Energy estimates for two-dimensional space-Riesz fractional wave equation

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Abstract The fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media which exhibits a power-law creep, and consequently provided a physical interpretation of this equation in the framework of dynamic viscoelasticity. In this paper, we first use the energy method to estimate the one-dimensional space-Riesz fractional wave equation. The stiff matrices are proved to be commutative for two-dimensional case, which ensures to carry out of the priori error estimates and the energy method. Then, the unconditional stability and convergence with the global truncation error $O(\tau^2 + h^2)$ are theoretically proved with the constant coefficients and numerically verified.

Keywords Riesz fractional wave equation · Nonlocal wave equation · Priori error estimates · Energy method · Numerical stability and convergence

1 Introduction

The fractional wave equation is obtained from the classical wave equation by replacing the second-order derivative with a fractional derivative of order $\alpha$, $1 < \alpha \leq 2$. Mainardi [18] pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media which exhibits a power-law

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creep, and consequently provided a physical interpretation of this equation in the framework of dynamic viscoelasticity. In this paper, we study a second-order accurate numerical method in both space and time for the two-dimensional space-Riesz fractional wave equation with the variable coefficients whose prototype is, for \(1 < \alpha, \beta < 2\),

\[
\frac{\partial^2 u(x, y, t)}{\partial t^2} = a(x, y) \frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} + b(x, y) \frac{\partial^\beta u(x, y, t)}{\partial |y|^\beta} + f(x, y, t), \quad (x, y) \in \Omega, \ t \in (0, T].
\]

(1.1)

The initial conditions are

\[
u(x, y, 0) = \varphi(x, y) \quad \text{for} \quad (x, y) \in \Omega,
\]

\[
u_t(x, y, 0) = \psi(x, y) \quad \text{for} \quad (x, y) \in \Omega,
\]

(1.2)

and the Dirichlet boundary condition

\[
u(x, y, t) = 0 \quad \text{for} \quad (x, y) \in \partial \Omega
\]

with \(\Omega = (0, x_r) \times (0, y_r)\). The function \(f\) is a source term and all the coefficients are positive, that is, \(0 < a_0 \leq a \leq a_1\) and \(0 < b_0 \leq b \leq b_1\) on \(\Omega\).

The space-Riesz fractional derivative appears in the continuous limit of lattice models with long-range interactions [30], for \(1 \leq \alpha < 2\), which is defined as [25]

\[
\frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} = -\kappa_\alpha \left(0D_\alpha^x + \chi D_\alpha^x\right) u(x, y, t) \quad \text{with} \quad \kappa_\alpha = \frac{1}{2 \cos(\alpha \pi / 2)},
\]

(1.3)

where

\[
0D_\alpha^x u(x, y, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_0^x (x - \xi)^{2-\alpha-1} u(\xi, y, t) d\xi,
\]

\[
\chi D_\alpha^x u(x, y, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_x^{x_r} (\xi - x)^{2-\alpha-1} u(\xi, y, t) d\xi.
\]

For the Caputo-Riesz time-space fractional wave equation \(cD_\gamma^t u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}\) with \(1 < \alpha, \gamma \leq 2\), Mainardi et al. obtained the fundamental solution of the space-time fractional diffusion equation [19]. Metzler and Nonnenmacher investigated the physical backgrounds and implications of a space- and time-fractional diffusion and wave equation [21]. For \(1 < \gamma < 2, \alpha = 2\), it has been proposed by various authors [5, 6, 16, 20, 23, 31, 34–36]. For example, based on the second-order fractional Lubich’s methods [17], Cuesta et al. derived the second-order error bounds of the time discretization in a Banach space with the \(\frac{\partial^2 u}{\partial x^2}\) a sectorial operator [6] and Yang et al. obtained the convergence schemes with \(1 \leq \gamma \leq 1.71832\) [34]. To the best of our knowledge, for problem (1.1), it is still lack of the stability and convergence analysis, although the numerical solution of space-time fractional diffusion-wave equations is discussed in [1, 7, 11]. This paper focuses on providing the weighted numerical scheme to solve the space-Riesz fractional wave equation with the variable coefficients in one-dimensional and two-dimensional cases for (1.1). The unconditional
stability and convergence with the global truncation error $O(\tau^2 + h^2)$ are theoretically proved and numerically verified by the energy method, which can be extended to the problems discussed in [1, 7, 11] and the nonlocal wave equation [10].

The rest of the paper is organized as follows. The next section proposes the second-order accurate scheme for (1.1). In Section 3, we carry out a detailed stability and convergence analysis with the second-order accuracy in both time and space directions for the derived schemes. To show the effectiveness of the schemes, we perform the numerical experiments to verify the theoretical results in Section 4. The paper is concluded with some remarks in the last section.

2 Discretization schemes

Let the mesh points $x_i = i h$, $i = 0, 1, \ldots, N_x$, and $t_k = k \tau$, $0 \leq k \leq N_t$ with $h = x_r/N_x$, $\tau = T/N_t$, where $h$ is the uniform space stepsize and $\tau$ the time stepsize. And $u^k_i$ denotes the approximated value of $u(x_i, t_k)$, $a_i = a(x_i)$, $f^k_i = f(x_i, t_k)$.

Nowadays, there are already many types of high-order discretization schemes for the Riemann-Liouville space fractional derivatives [2, 12, 14, 24, 27, 29, 32]. Here, we take the following schemes to approach (1.3) (see in [3, 32]):

$$0 D_x^\alpha u(x_i) = \delta_{x, +}^\alpha u(x_i) + O(h^2) \text{ with } \delta_{x, +}^\alpha u(x_i) = \frac{1}{h^\alpha} \sum_{m=0}^{i+1} \varphi_m^\alpha u(x_{i-m+1}),$$

$$x D_x^\alpha u(x_i) = \delta_{x, -}^\alpha u(x_i) + O(h^2) \text{ with } \delta_{x, -}^\alpha u(x_i) = \frac{1}{h^\alpha} \sum_{m=0}^{N_x-i+1} \varphi_m^\alpha u(x_{i+m-1}),$$

(2.1)

where

$$\varphi_0^\alpha = \frac{\alpha}{2} g_0^\alpha, \quad \varphi_m^\alpha = \frac{\alpha}{2} g_m^\alpha + \frac{2 - \alpha}{2} g_{m-1}^\alpha, \quad m \geq 1,$$

and

$$g_m^\alpha = (-1)^m \left( \frac{\alpha}{m} \right), \quad g_0^\alpha = 1, \quad g_m^\alpha = \left( 1 - \frac{\alpha + 1}{m} \right) g_{m-1}^\alpha, \quad m \geq 1.$$

Using (1.3) and (2.1), we obtain the approximation operator of the space-Riesz fractional derivative

$$\frac{\partial^\alpha u(x_i)}{\partial |x|^\alpha} = \nabla_h^\alpha u(x_i) + O(h^2)$$

(2.2)

with

$$\nabla_h^\alpha u(x_i) = -\kappa_\alpha \left( \delta_{x, +}^\alpha + \delta_{x, -}^\alpha \right) u(x_i) = -\frac{\kappa_\alpha}{h^\alpha} \sum_{l=0}^{N_x} \varphi_i^\alpha u(x_l),$$

$\kappa_\alpha$ Springer
where \( i = 1, \ldots, N_x - 1 \) (together with the zero Dirichlet boundary conditions) and

\[
\varphi^\alpha_{i,l} = \begin{cases} 
\varphi^\alpha_{i-1,l+1}, & l < i - 1, \\
\varphi^\alpha_0 + \varphi^\alpha_2, & l = i - 1, \\
2\varphi^\alpha_1, & l = i, \\
\varphi^\alpha_0 + \varphi^\alpha_2, & l = i + 1, \\
\varphi^\alpha_{l-i+1}, & l > i + 1. 
\end{cases}
\]

Taking \( u = [u(x_1), u(x_2), \cdots, u(x_{N_x - 1})]^T \), and using (2.1), (2.2), there exists

\[
\frac{1}{h^\alpha} \left[ \sum_{l=0}^{N_x} \varphi^\alpha_{1,l}u(x_l), \sum_{l=0}^{N_x} \varphi^\alpha_{2,l}u(x_l), \ldots, \sum_{l=0}^{N_x} \varphi^\alpha_{N_x-1,l}u(x_l) \right]^T = (\delta^\alpha_{x,+} + \delta^\alpha_{x,-}) u = \frac{1}{h^\alpha} A\alpha u,
\]

it yields

\[
\nabla^\alpha_h u = -\kappa_\alpha \left( \delta^\alpha_{x,+} + \delta^\alpha_{x,-} \right) u = -\frac{\kappa_\alpha}{h^\alpha} A\alpha u,
\]

where the matrix

\[
A\alpha = B\alpha + B^T\alpha \quad \text{with} \quad B\alpha = \begin{bmatrix}
\varphi^\alpha_1 & \varphi^\alpha_2 & \varphi^\alpha_3 & \cdots & \varphi^\alpha_{N_x-2} & \varphi^\alpha_{N_x-1} \\
\varphi^\alpha_0 & \varphi^\alpha_1 & \varphi^\alpha_2 & \cdots & \varphi^\alpha_{N_x-2} \\
\varphi^\alpha_0 & \varphi^\alpha_1 & \varphi^\alpha_2 & \cdots & \varphi^\alpha_{N_x-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\varphi^\alpha_0 & \varphi^\alpha_1 & \varphi^\alpha_2 & \cdots & \varphi^\alpha_1
\end{bmatrix}.
\]

### 2.1 Numerical scheme for one-dimensional space-Riesz fractional wave equation

We now examine the full discretization scheme to the one-dimensional space-Riesz fractional wave equation, that is,

\[
\frac{\partial^2 u(x, t)}{\partial t^2} = a(x) \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + f(x, t)
\]

with the homogeneous Dirichlet boundary condition and \( 0 < a_0 \leq a \leq a_1 \). The initial conditions are

\[
u(x, 0) = \varphi(x) \quad \text{for} \quad x \in \Omega,
\]

\[
u_t(x, 0) = \psi(x) \quad \text{for} \quad x \in \Omega.
\]

In the time direction derivative, we use the following center difference scheme

\[
\frac{\partial^2 u(x, t)}{\partial t^2} = \delta^2_t u(x_i, t_k) + \mathcal{O}(\tau^2) \quad \text{with} \quad \delta^2_t u(x_i, t_k) = \frac{u_{i}^{k+1} - 2u_{i}^{k} + u_{i}^{k-1}}{\tau^2}.
\]
In order to achieve an unconditional stable algorithm, we use the weighted algorithm for the space-Riesz fractional derivative, that is,

\[ \theta u_{i}^{k+1} + (1 - 2\theta)u_{i}^{k} + \theta u_{i}^{k-1}, \quad \frac{1}{4} \leq \theta \leq 1, \]

to approximate \( u(x_{i}, t_{k}) \). From (2.2) and the above equations, we can rewrite (2.5) as

\[ \frac{u(x_{i}, t_{k+1}) - 2u(x_{i}, t_{k}) + u(x_{i}, t_{k-1})}{\tau^{2}} = a(x_{i})\nabla_{h}^{\alpha} \left[ \theta u(x_{i}, t_{k+1}) + (1 - 2\theta)u(x_{i}, t_{k}) + \theta u(x_{i}, t_{k-1}) \right] + f(x_{i}, t_{k}) + R_{i}^{k} \]

(2.8)

with the local truncation error [32]

\[ R_{i}^{k} \leq C_{u,\alpha}(\tau^{2} + h^{2}), \]

(2.9)

where the constant \( C_{u,\alpha} \) is independent of \( h \) and \( \tau \). It should be mentioned that the truncation error in (2.9) holds true if the solution \( u \) is sufficiently regular. Therefore, the full discretization of (2.5) has the following form

\[ \delta_{i}^{2} u_{i}^{k} = \theta a_{i} \nabla_{h}^{\alpha} u_{i}^{k+1} + (1 - 2\theta)a_{i} \nabla_{h}^{\alpha} u_{i}^{k} + \theta a_{i} \nabla_{h}^{\alpha} u_{i}^{k-1} + f_{i}^{k}. \]

(2.10)

that is,

\[ u_{i}^{k+1} + \frac{\tau^{2}}{h^{\alpha} \kappa_{\alpha} a_{i}} \sum_{m=0}^{i+1} \varphi_{m}^{\alpha} u_{i-m+1}^{k+1} + \sum_{m=0}^{N_{x} - i + 1} \varphi_{m}^{\alpha} u_{i+m-1}^{k+1} \]

\[ = 2u_{i}^{k} - (1 - 2\theta) \frac{\tau^{2}}{h^{\alpha} \kappa_{\alpha} a_{i}} \sum_{m=0}^{i+1} \varphi_{m}^{\alpha} u_{i-m+1}^{k} + \sum_{m=0}^{N_{x} - i + 1} \varphi_{m}^{\alpha} u_{i+m-1}^{k} \]

\[ - u_{i}^{k-1} - \frac{\tau^{2}}{h^{\alpha} \kappa_{\alpha} a_{i}} \sum_{m=0}^{i+1} \varphi_{m}^{\alpha} u_{i-m+1}^{k-1} + \sum_{m=0}^{N_{x} - i + 1} \varphi_{m}^{\alpha} u_{i+m-1}^{k-1} \] + \tau^{2} f_{i}^{k} \]

(2.11)

with \( 0 \leq i \leq N_{x} \) and \( 0 \leq k \leq N_{t} \). Using (2.5), (2.6), and Taylor expansion with integral form of the remainder, there exists

\[ u(x_{i}, \tau) = u(x_{i}, 0) + \tau \frac{\partial u(x_{i}, 0)}{\partial t} + \frac{\tau^{2}}{2} \frac{\partial^{2} u(x_{i}, 0)}{\partial t^{2}} + \frac{1}{2} \int_{0}^{\tau} (\tau - t)^{2} \frac{\partial^{3} u(x_{i}, t)}{\partial t^{3}} dt \]

\[ = \varphi(x_{i}) + \tau \psi(x_{i}) + \frac{\tau^{2}}{2} \left[ a(x_{i}) \frac{\partial^{3} u(x_{i}, 0)}{\partial \|x\|^{3}} + f(x_{i}, 0) \right] + \frac{1}{2} \int_{0}^{\tau} (\tau - t)^{2} \frac{\partial^{3} u(x_{i}, t)}{\partial t^{3}} dt. \]

(2.12)

Then, we can obtain \( u_{i}^{1} \), that is,

\[ u_{i}^{1} = \varphi(x_{i}) + \tau \psi(x_{i}) + \frac{\tau^{2}}{2} \left[ a(x_{i}) \nabla_{h}^{\alpha} u(x_{i}, 0) + f(x_{i}, 0) \right] \]

(2.13)

with the local truncation error \( O(\tau^{3} + \tau^{2} h^{2}) \), which is proved in (3.11).
For the convenience of implementation, we use the matrix form of the grid functions

\[ U^k = \left[ u_1^k, u_2^k, \ldots, u_{N_x-1}^k \right]^T, \quad F^k = \left[ f_1^k, f_2^k, \ldots, f_{N_x-1}^k \right]^T. \]

Hence, the finite difference scheme (2.11) can be recast as

\[
\begin{bmatrix}
I + \theta \tau^2 h^\alpha \kappa \alpha DA \alpha \\
\end{bmatrix} U^{k+1} =
\begin{bmatrix}
2I - (1 - 2\theta) \tau^2 h^\alpha \kappa \alpha DA \alpha \\
\end{bmatrix} U^k
- \begin{bmatrix}
I + \theta \tau^2 h^\alpha \kappa \alpha DA \alpha \\
\end{bmatrix} U^{k-1} + \tau F^k,
\tag{2.14}
\]

where \( A_\alpha \) is defined by (2.4) and the diagonal matrix

\[
D = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{N_x-1}
\end{bmatrix}.
\tag{2.15}
\]

### 2.2 Numerical scheme for two-dimensional space-Riesz fractional wave equation

Let the mesh points \( x_i = ih_x, i = 0, 1, \ldots, N_x \) and \( y_j = jh_y, j = 0, 1, \ldots, N_y \) and \( t_k = k\tau, 0 \leq k \leq N_t \) with \( h_x = x_r/N_x, h_y = y_r/N_y, \tau = T/N_t \). Similarly, we take \( u_{i,j}^k \) as the approximated value of \( u(x_i, y_j, t_k) \), \( a_{i,j} = a(x_i, y_j) \), \( b_{i,j} = b(x_i, y_j) \), \( f_{i,j}^k = f(x_i, y_j, t_k) \). We use the center difference scheme to do the discretization in time direction derivative,

\[
\frac{\partial^2 u(x, y, t)}{\partial t^2} = \frac{u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1}}{\tau^2} + \mathcal{O}(\tau^2),
\]

and the weighted schemes for the space-Riesz fractional derivative, that is, \( \theta u_{i,j}^{k+1} + (1 - 2\theta)u_{i,j}^k + \theta u_{i,j}^{k-1} \) to approximate \( u(x_i, y_j, t_k) \). Therefore, (1.1) can be rewritten as

\[
\frac{u(x_i, y_j, t_{k+1}) - 2u(x_i, y_j, t_k) + u(x_i, y_j, t_{k-1})}{\tau^2} = a(x_i, y_j) \nabla_x^\alpha \left( \theta u(x_i, y_j, t_{k+1}) + (1 - 2\theta)u(x_i, y_j, t_k) + \theta u(x_i, y_j, t_{k-1}) \right)
+ b(x_i, y_j) \nabla_y^\beta \left( \theta u(x_i, y_j, t_{k+1}) + (1 - 2\theta)u(x_i, y_j, t_k) + \theta u(x_i, y_j, t_{k-1}) \right)
+ f(x_i, y_j, t_k) + R_{i,j}^k,
\tag{2.16}
\]

where the local truncation error is [32]

\[
R_{i,j}^k \leq C_{u,\alpha,\beta} \left( \tau^2 + h_x^2 + h_y^2 \right).
\tag{2.17}
\]
It should be mentioned that the truncation error in (2.17) holds true if the solution \( u \) is sufficiently regular. Similarly, we denote

\[
\nabla^\alpha_{h_x} u(x_i, y_j) = -\kappa_\alpha \left( \delta^\alpha_{x,+} + \delta^\alpha_{x,-} \right) u(x_i, y_j) \quad \text{and} \quad \nabla^\beta_{h_y} u(x_i, y_j) = -\kappa_\beta \left( \delta^\beta_{y,+} + \delta^\beta_{y,-} \right) u(x_i, y_j).
\]

Therefore, the resulting discretization of (1.1) has the following form

\[
\delta^2_t u^k_{i,j} = \theta a_{i,j} \nabla^\alpha_{h_x} u^{k+1}_{i,j} + (1 - 2\theta) a_{i,j} \nabla^\alpha_{h_x} u^k_{i,j} + \theta a_{i,j} \nabla^\alpha_{h_x} u^{k-1}_{i,j}
\]

\[
+ \theta b_{i,j} \nabla^\beta_{h_y} u^{k+1}_{i,j} + (1 - 2\theta) b_{i,j} \nabla^\beta_{h_y} u^k_{i,j} + \theta b_{i,j} \nabla^\beta_{h_y} u^{k-1}_{i,j} + f^k_{i,j},
\]

(2.19)

that is,

\[
\left[ 1 - \theta \tau^2 \left( a_{i,j} \nabla^\alpha_{h_x} + b_{i,j} \nabla^\beta_{h_y} \right) \right] u^{k+1}_{i,j}
\]

\[
= \left[ 2 + (1 - 2\theta) \tau^2 \left( a_{i,j} \nabla^\alpha_{h_x} + b_{i,j} \nabla^\beta_{h_y} \right) \right] u^k_{i,j} - \left[ 1 - \theta \tau^2 \left( a_{i,j} \nabla^\alpha_{h_x} + b_{i,j} \nabla^\beta_{h_y} \right) \right] u^{k-1}_{i,j} + \tau^2 f^k_{i,j}
\]

(2.20)

with 0 \( \leq i \leq N_x \), 0 \( \leq j \leq N_y \) and 0 \( \leq k \leq N_t \). Using (2.12) and (2.13), we can obtain

\[
u^1_{i,j} = \varphi(x_i, y_j) + \tau \psi(x_i, y_j) + \frac{\tau^2}{2} \left[ \left( a_{i,j} \nabla^\alpha_{h_x} + b_{i,j} \nabla^\beta_{h_y} \right) u^0_{i,j} + f^0_{i,j} \right]
\]

(2.21)

with the local truncation error \( \mathcal{O}(\tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2) \), which is proved in (3.23).

For the two-dimensional space-Riesz fractional wave equation (1.1), the relevant perturbation equation of (2.20) is of the form

\[
\left( 1 - \theta \tau^2 a_{i,j} \nabla^\alpha_{h_x} \right) \left( 1 - \theta \tau^2 b_{i,j} \nabla^\beta_{h_y} \right) u^{k+1}_{i,j}
\]

\[
= \left[ 2 \left( 1 - \theta \tau^2 a_{i,j} \nabla^\alpha_{h_x} \right) \left( 1 - \theta \tau^2 b_{i,j} \nabla^\beta_{h_y} \right) + \tau^2 a_{i,j} \nabla^\alpha_{h_x} + \tau^2 b_{i,j} \nabla^\beta_{h_y} \right] u^k_{i,j}
\]

\[- \left( 1 - \theta \tau^2 a_{i,j} \nabla^\alpha_{h_x} \right) \left( 1 - \theta \tau^2 b_{i,j} \nabla^\beta_{h_y} \right) u^{k-1}_{i,j} + \tau^2 f^k_{i,j}.
\]

(2.22)

Comparing (2.22) with (2.20), the splitting term is given by

\[
\theta^2 \tau^4 a_{i,j} b_{i,j} \nabla^\alpha_{h_x} \nabla^\beta_{h_y} \left( u^{k+1}_{i,j} - 2u^k_{i,j} + u^{k-1}_{i,j} \right);
\]
since $\left( u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1} \right)$ is an $O(\tau^2)$ term, it implies that the perturbation contributes an $O(\tau^6)$ error component to the truncation error of (2.20). Thus, we can rewrite (1.1) as

$$u(x_i, y_j, t_{k+1}) - 2u(x_i, y_j, t_k) + u(x_i, y_j, t_{k-1})$$

\[ \frac{\tau^2}{2} + \theta^2 \tau^4 a_{i,j} b_{i,j} \nabla_{h_x}^\alpha \nabla_{h_y}^\beta (u(x_i, y_j, t_{k+1}) - 2u(x_i, y_j, t_k) + u(x_i, y_j, t_{k-1})) \]

\[ = a(x_i, y_j) \nabla_{h_x}^\alpha (\theta u(x_i, y_j, t_{k+1}) + (1 - 2\theta)u(x_i, y_j, t_k) + \theta u(x_i, y_j, t_{k-1})) \]

\[ + b(x_i, y_j) \nabla_{h_y}^\beta (\theta u(x_i, y_j, t_{k+1}) + (1 - 2\theta)u(x_i, y_j, t_k) + \theta u(x_i, y_j, t_{k-1})) \]

\[ + f(x_i, y_j, t_k) + \tilde{R}_{i,j}^k \]

(2.23)

with

$$\tilde{R}_{i,j}^k = R_{i,j}^k + \theta^2 \tau^4 a_{i,j} b_{i,j} \nabla_{h_x}^\alpha \nabla_{h_y}^\beta (u(x_i, y_j, t_{k+1}) - 2u(x_i, y_j, t_k) + u(x_i, y_j, t_{k-1}))$$

\[ \leq \tilde{C}_{u,\alpha,\beta} (\tau^2 + h_x^2 + h_y^2). \]

(2.24)

Hence, the system (2.22) can be solved by the alternating direction implicit method (D-ADI) [8, 9]:

\[ \left( 1 - \theta \tau^2 a_{i,j} \nabla_{h_x}^\alpha \right) u_{i,j}^* = 2u_{i,j}^k - u_{i,j}^{k-1} + \tau^2 a_{i,j} \nabla_{h_x}^\alpha ((1 - 2\theta)u_{i,j}^k + \theta u_{i,j}^{k-1}) \]

\[ + \tau^2 b_{i,j} \nabla_{h_y}^\beta u_{i,j}^k + \tau^2 f_{i,j}^k, \]

\[ \left( 1 - \theta \tau^2 b_{i,j} \nabla_{h_y}^\beta \right) u_{i,j}^{k+1} = u_{i,j}^* + \theta \tau^2 b_{i,j} \nabla_{h_y}^\beta (-2u_{i,j}^k + u_{i,j}^{k-1}), \]

(2.25)

where $u_{i,j}^*$ is an intermediate solution. Take

$$U^k = [u_{1,1}^k, u_{2,1}^k, \ldots, u_{N_x-1,1}^k, u_{1,2}^k, u_{2,2}^k, \ldots, u_{N_x-1,2}^k, \ldots, u_{1,N_y-1}^k, u_{2,N_y-1}^k, \ldots, u_{N_x-1,N_y-1}^k]^T,$$

$$F^k = [f_{1,1}^k, f_{2,1}^k, \ldots, f_{N_x-1,1}^k, f_{1,2}^k, f_{2,2}^k, \ldots, f_{N_x-1,2}^k, \ldots, f_{1,N_y-1}^k, f_{2,N_y-1}^k, \ldots, f_{N_x-1,N_y-1}^k]^T,$$

and denote

$$A_{x} = I \otimes A_\alpha \quad \text{and} \quad A_{y} = A_\beta \otimes I.$$  

(2.26)
Here, $I$ denotes the unit matrix and the symbol $\otimes$ the Kronecker product [15], and $A_\alpha, A_\beta$ are defined by (2.4). Therefore, we can rewrite (2.25) as the following form

$$
\begin{align*}
(I + \theta \frac{\tau^2}{h_x^2} \kappa_\alpha D A_\alpha) U^* &= \left(2I - (1 - 2\theta) \frac{\tau^2}{h_x^2} \kappa_\alpha D A_\alpha - \frac{\tau^2}{h_y^2} \kappa_\beta E A_\beta\right) U^k \\
&- \left(I + \theta \frac{\tau^2}{h_x^2} \kappa_\alpha D A_\alpha\right) U^{k-1} + \tau^2 F^k,
\end{align*}
$$

$$
\begin{align*}
(I + \theta \frac{\tau^2}{h_y^2} \kappa_\beta E A_\beta) U^{k+1} &= \left(2\theta \frac{\tau^2}{h_y^2} \kappa_\beta E A_\beta - \left(\theta \frac{\tau^2}{h_y^2} \kappa_\beta E A_\beta\right) U^{k-1} + U^*,
\end{align*}
$$

(2.27)

where

$$
D = \begin{bmatrix}
D_1 \\
D_2 \\
\vdots \\
D_{N_y-1}
\end{bmatrix}
$$

with $D_j = \begin{bmatrix} a_{1,j} & a_{2,j} & \cdots & a_{N_x-1,j} \end{bmatrix}$

and

$$
E = \begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{N_y-1}
\end{bmatrix}
$$

with $E_j = \begin{bmatrix} b_{1,j} & b_{2,j} & \cdots & b_{N_x-1,j} \end{bmatrix}$.

### 3 Convergence and stability analysis

In this work, we use the energy method to estimate the space-Riesz fractional wave equation. For two-dimensional cases with the constant coefficients, the stiff matrices are proved to be commutative, which ensures to carry out of the priori error estimates.

Firstly, we introduce some relevant notations and properties of discretized inner product given in [13, 28]. Let $u^k = [u_0^k, u_1^k, \cdots, u_{N_x}^k]$, $0 \leq k \leq N_t$ and denote

$$
u_i^{k+1} = (u_i^k + u_i^{k+1}) / 2, \quad \nu_i^k = (u_i^k - u_i^{k-1}) / \tau, \quad (u^k, v^k) = h \sum_{i=1}^{N_x-1} u_i^k v_i^k, \quad ||u^k|| = (u^k, u^k)^{1/2}.
$$

(3.1)

Let us introduce the following lemmas.

**Lemma 3.1** [3, 33] Let $\nabla_h^\alpha$ be given in (2.3) and $1 < \alpha < 2$. Then, there exists an symmetric positive definite matrix $\Lambda_h^\alpha$ such that

$$
- (\nabla_h^\alpha u, u) \geq \frac{2\kappa_\alpha}{x_y^\alpha \Gamma(1 - \alpha)} ||u||^2 > 0 \quad \text{and} \quad -(\nabla_h^\alpha u, v) = (\Lambda_h^\alpha u, \Lambda_h^\alpha v) \quad \text{with} \quad -\nabla_h^\alpha = \Lambda_h^\alpha \cdot \Lambda_h^\alpha.
$$
Lemma 3.2 (Discrete Gronwall Lemma [26]) Assume that \( \{a_k\} \) and \( \{b_k\} \) is a nonnegative sequence, and the sequence \( \varphi^k \) satisfies
\[
\varphi^0 \leq c_0, \quad \varphi^k \leq c_0 + \sum_{l=0}^{k-1} b_l + \sum_{l=0}^{k-1} a_l \varphi^l, \quad k \geq 1,
\]
where \( c_0 \geq 0 \). Then, the sequence \( \{\varphi^k\} \) satisfies
\[
\varphi^k \leq \left( c_0 + \sum_{l=0}^{k-1} b_l \right) \exp \left( \sum_{l=0}^{k-1} a_l \right), \quad k \geq 1.
\]

Lemma 3.3 [15, p. 141] Let \( A \in \mathbb{R}^{n \times n} \) have eigenvalues \( \{\lambda_i\}_{i=1}^n \) and \( B \in \mathbb{R}^{m \times m} \) have eigenvalues \( \{\mu_j\}_{j=1}^m \). Then, the \( mn \) eigenvalues of \( A \otimes B \) are
\[
\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m.
\]

Lemma 3.4 [15, p. 140] Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{n \times p}, \) and \( D \in \mathbb{R}^{s \times t} \). Then,
\[
(A \otimes B)(C \otimes D) = AC \otimes BD \quad (\in \mathbb{R}^{mr \times pt}).
\]
Moreover, for all \( A \) and \( B \), \( (A \otimes B)^T = A^T \otimes B^T \).

Lemma 3.5 Let \( A_x = I \otimes A_\alpha \) and \( A_y = A_\beta \otimes I \) be defined by (2.26). Then,
\[
A_x A_y = A_y A_x, \quad \Lambda_x A_y = A_y \Lambda_x \quad \text{and} \quad \Lambda_x \Lambda_y = \Lambda_y \Lambda_x \quad \text{with} \quad -A_\alpha = \Lambda_\alpha \cdot \Lambda_\alpha, \nonumber
\]
\[
-\Lambda_\beta = \Lambda_\beta \cdot \Lambda_\beta \nonumber
\]
where we denote \( \Lambda_x := I \otimes \Lambda_\alpha \) and \( \Lambda_y := \Lambda_\beta \otimes I \).

Proof From Lemma 3.1, there exists \( -A_\alpha = \Lambda_\alpha \cdot \Lambda_\alpha \) and \( -A_\beta = \Lambda_\beta \cdot \Lambda_\beta \), since \( -A_\alpha \) and \( -A_\beta \) are the symmetric positive definite matrices. Taking \( \Lambda_x := I \otimes \Lambda_\alpha \) and \( \Lambda_y := \Lambda_\beta \otimes I \) and using Lemma 3.4, the results are obtained. \( \square \)

Lemma 3.6 Let \( \nabla^\alpha_h \) and \( \nabla^\beta_h \) be given in (2.18) with \( 1 < \alpha, \beta < 2 \). Then, there exist the symmetric positive definite matrices \( \Lambda^\alpha_h \) and \( \Lambda^\beta_h \), respectively, such that
\[
-(\nabla^\alpha_h \mathbf{U}, \mathbf{U}) > 0 \quad \text{and} \quad -(\nabla^\alpha_h \mathbf{U}, \mathbf{V}) = (\Lambda^\alpha_h \mathbf{U}, \Lambda^\alpha_h \mathbf{V}) \quad \text{with} \quad -\nabla^\alpha_h = \Lambda^\alpha_h \cdot \Lambda^\alpha_h,
\]
and
\[
-(\nabla^\beta_h \mathbf{U}, \mathbf{U}) > 0 \quad \text{and} \quad -(\nabla^\beta_h \mathbf{U}, \mathbf{V}) = (\Lambda^\beta_h \mathbf{U}, \Lambda^\beta_h \mathbf{V}) \quad \text{with} \quad -\nabla^\beta_h = \Lambda^\beta_h \cdot \Lambda^\beta_h.
\]

Proof According to (2.18) and (2.26), it implied that
\[
\nabla^\alpha_h \mathbf{U} = -\kappa_\alpha \left( \delta^\alpha_{x,+} + \delta^\alpha_{x,-} \right) \mathbf{U} = \frac{-\kappa_\alpha}{h^\alpha_x} A_\alpha \mathbf{U}.
\]
From Lemmas 3.3 and 3.5, we know that $A_x = I \otimes A_\alpha$ is a symmetric negative definite, which leads to $-\nabla_h^\alpha (\text{or } -\nabla_h^\beta)$ is the symmetric positive definite. The proof is completed. 

### 3.1 Convergence and stability for one-dimensional space-Riesz fractional wave equation

**Lemma 3.7** Let $\frac{1}{4} \leq \theta \leq 1$, $1 < \alpha < 2$ and $\{u_i^k\}$ be the solution of the difference scheme (2.14) with the constant coefficient and

- $u_i^0 = \varphi_i$, $0 \leq i \leq N_x$,
- $u_i^1 = \psi_i$, $0 \leq i \leq N_x$,
- $u_0^k = 0$, $u_{N_x}^k = 0$, $0 \leq k \leq N_t$.

Then,

$$E_u^k \leq e^{\frac{3}{2}k\tau} \left[ E_u^0 + \frac{3}{2} \tau \sum_{l=1}^{k} ||f_l||^2 \right],$$

where the energy norm is defined by

$$E_u = \| u_i^{k+1} \|^2 + \frac{1}{4} \| \sqrt{a} \left( \Lambda_h^\alpha u_i^{k+1} + \Lambda_h^\alpha u_i^{k} \right) \|^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{a} \left( \Lambda_h^\alpha u_i^{k+1} - \Lambda_h^\alpha u_i^{k} \right) \|^2.$$

**Proof** Multiplying (2.10) by $h(u_i^{k+1} - u_i^{k-1})$, it yields

$$\delta_i^2 u_i^k \cdot \left[ h(u_i^{k+1} - u_i^{k}) + h(u_i^{k} - u_i^{k-1}) \right] = h \left( u_i^{k+1} \right)^2 - h \left( u_i^{k} \right)^2,$$

and

$$\left[ \theta a \nabla_h^\alpha u_i^{k+1} + (1 - 2\theta) a \nabla_h^\alpha u_i^{k} + \theta a \nabla_h^\alpha u_i^{k-1} + f_i^k \right] \cdot h \left( u_i^{k+1} - u_i^{k-1} \right).$$

Then, summing up for $i$ from 1 to $N_x - 1$ for the above equations, respectively, there exists

$$\sum_{i=1}^{N_x-1} \left[ h \left( u_i^{k+1} \right)^2 - h \left( u_i^{k} \right)^2 \right] = \| u_i^{k+1} \|^2 - \| u_i^{k} \|^2,$$

and

$$\sum_{i=1}^{N_x-1} \left[ \theta a \nabla_h^\alpha u_i^{k+1} + (1 - 2\theta) a \nabla_h^\alpha u_i^{k} + \theta a \nabla_h^\alpha u_i^{k-1} + f_i^k \right] \cdot h \left( u_i^{k+1} - u_i^{k-1} \right) = I_1 + I_2 + (f^k, u_i^{k+1} - u_i^{k-1}),$$

where

$$I_1 = \theta \left( a \nabla_h^\alpha u_i^{k+1} + a \nabla_h^\alpha u_i^{k-1}, u_i^{k+1} - u_i^{k-1} \right), \quad I_2 = (1 - 2\theta) \left( a \nabla_h^\alpha u_i^{k}, u_i^{k+1} - u_i^{k-1} \right).$$
According to Lemma 3.1, which leads to

\[ I_1 = -\theta \left[ a \Lambda_h^\alpha (u^{k+1} + u^{k-1}), \Lambda_h^\alpha (u^{k+1} - u^{k-1}) \right] = -\theta \left( \| \sqrt{a} \Lambda_h^\alpha u^{k+1} \|^2 - \| \sqrt{a} \Lambda_h^\alpha u^{k-1} \|^2 \right), \]

and

\[ I_2 = -(1 - 2\theta) \left[ (a \Lambda_h^\alpha u^k, \Lambda_h^\alpha u^{k+1}) - (a \Lambda_h^\alpha u^k, \Lambda_h^\alpha u^{k-1}) \right] \\
\quad - \left( a \Lambda_h^\alpha u^k - a \Lambda_h^\alpha u^{k+1}, \Lambda_h^\alpha u^k - \Lambda_h^\alpha u^{k+1} \right) \\
\quad - \left( a \Lambda_h^\alpha u^{k-1} + a \Lambda_h^\alpha u^{k+1}, \Lambda_h^\alpha u^{k-1} + \Lambda_h^\alpha u^{k+1} \right) + \left( a \Lambda_h^\alpha u^{k-1} - a \Lambda_h^\alpha u^{k+1}, \Lambda_h^\alpha u^{k-1} - \Lambda_h^\alpha u^{k+1} \right) \\
= -\frac{(1 - 2\theta)}{4} \left( \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} + \Lambda_h^\alpha u^k \right) \|^2 - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} - \Lambda_h^\alpha u^k \right) \|^2 \right) \\
\quad - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} + \Lambda_h^\alpha u^{k+1} \right) \|^2 + \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} - \Lambda_h^\alpha u^{k+1} \right) \|^2. \]

Combining (2.10), (3.2), and (3.3), we obtain

\[ \| u_{i}^{k+1} \|^2 - \| u_{i}^{k} \|^2 - I_1 - I_2 = (f^k, u^{k+1} - u^{k-1}), \quad (3.4) \]

that is,

\[ \| u_{i}^{k+1} \|^2 + \theta \| \sqrt{a} \Lambda_h^\alpha u^{k+1} \|^2 + \frac{1 - 2\theta}{4} \]
\[ \times \left( \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} + \Lambda_h^\alpha u^k \right) \|^2 - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} - \Lambda_h^\alpha u^k \right) \|^2 \right) \\
= \| u_{i}^{k} \|^2 + \theta \| \sqrt{a} \Lambda_h^\alpha u^{k-1} \|^2 + \frac{1 - 2\theta}{4} \]
\[ \times \left( \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} + \Lambda_h^\alpha u^{k+1} \right) \|^2 - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} - \Lambda_h^\alpha u^{k+1} \right) \|^2 \right) \\
+ (f^k, u^{k+1} - u^{k-1}). \]

Adding \( \theta \| \sqrt{a} \Lambda_h^\alpha u^k \|^2 \) on both sides of the above equation, there exists

\[ \| u_{i}^{k+1} \|^2 + \theta \left( \| \sqrt{a} \Lambda_h^\alpha u^{k+1} \|^2 + \| \sqrt{a} \Lambda_h^\alpha u^k \|^2 \right) \\
\quad + \frac{1 - 2\theta}{4} \left( \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} + \Lambda_h^\alpha u^k \right) \|^2 - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k+1} - \Lambda_h^\alpha u^k \right) \|^2 \right) \\
= \| u_{i}^{k} \|^2 + \theta \left( \| \sqrt{a} \Lambda_h^\alpha u^{k-1} \|^2 + \| \sqrt{a} \Lambda_h^\alpha u^{k+1} \|^2 \right) \\
\quad + \frac{1 - 2\theta}{4} \left( \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} + \Lambda_h^\alpha u^{k+1} \right) \|^2 - \| \sqrt{a} \left( \Lambda_h^\alpha u^{k-1} - \Lambda_h^\alpha u^{k+1} \right) \|^2 \right) \\
+ (f^k, u^{k+1} - u^{k-1}). \]
Denoting
\[
E_u^k = \| u_i^{k+1} \|^2 + \theta \left( \| \sqrt{a} \Lambda_h u_i^{k+1} \|^2 + \| \sqrt{a} \Lambda_h u_i^k \|^2 \right)
+ \frac{1 - 2\theta}{4} \left( \| \sqrt{a} \left( \Lambda_h u_i^{k+1} + \Lambda_h u_i^k \right) \|^2 - \| \sqrt{a} \left( \Lambda_h u_i^{k+1} - \Lambda_h u_i^k \right) \|^2 \right),
\]
that is,
\[
E_u^k = \| u_i^{k+1} \|^2 + \frac{1}{4} \| \sqrt{a} \Lambda_h u_i^{k+1} \|^2 + \frac{1}{4} \left( 4\theta - 1 \right) \| \sqrt{a} \left( \Lambda_h u_i^{k+1} - \Lambda_h u_i^k \right) \|^2,
\]
where we use
\[
\| \sqrt{a} \Lambda_h u_i^k \|^2 + \| \sqrt{a} \Lambda_h u_i^{k-1} \|^2 = \frac{1}{2} \left( \| \sqrt{a} \left( \Lambda_h u_i^k + \Lambda_h u_i^{k-1} \right) \|^2 + \| \sqrt{a} \left( \Lambda_h u_i^k - \Lambda_h u_i^{k-1} \right) \|^2 \right).
\]
From
\[
(f^k, u_i^{k+1} - u_i^{k-1}) = h\tau \sum_{i=1}^{N_t-1} 2 f_i^k \left( \frac{u_i^{k+1} - u_i^{k-1}}{2\tau} \right)
\leq h\tau \sum_{i=1}^{N_t-1} \left[ \left( f_i^k \right)^2 + \left( \frac{u_i^{k+1} - u_i^{k-1}}{2\tau} \right)^2 \right]
\leq \frac{\tau}{2} \left( \| u_i^{k+1} \|^2 + \| u_i^{k-1} \|^2 \right) + \tau \| f^k \|^2,
\]
and (3.6), (3.4), we obtain
\[
E_u^k - E_u^{k-1} = (f^k, u_i^{k+1} - u_i^{k-1}) \leq \frac{\tau}{2} \left( E_u^k + E_u^{k-1} \right) + \tau \| f^k \|^2,
\]
that is,
\[
\left( 1 - \frac{\tau}{2} \right) E_u^k \leq \left( 1 + \frac{\tau}{2} \right) E_u^{k-1} + \tau \| f^k \|^2.
\]
Therefore, for \( \tau \leq 2/3 \), it yields
\[
E_u^k \leq \left( 1 + \frac{3\tau}{2} \right) E_u^{k-1} + \frac{3}{2} \tau \| f^k \|^2,
\]
that is,
\[
E_u^k \leq \sum_{l=0}^{k-1} \frac{3\tau}{2} E_u^l + \sum_{l=0}^{k-1} \frac{3\tau}{2} \| f^{l+1} \|^2.
\]
Hence, we get
\[
0 \leq E_u^0 \leq E_u^0, \quad E_u^k \leq E_u^0 + \sum_{l=0}^{k-1} \frac{3\tau}{2} \| f^{l+1} \|^2 + \sum_{l=0}^{k-1} \frac{3\tau}{2} E_u^l.
\]
Using the discrete Gronwall inequality (see Lemma 3.2), we have

\[ E^k_u \leq e^{\frac{3}{2}k\tau} \left[ E^0_u + \frac{3}{2} \tau \sum_{l=1}^{k} ||f^l||^2 \right]. \]

The proof is completed. \(\square\)

**Theorem 3.1** Let \(1 < \alpha < 2\), \(\frac{1}{4} \leq \theta \leq 1\) and \(u(x_i, t_k)\) be the exact solution of (2.5) with the constant coefficient; \(u^k_i\) be the solution of the finite difference scheme (2.14) and \(e^k_i = u(x_i, t_k) - u^k_i\). Then,

\[ E^k_e \leq \tilde{C}_\alpha e^{\frac{3}{2}k\tau} (\tau^2 + h^2)^2, \]

where \(\tilde{C}_\alpha\) is a positive constant and the energy norm is defined by

\[ E^k_e = \| e^{k+1}_i \|^2 + \frac{1}{4} \| \sqrt{a} \left( \Lambda^\alpha_h e^{k+1} + \Lambda^{-\alpha}_h e^k \right) \|^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{a} \left( \Lambda^\alpha_h e^{k+1} - \Lambda^{-\alpha}_h e^k \right) \|^2. \]  

(3.8)

**Proof** Subtracting (2.10) from (2.8), it yields

\[ \delta^2 e^k_i = \theta a \nabla^\alpha_h e^{k+1}_i + (1 - 2\theta) a \nabla^\alpha_h e^k_i + \theta a \nabla^\alpha_h e^{k-1}_i + R^k_i. \]  

(3.9)

Using Lemma 3.7, we obtain

\[ E^k_e \leq e^{\frac{3}{2}k\tau} \left[ E^0_e + \frac{3}{2} \tau \sum_{l=1}^{k} ||R^l||^2 \right]. \]  

(3.10)

Next, we estimate the local error truncation of \(E^0_e\). Since \(e^0_i = 0\) and

\[ e^1_i = \frac{\tau^2}{2} \left[ a \left( \frac{\partial^\alpha u(x_i, 0)}{\partial |x|^\alpha} - \nabla^\alpha_h \varphi(x_i) \right) \right] + \frac{1}{2} \int_0^\tau (\tau - t)^2 \frac{\partial^3 u(x_i, t)}{\partial t^3} dt \]

\[ = \frac{\tau^2}{2} a C_1.\alpha \frac{\partial^{\alpha+2} u(\xi_i, t)}{\partial |x|^{\alpha+2}} h^2 + \frac{1}{2} \int_0^\tau (\tau - t)^2 \frac{\partial^3 u(x_i, t)}{\partial t^3} dt \leq C_2.\alpha (\tau^3 + \tau^2 h^2), \]

(3.11)

where \(\xi_i \in (0, x_r)\) and

\[ C_{2,\alpha} = \max_{0 \leq x \leq x_r, 0 \leq t \leq T} \left\{ \frac{1}{2} a \left| C_1.\alpha \frac{\partial^{\alpha+2} u(\xi_i, t)}{\partial |x|^{\alpha+2}} \right|, \frac{1}{6} \int_0^\tau \left| \frac{\partial^3 u(x_i, t)}{\partial t^3} \right| dt \right\}, \]

it implies that

\[ ||e^1_i||^2 = \left| \frac{e^1_i - e^0_i}{\tau} \right|^2 \leq (N_x - 1) h \frac{1}{\tau^2} C_{2,\alpha} (\tau^3 + \tau^2 h^2) \cdot C_{2,\alpha} (\tau^3 + \tau^2 h^2) \leq C_{2,\alpha} (\tau^3 + \tau^2 h^2)^2. \]  

(3.12)

Here, the coefficients \(C_{l,\alpha}, 1 \leq l \leq 2\) are the constants independent of \(h\) and \(\tau\).
According to (2.2) and the above equations, there exists
\[ \| \sqrt{a} \Lambda_h e^1 \|_2 = - \left( a \nabla_h e^1, e^1 \right) = -h \sum_{i=1}^{N_t-1} a \nabla_h e_i^1 \cdot e_i^1 \]
\[ = -h \sum_{i=1}^{N_t-1} a \sum_{l=0}^{N_x} -\kappa_\alpha \frac{a}{h^2} \phi_{i,l} \left[ \frac{aC_{1,\alpha}}{2} \frac{\partial^{2\alpha+2}u(\xi_i,t)}{\partial |x|^2} \tau^2 h^2 + \frac{1}{2} \int_0^\tau (\tau-t)^2 \frac{\partial^{3\alpha}u(\xi_i,t)}{\partial t^3} dt \right] \cdot e_i^1 \]
\[ = -h \sum_{i=1}^{N_t-1} a \left[ \frac{aC_{1,\alpha}}{2} \frac{\partial^{2\alpha+2}u(\xi_i,t)}{\partial |x|^2} + C_{3,\alpha} \frac{\partial^{2\alpha+4}u(\xi_i,t)}{\partial |x|^2} \tau^2 h^4 + C_{4,\alpha} \frac{\partial^{2\alpha+5}u(\xi_i,t)}{\partial |x|^2} \tau^2 h^4 \right] \cdot e_i^1 \]
\[ \leq C_{5,\alpha} (\tau^3 + \tau^2 h^2) \cdot C_{2,\alpha} (\tau^3 + \tau^2 h^2), \]
where \( \xi_i, \xi_i, \xi_i \in (0, x_r) \) and \( C_{l,\alpha}, 1 \leq l \leq 5 \) are the constants independent of \( h \) and \( \tau \). Using (3.8), (3.12), and the above equation, we have
\[ E_{e}^0 \leq C_{\alpha} x_r (\tau^2 + \tau h^2)^2 \]
with a constant \( C_{\alpha} \). From (2.9), (3.10), and (3.13), it means that
\[ E_{e}^k \leq e^{3k\tau} \left[ C_{\alpha} x_r (\tau^2 + \tau h^2)^2 + \frac{3}{2} k \tau C_{\alpha} C_{u,\alpha} (\tau^2 + h^2)^2 \right] \leq \tilde{C}_{\alpha} e^{3T} (\tau^2 + h^2)^2 \]
with \( \tilde{C}_{\alpha} = 2 \max \{ C_{\alpha} x_r, \frac{3}{2} C_{u,\alpha} T \} \). The proof is completed. \( \square \)

**Theorem 3.2** Let \( 1 < \alpha < 2, \frac{1}{3} \leq \theta \leq 1 \) and \( u(x_i, t_k) \) be the exact solution of (2.5) with the constant coefficient; \( u^k_i \) be the solution of the finite difference scheme (2.10) and \( e_i^k = u(x_i, t_k) - u^k_i \). Then,
\[ \| e^k \| \leq C_e (\tau^2 + h^2), \quad 0 \leq k \leq N_t \]
with \( C_e = \left( \frac{\Gamma(1-\alpha)}{2a^\alpha} \tilde{C}_\alpha e^{3T} \right)^{1/2} > 0. \)

**Proof** From (3.1), (3.8), and Theorem 3.1, we have
\[ \| \Lambda_h^\alpha e^{k+1} \|_2^2 = \frac{1}{4} \| \Lambda_h^\alpha e^{k+1} + \Lambda_h^\alpha e^k \|_2^2 \leq \frac{1}{a} E_e^k \leq \frac{1}{a} \tilde{C}_\alpha e^{3T} (\tau^2 + h^2)^2, \quad 0 \leq k \leq N_t - 1. \]
For \( 0 \leq k \leq N_t \), we obtain
\[ \| \Lambda_h^\alpha e^k \|_2^2 = \frac{1}{4} \| \Lambda_h^\alpha e^{k+1} + \Lambda_h^\alpha e^{k-1} \|_2^2 \leq \frac{1}{2} \| \Lambda_h^\alpha e^{k+1} \|_2^2 + \frac{1}{2} \| \Lambda_h^\alpha e^{k-1} \|_2^2 \leq \frac{1}{a} \tilde{C}_\alpha e^{3T} (\tau^2 + h^2)^2. \]
According to the above equation and Lemma 3.1, there exists
\[ \|e^k\|^2 \leq \frac{x_r^\alpha \Gamma(1-\alpha)}{2\kappa_\alpha} (-\nabla_h^\alpha e^k, e^k) = \frac{x_r^\alpha \Gamma(1-\alpha)}{2\kappa_\alpha} \| \Lambda_h^\alpha e^k \|^2 \]
\[ \leq \frac{x_r^\alpha \Gamma(1-\alpha)}{2a\kappa_\alpha} \tilde{C}_\alpha e^{\frac{\tau^2}{2}} (\tau^2 + h^2). \]

The proof is completed.

\[ \text{Remark 3.1} \quad \text{For the case of variable coefficients, the finite difference method was recently studied in [22], whose prototype is, for } 0 < \beta < 1, \]
\[ -\partial_x(a(x)\partial_x^\beta u(x)) = f(x), \quad \forall x \in \Omega. \]

**Theorem 3.3** Let \( 0 < \alpha < 2 \) and \( \frac{1}{4} \leq \theta \leq 1 \). Then, the difference scheme (2.14) with the constant coefficient is unconditionally stable.

**Proof** From Lemma 3.7, the proof is completed.

### 3.2 Convergence and stability for two-dimensional space-Riesz fractional wave equation

Let
\[ u^k = [u^k_{1,1}, u^k_{2,1}, \ldots, u^k_{N_x,1}, u^k_{1,2}, u^k_{2,2}, \ldots, u^k_{N_x,2}, \ldots, u^k_{1,N_y}, u^k_{2,N_y}, \ldots, u^k_{N_x,N_y}], \quad 0 \leq k \leq N_t, \]
and denote
\[ u^k_{i,j} = (u^k_{i,j} - u^{k-1}_{i,j})/\tau, \quad (u^k, v^k) = h_x h_y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} u^k_{i,j} v^k_{i,j}, \quad ||u^k|| = (u^k, u^k)^{1/2}. \]

(3.14)

**Lemma 3.8** Let \( \frac{1}{4} \leq \theta \leq 1, 0 < \alpha, \beta < 2 \) and \( \{u^k_{i,j}\} \) be the solution of the difference scheme
\[ \delta^2_t u^k_{i,j} + \theta^2 \tau^4 a b \nabla_{h_x}^\alpha \nabla_{h_y}^\beta \left( u^k_{i,j} + 2u^k_{i,j} + u^{k-1}_{i,j} \right) = \theta a \nabla_{h_x}^\alpha u^{k+1}_{i,j} + (1 - 2\theta) a \nabla_{h_x}^\alpha u^k_{i,j} + \theta a \nabla_{h_x}^\alpha u^{k-1}_{i,j} \]
\[ + \theta b \nabla_{h_y}^\beta u^{k+1}_{i,j} + (1 - 2\theta) b \nabla_{h_y}^\beta u^k_{i,j} + \theta b \nabla_{h_y}^\beta u^{k-1}_{i,j} + f^k_{i,j} \]

(3.15)

with the constant coefficients and
\[ u^0_{i,j} = \varphi_{i,j}, \quad 0 \leq i \leq N_x, 0 \leq j \leq N_y, \]
\[ u^1_{i,j} = \psi_{i,j}, \quad 0 \leq i \leq N_x, 0 \leq j \leq N_y, \]
\[ u^k_{i,j} = 0, \quad (x_i, y_j) \in \partial \Omega, \quad 0 \leq k \leq N_t. \]

\[ \text{Springer} \]
Then,

$$E_u^k \leq e^{3k\tau} \left[ E_u^0 + \frac{3}{2} \tau \sum_{l=1}^{k} \| f_l \|^2 \right],$$

where the energy norm is defined by

$$E_u^k = \| u_t^{k+1} \|^2 + \frac{1}{4} \| \sqrt{a} \left( \Lambda^a_h u_t^{k+1} + \Lambda^a_h u_t^k \right) \| ^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{a} \left( \Lambda^a_h u_t^{k+1} - \Lambda^a_h u_t^k \right) \|^2$$

$$+ \frac{1}{4} \| \sqrt{b} \left( \Lambda^b_h u_t^{k+1} + \Lambda^b_h u_t^k \right) \| ^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{b} \left( \Lambda^b_h u_t^{k+1} - \Lambda^b_h u_t^k \right) \|^2$$

$$+ \theta^2 \tau^6 \| \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k+1} \|^2.$$  \hfill (3.15)

Proof Multiplying (3.15) by $h_x h_y (u_{i,j}^{k+1} - u_{i,j}^{k-1})$ and using Lemmas 3.5 and 3.6, there exists

$$\left( \delta_t^2 u_t^{k+1} + \theta^2 \tau^4 ab \nabla^a_h \nabla^b_h \left( u_{i,j}^{k+1} - 2u_{i,j}^{k} + u_{i,j}^{k-1} \right) \right) \left[ h_x h_y \left( u_{i,j}^{k+1} - u_{i,j}^{k} \right) + h_x h_y \left( u_{i,j}^{k} - u_{i,j}^{k-1} \right) \right]$$

$$= h_x h_y \left( u_{i,j}^{k+1} \right)^2 - h_x h_y \left( u_{i,j}^{k} \right)^2 + h_x h_y \theta^2 \tau^6 \left( \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k+1} \right)^2$$

$$- h_x h_y \theta^2 \tau^6 \left( \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k} \right)^2,$$

and

$$\left[ \theta a \nabla^a_h u_t^{k+1} + (1 - 2\theta) a \nabla^a_h u_t^k + \theta a \nabla^a_h u_t^{k-1} \right]$$

$$+ \theta u_t^{k+1} + (1 - 2\theta) b \nabla^b_h u_t^{k} + \theta b \nabla^b_h u_t^{k-1} + f_t^{k} \right] \cdot h_x h_y \left( u_{i,j}^{k+1} - u_{i,j}^{k-1} \right).$$

Then, summing up for $i$ from 1 to $N_x - 1$ and for $j$ from 1 to $N_y - 1$, we have

$$\sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \left[ h_x h_y \left( u_{i,j}^{k+1} \right)^2 - h_x h_y \left( u_{i,j}^{k} \right)^2 \right] = \| u_{i}^{k+1} \|^2 - \| u_{i}^{k} \|^2,$$

$$\sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \left( h_x h_y \theta^2 \tau^6 \left( \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k+1} \right)^2 - h_x h_y \theta^2 \tau^6 \left( \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k} \right)^2 \right)$$

$$= \theta^2 \tau^6 \| \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k+1} \|^2 - \theta^2 \tau^6 \| \sqrt{ab} \Lambda^a_h \Lambda^b_h u_t^{k} \|^2,$$ \hfill (3.16)

and

$$\sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \left[ \theta a_{i,j} \nabla^a_h u_{i,j}^{k+1} + (1 - 2\theta) a \nabla^a_h u_{i,j}^k + \theta a \nabla^a_h u_{i,j}^{k-1} \right]$$

$$+ \theta b \Lambda^b_h u_{i,j}^{k+1} + (1 - 2\theta) b \Lambda^b_h u_{i,j}^k + \theta b \Lambda^b_h u_{i,j}^{k-1} + f_{i,j}^{k} \right] \cdot h_x h_y \left( u_{i,j}^{k+1} - u_{i,j}^{k-1} \right)$$

$$= I_1 + I_2 + I_3 + I_4 + (f^k, u^{k+1} - u^{k-1}).$$ \hfill (3.17)
where
\[ I_1 = \theta \left( a \nabla^{\alpha}_{h_x} u^{k+1} + a \nabla^{\alpha}_{h_x} u^{k-1}, u^{k+1} - u^{k-1} \right), \]
\[ I_2 = (1 - 2\theta) \left( a \nabla^{\alpha}_{h_x} u^{k}, u^{k+1} - u^{k-1} \right), \]
\[ I_3 = \theta \left( b \nabla^{\beta}_{h_y} u^{k+1} + b \nabla^{\beta}_{h_y} u^{k-1}, u^{k+1} - u^{k-1} \right), \]
\[ I_4 = (1 - 2\theta) \left( b \nabla^{\beta}_{h_y} u^{k}, u^{k+1} - u^{k-1} \right). \]

According to Lemma 3.6, we have
\[ I_1 = -\theta \left( \| \sqrt{a} \Delta^{\alpha}_{h_x} u^{k+1} \|^2 - \| \sqrt{a} \Delta^{\alpha}_{h_x} u^{k-1} \|^2 \right), \]
\[ I_2 = -\frac{(1 - 2\theta)}{4} \left( \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k+1} + \Lambda^{\alpha}_{h_x} u^{k} \right) \|^2 - \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k+1} - \Lambda^{\alpha}_{h_x} u^{k} \right) \|^2 \right) \]
\[ - \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k} + \Lambda^{\alpha}_{h_x} u^{k-1} \right) \|^2 + \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k} - \Lambda^{\alpha}_{h_x} u^{k-1} \right) \|^2, \]
\[ I_3 = -\theta \left( \| \sqrt{b} \Delta^{\beta}_{h_y} u^{k+1} \|^2 - \| \sqrt{b} \Delta^{\beta}_{h_y} u^{k-1} \|^2 \right), \]
\[ I_4 = -\frac{(1 - 2\theta)}{4} \left( \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k+1} + \Lambda^{\beta}_{h_y} u^{k} \right) \|^2 - \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k+1} - \Lambda^{\beta}_{h_y} u^{k} \right) \|^2 \right) \]
\[ - \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k} + \Lambda^{\beta}_{h_y} u^{k-1} \right) \|^2 + \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k} - \Lambda^{\beta}_{h_y} u^{k-1} \right) \|^2. \]

From (3.16) and (3.17), we obtain
\[ \| u^{k+1}_i \|^2 - \| u^{k}_i \|^2 + \theta^2 \tau^6 \| \sqrt{a} b \Delta^{\alpha}_{h_x} \Delta^{\beta}_{h_y} u^{k+1}_i \|^2 - \theta^2 \tau^6 \| \sqrt{a} b \Delta^{\alpha}_{h_x} \Delta^{\beta}_{h_y} u^{k}_i \|^2 \]
\[ - I_1 - I_2 - I_3 - I_4 = (f^k, u^{k+1} - u^{k-1}), \]

that is,
\[ \| u^{k+1}_i \|^2 + \theta^2 \tau^6 \| \sqrt{a} b \Delta^{\alpha}_{h_x} \Delta^{\beta}_{h_y} u^{k+1}_i \|^2 + \theta \| \sqrt{a} \Delta^{\alpha}_{h_x} u^{k+1} \|^2 \]
\[ + \frac{1 - 2\theta}{4} \left( \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k+1} + \Lambda^{\alpha}_{h_x} u^{k} \right) \|^2 - \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k+1} - \Lambda^{\alpha}_{h_x} u^{k} \right) \|^2 \right) \]
\[ + \theta \| \sqrt{b} \Delta^{\beta}_{h_y} u^{k+1} \|^2 + \frac{1 - 2\theta}{4} \left( \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k+1} + \Lambda^{\beta}_{h_y} u^{k} \right) \|^2 \right) \]
\[ - \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k} + \Lambda^{\beta}_{h_y} u^{k-1} \right) \|^2 + \theta \| \sqrt{a} \Delta^{\alpha}_{h_x} u^{k-1} \|^2 \]
\[ + \frac{1 - 2\theta}{4} \left( \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k} + \Lambda^{\alpha}_{h_x} u^{k-1} \right) \|^2 - \| \sqrt{a} \left( \Lambda^{\alpha}_{h_x} u^{k} - \Lambda^{\alpha}_{h_x} u^{k-1} \right) \|^2 \right) \]
\[ + \theta \| \sqrt{b} \Delta^{\beta}_{h_y} u^{k} \|^2 + \frac{1 - 2\theta}{4} \left( \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k} + \Lambda^{\beta}_{h_y} u^{k-1} \right) \|^2 \right) \]
\[ - \| \sqrt{b} \left( \Lambda^{\beta}_{h_y} u^{k} - \Lambda^{\beta}_{h_y} u^{k-1} \right) \|^2 \]
\[ + (f^k, u^{k+1} - u^{k-1}). \]
Adding $\theta \parallel \sqrt{a} \Lambda_{h_x}^\alpha u^k \parallel^2 + \theta \parallel \sqrt{b} \Lambda_{h_y}^\beta u^k \parallel^2$ on both sides of the above equation, we have

$$
\| u_t^{k+1} \|^2 + \theta \left( \| \sqrt{a} \Lambda_{h_x}^\alpha u^{k+1} \|^2 + \| \sqrt{a} \Lambda_{h_x}^\alpha u^k \|^2 \right) + \frac{1-2\theta}{4} \left( \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} + \Lambda_{h_x}^\alpha u^k \right) \|^2 - \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} - \Lambda_{h_x}^\alpha u^k \right) \|^2 \right) + \frac{1-2\theta}{4} \left( \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} + \Lambda_{h_y}^\beta u^k \right) \|^2 - \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} - \Lambda_{h_y}^\beta u^k \right) \|^2 \right) + \theta^2 \tau^6 \| \sqrt{a} \Lambda_{h_x}^\alpha |_{h_y} \Lambda_{h_y}^\beta u_T^{k+1} \|^2 = \| u_t^k \|^2 + \theta \left( \| \sqrt{a} \Lambda_{h_x}^\alpha u^k \|^2 + \| \sqrt{a} \Lambda_{h_x}^\alpha u^{k-1} \|^2 \right) + \frac{1-2\theta}{4} \left( \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^k + \Lambda_{h_x}^\alpha u^{k-1} \right) \|^2 - \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^k - \Lambda_{h_x}^\alpha u^{k-1} \right) \|^2 \right) + \frac{1-2\theta}{4} \left( \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^k + \Lambda_{h_y}^\beta u^{k-1} \right) \|^2 - \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^k - \Lambda_{h_y}^\beta u^{k-1} \right) \|^2 \right) + \theta^2 \tau^6 \| \sqrt{a} \Lambda_{h_x}^\alpha |_{h_y} \Lambda_{h_y}^\beta u_T^k \|^2,
$$

there exists

$$
E_u^k = E_u^{k-1} + (f^k, u^{k+1} - u^{k-1}). \tag{3.19}
$$

We rewrite (3.18) as the following form

$$
E_u^k = \| u_t^{k+1} \|^2 + \frac{1}{4} \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} + \Lambda_{h_x}^\alpha u^k \right) \|^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} - \Lambda_{h_x}^\alpha u^k \right) \|^2 + \frac{1}{4} \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} + \Lambda_{h_y}^\beta u^k \right) \|^2 + \frac{1}{4} (4\theta - 1) \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} - \Lambda_{h_y}^\beta u^k \right) \|^2 + \theta^2 \tau^6 \| \sqrt{a} \Lambda_{h_x}^\alpha |_{h_y} \Lambda_{h_y}^\beta u_T^{k+1} \|^2, \tag{3.20}
$$

where we use

$$
\| \sqrt{a} \Lambda_{h_x}^\alpha u^{k+1} \|^2 + \| \sqrt{a} \Lambda_{h_x}^\alpha u^k \|^2 = \frac{1}{2} \left( \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} + \Lambda_{h_x}^\alpha u^k \right) \|^2 + \| \sqrt{a} \left( \Lambda_{h_x}^\alpha u^{k+1} - \Lambda_{h_x}^\alpha u^k \right) \|^2 \right),
$$
and
\[ \| \sqrt{b} \Lambda_{h_y}^\beta u^{k+1} \|^2 + \| \sqrt{b} \Lambda_{h_y}^\beta u^k \|^2 = \frac{1}{2} \left( \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} + \Lambda_{h_y}^\beta u^k \right) \|^2 \right. \]
\[ \left. + \| \sqrt{b} \left( \Lambda_{h_y}^\beta u^{k+1} - \Lambda_{h_y}^\beta u^k \right) \|^2 \right). \]

According to
\[ (f^k, u^{k+1} - u^{k-1}) = 2 h_x h_y \tau \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} f^k_{i,j} \left( \frac{u_{i,j}^{k+1} - u_{i,j}^{k-1}}{2\tau} \right) \]
\[ \leq h_x h_y \tau \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} \left[ \left( f^k_{i,j} \right)^2 \right. \]
\[ \left. + \left( \frac{u_{i,j}^{k+1} - u_{i,j}^{k-1}}{2\tau} \right)^2 \right] \]
\[ \leq \frac{\tau}{2} \left( \| u_i^{k+1} \|^2 + \| u_i^k \|^2 \right) + \tau || f^k ||^2, \]
and (3.20), (3.19), there exists
\[ E_u^k - E_u^{k-1} = (f^k, u^{k+1} - u^{k-1}) \leq \frac{\tau}{2} \left( E_u^k + E_u^{k-1} \right) + \tau || f^k ||^2, \]
that is,
\[ \left( 1 - \frac{\tau}{2} \right) E_u^k \leq \left( 1 + \frac{\tau}{2} \right) E_u^{k-1} + \tau || f^k ||^2. \]

For \( \tau \leq 2/3 \), which leads to
\[ E_u^k \leq \left( 1 + \frac{3\tau}{2} \right) E_u^{k-1} + \frac{3}{2} \tau || f^k ||^2. \]

From Lemma 3.2, there exists
\[ E_u^k \leq e^{3k\tau} \left[ E_u^0 + \frac{3}{2} \tau \sum_{l=1}^{k} \| f^l \|^2 \right]. \]
The proof is completed.

**Theorem 3.4** Let \( \frac{1}{4} \leq \theta \leq 1 \), \( 1 < \alpha, \beta < 2 \) and \( u(x_i, y_j, t_k) \) be the exact solution of (1.1) with the constant coefficients; \( u_{i,j}^k \) be the solution of (2.22) and \( e_{i,j}^k = u(x_i, y_j, t_k) - u_{i,j}^k \). Then,
\[ E_e^k = \mathcal{O} \left( \tau^2 + h_x^2 + h_y^2 \right)^2, \]
where the energy norm is defined by
\[
E_e^k = \| e_{i,j}^{k+1} \|^2 + \frac{1}{4} \left( \Lambda_{h_x}^\alpha e_{i,j}^{k+1} + \Lambda_{h_y}^\alpha e_{i,j}^k \right) \|^2 + \frac{1}{4} (4\theta - 1) \left\| \sqrt{a} \left( \Lambda_{h_x}^\alpha e_{i,j}^{k+1} - \Lambda_{h_x}^\alpha e_{i,j}^k \right) \right\|^2 \\
+ \frac{1}{4} \left\| \sqrt{b} \left( \Lambda_{h_y}^\beta e_{i,j}^{k+1} + \Lambda_{h_y}^\beta e_{i,j}^k \right) \right\|^2 + \frac{1}{4} (4\theta - 1) \left\| \sqrt{b} \left( \Lambda_{h_y}^\beta e_{i,j}^{k+1} - \Lambda_{h_y}^\beta e_{i,j}^k \right) \right\|^2 \\
+ \theta^2 \tau^6 \left( \sqrt{ab} \Lambda_{h_x}^\alpha \Lambda_{h_y}^\beta e_{i,j}^{k+1} \right) \| e_{i,j}^{k+1} \|^2. \]
Proof Subtracting (2.22) from (2.23), it yields
\[
\frac{1}{2} \delta^2 e_{i,j}^k + \theta^2 \tau^4 a b_\nu h_x \nabla^\beta e_{i,j}^k (e_{i,j}^{k+1} - 2e_{i,j}^k + e_{i,j}^{k-1})
\]
\[
= \theta a_{i,j} \nabla^\alpha e_{i,j}^{k+1} + (1 - 2\theta) a_{i,j} \nabla^\alpha e_{i,j}^k + \theta a_{i,j} \nabla^\alpha e_{i,j}^{k-1}
\]
\[
+ \theta b_{i,j} \nabla^\beta e_{i,j}^{k+1} + (1 - 2\theta) b_{i,j} \nabla^\beta e_{i,j}^k + \theta b_{i,j} \nabla^\beta e_{i,j}^{k-1} + \tilde{R}_{i,j}^k.
\]

Using Lemma 3.8, there exists
\[
E^k_e \leq e^{3k\tau} \left[ E^0_e + \frac{3}{2} \tau \sum_{l=1}^k ||\tilde{R}'||^2 \right],
\]
where the energy norm $E^k_e$ is given in (3.21). Next, we estimate the local error truncation of $E^0_e$. Since $e^0_{i,j} = 0$, $\theta^2 \tau^6 \| a b_\nu h_x \Lambda^\alpha h_y e^1_{i,j} \| = O(\tau^6)$ in (3.16) and
\[
ie_{i,j}^1 = \frac{\tau^2}{2} \left[ a(x_i, y_j) \left( \frac{\partial u(x_i, y_j, 0)}{\partial |x|^\alpha} - \nabla^\alpha h_x \varphi(x_i, y_j) \right) \right.
\]
\[
+ b(x_i, y_j) \left( \frac{\partial u(x_i, y_j, 0)}{\partial |y|^\beta} - \nabla^\beta h_y \varphi(x_i, y_j) \right) \left. \right] + \frac{1}{2} \int_0^\tau \frac{\partial^3 u(x_i, y_j, t)}{\partial t^3} dt
\]
\[
= \frac{\tau^2}{2} \left[ a(x_i, y_j) C_{1,\alpha} \frac{\partial^{a+2} u(\xi, y_j, t)}{\partial |x|^\alpha + 2} h_x^2 + b(x_i, y_j) C_{1,\beta} \frac{\partial^{\beta+2} u(\eta, y_j, t)}{\partial |y|^\beta + 2} h_y^2 \right]
\]
\[
+ \frac{1}{2} \int_0^\tau (\tau - t)^2 \frac{\partial^3 u(x_i, y_j, t)}{a^3} dt \leq C_{1,\alpha,\beta} (\tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2).
\]

Here, the coefficients $C_{1,\alpha}$ and $C_{1,\beta}$ are the constants independent of $h$, $\tau$ and
\[
C_{1,\alpha,\beta} = \max_{0 \leq x \leq x_r, 0 \leq y \leq y_r, 0 \leq t \leq T} \left\{ \frac{1}{2} a_1 \left| C_{1,\alpha} \frac{\partial^{a+2} u(\xi, y_j, t)}{\partial |x|^\alpha + 2} \right|, \right\}
\]
\[
\frac{1}{2} b_1 \left| C_{1,\beta} \frac{\partial^{\beta+2} u(\eta, y_j, t)}{\partial |y|^\beta + 2} \right|, \frac{1}{6} \int_0^\tau \frac{\partial^3 u(x_i, y_j, t)}{\partial t^3} dt \right\}.
\]

Then, we obtain
\[
\| e^1_{i,j} \|^2 \leq \left| \frac{e^1 - e^0}{\tau} \right|^2
\]
\[
\leq (N_x - 1) h_x (N_y - 1) h_y \frac{1}{\tau^2} C_{1,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right)
\]
\[
\cdot C_{1,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right)
\]
\[
\leq C_{1,\alpha,\beta} x_r y_r (\tau^2 + \tau h_x^2 + \tau h_y^2)^2. \quad (3.24)
\]
From (1.1) and the above equations, there exists

\[
\| \sqrt{a} \Lambda_0^\alpha e^1 \|_2 = - \left( a \nabla_{h^t} e^1, e^1 \right) = -h_x h_y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} a_{i,j} \left( \nabla_{h^t} e^1_{i,j} \right) \cdot e^1_{i,j}
\]

\[
= -h_x h_y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} a_{i,j} \sum_{l=0}^{N_x} -\kappa_{l,a} \frac{1}{h^l_x} \phi_{i,l} \left[ \frac{a_{i,j} C_{i,\alpha} }{2} \frac{\partial^{a+2} u(\xi_i, y_j, t)}{\partial |x|^{a+2}} \tau^2 h_x^2 + C_{2,\alpha} \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+4}} \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+4}} \tau^2 h_x^2 \right] + \frac{b_{i,j} C_{1,\beta} }{2} \frac{\partial^{\alpha+2} u(\xi_i, y_j, t)}{\partial |y|^{\beta+2}} \tau^2 h_y^2 + \frac{1}{2} \int_0^\tau (\tau - t)^2 \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+2}} \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+2}} \tau^2 h_x^2 \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+2}} \frac{\partial^{\alpha+4} u(\xi_i, y_j, t)}{\partial |x|^{\alpha+2}} \tau^2 h_x^2 \right] \cdot e^1_{i,j}
\]

\[
\leq C_{3,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right) \cdot C_{1,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right),
\]

where \( \xi_i, \tilde{\xi}_i, \bar{\xi}_i, \tilde{\xi}_i \in (0, x_r) \) and \( C_{l,\alpha} \) and \( C_{l,\alpha,\beta} \), \( 1 \leq l \leq 3 \) are the constants. Similarly, we have

\[
\| \sqrt{b} \Lambda_0^\beta e^1 \|_2 \leq \tilde{C}_{3,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right) \cdot \tilde{C}_{1,\alpha,\beta} \left( \tau^3 + \tau^2 h_x^2 + \tau^2 h_y^2 \right)
\]

with the constants \( \tilde{C}_{1,\alpha,\beta} \) and \( \tilde{C}_{3,\alpha,\beta} \).

According to (3.21), (3.24), and the above equations, we get

\[
E^0_e \leq C_{2,\alpha,\beta} x_{r} y_{r} \left( \tau^2 + \tau h_x^2 + \tau h_y^2 \right)^2,
\]

(3.25)

where \( C_{\alpha,\beta} \) is a constant. Hence, using (2.17), (3.22), and (3.25), there exists

\[
E^k_e \leq e^{2k\tau} \left[ C_{2,\alpha,\beta} x_{r} y_{r} \left( \tau^2 + \tau h_x^2 + \tau h_y^2 \right)^2 + 3k\tau C_{u,\alpha,\beta} \left( \tau^2 + h_x^2 + h_y^2 \right)^2 \right]
\]

\[
\leq \tilde{C}_{\alpha,\beta} e^{2\tau} \left( \tau^2 + h_x^2 + h_y^2 \right)^2
\]

with \( \tilde{C}_{\alpha,\beta} = 2 \max \left\{ C_{2,\alpha,\beta} x_{r} y_{r}, \frac{3}{2} C_{u,\alpha,\beta} T \right\} \). The proof is completed. \( \square \)
Theorem 3.5 Let $1 < \alpha, \beta < 2$, $\frac{1}{4} \leq \theta \leq 1$ and $u(x_i, y_j, t_k)$ be the exact solution of (1.1) with the constant coefficients; $u^k_{i,j}$ be the solution of the finite difference scheme (2.27) and $e^k_{i,j} = u(x_i, y_j, t_k) - u^k_{i,j}$. Then,

$$\| e^k \| \leq C e \left( \tau^2 + h_x^2 + h_y^2 \right), \quad 0 \leq k \leq N_t$$

with a positive constant $C e$.

Proof The similar arguments can be performed as Theorem 3.2, we omit it here. $\square$

Theorem 3.6 Let $1 < \alpha, \beta < 2$ and $\frac{1}{4} \leq \theta \leq 1$. Then, the difference scheme (2.27) with the positive constant coefficients is unconditionally stable.

Proof From Lemma 3.7, the result is obtained. $\square$

Remark 3.2 Consider the nonlocal wave equation [10]

$$\begin{cases}
\frac{\partial^2 u(x,t)}{\partial t^2} - L_\delta u(x, t) = f_\delta(x, t) & \text{on } \Omega, \ t > 0, \\
u(x, 0) = u_0 & \text{on } \Omega \cup \Omega_\delta, \\
u_t = g & \text{on } \Omega_\delta, \ t > 0.
\end{cases}$$

From [4], we known that the approximation operator of $-L_\delta$ is also the symmetric positive definite. Hence, the framework of the stability and convergence analysis are still valid for the nonlocal wave equation.

4 Numerical results

In this section, we numerically verify the above theoretical results and the discrete $L^2$-norm is used to measure the numerical errors.

Example 4.1 Consider the space-Riesz fractional wave equation (2.5), on a finite domain $0 < x < 1$, $0 < t \leq 1$ with the coefficient $a(x) = x^\alpha$, the forcing function is

$$f(x, t) = e^{-t}x^2(1 - x)^2$$

$$+ \frac{x^\alpha e^{-t}}{2 \cos(\alpha \pi / 2)} \left[ \Gamma(5) \frac{x^{4-\alpha} + (1 - x)^{4-\alpha}}{\Gamma(5 - \alpha)} - 2\Gamma(4) \frac{x^{3-\alpha} + (1 - x)^{3-\alpha}}{\Gamma(4 - \alpha)} \right]$$

$$+ \Gamma(3) \frac{x^{2-\alpha} + (1 - x)^{2-\alpha}}{\Gamma(3 - \alpha)} \right]$$

with the initial conditions $u(x, 0) = x^2(1 - x)^2$, $\frac{\partial}{\partial t} u(x, 0) = -x^2(1 - x)^2$, and the boundary conditions $u(0, t) = u(1, t) = 0$. The exact solution of the fractional PDEs is

$$u(x, t) = e^{-t}x^2(1 - x)^2.$$
Table 1 The discrete $L^2$-norm and convergent rates for (2.14) with $\tau = h$

| $\tau$ | $\alpha = 1.3, \theta = 0.25$ | Rate | $\alpha = 1.6, \theta = 0.5$ | Rate | $\alpha = 1.9, \theta = 1$ | Rate |
|--------|-------------------------------|------|-------------------------------|------|-------------------------------|------|
| 1/40   | 4.9759e−05                   |      | 5.1018e−05                   |      | 4.7788e−05                   |      |
| 1/80   | 1.2658e−05                   | 1.9749 | 1.2423e−05                   | 2.0380 | 1.1799e−05                   | 2.0180 |
| 1/160  | 3.2276e−06                   | 1.9715 | 3.0317e−06                   | 2.0348 | 2.9026e−06                   | 2.0233 |
| 1/320  | 8.2205e−07                   | 1.9732 | 7.4241e−07                   | 2.0298 | 7.1267e−07                   | 2.0260 |

Example 4.2 Consider the space-Riesz fractional wave equation (2.5), on a finite domain $0 < x < 1$, $0 < t \leq 1$ with the coefficient $a(x) = 1$, the forcing function is $f(x, t) = 0$ with the initial conditions $u(x, 0) = \sin(\pi x)$, $u(x, \tau) = (1 + \tau) \sin(\pi x)$, and the boundary conditions $u(0, t) = u(1, t) = 0$.

Since the analytic solutions is unknown for Example 4.2, the order of the convergence of the numerical results is computed by the following formula

$$\text{Convergence Rate} = \frac{\ln \left( \frac{||U_{Nt}^N - U_{Nt}^h||}{||U_{Nt}^N - U_{Nt}^{h/2}||} \right)}{\ln 2}.$$ 

Table 2 shows that the scheme (2.14) preserves the desired first-order convergence with nonhomogeneous initial conditions. And it is not possible to reach second-order convergence even with the high-order scheme (2.14) because of the weak regularity of the solution in the region close to the initial point and the boundaries.

Example 4.3 Consider the two-dimensional space-Riesz fractional wave equation (1.1), on a finite domain $0 < x < 1$, $0 < y < 1$, $0 < t \leq 1/2$ with the variable coefficients

$$a(x, y) = x^\alpha y, \quad b(x, y) = xy^\beta,$$

and the initial conditions $u(x, y, 0) = \sin(1)x^2(1 - x)^2y^2(1 - y)^2$, $\frac{\partial}{\partial t}u(x, y, 0) = \cos(1)x^2(1 - x)^2y^2(1 - y)^2$ with the zero Dirichlet boundary conditions on the rectangle. The exact solution of the PDEs is

$$u(x, y, t) = \sin(t + 1)x^2(1 - x)^2y^2(1 - y)^2.$$

Using the above conditions, it is easy to obtain the forcing function $f(x, y, t)$. Table 3 shows that the scheme (2.27) is second-order convergent in both space and time directions.

Table 2 The discrete $L^2$-norm and convergent rates for (2.14) with $\tau = h$

| $\tau$ | $\alpha = 1.3, \theta = 0.25$ | Rate | $\alpha = 1.6, \theta = 0.5$ | Rate | $\alpha = 1.9, \theta = 1$ | Rate |
|--------|-------------------------------|------|-------------------------------|------|-------------------------------|------|
| 1/40   | 1.5934e−02                   |      | 2.6548e−02                   |      | 3.2025e−02                   |      |
| 1/80   | 7.8998e−03                   | 1.0141 | 1.3239e−02                   | 1.0038 | 1.4572e−02                   | 1.1360 |
| 1/160  | 3.9323e−03                   | 1.0046 | 6.5937e−03                   | 1.0056 | 7.0120e−03                   | 1.0553 |
| 1/320  | 1.9650e−03                   | 1.0008 | 3.2897e−03                   | 1.0031 | 3.4292e−03                   | 1.0320 |
Table 3 The discrete $L^2$-norm and convergent rates for (2.27) with $\tau = h_x = h_y$ and $\theta = 0.75$.

| $\tau$ | $\alpha = 1.3, \beta = 1.7$ | Rate | $\alpha = 1.5, \beta = 1.5$ | Rate | $\alpha = 1.7, \beta = 1.3$ | Rate |
|--------|-----------------------------|------|-----------------------------|------|-----------------------------|------|
| 1/20   | 2.6206e− 05                 |      | 2.9756e− 05                 |      | 3.3634e− 05                 |      |
| 1/40   | 6.7726e− 06                 | 1.9521| 7.6528e− 06                 | 1.9591| 8.4875e− 06                 | 1.9865|
| 1/80   | 1.6017e− 06                 | 2.0801| 1.8303e− 06                 | 2.0639| 2.0690e− 06                 | 2.0364|
| 1/160  | 3.6930e− 07                 | 2.1167| 4.2848e− 07                 | 2.0948| 4.9445e− 07                 | 2.0650|

5 Conclusion

To the best of our knowledge, the convergence and stability are lack of study for the space-Riesz fractional wave equation. In our work, we first prove the unconditional stable for the proposed schemes by the energy method. We remark that the corresponding theoretical and algorithm can also be extended to the the nonlocal wave equation [10] and the fractional wave problems [1, 7, 11].

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