Infinite dimension of solutions of the Dirichlet problem

By the well–known Lindelöf maximum principle, see e.g. Lemma 1.1 in [3], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions $u$ on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. In general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under zero boundary data. In comparison with [7], here we give more elementary examples and constructions of solutions.

Many such nontrivial solutions $u$ for the Laplace equation can be given by the Poisson-Stieltjes integral

$$u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} P_r(\theta - t) \, d\Phi(t), \quad z = re^{i\theta}, \quad r < 1,$$

with an arbitrary singular function $\Phi : [0, 2\pi] \to \mathbb{R}$, i.e., where $\Phi$ is of bounded variation and $\Phi' = 0$ a.e., and where we use the standard notation for the Poisson kernel

$$P_r(\Theta) = \frac{1 - r^2}{1 - 2r \cos \Theta + r^2}, \quad r < 1.$$

Indeed, $u$ in (1) is harmonic for every function $\Phi : [0, 2\pi] \to \mathbb{R}$ of bounded variation and by the Fatou theorem, see e.g. Theorem I.D.3.1 in [6], $u(z) \to \Phi'(\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists. Thus, $u(z) \to 0$ as $z \to e^{i\Theta}$ for a.e. $\Theta \in [0, 2\pi]$ along any nontangential paths for every singular function $\Phi$.

**Example 1.1.** The first natural example is given by the formula (1) with $\Phi(t) = \varphi(t/2\pi)$ where $\varphi : [0, 1] \to [0, 1]$ is the well–known Cantor function, see e.g. [1] and further references therein.

**Example 1.2.** However, the simplest example of such a kind is given by nondecreasing step-like data $\Phi_{\theta_0}$ with values 0 and 2$\pi$ and with the jump at $\theta_0 \in (0, 2\pi)$:

$$u(z) = P_r(\rho - \theta_0) = \frac{1 - r^2}{1 - 2r \cos(\rho - \theta_0) + r^2}, \quad z = re^{i\rho}, \quad r < 1.$$

We see that $u(z) \to 0$ as $z \to e^{i\Theta}$ for all $\Theta \in (0, 2\pi)$ except $\Theta = \theta_0$.

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Note that the function $u$ is harmonic in the unit disk $\mathbb{D}$ because

$$u(z) = \Re \frac{\zeta_0 + z}{\zeta_0 - z} = \frac{1 - |z|^2}{2|z| \Re \zeta_0 + |z|^2}, \quad \zeta_0 = e^{i \theta_0}, \quad z \in \mathbb{D},$$

(3)

where the function $w = g(z) = g_{\zeta_0}(z) = (\zeta_0 + z)/(\zeta_0 - z)$ is analytic (conformal) in $\mathbb{D}$ and maps $\mathbb{D}$ onto half-plane $\Re w > 0$, $g(0) = 1$, $g(\zeta_0) = \infty$.

2 The main result

The formula (2) gives a continual set of such examples. Furthermore, one can prove the following result.

**Theorem 2.1.** The space of all harmonic functions in $\mathbb{D}$ with nontangential limit 0 at every point of $\partial \mathbb{D}$ except a countable collection of points in $\partial \mathbb{D}$ has the infinite dimension.

**Proof.** Indeed, let us consider the sequence of functions of the form (3):

$$u_n(z) = \Re \frac{\zeta_n + z}{\zeta_n - z} = \frac{1 - |z|^2}{2|z| \Re \zeta_n + |z|^2}, \quad \zeta_n = e^{i \theta_n}, \quad z \in \mathbb{D},$$

where

$$\theta_n = \pi (2^{-1} + \ldots + 2^{-n}), \quad n = 1, 2, \ldots$$

and denote by $\mathcal{H}_1$ the class of all series $u = \sum \gamma_n u_n$ whose sequences of coefficients $\gamma = \{\gamma_n\}$ belong to the space $l^1$ with the norm $\|\gamma\| = \sum_{n=1}^{\infty} |\gamma_n| < \infty$. Note that $\mathcal{H}_1$ consists of harmonic functions, see, e.g., Theorem I.3.1 in [5], because

$$0 < u_n(z) < \frac{1 + |z|}{1 - |z|} \quad \forall n = 1, 2, \ldots, z \in \mathbb{D}.$$

Note also that each function $u \in \mathcal{H}_1$ has nontangential limit 0 at every point $\zeta \in \partial \mathbb{D}$ except the points $\zeta_0 = -1 = e^{i \theta_0}$, $\zeta_0 = \pi$, and $\zeta_n, n = 1, 2, \ldots$. Indeed, let $\zeta = e^{i \theta}, \theta \in (0, 2\pi)$, $\zeta \neq \zeta_0, n = 0, 1, 2, \ldots$. Then, applying the formula (2), we have the estimate

$$u_n(z) \leq \frac{1 - r^2}{4r \sin^2 \frac{\theta - \theta_n}{4}} \leq C(1 - r), \quad z = re^{i \theta},$$

for all points $z = re^{i \theta}$ belonging to a sector $|\theta - \Theta| < c(1 - r)$ and for all $r$ which are close enough to 1 where $C < \infty$ does not depend on $n = 1, 2, \ldots$. Thus,

$$|u(z)| \leq C \|\gamma\|(1 - r) \to 0 \quad \text{as} \quad r \to 1, \quad z = re^{i \theta},$$

in any sector $|\theta - \Theta| < c(1 - r)$.

Now, let us show that $u_n, n = 1, 2, \ldots$, form a basis in the space $\mathcal{H}_1$ with the locally uniform convergence in $\mathbb{D}$ which is metrizable.

Indeed, firstly, $u = \sum_{n=1}^{\infty} \gamma_n u_n \neq 0$ if $\gamma \neq 0$. Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \ldots$. Then $u \neq 0$ because $u(z) \to \infty$ as $z = re^{i \theta_n} \to e^{i \theta_n}$. The latter follows because

$$u_n(re^{i \theta_n}) = \frac{1 + r}{1 - r} \to \infty \quad \text{as} \quad r \to 1,$$

and by the previous item

$$|\bar{u}(re^{i \theta_n})| \leq C \|\gamma\|(1 - r) \to 0 \quad \text{as} \quad r \to 1,$$

where $\bar{u} = u - \gamma_n u_n$. 


Secondly, \( u_m^* = \sum_{n=1}^{m} y_n u_n \to u \) locally uniformly in \( D \) as \( m \to \infty \). Indeed, elementary calculations give the following estimate of the remainder term

\[
|u(z) - u_m^*(z)| \leq \frac{1 + r}{1 - r} \cdot \sum_{n=m+1}^{\infty} |y_n| \to 0 \quad \text{as} \quad m \to \infty
\]

in every disk \( D(r) = \{z \in \mathbb{C} : |z| \leq r\} \), \( r < 1 \).

3 Corollaries and final remarks

**Corollary 3.1.** Given a measurable function \( \varphi : \partial D \to \mathbb{R} \), the space of all harmonic functions \( u : D \to \mathbb{R} \) with the limits \( \lim_{z \to \xi} u(z) = \varphi(\xi) \) for a.e. \( \xi \in \partial D \) along nontangential paths has the infinite dimension.

Indeed, the existence at least one such a harmonic function \( u \) follows from the known Gehring theorem in [4]. Combining this fact with Theorem 2.1, we obtain the conclusion of Corollary 3.1.

**Remark 3.2.** In view of Lemma 3.1 in [2], one can similarly prove the more refined result on harmonic functions than in Corollary 3.1 with respect to logarithmic capacity instead of the measure of the length on \( \partial D \).

Moreover, the statements on the infinite dimension of the space of solutions can be extended to the Riemann-Hilbert problem because the latter is reduced to the corresponding two Dirichlet problems as in papers [2] and [7].

Note also that harmonic functions \( u \) found in Theorem 2.1 and Corollary 3.1 themselves cannot be represented in the form of the Poisson integral with any integrable function \( \Phi : [0, 2\pi] \to \mathbb{R} \) because such integral would have nontangential limits \( \Phi \) a.e., see e.g. Corollary IX.1.1 in [5]. Consequently, \( u \) do not belong to the classes \( h_p \) for any \( p > 1 \), see e.g. Theorem IX.2.3 in [5].

However, the functions \( u \in \mathcal{H}_1 \) in the proof of Theorem 2.1 have the representation as the Poisson-Stiltjes integral \( \Phi = \sum y_n \Phi(\vartheta_n) \) where \( \Phi(\vartheta_n) : [0, 2\pi] \to \mathbb{R} \) are nondecreasing step-like functions with values \( 0 \) and \( 2\pi \) with jumps at the points \( \vartheta_n \), \( n = 1, 2, \ldots \). Thus, \( \Phi \) is of bounded variation and hence \( \mathcal{H}_1 \subset h_1 \), see e.g. Theorem IX.2.2 in [5].

**Problem 3.3.** It remains the open question whether the basis of the space of all such singular solutions of the Dirichlet problem for the Laplace equation has the power of the continuum.

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