A UNIVERSAL AXIOMATIZATION OF METROPOLIS-ROTA IMPLICATION ALGEBRAS

COLIN G. BAILEY AND JOSEPH S. OLIVEIRA

ABSTRACT. We show that the class of Metropolis-Rota implication algebras can be given
a universal axiomatization using an operation closely related to composition in oriented
matroids. Lastly we describe the role of our new operation in the collapse of an MR-
algebra.

1. INTRODUCTION

Metropolis-Rota implication algebras (MR-algebras) were first seen in [3] as an alge-
braic representation of the poset of faces of an n-cube. Therein was introduced the partial
reflection operator $\Delta$ that was used in conjunction with the lattice structure to characterize
these face lattices – in the finite case. They proved the following theorem.

Theorem 1.1. Let $L$ be a finite lattice with minimum $0$ and maximum $1$. For every $x \neq 0$,
let $\Delta_x$ be a function defined on the segment $[0, x]$ and taking values in $[0, x]$. Assume
(i) If $a \leq b \leq x$ then $\Delta_x(a) \leq \Delta_x(b)$;
(ii) $\Delta_x^2 = id$ (the identity map);
(iii) Let $a < x$ and $b < x$. Then the following two conditions are equivalent:
   $\Delta_x(a) \lor b < x$ and $a \land b = 0$.

Then $L$ is isomorphic to the lattice of faces of an n-cube, for some $n$.

Conversely, if $L$ is the lattice of faces of an n-cube, and $\Delta_x(y)$ is the antipodal face of
$y$ within the face $x$, then $L$ satisfies conditions (i) through (iii).

One can take an alternative approach to extending the Metropolis-Rota theorem by not-
 ing that the face lattice of the n-cube is exactly the lattice of closed intervals of the Boolean
algebra $2^n$, and the $\Delta$ operator is induced by local complementation. One can then ask
for an algebraic characterization of such lattices, which is provided by [1], wherein it was
shown that such lattices are essentially characterized by the Metropolis-Rota axioms plus
atomicity and the fact that every interval $[x, y]$, where $x > 0$, is a Boolean algebra.

In studying such lattices one quickly discovers that 0 plays a rather ambiguous role. It
makes the structure into a lattice, but it prevents the class of such structures from being
closed under cartesian products. It also makes the natural homomorphisms trivial (they are
forced to be embeddings).

The next obvious step then is to eliminate zero. The resulting structures are upper semi-
lattices, and (as is clear from the Bailey-Oliveira theorem) they have a natural implication
structure. (This was first pointed out and studied by Oliveira – see [4].) Furthermore the
class of such structures is now closed under cartesian products and homomorphic images.
We call the elements of this class MR-algebras.

1991 Mathematics Subject Classification. 06A06, 06E99.
Key words and phrases. cubes, Boolean algebras, implication algebras.
There are many ways to make this class into a variety--close under subobjects; add a
new constant naming an atom; or add new operations.

The second option is unsatisfactory, as the resulting category is naturally isomorphic to
the category of Boolean algebras. The first option gives rise to cubic implication algebras
which are studied in detail in [1]. A perhaps unfortunate side effect of the generality of
cubic algebras is that there are many finite cubic algebras that are not face lattices of cubes,
but only embed into such lattices. In this paper we look at the last option, providing a
new operation, \( \hat{\cdot} \), that simultaneously generalizes \( \Delta \) and provides certain important lower
bounds – and so partly encodes the MR-axiom. From this we get a univeral axiomatization.
That this encoding is ‘successful’ is illustrated by two things

- all algebras with this new operation satisfying the axioms below are exactly the cubic
  algebras satisfying the MR axiom;
- and hence the finite ones are exactly the face lattices of \( n \)-cubes.

This new operation can be considered a variation on the well-known operation of com-
position in oriented matroids.

The paper begins with background information on cubic and MR-algebras as developed
in [1], defines \( \hat{\cdot} \) and shows that it captures the MR-axiom. We then give purely universal
axioms for MR-algebras and show that all such algebras are cubic algebras with caret, and
hence are MR-algebras.

The latter part of the paper develops the notion of \textit{collapse} of an MR-algebra and shows
that caret is closely related to meet in an associated implication lattice.

2. Cubic & MR Algebras and the New Operation

First we recall some definitions.

\textbf{Definition 2.1.} A cubic algebra is a join semi-lattice with one and a binary operation \( \Delta \)
satisfying the following axioms:

\begin{enumerate}
  \item \( x \leq y \) implies \( \Delta(y, x) \lor x = y \);
  \item \( x \leq y \leq z \) implies \( \Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x)) \);
  \item \( x \leq y \) implies \( \Delta(y, \Delta(y, x)) = x \);
  \item \( x \leq y \leq z \) implies \( \Delta(z, x) \leq \Delta(z, y) \);
    \begin{enumerate}
      \item Let \( xy = \Delta(1, \Delta(x \lor y, y)) \lor y \) for any \( x, y \) in \( \mathcal{L} \).
      \item \( (xy)y = x \lor y \);
      \item \( x(yz) = y(xz) \);
    \end{enumerate}
\end{enumerate}

\textbf{Definition 2.2.} An MR-algebra is a cubic algebra satisfying the MR-axiom:

\begin{enumerate}
  \item if \( a, b < x \) then \( \Delta(x, a) \lor b < x \) iff \( a \land b \) does not exist.
\end{enumerate}

\textbf{Example 2.1.} Let \( X \) be any set, and

\[ \mathcal{S}(X) = \{ \langle A, B \rangle \mid A, B \subseteq X \text{ and } A \cap B = \emptyset \} . \]

Elements of \( \mathcal{S}(X) \) are called signed subsets of \( X \). The operations are defined by

\[ 1 = \langle \emptyset, \emptyset \rangle \]
\[ \langle A, B \rangle \lor \langle C, D \rangle = \langle A \cap C, B \cap D \rangle \]
\[ \Delta(\langle A, B \rangle, \langle C, D \rangle) = \langle A \cup D \setminus B, B \cup C \setminus A \rangle . \]

These are all atomic MR-algebras.
Example 2.2. Let $B$ be a Boolean algebra, then the interval algebra of $B$ is
\[ I(B) = \{ [a, b] \mid a \leq b \text{ in } B \} \]
ordered by inclusion. The operations are defined by
\[
1 = [0, 1] \\
[a, b] \lor [c, d] = [a \land c, b \lor d] \\
\Delta([a, b], [c, d]) = [a \lor (b \land \overline{c}), b \land (a \lor \overline{c})].
\]
These are all atomic MR-algebras. For further details see [1].

We note that $S(X)$ is isomorphic to $I(\mathcal{P}(X))$.

Example 2.3. Let $B$ be a Boolean algebra and $\mathcal{F}$ be a filter in $B$. Then the filter algebra is the set
\[ I(\mathcal{F}) = \{ [a, b] \mid a \leq b \text{ and } a, b \in \mathcal{F} \} \]
with the same order and operations as for interval algebras.

It is easy to show that if $\mathcal{F}$ is a principal filter then $I(\mathcal{F})$ is isomorphic to an interval algebra. Hence every finite filter gives only interval algebras and by the Metropolis-Rota theorem every finite MR-algebra is an interval algebra.

Definition 2.3. Let $\mathcal{L}$ be a cubic algebra. Then for any $x, y \in \mathcal{L}$ we define the (partial) operation $\hat{\cdot}$ (caret) by:
\[ x \hat{\cdot} y = x \land \Delta(x \lor y, y) \]
whenever this meet exists.

Lemma 2.4. In an MR-algebra the caret operation is total.

Proof. The meet will not exist iff $\Delta(x \lor y, \Delta(x \lor y, y)) \lor x < x \lor y$. But the left-hand-side is exactly $x \lor y$.

To get the converse we need the following result from [1] theorem 4.3.

Theorem 2.5. Let $\mathcal{L}$ be a cubic algebra. Let $a, b \in \mathcal{L}$. There is no pair of elements $x_1, x_2$ such that $a, b < x_1, x_2$ and
\[ \Delta(x_1, a) \lor b = x_1, \ \Delta(x_2, a) \lor b < x_2. \]

Using this we can get the desired connection between caret and the MR axiom.

Theorem 2.6. Let $\mathcal{L}$ be a cubic algebra on which caret is total. Then $\mathcal{L}$ satisfies the MR-axiom.

Proof. Let $a, b < x$.

First, if $a \land b$ exists then we have $x \geq a \lor \Delta(x, b) \geq (a \land b) \lor \Delta(x, a \land b) = x$.

Conversely suppose that $a \lor \Delta(x, b) = x$. There are two cases
- if $a \lor b$ is one of $a$ or $b$, then $a$ and $b$ are comparable and the meet clearly exists.
- Otherwise $a, b < a \lor b$. By theorem 2.5 we must have
\[
(1) \quad a \lor \Delta(a \lor b, b) = a \lor b.
\]

Then we have
\[
a \hat{\cdot} \Delta(a \lor b, b) = a \land \Delta(a \lor b, b) \Delta(a \lor b, b)) \quad \text{by definition}
= a \land \Delta(a \lor b, b)) \quad \text{by (1)}
= a \land b.
\]

□
3. Axiomatics

We turn now to providing an axiomatic description of cubic algebras with caret. The first version gives a caret-like operation that is enough to give all MR-algebras, but not strong enough to prove that caret satisfies definition 2.3. Following this result we consider how to improve the fit.

The fact that the axioms provided are universal shows that the class of MR-algebras forms a variety.

**Theorem 3.1.** Let \( L \) be a join-semilattice with 1 and a binary operation \( ^\wedge \) (caret) satisfying the following axioms:

(a) \( \forall x,y \ x \lor (y \wedge x) = x \lor y; \)
(b) \( \forall x,y \ (1 \wedge x) \wedge (1 \wedge y) = 1 \wedge (x \wedge y); \)
(c) \( \forall x \ 1 \wedge (1 \wedge x) = x; \)
(d) \( \forall x,y \text{ if } x \leq y \text{ then } 1 \wedge x \leq 1 \wedge y; \)
(e) If we define \( x \rightarrow y \) as equal to \( y \lor (1 \wedge (x \wedge y)) \) then
   i) \( (x \rightarrow y) \rightarrow y = x \lor y; \)
   ii) \( x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z); \)
(f) \( \forall x,y \ (x \lor y) \rightarrow ((x \lor y) \wedge y) = 1 \wedge (x \rightarrow y); \)
(g) \( \forall x,y \ x \wedge y \leq (x \lor y) \wedge y; \)
(h) \( \forall x,y \ x \leq y \leq x. \)

Then \( L \) is an MR-algebra.

**Proof.** We proceed via a series of lemmata. We aim to show that with these axioms we can define a \( \Delta \)-operator that makes \( L \) into a cubic algebra on which the operation \( \langle a, b \rangle \mapsto a \land \Delta(a \lor b, b) \) is total. From the results above this implies that \( L \) is an MR-algebra.

However it is not enough to show that the caret operations are the same, for that we need one more axiom as we see in lemma 3.13. This is given in theorem 3.14.

The next four lemmas establish that \( (L, \rightarrow) \) is an implication algebra, by checking each of the axioms in turn.

**Lemma 3.2.** \( (x \rightarrow y) \lor x = 1. \)

**Proof.** We recall that \( (x \rightarrow y) \lor x = (x \lor y) \lor 1 \wedge (x \wedge y) \). Also we have

\[
x \wedge y \leq (x \lor y) \wedge y \leq x \lor y \tag{by (g, h)}
\]

Therefore \( (x \rightarrow y) \lor x \geq (x \wedge y) \lor 1 \wedge (x \wedge y) \)

\[
= 1 \lor (x \wedge y) \tag{by (a)}
\]

\[
= 1.
\]

\[\square\]

**Lemma 3.3.** \( (x \rightarrow y) \wedge x \leq 1 \wedge x. \)

**Proof.** By axioms (c) and (d) the lemma is true iff \( 1 \wedge ((x \rightarrow y) \wedge x) \leq x. \)

\[
1 \wedge ((x \rightarrow y) \wedge x) = (1 \wedge (x \rightarrow y)) \wedge (1 \wedge x)
\]

\[
\leq ((1 \wedge (x \rightarrow y)) \lor (1 \wedge x)) \wedge (1 \wedge x) \tag{by (g)}
\]

\[
(1 \wedge (x \rightarrow y)) \lor (1 \wedge x) = 1 \wedge ((x \rightarrow y) \lor x)
\]

\[
= 1 \wedge 1 = 1.
\]
Hence
\[ 1 \land ((x \rightarrow y) \land x) \leq 1 \land (1 \land x) = x. \]

**Lemma 3.4.** \((x \rightarrow y) \rightarrow x = x.\)

**Proof.** By definition of \(\rightarrow\), the left-hand-side is greater than \(x\).

By the definition of \(\rightarrow\) and the proof of the last lemma we have \((x \rightarrow y) \rightarrow x = x \lor 1 \land ((x \rightarrow y) \land x) \leq x \lor x = x.\)

**Proposition 3.5.** \(⟨L, \rightarrow⟩\) is an implication algebra.

**Proof.** This clear, as the last lemma and axiom (e) give the axioms for implication algebras.

Now we turn to the verification that we have a cubic algebra by first defining \(\Delta\) and then checking each of the remaining axioms.

**Definition 3.6.** Let \(a \geq b\). Then \(\Delta(a, b) = a \land b\).

**Lemma 3.7.** Let \(c \leq b \leq a\). Then \(\Delta(a, c) \leq \Delta(a, b)\).

**Proof.** Since \(\rightarrow\) is an implication operation and \(c \leq b \leq a\) we have \(a \rightarrow c \leq b \rightarrow c\) and \(a \rightarrow b = b \lor (a \rightarrow c)\). Thus
\[
\Delta(a, b) = a \land \Delta(1, a \rightarrow b)
= a \land \Delta(1, b \lor (a \rightarrow c))
= a \land (\Delta(1, b) \lor \Delta(1, a \rightarrow c))
\geq a \land \Delta(1, a \rightarrow c)
= \Delta(a, c).
\]

**Lemma 3.8.** If \(c \leq b \leq a\) then \(\Delta(b, c) = b \land \Delta(a, (b \rightarrow c) \land a)\).

**Proof.** First we note that
\[
a \rightarrow ((b \rightarrow c) \land a) = ((b \rightarrow c) \land a) \lor (a \rightarrow c) \quad \text{as } (b \rightarrow c) \land a \geq c
= [(b \rightarrow c) \lor (a \rightarrow c)] \land [a \lor (a \rightarrow c)] \quad \text{in } [c, 1]
= (b \rightarrow c) \land 1
= b \rightarrow c.
\]
From this we can conclude that
\[
\Delta(a, (b \to c) \land a) = a \land \Delta(1, a \to ((b \to c) \land a)) \\
= a \land \Delta(1, b \to c).
\]
And so we get
\[
\Delta(b, c) = b \land \Delta(1, b \to c) \\
= (b \land a) \land \Delta(1, b \to c) \\
= b \land \Delta(a, (b \to c) \land a).
\]

Lemma 3.10. If \(c \leq b \leq a\) then \(\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), \Delta(a, c))\).

Proof. From the last lemma we have
\[
\Delta(a, \Delta(b, c)) = \Delta(a, b \land \Delta(a, (b \to c) \land a)) \\
= \Delta(a, b) \land \Delta(a, (b \to c) \land a)) \\
= \Delta(a, b) \land (b \to c) \land a.
\]
The other side gives
\[
\Delta(\Delta(a, b), \Delta(a, c)) = \Delta(a, b) \land \Delta(a, (\Delta(a, b) \to \Delta(a, c)) \land a). \\
(\Delta(a, b) \to \Delta(a, c)) \land a \text{ is the unique complement of } \Delta(a, b) \text{ in } [\Delta(a, c), a] \text{ and } \Delta(a, \bar{a}) \text{ is an automorphism on } [\bar{a}, a] \text{ so its image under } \Delta(a, \bar{a}) \text{ is the unique complement of } \Delta(a, \Delta(a, b)) = b \text{ in } [c, a]. \text{ But this is } (b \to c) \land a. \text{ Hence } (b \to c) \land a = \Delta(a, (\Delta(a, b) \to \Delta(a, c)) \land a). \]

Lemma 3.11. If \(b \leq a\) then \(a \to b = b \lor \Delta(1, \Delta(a, b))\).

Proof. This is immediate from the definitions.

Corollary 3.12. For any \(a, b\) we have \(a \to b = b \lor \Delta(1, \Delta(a \lor b, b))\).

Proof. Since \(a \to b = (a \lor b) \to b\).

It now follows from the above lemmas that the axioms of a cubic algebra are satisfied. To show that we have an MR-algebra it suffices to show that the 'new' operation
\[
(2) \quad a \circ b = a \land \Delta(a \lor b, b)
\]
is total.
In order to see that this is so, it suffices to note that \(a \land b \leq a\) and \(a \land b \leq \Delta(a \lor b, b)\) and so there is a lower bound to \(a\) and \(\Delta(a \lor b, b)\). Since the algebra is an implication algebra we know that the meet always exists.

Note that we have not proven that \(\circ\) and \(\ast\) are the same operation. Indeed we cannot do so as the next lemma shows, although axioms (e-h) put strong constraints on the possibilities.

Lemma 3.13. Let \(L\) be an MR-algebra, and for all \(x, y\) let \(p(x, y)\) be any element of \(L\) satisfying
\[(a) \text{ if } x \geq y \text{ then } p(x, y) = \Delta(x, y); \]
\[(b) \text{ } p(x, y) \leq x; \]
\[(c) \text{ } p(x, y) \leq y; \]
\[(d) \text{ } p(x, y) \leq x; \]
\[(e) \text{ } p(x, y) \leq \Delta(x, y); \]
\[(f) \text{ } p(x, y) \leq p(y, x); \]
\[(g) \text{ } p(x, y) \leq p(y, x) \lor p(y, x); \]
\[(h) \text{ } p(x, y) \leq p(y, x) \land p(y, x). \]
(c) \( p(x, y) \leq \Delta(x \lor y, y) \).

Then \( p \) is a caret operation on \( L \).

**Proof.** It suffices to show that

\[
y \lor \Delta(1, \Delta(x \lor y, y)) = y \lor \Delta(1, p(x, y))
\]

for any such \( p \).

Since every cubic algebra embeds into an interval algebra we may do all of the necessary computations in an interval algebra.

\[
y \lor \Delta(1, \Delta(x \lor y, y)) = \left[ y_0 \land \overline{x}_0, y_1 \lor \overline{x}_1 \right]
\]

\[
p(x, y) = [p_0, p_1] \leq x \land \Delta(x \lor y, y)
\]

\[
= \left[ x_0 \lor (x_1 \land \overline{y}_1), x_1 \land (x_0 \lor \overline{y}_0) \right]
\]

Therefore

\[
x_0 \lor (x_1 \land \overline{y}_1) \leq p_0 \leq p_1 \leq x_1 \land (x_0 \lor \overline{y}_0)
\]

\[
y \lor \Delta(1, p(x, y)) = \left[ y_0 \land \overline{p}_1, y_1 \lor \overline{p}_0 \right]
\]

\[
y_0 \land \overline{p}_1 \leq y_0 \land \overline{x}_0 \land (\overline{x}_1 \lor y_1)
\]

\[
= y_0 \land \overline{x}_0
\]

\[
y_0 \land \overline{p}_1 \geq y_0 \land (\overline{x}_1 \lor (\overline{x}_0 \land y_0))
\]

\[
= (y_0 \land \overline{x}_1) \lor (y_0 \land \overline{x}_0)
\]

\[
= y_0 \land \overline{x}_0
\]

\[
\text{by \(*)\)} \quad \text{as } y_0 \leq y_1
\]

\[
y_0 \land \overline{p}_1 \geq y_0 \land (\overline{x}_1 \lor (\overline{x}_0 \land y_0))
\]

\[
= (y_0 \land \overline{x}_1) \lor (y_0 \land \overline{x}_0)
\]

\[
= y_0 \land \overline{x}_0
\]

\[
\text{by \(*)\)} \quad \text{as } x_0 \leq x_1.
\]

Likewise

\[
y_1 \lor \overline{p}_0 = y_1 \lor \overline{x}_1.
\]

\[\square\]

This lemma shows that we need to add an additional axiom in order to ensure that caret is definable by \(\overline{2}\).

**Theorem 3.14.** Suppose that \( L \) is as in the above theorem, satisfying the additional axiom

(i) \((x \lor y) \wedge y \leq x \rightarrow (x \wedge y)\)

then \( L \) is an MR-algebra and \( x \wedge y = x \land \Delta(x \lor y, y) \) for all \( x, y \).

**Proof.** We only need to prove the last statement, and it is clear as \( x \wedge y \leq x \land (x \lor y) \land y = x \land \Delta(x \lor y, y) \leq x \land (x \rightarrow (x \wedge y)) = x \wedge y. \)

\[\square\]

Our results show that the class of MR-algebras form a variety that is contained in the variety of cubic algebras.

Any finite object in this variety satisfies the hypotheses of theorem 1.1 and is therefore isomorphic to the face lattice of an \( n \)-cube.

4. **What is caret?**

The \( \Delta \) operator on finite MR-algebras is very natural – \( \Delta(x, y) \) is the reflection of \( y \) through the centre of the face \( x \). But what of caret?
4.1. The signed set case. Earlier we gave the example of the MR-algebra of signed sets. It is well-known that oriented matroids arise as subposets of $\mathcal{S}(X)$, albeit with the reverse order, but meet does not usually correspond to our join. However there is a close connection between composition and caret as we see in the next theorem. First we recall the definition of composition as used in $\mathcal{S}(X)$. For notational convenience we write a signed set $A$ as the pair $\langle A^+, A^- \rangle$.

**Definition 4.1.** Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be two signed subsets of $X$. Then the composition of $A$ and $B$ is

$$A \circ B = \langle A^+ \cup (B^+ \setminus A^-), A^- \cup (B^- \setminus A^+) \rangle.$$  

**Theorem 4.2.** Let $A$ and $B$ be two signed sets in $\mathcal{S}(X)$. Then $A \hat{\circ} B = A \wedge (\Delta B)$.  

**Proof.**

$$A \hat{\circ} B = A \land \Delta(A \lor B, B)$$

$$A \lor B = \langle A^+ \cap B^+, A^- \cap B^- \rangle$$

$$\Delta(A \lor B, B) = \langle (A^+ \cap B^+) \cup B^- \setminus (A^- \cap B^-), (A^- \cap B^-) \cup B^+ \setminus (A^+ \cap B^+) \rangle$$

$$= \langle (A^+ \cap B^+) \cup B^- \setminus A^-, (A^- \cap B^-) \cup B^+ \setminus A^+ \rangle$$

$$A \hat{\circ} B = \langle A^+ \cup B^- \setminus A^-, A^- \cup B^+ \setminus A^+ \rangle$$

Therefore

$$A \hat{\circ} B = A \circ B.$$  

\[\Box\]

In a later paper ([2]) we study this connection between MR-algebras and oriented matroids in greater depth.

4.2. The general case. This gives us some idea about caret. But more can be seen by considering the collapse of an MR-algebra. To get this collapse we define two relations on $L$ that give us the collapsing relation.

**Definition 4.3.** Let $L$ be a cubic algebra and $a, b \in L$. Then $a \preceq b$ iff $\Delta(a \lor b, a) \leq b$

$a \simeq b$ iff $\Delta(a \lor b, a) = b$.

**Lemma 4.4.** Let $L$, $a, b$ be as in the definition. Then

$$a \preceq b$$

iff $b = (b \lor a) \land (b \lor \Delta(1, a))$.  

**Proof.** See [1] lemmas 2.7 and 2.12.  

\[\Box\]

Also from [1] (lemma 2.7c for transitivity) we know that $\simeq$ is an equivalence relation. In general it is not a congruence relation, but it does fit well with caret. In fact much more is true – the structure $L/\simeq$ is naturally an implication lattice. To show this we need to show that certain operations cohere with $\simeq$.

However before doing so we show that this relation actually describes a natural property of intervals of Boolean algebras.
Definition 4.5. Let $x = [x_0, x_1]$ be any interval in a Boolean algebra $B$. Then the length of $x$ is $x_0 \lor x_1 = \ell(x)$.

Lemma 4.6. Let $b, c$ be intervals in a Boolean algebra $B$. Then $b \simeq c \iff \ell(b) = \ell(c)$.

Proof. From $b \simeq c$ we have
\[
\Delta(b \lor c, c) = [(b_0 \land c_0) \lor (\overline{b}_1 \land c_1), (b_1 \lor c_1) \land (\overline{b}_0 \lor c_0)] = [c_0, c_1].
\]
Hence
\[
\ell(c) = c_0 \lor \overline{c}_1
\]
\[
= ((b_0 \land c_0) \lor (\overline{b}_1 \land c_1)) \lor ((b_1 \lor c_1) \land (\overline{b}_0 \lor c_0))
\]
\[
= (b_0 \land c_0) \lor (\overline{b}_1 \land c_1) \lor (\overline{b}_1 \land c_1) \lor (b_0 \land c_0)
\]
\[
= b_0 \lor c_0.
\]
Conversely, if $\ell(b) = \ell(c)$ then we have
\[
\Delta(b \lor c, c) = [(b_0 \land c_0) \lor (\overline{b}_1 \land c_1), (b_1 \lor c_1) \land (\overline{b}_0 \lor c_0)]
\]
\[
\overline{b}_1 \land c_1 = c_1 \land (\overline{b}_1 \land b_0) \land (\overline{b}_1 \lor b_0)
\]
\[
= c_1 \land (\overline{c}_1 \lor c_0) \land \overline{b}_0
\]
\[
= c_0 \land \overline{b}_0
\]
and therefore
\[
(b_0 \land c_0) \lor (\overline{b}_1 \land c_1) = (b_0 \land c_0) \lor (\overline{b}_0 \land c_0)
\]
\[
= c_0.
\]
Likewise $(b_1 \lor c_1) \land (\overline{b}_0 \lor c_0) = c_1$. □

First caret.

Lemma 4.7. Let $L, a, b$ be as in the definition. Then $a \triangleleft b \simeq b \triangleleft a$.

Proof.\[
\Delta(a \land b, a \triangleleft b) = \Delta(a \lor b, a \land \Delta(a \lor b, b)) = \Delta(a \lor b, a) \land \Delta(a \lor b, \Delta(a \lor b, b)) = \Delta(a \lor b, a) \land b = b \triangleleft a.
\]
It follows that $(a \triangleleft b) \lor (b \triangleleft a) = a \lor b$ and hence\[
\Delta((a \triangleleft b) \lor (b \triangleleft a), a \triangleleft b) = \Delta(a \lor b, a \triangleleft b) = b \triangleleft a.
\]
□
Lemma 4.8. Let \( \mathcal{L} \) be an MR algebra, and \( a, b, c \in \mathcal{L} \) with \( b \simeq c \). Then \( a \wedge b \simeq a \wedge c \).

Proof. Since we are working in an MR-algebra we may assume that \( a, b, c \) are all in an interval algebra, so let \( a = [a_0, a_1], b = [b_0, b_1], \) and \( c = [c_0, c_1] \).

Then
\[
a \wedge b = [a_0 \lor (a_1 \land \overline{b_1}), a_0 \lor (a_1 \land \overline{b_0})] = [s_0, s_1],
b \wedge c = [a_0 \lor (a_1 \land \overline{c_1}), a_0 \lor (a_1 \land \overline{c_0})] = [t_0, t_1]
\]
and
\[
\Delta(b \lor c, b) = [(b_0 \land c_0) \lor (b_1 \land \overline{c_1}), (b_1 \lor c_1) \land (b_0 \lor \overline{c_0})] = [c_0, c_1]
\]
\[
\Delta((a \wedge b) \lor (a \wedge c), a \wedge b) = [(s_0 \land t_0) \lor (s_1 \land \overline{t_1}), (s_1 \lor t_1) \land (s_0 \lor \overline{t_0})] = [a_0, a_1]
\]
Therefore
\[
(s_0 \land t_0) \lor (s_1 \land \overline{t_1}) = a_0 \lor ((a_1 \land \overline{t_1}) \land (a_1 \land \overline{t_0} \land c_0)) = a_0 \lor (a_1 \land [b_0 \land c_0] \lor (b_1 \land \overline{c_1})) = a_0 \lor (a_1 \land [(b_0 \lor \overline{c_0}) \land (b_1 \lor c_1)]) = a_0 \lor (a_1 \land \overline{c_1})
\]
Likewise (essentially dually) we have
\[
(s_1 \lor t_1) \land (s_0 \lor \overline{t_0}) = a_0 \lor (a_1 \land \overline{c_0}).
\]
\qed

Lemma 4.9. Let \( \mathcal{L} \) be an MR algebra, and \( a, b, c, d \in \mathcal{L} \) with \( a \simeq d \) and \( b \simeq c \). Then \( a \wedge b \simeq d \wedge c \).

Proof.
\[
a \wedge b \simeq a \wedge c \simeq c \wedge a \simeq c \wedge d \simeq d \wedge c.
\]
\qed

A similar proof shows us that caret is associative mod \( \simeq \). We leave this to the interested reader.

Lemma 4.10. Let \( \mathcal{L} \) be an MR algebra, and \( a, b, c \in \mathcal{L} \). Then \( a \wedge (b \wedge c) \simeq (a \wedge b) \wedge c \).

Now we consider the second operation that will give rise to joins on \( \mathcal{L}/ \simeq \):
\[
a * b = a \lor \Delta(a \lor b, b).
\]

Lemma 4.11. \( a * b \simeq b * a \).

Proof. We note that \( \Delta(a \lor b, a * b) = b * a \) and so we proceed as in lemma 4.7 \( \square \)

Lemma 4.12. \( b \simeq c \). Then \( a * b = a * c \) and \( b * a = c * a \).
Proof. As \( b \simeq c \) we have \( \ell(b) = \ell(c) \). Therefore we have

\[
a \ast b = a \lor \Delta(a \lor b, b) = [a_0 \land (b_0 \lor \bar{b}_1), a_1 \lor (b_1 \land \bar{b}_0)]
\]

\[
= [a_0 \land \ell(b), a_1 \lor \ell(b)]
\]

\[
= [a_0 \land \ell(c), a_1 \lor \ell(c)]
\]

\[
= a \lor \Delta(a \lor c, c) = a \ast c.
\]

The second result is left to the interested reader. \( \square \)

**Theorem 4.13.** Let \( a \simeq d \) and \( b \simeq c \). Then

\( a \ast b \simeq d \ast c. \)

**Proof.** Since \( a \ast b = a \ast c \simeq c \ast a \ast b \simeq a \ast d \simeq d \ast c. \) \( \square \)

As with caret we can also establish associativity, but this time we have true associativity.

**Theorem 4.14.** For all \( a, b, c \)

\( a \ast (b \ast c) = (a \ast b) \ast c. \)

Lastly we have to define an implication operation, and show that it satisfies the axioms for an implication operation. For any \( a, b \) we let

\( a \Rightarrow b = \Delta(a \lor b, a) \rightarrow b. \)

A direct computation on intervals gives us

\[
a \Rightarrow b = [b_0 \land \ell(a), b_1 \lor \ell(a)].
\]

First this goes through \( \simeq \):

**Lemma 4.15.** Let \( b \simeq c \). Then \( b \Rightarrow a = c \Rightarrow a \) and \( a \Rightarrow b \simeq a \Rightarrow c. \)

**Proof.**

\[
b \Rightarrow a = [a_0 \land \ell(b), a_1 \lor \ell(b)]
\]

\[
= [a_0 \land \ell(c), a_1 \lor \ell(c)] \quad \text{as } b \simeq c
\]

\[
= c \Rightarrow a.
\]

Let \( \alpha = \ell(a) \). Then

\[
a \Rightarrow b = [b_0 \land \overline{\alpha}, b_1 \lor \alpha]
\]

\[
a \Rightarrow c = [c_0 \land \overline{\alpha}, c_1 \lor \alpha]
\]

\[
(a \Rightarrow b) \lor (a \Rightarrow c) = [c_0 \land b_0 \land \overline{\alpha}, c_1 \lor b_1 \lor \alpha] = [x_0, x_1]
\]

\[
\Delta((a \Rightarrow b) \lor (a \Rightarrow c), a \Rightarrow b) = [x_0 \lor (x_1 \land \overline{b}_1 \land \overline{\alpha}), x_1 \land (x_0 \lor \overline{b}_0 \lor \alpha)]
\]

\[
x_1 \land \overline{b}_1 \land \overline{\alpha} = (b_1 \lor c_1 \lor \alpha) \land \overline{b}_1 \land \overline{\alpha}
\]

\[
= c_1 \land \overline{b}_1 \land \overline{\alpha}
\]

Therefore \( x_0 \lor (x_1 \land \overline{b}_1 \land \overline{\alpha}) = (b_0 \lor c_0 \lor \overline{\alpha}) \lor (c_1 \land \overline{b}_1 \land \overline{\alpha})
\]

\[
= \overline{\alpha} \lor ((b_0 \land c_0) \lor (c_1 \land \overline{b}_1))
\]

\[
= \overline{\alpha} \lor c_0 \quad \text{as } b \simeq c.
\]

Likewise we have \( x_1 \land (x_0 \lor \overline{b}_0 \lor \alpha) = c_1 \lor \alpha. \) \( \square \)

Now we need to check that the axioms for implication work:
Theorem 4.16. Let \( a, b, c \) be arbitrary. Then

(a) \( (a \Rightarrow b) \Rightarrow a = a \); 
(b) \( (a \Rightarrow b) \Rightarrow b = b \ast a \); 
(c) \( a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c) \).

Proof. Let \( \alpha = \ell(a) \) and \( \beta = \ell(b) \).

(a) 
\[
(a \Rightarrow b) \Rightarrow a = [b_0 \land \overline{\alpha}, b_1 \lor \alpha] \Rightarrow a
\]
\[
\ell(a \Rightarrow b) = (b_0 \land \overline{\alpha}) \lor b_1 \lor \alpha
\]
\[
= (b_0 \land \overline{\alpha}) \lor (b_1 \land \overline{\alpha})
\]
\[
= \ell(b) \land \overline{\alpha}.
\]
Thus \([b_0 \land \overline{\alpha}, b_1 \lor \alpha] \Rightarrow a = [a_0 \land \overline{\beta} \land \overline{\alpha}, a_1 \lor (\beta \land \overline{\alpha})] \)
\[
= [(a_0 \land \alpha) \lor (a_0 \land \overline{\beta}), (a_1 \lor \overline{\alpha}) \land (a_1 \lor \beta)]
\]
\[
a_0 \land \alpha = a_0 \land (a_0 \lor \overline{\alpha}) = a_0
\]
\[
a_1 \lor \overline{\alpha} = a_1 \lor (\overline{\alpha} \land a_1) = a_1
\]
and so \([a_0 \land \alpha) \lor (a_0 \land \overline{\beta}), (a_1 \lor \overline{\alpha}) \land (a_1 \lor \beta)] = [a_0 \lor (a_0 \land \overline{\beta}), a_1 \land (a_1 \lor \beta)]
\[
= a.
\]

(b) 
\[
(a \Rightarrow b) \Rightarrow b = [b_0 \land \overline{\alpha}, b_1 \lor \alpha] \Rightarrow b
\]
\[
\ell(a \Rightarrow b) = \beta \land \overline{\alpha}.
\]
Thus \([b_0 \land \overline{\alpha}, b_1 \lor \alpha] \Rightarrow b = [b_0 \land \overline{\beta} \land \overline{\alpha}, b_1 \lor (\beta \land \overline{\alpha})] \)
\[
b_0 \land \overline{\beta} = b_0 \land (\overline{b_0} \land b_1) = 0
\]
\[
b_1 \lor \beta = b_1 \lor b_0 \lor \overline{b_1} = 1
\]
and so \([b_0 \land \overline{\beta} \land \overline{\alpha}, b_1 \lor (\beta \land \overline{\alpha})] = [b_0 \land \alpha, b_1 \lor \overline{\alpha}] \)
\[
= b \ast a.
\]

(c) 
\[
a \Rightarrow (b \Rightarrow c) = [c_0 \land \beta, c_1 \lor \overline{\beta}]
\]
\[
= [(c_0 \land \beta) \land \alpha, (c_1 \lor \overline{\beta}) \land \overline{\alpha}].
\]
Since this is symmetric in \( \alpha \) and \( \beta \) we have the result.

We also note that \( a \Rightarrow a = \Delta(a \lor a, a) \rightarrow a = a \rightarrow a = 1 \) and so the 1 of \( L/\simeq \) is \([1]\). Lastly we have meets as expected. Firstly note that \( a \land b \leq a \) and is \( \leq \Delta(a \lor b, b) \simeq b \). And as \( a \leq b \) implies \( a \Rightarrow b = \Delta(a \lor b, a) \rightarrow b = \Delta(b, a) \rightarrow b = 1 \) (as \( \Delta(b, a) \leq b \)) we see that \([a] \leq [b] \). Thus \([a] \land \beta\) is a lower bound to both \([a] \) and \([b] \). Next we want to compute the meet. There are several steps:

Lemma 4.17. \( 1 \Rightarrow a = 1 \).

Proof. \( \ell(1) = 0 \) and so \( 1 \Rightarrow a = [a_0 \lor 0, a_1 \land 1] = a \).

Lemma 4.18. \( (a \Rightarrow b) \ast (b \Rightarrow a) = 1 \).
Proof. Let $\alpha = \ell(a)$ and $\beta = \ell(b)$.

\[
a \Rightarrow b = [b_0 \land \overline{\beta}, b_1 \lor \alpha]\]
\[
b \Rightarrow a = [a_0 \land \overline{\alpha}, a_1 \lor \beta]\]
\[
\ell(b \Rightarrow a) = (a_0 \land \overline{\beta}) \lor (\overline{a}_1 \land \overline{\beta}) = \alpha \land \overline{\beta}.
\]

Hence

\[
(a \Rightarrow b) \ast (b \Rightarrow a) = [b_0 \land \overline{\alpha} \land \alpha \land \overline{\beta}, b_1 \lor \alpha \lor \overline{\alpha} \lor \beta]
\]
\[
= [0, 1].
\]

\[\square\]

Theorem 4.19. $[a] \land [b] = [a \land b]$.

Proof. As in any implication algebra we compute

\[
((a \Rightarrow (a \land b)) \ast (b \Rightarrow (b \land a))) \Rightarrow (a \land b).
\]
\[
a \Rightarrow (a \land b) = [a_1 \land (a_0 \lor \overline{b}_1) \land \overline{\alpha}, a_0 \lor (a_1 \land \overline{b}_0) \lor \alpha]
\]
\[
= [a_1 \land \overline{a}_0 \land \overline{b}_1, a_0 \lor \overline{a}_1 \lor \overline{b}_1]
\]
\[
= [\overline{\alpha} \land \overline{b}_1, \alpha \lor \overline{b}_0]
\]
\[
= \Delta(1, a \Rightarrow b).
\]

Likewise we have

\[
b \Rightarrow (b \land a) = \Delta(1, b \Rightarrow a)
\]

and hence

\[
(a \Rightarrow (a \land b)) \ast (b \Rightarrow (b \land a)) = \Delta(1, (a \Rightarrow b) \ast (b \Rightarrow a))
\]
\[
= \Delta(1, 1) = 1.
\]

Therefore

\[
((a \Rightarrow (a \land b)) \ast (b \Rightarrow (b \land a))) \Rightarrow (a \land b) = 1 \Rightarrow (a \land b)
\]
\[
= a \land b.
\]

\[\square\]

Thus we have established that $\mathcal{L}/ \simeq$ is an implication lattice with the following operations:

\[
1 = [1]
\]
\[
[a] \land [b] = [a \land b]
\]
\[
[a] \lor [b] = [a \lor b]
\]
\[
[a] \Rightarrow [b] = [a \Rightarrow b].
\]

This implication algebra is very closely tied to $\mathcal{L}$. Locally it is exactly $\mathcal{L}$ as the next theorem shows us.

Theorem 4.20. On each interval $[a, 1]$ in $\mathcal{L}$ the mapping $x \mapsto [x]$ is an implication embedding.
We prove this via another short series of lemmas.

**Lemma 4.21.** Let $\mathcal{L}$ be a cubic algebra and $a \in \mathcal{L}$. Then on $[a, 1]$ we have

\[
\begin{align*}
&b * c = b \lor c \\
&b \hat{\ast} c = b \land c \\
&b \Rightarrow c = b \rightarrow c.
\end{align*}
\]

**Proof.** Without loss of generality we are in an interval algebra, and $a = [0, 0]$.

\[
\begin{align*}
b * c &= [0, b] * [0, c] \\
&= [0 \land c, b \lor c] \\
&= [0, b \lor c] \\
&= b \lor c.
\end{align*}
\]

\[
\begin{align*}
b \hat{\ast} c &= b \land c \quad \text{as the meet exists.}
\end{align*}
\]

\[
\begin{align*}
b \Rightarrow c &= [0 \land b, c \lor \bar{b}] \\
&= [0, c \lor \bar{b}] \\
&= b \rightarrow c.
\end{align*}
\]

□

**Lemma 4.22.** Let $\mathcal{L}$ be a cubic algebra and $a \in \mathcal{L}$. If $b, c \geq a$ then

\[
b \simeq c \iff b = c.
\]

**Proof.** If $b = \Delta(b \lor c, c)$ then we have $a \leq c$ and $a \leq b = \Delta(b \lor c, c)$ and so $b \lor c = a \lor \Delta(b \lor c, a) \leq c \lor c = c$. Likewise $b \lor c \leq b$ and so $b = c$. □

We note that a small variation of the proof shows that if $a \leq b, c$ then $b \leq c$ iff $b \leq c$.

What does all this say about caret? We’ve seen that caret collapses to meet. And that locally an MR-algebra is like an implication algebra. In some sense we can then view the MR-algebra as a family of connected implication algebras where $\Delta$ describes the interaction between the pieces. The operation $*$ describes the way the pieces link together (see [1] theorem 4.6) and caret is then describing local self-similarity in the following sense – the interval $[a \lor b]$ has intervals $[a \hat{\ast} b]$ which is similar via $\Delta$ to $[a \lor a]$ and these similar intervals occur densely.

**References**

[1] C.G.Bailey and J.S.Oliveira, An Axiomatization for Cubic Algebras, in ‘Mathematical Essays in Honor of Gian-Carlo Rota’ ed. B.E.Sagan & R.P.Stanley, Birkhäuser 1998.

[2] C.G.Bailey and J.S.Oliveira, Complementation & Orthogonality in Cubic Implication Algebras, in preparation.

[3] N. Metropolis and G.-C.Rota, Combinatorial Structure of the faces of the n-Cube, SIAM J.Appl.Math. 35 (1978) 689-694.

[4] J. S. Oliveira, The Theory of Cubic Lattices, Ph.D. thesis, MIT, 1992.