ON INDEPENDENCE OF ITERATED WHITEHEAD DOUBLES IN THE KNOT CONCORDANCE GROUP

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Abstract. Let $D(K)$ be the positively-clasped untwisted Whitehead double of a knot $K$, and $T_{p,q}$ be the $(p, q)$ torus knot. We show that $D(T_{2,2m+1})$ and $D^2(T_{2,2m+1})$ differ in the smooth concordance group of knots for each $m \geq 2$. In fact, they generate a $\mathbb{Z} \oplus \mathbb{Z}$ summand in the subgroup generated by topologically slice knots. We use the concordance invariant $\delta$ by Manolescu and Owens. More generally, we show that the same result holds for a knot $K$ with $|\delta(D(K))| > 8$ and give a formula to compute $\delta(D(K))$ for some classes of knots. Interestingly, these results are not easily shown using other concordance invariants such as $\tau$-invariants of the knot Floer theory and $s$-invariants of Khovanov homology. We also determine the infinity version of the knot Floer complex of $D(T_{2,2m+1})$ for any $m \geq 1$ generalizing a result for $T_{2,3}$ of Hedden, Kim and Livingston.

1. Introduction

A knotted circle $K$ in $S^3$ is called smoothly (resp. topologically) slice if it bounds a smoothly (locally flat) embedded disk in $B^4$. Two knots $K_1$ and $K_2$ are called smoothly (topologically) concordant if $K_1 \# - K_2$ is smoothly (topologically) slice, where $-K$ is the mirror of $K$ with reversed orientation. Modulo smooth concordance, the set of knots forms an abelian group, the (smooth) knot concordance group, $\mathcal{C}$. Note that every smoothly slice knot is topologically slice, but the converse is not true. We let $\mathcal{C}_{TS}$ be the subgroup of $\mathcal{C}$ generated by topologically slice knots.

The Whitehead double (positively-clasped untwisted) of a knot $K$, $D(K)$, is a satellite knot defined by the pattern in Figure 1. Whitehead doubles are interesting classes of knots in the study of the knot concordance. Any class of the Whitehead double of a knot is contained in $\mathcal{C}_{TS}$ since it has the same Alexander-Conway polynomial as the unknot and hence is topologically slice by a result of Friedman [4]. However, many of them are not smoothly slice, which show remarkable distinction between the smooth and topological categories in dimension four. It is thus important to understand their concordance properties as portrayed in the knot concordance group. It is also interesting to ask about the effect of $D$ on $\mathcal{C}$, and there is a long-standing conjecture.

Conjecture 1. [11] Problem 1.38] $D(K)$ is smoothly slice if and only if $K$ is smoothly slice.

Note that it is still unknown, as far as the author knows, if the conjecture is true even for some simple knots such as left-hand trefoil or figure-eight knot. One could study Whitehead doubles in $\mathcal{C}$ using homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$. The knot signature $\sigma$ gives one such homomorphism. Unfortunately the signature, indeed
Figure 1. The pattern of the positively-clasped untwisted Whitehead double and the Whitehead double of the (2, 5) torus knot. The $-5$ extra full twists arise from untwisting the writhe of the projection of the (2, 5) torus knot.

any invariant of topological concordance group, is not effective homomorphism for Whitehead double knots, since it vanishes for these knots [3]. Heegaard Floer theory provides manifestly smooth concordance invariants, and some of which give homomorphisms from $C$ to $\mathbb{Z}$. One is the $\tau$-invariant, defined using the knot Floer homology of Ozsváth-Szabó and Rasmussen [15, 19, 23]. Manolescu-Owens discovered another concordance invariant $\delta$, twice the Heegaard Floer correction term ($d$-invariant) of the double cover of $S^3$ branched over a knot [11]. More recently, Peters studied another concordance invariant $d(S^3_1(K))$ given by the correction term of 1-surgery on $K \subset S^3$ [22]. In contrast to the other two invariants, $dS^3_1$ does not induce a homomorphism to $\mathbb{Z}$. See the survey paper [10] of Jabuka for some applications of Heegaard Floer theory to the concordance group. Rasmussen’s $s$-invariant coming from Khovanov homology is also a powerful concordance homomorphism [24]. It is known that $-\sigma/2 = \tau = s/2$ for alternating knots, but they differ in general: see [8], [13] and [12].

Even though there are many concordance invariants developed, most of them are inefficient for distinguishing Whitehead doubles in $C$. The invariants $|\tau|$, $-dS^3_1/2$ and $|s/2|$ are known to be bounded above by the slice genus (four-ball genus) of the knot: the minimal genus of smoothly embedded surface in the 4-ball bounded by $K \subset \partial(B^4)$. Since the slice genus of $D(K)$ is at most one for any knot $K$, so are $|\tau|$, $-dS^3_1/2$ and $|s/2|$ of $D(K)$. Moreover, $\tau(D(K))$ is determined by $\tau(K)$ followed by the Theorem of Hedden below.

Theorem 2. [5] Theorem 1.5

$$\tau(D(K)) = \begin{cases} 0, & \text{for } \tau(K) \leq 0 \\ 1, & \text{for } \tau(K) > 0 \end{cases}$$

In particular, $\tau(D^n(K))$ is identically either 0 or 1 for any $n \geq 1$ and is determined by $\tau(K)$, where $D^n(K)$ denote the $n$th iterated positively clasped untwisted
Whitehead double of $K$. Therefore, it is interesting to ask if it is possible to distinguish the $D^n(K)$’s in $\mathcal{C}$. Using $\delta$-invariants, which are not constrained by the slice genus, we show:

**Theorem 3.** For each $m \geq 2$, $D(T_2, 2m+1)$ and $D^2(T_2, 2m+1)$ are not smoothly concordant. In fact, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.

In [13] it was computed that $\delta(D(T_2, 2m+1)) = -4m$. See also Section 5.2. Here, we show that $\delta(D^2(T_2, 2m+1)) = -4$ for any $m \geq 1$. A tool for our results is a computation of the infinity version of the knot Floer chain complex of $D(T_2, 2m+1)$:

**Theorem 4.** For any $m \geq 1$, the chain complex $CFK_\infty(D(T_2, 2m+1))$ is $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain homotopy equivalent to the chain complex $CFK_\infty(T_2, 3) \oplus A$, where $A$ is an acyclic complex.

More generally, we have the following:

**Theorem 5.** Suppose $K$ is a knot in $S^3$. If $|\delta(D(K))| > 8$, then $D(K)$ and $D^n(K)$ are not smoothly concordant for each $n \geq 2$. If, in addition, $\tau(K) > 0$, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.

**Corollary 6.** Suppose $K$ is an alternating knot in $S^3$. If $\tau(K) > 2$, then $D(K)$ and $D^n(K)$ are not smoothly concordant for each $n \geq 2$. In fact, they generate a $\mathbb{Z}^2$ summand of $\mathcal{C}_{TS}$.

Additionally, for some classes of knots including $(p, q)$ torus knots, we give an algorithmic formula for testing it in terms of its Alexander-Conway polynomial along with an implementation of a computer program.

Recently, Cochran-Harvey-Horn suggested a bipolar filtration of $\mathcal{C}$ and the induced filtration of $\mathcal{C}_{TS}$ [3].

$$\{0\} \subset \cdots \subset T_{n+1} \subset T_n \subset \cdots \subset T_0 \subset T = \mathcal{C}_{TS}.$$  

Since $\tau$ of $D(K)$ and $D^2(K)$ are nonzero for knots $K$ in Theorem [3] and [5] both of them are contained in $T/T_0$ by [3] Corollary 4.9. Therefore, their filtration cannot see the difference between $D(K)$ and $D^2(K)$. On the other hand, using those knots, we get the following corollary relating to the filtration. Let $\mathcal{C}_\Delta$ be the subgroup of $\mathcal{C}_{TS}$ generated by knots with trivial Alexander-Conway polynomial.

**Corollary 7.** There is a $\mathbb{Z}^2$ summand of $\mathcal{C}_\Delta/\mathcal{C}_\Delta \cap T_0$.

**Remark 8.** Recently, in [9] Hom showed there is $\mathbb{Z}^\infty$ summand of $\mathcal{C}_{TS}$, but her technique cannot see the difference between the iterated Whitehead doubles in $\mathcal{C}$ either.

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2. Preliminaries

In this section we briefly recall the Heegaard Floer and knot Floer theory, the staircase complexes and some invariants induced by them.
2.1. Heegaard Floer homology and knot Floer invariants. For simplicity, we work with coefficients in \( F \), the field of two elements. For a rational homology three-sphere \( Y \) equipped with a spin\(^c\) structure \( t \), one can associate to it a relatively \( Z \)-graded and filtered chain complex \( CF^\infty(Y, t) \), a finitely and freely generated \( F[U, U^{-1}] \)-module. In particular the filtration is given by the negative power of \( U \), and \( U \)-multiplication lowers the homological grading by 2. The filtered chain homotopy type of \( CF^\infty(Y, t) \) is known to be an invariant of \((Y, t)\). For more detailed and general exposition of the theory we refer to [17, 18].

We set \( CF^-(Y, t) := CF^\infty(Y, t)_{\{i < 0\}} \), the subcomplex consisting of the elements in \( CF^\infty(Y, t) \) whose filtration level \( i \) is less than 0, and also define the quotient complexes \( CF^+(Y, t) := CF^\infty(Y, t)_{\{i \geq 0\}} \) and \( CF(Y, t) := CF^\infty(Y, t)_{\{i = 0\}} \). The homology group of \( CF^\infty(Y, t) \) is denoted by \( HF^\infty(Y, t) \), and \( HF^-, \ HF^+ \) and \( \overline{HF} \) denote the homology of the other chain complexes. The various versions of Heegaard Floer homologies naturally fit into a long exact sequence:

\[
\cdots \rightarrow HF^-(Y, t) \rightarrow HF^\infty(Y, t) \rightarrow HF^+(Y, t) \rightarrow \cdots
\]

For \( S^3 \), it is known that \( HF^\infty(S^3) \cong F[U, U^{-1}] \) and \( \overline{HF}(S^3) \cong F \) [18]. We usually drop the spin\(^c\) structure in the notation if there is a unique one.

A knot \( K \) in an integer homology three-sphere \( Y \) has an associated \( Z \oplus Z \)-filtered chain complex \( CFK^\infty(Y, K) \) which reduces to \( CF^\infty(Y) \) after forgetting the second \( Z \) filtration. The \( U \)-multiplication decreases both of the filtration levels by 1. The filtered chain homotopy type of \( CFK^\infty(Y, K) \) is an invariant of the knot and we refer it as the knot Floer invariant of \((Y, K)\) [19, 23]. We denote by \( CFK^\infty(Y, K)_{\{(i, j)\}} \) the subgroup at \((i, j)\)-filtration level in \( CFK^\infty(Y, K) \) and define \( \widehat{CFK}(Y, K) := CFK^\infty(Y, K)_{\{i = 0\}} \). It is an easy fact that \( H_*(CFK^\infty(Y, K)) \cong HF^\infty(Y) \) and \( H_*(\widehat{CFK}(Y, K)) \cong \overline{HF}(Y) \). As a consequence, for a knot \( K \) in \( S^3 \), we obtain an induced sequence of maps:

\[
i_K^m : H_*(\widehat{CFK}(S^3, K)_{\{1 \leq m\}}) \rightarrow \overline{HF}(S^3) \cong F
\]

An invariant \( \tau \) for a knot \( K \) in \( S^3 \) is defined by

\[
\tau(K) := \min\{m \in Z | i_K^m \text{ is non-trivial}\}.
\]

In [15, 23] it was shown that \( \tau \) is a concordance invariant. For a knot \( K \) in \( S^3 \) we abbreviate the notations by \( CFK^\infty(K) := CFK^\infty(S^3, K) \) and \( \widehat{CFK}(K) := \widehat{CFK}(S^3, K) \).

It is useful to visualize a knot Floer complex as a collection of dots and arrows lying in a grid in the plane. In a diagram, a dot in \((i, j)\)-coordinate box represents an \( F \)-generator in \( CFK^\infty(K)_{\{(i, j)\}} \), and an arrow represents the non-trivial map, \( F \rightarrow F \). The differential is then the sum of the arrows, as a map of vector spaces. See Figure 2 for examples.

2.2. Staircase complex. For a given \((n-1)\)-tuple of positive integers \( \mathbf{v} \), a staircase complex of length \( n \), \( \text{St}(\mathbf{v}) \), is defined as a finitely generated \( Z \oplus Z \)-filtered chain complex over \( F \) with \( n \) generators, where the numbers in \( \mathbf{v} \) are the length of arrows, which alternate horizontal and vertical starting at the top left generator and moving to the bottom right generator in alternating right and downward steps in a grid diagram. We also locate the top left dot on the vertical axis \((i = 0)\) and the bottom right on the horizontal axis \((j = 0)\) on the diagram. See [1] for more detail.
Corollary 1.7. In particular, \( \tau \) is generated by a stair complex, where \( i \) and \( t \) denote that complex of \( \mathbb{T} \). There is a well-defined absolute grading, correction term and concordance invariants.

2.3. The absolute grading, correction term and concordance invariants. There is a well-defined \( \mathbb{Q} \)-valued map from generators of Heegaard Floer homology groups called the absolute grading with the following properties [21]:

- The absolute grading respects the homological grading of \( CF^\infty(Y) \).
- The absolute grading of a generator in \( \hat{HF}(S^3) \cong \mathbb{F} \) is 0.

For instance, complexes generated by \( \text{St}(1, 1, 1, 1) \) and \( \text{St}(1, 2, 2, 1) \) are shown in Figure 2.

The knot Floer invariant is a categorification of Alexander-Conway polynomial \( \Delta_K(t) \), in the following sense:

\[
\Delta_K(t) = \sum_{k \in \mathbb{Z}} \chi(CFK(K)_{(j-k)}) t^k,
\]

where \( \Delta_K(t) \) is the symmetrized Alexander-Conway polynomial of \( K \) and \( \chi \) is the Euler characteristic.

Conversely, for some classes of knots such as alternating knots and \( L \)-space knots (including torus knots), the knot Floer complex of a knot is determined by its Alexander-Conway polynomial [16] [20]. For an \( L \)-space knot, \( K \), the Alexander-Conway polynomial has the form \( \Delta_K(t) = \sum_{k=0}^{2m} (-1)^k t^k \), and \( CFK^\infty(K) \) is generated by a stair complex,

\[
CFK^\infty(K) \cong \text{St}(n_{i+1} - n_i) \otimes \mathbb{F}[U, U^{-1}],
\]

where \( i \) runs from 0 to \( 2m - 1 \) and \( U \)-multiplication is naturally extended: i.e. if \( x \) is a generator in \( (i, j) \)-filtration level, then \( U^n x \) sits in \((i - n, j - n)\)-filtration level, and \( \partial(U^n x) = U^n \partial(x) \). Denote \( \text{St}(K) := \text{St}(n_{i+1} - n_i) \).

For example, since the Alexander-Conway polynomial of \( T_{2, 2m+1} \) is \( \sum_{i=-m}^{m} (-1)^i t^i \), \( CFK^\infty(T_{2, 2m+1}) \) is generated by \( \text{St}(1, \cdots, 1) \) of length \( 2m + 1 \). The knot Floer complex of \( T_{3, 4} \) can be given by \( \text{St}(1, 2, 2, 1) \otimes \mathbb{F}[U, U^{-1}] \) from the fact that \( \Delta_{T_{3, 4}}(t) = t^{-3} - t^{-2} + 1 - t^2 + t^3 \). See Figure 2. Accordingly, it is easily obtained that \( \tau(T_{p, q}) = (p - 1)(q - 1)/2 \) from its staircase complex, see also [15], Corollary 1.7. In particular, \( \tau(T_{2, 2m+1}) \) is equal to \( m \).

Figure 2. Diagrams of \( CFK^\infty(T_{2, 5}) \) and \( CFK^\infty(T_{3, 4}) \) generated by staircase complexes \( \text{St}(1, 1, 1, 1) \) and \( \text{St}(1, 2, 2, 1) \) respectively.
If \((W, s)\) is a cobordism from \((Y_1, t_1)\) to \((Y_2, t_2)\), then
\[
gr(F_W^+(s, \xi)) - gr(\xi) = \frac{c_1^2(s) - 2\chi(W) - 3\sigma(W)}{4}
\]
for \(\xi \in HF^+(Y_1, t_1)\). Here, \(c_1\) is the first Chern class, \(\chi\) is the Euler characteristic, \(\sigma\) is the signature of the intersection form of \(W\), and \(F_W^+: HF^+(Y_1, t_1) \to HF^+(Y_2, t_2)\) is the map induced by the cobordism \(W\). See [21] for a discussion about the maps induced by a four-dimensional cobordism.

We usually write down the absolute grading using a subscript with parenthesis, for example, \(HF(S^3) \cong \mathbb{F}_{(0)}\). With the help of the absolute grading, the correction term \((d\text{-invariant})\) of \((Y, t)\) is defined [14]:
\[
d(Y, t) := \min \{gr(\xi) | \xi \in HF^\infty(Y, t) \text{ and } \pi(\xi) \text{ is nontrivial}\},
\]
where \(\pi\) is the map from \(HF^\infty(Y, t)\) to \(HF^+(Y, t)\) in [11].

In [13], Manolescu-Owens showed that the \(d\text{-invariant}\) of the double cover of \(S^3\) branched over a knot \(K\) is a concordance invariant for \(K\):
\[
d(K) := 2d(\Sigma(K), t_0),
\]
where \(\Sigma(K)\) is the branched double cover of \(S^3\) along \(K\), and \(t_0\) is the \(\text{Spin}^c\) structure of \(\Sigma(K)\) such that \(c_1(t_0) = 0 \in H^2(\Sigma(K); \mathbb{Z})\). They showed that \(d = \tau\) for alternating knots and knots with up to nine crossings. They also showed \(d\) of Whitehead double of an alternating knot \(K\) determined by \(\tau(K)\):

**Theorem 9.** [13 Theorem 1.5.] If \(K\) is alternating, then
\[
d(D(K)) = -4 \max\{\tau(K), 0\}.
\]

Therefore, it easily follows from the theorem and the computation of \(\tau\) in the previous section that \(d(D(T_{2, 2m+1})) = -4m\). See also Section 5.2.

The \(d\text{-invariant}\) of the \(1\text{-surgery}\) of \(S^3\) along the knot \(K\), \(d(S_1^3(K))\), is also a concordance invariant. In [22], Peters gives an algorithm of computing \(d(S_1^3(K))\), provided knowledge of \(CFK^\infty(K)\).

3. Infinity version of knot Floer complex of \(D(T_{2, 2m+1})\)

Recently, Hedden-Kim-Livingston showed that \(CFK^\infty(D(T_{2, 3}))\) is a chain homotopy equivalent to \(CFK^\infty(T_{2, 3}) \oplus A\) for some acyclic complex \(A\). [14 Proposition 6.1.]. Also see [23] section 9.1. Here, we will prove that the result can be generalized to the torus knots \(T_{2, 2m+1}\) for \(m \geq 1\), and furthermore \(CFK^\infty(D(T_{2, 2m+1}))\) will be completely determined.

Before proving the theorem, recall the following useful lemma regarding how a basis change in a filtered chain complex over \(\mathbb{F}\) affects the diagram of a knot Floer chain complex.

**Lemma 10.** [6 Lemma A.1.] Let \(C_*\) be a knot Floer complex with a 2-dimensional arrow diagram \(D\) given by an \(\mathbb{F}\text{-basis}\). Suppose that \(x\) and \(y\) are two basis elements of the same grading such that each of the \(i\) and \(j\) filtrations of \(x\) is not greater than that of \(y\). Then the \(\mathbb{Z} \oplus \mathbb{Z}\) filtered basis change given by \(y' = y + x\) gives rise to a diagram \(D'\) of \(C_*\) which differs from \(D\) only at \(y\) and \(x\) as follows:

- Every arrow from some \(z\) to \(y\) in \(D\) adds an arrow from \(z\) to \(x\) in \(D'\).
Every arrow from \( x \) to some \( w \) in \( D \) adds an arrow from \( y' \) to \( w \) in \( D' \)

We use the above lemma for the purpose of removing certain boundary arrows in chain complexes over \( \mathbb{F} \). For example, the proposition below will be useful for proving Theorem 4.

**Proposition 11.** Suppose \( C \) is one of the \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain complexes over \( \mathbb{F} \) given by the diagrams in Figure 3 with any possible combination of dotted arrows. Then all dotted arrows can be removed by a basis change.

**Proof.** First, consider the complex (I). Suppose that

\[
\begin{align*}
\partial a &= b + c + Ax + By, \\
\partial b &= d + Cz, \\
\text{and } \partial c &= d + Dz
\end{align*}
\]

for some \( A, B, C \) and \( D \) in \( \mathbb{F} \). Since \( \partial^2 = 0 \),

\[
0 = \partial^2 a = \partial(b + c + Ax + By) = (A + B + C + D)z.
\]

Therefore, the coefficients have to satisfy the equation that \( A + B + C + D = 0 \). Now, we consider every possible coefficient of \( A, B, C \) and \( D \) in \( \mathbb{F} \) satisfying the equation and show that each case can be transformed to have \( A = B = C = D = 0 \), as desired, after proper change of basis.

- If \( A = B = 1 \) and \( C = D = 0 \), change the basis by \( b' = b + x \), \( c' = c + y \) and \( d' = d + z \).
- If \( A = C = 1 \) and \( B = D = 0 \), change the basis by \( b' = b + x \).
- If \( A = D = 1 \) and \( B = C = 0 \), change the basis by \( b' = b + x \) and \( d' = d + z \).
- If \( B = C = 1 \) and \( A = D = 0 \), change the basis by \( c' = c + y \) and \( d' = d + z \).
- If \( B = D = 1 \) and \( A = C = 0 \), change the basis by \( c' = c + y \).
- If \( C = D = 1 \) and \( A = B = 0 \), change the basis by \( d' = d + z \).
- If \( A = B = C = D = 1 \), change the basis by \( b' = b + x \) and \( c' = c + y \).

Similar argument is applied to remove any combination of possible dotted arrows in the complexes (II) and (III).

**Proof of Theorem 4.** Let \( D \) be \( D(T_{2m+1}) \) for \( m \geq 1 \). Theorem 1.2 of [4] together with the computation of \( \overline{CFK}(T_{2m+1}) \) shows that

\[
\overline{\text{HFK}}_*(D, j) = \begin{cases} 
\mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-2)} \oplus \cdots \oplus \mathbb{F}^2_{(-2m+1)}, & j = 1 \\
\mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-4)} \oplus \cdots \oplus \mathbb{F}^2_{(-2m)}, & j = 0 \\
\mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-3)} \oplus \mathbb{F}^2_{(-5)} \oplus \cdots \oplus \mathbb{F}^2_{(-2m-1)}, & j = -1 \\
0 & \text{otherwise.}
\end{cases}
\]

We assign an \( \mathbb{F} \)-basis to each summand in the direct decomposition as below:

\[
\overline{\text{HFK}}_*(D, j) = \begin{cases} 
\langle x_0^0, \cdots, x_{2m}^0 \rangle \oplus \langle u_{1,1}^{-1}, u_{1,2}^{-1} \rangle \oplus \cdots \oplus \langle u_{m,1}^{-2m+1}, u_{m,2}^{-2m+1} \rangle & j = 1 \\
\langle y_1^{-1}, \cdots, y_{4m-1}^{-1} \rangle \oplus \langle v_1^{-2}, \cdots, v_1^{-4} \rangle \oplus \cdots \oplus \langle v_{m,1}^{-2m}, \cdots, v_{m,4}^{-2m} \rangle & j = 0 \\
\langle z_1^{-2}, \cdots, z_{2m}^{-2} \rangle \oplus \langle w_1^{-3}, w_1^{-3} \rangle \oplus \cdots \oplus \langle w_{m,1}^{-2m-1}, w_{m,2}^{-2m-1} \rangle & j = -1 \\
0 & \text{otherwise},
\end{cases}
\]

where the superscript of a generator represents its absolute grading.
for a diagram. In fact we will show that we can assume there are no $\partial^i$ since they are canonically determined by the grading consideration. See Figure 4

Figure 3. Any possible combination of dotted arrows can be removed by a basis change.

Since $\widehat{HFK}_*(D)$ is homotopic equivalent to the $\widehat{CFK}(D)$ [23 Lemma 4.5], we assume that $\widehat{CFK}^\infty(D)_{(0, j)} = \widehat{HFK}_*(D, j)$ and $\widehat{CFK}^\infty(D)_{(i, j)} \cong U^{-i} \widehat{CFK}^\infty(D)_{(0, j)} = \widehat{HFK}_*(D, j - i)$. Now, we investigate all differentials in $\widehat{CFK}^\infty(D)$ by using the facts, $\partial^2 = 0$, $H_*(\widehat{CFK}(D)) \cong \widehat{HF}(S^3) \cong \mathbb{F}(0)$ and $H_*(\widehat{CFK}^\infty(D)) \cong \widehat{HF}^\infty(S^3) \cong \mathbb{F}[U, U^{-1}]$.

First, note that there are no components of the boundary maps between generators of the same $(i, j)$-filtration since they would reduce $\widehat{HFK}_*(D)$. Thus, we can decompose the boundary maps $\partial$ to the vertical, horizontal and diagonal components, $\partial = \partial_V + \partial_H + \partial_D$. Also, we remark that it is enough to determine boundary maps of $\mathbb{F}[U, U^{-1}]$-generators in $\widehat{CFK}^\infty$ because the boundary map is $U$-equivariant.

Exactly the same argument as [6, Proposition 6.1] can be used to determine $\partial_V$ and $\partial_H$ i.e. by the fact that $H_*(\widehat{CFK}(D)) = \mathbb{F}(0)$ and using grading consideration, after changing basis, we can assume that

$$\begin{align*}
\partial_V(x^d_l) &= y^d_{k-1} \quad \text{for } k = 2, \cdots, 2m \\
\partial_V(y^d_{2m+1-1}) &= z^d_l \quad \text{for } l = 1, \cdots, 2m. \\
\partial_V(u^d_{p, i}) &= v^d_{p, i} \quad \text{and } \partial_V(v^d_{p, i+2}) = u^d_{p, i} \quad \text{for } p = 1, \cdots, m \text{ and } i = 1, 2,
\end{align*}$$

for $d \in 2\mathbb{Z}$ and $\partial_V$’s of other elements are trivial. Analogously, since $H_*(\widehat{CFK}^\infty(D)_{(j=0)})$ is isomorphic to $\widehat{HF}(S^3) \cong \mathbb{F}(0)$ and $\partial^2 = 0$, the horizontal boundary components can be assumed as following:

$$\begin{align*}
\partial_H(z^d_l) &= y^d_{k-1} \quad \text{for } k = 2, \cdots, 2m \\
\partial_H(x^d_{2m+1-1}) &= z^d_l \quad \text{for } l = 1, \cdots, 2m. \\
\partial_H(u^d_{p, i}) &= v^d_{p, i} \quad \text{and } \partial_H(v^d_{p, i+2}) = u^d_{p, i} \quad \text{for } p = 1, \cdots, m \text{ and } i = 1, 2,
\end{align*}$$

and $\partial_H$’s of other elements are trivial. We drop $\mathbb{F}[U, U^{-1}]$ coefficients of generators since they are canonically determined by the grading consideration. See Figure 4 for a diagram. In fact we will show that we can assume there are no $\partial_D$ components for any elements as shown in Figure 4.
Figure 4. A diagram of \( \text{CFK}^\infty(D(T_{2,5})) \). The superscript of each generator represents its grading. Note that the subcomplex generated by the bold arrows is a direct summand of the chain complex and isomorphic to \( \text{CFK}^\infty(T_{2,3}) \), and the homology of the complement of the summand is trivial.
We can split $CFK^\infty(D)$ as following disjoint subsets:

$$
A^d_{p,i} := \{v^d_{p,i+2}, u^d_{p,i}, u^{d-1}_{p,i}, v^{d-2}_{p,i}\}
$$

$$
B^d_q := \{y^d_{2m+q}, x^{d-2}_{q+1}, z^{d-2}_{q+1}, y^{d-3}_q\}
$$

and

$$
C^d := \{u^{d-1}_{2m}, x^{d-2}_1, z^{d-2}_1\},
$$

for $1 \leq p \leq m, 1 \leq q \leq 2m-1, i = 1, 2$ and $d \in 2\mathbb{Z}$. Note that any arrows between subsets must be diagonal. Disregarding the diagonal arrows between subsets, each complex of $A$'s and $B$'s has four generators with square-shaped grid diagram, and each complex of $C$'s has three generators which looks like $St(1,1)$. Therefore, if we remove all arrows between subsets i.e. $CFK^\infty(D)$ is a direct sum of $A$'s, $B$'s and $C$'s, the theorem follows.

Define a subset of $A$ as $A^d_{p,i} := \{v^d_{p,i+2}, u^d_{p,i}, u^{d-1}_{p,i}\}$. Due to grading constraints on the filtered complex, we observe the following:

- $\partial_D$ of any generator in $A^d_{p,i}$ has components only in $A^d_{p,j}, B^d_q$ and $C^d$ for $k < p, j = 1, 2$, and $q = 1, \cdots, 2m-1$ (i.e. diagonal arrows between $A$'s going from higher to smaller first index.)
- $\partial_D$ of the generators in $B$'s and $C$'s are zero.

These observations allow us to apply Corollary 11 inductively to remove all diagonal arrows in the complex.

We start to remove any diagonal arrows from $A^d_{m,1}$. First, we remove all diagonal arrows going from $A^d_{m,1}$ to $A^d_{m-1,1}$, using basis-change of the case (I) of Corollary 11. Differently from the corollary, there can be other components in $\partial_D$ of elements in $A^d_{m,1}$, not in $A^d_{m-1,1}$. However, considering the grading constraints again, one can easily check that the other components cannot induce $z$ component of $\partial^2u$ in the Equation 2, hence the equation that $A + B + C + D = 0$ in the proof of the corollary still holds and we remove diagonal arrows using a basis-change in the corollary.

After applying the basis-change, two types of newer diagonal arrows will be added due to arrows coming to $A^d_{m,1}$ and arrows going from $A^d_{m-1,1}$. First note that there are no diagonal arrows coming to $A^d_{m,1}$ by the observations above ($m$ is the greatest index for $A$.) Secondly, a diagonal arrow from $A^d_{m-1,1}$ to some generator adds an arrow going from $A^d_{m-1,1}$ to the generator after basis-change, but note that these arrows are going to the subsets $A^d_{p,i}$ with $p < m - 1$ and $i = 1, 2$, which we will remove later.

Now, we similarly change the basis for removing diagonal arrows from $A^d_{m,1}$ to $A^d_{m-1,2}$, $A^d_{m-2,1}$, $A^d_{m-2,2}$, $A^d_{1,1}$, and $A^d_{1,2}$ in sequence. Then, case (II) and (III) of the corollary will be applied to remove arrows from $A^d_{m,1}$ to $B^d_q$'s and $C^d$.

The induction ends with removing any $\partial_D$ from $A^d_{m,1}$, since there are no diagonal arrows from $A^d_{1,i}$'s, $B^d_q$'s and $C^d$.

Then, we remove $\partial_D$ of $A^d_{m,2}, A^d_{m-1,1}, A^d_{m-1,2}, \cdots, A^d_{1,1}$, and $A^d_{1,2}$ likewise. After removing the diagonal arrows from $A^d_{p,i}$ for all $p = 1, \cdots, m$ and $i = 1, 2$, the only remaining non-trivial $\partial_D$ are ones of $v_{p,1}$ and $v_{p,2}$. It is easy to see that $\partial_D$'s of $v_{p,1}$ and $v_{p,2}$ also vanish: $0 = \partial^2(u_{p,i}) = \partial(v_{p,i})$. Thus, we may assume that $\partial_D$'s of $CFK^\infty$ are all zero. □
4. $\delta$-invariant of $D^2(T_{2,2m+1})$ and proof of Theorem \[3\]

First, we present a lemma that relates the $\delta$-invariant of a Whitehead double to $dS^3_1$.

Lemma 12. For any knot $K$, $\delta(D(K)) = 2dS^3_1(K \# K^\ast)$.

Proof. Let $K_\ast$ denote the 3-manifold obtained by $p$-surgery of $S^3$ along a knot $K$. Manolescu-Owens showed that $d(K_{-1/2}) = d(K_{-1})$ for any knot in the proof of \[13\] Proposition 6.2; in fact both of them equal $2h_0(K)$, where $h_0(K)$ is an invariant defined by Rasmussen in \[22\]. Thus, using the behaviour of $d$-invariants under orientation reversal \[14\] Proposition 4.2, we have

$$d(K_{1/2}) = -d(K_{-1/2}) = -d(K_{-1}) = d(K_1),$$

where $\overline{K}$ is the mirror image of $K$.

Recall that the double cover of $S^3$ branched over $D(K)$, $\Sigma(D(K))$, can be obtained by $1/2$-surgery along $K \# K^\ast$ in $S^3$, where $K^\ast$ is the knot $K$ with its orientation reversed, see \[13\] Proposition 6.1. From the definition of $\delta$-invariant,

$$\delta(K) := 2d(\Sigma(K)) = 2d((K \# K^\ast)_{1/2}) = 2d((K \# K^\ast)_{1}),$$

Proof of Theorem \[3\]. By applying the lemma above to $D^2(T)$, we have

$$\delta(D^2(T)) = 2d((D(T) \# D(T)^\ast)_{1}),$$

where $T = T_{2,2m+1}$.

To compute $d(K_1)$, it suffices to understand $CFK^\infty(K)$ \[22\]. Let us recall the algorithm. Pick $\xi$, a generator of $H_*(CFK^\infty(K)) \cong \mathbb{F}$. Note that any generator become zero in $CFK^\infty_{i \geq 0}$ by the multiplication by high enough power of $U$. Then

\[3\]

$$d(K_1) = -2 \min \{n \geq 0 : [U^{n+1} \xi] = 0 \in H_*(CFK^\infty(K))_{i \geq 0} \}.$$ 

Observe that $d(K_1)$ is derived from a direct summand of $CFK^\infty(K)$ containing a generator of $H_*(CFK^\infty(K))_{i \geq 0} \cong \mathbb{F}$. In particular, if $CFK^\infty(K)$ and $CFK^\infty(K')$ are differ only by an acyclic complex, then $d(K_1) = d(K'_1)$.

Now, let us understand $CFK^\infty(D(T) \# D(T)^\ast)$. Since the knot Floer complex is unchanged under the orientation reversal \[19\] Proposition 3.9] and by the connected sum formula for knot Floer complexes in \[19\] Theorem 7.1],

$$CFK^\infty(D(T) \# D(T)^\ast) \cong CFK^\infty(D(T)) \otimes CFK^\infty(D(T)).$$

Thus, $d((D(T) \# D(T)^\ast)_{1})$ equals to $d((T_{2,3} \# T_{2,3})_{1})$ by Theorem \[4\]. It is computed that $d((T_{2,3} \# T_{2,3})_{1}) = -2$ as an example of the computer program, $d$Calc in \[22\], (it can also be computed by Proposition \[14\] so that $\delta(D^2(T)) = -4$. On the other hand, $\delta(D(T)) = -4m$ as stated in Section \[22\]. The first part of Theorem \[3\] follows.

Recall that $\delta \equiv \sigma \mod 4$ \[13\] (2.1) and $\sigma = 0$ for any knot in $C_{TS}$. Consider the homomorphism $\psi = (\tau, \delta/4 : C_{TS} \to \mathbb{Z} \oplus \mathbb{Z}$. Since $\psi(D(T)) = (1, -m)$ and
\[ \psi(D^2(T)) = (1, -1), \psi \text{ is surjective if } m \geq 2. \] Therefore, \( \mathcal{C}_{TS} \) has a \( \mathbb{Z}^2 \) summand generated by \( D(T) \) and \( D^2(T) \).

Proof of Corollary 7

By [3 Corollary 4.9, Corollary 6.11] both \( \tau \) and \( \delta \) vanish for the knots in \( \mathcal{C}_\Delta \cap \mathcal{T}_0 \). Now, consider the induced homomorphism \( (\tau, \delta/4) : \mathcal{C}_\Delta/\mathcal{C}_\Delta \cap \mathcal{T}_0 \to \mathbb{Z} \oplus \mathbb{Z} \), and the surjectivity of it can be shown by the knots, \( D(T) \) and \( D^2(T) \).

5. Generalization of the result

In this section we discuss how to generalize the result for \( T_{2,2m+1} \) to other knots. First, we use a genus-bound property of the concordance invariant \( dS^3_1 \) to show that, provided that \( |\delta(D(K))| > 8 \), \( D(K) \) and \( D^n(K) \) are not smoothly concordant for each \( n \geq 2 \). Secondly, we present formulas to compute \( dS^3_1(K) \) and \( \delta(D(K)) = 2dS^3_1(K\#K) \) for a given staircase complex of \( K \) introduced in Section 2.2.

5.1. Genus bound and proof of Theorem 5

Proof of Theorem 5. It is shown in [22 Theorem 1.5.] that \( -dS^3_1(K)/2 \) is a lower bound for the slice genus, \( g^*(K) \), of \( K \), and note that the slice genus of \( D(K) \) is at most 1 for any knot \( K \). Hence, for \( n \geq 2 \),
\[
-\delta(D^n(K)) = -2dS^3_1(D^n-1(K)\#D^n-1(K)^r) \leq 4g^*(D^n-1(K)\#D^n-1(K)^r) \leq 8.
\]
Therefore, if \( \delta(D(K)) < -8 \), equivalently \( dS^3_1(K\#K) < -4 \), \( D(K) \) and \( D^n(K) \) are not smoothly concordant. (According to [13 Theorem 1.5], \( \delta(D(K)) \) is nonpositive for any knot \( K \).)

If \( \tau(K) > 0 \), both \( \tau(D(K)) \) and \( \tau(D^n(K)) \) are 1 by Theorem 2. Now one can prove the second part of the theorem by considering the surjective homomorphism \( (\tau, \delta/4) : \mathcal{C}_{TS} \to \mathbb{Z} \oplus \mathbb{Z} \).

Proof of Corollary 7. This is obtained by Theorem 5 together with Theorem 9.

Remark 13. Since \( \delta(D(T_{2,2m+1})) = -4m \), Theorem 5 is a special case of Theorem 5 for \( m \geq 3 \), but we have shown it for the case \( m = 2 \) as well by computing \( \delta(D(2,2m+1)) = -4 \).

5.2. \( dS^3_1(K) \) and \( \delta(D(K)) \) of a Staircase Complex. If a knot admits a knot Floer complex generated by a staircase complex (equivalently, \( L \)-space knots), then its \( d \)-invariant can be easily obtained. For the \( d \)-invariants of higher surgery coefficients, we refer to [1 Section 4.2].

Proposition 14. Suppose the knot Floer complex of \( K \) can be given by a staircase complex \( St(K) \), then
\[
d(S^3_1(K)) = -2 \min_{(i,j) \in \text{Vert}(St(K))} \max\{i,j\}
\]
\[
\delta(D(K))/2 = d(S^3_1(K\#K)) = -2 \min_{(i,j), (k,l) \in \text{Vert}(St(K))} \max\{i+k,j+l\},
\]
where \( \text{Vert}(St(K)) \) is the set of the coordinates of the generators of \( St(K) \).

Proof. Suppose that \( CFK^\infty(K) \) is generated by \( St(K) \), then the top left element in \( St(K) \) represents the generator of \( H_*(CFK^\infty(K)_{(i=0)}) \cong \mathbb{F} \), say \( \xi \). The chain complex \( St(K) \) has the form \( 0 \to F_{i+1}^k \to F_{i+1}^{k+1} \to 0 \). Observe that any non-trivial generator \( \eta \) of \( F_{(0)}^{k+1} \) is homologous to \( \xi \). For \( \eta \) with \( (i,j) \)-coordinates, \( U^{\max\{i,j\}+1}\eta \)
lies in the subcomplex $\text{CFK}^\infty(K)_{\{i<0 \text{ and } j<0\}}$, whereas $U^\max\{i, j\}y$ does not. Note also that since $\text{St}(K)$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex, $\min\{i, j\} \in \text{Vert}(\text{St}(K))$ is realized by the elements in $\mathbb{F}^{k+1}_{(0)}$, not ones in $\mathbb{F}^{k}_{(+1)}$. Hence, the first formula follows from the Equation (3).

Although $\text{CFK}^\infty(K \# K)$ is not generated by a staircase complex, it can be constructed from the tensor complex, $\text{St}(K) \otimes \text{St}(K)$ by the connected-sum formula [19, Theorem 7.1]. The coordinates of the generators in $\text{St}(K) \otimes \text{St}(K)$ are given by the sums of a pair of coordinates of the generators of $\text{St}(K)$. The complex $\text{St}(K) \otimes \text{St}(K)$ has the form $0 \to \mathbb{F}^{k+1}_{(2)} \to \mathbb{F}^{2k+1}_{(+1)} \to \mathbb{F}^{k+1}_{(0)} \to 0$, and the generators with $(0)$-grading are homologous to the generator of $H_\ast(\text{CFK}^\infty(K \# K)_{\{i=0\}}) \cong \mathbb{F}$. See Figure 5 for the tensor complex of two copies of $\text{St}(1, 2, 2, 1)$. Therefore, we get the second formula similarly.

For example, since $\text{St}(T_{2,2m+1}) = (1, \cdots, 1)$ of length $2m + 1$, one can compute that $\delta(D(T_{2,2m+1})) = -4m$ again. In the case of $(3, 4)$ torus knot, $\text{St}(T_{3,4}) = (1, 2, 2, 1)$, and so

$$\text{Vert}(T_{3,4}) = \{(0, 3), (1, 3), (1, 1), (3, 1), (3, 0)\}.$$  

Thus $\delta(D(T_{3,4})) = -8$, and hence we cannot figure out if $D(T_{3,4})$ and $D^2(T_{3,4})$ are concordant, using Theorem 5. Note that there are many knots such that $\delta(D(K)) = -8$: for example, any knot $K$ whose $\text{CFK}^\infty$ is generated by $\text{St}(1, n, n, 1)$. However, one can similarly apply the arguments in Section 5 to show $\text{CFK}^\infty(D(T_{3,4})) \cong \text{CFK}^\infty(T_{2,3}) \oplus A$ for some acyclic complex $A$. Therefore, $\delta(D^2(T_{2,3})) = -4$, and we conclude that $D(T_{3,4})$ and $D^2(T_{3,4})$ are not concordant.

For the right-handed trefoil knot, as far as the author knows, all concordance invariants of $D(T_{2,3})$ and $D^2(T_{2,3})$ are same, so it is still mysterious if $D(T_{2,3})$ and $D^2(T_{2,3})$ are smoothly concordant.
Conjecture 15. \( D(T_{2,3}) \) and \( D^2(T_{2,3}) \) are not smoothly concordant.

Remark 16. This conjecture is possibly approached using gauge-theoretic invariants. See [7].

Implementation. We wrote a C++ program computing \( dS^3(K) \) and \( \delta(D(K)) \) for a \((p, q)\) torus knot or a staircase complex of \( K \). You may download the source file in the author’s webpage. In order to algorithmically obtain the staircase complex (equivalently Alexander-Conway polynomial) of a \((p, q)\) torus knot in the program, we used the subsemigroup of \( \mathbb{N} \) generated by \( p \) and \( q \), see [2].
ON INDEPENDENCE OF ITERATED DOUBLES IN THE CONCORDANCE GROUP

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