The Burkill-Cesari integral as a semivalue on subspaces of $AC$

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Abstract

We prove the symmetry of the Burkill-Cesari integral and discuss its continuity with respect to both the $BV$ norm of Aumann and Shapley and to the Lipschitz norm. As a consequence, we provide an existence result of a value, different from the Aumann and Shapley’s one, on a subspace of $AC_\infty$.

Key words: TU games, derivatives of set functions, Burkill-Cesari integral, value, semivalue, Lipschitz games.

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1 Introduction

Since the seminal Aumann and Shapley’s book [1], it is widely recognized that the theory of value of nonatomic games is strictly linked with different concepts of derivatives. A few papers, up to the recent literature, have investigated these relations (see for example [6], [13], [15]).

In the search of meaningful and simple formulas to express the value, the space $AC$ and its important subspace $pNA$ play a preeminent role, though in the literature other more general forms of absolute continuity for set functions are well known, and each of them gives rise to a space of set functions which is therefore worth investigating. Continuing in the framework initiated in [3], in Section 2 we introduce and study four different spaces of absolutely continuous games, corresponding to as many forms of absolute continuity for set functions, the smallest one, termed as $AC_4$, actually coinciding with the well known space $AC$ of Aumann and Shapley. We characterize scalar measure games $\nu = f \circ P$, with $P$ a nonatomic nonnegative measure, in each of these spaces: they belong to the $AC_i$’s under rather mild assumptions on $f$. Furthermore, we give a more general version of Theorem C of Aumann and Shapley.

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However, it should be mentioned that the spaces $AC_1 - AC_3$ turn out to be too large, for they are not even contained in $BV$. This is particularly undesirable for the purposes of this paper, where some form of continuity is part of the aim of the probe. For this reason, we bound our analysis to subspaces of $AC_4$, where suitable norms are available.

As for the link among derivatives of set functions and value theory, to the best of our knowledge the more recent and general contribution so far is that proposed in [15]. In [6] the question of the relation between the refinement derivative and the value is posed, and a possible direction sketched; in [15], the authors apply their more general (i.e. without the nonatomicity restriction) notion of refinement derivative to the study of the value on certain spaces of games by extending the potential approach of Hart and Mas-Colell ([8]) to infinite games.

In a previous paper ([3]) we had pointed out that, in a nonatomic context, the refinement derivative is connected with the classical Burkill-Cesari (BC) integral of set functions and, for BC integrable functions, the BC integral coincides with the refinement derivative at the empty set. Though less general, the BC integral is analytically more treatable.

Motivated by all these facts, in [3] we have started the study of the BC integral in the framework of transferable utility (TU) games. In this paper we extend and develop this approach, in particular in connection with the theory of value or of semivalue. In Section 3 we introduce the general class of Burkill-Cesari (BC) integrable games, and prove that under natural assumptions, a wide variety of measure games belongs to this class. In addition, on the subspace of those BC integrable games which belong to $AC_4$, the BC integral turns out to be a semivalue. Moreover, the class of BC integrable games contains a dense subspace of the largely used space $pNA$, where continuous values and semivalues are largely described in the literature (see for instance [5]). Unfortunately, the BC integral proves to be not continuous with respect to the $BV$ norm on these games. As continuity appears to be a crucial property for the value on subspaces of $BV$, in Section 4 we specialize to the subspace $AC_\infty \subset AC_4$ of so called Lipschitz games, where a suitable finer norm (the $\| \cdot \|_\infty$-norm) is defined and used as an alternative (see [14], [9]). We completely characterize the scalar measure games (where the measure is nonnegative) that belong to $AC_\infty$ and we show that the BC integral on an appropriate subspace is a Milnor (therefore $\| \cdot \|_\infty$-continuous) semivalue, and hence a value (though not unique) when restricted to the natural space of feasible games. Some convergence results for BC integrable games in the $\| \cdot \|_\infty$-norm are also given and motivated.

In every Section convenient examples and counterexamples complement the coverage. A deeper investigation on uniqueness of the value on alternative subspaces of $AC_\infty$ requires a better access to the $\| \cdot \|_\infty$-norm. This will be the object of a forthcoming paper.

2 Regularity and absolute continuity of measure games

From now on we will denote by $(\Omega, \Sigma)$ a standard Borel space (i.e. $\Omega$ is a Borel set of a Polish space, and $\Sigma$ the family of its Borel subsets). $\Omega$ represents a set of players, and $\Sigma$ the $\sigma$-algebra
of admissible coalitions.

A set function \( \nu : \Sigma \to \mathbb{R} \) such that \( \nu(\emptyset) = 0 \) is called a *transferable utility* (TU) game.

We refer the reader to [1] and to [10] for the terminology concerning TU games: in particular \( BV \) will denote the space of all bounded variation games, endowed with the variation norm \( || \cdot || \). The subspace of nonatomic countably additive measures will be denoted by \( NA \) and the cone of the nonnegative elements of \( NA \) by \( NA^+ \).

In [3] we have reminded several notion of absolute continuity between two games, and compared them.

Following [15], we shall say that \( \nu \ll_1 \mu \) iff \( N(\mu) \subset N(\nu) \) (where \( N(\mu) = \{ N \in \Sigma : \mu(N \cup A) = \mu(A), \text{for every } A \in \Sigma \} \}).

More classically, we shall say that \( \nu \) is \( \mu \)-absolutely continuous, and write \( \nu \ll_2 \mu \) iff for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that when \( |\mu(E)| < \delta \) there also holds \( |\nu(E)| < \varepsilon \).

In [15] the authors introduce the concept of \( \mu \)-continuity of a game, when \( \mu \) is a measure, (but it can be extended to the more general case of \( \mu \) monotone and subadditive). A game \( \nu \) is \( \mu \)-continuous (in symbols \( \nu \ll_3 \mu \)) when \( \nu \) is a continuous map from the pseudometric space \( (\Sigma, d_\mu) \), where \( d_\mu \) is the usual Fréchet pseudodistance \( d_\mu(A,B) = \mu(A \Delta B) \). Hence \( \nu \ll_3 \mu \) if for every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( \mu(A \Delta B) < \delta \) then \( |\nu(A) - \nu(B)| < \varepsilon \).

Finally we shall write \( \nu \ll_4 \mu \) when we shall mean that the absolute continuity by chains holds (see [1]). The space \( AC \) introduced in [1] is the space of all games \( \nu \) on \( \Sigma \) such that there exists an element \( \mu \in NA^+ \) such that \( \nu \ll_4 \mu \).

According to the notation of [3], we shall denote this space by \( AC_4 \), and consistently we shall denote by \( AC_1, AC_2, AC_3 \) the spaces of games \( \nu \) such that \( \nu \ll_1 \mu, \nu \ll_2 \mu, \nu \ll_3 \mu \) for some \( \mu \in NA^+ \).

As we always consider \( \mu \in NA^+ \), there follows from [3] that \( AC_4 \subset AC_3 \subset AC_2 \subset AC_1 \). It is also known (1 Proposition 5.2 page 35), that \( AC_4 \subset BV \). Note that in [3] we erroneously wrote that \( \nu \ll_2 \mu \) does not imply \( \nu \ll_1 \mu \).

It is therefore rather natural to ask whether the space \( BV \) is large enough to contain also \( AC_i \) for some \( i < 4 \). To show that this is not the case we need the following characterization of scalar measure games.

Recall that a game \( \nu : \Sigma \to \mathbb{R} \) is called a *vector measure game* if there exists a bounded, convex-ranged, finitely additive vector measure \( P = (P_1, \cdots, P_n) \), and a real valued function \( g : R(P) \to \mathbb{R} \), such that \( \nu = g \circ P \) on \( \Sigma \), where \( R(P) \) is the range of \( P \). If \( n = 1 \), \( \nu \) is called a *scalar measure game*.

**Proposition 2.1** Let \( P \in NA^+ \), and let \( f : R(P) \to \mathbb{R} \) be a function with \( f(0) = 0 \). Consider the game \( \nu = f \circ P \). Then

i. \( \nu \ll_1 P \)
ii. \( \nu \ll_2 P \) iff \( f \) is continuous at 0 iff there exists \( \mu \in NA^+ \) such that \( \nu \ll_2 \mu \);

iii. \( \nu \ll_3 P \) iff \( f \) is continuous iff there exists \( \mu \in NA^+ \) such that \( \nu \ll_3 \mu \);

iv. \( \nu \ll_4 P \) iff \( f \) is absolutely continuous iff there exists \( \mu \in NA^+ \) such that \( \nu \ll_4 \mu \);

v. \( \nu \) is monotone iff \( f \) is non increasing;

vi. \( \nu \in BV \) iff \( f \) is of bounded variation.

**Proof.** The proof of i. and ii. are straightforward.

Since \( P \) is additive, if \( N \in \mathcal{N}(P) \) then

\[
\nu(A \cup N) = f[P(A \cup N)] = f[P(A) + P(N)] = f[P(A)] = \nu(A).
\]

To prove ii. note first that if \( \delta = \delta(\varepsilon) \) is determined by the continuity of \( f \) at 0, and \( P(E) < \delta \) then \( |\nu(E)| = |f[P(E)]| < \varepsilon \) i.e. \( \nu \ll_2 P \). Conversely, let \( \delta = \delta(\varepsilon) \) be determined by the absolute continuity \( \nu \ll_2 P \); then for every \( t \in [0, \delta] \), by the non atomicity of \( P \) one applies the Darboux property; so there should exist \( E \in \Sigma \) with \( P(E) = t \) and hence \( |f(t)| = |\nu(E)| < \varepsilon \).

To prove assertion iii. note first that by Lyapounoff Theorem, the range of \( P \) is the compact interval \([0, P(\Omega)]\); hence the assumption on \( f \) implies that \( f \) is uniformly continuous on it. Let \( A \) and \( B \in \Sigma \) have \( P(A \Delta B) < \delta \), where \( \delta = \delta(\varepsilon) \) is determined by the uniform continuity of \( f \). Then clearly

\[
|P(A) - P(B)| = |P(A \setminus B) + P(A \cap B) - P(B \setminus A) - P(A \cap B)| \leq P(A \Delta B) < \delta
\]

and hence \( |\nu(A) - \nu(B)| = |f[P(A)] - f[P(B)]| < \varepsilon \); so \( \nu \ll_3 P \).

Suppose now that \( \nu \ll_3 \mu \) for some \( \mu \in NA^+ \) but assume by contradiction that \( f \) is not uniformly continuous on \([0, P(\Omega)]\); this is the same as saying that there exists \( \varepsilon > 0 \) such that for each \( \delta > 0 \) one can find two points \( x_\delta, y_\delta \in [0, P(\Omega)] \) with \( |x_\delta - y_\delta| < \delta \) but \( |f(x_\delta) - f(y_\delta)| > \varepsilon \). For the sake of simplicity we shall always choose \( x_\delta < y_\delta \); then by Lyapounoff Theorem there are sets \( A_\delta \subset B_\delta \)

in \( \Sigma \), with \( (P, \mu)(A_\delta) = x_\delta, (P, \mu)(B_\delta) = y_\delta \).

Choose now \( \delta = \delta\left(\frac{\varepsilon}{2}\right) \), determined according to the absolute continuity \( \nu \ll_3 \mu \); then \( \mu(A_\delta \Delta B_\delta) = \mu(B_\delta) - \mu(A_\delta) < \delta \) and hence \( |\nu(A_\delta) - \nu(B_\delta)| < \frac{\varepsilon}{2} \) which is a contradiction.

We turn now to assertion iv. To prove the necessary implication, fix \( \varepsilon > 0 \) and let \( \delta = \delta(\varepsilon) \) be determined by the absolute continuity of \( f \).

Let \( C \) be a chain, \( C = \{S_1, \cdots, S_n\} \) and let \( \Lambda = \{n_1, \cdots, n_k\} \) be a subchain with \( \|P\|_\Lambda < \delta \); set \( E = \bigcup_{k} (S_{n_k} \setminus S_{n_{k-1}}) \). Then we have that \( \sum_k P(S_{n_k}) - P(S_{n_{k-1}}) < \delta \) whence

\[
\sum_k |f[P(S_{n_k})] - f[P(S_{n_{k-1}})]| < \varepsilon \implies \|\nu\|_\Lambda < \varepsilon.
\]
Conversely, in Theorem C of \cite{1} the authors directly prove that if $\nu \ll_{\mu} \mu$ for some $\mu \in NA^+$, then $f$ is absolutely continuous.

Implication v. precisely coincides with Proposition 4.2 in \cite{3}.

To conclude the proof, we prove implication vi. The necessary condition is immediate (cfr \cite{1} page 14). For the “only if” implication, assume that $\nu$ is $BV$, and set $\|\nu\| = R < +\infty$.

Suppose by contradiction that $f$ is not of bounded variation, namely for each $k > 0$ there exists a decomposition of the interval, say $a = t_0 < t_1 < \cdots < t_n = b$ such that $\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| > K$.

Consider then any $K > R$ and again, by means of nonatomicity, construct a nested sequence of sets $C = \{A_0 \subset A_1 \subset \cdots \subset A_n\}$ with $P(A_k) = t_k$. Then

$$\|\nu\|_C = \sum_{k=1}^{n} |\nu(A_k) - \nu(A_{k-1})| = \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| > K > R$$

which is impossible.

Then we can provide a game $\nu' \in BV \setminus AC_1$ (hence, a fortiori, $(\nu' \in BV \setminus AC_i, i = 2, \cdots, 4)$, and a game $\nu'' \in AC_3 \setminus BV$ (hence, a fortiori, $\nu'' \in AC_2 \setminus BV$ and $\nu'' \in AC_1 \setminus BV$).

**Example 2.1** On $\Omega = [0,1]$ equipped with the Borel $\sigma$-algebra $\Sigma$ and the usual Lebesgue measure $\lambda \in NA^+$ consider the game

$$\nu'(I) = \begin{cases} 
\lambda(I) + 1 & \text{if } I \neq \emptyset \\
0 & \text{if } I = \emptyset 
\end{cases}$$

Then immediately $\nu'$ is monotone, and therefore $\nu' \in BV$. On the other side $\nu' \notin AC_1$; in fact one easily computes $N(\nu') = \{\emptyset\}$ and no element in $NA^+$ could have $\emptyset$ as the unique null set.

Consider the function $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 
x \sin \frac{1}{x} & \text{when } x \neq 0 \\
0 & \text{otherwise} 
\end{cases}$$

By Proposition 2.1 the game $\nu'' = f \circ \lambda$ is in $AC_3$, but $\nu'' \notin BV$.

It is interesting to note that some of the implications of Proposition 2.1 above extend to measure games.

**Proposition 2.2** Let $P : \Sigma \to \mathbb{R}^n$ be a nonatomic vector measure and $f : R(P) \to \mathbb{R}$ be a function that vanishes at 0. Consider the measure game $\nu = f \circ P$. Then, if $\overline{P}$ denotes the variation of $P$ ($\overline{P} = P_1 + \ldots + P_n$),

i. $\nu \ll_1 \overline{P}$

ii. $\nu \ll_2 \overline{P}$ iff $f$ is continuous at 0;

iii. if $f$ is continuous, then $\nu \ll_3 \overline{P}$;
iv. if \( f \) is nonincreasing (with respect to the vector partial ordering \( \mathbf{x} \preceq \mathbf{y} \) iff \( x_i \leq y_i, i = 1, \ldots, n \)), then \( \nu \) is monotone.

For a discussion of conditions ensuring \( \nu \preceq_4 \bar{P} \) for measure games we refer the reader to Section 5, where Lipschitz games are investigated. Although we have not proven a double implication in iii., as in the statement of Proposition 2.1, it is quite immediate to note that if a measure game \( \nu \preceq_3 \bar{P} \), then \( f \) is radially continuous, namely for every \( \mathbf{x} \in R(P) \), the restriction \( f|_{\mathbf{x}, t \in [0,1]} \) is a continuous map of \( t \).

The comparison between the statements iv. in the two above results is easier. Indeed here is a counterexample showing that in iv. only the direct implication holds.

**Example 2.2** Let \( R \) be a strictly convex closed zonoid in the positive orthant of \( \mathbb{R}^2 \); it is therefore known that one can find two measures \( P_1, P_2 \) such that \( R = R(P) \) where \( P \) is the pair \( P = (P_1, P_2) \).

Indeed, according to [2] (Corollary 4.5) one can find a pair of finitely additive measures, say \( \mu_1, \mu_2 \) such that \( R = R(\mu) \) with \( \mu = (\mu_1, \mu_2) \); then apply [11] (Theorem 3.2) to find an algebra \( \mathcal{F} \) on which the pair is also countably additive, and \( \mu(\mathcal{F}) = R \), and then use a Carathéodory procedure to extend \( \mu|_{\mathcal{F}} \) in a countably additive way to the generated \( \sigma \)-algebra \( \Sigma = \sigma(\mathcal{F}) \); a density argument will then show that \( \mu(\Sigma) = R \).

Le \( \partial^+ R \) be the upper boundary of the zonoid \( R \) (see picture), and let \( f : R \to \mathbb{R} \) be defined as

\[
    f(\mathbf{x}) = \begin{cases} 
        1 & \text{if } \mathbf{x} \in \partial^+ R \\
        0 & \text{otherwise}
    \end{cases}
\]

Then the measure game \( \nu = f \circ P \) is monotone, although \( f \) fails to be nondecreasing.

In fact let \( A, B \in \Sigma \) with \( A \subseteq B \); if \( P(B) \in \partial^+ R \) then clearly \( \nu(A) \leq 1 = \nu(B) \); if \( P(B) \notin \partial^+ R \) then the Hereditarily Overlapping Boundary Property ([11] Lemma 3.1) tells us that \( P(A) \notin \partial^+ R \) too, and hence \( \nu(A) = \nu(B) = 0 \).

3 A semivalue on a space of Burkill-Cesari integrable games

Recall first that a partition \( D \) of a set \( E \in \Sigma \) is a finite family of disjoint elements of \( \Sigma \), whose union is \( E \). By \( \Pi(E) \) we shall denote the set of all the partitions of \( E \). A partition \( \overline{D} \in \Pi(E) \) is
a refinement of another partition $D \in \Pi(E)$ if each element of $D$ is union of elements of $D$.

As in [3], given a monotone nonatomic game $\lambda$ one defines the mesh $\delta_\lambda$ as

$$
\delta_\lambda(D) = \max\{\lambda(I), I \in D\}.
$$

(1)

and the Burkill-Cesari (BC) integral of a game $\nu$ w.r.t. $\delta_\lambda$ as:

$$
E \mapsto \int_E \nu = \lim_{\delta(D) \to 0} \sum_{I \in D} \nu(I).
$$

(2)

We denote by $BC$ the space of games $\nu$ such that there exists $\lambda \in NA^+$ so that $\nu$ is Burkill-Cesari (BC) integrable with respect to the mesh $\delta_\lambda$.

According to Proposition 5.2 in [3], the BC integral does not depend upon the integration mesh; in other words, for every $\lambda \in NA^+$ such that $\nu$ is $\delta_\lambda$-BC integrable, the BC integral is the same. Moreover, the BC integral is a finitely additive measure and, as observed in [3], it coincides with the Epstein-Marinacci outer derivative at the empty set $\partial^+_{\emptyset}(\nu, \cdot)$. Hence, from now, on we shall use the notation $\partial^+_{\emptyset}(\nu, \cdot)$.

As we shall see, the space $BC$ contains many games which are of interest in the literature: we begin by recalling a sufficient condition for vector measure games to be in $BC$, which is an immediate consequence of Theorem 6.1 in [3].

**Proposition 3.1** Let $P : \Sigma \to \mathbb{R}^n$ be a nonatomic vector measure, and let $f : R(P) \to \mathbb{R}$ be a function with $f(0) = 0$. If $f$ is differentiable at 0, then the game $\nu = f \circ P \in BC$, and $\partial^+_{\emptyset}(\nu, F) = \nabla f(0) \cdot P(F), F \in \Sigma$.

Differently from the implications in **Proposition 2.1** here we do not have a double implication; indeed, consider as in [3] (Example 3.2) $f : \mathbb{R} \to \mathbb{R}$ to be any discontinuous solution to the functional equation

$$
f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.
$$

Then $\nu = f \circ P$ is additive, and therefore for each $F \in \Sigma$, and each $D \in \Pi(F)$ one has

$$
\sum_{I \in D} \nu(I) = \sum_{I \in D} f[P(I)] = f[P(F)] = \nu(F)
$$

and hence $\nu$ is BC integrable with respect to $P$ although $f$ is not differentiable at 0. A partial converse of this result will be given later in this work.

In this section we shall compare the spaces $BC$ versus $BV, AC_i, i = 1, \ldots, 4$.

It is immediate, by **Proposition 2.1** and **Proposition 3.1** to provide an example of a game $\nu \in BC \setminus BV$; in fact for any $f : [0,1] \to \mathbb{R}$ that vanishes at 0, admits derivative at 0 but is not bv, the corresponding scalar measure game $\nu = f \circ \lambda$ (where $\lambda$ is the Lebesgue measure on the unit interval) provides such an example.
To find an example of a game $\nu \in BV \setminus BC$, on $[0,1]$ equipped with the Borel $\sigma$-algebra consider the monotone game $\nu = \sqrt{x}$; then immediately $\nu \in BV$. To see that $\nu \notin BC$, one has to show that $\nu$ is not BC integrable with respect to any mesh $\delta_\mu$ determined by some $\mu \in NA^+$. Indeed $\nu$ is not refinement differentiable at $\emptyset$, and then it cannot be Burkill-Cesari integrable with respect to $\delta_\mu$; to get convinced that $\nu$ does not admit outer refinement derivative at $\emptyset$, observe that for every decomposition $D_o \in \Pi(\Omega) \forall \delta > 0$ we can provide a refinement $D' = \{I_1, \ldots, I_k, I_o\}$ such that $\lambda(I_1) + \ldots + \lambda(I_k) < \lambda(I_o) < \delta$ (see [12] Lemma 3.5). Clearly we can choose $k$ quite larger than say $\#D_o$. Also we can chose $\delta = \delta(\varepsilon)$ determined by the uniform continuity of $x \mapsto \sqrt{x}$ on $[0,1]$. Thus

$$\left| \sum_{I \in D'} \nu(I) - \sqrt{k(1 - \delta)} \right| < \varepsilon$$

which shows that the refinement limit does not exist.

**In fact** $\lambda(\Omega \setminus I_o) > 1 - \delta$, and therefore $\lambda(I_j) > \frac{1 - \delta}{k}$, $j = 1, \ldots, k$

whence $\nu(I_j) \geq \frac{1 - \delta}{k}$. $j = 1, \ldots, k$, while $\nu(I_o) < \varepsilon$.

So

$$k \cdot \frac{\sqrt{1 - \delta}}{\sqrt{k}} \leq \sum_{I \in D'} \nu(I) = \nu(I_o) + \sum_{j=1}^k \nu(I_j) \leq \varepsilon + k \cdot \frac{\sqrt{1 - \delta}}{\sqrt{k}}.$$

Note that, according to **Proposition 2.1** the game $\nu$ above is in $AC_4$; therefore we have also compared $BC$ versus $AC_i$, $i = 1, \ldots, 4$.

Also the assertion $AC_2 \not\subset BC$ is easy; take any strongly nonatomic finitely additive measure $\nu$ that is not countably additive; it is then clearly in $BC$, for $\partial_\emptyset = \nu$, but it cannot be in $AC_2$ since the condition $\nu \ll \mu$ with $\mu \in NA^+$ implies that $\nu$ is countably additive too. Again, since $AC_4 \subset AC_3 \subset AC_2$, such a game is therefore also in $BC \setminus AC_3$ and $BC \setminus AC_4$. To find an example of a game $\nu \in BC \setminus AC_1$ seems much more difficult, and our efforts so far have been unsuccessful.

Consider now the space $V = BC \cap AC_4$.

The following result states that the same measure can be used for the absolute continuity and the Burkill-Cesari integrability of a game in $V$.

**Proposition 3.2** The space $V$ can be equivalently defined as the space of games $\nu$ such that there exists $\mu \in NA^+$ such that $\nu \ll_{ABC} \mu$ and $\nu$ is Burkill-Cesari integrable with respect to $\delta_\mu$.

**Proof**. The fact that each game $\nu$ for which there exists $\mu \in NA^+$ such that $\nu \ll_{ABC} \mu$ and $\nu$ is Burkill-Cesari integrable with respect to $\delta_\mu$ lies in $V$ is straightforward.

Conversely, let $\nu \in V$; then there are $\mu_1, \mu_2 \in NA^+$ such that $\nu \ll_{ABC} \mu_1$ (since $\nu \in AC_4$) and $\nu$ is $\delta_{\mu_1}$-BC integrable. Then consider $\mu = \mu_1 + \mu_2$; evidently $\nu \ll_{ABC} \mu$ and, in force of Proposition 5.2 in [3], also $\nu$ is $\delta_\mu$-BC integrable. \qed
Definition 3.1 Let $\mathcal{G}$ denote the space of automorphisms of $(\Omega, \Sigma)$, that is isomorphisms of the space onto itself; then each $\vartheta \in \mathcal{G}$ induces a linear mapping $\vartheta_*$ of $BV$ onto itself, defined by

$$(\vartheta_\ast \nu)(E) = \nu(\vartheta(E))$$

for $E \in \Sigma$. A subspace that is invariant under $\vartheta_*$ for every $\vartheta \in \mathcal{G}$ is called symmetric.

Proposition 3.3 The space $V$ is symmetric.

Proof. We need to prove that for every $\vartheta \in \mathcal{G}$ and every game $\nu \in V$ the game $\vartheta_* \nu$ defined in (3) is in $V$, namely it is $\ll_2$ with respect to some nonatomic measure, and it is Burkill-Cesari integrable too.

Let $\mu$ be a measure in $NA^+$ with respect to which we have $\nu \ll_2 \mu$ and $\nu$ is $\delta_\mu$-Burkill-Cesari integrable. (remember that, thanks to Proposition 3.2 we can always assume that the default measure is the same).

Fix $\vartheta$; note that $\vartheta$ preserves set operations (unions, intersections, disjointness and so on). Therefore $\lambda = \vartheta_\ast \mu$ is in $NA^+$. In fact it is immediate to verify that $\lambda$ is finitely additive.

Then one proves that the Kolmogoroff axiom holds Let $(A_n)_n \downarrow \emptyset$; then the sequence $(\vartheta(A_n))_n$ is decreasing too and $\bigcap_{n \in \mathbb{N}} \vartheta(A_n) = \emptyset$. Thus $\lambda(A_n) = \vartheta_\ast \mu(A_n) = \mu[\vartheta(A_n)] \to 0$.

It remains to prove that $\lambda$ is nonatomic; since $\mu$ is nonatomic, it is strongly continuous, namely for every $\varepsilon > 0$ one can decompose $\Omega$ into finitely many pairwise disjoint subsets each of $\mu$-measure not exceeding $\varepsilon$; then the image decomposition $\vartheta^{-1}(\Omega_i)$ represents a decomposition of the whole space such that on each subset $\lambda$ does not exceed $\varepsilon$; in other words $\lambda$ enjoys the Darboux property which is equivalent to nonatomicity.

It is immediate to check that that $\vartheta_\ast \nu \ll_4 \lambda$, because $\vartheta$ transforms chains and subchains into chains and subchains as well.

It remains to prove that $\vartheta_\ast \nu$ is BC integrable with respect to the mesh $\delta_\lambda$. Indeed we shall prove that

$$\partial^+_{\vartheta}(\vartheta_\ast \nu, F) = \partial^+_{\vartheta}(\nu, \vartheta(F))$$

for every $F \in \Sigma$.

To this aim, for any $F \in \Sigma$ and any $\varepsilon > 0$ fixed, one has to find $\delta(\varepsilon, F) > 0$ such that for every decomposition $D \in \Pi(F)$ with $\delta_\lambda(D) < \delta$ there holds

$$\left| \sum_{I \in D} \vartheta_\ast \nu(I) - \partial^+_{\vartheta}(\nu, \vartheta(F)) \right| < \varepsilon.$$  

(5)

Since $\nu$ is BC integrable, to each $\varepsilon > 0$ there corresponds $\tau(\varepsilon, \vartheta(F)) > 0$ such that for each decomposition $D \in \Pi[\vartheta(F)]$ with $\delta_\mu(D) < \delta$ there follows

$$\left| \sum_{J \in D} \nu(J) - \partial^+_{\vartheta}(\nu, \vartheta(F)) \right| < \varepsilon.$$  

(4)
Clearly we can rewrite (5) as

\[ \left| \sum_{I \in D} \nu[\vartheta(I)] - \partial^+_{\varnothing}(\nu, \vartheta(F)) \right| < \varepsilon. \]

We choose \( \delta(\varepsilon, F) = \tau(\varepsilon, \vartheta(F)) \); thus if \( D \in \Pi(F) \) has \( \delta_\lambda(D) < \delta \), the corresponding decomposition \( D' = \{ \vartheta(I), I \in D \} \in \Pi[\vartheta(F)] \) has \( \delta_\mu(D') < \delta = \tau \) since, for each \( I \in D \) clearly \( \lambda(\vartheta(I)) = \vartheta_* \mu(\vartheta(I)) = \mu[\vartheta(I)] < \delta = \tau \) and hence

\[ \left| \sum_{I \in D} \vartheta(I) - \partial^+_{\varnothing}(\nu, \vartheta(F)) \right| = \left| \sum_{J \in D'} \nu(J) - \partial^+_{\varnothing}(\nu, \vartheta(F)) \right| < \varepsilon. \]

\[ \square \]

According to [5] we remind the following definition.

**Definition 3.2** A linear mapping \( \varphi : V \to NA \) on a symmetric subspace \( V \) of \( BV \) is called a semivalue provided it satisfies the properties

**V.1 (symmetry):** \( \vartheta_* \varphi = \varphi(\vartheta_*) \) for each \( \vartheta \in G \);

**V.2 (positivity):** \( \varphi \) is positive, that is for every monotone game \( \nu \) the measure \( \varphi(\nu) \) is non negative;

**V.3** \( \varphi \) is the identity operator on \( NA \cap V \).

When \( \varphi \) satisfies also

**V.4 (efficiency):** for each \( \nu \in V \) there holds \( \varphi(\nu)(\Omega) = \nu(\Omega) \)

is called a value on \( V \) (compare with [1]).

The following result immediately derives from (4) and the definition of \( \partial^+_{\varnothing} \)

**Corollary 3.1** The mapping \( \partial^+_{\varnothing} : V \to NA \) is a semivalue on \( V \).

However \( \partial^+_{\varnothing} \) is not continuous on \( V \) equipped with the variation norm, as the following construction shows.

Let \( B \) denote the linear span of powers of probability measures; more precisely let

\[ B = \left\{ \sum_{k=1}^{n} \alpha_k \mu_k^{r_k}, \alpha_k \in \mathbb{R}, r_k \in \mathbb{N}^+, n \in \mathbb{N}^+ \right\}. \]

Thus \( B \subset V \) since it can be interpreted as a space of measure games, determined by \( n \)-valued mappings that are differentiable at 0. (Moreover, as mentioned above, \( pNA = \bar{B} \) where the closure is meant in the \( BV \) norm).

In the sequel we shall prove that the operator \( \partial^+_{\varnothing} \) is not continuous on \( B \) with respect to this norm.
To this aim, we observe first that in Theorem 2 of [5], the authors prove that the only continuous semivalues on $pNA$ should be of the form

$$
\psi_g(\nu, S) = \int_0^1 \partial \nu^*(t, S) g(t) dt, \ S \in \Sigma
$$

(6)

for some $g \in L^\infty([0,1])$ with $\|g\|_1 = 1$. As noted in [1] page 145, if $f$ is continuously differentiable on $R(\mu), \mu = (\mu_1, \ldots, \mu_n)$ and $\nu = f \circ \mu$ then

$$
\partial \nu^*(t, S) = D_{\mu(S)} f[t\mu(\Omega)]
$$

(7)

(where $D_\cdot$ denotes the directional derivative of $f$).

Suppose that $\partial^+ \Theta(\cdot)$ were norm-continuous on $B$; then being a semivalue on $V$, it would remain a semivalue on $B$. By density we could extend $\partial^+ \Theta$ to the whole $pNA$ in a continuous way. Such an extension would remain a semivalue as well; in fact it is obvious that V.2 and V.3 would stay valid. As for symmetry, it is a standard computation to get convinced that for every $\vartheta \in G$, the map $\vartheta^*$ is an isometry. Hence the continuous extension of $\partial^+ \Theta$ to the whole $pNA$ would keep the symmetry of it on $B$.

Now relationship (6) would hold for some $g$.

Let $\nu \in B$. Then we can represent $\nu$ as $\nu = f \circ \mu$ for $f(x_1, \ldots, x_n) = \sum_{i=1}^n c_i x_i^r$ and $\mu = (\mu_1, \ldots, \mu_n)$ where the $\mu_i$'s are nonatomic probability measures on $\Sigma$. Then (6) and (7) together would read as

$$
\partial^+ \Theta(\nu, S) = \int_0^1 \sum_{i=1}^n c_i r_i t^{r_i-1} g(t) \mu_i(S) dt = \sum_{i=1}^n c_i r_i \left( \int_0^1 t^{r_i-1} g(t) dt \right) \mu_i(S), \ S \in \Sigma
$$

(8)

while, from Proposition 3.1

$$
\partial^+ \Theta(\nu, S) = \nabla f(0) \cdot \mu(S) = \sum_{r_i=1} c_i \mu_i(S).
$$

Hence the continuity of $\partial^+ \Theta$ would imply necessarily that $g$ satisfies

$$
\int_0^1 t^{r_i-1} g(t) dt = \begin{cases} 
1 & \text{when } r_1 = 1 \\
0 & \text{when } r_i > 1.
\end{cases}
$$

In other words the function $g$ should satisfy the moment problem (see for example [7])

$$
\int_0^1 t^n g(t) dt = \alpha_n = \begin{cases} 
1 & \text{when } n = 1 \\
0 & \text{when } n > 1.
\end{cases}
$$

This is impossible, since the requirements $g \in L^\infty([0,1])$ and $g \geq 0$ a.e. would imply that the integral function $\alpha(t) = \int_0^t g(s) ds$ is absolutely continuous and nondecreasing.

Then, as it is well known, the sequence $(\alpha_n)_n$ should satisfy at least the conditions

$$
\begin{vmatrix} 
\alpha_0 & \alpha_1 & \ldots & \alpha_n \\
\alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \\
\ldots & \ldots & \ldots & \ldots \\
\alpha_n & \alpha_{n+1} & \ldots & \alpha_{2n}
\end{vmatrix} > 0, \ n = 0, \ldots, n_0 - 1
$$

(9)
and

\[-1^n P_n(0) > 0, P_n(1) > 0, \quad n = 1, 2, \ldots, n_o \quad (10)\]

for some \( n_o \geq 1 \) where

\[P_n(x) = \begin{vmatrix}
\alpha_o & \alpha_1 & \ldots & \alpha_{n-1} & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n & x \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_n & \alpha_{n+1} & \ldots & \alpha_{2n-1} & x^n
\end{vmatrix}.\]

It is then clear that (9) is satisfied only for \( n_o = 1 \), and that \( P_1(x) = x \), while \( P_n(x) = 0 \) for \( n \geq 1 \).

4 The operator \( \partial_O^+ \) on subspaces of Lipschitz games

In [14] the author considers the class \( AC_\infty \) of Lipschitz games, that is games \( \nu \) in \( BV \) for which there exists a measure \( \mu \in NA^+ \) such that both \( \mu - \nu \) and \( \nu + \mu \) are monotone games. The reason why these games are called Lipschitz is the fact that the condition can be equivalently labelled in the following form: for every link \( S \subset T \) in \( \Sigma \) there holds

\[|\nu(T) - \nu(S)| \leq \mu(T) - \mu(S). \quad (11)\]

The connection to the Lipschitz condition is made even stronger by the following statement, which complements Proposition 2.1

**Proposition 4.1** For a scalar measure game \( \nu = f \circ \lambda \) the following are equivalent:

1. \( \nu \in AC_\infty \);

2. \( f \) is Lipschitz on the interval \([0, \lambda(\Omega)] \) (with \( \mu(\Omega) \) as Lipschitz constant, for each \( \mu \in NA^+ \) for which (11) above is satisfied);

3. (11) holds for \( \mu = L\lambda \), with \( L \) Lipschitz constant for \( f \).

The proof is rather elementary, so we include it only in the extended version for the sake of accuracy.

To prove that 1. implies 2. , assume that \( \nu = f \circ \lambda \) is a Lipschitz game, and let \( \mu \in NA^+ \) be a measure for which (11) holds. For simplicity we assume that \( \lambda \) is a probability measure. Let \( t, t' \in [0,1] \) with say \( t < t' \). Then, by Lyapounoff Theorem, there exist sets \( \Omega_t \subset \Omega_t' \subset \Omega \) such that

\[(\lambda, \mu)(\Omega_t) = t(\lambda, \mu)(\Omega), \quad (\lambda, \mu)(\Omega_t') = t'(\lambda, \mu)(\Omega).\]

Hence \( f(t) = f[\lambda(\Omega_t)] = \nu(\Omega_t) \) and \( f(t') = \nu(\Omega_t') \); then from (11)

\[|f(t) - f(t')| = |\nu(\Omega_t) - \nu(\Omega_t')| \leq \mu(\Omega_t \setminus \Omega_t') = (t' - t)\mu(\Omega).\]
The fact that a Lipschitz function $f$ generates a Lipschitz game (where one can precisely choose $L\lambda = \mu$ in (11)) is immediate, so 2. implies 3. Also 3. implies 1. trivially. 

It is immediate to note that $AC_\infty \subset AC_4$. However the smaller space can be equipped with an alternative norm defined in the following way: for every $\mu \in NA^+$ such that (11) holds, write $-\mu \preceq \nu \preceq \mu$. Then we set

$$\|\nu\|_\infty = \inf\{\mu(\Omega), \mu \in NA^+, -\mu \preceq \nu \preceq \mu\}.$$  

(12)

Then $AC_\infty$ is a Banach space when equipped with the above norm.

Again from [14] we quote the following definition.

**Definition 4.1** Let $\nu \in AC_\infty$ and define the following two subsets of $NA$:

$$D'\nu = \{\lambda \in NA|\nu \preceq \lambda\}, \quad D_\nu = \{\lambda \in NA|\lambda \preceq \nu\}.$$  

Then the following two measures exist: $\nu^* = g.l.b.D'\nu$, $\nu_* = l.u.b.D_\nu$, and immediately $\nu_* \leq \nu^*$ (although the symbol $\preceq$ should be distinguished from $\leq$, as the first one refers to setwise ordering, the second to the order induced by the cone of monotonic games, in the case of measures they actually assume the same meaning).

Let $NA \subseteq Q \subseteq AC_\infty$ be a linear subspace, and let $\psi : Q \rightarrow NA$ be a linear operator; we shall say that $\psi$ is a *Milnor operator* (MO) provided for every $\nu \in Q$ we have

$$\nu_* \leq \psi \nu \leq \nu^*.$$  

Consider now the vector subspace $Q = BC \cap AC_\infty$ of Lipschitz games that are Burkill-Cesari integrable.

$Q$ is strictly included in $BC$, for there are easy examples of games in $BC \setminus AC_\infty$.

For instance, consider the function $f : [0,1] \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2} \\
  \sqrt{1-x^2} & \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1
\end{cases}$$

and the scalar measure game $\nu = f \circ \lambda$ where $\lambda$ represents the usual Lebesgue measure. Then $\nu \in BC$ with $\partial_0^+(\nu) = f'(0)\lambda = \lambda$ thanks to Proposition 3.1 but $\nu \not\in AC_\infty$ since $f$ is not Lipschitz on $[0,1]$.

Also the inclusion $Q \subset AC_\infty$ is a strict one, for there are Lipschitz games that are not in $BC$. To see this we need the following result, which is a partial converse of Proposition 3.1.
**Proposition 4.2** Let the scalar measure game $\nu = f \circ \mu, \mu \in NA^+ \mu \neq 0$ be in $AC_\infty$; then the following are equivalent

1. $f$ admits right hand-side derivative at 0;
2. $\nu$ is $\delta_\mu$ BC integrable;
3. $\nu \in Q$.

**Proof.** The implication 1. $\implies$ 2. follows from **Proposition 3.1** while the fact that 2. implies 3. is trivial.

We turn then to the final implication 3. $\implies$ 1.

As $\nu \in Q$, there exists $\lambda \in N^+$ such that $\nu$ is $\delta_\lambda$ BC integrable. Since $\nu \in AC_\infty$ we already know that $f$ is Lipschitz; hence the ratios $f(x)/x$ are bounded. Assume by contradiction that $f'(0)$ does not exist. Then it can only happen that

$$-\infty < \ell_1 = \liminf_{x \to 0} \frac{f(x)}{x} < \limsup_{x \to 0} \frac{f(x)}{x} = \ell_2 < +\infty.$$

Choose then two decreasing sequences $\{x'_n\}, \{x''_n\} \in [0, \mu(\Omega)]$ with $\lim x'_n = \lim x''_n = 0$ and

$$\lim_n \frac{f(x'_n)}{x'_n} = \ell_1, \lim_n \frac{f(x''_n)}{x''_n} = \ell_2.$$

Fix $F \in \Sigma$ with $\mu(F) > 0$ and $\varepsilon \in [0, \mu(F)]$. Then there exists $\overline{n} \in \mathbb{N}$ such that for each $n > \overline{n}$

$$|\frac{f(x'_n)}{x'_n} - \ell_1| < \frac{\varepsilon}{3\mu(F)}, \quad |\frac{f(x''_n)}{x''_n} - \ell_2| < \frac{\varepsilon}{3\mu(F)}.$$

By means of the continuity of $f$ at 0, choose next $\overline{n} > \overline{n}$ such that $|f(x)| < \frac{\varepsilon}{3}$ whenever $x \leq x'_n$; also $\overline{n}$ can be chosen so that $|\ell_1|x''_n < \frac{\varepsilon}{3}$ and such that $x'_n \frac{\lambda(F)}{\mu(F)} < \delta \left(\frac{\varepsilon}{3}\right)$ where $\delta$ is that parameter of $\delta_\lambda$ BC integrability.

Choose now the following $D \in \Pi(F)$: by means of Lyapunov Theorem, divide $F$ into finitely many sets, say $I_1, \ldots, I_k$, each with $(\mu, \lambda)(I_j) = \left(\frac{x'_n}{F}, \frac{\lambda(F)}{\mu(F)}x'_n\right)$, until $\mu \left(F \setminus \bigcup_{j=1}^k I_j\right) \leq x'_n$ and then choose $I_{k+1} = F \setminus \bigcup_{j=1}^k I_j$; thus easily $\lambda(I_{k+1}) = \frac{\lambda(F)}{\mu(F)}\mu(I_{k+1})$.

$$\lambda(I_{k+1}) = \lambda(F) - \sum_{j=1}^k \lambda(I_j) = \lambda(F) - k \frac{\lambda(F)}{\mu(F)}x'_n = \frac{\lambda(F)}{\mu(F)}[\mu(F) - kx'_n] = \lambda(F) \mu(I_{k+1}).$$

Then for $D = \{I_1, \ldots, I_k, I_{k+1}\}$ one has $\delta_\lambda(D) < \delta \left(\frac{\varepsilon}{3}\right)$.

We have then, similarly to the computation in **Proposition 3.1**

$$\sum_{I \in D} |f[\mu(I)] - \ell_1 \mu(I) = \sum_{j=1}^k |f(x'_n) - \ell_1 x'_n| + |f[\mu(I_{k+1})] - \ell_1 \mu(I_{k+1})| \leq$$

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\[
\leq \sum_{j=1}^{k} |f(x_n^j) - \ell_1x_n^j| + |f[\mu(I_{k+1})]| + |\ell_1|x_n^j = \sum_{k=1}^{\infty} \left| \frac{f(x_n^j) - \ell_1x_n^j}{x_n^j} \right| x_n^j + \varepsilon \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.
\]

As for the first sum we have the following estimate
\[
\sum_{k=1}^{n} \left| \frac{f(x_n^j) - \ell_1x_n^j}{x_n^j} \right| x_n^j < \frac{\varepsilon}{3\mu(F)} \sum_{k=1}^{n} x_n^j = \frac{\varepsilon}{3} \cdot \frac{\mu(F \setminus I_{k+1})}{\mu(F)} < \frac{\varepsilon}{3}.
\]

In conclusion
\[
\sum_{I \in D} |f[\mu(I)] - \ell_1\mu(I)| < \varepsilon.
\]

Clearly we can repeat this construction with \(x_n''\) and find another decomposition \(D^* \in \Pi(F)\) with \(\delta_\lambda(D^*) < \delta \left( \frac{\varepsilon}{3} \right)\) as above; again
\[
\sum_{I \in D^*} |f[\mu(I)] - \ell_2\mu(I)| < \varepsilon.
\]

It is then clear that, since \(\ell_1 \neq \ell_2\), the game \(\nu\) is not \(\delta_\lambda\) BC integrable.

Therefore for instance, taking \(f : [0, 1] \to \mathbb{R}\) defined as
\[
f(x) = \begin{cases} x \sin \log x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}
\]
the game \(\nu = f \circ \lambda \in AC_\infty\), since for \(x \neq 0\) one has \(f'(x) = \sin \log x + \cos \log x \in L^\infty\), but, as \(f'(0)\) does not exist, according to the previous result, \(\nu \notin BC\).

We shall need in the sequel the following Lemma.

**Lemma 4.1** The space \(AC_\infty\) is symmetric and the following equality holds for every \(\vartheta \in G\)
\[
\|\vartheta_*\nu\|_\infty = \|\nu\|_\infty. \tag{13}
\]

**Proof.** Fix \(\varepsilon > 0\) and choose \(\mu \in NA^+\) such that \(-\mu \preceq \nu \preceq \mu\) and \(\mu(\Omega) < \|\nu\|_\infty + \varepsilon\).

Let \(\lambda = \vartheta_*\mu \in NA^+\). If \(A \subset B\) then \(\vartheta(\lambda) \subset \vartheta(\mu)\) and therefore, by monotonicity, \((\mu - \nu)[\vartheta(\lambda)] \leq (\mu - \nu)[\vartheta(\mu)]\), or else \(\vartheta_*\mu - \vartheta_*\nu(A) \leq \vartheta_*\mu(B) - \vartheta_*\nu(B)\), which is the same as to say that \(\vartheta_*\mu - \vartheta_*\nu\) is monotone, and hence \(\vartheta_*\nu \leq \vartheta_*\mu\).

In a completely analogous way, as \(\vartheta_*(-\mu) = -\vartheta_*(\mu)\), one reaches \(-\vartheta_*(\mu) \leq \vartheta_*(\nu)\). In conclusion \(-\lambda \leq \vartheta_*\nu \leq \lambda\).

Moreover, \(\lambda(\Omega) = \vartheta_*[\mu(\Omega)] = \mu[\vartheta(\Omega)] = \mu(\Omega)\) whence
\[
\|\vartheta_*\nu\|_\infty \leq \lambda(\Omega) = \mu(\Omega) < \|\nu\|_\infty + \varepsilon.
\]

To prove the converse inequality, first of all, for \(\lambda \in NA\) consider the game \(\vartheta_*^{-1}\lambda\) defined in the following fashion: for every \(B \in \Sigma\) set \(A = \vartheta^{-1}(B) \in \Sigma\) and set
\[
\vartheta_*^{-1}\lambda(B) = \lambda(A)
\]
so that $\partial_+\vartheta^{-1}\lambda = \lambda$. It is a routine computation, based on the properties of $\partial$, to show that $\partial_+^{-1}\lambda$ is a countably additive measure as well.

Again fix $\varepsilon > 0$ and choose $\lambda \in NA$ such that $-\lambda \leq \partial_+\nu \leq \lambda$ and $\lambda(\Omega) < \|\partial_+\nu\|_\infty + \varepsilon$.

Take $\mu = \partial_+^{-1}\lambda$ defined above.

Now $\partial_+\nu \succeq -\lambda = -\partial_+\mu$, or else $\partial_+\nu + \partial_+\mu$ monotone, implies in turn that $\nu + \mu$ is monotone too, and similarly $\nu \preceq \mu$.

Hence
\[
\|\nu\|_\infty \leq \mu(\Omega) = \lambda[\vartheta^{-1}(\Omega)] = \lambda(\Omega) < \|\partial_+\nu\|_\infty + \varepsilon
\]
which concludes the proof of relationship (13). $\blacklozenge$

In $Q$ we have the following result.

**Proposition 4.3** The Burkill-Cesari integral $\partial_+^\ast$ is a Milnor semivalue on $Q$.

**Proof.** Let $\nu$ be any game in $Q$, $\lambda \in D^\nu$; then $\lambda - \nu$ is a monotone game, and hence $(\lambda - \nu)(E) \geq 0$ for each $E \in \Sigma$, which in turn implies immediately that $\partial_+^\ast(\lambda - \nu) \geq 0$, namely $\lambda - \partial_+^\ast(\nu) \geq 0$ setwise in $\Sigma$. $\lambda - \partial_+^\ast(\nu)$ being a measure, this equivalently says that $\lambda - \partial_+^\ast(\nu)$ is monotone, that is $\lambda \preceq \partial_+^\ast(\nu)$. In complete analogy if $\lambda \in D\nu$ then $-\lambda \preceq \partial_+^\ast(\nu)$.

Hence for $\nu \in Q$ we have necessarily $\nu \preceq \partial_+^\ast(\nu) \preceq \nu^\ast$ which proves that $\partial_+^\ast$ is a MO.

From Lemma 1.6 in [14] then, $\partial_+^\ast$ is continuous with respect to the norm $\|\cdot\|_\infty$. Finally, we deduce from [4] the symmetry of the operator $\partial_+^\ast$, and the proof is thus complete. $\square$

According to Theorem 1.8 in [14], $\partial_+^\ast$ can be extended to the whole space $AC_\infty$ in such a way, that the extension, which we shall label as $\partial_+^\ast$, remains a linear continuous MO.

Let $Y$ denote the $\|\cdot\|_\infty$-closure of $Q$. Then on $Y$ we have

**Theorem 4.1** $\partial_+^\ast$ is a Milnor semivalue on $Y$.

**Proof.** If $\nu \in Y$, there exists a sequence in $Q$, say $(\nu_k)_k$ that $\|\cdot\|_\infty$-converges to $\nu$. Because of (13), for each $\vartheta \in G$, we have that $\vartheta_+\nu_k$ $\|\cdot\|_\infty$-converges to $\vartheta_+\nu$. But then $\partial_+^\ast(\vartheta_+\nu_k) \xrightarrow{\|\cdot\|_\infty} \partial_+^\ast(\vartheta_+\nu)$ too.

Similarly $\partial_+^\ast(\nu_k) \xrightarrow{\|\cdot\|_\infty} \partial_+^\ast(\nu)_k$ and then, again by (13), $\vartheta_+[\partial_+^\ast(\nu_k)] \xrightarrow{\|\cdot\|_\infty} \vartheta_+\partial_+^\ast(\nu)$.

In conclusion $\vartheta_+[\partial_+^\ast(\nu)] = \partial_+^\ast(\vartheta_+\nu)$. $\square$

Since powers of probabilities belong to $Q$, there immediately follows that

**Corollary 4.1** $\partial_+^\ast$ is a $\|\cdot\|_\infty$-continuous semivalue on $pNA_\infty$.

We point out that, as $Q$, $Y$, $pNA_\infty$ are symmetric subspaces of $AC_\infty$, there follows from [14] (Theorem 3.1) that $\partial_+^\ast$ and $\partial_+^\ast$ are diagonal.

Moreover from [14] (Theorem 2.1), there exists a Borel measure $\xi$ on $[0, 1]$ such that the following representation of $\partial_+^\ast$ on $pNA_\infty$ holds
\[
\partial_+^\ast(\nu, S) = \int_0^1 \nu^*(t1_\Omega, 1_S)d\xi, \quad S \in \Sigma,
\]
where $\partial \nu^*$ is the ideal extension of the game $\nu$ defined in [H], Theorem G.

Define now the space $\text{FEAS}$ as the set of games $\nu \in BC \cap \text{PNA}_\infty$ such that $\partial_\Omega^+(\Omega) = \nu(\Omega)$. Obviously, $\partial_\Omega^+$ is a value of $\text{FEAS}$. We next show that the value on $\text{FEAS}$ is not unique, in that $\partial_\Omega^+$ does not agree with the Aumann-Shapley (AS) value.

**Example 4.1** Let $f(x, y) = \phi_1(x) + \phi_2(y)$ where $\phi_1(x) = \frac{x^2}{2} + \frac{x}{2}$ and $\phi_1(y) = -\frac{y^2}{2} + y$. Let $\mu_1$ and $\mu_2$ be two linearly independent measures in $\text{NA}_1^+$, and $\nu = f \circ \mu$ where $\mu = (\mu_1, \mu_2)$.

We claim that $\nu \in \text{FEAS}$, but $\partial_\Omega^+(\nu) \neq \Phi(\nu)$, where $\Phi$ denotes the AS value.

The function $f \in C^1(\mathbb{R}^2)$ and we know that $\Phi(\nu) = \left[ \int_0^1 f'_x(t, t)dt \right] \mu_1 + \left[ \int_0^1 f'_y(t, t)dt \right] \mu_2$ and $\partial_\Omega^+(\nu) = f'_x(0)\mu_1 + f'_y(0)\mu_2$, namely $\partial_\Omega^+(\nu) = \frac{\mu_1}{2} + \mu_2$, while $\phi(\nu) = \mu_1 + \frac{\mu_2}{2}$. It is immediate to notice that $\nu \in \text{FEAS}$.

Define now $\text{FEAS}_\infty = \{ \nu \in \text{PNA}_\infty : \widetilde{\partial}_\Omega^+(\nu, \Omega) = \nu(\Omega) \}$. It is immediate to notice that the $\| \cdot \|_\infty$ closure of $\text{FEAS}$ is contained in $\text{FEAS}_\infty$, and that $\text{FEAS}_\infty$ is closed. $\partial_\Omega^+$ is a value on $\text{FEAS}_\infty$, and it is easy to check that, on the scalar measure games $\nu = f \circ \nu$ with $f$ continuously differentiable on $I$ and with $f'(0) = f(1)$, $\partial_\Omega^+$ agrees with the AS value.

Since all powers of $\text{NA}_+^+$ measures are contained in $Q$, it is clear that the subspace $G = Q \cap \text{PNA}_\infty$ is $\| \cdot \|_\infty$-dense in $\text{PNA}_\infty$. We have already given an example of a game in $BC \setminus \text{AC}_\infty$; hence the set $BC \setminus \text{PNA}_\infty$ is non empty. It is far more difficult to find an example in $\text{PNA}_\infty \setminus BC$; indeed, because of the above density, such a game should be the $\| \cdot \|_\infty$-limit of BC integrable games, without being itself in $BC$.

The next convergence result shows that, on the contrary, in many situations the limit of BC integrable games is itself BC integrable.

When dealing with a sequence of games $(\nu_k)_k \in BC$, we are meanwhile dealing with a sequence of measures $\lambda_k \in \text{NA}_+^+$ such that each game $\nu_k$ is $\delta_{\lambda_k}$ BC integrable. However we can always reduce it to the same mesh $\delta_{\lambda}$, where $\lambda = \sum_{n=1}^{\infty} \frac{\lambda_n}{2^n}$, thanks to Proposition 5.2 in [3].

We shall say that a sequence $(\nu_k)_k \in BC$ is *uniformly* BC integrable, if for every $F \in \Sigma$ and every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $D \in \Pi(F)$ with $\delta_{\lambda}(D) < \delta$ there follows

$$\left| \sum_{I \in D} \nu_k(I) - \partial_\Omega^+(\nu_k, F) \right| < \varepsilon,$$

for each $k \in \mathbb{N}$.

**Theorem 4.2** Let $(\nu_k)_k$ be a uniformly BC integrable sequence in $Q$ that $\| \cdot \|_\infty$-converges to $\nu$; then $\nu \in Q$. 

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Proof. Our aim is to prove that for every $F \in \Sigma$,
\[
\partial^+(\nu, F) = \lim_{\delta(\lambda(D)) \to 0} \sum_{I \in D} \nu(I),
\]
or else, that, for every $F \in \Sigma$ and $\varepsilon > 0$, there exists $\tilde{\delta}(\varepsilon) > 0$ such that for each $D \in \Pi(F)$ with $\delta(\lambda(D)) < \tilde{\delta}$,
\[
\left| \sum_{I \in D} \nu(I) - \partial^+(\nu, I) \right| < \varepsilon.
\]
Let then $F \in \Sigma$ and $\varepsilon > 0$ be fixed. By Theorem 4.1 we know that $\partial^+(\nu_k) \to \partial^+(\nu)$, which in turn implies the setwise convergence. Hence a $\tilde{k} \left( \frac{\varepsilon}{4} \right) \in \mathbb{N}$ can be found, such that for $k > \tilde{k}$
\[
\left| \partial^+(\nu_k, F) - \partial^+(\nu, F) \right| < \frac{\varepsilon}{4}.
\]
Choose now $\tilde{\delta} \left( \frac{\varepsilon}{4} \right) > 0$ such that for each $D \in \Pi(F)$ with $\delta(\lambda(D)) < \tilde{\delta}$ one has
\[
\left| \sum_{I \in D} \nu_k(I) - \partial^+(\nu_k, F) \right| < \frac{\varepsilon}{4}
\]
for all $k \in \mathbb{N}$. Let such a $D \in \Pi(F)$ be fixed, and take $k, p > \tilde{k}$; then
\[
\left| \sum_{I \in D} \nu_k(I) - \nu_p(I) \right| \leq \left| \sum_{I \in D} \nu_k(I) - \partial^+(\nu_k, F) \right| + |\partial^+(\nu_k, F) - \partial^+(\nu_p, F)| + \left| \sum_{I \in D} \nu_p(I) - \partial^+(\nu_p, F) \right| < \varepsilon.
\]
This proves that the sequence $k \mapsto \sum_{I \in D} \nu_k(I)$ is Cauchy, and therefore it converges in $\mathbb{R}$. On the other side, we have that $\nu_k \to \nu$ setwise; thus necessarily $\sum_{I \in D} \nu_k(I) \to \sum_{I \in D} \nu(I)$ and this is enough to deduce the assertion $\blacksquare$

It is then rather natural to seek conditions on the sequence $(\nu_k)_k$ that ensure its uniform BC-integrability; in the case of a sequence of scalar measure games, we have the following result

**Proposition 4.4** Let $f_k : [0, 1] \to \mathbb{R}$ be a sequence of Lipschitz functions, and let $(\lambda_k)_k$ be a sequence in $\text{NA}^1$; assume that
1. $\lim_{x \to 0^+} \frac{f_k(x)}{x} = f_k'(0)$ uniformly with respect to $k$;
2. $(\lambda_k)_k$ setwise converges to some $\mu \in \text{NA}^1$.

Then the sequence of games $\nu_k = f_k \circ \lambda_k$ is uniformly BC integrable.
Proof. Since each $\lambda_k \ll \lambda$, by 2. and the Vitali-Hahn-Saks Theorem, the absolute continuity is uniform with respect to $k$; let then $\delta = \delta(\cdot)$ be the uniform absolute continuity parameter.

Fix $F \in \Sigma, \varepsilon > 0$. Let $\rho = \rho(\varepsilon)$ be determined by the uniform limit in 1., and choose $D \in \Pi(F)$ with $\delta_\lambda(D) < \rho(\delta(\varepsilon))$. Hence for each $I \in D$ one has $\lambda(I) < \rho(\delta(\varepsilon))$ whence $\lambda_k(I) < \rho(\varepsilon)$ for each $k \in \mathbb{N}$. Therefore for every $k \in \mathbb{N}$

$$\left| \sum_{I \in D} (f_k \circ \lambda_k)(I) - f_k'(0)\lambda_k(F) \right| \leq \sum_{I \in D, \lambda_k(I) \neq 0} \left| \frac{(f_k \circ \lambda_k)(I)}{\lambda_k(I)} - f_k'(0) \right| \lambda_k(I) \leq \varepsilon \sum_{I \in D, \lambda_k(I) \neq 0} \lambda_k(I) < \varepsilon$$

(where the last inequality is justified by the fact that each $\lambda_k \in NA^1$). The proof is thus complete.

An easy case is therefore the case of $\lambda_k = P, k \in \mathbb{N}$, namely a sequence of scalar measure games $\nu_k = f_k \circ P$ with $f_k$ Lipschitz and satisfying assumption 1. of the previous Proposition is $\delta_P$ uniformly BC integrable.

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