Detecting automorphic orbits in free groups

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Abstract

We present an effective algorithm for detecting automorphic orbits in free groups, as well as a number of algorithmic improvements of train tracks for free group automorphisms.

Introduction

The following theorem is the main result of this paper.

Theorem 0.1. Let $\phi$ be an automorphism of a finitely generated free group $F_n$.

- There exists an explicit algorithm that, given two elements $u, v \in F_n$, decides whether there exists some exponent $N$ such that $u\phi^N = v$.

- There exists an explicit algorithm that, given two elements $u, v \in F_n$, decides whether there exists some exponent $N$ such that $u\phi^N$ is conjugate to $v$.

If such an exponent $N$ exists, then the algorithms will compute $N$ as well. The words $u, v$ are specified as words in the generators of $F_n$, and $\phi$ is specified in terms of the images of generators.

The results in this paper was motivated by work that first appeared in [Bri03]. Theorem 0.1 plays a role in the computation of fixed subgroups of free group automorphisms [Mas03], and it constitutes one part of the recent solution of the conjugacy problem in free-by-cyclic groups due to Bogopolski, Maslakova, Martino, and Ventura [BMMV06].
Our main technical tool is an algorithmic extension of the theory of relative train track maps \cite{BH92, BH95}. Specifically, we present algorithmic (and possibly even practical) ways of finding efficient relative train track maps that share many of the properties of improved relative train track maps as introduced (in a nonconstructive fashion) in \cite{BFH00}.

One intriguing aspect of our argument is that it suggests that the detection of orbits in free groups and the computation of efficient maps are closely related problems. Orbit detection and computation of efficient maps leapfrog each other, with orbit detection providing a crucial step in the computation of efficient maps, and efficient maps enabling the detection of orbits.

In Section 1, we review well-known results on homotopy equivalences of finite graphs, with an emphasis on computational aspects of the constants involved. Section 2 contains a brief review of the theory of relative train track maps, including first steps towards improvements. Section 3 contains the first part of our construction of efficient train track maps. Section 4 presents an algorithm that detects orbits of paths, and Section 5 builds upon the results of Section 3 and Section 4 to provide the last, and most difficult, step in our construction of efficient maps, the detection of fixed points of certain lifts of homotopy equivalences of finite graphs. Finally, in Section 6, we translate our results from the realm of homotopy equivalences of graphs to the realm of automorphisms of free groups.

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1 Quasi-isometries and bounded cancellation

The results in this section are well-known. We list them here, with detailed proofs, because explicit computations of the constants involved do not seem to appear in the literature.

Let $f: G \to H$ be a homotopy equivalence of finite connected graphs, which we equip with the usual path metric (denoted by $|.|$), and let $g: H \to G$ be a homotopy inverse of $f$. We denote the set of vertices of $G$ by $V(G)$ and the set of edges by $E(G)$. Throughout this paper, we only consider homotopy equivalences that map vertices to vertices and edges to edge paths of constant (but not necessarily identical) speed. We may assume that there exists some vertex $\bar{v}_0$ such that $\bar{v}_0 fg = \bar{v}_0$.

\footnote{Given $f$, we can easily compute $g$ (see, for instance, \cite{LS77}).}
Let $\tilde{f}: \tilde{G} \to \tilde{H}$ be a lift of $f$ to the universal covers, with a lift $v_0$ of $\bar{v}_0$. Given $x, y \in \tilde{G}$, we denote the unique geodesic path connecting $x$ and $y$ by $[x, y]$. For brevity, we write $|x, y|$ for $|[x, y]|$. We define $[x, y]\tilde{f} = [\bar{x}\tilde{f}, \bar{y}\tilde{f}]$.

The lift $\tilde{f}$ extends to a homeomorphism of the boundaries $\partial \tilde{G}, \partial \tilde{H}$. Let $\tilde{g}: \tilde{H} \to \tilde{G}$ be a lift of $g$ such that satisfies $v_0\tilde{f}\tilde{g} = v_0$, and note that $\tilde{f}\tilde{g}$ induces the identity on $\partial \tilde{G}$.

Arguments involving universal covers are generally nonconstructive. The universal cover of a finite connected graph, however, is a tree, and we can construct arbitrarily large subtrees as well as partial lifts of maps to these subtrees, which is enough for the computations we will encounter. We describe this construction here, with the tacit understanding that all computations in universal covers will require it as a preliminary step.

**Construction 1.1.** Fix some vertex $\bar{v}_0 \in G$. Let $v_0 \in \tilde{G}$ be a lift of $\bar{v}_0$ and $w_0 \in \tilde{H}$ a lift of $\bar{w}_0 = \bar{v}_0f$. We let $T_0 = \{v_0\}$ and $U_0 = \{w_0\}$ and define $\tilde{f}_0: T_0 \to U_0$ in the only possible way.

Now, suppose we have subtrees $T_0 \subseteq T_1 \subset \tilde{G}$ and $U_0 \subseteq U_1 \subset \tilde{H}$ as well as a partial lift $\tilde{f}_1: T_1 \to U_1$, i.e., $\tilde{f}_1|_{T_1} = \tilde{f}_1$. Our goal is to enlarge $T_1$ and $U_1$ and extend $\tilde{f}_1$ accordingly.

There is a bijective relationship between vertices of $\tilde{G}$ and edge paths in $G$ originating at $\bar{v}_0$. Let $\rho$ be an edge path in $G$ originating at $\bar{v}_0$. We want to construct $T_2$ so that it contains a lift of $\rho$. To this end, starting with $v_0$ and the first edge of $\rho$, we keep track of a current vertex $v$ and a current edge $E$. If $T_1$ already contains an edge $E'$ originating at $v$ that projects to $E$, we make the other endpoint of $E'$ our current vertex and move on to the next edge of $\rho$. If no such edge exists, we attach a new edge at $v$ and map it to $E$. Then we move on to the terminal endpoint of the new edge and the next edge in $\rho$.

Now, for each vertex $v$ of $T_2 \setminus T_1$, we compute the image $\rho_v$ of the path $[v_0, v]$ in $G$, and we construct a lift of $\rho_v\tilde{f}$ to the universal cover. Like before, we construct $U_2$ by extending $U_1$ such that it includes these lifts, obtaining a larger subtree of $\tilde{H}$ as well as a partial lift $\tilde{f}_2: T_2 \to U_2$.

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2Note that the composition of the path $[x, y]$ and $\tilde{f}$ is not, in general, an immersion. The path $[x\tilde{f}, y\tilde{f}]$ is the unique immersed path that is homotopic relative endpoints to this composition.

3In our computations, we will always be given such paths for those vertices of $\tilde{G}$ that we are interested in.

4An alternative approach is to attach an entire lift of $\rho$ at $v_0$ and then fold as necessary. [Sta83].
Proceeding in this fashion, we can build arbitrarily large subtrees of $\tilde{G}$ and $\tilde{H}$ along with partial lifts of $f$. If $G = H$, we can and will arrange that $T_2 \subseteq U_2$.

The lift $\tilde{f}$ is a quasi-isometry, i.e., there exist constants $K_f, D_f$ such that for all $x, y \in \tilde{G}$, we have

$$\frac{|x, y|}{K_f} - D_f \leq |x \tilde{f}, y \tilde{f}| \leq K_f|x, y| + D_f. \quad (1)$$

We need to compute suitable constants $K_f, D_f$. To this end, define the size of $f$ to be $S_f = \max_{E \in \mathcal{E}(G)} \{|E|\}$.

**Lemma 1.2.** We can compute a number $B_{fg}$ satisfying

$$B_{fg} \geq \max_{x \in \tilde{G}} \{|x, x \tilde{f} \tilde{g}|\}.$$  

**Proof.** We first compute $B = \max_{v \in \mathcal{V}(\tilde{G})} \{|v, v \tilde{f} \tilde{g}|\}$. Let $\gamma$ be a deck transformation of $\tilde{G}$. Since $\tilde{f} \tilde{g}$ extends to the identity on $\partial \tilde{G}$, we have $\gamma \tilde{f} \tilde{g} = \tilde{f} \tilde{g} \gamma$.

For $v \in \mathcal{V}(\tilde{G})$, we have $|v \gamma, v \gamma \tilde{f} \tilde{g}| = |v \gamma, v \tilde{f} \tilde{g} \gamma| = |v, v \tilde{f} \tilde{g}|$, so that we only need to check one representative of each orbit of vertices. The distance $|v, v \tilde{f} \tilde{g}|$ is the length of the path obtained by concatenating $[v, v_0]$ and $[v_0, v \tilde{f} \tilde{g}]$ and tightening. Hence, we can compute $B$.

Now consider some point $x \in \tilde{G}$. Then there exists some vertex $v \in \mathcal{V}(\tilde{G})$ such that $|x, v| < 1$, so that $|x, x \tilde{f} \tilde{g}| \leq 1 + |v, v \tilde{f} \tilde{g}| + S_{fg} \leq 1 + B + S_{fg}$. $\square$

**Lemma 1.3.** Inequality $(1)$ holds with $K_f = \max\{S_f, S_g\}$ and $D_f = \frac{2R_{fg}}{K_f}$.

**Proof.** Let $x, y \in \tilde{G}$. By definition of $K_f$, we have $|x \tilde{f}, y \tilde{f}| \leq K_f|x, y|$, so that the upper bound in Inequality $(1)$ holds.

Similarly, we have $|x \tilde{f} \tilde{g}, y \tilde{f} \tilde{g}| \leq K_f|x \tilde{f}, y \tilde{f}|$. The triangle inequality implies that $|x, y| \leq |x, x \tilde{f} \tilde{g}| + |x \tilde{f} \tilde{g}, y \tilde{f} \tilde{g}| + |y \tilde{f} \tilde{g}, y| \leq |x \tilde{f} \tilde{g}, y \tilde{f} \tilde{g}| + 2B_{fg} \leq K_f|x \tilde{f}, y \tilde{f}| + 2B_{fg}$. We conclude that $|x, y| - 2B_{fg} \leq K_f|x \tilde{f}, y \tilde{f}|$, and the claim follows. $\square$

Thurston’s *Bounded Cancellation Lemma* [Coo87] is a fundamental tool in the theory of free group automorphisms. We present a proof here because we require an explicit bound on the constant involved.

Let $p, x, y$ be points in $\tilde{G}$ and let $\alpha = [p, x]$ and $\beta = [p, y]$. We denote the common (possibly trivial) initial segment of $\alpha$ and $\beta$ by $\alpha \land \beta$. If $\alpha$ is a prefix of $\beta$, we write $\alpha \leq \beta$. 


Lemma 1.4 (Bounded Cancellation Lemma). Let $C_f = (B_f + D_g + S_g)K_g$. If $|\alpha \land \beta| = 0$, then $$|\alpha \tilde{f} \land \beta \tilde{f}| \leq C_f.$$ 

Proof. Let $L = |\alpha \tilde{f} \land \beta \tilde{f}|$. Inequality (1) implies that $$|(\alpha \tilde{f} \land \beta \tilde{f}) \tilde{g}| \geq \frac{L}{K_g} - D_g,$$ so that $$|\alpha \tilde{f} \tilde{g} \land \beta \tilde{f} \tilde{g}| \geq \frac{L}{K_g} - D_g - S_g.$$ Now Lemma 1.2 implies that $$|\alpha \land \beta| \geq \frac{L}{K_g} - D_g - S_g - B_{fg}.$$

Hence, if $L > C_f$, then $|\alpha \land \beta| > 0$. 

Finally, we record a basic property of homotopy equivalences of graphs. 

Lemma 1.5. Let $f: G \to G$ be a homotopy equivalence of a finite graph. If $\alpha$ is a path in $G$ whose endpoints are fixed by $f$, then there exists some path $\beta$ with the same endpoints satisfying $\beta f = \alpha$.

Proof. Let $v$ be the initial endpoint of $\alpha$. Then there exists some loop $\sigma$ based at $v$ so that $\alpha f$ is homotopic (relative endpoints) to the concatenation $\sigma \alpha$. Since $f$ is a homotopy equivalence, there exists a loop $\sigma'$ satisfying $\sigma' f = \sigma$, and we conclude that $(\sigma' \alpha) f = \alpha$. 

2 Relative train track maps

In this section, we review the theory of relative train tracks maps [BH92, DV96] as well as first steps towards our take on improvements of relative train track maps.

Given an automorphism $\phi \in \text{Aut}(F)$, we can find a based homotopy equivalence $f: G \to G$ of a finite connected graph $G$ such that $\pi_1(G) \cong F$ and $f$ induces $\phi$. This observation allows us to apply topological techniques to automorphisms of free groups. In many cases, it is convenient to work with outer automorphisms. Topologically, this means that we work with homotopy equivalences rather than based homotopy equivalences.

Oftentimes, a homotopy equivalence $f: G \to G$ will respect a filtration of $G$, i.e., there exist subgraphs $G_0 = \emptyset \subset G_1 \subset \cdots \subset G_k = G$ such that for each filtration element $G_r$, the restriction of $f$ to $G_r$ is a homotopy equivalence of $G_r$. The subgraph $H_r = G_r \setminus G_{r-1}$ is called the $r$-th stratum.
of the filtration. We say that a path $\rho$ has nontrivial intersection with a stratum $H_r$ if $\rho$ crosses at least one edge in $H_r$.

If $H_r = \{E_1, \cdots, E_m\}$, then the transition matrix of $H_r$ is the nonnegative $m \times m$-matrix $M_r$ whose $ij$-th entry is the number of times the $f$-image of $E_j$ crosses $E_i$, regardless of orientation. $M_r$ is said to be irreducible if for every tuple $1 \leq i, j \leq m$, there exists some exponent $n > 0$ such that the $ij$-th entry of $M_r^n$ is nonzero. If $M_r$ is irreducible, then it has a maximal real eigenvalue $\lambda_r \geq 1$ [Gan59]. We call $\lambda_r$ the growth rate of $H_r$.

Given a homotopy equivalence $f : G \to G$, we can always find a filtration of $G$ such that each transition matrix is either a zero matrix or irreducible. A stratum $H_r$ in such a filtration is called zero stratum if $M_r$ is a zero matrix. $H_r$ is called exponential if $M_r$ is irreducible with $\lambda_r > 1$, and it is called nonexponential if $M_r$ is irreducible with $\lambda_r = 1$.

An unordered pair of edges in $G$ originating from the same vertex is called a turn. A turn is called degenerate if the two edges are equal. We define a map $Df : \{\text{turns in } G\} \to \{\text{turns in } G\}$ by sending each edge in a turn to the first edge in its image under $f$. A turn is called illegal if its image under some iterate of $Df$ is degenerate; otherwise, it is called legal.

An edge path $\rho = E_1E_2\cdots E_s$ is said to contain the turns $(E_i^{-1}, E_{i+1})$ for $1 \leq i < s$; $\rho$ is legal if all its turns are legal, and it is $r$-legal if $\rho \subset G_r$ and no illegal turn in $\rho$ involves an edge in $H_r$.

Let $\rho$ be a path in $G$. In general, the composition $\rho \circ f^k$ is not an immersion, but there is a unique immersion that is homotopic to $\rho \circ f^k$ relative endpoints. We denote this immersion by $\rho f^k$, and we say that we obtain $\rho f^k$ from $\rho \circ f^k$ by tightening. If $\sigma$ is a circuit in $G$, then $\sigma f^k$ is the immersed circuit homotopic to $\sigma \circ f^k$.

**Theorem 2.1** ([BH92, Theorem 5.12]). Every outer automorphism of $F$ is represented by a homotopy equivalence $f : G \to G$ such that each exponential stratum $H_r$ has the following properties:

1. If $E$ is an edge in $H_r$, then the first and last edges in $Ef$ are contained in $H_r$.

2. If $\beta$ is a nontrivial path in $G_{r-1}$ with endpoints in $G_{r-1} \cap H_r$, then $\beta f$ is nontrivial.

3. If $\rho$ is an $r$-legal path, then $\rho f$ is an $r$-legal path.
We call $f$ a relative train track map. A detailed, explicit algorithm for computing relative train track maps appeared in [DV96].

We conclude this section with the introduction of some terminology that will be needed later.

A path $\rho$ is a (periodic) Nielsen path if $\rho f^k = \rho$ for some $k > 0$. In this case, the smallest such $k$ is the period of $\rho$. A Nielsen path $\rho$ is called indivisible if it cannot be expressed as a concatenation of shorter Nielsen paths.

A decomposition of a path $\rho = \rho_1 \cdot \rho_2 \ldots \rho_s$ into subpaths is called a $k$-splitting if $\rho f^k = \rho_1 f^k \cdot \rho_2 f^k \ldots \rho_s f^k$, i.e., there is no cancellation between $\rho_i f^k$ and $\rho_{i+1} f^k$ for $1 \leq i < s$. Such a decomposition is a splitting if it is a $k$-splitting for all $k > 0$. We will also use the notion of $k$-splittings of circuits $\sigma = \rho_1 \cdot \rho_2 \ldots \rho_s$, which requires, in addition, that there be no cancellation between $\rho_s f^k$ and $\rho_1 f^k$.

The $r$-length of a path $\rho$ in $G$, denoted by $|\rho|_r$, is the number of edges in $H_r$ that $\rho$ crosses. A path $\rho$ in $G$ is said to be of height $r$ if $\rho$ is contained in $G_r$ but not in $G_{r-1}$. If $H_r = \{E_r\}$ is a nonexponential stratum, then basic paths of height $r$ are of the form $E_r \gamma$ or $E_r \gamma E_r^{-1}$, where $\gamma$ is a path in $G_{r-1}$.

**Definition 2.2.** We say that a relative train track map $f : G \to G$ is normalized if the following properties hold:

1. For every vertex $v \in \mathcal{V}(G)$, $vf$ is a fixed vertex of $f$.

2. Every nonexponential stratum $H_r$ contains only one edge $E_r$ and $E_r f = E_r u_r$ for some path $u_r$ in $G_{r-1}$.

3. If $H_r = \{E_r\}$ is a nonexponential stratum, $u_r$ is of height $s$, and $s < t < r$, then $H_t$ is nonexponential and $u_t$ is also of height $s$.

4. If $E$ is an edge in an exponential stratum $H_r$, then $|Ef|_r \geq 2$.

5. Every isolated fixed point of $f$ is a vertex.

6. If $C$ is a noncontractible component of some filtration element $G_r$, then $C = Cf$.

**Lemma 2.3.** Every outer automorphism $\mathcal{O}$ has a positive power $\mathcal{O}^k$ that is represented by a normalized relative train track map $f : G \to G$. Both $k$ and $f$ can be computed.
Proof. First, we compute a relative train track map $f': G' \to G'$ representing $O$. We easily read off an exponent $k$ such that $f'^k$ satisfies the first, fourth, and sixth properties of normalized maps, and we have $Ef'^k = vEw$ for every edge $E$ in a nonexponential stratum $H_r$.

After replacing $f$ by a power $f^k$, we may need to refine the filtration of $G$ because an irreducible matrix may have reducible powers. We may also need to permute some filtration elements in order to achieve the desired alignment of nonexponential strata.

If $v$ is nontrivial and $w$ is trivial, we reverse the orientation of $E$. If both $v$ and $w$ are nontrivial, we split $E$ into two edges $E', E''$ such that $E = E'\bar{v}$ and $E''f'^k = E''w$.

By refining the filtration of $G'$ so that each nonexponential stratum contains exactly one edge and subdividing at isolated fixed points if necessary, we obtain a normalized representative $f: G \to G$ of $O_k$.

Lemma 2.4. Let $f: G \to G$ be a normalized relative train track map with an exponential stratum $H_r$. If $C$ is a noncontractible component of $G_{r-1}$ and $v$ is a vertex in $H_r \cap C$, then $v = vf$.

Proof. This argument is contained in the proof of [BFH99, Theorem 5.1.5]. We repeat it here because it is short.

Let $v$ be a vertex in $H_r \cap C$. Since $f$ is normalized, we have $C = Cf$, so that there exists a path $\alpha$ in $C$ that starts at $v$ and ends at $vf$. The vertex $vf$ is fixed, and there exists some path $\beta$ in $C$ that starts and ends at $vf$ such that $\alpha f = \beta f$. Then $(\alpha \beta)f$ is trivial, so that $\alpha \beta$ is trivial because of the second property of relative train track maps.

Lemma 2.5. Let $f: G \to G$ be a normalized train track map with a nonexponential stratum $H_r$. If $\rho$ is a path in $G_r$, then it splits as a concatenation of basic paths of height $r$ and paths in $G_{r-1}$.

Proof. This is essentially [BFH00, Lemma 4.1.4]. The lemma follows immediately from the second property of normalized train track maps.

Lemma 2.6. Let $f: G \to G$ be a normalized train track map with an exponential stratum $H_r$. If $\rho$ is a circuit or edge path of height $r$ containing an $r$-legal subpath of $r$-length $L > 2C_f$ (where $C_f$ is the bounded cancellation constant of $f$), then $\rho f$ contains an $r$-legal subpath of $r$-length greater than $L$.

Proof. This is an immediate consequence of Lemma 1.4 and the fourth property of normalized maps, which implies $\lambda_r \geq 2$. 

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We will need the following consequence of [Bri00, Proposition 6.2].

**Lemma 2.7.** Let \( f : G \to G \) be a relative train track map with an exponential stratum \( H_r \). If \( \rho \) is an edge path of height \( r \) and \( L_0 > 0 \), then at least one of the following three possibilities occurs:

- \( \rho f^M \) contains an \( r \)-legal segment of \( r \)-length greater than \( L_0 \).
- \( \rho f^M \) contains fewer \( r \)-illegal turns than \( \rho \).
- \( \rho f^M \) is a concatenation of indivisible Nielsen paths of height \( r \) and paths in \( G_{r-1} \).

\[ \square \]

## 3 Improving nonexponential strata

In [BFH00], the authors improve the behavior of nonexponential strata in a nonconstructive fashion. We retrace some of their steps here, replacing the nonconstructive parts by constructive arguments.

Let \( H_r = \{ E_r \} \) be a nonexponential stratum of a normalized train track map \( f : G \to G \), and let \( \rho \) be a path in \( G_{r-1} \) originating at the terminal vertex of \( E_r \). We define a new map \( f' : G' \to G' \) by removing \( E_r \) and adding an edge \( E'_r \) whose initial vertex is the initial endpoint of \( E_r \) and whose terminal vertex is the terminal vertex of \( \rho \). We obtain \( u'_r \) by tightening \( \bar{\rho} u_r (\rho f) \), so that \( E'_r f' = E'_r u'_r \). There is an obvious homotopy equivalence \( g : G \to G' \) that sends \( E_r \) to \( E'_r \bar{\rho} \). With this marking, \( f' \) induces the same outer automorphism as \( f \). We say the \( E'_r \) is obtained from \( E_r \) by *sliding along \( \rho \).*

Let \( \tilde{f} : \tilde{G} \to \tilde{G} \) be a lift of \( f \) that fixes the initial endpoint of a lift \( \tilde{E}_r \) of \( E_r \). Then \( \tilde{f} \) leaves invariant a copy \( \tilde{H} \) of the universal cover of the connected component of \( G_{r-1} \) that contains \( u_r \). Let \( h = \tilde{f} |_{\tilde{H}} \), and let \( v_0 \) be the terminal endpoint of \( \tilde{E}_r \). Note that \( v_0 \in H \), and that \([v_0, v_0 h] \) projects to \( u_r \).

**Lemma 3.1.** There exists a slide of \( E_r \) to \( E'_r \) with \( E'_r f' = E'_r \) if and only if \( h \) fixes a point in \( H \).

**Proof.** If \( h \) fixes \( v \in H \), then sliding \( E_r \) along \([v_0, v] \) yields a fixed edge \( E'_r \). Conversely, if there exists a path \( \rho \) such that sliding \( E_r \) along \( \rho \) yields a fixed edge, then the terminal endpoint of the lift of \( \rho \) is fixed by \( h \). \[ \square \]
In Section 5, we present an algorithm for detecting fixed points of \( h \).

**Lemma 3.2.** Assume that \( h \) has no fixed points. Let \( U_k = [v_0, v_0 h^k] \) and \( V_k = U_k \wedge U_{k+1} \) for \( k \geq 0 \). Then \( V_k \) is a proper prefix of \( V_{k+1} \).

*Proof.* This follows from the discussion of preferred edges in the proof of [BFH00, Proposition 5.4.3]. □

As an immediate consequence of Lemma 3.2, we obtain the following lemma.

**Lemma 3.3.** If \( h \) has a periodic point, then \( h \) has a fixed point. □

The following proposition is the main result of this section; it replaces a nonconstructive argument in [BFH00].

**Proposition 3.4.** Assume that \( h \) has no fixed points. We can compute a vertex in \( v \in H \) and an exponent \( m \geq 1 \) such that sliding \( E_r \) along \( [v_0, v] \) yields \( E'_r(f^m)' = E'_r \cdot u'_r \) and \( u'_r \) is a closed path starting and ending at a fixed vertex.

*Proof.* Let \( v_k \) equal the terminal vertex of the path \( V_k \) (Lemma 3.2)\footnote{This agrees with our original definition of \( v_0 \).} and let \( w_k = [v_k, v_{k+1}] \). The path \( w_{k+m} \) is a subpath of \( w_k h^m \) for all \( k, m \geq 0 \).

The idea of the proof is to compute \( w_0, w_1, w_2, \ldots, w_k \) until we identify a suitable vertex \( v \) in \( w_k \). Since \( w_{k+1} \) is a subpath of \( w_k h \), we have \( \text{height}(w_{k+1}) \leq \text{height}(w_k) \), so that the height of the paths \( w_k \) has to stabilize eventually. The following procedure assumes that the height remains constant; should the height drop while the procedure is in progress, we simply start over.

Assume the height stabilizes at \( r \). This means that \( H_r \) cannot be a zero stratum. Now, if \( H_r \) is nonexponential, we have \( |w_{k+1}|_r \leq |w_k|_r \). We keep iterating until we find \( w_k \) such that \( |w_k|_r = |w_{k+1}|_r \geq 1 \). Let \( v \) be the initial endpoint of an occurrence of \( E_r \) in \( w_k \). Then \( v \) has the desired properties (and we do not need to replace \( f \) by a higher power in this case).

Now, assume that \( H_r \) is exponential. If we encounter a path \( w_k \) that contains an \( r \)-legal subpath of \( r \)-length at least \( 2(C_r+1) \), then \( w_{k+1} \) contains a vertex \( v \) that projects to a fixed vertex of \( f \) and whose \( r \)-distance from the closest \( r \)-illegal turn is at least \( C_f \). Now Lemma 1.4 yields that \( v \) has the desired properties.
Assume that the length of $r$-legal subpaths remains bounded below $2(C_f + 1)$. The number of illegal turns cannot go up and must stabilize eventually, so that eventually we will end up in the third case of Lemma 2.7 and see a composition of Nielsen paths of height $r$ and paths in $G_{r-1}$. We can detect this case in a brute-force fashion, by checking all subpaths of $w_k$ in order to see whether they are Nielsen.

Let $v$ be the initial point of one of the Nielsen paths. Then $v$ is periodic of period $m$, so that sliding $E_r$ along $[v_0, v]$ yields the desired improvement of $f^m$.

Definition 3.5. Let $f : G \to G$ be a normalized relative train track map with a nonexponential stratum $H_r = \{E_r\}$. We say that $H_r$ is efficient if

1. $E_r f$ splits as $E_r \cdot u_r$ and $u_r$ is a closed path in $G_{r-1}$,

2. if $u_r$ is a periodic Nielsen path, then its period is one (in this case, we say that $E_r$ is linear), and

3. if $u_r$ is nontrivial, then there exists no slide of $E_r$ to $E'_r$ such that $E'_r f' = E'_r$.

We say that a relative train track map is efficient if it is normalized, all its nonexponential strata are efficient, and the nonexponential strata are sorted in such a way that if $u_r$ and $u_s$ are of the same height but $u_r$ is Nielsen and $u_s$ is not, then $s > r$.

Lemma 3.6. There exists a slide of $E_r$ to $E'_r$ with $E'_r f' = E'_r u'_r$ and $u'_r$ a periodic Nielsen path if and only if $h$ commutes with a nontrivial deck transformation.

Proof. This lemma follows from [BFH00, Proposition 5.4.3].

Remark 3.7. Lemma 3.6 implies that if $H_r$ is efficient and $u_r$ is nontrivial and non-Nielsen, then there exists no slide that takes $u_r$ to a periodic Nielsen path.

An infinite ray $\rho$ starting at a fixed vertex $v_0$ is a fixed ray if $\rho f = \rho$. It is attracting if there exists some $N$ such that if $\eta$ is a ray starting at $v_0$ and $|\rho \wedge \eta| > N$, then $\eta f^n$ converges to $\rho$, i.e., $|\rho \wedge \eta f^n|$ goes to infinity. A repelling fixed ray is an attracting fixed ray for a homotopy inverse of $f$. See [LL04] for a detailed discussion attracting and repelling fixed points for free group automorphisms.
Lemma 3.8. Let \( f : G \to G \) be an efficient relative train track map with a nonexponential stratum \( H_r = \{ E_r \} \) that is neither linear nor constant. Let

\[
R_r = E_r u_r( u_r f)( u_r f^2) \ldots
\]

Then \( R_r \) is the unique attracting fixed ray of the form \( E_r \gamma \), for \( \gamma \subset G_r \), and there are no Nielsen paths of the form \( E_r \gamma \). In particular, we have \( \lim_{k \to \infty} \rho f^k = R_r \) for all basic paths \( \rho \) of height \( r \).

Proof. This lemma follows from the proof of [BFH00, Lemma 5.5.1]. The assumptions of [BFH00] are stronger than our assumptions, but a close inspection of the proof shows that only our assumptions are needed for the results that we use here.

If \( \rho \) is a path starting and ending at fixed points, then we can find at most one path \( \rho' \) with the same endpoints such that \( \rho' f = \rho \). In this case, we write \( \rho' = \rho f^{-1} \). We define \( \rho f^{-k} \) in the obvious fashion. If \( \rho \) is closed, then \( \rho f^{-k} \) exists for all \( k \).

Lemma 3.9. Let \( f : G \to G \) be an efficient relative train track map with a nonexponential stratum \( H_r = \{ E_r \} \) that is neither linear nor constant. Let

\[
S_r = E_r(\bar{u}_r f^{-1})(\bar{u}_r f^{-2}) \ldots
\]

Then \( S_r \) is the unique repelling fixed ray of the form \( E_r \gamma \), for \( \gamma \subset G_r \). In particular, we have \( \lim_{k \to \infty} \rho f^{-k} = S_r \) for all basic paths \( \rho \) of height \( r \).

Proof. Lemma 3.8 implies that \( h \) only has one repelling fixed ray. Since \( S_r \) is clearly fixed, it is the unique repelling fixed ray.

4 Detecting orbits of paths

If \( H_r \) is an exponential stratum and \( \rho \) is a path of height \( r \), we let \( \iota_r(\rho) \) equal the number of \( r \)-illegal turns in \( \rho \).

Lemma 4.1. Let \( f : G \to G \) be an efficient relative train track map. If \( \rho \) is a circuit or edge path in \( G \), then we can determine algorithmically whether \( \rho \) is a periodic Nielsen path; if \( \rho \) is Nielsen, then we can compute its period as well.
Proof. Assume inductively that we can detect periodic Nielsen paths and circuits in $G_{r-1}$. We want to show that if $\rho$ is of height $r$, then we can determine whether $\rho$ is Nielsen.

We first assume that $H_r = \{E_r\}$ is nonexponential. Then $\rho$ splits as a concatenation of basic paths of height $r$ and paths in $G_{r-1}$ (Lemma 2.5), and it is Nielsen if and only if each of these constituent paths is Nielsen. Hence, we may assume that $\rho$ is a basic path of height $r$, i.e., $\rho = E_r \gamma$ or $\rho = E_r \gamma E_r$ for some $\gamma \in G_{r-1}$. If $E_r f = E_r$, then $\rho$ is Nielsen if and only if $\gamma$ is Nielsen so that we are done by induction. If $E_r$ is neither constant nor linear, then Lemma 3.8 yields that $\rho$ cannot be Nielsen.

This leaves the case that $E_r$ is linear. If $\rho = E_r \gamma$, then it cannot be Nielsen (if $E_r \gamma$ were Nielsen, then Lemma 3.3 would imply that we can slide $E_r$ to a constant edge, in violation of efficiency of $f$). Clearly, a path of the form $E_r \gamma E_r$ can only be Nielsen if $\gamma$ is a (possibly negative) power of $u_r$, which completes the proof for nonexponential $H_r$.

Now, assume that $H_r$ is exponential. If an endpoint of $\rho$ is not fixed, then $\rho$ cannot be Nielsen. If both endpoints of $\rho$ are fixed, we compute $\rho, \rho f, \rho f^2, \ldots$ until one of the following three cases occurs:

- We encounter some image $\rho f^k$ that contains an $r$-legal path whose length exceeds $2C_f$. Then Lemma 2.6 implies that $\rho$ is not Nielsen.

- We encounter some image $\rho f^k$ that contains fewer $r$-illegal turns than $\rho$. Since $f$ does not increase the number of $r$-illegal turns, $\rho$ is not Nielsen.

- We can express $\rho$ as $\rho = \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_m \beta_m$, where the $\alpha_i$ are Nielsen paths of height $r$, and the $\beta_i$ are subpaths in $G_{r-1}$, such that we encounter some $\rho f^k = \alpha_1 (\beta_1 f^k) \cdots \alpha_m (\beta_m f^k)$. In this case, $\rho$ is Nielsen if and only if the $\beta_i$ are Nielsen.

One of these three cases must occur eventually, and we can detect the third case in a brute-force way by checking all possible decompositions of $\rho$.

Finally, if $H_r$ is a zero stratum, then $\rho$ cannot possibly be Nielsen, so that the proof is complete.

If $u$ is a closed path and $\rho$ is an arbitrary edge path, we let $p_u(\rho)$ equal the largest exponent $m$ so that $u^m$ is a prefix of $\rho$. 

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Lemma 4.2. Let $f : G \to G$ be a relative train track map with an exponential stratum $H_r$ and a closed Nielsen path $u$ of height $r$. If $\rho$ is an edge path of height $r$ and $k \geq 0$ an exponent such that $p_u(\rho) = m$ and $p_u(\rho f^k) = l$, then $\Im_r(\rho) \geq (2m - l - 1)\Im_r(u)$.

Proof. We express $\rho$ as $\rho = u^m\gamma$. Since we have $p_u(\rho f^k) = l$, we conclude that $p_u(\gamma f^k) \geq m - l - 1$, so that $\Im_r(\gamma f^k) \geq (m - l - 1)\Im_r(u)$. Since $f$ does not introduce new illegal turns, we have $\Im_r(\gamma) \geq (m - l - 1)\Im_r(u)$, so that $\Im_r(\rho) \geq (2m - l - 1)\Im_r(u)$. $\square$

Lemma 4.3. Let $f : G \to G$ be an efficient train track map and let $\rho$ be a non-Nielsen path whose endpoints are fixed. Then for any $L > 0$, we can compute an exponent $k_0 > 0$ such that $|\rho f^k| > L$ and $|\rho f^{-k}| > L$ (if $\rho f^{-k}$ exists) for all $k \geq k_0$.

Proof. We assume inductively that the lemma holds for the restriction of $f$ to $G_{r-1}$. We first assume that $H_r = \{E_r\}$ is a nonexponential stratum. Then $\rho$ splits as a concatenation of basic paths of height $r$ and paths in $G_{r-1}$, so that we may assume that $\rho$ is a non-Nielsen basic path of height $r$, i.e., $\rho = E_r\gamma E_r$ or $\rho = E_r\gamma$.

Assume that $E_r$ is neither constant nor linear. Then we can find a prefix $R$ of $R_r$ as well as a prefix $S$ of $S_r$ (see Lemma 3.8 and Lemma 3.9) of length greater than $L$ for which $|Rf| > |R| + C_f$ and $|Sf^{-1}| > |S| + C_f$. Now Lemma 3.8, Lemma 3.9, and Lemma 1.4 imply that we can find some exponent $k_0$ such that $R$ is a prefix of $\rho f^k$ and $S$ is a prefix of $\rho f^{-k}$ for all $k \geq k_0$. We conclude that $|\rho f^{\pm k}| > L$ for all $k \geq k_0$.

If $E_r$ is constant, then the inductive hypothesis applied to $\gamma$ completes the proof. This leaves the case that $E_r$ is linear. Let $s$ be the height of $\gamma$. If $s$ is smaller than the height of $u_r$, we conclude that no copy of $u_r$ will cancel completely in $\rho f^k$ for any $k > 0$, so that we have $|\rho f^{\pm k}| > L$ for all $k > L$.

If $s$ equals the height of $u_r$ and $H_s$ is nonexponential, then no more than $|\gamma|$ copies of $u_r$ cancel in $|\rho f^k|$, so that we have $|\rho f^{\pm k}| > L$ for all $k > L + |\gamma|$. If $H_s$ is exponential, then for all $k \geq 0$, the number of copies of $u_r$ that cancel in $\rho f^k$ is bounded by $\Im_s(\gamma)$, so that $|\rho f^k| > L$ if $k > L + \Im_s(\gamma)$.

We still need to study the length of $\rho f^{-k}$ for $k \geq 0$. Let $m = p_{u_r}(\gamma f^{-k})$ and $l = p_{u_r}(\gamma)$. Then Lemma 1.2 implies that $\Im_r(\gamma f^{-k}) \geq (2m - l - 1)\Im_r(u_r)$. This implies that $\Im_r(\rho f^{-k}) \geq k\Im_r(u_r) + (2m - l - 1)\Im_r(u_r) - 2m\Im_r(u_r) = (k - l - 1)\Im_r(u_r)$, so that $|\rho f^{-k}| > L$ if $k > L + l + 1$. 14
If $s$ exceeds the height of $u_r$, then, by definition of efficiency, $H_s$ is also linear, and $\rho$ splits into subpaths of the form $E_r\eta$, $E_s\eta$, and $E_r\eta E_s$, where $\eta \subset G_{s-1}$. The first two cases are done by induction on $s$, so that we only need to consider the case $E_r\eta E_s$. This case is essentially the same as the previous one (we need to apply Lemma 4.2 to both $\eta$ and $\bar{\eta}$), except we need to consider the possibility that there is a closed Nielsen path $\tau$ such that $u_r = \tau^a$, $u_s = \tau^b$, and $\eta = \tau^c$. In this case, we have $a \neq b$ (or else $E_r E_s$ would be Nielsen, in violation of efficiency), so that $|(E_r\eta E_s)f^k| \geq k - c$, so that $|(E_r\eta E_s)f^k| > L$ if $k > L + c$.

Finally, assume that $H_r$ is exponential. In this case, we compute $\rho, \rho f, \ldots$ until we either find some $k_0$ such that $\rho f^{k_0}$ has an $r$-legal subpath of $r$-length greater than $L + 2C_f$ (in which case Lemma 2.6 yields that $|\rho f^k| > L$ for all $k \geq k_0$), or, by Lemma 2.7, we encounter some $k$ such that $\rho f^k$ is a composition of indivisible Nielsen paths of height $r$ and paths in $G_{r-1}$. Since $\rho$ is non-Nielsen, one of the subpaths in $G_{r-1}$ must be non-Nielsen, so that we are done by induction.

In order to understand lengths under backward iteration, we need to consider two cases: If $\rho$ is not a composition of indivisible Nielsen paths of height $r$ and paths in $G_{r-1}$, then Lemma 2.7 implies that the number of $r$-illegal turns has to go up under backward iteration. In this case, we simply compute $\rho, \rho f^{-1}, \rho f^{-2}, \ldots$ until we find some $k_0$ for which $\rho f^{-k_0}$ contains $L$ $r$-illegal turns, and we conclude that $|\rho f^{-k}| > L$ for all $k \geq k_0$.

If $\rho$ is a concatenation of indivisible Nielsen paths of height $r$ and paths in $G_{r-1}$, then one of the subpaths $\gamma$ in $G_{r-1}$ is not Nielsen, so that the inductive hypothesis applies to $\gamma$. Lemma 1.5 guarantees that $\gamma f^{-k}$ exists for all $k \geq 0$, so that we are done.

**Proposition 4.4.** Let $f : G \to G$ be an efficient train track map, and let $\rho_1$ and $\rho_2$ be paths whose endpoints are fixed. Then we can determine algorithmically whether $\rho_2$ is the image of $\rho_1$ under some power of $f^k$, and we can compute the exponent $k$ if it exists.

**Proof.** Using Lemma 4.1, we determine whether $\rho_1$ is a periodic Nielsen path. If it is, we simply enumerate all distinct images of $\rho_1$ and check whether $\rho_2$ is among them. If $\rho_1$ is not Nielsen, we apply Lemma 4.3 with $L = |\rho_2|$ to obtain an exponent $k_0$. Now we compute $\rho, \rho f, \ldots, \rho_1 f^{k_0}$ and check whether $\rho_2$ is contained in this list.

If $\rho_2$ is contained in this list, we obtain a positive answer as well as the desired exponent $k$. If not, we switch $\rho_1$ and $\rho_2$ and repeat the argument. \qed
Theorem 4.5. Let \( f : G \rightarrow G \) be an efficient train track map with an exponential stratum \( H_r \). Then we can compute all indivisible periodic Nielsen paths of height \( r \) as well as their periods.

Proof. Let \( \alpha \) be an indivisible Nielsen path of height \( r \). Then \( \alpha \) contains exactly one \( r \)-illegal turn, and the \( r \)-length of its two \( r \)-legal subpaths is bounded by \( C_f \) (Lemma \[1.3\]). Moreover, the first and last (possibly partial) edges of \( \alpha \) are contained in \( H_r \).

For an edge \( E \) in \( H_r \), let \( P_E \) be the set of maximal subpaths in \( G_{r-1} \) of \( Ef \), and let \( P = \bigcup_{E \in H_r} P_E \). If \( \beta \) is a maximal subpath in \( G_{r-1} \) of \( \alpha \), then there exists some \( \gamma \in P \) and \( k \geq 0 \) such that \( \beta = \gamma f^k \).

Let \( \gamma \) be a path in \( P \). If \( \gamma \) is Nielsen, we let \( L_\gamma = \max_k \{|\gamma f^k|\} \). If \( \gamma \) is not Nielsen, Lemma \[4.3\] with \( L = C_f \) yields an exponent \( k_0 \) such that \( |\rho f^k| > L \) for all \( k \geq k_0 \). We let \( L_\gamma = \max_{0 \leq k < k_0} \{|\rho f^k|\} \).

Let \( M = \max_{\gamma \in P} \{L_\gamma\} \) and observe that \( \alpha \) has no subpaths in \( G_{r-1} \) whose length exceeds \( M \). Let \( Q \) be the set of all edge paths \( \rho \) such that \( \rho \) contains exactly one \( r \)-illegal turn, the \( r \)-length of \( r \)-legal subpaths is bounded by \( C_f \), the length of subpaths in \( G_{r-1} \) is bounded by \( M \), and the first and last edges are contained in \( H_r \). Clearly, if \( \alpha \) is an indivisible Nielsen path of height \( r \), then \( \alpha \) is a subpath of some \( \rho \in Q \).

We define a map \( g : Q \rightarrow G \cup \{\ast\} \) by letting \( \rho g \) equal the unique maximal subpath of \( \rho f \) contained in \( Q \) if \( \rho f \) contains an \( r \)-illegal turn, and we let \( \rho g = \ast \) if \( \rho f \) contains no \( r \)-illegal turn.

For each \( \rho \in G \), we compute \( \rho, \rho g, \rho g^2, \ldots \) until we either encounter \( \ast \) (in which case \( \rho \) has no Nielsen subpath) or we find that \( \rho g^k = \rho g^m \) for some \( 0 \leq k < m \). Then \( \rho g^k \) contains an indivisible Nielsen subpath \( \alpha \), and we can easily compute the endpoints of \( \alpha \). Moreover, if \( k \) and \( m \) are as small as possible, then \( m - k \) is the period of \( \alpha \). Since all indivisible Nielsen paths of height \( r \) show up in this fashion, the proof is complete. \( \square \)

Corollary 4.6. Given an efficient relative train track map \( f : G \rightarrow G \), we can compute an exponent \( k \geq 1 \) such that all periodic Nielsen paths of \( f^k \) have period one.

\[ \ast \] is merely some termination symbol.
5 Detecting fixed points

Let $f: G \to G$ be a normalized relative train track map with a nonexponential stratum $H_r = \{E_r\}$. Assume that the restriction of $f$ to $G_{r-1}$ is efficient. The purpose of this section is to present an algorithm for determining whether $E_r$ has a slide to a constant edge (Proposition 5.6). This is the last missing piece in our computation of efficient maps (Theorem 5.7).

We have $E_rf = E_ru_r$, and we want to express $u_r$ as the path obtained by tightening $\bar{\rho}(\rho f)$ for some path $\rho$ in $G_{r-1}$, if possible. To this end, choose a fixed vertex $\bar{v}_0 \in G_{r-1}$. The main idea is to perform a breadth-first search of edge paths $\rho$ originating at $\bar{v}_0$, keeping track of the paths obtained by tightening $\bar{\rho}(\rho f)$ until we either encounter $u_r$ or we determine that further searching will not yield $u_r$. If we encounter $u_r$ along the way, then sliding $E_r$ along $\bar{\rho}$ will turn it into a constant edge.

It will be convenient to work in the universal cover $H$ of $G_{r-1}$, constructing partial lifts $h$ of $f$ as we go along (Construction 1.1), beginning with $T_0 = U_0 = \{v_0\}$. For a vertex $v$ in $H$, we define $\rho_v$ to be the path $[v_0, v]$ and $w_v$ to be the projection of $[v, vh]$. Note that $w_v$ is the projection of the path obtained by tightening $\bar{\rho}_v(\rho_v h)$.

We let $M_v = |v_0, v| - |[v_0, v] \cup [v_0, vh]|$ and $N_v = |v_0, vh| - |[v_0, v] \cup [v_0, vh]|$ (Figure 1). Note that $|w_v| = M_v + N_v$.

The following is a partial list of conditions under which we need not extend our search beyond a vertex $v$:

- The path $w_v$ was encountered before in our search. In this case, searching beyond $v$ will not yield any new results.
- If $|w_v| > |u_r| + C_f$, $M_{v'} > 0$ and $N_{v'} > C_f$ for some vertex $v' \in [v_0, v]$, then Lemma 1.4 implies that $|w_{v'}| > |u_r|$ for all vertices $v'$ beyond $v$.
so that we will not encounter \( u_r \) if we search beyond \( v \).

Assume that there exists an infinite sequence \( v_0, v_1, v_2, \ldots \) such that \( v_k \neq v_{k+2}, |v_k, v_{k+1}| = 1 \) for all \( k \), and none of the two cases above occurs. Then \( |w_{v_k}| \) goes to infinity (or else there would be some repetition along the way), and we have \( M_{v_k} = 0 \) or \( N_{v_k} \leq C_f \) for all \( k \). In fact, we have \( M_{v_k} = 0 \) for all \( k \) or \( N_{v_k} \leq C_f \) for all \( k \) (otherwise we would encounter a fixed interior vertex, i.e., a vertex \( v_k \neq v_0 \) for which \( w_{v_k} \) is trivial, so that we would have reached our first termination criterion because \( w_{v_0} \) is trivial). In the first case, the \( v_k \) define an attracting fixed ray of \( h \). In the second case, they define a repelling fixed ray of \( h \).

5.1 Attracting fixed rays

If \( v_0, v_1, v_2, \ldots \) is an attracting fixed ray with no interior fixed vertices, then this sequence is determined by \( v_0 \) and \( v_1 \) alone because the first edge of \( [v_k, v_kh] \) is the same as the edge \([v_k, v_{k+1}]\); otherwise we would encounter a trivial \( w_k \) along the way. For the same reason, the edge \([v_0, v_1]\) cannot project to a constant edge. In other words, we need to consider at most one attracting fixed ray for each nonconstant edge originating at \( v_0 \), and we can easily compute arbitrarily long prefixes of each ray.

In order to determine when to stop following an attracting ray, we will identify some \( k_0 \) such that \( |v_k, v_kh| > |u_r| + C_f \) for all \( k \geq k_0 \). This implies that \( |w_{v_k}| > |u_r| + C_f \) for all \( k \geq k_0 \). Moreover, if \( v \) is a vertex such that \( v_{k_0} \in [v_0, v] \), then Lemma 1.4 implies that \( |w_v| > |u_r| \), so that we can terminate our search at \( v_{k_0} \).

First, assume that the edge bounded by \( v_0 \) and \( v_1 \) is contained in an exponential stratum \( H_s \). Then \([v_0, v_k]\) projects to an \( r \)-legal path for all \( k \), and we have \( |v_k, v_kh|_r \geq |v_0, v_k|_r \) because \( f \) is normalized. Hence, we only need to compute \( v_0, \ldots, v_k \) until the \( r \)-length of \([v_0, v_k] \) exceeds \( |u_r| + C_f \).

Now, assume that \([v_0, v_1]\) projects to a nonexponential edge \( E_s \). Since \( v_0, v_1, \ldots \) is a fixed ray, \([v_0, v_1]\) cannot project to \( E_s \), and so \( \lim_{k \to \infty} [v_0, v_k] \) equals \( R_s \). If \( E_s \) is linear, then we reach our first termination criterion after at most \( |u_s| \) steps, so that we may assume that \( E_s \) is neither constant nor linear.

**Lemma 5.1.** Let \( L > 0 \) and assume that \( v \) is a vertex in \( H \) such that \( |v, vh| \geq L, |vh, vh^2| \geq L, \) and \( vh \in [v, vh^2] \). Then, for all \( x \in [v, vh] \), we
have
\[ |x, xh| \geq \frac{2L}{K_f+1} - D_f. \]

**Proof.** Let \( t = |x, v| \). Then Inequality 1 implies that 
\[ |xh, vh| \geq \frac{t}{K_f} - D_f \]
and
\[ |x, vh^2| \leq K_f(L - t) + D_f. \]
We conclude that
\[ |x, xh| \geq L - t + \max\{ \frac{t}{K_f} - D_f, L - K_f(L - t) - D_f \}. \]
The minimum of the right-hand side of this inequality is attained for 
\[ t = \frac{LK_f}{K_f+1}, \]
and substituting this value yields a lower bound of 
\[ \frac{2L}{K_f+1} - D_f. \]

We choose \( L \) such that \( \frac{2L}{K_f+1} - D_f > |u_v| + C_f \). Now Lemma 4.3 yields an exponent \( k_0 \) such that 
\[ |u_v f^k| > L \]
for all \( k \geq k_0 \). We only need to compute \( v_0, \ldots, v_k \) until \( [v_0, v_k] \) projects to \( E (u_s f^k u_s \cdots (u_s f^h u_s)) \), and Lemma 5.1 guarantees that 
\[ |w_v| > |u_v| + C_f \]
for all \( v \) beyond \( v_k \). This completes our algorithm in the case of attracting fixed rays.

### 5.2 Repelling fixed rays

In the attracting case, we construct fixed rays edge by edge, and an attracting fixed ray that contains no interior fixed points is determined by its first edge. In the repelling case, the situation is more complicated, but the following lemma still give us a way of computing successive edges in potential fixed rays given a sufficiently long prefix.

**Lemma 5.2.** Let \( v_0, v_1, \ldots, v_k \) be a sequence such that \( N_{v_j} \leq C_f \) for all \( 0 \leq j \leq k \) and \( M_{v_k} > C_f \). Then at most one vertex \( v \) adjacent to \( v_k \), other than \( v_{k-1} \), can be contained in a repelling ray originating at \( v_0 \), and we can find \( v \) algorithmically or determine that there is no such \( v \). Moreover, if \( v' \) is a vertex satisfying \( v_k \in [v_0, v'] \) and \( v \notin [v_0, v'] \), then \( M_{v'} \geq M_{v_k} |v_k, v'| - C_f \).

**Proof.** Using Inequality 1 we find some \( L > 0 \) such that if \( \rho \) is a path of length at least \( L \), then \( |\rho f| \geq 2C_f + 1 \). Now we enumerate all vertices \( p_1, \ldots, p_m \) such that \( [v_k, p_i] = L \) and \( v_k \in [v_0, p_i] \) for all \( i \) (Figure 2). Lemma 1.4 yields that 
\[ |[v_k, p_i] h \wedge [v_k, p_j] h| \leq C_f \] if 
\[ |[v_k, p_i] \wedge [v_k, p_j]| = 0. \]

If \( p_i \) and \( p_j \) are contained in fixed rays, then \( N_{p_i} < C_f \) and \( N_{p_j} < C_f \). This implies that 
\[ |[v_k, p_i] h \wedge [v_k, p_j] h| > C_f, \]
so that 
\[ |[v_k, p_i] \wedge [v_k, p_j]| > 0. \]
Hence, if there exists some \( p_i \) such that 
\[ |[v_k, p_i] h \wedge [v_k, v_k h]| > C_f, \]
then the second vertex \( v \) in \( [v_k, p_i] \) is uniquely determined by this property.

The last claim is an immediate consequence of Lemma 1.4.
Another complication in the repelling case is that the height may go up as we apply Lemma 5.2 to compute subsequent vertices. The following lemmas provide a means of handling this possibility.

**Lemma 5.3.** Assume that $H_s$ is an exponential stratum and let $C = S_f(1 + \#E(G))$. If $\eta$ is a repelling fixed ray of height $s$ with a maximal prefix $\alpha$ in $G_{s-1}$, then $|\alpha f| + C \geq |\alpha|$.

**Proof.** If the initial vertex $v_0$ is contained in a contractible component of $G_{s-1}$, then the claim is trivial, so that we may assume that $v_0$ is contained in a noncontractible component of $G_{s-1}$. By Lemma 2.4, the terminal endpoint of $\alpha$ is fixed.

Choose $\beta$ so that $\eta = \alpha \beta$. By definition, the first edge in $\beta$ is contained in $H_s$. Let $\gamma$ be the maximal subpath in $G_{s-1}$ of $\beta f$. It suffices to show that $|\gamma| \leq C$.

If $\gamma$ is a subpath of $Ef$ for some edge $E \subset H_s$, then $|\gamma| \leq S_f$. If $\gamma$ is the image of some subpath $\gamma' \subset G_{s-1}$ of $\beta$, then Lemma 2.4 implies that $\gamma'$ is contained in a contractible component of $G_{s-1}$ so that $|\gamma'| \leq \#E(G)$. This implies that $|\gamma| \leq S_f \#E(G)$. \qed

**Lemma 5.4.** If $H_s$ is an exponential stratum and the sequence $v_0, v_1, \ldots$ defines a repelling fixed ray $\eta$ of height $s$ without interior fixed points, then $\iota_s(w_{v_k})$ is an unbounded nondecreasing function of $k$.

**Proof.** Since $\eta$ has no interior fixed points, it cannot be a concatenation of Nielsen paths of height $r$ and subpaths in $G_{r-1}$. This implies that $\eta$

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7 Otherwise $\beta$ would have an initial subpath $\eta$ of height $s$, starting and ending at fixed vertices, so that $\eta f$ is trivial. This is impossible because $f$ is a homotopy equivalence.
contains infinitely many $r$-illegal turns. Now Lemma 2.6 implies that the distance between two $r$-illegal turns is bounded by some constant $L$. Since $\eta$ is repelling, $|w_{v_k}|$ is unbounded, which proves the claim.

Lemma 5.5. Assume that $H_\eta$ is a nonexponential stratum and that $\eta$ is a repelling fixed ray of height $s$ with no fixed interior vertices. Then $\eta = S_s$.

Proof. This is an immediate consequence of Lemma 2.5 and Lemma 3.3.

Lemma 5.5 implies that if the height goes up as we follow a potential repelling fixed ray, then the height must eventually stabilize at an exponential stratum.

We now continue our breadth-first traversal of vertices in $H$. If we encounter a vertex $v_k$ such that $[v_0, v_k]$ satisfies the hypotheses of Lemma 5.2, then we need to consider the possibility that $[v_0, v_k]$ is a prefix of a repelling fixed ray. In this case, we use Lemma 5.2 to compute subsequent vertices $v$. (In this process, $M_v$ may drop below $C_f$, so that Lemma 5.2 no longer applies; in this case, we simply continue our breadth-first search. This is not a problem, however, because it can only happen finitely many times before we encounter our first termination criterion.)

Let $s$ be the height of the potential repelling ray computed so far. If $H_\eta$ is nonexponential, then our ray must converge to $S_s$. Using arguments similar to those in the attracting case, we follow $S_s$ until we recognize a vertex $v_k$ such that for all vertices $v$ beyond $v_k$, we have $M_v > \max\{C, |u_r|\}$ (where $C$ is the constant from Lemma 5.3). $M_v > C$ guarantees that we are not following a prefix of a ray of greater height, and $M_v > |u_r|$ implies that we will not encounter $u_r$ as we follow the ray.

If $H_\eta$ is exponential, then we follow our ray until we encounter a vertex $v$ for which $\iota_s(w_v) > \max\{C, |u_r|\}$. Once again, Lemma 5.3 guarantees that the height will not go up if we continue following our ray, and we will not encounter $u_r$ if we continue our search. Hence, our algorithm terminates in all possible cases.

5.3 Picking up the pieces

Proposition 5.6. If $H_r = \{E_r\}$, then we can determine algorithmically whether there exists a path $\rho \subset G_{r-1}$ such that $u_r$ is obtained by tightening $\bar{\rho}(\rho f)$, and we can compute $\rho$ if it exists.
Proof. If \( \rho \) exists, then its initial vertex is a fixed vertex in \( G_{r-1} \). Repeating the procedure above for each fixed vertex in \( G_{r-1} \) yields the desired algorithm. \( \square \)

**Theorem 5.7.** Given an outer automorphism \( \mathcal{O} \) of \( F_n \), we can compute an efficient relative train track map \( f : G \to G \) as well as an exponent \( k \geq 1 \) such that \( f \) represents \( \mathcal{O}^k \).

**Proof.** We can compute an exponent \( k \geq 1 \) and a normalized relative train track map \( f : G \to G \) representing \( \mathcal{O}^k \). Now we assume inductively that the restriction of \( f \) to \( G_{r-1} \) is efficient. If \( H_r \) is zero or exponential, then there is nothing to do. If \( H_r = \{E_r\} \) is nonexponential, then we first use Proposition 5.6 to determine whether there exists a slide of \( E_r \) to a constant edge. If no such edge exists, we use Proposition 3.4 to achieve efficiency of \( H_r \). \( \square \)

### 6 Proof of the main result

**Lemma 6.1.** Let \( f : G \to G \) be an efficient relative train track map. There exists an algorithm that, given a circuit \( \sigma \) in \( G \) and a constant \( L > 0 \), determines whether \( \sigma \) is Nielsen. If \( \sigma \) is not Nielsen, then the algorithm finds an exponent \( k_0 \) such that \( |\sigma f^k| > L \) for all \( k \geq k_0 \).

**Proof.** Lemma 4.1 takes care of the detection of Nielsen circuits. If \( \sigma \) is not Nielsen, then we consider the height \( r \) of \( \sigma \). If \( H_r \) is nonexponential, then it splits as a concatenation of basic paths of height \( r \) (Lemma 2.5), so that Lemma 4.3 completes the proof in this case.

If \( H_r \) is exponential, then we compute \( \sigma, \sigma f, \sigma f^2, \ldots \) until we encounter an image \( \sigma' = \sigma f^k \) for some \( k > 0 \) such that \( \sigma' \) contains an \( r \)-legal path of length greater than \( 2(C_f + 1) \) or \( \sigma' \) is a concatenation of Nielsen paths of height \( r \) and paths in \( G_{r-1} \).

We can recognize both possibilities algorithmically. In the first case, \( \sigma' f \) splits at a fixed vertex in a long \( r \)-legal subpath. In the second case, \( \sigma' \) splits at the terminal endpoint of a subpath in \( G_{r-1} \). In either case, Lemma 4.3 completes the proof. \( \square \)

**Theorem 6.2.** Let \( \phi \) be an automorphism of \( F_n \). The exists an algorithm that, given two elements \( u,v \in F_n \), determines whether there exists some exponent \( N \) such that \( u \phi^N \) is conjugate to \( v \). If such an \( N \) exists, then the algorithm will compute \( N \) as well.
Proof. Theorem 5.7 yields an exponent $k$ and an efficient relative train track map $f: G \to G$ that represents the outer automorphism defined by $\phi^k$. We can find some constant $Q \geq 1$ such that if $\sigma$ is a circuit in $G$ representing a conjugacy class $\omega$ in $F_n$, then $\frac{1}{Q} |\omega| \leq |\sigma| \leq Q |\omega|$.  

Represent the conjugacy class of $u$ by a circuit $\sigma$. If $\sigma$ is a Nielsen circuit of period $p$, then we conclude that $u\phi^p$ is conjugate to $u$. Now we compute $u, u\phi, \ldots, u\phi^{kp-1}$ and check whether any conjugate of $v$ is in this list. If $\sigma$ is not Nielsen, we let $L = Q \cdot S^k_\phi |v|$, and we find some exponent $k_0$ such that $|\sigma^j| > L$ for all $j \geq k_0$. We conclude that the length of the conjugacy class of $u\phi^j$ exceeds $|v|$ for all $j \geq k_0$. Now we list $u, u\phi, u\phi^2, \ldots, u\phi^{kk_0-1}$ and check whether any conjugate of $v$ is in this list. If no conjugate is contained in this list, then we exchange $u$ and $v$ and repeat the argument. This completes the proof.  

Theorem 6.3. Let $\phi$ be an automorphism of $F_n$. The exists an algorithm that, given two elements $u, v \in F_n$, determines whether there exists some exponent $N$ such that $u\phi^N = v$. If such an $N$ exists, then the algorithm will compute $N$ as well.

Proof. We use a trick from [BFH97]. Let $F' = F_n \ast \langle a \rangle$ and define $\psi \in \text{Aut}(F')$ by letting $x\psi = x\phi$ if $x \in F_n$ and $a\psi = a$. If $w \in F_n$, then $wa$ is cyclically reduced in $F'$, so that $u\phi^N = v$ if and only if $(ua)\psi^N$ is conjugate to $va$. Now Theorem 6.2 completes the proof.  

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\footnote{The length of a conjugacy class $\omega$ is defined to be the length of the shortest element in $\omega$.}
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