs, that is, for all $R > 0$, $\lim_{n \to \infty} M_n(R) = 0$ up to a subsequence. By Lemma 2.11, $\{\|f_n\|\}_{n \geq 1}$ is bounded. Lemma 2.11 then implies that $\lim_{n \to \infty} \|f_n\|_{L^r} = 0$ for any $2 < r < 2N/(N-2\alpha)$. Since $2 < \frac{2N}{N+\beta} < \frac{2N}{N-2\alpha}$, it follows from (2.9) that

$$D_p(f_n) = \int_{\mathbb{R}^N} T^p_\beta(f_n)|f_n|^p \, dx \leq C\|f_n\|_{L^{2Np/(N+\beta)}}^{2p} \to 0,$$

as $n \to \infty$. Consequently, we obtain that

$$J(\sigma) = \lim_{n \to \infty} J(f_n) \geq \liminf_{n \to \infty} \int_{\mathbb{R}^N} |(-\nabla)^{\alpha/2}|^2 \, dx \geq 0,$$

which contradicts $J(\sigma) < 0$. Thus, we conclude from the concentration compactness principle (see Lemma I.1 of [16]) that one of the remaining two alternatives, namely
dichotomy or compactness, is the only option here. In what follows, for every \( \sigma > 0 \) and any minimizing sequence \( \{f_n\}_{n \geq 1} \subset H^\alpha(\mathbb{R}^N) \) of \( J(\sigma) \), we denote

\[
L = \lim_{R \to \infty} \left( \lim_{n \to \infty} M_n(R) \right) \in [0, \sigma].
\] (2.15)

Thus, we have established the following lemma.

**Lemma 2.12.** If \( \{f_n\}_{n \geq 1} \subset H^\alpha(\mathbb{R}^N) \) be any minimizing sequence for \( J(\sigma) \) and \( L \) be defined by (2.15), then \( L > 0 \).

The remaining two possibilities are \( L \in (0, \sigma) \) (dichotomy) and \( L = \sigma \) (compactness). The next step toward the proof of the relative compactness of minimizing sequences is to show that, we must have \( L = \sigma \).

Before we describe how energy minimizing sequences \( \{f_n\}_{n \geq 1} \) of \( J(\sigma) \) would behave in the case when \( L \in (0, \sigma) \), we need the following result.

**Lemma 2.13.** Let \( D^\alpha = (-\Delta)^{\alpha/2} \), \( 0 < \alpha < 1 \). If \( f, g \in \mathcal{S}(\mathbb{R}^N) \), then

\[
\| [D^\alpha, f] g \|_{L^2} \leq C_1 \| \nabla f \|_{L^2} \| D^{\alpha - 1} g \|_{L^2} + C_2 \| D^\alpha f \|_{L^2} \| g \|_{L^2},
\]

where \([X, Y] = XY - YX\) represents the commutator, \( s_1, s_2 \in [2, \infty) \), and

\[
\frac{1}{r_1} + \frac{1}{s_1} = \frac{1}{r_2} + \frac{1}{s_2} = 1/2.
\]

**Proof.** This is a variant of the commutator estimate result of Kato and Ponce in [10] and a proof is given in [9]. \( \square \)

**Lemma 2.14.** Suppose that \( f_n \in H^\alpha(\mathbb{R}^N), \|f_n\|_{L^2}^2 \to \sigma, J(f_n) \to J(\sigma) \) and \( L \in (0, \sigma) \), where \( L \) is as defined in (2.15). For some subsequence of \( \{f_n\}_{n \geq 1} \), which we continue to denote by the same, the following are true: For every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) and sequences \( \{v_n\}_{n \geq 1} \) and \( \{w_n\}_{n \geq 1} \) in \( H^\alpha(\mathbb{R}^N) \) such that for every \( n \geq n_0 \),

1. \( \int_{\mathbb{R}^N} |v_n(x)|^2 \, dx \in (L - \varepsilon, L + \varepsilon) \)
2. \( \int_{\mathbb{R}^N} |w_n(x)|^2 \, dx \in (\sigma - L - \varepsilon, \sigma - L + \varepsilon) \)
3. \( J(f_n) \geq J(v_n) + J(w_n) - \varepsilon. \)

**Proof.** Given \( \varepsilon > 0 \), by definition of \( L \), there exists \( R_0 \) such that if \( R > R_0 \), then \( L - \varepsilon < \lim_{n \to \infty} M_n(R) \leq L \). Therefore, after extracting a subsequence of \( \{M_n\} \) if needed, we can say that there exist \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \),

\[
L - \varepsilon < M_n(R) \leq M_n(2R) < L + \varepsilon.
\]

It then follows that for every \( n \geq n_0 \), there exists \( y_n \in \mathbb{R}^N \) such that

\[
L - \varepsilon < \int_{B_R(y_n)} \rho_n \, dx \leq \int_{B_{2R}(y_n)} \rho_n \, dx < L + \varepsilon.
\] (2.16)
Now introduce smooth cut-off functions $\phi$ and $\psi$, defined on $\mathbb{R}^N$, such that $\phi(x) \equiv 1$ for $|x| \leq 1$; $\phi(x) \equiv 0$ for $|x| \geq 2$; $\psi(x) \equiv 1$ for $|x| \geq 2$; and $\psi(x) \equiv 0$ for $|x| \leq 1$. Denote by $\phi_R$ and $\psi_R$ the functions $\phi_R(x) = \phi(x/R)$ and $\psi_R(x) = \psi(x/R)$, respectively. Define $v_n(x) = \phi_R(x - y_n)f_n(x)$ and $w_n(x) = \psi_R(x - y_n)f_n(x)$. With these definitions together with (2.13), Statements 1 and 2 are clear. To prove Statement 3, using Lemma 2.13 with $r_1 = 1$, $s_1 = 2$, and $s_2 = 1 + \frac{4a}{N}$, one obtains that

$$\int_{\mathbb{R}^N} |(\Delta)^{\alpha/2}(\phi_R f_n)|^2 \, dx \leq C_1 \|\nabla\phi_R\|_\infty \|(-\Delta)^{(\alpha-1)/2} f_n\|_{L^2}^2 + C_2 R^{\alpha(N-2\alpha)/(N+2\alpha)} \|(-\Delta)^{\alpha/2} \phi\|_{L^2(N+2\alpha)/(2\alpha-N)} \|f_n\|_{H^\alpha},$$

where, for ease of notation, we denote $\widetilde{\phi}_R(x) = \phi_R(x - y_n)$ for $x \in \mathbb{R}^N$. Using the fact $\|\nabla\phi_R\|_\infty = \|\nabla\phi\|_\infty/R$, it immediately follows from what we have just obtained that

$$\int_{\mathbb{R}^N} |(\Delta)^{\alpha/2}(\phi_R f_n)|^2 \, dx \leq \int_{\mathbb{R}^N} (\phi_R f_n)|^2 \, dx + C\varepsilon \quad (2.17)$$

for sufficiently large $R$. Similarly, we have the following estimate

$$\int_{\mathbb{R}^N} |(\Delta)^{\alpha/2}(\psi_R f_n)|^2 \, dx \leq \int_{\mathbb{R}^N} (\psi_R f_n)|^2 \, dx + C\varepsilon, \quad (2.18)$$

where $\widetilde{\psi}_R(x) = \psi_R(x - y_n)$ for $x \in \mathbb{R}^N$. Taking into account of $\phi_R^2 + \psi_R^2 \equiv 1$ on $\mathbb{R}^N$ and using the estimates (2.17) and (2.18), a direct computation yields

$$J(v_n) + J(w_n) \leq b \int_{\mathbb{R}^N} \left(\phi_R^2(x) + \psi_R^2(x)\right) |f_n|^s \, dx - b \int_{\mathbb{R}^N} |\phi_R f_n|^s \, dx$$

$$- b \int_{\mathbb{R}^N} |\psi_R f_n|^s \, dx + J(f_n) + \mu \int_{\mathbb{R}^N} \left(\phi_R^2(x) + \psi_R^2(x)\right) D_p(|f_n|, |f_n|) \, dxdy$$

$$- \mu \int_{\mathbb{R}^N} D_p(|\phi_R f_n|, |\phi_R f_n|) dxdy - \mu \int_{\mathbb{R}^N} D_p(|\psi_R f_n|, |\psi_R f_n|) dxdy. \quad (2.19)$$

Let us denote by $\mathbf{M}_b^s[f_n]$ the first three integrals on the right-hand side of (2.19) and by $\mathbf{F}_\mu^p[f_n]$ the last three integrals on the right-hand side of (2.19). Now let $B_R = B(y_k, 2R) - B(y_k, R)$, $2^*_\alpha = 2N/(N - 2\alpha)$, and $\lambda' = N(s - 2)/2\alpha s$. Then, it follows from the Interpolation inequality that

$$\mathbf{M}_b^s[f_n] \leq b \int_{\mathbb{R}^N} |\phi_R^2 - |\phi_R|^s| |f_n|^s \, dx + b \int_{\mathbb{R}^N} |\phi_R^2 - |\phi_R|^s| |f_n|^s \, dx$$

$$\leq 4b \int_B |f_n|^s \, dx \leq C \|f_n\|_{L^2(B_R)}^{1-\lambda'} \|f_n\|_{L^\lambda'_s(B_R)} \leq C \|f_n\|_{H^\alpha(B_R)} \|f_n\|_{L^2(B_R)}^{1-\lambda'} \leq C\varepsilon. \quad (2.20)$$

Similarly, if $\mu' = (Np - N - \beta)/2\alpha p$, then it follows from (2.9) that

$$\mathbf{F}_\mu^p[f_n] \leq C \|f_n\|_{H^\alpha(B_R)} \|f_n\|_{L^2(B_R)}^{2p(1-\mu')} \leq C\varepsilon, \quad (2.21)$$
where $C$ is independent of $R$ and $n$. Finally, taking into account of the estimates \(2.20\) and \(2.21\), Statement 3 follows from \(2.19\).

**Lemma 2.15.** If $0 < L < \sigma$, then $J(\sigma) \geq J(L) + J(\sigma - L)$.

**Proof.** First observe that if a function $v$ satisfies $\left| \int_{\mathbb{R}^N} |v|^2 \, dx - L \right| < \varepsilon$, then we have that $\int_{\mathbb{R}^N} |\eta v|^2 \, dx = L$, where $\eta = (L/\|v\|_{L^2})^{1/2}$ satisfies $|\eta - 1| < A_1 \varepsilon$ with $A_1 > 0$ independent of $v$ and $\varepsilon$. Thus

$$J(\sigma) \leq J(\eta v) \leq J(v) + A_2 \varepsilon,$$

where the constant $A_2$ depends only on $A_1$ and the power $\|v\|_{L^2}$. A similar estimate holds for the function $w$ such that $\left| \int_{\mathbb{R}^N} |w|^2 \, dx - (\sigma - L) \right| < \varepsilon$.

Taking into account of these observations and Lemma 2.14, it follows immediately that there exists a subsequence $\{f_{n_k}\}_{k \geq 1}$ of $\{f_n\}_{n \geq 1}$ and corresponding functions $v_{n_k}$ and $w_{n_k}$ for $k = 1, 2, \ldots$ such that for all $k$,

$$J(v_{n_k}) \geq J(L) - \frac{1}{k}, \quad J(w_{n_k}) \geq J(\sigma - L) - \frac{1}{k}, \quad \text{and}$$

$$J(f_{n_k}) \geq J(v_{n_k}) + J(w_{n_k}) - \frac{1}{k},$$

and so

$$J(v_{n_k}) \geq J(L) + J(\sigma - L) - \frac{1}{k}. \quad (2.22)$$

Passing the limit as $k \to \infty$ on both sides of \(2.22\) yields the desired result. \(\square\)

**Lemma 2.16.** If $\{f_n\}$ be any minimizing sequence of $J(\sigma)$, then $L \notin (0, \sigma)$.

**Proof.** Suppose to the contrary that $L \in (0, \sigma)$. Let $\sigma' = L$ and $\sigma'' = \sigma - L$. Then, $\sigma = \sigma' + \sigma''$. Lemma 2.15 implies that $J(\sigma') \geq J(L) + J(\sigma - L)$, which gives

$$J(\sigma' + \sigma'') \geq J(\sigma') + J(\sigma''). \quad (2.23)$$

To deduce a contradiction, we now claim the function $\sigma \mapsto J(\sigma)$ is strictly subadditive, i.e., for all $\sigma' > 0$ and $\sigma'' > 0$,

$$J(\sigma' + \sigma'') < J(\sigma') + J(\sigma''). \quad (2.24)$$

**Remark 2.17.** All results proved above remain true for the problem $\tilde{J}(\sigma)$. The proof of \(2.24\) below differs from the original ideas developed in [16, 17]. The advantage of this technique is that the same argument goes through unchanged to prove an analogue strict inequality for the problem $\tilde{J}(\sigma)$ related to equations with an arbitrary number of combined nonlinearities.
To see (2.24), let \( \{z_n\}_{n \geq 1} \) and \( \{w_n\}_{n \geq 1} \) be sequences of functions in \( H^\alpha(\mathbb{R}^N) \) such that
\[
\|z_n\|_{L^2}^2 \to \sigma', \quad J(z_n) \to J(\sigma'), \quad \|w_n\|_{L^2}^2 \to \sigma'', \quad J(w_n) \to J(\sigma'')
\] as \( n \to \infty \). Consider the sequence \( \{(K_1^n, K_2^n)\}_{n \geq 1} \subset \mathbb{R}^2 \) defined by
\[
K_1^n = \frac{1}{\|z_n\|_{L^2}^2} \left( \frac{1}{2} \int_{\mathbb{R}^N} \|(-\Delta)^{\alpha/2} z_n\|^2 \, dx - b \|z_n\|_{\dot{H}^s}^s - \mu \mathbb{D}_p(z_n) \right),
\]
\[
K_2^n = \frac{1}{\|w_n\|_{L^2}^2} \left( \frac{1}{2} \int_{\mathbb{R}^N} \|(-\Delta)^{\alpha/2} w_n\|^2 \, dx - b \|w_n\|_{\dot{H}^s}^s - \mu \mathbb{D}_p(w_n) \right).
\]
By passing to a subsequence, we can assume that \( (K_1^n, K_2^n) \to (K_1, K_2) \) in \( \mathbb{R}^2 \). Three cases are possible: \( K_1 < K_2, K_1 > K_2, \) and \( K_1 = K_2 \). Suppose first that \( K_1 < K_2 \).

Define a sequence \( \{F_n\}_{n \geq 1} \) in \( H^\alpha(\mathbb{R}^N) \) by \( F_n = T^{1/2} z_n \), where \( T = (\sigma' + \sigma'')/\sigma' \). Then \( F_n \to \sigma' + \sigma'' \) and consequently, \( J(\sigma' + \sigma'') \leq \lim_{n \to \infty} J(F_n) \). A straightforward calculation gives
\[
J(F_n) = \frac{T}{2} \|(-\Delta)^{\alpha/2} z_n\|_{L^2}^2 - b T^{s/2} \|z_n\|_{\dot{H}^s}^s - \mu T^p \int_{\mathbb{R}^N} D_p(|z_n|, |\nabla z_n|) \, dxdy. \tag{2.25}
\]
Since \( T > 1, p - 1 > 0, s - 2 > 0, \) and \( a, \lambda \) are nonnegative with \( a + \lambda > 0 \), it follows from (2.25) that \( J(F_n) \leq T J(z_n) \). This then implies that
\[
J(\sigma' + \sigma'') \leq T \lim_{n \to \infty} J(z_n) = T \sigma' K_1.
\]
Put \( \delta = \sigma''(K_2 - K_1) \). Since \( K_1 < K_2 \), we have \( \delta > 0 \). It follows from the inequality we have just obtained that \( J(\sigma' + \sigma'') \leq \sigma' K_1 + \sigma'' K_2 - \delta \). Thus, we obtain the strict inequality, \( J(\sigma' + \sigma_2) < J(\sigma') + J(\sigma'') \), which contradicts (2.23).

Using the similar argument as in the case \( K_1 < K_2 \), the case \( K_1 > K_2 \) also leads to a contradiction and will not be repeated. Finally, consider the case \( K_1 = K_2 \). As in the preceding paragraph, define \( F_n = T^{1/2} z_n \). If both \( a \) and \( \lambda \) are positive or \( a = 0 \) and \( \lambda > 0 \), using Lemma 2.10, there exists \( \delta > 0 \) such that
\[
\frac{1}{2} \|(-\Delta)^{\alpha/2} F_n\|_{L^2}^2 - \mu \mathbb{D}_p(F_n) \leq T \left( \frac{1}{2} \|(-\Delta)^{\alpha/2} z_n\|_{L^2}^2 - \mu \mathbb{D}_p(z_n) \right) - \delta.
\]
for sufficiently large \( n \). This in turn implies that
\[
J(F_n) \leq T J(z_n)_{a=0} - b T^{s/2} \|z_n\|_{\dot{H}^s}^s - \delta, \tag{2.26}
\]
where \( b = a/s \). In this case, since \( T > 1, b \geq 0, \lambda > 0, \) and \( s - 2 > 0 \), it follows from (2.26) that \( J(F_n) \leq T J(z_n) - \delta \) for sufficiently large \( n \). Thus, we obtain that
\[
J(\sigma' + \sigma'') \leq T \lim_{n \to \infty} J(z_n) - \delta = T K_1 \sigma' - \delta = K_1 \sigma' + \sigma'' K_1 - \delta \tag{2.27}
\]
Since the equality $K_1 = K_2$ holds, (2.27) gives the desired contradiction. If $a > 0$ and \( \lambda = 0 \), then making use of Lemma 2.10 again, there exists $\delta > 0$ such that

$$\frac{1}{2} \| (-\Delta)^{\alpha/2} F_n \|_{L^2}^2 - b \| F_n \|_{L^2}^\sigma \leq T \left( \frac{1}{2} \| (-\Delta)^{\alpha/2} z_n \|_{L^2}^2 - b \| z_n \|_{L^s}^\sigma \right) - \delta.$$ 

for sufficiently large $n$. Then, we again obtain that $J(F_n) \leq TJ(z_n) - \delta$ for sufficiently large $n$ and (2.27) gives the desired contradiction. \( \square \)

**Lemma 2.18.** Suppose the case $L = \sigma$. Then there exists \( \{y_n\} \subseteq \mathbb{R}^N \) such that

1. for any $\Lambda < \sigma$ there exists a number $R = R(\Lambda) > 0$ and $n_0 = n_0(\Lambda) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{B_R(y_n)} |f_n|^2 \, dx > \Lambda. \quad (2.28)$$

2. the shifted sequence $\tilde{f}_n(x) = f_n(x - y_n)$, $x \in \mathbb{R}^N$, converges (up to a subsequence) in $H^\alpha(\mathbb{R}^N)$ to some $u \in P_\sigma$.

**Proof.** Statement 1 is standard; we include the proof here for the readers convenience. Since $L = \sigma$, there exists $R_0$ and $n_0(R_0) \in \mathbb{N}$ such that for all $n \geq n_0(R_0)$, one has $M_n(R_0) > \sigma/2$. Consequently, by the definition of $M_n$, we can find \( \{y_n\} \subseteq \mathbb{R}^N \) such that $\| f_n \|_{L^2(B_R(y_n))}^2 > \sigma/2$ for sufficiently large $n$. Next let $\Lambda < \sigma$ be given. We may assume that $\Lambda > \frac{\sigma}{2}$. Since $L = \sigma$, we can find a number $R_0(\Lambda)$ and $n_0(\Lambda) \in \mathbb{N}$ such that if $n \geq n_0(\Lambda)$, then

$$\int_{B_{R_0(\Lambda)}(y_n(\Lambda))} |f_n|^2 \, dx > \Lambda \quad (2.29)$$

for some point $y_n(\Lambda) \in \mathbb{R}^N$. Since the power $\| f_n \|_{L^2}^2 = \sigma$ for each $n$, it follows that $B_{R_0(y_n)} \cap B_{R_0(\Lambda)}(y_n(\Lambda)) \neq \emptyset$, i.e., $|y_n(\Lambda) - y_n| \leq R_0 + R_0(\Lambda)$ for $\Lambda > \sigma/2$. Now define $R = R(\Lambda) = 2R_0(\Lambda) + R_0$, then we have that $B_{R_0(\Lambda)}(y_n(\Lambda)) \subset B_R(y_n)$, and so, (2.28) follows from (2.29) for all $n \geq n_0(\Lambda)$.

Statement 1 now ensures that for all $k \in \mathbb{N}$, there exists $R_k$ such that for sufficiently large $n$, we have that $\| \tilde{f}_n \|_{L^2(B_{R_k}(0))}^2 > L - 1/k$. Since \( \{ \tilde{f}_n \} \) is uniformly bounded in $H^\alpha(\mathbb{R}^N)$, and therefore also in $H^\alpha(\Omega)$ for any bounded domain $\Omega$ in $\mathbb{R}^N$. Thus, from Rellich-type lemma (see for example, Corollary 7.2 of [8]), it follows that \( \{ \tilde{f}_n \} \) converges in $L^2(\Omega)$ norm (up to a subsequence) to some $u \in L^2(\Omega)$ satisfying $\| u \|_{L^2(B_{R_k}(0))}^2 > L - 1/k$. By applying the standard Cantor diagonalization argument together with the fact that $\| \tilde{f}_n \|_{L^2} = \sigma$, all $n$, it can be shown that up to a subsequence $\tilde{f}_n \to u$ in $L^2(\mathbb{R}^N)$. FRACTIONAL SCHRÖDINGER SYSTEM OF CHOQUARD TYPE 17
satisfying \( \|u\|_{L^2}^2 = \sigma \). We now write
\[
\mathbb{D}_p(\tilde{f}_n) - \mathbb{D}_p(u) = \int_{\mathbb{R}^N} \frac{|\tilde{f}_n(x)|^p}{|x-y|^{N-\beta}} \left( |\tilde{f}_n(y)|^p - |u(y)|^p \right) \, dx dy \\
+ \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\beta}} \left( |\tilde{f}_n(x)|^p - |u(x)|^p \right) \, dx dy. \tag{2.30}
\]

Applying the Hardy-Littlewood-Sobolev inequality and using the fact that \( \{\tilde{f}_n\}_{n \geq 1} \) is bounded in \( H^\alpha(\mathbb{R}^N) \), the first term on the right-hand side of (2.30) satisfies
\[
|I_1| = \left| \int_{\mathbb{R}^N} \frac{|\tilde{f}_n(x)|^p}{|x-y|^{N-\beta}} \left( |\tilde{f}_n(y)|^p - |u(y)|^p \right) \, dx dy \right| \leq C \|\tilde{f}_n\|_{p}^{\alpha} \|u\|_{p}^{\alpha}. \tag{2.31}
\]

Next, for any two numbers \( S \) and \( T \) in \( \mathbb{R} \), one has for \( p \geq 1 \),
\[
\left| |T|^{p-1}T - |S|^{p-1}S \right| = \left| \int_0^1 \frac{d}{dt} |S + t(T - S)|^{p-1} (S + t(T - S)) \, dt \right| \\
\leq p|T - S| \int_0^1 |S + t(T - S)|^{p-1} \, dt \\
\leq p|T - S| \int_0^1 (t|T|^{p-1} + (1 - t)|S|^{p-1}) \, dt \\
= \frac{p}{2} |T - S| \left( |T|^{p-1} + |S|^{p-1} \right). \tag{2.32}
\]

Using (2.32) into (2.31) and applying Hölder inequality, we obtain that
\[
|I_1| \leq C \left( \int_{\mathbb{R}^N} \left( |\tilde{f}_n|^{p-1} + |u|^{p-1} \right) \frac{2N}{2N - \beta} |\tilde{f}_n - u|^{\frac{2N}{2N - \beta}} \, dx \right)^{\frac{N + \beta}{2N}} \\
\leq C \left( \int_{\mathbb{R}^N} \left( |\tilde{f}_n|^{\frac{2N}{\beta}} + |u|^{\frac{2N}{\beta}} \right) \, dx \right)^{\rho} \|\tilde{f}_n - u\|_{L^{\frac{2N}{\beta}}} \leq C \|\tilde{f}_n - u\|_{L^{\frac{2N}{\beta}}},
\]
where \( \rho = \frac{N + \beta}{2N} \left( 1 - \frac{1}{p} \right) \). Now, using the standard Interpolation inequality and the fractional Sobolev inequality, it follows that
\[
|I_1| \leq C \|\tilde{f}_n - u\|_{L^\alpha} \|\tilde{f}_n - u\|_{L^\alpha}^{1 - \frac{\alpha}{2N}} \leq C \|\tilde{f}_n - u\|_{L^\alpha}, \tag{2.33}
\]
where \( \alpha = (N + \beta - Np + 2\alpha)/2\alpha \). The right-hand side of (2.33) goes to zero since \( \tilde{f}_n \rightarrow u \) in \( L^2 \). Similarly, the second term on the right-hand side of (2.30) goes to zero. Thus, we have that \( \lim_{n \rightarrow \infty} \mathbb{D}_p(\tilde{f}_n) = \mathbb{D}_p(u) \). Using another application of the Interpolation inequality, one obtains that
\[
\|\tilde{f}_n - u\|_{L^\alpha} \leq \|\tilde{f}_n - u\|_{L^\alpha} \|\tilde{f}_n - u\|_{L^{2N/(N-2\alpha)}}^{1 - \frac{\alpha}{2N}} \leq C \|\tilde{f}_n - u\|_{L^\alpha}^{\alpha},
\]
where \( \lambda'' = (2N - sN + 2\alpha s) / 2\alpha s \). Thus, 
\[
\lim_{n \to \infty} \| \tilde{f}_n \|_{L^s} = \| u \|_{L^s}.
\]
Furthermore, as a consequence of the weak lower semi-continuity of the norm in a Hilbert space, we can assume, by extracting another subsequence if necessary, that \( \tilde{f}_n \rightharpoonup u \) weakly in \( H^\alpha \), and that
\[
\| u \|_{H^\alpha} \leq \liminf_{n \to \infty} \| \tilde{f}_n \|_{H^\alpha}.
\]
It follows then that
\[
J(u) \leq \lim_{n \to \infty} J(\tilde{f}_n) = J(\sigma),
\]
and since \( \tilde{f}_n \to f \) in \( L^2(\mathbb{R}^N) \), we also have that
\[
\| u \|_{L^2}^2 = \lim_{n \to \infty} \| \tilde{f}_n \|_{L^2}^2 = \sigma. \tag{2.38}
\]
By the definition of the infimum \( J(\sigma) \), we must have \( J(u) = J(\sigma) \) and \( u \in \Sigma_\sigma \). Finally, the facts \( J(u) = \lim_{n \to \infty} J(\tilde{f}_n) \), \( \| u \|_{L^s} = \lim_{n \to \infty} \| \tilde{f}_n \|_{L^s} \), \( \mathbb{D}_p(u) = \lim_{n \to \infty} \mathbb{D}_p(\tilde{f}_n) \), and \( \| u \|_{L^2} = \lim_{n \to \infty} \| \tilde{f}_n \|_{L^2} \) together imply that \( \| u \|_{H^\alpha} = \lim_{n \to \infty} \| \tilde{f}_n \|_{H^\alpha} \), and from a standard exercise in the elementary Hilbert space theory one then obtains that \( \tilde{f}_n \rightharpoonup u \) in \( H^\alpha \) norm. \( \square \)

We can now prove our main results. Since we ruled out the cases \( L = 0 \) and \( L \in (0, \sigma) \), the only option for any minimizing sequence of \( J(\sigma) \) is the compactness, i.e., \( L = \sigma \). Hence, by Lemma \( 2.18 \) the set \( P_\sigma \) is nonempty and Statement 1 of Theorem \( 2.11 \) holds.

Next suppose that \( u \in H^\alpha(\mathbb{R}^N) \) is a minimizer of \( J(\sigma) \), that is, \( \| u \|_{L^2}^2 = \sigma \) and \( J(u) = J(\sigma) \). Then it satisfies the Euler-Lagrange differential equation
\[
(-\Delta)^\alpha u - a|u|^{s-2}u - \lambda \left( |x|^{\beta-N} * |u|^p \right) |u|^{p-2}u = -\omega u \tag{2.34}
\]
for some \( \omega \in \mathbb{R} \). This gives that
\[
\|(-\Delta)^{\alpha/2}u\|_{L^2}^2 - a\|u\|_{L^s}^2 - \lambda \mathbb{D}_p(u) = -\omega \|u\|_{L^2}^2. \tag{2.35}
\]
We know that \( J(u) < 0 \). Since \( a = bs > b \) and \( \lambda = 2\mu p > \mu \), it follows that the left side of \( (2.35) \) is negative and so, \( \omega > 0 \).

We now show that \( |u|, |u|^s \in P_\sigma \) (for definition and properties about symmetric rearrangements, see, for example, Chapter 3 of \( [15] \)). Using the fact
\[
u \in H^\alpha(\mathbb{R}^N) \Rightarrow |u| \in H^\alpha(\mathbb{R}^N), \quad \|(-\Delta)^{\alpha/2}|u|\|_{L^2} \leq \|(-\Delta)^{\alpha/2}u\|_{L^2}, \tag{2.36}
\]
it follows that \( J(|u|) \leq J(u) \). Thus, \( P_\sigma \) also contains \( |u| \) and hence, the minimizer \( u \) can be chosen to be \( \mathbb{R} \)-valued. To prove \( |u|^s \in P_\sigma \), we need the following fact
\[
u \in H^\alpha(\mathbb{R}^N) \Rightarrow |u|^s \in H^\alpha(\mathbb{R}^N), \quad \|(-\Delta)^{\alpha/2}|u|^s\|_{L^2} \leq \|(-\Delta)^{\alpha/2}|u|\|_{L^2}. \tag{2.37}
\]
This is proved in Lemmas 3.4 and 3.5 of \( [5] \) for \( \alpha = 1/2 \) and such a proof for the case \( 0 < \alpha < 1 \) can be constructed by adapting the same argument. Moreover, it is well-known that the symmetric rearrangement preserve the \( L^p \) norm, i.e.,
\[
\|f|^s\|_{L^r} = \|f\|_{L^r}, \quad 1 \leq r \leq \infty. \tag{2.38}
\]
Furthermore, a classical rearrangement inequality of F. Riesz-S. L. Sobolev (see for example, Theorem 3.7 of [15]) gives
\[ \int_{\mathbb{R}^N} D_p(|u|^*, |u|^*) \, dx \, dy \geq \int_{\mathbb{R}^N} D_p(|u|, |u|) \, dx \, dy. \] (2.39)
Taking into account of (2.37), (2.38), and (2.39), it follows that
\[ \| |f|^* \|_{L^2}^2 = \| f \|_{L^2}^2 \] and \[ J(|f|^*) \leq J(f), \forall f \in H^\alpha(\mathbb{R}^N), \]
which shows that \( P_\sigma \) contains \( |u|^* \) whenever it does \( u \).

To show that \( |u| > 0 \) on \( \mathbb{R}^N \), observe that \( \tilde{u} = |u| \in \Sigma_\sigma \) satisfies the Euler-Lagrange differential equation
\[ (-\Delta)^\alpha \tilde{u} + \omega \tilde{u} = f(\tilde{u}), \quad \text{where} \quad f(\tilde{u}) = a|\tilde{u}|^{s-2}\tilde{u} + \lambda(|x|^{\beta-N} * |\tilde{u}|^p)|\tilde{u}|^{p-2}\tilde{u}. \] (2.40)
The Lagrange multiplier in (2.40) stays same because it is determined by (2.34). Since \( \omega > 0 \), we have the convolution formula
\[ \tilde{u} = \mathcal{W}_\omega \ast f(\tilde{u}) = \int_{\mathbb{R}^N} \mathcal{W}_\omega(x-x') f(\tilde{u})(x') \, dx', \]
where for any \( \tau > 0 \), the Bessel kernel \( \mathcal{W}_\tau(x) \) is given by
\[ \mathcal{W}_\tau(x) = \mathcal{F}^{-1}\left( \frac{1}{\tau + |\xi|^{2\alpha}} \right). \] (2.41)
Since \( \tilde{u} \) is the convolution of \( \mathcal{W}_\omega \) with the function \( f(\tilde{u}) \) which is nonnegative and not identically zero, it follows that \( \tilde{u} > 0 \) on \( \mathbb{R}^N \).

To prove Statement 2 of Theorem 2.1 suppose it does not hold. Then there would exist a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) and \( \varepsilon_0 > 0 \) such that
\[ \inf_{u \in P_\sigma} \inf_{y \in \mathbb{R}^N} \| f_{n_k}(\cdot + y) - u \|_{H^\alpha} \geq \varepsilon_0 \]
for all \( k \in \mathbb{N} \). But since \( \{f_{n_k}\} \) would itself enjoy being a minimizing sequence for \( J(\sigma) \). Consequently, there would exist \( \{y_k\} \subset \mathbb{R}^N \) and \( u_0 \in P_\sigma \) such that
\[ \lim_{k \to \infty} \| f_{n_k}(\cdot + y_k) - u_0 \|_{H^\alpha} = 0, \]
which is a contradiction and hence, Statement 2 follows. Because of translation invariance of the functional \( J(u) \) and the power \( \int_{\mathbb{R}^N} |u|^2 \, dx \), Statement 3 is an immediate consequence of Statement 2.

To prove the stability of the set \( P_\sigma \), suppose the contrary. Then there would exist \( \varepsilon_0 > 0 \), a sequence \( \{v_n\} \subset H^\alpha(\mathbb{R}^N) \), and times \( t_n \) enjoying
\[ \inf_{h \in P_\sigma} \| v_n - h \|_{H^\alpha} < \frac{1}{n} \] (2.42)
and
\[ \inf_{h \in P_\sigma} \| u_n(\cdot, t_n) - h \|_{H^\alpha} \geq \varepsilon_0. \]
for all \( n \), where \( u_n(x,t) \) solves the equation (2.1) with \( u_n(x,0) = v_n \). Now taking into account of the convergence \( v_n \to P_\sigma \) in \( H^\alpha(\mathbb{R}^N) \), and \( J(h) = J(\sigma), \|h\|_{L^2}^2 = \sigma \) for \( h \in P_\sigma \), we would have \( J(v_n) \to J(\sigma) \) and \( \|v_n\|_{L^2}^2 \to \sigma \). Take the numbers \( \{\zeta_n\} \subset \mathbb{R} \) such that \( \|\zeta_n v_n\|_{L^2}^2 = \sigma \) for all \( n \). Then \( \zeta_n \to 1 \). Let us denote \( w_n = \zeta_n u_n(\cdot,t_n) \). Then \( w_n \in \Sigma_\sigma \) for each \( n \) and

\[
\lim_{n \to \infty} J(w_n) = \lim_{n \to \infty} J(u_n(\cdot,t_n)) = \lim_{n \to \infty} J(v_n) = J(\sigma).
\]

Thus, \( \{w_n\} \) enjoys being a minimizing sequence of \( J(\sigma) \). In consequence, by Theorem 2.1 there would exists \( h_n \in P_\sigma \) such that \( \|w_n - h_n\|_{H^\alpha} < \varepsilon_0/2 \). But then

\[
\varepsilon_0 \leq \|u_n(\cdot,t_n) - h_n\|_{H^\alpha} \leq \|u_n(\cdot,t_n) - w_n\|_{H^\alpha} + \|w_n - h_n\|_{H^\alpha} \leq |1 - \zeta_n| \|u_n(\cdot,t_n)\|_{H^\alpha} + \frac{\varepsilon_0}{2},
\]

which after passing the limit \( n \to \infty \) yields \( \varepsilon_0 \leq \varepsilon_0/2 \), a contradiction.

### 3. Standing waves for Choquard type systems

In this section, we prove existence and stability of standing waves for the coupled fNLS system with Choquard-type nonlinearities

\[
\begin{aligned}
    i\partial_t \Psi_1 + (-\Delta)^\alpha \Psi_1 &= \lambda_1 I_\beta^p_1(\Psi_1)|\Psi_1|^{p_1-2}\Psi_1 + cI_\beta^q_2(\Psi_2)|\Psi_1|^{q-2}\Psi_1, \\
    i\partial_t \Psi_2 + (-\Delta)^\alpha \Psi_2 &= \lambda_2 I_\beta^p_2(\Psi_2)|\Psi_2|^{p_2-2}\Psi_2 + cI_\beta^q_2(\Psi_1)|\Psi_2|^{q-2}\Psi_2,
\end{aligned}
\]

where \( (x,t) \in \mathbb{R}^N \times (0,\infty) \) and \( \lambda_1, c > 0 \). Throughout this section, we assume that the following conditions hold for the powers \( \alpha, \beta, p_1, p_2, \) and \( q \).

\[
N \geq 2, \quad 0 < \alpha < 1, \quad \beta \in (0,N), \quad 2 \leq p_1, p_2, q < \frac{N + 2\alpha + \beta}{N}. \quad (3.2)
\]

A standing wave solution of (3.1) is a solution of the form

\[
(\Psi_1(x,t), \Psi_2(x,t)) = (e^{-i\omega_1 t} u_1(x), e^{-i\omega_2 t} u_2(x))
\]

for some \( \omega_1, \omega_2 \in \mathbb{R} \) and \( (u_1, u_2) \) solves the Choquard system (1.9). The associated energy functional is

\[
E(u) = \frac{1}{2} \sum_{j=1}^{2} \|(-\Delta)^{\alpha/2} u_j\|_{L^2}^2 - \sum_{j=1}^{2} \mu_j |D_{p_j}(u_j)| - \mu |D_q(u_1, u_2)|,
\]

where \( \mu_j = \lambda_j/2p_j \) for \( j = 1, 2 \) and \( \mu = c/q \). We look for the profile function \( (u_1, u_2) \) in the space \( Y^\alpha(\mathbb{R}^N) = H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \) satisfying \( \|u_1\|_{L^2}^2 = \sigma_1 \) and \( \|u_2\|_{L^2}^2 = \sigma_2 \) for given \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \).

To describe the main results of this section, let us first fix some definitions and notation. We use the notations: \( \mathbb{R}_{>0} = (0,\infty), \mathbb{R}_{\geq0} = [0,\infty), \) and similar meanings for \( \mathbb{R}_{>0}^2 \) and \( \mathbb{R}_{\geq0}^2 \). We write the ordered pairs in \( \mathbb{R}^2 \) as \( \sigma = (\sigma_1, \sigma_2), \sigma' = (\sigma'_1, \sigma'_2), \) etc.
Vectors in $Y^\alpha(\mathbb{R}^N)$ are written as $u = (u_1, u_2), f = (f_1, f_2)$, etc. A sequence $\{f_n\}_{n \geq 1}$ in $Y^\alpha(\mathbb{R}^N)$ is always understood as $f_n = (f_1^n, f_2^n)$.

For any $\sigma \in \mathbb{R}^2_{>0}$, denote by $\Sigma_\sigma = \Sigma_{\sigma_1} \times \Sigma_{\sigma_2}$ the product of $L^2(\mathbb{R}^N)$ spheres and let $M_\sigma$ denotes the set of coupled standing wave solutions

$$M_\sigma = \{ u \in Y^\alpha(\mathbb{R}^N) : u \in \Sigma_\sigma, \ E(u) = E(\sigma) \} , \ E(\sigma) := \inf_{f \in \Sigma_\sigma} E(f). \tag{3.3}$$

The following analogues of Theorem 2.1 and its Corollary 2.2 are the main results of this section.

**Theorem 3.1.** Suppose that (3.2) holds. Then, for any $\sigma \in \mathbb{R}^2_{>0}$, the set $M_\sigma$ defined in (3.3) is nonempty. Moreover, the following statements hold:

1. If $f_n \in Y^\alpha(\mathbb{R}^N), \ \|f_1^n\|_{L^2}^2 \to \sigma_1, \ \|f_2^n\|_{L^2}^2 \to \sigma_2,$ and $E(f_n) \to E(\sigma)$, then the sequence $\{f_n\}_{n \geq 1}$ is relatively compact in $Y^\alpha(\mathbb{R}^N)$ up to a translation.

2. $f_n \to M_\sigma$ in the following sense,

$$\lim_{n \to \infty} \inf_{u \in M_\sigma} \|f_n - u\|_{Y^\alpha} = 0.$$

Furthermore, the solution set $M_\alpha$ has the following properties

3. The Lagrange multiplier $(\omega_1, \omega_2)$ associated with $u = (u_1, u_2)$ on $\Sigma_\alpha$ satisfies $\omega_1 > 0$ and $\omega_2 > 0$.

4. If $(u_1, u_2) \in M_\sigma$, then $(|u_1|, |u_2|) \in M_\sigma$ and $|u_1| > 0, |u_2| > 0$ on $\mathbb{R}^N$. One also has $(|u_1|^*, |u_2|^*) \in M_\sigma$ whenever $(u_1, u_2) \in M_\sigma$.

As an immediate consequence we can get the stability result:

**Corollary 3.2.** The set $M_\sigma$ is stable in the same sense as in Corollary 2.2.

**Remark 3.3.** In Corollary 3.2, we made the assumption that for any $(\Psi_1^0, \Psi_2^0) \in Y^\alpha(\mathbb{R}^N)$ and every $T > 0$, 3.1 has a unique solution $(\Psi_1, \Psi_2) \in C([0, T], Y^\alpha(\mathbb{R}^N))$ with $(\Psi_1(x, 0), \Psi_2(x, 0)) = (\Psi_1^0(x), \Psi_2^0(x))$. The map $(\Psi_1^0, \Psi_2^0) \mapsto (\Psi_1, \Psi_2)$ is locally Lipschitz from $Y^\alpha(\mathbb{R}^N)$ to $C([0, T], Y^\alpha(\mathbb{R}^N))$. Moreover, the functional $E(\Psi_1(t), \Psi_2(t))$ and the powers $\|\Psi_1(t)\|_{L^2}^2, \ \|\Psi_2(t)\|_{L^2}^2$ are independent of $t \in [0, T]$. Moreover, the functional $E(\Psi_1(t), \Psi_2(t))$ and the powers $\|\Psi_1(t)\|_{L^2}^2, \ \|\Psi_2(t)\|_{L^2}^2$ are independent of $t \in [0, T]$.

**Remark 3.4.** In 2.2 and 3.2, one can also include $\alpha = 1$, in which case the same argument works and analogues of Theorem 2.1, Theorem 2.3, Theorem 3.1 and their corollaries remain true. We fix $\alpha \in (0, 1)$ only to avoid providing some additional technical details.

In what follows we use the following notation

$$\mathbb{F}_1(f) = \frac{1}{2}\|(-\Delta)^{\alpha/2} f\|_{L^2}^2 - \mu_1 \|D_{p_1} f\|_{L^2}^2, \ \forall f \in H^\alpha(\mathbb{R}^N), \tag{3.4}$$

$$\mathbb{F}_2(f) = \frac{1}{2}\|(-\Delta)^{\alpha/2} f\|_{L^2}^2 - \mu_2 \|D_{p_2} f\|_{L^2}^2, \ \forall f \in H^\alpha(\mathbb{R}^N).$$
We will prove Theorem 3.1 and its corollary following the same steps as used in the preceding section to prove Theorem 2.1. As in the case of one parameter problem, we first prove some preliminaries lemmas.

Analogue of Lemma 2.8 is the following.

**Lemma 3.5.** Suppose the conditions (3.2) hold. Then, for any \( \sigma \in \mathbb{R}^2 \),

(i) If \( f_n \in Y^\alpha(\mathbb{R}^N), \|f_n^1\|_{L^2} \to \sigma_1, \|f_n^2\|_{L^2}^2 \to \sigma_2, \) and \( E(f_n) \to E(\sigma) \), then the sequence \( \{f_n\}_{n \geq 1} \) is bounded in \( Y^\alpha(\mathbb{R}^N) \).

(i) One has \( E(\sigma) \in (-\infty, 0) \).

**Proof.** To prove (i), we use the estimate (2.9) to obtain

\[
\mathbb{D}_p(f_n^1) = \int_{\mathbb{R}^N} T_p^1(f_n^1)|f_n^1|^p dx \leq C\|f_n\|_{H^\alpha}^{2p_1\beta_1},
\]

\[
\mathbb{D}_p(f_n^2) = \int_{\mathbb{R}^N} T_p^2(f_n^2)|f_n^2|^p dx \leq C\|f_n\|_{H^\alpha}^{2p_2\beta_2},
\]

where \( \beta_j = (Np_j - N - \beta)/2\alpha p_j \) for \( j = 1, 2 \). Since \( \|f_n^1\|_{L^2} \) and \( \|f_n^2\|_{L^2} \) are bounded, using the Hardy-Littlewood-Sobolev inequality, the Young’s inequality, and the fractional Gagliardo-Nirenberg inequality, we have that

\[
\mathbb{D}_q(f_n^1, f_n^2) \leq C\|f_n\|_{L^{2Nq/(N+\beta)}}^{2q} \|f_n^1\|_{L^{2Nq/(N+\beta)}}^{2q} \leq C\|f_n\|_{H^\alpha}^{2q},
\]

where \( q = (Nq - N - \beta)/2\alpha q \). As in the proof of Lemma 2.8, we now write

\[
\frac{1}{2}\|f_n\|_{Y^\alpha}^2 = E(f_n) + \sum_{j=1}^2 \mu_j \mathbb{D}_p(f_n^j) + \mu \mathbb{D}_q(f_n) + \frac{1}{2}\|f_n^1\|_{L^2}^2 + \frac{1}{2}\|f_n^2\|_{L^2}^2.
\]

Since the sequence of numbers \( \{E(f_n)\}_{n \geq 1} \) is bounded, using the estimates (3.5) and (3.6), we obtain that

\[
\frac{1}{2}\|f_n\|_{Y^\alpha}^2 \leq C \left(1 + \|f_n\|_{H^\alpha}^{2p_1\beta_1} + \|f_n\|_{H^\alpha}^{2p_2\beta_2} + \|f_n\|_{H^\alpha}^{2q}\right).
\]

Since \( 2p_j\beta_j < 2 \) for \( j = 1, 2 \) and \( 2\alpha q < 2 \), it follows from (3.7) that the sequence \( \{f_n\}_{n \geq 1} \) is bounded in \( Y^\alpha(\mathbb{R}^N) \). The statement \( E(\sigma) > -\infty \) can be easily proved using the estimates (3.5) and (3.6).

To see \( E(\sigma) < 0 \), we use the fact that \( \mu > 0 \) to obtain

\[
E(\sigma) \leq E(\sigma_1, 0) + E(0, \sigma_2).
\]

Since \( \sigma_1, \sigma_2 \in \mathbb{R}_{>0} \), as in Lemma 2.8 we have that \( E(\sigma_1, 0) < 0 \) and \( E(0, \sigma_2) < 0 \) and hence, \( E(\sigma) < 0 \). \( \square \)

Analogue of Lemma 2.10 holds in the present context without change of statement and an obvious modification in the proof. One also has the following
Lemma 3.6. For any $\sigma \in \mathbb{R}^2$, suppose that $f_n \in Y^\alpha(\mathbb{R}^N)$, $\|f_1^n\|_{L^2}^2 \to \sigma_1$, $\|f_2^n\|_{L^2}^2 \to \sigma_2$, and $E(f_n) \to E(\sigma)$. Then there exist numbers $\delta_1 > 0, \delta_2 > 0$, and $n_0 = n_0(\delta_j) \in \mathbb{N}$ such that for all $n \geq n_0$,
\[ F_j(f_j^n) - \mu D_q(f_n) \leq -\delta_j, \quad j = 1, 2. \]

Proof. Suppose to the contrary that there is a sequence $\{f_n\}_{n \geq 1}$ in $Y^\alpha(\mathbb{R}^N)$ satisfying the hypotheses of the lemma and that
\[ \liminf_{n \to \infty} (F_1(f_1^n) - \mu D_q(f_n)) \geq 0. \]
This implies that
\[ E(\sigma) = \lim_{n \to \infty} E(f_n) \geq \liminf_{n \to \infty} F_2(f_2^n). \]
Using Lemma 2.1, let $\bar{u}_2 \in H^\alpha(\mathbb{R}^N)$ be such that $E_2(\bar{u}_2) = \inf_{f \in \Sigma_{\sigma}} E_2(f)$. Then (3.8) gives that $E(\sigma) \geq E(0, \bar{u}_2)$. To deduce a contradiction, let $\bar{u}_1 \in \Sigma_{\sigma}$, be such that $E_1(\bar{u}_1) - \mu D_q(\bar{u}_1) \bar{u}_2 = 0$. Then it follows that
\[ E(\sigma) \leq E(\bar{u}_1, \bar{u}_2) = E_1(\bar{u}_1) - \mu D_q(\bar{u}_1) \bar{u}_2 + E(0, \bar{u}_2) < E(0, \bar{u}_2), \]
which is a contradiction. The proof of the statement involving $f_2^n$ follows the same lines and we will not repeat it.

For any $\sigma \in \mathbb{R}^2$, let $\{(f_1^n, f_2^n)\}_{n \geq 1}$ be any minimizing sequence for $E(\sigma)$. We now employ the concentration compactness principle to the sequence of non-negative functions $\sigma_n = |f_1^n|^2 + |f_2^n|^2$. As in the preceding case, consider the associated concentration function $P_n(R)$ defined by
\[ P_n(R) = \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \sigma_n \, dx, \quad n = 1, 2, \ldots, \text{ and } R > 0. \]
In what follows, we continue to denote $f_n = (f_1^n, f_2^n)$. Suppose that evanescence of the energy minimizing $\{f_n\}_{n \geq 1}$ occurs, that is, $\lim_{n \to \infty} P_n(R) = 0$ for all $R > 0$. Then, as before, we see from Lemma 2.11 that $D_{p_j}(f_2^n) \leq C \|f_2^n\|_{L^{2Np_j/(N+p)}}^{2p_j} \to 0$ as $n \to \infty$ for $j = 1, 2$. Using (3.6) and Lemma 2.11 it also follows that $D_q(f_2^n) \to 0$ as $n \to \infty$. Then, we obtain that
\[ E(\sigma) = \lim_{n \to \infty} E(f_n) \geq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} f_1^n|^2 + |(-\Delta)^{\alpha/2} f_2^n|^2 \right) \, dx \geq 0, \]
which contradicts $E(\sigma) < 0$. Let us denote
\[ M = \lim_{R \to \infty} \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \sigma_n \, dx \right) \in [0, \sigma_1 + \sigma_2]. \]
Thus, we have established the following lemma.

Lemma 3.7. If $\{f_n\} \subset Y^\alpha(\mathbb{R}^N)$ be any minimizing sequence for $E(\sigma)$ and $M$ be as defined in (3.9), then $M > 0$. 

Next, we rule out the possibility of the case $M \in (0, \sigma_1 + \sigma_2)$. Analogue of Lemma 2.14 is the following.

**Lemma 3.8.** For some subsequence of $\{f_n\}_{n \geq 1}$, which we denote by the same, the following are true. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and the sequences $\{(v_1^n, v_2^n)\}_{n \geq 1}$ and $\{(w_1^n, w_2^n)\}_{n \geq 1}$ of functions in $Y^u(\mathbb{R}^N)$ such that for every $n \geq n_0$,

1. $\left| \int_{\mathbb{R}^N} (|v_1^n|^2 + |v_2^n|^2) \, dx - M \right| < \varepsilon$
2. $\left| \int_{\mathbb{R}^N} (|w_1^n|^2 + |w_2^n|^2) \, dx - ((\sigma_1 + \sigma_2) - M) \right| < \varepsilon$
3. $E(f_n) \geq E(v_1^n, v_2^n) + E(w_1^n, w_2^n) - C\varepsilon$.

**Proof.** As in the proof of Lemma 2.14, we choose $R \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$M - \varepsilon < P_n(R) \leq P_n(2R) < M + \varepsilon.$$ 

Define the sequences $\{(v_1^n, v_2^n)\}$ and $\{(w_1^n, w_2^n)\}$ as follows

$$(v_1^n, v_2^n) = \phi_R(x - y_n)f_n(x), \quad (w_1^n, w_2^n) = \psi_R(x - y_n)f_n(x), \quad x \in \mathbb{R}^N,$$

where $\phi$ and $\psi$ are as in the proof of Lemma 2.14 and $y_n \in \mathbb{R}^N$ is chosen so that (2.16) holds with $\rho_n$ replaced by $\sigma_n$ and $L$ replaced by $M$. Since the sequences $\{v_1^n\}_{n \geq 1}$, $\{v_2^n\}_{n \geq 1}$, $\{w_1^n\}_{n \geq 1}$, and $\{w_2^n\}_{n \geq 1}$ are all bounded in $L^2(\mathbb{R}^N)$, so there exist $\sigma_1' \in [0, \sigma_1]$ and $\sigma_2' \in [0, \sigma_2]$ such that up to a subsequence $\|v_1^n\|^2_{L^2} \to \sigma_1'$ and $\|v_2^n\|^2_{L^2} \to \sigma_2'$, whence one also has $\|w_1^n\|^2_{L^2} \to \sigma_1 - \sigma_1'$ and $\|w_2^n\|^2_{L^2} \to \sigma_2 - \sigma_2'$. Then, in view of (2.16), Statements 1 and 2 of Lemma 3.8 follow.

Analogue of the inequality (2.17) holds for $\tilde{\phi}_Rf_1^n$, $\tilde{\phi}_Rf_2^n$, $\tilde{\psi}_Rf_1^n$, and $\tilde{\psi}_Rf_2^n$. The inequality (2.19) takes the form

$$E(v_n) + E(w_n) \leq E(f_n) + \sum_{j=1}^2 F^\phi_{\mu_j}[f_j^n] + F^\psi_{\mu}[f_n] + C\varepsilon,$$  

(3.10) where $F^\phi_{\mu}[f]$ is as defined in the proof of Lemma 2.14. An estimate similar to (2.21) holds in the present context as well. Then, Statement 3 follows from (3.10).  

**Lemma 3.9.** For every $\sigma \in \mathbb{R}^2_{>0}$, there exists the numbers $\sigma_1' \in [0, \sigma_1]$ and $\sigma_2' \in [0, \sigma_2]$ such that $M = \sigma_1' + \sigma_2'$ and

$$E(\sigma) \geq E(\sigma') + E(\sigma - \sigma').$$  

(3.11)

**Proof.** With the notation of Lemma 3.8 and its proof, for any $\varepsilon > 0$, since $\{E(v_n)\}_{n \geq 1}$ and $\{E(w_n)\}_{n \geq 1}$ are bounded, we can assume that $E(v_n) \to \Lambda_1$ and $E(w_n) \to \Lambda_2$. Statement 3 of Lemma 3.8 then implies that

$$\Lambda_1 + \Lambda_2 \leq E(\sigma) + C\varepsilon.$$
Next, for every \( m \in \mathbb{N} \), choose the sequences \( \{(v_{1}^{n,m}, v_{2}^{n,m})\} \) and \( \{(w_{1}^{n,m}, w_{2}^{n,m})\} \) in \( Y^{\alpha}(\mathbb{R}^{N}) \) such that for \( j = 1, 2 \),
\[
\|v_{j}^{n,m}\|_{L^{2}}^{2} \rightarrow \sigma_{j}'(m), \quad E(v_{1}^{n,m}, v_{2}^{n,m}) = \Lambda_{1}(m),
\]
\[
\|w_{j}^{n,m}\|_{L^{2}}^{2} \rightarrow \sigma_{j}'(m), \quad E(w_{1}^{n,m}, w_{2}^{n,m}) = \Lambda_{2}(m),
\]
where \( \sigma_{j}'(m) \) is in \([0, \sigma_{1}], \sigma_{2}'(m) \) is in \([0, \sigma_{2}] \),
\[
|\sigma_{1}'(m) + \sigma_{2}'(m) - M| \leq \epsilon, \quad \text{and} \quad \Lambda_{1}(m) + \Lambda_{2}(m) \leq E(\sigma) + \frac{1}{m}. \tag{3.12}
\]

We may further extract a subsequence and assume that \( \sigma_{j}'(m) \rightarrow \sigma_{j}' \in [0, \sigma_{1}], \sigma_{2}'(m) \rightarrow \sigma_{2}' \in [0, \sigma_{2}] \), \( \Lambda_{1}(m) \rightarrow \Lambda_{1} \), and \( \Lambda_{2}(m) \rightarrow \Lambda_{2} \). Moreover, after redefining \( v_{n} \) and \( w_{n} \) to be diagonal entries \( v_{n} = (v_{1}^{n,n}, v_{2}^{n,n}) \) and \( w_{n} = (w_{1}^{n,n}, w_{2}^{n,n}) \), we can say that \( \|v_{j}'\|_{L^{2}}^{2} \rightarrow \sigma_{j}' \),
\[
\|w_{j}'\|_{L^{2}}^{2} \rightarrow \sigma_{j}' - \sigma_{j}'', \quad E(v_{n}) \rightarrow \Lambda_{1}, \quad \text{and} \quad E(w_{n}) \rightarrow \Lambda_{2}.
\]
Now, letting \( m \) tend to infinity on both sides of the first inequality in \( (3.12) \), we deduce that \( M = \sigma_{1}' + \sigma_{2}' \). Next, we claim that
\[
\Lambda_{1} \geq E(\sigma') \quad \text{and} \quad \Lambda_{2} \geq E(\sigma - \sigma'). \tag{3.13}
\]
To see \( \Lambda_{1} \geq E(\sigma') \), suppose first that \( \sigma_{1}' > 0 \) and \( \sigma_{2}' > 0 \). Define
\[
\beta_{1}^{n} = \frac{(\sigma_{1}')^{1/2}}{\|v_{1}^{n}\|_{L^{2}}} \quad \text{and} \quad \beta_{2}^{n} = \frac{(\sigma_{2}')^{1/2}}{\|v_{2}^{n}\|_{L^{2}}}.
\]
Then, we get \( E(\beta_{1}^{n}v_{1}^{n}, \beta_{2}^{n}v_{2}^{n}) \geq E(\sigma') \). Since \( \beta_{1}^{n} \rightarrow 1 \) and \( \beta_{2}^{n} \rightarrow 1 \) as \( n \rightarrow \infty \), one has that \( E(\beta_{1}^{n}v_{1}^{n}, \beta_{2}^{n}v_{2}^{n}) \rightarrow \Lambda_{1} \) and hence, \( \Lambda_{1} \geq E(\sigma') \). Suppose now that \( \sigma_{1}' = 0 \) and \( \sigma_{2}' > 0 \) (the same argument applies for the case \( \sigma_{1}' > 0 \) and \( \sigma_{2}' = 0 \)). Then, using the Hardy-Littlewood-Sobolev and interpolation inequalities, one can see that \( D_{p_{1}}(v_{1}^{n}) \rightarrow 0 \) and \( D_{q}(v_{1}^{n}, v_{2}^{n}) \rightarrow 0 \) as \( n \rightarrow \infty \). Consequently, we get
\[
\Lambda_{1} = \lim_{n \rightarrow \infty} E(v_{n}) = \lim_{n \rightarrow \infty} \left( F_{2}(v_{2}^{n}) + \int_{\mathbb{R}^{N}} \left| (-\Delta)^{n/2}v_{1}^{n}\right|^{2} dx \right) \geq E(0, \sigma_{2}').
\]
This finishes the proof of \( \Lambda_{1} \geq E(\sigma') \). The proof of \( \Lambda_{2} \geq E(\sigma - \sigma') \) uses the same arguments and so will not be repeated here. Finally, with \( (3.13) \) in hand, \( (3.11) \) follows from the second inequality of \( (3.12) \).

Analogue of Lemma 2.16 is the following.

**Lemma 3.10.** For any minimizing sequence \( \{f_{n}\}_{n \geq 1} \) of \( E(\sigma) \), let \( M \) be defined by \( (3.19) \). Then \( M \) satisfies \( M \notin (0, \sigma_{1} + \sigma_{2}) \).

**Proof.** To rule out the dichotomy, suppose to the contrary that \( M \in (0, \sigma_{1} + \sigma_{2}) \). Let \( \sigma' \) be as in Lemma 3.9 and denote \( \sigma'' = \sigma - \sigma' \). Lemma 3.9 implies that \( E(\sigma) \geq E(\sigma') + E(\sigma - \sigma') \), which is same as
\[
E(\sigma' + \sigma'') \geq E(\sigma') + E(\sigma''). \tag{3.14}
\]
We now deduce a contradiction. Since $\sigma', \sigma'' \in \mathbb{R}_{>0}$ with $\sigma', \sigma'' \neq \{0\}$ and $\sigma' + \sigma'' \in \mathbb{R}_{>0}$, we consider the following cases: (i) $\sigma', \sigma'' \in \mathbb{R}_{>0}$; (ii) $\sigma' \in \mathbb{R}_{>0}$ and $\sigma'' \in \{0\} \times \mathbb{R}_{>0}$; (iii) $\sigma' \in \mathbb{R}_{>0} \times \{0\}$ and $\sigma'' \in \mathbb{R}_{>0}^2$; and (iv) $\sigma' \in \mathbb{R}_{>0} \times \{0\}$ and $\sigma'' \in \{0\} \times \mathbb{R}_{>0}$. All other cases coincide with one of these cases after switching the roles of $\sigma'$ and $\sigma''$. We consider each case separately.

**Case 1.** $\sigma', \sigma'' \in \mathbb{R}_{>0}^2$. Let $\{(z_n^1, z_n^2)\}_{n \geq 1}$ and $\{(w_n^1, w_n^2)\}_{n \geq 1}$ be sequences in $Y^\alpha(\mathbb{R}^N)$ such that 
\[
\|z_n^1\|_{L^2}^2 \to \sigma_1', \quad \|z_n^2\|_{L^2}^2 \to \sigma_2', \quad E(z_n^1, z_n^2) \to E(\sigma'), \\
\|w_n^1\|_{L^2}^2 \to \sigma_1'', \quad \|w_n^2\|_{L^2}^2 \to \sigma_2'', \quad E(w_n^1, w_n^2) \to E(\sigma'').
\]

To deduce a contradiction in this case, consider the sequences $\{(e_n^1, e_n^2)\}$ and $\{(d_n^1, d_n^2)\}$ in $\mathbb{R}^2$ defined as follows 
\[
e_1^n = \frac{1}{\|z_n^1\|_{L^2}^2} \left( \mathbb{F}_1(z_n^1) - \mu \int_{\mathbb{R}^N} D_q(|z_n^1|, |z_n^2|) \, dxdy \right),
\]
\[
d_1^n = \frac{1}{\|z_n^2\|_{L^2}^2} \mathbb{F}_1(z_n^2),
\]
\[
e_2^n = \frac{1}{\|w_n^1\|_{L^2}^2} \left( \mathbb{F}_2(w_n^1) - \mu \int_{\mathbb{R}^N} D_q(|w_n^1|, |w_n^2|) \, dxdy \right),
\]
\[
d_2^n = \frac{1}{\|w_n^2\|_{L^2}^2} \mathbb{F}_2(w_n^2).
\]

Then, we can assume that $(e_n^1, e_n^2) \to (e_1, e_2)$ in $\mathbb{R}^2$ and $(d_n^1, d_n^2) \to (d_1, d_2)$ in $\mathbb{R}^2$. As in the proof of (2.24), three cases may arise: $e_1 < e_2, e_1 > e_2$, and $e_1 = e_2$. Suppose that $e_1 < e_2$. Without loss of generality, we may assume that $z_n^1, z_n^2, w_n^1,$ and $w_n^2$ are $\mathbb{R}$-valued, non-negative, and have compact supports. Let $\nu$ be a unit vector in $\mathbb{R}^N$ and define the function $f_n^2$ as follows 
\[
f_n^2(\cdot) = z_n^2(\cdot - b_n\nu), \quad f_n^2 = z_n^2 + w_n^2,
\]
where $b_n$ is such that $z_n^2$ and $w_n^2$ have disjoint supports. Then $\|f_n^2\|_{L^2}^2 \to \sigma_1' + \sigma_2'$. Define $\{f_1^n\} \subset H^\alpha(\mathbb{R}^N)$ as $f_1^n = Tz_n^1$, where $T = 1 + \sigma''/\sigma_1'$ and put $F_n = (f_1^n, f_2^n)$. Then, we have that 
\[
E(\sigma' + \sigma'') \leq \lim_{n \to \infty} E(F_n). \tag{3.15}
\]

Now, since $\mu > 0$ and $q-2 > 0$, using the same argument as in the proof of Lemma 2.16 we deduce that 
\[
\mathbb{F}_1(f_1^n) - \mu \mathbb{D}_q(F_n) \leq \mathbb{F}_1(f_1^n) - \mu \int_{\mathbb{R}^N} D_q(|f_1^n|, |z_n^2|) \, dxdy \leq T \left( \mathbb{F}_1(z_n^1) - \mu \int_{\mathbb{R}^N} D_q(|z_n^1|, |z_n^2|) \, dxdy \right).
\]

Letting $n$ tend to $+\infty$ on both sides of this last inequality, we get 
\[
\lim_{n \to \infty} (\mathbb{F}_1(f_1^n) - \mu \mathbb{D}_q(F_n)) \leq T \sigma_1' e_1 = \sigma_1' e_1 + \sigma'' e_1. \tag{3.16}
\]
Using (3.16) and the assumption $e_1 < e_2$, we see from (3.15) that

$$E(\sigma' + \sigma'') \leq \sigma'_1 e_1 + \sigma''_1 e_1 + d_1 \sigma'_2 + d_2 \sigma''_2$$

$$= E(\sigma') + \sigma'_1 e_1 + d_2 \sigma''_2 < E(\sigma') + E(\sigma''),$$

which contradicts (3.14). The proof in the case $e_1 > e_2$ uses the same argument and so will not be repeated. Suppose now that $e_1 = e_2$. Two subcases may arise: $d_1 \leq d_2$ or $d_1 \geq d_2$. Since both subcases use the same arguments, we only consider the subcase $d_1 \leq d_2$. Let $f^n_1$ be as in the preceding paragraph and define $f^n_2$ by $f^n_2 = S z^n_2$, where $S = 1 + \sigma''/\sigma'_2$. Then, using the second part of Lemma 2.10, we can find \( \delta > 0 \) such that

$$\mathbb{P}_2(f^n_2) \leq S \mathbb{P}_2(z^n_2) - \delta.$$  \hspace{1cm} (3.17)

Using the same argument as in the proof of Lemma 2.16, we also have

$$\mathbb{F}_1(f^n_1) - \mu \mathbb{D}_q(f^n_1, f^n_2) \leq T(\mathbb{F}_1(z^n_1) - \mu \int_{\mathbb{R}^N} D_q(|z^n_1|, |z^n_2|) \, dx \, dy).$$ \hspace{1cm} (3.18)

Now, since $\|f^n_1\|_{L^2} \to \sigma'_1 + \sigma''_1$ and $\|f^n_2\|_{L^2} \to \sigma'_2 + \sigma''_2$, as a consequence of inequalities (3.17) and (3.18), one obtains that

$$E(\sigma' + \sigma'') \leq \lim_{n \to \infty} E(f^n_1, f^n_2) \leq T \sigma'_1 e_1 + S \sigma'_2 d_1 - \delta$$

$$= e_1 \sigma'_1 + \sigma'_2 d_1 + \frac{\sigma''_1}{\sigma'_1} e_1 \sigma'_1 + \frac{\sigma''_2}{\sigma'_2} \sigma'_2 d_1 - \delta = E(\sigma') + \frac{\sigma''_1}{\sigma'_1} e_1 + \frac{\sigma''_2}{\sigma'_2} \sigma'_2 d_1 - \delta.$$

Using $e_1 = e_2$ and $d_1 \leq d_2$, this last inequality implies that

$$E(\sigma' + \sigma'') \leq E(\sigma') + E(\sigma'') - \delta,$$

which gives, $E(\sigma' + \sigma'') < E(\sigma') + E(\sigma'')$, which again contradicts (3.14).

**Case 2.** $\sigma' \in \mathbb{R}^2_0$ and $\sigma'' \in \{0\} \times \mathbb{R}_{>0}$. Let \( \{(z^n_1, z^n_2)\} \) and \( \{(w^n_1, w^n_2)\} \) be sequences in $Y^\alpha(\mathbb{R}^N)$ such that

$$\|z^n_1\|_{L^2} \to \sigma'_1, \|z^n_2\|_{L^2} \to \sigma'_2, E(z^n_1, z^n_2) \to E(\sigma'),$$

$$\|w^n_1\|_{L^2} \to 0, \|w^n_2\|_{L^2} \to \sigma''_2, E(w^n_1, w^n_2) \to E(\sigma'').$$

Define \( \{(K^n_1, K^n_2)\}_{n \geq 1} \subset \mathbb{R}^2 \) as follows

$$K^n_1 = \frac{1}{\|w^n_2\|_{L^2}} \left( \mathbb{F}_2(w^n_2) - \mu \int_{\mathbb{R}^N} D_q(|z^n_1|, |w^n_2|) \, dx \, dy \right),$$

$$K^n_2 = \frac{1}{\|z^n_2\|_{L^2}} \left( \mathbb{F}_2(z^n_2) - \mu \int_{\mathbb{R}^N} D_q(|z^n_1|, |z^n_2|) \, dx \, dy \right).$$

Then, as before, we can assume that \( (K^n_1, K^n_2) \to (K_1, K_2) \) in \( \mathbb{R}^2 \). Three cases may arise: $K_1 < K_2, K_1 > K_2$, and $K_1 = K_2$. Suppose that $K_1 < K_2$. Let $f^n_2 = B w^n_2$, where $B = 1 + \sigma'_2/\sigma''_2$. Then

$$E(\sigma' + \sigma'') \leq \lim_{n \to \infty} E(z^n_1, f^n_2).$$  \hspace{1cm} (3.19)
Using the same argument as in the proof of Lemma 2.16, we can show that
\[ \mathbb{F}_2(f_2^n) - \mu \mathbb{D}_q(z_1^n, f_2^n) \leq B \left( \mathbb{F}_2(w_2^n) - \mu \iint_{\mathbb{R}^N} D_q(|z_1^n|, |w_2^n|) \, dx \, dy \right). \]
Using this and the assumption \( K_1 < K_2 \) into (3.19), one obtains that
\[ E(\sigma' + \sigma'') \leq \lim_{n \to \infty} \mathbb{F}_1(z_1^n) + B \lim_{n \to \infty} \left( \mathbb{F}_2(w_2^n) - \mu \iint_{\mathbb{R}^N} D_q(|z_1^n|, |w_2^n|) \, dx \, dy \right) \]
\[ = E(\sigma'') + \lim_{n \to \infty} \mathbb{F}_1(z_1^n) + \sigma'_1 K_1 < E(\sigma'') + E(\sigma'), \]
which is a contradiction. The case \( K_1 > K_2 \) also leads to a contradiction with the same argument. Finally, consider the case \( K_1 = K_2 \). Let \( f_2^n = Bw_2^n \) as above. Then we have that
\[ E(\sigma' + \sigma'') \leq \lim_{n \to \infty} E(z_1^n, f_2^n). \] (3.20)
Making use of Lemma 2.10 and using the same argument that we used in the case \( \epsilon_1 = \epsilon_2 \) before, it follows that
\[ E(z_1^n, f_2^n) \leq \mathbb{F}_1(z_1^n) + B \mathbb{F}(w_2^n) - \mu B \iint_{\mathbb{R}^N} D_q(|z_1^n|, |w_2^n|) \, dx \, dy - \delta. \]
Using this and \( K_1 = K_2 \) into (3.20), we obtain that
\[ E(\sigma' + \sigma'') \leq E(\sigma'') + \lim_{n \to \infty} \mathbb{F}_1(z_1^n) + \frac{\sigma'_2}{\sigma''_2} K_1 \sigma''_2 - \delta = E(\sigma'') + E(\sigma') - \delta, \]
which gives \( E(\sigma' + \sigma'') < E(\sigma'') + E(\sigma') \), this contradicts (3.14).

**Case 3.** \( \sigma' \in \mathbb{R}_{>0} \times \{0\} \) and \( \sigma'' \in \mathbb{R}_{>0}^2 \). The proof in this case uses the same argument as in the Case 2 and so will not be repeated here.

**Case 4.** \( \sigma' \in \mathbb{R}_{>0} \times \{0\} \) and \( \sigma'' \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \). In this case, we have that \( E(\sigma') = J_1(\sigma'_1) \) and \( E(\sigma'') = J_2(\sigma''_2) \), where
\[ J_1(\sigma'_1) = \inf_{f \in \Sigma_{\sigma'_1}} J(f)|_{(a, \lambda, \rho) = (0, \lambda_1, \rho_1)} \quad \text{and} \quad J_2(\sigma''_2) = \inf_{f \in \Sigma_{\sigma''_2}} J(f)|_{(a, \lambda, \rho) = (0, \lambda_2, \rho_2)}. \]
For \( \sigma'_1 > 0 \) and \( \sigma''_2 > 0 \), let \( u' \) and \( u'' \) be such that \( J_1(\sigma'_1) = J(u')|_{(a, \lambda, \rho) = (0, \lambda_1, \rho_1)} \) and \( J_2(\sigma''_2) = J(u'')|_{(a, \lambda, \rho) = (0, \lambda_2, \rho_2)} \). Then, it is obvious that \( \mathbb{D}_q(u', u'') > 0 \) and
\[ E(\sigma'_1, \sigma''_2) < J(u')|_{(a, \lambda, \rho) = (0, \lambda_1, \rho_1)} + J(u'')|_{(a, \lambda, \rho) = (0, \lambda_2, \rho_2)} = J_1(\sigma'_1) + J_2(\sigma''_2), \]
which again contradicts (3.14). ☐

Analogue of Lemma 2.18 hold in the present context without change of statement and an obvious modification in the proof. Thus, all the preliminaries for the proof of Theorem 3.1 and its corollary have been established. The proofs of Theorem 3.1 except Statements 3 and 4, and its Corollary are now standard and so will not be repeated here.
To prove Statement 3 of Lemma 3.1 let \((u_1, u_2) \in M_\sigma\) for any \(\sigma \in \mathbb{R}_{>0}\). Then there exists a pair \(\omega = (\omega_1, \omega_2) \in \mathbb{R}^2\) such that \((\omega, u_1, u_2)\) solves the Euler-Lagrange differential equations

\[
\begin{aligned}
(-\Delta)^{\alpha\beta} u_1 + \omega_1 u_1 &= \lambda_1 (\mathcal{K}_\beta \ast |u_1|^p) |u_1|^{p-2} u_1 + c (\mathcal{K}_\beta \ast |u_2|^q) |u_1|^{q-2} u_1, \\
(-\Delta)^{\alpha\beta} u_2 + \omega_2 u_2 &= \lambda_2 (\mathcal{K}_\beta \ast |u_2|^p) |u_2|^{p-2} u_2 + c (\mathcal{K}_\beta \ast |u_1|^q) |u_2|^{q-2} u_2,
\end{aligned}
\]

where \(\mathcal{K}_\beta(x) = |x|^{\beta-N}\) for \(x \in \mathbb{R}^N\). The first equation of (3.21) gives

\[
||(-\Delta)^{\alpha/2} u_1||^2_{L^2} - \lambda_1 \mathbb{D}_{p_1}(u_1) - c \mathbb{D}_q(u_1, u_2) = -\omega_1 ||u_1||^2_{L^2}
\]

Applying Lemma 3.6 for \((f_1^n, f_2^n) = (u_1, u_2)\), we have that

\[
||(-\Delta)^{\alpha/2} u_1||^2_{L^2} - \mu_1 \mathbb{D}_{p_1}(u_1) - \mu \mathbb{D}_q(u_1, u_2) < 0.
\]

Since \(\lambda_1 = 2\mu_1 p_1 > \mu_2\) and \(c = \mu q > \mu\), it follows from (3.22) that \(-\omega_1 \sigma_1 < 0\). This implies that \(\omega_1 > 0\). Similarly, we have that \(\omega_2 > 0\).

The proofs that \((|u_1|, |u_2|) \in M_\sigma\) and \((|u_1|^*, |u_2|^*) \in M_\sigma\) whenever \(u \in M_\sigma\) follow from the facts (2.39)–(2.39). To show that \(|u_1| > 0\) and \(|u_2| > 0\) on \(\mathbb{R}^N\), denote \(\bar{u}_1 = |u_1|\) and \(\bar{u}_2 = |u_2|\). Then, the function \((\bar{u}_1, \bar{u}_2)\) satisfies the system of the form

\[
\begin{aligned}
(-\Delta)^{\alpha\beta} \bar{u}_1 + \omega_1 \bar{u}_1 &= f_1(\bar{u}_1, \bar{u}_2) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^{\alpha\beta} \bar{u}_2 + \omega_2 \bar{u}_2 &= f_2(\bar{u}_1, \bar{u}_2) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

where \((\omega_1, \omega_2)\) is the same pair of numbers as in (3.21). Since \(\omega_1 > 0\) and \(\omega_2 > 0\), we have the convolution representation

\[
\begin{aligned}
\bar{u}_1 = \mathcal{W}_1 \ast f_1(\bar{u}_1, \bar{u}_2) &= \int_{\mathbb{R}^N} \mathcal{W}_1(x - x') f_1(\bar{u}_1, \bar{u}_2)(x') \, dx', \\
\bar{u}_2 = \mathcal{W}_2 \ast f_2(\bar{u}_1, \bar{u}_2) &= \int_{\mathbb{R}^N} \mathcal{W}_2(x - x') f_2(\bar{u}_1, \bar{u}_2)(x') \, dx',
\end{aligned}
\]

where \(\mathcal{W}_x(x)\) is as defined in (2.41). Since the functions \(f_1, f_2\) are everywhere nonnegative and not identically zero, it follows that \(\bar{u}_1 > 0\) and \(\bar{u}_2 > 0\) on \(\mathbb{R}^N\). This concludes the proof of Statement 4.

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