On majorization of closed walks vector of trees with given degree sequences

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Abstract

Let \( C_v(k; T) \) be the number of the closed walks of length \( k \) starting at vertex \( v \) in a tree \( T \). We prove that for a given tree degree sequence \( \pi \), then for any tree with degree sequence \( \pi \), the sequence \( C(k; T) \equiv (C_v(k; T), v \in V(T)) \) is weakly majorized by the sequence \( C(k; T^*_\pi) \equiv (C_v(k; T^*_\pi), v \in V(T^*_\pi)) \), where \( T^*_\pi \) is the greedy tree corresponding to \( \pi \). In addition, for two trees degree sequences \( \pi, \pi' \), if \( \pi \) is majorized by \( \pi' \), then \( C(k; T^*_\pi) \) is weakly majorized by \( C(k; T^*_\pi) \).

1 Introduction

Let \( G = (V(G), E(G)) \) be a simple graph of order \( n \). A walk of \( G \) is a sequence of vertices and edges, i.e., \( w_1e_1w_2e_2\cdots e_{k-1}w_k \) such that \( e_iw_iw_{i+1} \in E(G), \ i = 1, 2, \ldots, k-1 \). Moreover, if \( w_1 = w_k \), then this walk is called closed walk with length \( k - 1 \). Further, denote by \( C_v(k; G) \) be the number of the closed walks of length \( k \) starting at vertex \( v \) in \( G \) and the vector \( C(k; G) \equiv (C_v(k; G), v \in V(G)) \). Moreover, denote by \( M_k(G) \) the number of the closed walks of length \( k \) in \( G \). The number of closed walks of length \( k \) in \( G \) has been intensively studied. For example, Dress etc. [8] studied when \( M_{k-1}M_{k+1}(T) - M_k^2(T) \) is positive, zero, or negative. Taubig etc. [15] investigate the growth of the number \( M_k(G) \) and related inequalities. Further, the number of closed walks may be used to

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characterize the complexity in the model of the symmetric Turing machine (see [15]) and to study the Dense $r$–Subgraph Problem (see [21]). Since the dense $r$–subgraph maximization problem is of computing the dense $r$–vertex subgraph of a given graph, it may be an interesting problem to study the the number of closed walks of length $k$ with starting at vertices in any vertex subset $U$ of $V(G)$ with $|U| = r \leq n$. If $r = n$, Csikvari [6] proved that the star has the maximum number of closed walks of length $k$ among all the trees on $n$ vertices, which confirm a conjecture of Nikiforov concerning the number of closed walks on trees. Further, Bollobas and Tyomkyn [5] proved that the KC$-$ transformation on tree increases the number of closed walks of length $k$. In addition, Andriantian and Wagner [2] characterized the extremal trees with the maximum $M_k(T)$ among all trees with a given tree degree sequence $\pi$. If $r < n$, there are no any results on the problem.

On the other hand, the number of closed walks is direct relationship to the spectral radius of the adjacency matrix. Let $A(G) = (a_{ij})$ be the adjacent matrix of $G$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and 0 otherwise, then $A(G)$ has $n$ real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Since the trace of $A^k(G)$ is equal to the number of closed walks of length $k$ in $G$, it is easy to see that

$$M_k(G) = \sum_{i=1}^{n} \lambda_i^k,$$

which is also called The $k$-th spectral moment of $G$. Moreover, the Estrada index [13] of $G$, which is relative to $M_k(G)$ and proposed by Estrada, is defined to be

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$ (2)

It is easy to see

$$EE(G) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{k=0}^{\infty} \frac{M_k(G)}{k!}.$$ (3)

The Estrada index may have many applications in the study of molecular structures and complex network, etc. For more about the Estrada index, the reader may refer to the excellent survey [11]. A nonincreasing sequence of nonnegative integers $\pi = (d_0, d_1, \cdots, d_{n-1})$ is called graphic if there exists a simple connected graph having $\pi$ as its vertex degree sequence. For a given tree degree sequence $\pi = (d_0, d_1, \cdots, d_{n-1})$, let

$$\mathcal{T}_\pi = \{ T \mid T \text{ is any tree with } \pi \text{ as its degree sequence} \}.$$

There are several papers which investigated the graph parameters, such as Energy, Hosoya index and Merrifield-Simmons index in [1]; the Estrada index in [2]; the Wiener index in [10] and [19]; the largest spectral radius in [4]; the Laplacian spectral radius in [18]; the number of subtrees in [20, 21], etc.

In this paper, motivated by the Dense $r$–Subgraph Problem and the research in the class $\mathcal{T}_\pi$, we consider the following problem: determine

$$\max_{T \in \mathcal{T}_\pi} \max_{U \subseteq V(T), |U| = r} \sum_{v \in U} C_k(v, T)$$

for a given tree degree sequence $\pi$. The rest of this paper is arranged as follows. In Section 2, after introducing some notations, we present the main results of this paper. In the sections 3 and 4, the proofs of Theorems [23] and [24] are given respectively.
2 Preliminary and main results

In order to present our main results, we first introduce some notations. Let \( G = (V(G), E(G)) \) be a simple graph with a root set \( V_0 = \{v_0, ..., v_r\} \subseteq V(G) \). The height \( h(v) \) of a vertex \( v \) in \( G \) is defined by

\[
h(v) = \text{dist}(v, V_0) = \min\{\text{dist}(v, w)\},
\]

where \( \text{dist}(v, w) \) is the distance between vertices \( v \) and \( w \) in \( V(G) \). Moreover, we say that \( v \) is at the \( h(v) \)-th level. Further, we need the following notation from [2].

**Definition 2.1** [2] Let \( F \) be a forest with the root set \( V_{\text{root}} = \{v_0, ..., v_r\} \) and the maximum height of all components is \( l - 1 \). Then the sequence

\[
\pi = (V_0, ..., V_{l-1})
\]

is called the leveled degree sequence of \( F \), if \( V_i \) is the non-increasing sequence formed by the degrees of the vertices of \( F \) at the \( i \)-th level for any \( i = 0, 1, \ldots, l - 1 \).

**Definition 2.2** Let \( F \) be a forest with the following leveled degree sequence

\[
\pi = (V_0, ..., V_{l-1}).
\]

A well-ordering \( \prec \) of the vertices in \( F \) is called breadth-first search ordering (BFS-ordering for short) if the following holds for all vertices \( u, v \) in the same level:

1. \( u \prec v \) implies \( d(u) \geq d(v) \);
2. If there are two edges \( uu_1 \in E(F) \) and \( vv_1 \in E(F) \) such that \( u \prec v \), \( h(u) = h(u_1) + 1 \) and \( h(v) = h(v_1) + 1 \), then \( u_1 \prec v_1 \).

Moreover, a forest with BFS-ordering is called level greedy forest. If the forest is a tree, then it is called level greedy tree. If \( |V_0| = 2 \) and add an edge to the vertices in \( V_0 \), then it is called edge-rooted level greedy tree. If \( |V_0| = 1 \) and \( \min\{d\} \geq \max \{d'\} \), \( 0 \leq i \leq l - 2 \), then it is called greedy tree. It is easy, but boring, to check the above definitions is equivalent to the level greedy forest (tree, etc) defined in [2]. For a given tree degree sequence \( \pi \), there exists exactly one greedy tree with degree sequence \( \pi \). Moreover, this greedy tree is denoted by \( T_\pi^* \) (see [18]).

In addition, we also need the notation of majorization. Let \( \alpha = (x_0, x_1, ..., x_{n-1}) \) and \( \beta = (y_0, y_1, \ldots, y_{n-1}) \) be two nonnegative sequences. We arrange the entries of \( \pi \) and \( \tau \) in nonincreasing order \( \pi_\downarrow = (x_{[0]}, \cdots, x_{[n-1]}) \) and \( \tau_\downarrow = (y_{[0]}, \cdots, y_{[n-1]}) \) with \( x_{[0]} \geq x_{[1]} \geq \cdots \geq x_{[n-1]} \) and \( y_{[0]} \geq y_{[1]} \geq \cdots \geq y_{[n-1]} \). Then we say that \( \alpha \) is weakly majorized by \( \beta \), denoted by \( \alpha \preceq_w \beta \), if

\[
\sum_{i=0}^{t} x_{[i]} \leq \sum_{i=0}^{t} y_{[i]} \quad \text{for} \quad t = 0, 1, \ldots, n - 1.
\]

Furthermore, if

\[
\sum_{i=0}^{n-1} x_{[i]} = \sum_{i=0}^{n-1} y_{[i]}
\]
then $\alpha$ is majorized by $\beta$, denoted by $\alpha \triangleleft \beta$. If some inequality in (1) is strict, then the majorization (weak majorization, respectively) is strict. For more about the majorization, the reader may refer to [2].

Now we are ready to present the main results of this paper.

**Theorem 2.3** Let $\pi$ be a tree degree sequence. Then, for any $T \in \mathcal{T}_\pi$,

$$C(k; T) \triangleleft_w C(k; T^*_\pi),$$

where $C_v(k; T)$ is the number of the closed walks of length $k$ starting at vertex $v$ in a tree $T$ and $C(k; T) \equiv (C_v(k; T), v \in V(T))$. In other words,

$$\max_{T \in \mathcal{T}_\pi} \max_{U \subseteq V(T), |U| = r} \sum_{v \in U} C_k(v, T) = \sum_{v \in U^*} C_k(v, T^*_\pi), \text{ for } r = 0, \ldots, n - 1,$$

where $U^*$ is the first $r$ vertices in the greedy tree $T^*_\pi$ with a BFS-ordering.

**Theorem 2.4** Let $\pi$, $\pi'$ be two tree degree sequences with $\pi \triangleleft_w \pi'$. Then

$$C(k; T^*_\pi) \triangleleft_w C(k; T^*_\pi').$$

In other words, if $U^*_\pi$ and $U^*_\pi'$, are the first $r$ vertices in the greedy tree $T^*_\pi$ and $T^*_\pi'$, with the BFS-ordering respectively, then

$$\sum_{v \in U^*_\pi} C_k(v, T^*_\pi) \leq \sum_{v \in U^*_\pi'} C_k(v, T^*_\pi').$$

### 3 The proof of Theorem 2.3

Let $F$ be a rooted forest. Denote by $C_v(k; F)$ the set of closed walks of length $k$ starting at $v$ in $T$. Clearly, $|C_v(k; F)| = C_v(k; F)$. If $W = w_1e_1w_2e_2 \cdots e_{k-1}w_k$ is a walk in a rooted forest, $(i_1, i_2, \ldots, i_k)$ is called level sequence of $W$ if $w_t$ is in the $i_t$-th level for $1 \leq t \leq k$. Denote by $C_v(i_1, \ldots, i_k; F)$ the number of closed walks of length $k$ starting at $v$ in $F$ and the level sequences of the closed walks are $(i_1, i_2, \ldots, i_k)$. Denote by $S(v^j_i; k; F)$ the set of level sequences of walks of length $k$ in $F$ starting at vertex $v^j_i$ in the $i$-th level and by $S_i(k; F) = \bigcup_{p=1}^{|U_i|} S(v^j_i; k; F)$, where $U_i = \{v^j_1, \cdots, v^j_{l_i}\}$ is the set of all vertices in the $i$-th level. Denote by $C_{v,w}(k; F)$ be the number of closed walks of length $k$ starting from the edge $vw$ in $F$. For $v \in V(F)$, denote the father of $v$ by $f(v)$ if $v$ has father. Moreover, denote by $\mathcal{F}_\pi$ the set of all rooted forests with leveled degree sequence $\pi$. Before presenting the proof of Theorem 2.3 we need some Lemmas.

**Lemma 3.1** [2] Let $F \in \mathcal{F}_\pi$ for some leveled degree sequence $\pi$ of a vertex-rooted forest and $G = F_\pi$ be the associated leveled greedy forest. Let $v^1_i, \cdots, v^j_i$ and $g^1_i, \cdots, g^j_i$ be the vertices of $F$ and $G$ at the $i$-th level, respectively. Then the following relations hold for all $i$:

$$(C_{v^1_i}(i_1, \cdots, i_t; F), \cdots, C_{v^j_i}(i_1, \cdots, i_t; F)) \triangleleft_w (C_{g^1_i}(i_1, \cdots, i_t; G), \cdots, C_{g^j_i}(i_1, \cdots, i_t; G))$$

and

$$C_{g^1_i}(i_1, \cdots, i_t; G) \geq \cdots \geq C_{g^j_i}(i_1, \cdots, i_t; G)$$
Lemma 3.2 Let \( T \in F_\pi \) for some leveled degree sequence \( \pi \) of a vertex-rooted forest and \( G = F_\pi \) be the associated leveled greedy forest. Let \( v_1^i, \ldots, v_i^i \) and \( g_1^i, \ldots, g_i^i \) be the vertices of \( F \) and \( G \) at the \( i \)-th level, respectively. Then the following relations hold for all \( i \):

\[
(C_{v_1^i} (k; F), \ldots, C_{v_i^i} (k; F)) \prec \prec \pi \ (C_{g_1^i} (k; G), \ldots, C_{g_i^i} (k; G))
\]

(5)

and

\[
C_{g_1^i} (k; G) \geq C_{g_2^i} (k; G) \geq \cdots \geq C_{g_i^i} (k; G).
\]

(6)

**Proof.** Since

\[
C_{v_j} (k; F) = \sum_{i_1, i_2, \ldots, i_{k+1} \in S(v^j, k; F)} C_{v_j} (i_1, \ldots, i_{k+1}; F) = \sum_{i_1, i_2, \ldots, i_{k+1} \in S(k; F)} C_{v_j} (i_1, \ldots, i_{k+1}; F),
\]

we have

\[
\sum_{j=1}^t C_{v_j} (k, F) = \sum_{i_1, i_2, \ldots, i_{k+1} \in S(k; F)} \sum_{j=1}^t C_{v_j} (i_1, \ldots, i_{k+1}; F) \leq \sum_{i_1, i_2, \ldots, i_{k+1} \in S(k; F)} \sum_{j=1}^t C_{g_j} (i_1, \ldots, i_{k+1}; G) \leq \sum_{i_1, i_2, \ldots, i_{k+1} \in S(k; F)} \sum_{j=1}^t C_{g_j} (i_1, \ldots, i_{k+1}; G) \leq \sum_{j=1}^t \sum_{i_1, i_2, \ldots, i_{k+1} \in S(k; G)} C_{g_j} (i_1, \ldots, i_{k+1}; G) = \sum_{j=1}^t C_{g_j} (k; G)
\]

for \( 1 \leq t \leq l_i \). And

\[
C_{g_j} (k; G) = \sum_{i_1, i_2, \ldots, i_{k+1} \in S(g_j, k; G)} C_{g_j} (i_1, \ldots, i_{k+1}; G) \geq \sum_{i_1, i_2, \ldots, i_{k+1} \in S(g_j+1, k; G)} C_{g_j+1} (i_1, \ldots, i_{k+1}; G) = C_{g_{j+1}} (k; G)
\]

for \( 1 \leq j \leq l_i - 1 \), since \( S(g_j^i; k; G) \supseteq S(g_{j+1}^i, k; G) \).

Denote by \( \hat{C}_{v_j^i, v_j^{i+1}} (2k; F) \), \( \hat{C}_{v_j} (2k; F) \) the number of closed walks of length \( 2k \) in \( F \) starting from \( v_j^i \) and \( v_j^{i+1} \) respectively, and the level sequences of the closed walks do not contain pairs \((0,0)\) and \((i, i-1)\) if \( i > 0 \).

Lemma 3.3 Let \( F \in F_\pi \) for some leveled degree sequence \( \pi \) of a vertex-rooted forest, and let \( G = F_\pi \) be the associated leveled greedy forest. Let \( v_1^i, \ldots, v_i^i \) and \( g_1^i, \ldots, g_i^i \) be the vertices of \( F \) and \( G \) at the \( i \)-th level, respectively. Then the following relations hold for all \( i \):

\[
(\hat{C}_{f(v_1^i), v_1^i} (2k; F), \ldots, \hat{C}_{f(v_i^i), v_i^i} (2k; F)) \prec \prec \pi \ (\hat{C}_{f(g_1^i), g_1^i} (2k; G), \ldots, \hat{C}_{f(g_i^i), g_i^i} (2k; G))
\]
and
\[ \tilde{C}_{f(g'_1),g'_1}(2k; G) \geq \tilde{C}_{f(g'_2),g'_2}(2k; G) \geq \cdots \geq \tilde{C}_{f(g'_m),g'_m}(2k; G). \]

**Proof.** We use induction on \( k \). If \( k = 1 \), then it is easy to find the assertion holds. Suppose that the assertion holds for the number not more than \( k (k \geq 1) \). Since
\[ \tilde{C}_{f(v'_j),v'_j}(2k + 2; F) = \sum_{t=0}^{k} \tilde{C}_{v'_j}(2t; F) \cdot \tilde{C}_{f(v'_j)}(2k - 2t; T), \]
by Lemma 8 in [2] and Lemma 3.1, we have
\[ \sum_{j=1}^{m} \tilde{C}_{f(v'_j),v'_j}(2k + 2; F) \leq \sum_{t=0}^{k} \sum_{j=1}^{m} \tilde{C}_{g'_j}(2t; G) \tilde{C}_{f(g'_j)}(2k - 2t; G) = \sum_{j=1}^{m} \tilde{C}_{f(g'_j),g'_j}(2k + 2; G) \]
and
\[ \tilde{C}_{f(g'_i),g'_i}(2k + 2; G) \geq \sum_{t=0}^{k} \tilde{C}_{g'_{i+1}(2t; G) \tilde{C}_{g'_{i+1}}(2k - 2t; G) = \tilde{C}_{f(g'_{i+1}),g'_{i+1}}(2k + 2; G). \]
This completes the proof. 

**Lemma 3.4** [2] Let \( \pi \) be a leveled degree sequence of an edge-rooted tree and \( G = T_\pi \) be the associated edge-rooted greedy tree. For any element \( T \in T_\pi \), we have
\[ C_{v'_1,v'_2}(k; T) = C_{v'_2,v'_1}(k; T) \leq C_{g'_2,g'_1}(k; G) = C_{g'_1,g'_2}(k; G) \]
for any nonnegative integer \( k \), where \( v'_1 \), \( v'_2 \) and \( g'_1 \), \( g'_2 \) are the roots of \( T \) and \( G \), respectively.

**Lemma 3.5** Let \( F \in F_\pi \) for some leveled degree sequence \( \pi \) of an edge-rooted forest and \( G = F_\pi \) be the associated leveled greedy forest. Let \( v'_1, \ldots, v'_{i} \) and \( g'_1, \ldots, g'_i \) be the vertices of \( T \), \( G \) at the \( i \)-th level, respectively. Then the following relations hold for all \( i \):
\[ (C_{f(v'_1),v'_1}(2k; F), \ldots, C_{f(v'_i),v'_i}(2k; F)) \prec_w (C_{f(g'_1),g'_1}(2k; G), \ldots, C_{f(g'_i),g'_i}(2k; G)) \]
and
\[ C_{f(g'_i),g'_i}(2k; G) \geq C_{f(g'_1),g'_1}(2k; G) \geq \cdots \geq C_{f(g'_m),g'_m}(2k; G). \]

**Proof.** Induction on \( k \), if \( k = 1 \), then it is easy to find the assertion holds. Suppose that the assertion holds for the number not more than \( k (k \geq 1) \). Without loss of generality, we can suppose that \( C_{f(v'_1),v'_1}(2k + 2; F) \geq \cdots \geq C_{f(v'_i),v'_i}(2k + 2; F) \), otherwise, we can change the label of the vertex in \( T \). We divide the following two cases:

**Case 1:** \( i = 1 \), we need to prove
\[ (C_{f(v'_1),v'_1}(2k + 2; F), \ldots, C_{f(v'_i),v'_i}(2k + 2; F)) \prec_w (C_{f(g'_1),g'_1}(2k + 2; G), \ldots, C_{f(g'_i),g'_i}(2k + 2; G)) \]
and
\[ C_{f(g'_1),g'_1}(2k + 2; G) \geq C_{f(g'_2),g'_2}(2k + 2; G) \geq \cdots \geq C_{f(g'_m),g'_m}(2k + 2; G). \]
If $f(v_j^1) = v_j^0$, then
\[
C_{v_j^0, v_j^1}(2k + 2; T) = \sum_{t=1}^{k} \tilde{C}_{v_j^0, v_j^1}(2t; T) \cdot C_{v_j^0, v_j^1}(2k + 2 - 2t; T) + \tilde{C}_{v_j^0, v_j^1}(2k + 2; T).
\]

If $f(v_j^1) = v_j^0$, then
\[
C_{v_j^0, v_j^1}(2k + 2; T) = \sum_{t=1}^{k} \tilde{C}_{v_j^0, v_j^1}(2t; T) \cdot C_{v_j^0, v_j^1}(2k + 2 - 2t; T) + \tilde{C}_{v_j^0, v_j^1}(2k + 2; T).
\]

By the Lemmas 3.3, 3.4 and the induction hypothesis, we have
\[
\sum_{j=1}^{m} C_{f(v_j^1), v_j^1}(2k + 2; T)
\]
\[
= \sum_{j=1}^{m} \left[ \sum_{t=1}^{k} \tilde{C}_{f(v_j^1), v_j^1}(2t; T) \cdot C_{v_j^0, v_j^1}(2k + 2 - 2t; T) + \tilde{C}_{f(v_j^1), v_j^1}(2k + 2; T) \right]
\]
\[
\leq \sum_{j=1}^{m} \left[ \sum_{t=1}^{k} \tilde{C}_{f(g_j^1), g_j^1}(2t; G) \cdot C_{g_j^0, g_j^1}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j^1), g_j^1}(2k + 2; G) \right]
\]
\[
= \sum_{j=1}^{m} C_{f(g_j^1), g_j^1}(2k + 2; G)
\]

and
\[
C_{f(g_j^1), g_j^1}(2k + 2; G)
\]
\[
= \sum_{t=1}^{k} \tilde{C}_{f(g_j^1), g_j^1}(2t; G) \cdot C_{g_j^0, g_j^1}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j^1), g_j^1}(2k + 2; G)
\]
\[
\geq \sum_{t=1}^{k} \tilde{C}_{f(g_j^1), g_j^1}(2t; G) \cdot C_{g_j^0, g_j^1}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j^1), g_j^1}(2k + 2; G)
\]
\[
= C_{f(g_j^1), g_j^1}(2k + 2; G)
\]

Case 2: $i \geq 2$. Since
\[
C_{f(v_j^1), v_j^1}(2k + 2; F) = \sum_{t=1}^{k} \tilde{C}_{f(v_j^1), v_j^1}(2t; F) \cdot C_{f(v_j^1), f^2(v_j^1)}(2k + 2 - 2t; F) + \tilde{C}_{f(v_j^1), v_j^1}(2k + 2; F),
\]
by Lemmas 6.3 [3.4] and the induction hypothesis, we have

\[
\sum_{j=1}^{m} C_{f(v_j),v_j}(2k + 2; T) = \sum_{i=1}^{k} \sum_{j=1}^{m} \tilde{C}_{f(v_j),v_j}(2t; T)C_{f(v_j),v_j}(2k + 2 - 2t; T) + \sum_{j=1}^{m} \tilde{C}_{f(v_j),v_j}(2k + 2; T)
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{m} \tilde{C}_{f(g_j'),g_j'}(2t; G)C_{f(g_j'),g_j'}(2k + 2 - 2t; G) + \sum_{j=1}^{m} \tilde{C}_{f(g_j'),g_j'}(2k + 2; G)
\]

\[
= \sum_{j=1}^{m} \sum_{i=1}^{k} \tilde{C}_{f(g_j'),g_j'}(2t; G)C_{f(g_j'),g_j'}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j'),g_j'}(2k + 2; G)
\]

\[
= \sum_{j=1}^{m} C_{f(g_j'),g_j'}(2k + 2; G)
\]

and

\[
\tilde{C}_{f(g_j'),g_j'}(2k + 2; G)
\]

\[
= \sum_{i=1}^{k} \tilde{C}_{f(g_j'),g_j'}(2t; G)\tilde{C}_{f(g_j'),g_j'}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j'),g_j'}(2k + 2; G)
\]

\[
\geq \sum_{i=1}^{k} \tilde{C}_{f(g_j',g_j')}(2t; G)\tilde{C}_{f(g_j',g_j')}(2k + 2 - 2t; G) + \tilde{C}_{f(g_j',g_j')}(2k + 2; G)
\]

\[
= \tilde{C}_{f(g_j',g_j')}(2k + 2; G).
\]

This completes the proof. ■

**Lemma 3.6** Let \( T \in F_\pi \) for some leveled degree sequence \( \pi \) of an edge-rooted forest, and let \( G = F_\pi \) be the associated leveled greedy forest. Let \( v_1, \ldots, v_i \) and \( g_1, \ldots, g_i \) be the vertices of \( F \) and \( G \) at the \( i \)-th level, respectively. Then

\[
(C_{v_1}(2k; F), \ldots, C_{v_i}(2k; F)) < w (C_{g_1}(2k; G), \ldots, C_{g_i}(2k; G))
\]

and

\[
C_{g_i}(2k; G) \geq C_{g_2}(2k; G) \geq \cdots \geq C_{g_1}(2k; G).
\]

**Proof.** Induction on \( k \). If \( k = 1 \), then it is easy to find the assertion holds. Suppose that the assertion holds for the number not more than \( k \) (\( k \geq 1 \)). Without loss of generality, we can suppose that \( C_{v_1}(2k + 2; T) \geq \cdots \geq C_{v_i}(2k + 2; T) \), otherwise, we can change the label of the vertex in \( T \). We divide the following two cases:

**Case 1:** \( i = 0 \). Since

\[
C_{v_1}(2k + 2; T) = \sum_{t=1}^{k} \tilde{C}_{v_1}(2t; T)C_{v_1,v_1}(2k + 2 - 2t; T) + \tilde{C}_{v_1}(2k + 2; T) + C_{v_1,v_1}(2k + 2; T)
\]

\[
C_{v_2}(2k + 2; T) = \sum_{t=1}^{k} \tilde{C}_{v_2}(2t; T)C_{v_2,v_2}(2k + 2 - 2t; T) + \tilde{C}_{v_2}(2k + 2; T) + C_{v_2,v_1}(2k + 2; T),
\]

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by Lemmas 3.2, 3.5 and the induction hypothesis, we have

\[ C_{v_j}(2k + 2; T) \]
\[ = \sum_{t=1}^{k} \tilde{C}_{v_j}(2t; T)C_{v_j,v_j}^{(2k + 2 - 2t; T)} + \tilde{C}_{v_j}(2k + 2; T) + C_{v_j,v_j}^{(2k + 2; T)} \]
\[ \leq \sum_{t=1}^{k} \tilde{C}_{v_j}(2t; G)C_{v_j,v_j}^{(2k + 2 - 2t; G)} + \tilde{C}_{v_j}(2k + 2; G) + C_{v_j,v_j}^{(2k + 2; G)} \]
\[ = C_{g_j}(2k + 2; G) \]

\[
\text{and} \]
\[ \]
\[ C_{v_j}(2k + 2; T) + C_{v_j}^{(2k + 2; T)} \]
\[ = \sum_{j=1}^{k} \sum_{t=1}^{k} \tilde{C}_{v_j}(2t; T)C_{v_j,v_j}^{(2k + 2 - 2t; T)} + \sum_{j=1}^{k} \tilde{C}_{v_j}(2k + 2; T) + \sum_{j=1}^{k} C_{v_j,v_j}^{(2k + 2; T)} \]
\[ \leq \sum_{j=1}^{k} \sum_{t=1}^{k} \tilde{C}_{v_j}(2t; G)C_{v_j,v_j}^{(2k + 2 - 2t; G)} + \sum_{j=1}^{k} \tilde{C}_{v_j}(2k + 2; G) + \sum_{j=1}^{k} C_{v_j,v_j}^{(2k + 2; G)} \]
\[ = C_{g_j}(2k + 2; G) + C_{g_j}^{(2k + 2; G)}. \]

**Case 2:** \( i \geq 2 \). Since

\[ C_{v_j}(2k + 2; F) = C_{f(v_j),v_j}^{(2k + 2; F)} + \sum_{t=1}^{k} \tilde{C}_{v_j}(2t; F)C_{f(v_j),v_j}^{(2k + 2 - 2t; F)} + \tilde{C}_{v_j}(2k + 2; F), \]

by Lemmas 3.2, 3.5 and the induction hypothesis, we have

\[ \sum_{j=1}^{m} C_{v_j}(2k + 2; F) \]
\[ = \sum_{j=1}^{m} C_{f(v_j),v_j}^{(2k + 2; F)} + \sum_{t=1}^{k} \sum_{j=1}^{m} \tilde{C}_{v_j}(2t; F)C_{f(v_j),v_j}^{(2k + 2 - 2t; F)} + \sum_{j=1}^{m} \tilde{C}_{v_j}(2k + 2; F) \]
\[ \leq \sum_{j=1}^{m} C_{f(v_j),v_j}^{(2k + 2; G)} + \sum_{t=1}^{k} \sum_{j=1}^{m} \tilde{C}_{v_j}(2t; G)C_{f(v_j),v_j}^{(2k + 2 - 2t; G)} + \sum_{j=1}^{m} \tilde{C}_{v_j}(2k + 2; G) \]
\[ = \sum_{j=1}^{m} C_{g_j}(2k + 2; G) \]

and

\[ C_{g_j}(2k + 2; G) \]
\[ = \sum_{t=1}^{k} \tilde{C}_{g_j}(2t; G)C_{f(g_j),g_j}^{(2k + 2 - 2t; G)} + \tilde{C}_{g_j}(2k + 2; G) \]
\[ \geq \sum_{t=1}^{k} \tilde{C}_{g_j}(2t; G)C_{f(g_j+1),g_j+1}^{(2k + 2 - 2t; G)} + \tilde{C}_{g_j+1}(2k + 2; G) \]
\[ = C_{g_{j+1}}(2k + 2; G). \]
This completes the proof. □

**Theorem 3.7** Let $T \in T_\pi$ for the leveled degree sequence $\pi$, and let $G = T^*_\pi$ be the associated greedy tree. Then for any positive integer $k$,

$$C(2k; T) \prec_w C(2k; G)$$

Moreover, the majorization is strict for sufficiently large even $k$ if $T$ and $T^*_\pi$ are not isomorphic.

**Proof.** If possible to choose an edge or a vertex as root such that $T$ is not level greedy, then choose the edge or vertex as root to get $T_1$ being the level greedy tree with the same leveled degree sequence $\pi$. We iterate this process: if an edge or a vertex root can be chosen such that $T_1$ is not level greedy, choose the edge or vertex as root to get a level greedy tree, which we denote by $T_{i+1}$. Then, by Theorem 20 in [2], no infinite loops are possible in this process. By Lemma 3.2, Lemma 3.6 and Theorems 15, 19 in [2], we have

$$C(2k; T_i) \prec_w C(2k; T_{i+1})$$

Moreover, the Majorization is strict for sufficiently large $k$. Hence there exists an integer $m$ such that $T_m$ is level greedy with respect to any choice of vertex or edge root. This tree $T_m$ satisfies the semi-regular property defined in [14], and hence it is a greedy tree. This completes the proof. □

Since the number of closed walks in a tree with length odd is zero, by Theorem 3.7, Theorem 2.3 holds. Therefore, we finish the proof of Theorem 2.3.

### 4 The Proof of Theorem 2.4

In order to prove Theorem 2.4, we need the following Lemma.

**Lemma 4.1** Let $\alpha = (a_1,\ldots,a_n)$, $\beta = (b_1,\ldots,b_n)$, $V_1 \subseteq \{1,\ldots,n\}$ and $\varphi$ be a bijective map from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$ such that (1) $\varphi(V_1) \cap V_1 = \emptyset$. (2) $a_i \leq b_i$ for $i \notin V_1$; $a_i + a_{\varphi(i)} \leq b_i + b_{\varphi(i)}$ and $a_i \leq a_{\varphi(i)}$ for $i \in V_1$. Then $\alpha \prec_w \beta$.

**Proof.** Induction on $|V_1|$ which is the size of $V_1$, let $k = |V_1|$. If $k = 1$, then the assertion holds by considering that the sum of the first $l$ largest elements of $\alpha$ contains $a_{\varphi(i)}$, $i \in V_1$ or does not contain. Next suppose the assertion holds for $k > 2$, we will prove that the assertion holds for $k + 1$. Let $i_0 \in V_1$, and $\alpha' = (a'_1,\ldots,a'_n)^T$ where $a'_i = a_i$ for $i \neq i_0, \varphi(i_0)$ and $a'_i = b_i$ for $i = i_0, \varphi(i_0)$. By induction,

$$\alpha \prec_w \alpha' \prec_w \beta.$$ 

This completes the proof. □

Denoted by $C^c_u(k; G)$ be the set of the closed walks of length $k$ in $G$ starting at $u$ and going through $e$. The cardinality of $C^c_u(k; G)$ is denoted by $C^c_u(k; G)$.

**Theorem 4.2** Let $D$ be a leveled degree sequence of rooted tree and $e = xx_1 \in E(T^*_\pi)$, $B$ is a branch of the level greedy tree $G = T^*_\pi$ by deleting the edge $e$, which does not contain the root. Let $T = G - xx_1 + x'x_1$ where $x$, $x'$ are in the same level(see Fig.1), then $C(2k; G) \prec_w C(2k; T)$.
Next we will verify $F$.

Suppose $H$ same. Let $w$, and $w'$ are such that $h(x) = x'$, $h(e) = e'$, $h(w) = w$ and keeps the level.

Define

$$F : C_w(k, G) \rightarrow C_w(k, T).$$

Let $W = w_1w_2 \cdots w_{k+1} \in C_w(k, G)$ and $m, M$ be the minimal and maximal index such that $w_m = w_M = w, 1 < m \leq M < k + 1$, if there exist such integers. Then

- If $w \notin \{w_2, \cdots, w_k\}$ and $w, w_{s+1} \neq e, s = 2, 3, \cdots, k$, then $H(W) = W$.
- If $w \notin \{w_2, \cdots, w_k\}$ and $w, w_{s+1} = e$, for some $s \in \{2, 3, \cdots, k\}$, then $H(W) = h(w_1)h(w_2)h(w_3) \cdots h(w_k)h(w_{k+1})$.
- Otherwise, then $H(W) = \phi(w_1 \cdots w_{k-1})H(w_m \cdots w_M)\phi(w_{M+1} \cdots w_{k+1})$, where $\phi(w_1 \cdots w_{k-1}) = h(w_1)h(w_2) \cdots h(w_{k-1})$ and $w, w_{s+1} = e$, for some $s \in \{1, 2, \cdots, s - 2\}$ and $\phi(w_1 \cdots w_{k-1}) = w_1w_2 \cdots w_{k-1}$ otherwise.

That is, break a walk into pieces divided by visiting the vertex $w$, each pieces is either kept the same or replaced by its image under the injective $h$ depending on whether it contains $e$. By the uniqueness of the decomposition of the walks and $h$ is injective, then $H$ is also a injective map. By Lemma 11 It is sufficient to prove the following two cases:

**Case 1:** If $w' \notin V(T) - V(T_2)$, then $C_{w'}(k; T) \geq C_{w'}(k; G)$.

It is sufficient if there exists an injective map form $C_{w'}(k; G)$ to $C_{w'}(k; T)$. Suppose $W = w_1w_2 \cdots w_{k+1} \in C_w(k; G)$, and $m, M$ defined as before. Define

$$F_1(W) = \phi(w_1 \cdots w_{m-1})H(w_m \cdots w_M)\phi(w_{M+1} \cdots w_{k+1}).$$

Next we will verify $F_1$ is injective.

Suppose $F_1(W_1) = F_1(W_2), W_1, W_2 \in C_w(k; G)$, then the positions of $w$ in $W_1$ and $W_2$ are same. Let $W_i = w_1^i \cdots w_{k+1}^i, i = 1, 2$, then $\phi(w_1^I \cdots w_{k+1}^1) = \phi(w_1^2 \cdots w_{k+1}^2), H(w_m^1 \cdots w_M^1) = H(w_m^2 \cdots w_M^2), \phi(w_{M+1}^1 \cdots w_{k+1}^1) = \phi(w_{M+1}^2 \cdots w_{k+1}^2)$. This implies that $W_1 = W_2$. 

**Fig.1**

![Diagram showing the decomposition of walks and trees](image)
Case 2: If \( w' \in V(T_1) \), then \( C_{w'}(k;T) + C_{h(w')}h(k;T) \geq C_{w'}(k;G) + C_{h(w')}h(k;G) \).

For simplicity, let \( u = h(w') \), \( v = w' \), since \( T - B = G - B \) and

\[
\begin{align*}
C_u(k;T) &= C_u^e(k;T) + (C_u(k;T) - C_u^e(k;T)), \\
C_u(k;G) &= C_u^e(k;G) + (C_u(k;G) - C_u^e(k;G)), \\
C_v(k;T) &= C_v^e(k;T) + (C_v(k;T) - C_v^e(k;T)), \\
C_v(k;G) &= C_v^e(k;G) + (C_v(k;G) - C_v^e(k;G)),
\end{align*}
\]

then

\[
\begin{align*}
C_u(k;T) - C_u(k;G) &= C_u^e(k;T) - C_u^e(k;G), \\
C_v(k;T) - C_v(k;G) &= C_v^e(k;T) - C_v^e(k;G).
\end{align*}
\]

Thus \( C_u(k;T) + C_v(k;T) \geq C_u(k;G) + C_v(k;G) \) holds if and only if \( C_u^e(k;T) + C_v^e(k;T) \geq C_u^e(k;G) + C_v^e(k;G) \). So it is sufficient if there exists an injective map from \( C_u^e(k;G) \cup C_v^e(k;G) \) to \( C_u^e(k;T) \cup C_v^e(k;T) \). Define the following injective map:

\[
F_2 : C_u^e(k;G) \cup C_v^e(k;G) \rightarrow C_u^e(k;T) \cup C_v^e(k;T).
\]

Let \( W = \tilde{w}W_1wW_2wW_3\tilde{w}, w \notin V(W_1), w \notin V(W_3) \), suppose that the following closed walk has the same form.

Subcase 2.1: If \( W = uW_1wW_2wW_3u \in C_u^e(k;G) \), define \( F_2(W) = uW_1H(wW_2w)W_3u \).

Subcase 2.2: If \( W = vW_1wW_2wW_3v \in C_v^e(k;G) \), divide it into the followings:

- If \( e \notin E(W_1) \cup E(W_3) \), then \( F_2(W) = vW_1H(wW_2w)W_3v \).
- If \( e \in E(W_1), e \notin E(W_3) \), then \( F_2(W) = uh(W_1)H(wW_2w)h(W_3)u. \)
- If \( e \notin E(W_1), e \in E(W_3) \), then \( F_2(W) = uh(W_1)H(wW_2w)h(W_3)u. \)
- If \( e \in E(W_1), e \in E(W_3) \), then \( F_2(W) = uh(W_1)H(wW_2w)h(W_3)u. \)

Next we verify that \( F_2 \) is injective. If \( W, \tilde{W} \) in the same case, then \( F_2(W) = F_2(\tilde{W}) \) implies that \( W = \tilde{W} \), since \( H \) is injective. If \( W \in C_u^e(k;G), \tilde{W} \in C_v^e(k;G) \). Then \( F_2(W) = F_2(\tilde{W}) \), since they do not have the same initial vertex or \( W_1 \neq h(\tilde{W}_1) \) or \( W_3 \neq h(\tilde{W}_3) \) by \( e' \notin E(W_1) \cup E(W_3) \) and \( e' \in E(h(\tilde{W}_1)) \cup E(h(\tilde{W}_3)) \).

**Theorem 4.3** Let \( D \) be a leveled degree sequence of edge rooted tree and \( e = xx_1 \in E(T_n^*) \), \( B \) is a branch of the level greedy tree \( G = T_n^* \) by deleting the edge \( e \), which does not contain the root. Let \( T = G - xx_1 + x'x \), where \( x, x' \) are in the same level (see Fig.2), then \( C(k;G) \leq_C C(k;T) \).
Fig. 2

**Proof.** Let $G_1, G_2$ (respectively, $T_1, T_2$) be the two components of $G - yy'$ (respectively, $T - yy'$), which contain $x, x'$, respectively. Then we define a isomorphism $h$ from $G_1$ to $T_2$, such that $y' = h(y), e' = h(e)$ and keeps the level. Then we define the following injective map:

$$F : C_r(G) \rightarrow C_r(T)$$

Let $W = w_1w_2w_1w_2 \cdots w_1w_2w_{2k+1}w_2w_{2k+2}w_1$, where $w_1w_2 = yy'$ or $y'y, w_1w_2, w_2w_1 \notin \cup_{i=1}^{2k+2} E(W_i)$ and $w_1 \notin \cup_{i=0}^{k} W_{2i+1}$. If $W_i$ is a empty set, then denote $w_iw = w$. Define

- If $w_1w_2 = y'y$, then $H(W) = \phi(y'yW_1yy'W_2) \cdots \phi(y'yW_{2k+1}yy'W_{2k+2})y'$. Where $\phi(y'yW_1yy'W_2) = y'yW_1yy'W_2$ if $e \notin W_1$, $\phi(y'yW_1yy'W_2) = y'yh(W_1)y/W_2$ otherwise.

- If $w_1w_2 = yy'$, then $H(W) = yy'W_1H(y'yW_{2}yy' \cdots y'yW_{2k+1}yy')\phi(W_{2k+1}yy'W_{2k+2})y, \phi(W_{2k+1}yy'W_{2k+2})y = W_{2k+1}yy'h(W_{2k+2})y'$ otherwise.

In words, break a walk into pieces divided by edges $yy', y'y$, each piece is kept the same or replaced by its image under the injective map $h$ if it contains $e$. Since the decomposition of the walks is unique and $h$ is injective, so $H$ is also injective. By Lemma 4.1, it is sufficient to prove the following.

**Case 1:** If $w' \notin V(T) - V(T_1)$, then $C_{w'}(k; T) \geq C_{w'}(k; G)$.

If $w' \notin V(T) - V(G_1)$, then it is sufficient if there exists an injective map $F_1$ from $C_{w'}(k; T)$ to $C_{w'}(k; G)$. Let $W = w_1w_2 \cdots w_{k+1}$ and $m'$ (respectively, $M'$) be the smallest (respectively, largest) integer such that $w_mw_{m'+1} = y'y$ (respectively, $w_{M'-1}w_{M'} = yy'$). Define

$$F_1(W) = \phi(w_1w_2 \cdots w_{m-1})H(w_mw_{m+1} \cdots w_M)\phi(w_{M+1} \cdots w_{k+1}).$$

Since $H$ is injective, then $F_1$ is also injective.

If $w' \notin V(B)$. Let $m'$ (respectively, $M'$) be the smallest (respectively, largest) integer such that $w_mw_{m'+1} = yy'$ (respectively, $w_{M'-1}w_{M'} = yy'$). Define

$$F_1(W) = \phi(w_1w_2 \cdots w_{m'-1})H(w_mw_{m+1} \cdots w_{M'})\phi(w_{M'+1} \cdots w_{k+1}).$$

Since $H$ is injective, then $F_1$ is also injective.

**Case 2:** If $w' \in V(T_1)$, then $C_{w'}(k; T) + C_{h(w')}(k; T) \geq C_{w'}(k; G) + C_{h(w')}(k; G)$.

For simplicity, let $u = h(w'), v = w'$, since $T - B = G - B$ and

$$\begin{align*}
C_u(k; T) &= C_u^v(k; T) + (C_u(k; T) - C_u^v(k; T)), \\
C_u(k; G) &= C_u^v(k; G) + (C_u(k; G) - C_u^v(k; G)), \\
C_v(k; T) &= C_v^u(k; T) + (C_v(k; T) - C_v^u(k; T)), \\
C_v(k; G) &= C_v^u(k; G) + (C_v(k; G) - C_v^u(k; G)),
\end{align*}$$

then

$$\begin{align*}
C_u(k; T) - C_u(k; G) &= C_u^v(k; T) - C_u^v(k; G), \\
C_v(k; T) - C_v(k; G) &= C_v^u(k; T) - C_v^u(k; G).
\end{align*}$$
Thus \( C_u(k; T) + C_v(k; T) \geq C_u(k; G) + C_v(k; G) \) holds if and only if \( C'_u(k; T) + C'_v(k; T) \geq C'_u(k; G) + \mathcal{C}'_v(k; G) \). So it is sufficient if there exists an injective map \( \phi : \mathcal{C}'_u(k; G) \cup \mathcal{C}'_v(k; G) \to \mathcal{C}'_u(k; T) \cup \mathcal{C}'_v(k; T) \). Define the following injective map:

\[
F_2 : \mathcal{C}'_u(k; G) \cup \mathcal{C}'_v(k; G) \rightarrow \mathcal{C}'_u(k; T) \cup \mathcal{C}'_v(k; T).
\]

Let \( W = wW_1y'yW_2yy'W_3w, y'y \notin E(W_1), yy' \notin E(W_3) \), suppose that the following closed walk has the same form.

**Subcase 2.1:** If \( W = uW_1y'yW_2yy'W_3u \in C'_u(k; G) \), then define \( F_2(W) = uW_1H(y'yW_2yy')W_3u \).

**Subcase 2.2:** If \( W = vW_1y'yW_2yy'W_3v \in C'_v(k; G) \), then divide it into the followings:

- If \( e \notin E(W_1) \cup E(W_3) \), then \( F_2(W) = vW_1H(y'yW_2yy')W_3v \).
- If \( e \in E(W_1), e \notin E(W_3) \), then \( F_2(W) = uh(W_1)y'yW_21H(y'yW_22yy')h(W_3)u \).
- If \( e \notin E(W_1), e \in E(W_3) \), then \( F_2(W) = uh(W_1)y'yW_21H(y'yW_22yy')h(W_3)u \).
- If \( e \in E(W_1), e \in E(W_3) \), then \( F_2(W) = uh(W_1)y'yW_21H(y'yW_22yy')h(W_3)u \).

where \( y'yW_2 = y'yW_22y'yW_222, y'y \notin W_22 \). Next we verify that \( F_2 \) is injective. If \( W, \tilde{W} \) in the same case, then \( F_2(W) = F_2(\tilde{W}) \) implies that \( W = \tilde{W} \), since \( H \) is injective. If \( W \in C'_u(k; G), \tilde{W} \in C'_v(k; G) \).

Then \( F_2(W) \neq F_2(\tilde{W}) \), since they do not have the same initial vertex or \( W_1 \neq h(\tilde{W}_1) \) or \( W_3 \neq h(\tilde{W}_3) \) by \( e' \notin E(W_1) \cup E(W_3) \) and \( e' \in E(h(W_1)) \cup E(h(\tilde{W}_3)) \). 

Now we are ready to prove Theorem 4.4.

**Theorem 4.4** Let \( \pi = (d_0, d_1, ..., d_{n-1}) \) and \( \pi' = (d'_0, d'_1, ..., d'_{n-1}) \) be decreasing degree sequences of trees of the same order such that \( \pi \preceq \pi' \). Then for any integer \( k \geq 1 \) we have

\[
C(2k; T^*_\pi) \prec_w C(2k; T^*_\pi').
\]

If \( \pi \neq \pi' \) and \( k \geq 1 \), then the majorization is strict.

**Proof.** If \( \pi = \pi' \), the assertion holds. Next suppose \( \pi \neq \pi' \), then there exists an integer \( i \) such that \( l_i \neq l'_i \). Set \( \{ i : l_i \neq l'_i \} \), by \( \sum_{i=0}^{n-1} l_i = \sum_{i=0}^{n-1} l'_i \), we will find that \( \{ i : l_i \neq l'_i \} \) has at least two elements.

Let \( l = \min \{ i : l_i \neq l'_i \} \), \( L = \max \{ i : l_i \neq l'_i \} \). Then \( d_l < d'_l, d_L > d'_L \). Let

\[
\pi_1 = (d_0, d_1, ..., d_l, d_l + 1, ..., d_L, ..., d_{n-1}).
\]

We will find that \( \pi_1 \) is decreasing and \( \pi \prec \pi_1 \preceq \pi' \). Next consider the two vertices \( u \) and \( v \) in \( T_\pi \) such that \( d(u) = d_l, d(v) = d_L \), then divide the following two cases:

**Case 1:** If the distance between \( u \) and \( v \) is even. Let \( w \) be the middle vertex in the path from \( u \) to \( v \), consider \( T_\pi \) as a rooted tree with root \( w \). Then \( u, v \) are in the same level, let \( v' \) be a children of \( v \), Consider the tree \( T = T^*_\pi - uv' + uv' \). By Lemma 3.2 and Lemma 4.2 we have

\[
C(2k; T^*_\pi) \prec_w C(2k; T^*_\pi). \]

**Case 2:** If the distance between \( u \) and \( v \) is odd. Let \( yy' \) be the middle in the path from \( u \) to \( v \), consider \( T_\pi \) as a edge-rooted tree with edge root \( yy' \). Then \( u, v \) are in the same level, let \( v' \) be a children of \( v \), Consider the tree \( T = T^*_\pi - uv' + uv' \). By Lemma 3.2 and Lemma 4.3 we have

\[
C(2k; T^*_\pi) \prec_w C(2k; T^*_\pi). \]
By the two cases, we find that $C(2k; T^*_\pi^0) \triangleleft_w C(2k; T^*_\pi^1)$. By repeating the above process we can get

$$\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_m = \pi'$$

such that

$$C(2k; T^*_\pi) = C(2k; T^*_\pi^0) \triangleleft_w C(2k; T^*_\pi^1) \triangleleft_w \cdots \triangleleft_w C(2k; T^*_\pi^m) = C(2k; T^*_\pi')$$

This completes the proof. ■

**Corollary 4.5** For any $n$ vertex tree. Then

$$C(2k; T) \triangleleft_w C(2k; S_n),$$

for any positive integer $k$, where $S_n$ is a star of order $n$.

**Corollary 4.6** For any $n$ vertex tree $T$ with maximal degree is $\Delta$. Then

$$C(2k; T) \triangleleft_w C(2k; T^*_\pi),$$

where $\pi = (\Delta, ..., \Delta, r, 1, ..., 1)$, $1 \leq r < \Delta$, the sum of the elements of $\pi$ is $2n - 2$.

**Corollary 4.7** For any tree $T$ of order $n$ with $s$ leaves. Then

$$C(2k; T) \triangleleft_w C(2k; T^*_\pi),$$

for any positive integer $k$, where $\pi = (s, 2, 2, ..., 2, 1, 1, ..., 1)$ (the number of 2 is $n - s - 1$, the number of 1 is $s$).

**Corollary 4.8** For any tree $T$ of order $n$ with independence number $\alpha \geq n/2$ or with matching number $n - \alpha \leq n/2$. Then

$$C(2k; T) \triangleleft_w C(2k; T^*_\pi),$$

for any positive integer $k$, where $\pi = (\alpha, 2, 2, ..., 2, 1, 1, ..., 1)$ (the number of 2 is $n - \alpha - 1$, the number of 1 is $\alpha$).

For a given tree degree sequence $\pi$, we determined the maximum value of the number of the closed walks of length $k$ starting at any vertex $v \in U \subseteq V(T)$ in any tree $T = (V(T), E(T))$ with $|U| = r$. It is interesting to determine the minimum value of them.

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