A Toy Model of the M5-brane: Anomalies of Monopole Strings in Five Dimensions.

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We study a five-dimensional field theory which contains a monopole (string) solution with chiral fermion zero modes. This monostring solution is a close analog of the fivebrane solution of M-theory. The cancellation of normal bundle anomalies parallels that for the M-theory fivebrane, in particular, the presence of a Chern-Simons term in the low-energy effective $U(1)$ gauge theory plays a central role. We comment on the relationship between the the microscopic analysis of the world-volume theory and the low-energy analysis and draw some cautionary lessons for M-theory.

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1 Introduction

The question of anomaly cancellation in the low-energy effective action of M-theory in the presence of M5-brane was discussed in [1] where non-cancellation of the normal bundle anomaly was first found. This issue was addressed later by several authors from different points of view (see e.g. [2, 3, 4] and discussions therein). In this paper we are continuing to follow the path of [3].

The spectrum of a theory containing solitonic brane solutions can be decomposed into normalizable modes living on the brane, and non-normalizable bulk modes. If one approximates these as decoupled theories, then one can find inconsistencies at the very beginning. In particular, if some of the zero modes are chiral and the dimension of the world-volume is even (as for the M5-brane or solitonic string case), the world volume theory suffers from gravitational and gauge anomalies. Thus it cannot be considered by itself and the two theories – bulk and world-volume – interact in such a way that the whole theory is anomaly free. In general one finds that the bulk action has a gauge variation which is non-zero only on the brane, and this variation cancels the gauge variation arising from the localized chiral zero modes. Such cancellation of anomalies is sometimes referred to as the “inflow mechanism” [5].

As far as gravitational anomalies are concerned, the world-volume zero modes transform under local Lorentz transformations with the structure groups of tangent and normal bundles of the brane’s world-volume. The latter can be treated as a gauge symmetry from the world-volume theory point of view. Anomalies of the chiral fields of this theory are in general derived from anomaly polynomials, given by Atiyah-Singer index densities. This expression is not symmetric with respect to normal and tangent bundles. On the other hand, the naive variation of gravitational Chern-Simons terms in the bulk effective action is symmetric with respect to normal and tangent bundles. Therefore some other contribution to gravitational anomaly inflow from the bulk is required.

The work [3] was based on the idea that for the M5-brane the additional contribution comes from the $C \wedge dC \wedge dC$ Chern-Simons term of eleven-dimensional supergravity when properly defined in the presence of an M5-brane. The Chern-Simons is defined in terms of descent of a globally defined gauge invariant closed form on a manifold of one dimension higher. In the presence of an M5-brane, the differential of the naive Chern-Simons is no longer gauge-invariant. Therefore we should construct a globally defined, closed, gauge invariant form in one dimension higher first. This was done
in [3] regularizing the M5-brane source, solving the regularized magnetic coupling equation (Bianchi identity) and constructing a closed form out of this solution. The Chern-Simons term defined by such a procedure also gets regularized – it changes near the brane – and this regularization is not invariant under normal bundle gauge transformations. This gives an additional contribution to the normal bundle anomaly inflow only and cancels the anomaly.

The price paid for this anomaly canceling mechanism was the presence in the action of a \textit{bump function}: an arbitrary function of the distance from the brane with definite boundary values. This treatment left open the question of the physical origin of the bump function $\rho(r)$ and a microscopic derivation of the modified Chern-Simons term. The origin of the bump function was further addressed in a simple field theory model in [11].

In other examples of anomaly cancellation, one can often understand the cancellation in terms of the anomaly freedom of an underlying short distance theory. In the example of axion strings [3, 4], there is a low-energy effective theory containing chiral zero modes on the axion string, and a coupling of the axion to $F \wedge F$ which cancels the anomaly in the presence of a topologically non-trivial string. In the underlying theory, the fermion measure is gauge invariant, and the anomaly cancellation reflects the decomposition of this measure into a localized zero mode part and the remaining massive, bulk degrees of freedom. The axion-gauge coupling arises from integrating out the massive fermion degrees of freedom and clearly must cancel the zero mode anomaly since the full fermion measure is gauge invariant.

In M-theory, no such corresponding microscopic picture that would explain the observed anomaly cancellation is known. It would thus be interesting to know if there are other models where cancellation of normal bundle anomalies arises from a similar modification of a Chern-Simons term, and where this term can be derived from a consistent underlying theory.

Another interesting question raised by anomaly cancellation using Chern-Simons terms is the question of the scaling of the anomaly of the world-volume theory with the magnetic charge $N$ of the brane [2]. The Chern-Simons term is cubic in the gauge field which, naively speaking, should mean that the anomaly inflow coming from the Chern-Simons term and, therefore, the anomaly of the world-volume theory, should scale like $N^3$. The M5-brane world volume theory, the so called (2,0) theory, is very special and there are independent arguments for an $N^3$ dependence of related quantities [8, 9, 10]. All these arguments (including the inflow argument itself) are indirect. It is very interesting to see another example of the same kind,
with anomaly inflow coming from cubic Chern-Simons terms, but where the theory of the zero modes is known explicitly and the corresponding anomaly can be calculated directly.

To analyze these and other questions raised in [3], and to try to get some possible insight on the high-energy structure of M-theory, we are going to consider a simple field theory model, which however turns out to be very similar to the original M-theory setup. The theory in hand (Section 2) is just the theory of the 'tHooft-Polyakov monopole lifted to 5-dimensional space-time and coupled to fermions. The monopole solution, viewed as a solution in five dimensions which is static and independent of the fifth coordinate $x_4$, looks like a magnetically charged string (monostring). We analyze the theory of zero modes on the string and find that it is anomalous (Section 3). We show that gravitational and gauge Chern-Simons terms do appear in the low energy effective action of the bulk theory although no $SU(2)$ Chern-Simons in five dimensions is allowed (Section 4.2). We discuss the correct definition of the Chern-Simons term on in the monostring background and show that it varies under normal bundle transformations even in the limit of zero monopole core i.e. without regularization as in [3, 11] (which solves the above stated problem of appearance of an arbitrary bump function in the effective action\footnote{A similar construction for the M5-brane case has been recently presented in [11]}). This gives the possibility to have a non-symmetric inflow of anomalies with respect to normal and tangent bundles which indeed cancels the anomalies of the zero modes. This situation closely resembles the M5-brane, yet is not precisely the same, because in our case we know the full underlying $SU(2)$ theory and not just an effective description. We also discuss the anomaly analysis in the low-energy effective theory (Section 8) and discuss the relation to the picture in M-theory. Apart from that we analyze the situation of monopole solution with charge $N$ (Section 7), and show that in this case there is no argument for an $N^3$ dependence of the anomaly inflow (despite the presence of a cubic Chern-Simons term!), in contrast to the situation in M-theory.

2 The Model

We start by considering a five-dimensional theory with $SU(2)$ gauge group, Higgs fields in the adjoint representation, and fermions in either the fundamental or adjoint representation (later we briefly consider fermions in arbitrary
trary representations of $SU(2)$). This theory is not renormalizable, but we will only be concerned with the anomaly structure for which renormalizability is irrelevant. We take the Lagrangian to be (in the absence of gravity):

$$\mathcal{L} = -\frac{1}{4} (\text{Tr} \ G_{MN})^2 + \frac{1}{2} (D_M \Phi)^2 - \frac{1}{g^2} U(g \Phi) + i \bar{\psi}_n \slashed{D} \psi_n - G g \bar{\psi}_n T^a_{nm} \psi_m \Phi_a \quad (1)$$

Here $M, N = 0, \ldots, 4$ label coordinates in the bulk; indices $i, j, k = 1, 2, 3$ will be used later for bulk indices transverse to the string solution; $m, n = 1, \ldots, \text{dim}(r)$ are flavor indices and run up to the dimension of the fermion representation $r$; $a = 1, 2, 3$ label generators of $su(2)$ with commutation relations $[T^a, T^b] = i \epsilon^{abc} T^c$. We take the potential $U(\Phi)$ to be

$$U(g \Phi) = \frac{\lambda^2 v^2 g^2}{2} \left( 1 - \frac{1}{v^2} \Phi^a \Phi_a \right)^2 \quad (2)$$

In the fundamental representation we can choose

$$T^3 = \text{diag}(1/2, -1/2) \quad (3)$$

while in the adjoint representation we take

$$T^3 = \text{diag}(1, 0, -1) \quad (4)$$

This theory can also be coupled to gravity in the obvious way.

The Lagrangian (1) admits a monopole solution [12] which is independent of $x^0, x^4$ and varies only in the $y^i$ directions:

$$A^a_0 = 0 \quad (5)$$
$$A^a_i = -\epsilon_{aij} \hat{y}^j A(y)/g \quad (6)$$
$$\Phi^a = \hat{y}^a \phi(y)/g \quad (7)$$

Both $\phi(y)$ and $A(y)$ vanish at $y = 0$; for large $y \phi(y) \rightarrow g v$ and $A(y) \rightarrow -1/y$ exponentially. In $4 + 1$ dimensions this solution, supplemented with the condition $A^a_4 = 0$, describes a magnetically charged string, stretched along the $x^4$ direction. Note that the solution \( (5-7) \) is invariant under a diagonal $SO(3)_{\text{diag}}$ transformation with generators $K = \vec{J} + \vec{T}$ with $\vec{J}$ the generators of spatial rotations and $\vec{T}$ the generators of $SU(2)$ gauge transformations. This symmetry will play an important role in the later analysis.

\footnote{see Appendix \[3\] for conventions}
We want to consider the theory (1) on the monopole background (5-7) and explore the close analogy between anomaly cancellation in this theory with that of the M5-brane theory [3]. We will see that by integrating out massive fermions in this background we will obtain a theory closely resembling $D = 11$ SUGRA.

2.1 Abelian Gauge

Throughout this paper we will use the Abelian gauge condition (see e.g. [12, 13]). By performing the gauge rotation $G = \exp(-i\varphi T^3) \exp(i\theta T^2) \exp(i\varphi T^3)$ (where $\theta, \phi$ are spherical angles in transverse dimensions, $G \in SU(2)$) we rotate the field $\Phi^a$ given by (7) into $\tilde{\Phi}^a = \phi(y)\delta^a_3$. Such a transformation (which effectively can be implemented only outside the core i.e. when the profile of the gauge field $\phi(y)$ is not zero) is of course singular (it changes the topological charge of the field $\Phi$ configuration from 1 to zero):

$$Q_{\text{top}} = \frac{1}{4\pi} \int_{S^2} \epsilon_{abc} \Phi^a d\Phi^b \wedge d\Phi^c$$

(8)

This gauge transformation also rotates the gauge field (6) into the following configuration (outside the core): $\tilde{A}_i^{1,2}$ is equal to zero and $\tilde{A}_i^3$ is given by the gauge potential for a Dirac monopole (with the Dirac string along say the positive $y^3$ semi-axis):

$$\tilde{A}_i^3 = \frac{1}{g} \epsilon_{ijk} \hat{y}^k |y| - y^3$$

(9)

Following ’tHooft [12] we introduce the $U(1)$ field strength

$$F_{MN} = \tilde{\Phi}^a G_a^{MN} - \frac{1}{g} \epsilon_{abc} \tilde{\Phi}^a D_M \tilde{\Phi}^b D_M \tilde{\Phi}^c$$

$$= \partial_M (\tilde{\Phi}^a A^a_N) - \partial_N (\tilde{\Phi}^a A^a_M) + \frac{1}{g} \epsilon_{abc} \tilde{\Phi}^a \partial_M \tilde{\Phi}^b \partial_N \tilde{\Phi}^c$$

(10)

where $\tilde{\Phi}^a = \Phi^a/v$. This object is $SU(2)$ invariant, defined outside the core, and obeys the anomalous Bianchi identity:

$$dF = \frac{4\pi}{g} \delta^3(\Sigma_2 \hookrightarrow M_5)$$

(11)

showing that there is a magnetic monopole source for $F$. For brevity we are using the language of differential forms in (11). The precise meaning of the source term $\delta^3(\Sigma_2 \hookrightarrow M_5)$ will be explained later (Section 5.1).
3 Jackiw-Rebbi zero modes

Jackiw and Rebbi [14] considered the Hamiltonian for the Dirac operator (in 3 + 1 dimensions) in the monopole background (37). It has the following form:

\[ \hat{H}_D \psi = [\vec{\alpha} \cdot \vec{p} + A(y)T^a_{nm}(\vec{\alpha} \times \hat{y})_a + G\phi(y)T^a_{nm}\hat{y}^a\beta] \psi_m = E\psi_n \] (12)

Here \( \vec{\alpha} = \gamma^0 \vec{\gamma} \), with the gamma-matrix conventions given in Appendix A. This equation has zero modes – solutions with \( E = 0 \) for fermions in both the fundamental and adjoint representations. The Jackiw-Rebbi zero mode for both representations has the following form:

\[ \psi^\text{JR}_{n} = \begin{pmatrix} \chi_n^+ \\ \chi_n^- \end{pmatrix} \] (13)

where \( \chi_n^\pm \) are Weyl spinors. The zero energy solutions for both representations have \( \chi_n^- = 0 \). For fundamental fermions we have:

\[ \chi_{\alpha n}^+ = c \sigma_{\alpha n}^2 f(y) \] (14)

Here \( n = 1, 2 \) is a color index, \( \alpha = 1, 2 \) is a spinor index, and \( c \) is a normalization constant. The function \( f(y) \) is given by (we denote \( |y| \) by \( y \))

\[ f(y) = \exp \left( - \int_0^y dy' \left( \frac{1}{2} G\phi'(y') - A(y') \right) \right) \] (15)

For the adjoint representation the zero modes are given by (see [14] for details):

\[ \chi_n^+ = N \left[ \tilde{f}_1(y)\hat{y}^n(\sigma \cdot \hat{y}) + f_2(y)\sigma^n \right] s \] (16)

with \( N \) a normalization constant and \( s \) an arbitrary two-component spinor on which the \( \sigma \)-matrices act. We won’t need the explicit functions \( f_{1,2}(y) \), see [14], eqs. (3.8)-(3.10)), the only important property being that they depend on \( |y| \) and not on \( \hat{y} \). Also, for convenience we have introduced a new function (compared with \( f_1, f_2 \) from [14]): \( \tilde{f}_1(y) \equiv f_1(y) - f_2(y) \).

Note, that the existence and the number of these zero modes is dictated by the Callias index theorem [15].

We note the following: if (14), (16) are zero modes of the equation (12) (which we can think of as the Dirac equation in 3 dimensions), then we can
use them to build a solution of the Dirac equation in 4 + 1 dimensions in the background (5-7). This is done in the following way.

Consider the fundamental case first. Construct the following spinor out of the solution (14):

$$\Psi_n(x, y) = c(x) \psi^R_n(y)$$  \hspace{1cm} (17)

Here $\Psi$ is a five-dimensional spinor, $c(x)$ an arbitrary (complex) scalar function, and $\psi^R_n(y)$ is given by (13–14). The fermionic e.o.m.’s, following from (1) can be written as

$$\not \! D \Psi = i\gamma^0 (\partial_0 + \gamma^{int} \partial_4 - i\hat{H}_D) \Psi = 0$$  \hspace{1cm} (18)

The definition of $\gamma^{int}$ as the chirality matrix in the two-dimensions of the string world-sheet is given by (A5). Now substitute the ansatz (17) into (18). Note once again that both spinor and color indices are carried by $\psi^R_n$. Also note that $\gamma^{int} \psi^R = -\psi^R$. As a result, the equation $\not \! D \Psi = 0$ boils down to

$$\left( \partial_0 - \partial_4 \right) c(x) = 0$$  \hspace{1cm} (19)

We see that our model possesses a chiral (left-moving) zero mode, localized on the string and falling off exponentially (as $e^{-\nu y}$ at large $|y|$) in the transverse dimensions.

In a totally similar manner one can build the zero modes of (18) for adjoint fermions. The difference being that in this case there are two zero modes (16). The ansatz has the following form then:

$$\Psi_n(x, y) = \sum_{A=1}^{2} s_A(x) \psi^R_{n,A}$$  \hspace{1cm} (20)

The index $A$ is a spinor index of the transverse $SO(3)$. Substituting the ansatz (20) into eq. (18) we will get an answer similar to (19):

$$\left( \partial_0 - \partial_4 \right) s_A = 0 \quad A = 1, 2$$  \hspace{1cm} (21)

We will be also interested in finding the action for these zero modes. Substitute (17–20) into the action for fermions:

$$S_D = \int d^5 x \, \bar{\Psi} (i \not \! D - Gg\Phi) \Psi$$  \hspace{1cm} (22)

Here we consider $\not \! D$ in the background of the monopole gauge field plus some (small) fluctuations in the $(x^0, x^4)$ directions, which we call $a^\mu$ (recall that indices $\mu, \nu$ run along the world-sheet directions of the string).
First, consider fundamental fermions. Then eq. (22) can be written as:

\[ S_D[\Psi] = \int d^5x i\Psi^+(D_0 + \gamma^i D_i + \alpha^i + iG\gamma^0\Phi)\Psi \]

\[ = \int d^2x \left( \int d^3y f^2(y)\bar{\sigma}^2_{an}\sigma^2_{am}ic^*(x)(D_0 - D_4)_{nm}c(x) \right) \]

(we have \( \chi^+_n \chi_m = f^2(y)\sigma^2_{an}\sigma^2_{am} = f^2(y)\delta_{nm} \)). This means that eq. (23) is equivalent to:

\[ S_D[c] = N \int d^2x ic^*(\partial_0 - \partial_4)c \] (24)

Here \( N = 2 \int d^3y f^2(y) \) is a finite normalization constant. The gauge field \( a^a_\mu \) has dropped out of \((\tilde{\Phi}_{04})_{nm}\) because of \( \delta_{nm}\). We see that the 1+1 dimensional fermions effectively couple to \( \text{Tr} a^a_\mu \), which is zero for any \( SU(2) \) connection.

We see that fundamental fermions are singlets with respect to the diagonal \( SO(3)_{\text{diag}} \) defined earlier (c.f. [14]). This fact can be checked directly: acting on the fermion [14] with the transformation of \( SO(3)_{\text{diag}} \) leaves it invariant.

For adjoint fermions we substitute the ansatz (20) into the action (22). This gives:

\[ S_D[\Psi] = \int d^2x \sum_{A,B} 2_{nm} \int d^3y \chi^+_n \chi_m,A \]

we note that \( \sum_n \chi^+_n \chi_m,A \sim \delta_{AB} \) and that the indices \( A,B \) are acted upon by \( \sigma \)-matrices from (16). As a result one obtains:

\[ S_D[s] = \int d^2x c_1s^*_A(\partial_0 - \partial_4)s_A + s^*_A A^a_\mu \sigma^a_{AB} s_B \] (26)

here \( c_1 = \int d^3y (f^2_1 + f^2_2) \) and the “gauge field” on the brane is given by the expression:

\[ A^a_\mu = \int d^3y 2i\bar{f}_1f_2(\tilde{g}^b a^b_\mu)\tilde{y}^a + 2i\bar{f}_1f_2a^a_\mu \] (27)

From (26)-(27) we see that in this case the zero modes become charged with respect to \( SO(3)_{\text{diag}} \) and transform as (iso)spinor representation of \( SO(3) \) (c.f. [14]). The original spinor indices of \( \sigma_{AB} \) become gauge indices of \( SU(2) \). This is another exhibition that the zero modes provide a representation of the diagonal \( SO(3)_{\text{diag}} \).

Theories of chiral fermions in two dimensions suffer from gauge and gravitational anomalies. The magnetic string solution decomposes the tangent bundle to the five-dimensional space-time \( M_5 \) in the usual way as \( TM_5|_\Sigma = \)
$T\Sigma \oplus N$ and correspondingly the symmetry group of the tangent bundle $SO(4,1)$ breaks into $SO(1,1) \times SO(3)$. The group $SO(3)$, the structure group of the normal bundle to the string, becomes a gauge group from the point of view of two-dimensional fermions (the same as $SO(3)_{\text{diag}}$). Using the standard descent formalism [16, 17, 18, 19] we can write down anomalies associated with the fermion zero modes on the string. We find that the tangent bundle anomaly on string world-sheet plus $SO(3)_{\text{diag}}$ anomaly with respect to the group of normal bundle $N$ is given by

$$I_{\text{z.m.}}^{\text{fund}} = \frac{\pi}{12} \int_{\Sigma_2} p_1(T\Sigma)$$
$$I_{\text{z.m.}}^{\text{adj}} = \int_{\Sigma_2} \left( \frac{\pi}{6} p_1(T\Sigma) - \frac{\pi}{2} p_1(N) \right) \tag{28}$$

Note that the anomalies are non-symmetric between tangent and normal bundles. On the other hand, the corresponding terms in the bulk effective action, which are due to interaction with gravity, are expected to depend on the full 4 + 1 dimensional spin connection because local Lorentz symmetry is not broken in the bulk. Thus the contribution to inflow from such terms will be symmetric with respect to tangent and normal bundles as it was for the M5-brane case (as we will see below, this is also true in our model). This means that some other sources of anomalies should be found. This problem was originally resolved in the context of an analysis of the M5 brane by showing that the Chern-Simons term of $D = 11$ SUGRA has an additional anomalous variation. At first sight such a resolution cannot be applied in our case because no Chern-Simons term is allowed in 4 + 1 dimensions for an $SU(2)$ gauge theory. Any such term would arise from descent of $\text{Tr} F^3$ but this vanishes in $SU(2)$ (since there is no $d^{abc}$ coefficient in $SU(2)$). In Section 4.1 we will see what actually happens.

## 4 Effective Action in the Bulk

In this section we will implement the procedure, described in the introduction, to obtain a low-energy effective action. As mentioned earlier, the monopole solution breaks the $SU(2)$ gauge symmetry down to $U(1)$ and makes the fermions massive, with masses given by $Gv$ (c.f. (30–31)). We consider the low-energy effective action after integrating out these massive (charged) fermions. This action will be a function of the $U(1)$ gauge field and
of the gravitational field if we couple the theory to gravity. We will assume we have done this in the standard way.

Integrating out the massive fermions exactly is a difficult task, but we can go to Abelian gauge (Section 2.1) first and it will reduce the theory down to a $U(1)$ gauge theory coupled to massive charged fermions as described below (Section 4.1), at least outside the core of the monopole. All massive modes vanish near the core of monopole, and thus far from defect the theory looks topologically trivial (in Abelian gauge). Therefore we believe that in integrating over massive fermions we can use the standard (perturbative) approach (as in [21, 22]). In addition, we will only be interested in the parity non-invariant contributions to the effective action which play a role in anomaly cancellation.

4.1 Parity Violation

As is usual in odd-dimensions, a parity transformation changes the sign of an odd number of spatial directions and the fermion mass term changes sign under this transformation (see e.g. the Appendix of [20]). Thus, (taking into account that $U(\Phi)$ is an even function of $\Phi$), the above action has a parity symmetry given by

$$x_4 \to -x_4, \quad \psi \to \gamma^4 \psi, \quad \Phi^a \to -\Phi^a \quad (29)$$

The monopole solution (5-7) not only breaks $SU(2)$ down to $U(1)$, but it also breaks the parity symmetry. However the product of $P$ and charge conjugation $C$ (which takes $A_M \to -A_M$ with $A_M$ the $U(1)$ gauge potential and interchanges positive and negative charged states) remains a symmetry. For example, after symmetry breaking $SU(2) \to U(1)$ we have a mass term for fundamental fermions $\chi_\pm$ with $U(1)$ charge $\pm g/2$ of the form

$$Gv(\bar{\chi}_+\chi_+ - \bar{\chi}_-\chi_-) \quad \tag{30}$$

while for adjoint fermions $\psi_+, \psi_0, \psi_-$ with charges $g, 0, -g$ we have

$$Gv(\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_-). \quad \tag{31}$$

In either case we have two fermions with opposite charge and opposite mass. These mass terms are invariant under parity plus the interchange of $\chi_\pm$ or $\psi_\pm$. For adjoint fermions one field will remain massless and uncharged and will be present in the massless spectrum of the bulk effective theory.
Note that after symmetry breaking a Chern-Simons term for the $U(1)$ gauge field $\int A \wedge F \wedge F$ is allowed, and is consistent with the unbroken $CP$ symmetry. This does not contradict the comment made at the end of Section 3. The Chern-Simons term cannot be made out of the $SU(2)$ connection alone, but must also involve the scalar Higgs fields. We will see that the Chern-Simons term and other terms are induced at the one-loop level by integrating out the massive fermions.

### 4.2 Induced Chern-Simons Term

Following the results of [21, 22, 23], integrating out the massive fermions will induce, among other terms, parity odd Chern-Simons (CS) terms in the effective action. A fermion of mass $m$ in a representation $\mathbf{r}$ generates the C-S term

$$S_{\text{cs}} = -\pi \frac{m}{|m|} \int Q_5$$

where $dQ_5$ is given by the six-form part of the Dirac index density

$$dQ_5 = \hat{A}(R)\text{ch}_\mathbf{r}(F)|_6$$

We now apply this to our model with fermions of charge $\pm q$ with $q = g/2$, $g$ for fundamental or adjoint fermions and with opposite sign masses to obtain

$$S_{\text{cs}} = \pi \int Q_5^- - Q_5^+$$

where

$$dQ_5^\pm = \hat{A}(R)\text{ch}_\pm(F)|_6$$

$$= (1 - \frac{1}{24}p_1 + \cdots)(1 \pm \frac{q}{2\pi} F + \frac{1}{2!}(\frac{q}{2\pi})^2 F^2 \pm \frac{1}{3!}(\frac{q}{2\pi})^3 F^3 + \cdots)$$

which gives

$$S_{\text{cs}} = \int_{M_5} -\frac{q^3}{24\pi^2} A \wedge F \wedge F + \frac{q}{24} F \wedge p_1^{(0)}(R)$$

(In this expression we take the $U(1)$ gauge field to be Hermitian.)

Note the close analogy between this action and the action for M theory [29, 1]. The latter contains a CS term $\int C_3 \wedge G_4 \wedge G_4$ as well as the term...
\[ \int G_4 \wedge X_8^{(0)} \] with \( X_8 \) being an eight-form in curvature. We might also expect in analogy to M theory, that the \( F \wedge p_1(R)^{(0)} \) term is involved in cancellation of both tangent and normal bundle anomalies of the world-sheet zero modes while the \( A \wedge F \wedge F \) term will play a role in the cancellation of normal bundle anomaly. We will see that this is indeed the case.

5 Anomaly cancellation for \( N = 1 \)

There are three sources of anomalies: the fermion zero modes, inflow from the \( F \wedge p_1^{(0)}(R) \) coupling, and a contribution to the normal bundle anomaly from the CS term as in [3]. In the following we discuss the structure of the CS terms and the cancellation of anomalies in this model.

5.1 Discussion of the Chern-Simons terms

Following [3, 4] we want to correctly define the Chern-Simons term in the bulk effective action. It was implied in eqs. (33–36) that \( dF = 0 \) and so (at least locally) \( F = dA \). We would like to consider gauge fields in the same topological class as the background monopole solution \( F = F_{bg} + d\tilde{A} = d(A_{bg} + \tilde{A}) \), where \( A_{bg} \) could be defined only locally. To specify it explicitly, we take a closer look at (10). On the background (5-7) it has the form (again in the language of differential forms):

\[ F = F_{bg} = \frac{1}{g} \epsilon_{abc} \hat{y}^a d\hat{y}^b \wedge d\hat{y}^c \]  

(37)

(from now on we will put \( g = 1 \)). One can recognize the right hand side of this expression as being the volume form on the two-sphere: \( \Omega_2 : \int_{S^2} \Omega_2 = 4\pi \).

We generalize it for the case of non-trivial gravitational field and curved string world-sheet volume form as \( F_{bg} = \frac{2\pi}{g} e_2 \). Here by \( e_2 \) is the global angular form:

\[ e_2 = \frac{1}{4\pi} \epsilon_{abc} \left( d\hat{y}^a d\hat{y}^b \hat{y}^c - d(\Theta^{ab} \hat{y}^c) \right) = \frac{1}{4\pi} \epsilon_{abc} (D\hat{y}^a \wedge D\hat{y}^b \hat{y}^c - R^{ab}(\Theta)\hat{y}^c) \]  

(38)

where \( \Theta^{ab} \) is an \( SO(3) \) connection on the normal bundle and \( R(\Theta) \) is the corresponding curvature (see [24] for rigorous definitions or Appendix in [25] for a list of useful formulae and properties). Note also the striking resemblance of
the expression (38) to ’t Hooft’s definition of the Abelian field strength (10)!
For \( F_{bg} = 2\pi e_2 \) the one-form \( A \) would be given by
\[
A_{bg} = 2\pi e_1^{(0)}
\]  
(39)

Obviously, the form \( e_1^{(0)} \) is not globally defined.

Now, let us repeat formulae (33–36) substituting explicitly \( F = F_{bg} + d\tilde{A} \)
and \( A = A_{bg} + \tilde{A} \) (we use \( \tilde{A} \) to denote only the globally well defined part
of the gauge potential). Then (36) will be modified to:
\[
S_{cs} = \int_{M_5} -\frac{q^3}{24\pi^2} (2\pi e_1^{(0)} + \tilde{A}) \wedge (2\pi e_2 + d\tilde{A}) \wedge (2\pi e_2 + d\tilde{A})
\]
\[
+ \frac{q}{24} (2\pi e_2 + d\tilde{A}) \wedge p_1^{(0)}(R)
\]  
(40)

Let us recall that the one form \( e_1^{(0)} \) is not gauge invariant under \( SO(3) \)
rotations of normal bundle. Therefore part of the Chern-Simons term is not
invariant as well. Its gauge variation will be calculated in the Section 5.2
and we will see that it gives additional normal bundle anomaly inflow we
were looking for. The reason for this seems to be the following – with local
definition of \( A_{bg} \), Chern-Simons is usually defined patch by patch. When we
do \( SO(3) \) rotation of the normal bundle, we mix patches and Chern-Simons
needs to be transformed. To see this effect we need some topologically non-
trivial field configuration which in case at hand is represented by fluctuations
around monopole (monostring) background (37). Technically this effect is
reflected in the fact that gauge potential of any field from this topological
class contains the form \( e_1^{(0)} \) which has non-zero \( SO(3) \) variation.

All this was done in the bulk, outside the monopole core where the
Abelian theory is defined. We see that for the Chern-Simons term it is
not necessary to be regularized to give normal bundle anomaly inflow – it
just a property of topologically non-trivial background. The same procedure
can be done also for the M5-brane case [11].

Let us see now what happens if we want to regularize the Chern-Simons
term nevertheless and compare with [3]. There the approach was the fol-
lowing. First, to have forms defined everywhere we should regularize the
magnetic source \( \delta^3(\Sigma_2 \hookrightarrow M_5) \). The natural way to do it is to consider a
Poincare dual [24] of the submanifold \( \Sigma \). It is given by
\[
\tau(\Sigma) = \frac{1}{2} d\rho \wedge e_k
\]  
(41)
where $e_k$ is global angular $k$-form, and the monotonic function $\rho(|y|)$ equals $-1$ on the submanifold, then goes to zero fast enough and is zero outside tubular neighborhood of the submanifold $\Sigma$. The equation of magnetic coupling (11) (or the modified Bianchi identity) then has the form:

$$dF = 2\pi d\rho \wedge e_k$$

(42)

The above formulae are the same for any magnetic source of even codimension. The only difference between M5 and our case is that $k = 4$ in the former case while here $k = 2$ and $F = dC_3$ instead of $F = dA_1$. Now we have to find the solution of magnetic coupling equation (11) defined everywhere. The general solution is given by $F = Ae_k + B\rho e_k + Cd\rho e_{k-1} + \text{(exact form)}$, where $A, B, C$ are arbitrary constants. In [3] the solution with $A = 0, B = 0, C = 1$ was chosen. To compare this with our case let’s recall that we already have the regularized version of our string – this is the ’t Hooft-Polyakov monopole (monostring) of the 5-d $SU(2)$ theory. Indeed, let’s consider (11) evaluated on the monopole background (5-7):

$$F = \frac{1}{v^3} \epsilon_{abc} \Phi^a d\Phi^b \wedge d\Phi^c = 2\pi \phi^3(y) v^3 e_2$$

(43)

Comparing this with the expression (11) for the Poincare dual we find the relation:

$$1 + \rho(y) = \frac{\phi^3(y)}{v^3}$$

(44)

As a result we come to the conclusion that in the monopole background, the (regularized) Abelian field strength is given by:

$$F = 2\pi (1 + \rho) e_2$$

(45)

This corresponds to a solution of the generalized Bianchi identity with $A = 1, B = 1, C = 0$. Let us take this solution and proceed further along the lines of [3]. We have to construct out of this solution a closed form. Exactly as in [3] and by the same reason we can choose it to be

$$dA = F - 2\pi \rho e_k$$

(46)

(Again, as it was above, the form $A$ here is defined only locally because $dA$ does not need to be (and, in fact, is not) exact, but only closed). If we substitute our choice for the $F$ we see that $dA = 2\pi e_k + \text{(exact form)}$ and
$A = 2\pi \epsilon_k^{(0)} + (\text{globally defined form})$. As a result the “regularized” Chern-Simons term (obtained by the same analysis as in \cite{3} but applied to another solution of magnetic coupling equation) is the same as it was above \cite{3}, before regularization, and does not contain function $\rho(y)$ (as opposed to the case \cite{3}). This seems to be a solution of one of the problems formulated in the Introduction.

### 5.2 Anomaly Cancellation I

Let us now calculate the anomaly inflow from the bulk and then put all the different contributions to the anomaly together.

First consider the term $dA \wedge p_1^{(0)}(TM)$:

$$\delta_{TM} \left[ \int_{M_5} dA \wedge p_1^{(0)}(TM) \right] = \int_{M_5} dA \wedge dp_1^{(1)}(TM) = \int_{M_5} d \left( dA \wedge p_1^{(1)} \right) \quad (47)$$

Recall that the integration in (47) goes along $M_5$ minus the tubular neighborhood of the string world-sheet $D_\epsilon(\Sigma)$. Thus the integration in eq. (47) reduces to that on the boundary of $S_\epsilon(\Sigma_2) = \partial D_\epsilon(\Sigma)$:

$$-\int_{S_\epsilon(\Sigma_2)} dA \wedge p_1^{(1)} = -4\pi \int_{\Sigma_2} p_1^{(1)}(TM) = -4\pi \int_{\Sigma_2} \left( p_1^{(1)}(T\Sigma) \oplus p_1^{(1)}(N) \right) \quad (48)$$

where we used the explicit form of $A$ (eq. (39)) and the fact that integral of $e_2$ along the fiber is equal to 2. The minus sign in eq. (48) comes from the fact that the positive orientation of the $S_\epsilon(\Sigma_2)$ is with respect to the brane $\Sigma_2$ and not to the bulk. As already noted at the end of Section 3 this inflow is symmetric between tangent and normal bundle to the string and so no values of the coefficient would cancel the anomaly (28) of the zero modes.

Now we consider the $A \wedge dA \wedge dA$ term. It has a non-zero variation only with respect to the $SO(3)$ normal bundle due to $\delta_{N} e_1^{(0)} = de_0^{(1)}$. Following \cite{26}, \cite{3} we write:

$$\delta_{N} \left[ \int_{M_5} A(dA)^2 \right] \equiv (2\pi)^3 \int_{M_5} de_0^{(1)} e_2^2$$

$$\quad = -(2\pi)^3 \int_{S_\epsilon(\Sigma_2)} e_0^{(1)} e_2^2 = -2(2\pi)^3 \int_{\Sigma_2} p_1^{(1)}(N)$$

\footnote{This last part of the analysis is also exactly the same for the M5 case \cite{11}.}
Bringing the results of (48), (49) together with (40) we can write:

\[ I_{cs} = \frac{2\pi q^3}{3} \int_{\Sigma_2} p_1^{(1)}(N) \]

\[ I_{\text{inflow}} = -\frac{q\pi}{6} \int_{\Sigma_2} \left( p_1^{(1)}(T\Sigma) + p_1^{(1)}(N) \right) \]  

where \( q = 1/2, 1 \) for fundamental or adjoint fermions. Remarkably, the anomalies cancel for both fundamental and adjoint fermions (28).

We stress once again that the bump function \( \rho(y) \) does not enter directly in the anomaly cancellation procedure. The anomalous variation of the Chern-Simons term is simply a consequence of evaluating it on the topologically nontrivial monopole background (\( e_2 \) in this case). This evaluation can always be done outside the core and will not involve the function \( \rho \). We see, that the additional normal bundle anomaly inflow comes not because of microscopic modifications inside the core, but rather due to the fact that we have to carefully define the Chern-Simons in the presence of the topologically non-trivial configuration.

6 Anomaly Cancellation II
Arbitrary Fermion Representation

To make the story even more convincing we would like to demonstrate that this anomaly cancellation actually works for arbitrary fermion representation. Consider fermions with the arbitrary isospin \( T \). To compute the tangent bundle anomaly it suffices to know the total number of zero modes which is given by the index theorem of Callias [15]. It tells us that the total number of zero modes for the fermions in the arbitrary representation \( T \) is:

\[ N_{z.m.} = \begin{cases} 
T(T+1) & T \text{ - integer} \\
T(T+1) - \frac{1}{4} & T \text{ - half-integer} 
\end{cases} \]  

\[ (51) \]

However, to determine the normal bundle anomaly we also need to know how these zero modes transform under \( SO(3)_{\text{diag}} \). As far as we are aware, this information is not available in the literature, and thus we cannot provide a definitive test of anomaly cancellation. We will however show that a simple and natural guess for the structure of the fermion zero modes does lead to
complete anomaly cancellation. It would be interesting to try to verify this
guess by explicit computations.

Any wave-function $\psi_{\alpha n}(x)$ realizes the tensor product of two represen-
tations: rotational $SO(3)$ with a spin $J$ (where $J = L + 1/2$ is the total
angular momentum) and $SU(2)$ gauge symmetry with isospin $T$. The ten-
sor product $J \otimes T$ can be decomposed in terms of $SO(3)$ representations as
$|J - T| \oplus \ldots \oplus |J + T|$. To find what representation of $SO(3)_{\text{diag}}$ the zero
modes realize, we choose the following strategy: adjust the angular momen-
tum $L$ of the zero modes so as to achieve the lowest possible total spin of
in the tensor product. It is clear from (51) that the zero modes will then
transform in a reducible representation of $SO(3)$.

**Integer Isospin:** Consider the case of integer $T$ first. The wave-function
of zero modes transforms as $J \otimes T$ and we choose $J = T + 1/2$. Then

$$(T + 1/2) \otimes T = 1/2 \oplus 3/2 \oplus \ldots \oplus (2T + 1/2) \quad (52)$$

One can easily check that the total dimension of the first $T$ terms in (52) (up
to the spin $(T - 1/2)$) is equal to $T(T + 1)$, as we need in view of (51). Let
us check that the anomaly indeed cancels against the bulk.

The CS term in the action is just the sum of those similar to (40) for
$q = 1, 2, \ldots, T$. As a result, anomaly inflow is given by the sum of analogs
of (50) with different $q$’s:

$$I_{\text{cs}} = \frac{\pi}{6} T^2 (T + 1)^2 \int_{\Sigma_2} p_1^{(1)}(N)$$

$$I_{\text{inflow}} = -\frac{\pi}{12} T(T + 1) \int_{\Sigma_2} \left( p_1^{(1)}(T\Sigma) + p_1^{(1)}(N) \right) \quad (53)$$

The anomaly of the zero modes is not difficult to find using the usual descent
formalism.

$$I_{z.m.} = -2\pi \int_{\Sigma_2} \left[ \hat{A}(R) \, \text{ch}_r(F) \right]^{(1)}$$

where the Chern character $\text{ch}_r(F) = \text{Tr}_r \exp \frac{F}{2\pi}$. And the Chern character
for arbitrary representation is $r$ with the spin $\frac{2t - 1}{2}$ is:

$$\text{ch}_{\frac{2t - 1}{2}}(F) = 2t + \frac{t(4t^2 - 1)}{12} p_1(N) + \cdots \quad (54)$$
The total anomaly is the sum of anomalies for \( t = 1, \ldots, T \), and is equal to:

\[
I^T_{z.m.} = \int_{\Sigma_2} \left( \frac{\pi}{12} T(T + 1) p_1^{(1)}(T) - \frac{\pi}{12} T(T + 1)(2T^2 + 2T - 1) p_1^{(1)}(N) \right)
\]

We see that this expression cancels \([53]\).

**Half-integer Isospin:** This case is very similar to the previous one. The total number of zero modes is given by \([51]\), and if we write \( T = n + 1/2 \), where \( n \) is a natural number, then \( N_{z.m.} = n^2 \). We choose that \( J = T \) in this case (again, to be able to get the lowest spin with respect to \( SO(3)_{\text{diag}} \) in \( J \otimes T \)). As a result the zero modes form the reducible representation with the \( n \) lowest spins

\[
T \otimes T = 0 \oplus 1 \oplus \ldots \oplus (n - 1) \oplus \ldots \oplus 2n - 1
\]

We repeat the procedure described for the integer isospin. The inflow terms give us:

\[
I_{\text{CS}} = \frac{\pi}{12} n^2 (2n^2 - 1) \int_{\Sigma_2} p_1^{(1)}(N)
\]

\[
I_{\text{inflow}} = -\frac{\pi}{12} n^2 \int_{\Sigma_2} \left( p_1^{(1)}(T\Sigma) + p_1^{(1)}(N) \right)
\]

On the zero mode side, the Chern character for the (integer) representation of spin \( t \) is given by

\[
\text{ch}_t(F) = (2t + 1) + \frac{t(t + 1)(2t + 1)}{6} p_1(N) + \cdots
\]

and the total anomaly (sum over \( t \) from 0 to \( n - 1 \)) is

\[
I_{z.m.} = \int_{\Sigma_2} \left( \frac{\pi}{12} n^2 p_1^{(1)}(T) - \frac{\pi}{6} n^2(n^2 - 1) p_1^{(1)}(N) \right)
\]

We see that for arbitrary fermion representation the anomaly cancels for a simple choice of \( SO(3)_{\text{diag}} \) representation for the zero modes.

### 7 Monopole with Charge \( N > 1 \)

Recall \([7]\) that for \( N \) M5 branes the cancellation of normal bundle anomalies from the modified Chern-Simons term predicts that the anomaly on the brane
scales like $N^3$ for large $N$. This is related to the simple fact that the Chern-Simons term in 11-d SUGRA is cubic in the three-form gauge field. If the magnetic charge of the field is $N$ one expects that the magnetic coupling equation will have the form $dF_k = Nd\rho \wedge e_k$ and naively its solution will be just $F_k = NF_k(1)$ (where $F_k(1)$ is the charge one solution), so that the Chern-Simons evaluated on such a field will be proportional to $N^3$. In [7] this information was used to deduce some properties of the correlators of the $(2,0)$ theory. In our case, however, the structure of fermionic zero modes is known, specifically it is known that the number of zero modes grows as $N$ [15]. Thus it is clear that the anomaly computed in the world-volume theory on the monostring will only scale as $N$, and not as $N^3$.

To explain this difference between the M5-brane and the monostring, we note that for the charge $N > 1$ there are no spherically symmetric solutions in our model [27]. The background and therefore the theory are not symmetric inside the monopole core. As a result there is no $SO(3)_{\text{diag}}$ symmetry of the world-volume theory and no anomaly to be computed.

To understand this fact better one can use the following example. We can build an explicit mapping $\Phi$ of degree $N$ from $S^2_\infty \to S^2_{SU(2)}$, such that the topological charge $Q_{\text{top}} = N$. One way to construct such a mapping is to use stereographic projection from $S^2$ on the complex plane and consider a map \( w = z^N \) in complex coordinates. It is easy to see that this mapping is not spherically symmetric – the Jacobian of the map depends on angles and $\text{Tr}(\Phi \wedge d\Phi \wedge d\Phi) \neq Ne_2$ (although it is equal to $N$ after integration over the sphere). Therefore, the field strength will not scale as $Ne_2$ and the Chern-Simons will not necessarily give an $N^3$ contribution.

We see that in our model, the fundamental theory which lives in the core of the monopole does not possess the symmetry with respect to normal bundle rotations for $N > 1$ and this fact is not obvious from the effective theory in the bulk. This raises the question of whether the analysis of [7] should really be taken as evidence for $N^3$ scaling.

For $N$ M5-branes there are independent arguments which strongly suggest the $N^3$ dependence. First, as usual for extremal p-brane solutions the area of the horizon is related to the charge (the number of the branes) $N$ and thus the Bekenstein-Hawking entropy depends on $N$. It is easy to see that for the case of $N$ M5 branes it scales like $N^3$ [8]. Similar arguments, involving absorptions cross-section of low energy gravitons in the background of $N$ coincident M5-branes, were discussed in [9]. This provides an indication that the $(2,0)$ theory has a total number of degrees of freedom that scales like $N^3$.
This of course does not give a direct proof that the normal bundle anomaly scales like $N^3$. The theory for $N > 1$ is interacting and its structure is still unclear.

However, independent information also arises from the calculation of the Weyl anomaly in the $(2,0)$ theory, which also scales as $N^3$ [14], and the fact that the Weyl anomaly and the $SO(5)$ anomaly are in the same supersymmetry multiplet.

8 Comparison with M-theory and Open Questions

It is interesting to push the analogy between the M5-brane and the monostring as far as possible. We can think of our $SU(2)$ theory [1] as an analog of M-theory in $D = 11$, and the effective $U(1)$ theory with terms (40) – as an analog of eleven-dimensional supergravity together with the higher-derivative correction $G_4 \wedge X_8^{(0)}$ [1, 29]. In the case of eleven-dimensional supergravity information about zero modes living on one M5 brane comes from a symmetry analysis, that is they can be viewed as Nambu-Goldstone zero modes [30, 31, 28]. To make the analogy even more transparent, we consider the supersymmetric version of our problem and then comment on some other aspects.

We can easily construct a supersymmetric version of both the $SU(2)$ and $U(1)$ theories. It is given by starting from $(1,0)$ supersymmetric Yang-Mills theory in $D = 6$ and reducing it on an $S^1$. This gives a supersymmetric theory in $D = 5$ with adjoint fermions, an adjoint scalar, and gauge fields. The potential for the scalar field vanishes, and we can then consider monopole solutions in the BPS limit of vanishing potential.

It is clear from thinking about this from the $D = 6$ point of view that the monopole-string breaks half of the supersymmetry and should lead to a world-sheet sigma-model with $(4,0)$ supersymmetry. Repeating the analysis analogous to that in [30, 28], we see that the $(4,0)$ theory on the magnetic string consists of the following modes: 3 (non-chiral) Nambu-Goldstone scalars from breaking of translational invariance; 4 Majorana-Weyl fermions from broken SUSY; one dyon from broken large gauge transformations. We see that the field content of this two-dimensional $(4,0)$ theory closely resembles that of the six-dimensional theory on the world volume of one M5 brane.
In the latter case the zero modes consist of five scalars, 4 Weyl fermions and one self-dual two-form. To make the analogy complete, we would need a self-dual one-form in two dimensions, that is, a chiral dyon zero mode. The issue of whether the dyon mode is chiral is in fact somewhat intricate.

First of all, from the classical analysis of the full $SU(2)$ model there is no chiral dyon (c.f. (C10)) in the spectrum of zero modes of the theory. The question arises as to whether the dyon zero mode might be chiral in the quantum theory. It is interesting to note that $(4,0)$ SUSY itself does not itself require a chiral dyon. Indeed, every scalar field can be divided into left and right moving parts. $(4,0)$ supersymmetry acts only on the left-moving components (of both bosons and fermions), leaving the right-movers invariant. Therefore the presence of right-moving scalar degrees of freedom is not controlled by supersymmetry in any way.

For the M5-brane an analysis of the classical theory shows that the two-form zero mode is chiral. Both chiralities appear as zero modes, but because of the Chern-Simons term, only one of them is normalizable.

On the other hand, a chiral dyon can be found in our model as well. It arises as a Goldstone zero mode, precisely as it appeared in the case of the M5 brane. Indeed, let us consider the effective $U(1)$ theory with added Chern-Simons term ($\Omega$). Recall that it was obtained as a quantum correction from integrating out massive fermions. Note that this is in contrast to the $D = 11$ case, where the Chern-Simons term was present in the action at tree level (it was required by supersymmetry) and therefore played a role in the classical analysis of the zero modes. We can formally repeat the analysis of [30, 31, 28] for our case with the Chern-Simons corrections ($\Omega$) and find that there is a localized massless zero mode coming from large gauge transformations and that only part with one chirality is normalizable (has decreasing radial profile). For details we refer to Appendix C.

How should we interpret this result and its apparent discrepancy with the classical analysis (C1I)? To begin with, let us stress once again that it is necessary to have a Chern-Simons (or some other parity-odd) term in the action in order to obtain a chiral dyon zero mode. This term was absent in the original $SU(2)$ theory (and therefore the analysis there gave a dyon zero mode with both chiralities present) and appeared only in the quantum effective action as a result of interaction between the gauge field and massive fermions.

One of the arguments in favor of the chiral two-form for the M5-brane was anomaly cancellation: chiral two-form gives an additional contribution to the
tangent bundle anomaly and without it cancellation would not be possible \[^1\]. However in our case the situation is precisely the opposite. Indeed, let us suppose that the dyon is chiral in the $U(1)$ effective theory. Then together with 4 Majorana-Weyl fermions it produces the following anomaly:

$$I_{z.m.} = \int_{\Sigma_2} \left( \frac{\pi}{4} p_1^{(1)}(T\Sigma) - \frac{\pi}{2} p_1^{(1)}(N) \right)$$  \hspace{1cm} (61)

This anomaly differs by $\frac{\pi}{2} p_1^{(1)}(T\Sigma)$ from anomaly inflow (50).

Before discussing this further, let us emphasize precisely what was involved in studying anomaly cancellation so far. In Sections 2–5 we made an attempt to compute the log det $\hat{\mathcal{D}}$ – fermionic determinant in the background of monopole (57). We split the integration over the fermions $\Psi$ into two parts:

$$\int \mathcal{D}\Psi \ e^{iS[A,\Psi]} \equiv \int \mathcal{D}\Psi_0 \mathcal{D}\Psi' e^{iS[A,\Psi]} = \int \mathcal{D}\Psi_0 e^{iS^{(5)}[A] + iS^{(2)}[\Psi_0]}$$  \hspace{1cm} (62)

Here by $\mathcal{D}\Psi'$ we mean integration over all but zero modes, and by $\mathcal{D}\Psi_0$ we mean integration over the set of Jackiw-Rebbi zero modes. Action $S^{(5)}_{\text{eff}}[A]$ was computed under assumption that $\Psi'$ is the full system of functions and that the background is topologically trivial. Such assumption had its price: we saw that Chern-Simons terms (40) in $S^{(5)}_{\text{eff}}[A]$ was anomalous, even though the original Lagrangian (11) was anomaly-free.

The action $S^{(2)}$ is the action of Jackiw-Rebbi zero modes on the string described by eq. (24) or (26), depending on the representation.

We would like to stress once again that eq. (62) is just an approximation, widely used in analyzing different brane constructions (c.f. \[1\], \[2\], \[3\], \[4\], \[5\]). The aim of this paper is in particular to check the consistency of such approximation and to show how many subtleties are encountered on the way of realization of this idea. Thus, in Section 5 we explicitly analyzed all anomalies coming from representing the fermionic determinant as the sum of these two actions and demonstrated that anomalies indeed cancel. This is the reflection of the fact that in five dimensions there exists a gauge-invariant regularization of fermions and log det $\hat{\mathcal{D}}$ is non-anomalous.

Having said all that, let’s return back to the discussion of the dyon anomaly. In the manner similar to that of described above we would in principle need to perform integration in two stages – integration over the zero mode (dyon) and integration over the gauge fields in the bulk. Such integration in the bulk could give rise (among other things) to renormalization
of coefficients in (40) and as a result change of the inflow (50). We claim, however, that this is not the case. Indeed, renormalization of parity-odd terms can come from parity-odd interactions only, and Chern-Simons terms itself is the only candidate. However, a simple analysis (Appendix [3]) shows that this is not the case.

The dyon anomaly computation (i.e. integration over the dyon zero mode) is intrinsically different from the fermion one. In contrast to the former case dyon (as “zero mode” of gauge field) cannot be clearly separated from the rest of the fields in path integral, as part of the $SU(2)$ gauge field still remains massless. In this sense it is not even clear whether the dyon zero mode ever decouples from the bulk in a sense in which Jackiw-Rebbi zero modes did. This might suggest, that one cannot even write a local counterterm for the dyon anomaly cancellation. However, we are not addressing this issue in the current paper.

Another important difference between our model and the M-theory situation is the origin of the Chern-Simons term. In the present analysis the Abelian Chern-Simons arises as the one-loop quantum correction, on the same footing as the term containing first Pontriagin class in (10). As it was discussed above, this Chern-Simons term is the bulk part of the fermionic determinant. Because of this it is clear why it has exactly the correct coefficient in front of it to cancel the anomaly of normal bundle, which came from computation of the same determinant. In the M5-brane case, however, the situation is different. The “Gravitational” term $G_4 \wedge X_8^{(0)}$ did arise as a quantum correction, while the Chern-Simons term was the (required) part of the “tree-level” $D = 11$ SUGRA action and the coefficient on front of it was fixed by supersymmetry. At first sight it is not clear at all why this terms would “know” anything about quantum effects – anomalies. Note, that this problem is directly related to the problem of chiral dyon discussed above. If the parity-odd Chern-Simons term is present from the very beginning at tree-level, the “dyon” (two-form) zero modes is chiral and plays its role in the cancellation of anomalies. If it appears only as a one-loop correction, then the classical dyon is non-chiral and does not contribute to anomalies.

There exists a direct five-dimensional analog of $D = 11$ supergravity. $N = 1 \; D = 5$ SUSY has a gravitational multiplet, with bosonic degrees of freedom consisting of the graviton and a $U(1)$ vector field. The vector field part of such a supergravity Lagrangian has a Chern-Simons term $A \wedge F \wedge F$ and higher-derivative term $F \wedge p_1^{(0)}(R)$ as in (40) but with different coefficients, dictated by supersymmetry. This SUGRA has a magnetic p-
string solution. Goldstone zero modes analysis of this solution gives the same \((4, 0)\) theory on the string (3 scalars, 4 Majorana-Weyl fermions, 1 chiral scalar). But in this case anomaly of \((4, 0)\) theory is precisely canceled by anomaly inflow \[3, 33\].

The situation with the origin of Chern-Simons term in M-theory and its role in anomaly cancellation resembles Green-Schwarz anomaly cancellation \[32\]. From the viewpoint of the present analysis it could be interpreted as one more evidence of the existence of some fundamental theory which would explain this phenomenon naturally, as for string theory.

9 Conclusions

In this paper we have studied the following field theory: Yang-Mills coupled to a scalar Higgs field and fermions in five dimensions. This theory possesses a monopole solution, which looks like a magnetically charged string. We considered this theory as an example, which reproduced many features of the M5 brane anomaly cancellation. For example, we showed that both gauge and gravitational anomalies of the fermion zero modes are canceled by anomaly inflow, coming from the Chern-Simons terms which arise from integrating out massive fermions in the bulk. We showed that the Chern-Simons terms, defined on a topologically non-trivial background, possess additional non-trivial variation under the normal bundle transformation and thus contribute to the normal bundle anomaly inflow (both in our example and in supergravity).

This analysis allowed us to make some predictions regarding the structure of the fermion zero modes of the 't Hooft-Polyakov background for arbitrary fermion representation.

Apart from that we were able to see that effective theory \((U(1)\) gauge theory in our example) does not allow to resolve some important features of the full \((SU(2))\) theory. Symmetry of the solution in the effective theory (Dirac monopole in our case) does not coincide with the symmetry of the non-Abelian monopole solution and therefore naive anomaly analysis gives the wrong answer. This shows once again that in case of several M5 branes the conclusions drawn from analyzing the anomaly inflow in the effective theory should be taken with the grain of salt.

The part in which our example differed the most from the original five-brane case was the question of the chiral bosonic zero modes, arising as the
Nambu-Goldstone boson of large gauge transformations. In our case we were not able to show what cancels the anomaly coming from the chiral dyon on the world-volume on the string. We did not address this issue in detail, but we argued that it was related to the fact, that in our case Chern-Simons was generated at the one-loop level via integrating out massive fermions, while in the eleven dimensional supergravity it was present from the very beginning. We leave this and related questions for future investigations.

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A Gamma Matrices

Following Jackiw and Rebbi [14] we take the following basis of gamma-matrices (as usual: \( \beta = \gamma^0; \quad \bar{\alpha} = \gamma^0 \cdot \vec{\gamma} \)):

\[
\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \quad (A1)
\]

\[
\gamma^k = -i \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \quad \quad (A2)
\]

where \( \sigma \) are the usual Pauli matrices:

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \quad (A3)
\]

One can easily check that we are working in \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) signature. Define also

\[
\gamma^4 = \prod_{n=0}^{3} \gamma^n = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \quad (A4)
\]

Which also obeys \((\gamma^4)^2 = -1\) and \(\{\gamma^4, \gamma^k\} = \{\gamma_4, \gamma_0\} = 0\) Let us also introduce the matrix \(\gamma^{int}\):

\[
\gamma^{int} \equiv \gamma^0 \gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \quad (A5)
\]
B Non-renormalizability of the Coefficient of the Chern-Simons.

In this Section we show that coefficient of Chern-Simons does not get renormalized due to the Chern-Simons interaction at one-loop order. The analysis is very simple. Interaction term $AdAdA$ gives the following vertex:

$$V(k^{(1)}, k^{(2)}, k^{(3)}) = (2\pi)^5 \delta(k^{(1)} + k^{(2)} + k^{(3)}) \epsilon_{\mu\nu\lambda\rho\sigma} A_\mu(k^{(1)}) k_\nu(k^{(2)}) A_\lambda(k^{(2)}) k_\rho(k^{(3)}) A_\sigma(k^{(3)})$$  \hspace{2cm} (B6)

Renormalization of the Chern-Simons term comes from a triangle diagram with the insertion of (B6) in each of three vertices. This diagram is proportional to

$$S_{\text{triang}} \sim k^{(1)} A_\mu(k^{(1)}) k^{(2)} A_\lambda(k^{(2)}) k^{(3)} A_\sigma(k^{(3)})$$  \hspace{2cm} (B7)

So, we see that interaction due to the Chern-Simons term cannot produce counter-terms proportional to (B6).

C Classical and Goldstone dyon

We once again consider the Lagrangian (I). Our conventions for the field strength and covariant derivatives are

$$G_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + g \epsilon_{abc} A_M^b A_N^c$$

$$(D_M \Phi)^a = \partial_M \Phi^a + g \epsilon_{abc} A_M^b \Phi^c$$

$$(D_M \psi)_n = \partial_M \psi_n - ig T^a_{nm} A_M^a \psi_m$$  \hspace{2cm} (C8)

In the absence of fermions, the equations of motion are:

$$D^N G_{NM} = [D_M \Phi, \Phi]$$

$$D^2 \Phi = -2\Phi U'(g\Phi)$$  \hspace{2cm} (C9)

We consider an ansatz for the zero mode in the following form:

$$\begin{cases}
\delta_\alpha A^a_\mu = 0 \\
\delta_\alpha \Phi^a = 0 \\
\delta_\alpha A^a_i(x, y) = \alpha(x) D_i(f(y)\Phi^a)
\end{cases}$$  \hspace{2cm} (C10)
(recall that \( y = \sqrt{y_i y^i} \) is the radial coordinate in transversal directions). Here \( f(y) \) is a profile function which needs to be determined. Substituting this ansatz into the action, we find that variation (C10) over the background (5-7) is still a solution of (C9) if
\[
\begin{cases}
\partial_x^2 \alpha = 0 \\
f \sim 1/y, \quad y \to \infty
\end{cases}
\]
Thus, we see that zero mode (scalar \( \alpha(x) \)) enjoys the following properties: (1) it behaves like massless scalar in 1 + 1 dimensions; (2) it is localized inside the core, i.e. its transversal profile is given by \( D(f \Phi) \) and decays as 1/y at infinity. One should note that this derivation will not be true in BPS case, when \( m_H = 0 \) and \( \Phi \) falls of much slower. In that case, however, the dyon ansatz is also given by (C10) with \( f = 1 \) (see, e.g. [34]). Note, that the dyon \( \alpha(x) \) is non-chiral from the two-dimensional point of view!

However, things change once we consider this zero mode in the effective \( U(1) \) theory (36). In this case a dyon ansatz similar to (C10) is given by:
\[
\delta \alpha A = \alpha(x) df(y); \quad \delta \alpha F = d\alpha(x) \wedge df(y)
\]
with \( f(y) \) being some normalizable function in the transverse directions. We want to repeat the analysis of [30, 31, 28]. Let us forget about gravity for simplicity (i.e. put \( p_1(R) = 0 \) in (36)) and try to find the equations of motion for \( \alpha(x) \), using the ansatz (C12). We will be somewhat schematic through the end of this Section, concentrating only on the details and ignoring all numerical factors.

Consider the \( U(1) \) action
\[
S = \int F \wedge * F + \frac{c}{3} A \wedge F \wedge F
\]
where all the numerical factors are absorbed into the constant \( c \). Equations of motion reads:
\[
d * F - cF \wedge F = 0
\]
Substituting in (C14) \( F = 2\pi e_2 + d\alpha(x) \wedge df(y) \) we get (note that in the case of trivial normal bundle connection \( e_2 \) reduces down to the volume form on two sphere \( d\Omega_2 = \sin \theta \ d\theta \wedge d\varphi \)):
\[
d * (d\alpha \wedge df) - 2c d\alpha \wedge df \wedge e_2
\]
\[ d \ast (d\alpha \wedge df) = c_1 d((f'(y)y^2)\epsilon_{\mu\nu}\partial_\mu \alpha \ dx^\nu \wedge d\Omega_2) \]
\[ = c_1 \left( \partial^2 \alpha \epsilon_{\mu\nu}dx^\mu \wedge dx^\nu + \right. \]
\[ \left. + \ (y^2 f')' dy \wedge \epsilon_{\mu\nu}\partial_\mu \alpha \ dx^\nu \right) \wedge d\Omega_2 \]  
(C16)

Prime denotes the differentiation with respect to \(|y|\) (we are working in spherical coordinates in the \(y\) directions). Constant \(c_1\) absorbed some numerical factors and/or signs, coming from performing operation \(\ast\) and differentiation. We are not following them, as it is not really important, because we are not interested in the precise chirality of the dyon, we only want to know whether the object is chiral.

Now consider second term in (C15):
\[ -2c_1 \partial_{\nu} \alpha f'(y) dx^\nu \wedge dy \wedge d\Omega_2 \]  
(C17)

we see that for eq. (C16) to cancel (C17) one needs two conditions. First, form proportional to \(dx^\mu \wedge dx^\nu \wedge d\Omega_2\) should be equal to zero, which means to \(\partial^2 \alpha = 0\). For the second term in (C16) we get
\[ c_1 \epsilon_{\mu\nu}\partial_\nu \alpha (y^2 f')' - 2ca_\mu f' = 0 \]  
(C18)

which can be solved in the following way:
\[ \epsilon_{\mu\nu}\partial_\nu \alpha = \pm \partial_\mu \alpha \quad \text{(C19)} \]
\[ (y^2 f')' = \pm \frac{2c}{c_1} f' \]  
(C20)

Solution of eq. (C20) is given by:
\[ f(y) = f_0 \left( 1 - \exp \left( \pm \frac{2c}{cy} \right) \right) \]  
(C21)

Only for one choice of sign in (C19) ansatz (C12) describes normalizable zero mode (chiral or anti-chiral, depending on the initial sign of \(c_1\)). Resulting dyon falls off as \(1/y\) for \(y \to \infty\).

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