State-independent experimental test of quantum contextuality in an indivisible system

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We report the first state-independent experimental test of quantum contextuality on a single photonic qutrit (three-dimensional system), based on a recent theoretical proposal [Yu and Oh, Phys. Rev. Lett. 108, 030402 (2012)]. Our experiment spotlights quantum contextuality in its most basic form, in a way that is independent of either the state or the tensor product structure of the system.

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Contextuality represents a major deviation of quantum theory from classical physics\textsuperscript{[1,2]}. Non-contextual realism is a pillar of the familiar worldview of classical physics. In a non-contextual world, observables have pre-defined values, which are independent of our choices of measurements. Non-contextuality plays a role also in the derivation of Bell’s inequalities, as the property of local realism therein can be seen as a special form of non-contextuality, where the independence of the measurement context is enforced by the no-signalling principle\textsuperscript{[3,6]}. In an attempt to save the non-contextuality of the classical worldview, non-contextual hidden variable theories have been proposed as an alternative to quantum mechanics. In these theories, the outcomes of measurements are associated to hidden variables, which are distributed according to a joint probability distribution. However, the celebrated Kochen-Specker theorem\textsuperscript{[1-4]} showed that non-contextual hidden variable theories are incompatible with the predictions of quantum theory. The original Kochen-Specker theorem is presented in the form of a logical contradiction, which is conceptually striking, but experimentally unfriendly: the presence of unavoidable experimental imperfections motivated a debate on whether or not the non-contextual features highlighted by Kochen-Specker theorem can be actually tested in experiments\textsuperscript{[7,8]}. As a result of the debate, new Bell-type inequalities have been proposed in the recent years, with the purpose of pinpointing the contextuality of quantum mechanics in an experimentally testable way. These inequalities are generally referred to as the KS inequalities\textsuperscript{[5]}. Violation of the KS inequalities confirms quantum contextuality and rules out the non-contextual hidden variable theory. Different from the Bell inequality tests, violation of the KS inequality can be achieved independently of the state of quantum systems\textsuperscript{[1,2,5,6]}, showing that the conflict between quantum theory and non-contextual realism resides in the structure of quantum mechanics instead of particular quantum states. The KS inequalities have been tested in experiments for two qubits, using ions\textsuperscript{[9]}, photons\textsuperscript{[10,11]}, neutrons\textsuperscript{[12]}, or an ensemble nuclear magnetic resonance system\textsuperscript{[13]}. A single qutrit represents the simplest system where it is possible to observe conflict between quantum theory and non-contextual realistic models\textsuperscript{[6,14-16]}. A recent experiment has demonstrated quantum contextuality for photonic qutrits in a particular quantum state\textsuperscript{[14]}, based on a version of the KS inequality proposed by Klyachko, Can, Binicioglu, and Shumovsky\textsuperscript{[15]}.

A state-independent test of quantum contextuality for a single qutrit, in the spirit of the original KS theorem, is possible but complicated as one needs to measure many experimental configurations\textsuperscript{[4,6,16]}. A recent theoretical work by Yu and Oh proposes another version of the KS inequality, which requires to measure 13 variables and 24 of their pair correlations\textsuperscript{[6]}. This is a significant simplification compared with the previous KS inequalities for single qutrits, and the number of variables cannot be further reduced as proven recently by Cabello\textsuperscript{[17]}. Our experiment confirms quantum contextuality in a state-independent fashion using the Yu-Oh version of the KS inequality for qutrits represented by three distinctive paths of single photons. The maximum violation of this inequality by quantum mechanics is only 4\% beyond the bound set by the non-contextual hidden variable theory, so we need to accurately control the paths of single photons in experiments to measure the 13 variables and their correlations for different types of input states. We have achieved a violation of the KS inequality by more than five standard deviations for all the nine different states that we tested.

For a single qutrit with basis vectors \{\(|0\rangle,|1\rangle,|2\rangle\)}, we detect projection operators to the states \(i|0\rangle+j|1\rangle+k|2\rangle\) specified by the 13 unit vectors \((i,j,k)\) in Fig. 1. The 13 projectors have eigenvalues either 0 or 1. In the hidden variable theory, the corresponding observables are assigned randomly with values 0 or 1 according to a (generally unknown) joint probability distribution. When two states are orthogonal, the projectors onto them commute, and the corresponding observables are called compatible, which means that they can be measured simultaneously. Non-contextuality means that the assignment of values to an observable should be independent of the choice of compatible observables that are mea-
sured jointly with it. For instance, $z_1$ in Fig. 1 should be assigned the same value in the correlators $z_1 z_2$ and $z_1 y_1^\pm$ for each trial of measurement. For each observable $b_i \in \{z_\mu, y_\mu, h_\alpha, \mu = 1, 2, 3; \alpha = 0, 1, 2, 3\}$ defined in Fig. 1, we introduce a new variable $a_i \equiv 1 - 2b_i$, which takes values of $\pm 1$. For the 13 observables $a_i$ with two outcomes $\pm 1$, it is shown in Ref. [6] that they satisfy the inequality

$$\sum_i a_i - \frac{1}{4} \sum_{(i,j)} a_ia_j \leq 8,$$

where $(i, j)$ denotes all pairs of observables that are compatible with each other. There are 24 compatible pairs among all the $13 \times 13$ combinations, and a complete list of them is given in Table 1 for the corresponding operator correlations. The inequality (1) can be proven either through an exhaustive check of all the possible $2^{13}$ value assignments of $a_i$ ($i = 1, 2, \cdots, 13$) or by a more elegant analytic argument as shown in Ref. [6]. In quantum theory, each $a_i$ corresponds to an operator $A_i$ with eigenvalues $\pm 1$ in quantum mechanics. In the hidden variable theory, the value $a_i$ corresponds to a random variable $A_i$, and the different values are distributed according to a (possibly correlated) joint probability distribution. Hence, for the hidden variable theory the expectation values of $A_i$ must satisfy the inequality

$$\sum_i \langle A_i \rangle - \frac{1}{4} \sum_{(i,j)} \langle A_i A_j \rangle \leq 8,$$

which follows by taking the average of (1) over the joint probability distribution of the values $a_i$. On the other hand, quantum theory gives a different prediction: From the definition $A_i \equiv I - 2B_i$, where $B_i$ is the projection operator to the 13 states in Fig. 1, we find that $S = \sum_i A_i - \frac{1}{4} \sum_{(i,j)} A_iA_j = 25I/3$, where $I$ is the unity operator. Hence, for any state of the system, quantum theory predicts the inequality $\langle S \rangle = 25/3 \leq 8$, which violates the inequality (2) imposed by the non-contextual hidden variable theory and rules out any non-contextual realistic model.

Since the quantum mechanical prediction $\langle S \rangle = 25/3$ is close to the upper bound $\langle S \rangle \leq 8$ set by the non-contextual realism, we need to achieve accurate control in experiments to violate the inequality (2). Yu and Oh also derived another simpler inequality in Ref. [6] by introducing an additional assumption (as proposed in the original KS proof [1, 4]) that the algebraic structure of compatible observables is preserved at the hidden variable level, that is, that the value assigned to the product (or sum) of two compatible observables is equal to the product (or sum) of the values assigned to these observables. Under this assumption, it is shown in [6] that

$$\sum_{\alpha = 0, 1, 2, 3} \langle B_{h_\alpha} \rangle \leq 1$$

for non-contextual hidden variable theory, while quantum mechanically $\sum_{\alpha = 0, 1, 2, 3} B_{h_\alpha} \equiv 43I$, and thus

$$\sum_{\alpha = 0, 1, 2, 3} \langle B_{h_\alpha} \rangle = 4/3 > 1.$$
sition of the three basis-vectors. The theoretical values in the quantum mechanical case are calculated using the Born rule with the ideal state $|s\rangle$. Each of the experimental correlations is constructed from the joint probabilities $P(A_i = \pm 1; A_j = \pm 1)$ registered by the detectors. As an example to show the measurement method, in section 2 of the supplementary information, we give detailed data for the registered joint probabilities under different measurement configurations, which together fix all the correlations in Table 1. The expectation value $\langle B_i \rangle$ (or $\langle A_i \rangle \equiv 1 - 2 \langle B_i \rangle$) is directly determined by the relative probability of the photon firing in the corresponding detector. From the data summarized in Table 1, we find both of the inequalities (2) and (3) are significantly violated in experiments, in agreement with the quantum mechanics prediction and in contradiction with the non-contextual realistic models. Even the tough inequality (2) is violated by more than five times the error bar (standard deviation).

To verify that the inequalities (2) and (3) are experimentally violated independently of the state of the system, we have tested them for different kinds of input states. The set of states tested include the three basis-vectors $\{|0\rangle, |1\rangle, |2\rangle\}$, the two-component superposition states $\{|(0) + |1\rangle + |2\rangle) / \sqrt{2}, (|0\rangle + |2\rangle) / \sqrt{2}, (|1\rangle + |2\rangle) / \sqrt{2}\}$, the three-component superposition state $|s\rangle$, and two mixed states $\rho_{\pm} = (|0\rangle + |2\rangle) / 2$ and $\rho_0 = (|0\rangle + |1\rangle + |2\rangle) / 3 \equiv I / 3$. The detailed configurations of the wave plates to prepare these different input states are summarized in section 1 of the supplementary information. To generate the mixed states, we first produce photon pairs entangled in polarization using the type-I phase matching in the BBO crystal [18]. After tracing out the idler photon by the detection at D0, we get a mixed state in polarization for the signal photon, which is then transferred to a mixed qutrit state represented by the optical paths through the PBS. For various input states, we measure correlations of all the observables in the inequality (2) and the detailed results are presented in section 3 of the supplementary information. Although the expectation values $\langle A_i \rangle$ and the correlations $\langle A_i A_j \rangle$ strongly depend on the input states, the inequalities (2) and (3) are state-independent and significantly violated for all the cases tested in experiments. In Fig. 3, we present the measurement outcomes of these two inequalities for nine different input states. The results violate the boundary set by the non-contextual hidden variable theory and are in excellent agreement with quantum mechanics predictions.

In this work, we have observed violation of the KS inequalities (2) and (3) for a single photonic qutrit, which represents the first state-independent experimental test of quantum contextuality in an indivisible quantum system. The experiment confirmation of quantum contextuality in its most basic form, in a way that is independent of either the state or the tensor product structure of the system, sheds new light on the contradiction between quantum mechanics and non-contextual realistic models.

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FIG. 2: Illustration of the experimental setup to detect the KS inequalities. The setup in the box (a) is for state preparation of a single photonic qutrit. Ultrafast laser pulses (with a repetition rate of 76 MHz) at the wavelength of 400 nm from a frequency doubled Ti:sapphire laser pump two joint beta-barium-borate (BBO) crystals, each of 0.6 mm depth with perpendicular optical axis, to generate correlated (entangled) photon pairs at the wavelength of 800 nm. With registration of a photon-count at the detector D0, we get a heralded single photon source in the other output mode. This photon is split by two polarizing beam splitters (PBS) into three optical paths, representing a single photonic qutrit. By adjusting the angle of the half wave plates (HWP1 and HWP2), we can control the superposition coefficients of this qutrit state. The setup in box (b) is for measurement of the qutrit state along compatible projections to three orthogonal states. By tuning the wave plates (HWP3 and HWP4), we choose these projection operators to be along the directions specified by the Aₖ operators to measure the correlations of the compatible Aᵢ. The wave plates HWP3 and HWP4 are used to balance the Mach-Zender interferometers and can be tilted for fine tuning of the relative phase.

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|-------------------|------------------------|-------------|-------------------|------------------------|-------------|-------------------|------------------------|
| (Aₓ₁)       | 0.328(18)         | 0.333                  | (Aₓ₂, Aₓ₂)  | -0.348(18)        | -0.333                 | (Aᵧ₁, Aᵧ₁)  | -0.628(17)        | -0.555                 |
| (Aₓ₂)       | 0.324(18)         | 0.333                  | (Aₓ₁, Aₓ₁)  | 0.348(18)         | 0.333                  | (Aᵧ₂, Aᵧ₂)  | -0.481(16)        | -0.555                 |
| (Aᵧ₁)       | 0.348(18)         | 0.333                  | (Aₓ₁, Aₓ₁)  | -0.324(18)        | -0.333                 | (Aᵧ₁, Aᵧ₁)  | -0.317(17)        | -0.333                 |
| (Aᵧ₂)       | -0.320(17)        | -0.333                 | (Aₓ₁, Aₓ₁)  | -0.320(17)        | -0.333                 | (Aᵧ₂, Aᵧ₂)  | -0.940(6)         | -1                     |
| (Aᵧ₃)       | 0.876(9)          | 1                      | (Aₓ₁, Aₓ₁)  | -0.320(17)        | 0.333                  | (Aᵧ₂, Aᵧ₂)  | -0.435(16)        | -0.555                 |
| (Aᵧ₄)       | -0.220(17)        | -0.333                 | (Aₓ₁, Aₓ₁)  | -0.903(8)         | -1                     | (Aᵧ₂, Aᵧ₂)  | 0.682(13)         | 0.778                  |
| (Aᵧ₅)       | 0.903(8)          | 1                      | (Aₓ₁, Aₓ₁)  | 0.220(17)         | 0.333                  | (Aᵧ₂, Aᵧ₂)  | -0.572(15)        | -0.555                 |
| (Aᵧ₆)       | -0.255(18)        | -0.333                 | (Aₓ₁, Aₓ₁)  | -0.939(6)         | -1                     | (Aᵧ₂, Aᵧ₂)  | -0.315(18)        | -0.333                 |
| (Aᵧ₇)       | 0.939(6)          | 1                      | (Aₓ₁, Aₓ₁)  | 0.255(18)         | 0.333                  | (Aᵧ₂, Aᵧ₂)  | -0.979(4)         | -1                     |
| (Aᵧ₈)       | -0.943(6)         | -1                     | (Aₓ₁, Aₓ₁)  | -0.444(17)        | -0.333                 | (Aᵧ₂, Aᵧ₂)  | -0.480(16)        | -0.555                 |
| (Aᵧ₉)       | 0.745(12)         | 0.778                  | (Aₓ₁, Aₓ₁)  | -0.938(6)         | -1                     | (Aᵧ₂, Aᵧ₂)  | -0.510(16)        | -0.555                 |
| (Aᵧ₁₀)      | 0.817(10)         | 0.778                  | (Aₓ₁, Aₓ₁)  | 0.698(12)         | 0.778                  | (Aᵧ₂, Aᵧ₂)  | 0.710(13)         | 0.778                  |
| (Aᵧ₁₁)      | 0.746(12)         | 0.778                  | (Aₓ₁, Aₓ₁)  |                      |                        | (Aᵧ₂, Aᵧ₂)  |                        |                        |

K-S Inequality (2) \[ \sum_i \langle A_i \rangle - \frac{1}{3} \sum_{i,j} \langle A_i A_j \rangle = 8.226 \pm 0.041 \]

K-S Inequality (3) \[ \sum_{\alpha=0,1,2,3} \langle B_{\alpha} \rangle = 1.317 \pm 0.011 \]

FIG. 3: Table 1: (A) The measured expectation values \( \langle A_i \rangle \) and the correlations \( \langle A_i A_j \rangle \) for all the compatible pairs under a particular input state \( |s\rangle = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3} \). For the experimental values, the numbers in the bracket represent the statistical error associated with the photon detection under the assumption of a Poissonian distribution for the photon counts, for instance, \( \langle A_{\alpha i} \rangle = 0.328(18) \approx 0.328 \pm 0.018 \). Both of the inequalities (2) and (3) are significantly violated by the experimental data.

SUPPLEMENTARY INFORMATION:
STATE-INDEPENDENT EXPERIMENTAL TEST OF QUANTUM CONTEXTUALITY IN AN INDIVISIBLE SYSTEM

This supplementary information gives the detailed configurations and data for the experimental test of state-independent quantum contextuality on a single photonic qutrit. In Sec. I, we first give the configurations of the wave plates to prepare different input states for a single photonic qutrit, and then summarize the configurations of the experiment to measure all the observables and their
angles of three half wave plates HWP0, HWP1, and HWP2 in the experimental setup shown in Fig. 2 of the manuscript. To prepare pure input state, the polarization of the pumping laser is set to $|V\rangle$ (vertically polarized) by the HWP0. With the type-I phase matching in the BBO nonlinear crystal, the generated signal and idler photons are both in the polarization state $|H\rangle$. After the heralding measurement of the idler photon, the polarization of the signal photon is rotated by the HWP1 and HWP2. The polarization beam splitter (PBS) transmits the photon when it is in $|H\rangle$ polarization and reflects it when it is in $|V\rangle$ polarization. The half-wave plate with angle $\theta$ transfers the polarization basis-states $|H\rangle$ and $|V\rangle$ by the formula $|H\rangle \rightarrow \cos (2\theta) |H\rangle + \sin (2\theta) |V\rangle$ and $|V\rangle \rightarrow \cos (2\theta) |V\rangle - \sin (2\theta) |H\rangle$. To prepare an arbitrary input state $c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, the angle of the HWP1 sets the branching ratio $c_0/\sqrt{c_1^2 + c_2^2}$, and the angle of the HWP2 then determines $c_1/c_2$. For the seven pure input states in the experiment, the corresponding angles of the HWP1 and HWP2 are listed in Table I.

It is more striking to see that the KS inequalities are violated even for completely mixed states. To prepare a mixed state for the signal photon, we rotate the polarization of the pumping laser to $(|H\rangle + |V\rangle)/\sqrt{2}$ by setting the angle of HWP0 at 22.5°. The output state for the signal and the ideal photon after the BBO crystal is a maximally entangled one with the form $\Psi_{si} = (|HH\rangle + e^{i\varphi}|VV\rangle)/\sqrt{2}$, where $\varphi$ is a relative phase of the two polarization components. After the heralding measurement of the idler photon, the state of the signal photon is described by the reduced density matrix $(|H\rangle \langle H| + |V\rangle \langle V|)/2$. If we set the HWP1 and HWP2 respectively at the angle of 0° and 45°, this polarization mixed state is transferred to the qutrit mixed state $\rho_8 = (|0\rangle \langle 0| + |2\rangle \langle 2|)/2$ as shown in Table 1.

To prepare the mixed state $\rho_9$ (the most noisy qutrit state), we set the HWP0 at the angle 27.37° and the density operator for the signal photon right after the BBO crystal becomes $(|H\rangle \langle H| + 2|V\rangle \langle V|)/3$. When we set the HWP1 and HWP2 respectively at 0° and 22.5°, the photonic qutrit is described by the state $|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| + (e^{i\phi}|1\rangle \langle 2| + H.c.|)/3$, where the relative phase $\phi = 0$ in the ideal case. However, we randomly tilt the HWP3 in this experiment which sets the phase $\phi$ to a random value. After average over many experimental runs to measure the correlation, the effective state for the qutrit is described by $\rho_9 = (|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|)/3 = I/3$, the completely mixed state.

To detect the KS inequalities, we need to measure the 13 observables $A_i$ and 24 compatible combinations of their pair-wise correlations $\langle A_i A_j \rangle$. By rotating the angles of the HWP5 and HWP6, We choose the measurement basis so that the photon count at the detectors D1, D2, or D3 correspond to a measurement of the compatible combinations of the projection operators $A_i$. In

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Input State & HWP0 & HWP1 & HWP2 \\
\hline
$|\Psi_1\rangle = |0\rangle$ & 0° & 0° & 0° \\
$|\Psi_2\rangle = |1\rangle$ & 0° & 45° & 0° \\
$|\Psi_3\rangle = |2\rangle$ & 0° & 22.5° & -45° \\
$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ & 0° & 22.5° & 0° \\
$|\Psi_5\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$ & 0° & 22.5° & -45° \\
$|\Psi_6\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ & 0° & 45° & -22.5° \\
$|\Psi_7\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + |2\rangle)$ & 0° & 27.37° & -22.5° \\
$\rho_0 = |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|$ & 22.5° & 0° & -45° \\
$\rho_9 = \frac{1}{3}(|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|)$ & 27.37° & 0° & -22.5° \\
\hline
\end{tabular}
\caption{Angles of half-wave plates (HWP) to prepare different states for a single photonic qutrit.}
\end{table}
In this section, we list the measured expectation values in calculation of the correlation \(\langle A_iA_j \rangle\) for a particular input state \(\Psi_2 = |s\rangle = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}\). We have tested the KS inequalities for 9 different input states, ranging from the simple basis vectors, to the superposition states, and to the most noisy mixed states. For all the input states, we have observed significant violation of the KS inequalities for single photonic qutrits.

In this section, we list the measured expectation values \(\langle A_i \rangle\) and the correlations \(\langle A_iA_j \rangle\) for the other 8 input states (shown in Table IV, V, VI, and VII).
| Joint Probability | Value  | Joint Probability | Value  | Joint Probability | Value  | Term       | Value  |
|-------------------|--------|-------------------|--------|-------------------|--------|------------|--------|
| \( P(A_y = 1, A_z = -1) \) | 0.332(9) | \( P(A_y = -1, A_z = 1) \) | 0.336(9) | \( P(A_y = 1, A_z = 1) \) | 0.326(9) | \( (A_z, A_{zh}) \) | -0.348(18) |
| \( P(A_y = 1, A_z = -1) \) | 0.326(9) | \( P(A_y = -1, A_z = 1) \) | 0.336(9) | \( P(A_y = 1, A_z = 1) \) | 0.338(9) | \( (A_z, A_{zh}) \) | -0.324(18) |
| \( P(A_y = 1, A_z = 1) \) | 0.326(9) | \( P(A_y = -1, A_z = 1) \) | 0.336(9) | \( P(A_y = 1, A_z = 1) \) | 0.336(9) | \( (A_z, A_{zh}) \) | -0.328(18) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.660(9) | \( P(A_y = -1, A_{z1} = 1) \) | 0.278(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.062(5) | \( (A_{z1}, A_{z1}) \) | -0.876(9) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.062(5) | \( P(A_y = -1, A_{z1} = 1) \) | 0.278(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.660(9) | \( (A_{z1}, A_{z1}) \) | 0.320(17) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.610(9) | \( P(A_y = -1, A_{z1} = 1) \) | 0.341(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.048(4) | \( (A_{z1}, A_{z1}) \) | -0.903(8) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.048(4) | \( P(A_y = -1, A_{z1} = 1) \) | 0.341(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.610(9) | \( (A_{z1}, A_{z1}) \) | 0.220(17) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.610(9) | \( P(A_y = -1, A_{z1} = 1) \) | 0.048(4) | \( P(A_y = 1, A_{z1} = 1) \) | 0.341(9) | \( (A_{z1}, A_{z1}) \) | -0.317(17) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.627(9) | \( P(A_y = -1, A_{z1} = 1) \) | 0.342(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.030(5) | \( (A_{z1}, A_{z1}) \) | -0.939(6) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.030(5) | \( P(A_y = -1, A_{z1} = 1) \) | 0.342(9) | \( P(A_y = 1, A_{z1} = 1) \) | 0.627(9) | \( (A_{z1}, A_{z1}) \) | 0.255(18) |
| \( P(A_y = 1, A_{z1} = -1) \) | 0.627(9) | \( P(A_y = -1, A_{z1} = 1) \) | 0.030(5) | \( P(A_y = 1, A_{z1} = 1) \) | 0.342(9) | \( (A_{z1}, A_{z1}) \) | -0.315(18) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.935(4) | \( P(A_y = -1, A_{zh} = 1) \) | 0.034(3) | \( P(A_y = 1, A_{zh} = 1) \) | 0.031(3) | \( (A_{zh}, A_{zh}) \) | -0.938(6) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.111(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.040(6) | \( P(A_y = 1, A_{zh} = 1) \) | 0.849(3) | \( (A_{zh}, A_{zh}) \) | 0.698(12) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.132(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.682(9) | \( P(A_y = 1, A_{zh} = 1) \) | 0.186(7) | \( (A_{zh}, A_{zh}) \) | -0.628(17) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.140(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.601(9) | \( P(A_y = 1, A_{zh} = 1) \) | 0.259(8) | \( (A_{zh}, A_{zh}) \) | -0.481(16) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.917(5) | \( P(A_y = -1, A_{zh} = 1) \) | 0.053(4) | \( P(A_y = 1, A_{zh} = 1) \) | 0.030(3) | \( (A_{zh}, A_{zh}) \) | -0.940(6) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.133(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.585(9) | \( P(A_y = 1, A_{zh} = 1) \) | 0.283(8) | \( (A_{zh}, A_{zh}) \) | -0.435(16) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.138(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.021(3) | \( P(A_y = 1, A_{zh} = 1) \) | 0.841(7) | \( (A_{zh}, A_{zh}) \) | 0.682(13) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.122(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.664(8) | \( P(A_y = 1, A_{zh} = 1) \) | 0.214(7) | \( (A_{zh}, A_{zh}) \) | -0.435(16) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.971(3) | \( P(A_y = -1, A_{zh} = 1) \) | 0.018(3) | \( P(A_y = 1, A_{zh} = 1) \) | 0.010(2) | \( (A_{zh}, A_{zh}) \) | -0.979(4) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.128(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.613(9) | \( P(A_y = 1, A_{zh} = 1) \) | 0.260(8) | \( (A_{zh}, A_{zh}) \) | -0.480(16) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.092(5) | \( P(A_y = -1, A_{zh} = 1) \) | 0.664(8) | \( P(A_y = 1, A_{zh} = 1) \) | 0.245(8) | \( (A_{zh}, A_{zh}) \) | -0.510(16) |
| \( P(A_y = 1, A_{zh} = -1) \) | 0.127(6) | \( P(A_y = -1, A_{zh} = 1) \) | 0.018(2) | \( P(A_y = 1, A_{zh} = 1) \) | 0.855(6) | \( (A_{zh}, A_{zh}) \) | 0.710(13) |

**TABLE III:** The measured joint probabilities for all the compatible pairs \( A_i \) and \( A_j \) in the KS inequality under the input state \( |\psi_i\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle + |2\rangle) \)
### Input State $|\Psi_1\rangle = |0\rangle$

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|--------------------|------------------------|-------------|--------------------|------------------------|-------------|--------------------|------------------------|
| $\langle A_{z_1}\rangle$ | -0.996(2) | -1 | $\langle A_{z_2}, A_{z_2}\rangle$ | -0.997(1) | -1 | $\langle A_{y_1} A_{h_3}\rangle$ | 0.312(17) | 0.333 |
| $\langle A_{z_2}\rangle$ | 0.997(2) | 1 | $\langle A_{z_1}, A_{z_2}\rangle$ | -0.997(2) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.310(16) | 0.333 |
| $\langle A_{z_3}\rangle$ | 0.999(1) | 1 | $\langle A_{z_2}, A_{z_2}\rangle$ | 0.996(2) | 1 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.999(1) | -1 |
| $\langle A_{y_1}\rangle$ | 0.999(1) | 1 | $\langle A_{z_1}, A_{y_1}\rangle$ | -0.999(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.653(15) | -0.667 |
| $\langle A_{y_2}\rangle$ | 0.041(21) | 0 | $\langle A_{z_2} A_{y_2}\rangle$ | 0.040(21) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.651(15) | -0.667 |
| $\langle A_{y_2}\rangle$ | -0.040(21) | 0 | $\langle A_{z_2} A_{y_2}\rangle$ | -0.041(21) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.633(15) | -0.667 |
| $\langle A_{y_2}\rangle$ | -0.006(21) | 0 | $\langle A_{z_2} A_{y_2}\rangle$ | -0.007(21) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.999(1) | -1 |
| $\langle A_{y_2}\rangle$ | 0.007(21) | 0 | $\langle A_{z_2} A_{y_2}\rangle$ | 0.006(21) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.636(15) | -0.667 |
| $\langle A_{h_0}\rangle$ | 0.331(19) | 0.333 | $\langle A_{y_2} A_{y_2}\rangle$ | 0.998(1) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.676(14) | -0.667 |
| $\langle A_{h_1}\rangle$ | 0.362(19) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.339(16) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.624(16) | -0.667 |
| $\langle A_{h_2}\rangle$ | 0.325(18) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.354(17) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.652(15) | -0.667 |
| $\langle A_{h_3}\rangle$ | 0.335(19) | 0.333 | | | | | |

K-S Inequality (2) \[ \sum A_i - \frac{1}{4} \sum_{i,j} A_i A_j = 8.278 \pm 0.041 \] K-S Inequality (3) \[ \sum_{\alpha=0,1,2,3} \langle B_{h\alpha} \rangle = 1.323 \pm 0.018 \]

### Input State $|\Psi_2\rangle = |1\rangle$

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|--------------------|------------------------|-------------|--------------------|------------------------|-------------|--------------------|------------------------|
| $\langle A_{z_1}\rangle$ | 0.999(1) | 1 | $\langle A_{z_2}, A_{z_2}\rangle$ | -0.999(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.647(14) | -0.667 |
| $\langle A_{z_2}\rangle$ | -0.999(1) | -1 | $\langle A_{z_1}, A_{z_2}\rangle$ | 0.999(1) | 1 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.649(14) | -0.667 |
| $\langle A_{z_3}\rangle$ | 0.999(1) | 1 | $\langle A_{z_2}, A_{z_2}\rangle$ | -0.999(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.999(1) | 1 |
| $\langle A_{y_1}\rangle$ | 0.023(19) | 0 | $\langle A_{z_1}, A_{y_1}\rangle$ | 0.021(19) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.340(17) | 0.333 |
| $\langle A_{y_2}\rangle$ | -0.021(19) | 0 | $\langle A_{z_2}, A_{y_2}\rangle$ | -0.023(19) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.339(16) | 0.333 |
| $\langle A_{y_2}\rangle$ | 0.999(1) | 1 | $\langle A_{z_2}, A_{y_2}\rangle$ | -0.999(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.356(16) | 0.333 |
| $\langle A_{y_2}\rangle$ | 0.999(1) | 1 | $\langle A_{z_2}, A_{y_2}\rangle$ | -0.999(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.312(16) | 0.333 |
| $\langle A_{y_2}\rangle$ | 0.015(17) | 0 | $\langle A_{z_2}, A_{y_2}\rangle$ | 0.014(17) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.999(1) | -1 |
| $\langle A_{y_2}\rangle$ | -0.014(17) | 0 | $\langle A_{z_2}, A_{y_2}\rangle$ | -0.015(17) | 0 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.635(14) | -0.667 |
| $\langle A_{h_0}\rangle$ | 0.368(16) | 0.333 | $\langle A_{y_2} A_{y_2}\rangle$ | -0.998(1) | -1 | $\langle A_{y_2} A_{h_2}\rangle$ | 0.666(14) | -0.667 |
| $\langle A_{h_1}\rangle$ | 0.337(17) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.652(13) | -0.667 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.642(14) | -0.667 |
| $\langle A_{h_2}\rangle$ | 0.354(17) | 0.333 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.689(13) | -0.667 | $\langle A_{y_2} A_{h_2}\rangle$ | -0.682(13) | -0.667 |
| $\langle A_{h_3}\rangle$ | 0.304(18) | 0.333 | | | | | |

K-S Inequality (2) \[ \sum A_i - \frac{1}{4} \sum_{i,j} A_i A_j = 8.321 \pm 0.039 \] K-S Inequality (3) \[ \sum_{\alpha=0,1,2,3} \langle B_{h\alpha} \rangle = 1.321 \pm 0.016 \]

**TABLE IV:** The measured expectation values $\langle A_i \rangle$ and the correlations $\langle A_i A_j \rangle$ for all the compatible pairs under input states $|\Psi_1\rangle = |0\rangle$ and $|\Psi_2\rangle = |1\rangle$. 


### Input State $|\Psi_3\rangle = |2\rangle$

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|-------------------|------------------------|-------------|-------------------|------------------------|-------------|-------------------|------------------------|
| $\langle A_{z_1} \rangle$ | 0.999(1) | 1 | $\langle A_{z_2} \rangle$ | 0.984(3) | 1 | $\langle A_{z_3} \rangle$ | -0.983(3) | -1 | $\langle A_{y_1} \rangle$ | -0.009(17) | 0 | $\langle A_{y_2} \rangle$ | 0.020(17) | 0 | $\langle A_{y_3} \rangle$ | -0.021(18) | 0 | $\langle A_{y_4} \rangle$ | 0.037(18) | 0 | $\langle A_{y_5} \rangle$ | 0.982(3) | 1 |
| $\langle A_{y_1} \rangle$ | 0.999(1) | 1 | $\langle A_{y_2} \rangle$ | -0.982(3) | -1 | $\langle A_{y_3} \rangle$ | -0.988(3) | -1 | $\langle A_{y_4} \rangle$ | 0.333(16) | 0.333 | $\langle A_{y_5} \rangle$ | 0.325(17) | 0.333 | $\langle A_{y_6} \rangle$ | 0.361(16) | 0.333 |

K-S Inequality (2) \[ \sum_i \langle A_i \rangle - \frac{1}{4} \sum_{(j,k)} \langle A_i A_j \rangle = 8.313 \pm 0.028 \]

K-S Inequality (3) \[ \sum_{\alpha=0,1,2,3} \langle B_{\alpha \alpha} \rangle = 1.314 \pm 0.017 \]

### Input State $|\Psi_4\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|-------------------|------------------------|-------------|-------------------|------------------------|-------------|-------------------|------------------------|
| $\langle A_{z_1} \rangle$ | 0.018(18) | 0 | $\langle A_{z_2} \rangle$ | -0.999(1) | -1 | $\langle A_{z_3} \rangle$ | 0.528(15) | 0.5 | $\langle A_{y_1} \rangle$ | 0.497(16) | 0.5 | $\langle A_{y_2} \rangle$ | 0.511(15) | 0.5 |
| $\langle A_{y_1} \rangle$ | 0.018(18) | 0 | $\langle A_{y_2} \rangle$ | -0.497(16) | -0.5 | $\langle A_{y_3} \rangle$ | -0.528(16) | -0.5 | $\langle A_{y_4} \rangle$ | 0.462(15) | 0.5 | $\langle A_{y_5} \rangle$ | -0.961(5) | -1 |
| $\langle A_{y_5} \rangle$ | 0.461(5) | 1 | $\langle A_{y_6} \rangle$ | -0.961(5) | -1 | $\langle A_{y_7} \rangle$ | 0.961(5) | 1 | $\langle A_{y_8} \rangle$ | -0.276(17) | -0.333 | $\langle A_{y_9} \rangle$ | -0.977(4) | 1 |
| $\langle A_{y_9} \rangle$ | -0.973(4) | 1 | $\langle A_{y_10} \rangle$ | 0.453(15) | 0.5 | $\langle A_{y_11} \rangle$ | -0.267(17) | -0.333 |

K-S Inequality (2) \[ \sum_i \langle A_i \rangle - \frac{1}{4} \sum_{(j,k)} \langle A_i A_j \rangle = 8.285 \pm 0.036 \]

K-S Inequality (3) \[ \sum_{\alpha=0,1,2,3} \langle B_{\alpha \alpha} \rangle = 1.296 \pm 0.013 \]

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**TABLE V:** The measured expectation values $\langle A_i \rangle$ and the correlations $\langle A_i A_j \rangle$ for all the compatible pairs under input states $|\Psi_3\rangle = |2\rangle$ and $|\Psi_4\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. 
\( |\Psi_5\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |2\rangle) \)

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|--------------|--------------------|------------------------|--------------|--------------------|------------------------|--------------|--------------------|------------------------|
| \( \langle A_{z_1} \rangle \) | 0.019(18) | 0 | \( \langle A_{z_2} A_{z_2} \rangle \) | -0.500(18) | 0 | \( \langle A_{y_1} A_{y_2} \rangle \) | -0.767(12) | -0.833 |
| \( \langle A_{z_2} \rangle \) | 0.991(2) | 1 | \( \langle A_{z_2} A_{z_2} \rangle \) | 0.019(2) | -1 | \( \langle A_{y_1} A_{y_2} \rangle \) | 0.519(15) | 0.5 |
| \( \langle A_{z_3} \rangle \) | -0.010(18) | 0 | \( \langle A_{z_2} A_{z_2} \rangle \) | 0.019(18) | 0 | \( A_{y_1} A_{y_2} \) | -0.986(15) | -1 |
| \( \langle A_{y_1} \rangle \) | 0.520(16) | 0.5 | \( A_{z_3} A_{y_1} \) | 0.514(15) | -0.5 | \( A_{y_1} A_{y_2} \) | -0.992(2) | -1 |
| \( \langle A_{y_2} \rangle \) | 0.514(15) | 0.5 | \( A_{z_3} A_{y_1} \) | -0.520(16) | -0.5 | \( A_{y_1} A_{y_2} \) | -0.360(17) | -0.333 |
| \( \langle A_{y_3} \rangle \) | -0.964(5) | -1 | \( A_{z_3} A_{y_1} \) | 0.978(4) | 1 | \( A_{y_1} A_{y_2} \) | 0.959(5) | -1 |
| \( \langle A_{y_1} \rangle \) | 0.503(15) | 0.5 | \( A_{z_3} A_{y_1} \) | 0.978(4) | 1 | \( A_{y_1} A_{y_2} \) | -0.959(5) | -0.333 |
| \( \langle A_{y_2} \rangle \) | 0.478(17) | 0.5 | \( A_{z_3} A_{y_1} \) | -0.503(15) | -0.5 | \( A_{y_1} A_{y_2} \) | -0.959(5) | -0.333 |
| \( \langle A_{y_3} \rangle \) | 0.514(15) | 0.5 | \( A_{z_3} A_{y_1} \) | 0.978(4) | 1 | \( A_{y_1} A_{y_2} \) | 0.959(5) | -0.333 |
| \( \langle A_{y_3} \rangle \) | 0.514(15) | 0.5 | \( A_{z_3} A_{y_1} \) | 0.978(4) | 1 | \( A_{y_1} A_{y_2} \) | 0.959(5) | -0.333 |

K-S Inequality (2) \( \sum_i \langle A_i \rangle - \frac{1}{4} \sum_{i,j} \langle A_i A_j \rangle = 8.275 \pm 0.041 \)

K-S Inequality (3) \( \sum_{a=0,1,2,3} \langle B_{h_a} \rangle = 1.299 \pm 0.013 \)

\( |\Psi_6\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \)

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|--------------|--------------------|------------------------|--------------|--------------------|------------------------|--------------|--------------------|------------------------|
| \( \langle A_{z_1} \rangle \) | 0.999(1) | 1 | \( \langle A_{z_2} A_{z_2} \rangle \) | 0.009(18) | 0 | \( \langle A_{y_1} A_{y_2} \rangle \) | -0.994(3) | -1 |
| \( \langle A_{z_2} \rangle \) | -0.008(17) | 0 | \( A_{z_2} A_{z_2} \) | 0.008(17) | 0 | \( \langle A_{y_1} A_{y_2} \rangle \) | -0.983(3) | -1 |
| \( \langle A_{z_3} \rangle \) | 0.009(18) | 0 | \( A_{z_2} A_{z_2} \) | -0.999(1) | -1 | \( A_{y_1} A_{y_2} \) | -0.013(18) | 0 |
| \( \langle A_{y_1} \rangle \) | -0.963(5) | -1 | \( A_{z_3} A_{y_1} \) | -0.978(4) | -1 | \( A_{y_1} A_{y_2} \) | -0.793(11) | -0.833 |
| \( \langle A_{y_2} \rangle \) | 0.472(16) | 0.5 | \( A_{z_3} A_{y_1} \) | 0.963(5) | 1 | \( \langle A_{y_1} A_{y_2} \rangle \) | -0.771(11) | -0.833 |
| \( \langle A_{y_3} \rangle \) | 0.515(15) | 0.5 | \( A_{z_3} A_{y_1} \) | -0.515(15) | -0.5 | \( A_{y_1} A_{y_2} \) | 0.470(15) | 0.5 |
| \( \langle A_{y_2} \rangle \) | 0.491(15) | 0.5 | \( A_{z_3} A_{y_1} \) | -0.500(15) | -0.5 | \( A_{y_1} A_{y_2} \) | -0.800(17) | 0 |
| \( \langle A_{y_3} \rangle \) | 0.500(15) | 0.5 | \( A_{z_3} A_{y_1} \) | -0.491(15) | -0.5 | \( A_{y_1} A_{y_2} \) | -0.810(10) | -0.833 |
| \( \langle A_{y_3} \rangle \) | -0.279(16) | -0.333 | \( A_{y_1} A_{z_1} \) | -0.985(3) | -1 | \( A_{y_1} A_{y_2} \) | -0.814(10) | -0.833 |
| \( \langle A_{y_3} \rangle \) | -0.322(17) | -0.333 | \( A_{y_1} A_{z_1} \) | -0.313(17) | -0.333 | \( A_{y_1} A_{y_2} \) | 0.510(16) | 0.5 |
| \( \langle A_{y_3} \rangle \) | 0.977(4) | 1 | \( A_{y_1} A_{z_1} \) | -0.359(16) | -0.333 | \( A_{y_1} A_{y_2} \) | 0.473(15) | 0.5 |
| \( \langle A_{y_3} \rangle \) | 0.973(4) | 1 | \( A_{y_1} A_{z_1} \) | -0.359(16) | -0.333 | \( A_{y_1} A_{y_2} \) | 0.473(15) | 0.5 |

K-S Inequality (2) \( \sum_i \langle A_i \rangle - \frac{1}{4} \sum_{i,j} \langle A_i A_j \rangle = 8.278 \pm 0.039 \)

K-S Inequality (3) \( \sum_{a=0,1,2,3} \langle B_{h_a} \rangle = 1.325 \pm 0.012 \)

| TABLE VI: The measured expectation values \( \langle A_i \rangle \) and the correlations \( \langle A_i A_j \rangle \) for all the compatible pairs under input states | | |
|-------------------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \( |\Psi_5\rangle = (|0\rangle + |2\rangle)/\sqrt{2} \) and \( |\Psi_6\rangle = (|1\rangle + |2\rangle)/\sqrt{2} \). | | | | | | | | | | | | | | |
\[
\text{Input State } \rho_8 = \frac{1}{2} (|0\rangle\langle 0| + |2\rangle\langle 2|)
\]

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|-------------------|------------------------|-------------|-------------------|------------------------|-------------|-------------------|------------------------|
| \langle A_{z_1} \rangle | 0.019(17) | 0 | \langle A_{z_2} A_{z_2} \rangle | 0.014(17) | 0 | \langle A_{y_2} A_{h_2} \rangle | -0.151(17) | -0.167 |
| \langle A_{z_2} \rangle | 0.996(2) | 1 | \langle A_{z_2} A_{z_2} \rangle | -0.996(2) | -1 | \langle A_{y_2} A_{h_2} \rangle | -0.166(16) | -0.167 |
| \langle A_{z_3} \rangle | -0.014(17) | 0 | \langle A_{z_2} A_{z_2} \rangle | -0.019(17) | 0 | \langle A_{y_2} A_{y_2} \rangle | -0.997(1) | -1 |
| \langle A_{y_1} \rangle | 0.477(15) | 0.5 | \langle A_{z_1} A_{y_1} \rangle | -0.494(15) | -0.5 | \langle A_{y_2} A_{h_2} \rangle | -0.611(19) | -0.667 |
| \langle A_{y_2} \rangle | 0.494(15) | 0.5 | \langle A_{z_1} A_{y_1} \rangle | -0.477(15) | -0.5 | \langle A_{y_2} A_{h_2} \rangle | -0.630(17) | -0.667 |
| \langle A_{y_3} \rangle | 0.008(17) | 0 | \langle A_{z_2} A_{y_2} \rangle | 0.005(17) | 0 | \langle A_{y_2} A_{h_2} \rangle | -0.666(17) | -0.667 |
| \langle A_{y_3} \rangle | -0.005(17) | 0 | \langle A_{z_2} A_{y_2} \rangle | -0.008(17) | 0 | \langle A_{y_2} A_{h_2} \rangle | -0.647(18) | -0.667 |
| \langle A_{y_4} \rangle | 0.496(15) | 0.5 | \langle A_{z_2} A_{y_2} \rangle | -0.477(15) | -0.5 | \langle A_{y_2} A_{h_2} \rangle | -0.027(17) | 0 |
| \langle A_{y_3} \rangle | 0.477(15) | 0.5 | \langle A_{z_2} A_{y_2} \rangle | -0.496(15) | -0.5 | \langle A_{y_2} A_{h_2} \rangle | -0.116(17) | -0.167 |
| \langle A_{h_1} \rangle | 0.567(17) | 0.333 | \langle A_{y_1} A_{y_1} \rangle | -0.503(17) | 0 | \langle A_{y_2} A_{h_2} \rangle | -0.135(16) | -0.167 |
| \langle A_{h_2} \rangle | 0.373(16) | 0.333 | \langle A_{y_1} A_{y_1} \rangle | -0.164(17) | -0.167 | \langle A_{y_2} A_{h_2} \rangle | -0.124(17) | -0.167 |
| \langle A_{h_3} \rangle | 0.343(16) | 0.333 | \langle A_{y_1} A_{h_1} \rangle | -0.187(16) | -0.167 | \langle A_{y_2} A_{h_2} \rangle | -0.177(17) | -0.167 |
| \langle A_{h_3} \rangle | 0.318(16) | 0.333 |

K-S Inequality (2) \[\sum_i \langle A_i \rangle - \frac{1}{4} \sum_{(i,j)} \langle A_i A_j \rangle = 8.236 \pm 0.043\]

K-S Inequality (3) \[\sum_{a=1,2,3} \langle B_{h_a} \rangle = 1.299 \pm 0.016\]

| Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction | Observables | Experimental value | Theoretical prediction |
|-------------|-------------------|------------------------|-------------|-------------------|------------------------|-------------|-------------------|------------------------|
| \langle A_{z_1} \rangle | 0.316(19) | 0.333 | \langle A_{z_2} A_{z_2} \rangle | -0.333(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.350(19) | -0.333 |
| \langle A_{z_2} \rangle | 0.352(17) | 0.333 | \langle A_{z_2} A_{z_2} \rangle | -0.352(17) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.314(18) | -0.333 |
| \langle A_{z_3} \rangle | 0.333(18) | 0.333 | \langle A_{z_2} A_{z_2} \rangle | -0.316(19) | -0.333 | \langle A_{y_2} A_{y_2} \rangle | -0.388(18) | -0.333 |
| \langle A_{y_1} \rangle | 0.333(18) | 0.333 | \langle A_{z_2} A_{y_1} \rangle | -0.304(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.305(17) | -0.333 |
| \langle A_{y_2} \rangle | 0.304(18) | 0.333 | \langle A_{z_2} A_{y_1} \rangle | -0.333(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.315(18) | -0.333 |
| \langle A_{y_3} \rangle | 0.309(18) | 0.333 | \langle A_{z_2} A_{y_1} \rangle | -0.303(19) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.283(18) | -0.333 |
| \langle A_{y_4} \rangle | 0.303(18) | 0.333 | \langle A_{z_2} A_{y_2} \rangle | -0.309(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.348(18) | -0.333 |
| \langle A_{y_5} \rangle | 0.356(19) | 0.333 | \langle A_{z_2} A_{y_2} \rangle | -0.337(18) | -0.333 | \langle A_{y_2} A_{y_2} \rangle | -0.307(18) | -0.333 |
| \langle A_{y_3} \rangle | 0.337(18) | 0.333 | \langle A_{z_2} A_{y_2} \rangle | -0.356(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.310(18) | -0.333 |
| \langle A_{h_1} \rangle | 0.365(18) | 0.333 | \langle A_{y_1} A_{y_1} \rangle | -0.363(19) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.293(18) | -0.333 |
| \langle A_{h_2} \rangle | 0.355(18) | 0.333 | \langle A_{y_1} A_{h_1} \rangle | -0.355(17) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.326(17) | -0.333 |
| \langle A_{h_3} \rangle | 0.281(18) | 0.333 | \langle A_{y_1} A_{h_1} \rangle | -0.357(18) | -0.333 | \langle A_{y_2} A_{h_2} \rangle | -0.294(17) | -0.333 |
| \langle A_{h_3} \rangle | 0.383(17) | 0.333 |

K-S Inequality (2) \[\sum_i \langle A_i \rangle - \frac{1}{4} \sum_{(i,j)} \langle A_i A_j \rangle = 8.248 \pm 0.050\]

K-S Inequality (3) \[\sum_{a=1,2,3} \langle B_{h_a} \rangle = 1.309 \pm 0.018\]

TABLE VII: The measured expectation values \(\langle A_i \rangle\) and the correlations \(\langle A_i A_j \rangle\) for all the compatible pairs under input states \(\rho_8 = (|0\rangle\langle 0| + |2\rangle\langle 2|)/2\) and \(\rho_9 = (|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)/3\).