Hořava Gravity with Mixed Derivative Terms:
Power-Counting Renormalizability with Lower-Order Dispersions

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It has been argued that Hořava gravity needs to be extended to include terms that mix spatial and time derivatives in order to avoid unacceptable violations of Lorentz invariance in the matter sector. In an earlier paper we have shown that including such mixed derivative terms generically leads to 4th instead of 6th order dispersion relations and this could be (naively) interpreted as a threat to renormalizability. We have also argued that power-counting renormalizability is not actually compromised, but instead the simplest power-counting renormalizable model is not unitary. In this note we consider the Lifshitz scalar as a toy theory and we generalize our analysis to include higher order operators. We show that models which are power-counting renormalizable and unitary do exist. Our results suggest the existence of a new class of Hořava theories with mixed derivative terms.

I. INTRODUCTION

The gravity theory proposed by Hořava in Ref. [1] has acquired significant attention since its introduction. The basic idea is to improve the UV behaviour of the theory by modifying the dispersion relations, and hence the propagators. This is achieved by introducing a preferred foliation and constructing the action of the theory in such a way so that the ‘kinetic’ terms contain only two time derivatives but there are also ‘potential terms’ with higher-order spatial derivatives. This introduces an anisotropic scaling between time and space at high energies,

\[ t \rightarrow [k]^{-m} t, \quad x^i \rightarrow [k]^{-1} x^i, \]

where the latin indices span the \( D \) dimensional spatial directions and \([k]\) is the momentum dimension. It has been argued in Ref. [1] that power counting renormalizability requires \( m \geq D \), so in 3 + 1 dimensions \( m \) has to be at least equal to 3 and the dispersion relations would be of the type \( \omega^2 \sim k^6 \) in the UV.

The existence of a preferred foliation leads to violations of Lorentz invariance in the gravity sector. One of the main challenges that the theory confronts with is the percolation of Lorentz violations at low energy into the matter sector, where Lorentz symmetry is very stringent constrained (see e.g. [2]). If dimension four Lorentz violating operators are present in the matter sector, the propagation speeds of different species of particles run to the universal value logarithmically, indicating a severe fine-tuning problem [3]. Even if such terms are absent (or tuned away) and the Lorentz violating operators are generated at higher dimensions, the latter are heavily constrained from synchrotron radiation in the Crab nebula [5].

A possible mechanism to suppress the Lorentz violations in the matter sector was proposed in [6]. Lorentz violations were restricted to the gravity sector at tree level and percolation to the matter sector though graviton loops was considered. It was shown that the Lorentz-violating terms that are generated in the matter sector end up being suppressed by powers of \( M_\star/M_p \), where \( M_\star \) is the UV scale above which the dispersion relations in the gravity sector cease to be relativistic and \( M_p \) is the Planck scale. Lorentz violation constraints in the gravity sector are quite weak as we do not test gravity at energies above \( 10^{-2} \text{eV} \). Hence, one can choose \( M_\star \ll M_p \) and this can push the Lorentz violations in the matter sector below the experimental constraints.

However, the analysis of Ref. [6] also revealed a naturaness problem, stemming from the fact that the vector mode propagators are unaffected by the higher dimensional Hořava terms. As a result, the vector loops lead to quadratic divergences in the correction to the difference of propagation speeds between different matter species. The proposed resolution was to add the mixed derivative term \( \nabla_i K_{jk} \nabla^i K^{jk} \), where \( K^{ij} \) is the extrinsic curvature of the leaves of the preferred foliation and \( \nabla_i \) is the 3-dimensional covariant derivative operator on a leaf. Including this term in the action improves the behavior of the vector mode.

This term is not the only operator with two temporal and two spatial derivatives that one could add. In Ref. [6] we considered all possible such terms and we performed the complete perturbative analysis of the most general extension of Hořava gravity along these lines. The dispersion relation of the scalar and tensor modes in the UV turned out to be of 4th order, i.e. \( \omega^4 \propto k^4 \), as opposed to the 6th order ones in standard Hořava gravity. One could interpret this as a threat to renormalizability, based on the standard power counting of Hořava gravity. In fact, a specific tuning of coefficients that can restore the sixth order dispersion relations and still provide the sought for modification to the vector mode propagator does exist, so it is rather tempting to conclude that this tuning is necessary. However, as shown in Ref. [7], by studying the Lifshitz scalar as a toy model, counting rules get modified once the mixed-derivative terms are considered and one can have a renormalizable theory even with lower-order dispersion relations. Surprisingly, the sim-
ples power counting renormalizable theory of this type exhibits relativistic, instead of anisotropic, scaling and this, unfortunately, leads to problems with unitarity \cite{7}.

In this note, we revisit the Lifshitz scalar with mixed derivative terms as a proxy for the behaviour of Hořava gravity and show that, if a larger number of operators is taken into account, models that are power-counting renormalizable and unitary exist and can have lower order dispersion relations than the standard Lifshitz scalar (or standard Hořava gravity).

\section{Lifshitz Scalar with Arbitrary Mixed Derivative Terms}

In order to consider the mixed derivative case, we use, following Ref. \cite{7}, the Lagrangian

\[ \mathcal{L} = \alpha \phi^2 + \beta \phi (\Delta)^y \dot{\phi} - \gamma \phi (-\Delta)^z \phi. \]  

The anisotropic scaling is given in equation (1). The dimensions of the coupling constants are related through

\[ [\alpha] = [\beta][k]^{2y}, \quad [\gamma] = [\beta][k]^{2(m+y-z)}. \]  

Hence, we can rewrite the Lagrangian (2) as

\[ \mathcal{L} = \beta \left[ \lambda M^2 y \dot{\phi}^2 + \phi (\Delta)^y \dot{\phi} - M^2 (m+y-z) \phi (-\Delta)^z \phi \right], \]  

where \([M] = [k]\) and \([\lambda] = [k]^0\).

\begin{subsection}{Power-counting renormalizability: Dimensional counting}

Here, we choose the normalization such that \(\beta = 1\) and fix the units such that the last two terms, which are expected to dominate in the UV, have the same dimensions. The latter gives the relation

\[ z = m + y. \]  

Requiring that the action is dimensionless, the dimension of the Lifshitz scalar is found to be

\[ [\phi] = [k]^{(D-z-y)/2}. \]  

If \(\phi\) is dimensionless or has negative momentum dimensions, then any interaction term of the form \(\phi^n\) is power-counting renormalizable. This gives us the first condition

\[ z = m + y \geq D - y. \]  

In Ref. \cite{7}, only the case where \(D = 3, \ y = 1\) and \(z\) has the smallest possible value that satisfies the above condition (the theory with the lowest number of operators) was considered. This implies \(z = 2\) and \(m = 1\), i.e. relativistic scaling. The drawback of having relativistic scaling is that terms with 4 time derivatives, such as \(\delta^2\) come at the same order as terms with 4 spatial derivatives that are already present in the action. Hence, one expects that they would be generated by radiative corrections and lead to a loss of unitarity.

Clearly, one could relax the restrictions of Ref. \cite{7} and construct power-counting renormalizable theories with different values of \(z\) and \(y\), as we will discuss in more detail below. In Sec. \[HQ\] we will further impose conditions to preserve unitarity and show that for \(y > 0\), the minimum value for \(m\) is not allowed.

\begin{subsection}{Power-counting renormalizability: Superficial degree of divergence}

Since the accuracy of the dimensional counting method above relies on the correct choice of units and normalizations (see the discussion Ref. \cite{7}), we also provide the power-counting renormalizability condition by calculating the superficial degree of divergence.

Starting with the Lagrangian (1), the Green’s function for the Lifshitz scalar in the UV \((k \gg \lambda^{1/2} M)\) can be calculated as

\[ G_{\omega,k} = \frac{1}{\beta k^{2y} \left[ \omega^2 - M^2 (m-z-y) k^2 (z-y) \right]} \]  

For the momentum-cutoff \(\Lambda_k\), each internal line contributes

\[ G_{\omega,k} \to \beta^{-1} M^{-2(m-z-y)} \Lambda_k^{-2z} \]  

Due to the anisotropic scaling the energy cutoff is different than the momentum cutoff and can be obtained through the dispersion relations as \(\Lambda_m = M^{m-z+y} \Lambda_k^{z-y}\).

Thus, each loop integral contributes

\[ \int d\omega d^D k \to \Lambda_{\omega} \Lambda_k^{D} = M^{m-z+y} \Lambda_k^{2+y}. \]  

For a theory with non-derivative interactions, the cutoff dependence from the Feynman diagram with \(I\) internal lines and \(L\) loops is

\[ \beta^{-I} M^{(m-z+y)(L-2I)} \Lambda_k^{L(D+z-y)-2Lz} \]  

giving the superficial degree of divergence as

\[ \delta = (D+z-y) L - 2I z = (D-z-y)L - 2(I-L)z. \]  

Using \(L - I \leq 0\), we obtain

\[ \delta \leq (D - z - y)L, \]  

which implies that if \(z \geq D - y\), the theory is power-counting renormalizable. This verifies the result derived in the previous section in Eq. \[7\].

Since our goal is to apply the power-counting renormalizability arguments to a gravity theory, it is important to
include derivative interactions. To this end, we consider the Lagrangian
\[
\mathcal{L} = -\dot{\phi}(-\triangle)^{\eta}\phi + P(\nabla^{2z}, \phi),
\]
where \( P(\nabla^{2z}, \phi) \) is an infinite order polynomial with derivatives up to \( 2z \). For a free field, the contribution from the second term will be dominated by \( \phi\triangle^2\phi \) in the UV. Therefore, for both the internal lines and the loop integrals will give the same contributions as above [9]-[10], with \( \beta = 1 \) and \( m = z - y \). However, each of the \( V \) vertices can now bring up to \( 2z \) powers of momentum to the superficial degree of divergence, which now becomes
\[
\delta \leq (D - z - y)L - 2(I - L - V)z,
\]
or using \( V + L - I = 1 \), reduces to
\[
\delta \leq (D - z - y)L + 2z. \tag{16}
\]
If the condition (17) holds, the above inequality reduces to \( \delta < 2z \), i.e. the superficial degree of divergence becomes bounded from above by the canonical dimension of the operators.

C. Unitarity

We now impose further restrictions in the power-counting renormalizability condition (7) by requiring that no unitarity breaking term can be generated. This is ensured if the dimensions of such terms are higher than the dimension of the last two terms in Eq. (4). If the condition (7) is not saturated, the scalar field has negative dimensions, so adding more powers of scalar fields might pull the dimension of the high derivative term to lower values. We also impose that the action is invariant under time reversal \( t \rightarrow -t \).

As a starting point, we consider the addition of a term \( \phi^2 \) and require that it has higher dimension than the kinetic term \( \dot{\phi}(-\triangle)^{\eta}\phi \). This gives the condition:
\[
4m > 2y + 2m
\]
or simply
\[
m > y. \tag{18}
\]
Notice that if this condition is satisfied, the momentum dimension of the second derivative
\[
[\dot{\phi}] = [k]^{(D-2y+3m)/2}, \tag{19}
\]
is manifestly positive. Therefore, terms of the form \( \phi^{2n} \) are also guaranteed to have high dimensions.

Before we move on, let us also note that the first time derivative has dimension
\[
[\phi] = [k]^{(D-2y+m)/2}. \tag{20}
\]
Therefore, it can have a negative dimension if \( m < 2y - D \). In that case, a term like \( \phi^2 \dot{\phi}^{2n} \) with high enough \( n \) can be made relevant. For example, consider the case where \( D = 3, m = 7, y = 6 \). This setup is power-counting renormalizable and \( \phi^2 \) term is forbidden. However, \( \phi^2 \dot{\phi}^{2n} \) has dimension 4, while the mixed term \( \phi \Delta^{y}\dot{\phi} \) has dimension 7, so the former term can be generated. To avoid unitarity breaking from such terms, we further impose
\[
m \geq 2y - D. \tag{21}
\]

D. Allowed region

Collecting together what we found until now, we obtain the following conditions:
\[
\begin{align*}
z &= m + y, \\
m &\geq |D - 2y|, \\
m &> y. \tag{22}
\end{align*}
\]
For \( y = 0 \), we obtain the standard relations of Hořava theory
\[
z = m, \quad m \geq D, \tag{23}
\]
along with the trivially satisfied condition \( m > 0 \).

For \( D = 3 \) and \( y = 1 \), we obtain
\[
z = m + 1, \quad m \geq 1, \quad m > 1. \tag{24}
\]
As we have already seen, the last condition forbids the relativistic scaling on the grounds of unitarity.

In Fig. 1 we show the allowed \((m, n)\) region in \( D = 3 \). For a given mixed derivative term with arbitrary spatial derivatives, one can always satisfy the power-counting renormalizability and the unitarity conditions, provided that enough powers of gradient terms are included in the action.

III. DISCUSSION

In this note, we have revisited the power counting arguments for a Lifshitz scalar with mixed derivative terms. We have gone beyond the analysis of Ref. [7] by considering the full class of theories with mixed derivative terms. Our result suggests that the addition of mixed derivative terms to the standard Lifshitz scalar can lead to theories that are power-counting renormalizable and unitary, even though their dispersion relations and scaling properties are different than the standard Lifshitz scalar. We have identified the precise conditions, see Eq. (22), that define the subclass of theories with these characteristics. As discussed in the introduction, mixed derivative terms have been used in the context of Hořava gravity in order to regulate divergencies in vector mode loop integrals. These divergencies would otherwise introduce a naturalness problem in the suppression of Lorentz violations in the matter sector [5]. As shown in Ref. [7], when all mixed derivative terms are consistently taken
into account, the dispersion relations generically become 4th order. One could tune the coefficients in order to recover the 6th order dispersion relations (while still having the sought for modification to the vector mode propagator). However, to the extent that the Lifshitz scalar is a good proxy for Hořava gravity, our results suggest that such tuning is not actually necessary. In 3 + 1 dimensions, in particular a theory with $y = 1, z = 3$, i.e. standard Hořava gravity with terms that are 6th order in spatial derivatives, supplemented with mixed derivative term with two temporal and two spatial derivatives, is both power-counting renormalizable and unitary, even though it has anisotropic index $m = 2$ and 4th order dispersion relations.

We close with a note of caution: even though in all of the theories with mixed derivatives terms, the behaviour of the vector mode gets modified, the dispersion relations for the scalar and tensor modes are not necessarily of sixth order. The analysis of Ref. [6] regarding the suppression of Lorentz violations in the matter sector assumed sixth order dispersion relations, so it is not straightforward to conclude that its results would be applicable to theories with a different anisotropic index. One would have to revisit the problem in order to reach a final conclusion.

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[1] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].
[2] V. A. Kostelecky and N. Russell, Rev. Mod. Phys. 83, 11 (2011) [arXiv:0801.0287 [hep-ph]].
[3] J. Collins, A. Perez, D. Sudarsky, L. Urrutia and H. Vucetich, Phys. Rev. Lett. 93, 191301 (2004) [gr-qc/0403053].
[4] R. Iengo, J. G. Russo and M. Serone, JHEP 0911, 020 (2009) [arXiv:0906.3477 [hep-th]].
[5] S. Liberati, L. Maccione and T. P. Sotiriou, Phys. Rev. Lett. 109, 151602 (2012) [arXiv:1207.0670 [gr-qc]].
[6] M. Pospelov and Y. Shang, Phys. Rev. D 85, 105001 (2012) [arXiv:1010.5249 [hep-th]].
[7] M. Colombo, A. E. Gümrükçüoğlu and T. P. Sotiriou, Phys. Rev. D 91, no. 4, 044021 (2015) [arXiv:1410.6360 [hep-th]].