A relativistic description of MOND using the Palatini formalism in an extended metric theory of gravity

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(Dated: October 18, 2016)

We construct a relativistic metric description of MOND using the Palatini formalism following the $f(\chi) = \chi^b$ description of [1]. We show that in order to recover the non-relativistic MOND regime where, for circular orbits the Tully-Fisher law replaces Kepler’s third law, the value of the parameter $b = 3/2$, which is coincident with the value found using a pure metric formalism [1]. Unlike this pure metric formalism, which yields fourth order field equations, the Palatini approach yields second order field equations, which is a desirable requirement from a theoretical perspective. Thus, the phenomenology associated to astrophysical phenomena with Tully-Fisher scalings can be accounted for using this proposal, without the need to introduce any non-baryonic dark matter particles.

PACS numbers: 04.50.Kd, 04.20.Fy, 11.25.Hf, 98.80.Jk, 04.25.-g, 04.20.-q
Keywords: Modified theories of gravity; Variational methods in general relativity; Conformal field theory; Relativistic astrophysics; Approximations methods in relativity; Einstein equation

I. INTRODUCTION

Newton developed the first successful mathematical theory of gravitation. This non-relativistic theory of gravity is based on the empirical foundations of Kepler laws. In particular, Kepler’s third law of motion describes the orbital velocity $v$ of a planet about the sun as a function of its mass $M$ and separation $r$ between a given planet:

$$v \propto M^{1/2} r^{-1/2},$$

for circular orbits. Since a particular planet is in centrifugal equilibrium with the force of gravity, and has a centrifugal acceleration $a = v^2/r$, the end result is that the force per unit mass exerted to a given planet is given by: $a \propto -M/r^2$. Newton’s idea of gravity was the result of a mathematical language of forces and accelerations associated to the empirical observations of Kepler’s laws.

The general relativistic theory of gravity described by Einstein was built as a wider description of gravitational phenomena embracing standard Newtonian gravity as its weak field limit of approximation. It has proven extremely well at mass to length ratio scales similar to the ones associated to our solar system [2–11]. For these reasons, general relativity has been taken as the correct theory to describe gravitation at such scales.

Observational data of astrophysical systems, including individual, groups and clusters of galaxies, and the universe in a cosmological context, show that in order to maintain the standard gravitational field equations of general relativity, including their Newtonian non-relativistic weak field limit, it is necessary to postulate the existence a new kind of non-baryonic dark matter [12–23]. Although, current research is usually done assuming the existence of this non-detected dark matter, the alternative scenario consists on changing the field equations of gravitation at those scales. It was under this point of view that [24, 25] developed a MOdified Newtonian Dynamics (MOND) approach to non-relativistic gravity.

Shortcomings between the theoretical predictions of general relativity and astronomical observations have led to propose alternative theories in order to explain such observations. Amongst these ideas we can name the $f(R)$ theories [26–30], Tensor-Vector-Scalar theories [31–35], galleions [36], bimetric theories [37, 38], modified energetic theories [39], dipolar dark matter [40–42] and non-local theories [43, 44].

In recent years, through dynamical observations of spiral, elliptical and dwarf spheroidal galaxies [43, 46], globular clusters [47, 48] and even wide open binaries [49], it has became clear that at certain scales of mass and length, where the induced gravitational accelerations on test particles are smaller than a certain value $a_0$, Kepler’s third law appears not to hold in its classical form on these systems, but rather obey the Tully-Fisher law.

1 To be more precise, the baryonic Tully-Fisher relation is only observed in spiral galaxies. For elliptical galaxies an analogous relation is also observed and is known as the Faber-Jackson relation [50]. We are using both relations as to mean the same physical idea, that the scaling with velocity -or velocity disper-
\[ v \propto M^{1/4}. \] (2)

Following [50, 51], we assume that at some regime, gravity follows Kepler’s third law [1], and at some other it follows the Tully-Fisher law. As such, when the Tully-Fisher regime is reached, the acceleration exerted by a test particle at a distance \( r \) from a point mass source \( M \) generating a gravitational field is given by

\[ a = \frac{v^2}{r} \propto -\frac{M^{1/2}}{r}. \]

As noted by [50, 51], the proportionality constant can be written as \( \sqrt{G}a_0 \), where \( a_0 \approx 10^{-10}\text{ m s}^{-2} \) is Milgrom’s acceleration constant. Using this, the previous relation can be written as:

\[ a = \frac{\sqrt{a_0}GM}{r}. \] (3)

All current observations [52–59] show that Newtonian gravity is reached when test particles acquire an acceleration greater than \( a_0 \) and a full MONDian regime is obtained when those accelerations are smaller than \( a_0 \). View in this way, all systems with accelerations \( a \lesssim a_0 \) are the ones that are commonly viewed as systems where non-baryonic dark matter is required to explain the observed dynamics.

In [1], a construction of an extended relativistic metric theory of gravity that recovers MOND on its weak field limit of approximation was made. This construction, has been tested to yield the correct bending angle of light for gravitational lensing in individual, groups and clusters of galaxies [60] as well as for the dynamics of clusters of galaxies [61] and for the accelerated expansion of the universe [62].

In this article, we search for a possible extended metric theory of gravity using the Palatini formalism, which recovers MOND on its weak field limit of approximation. In sect. II we briefly introduce the relevant equations for the Palatini metric formalism useful for our further developments. In sect. III we propose a power of the Ricci scalar for the gravitational action and we find an expression for the Ricci scalar curvature as function of the trace of the energy-momentum tensor. In sect. IV we explore the non-relativistic weak-field limit of the theory and expand the metric as Minkowskian plus a second order perturbation to arrive at a non-relativistic equation for the acceleration as function of the energy-momentum tensor. In sect. V we fix the free parameters of our theory such that in the weak-field limit of approximation the acceleration converges to the simplest MONDian description of eq. (3). In sect. VI we perform a Parametrised Post Newtonian (PPN) analysis to second order of our field equations in order to complement the results of sect. IV. In sect. VII we conclude and discuss our results.

II. \( f(\chi) \) IN PALATINI FORMALISM

Many of the results mentioned in this section are well known on the studies of the Palatini formalism for metric \( F(R) \) theories of gravity. For further information, the reader is referred to the excellent introductory texts by [26–30].

Let us start with a dimensionally correct action for the gravitational field motivated by the one built by [1]:

\[
S = -\frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g} \, d^4x - \frac{1}{c} \int \mathcal{L}_{\text{mat}} \sqrt{-g} \, d^4x,
\] (4)

where \( L_M \) is a coupling constant with dimensions of length and the dimensionless Ricci scalar \( \chi \) is given by:

\[
\chi := L_M^2 \mathcal{R},
\] (5)

where \( \mathcal{R} \) is a non-traditional Ricci scalar, not to be confused with the standard Levi-Civita Ricci’s one \( R \). Both are related to each other by the following relation:

\[
\mathcal{R} := g^{\mu\nu} \mathcal{R}_{\mu\nu}.
\] (6)

In the previous equation, and in what follows, we use Einstein’s summation convention, greek and latin indices take values from 0 to 4 and from 0 to 3 respectively. The tensor \( g_{\mu\nu} \) represents the metric tensor and \( \mathcal{R}_{\mu\nu} \) is a non-traditional Ricci tensor defined exclusively in terms of the affine connection \( \Gamma^\alpha_{\mu\nu} \) through the following equation:

\[
\mathcal{R}_{\mu\nu} := \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\mu\rho} \Gamma^\lambda_{\nu\lambda}.
\] (7)

In the Palatini formalism, the connection \( \Gamma^\alpha_{\mu\nu} \) has no relation with the standard Levi-Civita connection \( \Gamma^\alpha_{\mu\nu} \).

The null variations of the action \( \mathcal{A} \) with respect to the metric \( g_{\mu\nu} \) yield the following field equations:

\[
f'(\chi) \chi_{\mu\nu} - \frac{1}{2} f(\chi) g_{\mu\nu} = \frac{8\pi G L_M^2}{c^4} T_{\mu\nu},
\] (8)

where the dimensionless tensor

\[
\chi_{\mu\nu} := L_M^2 \mathcal{R}_{\mu\nu}
\] (9)

and \( f'(\chi) := df(\chi)/d\chi \). The energy-momentum tensor \( T_{\mu\nu} \) is given by [63]:

\[
\cdots
\]
\[
T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta (\mathcal{L}_{\text{matter}} \sqrt{-g})}{\delta g^{\mu\nu}}. \tag{10}
\]

The contraction of eq. (8) with \( g^{\mu\nu} \) yields:

\[
L_M^2 f'(\chi) R - 2f(\chi) = \frac{8\pi G L_M^2}{c^4} T, \tag{11}
\]

for all \( f(\chi) \neq \chi^2 \). Under the assumption of a torsion free connection, i.e. imposing a symmetric connection \( \Gamma^\alpha{}_{\mu\nu} \), the null variations of the action \( \mathcal{I} \) with respect to this affine connection yield:

\[
\nabla_\lambda (\sqrt{-g} f'(\chi) g^{\mu\nu}) = 0. \tag{12}
\]

The usual approach to solve this equation, consists on performing the following conformal transformation to the metric tensor:

\[
h_{\mu\nu} = f'(\chi) g_{\mu\nu}. \tag{13}
\]

Substitution of this last equation into relation (12) gives:

\[
\nabla_\lambda (\sqrt{-h} h^{\mu\nu}) = 0, \tag{14}
\]

where \( h := h^{\mu\nu} \). Equation (14) is known as the metricity condition and states that \( \Gamma^\alpha{}_{\mu\nu} \) is the Levi-Civita connection of the \( h_{\mu\nu} \) metric, i.e.:

\[
\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} h^{\lambda\rho} (h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}). \tag{15}
\]

For the conformal transformation \( (13) \), the tensor \( R_{\mu\nu}(\Gamma) \) is related to the usual Ricci tensor \( R_{\mu\nu}(\Gamma) \) defined in terms of the Levi-Civita connection of the \( g_{\mu\nu} \) by \cite{28, 64}:

\[
R_{\mu\nu} = R_{\mu\nu} - \frac{1}{f'} \nabla_\mu \nabla_\nu f' - \frac{1}{2f^2} g_{\mu\nu} \Delta f' + \frac{3}{2f^2} \nabla_\mu f' \nabla_\nu f', \tag{16}
\]

The contraction of this last result with the metric \( g^{\mu\nu} \) yields:

\[
R = R - \frac{3}{f'} \Delta f' + \frac{3}{2f'^2} \nabla_\mu f' \nabla^\mu f'. \tag{17}
\]

Note that \( R \) is not the Ricci scalar for the \( h_{\mu\nu} \) metric, since it is built by its contraction with the conformal metric \( h^{\mu\nu} \).

In what follows we are going to work extensively with eqs. (11) and (17), since using the former it is possible to find \( R \) as a function of the trace of the energy-momentum tensor, i.e.: \( R = R(T) \). Substitution of this result on the latter, and bearing in mind the fact that \( f' \) is a function of \( R \) and hence of \( T \), the solution \( R = R(T) \) can be found.

### III. \( f(\chi) \) AS A POWER LAW

Let us now assume that:

\[
f(\chi) = \chi^b. \tag{18}
\]

and substitute it into relations (17) and (11) to obtain:

\[
R = R + \frac{3}{2} R^{2-2b} \nabla_\mu R^{b-1} \nabla^\mu R^{b-1} - 3 R^{1-b} \Delta R^{b-1}, \tag{19}
\]

and

\[
R^b = \frac{\alpha T}{b - 2}, \tag{20}
\]

where

\[
\alpha := \frac{8\pi G L_M^2}{c^4}. \tag{21}
\]

In order to obtain an equation that relates the curvature \( R \) with the trace \( T \) of the energy-momentum tensor, eq. (20) must be substituted into (19). Since this procedure yields a complex equation, we will tackle the problem in a different manner.

Let us then proceed by expressing relation (11) as:

\[
R_{\mu\nu} = R_{\mu\nu} + H_{\mu\nu}(R), \tag{22}
\]

where we have used the fact that \( f' = f'(R) \). The tensor \( H_{\mu\nu}(R) \) has a complicated algebraic form which will be determined in sect. \( \text{VI} \) and we will show an explicit functional form which allows our proposal to have full consistency at the lowest second perturbation order. The trace of eq. (22) is:

\[
R = R + H(R), \tag{23}
\]

which is another way to express eq. (17). A Taylor expansion of the function \( H(R) \) yields the following linear relation:

\[
H(R) = \kappa R + O(R^2), \tag{24}
\]

since \( H(R = 0) = 0 \) according to eq. (23). Substitution of eq. (24) into eq. (23) yields:

\[
R = \kappa' R, \quad \text{where} \quad \kappa' := 1 - \kappa. \tag{25}
\]

Using this result in eq. (20), we obtain:

\[
R = \kappa' \left[ \frac{\alpha T}{b - 2} \right]^{1/b}. \tag{26}
\]
IV. WEAK FIELD LIMIT

Our main target is to find $b$ such that in the weakest (non-relativistic) limit of the theory, the acceleration of a test particle in a gravitational field produced by a point mass source $M$ is reduced to the MONDian one \[4\].

For this purpose we take the background metric as the Minkowsky space-time plus a small perturbation expanded in powers of $1/c$, which we call perturbation orders. As an example, a second perturbation order is proportional to $1/c^2$ and zeroth order terms have no dependence on the speed of light. The next perturbation expansion to the Minkowsky background metric is of second order \[65\], and since we are interested in the weakest limit of the theory describing the motion of non-relativistic massive test particles, this correction is enough for our study.

At this point we stress that in eq. \[20\] we have returned to the original metric $g_{\mu\nu}$. The conformal transformation was just a mathematical tool in order to manipulate more easily the resulting equations. Therefore, the expansion used below is justified.

For the second perturbation order, we take as base the work of \[51\], in which they proved that, to be in accordance with dimensional analysis in the description of a point mass source for a relativistic version of MOND, it is possible to construct two independent fundamental lengths and it is expected that the length $L_M$ should be a function of those two lengths, in other words:

\[ L_M \propto c. \]

As noted by \[1\], using dimensional analysis in the description of a point mass source for a relativistic version of MOND, it is possible to construct two independent fundamental lengths and it is expected that the length $L_M$ should be a function of those two lengths, in other words:

\[ L_M := \xi r_g \sqrt{r_M}, \]

where $r_g := \frac{GM}{c^2}$, $l_M := \left(\frac{GM}{a_0}\right)^{1/2}$, and $a_0$ is the mass density. Thus, eq. \[31\] turns into:

\[ \frac{a}{c^2 r} \approx \left[ \frac{G L^2_{\rho} \rho}{c^2} \right]^{1/b}. \]

This last equation will allow us to fix the parameter $b$ such that it is possible to recover a MONDian acceleration \[4\].

V. RECOVERING MOND

At order of magnitude, eq. \[31\] turns into:

\[ \frac{a}{c^2 r} \approx \left[ \frac{G L^2_{\rho} \rho}{c^2} \right]^{1/b}. \]

For a point mass source located at the origin, the density $\rho$ is given by:

\[ \rho = M \delta (r), \]

where $\delta (r)$ is the three-dimensional Dirac’s delta distribution in spherical coordinates. Approximating the previous equation to the same order of magnitude yields $\rho \approx M/r^3$, and so expression \[32\] reduces to:

\[ a \approx (GM)^{1/b} L^2_{\rho} (1-b)/b c^{2(b-1)/b} r(b-3)/b. \]

On the one hand, the flattening of rotation curves requires $a \propto r^{-1}$, and so:

\[ b = 3/2. \]

On the other hand, the weakest field limit of approximation yields a non-relativistic description of gravity and as such, the velocity of light should not appear on eq. \[34\]. In other words,

\[ L_M \propto c. \]

Since the Tully-Fisher law describes the motion of non-relativistic dust particles, then the energy-momentum tensor trace is

\[ T = \rho c^2, \]

where $\rho$ is the mass density. Thus, eq. \[20\] turns into:

\[ -\frac{2 \nabla^2 \phi}{c^2} = \kappa' \left[ \frac{8\pi G L^2_{\rho} \rho}{c^2(b-2)} \right]^{1/b}. \]
represent the gravitational radius and a MONDian “mass-length” scale respectively. The constant $\zeta$ is a proportionality factor and the exponents $\alpha$ and $\beta$ must satisfy the condition $\alpha + \beta = 1$ so that eq. (37) is dimensionally correct. With the aid of eq. (36) it follows that $\alpha = -1/2$, and so, $\beta = 3/2$. In other words:

$$L_M = \zeta \left( \frac{GM}{a_0^2} \right)^{1/4} c.$$  \hspace{1cm} (39)

Using this expression for $L_M$ and the value for $b$ previously found, at order of magnitude the acceleration (34) reaches a MONDian value: $a \approx (GMa_0)^{1/2}/r$.

In order to fully show that a MONDian non-relativistic limit is obtained in the weak-field limit of the theory, we proceed as follows. Direct substitution of the values obtained for $b$ and $L_M$, into eq. (31) yields:

$$-2\nabla \cdot a = \kappa' (a_0 GM)^{1/2} \left[ \frac{4\delta(r)}{\zeta r^2} \right]^{2/3}.$$  \hspace{1cm} (40)

where we have used the fact that the three-dimensional Dirac’s delta function is given by $\delta(r) = \delta(r)/4\pi r^2$.

For Schwartz distributions it is impossible in general terms, to define a product in such a way that the resulting distribution forms an algebra with acceptable topological properties. Schwartz’s impossibility result states that it is not possible to have a differential algebra that contains the space of distributions and preserves the product of continuous functions. To overcome these disadvantages, has developed a theory of generalised functions, which allows to define a fully consistent product of distributions. As such, we can consider Dirac’s delta distribution as a standard function so that we can write the following identity

$$[\delta(r)]^{2/3} = [\delta(r)]^{-1/3} \delta(r).$$  \hspace{1cm} (41)

With this relation, eq. (40) turns into:

$$-2\nabla \cdot a = \kappa' (a_0 GM)^{1/2} \left( \frac{4}{\zeta} \right)^{2/3} \left[ \frac{1}{r^3 \delta(r)} \right]^{1/3} \delta(r).$$  \hspace{1cm} (42)

Since we are searching for a MONDian value for the acceleration, let us assume it obeys the following general power law:

$$a = \lambda r^\sigma e_r,$$  \hspace{1cm} (43)

where $e_r$ is a unitary vector in the radial direction, $\lambda$ and $\sigma$ are constants so that:

$$\nabla \cdot a = \lambda (\sigma + 2) r^{\sigma - 1}.$$  \hspace{1cm} (44)

Substitution of this last equation into (42) and performing an integration over $r$ yields:

$$-2\lambda (\sigma + 2) r^\sigma \bigg|_{r=0}^{r=\infty} = \kappa' (a_0 GM)^{1/2} \left( \frac{4}{\zeta} \right)^{2/3} \left[ \frac{1}{r^3 \delta(r)} \right]^{1/3} \bigg|_{r=0}.$$  \hspace{1cm} (45)

Let us now use the fact that $\delta(0)$ can be obtained from the following relation [69]:

$$\delta(r = 0) = \lim_{r \to 0} \frac{1}{2\pi r^2},$$  \hspace{1cm} (46)

and substitute it into eq. (45) in order to obtain:

$$-\frac{2\lambda (\sigma + 2)}{\gamma} r^\sigma \bigg|_{r=0}^{r=\infty} = \kappa' (a_0 GM)^{1/2} \left( \frac{4\pi}{\zeta} \right)^{1/3} \frac{1}{r}.$$  \hspace{1cm} (47)

Since $\zeta$ and $\lambda$ are constants, the following relation is necessarily satisfied:

$$\sigma = -1,$$  \hspace{1cm} (48)

which is an expected result from the order of magnitude analysis developed above in order to obtain flat rotation curves. Equation (47) is then reduced to:

$$-\lambda = \kappa' (a_0 GM)^{1/2} \left( \frac{4\pi}{\zeta^2} \right)^{1/3}.$$  \hspace{1cm} (49)

In order to recover a MONDian acceleration (3) limit, it is necessary that $\lambda = - (a_0 GM)^{1/2}$ and so:

$$\zeta = 2 \left( k' \pi \right)^{1/2}.$$  \hspace{1cm} (50)

VI. SECOND ORDER PERTURBATION ANALYSIS

In order to show that a MONDian solution is directly obtained from the field equations of the previous analysis, let us proceed as follows. Substituting the value $b = 3/2$ in eq. (5) and (20), the field equations and the trace take the following form:

$$3R^{1/2} R_{\mu\nu} - g_{\mu\nu} R^{3/2} = \frac{16\pi G}{c^4 L_M} T_{\mu\nu},$$  \hspace{1cm} (51)

and

$$-R^{3/2} = \frac{16\pi G}{c^4 L_M} T.$$  \hspace{1cm} (52)

This last equation is meaningless unless the energy-momentum tensor is defined with a minus sign on the
right-hand side of eq. (10). This fact is closely related to the multiple branches that the solution space of any \( F(R) \) theory of gravity has, which is usually ascribed to the choice of the Riemann tensor (see e.g. the discussion on the appendix of [60]). Quite curiously for the previous and following discussions it is not necessary at all to enter into further discussions about this, since the obtained results require only the square of eq. (52). Substituting eqs. (52), (39) and (22) into (51), we obtain the following field equations:

\[
3(R_{\mu\nu} + H_{\mu\nu}) = \left(\frac{16\pi}{c^4\zeta}\right)^{2/3} \frac{(a_0 G)^{1/2}}{M^{1/6}} \frac{(T g_{\mu\nu} - T_{\mu\nu})}{T^{1/3}}.
\]

(53)

If we now perturb the metric \( g_{\mu\nu} \) about a flat Minkowsky space-time \( \eta_{\mu\nu} \) we obtain:

\[
g_{\mu\nu} = \eta_{\mu\nu} + \xi_{\mu\nu}.
\]

(54)

The quantity \( \xi_{\mu\nu} \) is the perturbation expanded in powers of \( 1/c \). To first order in \( \xi \) (second order in \( 1/c \)), the time and space components of \( R_{\mu\nu} \) are \[65\]:

\[
(2) R_{00} = \frac{1}{2} \nabla^2 \xi_{00},
\]

(55)

and

\[
(2) R_{ij} = \frac{1}{2} \nabla^2 \xi_{ij},
\]

(56)

where we have suppressed the upper index in \( \xi \) in the understanding that only the second perturbation order is relevant in our analysis. We have also chosen the PPN gauge for which: \( \xi^\mu_{\ i,\mu} - 1/2 \xi^\mu_{\ \mu,\ i} = 0 \).

The constraint eq. (24) implies that \( H_{\mu\nu} \) is a linear function in \( \mathcal{R} \) and so by eq. (25) it is also linear \( R \), but now the proportionality constant is a second rank tensor \( \kappa_{\mu\nu} \), i.e.:

\[
H_{\mu\nu} = \kappa_{\mu\nu} \mathcal{R}.
\]

(57)

Thus, if the first perturbative term of \( R \) is a second order term, \( H_{\mu\nu} \) would also be as such.

The spatial components of eq. (53) are:

\[
3 \left(\frac{1}{2} \nabla^2 \xi_{ij} + (2) H_{ij}\right) = \left(\frac{16\pi}{c^4\zeta}\right)^{2/3} \frac{(a_0 G)^{1/2}}{M^{1/6}} \frac{(T g_{ij} - T_{ij})}{T^{1/3}}.
\]

(58)

The left-hand side of this relation is of second order and so, to obtain a second order term on the right-hand side, the last factor involving only \( T \) must be of order \( \mathcal{O}(c^{4/3}) \).

For dust, the lowest perturbation order on \( T \) implies that: \( T = \rho c^2 \) and \( T_{ij} = 0 \), satisfying the previous requirement. This is a consistency check that our proposal is coherent at the lowest perturbation order. Thus, for dust and a point mass source, eq. (58) turns into:

\[
3 \left(\frac{1}{2} \nabla^2 \xi_{ij} + (2) H_{ij}\right) = -\left(\frac{4d(r)}{r^2}\right)^{2/3} \frac{(a_0 G M)^{1/2}}{c^2} \delta_{ij}.
\]

(59)

Comparison of this expression with (10), yields:

\[
3 \left(\frac{1}{2} \nabla^2 \xi_{ij} + (2) H_{ij}\right) = \frac{2 \nabla^2 \phi}{\kappa' c^2} \delta_{ij}.
\]

(60)

In order to recover the value of \( \xi_{ij} \) consistent with an isotropic metric, i.e. \( \xi_{ij} = 2\phi/c^2 \delta_{ij} \), the following value of \( H_{ij} \) is obtained:

\[
(2) H_{ij} = \frac{\nabla^2 \phi}{2} \delta_{ij} \left( \frac{2}{3\kappa'} - 1 \right).
\]

(61)

An analogous procedure for the time component yields:

\[
(2) H_{00} = -\frac{\nabla^2 \phi}{c^2}.
\]

(62)

Physically in a weak field limit it is expected that the Jordan and the Einstein frames, with metrics \( g_{\mu\nu} \) and \( h_{\mu\nu} \) respectively, lead to the same physical results. This means that the contributions of the tensor \( H_{\mu\nu} \) must be sufficiently small. Bearing this in mind and the arbitrariness of the constant \( \kappa' \), let us choose:

\[
\kappa' = \frac{2}{3}, \quad \text{for which:} \quad \kappa = \frac{1}{3}.
\]

(63)

Using these results together with eqs. (57), (61), 24, and 62, we obtain that the only non vanish component of the tensor \( \kappa_{\mu\nu} \) is \( \kappa_{00} = 1/3 \).

Finally, from eq. (60), we find:

\[
\zeta = \left(\frac{32\pi}{27}\right)^{1/2}.
\]

(64)

VII. DISCUSSION

It has become quite challenging to find a general expression that could potentially yield MOND on the weak-field limit of approximation [31, 33, 34, 36, 37, 70, 72]. Many of the proposal fail since the metric coefficients (27) at second perturbation order are in no agreement with the mathematical particularities of the theories involved. Most importantly, it has always been desired that the field equations of a relativistic version of MOND are of the second order and involve only a power law function of the Ricci scalar. In this article, we have shown how to build such a second order field equations theory based on
the metric coefficients [27] that converges to the simplest form of MOND [3] on its weakest limit of approximation.

It is worth noticing at this point that the developed formalism in this article is such that the “coupling constant length” $L_M$ of the gravitational action [41] is proportional to $M^{1/4}$. [1, 72, 72] have all encountered this particularity when trying to build relativistic versions of MOND for metric formulations of gravity. Since it is customary that the gravitational action does not depend on the mass (or the energy-momentum tensor) then these authors have noticed that “one should not be surprised if some of the commonly accepted notions, even at the fundamental level of the action, require generalisations and re-thinking”. An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we should change some of our standard views regarding its fundamental principles. Accepting this we can formally write the gravitational action $S_g$—first term on the right-hand side of eq. (4)—inspired by the generalisations made by [78–82] and following a similar approach as that of [83]:

$$S_g = -\frac{c^3}{16\pi G} \int \frac{f(\chi)}{L_M^2} \sqrt{-g} \, dx^4,$$

where following the results of eq. (59):

$$L_M = c \left( \frac{G}{\dot{a}_0^3} \right)^{1/4} \int \rho d^3x,$$

and we have used the fact that the matter Lagrangian $L_{matt} = \rho c^2$ for dust, and for systems with sufficient degree of symmetry, e.g. isotropic or spherically symmetric space-times, the integral is taken over all the causally connected masses related to a particular problem. For the single point mass source discussed in this article, $\rho = M \delta(r)$ and in this case, eq. (65) converges to the gravitational action [4].

At this point, it is important to note that usually in the analysis of $F(R)$ theories on a Post-Newtonian frame, the comparison with a Brans-Dicike-like scalar-tensor theory can be achieved [84]. In our work, we do not appeal to this analogy and we keep the original equations throughout our analysis.

We choose to work in the frame of the Palatini formalism since it provides a deeper understanding of our proposal than the metric formalism because we do not restrict to a special kind of connection. While it is true that in standard general relativity, the Palatini formalism does not seem to bring something new, its use in areas where general relativity is not tested has been extended [55, 57].

The value of the parameter $b = 3/2$ required for an extended metric theory of gravity $f(\chi) = \chi^{3/2}$ in the Palatini formalism to yield a MONDian behaviour has appeared on many other works related to the cosmology [88, 89] and to MOND using a pure metric approach to the problem [1] using Noether’s symmetry. It is quite interesting that this value also appears in the Palatini formalism presented in this article and together with the previous findings sheds some light into a deepest understanding of gravitational phenomena beyond Einstein’s general relativity.

The analysis performed in this article shows that it is possible to explain the flattening of rotation curves and the Tully-fisher law from our $f(\chi) = \chi^{3/2}$ theory using the Palatini formalism. By construction it not only reproduces the dynamics of material particles required to flatten rotation curves which show a Tully-Fisher scaling, but also reproduces bending of light associated to individual, groups and clusters of galaxies. This approach can be tested in cosmological models dealing with the accelerated expansion of the universe and in complex gravitational lensing, such as for example the ones produced by collisional clusters of galaxies. The fourth perturbation order of the theory can be also used to model the dynamics of clusters of galaxies in a completely analogous way as it was done in [61]. These analyses are beyond the scope of this article and will be studied elsewhere.

**ACKNOWLEDGEMENTS**

This work was supported by DGAPA-UNAM (IN112616) and CONACyT (240512) grants. EB and SM acknowledge economic support from CONACyT (517586 and 26344).
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