On Generalizations of Mean–Variance Model

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Abstract. Mean-variance analysis is a mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk. This paper is closely related to the mean-variance model, considers the variation of market parameters and the corresponding influence on efficient frontier and generalizes the mean-variance model to separable utility function, then compares the efficient frontier with the original mean-variance model and tests the conclusions on real world financial dataset.

1. Introduction

1.1 Mean-Variance model

Harry Markowitz published his paper titled “Portfolio Selection” in the Journal of Finance in 1952. His work formed the foundations of “what is now known as mean–variance analysis, mean–variance optimization, and modern portfolio theory (MPT)” [1]. The main idea of MV model is to handle the returns of individual security as random variables and adopt the expected value and variance to quantify the investment return and investment risk, respectively. MPT assumes that investors are risk averse, meaning that a rational investor minimizes the risk level for a fixed expected return value or maximize the expected return value for a given risk level.

Given $n$ assets, denote their mean of returns as $e = (e_1, \ldots, e_n)^T$ and covariance of returns as $V$ in a time period, the optimization problem can be formulated as follow [2]:

$$\begin{align*}
\min_w & \frac{1}{2} w^T V w \\
\text{s.t.} & w^T e = E \\
& w^T 1 = 1
\end{align*}$$

where $w = (w_1, \ldots, w_n)^T$ is the portfolio.

Since (1) is convex optimization problem, it can be solved by KKT condition.

The Lagrange function of (1) is

$$L = \frac{1}{2} w^T V w - \lambda (E - w^T e) - \gamma (1 - w^T 1)$$

The KKT condition is

$$\frac{\partial L}{\partial w} = V w + \lambda e + \gamma 1 = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda} = E - e^T w = 0 \quad (4)$$

$$\frac{\partial L}{\partial \gamma} = 1 - w^T 1 = 0 \quad (5)$$
The solution is

\[ \lambda = \frac{EE-A}{D} \quad (6) \]

\[ \gamma = \frac{B - AE}{D} \quad (7) \]

\[ w = \frac{1}{D} [B(V^{-1}1) - A(V^{-1}e)] + \frac{1}{D} [C(V^{-1}e) - A(V^{-1}1)]E \quad (8) \]

where

\[ A = 1^TV^{-1}e, B = e^TV^{-1}e, C = 1^TV^{-1}1, D = BC - A^2 \]

The efficient frontier is an investment portfolio which is the boundary of the risk-return spectrum. Formally, it is the set of portfolios which satisfy the condition that no other portfolios exists with a higher expected return with the same standard deviation of return.

By (8), the portfolio efficient frontier of Markowitz model satisfies:

\[ \frac{\sigma(r)^2}{\mu/C} - \frac{(E(r) - A/C)^2}{B/C^2} = 1 \quad (9) \]

where \( \sigma(r) \) is the variance and \( E(r) \) is the mean of portfolio \( r \). Below is an illustration of (9):

![Efficient Frontier Illustration](image-url)

**Figure 1.** The efficient frontier is the upper half of above hyperbolic curve.

### 1.2 Trade-off mean-variance model

In practice, one would like to have a better understanding of the return-risk trade-off since one want to both maximize return while minimizing risk. An alternative strategy is to try to balance these two objectives in a single objective function. One way to do this is to solve the Quadratic Programming [3]

\[ \min_w \frac{1}{2} w^T V w - \lambda e^T w \quad (10) \]

s. t. \( w^T 1 = 1 \)

name it as trade-off mean-variance model. By solving its KKT condition can obtain the solution

\[ w = (1 - \alpha) \frac{V^{-1}1}{e^TV^{-1}1} + \alpha \frac{V^{-1}e}{e^TV^{-1}e} \quad (11) \]

where \( \alpha = \lambda (e^TV^{-1}1) \)

### 1.3 Vary of market parameters

Investors may concern, with the change of market parameters (i.e. \( V \) and \( e \)), whether it is possible for every feasible portfolio to be optimal, since if they are not, investors may be able to avoid certain choice that is impossible to obtain optimal. The next section will focus on a specific kind of change to see the efficient frontier.
1.4 Generalization of mean-variance model

The utility function of mean-variance model is the variance and that of the trade-off mean-variance model is the linear combination of mean and variance, but in reality, investors may favour various kinds of utility functions [4], thus it is necessary to extend the mean-variance model to wider range of utility functions. The next section will extend the mean-variance model to separable utility functions and will also show that (1) and (10) have the same efficient frontier, but this may not hold in the generalized model.

2. Method

2.1 Vary of efficient frontier

The problem consider here is that for \( n \) given assets with expectation of return \( e = (e_1, ..., e_n)^T \) and variance \( V \) in a time period, whether it is possible for every feasible portfolio to be optimal by changing market parameters (i.e. \( V \) and \( e \)). Specifically, for given portfolio \( w \), its expectation \( E \) and variance \( var \), whether it is possible to vary market parameters \( e \) and \( V \) to let \( w \) on the efficient frontier.

Since the efficient frontier is

\[
\frac{\sigma(r)^2}{\bar{r}/c} = \frac{(E(r)-A/c)^2}{D/c^2} = 1
\]

as Figure 1 shows.

A way of parameter change is to fix the apex and the symmetry axis of the parabolic and let others change. To fix the apex, i.e. \( \bar{r}, \bar{c} \), one should fix \( A, C \). Since \( C = V^{-1}1 \), this means \( V \) is fixed.

Let \( e \) change, then \( B \) is the only variable in the efficient frontier, thus only have to get the range of \( B \).

The minimization problem is formulated as follows:

\[
\min_{e} \quad e^T V^{-1} e
\]

s.t. \( w^T e = E \)
\[
1^T V^{-1} e = A
\]

Notice that the objective function is convex for \( e \), the Lagrangian function is \( L(e, \lambda, \gamma) = e^T V^{-1} e + \lambda (E - w^T e) + \gamma (A - 1^T V^{-1} e) \).

Solve the following equations:

\[
\frac{\partial L}{\partial e} = 2 V^{-1} e - \lambda w - \gamma V^{-1} 1 = 0
\]
\[
\frac{\partial L}{\partial \lambda} = E - w^T e = 0
\]
\[
\frac{\partial L}{\partial \gamma} = A - 1^T V^{-1} e = 0
\]

obtain:

\[
\lambda = 2A - \gamma
\]
\[
\gamma = \frac{2(E - w^T V w A)}{1 - w^T V W c}
\]
\[
e = \frac{\lambda}{2} V w + \gamma 1
\]
The problem of maximization of $B$ is formulated as:

$$\max_e e^T V^{-1} e \quad (21)$$

s. t. $w^T e = E$

$$1^T V^{-1} e = A \quad (21)$$

Next will show that the objective value can tend to $\infty$. Let $e_1, e_2$ be 2 different vector that satisfy the two constrains of (21) then for any $\lambda \in R, \lambda e_1 + (1 - \lambda)e_2$ satisfy the two constrains of (21)

$$\begin{align*}
(\lambda e_1 + (1 - \lambda)e_2)^T V^{-1}(\lambda e_1 + (1 - \lambda)e_2) \\
= (\lambda(e_1 - e_2) + e_2)^T V^{-1}(\lambda(e_1 - e_2) + e_2) \\
= \lambda^2(e_1 - e_2)^T V^{-1}(e_1 - e_2) + 2\lambda(e_1^T V^{-1}(e_1 - e_2)) + e_2^T V^{-1} e_2
\end{align*} \quad (22)$$

Since $V$ is positive-definite, $V^{-1}$ is positive definite, thus, have $(e_1 - e_2)^T V^{-1}(e_1 - e_2) > 0$. When $\lambda \to \infty$, the value of the objective function of (21) tend to $\infty$. Thus the upper-bound of the objective function of (21) is $\infty$.

Now with the range of $B$, next will check if $B$ falls into this range when $w$ is on the efficient frontier.

If $w$ is on the efficient frontier, then by (9), obtain:

$$D = \frac{(CE-A)^2}{Cw^T Vw - 1} \quad (23)$$

the definition of $D$ yeilds

$$B = \frac{(CE-A)^2}{C^2 w^T Vw - C} + \frac{A^2}{C} = \frac{C^2 E^2 - 2CEA + CA^2 w^T Vw}{C^2 w^T Vw - C} = \frac{2AE - CE^2 - A^2 w^T Vw}{1 - Cw^T Vw} \quad (24)$$

Notice that (24) is identical to (20), can conclude that with the change of $e$, there is one and only one $e$ to make every $(E, Var)$ on the efficient frontier.

### 2.2 Generalized mean-variance model

Despite mean-variance model’s theoretical importance, it does have some limitations. In reality, different investors may choose different utility function other than the simple linear combination of mean and variance. To generalize mean-variance model, considering transform $F$ of mean and variance, ideally, $F$ is increasing for $w^T e$ while decreasing for $w^T V w$, the mean-variance model can be written as

$$\max_w F(w^T e, w^T V w) \quad (25)$$

s. t. $w^T 1 = 1$

The first order KKT condition [5] of

$$\frac{\partial F(w^T e, w^T V w)}{\partial w} - y 1 = 0 \quad (26)$$
The second order sufficient condition is

\[ w^T \frac{\partial^2 F(w^T e, w^T Vw)}{\partial w^2} \geq 0, \quad w \text{ satisfy KKT condition} \]  

(28)

### 2.2.1 Separable generalized mean-variance model

Specifically, consider the situation when \( F \) is separable for \( w^T e \) and \( w^T Vw \), i.e. \( f \) and \( g \) are inverseable increasing functions s.t. \( F(w^T e, w^T Vw) = f(w^T e) - \lambda g(w^T Vw) \).

Then the optimization problem is

\[
\min_w f(w^T Vw) - \lambda g(w^T e) \\
\text{s.t. } w^T 1 = 1
\]  

(29)

The KKT condition of this problem is

\[
\frac{\partial f(w^T Vw)}{\partial w} - \lambda \frac{\partial g(w^T e)}{\partial w} - \gamma 1 = 0 \\
w^T 1 = 1
\]  

(30)

(31)

consider following optimization problem:

\[
\min_w f(w^T Vw) \\
\text{s.t. } w^T 1 = 1 \\
g(w^T e) = E
\]  

(32)

The KKT condition of this problem is:

\[
\frac{\partial f(w^T Vw)}{\partial w} - \lambda_2 \frac{\partial g(w^T e)}{\partial w} - \gamma 1 = 0 \\
w^T 1 = 1 \\
g(w^T e) = E
\]  

(33)

(34)

(35)

This problem is equivalent as

\[
\min_w w^T Vw \\
\text{s.t. } w^T 1 = 1 \\
w^T e = g^{-1}(E)
\]  

(36)

which is the Markowitz problem that has the solution:

\[
w = \frac{1}{d} [B(V^{-1} 1) - A(V^{-1} e)] + \frac{1}{d} [C(V^{-1} e) - A(V^{-1} 1)] g^{-1}(a)
\]  

(37)

If \( f(w^T Vw) - \lambda_1 g(w^T e) \) is convex for \( w \), Denote the solution to (33) ~ (35) as \((w, \lambda_2, \gamma_2)\), if \( \lambda_2 = \lambda_1 \) then \((w, \gamma_2)\) is the solution to (20), (21). Since KKT condition is sufficient for convex optimization problem, every solution of (32) is a solution of (29) In this case, the efficient frontier of (32) is part of the trade-off efficient frontier of (29).

Notice that if \((w_1)\) is a solution to (29), (29) can be written as
\[
\min_w f(w^T Vw) - \lambda g(w^T e) \\
\text{s.t. } w^T 1 = 1 \\
w^T e = w_1^T e
\]  
(38)

which is equivalent to

\[
\min_w f(w^T Vw) \\
\text{s.t. } w^T 1 = 1 \\
w^T e = w_1^T e
\]  
(39)

Thus the efficient frontier of (29) is part of the efficient frontier of (32).

Now can conclude that the efficient frontier of (29) is identical to the efficient frontier of (32).

If \( f(w^T Vw) - \lambda g(w^T e) \) is not convex for \( w \), satisfying KKT condition dose not guarantee the solution to the optimization problem but the analysis of (38), (39) still hold. In this case, the efficient frontier of (29) is part of the efficient frontier of (32).

Specially, If \( f \) and \( g \) are both identity function, then (32) is the classic Markowitz model. By (4), when \( E \) increase from 0 to \( \infty \), \( \lambda \) will increase from 0 to \( \infty \). Thus, for \( \lambda \in (0, \infty) \), (8) is the solution to (1), i.e. as \( \lambda \) varies from 0 to \( \infty \), the trade-off efficient frontier of (7) is identical to the efficient frontier of (1). This result is intuitive in geometric point of view. The feasible region of the mean-variance model is bounded by a hyperbolic curve, the objective function of the trade-off mean-variance model is the linear combination with \( E \) and \( var \). It reaches the optimal at the tangent point of the hyperbolic curve. With the change of parameter \( \lambda \) from 0 to \( \infty \), the slope of the objective function varies from 0 to \( \infty \). On the other hand, every point on the upper half of the hyperbolic curve has continuous slope, thus every point is the optimal if and only if \( \lambda \) equals to its slope.

The next section compare the efficient frontier of the separable generalized trade-off model for different \( f \) and \( g \) to that of mean-variance model.

3. Results

To show the efficient frontier of the separable generalized trade-off mean-variance model, randomly choose 25 stocks in Yahoo Financial and collected their daily return from year 2007 to 2018[6].

The figures show the efficient frontier of mean-variance model and the trade-off mean-variance model with the following three different choices of \( g \) for the original, non-convex, convex cases that discussed in the method section while \( f \) remains identical.

1) \( f(w^T e) = w^T e, g(w^T Vw) = w^T Vw \)
2) \( f(w^T e) = w^T e, g(w^T Vw) = \exp(w^T Vw) \)
3) \( f(w^T e) = w^T e, g(w^T Vw) = \ln(w^T Vw) \)
Figure 2 Efficient Frontier of (1) and (2)

The upper side of Figure 2 shows part of the efficient frontier of the original mean-variance model and the trade-off mean-variance model in which the parameter $\lambda$ is set as the dual parameters in the Lagrange function of the mean-variance model. As the figure shows, the two models have identical efficient frontier.

Figure 3 Efficient Frontier of (3)

In the lower side of Figure 2, the black points illustrate part of efficient frontier of the mean-variance model with (3) and the red curve represents the efficient frontier of the trade-off model in which $\lambda$ is set as the dual parameter in the Lagrange function of mean-variance model with (3).

According to the method section, the efficient frontier of the trade-off model should be the same as the mean-variance model, but the red curve only covers part of it, and also notice that some points are not on the efficient frontier of (1).

It is because that the Lagrangian dual parameter in solving these two optimization problems increases to so large that causes computational error.

Both pictures in Figure 3 shows the efficient frontier of the trade-off mean-variance model with (2).

The upper shows the efficient frontier of the trade-off model in which the parameter $\lambda$ is set as the dual parameters in the Lagrange function of the mean-variance model with (2). It is different with the efficient frontier of (1), this is due to the non-convexity of $g$ as discussed in last section.
The lower shows the efficient frontier of the trade-off model which is part of the efficient frontier of the mean-variance model. This is in accordance with the non-convex case that concluded in last section.

4. Conclusion
For given assets with their returns and variance in a time period, every feasible point on the plane can become optimal of the mean-variance model (i.e. lies on the efficient frontier) with the change of market parameters. Specifically, fixing the apex and the symmetric axis of the original efficient frontier, there exists one and only one expectation parameter of assets returns to let a feasible mean-variance point lies on the efficient frontier after parameter change.

The generalized mean-variance model aims to extend the utility function of the mean-variance trade-off model (i.e. the linear combination of mean and variance) to larger variety of functions. The generalized separable mean-variance model focuses on the case that the utility function is separable for mean and variance. The efficient frontier of generalized separable mean-variance model is part of that of the mean-variance model. In this perspective, can be said that the generalized separable mean-variance model is generally less sensitive than the original trade-off mean-variance model. If the objective function is convex for \( w \), the efficient frontier is identical to that of the mean-variance model, but one should notice that the generalized separable mean-variance model may suffer from high computational cost when the expectation of assets return is large enough.

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