ON STABLE EMBEDDABILITY OF PARTITIONS

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Abstract. Several natural partial orders on integral partitions, such as the embeddability, the stable embeddability, the bulk embeddability and the supermajorization, raise in the quantum computation, bin-packing and matrix analysis. We find the implications between these partial orders. For integral partitions whose entries are all powers of a fixed number $p$, we show that the embeddability is completely determined by the supermajorization order and we find an algorithm to determine the stable embeddability.

1. introduction

A partition $\lambda$ is a finite sequence of nonincreasing positive real numbers, denoted by $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ where $\lambda_i \geq \lambda_j$ for all $i \leq j$. $\lambda_i$ is called an entry of $\lambda$. A partition $\lambda$ is an integral partition if all $\lambda_i \in \mathbb{N}$. Throughout the article, we assume all partitions are integral unless we state differently. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$, $\mu = [\mu_1, \mu_2, \ldots, \mu_n]$ be two partitions. We can naturally define an addition of two partitions, $\lambda + \mu$ by a reordered juxtaposition, a product of two partitions, $\lambda \times \mu$ by $[\lambda_i \cdot \mu_j]$ and a scalar multiplication, $\alpha \lambda$ by $[\alpha \cdot \lambda_i]$.

We denote $\lambda \times \lambda \times \ldots \times \lambda$ by $\lambda^n$. We recall definitions of partial orders on partitions. For more terms and notations, we refer to [2, 7]. A partition $\lambda$ supermajorizes a partition $\mu$, or $\lambda \succeq_{S} \mu$, if for every $x \in \mathbb{N}$

$$\sum_{\lambda_i \geq x} \lambda_i \geq \sum_{\mu_j \geq x} \mu_j.$$ 

A partition $\lambda$ embeds into $\mu$ if there exists a map $\varphi : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in \varphi^{-1}(j)} \lambda_i \leq \mu_j$$

for all $j$, denoted by $\lambda \hookrightarrow \mu$. This embedding problem can be interpolated as a bin-packing problem by replacing the entries of a partition $\lambda$ by the sizes of the blocks and the entries of a partition $\mu$ by the sizes of the bins. It is well known that the question of whether $\lambda$ embeds into $\mu$ is computable but NP-hard.

Kuperberg found an interesting embeddability, $\lambda$ bulk-embeds into $\mu$, or $\lambda \overset{b}{\rightarrow} \mu$, if for every rational $\epsilon > 0$, there exists an $N$ such that $\lambda \times N \hookrightarrow \mu \times N(1+\epsilon)$ [4]. He showed the following theorem.

**Theorem 1.1.** [4] Let $\lambda$ and $\mu$ are two partitions, then $\lambda \overset{b}{\rightarrow} \mu$ if and only if

$$||\lambda||_p \leq ||\mu||_p$$

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for all $p \in [1, \infty]$.

He also showed the following implications,

$$\lambda \hookrightarrow \mu \Rightarrow \lambda \preceq S \mu \Rightarrow \lambda \hookrightarrow^b \mu,$$

(1)

$$\lambda \hookrightarrow^b \mu \Rightarrow \lambda \preceq S \mu \Rightarrow \lambda \hookrightarrow \mu.$$

One can consider a partition as the capacity of a quantum memory [6]. Kuperberg introduced a stable embeddability in the presence of an auxiliary memory [4]. A partition $\lambda$ stably embeds into a partition $\mu$ if there exist a partition $\nu$ such that $\lambda \times \nu \hookrightarrow \mu \times \nu$, denoted by $\lambda \hookrightarrow^s \mu$. Then he asked the relation between the stable embeddability and the supermajorization order. We answer the question and compare these embeddabilities in section 2. A complete classification of the stable embeddability is unknown. Since the sizes of the classical memories are all powers of 2, it is natural to study the case all entries of partitions are powers of a fixed positive integer $p$. For these partitions, we find that the embeddability is completely determined by the supermajorization order. Also we find an algorithm to determine the stable embeddability in section 3.1.

2. Comparison of embeddabilities

For partitions $\lambda, \mu$, we find the following diagram about the implications of these embeddings.

$$\lambda \hookrightarrow \mu \Rightarrow \lambda \preceq S \mu \quad \downarrow \quad \downarrow$$

$$\lambda \hookrightarrow^s \mu \Rightarrow \lambda \hookrightarrow^b \mu$$

The converse of all implications are false. We provide counterexamples in Example 2.4. Moreover, there is no relation between the stable embeddability and the supermajorization order, which address the question arose in [4]. For these counterexamples, we need to show a few facts about these embeddabilities. One can see that if $\lambda \hookrightarrow \mu$, then $||\lambda||_p \leq ||\mu||_p$ for all $p \in [1, \infty]$.

**Theorem 2.1.** Let $\lambda, \mu$ be partitions. If $\lambda \neq \mu$ and $\lambda \hookrightarrow \mu$, then $||\lambda||_p < ||\mu||_p$ for all $1 < p < \infty$.

**Proof.** Let

$$\lambda = [a_1, a_2, \ldots, a_l], \mu = [b_1, b_2, \ldots, b_m].$$

We will prove it by a contradiction. Suppose $\lambda \hookrightarrow \mu$, $\lambda \neq \mu$ and $||\lambda||_p = ||\mu||_p$ for some $1 < p < \infty$. Then there exists a map $\varphi : \{1, 2, \ldots, l\} \rightarrow \{1, 2, \ldots, m\}$ presenting the embedding. We divide cases by the sizes of $l, m$. If $l > m$, then there exist $i_1, i_2$ and $j$ such that $\{i_1, i_2\} \subset \varphi^{-1}(j)$. Since

$$\alpha^p + \beta^p < (\alpha + \beta)^p$$

for all $p > 1$ and nonzero $\alpha, \beta$, we have

$$a_{i_1} + a_{i_2} \leq b_j \Rightarrow a_{i_1}^p + a_{i_2}^p < b_j^p \Rightarrow ||\lambda||_p < ||\mu||_p.$$

If $l < m$, then there is $k$ such that $\varphi^{-1}(k) = \emptyset$ and hence

(2)
Proof. Obviously we know \( \lambda \) of the size \( b \) and \( \mu \) of the size \( b \), since \( \lambda \times \mu = \mu \times \lambda \), either two or more boxes embed into the box of the size \( b \) or a part of the box of size \( b \) has not been used. If two or more boxes of \( \lambda \times \mu \) embed into the box of the size \( b \), then we find a contradiction by equation (2). If a part of the box of size \( b \) has not been used, then we find a contradiction by equation (3). \( \square \)

The following corollary shows the essentiality of \( \epsilon \) in Theorem 2.1.

**Corollary 2.2.** Let \( \lambda, \mu \) be partitions. If \( ||\lambda||_p = ||\mu||_p \) for some \( 1 < p < \infty \) and \( \lambda \neq \mu \), then \( \lambda \not\sim^{S} \mu \) for all \( n \).

**Proof.** Suppose \( \lambda \times n \not\sim \mu \) for some \( n \). Since \( \lambda \neq \mu \), we find \( \lambda \times n \neq \mu \times n \). By Theorem 2.1 if \( \lambda \times n \neq \mu \times n \) and \( \lambda \times n \not\sim \mu \) for some \( n \), then \( ||\lambda \times n||_p < ||\mu \times n||_p \) for all \( 1 < p < \infty \). But one can observe that for any partition \( \lambda \),

\[
||\lambda \times n||_p = (||\lambda||_p)^n.
\]

Thus we find a contradiction that for all \( 1 < p < \infty \),

\[
||\lambda||_p < ||\mu||_p.
\]

**Corollary 2.3.** Let \( \lambda, \mu \) be two partitions. If \( ||\lambda||_p = ||\mu||_p \) for some \( 1 < p < \infty \) and \( \lambda \neq \mu \), then \( \lambda \not\sim^{S} \mu \).

**Proof.** Suppose \( \lambda \not\sim^{S} \mu \). There exists a partition \( \nu \) such that \( \lambda \times \nu \not\sim \mu \times \nu \). Since \( \lambda \times \nu \neq \mu \times \nu \) by Theorem 2.1 for all \( 1 < p < \infty \),

\[
||\lambda \times \nu||_p < ||\mu \times \nu||_p.
\]

One can easily see that \( ||\lambda||_p = ||\mu||_p \) for some \( 1 < p < \infty \) implies that for the same \( p \),

\[
||\lambda \times \nu||_p = ||\lambda||_p ||\nu||_p = ||\mu||_p ||\nu||_p = ||\mu \times \nu||_p.
\]

Therefore, \( \lambda \not\sim^{S} \mu \). \( \square \)

**Example 2.4.** Let \( \lambda_1 = [2, 2, 2, 2] \), \( \lambda_2 = [8, 8, 8, 8, 4, 4, 4, 4] \), \( \lambda_3 = [4, 2, 2] \), \( \mu_1 = [4, 1, 1, \ldots, 1] \), \( \mu_2 = [3, 3, 3] \), \( \mu_3 = [16, 2, 2, \ldots, 2, 1, 1, \ldots, 1] \) and \( \mu_4 = [5, 3] \). Then \( \lambda_1 \not\sim^{S} \mu_1 \) but \( \lambda_1 \not\sim^{S} \mu_1 \) and \( \lambda_1 \not\sim^{S} \mu_2 \) but \( \lambda_1 \not\sim^{S} \mu_2 \). \( \lambda_2 \not\sim^{b} \mu_3 \) but \( \lambda_2 \not\sim^{S} \mu_3 \) and \( \lambda_2 \not\sim^{b} \mu_3 \) and \( \lambda_2 \not\sim^{S} \mu_3 \). \( \lambda_3 \not\sim^{S} \mu_4 \) but \( \lambda_3 \not\sim^{S} \mu_4 \).

**Proof.** If we set \( \lambda = [2, 1, 1] \), we get

\[
\lambda_1 \times \nu = [4, 4, 4, 4, 2, 2, 2, \ldots, 2] \quad \text{and} \quad \mu_1 \times \nu = [8, 4, 4, 4, 2, 2, 2, \ldots, 2, 1, 1, \ldots, 1].
\]

Then one can see that \( \lambda_1 \not\sim^{a} \mu_1 \). Since

\[
\sum_{(\lambda_1)_i \geq 2} (\lambda_1)_i = 8 > 4 = \sum_{(\mu_1)_j \geq 2} (\mu_1)_j,
\]

we have

\[
\sum_{(\lambda_1)_i \geq 2} (\lambda_1)_i > \sum_{(\mu_1)_j \geq 2} (\mu_1)_j.
\]
we see $\lambda_1 \not\preceq_s \mu_1$. It is clear that $\lambda_1 \not\rightarrow_s \mu_1$. To show $\lambda_2 \not\rightarrow_s \mu_3$, one can check that
$$||\lambda_2||_p \leq ||\mu_3||_p$$
for all $p \in [1, \infty]$ and the equality holds at
$$p = \frac{\ln(1 + \sqrt{5})}{\ln(2)} > 1.$$
Since $\lambda_2 \neq \mu_3$, we find that $\lambda_2 \not\rightarrow_s \mu_3$ by Corollary 2.8. Clearly $\lambda_3 \preceq_s \mu_4$. Suppose $\lambda_3 \not\rightarrow \mu_4$, there exists a partition $\nu = [\nu_1, \nu_2, \ldots, \nu_n]$ such that $\lambda_3 \times \nu \hookrightarrow \mu_4 \times \nu$. Let $p$ be the power of 2 in the prime factorization of the greatest common divisor $(\nu_1, \nu_2, \ldots, \nu_n)$ of $\nu_1, \nu_2, \ldots, \nu_n$. First we looks at entries of $\lambda_3 \times \nu$, all these entries are multiples of $2^{p+1}$. Since $||\lambda_3 \times \nu||_1 = ||\mu_4 \times \nu||_1$, there will be no space in $\mu_4 \times \nu$ which was not used in the embedding, i.e., for all $j$,
$$\sum_{i \in \varphi^{-1}(j)} (\lambda_3 \times \nu)_i = (\mu_4 \times \nu)_j.$$ 
Therefore, all entries of $\mu_4 \times \nu$ have to be multiples of $2^{p+1}$. Since all entries of $\mu_4$ are odd numbers, $\nu_1$ has to be a multiple of $2^{p+1}$ and so does the greatest common divisor of $\nu_1, \nu_2, \ldots, \nu_n$. It contradicts the hypothesis of $p$. All others should be straightforward. 

\section{Stable embeddability}

Let $\lambda, \mu$ be two partitions. Let us consider the following algorithm which is called a \textit{first fit} algorithm [1]. From $\lambda_1$ of $\lambda$, place it to any entry of $\mu$ in which it fits. Then repeat this step for $\lambda_2$ and so on. Usually this is not an efficient algorithm [3]. It is obvious that if the first fit algorithm works, then $\lambda \hookrightarrow \mu$. The converse is not true in general. But with some conditions on $\lambda$ we can show that it determines the embeddability of $\lambda$ into $\mu$.

\textbf{Theorem 3.1.} Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_s], \mu = [\mu_1, \mu_2, \ldots, \mu_t]$ be partitions with $\lambda_i | \lambda_j$ for all $i \geq j$. If $\lambda \hookrightarrow \mu$, then the first fit algorithm works.

\textit{Proof.} Let us induct on $s$. It is trivial for $s = 1$ because this is the first step of the algorithm. Suppose this is true for $s = n$, we look at the case $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{n+1}]$. Since $\lambda \hookrightarrow \mu$, there is a map $\varphi : \{1, 2, \ldots, n+1\} \rightarrow \{1, 2, \ldots, t\}$ which represents the embedding of $\lambda$ into $\mu$, let us denote $\varphi(1) = j$. Then we will construct another embedding representing map $\psi$ after we decide where we put $\lambda_1$, say $\mu_k$, i.e., $\psi(1) = k$. To construct $\psi$, let us compare $\varphi(1)$ and $\psi(1)$. If $\varphi(1) = \psi(1)$, we pick $\psi = \varphi$. If $\varphi(1) = j \neq k = \psi(1)$, first we need to prove that there exists a subset $P$ of $\varphi^{-1}(k) = \{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_l}\}$ such that
$$\sum_{j \in P} \lambda_j \leq \lambda_1 \quad \text{and} \quad \sum_{j \in P^c} \lambda_j \leq \mu_k - \lambda_1.$$ 
First we divide cases by the sizes of $\lambda_1$ and $\lambda_2$. If $\lambda_1 = \lambda_2$, then we pick $P = \{2\}$. If $\lambda_1 \neq \lambda_2$, then $\lambda_1 > \lambda_2$. If
$$\sum_{j=2}^l \lambda_j \leq \lambda_1,$$
we can pick \( P = \{2, 3, \ldots, l\} \). Otherwise, there exist an integer \( m \) such that
\[
\sum_{j=2}^{m} \lambda_j \leq \lambda_1 < \sum_{j=2}^{m+1} \lambda_j = (\sum_{j=2}^{m} \lambda_j) + \lambda_{m+1}.
\]

If we divide by \( \lambda_{m+1} \), we have
\[
\sum_{j=2}^{m} \frac{\lambda_j}{\lambda_{m+1}} \leq \frac{\lambda_1}{\lambda_{m+1}} < \sum_{j=2}^{m} \frac{\lambda_j}{\lambda_{m+1}} + 1.
\]

Since \( \lambda_i | \lambda_j \) for all \( i \geq j \), all these three numbers are integers, so the first two have to be the same. We choose \( P = \{2, 3, \ldots, m\} \). Once we have such a \( P \), we can define
\[
\psi(i) = \begin{cases} 
  k & \text{if } i = 1, \\
  j & \text{if } i \in \varphi^{-1}(k) \cap P, \\
  \varphi(i) & \text{if } i \not\in \varphi^{-1}(k) \cup \{1\} \text{ or } i \not\in \varphi^{-1}(k) \cap P^c.
\end{cases}
\]

Then \( \psi|_{\lambda} \) shows \( \tilde{\lambda} = [\lambda_2, \ldots, \lambda_{n+1}] \hookrightarrow \tilde{\mu} = [\mu_1, \ldots, \mu_k - \lambda_1, \ldots, \mu_t] \). By the induction hypothesis, the first fit algorithm works.

Let \( \mathcal{P} \) be the set of all partitions whose entries are all powers of a fixed number \( p \). For these partitions, we can show that the supermajorization completely determine the embeddability. Instead of the standard notation, we can use
\[
\lambda = [a_0, a_1, a_2, \ldots, a_s]_p
\]
where \( a_i \) is the number of entries \( p^i \).

**Theorem 3.2.** Let \( \lambda, \mu \) be partitions in \( \mathcal{P} \). \( \lambda \hookrightarrow \mu \) if and only if \( \lambda \preceq_S \mu \).

**Proof.** We only need to show that if \( \lambda \preceq_S \mu \), \( \lambda \hookrightarrow \mu \) because of equation\( \Box \). Suppose \( \lambda \preceq_S \mu \). Let
\[
\lambda = [a_0, a_1, a_2, \ldots, a_s]_p, \ \mu = [b_0, b_1, b_2, \ldots, b_t]_p.
\]

Without loss of generality, we assume \( a_s \neq 0 \neq b_t \). Obviously \( s \leq t \). We induct on the number of the boxes of \( \lambda \), say \( k \). If \( k = 1 \), then \( a_s = 1 \)
\[
1p^s = \lambda_{\geq p^s} \leq \mu_{\geq p^s} = \sum_{j=s}^{t} b_j p^j
\]
implies \( \lambda \hookrightarrow \mu \). For nonzero \( a_s \), we pick a box of size \( p^s \), put it into a box of size \( p^j \) in \( \mu \). Then for \( \lambda \) we subtract 1 from \( a_s \) and for \( \mu \), we subtract 1 from \( b_t \) and distribute the reminder of \( p^j - p^s \) in base \( p \) into \( \mu \). One can observe that all these numbers which have been distributed are bigger than or equal to \( p^s \). Thus resulting partitions still have the same supermajorization order. By the induction hypothesis, we find an embedding of \( \lambda' = [a_0, a_1, \ldots, a_{s-1}, a_s - 1]_p \) into \( \mu' = [b_0', b_1', \ldots, b'_{t-1}, b_t - 1]_p \). But, it is easy to recover an embedding of \( \lambda \) into \( \mu \). \( \Box \)

Now we look the stable embeddability for partitions in \( \mathcal{P} \).

**Theorem 3.3.** Let \( \lambda, \mu \) be partitions in \( \mathcal{P} \). If \( \lambda \rightarrow \mu \), then there exists a partition \( \nu \) in \( \mathcal{P} \) such that \( \lambda \times \nu \hookrightarrow \mu \times \nu \). 

Proof. Suppose $\lambda \rightarrow \mu$, then there is a partition $\nu$ such that $\lambda \times \nu \rightarrow \mu \times \nu$ and $\nu = [c_1, c_2, \ldots, c_k]$. We can uniquely rewrite $c_j$ in the base $p$ such as

$$c_j = c_{j,0}p^0 + c_{j,1}p^1 + c_{j,2}p^2 + \ldots + c_{j,l(j)}p^{l(j)}$$

where $c_{j,i}$ are nonnegative integers less than $p$ and $c_{j,l(j)} \neq 0$. Using these expressions we can subdivide $\nu$ to get a refinement $\tilde{\nu} = [\sum_j c_{j,0}, \sum_j c_{j,1}, \ldots, \sum_j c_{j,i}, \ldots, \sum_j c_{j,t}]_p$ where the sum runs over all nonzero $c_{j,i}$ for each $i$. If the boxes $\sum [c_i \times p^{j_k}]$ of $\lambda \otimes \nu$ were embedded into $[m \times p^{m'}]$ in $\mu \otimes \nu$, We can show that the refinement of $\sum [c_i \times p^{j_k}]$ can be embedded in the refinement of $[m \times p^{m'}]$. Precisely if

$$p^{j_1}c_{i_1} + p^{j_2}c_{i_2} + \ldots + p^{j_n}c_{i_n} \leq p^{m'}c_m$$

where $j_1 \leq j_2 \leq \ldots \leq j_n$, $c_{j_l} \neq 0$ and 

$$c_{i_t} = c_{i_t,0}p^0 + c_{i_t,1}p^1 + \ldots + c_{i_t,l(i_t)}p^{l(i_t)}$$

for all $t$, then

$$\sum_{\alpha=1}^{n} \sum_{\beta=0}^{l(\beta)} [c_{i_{\alpha},\beta} \times p^{j_{\alpha+\beta}}] \leftrightarrow \sum_{\gamma} [c_{m,\gamma} \times p^{m'+\gamma}].$$

First we look at the case, $n = 1$. If $p^{j_1}c_{i_1} \leq p^{m'}c_m$, one can easily see that

$$\sum_{\beta} [c_{i_1,\beta} \times p^{j_1+\beta}] \approx [c_{m,\gamma} \times p^{m'+\gamma}]$$

because we are comparing two integers in base $p$. By Theorem 3.2

$$\sum_{\beta} [c_{i_1,\beta} \times p^{j_1+\beta}] \leftrightarrow \sum_{\gamma} [c_{m,\gamma} \times p^{m'+\gamma}].$$

For the case $n > 1$, we look at the integer

$$\sum_{\alpha=1}^{n} \sum_{\beta=0}^{l(\beta)} c_{i_{\alpha},\beta} \times p^{j_{\alpha+\beta}}$$

as a sum of integers

$$\sum_{\beta=0}^{l(\beta)} c_{i_{\alpha},\beta} \times p^{j_{\alpha+\beta}}$$

in base $p$. Then this returns to the case $n = 1$. If we keep on tracking the addition, we can recover the embedding of

$$\sum_{\alpha=1}^{n} \sum_{\beta=0}^{l(\beta)} [c_{i_{\alpha},\beta} \times p^{j_{\alpha+\beta}}] \leftrightarrow \sum_{\gamma} [c_{m,\gamma} \times p^{m'+\gamma}].$$

Moreover, this process does not involve with other terms. Therefore, we can rewrite $\nu$ as the shape we desired. \qed
Corollary 3.4. Let \( \lambda = [a_i]_p, \mu = [b_i]_p \) be partitions in \( \mathcal{P} \). If \( 0 \leq a_i \leq b_i (0 \leq b_i < a_i) \), we have two new partitions \( \tilde{\lambda}, \tilde{\mu} \) which are obtained from \( \lambda, \mu \) by replacing \( a_i, b_i \) by 0 and from \( \mu, \lambda \) by replacing by \( b_i - \text{Min}\{a_i, b_i\}(a_i - \text{Min}\{a_i, b_i\}, \text{respectively}) \). Then \( \lambda \rightarrow \tilde{\mu} \) if and only if \( \lambda \rightarrow \tilde{\mu} \).

Proof. We assume \( a_i \leq b_i \) for a fixed \( i \). Suppose \( \lambda \rightarrow \mu \). By Theorem 3.3, we can find
\[
\nu = [\nu_0, \nu_1, \ldots, \nu_n]_p
\]
such that all entries of \( \nu \) are all powers of a fixed number \( p \) and \( c_k \) is the number of the boxes of size \( p^k \). Now \( \lambda \otimes \nu \leftarrow \mu \otimes \nu \) and \( \lambda \otimes \nu, \mu \otimes \nu \) satisfy the hypothesis of Theorem 3.3. We can use the first fit algorithm. We put all boxes whose sizes are bigger than \( \nu \) such that all entries of \( \nu, \lambda \) and \( \mu \) are all powers of a fixed number \( p \). Suppose \( \nu = \nu_0 \neq 0 \). Initially, we will start \( \nu_0 = 1 \). There are \( a_n \) boxes of size \( 2^n \) in \( \lambda \times [c_0 \times 1] \) and none of blocks of size \( 2^n \) in \( \mu \times [c_0 \times 1] \). But there are rooms for
\[
b_m \times 2^{m-n} + b_{m-1} \times 2^{m-n-1} + \ldots + b_{n+1} \times 2
\]
many boxes of size \( 2^n \) in \( \mu \times [c_0 \times 1] \). If
\[
b_m \times 2^{m-n} + b_{m-1} \times 2^{m-n-1} + \ldots + b_{n+1} \times 2 \geq a_n,
\]
we set \( c_1 \) to zero and keep the difference for the next step, say \( M \). Otherwise we set
\[
c_1 = \left[ \frac{a_n - (b_m \times 2^{m-n} + b_{m-1} \times 2^{m-n-1} + \ldots + b_{n+1} \times 2)}{b_m} \right]
\]
and \( M = 0 \), where \([x]\) is the smallest natural number which is bigger than or equal to \( x \). Then we look at \( \lambda \times [c_0 \times 1, c_1 \times c_0, \frac{1}{2}] \), \( \mu \times [c_0 \times 1, c_1 \times \frac{1}{2}] \). We have \( a_{n-1} \cdot c_0 + a_n \cdot c_1 \) many boxes of size \( 2^{n-1} \) in \( \lambda \times [c_0 \times 1] \), then we compare it with
\[
2 \times M + c_1 \times (b_m \times 2^{m-n} + b_{m-1} \times 2^{m-n-2} + \ldots + b_{n+1} \times 2) + b_{n-1} \cdot c_0
\]
and we repeat exactly the same process. For \( N \geq m \), we find \( c_N \) by comparing two terms
\[
\alpha = b_{m-1} \times c_{N+m-1} + b_{m-2} \times c_{N+m-2} + \ldots + b_0 \times c_N + 2 \times M
\]
and
\[
\beta = a_n \times c_{N+n} + a_{n-1} \times c_{N+n-1} + \ldots + a_0 \times c_N
\]
because these numbers count exactly how many blocks of size $2^{-N}$ in the product

$$\lambda \times [c_0 \times 1, c_1 \times \frac{1}{2}, \ldots, c_N \times \frac{1}{2^N}]$$

and

$$\mu \times [c_0 \times 1, c_1 \times \frac{1}{2}, \ldots, c_N \times \frac{1}{2^N}]$$

where $M$ is the number of boxes that were left in the previous step. Then $C_{N+1}^N$ is $\left\lceil (\beta - \alpha)/b_m \right\rceil$ if $\beta - \alpha > 0$ (and set $M = 0$) 0 otherwise (set $M = \alpha - \beta$, respectively). Then we compare the next biggest boxes. We stop if we get $n$ consecutive 0’s for $c_i$. Let $N$ be the largest integer that $c_N$ is non-zero. We repeat the process starting $c_0 = (b_m)^{N+1}$. One can easily see that we no longer have to use $\lceil \rceil$ because $(b_m)^{N+1-k}|c_k$ for all $0 \leq k \leq N + 1$. Finally we multiply $2^N$ to make $\nu$ an integral partition.

To compare the optimality of such $\nu$‘s, we define the length of $\nu = [c_0, c_1, \ldots, c_n]_p$ to be $n+1$ where $c_0 \neq 0 \neq c_n$. From the given $\lambda, \mu$ we collect all possible $\nu \in \mathcal{P}$ and $\lambda \otimes \nu \hookrightarrow \mu \otimes \nu$, say $\mathcal{T}$. Then we define a partial order on $\mathcal{T}$ by a lexicographic order,

$$(\text{length of } \hat{\lambda}(\nu), \frac{c_1}{c_0}, \ldots, \frac{c_n}{c_0}).$$

Moreover, $\mathcal{T}$ is closed under an addition, a tensor and a scalar multiplication.

**Theorem 3.5.** 1) The algorithm stops at finite time if and only if $\lambda \hookrightarrow^s \mu$.

2) Let $D$ be a partition which is obtained from the algorithm. Then $D$ is a minimal element with respect to the partial order we defined on $\mathcal{T}$.

**Proof.** We want to show that if $\lambda \hookrightarrow^s \mu$, then the algorithm must stop at finite steps and the one we find by the algorithm has the smallest length. Since $\lambda \hookrightarrow^s \mu$, $\mathcal{T}$ is nonempty and we find a minimal element in $\mathcal{T}$, say $t = [t_0, t_1, \ldots, t_l]_p$.

First we assume the existence, i.e., the algorithm gives us an integral partition

$$\nu = [c_0, c_1, \ldots, c_n]_p.$$ 

By the minimality, we have $l \leq m$. But the process itself provides us $l \geq m$. We compare

$$c_0 t = [c_0 \cdot t_0, c_0 \cdot t_1, \ldots, c_0 \cdot t_l]_p$$

and

$$t_0 \nu = [t_0 \cdot c_0, t_0 \cdot c_1, \ldots, t_0 \cdot c_m]_p.$$ 

Suppose $c_0 t \neq t_0 \nu$. There is a $j$ such that

$$c_0 \cdot t_j < t_0 \cdot c_j.$$ 

But this obviously contradicts the process of the algorithm. Therefore, $c_0 t = t_0 \nu$ and it does also prove the existence.  \(\square\)
4. Discussions

4.1. Algebraic embeddabilities. Let $\mathcal{A}$ be a finite dimensional semisimple algebra over an algebraically closed field $K$. By a simple application of Webberburn-Artin theorem, we can decompose $\mathcal{A}$ into a direct sum of matrix algebras. From a direct sum of matrix algebras $\mathcal{A}$, we can find a unique integral partition $\lambda$, denoted by $\lambda(\mathcal{A})$. For an integral partition $\lambda$, one can assign a direct sum of matrix algebras

$$\mathcal{A}(\lambda) = \bigoplus_{i=1}^{m} \mathcal{M}_{\lambda_i},$$

where $\mathcal{M}_{\lambda_i}$ is the set of all $\lambda_i$ by $\lambda_i$ matrices over $K$. For integral partitions, one can see that $\lambda \hookrightarrow \mu$ if and only if $\mathcal{A}(\lambda)$ embeds into $\mathcal{A}(\mu)$ as $K$ algebras. All other partial orders can be naturally defined for a direct sum of matrix algebras. The question of the embeddability between algebraic objects such as groups, rings, modules and etc, is a long standing difficult question. For some algebraic objects such as sets, vector spaces, the question is straightforward. The embeddability between the modules over a complex simple Lie algebra is completely determined by Littlewood-Richardson formula and Schur’s lemma. Authors have made a few progress on stable embeddability, the product is replaced by the tensor product, between the modules over a complex simple Lie algebra [5]. The stable embeddability between other algebraic objects should be an interesting question.

4.2. Analytic embeddabilities. Let $\lambda, \mu$ be partition in $\mathcal{P}$. The algorithm we defined in section 3.1 brings us a new embeddability, $\lambda$ weakly stably embeds into $\mu$, denoted by $\lambda^{w.s} \hookrightarrow \mu$, if there exists a rational partition $\nu$ of infinite length such that all entries of $\nu$ are nonpositive powers of the fixed number $p$ and

$$\sum_{i=0}^{\infty} c_i p^{-i} < \infty,$$

where $c_i$ is the number of the entries $p^{-i}$. One can see that

$$\lambda^{s} \hookrightarrow \mu \quad \Rightarrow \quad \lambda^{w.s} \hookrightarrow \mu \quad \Rightarrow \quad \lambda^{b} \hookrightarrow \mu$$

$$||\lambda||_p < ||\mu||_p, \ \forall p \in (1, \infty) \quad \Rightarrow \quad ||\lambda||_p \leq ||\mu||_p, \ \forall p \in [1, \infty]$$

It is not known that the converses of the first row of equation [4] are true or not for partitions in $\mathcal{P}$. Authors have written a program that performs the algorithm described in section 3.1 to see $||\lambda||_p < ||\mu||_p, \ \forall p \in (1, \infty)$ and equality holds for $p = 1$ and $\infty$ implies $\lambda^{s} \hookrightarrow \mu$. We have not found any answer yet.

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