Dynamics of a quantum measurement

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Abstract

We work out an exactly solvable hamiltonian model which retains all the features of realistic quantum measurements. In order to use an interaction process involving a system and an apparatus as a measurement, it is necessary that the apparatus is macroscopic. This implies to treat it with quantum statistical mechanics. The relevant time scales of the process are exhibited. It begins with a very rapid disappearance of the off-diagonal blocks of the overall density matrix of the tested system and the apparatus. Possible recurrences are hindered by the large size of the latter. On a much larger time scale the apparatus registers the outcome: Correlations are established between the final values of the pointer and the initial diagonal blocks of the density matrix of the tested system. We thus derive Born’s rule and von Neumann’s reduction of the state from the dynamical process.

Key words: quantum measurements, quantum dynamics

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1 Properties of quantum measurement processes.

The interpretation of quantum mechanics is tightly connected with the understanding of quantum measurements. The textbooks and most articles devoted to this subject focus on the initial and final states of the system S and the apparatus A, without describing in detail the coupled dynamics of S and A during the measurement process [1,2]. Our purpose is to study such a dynamics. We shall present a model which displays all the features of a realistic quantum measurement and is exactly solvable [3]. This will allow us to explain...
how these features arise from the microscopic equations of motion. We have therefore to solve a problem of quantum statistical mechanics for which it is crucial to take into account both the microscopic nature of the object S and the macroscopic nature of the apparatus A.

A quantum measurement shares with classical measurements several general properties. It is an *experiment* during which the system S to be tested and the apparatus A, separately prepared at an initial time $t_i = 0$, *interact*. This dynamical process creates correlations between the state of A at the final time $t_f$ and the initial state of S. Information can thereby be gained on S through observation, at (or after) the time $t_f$, of some pointer variable belonging to A. Provided the value of the pointer variable is *registered*, the rôle of the observer is eliminated.

Quantum measurements differ from classical ones in two respects. On the one hand, quantum mechanics is an irreducibly *probabilistic* theory. What is called “state of a system”, whether it is represented by a wavefunction, a ket or a density operator, refers to a *statistical ensemble* of systems, all prepared under the same conditions as the system in hand. This object gathers our whole information about the preparation of any system belonging to this ensemble. However, in contrast to a state in classical physics, a quantum state necessarily involves statistical fluctuations due to the non-commutation of the observables and is therefore akin to a probability distribution. Indeed, if two physical quantities are represented by observables $A$ and $B$ with commutator $[A, B] = 2iC$, they must statistically fluctuate in agreement with Heisenberg’s inequality $\Delta A \Delta B \geq |\langle C \rangle|$ if the expectation value $\langle C \rangle$ is non-zero in the considered state. Accordingly, a quantum measurement must in general involve statistical fluctuations. In classical physics, probabilities may occur in measurements due to uncertainties in the preparation of the initial state and to measurement errors; however one can imagine more and more precise preparations, and more and more precise measurements, so that nothing forbids to find the outcome within a negligible error. In a quantum measurement, even under the most perfect conditions, the outcome is always probabilistic.

Consider a measurement of the observable $\hat{s} = \sum_i s_i \hat{\Pi}_i$ of S; we denote as $s_i$ its eigenvalues, as $\hat{\Pi}_i$ its eigenprojections in the Hilbert space of S, and as $\hat{A}$ the pointer observable of the apparatus A which is coupled to $\hat{s}$. (For a non-degenerate eigenvalue of $\hat{s}$ with eigenfunction $|\psi_i\rangle$, $\hat{\Pi}_i = |\psi_i\rangle\langle \psi_i|$. ) If the system is prepared initially in an eigenstate of $\hat{s}$ corresponding to the eigenvalue $s_i$, the pointer variable takes the well defined value $A_i$. However, for an arbitrary initial state of S represented at the initial time by the density operator $r(0)$, different runs of the experiment, performed on systems identically prepared in this state $r(0)$, may yield different outcomes $A_i$. The probability $p_i$ of finding $A_i$ at the time $t_f$, for the statistical ensemble described by $r(0)$, is given by *Born’s rule*

$$p_i = \text{Tr}_S \hat{\Pi}_i r(0),$$ (1.1)
which exhibits the irreducibly probabilistic nature of the measurement. (The situation is not different for a pure state \( r(0) = |\phi\rangle\langle\phi| \), in which case \( p_i = \langle \phi|\hat{\Pi}_i|\phi\rangle \).)

On the other hand, the *perturbation* of \( S \) induced by a measurement cannot be neglected in quantum physics, whereas nothing prevents in classical physics to make it smaller and smaller. Consider first the apparatus. If we include in \( A \) the registration device, any measurement, whether classical or quantal, must perturb \( A \) so as to be informative. This perturbation, when induced by a microscopic system \( S \), should be sufficiently strong so as to let \( A \) undergo a macroscopic change. In many real measurements, this interaction process drastically modifies the system \( S \) itself and may even destroy it. We wish to focus on *ideal measurements*, those which perturb \( S \) as little as possible. This is achieved by preparing \( A \) at the initial time in a metastable state, with density operator \( \mathcal{R}(0) \). During the measurement, the interaction with \( S \) triggers \( A \) in the same way as a small source, leading it towards one or another among several possible stable states \( \mathcal{R}_i \), each characterized by a value \( A_i \) of the pointer variable. The fact that these possible final states are exclusive is expressed by \( \text{Tr} \mathcal{R}_i \mathcal{R}_j = \delta_{ij} \), which for positive matrices \( \mathcal{R}_i \) and \( \mathcal{R}_j \) implies

\[
\mathcal{R}_i \mathcal{R}_j = 0, \quad \text{for} \quad i \neq j.
\]

Such a property holds, in particular, if \( A \) is a large system with spontaneously broken invariance, \( \mathcal{R}_i \) being then an equilibrium state characterized by the value \( \text{Tr}(\hat{A} \mathcal{R}_i) = A_i \) of its order parameter.

In an ideal measurement, the occurrence of \( A_i \) is correlated with the fact that \( S \) lies at the time \( t_f \) in an eigenstate of \( \hat{s} \) associated with the eigenvalue \( s_i \). More precisely, *von Neumann’s reduction* expresses that the initial joint density operator \( \mathcal{D}(0) = r(0) \otimes \mathcal{R}(0) \) of the compound system \( S + A \) is transformed at the final time \( t_f \) into

\[
\mathcal{D}(0) \mapsto \mathcal{D}(t_f) = \sum_i \left[ \hat{\Pi}_i r(0) \hat{\Pi}_i \right] \otimes \mathcal{R}_i . \tag{1.2}
\]

Full memory is kept of the diagonal blocks \( \hat{\Pi}_i r(0) \hat{\Pi}_i \) of the initial state \( r(0) \) of \( S \). This means that the statistics of the *any* observable which *commutes* with \( \hat{s} \) is left unchanged by the measurement process. However, the initial information carried by the off-diagonal blocks of \( r(0) \) is lost in spite of the ideal nature of the measurement.

The expression (1.2) encompasses (1.1), which is obtained by taking the expectation value of \( \hat{\Pi}_i \) over the state \( \mathcal{D}(t_f) \). It is more detailed since it describes the overall properties of the final state, but it applies only to the ideal measurements for which the perturbation of \( S \) is the weakest. It exhibits a correlation
between the possible final states of $A$ and those of $S$, which is expressed by

$$\langle \hat{\Pi}_i (\hat{A} - A_i)^2 \rangle = 0, \quad (1.3)$$

meaning that $\hat{A}$ takes the value $A_i$, when $\hat{s}$ equals $s_i$.

From the form (1.2) of $D(t_f)$ we can infer that the observation of the pointer variable allows us to split the statistical ensemble initially described by $D(0)$ into a set of subensembles, each characterized by a value $A_i$ of the pointer variable. In each subensemble, the state of $S + A$ is factorized, with $S$ represented by the normalized density operator $\hat{\Pi}_r(0) \hat{\Pi}_r/p_i$ and $A$ by $\mathcal{R}_i$. Splitting the ensemble into subensembles labeled by $i$ thus decorrelates $S$ from $A$. Selecting one subensemble indexed by $A_i$ constitutes a preparation of $S$ in the projected state $\hat{\Pi}_r(0) \hat{\Pi}_r/p_i$, which will characterize for future experiments the statistics of the subensemble. A further ideal measurement of $\hat{s}$ will then leave this state unchanged while providing a perfect prediction of the outcome $A_i$.

It is therefore crucial to loose the information included in the off-diagonal terms $\hat{\Pi}_r(0) \hat{\Pi}_j (i \neq j)$ present in the initial state $r(0)$ to achieve the measurement of $\hat{s}$: This is a price to pay in order to gain information about the eigenvalues $s_i$ of the observable $\hat{s}$. It is also crucial to register $A_i$ so as to filter the final ensemble into subensembles: This is needed to determine the probabilities $p_i$, proportional to the number of counts of $A_i$, and to prepare $S$ in an eigenstate of $\hat{s}$ if the measurement is ideal.

2 Irreversibility of measurements

Due to the above two unavoidable features, the transformation from $D(0)$ to $D(t_f)$ is irreversible. This property is exhibited, in particular, by the entropy balance. The von Neumann entropy

$$S[D(t_f)] = S \left[ \sum_i \hat{\Pi}_r(0) \hat{\Pi}_i \right] + \sum_i p_i S[\mathcal{R}_i] \quad (2.1)$$
of the final state \(^2\) is indeed larger than that

\[
S[D(0)] = S[r(0)] + S[R(0)]
\]  \hspace{1cm} (2.2)

of the initial state for two reasons. On the one hand, when the density operator \(r(0)\) includes off-diagonal blocks \(\bar{\Pi}_i r(0) \bar{\Pi}_j (i \neq j)\), their truncation raises the entropy. On the other hand, a robust registration requires that the possible final states \(R_i\) of \(A\) are more stable than the initial state \(R_i(0)\), so that their entropy is larger. The latter effect dominates because the apparatus is necessarily macroscopic, as discussed below.

Another manifestation of the irreversibility of the process is the fact that two different initial states of \(S\) which have the same diagonal blocks lead to the same final state \(D(t_f)\).

We aim at explaining how the state of \(S + A\) switches from \(D(0)\) to \(D(t_f)\) during the time lapse \(t_f\). We wish to rely only on the basic laws of quantum mechanics, applied to the system \(S + A\). Namely, (i) the physical quantities are represented by hermitian operators \(\hat{X}\) acting in the Hilbert space of \(S + A\); (ii) a state of the system \(S + A\) (or more precisely of the statistical ensemble to which this system belongs) is represented at each time by a density operator \(D\), and this state implements the correspondence between any observable and its expectation value as \(\langle \hat{X} \rangle = \text{Tr} \hat{X} D\); (iii) since \(S + A\) is isolated, its joint density operator \(D\) evolves according to the Liouville – von Neumann equation

\[
 i\hbar \frac{dD}{dt} = [\hat{H}, D],
\]  \hspace{1cm} (2.3)

generated by the Hamiltonian \(\hat{H}\). This equation is expected to govern the dynamical process leading from \(D(0)\) to \(D(t_f)\), which we wish to analyze. One cannot hope to solve this question in the general case of real measurements, and this led many authors to consider models [8]. Here we work out a model that retains all the features of realistic measurements, so as to understand how the process can be interpreted as a measurement.

We have, however, to cope with the celebrated measurement problem. The equation of motion (2.3) generates for \(D(t)\) a unitary evolution which con-
serves von Neumann’s entropy $S = -\text{Tr} \mathcal{D} \ln \mathcal{D}$ of $S + A$, in contradiction with the consequence (2.1-2) of von Neumann’s surmise (1.2).

This problem is akin to the paradox of irreversibility in statistical mechanics, which relies on the contradiction between the reversibility of the microscopic evolution of a system and the irreversibility of its macroscopic behaviour. The solution of this paradox is based as well known on the large number $P$ of particles of the system. Remember that $\mathcal{D}(t)$ represents at each time our whole statistical information about this system, including the correlations between any number of particles. We then associate with $\mathcal{D}$ for given $p < P$ a reduced coarse-grained density operator $\tilde{\mathcal{D}}^{(p)}$, which is equivalent to $\mathcal{D}$ as regards all the “simple” observables, that is, all the expectation values of quantities correlating at most $p$ particles, but which is completely random as regards the higher order correlations [5]. This density operator $\tilde{\mathcal{D}}^{(p)}$ should contain no more amount of information than the minimum required to account for the “simple” variables; its entropy is therefore larger than that of $\mathcal{D}$ and may increase. Although $\mathcal{D}$ and $\tilde{\mathcal{D}}^{(p)}$ are indistinguishable for all practical purposes since high order correlations are inobservable, their entropies can differ significantly for $P \gg 1$ because these irrelevant correlations are numerous. As the time $t$ flows, the interactions build up in $\mathcal{D}(t)$ through the evolution (2.3) correlations between larger and larger numbers of particles. For nearly all physical systems and models, provided $1 \ll p \ll P$, the high order correlations are so intricate that they do not affect the subsequent evolution of the “simple” variables, at least over any reasonable time scale. It is then legitimate to replace $\mathcal{D}(t)$ by its reduction $\tilde{\mathcal{D}}^{(p)}(t)$, which evolves irreversibly. This irreversibility is interpreted as a leak of information in $\mathcal{D}(t)$ from the “simple” variables towards the correlations of more than $p$ particles. Such correlations are ineffective, and the order that they carry cannot return in practice to the “simple” variables, because recurrence times are inaccessibly large even for a few tens of particles. This argument can be made mathematically rigorous in solvable models by letting first $P$, then $p$ tend to infinity [5].

We shall propose a similar solution for the measurement problem. It will be essential to deal with the microscopic system $S$ exactly. However, here again, the apparatus $A$ will be a macroscopic object, with $P \gg 1$ degrees of freedom. For finite $P$ we shall perform approximations, which amount to the above-mentioned replacement of $\mathcal{D}$ by $\tilde{\mathcal{D}}^{(p)}$, and which are necessary to account for irreversibility. It can be shown [3] that the errors thus introduced in the solution of (2.3) become negligible in the limit $P \rightarrow \infty$ as regards all the observables involving a finite number of degrees of freedom. The expression (1.2) that we shall derive below for the final state $\mathcal{D}(t_f)$ will actually be valid for all components of $\mathcal{D}$, except for those describing the irrelevant correlations between an extremely large number of degrees of freedom. Discarding those imperceptible but numerous correlations is legitimate, although they are responsible for the difference between the values of the entropies (2.1) and (2.2).
The model

In our model, the system $S$ is the simplest possible: a spin $\frac{1}{2}$. The observable $\hat{s} = \hat{s}_z$ to be measured is the $z$-component of this spin, with eigenvalues $s_i^\uparrow = +1$, $s_i^\downarrow = -1$. In the $i = \uparrow\downarrow$ basis, the initial state of $S$ is represented by the $2 \times 2$ density matrix $\rho (0)$.

The apparatus $A$ simulates a magnetic dot, a collection of spins in a small solid grain. It can be analyzed into two elements, the magnet $M$ and the phonon bath $B$. The magnet $M$ consists of $N \gg 1$ spins with Pauli operators $\hat{\sigma}_a^{(n)}$ ($a = x,y,z$ ; $n = 1,\ldots,N$). These spins interact according to a Curie–Weiss type of Hamiltonian

$$\hat{H}_M = -\frac{1}{4} J N \hat{m}^4,$$  

(3.1)

where $\hat{m} \equiv \frac{1}{N} \sum_n \hat{\sigma}_z^{(n)}$ is the magnetization per spin in the $z$-direction. This is a model for superexchange interactions in metamagnets, suited for a small anisotropic grain.

This part of the apparatus will describe the pointer. The choice of (3.1) as its Hamiltonian relies on the following remarks. The measurement process will be governed by a coupling between the system $S$ and the apparatus $A = M + B$, which we represent as the spin-spin interaction

$$\hat{H}_{SA} = -g \hat{s}_z \sum_n \hat{\sigma}_z^{(n)} \hat{s}_z = -g \hat{s}_z N \hat{m}.$$  

(3.2)

Seen from the viewpoint of $M$, (3.2) looks like the effect of an operator-valued magnetic field along $z$.

As a preliminary step, let us consider the equilibrium of the system $M$ alone at the temperature $T$. If $M$ is submitted to an external magnetic field, its Hamiltonian is the sum of (3.1) and (3.2) where $\hat{s}_z$ is replaced by $s_i = +1$ or $-1$ for $i = \uparrow$ or $\downarrow$. In the limit $N \to \infty$ the static mean-field approach is exact, a property which suggests that the dynamics of a model including $S$ and $A = M + B$ could also be solved exactly. The equilibrium of $M$ is found as well known by looking for the minimum of the free energy per spin

$$F_i (m) = -s_i g m - \frac{1}{4} J m^4 - T S (m),$$  

(3.3)

$$S (m) = -\frac{1 + m}{2} \ln \frac{1 + m}{2} - \frac{1 - m}{2} \ln \frac{1 - m}{2},$$  

(3.4)

expressed in terms of the order parameter $m$. The index $i$ refers to the direction $s_i = \pm 1$ of the field. For $g \neq 0$, there is a single true equilibrium state
(a property that we shall use in section 7), which corresponds to the absolute minimum of (3.3). It is reached for the solution with largest $|m|$ of the equations

$$m_i = \tanh \left( \frac{h_i}{T} \right), \quad h_i = \pm g + Jm^3$$

($m_+ > 0, m_- < 0$). For $g = 0$, below the Curie temperature $T = 0.36$ J, the invariance $m \leftrightarrow -m$ is spontaneously broken: there are two stable ferromagnetic states, with $m_+ = -m_- = m^f$ very close to 1; for $T \ll J$, $1 - m^f \sim 2e^{-J/T}$. The paramagnetic state $m^p = 0$ is still a local minimum of (3.3); this is why we chose a quartic rather than a quadratic interaction. As $g$ increases, this paramagnetic state is shifted, with $m^p \sim g/T$, but it remains metastable. It disappears only when $g$ becomes sufficiently large, in which case the single minimum of $F^f (m^f)$ is the ferromagnetic one; the limiting value $g_c$ is found as function of $J$ and $T$ by eliminating $m$ from

$$m = \tanh \left( \frac{g_c}{T} + Jm^3/T \right),$$

$$2m^2 = 1 - \sqrt{1 - 4T/3J},$$

which yields $m^2 \sim \frac{T}{3J}, g^2_c \sim \frac{4T^3}{27J}$ when $T \ll J$.

These features agree with what we expect for a measuring apparatus. As the initial metastable state of A we can take $\mathcal{R} (0) = R_M (0) \otimes R_B$ where $R_M (0) = 1/2^N$ is the paramagnetic state of the magnet, with $m = \langle \hat{m} \rangle = 0$, and $R_B$ is the equilibrium density operator $R_B \propto e^{-\hat{H}_0/T}$ of the phonon bath. This state has a lifetime long as an exponential of $N$ for $N \gg 1$ since (3.3) is a minimum, and can easily be prepared by cooling A before the measurement. The transition towards either one or the other of the two stable ferromagnetic states $\mathcal{R}_i = R_{Mi} \otimes R_B$ ($i = \uparrow$ or $\downarrow$) is expected to be triggered by the interaction (3.2), depending on $r (0)$. The macroscopic size of A allows such a relaxation to be irreversible, and the stability of the two states $\mathcal{R}_i$ ensures a robust and permanent registration. The fact that, in each state $R_{Mi}$, $m$ has statistical fluctuations around $\pm m^f$ small as $1/\sqrt{N}$ ensures a clear distinction between the two possible outcomes $\pm m^f$ since $m^f$ is close to $\pm 1$. Thus $\langle \hat{m} \rangle$, which in the final state may take two values in correspondence with the eigenvalues of $\hat{s}_z$, is a good candidate for a pointer variable. Finally the fact that M displays for $N \gg 1$ a phase transition with broken symmetry makes the two outcomes equally probable a priori, and thus prevents bias of the apparatus.

The spin system M, although large, cannot reach equilibrium by itself since its Hamiltonian is too simple. This will be achieved owing to the phonon bath B, described by an independent phonon Hamiltonian $\hat{H}_B$ with a dense, quasi continuous spectrum and a Debye frequency cutoff $\Gamma$. A weak interaction ($\gamma \ll 1$)

$$\hat{H}_{MB} = \sqrt{\gamma} \sum_{n,a} \hat{s}_a^{(n)} \hat{B}_a^{(n)}$$

(3.7)
involving phonon operators $\hat{B}^{(n)}_a (a = x, y, z; n = 1, \ldots N)$ is able to thermalize the spins after some rather long delay of order $\hbar/\gamma T$. Since $\gamma$ is small and the bath large, the correlations between $M$ and $B$ are negligible in the initial state $\mathcal{R} (0)$, and the marginal state of $B$, initially at equilibrium $R_B \propto e^{-\hat{H}_B/T}$, is not deeply affected by the evolution.

Altogether the full Hamiltonian of $S+A$ is given by $\hat{H} = \hat{H}_{SA} + \hat{H}_M + \hat{H}_B + \hat{H}_{MB}$. It commutes with the measured observable $\hat{s}_z$; this is a standard requirement [6], which ensures that the measured quantity does not change during the measurement process. In section 6 we shall replace (3.2) by the more general spin-apparatus interaction

$$\hat{H}'_{SA} = - \sum_n g_n \hat{s}_z \hat{\sigma}^{(n)}_z,$$

for which the coupling constants $g_n$ between the measured spin $\hat{s}_z$ and the various apparatus spins $\hat{\sigma}^{(n)}_z$ are not the same, due to the different locations of the spins of $M$ in the magnetic grain. We assume the deviation $\delta g^2 = N^{-1} \sum_n (g_n - g)^2$ of the couplings $g_n$ around their average $g$ to be small, $\delta g \ll g$.

4 Dynamical equations

Our purpose is to solve the equation of motion (2.3), starting from the initial condition $\mathcal{D} (0) = r (0) \otimes R_M (0) \otimes R_B$. The large value of $N$ and the form of $\hat{H}_M$ and $\hat{H}_{SA}$ suggest us to rely on a time-dependent mean-field approach, which we expect to become exact as $N \to \infty$. We only sketch below the main results and the main steps of the derivation; details, proofs and discussions will be found in [4].

We first note that owing to the conservation of $\hat{s}_z$, the four blocks of $\mathcal{D} (t)$ labeled by the eigenvalues $i = \uparrow$ or $\downarrow$ of $\hat{s}_z$, initially proportional to $r_{\uparrow\uparrow} (0)$, $r_{\uparrow\downarrow} (0)$, $r_{\downarrow\uparrow} (0)$, $r_{\downarrow\downarrow} (0)$, evolve independently. The equations of motion in each sector have in the Hilbert space of $A$ the form

$$i\hbar \frac{d\mathcal{D}_{ij}}{dt} = - g N (s_i \hat{m}\mathcal{D}_{ij} - \mathcal{D}_{ij} s_j \hat{m}) + \left[ \hat{H}_A, \mathcal{D}_{ij} \right], \quad (4.1)$$

where $\hat{H}_A = \hat{H}_M + \hat{H}_B + \hat{H}_{MB}$. Our mean-field approximation consists in replacing at each time $\hat{H}_M$ in each of the four equations (4.1) by $m_{ij}^4 + 4m_{ij}^3 (\hat{m} - m_{ij})$, and using the self-consistency condition

$$m_{ij} = \frac{\text{Tr}_A \hat{m} |\mathcal{D}_{ij}|}{\text{Tr}_A |\mathcal{D}_{ij}|}, \quad |\mathcal{D}_{ij}| \equiv \sqrt{\mathcal{D}_{ij} \mathcal{D}_{ij}^\dagger}. \quad (4.2)$$
This approach differs from standard mean-field approaches through the occurrence of different parameters $m_{ij}$ in the four sectors. They have only the status of mathematical tools, and their simultaneous occurrence in the equations of motion shows that they are not directly related to the expectation value of $\hat{m}$ in the state $\mathcal{D}(t)$, contrary to what happens in usual mean-field theories such as (3.5). Moreover the unconventional form of (4.2) is related to the non-hermiticity of $\mathcal{D}_{ij} = \mathcal{D}_{ji}^\dagger$. The present approach can be justified by showing that the corrections are negligible in the large $N$ limit.

The phonon variables are eliminated through replacement of (4.1) by an equation for the partial trace $D_{ij}(t) = \text{Tr}_\mathcal{B} \mathcal{D}_{ij}(t)$, an operator in the Hilbert space of $\mathcal{M}$. This is achieved, as usual in the limit of a very large bath $\mathcal{B}$ weakly coupled ($\gamma \ll 1$) with $\mathcal{M}$, by noting that the bath occurs only through the memory kernel $\text{Tr}_\mathcal{B} \hat{B}_a \hat{B}_b \hat{B}_c \hat{B}_d (t, t')$. For simplicity we choose the bath Hamiltonian $\hat{H}_\mathcal{B}$ in such a way that this kernel has the form $\delta_{a,b} \delta_{n,m} K(t - t')$, with a quasi-ohmic spectrum:

$$K(t) = \hbar^2 \int_{-\infty}^{+\infty} \frac{d\omega}{16\pi} e^{i\omega t} \left[ \coth \left( \frac{\hbar\omega}{2T} \right) - 1 \right] e^{-|\omega|/\Gamma}.$$  

The memory time $\hbar/T$ of $\mathcal{B}$ is taken much smaller than the equilibration time $\hbar/\gamma T$ of $\mathcal{M}$.

According to the mean-field surmise, the operator $D_{ij}$ for each pair $ij$ can be factorized as

$$D_{ij}(t) = r_{ij}(0) \times \rho_{ij}^{(1)}(t) \otimes \cdots \otimes \rho_{ij}^{N}(t) ,$$

where each factor

$$\rho_{ij}^{(n)} = \frac{1}{2} \left( \zeta_{0,ij} + \sum_a \zeta_{a,ij} \hat{\sigma}_a^{(n)} \right) = \rho_{ji}^{(n)}$$

is an operator in the 2-dimensional Hilbert space of the spin $n$ of $\mathcal{M}$, parametrized by the 4 functions $\zeta_{0,ij}$, $\zeta_{a,ij}$ ($a = x, y, z$) of time. The elimination of the bath variables then provides for these functions equations of motion which depend on $K(t)$ in a rather complicated way. We write them here in a simplified form, which is valid for the time scales over which they will be relevant, namely

$$\dot{\zeta}_{0,\uparrow\downarrow} = \frac{2ig}{\hbar} \zeta_{z,\uparrow\downarrow},$$

$$\dot{\zeta}_{z,\uparrow\downarrow} = \frac{2ig}{\hbar} \left( 1 + \frac{\gamma \Gamma^2 t^2}{2\pi} \right) \zeta_{0,\uparrow\downarrow} - \frac{\gamma \Gamma^2 t}{\pi} \zeta_{z,\uparrow\downarrow},$$

for short times ($t \ll 1/\Gamma$), and

$$\dot{\zeta}_{0,ii} = 0 , \quad \dot{m}_i = \frac{\gamma h_i}{\hbar} \left( 1 - \frac{m_i}{\text{tanh} \frac{h_i}{T}} \right),$$

for short times ($t \ll 1/\Gamma$), and
for larger times \((t \gg \hbar/T)\). The parameters \(m_i \equiv \zeta_{a,ii}/\zeta_{0,ii}\) introduced in (4.7) are the time-dependent magnetizations in the two diagonal sectors and we denote here as \(h_i = gs_i + Jm_i^3\) with \(s_i = \pm 1\) the associated effective fields. The parameters \(m_{\uparrow\downarrow}^1\) and \(m_{\downarrow\uparrow}^1\) are found to vanish, as well as \(\zeta_{x,ij}\) and \(\zeta_{y,ij}\). The initial paramagnetic conditions are \(\zeta_{0,ij} = 1, \zeta_{a,ij} = 0\). It can also be shown that in the limit of a large bath B and of a weak coupling \(\gamma\), the state of B is hardly affected and that the correlations between B and M are negligible.

We stress that, although our equations (4.6) and (4.7) have been obtained through approximations, they become exact in the limit of a large apparatus. On the one hand, the large value on \(N\) together with the long range of the interaction \(\hat{H}_M\) ensure the validity of the mean-field approach and the existence of different phases for M. On the other hand, the quasi-continuity of the phonon spectrum entails that the energy exchanges between M and B dissipate entropy on any reasonable time lapse.

# 5 Initial collapse of S

Let us first solve eq. (4.6) for \(\mathcal{D}_{\uparrow\downarrow} (t)\) and for \(\mathcal{D}_{\downarrow\uparrow} (t) = \mathcal{D}_{\uparrow\downarrow} (t)\)†. Over very short times \((t < \hbar/g)\), the evolution is governed only by the interaction \(\hat{H}_{SA}\), and we have

\[
\zeta_{0,\uparrow\downarrow} = \cos \frac{2gt}{\hbar}, \quad \zeta_{z,\uparrow\downarrow} = i \sin \frac{2gt}{\hbar}.
\]  

(5.1)

Hence, the off-diagonal block \(\mathcal{D}_{\uparrow\downarrow} \) of \(\mathcal{D} \) behaves on this time scale as

\[
\mathcal{D}_{\uparrow\downarrow} = \frac{r_{\uparrow\downarrow} (0)}{2N} D_B \otimes \prod_{n=1}^{N} \left( \cos \frac{2gt}{\hbar} + i \sin \frac{2gt}{\hbar} \hat{\sigma}_z^{(n)} \right).
\]  

(5.2)

As a consequence, the element \(r_{\uparrow\downarrow} (t)\) of the marginal density operator of S rapidly decreases as

\[
r_{\uparrow\downarrow} (t) = r_{\uparrow\downarrow} (0) \cos^N \frac{2gt}{\hbar} \sim r_{\uparrow\downarrow} (0) e^{-t^2/\tau_{\text{red}}^2},
\]  

(5.3)

over a reduction time

\[
\tau_{\text{red}} = \frac{\hbar}{\sqrt{2Ng}}.
\]  

(5.4)

which will be the shortest of all the characteristic times that we shall encounter in the dynamics of S + A. This decay, which describes a very rapid collapse of the off-diagonal terms of \(r(t)\), leads to negligible values of (5.3), small as \(e^{-N}\) for \(t = \hbar/g\sqrt{2}\).

From the viewpoint of the system S, such a behaviour is reminiscent of a decoherence. However, instead of the factor \(g\) occurring in the denominator of
the reduction time (5.4), environment induced decoherence times involve the temperature of the environment. More crucially, whereas a usual decoherence takes place in a given basis, the basis in which the off-diagonal components disappear is determined here by the observable which is being measured. A rotation of the apparatus, which is anisotropic, modifies this basis in a controlled way.

Eq. (5.2) shows that the disappearance of the off-diagonal elements of $r(t)$, that is, of $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$, is accompanied by the creation of correlations between $\hat{s}_x$ (or $\hat{s}_y$) and an arbitrary number of $\hat{s}_z^{(n)}$ operators. As usual in relaxation processes, the relaxation of $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ is compensated for by a transfer of order towards more and more complicated observables coupling S and M. The number of these correlations is extremely large, but each one is very small; indeed, if they involve a number $p \ll N$ of spins of the apparatus, their value $r_{\uparrow \downarrow}^{(0)} e^{-t^4/\tau_{\text{red}}^4} (2gt/\hbar)^p$ is hindered either due to the exponential or due to the small value of $t$ in the range $\tau_{\text{red}}$. This is an example of transfer of order towards many-spin correlations (of order $N$) that we alluded to when recalling the mechanism of irreversibility.

6 Suppression of recurrences

The expression (5.2) for $D_{\uparrow \downarrow} (t)$, valid for the very beginning $t < \hbar/g$ of the measurement process, would exhibit a periodic structure, with period $t = \pi \hbar/g$, if we extrapolated it towards larger times. According to (5.3), $r_{\uparrow \downarrow} (t)$, which nearly vanishes after a time $\tau_{\text{red}}$, would present later on, for integer values of $2gt/\pi \hbar$, a sequence of narrow gaussian peaks, all with height $r_{\uparrow \downarrow} (0)$. The collapse would then not be irreversible.

However, when establishing (5.1), we have dropped the bath terms of (4.6), which must be taken into account as $t$ increases. Their effect, apart from slightly shifting the position of the peaks, is to multiply (5.1) by $e^{-\chi(t)}$, with $\chi(t) \sim \gamma \Gamma^2 g^2 t^4/2\pi \hbar^2$. Hence, the off-diagonal parts of not only the marginal density operator of S but also of the full state of S + A, that is, both (5.3) and (5.2) are damped by the factor

$$e^{-t^4/\tau_2^4}, \quad \tau_2 = \left(\frac{2\pi}{\gamma N}\right)^{1/4} \left(\frac{\hbar}{\Gamma g}\right)^{1/2}. \tag{6.1}$$

This decay time $\tau_2$, which has the same nature as the off-diagonal spin-lattice relaxation time in NMR, is much larger than the collapse time (5.4) if $\gamma \ll 1$ and $N \gg 1$. We choose the parameters of the model in such a way that it is much smaller than the first recurrence time $\pi \hbar/2g$, that is, $\gamma \gg g^2/N \hbar^2 \Gamma^2$. All recurrent peaks are therefore cancelled; the height of the first one in $r_{\uparrow \downarrow} (t)$
is small as \( r_{\uparrow\downarrow}(0) e^{-N\pi^3\hbar^2\Gamma^2/32g^2} \) and the full matrix \( D_{\uparrow\downarrow}(t) \) disappears on the time scale \( \tau_2 \).

The presence of the phonon bath thus makes the collapse irreversible. During the first stage, over the characteristic time \( \tau_{\text{red}} \), we have seen that the off-diagonal order initially present in \( S \), which is expressed by a finite value of \( \langle \hat{s}_x \rangle \) or \( \langle \hat{s}_y \rangle \), dissolves into correlations between \( S \) and \( M \), but this transfer is reversible as shown by the possibility of recurrences. During the next stage, over the time scale \( \tau_2 \), the order is transferred further to the phonon bath, but now can no longer come back after any reasonable time lapse.

In fact, the recurrent peaks in \( r_{\uparrow\downarrow}(t) \) may disappear even if there is no bath, provided the interaction between \( S \) and \( A \) has the form \( \hat{H}_{SA}' \) given by (3.8). The solution for \( \gamma = 0 \) of the equations of motion for \( D_{\uparrow\downarrow}(t) \) is then given at all times by

\[
D_{\uparrow\downarrow}(t) = \frac{r_{\uparrow\downarrow}(0)}{2^N} D_B \otimes \prod_{n=1}^{N} \left( \cos \frac{2g_n t}{\hbar} + i \sin \frac{2g_n t}{\hbar} \sigma_n(z) \right). \tag{6.2}
\]

Instead of (5.3) we find a destructive interference of the cos factors entering

\[
r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \prod_{n=1}^{N} \cos \frac{2g_n t}{\hbar} \sim r_{\uparrow\downarrow}(0) e^{-t^2/\tau_2^2} \cos \frac{2gt}{\hbar}, \tag{6.3}
\]

which again produces a decay, with the alternative characteristic time

\[
\tau'_2 = \frac{1}{\sqrt{2N} \delta g}. \tag{6.4}
\]

The height of the first peak is here small as \( r_{\uparrow\downarrow}(0) e^{-N\pi^3\delta g^2/2g^2} \). A dispersion such that \( 1 \gg \delta g/g \gg 1/\sqrt{N} \) is therefore sufficient to make all the recurrent peaks disappear. Provided \( N \) is large and the couplings are slightly different, the bath is not necessary to ensure the irreversible disappearance of the off-diagonal elements \( r_{\uparrow\downarrow}(t) \).

The existence of two alternative mechanisms which make the rapid disappearance (5.3) of the off-diagonal parts of \( r(t) \) irreversible is reminiscent of two alternative relaxation mechanisms in NMR. The decay (6.1) due to the coupling with the phonons looks like a spin-lattice relaxation, while the decay (6.2) due to a dispersion in the couplings \( g_n \) looks like the relaxation due to the spread of the Larmor frequencies in the transverse motion of non interacting spins in an inhomogeneous magnetic field. As in the latter case, the initial order associated with the non vanishing value of \( \langle \hat{s}_x \rangle \) or \( \langle \hat{s}_y \rangle \), which has escaped towards correlations, gets trapped there due to the inhomogeneity of the couplings \( g_n \) as exhibited by (6.2). However, we may imagine to retrieve this order, as currently done in NMR by means of spin-echo experiments. Sup-
pose it is possible to apply on M an external magnetic field which acts on the spins $\hat{\sigma}^{(n)}$ without affecting S. Then a brief pulse $\pi$ around $y$ applied at the time $\theta$ suddenly changes the signs of the operators $\hat{\sigma}^{(n)}_z$ in the expression (6.2) for the state $D_{\uparrow\downarrow}(\theta)$. The subsequent evolution generated by $\hat{H}_{SA}$ will lead to $D_{\uparrow\downarrow}(2\theta) = D_{\uparrow\downarrow}(0)$ and hence to $r_{\uparrow\downarrow}(2\theta) = r_{\uparrow\downarrow}(0)$. The second relaxation mechanism is therefore less effective than the first one.

Anyhow, even though we do not need a phonon bath to explain why the final state (1.2) of $S + A$ involves no off-diagonal block, we must resort to it to account for the final form of the diagonal blocks, which results from energy exchanges between M and B.

7 Registration

Let us now turn to the evolution of $D_{ii}(t)$ for $i = \uparrow$ and $i = \downarrow$, which is governed by (4.7).

We first have at all time $\zeta_{0,ii}(t) = 1$ and hence $\zeta_{z,ii} = m_i$. We can rewrite the equation for $m_\uparrow(t)$, with $m_\uparrow(0) = 0$, as

$$\frac{h \, dm_\uparrow}{\gamma \, dt} = h_\uparrow \left( 1 - \frac{m_\uparrow}{\tanh h_\uparrow/T} \right) = -\frac{dF_\uparrow}{dm_\uparrow} \frac{1 - \frac{m_\uparrow}{\tanh h_\uparrow/T}}{1 - \frac{\tanh^{-1} m_\uparrow}{h_\uparrow/T}}, \quad (7.1)$$

where $h_\uparrow \equiv g + Jm^2_i$ and where the function $F_\uparrow(m_\uparrow)$ is defined by (3.3); for $m_\downarrow(t)$, $g$ will be changed into $-g$. The last factor in (7.1) is a positive function of $m_\uparrow$. Hence, $m_\uparrow(t)$ relaxes by increasing up to the smallest positive value of $m$ such that $F_\uparrow(m)$ is minimal. As discussed in section 3, this minimum is the paramagnetic one, not only for $T$ above the transition temperature, but also below if $g$ is smaller than $g_c$ given by (3.6). In such a case, the measurement fails since suppression of the interaction term $\hat{H}_{SA}$ would bring back the apparatus to its initial state with $m_\uparrow = 0$.

Let us therefore restrict to $g > g_c$. Numerically, if we take $T = 0.34J$, slightly below the transition temperature 0.36J, we have $g_c = 0.08J$. In this case, $m_\uparrow(t)$ increases up to the value $m^f$ where $F_\uparrow(m)$ has its lowest minimum, so that M reaches the ferromagnetic state with $m$ very close to $+1$ ($m^f = 0.996J$ for $T = 0.34J$ and $g = 0.09J$). If the coupling between S and A is then switched off, $m_\uparrow$ remains practically unchanged near $m^f$.

In the sector $D_{\downarrow\downarrow}$ of the density matrix, M symmetrically reaches the ferromagnetic state with $m_\downarrow(t)$ tending to $-m^f$. The memory of the triggering of A by S is kept forever. The overall density matrix $D(t)$ therefore reaches the
expected form (1.2) and all the features of ideal measurements listed above are obtained.

The time dependence of \( m_\uparrow(t) \) or \( m_\downarrow(t) \) is found from (7.1) by integration. Contrary to the characteristic time scales (5.4), (6.1) and (6.4) for \( D_{\uparrow\downarrow} \) and \( D_{\downarrow\uparrow} \), the time scales for \( D_{\uparrow\uparrow} \) and \( D_{\downarrow\downarrow} \) are not divided by a power of \( N \) and are thus much longer. We illustrate the behaviour as function of \( m_\uparrow \) of the right-hand side of (7.1) by considering the regime \( g \ll T \ll J \). Equal to \( g \) for \( m_\uparrow = 0 \), this right-side first decreases down to \( g - g_c \), \( g_c^2 = 4T^3/27J \), a minimum reached for \( m_\uparrow^2 = T/3J \). It then increases up to the value \( 27J/256 \) attained for \( m_\uparrow = 3/4 \), and decreases again to vanish for \( m_\uparrow = m^f \) as \( J (m^f - m_\uparrow) \). Hence \( m_\uparrow(t) \) approaches \( m^f \) for large \( t \) asymptotically as \( e^{-\gamma Jt/\hbar} \). Although strictly speaking the full relaxation time is thus infinite, \( m_\uparrow(t) \) reaches values nearly equal to \( m^f \) (with \( m^f - m \) of order \( g/J \)) after a finite delay \( \tau_{\text{reg}} \) governed by the region \( m^3 \ll T/J \). Integration of (7.1) then provides

\[
\frac{\gamma t}{\hbar} = \frac{1}{g + Jm^3 - Tm} \int_0^{m_\uparrow(t)} \frac{dm}{dx}, \tag{7.2}
\]

The registration time thus found

\[
\tau_{\text{reg}} = \frac{3\hbar}{\gamma T} \int_0^{\infty} \frac{dx}{(x-1)^2(x+2) + 2(g-g_c)/g_c},
\]

\[
g_c = \frac{2T}{3} \sqrt{\frac{T}{3J}}, \tag{7.3}
\]

behaves as

\[
\tau_{\text{reg}} = \frac{\pi \hbar}{\gamma T} \sqrt{\frac{3g_c}{2(g-g_c)}}, \tag{7.4}
\]

for \( g - g_c \ll g_c \). For \( g - g_c \) of order \( g_c \), \( \tau_{\text{reg}} \) is proportional to \( \hbar/\gamma T \) and thus depends only on the bath; it becomes large for a weak coupling \( \gamma \) between M and B and at low bath temperature.

8 Conclusion

In spite of its simplicity, the model gives rise to an elaborate scenario. It produces all the required features of a quantum measurement, and exhibits several time scales. During the very first stages, the large apparatus, without changing much, destroys the off-diagonal elements of the density matrix of S. This takes place very rapidly, over the reduction time (5.4), and the possible recurrences are hindered owing to interaction with the phonon bath. In spite
of the weakness of this interaction, the corresponding decay time (6.1) can be short. The inertia of the large apparatus implies that its changes are much slower; they occur significantly only through correlations with the diagonal elements of the state of S. They are triggered by the coupling \( g \) between S and A, and take place on the registration time scale (7.3) or (7.4) independent of the size of M. After this time, the coupling \( g \) is ineffective.

The dynamical process that we described shows that the experiment can behave as a measurement only if the parameters satisfy the inequalities

\[
N \gg 1 , \quad N \gg \frac{1}{\gamma} \left( \frac{g}{\hbar \Gamma} \right)^2 \quad \text{or} \quad N \gg \frac{g^2}{\delta g^2},
\]

(8.1)

which ensure the disappearance of the off-diagonal blocks, and

\[
\hbar \Gamma \gg T \gg \gamma J , \quad \hbar \Gamma \gg J > g,
\]

(8.2)

which were used to establish the relaxation equations.

The above solution enforces the statistical interpretation of quantum mechanics, according to which a density operator (even when it reduces to the projection on a pure state) plays with respect to the non-commuting observables the same rôle as the probability density with respect to the commuting physical variables in classical statistical mechanics. In this interpretation, a quantum “state” does not refer to a single object but only characterizes the statistics of an ensemble of identically prepared objects. Only a probabilistic description of the microscopic world is available to us. Moreover, since a quantum measurement requires a macroscopic apparatus, it can be described only by means of quantum statistical physics. The irreversible aspects of quantum measurements are then explained by the large size of the apparatus, as in the solution of the irreversibility paradox. Here too, we need in practice to rely on approximation schemes but our approach becomes exact in the limit of a large apparatus over any reasonable delay; the consideration of time scales is crucial.

A specific feature arises from the dynamics of the measurement process. The laws of quantum mechanics involve a special type of probabilities associated with the non commutation of the observables, as exemplified by Bell’s inequalities. These laws even violate standard logical reasoning. In the GHZ paradox [7] three statements separately true, but which can be checked experimentally only by means of different measurement settings, are not true together: if they are put together, they have a common consequence which can be shown experimentally to be wrong. The above solution of a model for measurement shows that quantum mechanics is consistent, although its assertions are contextual: they are valid only in a given experimental setting governed by the measuring apparatus. We have shown how the process generates a final state of the form
(1.2), which does not involve off-diagonal blocks. Their disappearance, which is a real dynamical phenomenon, is conceptually important since they have no classical meaning, nor even ordinary logical interpretation. Thus, classical probabilities emerge from an initial state \( r(0) \) of \( S \) which cannot be described classically. This possibility of interpreting the outcome of a measurement in a classical language arises owing to the change of scale, just as continuity of matter or phase transitions or irreversibility emerge from a large number of degrees of freedom. Moreover, a change of orientation of the apparatus allows us to explore other components of \( r(0) \), in agreement with the contextual nature of quantum mechanics.

We have also seen how classical correlations emerge in the diagonal blocks of (1.2) for an ideal measurement. These correlations allow us to get information on \( S \) through registration in \( M \). The non standard features of quantum correlations have been lost together with the off-diagonal terms. This loss of off-diagonal information is the price to be paid for gaining classical information about the diagonal elements of \( r(0) \). Let us stress that the Born rule and the von Neumann reduction have been recovered in our approach because we could interpret the outcomes of the quantum pointer variable as a classical distribution of a macroscopic classical random variable, correlated with \( S \).

Finally, an ideal measurement followed by a selection of one among the possible outcomes is identified as a preparation of \( S \). A subensemble with properties controlled by the measurement apparatus is thus extracted from the whole statistical ensemble. Here again it is the macroscopic size of the apparatus which allows us to distinguish a selected value of the pointer variable; we can thus rely on its classical correlation with the system \( S \) so as to set it into a controlled new state.

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