QUADRATIC ESTIMATES FOR PERTURBED DIRAC TYPE OPERATORS ON DOUBLING MEASURE METRIC SPACES

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Abstract. We consider perturbations of Dirac type operators on complete, connected metric spaces equipped with a doubling measure. Under a suitable set of assumptions, we prove quadratic estimates for such operators and hence deduce that these operators have a bounded functional calculus. In particular, we deduce a Kato square root type estimate.

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1. Introduction

Let $\mathcal{X}$ be a complete, connected metric space and $\mu$ a Borel-regular doubling measure. We consider densely-defined, closed, nilpotent operators $\Gamma$ on $L^2(\mathcal{X}, \mathbb{C}^N)$ and perturbed Dirac type operators $\Pi_B = \Gamma + B_1\Gamma^*B_2$, where $B_i$ are strictly accretive $L^\infty$ matrix valued functions. We prove quadratic estimates

$$\int_0^\infty \|t\Pi_B(1 + t^2\Pi_B)^{-1}u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for $u \in \mathcal{R}(\Pi_B)$ under a set of hypotheses (H1)-(H8). These estimates are equivalent to $\Pi_B$ having a bounded holomorphic functional calculus. This allows us to conclude that $\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1\Gamma^*B_2)$ and that $\|\sqrt{\Pi_B^2}u\| \simeq \|\Pi_Bu\| \simeq \|\Gamma u\| + \|B_1\Gamma^*B_2u\|$. When $\mathcal{X} = \mathbb{R}^n$ and $\mu$ is the Lebesgue measure, it is shown by Axelsson, Keith and McIntosh in [5]

2010 Mathematics Subject Classification. 47B44, 42B35, 42B37.
Key words and phrases. Quadratic estimates, holomorphic functional calculi, doubling measure, measure metric space, Dirac type operators, Kato square root problem, Carleson measure, maximal function.
that this implies $D(\sqrt{-\text{div} A \nabla}) = D(\nabla)$ and $\|\sqrt{-\text{div} A \nabla u}\| \simeq \|\nabla u\|$ for an appropriate class of perturbations $A$. Thus, we are justified in calling this a Kato square root type estimate.

We proceed to prove our theorem based on the ideas presented in [5]. These ideas date back to the resolution of the Kato conjecture by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [2]. The exposition [10] by Hofmann is an excellent survey of the history and resolution of the Kato conjecture. Further historical references include the article [13] by McIntosh and [3] by Auscher and Tchamitchian. More recently, the proof in [5] was generalised by Morris in [15] for complete Riemannian manifolds with exponential volume growth. This work is beneficial to us since we rely upon the same abstract dyadic decomposition of Christ in [7].

The main novelty of the work presented here is that we have separated the assumptions on the operator $\Gamma$ from the underlying differentiable structure of the space. In general, the spaces we consider may not admit a differentiable structure. However, we are motivated by the existence of measure metric spaces more general than Riemannian manifolds admitting such structures. See the work of Cheeger in [6] and of Keith in [12].

In our exposition, we follow the structure of the proof in [5]. We rephrase the proof purely in terms of Lipschitz functions. We use an upper gradient quantity, namely the pointwise Lipschitz constant, as a replacement for a gradient. This is the key feature that allows us to generalise the proof in [5].

The structure of this paper is as follows. In §2, we state the hypotheses (H1)-(H8) under which we obtain the quadratic estimates and state the main results. We devote §3 to illustrating some important consequences of the dyadic decomposition in [7]. In §4, we present some results about Carleson measures and maximal functions on doubling measure metric spaces. These tools are crucial since the proof of the main result proceeds by reducing the main estimate to a Carleson measure estimate. Lastly, we give a proof of the main theorem in §5, taking care to avoid unnecessary repetition of the work of [5] and [15], and highlight the key differences which we have introduced.

Acknowledgements

This work was undertaken at the Centre for Mathematics and its Applications at the Australian National University and supported by this institution and an Australian Postgraduate Award.

I am indebted to Alan McIntosh for his support, insight, and excellent supervision which made this work possible. I would also like to thank Andrew Morris, Rob Taggart, and Pierre Portal for their encouragement and helpful suggestions.
We list a set of hypotheses (H1)-(H8). These assumptions are similar those in [5], with the exception of (H6) and (H8) which require modification due to the lack of a differentiable structure in our setting. The assumptions (H1)-(H3) are purely operator theoretic and thus hold in sufficient generality. They are taken in verbatim from [5] but we list them here for completeness. We emphasise that here, $\mathcal{H}$ denotes an abstract Hilbert space.

1. The operator $\Gamma : \mathcal{D}(\Gamma) \to \mathcal{H}$ is closed, densely-defined and nilpotent ($\Gamma^2 = 0$).
2. The operators $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ satisfy
   \[ \text{Re} \langle B_1u, u \rangle \geq \kappa_1 \|u\| \quad \text{whenever } u \in \mathcal{R}(\Gamma^*), \]
   \[ \text{Re} \langle B_2u, u \rangle \geq \kappa_2 \|u\| \quad \text{whenever } u \in \mathcal{R}(\Gamma) \]
   where $\kappa_1, \kappa_2 > 0$ are constants.
3. The operators $B_1, B_2$ satisfy
   \[ B_1B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma) \text{ and } B_2B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*). \]

The full implications of these assumptions are listed in §4 in [5]. However, for the sake of convenience, we include some relevant details from this reference. Define $\Gamma^* B = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma^* B$, and $\Pi = \Gamma + \Gamma^*$. Furthermore, define the following associated bounded operators:

\[ R^B_t = (1 + it\Pi_B)^{-1}, \quad P^B_t = (1 + t^2\Pi_B^2)^{-1}, \quad Q^B_t = t\Pi_B(1 + t^2\Pi_B^2)^{-1}, \]

\[ \Theta^B_t = t\Gamma^* B(1 + t^2\Pi_B^2)^{-1}, \]

and write $R_t, P_t, Q_t, \Theta_t$ by setting $B_1 = B_2 = 1$. With this in mind, we bring the attention of the reader to the following important proposition.

**Proposition 2.1** (Proposition 4.8 of [5]). Suppose that $(\Gamma, B_1, B_2)$ satisfy the hypotheses (H1)-(H3) and that there exists $c > 0$ such that

\[ \int_0^\infty \|\Theta^B_t P_t u\|^2 \frac{dt}{t} \leq c \|u\|^2 \]

for all $u \in \mathcal{R}(\Gamma)$, together with three similar estimates obtained by replacing $(\Gamma, B_1, B_2)$ by $(\Gamma^*, B_2, B_1)$, $(\Gamma^*, B_2^*, B_1^*)$ and $(\Gamma, B_1^*, B_2^*)$. Then, $\Pi_B$ satisfies

\[ \int_0^\infty \|Q^B_t u\|^2 \frac{dt}{t} \simeq \|u\|^2 \]

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathcal{H}$. Thus, $\Pi_B$ has a bounded $H^\infty$ functional calculus.

For a fuller treatment of the theory of sectorial operators and holomorphic functional calculi, see [1] by Albrecht, Duong and McIntosh, and [11] by Kato. Furthermore, Morris deals with local quadratic estimates and their functional calculus implications in [14].
It is the conclusion of the above proposition that is our primary objective. We note as do the authors of [5] that we require additional assumptions on $\mathcal{X}$ and $(\Gamma, B_1, B_2)$ in order to satisfy the hypothesis of the proposition. Thus, we start with the following definition.

**Definition 2.2 (Doubling measure).** We say that $\mu$ is a doubling measure on $\mathcal{X}$ if there exists a constant $C_D \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) < \infty.$$

We call $C_D$ the doubling constant and we let $p = \log_2(C_D)$.

It is, in fact, easy to show that a measure is doubling if and only if

$$\mu(B(x, \kappa r)) \leq C_D \kappa^p \mu(B(x, r))$$

whenever $\kappa > 1$.

We are now in a position to list (H4) and (H5).

(H4) Let $\mathcal{X}$ be a complete, connected metric space and $\mu$ a Borel-regular measure on $\mathcal{X}$ that is doubling. Then set $\mathcal{H} = L^2(\mathcal{X}, C^N; d\mu)$.

(H5) $B_i \in L^\infty(\mathcal{X}, L(C^N))$ for $i = 1, 2$.

For convenience, we write $\mathcal{H} = L^2(\mathcal{X})$ or $L^2(\mathcal{X}, C^N)$.

Note that the two hypotheses above are the obvious adaptations of (H4) and (H5) in [5]. The matter of (H6) is a little more complicated since (H6) of [5] and [15] involves $\nabla$ which in general does not exist for us. To circumvent this obstacle, we define the following quantity.

**Definition 2.3 (Pointwise Lipschitz constant).** For $\xi : \mathcal{X} \to C^N$ Lipschitz, define Lip$_\xi : \mathcal{X} \to \mathbb{R}$ by

$$\text{Lip}_\xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x, y)}.$$

We take the convention that $\text{Lip}_\xi(x) = 0$ when $x$ is an isolated point.

Letting Lip$_\xi$ denote the Lipschitz constant of $\xi$, we note that by construction, Lip$_\xi(x) \leq \text{Lip}_\xi$ for all $x \in \mathcal{X}$. Also, Lip$_\xi$ is a Borel function and therefore measurable. Many of the properties of Lip$_\xi$ are described in greater detail in [6]. We note that it is from this reference that we have borrowed this notation and the term “pointwise Lipschitz constant.”

(H6) For every bounded Lipschitz function $\xi : \mathcal{X} \to \mathbb{C}$, multiplication by $\xi$ preserves $\mathcal{D}(\Gamma)$ and $M_\xi = [\Gamma, \xi I]$ is a multiplication operator. Furthermore, there exists a constant $m > 0$ such that $|M_\xi(x)| \leq m |\text{Lip}_\xi(x)|$ for almost all $x \in \mathcal{X}$.

We note that this implies the same hypothesis when $\Gamma$ is replaced by $\Gamma^*$ and $\Pi$. This observation is made in [15] and originated in [4].
When $\mathcal{X} = \mathbb{R}^n$ and $\mu$ is the Lebesgue measure (the setting in [5]), our (H6) is automatically satisfied since $|\nabla \xi(x)| = |\text{Lip} \xi(x)|$ for almost all $x \in \mathbb{R}^n$.

The following is called the cancellation hypothesis. In the work of [15] and [4], this hypothesis is replaced by a weaker estimate which is applicable for local quadratic estimates as described by Morris in [14]. The estimates we require are global and thus we assume the cancellation hypothesis in [5]. We denote the support of a function $f$ by $\text{spt} f$.

(H7) For each open ball $B$, we have
\[
\int_B \Gamma u \, d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \, d\mu = 0
\]
for all $u \in \mathcal{D}(\Gamma)$ with $\text{spt} u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\text{spt} v \subset B$.

The last assumption is a Poincaré hypothesis. In [15], a Poincaré inequality on balls is assumed as a separate hypothesis. Their (H8) is a coercivity assumption following [5]. In our work, we find that a Poincaré type hypothesis with respect to the unperturbed operator $\Pi$ is a sensible substitution.

(H8) There exists $C' > 0$ and $c > 0$ such that for all balls $B = B(y,r)$
\[
\int_B |u(x) - u_B|^2 \, d\mu(x) \leq C' r^2 \int_{cB} |\Pi u(x)|^2 \, d\mu(x)
\]
for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$.

The authors of [5] reveal that (H1)-(H3) are adequate to set up the necessary operator theoretic framework. However, as we have noted before, the full set of assumptions (H1)-(H8) are necessary to obtain the desired estimates. It is under these assumptions that we present the main theorem of this paper.

**Theorem 2.4.** Let $\mathcal{X}$, $(\Gamma, B_1, B_2)$ satisfy (H1)-(H8). Then, $\Pi_B$ satisfies the quadratic estimate
\[
\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2
\]
for all $u \in \mathcal{R}(\Pi_B) \subset L^2(\mathcal{X}, \mathbb{C}^N)$ and hence has a bounded $H^\infty$ functional calculus.

Let $E_B^\pm = \chi^\pm(\Pi_B)$, where $\chi^+(\zeta) = 1$ when $\Re(\zeta) > 0$ and 0 otherwise, and similarly, $\chi^-(\zeta) = 1$ when $\Re(\zeta) < 0$ and 0 otherwise. We have the following corollary resembling Corollary 2.11 in [5].

**Corollary 2.5** (Kato square root type estimate).

(i) There is a spectral decomposition
\[
L^2(\mathcal{X}, \mathbb{C}^N) = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-
\]
(where the sum is in general non-orthogonal), and
(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^\ast) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$ with

$$||\Gamma u|| + ||\Pi_B u|| \simeq ||\Pi_B u|| \simeq \left\| \Pi_B u \right\|$$

for all $u \in \mathcal{D}(\Pi_B)$.

3. Abstract dyadic decomposition

We begin this section by quoting Theorem 11 in [7].

**Theorem 3.1.** There exists a countable collection of open subsets

$$\left\{ Q_k^\alpha : k \in \mathbb{Z}, \alpha \in I_k \right\}$$

with each $z_k^\alpha \in Q_k^\alpha$, where $I_k$ are index sets (possibly finite), and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ and $C_1, C_2 < \infty$ satisfying:

(i) For all $k \in \mathbb{Z}$, $\mu(\mathcal{X} \setminus \bigcup_{\alpha} Q_k^\alpha) = 0$,

(ii) If $l \geq k$, either $Q_l^\beta \subset Q_k^\alpha$ or $Q_l^\beta \cap Q_k^\alpha = \emptyset$,

(iii) For each $(k, \alpha)$ and each $l < k$ there exists a unique $\beta$ such that $Q_k^\alpha \subset Q_l^\beta$,

(iv) $\text{diam} Q_k^\alpha \leq C_1 \delta^k$, 

(v) $B(z_k^\alpha, a_0 \delta^k) \subset Q_k^\alpha$, 

(vi) For all $k, \alpha$ and for all $t > 0$, $\mu \left\{ x \in Q_k^\alpha : d(x, \mathcal{X} \setminus Q_k^\alpha) \leq t \delta^k \right\} \leq C_2 t^\eta \mu(Q_k^\alpha)$.

Define $\mathcal{Q}^k = \{ Q_k^\alpha : \alpha \in I_k \}$ to be the level $k$ dyadic cubes and $\mathcal{Q} = \bigcup_k \mathcal{Q}^k$ to be the collection of dyadic cubes. For $Q_k^\alpha \in \mathcal{Q}^k$, define the length as $\ell(Q_k^\alpha) = \delta^k$ and the centre as $z_k^\alpha$.

It is easy to see that each $\mathcal{Q}^k$ is a mutually disjoint collection. Furthermore, we have $\partial(\bigcup \mathcal{Q}^k) = \bigcup_{Q \in \mathcal{Q}^k} \partial Q$. These facts coupled with the assumption $\mu(B(x, r)) > 0$ implies that $\mathcal{X} = \bigcup \mathcal{Q}^k$.

Fix a cube $Q \in \mathcal{Q}^j$ and denote the centre of this cube by $z$. We are interested in counting the number of cubes inside “shells” centred from this cube. We begin with the following definition.

**Definition 3.2.** Whenever $k \geq 1$, define

$$\mathcal{C}_k = \left\{ Q_k^\alpha : (k - 1)C_1 \delta^j \leq d(z, z_k^\alpha) \leq kC_1 \delta^j \right\} .$$

Also, let $\tilde{\mathcal{C}}_k = \left\{ Q_k^\alpha : d(z, z_k^\alpha) \leq kC_1 \delta^j \right\}$.

It is easy to see that $\mathcal{Q}^j = \bigcup_{k \geq 1} \mathcal{C}_k$. We compute a bound for $\text{card} \mathcal{C}_k$ (where $\text{card} S$ denotes the cardinality of a set $S$). First, we have the following proposition describing the distance of points in $\bigcup \mathcal{C}_k$ to $z$. 
Proposition 3.3. Let $Q^j_\alpha \in \mathcal{C}_k$. Then,

(i) $0 \leq d(z, x) \leq (k + 1)C_1 \delta^j$ for all $x \in Q^j_\alpha$ when $k \leq 2$, and

(ii) $\frac{1}{3}kC_1 \delta^j \leq d(z, x) \leq (k + 1)C_1 \delta^j$ for all $x \in Q^j_\alpha$ when $k \geq 3$.

Proof. Fix $Q^j_\alpha \in \mathcal{C}_k$ and fix $x \in Q^j_\alpha$. Then,

\[d(x, z) \leq d(x, z^j_\alpha) + d(z^j_\alpha, z) \leq \text{diam } Q^j_\alpha + kC_1 \delta^j \leq (k + 1)C_1 \delta^j.\]

Also,

\[(k - 1)C_1 \delta^j \leq d(z, z^j_\alpha) \leq d(x, z) + d(x, z^j_\alpha) \leq d(x, z) + C_1 \delta^j.\]

Combining these two estimates we have

\[(k - 2)C_1 \delta^j \leq d(z, x) \leq (k + 1)C_1 \delta^j.\]

This gives us (i). To obtain (ii), note that whenever $k \geq 3$ we have $\frac{1}{3}k \leq k - 2$. \qed

Next, we compare two balls which are separated by an arbitrary distance. In the following proposition (and indeed the rest of the paper), let us fix $p = \log_2(C_D)$, where $C_D$ is the doubling constant.

Proposition 3.4. Fix balls $B(x, r), B(y, r) \subset X$. Then, for all $\varepsilon > 0$,

\[2^{-p} \left( \frac{d(x, y) + r + \varepsilon}{r} \right)^{-p} \mu(B(y, r)) \leq \mu(B(x, r)) \leq 2^p \left( \frac{d(x, y) + r + \varepsilon}{r} \right)^p \mu(B(y, r)).\]

Proof. Fix $\varepsilon > 0$ and note that

\[B(x, r), B(y, r) \subset B(x, d(x, y) + r + \varepsilon), B(y, d(x, y) + r + \varepsilon).\]

Therefore,

\[\mu(B(y, r)) \leq \mu\left( B\left( x, \frac{d(x, y) + r + \varepsilon}{r} \right) \right) \leq 2^p \left( \frac{d(x, y) + r + \varepsilon}{r} \right)^p \mu(B(x, r)).\]

Similarly, we have

\[\mu(B(x, r)) \leq \mu\left( B\left( y, \frac{d(x, y) + r + \varepsilon}{r} \right) \right) \leq 2^p \left( \frac{d(x, y) + r + \varepsilon}{r} \right)^p \mu(B(y, r)).\]

which establishes the claim. \qed
We make a parenthetical remark that our assumption $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$ is not strong since by the previous proposition, coupled with the doubling property, allow us to recover this assumption if we only required $0 < \mu(B(x_0, r_0)) < \infty$ to hold for some $x_0 \in X$ and $r_0 > 0$.

We now return back to the problem of estimating $\text{card } \mathcal{C}_k$. The reader will observe that we have been generous in our calculations.

**Proposition 3.5.** We have $\text{card } \tilde{\mathcal{C}}_k \leq C k^{2p}$ where

$$C = 4^p \left( \frac{C_1 + 2a_0}{a_0} \right)^p \left( \frac{2C_1}{a_0} \right)^p.$$

In particular, $\text{card } \mathcal{C}_k \leq C k^{2p}$.

**Proof.** Fix $k \geq 1$. Set $\varepsilon = r = a_0 \delta^j$ and then

$$d(z, z^j_k) + r + \varepsilon \leq kC_1 \delta^j + 2a \delta^j \leq (C_1 + 2a \delta^j) \delta^j k$$

when $Q^l_j \in \tilde{\mathcal{C}}_k$. By Proposition 3.4,

$$2^{-p} \left( \frac{C_1 + 2a_0}{a_0} \right)^{-p} k^{-p} \mu(B(z, a_0 \delta^j)) \leq \mu(B(z^k, a_0 \delta^j)).$$

Now, note that by Proposition 3.3, we have $\sup_{x \in Q^l_j} d(x, z) \leq (k + 1)C_1 \delta^j$ and so $\bigcup \tilde{\mathcal{C}}_k \subset B(z, (k + 1)C_1 \delta^j)$. Then,

$$\mu(B(z, (k + 1)C_1 \delta^j)) \leq 2^p \left( \frac{(k + 1)C_1}{a_0} \right)^p \mu(B(z, a_0 \delta^j)) \leq 2^p \left( \frac{2C_1}{a_0} \right)^p k^p \mu(B(z, a_0 \delta^j)).$$

Since $\mu(B(z, a_0 \delta^j)) < \infty$ and by combining the two estimates, and the fact that $B(z^k, a_0 \delta^j) \subset Q^l_j$ for each $Q^l_j \in \tilde{\mathcal{C}}_k$, we compute

$$\text{card } \mathcal{C}_k \leq 2^p \left( \frac{2C_1}{a_0} \right)^p k^p 2^p \left( \frac{C_1 + 2a_0}{a_0} \right)^p k^p = 4^p \left( \frac{C_1 + 2a_0}{a_0} \right)^p \left( \frac{2C_1}{a_0} \right)^p k^p.$$

The observation that $\mathcal{C}_k \subset \tilde{\mathcal{C}}_k$ completes the proof. \qed

We have the following important consequences. They are useful in many of the calculations in \S 5. Following the notation in [5], we write $\langle x \rangle = 1 + |x|$.

**Corollary 3.6.** Fix $\delta^{j+1} < t \leq \delta^j$ and a cube $Q \in 2^j$. Then,

$$\sum_{R \in 2^j} \left\langle \frac{\text{dist}(R, Q)}{t} \right\rangle^{-M} \leq C \left( 1 + 4^p + \left( \frac{3}{C_1} \right)^M \sum_{k=5}^{\infty} k^{2p-M} \right)$$

with $C$ being the constant in the previous proposition.
Proof. First, we note that
\[ 1 \leq 1 + \frac{\text{dist}(R, Q)}{t} \quad \text{and} \quad \frac{\text{dist}(R, Q)}{\delta_j} \leq 1 + \frac{\text{dist}(R, Q)}{t}. \]

Then,
\[
\sum_{R \in \mathcal{Q}_j} \left\langle \frac{\text{dist}(R, Q)}{t} \right\rangle^{-M} \leq \text{card } \mathcal{C}_1 + \text{card } \mathcal{C}_2 + \sum_{k=3}^{\infty} \sum_{R \in \mathcal{C}_k} \left( \frac{\delta_j}{d(R, Q)} \right)^M \\
\leq C + C2^{2p} + \sum_{k=3}^{\infty} \text{card } \mathcal{C}_k \left( \frac{\delta_j}{3kC_1\delta_j} \right)^M \\
\leq C \left( 1 + 4^p + \left( \frac{3}{C_1} \right)^M \sum_{k=3}^{\infty} k2^{p-M} \right).
\]

\[ \square \]

Corollary 3.7. For each \( M > 2p + 1 \), there exists a constant \( A_M > 0 \) such that
\[
\sup_Q \sum_{R \in \mathcal{Q}_j} \left\langle \frac{\text{dist}(R, Q)}{t} \right\rangle^{-M} \leq A_M.
\]

4. Maximal functions and Carleson Measures

A full treatment of the classical theory of maximal functions and Carleson measures can be found in §4 of [16] by Stein. The objects of interest that we define in this section are taken from this book \textit{mutatis mutandis}. Furthermore, we refer the reader to [9] by Heinonen and [8] by Coifman and Weiss as two excellent expositions that touch on some of the issues and ideas presented here.

For a measurable subset \( S \) with \( 0 < \mu(S) < \infty \) and \( f \in L^1_{\text{loc}}(\mathcal{X}, \mathbb{C}^N) \), we define the \textit{average} of \( f \) on \( S \) by \( \bar{f}_S f = \mu(S)^{-1} \int_S f \). Then, we make the following definition.

Definition 4.1 (Maximal function). Let \( f \in L^1_{\text{loc}}(\mathcal{X}, \mathbb{C}^N) \). Define the \textit{un-centred maximal function} of \( f \) by:
\[
\mathcal{M}f(x) = \sup_{B \ni x} \int_B |f| \, d\mu
\]
where the supremum is taken over all balls \( B \) containing \( x \).

We want to deduce that this \( \mathcal{M} \) exhibits a weak type \((1,1)\) estimate and is bounded in \( L^p(\mathcal{X}, \mathbb{C}^N) \) for \( p > 1 \). The proof of the following theorem is standard via the Vitali type covering Theorem 1.2 in [8].

Theorem 4.2 (Maximal theorem). There exists a constant \( C_1 > 0 \) such that whenever \( f \in L^1(\mathcal{X}, \mathbb{C}^N) \), we have
\[
\mu(\{x \in \mathcal{X} : \mathcal{M}f(x) > \alpha\}) \leq \frac{C_1}{\alpha} \int_{\mathcal{X}} |f| \, d\mu.
\]
Whenever \( f \in L^q(\mathcal{X}, \mathbb{C}^N) \) with \( q > 1 \),

\[
\|Mf\|_q \leq C_q \|f\|_q
\]

where \( C_q > 0 \) is a constant.

In order to set up a theory of Carleson measures, we require an upper half space. We define this to be \( \mathcal{X}_+ = \mathcal{X} \times \mathbb{R}^+ \) where \( \mathbb{R}^+ = (0, \infty) \). The cone over a point \( x \in \mathcal{X} \) is then defined as \( \Gamma(x) = \{(y, t) \in \mathcal{X}_+ : d(x, y) < t\} \) and this leads to the following.

**Definition 4.3** (Nontangential maximal function). Let \( f \in L^1_{loc}(\mathcal{X}_+, \mathbb{C}^N) \). Define

\[
\mathcal{M}^* f(x) = \sup_{(y,t) \in \Gamma(x)} |f(y,t)|.
\]

Like its classical counterpart, this maximal function is measurable. This is the content of the following proposition.

**Proposition 4.4.** The set \( \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\} \) is open and hence \( \mathcal{M}^* f \) is measurable.

**Proof.** Fix \( x \in \mathcal{X} \) with \( \mathcal{M}^* f(x) > \alpha \). Then, there exists a \( (y, t) \in \Gamma(x) \) such that \( |f(y,t)| > \alpha \). Consider the ball \( B(y,t) \) and take any \( z \in B(y,t) \). Note that since \( d(z,y) < t \) we have \( (y,t) \in \Gamma(z) \) and so \( \mathcal{M}^* f(z) > \alpha \). Therefore, \( x \in B(y,t) \subset \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\} \). \( \square \)

Therefore, we define the following function space in an analogous way to the classical theory.

**Definition 4.5** (Nontangential function space). Let \( \mathcal{N} \) denote the space of Borel measurable functions \( f : \mathcal{X}_+ \to \mathbb{C} \) such that \( \mathcal{M}^* f \in L^1(\mathcal{X}) \). We equip this space with the norm \( \|f\|_{\mathcal{N}} = \|\mathcal{M}^* f\|_1 \).

Now, let \( B = B(x, r) \) and define the tent over \( B \) as

\[
T(B) = \{(y,t) \in \mathcal{X}_+ : d(x, y) \leq r - t\}.
\]

For an arbitrary open set \( O \subset \mathcal{X} \), we define the tent over \( O \) by \( T(O) = \mathcal{X}_+ \setminus \cup_{x \in \mathcal{X} \setminus \Gamma(y)} \Gamma(x) \). The following is an equivalent characterisation of \( T(O) \).

**Proposition 4.6.** Whenever \((x, t) \in T(O)\) we have that

\[
(x, t) \in T(B(x, d(x, \mathcal{X} \setminus O)))
\]

and in particular, \( T(O) = \cup_{x \in O} T(B(x, d(x, \mathcal{X} \setminus O))) \).

**Proof.** First, note that by de Morgen’s law, we can conclude that \( T(O) = \cap_{y \in \mathcal{X} \setminus O} \mathcal{X}_+ \setminus \Gamma(y) \). Fix \((x, t) \in T(O) \). So, \((x, t) \in \mathcal{X}_+ \setminus \Gamma(y)\) for all \( y \in \mathcal{X} \setminus O \). That is, for all \( y \notin O \), we have \((x, t) \notin \Gamma(y)\) which implies \( d(x, y) \geq t \). Therefore, \( d(x, \mathcal{X} \setminus O) \geq t \). Then, by the definition of \( T(B(x, r)) \) and setting \( r = d(x, \mathcal{X} \setminus O) \), we conclude \((x, t) \in T(B(x, d(x, \mathcal{X} \setminus O)))\). The converse inclusion is easy since \( B(x, d(x, \mathcal{X} \setminus O)) \subset O \). \( \square \)
Definition 4.7 (Carleson function). Let \( \nu \) be any Borel measure on \( X_+ \). Define
\[
C(\nu)(x) = \sup_{B \ni x} \frac{\nu(T(B))}{\mu(B)}.
\]

Definition 4.8 (Space of Carleson measures). We define \( \mathcal{C} \) to be the space of measures \( \nu \) that are Borel on \( X_+ \) and such that \( C(\nu) \) is bounded. Such a measure is called a Carleson measure and we define
\[
\|\nu\|_\mathcal{C} = \sup_{x \in X} C(\nu)(x)
\]
to be the Carleson norm.

Since we have a dyadic structure, we define the Carleson box over \( Q \in \mathcal{D} \) by \( R_Q = Q \times (0, \ell(Q)] \). Unlike the classical definition, we are forced to take \( Q \) since \( \mathcal{D} \) is only guaranteed to cover \( X \) almost everywhere. The importance of this subtlety will become apparent in the proof of the following proposition that provides an alternative characterisation of a Carleson measure.

Proposition 4.9. Let \( \nu \) be a Borel measure on \( X_+ \). Then the statement
\[
\sup_B \frac{\nu(T(B))}{\mu(B)} < \infty \quad \text{for every ball } B
\]
is equivalent to the statement
\[
\sup_Q \frac{\nu(R_Q)}{\mu(Q)} < \infty \quad \text{for every } Q \in \mathcal{D}.
\]

Proof. First, fix \( Q \in \mathcal{D} \) and let \( x_Q \) be its centre. Then, we have that \( Q \subset B(x_Q, C_1 \delta^j) \). Then, certainly, \( R_Q \subset T(B(x_Q, (C_1 + 2)\delta^j)) \). So,
\[
\nu(R_Q) \leq \nu(T(B(x_Q, (C_1 + 2)\delta^j))) \leq \|\nu\|_\mathcal{C} \mu(B(x_Q, (C_1 + 2)\delta^j)) \leq 2^p \left( \frac{C_1 + 2}{a_0} \right)^p \|\nu\|_\mathcal{C} \mu(B(x_Q, a_0\delta^j)) \leq 2^p \left( \frac{C_1 + 2}{a_0} \right)^p \|\nu\|_\mathcal{C} \mu(Q).
\]
The converse is harder. Fix \( B = B(x, r) \) and let \( j \in \mathbb{Z} \) such that \( \delta^{j+1} < r \leq \delta^j \). Let \( N(B) = \{ Q \in \mathcal{D} : Q \cap B \neq \emptyset \} \). It is an easy fact that \( N(B) \neq \emptyset \).

(i) First, we claim that \( B \subset \bigcup_{Q \in N(B)} \overline{Q} \). Suppose \( y \in B \) but \( y \notin \bigcup N(B) \). That is, \( y \not\in Q \) for all \( Q \in \mathcal{D} \). Thus, there exists a \( Q \in \mathcal{D} \) such that \( y \in \partial Q \). That is, for every \( \varepsilon > 0 \), \( B(y, \varepsilon) \cap Q \neq \emptyset \). But there exists an \( \varepsilon > 0 \) such that \( B(y, \varepsilon) \subset B \), and so \( Q \cap B \neq \emptyset \). This means that \( Q \in N(B) \) and establishes the claim.

(ii) Fix \( Q \in N(B) \) as a reference cube and let \( Q' \in N(B) \) be any other cube. Since \( r < \delta^j \), we note that \( d(x, x_Q), d(x, x_{Q'}) \leq \delta^j + C_1 \delta^j \). Therefore, \( d(x_Q, x_{Q'}) \leq 2(C_1 + 1)\delta^j \). That is, all the centres of cubes \( Q' \in N(B) \) are inside the ball \( B(x_Q, 2(C_1 + 1)\delta^j) \) and hence \( \hat{E}_{2(C_1+1)} \). Thus, by Proposition 3.5,
\[
\text{card } N(B) \leq \text{card } \hat{E}_{2(C_1+1)} \leq C2^p(C_1 + 1)^{2p}.
\]
(iii) Now, suppose that \((y, t) \in T(B)\). That is, \(y \in B\) and we have \(d(y, t) \leq r - t \leq \delta^j\). By (i), there exists a cube \(Q \in N(B)\) such that \(y \in Q\). Therefore, \((y, t) \in R_Q = \overline{Q}\) and shows that \(T(B) \subset \bigcup Q \in N(B) R_Q\).

(iv) Fix \(Q \in N(B)\) and so \(d(x, x_Q) \leq (C_1 + 1) \delta^j\). Set \(\varepsilon = r = \delta^{j+1}\) in Proposition 3.4 so that

\[
\mu(B(x_Q, \delta^{j+1})) \leq 2^p \left( \frac{(C_1 + 1) \delta^j + 2 \delta^{j+1}}{\delta^{j+1}} \right) \mu(B(x, \delta^{j+1})) \leq 2^p ((C_1 + 1) \delta^{-1} + 2)^p \mu(B(x, r)).
\]

Now, by combining (i) - (iv),

\[
\nu(T(B)) \leq \sum_{Q \in N(B)} \nu(R_Q) \lesssim \sum_{Q \in N(B)} \mu(B(x_Q, C_1 \delta^j)) \lesssim \sum_{Q \in N(B)} \mu(B(x_Q, \delta^{j+1})) \lesssim \text{card}(N(B)) \mu(B(x, \delta^{j+1})) \lesssim \mu(B(x, r)),
\]

which completes the proof.

We quote the following covering theorem of Whitney given as Theorem 1.3 in [8].

**Theorem 4.10 (Whitney Covering Theorem).** Let \(O \subseteq X\) be open. Then, there exists a set of balls \(\mathcal{E} = \{B_j\}_{j \in \mathbb{N}}\) and a constant \(c_1 < \infty\) independent of \(O\) such that

(i) The balls in \(\mathcal{E}\) are mutually disjoint,

(ii) \(O = \bigcup_{j \in \mathbb{N}} c_1 B_j\),

(iii) \(4c_1 B_j \not\subset O\).

This allows us to prove the following theorem of Carleson.

**Theorem 4.11 (Carleson’s Theorem).** Let \(f \in \mathcal{N}\) and \(\nu \in \mathcal{C}\). Then,

\[
\int_{\mathcal{X}_+} |f(x, t)| \, d\nu(x, t) \lesssim \|f\|_{\mathcal{N}} \|\nu\|_{\mathcal{C}}
\]

where the constant depends only on \(p\) and the Whitney constant \(c_1\).

**Proof.** (i) We prove \(\{(x, t) \in \mathcal{X}_+ : |f(x, t)| > \alpha\} \subset T(E_\alpha)\) where \(E_\alpha = \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}\). Fix \((x, t) \in \mathcal{X}_+\) such that \(|f(x, t)| > \alpha\). Then, whenever \(y \in B(x, t)\), we also have \(x \in B(y, t)\) and

\[
\mathcal{M}^* f(y) = \sup_{t > 0} \sup_{z \in B(y, t)} |f(z, t)| > |f(x, t)| > \alpha.
\]

Therefore, \(B(x, t) \subset E_\alpha\) and \((x, t) \in T(B(x, t)) \subset T(E_\alpha)\).

(ii) Let \(O \subseteq X\) be an open set, and let \(\mathcal{E} = \{B_j\}_{j \in \mathbb{N}}\) be the Whitney covering guaranteed by Theorem 4.10. We prove that \(T(O) \subset \bigcup_j T(9c_1 B_j)\).
Let \( Q = (x,t) \in T(B(x, d(x, \mathcal{X} \setminus O))) \). Then, there exists a ball \( B_j = B_j(x_j, r_j) \in \mathcal{S} \) such that \( x \in c_1 B_j \). Let \( y \in B(x, d(x, \mathcal{X} \setminus O)) \). Since \( 4c_1 B_j \cap \mathcal{X} \setminus O \), for any \( z \in \mathcal{X} \setminus O \) \( d(y, \mathcal{X} \setminus O) \leq d(x, z) \leq 8c_1 r_j \).

Then,

\[
d(y, x_j) \leq d(y, x) + d(x, x_k) \leq d(x, \mathcal{X} \setminus O) + d(x, x_k) < 8c_1 r_j + c_1 r_j = 9c_1 r_j.
\]

This proves that \( B(x, d(x, \mathcal{X} \setminus O)) \subset 9c_1 B_j \) and so \( T(B(x, d(x, \mathcal{X} \setminus O))) \subset T(9c_1 B_j) \). We apply Proposition 4.6 to conclude that \( T(O) \subset \bigcup_j T(9c_1 B_j) \).

(iii) Now, we prove that there exists a constant \( C > 0 \) such that for all open sets \( O \subset \mathcal{X} \),

\[
\nu(T(O)) \leq C \| \nu \|_C \mu(O).
\]

First assume that \( O = \mathcal{X} \). If \( \mu(\mathcal{X}) = \infty \), then there is nothing to prove. So suppose otherwise. Now, for any \( x \in \mathcal{X} \) and any ball \( B_r = B(x, r) \),

\[
\frac{1}{\mu(B_r)} \nu(T(B_r)) \leq C(\nu)(x) \leq \| \nu \|_C
\]

and therefore, \( \nu(T(B_r)) \leq \| \nu \|_C \mu(\mathcal{X}) \) for every ball \( B_r \) of radius \( r \).

Now, \( \chi_{T(\mathcal{B}(n))} \leq 1 \) for each \( n \in \mathbb{N} \) and \( \chi_{T(\mathcal{B}(n))} \to \chi_{T(\mathcal{X})} \) and \( n \to \infty \) pointwise. Then, by application of Dominated Convergence Theorem,

\[
\nu(T(\mathcal{X})) = \int_{\mathcal{X}^+} \lim_{n \to \infty} \chi_{T(\mathcal{B}(n))} \, d\nu = \lim_{n \to \infty} \int_{\mathcal{X}^+} \chi_{T(\mathcal{B}(n))} \, d\nu \leq \| \nu \|_C \mu(\mathcal{X}).
\]

Now, consider the case when \( O \not\subset \mathcal{X} \). Then, by (ii) and the subadditivity of the measure,

\[
\nu(T(O)) \leq \sum_j \nu(T(9c_1 B_j)) \leq \| \nu \|_C \sum_j \mu(9c_1 B_j) \leq 2^p (9c_1)^p \| \nu \|_C \sum_j \mu(B_j) \leq (18c_1)^p \| \nu \|_C \mu(O).
\]

(iv) By (i) and (iii),

\[
\nu \{ (x, t) \in \mathcal{X}^+ : |f(x, t)| > \alpha \} \lesssim \| \nu \|_C \mu \{ x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha \}
\]

and integrating both sides with respect to \( \alpha \) completes the proof.

\[ \square \]

5. Harmonic Analysis of \( \Pi_B \)

Let \( \mathcal{Q}_t = \mathcal{Q}_t^j \) for \( \delta^j+1 < t \leq \delta^j \). Following the structure of the proof in [5], for \( t \in \mathbb{R}^+ \), we define the dyadic averaging operator \( \mathcal{A}_t : \mathcal{H} \to \mathcal{H} \) as

\[
\mathcal{A}_t(x) = \sum_{Q \in \mathcal{Q}_t} \chi_Q(x) \int_Q u \, d\mu
\]
when \( x \in \cup \mathcal{Q} \) and 0 elsewhere. A straightforward calculation shows that \( \mathcal{A}_t \in \mathcal{L}(\mathcal{H}) \) and \( \| \mathcal{A}_t \| \leq 1 \) uniformly in \( t \). Then, the principal part is defined as \( \gamma_t(x)w = (\Theta_t^B \omega)(x) \) for \( w \in \mathbb{C}^N \) and where \( \omega(x) = w \) for all \( x \in \mathcal{X} \).

Following [5], to prove Theorem 2.4 as a consequence of Proposition 2.1, we need to show that

\[
\int_0^\infty \| \Theta_t^B P_t u \|^2 \frac{dt}{t} \lesssim \| u \|^2
\]

for \( u \in \mathcal{R}(\Pi) \). Thus, we follow the paradigm in [5], [4] and [15] and decompose this problem in the following way:

\[
\int_0^\infty \| \Theta_t^B P_t u \|^2 \frac{dt}{t} \leq \int_0^\infty \| \Theta_t^B P_t u - \gamma_t \mathcal{A}_t u \|^2 \frac{dt}{t} + \int_0^\infty \| \gamma_t \mathcal{A}_t (P_t - I) u \|^2 \frac{dt}{t} + \int_{\mathcal{X}^+} |\mathcal{A}_t u(x)|^2 |\gamma_t(x)|^2 \frac{d\mu(x) dt}{t}.
\]

The purpose of the first two terms is to reduce the estimate down to the third term which can be dealt with a Carleson measure estimate.

5.1. Off-Diagonal Estimates. The following lemma is a primary tool in our argument. Certainly, it was known to the authors of [5] since they use a similar result in the proof of their Proposition 5.2. The key difference is that we use \( \text{Lip} \xi \) instead of \( \| \nabla \xi \|_\infty \) to control the “slope” of our cutoff. Furthermore, this lemma is used later in our work to construct Lipschitz substitutions where [5], [4] and [15] use smooth cutoff functions. We include a detailed proof of this lemma since it is central to our work.

**Lemma 5.1** (Lipschitz separation lemma). Let \((X, d)\) be a metric space and suppose \( E, F \subset X \) satisfy \( d(E, F) > 0 \). Then, there exists a Lipschitz function \( \eta : X \to [0, 1] \), and a set \( \tilde{E} \supset E \) with \( d(\tilde{E}, F) > 0 \) such that

\[
\eta|_E = 1, \quad \eta|_{X \setminus \tilde{E}} = 0 \quad \text{and} \quad \text{Lip} \eta \leq 4/d(E, F).
\]

**Proof.** Define \( \tilde{E} = \{ x \in X : d(x, E) < 1/4d(E, F) \} \). By construction, \( E \subset \tilde{E} \) and from the triangle inequality for \( d \) and taking infima,

\[
d(\tilde{E}, F) + \sup_{x \in \tilde{E}} d(x, E) \geq d(E, F),
\]

and since \( \sup_{x \in \tilde{E}} d(x, E) \leq \frac{1}{4} d(E, F) \), it follows that \( d(\tilde{E}, F) \geq \frac{3}{4} d(E, F) > 0 \).

Now, define:

\[
\eta(x) = \begin{cases} 
1 - \frac{4d(x, E)}{d(E, F)} & x \in \tilde{E} \\
0 & x \notin \tilde{E}.
\end{cases}
\]

We consider the three possible cases.
(i) First, suppose that \(x, y \notin \tilde{E}\). Then,
\[
|\eta(x) - \eta(y)| = 0 \leq \frac{4d(x, y)}{d(E, F)}.
\]

(ii) Now, suppose that \(x, y \in \tilde{E}\). By the triangle inequality, we have \(d(x, z) \leq d(x, y) + d(y, z)\) and by taking an infimum over \(z \in E\) and invoking the symmetry of distance, \(|d(x, E) - d(y, E)| \leq d(x, y)\). Therefore,
\[
|\eta(x) - \eta(y)| = 1 - \frac{4d(x, E)}{d(E, F)} + \frac{4d(y, E)}{d(E, F)} - \frac{4d(x, E)}{d(E, F)} \leq \frac{4}{d(E, F)} d(x, y).
\]

(iii) Lastly, suppose that \(x \in \tilde{E}\) and \(y \notin \tilde{E}\). Then \(\eta(y) = 0\) and since \(d(x, E) \leq \frac{1}{4}d(E, F)\),
\[
|\eta(x) - \eta(y)| = |\eta(x)| = \eta(x) = 1 - \frac{4d(x, E)}{d(E, F)} = \frac{d(E, F) - 4d(x, E)}{d(E, F)}.
\]

But we also have the triangle inequality \(d(E, x) + d(x, y) \geq d(y, E)\) and by the choice of \(y\) we have that \(d(y, E) \geq \frac{1}{4}d(E, F)\). Therefore, \(d(x, y) \geq d(y, E) - d(x, E) \geq \frac{1}{4}d(E, F) - d(x, E)\) which implies that
\[
\frac{4d(x, y)}{d(E, F)} \geq \frac{d(E, F) - d(x, E)}{d(E, F)} = |\eta(x) - \eta(y)|.
\]

□

A preliminary and immediate consequence is the following off-diagonal estimates resembling those in §5.1 in [5].

**Proposition 5.2** (Off-diagonal estimates). Let \(U_t\) be either \(R_t^B\) for \(t \in \mathbb{R}\) or \(P_t^B, Q_t^B, \Theta_t^B\) for \(t > 0\). Then, for each \(M \in \mathbb{N}\), there exists a constant \(C_M > 0\) (that depends only on \(M\) and the constants in (H1)-(H6)) such that
\[
\|U_t u\|_{L^2(E)} \leq C_M \left(\frac{\text{dist}(E, F)}{t}\right)^{-M} \|u\|_{\mathcal{H}}
\]
whenever \(E, F \subset X\) are Borel sets and \(u \in \mathcal{H}\) with \(\text{spt} u \subset F\).

We omit the proof since it is essentially the same as that of Proposition 5.2 in [5]. The following is an immediate consequence.

**Corollary 5.3.** Let \(Q \in \mathcal{D}_t\) and \(0 < s \leq t\) with \(U_s\) as specified in the proposition. Then,
\[
\|U_s u\|_{L^2(Q)} \leq C_M \sum_{R \in \mathcal{D}_t} \left(\frac{\text{dist}(R, Q)}{s}\right)^{-M} \|u\|_{L^2(R)}
\]
whenever \(u \in \mathcal{H}\).

In our setting, it is more convenient to deal with the following function space rather than \(L^2_{\text{loc}}\) as used in [5].
Definition 5.4. We define $L^2_{\mathcal{A}}(\mathcal{X}, \mathbb{C}^N)$ to be the space of measurable functions $f : \mathcal{X} \to \mathbb{C}^N$ such that on each $Q \in \mathcal{D}$,

$$\int_Q |f|^2 \, d\mu < \infty.$$  

We equip this space with the seminorms $\|\cdot\|_{L^2(Q)}$ indexed by $\mathcal{D}$.

We have the following observations analogous to those on page 478 in [5]. It follows from Propositions 3.3, 3.4, 3.5 coupled with the off-diagonal estimates and by choosing $M > \frac{5p}{2} + 1$. We remind the reader that $p = \log_2(CD)$ where $CD$ is the doubling constant.

Corollary 5.5. There exists a $C' > 0$ such that for all $t > 0$, $U_t$ extends to a continuous map $U_t : L^\infty(\mathcal{X}, \mathbb{C}^N) \to L^2_{\mathcal{A}}(\mathcal{X}, \mathbb{C}^N)$ with

$$\|U_t u\|_{L^2(Q)} \leq C' \mu(Q)^\frac{1}{2} \|u\|_{L^\infty}.$$  

Corollary 5.6. We have $\gamma_t \in L^2_{\mathcal{A}}(\mathcal{X}, L(\mathcal{C}^N))$ and for all $Q \in \mathcal{D}$ satisfy

$$\int_Q |\gamma_t(x)|^2 \, d\mu(x) \leq C'^2$$  

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$, where the constant depends on $M$.

Proof. Observe that for $M > 1$, we have

$$\left\langle \frac{d(x,Q)}{t} \right\rangle^{-M} \leq 2C_1 \delta \left\langle \frac{d(x,Q)}{t} \right\rangle^{-M}.$$
By evaluating the integral
\[ \int_X \int_0^{\infty} |u(x) - u_Q| \, d\nu(r) \, d\mu(x), \]
where \( d\nu(r) = M r^{-M-1} \, dr \), and invoking Lemma 5.7 along with Fubini’s Theorem establishes the claim.

This leads to the following proposition which bounds the first term.

**Proposition 5.9 (First term inequality).** Whenever \( u \in \mathcal{R}(\Pi) \), we have
\[ \int_0^{\infty} \| \Theta_t^B P_t u - \gamma_t A_t P_t u \|^2 \lesssim \| u \|^2. \]

We omit the proof since it is very similar to the proof of Proposition 5.5 in [5]. It is a simple matter of verification using Corollary 3.7 and invoking the weighted Poincaré inequality.

### 5.3. Bounding the second term

The bounding of the second term relies on a suitable substitution for Lemma 5.6 in [5]. The crux of the argument is to be able to perform a cutoff “close” to the boundary of the dyadic cube in question. First, we define the following sets.

**Definition 5.10 \((\mathcal{E}_\tau, \tilde{\mathcal{E}}_\tau)\).** Let \( Q \in \mathcal{Q}_t \) and \( \tau \leq t \) Define
\[ \mathcal{E}_\tau = \left\{ x \in Q : d(x, X \setminus Q) > \frac{a_0 \tau}{2} \right\}, \quad \tilde{\mathcal{E}}_\tau = \left\{ x \in Q : d(x, X \setminus Q) \leq \frac{a_0 \tau}{2} \right\}. \]

The following proposition renders a suitable Lipschitz substitution to the smooth cutoff used in Lemma 5.6 in [5] and Lemma 5.7 in [15].

**Proposition 5.11.** There exists a Lipschitz function \( \xi : Q \to [0,1] \) such that \( \xi = 1 \) on \( \mathcal{E}_\tau \), \( \text{spt} (\text{Lip} \xi) \subset \tilde{\mathcal{E}}_\tau \), and
\[ \text{Lip} \xi \leq \frac{16}{a_0 \tau}. \]

**Proof.** Set
\[ F = \left\{ x \in Q : d(x, X \setminus Q) \leq \frac{a_0 \tau}{4} \right\} \]
and note that \( F \subset \tilde{\mathcal{E}}_\tau \). Then,
\[ \frac{a_0 \tau}{2} \leq \text{dist}(X \setminus Q, \mathcal{E}_\tau) \leq \text{dist}(\mathcal{E}_\tau, F) + \text{dist}(X \setminus Q, F) \leq \text{dist}(\mathcal{E}_\tau, F) + \frac{a_0 \tau}{4} \]
and so \( \text{dist}(\mathcal{E}_\tau, F) > \frac{a_0 \tau}{4} \). By application of Lemma 5.1, we find \( \xi = 1 \) on \( \mathcal{E}_\tau \), \( \xi = 0 \) on \( Q \setminus F \) and
\[ \text{Lip} \xi \leq \frac{4}{a_0 \tau} = \frac{16}{a_0 \tau}. \]
Now, fix $x \in \mathcal{E}_\tau$. It is a simple matter to verify that $\mathcal{E}_\tau$ is open and nonempty. So there exists an $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subset \mathcal{E}_\tau$. Therefore,

$$\text{Lip} \xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x, y)} = \limsup_{\varepsilon \to 0} \left\{ \frac{|\xi(x) - \xi(y)|}{d(x, y)} : y \in \mathcal{E}_\tau \cap B(x, \varepsilon) \setminus \{a\} \right\} = 0.$$ 

Thus, $\text{spt} \xi \subset \tilde{\mathcal{E}}_\tau$. 

This enables us to prove the following lemma. It is of key importance in bounding the second term, as well as in the Carleson measure estimate which allows us to bound the last term.

**Lemma 5.12.** Let $\Upsilon$ be $\Gamma, \Gamma^*$ or $\Pi$. Then, whenever $Q \in \mathcal{Q}$,

$$\left| \int_Q \Upsilon \, d\mu \right|^2 \leq \frac{1}{16} \left( \int_Q |u|^2 \, d\mu \right)^{\frac{3}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}}$$

where the constant depends only on $C_1, C_2, a_0, \eta$ and $p$.

**Proof.** Let $\tau = \left( \int_Q |u|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}}$. The case of $t \leq \tau$ is easy.

So, suppose that $\tau \leq t \leq \delta^j$ and let $\xi$ be the Lipschitz function guaranteed in Proposition 5.11 extended to 0 outside of $Q$, and so write

$$\left| \int_Q \Upsilon \, d\mu \right| \leq \left| \int_Q (1 - \xi) \Upsilon \, d\mu \right| + \left| \int_Q \xi \Upsilon \, d\mu \right| + \left| \int_Q \Upsilon (\xi u) \, d\mu \right|.$$ 

The last term is 0 by (H7) and so we are left with estimating the two remaining terms. First, noting that $\text{spt} \leq \mathcal{E}_\tau$ we compute

$$\left| \int_Q (1 - \xi) \Upsilon \, d\mu \right| \leq \left| \int_{\tilde{\mathcal{E}}_\tau} (1 - \xi) \Upsilon \, d\mu \right| \leq \left( \int_{\tilde{\mathcal{E}}_\tau} |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}} \mu(\tilde{\mathcal{E}}_\tau) \leq C \left( \frac{a_0 \tau}{2t} \right)^{\frac{1}{2}} \mu(Q)^{\frac{1}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}} \leq C \left( \frac{a_0 \tau}{2t} \right)^{\frac{1}{2}} \mu(Q)^{\frac{1}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}}.$$ 

Now, for the second term. We note that $\text{spt} \, M_\xi \subset \text{spt} \, \text{Lip} \xi \subset \tilde{\mathcal{E}}_\tau$ and compute

$$\left| \int_Q \xi \Upsilon \, d\mu \right| = \left| \int_{\tilde{\mathcal{E}}_\tau} M_\xi (x) u(x) \, d\mu(x) \right| \leq \left( \int_{\tilde{\mathcal{E}}_\tau} |M_\xi|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_{\tilde{\mathcal{E}}_\tau} |u|^2 \, d\mu \right)^{\frac{1}{2}} \leq \text{Lip} \xi \mu(\tilde{\mathcal{E}}_\tau)^{\frac{1}{2}} \left( \int_Q |u|^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{16}{a_0} C \left( \frac{a_0 \tau}{2t} \right)^{\frac{1}{2}} \mu(Q)^{\frac{1}{2}} \left( \int_Q |u|^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{16}{a_0} C \left( \frac{a_0 \tau}{2t} \right)^{\frac{1}{2}} \mu(Q)^{\frac{1}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}}.$$
where we have used Cauchy-Schwarz inequality to obtain the first inequality, (H6) in the second, the condition (vi) of Theorem 3.1 in the third, and substitution for $\frac{1}{\tau}$ in the last. Combining these estimates, we have

$$\left| \int_Q \Upsilon u \, d\mu \right| \leq D \frac{1}{t^2} \tau^2 \mu(Q)^{\frac{1}{2}} \left( \int_Q |\Upsilon u|^2 \, d\mu \right)^{\frac{1}{2}}$$

where

$$D = C_2 \left( \frac{a_0}{2} \right)^{\frac{3}{2}} + \frac{16}{a_0} C_2 \left( \frac{a_0}{2} \right)^{\frac{3}{2}} \quad \text{and} \quad \tilde{D} = C(2^p C_1 a_0^{-p})^{\frac{1}{2}}.$$ 

By Cauchy-Schwartz and multiplying both sides by $\mu(Q)$, we find

$$\left| \int_Q \Upsilon u \, d\mu \right|^2 \leq 2 D^2 \frac{1}{t^4} \tau^2 \int_Q |\Upsilon u|^2 \, d\mu.$$ 

The proof is complete by making a substitution for $\tau^2$. □

**Proposition 5.13 (Second term estimate).** For all $u \in H$, we have

$$\int_0^\infty \|\gamma_t A_t (P_t - I) u\| \, \frac{dt}{t} \lesssim \|u\|^2.$$ 

Again, the proof of this proposition is omitted since it resembles the proof of Proposition 5.7 in [5] with minor differences.

### 5.4. Carleson measure estimate

We begin this section with the following proposition which illustrates that the final term can be dealt with a Carleson measure estimate.

**Proposition 5.14.** For all $u \in H$, we have

$$\iint_{\mathcal{X}^+} |A_t u(x)|^2 \, d\nu(x,t) \lesssim |\nu|_C \|u\|^2$$

for every $\nu \in C$.

**Proof.** First, we show that for almost every $x \in \mathcal{X}$,

$$\mathcal{M}^* |A_t u(x)|^2 \lesssim \mathcal{M} u(x)^2$$

where the constant depends only on $p, C_1, \delta$ and $a_0$. Let $f \in L^1_{\text{loc}}(\mathcal{X}^+, \mathbb{C}^N)$. Then, we note that

$$\mathcal{M}^* f(x) = \sup_{t > 0} \sup_{y \in B(x,t)} |f(y, t)|.$$

Fix $t$ such that $\delta_j + 1 < t \leq \delta_j$ and fix $x \in \cup \mathcal{D}_t$. Since $A_t u(z) = 0$ when $z \notin \cup \mathcal{D}_t$, take $y \in \cup \mathcal{D}_t$ such that $d(x, y) < t$. Let $Q \in \mathcal{D}_t$ be the unique cube with $y \in Q$ and let $y_Q \in Q$ such that $B(y_Q, a_0 \delta^j) \subset Q \subset B(y_Q, C_1 \delta^j)$. Then, $d(y_Q, x) \leq d(y_Q, y) + d(y, x) \leq C t$, where $C = (C_1 \delta^{-1} + 1)$.

Also

$$\mu(B(y_Q, C t)) \leq \mu(B(y_Q, C \delta^j)) \leq 2^p C^p a_0^{-p} \mu(B(y_Q, a_0 \delta^j)) \leq 2^p C^p a_0^{-p} \mu(Q)$$
and therefore,
\[ |A_tu(y)| \leq \int_Q |u| \, d\mu \leq 2^p C^p a_0^{-p} \int_{B(y_Q, Ct)} |u| \, d\mu. \]
Moreover,
\[ |A_tu(y)|^2 \leq C' \left( \int_{B(y_Q, Ct)} |u| \, d\mu \right)^2 \]
where \( C' = 2^{2p} C^{2p} a_0^{-2p} \).

Now, since we have established that \( x \in B(y_Q, Ct) \),
\[ \sup_{y \in B(x, t)} |A_tu(y)|^2 \leq C' \sup_{y \in B(x, t)} \left( \int_{B(y_Q, Ct)} |u| \, d\mu \right)^2 \leq C'(Mu(x))^2. \]
Let \( \tilde{X} = \cap_j \cup 2^j \) and so \( \mu(\mathcal{X} \setminus \tilde{X}) = \mu(\cup_j (\mathcal{X} \setminus 2^j)) \leq \sum_j \mu(\mathcal{X} \setminus 2^j) = 0. \)
Therefore, \( x \in \tilde{X} \), then \( x \in \cup 2^j \) for all \( t > 0 \). So, fix \( x \in \tilde{X} \). Then,
\[ \mathcal{M}^* |A_t u|^2 (x) = \sup_{t > 0} \sup_{y \in B(x, t)} |A_t u(y)|^2 \leq C' Mu(x)^2 \]
which completes the proof.

Next, let \( f(x, t) = |A_t u(x)|^2 \). Then, \( \|f\|_{\mathcal{X}} = \|\mathcal{M}^* f\|_1 \lesssim \|Mu\|^2 < \infty \) by the Maximal Theorem 4.2. Invoking Carleson’s Theorem 4.11 completes the proof.

Thus, to bound the final term, it suffices to prove
\[ A \mapsto \int_A |Y_t(x)|^2 \, d\mu(x) \frac{dt}{t} \]
is a Carleson measure. We follow [5] and fix \( \delta > 0 \) to be chosen later. Let
\[ K_\nu = \left\{ \nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left| \frac{\nu'}{\nu'} - \nu \right| \leq \sigma \right\} \]
and let \( \mathcal{F} \) be a finite set of \( \nu \in \mathcal{L}(\mathbb{C}^N) \) with \( |\nu| = 1 \) such that \( \cup_{\nu \in \mathcal{F}} K_\nu = \mathcal{L}(\mathbb{C}^N) \setminus \{0\} \). We note as do the authors of [5] that it is enough to show
\[ \int_{(x, t) \in R_Q, Y_t \in K_\nu} |Y_t(x)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \mu(Q) \]
for each \( \nu \in \mathcal{F} \). A stopping time argument allows us to reduce this to the following.

**Proposition 5.15.** There exists a \( 0 < \beta < 1 \) such that for every dyadic cube \( Q \in \mathcal{D} \) and \( \nu \in \mathcal{L}(\mathbb{C}^N) \) with \( |\nu| = 1 \), there exists a collection \( \{Q_k\} \subset \mathcal{D} \) of disjoint subcubes of \( Q \) satisfying \( \mu(E_{Q, \nu}) > \beta \mu(Q) \) and such that
\[ \int_{(x, t) \in E_{Q, \nu}, Y_t \in K_\nu} |Y_t(x)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \mu(Q) \]
where \( E_{Q, \nu} = Q \setminus \cup_k Q_k \) and \( E_{Q, \nu}^* = R_Q \setminus \cup_k RQ_k \).
We prove this via defining a test function similar to the one found on page 484 in [5]. Here, the authors use a smooth cutoff function in their construction. Again, we rephrase this in terms of a Lipschitz cutoff function whose existence is guaranteed by the following lemma.

**Lemma 5.16.** Let $Q \in \mathcal{Q}$. Then, there exists a Lipschitz function $\eta : \mathcal{X} \to [0, 1]$ such that $\eta = 1$ on $B(x_Q, \tau C_1 \ell(Q))$ and $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, 2\tau C_1 \ell(Q))$ with

$$\text{Lip} \eta \leq \frac{4}{\tau C_1 \ell(Q)}$$

whenever $\tau > 1$.

**Proof.** Fix $Q \in \mathcal{Q}$, and we have $Q \subset B(x_Q, \tau C_1 \delta^j) \subset B(x_Q, 2\tau C_1 \delta^j)$. Also,

$$d(B(x_Q, \tau C_1 \delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j)) \geq (2\tau C_1 - \tau C_1)\delta^j = \tau C_1 \delta^j.$$

Now, we invoke Lemma 5.1 with $E = B(x_Q, \tau C_1 \delta^j)$ and $F = \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j)$ to find a Lipschitz $\eta : \mathcal{X} \to [0, 1]$ with $\eta = 1$ on $B(x_Q, \tau C_1 \delta^j)$, $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, \tau C_1 \delta^j)$ and

$$\text{Lip} \eta \leq \frac{4}{d(B(x_Q, \tau C_1 \delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1 \delta^j))} \leq \frac{4}{\tau C_1 \delta^j} \leq \frac{4}{\tau C_1 \ell(Q)}$$

which completes the proof. \qed

The test function is now defined as follows. Let $Q \in \mathcal{Q}$ and fix $\nu \in \mathcal{L}(C^N)$ with $|\nu| = 1$. Let $\eta_Q$ be the Lipschitz map guaranteed by Lemma 5.16 and let $w, \tilde{w} \in C^N$ such that $\nu^* \tilde{w} = w$ with $|w| = |\tilde{w}| = 1$. Furthermore, let $w_Q = \eta_Q w$ and define

$$f_{Q,\varepsilon}^w = w_Q - \varepsilon \ell(Q)\mu(I + \varepsilon \ell(Q)\mu \Pi_B)^{-1}w_Q = (1 + \varepsilon \ell(Q)\mu \Pi_B)^{-1}w_Q - \varepsilon \ell(Q)\mu \Pi_B^{-1}w_Q.$$

It is then an easy fact that $\|w_Q\|^2 \leq (4\tau C_1 a_0^{-1})^p \mu(Q)$ and we obtain the following lemma analogous to Lemma 5.10 in [5].

**Lemma 5.17.** There exists $c > 0$ such that for all $\varepsilon > 0$, $\left\|f_{Q,\varepsilon}^w\right\| \leq c \mu(Q)^{\frac{1}{2}}$, $\int_{R_Q} \left| \Theta_t^B f_{Q,\varepsilon}^w \right|^2 d\mu(x) dt \leq c \varepsilon^{\frac{1}{2}} \mu(Q)$, and $\left| f_Q f_{Q,\varepsilon}^w - w \right| \leq c \varepsilon^{\frac{7}{4}}$.

**Proof.** The proof of the first two estimates are essentially the same as that of Lemma 5.10 in [5]. To prove the last estimate, note that since $\eta_Q = 1$ on $Q$, we have on $Q$ that

$$f_{Q,\varepsilon}^w - w = w_Q - \varepsilon \ell(Q)\mu(I + \varepsilon \ell(Q)\mu \Pi_B)^{-1}w_Q - w = (\eta_Q - 1)w - \varepsilon \ell(Q)\mu(I + \varepsilon \ell(Q)\mu \Pi_B)^{-1}w_Q = -\varepsilon \ell(Q)\mu(I + \varepsilon \ell(Q)\mu \Pi_B)^{-1}w_Q.$$

This completes the proof. \qed
Setting $u = (1 + \epsilon \ell(Q)\Pi_B)^{-1}w_Q$ and $\Upsilon = \Gamma$, we apply Lemma 5.12

$$\left| \int_Q \int_{Q,\epsilon} w - w \right| = \left| \int_Q \epsilon \ell(Q)(1 + \epsilon \ell(Q)\Pi_B)^{-1}w_Q \right|$$

$$= \epsilon \ell(Q) \left| \int_Q \Pi(1 + \epsilon \ell(Q)\Pi_B)^{-1}w_Q \right|$$

$$\lesssim \frac{\epsilon \ell(Q)}{t^\frac{4}{3}} \left( \int_Q |(1 + \epsilon \ell(Q)\Pi_B)^{-1}w_Q| \, d\mu \right) \frac{1}{2} \frac{1}{2 - \frac{4}{3}}$$

$$= \left( \frac{\epsilon \ell(Q)}{t} \right)^\frac{2}{3} \left( \int_Q |(1 + \epsilon \ell(Q)\Pi_B)^{-1}w_Q| \, d\mu \right) \frac{1}{2} \frac{1}{2 - \frac{4}{3}}$$

The proof is completed by noting $t \simeq \ell(Q)$ and invoking Proposition 2.5 and Lemma 4.2 of [5].

The proof of Proposition 5.15 then follows a procedure similar to that which is used to prove Lemma 5.12 in [5].

We note that our hypotheses (H1)-(H8) remain unchanged upon replacing $(\Gamma, B_1, B_2)$ by $(\Gamma^*, B_2, B_1)$, $(\Gamma^*, B_2^*, B_1^*)$ and $(\Gamma, B_1^*, B_2^*)$. Thus, the hypothesis of Proposition 2.1 is satisfied and Theorem 2.4 is proved.

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