A Unified Framework for Analyzing Closed Queueing Networks in Bike Sharing Systems*

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Abstract

During the last decade bike sharing systems have emerged as a public transport mode in urban short trips in more than 500 major cities around the world. For the mobility service mode, many challenges from its operations are not well addressed yet, for example, how to develop the bike sharing systems to be able to effectively satisfy the fluctuating demands both for bikes and for vacant lockers. To this end, it is a key to give performance analysis of the bike sharing systems. This paper first describes a large-scale bike sharing system. Then the bike sharing system is abstracted as a closed queueing network with multi-class customers, where the virtual customers and the virtual nodes are set up, and the service rates as well as the relative arrival rates are established. Finally, this paper gives a product-form solution to the steady state joint probabilities of queue lengths, and gives performance analysis of the bike sharing system. Therefore, this paper provides a unified framework for analyzing closed queueing networks in the study of bike sharing systems. We hope the methodology and results of this paper can be applicable in the study of more general bike sharing systems.

Keywords: Bike sharing system; closed queueing network; product-form solution; problematic station.

*Interpretation: After this paper was published, we find an interesting work by Rick Zhang and Marco Pavone “R. Zhang and M. Pavone (2014). A queueing network approach to the analysis and control of mobility-on-demand systems. Published Online: arXiv:1409.6775 Pages 1-9.”
1 Introduction

During the last decade the bike sharing systems are fast increasing as a public transport mode in urban short trips, and have been launched in more than 500 major cities around the world. Also, the bike sharing systems offer a low cost and environmental protection mobility service through sharing one-way use. Now, the bike sharing systems are regarded as an effective way to jointly solve traffic congestion, parking difficulties, traffic noise, air pollution and so forth. DeMaio [3] reviewed the history, impacts, models of provision and future of the bike sharing systems. Larsen [12] reported that over 500 major cities host advanced bike sharing systems with a combined fleet of more than half a million bikes up to April 2013. A synthesis of the literature for the bike sharing systems was given by Fishman et al. [5] and Labadi et al. [11]. At the same time, for some countries or cities developing the bike sharing systems, readers may refer to, such as, Europe, the Americas and Asia by Shaheen et al. [22], the European OBIS Project by Janett and Hendrik [10], the France by Faye [4], China by Tang et al. [24], London by Lathia et al. [13], Montreal by Morency et al. [16], and a number of famous cities by Shu et al. [23].

In operations of the bike sharing systems, a crucial question is the ability not only to meet the fluctuating demand for renting bikes at each station but also to provide enough vacant lockers to allow the renters to return the bikes at their destinations. Since the number of bikes packed in each station is always randomly dynamically changed, this causes an unpredictable imbalance, such as, some stations contain more bikes but the others are seriously short of available bikes. Such a randomly dynamic unbalance of bikes distributed among the stations often leads to occurrence of the problematic stations (i.e., full or empty stations). Notice that the problematic stations reflect a common challenge faced by the bike sharing systems in practice due to the stochastic and time-inhomogeneous nature of both the customer arrivals and the bike returns, thus the probability of problematic stations has been regarded as a main factor to measure the satisfaction of customers and even to estimate the quality of service. Obviously, how to effectively reduce the probability of problematic stations becomes a key way to improve the satisfaction of customers and further to promote the quality of system service. Therefore, it is a major task to develop effective methods for computing the probability of problematic stations in the study of bike sharing systems.

Queueing theory and Markov processes are very useful for computing the probability
of problematic stations, and more generally, analyzing performance measures of the bike sharing systems. However, available works on such a research line are still fewer up to now. We would like to refer readers to four classes of recent literature as follows. **(a) Simple queues:** Leurent [14] used the $M/M/1/C$ queue to study a vehicle-sharing system in which each station contains an additional waiting room which helps those customers arriving at a problematic station, and analyzed performance measures of this system in terms of a geometric distribution. Schuijbroek et al. [20] evaluated the service level by means of the transient distribution of the $M/M/1/C$ queue, and the service level is used to establish some optimal models to discuss the inventory rebalancing and vehicle routing. Raviv et al [18] and Raviv and Kolka [17] employed the transient distribution of a time-inhomogeneous $M(t)/M(t)/1/C$ queue to compute the expected number of bike shortages at each station. **(b) The mean-field theory:** Fricker et al. [7] considered a space inhomogeneous bike sharing system with different clusters, and expressed the minimal proportion of problematic stations within each cluster. For a space homogeneous bike sharing system, Fricker and Gast [6] used the $M/M/1/K$ queue to provide a more detailed analysis for some simple mean-field models (including the power of two choices), derived a closed-form solution to the minimal proportion of problematic stations, and compared the incentives and redistribution mechanisms. Fricker and Tibi [8] studied the central limit and local limit theorems for the independent (perhaps non identically distributed) random variables which effectively support analysis of a generalized Jackson network with product-form solution; and used these obtained results to evaluate performance measures of the space inhomogeneous bike sharing systems, where its asymptotics gives a complete picture for equilibrium state analysis of the locally space homogeneous bike sharing systems. Li et al. [15] provided a mean-field queueing method to study a large-scale bike sharing system through using a combination of, such as, the virtual time-inhomogeneous queue, the mean-field equations, the martingale limit, the nonlinear birth-death process, numerical computation of the fixed point, and numerical analysis for the steady state probability of the problematic stations. **(c) Queueing networks:** Savin et al. [19] used a loss network as well as admission control to discuss capacity allocation of a rental model with two classes of customers, and studied the revenue management and fleet sizing decision in the rental system. Adelman [1] applied a closed queueing network to set up an internal pricing mechanism for managing a fleet of service units, and also used a nonlinear flow model to discuss the price-based policy for the vehicle redistribution. George and
Xia [9] provided a queueing network method in the study of vehicle rental systems, and determined the optimal number of parking spaces for each rental location. (d) Markov decision processes: Stochastic optimization and Markov decision processes are applied to analysis of the bike sharing systems. From a dynamic price mechanism, Waserhole and Jost [25] used the closed queueing networks to propose a Markov decision model of a bike sharing system. To overcome the curse of dimensionality in the Markov decision process with a high dimension, they established a fluid approximation that computes a static policy and gave an upper bound on the potential optimization. Such a fluid approximation for the Markov decision processes of the bike sharing systems was further developed in Waserhole and Jost [26] [27] and Waserhole et al. [28].

The main purposes of this paper are to provide a unified framework for analyzing closed queueing networks in the study of bike sharing systems. This framework of closed queueing networks is interesting, difficult and challenging from three crucial features: (a) Stations and roads have very different physical attributes, but all of them are abstracted as indistinguishable nodes in the closed queueing networks; (b) the service discipline of the stations is First Come First Service (abbreviated as FCFS), while the service discipline of the roads is Processor Sharing (abbreviated as PS); and (c) the virtual customers (i.e., bikes) in the stations are of a single class, while the virtual customers (i.e., bikes) in the roads are of two classes, and their classes may change on the roads according to the first bike-return or the at least two successive bike-returns due to the full stations, respectively. For such a closed queueing network, this paper provides a detailed analysis both for establishing a product-form solution to the steady state joint probabilities of queue lengths, and for computing the steady state probability of problematic stations, more generally, for analyzing performance measures of the bike sharing system. The main contributions of this paper are twofold. The first contribution is to describe a large-scale bike sharing system and to provide a unified framework for analyzing closed queueing networks through establishing some basic factors: The service rates from stations or roads; and the routing matrix as well as the relative arrival rates to stations or roads. Notice that the basic factors play a key role in the study of closed queueing networks. The second contribution of this paper is to provide a product-form solution to the steady state joint probabilities of queue lengths in the closed queueing network, and give performance analysis of the bike sharing system in terms of the steady state joint probabilities.

The remainder of this paper is organized as follows. In Section 2, we describe a large-
scale bike sharing system with $N$ different stations and with at most $N(N - 1)$ different roads. In Section 3, we provide a unified framework for analyzing closed queueing networks in the study of bike sharing systems, and also compute the service rates, the routing matrix, and the relative arrival rates. In Section 4, we give a product-form solution to the steady state joint probabilities of queue lengths in the closed queueing network, and analyze performance measures of the bike sharing system by means of the steady state joint probabilities. Some concluding remarks are given in Section 5.

2 Model Description

In this section, we describe a large-scale bike sharing system with $N$ different stations and with at most $N(N - 1)$ different roads due to the riding-bike directed connection between any two stations. To analyze such a bike sharing system, we provide a unified framework for analyzing closed queueing networks in the study of bike sharing systems.

In a large-scale bike sharing system, a customer arrives at a station, rents a bike, and uses it for a while; then she returns the bike to a destination station, and immediately leaves this system. Obviously, for any customer renting and using a bike, her first return-bike time is different from those return-bike times that she has successively returned the bike for at least two times due to arriving at the full stations. At the same time, it is easy to understand that for any customer, her first road selection as well as her first riding-bike speed are different from those of having successively returned her bike for at least two times. Also, it is noted that the customer must return her bike to a station, then she can immediately leave the bike sharing system.

Now, we describe the bike sharing system, including operation mechanism, system parameters and mathematical notation, as follows:

(1) **Stations and roads:** There are $N$ different stations and at most $N(N - 1)$ different roads, where the $N(N - 1)$ roads are observed from the fact that there must exist a directed road from a station to another station. In addition, we assume that at the initial time $t = 0$, every station has $C$ bikes and $K$ parking places, where $1 \leq C < K < \infty$; and $NC \geq K$, which makes that some of the $NC$ bikes can result in at least a full station.

(2) **Customer arrival process:** The arrivals of the outside customers at the $i$th station are a Poisson process with arrival rate $\lambda_i > 0$ for $1 \leq i \leq N$.

(3) **The first riding-bike time:** Once an outside customer arrives at the $i$th station,
she immediately goes to rent a bike. If there is no bike in the $i$th station (i.e., the $i$th station is empty), then the customer directly leaves this bike sharing system. If there is at least one bike in the $i$th station, then the customer rents a bike, and then goes to Road $i \rightarrow j$. We assume that for $j \neq i$ with $1 \leq i, j \leq N$, the customer at the $i$th station rides the bike into Road $i \rightarrow j$ with probability $p_{i,j}$ for $\sum_{j \neq i}^{N} p_{i,j} = 1$; and her riding-bike time from the $i$th station to the $j$th station (i.e., riding on Road $i \rightarrow j$) is an exponential random variable with riding-bike rate $\mu_{i,j} > 0$, where the expected riding-bike time is $1/\mu_{i,j}$.

(4) The bike return times:

The first return – When the customer completes her short trip on the above Road $i \rightarrow j$ (see Assumption (3)), she needs to return her bike to the $j$th station. If there is at least one available parking position (i.e., a vacant dock), then the customer directly returns her bike to the $j$th station, and immediately leaves this bike sharing system.

The second return – If no parking position is available at the $j$th station, then she has to ride the bike to another station $l_1$ with probability $\alpha_{j,l_1}$ for $l_1 \neq j$ for $\sum_{l_1 \neq j}^{N} \alpha_{j,l_1} = 1$; and her riding-bike time from the $j$th station to the $l_1$th station (i.e., riding on Road $j \rightarrow l_1$) is an exponential random variable with riding-bike rate $\xi_{j,l_1} > 0$. If there is at least one available parking position, then the customer directly returns her bike to the $l_1$th station, and immediately leaves this bike sharing system.

The third return – If no parking position is available at the $l_1$th station, then she has to ride the bike to another station $l_2$ with probability $\alpha_{l_1,l_2}$ for $l_2 \neq l_1$ for $\sum_{l_2 \neq l_1}^{N} \alpha_{l_1,l_2} = 1$; and her riding-bike time from the $l_1$th station to the $l_2$th station (i.e., riding on Road $l_1 \rightarrow l_2$) is an exponential random variable with riding-bike rate $\xi_{l_1,l_2} > 0$. If there is at least one available parking position, then the customer directly returns her bike to the $l_2$th station, and immediately leaves this bike sharing system.

The $(k+1)$st return for $k \geq 3$ – We assume that this bike has not been returned at any station yet through $k$ consecutive return processes. In this case, the customer has to try her $(k+1)$st lucky return. Notice that the customer goes to the $l_k$th station from the $l_{k-1}$th full station with probability $\alpha_{l_{k-1},l_k}$ for $l_k \neq l_{k-1}$ for $\sum_{l_k \neq l_{k-1}}^{N} \alpha_{l_{k-1},l_k} = 1$; and her riding-bike time from the $l_{k-1}$th station to the $l_k$th station (i.e., riding on Road $l_{k-1} \rightarrow l_k$) is an exponential random variable with riding-bike rate $\xi_{l_{k-1},l_k} > 0$. If there is at least one available parking position, then the customer directly returns her bike to the $l_k$th station, and immediately leaves this bike sharing system; otherwise she has to continuously try
another station again.

We further assume that the returning-bike process is persistent in the sense that the customer must find a station with an empty position to return her bike, because the bike is the public property so that no one can make it her own.

It is seen from the above description that the parameters: $p_{i,j}$ and $\mu_{i,j}$ for $j \neq i$ and $1 \leq i, j \leq N$, of the first return, may be different from the parameters: $\alpha_{i,j}$ and $\xi_{i,j}$ for $j \neq i$ and $1 \leq i, j \leq N$, of the $k$th return for $k \geq 2$. Notice that such an assumption with respect to these different parameters is actually reasonable because the customer possibly has more things (for example, tourism, shopping, visiting friends and so on) in the first return process, but she become to have only one return task during the $k$ successive return processes for $k \geq 2$.

(5) **The departure discipline:** The customer departure has two different cases: (a) An outside customer directly leaves the bike sharing system if she arrives at an empty station; or (b) if one customer rents and uses a bike, and she finally returns the bike to a station, then the customer completes her trip, and immediately leaves the bike sharing system.

We assume that the customer arrival and riding-bike processes are independent, and also all the above random variables are independent of each other. For such a bike sharing system, Figure 1 provides some physical interpretation.

### 3 A Closed Queueing Network

In this section, we first provide a closed queueing network to express the bike sharing system, as seen in Figure 1. Then we determine the service rates, the routing matrix, and the relative arrival rates of the closed queueing network. Notice that there are two classes of customers in the $N(N-1)$ roads.

In the bike sharing system described in the above section, there are $NC$ bikes, $N$ stations and $N(N-1)$ roads. Now, we abstract the bike sharing system as a closed queueing network as follows:

(1) **Virtual nodes:** Although the stations and roads have different physical attributes such as functions, and geographical topologies, the stations and roads are all regarded as the same nodes in the closed queueing network.
(2) Virtual customers: The bikes at the stations or roads are described as follows:

**Abstract:** The virtual customers are abstracted by the bikes, which are either parked in the stations or ridden on the roads. Notice that the total number of bikes in the bike sharing system is fixed as $NC$ due to the fact that bikes can neither enter nor leave this system, thus the bike sharing system can be regarded as a closed queueing network.

**Multiple classes:** From Assumption (2) in Section 2, it is seen that there are only one class of customers in the nodes abstracted from the stations. From Assumptions (3) and (4) in Section 2, we understand that there are two different classes of customers in the nodes abstracted from the roads, where the first class of customers are the bikes ridden on the roads for the first time; while the second class of customers are the bikes which are successively ridden on the at least two different roads due to the full station.

(3) Service disciplines: The First Come First Service (or FCFS) is used in the nodes abstracted from the stations; while a new processor sharing (or PS) is used in the nodes abstracted from the roads.

In the above closed queueing network, let $Q_i(t)$ be the number of bikes parked in $i$th station at time $t \geq 0$ for $1 \leq i \leq N$, and $R_{k,l}^{(r)}(t)$ the number of bikes of class $r$ ridden on
Road \( k \to l \) at time \( t \) for \( r = 1, 2 \), and \( k \neq l \) with \( 1 \leq k, l \leq N \). We write

\[
\mathbf{X}(t) = (\mathbf{L}_1(t), \mathbf{L}_2(t), \ldots, \mathbf{L}_{N-1}(t), \mathbf{L}_N(t)),
\]

where for \( 1 \leq i \leq N \)

\[
\mathbf{L}_i(t) = \left( Q_i(t); R_{i,1}^{(1)}(t), R_{i,1}^{(2)}(t); R_{i,2}^{(1)}(t), R_{i,2}^{(2)}(t); \ldots; R_{i,1}^{(1)}(t), R_{i,2}^{(1)}(t); R_{i,1}^{(1)}(t), R_{i,2}^{(2)}(t) \right).
\]

Obviously, \( \{\mathbf{X}(t) : t \geq 0\} \) is a Markov process of size \( N(2N - 1) \) due to the exponential and Poisson assumptions of this bike sharing system.

Now, we describe the state space of the Markov process \( \{\mathbf{X}(t) : t \geq 0\} \). It is seen from Section 2 that

\[
0 \leq Q_i(t) \leq K, \quad 1 \leq i \leq N, \quad (1)
\]

\[
0 \leq R_{k,l}^{(r)}(t) \leq NC, \quad r = 1, 2, \ k \neq l, \ 1 \leq k, l \leq N, \quad (2)
\]

and

\[
\sum_{i=1}^{N} Q_i(t) + \sum_{k=1}^{N} \sum_{l \neq k} R_{k,l}^{(1)}(t) + \sum_{k=1}^{N} \sum_{l \neq k} R_{k,l}^{(2)}(t) = NC. \quad (3)
\]

From (1) to (3), it is seen the state space of Markov process \( \{\mathbf{X}(t) : t \geq 0\} \) of size \( N(2N - 1) \) is given by

\[
\Omega = \left\{ \overrightarrow{n} : 0 \leq n_i \leq K, 0 \leq m^{(1)}_{k,l}, m^{(2)}_{k,l} \leq NC, \right. \\
\left. \sum_{i=1}^{N} n_i + \sum_{k=1}^{N} \sum_{l \neq k} m^{(1)}_{k,l} + \sum_{k=1}^{N} \sum_{l \neq k} m^{(2)}_{k,l} = NC \right\},
\]

where

\[
\overrightarrow{n} = (n_1, n_2, \ldots, n_{N-1}, n_N),
\]

and for \( 1 \leq i \leq N \)

\[
n_i = \left( n_i; m^{(1)}_{i,1}, m^{(2)}_{i,1}; m^{(1)}_{i,2}, m^{(2)}_{i,2}; \ldots; m^{(1)}_{i,i-1}, m^{(2)}_{i,i-1}; m^{(1)}_{i,i+1}, m^{(2)}_{i,i+1}; m^{(1)}_{i,i+2}, m^{(2)}_{i,i+2}; \ldots; m^{(1)}_{i,N}, m^{(2)}_{i,N} \right).
\]

Notice that \( m_{k,l} = m^{(1)}_{k,l} + m^{(2)}_{k,l} \) is the total number of bikes being ridden on Road \( k \to l \) for \( k \neq l \) with \( 1 \leq k, l \leq N \), and also the state space \( \Omega \) contains \( (K+1)^N (NC+1)^{2N(N-1)} \) elements.
To compute the steady state joint probabilities of \(N(2N - 1)\) queue lengths in the bike sharing system, it is seen from Chapter 7 in Bolch et al. [2] that we need to determine the service rate, the routing matrix and the relative arrival rate for each node in the closed queueing network.

(a) The service rates

From Figure 2, it is seen that the service rates of the closed queueing network are given from two different cases as follows:

Case one: The node is one of the \(N\) stations

The departure process of bikes from the \(i\)th station, renting at the \(i\)th station and immediately ridden on one of the \(N - 1\) roads (such as, Road \(i \rightarrow l\) for \(l \neq i\) with \(1 \leq l \leq N\)), is Poisson with service rate

\[
b_i = \lambda_i \mathbf{1}_{\{1 \leq n_i \leq K\}} \sum_{l \neq i} p_{i,l} = \lambda_i \mathbf{1}_{\{1 \leq n_i \leq K\}}
\]

by means of the condition: \(\sum_{l \neq i} p_{i,l} = 1\).

Case two: The node is one of the \(N(N - 1)\) roads

In this case, two different processor sharing queueing processes of Road \(i \rightarrow l\) (with two classes of different customers) are explained in Figure 2. Now, we describe the service rates with respect to the two classes of different customers as follows:
The first class of customers: The departure process of bikes from Road $i \to l$, rented from Station $i$ and being ridden on Road $i \to l$ for the first time, is Poisson with service rate
\[ b^{(1)}_{i,l} = m^{(1)}_{i,l} \mu_{i,l}. \] (5)

The second class of customers: The departure process of bikes from Road $i \to l$, having successively been ridden on the roads for at least two times but now on Road $i \to l$, is Poisson with service rate
\[ b^{(2)}_{i,l} = m^{(2)}_{i,l} \xi_{i,l}. \] (6)

(b) The routing matrix and the relative arrival rates

Now, we compute the relative arrival rate of each node in the closed queueing network. Differently from the service rates analyzed above, it is more complicated to determine the relative arrival rates by means of the routing matrix.

Based on Chapter 7 in Bolch et al. [2], we denote by $e_i(n_i)$ and $e_i^{(r)}(m_i^{(r)})$ the relative arrival rates of the $i$th station with $n_i$ parking bikes, and of Road $i \to l$ with $m_i^{(r)}$ riding bikes of class $r$, respectively. We write
\[ \mathcal{E} = \{ \overrightarrow{e}(\overrightarrow{n}) : \overrightarrow{n} \in \Omega \}, \]
where
\[ \overrightarrow{e}(\overrightarrow{n}) = (e_1(\overrightarrow{n}), e_2(\overrightarrow{n}), \ldots, e_{N-1}(\overrightarrow{n}), e_N(\overrightarrow{n})), \]
and for $1 \leq i \leq N$
\[ e_i(\overrightarrow{n}) = \left( e_{i_1}(n_{i_1}) ; e_{i_1}^{(1)}(m_{i_1}^{(1)}) ; e_{i_1}^{(2)}(m_{i_1}^{(2)}) ; \ldots ; e_{i-1}(m_{i-1}^{(1)}) ; e_{i-1}^{(2)}(m_{i-1}^{(2)}) ; e_{i+1}(n_{i+1}) ; e_{i+1}^{(1)}(m_{i+1}^{(1)}) ; e_{i+1}^{(2)}(m_{i+1}^{(2)}) ; \ldots ; e_N(\overrightarrow{m}), e_N^{(1)}(\overrightarrow{m}) ; e_N^{(2)}(\overrightarrow{m}) \right). \]

Now, we introduce two useful notations: $\overrightarrow{g}_i$ and $\overrightarrow{g}_{i,l}^{(r)}$ as follows:

$\overrightarrow{g}_i$: A unit row vector of size $N(2N-1)$, which is given by a method of replacing elements from $\overrightarrow{n}$ to $\overrightarrow{0}$, that is, corresponding to the row vector $\overrightarrow{n}$, the element $n_i$ is replaced by one, while all other elements of the vector $\overrightarrow{n}$ are replaced by zeros.

$\overrightarrow{g}_{i,l}^{(r)}$: A unit row vector of size $N(2N-1)$, which is given by a method of replacing elements from $\overrightarrow{n}$ to $\overrightarrow{0}$, that is, corresponding to the row vector $\overrightarrow{n}$, the element $m_{i,l}^{(r)}$ is replaced by one, while all other elements of the vector $\overrightarrow{n}$ are replaced by zeros.
To compute the vector $\vec{d}(\vec{n})$, we first need to give the routing matrix $P$ of the closed queueing network as follows:

$$P = (P_{\vec{n}, \vec{n}^{'}})_{\vec{n}, \vec{n}^{'} \in \Omega},$$

where the routing matrix $P$ is of order $(K + 1)^N (NC + 1)^{2N(N-1)}$, and the element $P_{\vec{n}, \vec{n}^{'}}$ is computed from the following three cases:

**Case one: From a station to a road**

For $1 \leq i, l \leq N$ with $l \neq i$, we observe a transition route from the $i$th station to Road $i \rightarrow l$. If a rented bike leaves the $i$th station and enters Road $i \rightarrow l$, then $1 \leq n_i \leq K$, and there is a two-element change: $(n_i, m_{i,l}^{(1)}) \rightarrow (n_i - 1, m_{i,l}^{(1)} + 1)$. Thus we obtain that for $1 \leq n_i \leq K$

$$P_{\vec{n}, \vec{n}^{'}_{n,l}} = P_{\vec{n}, \vec{n} - \vec{g}_{i,l} + \vec{g}_{i,l}^{(1)}} = p_{i,l}$$

by means of Assumption (3) of Section 2. There are $NK(N - 1)$ such elements with $P_{\vec{n}, \vec{n}^{'}} = P_{\vec{n}, \vec{n} - \vec{g}_{i,l} + \vec{g}_{i,l}^{(1)}} = p_{i,l}$ in the closed queueing network.

**Case two: From a road to a station**

For $r = 1, 2$ and $1 \leq k, i, l \leq N$ with $i \neq k$ and $l \neq i$, we observe a transition route from Road $k \rightarrow i$ to the $i$th station. If a riding bike of class $r$ leaves Road $k \rightarrow i$, then either it enters the $i$th station if $0 \leq n_i \leq K - 1$; or it goes to Road $i \rightarrow l$ if $n_i = K$.

In the former case (the riding bike of class $r$ enters the $i$th station if $0 \leq n_i \leq K - 1$), we obtain that for $0 \leq n_i \leq K - 1$, there is a two-element change: $(m_{k,i}^{(r)}, n_i) \rightarrow (m_{k,i}^{(r)} + 1, n_i + 1)$, hence this gives that for $0 \leq n_i \leq K - 1$

$$P_{\vec{n}, \vec{n}^{'}_{r,n}} = P_{\vec{n}, \vec{n} - \vec{g}_{k,i}^{(r)} + \vec{g}_{i}} = 1,$$

since the end of Road $k \rightarrow i$ is only the $i$th station. There are $2N^2(N - 1)CK$ such elements with $P_{\vec{n}, \vec{n}^{'}} = P_{\vec{n}, \vec{n} - \vec{g}_{k,i}^{(r)} + \vec{g}_{i}} = 1$ in the closed queueing network.

**Case three: From a road to another road**

In the latter case (the riding bike of class $r$ goes to Road $i \rightarrow l$ if $n_i = K$), we get that there is a two-element change: $(m_{k,i}^{(r)}, m_{i,l}^{(2)}) \rightarrow (m_{k,i}^{(r)} - 1, m_{i,l}^{(2)} + 1)$. Thus we obtain that for $n_i = K$

$$P_{\vec{n}, \vec{n}^{'}_{r,n}} = P_{\vec{n}, \vec{n} - \vec{g}_{k,i}^{(r)} + \vec{g}_{i,l}^{(2)}} = \alpha_{i,l}$$

by means of Assumption (4) of Section 2. There are $2N^3(N - 1)^2C^2$ such elements with $P_{\vec{n}, \vec{n}^{'}} = P_{\vec{n}, \vec{n} - \vec{g}_{k,i}^{(r)} + \vec{g}_{i,l}^{(2)}} = \alpha_{i,l}$ in the closed queueing network.
In summary, the above analysis gives

\[
P_{\pi', \pi} = \begin{cases} 
P_{\pi', \pi} - \gamma_{i} + \gamma_{i}^{(1)} = p_{i, l}, & \text{if } 1 \leq n_{i} \leq K, \quad \text{(station \rightarrow road)} \\
P_{\pi', \pi} - \gamma_{k_{i}}^{(1)} + \gamma_{k_{i}}^{(2)} = 1, & \text{if } 0 \leq n_{i} \leq K - 1, \quad \text{(road \rightarrow station)} \\
P_{\pi', \pi} - \gamma_{k_{i}}^{(1)} + \gamma_{k_{i}}^{(2)} = \alpha_{i, l}, & \text{if } n_{i} = K, \quad \text{(road \rightarrow road, a full station)} \\
0, & \text{otherwise.}
\end{cases}
\]

At the same time, the minimal number of zero elements in the routing matrix \( P \) is given by

\[
\left[ (K + 1)^{N} (NC + 1)^{2N(N-1)} \right]^{2} - NK(N - 1) - 2N^{2}(N - 1)CK - 2N^{3}(N - 1)^{2}C^{2}
\]

This also shows that there exist more zero elements in the routing matrix \( P \).

We write a row vector

\[
\overrightarrow{\mathcal{R}} = (\overrightarrow{e} (\overrightarrow{n}) : \overrightarrow{e} (\overrightarrow{n}) \in \mathcal{E}),
\]

where

\[
\mathcal{E} = \{\overrightarrow{e} (\overrightarrow{n}) : \overrightarrow{n} \in \Omega\}.
\]

**Theorem 1** The routing matrix \( P \) is irreducible and stochastic (i.e., \( P1 = 1 \), where \( 1 \) is a column vector of ones), and there exists a unique positive solution to the following system of linear equations

\[
\begin{cases} 
\overrightarrow{\mathcal{R}} = \overrightarrow{\mathcal{R}} P, \\
\left( \overrightarrow{\mathcal{R}} \right)_{1} = 1,
\end{cases}
\]

where \( \left( \overrightarrow{\mathcal{R}} \right)_{1} \) is the first element of the row vector \( \overrightarrow{\mathcal{R}} \).

**Proof:** The outline of this proof is described as follows. It is well-known that the routing structure of the closed queueing network indicates that the routing matrix \( P \) is stochastic, and the accessibility of each station or road of the bike sharing system shows that the routing matrix \( P \) is irreducible. Thus the routing matrix \( P \) is not only irreducible but also stochastic. Notice that the size of the routing matrix \( P \) is \( (K + 1)^{N} (NC + 1)^{2N(N-1)} \), it follows from Theorem 1.1 (a) and (b) of Chapter 1 in Seneta [21] that the left eigenvector \( \overrightarrow{\mathcal{R}} \) of the irreducible stochastic matrix \( P \) corresponding to the maximal eigenvalue 1 is more than 0, that is, \( \overrightarrow{\mathcal{R}} > 0 \), and \( \overrightarrow{\mathcal{R}} \) is unique for \( \left( \overrightarrow{\mathcal{R}} \right)_{1} = 1 \). This completes this proof. \( \blacksquare \)
4 A Product-Form Solution and Performance Analysis

In this section, we first provide a product-form solution to the steady state joint probabilities of $N(2N-1)$ queue lengths in the closed queueing network. Then we analyze performance measures of the bike sharing system by means of the steady state joint probabilities.

Notice that

$$X(t) = (L_1(t), L_2(t), \ldots, L_{N-1}(t), L_N(t),$$

where for $1 \leq i \leq N$

$$L_i(t) = \left( Q_i(t); R_{i,1}^{(1)}(t), R_{i,1}^{(2)}(t); R_{i,2}^{(1)}(t), R_{i,2}^{(2)}(t); \ldots; R_{i,i-1}^{(1)}(t), R_{i,i-1}^{(2)}(t) \right).$$

At the same time, $\{X(t): t \geq 0\}$ is an irreducible continuous-time Markov process on state space $\Omega$ which contains $(K + 1)^N (NC + 1)^{2N(N-1)}$ states. Therefore, the Markov process $\{X(t): t \geq 0\}$ is irreducible and positive recurrent. In this case, we set

$$\pi(\vec{n}) = \lim_{t \to +\infty} P \left\{ Q_i(t) = n_i, 1 \leq i \leq N; R_{k,l}^{(1)}(t) = m_{k,l}^{(1)}, R_{k,l}^{(2)}(t) = m_{k,l}^{(2)}; 1 \leq k, l \leq N \text{ with } k \neq l, \sum_{i=1}^{N} n_i + \sum_{r=1,2}^{N} \sum_{k=1}^{N} \sum_{l \neq k}^{N} m_{k,l}^{(r)} = NC \right\}.$$  

(a) A product-form solution to the steady state joint probabilities

The following theorem provides a product-form solution to the steady state joint probability $\pi(\vec{n})$ for $\vec{n} \in \Omega$; while its proof is easy by means of Chapter 7 in Bolch et al. [2] and is omitted here.

**Theorem 2** For the closed queueing network of the bike sharing system, the steady state joint probability $\pi(\vec{n})$ is given by

$$\pi(\vec{n}) = \frac{1}{G} \prod_{i=1}^{N} F(n_i) \prod_{k=1}^{N} \prod_{l \neq k}^{N} m_{k,l}^{(1)} H^{(1)} \left( m_{k,l}^{(1)} \right) H^{(2)} \left( m_{k,l}^{(2)} \right),$$

where $\vec{n} \in \Omega$, $m_{k,l} = m_{k,l}^{(1)} + m_{k,l}^{(2)}$, and

$$F(n_i) = \begin{cases} \left[ \frac{e^{(n_i)/\lambda_i}}{\lambda_i} \right]^{n_i}, & 1 \leq n_i \leq K, \\ 1, & n_i = 0, \end{cases}$$

$$H^{(1)} \left( m_{k,l}^{(1)} \right) = \frac{1}{m_{k,l}^{(1)}}, H^{(2)} \left( m_{k,l}^{(2)} \right) = m_{k,l}^{(2)}.$$
\[ H^{(1)} \left( m_{k,l}^{(1)} \right) = \begin{cases} \frac{1}{m_{k,l}^{(1)}} \left[ e_{k,l}^{(1)}(m_{k,l}^{(1)}) \right] m_{k,l}^{(1)}, & 1 \leq m_{k,l}^{(1)} \leq NC, \\ 1, & m_{k,l}^{(1)} = 0, \end{cases} \]
\[ H^{(2)} \left( m_{k,l}^{(2)} \right) = \begin{cases} \frac{1}{m_{k,l}^{(2)}} \left[ e_{k,l}^{(2)}(m_{k,l}^{(2)}) \right] m_{k,l}^{(2)}, & 1 \leq m_{k,l}^{(2)} \leq NC, \\ 1, & m_{k,l}^{(2)} = 0, \end{cases} \]

and \( G \) is a normalization constant, given by

\[ G = \sum_{\vec{n} \in \Omega} \prod_{i=1}^{N} F(n_i) \prod_{k=1}^{N} \prod_{l \neq k}^{N} m_{k,l}^{(1)} H^{(1)} \left( m_{k,l}^{(1)} \right) H^{(2)} \left( m_{k,l}^{(2)} \right). \]

(b) Performance analysis

Now, we consider three key performance measures of the bike sharing system in terms of the steady state joint probability \( \pi(\vec{n}) \) for \( \vec{n} \in \Omega \).

1. The steady state probability of problematic stations

In the study of bike sharing systems, it is a key task to compute the steady state probability of problematic stations. To this end, our aim is to care for the \( i \)th station with respect to its full or empty cases. Thus the steady state probability \( \Im \) of problematic stations is given by

\[ \Im = P\{n_i = 0 \text{ or } n_i = K\} = P\{n_i = 0\} + P\{n_i = K\} \]
\[ = \sum_{\vec{n} \in \Omega \text{ & } n_i = 0} \pi(\vec{n}) + \sum_{\vec{n} \in \Omega \text{ & } n_i = K} \pi(\vec{n}). \]

2. The means of steady state queue lengths

The steady state mean of the number of bikes parked at the \( i \)th station is given by

\[ Q_i = \sum_{\vec{n} \in \Omega \text{ & } 1 \leq n_i \leq K} n_i \pi(\vec{n}), \quad 1 \leq i \leq N, \]

and the steady state mean of the number of bikes ridden on the \( N(N-1) \) roads is given by

\[ Q_0 = NC - \sum_{i=1}^{N} \left[ \sum_{\vec{n} \in \Omega \text{ & } 1 \leq n_i \leq K} n_i \pi(\vec{n}) \right], \]

or

\[ Q_0 = \sum_{r=1,2} \sum_{k=1}^{N} \sum_{l \neq k}^{N} \sum_{\vec{n} \in \Omega \text{ & } 1 \leq m_{k,l}^{(r)} \leq NC} m_{k,l}^{(r)} \pi(\vec{n}). \]
5 Concluding Remarks

In this paper, we provide a unified framework for analyzing closed queueing networks in the study of bike sharing systems, and show that this framework of closed queueing networks is interesting, difficult and challenging. We describe and analyze a closed queueing network corresponding to a large-scale bike sharing system, and specifically, we provide a product-form solution to the steady state joint probabilities of \( N(2N - 1) \) queue lengths, which leads to be able to calculate the steady state probability of problematic stations, and more generally, to analyze performance measures of this bike sharing system. We hope the methodology and results of this paper can be applicable in the study of more general bike sharing systems by means of the closed queueing networks. Along these lines, there are a number of interesting directions for potential future research, for example:

- Developing effective algorithms for computing the routing matrix, the relative arrival rates, and the steady state joint probabilities of queue lengths;

- analyzing bike sharing systems with Markovian arrival processes (MAPs) of customers to rent bikes, and phase type (PH) riding-bike times on the roads;

- considering heterogeneity of bike sharing systems under an irreducible graph with stations, roads and their connections;

- discussing repositioning bikes by trucks in bike sharing systems with information technologies; and

- applying periodic MAPs, periodic PH distributions, or periodic Markov processes to studying time-inhomogeneous bike sharing systems.

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References

[1] Adelman, D.: Price-directed control of a closed logistics queueing network. Operations Research, 55(6), 1022–1038 (2007)

[2] Bolch, G., Greiner, S., de Meer, H., Trivedi, K.S.: Queueing Networks and Markov Chains: Modeling and Performance Evaluation with Computer Science Applications. John Wiley & Sons (2006).

[3] DeMaio, P.: Bike-sharing: history, impacts, models of provision, and future. Journal of Public Transportation, 12(4), 41–56 (2009)

[4] Faye, V.: French Network of Bike: Cities and Bikesharing Systems in France. le Club des Villes Cyclables, Paris (2008)

[5] Fishman, E., Washington, S., Haworth, N.: Bike share: a synthesis of the literature. Transport Reviews, 33(2), 148–165 (2013)

[6] Fricker, C., Gast, N.: Incentives and redistribution in homogeneous bike-sharing systems with stations of finite capacity. EURO Journal on Transportation and Logistics, Published online June 7, 2014, pp. 1–31 (2014)

[7] Fricker, C., Gast, N., Mohamed, A.: Mean field analysis for inhomogeneous bikesharing systems. In: DMTCS Proceedings, vol. 1, 365–376 (2012)

[8] Fricker, C., Tibi, D.: Equivalence of ensembles for large vehicle-sharing models. arXiv Preprint: arXiv:1507.07792 pp. 1–28 (2015)

[9] George, D.K., Xia, C.H.: Fleet-sizing and service availability for a vehicle rental system via closed queueing networks. European Journal of Operational Research, 211(1), 198–207 (2011)

[10] Janett, B., Hendrik, M.: Optimising Bike-Sharing in European Cities: A Handbook. OBIS Project (2011)

[11] Labadi, K., Benarbia, T., Barbot, J.P., Hamaci, S., Omari, A.: Stochastic Petri net modeling, simulation and analysis of public bicycle sharing systems. IEEE Transactions on Automation Science and Engineering, 12(4), 1380–1395 (2015)
[12] Larsen, J.: Bike-sharing programs hit the streets in over 500 cities worldwide. Earth Policy Institute, 25 (2013)

[13] Lathia, N., Ahmed, S., Capra, L.: Measuring the impact of opening the London shared bicycle scheme to casual users. Transportation Research Part C, 22(1), 88–102 (2012)

[14] Leurent, F.: Modelling a vehicle-sharing station as a dual waiting system: stochastic framework and stationary analysis. HAL Id: hal-00757228, pp. 1–19 (2012)

[15] Li, Q.L., Chen, C., Fan, R.N., Xu, L., Ma, J.Y.: Queueing analysis of a large-scale bike sharing system through mean-field theory. arXiv Preprint: arXiv:1603.09560, pp. 1–50 (2016)

[16] Morency, C., Trépanier, M., Godefroy, F.: Insight into the Montreal bikesharing system. In: TRB-Transportation Research Board Annual Meeting, Washington, USA (2011)

[17] Raviv, T., Kolka, O.: Optimal inventory management of a bikesharing station. IIE Transactions, 45(10), 1077–1093 (2013)

[18] Raviv, T., Tzur, M., Forma, I.A.: Static repositioning in a bike-sharing system: models and solution approaches. EURO Journal on Transportation and Logistics, 2(3), 187–229 (2013)

[19] Savin, S., Cohen, M., Gans, N., Katala, Z.: Capacity management in rental businesses with two customer bases. Operations Research, 53(4), 617–631 (2005)

[20] Schuijbroek, J., Hampshire, R., van Hoeve, W.J.: Inventory rebalancing and vehicle routing in bike-sharing systems. Technical Report 2013-2, Tepper School of Business, Carnegie Mellon University, pp. 1–27 (2013)

[21] Seneta, E.: Non-negative Matrices and Markov Chains. Springer-Verlag (1981)

[22] Shaheen, S., Guzman, S., Zhang, H.: Bikesharing in Europe, the Americas, and Asia: past, present, and future. Transportation Research Record: Journal of the Transportation Research Board, No. 2143, 159–167 (2010)
[23] Shu, J., Chou, M.C., Liu, Q., Teo, C.P., Wang, I.L.: Models for effective deployment and redistribution of bicycles within public bicycle-sharing systems. Operations Research, 61(6), 1346–1359 (2013)

[24] Tang, Y., Pan, H., Shen, Q.: Bike-sharing systems in Beijing, Shanghai, and Hangzhou and their impact on travel behavior. In: The 90th Annual Meeting of the Transportation Research Board, Washington, D.C. (2011)

[25] Waserhole, A., Jost, V.: Vehicle sharing system pricing regulation: transit optimization of intractable queuing network. HAL Id: hal-00751744, pp. 1–20 (2012)

[26] Waserhole, A., Jost, V.: Vehicle sharing system pricing regulation: A fluid approximation. HAL Id: hal-00727041, pp. 1–35 (2013)

[27] Waserhole, A., Jost, V.: Pricing in vehicle sharing systems: Optimization in queuing networks with product forms. EURO Journal on Transportation and Logistics, Published online: November 4, 2014, pp. 1–28 (2014)

[28] Waserhole, A., Jost, V., Brauner, N.: Pricing techniques for self regulation in vehicle sharing systems. Electronic Notes in Discrete Mathematics, vol. 41, 149–156 (2013)