(Almost) Efficient Mechanisms for Bilateral Trading

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Abstract

We study the simplest form of two-sided markets: one seller, one buyer and a single item for sale. It is well known that there is no fully-efficient mechanism for this problem that maintains a balanced budget. We characterize the quality of the most efficient mechanisms that are budget balanced, and design simple and robust mechanisms with these properties. We also show that even minimal use of statistical data can yield good results. Finally, we demonstrate how a mechanism for this simple bilateral-trade problem can be used as a “black-box” for constructing mechanisms in more general environments.

1 Introduction

Computerized markets have become more and more complex over the years. These platforms now involve numerous strategic players with different roles and characteristics. In particular, unlike classic auctions, many of these large-scale markets are “two-sided”. For example, online advertisements are sold via exchange markets (e.g., Google’s Doubleclick, Micorosoft’s AdECN, etc), where advertisers wish to buy advertising slots on one side of the market, and web sites sell their advertising opportunities on the other side. eBay is another prominent example for a trade platform that accommodates multiple buyers and sellers. An interesting recent example is the FCC incentive auctions, which is an attempt to reallocate frequencies currently held by TV broadcasters to wireless phone companies that wish to buy them (see [13]); A major challenge in these FCC two-sided auctions is providing incentives to TV broadcasters, that own the frequencies, to relinquish their licenses (see also [1]). These markets – as well as many other examples – are multi-billion dollar two-sided markets.

This paper focuses on the design of two-sided markets and exchanges that foster efficient trading. This standard goal is natural in some markets (like the FCC spectrum auctions and stock exchanges), and in other markets it may serve as a reasonable approximation to the long-term goal of the trade platform of generating value for its customers. The literature on designing two-sided markets and exchanges is deep and rich. Yet, to a large extent, some basic design principles are still not well understood.

In this paper, we take a bottom-up approach by first revisiting the well-studied, simplest model of two-sided markets: Bilateral Trade [15]. We show how to design simple mechanisms with a guaranteed level of efficiency for this basic setup. Furthermore, the mechanisms that we construct for this fundamental problem yield – in a black-box manner – nearly efficient mechanisms for more complex settings.

1.1 Almost Efficient Mechanisms for Bilateral Trade

In the Bilateral trade problem, a single seller is the owner of one indivisible item which can be traded with a single buyer. Both the seller and the buyer have privately known values for consuming the

\footnote{See, e.g., [13] [17] [5] [8] [7].}
These values are unknown to the market designer who only has access to their probability distributions. Technically, designing mechanisms for the bilateral trade problem translates to three requirements. The first one is individual rationality: the participation of the agents is voluntary and at any point they may leave the market and consume their initial endowments. The second requirement is budget balance: the mechanism is not allowed to subsidize the agents or to make any profits. The third requirement is incentive compatibility (IC). While most of the literature focuses on Bayes-Nash incentive compatibility, our mechanisms will possess the stronger (and thus more desired) property of dominant-strategy incentive compatibility (DSIC).

The seminal paper by Myerson and Satterswaite [15] analyzed the bilateral trade problem. Their celebrated impossibility result states that even in this simple setting, there is no mechanism which is fully efficient, individually rational and budget balanced. In the same paper, Myerson and Satterswaite characterized the “second best” mechanism, i.e., the mechanism that maximizes efficiency subject to the other constraints mentioned above. However, this “second best” mechanism is not given as a closed-form formula, and it is a non-trivial task to explicitly describe it even for simple distributions.

In contrast, our work is inspired by a recent line of research that studies the power of simple mechanisms in comparison to the performance of the optimal, but complex, mechanisms (see, e.g., [2, 10, 6, 11, 12] and the reference within.). We restrict our attention to mechanisms that are easier to understand and implement. Specifically, we focus on mechanisms that simply post a take-it-or-leave-it price; A trade occurs only if both agents accept this price. In such mechanisms, the agents have obvious dominant strategies, and in the equilibrium analysis we do not need to speculate whether the agents compute the equilibrium correctly or converge to a particular Bayes-Nash equilibrium.

The simplicity of our mechanisms comes at the obvious price of sub-optimality, as the efficiency of our mechanisms might be inferior to the second-best mechanism. However, we are able to quantify this loss and show that our simple mechanisms perform quite well even with respect to a stronger benchmark: the optimal ex-post efficiency (i.e., “first-best” efficiency or simply the expected value of the maximum between the seller’s and the buyer’s value for the item). We measure the efficiency of our mechanisms by the fraction of the optimal efficiency that they guarantee. Our main result is as follows:

**Theorem:** For every pair of distributions, there is take-it-or-or-leave-it mechanism that achieves at least $\frac{2}{3}$ of the optimal efficiency in dominant strategies. The mechanism is obviously individually rational and budget balanced.

We stress that this is a worst-case bound over all possible distributions; typically the approximation achieved by our mechanism is much higher. A recent lower bound by [3] shows that no dominant-strategy mechanism (which is also individually rational and budget balanced) can guarantee more than 0.749 approximation to the optimal welfare. Together with our 2/3 approximation this leaves a relatively small gap. One immediate corollary from the above theorem is a bound on the performance of the most efficient (second-best) mechanism with Bayes-Nash incentive compatibility. That is, it shows that this mechanism that was characterized by [15] achieves at least 2/3 of the optimal efficiency even for the worst pair of distributions.

We then proceed to designing mechanisms that perform well despite limited information on the market. While one cannot ignore the wealth of distributional information that retailers have at their disposal, this knowledge is often incomplete. In some scenarios, the designer may have limited

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2 In an earlier version of this paper we gave the first bound for this problem of 0.51. [3] improved this bound to 0.52, and in this version we improve it further to 2/3.
statistics on some participants that are new to this market. In other cases, the designer can only
accurately estimate some statistics of the distributions (expectation, median, etc.) rather than the
full details of the distribution.

We consider settings where designers have pretty accurate statistical information about one side
of the market but know little about the other side. We start by presenting two simple mechanisms
that achieve at least 1/2 of the full efficiency \(^3\). The first mechanism posts a price equal to the median
value of the seller’s distribution, and the other mechanism computes some sort of a weighted median
for the buyer and offers this price to the agents.

We also show that richer statistics on the seller’s distribution suffices to achieve improved efficiency:

**Theorem:** For every distribution of the seller, there is a distribution \( G \) of prices with the following
property: if we post a take-it-or-leave-it price sampled from \( G \) then for *every* distribution of the
buyer we obtain a fraction of at least \((1 - 1/e) \approx 0.63\) of the optimal efficiency. This mechanism is
individually rational, budget balanced and dominant-strategy incentive compatible.

### 1.2 Using Solutions for Bilateral Trade as “Black-Boxes”

The above results study the fundamental problem of Bilateral Trade, and insights from the study of
this basic problem should alleviate the design of mechanisms in more complex trade environments.
In the second part of the paper, we show how results for Bilateral Trade can be directly applied
to construct approximately-optimal mechanisms in other settings in a black box manner. More
specifically, we show reductions of the following form: given a mechanism that guarantees some
\( \alpha \) approximation to Bilateral Trade, we design a mechanism for the other problem that obtains
\( f(\alpha) \) approximation for some function \( f \). We assume that we are given a DSIC, ex-post budget
balanced and individually-rational mechanism as input, and the mechanism we output maintains
these desired properties. Given an \( \alpha \)-approximation mechanism for Bilateral Trade, we give the
following reductions:

- **(Divisible good with general monotone valuations.)** Consider a seller and a buyer, where the
  seller initially owns a fully divisible good. The players have monotonically non-decreasing
  valuations. We design a \( \frac{\alpha}{\alpha+1} \) approximation to the optimal efficiency in this setting.

- **(Divisible good with convex valuations.)** Consider a 2-player environment, where each player
  \( i \) initially holds some fraction \( r_i \) of a divisible good. The players have convex valuations,
  i.e., preferences with decreasing marginal valuations. We design a \( \frac{\alpha}{\alpha+1} \) approximation to the
  optimal efficiency in this setting.

- **(Partnership Dissolving )** Consider \( n \) players, each player \( i \) initially owns a fraction \( r_i \) of a
  divisible good. Players have linear valuations, such that the value for player \( i \) for a fraction \( c \) of
  the item is \( c \cdot v_i \). This is exactly the classic model by Cramton, Gibbons and Klemperer \(^4\). We
  design a mechanism that achieves an \( \alpha \) approximation to the optimal efficiency. This result
  proves that the extreme-ownership scenario of the partnership dissolving problem is actually
  the hardest to solve, and any approximation for the first implies the same approximation
  factor for the latter problem.

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\(^3\) At first glance, achieving 1/2 of the optimal efficiency looks trivial: we can allocate the item to the player
with the higher expected efficiency and that should give us at least 1/2 of the optimal efficiency. However, this
overly-optimistic argument fails since in bilateral trade the two agents are not symmetric: while it is easy to leave
the item with the seller, we need to convince the seller to relinquish the item (via a sufficiently high payment) for the
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two agents to trade.
Applying our 2/3-approximation mechanisms mentioned above, we get a 0.4 approximation for the 2-player trade problems with a divisible good, and a 2/3 approximation to the n-player partnership dissolving problem. Any future improvements to the 2/3 approximation would immediately imply an improvement to the above bound.

More related research. The paper of McAfee [12] is the closest in spirit to ours. McAfee showed how simple posted-price mechanisms can approximate the efficient outcome. More specifically, he showed that for distributions such that the median of the buyer is greater than the median of the seller, posting any price between these two medians achieves at least half of the optimal expected gain from trade. In a sense we show how this median technique can be developed to approximate the social welfare for every pair of distributions. A recent paper by Garratt and Pycia [9] shows that with non quasi-linear preferences, e.g., with risk aversion and wealth effects, efficient trade is possible under some general conditions on the information structure. The partnership dissolving model was first studied by Cramton, Gibbons and Klemperer [4] who showed that if the shares are close enough to equal holdings, there exists a fully efficient Bayes-Nash incentive compatible mechanism (see also [11], and a survey [14]).

2 Model

The bilateral trade problem involves two agents, a seller and a buyer. The seller owns an indivisible item, and his privately known valuation for consuming this item is \( s \). If a trade occurs and the item is allocated to the buyer, then the buyer has a privately known value of \( b \). \( s \) and \( b \) are independently distributed according to distributions \( F_s \) and \( F_b \), respectively. All our results hold for any pair of distributions, but for the simplicity of presentation we assume the existence of density functions \( f_s \) and \( f_b \), for the seller and the buyer respectively, which are always positive on a support \( [a, \infty) \) (\( a \geq 0 \)) and atomless.

When trade occurs at price \( p \), the seller’s utility is \( p \) and the buyer’s utility is \( b - p \). With no trade, the seller’s utility is \( s \) and the buyer’s utility is 0. Both agents are risk neutral.

Ideally, a fully efficient mechanism will initiate a trade whenever \( s > b \). Therefore, we define the optimal efficiency as

\[
E_{s \sim F_s, b \sim F_b} \left[ \max\{s, b\} \right]
\]

We will measure the performance of our mechanisms by the fraction of the optimal efficiency that they obtain. For every realization of \( s \) and \( b \), an allocation \( X \) of a (deterministic) mechanism determines the allocation of the good, where \( X_s(s, b) = 1 \) in the case of no trade and otherwise \( X_s(s, b) = 0 \) (and the opposite for \( X_b \)). Therefore, we denote the expected efficiency achieved by a mechanism an allocation function \( X \) as

\[
MECH = E_{s \sim F_s, b \sim F_b} \left[ X_s(s, b) \cdot s + X_b(s, b) \cdot b \right]
\]

Consider a mechanism with an allocation function \( X(s, b) \) and payment \( p(s, b) \). We say that a mechanism is (ex-post) individually rational if upon trade (\( X_s = 0 \) and \( X_b = 1 \)) we have that \( p_s(s, b) \geq s \) and \( p_b(s, b) \leq b \), and with no trade \( p_s(s, b) = p_b(s, b) = 0 \).

We say that a mechanism is (ex-post and strongly) budget balanced if for all \( s, b \) we have that \( p_s(s, b) = p_b(s, b) \).

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4 A further discussion on approximating social efficiency vs. approximating the gain-from-trade is given in Section
3 Warm Up: Two Simple Mechanisms for Bilateral Trade

We begin our investigations by introducing two simple mechanisms that guarantee at least half of the optimal efficiency. In the next sections we provide improved approximation ratios that rely on ideas that are presented here in their most crystallized form.

The mechanisms that are developed in this section – as well as all of our mechanisms for the bilateral trade problem – simply set a price \( p \). The item is traded if and only if \( b \geq p \) and \( s \leq p \). In case of trade, the buyer pays \( p \) and the seller receives the same amount. Since the choice of \( p \) will depend only on the the distributions of the players and not on the actual realizations, this family obviously yields dominant strategy mechanisms that are individually rational and budget balanced.

Our first mechanism is the median mechanism which sets the price \( p \) to be the median of seller’s distribution, i.e., the point \( M_s \) such that \( F_b(M_s) = \frac{1}{2} \).

**Theorem 3.1.** The median mechanism is DSIC, individually rational, budget balanced and always achieves at least \( 1/2 \) of the optimal social welfare.

**Proof.** By the discussion above, the mechanism is obviously DSIC, individually rational, and budget balanced. We now analyze its approximation ratio. We start with some notation. Since the trade price depends only on the distribution of the seller, to analyze the approximation ratio we may fix the value \( b \) of the buyer and show that for every fixed \( b \) the expected approximation ratio is \( 1/2 \) in expectation over the seller’s distribution. We divide our analysis to two disjoint cases. In both cases we use the observation that the expected optimal welfare is at most \( \Pr[b \geq s] \cdot b + \Pr[s > b] \cdot E[s > b] \).

1. \( b \geq M_s \). In this case the item is sold with probability \( \frac{1}{2} \), when \( s \leq M_s \). We get that the approximation ratio of the mechanism (i.e., the expected optimal efficiency divided by the expected efficiency of the median mechanism) is at most:

\[
\frac{\Pr[b \geq s] \cdot b + \Pr[s > b] \cdot E[s > b]}{\Pr[s < M_s] \cdot b + \Pr[s \geq M_s] \cdot E[s > b]} = \frac{\Pr[b \geq s] \cdot b + \Pr[s > b] \cdot E[s > b]}{\Pr[s < M_s] \cdot b + \Pr[s \geq b] \cdot E[s > b] + \Pr[M_s < s \leq b] \cdot E[s|M_s < s \leq b]} \leq \frac{\frac{1}{2} \cdot b + \frac{1}{2} \cdot E[s > b]}{2} \leq 2
\]

2. \( b < M_s \). Here the item is never sold and the expected welfare is \( E[s] \). We bound the expected approximation ratio:

\[
\frac{\Pr[b \geq s] \cdot b + \Pr[s > b] \cdot E[s > b]}{E[s]} \leq \frac{\frac{1}{2} M_s + \Pr[s > b] \cdot E[s > b]}{\Pr[s \leq b] \cdot E[s \leq b] + \Pr[s > b] \cdot E[s > b]} \leq \frac{\frac{1}{2} M_s + \Pr[s > b] \cdot E[s > b]}{\Pr[s > b] \cdot E[s > b]} \leq \frac{\frac{1}{2} M_s}{\frac{1}{2} M_s} = 2
\]

where in the last transition we use the fact that \( \Pr[s > b] \cdot E[s > b] = \Pr[M_s \geq s > b] \cdot E[s|M_s \geq s > b] + \Pr[s > M_s] \cdot E[s > M_s] \geq \frac{1}{2} M_s \).
One could hope that the “symmetric” mechanism that sets the trade price to be \( M_b \) (the median of the distribution of the buyer) will provide a good approximation ratio as well. Unfortunately, this is not the case. To see this, consider the distribution where the buyer’s value is \( \epsilon \) with probability \( \frac{1}{2} + \epsilon \) and \( t > 1 \) with probability \( \frac{1}{2} - \epsilon \). Let the seller’s value be 1 with probability 1. When we set the trade price to be \( M_b = 1 \) the item is never sold since the price is always too low for the seller. The welfare of the mechanism is 1 while ideally we would like to sell the item whenever the buyer’s value is \( t > 1 \). The approximation ratio. Unfortunately, this does not guarantee any constant approximation ratio.

However, we do show that a more careful choice of a trade price \( p \) (using only the distribution of the buyer) results in an approximation ratio of \( 1/2 \). The weighted median mechanism sets the trade price \( p \) to be the point \( W_b \) for which \( F_b(W_b) \cdot E[b|b < W_b] = (1 - F_b(W_b)) \cdot E[b|b \geq W_b] \). In particular, since \( E[b|b \geq W_b] \geq E[b|b < W_b] \) we get that \( \Pr[b > W_b] \leq \frac{1}{2} \). In addition:

\[
(1 - F_b(W_b)) \cdot E[b|b \geq W_b] = F_b(W_b) \cdot E[b|b < W_b] = \frac{E[b]}{2}
\]

**Theorem 3.2.** The weighted median mechanism is DSIC, individually rational, budget balanced and always achieves at least \( 1/2 \) of the optimal social welfare.

*Proof.* Again, we only analyze the approximation ratio of the mechanism. Similarly to the analysis of the median mechanism, since the trade price \( p \) depends only on the buyer’s distribution we may analyze the approximation ratio assuming that \( s \) is fixed. We divide our analysis into two cases:

1. \( s < W_b \). In this case the item is sold whenever the buyer’s value is at least \( W_b \). The approximation ratio is at most:

\[
\frac{\Pr[b \geq s] \cdot E[b|b \geq s] + \Pr[s > b]}{\Pr[b \geq W_b] \cdot E[b|b \geq W_b] + \Pr[b < W_b]} \cdot s \leq \frac{E[b]}{2} + \frac{s}{2} \leq 2
\]

where we use Equation (1) and \( \Pr[b < W_b] \geq \frac{1}{2} \).

2. \( s \geq W_b \). In this case the item is never sold. We will make use of \( E[b] = \Pr[b > W_b] \cdot E[b|b > W_b] + \Pr[b \leq W_b] \cdot E[b|b \leq W_b] \leq \frac{1}{2} \cdot E[b] + W_b \) which implies that \( \frac{E[b]}{2} \leq W_b \leq s \). Therefore:

\[
\frac{\Pr[b \geq s] \cdot E[b|b \geq s] + \Pr[s > b]}{s} \leq \frac{\Pr[b \geq W_b] \cdot E[b|b \geq W_b] + \Pr[s > b]}{s} \leq \frac{E[b]}{2} + \frac{s}{2} \leq 2
\]

where we again use Equation (1).

The two simple mechanisms that we presented in this section prove that one can achieve good approximation to the optimal efficiency while preserving budget balance and individual rationality. Moreover, this can be done having statistical knowledge only on one side of the market. In the next sections we give improved approximation mechanisms extending the ideas in the above mechanisms.
4 A 2/3 Approximation for Bilateral Trade

In this section we present the main result of the paper. We present a mechanism that computes a single fixed price from both distributions. The mechanism combines ideas from the two simple mechanisms in the previous section and guarantees at least 2/3 of the optimal efficiency.

The Median mechanism from Section 3 computes the 1/2-quantile of the seller’s distribution, that is, a price \( p \) such that the probability \( F_s(p) \) that the seller rejects the price \( p \) is exactly 1/2. The Weighted-Median mechanism chooses a price \( q \) such that the contribution \( (1 - F_b(q)) \cdot E[b|b > q] \) of values above \( q \) is exactly half of the expectation of \( b \). Our next mechanism takes a hybrid approach, and chooses a price \( p^* \) such that the probability that the seller has values below \( p^* \) equals the percentage of the contribution of values above \( p^* \) to the expectation of \( b \). This hybrid approach may look nonintuitive at first glance as it compares “apples and oranges”, but it actually enables us to improve the approximation ratio from 1/2 to 2/3.

Consider the following Hybrid-Median Mechanism: The mechanism offers a take-it-or-leave-it price \( p^* \) such that

\[
F_s(p^*) = \frac{(1 - F_b(p^*)) E[b|b > p^*]}{E[b]}
\]

We first note that when the two density functions are always positive and atomless, such a price uniquely exists. To see this, note that in such cases \( F_s(p) \) is a continuous function which is increasing in \( p \) (from 0 to 1), and that \( \frac{(1 - F_b(p)) E[b|b > p]}{E[b]} \) is continuous and decreasing in \( p \) (from 1 to 0). Therefore, the difference of these two function must cross 0 exactly once, at \( p^* \).

**Theorem 4.1.** For every \( F_s \) and \( F_b \), the Hybrid-Median mechanism achieves at least a 2/3 fraction of the optimal welfare.

Before we prove the theorem, we prove some of the tools that we use in the proof. We start with a trivial fact:

**Observation 4.2.** Let \( 0 \geq c_1 \geq c_2 \) and \( a, b > 0 \). Then, \( \frac{c_1}{c_2} \leq \frac{a - c - b}{c_2 - a} \).

**Proof.** Follows since \( a(c_2 - c_1) + c_2 b \geq 0 \). \( \square \)

We now present a lemma that claims that we can change the distribution of the seller to be a 2-point distribution, and the approximation achieved by the mechanism will only get worse. Specifically, let \( F^*_s \) a distribution for the seller such that \( \Pr[s = 0] = F_s(p^*), \Pr[s = p^*] = 1 - F_s(p^*) \).

Let \( HYB(G, H) \) and \( OPT(G, H) \) denote the expected efficiency achieved by the Hybrid-Median mechanism and the optimal (ex-post efficient) mechanism, respectively, when the distributions of the seller and the buyer are \( G, H \). The proof of this lemma is given in Appendix A.1.

**Lemma 4.3.** \( \frac{HYB(F^*_s, F_b)}{OPT(F^*_s, F_b)} \geq \frac{HYB(F_s, F_b)}{OPT(F_s, F_b)} \).

The next observation claims that the price offered by the mechanism cannot be too large compared to the expected value of the buyer. For the rest of this section, we will use the abbreviated notations \( x = F_s(p^*) \) and \( y = 1 - F_b(p^*) \).

**Observation 4.4.**

\[
p \geq \frac{1 - x}{1 - y} E[b] \geq (1 - x) E[b]
\]
Proof. By the definition of $p^*$, $yE[b|b \geq p^*] = E[b|x]$. Then,

$$
E(b) = (1 - y)E[b|b \leq p^*] + yE[b|b \geq p^*]
= (1 - y)E[b|b \leq p^*] + E[b|x]
\leq (1 - y)p^* + E[b|x]
\leq p^* + E[b|x]
$$

We are now ready to give the proof of Theorem 4.1:

Proof. In the Hybrid-Median mechanism, with probability $1 - x$ the seller rejects the price and keeps the item with a value of $p^*$, and with probability $x$ the seller accepts the price what results in efficiency of $yE[b|b > p^*] = xE[b]$.

$$
HYB(F^*_s, F^*_b) = x^2 E[b] + (1 - x)p^*
$$

In a similar way we compute the optimal efficiency and get:

$$
OPT(F^*_s, F^*_b) = xE[b] + (1 - x)(yxE[b] + (1 - y)p^*)
= xE[b] + (1 - x)(y(xE[b] - p^* + p^*)
$$

We need to prove a lower bound on the ratio (which we denote for simplicity $HYB/OPT$ from now on):

$$
\frac{HYB(F^*_s, F^*_b)}{OPT(F^*_s, F^*_b)} = \frac{x^2 E[b] + (1 - x)p^*}{xE[b] + (1 - x)(y(xE[b] - p^*) + p^*)}
$$

We will prove the theorem for two different cases.

**Case 1:** $p^* \geq xE[b]$.

The ratio is minimized when $y$ is minimal (has a non-positive coefficient). Thus, we substitute $y = 0$ and the ratio cannot increase:

$$
\frac{HYB}{OPT} \geq \frac{x^2 E[b] + (1 - x)p^*}{xE[b] + (1 - x)p^*}
$$

This ratio is decreasing in $E[b]$ (as $x^2 \leq x$), we can therefore substitute $E[b] = p^*/(1 - x)$ (Observation 4.3) and have:

$$
\frac{HYB}{OPT} \geq \frac{x^2 \frac{p^*}{1 - x} + (1 - x)p^*}{\frac{p^*}{1 - x} + (1 - x)p^*}
= \frac{x^2 + (1 - x)^2}{x + (1 - x)^2}
$$

This function is minimized when $x = 0.5$, with value $2/3$, hence $\frac{HYB}{OPT} \geq \frac{2}{3}$.

**Case 2:** $p^* < xE[b]$.
We can write \( \frac{\text{HYB OPT}}{\text{OPT}} = \frac{x^2E[b]+(1-x)p^*}{xE[b]+(1-y)(1-x)p^*} \), and observe that it is increasing in \( p^* \) (due to Observation 4.2). Due to Observation 4.4 we can substitute \( p^* = \frac{1-x}{y}E[b] \) and the ratio will not increase. \( E[b] \) then cancels, and it is left to show that

\[
\frac{x^2 + \frac{(1-x)^2}{1-y}}{x + (1-x)yx + (1-x)^2} \geq \frac{2}{3}
\]

With some simple algebra, we need to show that:

\[
3x^2 + \frac{3(1-x)^2}{1-y} - 2x - 2(1-x)yx - 2(1-x)^2 \geq 0 \tag{2}
\]

We now observe that \( x \geq y \) (as \( x = y \cdot \frac{E[b|b>p^*]}{E[b]} \geq y \)). Thus, we can show the following inequality:

\[
\frac{3(1-x)^2}{1-y} - 2(1-x)^2 = \frac{1-x}{1-y}(1-x)(1+2y) \geq (1-x)(1+2y) \tag{3}
\]

We use Equation (3) to show that the LHS in Equation (2) is at least

\[
3x^2 + (1-x)(1+2y) - 2x - 2(1-x)yx
\]

\[
= (x-1)^2 + (2x^2-x) + 2y(1-x)^2 \geq 0
\]

Where the inequality holds since we consider the case where \( p^* < xE[b] \), and together with Observation 4.4 we get that \( x \geq \frac{1}{2} \) and thus \( 2x^2 - x \geq 0 \). \( \square \)

The Hybrid-Median mechanism may clearly be sub-optimal. Given two distributions, one can compute the optimal price to post and it typically will be different than the price in the Hybrid-Median mechanism. However, an immediate corollary of Theorem 4.1 is that the optimal posted price mechanism achieves a 2/3 approximation for every pair of distributions. Furthermore, the most efficient Bayes-Nash incentive-compatible mechanism (the “second-best” mechanism from [15]) is also at least as good as the Hybrid-Median mechanism and is thus guaranteed to achieve at least 2/3 of the optimal efficiency. As far as we know, we are the first to bound the efficiency of this second-best mechanism for the bilateral trade problem.

In the next section, we show that randomization improves the approximation bounds when the social planner has partial distributional knowledge. We note that when the planner fully knows the two distributions (i.e., in the setting of Theorem 4.1) then randomized mechanisms cannot outperform deterministic mechanisms. To see this, observe that instead of picking prices randomly the planner simply posts a price in the support that attains the highest expected efficiency. When the planner has access only to one of the distributions, then randomization can help achieving worst-case guarantees against all possible distributions of the other agent.

5 An Improved Mechanism with Partial Statistical Knowledge

In Section 3 we showed that one can compute a price that depends only on the distribution of the seller and this price will guarantee at least half of the optimal welfare no matter what is the distribution of the buyer. We can show that this bound of \( \frac{1}{2} \) is tight for all deterministic dominant-strategy mechanisms (see Appendix A.2). However, this bound could be improved by randomly choosing a price.
We let \( q() \) denote the quantile function of the seller, that is, \( F(q(x)) = x \) for every \( x \in [\underline{a}, \bar{a}] \). Consider the following mechanism that posts a price to both players, but chooses the price randomly. We call this mechanism the Random-Quantile mechanism:

- Choose a number \( x \in [1/e, \ldots 1] \) according to the distribution with PDF \( G(x) = \ln(e \cdot x) \).
- Set the price to be \( q(x) \).

**Theorem 5.1.** The Random-Quantile mechanism is DSIC, individually rational, budget balanced, and always achieves at least \( 1 - \frac{1}{e} \) fraction of the optimal welfare.

**Proof.** We prove the theorem for every fixed value \( b \) of the buyers, and it will clearly hold for every distribution of \( b \)'s as well.

We will first prove that if we truncate the seller’s distribution at \( b \), the approximation ratio can only get worse.

**Claim 5.2.** Consider a mechanism that posts a price \( p \) and fixing \( b \). Let \( \text{RAND}(F_s, b) \) be the expected efficiency of the Random-Quantile mechanism, and let \( F_s^* \) be a distribution for the seller such that \( F_s^*(x) = Fs(x) \) for \( x < b \), \( F_s^*(x) = 0 \) for \( x > b \) and \( F_s^*(b) = 1 - F_s(b) \). Then,

\[
\frac{\text{RAND}(F_s, b)}{\text{OPT}(F_s, b)} > \frac{\text{RAND}(F_s^*, b)}{\text{OPT}(F_s^*, b)}
\]

**Proof.** We will write the efficiency of our mechanism when the seller’s distributions are \( F_s \) and \( F_s^* \):

\[
\text{RAND}(F_s, b) = F_s(p)b + (1 - F_s(p))E[s|s \geq p]
\]
\[
\text{RAND}(F_s^*, b) = F_s(p)b + (F_s(b) - F_s(p))E[p \leq s \leq b] + (1 - F_s(p))b
\]

We can see that their difference is exactly \((1 - F_s(b))(E[s|s \geq b] - b)\).

We will write the optimal efficiency when the seller’s distributions are \( F_s \) and \( F_s^* \):

\[
\text{OPT}(F_s, b) = F_s(b)b + (1 - F_s(b))E[s|s \geq b]
\]
\[
\text{OPT}(F_s^*, b) = F_s(b)b + (1 - F_s(b))b
\]

Their difference is also exactly \((1 - F_s(b))(E[s|s \geq b] - b)\).

Therefore, moving from \( \frac{\text{RAND}(F_s, b)}{\text{OPT}(F_s, b)} \) to \( \frac{\text{RAND}(F_s^*, b)}{\text{OPT}(F_s^*, b)} \) the enumerator and the denominator decrease by the same additive term. As the two fractions are at most 1, the claim follows. \( \square \)

We now analyze the approximation ratio assuming a constant buyer valuation \( b \) and the seller distribution \( F_s^* \), for which \( \text{OPT} = b \). This will give us the same bound for the original distributions. Let \( y = q(b) \) denote the probability that the seller’s value is at most \( b \). Note that the seller accepts the price \( q(x) \) with probability \( x \) by definition, and that the density of the price distribution \( G(x) \) is \( \frac{1}{e} \).

The expected welfare of the mechanism is \( b \) when the seller value is \( b \), and if the seller value is less than \( b \) the expected welfare is \( b \) if the seller accepts the price and \( E[s|s < b] \) if the seller rejects the price and keeps the item. Thus,
\[ RAND(F_s^*, b) = (1 - F_s(b))b + F_s(b) \left( \Pr[s < q(x)|s < b] \cdot b + \Pr[s \geq q(x)|s < b] \cdot E[s|s < b] \right) \]
\[ \geq (1 - F_s(b))b + F_s(b) \Pr[s < q(x)|s < b] \cdot b \]
\[ \geq (1 - F_s(b))b + F_s(b) E_x[\Pr[s < q(x)|s < b]] \cdot b \]
\[ = (1 - F_s(b))b + F_s(b) \left( \int_{b}^{y} x \cdot \frac{1}{x} dx \right) \cdot b \]
\[ = (1 - y)b + \int_{\frac{1}{e}}^{y} x \cdot \frac{1}{x} dx \cdot b \]
\[ = (1 - y)b + (y - 1/e)b = (1 - 1/e)b \]  

Equation (4) is due to the fact that the price \( q(x) \) is posted with probability \( 1/x \) and it is accepted by the seller with probability \( x \).

\[ \square \]

6 Black-Box Reductions

The Bilateral Trade problem is one of the most fundamental and well-studied problems in mechanism design. So far, we analyzed the Bilateral Trade problem and designed mechanisms that approximate the optimal social welfare for this problem. In this section, we show that the simple Bilateral Trade environment can be used as a building block for the construction of mechanisms in more general environments. We design the mechanisms using black-box reductions: we show that given any \( \alpha \) approximation mechanism for Bilateral Trade we can construct a mechanism with an approximation ratio which is some function of this \( \alpha \) that maintains all the desired economic properties.

In the rest of this section we will consider a more general model that reallocates a fully divisible good. In section 6.1 we study the \( n \)-player Partnership Dissolving setting, where players initially own fractions of the good and have linear utilities. In section 6.2 we study 2-player settings with general monotone valuations, and in Section 6.3 we discuss a similar 2-player setting with the restriction to convex valuations (but with an arbitrary initial allocation).

6.1 Dissolving Partnerships

In the partnership dissolving problem, there are \( n \) agents, each agent \( i \) owns a share \( r_i \) of an asset, and \( \sum_{i=1}^{n} r_i = 1 \). Each agent \( i \) has a value \( v_i \) for holding the entire asset, or a value \( c \cdot v_i \) for holding a fraction \( c \geq 0 \) of the asset. Let \( r_{max} = \max\{r_1, ..., r_n\} \) be the largest share held by an agent.

We show that any approximation mechanism for bilateral trade can be used for constructing a mechanism for partnership dissolving with exactly the same guarantee on the approximation ratio. This reduction from partnership dissolving to bilateral trade holds for all distributions of the buyers, and irrespectively of the size of the initial shares. The idea is that each agent sells his share to the other agents via a second-price auction, where the price taken from the bilateral trade mechanisms for this setting serves as a reserve price. We show that for each player, and thus for the whole economy, this mechanism can only improve the efficiency of the bilateral trade mechanism.
Theorem 6.1. Let $M$ be some DSIC, individually rational, and budget balanced mechanism for bilateral trade that achieves an $\alpha$-approximation to the welfare. There is a DSIC, individually-rational and budget-balanced mechanism for partnership dissolving which also achieves an $\alpha$-approximation to the optimal efficiency.

Proof. In the proof we use the already-mentioned fact that any truthful mechanism for bilateral trade simply sets a trade price $p$ that does not depend on the values of the bidders. We develop our mechanism for partnership dissolving in two stages.

First Stage: A Mechanism where only Bidder $i$ may sell. We will “run” $M$ with bidder $i$ and a hypothetical buyer whose value is distributed according to the distribution of $\max_{k \neq i} v_k$. Let $p$ be the price that $M$ posts. Let $M'$ be the following mechanism:

- Let $j \in \arg\max_{k \neq i} \{v_k\}$ and let $m_2$ be the second highest value of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$, that is, $m_2 = \max_{k \in \mathbb{N} \setminus \{j,i\}} v_k$.
- Let $p^* = \max\{p, m_2\}$. If $v_i \leq p^*$ and $v_j \geq p^*$ then bidder $j$ pays to bidder $i$ the amount of $p^*$ and receives the item. Otherwise bidder $i$ keeps his item and does not get paid.

To see that the mechanism is truthful, observe that if a sale is made neither the winning buyer nor bidder $i$ cannot affect the price by changing their bid. A losing buyer $k$ can only turn into a winner by overbidding the winning buyer and paying $p^* \geq v_j \geq v_k$, and therefore cannot gain a positive payoff. If the item is not sold, it is either because the seller’s value exceeds $p^*$ (and winning by underbidding induces a payment below $v_i$) or all of the buyers’ values are below $p^*$ (and again, overbidding results in a payment higher than $v_j$).

Now for the approximation ratio. Notice that whenever there is a trade in $M$ there is a trade in $M'$ (but the opposite is not true; for example, a trade takes place when $p < v_i \leq m_2$). $M$ achieves an $\alpha$-approximation to the optimal solution (that may only allocate bidder $i$’s share) which equals $\max\{v_i, \max\{v_{-i}\}\} = \max_k \{v_k\}$. $M'$ achieves at least the same expected welfare as $M$, thus it is an $\alpha$-approximation to the optimal welfare (that may only allocate bidder $i$’s share) as well.

The mechanism is individually rational since we always have that $v_i \leq p^* \leq v_j$. In addition, payments are transferred from one player to another, hence the mechanism is budget balanced.

Second Stage: The Final Mechanism. At an arbitrary order, use $M'$ to sell to the other bidders the endowment $r_i$ of each bidder $i$ as a single indivisible item.

We now analyze the approximation ratio. Let $v_{max} = \max_k v_k$ be the highest value. By selling the endowment of bidder $i$ the expected social welfare is at least $\frac{r_i v_{max}}{\alpha}$. Since the valuations of the bidders are linear, after selling all endowments the expected social welfare of at least $\frac{v_{max}}{\alpha}$.

The truthfulness of the mechanism also follows from the linearity of the valuations of the bidders: at every stage they will maximize their payoff from the item independently of the other sales. Therefore, truthfulness follows from the truthfulness of $M'$. Similarly, the mechanism is individually rational and budget balanced.

As our best approximation for the bilateral-trade problem is $\frac{2}{3}$, the reduction guarantees the same approximation ratio for Partnership Dissolving.

Corollary 6.2. There is a truthful, individually rational, and budget balanced mechanism the Partnership Dissolving problem which is a $\frac{2}{3}$ approximation to the optimal social welfare.
6.2 2-player Markets with Monotone Valuations

We now consider a 2-player market where a seller initially owns the whole quantity of a divisible good. The seller and the buyer may have arbitrary monotone valuations, that is, the only restriction on the valuations is that for fractions $x > y$ we have $v(x) \geq v(y)$. In this setting, we show that given an $\alpha$-approximation mechanism to the bilateral trade problem, we can design a mechanism with an approximation guarantee of $\frac{\alpha}{\alpha+1}$ that preserves the economic properties of the original mechanism.

**Theorem 6.3.** Let $A$ be a budget balanced and DSIC mechanism for bilateral trade with one indivisible good that achieves an approximation ratio of $\alpha < 1$. There is a DSIC and budget balanced mechanism $A'$ for bilateral trade with one divisible good that guarantees an approximation ratio of $\frac{\alpha}{\alpha+1}$, as long as the players’ valuations are monotonically increasing.

**Proof.** Let $S$ be the expected contribution of the seller to the welfare-maximizing solution, and let $B$ be the expected contribution of the buyer $B$. Thus, if we let $OPT$ denote the expected optimal welfare, we have that $OPT = B + S$.

Consider some instance and let $s$ be the value of the seller for getting all the item and similarly let $b$ be the value of the buyer for getting all the item. Let $D_s$ be the distribution of $s$ given the seller’s distribution and $D_b$ the distribution of $b$. Our mechanism $A'$ will simply run $A$ to obtain a trade price $p$. If $b \geq p \geq s$ the item is re-allocated in full to the buyer and the buyer transfers a payment of $p$ to the seller. Otherwise the seller keeps the full item. Notice that $A'$ inherits from $A$ its truthfulness, budget balance, and individual rationality.

We divide the analysis of the approximation ratio into two cases. In the first case we assume that $S \geq \frac{\alpha}{\alpha+1}OPT$. Observe that in this case the approximation ratio is $\frac{\alpha}{\alpha+1}$: the expected welfare of any mechanism is at least $E[s]$ since every reallocation cannot decrease the welfare. To conclude this case, observe that clearly we have that by the monotonicity of the valuations, $E[s] \geq S$.

Thus assume that $S < \frac{\alpha}{\alpha+1}OPT$. In other words, $B \geq \frac{OPT}{\alpha+1}$. Notice that if we restrict our attention to instances where the item can only be reallocated in full, then the optimal welfare is at least $E[b] \geq B$ (again, using the monotonicity of the valuations). Thus, using the approximation guarantee of $A$, the expected welfare of $A'$ is at least $\alpha B \geq \frac{\alpha}{\alpha+1}OPT$, as needed. \qed

6.3 Trading a Divisible Good with Convex Valuations

We consider a 2-player market, where each player owns some arbitrary share of a fully divisible good. The initial shares of players 1, 2 are denoted by $r_1, r_2$, where $r_1 + r_2 = 1$. Each player $i$ has a valuation function $v_i : [0, 1] \to \mathbb{R}$, and for every $x, y$ we define the marginal valuation $v_i(x|y) = v_i(x + y) - v_i(y)$. We assume that the valuation functions are normalized ($v_i(0) = 0$), non-decreasing, and have decreasing marginal valuations (i.e., $v_i(\epsilon|x) = v_i(\epsilon|y)$ for every $\epsilon > 0, y > x$).

In this model, we show that given any DSIC and budget balanced $\alpha$-approximation mechanism, we can use it to create an $\frac{\alpha}{\alpha+1}$-approximation mechanism for the 2-player market with convex valuations.

**Theorem 6.4.** Let $A$ be a budget balanced and truthful mechanism for bilateral trade that achieves an $\alpha < 1$ approximation to the optimal social welfare. There is a truthful and budget balanced mechanism for the 2-player reallocation problem with arbitrary endowments when the players’ valuations are monotone and have decreasing marginals that guarantees an approximation ratio of $\frac{\alpha}{\alpha+1}$.

\footnote{When $v_i(\cdot)$ is twice differentiable, we simply assume that $v_i''(x) \leq 0$ for every $x$.}
Proof. Consider the following mechanism: give the seller for free a fraction $x$ of the item ($x \leq 1$ will be specified later, one of the players that has a share of at least $x$ will be viewed as the seller and we will claim that there is such a player). Denote by $D_s$ the marginal distribution that specifies the value of the seller for receiving an additional $y \in [0, 1-x]$ fraction of the item and by $D_b$ the distribution that specifies the value of the buyer for receiving $y \in [0, 1-x]$ of the item. Let $i$ be the player that received the item when running the mechanism $A$ on the distributions $D_s$ and $D_b$, and let $p_s \geq 0$ be the payment that the seller receives and $p_b$ the amount that the buyer is charged. Our mechanism give player $i$ a fraction of $1-x$ of the item, additionally allocates the seller $x$, charges the buyer $p_b$ and gives the seller a payment $p_s$. The mechanism is clearly truthful and budget balanced. All that is left is to analyze its approximation ratio.

Let $S$ be the expected value of the seller for receiving all the item, and similarly let $B$ denote the expected value of the buyer for receiving all the item. Clearly, $S + B$ is an upper bound on the expected value of the optimal solution. We will show that for $x = \alpha / (\alpha + 1)$ the expected welfare of our mechanism is $x \cdot (S + B)$ and the claim regarding the approximation ratio will follow.

We first bound the contribution of the seller for receiving a fraction $x$ of the item in the first step: Since the valuation of the seller exhibits decreasing marginals the expected contribution is at least $x \cdot S$. We now bound the contribution of the second step. Ideally, we could have calculated the expected optimal value $OPT_2$ of the secondary problem (with the distributions $D_s$ and $D_b$) and claim that the when running the mechanism $A$ the expected welfare is at least $\alpha \cdot OPT_2$. However, since we do not know how to compute $OPT_2$ exactly we will only give a rough bound to it. Specifically, we claim that $OPT_2 \geq (1-x) \cdot B$, which is the welfare we when always allocating the buyer a fraction of $(1-x)$ of the item, since the buyer’s valuation function exhibits decreasing marginals.

We get that the overall expected welfare of our mechanism is $x \cdot S + \alpha \cdot (1-x) \cdot B$. Thus, for $x = \frac{\alpha}{\alpha + 1}$ the expected welfare is $\frac{\alpha S}{1+\alpha} + \frac{\alpha B}{1+\alpha}$, which gives us an $\frac{\alpha}{\alpha + 1}$ approximation ratio. As there is always a player with at least $\alpha / (\alpha + 1)$ fraction of the good (this value is at most 1/2), this player will be the seller.

7 Discussion

In this paper, we constructed mechanisms that approximate efficiency in bilateral-trade settings, and showed how such mechanisms can be used as building blocks of mechanisms in more complex scenarios. Our main result is a simple, fixed-price, dominant-strategy incentive compatible, budget balanced and ex-post individually rational mechanism that achieves at least $2/3$ of the optimal efficiency. This mechanism computes a price as a function of the distribution functions of the seller and the buyer, and achieves the approximation guarantee for every pair of such distributions. We then show how this approximation ratio implies a $0.4$ approximation for a similar setting with a divisible good and arbitrary monotone valuations, or a 2-player exchange setting for general convex valuations. We also prove the same $2/3$ approximation factor for the $n$ player partnership-dissolving problem. These three approximation results are given as “black-box” reductions, such that if the $2/3$ bound is improved in the future (we know that it cannot be larger than 0.749 [3]), these bounds will be immediately improved as well.

In our paper we approximate the expected efficiency in the economy. Another possible objective function may be approximating the expected gain from trade (GFT) in the efficient solution. A
mechanism that maximizes gain-from-trade is clearly fully efficient as well. However, from an approximation point of view, the two objectives are different. It turns out that maximizing the social welfare is an easier task, as any \( c \) approximation to the GFT is also at least a \( c \) approximation to the efficiency (the converse is false). McAfee \cite{McAfee2008} constructed a mechanism that achieves a \( 1/2 \) approximation to the optimal GFT for distributions that satisfy a certain condition (that the median of the seller’s distribution is no greater than the median of the seller). In Appendix \textbf{B.1} we show that this result by McAfee cannot be extended to all distributions, as no DSIC mechanism can guarantee a constant approximation to the optimal GFT. This result is proved via examples where the optimal GFT is a negligible fraction of the social welfare, demonstrating the fact that when the expected GFT is a small fraction of the overall efficiency then approximating it may not be very informative. We also make a straightforward observation that if the fraction of the optimal GFT out of the optimal overall efficiency is large, then an approximation to the optimal efficiency implies a good approximation ratio to the GFT as well (see Appendix \textbf{B.2}).

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A.1 Hybrid-Median Mechanism

Following is the proof of Lemma 4.3 which is used in the proof of Theorem 4.1 (2/3 approximation by the Hybrid-Median mechanism).

Proof. by computing the expected efficiency in the Hybrid-Median mechanism and the optimal efficiency for distributions $F_s$ and $F_b$.

$$HYB(F_s, F_b) = F_s(p^*) \left( F_b(p^*)E[s|s < p^*] + (1 - F_b(p^*))E[b|b \geq p^*] \right)$$
$$+ (1 - F_s(p^*))E[s|s \geq p^*]$$

$$OPT(F_s, F_b) = F_s(p^*) \left( F_b(p^*)E[\max\{s, b\}|s, b < p^*] + (1 - F_b(p^*))E[b|b \geq p^*] \right)$$
$$+ (1 - F_s(p^*))E[\max\{s, b\}|s \geq p^*]$$

We now define the following artificial function of the two distributions.

$$OPT^*(F_s, F_b) = F_s(p^*) \left( F_b(p^*)(E[s|s < p^*] + E[b|b < p^*]) + (1 - F_b(p^*))E[b|b \geq p^*] \right)$$
$$+ (1 - F_s(p^*))\left(E[s|s \geq p^*] + E[b]\right)$$

Clearly, $OPT^*(F_s, F_b) > OPT(F_s, F_b)$. Therefore,

$$OPT(F_s, F_b) - OPT(F_s^*, F_b)$$
$$\leq OPT^*(F_s, F_b) - OPT(F_s^*, F_b)$$
$$\leq F_s(p^*)F_b(p^*)E[s|s < p^*] + (1 - F_s(p^*))E[s|s > p^*] - p^*$$
$$= HYB(F_s, F_b) - HYB(F_s^*, F_b)$$

We see that when moving from $HYB(F_s, F_b)$ to $HYB(F_s^*, F_b)$ the enumerator decreases more than the denominator, thus from Observation 4.2 the theorem follows. □
A.2 1/2 Approximation by the Median Mechanism is Tight

Proposition A.1. No truthful, individually rational and budget balanced mechanism can obtain an approximation ratio better than 2 if the mechanism uses only the distribution of the seller or only the distribution of the buyer.

Proof. We start with the first part. Consider the following distribution $D_s$ of the seller: with probability $\frac{1}{2}$, $v_s$ gets a value (uniformly at random) in $(0, \epsilon)$. With probability $\frac{1}{2}$, $v_s$ gets a value (uniformly at random) between $(1, 1 + \epsilon)$. Observe that the median of $D_s$ is 1.

There are two possible cases, depending on the trade price $r$:

1. $r \leq 1$: let $v_b = \infty$ with probability 1. The optimal solution always sells the item to the buyer, but the mechanism will sell the item with probability $\frac{1}{2}$. We get an approximation of 2 since $v_b >> v_s$.

2. $r > 1$: let $v_b = 1 - \epsilon$ with probability 1. The value of the optimal solution is at least $1 - \epsilon$ (always sell the item to the buyer). However, the mechanism sells the item only when $v_s \in (0, \epsilon)$, which happens with probability $\frac{1}{2}$. The approximation is 2 also in this case.

We now prove the second part. Consider the following distribution $D_b$ of the buyer: with probability 0.99, $v_b$ gets a value (uniformly at random) in $(1, 1 + \epsilon)$. With probability 0.01, $v_b$ gets a value (uniformly at random) between $(100, 100 + \epsilon)$.

There are several possible cases, depending on the trade price $r$:

1. $r \leq 1 + \epsilon$: let $v_s = 1 + \epsilon$ with probability 1. The optimal solution sells the item to the buyer with probability $\frac{1}{100}$ and the expected welfare is about 2. The mechanism that posts the price $r$ however will never sell the item and will generate an expected welfare of $1 + \epsilon$.

2. $1 + \epsilon < r \leq 100 + \epsilon$: let $v_s = 0$ with probability 1. The expected value of the optimal solution is about 2 (always sell the item to the buyer). However, the mechanism sells the item only when $v_b \in (100, 100 + \epsilon)$, so the expected welfare is about 1. The approximation is 2 also in this case.

3. $r > 100 + \epsilon$: let $v_s = 0$. The expected value of the optimal solution is 2, but the mechanism achieves welfare of 0.

\[ \square \]

B Approximating Gain-from-Trade

B.1 Lower bound for approximating gain-from-trade

Consider a buyer and a seller with values on the support $[0, \ldots, t]$, and let $\lambda = \frac{1}{1-e^{-t}}$. Let $F_b(x) = \lambda(1 - e^{-x})$ with $f_b(x) = \lambda e^{-x}$ and $F_s(x) = \lambda(e^{x-t} - e^{-t})$ with $f_s(x) = \lambda e^{x-t}$.

Proposition B.1. For the above distributions, every fixed price mechanism achieves at most $O(1/t)$ approximation to the optimal gain from trade.
The optimal gain from trade (divided by the normalization factor of the distributions):

\[
\frac{1}{\lambda^2} GFT = \int_0^t \int_0^v (v-c)e^{-t} dc e^{-v} dv \\
= \int_0^t [e^{-t}(v-c+1)]_0^v e^{-v} dv \\
= \int_0^t (e^{v-t} - e^{-t}(v+1)) e^{-v} dv \\
= e^{-t}t - \int_0^t e^{-v-t}(v+1) dv \\
= e^{-t}t + [e^{-t-v}(v+2)]_0^t dv \\
= e^{-t}t + e^{-2t}(t+2) - 2e^{-t} \\
= \frac{t - 2}{e^t} + \frac{t + 2}{e^{2t}}
\]

The gain from trade from posting a price \( p \):

\[
\frac{1}{\lambda^2} GFT(p) = \int_p^t \int_p^v (v-c)e^{-v} dv e^{-t} dc \\
= \int_0^p [(c-v-1)e^{-v}]_p^t e^{-t} dc \\
= \int_0^p ((c-t-1)e^{-t} - (c-p-1)e^{-p}) e^{-t} dc \\
= \int_0^p (c - t - 1)e^{-2t} - (c - p - 1)e^{c-t-p} dc \\
= [e^{-2t}(c-t-2)]_0^p - [e^{c-p-t}(c-p-2)]_0^p \\
= e^{p-2t}(p - t - 2) - e^{-2t}(-t - 2) + 2e^{-t} + e^{p-t}(-p - 2) \\
= \frac{t}{e^{2t}} + \frac{p}{e^{2t}} - \frac{e^p(t + 2 - p)}{e^{2t}} \\
< \frac{t + 2}{e^{2t}} + \frac{2}{e^t}
\]

Overall, the optimal gain from trade is about \( \frac{t}{e^t} \) which is \( O(t) \) more than the gain from trade from any price \( p \) which is at most \( \frac{2}{e^t} \). (Note that the terms that are \( O\left(\frac{t}{e^{2t}}\right) \) are negligible for large \( t \)’s.)

**B.2 A Simple Reduction from Efficiency to Gain From Trade**

**Observation B.2.** Let \( c \) be the fraction of the optimal expected gain-from-trade out of the optimal expected efficiency. If a mechanism achieves a \( x \) fraction of the optimal efficiency, then it achieves at least a \( x + \frac{c-1}{c} \) fraction of the optimal gain-from-trade.

**Proof.** We know that \( \frac{OPTGF}{OPT} = c \), where \( OPTGF \) is the optimal expected gain from trade (in the first-best outcome) and \( OPT \) is the optimal efficiency.

\( OPT = OPTGF + E[s] \), where \( E[s] \) is the expected value of the seller. Thus, \( \frac{E[s]}{OPT} = 1 - c \).
Now, the approximation we get for the GFT is ($MECH$ is the expected efficiency of the $x$-approximation mechanism):

\[
\frac{MECH - E[s]}{OPT - E[s]} = \frac{MECH_{OPT} - E[s]_{OPT}}{1 - E[s]_{OPT}} \geq \frac{x - (1 - c)}{c}
\]

For example, for the uniform distribution we have $OPT_{GFT} = 1/6, OPT = 2/3$ and then $c = 1/4$. If we have a mechanism that gains us 90 percent of the welfare, we know it gains at least $(0.9 + 0.25 - 1)/0.25 = 0.6$ of the GFT.