G-parity breaking contributions in $\tau^- \rightarrow \eta^{(i)}\pi^-\nu_\tau$ decays induced at one loop level by the anomalous $\gamma\gamma\eta^{(i)}$ vertices

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Abstract. Second class current processes (SCC) are forbidden in the limit of exact isospin symmetry by G-parity. Since G-parity invariance is slightly broken by the $u – d$ mass difference and electromagnetic effects, SCC can appear but they are naturally suppressed. $\tau^- \rightarrow \eta^{(i)}\pi^-\nu_\tau$ decays represent a clean test of SCC and according to theoretical estimations and thanks to the increased luminosity contemplated at the Belle-II collaboration, their discovery should be finally possible. Consequently, a careful evaluation of G-parity breaking contributions results mandatory in order to isolate possible new physics contributions. In this work, we evaluate the G-parity breaking contributions in $\tau^- \rightarrow \eta^{(i)}\pi^-\nu_\tau$ decays induced at one loop level by the anomalous electromagnetic $\gamma\gamma\eta^{(i)}$ vertices.

1. Introduction
The study of the G-parity\(^1\) invariance plays a fundamental role in considering the effects of strong interactions on weak processes, based on that Weinberg established a classification for non-strange weak (V-A) hadronic currents\([1]\). According to this classification, weak hadronic currents with isospin 0, 1 quantum numbers can be split into two classes. The first class includes currents with quantum numbers $J^{PG} = 0^{++}, 0^{-+}, 1^{++}, 1^{+-}$, whereas the second class currents (SCC) have opposite G-parity $J^{PG} = 0^{-+}, 0^{++}, 1^{+-}, 1^{--}$. SCC can not occur if G-parity were an exact symmetry, but since G-parity invariance is slightly broken by isospin non-conservation, electromagnetic effects and the mass difference of $u – d$ quarks can induce the hadronization of the standard model (SM) currents into states that mimic the effects of a genuine SCC\(^2\).

Several channels for SCC have been proposed involving nuclear $\beta$ decay processes, where the many form factors and small momentum transfer have driven to a complicated search\([4]\). On the other hand, a clean test for the existence of SCC would be provided by the observation of the semileptonic transitions $\tau^- \rightarrow \eta^{(i)}\pi^-\nu_\tau$\([6]\), since the G-parity of the system is $-1$, which is opposed to the vector current that drives the decay in the SM. Consequently, the S(P)-wave of the $\pi^-\eta^{(i)}$ system gives $J^{PG} = 0^{+-}(1^{--})$, which as mentioned before corresponds to

\(^1\) G operator is defined as $G = C e^{i\pi I_2}$, with $C$ the charge conjugation operator and $I_2$ the second component of the isospin rotation operator.

\(^2\) For genuine SCC we mean effects due to New Physics; for example, the ones induced by the exchange of charged Higgs or leptoquarks particles. A recent study of the effects of New Physics in $\tau^- \rightarrow \eta^{(i)}\pi^-\nu_\tau$ decays using the framework of a general effective field theory can be found in [5].
a SCC independently of possible intermediate resonant states. Regarding experimental search of \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \) decays, CLEO and Belle collaborations reported previous limits on their branching fractions, but the most stringent bounds available are based on searches by the BaBar collaboration [7] corresponding to \( BR(\tau^- \to \eta \pi^- \nu_\tau) < 9.9 \times 10^{-5} \) and \( BR(\tau^- \to \eta' \pi^- \nu_\tau) < 7.2 \times 10^{-6} \) [8], which lie close to the estimates based on isospin symmetry breaking [9] for the \( BR(\tau^- \to \eta^{(i)} \pi^- \nu_\tau) \) decays mainly induced by the \( u - d \) quark mass difference [10, 11].

We are interested in the study of the G-parity breaking contribution in \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \) decays induced at one loop level by the anomalous electromagnetic \( \gamma \gamma \eta^{(i)} \) vertices. In spite of this kind of contribution is expected to give only a minor correction with respect to tree level (\( (\pi^0 - \eta - \eta' \) mixing) contributions), it is crucial to have a reliable estimate of these effects in order to eliminate a possible source of background for a genuine SCC \(^3\). In order to compare the different contributions to \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \) decays, we show in section 2 the relevant formulas for the tree level result \(^4\), whereas a brief explanation of the approach used in order to calculate the one-loop contribution is given in section 3. Then, an evaluation of our results is presented in section 4. Finally, the conclusions are presented in section 5.

2. Tree level contribution (Hadronic matrix elements)

Into the SM, the semileptonic \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \) decays are induced at tree level due to the charged weak interaction between the tau lepton and its corresponding neutrino and later hadronization of the current. Since parity conservation implies that only weak vector current contributes to the hadronic part, the amplitude of the decay \( \tau^- \to \pi^- \eta^{(i)} \nu_\tau \) it can be written as

\[
\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} \bar{u}(p_\nu_\tau) \gamma_\mu (1 - \gamma_5) u(p_\tau) \langle \pi^- \eta^{(i)} | \bar{d} \gamma^\mu u | 0 \rangle, \tag{1}
\]

where the hadron matrix element is

\[
\langle \pi^- \eta^{(i)} | \bar{d} \gamma^\mu u | 0 \rangle = \left[ (p_{\eta^{(i)}} - p_\pi)^\mu + \frac{\Delta \pi - \eta^{(i)}}{s} q^\mu \right] c_{\pi - \eta^{(i)}}^V F_{\pi \eta^{(i)}}^V(s) + \frac{\Delta_{K^0 K^+} q^\mu c_{\pi - \eta^{(i)}}^S F_{0}^{\pi - \eta^{(i)}}(s)}{s} \tag{2}
\]

and the following definitions have been used \( s = q^2 = (p_\pi + p_{\eta^{(i)}})^2 \), \( c_{\pi - \eta^{(i)}}^V = \sqrt{2} \), \( c_{\pi - \eta}^S = \frac{2}{\sqrt{3}} \), \( \Delta_{\pi - \eta} = m^2_\pi - m^2_\eta \) and the superscript \( QCD \) indicates that the \( K^0 K^+ \) electromagnetic mass difference does not contribute to eq. (2).

The differential partial decay width, as a function of the \( \pi^- \eta^{(i)} \) invariant mass, is

\[
\frac{d\Gamma(\tau^- \to \pi^- \eta^{(i)} \nu_\tau)}{ds} = \frac{G_F^2 M_\tau^3}{24 \pi^3} S_{EW} \left| V_{ud} F_{\pi \eta^{(i)}}^{\pi - \eta^{(i)}}(0) \right|^2 \left( 1 - \frac{s}{M_\tau^2} \right)^2 \times \left\{ \left( 1 + \frac{2s}{M_\tau^2} \right) q_{\pi - \eta^{(i)}}^2 \bar{F}_{\pi \eta^{(i)}}(s)^2 + \frac{3 \Delta_{\pi - \eta^{(i)}}^2}{4s} q_{\pi - \eta^{(i)}}(s) \bar{F}_{0}^{\pi - \eta^{(i)}}(s)^2 \right\}, \tag{3}
\]

\(^3\) Another possible source of background to \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \) decays is given by the \( \tau^- \to \eta^{(i)} \pi^- \nu_\tau \gamma \) channels since they are not suppressed by G-parity considerations, however, this kind of background can be eliminated considering appropriate cuts in the energy of the photons [12].

\(^4\) A complete explanation of the theoretical construction of the participant vector and scalar form factors in eqs. (2) and (3) can be found in Ref. [13], where the authors employ the state-of-the-art analysis of meson-meson scattering within unitarized Chiral Perturbation Theory [15] in order to obtain the scalar form factors.
Figure 1. Feynman diagrams for the $\tau^{-} \to \eta^{(l)} \pi^{-} \nu_{\tau}$ decays at one loop level induced by $\gamma \gamma \eta^{(l)}$ vertices. It is easy to check that the contributions from diagrams (c), (e) and (f) are identically zero (see text).

where $F_{\pi^{-}\eta^{(l)}}(s) = \frac{F_{\pi^{-}\eta^{(l)}}(s)}{F_{\pi^{-}\eta^{(l)}}(0)}$ stand for the two form factors normalized to unity at the origin. In eq. (3) the short-distance electroweak correction factor, $S_{\text{EW}} = 1.0201$ [26], has been introduced and

$$q_{PQ}(s) = \frac{\sqrt{s^2 - 2s\Sigma_{PQ} + \Delta_{PQ}^2}}{2\sqrt{s}}, \quad \Sigma_{PQ} = m_{P}^2 + m_{Q}^2.$$ (4)

3. $\tau^{-} \to \eta^{(l)} \pi^{-} \nu_{\tau}$ decays induced at one loop level

Besides the contribution given by the eq. (1), electromagnetic interactions also break isospin symmetry and will contribute to $\tau^{-} \to \eta^{(l)} \pi^{-} \nu_{\tau}$ decays. This can occur at the one-loop level, via the emission of a pair of photons from $\tau^{-} \to \pi^{-} \nu_{\tau}$ decays and their later conversion into an $\eta^{(l)}$ meson through the anomalous $\gamma \gamma \eta^{(l)}$ vertices as shown in Figure 1. In order to calculate this kind of contribution, we have considered, in the low-energy limit, a point-like interaction for the $\tau^{-} \to \nu_{\tau} \pi^{-}$ vertex, which can be described by the Lagrangian density

$$\mathcal{L} = G_{F}V_{ud}f_{\pi}\bar{\nu}_{\ell}\gamma_{\mu}(1-\gamma_{5})\ell\partial^{\mu}\pi^{+} + \text{h.c.},$$ (5)

where $f_{\pi} = F_{\pi}/\sqrt{2} = 92.2$ MeV. A detailed explanation of the low- and high-energy limits of QCD for the structure of the pion electromagnetic coupling and the two-photon coupling of the neutral meson is discussed in appendix A of [21], where we also explain the cancellation of ultraviolet divergences in the loop integrals.

We have verified that the contributions from diagrams (c) and (f) vanish due to the conservation of P and CP by strong and electromagnetic interactions, whereas the contribution from diagram (e) vanishes when considering the loop integration because it is odd in the

5 Obviously, this type of vertex would also contribute to the $\tau^{-} \to \pi^{-}\pi^{0} \nu_{\tau}$ decays. However, as we will check, it is negligible in this case because the tree level contribution is not suppressed.
integration variable. On the other hand, after performing the loop integration for diagrams in Fig. 1 (a), (b) and (d) and employing the Chisholm identity \(^6\) for the Levi-Civita tensor contracted with a gamma matrix, we get the following generic form for each of the non-vanishing amplitudes

\[
\mathcal{M}^k = \frac{e^4 G_F V_{ud} f_\pi}{16 \pi^2} g_{\gamma\gamma\eta}(q) \bar{u}(p_\nu) \left[ F_0^k P_R + F_1^k \gamma_\pi P_L \right] u(p_\tau),
\]

where the superindex \(k = a, b, d\) labels the non-vanishing Feynman diagrams in Figure 1.

The form factors \(F_{0,1}^k\) are generated by the loop integration, they have been obtained using the Mathematica packages FeynCalc \(^22\) and are given in terms of the invariant Passarino-Veltman (PaVe) scalar functions \(^20\). Further, they can be understood as corrections to the form factors appearing in eq. (2), the relations between both will be presented in the next section. The factor \(g_{\gamma\gamma\eta}(q)\) stands for the value of the \(\gamma(\gamma)\gamma(\gamma)\eta(\gamma)\) form factor for on-shell photons, which is a global dependence of the matrix element (6) and is given by

\[
g_{\gamma\gamma\eta} = \left( \frac{5}{3} C_\gamma - \frac{\sqrt{3}}{3} C_s \right) g_{\gamma\gamma\eta}, \quad g_{\gamma\gamma\eta'} = g_{\gamma\gamma\eta} (C_q \rightarrow C_{q'}, C_s \rightarrow -C_{s'}) ,
\]

where the input values for the mixing coefficients can be found in Ref. \(^17\).

The square of the total decay amplitude (\(\mathcal{M} = \sum_k \mathcal{M}^k\)) is given by

\[
|\mathcal{M}|^2 = \left( \frac{e^4 |V_{ud}| G_F f_\pi g_{\gamma\gamma\eta}(q)}{128 \pi^4} \right)^2 \left[ p_\nu \cdot p_\tau \left( |F_0|^2 - |F_1|^2 m_\pi^2 \right) \right] + 2 p_\nu \cdot p_\tau \left( |F_1|^2 p_\tau \cdot p_\pi + m_\pi \text{Re}[F_0 F_1^*] \right),
\]

where \(F_0 = \sum_{k=a,b,d} F_{0}^k\) and \(F_1 = \sum_{k=a,b,d} F_{1}^k\). The explicit expression for these form factors can be found as well in appendix B of \(^21\), they are given in terms of the independent kinematical scalars \(s_{12} = (p_\pi + p_\eta)^2 = (p_\tau - p_\nu)^2\) (the square of the invariant-mass of the hadronic system) and \(s_{13} = (p_\pi + p_\nu)^2 = (p_\tau - p_\eta)^2\).

### 4. Evaluation

An evaluation of the individual contribution of eq. (8) can be done straightforwardly considering the well know kinematics for a three body decay. In order to distinguish this individual contribution, we will employ the notation \(BR_{\pi\eta}^{\gamma\gamma} \equiv BR(\tau^{-} \rightarrow P^0 \pi^{-} \nu_{\tau})\) when \(P^0\) is produced from a two-photon intermediate state \((P = \pi, \eta, \eta')\), we obtain

\[
BR_{\pi}^{\gamma\gamma} = 5.3 \cdot 10^{-13}, \quad BR_{\eta}^{\gamma\gamma} = 5.2 \cdot 10^{-13}, \quad BR_{\eta'}^{\gamma\gamma} = 0.8 \cdot 10^{-16}.
\]

Notice that in the case of \(P = \pi^0\) the ratio between the number in eq. (9) and the corresponding measured branching fraction \((BR_{\pi^0} \sim 25\%)\) is at the level 10\(^{-11}\), which validates neglecting the one-loop contribution in the \(\tau^{-} \rightarrow \pi^{-} \pi^{-} \nu_{\tau}\) decays, as we anticipated. On the other hand, the ratio for the \(\eta(\gamma)\) modes between the numbers in eq. (9) and their respective branching fractions predicted at the tree level in Ref. \(^13\) \((BR_{\eta}^{\gamma\gamma} \sim 1.7 \cdot 10^{-5}\) and \(10^{-7} \leq BR_{\eta'}^{\gamma\gamma} \leq 10^{-8}\)) are of the order of 10\(^{-8}\).

Now, we turn to the evaluate the branching ratio including the sum of the tree level \(^13\) and the one-loop contribution. We note that there is an equivalence between the \(F_{0,1}\) form factors appearing in eq. (2) and our \(F_{0,1}\) form factors in Eq. (6). It turns out appropriate to use an upper index \(\gamma\gamma\) for the \(F_{0,1}\) form factors defined in eq. (6) in order to avoid confusion with the \(F_0\)

\(^6\) Chisholm identity reads \(\gamma_{\mu} \epsilon^{\mu\nu\rho\sigma} = i (\gamma^{\nu} \gamma^{\rho} \gamma^{\sigma} - g^{\nu\rho} \gamma^{\sigma} - g^{\nu\sigma} \gamma^{\rho} + g^{\rho\sigma} \gamma^{\nu}) \gamma^{\gamma}\).
form factor appearing in eq. (2). In this way, we can include the electromagnetic contribution into the vector and scalar form factors by shifting

$$F_{\pi \eta'}(s_{12}) \rightarrow F_{\pi \eta'}(s_{12}) + F_{\gamma \gamma}(s_{12}, s_{13}),$$

where:

$$F_{\gamma \gamma} = -\frac{e^4 f_\pi g_{\gamma \gamma \eta} F_1}{64 \pi^2},$$

$$F_{\gamma \gamma}^0 = \frac{f_\pi}{16 \sqrt{2} \pi} e^4 g_{\gamma \gamma \eta} \left[ \frac{F_0}{m_\tau} + \frac{F_1}{2} \left( 1 + \frac{\Delta_{\pi \eta}(s_{12})}{s_{12}} \right) \right],$$

with $c_{\eta}^S$ and $\Delta_{PQ}$ defined previously. Now, using the fortran version of LoopTools we have obtained

$$BR_{\eta \gamma \gamma}^{\text{tree}} - BR_{\eta}^{\text{tree}} \in \left[ -5 \cdot 10^{-9}, 2 \cdot 10^{-9} \right],$$

$$BR_{\eta' \gamma \gamma}^{\text{tree}} - BR_{\eta'}^{\text{tree}} \in \left[ -3 \cdot 10^{-12}, 3 \cdot 10^{-12} \right],$$

here the difference comes mainly from the interference or tree- and loop-level contributions and --as in Ref. [13]-- the quoted errors only arise from the uncertainties in the overall normalization factor $F_{\pi \eta}(0)$.

5. Conclusions

$\tau^{\pm} \rightarrow \eta(0)^{\pm} \pi^{\mp} \nu_\tau$ decays belong to the SCC processes, these modes are naturally suppressed by approximate $G$-parity symmetry in the SM. This suppression provides an ideal place to look for New Physics contributions, which can be of the similar size as the SM effects. Consequently, a precise evaluation of the possible $G$-parity breaking contributions is required in order to isolate the New Physics.

In this work, we have calculated the one-loop contribution to the $\tau^{\pm} \rightarrow \eta(0)^{\pm} \pi^{\mp} \nu_\tau$ decays induced by the $\gamma \gamma \eta(0)$ vertices; as far as we know, this is the first time in considering this contributions to these decays. In the case of final state with an $\eta$ meson, the two-photon intermediate states contribute -at most- with corrections at the $10^{-4}$ level and in the case of a decay with an $\eta'$ their maximum relative size can vary between $3 \cdot 10^{-4}$ and $3 \cdot 10^{-5}$ depending on the value of the tree level branching ratio. At the same time, we have verified that the analogous contributions for the $\tau \rightarrow \pi^0 \pi^- \nu_\tau$ channel are negligible, as was expected.

We can conclude that searches at forthcoming flavor factories will not be sensitive to the one loop effects in $\tau^{\pm} \rightarrow \eta(0)^{\pm} \pi^{\mp} \nu_\tau$ decays considered in this work. Even if $\tau^{\pm} \rightarrow \eta(0)^{\pm} \pi^{\mp} \nu_\tau$ decays are finally observed at Belle-II it will be very difficult to achieve a measurement with a few percent accuracy even with the complete Belle-II data sample.

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7 This is due to the fact that the tree level contribution to the $\eta'$ channel has a big uncertainty [13] ($\sim 90\%$) which is dominated by the error on $F_{\pi \eta}(0)$. 

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