1. IMPROVED OVERLAP FERMIONS

The Ginsparg-Wilson relation (GWR) for a lattice Dirac operator \( D \) reads \(^1\)

\[ \{ D, \gamma_5 \} = 2 D R \gamma_5 D, \quad (R \text{ local}, \{ R, \gamma_5 \} \neq 0). \quad (1) \]

We choose \( R_{x,y} = \frac{1}{2} \delta_{x,y} \), \( (\mu > 0) \), and \( D^\dagger = \gamma_5 D \gamma_5 \), hence the GWR simplifies to

\[ A^\dagger A = \mu^2, \quad A := D - \mu. \quad (2) \]

It guarantees an exact — though lattice modified — chiral symmetry in even dimensions \(^3\), and parity symmetry in odd dimensions \(^4\). In both cases, the anomaly arises from the measure.

Given some Dirac operator \( D_0 \) (local, no doublers), we can enforce the GWR on \( A_0 = D_0 - \mu \) by the overlap formula \( A_{ov} = \mu A_0 / \sqrt{A_0^\dagger A_0} \), so that \( D_{ov} = A_{ov} + \mu \) obeys the GWR (for \( \mu \) in some allowed interval). The standard Neuberger fermion uses the Wilson operator \( D_0 = D_W \) \(^5\).

The guide-line for our concept to improve overlap fermions \(^6\) is the following observation: if \( D_0 \) is already a GW fermion (with respect to a fixed kernel \( R \)), then \( D_{ov} = D_0 \). We can explicitly construct an approximate solution to the GWR, use it as \( D_0 \), and then we expect that the overlap formula only causes a small “chiral correction”.

\[ D_{ov} \approx D_0. \quad (3) \]

Therefore, we first construct a short-ranged approximate GW fermion, which includes elements of a truncated perfect action, and then we insert it as \( D_0 \) into the overlap formula. The resulting \( D_{ov} \) has an exact chiral symmetry, and we expect a high level of locality, good scaling and approximate rotation invariance, because these properties hold for \( D_0 \), and we rely on relation \(^3\). Moreover, we predict a fast convergence under iterative evaluation of \( D_{ov} \), because we start off in the right vicinity.

All these properties have been tested and confirmed in the 2-flavour Schwinger model \(^6\). Here we report on convergence and locality in quenched QCD.

2. THE HYPERCUBE FERMION (HF)

Our construction of a short-ranged approximate GW fermion is based on the perfect free fermion \(^6\), which is truncated to a unit hypercube. It consists of a vector term and a scalar term, \( D(x, y) = \rho_\mu (x - y) \gamma_\mu + \lambda (x - y) \), where \( \rho_\mu (x - y), \lambda (x - y) \neq 0 \) only for \( |x_i - y_i| \leq 1, (i = 1 \ldots 4) \). The couplings are given in Ref. \(^7\).

The first step is a “minimal gauging” of this HF, which means that the free couplings are attached to the shortest lattice paths only. This formulation provides already an excellent rotation behaviour, but it suffers from strong additive mass renormalization \(^8\).

To cure this problem, we attach a “link amplification factor” \( 1/u \) to each link and tune it to \( u_{crit} < 1 \) to approach at the chiral limit. Using different factors for the vector term and the scalar term further improves the GW approximation.

Next we add a fat link, i.e. each link variable is substituted by a linear combination: \( (1 - \alpha) \) “direct link” + \( \alpha/6 \sum \) “staple terms”. Finally we also include a clover term.

With this modest set of parameters we optimize the QCD spectrum at \( \beta = 6 \). In Ref. \(^8\) we show...
the full spectrum for typical configurations on a $4^4$ lattice, as well as the eigenvalues with the least real parts on a $8^4$ lattice.

3. POLYNOMIAL OVERLAP EVALUATION AND LOCALITY

We consider two ways to evaluate the overlap Dirac operator by means of polynomial expansions, which have appeared in the literature.

1) We introduce the Hermitean operator $H_0 = \gamma_5 A_0$, and evaluate

$$D_{ov} = \mu \left(1 + \gamma_5 \frac{H_0}{\sqrt{H_0^2}}\right).$$

The last term corresponds to a sign function $\epsilon(x) = 1 (-1)$ for $x > 0 (x < 0)$, which is approximated by a polynomial.

2) We simply expand $1/\sqrt{\tau}$ by a polynomial, and insert $A_0^\dagger A_0$ for $x$.

For the sign expansion, the convergence in the degree of the polynomial is exponential, if $x \in [-1, 1]$. Hence we first re-scale $H_0 \rightarrow \tilde{H}_0$, so that the spectrum $\sigma(\tilde{H}_0) \subset [-1, 1]$. What matters is the density of eigenvalues (EVs) of $\tilde{H}_0$ around 0, where the polynomial converges most painfully due to the discontinuity. For the HF, $\sigma(H_0)$ has two peaks around $\pm 1$, which are shifted slightly towards the origin for $\tilde{H}_0$, leaving typically (at $\beta = 6$) a wide gap without EVs of $\tilde{H}_0$ between about $[-0.3, 0.3]$. On the other hand, $\tilde{H}_{0W}$ has a broad spectrum, ranging from about $[-6, 6]$, and after re-scaling one does obtains EVs near 0. Histograms are shown in Ref. [1].

For the $1/\sqrt{\tau}$ expansion the situation is similar: here we re-scale so that $\tilde{x} \in [\delta, 1]$, ($\delta > 0$). It is crucial that $\delta$ (the lower bound of the re-scaled spectrum) does not become very small; getting close to the singularity slows down the convergence. Again, the Wilson spectrum is very broad to start with, and re-scaling usually leads to small $\delta$, whereas it is kept larger for the HF.

In Fig. 1 we illustrate the convergence rate in QCD at $\beta = 6$ using Chebyshev polynomials. As an example, we discuss the $\epsilon(x)$ expansion on the $4^4$ lattice. For $D_0 = D_W$, the maximal (mean) deviation of the EVs of $\sqrt{\tau} D$ from the unit circle in $\mathbb{C}$ (with center 1) behaves

Figure 1. The maximal deviation of an energy eigenvalue of approximate overlap Dirac operators (after re-scaling) from the unit circle in $\mathbb{C}$, typically as $d_{\text{max}} = \exp(-0.134n)$ ($d_{\text{mean}} = 0.13\exp(-0.134n)$), where $n$ is the degree of the Chebyshev polynomials. The corresponding decays for $D_0 = D_{\text{HF}}$ amount to $d_{\text{max}} = \exp(-0.737n)$ ($d_{\text{mean}} = 0.1\exp(-0.737n)$).

If we fix some affordable degree $n$, the precision of the GW approximation is superior for the HF by a factor $d_{\text{max}} / d_{\text{max}} = \exp(0.6n)$, which takes a very considerable magnitude for realistic numbers like $n = 20 \ldots 100$.

On the other hand, we could fix a certain accuracy $d_{\text{max}}$ which we consider necessary, so the required degree $n$ compares as $n_W / n_{HF} = 5.5$. Referring to this number, the computational overhead of the HF, which amounts to a factor $\leq 20$, is compensated in part. It now takes only a relatively small progress in the scaling behaviour to make up for the remaining overhead.

Fig. 2 shows that locality is clearly improved for the overlap-HF compared to the Neuberger fermion. Following Ref. [1], we measure the “maximal correlation” $f(r)$ over a distance $r$. 


Figure 2. Comparison of the degree of locality for different overlap fermions.

We stay on the $12^4$ lattice and we observe also here a clear progress in the speed of convergence. This is related to the condition number $c$, i.e. the ratio of the upper and the lower bound of the spectrum of $A_0^\dagger A_0$. Fig. 3 (on top) shows that the improved condition number for the HF is essentially due to the decrease of the upper bound. To illustrate the effect of $c$ on the convergence rate, Fig. 3 (below) shows the deviation of $f(r = 24)$ from the precise result if we use a moderate polynomial of degree $n = 60$. This deviation depends smoothly on $c$, up to an extra gap which makes the deviation for $D_w = D_W$ yet a bit worse than $c$ suggests, even at the optimal parameter $\mu = 1.64$.

4. CONCLUDING REMARKS

Compared to the standard Neuberger fermion, the overlap-HF gains significantly in the convergence rate under polynomial evaluation. We have studied the evolution of the full operator in the polynomial at $\beta = 6$, which might be roughly comparable to the situation at stronger coupling, say $\beta = 5.75$, after treating the $O(10)$ worst modes separately, as it is often done.

Moreover, we observed a superior degree of locality. This suggests that the overlap formula is applicable up to stronger coupling (the limit for $\beta$ was discussed in Ref. [3]).

The scaling quality remains to be tested; we are currently measuring meson dispersions. Another important question is the applicability of preconditioning techniques. They were already applied successfully to the HF without overlap [11].

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