On the Kurzweil-Henstock Integral in Probability

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Abstract
By using the method in [5], the aim of the present note is to generalize the Riemann integral in probability introduced in [7], to Kurzweil-Henstock integral in probability. Properties of the new integral are proved.

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1 Introduction
Let \((E, B, P)\) be a field of probability, where \(E\) is a nonempty set, \(B\) a field of parts on \(E\) and \(P\) a composite probability on \(B\). Let us denote by \(L(E, B, P)\) the set of all real random variables defined on \(E\) and a.e. finite.

It is well-known the following concept:

Definition 1.1. (see e.g. [7], p. 50) We say that the random function \(f : [a, b] \to L(E, B, P)\) is Riemann integrable in probability on \([a, b]\), if there exists a random variable \(I = I(\omega) \in L(E, B, P)\) satisfying:

- for all \(\varepsilon > 0, \eta > 0\), there exists \(\delta = \delta(\varepsilon) > 0\), such that for all divisions \(d : a = x_0 < x_1 < \ldots < x_n = b\) with the norm \(\nu(d) < \delta\) and all \(\xi_i \in [x_i, x_{i+1}]\), \(i \in \{0, \ldots, n-1\}\), we have

\[P(\{\omega \in E; |S(f; d, \xi)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta,\]

where \(\nu(d) = \max\{x_{i+1} - x_i; i = 0, 1, \ldots, n-1\}\) and

\[S(f; d, \xi)(\omega) = \sum_{i=0}^{n-1} f(\xi_i, \omega)(x_{i+1} - x_i).\]

In this case, \(I(\omega)\) is called the Riemann integral in probability of \(f\) on \([a, b]\) and it is denoted by \(I(\omega) = (P) \int_a^b f(t, \omega) dt\).

Remark. As it was proved in [7, p. 50], if \(I_1(\omega), I_2(\omega)\) are Riemann integrals in probability of \(f\) on \([a, b]\), then \(P(\{\omega \in E; I_1(\omega) \neq I_2(\omega)\}) = 0.\)
Using the method in [5], in Section 2 we introduce the so-called Kurzweil-Henstock (Riemann generalized) integral in probability. Section 3 contains basic properties of this generalized integral.

2 The Kurzweil-Henstock Integral in Probability

Firstly, we recall some concepts in [5] we need for our purpose.

A tagged division of \([a, b]\) is of the form

\[
d_S : x_0 \leq \xi_0 \leq x_1 < \ldots < x_i \leq \xi_i \leq x_{i+1} < \ldots < x_{n-1} \leq \xi_{n-1} \leq x_n = b.
\]

A gauge on \([a, b]\) is an open interval valued function \(\gamma\) defined on \([a, b]\), such that \(t \in \gamma(t), t \in [a, b]\). A tagged division \(d_S\) of \([a, b]\) is called \(\gamma\)-sharp if \([x_i, x_{i+1}] \subset \gamma(\xi_i),\) for all \(i \in \{0, \ldots, n-1\}\).

Now, we are in position to introduce the following.

**Definition 2.1.** Let \(f : [a, b] \to L(E, B, P)\). A random variable \(I(\omega) \in L(E, B, P)\) is called a Kurzweil-Henstock (shortly (KH)) integral in probability of \(f\) on \([a, b]\), if for all \(\varepsilon > 0, \eta > 0,\) there exists \(\gamma_{\varepsilon, \eta}\)-gauge on \([a, b]\), such that for any tagged division \(d_S\) which is \(\gamma_{\varepsilon, \eta}\)-sharp, we have

\[
P\{|\omega \in E; |S(f; d_S, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\} < \eta,
\]

where \(S(f; d_S, \xi_i)(\omega) = \sum_{i=0}^{n-1} f(\xi_i, \omega)(x_{i+1} - x_i)\).

In this case, we write \(I(\omega) = (KH) \int_a^b f(t, \omega)dt\).

**Theorem 2.2.** If \(I_1(\omega), I_2(\omega)\) are (KH) integrals in probability of \(f\) on \([a, b]\), then \(P\{|\omega \in E; I_1(\omega) - I_2(\omega)| > \varepsilon\} = 0\).

**Proof.** We will prove that \(P\{|\omega \in E; |I_1(\omega) - I_2(\omega)| > \varepsilon\} = 0\). Let \(\varepsilon, \eta > 0\). There exists the gauges \(\gamma_{\varepsilon, \eta}^{(1)}, \gamma_{\varepsilon, \eta}^{(2)}\) on \([a, b]\), such that for any tagged divisions of \([a, b]\), \(d^{(1)}_S, d^{(2)}_S\) which are \(\gamma_{\varepsilon, \eta}^{(1)}\)-sharp and \(\gamma_{\varepsilon, \eta}^{(2)}\)-sharp, respectively, we have

\[
P\{|\omega \in E; |S(f; d_S^{(1)}, \xi_i)(\omega) - I_1(\omega)| \geq \varepsilon/2\} < \eta/2,
\]

\[
P\{|\omega \in E; |S(f; d_S^{(2)}, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\} < \eta/2.
\]

Let us define a new gauge on \([a, b]\) by \(\gamma(t) = \gamma_{\varepsilon, \eta}^{(1)}(t) \cap \gamma_{\varepsilon, \eta}^{(2)}(t), t \in [a, b]\). By [5, Section 1.8], there exists a \(\gamma\)-sharp tagged division \(d_S\) of \([a, b]\).

Since \(\gamma(t) \subset \gamma_{\varepsilon, \eta}^{(1)}(t), \gamma(t) \subset \gamma_{\varepsilon, \eta}^{(2)}(t), t \in [a, b]\), obviously that \(d_S\) is \(\gamma_{\varepsilon, \eta}^{(1)}\)-sharp and \(\gamma_{\varepsilon, \eta}^{(2)}\)-sharp too.

We have

\[
|I_1(\omega) - I_2(\omega)| \leq |I_1(\omega) - S(f; d_S, \xi_i)(\omega)| + |S(f; d_S, \xi_i)(\omega) - I_2(\omega)|,
\]

which immediately implies

\[
\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon\} \subset \{\omega \in E; |I_1(\omega) - S(f; d_S, \xi_i)(\omega)| \geq \varepsilon/2\}
\]
and
\[ P(\{\omega \in E; |S(f; dS, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\}) \leq P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon/2\}) + P(\{\omega \in E; |S(f; dS, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\}) < \eta/2 + \eta/2 = \eta. \]

Now, considering \( \varepsilon > 0 \) fixed and passing to limit with \( \eta \to 0 \), we get \( P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon\}) = 0. \)

For \( \varepsilon = \frac{1}{n} \), let us denote \( A_n = \{\omega \in E; |I_1 - I_2| \geq 1/n\}. \) Obviously \( A_n \subset A_{n+1} \) and \( \bigcup_{n=1}^{\infty} A_n = \{\omega \in E; |I_1(\omega) - I_2(\omega)| > 0\}. \) Then,
\[ P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| > 0\}) = \lim_{n \to \infty} P(A_n) = 0, \]
which proves the theorem. \( \square \)

As in the case of usual real functions, another definition for the (KH) integral can be the following.

**Definition 2.3.** Let \( f : [a, b] \to L(E, B, P) \). We say that \( f \) is Kurzweil-Henstock integrable in probability on \([a, b]\), if there exists \( I \in L(E, B, P) \) with the property: for all \( \varepsilon > 0 \) and \( \eta > 0 \), there exists \( \delta_{\varepsilon, \eta} : [a, b] \to \mathbb{R}_+ \), such that for any division \( d_S : a = x_0 < x_1 < \ldots < x_n = b \) and any \( \xi_i \in [x_i, x_{i+1}] \) with \( x_{i+1} - x_i < \delta_{\varepsilon, \eta}(\xi_i) \), \( i = 0, \ldots, n - 1 \), we have
\[ P(\{\omega \in E; |S(f; dS, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta. \]

**Remarks.** 1) The Definitions 2.1 and 2.3 are equivalent. Indeed, this easily follows from the fact that any function \( \delta_{\varepsilon, \eta} : [a, b] \to \mathbb{R}_+ \), generates the gauge \( \gamma_{\varepsilon, \eta}(t) = (t - \delta_{\varepsilon, \eta}(t)/2, t + \delta_{\varepsilon, \eta}(t)/2), \ t \in [a, b] \) and conversely, any gauge \( \gamma_{\varepsilon, \eta} \) on \([a, b]\) (which obviously can be written in the form \( \gamma_{\varepsilon, \eta}(t) = (t - \alpha(t), t + \beta(t)) \), \( \alpha(t), \beta(t) > 0, t \in [a, b] \)) generates the function \( \delta_{\varepsilon, \eta}(t) = \alpha(t) + \beta(t), t \in [a, b] \), such that the (KH)-integrability which uses the function \( \delta_{\varepsilon, \eta} \) is equivalent with the (KH)-integrability which uses the gauge \( \gamma_{\varepsilon, \eta} \).

2) If \( \delta_{\varepsilon, \eta} \) is a constant function, Definition 2.3 reduces to Definition 1.1.

## 3 Properties of the (KH)-Integral in Probability

In this section, we will prove some properties of the (KH)-integral in probability. Firstly, we need the following.

**Definition 3.1.** (see e.g. [3, p. 82], [4]). We say that \( \varphi : [a, b] \to \mathbb{R} \) is Kurzweil-Henstock integrable on \([a, b]\), if there exists \( I \in \mathbb{R} \), such that for all \( \varepsilon > 0 \), there exists \( \delta_\varepsilon : [a, b] \to \mathbb{R} \), such that for any division \( d : a = x_0 < x_1 < \ldots < x_n = b \) and any \( \xi_i \in [x_i, x_{i+1}] \) with \( x_{i+1} - x_i < \delta_\varepsilon(\xi_i) \), we have
\[ |I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| < \varepsilon. \] We write \( I = (KH) \int_a^b \varphi(t)dt. \)

The following result holds.

**Theorem 3.2.** If \( f : [a, b] \to L(E, B, P) \) is of the form \( f(t, \omega) = \sum_{k=1}^{P} C_k(\omega) \cdot \varphi_k(t) \), where \( C_k \in L(E, B, P) \) and \( \varphi_k \) are Kurzweil-Henstock integrable on \([a, b]\),
\[ k = 1, \ldots, p, \text{ then } f \text{ is Kurzweil-Henstock integrable in probability on } [a, b] \text{ and we have} \]

\[ (KH) \int_a^b f(t, \omega)dt = \sum_{k=1}^p C_k(\omega) \cdot (KH) \int_a^b \varphi_k(t)dt. \]

**Proof.** Obviously that it is sufficient to consider only the case when \( f(t, \omega) = C(\omega) \cdot \varphi(t), \omega \in E, t \in [a, b]. \)

If \( d : a = x_0 < \ldots < x_n = b, \xi_i \in [x_i, x_{i+1}], i = 0, \ldots, n - 1, \) then it is easy to see that \( S(f; d, \xi_i)(\omega) = C(\omega) \cdot \sum_{i=0}^{n-1} \varphi(\xi_i) \cdot (x_{i+1} - x_i). \) Let us denote \( I = (KH) \int_a^b \varphi(t)dt \) and \( A_m = \{ \omega \in E; |C(\omega)| \geq m \}. \) Obviously, \( A_{m+1} \subset A_m, m \in \mathbb{N}. \) Denoting \( A = \bigcap_{m=1}^{\infty} A_m, \) since \( C \in L(E, B, P) \) we get \( P(A) = 0 \) and \( \lim_{m \to \infty} P(A_m) = P(A) = 0. \) Consequently, if \( \eta > 0, \) there exists \( N(\eta) \in \mathbb{N}, \) such that

\[ P(\{ \omega \in E; |C(\omega)| \geq m \}) < \eta, m \in \mathbb{N}, m \geq N(\eta). \]

For fixed \( m \geq N(\eta), \) let us consider \( \varepsilon > 0, \) such that \( 1/\varepsilon \geq m. \) Now, for \( \varepsilon^2 > 0, \) since \( \varphi \) is Kurzweil-Henstock integrable on \( [a, b], \) by Definition 3.1, there exists \( \delta_{\varepsilon^2} : [a, b] \to \mathbb{R}, \) such that for any division \( d : a = x_0 < \ldots < x_n = b \) and any \( \xi_i \in [x_i, x_{i+1}] \) with \( x_{i+1} - x_i < \delta_{\varepsilon^2}(\xi_i), i = 0, \ldots, n - 1, \) we have \( |I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| < \varepsilon^2. \)

We have

\[ \{ \omega \in E; |S(f; d, \xi_i)(\omega) - C(\omega) \cdot I| \geq \varepsilon \} \]

\[ = \{ \omega \in E; |C(\omega)| \cdot |I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| \geq \varepsilon \} \]

\[ \subset \{ \omega \in E; |C(\omega)| \geq 1/\varepsilon \} \subset \{ \omega \in E; |C(\omega)| \geq m \}, \]

i.e. \( P(\{ \omega \in E; |S(f; d, \xi_i)(\omega) - C(\omega) \cdot I| \geq \varepsilon \}) < \eta, \) for any division \( d : a = x_0 < \ldots < x_n = b \) and any \( \xi_i \in [x_i, x_{i+1}] \) with \( x_{i+1} - x_i < \delta_{\varepsilon^2}(\xi_i) \) (in fact, \( \varepsilon \) depends on \( m, \) which depends on \( \eta, \) therefore \( \delta_{\varepsilon^2}(\xi_i) \) depends on \( \eta \) too).

Then, by Definition 2.3, we get

\[ (KH) \int_a^b f(t, \omega)dt = C(\omega) \cdot (KH) \int_a^b \varphi(t)dt, \]

which proves the theorem.

\[ \square \]

**Remark.** Since the Kurzweil-Henstock integrability of a function \( \varphi : [a, b] \to \mathbb{R} \) is more general than the Riemann integrability (in fact, it is equivalent with the so-called Denjoy-Perron integrability, see [2], [8]), Theorem 3.2 gives examples of random functions which are Kurzweil-Henstock integrable in probability on \([a, b]\) but are not Riemann integrable in probability in the sense of Definition 1.1.

For \( p \geq 1, \) let us consider

\[ L^p(E, B, P) = \{ g \in L(E, B, P); \int_E |g(\omega)|^pdP(\omega) < +\infty \}, \]
where \( \int_E |g(\omega)|^q dP(\omega) \) represents the \( q \)-th moment of the random variable \( g \).

The following Fubini-type result holds.

**Theorem 3.3.** Let \( f : [a, b] \to L(E, B, P) \) be Kurzweil-Henstock integrable in probability on \([a, b]\) and such that there exists \( A \in L^1(E, B, P) \), \( A(\omega) \geq 0 \), a.e. \( \omega \in E \) with \( P(\{ \omega \in E : |f(t, \omega)| \leq A(\omega) \}) = 1 \), for all \( t \in [a, b] \). Then, \( \varphi(t) = \int_E f(t, \omega) dP(\omega), t \in [a, b] \), is Kurzweil-Henstock integrable on \([a, b]\) and

\[
(KH) \int_a^b \left[ \int_E f(t, \omega) dP(\omega) \right] dt = \int_E \left[ (KH) \int_a^b f(t, \omega) dt \right] dP(\omega).
\]

**Proof.** Let us denote \( I(\omega) = (KH) \int_a^b f(t, \omega) dt \in L(E, B, P) \). Since \( f \) is Kurzweil-Henstock integrable on \([a, b]\), for \( \varepsilon > 0 \) and \( \eta = 1/m, m \in \mathbb{N} \), there exists \( \delta_{\varepsilon, m} : [a, b] \to \mathbb{R} \), such that for any division \( d_m : a = x_0^{(m)} < x_1^{(m)} < \ldots < x_{n_m}^{(m)} = b \) and any \( \xi_i^{(m)} \in [x_i^{(m)}, x_{i+1}^{(m)}] \), with \( x_{i+1}^{(m)} - x_i^{(m)} < \delta_{\varepsilon, m}(\xi_i^{(m)}) \), \( i = 0, \ldots, n_m - 1 \), we have

\[
P(\{ \omega \in E : |S(f; d_m, \xi_i^{(m)})(\omega) - I(\omega)| \geq \varepsilon \}) < 1/m, m \in \mathbb{N}.
\]

This means that \( S(f; d_m, \xi_i^{(m)})(\omega) \to I(\omega) \) in probability, as \( m \to \infty \).

On the other hand,

\[
|S(f; d_m, \xi_i^{(m)})(\omega)| = \left| \sum_{i=0}^{n_m-1} f(\xi_i^{(m)}; \omega)(x_{i+1}^{(m)} - x_i^{(m)}) \right| \leq A(\omega) \cdot (b-a), \text{ a.e. } \omega \in E
\]

and taking into account the well-known property of the integral with respect to \( P \), we immediately get

\[
\int_E I(\omega) dP(\omega) = \lim_{m \to \infty} \int_E S(f; d_m, \xi_i^{(m)})(\omega) dP(\omega)
\]

\[
= \lim_{m \to \infty} \sum_{i=0}^{n_m-1} \left[ \int_E f(\xi_i^{(m)}; \omega) dP(\omega) \right] \cdot (x_{i+1}^{(m)} - x_i^{(m)}).
\]

Now, reasoning exactly as in the case of the definitions of Riemann integrability (see e.g. [6, p. 379-380 and p. 383-384]), it is easy to obtain that the Kurzweil-Henstock integrability in Definition 3.1 is equivalent with the fact that there exists a sequence \( \delta_m : [a, b] \to \mathbb{R}, m \in \mathbb{N} \), such that for any sequence of divisions \( (d_m)_{m \in \mathbb{N}}, d_m : a = x_0^{(m)} < x_1^{(m)} < \ldots < x_{n_m}^{(m)} = b \), and any sequence \( (\xi_i^{(m)})_{i=0,n_m-1} \) with \( \xi_i^{(m)} \in [x_i^{(m)}, x_{i+1}^{(m)}] \), \( x_{i+1}^{(m)} - x_i^{(m)} < \delta_m(\xi_i^{(m)}) \), \( i = 0, \ldots, n_m - 1 \), we have

\[
\lim_{m \to \infty} S(\varphi; d_m, \xi_i^{(m)}) = I = (KH) \int_a^b \varphi(t) dt.
\]

But, denoting \( \varphi(t) = \int_E f(t, \omega) dP(\omega), t \in [a, b] \), by the previous reasonings we immediately get that \( \varphi \) is Kurzweil-Henstock integrable on \([a, b]\) and

\[
(KH) \int_a^b \left[ \int_E f(t, \omega) dP(\omega) \right] dt = \int_E I(\omega) dP(\omega).
\]
\[ = \int_{E} \left( (KH) \int_{a}^{b} f(t, \omega) dt \right) dP(\omega), \]

which proves the theorem. \( \square \)

**Remarks.** 1) Theorem 3.3 is an analogue of Theorem III.8 in [7, p. 55].

2) Let \( f, F : [a, b] \to \mathbb{R} \) be such that \( F'(x) = f(x), x \in (a, b) \). It is known (see e.g. [1]) that in this case \( f \) is (KH)-integrable on \([a, b]\) and \((KH) \int_{a}^{b} f(x)dx = F(b) - F(a)\).

Now, let \( f, F : [a, b] \to L(E, B, P) \) be such that in each \( t_{0} \in (a, b) \), \( f \) is the derivative in probability of \( F(t_{0}, w) \), i.e. for all \( \varepsilon, \eta > 0 \), there exists \( \delta(\varepsilon, \eta) > 0 \), such that for all \( t \in [a, b], t \neq t_{0}, |t - t_{0}| < \delta(\varepsilon, \eta) \), we have

\[ P\{\omega \in E; \left| \left[ F(t_{0}, \omega) - F(t, \omega) \right]/(t - t_{0}) - f(t_{0}, \omega) \right| \geq \varepsilon \} < \eta, \]

holds.

Then, the following question arises: in what conditions \( f(t, \omega) \) is (KH)-integrable in probability on \([a, b]\) and

\[ (KH) \int_{a}^{b} f(t, \omega)dP(\omega) = F(b, \omega) - F(a, \omega), \text{ a.e. } \omega \in E. \]

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