Toroidality and projective-planarity of intersection graphs of subgroups of finite groups

R. Rajkumar∗  P. Devi†
Department of Mathematics, The Gandhigram Rural Institute – Deemed University,
Gandhigram – 624 302, Tamil Nadu, India.

Abstract

Let $G$ be a group. The intersection graph of subgroups of $G$, denoted by $\mathcal{I}(G)$, is a graph with all the proper subgroups of $G$ as its vertices and two distinct vertices in $\mathcal{I}(G)$ are adjacent if and only if the corresponding subgroups having a non-trivial intersection in $G$. In this paper, we classify the finite groups whose intersection graph of subgroups are toroidal or projective-planar. In addition, we classify the finite groups whose intersection graph of subgroups are one of bipartite, complete bipartite, tree, star graph, unicyclic, acyclic, cycle, path or totally disconnected. Also we classify the finite groups whose intersection graph of subgroups does not contain one of $K_5$, $K_4$, $C_5$, $C_4$, $P_4$, $P_3$, $P_2$, $K_{1,3}$, $K_{2,3}$ or $K_{1,4}$ as a subgraph. We estimate the girth of the intersection graph of subgroups of finite groups. Moreover, we characterize some finite groups by using their intersection graphs. Finally, we obtain the clique cover number of the intersection graph of subgroups of groups and show that intersection graph of subgroups of groups are weakly $\alpha$-perfect.

Keywords: Intersection graph, finite groups, genus, toroidal graph, nonorientable genus, projective-planar graph.

2010 Mathematics Subject Classification: 05C25, 05C10, 05E15, 20E99.

1 Introduction

Let $\mathcal{F} = \{S_i \mid i \in I\}$ be an arbitrary family of sets. The intersection graph of $\mathcal{F}$ is a graph having the elements of $\mathcal{F}$ as its vertices and two vertices $S_i$ and $S_j$ are adjacent if

∗e-mail: rrajmaths@yahoo.co.in
†e-mail: pdevigri@gmail.com
and only if $i \neq j$ and $S_i \cap S_j \neq \{\emptyset\}$. For the properties of these graphs and some special class of intersection graphs, we refer the reader to [20]. In the past fifty years, it has been an interesting topic for mathematicians, when the members of $\mathcal{F}$ have some specific algebraic structures. Especially, they investigate on the interplay between the algebraic properties of algebraic structures, and the graph theoretic properties of their intersection graphs. In this direction, in 1964 Bosak [7] initiated the study of the intersection graphs of semigroups. Later, Csákány and Pollák [12] defined the intersection graph of subgroups of a finite group.

Let $G$ be a group. The intersection graph of subgroups of $G$, denoted by $\mathcal{I}(G)$, is a graph with all the proper subgroups of $G$ as its vertices and two distinct vertices in $\mathcal{I}(G)$ are adjacent if and only if the corresponding subgroups have a non-trivial intersection in $G$.

In [31] Zelinka made some investigations on the intersection graphs of subgroups of finite abelian groups. Motivated by these, many authors have defined, and studied the intersecting graphs on several algebraic structures, viz., rings, vector spaces, modules, and contributed interesting results. See, for instance [1, 2, 3, 9, 18, 23, 30] and the references therein.

Embeddability of graphs, associated with algebraic structures, on topological surfaces is considered in several recent papers [10, 24, 25, 26]. Planarity of intersection graphs of subgroups finite groups were studied by Selçuk Kayacan et al. in [22], and by H. Ahmedi et al. in [15]. Planarity of intersection graphs of ideals of rings, and submodules of modules were studied in [16, 30].

A natural question arises in this direction is the following: Which groups have their intersection graph of subgroups is of genus one, that is toroidal, or of nonorientable genus one, that is projective-planar?. In this paper, we answer this question in the case of finite groups by classifying the finite groups whose intersection graph of subgroups is toroidal or projective-planar (see Theorem 5.1 in Section 5 below). As a consequence of this research, we also classify finite groups whose intersection graph of subgroups is in some class of graphs (see Theorem 5.2 and Corollary 5.1 in Section 5 below) and characterize some finite groups by using their intersection graphs (see Corollary 5.2 in Section 5 below), which are some of the main applications of these results for the group theory. Also we estimate the grith of the intersection graph of finite groups. Finally, we obtain the clique covering number of the intersection graph of subgroups of groups and show that intersection graph of subgroups of groups are weakly $\alpha$-perfect.

## 2 Preliminaries and notations

In this section, we first recall some notation, and results in graph theory, which are used later in the subsequent sections. We use standard basic graph theory terminology and notation (e.g., see [29]). Let $G$ be a simple graph with a vertex set $V$ and an edge set
E. $G$ is said to be bipartite if $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins a vertex of $V_1$ to a vertex of $V_2$. Then $(V_1, V_2)$ is called a bipartition of $G$. Moreover, if every vertex of $V_1$ is adjacent to every vertex of $V_2$, then $G$ is called complete bipartite and is denoted by $K_{m,n}$, where $|V_1| = m$, $|V_2| = n$. In particular, $K_{1,n}$ is a star graph. $G$ is said to be complete if each pair of distinct vertices in $G$ are adjacent. The complete graph on $n$ vertices is denoted by $K_n$. $G$ is said to be totally disconnected if its edge set is empty. A path connecting two vertices $u$ and $v$ in $G$ is a finite sequence $(u =) v_0, v_1, \ldots, v_n (= v)$ of distinct vertices (except, possibly, $u$ and $v$) such that $u_i$ is adjacent to $v_{i+1}$ for all $i = 0, 1, \ldots, n - 1$. A path is a cycle if $u = v$. The length of a path or a cycle is the number of edges in it. A path or a cycle of length $n$ is denoted by $P_n$ or $C_n$, respectively. A graph is unicyclic if it has exactly one cycle, and is acyclic if it has no cycles. The girth of a graph $G$, denoted by $girth(G)$, is the length of the shortest cycle in $G$, if it exist; otherwise $girth(G) = \infty$.

We define a graph $G$ to be $X$-free if $G$ has no subgraph isomorphic to a given graph $X$. An independent set of a graph $G$ is a subset of the vertices of $G$ such that no two vertices in the subset are adjacent. The independence number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of $G$. A clique of a graph $G$ is a complete subgraph of $G$. The clique cover number of $G$, denoted by $\theta(G)$, is the minimum number of cliques in $G$ which cover all the vertices of $G$ (not necessarily all the edges of $G$). $G$ said to be weakly $\alpha$-perfect if $\alpha(G) = \theta(G)$. For two graphs $G$ and $H$, $G \cup H$ denotes disjoint union of $G$ and $H$, $G + H$ denotes a graph with the vertex set consist of vertices of $G$ and $H$ and edge set having all the lines joining vertices of $G$ to vertices of $H$. $\overline{G}$ denotes the complement of a graph $G$, and, for an integer $n \geq 1$, $nG$ denotes the graph having $q$ disjoint copies of $G$.

Let $S_n$ denote the surface obtained from the sphere by attaching $n$ handles. A graph is said to be embeddable on a topological surface if it can be drawn on the surface such that no two edges cross. The (orientable) genus of a graph $G$, denoted by $\gamma(G)$, is the smallest non-negative integer $n$ such that $G$ can be embedded on $S_n$. $G$ is planar if $\gamma(G) = 0$ and toroidal if $\gamma(G) = 1$. A crosscap is a circle (on the surface) such that all its pairs of opposite points are identified, and the interior of this circuit is removed. Let $N_k$ denote the sphere with $k$ added crosscaps. For non-orientable topological surfaces (e.g., the projective plane, Klein bottle, etc.), the nonorientable genus of $G$ is the smallest integer $k$ such that $G$ can be embedded on $N_k$, and it is denoted by $\gamma(G)$. $N_1$ is the projective plane. Respectively, a graph $G$ is projective-planar if $\tau(G) = 1$.

A topological obstruction for a surface is a graph $G$ of minimum vertex degree at least three such that $G$ does not embed on the surface, but $G - e$ is embeddable on the surface for every edge $e$ of $G$. A minor-order obstruction $G$ is a topological obstruction with the additional property that, for each edge $e$ of $G$, $G$ with the edge $e$ contracted embeds on the surface.

The following results are used in the subsequent sections.
Theorem 2.1. ([22, Theorems 6.37, 6.38 and 11.19, 11.23])

(1) \(\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, n \geq 3;\)
\(\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, m, n \geq 2.\)

(2) \(\overline{\gamma}(K_n) = \begin{cases} 
\left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, & \text{if } n \geq 3, n \neq 7; \\
3, & \text{if } n = 7; \\
\left\lceil \frac{(m-2)(n-2)}{2} \right\rceil, & \text{if } n \geq 5 \text{ or } m = n = 4. 
\end{cases}\)
\(\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil, m, n \geq 2.\)

By Theorem 2.1, one can see that \(\gamma(K_n) > 1\) for \(n \geq 8\), \(\overline{\gamma}(K_n) > 1\) for \(n \geq 7\), \(\gamma(K_{m,n}) > 1\) if either \(m \geq 4\), \(n \geq 5\) or \(m \geq 3\), \(n \geq 7\), and \(\overline{\gamma}(K_{m,n}) > 1\) if either \(m \geq 3\), \(n \geq 5\) or \(m = n = 4\). Neufeld and Myrvold [21] have shown the following.

Theorem 2.2. ([27, Figure 4, p. 578]) There are exactly three eight-vertex obstructions \(A_1, A_2, A_3\) for the torus, each of them being topological and minor-order (see Figure 1).

Note that the graph \(A_1\) in Figure 1 has \(K_{3,5}\) as a subgraph, so it follows that if a graph \(G\) has \(A_1\) as a subgraph, then \(\gamma(G) > 1\) and \(\overline{\gamma}(G) > 1\). In this paper, we repeatedly use this fact.

Gagarin et al. [4] have found all the toroidal obstructions for the graphs containing no subdivisions of \(K_{3,3}\) as a subgraph. These graphs coincide with the graphs containing no \(K_{3,3}\)-minors and are called with no \(K_{3,3}\)'s.

Theorem 2.3. ([4, Theorem 3, p. 3628]) There are exactly four minor-order obstructions with no \(K_{3,3}\)'s for the torus, precisely, \(B_1, B_2, B_3, B_4\) shown in Figure 2 as a minor.

Notice that all the obstructions in Theorems 2.2 and 2.3 are obstructions for toroidal graphs in general, which are very numerous (e.g., see [4]).

In [9], Chakarabarthy et al. proved the following results on the intersection graph of ideals of the ring \(\mathbb{Z}_n\).
Theorem 2.4. (22) Let \( p, q, r \) be distinct primes. Then

1. \( \mathcal{I}(\mathbb{Z}_n) \) is planar if and only if \( n \) is one of \( p^\alpha (\alpha = 2, 3, 4, 5), p^\alpha q (\alpha = 1, 2) \), or \( pqr \).

2. \( \mathcal{I}(\mathbb{Z}_n) \) is \( K_5 \)-free if and only if it is planar.

3. \( \mathcal{I}(\mathbb{Z}_n) \) is bipartite if and only if \( n \) is either \( p^3 \) or \( pq \).

4. \( \mathcal{I}(\mathbb{Z}_n) \) is \( C_3 \)-free if and only if \( n \) is either \( p^\alpha (\alpha = 2, 3) \) or \( pq \).

In [22] Selçuk Kayacan et al. have classified all the finite groups whose intersection graphs of subgroups are planar.

Theorem 2.5. (22) Let \( G \) be a finite group and \( p, q, r \) be distinct primes. Then \( \mathcal{I}(G) \) is planar if and only if \( G \) is one of the following groups:

1. \( \mathbb{Z}_{p^\alpha} (\alpha = 2, 3, 4, 5), \mathbb{Z}_{p^\alpha q} (\alpha = 1, 2), \mathbb{Z}_{pqr}, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_6 \times \mathbb{Z}_2; \)

2. \( Q_8, M_8, \mathbb{Z}_q \times \mathbb{Z}_p, \mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2}, A_4; \)

3. \( G_1 \cong \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1 b^1 \rangle \), where \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right) \) has order \( q \) in \( GL_2(p) \), where \( q \mid (p + 1); \)

4. \( G_2 \cong \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b^{-1}, cbc^{-1} = a^1 b^1 \rangle \), where \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right) \) has order \( q^2 \) in \( GL_2(p) \), where \( q^2 \mid (p + 1); \)

5. \( G_3 \cong \langle a, b, c \mid a^p = b^q = c^r = 1, b^{-1} ab = a^\mu, c^{-1} ac = a^\nu, bc = cb \rangle, q, r \) are divisors of \( (p - 1), v, \mu \neq 1. \)

The intersection graph of subgroups of the groups listed in Theorem 2.5, are given below for use in the subsequent sections.

\[ \mathcal{I}(\mathbb{Z}_{p^\alpha}) \cong K_{\alpha - 1}, \alpha = 2, 3, 4, 5. \]  \hspace{1cm} (2.1)

\[ \mathcal{I}(\mathbb{Z}_{pq}) \cong K_2. \]  \hspace{1cm} (2.2)

\[ \mathcal{I}(\mathbb{Z}_{p^2q}) \cong K_1 + (K_2 \cup K_1). \]  \hspace{1cm} (2.3)

\[ \mathcal{I}(\mathbb{Z}_p \times \mathbb{Z}_q) \cong K_{p+1}. \]  \hspace{1cm} (2.4)

Figure 2: The minor-order obstructions with no \( K_{3,3} \)’s for the torus.
Intersection graph of subgroups

\[ \mathcal{I}(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong K_1 + (K_4 \cup \overline{K}_2). \]  
(2.5)

\[ \mathcal{I}(Q_8) \cong K_4. \]  
(2.6)

\[ \mathcal{I}(\mathbb{Z}_q \times \mathbb{Z}_p) \cong \overline{K}_{q+1} \]  
(2.7)

\[ \mathcal{I}(\mathbb{Z}_q \times_2 \mathbb{Z}_{p^2}) \cong K_1 + (K_1 \cup qK_2), \]  
(2.8)

\[ \Gamma(\mathcal{G}_1) \cong K_{1,p+1} \cup \overline{K}_{p^2}. \]  
(2.9)

\[ \mathcal{I}(A_4) \cong K_{1,3} \cup \overline{K}_4. \]  
(2.10)

\[ \mathcal{I}(\mathcal{G}_2) \cong K_1 + (K_{1,p+1} \cup p^2K_2). \]  
(2.11)

We summarize some of the results of group theory which we use in forthcoming sections.

![Graphs](image.png)

Figure 3: (a) \( \mathcal{I}(\mathbb{Z}_{pqr}) \), (b) \( \mathcal{I}(\mathbb{Z}_6 \times \mathbb{Z}_2) \), (c) \( \mathcal{I}(M_8) \), (d) \( \mathcal{I}(\mathcal{G}_3) \).

**Theorem 2.6.**  
(i) ([8, Theorem IV, p.129]) If \( G \) is a \( p \)-group of order \( p^n \), then the number of subgroups of order \( p^s \), \( 1 \leq s \leq n \) is congruent to 1 (mod \( p \)).

(ii) ([27, Proposition 1.3]) If \( G \) is a \( p \)-group of order \( p^n \) and it has a unique subgroup of order \( p^s \), \( 1 < s \leq n \), then \( G \) is cyclic or \( s = 1 \) and \( p = 2 \), \( G \cong \mathbb{Z}_2^n \).

(iii) ([8, Corollary IV, p. 53]) If \( G \) has a normal subgroup \( H \) of order \( mn \), where \( m \) and \( n \) are relatively prime and \( N \) is a normal subgroup of \( H \) of order \( n \), then \( N \) is also normal in \( G \).

3 Finite abelian groups

In this section, first we classify the finite abelian groups whose intersection graphs of subgroups are toroidal or projective-planar, and next we classify the finite non-cyclic abelian groups whose intersection graphs of subgroups are one of \( K_5 \) free, \( C_3 \)-free, acyclic or bipartite.

**Proposition 3.1.** Let \( G \) be a finite cyclic group and \( p, q \) be distinct primes. Then
Intersection graph of subgroups

1. $\mathcal{I}(G)$ is toroidal if and only if $G$ is one of $\mathbb{Z}_{p^\alpha}(\alpha = 6, 7, 8)$, $\mathbb{Z}_{p^\alpha q}(\alpha = 3, 4)$ or $\mathbb{Z}_{p^2 q^2}$;

2. $\mathcal{I}(G)$ is projective-planar if and only if $G$ is one of $\mathbb{Z}_{p^\alpha}(\alpha = 6, 7)$ or $\mathbb{Z}_{p^3 q}$.

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_i$’s are distinct primes, and $\alpha_i \geq 1$ are integers.

Note that to prove this result, it is enough to investigate the finite cyclic groups other than those listed in Theorem 2.4(1). We have to deal with the following cases.

Case 1: If $k = 1$, then the number of proper subgroups of $G$ is $\alpha_1 - 1$ and so

$$\mathcal{I}(G) \cong K_{\alpha_1 - 1}.$$ (3.1)

It follows that $\gamma(\mathcal{I}(G)) = 1$ if and only if $\alpha_1 = 6, 7, 8$ and $\gamma(\mathcal{I}(G)) = 1$ if and only if $\alpha_1 = 6, 7$.

Case 2: If $k = 2$ and $\alpha_2 = 1$, then we consider the following subcases:

Sub case 2a: $\alpha_1 = 3$. Let $H_i$, $i = 1, \ldots, 6$ be the subgroups of $G$ of order $p_1, p_1^2, p_1^3$, $p_2, p_1 p_2, p_1^2 p_2$ respectively. Here $H_1$ is a subgroup of $H_2, H_3, H_5, H_6$; $H_4$ is a subgroup of $H_5, H_6$; $H_4$ has trivial intersection with $H_i$, $i = 1, 2, 3$. Therefore,

$$\mathcal{I}(G) \cong K_2 + (K_3 \cup K_1),$$ (3.2)

which is a subgraph of $K_6$ and it contains $K_5$. So $\gamma(\mathcal{I}(G)) = 1$ and $\gamma(\mathcal{I}(G)) = 1$.

Sub case 2b: $\alpha_1 = 4$. Let $H_i$, $i = 1, \ldots, 8$ be the subgroups of $G$ of order $p_1, p_1^2, p_1^3, p_1^4$, $p_2, p_1 p_2, p_1^2 p_2, p_1^3 p_2$ respectively. Here $H_1$ is a subgroup of $H_2, H_3, H_4, H_6, H_7, H_8$; $H_5$ is a subgroup of $H_i$, $i = 6, 7, 8$; $H_5$ has trivial intersection with $H_i$, $i = 1, 2, 3, 4$. It follows that

$$\mathcal{I}(G) \cong K_3 + (K_4 \cup K_1).$$ (3.3)

Also $\gamma(\mathcal{I}(G)) = 1$ and the corresponding toroidal embedding is shown in Figure 4. Since $\mathcal{I}(G)$ contains $K_7$ as a subgraph, it follows that $\gamma(\mathcal{I}(G)) > 1$.

Figure 4: An embedding of $\mathcal{I}(\mathbb{Z}_{p_1 p_2})$ in torus.
Sub case 2c: \(\alpha_1 \geq 5\). Let \(H_i, i = 1, \ldots, 8\) be the subgroups of \(G\) of order \(p_1, p_1^2, p_1^3, p_1^4, p_1^5, p_1p_2, p_1^3p_2, p_1^4p_2\) respectively. Since \(H_1\) is a subgroup of \(H_i, i = 2, \ldots, 8\), it follows that they form \(K_8\) as a subgraph of \(\mathcal{I}(G)\) and so \(\gamma(\mathcal{I}(G)) > 1, \overline{\gamma}(\mathcal{I}(G)) > 1\).

Case 3: \(k = 2\) and \(\alpha_2 \geq 2\). Now we have to deal with the following sub cases:

Sub case 3a: \(\alpha_1 = \alpha_2 = 2\). Let \(H_i, i = 1, \ldots, 7\) be the subgroups of \(G\) of order \(p_1, p_1^2, p_1^2, p_1p_2, p_1^3p_2, p_1^2p_2, p_1p_2^2\) respectively. Here \(H_1\) is a subgroup of \(H_2, H_3, H_6, H_7, H_8\); \(H_4\) is a subgroup of \(H_5, H_6, H_7, H_8\). It follows that \(\mathcal{I}(G) \cong K_3 + 2K_2\).

Moreover \(\gamma(\mathcal{I}(G)) = 1\), since \(\mathcal{I}(G)\) is a subgraph of \(K_7\) and it contains \(K_5\).

Also \(\overline{\gamma}(\mathcal{I}(G)) > 1\), since \(\mathcal{I}(G)\) has a subgraph isomorphic to the graph shown in Figure 5 which is one of the obstruction for projective-plane (e.g., see Theorem 0.1 and graph \(B_1\) of case (3.6) on p. 340 in [13]).

Sub case 3b: Either \(\alpha > 2\) or \(\beta > 2\). Let \(H_i, i = 1, \ldots, 8\) be subgroups of \(G\) of order \(p_1, p_1^2, p_2, p_1p_2, p_1^3p_2, p_1^2p_2, p_2^2\) respectively. Here \(H_1\) is a subgroup of \(H_2, H_3, H_6, H_7, H_8\); \(H_4\) is a subgroup of \(H_5, H_6, H_7, H_8\). It follows that \(\mathcal{I}(G)\) has a subgraph \(A_1\) as shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

Also \(\mathcal{I}(G)\) contains \(K_{3,5}\) as a subgraph with bipartition \(X := \{H_3, H_6, H_7\}\) and \(Y := \{H_1, H_2, H_4, H_5, H_8\}\) and so \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

Case 4: If \(k = 3\), then we need to consider only for \(\alpha_1 \geq 2, \alpha_2, \alpha_3 \geq 1\). Let \(H_i, i = 1, \ldots, 8\) be subgroups of \(G\) of order \(p_1, p_1^2, p_1p_2p_3, p_1p_2, p_1p_3, p_2p_3, p_1^2p_2, p_1^2p_3\) respectively. Here \(H_1\) is a subgroup of \(H_2, H_3, H_4, H_5, H_7, H_8\); \(H_i, i = 3, \ldots, 8\) intersect non-trivially with each other. It follows that \(\mathcal{I}(G)\) has a subgraph \(A_1\) as shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

Case 5: If \(k \geq 4\), then let \(H_i, i = 1, \ldots, 8\) be subgroups of \(G\) of orders \(p_1, p_1p_2, p_1p_2p_3, p_1p_2p_4, p_1p_2p_4, p_1p_3p_4, p_2p_3p_4, p_1p_3, p_1p_4\) respectively. Here \(H_1\) is a subgroup of \(H_2, H_3, H_4, H_5, H_6, H_7, H_8; H_5, H_3, H_4, H_6, H_7, H_8\) intersect non-trivially with each other. It follows that \(\mathcal{I}(G)\) has a subgraph \(A_1\) as shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

Combining all the cases together the proof follows.

\[\square\]

**Proposition 3.2.** Let \(G\) be a finite non-cyclic abelian group and \(p, q\) be distinct primes. Then
Intersection graph of subgroups

(1) \( \mathcal{I}(G) \) is \( K_5 \)-free if and only if \( G \) is one of \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_4 \times \mathbb{Z}_2 \), or \( \mathbb{Z}_6 \times \mathbb{Z}_2 \);

(2) The following are equivalent:
   (a) \( G \cong \mathbb{Z}_p \times \mathbb{Z}_p \);
   (b) \( \mathcal{I}(G) \) is \( C_3 \)-free;
   (c) \( \mathcal{I}(G) \) is acyclic;
   (d) \( \mathcal{I}(G) \) is bipartite.

(3) \( \mathcal{I}(G) \) is toroidal if and only if \( G \) is one of \( \mathbb{Z}_{p^2} \times \mathbb{Z}_p(p = 3, 5) \) or \( \mathbb{Z}_{3q} \times \mathbb{Z}_3 \);

(4) \( \mathcal{I}(G) \) is projective-planar if and only if \( G \) is either \( \mathbb{Z}_9 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_{3q} \times \mathbb{Z}_3 \).

Proof. We split the proof in to several cases:

Case 1: \( G \cong \mathbb{Z}_p \times \mathbb{Z}_p \). Then by Theorem 2.5 \( \mathcal{I}(G) \) is planar and by (2.4), it is acyclic.

Case 2: \( G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \). Here \( \langle (1,0), (1,1), \ldots, (1,p-1), (p,0), (0,1) \rangle \) are the only proper subgroups of \( G \). Note that \( \langle (p,0) \rangle \) is a subgroup of \( \langle (1,0), (1,1), \ldots, (1,p-1), (p,0), (0,1) \rangle \); no two remaining subgroups intersect non-trivially. It follows that

\[
\mathcal{I}(G) \cong K_1 + (K_{p+1} \cup K_p). \tag{3.5}
\]

So \( \mathcal{I}(G) \) contains \( C_3 \) as a subgraph. Note that \( \mathcal{I}(G) \) is a graph obtained by attaching \( p \) pendent edges to any one of the vertices of \( K_{p+2} \). So \( \gamma(\mathcal{I}(G)) = 1 \) if and only if \( p = 3, 5 \); \( \overline{\gamma}(\mathcal{I}(G)) = 1 \) if and only if \( p = 3 \); \( \mathcal{I}(G) \) is \( K_5 \)-free if and only if \( p = 2 \).

Case 3: \( G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p \). If \( p = 2 \), then by Theorem 2.5 \( \mathcal{I}(G) \) is planar and by Figure 3(b), it contains \( C_3 \). If \( p = 3 \), then \( H := \mathbb{Z}_p \times \mathbb{Z}_p \) is a subgroup of \( G \). \( H \) has four proper subgroups, let them be \( H_1, H_2, H_3, H_4 \). Now \( H \) and its proper subgroups, \( H_5 := \mathbb{Z}_q \times \{e\} \), \( H_iH_5 \), \( i = 1, 2, 3, 4 \) are the only proper subgroups of \( G \). Here \( H_5 \) is a subgroup of \( H_iH_5 \), \( i = 1, 2, 3, 4 \); \( H_iH_5 \), \( i = 1, 2, 3, 4 \) has non-trivial intersection with \( H \); \( H_i \), \( i = 1, 2, 3, 4 \) are subgroups of \( H \); no two remaining subgroups intersect nontrivially. A toroidal embedding of \( \mathcal{I}(G) \) is shown in Figure 4 and an embedding of \( \mathcal{I}(G) \) in the projective-plane is shown in Figure 7. Note that \( \mathcal{I}(G) \) contains \( K_5 \).

If \( p = 5 \), then \( H := \mathbb{Z}_p \times \mathbb{Z}_p \) is a subgroup of \( G \). \( H \) has six proper subgroups, let them be \( H_i, i = 1, \ldots, 6 \). Now \( H \) and its proper subgroups, \( H_7 := \mathbb{Z}_q \times \{e\} \), \( B_i = H_iH_7 \), \( i = 1, \ldots, 6 \) are the only proper subgroups of \( G \). Here \( H_7 \) is a subgroup of \( B_i, i = 1, \ldots, 6 \); \( A_i, i = 1, \ldots, 6 \) and \( H \) intersect non-trivially; \( H_i, i = 1, \ldots, 6 \) are subgroups of \( H \); no two remaining subgroups have non-trivial intersection. It follows that \( \mathcal{I}(G) \) has a subgraph \( A_1 \) as shown in Figure 11 so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

If \( p \geq 7 \), then \( H := \mathbb{Z}_p \times \mathbb{Z}_p \), \( H_1 := \langle (1,0) \rangle \), \( H_2 := \langle (1,1) \rangle \), \( H_3 := \langle (2,1) \rangle \), \( H_4 := \langle (3,1) \rangle \), \( H_5 := \langle (4,1) \rangle \), \( H_6 := \langle (5,1) \rangle \), \( H_7 := \langle (0,1) \rangle \), \( H_8 := \mathbb{Z}_q \times \{e\} \), \( H_iH_8, i = 1, \ldots, 7 \)
are proper subgroups of $G$. Here $H_8$ is a subgroup of $H_iH_8$, $i = 1, \ldots, 7$, so they form $K_8$ as a subgraph of $\mathcal{I}(G)$ and hence $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$.

**Case 4:** $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. Here $H := \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ is a subgroup of $G$. As in Case 2, $H$ has at least three proper subgroups of order $p^2$, say $H_i$, $i = 1, 2, 3$ and has at least three subgroups of order $p$, say $H_j$, $j = 4, 5, 6$ are subgroups of $H_i$ for some $i \in \{1, 2, 3\}$, let it be $H_3$; also for some $j \in \{4, 5, 6\}$, $H_j$ is a subgroup of $H_i$, for every $i = 1, 2, 3$, let it be $H_4$. Let $H_7$ be a subgroup of $G$ of order $q$. Here $H_iH_7$, $H_jH_7$, $i = 1, 2, 3$ and $j = 4, 5, 6$ are subgroups of $G$ and they have $H_7$ as their intersection, so $H_7$, $H_iK$, $H_jK$, $i = 1, 2$, $j = 4, 5, 6$ form $K_6$ as a subgraph of $\mathcal{I}(G)$; $H$ and $H_3H_7$ intersect non-trivially and they also intersect with $H_jH_7$, $j = 4, 5, 6$ non-trivially. It follows that $\mathcal{I}(G)$ has a subgraph $A_1$ as shown in Figure 6, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

**Case 5:** $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} = \langle a, b \mid a^{p_2} = b^{p_2} = 1, ab = ba \rangle$. Then $H_1 := \langle a \rangle$, $H_2 := \langle ab \rangle$, $H_3 := \langle a^p, b^p \rangle$, $H_4 := \langle a, b^p \rangle$, $H_5 := \langle a^p, a^p b \rangle$, $H_6 := \langle a^p \rangle$, $H_7 := \langle b \rangle$, $H_8 := \langle b^p \rangle$ are subgroups of $G$. Also $H_6$ is a subgroup of $H_i$, $i = 1, \ldots, 5$; $H_8$ is a subgroup of $H_3$, $H_4$, $H_5$, $H_7$. It follows that $\mathcal{I}(G)$ has a subgraph $A_1$ as shown in Figure 6, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

**Case 6:** $G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, $k, l \geq 3$. Then $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ is a proper subgroup of $G$ and so by
Case 4, we have \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

**Case 7:** \( G \cong \mathbb{Z}_p \times \mathbb{Z}_{pq} \). Then \( H := \mathbb{Z}_p \times \mathbb{Z}_p \) is a subgroup of \( G \) and as in Case 1, \( H \) has at least three proper subgroups of order \( p \), say \( H_i, i = 1, 2, 3 \). Let \( H_1, H_2 \) be subgroups of \( G \) of orders \( q, r \) respectively. Here \( H_i H_j, i = 1, 2, 3, j = 4, 5 \) are also proper subgroups of \( G \). So \( H_i H_j, i = 1, 2, 3 \), \( HH_i H_j, H_4 \) forms \( K_5 \) as a subgraph of \( \mathcal{I}(G) \); \( H_i H_j, i = 1, 2, 3, HH_5, H_5 \) also forms \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). Thus \( \mathcal{I}(G) \) has a subgraph \( B_1 \) as shown in Figure 2, so \( \gamma(\mathcal{I}(G)) > 1 \). \( B_1 \) is also a topological obstruction for projective-plane embedding (e.g., see Theorem 0.1 and the graph \( A_5 \) of case (3.15) on p. 343 in [13]), so \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

**Case 8:** \( G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times Z_{p_3} \). Here \( \langle (1, 0, 0), (0, 1, 0) \rangle, \langle (1, 0, 0), (0, 0, 1) \rangle, \langle (1, 0, 0), (1, 1, 0) \rangle, \langle (1, 0, 0), (0, 1, 1) \rangle \) are proper subgroups of \( G \), which have \( \langle (1, 0, 0) \rangle \) as their intersection. It follows that these five subgroups form \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). Also \( \langle (1, 1, 1), (0, 1, 0) \rangle, \langle (0, 1, 0), (0, 0, 1) \rangle, \langle (0, 1, 0), (1, 1, 0) \rangle, \langle (0, 1, 0), (0, 1, 1) \rangle \) are proper subgroups of \( G \), which have \( \langle (0, 1, 0) \rangle \) as their intersection and so they also form \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). Thus \( \mathcal{I}(G) \) has a subgraph \( B_1 \) as shown in Figure 2, so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

**Case 9:** \( G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_k} \), where \( p_i \)'s are primes with at least two \( p_i \)'s are equal and \( \alpha_i \geq 1 \). If \( k \geq 2 \), then \( G \) has one of the following groups as its subgroup: \( \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2}, \mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2}, \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \), for some \( i, j \). So by Cases 3, 4, 7, the intersection graph of subgroups of these subgroups are toroidal, projective planar and contains \( K_5 \). So it follows that \( \gamma(\mathcal{I}(G)) > 1, \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \) as a subgraph.

The result follows by combining all the above cases.

4 Finite non-abelian groups

In this section, we classify the finite non-abelian groups whose intersection graphs of subgroups are one of toroidal, projective-planar, \( K_5 \) free, \( C_3 \)-free, acyclic or bipartite. We first investigate the non-abelian solvable groups and then we deal with the nonsolvable groups.

For any integer \( n \geq 3 \), the dihedral group of order \( 2n \) is given by \( D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle \); For any integer \( n \geq 2 \), the generalized quaternion group of order \( 2^n \) is given by \( Q_{2n} = \langle a, b \mid a^{2^{n-1}} = b^4 = 1, a^{2^{n-2}} = b^2 = 1, bab^{-1} = a^{-1} \rangle \); For any \( \alpha \geq 3 \) and \( p \) is a prime, the modular group of order \( p^{\alpha} \) is given by \( M_{p^\alpha} = \langle a, b \mid a^{p^{\alpha-1}} = b^p = 1, bab^{-1} = a^{p^{\alpha-2}+1} \rangle \); \( S_n \) and \( A_n \) are symmetric and alternating groups of degree \( n \) acting on \( \{1, 2, \ldots, n\} \) respectively. We denote the order of an element \( a \in \mathbb{Z}_n \) by \( \text{ord}_n(a) \). The number of Sylow \( p \)-subgroups of a group \( G \) is denoted by \( n_p(G) \).
4.1 Finite non-abelian solvable groups

Proposition 4.1. Let $G$ be a non-abelian group of order $p^\alpha$, where $p$ is a prime and $\alpha \geq 3$. Then

1. $\mathcal{I}(G)$ is $K_5$-free if and only if $G$ is either $Q_8$ or $M_8$;
2. $\mathcal{I}(G)$ contains $C_3$;
3. $\mathcal{I}(G)$ is toroidal if and only if $G$ is one of $M_{p^3}(p = 3, 5)$ or $M_{16}$.
4. $\mathcal{I}(G)$ is projective-planar if and only if $G \cong M_{27}$.

Proof. We prove the result in the following cases:

Case 1: $\alpha = 3$. If $p = 2$, then the only non-abelian groups of order 8 are $Q_8$ and $M_8$. By Theorem 2.5, the intersection graphs of subgroups of these two groups are planar and by (2.6) and Figure 3(c), they contain $C_3$.

If $p \neq 2$, then up to isomorphism the only non-abelian groups of order $p^3$ are $M_{p^3}$ and $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.

(i) If $G \cong M_{p^3}$, then the subgroup lattices of $M_{p^3}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ are isomorphic, so their intersection graphs of subgroups are also isomorphic. By Case 2 in the proof of Proposition 3.2, we have

$$\mathcal{I}(G) \cong K_1 + (K_{p+1} \cup \overline{K}_p).$$

Also, $\gamma(\mathcal{I}(G)) = 1$ if and only if $p = 3, 5$; $\tau(\mathcal{I}(G)) = 1$ if only if $p = 3$; $\mathcal{I}(G)$ contains $K_5$.

(ii) If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, then consider its proper subgroups $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle ab, c \rangle$, $H_4 := \langle ab^2, c \rangle$, $H_5 := \langle b, ac \rangle$, $H_6 := \langle b, a^2c \rangle$, $H_7 := \langle b, c \rangle$, $H_8 := \langle b \rangle$, $H_9 := \langle c \rangle$. Here $H_9$ is a subgroup of $H_2$, $H_3$, $H_4$, $H_7$, so these five subgroups forms $K_5$ as a subgraph of $\mathcal{I}(G)$; $H_5$ is a subgroup of $H_1$, $H_3$, $H_4$, $H_6$, so these four subgroups forms $K_4$ as a proper subgraph of $\mathcal{I}(G)$; $H_1$, $H_4$ intersects non-trivially; $H_8$ is a subgroup of $H_7$, $H_5$, $H_6$. It follows that $\mathcal{I}(G)$ has a subgraph $\mathcal{B}_3$ as shown in Figure 2, so $\gamma(\mathcal{I}(G)) > 1$.

Also $\mathcal{I}(G)$ has a subgraph as shown in Figure 8 which is an obstruction for projective-plane (e.g., see Theorem 0.1 and graph $C_7$ of case (3.20) on p. 344 in [13]). Therefore, $\tau(\mathcal{I}(G)) > 1$.

Case 2: $\alpha = 4$. According to Burnside [8], up to isomorphism, there are fifteen groups of order $p^4$, $p > 2$ and there are nine groups of order $2^4$. In following we investigate the genus of the intersection graphs of subgroups of each of them:

Sub case 2a: $p > 2$. 

(i) $G \cong M_p$. Here $H_1 := \langle a \rangle$, $H_2 := \langle ab \rangle$, $H_3 := \langle ab^2 \rangle$, $H_4 := \langle a^p \rangle$, $H_5 := \langle a^p b \rangle$, $H_6 := \langle a^p b^2 \rangle$, $H_7 := \langle a^p, b \rangle$, $H_8 := \langle a^p \rangle$ are proper subgroups of $G$. Also $H_8$ is a subgroup of $H_i$, $i = 1, \ldots, 7$. It follows that they form $K_8$ as a subgraph of $\mathcal{I}(G)$, so $\gamma(\mathcal{I}(G)) > 1$.

(ii) $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, cb = a^p bc, ab = ba, ac = ca \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle a \rangle$, $H_4 := \langle a^p, c \rangle$, $H_5 := \langle a^p, b \rangle$, $H_6 := \langle ab \rangle$, $H_7 := \langle ac \rangle$, $H_8 := \langle a^p \rangle$ are proper subgroups of $G$. Also $H_8$ is a subgroup of $H_i$, $i = 1, \ldots, 7$. It follows that they form $K_8$ as a subgraph of $\mathcal{I}(G)$, so $\gamma(\mathcal{I}(G)) > 1$.

(iii) $G \cong \langle a, b \mid a^p = b^p = 1, bab^{-1} = a^{1+p} \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^p \rangle$, $H_3 := \langle a, b \rangle$, $H_4 := \langle a^p, b^p \rangle$, $H_5 := \langle a, b^p \rangle$, $H_6 := \langle b \rangle$, $H_7 := \langle b^p \rangle$, $H_8 := \langle ab, b^p \rangle$, $H_9 := \langle a^2 b, b^p \rangle$ are proper subgroups of $G$. Here $H_i$, $i = 1, \ldots, 5$ intersect with each other non-trivially and so they form $K_5$ as a subgraph of $\mathcal{I}(G)$; $H_7$ is a subgroup of $H_4$, $H_6$, $H_8$, $H_9$; $H_6$, $H_8$, $H_9$ intersect with $H_5$ non-trivially. It follows that $\mathcal{I}(G)$ has a subgraph $B_3$ as shown in Figure 2, so $\gamma(\mathcal{I}(G)) > 1$.

(iv) $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, ca = a^{1+p} c, ab = ba, cb = bc \rangle$. We can take the proper subgroups of $G$ as in (ii), except by taking $H_7 := \langle a, bc \rangle$ instead of $H_7 := \langle ac \rangle$. Then $\mathcal{I}(G)$ has $K_8$ as a subgraph and so $\gamma(\mathcal{I}(G)) > 1$.

(v) $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, ca = abc, ab = ba, cb = bc \rangle$. We can use a similar argument as in (iv) to show $\mathcal{I}(G)$ has $K_8$ as a subgraph and so $\gamma(\mathcal{I}(G)) > 1$.

(vi) $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, ba = a^{1+p} b, ca = abc, cb = bc \rangle$. Here $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle b, c \rangle$, $H_4 := \langle a^p, c \rangle$, $H_5 := \langle a^p, b \rangle$, $H_6 := \langle a^p \rangle$, $H_7 := \langle a \rangle$, $H_8 := \langle a^p, bc \rangle$, $H_9 := \langle a^p, b^2 c \rangle$ are proper subgroups of $G$. Also $H_i$, $i = 1, \ldots, 5$ intersect with each other non-trivially and so they form $K_5$ as a subgraph of $\mathcal{I}(G)$; $H_6$ is a subgroup of $H_1$, $H_8$, $H_6$; $H_6$ is a subgroup of $H_1$; $H_7$, $H_8$, $H_9$ intersect with $H_4$ non-trivially. It follows that $\mathcal{I}(G)$ has a subgraph $B_3$ as shown in Figure 2 so $\gamma(\mathcal{I}(G)) > 1$.

(vii) If $p = 3$, then $G \cong \langle a, b, c \mid a^3 = b^3 = c^3 = 1, c^p = a^p, ab = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle$. Here $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle b, c \rangle$, $H_4 := \langle a^p, b \rangle$, $H_5 := \langle c \rangle$, $H_6 := \langle a \rangle$, $H_7 := \langle a^p b \rangle$, $H_8 := \langle a^p, b^2 \rangle$, $H_9 := \langle a^{2+p}, b^3 \rangle$. We can use a similar argument as in (iv) to show $\mathcal{I}(G)$ has $K_8$ as a subgraph and so $\gamma(\mathcal{I}(G)) > 1$.
If $p > 3$, then $G \cong \langle a, b, c | a^p = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^pb \rangle$.

Here $\langle a, b \rangle \cong M_p^3$ is a subgroup of $G$, so by (4.1), $M_p^3$ together with its proper subgroups forms $K_3$ as a subgraph of $\mathcal{I}(G)$. Hence $\gamma(\mathcal{I}(G)) > 1$.

(viii) If $p = 3$, then $G \cong \langle a, b, c | a^p = b^p = c^p = 1, ba = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle$. We can use a similar argument as in (vii), to show $\gamma(\mathcal{I}(G)) > 1$.

If $p > 3$, then $G \cong \langle a, b, c | a^p = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^pb \rangle$. Here $\langle a, b \rangle \cong M_p^3$ is a subgroup of $G$, so by (4.1), $M_p^3$ together with its proper subgroups forms $K_3$ as a subgraph of $\mathcal{I}(G)$. Thus $\gamma(\mathcal{I}(G)) > 1$.

(ix) $G \cong \langle a, b, c, d | a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca \rangle$.

Here $\langle a, b, c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of $G$ and so by Case 7 in the proof of Proposition 3.2 it follows that $\gamma(\mathcal{I}(G)) > 1$.

(x) If $p = 3$, then $G \cong \langle a, b, c | a^p = b^p = c^p = 1, ab = ba, ac = cab, cb = ca^{-p}b \rangle$.

Here $H_1 := \langle a, b \rangle, H_2 := \langle a, c \rangle, H_3 := \langle b, c \rangle, H_4 := \langle a^p, b \rangle, H_5 := \langle a^p, c \rangle, H_6 := \langle a \rangle, H_7 := \langle a^p \rangle, H_8 := \langle ab \rangle, H_9 := \langle a^2b \rangle$ are proper subgroups of $G$. Also $H_i, i = 1, \ldots, 5$ intersect with each other non-trivially and so they form $K_5$ as a subgraph of $\mathcal{I}(G)$; $H_7$ is a subgroup of $H_1, H_2, H_3, H_4; H_5, H_6, H_8, H_9$ intersect with $H_5$ non-trivially. It follows that $\mathcal{I}(G)$ has a subgraph $B_3$ as shown in Figure 2 so $\gamma(\mathcal{I}(G)) > 1$.

If $p > 3$, then $G \cong \langle a, b, c, d | a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca \rangle$. Here $\langle a, b, c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of $G$ and so by Case 7 in the proof of Proposition 3.2 it follows that $\gamma(\mathcal{I}(G)) > 1$.

Sub case 2b: $p = 2$.

(i) $G \cong M_{2^4}$. Here $H_1 := \langle a \rangle, H_2 := \langle ab \rangle, H_3 := \langle a^2, b \rangle, H_4 := \langle a^2 \rangle, H_5 := \langle a^2 b, c \rangle, H_6 := \langle a^4, b \rangle, H_7 := \langle a^4 \rangle, H_8 := \langle a^4 b \rangle, H_9 := \langle b \rangle$ are the only subgroups of $M_{2^4}$. Further $H_7$ is a subgroup of $H_1, i = 1, \ldots, 6; H_8, H_9$ are proper subgroups of $H_6$; no two remaining subgroups intersect non-trivially. It follows that

$$\mathcal{I}(G) \cong K_1 + (K_6 \cup K_2).$$

(4.2)

Note that $\mathcal{I}(G)$ is a graph obtained by attaching 2 pendent edges to any one of the vertices of $K_7$, so $\gamma(\mathcal{I}(G)) = 1$. 
Intersection graph of subgroups

(ii) $G \cong \langle a, b, c \mid a^4 = b^4 = c^2 = 1, bab^{-1} = a^{-1}, a^2 = b^2, bc = cb, ac = ca \rangle$. Here $H_1 := \langle a, c \rangle$, $H_2 := \langle a \rangle$, $H_3 := \langle ac \rangle$, $H_4 := \langle a^2 \rangle$, $H_5 := \langle a^2, c \rangle$, $H_6 := \langle bc \rangle$, $H_7 := \langle b \rangle$, $H_8 := \langle b, c \rangle$ are proper subgroups of $G$ and these subgroups intersect with each other non-trivially, so they form $K_8$ as a subgraph of $\mathcal{J}(G)$. It follows that $\gamma(\mathcal{J}(G)) > 1$.

(iii) $G \cong \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^{-1} \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^2 \rangle$, $H_3 := \langle a^4 \rangle$, $H_4 := \langle a^2, b \rangle$, $H_5 := \langle a^4, b \rangle$, $H_6 := \langle a^4, ab \rangle$, $H_7 := \langle a^4, a^2b \rangle$, $H_8 := \langle a^2, ab \rangle$ are proper subgroups of $G$ and these eight subgroups intersect with each other non-trivially, so they form $K_8$ as a subgraph of $\mathcal{J}(G)$. It follows that $\gamma(\mathcal{J}(G)) > 1$.

(iv) $G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^2 \rangle$, $H_3 := \langle a^4 \rangle$, $H_4 := \langle a^2, b \rangle$, $H_5 := \langle a^4, b \rangle$, $H_6 := \langle a^4, ab \rangle$, $H_7 := \langle a^4, a^2b \rangle$, $H_8 := \langle a^4, a^3b \rangle$ are proper subgroups of $G$ and these eight subgroups intersect with each other non-trivially. Hence they form $K_8$ as a subgraph of $\mathcal{J}(G)$. It follows that $\gamma(\mathcal{J}(G)) > 1$.

(v) $G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^{-1}, b^2 = a^4 \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^2 \rangle$, $H_3 := \langle a^4 \rangle$, $H_4 := \langle b \rangle$, $H_5 := \langle a^2, b \rangle$, $H_6 := \langle ab \rangle$, $H_7 := \langle a^3b \rangle$, $H_8 := \langle a^3b \rangle$ are proper subgroups of $G$ and these eight subgroups intersect with each other non-trivially, so they form $K_8$ as a subgraph of $\mathcal{J}(G)$. It follows that $\gamma(\mathcal{J}(G)) > 1$.

The remaining groups in this subcase are identical with the groups described in (ii), (iii), (iv), (v) of Subcase 2a.

Case 3: $\alpha = 5$. By Theorem 2.6 (i), $G$ has at least three subgroups of order $p^4$, say $H_i$, $i = 1, 2, 3$ and at least three subgroups of order $p^3$, say $H_i$, $i = 4, 5, 6$. Here for each $i = 1, 2, 3$ and $j = 4, 5, 6$, $H_i$ and $H_j$ have a non-trivial intersection. For otherwise, $|H_iH_j| = p^k$, $k = 7$ or 8, which is not possible. Let $H_7$ be a common subgroup of order $p^2$ for both $H_4$ and $H_5$. Let $H_8$ be a subgroup of $H_7$ of order $p$. Here $H_1$, $H_4$, $H_5$, $H_7$, $H_8$ intersects with each other non-trivially, so they form $K_5$ as a subgraph of $\mathcal{J}(G)$. Now $H_2$ and $H_3$ have a common subgroup of order $p^3$, say $H_9$; let $H_{10}$ be a subgroup of $H_9$ of order $p^2$; let $H_{11}$ be a subgroup of $H_{10}$ of order $p$. Then $H_2$, $H_3$, $H_9$, $H_{10}$, $H_{11}$ intersect with each other non-trivially. Therefore, $\mathcal{J}(G)$ contains a subgraph $B_1$ as shown in Figure 2 so $\gamma(\mathcal{J}(G)) > 1$.

Case 4: $\alpha \geq 6$. By Theorem 2.6 (i), (ii), $G$ has at least two subgroups of order $p^{\alpha-1}$, say $H_1$, $H_2$. Let $H_i$, $i = 3, 4, 5, 6$ be subgroups of $H_1$ of order $p^{\alpha-2}$, $p^3$, $p^2$, $p$ respectively. Also let $H_i$, $i = 7, 8, 9, 10$ be subgroups of $H_2$ of orders $p^{\alpha-2}$, $p^3$, $p^2$, $p$ respectively. It follows that $H_1$, $H_3$, $H_4$, $H_5$, $H_6$ forms $K_5$ as a subgraph of $\mathcal{J}(G)$ and $H_2$, $H_7$, $H_8$, $H_9$, $H_{10}$ forms another copy of $K_5$ as a subgraph of $\mathcal{J}(G)$. Thus $\mathcal{J}(G)$ has a subgraph $B_1$ as shown in Figure 2 so $\gamma(\mathcal{J}(G)) > 1$.

Now we investigate the projective-plane embedding of the intersection graphs of subgroups of $G$, when $\alpha \geq 4$. We need to consider the following cases:

Case a: $p > 2$. Then by Theorem 2.6 (i), $G$ has at least four subgroups of order $p^3$, let them be $H_i$, $i = 1, 2, 3, 4$; and has at least four subgroups of order $p^2$, say $B_i$, $i = 1, 2, 3, 4$. 
Here \( H_i \cap B_j \neq \emptyset \) for all \( i, j = 1, 2, 3, 4 \) with \( i \neq j \) and so they form \( K_{4,4} \) as a subgraph of \( \mathcal{I}(G) \) with bipartition \( X := \{ H_1, H_2, H_3, H_4 \} \) and \( Y := \{ B_1, B_2, B_3, B_4 \} \). It follows that \( \gamma(\mathcal{I}(G)) > 1 \).

**Case b:** \( p = 2 \). If \( G \) has an unique subgroup of order 2, then by Theorem \( 2.6 \) (i), (ii), \( G \cong Q_{2^s} \). Then \( G \) has at least seven proper subgroups and they have a unique subgroup of order 2 in common. It follows that \( \mathcal{I}(G) \) has \( K_7 \) as a subgraph, so \( \gamma(\mathcal{I}(G)) > 1 \).

If \( G \neq Q_{2^s} \), then it has at least three subgroups of order \( p^{\alpha-1} \), let them be \( H_1, H_2, H_3 \); \( G \) has at least three subgroups, say \( H_4, H_5, H_6 \) of order \( p^{\alpha-2} \); at least two subgroups of order \( p^{\alpha-3} \), say \( H_7, H_8 \). Here \( H_i \cap H_j \neq \emptyset \), for every \( i = 1, 2, 3 \) and \( j = 4, 5, 6, i \neq j \); \( H_7, H_8 \) are subgroups of \( H_i, i = 1, 2, 3 \). It follows that \( K_{3,5} \) is a subgraph of \( \mathcal{I}(G) \) with bipartition \( X := \{ H_1, H_2, H_3 \} \) and \( Y := \{ H_4, H_5, H_6, H_7, H_8 \} \), so \( \gamma(\mathcal{I}(G)) > 1 \).

The proof follows by putting together all the cases. \( \square \)

If \( G \) is the non-abelian group of order \( pq \), where \( p < q \) and \( p, q \) are two distinct primes, then by Theorem 2.5 \( \gamma(\mathcal{I}(G)) \) is planar and by (2.7) it is acyclic.

Next we investigate the groups of order greater than \( pq \).

Consider the semi-direct product \( \mathbb{Z}_q \rtimes \mathbb{Z}_{p^\alpha} = \langle a, b | a^p = b^{p^\alpha} = 1, bab^{-1} = a', \text{ord}_q(i) = p^t \rangle \), where \( p, q \) are distinct primes with \( p^t | (q-1), t \geq 0 \). Then every semi-direct product \( \mathbb{Z}_q \rtimes \mathbb{Z}_{p^\alpha} \) is one of these types [6 Lemma 2.12]. So here after, when \( t = 1 \) we will suppress the subscript.

**Proposition 4.2.** Let \( G \) be a non-abelian group of order \( p^2q \), where \( p \) and \( q \) are distinct primes. Then

1. \( \mathcal{I}(G) \) is \( K_{5^2} \)-free if and only if \( G \) is either \( \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} = \langle a, b, c | a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1b^1 \rangle \), where \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) has order \( q \) in \( GL_2(p) \), \( q | (p+1) \) or \( A_4 \).

2. The following are equivalent:
   (a) \( G \cong \langle a, b, c | a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b \rangle \), where \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)
   has order \( q \) in \( GL_2(p) \), \( q | (p+1) \) or \( A_4 \);
   (b) \( \mathcal{I}(G) \) is \( C_3 \)-free;
   (c) \( \mathcal{I}(G) \) is acyclic;
   (d) \( \mathcal{I}(G) \) is bipartite.

3. \( \mathcal{I}(G) \) is toroidal if and only if \( G \) is one of \( \mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \) or \( \mathbb{Z}_9 \rtimes \mathbb{Z}_2, \mathbb{Z}_{25} \rtimes \mathbb{Z}_2 \).

4. \( \mathcal{I}(G) \) is projective-planar if and only if \( G \) is either \( \mathbb{Z}_3 \rtimes \mathbb{Z}_4 \) or \( \mathbb{Z}_9 \rtimes \mathbb{Z}_2 \).

**Proof.** To prove the result we use the classification of groups of order \( p^2q \) given in [8] p. 76-80. We have the following cases to consider.

**Case 1:** \( p < q \);

**Case 1a:** \( p \uparrow (q-1) \). By Sylow’s Theorem, there is no non-abelian group in this case.
Case 1b: $p \mid (q - 1)$ but $p^2 \nmid (q - 1)$. In this case, there are two non-abelian groups.

The first group is $G_1 := \mathbb{Z}_q \times \mathbb{Z}_{p^2} = \langle a, b | a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle$. We have $\langle a \rangle, \langle a^i b \rangle, i = 1, 2, \ldots, q$, $\langle b \rangle$ and $\langle ab \rangle$ are the only proper subgroups of $G_1$. Here $\langle b \rangle$ is a subgroup of the remaining proper subgroups, except $\langle a \rangle$. Also $\langle a \rangle$ is a subgroup of $\langle ab \rangle$. It follows that

$$\mathcal{I}(G_1) \cong K_1 + (K_1 \cup K_{q+1}). \quad (4.3)$$

Note that $q = 5$ is not possible here, since $p = 2$ is such that $p \mid (q - 1)$ but $p^2 \nmid (q - 1)$. Note that $\mathcal{I}(G)$ is a graph obtained by attaching one pendant edge to any one of the vertices of $K_{q+2}$. So $\gamma(\mathcal{I}(G)) = 1$ if and only if $q = 3$; $\overline{\gamma}(\mathcal{I}(G)) = 1$ if and only if $q = 3$; $\mathcal{I}(G)$ contains $K_5$.

The second group is $G_2 := \langle a, b, c | a^q = b^p = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb, \text{ord}_q(i) = p \rangle$. Here $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle b, c \rangle$, $H_4 := \langle ab, c \rangle$, $H_5 := \langle a^2 b, c \rangle$, $H_6 := \langle a^3 b, c \rangle$, $H_7 := \langle c \rangle$, $H_8 := \langle a^4 b, c \rangle$ are proper subgroups of $G$. Also $H_i, i = 3, 4, \ldots, 8$ intersect with each other non-trivially and so they form $K_6$ as a subgraph of $\mathcal{I}(G_2)$; $H_1$ and $H_2$ intersect non-trivially; also they intersect non-trivially with $H_3, H_4, H_5$. It follows that $\mathcal{I}(G_2)$ has a subgraph $A_1$ as shown in Figure 1, so $\gamma(\mathcal{I}(G_2)) > 1$ and $\overline{\gamma}(\mathcal{I}(G_2)) > 1$.

Case 1c: $p^2 \mid (q - 1)$. In this case, we have both groups $G_1$ and $G_2$ from Case 1b together with the group $G_3 := \mathbb{Z}_q \times_2 \mathbb{Z}_{p^2} = \langle a, b | a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_q(i) = p^2 \rangle$. Note that here $q = 5, p = 2$ is possible for the group $G_1$. By (4.3), $\gamma(\mathcal{I}(G_1)) = 1, \overline{\gamma}(\mathcal{I}(G_1)) > 1$. $\mathcal{I}(G_2)$ is already discussed in Case 1b. By Theorem 2.5, $\mathcal{I}(G_3)$ is planar and by (2.8), it contains $C_3$.

Case 2: $p > q$

Case 2a: $q \mid (p^2 - 1)$. Then there is no non-abelian subgroups.

Case 2b: $q \nmid (p - 1)$. In this case, we have two groups.

The first one is $G_4 := \langle a, b | a^q = b^q = 1, bab^{-1}, \text{ord}_p(i) = q \rangle$. By Sylow’s Theorem, $G_4$ has a unique subgroup say $N$, of order $p^2$ and has a unique subgroup $N'$ of order $p$; $G_4$ has $p^2$ Sylow $q$-subgroups of order $q$, say $H_i, i = 1, 2, \ldots, p^2$; it has $p$ subgroups of order $pq$, say $A_i = \langle a^p, a^i b \rangle, i = 0, 1, \ldots, p - 1$. These are the only proper subgroups of $G_4$. Here $N'$ is a subgroup of $N$ and $A_i, i = 0, 1, \ldots, p - 1$; $H_i, H_{i+p}, \ldots, H_{i+p(p-1)}$ are subgroups of $A_i$ for each $i = 0, 1, \ldots, p - 1$. It follows that $\mathcal{I}(G_4)$ is the graph obtained by attaching a single pendant edge to any $p$ vertices of $K_{p+2}$. Thus, $\gamma(\mathcal{I}(G_4)) = 1$ if and only if $p = 3, 5; \overline{\gamma}(\mathcal{I}(G_4)) = 1$ if and only if $p = 3; \mathcal{I}(G_4)$ contains $K_5$.

Next we have the family of groups $\langle a, b, c | a^p = b^p = c^p = 1, cac^{-1} = a^i, abc^{-1} = b^i, ab = ba, \text{ord}_p(i) = q \rangle$. There are $(q + 3)/2$ isomorphism types in this family (one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in $F_{p^2}$). We will refer to all of these groups as $G_{5(t)}$ of order $p^2 q$. Here $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle a, bc \rangle$, $H_4 := \langle a, b^2 c \rangle$, $H_5 := \langle ab, c \rangle$, $H_6 := \langle b, c \rangle$, $H_7 := \langle a^2 b, c \rangle$, $H_8 := \langle c \rangle$ are proper subgroups of $G$. Further $H_i, i = 1, \ldots, 6$ intersect non-trivially with each other and so they form $K_6$ as a subgraph of $\mathcal{I}(G_{5(t)})$; $H_7, H_8$ intersect non-trivially and they intersect with $H_2, H_3, H_6$. It follows that $\mathcal{I}(G_{5(t)})$ has a subgraph $A_1$ as shown in Figure 1, so $\gamma(\mathcal{I}(G_{5(t)})) > 1, \overline{\gamma}(\mathcal{I}(G_{5(t)})) > 1$ and $\mathcal{I}(G_{5(t)})$. 


contains $K_5$ as a subgraph.

**Case 2c:** $q | (p + 1)$. In this case, we have only one group of order $p^2 q$, given by $G_6 := (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q = \langle a, b, c | a^p = b^q = c^q = 1, ab = ba, cac^{-1} = a^i b^j, bbc^{-1} = a^k b^l \rangle$, where $(i, j)$ has order $q$ in $GL_2(p)$.

(i) If $G_6$ has a subgroup of order $pq$, then $H_1 := \langle a, b \rangle$, $H_2 := \langle a, c \rangle$, $H_3 := \langle a, bc \rangle$, $H_4 := \langle a^2 b_c \rangle$, $H_5 := \langle a^2 b, c \rangle$, $H_6 := \langle b, c \rangle$, $H_7 := \langle a^3 b_c, c \rangle$, $H_8 := \langle c \rangle$ are proper subgroups of $G$. Also $H_i, i = 1, \ldots, 6$ intersect non-trivially with each other and so they form $K_6$ as a subgraph of $\mathcal{I}(G)$; $H_7$, $H_8$ intersect non-trivially and they intersect with $H_2, H_6, H_5$. Thus $\mathcal{I}(G_6)$ has a subgraph $A_1$ as shown in Figure 11 so $\gamma(\mathcal{I}(G_6)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

(ii) If $G_6$ has no subgroup of order $pq$, then $G_6 := \langle a, b, c | a^p = b^q = c^q = 1, ab = ba, cac^{-1} = b, bbc^{-1} = a^i b^j \rangle$, where $\langle 0, -1 \rangle$ has order $q$ in $GL_2(p)$. By Theorem 2.5 $\mathcal{I}(G_6)$ is planar and by (2.9), it is acyclic.

Note that if $(p, q) = (2, 3)$, the Cases 1 and 2 are not mutually exclusive. Up to isomorphism, there are three non-abelian groups of order 12: $\mathbb{Z}_3 \times \mathbb{Z}_4$, $D_{12}$ and $A_4$. Here the intersection graph of subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_4$ (the group $G_1$) and $D_{12}$ (the group $G_2$) contains $K_5$. But by Theorem 2.5, $\mathcal{I}(A_4)$ is planar and by (2.10), it is acyclic.

Combining all the cases together, the proof follows.

**Proposition 4.3.** If $G$ is a non-abelian group of order $p^\alpha q$, where $p$, $q$ are distinct primes and $\alpha \geq 3$, then $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $K_5$.

**Proof.** Let $P$ denote a Sylow $p$-subgroup of $G$. We shall prove this result by induction on $\alpha$. First we prove this result when $\alpha = 3$. If $p > q$, then $n_p(G) = 1$, by Sylow’s theorem and our group $G \cong P \times \mathbb{Z}_q$. Suppose $\gamma(\mathcal{I}(P)) > 1$, $\overline{\gamma}(\mathcal{I}(P)) > 1$ and $\mathcal{I}(P)$ contains $K_5$, then the same holds for $\mathcal{I}(G)$ also. So it is enough to consider the cases when $\mathcal{I}(P)$ is one of planar, toroidal, projective-planar, $K_5$-free, $C_3$-free or bipartite.

By Propositions 3.1, 3.2, 4.1 and Theorem 2.5, $P$ is isomorphic to one of $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ ($p = 2, 3, 5$) or $M_{p^3}$ ($p = 3, 5$).

- If $P \cong \mathbb{Z}_{p^3}$, then $G \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q = \langle a, b | a^p = b^q = 1, bab^{-1}, ord_p(i) = q \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^p, b \rangle$, $H_3 := \langle a^p, b \rangle$, $H_4 := \langle a^p, ab \rangle$, $H_5 := \langle a^p, ab \rangle$, $H_7 := \langle a^p, a^2 b \rangle$, $H_8 := \langle a^p \rangle$ are proper subgroups of $G$. Further $H_8$ is a subgroup of $H_i$, for every $i = 1, \ldots, 7$. It follows that they form $K_8$ as a subgraph of $\mathcal{I}(G)$, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

- If $P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $p = 2, 3, 4$, then $G \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$. Here $H_1 := \langle a, c \rangle$, $H_2 := \langle ab, c \rangle$, $H_3 := \langle a^p, b, c \rangle$, $H_4 := \langle a^p, c \rangle$, $H_5 := \langle a \rangle$, $H_6 := \langle ab \rangle$, $H_7 := \langle a^p, b \rangle$, $H_8 := \langle a^p \rangle$ are subgroups of $G$, where $\langle a, b \rangle = P$ and $\langle c \rangle = \mathbb{Z}_q$. Also $H_8$ is a subgroup of $H_i$, $i = 1, \ldots, 7$. Thus $\mathcal{I}(G)$ has $K_8$ as a subgraph, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$. 


If \( P \cong M_{p^3}, p = 3, 5 \), then \( G \cong M_{p^3} \times \mathbb{Z}_q \). Here \( H_1 := \langle a, c \rangle, H_2 := \langle ab, c \rangle, H_3 := \langle a^p, b, c \rangle, H_4 := \langle a^2, c \rangle, H_5 := \langle a \rangle, H_6 := \langle ab \rangle, H_7 := \langle a^p, b \rangle, H_8 := \langle a^p \rangle \) are subgroups of \( G \), where \( \langle a, b \rangle = P \) and \( \langle c \rangle = \mathbb{Z}_q \). Also \( H_8 \) is a subgroup of \( H_i, i = 1, \ldots, 7 \). Therefore, \( \mathcal{J}(G) \) has \( K_8 \) as a subgraph and \( \gamma(\mathcal{J}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{J}(G)) > 1 \).

Now, let us consider the case \( p < q \) and \( (p, q) \neq (2, 3) \). \( n_q(G) = p \) is not possible. If \( n_q = p^2 \), then \( q|(p + 1)(p - 1) \), which implies that \( q|(p + 1) \) or \( q|(p - 1) \). But this only leaves \( p^3 - p^3(q-1) = p^3 \) elements and our Sylow p-subgroup must be normal, a case we already considered. Therefore, the only remaining possibility is that \( G \cong \mathbb{Z}_q \times P \). Suppose \( \gamma(\mathcal{J}(P)) > 1, \overline{\gamma}(\mathcal{J}(P)) > 1 \) and \( \mathcal{J}(P) \) contains \( K_5 \), then the same holds for \( \mathcal{J}(G) \) also. So it is enough to consider the cases when \( \mathcal{J}(P) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Propositions 3.1, 3.2, 4.1 and Theorem 2.5 \( P \) is isomorphic to one of \( \mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p (p = 2, 3, 5), Q_8, M_8 \) or \( M_{p^3} \).

- If \( P \cong \mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p \) or \( p = 3, 5 \), then \( G \cong \mathbb{Z}_q \times (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \) or \( \mathbb{Z}_q \times M_{p^3} \). Here \( H_1 := \langle a, c \rangle, H_2 := \langle ab, c \rangle, H_3 := \langle a^p, b, c \rangle, H_4 := \langle a^p, c \rangle, H_5 := \langle a \rangle, H_6 := \langle ab \rangle, H_7 := \langle a^p, b \rangle, H_8 := \langle a^p \rangle \) are subgroups of \( G \), where \( \langle a, b \rangle = P \) and \( \langle c \rangle = \mathbb{Z}_q \). Also \( H_8 \) is a subgroup of \( H_i, i = 1, \ldots, 7 \). Thus \( \mathcal{J}(G) \) has \( K_8 \) as a subgraph and \( \gamma(\mathcal{J}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{J}(G)) > 1 \).

- If \( \mathcal{J}(G) \cong Q_8 \), then \( G \cong \mathbb{Z}_q \times Q_8 \). Here \( H_1 := \langle a, c \rangle, H_2 := \langle b, c \rangle, H_3 := \langle ab, c \rangle, H_4 := \langle a^2, c \rangle, H_5 := \langle a \rangle, H_6 := \langle b \rangle, H_7 := \langle ab \rangle, H_8 := \langle a^2 \rangle \) are proper subgroups of \( G \), where \( \langle a, b \rangle = P \) and \( \langle c \rangle = \mathbb{Z}_q \). Also \( H_8 \) is a subgroup of \( H_i, i = 1, \ldots, 7 \). Thus these eight subgroups forms \( K_8 \) as a subgraph of \( \mathcal{J}(G) \) and this implies that \( \gamma(\mathcal{J}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{J}(G)) > 1 \).

- If \( P \cong M_8 \). Here \( \mathbb{Z}_q \times M_8, H_1 := \langle a, c \rangle, H_2 := \langle a^2, ab, c \rangle, H_3 := \langle a^2, b, c \rangle, H_4 := \langle a^2, c \rangle, H_5 := \langle a \rangle, H_6 := \langle a^2, b \rangle, H_7 := \langle a^2, ab \rangle, H_8 := \langle a^2 \rangle \) are proper subgroups of \( G \), where \( \langle a, b \rangle = P \) and \( \langle c \rangle = \mathbb{Z}_q \). Also \( H_8 \) is a subgroup of \( H_i, i = 1 \) to 7. Thus these eight subgroups form \( K_8 \) as a subgraph of \( \mathcal{J}(G) \) and so \( \gamma(\mathcal{J}(G)) > 1, \overline{\gamma}(\mathcal{J}(G)) > 1 \).

If \( (p, q) = (2, 3) \), then \( G \cong S_4 \). In this case, \( G \) has at least two copies of \( D_8 \). So by Figure 3(c), \( D_8 \) together with its proper subgroups form \( K_5 \) as a subgraph of \( \mathcal{J}(G) \). Then \( \mathcal{J}(G) \) has a subgraph \( B_1 \) as shown in Figure 2 so \( \gamma(\mathcal{J}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{J}(G)) > 1 \). Thus from the above arguments the result is true when \( \alpha = 3 \).
Intersection graph of subgroups

Assume that the result is true for all non-abelian group of order $p^m q$, with $m < \alpha$. We prove the result when $\alpha > 3$. If $n_p(G) = 1$, then $G \cong P \times \mathbb{Z}_q$. Suppose $\gamma((\mathcal{I}(P)) > 1$, $\overline{\gamma}(\mathcal{I}(P)) > 1$ and $\mathcal{I}(P)$ contains $K_5$, then the same holds for $\mathcal{I}(G)$ also. So it is enough to consider the cases when $\mathcal{I}(P)$ is one of planar, toroidal, projective-planar, $K_5$-free, $C_3$-free and bipartite. By Propositions 3.1, 3.2, 4.1 and Theorem 2.5, $P \cong \mathbb{Z}_{p^m}$ or $M_{24}$.

- If $P \cong \mathbb{Z}_{p^m}$, then $G$ has a subgroup $\langle a^p, b \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_q$. So by induction hypothesis $\gamma(\mathcal{I}(\langle a^p, b \rangle)) > 1$, $\overline{\gamma}(\mathcal{I}(\langle a^p, b \rangle)) > 1$, and so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

- If $P \cong M_{24}$, then by (4.2), $P$ together with its proper subgroups form $K_8$ as a subgraph of $\mathcal{I}(G)$ and so $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\gamma(\mathcal{I}(G)) > 1$.

Let $n_p(G) \neq 1$. Since $G$ is solvable, $G$ has a normal subgroup $N$ of order $p^{\alpha-1}q$. Suppose $\gamma(\mathcal{I}(N)) > 1$, $\overline{\gamma}(\mathcal{I}(N)) > 1$ and $\mathcal{I}(N)$ contains $K_5$, then the same holds for $\mathcal{I}(G)$ also. So it is enough to consider the cases when $\mathcal{I}(N)$ is one of planar, toroidal, projective-planar, $K_5$-free, $C_3$-free and bipartite. By Propositions 3.1, 3.2, 4.2 and Theorem 2.5, $N \cong \mathbb{Z}_{p^m} q$.

Suppose $G$ has two elements, say $a, b$ of orders $p, q$ respectively with $b \in N$, $a \notin N$, then $H := \langle a, b \rangle$ is a proper subgroup of $G$. Consider $N$ together with its proper subgroups $H_1, H_2, H_3, H_4, H_5, H_6, H_7$, of order $p, p^2, p^3, p, p^2, p^3, p^2, q, q$ respectively. Here $H_i \cap H_j \neq \{e\}$, for every $i, j = 1$ to $6$; $H_7$ is a subgroup of $H_i$, $i = 4, 5, 6, H$; $H \cap H_i = H_7, i = 4, 5, 6$. It follows that $\mathcal{I}(G)$ has a subgraph $A_1$ as shown in Figure 1, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

Suppose every subgroup of $G$ of order $p$ is contained in $N$, then $G$ has at least three Sylow $p$-subgroups, let them be $P_1, P_2, P_3$. Consider $N$ together with its proper subgroups $H_1, H_2, H_3, H_4, H_5, H_6$ of order $p, p^2, p^3, p, p^2, p^3$, respectively. Here $H_1$ is a subgroup of $H_i, i = 2, \ldots, 6, N, P_1$, so they form $K_8$ as a subgraph of $\mathcal{I}(G)$. Therefore, $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

Thus the result is true when $\alpha > 3$, it follows that the result is true for all $\alpha \geq 3$.

**Proposition 4.4.** If $G$ is a non-abelian group of order $p^2 q^2$, where $p, q$ are distinct primes, then $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $C_3$. Also $\mathcal{I}(G)$ is $K_5$-free if and only if $G \cong \langle a, b, c \mid a^p = b^q = c^2 = 1, ab = ba, cac^{-1} = ab^{-1}, cbc^{-1} = ab \rangle$, where $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order $q^2$ in $GL_2(p)$, $q^2 | (p+1)$.

**Proof.** We use the classification of groups of order $p^2 q^2$ given in [19].

Let $P$ and $Q$ denote a Sylow $p, q$-subgroups of $G$ respectively, with out loss of generality, we assume that $p > q$. By Sylow’s theorem, $n_p(G) = 1, q, q^2$. But $n_p(G) = q$ is not possible, since $p > q$. If $n_p(G) = q^2$, then $p | (q+1)(q-1)$, this implies that $p | (q+1)$, which is true only when $(p,q) = (3, 2)$.

**Case 1:** $(p,q) \neq (3,2)$. Then $G \cong P \times Q$.

**Subcase 1a:** If $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^i, id \equiv 1(mod(p^2)) \rangle$, then we have $H := \langle a^i, b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_q$, so by (4.3), $H$ together with its proper subgroups forms $K_8$ as a subgraph of $\mathcal{I}(G)$. Hence $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$. 

Subcase 1b: If $G \cong \mathbb{Z}_p^2 \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $H_1 := \langle a \rangle = \mathbb{Z}_p^2$, $H_2 := \langle a^p \rangle$, $H_3 := \langle a, c \rangle$, $H_4 := \langle a^p, b \rangle$, $H_5 := \langle a, b \rangle$, $H_6 := \langle a^p, c \rangle$, $H_7 := \langle b, c \rangle = \mathbb{Z}_q \times \mathbb{Z}_q$, $H_8 := \langle a^p, b, c \rangle$ are proper subgroups of $G$, where $\langle a \rangle = \mathbb{Z}_p^2$ and $\langle b, c \rangle = \mathbb{Z}_q \times \mathbb{Z}_q$. Here $H_i, i = 3, \ldots, 8$ intersect with each other nontrivially; $H_2$ is a subgroup of $H_1, H_3, H_4$; $H_1 \cap H_4 = H_2$; $H_1$ is a subgroup of $H_3$. It follows that $\mathscr{S}(G)$ has a subgraph $A_1$ as shown in Figure 11 so $\gamma(\mathscr{S}(G)) > 1$ and $\overline{\gamma}(\mathscr{S}(G)) > 1$.

Subcase 1c: If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q^2 := \langle a, b, c \mid a^p = b^q = c^2 = 1, ab = ba, cac^{-1} = a^ib^j, bcb^{-1} = a^kb^l \rangle$, where $\binom{ij}{kl}$ has order $q^2$ in $GL_2(p)$, $q^2 | (p + 1)$. Then we have two possibilities:

- Suppose $G$ has a subgroups of order $pq^2$ and $pq$, then $H_1 := \langle c \rangle = \mathbb{Z}_q^2$, $H_2 := \langle c^3 \rangle$, $H_3 := \langle a, c \rangle$, $H_4 := \langle b, c^p \rangle$, $H_5 := \langle b, c \rangle$, $H_6 := \langle a, c^p \rangle$, $H_7 := \langle a, b \rangle = \mathbb{Z}_p \times \mathbb{Z}_p$, $H_8 := \langle a, b, c^p \rangle$ are proper subgroups of $G$. Here $H_i, i = 3, \ldots, 8$ intersect with each other nontrivially; $H_2$ is a subgroup of $H_1, H_3, H_4$; $H_1$ is a subgroup of $H_3$; $H_1 \cap H_4 = H_2$. It follows that $\mathscr{S}(G)$ has a subgraph $A_1$ as shown in Figure 11 so $\gamma(\mathscr{S}(G)) > 1$, $\overline{\gamma}(\mathscr{S}(G)) > 1$.

- Suppose $G$ has a no subgroups of order $pq^2$ and $pq$, then by Theorem 2.3, $\mathscr{S}(G)$ is planar and by (2.11), $\mathscr{S}(G)$ contains $C_3$.

Subcase 1d: If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $H_1 := \langle a, b \rangle = \mathbb{Z}_p \times \mathbb{Z}_p$, $H_2 := \langle a, c \rangle$, $H_3 := \langle a, d \rangle$, $H_4 := \langle b, c \rangle$, $H_5 := \langle a, b, c \rangle$, $H_6 := \langle a, b, d \rangle$, $H_7 := \langle a, c, d \rangle$, $H_8 := \langle b, c, d \rangle$ are proper subgroups of $G$, where $\langle a, b \rangle = \mathbb{Z}_p \times \mathbb{Z}_p$ and $\langle c, d \rangle = \mathbb{Z}_q \times \mathbb{Z}_q$. Here $H_i, i = 1, \ldots, 8$ intersect with each other nontrivially. It follows that $\mathscr{S}(G)$ contains $K_8$ as a subgraph and hence $\gamma(\mathscr{S}(G)) > 1$ and $\overline{\gamma}(\mathscr{S}(G)) > 1$.

Case 2: $(p, q) = (3, 2)$. Up to isomorphism, there are nine groups of order 36. In the following we consider each of these groups.

(i) $G \cong D_{18}$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a^2 \rangle$, $H_3 := \langle a^6 \rangle$, $H_4 := \langle a^6, b \rangle$, $H_5 := \langle a^6, ba \rangle$, $H_6 := \langle a^6, ba^3 \rangle$, $H_7 := \langle a^6, ba^3 \rangle$, $H_8 := \langle a^6, ba^4 \rangle$ are proper subgroups of $G$. Also $H_3$ is a subgroup of $H_i, i = 1, \ldots, 8$. It follows that $\mathscr{S}(G)$ contains $K_8$ as a subgraph, so $\gamma(\mathscr{S}(G)) > 1$ and $\overline{\gamma}(\mathscr{S}(G)) > 1$.

(ii) $G \cong S_3 \times S_3$. Here $H_1 := S_3 \times \{e\}$, $H_2 := S_3 \times \langle (123) \rangle$, $H_3 := S_3 \times \langle (12) \rangle$, $H_4 := S_3 \times \langle (13) \rangle$, $H_5 := S_3 \times \langle (23) \rangle$, $H_6 := \langle (13) \rangle \times S_3$, $H_7 := \langle (123) \rangle \times S_3$, $H_8 := \langle (12) \rangle \times S_3$ are proper subgroups of $G$. Also $H_i, i = 1, \ldots, 8$ intersect with each other nontrivially, it follows that $\mathscr{S}(G)$ contains $K_8$ as a subgraph and so $\gamma(\mathscr{S}(G)) > 1$ and $\overline{\gamma}(\mathscr{S}(G)) > 1$.

(iii) $G \cong \mathbb{Z}_3 \times A_4$. Here $H_1 := \mathbb{Z}_3 \times \{e\}$, $H_2 := \mathbb{Z}_3 \times \langle (123) \rangle$, $H_3 := \mathbb{Z}_3 \times \langle (124) \rangle$, $H_4 := \mathbb{Z}_3 \times \langle (134) \rangle$, $H_5 := \mathbb{Z}_3 \times \langle (234) \rangle$, $H_6 := \mathbb{Z}_3 \times \langle (12)(34) \rangle$, $H_7 := \mathbb{Z}_3 \times \langle (13)(24) \rangle$, $H_8 := \mathbb{Z}_3 \times \langle (14)(23) \rangle$ are proper subgroups of $G$. Also $H_1$ is a subgroup of $H_i, i = 2, \ldots, 8$, it follows that $\mathscr{S}(G)$ contains $K_8$ as a subgraph and so $\gamma(\mathscr{S}(G)) > 1$ and $\overline{\gamma}(\mathscr{S}(G)) > 1$. 

\begin{thebibliography}{99}

\end{thebibliography}
(iv) \( G \cong \mathbb{Z}_6 \times S_3 \). Here \( H_1 := \mathbb{Z}_6 \times \langle (123) \rangle, H_2 := \mathbb{Z}_6 \times \langle (12) \rangle, \) \( H_3 := \mathbb{Z}_6 \times \langle (13) \rangle, \) \( H_4 := \mathbb{Z}_6 \times \langle (23) \rangle, H_5 := \mathbb{Z}_3 \times \langle (123) \rangle, H_6 := \mathbb{Z}_3 \times \langle (12) \rangle, H_7 := \mathbb{Z}_3 \times \langle (13) \rangle, \) \( H_8 := \mathbb{Z}_3 \times \langle (23) \rangle \) are proper subgroups of \( G \). Also \( H_i, i = 1, \ldots, 8 \) intersect with each other non-trivially, it follows that \( \mathcal{I}(G) \) contains \( K_8 \) as a subgraph and hence \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

(v) \( G \cong \mathbb{Z}_6 \times \mathbb{Z}_4 \times \mathbb{Z}_4 = \langle a, b \mid a^9 = b^4 = 1, bab^{-1} = a^i, i^4 \equiv 1 \text{ (mod 9)} \rangle \). Here \( H_1 := \langle a \rangle, H_2 := \langle a, b^2 \rangle, H_3 := \langle a^3 \rangle, H_4 := \langle a^3, b \rangle, H_5 := \langle a^3, b^3 \rangle, H_6 := \langle a^3, ab^2 \rangle, H_7 := \langle a^3, a^3b^2 \rangle, H_8 := \langle b \rangle \) are proper subgroups of \( G \). Also \( H_3 \) is a subgroup of \( H_i, i = 1, \ldots, 7; H_8 \) intersect with \( H_2, H_4, H_5 \) non-trivially. It follows that \( \mathcal{I}(G) \) has a subgraph \( \mathcal{A}_1 \) as shown in Figure 1, so \( \gamma(\mathcal{I}(G)) > 1, \overline{\gamma}(\mathcal{I}(G)) > 1 \).

(vi) \( G \cong \mathbb{Z}_3 \times (\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)) = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, ac = ca, abc^{-1} = b^i, \text{ord}_2(i) = 3 \rangle \). Here \( H := \mathbb{Z}_3 \times \mathbb{Z}_4 \) is a subgroup of \( G \), and as in Case 1b, in the proof of Theorem 4.2, \( H \) has a unique subgroup of order 2, let it be \( H_1; H_i, i = 2, 3, 4 \) be subgroups of \( H \) of order 4; \( H_5 \) be a subgroup of \( H \) of order 6; it follows that \( H \) together with its proper subgroups forms \( K_6 \) as a subgraph of \( \mathcal{I}(H) \). Moreover \( H_6 := \mathbb{Z}_3 \times H_1 \) and \( H_7 := \mathbb{Z}_4 \times H_2 \) are subgroups of \( G \). Thus \( H_1 \) is a subgroup of \( H, H_i, i = 1, 2, 3, 4, 6, 7 \). It follows that \( G \) contains \( K_8 \) as a subgraph, and so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

(vii) \( G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_4 = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cac^{-1} = a^i b^j, bcb^{-1} = a^k b^l \rangle \), where \( \binom{i}{k} j \) has order 4 in \( GL_2(3) \). We already discussed the intersection graph of subgroups of this group in Subcase 1c.

(viii) \( G \cong \mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \). Here \( H_1 := \langle a, b, c \rangle, H_2 := \langle a, b, d \rangle, H_3 := \langle a, c, d \rangle, H_4 := \langle b, c, d \rangle, H_5 := \langle a, b \rangle, H_6 := \langle a, c \rangle, H_7 := \langle a, d \rangle, H_8 := \langle b, c \rangle \) are proper subgroups of \( G \). Also these subgroups intersect with each other non-trivially. It follows that \( \mathcal{I}(G) \) contains \( K_8 \) as a subgraph, so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

(ix) \( G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_4 \). Here \( H_1 := \langle a, b \rangle, H_2 := \langle a, c \rangle, H_3 := \langle a, c^2 \rangle, H_4 := \langle b, c \rangle, H_5 := \langle b, c^3 \rangle, H_6 := \langle a, b, c^3 \rangle, H_7 := \langle c \rangle, H_8 := \langle c^3 \rangle \) are proper subgroups of \( G \). Also \( H_i, i = 1, \ldots, 6 \) intersect with each other non-trivially; \( H_7 \) is a subgroup of \( H_8, H_3, H_4; H_1 \) intersect with \( H_3, H_4 \) non-trivially. It follows that \( \mathcal{I}(G) \) has a subgraph \( \mathcal{A}_1 \) as shown in Figure 1, so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

The proof follows by combining all the above cases.

□

**Proposition 4.5.** If \( G \) is a non-abelian group of order \( p^\alpha q^\beta \), where \( p, q \) are distinct primes and \( \alpha, \beta \geq 2 \), then \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_8 \).

**Proof.** We prove the result by induction on \( \alpha + \beta \). If \( \alpha + \beta = 5 \), then \( |G| = p^\alpha q^\beta \). Since \( G \) is solvable, it has a normal subgroup \( N \) of prime index.
**Case 1:** If \([G : N] = p\), then \(|N| = p^2q^2\). Suppose \(\gamma(\mathcal{I}(N)) > 1\), \(\overline{\gamma}(\mathcal{I}(N)) > 1\) and \(\mathcal{I}(N)\) contains \(K_5\), then the same holds for \(\mathcal{I}(G)\) also. So it is enough to consider the cases when \(\mathcal{I}(N)\) is one of planar, toroidal, projective-planar, \(K_5\)-free, \(C_3\)-free and bipartite. By Propositions 3.1, 3.2, 4.1 and Theorem 2.5 \(N \cong \mathbb{Z}_{p^6q^2}\) or \((a, b, c \mid a^p = b^2 = c^2 = 1, ab = ba, cac^{-1} = b^{-1}, bcbc^{-1} = a^{-1}b)\), where \((\frac{1}{1} \frac{-1}{1})\) has order \(q^2\) in \(GL_2(p)\), \(q^2 \mid (p+1)\). If \(N \cong \mathbb{Z}_{p^6q^2}\), then \(N\) together with its proper subgroups forms a subgraph in \(\mathcal{I}(G)\), which is isomorphic to \(A_1\) as shown in Figure 1 and so \(\gamma(\mathcal{I}(G)) > 1\), \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

If \(N\) is isomorphic to the second group, then \(N\) has unique subgroups of order \(p^2\) and \(p^2q\), let them be \(H_1, H_2\); also there are \(p+1\) subgroups of order \(p\), let them be \(B_i\), \(i = 1, 2, \ldots, p+1\). Here \(p > q\). Suppose \(p \geq 5\), then \(\mathcal{I}(G)\) contains \(K_{4,5}\) as a subgraph with bipartition \(X := \{P, H_1, H_2, N\}\) and \(Y := \{B_1, B_2, B_3, B_4, B_5\}\), where \(P\) is a Sylow \(p\)-subgroup of \(G\) containing \(H_1\) and \(H_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p\). Here \(B_1\) is a subgroup of \(N, H_1, H_2, P\) and so they form \(K_{5}\) as a subgraph of \(\mathcal{I}(G)\). If \(p = 3\), then \(P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, M_3\), or \((\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_p\). If \(P \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, M_3\), then by (3.5), (4.1), \(H_1, H_2\) together with the proper subgroups of \(H_1\) forms a subgraph of \(\gamma(\mathcal{I}(G))\), which is isomorphic to \(A_1\) shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\). If \(P \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_p\), then by Proposition 4.1 \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

**Case 2:** If \([G : N] = q\), then \(|N| = p^3q\). Suppose \(\gamma(\mathcal{I}(N)) > 1\), \(\overline{\gamma}(\mathcal{I}(N)) > 1\) and \(\mathcal{I}(N)\) contains \(K_5\), then the same holds for \(\mathcal{I}(G)\) also. So it is enough to consider the cases when \(\mathcal{I}(P)\) is one of planar, toroidal, projective-planar, \(K_5\)-free, \(C_3\)-free and bipartite. By Propositions 3.1, 3.2, 4.1 and Theorem 2.5 \(N \cong \mathbb{Z}_{p^6q^2}\). Let \(H_i, i = 1, 2, 3, 4, 5, 6\) be subgroups of \(N\) of order \(p^2, p^3, pq, p^2q, q\) respectively. Let \(H\) be a subgroup of \(G\) of order \(q^2\) such that \(H\) contains \(H_6\). By (3.5), \(N\) together with its proper subgroups and \(H\) forms a subgraph in \(\mathcal{I}(G)\), which is isomorphic to \(A_1\) as shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

Now we assume that the result is true for all non-abelian groups of order \(p^m q^n\), where \(m + n < \alpha + \beta\) \((m + n \geq 5, m, n \geq 2)\). We prove the result when \(\alpha + \beta > 5\). Since \(G\) is solvable, \(G\) has a subgroup \(H\) of prime index, with out loss of generality, say \(q\), and so \(|H| = p^\alpha q^\beta - 1\). If \(H\) is cyclic, then \(H\) together with its proper subgroups forms \(K_6\) as a subgraph of \(\mathcal{I}(G)\). Now let \(K\) be the subgroup of \(H\) of order \(q^\beta - 1\) and let \(Q\) be a \(q\)-Sylow subgroup of \(G\) containing \(K\). Also let \(H_i, i = 1, 2, 3\) be subgroups of \(H\) of order \(pq, p^2q, p^3q\) respectively. It follows that \(\mathcal{I}(G)\) has a subgraph \(A_1\) as shown in Figure 1 so \(\gamma(\mathcal{I}(G)) > 1\) and \(\overline{\gamma}(\mathcal{I}(G)) > 1\).

If \(H\) is non-cyclic abelian, then by Proposition 3.2 \(\gamma(\mathcal{I}(H)) > 1\), \(\overline{\gamma}(\mathcal{I}(H)) > 1\) and \(\mathcal{I}(H)\) contains \(K_5\) as a subgraph, so \(\gamma(\mathcal{I}(G)) > 1\), \(\overline{\gamma}(\mathcal{I}(G)) > 1\) and \(\mathcal{I}(G)\) contains \(K_5\). If \(H\) is non-abelian, then we have the following cases to consider:

**Case a:** If \(\beta = 2\), then \(\alpha > 3\). So by Proposition 4.1 \(\gamma(\mathcal{I}(H)) > 1\), \(\overline{\gamma}(\mathcal{I}(H)) > 1\) and \(\mathcal{I}(H)\) contains \(K_5\) as a subgraph.

**Case b:** If \(\beta > 2\), then by induction hypothesis, \(\gamma(\mathcal{I}(H)) > 1\), \(\overline{\gamma}(\mathcal{I}(H)) > 1\) and \(\mathcal{I}(H)\) contains \(K_5\) as a subgraph.
**Case c:** If $\alpha = 2$, then $\beta > 2$. By Case b, $\gamma(\mathcal{I}(H)) > 1$, $\overline{\gamma}(\mathcal{I}(H)) > 1$ and $\mathcal{I}(H)$ contains $K_5$ as a subgraph.

**Case d:** If $\alpha > 2$, then by induction hypothesis, $\gamma(\mathcal{I}(H)) > 1$, $\overline{\gamma}(\mathcal{I}(H)) > 1$ and $\mathcal{I}(H)$ contains $K_5$ as a subgraph.

It follows that $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $K_5$ as a subgraph. □

**Proposition 4.6.** Let $G$ be a non-abelian solvable group of order $pqr$, where $p$, $q$, $r$ are distinct primes, $p > q > r$. Then

1. $\mathcal{I}(G)$ contains $C_3$;
2. $\mathcal{I}(G)$ is $K_5$-free if and only if $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, b^{-1}ab = a^\mu, c^{-1}ac = a^\nu, bc = cb \rangle$, where $r$, $q$ are divisor of $(p - 1)$ and $v$, $\mu \neq 1$;
3. $\mathcal{I}(G)$ is non-toroidal and non-projective-planar;

**Proof.** By [11, p. 215], up to isomorphism there are four groups of order $pqr$. In the following we deal with each of these groups.

**Case a:** If $r \nmid (p - 1)$ and $q \nmid (p - 1)$, then $G \cong \mathbb{Z}_{pqr}$, which is not possible.

**Case b:** If $r \mid (p - 1)$, $r \nmid (q - 1)$ and $q \mid (p - 1)$, then $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, ac = ca, c^{-1}bc = b^r \rangle, v \neq 1$. Here $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle a, b \rangle$, $H_4 := \langle b, c \rangle$, $I_1 := \langle c \rangle$, $I_2 := \langle bc \rangle$, $I_3 := \langle b^2c \rangle$, ..., $I_q := \langle b^{q-1}c \rangle$, $B_1 := \langle a, c \rangle$, $B_2 := \langle a, bc \rangle$, ..., $B_q := \langle a, b^{q-1}c \rangle$ are the only subgroups of $G$. If $q = 3$, then $r = 2$, and so $r$ divides $(q - 1)$, which is not possible.

If $q \geq 5$, then $B_i, i = 1, \ldots, 5$, $H_1$ intersect with each other non-trivially; $H_2$ is a subgroup of $H_3$, $H_4$; $H_3 \cap B_i = H_1$. It follows that $\mathcal{I}(G)$ has a subgraph $A_1$ as shown in Figure 1, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

**Case c:** If $r \mid (p - 1)$, $r \nmid (q - 1)$ and $q \mid (p - 1)$, then the group is different from the group given in Case b only in the exchange the roles of $a$, $b$, so $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, bc = cb, c^{-1}ac = a^\nu, \mu \neq 1 \rangle$. Here $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle a, b \rangle$, $I_1 := \langle c \rangle$, $I_2 := \langle ac \rangle$, $I_3 := \langle a^2c \rangle$, ..., $I_q := \langle a^{q-1}c \rangle$, $B_1 := \langle a, c \rangle$, $B_2 := \langle b, ac \rangle$, ..., $B_q := \langle b, a^{q-1}c \rangle$ are the only subgroups of $G$. Here also $q = 3$ not possible. If $q \geq 5$, then $B_i, i = 1$ to $5$, $H_1$ intersect with each other non-trivially; $H_2$ is a subgroup of $H_3$, $H_4$; $H_3 \cap B_i = H_1$. It follows that $\mathcal{I}(G)$ has a subgraph $A_1$ as in Figure 1, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$.

**Case d:** If $r \mid (p - 1)$, $r \nmid (q - 1)$ and $q \mid (p - 1)$, then $G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, c^{-1}ac = a^\mu, \mu \neq 1 \rangle$, $v \neq 1$. Here $H_1 := \langle a \rangle$, $H_2 := \langle a, b \rangle$, $H_3 := \langle a, c \rangle$, $H_4 := \langle a, bc \rangle$, $H_5 := \langle a, b^2c \rangle$, $H_6 := \langle b, c \rangle$, $H_7 := \langle b, ac \rangle$, $H_8 := \langle b, a^2c \rangle$, $H_9 := \langle b, a^4c \rangle$, $H_{10} := \langle b, a^{q-1}c \rangle$ are proper subgroups of $G$. Also $H_1$ is a subgroup of $H_i$, $i = 2, 3, 4, 5$ and so they form $K_5$ as a subgraph of $\mathcal{I}(G)$. $\langle b \rangle$ is a subgroup of $H_i, i = 6, 7, 8, 9, 10$, so they form $K_5$ as a subgraph of $\mathcal{I}(G)$. It follows that $\mathcal{I}(G)$ has a subgraph $B_1$ as in Figure 2, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$. 
**Case e:** If \( q | (p - 1) \), then we have the group is of essentially the same form as Case 1b and Case 1c. The elements \( c, a, b \) here playing the same role as \( a, b, c \) in Case 1b. The toroidality and the projective-planarity of the intersection graphs of subgroups of this group are as described in Case 1b and Case 1c.

**Case f:** If \( r | (p - 1) \) and \( q | (p - 1) \), then \( G \cong \langle a, b, c \mid b^p = c^q = 1, b^{-1}ab = c^{-1}ac = a^r, bc = cb \rangle, \) \( v, \mu \neq 1 \). By Theorem 2.5, \( \mathcal{I}(G) \) is planar and by Figure 3(d), it contains \( C_3 \).

Combining together all the cases, the proof follows. \( \square \)

**Proposition 4.7.** Let \( G \) be a non-abelian solvable group of order \( p^2qr \), where \( p, q, r \) are distinct primes. Then

1. \( \mathcal{I}(G) \) contains \( C_3 \);
2. \( \mathcal{I}(G) \) is \( K_5 \)-free if and only if \( G \cong \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, cac^{-1} = b, cbc^{-1} = ab \rangle \), where \( l \) is any integer with \( \binom{l}{1} \) has order \( qr \) in \( GL_2(p), qr|(p + 1) \).
3. \( \mathcal{I}(G) \) is toroidal if and only if \( G \cong \langle a, b, c \mid a^5 = b^5 = c^6 = 1, ab = ba, cac^{-1} = b, cbc^{-1} = ab \rangle, \) where \( l \) is any integer with \( \binom{l}{1} \) has order \( 6 \) in \( GL_2(5) \).
4. \( \mathcal{I}(G) \) is non projective-planar.

**Proof.** Since \( G \) is solvable and it has a Sylow basis \( \{P, Q, R\} \), where \( P, Q, R \) denotes the Sylow p,q,r-subgroups of \( G \) respectively. Then \( PQ, PR, QR \) are subgroups of \( G \) of order \( p^2q, p^2r, qr \) respectively.

**Case 1:** A subgroup of \( G \) of index \( r \) is normal in \( G \): With out loss of generality, we may assume that \( PQ \) is normal in \( G \).

**Subcase 1a:** Suppose \( P \) is normal in \( PQ \), then by Theorem 2.6 \( P \) is normal in \( G \). Then two possibilities arise:

1. \( P \cong Z_{p^2} \): Let \( H_1 \) be the subgroup of \( P \) of order \( p \). Then \( H_1 \) is also normal in \( G \). Here \( H_1R, H_1Q, H_1QR \) are subgroups of \( G \). Also \( H_1 \) is a subgroup of \( H_1Q, H_1R, P, PQ, PR, QR \) intersect with \( PQ, PR, H_1Q, H_1QR \) non-trivially. It follows that \( \mathcal{I}(G) \) has a subgraph isomorphic to \( A_1 \) as shown in Figure 1 so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\Gamma(G)) > 1 \).

2. \( P \cong Z_p \times Z_p \): Then \( P \) has atleast three subgroups of order \( p \), let them be \( H_i, i = 1, 2, 3 \). Suppose \( \gamma(\mathcal{I}(PQ)) > 1, \overline{\gamma}(\mathcal{I}(PQ)) > 1 \) and \( \mathcal{I}(PQ) \) contains \( K_5 \), then the same holds for \( \mathcal{I}(G) \) also. So it is enough to consider the cases when \( \mathcal{I}(PQ) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Theorem 2.5 Propositions 3.2 and 4.2 the only possibilities for \( PQ \) such that \( \mathcal{I}(PQ) \) satisfying the above properties are \( Z_{pq} \times Z_p \) or \( \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b \rangle \), where \( \binom{l}{1} \) has order \( q \) in \( GL_2(p), q | (p + 1) \).

By a similar argument, we can show that the only possibilities for \( PR \) are \( Z_{pr} \times Z_p \).
Intersection graph of subgroups

or \( \langle a, b, c \mid a^p = b^p = c^r = 1, ab = ba, cac^{-1} = b,cbc^{-1} = a^{-1}b^k \rangle \), where \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & k \end{smallmatrix} \right) \) has order \( r \) in \( GL_2(p) \), \( q \mid (p + 1) \).

- Suppose either \( PQ \) or \( PR \) is isomorphic to the respective first group, with out loss of generality, let us take \( PQ \). Then \( PQ \) has at least three subgroups \( H_iQ \), \( i = 1, 2, 3 \) of order \( pq \). Here \( PQ, PR, QR, H_1Q, H_2Q, H_3Q \) intersect with each other non-trivially; \( H_1 \) is a subgroup of \( PQ, PR, H_1Q, P \). It follows that these eight subgroups forms a subgraph of \( \mathcal{I}(G) \), which is isomorphic to \( A_1 \) as shown in Figure \( \text{I} \). Therefore, \( \gamma(\mathcal{I}(G)) > 1 \) and \( \tau(\mathcal{I}(G)) > 1 \).

- Now assume that \( PQ \) and \( PR \) are isomorphic to the respective second group.

Suppose that \( PQ \) is the only subgroup of order \( p^2q \), but the subgroup of order \( p^2r \) is not unique. If \( n_r(G) = 1 \), then \( R \) is normal in \( G \). Then \( H_1Q, H_1R, H_1QR \) are subgroups of \( G \) of order \( pq, pr, pqr \) respectively, where \( H_1 \) is subgroup of \( P \) of order \( p \). Here \( PQ, PR, H_1Q, H_1R, H_1QR \) intersect non-trivially; \( Q \) is a subgroup of \( PQ, H_1Q, H_1QR, QR \). It follows that \( \mathcal{I}(G) \) has a subgraph isomorphic to \( A_1 \) as shown in Figure \( \text{I} \). If \( n_r(G) \neq 1 \), then by Sylow’s theorem \( n_r(G) = p, q, pq, p^2 or p^2q \). Here \( n_r(G) = p, q, pq \) are not possible, since \( G \) has at least \( 2p^2 \) subgroups of order \( r \). Assume that \( n_r(G) = p^2q \). Then for \( q \geq 5 \), \( G \) has at least four subgroups of order \( p^2r \), let them be \( PR_1, PR_2, PR_3, PR_4 \). It follows that \( \mathcal{I}(G) \) contains \( K_{5,4} \) as a subgraph with bipartition \( X := \{ PQ, PR_1, PR_2, PR_3, PR_4 \} \) and \( Y := \{ P, H_1, H_2, H_3 \} \), so \( \gamma(\mathcal{I}(G)) > 1, \tau(\mathcal{I}(G)) > 1 \), where \( H_i, i = 2, 3 \) are subgroups of \( P \) of order \( p \). Also \( PQ, P, H_1, PR_1, PR_2 \) intersect non-trivially and so they form \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). If \( p > 2, q = 3 \), then \( G \) has at least three subgroups of order \( p^2r \) and so \( \mathcal{I}(G) \) contains \( K_{5,4} \) as a subgraph with bipartition \( X := \{ PQ, PR_1, PR_2, PR_3, P \} \) and \( Y := \{ H_1, H_2, H_3, H_4 \} \), so \( \gamma(\mathcal{I}(G)) > 1, \tau(\mathcal{I}(G)) > 1 \), where \( H_4 \) is a subgroup of \( P \) of order \( p \). Also \( PQ, P, H_1, PR_1, PR_2 \) intersect non-trivially and so they form \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). If \( p = 2, q = 3 \), then \( n_r(G) = 12 \), which implies \( r = 11 \). Thus we have \( n_p(G) = 1, n_q(G) = 4, n_r(G) = 12 \), and this implies that \( G \) has subgroup \( QR \) of order 33. Suppose \( QR \) is cyclic, then \( n_q(G) = n_r(G) \), which is not possible. Suppose \( QR \) is non-cyclic, then \( G \) has at least 11 subgroups of order 3, which is also not possible. If \( q = 2, p \geq 5 \), then \( \mathcal{I}(G) \) contains \( K_{5,4} \) as a subgraph with bipartition \( X := \{ PQ, PR_1, PR_2, P \} \) and \( Y := \{ H_1, H_2, H_3, H_4, H_5 \} \), so \( \gamma(\mathcal{I}(G)) > 1, \tau(\mathcal{I}(G)) > 1 \), where \( H_i, i = 2, 3 \) of \( P \) of order \( p \). Also \( PQ, P, H_1, PR_1, PR_2 \) intersect non-trivially and so they form \( K_5 \) as a subgraph of \( \mathcal{I}(G) \). If \( q = 2, p = 3 \), then \( n_r(G) = 18 \), which implies \( r = 17 \). Here \( n_r(G) = 18, n_q(G) = 9 \) and by a similar argument as above, we can show that such a group does not exist. If \( n_r(G) = p^2 \), then \( G \) has a unique subgroup of order \( p^2r \), which is a contradiction to our assumption.
Suppose the subgroup of order $p^2q$ and $p^2r$ in $G$ are not unique, then we can use the previous argument to show $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $K_5$.

Suppose the subgroup of order $p^2q$ and $p^2r$ in $G$ are unique, then the subgroup structure of $G$ is as follows: $G$ has unique subgroup of order $p$; $G$ has $p + 1$ subgroups of order $p$, let them be $H_i$, $i = 1, 2, \ldots, p + 1$. Now $PQ$ has $p^2$ subgroups of order $q$, say $Q_i$, $i = 1, \ldots, p^2$ and $PR$ has $p^2$ subgroups of order $r$, say $R_i$, $i = 1, \ldots, p^2$. Now we show that the subgroup of order $qr$ in $G$ is not unique. If it is unique, then this subgroup is isomorphic to one of $\mathbb{Z}_{qr}, \mathbb{Z}_q \rtimes \mathbb{Z}_r$ or $\mathbb{Z}_r \rtimes \mathbb{Z}_q$. But in either case, $G$ has a unique subgroup of order either $q$ or $r$, which is not possible.

Note that the normalizer in $G$ of $Q$, $N_G(Q)$ is a subgroup of $G$ of order $qr$. Similarly consider $R$, then $N_G(R)$ is a subgroup of $G$ of order $qr$. If $N_G(Q)$ is non-abelian, then necessarily $r < q$. Likewise if $N_G(R)$ is non-abelian, then necessarily $q < r$. As these inequalities cannot both be true, it follows that $G$ has an abelian (and hence cyclic) subgroup of order $qr$, and we may choose $Q$ and $R$ so that $N_G(Q) = N_G(R) = Q \times R$ is this cyclic subgroup. It follows that $G$ has $p^2$ subgroups of order $qr$, let them be $L_i, i = 1, \ldots, p^2$. This completes the subgroup structure of $G$.

By the presentation of the subgroup of order $p^2q$, an element $c$ of order $q$ acts on the $p$-Sylow subgroup $P$ via a matrix of determinant 1 which is not $-1$. Hence $q$ is not 2. Likewise $r$ is not 2. So, $Q \times R$ is uniquely determined up to conjugacy in $GL_2(p)$. Hence, $G$ is uniquely determined up to isomorphism.

A presentation of this group $G$ is $\langle a, b, c \mid a^p = b^q = c^{qr} = 1, ab = ba, cac^{-1} = b, cbc^{-1} = ab^l \rangle$, where $l$ is an integer with $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ has order $qr$ in $GL_2(p)$, $qr \mid (p + 1)$. The structure of $\mathcal{I}(G)$ is shown in Figure 9. Note that $\mathcal{I}(G)$ is $K_5$-free. If $p \geq 7$, then $\mathcal{I}(G)$ contains $K_{3,7}$ as a subgraph with bipartition $X := \{PQ, PR, P\}$ and $Y := \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$, so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\gamma}(\mathcal{I}(G)) > 1$. If $p = 5$, then $\mathcal{I}(G)$ is toroidal and the corresponding toroidal embedding is shown in Figure 10. Moreover, $\mathcal{I}(G)$ contains $K_{3,5}$ as a subgraph with bipartition $X := \{PQ, PR, P\}$ and $Y := \{H_1, H_2, H_3, H_4, H_5\}$, so $\gamma(\mathcal{I}(G)) > 1$. Since $qr|(p + 1)$, so $p \leq 3$ is not possible.

**Subcase 1b:** Suppose $P$ is non-normal in $PQ$, then $Q$ is normal in $PQ$. So, by Theorem 2.6(iii), $Q$ is normal in $G$. Then $H_1Q$, $H_1R$, $H_1QR$ are subgroups of $G$ of order $pq$, $pr$, $pqr$ respectively, where $H_1$ is subgroup of $P$ of order $p$. Here $PQ$, $PR$, $P$, $H_1Q$, $H_1R$, $H_1QR$ intersect non-trivially; $Q$ is a subgroup of $PQ$, $H_1Q$, $H_1QR$, $QR$. It follows that $\mathcal{I}(G)$ has a subgraph isomorphic to $A_4$ as shown in Figure 11.

**Case 2:** A subgroup of $G$ of index $q$ is normal in $G$. With out loss of generality, we may
assume that, $PR$ is normal in $G$. Here we can use a similar argument as in Subcase 1a, by taking $R$ instead of $Q$ and end up with the same group which we have obtained in Subcase 1a, whose intersection graph is toroidal, non-projective-planar and $K_5$-free.

**Case 3:** A subgroup of $G$ of index $p$ is normal in $G$: With out loss of generality, we may assume that, $H_1QR$ is normal in $G$, where $H_1$ is a subgroup of $P$ of order $p$. Then by Theorem 2.6(iii), $H_1$ is normal in $G$. So, $H_1Q, H_1R, H_1QR$ are subgroups of $G$. Here $PQ, PR, P, H_1Q, H_1R, H_1QR$ intersect non-trivially; $Q$ is a subgroup of $PQ, H_1Q, H_1R, QR$. It follows that $\mathcal{I}(G)$ has a subgraph isomorphic to $A_1$ as shown in Figure 9.

Putting together all the cases, the result follows.

**Proposition 4.8.** Let $G$ be a non-abelian solvable group of order $p^3qr$, where $p, q, r$ are distinct primes. Then $\gamma(\mathcal{I}(G)) > 1$, $\overline{\gamma}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $K_5$.

**Proof.** Since $G$ is solvable, so it has a Sylow basis $\{P, Q, R\}$, where $p, Q, R$ are Sylow $p, q, r$-subgroup of $G$ respectively. Then $PQ, PR$ are subgroups of $G$. Suppose $\gamma(\mathcal{I}(PQ)) > 1$, $\overline{\gamma}(\mathcal{I}(PQ)) > 1$ and $\mathcal{I}(PQ)$ contains $K_5$, then the same holds for $\mathcal{I}(G)$ also. So it is enough to consider the cases when $\mathcal{I}(PQ)$ is one of planar, toroidal, projective-planar, $K_5$-free, $C_3$-free and bipartite. By Theorem 2.5, Propositions 3.1, 3.2 and 4.3, $PQ \cong \mathbb{Z}_{p^3q}$. By a similar argument, we can show that the only possibility of $PR$ is $\mathbb{Z}_{p^2q}$. Let $H_1, H_2, H_3, H_4, H_5$ be subgroups of $PQ$ of orders $p, p^2, p^3, pq, p^2q$
respectively; \( H \) be a subgroup of \( PR \) of order \( p^2r \). Here \( H_i \) is a subgroup of \( H, PR, H, i = 2 \) to \( 5 \). It follows that, these five subgroups intersect with each other non-trivially and so \( \mathcal{I}(G) \) contains \( K_8 \) as a subgraph. Hence, \( \gamma(\mathcal{I}(G)) > 1 \) and \( \tau(\mathcal{I}(G)) > 1 \). □

**Proposition 4.9.** Let \( G \) be a non-abelian solvable group of order \( p^2q^2r \), where \( p, q, r \) are distinct primes. Then \( \gamma(\mathcal{I}(G)) > 1 \), \( \tau(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \).

**Proof.** Since \( G \) is a solvable group, it has subgroups of order \( p^2r \) and \( q^2r \). Now we split the proof into the following cases:

**Case 1:** A subgroup of \( G \) of order either \( p^2r \) or \( q^2r \) is unique: Then \( G \) has a subgroup of order either \( p^2qr \) or \( pq^2r \), let it be \( H \), and without loss of generality, we assume that \( |H| = p^2qr \). Suppose \( \gamma(\mathcal{I}(H)) > 1 \), \( \tau(\mathcal{I}(H)) > 1 \) and \( \mathcal{I}(H) \) contains \( K_5 \), then the same holds for \( \mathcal{I}(G) \) also. So it is enough to consider the cases when \( \mathcal{I}(H) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Theorem 2.5, Propositions 3.1, 3.2 and 4.7 \( H \cong \langle a, b, c | a^p = b^q = c^r = 1, ab = ba, cac^{-1} = b, bbc^{-1} = ab \rangle \), where \( \left[ \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right] \) has order \( qr \) in \( GL(2; p), qr|(p + 1) \). So by Figure 9 \( H \) together with its proper subgroups forms \( K_{3.7} \) as a subgraph of \( \mathcal{I}(G) \) and it contains \( K_5 \) as a subgraph.

**Case 2:** Subgroups of \( G \) of order \( p^2r, q^2r \) are not unique:

Since \( G \) is a solvable group, so it has a subgroup, say \( N \) of prime index. Now we need to consider the following subcases:

**Subcase 2a:** Let \( [G : N] = p \) or \( q \), without loss of generality, we assume that \( [G : N] = q \), and so \( |N| = p^2qr \). Then by the argument mentioned in Case 1, \( \gamma(\mathcal{I}(G)) > 1 \), \( \tau(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \) as a subgraph.

**Subcase 2b:** Let \( [G : N] = r \). Then \( |N| = p^2q^2 \). Suppose \( \gamma(\mathcal{I}(N)) > 1 \), \( \tau(\mathcal{I}(N)) > 1 \) and \( \mathcal{I}(N) \) contains \( K_5 \), then the same holds for \( \mathcal{I}(G) \) also. So it is enough to consider the cases when \( \mathcal{I}(N) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Theorem 2.5, Propositions 3.1, 3.2 and 4.4 \( N \cong \mathbb{Z}_{p^2q^2} \) or \( N \cong \mathbb{Z}_{p^2} \) or \( a, b, c | a^p = b^q = c^r = 1, ab = ba, cac^{-1} = b, bbc^{-1} = ab \rangle \), where \( \left[ \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right] \) has order \( q^2 \) in \( GL(2; p), q^2|(p + 1) \). If \( N \cong \mathbb{Z}_{p^2q^2} \), then \( N \) together with its proper subgroups forms a subgraph isomorphic to \( A_1 \) shown in Figure 1. Therefore, \( \gamma(\mathcal{I}(G)) > 1 \) and \( \tau(\mathcal{I}(G)) > 1 \).

If \( N \) isomorphic to the second group, then it has a subgroup of order \( p^2q \), let it be \( I \). Suppose \( G \) has unique subgroup of order \( p^2r \), then \( G \) has a subgroup of order \( p^2qr \), let it be \( H \). Suppose \( \gamma(\mathcal{I}(H)) > 1 \), \( \tau(\mathcal{I}(H)) > 1 \) and \( \mathcal{I}(H) \) contains \( K_5 \), then the same holds for \( \mathcal{I}(G) \) also. So it is enough to consider the cases when \( \mathcal{I}(H) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Theorem 2.5, Propositions 3.1, 3.2 and 4.7 \( H \cong \langle a, b, c | a^p = b^q = c^r = 1, ab = ba, cac^{-1} = b^{-1}, bbc^{-1} = ab \rangle \), where \( \left[ \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right] \) has order \( qr \) in \( GL(2; p), qr|(p + 1) \). By Figure 9 \( H \) together with its subgroups forms \( K_{3.7} \) as a subgraph of \( \mathcal{I}(G) \) and it contains \( K_5 \). Suppose the subgroup of order \( p^2r \) is not unique, then \( G \) has at least two subgroups of order \( p^2r \), let them be \( PR_1, PR_2 \). Let \( L \) be a subgroup of \( Q \) of order \( q \). Here \( H_1 \) is a subgroup of \( P, PR_1, PR_2, N \); \( L \) is a subgroup of \( Q, N, I, QR \). It follows that \( \mathcal{I}(G) \) has a subgraph isomorphic to \( B_2 \) as
shown in Figure [1]. Therefore, \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \) as a subgraph.

Proof follows by combining all the cases together. \( \square \)

Proposition 4.10. Let \( G \) be a non-abelian solvable group of order \( p^\alpha q^\beta r^\delta \), where \( p, q, r \) are distinct primes and \( \alpha + \beta + \delta = 6 \). Then \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \).

Proof. Here \( |G| = p^\alpha q^\beta r^\delta \). First let us assume that \( |G| \) is either \( p^\alpha q^\beta r^\delta \) or \( p^\beta q^\alpha r^\delta \). Then \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \). Suppose \( \gamma(\mathcal{I}(H)) > 1 \), \( \overline{\gamma}(\mathcal{I}(H)) > 1 \) and \( \mathcal{I}(H) \) contains \( K_5 \), then the same holds for \( \mathcal{I}(G) \) also. So it is enough to consider the cases when \( \mathcal{I}(H) \) is one of planar, toroidal, projective-planar, \( K_5 \)-free, \( C_3 \)-free and bipartite. By Theorem 2.5, Propositions 3.1, 3.2 and 4.3 and 4.5 we have \( H \cong \mathbb{Z}_{pq^r} \). Here \( H \) together with its proper subgroups forms \( K_5 \), so \( \gamma(\mathcal{I}(G)) > 1 \) and \( \overline{\gamma}(\mathcal{I}(G)) > 1 \).

If \( |G| = p^\beta q^\gamma r^\delta \), then \( G \) has a normal subgroup \( N \) with prime index, without loss of generality, say \( r \). Since \( |N| = p^\beta q^\gamma r^\delta \), and so by Theorem 2.5, Propositions 3.1, 3.2 and 4.9 \( \gamma(\mathcal{I}(N)) > 1 \), \( \overline{\gamma}(\mathcal{I}(N)) > 1 \) and \( \mathcal{I}(N) \) contains \( K_5 \). Therefore, the same holds for \( \mathcal{I}(G) \) also. Hence \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \) as a subgraph.

This completes the proof. \( \square \)

Proposition 4.11. Let \( G \) be a non-abelian solvable group of order \( p^\alpha q^\beta r^\delta \), where \( p, q, r \) are distinct primes and \( \alpha + \beta + \delta \geq 7 \). Then \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \).

Proof. Without loss of generality, we assume that \( \alpha \geq \beta \geq \delta \). Since \( G \) is solvable, \( G \) has a Sylow basis containing \( P, Q, R \), where \( P, Q, R \) are Sylow \( p, q, r \)-subgroups of \( G \) respectively and so \( PQ \) is a subgroup of \( G \). By Theorem 2.5, Propositions 3.1, 3.2 and 4.3, \( \gamma(\mathcal{I}(PQ)) > 1 \), \( \overline{\gamma}(\mathcal{I}(PQ)) > 1 \), \( \mathcal{I}(PQ) \) contains \( K_5 \), and so \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \) as a subgraph. \( \square \)

Proposition 4.12. If \( G \) is a solvable group, whose order has more than three distinct prime factors, then \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \).

Proof. Since \( G \) is solvable, it has a Sylow basis containing \( P, Q, R, S \), where \( P, Q, R, S \) are Sylow \( p, q, r, s \)-subgroups of \( G \) respectively. Then \( P, PQ, PR, PS, PQR, PQS, PRS, QRS \) are subgroups of \( G \). Here \( P \) is a subgroup of \( PQ, PR, PS, PQR, PQS, PRS; QRS, PR, PS, PQR, PQS, PRS; QRS, PR, PS \) intersect with each other nontrivially. It follows that \( \mathcal{I}(G) \) has a subgraph \( A_1 \) as shown in Figure [1] so \( \gamma(\mathcal{I}(G)) > 1 \), \( \overline{\gamma}(\mathcal{I}(G)) > 1 \) and \( \mathcal{I}(G) \) contains \( K_5 \). \( \square \)

4.2 Finite non-solvable groups

If \( G \) is a group and \( N \) is a normal subgroups of \( G \), then \( \mathcal{I}(G/N) \) is isomorphic (as a graph) to a subgraph of \( \mathcal{I}(G) \). It is well known that any non-solvable group has a simple
group as a sub-quotient and every simple group has a minimal simple group as a sub-quotient. Therefore, if we can show that the minimal simple groups have non-toroidal (non-projective-planar) intersection graphs, then the intersection graph of a non-solvable group is non-toroidal (resp., non-projective-planar).

Recall that $SL_m(n)$ is the group of $m \times m$ matrices having determinant 1, whose entries are lie in a field with $n$ elements and that $L_m(n) = SL_m(n)/H$, where $H = \{ kI | k^m = 1 \}$. For any prime $q > 3$, the Suzuki group is denoted by $Sz(2^q)$

**Lemma 4.1.** If $n > 2$, then $\gamma(\mathcal{I}(D_{4n})) > 1$, $\overline{\eta}(\mathcal{I}(D_{4n})) > 1$ and $\mathcal{I}(G)$ contains $K_5$.

**Proof.** Here $H_1 := \langle a \rangle$, $H_2 := \langle a^n, b \rangle$, $H_3 := \langle a^n, ba \rangle$, $H_4 := \langle a^n, ba^2 \rangle$, $H_5 := \langle a^n \rangle$, $H_6 := \langle a^{n+2}, b \rangle$, $H_7 := \langle a^{2n} \rangle$, $H_8 := \langle a^{2n}, ba \rangle$, $H_9 := \langle a^{2n}, ba^2 \rangle$, $H_{10} := \langle a^{2n} \rangle$ are proper subgroups of $D_{4n}$. Also $H_5$ is a subgroup of $H_i$, $i = 1, 2, 3, 4$; $H_{10}$ is a subgroup of $H_i$, $i = 6, 7, 8, 9$. It follows that $\mathcal{I}(D_{4n})$ has a subgraph $A_1$ as shown in Figure 11 so $\gamma(\mathcal{I}(D_{4n})) > 1$, $\overline{\eta}(\mathcal{I}(D_{4n})) > 1$.

**Proposition 4.13.** If $G$ is a finite non-solvable group, then $\gamma(\mathcal{I}(G)) > 1$, $\overline{\eta}(\mathcal{I}(G)) > 1$ and $\mathcal{I}(G)$ contains $K_5$.

**Proof.** The classification of minimal simple groups is given in [28 Corollary 1]. As mentioned above, to prove the result, it is enough to show that these minimal simple groups are non-toroidal, non-projective-planar and contains $K_5$. Now we investigate each of these groups. Her we will denote the image of a matrix $A$ in $L_m(n)$ by $\overline{A}$.

**Case 1:** $G \cong L_2(q^p)$, where $q = 2, 3$ and $p$ is any prime. If $p = 2$, then the only non-solvable group is $L_2(4)$. Also $L_2(4) \cong A_5$ (see [3]). $A_5$ has five copies of $A_4$ and any two $A_4$ in $A_5$ have non-trivial intersection, for otherwise $|A_4A_4| = 144$, which is not possible. Also $H_1 := \langle (12, 34) \rangle$, $H_2 := \langle (12, 34), (13, 24) \rangle$, $H_3 := \langle (12, 34), (12354) \rangle$, $H_4 := \langle (12, 34), (12453) \rangle$, $H_5 := \langle (12, 34), (345) \rangle$ are proper subgroups of $A_5$. Here $H_1$ is a subgroup of $H_i$, for every $i = 1, 2, 3, 4$. It follows that $\mathcal{I}(A_5)$ has a subgraph $B_1$ as in Figure 2 so $\gamma(\mathcal{I}(G)) > 1$ and $\overline{\eta}(\mathcal{I}(G)) > 1$.

If $p > 2$, then $L_2(q^p)$ contains a subgroup isomorphic to $(\mathbb{Z}_q)^p$, namely the subgroup of matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{F}_{q^p}$. By Proposition 4.12, $\gamma(\mathcal{I}((\mathbb{Z}_q)^p)) > 1$, and $\overline{\eta}(\mathcal{I}((\mathbb{Z}_q)^p)) > 1$, $p > 2$ and $\mathcal{I}((\mathbb{Z}_q)^p)$ contains $K_5$.

**Case 2:** $G \cong L_3(3)$. In $SL_3(3)$ the only matrix in the subgroup $H$ is the identity matrix, so $L_3(3) \cong SL_3(3)$. Let us consider the subgroup consisting of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in \mathbb{F}_3$. This subgroup is isomorphic to the group $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ with $p = 3$. By Proposition 4.11, $\gamma(\mathcal{I}((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)) > 1$, $\overline{\eta}(\mathcal{I}((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)) > 1$ and $\mathcal{I}((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$ contains $K_5$.

**Case 3:** $G \cong L_2(p)$, where $p$ is any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod{5}$. We have to consider two subcases:

**Subcase 3a:** $p \equiv 1 \pmod{4}$. It is shown in [6, p. 222] that $L_2(p)$ has a subgroup isomorphic to $D_{p-1}$. So by Lemma 4.11, $\gamma(\mathcal{I}(D_{p-1})) > 1$, $\overline{\eta}(\mathcal{I}(D_{p-1})) > 1$ and $\mathcal{I}(D_{p-1})$ contains $K_5$. 
If \( p = 5 \), then \( L_2(5) \cong A_5 \cong L_2(4) \). By Case 1, \( \gamma(I(A_5)) > 1 \), \( \overline{\gamma}(I(A_5)) > 1 \) and \( I(A_5) \) contains \( K_5 \).

**Subcase 3b:** \( p \equiv 3 \pmod{4} \). \( L_2(p) \) has a subgroup isomorphic to \( D_{p+1} \) \[\text{p. 222}\]. By Lemma 4.1, \( \gamma(I(D_{p+1})) > 1 \), \( \overline{\gamma}(I(D_{p+1})) > 1 \) and \( I(D_{p+1}) \) contains \( K_5 \). If \( p = 7 \), then \( S_4 \) is a maximal subgroup of \( L_2(7) \). Also by Theorem 4.3, \( \gamma(I(S_4)) > 1 \), \( \overline{\gamma}(I(S_4)) > 1 \) and \( I(S_4) \) contains \( K_5 \).

**Case 4:** \( G \cong Sz(2^r) \), where \( q \) is any odd prime. Then \( Sz(2^r) \) has a subgroup isomorphic to \( (\mathbb{Z}_2)^q \), \( q \geq 3 \) \[\text{p. 466}\]. But by Proposition 5.2, \( \gamma(I((\mathbb{Z}_2)^q)) > 1 \), \( \overline{\gamma}(I((\mathbb{Z}_2)^q)) > 1 \), \( q \geq 3 \) and \( I((\mathbb{Z}_2)^q) \) contains \( K_5 \).

Combining all these cases together, the proof follows.

\[ \square \]

## 5 Main results

By combining all the results obtained in Sections 3 and 4 above, we have the following general main results, which classifies the finite groups whose intersection graphs of subgroups are toroidal or projective-planar, and classifies the finite non-cyclic groups whose intersection graphs of subgroups are one of \( K_5 \) free, \( C_3 \)-free, acyclic or bipartite.

**Theorem 5.1.** Let \( G \) be a finite group and \( p, q, r \) be distinct primes. Then

1. \( I(G) \) is toroidal if and only if \( G \) is isomorphic to one of the following:
   - \( \mathbb{Z}_{p^r}(\alpha = 6, 7, 8), \mathbb{Z}_{p^r,q}(\alpha = 3, 4), \mathbb{Z}_{p^r,q^2}, \mathbb{Z}_{p^r,q^2} \times \mathbb{Z}_3, \mathbb{Z}_{25} \times \mathbb{Z}_5, \mathbb{Z}_{3q} \times \mathbb{Z}_3, M_{p^r}(p = 3, 5), M_{16}, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_9 \times \mathbb{Z}_4, \mathbb{Z}_9 \times \mathbb{Z}_2, \mathbb{Z}_{25} \times \mathbb{Z}_2; \)
   - \( \langle a, b, c \mid a^5 = b^5 = c^5 = 1, ab = ba, cac^{-1} = b, cbc^{-1} = ab \rangle \), where \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) has order 6 in \( GL_2(5) \).
2. \( I(G) \) is projective-planar if and only if \( G \) is isomorphic to one of \( \mathbb{Z}_{p^r}(\alpha = 6, 7), \mathbb{Z}_{p^r,q}, \mathbb{Z}_9 \times \mathbb{Z}_3, \mathbb{Z}_{3q} \times \mathbb{Z}_3, M_{27}, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_9 \times \mathbb{Z}_2 \).

**Theorem 5.2.** Let \( G \) be a finite non-cyclic group and \( p, q, r \) be distinct primes. Then

1. \( I(G) \) is \( K_5 \)-free if and only if \( G \) is isomorphic to one of the following groups:
   - \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_6 \times \mathbb{Z}_2, \mathbb{Q}_8, M_8, \mathbb{Z}_9 \times \mathbb{Z}_p, \mathbb{Z}_9 \times \mathbb{Z}_2 \mathbb{Z}_{p^2}, A_4; \)
   - \( \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^b \rangle \), where \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) has order \( q \) in \( GL_2(p) \), \( q|(p+1); \)
   - \( \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = b^{-1}, cbc^{-1} = a^b \rangle \), where \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) has order \( q^2 \) in \( GL_2(p) \), \( q^2|(p+1); \)
   - \( \langle a, b, c \mid a^p = b^q = c^r = 1, b^{-1}ab = a^q, c^{-1}ac = a^r, bc = cb \rangle \), where \( r, q \) are divisor of \( p \) and \( v, \mu \neq 1; \)
Corollary 5.1. Let $G$ be a finite group and $p, q, r$ are distinct primes. Then

1. $\mathcal{I}(G)$ is unicyclic if and only if $G$ is either $\mathbb{Z}_{p^4}$ or $\mathbb{Z}_{p^{2}q}$;

2. $\mathcal{I}(G) \cong C_n$ if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^4}$;

3. $\mathcal{I}(G) \cong P_n$ if and only if $n = 1$ and $G \cong \mathbb{Z}_{p^3}$;

4. $\mathcal{I}(G)$ is $C_5$-free if and only if $G$ is one of $\mathbb{Z}_{p^5}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{p^{2}q}(\alpha = 1, 2)$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{4} \times \mathbb{Z}_{2}$, $Q_8$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$, $A_4$, $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, b^c = a^p \rangle$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ has order $q$ in $GL_2(p)$, $q \mid (p + 1)$;

5. $\mathcal{I}(G)$ is $C_4$-free if and only if $G$ is one of $\mathbb{Z}_{p^4}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{p^{2}q}(\alpha = 1, 2)$, $Q_8$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p^2}$, $A_4$, $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, b^c = a^p \rangle$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ has order $q$ in $GL_2(p)$, $q \mid (p + 1)$;

6. $\mathcal{I}(G)$ is $P_4$-free if and only if $G$ is one of $\mathbb{Z}_{p^4}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{p^{2}q}(\alpha = 1, 2)$, $Q_8$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p^2}$, $A_4$, $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, b^c = a^p \rangle$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ has order $q$ in $GL_2(p)$, $q \mid (p + 1)$ or $A_4$;

7. $\mathcal{I}(G)$ is $P_3$-free if and only if $G$ is one of $\mathbb{Z}_{p^3}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p^2}$, $A_4$, $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, b^c = a^p \rangle$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ has order $q$ in $GL_2(p)$, $q \mid (p + 1)$;

8. $\mathcal{I}(G)$ is $P_2$-free if and only if $G$ is one of $\mathbb{Z}_{p^2}(\alpha = 2, 3)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$ or $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$;

9. $\mathcal{I}(G)$ is totally disconnected if and only if $G$ is isomorphic to one of $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$ or $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$;

10. $\mathcal{I}(G)$ is $K_{2,3}$-free if and only if $G$ is one of $\mathbb{Z}_{p^3}(\alpha = 2, 3, 4, 5)$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_{q}$, $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p}$, $\mathbb{Z}_{q} \times \mathbb{Z}_{p^2}$, $A_4$ or $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, b^c = a^p \rangle$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ has order $q$ in $GL_2(p)$, $q \mid (p + 1)$;
(11) \( \mathcal{I}(G) \) is \( K_4 \)-free if and only if \( G \) is one of \( \mathbb{Z}_{p^\alpha} (\alpha = 2, 3, 4), \mathbb{Z}_{p^\alpha q} (\alpha = 1, 2), \mathbb{Z}_{pq}, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}, A_4 \) or \( \langle a, b, c \mid a^p = b^q = c^a = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1b^1 \rangle \), where \( (0_{1 \times 1}) \) has order \( q \) in \( GL_2(p) \), \( q \mid (p + 1) \).

(12) \( \mathcal{I}(G) \) is \( K_{1,4} \)-free if and only if \( G \) is one of \( \mathbb{Z}_{p^\alpha} (\alpha = 2, 3, 4, 5), \mathbb{Z}_{p^\alpha q} (\alpha = 1, 2), \mathbb{Z}_p \times \mathbb{Z}_p, Q_8, \mathbb{Z}_q \rtimes \mathbb{Z}_p \) or \( A_4 \).

(13) \( \mathcal{I}(G) \) is claw-free if and only if \( G \) is one of \( \mathbb{Z}_{p^\alpha} (\alpha = 2, 3, 4), \mathbb{Z}_{pq}, \mathbb{Z}_p \times \mathbb{Z}_p \) or \( \mathbb{Z}_q \rtimes \mathbb{Z}_p \).

(14) The following are equivalent:

(a) \( G \cong \mathbb{Z}_{p^\alpha} (\alpha = 2, 3) \);

(b) \( \mathcal{I}(G) \) is a tree;

(c) \( \mathcal{I}(G) \) is a star graph;

(d) \( \mathcal{I}(G) \) is complete bipartite.

(15) \( \text{girth}(\mathcal{I}(G)) \) is \( \infty \), if \( G \) is one of \( \mathbb{Z}_{p^\alpha} (\alpha = 2, 3), \mathbb{Z}_{pq}, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_q \rtimes \mathbb{Z}_p, A_4 \) or \( \langle a, b, c \mid a^p = b^q = c^a = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^1b^1 \rangle \), where \( (0_{1 \times 1}) \) has order \( q \) in \( GL_2(p) \), \( q \mid (p + 1) \); otherwise \( \text{girth}(\mathcal{I}(G)) = 3 \).

Proof. Note that if the intersection graph of subgroups of a group has \( K_5 \) as a subgraph, then it is none of the following graphs: unicyclic, cycle, path, claw-free, \( C_5 \)-free, \( C_4 \)-free, \( P_3 \)-free, \( P_2 \)-free, totally disconnected, \( K_{2,3} \)-free, \( K_4 \)-free, \( K_{1,4} \)-free. So to classify the finite groups whose intersection graph of subgroups is one of outerplanar, unicyclic, claw-free, path, cycle, \( C_4 \)-free, \( C_3 \)-free, totally disconnected, \( P_2 \)-free, \( P_3 \)-free, \( P_4 \)-free, \( K_{1,4} \)-free, \( K_{2,3} \)-free, it is enough to consider the finite groups whose intersection graph of subgroups are \( K_5 \)-free. Thus, we need to investigate these properties only for groups given in Theorems \( 2.4(2) \) and \( 5.2(1) \).

By Theorem \( 2.4(2) \) and using \( (2.1) \), \( (2.3) \), the only groups \( G \) such that \( \mathcal{I}(G) \) is unicyclic are \( \mathbb{Z}_{p^4} \) and \( \mathbb{Z}_{p^2 q} \). By Theorem \( 5.2(1) \) and using \( (2.4) \), \( (2.11) \), Figure \( 3(b) \), \( 3(c) \), \( 3(d) \), Figure \( 10 \) there is no group \( G \) such that \( \mathcal{I}(G) \) is unicyclic. Thus, the proof of (1) follows.

Proof of parts (2) to (14) of this Corollary are similar to the proof of part (1).

If \( \mathcal{I}(G) \) contains \( C_3 \), then obviously \( \text{girth}(G) = 3 \). Now assume that \( \mathcal{I}(G) \) is \( C_3 \)-free. If \( G \) is cyclic, then by Theorem \( 2.4(4) \), \( \mathcal{I}(G) \) is acyclic, since \( \mathcal{I}(\mathbb{Z}_{p^\alpha}) \cong K_{p^\alpha-1} (\alpha = 2, 3) \) and \( \mathcal{I}(\mathbb{Z}_{pq}) \cong K_2 \). If \( G \) is non-cyclic, then by Theorem \( 5.2(2) \), \( \mathcal{I}(G) \) is also acyclic. So in both these cases \( \text{girth}(G) = \infty \). This completes the proof.

Remark 5.1. (i) In Theorem \( 5.1 \) we show that all the projective-planar intersection graphs of subgroups of groups are toroidal, which is not the case for arbitrary graphs (e.g., see pp. 367-368 and Figure 13.33 in \( [17] \)).
(ii) In [22], Selcuk Kayacan et al. showed that the intersection graph of subgroups of groups other than those listed in Theorem 2.5, the group $G_2$ given in Case 1b in the proof of Proposition 4.2 and the group of order $p^2qr$ given in part (2) of Proposition 4.6 contains $K_5$ as a subgraph. Also they showed that the intersection graph of subgroups of these two groups contains $K_{3,3}$ as a subgraph. In this paper, we proved by using another method that the intersection graph of subgroups of the first group contains $K_5$ as a subgraph, but not the second group. Thus it follows that the class of all groups having $K_5$-free intersection graph of subgroups properly contains the class of all groups having planar intersection graph of subgroups.

(iii) In [1] Akbari et al. classified all groups whose intersection graphs of subgroups are one of $C_3$-free, tree, totally disconnected. Also they obtained the girth of the intersection graph of groups. In this paper, we proved these results for finite groups in a different method.

In the next result we characterize some finite groups by using their intersection graph of subgroups.

Corollary 5.2. Let $G$ be a group and $p, q, r$ are distinct primes. Then

1. $\mathcal{I}(G) \cong \mathcal{I}(M_8)$ if and only if $G \cong M_8$.
2. $\mathcal{I}(G) \cong \mathcal{I}(\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2})$ if and only if $G \cong \mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2}$;
3. $\mathcal{I}(G) \cong \mathcal{I}(G_1)$ if and only if $G \cong G_1$, where $G_1$ is the group described in Theorem 2.5.
4. $\mathcal{I}(G) \cong \mathcal{I}(A_4)$ if and only if $G \cong A_4$;
5. $\mathcal{I}(G) \cong \mathcal{I}(G_2)$ if and only if $G \cong G_2$, where $G_2$ is the group described in Theorem 2.5.
6. $\mathcal{I}(G) \cong \mathcal{I}(G_3)$ if and only if $G \cong G_3$, where $G_3$ is the group described in Theorem 2.5.
7. $\mathcal{I}(G) \cong \mathcal{I}(G_1)$ if and only if $G \cong G_1$, where $G_1$ is the group of order $p^2qr$ described in Theorem 5.2(1).

Proof. (1)-(7): By Theorem 5.2(1), we can see that the groups mentioned in this Corollary are having $K_5$-free intersection graph of subgroups. Also by (2.4)-(2.11), Figure 3(b)-(d), Figure 10 the intersection graphs of subgroups of these groups are unique. Since an infinite group has infinite number of subgroups, so its intersection graph of subgroups can not be isomorphic to any of the intersection graphs of subgroups of these groups. If the intersection graph of a given finite group is isomorphic to any of these intersection graphs of subgroups of these groups, then by the uniqueness of these graphs, the given group must be isomorphic to that corresponding group. Proof of the result follows from this fact.

\qed
**Theorem 5.3.** Let $G$ be a group. Then $\theta(I(G)) = m$, where $m$ is the number of prime order subgroups of groups of $G$.

**Proof.** Let $G$ be finite and $|G| = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i$’s are distinct primes, and $\alpha_i \geq 1$. For each $i = 1, 2, \ldots, k$, let $t_i$ be the number of subgroups of order $p_i$ and let $H(j, p_i)$, $j = 1, 2, \ldots, t_i$ be a subgroup of $G$ of order $p_i$. For each $j = 1, 2, \ldots, t_i$, let $S(j, p_i)$ be the set of all subgroups of $G$ having $H(j, p_i)$ in common. Clearly $S(j, p_i)$ forms a clique in $I(G)$ and $S := \{S(j, p_i) \mid i = 1, 2, \ldots, k \text{ and } j = 1, 2, \ldots, t_i\}$ forms a clique cover of $I(G)$ with $|S| = t_1 + t_2 + \ldots + t_k$. Therefore, $\Theta(I(G)) \leq |S|$. Let $T$ be a clique cover of $I(G)$ such that $\Theta(I(G)) = |T|$. If $|T| < |S|$, then by pigeonhole principle, $T$ has a clique which contains at least two subgroups, say $H(j, p_i)$, $H(l, p_r)$, for some $i \neq r$, $j \in \{1, 2, \ldots, t_i\}$, $l \in \{1, 2, \ldots, t_r\}$, which is not possible, since $H(j, p_i)$ and $H(l, p_r)$ are not adjacent in $I(G)$. So $|T| = |S| = \text{the number of prime order subgroups of } G$. A similar argument also works when $G$ is infinite. □

Zelinka shown in [31] that for any group $G$, $\alpha((G)) = m$, where $m$ is the number of prime order proper subgroups of $G$. This fact together with Theorem 5.3 gives the following result.

**Corollary 5.3.** If $G$ is a group, then $I(G)$ is weakly $\alpha$-perfect.

**Acknowledgement**

The authors would like to thank Professor Ronald Solomon, Department of Mathematics, Ohio State University, Columbus, United States for his valuable suggestions to complete this work.

**References**

[1] S. Akbari, F. Heydari and M. Maghasedi, The intersection graph of group, *J. Algebra. Appl.*, 5 (2015) Article No. 1550065, 9 pp.

[2] S. Akbari, R. Nikandish and M. J. Nikmehr, Some results on the intersection graphs of ideals of rings, *J. Algebra. Appl.* 12 (2013) Article No. 1250200, 13 pp.

[3] S. Akbaria, H. A. Tavallaeeb and S. Khalashi Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.* 10 (2011) Article No. 1250019, 8 pp.

[4] Andrei Gagarin, Wendy Myrvold, John Chamber, The obstructions for toroidal graphs with no $K_{3,3}$’s, *Discrete Mathematics*. 309 (2009) 3625–3631.

[5] Atlas of Finite Group Representations, [http://web.mat.bham.ac.uk/atlas/v2.0/](http://web.mat.bham.ac.uk/atlas/v2.0/).
[6] J. P. Bohanon and Les Reid, Finite groups with planar subgroup lattices, *J. Algebraic Combin.* 23 (2006) 207–223.

[7] J. Bosak, The graphs of semigroups in: Theory of graphs and application, Academic Press, New York, 1964, pp. 119–125.

[8] W. Burnside, *Theory of Groups of Finite Order*, Dover Publications, Cambridge, 1955.

[9] I. S. Chakrabarty, T. K. Ghosh, Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* 309 (2009) 5381–5392.

[10] H. J. Chiang-Hsieh, Classification of rings with projective zero-divisor graphs, *J. Algebra* 319(7) (2008) 2789–2802.

[11] F. N. Cole and J.W. Glover. On groups whose orders are products of three prime factors, *American Journal of Mathematics*, 15 (1893) 191–220.

[12] B. Csákány and G. Pollák, The graph of subgroups of a finite group (Russian), *Czechoslovak Math. J.* 19 (1969) 241–247.

[13] H. Glover, J. P. Huneke and C. S. Wang, 103 graphs that are irreducible for the projective plane, *J. Combin. Theory Ser. B* 27 (1979) 332–370.

[14] D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

[15] Hadi Ahmadi and Bijan Taeri, Planarity of the intersection graph of subgroups of a finite group, *J. Algebra Appl.*, Accepted for publication. DOI: 10.1142/S0219498816500407.

[16] S. H. Jafari and N. Jafari Rad, Planarity of intersection graphs of ideals of rings, *Int. Electron J. Algebra.* 8 (2010) 161–166.

[17] W. Kocay, D. Kreher, *Graphs, Algorithms, and Optimization*, CRC Press, Boca Raton, 2005.

[18] J. D. Laison and Y. Qing, Subspace intersection graphs, *Discrete Math.* 310 (2010) 3413–3416.

[19] H. L. Lin, On groups of order $p^2q, p^3q^2$, *Tamkang J. Math.* (1974) 167–190.

[20] T. A. McKee and F. R. McMorris, *Topics in Intersection Graph Theory*, SIAM Monographs on Discrete Mathematics and Application, SIAM, USA, 1999.

[21] E. Neufeld and W. Myrvold, Practical toroidality testing in: 8th Annual ACM-SIAM symposium on discrete algorithms, (SODA) 1997, pp. 574–580.
Intersection graph of subgroups

[22] Selçuk Kayacan and Ergun Yaraneri, Finite groups whose intersection graphs are planar, J. Korean Math. Soc. 52 (2015) 81–96.

[23] Rulin Shen, Intersection graphs of subgroups of finite groups, Czechoslovak Math J. 60 (2010) 945–950.

[24] Z.S. Pucanovic, Z. Z. Petrovi, Toroidality of intersection graphs of ideals of commutative rings, Graphs and Combinatorics, 30 (2013) 707–716.

[25] Z.S. Pucanovic, Marko Radovanovic and Aleksandra Lj. Eric, On the genus of the intersection graph of ideals of a commutative ring, J. Algebra Appl. 13 (2014) Article No. 1350155, 20 pp.

[26] R. Rajkumar and P. Devi and Andrei Gagarin, Classification of finite groups with toroidal or projective-planar permutability graphs, 2015. [arXiv:1505.03462]

[27] W. R. Scott, Group Theory, Dover, New York, 1964.

[28] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable I, Bull. Amer. Math. Soc. (1968) 383–437.

[29] A. T. White, Graphs, Groups and Surfaces, North-Holland Mathematics Studies, no. 8. American Elsevier Publishing Co. Inc., New York, 1973.

[30] E. Yaraneri, Intersection graph of a module, J. Algebra Appl. 12 (2013) Article No. 1250218, 30 pp.

[31] B. Zelinka, Intersection graphs of finite abelian groups, Czech. Math. J. 25 (1975) 171–174.