VALUE-AT-RISK SUBSTITUTE FOR NON-RUIN CAPITAL IS FALLACIOUS AND REDUNDANT

Vsevolod K. Malinovskii

Abstract. This seemed impossible to use a theoretically adequate but too sophisticated risk measure called non-ruin capital, whence its widespread (including regulatory documents) replacement with an inadequate, but simple risk measure called Value-at-Risk. Conflicting with the idea by Albert Einstein that “everything should be made as simple as possible, but not simpler”, this led to fallacious, and even deceitful (but generally accepted) standards and recommendations. Arguing from the standpoint of mathematical theory of risk, we aim to break this impasse.

1. Introduction

The basis of Solvency II system (see Directives [5], [6]) is the Value-at-Risk set as the main measure of risk. Various criticisms (see, e.g., [1], [3], [4], [7], [8], [9], [11], [12], [23], [24], [25]) are directed against this basis. Floreani (see [11]) expressed it categorically:

the Solvency II regime uses an inadequate risk measure to compute the Solvency Capital Requirement . . . The metric used by regulators, which is based on a total risk measure such as the Value-at-Risk, is not a balanced solution between effectiveness and simplicity, but is simply wrong and could lead to significant adverse side effects, ultimately resulting in a generalized European insurance industry crisis in the case of a hard market shortfall.

Surprisingly, the text of Directives [5], [6] contains clear evidence of controversy regarding risk measures: the phrase from Directive [5] that the Solvency Capital Requirement (SCR) “shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period” dramatically differs from the phrase in the same Directive [5], that the SCR determines the economic capital which an insurance company must hold in order to guarantee a one-year ruin probability of at most 0.5%.

Every risk theory expert knows that, dealing with solvency, it is appropriate to address the occurrence of ruin within a year (and its probability), rather than the capital

Key words and phrases. Insurance solvency, risk measures, Value-at-Risk, non-ruin capital.

This work was supported by RFBR (grant No. 19-01-00045).

1See Directive [5], Article 101: Calculation of the Solvency Capital Requirement.

2In the preambular paragraph (64) for Directive [5], it is said as follows: “the Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases”. 
deficit at the end of this year (and its probability). Stepping back one step more, the aggregate claim amount distribution and the probability of ruin within finite time, which are the basis for determining the Value-at-risk and non-ruin capital respectively, were always clearly distinguished in the risk theory.

In this paper, relying on inverse Gaussian approximations in the problem of level crossing by a compound renewal processes and on associated results for the level which a compound renewal process crosses with a given probability, obtained in [18] and [19], we show that the non-ruin capital, being a theoretically sound risk measure, is not inferior to the Value-at-risk in simplicity even in a fairly general risk model.

The rest of this paper is arranged as follows. In Section 2 we introduce the risk model, focussing on the difference between the Value-at-Risk and non-ruin capital. In Section 3 using a series of well-known results, we show that in the exponential risk model the analysis of non-ruin capital is not much harder than the analysis of Value-at-Risk. In Section 4 we show that in the general risk model most results so much discussed in the literature are fit for the analysis of Value-at-Risk, but not of non-ruin capital. We turn to recent advances in the direct and inverse level crossing problems (see [18], [19]) which are suitable for a deep insight into the structure of non-ruin capital. In Section 5 we present numerical results obtained by both analytical technique and direct simulation, in order to illustrate the non-ruin capital’s structure. The final conclusion of this paper, given in Section 6, is that the analytical structure of non-ruin capital is simple enough, and this measure of risk can be used per se, without resorting to any substitute.

2. Model and main definitions

In the risk theory, the quantitative analysis is based on the annual model that formalizes the concept of collective risk. Given that time is operational and \( t \) is the length of the year, for \( 0 \leq s \leq t \) the claim arrival process is

\[
N_s = \max \left\{ n > 0 : \sum_{i=1}^{n} T_i \leq s \right\}, \tag{2.1}
\]

or 0, if \( T_1 > s \), the cumulative claim payout process is

\[
V_s = \sum_{i=1}^{N_s} Y_i, \tag{2.2}
\]

or 0, if \( T_1 > s \), and the balance of income and outcome is modeled by the risk reserve process

\[
R_s = u + cs - V_s, \tag{2.3}
\]

which starts at time zero at the point \( u > 0 \), called initial capital. Here \( c \geq 0 \) is called premium intensity (or, for the sake of brevity, price), \( T_i \overset{d}{=} T, \ i = 1, 2, \ldots \), are i.i.d. intervals between claims, and \( Y_i \overset{d}{=} Y, \ i = 1, 2, \ldots \), are i.i.d. claim sizes. It is generally assumed that these sequences are independent of each other.

In what follows, \( \alpha \) is a reasonably small positive real number, e.g., \( \alpha = 0.05 \).

**Definition 2.1.** The Value-at-Risk \( u_{\alpha,t}(c), \ c \geq 0, \) is a positive solution to the equation

\[
P\{R_t < 0\} = \alpha; \tag{2.4}
\]

3 This model, traditionally called Lundberg’s collective risk model, is most useful (see [20]) as a building block for multi-year models. From the angle of Directive [5], this modeling is very close to building an internal model.
VALUE-AT-RISK SUBSTITUTE FOR NON RUIN CAPITAL

Figure 1. Graphs (X-axis is c) of $u_{\alpha,t}^{[\text{VaR}]}(c)$ (red) and $u_{\alpha,t}(c)$ (blue), drawn for $T$ and $Y$ exponentially distributed with parameters $\delta = 1$, $\rho = 1$, and $\alpha = 0.05$, $t = 200$. Horizontal grid line: $u_{\alpha,t}(\delta/\rho) = 40.0844$. Vertical grid line: $\delta/\rho = 1$. for those $c$, for which this solution is negative, we set $u_{\alpha,t}^{[\text{VaR}]}(c)$ equal to zero. The non-ruin capital $u_{\alpha,t}(c)$, $c \geq 0$, is a positive solution to the equation

$$P\{\inf_{0 \leq s \leq t} R_s < 0\} = \alpha; \quad (2.5)$$

for those $c$, for which this solution is negative, we set $u_{\alpha,t}(c)$ equal to zero.

Note that the left-hand side of (2.5) is the probability of ruin within time $t$, i.e.,

$$\psi_t(u,c) = P\{\inf_{0 \leq s \leq t} R_s < 0\} = P\{\Upsilon_{u,c} \leq t\},$$

where $\Upsilon_{u,c} = \inf\{s > 0 : V_s - cs > u\}$, or $+\infty$, if $V_s - cs \leq u$ for all $s \geq 0$, is the time of the first ruin. In these terms, equation (2.5) rewrites as

$$P\{\Upsilon_{u,c} \leq t\} = \alpha. \quad (2.6)$$

Obviously, to investigate solvency in the usual sense of non-ruin 4, we must focus on $u_{\alpha,t}(c)$, rather than on $u_{\alpha,t}^{[\text{VaR}]}(c)$. Since $\inf_{0 \leq s \leq t} R_s$ is always less than or equal to $R_t$, we have

$$u_{\alpha,t}^{[\text{VaR}]}(c) \leq u_{\alpha,t}(c), \quad c \geq 0, \quad (2.7)$$

and $u_{\alpha,t}^{[\text{VaR}]}(c)$ always underestimates $u_{\alpha,t}(c)$. The underestimating of $u_{\alpha,t}(c)$ by $u_{\alpha,t}^{[\text{VaR}]}(c)$ can be significant (see Fig. 1). To deal with this problem quantitatively, rather than qualitatively, we must calculate both $u_{\alpha,t}(c)$ and $u_{\alpha,t}^{[\text{VaR}]}(c)$ in the risk model (2.1)–(2.3), striving for the most general assumptions about $T$ and $Y$. Traditionally, it has been considered possible to achieve success in this endeavor for $u_{\alpha,t}^{[\text{VaR}]}(c)$, but not for $u_{\alpha,t}(c)$; our aim is to break this impasse.

Remark 1. Let us write $c^* = EY/ET$ and introduce the ultimate ruin probability

$$\psi_\infty(u,c) = P\{\inf_{s \geq 0} R_s < 0\},$$

which is equivalently written as $P\{\Upsilon_{u,c} < \infty\}$. It has been studied in detail (see, e.g., [26]). Plainly, $P\{\Upsilon_{u,c} \leq t\} \leq P\{\Upsilon_{u,c} < \infty\}.$

4In risk theory, the event of ruin is traditionally synonymous with bankruptcy, and solvency is usually measured by the probability of ruin.
Figure 2. Graphs (X-axis is $c$) of $u_{\alpha,t}(c)$ (blue) and $u_{\alpha}(c)$ (red), drawn for $T$ and $Y$ exponentially distributed with parameters $\delta = 1$, $\rho = 1$, and $\alpha = 0.05$, $t = 200$. Horizontal grid line: $u_{\alpha,t}(c^*) = 40.08$. Vertical grid line: $c^* = 1$.

In [13], Chapter 6, Section 2, the insurer’s risk is measured by the ultimate ruin probability $P\{\Upsilon_{u,c}<\infty\}$ and the “minimal admissible initial capital” is introduced as a solution to the equation

$$P\{\Upsilon_{u,c}<\infty\} = \alpha.$$  (2.8)

In our notation, this is $u_{\alpha}(c)$, $c > c^*$. Plainly (see Fig. 2), $u_{\alpha,t}(c) \leq u_{\alpha}(c)$, $c > c^*$, and $u_{\alpha}(c)$ is tending to infinity, as $c \to c^*$.

Assuming that the “insurer wants to attract as many clients as possible keeping the relative safety loading at the lowest possible level” ([13], pp. 172–173), in [13] focussed is $u_{\alpha}(c)$, as $c \to c^*$. Thus, the problem to explore “the initial capital securing a prescribed risk level when the relative safety loading tends to zero” ([13], p. 27) is put forth.

In our opinion, the focus on $u_{\alpha}(c)$, as $c \to c^*$, does not help the insurer “to attract as many clients as possible keeping the relative safety loading at the lowest possible level” ([13], pp. 172–173), given that “the insurer accepts at most $\alpha$ as an acceptable risk level” ([13], p. 172). And even worse, this is hard to accept that it “can help the insurer to determine whether the initial capital suffices to start the business” ([13], p. 175) because the theory developed in [13] claims that when the insurer’s price $c$ decreases to the equilibrium price $c^*$, what often happens in some years of the real insurance business and what is far from tragic, “the initial capital securing a prescribed risk level” is tending to infinity.

3. Value-at-Risk and non-ruin capital in exponential case

The additional assumption that $T$ and $Y$ in the model (2.1)–(2.3) are exponentially distributed with parameters $\delta$ and $\rho$, yields many items of our interest in the analytical form, in terms of elementary or special functions, such as modified Bessel functions $I_k(x)$, $x \geq 0$, of the first kind of order $k$.

In what follows, we denote the cumulative distribution function (c.d.f.) of a standard Gaussian distribution by $\Phi_{(0,1)}(x)$, $x \in \mathbb{R}$. The corresponding probability density function (p.d.f.) is denoted by $\varphi_{(0,1)}(x)$, $x \in \mathbb{R}$. The $(1-\alpha)$-quantile of this distribution is denoted by $\kappa_\alpha = \Phi_{(0,1)}^{-1}(1-\alpha)$. Plainly, $0 < \kappa_\alpha < \kappa_{\alpha/2}$ for $0 < \alpha < 1/2$.

3.1. Value-at-Risk and aggregate claim amount distribution. In the exponential case, for the aggregate claim amount $V_t$ we have the following widely known
closed-form results:

\[ \mathbb{E}(V_t) = (\delta/\rho) t, \quad \mathbb{D}(V_t) = 2 (\delta/\rho^2) t, \]  

(3.1)

and

\[ P\{V_t \leq x\} = e^{-t\delta} + e^{-t\delta} \sum_{n=1}^{\infty} \frac{(t\delta)^n}{n!} \frac{\rho^n}{\Gamma(n)} \int_0^x e^{-\rho z} z^{n-1} dz \]

\[ = e^{-t\delta} + e^{-t\delta} (\delta \rho t)^{1/2} \int_0^x z^{-1/2} I_1(2\sqrt{\delta \rho tz}) e^{-\rho z} dz \]  

(3.2)

for \( x > 0 \), and zero otherwise.

An important observation is that equation (2.4) rewrites as

\[ P\{V_t > u + ct\} = \alpha, \]  

(3.3)

and its solution \( u_{\alpha,t}^{[VaR]}(c) \) is (see, e.g., [27], Section 14.3.2) a percentile (or quantile) of the distribution of the aggregate claim amount distribution at the ye ar-end time point \( t \).

Using equality (3.2), we express equation (3.3) in a closed form, when \( u_{\alpha,t}^{[VaR]}(c) \) is an implicit function defined by the equation

\[ e^{-t\delta} + e^{-t\delta} (\delta \rho t)^{1/2} \int_0^{u + ct} z^{-1/2} I_1(2\sqrt{\delta \rho tz}) e^{-\rho z} dz = 1 - \alpha. \]  

(3.4)

This implicit function can not be found in a closed form, but it can be calculated numerically. The graph of \( u_{\alpha,t}^{[VaR]}(c) \), \( c \geq 0 \), was drawn in Fig. 1 in this way.

Since \( V_t \) is (see (2.2)) the sum of \( N_t \) i.i.d. random variables, where \( \mathbb{E}(N_t) = \delta t \), it seems natural to address the asymptotic analysis of \( u_{\alpha,t}^{[VaR]}(c) \), as \( t \to \infty \). The assumption that \( t \) is large, which allows us to turn to the central limit theory, is sensible in terms of applications for the following reasons: time in the model (2.1)–(2.3) is operational (see, e.g., [28], p. 219), rather than calendar. This time, measured in monetary units, is proportional to the ball-park figure of the annual financial transactions of the company. Consequently, the assumption that \( t \to \infty \) means that this ball-park figure is large, i.e., the insurer’s portfolio size is large.

Since \( V_t \) is asymptotically normal with mean and variance given in (3.1), equation (3.4) is closely related to the equation

\[ \Phi_{(0,1)} \left( \frac{u + ct - (\delta/\rho) t}{\sqrt{2(\delta/\rho^2) t}} \right) = 1 - \alpha, \]  

(3.5)

whose solution \( (\delta/\rho - c) t + (\sqrt{2\delta}/\rho) \sqrt{t} \) is straightforward. Applying simple arguments based on the proximity of two implicit functions, we conclude that for all \( c \geq 0 \)

\[ u_{\alpha,t}^{[VaR]}(c) = \max \left\{ 0, (\delta/\rho - c) t + \sqrt{\frac{2\delta}{\rho}} \sqrt{t} (1 + o(1)) \right\}, \quad t \to \infty. \]  

(3.6)

3.2. Probability of ruin and non-ruin capital. In the exponential case, we have the following widely known (see, e.g., [14], Remark 2) closed-form result:

\[ P\{Y_{u,c} \leq t\} = P\{Y_{u,c} < \infty\} - \frac{1}{\pi} \int_0^t f(x) \, dx, \]  

(3.7)

where

\[ P\{Y_{u,c} < \infty\} = \begin{cases} 
1, & \delta/(cp) \geq 1, \\
\frac{\delta}{cp} \exp\left\{ -u \frac{\delta}{cp} - \delta \right\}, & \delta/(cp) < 1,
\end{cases} \]
Figure 3. Graph (X-axis is x) of $z_{\alpha,t}(x)$, drawn for $T$ and $Y$ exponentially distributed with parameters $\rho = 1$, $\delta = 1$, and $\alpha = 0.05$, $t = 200$. Horizontal grid lines: $\kappa_\alpha = 1.645$ and $\kappa_{\alpha/2} = 1.960$.

and

$$f(x) = \left(\frac{\delta}{(\rho c)}\right) \left(1 + \frac{\delta}{(\rho c)} - 2\sqrt{\frac{\delta}{(\rho c)}} \cos x\right)^{-1} \times \exp\left\{u_\rho \left(\sqrt{\frac{\delta}{(\rho)}} \cos x - 1\right) - t\delta(c\rho/\delta) \times \left(1 + \frac{\delta}{(\rho c)} - 2\sqrt{\frac{\delta}{(\rho c)}} \cos x\right) \times \left(\cos (u_\rho \sqrt{\frac{\delta}{(\rho)}} \sin x) - \cos (u_\rho \sqrt{\frac{\delta}{(\rho)}} \sin x + 2x)\right)\right\}.$$  

Using equality (3.7), we express the left-hand side of equation (2.6) in a closed form, whence $u_{\alpha,t}(c)$ is an implicit function defined by this equation. The same as $u_{\alpha,t}^{[\text{VaR}]}(c)$, this implicit function can not be found in a closed form, but can be calculated numerically. The graph of $u_{\alpha,t}(c)$, $c \geq 0$, was drawn in Fig. 1 in this way.

The function $u_{\alpha,t}(c)$, $c \geq 0$, can be analyzed asymptotically (see, e.g., [15], Theorems 3.1 and 3.2; this analysis was based on the properties of Bessel functions). In particular, we have (see [15], Theorem 3.2)

$$u_{\alpha,t}(c) = \begin{cases} \frac{(\rho)(\delta/\rho - c)}{\rho} t + \frac{\sqrt{\delta}}{\rho} z_{\alpha,t}(\frac{\rho}{\sqrt{\delta}})(\frac{\rho}{\sqrt{\delta}})(1 + o(1)), & 0 \leq c \leq c^*, \\ \frac{\sqrt{2\delta}}{\rho} z_{\alpha,t}(\frac{\rho}{\sqrt{2\delta}})(1 + o(1)), & c > c^*, \end{cases}$$ (3.8)

where $c^* = \delta/\rho$ and the function $z_{\alpha,t}(x)$, $x \in \mathbb{R}$, is (see Fig. 3) continuous, monotone increasing, as $x$ increases from $-\infty$ to 0, monotone decreasing, as $x$ increases from 0 to $\infty$, and such that $\lim_{x \to -\infty} z_{\alpha,t}(x) = 0$, $\lim_{x \to \infty} z_{\alpha,t}(x) = \kappa_\alpha$, and $z_{\alpha,t}(0) = \kappa_{\alpha/2}(1 + o(1))$, as $t \to \infty$. In particular, we have

$$u_{\alpha,t}(0) = \frac{(\delta/\rho)t + \sqrt{2\delta}}{\rho} \kappa_\alpha \sqrt{t}(1 + o(1)), \quad t \to \infty,$$ $u_{\alpha,t}(c^*) = \frac{\sqrt{2\delta}}{\rho} \kappa_{\alpha/2} \sqrt{t}(1 + o(1)), \quad t \to \infty.$ (3.9)

First equality in (3.9) is straightforward; see, e.g., (2.4) in [21]. Second equality in (3.9) is Theorem 3.1 in [15].
4. Value-at-Risk and non-ruin capital in general case

It is widely believed that in the general case, the situation described in Section 3 deteriorates dramatically, and the non-ruin capital becomes intractable. First, we clarify the reasons for this belief. Second, we show that the situation in the general case is not so bad due to several innovative approaches.

4.1. Value-at-Risk and aggregate claim amount distribution. In the general case, there is no hope to get explicit equalities like (3.1) or (3.2) for all \( t \). But since the asymptotic analysis, as \( t \to \infty \), is based on the fairly general central limit theory, it is easy to obtain analogues for (3.5) and (3.6). To be specific, \( P\{V_t \leq x\} \) is approximated by \( \Phi(M_{Vt,DVt}(x)) \), as \( t \to \infty \), where

\[
M_N = 1/ET, \quad M_V = EY/ET, \\
D^2_N = DT/(ET)^3, \quad D^2_V = E(TEY - YET)^2/(ET)^3.
\]

This approximation, being a version of the central limit theorem, is valid under well-known mild technical condition on \( T \) and \( Y \) and can be applied to (3.3). Therefore, though in the general case equation (3.3) cannot be written in terms of elementary or special functions, as it was done (see (3.4)) in the exponential case, for sufficiently large \( t \) (3.3) is close to the equation (cf. (3.5))

\[
\Phi(0,1)\left(\frac{u + ct - M_V t}{D_V \sqrt{t}}\right) = 1 - \alpha,
\]

whose closed-form solution \((M_V - c) t + \kappa \alpha D_V \sqrt{t}\) is straightforward. Applying simple arguments based on the proximity of two implicit functions, we conclude that (cf. (3.6)) for all \( c \geq 0 \)

\[
u\alpha,t|\|0(c) = \max\left\{0, (M_V - c) t + \kappa \alpha D_V \sqrt{t} (1 + o(1))\right\}, \quad t \to \infty.
\]

It is easy to see that the analysis in the general case differs a little, regarding both applied technique and results, from the analysis in the exponential case.

4.2. Standard results for ruin probability. In the general case (except for some very special subcases), there is no hope to express \( P\{Y_u,c \leq t\} \) in terms of elementary or special functions for all \( t \). There is even less hope of finding in a closed form for all \( t \) the implicit function \( u\alpha,t(c), c \geq 0 \), defined by the corresponding equation (2.6), even if its left-hand side could be represented in such terms.

Moreover, the results of asymptotic analysis, as \( u \to \infty \), so much discussed in the literature, are unsatisfactory from the angle of their further application to asymptotical, as \( t \to \infty \), analysis of the non-ruin capital \( u\alpha,t(c), c \geq 0 \). We will show this by referring to the normal (or Cramér’s) and diffusion approximations that are best known. We start with the former\(^5\) and point out its deficiencies.

\(^5\)It is noteworthy that \( E(N_t) = MN t + \frac{DT - (ET)^2}{2(ET)^2} + o(1), D(N_t) = D^2_N t + o(t), E(V_t) = M_V t + EY\left(\frac{DT - (ET)^2}{2(ET)^2}\right) + o(1), D(V_t) = D^2_V t + o(t), t \to \infty.\)

\(^6\)What is said below about this approximation is folklore of the risk theory and can be found in many standard textbooks, e.g., in [26].
4.2.1. Normal approximation. The primary assumption is that there exists a positive solution \( \kappa \), called adjustment coefficient, to the equation (w.r.t. \( r \))

\[
M_X(r) = 1, \tag{4.3}
\]
called Lundberg’s equation. Here \( M_X(r) = \mathbb{E}(e^{rX}) \) is the moment generating function of \( X \stackrel{d}{=} Y - cT \); plainly, \( M_X(0) = 1 \). This assumption is a significant limitation of the model. It implies that \( M_X(r) \) has to exist in a neighborhood of 0 or, in other words, that the right tail of c.d.f. \( F_X \) is exponentially bounded above. The latter follows from Markov’s inequality

\[
1 - F_X(x) \leq e^{-\kappa x} \mathbb{E}(e^{\kappa X}) = e^{-\kappa x}, \quad x > 0. \tag{4.4}
\]

Starting with c.d.f. \( F_{XT}(x, t) = \mathbb{P}\{X < x, T \leq t\} \) and having \( \kappa > 0 \) found, we introduce the associated joint distribution\( ^7 \), whose c.d.f. \( F_{\bar{X}\bar{T}}(x, t) = \mathbb{P}\{\bar{X} < x, \bar{T} \leq t\} \) is defined by the equality\( ^8 \)

\[
F_{\bar{X}\bar{T}}(x, t) = \int_{-\infty}^{x} \int_{0}^{t} e^{\kappa z} F_{XT}(dz, dw).
\]

Plainly, this is a proper probability distribution.

Recall that \( c^* = \mathbb{E}Y/\mathbb{E}T \). The normal (or Cramér’s) approximation is formulated separately for \( 0 \leq c < c^* \) and for \( c > c^* \), with the case \( c = c^* \) excluded. For \( 0 \leq c < c^* \), i.e., for \( \mathbb{E}X = \mathbb{E}Y - c\mathbb{E}T > 0 \), we write

\[
m_\gamma = \mathbb{E}(X\mathbb{E}T - TE\bar{X})^2/(\mathbb{E}X)^3.
\]

Plainly, we have \( m_\gamma > 0 \) and \( D^2_\gamma > 0 \).

**Proposition 4.1 (Case \( 0 \leq c < c^* \)).** Assume that p.d.f. of the random vector \( (T, Y) \) is bounded above by a finite constant and \( 0 < D^2_\gamma < \infty \). Then

\[
d_u = \sup_{t>0} |P\{\Upsilon_{u,c} \leq t\} - \Phi(m_\gamma u, D^2_\gamma u)(t) | = o(1), \quad u \to \infty.
\]

If, in addition, \( \mathbb{E}(Y^3) < \infty, \mathbb{E}(T^3) < \infty \), then \( d_u = O(u^{-1/2}) \), as \( u \to \infty \).

For \( c > c^* \), i.e., for \( \mathbb{E}X = \mathbb{E}Y - c\mathbb{E}T < 0 \), we write

\[
m_\delta = \mathbb{E}\bar{T}/\mathbb{E}\bar{X}, \quad D^2_\delta = \mathbb{E}(\bar{X}\mathbb{E}\bar{T} - \bar{T}E\bar{X})^2/(\mathbb{E}\bar{X})^3,
\]

\[
C = \frac{1}{\kappa \mathbb{E}\bar{X}} \exp \left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\{S_n > 0\} - \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\{S_n \leq 0\} \right\}.
\]

where \( \bar{X}_i \stackrel{d}{=} \bar{X}, \ i = 1, 2, \ldots \), and \( \bar{T}_i \stackrel{d}{=} \bar{T}, \ i = 1, 2, \ldots \), are associated random variables, and \( S_n = \sum_{i=1}^{n} \bar{X}_i \), and \( Z_n = \sum_{i=1}^{n} \bar{T}_i \), \( i = 1, 2, \ldots \), are associated random walks.

**Proposition 4.2 (Case \( c > c^* \)).** Assume that a solution \( \kappa > 0 \) to equation \( ^7 \) exists, p.d.f. of the random vector \( (T, Y) \) is bounded above by a finite constant, and \( 0 < D^2_\delta < \infty \). Then

\[
d_u = \sup_{t>0} |e^{\kappa u} P\{\Upsilon_{u,c} \leq t\} - C \Phi(m_\gamma u, D^2_\delta u)(t) | = o(1), \quad u \to \infty.
\]

If, in addition, \( \mathbb{E}(T^3) < \infty \), then \( d_u = O(u^{-1/2}) \), as \( u \to \infty \).

\(^7\)See, e.g., Example (b) in \( ^{10} \), Chapter XII, Section 4.

\(^8\)Commonly used shorthand notation for it is \( F_{\bar{X}\bar{T}}(dx, dt) = e^{\kappa x} F_{XT}(dx, dt) \).
In the exponential case, when $T$ and $Y$ are exponentially distributed with parameters $\delta > 0$ and $\rho > 0$, straightforward calculations (see [21], Proposition 2.3) yield $c^* = \delta/\rho$,

$$
C = \delta/(cp), \quad \kappa = \rho(1 - \delta/(cp)),
$$

$$
m_x = -\frac{1}{c(1 - \delta/(cp))}, \quad D_x^2 = -\frac{2(\delta/(cp))}{c^2 \rho(1 - \delta/(cp))^3},
$$

$$
m_\delta = \frac{\delta/(cp)}{c(1 - \delta/(cp))}, \quad D_\delta^2 = \frac{2(\delta/(cp))}{c^2 \rho(1 - \delta/(cp))^3},
$$

and Propositions 4.1 and 4.2 are fused together, as follows.

**Proposition 4.3.** In the renewal model with $T$ and $Y$ exponentially distributed with parameters $\delta > 0$ and $\rho > 0$, we have for $0 \leq c < c^*$

$$
\sup_{t > 0} \left| P\{T_{u,c} \leq t\} - \Phi(m_x, D_x)(t) \right| = o(1), \quad u \to \infty,
$$

$$
\sup_{t > 0} \left| P\{T_{u,c^*} \leq t\} - \Phi(m_\delta, D_\delta)(t) \right| = o(1), \quad u \to \infty.
$$
where \( m_\gamma > 0 \), \( D_\gamma^2 > 0 \) are defined in (4.5), and for \( c > c^* \)

\[
\sup_{t>0} \left| e^{ux} \mathbb{P}\{\Upsilon_{u,c} \leq t\} - C \Phi_{(m_\gamma,u,D_\gamma^2)}(t) \right| = o(1), \quad u \to \infty,
\]

where \( \kappa > 0 \), and \( 0 < C < 1 \), \( m_\gamma > 0 \), \( D_\gamma^2 > 0 \) are defined in (4.5).

These results, if used for an asymptotic analysis of non-ruin capital, \( u_{\alpha,t}(c), c \geq 0 \), have several deficiencies.

Deficiency 1 (Limited applicability). Proposition 4.2 requires restrictive technical condition: \( Y \) must be (see (4.3)) light-tailed.

Deficiency 2 (Flaw near \( c = c^* \)). Besides the fact that the case \( c = c^* \) is formally excluded, the normal (or Cramér's) approximation fails for \( c \) in a neighborhood of \( c^* \). This is illustrated in Fig. 4.

Deficiency 3 (Structural imbalance). The structure of the approximation in Propositions 4.1 and 4.2 is significantly different from the structure of CLT-type approximation used to get (4.1) and (4.2). This is particularly evident when \( c = 0 \), i.e., when the left-hand sides of equations (2.4) and (2.6) are the same and can be written as \( \mathbb{P}\{V_t > u\} \).

It is clear that, being solutions to the same equation, \( u_{\alpha,t}(0) \) and \( u_{\alpha,t}^{\text{VaR}}(0) \) coincide with each other. The CLT-type approximation is

\[
\mathbb{P}\{V_t > u\} \approx \Phi_{(0,1)} \left( \frac{u - M_V|_{c=0} t}{D_V|_{c=0} \sqrt{t}} \right), \quad t \to \infty,
\]

whereas the approximation in Proposition 4.1 is

\[
\mathbb{P}\{V_t > u\} \approx \Phi_{(0,1)} \left( \frac{t - m_\gamma|_{c=0} u}{D_t|_{c=0} \sqrt{u}} \right), \quad u \to \infty.
\]

When \( T \) and \( Y \) are exponentially distributed with parameters \( \delta \) and \( \rho \), \( m_\gamma|_{c=0} = \rho/\delta \), \( D_\gamma^2|_{c=0} = 2\rho/\delta^2 \) in (4.6), and \( M_V|_{c=0} = \delta/\rho \), \( D_t^2|_{c=0} = 2\delta/\rho^2 \) in (4.7).

It is worth noting that if \( u_{\alpha,t}(0) \) would not tend to infinity, as \( t \to \infty \), the approximation (4.7) (unlike (4.6)) would be useless to get information about \( u_{\alpha,t}(0) \), as \( t \to \infty \).

Fortunately, \( u_{\alpha,t}(0) \approx M_V|_{c=0} t \), which tends to infinity, as \( t \to \infty \). Indeed, \( u_{\alpha,t}(0) \) is equal to \( u_{\alpha,t}^{\text{VaR}}(0) \) (and see (4.2))

\[
u_{\alpha,t}^{\text{VaR}}(0) = M_V|_{c=0} t + \kappa_\alpha D_V|_{c=0} \sqrt{1 + o(1)}, \quad t \to \infty.
\]

For \( c \) sufficiently larger than \( c^* \), i.e., for \( c > Kc^* \) with \( K > 1 \) sufficiently large, \( u_{\alpha,t}(c) \) is finite regardless of \( t \). Therefore, the structural difference between the approximations of Propositions 4.1 and 4.2 matters.

4.2.2. Diffusion approximation. Because of the space limitations, we describe the deficiencies of simple and corrected diffusion approximations for \( \mathbb{P}\{\Upsilon_{u,c} \leq t\} \) verbally. The idea of a simple diffusion approximation\(^9\) is that the original risk reserve process \( \mathbb{P}\{X_{t-1} \leq \Delta t\} \) has some similarities (regarding the properties of distributions, rather than trajectories) with the diffusion process, although its trajectories are continuous. Matching the original and the auxiliary diffusion processes, one finds\(^10\) that the distribution of the first level crossing time for the former process is approximated by the distribution of the first level crossing time for the auxiliary process.
crossing time for the latter process. This observation is productive because the first passage probabilities in the diffusion model are found in a closed form.

The diffusion process is skip-free; the idea behind the simple diffusion approximation ignores the presence of overshoot in the original process with discontinuous trajectories. The corrected diffusion approximation takes into account this and other similar features of the initial process.

Congenital deficiency of both simple and adjusted diffusion approximations is Deficiency 1: these results often require that the distribution of $Y$ has a light tail. In addition, these results are valid under the assumption $\lim_{c \to c^*} + 0$. Such regime is certainly a structural drawback; this impedes the analysis of $u_{\alpha, t}(c), c \geq 0$.

4.3. Innovative results for ruin probability. For

$$M = \frac{\mathbb{E} T}{\mathbb{E} Y}, \quad D^2 = \frac{((\mathbb{E} T)^2 \mathbb{D} Y + (\mathbb{E} Y)^2 \mathbb{D} T)}{\mathbb{E} Y^3},$$

\[ (4.8) \]

\[ ^{11} \text{Often formulated as “the safety loading is small and positive”; it is often added that this is just the same as the heavy traffic in the queuing theory.} \]
we write

\[ M_{u,c}(t) = \int_0^{ct} \frac{1}{x+1} \varphi \left( \frac{cM(x+1)}{x+2D_2(x+1)} \right) (x) \, dx. \]  

(4.9)

Bearing in mind that \( c^* = \frac{EY}{ET} \) equals to \( \frac{1}{M} \) and denoting p.d.f. of inverse Gaussian distribution by

\[ F(x; \mu, \lambda) = \Phi_{(0,1)} \left( \sqrt{\frac{\lambda}{x}} \left( \frac{x}{\mu} - 1 \right) \right) + \exp \left\{ - \frac{2\lambda}{\mu} \right\} \Phi_{(0,1)} \left( - \sqrt{\frac{\lambda}{x}} \left( \frac{x}{\mu} + 1 \right) \right), \quad x > 0, \]

we can show by elementary calculations that

\[ M_{u,c}(t) = \begin{cases} 
\left( F \left( \frac{ct}{u} + 1; \mu, \lambda \right) - F \left( 1; \mu, \lambda \right) \right) \bigg|_{\mu=\frac{ct}{u}, \lambda=\frac{D_2}{ct}}, & 0 < c \leq c^*, \\
\exp \left\{ - \frac{2\lambda}{\mu} \right\} \left( F \left( \frac{ct}{u} + 1; \hat{\mu}, \lambda \right) - F \left( 1; \hat{\mu}, \lambda \right) \right) \bigg|_{\hat{\mu}=\frac{ct}{u}, \lambda=\frac{D_2}{ct}}, & c > c^*. 
\end{cases} \]

The following theorem, as well as its refinements like Edgeworth expansions (see [18] and [16], [17], [22]), is called the inverse Gaussian approximation for \( P \{ \Upsilon_{u,c} \leq t \} \).

**Theorem 4.1.** Assume that p.d.f. \( f_T(x) \) and \( f_Y(x) \) are bounded above by a finite constant, \( D^2 > 0 \), \( E(T^3) < \infty \), \( E(Y^3) < \infty \). Then for any \( c \geq 0 \)

\[ \sup_{t > 0} | P \{ \Upsilon_{u,c} \leq t \} - M_{u,c}(t) | = o(1), \quad t, u \to \infty. \]

In a nutshell, the proof of Theorem 4.1 is based on Kendall’s identity which represents the first level crossing time’s distribution in terms of the convolution powers of p.d.f. \( f_T(x) \) and \( f_Y(x) \); then the well-developed central limit theory is applied to these convolution powers.

**Remark 2.** The inverse Gaussian distribution \( F(x; \mu, \lambda) \) is concentrated on the positive half-line; its mean is \( \mu \), variance is \( \mu^3/\lambda \), and the third central moment is \( 3\mu^5/\lambda^2 \). The appearance of this skewed distribution in Theorem 4.1 sheds light on numerous claims by many practitioners (see, e.g., [11], [24], [25]) that the “world of

\[ \text{Figure 8. Graph (X-axis is c) of } M_{u,c}(t) \text{ (red) and of simulated values } (\Delta c = 0.05, N = 1000) \text{ of } P \{ \Upsilon_{u,c} \leq t \} \text{ (blue), drawn for } T \text{ which is Pareto with parameters } a_T = 4.0, b_T = 0.4. \text{ Y which is Pareto with parameters } a_Y = 4.0, b_Y = 0.4, \text{ and } t = 1000, u = 40. \]
Conditions of Theorem 4.1 are very general for both $T$ and $Y$, and the accuracy of approximation of $P\{Y_{u,c} < t\}$ by $M_{u,c}(t)$ is high for all $c$ for which these values are not negligibly small; this is a satisfactory approximation for the left-hand side of equation (2.6) that defines $u_{a,t}(c)$, $c \geq 0$, as an implicit function.

To demonstrate this in a spectacular way, we compare Figs. 4 and 5. This shows the difference in the accuracy of the normal (or Cramér’s) and inverse Gaussian approximations. The advantages of the latter are especially noticeable in that domain (including the point $c^*$ and its neighborhood) where $P\{Y_{u,c} < t\}$ assumes not too small values.

To emphasize that the inverse Gaussian approximation works well for heavy-tailed $Y$, we address Figs. 6–8 (see Table 1), where $F_{u,c}$ assumes not too small values.

In Fig. 6, we draw the graph of $M_{u,c}(t)$ calculated by means of numerical integration in (4.9) for $T$ 2-mixture of exponential with parameters $\delta_1 = 1$, $\delta_2 = 2$, $p = 2/3$ and $Y$ Pareto with parameters $\alpha_Y = 4.0$, $b_Y = 0.35$, whence $c^* = 1.143$, $M = 0.8750$, and $D^2 = 2.3042$. In Fig. 7 we do this for $T$ Erlang with parameters $\delta = 6.0$, $k = 4$ and $Y$ Pareto with parameters $\alpha_Y = 4.0$, $b_Y = 0.4$, whence $c^* = 1.25$, $M = 0.8$, and $D^2 = 1.2$. In Fig. 8 we do this for $T$ Pareto with parameters $\alpha_T = 4.0$, $b_T = 0.4$ and $Y$ Pareto with parameters $\alpha_Y = 4.0$, $b_Y = 0.4$, whence $c^* = 1$, $M = 1$, and $D^2 = 1.3333$.

4.4. Non-ruin capital. The analytical technique which yields Theorem 4.1 (see [18] and [16], [17], [22]) is suitable for asymptotic analysis of non-ruin capital in the general risk model. The following theorem (see [19], Theorem 1) gives an asymptotic representation for $u_{a,t}(c)$ at the points $c = 0$ and $c = c^*$, where (see (4.8)) $c^* = 1/M$; this generalizes asymptotic equalities (3.9).

**Theorem 4.2.** Assume that p.d.f. $f_T(x)$ and $f_Y(x)$ are bounded above by a finite constant, $D^2 > 0$, $E(T^3) < \infty$, $E(Y^3) < \infty$. Then

$$u_{a,t}(0) = \frac{t}{M} + \frac{D}{M^{3/2}} \kappa_a \sqrt{t} (1 + o(1)), \quad t \to \infty,$$

$$u_{a,t}(c^*) = \frac{D}{M^{3/2}} \kappa_a \sqrt{t} (1 + o(1)), \quad t \to \infty.$$
The following theorem (see [19], Theorem 2) gives an asymptotic representation for $u_{α,t}(c), c ≥ 0$, which generalizes asymptotic equality (3.8).

**Theorem 4.3.** Assume that p.d.f. $f_T(x)$ and $f_Y(x)$ are bounded above by a finite constant, $D^2 > 0$, $E(T^3) < ∞$, $E(Y^3) < ∞$. Then

$$u_{α,t}(c) = \begin{cases} (c^* - c)t + \frac{D}{M^{3/2}} z_{α,t} \left( \frac{M^{3/2}(c^* - c)}{D} \sqrt{t} \right) \sqrt{t}, & 0 ≤ c ≤ c^*, \\ \frac{D}{M^{3/2}} z_{α,t} \left( \frac{M^{3/2}(c^* - c)}{D} \sqrt{t} \right) \sqrt{t}, & c > c^*, \end{cases}$$

where for $t$ sufficiently large the function $z_{α,t}(x), x ∈ R$, is continuous, monotone increasing, as $x$ increases from $−∞$ to $0$, monotone decreasing, as $x$ increases from $0$ to $∞$, and such that

$$\lim_{x→−∞} z_{α,t}(x) = 0, \lim_{x→∞} z_{α,t}(x) = κ_α$$

and $z_{α,t}(0) = κ_{α/2}(1 + o(1)), t → ∞$.

Let us construct simple bounds for $u_{α,t}(c), c ≥ 0$. First, Theorem 4.3 yields the following bilateral asymptotic bounds: for $0 ≤ c ≤ c^*$, we have

$$(c^* - c)t + \frac{D}{M^{3/2}} K_α \sqrt{t} (1 + o(1)) ≤ u_{α,t}(c) \leq (c^* - c)t + \frac{D}{M^{3/2}} K_{α/2} \sqrt{t} (1 + o(1)), \quad t → ∞. \tag{4.10}$$

Second, looking for upper bounds for $u_{α,t}(c), c > c^*$, we note that for $c$ sufficiently larger than $c^*$, i.e., for $c > Kc^*$ with $K > 1$ sufficiently large, $u_{α,t}(c)$ is finite regardless of $t$. Thus, we focus (see Remark 1) on $u_0(c), c > c^*$, which is a natural upper bound for $u_{α,t}(c)$, or [14] on any sensible upper bound for $u_α(c)$.

Bearing in mind the widely known theory (see, e.g., [26]) built for the ultimate ruin probability $P\{Y_{u,c} < ∞\}$, let us focus on the following cases.

**M(i): exponential case.** When $T$ and $Y$ are exponentially distributed with parameters $δ$ and $ρ$, we have $c^* = δ/ρ$, $κ = ρ - δ/c$. For $c > δ/ρ$, we have (see, e.g., [26])

$$P\{Y_{u,c} < ∞\} = (1 - κ/ρ) e^{-κu} \text{ for all } u ≥ 0. \tag{4.11}$$

This rewrites as

$$P\{Y_{u,c} < ∞\} = (δ/(cp)) \exp\{-(ρ - δ/c)u\}, \quad c > δ/ρ,$$

and by simple calculations we have

$$u_{α,t}(c) ≤ \max \left\{ 0, \frac{\ln(αc/δ)}{ρ - δ/c} \right\}, \quad c > δ/ρ. \tag{4.11}$$

**M(ii): Poisson claims arrival and $Y$ light-tailed.** When $T$ is exponentially distributed with parameter $δ$ and the distribution of $Y$ is light-tailed, but non-exponential, special cases of which are, e.g.,

(a) $T$ exponentially distributed and $Y$ 2-mixture,

(b) $T$ exponentially distributed and $Y$ Erlang,

we have $c^* = δEY$, equation (4.3) rewrites as $E \exp\{κY\} = 1 + cκ/δ$, and $κ$ is its positive solution. For $c > c^*$, we have (see, e.g., [26])

$$P\{Y_{u,c} < ∞\} ≤ e^{-κu} \text{ for all } u ≥ 0.$$

Therefore, by simple calculations we have

$$u_{α,t}(c) ≤ -\ln(α/κ), \quad c > δEY,$$

and the problem comes down to finding $κ$ in a closed form.

---

13 In order to have more freedom of action, especially for finding compact formulas.
M(iii): Poisson claims arrival and $Y$ heavy-tailed. Special cases are, e.g.,

(a) $T$ exponentially distributed and $Y$ Pareto,
(b) $T$ exponentially distributed and $Y$ Kummer.

Any upper bounds for $u_{\alpha,\ell}(c)$, $c > c^*$, which assumes small, rather than large values, is tightly related to particulars of the probability $P\{Y_{u,c} < \infty\}$ for small, rather than large values of $u$, which is a problem beyond the scope of this article.

$M(iv)$: renewal claims arrival and $Y$ exponentially distributed. When $Y$ is exponentially distributed with parameter $\rho$ and the problem comes down to finding $\kappa$, we have (see [26, Corollary 6.5.2])

$$u_{\alpha,\ell}(c) = \rho \rho_{\alpha,\ell} - \ln \alpha, \quad c > 1/(\rho \mathbb{E} T),$$

(4.12)

and the problem comes down to finding $\alpha$ in a closed form.

$M(v)$: renewal claims arrival and $Y$ light-tailed. When $Y$ is light-tailed, but non-exponential, and the distribution of $T$ is arbitrary, special cases of which are, e.g.,

(a) $T$ Erlang and $Y$ Erlang,
(b) $T$ Erlang and $Y$ 2-mixture.

we address $X \overset{d}{=} Y - cT$, whose c.d.f. is $F_X$, denote by $F_X(x) = 1 - F_X(x)$ is tail function, and write $x_0 = \sup\{x : F_X(x) < 1\}$. For $c > c^*$, we have $P\{Y_{u,c} < \infty\} = (1 - \alpha/\rho) e^{-\alpha u}$ for all $u \geq 0$. Bearing in mind that $1 - \alpha/\rho \leq 1$, we have (see [26, Theorem 6.5.4])

$$b_\oplus e^{-\alpha u} \leq P\{Y_{u,c} < \infty\} \leq b_\ominus e^{-\alpha u} \quad \text{for all} \quad u \geq 0,$$

(4.13)

where $\alpha$ is a positive solution to (4.3) and

$$b_\ominus = \inf_{x \in [0,x_0]} \frac{e^{\alpha x} F_X(x)}{\int_x^\infty e^{\alpha y} dF_X(y)}, \quad b_\oplus = \sup_{x \in [0,x_0]} \frac{e^{\alpha x} F_X(x)}{\int_x^\infty e^{\alpha y} dF_X(y)}.$$

Alternatively (see [26, Theorem 6.5.5]), the inequalities (4.13) hold with

$$b_\ominus^* = \inf_{x \in [0,x_0]} \frac{e^{\alpha x} F_Y(x)}{\int_x^\infty e^{\alpha y} dF_Y(y)}, \quad b_\oplus^* = \sup_{x \in [0,x_0]} \frac{e^{\alpha x} F_Y(x)}{\int_x^\infty e^{\alpha y} dF_Y(y)},$$

where $x_0^* = \sup\{x : F_Y(x) < 1\}$; the inequalities $0 \leq b_\ominus^* \leq b_\ominus \leq b_\ominus \leq b_\ominus \leq 1$ hold.

Both upper and lower bounds for $u_{\alpha,\ell}(c)$, $c > c^*$, which is a solution to equation (2.8), and therefore upper bounds for $u_{\alpha,\ell}(c)$, $c > c^*$, is easy to get from (4.13), and we leave this to the reader.

$M(vi)$: renewal claims arrival and $Y$ heavy-tailed. Special cases are, e.g.,

(a) $T$ is 2-mixture and $Y$ is Pareto,
(b) $T$ is Erlang and $Y$ is Pareto,
(c) $T$ is Pareto and $Y$ is Pareto.

---

14Definition of the Kummer distribution see, e.g., in [14].
Any upper bounds for $u_{\alpha,t}(c)$, $c > c^*$, which assumes small, rather than large values, is tightly related to particulars of the probability $P\{Y_{u,c} < \infty\}$ for small, rather than large values of $u$, which is a problem beyond the scope of this article.

5. Numerical illustrations of non-ruin capital’s structure

Let us compare numerically the results formulated in Section 4.4 with the simulation results taken as exact values; the algorithm of simulation is the same as in [21], or in [22]. For completeness, we return to Section 3.2 and start with $T$ and $Y$ exponentially distributed with parameters $\delta$ and $\rho$, whose p.d.f. are

$$f_T(x) = \delta e^{-\delta x}, \quad f_Y(x) = \rho e^{-\rho x}, \quad x > 0.$$ 

5.1. Model $M(i)$: exponential case. Elementary calculations yield $E(T^k) = k!/\delta^k$, $E(Y^k) = k!/\rho^k$, $k = 1, 2, \ldots$, whence

$$E T = 1/\delta, \quad D T = 1/\delta^2,$$
$$E Y = 1/\rho, \quad D Y = 1/\rho^2,$$

and

$$E e^{-\kappa c T} = \delta \int_0^\infty e^{-(\kappa c+\delta)x} \, dx = \delta/(\delta + c\kappa),$$
$$E e^{\kappa Y} = \rho \int_0^\infty e^{(\kappa-\rho)x} \, dx = \rho/(\rho - \kappa).$$

Plainly, $c^* = E Y/E T$ is equal to $\delta/\rho$, the constants defined in (4.3) are

$$M = E T/E Y = \rho/\delta,$$
$$D^2 = ((E T^2 D Y + (E Y)^2 D T)/(E Y)^3 = 2 \rho/\delta^2,$$

and for $c > \delta/\rho$ the positive solution $\kappa$ to the Lundberg equation (4.3), which rewrites as the quadratic equation $$(\rho - \kappa)(\delta + c\kappa) - \delta \rho = 0,$$ is $\kappa = \rho - \delta/c$.

In Fig. 9, the upper bounds (4.10) in the case $0 \leq c \leq \delta/\rho$, and (4.11) in the case $c > \delta/\rho$, are drawn for $t = 200$, $\alpha = 0.05$, $\delta = 4/5$, $\rho = 3/5$, whence $c^* = 1.3333$, $M = 0.75$, and $D^2 = 1.875$. In Fig. 9 by dots are drawn the simulated values of $u_{\alpha,t}(c)$. 

Figure 9. Model $M(i)$: upper bound (X-axis is $c$) on $u_{\alpha,t}(c)$ and simulated values of $u_{\alpha,t}(c)$, drawn for $T$ and $Y$ exponentially distributed with parameters $\delta = 3/5$, $\rho = 4/5$, and $\alpha = 0.05$, $t = 200$. Vertical grid line: $c^* = 4/3$. Horizontal grid line: $u_{\alpha,t}(c^*) = 59.9033.$
5.2. Model M(iv): Erlang \( T \) and exponentially distributed \( Y \). This model is a particular case of Model M(v), where \( T \) is Erlang with parameters \( k \) integer and \( \delta > 0 \) and \( Y \) is Erlang with parameters \( m \) integer and \( \rho > 0 \), whose p.d.f. are

\[
\begin{align*}
    f_T(x) &= \frac{\delta^k x^{k-1}}{\Gamma(k)} e^{-\delta x}, \\
    f_Y(x) &= \frac{\rho^m x^{m-1}}{\Gamma(m)} e^{-\rho x}, \quad x > 0.
\end{align*}
\]

Elementary calculations yield

\[
\begin{align*}
    \mathbb{E} T &= k/\delta, \quad D T = k/\delta^2, \\
    \mathbb{E} Y &= m/\rho, \quad D Y = m/\rho^2, \nonumber
\end{align*}
\]

and

\[
\begin{align*}
    \mathbb{E} e^{-xe^T} &= \frac{\delta^k}{\Gamma(k)} \int_0^\infty e^{-(\delta + \rho)x} x^{k-1} dx = \frac{\delta^k}{(\delta + \rho)^k}, \\
    \mathbb{E} e^{\ast Y} &= \frac{\rho^m}{\Gamma(m)} \int_0^\infty e^{(\rho - \ast)x} x^{m-1} dx = \frac{\rho^m}{(\rho - \ast)^m}.
\end{align*}
\]

Plainly, \( \ast = \mathbb{E} Y/\mathbb{E} T \) is equal to \((m\delta)/(k\rho)\), the constants defined in (4.8) are

\[
\begin{align*}
    M &= \mathbb{E} T/\mathbb{E} Y = k\rho/(m\delta), \\
    D^2 &= ((\mathbb{E} T)^2 \mathbb{D} Y + (\mathbb{E} Y)^2 \mathbb{D} T)/(\mathbb{E} Y)^3 \\
    &= k(k + m)\rho/(m^2\delta^2),
\end{align*}
\]

and for \( c > (m\delta)/(k\rho) \) the positive solution \( \ast \) to the Lundberg equation (4.5), which rewrites as \((\rho - \ast)^m(\delta + c\ast)^k - \delta^k\rho^m = 0\), is easy to find numerically; this \( \ast \) is not explicit, except for \( m = 1 \).

In Fig. 10 the upper and lower bounds (4.10) in the case \( 0 \leq c \leq \delta/\rho \), and the upper bound (4.12) in the case \( c > \delta/\rho \), are drawn for \( t = 200, \alpha = 0.05, \delta = 8/5, k = 2, \rho = 3/5 \), whence \( \ast = \mathbb{E} Y/\mathbb{E} T = 1.3333, M = 0.75, \) and \( D^2 = 1.40625 \). In Fig. 10 by dots are drawn the simulated values of \( u_{n,t}(c) \).
5.3. Model M(iii): exponentially distributed T and Pareto Y. For T exponentially distributed with parameter $\delta > 0$ and $Y$ whose distribution is Pareto with parameters $a_Y > 0$, $b_Y > 0$, p.d.f. are

$$f_T(x) = \delta e^{-\delta x}, \quad f_Y(x) = \frac{a_Y b_Y}{(x b_Y + 1)^{a_Y + 1}}, \quad x > 0,$$

elementary calculations yield

$$E_T = 1/\delta, \quad D_T = 1/\delta^2,$$

$$E_Y = 1/((a_Y - 1) b_Y), \quad D_Y = a_Y/((a_Y - 1)^2 (a_Y - 2) b_Y^2).$$

Plainly, $c^* = E_Y/E_T$ is equal to $\delta/((a_Y - 1) b_Y)$, the constants defined in (4.8) are

$$M = E_T/E_Y = \frac{(a_Y - 1) b_Y}{\delta},$$

$$D^2 = ((E_T)^2 D Y + (E Y)^2 D T)/(E Y)^3 = \frac{2(a_Y - 1)^2 b_Y}{\delta^2 (a_Y - 2)},$$

and the adjustment coefficient does not exist.

In Fig. 11, the upper and lower bounds (4.10) in the case $0 \leq c \leq c^*$ are drawn. Bounds for $c > c^*$ are beyond the scope of this article and are not considered, although the essence of the complexity in their construction is clear. By dots, drawn are simulated values of $u_{\alpha,t}(c)$, $c \geq 0$. We note that for $a_Y = 3$, $b_Y = 0.3$ (crosses), the third moment $E(Y^3)$ is not finite, and Fig. 11 suggests that the moment conditions in Theorems 4.2 and 4.3 may be somewhat relaxed. However, this will significantly complicate the proof, which lies beyond the scope of this article.

5.4. Model M(iii): exponentially distributed T and Kummer Y. For T exponentially distributed with parameter $\delta > 0$ and Y whose distribution is Kummer
Figure 12. Model $M(iii)$: graph (X-axis is $c$) of two-sided bounds (4.10) for $0 \leq c \leq c^*$ and of simulated values of $u_{a,t}(c)$, drawn for $T$ exponentially distributed with parameter $\delta = 4/5$ and $Y$ which is Kummer with parameters $k_Y = 5$, $l_Y = 5$ (dots), $k_Y = 200$, $l_Y = 200$ (crosses), $\alpha = 0.05$, and $t = 200$. Vertical grid line: $c^* = 1.3333$ (dots) and $c^* = 0.8081$ (crosses). Horizontal grid lines: simulated $u_{a,t}(c^*) = 102$ (dots) and $u_{a,t}(c^*) = 36$ (crosses).

with parameters $k_Y > 0$, $l_Y > 0$, p.d.f. are

$$f_T(x) = \delta e^{-\delta x}, \quad f_Y(x) = \frac{k_Y \Gamma \left( \frac{k_Y + l_Y}{2} \right)}{\Gamma \left( \frac{k_Y}{2} \right) \Gamma \left( \frac{l_Y}{2} \right)} U \left( 1 + \frac{l_Y}{2}, 2 - \frac{k_Y}{2}, \frac{k_Y}{l_Y} x \right), \quad x > 0.$$  

Elementary calculations yield

$$E_T^k = \frac{k!}{\delta^k}, \quad E_Y^k = \frac{\Gamma \left( \frac{k_Y + l_Y}{2} \right)}{\Gamma \left( \frac{k_Y}{2} \right) \Gamma \left( \frac{l_Y}{2} \right)} l_Y^{2-k} k_Y^{-k}, \quad 2k < l_Y, \quad k = 1, 2, \ldots.$$  

In particular,

$$E_T = 1/\delta, \quad D_T = 1/\delta^2, \quad E_Y = \frac{l_Y}{l_Y - 2}, \quad D_Y = \frac{l_Y^2 (4(l_Y - 2) + k_Y l_Y)}{k_Y (l_Y - 2)^2(l_Y - 4)}.$$  

Plainly, $c^* = E_Y/E_T$ is equal to $l_Y/(l_Y - 2)$, the constants defined in (4.8) are

$$M = E_T/E_Y = \frac{(l_Y - 2)}{\delta l_Y},$$  

$$D^2 = \frac{(E_T^2 D_Y + (E_Y)^2 D_T)/E_Y^3}{\delta^2 k_Y (l_Y - 4)/l_Y},$$  

and the adjustment coefficient does not exist.

In Fig. 12 the upper and lower bounds (4.10) in the case $0 \leq c \leq c^*$ are drawn. Bounds for $c > c^*$ are beyond the scope of this article and are not considered, although the essence of the complexity in their construction is clear. By dots, drawn are the simulated values of $u_{a,t}(c)$, $c \geq 0$.

6. Conclusion

This paper provides a mathematical investigation of an observation (see, e.g., [11]) made empirically, that the Value-at-Risk is not a good solution to the problem of risk

\[^{15}\text{For other equivalent formulas for } f_Y(x) \text{ see [14].}\]
measure’s choice balanced for efficiency and simplicity. Regarding efficiency, the Value-at-Risk is not a good substitute for non-ruin capital, which can be seen even from definitions. Regarding simplicity, it turns out in fairly general risk models that the structure of non-ruin capital is as simple as the structure of Value-at-Risk.

References

[1] Artzner, P., Delbaen, F., Eber, J.M., and Heath, D. (1999) Coherent measures of risk. Mathematical Finance, Vol. 9 (3), 203–228.
[2] Billingsley, P. (1999) Convergence of Probability Measures. 2-nd ed., John Wiley & Sons, New York.
[3] Culp, C.L., Miller, M.H., and Neves, A.M.P. (1997) Value-at-Risk: uses and abuses, Journal of Applied Corporate Finance, Vol. 10, 26–38.
[4] Cummins, D., Harrington, S. and Niehaus, G. (1994) An economic overview of risk-based capital requirements for the property-liability industry, Journal of Insurance Regulation, Vol. 11, 427–447.
[5] Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II), Brussels, 25 November 2009.
[6] Directive 2014/51/EU of the European Parliament and of the Council of 16 April 2014 amending Directives 2003/71/EC and 2009/138/EC and Regulations (EC) No 1060/2009, (EU) No 1094/2010 and (EU) No 1095/2010 in respect of the powers of the European Supervisory Authority (European Insurance and Occupational Pensions Authority) and the European Supervisory Authority (European Securities and Markets Authority), Brussels, 16 April 2016.
[7] Doff, R.R. (2008) A critical analysis of the Solvency II proposals, The Geneva Papers on Risk and Insurance – Issues and Practice, Vol. 33, Issue 2, 193–206.
[8] Eling, M., and Schmeiser, H. (2010) Insurance and the credit crisis: impact and ten consequences for risk management and supervision, The Geneva Papers on Risk and Insurance – Issues and Practice, Vol. 35, Issue 1, 9–34.
[9] Eling, M., Schmeiser, H., and Schmit, J. (2007) The Solvency II process: overview and critical analysis, Risk Management and Insurance Review, Vol. 10, 69–85.
[10] Feller, W. (1971) An Introduction to Probability Theory and its Applications. Vol. II. 2-nd ed., John Wiley & Sons, New York, etc.
[11] Floreani, A. (2013) Risk measures and capital requirements: a critique of the Solvency II approach, The Geneva Papers on Risk and Insurance – Issues and Practice, Vol. 38, 189–212.
[12] Heep-Altiner, M., Mullins, M., and Rohlf, T., Eds. (2018) Solvency II in the Insurance Industry Application of a Non-Life Data Model. Springer.
[13] Kalashnikov, V. (1997) Geometric Sums: Bounds for Rare Events with Applications. Kluwer Academic Publishers, Dordrecht.
[14] Malinovskii, V.K. (1998) Non-Poissonian claims arrivals and calculation of the probability of ruin. Insurance: Mathematics and Economics, Vol. 22, 123–138.
[15] Malinovskii, V.K. (2012) Equitable solvent controls in a multi-period game model of risk, Insurance: Mathematics and Economics, Vol. 51, 599–616.
[16] Malinovskii, V.K. (2017) On the time of first level crossing and inverse Gaussian distribution; https://arxiv.org/pdf/1708.08665.pdf
[17] Malinovskii, V.K. (2017) Generalized inverse Gaussian distributions and the time of first level crossing; https://arxiv.org/pdf/1708.08671.pdf
[18] Malinovskii, V.K. (2018) Approximations in the problem of level crossing by a compound renewal process, Doklady Akademii Nauk, Vol. 483, No. 5, 622–625.
[19] Malinovskii, V.K. (2020) The level which a compound renewal process crosses with a given probability, Doklady Akademii Nauk, To appear.
[20] Malinovskii, V.K. (2020) Insurance Planning Models. Price Competition and Regulation of Financial Stability. World Scientific. To appear.
[21] Malinovskii, V.K., and Kosova, K.O. (2014) Simulation analysis of ruin capital in Sparre Andersen’s model of risk, Insurance: Mathematics and Economics, Vol. 59, 184–193.
[22] Malinovskii, V.K., and Malinovskii, K.V. (2017) On approximations for the distribution of first level crossing time; https://arxiv.org/pdf/1708.08676.pdf.
[23] Marano, P., and Siri, M., Eds. (2017) Insurance Regulation in the European Union Solvency II and Beyond. Palgrave Macmillan.
[24] Mittnik, S. (2011) Solvency II calibrations: where curiosity meets spuriousity. Munich: Center for Quantitative Risk Analysis (CEQUA), Department of Statistics, University of Munich.

[25] Pfeifer, D., and Straßburger, D. (2008) Solvency II: stability problems with the SCR aggregation formula, *Scandinavian Actuarial Journal*, Vol. 1, 61–77.

[26] Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. (1999) Stochastic Processes for Insurance and Finance. John Wiley & Sons, Chichester, etc.

[27] Sandström, A. (2011) Handbook of Solvency for Actuaries and Risk Managers: Theory and Practice. Chapman & Hall/CRC, Taylor & Francis Group. Boca Raton, etc.

[28] Sparre-Andersen, E. (1957) On the collective theory of risk in case of contagion between the claims. In book: Transactions of the XV-th International Congress of Actuaries, Vol. 2, 219–229.

Central Economics and Mathematics Institute (CEMI) of Russian Academy of Science, 117418, Nakhimovsky prosp., 47, Moscow, Russia

E-mail address: Vsevolod.Malinovskii@mail.ru, admin@actlab.ru

URL: http://www.actlab.ru