More on the skew-spectra of bipartite graphs and Cartesian products of graphs*

Xiaolin Chen, Xueliang Li, Huishu Lian
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
E-mail: chxlnk@163.com; lxl@nankai.edu.cn; lhs6803@126.com

Abstract

Given a graph $G$, let $G^\sigma$ be an oriented graph of $G$ with the orientation $\sigma$ and skew-adjacency matrix $S(G^\sigma)$. Then the spectrum of $S(G^\sigma)$ is called the skew-spectrum of $G^\sigma$, denoted by $Sp_S(G^\sigma)$. It is known that a graph $G$ is bipartite if and only if there is an orientation $\sigma$ of $G$ such that $Sp_S(G^\sigma) = iSp(G)$. In [D. Cui, Y. Hou, On the skew spectra of Cartesian products of graphs, Electron. J. Combin. 20(2013), #P19], Cui and Hou conjectured that such orientation of a bipartite graph is unique under switching-equivalence. In this paper, we prove that the conjecture is true. Moreover, we give an orientation of the Cartesian product of a bipartite graph and a graph, and then determine the skew-spectrum of the resulting oriented product graph, which generalizes Cui and Hou’s result, and can be used to construct more oriented graphs with maximum skew energy.

Keywords: oriented graph, skew-spectrum, skew energy, bipartite graph, Cartesian product.

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1 Introduction

Let $G$ be a simple undirected graph on order $n$ with vertex set $V(G)$ and edge set $E(G)$. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the adjacency matrix of $G$ is the $n \times n$
symmetric matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if the vertices $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. The spectrum of $G$, denoted by $Sp(G)$, is defined as the spectrum of $A(G)$.

Let $G^\sigma$ be an oriented graph of $G$ with the orientation $\sigma$, which assigns to each edge of $G$ a direction so that the induced graph $G^\sigma$ becomes an oriented graph or a directed graph. Then $G$ is called the underlying graph of $G^\sigma$. The skew-adjacency matrix of $G^\sigma$ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $(v_i, v_j)$ is an arc of $G^\sigma$, otherwise $s_{ij} = s_{ji} = 0$. Obviously, $S(G^\sigma)$ is a skew-symmetric matrix, and thus all the eigenvalues are purely imaginary numbers or 0, which form the spectrum of $S(G^\sigma)$ and are said to be the skew-spectrum of $G^\sigma$. The eigenvalues of $S(G^\sigma)$ are called the skew eigenvalues of $G^\sigma$.

The energy $\mathcal{E}(G)$ of an undirected graph is defined as the sum of the absolute values of all the eigenvalues of $G$, which was introduced by Gutman in [5]. We refer the survey [6] and the book [10] to the reader for details. The skew energy of an oriented graph, as one of various generalizations of the graph energy, was first proposed by Adiga et al. [1]. It is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$, denoted by $\mathcal{E}_S(G^\sigma)$. Most of the results on the skew energy are collected in our recent survey [9].

In [1], Adiga et al. obtained some properties of the skew energy and proposed some open problems, such as the following two problems.

**Problem 1.1** Find new families of oriented graphs $G^\sigma$ with $\mathcal{E}_S(G^\sigma) = \mathcal{E}(G)$.

**Problem 1.2** Which $k$-regular graphs on $n$ vertices have orientations $G^\sigma$ with $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$, or equivalently, $S(G^\sigma)^T S(G^\sigma) = kI_n$?

For Problem 1.1, it is clear that if an oriented graph $G^\sigma$ satisfies $Sp_S(G^\sigma) = iSp(G)$ then $\mathcal{E}_S(G^\sigma) = \mathcal{E}(G)$. In [7][11], they proved that $Sp_S(G^\sigma) = iSp(G)$ for any orientation $\sigma$ of $G$ if and only if $G$ is a tree, and $Sp_S(G^\sigma) = iSp(G)$ for some orientation $\sigma$ of $G$ if and only if $G$ is a bipartite graph. They also pointed out that the elementary orientation of a bipartite graph $G = G(X,Y)$, which assigns each edge the direction from $X$ to $Y$, is such an orientation that $Sp_S(G^\sigma) = iSp(G)$. Are there any other orientations?

Let $W$ be a vertex subset of an oriented graph $G^\sigma$ and $\overline{W} = V(G^\sigma) \setminus W$. Another oriented graph $G^\sigma'$ of $G$, obtained from $G^\sigma$ by reversing the orientations of all arcs between $W$ and $\overline{W}$, is said to be obtained from $G^\sigma$ by switching with respect to $W$. Two oriented graphs $G^\sigma$ and $G^\sigma'$ are said to be switching-equivalent if $G^\sigma'$ can be obtained from $G^\sigma$ by a sequence of switchings. Note that if $G^\sigma$ and $G^\sigma'$ are switching-equivalent, then
Conjecture 1.3 Let $G = G(X, Y)$ be a bipartite graph and $\sigma$ be an orientation of $G$. Then $Sp_S(G^\sigma) = iSp(G)$ if and only if $\sigma$ is switching-equivalent to the elementary orientation of $G$.

In Section 2, we prove that the Conjecture 1.3 is true. We also obtain some other results of the skew-spectrum of an oriented bipartite graph.

For Problem 1.2, it is known that $E_S(G^\sigma) \leq n\sqrt{\Delta}$ and equality holds if and only if $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$, which implies that $G$ is regular. Some families of oriented graphs with maximum skew energy were characterized in [1,4,12]. Moreover, in [3] Cui and Hou constructed a new family of oriented graphs with maximum skew energy by considering the skew-spectrum of Cartesian product $P_2 \square G$. In Section 3 we extend their result of the product graph $P_2 \square G$ to that of the product graph $H \square G$, where $H$ is a bipartite graph. Using it, we obtain a larger new family of oriented graphs with maximum skew energy.

2 Oriented bipartite graphs with $Sp_S(G^\sigma) = iSp(G)$

In this section, we consider bipartite graphs and their orientations. We prove that Conjecture 1.3 is true and also obtain the characterizations of orientations of a bipartite graph $G$ with $Sp_S(G^\sigma) = iSp(G)$.

First, let’s recall some definitions. Let $G^\sigma$ be an oriented graph and $C_{2\ell}$ be an undirected even cycle of $G$. Then $C_{2\ell}$ is said to be even oriented relative to $G^\sigma$ if it has an even number of edges oriented in clockwise direction (and now it also has an even number of edges oriented in anticlockwise direction, since $C_{2\ell}$ is an even cycle); otherwise $C_{2\ell}$ is called odd oriented. The even cycle $C_{2\ell}$ is said to be oriented uniformly if $C_{2\ell}$ is oddly (resp., evenly) oriented relative to $G^\sigma$ when $\ell$ is odd (resp., even). It should be noted that if an even cycle $C_{2\ell}$ is oriented uniformly in $G^\sigma$, then after switching operations on $G^\sigma$, $C_{2\ell}$ is also oriented uniformly. The following lemma was obtained in [3].

Lemma 2.1 [3] Let $G$ be a bipartite graph and $\sigma$ be an orientation of $G$. Then $Sp_S(G^\sigma) = iSp(G)$ if and only if every even cycle is oriented uniformly in $G^\sigma$.

Now we prove the following result which implies that Conjecture 1.3 is true.

Theorem 2.2 Let $G = G(X, Y)$ be a bipartite graph and $\sigma$ be an orientation of $G$. Then
\[ \text{Sp}_S(G^\sigma) = i\text{Sp}(G) \] if and only if \( \sigma \) is switching-equivalent to the elementary orientation of \( G \).

**Proof.** By Lemma 2.1, we can easily get the sufficiency of the theorem. We prove the necessity by induction on the number \( m \) of edges in \( G \).

When \( m = 1 \), the case is trivial. Assume that the result holds for \( m - 1 \). Let \( G = G(X, Y) \) be a bipartite graph with \( m \) edges and \( \sigma \) be an orientation of \( G \) such that \( \text{Sp}_S(G^\sigma) = i\text{Sp}(G) \). Then by Lemma 2.1, every even cycle of \( G^\sigma \) is oriented uniformly. Suppose that \( e = xy \) is an edge of \( G \) with \( x \in X \), \( y \in Y \) and \( \hat{e} \) is the corresponding arc of \( G^\sigma \). Consider the oriented bipartite graph \( G^\sigma - \hat{e} \). Note that every even cycle of \( G^\sigma - \hat{e} \) is also oriented uniformly, and thus \( \text{Sp}_S(G^\sigma - \hat{e}) = i\text{Sp}(G - e) \). By induction, we obtain that the orientation of \( G^\sigma - \hat{e} \) is switching-equivalent to the elementary orientation of \( G - e \). That is, there exists a sequence of switchings which transform the orientation of \( G^\sigma - \hat{e} \) to the elementary orientation of \( G - e \). Applying the same switching operations on \( G^\sigma \), we obtain an oriented graph \( G^\sigma' \), whose arcs except the arc corresponding to \( e \) have directions from \( X \) to \( Y \).

If the direction of the arc corresponding to \( e \) in \( G^\sigma' \) is from \( x \) to \( y \), it follows that \( \sigma \) is switching-equivalent to the elementary orientation of \( G \).

If the direction of the arc corresponding to \( e \) in \( G^\sigma' \) is from \( y \) to \( x \). We claim that \( e \) is a cut edge of \( G \). Otherwise, there exists a cycle containing \( e \), say \( C_{2k} = x_0y_0x_1y_1 \cdots x_{k-1}y_{k-1}x_0 \) in clockwise direction. We find that \( C_{2k} \) contains precisely \( k - 1 \) arcs in clockwise direction, which contradicts that \( C_{2k} \) is oriented uniformly. Hence \( e \) is a cut edge of \( G \). Then \( V(G) \) can be partitioned into two parts \( W_1 \) and \( W_2 \) such that \( e \) is the only edge between them. By switching with respect to \( W_1 \) in \( G^\sigma' \), the direction of the arc corresponding to \( e \) is reversed and the directions of other arcs keep unchanged. Then we conclude that \( \sigma \) is switching-equivalent to the elementary orientation of \( G \). The proof is thus complete.

Lemma 2.1 provides a good characterization of the oriented bipartite graphs \( G^\sigma \) with \( \text{Sp}_S(G^\sigma) = i\text{Sp}(G) \). But it requires one to check that every even cycle is oriented uniformly. A natural question is how to simplify the checking task, such as only to check all chordless cycles. Based on this, we get the following result.

**Theorem 2.3** Let \( G \) be a bipartite graph and \( \sigma \) be an orientation of \( G \). If every chordless cycle of \( G^\sigma \) is oriented uniformly, then \( \text{Sp}_S(G^\sigma) = i\text{Sp}(G) \).

**Proof.** We prove the theorem by contradiction. Let \( G \) be a bipartite graph and \( \sigma \) be an orientation of \( G \) such that every chordless cycle is oriented uniformly. But \( \text{Sp}_S(G^\sigma) \neq i\text{Sp}(G) \).
As an application of this orientation, we construct a larger new family of oriented graphs with maximum skew energy, which generalizes the construction in [3].

We can prove that for an oriented bipartite graph $G_{bipartite}$, by extending the orientation of $P_r$ is called a $C_3$ The skew-spectrum of $H$ of cycles $C_3$ (cycles of a graph is a generating set of the set of all cycles of $G$ is oriented uniformly, a contradiction. The proof is now complete. □

Combining with Lemma 2.1, we immediately obtain the following corollary.

**Corollary 2.4** Let $G$ be a bipartite graph and $\sigma$ be an orientation of $G$. Then $Sp_S(G^\sigma) = iSp(G)$ if and only if all chordless cycles are oriented uniformly in $G^\sigma$.

**Remark 2.5** Let $\mathcal{C}$ denote the set of all cycles of a bipartite graph $G$. A subset $\mathcal{S}$ of $\mathcal{C}$ is called a generating set of $\mathcal{C}$ if for any cycle $C$ of $\mathcal{C}$, $C \in \mathcal{S}$ or there is a sequence of cycles $C_1, C_2, \ldots, C_k$ in $\mathcal{S}$ such that $C = ((C_1 \Delta C_2) \Delta C_3) \cdots \Delta C_k$ and $C_1 \Delta C_2, (C_1 \Delta C_2) \Delta C_3, \ldots, ((C_1 \Delta C_2) \Delta C_3) \cdots \Delta C_{k-1}$ all are cycles. With this notation, one can prove that for an oriented bipartite graph $G^\sigma$, $Sp_S(G^\sigma) = iSp(G)$ if and only if every cycle in a generating set $\mathcal{S}$ of $\mathcal{C}$ is oriented uniformly in $G^\sigma$. Actually, the set of chordless cycles of a graph is a generating set of the set of all cycles of $G$.

### 3 The skew-spectrum of $H \Box G$ with $H$ bipartite

In this section, we give an orientation of the Cartesian product $H \Box G$, where $H$ is bipartite, by extending the orientation of $P_m \Box G$ in [3], and we calculate its skew-spectrum. As an application of this orientation, we construct a larger new family of oriented graphs with maximum skew energy, which generalizes the construction in [3].

Let $H$ and $G$ be graphs with $m$ and $n$ vertices, respectively. The Cartesian product $H \Box G$ of $H$ and $G$ is a graph with vertex set $V(H) \times V(G)$ and there exists an edge between $(u_1, v_1)$ and $(u_2, v_2)$ if and only if $u_1 = u_2$ and $v_1v_2$ is an edge of $G$, or $v_1 = v_2$ and $u_1u_2$
is an edge of \( H \). Assume that \( H^r \) is any orientation of \( H \) and \( G^\sigma \) is any orientation of \( G \). There is a natural way to give an orientation \( H^r \square G^\sigma \) of \( H^r \) and \( G^\sigma \). There is an arc from \((u_1, v_1)\) to \((u_2, v_2)\) if and only if \( u_1 = u_2 \) and \((v_1, v_2)\) is an arc of \( G^\sigma \), or \( v_1 = v_2 \) and \((u_1, u_2)\) is an arc of \( H^r \).

When \( H \) is a bipartite graph with bipartition \( X \) and \( Y \), we modify the above orientation of \( H^r \square G^\sigma \) with the following method. If there is an arc from \((u, v_1)\) to \((u, v_2)\) in \( H^r \square G^\sigma \) and \( u \in Y \), then we reverse the direction of the arc. The other arcs keep unchanged. This new orientation of \( H \square G \) is denoted by \((H^r \square G^\sigma)^o\).

**Theorem 3.1** Let \( H^r \) be an oriented bipartite graph of order \( m \) and let the skew eigenvalues of \( H^r \) be the non-zero values \( \pm \mu_1 i, \pm \mu_2 i, \ldots, \pm \mu_i i \) and \( m - 2t \) 0’s. Let \( G^\sigma \) be an oriented graph of order \( n \) and let the skew eigenvalues of \( G^\sigma \) be the non-zero values \( \pm \lambda_1 i, \pm \lambda_2 i, \ldots, \pm \lambda_r i \) and \( n - 2r \) 0’s. Then the skew eigenvalues of the oriented graph \((H^r \square G^\sigma)^o\) are \( \pm i \sqrt{\mu_j^2 + \lambda_k^2} \) with multiplicities 2, \( j = 1, \ldots, t \), \( k = 1, \ldots, r \), \( \pm \mu_i i \) with multiplicities \( n - 2r \), \( j = 1, \ldots, t \), \( \pm \lambda_k i \) with multiplicities \( m - 2t \), \( k = 1, \ldots, r \), and 0 with multiplicities \((m - 2t)(n - 2r)\).

**Proof.** Let \( H = H(X, Y) \) be a bipartite graph with \(|X| = m_1\) and \(|Y| = m_2\). With suitable labeling of the vertices of \( H \square G \), the skew-adjacency matrix \( S = S((H^r \square G^\sigma)^o) \) can be formulated as follows:

\[
S = I'_{m_1+m_2} \otimes S(G^\sigma) + S(H^r) \otimes I_n,
\]

where \( I'_{m_1+m_2} = (a_{ij}) \), \( a_{ii} = 1 \) if \( 1 \leq i \leq m_1 \), \( a_{ii} = -1 \) if \( m_1 + 1 \leq i \leq m \), and \( a_{ij} = 0 \) for \( i \neq j \); \( S(H^r) \) is the partition matrix \(
\begin{pmatrix}
0 & B \\
-B^T & 0
\end{pmatrix}
\) and \( B \) is an \( m_1 \times m_2 \) matrix.

We first determine the singular values of \( S \). Note that \( S \), \( S(H^r) \) and \( S(G^\sigma) \) are all skew symmetric. By calculation, we have

\[
SS^T = (I'_{m_1+m_2} \otimes S(G^\sigma) + S(H^r) \otimes I_n) (I'_{m_1+m_2} \otimes (-S(G^\sigma)) + (-S(H^r)) \otimes I_n)
\]

\[
= - \left[(I_{m_1+m_2} \otimes S^2(G^\sigma) + S^2(H^r) \otimes I_n) + (I'_{m_1+m_2} \otimes S(G^\sigma))(S(H^r) \otimes I_n)
\right]
\]

\[
+ (S(H^r) \otimes I_n)(I'_{m_1+m_2} \otimes S(G^\sigma))].
\]

Define \( \sigma_i = 1 \) for \( i = 1, 2, \ldots, m_1 \) and \( \sigma_i = -1 \) for \( i = m_1 + 1, m_1 + 2, \ldots, m \). Denote \( M^1 = [M^1_{ij}] = (I'_{m_1+m_2} \otimes S(G^\sigma))(S(H^r) \otimes I_n) \) and \( M^2 = [M^2_{ij}] = (S(H^r) \otimes I_n)(I'_{m_1+m_2} \otimes S(G^\sigma)) \). Note that \( M^1 \) and \( M^2 \) are both \( m \times m \) partition matrix in which every entry is an \( n \times n \)
submatrix. Direct computing gives
\[ M^1_{ij} + M^2_{ij} = S(H^\tau)_{ij}S(G^\sigma)((-1)^{\sigma_i} + (-1)^{\sigma_j}). \]

For any \(1 \leq i, j \leq m\), if \(S(H^\tau)_{ij} = 0\), then \(M^1_{ij} + M^2_{ij} = 0\). Otherwise the vertices corresponding to \(i\) and \(j\) in \(H^\tau\) are in different parts of the bipartition. That is, \(1 \leq i \leq m_1, m_1 + 1 \leq j \leq m\) or \(1 \leq j \leq m_1, m_1 + 1 \leq i \leq m\). Then \((-1)^{\sigma_i} + (-1)^{\sigma_j} = 0\), an thus \(M^1_{ij} + M^2_{ij} = 0\). It follows that \(M^1 + M^2 = 0\). Hence,
\[
SS^T = - \left( I_{m_1+m_2} \otimes S^2(G^\sigma) + S^2(H^\tau) \otimes I_n \right).
\]

Therefore, the eigenvalues of \(SS^T\) are \(\mu(H^\tau)^2 + \lambda(G^\sigma)^2\), where \(\mu(H^\tau)i \in Sp_S(H^\tau)\) and \(\lambda(G^\sigma)i \in Sp_S(G^\sigma)\). Then the skew-spectrum of \((H^\tau \square G^\sigma)^o\) follows. The proof is thus complete.

As an application of Theorem 3.1, we can now construct a new family of oriented graphs with maximum skew energy.

**Theorem 3.2** Let \(H^\tau\) be an oriented \(\ell\)-regular bipartite graph on \(m\) vertices with maximum skew energy \(E_S(H^\tau) = m\sqrt{\ell}\) and \(G^\sigma\) be an oriented \(k\)-regular graph on \(n\) vertices with maximum skew energy \(E_S(G^\sigma) = n\sqrt{k}\). Then the oriented graph \((H^\tau \square G^\sigma)^o\) of \(H \square G\) has the maximum skew energy \(E_S((H^\tau \square G^\sigma)^o) = mn\sqrt{\ell + k}\).

**Proof.** Since \(H^\tau\) and \(G^\sigma\) have maximum skew energy, \(S(H^\tau)S(H^\tau)^T = \ell I_m\) and \(S(G^\sigma)S(G^\sigma)^T = k I_n\). Then the skew eigenvalues of \(H^\tau\) are all \(\pm i\sqrt{\ell}\) and the skew eigenvalues of \(G^\sigma\) are all \(\pm i\sqrt{k}\). By Theorem 3.1, \((H^\tau \square G^\sigma)^o\) have all skew eigenvalues \(\pm imn\sqrt{\ell + k}\).

The following result was obtained in [3], which can be viewed as an immediate corollary of Theorem 3.2.

**Corollary 3.3** [3] Let \(G^\sigma\) be an oriented \(k\)-regular graph on \(n\) vertices with maximum skew energy \(E_S(G^\sigma) = n\sqrt{k}\). Then the oriented graph \((P_2 \square G^\sigma)^o\) of \(P_2 \square G\) has maximum skew energy \(E_S((P_2 \square G^\sigma)^o) = 2n\sqrt{k + 1}\).

Adiga et al. [1] showed that a 1-regular connected graph that has an orientation with maximum skew energy is \(K_2\); while a 2-regular connected graph has an orientation with maximum skew energy if and only if it is \(C_4\) with oddly orientation. Tian [12] proved that there exists a \(k\)-regular graph with \(n = 2^k\) vertices having an orientation \(\sigma\) with maximum skew energy. Cui and Hou [3] constructed a \(k\)-regular graph of order \(n = 2^{k-1}\) having an
orientation $\sigma$ with maximum skew energy. The following examples provide new families of oriented graphs with maximum skew energy that have much less vertices.

**Example 3.4** Let $G_1 = K_{4,4}$, $G_2 = K_{4,4} \Box G_1, \ldots, G_r = K_{4,4} \Box G_{r-1}$. Because there is an orientation of $K_{4,4}$ with maximum skew energy 16; see [2]. Thus, we can get an orientation of $G_r$ with maximum energy $2^{3r} \sqrt{4r}$. This provides a family of $4r$-regular graphs of order $n = 2^{3r}$ having an orientation with skew energy $2^{3r} \sqrt{4r}$ for $r \geq 1$.

**Example 3.5** Let $G_1 = K_4$, $G_2 = K_{4,4} \Box G_1, \ldots, G_r = K_{4,4} \Box G_{r-1}$. It is known that $K_4$ has an orientation with maximum skew energy; see [3]. Thus we can get an orientation of $G_r$ with maximum energy $2^{3r-1} \sqrt{4r-1}$. This provides a family of $4r-1$-regular graphs of order $n = 2^{3r-1}$ having an orientation with skew energy $2^{3r-1} \sqrt{4r-1}$ for $r \geq 1$.

**Example 3.6** Let $G_1 = C_4$, $G_2 = K_{4,4} \Box G_1, \ldots, G_r = K_{4,4} \Box G_{r-1}$. Thus, we can get an orientation of $G_r$ with maximum energy $2^{3r-1} \sqrt{4r-2}$. This provides a family of $4r-2$-regular graphs of order $n = 2^{3r-1}$ having an orientation with skew energy $2^{3r-1} \sqrt{4r-2}$ for $r \geq 1$.

**Example 3.7** Let $G_1 = P_2$, $G_2 = K_{4,4} \Box G_1, \ldots, G_r = K_{4,4} \Box G_{r-1}$. Thus, we can get an orientation of $G_r$ with maximum energy $2^{3r-2} \sqrt{4r-3}$. This provides a family of $4r-3$-regular graphs of order $n = 2^{3r-2}$ having an orientation with skew energy $2^{3r-2} \sqrt{4r-3}$ for $r \geq 1$.

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