We give a short review of the spin foam models of quantum gravity, with an emphasis on the Barret-Crane model. After explaining the shortcomings of the Barret-Crane model, we briefly discuss two new approaches, one based on the 3d spin foam state sum invariants for the embedded spin networks, and the other based on representing the string scattering amplitudes as 2d spin foam state sum invariants.

1. Introduction

The spin foam models originate from the Ponzano-Regge model of 3d Euclidian quantum gravity [1]. The idea there was to use the simplicial complex, i.e. the spacetime triangulation, whose triangles had integer lengths, which were proportional to the spins of the SU(2) group. Then the 3d gravity path integral (PI) was defined as a sum over the spins of the products of the $6j$ symbols which were associated to the tetrahedrons of the simplicial complex. A length cut-off was introduced in order to regularize the path integral. Since 3d gravity is a topological theory, the corresponding path integral would be a topological invariant of the 3d manifold. However, the Ponzano-Regge path integral was not a topological invariant, because the topological invariance required the quantum SU(2) group at a root of unity, which was discovered by Turaev and Viro [2].

Still, the idea of the Ponzano-Regge model was useful, because the model can be understood as a path integral for the SU(2) BF theory [3]. This then inspired Ooguri to consider a 4d version of the model, as a PI for the 4d SU(2) BF theory [4]. In this case the areas of the triangles are integer valued, i.e. proportional to the spins. The PI was formally topologically invariant, but divergent. A well-defined topological invariant was obtained by Crane, Yetter and Kauffman [5], who replaced the SU(2) group by the quantum SU(2) group at a root of unity, so that the SU(2) spins become bounded by a maximal spin.

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However, the corresponding invariant was not new, contrary to the 3d case, since it gave a signature of the 4-manifold.

At the same time, Baez proposed the idea of the spin foams [6], as a way of understanding the results of loop quantum gravity [7], from a spacetime perspective, i.e. a spin foam is a time evolution surface of a spin network, so that the spins are naturally associated to the faces of the spin foam. Given that the Einstein-Hilbert (EH) action of GR can be understood as a constrained BF theory, this then prompted Crane and Barrett to look for a constrained version of the CYK topological spin foam state sum [8, 9].

2. The Barret-Crane model

The EH action can be written as the $SO(3,1)$ BF theory action

$$\int_M Tr(B \wedge F) \ ,$$

where $F_{ab} = d\phi_{ab} + \phi_c^a \wedge \phi_e^b$ is the curvature two-form for the spin connection $\phi$, and the two-form $B$-field is constrained by

$$B_{ab} = \epsilon_{abcd} e^c \wedge e^d \ ,$$

where the $e$’s are the tetrad one-forms. The BF theory path integral can be written as

$$Z = \int DA DB \exp \left( i \int_M Tr(B \wedge F) \right)$$

$$= \prod_l dA_l \prod_{\Delta} dB_{\Delta} \exp \left( i \sum_f Tr(B_{\Delta} F_f) \right) \ ,$$

where $l$ and $f$ are the edges and the faces of the dual two-complex $F$ for the simplicial complex $T(M)$, while $\Delta$ are the triangles of $T$. The variables $A_l$ and $B_{\Delta}$ are defined as $\int_l A$ and $\int_{\Delta} B$ respectively, while $F_f = \int_f F$.

By performing the $B$ integrations one obtains

$$Z = \int \prod_l dA_l \prod_f \delta(F_f) \ ,$$

which can be defined as

$$Z = \int \prod_l dg_l \prod_f \delta(g_f) \ ,$$

where $g_f = \prod_{l \in \partial f} g_l$. By using the well-known identity

$$\delta(g) = \sum_{\Lambda} \dim \Lambda \chi_{\Lambda}(g) \ ,$$

where $\chi_{\Lambda}(g) = \sum_{\Lambda} \dim \Lambda \chi_{\Lambda}(g)$.  

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where \( \Lambda \)'s are the irreducible representations (irreps) of the group and \( \chi \)'s are the characters, one obtains

\[
Z = \sum_{\Lambda_f, i} \prod_f \dim \Lambda_f \prod_v A_v(\Lambda_f, i_l) ,
\]

where \( A_v \) is the vertex amplitude associated to the 4-simplex dual to the vertex \( v \). This amplitude is given by the evaluation of the corresponding 4-simplex spin network, known as the 15\( j \) symbol. The sum (7) is called a spin foam state sum, because it is a sum of the amplitudes for the colored two complex \( F \), i.e. a spin foam.

One can now conjecture that exists a quantization procedure such that the quantities \( B_\Delta \) become the 4d rotations algebra operators \( J_\Delta \), since the 4d rotation group irreps are labelling the triangles \( \Delta \), or the dual faces \( f \). Then one can show that the constraint (2) becomes a constraint on the representations labelling the triangles \( \Delta \), given by

\[
\epsilon^{abcd} J_{ab} J_{cd} = 0 \tag{8}
\]

[8, 9]. In the Euclidian case the irreps are given by the pairs of the \( SU(2) \) spins \((j, j')\), so that the constraint (8) implies \( j = j' \). In the Minkowski case, requiring the hermiticity of the \( B \) operators implies that one needs the unitary irreps of the Lorentz group. These are infinite-dimensional irreps and they are given by the pairs \((j, p)\) where \( j \) is the \( SU(2) \) spin and \( p \) is a continuous label. The constraint (8) implies that \( \Lambda = (0, p) \) or \( \Lambda = (j, 0) \).

One can argue that the spacelike triangles should be labelled by the \((0, p)\) irreps, while the time-like triangles should be labelled by the \((j, 0)\) irreps. Since a spacetime triangulation can be built from the spacelike triangles, Barrett and Crane have proposed the following spin foam state sum (integral) for the quantum general relativity [9]

\[
Z_{BC} = \int \prod_f p_f dp_f \prod_v \tilde{A}_v(p_f) , \tag{9}
\]

where \( \tilde{A}_v \) is an amplitude for the corresponding 4-simplex spin network, given by

\[
\tilde{A}(p_1, \ldots, p_{10}) = \int_{H^5} \prod_{i=1}^5 dx_i \delta(x_1 - x_0) \prod_{i<j} K_{p_{ij}}(x_i, x_j) . \tag{10}
\]

This is as an integral over the fifth power of the hyperboloid \( H = SO(3, 1)/SO(3) \) of a propagator \( K_p(x, y) \) on that space. The propagator is given by

\[
K_p(x, y) = \frac{\sin (pd(x, y))}{p \sinh d(x, y)} , \quad \cosh d(x, y) = x \cdot y . \tag{11}
\]

The expression (9) is not finite for all triangulations, but after a slight modification, consisting of including a non-trivial edge amplitude \( A(p_1, \ldots, p_4) \), the
partition function becomes finite for all non-degenerate triangulations [10]. This was a remarkable result, because it gave a perturbatively finite\(^2\) quantum theory of gravity, which was not based on string theory.

The main difficulties with the BC type models are:

1) It is difficult to see what is the semi-classical limit, i.e. what is the corresponding effective action, and is it given by the EH action plus the \(O(l_P)\) corrections, where \(l_P\) is the Planck length.

2) Coupling of matter: since matter couples to the gravitational field through the tetrads, one would need a formulation where a basic field is a tetrad and not the \(B\) 2-form. In the case of the YM field, the coupling can be expressed in terms of the \(B\) field [11], so that one can formulate a BC type models [12, 13]. However, for the fermions this is not possible, and a tetrad based formulation is necessary. In [11] an algebraic approach was proposed in order to avoid this problem, and the idea was to use a result from the loop quantum gravity, according to which the fermions appear as free ends of the spin networks. Hence including open spin networks gives a new type of spin foams [14], and this opens a possibility of including matter in the spin foam formalism. However, what is the precise form of the matter spin foam amplitudes remains an open question.

3. New directions

Given the difficulties of the BC model, we have proposed two new directions how to use the spin foam state sum formalism in order to arrive at a desirable quantum theory of gravity.

In [15] it was proposed to use the 3d spin foam state sum invariants in order to define the relevant quantities in the loop quantum gravity formalism. The idea is to use the representation of a quantum gravity state \(|\Psi\rangle\) in the spin network basis

\[
|\Psi\rangle = \sum_\gamma |\gamma\rangle \langle\gamma|\Psi\rangle.
\] (12)

The expansion coefficients are then invariants of the embedded spin networks in the spatial manifold \(\Sigma\), and can be formally expressed as

\[
\langle\gamma|\Psi\rangle = \int DA \langle\gamma|A\rangle \langle A|\Psi\rangle = \int DA W_\gamma[A] \Psi[A],
\] (13)

where \(A\) is a 3d complex \(SU(2)\) connection, \(W_\gamma[A]\) is the spin network wave-functional (generalization of the Wilson loop functional) and \(\Psi[A]\) is a holomorphic wave-functional satisfying the quantum gravity constraints in the Ashtekar representation.

\(^2\)\(Z_{BC}\) depends on a triangulation, in accordance with the fact that 4d gravity is non-topological, and hence one should also sum over the triangulations in order to obtain a well-defined quantity. How to do this it is not clear at present, so that one can obtain only the perturbative results.
In the case of non-zero cosmological constant $\lambda$, a non-trivial solution is known, i.e. the Kodama wavefunction

$$\Psi[A] = e^{\frac{i}{2} \int_{\Sigma} Tr (\wedge dA + \frac{\lambda}{2} A \wedge A \wedge A)} ,$$

while in the $\lambda = 0$ case a class of formal solutions is given by

$$\Psi[A] = \prod_{x \in \Sigma} \delta(F_x) \Psi_0[A] ,$$

i.e. a flat-connection wavefunction [15]. In the $\lambda = 0$ case one can show that the corresponding spin network invariant is given by a 3d spin foam state sum for the quantum $SU(2)$ at a root of unity [15].

In the $\lambda \neq 0$ case, it is conjectured that the corresponding spin network invariant is given in the Euclidean gravity case by the Witten-Reshetikhin-Turaev invariant for $q = e^{2\pi i/(k+2)}$, where $k \in \mathbb{N}$ and $\lambda = k/l^2$, while in the Minkowski case, the invariant is given by an analytical continuation of the Euclidean one, as $k \to ik$ [17].

In [16] it was proposed to use the 2d spin foam state sums in order to define a string theory as a quantum theory of gravity. The main idea is to use the string theory formal expression for the scattering amplitude of $n$ gravitons (or any other massless string modes), given as

$$A(p_1, \ldots, p_n) = \int_\Sigma d\sigma_1 \cdots \int_\Sigma d\sigma_n \langle V_{p_1}(\sigma_1) \cdots V_{p_n}(\sigma_n) \rangle ,$$

where

$$\langle V_{p_1}(\sigma_1) \cdots V_{p_n}(\sigma_n) \rangle = \sum_{2d \text{metrics}} \int DX e^{i \int_{\Sigma} (\partial X)^2 V_{p_1}(\sigma_1) \cdots V_{p_n}(\sigma_n) ,}$$

and argue that (16) should represent a 2d BF theory invariant for the $\theta_n$ spin network embedded in the string world-sheet manifold $\Sigma$. The BF theory group is given by the group of isometries of the spacetime background metric, and in [16] a simple possibility for the amplitude was considered

$$A(p_1, \ldots, p_n) = \int DA DB e^{i \int_{\Sigma} Tr (B \wedge F) W_{\theta_n}[A] ,}$$

where the isometry group was taken to be $SU(2)$. Then the labels $p_1, \ldots, p_n$ become the $SU(2)$ spins, and there is a maximal spin, because the PI (18) becomes a state sum for the quantum $SU(2)$ at a root of unity. Because the $BF$ theory is a topological theory, one can expect that the amplitude (18) will correspond to a topological string theory.

References

[1] G. Ponzano and T. Regge, in Spectroscopy and Group Theoretical Methods in Physics, ed F. Block et al, North-Holland, Amsterdam (1968)
[2] V.G. Turaev and O.Y. Viro, *Topology* **31**, 865 (1992).

[3] D. V. Boulatov, *Mod. Phys. Lett.* **A7**, 1629 (1992).

[4] H. Ooguri, *Mod. Phys. Lett.* **A7**, 2799 (1992).

[5] L. Crane, D. N. Yetter and D. H. Kauffman, *Knot Theor. Ramif.* **6**, 177 (1997).

[6] J. Baez, *Lect. Notes Phys.* **543**, 25 (2000).

[7] C. Rovelli, *Living Rev. Rel.* **1**, 1 (1995).

[8] J. W. Barrett and L. Crane, *J. Math. Phys.* **39**, 3296 (1998).

[9] J. W. Barrett and L. Crane, *Class. Quant. Grav.* **17**, 3101 (2000).

[10] L. Crane, A. Perez and C. Rovelli, *Phys. Rev. Lett.* **87**, 181301 (2001).

[11] A. Miković, *Class. Quant. Grav.* **19**, 2335 (2003).

[12] A. Miković, *Class. Quant. Grav.* **20**, 239 (2003).

[13] D. Oriti and H. Pfeiffer, *Phys. Rev.* **D66**, 124010 (2002).

[14] A. Miković, *Int. J. Mod. Phys.* **A18S2**, 83 (2003).

[15] A. Miković, *Class. Quant. Grav.* **20**, 3483 (2003).

[16] A. Miković, *String Theory and Quantum Spin Networks*, preprint hep-th/0307141.

[17] L. Smolin, *Quantum Gravity with a Positive Cosmological Constant*, preprint hep-th/0209079.