ON THE SCALE DYNAMICS OF THE TROPICAL CYCLONE INTENSITY

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Abstract. This study examines the dynamics of tropical cyclone (TC) development in a TC scale framework. It is shown that this TC-scale dynamics contains the maximum potential intensity (MPI) limit as an asymptotically stable point for which the Coriolis force and the tropospheric stratification are two key parameters responsible for the bifurcation of TC development. In particular, it is found that the Coriolis force breaks the symmetry of the TC development and results in a larger basin of attraction toward the cyclonic (anticyclonic) stable point in the Northern (Southern) Hemisphere. Despite the sensitive dependence of intensity bifurcation on these two parameters, the structurally stable property of the MPI critical point is maintained for a wide range of parameters.

1. Introduction. Tropical cyclones (TC) are essentially multi-scale dynamical systems. The difficulty in understanding TC development roots in the highly nonlinear nature of the Navier-Stoke equations, along with our inadequate understanding of the TC physics and thermodynamic feedbacks. Numerous modeling and observational studies of TC development have documented the existence a maximum intensity limit that a TC can attain in both idealized and real-data experiments [1, 2, 4, 8, 9, 18], which is fairly consistent with the theoretical estimation of the maximum potential intensity (MPI) limit given by (see, e.g., [5, 6])

$$V_{MPI}^2 = \frac{C_h (T_s - T_o)}{C_d T_s} (s^*_s - s_a),$$

where $T_s$ is the sea surface temperature, $T_o$ is the outflow temperature near the tropopause, $s_s,a$ are the saturated enthalpy at the ocean surface and the actual

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enthalpy at the atmospheric layer right above the ocean surface, and $C_h/C_d$ is the ratio of enthalpy over momentum exchange coefficients.

An important question about the MPI equilibrium is whether this equilibrium is stable, and how its stability varies with different environmental conditions. Despite numerous evidence of the existence of the MPI limit from observational and modeling studies, this stability question is still elusive, at least from theoretical standpoint. As a step to understand the stability of the MPI equilibrium, a simplified hurricane scale-dynamical (HSD) model was recently proposed in [11], which is based on few particular characteristics of TC scales. Stability analyses of the HSD system revealed that the MPI equilibrium is asymptotically stable and unique under the wind-induced surface heat exchange (WISHE) feedback. The stable MPI equilibrium as captured in the HSD model explains why numerous idealized simulations of TCs tend to produce a similar TC structure and MPI equilibrium at their peak intensity under the same ambient environment, regardless of modeling configurations and parameterizations.

Although the HSD model could capture the important stability of the MPI equilibrium, a caveat of this HSD model is the neglect of the atmospheric stratification and the Coriolis force [11]. In this study, we present a different dynamical system to study the MPI equilibrium, which permits the gradient wind imbalance in the planetary boundary layer (PBL) that plays an essential role in TC development [3, 13, 20]. Our particular focuses will be on the stability and bifurcation of the MPI limit in the general presence of both the Coriolis force and the tropospheric stratification, thus rigorously establishing the stable property of the MPI equilibrium.

2. TC-scale model.

2.1. Formulation. Under the approximated axisymmetry of TC circulation, the dynamics of TCs can be most conveniently examined using a system of anelastic equations in the pseudo-height coordinate defined as $z \equiv -H \ln(p/p_s)$, where $H$ the scale height of the atmosphere and $p_s$ the surface pressure [17, 21, 22, 23]. In the explicit form, this anelastic system is given as follows.

$$u_t + uu_r + \frac{v}{r} u_\phi + uw_z - \frac{v^2}{r} = -\phi_r + fv + F_u \tag{2}$$

$$v_t + uv_r + \frac{v}{r} v_\phi + wv_z + \frac{w w_r}{r} = -\frac{1}{r} \phi_\phi - fu + F_v \tag{3}$$

$$w_t + uw_r + \frac{v}{r} w_\phi + w w_z = -\phi_z + b + F_w \tag{4}$$

$$\frac{1}{r} (ur)_r + \frac{1}{r} v_\phi + w_z - \frac{w}{H} = 0 \tag{5}$$

$$b_t + ub_r + \frac{v}{r} b_\phi + Sw = Q, \tag{6}$$

where $u, v, w$ are wind components in the radial, azimuthal, and vertical directions $(r, \phi, z)$ respectively; $\phi$ is the geopotential height perturbation from a reference value $\bar{\phi}(z)$; $b \equiv gT'/\bar{T}(z)$ is buoyancy; $f$ is the Coriolis parameter; $S$ denotes the atmospheric stratification, $F_u,v,w$ are the frictional force, and $Q$ represents the total diabatic heating rate associated with all sources of diabatic heating. In the above system, the subscripts $(t, r, \phi, z)$ denote the partial derivatives in the corresponding directions.
Consider next a set of scales for TCs, which include the scales for the maximum radial wind in the PBL ($U$), the maximum tangential wind near the surface ($V$), the maximum vertical motion in the eyewall ($W$), the absolute geopotential deficit between the vortex center and the eyewall ($\Delta \Phi$), the radius of the maximum wind ($R$), and the maximum buoyancy at the vortex center ($B$). Direct scale analyses of the above equations for the TC inner-core region lead to the following reduced model for TC scale dynamics:

$$\frac{dU}{dt} = \gamma V^2 - \frac{\gamma}{\alpha} B - \beta U|V| + fV$$  \hspace{1cm} (7)

$$\frac{dV}{dt} = -\gamma U V - \beta V|V| - fU$$  \hspace{1cm} (8)

$$\frac{dB}{dt} = \gamma U B + \sigma U + Q,$$  \hspace{1cm} (9)

where $\gamma \equiv 1/R$, $\beta \equiv C_D/h$, $\alpha = 1/H$, and $\sigma \equiv SH/R$. Unlike the HSD system studied in [11] that relies on the gradient wind balance, the above TC-scale dynamic system (7)-(9) makes a full use of the radial momentum equation so that the gradient wind imbalance is taken into account in (7). Note that the scale of the PBL fictional forces in the above scale analyses assumes the familiar bulk formula for both the radial and tangential momentum equations such that

$$F_u = \frac{\partial \tau}{\partial z} \approx \frac{\tau u,v}{h} \approx C_D U \sqrt{U^2 + V^2}/h \approx C_D U|V|/h,$$

where $h$ is the PBL depth and $C_D$ is the surface drag coefficient, and similar for $F_v$.

Following [11], we will close the above system (7)-(9) by utilizing a bulk parameterization for the diabatic heating source $Q$, using the WISHE mechanism and the Newtonian cooling relaxation forcing so that the diabatic heating source term $Q$ is given by

$$Q \sim \eta gC_h (s^*_s - s_a) \frac{|V|}{C_p \bar{T}} + \kappa B, \hspace{1cm} (10)$$

where $C_p$ is the specific heat constant, and $\kappa < 0$ represents the Newton radiative relaxation rate. Following [5], a factor $\eta \equiv (T_s - T_o)/T_s$ is also introduced in (10) to represent the efficiency of the energy conversion. More details about this parameterization can be found in [5, 6].

Substituting (10) into (9), we obtain a system of equations describing the evolution of the TC scales as follows:

$$\frac{dU}{dt} = \gamma V^2 - \frac{\gamma}{\alpha} B - \beta U|V| + fV$$  \hspace{1cm} (11)

$$\frac{dV}{dt} = -\gamma U V - \beta V|V| - fU$$  \hspace{1cm} (12)

$$\frac{dB}{dt} = \gamma U B + \sigma U + \delta |V| + \kappa B,$$  \hspace{1cm} (13)

where $\delta \equiv \eta gC_h (s^*_s - s_a) C_p \bar{T}$. For the sake of discussion, the system of equations (11)-(13) will be hereinafter referred to as a modified TC-scale dynamical (MSD) system to distinguish with the HSD system presented in [11].

2.2. MPI critical point. To establish that the MSD system (11)-(13) correctly describes the TC dynamics, we will first prove an important Proposition concerning the existence of the MPI equilibrium. This will lay out some further simplification for the MSD system, in which the MPI equilibrium can be used to nondimensionalize the MSD system and facilitate our subsequent stability analysis.
Proposition 1. In a special limit of \( f = \sigma = \kappa = 0 \), the MSD system contains the MPI equilibrium as one of its critical points.

Proof. For an atmosphere in which the tropospheric stratification is close to neutral such that \( s \approx 0 \) and the impacts of both Coriolis forcing and radiative cooling are negligible in the TC inner-core region (see [5]), the MSD system (11)-(13) is reduced to

\[
\begin{align*}
\frac{dU}{dt} &= \gamma V^2 - \frac{\gamma}{\alpha} B - \beta U|V|, \\
\frac{dV}{dt} &= -\gamma UV - \beta V|V|, \\
\frac{dB}{dt} &= \gamma UB + \delta |V|, 
\end{align*}
\]

(14)

(15)

(16)

Apparenty, (14)-(16) possess three critical points including one at the origin, and two other nonzero critical points given as follows:

\[
U_1 = -\frac{\beta V_1}{\gamma}, \quad V_1 = \sqrt{\frac{\delta}{\alpha \beta \gamma^2 + \beta^2}}, \quad B_1 = \frac{\delta}{\beta}, \quad V \in \Omega_+ \equiv \{V|V \geq 0\},
\]

and

\[
U_2 = \frac{\beta V_2}{\gamma}, \quad V_2 = -\sqrt{\frac{\delta}{\alpha \beta \gamma^2 + \beta^2}}, \quad B_2 = \frac{\delta}{\beta}, \quad V \in \Omega_- \equiv \{V|V < 0\}.
\]

Note that \( V_1 = -V_2 \) and its absolute value after substituting explicit expressions for all parameters \( \delta, \alpha, \beta \) and \( \gamma \) is:

\[
V_1^2 = V_2^2 = \frac{C_h}{C_d} \frac{T_s - T_a}{T_s} (s_s^* - s_a) \frac{\gamma^2}{\gamma^2 + \beta^2},
\]

(19)

where the relationship between the scale height \( H \) and the mean temperature of the troposphere \( \bar{T} \) (i.e., \( gH = C_p \bar{T} \)) has been used in (19). If one notes further that because \( \beta \approx 10^{-6}, \gamma \approx 10^{-4} \) for a typical TC, it is immediate to see that \( V_1^2 \approx V_{MPI}^2 \), thus proving the existence of the MPI equilibrium in the MSD system (11)-(13).

That the MSD system contains the MPI equilibrium as one of its critical points as stated in Proposition 1 is remarkable, because it confirms the validity of the MSD system in capturing the dynamics of TCs and its asymptotic limit. Specifically, the above derivation of the MPI solution in Proposition 1 is purely from the perspective of the TC-scale dynamics, which is different from the energy method or the trajectory integration in [5, 6]. Despite drastically different approaches, it turns out that these different approaches all lead to the same approximation of the MPI solution, thus supporting the existence of such a limit in TC development.

Following [11], we use the values of the scales at the critical point \( (U_1, V_1, B_1) \) given by (17) to nondimensionalize the MSD system by setting \( U = |U_1|u,v, V = V_1v, B = B_1b, t = Tt \), where the prime denotes nondimensionalized variables, and obtain:

\[
\frac{du}{dt} = \frac{\gamma^2 TV_1}{\beta} - \frac{TV_1(\gamma^2 + \beta^2)}{\beta} b - \beta TV_1 uv + \frac{f_0 TV_1}{|U_1|} v.
\]

(20)
\[
\frac{dv}{dt} = -\beta TV_1 (uv + v^2) - \frac{f_0 T |U_1|}{V_1} u \tag{21}
\]
\[
\frac{db}{dt} = \beta TV_1 b u + \frac{\sigma \beta^2 TV_1}{\gamma \delta} u + \beta TV_1 v + \kappa Tb, \quad \forall V \in \Omega_+ \tag{22}
\]
with an implicit convention that the prime will be hereinafter neglected in (20)-(22) for a simplification of notation. The above system can be further simplified if we choose a time scale \( T \) such that \( T \beta V_1 = 1 \). Substitute \( T = 1/\beta V_1 \) in the system (20)-(22), we have
\[
\frac{du}{dt} = pv^2 - (p + 1)b - uv + pf v \tag{23}
\]
\[
\frac{dv}{dt} = -uv - v^2 - fu \tag{24}
\]
\[
\frac{db}{dt} = bu + su + v - rb, \quad \forall v \in \Omega_+ \tag{25}
\]
where \( p \equiv (\gamma/\beta)^2, s \equiv (\sigma \beta)/(\gamma \delta), r = -\kappa T, f = f_0 |U_1|/|\beta v_1^2| \). For the sake of convenience, we will hereinafter refer the system (23)-(25) as an MSD system, because this system of equations are applied only for \( v \in \Omega_+ \). Similar nondimensionalization for the domain \( v \in \Omega_- \) will lead to an MSD system as follows:
\[
\frac{du}{dt} = pv^2 - (p + 1)b + uv + pf v \tag{26}
\]
\[
\frac{dv}{dt} = -uv + v^2 - fu \tag{27}
\]
\[
\frac{db}{dt} = bu + su - v - rb. \tag{28}
\]

Remark 1. Given the typical TC scales, the parameters \( (p, r, s, f) \) have a range of \( p \in [50, 200], r \in [0, 1], s \in [0, 1], \) and \( f \in [0, 0.01] \). It is worth emphasizing that even though the Coriolis parameter \( f \) is the same in (2)-(3), its impacts in the nondimensionalized form are much different between (23) and (24) due to the larger scale of the tangential wind \( V_1 \) as compared to the scale of the radial wind \( U_1 \) (i.e., \( V_1 \gg U_1 \)).

Remark 2. In the Northern Hemisphere where \( f > 0 \), the system (23)-(25) is applied for cyclonic TCs, while (26)-(28) is for anticyclonic TCs. Notice that these two systems (23)-(25) and (26)-(28) are nevertheless symmetric under the transformation
\[
(u, v, b, f) \rightarrow (u, -v, b, -f), \tag{29}
\]
because such a transformation does not effect the structure of forcing vector field on their right hand side. Because of this symmetry, all analyses for the MSD system with \( v < 0, f > 0 \) can be derived from the results of the MSD system by simple applying the transformation (29) for \( v > 0, f < 0 \), and vice versa.

3. Stability analysis. Before examining the critical points of the MSD system and their associated stability, it is important to note a property of the MSD system in the absence of the Coriolis force that allows for much simplified stability analysis of the MSD system. We have the following Lemma:

Lemma 3.1. In the absence of the Coriolis force, the plane \( v = 0 \) acts as a separatrix in the phase space of \( (u, v, b) \), where the domain \( v > 0 \) is completely separated
from the domain with \( v < 0 \). Hence, there is no possibility for an orbit starting in the domain \( \Omega_+ \) to cross the plane \( v = 0 \) and move to the other domain \( \Omega_- \).

**Proof.** We only need to prove that \( v(t) \equiv 0 \) \( \forall t \) for any initial value \((u_0, v_0, b_0)\) with \( v_0 = 0 \). This is indeed immediately seen from (23)–(25), which reduce to

\[
\frac{du}{dt} = -(p + 1)b \\
\frac{db}{dt} = bu + su - rb,
\]

for an initial condition \((u_0, 0, b_0)\). Obviously, (30)–(31) has a unique solution with any initial value \((u_0, b_0)\), thus proving that (23)–(25) has a solution \((u(t), 0, b(t))\) \( \forall t \).

Based on the uniqueness of solutions, the Lemma holds true. \( \square \)

**Remark 3.** Physically, Lemma 3.1 states that in the absence of the Coriolis force, a cyclonic TC cannot evolve into an anticyclonic TCs. As such, an initial vortex with either cyclonic flow (i.e., \( v > 0 \) at \( t = 0 \)) or anticyclonic flow (i.e., \( v < 0 \) at \( t = 0 \)) will be guaranteed to evolve entirely within the domain \( \Omega_+ \) or \( \Omega_- \) forever. Because of this property, all absolute signs in (20)-(22) can be neglected if we restrict our analyses in one specific domain \( \Omega_+ \) or \( \Omega_- \) for \( f = 0 \).

### 3.1. The MSD\(^+\) system with \( f = 0 \)

Given the special role of the Coriolis parameter \( f \) in the MSD system, let us examine first the dynamics of the MSD system in the absence of the Coriolis forcing, i.e. \( f = 0 \). As will be seen later, these analyses for the MSD system with \( f = 0 \) will serve as a first order approximation for subsequent stability analyses for the more general MSD system with \( f \neq 0 \). Because of the Lemma 3.1, we will hereinafter consider only TC development governed by the MSD\(^+\) system in the domain \( \Omega_+ \), which is applied for cyclonic TCs in the Northern Hemisphere. By Remark 2, all analyses for the anticyclonic TCs in the Southern Hemisphere can be reproduced by a simple transformation \( v \rightarrow -v \).

#### 3.1.1. Critical points

In the absence of the Coriolis force, the critical points of the MSD\(^+\) system (23)–(25) consist of one zero point \((0, 0, 0)\), and two other nonzero critical points whose \( v \) components satisfy the following characteristic equation

\[
v^2 + rv + s - 1 = 0.
\]

The roots of (32) are given by

\[
v_1 = \frac{-r + \sqrt{r^2 + 4s - 4}}{2},
\]

\[
v_2 = \frac{-r - \sqrt{r^2 + 4s - 4}}{2},
\]

from which one can readily obtain the other values for \( u \) and \( b \) at the critical points. As such, the MSD\(^+\) system (23)–(25) has a total of three critical points as follows

\[
x_0 = (0, 0, 0), x_1 = \left( -v_1, v_1, \frac{(1 - s)v_1}{r + v_1} \right), x_2 = \left( -v_2, v_2, \frac{(1 - s)v_2}{r + v_2} \right), s \neq 1.
\]

We observe from the above expression for the critical point \( x_1 \) that \( x_1 \) will approach \( x_0 \) as \( s \rightarrow 1^- \), and the MSD\(^+\) system thus has only two critical points if \( s = 1, r \neq 0 \). If the radiative forcing \( r \) is further set to zero as often assumed in previous modeling studies of TC inner-core region, all critical points will in fact reduce to a single zero critical point for \( s = 1 \), i.e. there is no development of TCs regardless of their initial conditions if the stratification is sufficiently large. In this regard, the parameter \( s \) plays a role of a bifurcation parameter, which controls the number of
the critical points that the MSD model could have in the absence of the Coriolis forcing.

We examine next the stability of all critical point \( x_{0,1,2} \) given by (35). Our goal is to show that only one of these critical points is asymptotically stable, and its stability is continuous in the parameter space \((p, r, s)\), thus proving that this stable critical point is locally structurally stable. This result will establish the stable property for the MPI equilibrium in the MSD system with \( f = 0 \). We have the following result:

**Theorem 3.2.** For the MSD system (23)-(25) with \( 0 \leq s < 1 \) and \( f = 0 \), the critical point \( x_1 \) with the positive \( v \) component (i.e., \( v_1 > 0 \)) is stable, whereas the critical pooint \( x_2 \) is unstable.

**Proof.** Direct evaluation of the Jacobian matrix for the MSD system (23)-(25) at \( x_c \) gives:

\[
\frac{\partial F(x)}{\partial y} \bigg|_{x_c} = \begin{pmatrix} -v_c & 2pv_c - u_c & -p - 1 \\ -v_c & -u_c - 2v_c & 0 \\ b_c + s & 1 & u_c - r \end{pmatrix},
\]

whose characteristic equation is

\[
\lambda^3 + A\lambda^2 + B\lambda + C = 0,
\]

where

\[
A(x_c, p, r, s) = 3v_c + r, \quad B(x_c, p, r, s) = 2(p + 1)v_c^2 + ru_c - u_c^2 + 3rv_c - 3uv_c + (p + 1)(b_c + s), \quad C(x_c, p, r, s) = (p + 1)(bu_c + 2bv_c + sv_c - v_c + 2sv_c + 2sv_c^2 - 2uv_c^2),
\]

A critical point \( x_c \) is stable if the cubic equation (37) possesses all three roots with negative real parts. It is known that the existence of three roots with negative real parts is guaranteed if the following conditions are applied

\[
A(x_c, p, r, s) > 0, \quad C(x_c, p, r, s) > 0, \quad A(x_c, p, r, s)B(x_c, p, r, s) - C(x_c, p, r, s) > 0.
\]

With the critical points given by (35), the explicit expressions for the above coefficients \( A, B, C \) are:

\[
A(x_1, p, r, s) = 3\sqrt{\frac{r^2}{4} + 1 - s - \frac{r}{2}} > 0,
\]

\[
B(x_1, p, r, s) = (p + 1)\left[\frac{3v_1^2 - 2sv_1^2 + rsv_1}{1 - s} + \frac{2 - 2s}{p + 1}\right] > 0,
\]

\[
C(x_1, p, r, s) = (p + 1)(v_1^3 + v_1 - sv_1) > 0,
\]

and so

\[
A(x_1, p, r, s)B(x_1, p, r, s) - C(x_1, p, r, s)
\]

\[
= (p + 1)\left[3\sqrt{\frac{r^2}{4} + 1 - s - \frac{r}{2}}\left(\frac{3v_1^2 - 2sv_1^2 + rsv_1}{1 - s} + \frac{2 - 2s}{p + 1}\right) - \left(v_1^3 + v_1 - sv_1\right)\right]
\]

\[
\geq (p + 1)\left[(3v_1 + r)2v_1^2 - v_1^3 + sv_1^3 > 0. \quad (46)
\]
Apparently, both conditions (41)-(42) are satisfied at the critical point \( x_1 \), and the characteristic equation (37) thus possesses all roots with negative real parts. As a result, the critical point \( x_1 \) is asymptotically stable as expected. Note further from the above expressions that the eigenvalues of the Jacobian matrix (36) are continuously dependent on \((p, r, s)\). Because of this smoothness and the hyperbolic property of \( x_1 \), we conclude that \( x_1 \) is not only asymptotically stable but also locally structural stable, based on the Hartman-Grobman Theorem.

For the critical point \( x_2 \), it is easy to see that

\[
A(x_2, p, r, s) = -3\sqrt{\frac{r^2}{4} + 1 - s - \frac{r}{2}} < 0,
\]

\[
C(x_2, p, r, s) = 2(p + 1)v_2^2\left(\frac{r}{2} - \sqrt{\frac{r^2}{4} + 1 - s}\right) - (p + 1)rv_2^2 < 0,
\]

which implies that the characteristic equation (37) possesses at least one root with positive real part, i.e., \( x_2 \) is unstable.

Regarding the zero critical point \( x_0 \), we have the following result about its stability.

**Proposition 2.** *In the absence of the Coriolis force, the critical point \( x_0 \) is unstable. In particular, if \( r = s = 0 \), all points of the form \((\alpha, 0, 0) (\alpha \neq 0)\) are critical points and they are also unstable.*

**Proof.** For a general case with \( r > 0 \) and \( s > 0 \), a direct linearization of the \( MSD_+ \) system at \( x_0 \) shows that the \( MSD_+ \) system reduces to a simple forced harmonic oscillation that is always pulled away from the zero critical point, and so \( x_0 \) is unstable. This unstable property of \( x_0 \) can be further seen from the bifurcation of the \( MSD_+ \) system at \( f = 0 \) as will be seen later (cf. Figure 2).

For the special case of \( r = s = 0 \), it can be checked that all points of the form \((\alpha, 0, 0) (\alpha \neq 0)\) are critical points of the \( MSD_+ \) system by directly verifying the forcing fields on the right hand side of (23)-(25). To show that these critical points are unstable, we note that the corresponding linear matrix at the critical point \((\alpha, 0, 0)\) of the \( MSD_+ \) system is given by

\[
\begin{pmatrix}
0 & -\alpha & -p - 1 \\
0 & -\alpha & 0 \\
0 & 1 & \alpha
\end{pmatrix}.
\]

Applying this linearized matrix has eigenvalues \( \alpha \) and \(-\alpha\), one of them is guaranteed to be positive, and so \((\alpha, 0, 0)\) is unstable.

**Remark 4.** While the above stability examination includes both critical points \( x_{1,2} \), it should be noted that the critical point \( x_2 \) has to be ultimately discarded because of the restriction of the \( MSD_+ \) system in the domain \( \Omega_+ \). The proof of the unstable property for \( x_2 \) is nevertheless important, because it indicates that the TC development has no other choice except for the unique stable point \( x_1 \), which corresponds to the cyclonic MPI limit that a TC must attain. Without this unique and stable property of \( x_1 \), the TC development would settle in either \( x_0 \) or \( x_2 \), which is not consistent with both observational and modeling studies of TC development. In this regard, the existence of the stable critical point \( x_1 \) along with the unstable property of both the zero critical points \( x_0 \) and the critical point \( x_2 \) is strictly required so that the MSD system could indeed describe the correct development of
TCs as expected. Because the magnitude of the \( v \) component of the critical point \( x_1 \) matches the MPI expression given by (1), we will hereinafter refer to \( x_1 \) as the MPI critical point.

### 3.1.2. Topological index for nonzero critical points

To later facilitate the analyses of bifurcation, we establish in this section a noteworthy property related to the index of the vector field of the \( MSD_+ \) system in the absence of the Coriolis parameter. With the forcing vector field on the right hand side of (23)–(25) given by

\[
F_{rs}(x) = \left( \begin{array}{c} \text{component of the critical point} \\
\text{is determined by an equation}
\end{array} \right), x = (u, v, b), \tag{50}
\]

where the subscripts \((r, s)\) indicate given values of the parameters \( r \) and \( s \), we have the following Lemma.

**Lemma 3.3.** For the critical points \( x_{01,2} \) given by (35), we have

\[
\text{ind}(F_{rs}(x_1)) = -1, \text{ind}(F_{rs}(x_2)) = 1, \text{ind}(F_{rs}(x_0)) = 1. \tag{53}
\]

**Proof.** Based on the definition of the index of isolate singular points (see [15, 16]) and the explicit form of the Jacobian matrix of the forcing \( F_{rs} \), it is directly seen that

\[
\text{ind}(F_{rs}(x_1)) = \text{sign}(\text{det} D F_{rs}(x_1)) = \text{sign}(-C(x_1, p, r, s)) = -1, \tag{54}
\]

\[
\text{ind}(F_{rs}(x_2)) = \text{sign}(\text{det} D F_{rs}(x_2)) = \text{sign}(-C(x_2, p, r, s)) = 1, \tag{55}
\]

where \( C(x_{1,2}, p, r, s) \) is a function given by (48).

For the zero critical point \( x_0 \), we note that \( x_0 \) is a degenerate singular point, and one therefore has to use a regular point near \( 0 \) to determine the index of \( x_0 \). Because of the homotopy invariance of the index, it is sufficient to determine the index for \( x_0 \) for the special case of \( s = r = 0 \). Let \( y = (0, 0, \varepsilon) \) be a regular point near \( 0 \equiv (0, 0, 0) \) such that

\[
F_{00}(x) = y. \tag{56}
\]

The \( v \)-component of (53) is determined by an equation

\[
x^3 - x + \varepsilon = 0. \tag{57}
\]

Obviously, (54) has three distinct real roots if \( \varepsilon \) is very small. If \( \varepsilon = 0 \), (54) has in fact three roots \((1, -1, 0)\). As such, for very small \( \varepsilon \), one root of (54) is close to 1, one root is close to -1, the other one has to be near 0. Denote the three points \( x_{\varepsilon i} = (u_{\varepsilon i}, v_{\varepsilon i}, b_{\varepsilon i})(i=1,2,3) \) satisfying (53), and

\[
\lim_{\varepsilon \to 0}(u_{\varepsilon 1}, v_{\varepsilon 1}, b_{\varepsilon 1}) = (-v_1, v_1, v_1), \tag{58}
\]

\[
\lim_{\varepsilon \to 0}(u_{\varepsilon 2}, v_{\varepsilon 2}, b_{\varepsilon 2}) = (-v_2, v_2, v_2), \tag{59}
\]

\[
\lim_{\varepsilon \to 0}(u_{\varepsilon 3}, v_{\varepsilon 3}, b_{\varepsilon 3}) = (0, 0, 0). \tag{60}
\]

The conditions (55)–(56) imply that

\[
\text{ind}(F_{rs}(x_{1})) = \text{sign}(\text{det} D F_{rs}(x_{1})) = -1, \tag{56}
\]

\[
\text{ind}(F_{rs}(x_{2})) = \text{sign}(\text{det} D F_{rs}(x_{2})) = 1. \tag{57}
\]

For \( x_{\varepsilon 3} \), because \( \varepsilon \) is very small, it is easy to deduce from (54) that

\[
v_{\varepsilon 3} = \varepsilon + o(\varepsilon). \tag{58}
\]
Combining (40) and (60), one obtains
\[ C(x_3, p, 0, 0) = 2v_x - 3\varepsilon + o(\varepsilon) < 0. \tag{61} \]
Therefore, we have
\[ \text{ind}(F_{rs}(x_0)) = 1. \tag{62} \]
Because of the special case of \( s = 1 \) that reduces all critical points of (32) to a single point \((0, 0, 0)\), we consider the case \( s = 1 \) separately. In this case, we have
\[ \text{ind}(F_{r1}(x)) = \text{ind}(F_{r1}(x_0)). \tag{63} \]
Based again on the homotopy invariance of index, we only need to compute
\[ \text{ind}(F_{01}(x)) = \text{ind}(F_{01}(x_0)). \tag{64} \]
Similarly to (53), we solve
\[ F_{01}(x) = y. \tag{65} \]
to get
\[ x_{c0} = (\varepsilon^\frac{1}{3}, -\varepsilon^\frac{2}{3}, \varepsilon^\frac{2}{3}), \tag{66} \]
Plugging (66) into (40), we get
\[ C(x_{c0}, p, 0, 1) = -\varepsilon - 2\varepsilon^\frac{2}{3}, \tag{67} \]
For very small \( \varepsilon \), \(-\varepsilon - 2\varepsilon^\frac{2}{3}\) is negative, and so
\[ \text{ind}(F_{01}(x)) = \text{ind}(F_{01}(x_0)) = 1. \tag{68} \]

Given the above indices of the forcing vector field in the \( MSD_+ \) system evaluated at all critical points, it is noted that any abrupt change in the index of the forcing vector field as parameters \((r, s, f)\) vary will indicate a point in the parameter space where bifurcations in TC development may occur. Because of this property, any variation in the index of the forcing vector field will provide useful information about possible bifurcation in TC development as will be presented in Section 3.3.

3.2. General \( MSD \) system. With the analyses for a special case in which the impacts of the Coriolis force in the \( MSD_+ \) system are neglected in the previous section, we will examine in this section the general MSD system with \( f \neq 0 \). While the absolute value of the Coriolis parameter \( f \) is small, inclusion of the Coriolis force in the MSD system results in much more complex analyses because of potential bifurcation of the MSD system as the Coriolis parameter \( f \) varies. In particular, the separatrix plane \( v = 0 \) no longer exists, and it is now possible that a flow can cross the plane \( v = 0 \) in the presence of the Coriolis force. Because of the complication of the absolute value signs in the general MSD model, we have to therefore consider both the \( MSD_+ \) system (23)-(25) and the \( MSD_- \) system (26)-(28) simultaneously in all analyses. For the sake of convenience, recall again that in the non-dimensional form, the \( MSD_+ \) system is given by
\[
\begin{align*}
\dot{u} &= pv^2 - (p + 1)b - uv + pfv, \tag{69} \\
\dot{v} &= -uv - v^2 - fu, \tag{70} \\
\dot{b} &= bu + su + v - rb, \quad v \in \Omega_+,
\end{align*}
\]
whereas the $MSD_-$ (26)–(28) is

$$
\dot{u} = pv^2 - (p + 1)b + uv + pfv, \quad (72)
$$

$$
\dot{v} = -uv + v^2 - fu, \quad (73)
$$

$$
\dot{b} = bu + su - v - rb \quad v \in \Omega_-. \quad (74)
$$

Basically, (69)–(71) describes cyclonic TCs and (72)–(74) describes anticyclonic TCs in the Northern Hemisphere for $f > 0$ or in the Southern Hemisphere for $f < 0$. Given two different types of circulations (cyclonic and anticyclonic flows) in each Hemisphere, we thus have a total of four different combinations:

- $v > 0, f > 0$: cyclonic TCs in the Northern Hemisphere, which is governed by (69)–(71);
- $v < 0, f > 0$: anticyclonic TCs in the Northern Hemisphere, which is governed by (72)–(74);
- $v > 0, f < 0$: cyclonic TCs in the Southern Hemisphere, which is governed by (69)–(71); and
- $v < 0, f < 0$: anticyclonic TCs in the Southern Hemisphere, which is governed by (72)–(74);

Because of the symmetry of the MSD system under the transformation $(u, v, b, f) \rightarrow (u, -v, b, -f)$ as in Remark 2, all stability of critical points and structure of vector fields for the MSD system in the Southern Hemisphere with $f < 0$ can be immediately obtained from the analyses for the MSD system in the Northern Hemisphere under such transformations. This symmetry between the $MSD_+$ and $MSD_-$ systems under the transformation $(u, v, b, f) \rightarrow (u, -v, b, -f)$ is critical, because analyses for $MSD_-$ with $f > 0, v < 0$ are also mathematically equivalent to the $MSD_+$ with $f < 0, v > 0$. Thus, this transformation allows us to focus exclusively on the $MSD_+$ system in which the parameter $f$ can take both negative and positive values, instead of working simultaneously with both the $MSD_+$ and $MSD_-$ systems with $f > 0$. As will be seen in Section 4, this observation drastically simplifies all bifurcation analyses as $f$ varies. Because of this symmetry, we will hereinafter focus only on the $MSD_+$ system with an implicit convention that $f$ can take both positive and negative values instead of working simultaneously with both the $MSD_+$ and $MSD_-$ systems.

### 3.2.1. Critical points

The critical points of the $MSD_+$ system (69)–(71) in the presence of the Coriolis force satisfy the following algebra equations

$$
pv^2 - (p + 1)b - uv + pfv = 0, \quad (75)
$$

$$
-uv - v^2 - fu = 0, \quad (76)
$$

$$
bu + su + v - rb = 0. \quad (77)
$$

It is easy to see that $(0, 0, 0, 0)$ is one root of (75)–(77). Because $f \ll 1$ for a wide range of the scale of the maximum surface tangential wind $v$ in the $MSD_+$ system (69)–(71), we have the following important Lemma.

**Lemma 3.4.** $x_0 \equiv (0, 0, 0, 0)$ is a zero critical point of (69)–(71), and

$$
\text{ind}(F^*_r(s(x_0))) = -1 \text{ for } f < 0, \text{ind}(F^*_r(s(x_0))) = 1 \text{ for } f > 0. \quad (78)
$$

Hence, $f = 0$ is a bifurcation point of the $MSD_+$ system at $x_0$. 
Proof. The linear matrix of the $MSD_+$ system evaluated at (0,0,0) is given by
\[
\begin{pmatrix}
0 & pf & -p - 1 \\
-f & 0 & 0 \\
s & 1 & -r
\end{pmatrix},
\]
from which we obtain the following characteristic polynomial
\[
x^3 + rx^2 + ((p + 1)s + pf^2)x - f((p + 1) - prf) = 0.
\]
Because $(p + 1 - prf) > 0$ due to $|rf| \ll 1$, it is immediate to see from (80) that the sign of the last term in (80) will depend on the sign of $f$. As a result, the index of the zero critical point will be given by the sign of this term, and so Lemma 3.4 holds true.

Direct inspection of (75)–(77) shows that all other nonzero critical points of the system $MSD_+ (69)$–(71) have their $v$-components satisfying the following equation
\[
v^4 + av^3 + bv^2 + cv + d = 0,
\]
where coefficients $a, b, c$ and $d$ are functions of the parameters $(p, r, s, f)$ as follows:
\[
a = \frac{2pf}{p + 1} + r, \quad (82)
\]
\[
b = \frac{2pf}{p + 1} + fr + s - 1 + \frac{pf^2}{p + 1}, \quad (83)
\]
\[
c = \frac{3pf^2}{p + 1} + f(s - 2), \quad (84)
\]
\[
d = -f^2 + \frac{prf^3}{p + 1}. \quad (85)
\]

It should be mentioned again that in the absence of the Coriolis force, the $MSD_+$ system has a unique stable critical point with $v > 0$, which describes an asymptotic limit of the maximum intensity limit that a TC can attain in the Northern Hemisphere, while all other nonzero critical points are not physical solutions. With inclusion of the Coriolis force, it is highly expected that the $MSD_+$ system will continue to maintain this very important property with a unique stable critical point that has $v > 0$. This property of a unique asymptotic limit in TC development is well captured in previous observational and modeling studies [1, 4, 8, 14, 18], and it would be serious if inclusion of the Coriolis force destroys this property. To this end, we need to analyze first the number of critical points of the $MSD_+$ system (69)–(71), along with explicit estimations of these critical points for subsequent stability analyses. Indeed, we have the following Theorem.

**Theorem 3.5.** For the $MSD_+$ system of (69)–(71), there always exists a sufficiently small $\delta_0 > 0$ such that

1. If $0 < r < \delta_0$, $0 < s < \delta_0$, and $0 < |f| < \delta_0$, the system (75)–(77) has four nonzero critical points, one has $v > 0$, one has $v < 0$, and two other critical points have their component $v$ depending on the sign of $f$.
2. If $0 < r < \delta_0$, $0 < 1 - \delta_0 < s < 1$, and $0 < |f| < \delta_0$, (75)–(77) have only two critical points, one with positive $v$ component.
3. For a fixed small value of $f$, there exist $s_{f_0} > 0$ such that (75)–(77) have three nonzero critical points if $s = s_{f_0}$. 


Proof. We will use the Implicit Function Theorem and the discriminant of the quadratic function to prove above results. Denote

\[ F(x, f) = \begin{bmatrix} pv^2 - (p + 1)b - uv + pf v \\ -uv - v^2 - fu \\ bu + su + v - rb \end{bmatrix}, x = \begin{bmatrix} u \\ v \\ b \end{bmatrix}. \]  

Let \( x = y + x_i (i = 1 \text{ or } 2) \), where \( y \) is a small perturbation around the nonzero critical points \( x_i \) of the \( MSD_+ \) system in the absence of the Coriolis force as given by (35), and \( M \) be the Jacobian matrix defined as follows

\[ M = \begin{bmatrix} -v_i & 2pv_i + v_i + pf & -p - 1 \\ -v_i - f & -v_i & 0 \\ (1-s)v_i & 1 & -v_i - r \end{bmatrix} \]  

then

\[ G(y, f) \equiv F(x_i + y, f) = My^T + h(y), \]  

where

\[ h(y) = \begin{bmatrix} Py^2 - y_1y_2 + pf v_i \\ -y_1y_2 - y_2^2 + f v_i \\ y_3y_1 \end{bmatrix}. \]  

Furthermore, we have

\[ \frac{\partial G(y, f)}{\partial y} \bigg|_{(y, f) = (0, 0, 0, 0)} = M = \begin{bmatrix} -v_i & 2pv_i + v_i & -p - 1 \\ -v_i & -v_i & 0 \\ v_i^2 + s & 1 & -v_i - r \end{bmatrix}, \]  

and

\[ \det \left( \frac{\partial G(y, f)}{\partial y} \bigg|_{(y, f) = (0, 0, 0, 0)} \right) = \begin{vmatrix} -v_i & 2pv_i + v_i & -p - 1 \\ -v_i & -v_i & 0 \\ v_i^2 + s & 1 & -v_i - r \end{vmatrix} = -(p + 1)v_i(v_i^2 + 1 - s) \neq 0 \text{ if } s \neq 1. \]  

By the Implicit Function Theorem, there exists a positive number \( 1 \gg \lambda_0 > 0 \) such that for any \( |y| < \lambda_0 \), the system (75)-(77) has two roots \( x_{fi} \) given by

\[ x_{fi} = x_i + \varphi_i(f), \varphi_i(0) = 0, i = 1, 2. \]  

As a result, the system (75)-(77) has two critical points for \( 0 < |f| < \delta \ll 1 \), which are in the small neighborhood of points \( x_1 \) and \( x_2 \), respectively.

Next, we will use the discriminant to show that the other two roots of (81) are complex if \((r, s, f)\) are near the point \((0, 1, 0)\), but real if \((r, s, f)\) are near \((0, 0, 0)\). Indeed, the discriminant of (81) is given by

\[ \Delta(p, r, s, f) = -\frac{1}{27} (\Delta_1^2 - 4 \Delta_0^3), \]  

where \( \Delta_i, i = 0, 1 \) are defined as

\[ \Delta_0 = b^2 - 3ac + 12d, \Delta_1 = 2b^3 - 9abc + 27a^2d + 27c^2 - 72bd, \]  

\[ \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = \Delta_7 = \Delta_8 = \Delta_9 = 0. \]
Using the fact that
\[
\Delta(p, 0, 0) = \frac{16f^4}{p+1} + f^6 \left( \frac{448}{p+1} - \frac{176p^2}{(p+1)^2} - \frac{16p^3}{(p+1)^3} - 256 \right) + f^8 \left( \frac{304p^3}{(p+1)^3} - \frac{176p^4}{(p+1)^4} - \frac{128p^2}{(p+1)^2} \right) + f^{10} \left( \frac{16p^5}{(p+1)^5} - \frac{16p^4}{(p+1)^4} \right),
\]
(95)
and continuity of the discriminant function, there must exist a \(\lambda\) such that
\[
0 < |f| < \lambda, 0 < r < \lambda, 1 - \lambda < s < 1, \Delta(p, r, s, f) < 0,
\]
(97)
\[
0 < |f| < \lambda, 0 < r < \lambda, 0 < s < \lambda, \Delta(p, r, s, f) > 0,
\]
(98)
(97)-(98) state that the other two roots are complex if parameters \((r, s, f)\) are in neighborhood of \((0, 1, 0)\), whereas the four roots of \((81)\) are real if the parameters \((r, s, f)\) are in neighborhood \((0, 0, 0)\) with one close to \(x_1\), one close to \(x_2\), and the other two close to \(0\).

As a final step, we need to prove that the two roots close \(0\) are negative if \(f\) is positive, but they are positive if \(f\) is negative. Let
\[
v_{f1} = \sum_{n=0}^{\infty} a_n f^n, v_{f2} = \sum_{n=0}^{\infty} b_n f^n
\]
(99)
be two roots close to \(0\), and substitute (99) into \((81)\) to get
\[
v_{f1} = \sqrt{1-s} + \frac{ps + 2 - s}{(2 - 2s)(p + 1)} f + o(f),
\]
(100)
\[
v_{f2} = -\sqrt{1-s} + \frac{ps + 2 - s}{(2 - 2s)(p + 1)} f + O(f^2),
\]
(101)
and
\[
v_{f1} + v_{f2} = \frac{ps + 2 - s}{(1-s)(p+1)} f + O(f^2).\]
(102)
Denote the other roots of \((81)\) as \(v_{fi}(i = 3, 4)\). Based on the relationship between roots and coefficient, one have
\[
v_{f1} + v_{f2} + v_{f3} + v_{f4} = -\frac{2pf}{p+1}, v_{f1}v_{f2}v_{f3}v_{f4} = -f^2.
\]
(103)
It is apparent from (103) that the sign of the two roots \(v_{f3}\) and \(v_{f4}\) of \((81)\) that are close to \(0\) depends on sign of \(f\). For \(f > 0\), conditions (102)-(103) indicate that \(v_{f3}\) and \(v_{f4}\) must be negative. Likewise, conditions (102)-(103) indicate that \(v_{f3}\) and \(v_{f4}\) must be positive for \(f < 0\) as stated.
Regarding the existence of a value \( s_0 \) such that there are exactly three critical points as stated, it is seen from (97)–(98) that there must exist \( s_0 \) such that
\[
\Delta(p, r, s_0, f) = 0, \quad (104)
\]
By Implicit Function Theorem, we know (81) must then have two different roots with different sign. Along with conditions (99)–(104), the other two roots of (81) must therefore be the same and negative. This constraint on the roots of (81) implies that (69)–(71) have three nonzero critical points, only one with a positive component \( v > 0 \).

Although Theorem 3.5 could reveal the number of the critical points that the MSD system could have in the general case with \( f \neq 0 \), the theorem does not give explicit expressions for these critical points, which are necessary to evaluate their stability. Depending on the values of various parameters, it is seen from Theorem 3.5 that the MSD system (69)–(71) may have two, three, or four nonzero critical points. In general, the explicit expressions for the critical of (69)–(71) are hard to obtain for arbitrary parameters \( (r, s, f) \). Nonetheless, we observe a particular property of the Coriolis parameter \( f \) that \( f \ll 1 \), and that the critical points of the MSD system will approach \( x_{1,2} \) as \( f \to 0 \). To obtain the approximate expressions for these critical points for \( f \neq 0 \), we can therefore employ the Taylor expansion in terms of \( f \) for few special limits of parameters \( (r, s, f) \) that the MSD system (69)–(71) can be simplified. Specifically for \( r < 1, s < 1, \) and \( 0 < f \ll 1 \), the MSD system will have four nonzero critical points \( x_{fi}(i = 1 \cdots 4) \) with their \( v \) component \( v_{fi}(i = 1 \cdots 4) \) obtained directly from (75)–(77) and (81) as follows
\[
x_{fi} = \left( \frac{-v_{fi}^2}{f + v_{fi}}, \frac{v_{fi}}{v_{fi}}, f + o(f) \right), i = 1 \cdots 4, \quad (105)
\]
where
\[
v_{f1} = \sqrt{1 - s} + \frac{p + 2 - s}{(2 - 2s)(p + 1)} f + o(f), \quad (106)
\]
\[
v_{f2} = -\sqrt{1 - s} + \frac{p + 2 - s}{(2 - 2s)(p + 1)} f + O(f^2), \quad (107)
\]
\[
v_{f3} = \begin{cases} 
-f + \frac{1}{2 + 2p} (r + \sqrt{r^2 + 4 + 4p}) f^2 + o(f^2), & s = 0, \\
-f + \frac{1}{s(p + 1)} f^3 + o(f^3), & 0 < s < 1,
\end{cases} \quad (108)
\]
\[
v_{f4} = \begin{cases} 
-f + \frac{1}{2 + 2p} (r - \sqrt{r^2 + 4 + 4p}) f^2 + o(f^2), & s = 0, \\
-f + \frac{1}{s(1 - s)} \left( \frac{2p}{(p + 1)(1 - s)^2} - \frac{1}{(1 - s)^3} - \frac{p}{(p + 1)(1 - s)} \right) f^2 + o(f^2), & 0 < s < 1.
\end{cases} \quad (109)
\]

For a special case of a very stable atmospheric stratification such that \( s = 1, \) the Taylor expansion techniques will not be applicable due to potential emergence of degenerate solutions, and so this case will not possess explicit solutions. Note however that if the radiative forcing is negligible as often assumed in previous studies such as \( r \approx 0, \) the special case of \( s = 1 \) will possess approximate expressions for
two critical points, whose $v$ components can be derived from (81) and (75)–(77) as follows
\[ v_{f_1}^* = -f + \frac{1}{p+1} f^3 + o(f^3), \quad v_{f_2}^* = f^\frac{1}{2} \]
and the complete critical points $x_{f_1,2}^*$ are therefore given by
\[ x_{f_1}^* \approx \left( -\frac{p+1}{f} f^3 + \frac{1}{p+1} f^3 + o(f^3), -1 + \frac{f^2}{p+1} \right), \]
\[ x_{f_2}^* \approx \left( -f^\frac{1}{2}, f^\frac{1}{2}, f^\frac{1}{2} \right). \]

Obviously, as $f \to 0^+$, the first component of $x_{f_1}^*$ goes to infinity and the other critical point $x_{f_2}^*$ approaches 0. This asymptotic limit implies that $x_{f_1}^*$ is not a physical critical point, and the $MSD_+$ system has a unique zero critical point for the set of parameters $r \approx 0, s = 1, f = 0$, consistent with the result (35) obtained from Section 3.1.1.

3.2.2. **Linear stability.** Given the above expressions for the critical points in the two special cases of either $(s < 1, r < 1, |f| \ll 1)$, or $(r = 0, s = 1, |f| \ll 1)$, we have the following important results regarding the stability of the critical points.

**Theorem 3.6.** For general parameters $(r, s, f)$ with $(0 \leq s < 1, r \ll 1, |f| \ll 1)$, the $MSD_+$ system (69)–(71) has one unique stable critical point with positive component $v$. All other three critical points are unstable.

**Proof.** Because of the complication of different numbers of critical points for different signs of the Coriolis parameter $f$, we will divide our proof into two different cases:

**Case 1.** $f > 0$

By Theorem 3.5, there exists $f_0$ such that $0 < r < f_0, 0 < s < f_0$, and the MSD system has only one critical point $x_1$ with $v_{f_1} > 0$. All other three critical points have $v_{f_2,3,4} < 0$. Let $x_c = (v_c, w_c, b_c)$ be any critical point, the Jacobian of $F$ at $x_c$ is as follows
\[ \frac{\partial F(x)}{\partial y} |_{(x_c) = (v_c, w_c, b_c)} = \begin{pmatrix} -v_c & 2pv_c - u_c + pf & -p - 1 \\ -v_c - f & -u_c - 2fv_c & 0 \\ b_c + s & -u_c + r & -u_c - r \end{pmatrix}, \]
Direct calculation of the determinant matrix $F$ yields the characteristic equation as follows
\[ \lambda^3 + A\lambda^2 + B\lambda + C = 0, \]
where
\[ A(x_c, p, r, s, f) = 3v_c + r, \]
\[ B(x_c, p, r, s, f) = 2(p + 1)v_c^2 + 3pfv_c - fu_c - (u_c - r)(3v_c + u_c) + (p + 1)(b_c + s) + pf^2, \]
\[ C(x_c, p, r, s, f) = (p + 1)(b_c u_c + 2b_c v_c + su_c + 2sv_c) - (u_c - r)(2v_c^2 + 2pv_c^2 + 3pfv_c - fu_c) - (p + 1)(v_c + f) - p(f^2 u_c - rf^2), \]
To determine the possible roots of (114) using the coefficients \(A, B, C\), we employ the smoothness of these coefficients with respect to the parameters \((r, s, f)\) and recall from Theorem 3.2 that in the limit of \(f = 0\), we have

\[
A(x_1, p, r, s, 0) > 0, B(x_1, p, r, s, 0) > 0, C(x_1, p, r, s, f) > 0,
\]

\[
A(x_1, p, r, s, 0)B(x_1, p, r, s, 0) - C(x_1, p, r, s, 0) \geq (p + 1)[(3v_1 + r)2v_1^2 - v_1^3 - v_1 + sv_1] > 0. \tag{118}
\]

For any given parameters \((r, s)\), it is known further from Theorem 3.5 that

\[
\lim_{f \to 0} x_{f1} = x_1, \lim_{f \to 0} x_{f2} = x_2. \tag{119}
\]

By the fact that eigenvalue of matrix (113) is continuously dependent on \(f\) and (118)–(119), it is therefore guaranteed that there must exist a value \(f_0(r, s)\) such that condition (118) will be applied for the critical point \(x_{f1}\) for \(0 < f < f_0(r, s) \ll 1\). Hence, the critical point \(x_{f1}\) is stable. Similar argument for the other critical point \(x_{f2}\) confirms also that \(x_{f2}\) is unstable for \(0 < f < f_0(r, s) \ll 1\) as well.

Regarding the other two critical points \(x_{f3}\) and \(x_{f4}\) near 0, one notices from (108) and (109) that the approximate expressions for \(x_{f3}\) and \(x_{f4}\) with \(r = s = 0\) can be explicitly obtained as follows

\[
x_{f3} = \left( -\sqrt{p+1} + O(f), -f + \frac{1}{p+1} f^2 + o(f^2), -\sqrt{\frac{1}{p+1} f + o(f)} \right), \tag{120}
\]

\[
x_{f4} = \left( \sqrt{p+1} + O(f), -f - \frac{1}{p+1} f^2 + o(f^2), \sqrt{\frac{1}{p+1} f + o(f)} \right). \tag{121}
\]

The explicit expressions (120)–(121) indicate that there must exist \(f_0\) such that \(0 < r < f_0\) and \(0 < s < f_0\), and \(A(x_{f1}, p, r, s, f)(i = 3, 4) = 3v_{f4} + r < 0\). As a result, the corresponding Jacobian matrix has at least one positive eigenvalue, and so both critical points \(x_{f3}\) and \(x_{f4}\) are unstable for the general values of \(r \neq 0, s \neq 0\).

**Case 2.** \(f < 0\)

As indicated by Theorem 3.5, the MSD\(_+\) system will have four critical points given by (105), three of which \(x_{1,3,4}\) have positive \(v\) component, and the other \(x_2\) has a negative \(v\) component. Note again that the eigenvalue of matrix (113) is continuously dependent on \(f\). Hence, direct check of (115)–(117) will show that there exists also \(f^* < 0\) such that if \(f^* \ll f < 0\), \(x_{f1}\) is stable, and \(x_{f2}\) is unstable similar to the case of \(f > 0\).

For the other two critical points \(x_{f3}\) and \(x_{f4}\), substituting (120)–(121) into (115)–(117), we have

\[
B(x_{f3}, p, 0, 0, f) = -3\sqrt{p+1} f - (p + 1) + o(f), \tag{122}
\]

\[
B(x_{f4}, p, 0, 0, f) = 3\sqrt{p+1} f - (p + 1) + o(f). \tag{123}
\]

Given (122)–(123), we conclude that \(x_{fj}(j = 3, 4)\) are also unstable for \(f < 0\) as expected.

**Remark 5.** The Implicit Function Theorem used in proving of Theorem 3.6 requires the regularity of the Jacobian matrix. It is easy to see from (91) that for the case of \(s = 1\), the Implicit Function Theorem may however fail if the radiative forcing parameter \(r = 0\). Indeed, Theorem 3.5 states that the MSD\(_+\) system (69)–(71) has only two nonzero critical points for sufficiently large value of \(s\) in the absence of the radiative forcing. In the following, we will use the approximate expressions for the
two critical points given by (111)–(112) to examine their stability separately for the special case of \( s = 1, r = 0 \). We have the following Proposition:

**Proposition 3.** For \( s = 1, 0 < r < 1 \), and \( 1 > f > 0 \), the \( MSD_+ \) system has two critical points, and only critical point with \( v > 0 \) is stable. For \( f < 0 \), both critical points are unstable.

**Proof.** Substituting (111)–(112) into (115)–(117) and note that for very small value of \( f \), we have

\[
A(x_{f1}^*, p, 0, 1, f) > 0, C(x_{f1}^*, p, 0, 1, f) > 0, B(x_{f1}^*, p, 0, 1, f) > 0, \quad (124)
\]

\[
A(x_{f1}^*, p, 0, 1, f)B(x_{f1}^*, p, 0, 1, f) - C(x_{f1}^*, p, 0, 1, f) > 0. \quad (125)
\]

Apparently, conditions (124)–(125) indicate that \( x_{f1}^* \) is stable because all eigenvalues of the Jacobian matrix will have negative real parts. On the contrary, a direct calculation of the coefficients \( A, B, C \) for the critical point \( x_{f2}^* \) given by (112) shows that \( B(x_{f2}^*, p, 0, 1, f) < 0 \), and the so condition (124) will not be ensured, thus proving that \( x_{f2}^* \) is unstable due to the existence of at least one eigenvalue with a positive real part. Similar calculations for the case of \( f > 0 \) shows that both nonzero critical points are unstable because \( B(x_{f1,2}^*, p, 0, 1, f) < 0 \) for \( f < 0 \). \( \square \)

Regarding the zero critical point \( x_0 \) of the \( MSD_+ \) system that is not addressed in Theorem 3.6 and the Proposition 3 above, we have the following result about its stability in the general presence of the Coriolis force.

**Proposition 4.** If \( f > 0 \), the zero critical point \( x_0 \) of the \( MSD_+ \) system is unstable. If \( f < 0 \), \( x_0 \) can be however stable if the constraint \( rs + f > 0 \) is held.

**Proof.** Substituting the zero critical point \( x_0 \) into (115)–(117), we have

\[
A(x_0, p, r, s, f) = r, \quad (126)
\]

\[
B(x_0, p, r, s, f) = (p + 1)s + pf^2, C(x_0, p, r, s, f) = -(p + 1)f + prf^2. \quad (127)
\]

\( x_0 \) stable if and only if

\[
A(x_0, p, r, s, f)B(x_0, p, r, s, f) - C(x_0, p, r, s, f) > 0, \quad (128)
\]

which is equal to

\[
rs + f > 0. \quad (129)
\]

The condition for all negative real parts can be only satisfied if \( f < 0 \) such that \( C(x_0, p, r, s, f) > 0 \), provided that the amplitude of \( f \) must be sufficiently small so that the constraint (129) is valid as well. \( \square \)

A remarkable implication of Theorem 3.6, Propositions 3 and 4 is that the general \( MSD_+ \) has a unique stable point for \( f > 0 \), and potentially another stable point at \( (0, 0, 0) \) for \( f < 0 \) if \( rs + f > 0 \). Recall the convention from Remark 2 that any analysis of the \( MSD_+ \) system with \( f < 0, v > 0 \) implies similar results for an anticyclonic vortex in the \( MSD_- \) system with \( f > 0, v < 0 \). Because of this convention, the potential existence of the new stable point at \( (0, 0, 0) \) for the case of \( f < 0 \) as stated in Proposition 4 indicates that an initially weak anticyclonic flow with \( v < 0 \) in the Northern Hemisphere may be quickly pulled toward the origin \((0, 0, 0)\), and possibly cross the plane \( v = 0 \) to enter the cyclonic domain \( \Omega_- \). Depending on the relative basins of attraction between the stable point \( x_{f1} \) and \((0, 0, 0)\), the flow may then be either trapped to the zero state or proceed to \( x_{f1} \). While Proposition 4 could not tell us how the basins of attraction towards each
critical point look like, the possibility of an initial anticyclonic vortex to switch to a cyclonic vortex for \( f < 0 \) is a very remarkable result, because it implies that the Coriolis force breaks the symmetry between the cyclonic \((v > 0)\) and anticyclonic \((v < 0)\) vortex development. Specifically, the TC development in the Northern Hemisphere will favor the cyclonic state, even if an initial vortex is anticyclonic, provided that the initial strength of the anticyclonic vortex is not too strong (i.e., \(|v| \ll 1|s| < rs\). Details of this symmetry breaking can be verified by numerical integrations of the MSD system, for which we will present in Section 4.

3.3. **Intensity Bifurcation.** The examination of the critical points and their topological index presented in Section 3.1 and 3.2 allows us to examine next a number of interesting aspects of the bifurcation of the MSD system as parameters \((r, s, f)\) vary. Among all parameters of the MSD system, the Coriolis parameter \(f\) and the atmospheric stratification parameter \(s\) are most important as indicated in Lemma 3.3, Theorem 3.6, and Proposition 4. As such, we will present in this section further analyses of possible bifurcations of the \(MSD_+\) system with a note that the \(MSD_+\) system with \(v > 0, f < 0\) will represent the \(MSD_-\) system with \(v < 0, f > 0\) (i.e., anticyclonic flow).

Consider the first the bifurcation of the \(MSD_+\) system with a fixed value of \(f > 0\) and parameter \(s\) varying in \([0,1]\). Theorem 3.5 states that there exists a limit \(\delta_0\) such that the \(MSD_+\) system has four nonzero critical points if \(0 < s < \delta_0\), and two nonzero critical points if \(1 - \delta_0 < s < 1\). An immediate consequence of this Proposition is that there must exist a value \(s_0\) that acts as a bifurcation threshold where the number of the critical points jumps from 4 to 2. In fact, the third statement of Theorem 3.5 indicates that the \(MSD_+\) system has exactly three nonzero critical points due to the merging of the two critical points for \(s = s_0\).

In Figure 1, one starts with four critical points for \(s \ll 1\) and ends with two critical points for \(s = 1\) as seen from Theorem 3.5. Somewhere between \([0,1]\), there exists a value \(s_0\) at which two unstable critical points \(x_{f2,4}\) merge, afterward the \(MSD_+\) system possesses just two critical points as stated in Theorem 3.5.

From the physical point of view, the bifurcation of the \(MSD_+\) system with the stratification parameter \(s\) reveals the sensitive dependence of TC development on the atmospheric stratification that has been recently noted in a number of modeling and statistical studies [19, 10, 7]. In fact, the modeling study in [10] suggested that the impacts of the atmospheric stratification could offset as large as 50% the positive influence of warmer sea surface temperature, thus likely accounting for an overall decrease of the TC intensity even with an increase of sea surface temperature. Our detailed examination of the \(v\) component of the stable critical point \(x_{f1}\) given by (106) indeed confirms the decrease of \(v\) as \(s\) increases to 1, beyond which the critical point \(x_{f1}\) no longer exists. In this regard, the MSD system could indeed capture well the dependence of the MPI limit on the tropospheric stratification that has not been previously understood.

Regarding the bifurcation with respect to the Coriolis parameter \(f\), we note from Lemma 3.4 that \(f = 0\) is a bifurcation point of the \(MSD_+\) system due to the sudden jump of the index of the forcing vector field of the \(MSD_+\) system at \(f = 0\). Note however from Theorem 3.5 that for \(s \ll 1\) the \(MSD_+\) system will have four
nonzero critical points, two of which are in neighborhood of 0 whose the sign of their \(v\)-components depends on the sign of the Coriolis parameter \(f\). To confirm that \(f = 0\) is indeed a bifurcation point for the \(MSD_+\) system in the presence of these two critical points, recall that the forcing vector field of the \(MSD_+\) system is given as follows:

\[
F^*_{rs}(x) = \begin{pmatrix} \frac{p v^2 - (p + 1)b - uv + pf v}{b u - v^2 - fu} \\ -uv - v^2 - fu \\ bu + su + v - rb \end{pmatrix}, \quad x = (u, v, b),
\]

(130)

Given this forcing vector field, we have the following Proposition concerning the index of the \(MSD_+\) forcing vector field at all four nonzero critical points discussed in Theorem 3.6:

**Proposition 5.** For the critical points given in Theorem 3.6 with small values of parameters \((r, s, f)\), we have

\[
\text{ind}(F^*_{rs}(x_{f1})) = -1, \quad \text{ind}(F^*_{rs}(x_{f2})) = 1,
\]

\[
\text{ind}(F^*_{rs}(x_{f3})) = \text{sign}(-f), \quad \text{ind}(F^*_{rs}(x_{f4})) = \text{sign}(-f),
\]

Proof. By Theorem 3.5, we know that if parameters \((r, s, f)(1 \gg f > 0)\) are in a small neighborhood of \((0,0,0)\), the characteristic equation (81) has four critical points. In addition, the index of critical point close to \(x_1\) is 1, and the index of the critical point close to \(x_2\) is \(-1\) due to the smoothness of the determinant of the Jacobian and Lemma 3.3. For the index of other critical points \(x_{fi}(i = 3, 4)\) near \((0,0,0)\), it suffices to consider \(r = s = 0\) due to the smoothness of the Jacobian matrix with respect to \(r, s\) so that one can obtained from (120)–(121) and (40) the following evaluation

\[
\text{ind}(F_{00}(x_{fi}))(i = 3, 4) = \text{sign}(-C(x_{fi}, p, 0, 0, f)) = \text{sign}(-2(p + 1)f + o(f)) = \text{sign}(-f).
\]

(131)
A direct implication of Proposition 5 is that the total index of the forcing vector field of the $MSD_+$ system will jump from -1 to 1 as parameter $f$ goes from negative to positive values (see Fig. 2). If one notes further from Eqs. (108)–(109) that
\[
\lim_{f \to 0} x_f^3 \neq (0, 0, 0),
\]
\[
\lim_{f \to 0} x_f^4 \neq (0, 0, 0),
\]
it is then evident that the bifurcation of system at $f = 0$ is indeed a jump as expected. The bifurcation diagram of the $MSD_+$ system as a function of $f$ is given in Figure 2.

4. **Numerical analysis.** To demonstrate the stability of the MPI equilibrium and bifurcation of the $MSD_+$ system analyzed in Section 3, Figure 3 shows several examples of flow orbits in the phase space of $(u, v, b)$ for a number of different initial conditions in numerical integrations of the $MSD_+$ system (69)-(71). Here, all numerical experiments with the $MSD_+$ system are carried out, using the Runge-Kutta 4th-order scheme with time step $\delta t = 0.001$ and integrated for 10000 time steps. All other parameters $(r, s, f)$ are set within a typical range of the TC scales in the real atmosphere, i.e. $r = 0.1$, $s = 0.1$, and $|f| = 0.01$.

Despite much different initial conditions, it is seen in Figure 3 that all orbits converge to the same MPI critical point regardless of the initial conditions, thus confirming the linear stability analyses of the $MSD_+$ system similar to the HSD system [11, 12]. Note that the stability analysis in Section 3 by no mean ensures the global stability or dictates the basin of attraction around critical points. An important question regarding the basin of the attraction of the MPI point is much harder to quantify, and can only be addressed from a numerical perspective for
Figure 3. a) Flow trajectories for four different initial points in the phase space of \((u, v, b)\) that represent an incipient weak anticyclonic vortex \((-0.1, -0.1, 0.1)\) (red); a mature TC near the MPI equilibrium with a weak warm core \((-1, 1, 0.5)\) (cyan); a mature TC with intensity significant above the MPI equilibrium limit \((-1, 1.4, 1)\) (green); and a mature TC near the MPI equilibrium limit with too weak low level convergence \((-0.1, 1, 1)\) (blue) for the case of \(f = 0.05\); (b) Time series of \(v\) during the entire simulation; and (c)-(d) Similar to (a)-(b) but for the case of \(f = 0\).

For a large number of numerical experiments with different initial conditions. Our experiments with various initial conditions confirm that the basin of the attraction toward the MPI stable point \(x_{f1}\) is ensured for the domain \(v \geq \epsilon, b \geq 0\), where the lower bound \(\epsilon\) depends on the value of \(f\) and can be negative for \(f < 0\) (see the red orbit in Figure 3a,b).

Of particular importance from this numerical experiment is the orbit starting from the initial point \((-0.1, -0.1, 0.1)\), which represents an incipient anticyclonic flow (red curve). In the absence of the Coriolis force, this initial point would immediately converge to the negative MPI stable point in domain \(\Omega_-\) due to its stable property as seen from Theorem 3.2 and Remark 2 (see Figure 3c-d). In the presence of the Coriolis force, this initial vortex however evolves in such a way that the tangential flow \(v\) first crosses the plane \(v = 0\) and eventually approaches the positive MPI critical point in the domain \(\Omega_+\) (Figure 3a-b). This indicates that Coriolis force shifts the domain of TC development more toward cyclonic flows in the Northern Hemisphere due to its enlarged basin of attraction, and explains the dominance of the cyclonic TCs in the Northern Hemisphere, even if one starts with an initially weak anticyclonic vortex. In this regard, it is remarkable to see from these numerical experiments a larger basin of attraction towards the unique MPI stable equilibrium in the presence of the Coriolis force as suggested by Proposition
5, which justifies the dominance of cyclonic TCs in the Northern Hemisphere as observed.

5. Conclusion. In this study, a low-order model based on the TC basic scales has been presented to study the stability of the MPI equilibrium in the presence of both Coriolis force and tropospheric stratification. Unlike the previous model in which the exact gradient wind balance is assumed [11], the extended model in this study allows the gradient wind imbalance in the PBL in the presence of both the Coriolis force and atmospheric stratification. Stability analyses of this extended model confirmed the existence of a unique stable point that corresponds to the MPI equilibrium under the WISHE feedback mechanism. That different balance approximations could lead to the same unique MPI point and its stable behaviors indicates that the long term stability of the MPI equilibrium is mostly governed by the gradient wind balance and the hydrostatic balance.

Of particular significance in this study is the findings that both Coriolis force and the atmospheric stratification act as bifurcation parameters in TC development. Specifically, our detailed analyses of the MPI stability showed that a more stable atmosphere would be inimical to TC development, and results in a weaker MPI limit. As the atmospheric stratification becomes more stable than a given threshold, the MPI limit will vanish and any initial TC-like vortex will not be able to attain their maximum intensity limit. This new factor related to the tropospheric stratification is beyond the traditional TC maximum intensity theory in which the MPI limit is solely determined by sea surface temperature and outflow temperature near the tropopause. Our results reveal explicitly how the MPI depends on the stratification of the atmosphere, which to our knowledge has not been obtained previously.

Similar to the bifurcation related to the atmospheric stratification, it turns out that Coriolis force also imposes a strong bifurcation property on TC development, despite its small magnitude in the tropical region where TCs reside. Detailed analyses of the MSD system in the presence of the Coriolis force revealed the existence of another stable point at the origin \((0,0,0)\) in the phase space of \((u,v,b)\), which corresponds to no TC development at all. While the basin of attraction of this zero critical point cannot be obtained analytically due to the complex dynamics near \((0,0,0)\), the emergence of this stable point results in a very important symmetry breaking in the development between cyclonic and anticyclonic TCs in the Northern Hemisphere. Specifically, the Coriolis force allows an initially weak anticyclonic vortex to be attracted toward the zero critical point \((0,0,0)\), and enter the domain of cyclonic flows. Due to a larger basin of attraction towards the MPI stable point, this initially weak anticyclonic vortex then quickly approaches the cyclonic MPI stable point, thus creating an asymmetry in TC development with more favorable development of cyclonic TCs than anticyclonic TCs in the Northern Hemisphere as observed. Our additional numerical integration of the MSD system confirmed this role of the Coriolis force in breaking the symmetry between cyclonic and anticyclonic TCs. Such intricate impacts of the Coriolis force in TC development explain why TCs are dominantly cyclonic in the Northern Hemisphere, thus offering new insight into TC development beyond the traditional explanation that is merely based on TC initial conditions.

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