GENERALIZED HYDRODYNAMIC REDUCTIONS OF THE KINETIC EQUATION FOR A SOLITON GAS

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We derive generalized multiflow hydrodynamic reductions of the nonlocal kinetic equation for a soliton gas and investigate their structure. These reductions not only provide further insight into the properties of the new kinetic equation but also could prove to be representatives of a novel class of integrable systems of hydrodynamic type beyond the conventional semi-Hamiltonian framework.

Keywords: kinetic equation, Riemann invariant, hydrodynamic reduction

1. Introduction

The generalized soliton-gas kinetic equation is the integro-differential system [1]

\[ \frac{f_t}{t} + (sf)_x = 0, \]

\[ s(\eta) = S(\eta) + \frac{1}{\eta} \int_0^\infty G(\eta, \mu) f(\mu) [s(\mu) - s(\eta)] d\mu. \]

Here, \( f(\eta) \equiv f(\eta, x, t) \) is the distribution function, and \( s(\eta) \equiv s(\eta, x, t) \) is the associated transport velocity. The given functions \( S(\eta) \) and \( G(\eta, \mu) \) are independent of \( x \) and \( t \). The function \( G(\eta, \mu) \) is assumed to be symmetric, i.e., \( G(\eta, \mu) = G(\mu, \eta) \).

System (1), (2), where

\[ S(\eta) = 4\eta^2, \quad G(\eta, \mu) = \log \left( \frac{\eta - \mu}{\eta + \mu} \right), \]

was derived in [2] as the thermodynamic limit (when the number of bands appropriately tends to infinity) of the Whitham modulation equations associated with the Korteweg-de Vries (KdV) equation \( \varphi_t - 6 \varphi \varphi_x + \varphi_{xxx} = 0 \). It was shown that this system describes the macroscopic dynamics of a soliton gas, a disordered infinite-soliton ensemble of finite density [3]. In the KdV context, \( \eta \geq 0 \) is a real-valued spectral parameter, and the function \( f(\eta, x, t) \) is the distribution function of solitons over the spectrum such that \( \kappa = \int_0^\infty f(\eta) d\eta = O(1) \) is the spatial density of solitons. If \( \kappa \ll 1 \), the Zakharov kinetic equation for a dilute gas of KdV solitons in the first-order approximation is obtained from (2) and (3) (see [4]). The quantity \( S(\eta) \) in (2) has a natural meaning of the velocity of an isolated (free) soliton with the spectral parameter \( \eta \), and the function \( G(\eta, \mu) / \eta \) is the expression for a phase shift of this soliton occurring after

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its collision with another soliton with the spectral parameter $\mu < \eta$. Then $s(\eta, x, t)$ acquires the meaning of the self-consistently defined mean local velocity of solitons with the spectral parameter close to $\eta$. A straightforward physical derivation of kinetic equation (1), (2) for integrable systems based on using the soliton phase shifts (as was done in Zakharov’s original work [4]) was proposed in [5].

Multiflow hydrodynamic reductions of kinetic equation (1), (2) were recently studied using the so-called cold-gas ansatz

$$f(\eta, x, t) = \sum_{m=1}^{N} f^m(x, t) \delta(\eta - \eta^m),$$  \hspace{1cm} (4)

where the “spectral” components $\eta^N > \eta^{N-1} > \cdots > \eta^1 > 0$ are arbitrary numbers [1]. These “isospectral” cold-gas reductions were shown to have the form of systems of hydrodynamic conservation laws

$$u^i_t = (u^i v^i)_x, \quad i = 1, \ldots, N,$$  \hspace{1cm} (5)

where the conserved “densities” $u^i = \eta^i f(\eta^i, x, t)$ and the associated velocities $v^i = -s(\eta^i, x, t)$ are related algebraically,

$$v^i = \xi_i + \sum_{m \neq i} \epsilon^{im} u^m (v^m - v^i), \quad \epsilon^{ik} = \epsilon^{ki}.$$  \hspace{1cm} (6)

Here,

$$\xi_i = -S(\eta^i), \quad \epsilon^{ik} = \frac{G(\eta^i, \eta^k)}{\eta^i \eta^k}, \quad i \neq k.$$  \hspace{1cm} (7)

It was proved in [1] that isospectral cold-gas reductions (5), (6) are integrable (semi-Hamiltonian [6]) linearly degenerate systems of hydrodynamic type (see [7], [8]) for arbitrary $N$. This is a strong indication that the full kinetic equation (1), (2) could turn out to be integrable in a sense yet to be explored.

The present paper is devoted to a more general multiflow hydrodynamic approximation of kinetic equation (1), (2), which we derive by considering the ansatz (see, e.g., [9])

$$f(\eta, x, t) = \sum_{m=1}^{N} f^m(x, t) \delta(\eta - \eta^m(x, t)),$$  \hspace{1cm} (8)

where the “spectral components” $\eta^k = \eta^k(x, t)$ are (unknown) functions of $x$ and $t$ in contrast to the arbitrary constants in (4). We show that the corresponding $N$-flow nonisospectral hydrodynamic reductions have the form of $2N$-component systems of hydrodynamic type

$$u^i_t = (u^i v^i)_x, \quad \eta^i_t = v^i \eta^i_x, \quad i = 1, 2, \ldots, N,$$  \hspace{1cm} (9)

where the functions $u^i(x, t)$, $v^i(x, t)$, and $\eta^i(x, t)$ are algebraically related by the same equations (6), (7) if certain restrictions on the behavior of the kernel function $G(\eta, \mu)$ as $\eta \rightarrow \mu$ are satisfied.

System (5), (6), (9) is not integrable by the standard Tsarev generalized hodograph method, because it has only $N$ Riemann invariants and has double characteristic velocities. But taking into account that this system is obtained as an exact reduction of an integrable system (at least, for $S(\eta)$ and $G(\eta, \mu)$ defined in KdV case (3)), we can expect that multiflow reductions (9) are integrable by some modification of the generalized hodograph method [6]. This could lead to an extension of the conventional notion of an integrable system of hydrodynamic type. We plan to investigate this problem in the future.
2. Generalized hydrodynamic reductions

2.1. Evolution equations. Substituting (8) in (1), we obtain

\[ \frac{\partial}{\partial t} \left( \sum_{i=1}^{N} f^i(x,t)\delta(\eta - \eta^i) \right) + \frac{\partial}{\partial x} \left( s(\eta, x, t) \sum_{i=1}^{N} f^i(x,t)\delta(\eta - \eta^i) \right) = 0 \]

(here and hereafter, we use the shorthand notation \( \eta^i \) for \( \eta^i(x,t) \)). Differentiating and collecting the terms for \( \delta(\eta - \eta^i) \) and \( \delta'(\eta - \eta^i) \), we obtain

\[ \sum_{n=1}^{N} [f^n_i + (s(\eta, x, t)f^n)x] \delta(\eta - \eta^n) - \sum_{n=1}^{N} [f^n_i \eta^n_t + s(\eta, x, t)f^n \eta^n_x] \delta'(\eta - \eta^n) = 0. \] (10)

Here, \( f^i \equiv f^i(x,t) \). Evaluating the asymptotic behavior of this expression near each point \( \eta^i \), we obtain a 2N-component system of hydrodynamic type (cf. (5))

\[ f^n_t + (s(\eta^i, x, t)f^n)x = 0, \quad \eta^n_t + s(\eta^i, x, t)\eta^n_x = 0, \quad n = 1, \ldots, N. \] (11)

It is instructive to derive hydrodynamic reduction (11) by a direct calculation. This is done by integrating (10) with respect to \( \eta \) over a small vicinity of each point \( \eta = \eta^i \) with the corresponding weights 1 and \( \eta - \eta^i \).

We set \( x = x_0 \). Choosing \( \eta^N > \eta^{N-1} > \cdots > \eta^1 > 0 \) for all \( t \) in a small vicinity of \( x_0 \) for definiteness, we then introduce \( N \) closed intervals \( \sigma_i = [\eta^i - \varepsilon_i, \eta^i + \varepsilon_i] \) choosing \( \varepsilon_i > 0 \) such that \( \eta^i(x,t) \in \sigma_i \) iff \( j = i \) in the vicinity of \( x_0 \).

We now integrate (10) over the interval \( \sigma_i \),

\[ \int_{\sigma_i} \left[ \sum_{n=1}^{N} [f^n_i + (s(\eta, x, t)f^n)x] \delta(\eta - \eta^n) - \sum_{n=1}^{N} [f^n_i \eta^n_t + s(\eta, x, t)f^n \eta^n_x] \delta'(\eta - \eta^n) \right] d\eta = 0, \]

which after integrating the term with \( \delta'(\eta - \eta^i) \) by parts leads to

\[ \int_{\sigma_i} \left[ \sum_{n=1}^{N} [f^n_i + s(\eta, x, t)f^n] + \frac{\partial s(\eta, x, t)}{\partial x} f^n + \frac{\partial s(\eta, x, t)}{\partial \eta} f^n \eta^n_x \right] \delta(\eta - \eta^n) \right] d\eta = 0. \] (12)

Further, integrating over \( \sigma_i \) leads directly to the hydrodynamic conservation law

\[ f^n_t + (s(\eta^i, x, t)f^n)x = 0, \] (13)

which holds in the small vicinity of \( x_0 \). Under the assumption that the above restrictions on the behavior of the functions \( \eta^i(x,t) \) hold for all \( x = x_0 \in \mathbb{R} \) and \( t > 0 \), Eq. (13) is satisfied on the entire real line. Setting \( i = 1, \ldots, N \) in (12), we immediately obtain the first \( N \) equations in system (11).

To derive the second half of the equations in (11), we multiply (10) by \( (\eta - \eta^i) \) and integrate over the interval \( \sigma_j \). We then obtain

\[ \int_{\sigma_j} \left[ \sum_{n=1}^{N} [f^n_i + s(\eta, x, t)f^n)x] \delta(\eta - \eta^n)(\eta - \eta^i) \right] d\eta - \int_{\sigma_j} \left[ \sum_{n=1}^{N} [f^n_i \eta^n_t + s(\eta, x, t)f^n \eta^n_x](\eta - \eta^j) \delta'(\eta - \eta^n) \right] d\eta = 0. \] (14)
If \( j = i \), then the first integral vanishes, and the second one after integration by parts leads to

\[
\int_{\sigma_i} \frac{\partial}{\partial \eta} \left( (\eta - \eta^i_t)[f^i \eta_t^i + s(\eta, x, t)f^i \eta^i_x] \right) \delta(\eta - \eta^i_t) \, d\eta = \\
= \int_{\sigma_i} \left( f^i \eta_t^i + s(\eta, x, t)f^i \eta^i_x \right) \frac{\partial s(\eta, x, t)}{\partial \eta} f^i \eta_t^i \delta(\eta - \eta^i_t) \, d\eta = 0,
\]

(15)

where we take into account that each interval \( \sigma_i \) contains only its “own” value \( \eta^i_t \). Evaluating the integral in (15), we obtain

\[
\eta^i_t + s(\eta^i, x, t)\eta^i_x = 0, \quad i = 1, \ldots, N.
\]

(16)

It is easy to see that if \( j \neq i \), then we recover Eqs. (13). The compatibility of nonisospectral ansatz (8) with kinetic equation (1), (2) thus leads to (16) for the functions \( \eta^i(x, t) \).

Altogether, 2\( N \)-component hydrodynamic-type system (11) has \( N \) conservation laws

\[
\partial_t (\varphi_i(\eta^i)f^i) + (s(\eta^i, x, t)\varphi_i(\eta^i)f^i)_x = 0,
\]

where \( \varphi_i(\eta^i) \) are arbitrary functions of one variable. For subsequent calculations, it is convenient to choose \( \varphi_i(\eta^i) = \eta^i \), and (11) then leads to (cf. (5))

\[
u^i = (u^i v^i)_x, \quad \eta^i_t = v^i \eta^i_x, \quad i = 1, \ldots, N,
\]

(17)

where \( u^i = \eta^i f^i \) and \( v^i = -s(\eta^i, x, t) \).

2.2. Closure relations. The closure relations connecting the field variables \( u^i, \eta^i, \) and \( v^i \) in (17) are obtained by substituting the same ansatz (8) in integral equation (2). Because we use the variables \( u^i \) instead of \( f^i \), we slightly modify ansatz (8):

\[
\eta f(\eta, x, t) = \sum_{i=1}^N u^i(x, t) \delta(\eta - \eta^i).
\]

(18)

Substituting (18) in (2) yields

\[
s(\eta, x, t) = S(\eta) + \sum_{m=1}^N u^m \frac{G(\eta, \eta^m)}{\eta \eta^m} [s(\eta^m, x, t) - s(\eta, x, t)].
\]

(19)

As in [1], we introduce (see (7))

\[
\epsilon^{ik} = \frac{G(\eta^i, \eta^k)}{\eta^i \eta^k}, \quad i \neq k.
\]

(20)

There is an important point to be made. In linearly degenerate reductions (5) and (6) associated with isospectral ansatz (4) involving arbitrary constants \( \eta^i \), the dependence of \( \xi^i = -S(\eta^i) \) and \( \epsilon^{ik}(\eta^i, \eta^k) \) on the relevant components of the vector \( \eta = \{ \eta^1, \eta^2, \ldots, \eta^N \} \) is not important from the integrability standpoint: it only provides the connection with the original nonlocal equation (2) (see [1]). But with generalized ansatz (8), the \( \eta^i \) become dependent variables, \( \eta^i = \eta^i(x, t) \), and that dependence becomes essential for the structure of the corresponding hydrodynamic reductions.

We now pass to the limit \( \eta \rightarrow \eta^i \) in (19):

\[
v^i = \sum_{m \neq i} \epsilon^{im} u^m (v^i - v^m) - S(\eta^i) + \frac{u^i}{(\eta^i)^2} \lim_{\eta \rightarrow \eta^i} G(\eta, \eta^i)(s(\eta, x, t) - s(\eta^i, x, t)),
\]

(21)
where \( v^i = \lim_{\eta \to \eta^i} s(\eta, x, t) \) (continuity). If

\[
\lim_{\eta \to \eta^i} G(\eta, \eta^j)(s(\eta, x, t) - s(\eta^j, x, t))
\]

exists, then (21) becomes

\[
v^i = \sum_{m \neq i} \epsilon^{im} u^m (v^i - v^m) - S(\eta^i) + g_i(u, v, \eta),
\]

(23)

where

\[
g_i(u, v, \eta) = \frac{u^i}{(\eta^i)^2} \lim_{\eta \to \eta^i} G(\eta, \eta^i)[s(\eta, x, t) - s(\eta^i, x, t)].
\]

The existence of limit (22) implies that \( G(\eta, \mu) \) has at most a simple pole singularity on the diagonal \( \mu = \eta \).

If limit (22) vanishes for all \( i = 1, \ldots, N \) (which happens if \( G(\eta, \mu) \) either vanishes itself or has a singularity weaker than a simple pole as \( \mu \to \eta \)) then \( g_i \equiv 0 \), and Eq. (23) reduces to closure conditions (6) and (7) obtained for the isospectral cold-gas reduction. Below, we restrict our consideration to only this most important case, which in particular arises in the case of the kinetic equation for the KdV solitons [2], where the kernel function \( G(\eta, \mu) \) has only a logarithmic singularity on the diagonal (see (3)).

Concluding this section, we note that the nonexistence of limit (22) for some given \( G(\mu, \eta) \) signifies that delta-function ansatz (18) is incompatible with integral equation (2) for the corresponding kernel \( G(\mu, \eta) \).

3. The structure of generalized multiflow hydrodynamic reductions

Using the results in [1] for isospectral cold-gas hydrodynamic reductions (5) and (6), we introduce a symmetric matrix \( \hat{\epsilon} = [\epsilon^{mn}]_{N \times N} \) whose off-diagonal elements \( \epsilon^{ik}(\eta) \) are defined by (20) and diagonal elements \( \epsilon^{kk} \) are some new field variables \( r^k(u, \eta) \).

**Theorem 1** (see [1]). Algebraic system (6) admits the parametric solution

\[
u^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi_m \beta_{mi}, \quad v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi_m \beta_{mi},
\]

(24)

where the symmetric functions \( \beta_{ik}(r, \eta) \) are elements of the matrix \( \hat{\beta} = [\beta_{mn}]_{N \times N} \) such that \( \hat{\beta} \hat{\epsilon} = -\mathbf{1} \).

**Proof.** We replace (6) with the equivalent system

\[
v^i = \xi_i + \sum_{m=1}^{N} \epsilon^{im} u^m (v^m - v^i)
\]

(25)

(we note that the summation ranges all \( m \) including \( m = i \); cf. (6)), which can be rewritten as

\[
v^i \left( 1 + \sum_{m=1}^{N} \epsilon^{im} u^m \right) = \xi_i + \sum_{m=1}^{N} \epsilon^{im} u^m v^m.
\]

Substituting (24) in this formula, we obtain

\[
v^i \left( 1 + \sum_{m=1}^{N} \sum_{k=1}^{N} \beta_{mk} \epsilon^{ki} \right) = \xi_i + \sum_{m=1}^{N} \sum_{k=1}^{N} \xi_m \beta_{mk} \epsilon^{ki}.
\]

(26)

Taking \( \hat{\beta} \hat{\epsilon} = -\mathbf{1} \) into account, we can see that the two sides of (26) vanish independently. Therefore, (26) is an identity, and parametric representation (24) is hence consistent with system (6). The theorem is proved.
Corollary 1. The field variables \( r^k(u, \eta) \) are rational functions of the conserved densities \( u^m \), namely,
\[
r^k = -\frac{1}{u^k} \left( 1 + \sum_{m \neq k} u^m \epsilon^{mk}(\eta) \right), \quad k = 1, 2, \ldots, N.
\]
Indeed, multiplying both sides of the first relation in (24) by \( \epsilon^{ik} \) and summing over \( i \), we obtain
\[
\sum_{m=1}^{N} u^m \epsilon^{mk} = \sum_{m=1}^{N} \sum_{n=1}^{N} \beta_{mn} \epsilon^{nk} = -1.
\]
Hence,
\[
\sum_{m \neq k} u^m \epsilon^{mk} + r^k u^k = -1,
\]
which immediately yields (27).

Theorem 2. Under parameterization (24), 2N-component hydrodynamic-type system (17), (6), (7) reduces to a quasidiagonal form:
\[
\eta^i_t = v^i \eta^i_x, \quad i = 1, \ldots, N,
\]
\[
r^k_t = v^k r^k_x + \frac{1}{u^k} \left( \sum_{n \neq k} u^n (v^n - v^k) \frac{\partial \epsilon^{nk}}{\partial \eta^k} + \xi^k \right) \eta^k_x, \quad k = 1, \ldots, N.
\]

Proof. Evolution equations (28) for \( \eta^i(x, t) \) are the same as in (17), and we therefore need only derive Eqs. (29) for \( r^k(x, t) \), \( k = 1, \ldots, N \). Substituting parametric representation (24) in conservation laws (17), we obtain
\[
\partial_t \left( \sum_{m=1}^{N} \beta_{mi} \right) = \partial_x \left( \sum_{m=1}^{N} \xi_m \beta_{mi} \right).
\]
Multiplying both sides by \( \epsilon^{ik} \), summing over the repeated index \( i \), and using the relation \( \hat{\beta} \hat{\epsilon} = -1 \), we obtain the equation
\[
\sum_{i=1}^{N} \sum_{m=1}^{N} \beta_{mi} \partial_t \epsilon^{ik} = \sum_{i=1}^{N} \sum_{m=1}^{N} \xi_m \beta_{mi} \partial_x \epsilon^{ik} + \partial_x \xi^k. \tag{30}
\]
With the obvious property of the matrix \( \epsilon(r, \eta) \) that
\[
\frac{\partial \epsilon^{nk}}{\partial \eta^s} = \delta_{nk} \delta_{ks}
\]
and also evolution equations (28) for \( \eta^k \) taken into account, a simple but not entirely trivial calculation shows that (30) reduces to
\[
r^k_t = v^k r^k_x + \frac{1}{u^k} \left( \sum_{s=1}^{N} \sum_{n=1}^{N} \beta_{mn} (\xi_m - \nu^s) \frac{\partial \epsilon^{nk}}{\partial \eta^s} + \frac{\partial \xi^k}{\partial \eta^s} \right) \eta^s_x, \quad k = 1, \ldots, N. \tag{31}
\]
Because
\[
\sum_{m=1}^{N} \beta_{mi} = u^i, \quad \sum_{m=1}^{N} \xi_m \beta_{mi} = u^i \nu^i
\]
(see (24)), Eqs. (31) reduce to form (29). The theorem is proved.
Eliminating $u^i$ from (24), we obtain expressions relating $v$ to $\eta$ and $r$, 

$$
v^i(r, \eta) = \frac{\sum_{m=1}^{N} \xi_m \beta_{mi}}{\sum_{m=1}^{N} \beta_{mi}},
$$

(32)

while $u^i(r, \eta)$ are given by the first equation in (24). System (28), (29) is now closed.

In the isospectral case, where $\eta^i$, $i = 1, 2, \ldots, N$, are constants (i.e., $\partial_t \eta^i = \partial_x \eta^i = 0$), from (29), we obtain the representation of system (5), (6) in the diagonal form

$$
r^i_t = v^i(r)r^i_x,
$$

(33)

which was found in [1] using the linear degeneracy of the isospectral cold-gas hydrodynamic reductions. In fact, it can now be seen that Riemann invariant equations (33) can be obtained directly from system (5), (6) using parameterization (24).

Therefore, 2N-component hydrodynamic reduction (17) admits parameterization (24) of algebraic system (6), leading to the evolution equations of form (28), (29). System (28), (29) has the double characteristic velocities $v^k(r, \eta)$ given by (32). But in the general case, just $N$ functions $\eta^k(x, t)$ are Riemann invariants (i.e., only half of the complete hydrodynamic system (17) can be written in diagonal form), and the field variables $r^k(x, t)$ become Riemann invariants only if the corresponding $\eta^k = \text{const}$.

Concluding this section, we note that the linear degeneracy of system (33), proved in [1], means that $\partial v^i(r)/\partial r^i = 0$ for all $i = 1, \ldots, N$. The last property clearly holds for the characteristic velocities $v^i(r, \eta)$ of generalized hydrodynamic reductions (28), (29). But this is now no longer associated with the notion of linear degeneracy of systems of hydrodynamic type in the classical sense [10], [11], because $r^k$ are no longer Riemann invariants and also $\partial v^i(r, \eta)/\partial \eta^i \neq 0$.

4. Conclusion

We have here derived the generalized hydrodynamic reductions of nonlocal kinetic equation (1), (2) for a soliton gas using nonisospectral multiflow ansatz (8) for the distribution function. These new reductions turned out to have a rather unusual structure, which we revealed using parameterization (24) applied to algebraic closure conditions (6) and (7). More precisely, we showed that the nonisospectral $N$-flow hydrodynamic reductions of the kinetic equation represent 2N-component half-diagonal hydrodynamic-type system (28), (29) with $N$ Riemann invariants and $N$ double characteristic velocities. Although the derived generalized hydrodynamic reductions are clearly not integrable by the generalized hodograph method [6], they could still prove to be integrable in some new sense yet to be understood. Indeed, having in mind that system (28), (29) can be derived as a generalized hydrodynamic reduction of the kinetic equation associated with an integrable equation (e.g., with the KdV equation for $S(\eta)$ and $G(\eta, \mu)$ defined by (3)), we can expect that this reduction is integrable by some nontrivial extension of the generalized hodograph method.

A by-product of our calculations is a demonstration that isospectral hydrodynamic cold-gas reductions can be brought to form (5)–(7), previously obtained in [1]. We note that the compact derivation presented here, in contrast to that used in [1], is not based on the linear degeneracy of the studied hydrodynamic reductions.

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