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Numerical methods in large-scale optimization: inexact oracle and primal-dual analysis

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1 Introduction

Numerical optimization remains an active area of research since 1980’s, motivated by a vast range of applications, e.g., operations research, optimal control. Starting with the works [Karmarkar 1984, Nesterov and Nemirovskii 1994] one of the main areas of research in numerical optimization became interior-point methods. These methods combine Newton steps with penalty approach and allow to solve a very general class of convex problems in polynomial-time, which is justified both theoretically and practically. The new century introduced new challenges for numerical methods in optimization. Thanks to increasing amount of available data and more powerful computational resources, machine learning became an area of intensive research. A cornerstone optimization problem in machine learning is the empirical risk minimization with the key aspect being large dimension of the decision variable and large number of components used in the objective function. In this setting the Newton iteration becomes expensive in general since it requires matrix inversion. This motivated a sacrifice of logarithmic dependence on the accuracy to a cheap iteration and the use of first-order methods to solve such problems. Another reason was that the data is usually noisy and there is no need to solve the optimization problem to a high accuracy in this setting. Another main application for first-order methods is signal processing and image analysis, where the goal is to reconstruct a high-dimensional signal from high-dimensional data, e.g., noisy images.

Yet, known already for a long time Cauchy [1847], Polyak [1963], Robbins and Monro [1951], first-order methods entered their renaissance in 2000’s. Some important facts on these methods were already known for 15 years. In particular, the concept of black-box oracle Nemirovsky and Yudin [1983] allowed to obtain lower worst-case complexity bounds for different classes of problems and methods. In particular, a gap was discovered between the lower bound $O(1/k^2)$ and an upper bound $O(1/k)$ for gradient method for minimizing convex smooth functions. Here $k$ is the iteration counter. This gap led to an important phenomenon of acceleration for first-order methods and accelerated gradient method Nesterov [1983]. In the new century many extensions of this algorithm were proposed motivated by image processing problems and machine learning, including composite versions Beck and Teboulle [2009], Nesterov [2013], accelerated stochastic gradient method Lan [2012], accelerated variance reduction methods Johnson and Zhang [2013], Lin et al. [2014, 2015], Lan and Zhou [2017], Shalev-Shwartz and Zhang [2014]. In addition to accelerated stochastic gradient methods for finite-sum problems, which use a random choice of the gradient of the component, acceleration was introduced for other randomized methods such as random coordinate descent Nesterov [2012] and random gradient-free optimization Nesterov and Spokoiny [2017]. The latter is motivated by problems, in which only zero-order oracle is available, e.g., when the objective is given as a solu-
tion of some auxiliary problem. For this setting, it is important to analyze zero-order methods with inexact function values since this auxiliary problem may be possible to solve only inexactly. In the setting of first-order methods inexactness may also be encountered in practice. Accelerated gradient method with inexact gradients was analyzed in d’Aspremont [2008], and an important framework of inexact first-order oracle was introduced in Devolder et al. [2014] and several extensions are discussed in Gasnikov et al. [2016]. Another important extension of accelerated first-order methods are accelerated methods for problems with linear constraints, which was proposed in Beck and Teboulle [2014], yet with a non-optimal rate $O(1/k)$ for the constraints feasibility.

**Object and goals of the dissertation.** The goal of the dissertation is twofold. The first goal is to further extend the existing first and zero-order methods for problems with inexactness in function and gradient values, the inexactness being deterministic or stochastic. The second goal is to construct new primal-dual first-order methods, which allow to solve simultaneously the primal and dual problem with optimal convergence rates. A particular focus is made on problems with linear constraints and the application of the proposed methods to optimal transport distance and barycenter problems.

**The obtained results:**

1. We propose a stochastic intermediate gradient method for convex problems with stochastic inexact oracle.

2. We develop a gradient method with inexact oracle for deterministic non-convex optimization.

3. We develop gradient-free method with inexact oracle for deterministic convex optimization.

4. We develop a method to calculate the derivative of the pagerank vector and in combination with the above two methods propose gradient-based and gradient-free optimization methods for learning supervised pagerank model.

5. We develop a concept of inexact oracle for the methods which use directional derivatives and propose accelerated directional derivative method for smooth stochastic convex optimization. We also develop an accelerated and non-accelerated directional derivative method for strongly convex smooth stochastic optimization.

6. We develop primal-dual methods for solving infinite-dimensional games in convex-concave and strongly convex-concave setting.

7. We develop non-adaptive and adaptive accelerated primal-dual gradient method for strongly convex minimization problems with linear equality and inequality constraints.
8. We apply this algorithm to the optimal transport problem and obtain new complexity estimates for this problem, which in some regime are better than the ones for the Sinkhorn’s algorithm.

9. We propose a stochastic primal-dual accelerated gradient method for problems with linear constraints and apply it to the problem of approximation of Wasserstein barycenter.

10. We propose a primal-dual extension of accelerated methods which use line-search to define the stepsize and to be adaptive to the Lipschitz constant of the gradient.

**Author’s contribution** includes the development of the listed above optimization methods, proving convergence rates and complexity result theorems for these methods and their applications to optimal transport problems and learning problem for a supervised pagerank model.

**Novelties.** The proposed versions of accelerated first and zero-order methods for convex optimization under different types of inexactness are novel. The proposed primal-dual methods for the listed setups are also novel, and allow to obtain new methods for optimal transport problems. In particular, we obtain new complexity results for non-regularized optimal transport problem and a new distributed algorithm for approximating Wasserstein barycenter of a set of measures using samples from these measures.

As a result of the work on this dissertation, 10 papers were published:

**First-tier publications:**

1. Dvurechensky, P., and Gasnikov, A. Stochastic intermediate gradient method for convex problems with stochastic inexact oracle. Journal of Optimization Theory and Applications 171, 1 (2016), 121–145, Scopus Q1 (main co-author; the author of this thesis proposed main algorithms, formulated and proved convergence rate theorems for the proposed methods).

2. Gasnikov, A. V., and Dvurechensky, P. E. Stochastic intermediate gradient method for convex optimization problems. Doklady Mathematics 93, 2 (2016), 148–151, Scopus Q2 (main co-author; the author of this thesis proposed main algorithms, formulated and proved convergence rate theorems for the proposed methods).

3. Bogolubsky, L., Dvurechensky, P., Gasnikov, A., Gusev, G., Nesterov, Y., Raigorodskii, A. M., Tikhonov, A., and Zhukovskii, M. Learning supervised pagerank with gradient-based and gradient-free optimization methods. In Advances in Neural Information Processing Systems 29, D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, Eds. Curran Associates, Inc., 2016, pp. 4914–4922, CORE A* (the author of this thesis proposed general gradient-free (Algorithm
1,2) and gradient (Algorithm 3,4) methods with inexact oracle, proposed a method for approximating the derivative of the pagerank vector, formulated and proved convergence rate theorems for the proposed methods: Lemma 1,2, Theorem 1-4).

4. Dvurechensky, P., Gorbunov, E., and Gasnikov, A. An accelerated directional derivative method for smooth stochastic convex optimization. European Journal of Operational Research (2020), https://doi.org/10.1016/j.ejor.2020.08.027, Scopus Q1 (main co-author; the author of this thesis proposed a concept of inexact oracle for directional derivatives in stochastic convex optimization, proved (in inseparable cooperation with E. Gorbunov) convergence rate Theorem 1 for the accelerated directional derivative method, proved convergence rate Theorems 3,4 for strongly convex problems).

5. Dvurechensky, P., Nesterov, Y., and Spokoiny, V. Primal-dual methods for solving in infinite-dimensional games. Journal of Optimization Theory and Applications 166, 1 (2015), 23–51, Scopus Q1 (main co-author; the author of this thesis developed main algorithms and proved convergence rate theorems).

6. Dvurechensky, P., Gasnikov, A., and Kroshnin, A. Computational optimal transport: Complexity by accelerated gradient descent is better than by Sinkhorn’s algorithm. In Proceedings of the 5th International Conference on Machine Learning (2018), J. Dy and A. Krause, Eds., vol. 80 of Proceedings of Machine Learning Research, pp. 1367–1376, CORE A* (main co-author; the author of this thesis proposed general primal-dual adaptive accelerated gradient method (Algorithm 3) for problems with linear constraints, proved convergence rate Theorem 3, proposed an algorithm for approximating optimal transport (OT) distance (Algorithm 4), obtained complexity bound for approximating OT distance (Theorem 4), performed numerical experiments for comparison of this method with the Sinkhorn’s method).

7. Dvurechensky, P., Dvinskikh, D., Gasnikov, A., Uribe, C. A., and Nedić, A. Decentralize and randomize: Faster algorithm for Wasserstein barycenters. In Advances in Neural Information Processing Systems 31 (2018), S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, Eds., NeurIPS 2018, Curran Associates, Inc., pp. 10783–10793, CORE A* (main co-author; the author of this thesis proposed the general idea of the paper, general primal-dual accelerated stochastic gradient method (Algorithm 2) for problems with linear constraints, proved convergence rate Theorem 2, proposed an algorithm for approximating Wasserstein barycenter (Algorithm 4),
proved (in inseparable cooperation with D. Dvinskikh) its complexity Theorem 3).

8. Guminov, S. V., Nesterov, Y. E., Dvurechensky, P. E., and Gasnikov, A. V. Accelerated primal-dual gradient descent with linesearch for convex, nonconvex, and nonsmooth optimization problems. Doklady Mathematics 99, 2 (2019), 125-128, Scopus Q2 (the author of this thesis proposed a primal-dual variant of the accelerated gradient method with linesearch for problems with linear constraints, proved convergence rate Theorem 3).

9. Nesterov, Y., Gasnikov, A., Guminov, S., and Dvurechensky, P. Primal-dual accelerated gradient methods with small-dimensional relaxation oracle. Optimization Methods and Software (2020). [https://doi.org/10.1080/10556788.2020.1731747] Scopus Q1 (the author of this thesis proposed a primal-dual variant of the universal accelerated gradient method with small-dimensional relaxation (Algorithm 7) for problems with linear constraints, proved its convergence rate Theorem 4.1).

Second-tier publications:

1. Chernov, A., Dvurechensky, P., and Gasnikov, A. Fast primal-dual gradient method for strongly convex minimization problems with linear constraints. In Discrete Optimization and Operations Research: 9th International Conference, DOOR 2016, Vladivostok, Russia, September 19-23, 2016, Proceedings (2016), Y. Kochetov, M. Khachay, V. Beresnev, E. Nurminski, and P. Pardalos, Eds., Springer International Publishing, pp. 391–403, Web of Science and Scopus (main co-author; the author of this thesis developed main algorithm and proved convergence rate theorem).

Reports at conferences and seminars:

1. International Workshop "Advances in Optimization and Statistics", Berlin, 15.05.2014–16.05.2014, "Stochastic Intermediate Gradient Method for Convex Problems with Inexact Stochastic Oracle".

2. Seminar "Modern Methods in Applied Stochastics and Nonparametric Statistics", Berlin, 03.06.2014, "Gradient methods for convex problems with stochastic inexact oracle".

3. V International Conference on Optimization Methods and Applications (OPTIMA-2014), Petrovac, Montenegro, 28.09.2014–04.10.2014, "Gradient-free optimization methods with ball randomization".
4. VI traditional school for young scientists "Control, information, optimization", Moscow, 22.06.2014-29.06.2014, "Gradient methods for convex problems with stochastic inexact oracle".

5. 38-th conference-school of IITP RAS "Information technologies and systems", Nizhni Novgorod, 01.09.2014–05.09.2014, "Stochastic Intermediate Gradient Method for Convex Problems with Inexact Stochastic Oracle".

6. Workshop “Frontiers of High Dimensional Statistics, Optimization, and Econometrics”, Moscow, 26.02.2015–27.02.2015 г., "Random gradient-free methods for random walk based web page ranking functions learning".

7. VII traditional school for young scientists "Control, information, optimization", Moscow, 14.06.2014-20.06.2014, "Semi-Supervised PageRank Model Learning with Gradient-Free Optimization Methods".

8. 29-th conference-school of IITP RAS "Information technologies and systems", Sochi, 07.09.2014–11.09.2015, "Stochastic Intermediate Gradient Method: convex and strongly-convex case".

9. 30th annual conference of Belgian Operational Research Society (ORBEL 30), Louvain-la-Neuve, Belgium, 28.01.2016–29.01.2016, "Random gradient-free methods for ranking algorithm learning".

10. Workshop on Modern Statistics and Optimization, Moscow, 23.02.2016–24.02.2016, "Gradient and gradient-free methods for pagerank algorithm learning".

11. VII International Conference Optimization and Applications (OPTIMA 2016), Petrovac, Montenegro, 25.09.2016–02.10.2016, "Accelerated Primal-Dual Gradient Method for Linearly Constrained Minimization Problems".

12. VIII Moscow International Conference on Operations Research (ORM 2016), Moscow, 17.10.2016–22.10.2016, "Accelerated Primal-Dual Gradient Method for Composite Optimization with Unknown Smoothness Parameter".

13. Conference on Neural Information Processing Systems (NIPS 2016), Barcelona, 05.12.2016–10.12.2016, "Learning Supervised PageRank with Gradient-Based and Gradient-Free Optimization Methods".

14. Workshop Shape, Images and Optimization, Münster, Germany, 28.02.2017–03.03.2017, "Gradient Method With Inexact Oracle for Composite Non-Convex Optimization".
15. Optimization and Statistical Learning, Les Houches, France, 10.04.2017-14.04.2017, "Gradient Method With Inexact Oracle for Composite Non-Convex Optimization".

16. Foundations of Computational Mathematics, Barcelona, Spain, 10.07.2017-19.07.2017, "Gradient Method With Inexact Oracle for Composite Non-Convex Optimization".

17. Co-Evolution of Nature and Society Modelling, Problems & Experience. Devoted to Academician Nikita Moiseev centenary (Moiseev-100), Moscow, 07.11.2017–10.11.2017, "Adaptive Similar Triangles Method: a Stable Alternative to Sinkhorn’s Algorithm for Regularized Optimal Transport".

18. 18th French-German-Italian Conference on Optimization, Germany, 25.09.2017–28.09.2017, Paderborn, Germany, "Gradient method with inexact oracle for composite non-convex optimization".

19. 3. International Matheon Conference on Compressed Sensing and its Applications, Berlin, 04.12.2017–08.12.2017, "Adaptive Similar Triangles Method: a Stable Alternative to Sinkhorn’s Algorithm for Regularized Optimal Transport".

20. Games, Dynamics and Optimization (GDO2018), Vienna, Austria, 13.03.2018–15.03.2018, "Primal-Dual Methods for Solving Infinite-Dimensional Games".

21. International Conference on Machine Learning (ICML 2018), Stockholm, Sweden, 10.07.2018–15.07.2018, "Computational optimal transport: Complexity by accelerated gradient descent is better than by Sinkhorn’s algorithm".

22. 23rd International Symposium on Mathematical Programming, 01.07.2018–06.07.2018, "Computational Optimal Transport: Accelerated Gradient Descent vs Sinkhorn".

23. Grenoble Optimization Days 2018: Optimization algorithms and applications in statistical learning, Grenoble, France, 28.06.2018–29.06.2018, "Faster algorithms for (regularized) optimal transport".

24. Statistical Optimal Transport Conference, Moscow, 24.07.2018–25.07.2018, "Computational Optimal Transport: Accelerated Gradient Descent vs Sinkhorn’s Algorithm".

25. Conference on Neural Information Processing Systems (NIPS 2018), Montreal, Canada, 02.12.2018–08.12.2018, "Decentralize and randomize: Faster algorithm for Wasserstein barycenters".
26. Optimization and Statistical Learning, Les Houches, France, 24.03.2019–29.03.2019, "Distributed optimization for Wasserstein barycenter".

27. International Conference on Machine Learning (ICML 2019), Long Beach, USA, 09.06.2019–15.06.2019, "On the Complexity of Approximating Wasserstein Barycenters".

28. International Conference on Continuous Optimization (ICCOPT 2019), Berlin, Germany, 03.08.2019–08.08.2019, "A Unifying Framework for Accelerated Randomized Optimization Methods".

29. Workshop on optimization and applications, Moscow, 27.09.2019, "Accelerated Alternating Minimization for Optimal Transport".

30. Recent advances in mass transportation, Moscow, 23.09.2019–27.09.2019, "On the complexity of optimal transport problems".

31. Workshop by the GAMM Activity Group on Computational and Mathematical Methods in Data Science, Berlin, Germany, 24.10.2019–25.10.2019, "On the complexity of optimal transport problems".

32. HSE-Yandex autumn school on generative models, Moscow, 26.11.2019–29.11.2019, "Optimization methods for optimal transport".

33. Workshop on Mathematics of Deep Learning 2019, Berlin, Germany, 03.12.2019–05.12.2019, "On the complexity of optimal transport problems".

34. Workshop on PDE Constrained Optimization under Uncertainty and Mean Field Games, Berlin, Germany, 28.01.2020–30.01.2020, "Distributed optimization for Wasserstein barycenters".

2 Optimization with inexact oracle

In this section we briefly describe the methods and their convergence properties for optimization problems under inexact information. We consider first-order methods and directional derivative methods.

2.1 Stochastic intermediate gradient method for convex problems with stochastic inexact oracle

The results of this subsection are published in [Dvurechensky and Gasnikov (2016), Gasnikov and Dvurechensky (2016)].

Let $E$ be a finite-dimensional real vector space and $E^*$ be its dual. We denote the value of a linear function $g \in E^*$ at $x \in E$ by $\langle g, x \rangle$. Let $\| \cdot \|$ be some norm on $E$. We denote by $\| \cdot \|_*$ the dual norm for $\| \cdot \|_E$, i.e.
\[ \|g\|_* = \sup_{y \in E} \{ \langle g, y \rangle : \|y\|_E \leq 1 \}. \]

By \( \partial f(x) \) we denote the subdifferential of the function \( f(x) \) at a point \( x \). In this subsection, we consider the composite optimization problem of the form

\[ \min_{x \in Q} \{ \varphi(x) := f(x) + h(x) \} , \tag{1} \]

where \( Q \subset E \) is a closed and convex set, \( h(x) \) is a simple convex function, \( f(x) \) is a convex function with stochastic inexact oracle \cite{Devolder2011}. This means that, for every \( x \in Q \), there exist \( f_{\delta, L}(x) \in \mathbb{R} \) and \( g_{\delta, L}(x) \in \mathbb{E}^* \), such that

\[ 0 \leq f(y) - f_{\delta, L}(x) - \langle g_{\delta, L}(x), y - x \rangle \leq \frac{L}{2} \|x - y\|^2 + \delta , \quad \forall y \in Q , \tag{2} \]

and also that, instead of \((f_{\delta, L}(x), g_{\delta, L}(x))\) (we will call this pair a \((\delta, L)\)-oracle), we use their stochastic approximations \((F_{\delta, L}(x, \xi), G_{\delta, L}(x, \xi))\). The latter means that, for any point \( x \in Q \), we associate with \( x \) a random variable \( \xi \) whose probability distribution is supported on a set \( \Xi \subset \mathbb{R} \) and such that \( \mathbb{E}_x F_{\delta, L}(x, \xi) = f_{\delta, L}(x) \), \( \mathbb{E}_x G_{\delta, L}(x, \xi) = g_{\delta, L}(x) \) and \( \mathbb{E}_x (\|G_{\delta, L}(x, \xi) - g_{\delta, L}(x)\|_*)^2 \leq \sigma^2 \).

To deal with such problems we will need a prox-function \( d(x) \), which is differentiable and strongly convex with parameter 1 on \( Q \) with respect to \( \| \cdot \| \). Let \( x_0 \) be the minimizer of \( d(x) \) on \( Q \). By translating and scaling \( d(x) \), if necessary, we can always ensure that \( d(x_0) = 0 \), \( d(x) \geq \frac{1}{2} \|x - x_0\|^2 \), \( \forall x \in Q \). We define also the corresponding Bregman distance: \( V(x, z) = d(x) - d(z) - \langle \nabla d(z), x - z \rangle \). Let \( \{\alpha_i\}_{i \geq 0} \), \( \{\beta_i\}_{i \geq 0} \), \( \{B_i\}_{i \geq 0} \subset \mathbb{R} \) be three sequences of coefficients satisfying

\[
\alpha_0 \in [0, 1] , \quad \beta_{i+1} \geq \beta_i > L , \quad \forall i \geq 0 , \tag{3}
\]

\[
0 \leq \alpha_i \leq B_i , \quad \forall i \geq 0 , \tag{4}
\]

\[
\alpha_i \beta_k \leq B_k \beta_{k-1} \leq \left( \sum_{i=0}^k \alpha_i \right) \beta_{k-1} , \quad \forall k \geq 1 . \tag{5}
\]

\[
A_k := \sum_{i=0}^k \alpha_i , \quad \tau_i := \frac{\alpha_{i+1}}{B_{i+1}} . \tag{6}
\]

The Stochastic Intermediate Gradient Method (SIGM) is described below as Algorithm 1. Let \( a \geq 1 \) and \( b \geq 0 \) be some parameters. Let us assume that we know a number \( R \) such that \( \sqrt{2d(x_0)} \leq R \). We set for \( p \in [1, 2] \)

\[
\alpha_i = \frac{1}{a} \left( \frac{i + p}{p} \right)^{p-1} , \quad \forall i \geq 0 , \tag{7}
\]

\[
\beta_i = L + \frac{b \tau_i}{R} (i + p + 1)^{2p-1} , \quad \forall i \geq 0 , \tag{8}
\]

\[
B_i = a \alpha_i^2 = \frac{1}{a} \left( \frac{i + p}{p} \right)^{2p-2} , \quad \forall i \geq 0 . \tag{9}
\]
Theorem 2.1. If the sequences \( \{\alpha_i\}_{i \geq 0}, \{\beta_i\}_{i \geq 0}, \{B_i\}_{i \geq 0} \) are chosen according to (7), (8), (9) with \( a = 2^{\frac{2p-1}{2}} \) and \( b = 2^{-\frac{1}{2}} \), then the sequence \( y_k \) generated by the SIGM satisfies

\[
\mathbb{E}\xi_0,\ldots,\xi_k \varphi(y_k) - \varphi^* 
\leq \frac{LR^2p\|B\|^22^{\frac{p-1}{2}}}{(k+p)p} + \frac{\sigma R2^{\frac{3+2p}{2}}}{(k+p)p} \sqrt{p(k+p+2)^{p-\frac{1}{2}}} + 2^{2p-1} \left( \frac{p+k}{p} \right)^{p-1} + 1 \delta \leq \frac{C_1LR^2}{k^p} + \frac{C_2\sigma R}{\sqrt{k}} + C_3k^p\delta = \Theta \left( \frac{LR^2}{k^p} + \frac{\sigma R}{\sqrt{k}} + k^{p-1}\delta \right),
\]

where \( C_1 = 4\sqrt{2}, \ C_2 = 16\sqrt{2}, \ C_3 = 48. \)

**Algorithm 1 Stochastic Intermediate Gradient Method (SIGM)**

**Require:** The sequences \( \{\alpha_i\}_{i \geq 0}, \{\beta_i\}_{i \geq 0}, \{B_i\}_{i \geq 0} \), functions \( d(x), V(x,z) \).

**Ensure:** The point \( y_k \).

1. Compute \( x_0 := \arg\min_{x \in Q} \{d(x)\} \). Let \( \xi_0 \) be a realization of the random variable \( \xi \). Calculate \( G_{\delta,L}(x_0, \xi_0) \). Set \( k = 0 \).
2. \( y_0 := \arg\min_{x \in Q} \{\beta_0 d(x) + \alpha_0 \langle G_{\delta,L}(x_0, \xi_0), x - x_0 \rangle + \alpha_0 h(x)\} \).
3. repeat
   4. \( z_k := \arg\min_{x \in Q} \{\beta_k d(x) + \sum_{i=0}^{k} \alpha_i \langle G_{\delta,L}(x_i, \xi_i), x - x_i \rangle + A_k h(x)\} \).
   5. \( x_{k+1} := \tau_k z_k + (1 - \tau_k) y_k \).
   6. Let \( \xi_{k+1} \) be a realization of the random variable \( \xi \). Calculate \( G_{\delta,L}(x_{k+1}, \xi_{k+1}) \).
   7. \( \hat{x}_{k+1} := \arg\min_{x \in Q} \{\beta_k V(x, z_k) + \alpha_{k+1} \langle G_{\delta,L}(x_{k+1}, \xi_{k+1}), x - z_k \rangle + \alpha_{k+1} h(x)\} \).
   8. \( w_{k+1} := \tau_k \hat{x}_{k+1} + (1 - \tau_k) y_{k} \).
   9. \( y_{k+1} := \frac{A_{k+1} - B_{k+1}}{A_{k+1}} y_{k} + \frac{B_{k+1}}{A_{k+1}} w_{k+1} \).
10. until

It is possible to obtain an upper bound on the probability of large deviations for the \( \varphi(y_k) - \varphi^* \). To do that, we make the following additional assumptions.

1. \( \xi_0, \ldots, \xi_k \) are i.i.d random variables.
2. \( G_{\delta,L}(x, \xi) \) satisfies the light-tail condition

\[
\mathbb{E}_\xi \left[ \exp \left( \frac{\|G_{\delta,L}(x, \xi) - g_{\delta,L}(x)\|^2}{\sigma^2} \right) \right] \leq \exp(1).
\]

3. Set \( Q \) is bounded, and we know a number \( D > 0 \), such that \( \max_{x,y \in Q} \|x - y\| \leq D \).
Theorem 2.2. If the sequences \( \{\alpha_i\}_{i \geq 0} \), \( \{\beta_i\}_{i \geq 0} \), \( \{B_i\}_{i \geq 0} \) are chosen according to (17), (18), (19) with \( a = 2^{2p-1} \) and \( b = 2^{-2p} + 1 \), then the sequence \( y_k \) generated by the SIGM satisfies

\[
\mathbb{P}\left( \varphi(y_k) - \varphi^* > \frac{C_1 LR^2}{kp} + \frac{C_2 (1 + \Omega)\sigma R}{\sqrt{k}} + \frac{C_3 k^p - 1}{\sqrt{k}} \delta + \frac{C_4 D\sigma \sqrt{\Omega}}{\sqrt{k}} \right) \\
\leq \mathbb{P}\left( \varphi(y_k) - \varphi^* > \frac{LR^2 p^2 2^{2p-1} x}{(k+p)^p} + \frac{(1 + \Omega)\sigma R_2 2^{1+p} \sqrt{k}(k+p)^2}{(k+p)^p} \right) \\
+ 2^{2p-1} \left( \frac{(k+p)}{p} \right)^{p-1} + 1 \delta + \frac{2D\sigma \sqrt{\Omega} l p}{\sqrt{k} + p} \leq 3 \exp(-\Omega),
\]

where \( C_1 = 4\sqrt{2} \), \( C_2 = 16\sqrt{2} \), \( C_3 = 48 \), \( C_4 = 4\sqrt{3} \).

Next, we consider two modifications of the SIGM for strongly convex problems. For the first modification, we obtain the rate of convergence in terms of the non-optimality gap expectation and for the second we bound the probability of large deviations from this rate. We additionally assume that \( E \) is a Euclidean space with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \|x\| := \sqrt{\langle x, Hx \rangle} \), where \( H \) is a symmetric positive definite matrix. Without loss of generality, we assume that the function \( d(x) \) satisfies conditions \( 0 = \arg \min_{x \in Q} d(x) \) and \( d(0) = 0 \). Also we assume that the function \( \varphi(x) \) is strongly convex, i.e.

\[
\frac{1}{2} \|x - y\|^2 \leq \varphi(y) - \varphi(x) - \langle g(x), y - x \rangle \quad \text{for all } x, y \in Q, g(x) \in \partial \varphi(x).
\]

As a corollary, we have

\[
\varphi(x) - \varphi(x^*) \geq \frac{\mu}{2} \|x - x^*\|^2, \quad \forall x \in Q,
\]

where \( x^* \) is the solution of the problem (11). We also assume that \( d(x) \) satisfies the following property. If \( x_0 \) is a random vector such that \( \mathbb{E}_{x_0} \|x - x_0\|^2 \leq R_0^2 \) for some fixed point \( x \) and number \( R_0 \), then, for some \( V > 0 \),

\[
\mathbb{E}_{x_0} d\left( \frac{x - x_0}{R_0} \right) \leq \frac{V^2/2}{2}.
\]

Theorem 2.3. After \( k \geq 1 \) outer iterations of Algorithm 2, we have

\[
\mathbb{E}_x \varphi(u_k) - \varphi^* \leq \frac{\mu R_0^2}{2} e^{-k} + \frac{C_3 e 2^{p-1}}{\mu(e - 1)} \left( \frac{4 e C_1 L V^2}{\mu} \right)^{\frac{p-1}{p}} d, \quad (15)
\]

\[
\mathbb{E}_x \|u_k - x^*\|^2 \leq R_0^2 e^{-k} + \frac{C_3 e^2 p}{\mu(e - 1)} \left( \frac{4 e C_1 L V^2}{\mu} \right)^{\frac{p-1}{p}} d. \quad (16)
\]

As a consequence, if we choose the error \( \delta \) of the oracle satisfying

\[
\delta \leq \frac{\varepsilon(e - 1)}{2^p C_3 e} \left( \frac{4 e C_1 L V^2}{\mu} \right)^{\frac{p-1}{p}}, \quad (17)
\]

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Algorithm 2 Stochastic Intermediate Gradient Method for Strongly Convex Problems

Require: The function \( d(x) \), point \( u_0 \), number \( R_0 \) such that \( \|u_0 - x^*\| \leq R_0 \), number \( p \in [1,2] \).

Ensure: The point \( u_{k+1} \).

1: Set \( k = 0 \).
2: Calculate
\[
N_k := \left\lceil \left( \frac{4eC_1LV^2}{\mu} \right)^\frac{1}{p} \right\rceil. \tag{12}
\]

3: repeat
4: Calculate
\[
m_k := \max \left\{ 1, \left\lceil \frac{16e^{k+2}C_2^2\sigma^2V^2}{\mu^2R_0^2N_k} \right\rceil \right\}, \tag{13}
\]
\[
R_k^2 := R_0^2e^{-k} + \frac{2peC_3\delta}{\mu(e - 1)} \left( \frac{4eC_1LV^2}{\mu} \right)^\frac{1}{p} \left( 1 - e^{-k} \right). \tag{14}
\]
5: Run Algorithm 1 with \( x_0 = u_k \) and prox-function \( d\left( \frac{x-u_k}{R_k} \right) \) for \( N_k \) steps, using oracle \( \tilde{G}_{\delta,L}(x) := \frac{1}{m_k} \sum_{i=1}^{m_k} G_{\delta,L}(x,\xi^i) \), where \( \xi^i \), \( i = 1,\ldots,m_k \) are i.i.d. on each step and sequences \( \{\alpha_i\}_{i \geq 0}, \{\beta_i\}_{i \geq 0}, \{B_i\}_{i \geq 0} \) defined in Theorem 2.1.
6: Set \( u_{k+1} = y_{N_k}, k = k + 1 \).
7: until

then we need \( N = \left\lceil \ln \left( \frac{\mu R_0^2}{\varepsilon} \right) \right\rceil \) outer iterations and no more than
\[
\left( 1 + \left( \frac{4eC_1LV^2}{\mu} \right)^\frac{1}{p} \right) \left( 1 + \ln \left( \frac{\mu R_0^2}{\varepsilon} \right) \right) + \frac{16e^3C_2^2\sigma^2V^2}{\mu \varepsilon (e - 1)}
\]
oracle calls to guarantee that \( \mathbb{E} \varphi(u_N) - \varphi^* \leq \varepsilon \).

To obtain complexity in terms of large deviations probability, we assume that the prox-function has quadratic growth with parameter \( V^2 \) with respect to the chosen norm, i.e.
\[
d(x) \leq \frac{V^2}{2}\|x\|^2, \quad \forall x \in \mathbb{R}^n. \tag{18}\]

Now we present a modification of Algorithm 2 and a theorem with a bound for the probability of large deviations for the non-optimality gap of this algorithm.
Algorithm 3 Stochastic Intermediate Gradient Method for Strongly Convex Problems

Require: The function $d(x)$, point $u_0$, number $R_0$ such that $\|u_0 - x^*\| \leq R_0$, number $p \in [1, 2]$, number $N \geq 1$ of outer iterations, confidence level $\Lambda$.

Ensure: The point $u_N$.

1: Set $k = 0$.
2: Calculate
$$N_k := \left\lceil \left( \frac{6e C_1 LV^2}{\mu} \right)^{\frac{1}{p}} \right\rceil.$$

3: repeat
4: Calculate
$$m_k := \max \left\{ 1, \left[ \frac{36e^{k+2} C_2^2 \sigma^2 V^2 (1 + \ln \left( \frac{3N}{\Lambda} \right))^2}{\mu^2 R_0^2 N_k} \right], \left[ \frac{144e^{k+2} C_2^2 \sigma^2 \ln \left( \frac{3N}{\Lambda} \right)}{\mu^2 R_0^2 N_k} \right] \right\},$$

$$R_k^2 := R_0^2 e^{-k} + \frac{2\epsilon C_3 \delta}{\mu (e-1)} \left( \frac{6e C_1 LV^2}{\mu} \right)^{\frac{p-1}{p}} \left( 1 - e^{-k} \right),$$

$$Q_k := \{ x \in Q : \|x - u_k\|^2 \leq R_k^2 \}.$$

5: Run Algorithm 1 applied to the problem $\min_{x \in Q_k} \varphi(x)$ with $x_0 = u_k$ and prox-function $d\left( \frac{x - u_k}{R_k^2} \right)$ for $N_k$ steps using oracle $\tilde{G}^{k}_{\delta,L}(x) := \frac{1}{m_k} \sum_{i=1}^{m_k} G_{\delta,L}(x, \xi^i)$, where $\xi^i$, $i = 1, \ldots, m_k$ are i.i.d, on each step and sequences $\{\alpha_i\}_{i \geq 0}$, $\{\beta_i\}_{i \geq 0}$, $\{B_i\}_{i \geq 0}$ defined in Theorem 2.1.

6: Set $u_{k+1} = y_{N_k}$, $k = k + 1$.
7: until $k = N - 1$

Theorem 2.4. After $N$ outer iterations of Algorithm 3, we have
$$\mathbb{P} \left\{ \varphi(u_N) - \varphi^* > \frac{\mu R_0^2}{2} e^{-N} + \frac{2^{p-1} e C_3 \delta}{(e-1)} \left( \frac{6e C_1 LV^2}{\mu} \right)^{\frac{p-1}{p}} \delta \right\} \leq \Lambda.$$

As a consequence, if we choose error of the oracle $\delta$ satisfying
$$\delta \leq \frac{\epsilon (e-1)}{2^p C_3 e} \left( \frac{6e C_1 LV^2}{\mu} \right)^{\frac{1-e}{p}},$$
then we need no more than $N = \left\lceil \ln \left( \frac{\mu R_0^2}{\epsilon} \right) \right\rceil$ outer iterations and no more
\[
\left(1 + \left(\frac{6eC_1LV^2}{\mu}\right)^\frac{1}{2}\right) \left(1 + \ln \left(\frac{\mu R_0^2}{\varepsilon}\right)\right) + \\
\frac{36e^3C_2^2\sigma^2V^2}{\mu(e-1)\varepsilon} \left(1 + \ln \left(\frac{3}{\Lambda} \left(1 + \ln \left(\frac{\mu R_0^2}{\varepsilon}\right)\right)^\frac{1}{2}\right)\right)^2 + \\
\frac{144e^3C_2^2\sigma^2}{\mu\varepsilon(e-1)} \ln \left(\frac{3}{\Lambda} \left(1 + \ln \left(\frac{\mu R_0^2}{\varepsilon}\right)\right)^\frac{1}{2}\right)
\]

(25)

oracle calls to guarantee that \(\mathbb{P}\{\phi(u_N) - \phi^* > \varepsilon\} \leq \Lambda\).

2.2 Learning supervised pagerank with gradient-based and gradient-free optimization methods.

In this subsection we consider a parametric model for web-page ranking and learning the parameters of this model in a supervised setting. The results of this subsection are published in Bogolubsky et al. [2016].

2.2.1 Model description

Let \(\Gamma = (V,E)\) be a directed graph. We suppose that for any \(i \in V\) and any \(i \to j \in E\), a vector of node’s features \(V_i \in \mathbb{R}^{m_1}\) and a vector of edge’s features \(E_{ij} \in \mathbb{R}^{m_2}\) are given. Let \(\varphi_1 \in \mathbb{R}^{m_1}, \varphi_2 \in \mathbb{R}^{m_2}\) be two vectors of parameters. We denote \(m = m_1 + m_2, p = |V|, \varphi = (\varphi_1, \varphi_2)^T\). Let us describe the random walk on the graph \(\Gamma\). A surfer starts a random walk from a random page \(i \in U\) (\(U\) is some subset in \(V\) called seed set, \(|U| = n\)). We assume that \(\varphi_1\) and node features are chosen in such way that \(\sum_{l \in U} \langle \varphi_1, V_l \rangle\) is non-zero. The initial probability of being at vertex \(i \in V\) is called the restart probability and equals

\[
[\pi^0(\varphi)]_i = \frac{\langle \varphi_1, V_i \rangle}{\sum_{l \in U} \langle \varphi_1, V_l \rangle}, \quad i \in U
\]

(26)

and \([\pi^0(\varphi)]_i = 0\) for \(i \in V \setminus U\). At each step, the surfer (with a current position \(i \in V\)) either chooses with probability \(\alpha \in (0,1)\), which is called the damping factor, to go to any vertex from \(V\) in accordance with the distribution \(\pi^0(\varphi)\) (makes a restart) or chooses to traverse an outgoing edge (makes a transition) with probability \(1 - \alpha\). We assume that \(\varphi_2\) and edges features are chosen in such way that \(\sum_{l \in U} \langle \varphi_2, E_{ij} \rangle\) is non-zero for all \(i\) with non-zero outdegree. For \(i\) with non-zero outdegree, the probability

\[
[P(\varphi)]_{i,j} = \frac{\langle \varphi_2, E_{ij} \rangle}{\sum_{l \in U} \langle \varphi_2, E_{il} \rangle}
\]

(27)

of traversing an edge \(i \to j \in E\) is called the transition probability. If an outdegree of \(i\) equals 0, then we set \([P(\varphi)]_{i,j} = [\pi^0(\varphi)]_j\) for all \(j \in V\) (the
surfer with current position \( i \) makes a restart with probability 1). Finally, by Equations (26) and (27) the total probability of choosing vertex \( j \in V \) conditioned by the surfer being at vertex \( i \) equals \( \alpha [\pi^0(\varphi)]_j + (1 - \alpha) [P(\varphi)]_{i,j} \).

Denote by \( \pi(\varphi) \in \mathbb{R}^p \) the stationary distribution of the described Markov process. It can be found as a solution of the system of equations

\[
\pi = \alpha \pi^0(\varphi) + (1 - \alpha) P^T(\varphi) \pi
\]  

(28)

We learn the ranking algorithm, which orders the vertices \( i \) by their probabilities \( \pi_i \) in the stationary distribution \( \pi \).

### 2.2.2 Loss-minimization problem statement

Let \( Q \) be a set of queries and, for any \( q \in Q \), a set of nodes \( V_q \) which are relevant to \( q \) be given. We are also provided with a ranking algorithm which assigns nodes ranking scores \( \pi_q(i), i \in V_q, \pi_q = \pi_q(\varphi) \), as its output. For example, in web search, the score \( \pi_q(i) \) may represent relevance of the page \( i \) w.r.t. the query \( q \). Our goal is to find the parameter vector \( \varphi \) which minimizes the discrepancy of the ranking scores from the ground truth scoring defined by assessors. For each \( q \in Q \), there is a set of nodes in \( V_q \) manually judged and grouped by relevance labels \( 1, \ldots, \ell \). We denote \( V_{q,j} \) the set of nodes annotated with label \( \ell + 1 - j \) (i.e., \( V_{q,1} \) is the set of all nodes with the highest relevance score). Following the common approach, we consider the square loss function and minimize

\[
f(\varphi) = \frac{1}{|Q|} \sum_{q=1}^{|Q|} \| (A_q \pi_q(\varphi))_+ \|^2_2
\]  

(29)

as a function of \( \varphi \) over some set of feasible values \( \Phi \), where vector \( x_+ \) has components \( [x_+]_i = \max\{x_i, 0\} \), the matrices \( A_q \in \mathbb{R}^{r_q \times p_q}, q \in Q \) represent assessor’s view of the relevance of pages to the query \( q \), \( r_q \) equals \( \sum_{1 \leq j < l \leq \ell} |V_{q,j}||V_{q,l}| \). We denote \( r = \max_{q \in Q} r_q \). By definition each row of matrix \( A_q \) corresponds to some pair of pages \( i_1 \in V_{q,j}, i_2 \in V_{q,l} \), where \( j < l \), and the \( i_1 \)-th element of this row is equal to \(-1\), \( i_2 \)-th element is equal to \(1\), and all other elements are equal to \(0\).

We consider the ranking algorithm based on scores (28) in Markov random walk on a graph \( \Gamma_q = (V_q, E_q) \). We assume that feature vectors \( V_q(i), i \in V_q, E_{ij}^q, i \rightarrow j \in E_q \), depend on \( q \) as well. For example, vertices in \( V_q \) may represent web pages which were visited by users after submitting a query \( q \) and features may reflect different properties of query–page pair.

For fixed \( q \in Q \), we consider all the objects related to the graph \( \Gamma_q \) introduced in the previous section: \( U_q := U, \pi^0_q := \pi^0, P_q := P, p_q := p, n_q := n, \pi_q := \pi \). This allows ranking model to capture common (“static”) dependencies, which do vary between different queries. In this way, the
ranking scores depend on query via the “dynamic” (query-dependent) features, but the parameters of the model \( \alpha \) and \( \varphi \) are not query-dependent. We also denote \( p = \max_{q \in Q} p_q \), \( n = \max_{q \in Q} n_q \), \( s = \max_{q \in Q} s_q \), where \( s_q = \max_{i \in V_q} \{ j : i \to j \in E_q \} \). In order to guarantee that the probabilities in (26) and (27) are non-negative and that they do not blow up due to zero value of the denominator, we need appropriately choose the set \( \Phi \) of possible values of parameters \( \varphi \). Thus we choose some \( \hat{\varphi} \) and \( R > 0 \) such that the set \( \Phi = \{ \varphi \in \mathbb{R}^m : \| \varphi - \hat{\varphi} \|_2 \leq R \} \) lies in the set of vectors with positive components \( \mathbb{R}^m_{++} \). The loss-minimization problem which we solve is as follows

\[
\min_{\varphi \in \Phi} f(\varphi), \quad \Phi = \{ \varphi \in \mathbb{R}^m : \| \varphi - \hat{\varphi} \|_2 \leq R \}. \tag{30}
\]

From (28), we obtain the following equation for \( p \times m \) matrix \( \frac{d\pi_q(\varphi)}{d\varphi} \) which is the derivative of stationary distribution \( \pi_q(\varphi) \) with respect to \( \varphi \)

\[
\frac{d\pi_q(\varphi)}{d\varphi} = \alpha \frac{d\pi_q^0(\varphi)}{d\varphi} + (1 - \alpha) \sum_{i=1}^{p_q} \frac{d\pi_i(\varphi)}{d\varphi}[\pi_q(\varphi)]_i + (1 - \alpha) P_q^T(\varphi) \frac{d\pi_q(\varphi)}{d\varphi},
\]

where \( p_i(\varphi) \) is the \( i \)-th column of the matrix \( P_q^T(\varphi) \). Then the gradient of the function \( f(\varphi) \) is easy to derive:

\[
\nabla f(\varphi) = 2 \left| \sum_{q=1}^{|Q|} \left( \frac{d\pi_q(\varphi)}{d\varphi} \right)^T A_q \pi_q(\varphi) \right|_+.
\]

The method [Nesterov and Nemirovski 2015] for approximation of \( \pi_q(\varphi) \) for any fixed \( q \in Q \) constructs a sequence \( \pi_k \) and the output \( \tilde{\pi}_q(\varphi, N) \) (for some fixed non-negative integer \( N \)) by the following rule

\[
\pi_0 = \pi_q^0(\varphi), \quad \pi_{k+1} = P_q^T(\varphi) \pi_k, \quad \tilde{\pi}_q(\varphi, N) = \frac{\alpha}{1 - (1 - \alpha)^{N+1}} \sum_{k=0}^{N} (1 - \alpha)^k \pi_k.
\]

**Lemma 2.1.** Assume that for some \( \delta_1 > 0 \) Method \( \text{[33]} \) with \( N = \left\lfloor \frac{1}{\alpha} \ln \frac{8}{\delta_1} \right\rfloor + 1 \) is used to calculate the vector \( \tilde{\pi}_q(\varphi, N) \) for every \( q \in Q \). Then

\[
\tilde{f}(\varphi, \delta_1) = \frac{1}{|Q|} \sum_{q=1}^{|Q|} \|(A_q \tilde{\pi}_q(\varphi, N))_+\|_2^2
\]

satisfies

\[
|\tilde{f}(\varphi, \delta_1) - f(\varphi)| \leq \delta_1.
\]

Moreover, the calculation of \( \tilde{f}(\varphi, \delta_1) \) requires not more than \( |Q|(3mps + 3psN + 6r) \) a.o. and not more than 3ps memory items.
Our generalization of the method [Nesterov and Nemirovski 2015] for calculation of $\frac{d\pi_0(\varphi)}{d\varphi^T}$ for any $q \in Q$ is the following. Choose some non-negative integer $N_1$ and calculate $\tilde{\pi}_q(\varphi, N_1)$ using (33). Calculate a sequence $\Pi_k$

$$\Pi_0 = \alpha \frac{d\pi_0(\varphi)}{d\varphi^T} + (1 - \alpha) \sum_{i=1}^{p_q} \frac{dp_i(\varphi)}{d\varphi^T} [\tilde{\pi}_q(\varphi, N_1)]_i, \quad \Pi_{k+1} = P_q^T(\varphi)\Pi_k. \quad (36)$$

The output is (for some fixed non-negative integer $N_2$)

$$\tilde{\Pi}_q(\varphi, N_2) = \frac{1}{1 - (1 - \alpha)^{N_2+1}} \sum_{k=0}^{N_2} (1 - \alpha)^k \Pi_k. \quad (37)$$

In what follows, we use the following norm on the space of matrices $A \in \mathbb{R}^{n_1 \times n_2}$: $\|A\|_1 = \max_{j=1, \ldots, n_2} \sum_{i=1}^{n_1} |a_{ij}|$.

**Lemma 2.2.** Let $\beta_1$ be a number (explicitly computable, see Bogolubsky et al. [2016]) such that for all $\varphi \in \Phi$

$$\alpha \left\| \frac{d\pi_0(\varphi)}{d\varphi^T} \right\|_1 + (1 - \alpha) \sum_{i=1}^{p_q} \left\| \frac{dp_i(\varphi)}{d\varphi^T} \right\|_1 \leq \beta_1. \quad (38)$$

Assume that Method (33) with $N_1 = \left\lceil \frac{1}{\alpha} \ln \frac{24\beta_1 r}{\alpha \delta_2} \right\rceil - 1$ is used for every $q \in Q$ to calculate the vector $\tilde{\pi}_q(\varphi, N_1)$ and Method (36), (37) with $N_2 = \left\lceil \frac{1}{\alpha} \ln \frac{8\beta_1 r}{\alpha \delta_2} \right\rceil - 1$ is used for every $q \in Q$ to calculate the matrix $\tilde{\Pi}_q(\varphi, N_2)$ (37). Then the vector

$$\tilde{g}(\varphi, \delta_2) = \frac{2}{|Q|} \sum_{q=1}^{|Q|} \left( \tilde{\Pi}_q(\varphi, N_2) \right)^T A_q^T(A_q \tilde{\pi}_q(\varphi, N_1))_+ \quad (39)$$

satisfies

$$\|\tilde{g}(\varphi, \delta_2) - \nabla f(\varphi)\|_\infty \leq \delta_2. \quad (40)$$

Moreover the calculation of $\tilde{g}(\varphi, \delta_2)$ requires no more than $|Q|(10mps + 3psN_1 + 3mpsN_2 + 7r)$ a.o. and not more than $4ps + 4mp + r$ memory items.

As we see, there is an inexact oracle available for the considered supervised learning problem. Thus, in the next subsections, we consider a general problem with inexact oracle and solve it by zero-order and first-order methods.
2.2.3 Solving the learning problem by zero-order method

First, we consider a general zero-order method with inexact function evaluations and then we apply it to solve the learning problem. Let $\mathcal{E}$ be an $m$-dimensional vector space. First, we consider a general function $f(\cdot) : \mathcal{E} \to \mathbb{R}$ and denote its argument by $x$ or $y$ to avoid confusion with the above text. We denote the value of linear function $g \in \mathcal{E}^*$ at $x \in \mathcal{E}$ by $\langle g, x \rangle$. We choose some norm $\| \cdot \|$ in $\mathcal{E}$ and say that $f \in C_{L}^{1,1}(\| \cdot \|)$ iff

$$|f(x) - f(y) - \langle \nabla f(y), x - y \rangle| \leq \frac{L}{2} \|x - y\|^2, \quad \forall x, y \in \mathcal{E}. \quad (41)$$

The problem of our interest is to find $\min_{x \in X} f(x)$, where $f \in C_{L}^{1,1}(\| \cdot \|)$, $X$ is a closed convex set and there exists a number $D \in (0, +\infty)$ such that $\text{diam} X := \max_{x,y \in X} \|x - y\| \leq D$. Also we assume that the inexact zero-order oracle for $f(x)$ returns a value $\tilde{f}(x, \delta) = f(x) + \delta(x)$, where $\delta(x)$ is the error satisfying for some $\delta > 0$ (which is known) $|\delta(x)| \leq \delta$ for all $x \in X$. Let $x^* \in \arg\min_{x \in X} f(x)$. Denote $f^* = \min_{x \in X} f(x)$.

Unlike Nesterov and Spokoiny [2017], we define the biased gradient-free oracle $g_{\tau}(x, \delta) = \frac{\tau}{T}(\tilde{f}(x + \tau \xi, \delta) - \tilde{f}(x, \delta))\xi$, where $\xi$ is a random vector uniformly distributed over the unit sphere $S = \{ t \in \mathbb{R}^m : \|t\|_2 = 1 \}$, $\tau$ is a smoothing parameter.

Algorithm 4 below is the variation of the projected gradient descent method. Here $\Pi_X(x)$ denotes the Euclidean projection of a point $x$ onto the set $X$.

**Algorithm 4** Gradient-type method

1. **Input:** Point $x_0 \in X$, stepsize $h > 0$, number of steps $M$.
2. Set $k = 0$.
3. **repeat**
   4. Generate $\xi_k$ and calculate corresponding $g_{\tau}(x_k, \delta)$.
   5. Calculate $x_{k+1} = \Pi_X(x_k - hg_{\tau}(x_k, \delta))$.
   6. Set $k = k + 1$.
4. **until** $k > M$
5. **Output:** The point $y_M = \arg\min_{x \in \{x_0, \ldots, x_M\}} f(x)$.

Next theorem gives the convergence rate of Algorithm 4. Denote by $\Xi_k = (\xi_0, \ldots, \xi_k)$ the history of realizations of the vector $\xi$ generated on each iteration of the algorithm.

**Theorem 2.5.** Let $f \in C_{L}^{1,1}(\| \cdot \|_2)$ and convex. Assume that $x^* \in \text{int} X$, and the sequence $x_k$ is generated by Algorithm 4 with $h = \frac{1}{8mL}$. Then for any $M \geq 0$, we have

$$\mathbb{E}_{\Xi_{M-1}} f(y_M) - f^* \leq \frac{8mLD^2}{M+1} + \frac{\tau^2 L(m+8)}{8} + \frac{\delta m D}{4\tau} + \frac{\delta^2 m}{L\tau^2}. \quad (42)$$
Algorithm 5 Gradient-free method for Problem (30)

1: **Input:** Point $\varphi_0 \in \Phi$, $L$ – Lipschitz constant for the function $f(\varphi)$ on $\Phi$, accuracy $\varepsilon > 0$. 
2: Define $M = \left\lceil \frac{128mL^2}{\varepsilon^2} \right\rceil$, $\delta = \frac{\varepsilon^2\sqrt{2}}{16mR\sqrt{L(m+8)}}$, $\tau = \sqrt{\frac{2\varepsilon}{L(m+8)}}$. 
3: Set $k = 0$. 
4: repeat 
5: Generate random vector $\xi_k$ uniformly distributed over a unit Euclidean sphere $S$ in $\mathbb{R}^m$. 
6: Calculate $\tilde{f}(\varphi_k + \tau\xi_k, \delta), \tilde{f}(\varphi_k, \delta)$ using Lemma 2.1 with $\delta_1 = \delta$. 
7: Calculate $g_\tau(\varphi_k, \delta) = \frac{m}{8mLg_\tau(\varphi_k, \delta)}$. 
8: Calculate $\varphi_{k+1} = \Pi_\Phi(\varphi_k - \frac{1}{8mL}g_\tau(\varphi_k, \delta))$. 
9: Set $k = k + 1$. 
10: until $k > M$ 
11: **Output:** The point $\hat{\varphi}_M = \arg\min_{\varphi \in \{\varphi_0, \ldots, \varphi_M\}} f(\varphi)$.

Next, we apply the above method to solve the learning problem (30). The resulting algorithm is listed as Algorithm 5. The most computationally hard on each iteration of the main cycle of this method are calculations of $\tilde{f}(\varphi_k + \tau\xi_k, \delta), \tilde{f}(\varphi_k, \delta)$. Using Lemma 2.1 we obtain that each iteration of Algorithm 5 needs no more than

$$2|Q| \left(3mps + \frac{3ps}{\alpha} \ln \frac{128mR^2 \sqrt{L(m+8)}}{\varepsilon^{3/2} \sqrt{2}} + \frac{1}{6r} \right)$$

a.o. So, we obtain the following result, which gives the complexity of Algorithm 5.

**Theorem 2.6.** Assume that the set $\Phi$ in (30) is chosen in a way such that $f(\varphi)$ is convex on $\Phi$ and some $\varphi^* \in \arg\min_{\varphi \in \Phi} f(\varphi)$ belongs also to $\text{int}\Phi$. Then the mean total number of arithmetic operations of the Algorithm 5 for the accuracy $\varepsilon$ (i.e. for the inequality $\mathbb{E}_{\Xi_{M-1}} f(\hat{\varphi}_M) - f(\varphi^*) \leq \varepsilon$ to hold) is no more than

$$768mps|Q| L^2 \varepsilon \left(m + \frac{1}{\alpha} \ln \frac{128mR^2 \sqrt{L(m+8)}}{\varepsilon^{3/2} \sqrt{2}} + \frac{1}{6r} \right).$$

2.2.4 Solving the learning problem by first-order method

First we consider a general first-order method with inexact function values and inexact gradient, and then we apply it to solve the learning problem. We generalize the approach in Ghadimi et al. [2016] for constrained non-convex optimization problems. Our main contribution consists in developing this
framework for an inexact first-order oracle and unknown "Lipschitz constant" of this oracle.

Let $E$ be a finite-dimensional real vector space and $E^*$ be its dual. We denote the value of linear function $g \in E^*$ at $x \in E$ by $\langle g, x \rangle$. Let $\| \cdot \|$ be some norm on $E$, $\| \cdot \|_*$ be its dual. Our problem of interest in this subsection is a composite optimization problem of the form

$$\min_{x \in X} \{ \psi(x) := f(x) + h(x) \},$$

where $X \subset E$ is a closed convex set, $h(x)$ is a simple convex function, e.g. $\| x \|_1$. We assume that $f(x)$ is a general function endowed with an inexact first-order oracle in the following sense. There exists a number $L \in (0, +\infty)$ such that for any $\delta \geq 0$ and any $x \in X$ one can calculate $\tilde{f}(x, \delta) \in \mathbb{R}$ and $\tilde{g}(x, \delta) \in E^*$ satisfying

$$|f(y) - (\tilde{f}(x, \delta) - \langle \tilde{g}(x, \delta), y - x \rangle)| \leq \frac{L}{2} \| x - y \|^2 + \delta. \quad (44)$$

for all $y \in X$. The constant $L$ can be considered as "Lipschitz constant" because for the exact first-order oracle for a function $f \in C^{1,1}_L(\| \cdot \|)$ holds with $\delta = 0$. This is a generalization of the concept of $(\delta, L)$-oracle considered in ? for convex problems.

We choose a prox-function $d(x)$ which is continuously differentiable and 1-strongly convex on $X$ with respect to $\| \cdot \|$. This means that for any $x, y \in X$

$$d(y) - d(x) - \langle \nabla d(x), y - x \rangle \geq \frac{1}{2} \| y - x \|^2. \quad \text{We define also the corresponding Bregman distance} \quad V(x, z) = d(x) - d(z) - \langle \nabla d(z), x - z \rangle. \quad \text{Theorem 2.7.} \quad \text{Assume that} \quad f(x) \text{\ is endowed with the inexact first-order oracle in the sense of (44) and that there exists a number} \quad \psi^* > -\infty \text{\ such that} \quad \psi(x) \geq \psi^* \text{\ for all} \quad x \in X. \quad \text{Then after} \quad M \text{\ iterations of Algorithm (6) it holds that} \quad \| M_K(x_K - x_{K+1}) \|^2 \leq \frac{4L(\psi(x_0) - \psi^*)}{M + 1} + \frac{\varepsilon}{2}. \quad (45)$$

Moreover, the total number of inner steps is no more than $M + \log_2 \frac{2L}{\varepsilon_0}$.\footnote{Theorem 2.7.}

Next we apply the general method to the learning problem. Since $\Phi$ is the Euclidean ball, it is natural to set $E = \mathbb{R}^m$ and $\| \cdot \| = \| \cdot \|_2$. Then the Bregman distance is $V(\varphi, \omega) = \frac{1}{2} \| \varphi - \omega \|^2$. Algorithm (7) is a formal record of the algorithm.

The most computationally consuming operations of the inner cycle of Algorithm (7) are calculations of $\tilde{f}(\varphi_k, \delta_1)$, $\tilde{f}(\omega_k, \delta_1)$ and $\tilde{g}(\varphi_k, \delta_2)$. Using Lemma 2.1 and Lemma 2.2 we obtain that each inner iteration of Algorithm (7) needs no more than

$$7r |Q| + \frac{6mp|Q|}{\alpha} \ln \frac{1024\beta_1 r RL \sqrt{m}}{\alpha \varepsilon}$$

a.o. Using Theorem 2.7, we obtain the following result, which gives the complexity of Algorithm (7).
Algorithm 6 Adaptive projected gradient algorithm

1: **Input:** Point $x_0 \in X$, number $L_0 > 0$.
2: Set $k = 0$, $z = +\infty$.
3: **repeat**
4: Set $M_k = L_k$, flag = 0.
5: **repeat**
6: Set $\delta = \frac{1}{16M_k}$. Calculate $\tilde{f}(x_k, \delta)$ and $\tilde{g}(x_k, \delta)$.
7: $w_k = \arg\min_{x \in Q}\{\langle \tilde{g}(x_k, \delta), x \rangle + M_k V(x, x_k) + h(x)\}$
8: If the inequality $\tilde{f}(w_k, \delta) \leq \tilde{f}(x_k, \delta) + \langle \tilde{g}(x_k, \delta), w_k - x_k \rangle + \frac{M_k}{2}\|w_k - x_k\|^2 + \frac{\varepsilon}{8M_k}$ holds, set flag = 1. Otherwise set $M_k = 2M_k$.
9: **until** flag = 1
10: Set $x_{k+1} = w_k$, $L_{k+1} = \frac{M_k}{2}$.
11: If $\|M_k(x_k - x_{k+1})\| < z$, set $z = \|M_k(x_k - x_{k+1})\|$, $K = k$.
12: Set $k = k + 1$.
13: **until** $z \leq \varepsilon$
14: **Output:** The point $x_{K+1}$.

**Theorem 2.8.** The total number of arithmetic operations in Algorithm 7 for the accuracy $\varepsilon$ (i.e. for the inequality $\|M_K(\varphi_K - \varphi_{K+1})\|^2 \leq \varepsilon$ to hold) is no more than

$$\left(\frac{8L(f(\varphi_0) - f^*)}{\varepsilon} + \log_2 \frac{2L}{L_0}\right) \cdot \left(7r|Q| + \frac{6mps|Q|}{\alpha} \ln \frac{1024\beta_1 r RL_{\alpha \varepsilon}}{\alpha \varepsilon}\right).$$

2.3 An accelerated directional derivative method for smooth stochastic convex optimization.

In this section we consider directional derivatives methods with inexact oracle for stochastic convex optimization. The results of this subsection are published in [Dvurechensky et al., 2020]. Motivated by potential presence of non-stochastic noise in an optimization problem, we assume that the noise in the directional derivative consists of two parts. Similar to stochastic optimization problems, the first part is of a stochastic nature. On the opposite, the second part is an additive noise of an unknown nature, but bounded in the absolute value. More precisely, we consider the following optimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) := \mathbb{E}_\xi[F(x, \xi)] = \int_X F(x, \xi) dP(x) \right\}, \quad (46)$$

where $\xi$ is a random vector with probability distribution $P(\xi)$, $\xi \in \mathcal{X}$, and for $P$-almost every $\xi \in \mathcal{X}$, the function $F(x, \xi)$ is closed and convex. Moreover,
Algorithm 7 Adaptive gradient method for Problem (30)

1: **Input:** Point $\varphi_0 \in \Phi$, number $L_0 > 0$, accuracy $\varepsilon > 0$.
2: Set $k = 0$, $z = +\infty$.
3: **repeat**
4: Set $M_k = L_k$, flag = 0.
5: **repeat**
6: Set $\delta_1 = \frac{\varepsilon}{2M_k}$, $\delta_2 = \frac{\varepsilon}{64M_k R^m}$.
7: Calculate $\tilde{f}(\varphi_k, \delta_1)$ using Lemma 2.1 and $\tilde{g}(\varphi_k, \delta_2)$ using Lemma 2.2.
8: Find $\omega_k = \arg \min_{\varphi \in \Phi} \left\{ \langle \tilde{g}(\varphi_k, \delta_2), \varphi \rangle + \frac{M_k}{2} \| \varphi - \varphi_k \|^2 \right\}$
9: Calculate $\tilde{f}(\omega_k, \delta_1)$ using Lemma 2.1.
10: If the inequality
      $$\tilde{f}(\omega_k, \delta_1) \leq \tilde{f}(\varphi_k, \delta_1) + \langle \tilde{g}(\varphi_k, \delta_2), \omega_k - \varphi_k \rangle + \frac{M_k}{2} \| \omega_k - \varphi_k \|^2 + \frac{\varepsilon}{8M_k}$$
      holds, set flag = 1. Otherwise set $M_k = 2M_k$.
11: **until** flag = 1
12: Set $\varphi_{k+1} = \omega_k$, $L_{k+1} = \frac{M_k}{2}$.
13: If $\| M_k(\varphi_k - \varphi_{k+1}) \|_2 < \varepsilon$, set $z = \| M_k(\varphi_k - \varphi_{k+1}) \|_2$, $K = k$.
14: Set $k = k + 1$.
15: **until** $z \leq \varepsilon$
16: **Output:** The point $\varphi_{K+1}$.

we assume that, for $P$ almost every $\xi$, the function $F(x, \xi)$ has gradient $g(x, \xi)$, which is $L(\xi)$-Lipschitz continuous with respect to the Euclidean norm and there exists $L_2 \geq 0$ such that $\sqrt{\mathbb{E}_\xi L(\xi)^2} \leq L_2 < +\infty$. Under this assumptions, $\mathbb{E}_\xi g(x, \xi) = \nabla f(x)$ and $f$ has $L_2$-Lipschitz continuous gradient with respect to the Euclidean norm. Also we assume that

$$\mathbb{E}_\xi [\| g(x, \xi) - \nabla f(x) \|_2^2] \leq \sigma^2,$$

where $\| \cdot \|_2$ is the Euclidean norm.

Finally, we assume that an optimization procedure, given a point $x \in \mathbb{R}^n$, direction $e \in S_2(1)$ and $\xi$ independently drawn from $P$, can obtain a noisy stochastic approximation $\tilde{f}(x, \xi, e)$ for the directional derivative $\langle g(x, \xi), e \rangle$:

$$\tilde{f}(x, \xi, e) = \langle g(x, \xi), e \rangle + \zeta(x, \xi, e) + \eta(x, \xi, e),$$

$$\mathbb{E}_\xi (\zeta(x, \xi, e))^2 \leq \Delta_\zeta, \ \forall x \in \mathbb{R}^n, \forall e \in S_2(1),$$

$$|\eta(x, \xi, e)| \leq \Delta_\eta, \ \forall x \in \mathbb{R}^n, \forall e \in S_2(1), \ a.s. \ in \ \xi,$$

(48)
where $S_2(1)$ is the Euclidean sphere of radius one with the center at the point zero and the values $\Delta \zeta$, $\Delta \eta$ are controlled and can be made as small as it is desired. Note that we use the smoothness of $F(\cdot, \xi)$ to write the directional derivative as $\langle g(x, \xi), e \rangle$, but we do not assume that the whole stochastic gradient $g(x, \xi)$ is available. We choose a prox-function $d(x)$ which is continuous, convex on $\mathbb{R}^n$ and is $1$-strongly convex on $\mathbb{R}^n$ with respect to $\|\cdot\|_p$, i.e., for any $x, y \in \mathbb{R}^n$ $d(y) - d(x) - \langle \nabla d(x), y - x \rangle \geq \frac{1}{2\kappa} \|y - x\|_p^2$. Without loss of generality, we assume that $\min_{x \in \mathbb{R}^n} d(x) = 0$. We define also the corresponding Bregman divergence $V[z](x) = d(x) - d(z) - \langle \nabla d(z), x - z \rangle$, $x, z \in \mathbb{R}^n$. For the case $p = 1$, we choose the following prox-function 

$$d(x) = \frac{\epsilon p(\kappa - 1)/(2 - \kappa)/n}{2} \ln n \|x\|_\kappa^2, \quad \kappa = 1 + \frac{1}{\ln n}$$

and, for the case $p = 2$, we choose the prox-function to be the squared Euclidean norm $d(x) = \frac{1}{2} \|x\|_2^2$.

Based on the noisy stochastic observations (48) of the directional derivative, we form the following stochastic approximation of $\nabla f(x)$

$$\tilde{\nabla}^m f(x) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}'(x, \xi_i, e),$$

where $e \in R S_2(1)$, $\xi_i, i = 1, ..., m$ are independent realizations of $\xi$, $m$ is the batch size.

### 2.3.1 Algorithms and main results for convex problems

Our Accelerated Randomized Directional Derivative (ARDD) method is listed as Algorithm 8.

**Theorem 2.9.** Let ARDD method be applied to solve problem (46). Then

$$\mathbb{E}[f(y_N)] - f(x^*) \leq \frac{384 \Theta_p n^2 \rho_n L_2}{N^2} + \frac{4N}{m} \frac{\epsilon^2}{m} + \frac{61N}{24L_2} \Delta \zeta + \frac{123N}{48L_2} \Delta \eta^2 + \frac{12 \sqrt{2n} \sigma}{N} \left( \frac{\sqrt{2} \Delta \zeta}{2} + 2 \Delta \eta \right) + \frac{N^2}{12\rho_n L_2} \left( \frac{\sqrt{2} \Delta \zeta}{2} + 2 \Delta \eta \right)^2,$$

where $\Theta_p = V[z_0](x^*)$ is defined by the chosen proximal setup and $\mathbb{E} [\cdot] = \mathbb{E}_{\xi_1, ..., \xi_N, \xi_{i_1}, ..., \xi_{N,m}} [\cdot]$.

Before we proceed to the non-accelerated method, we give the appropriate choice of the ARDD method parameters $N$, $m$, and accuracy of the directional derivative evaluation $\Delta \zeta$, $\Delta \eta$. These values are chosen such that the r.h.s. of (51) is smaller than $\varepsilon$. For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 1 below. The last row represents the total number $Nm$ of oracle calls.

Our Randomized Directional Derivative (RDD) method is listed as Algorithm 9.
Algorithm 8  Accelerated Randomized Directional Derivative (ARDD) method

Require: $x_0$ — starting point; $N \geq 1$ — number of iterations; $m \geq 1$ — batch size.

Ensure: point $y_N$.

1: $y_0 \leftarrow x_0$, $z_0 \leftarrow x_0$.
2: for $k = 0, \ldots, N - 1$. do
3: \hspace{1em} $\alpha_{k+1} \leftarrow \frac{k+2}{96n^2\rho_nL_2}$; $\tau_k \leftarrow \frac{1}{48\alpha_{k+1}n^2\rho_nL_2} = \frac{2}{k+2}$.
4: \hspace{1em} Generate $e_{k+1} \in RS_2(1)$ independently from previous iterations and $\xi_i, i = 1, \ldots, m$ — independent realizations of $\xi$.
5: \hspace{1em} $\tilde{\nabla}^m f(x_{k+1}) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}'(x_{k+1}, \xi_i, e)e$.
6: \hspace{1em} $x_{k+1} \leftarrow \tau_k z_k + (1 - \tau_k)y_k$.
7: \hspace{1em} $y_{k+1} \leftarrow x_{k+1} - \frac{1}{2L_2} \tilde{\nabla}^m f(x_{k+1})$.
8: \hspace{1em} $z_{k+1} \leftarrow \arg\min_{z \in \mathbb{R}^n} \left\{ \alpha_{k+1}n \left\langle \tilde{\nabla}^m f(x_{k+1}), z - z_k \right\rangle + V[z_k](z) \right\}$.
9: end for
10: return $y_N$

Algorithm 9 Randomized Directional Derivative (RDD) method

Require: $x_0$ — starting point; $N \geq 1$ — number of iterations; $m \geq 1$ — batch size.

Ensure: point $\bar{x}_N$.

1: for $k = 0, \ldots, N - 1$. do
2: \hspace{1em} $\alpha \leftarrow \frac{1}{48\rho_nL_2}$.
3: \hspace{1em} Generate $e_{k+1} \in RS_2(1)$ independently from previous iterations and $\xi_i, i = 1, \ldots, m$ — independent realizations of $\xi$.
4: \hspace{1em} $\tilde{\nabla}^m f(x_k) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}'(x_k, \xi_i, e)e$.
5: \hspace{1em} $x_{k+1} \leftarrow \arg\min_{x \in \mathbb{R}^n} \left\{ \alpha n \left\langle \tilde{\nabla}^m f(x_k), x - x_k \right\rangle + V[x_k](x) \right\}$.
6: end for
7: return $\bar{x}_N \leftarrow \frac{1}{N} \sum_{k=0}^{N-1} x_k$
\[ \frac{\sqrt{n \ln n L n \Theta_1}}{\varepsilon} \begin{array}{c|c}
 p = 1 & p = 2 \\
 \hline
 N & \sqrt{\frac{n^2 L_2 \Theta_2}{\varepsilon}} \\
 m & \max \left\{1, \frac{\sqrt{n}}{\varepsilon \sqrt{\ln n}} \cdot \frac{\sigma^2}{\varepsilon}, \frac{\sqrt{L_2}}{\Theta_1} \right\} \\
 \Delta_\zeta & \min \left\{n (\ln n)^2 L_2^2 \Theta_1, \frac{\varepsilon^2}{\ln \ln n} \cdot \frac{L_2}{\varepsilon}, \frac{L_2}{\Theta_1} \right\} \\
 \Delta_\eta & \min \left\{n \ln n L_2 \sqrt{\Theta_1}, \frac{\varepsilon}{\sqrt{n \ln n}} \cdot \frac{L_2}{\Theta_1} \right\} \\
 \text{Calls} & \max \left\{\frac{n \ln n L_2 \Theta_1}{\varepsilon}, \frac{\sigma^2}{\varepsilon^2} \ln n \right\} \\
 \end{array} \]

Table 1: Algorithm parameters for the cases \( p = 1 \) and \( p = 2 \).

**Theorem 2.10.** Let RDD method be applied to solve problem \((46)\). Then

\[
\mathbb{E}[f(\hat{x}_N)] - f(x^*) \leq \frac{384 n \rho_n L_2 \Theta_2}{N} + \frac{2 \sigma^2}{\varepsilon L_2 m} + \frac{n}{12 L_2} \Delta_\zeta + \frac{4n}{3L_2} \Delta_\eta^2 + \frac{8 \sqrt{2n^2 \Theta_2}}{N} \\
+ \frac{\sqrt{\Delta_\zeta}}{2} + \frac{2 \Delta_\eta}{N} \left(\frac{\sqrt{\Delta_\zeta}}{2} + \frac{2 \Delta_\eta}{N} \right)^2, \tag{52}
\]

where \( \Theta_p = \mathbb{V}[z_0](x^*) \) is defined by the chosen proximal setup and \( \mathbb{E}[\cdot] = \mathbb{E}_{\varepsilon_1, \ldots, \varepsilon_N, \xi_1, \ldots, \xi_N, m}[\cdot] \).

Before we proceed, we give the appropriate choice of the RDD method parameters \( N, m, \) and accuracy of the directional derivative evaluation \( \Delta_\zeta, \Delta_\eta \). These values are chosen such that the r.h.s. of \((52)\) is smaller than \( \varepsilon \). For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 2 below. The last row represents the total number \( Nm \) of oracle calls, that is, the number of directional derivative evaluations.

### 2.3.2 Algorithms and main results for strongly convex problems.

To obtain faster rates, we assume additionally that \( f \) is \( \mu_p \)-strongly convex w.r.t. \( p \)-norm. Our algorithms and proofs rely on the following assumption. Let \( x_s \) be some fixed point and \( x \) be a random point such that \( \mathbb{E}_x \left[\|x - x_s\|_p^2\right] \leq R_p^2 \), then

\[
\mathbb{E}_x \left[\frac{x - x_s}{R_p}\right] \leq \frac{\Omega_p}{2}, \tag{53}
\]

where \( \mathbb{E}_x \) denotes the expectation with respect to random vector \( x \) and \( \Omega_p \) is defined as follows. For \( p = 1 \) and our choice of the prox-function \((49)\), \( \Omega_p = \)
Table 2: Algorithm 9 parameters for the cases $p = 1$ and $p = 2$.

|   | $p = 1$                                                                 | $p = 2$                                                                 |
|---|------------------------------------------------------------------------|------------------------------------------------------------------------|
| $N$ | $\frac{L_2 \Theta_1 \ln n}{\varepsilon}$                           | $nL_2 \Theta_2$                                                        |
| $m$ | $\max \left\{ 1, \frac{\sigma^2}{\varepsilon L_2} \right\}$         | $\max \left\{ 1, \frac{\sigma^2}{\varepsilon L_2} \right\}$         |
| $\Delta_\zeta$ | $\min \left\{ \frac{\ln n^2}{n^2} L_2^2 \Theta_1, \frac{\varepsilon^2}{n \Theta_1}, \frac{\varepsilon L_2}{n} \right\}$ | $\min \left\{ nL_2^2 \Theta_2, \frac{\varepsilon^2}{n \Theta_2}, \frac{\varepsilon L_2}{n} \right\}$ |
| $\Delta_\eta$ | $\min \left\{ \frac{\ln n}{\sqrt{n}} L_2 \sqrt{\Theta_1}, \frac{\varepsilon}{\sqrt{n} \epsilon_1}, \frac{\varepsilon L_2}{n} \right\}$ | $\min \left\{ \sqrt{nL_2 \sqrt{\Theta_2}}, \frac{\varepsilon}{\sqrt{n} \epsilon_2}, \frac{\varepsilon L_2}{n} \right\}$ |
| $Nm$ | $\max \left\{ \frac{L_2 \Theta_1 \ln n}{\varepsilon}, \frac{\sigma^2 \Theta_1 \ln n}{\varepsilon^2} \right\}$ | $\max \left\{ nL_2 \Theta_2, \frac{\sigma^2 \Theta_2}{\varepsilon^2} \right\}$ |

Algorithm 10 Accelerated Randomized Directional Derivative method for strongly convex functions (ARDDsc)

**Require:** $x_0$ — starting point s.t. $||x_0 - x^*||_p^2 \leq R_p^2$; $K$ — number of iterations; $\mu_p$ — strong convexity parameter.

**Ensure:** point $u_K$.

1. Set $N_0 = \left\lceil \sqrt{\frac{8aL_2 \Theta_2}{\mu_p}} \right\rceil$, where $a = 384n^2 \rho_n$.
2. for $k = 0, \ldots, K - 1$ do
3. \quad $m_k := \max \left\{ 1, \frac{32\sigma^2 N_0^2}{nL_2 \mu_p R_p^2} \right\}$, \quad $R_k^2 := R_p^2 2^{-k} + \frac{4\Delta}{\mu_p} (1 - 2^{-k})$.
4. \quad Set $d_k(x) = R_k^2 d\left( \frac{x - u_k}{R_k} \right)$.
5. Run ARDD with starting point $u_k$ and prox-function $d_k(x)$ for $N_0$ steps with batch size $m_k$.
6. Set $u_{k+1} = y_{N_0}$, $k = k + 1$.
7. end for
8. return $u_K$.

**Theorem 2.11.** Let $f$ in problem (40) be $\mu_p$-strongly convex and ARDDsc method be applied to solve this problem. Then

$$
\mathbb{E}f(u_K) - f^* \leq \frac{\mu_p R_p^2}{2} \cdot 2^{-K} + 2\Delta.
$$

where $\Delta = \frac{61N_0}{24L_2^2} \Delta_\zeta + \frac{12N_0^2 \Delta_\eta}{2L_2} + \frac{12\sqrt{2nR_p^2 \Omega_p}}{N_0^2} \left( \frac{\sqrt{\Delta_\zeta}}{2} + 2\Delta_\eta \right) + \frac{N_0^2}{4n^2 \rho_n L_2} \left( \frac{\sqrt{\Delta_\eta}}{2} + 2\Delta_\eta \right)^2$.
Moreover, under an appropriate choice of $\Delta_\zeta$ and $\Delta_\eta$ s.t. $2\Delta \leq \varepsilon/2$, the oracle complexity to achieve $\varepsilon$-accuracy of the solution is

$$
\tilde{O} \left( \max \left\{ n^{\frac{1}{2}+\frac{1}{p}} \sqrt{\frac{L_2 \Omega_p}{\mu_p}}, \log_2 \frac{\mu_p R^2_p}{\varepsilon}, \frac{\sigma^2 \Omega_p}{\mu_p \varepsilon} \right\} \right).
$$

Before we proceed to the non-accelerated method, we give the appropriate choice of the accuracy of the direction derivative evaluation $\Delta_\zeta, \Delta_\eta$ for ARDDsc to achieve an accuracy $\varepsilon$ of the solution. These values are chosen such that the r.h.s. of (54) is smaller than $\varepsilon$. For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 3 below. The last row represents the total number of oracle calls, that is, the number of directional derivative evaluations.

| $p$ | $\Delta_\zeta$ | $\Delta_\eta$ | Calls |
|-----|----------------|----------------|-------|
| 1   | $\min \left\{ \varepsilon \sqrt{\frac{L_2 \mu_1}{n \ln n \mu_1}}, \varepsilon \sqrt{\frac{n (\ln n)^2 L_2 \Omega_1}{R^2_1 \mu_1}}, \varepsilon \cdot \frac{\mu_1}{n \Omega_1} \right\}$ | $\min \left\{ \varepsilon \sqrt{\frac{L_2 \mu_2}{n^2 \Omega_2}}, \varepsilon \sqrt{\frac{\ln n L_2 \sqrt{\Omega_1}}{R^2_1 \mu_1}}, \varepsilon \cdot \frac{\mu_1}{n \Omega_1} \right\}$ | $\max \left\{ n^{\frac{1}{2}+\frac{1}{p}} \sqrt{\frac{L_2 \Omega_p}{\mu_p}}, \log_2 \frac{\mu_p R^2_p}{\varepsilon}, \frac{\sigma^2 \Omega_p}{\mu_p \varepsilon} \right\}$ |
| 2   | $\min \left\{ \varepsilon \sqrt{\frac{L_2 \mu_1}{n \ln n \mu_1}}, \varepsilon \sqrt{\frac{n (\ln n)^2 L_2 \Omega_1}{R^2_1 \mu_1}}, \varepsilon \cdot \frac{\mu_1}{n \Omega_1} \right\}$ | $\min \left\{ \varepsilon \sqrt{\frac{L_2 \mu_2}{n^2 \Omega_2}}, \varepsilon \sqrt{\frac{\ln n L_2 \sqrt{\Omega_1}}{R^2_1 \mu_1}}, \varepsilon \cdot \frac{\mu_1}{n \Omega_1} \right\}$ | $\max \left\{ n^{\frac{1}{2}+\frac{1}{p}} \sqrt{\frac{L_2 \Omega_p}{\mu_p}}, \log_2 \frac{\mu_p R^2_p}{\varepsilon}, \frac{\sigma^2 \Omega_p}{\mu_p \varepsilon} \right\}$ |

Table 3: Algorithm parameters for the cases $p = 1$ and $p = 2$.

Our Randomized Directional Derivative method for strongly convex problems (RDDsc) is listed as Algorithm 11.

**Algorithm 11** Randomized Directional Derivative method for strongly convex functions (RDDsc)

**Require:** $x_0$ — starting point s.t. $\|x_0 - x^\ast\|_p^2 \leq R^2_p$, $K \geq 1$ — number of iterations; $\mu_p$ — strong convexity parameter.

**Ensure:** point $u_K$.

1. Set $N_0 = \left\lfloor \frac{8aL_2 \Omega_1}{\mu_p} \right\rfloor$, where $a = 384n\rho n$.
2. for $k = 0, \ldots, K - 1$ do
3. \hspace{1em} $m_k := \max \left\{ 1, \left[ \frac{16 \varepsilon^2 \mu_p}{R^2_k \mu_p} \right] \right\}$; \hspace{1em} $R^2_k := R^2_p 2^{-k} + \frac{4\Delta \mu_p}{\mu_p} (1 - 2^{-k})$;
4. \hspace{1em} Set $d_k(x) = R^2_k d \left( \frac{x - u_k}{R^2_k} \right)$.
5. Run RDD with starting point $u_k$ and prox-function $d_k(x)$ for $N_0$ steps with batch size $m_k$.
6. Set $u_{k+1} = y_{N_0}$, $k = k + 1$.
7. **end for**
8. return $u_K$
Theorem 2.12. Let $f$ in problem (46) be $\mu_p$-strongly convex and RDDsc method be applied to solve this problem. Then

$$E f(u_K) - f^* \leq \frac{\mu_p R_n^2}{2} \cdot 2^{-K} + 2\Delta. \quad (55)$$

where $\Delta = \frac{n}{2L_2} \Delta_\xi + \frac{4n}{3L_2} \Delta_\eta^2 + \frac{8\sqrt{2nR_n^2\Omega_p}}{N_0} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2\Delta_\eta \right) + \frac{N_0}{3L_2 p n} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2\Delta_\eta \right)^2$.

Moreover, under an appropriate choice of $\Delta_\xi$ and $\Delta_\eta$ s.t. $2\Delta \leq \varepsilon/2$, the oracle complexity to achieve $\varepsilon$-accuracy of the solution is

$$\tilde{O} \left( \max \left\{ \frac{n^{2} L_2 \Omega_p}{\mu_p} \log_2 \frac{\mu_p R_n^2}{\varepsilon}, \frac{n^{2} \sigma^2 \Omega_p}{\mu_p \varepsilon} \right\} \right) \cdot \max \left\{ \frac{L_2 \Omega \ln \mu_1}{\mu_1} \log_2 \frac{\mu_2 R_n^2}{\varepsilon}, \frac{L_2 \Omega}{\mu_2} \log_2 \frac{\mu_1 R_n^2}{\varepsilon}, \frac{\sigma^2 \Omega_2}{\mu_2 \varepsilon} \right\}. \quad (56)$$

Before we proceed, we give the appropriate choice of the accuracy of the directional derivative evaluation $\Delta_\xi$, $\Delta_\eta$ for RDDsc to achieve an accuracy $\varepsilon$ of the solution. These values are chosen such that the r.h.s. of (55) is smaller than $\varepsilon$. For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 4 below. The last row represents the total number of oracle calls, that is, the number of directional derivative evaluations.

|                | $p = 1$                                                                 | $p = 2$                                                                 |
|----------------|-------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $\Delta_\xi$  | $\min \left\{ \frac{n^{2} L_2 \Omega_p}{\mu_1}, \varepsilon \ln n \frac{L_2^2 \Omega_p}{\mu_1^2}, \varepsilon \frac{\mu_1}{n^{2} \Omega_1} \right\}$ | $\min \left\{ \frac{n^{2} L_2 \Omega_p}{\mu_2}, \varepsilon \ln n \frac{L_2^2 \Omega_p}{\mu_2^2}, \varepsilon \frac{\mu_2}{n^{2} \Omega_2} \right\}$ |
| $\Delta_\eta$ | $\min \left\{ \frac{n^{2} L_2 \Omega_p}{\mu_1}, \varepsilon \ln n \frac{L_2^2 \Omega_p}{\mu_2}, \varepsilon \frac{\mu_1}{n^{2} \Omega_1} \right\}$ | $\min \left\{ \frac{n^{2} L_2 \Omega_p}{\mu_2}, \varepsilon \ln n \frac{L_2^2 \Omega_p}{\mu_2}, \varepsilon \frac{\mu_2}{n^{2} \Omega_2} \right\}$ |
| Calls         | $\max \left\{ \frac{L_2 \Omega \ln \mu_1}{\mu_1} \log_2 \frac{\mu_2 R_n^2}{\varepsilon}, \frac{\sigma^2 \Omega_1}{\mu_1 \varepsilon} \right\}$ | $\max \left\{ \frac{n L_2 \Omega_2}{\mu_2} \log_2 \frac{\mu_1 R_n^2}{\varepsilon}, \frac{n \sigma^2 \Omega_2}{\mu_2 \varepsilon} \right\}$ |

Table 4: Algorithm parameters for the cases $p = 1$ and $p = 2$.

3 Primal-dual methods

In this section, we focus on the developed primal-dual first-order methods for convex problems with linear constraints.

3.1 Primal-dual methods for solving infinite-dimensional games

The results of this subsection are published in Dvurechensky et al. [2015]. Consider two moving objects with dynamics given by the following equations:

$$\dot{x}(t) = A_x(t)x(t) + B(t)u(t), \dot{y}(t) = A_y(t)y(t) + C(t)v(t),$$

$$(x(0), y(0)) = (x_0, y_0). \quad (56)$$
Here \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^m \) are the phase vectors of these objects, \( u(t) \) is the control of the first object (pursuer), and \( v(t) \) is the control of the second object (evader). Matrices \( A_x(t), A_y(t), B(t), \) and \( C(t) \) are continuous and have appropriate sizes. The system is considered on the time interval \([0, \theta]\). Controls are restricted in the following way \( u(t) \in P \subseteq \mathbb{R}^p \), \( v(t) \in Q \subseteq \mathbb{R}^q \) \( \forall t \in [0, \theta] \). We assume that \( P, Q \) are closed, convex sets.

The goal of the pursuer is to minimize the value of the functional:

\[
F(u, v) + \Phi(x(\theta), y(\theta)) := \int_0^\theta \tilde{F}(\tau, u(\tau), v(\tau)) d\tau + \Phi(x(\theta), y(\theta)).
\] (57)

The goal of the evader is the opposite. We need to find an optimal guaranteed result for each object, which leads to the problem of finding the saddle point of the above functional. We assume the following:

- \( u(\cdot) \in L^2([0, \theta], \mathbb{R}^p) \), and \( v(\cdot) \in L^2([0, \theta], \mathbb{R}^q) \) (for the notation simplification we denote \( L^2([0, \theta], \mathbb{R}^p) \) by \( L^2_p \) and \( L^2([0, \theta], \mathbb{R}^q) \) by \( L^2_q \)),
- the saddle point in this class of strategies exists,
- the function \( F(u, v) \) is upper semi-continuous in \( v \) and lower semi-continuous in \( u \),
- \( \Phi(x, y) \) is continuous.

Denote by \( V_x(t, \tau) \) the transition matrix of the first system in (56). It is the unique solution of the following matrix Cauchy problem

\[
\frac{dV_x(t, \tau)}{dt} = A_x(t)V_x(t, \tau), \quad t \geq \tau, \quad V_x(\tau, \tau) = E.
\]

Here \( E \) is the identity matrix. If the matrix \( A_x(t) \) is constant, then \( V_x(t, \tau) = e^{(t-\tau)A} \).

If we solve the first differential equation in (56), then we can express \( x(\theta) \) as a result of the application of the linear operator \( B : L^2_p \to \mathbb{R}^n \):

\[
x(\theta) = V_x(\theta, 0)\xi_0 + \int_0^\theta V_x(\theta, \tau)B(\tau)u(\tau) d\tau := \tilde{x}_0 + Bu.
\] (58)

Below, we will use the conjugate operator \( B^* \) for the operator \( B \). Let us find it explicitly. Let \( \mu \) be a \( n \)-dimensional vector. Then

\[
\langle \mu, Bu \rangle = \langle \mu, \int_0^\theta V_x(\theta, \tau)B(\tau)u(\tau) d\tau \rangle = \int_0^\theta \langle \mu, V_x(\theta, \tau)B(\tau)u(\tau) \rangle d\tau = \int_0^\theta \langle B^T(\tau)V_x^T(\theta, \tau)\mu, u(\tau) \rangle d\tau = \langle B^* \mu, u \rangle.
\]

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Note that the vector $\zeta(t) = V_x^T(\theta,t)\mu$ is the solution of the following Cauchy problem:

$$\dot{\zeta}(t) = -A_x^T(t)\zeta(t), \quad \zeta(\theta) = \mu, \quad t \in [0, \theta].$$

So we can solve this ODE and find $B^*\mu$ using the obtained solution $\zeta(t)$ as $B^*\mu(t) = B^T(t)\zeta(t)$.

In the same way, we introduce the transition matrix $V_y(t, \tau)$ of the second system in (56), the operator $C : \mathbb{L}_p^2 \to \mathbb{R}^m$ defined by the formula

$$C \nu := \int_0^\theta V_y(\theta, \tau)C(\tau)\nu(\tau)d\tau,$$

and the vector $\tilde{y}_0 := V_y(\theta, 0)y_0$. The adjoint operator $C^*$ also can be computed using the solution of some ODE.

So below we study differential game problem in the following form:

$$\min_{u \in \mathcal{U}} \left[ \max_{v \in \mathcal{V}} \{ F(u, v) + \Phi(x, y) : y = \tilde{y}_0 + Cv \} : x = \tilde{x}_0 + Bu \right], \quad (59)$$

where

$$\mathcal{U} := \{ u(\cdot) \in \mathbb{L}_p^2 : u(t) \in P \quad \forall t \in [0, \theta] \}, \quad \mathcal{V} := \{ v(\cdot) \in \mathbb{L}_p^2 : v(t) \in Q \quad \forall t \in [0, \theta] \}$$

are sets of admissible strategies of the players and $u \in \mathcal{U}$, $v \in \mathcal{V}$ mean $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$. Our goal is to introduce a computational method for finding an approximate solution of the problem (59).

First, we consider the problem (59) under two assumptions.

**A1** The sets $P$ and $Q$ are bounded.

**A2** In (57), the functional $F(\cdot, v)$ is convex for any fixed $v$, $F(u, \cdot)$ is concave for any fixed $u$, $\Phi(\cdot, y)$ is convex for any fixed $y$, and $\Phi(x, \cdot)$ is concave for any fixed $x$.

From **A1**, since the norms of the operators $B, C$ are bounded, $x(\theta), y(\theta)$ are also bounded and we can equivalently reformulate the problem (59) in the following way:

$$\min_{u \in \mathcal{U}, x \in X} \left[ \max_{v \in \mathcal{V}, y \in Y} \{ F(u, v) + \Phi(x, y) : y = \tilde{y}_0 + Cv \} : x = \tilde{x}_0 + Bu \right] =$$

$$\max_{v \in \mathcal{V}, y \in Y} \left[ \min_{u \in \mathcal{U}, x \in X} \{ F(u, v) + \Phi(x, y) : x = \tilde{x}_0 + Bu \} : y = \tilde{y}_0 + Cv \right], \quad (60)$$

where the sets $X$ and $Y$ are closed, convex and bounded. Let us introduce the spaces of dual variables $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^n$ corresponding to the linear constraints in the problem (60), and some norms $\| \cdot \|_\lambda$ and $\| \cdot \|_\mu$ in these spaces. We define the norms in the dual space in the standard way

$$\| s_\lambda \|_{\lambda,*} := \max\{ \langle s_\lambda, \lambda \rangle : \| \lambda \|_\lambda \leq 1 \}, \quad \| s_\mu \|_{\mu,*} := \max\{ \langle s_\mu, \mu \rangle : \| \mu \|_\mu \leq 1 \}.$$
Lemma 3.1. Let the Assumptions A1, A2 hold. Also assume that the function $F(u,v)$ is upper semi-continuous in $v$ and lower semi-continuous in $u$, the function $\Phi(x,y)$ is continuous, and that the sets $P$ and $Q$ are convex and closed. Then the problem (60) is equivalent to the problem

$$\min_{\lambda} \max_{\mu} \left\{ \min_{u \in U} \max_{v \in V} \left[ F(u,v) - \langle \mu, Bu \rangle + \langle \lambda, Cv \rangle \right] \right\},$$

which we call the conjugate problem to (60).

We assume that the problems

$$\psi_1(\lambda, \mu) := \min_{u \in U} \max_{v \in V} \left[ F(u,v) - \langle \mu, Bu \rangle + \langle \lambda, Cv \rangle \right],$$

$$\psi_2(\lambda, \mu) := \min_{x \in X} \max_{y \in Y} \left[ \Phi(x,y) + \langle \mu, x \rangle - \langle \lambda, y \rangle \right] - \langle \mu, \tilde{x}_0 \rangle + \langle \lambda, \tilde{y}_0 \rangle,$$

are rather simple so that they can be solved efficiently or in a closed-form.

Note that the conjugate problem is finite-dimensional and the saddle point in the problems (62), (63) exists for all $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^n$.

Note that the problem (62) has the following form

$$\min_{u \in U} \max_{v \in V} \left[ \int_0^\theta \left\{ \tilde{F}(\tau, u(\tau), v(\tau)) - \langle B^* \mu(\tau), u(\tau) \rangle + \langle C^* \lambda(\tau), v(\tau) \rangle \right\} d\tau \right] = \int_0^\theta \left\{ \min_{u \in U} \max_{v \in V} \left[ \tilde{F}(\tau, u(\tau), v(\tau)) - \langle B^* \mu(\tau), u(\tau) \rangle + \langle C^* \lambda(\tau), v(\tau) \rangle \right] \right\} d\tau,$$

and it can be solved pointwise.

3.1.1 Algorithm for convex-concave problem

We assume that we are given some prox-function $d_\lambda(\lambda)$ with prox-center $\lambda_0$, which is strongly convex with convexity parameter $\sigma_\lambda$ in the given norm $\| \cdot \|_{\lambda}$. For $\mu$ we introduce the similar assumptions. Since $(\lambda^*, \mu^*)$ is the saddle point, $(\lambda^*, \mu^*)$ is a weak solution to the following variational inequality

$$\langle g(\lambda, \mu), (\lambda - \lambda^*, \mu - \mu^*) \rangle \geq 0, \quad \forall \lambda, \mu, \text{ where } g(\lambda, \mu) := (\psi'_\lambda(\lambda, \mu), -\psi'_\mu(\lambda, \mu)).$$

We apply the method of Simple Dual Averages (SDA) from [Nesterov 2009] for finding an approximate solution of the finite-dimensional problem (61).

Let us choose some $\kappa \in [0, 1]$. We consider a space of $z := (\lambda, \mu)$ with the norm

$$\|z\|_2 := \sqrt{\kappa \sigma_\lambda \|\lambda\|_{\lambda}^2 + (1 - \kappa) \sigma_\mu \|\mu\|_{\mu}^2},$$

an oracle $g(z) := (g_\lambda(z), -g_\mu(z))$, a new prox-function $d(z) := \kappa d_\lambda(\lambda) + (1 - \kappa) d_\mu(\mu)$, which is strongly convex with constant
\[ \sigma_0 = 1 \] with respect to the norm \( \| \cdot \|_{z,*} \). We define \( W := \mathbb{R}^m \times \mathbb{R}^n \). The conjugate norm for \( \| \cdot \|_{z,*} \) is 
\[ \| g \|_{z,*} := \sqrt{\frac{1}{\sigma_0} \| g \|_2 + \frac{1}{(1-\kappa)\sigma_0} \| g \|_{\mu,*}^2} \]. So we have a uniform upper bound for the answers of the oracle 
\[ \| g(\lambda,\mu) \|_{z,*}^2 \leq L^2 := \frac{L_2^2}{(1-\kappa)\sigma_0} + \frac{L_2^2}{\sqrt{\theta}}, \]
where \( L_\lambda := \sqrt{\theta} \| C \|_{\lambda,2} \) \( \text{diam}_2 Q + \text{diam}_{\lambda,*} Y + \| \tilde{y}_0 \|_{\lambda,*} \) and \( L_\mu := \sqrt{\theta} \| B \|_{\mu,L_2^2} \) \( \text{diam}_2 P + \text{diam}_{\mu,*} X + \| x_0 \|_{\mu,*} \).

The SDA method for solving \( (61) \) is the following:
1. Initialization: Set \( s_0 = 0 \). Choose \( z_0, \gamma > 0 \).
2. Iteration \((k \geq 0)\):
   \[ \text{Compute } g_k = g(z_k). \text{ Set } s_{k+1} = s_k + g_k. \] \( (\text{M1}) \)
   \[ \beta_{k+1} = \gamma \hat{\beta}_{k+1}. \text{ Set } z_{k+1} = \pi_{\beta_{k+1}}(\tilde{s}_{k+1}). \]
Here the sequence \( \tilde{\beta}_{k+1} \) is defined by relations \( \tilde{\beta}_0 = \tilde{\beta}_1 = 1, \tilde{\beta}_{i+1} = \tilde{\beta}_i + \frac{1}{\beta_i}, \)
for \( i \geq 1 \). The mapping \( \pi_{\beta}(s) \) is defined in the following way 
\[ \pi_{\beta}(s) := \arg \min_{z \in \mathcal{W}} \{ \langle s, z \rangle + \beta d(z) \}. \]
Since the saddle point in the problem \( (60) \) does exist, there exists a saddle point \((\lambda^*, \mu^*)\) in the conjugate problem \((61)\). According to the Theorem 1 in Nesterov [2009], the method \((\text{M1})\) generates a bounded sequence \( \{ z_i \}_{i \geq 0} \). Hence, the sequences \( \{ \lambda_i \}_{i \geq 0}, \{ \mu_i \}_{i \geq 0} \) are also bounded. So we can choose 
\( D_\lambda, D_\mu \) such that \( d_\lambda(\lambda_i) \leq D_\lambda, d_\mu(\mu_i) \leq D_\mu \) for all \( i \geq 0 \) and also, the pair \((\lambda^*, \mu^*)\) is an interior solution: 
\[ \mathfrak{B}_{r/\sqrt{\theta}}(\lambda^*) \subseteq W_\lambda := \{ \lambda : d_\lambda(\lambda) \leq D_\lambda \}, \]
and \( \mathfrak{B}_{r/\sqrt{(1-\kappa)\sigma_0}}(\mu^*) \subseteq W_\mu := \{ \mu : d_\mu(\mu) \leq D_\mu \} \) for some \( r > 0 \).
Then we have \( z^* := (\lambda^*, \mu^*) \in \mathcal{F}_D := \{ z \in \mathcal{W} : d(z) \leq D \} \) with 
\[ D := \kappa D_\lambda + (1 - \kappa) D_\mu \]
and \( \mathfrak{B}_{r/\sqrt{\theta}}(z^*) \subseteq \mathcal{F}_D \).
Let us introduce a gap function
\[ \delta_k(D) := \max_{z} \left\{ \sum_{i=0}^{k} \langle g_i, z_i - z \rangle : z \in \mathcal{F}_D \right\}. \]
From the Theorem 2 in Nesterov [2009] we have
\[ \frac{1}{k+1} \delta_k(D) \leq \frac{\beta_{k+1}}{k+1} \left( \gamma D + \frac{L^2}{2\gamma} \right). \]
Denote
\[ (\hat{u}_{k+1}, \hat{v}_{k+1}, \hat{x}_{k+1}, \hat{y}_{k+1}) := \frac{1}{k+1} \sum_{i=0}^{k} (u_i, v_i, x_i, y_i), \]
where \((u_i, v_i), (x_i, y_i)\) are the saddle points at the point \((\lambda_i, \mu_i)\) in \( (62) \) and \( (63) \) respectively. We define a function
\[ \phi(u, x, v, y) := \min_{\lambda} \max_{\mu} \{ F(u, v) + \Phi(x, y) + \langle \mu, x - \tilde{x}_0 - Bu \rangle + + \langle \lambda, Cv + \tilde{y}_0 - y \rangle : d_\lambda(\lambda) \leq D_\lambda, d_\mu(\mu) \leq D_\mu \}. \]
Since \( d_\lambda(\lambda^*) \leq D_\lambda, d_\mu(\mu^*) \leq D_\mu, \) and the conjugate problem is equivalent to the initial one, we conclude that the initial problem is equivalent to the problem

\[
\min_{u \in \mathcal{U}, x \in X} \max_{v \in \mathcal{V}, y \in Y} \phi(u, x, v, y). \tag{70}
\]

Let us introduce two auxiliary functions:

\[
\xi(u, x) := \max_{v \in \mathcal{V}, y \in Y} \phi(u, x, v, y), \tag{71}
\]

\[
\eta(v, y) := \min_{u \in \mathcal{U}, x \in X} \phi(u, x, v, y). \tag{72}
\]

Note that \( \xi(u, x) \) is convex, \( \eta(v, y) \) is concave, and \( \xi(u, x) \geq \phi(x^*, v^*, y^*) \geq \eta(v, y) \) for all \( u \in \mathcal{U}, v \in \mathcal{V}, x \in X, y \in Y \), where \( \phi(x^*, v^*, y^*) \) is the solution to (70).

**Theorem 3.1.** Let the assumptions A1 and A2 be true. Then the points \( (\hat{u}_k, \hat{x}_k) \) generated by the method (M1) satisfy:

\[
\xi(\hat{u}_{k+1}, \hat{x}_{k+1}) - \eta(\hat{v}_{k+1}, \hat{y}_{k+1}) \leq \frac{\beta_{k+1}}{k+1} \left( D + \frac{L^2}{2\gamma} \right), \tag{73}
\]

\[
\| \hat{x}_0 + \mathcal{B} \hat{u}_k - \hat{x}_k \|_{\mu_\ast} \leq \frac{\beta_{k+1} \sqrt{\sigma_{x^*}}}{\gamma((k+1)^2)} \left( D + \frac{L^2}{2\gamma} \right), \tag{74}
\]

\[
\| \hat{y}_0 + \mathcal{C} \hat{v}_k - \hat{y}_k \|_{\lambda_\ast} \leq \frac{\beta_{k+1} \sqrt{\sigma_{x^*}}}{\gamma((k+1)^2)} \left( D + \frac{L^2}{2\gamma} \right). \tag{75}
\]

### 3.1.2 Algorithm for strongly convex-concave problem

In this subsection, we consider the problem (59), under stronger assumptions and obtain faster convergence rates.

**A3** The function \( F(\cdot, v) \) is strongly convex for any fixed \( v \) with constant \( \sigma_{F_v} \) which does not depend on \( v \), and function \( F(u, \cdot) \) is strongly concave for any fixed \( u \) with constant \( \sigma_{F_u} \) which does not depend on \( u \). Assume that:

\[
\| \nabla_u F(u, v_1) - \nabla_u F(u, v_2) \|_{L_p^2} \leq L_{uv} \| v_1 - v_2 \|_{L_q^2}, \tag{75}
\]

\[
\| \nabla_v F(u_1, v) - \nabla_v F(u_2, v) \|_{L_q^2} \leq L_{vu} \| u_1 - u_2 \|_{L_p^2}. \tag{76}
\]

**A4** \( \Phi(\cdot, y) \) is strongly convex for any fixed \( y \) with respect to the norm \( \| \cdot \|_{\mu_\ast} \) with constant \( \sigma_{\Phi_y} \) which doesn’t depend on \( y \) and \( \Phi(x, \cdot) \) is strongly concave for any fixed \( x \) with respect to the norm \( \| \cdot \|_{\lambda_\ast} \) with constant \( \sigma_{\Phi_x} \) which doesn’t depend on \( x \). Also we assume that:

\[
\| \nabla_x \Phi(x, y_1) - \nabla_x \Phi(x, y_2) \|_{\mu} \leq L_{xy} \| y_1 - y_2 \|_{\lambda_\ast}, \tag{77}
\]

\[
\| \nabla_y \Phi(x_1, y) - \nabla_y \Phi(x_2, y) \|_{\lambda} \leq L_{yx} \| x_1 - x_2 \|_{\mu_\ast}. \tag{78}
\]
Using the explicit expression for \( d \) the variational inequality are differentiable everywhere. saddle point \((\lambda, \mu)\) form. Note that the saddle points in the problems \((82), (83)\) exist for all are simple, which means that they can be solved efficiently or in a closed

reduce the prox-function \( d \) Here \( \lambda \) form. We assume that the norms \( u, v \) are closed, convex and bounded. Similarly to the proof of the Lemma 3.1, we get that the conjugate problem for \((59)\) is

\[
\min_{\lambda} \max_{\mu} \left\{ \min_{u \in U} \max_{v \in V} \left[ F(u, v) - \langle \mu, Bu \rangle + \langle \lambda, Cv \rangle \right] + \max_{x} \max_{y} \left[ \Phi(x, y) + \langle \mu, x \rangle - \langle \lambda, y \rangle \right] - \langle \lambda, \tilde{x}_0 \rangle - \langle \lambda, \tilde{y}_0 \rangle \right\}.
\]

Here \( \lambda \in \mathbb{R}^m \) and \( \mu \in \mathbb{R}^n \).

We assume that the problems

\[
\psi_1(\lambda, \mu) := \min_{u \in U} \max_{v \in V} \left[ F(u, v) - \langle \mu, Bu \rangle + \langle \lambda, Cv \rangle \right],
\]

\[
\psi_2(\lambda, \mu) := \max_{x} \max_{y} \left[ \Phi(x, y) + \langle \mu, x \rangle - \langle \lambda, y \rangle \right]
\]

are simple, which means that they can be solved efficiently or in a closed form. Note that the saddle points in the problems \((82), (83)\) exists for all \( \lambda \in \mathbb{R}^m \) and \( \mu \in \mathbb{R}^n \).

We assume that the norms \( \| \cdot \|_\lambda \) and \( \| \cdot \|_\mu \) are Euclidian. Let us introduce the prox-function \( d_\lambda(\lambda) := \frac{\mu}{2} \| \lambda \|_\lambda^2 \). The function \( d_\lambda(\lambda) \) is strongly convex in this norm with the convexity parameter \( \sigma_\lambda \). For the variable \( \mu \) we introduce the prox-function \( d_\mu(\mu) := \frac{\mu}{2} \| \mu \|_\mu^2 \), which is strongly convex with the convexity parameter \( \sigma_\mu \) with respect to the norm \( \| \cdot \|_\mu \). These prox-functions are differentiable everywhere.

For any \( \lambda_1, \lambda_2 \in \mathbb{R}^m \) we can define the Bregman distance:

\[
\omega_\lambda(\lambda_1, \lambda_2) := d_\lambda(\lambda_2) - d_\lambda(\lambda_1) - \langle \nabla d_\lambda(\lambda_1), \lambda_2 - \lambda_1 \rangle.
\]

Using the explicit expression for \( d_\lambda(\lambda) \), we get \( \omega_\lambda(\lambda_1, \lambda_2) = \frac{\mu}{2} \| \lambda_1 - \lambda_2 \|^2 \). Let us choose \( \tilde{\lambda} = 0 \) as the center of the space \( \mathbb{R}^m \). Then we have \( \omega_\lambda(\lambda, \lambda) = d_\lambda(\lambda) \). For \( \mu \) we introduce the similar settings.

In the same way as it was done above, we conclude that finding the saddle point \((\lambda^*, \mu^*)\) for the conjugate problem \((81)\) is equivalent to solving the variational inequality

\[
\langle g(\lambda, \mu), (\lambda - \lambda^*, \mu - \mu^*) \rangle \geq 0, \quad \forall \lambda, \mu,
\]

where \( g(\lambda, \mu) := (\nabla_\lambda \psi(\lambda, \mu), -\nabla_\mu \psi(\lambda, \mu)) \).

Let us choose some \( \kappa \in ]0, 1[\). Consider a space of \( z := (\lambda, \mu) \) with the norm

\[
\| z \|_z := \sqrt{\kappa \sigma_\lambda \| \lambda \|_\lambda^2 + (1 - \kappa) \sigma_\mu \| \mu \|_\mu^2}.
\]

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an oracle \( g(z) := (\nabla_\lambda \psi(\lambda, \mu), -\nabla_\mu \psi(\lambda, \mu)) \), a new prox-function

\[
d(z) := \kappa d_\lambda(\lambda) + (1 - \kappa)d_\mu(\mu)
\]

which is strongly convex with constant \( \sigma_0 = 1 \). We define \( W := \mathbb{R}^m \times \mathbb{R}^n \), the Bregman distance

\[
\omega(z_1, z_2) := \kappa \omega_\lambda(\lambda_1, \lambda_2) + (1 - \kappa)\omega_\mu(\mu_2, \mu_2)
\]

which has an explicit form of \( \omega(z_1, z_2) = d(z_1 - z_2) \), and center \( \bar{z} = (0, 0) \). Then, \( \omega(\bar{z}, z) = d(z) \). Note that the norm in the dual space is defined as

\[
\| g \|_{x,*} := \sqrt{\frac{1}{\kappa \sigma_\lambda} \| g_\lambda \|_{x,*}^2 + \frac{1}{(1 - \kappa)\sigma_\mu} \| g_\mu \|_{x,*}^2}.
\]

In accordance to Nesterov [2007] for solving (84), we can use the following method:

1. Initialization: Fix \( \beta = L \). Set \( s_{-1} = 0 \).

2. Iteration \((k \geq 0)\):

   \begin{align*}
   &\text{Compute } x_k = T_\beta(\bar{z}, s_{k-1}), \quad \text{(M2)} \\
   &\text{Compute } z_k = T_\beta(x_k, -g(x_k)), \\
   &\text{Set } s_k = s_{k-1} - g(z_k).
   \end{align*}

Here \( T_\beta(z, s) := \arg \max_{x \in \mathcal{W}} \{\langle s, x - z \rangle - \beta \omega(z, x)\} \).

Similarly to Nesterov [2009], we can prove that the method (M2) generates a bounded sequence \( \{z_i\}_{i \geq 0} \). Hence the sequences \( \{\lambda_i\}_{i \geq 0}, \{\mu_i\}_{i \geq 0} \) are also bounded. Also, since the saddle point in the problem (81) exists, there exists a saddle point \((\lambda^*, \mu^*)\) for the conjugate problem (84). These arguments allow us to choose \( D_\lambda, D_\mu \) such that \( d_\lambda(\lambda_i) \leq D_\lambda, d_\mu(\mu_i) \leq D_\mu \) for all \( i \geq 0 \), which also ensure that \((\lambda^*, \mu^*)\) is an interior solution:

\[
\mathcal{W}_\lambda := \{\lambda : d_\lambda(\lambda) \leq D_\lambda\},
\]

\[
\mathcal{W}_\mu := \{\mu : d_\mu(\mu) \leq D_\mu\}
\]

for some \( r > 0 \). Then we have \( z^* := (\lambda^*, \mu^*) \in \mathcal{F}_D := \{z \in \mathcal{W} : d(z) \leq D\} \) with \( D := \kappa D_\lambda + (1 - \kappa)D_\mu \) and \( \mathcal{F}^*_D := \{z^*\} \subseteq \mathcal{F}_D \).

**Theorem 3.2.** Let the Assumptions A3 and A4 be true, \( \kappa = \frac{\sigma_\mu}{\sigma_\mu + \sigma_\lambda} \), and

\[
L = \frac{\sigma_\lambda + \sigma_\mu}{\sigma_\mu \sigma_\lambda} \sqrt{2 \left( \frac{\|C\|^2_{L^2}}{\sigma_{F_v}} + \frac{1}{\sigma_{\Phi_y}} + \frac{\|B\|^2_{L^2} \|C\|^2_{\lambda, L^2} L_{uv}}{\sigma_{F_v} \sigma_{F_v}} + \frac{L_{uv}}{\sigma_{\Phi_y} \sigma_{\Phi_y}} \right)}
\]

\[
\left( \frac{\|B\|^2_{L^2} \|C\|^2_{\lambda, L^2} L_{uv}}{\sigma_{F_v} \sigma_{F_v}} + \frac{L_{uv}}{\sigma_{\Phi_y} \sigma_{\Phi_y}} + \frac{\|B\|^2_{L^2} \|C\|^2_{\mu, L^2} L_{uv}}{\sigma_{F_v} \sigma_{F_v}} + \frac{1}{\sigma_{\Phi_y}} \right).
\]
Let the points $z_i = (\lambda_i, \mu_i), i \geq 0$ be generated by the method (M2). Let the points in (68) be defined by points $(u_i, v_i), (x_i, y_i)$ which are the saddle points at the points $(\lambda_i, \mu_i)$ in (82) and (83) respectively. Then for functions $\xi(u, x), \eta(v, y)$ defined in (71) and (72) we have:

$$\xi(\hat{u}_{k+1}, \hat{x}_{k+1}) - \eta(\hat{v}_{k+1}, \hat{y}_{k+1}) \leq \frac{LD}{k + 1}. \quad (87)$$

Also the following is true:

$$\|B\hat{u}_{k+1} + \tilde{x}_0 - \hat{x}_{k+1}\|_{\mu,*} \leq \frac{LD\sqrt{\sigma_\mu}}{r(k + 1)}, \quad \|C\hat{v}_{k+1} + \tilde{y}_0 - \hat{y}_{k+1}\|_{\lambda,*} \leq \frac{LD\sqrt{\sigma_\lambda}}{r(k + 1)}.$$

### 3.2 Accelerated primal-dual gradient method for strongly convex problems with linear constraints

The results of this subsection are published in Chernov et al. [2016], Dvurechensky et al. [2018b]. See also a close work Dvurechensky et al. [2016].

The main motivation for the algorithms in this subsection is approximating the optimal transport (OT) distance, which amounts to solving the OT problem Kantorovich [1942]:

$$\min_{X \in U(r,c)} \{ C, X \},$$

$$U(r,c) := \{ X \in \mathbb{R}_+^{n \times n} : X1 = r, X^T1 = c \}, \quad (88)$$

where $X$ is transportation plan, $C \in \mathbb{R}_+^{n \times n}$ is a given ground cost matrix, $r, c \in \mathbb{R}^n$ are given vectors from the probability simplex $\Delta^n$, $1$ is the vector of all ones. The regularized OT problem is

$$\min_{X \in U(r,c)} \{ C, X \} + \gamma R(X), \quad (89)$$

where $\gamma > 0$ is the regularization parameter and $R(X)$ is a strongly convex regularizer, e.g. negative entropy or squared Euclidean norm. Our goal is to find $\hat{X} \in U(r,c)$ such that

$$\langle C, \hat{X} \rangle \leq \min_{X \in U(r,c)} \langle C, X \rangle + \varepsilon. \quad (90)$$

In this case, $\langle C, \hat{X} \rangle$ is an $\varepsilon$-approximation for the OT distance and $\hat{X}$ is an approximation for the transportation plan.

Let us introduce some notation. For a general finite-dimensional real vector space $E$, we denote by $E^*$ its dual, given by linear pairing $\langle g, x \rangle$, $x \in E$, $g \in E^*$; by $\| \cdot \|_E$ the norm in $E$ and by $\| \cdot \|_{E,*}$ the norm in $E^*$, which is dual to $\| \cdot \|_E$. For a linear operator $A : E \to H$, we define its norm as $\| A \|_{E \to H} = \max_{x \in E, u \in H^*} \{ \langle u, Ax \rangle : \| x \|_E = 1, \| u \|_{H^*,*} = 1 \}$. We say that a function $f : E \to \mathbb{R}$ is $\gamma$-strongly convex on a set $Q \subseteq E$ w.r.t. a norm in
iff, for any $x, y \in \mathcal{Q}$, $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\gamma}{2} \|x - y\|_E^2$, where $
abla f(x)$ is any subgradient of $f(x)$ at $x$.

For a matrix $A$ and a vector $a$, we denote $e^A$, $e^a$, $\ln A$, $\ln a$ their entrywise exponents and natural logarithms respectively. For a vector $a \in \mathbb{R}^n$, we denote by $\|a\|_1$ the sum of absolute values of its elements, and by $\|a\|_\infty$ the maximum absolute value of its elements. Given a matrix $A \in \mathbb{R}^{n \times n}$, we denote by $\|A\|_1 = \|\text{vec}(A)\|_1$ and $\|A\|_\infty = \|\text{vec}(A)\|_\infty$. Further, we define the entropy of a matrix $X \in \mathbb{R}^{n \times n}^+$ by

$$H(X) := -\sum_{i,j=1}^n X_{ij} \ln X_{ij}.$$  

(91)

For two matrices $A, B$, we denote their Frobenius inner product by $\langle A, B \rangle$.

We denote by $\Delta_n := \{a \in \mathbb{R}^n_+ : a^T \mathbf{1} = 1\}$ the probability simplex in $\mathbb{R}^n$.

We start by considering a general minimization problem of strongly convex objective with linear constraints

$$\min_{x \in \mathcal{Q} \subseteq E} \{f(x) : Ax = b\},$$

(92)

where $E$ is a finite-dimensional real vector space, $\mathcal{Q}$ is a simple closed convex set, $A$ is a given linear operator from $E$ to some finite-dimensional real vector space $H$, $b \in H$ is given, $f(x)$ is a $\gamma$-strongly convex function on $\mathcal{Q}$ with respect to some chosen norm $\|\cdot\|_E$ on $E$.

The Lagrange dual problem for (92), written as a minimization problem, is

$$\min_{\lambda \in H^*} \left\{ \varphi(\lambda) := \langle \lambda, b \rangle + \max_{x \in \mathcal{Q}} \left( -f(x) - \langle A^T \lambda, x \rangle \right) \right\}.$$  

(93)

Note that $\nabla \varphi(\lambda) = b - A x(\lambda)$ is Lipschitz-continuous [Nesterov, 2005]

$$\|\nabla \varphi(\lambda_1) - \nabla \varphi(\lambda_2)\|_H \leq L \|\lambda_1 - \lambda_2\|_{H^*},$$

where $x(\lambda) := \arg\min_{x \in \mathcal{Q}} \left( -f(x) - \langle A^T \lambda, x \rangle \right)$ and $L \leq \frac{\|A\|_E^2}{\gamma}$. This estimate can be pessimistic and our algorithm does not use it and adapts automatically to the local value of the Lipschitz constant.

We assume that the dual problem (93) has a solution and there exists some $R > 0$ such that $\|\lambda^*\|_2 \leq R < +\infty$, where $\lambda^*$ is the solution to (93) with minimum value of $\|\lambda^*\|_2$. Note that the algorithm does not need any estimate of $R$ and the value $R$ is used only in the convergence analysis.

**Theorem 3.3.** Assume that the objective in the primal problem (92) is $\gamma$-strongly convex and that the dual solution $\lambda^*$ satisfies $\|\lambda^*\|_2 \leq R$. Then, for
Algorithm 12 Adaptive Primal-Dual Accelerated Gradient Descent (APDAGD)

Require: Accuracy $\varepsilon_f, \varepsilon_{eq} > 0$, initial estimate $L_0$ s.t. $0 < L_0 < 2L$.
1: Set $i_0 = k = 0$, $M_{-1} = L_0$, $\beta_0 = \alpha_0 = 0$, $\eta_0 = \zeta_0 = \lambda_0 = 0$.
2: repeat {Main iterate}
3: repeat {Line search}
4: Set $M_k = 2^{i_k+1} M_{k}$, find $\alpha_{k+1}$ s.t. $\beta_{k+1} := \beta_k + \alpha_{k+1} = M_k \alpha_{k+1}^2$.
5: Set $\lambda_{k+1} = \tau_k \lambda_k + (1 - \tau_k) \eta_k$.
6: $\zeta_{k+1} = \zeta_k - \alpha_{k+1} \nabla \varphi(\lambda_{k+1})$.
7: $\eta_{k+1} = \tau_k \zeta_{k+1} + (1 - \tau_k) \eta_k$.
8: until
9: $\hat{x}_{k+1} = \tau_k x(\lambda_{k+1}) + (1 - \tau_k) \hat{x}_k$.
10: Set $i_{k+1} = 0$, $k = k + 1$.
11: until $f(\hat{x}_{k+1}) + \varphi(\eta_{k+1}) \leq \varepsilon_f$, $\|A \hat{x}_{k+1} - b\|_2 \leq \varepsilon_{eq}$.
Ensure: $\hat{x}_{k+1}, \eta_{k+1}$.

$k \geq 1$, the points $\hat{x}_k, \eta_k$ in Algorithm 12 satisfy

$$f(\hat{x}_k) - f^* \leq f(\hat{x}_k) + \varphi(\eta_k) \leq \frac{16\|A\|_{E\rightarrow H}^2 R^2}{\gamma k^2}, \quad (94)$$

$$\|A \hat{x}_k - b\|_2 \leq \frac{16\|A\|_{E\rightarrow H}^2 R}{\gamma k^2}, \quad (95)$$

$$\|\hat{x}_k - x^*\|_E \leq \frac{8\|A\|_{E\rightarrow H} R}{\gamma k}, \quad (96)$$

where $x^*$ and $f^*$ are respectively an optimal solution and the optimal value in (92). Moreover, the stopping criterion in step 11 is correctly defined.

Now we apply the general method to derive a complexity estimate for finding $\hat{X} \in \mathcal{U}(r, c)$ satisfying (90). We use entropic regularization of problem \cite{85} and consider the regularized problem \cite{85} with the regularizer $\mathcal{R}(X) = -H(X)$, where $H(X)$ is given in (91). We define $E = \mathbb{R}^{n^2}$, $\|\cdot\|_E = \|\cdot\|_1$, and variable $x = \text{vec}(X) \in \mathbb{R}^{n^2}$ to be the vector obtained from a matrix $X$ by writing each column of $X$ below the previous column. Also we set $f(x) = \langle C, X \rangle - \gamma H(X)$, $Q = \Delta n^2$, $b^T = (r^T, c^T)$ and $A : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{2n}$ defined by the identity $(A \text{vec}(X))^T = ((X1)^T, (X^T 1)^T)$. With this setting, we solve problem (92) by our APDAGD. Let $\hat{X}_k$ be defined by identity $\text{vec}(\hat{X}_k) = \hat{x}_k$, where $\hat{x}_k$ is generated by APDAGD. We also define $\hat{X} \in \mathcal{U}(r, c)$ to be the projection of $\hat{X}_k$ onto $\mathcal{U}(r, c)$ constructed by Algorithm 2 in \cite{Altschuler et al}.
Algorithm 13 Approximate OT by APDAGD

Require: Accuracy $\varepsilon$.

1: Set $\gamma = \frac{\varepsilon}{3\ln n}$.
2: for $k = 1, 2, \ldots$ do
3: Make step of APDAGD and calculate $\hat{X}_k$ and $\eta_k$.
4: Find $\hat{X}$ as the projection of $\hat{X}_k$ on $U(r, c)$ by Algorithm 2 in Altschuler et al. [2017].
5: if $\langle C, \hat{X} - \hat{X}_k \rangle \leq \frac{\varepsilon}{6}$ and $f(\hat{x}_k) + \varphi(\eta_k) \leq \frac{\varepsilon}{6}$ then
6: Return $\hat{X}$.
7: else
8: $k = k + 1$ and continue.
9: end if
10: end for

The pseudo-code of our procedure for approximating the OT distance is listed as Algorithm 13.

Theorem 3.4. Algorithm 13 outputs $\hat{X} \in U(r, c)$ satisfying (90) in

$$O\left(\min\left\{\frac{n^{9/4} \sqrt{R \|\| C \|\|_\infty \ln n}}{\varepsilon}, \frac{n^2 R \|\| C \|\|_\infty \ln n}{\varepsilon^2}\right\} \right)$$

arithmetic operations.

3.3 Distributed primal-dual accelerated stochastic gradient method

The results of this subsection are published in Dvurechensky et al. [2018a].

In this subsection we are motivated by regularized semi-discrete formulation of the optimal transport problem. We start with some notation. We define $\mathcal{M}_1^+(\mathcal{X})$ – the set of positive Radon probability measures on a metric space $\mathcal{X}$, and $S_1(n) = \{a \in \mathbb{R}^n_+ \mid \sum_{l=1}^n a_l = 1\}$ the probability simplex. We use $C(\mathcal{X})$ as the space of continuous functions on $\mathcal{X}$. We denote by $\delta(x)$ the Dirac measure at point $x$. We refer to $\lambda_{\max}(W)$ as the maximum eigenvalue of matrix $W$. We also use bold symbols for stacked vectors $p = [p_1^T, \ldots, p_m^T]^T \in \mathbb{R}^{mn}$, where $p_1, \ldots, p_m \in \mathbb{R}^n$. In this case $[p]_i = p_i$ – the $i$-th block of $p$. For a vector $\lambda \in \mathbb{R}^n$, we denote by $[\lambda]_i$ its $i$-th component.

We refer to the Euclidean norm of a vector $\|p\|_2 := \sqrt{\sum_{l=1}^n \|p_l\|^2}$ as 2-norm.

Following the line of work started by Cuturi [2013], we consider entropic regularization for the optimal transport problem and the corresponding regularized Wasserstein distance and barycenter. Assume that we are given a positive Radon probability measure $\mu$ with density $q(y)$ on a metric space $\mathcal{Y}$, and a discrete probability measure $\nu = \sum_{i=1}^n p_i \delta(z_i)$ with weights $p$ and finite support given by points $z_1, \ldots, z_n \in Z$ from a metric space $\mathcal{Z}$. The
introducing a single equality constraint instead of the constraints (also referred to as an interaction matrix) by $W$. $\Pi(\mu, \nu)$ is defined as follows:

$$\Pi(\mu, \nu) = \left\{ \pi \in \mathcal{M}_+^1(\mathcal{Y}) \times S_1(n) : \sum_{i=1}^n \pi_i(y) = q(y), y \in \mathcal{Y}, \int_{\mathcal{Y}} \pi_i(y) dy = p_i, \forall i = 1, \ldots, n \right\}.$$ 

For a set of positive Radon probability measures $(\mu_1, \ldots, \mu_m)$ the regularized Wasserstein barycenter in the semi-discrete setting is defined as the solution $p$ to the following convex optimization problem

$$\min_{p \in S_1(n)} \sum_{i=1}^m W_{\gamma, \mu_i}(p) = \min_{p_1, \ldots, p_m \in S_1(n)} \sum_{i=1}^m W_{\gamma, \mu_i}(p_i), \quad (98)$$

where we fixed the support $z_1, \ldots, z_n \in \mathcal{Z}$ of the barycenter $\nu$ and characterize it by the vector $p \in S_1(1)$, i.e., $\nu = \sum_{i=1}^n p_i \delta(z_i)$ and $W_{\gamma, \mu}(p) := W_{\gamma}(\mu, \nu)$.

We now describe the distributed optimization setting for solving the second problem in $(98)$. We assume that each measure $\mu_i$ is held by an agent $i$ on a network and this agent can sample from this measure. We model such a network as a fixed connected undirected graph $\mathcal{G} = (V, E)$, where $V$ is the set of $m$ nodes and $E$ is the set of edges. We assume that the graph $\mathcal{G}$ does not have self-loops. The network structure imposes information constraints, specifically, each node $i$ has access to $\mu_i$ only and a node can exchange information only with its immediate neighbors, i.e., a node $i$ can communicate with node $j$ if and only if $(i, j) \in E$.

We represent the communication constraints imposed by the network by introducing a single equality constraint instead of the constraints $p_1 = \cdots = p_m$ in $(98)$. To do so, we define the Laplacian matrix $W \in \mathbb{R}^{m \times m}$ of the graph $\mathcal{G}$ such that a) $[W]_{ij} = -1$ if $(i, j) \in E$, b) $[W]_{ij} = \deg(i)$ if $i = j$, c) $[W]_{ij} = 0$ otherwise. Here $\deg(i)$ is the degree of the node $i$, i.e., the number of neighbors of the node. Finally, define the communication matrix (also referred to as an interaction matrix) by $W := \hat{W} \otimes I_n$.

In this setting, $\sqrt{W}p = 0$ if and only if $p_1 = \cdots = p_m$, where we defined stacked column vector $p = [p_1^T, \ldots, p_m^T]^T \in \mathbb{R}^{mn}$. Using this fact, we equivalently rewrite problem $(98)$ as the maximization problem with linear
equality constraint

$$\max_{p_1, \ldots, p_m \in S_1(n)} \frac{\sqrt{W}p}{\sqrt{W}} = 0$$

Given that problem (99) is an optimization problem with linear constraints, we introduce a stacked vector of dual variables $\lambda = [\lambda_1^T, \ldots, \lambda_m^T]^T \in \mathbb{R}^{mn}$ for the constraints $\sqrt{W}p = 0$ in (99). Then, the Lagrangian dual problem for (99) is

$$\min_{\lambda \in \mathbb{R}^{mn}} \max_{p_1, \ldots, p_m \in S_1(n)} \left\{ \sum_{i=1}^{m} \langle \lambda_i, [\sqrt{W}p]_i \rangle - \mathcal{W}_{\gamma, \mu_i}(p_i) \right\} = \min_{\lambda \in \mathbb{R}^{mn}} \sum_{i=1}^{m} \mathcal{W}_{\gamma, \mu_i}([\sqrt{W} \lambda]_i), \quad (100)$$

where $[\sqrt{W}p]_i$ and $[\sqrt{W} \lambda]_i$ denote the $i$-th $n$-dimensional block of vectors $\sqrt{W}p$ and $\sqrt{W} \lambda$ respectively, and $\mathcal{W}_{\gamma, \mu_i}(\cdot)$ is the Fenchel-Legendre transform of $\mathcal{W}_{\gamma, \mu_i}(p_i)$.

Next, we consider a general smooth stochastic convex optimization problem which is dual to some optimization problem with linear equality constraints. For any finite-dimensional real vector space $E$, we denote by $E^*$ its dual. Let $\| \cdot \|_E$ denote some norm on $E$ and $\| \cdot \|_{E^*}$ denote the norm on $E^*$ which is dual to $\| \cdot \|_E$ $\| \lambda \|_{E^*} = \max \{ \| x \|_E : \langle \lambda, x \rangle \leq 1 \}$. For a linear operator $A : E_1 \to E_2$, we define the adjoint operator $A^T : E_2^* \to E_1^*$ in the following way $\langle u, Ax \rangle = \langle A^T u, x \rangle$, $\forall u \in E_2^*$, $x \in E_1$. We say that a function $f : E \to \mathbb{R}$ has a $L$-Lipschitz-continuous gradient w.r.t. norm $\| \cdot \|_{E^*}$ if it is differentiable and its gradient satisfies Lipschitz condition $\| \nabla f(x) - \nabla f(y) \|_{E^*} \leq L \| x - y \|_E$, $\forall x, y \in E$.

Our next goal is to provide an algorithm for a primal-dual pair of problems

$$(P) \min_{x \in Q \subseteq E} \{ f(x) : Ax = b \}, \quad (D) \min_{\lambda \in \Lambda} \left\{ \langle \lambda, b \rangle + \max_{x \in Q} (-f(x) - \langle A^T \lambda, x \rangle) \right\},$$

where $Q$ is a simple closed convex set, $A : E \to H$ is given linear operator, $b \in H$ is given, $\Lambda = H^*$. We define

$$\varphi(\lambda) := \langle \lambda, b \rangle + \max_{x \in Q} (-f(x) - \langle A^T \lambda, x \rangle) = \langle \lambda, b \rangle + f^*(-A^T \lambda) \quad (101)$$

and assume it to be smooth with $L$-Lipschitz-continuous gradient. Here $f^*$ is the Fenchel-Legendre dual for $f$. We also assume that $f^*(-A^T \lambda) = \mathbb{E}_\xi F^*(-A^T \lambda, \xi)$, where $\xi$ is random vector. Also, we define $F(x, \xi)$ to be the Fenchel-Legendre conjugate function to $F^*$, i.e. it satisfies $F^*(-A^T \lambda, \xi) = \max_{x \in Q} \{ \langle -A^T \lambda, x \rangle - F(x, \xi) \}$ and $x(\lambda, \xi)$ to be the solution of this maximization problem. Under these assumptions, the dual problem $(D)$ can be
accessed by a stochastic oracle \((\Phi(x, \xi), \nabla \Phi(\lambda, \xi)) = (F^*(-A^T\lambda, \xi), \nabla F^*(-A^T\lambda, \xi))\) satisfying \(\mathbb{E}_{\xi}\Phi(\lambda, \xi) = \varphi(\lambda), \mathbb{E}_{\xi}\nabla \Phi(\lambda, \xi) = \nabla \varphi(\lambda)\), which we use in our algorithm. Finally, we assume that dual problem \((D)\) has a solution \(\lambda^*\) and there exists some \(R > 0\) such that \(\|\lambda^*\|_2 \leq R < +\infty\).

We additionally assume that the variance of the stochastic approximation \(\nabla \Phi(\lambda, \xi)\) for the gradient of \(\varphi\) can be controlled and made as small as we desire. This can be done, for example by mini-batching the stochastic approximation. Also, since \(\nabla \Phi(\lambda, \xi) = b - A\nabla F^*(-A^T\lambda, \xi) = b - Ax(\lambda, \xi)\), on each iteration, to find \(\nabla \Phi(\lambda, \xi)\) we find the vector \(x(\lambda, \xi)\) and use it for the primal iterates.

**Algorithm 14** Accelerated Primal-Dual Stochastic Gradient Method (APDSGM)

**Require:** Number of iterations \(N\).

1. \(C_0 = \alpha_0 = 0, \eta_0 = \zeta_0 = \lambda_0 = 0\).
2. **for** \(k = 0, \ldots, N - 1\) **do**
3. \(\alpha_{k+1}\) as the largest root of the equation \(C_{k+1} := C_k + \alpha_{k+1} = 2L\alpha^2_{k+1}, \tau_{k+1} = \alpha_{k+1}/C_{k+1}\).
4. \(\lambda_{k+1} = \tau_{k+1}\zeta_k + (1 - \tau_{k+1})\eta_k\)
5. \(\zeta_{k+1} = \zeta_k - \alpha_{k+1}\nabla \Phi(\lambda_{k+1}, \xi_{k+1})\).
6. \(\eta_{k+1} = \tau_{k+1}\zeta_k + (1 - \tau_{k+1})\eta_k\).
7. \(\hat{x}_{k+1} = \tau_{k+1}x(\lambda_{k+1}, \xi_{k+1}) + (1 - \tau_{k+1})\hat{x}_k\).
8. **end for**

**Ensure:** The points \(\hat{x}_{k+1}, \eta_{k+1}\).

**Theorem 3.5.** Let \(\varphi\) have \(L\)-Lipschitz-continuous gradient w.r.t. 2-norm and \(\|\lambda^*\|_2 \leq R\), where \(\lambda^*\) is a solution of dual problem \((D)\). Given desired accuracy \(\varepsilon\), assume that, at each iteration of Algorithm [14], the stochastic gradient \(\nabla \Phi(\lambda_k, \xi_k)\) is chosen in such a way that \(\mathbb{E}_{\xi}\|\nabla \Phi(\lambda_k, \xi_k) - \nabla \varphi(\lambda_k)\|_2^2 \leq \varepsilon L\alpha_{k}\). Then, for any \(\varepsilon > 0\) and \(N \geq 0\), and expectation \(\mathbb{E}\) w.r.t. all the randomness \(\xi_1, \ldots, \xi_N\), the outputs \(\eta_N\) and \(\hat{x}_N\) generated by the Algorithm [14] satisfy

\[
f(\mathbb{E}\hat{x}_N) - f^* \leq \frac{32LR^2}{N^2} + \frac{\varepsilon}{2} \quad \text{and} \quad \|A\mathbb{E}\hat{x}_N - b\|_2 \leq \frac{32LR}{N^2} + \frac{\varepsilon}{2R}.
\]

Next, we apply the general algorithm to solve the primal-dual pair of problems \((99)-(100)\) and approximate the regularized Wasserstein barycenter which is a solution to \((99)\).

**Lemma 3.2.** The gradient of the objective function \(W^*_\gamma(\lambda)\) in the dual problem \((100)\) is \(\lambda_{\max}(W)/\gamma\)-Lipschitz-continuous w.r.t. 2-norm. If its stochastic
approximation is defined as

\[
[\tilde{\nabla}_W^* \gamma(\lambda)]_i = \sum_{j=1}^{m} \sqrt{W}_{ij} \tilde{\nabla}_W^*_{\gamma,\mu_j}(\tilde{\lambda}_j), \quad i = 1, \ldots, m, \quad \text{with}
\]

\[
\tilde{\nabla}_W^*_{\gamma,\mu_j}(\tilde{\lambda}_j) = \frac{1}{M} \sum_{r=1}^{M} p_j(\tilde{\lambda}_j), \quad \text{and} \quad [p_j(\tilde{\lambda}_j)]_i = \frac{\exp(([\tilde{\lambda}_j]_i - c(Y^j_i))/\gamma)}{\sum_{\ell=1}^{n} \exp(([\tilde{\lambda}_j]_\ell - c(Y^\ell_i))/\gamma)}. \tag{103}
\]

where \( M \) is the batch size, \( \tilde{\lambda}_j := \sqrt{W} \lambda_j, \ j = 1, \ldots, m, \ Y^j_1, \ldots, Y^j_r \) is a sample from the measure \( \mu_j, \ j = 1, \ldots, m \). Then \( \mathbb{E}_{Y^j_1, \ldots, Y^j_r \sim \mu_j, j = 1, \ldots, m, r = 1, \ldots, M} \tilde{\nabla}_W^* \gamma(\lambda) = \nabla W^* \gamma(\lambda) \) and

\[
\mathbb{E}_{Y^j_1, \ldots, Y^j_r \sim \mu_j, j = 1, \ldots, m, r = 1, \ldots, M} \| \tilde{\nabla}_W^* \gamma(\lambda) - \nabla W^* \gamma(\lambda) \|^2 \leq \frac{\lambda_{\max}(W)}{M}, \quad \lambda \in \mathbb{R}^{mn}. \tag{104}
\]

Based on this lemma, we see that if, on each iteration of Algorithm 14, the mini-batch size \( M_k \) satisfies \( M_k \geq \frac{\lambda_{\max}(W)C_k}{\alpha_k \varepsilon} \), the assumptions of Theorem 3.5 hold.

For the particular problem (100) the step 5 of Algorithm 14 can be written block-wise \( [\tilde{\zeta}_{k+1}]_i = [\tilde{\zeta}_k]_i - \alpha_{k+1} \sum_{j=1}^{m} \sqrt{W}_{ij} \tilde{\nabla}_W^*_{\gamma,\mu_j}(\sqrt{W} \lambda_{k+1}) \), \( i = 1, \ldots, m \). Unfortunately, this update cannot be made in the decentralized setting since the sparsity pattern of \( \sqrt{W}_{ij} \) can be different from \( W_{ij} \) and this will require some agents to get information not only from their neighbors. To overcome this obstacle, we change the variables and denote \( \tilde{\lambda} = \sqrt{W} \lambda, \ \tilde{\eta} = \sqrt{W} \eta, \ \tilde{\zeta} = \sqrt{W} \zeta \). Then the step 5 of Algorithm 14 becomes \( [\tilde{\zeta}_{k+1}]_i = [\tilde{\zeta}_k]_i - \alpha_{k+1} \sum_{j=1}^{m} W_{ij} \tilde{\nabla}_W^*_{\gamma,\mu_j}[\sqrt{W} \lambda_{k+1}]_j \), \( i = 1, \ldots, m \).
Algorithm 15 Distributed computation of Wasserstein barycenter

Require: Each agent $i \in V$ is assigned its measure $\mu_i$.

1: All agents set $[\bar{\eta}_0]_i = [\bar{\zeta}_0]_i = [\bar{\lambda}_0]_i = 0 \in \mathbb{R}^n$, $C_0 = \alpha_0 = 0$ and $N$
2: For each agent $i \in V$
3: for $k = 0, \ldots, N - 1$ do
4: \hspace{1em} Find $\alpha_{k+1}$ as the largest root of the equation
5: \hspace{1.5em} $C_{k+1} := C_k + \alpha_{k+1} = 2L\alpha_{k+1}^2$.
6: \hspace{1.5em} $\tau_{k+1} = \alpha_{k+1}/C_{k+1}$.
7: \hspace{1em} Set $M_{k+1} = \max\{1, \lambda_{\max}(W)C_{k+1}/(L\alpha_{k+1}\varepsilon)\}$
8: \hspace{1em} $[\bar{\lambda}_{k+1}]_i = \tau_{k+1}[\bar{\zeta}_k]_i + (1 - \tau_{k+1})[\bar{\eta}_k]_i$
9: \hspace{1em} $\bar{\lambda}_{k+1} = \tau_{k+1}[\bar{\zeta}_{k+1}]_i + (1 - \tau_{k+1})[\bar{\eta}_{k+1}]_i$
10: \hspace{1em} $[\bar{\eta}_{k+1}]_i = \tau_{k+1}p_i([\bar{\lambda}_{k+1}]_i) + (1 - \tau_{k+1})[\bar{\eta}_{k+1}]_i,$ where $p_i(\cdot)$ is defined in (103).
11: \hspace{1em} $\bar{\lambda}_{k+1} = \tau_{k+1}p_i([\bar{\lambda}_{k+1}]_i) + (1 - \tau_{k+1})[\bar{\eta}_{k+1}]_i$
12: end for

Ensure: $\bar{\lambda}_{N}$.

Theorem 3.6. Under the above assumptions, Algorithm 15 after $N = \sqrt{16\lambda_{\max}(W)R^2/\varepsilon\gamma}$
iterations returns an approximation $\bar{\lambda}_{N}$ for the barycenter, which satisfies

$$\sum_{i=1}^m W_{\gamma,\mu_i}(E[\bar{\lambda}_{N}]_i) - \sum_{i=1}^m W_{\gamma,\mu_i}(\bar{\lambda}_i) \leq \varepsilon, \quad \|\bar{\lambda}_{N}\|_2 \leq \varepsilon/R. \quad (105)$$

Moreover, the total complexity is $O\left(n \max\lambda_{\max}(W)R^2/\varepsilon^2, \sqrt{\lambda_{\max}(W)R^2/\varepsilon\gamma}\right)$
arithmetic operations.

3.4 Primal-dual accelerated gradient method with small-dimensional relaxation oracle

The results of this subsection are published in Guminov et al., 2019a, Nesterov et al., 2020. See also a follow-up work Guminov et al., 2019b.

In this subsection, we consider a minimization problem with linear equality constraints.

Specifically, we consider the following minimization problem

$$(P_1) \min_{x \in Q \subseteq E} \{f(x) : Ax = b\},$$

where $E$ is a finite-dimensional real vector space, $Q$ is a simple closed convex set, $A$ is given linear operator from $E$ to some finite-dimensional real vector

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space $H$, $b \in H$ is given. The Lagrange dual problem to Problem $(P_1)$ is

$$
(D_1) \quad \max_{\lambda \in \Lambda} \left\{ -\langle \lambda, b \rangle + \min_{x \in Q} \left( f(x) + \langle A^T \lambda, x \rangle \right) \right\}.
$$

Here we denote $\Lambda = H^*$. It is convenient to rewrite Problem $(D_1)$ in the equivalent form of a minimization problem

$$
(P_2) \quad \min_{\lambda \in \Lambda} \left\{ \langle \lambda, b \rangle + \max_{x \in Q} \left( -f(x) - \langle A^T \lambda, x \rangle \right) \right\}.
$$

We denote

$$
\varphi(\lambda) = \langle \lambda, b \rangle + \max_{x \in Q} \left( -f(x) - \langle A^T \lambda, x \rangle \right).
$$

(106)

Since $f$ is convex, $\varphi(\lambda)$ is a convex function and, by Danskin’s theorem, its subgradient is equal to (see e.g. Nesterov [2005])

$$
\nabla \varphi(\lambda) = b - Ax(\lambda)
$$

(107)

where $x(\lambda)$ is some solution of the convex problem

$$
\max_{x \in Q} \left( -f(x) - \langle A^T \lambda, x \rangle \right).
$$

(108)

In what follows, we make the following assumptions about the dual problem $(D_1)$

- Subgradient of the objective function $\varphi(\lambda)$ satisfies Hölder condition with constant $M_\nu$, i.e., for all $\lambda, \mu \in \Lambda$ and some $\nu \in [0, 1]$

$$
\|\nabla \varphi(\lambda) - \nabla \varphi(\mu)\|_* \leq M_\nu \|\lambda - \mu\| ^\nu.
$$

(109)

- The dual problem $(D_1)$ has a solution $\lambda^*$ and there exist some $R > 0$ such that

$$
\|\lambda^*\|_2 \leq R < +\infty.
$$

(110)

It is worth noting that the quantity $R$ will be used only in the convergence analysis, but not in the algorithm itself. As it was pointed in Yurtsever et al. [2015], the first assumption is reasonable. Namely, if the set $Q$ is bounded, then $\nabla \varphi(\lambda)$ is bounded and Hölder condition holds with $\nu = 0$. If $f(x)$ is uniformly convex, i.e., for all $x, y \in Q$, $\langle \nabla f(x) - \nabla f(y) \rangle \geq \mu \|x - y\| ^\rho$, for some $\mu > 0$, $\rho \geq 2$, then $\nabla \varphi(\lambda)$ satisfies Hölder condition with $\nu = \frac{1}{\rho-1}$.

$$
M_\nu = \left( \frac{\|A\|_{E_1 \rightarrow E_2}}{\mu} \right) ^{\frac{1}{\rho-1}}.
$$

Here the norm of an operator $A : E_1 \rightarrow E_2$ is defined as follows

$$
\|A\|_{E_1 \rightarrow E_2} = \max_{x \in E_1, u \in E_2} \{ \langle u, Ax \rangle : \|x\|_{E_1} = 1, \|u\|_{E_2, *} = 1 \}.
$$

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The prox-function $d(\cdot)$ that we introduce Euclidean norm $A$ is smaller than 3.7, whenever $A$ starting point is.

**Algorithm 16** PDUGDsDR

Require: starting point $\lambda_0 = 0$, accuracy $\bar{\varepsilon}_f, \bar{\varepsilon}_{eq} > 0$.

1: Set $k = 0$, $A_0 = \alpha_0 = 0$, $\eta_0 = \zeta_0 = \lambda_0 = 0$.

2: repeat

3: $\beta_k = \text{argmin}_{\beta \in [0, 1]} \varphi(\zeta_k + \beta(\eta_k - \zeta_k))$; $\lambda_k = \zeta_k + \beta_k(\eta_k - \zeta_k)$

4: $h_{k+1} = \text{argmin}_{t \geq 0} \varphi(\lambda_k - h\nabla \varphi(\lambda_k))$; $\eta_{k+1} = \lambda_k - h_{k+1}\nabla \varphi(\lambda_k)$ // Choose

5: $\nabla \varphi(\lambda_k) : (\nabla \varphi(\lambda_k), \zeta_k - \lambda_k) \geq 0$

6: Choose $a_{k+1}$ from $\varphi(\eta^{k+1}) = \varphi(\lambda_k) - \frac{\alpha_{k+1}}{2A_{k+1}}\|\nabla \varphi(\lambda_k)\|_2^2 + \frac{\sigma_{k+1}}{2A_{k+1}}$ // $A_{k+1} = A_k + a_{k+1}$

7: $\zeta^{k+1} = \zeta_k - a_{k+1}\nabla \varphi(\lambda_k)$

8: Set

9: $|f(\hat{x}^{k+1}) + \varphi(\eta^{k+1})| \leq \bar{\varepsilon}_f, \|A\hat{x}^{k+1} - b\|_2 \leq \bar{\varepsilon}_{eq}$.

Ensure: The points $\hat{x}^{k+1}, \eta^{k+1}$.

We choose Euclidean proximal setup in the dual space, which means that we introduce Euclidean norm $\|\cdot\|_2$ in the space of vectors $\lambda$ and choose the prox-function $d(\lambda) = \frac{1}{2}\|\lambda\|_2^2$. Then, we have for the Bregman distance $V(\zeta)(\lambda) = \frac{1}{2}\|\lambda - \zeta\|_2^2$. Our primal-dual algorithm for Problem $(P_1)$ is listed below as Algorithm 16.

**Theorem 3.7.** Let the objective $\varphi$ in the problem $(P_2)$ have Hölder-continuous subgradient and the solution of this problem be bounded, i.e. $\|\lambda^*\|_2 \leq R$. Then, for the sequence $\hat{x}^{k+1}, \eta^{k+1}$, $k \geq 0$, generated by Algorithm 16

$$\|A\hat{x}^k - b\|_2 \leq \frac{2R}{A_k} + \frac{\varepsilon}{2R}, \quad |\varphi(\eta^k) + f(\hat{x}^k)| \leq \frac{2R^2}{A_k} + \frac{\varepsilon}{2},$$

(111)

where $A_k \geq \left[1 + \frac{\nu}{1 - \nu} \right]^{-1} \left[\frac{1 + \frac{\nu}{2}}{1 + \frac{\nu}{2}} \right]^{\frac{1}{1 + \frac{\nu}{2}}} \left[\frac{1 + \frac{\nu}{2}}{1 + \frac{\nu}{2}} \right]^{\frac{1}{1 + \frac{\nu}{2}}} \frac{M^2_\nu}{\varepsilon} \frac{R^2}{\varepsilon} \frac{1}{\nu + 2

\nu}$

Let us make a remark on complexity. As it can be seen from Theorem 3.7 whenever $A_k \geq 2R^2/\varepsilon$, the error in the objective value and equality constraints is smaller than $\varepsilon$. At the same time, using the lower bound for $A_k$, we obtain that the number of iterations to achieve this accuracy is

$$O \left( \left( \frac{M^2_\nu R^2}{\varepsilon} \right)^{\frac{1}{\nu + 2}} \right)$$

Since the algorithm does not use the value of $\nu$, we can take infimum in $\nu \in [0, 1]$ of this complexity. This means that the method is uniformly optimal for the class of problems with Hölder-continuous gradient.

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4 Conclusion

This thesis is based on published papers Dvurechensky and Gasnikov [2016], Gasnikov and Dvurechensky [2016], Bogolubsky et al. [2016], Dvurechensky et al. [2020, 2015], Chernov et al. [2016], Dvurechensky et al. [2018b,a], Guminov et al. [2019a], Nesterov et al. [2020].

In papers Dvurechensky and Gasnikov [2016], Gasnikov and Dvurechensky [2016], Bogolubsky et al. [2016], Dvurechensky et al. [2015, 2016], Chernov et al. [2016], Dvurechensky et al. [2018b,a], Guminov et al. [2019a], Nesterov et al. [2020] we developed optimization methods with (stochastic) inexact first-order oracle, inexact zero-order oracle, inexact directional derivative oracle. We also considered a particular application to learning a parametric model for web-page ranking.

Papers Dvurechensky et al. [2015], Chernov et al. [2016], Dvurechensky et al. [2018b,a], Guminov et al. [2019a], Nesterov et al. [2020] devoted to primal dual methods for convex problems with linear constraints. In particular, we consider infinite-dimensional problems and propose dimension-independent convergence rates for this problem. We also consider (stochastic) convex problems with linear constraints and propose accelerated gradient methods with optimal convergence rates. We apply these methods for approximating optimal transport distance and barycenters.

Let us list the main results that are obtained in this thesis and submitted for defense.

1. Stochastic intermediate gradient method for convex problems with stochastic inexact oracle.

2. Gradient method with inexact oracle for deterministic non-convex optimization and gradient-free method with inexact oracle for deterministic convex optimization.

3. A concept of inexact oracle for the methods which use directional derivatives, accelerated and non-accelerated inexact directional derivative method for strongly convex smooth stochastic optimization.

4. Primal-dual methods for solving infinite-dimensional games in convex-concave and strongly convex-concave setting.

5. Non-adaptive and adaptive accelerated primal-dual gradient method for strongly convex minimization problems with linear equality and inequality constraints.

6. New complexity estimates for the optimal transport problem.

7. Stochastic primal-dual accelerated gradient method for problems with linear constraints and its application to the problem of approximation of Wasserstein barycenter.

8. A universal primal-dual accelerated gradient method with line-search.
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