Symmetric entanglement classes for \( n \) qubits

Martin Aulbach

Department of Physics, University of Oxford, Clarendon Laboratory, Oxford OX1 3PU, United Kingdom

The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom

Permutation-symmetric \( n \) qubit pure states can be represented by \( n \) points on the surface of the unit sphere by means of the Majorana representation. Here this representation is employed to characterize and compare the three entanglement classification schemes LOCC, SLOCC and the Degeneracy Configuration. Symmetric SLOCC operations are found to be described by Möbius transformations, and an intuitive visualization of their freedoms is presented. For symmetric states of up to 5 qubits explicit forms of representative states for all SLOCC classes are derived. The symmetric 4 qubit entanglement classes are compared to the entanglement families introduced in [PRA 65, 052112 (2002)], and examples are given how the SLOCC-inequivalence of symmetric states can be quickly determined from known results about Möbius transformations.

PACS numbers: 03.67.Mn, 03.65.Ud, 02.40.Tt, 02.40.Dr

I. INTRODUCTION

Multpartite entanglement is an essential resource in quantum information science, and therefore it is desirable to categorize the states of a given Hilbert space into groups of states with similar entanglement. The object of interest in this paper are permutation-symmetric states. These kind of states have prominently featured in several recent works, such as the characterization of SLOCC entanglement classes [1–3], the determination of maximal entanglement in terms of the geometric measure [4–6], entanglement classes [1–3], the determination of maximal entanglement of the same model [12].

The central tool in all of these studies was the Majorana representation [13], a generalization of the Bloch sphere representation which allows symmetric \( n \) qubit states to be uniquely represented by \( n \) undistinguishable points on the sphere. Here this paradigm is employed to discuss three different entanglement classification schemes, namely, LOCC, SLOCC and the recently introduced Degeneracy Configuration [2], for symmetric \( n \) qubit states. It is seen that symmetric SLOCC operations can be described by the Möbius transformations of complex analysis, a result that is not only of theoretical interest but also of practical value, e.g., to determine whether two symmetric states belong to the same SLOCC class. Intriguingly, SLOCC operations can be uniquely decomposed into affine Möbius transformations and LOCC operations, thus allowing for a straightforward visualization of the innate SLOCC freedoms. A study of all symmetric SLOCC and DC classes for up to 5 qubits will yield the analytical form of representative states for each class. For the 4 qubit case the results are put into relation to the concept of entanglement families introduced in [14].

II. MAJORANA REPRESENTATION

Permutation-symmetric quantum states are defined as being invariant under any permutation of their subsystems. For an \( n \)-partite state \(|\psi\rangle\) this is the case iff \(P|\psi\rangle=|\psi\rangle\) for all \( P \in S_n \), where \( S_n \) is the symmetric group of \( n \) elements. For \( n \) qubits the symmetric sector of the Hilbert space is spanned by the \( n+1 \) Dicke states \(|S_k\rangle\), \( 0 \leq k \leq n \), the equally weighted sums of all permutations of computational basis states with \( n-k \) qubits being \(|0\rangle\) and \( k \) being \(|1\rangle\):

\[
|S_k\rangle = \binom{n}{k}^{-1/2} \sum_{\text{perm}} |0\rangle|0\rangle\cdots|1\rangle|1\rangle\cdots|1\rangle .
\]

By means of the Majorana representation any permutation-symmetric state \(|\psi^\sigma\rangle\) of \( n \) spin-\( \frac{1}{2} \) particles can be uniquely represented, up to an unphysical global phase, by a multiset of \( n \) points on \( S^2 \), with an isomorphism mediating between the pure states of the symmetric subspace and the set of \( n \) unit vectors in \( \mathbb{R}^2 \). Mathematically, this is expressed as

\[
|\psi^\sigma\rangle = \frac{e^{i\delta}}{\sqrt{K}} \sum_{\text{perm}} |\phi_{P(1)}\rangle|\phi_{P(2)}\rangle\cdots|\phi_{P(n)}\rangle ,
\]

where \( e^{i\delta} \) is a global phase, \( K \) the normalization factor, and the sum runs over all permutations of \( n \) single qubit states \(|\phi_i\rangle = \cos \frac{\delta_i}{2} |0\rangle + e^{i\phi_i} \sin \frac{\delta_i}{2} |1\rangle \). Thus the multiqubit state \(|\psi^\sigma\rangle\) can be visualized by \( n \) Bloch vectors \(|\phi_i\rangle\) on the surface of a sphere. These points are called the Majorana points (MP), and the sphere is called the Majorana sphere. See, for example, [14, 15] for some examples of Majorana representations.

By means of a stereographic projection the MPs can be projected from the sphere onto the complex plane, where they coincide with the roots of the Majorana polynomial

\[
\psi(z) = \sum_{k=0}^{n} (-1)^{k-n} a_k \sqrt{\binom{n}{k}} z^k \propto \prod_{i=1}^{n} (z - z_i) .
\]
The function $\psi(z)$ represents symmetric states in terms of spin coherent states [12], and is also known as the characteristic polynomial, amplitude function [17], or coherent state decomposition [18].

III. ENTAILGEMENT CLASSES

In order to categorize different types of entanglement, the given Hilbert space can be partitioned into equivalence classes. For LOCC operations the equivalence classes contain those states that can be deterministically interconverted by means of local operations and classical communication. A coarser partition is achieved by SLOCC operations [3, 19]. In the symmetric sector, a yet more coarse partition is the Degeneracy Configuration (DC), which depends on the number of coinciding MPs of symmetric states [2]. These three entanglement classification schemes will now be outlined.

A. LOCC

It is known (Corollary 1 of [19]) that two states are LOCC-equivalent iff they are LU-equivalent. For multiqubit symmetric states the condition for LOCC-equivalence of two states $|\psi^p\rangle$ and $|\phi^p\rangle$ reads:

$$|\psi^p\rangle \overset{\text{LOCC}}{\leftrightarrow} |\phi^p\rangle \Leftrightarrow \exists A \in SU(2) : |\psi^p\rangle = A^{\otimes n}|\phi^p\rangle. \quad (4)$$

The LU can be restricted to the form $A^{\otimes n}$, because there always exists a fully symmetric LU that mediates between two LOCC-equivalent symmetric states [2, 20]. The special unitary group SU(2) has 3 real degrees of freedom (d.f.) that can be identified with the three rotation axes on the Bloch sphere. The effect of $A^{\otimes n}$ on $|\psi^p\rangle$ can then be understood as a rotation of the Majorana sphere which changes the location of MPs, but leaves the relative MP distribution (i.e., distances and angles) intact [1].

B. SLOCC

SLOCC operations are mathematically expressed as invertible local operations [21]. In the case of two $n$ qubit symmetric states $|\psi^p\rangle$ and $|\phi^p\rangle$, the condition for SLOCC-equivalence can be cast as:

$$|\psi^p\rangle \overset{\text{SLOCC}}{\leftrightarrow} |\phi^p\rangle \Leftrightarrow \exists B \in SL(2, \mathbb{C}) : |\psi^p\rangle = B^{\otimes n}|\phi^p\rangle. \quad (5)$$

This operation can be chosen to be fully symmetric [2], and from Eq. (2) it is clear that $B$ acts on each MP individually. In the following we will therefore always consider single-qubit operations $B$ (or $A$) instead of the tensor product $B^{\otimes n}$ (or $A^{\otimes n}$) whenever referring to symmetric SLOCC (or LOCC) operations. The special linear group SL($2, \mathbb{C}$) which contains the $2 \times 2$ complex matrices with unit determinant has six real d.f., and because of SU($2$) $\subset$ SL($2, \mathbb{C}$) three of them can be identified as rotations of the Bloch sphere. The Lie group SL($2, \mathbb{C}$) is a double cover of the Möbius group, the automorphism group on the Riemann sphere. Therefore the transformations of MPs under symmetric SLOCC operations are described by the Möbius transformations of complex analysis, with the Majorana sphere in lieu of the Riemann sphere (see Fig. 1 and Fig. 2). The concept of Möbius transformations will be outlined in detail in Section IV.

C. Degeneracy Configuration

The Degeneracy Configuration (DC) of a symmetric $n$ qubit state is characterized by the number of coinciding MPs [2]. The DC class $D_{n_1, \ldots, n_d}$ with $n = n_1 + \ldots + n_d$ ($n_1 \geq \ldots \geq n_d$) encompasses those states where $n_1$ MPs coincide on one point of the Bloch sphere, $n_2$ on a different point, and so on. The number $d$ is called the diversity degree, and the number of DC classes for $n$ qubit symmetric states is given by the partition function $p(n)$. The DC class of a given symmetric state does not change under symmetric SLOCC operations, because of the automorphism nature of the Möbius group. On the other hand, two states that belong to the same DC class do not necessarily belong to the same SLOCC class [2].

D. Hierarchy of classification schemes

Given two partitions $A$ and $B$ of a set $M$, the partition $A$ is called a refinement of $B$ ($A \leq B$) if every element of $A$ is a subset of some element of $B$. Since LOCC is a special case of SLOCC, and because the DC is invariant under SLOCC, the following statement can be made:

**Theorem 1.** The symmetric subspace of every pure $n$ qubit Hilbert space has the following refinement hierarchy of entanglement partitions:

$$\text{LOCC} \leq \text{SLOCC} \leq \text{DC}. \quad (6)$$
The existence of a factorization of each SLOCC operation is facilitated by the transformation $f(z) = z/2$ which maps the set of roots $\{z_1, z_2, z_3\}$ onto the set $\{z_1', z_2', z_3'\}$, thus lowering the ring of MPs. On the sphere Möbius transformations always project circles onto other circles.

IV. Möbius Transformations

As outlined in the previous section, SLOCC operations between multiqubit symmetric states can be understood as Möbius transformations. These isomorphic functions $f : \mathbb{C} \to \mathbb{C}$ are defined on the extended complex plane $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ as the rational functions

$$f(z) = \frac{az + b}{cz + d}, \quad (7)$$

with $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. The latter condition ensures that $f$ is invertible. The coefficients give rise to the matrix representation $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Möbius group, and from Eq. $7$ it is clear that it suffices to consider those $B$ with determinant one (i.e., $ad - bc = 1$). Since $+B$ and $-B$ describe the same transformation $f$, the Möbius group is isomorphic to the projective special linear group $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$.

By means of an inverse stereographic projection all points of $\hat{\mathbb{C}}$ can be projected onto the Riemann sphere. As seen in Fig. 2 the complex plane is projected to the surface of the sphere along rays originating from the north pole, and by convention the infinitely remote point $\infty \in \mathbb{C}$ is projected onto the north pole. By the same projection the roots $\{z_1, \ldots, z_n\}$ of the Majorana polynomial $\mathcal{R}$ are associated with the MPs on the surface of the Majorana sphere. Therefore the Riemann sphere can be employed as the Majorana sphere, and symmetric SLOCC operations have the effect of transforming one set of Majorana roots (or equivalently MPs) to another:

**Theorem 2.** Two symmetric $n$ qubit states are SLOCC-equivalent iff there exists a Möbius transformation (7) between their Majorana roots.

Möbius transformations can be categorized into different types, namely, parabolic, elliptic, hyperbolic and loxodromic, but a unifying feature is that two (not necessarily diametral) points on the sphere are left invariant. This generalizes the SU(2) rotations where the diametrically opposite intersections of the rotation axis with the sphere are left invariant. The SLOCC operation from Fig. 1 mediated by the Möbius transformation $f(z) = z/2$, is shown in detail in Fig. 2. This transformation is hyperbolic, which means that the two invariant points (here the north and south pole) act as attractive and repulsive poles, with the MPs moving away from the repulsive pole towards the attractive one.

A well-known property of Möbius transformations is that for any two ordered sets of three pairwise distinct points $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ there always exists exactly one Möbius transformation that maps one set to the other. With this it immediately becomes clear why DC classes $\mathcal{D}_{n_1, \ldots, n_d}$ with a diversity degree $d \leq 3$ consist of a single SLOCC class $2$.

In the following the three d.f. of Möbius transformations $4$ which genuinely belong to SLOCC operations (i.e., which cannot be realized by LOCC operations) are isolated, and a visual interpretation in terms of the Majorana representation is given.

**Theorem 3.** Every SLOCC operation between two symmetric $n$ qubit states can be factorized into an affine Möbius transformation of the form

$$f(z) = Az + B, \quad A > 0, \quad B \in \mathbb{C}, \quad (8)$$

and a LOCC operation. This decomposition is unique, and the set of transformations forms a group that is isomorphic to $\text{SL}(2, \mathbb{C})/\text{SU}(2)$.

**Proof.** First the existence of a factorization of each SLOCC operation into a transformation $\tilde{f}$ and a LOCC operation is shown. For each $B \in \text{SL}(2, \mathbb{C})$ we define $\tilde{B} = \lambda B$ with $\lambda = \sqrt{aa^* + cc^*} > 0$. Since $\tilde{B}$ describes the same SLOCC operation as $B$, it suffices to show that $\tilde{B}$ can be decomposed into a LOCC operation $A \in \text{SU}(2)$ and a Möbius transformation of the form $5$:

$$\begin{pmatrix} \alpha a & \beta b \\ \alpha c & \beta d \end{pmatrix} = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}, \quad (9)$$

with $A > 0$ and $\alpha, \beta, B \in \mathbb{C}, \alpha a^* + \beta^* = 1$. For given parameters $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$, this is fulfilled for $\alpha = \frac{a}{\sqrt{a^2 + c^2}}, \beta = \frac{b}{\sqrt{a^2 + c^2}}, A = \lambda^2$ and $B = \frac{\sqrt{a^2 + c^2}}{a} = \frac{\sqrt{a^2 + c^2}}{\lambda^2}$. This proves the existence of a factorization.

To show the uniqueness of factorizations, it is assumed that a given SLOCC operation $B \in \text{SL}(2, \mathbb{C})$ can be factorized, up to scalar prefactors $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, in the above way by two sets of parameters $\{a_1, b_1, A_1, B_1\}$ and $\{a_2, b_2, A_2, B_2\}$. Elimination of $B$ from the resulting matrix equations yields the condition

$$\lambda_2 \begin{pmatrix} a_1 - b_1^* \\ \beta_2^* \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_2 - \beta_2^* \\ \alpha_2^* \beta_2 \end{pmatrix} \otimes \begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix}. \quad (10)$$
A straightforward calculation yields $|\frac{\lambda z}{A z}| = 1$, and from this it readily follows that the two sets of parameters must coincide. This uniqueness implies that the set of transformations $\tilde{f}$ is isomorphic to $\text{SL}(2, \mathbb{C})/\text{SU}(2)$, and their group properties are easily verified explicitly. □

Theorem 3 is closely related to the polar decomposition of matrices which states that every invertible complex matrix can be uniquely decomposed into a unitary matrix and a positive-semidefinite Hermitian matrix [23]. However, while the matrices of the affine transformations $\tilde{f}$ are positive, they are in general not Hermitian, and the introduction of the prefactor $\lambda$ in the proof is necessary because $A$ and $B$ are defined to have unit determinants.

The six d.f. of the Möbius transformations are then split into three translational freedoms (rotation of sphere around its axes). By considering these elementary operations it can be verified by calculation that this is an equivalent way of viewing the change of points on the sphere under the action of Möbius transformations. In this approach the affine transformations $\tilde{f}$ are easily identified as the set of all translations in $\mathbb{R}^3$ which leave the sphere’s north pole above the complex plane. A general SLOCC operation between symmetric states can therefore be described as a translation of the Majorana sphere in $\mathbb{R}^3$, followed by a rotation. The parameters of the affine function $\tilde{f}(z) = Az + B$ are connected to the transformation as follows: The parameter $A = \frac{\lambda}{\bar{\lambda}}$ is the ratio of the heights of the north pole before ($h_1$) and after ($h_2$) the transformation, and $B$ is the horizontal displacement vector (cf. Fig. 3).

**V. REPRESENTATIVE STATES FOR SYMMETRIC ENTANGLEMENT CLASSES**

Multiqubit entanglement classes have been well studied before, in particular, the SLOCC-equivalent classes for a single copy of a pure $n$ qubit state. For 2 qubits every entangled state can be turned into a singlet by a SLOCC operation, while for 3 qubits there exist three classes with non-symmetric bipartite entanglement as well as two classes for GHZ-type and W-type entanglement [21, 24]. For as few as 4 qubits, however, the number of SLOCC classes becomes infinite [21]. Verstrate et al. [14] suggested to solve this dilemma by identifying nine different families of 4 qubit entanglement, and a similar approach was pursued by Lamata et al. [22]. In the symmetric sector a different approach is the introduction of the aforementioned DC classes.

![Alternative visualization of Möbius transformations](image)

**FIG. 3.** Alternative visualization of Möbius transformations where a fixed set of complex points is projected onto the surface of a moving sphere. The three innate freedoms of SLOCC operations not present in LOCC operations are then described by the translations of the Majorana sphere in $\mathbb{R}^3$. The north pole of sphere $M_1$ (with the MP distribution of the 5 qubit “square pyramid” state outlined in [3]) lies 2 units above the origin of the complex plane, while the one of $M_2$ lies 5 units above, and $M_3$ is additionally displaced horizontally by a vector $5 - 5i$. The parameters $(A, B)$ of Eq. (5) for the transformation of $M_1$ to $M_2$ and $M_3$ are $(\frac{5}{2}, 0)$ and $(\frac{5}{2}, 5 - 5i)$, respectively.

The orthodox way to visualize Möbius transformations is to fix the Riemann sphere in $\mathbb{R}^3$ (usually with the sphere’s center or south pole coinciding with the complex plane’s origin), and points $z_1, \ldots, z_n$ on the complex plane are transformed to different points $z_1', \ldots, z_n'$ under the action of the functions $f$. By means of the inverse stereographic projection, this transformation can then be observed on the sphere too, as seen in Fig. 2.

Alternatively, the points in the plane can be considered fixed, and instead the Riemann sphere moves in $\mathbb{R}^3$, as shown in Fig. 3. The six d.f. of the Möbius transformations are then split into three translational freedoms (movement of sphere in $\mathbb{R}^3$) and three rotational freedoms (rotation of sphere around its axes). By considering these elementary operations it can be verified by calculation that this is an equivalent way of viewing the change of points on the sphere under the action of Möbius transformations. In this approach the affine transformations $\tilde{f}$ are easily identified as the set of all translations in $\mathbb{R}^3$ which leave the sphere’s north pole above the complex plane. A general SLOCC operation between symmetric states can therefore be described as a translation of the Majorana sphere in $\mathbb{R}^3$, followed by a rotation. The parameters of the affine function $\tilde{f}(z) = Az + B$ are connected to the transformation as follows: The parameter $A = \frac{\lambda}{\bar{\lambda}}$ is the ratio of the heights of the north pole before ($h_1$) and after ($h_2$) the transformation, and $B$ is the horizontal displacement vector (cf. Fig. 3).

![All DC classes of 2 and 3 qubit symmetric states](image)

**FIG. 4.** All DC classes of 2 and 3 qubit symmetric states are listed together with representative states and their MP distribution. Each DC class is also a SLOCC class, which implies that every state of a DC class can be reached from the representative state by a SLOCC operation.

In the following the SLOCC and DC classes of symmetric states of up to 5 qubits are characterized, and representative states with simple MP distributions are given for each equivalence class. Since all DC classes of 2 and 3 qubit states have a diversity degree of 3 or less, their DC classes are identical to SLOCC classes. In Fig. 4 these classes are listed together with a representative state for each class. For three qubits the class $D_3$ contains the separable states, $D_{2,1}$ the W-type entangled states and
Every symmetric state of 4 qubits is SLOCC-equivalent to exactly one state of the set
\[
\{|S_0\}, \ |S_1\rangle, \ |S_2\rangle, \ 2|S_0\rangle + t|S_1\rangle + |S_2\rangle + 2t|S_4\rangle, \quad \text{with} \quad t = e^{i\theta} \tan \frac{\phi}{2}, \quad \text{and} \quad (\theta, \phi) \in \left\{(0, \frac{\pi}{2}) \times [0, 2\pi]\right\} \cup \left\{\left\{0, \frac{\pi}{2}\right\} \times (0, \frac{\pi}{2})\right\}.
\]

**Proof.** First it will be shown that every symmetric 4 qubit state \(|\psi^s\rangle\) can be transformed by SLOCC into one of the above states. From the previous discussion and Fig. 5 this is clear for all DC classes except \(D_{1,1,1}\), where always exists a Möbius transformation \(f : |\psi^s\rangle \rightarrow |\psi^p\rangle\) s.t. three of the distinct MPs are projected onto the corners of an equilateral triangle in the equatorial plane. If the fourth MP \(|\phi_4\rangle\) is not projected into the area parameterized by \((\theta, \phi) \in \left\{(0, \frac{\pi}{2}) \times [0, 2\pi]\right\} \cup \left\{\left\{0, \frac{\pi}{2}\right\} \times (0, \frac{\pi}{2})\right\}\) (cf. Fig. 5), then it can be projected into that area through a combination of \(R_\theta(\pi), R_\pi(\frac{2\pi}{3})\)-rotations of the Majorana sphere (which preserve the equatorial MP distribution).

It remains to show that this set of states is unique, i.e., two different MPs \(|\phi_4\rangle\) and \(|\phi'_4\rangle\) within the aforementioned parameter range give rise to two different states \(|\psi^s\rangle \neq |\psi'^s\rangle\) which are SLOCC-inequivalent. This can be verified by the cross-ratio preservation of Möbius transformations \(M\), namely, that a projection of an ordered quadruple of distinct complex numbers \((v_1, v_2, v_3, v_4)\) onto another quadruple \((w_1, w_2, w_3, w_4)\) requires that
\[
\frac{(v_1 - v_3)(v_2 - v_4)}{(v_2 - v_3)(v_1 - v_4)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)}.
\]

By considering all 4! possible projections between the MPs of \(|\psi^s\rangle\) and \(|\psi'^s\rangle\) it can be explicitly verified that a transformation is possible only if \(|\phi_4\rangle = |\phi'_4\rangle\). \(\Box\)

The DC classes of 5 qubits and representative states for the SLOCC classes can be seen in Fig. 6. The SLOCC classes of the generic class \(D_{1,1,1,1,1}\) can be parameterized by two complex variables, corresponding to two MPs in the black and white area, respectively. Unlike the 4 qubit case, however, this parameterization is neither unique, nor confined to the generic DC class. Different sets of parameters \((\theta_1, \varphi_1, \theta_2, \varphi_2) \neq (\theta'_1, \varphi'_1, \theta'_2, \varphi'_2)\) can give rise to SLOCC-equivalent states, and for \((\theta_1, \varphi_1) = (\theta_2, \varphi_2)\) the corresponding state does not even belong to \(D_{1,1,1,1,1}\) because of coinciding MPs. A unique set of representative states is therefore provided for the subset of symmetric states with a MP degeneracy:

**Theorem 5.** Every symmetric state of 5 qubits is SLOCC-equivalent to exactly one state of the set
\[
\{|S_0\}, \ |S_1\rangle, \ |S_2\rangle, \ \sqrt{\frac{t}{1 + t}}(|S_0\rangle + t|S_1\rangle) + t|S_2\rangle + |S_3\rangle + \sqrt{\frac{1}{1 + t}}(|S_1\rangle + |S_3\rangle), \quad \text{with} \quad t = e^{i\theta} \tan \frac{\phi}{2}, \quad \text{and} \quad (\theta, \phi) \in \left\{(0, \frac{\pi}{2}) \times [0, 2\pi]\right\} \cup \left\{\left\{0, \frac{\pi}{2}\right\} \times (0, \frac{\pi}{2})\right\}.
\]

**Proof.** The proof runs analogous to the one of Theorem 4 with the observation that the representative states of the \(D_{3,1,1}\) and \(D_{2,2,1}\) class are readily subsumed in the parameter range of \(D_{2,1,1,1}\). The fixed MPs of \(D_{2,1,1,1}\) are left
invariant under a $R_x(\pi)$-rotation, thus ensuring that the remaining MP can be projected into the desired parameter range. The uniqueness is again verified by considering all possible cross-ratios.

An over-complete set of representative states for the general case can then be given as follows:

**Corollary 6.** Every symmetric state of 5 qubits is SLOCC-equivalent to one or more state of the set
\[
\{|S_0\}, \{|S_1\}, \{|S_2\}, \sqrt{10}(|S_0| + t_1 t_2|S_3|) + t_1 t_2|S_2| + |S_0| + \sqrt{2}(t_1 + t_2)(|S_1| + |S_1|),
\]
with $t_i = e^{i\phi_i} \tan \frac{\theta_i}{2}$, and
\[
(\theta_1, \phi_1) \in \left\{ \{0, \frac{\pi}{2}\} \times [0, 2\pi) \cup \{\frac{\pi}{2}\} \times (0, \pi) \right\},
\]
\[
(\theta_2, \phi_2) \in \left\{ \{0, \pi\} \times [0, 2\pi) \right\}.
\]

**Proof.** Only the generic class $D_{1,1,1,1,1}$ needs to be considered. Given an arbitrary state of this class, three of its MPs can be projected onto the corners of an equilateral triangle by means of a Möbius transformation. These MPs are left invariant under $\{R_x(\pi), R_y(\pi)\}$-rotations. If the fourth MP does not lie in the $(\theta_1, \phi_1)$-area, it can be projected there by a $R_x(\pi)$-rotation. Subsequent $R_y(\pi)$-rotations can project the fifth MP into the $(\theta_2, \phi_2)$-area, while leaving the fourth MP in the $(\theta_1, \phi_1)$-area.

As the number of qubits increases, the picture gradually becomes more complicated, because DC classes with diversity degree $n$ contain a continuous range of SLOCC classes that is parameterized by $n - 3$ variables [2].

**VI. APPLICATIONS AND CONNECTIONS**

A. Four qubit entanglement families

To describe the behavior of 4 qubit states under SLOCC operations, the concept of entanglement families (EF) was introduced in [14]. Nine different EFs were identified, and every 4 qubit state is SLOCC-equivalent to one of these families. Hence, SLOCC is a refinement of the entanglement families: SLOCC < EF.

It will now be determined in which EFs the symmetric SLOCC and DC classes are located. The separable state $|S_0\rangle$, and therefore the entire $D_3$ class, is present (up to LU) in the family $L_{ab_2}$, namely, by setting $a = b = c = 0$. The W state $|S_1\rangle$ is LU-equivalent to the family $L_{ab_2}$ for $a = b = 0$. The state $|S_2\rangle$ can be found in the general family $G_{abcd}$ by setting $a = 1, b = 2, c = 0, d = -1$. The continuum of SLOCC classes present in the generic family $D_{1,1,1,1,1}$ has previously been parameterized in [2] as $|S_0\rangle + |S_3\rangle + \mu|S_2\rangle$, with $\mu \in \mathbb{C} \setminus \{ \pm \frac{i}{\sqrt{2}} \}$. These states are easily recovered from the general family $G_{abcd}$ with $a = 1 + \frac{i}{\sqrt{2}}, b = \mu, c = 0, d = 1 - \frac{i}{\sqrt{2}}$.

It is noteworthy that two different DC classes, namely, $D_{2,2}$ and $D_{1,1,1,1,1}$, belong to the same entanglement family $G_{abcd}$. On the other hand, all states of a given DC class belong to only one EF [22]. Thus Theorem 1 can be stated more precisely for the four qubit case:

**Theorem 7.** The symmetric subspace of the pure 4 qubit Hilbert space has the following refinement hierarchy of entanglement partitions:

\[
LOCC < SLOCC < DC < EF.
\]

B. Determination of SLOCC inequivalence from the MP distribution

The known properties of Möbius transformations can be utilized to determine from the MP distributions whether symmetric states with the same degeneracy configuration could be SLOCC-equivalent. For example, circles on the surface of the Majorana sphere are always projected onto circles [22], and this trait can be exploited by looking for circles with a certain number of MPs.

As an example, the two 5 qubit states shown in Fig. 7 are not SLOCC-equivalent, because $|\Psi_5\rangle$ exhibits a ring with 4 MPs, while such a ring is not present in $|\psi_5\rangle$. Similarly one can show that for the maximally entangled symmetric states (in terms of the geometric measure) of 10 and 11 qubits, as discussed in [4], the presumed solutions for the general case are not SLOCC-equivalent to those for the subset of states with positive coefficients. For 12 qubits it is not as obvious that the general and positive solutions, shown in Fig. 7, are SLOCC-in equivalent, since both states have several rings with 4 or 5 MPs each. For the highly symmetric icosahedron state $|\Psi_{12}\rangle$ it is possible to identify twenty different circles, each through three adjacent MPs (the corners of all faces of the icosahedron), so that the interior of each circle contains no MPs. This property must be preserved under Möbius transformations, but for $|\psi_{12}\rangle$ it is not possible to find such twenty distinct circles that are all free of other MPs in their interior.

**VII. CONCLUSION**

In this paper the three entanglement classification schemes LOCC, SLOCC and Degeneracy Configuration were employed to characterize and explore symmetric
multiqubit states. It was found that the Möbius transformations from complex analysis do not only allow for a simple and complete description of the freedoms present in SLOCC operations, but also provide a straightforward visualization of these freedoms by means of the Majorana sphere. In particular, it would be promising to study how the entanglement and interconversion probabilities changes under the action of the Möbius transformations \( \tilde{f} \) which translate the Majorana sphere in \( \mathbb{R}^3 \). For example, in Fig. 3 the maximally entangled symmetric 5 qubit state in terms of the geometric measure is displayed at \( M_1 \), and any translation of the sphere decreases the entanglement of the underlying state.

The symmetric SLOCC classes of up to 5 qubits were fully characterized by representative states whose MP distributions are of a particularly simple form, or can be easily parameterized by well-defined areas on the sphere for the variable MPs. For 4 qubits the concept of entanglement families was fitted into the hierarchy of symmetric entanglement classification schemes, and it was demonstrated how the existing theory of Möbius transformations can prove helpful to easily determine whether two symmetric states are SLOCC-equivalent or not.

ACKNOWLEDGMENTS

The author thanks D Markham, J Biamonte, V Vedral, J Dunningham and M Williamson for very helpful discussions. This work is supported by the National Research Foundation & Ministry of Education, Singapore.

Note added. During the completion of this paper I became aware of a similar work which also points out the relationship between symmetric SLOCC operations and Möbius transformations \([27]\).

[1] D. Markham [arXiv:1001.0343]
[2] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. 103, 070503 (2009).
[3] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A 81, 052315 (2010).
[4] M. Aulbach, D. Markham, and M. Murao, New J. Phys. 12, 073025 (2010).
[5] J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A 81, 062347 (2010).
[6] M. Aulbach, D. Markham, and M. Murao, in Proceedings of the 5th Conference on Theory of Quantum Computation, Communication and Cryptography, edited by W. van Dam, V. M. Kendon, and S. Severini (LNCS, Berlin, 2010) pp. 141–158, [arXiv:1010.4777]
[7] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. 95, 120502 (2005).
[8] J. K. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C. F. Roos, and R. Blatt, Phys. Rev. A 74, 052319 (2006).
[9] R. Prevedel, G. Crowenber, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. 103, 020503 (2009).
[10] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Phys. Rev. Lett. 103, 020504 (2009).
[11] P. Ribeiro, J. Vidal, and R. Mosseri, Phys. Rev. E 78, 021106 (2008).
[12] R. Orús, S. Dusuel, and J. Vidal, Phys. Rev. Lett. 101, 025701 (2008).
[13] E. Majorana, Nuovo Cimento 9, 43 (1932).
[14] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).
[15] H. Bacry, J. Math. Phys. 15, 1686 (1974).
[16] P. Kolenderski and R. Demkowicz-Dobrzański, Phys. Rev. A 78, 052333 (2008).
[17] J. M. Radcliffe, J. Phys. A: Math. Gen. 4, 313 (1971).
[18] P. Leboeuf, J. Phys. A: Math. Gen. 24, 4575 (1991).
[19] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2000).
[20] C. D. Cenci, D. W. Lyons, and S. N. Walck [arXiv:1011.5229]
[21] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[22] K. Knopp, Elements of the Theory of Functions (Dover Publications, New York, 1952).
[23] G. H. Golub and C. F. van Loan, Matrix Computations (Johns Hopkins University Press, Baltimore, 1996).
[24] A. Acín, E. Jané, W. Dür, and G. Vidal, Phys. Rev. Lett. 85, 4811 (2000).
[25] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A 75, 022318 (2007).
[26] It is not known yet to which EF the DC class \( D_{2,1,1} \) belongs. However, since \( D_{2,1,1} \) consists of only one SLOCC class, all of its states must belong to a single EF.
[27] P. Ribeiro and R. Mosseri [arXiv:1101.2828]