On ordered Ramsey numbers of bounded-degree graphs

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Ramsey theory

Ramsey's theorem for graphs
For every graph \( G \) there is an integer \( N = N(G) \) such that every 2-coloring of the edges of \( K_N \) contains a monochromatic copy of \( G \).

Ramsey number \( R(G) \) of \( G \) is the smallest such \( N \).

Example: \( R(C_4) = 6 \)
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Ordered Ramsey numbers

An ordered graph $G$ is a pair $(G, \prec)$ where $G$ is a graph and $\prec$ is a total ordering of its vertices. $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.

The ordered Ramsey number $R(G)$ of an ordered graph $G$ is the least number $N$ such that every 2-coloring of edges of $K_N$ contains a monochromatic copy of $G$ as an ordered subgraph.
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![Diagram of ordered Ramsey numbers](image-url)
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The ordered Ramsey number $\overline{R}(G)$ of an ordered graph $G$ is the least number $N$ such that every 2-coloring of edges of $K_N$ contains a monochromatic copy of $G$ as an ordered subgraph.

- $\overline{R}(C_A) = 10$
- $\overline{R}(C_B) = 11$
- $\overline{R}(C_C) = 14$
Bounded-degree graphs

We consider graphs with the maximum degree bounded by a constant. There is a substantial difference between ordered and unordered case.

**Theorem (Chvátal, Rödl, Szemerédi, Trotter, 1983)**
Every graph $G$ on $n$ vertices with bounded maximum degree satisfies $R(G) \leq O(n)$.

**Theorem (B., Cibulka, Kráľ, Kynčl and Conlon, Fox, Lee, Sudakov, 2014)**
There are arbitrarily large ordered matchings $M_n$ on $n$ vertices such that $R(M_n) \geq n \Omega(\log n \log \log n)$.

Conlon et al. showed that this holds for almost every ordered matching.
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- Conlon et al. showed that this holds for almost every ordered matching.
There are $n$-vertex ordered matchings $M$ with $R(M)$ linear in $n$. Which orderings have asymptotically smallest ordered Ramsey numbers? When can we attain linear ordered Ramsey numbers?

Problem (Conlon, Fox, Lee, Sudakov, 2014)
Do random 3-regular graphs have superlinear ordered Ramsey numbers for all orderings?
Smallest ordered Ramsey numbers

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$\overline{R}(\mathcal{M}_1), \overline{R}(\mathcal{M}_2) \leq 2n - 2$
Smallest ordered Ramsey numbers

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Superlinear lower bound

We give a positive answer to the problem of Conlon et al. $\min_{G} R(G)$ is the minimum of $R(G)$ over all orderings $G$ of $G$.

**Theorem**
For every $d \geq 3$, almost every $d$-regular graph $G$ on $n$ vertices satisfies

$$\min_{G} R(G) \geq \frac{n}{3} - \frac{1}{d} \frac{4 \log n \log \log n}{\log n}.$$ 

For 3-regular graphs, we obtain

$$\min_{G} R(G) \geq \frac{n}{6} \frac{4 \log n \log \log n}{\log n}.$$
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$\min - \bar{R}(G)$ is the minimum of $\bar{R}(G)$ over all orderings $G$ of $G$.
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For 3-regular graphs, we obtain \( \min \overline{R}(G) \geq \frac{n^{7/6}}{4 \log n \log \log n} \).
For $d \geq 3$, let $G$ be a $d$-regular graph on $n$ vertices.

Key lemma: Almost every such $G$ satisfies the following: for every partition of $V(G)$ into few sets $X_1, \ldots, X_t$, each of size at most $s$, there are many pairs $(X_i, X_j)$ with an edge between them. Here, $t = n/\left(2 \log n \log \log n\right)$ and $s = n^{1/2 - 1/d/2}$. We use an estimate by Bender and Canfield and by Wormald for the number of $d$-regular graphs on $n$ vertices.
Sketch of the proof

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![Graph images]
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- Each \( n \)-vertex 1-regular graph has an ordering \( \mathcal{M} \) with \( \overline{R}(\mathcal{M}) \) linear in \( n \).
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- How about 2-regular graphs?
Theorem

Every graph $G$ with $n$ vertices and with maximum degree at most two satisfies $\min R(G) \leq O(n)$.

First, for every $n$, we find an ordering $C_n$ of $C_n$ with $R(C_n)$ linear in $n$.

Second, we find an ordering of a disjoint union $G$ of these ordered cycles with linear $R(G)$.

Placing cycles sequentially does not work.
2-regular graphs

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Every graph $G$ with $n$ vertices and with maximum degree at most two satisfies

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$\mathcal{P}_5$
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Sketch of the proof II

Orderings of disjoint union of cycles are constructed as follows. For bipartite 2-regular graphs we obtain a stronger Turán-type result.

Theorem

For each $\varepsilon > 0$, there is $C(\varepsilon)$ such that, for every $n \in \mathbb{N}$, every bipartite graph $G$ on $n$ vertices with maximum degree 2 admits an ordering $G$ of $G$ that is contained in every ordered graph with $N \geq C(\varepsilon)n$ vertices and with at least $\varepsilon N^2$ edges.

No longer true if $G$ contains an odd cycle.
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Final remarks

A result of Conlon, Fox, Lee, Sudakov gives a polynomial upper bound.

Corollary

Every graph $G$ with $n$ vertices and with maximum degree $d$ satisfies

$$\min R(G) \leq O(n(d+1)\lceil \log(d+1) \rceil + 1).$$

The upper and lower bounds for $\min R(G)$ are far apart.

Problem

Improve the upper and lower bounds on $\min R(G)$ for 3-regular graphs $G$.

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**Corollary**

Every graph $G$ with $n$ vertices and with maximum degree $d$ satisfies

$$\min-\overline{R}(G) \leq O(n^{(d+1)[\log(d+1)]+1}).$$

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Improve the upper and lower bounds on $\min-\overline{R}(G)$ for 3-regular graphs $G$. 
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