Ulam Floating Functions

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Abstract
We extend the notion of Ulam floating sets from convex bodies to Ulam floating functions. We use the Ulam floating functions to derive a new variational formula for the affine surface area of log-concave functions.

Keywords Affine surface area · Floating body · Log-concave functions · Ulam floating function · Ulam floating set

Mathematics Subject Classification 52A20 · 52A38 · 52A41

1 Introduction

The study of affine surface area was initiated by Blaschke [4] for smooth convex bodies in Euclidean space of dimension two and three, and extended to \( \mathbb{R}^n \) and gen-
eral convex bodies by Leichtweiss [16], Lutwak [18], and Schütt and Werner [27].

The affine surface area has remarkable properties. Aside from affine invariance and translation invariance, we mention only the affine isoperimetric inequality which is a powerful tool in locating extremizers. Therefore it is not surprising that the affine surface area has proved to be useful in many problems, e.g., Plateau problems [32–34], the approximation theory of convex bodies by polytopes [5, 11, 20, 21, 26, 28], and affine curvature flow [2, 14, 15, 30, 31].

Blaschke used Dupin’s [9] notion of the floating body for his definition of the affine surface area. Dupin’s floating body needs not to be convex. Therefore, Schütt and Werner in [27] used the (convex) floating body, introduced independently in [3, 27], in their definition of the affine surface area.

An isomorphic variant of the (convex) floating body is the metronoid introduced by Huang and Slomka [12]. The metronoid is also called the Ulam floating body as it is intimately related to Ulam’s long-standing floating body problem which asks whether Euclidean balls are the only convex bodies that float in equilibrium in any orientation. A negative answer to this problem was recently given by Ryabogin [24]. A close connection between Ulam floating bodies and the affine surface area was proved by Huang et al. [13].

In recent years, considerable effort has been devoted to develop a geometric theory of log-concave functions. A major goal in this area is to extend notions from convex geometry to a functional setting. For the functional analog of the (convex) floating body, this was achieved by Li et al. [17]. They introduced the notion of floating functions and used it to define an affine surface area for log-concave functions $f = e^{-\psi}$, $\psi$ convex, as follows [17]:

$$\text{as}(f) = \int_{\mathbb{R}^n} \left( \det(\nabla^2 \psi(x)) \right)^{\frac{1}{n+2}} e^{-\psi(x)} \, dx,$$

(1)

where $\det(\nabla^2 \psi(x))$ denotes the determinant of $\nabla^2 \psi$, the Hessian of the convex function $\psi$. In Sect. 2, we explain why it is natural to call this expression affine surface area of $f$. A slightly different definition of the affine surface area for log-concave functions was given in [8].

It is thus natural to ask whether the notion of Ulam floating body can also be extended to a functional setting and whether a connection to the functional affine surface area can be established. This is carried out in this paper.

In Sect. 3, we introduce the Ulam floating set $M_\delta(C)$ for an unbounded convex set $C$ in the same way as for a bounded set. We apply that to the epigraph $\text{epi}(\psi)$ of a convex function, which is the unbounded set we are interested in. It is easy to see that there is a unique convex function $M_\delta(\psi) : \mathbb{R}^n \to \mathbb{R}$ such that $\text{epi}(M_\delta(\psi)) = M_\delta(\text{epi}(\psi))$. This leads to the definitions of Ulam floating functions $M_\delta(\psi)$ and $U_\delta(f) = e^{-M_\delta(\psi)}$ for a convex function $\psi$ and a log-concave function $f = e^{-\psi}$. Taking the right derivatives of the integral difference of a log-concave function and its Ulam floating function gives rise to the affine surface area of the log-concave function. This is the main result in this paper. One of the difficulties when dealing with functions instead of convex bodies is that the convex epigraph of a convex function is not necessarily bounded anymore.
Theorem 1.1 Let \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function such that \( 0 < \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx < \infty \). Then

\[
\lim_{\delta \to 0^+} \left( \delta - \frac{2}{\pi^2} \int_{\mathbb{R}^n} (e^{-\psi(x)} - e^{-M_\delta(\psi)(x)}) \, dx \right) = \lim_{\delta \to 0^+} \left( \delta - \frac{2}{\pi^2} \int_{\mathbb{R}^n} |M_\delta(\psi)(x) - \psi(x)| e^{-\psi(x)} \, dx \right) = c_{n+1} \int_{\mathbb{R}^n} (\det(\nabla^2 \psi(x)))^{\frac{1}{n+2}} e^{-\psi(x)} \, dx,
\]

where \( dx \) is the Lebesgue measure on \( \mathbb{R}^n \) and \( c_{n+1} \) is a constant given by

\[
c_{n+1} = \frac{n + 2}{2(n + 4)} \left( \frac{n + 2}{\text{vol}_n(B_{2}^n)} \right)^{\frac{2}{n+2}},
\]

and \( \text{vol}_n(B_{2}^n) \) is the volume of \( B_{2}^n \), the Euclidean unit ball centered at the origin in \( \mathbb{R}^n \). This theorem provides a variational interpretation of the affine surface area of log-concave functions via Ulam floating functions, and reveals the deep differential properties of the Ulam floating functions. Moreover, it follows from results in [8] that an affine isoperimetric inequality holds for this affine surface area for 2-homogeneous convex functions. Thus, this notion will in particular be useful for applications related to extremal problems. Yet another aspect is that these results give indications how to establish affine isoperimetric inequalities for general convex functions as well as geometric or variational interpretations of \( L_p \) affine surface areas and geominimal surface areas [8, 13] for log-concave functions.

The paper is organized as follows. Section 2 will provide background and notations. Ulam floating functions will be introduced in Sect. 3, where we also prove some of their basic properties. The proof of the relation between Ulam floating function and the affine surface area (i.e., Theorem 1.1) will be established in Sect. 4.

2 Background and Notations

In this section, we collect background and notations that will be used throughout the paper. We refer the reader to the books [10, 22, 23, 25], and the articles [3, 12, 13, 17, 27] for more details.

Let \( \mathbb{R}^n, n \geq 2 \), be the \( n \)-dimensional Euclidean space, \( \|x\| \) be the Euclidean norm of \( x \in \mathbb{R}^n \), and \( \langle x, y \rangle \) be the inner product of \( x, y \in \mathbb{R}^n \). The standard basis of \( \mathbb{R}^n \) is denoted by \( \{e_1, \cdots, e_n\} \). By \( \partial E, E^c \), and \( \text{int}(E) \), we mean the boundary, complement, and interior of \( E \subset \mathbb{R}^n \), respectively. Also, \( \text{vol}_n(E) \) stands for the \( n \)-dimensional volume of \( E \subset \mathbb{R}^n \). The distance between \( x \in \mathbb{R}^n \) and \( E \subset \mathbb{R}^n \) is defined by

\[
dist(x, E) = \inf_{y \in E} \|x - y\|.
\]

Let \( B_{2}^n(x, \rho) \) stand for the closed Euclidean ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( \rho \). In particular, we write in short \( B_{2}^n = B_{2}^n(o, 1) \) for the Euclidean unit ball centered at the origin \( o \), and \( S^{n-1} = \partial B_{2}^n \) for the unit sphere. We would like to point out that the
notation $o$ always means the origin but its dimension may vary in the later context. Let $[x, y)$ denote the ray from $x \in \mathbb{R}^n$ (inclusively) to $y \in \mathbb{R}^n$.

Let $\mathcal{C}$ be the collection of convex functions $\psi : \mathbb{R}^n \to \mathbb{R}$. Throughout this paper, we say that $f : \mathbb{R}^n \to (0, \infty)$ is a log-concave function if $f = e^{-\psi}$ with $\psi \in \mathcal{C}$.

Denote by $\nabla \psi$ and $\nabla^2 \psi$ the gradient and the generalized Hessian of $\psi$, respectively. It is well known that $\nabla \psi$ exists almost everywhere by Rademacher’s theorem, e.g., [6], and $\nabla^2 \psi$ exists almost everywhere in $\mathbb{R}^n$ by Alexandrov [1] and Busemann and Feller in [7].

We now give the definition of the generalized Hessian (see [28]). It coincides with the usual Hessian if $\psi$ is smooth enough. A vector $y \in \mathbb{R}^n$ is said to be a subgradient of $\psi$ at $x_0 \in \mathbb{R}^n$ if $\psi(z) - \psi(x_0) \geq \langle y, z - x_0 \rangle$ for all $z \in \mathbb{R}^n$. For a convex function $\psi \in \mathcal{C}$, the subgradient exists at every point in $\mathbb{R}^n$. We say that $\psi \in \mathcal{C}$ is twice differentiable in a generalized sense at $x_0 \in \mathbb{R}^n$ (see e.g., [28]), if there exists a linear map $\nabla^2 \psi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ such that, for all subgradients $\partial \psi$ of $\psi$,

$$\|\partial \psi(x) - \partial \psi(x_0) - \nabla^2 \psi(x_0)(x - x_0)\| \leq \omega(\|x - x_0\|)\|x - x_0\|,$$

holds in a neighborhood $\mathcal{U}(x_0) \subseteq \mathbb{R}^n$, where $\omega(\cdot) : (0, \infty) \to (0, \infty)$ is a properly chosen function with $\lim_{t \to 0^+} \omega(t) = 0$. In this case, we call $\nabla^2 \psi(x_0)$ the generalized Hessian matrix of $\psi$ at $x_0$.

Let $\theta \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}$. Then

$$H(\theta, a) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = a\}$$

is a hyperplane with normal vector $\theta$. The hyperplane $H(\theta, a)$ defines two closed halfspaces

$$H^+(\theta, a) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geq a\} \text{ and } H^-(\theta, a) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq a\}.$$

We often write $H$, $H^+$, and $H^-$ instead of $H(\theta, a)$, $H^+(\theta, a)$, and $H^-(\theta, a)$ if no confusion occurs.

A subset $K$ of $\mathbb{R}^n$ is a convex body if $K$ is a compact convex set with nonempty interior. The support function of a convex body $K$, $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, is defined by

$$h_K(\theta) = \max_{x \in K} \langle x, \theta \rangle \text{ for } \theta \in \mathbb{S}^{n-1}.$$

Note that any convex body $K$ is uniquely determined by its support function.

By $N_K(z)$, and $\mu_{\partial K}$, we mean the unit outer normal at $z \in \partial K$, and the surface area measure on $\partial K$, respectively. Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \partial K$ and $N_K(o) = -e_n$. The boundary of $K$ (around the origin) can be represented by $x_n = \varphi(x_1, \ldots, x_{n-1})$ for some convex function $\varphi : \mathbb{R}^{n-1} \to [0, \infty)$ such that $\varphi(o) = 0$. The indicatrix of Dupin is the quadratic form

$$\left\{y \in \mathbb{R}^{n-1} : \langle y, (\nabla^2 \varphi(o))y \rangle = 1\right\},$$
where $\nabla^2 \varphi(o)$ is the generalized Hessian matrix of $\varphi$ at $o$. The eigenvalues of $\nabla^2 \varphi(o)$ are the lengths of the principal axes of the indicatrix raised to the power $-2$. They are called the generalized principal curvatures and their product is called the generalized Gauss curvature $\kappa_K$. If the generalized Gauss curvature $\kappa_K(z)$ at a boundary point $z \in \partial K$ exists, we translate and rotate $K$ so that we may assume that $z = o$ and the outer normal $N_K(z) = -e_n$. Denote the generalized Gauss curvature $\kappa_K(z)$ as the Gauss curvature of the convex function $\varphi$ whose graph in the neighborhood of $o$ (locally) represents $\partial K$, see reference [28] for more details.

An important notion in affine, convex, and differential geometry is the affine surface area, which was introduced by Blaschke [4] in dimensions 2 and 3 for smooth enough convex bodies. For a convex body $K \subset \mathbb{R}^n$, it is defined as

$$as(K) = \int_{\partial K} (\kappa_K(z))^{-\frac{1}{n+1}} d\mu_{\partial K}(z). \quad (4)$$

It was shown in [27] that the affine surface area can be obtained by a first-order variation of the volume of $K$ via a family of convex floating bodies. Let $K$ be a convex body in $\mathbb{R}^n$ and $0 < \delta < \text{vol}_n(K)$ be small enough. The convex floating body $K_\delta$ of $K$ is a variant of Dupin’s floating body [9] and was introduced in [3, 27] as the intersection of all halfspaces $H^+$ whose defining hyperplanes $H$ cut off sets of volume $\delta$ from $K$, i.e.,

$$K_\delta = \bigcap_{\{H: \text{vol}_n(H^+ \cap K) = \delta\}} H^+. \quad (5)$$

It has been proved in [27] that

$$as(K) = d_n^{-1} \lim_{\delta \to 0^+} \frac{\text{vol}_n(K) - \text{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}}, \quad (6)$$

where $d_n$ is the constant given by

$$d_n = \frac{1}{2} \left( \frac{n + 1}{\text{vol}_{n-1}(B_{n-1}^\infty)} \right)^{\frac{2}{n+1}}.$$

The following lemma was proved in [27, Lemma 11 (ii)].

**Lemma 2.1** [27] Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \partial K$ and $N_K(o) = -e_n$. Suppose that the indicatrix of Dupin at the origin $o$ exists and is an $(n-1)$ dimensional sphere with radius $\sqrt{\rho}$. Let $\xi$ be an interior point of $K$. Then there is $\varepsilon_0 > 0$ such that, for all $z \in [o, \xi]$ with $\|z\| < \varepsilon_0$,

$$\text{dist}(z, B^n_2(\rho e_n, \rho)^c) \leq z_n \leq \text{dist}(z, B^n_2(\rho e_n, \rho)^c) \left( 1 + \frac{2 \text{dist}(z, B^n_2(\rho e_n, \rho)^c)}{1 + \frac{\rho}{\|\xi\|} N_K(o)^2 \rho} \right).$$
The identity (6) shows that $as(K)$ can be expressed as a derivative of volume, using floating bodies. That is just one example of this phenomenon and in fact floating bodies can be replaced by other families of bodies constructed from $K$. We refer to e.g., [19, 29, 35–38] and only mention in more detail the Ulam floating body (or metronoid) $M_\delta(K)$. These bodies were introduced in [12] (see also [13]), by

$$M_\delta(K) := M(v_{K,\delta}) = \bigcup_{g \in \mathcal{F}_K} \left\{ \int_{\mathbb{R}^n} yg(y)d\nu_{K,\delta}(y) \right\},$$

where $d\nu_{K,\delta} = \delta^{-1} \mathbf{1}_K dx$ with $\mathbf{1}_K$ being the characteristic function of $K$ and $dx$ the Lebesgue measure on $\mathbb{R}^n$, and

$$\mathcal{F}_K = \left\{ g : \mathbb{R}^n \rightarrow [0, 1] : \int_{\mathbb{R}^n} g d\nu_{K,\delta} = 1 \text{ and } \int_{\mathbb{R}^n} yg(y)d\nu_{K,\delta}(y) \text{ exists} \right\}.$$

It has been noted in [12] that $M_\delta(K)$ is convex and in [13, Theorem 1.1] that for $0 < \delta < \text{vol}_n(K)$ small enough, $M_\delta(K)$ is a isomorphic to $K_\delta$ in the sense that

$$K_{(1-\frac{1}{2})\delta} \subseteq M_\delta(K) \subseteq K_{\frac{1}{2}\delta}.$$

As shown in [12, Proposition 2.1], the support function of $M_\delta(K)$ can be calculated by $h_{M_\delta(K)}(\theta) = \langle z_\theta, \theta \rangle$ for $\theta \in S^{n-1}$, where

$$z_\theta = \frac{1}{\delta} \int_{\{x \in K : \langle x, \theta \rangle \geq R_\delta(\theta)\}} x \, dx = \frac{1}{\delta} \int_{K \cap H^+ (\theta, R_\delta(\theta))} x \, dx$$

is the barycenter of $K \cap H^+ (\theta, R_\delta(\theta))$ and $R_\delta(\theta) \in \mathbb{R}$ satisfies that $\text{vol}_n(K \cap H^+ (\theta, R_\delta(\theta))) = \delta$.

Moreover, it was shown in [13] that a variational argument involving the Ulam floating bodies leads to the affine surface area, namely

$$\lim_{\delta \to 0^+} \frac{\text{vol}_n(K) - \text{vol}_n(M_\delta(K))}{\frac{2}{\delta^{n+1}}} = c_n as(K),$$

where $c_n$ is the constant given by

$$c_n = \frac{n + 1}{2(n + 3)} \left( \frac{n + 1}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n+1}}.$$

We will make use of the following lemma which has been proved in [13, Lemma 3.5].
Lemma 2.2 [13] Let $K \subseteq \mathbb{R}^n$ be a convex body. Assume that $o \in \partial K$ and $N_K(o) = -e_n$ is the unique outer normal vector to $\partial K$ at $o$. Then, for each $t > 0$, there exists $r > 0$ such that for any $\delta > 0$,

$$M_\delta(K) \cap B^+_2(o, r) = M_\delta(K \cap H^+(-e_n, -t)) \cap B^+_2(o, r) = M_\delta(K \cap H^-(e_n, t)) \cap B^+_2(o, r).$$

For $\psi \in \mathcal{C}$, let $\text{epi}(\psi)$ stand for the epigraph of $\psi$, namely,

$$\text{epi}(\psi) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \geq \psi(x)\}.$$ 

Clearly, $\text{epi}(\psi)$ is a closed convex set in $\mathbb{R}^{n+1}$. For each $x \in \mathbb{R}^n$, let $z_x = (x, \psi(x))$ be the point on the boundary of $\text{epi}(\psi)$. It is well known that the Gauss curvature $\kappa_{\psi}(z_x)$ and the outer unit normal $N_{\psi}(z_x)$ of $\partial \text{epi}(\psi)$ at $z_x$ are given by (see, e.g., [8]),

$$\kappa_{\psi}(z_x) = \frac{\det(\nabla^2 \psi(x))}{(1 + \|\nabla \psi(x)\|^2)^{\frac{n+2}{2}}} \quad \text{and} \quad N_{\psi}(z_x) = \frac{(\nabla \psi(x), -1)}{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}. \quad (8)$$

Moreover, the following formula holds:

$$(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}} \, dx = d\mu_{\partial \text{epi}(\psi)}(z_x). \quad (9)$$

Let $f = e^{-\psi}$ be a log-concave function. It is also well known (see e.g., [23]) that $\int e^{-\psi} \, dx < \infty$ holds if and only if there exist constants $a > 0$ and $b \in \mathbb{R}$ such that $\psi(x) \geq a\|x\| + b$ for all $x \in \mathbb{R}^n$. As remarked above, the quantity

$$\text{as}(f) = \int_{\mathbb{R}^n} (\det(\nabla^2 \psi(x)))^{\frac{1}{n+2}} e^{-\psi(x)} \, dx, \quad (10)$$

is called the affine surface area of the log-concave function $f = e^{-\psi}$. This is justified as it shares many properties with the affine surface area for convex bodies. For one, this quantity is an affine invariant. We have for all affine transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det T$, the determinant of $T$, is nonzero,

$$\text{as}(f \circ T) = |\det T|^{-\frac{n}{n+2}} \text{as}(f).$$

Then, this quantity is a valuation, namely,

$$\text{as}(f_1) + \text{as}(f_2) = \text{as}(\max(f_1, f_2)) + \text{as}(\min(f_1, f_2)),$$

holds for log-concave functions $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$ such that $\min(\psi_1, \psi_2)$ is convex.

Moreover, it follows from (8) and (9) that

$$\text{as}(f) = \int_{\partial \text{epi}(\psi)} (\kappa_{\text{epi}(\psi)}(z_x))^{\frac{1}{n+2}} \, e^{-\langle z_x, e_n + 1 \rangle} \, d\mu_{\partial \text{epi}(\psi)}(z_x), \quad (11)$$

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and the expression for $as(f)$ in (11) resembles the one of the affine surface area $as(K)$ of a convex body $K \subseteq \mathbb{R}^{n+1}$ when the bounded convex body $K$ is replaced by the unbounded convex set $\text{epi}(\psi)$. Readers are referred to [17] for more details of the affine surface area $as(f)$ of log-concave functions $f$. We also refer to [8] for a slightly different definition of the affine surface area of a log-concave function and the related properties.

Motivated by the convex floating body, Li et al. [17] introduced the floating function for a log-concave function. For a log-concave function $f = e^{-\psi}$ and $\delta > 0$ they first defined the floating set of $\text{epi}(\psi)$ by

$$
(\text{epi}(\psi))_\delta = \bigcap_{\{H: \text{vol}_{n+1}(H^+ \cap \text{epi}(\psi)) \leq \delta\}} H^+.
$$

(12)

Then they defined the floating function $\psi_\delta$ of $\psi$ via its epigraph as follows:

$$
\text{epi}(\psi_\delta) = (\text{epi}(\psi))_\delta.
$$

(13)

And finally, the floating function $f_\delta$ of $f = e^{-\psi}$ is given by $f_\delta = e^{-\psi_\delta}$.

It was proved in [17, Theorem 1] that

$$
as(f) = \int_{\mathbb{R}^n} (\det(\nabla^2 \psi(x)))^{\frac{1}{n+2}} e^{-\psi(x)} \, dx
= \lim_{\delta \to 0^+} \left( (d_{n+1} \cdot \delta \frac{2}{n+2})^{-1} \int_{\mathbb{R}^n} (e^{-\psi(x)} - e^{-\psi_\delta(x)}) \, dx \right),
$$

(14)

where the constant $d_{n+1}$ is given by

$$
d_{n+1} = \frac{1}{2} \left( \frac{n + 2}{\text{vol}_n(B_2^n)} \right)^{\frac{2}{n+2}}.
$$

(15)

We shall need the following lemma, whose proof can be found in e.g., [17, 28].

**Lemma 2.3** Let $r > 0$ be a constant and $\mathcal{E} \subseteq \mathbb{R}^{n+1}$ be an ellipsoid given by

$$
\mathcal{E} = \left\{ z \in \mathbb{R}^{n+1} : \left( \frac{z_{n+1} - r}{r} \right)^2 + \sum_{i=1}^{n} \left( \frac{z_i}{r} \right)^2 \leq 1 \right\}.
$$

Then, for all $h \in (0, 2r)$, one has

$$
\left( 1 - \frac{h}{2r} \right)^{\frac{n}{n+2}} h \leq d_{n+1} \cdot \left( \frac{\text{vol}_{n+1}(\mathcal{E} \cap H^-(e_{n+1}, h))}{r^{\frac{n}{2}}} \right)^{\frac{2}{n+2}} \leq h.
$$

(16)
3 Ulam Floating Functions

Motivated by Ulam floating bodies, we will now introduce Ulam floating functions. We define first the Ulam floating set $M_\delta(C)$ for a convex, closed, not necessarily bounded set $C \subseteq \mathbb{R}^{n+1}$. For convenience, we denote $d\nu_{C,\delta} = \delta^{-1}1_C d\mathbf{x}$. Let

$$
\mathcal{F}_C = \left\{ g : \mathbb{R}^{n+1} \to [0, 1] : \int_{\mathbb{R}^{n+1}} g \, d\nu_{C,\delta} = 1 \text{ and } \int_{\mathbb{R}^{n+1}} yg(y) \, d\nu_{C,\delta}(y) \text{ exists} \right\}.
$$

**Definition 3.1** (Ulam floating set) Let $C$ be a closed convex set in $\mathbb{R}^{n+1}$ with nonempty interior. For $\delta > 0$, the Ulam floating set of $C$ is defined by

$$
M_\delta(C) = M(\nu_{C,\delta}) = \bigcup_{g \in \mathcal{F}_C} \left\{ \int_{\mathbb{R}^{n+1}} yg(y) \, d\nu_{C,\delta}(y) \right\}.
$$

Clearly, the Ulam floating set $M_\delta(C)$ is closed and convex. As we are mainly interested in Ulam floating functions, from now on, our discussion will be concentrated on the case when $C = \text{epi}(\psi)$ with $\psi \in \mathcal{C}$. As $\text{epi}(\psi)$ is a closed convex set in $\mathbb{R}^{n+1}$ with nonempty interior, we define

$$
M_\delta(\text{epi}(\psi)) = M(\nu_{\text{epi}(\psi),\delta}) = \bigcup_{g \in \mathcal{F}_{\text{epi}(\psi)}} \left\{ \int_{\mathbb{R}^{n+1}} yg(y) \, d\nu_{\text{epi}(\psi),\delta}(y) \right\},
$$

where the set $\mathcal{F}_{\text{epi}(\psi)}$ is given by

$$
\mathcal{F}_{\text{epi}(\psi)} = \left\{ g : \mathbb{R}^{n+1} \to [0, 1] : \int_{\mathbb{R}^{n+1}} g \, d\nu_{\text{epi}(\psi),\delta} = 1 \text{ and } \int_{\mathbb{R}^{n+1}} yg(y) \, d\nu_{\text{epi}(\psi),\delta}(y) \text{ exists} \right\}.
$$

Since $M_\delta(\text{epi}(\psi))$ is closed, convex, and containing the ray emanating from the origin in direction $e_n+1$, there is a unique convex function $M_\delta(\psi) : \mathbb{R}^n \to \mathbb{R}$ such that

$$
M_\delta(\text{epi}(\psi)) = \text{epi}(M_\delta(\psi)).
$$

This leads to the following definitions of Ulam floating functions for convex and log-concave functions.

**Definition 3.2** Let $\psi \in \mathcal{C}$ and $\text{epi}(\psi)$ be its epigraph. Let $\delta \in (0, \infty)$ and $f = e^{-\psi}$ be a log-concave function.

(i) Ulam floating function $M_\delta(\psi)$ of $\psi$ is defined by

$$
M_\delta(\text{epi}(\psi)) = \text{epi}(M_\delta(\psi)).
$$

(ii) Ulam floating function $U_\delta(f)$ of $f$ is defined as

$$
U_\delta(f) = e^{-M_\delta(f)}.
$$
As $M_\delta(e_{\pi}(\psi)) \subseteq e_{\pi}(\psi)$ for all $\delta > 0$, one sees that $e_{\pi}(M_\delta(\psi)) \subseteq e_{\pi}(\psi)$ which further yields $M_\delta(\psi) \geq \psi$. More generally, if $0 < \delta_1 < \delta_2$, then $M_{\delta_2}(e_{\pi}(\psi)) \subseteq M_{\delta_1}(e_{\pi}(\psi))$. Consequently, $M_{\delta_2}(\psi) \geq M_{\delta_1}(\psi)$ and $U_{\delta_2}(f) \leq U_{\delta_1}(f) \leq f$ for $0 < \delta_1 < \delta_2$. It can also be checked easily that for all $x \in \mathbb{R}^n$,

$$\lim_{\delta \to 0^+} M_\delta(\psi) = \psi \quad \text{and} \quad \lim_{\delta \to 0^+} U_\delta(f) = f.$$ 

Following the discussion in [13] for properties of Ulam floating bodies, we now describe some basic properties for the Ulam floating set $M_\delta(e_{\pi}(\psi))$.

Let the function $\eta : \mathbb{S}^n \times \mathbb{R} \to [0, \infty]$ be defined by

$$\eta(\theta, a) = \text{vol}_{n+1}(e_{\pi}(\psi) \cap H^-(\theta, a)),$$

for $\theta \in \mathbb{S}^n$ and $a \in \mathbb{R}$. \hspace{1cm} (18)

Let $\eta_\theta(\cdot) = \eta(\theta, \cdot)$ for any fixed $\theta \in \mathbb{S}^n$ and let

$$\Sigma_\psi = \{ \theta \in \mathbb{S}^n : \eta_\theta(a) < \infty \text{ for all } a \in \mathbb{R} \}.$$

Note that if $\theta \in \Sigma_\psi$, then $\eta_\theta : [-h_{e_{\pi}(\psi)}(-\theta), \infty) \to [0, \infty)$ is continuous and strictly increasing. Indeed, it is obvious that

$$e_{\pi}(\psi) \cap H^-(\theta, a) \subseteq e_{\pi}(\psi) \cap H^-(\theta, b)$$

for $-h_{e_{\pi}(\psi)}(-\theta) < a < b < \infty$, and then $\eta_\theta(a) < \eta_\theta(b)$. Let $a_0 \in [-h_{e_{\pi}(\psi)}(-\theta), \infty)$ and $\zeta \in (0, 1)$. Then, whenever $|a - a_0| \leq \zeta$ and $a \in [-h_{e_{\pi}(\psi)}(-\theta), \infty)$, one has

$$e_{\pi}(\psi) \cap H^-(\theta, a_0 - \zeta) \subseteq e_{\pi}(\psi) \cap H^-(\theta, a) \subseteq e_{\pi}(\psi) \cap H^-(\theta, a_0 + \zeta).$$

This further gives

$$|\eta_\theta(a) - \eta_\theta(a_0)| \leq \int_{a_0 - \zeta}^{a_0 + \zeta} \text{vol}_n(\{ x \in e_{\pi}(\psi) : (x, \theta) = t \}) \, dt \leq 2D\zeta,$$

where $D < \infty$ is given by

$$D = \max \left\{ \text{vol}_n(\text{epi}(\psi) \cap H(\theta, t)) : t \in [a_0 - 1, a_0 + 1] \cap [-h_{e_{\pi}(\psi)}(-\theta), \infty) \right\}.$$

Hence, for all $\zeta \in (0, \frac{\epsilon}{2D})$, one gets

$$|\eta_\theta(a) - \eta_\theta(a_0)| < \epsilon, \text{ for all } a \in (a_0 - \zeta, a_0 + \zeta) \cap [-h_{e_{\pi}(\psi)}(-\theta), \infty).$$

This shows that $\eta_\theta(\cdot)$ is continuous on $[-h_{e_{\pi}(\psi)}(-\theta), \infty)$. Consequently, the inverse function of $\eta_\theta$ exists and will be denoted by $\eta_\theta^{-1} : [0, \infty) \to [-h_{e_{\pi}(\psi)}(-\theta), \infty)$. Clearly, $\eta_\theta^{-1}(\cdot)$ is also continuous and strictly increasing. Let $\delta > 0$ be fixed. Then,
\[ \eta_\theta^{-1}(\delta) = \min \left\{ a \in \mathbb{R} : \nu_{\text{epi}(\psi),\delta}(\{x \in \mathbb{R}^{n+1} : \langle x, \theta \rangle \leq a\}) \geq 1 \right\} \]

\[ = \min \left\{ a \in \mathbb{R} : \text{vol}_{n+1}(\text{epi}(\psi) \cap H^-(\theta, a)) \geq \delta \right\} \]

\[ = \min \left\{ a \in \mathbb{R} : \eta_\theta(a) \geq \delta \right\} . \]

Moreover for \( \delta > 0, \)

\[ \eta_\theta(\eta_\theta^{-1}(\delta)) = \text{vol}_{n+1}(\text{epi}(\psi) \cap H^-(\theta, \eta_\theta^{-1}(\delta))) = \delta. \]

Define \( g_\theta : \mathbb{R} \to [0, 1] \) as follows:

\[ g_\theta(t) = \begin{cases} 0, & t \geq \eta_\theta^{-1}(\delta) \\ 1, & t < \eta_\theta^{-1}(\delta) \end{cases} \]

Clearly, \( 0 \leq g_\theta \leq 1 \) and

\[ \int_{\mathbb{R}^{n+1}} g_\theta(\langle x, \theta \rangle) d\nu_{\text{epi}(\psi),\delta}(x) = \frac{1}{\delta} \int_{\{x \in \text{epi}(\psi) : \langle x, \theta \rangle < \eta_\theta^{-1}(\delta)\}} dx = \frac{\text{vol}_{n+1}(\text{epi}(\psi) \cap H^-(\theta, \eta_\theta^{-1}(\delta)))}{\delta} = 1. \] (19)

Note that \( g_\theta(\langle \theta, x \rangle) \in \mathcal{F}_{\text{epi}(\psi)} \) because \( y_\theta \) exists (and indeed \( y_\theta \in M_\delta(\text{epi}(\psi)) \)), where

\[ y_\theta = \int_{\mathbb{R}^{n+1}} x g_\theta(\langle x, \theta \rangle) d\nu_{\text{epi}(\psi),\delta}(x) = \frac{1}{\delta} \int_{\{x \in \text{epi}(\psi) : \langle x, \theta \rangle < \eta_\theta^{-1}(\delta)\}} x \ dx \]

\[ = \frac{1}{\text{vol}_{n+1}(\text{epi}(\psi) \cap H^-(\theta, \eta_\theta^{-1}(\delta)))} \int_{\text{epi}(\psi) \cap H^-(\theta, \eta_\theta^{-1}(\delta))} x \ dx. \] (20)

In other words, \( y_\theta \) is the barycenter of \( \text{epi}(\psi) \cap H^-(\theta, \eta_\theta^{-1}(\delta)) \), which is illustrated in Fig. 1.

We now prove the following result.

**Proposition 3.3** Let \( \psi \in \mathcal{C} \). For \( \delta > 0 \) and \( \theta \in \Sigma_\psi \), one has \( y_\theta \in \partial(M_\delta(\text{epi}(\psi))) \) and

\[ h_{M_\delta(\text{epi}(\psi))}(-\theta) = (y_\theta, -\theta). \] (21)

**Proof** Let \( \delta > 0 \) and \( \theta \in \Sigma_\psi \). Clearly, \( y_\theta \in M_\delta(\text{epi}(\psi)) \). The desired result in Proposition 3.3 follows immediately once (21) is verified. Let \( y \in M_\delta(\text{epi}(\psi)) \). There exists a function \( g \in \mathcal{F}_{\text{epi}(\psi)}, g : \mathbb{R}^{n+1} \to [0, 1] \) such that

\[ 1 = \int_{\mathbb{R}^{n+1}} g \ d\nu_{\text{epi}(\psi),\delta} = \frac{1}{\delta} \int_{\text{epi}(\psi)} g \ dx, \] (22)

\[ y = \int_{\mathbb{R}^{n+1}} x g(x) \ d\nu_{\text{epi}(\psi),\delta}(x) = \frac{1}{\delta} \int_{\text{epi}(\psi)} x g(x) \ dx. \] (23)
Fig. 1 It illustrates that $y_\theta$ is a boundary point of $M_\delta(e^\psi)$. The halfspace $H^-(\theta, \eta^{-1}_\theta(\delta))$, defined by the support hyperplane $H = H(-\theta, h_{M_\delta(e^\psi)}(-\theta))$ to $M_\delta(e^\psi)$ in the direction of $(-\theta)$, cuts off a set of volume $\delta$ from $e^\psi$.

These, together with (19), further imply that

$$0 = \int_{e^\psi} g \, dx - \int_{e^\psi} g_\theta(\langle x, \theta \rangle) \, dx$$

$$= \int_{\{x \in e^\psi : \langle x, \theta \rangle \geq \eta^{-1}_\theta(\delta)\}} g(x) \, dx + \int_{\{x \in e^\psi : \langle x, \theta \rangle < \eta^{-1}_\theta(\delta)\}} (g(x) - 1) \, dx.$$

Dividing both sides by $\eta_\theta(\delta)$, it follows from (20), (22), (23) and $0 \leq g \leq 1$ that

$$0 = \int_{\{x \in e^\psi : \langle x, \theta \rangle \geq \eta^{-1}_\theta(\delta)\}} g(x) \eta^{-1}_\theta(\delta) \, dx + \int_{\{x \in e^\psi : \langle x, \theta \rangle < \eta^{-1}_\theta(\delta)\}} (g(x) - 1) \eta^{-1}_\theta(\delta) \, dx$$

$$\leq \int_{\{x \in e^\psi : \langle x, \theta \rangle \geq \eta^{-1}_\theta(\delta)\}} g(x) \langle x, \theta \rangle \, dx + \int_{\{x \in e^\psi : \langle x, \theta \rangle < \eta^{-1}_\theta(\delta)\}} (g(x) - 1) \langle x, \theta \rangle \, dx$$

$$= \int_{e^\psi} g(x) \langle x, \theta \rangle \, dx - \int_{\{x \in e^\psi : \langle x, \theta \rangle < \eta^{-1}_\theta(\delta)\}} \langle x, \theta \rangle \, dx$$

$$= \delta \cdot \langle y_\theta - (y_\theta, \theta) \rangle.$$

This gives $\langle y_\theta, -\theta \rangle \geq \langle y, -\theta \rangle$ for all $y \in M_\delta(e^\psi)$, as desired. 

The following result provides a way to calculate the support function of $M_\delta(e^\psi)$ when $\psi \in C$ is a supercoercive convex function, that is $\psi \in C$ satisfies

$$\lim_{\|x\| \to \infty} \frac{\psi(x)}{\|x\|} = +\infty.$$ (24)

### Corollary 3.4

Suppose that $\psi \in C$ is a supercoercive convex function. Then, for any $\theta \in S^n$ such that $\langle \theta, e_{n+1} \rangle > 0$ and any $\delta > 0$, one has that $y_\theta \in \partial(M_\delta(e^\psi))$ and (21) holds.
Proof In view of Proposition 3.3, it is enough to prove

\[ \Sigma_\psi = \{ \theta \in \mathbb{S}^n : \langle \theta, e_{n+1} \rangle > 0 \}. \]

To this end, let \( \theta = (\tilde{\theta}, \theta_{n+1}) \) with \( \theta_{n+1} > 0 \) and \( a \in \mathbb{R} \) be fixed. If \( \text{epi}(\psi) \cap H^{-1}(\theta, a) \neq \emptyset \) and \( (x, t) \in \text{epi}(\psi) \cap H^{-1}(\theta, a) \), then \( t \geq \psi(x) \) and

\[ \langle (x, \psi(x)), (\tilde{\theta}, \theta_{n+1}) \rangle = \langle x, \tilde{\theta} \rangle + \theta_{n+1} \psi(x) \leq \langle x, \tilde{\theta} \rangle + \theta_{n+1} t = \langle (x, t), (\tilde{\theta}, \theta_{n+1}) \rangle \leq a. \]

Dividing by \( \|x\| \) and letting \( \|x\| \to \infty \), it follows from (24) that

\[ +\infty = \lim_{\|x\| \to \infty} \left( \left( \frac{x}{\|x\|}, \frac{\tilde{\theta}}{\|x\|} \right) + \frac{\theta_{n+1}}{\|x\|} \cdot \frac{\psi(x)}{\|x\|} \right) \leq \lim_{\|x\| \to \infty} \frac{a}{\|x\|} = 0. \]

This is impossible. Hence, \( \text{epi}(\psi) \cap H^{-1}(\theta, a) \), if not an empty set, contains all \( (x, t) \) with \( x \in E \) for some bounded set \( E \subset \mathbb{R}^n \). This in turn implies that

\[ \psi(x) \leq t \leq \frac{a}{\theta_{n+1}} + \frac{\|\tilde{\theta}\|}{\theta_{n+1}} \cdot \sup_{x \in E} \|x\| < \infty. \]

Consequently, either \( \text{vol}_{n+1}(\text{epi}(\psi) \cap H^{-1}(\theta, a)) = 0 \) if \( \text{epi}(\psi) \cap H^{-1}(\theta, a) = \emptyset \), or \( \text{vol}_{n+1}(\text{epi}(\psi) \cap H^{-1}(\theta, a)) < \infty \) if the set \( \text{epi}(\psi) \cap H^{-1}(\theta, a) \) is a bounded nonempty set. \( \square \)

The next result establishes the affine invariance of Ulam floating functions for convex and log-concave functions.

**Proposition 3.5** Let \( \psi \in \mathcal{C} \) and \( T : \mathbb{R}^n \to \mathbb{R}^n \) be an invertible linear map. Assume that \( f = e^{-\psi} \) is integrable. Define \( (f \circ T)(x) = f(Tx) \). For any \( \delta > 0 \), one has

\[ M_\delta(f \circ T) = (M_\delta | \det T|(\psi)) \circ T \quad \text{and} \quad U_\delta(f \circ T) = (U_\delta | \det T|(f)) \circ T. \]

In particular, if \( | \det T | = 1 \), then

\[ M_\delta(f \circ T) = (M_\delta(f)) \circ T \quad \text{and} \quad U_\delta(f \circ T) = (U_\delta(f)) \circ T. \]

**Proof** It is enough to only prove the statement

\[ M_\delta(f \circ T) = (M_\delta | \det T|(\psi)) \circ T. \]

Let \( \tilde{T} \) be the map defined by \( \tilde{T}(x, x_{n+1}) = (Tx, x_{n+1}) \). It can be checked that

\[ \tilde{T}^{-1}(x, x_{n+1}) = (T^{-1}x, x_{n+1}). \]
Thus, \( \text{epi}(\psi \circ T) = \tilde{T}^{-1}( \text{epi}(\psi)) \). According to Definition 3.1, if \( g \in \mathcal{F}_{\text{epi}(\psi \circ T)} \), then \( \tilde{g} = g \circ (\tilde{T}^{-1}) \) satisfies that \( \tilde{g} : \mathbb{R}^{n+1} \to [0, 1], \int_{\text{epi}(\psi)} \tilde{g}(\tilde{z}) \, d\tilde{z} = \delta \cdot |\det T|, \) and

\[
\int_{\text{epi}(\psi \circ T)} zg(z) \, d\tilde{z} = \tilde{T}^{-1} \left( \int_{\text{epi}(\psi)} |\det T| \tilde{g}(\tilde{z}) \, d\tilde{z} \right).
\]

This further yields that

\[
M_\delta(\text{epi}(\psi \circ T)) = M_\delta(\tilde{T}^{-1}( \text{epi}(\psi))) = \tilde{T}^{-1}(M_\delta|\det T|( \text{epi}(\psi))).
\]

It follows from the definition of \( M_\delta(\psi) \) that

\[
\text{epi}(M_\delta(\psi \circ T)) = M_\delta(\text{epi}(\psi \circ T)) = \tilde{T}^{-1}(M_\delta|\det T|( \text{epi}(\psi))) = \tilde{T}^{-1}(\text{epi}(M_\delta|\det T|(\psi))).
\]

Thus, for any \( z = (x, s) \in \text{epi}(M_\delta(\psi \circ T)) \), one obtains that

\[
s \geq (M_\delta(\psi \circ T))(x).
\]

The point \( (x, s) \) is also in \( \tilde{T}^{-1}(\text{epi}(M_\delta|\det T|(\psi))) \), which further implies

\[
\tilde{T}(x, s) = (Tx, s) \in \text{epi}(M_\delta|\det T|(\psi)),
\]

and thus \( s \geq (M_\delta|\det T|(\psi))(Tx) \). This can hold if and only if

\[
(M_\delta(\psi \circ T))(x) = (M_\delta|\det T|(\psi))(Tx).
\]

As \( x \) is arbitrary, one immediately gets

\[
M_\delta(\psi \circ T) = (M_\delta|\det T|(\psi)) \circ T.
\]

This completes the proof. \( \square \)

## 4 Ulam Floating Function and Affine Surface Area

The section is devoted to the proof of Theorem 1.1. We will need more preparation and several preliminary results.

We first note that, under the assumptions of Theorem 1.1, the integral term in Theorem 1.1 is finite. To see that we recall the definition of the rolling function \( r_C : \partial C \to [0, \infty) \) of a closed convex set \( C \) in \( \mathbb{R}^{n+1} \), which was introduced in [27], see also [17], as follows: If the outer unit normal \( N_C(z) \) at \( z \in \partial C \) is unique, then \( r_C(z) \) is the radius of the biggest Euclidean ball contained in \( C \) that touches \( C \) at \( z \),

\[
r_C(z) = \max\{\rho : \mathcal{B}_2^{n+1}(z - \rho N_C(z), \rho) \subseteq C\}.
\]
If $N_C(z)$ is not unique, then $r_C(z) = 0$. In particular, if $C = \text{epi}(\psi)$ with $\psi \in \mathcal{C}$, we write

$$r_\psi(x) = r_{\text{epi}(\psi)}(z_x),$$

where $z_x = (x, \psi(x)) \in \partial \text{epi}(\psi)$.

Let $\psi \in \mathcal{C}$. Since $\psi$ is continuous, $\text{epi}(\psi)$ is a closed set. For functions $\psi$ such that $e^{-\psi}$ is integrable, we have that $r_\psi(z)$ is bounded for every $z \in \partial \text{epi}(\psi)$. It follows that $\text{epi}(\psi)$ contains a Euclidean ball with radius $r_\psi(z)$ that has $z$ as an element.

Let now $\psi \in \mathcal{C}$ be a convex function satisfying the above assumptions and let $z_x = (x, \psi(x))$. Recall that $\kappa_\psi(z_x)$ and $N_\psi(z_x)$ are the Gauss curvature and the outer unit normal of $\partial \text{epi}(\psi)$ at $z_x$, respectively. We have for almost all $x \in \mathbb{R}^n$ (see e.g., [17]) that

$$r_\psi(x) \leq (\kappa_\psi(z_x))^{-\frac{1}{n}} = \left(1 + \|\nabla \psi(x)\|^2\right)^{\frac{n+2}{2n}} \left(\frac{\text{det} \nabla^2 \psi(x)}{2n}\right)^{\frac{1}{2n}}. \quad (26)$$

It has been proved in [17, Lemma 8] that, if $\psi : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $e^{-\psi}$ is integrable, then for all $0 \leq \alpha < 1$,

$$\int_{\mathbb{R}^n} \frac{1 + \|\nabla \psi(x)\|^2}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx < \infty. \quad (27)$$

In particular, (27) holds for $\alpha = \frac{n}{n+2}$. Together with (26), one gets

$$\int_{\mathbb{R}^n} \left(\text{det} \nabla^2 (\psi(x))\right)^{\frac{1}{n+2}} e^{-\psi(x)} \, dx \leq \int_{\mathbb{R}^n} \frac{1 + \|\nabla \psi(x)\|^2}{r_\psi(x)^{\frac{n}{n+2}}} e^{-\psi(x)} \, dx < \infty. \quad (28)$$

If the Gauss curvature $\kappa_\psi(z_x)$ is 0 almost everywhere, then all expressions in the identities of Theorem 1.1 are 0. This is in particular the case when $\psi$ is piecewise affine.

In the following lemma we write in short $\rho \mathbb{B}^{n+1}_2 = \mathbb{B}^{n+1}_2(o, \rho)$. Note that, for $\delta > 0$ small enough, $M_\delta(\rho \mathbb{B}^{n+1}_2)$ is again a Euclidean ball centered at the origin $o$.

**Lemma 4.1** Let $\delta > 0$ be small enough and $\Delta_h$ be the height of a cap of $\rho \mathbb{B}^{n+1}_2$ of volume $\delta$. Let $\Delta_\rho$ be the difference of the radii of $\rho \mathbb{B}^{n+1}_2$ and $M_\delta(\rho \mathbb{B}^{n+1}_2)$. Then,

$$\lim_{\delta \to 0^+} \frac{\Delta_h}{\Delta_\rho} = \frac{n + 4}{n + 2}. \quad (29)$$

**Proof** Let $\Delta_h$ be the height of a cap of $\rho \mathbb{B}^{n+1}_2$ of volume $\delta$. With the change of variable $y = (z, \rho - t) \in \rho \mathbb{B}^{n+1}_2$ with $t \in (0, \rho)$ and $z \in (\rho^2 - (\rho - t)^2)^{\frac{1}{2}} \mathbb{B}^n_2 = (2\rho t - t^2)^{\frac{1}{2}} \mathbb{B}^n_2$, we have

$$\Delta_h = \frac{1}{2} \rho \int_{\mathbb{B}^n_2} \left(\frac{1}{2} \rho^2 - t^2\right)\frac{1}{2} \rho \int_{\mathbb{B}^n_2} \left(\frac{1}{2} \rho^2 - t^2\right) \, dt.$$
the following holds:

\[
\delta = \text{vol}_{n+1}(\rho B_2^{n+1} \cap H^+(e_{n+1}, \Delta_h)) = \text{vol}_n(B_2^n) \int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} \, dt,
\]

\[
\int_{\rho B_2^{n+1} \cap H^+(e_{n+1}, \Delta_h)} y \, dy = e_{n+1} \cdot \text{vol}_n(B_2^n) \int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} (\rho - t) \, dt.
\]

These, together with \((17)\) and \(d_{\text{epi}(\psi)} = \delta^{-1} 1_{\text{epi}(\psi)} \, dx\), result in

\[
\Delta_{\rho} = \frac{\int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} t \, dt}{\int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} \, dt}.
\]

Clearly, if \(\delta \to 0^+,\) then \(\Delta_h \to 0^+.\) It follows from L’Hospital’s rule that

\[
\lim_{\delta \to 0^+} \frac{\Delta_h}{\Delta_{\rho}} = \lim_{\Delta_h \to 0^+} \frac{\Delta_h \int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} \, dt}{\int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} \, dt} = 1 + \lim_{\Delta_h \to 0^+} \frac{\int_0^{\Delta_h} (2\rho t - t^2)^{\frac{n}{2}} \, dt}{(2\rho \Delta_h - \Delta_h^2)^{\frac{n}{2}}} \Delta_h
\]

\[
= 1 + \lim_{\Delta_h \to 0^+} \frac{(2\rho \Delta_h - \Delta_h^2)^{\frac{n}{2}}}{(2\rho - \Delta_h) + n (\rho - \Delta_h)} = \frac{n + 4}{n + 2}.
\]

This completes the proof. \(\Box\)

The following lemma will be needed in order to use the Dominated Convergence Theorem in the proof of Theorem 1.1.

**Lemma 4.2** Let \(\psi \in \mathcal{C}\) be such that \(0 < \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx < \infty.\) There exist \(\delta_0 > 0\) and a constant \(\alpha_n > 0\) such that

\[
0 \leq \frac{M_\delta(\psi)(x) - \psi(x)}{\delta^{\frac{n}{2}} \pi^{\frac{n}{2}}} \leq \frac{\alpha_n \left(1 + \|\nabla \psi(x)\|^2\right)^{\frac{1}{2}}}{r_\psi(x) \delta^{\frac{n}{2}} \pi^{\frac{n}{2}}},
\]

holds for all \(\delta \in (0, \delta_0)\) and for all \(x \in \mathbb{R}^n\), where \(r_\psi(x)\) is given by \((25)\).

**Proof** Let \(\psi_\delta\) be as in \((13)\). It has been proved in [17, Lemma 5] that there exist \(\delta_0 > 0\) and a constant \(\alpha_n > 0\) such that

\[
0 \leq \frac{\psi_\delta(x) - \psi(x)}{\delta^{\frac{n}{2}} \pi^{\frac{n}{2}}} \leq \frac{\alpha_n \left(1 + \|\nabla \psi(x)\|^2\right)^{\frac{1}{2}}}{r_\psi(x) \delta^{\frac{n}{2}} \pi^{\frac{n}{2}}},
\]

holds for all \(\delta \in (0, \delta_0)\) and for all \(x \in \mathbb{R}^n\). It can be checked that for any \(\delta \in (0, \delta_0)\), \((\text{epi}(\psi))_\delta \subseteq M_\delta(\text{epi}(\psi))\), where \((\text{epi}(\psi))_\delta\) is the floating set of \text{epi}(\psi)\) defined in

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(12). Hence, $\psi_\delta \geq M_\delta(\psi)$, and this in turn further yields

$$0 \leq \frac{M_\delta(\psi)(x) - \psi(x)}{\delta \frac{2}{\pi^2}} \leq \alpha_n \frac{(1 + \| \nabla \psi(x) \|^2)^{\frac{1}{2}}}{r_\psi(x) \frac{2}{\pi^2}} ,$$

for all $\delta \in (0, \delta_0)$ and for all $x \in \mathbb{R}^n$. \hfill \Box

We apply Lemma 4.2 to the log-concave function $f = e^{-\psi}$, where $\psi \in \mathcal{C}$ is such that $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx < \infty$. As $1 - e^{-t} \leq t$ for all $t \in (0, \infty)$, one gets

$$f - U_\delta(f) = e^{-\psi} - e^{-M_\delta(\psi)} = e^{-\psi} \cdot (1 - e^{-(M_\delta(\psi) - \psi)}) \leq e^{-\psi} \cdot (M_\delta(\psi) - \psi).$$

It then follows from Lemma 4.2 that there exist $\delta_0 > 0$ and a constant $\alpha_n > 0$ such that

$$0 \leq \frac{f - U_\delta(f)}{\delta \frac{2}{\pi^2}} \leq \alpha_n \frac{(1 + \| \nabla \psi(x) \|^2)^{\frac{1}{2}}}{r_\psi(x) \frac{2}{\pi^2}} \cdot f(x),$$

holds for all $\delta \in (0, \delta_0)$ and for all $x \in \mathbb{R}^n$. Together with (27) and the Dominated Convergence Theorem, one gets

$$\lim_{\delta \to 0^+} \left( \frac{\delta}{\frac{2}{\pi^2}} \int_{\mathbb{R}^n} (f - U_\delta(f)) \, dx \right) = \lim_{\delta \to 0^+} \left( \delta \frac{2}{\pi^2} \int_{\mathbb{R}^n} (e^{-\psi(x)} - e^{-M_\delta(\psi)(x)}) \, dx \right) = \int_{\mathbb{R}^n} \lim_{\delta \to 0^+} \frac{(e^{-\psi(x)} - e^{-M_\delta(\psi)(x)})}{\delta \frac{2}{\pi^2}} \, dx. \quad (29)$$

Similarly, the following formulas hold:

$$\lim_{\delta \to 0^+} \left( \delta \frac{2}{\pi^2} \int_{\mathbb{R}^n} |M_\delta(\psi)(x) - \psi(x)| \cdot e^{-\psi(x)} \, dx \right) = \int_{\mathbb{R}^n} \lim_{\delta \to 0^+} \frac{|M_\delta(\psi)(x) - \psi(x)| \cdot e^{-\psi(x)}}{\delta \frac{2}{\pi^2}} \, dx = \int_{\mathbb{R}^n} \lim_{\delta \to 0^+} \frac{(e^{-\psi(x)} - e^{-M_\delta(\psi)(x)})}{\delta \frac{2}{\pi^2}} \, dx. \quad (30)$$

Let us recall the main theorem.

**Theorem 1.1.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that

$$0 < \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx < \infty.$$

Then

$$\lim_{\delta \to 0^+} \left( \delta \frac{2}{\pi^2} \int_{\mathbb{R}^n} (e^{-\psi(x)} - e^{-M_\delta(\psi)(x)}) \, dx \right) = \lim_{\delta \to 0^+} \left( \delta \frac{2}{\pi^2} \int_{\mathbb{R}^n} |M_\delta(\psi)(x) - \psi(x)| \cdot e^{-\psi(x)} \, dx \right) = c_{n+1} \int_{\mathbb{R}^n} (\det(\nabla^2 \psi(x))) \frac{1}{\pi^2} e^{-\psi(x)} \, dx, \quad (31)$$
where \( c_{n+1} \) is the constant given in (2), i.e.,

\[
  c_{n+1} = \frac{n+2}{2(n+4)} \left( \frac{n+2}{\text{vol}_n(B^2)} \right)^{\frac{2}{n+2}}.
\]

**Proof** Let \( z_x = (x, \psi(x)) \) as above. According to (29) and (30), the desired formula (31) will hold if the following limit is verified. For almost all \( x \in \mathbb{R}^n \),

\[
  \lim_{\delta \to 0^+} \frac{M_\delta(\psi)(x) - \psi(x)}{\delta^{\frac{n}{n+2}}} = c_{n+1} \cdot \left( \det(\nabla^2 \psi(x)) \right)^{\frac{1}{n+2}}.
\]

(32)

As \( \nabla^2 \psi \) exists almost everywhere, the proof of (32) can be separated into two cases.

**Case 1:** The Hessian matrix \( \nabla^2 \psi(x) \) is positive definite and \( N_{\psi}(z_x) \) exists uniquely at \( x \in \mathbb{R}^n \).

In this case, (32) follows immediately from the following claim: for all \( \varepsilon > 0 \) small enough, there exists \( \delta_1(\varepsilon) \) such that for all \( 0 < \delta < \delta_1 \),

\[
  (1 - \varepsilon)^3 \leq \frac{M_\delta(\psi)(x) - \psi(x)}{c_{n+1} \cdot \delta^{\frac{n}{n+2}} \cdot \det(\nabla^2 \psi(x))^{\frac{1}{n+2}}} \leq (1 - \varepsilon)^{-5}.
\]

(33)

We first assume that the indicatrix of Dupin at \( z_x \in \partial \text{epi}(\psi) \) is a sphere of radius \( \sqrt{\rho} \). Then (see, e.g., [28, equation (5)])

\[
  \rho = (k_{\psi}(z_x))^{-\frac{1}{n}}.
\]

(34)

It follows from (8) that

\[
  \langle -N_{\psi}(z_x), e_{n+1} \rangle = (1 + \| \nabla \psi(x) \|)^2 \cdot \rho^{-\frac{n}{n+2}} (1 + \| \nabla \psi(x) \|)^{\frac{1}{2}} = \det(\nabla^2 \psi(x))^{\frac{1}{n+2}}.
\]

(35)

And, see [28], locally around \( z_x \), \( \text{epi}(\psi) \) can be approximated by \( B^{n+1}_2(z_x - \rho N_{\psi}(z_x), \rho) \).

We now describe the approximation in more details. Let \( \varepsilon > 0 \) be a given small enough number. For simplicity, let

\[
  B = B^{n+1}_2(z_x - \rho N_{\psi}(z_x), \rho),
\]

\[
  B(\varepsilon^-) = B^{n+1}_2(z_x - (1 - \varepsilon)\rho N_{\psi}(z_x), (1 - \varepsilon)\rho),
\]

\[
  B(\varepsilon^+) = B^{n+1}_2(z_x - (1 + \varepsilon)\rho N_{\psi}(z_x), (1 + \varepsilon)\rho).
\]

It is easily checked that there is a \( \Xi_\varepsilon > 0 \) such that, (see e.g., the proof of [29, Lemma 23]), for all \( 0 < t \leq \Xi_\varepsilon \),

\[
  H^-(-N_{\psi}(z_x), t) \cap B(\varepsilon^-) \subseteq H^-(-N_{\psi}(z_x), t) \cap \text{epi}(\psi) \subseteq H^-(-N_{\psi}(z_x), t) \cap B(\varepsilon^+).
\]

(36)

This further implies
\[ M_\delta(H^-(-N_\psi(z_x), t) \cap B(e^-)) \subseteq M_\delta(H^-(-N_\psi(z_x), t) \cap \text{epi}(\psi)) \subseteq M_\delta(H^-(-N_\psi(z_x), t) \cap B(e^+)). \] (37)

Although Lemma 2.2 was proved for convex bodies, it can be checked that the same result also holds for \( \text{epi}(\psi) \) with \( z_x \in \partial \text{epi}(\psi) \). Indeed, Lemma 2.2 is local around the point \( z_x \). Thus we apply Lemma 2.2 with \( t = \Xi_\epsilon \) to \( B(e^-), \text{epi}(\psi), \) and \( B(e^+) \), respectively. Hence, there exists \( r > 0 \) such that for all \( \delta > 0 \),

\[ M_\delta(B(e^-) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap B_2^{\delta+1}(z_x, r) = M_\delta(B(e^-)) \cap B_2^{\delta+1}(z_x, r), \] (38)

\[ M_\delta(\text{epi}(\psi) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap B_2^{\delta+1}(z_x, r) = M_\delta(\text{epi}(\psi)) \cap B_2^{\delta+1}(z_x, r). \] (39)

and

\[ M_\delta(B(e^+) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap B_2^{\delta+1}(z_x, r) = M_\delta(B(e^+)) \cap B_2^{\delta+1}(z_x, r). \] (40)

We choose \( \Theta_\delta < \Xi_\epsilon \) small enough that

\[ H^-(-N_\psi(z_x), \Theta_\delta) \subseteq H^-(-N_\psi(z_x), \Xi_\epsilon) \text{ and } H^-(-N_\psi(z_x), \Theta_\delta) \cap B(e^+) \subseteq B_2^{\delta+1}(z_x, r). \] (41)

We intersect both sides of (38), (39), and (40) with sets \( H^-(-N_\psi(z_x), \Theta_\delta) \cap B(e^-), H^-(-N_\psi(z_x), \Theta_\delta) \cap \text{epi}(\psi), \) and \( H^-(-N_\psi(z_x), \Theta_\delta) \cap B(e^+) \), respectively. Thus

\[ M_\delta(B(e^-) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap H^-(-N_\psi(z_x), \Theta_\delta) = M_\delta(B(e^-)) \cap H^-(-N_\psi(z_x), \Theta_\delta), \]

\[ M_\delta(\text{epi}(\psi) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap H^-(-N_\psi(z_x), \Theta_\delta) = M_\delta(\text{epi}(\psi)) \cap H^-(-N_\psi(z_x), \Theta_\delta), \]

and

\[ M_\delta(B(e^+) \cap H^-(-N_\psi(z_x), \Xi_\epsilon)) \cap H^-(-N_\psi(z_x), \Theta_\delta) = M_\delta(B(e^+)) \cap H^-(-N_\psi(z_x), \Theta_\delta). \]

By (37) we get

\[ M_\delta(B(e^-)) \cap H^-(-N_\psi(z_x), \Theta_\delta) \subseteq M_\delta(\text{epi}(\psi)) \cap H^-(-N_\psi(z_x), \Theta_\delta) \subseteq M_\delta(B(e^+)) \cap H^-(-N_\psi(z_x), \Theta_\delta). \]

Let \( z_\delta = (x, M_\delta(\psi)(x)) \). Choose \( \delta \) so small that \( z_\delta \in H^-(-N_\psi(z_x), \Theta_\delta) \) and \( z_\delta \in B(e^-) \). Clearly, in Fig. 2 we see that

\[ z_\delta \in \text{int}(M_\delta(B(e^+)) \cap H^-(-N_\psi(z_x), \Theta_\delta)) \text{ and } z_\delta \notin \text{int}(M_\delta(B(e^-)) \cap H^-(-N_\psi(z_x), \Theta_\delta)). \] (42)

Denote by \( \Delta_{\rho^+} \) the difference of the radii of \( B(e^+) \) and \( M_\delta(B(e^+)) \). Let \( \Delta_{h^+} \) be the height of a cap of \( B(e^+) \) which has volume \( \delta \). It follows from Lemma 4.1 that

\[ \lim_{\delta \to 0^+} \frac{\Delta_{h^+}}{\Delta_{\rho^+}} = \frac{n + 4}{n + 2}. \]
Fig. 2 It illustrates the position of $z_\delta$

Hence, for $\varepsilon > 0$ small enough, there is a constant $\delta_2 = \delta_2(x, \varepsilon)$ such that for all $0 < \delta \leq \delta_2$,

$$\frac{n + 4}{n + 2} - \varepsilon \leq \frac{\Delta h^+}{\Delta \rho^+} \leq \frac{n + 4}{n + 2} + \varepsilon. \quad (43)$$

Lemma 2.3 yields that

$$\delta \leq (d_{n+1})^{-\frac{n^2 + 2}{2}} ((1 + \varepsilon) \rho)^\frac{n}{2} (\Delta h^+)^{-\frac{n + 2}{2}}. \quad (44)$$

Moreover, we assume that for all $0 < \delta \leq \delta_2$,

$$\text{dist}(z_\delta, (B(\varepsilon^+))^\circ) - \text{dist}(z_\delta, B^c) \leq \varepsilon \text{dist}(z_\delta, B^c).$$

From (42), one gets

$$\Delta \rho^+ \leq \text{dist}(z_\delta, (B(\varepsilon^+))^\circ) \leq (1 + \varepsilon) \text{dist}(z_\delta, B^c). \quad (45)$$

This, together with (43), (44), and (45), imply that

$$d_{n+1}\delta^{\frac{2}{n+2}} \leq ((1 + \varepsilon) \rho)^\frac{n}{2} \left( \frac{n + 4}{n + 2} + \varepsilon \right) ((1 + \varepsilon) \text{dist}(z_\delta, B^c)). \quad (46)$$

Applying Lemma 2.1 with $\alpha$, $z$, $-e_{n+1}$, and $z_n$ replaced by $z_x$, $z_\delta$, $N_\psi(z_x)$, and $(M_\delta(\psi)(x) - \psi(x))(-N_\psi(z_x), e_{n+1})$, respectively, one gets that for $\delta > 0$ small enough,

$$(M_\delta(\psi)(x) - \psi(x))\left(-N_\psi(z_x), e_{n+1}\right) \geq \text{dist}(z_\delta, B^c). \quad (47)$$
Together with (35) and (46), we obtain the following inequality:

\[
\frac{M_\delta(\psi)(x) - \psi(x)}{d_{n+\delta}^{\frac{n+2}{2}}} \geq \left( (1 + \varepsilon)\rho - \frac{n}{\pi^{\frac{n}{2}}} \right)^{-1} \left[ \left( \frac{n+4}{n+2} + \varepsilon \right) \cdot \frac{(1 + \varepsilon)\text{dist}(z_\delta, B^c)}{\text{dist}(z_\delta, B^c)(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}} \right]^{-1} \\
\geq \left( (1 + \varepsilon)\rho - \frac{n}{\pi^{\frac{n}{2}}} \right)^{-1} (1 - \frac{n+2}{n+4} \varepsilon) \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{1 + \varepsilon} \\
\geq (1 + \varepsilon)^{\frac{-n+4}{n+2}} \left( \frac{n+4}{n+2} \right)^{-1} \left( 1 - \frac{n+2}{n+4} \varepsilon \right) \text{det}(\nabla^2 \psi(x))^{\frac{1}{n+2}}.
\]

After rearrangement, and using the relation between \(c_{n+1}\) and \(d_{n+1}\) (see (2) and (15), respectively), one gets for all \(0 < \delta \leq \delta_2\),

\[
\frac{M_\delta(\psi)(x) - \psi(x)}{c_{n+\delta}^{\frac{n+2}{2}}} \geq (1 - \varepsilon)^{\frac{1}{n+2}} \left( \text{det}(\nabla^2 \psi(x)) \right)^{\frac{1}{n+2}},
\]

(48)

where we have used \(\frac{1}{1-a} \geq 1 - a\) for all \(a \in (0, 1)\).

Let us now prove the upper bound in (33). Denote by \(\Delta_{\rho^-}\) the difference of the radii of \(B(\varepsilon^-)\) and \(M_\delta(B(\varepsilon^-))\). Let \(\Delta_{h^-}\) be the height of a cap of \(B(\varepsilon^-)\) which has volume \(\delta\). It follows from Lemma 4.1 that

\[
\lim_{\delta \to 0^+} \frac{\Delta_{h^-}}{\Delta_{\rho^-}} = \frac{n+4}{n+2}.
\]

Hence, for \(\varepsilon > 0\) small enough, there is a constant \(\delta_3 = \delta_3(x, \varepsilon)\) such that for all \(0 < \delta \leq \delta_3\),

\[
\frac{n+4}{n+2} - \varepsilon \leq \frac{\Delta_{h^-}}{\Delta_{\rho^-}} \leq \frac{n+4}{n+2} + \varepsilon.
\]

(49)

Lemma 2.3 yields that

\[
\delta \geq \left( d_{n+1} \right)^{-\frac{n+2}{2}} \rho^{\frac{n}{2}} (1 - \varepsilon)^{\frac{n}{2}} \left( 1 - \frac{\Delta_{h^-}}{2(1 - \varepsilon)\rho} \right)^{\frac{n}{2}} (\Delta_{h^-})^{\frac{n+2}{2}}.
\]

(50)

Moreover, we assume that for all \(0 < \delta \leq \delta_3\),

\[
\text{dist}(z_\delta, B^c) - \text{dist}(z_\delta, (B(\varepsilon^-))^c) \leq \varepsilon \text{dist}(z_\delta, B^c)
\]

and \(\Delta_{h^-} < \rho \varepsilon\). From (42), one gets

\[
\Delta_{\rho^-} \geq \text{dist}(z_\delta, (B(\varepsilon^-))^c) \geq (1 - \varepsilon)\text{dist}(z_\delta, B^c).
\]

(51)
This, together with (49), (50), (51), $\Delta_{h^-} < \rho \varepsilon$ and $\varepsilon \in (0, (n + 1)^{-2})$, imply that
\[
d_{n+1} \delta_{\frac{n}{n+2}}^2 \geq \rho \frac{n}{n+2} (1 - \varepsilon) \frac{n}{n+2} \left(1 - \frac{\Delta_{h^-}}{2(1 - \varepsilon)\rho}\right) \frac{n}{n+2} \left(\frac{n+4}{n+2} - \varepsilon\right) \left((1 - \varepsilon)\operatorname{dist}(z_\delta, B^c)\right)
\geq \rho \frac{n}{n+2} (1 - \varepsilon) \frac{2n+2}{n+2} \left(1 - \frac{\Delta_{h^-}}{\rho}\right) \frac{n}{n+2} \left(1 - \frac{(n+2)\varepsilon}{n+4}\right) \operatorname{dist}(z_\delta, B^c)
\geq \rho \frac{n}{n+2} (1 - \varepsilon) \frac{4n+4}{n+2} \frac{n+4}{n+2} \operatorname{dist}(z_\delta, B^c).\tag{52}
\]
Applying again Lemma 2.1, one gets that for $\delta > 0$ small enough,
\[
\left(M_\delta(\psi)(x) - \psi(x)\right) \left(-N_\psi(z_x, e_{n+1})\right) \leq \operatorname{dist}(z_\delta, B^c) \left(1 + \frac{2\operatorname{dist}(z_\delta, B^c)}{\rho \left(-N_\psi(z_x, e_{n+1})\right)^2}\right)
\leq \operatorname{dist}(z_\delta, B^c)(1 + \varepsilon).\tag{53}
\]
Together with (35), (52) and the relation between $c_{n+1}$ and $d_{n+1}$ (see (2) and (15), respectively), one has for all $0 < \delta \leq \delta_3$,
\[
\frac{\left.M_\delta(\psi)(x) - \psi(x)\right)}{c_{n+1} \delta_{\frac{n}{n+2}}^2} \leq \rho^{-\frac{n}{n+2}} (1 - \varepsilon) \frac{4n+4}{n+2} (1 + \varepsilon) (1 + \|\nabla \psi(x)\|^2) \frac{1}{2}
\leq (1 - \varepsilon)^{-5} \det(\nabla^2 \psi(x)) \frac{1}{\pi^{\frac{n}{2}}},\tag{54}
\]
where we have in the second inequality again used that $\frac{1}{1+a} \geq 1 - a$ for all $a \in (0, 1)$. In conclusion, (48) and (54) give the desired inequality (33) with $\delta_1 = \min\{\delta_2, \delta_3\}$ under the assumption that the indicatrix of Dupin at $z_x$ is a sphere.

If the indicatrix of Dupin at $z_x \in \partial\operatorname{epi}(\psi)$ is an ellipsoid instead of a sphere, one can find a volume preserving affine transformation which maps the indicatrix of Dupin at $z_x \in \partial\operatorname{epi}(\psi)$ into a sphere. As the determinant remains unchanged under a volume preserving affine transformation, it is easily checked that the desired inequality (33) still holds if the indicatrix of Dupin at $z_x \in \partial\operatorname{epi}(\psi)$ is the boundary of an ellipsoid instead of a sphere, due to, for instance, Proposition 3.5. This completes the proof of the desired inequality (33), which in turn implies the limit (32).

**Case 2**: Assume that $\det(\nabla^2 \psi(x)) = 0$ and $N_\psi(z_x)$ exists uniquely at $x \in \mathbb{R}^n$.

In this case, (32) follows immediately from the following claim: for all $\varepsilon > 0$ small enough, there is $\delta_4 = \delta_4(x, \varepsilon)$ such that for all $0 < \delta \leq \delta_4$, one has
\[
0 \leq \frac{M_\delta(\psi)(x) - \psi(x)}{\delta_{\frac{n}{n+2}}^2} \leq b_0 \varepsilon \frac{n}{\pi^{\frac{n}{2}}} (1 - \varepsilon)^{-3}.\tag{55}
\]
As $\det(\nabla^2 \psi(x)) = 0$, the indicatrix of Dupin at $z_x$ is the surface of an elliptic cylinder. This provides again an approximation of $\partial(\operatorname{epi}(\psi))$ around $z_x$. Indeed, for any $\varepsilon > 0$ small enough, there is an ellipsoid $\tilde{E}$ and a constant $\Xi_\varepsilon > 0$ such that, (without loss of generality) the lengths of the first $k$ principal axes of $\tilde{E}$ are larger than $\varepsilon^{-1}$, and for all $0 < s \leq \Xi_\varepsilon$,
\[ \mathcal{E} \cap H^{-}(-N_{\psi}(z_{x}), s) \subseteq \text{epi}(\psi) \cap H^{-}(-N_{\psi}(z_{x}), s). \]  

(56)

A proof of this argument can be found in the proof of [29, Lemma 23]. There, the corresponding statement was proved for convex bodies but the argument given there works for an unbounded convex set \( \text{epi}(\psi) \) as well.

Let \( \delta > 0 \) be small enough and \( 0 < \Theta_{\delta} < \Xi_{\varepsilon} \) be such that

\[
\tilde{z}_{\delta} = (x, M_{\delta}(\psi)(x)) \in \partial M_{\delta}(\text{epi}(\psi)) \cap H^{-}(-N_{\psi}(z_{x}), \Theta_{\delta}).
\]

It is enough to prove the claim when \( \tilde{z}_{\delta} \in \text{int}(\text{epi}(\psi)) \) for all \( \delta > 0 \). If that is not the case, there is nothing to prove as one finds a constant \( \delta_{4} > 0 \) such that \( M_{\delta}(\psi)(x) - \psi(x) = 0 \) for all \( \delta \in (0, \delta_{4}) \).

Let \( \tilde{z}_{\delta} \in \text{int}(\text{epi}(\psi)) \) for all \( \delta > 0 \) small enough. Assume that the approximating ellipsoid \( \tilde{E} \) is a Euclidean ball with radius \( \rho \) such that \( \rho \geq \varepsilon^{-1} \). Let \( \Delta_{h} \) denote the height of a cap of \( \tilde{E} \) which has volume \( \delta \) and let \( \tilde{\Delta}_{\rho} \) be the difference of the radii of \( \tilde{E} \) and \( M_{\delta}(\tilde{E}) \). There exists \( \delta_{4} = \delta_{4}(x, \varepsilon) \) such that, for all \( 0 < \delta < \delta_{4} \),

\[
\frac{n + 4}{n + 2} - \varepsilon \leq \frac{\tilde{\Delta}_{h}}{\Delta_{\rho}} \leq \frac{n + 4}{n + 2} + \varepsilon.
\]

(57)

Moreover, as \( \rho \geq \varepsilon^{-1} \) and by Lemma 2.3, one gets

\[
\left(1 - \frac{\tilde{\Delta}_{h}}{2\rho}\right)^{\frac{n}{n+2}} \leq d_{n+1} \cdot \delta \frac{2}{n+2} \left(\frac{1}{\rho}\right)^{\frac{n}{n+2}} \leq d_{n+1} \cdot \delta \frac{2}{n+2} e^{\frac{n}{n+2}}.
\]

(58)

Thus, we can also assume that \( \tilde{\Delta}_{h} < 2\varepsilon \) for all \( 0 < \delta < \delta_{4} \). Appplying again Lemma 2.1, (53) holds with \( B \) replaced by \( \tilde{E} \). That is, for \( \delta > 0 \) small enough,

\[
(M_{\delta}(\psi)(x) - \psi(x)) \cap (-N_{\psi}(z_{x}), e_{n+1}) \leq \text{dist}(z_{\delta}, \tilde{E}) \cdot \left(1 + \frac{2\text{dist}(z_{\delta}, \tilde{E})}{\rho(-N_{\psi}(z_{x}), e_{n+1})}\right)
\]

\[
\leq \text{dist}(z_{\delta}, \tilde{E})(1 + \varepsilon).
\]

Note that \( \tilde{\Delta}_{\rho} > \text{dist}(z_{\delta}, \tilde{E}) \). For \( \varepsilon > 0 \) small enough and as \( \rho \geq \varepsilon^{-1} \), one has with (2), (15), (57), and (58) that for all \( 0 < \delta \leq \delta_{4} \),

\[
\frac{M_{\delta}(\psi)(x) - \psi(x)}{c_{n+1}\delta^{\frac{2}{n+2}}} \leq b_{0} \left(1 - \frac{\tilde{\Delta}_{h}}{2\rho}\right)^{-\frac{n}{n+2}} \left(\frac{d_{n+1}\text{dist}(z_{\delta}, \tilde{E})}{c_{n+1}\Delta_{h}}\right)(1 + \varepsilon)
\]

\[
\leq b_{0} \left(1 - \varepsilon\right)^{-\frac{n}{n+2}} \left(1 - \frac{(n+2)\varepsilon}{n+4}\right)^{-1}(1 + \varepsilon)
\]

\[
\leq b_{0} \left(1 - \varepsilon\right)^{-\frac{3n+4}{n+2}} \leq b_{0} \left(1 - \varepsilon\right)^{-3},
\]

(59)

where \( b_{0} = (-N_{\psi}(z_{x}), e_{n+1})^{-1} \). This implies the desired inequality (55) for all \( 0 < \delta \leq \delta_{4} \) under the assumption that \( \tilde{E} \) is a Euclidean ball. If \( \tilde{E} \) is an ellipsoid, then a
volume preserving affine transformation maps $\tilde{\mathcal{E}}$ into a Euclidean ball and we conclude as above.

This finishes the proof of inequality (55), which in turn implies the limit (32).  

\section*{Declarations}

\subsection*{Conflict of interest}

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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