THE COTANGENT COMPLEX AND THOM SPECTRA

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ABSTRACT. We first prove, in the context of ∞-categories and using Goodwillie’s calculus of functors, that various definitions of the cotangent complex of a map of E_∞-ring spectra that exist in the literature are equivalent. We then prove the following theorem: if R is an E_∞-ring spectrum and f : G → Pic(R) is a map of E_∞-groups, then the cotangent complex over R of the Thom E_∞-R-algebra of f is equivalent to the smash product of Mf and the connective spectrum associated to G.

1. INTRODUCTION

1.1. Deformation theory. The cotangent complex arises as a central object of deformation theory. One wishes to understand extensions of functions between geometric spaces to infinitesimal thickenings of those spaces. Working algebraically, the simplest example of an infinitesimal thickening of a commutative R-algebra S is given by a square-zero extension: that is, a surjective map of commutative R-algebras φ : \tilde{S} → S such that the product of any two elements in the kernel of φ is zero.

One way to construct square-zero extensions is using derivations. An R-linear derivation S → M is a map of R-modules which satisfies the Leibniz condition. The S-module of R-linear derivations Der_R(S, M) is co-represented by the S-module of Kähler differentials Ω_S/R. It is also represented by the commutative R-algebra over S given by the trivial square-zero extension S ⊕ M with multiplication given by (s, m)(s', m') = (ss', sm' + ms'). Therefore, we have bijections
\[
\text{Hom}_{\text{Mod}_R}(\Omega_S/R, M) \cong \text{Der}_R(S, M) \cong \text{Hom}_{\text{CAlg}_R/S}(S, S ⊕ M)
\]
where CAlg_{R//S} denotes the category of commutative rings C with maps of commutative rings R → C → S composing to the unit of S.

Starting from a class in Ext^1_S(Ω_S/R, M) represented by an extension of S-modules
\[
M → \tilde{M} → \Omega_S/R,
\]
one can build a square-zero extension of S by M by pulling back the second map along the universal derivation S → Ω_S/R in Mod_R:
\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{=} & S \\
\downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{=} & \Omega_S/R.
\end{array}
\]
The pullback \tilde{S} gets a commutative R-algebra structure, and the map \tilde{S} → S is surjective with kernel isomorphic to M as an R-module.

However, this does not produce all square-zero extensions: for that, one needs to derive the module of Kähler differentials. The way this was originally achieved independently by André [And67] and Quillen [Qui68] was by simplicial methods. Quillen placed a model structure on the category of simplicial commutative A-algebras. This produces a B-module LΩ_S/R (called...
the algebraic cotangent complex by Lurie [Lur18b, 25.3]), and the first André-Quillen cohomology group \( \text{Ext}^1_\mathbb{L}(\Omega S/R, M) \) is isomorphic to the \( S \)-module of equivalence classes of square-zero extensions of \( S \) by \( M \).

We shall not take this approach, but rather work with the more general \( E_\infty \)-ring spectra. The module of Kähler differentials in this context is replaced by the cotangent complex: if \( A \to B \) is a map of \( E_\infty \)-ring spectra, the cotangent complex is a \( B \)-module \( L_{B/A} \). Using \( L_{S/R} \) instead of \( \Omega S/R \), André-Quillen (co)homology is replaced by topological André-Quillen (co)homology, also known as TAQ. The precise relation between \( \Omega B/A \) and \( L_{B/A} \) can be found in [Lur18b, 25.3.3.7/25.3.5.4].

One can define square-zero extensions of \( E_\infty \)-ring spectra. The cotangent complex helps classify them. As is usual in homotopy theory, one does not merely want to describe equivalence classes of square-zero extensions. One would like to prove that the whole \( \infty \)-category of square-zero extensions of an \( E_\infty \)-\( A \)-algebra \( B \) is equivalent to the \( \infty \)-category of \( A \)-linear derivations \( B \to \Sigma M \) where \( M \) is a \( B \)-module.\(^2\) This is proven in [Lur17, 7.4.1.26], provided one restricts the \( \infty \)-categories a bit. Note that the traditional definition of derivations as linear maps that satisfy the Leibniz rule does not work in this setup, whereas the interpretations of (1.1) do.

Apart from the connections to deformation theory, the cotangent complex \( L_{B/A} \) helps detect useful properties of the map \( f : A \to B \). For example, if \( A \) and \( B \) are connective, then \( f \) is an equivalence if and only if \( \pi_0(f) : \pi_0(A) \to \pi_0(B) \) is an isomorphism and \( L_{B/A} \) vanishes [Lur17, 7.4.3.4]. The theory of the cotangent complex also helps in proving theorems that do not mention it at all: for example, if \( A \) is an \( E_\infty \)-ring spectrum and a map of commutative rings \( \pi_0(A) \to B_0 \) is étale, then it lifts in an essentially unique way to an étale map of \( E_\infty \)-ring spectra \( A \to B \) [Lur17, 7.5.0.6].

1.2. Different approaches to the cotangent complex. Let us trace the history of the \( E_\infty \) cotangent complex of a map of \( E_\infty \)-ring spectra \( A \to B \), since it has been defined in different ways in the literature.

The first definition can be found in a preprint by Kriz [Kri93]. It was defined as a sequential colimit built out of tensoring \( B \wedge_A B \) with spheres, in a certain way.

Basterra [Bas99] took another approach: she defined the cotangent complex to be the indecomposables of the augmentation ideal \( I(B \wedge_A B) \) of \( B \wedge_A B \), which is the fiber of the multiplication map \( B \wedge_A B \to B \). She did not prove the equivalence with Kriz’s approach.

Later, herself and Mandell [BM05] established the connection between Basterra’s definition of the cotangent complex and stabilization. Just as Beck had observed in the sixties that the category of modules over a commutative ring \( R \) was equivalently given by the abelian group objects in augmented commutative \( R \)-algebras [Bec03], they proved the \( E_\infty \)-analog. Abelian group objects have to be replaced by spectra objects. They proved that to get \( L_{B/A} \), one can start from \( B \wedge_A B \) considered as an augmented commutative \( B \)-algebra, then stabilize it, i.e. apply \( \Omega^{\infty, \Sigma^{\infty}} \) to it, then take its augmentation ideal.

It was known to the experts that one could extract from the results of Basterra and Mandell an expression of the cotangent complex as a sequential colimit, similar to Kriz’s expression, see e.g. [Sch11, Page 164]. However, we think a full description of how these approaches are connected has not appeared in the literature. We take the opportunity to expand on them in Section 3.

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1When treating a discrete commutative ring as an \( E_\infty \)-ring spectrum, the Eilenberg–Mac Lane functor shall be understood.

2Recall that \( \text{Ext}^1_\mathbb{L}(L_{B/A}, M) \simeq \pi_0(\text{Map}_{\text{Mod}_{\mathbb{L}}} (L_{B/A}, \Sigma M)) \).
Theorem 4.3. Let $R$ be an $E$-ring spectrum. This allows for generalized, possibly non-connective Thom algebras: if $M_f$ is an $E$-algebra, then we get that $\Omega^1(\otimes_A B) \to \Omega^2(\otimes_A B) \to \cdots$, and with this additional hypothesis the cotangent complex $L_{B/A}$ becomes a Thom $E$-group and returns a $B$-module.

In summary, we prove in the $\infty$-categorical context of [Lur17] that the cotangent complex $L_{B/A}$ can be presented in the following ways:

- As the augmentation ideal of the stabilization of $B \wedge_A B$, i.e. $I(\Omega^\infty \Sigma^\infty (B \wedge_A B))$ (3.9/3.10),
- As the excisive approximation of $I$ evaluated in $B \wedge_A B$, i.e. $(P_I)(B \wedge_A B)$ (3.11),
- As the sequential colimit of $B$-modules (3.22)

$$L_{B/A} \simeq \text{colim}_{\text{Mod}_R} (S^0 \otimes_A B \to \Omega(S^1 \otimes_A B) \to \Omega^2(S^2 \otimes_A B) \to \cdots),$$

- As the module of indecomposables of the augmentation ideal of $B \wedge_A B$ (3.30).

Here $\otimes -$ is a certain operation to be introduced in Notation 3.20 that takes a pointed space and an $E_\infty$-$A$-algebra $B$ and returns a $B$-module.

1.3. Thom spectra. The main result of this paper is the determination of the cotangent complex of Thom $E_\infty$-ring spectra. Let us quickly recall what these are, following the $\infty$-categorical approach of [ABG+14]. Let $G$ be a space, $R$ be an $E_\infty$-ring spectrum, $\text{Pic}(R)$ be the Picard space of $R$ (the subspace of $\text{Mod}_R$ spanned by the invertible $R$-modules), and $f : G \to \text{Pic}(R)$ be a map. The colimit of $G \to \text{Pic}(R) \to \text{Mod}_R$ is the Thom $R$-module of $f$, denoted $M_f$. In fact, $\text{Pic}(R)$ is an $E_\infty$-group. When $G$ is an $E_\infty$-group and $f$ is a map of $E_\infty$-groups, then $M_f$ gets the structure of an $E_\infty$-$R$-algebra [ABG18], [ACB19]. Examples of Thom $E_\infty$-ring spectra include complex cobordism $MU$ and periodic complex cobordism $MUP$.

Theorem 4.3. Let $R$ be an $E_\infty$-ring spectrum. Let $f : G \to \text{Pic}(R)$ be a map of $E_\infty$-groups. There is an equivalence of $M_f$-modules

$$L_{M_f/R} \simeq M_f \wedge B^\infty G.$$

Here $B^\infty G$ denotes the connective spectrum associated to $G$.

A model-categorical version had first appeared in [BM05]. For example, we recover the equivalence of $MUI$-modules

$$L_{MUI} \simeq MUI \wedge bu$$

of that paper. The result of Basterra and Mandell, however, only applies to Thom spectra of maps to $BGL_1(S)$, whereas ours applies to maps to $\text{Pic}(R)$ where $R$ is any $E_\infty$-ring spectrum. This allows for generalized, possibly non-connective Thom $E_\infty$-ring spectra. For example, we get that

$$L_{MUP} \simeq MUP \wedge ku$$

as $MUP$-modules. In fact, $L_{MUP}$ is actually a Thom $E_\infty$-$ku$-algebra. More generally, we observe in Proposition 4.9 that when $G$ is an $E_\infty$-ring space, then the $R$-module $L_{M_f/R}$ underlies a Thom $E_\infty$-$(R \wedge B^\infty G)$-algebra. In other words, with this additional hypothesis the cotangent complex of a Thom $E_\infty$-algebra becomes a Thom $E_\infty$-algebra.

In Section 5 we extend Theorem 4.3 to cotangent complexes of étale extensions of Thom algebras: if $M_f \to B$ is an étale map of $E_\infty$-$R$-algebras, then

$$L_{B/R} \simeq B \wedge B^\infty G.$$

This allows us, for example, to recover the equivalence $L_{KU} \simeq KU \wedge HQ$ from [Sto19].
1.4. **Notation and conventions.** We will freely use the language of $\infty$-categories as developed in [Lur09], [Lur17].

Let $\mathcal{C}$ be an $\infty$-category. Given a fixed map $f : A \to B$, we denote by $\mathcal{C}_{A//B}$ the $\infty$-category $(\mathcal{C}/B)_{f/}$ of objects $C \in \mathcal{C}$ together with maps $A \to C \to B$ which compose to $f$. Similarly, we denote by $\mathcal{C}_{B//B}$ the $\infty$-category $(\mathcal{C}/B)_{\text{id}_B/}$.

If $\mathcal{C}$ is an $\infty$-category with terminal object $T$, we let $\mathcal{C}_+$ denote the undercategory $\mathcal{C}_{T/}$.

The $\infty$-category of spaces will be denoted by $S$, and that of spectra by $\text{Sp}$. Its full subcategory of connective spectra is denoted by $\text{Sp}^\text{conn}$. We denote by $\text{CAlg}$ the $\infty$-category $\text{CAlg}(\text{Sp})$ of $E_\infty$-ring spectra, and by $\text{CAlg}_R$ that of $E_\infty$-algebras over an $E_\infty$-ring spectrum $R$, i.e. $\text{CAlg}(\text{Mod}_R)$. The suspension spectrum functor $S \to \text{Sp}$ is denoted $\Sigma_\infty^*$, and if $G \in \text{CAlg}(R)$, then $S[G]$ denotes the $E_\infty$-ring spectrum $\Sigma_\infty^*(G)$.

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2. **Short review of Goodwillie calculus**

Let us summarize some notions from the Goodwillie calculus of functors which we shall be using. We shall work with the $\infty$-categorical version of it as in [Lur17, Chapter 6]. For simplicity, let us assume that $\mathcal{C}$ and $\mathcal{D}$ are $\infty$-categories which are pointed and presentable.

1. A functor $F : \mathcal{C} \to \mathcal{D}$ is reduced if it takes a final object to a final object. It is excisive if it takes pushout squares to pullback squares. The full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the excisive functors is denoted $\text{Exc}(\mathcal{C}, \mathcal{D})$, and the one spanned by the reduced, excisive functors is denoted $\text{Exc}_r(\mathcal{C}, \mathcal{D})$.

2. The $\infty$-category of spectra in $\mathcal{C}$ is defined by $\text{Sp}(\mathcal{C}) = \text{Exc}_r(S^\text{fin}_\mathcal{C}, \mathcal{C})$ where $S^\text{fin}_\mathcal{C}$ is the $\infty$-category of pointed finite spaces [Lur17, 1.4.2.8]. Evaluation at the sphere $S^0$ defines a functor $\Omega^\infty : \text{Sp}(\mathcal{C}) \to \mathcal{C}$ which is an equivalence when $\mathcal{C}$ is stable [Lur17, 1.4.2.21]; since $\mathcal{C}$ is pointed and presentable, $\Omega^\infty$ admits a left adjoint $\Sigma^\infty : \mathcal{C} \to \text{Sp}(\mathcal{C})$ [Lur17, 1.4.4.4]. If $\mathcal{C}$ is presentable but not pointed, the left adjoint to $\Omega^\infty$ is denoted $\Sigma^\infty_r$ and it factors into two left adjoint functors $\mathcal{C} \xrightarrow{(-)_{+}} \mathcal{C}_+ \xrightarrow{\Sigma^\infty_r} \text{Sp}(\mathcal{C}_+) \simeq \text{Sp}(\mathcal{C})$ [Lur17, 7.3.3.9].

3. The inclusion $\text{Exc}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})$ has a left adjoint $P_1$, called the excisive approximation functor [Lur17, 6.1.1.10]. The unit natural transformation $F \Rightarrow P_1 F$ is said to exhibit $P_1 F$ as the excisive approximation to $F$. If $F : \mathcal{C} \to \mathcal{D}$ is reduced, then $P_1 F$ is reduced, and by [Lur17, 6.1.1.28],

$$P_1 F \simeq \colim_n (\Omega^\infty_D \circ F \circ \Sigma^\infty_r).$$

4. [Lur17, 6.2.1.4] If $F : \mathcal{C} \to \mathcal{D}$ is left exact (i.e. preserves finite limits), then composition with $F$ defines a functor $\partial F : \text{Sp}(\mathcal{C}) \to \text{Sp}(\mathcal{D})$, the (Goodwillie) derivative which makes the following diagram commute

$$\begin{array}{ccc}
\text{Sp}(\mathcal{C}) & \xrightarrow{\partial F} & \text{Sp}(\mathcal{D}) \\
\Omega^\infty_C \downarrow & & \downarrow \Omega^\infty_D \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}.
\end{array}$$

5. In the case of an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$, derivatives admit a general definition [Lur17, 6.2.1.1]: they consist of a functor $\partial F : \text{Sp}(\mathcal{C}) \to \text{Sp}(\mathcal{D})$ and a natural transformation $F \circ \Omega^\infty_C \Rightarrow \Omega^\infty_D \circ \partial F$ satisfying some properties. Derivatives are unique up to
canonical equivalence [Lur17, 6.2.1.2], and there are very general existence results: in particular, under the conditions on \( \mathcal{C} \) and \( \mathcal{D} \) which we have imposed, every \( F : \mathcal{C} \to \mathcal{D} \) admits a derivative [Lur17, 6.2.1.9/6.2.3.13].

(6) [Lur17, Page 1071] If \( F : \mathcal{C} \to \mathcal{D} \) is reduced and preserves filtered colimits, then

\[ P_1 F \simeq \Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}. \]

(7) \( \partial \text{id}_C \simeq \text{id}_{Sp(C)} \) and \( P_1 \text{id}_C \simeq \Omega^\infty_{\mathcal{C}} \circ \Sigma^\infty_{\mathcal{C}} \simeq \text{colim}_n (\Omega^n_{\mathcal{C}} \circ \Sigma^n_{\mathcal{C}}) \). In particular, if \( \mathcal{C} \) is stable then \( \text{id}_C \simeq \text{colim}_n (\Omega^n_{\mathcal{C}} \circ \Sigma^n_{\mathcal{C}}) \).

(8) If \( F : \mathcal{C} \to \mathcal{D} \) is reduced, left exact and preserves filtered colimits, then \( P_1 F \simeq F \circ \Omega^\infty_{\mathcal{C}} \circ \Sigma^\infty_{\mathcal{C}} \), as follows from (6) and (4).

3. THE COTANGENT COMPLEX

In this section, we introduce the cotangent complex of a map of \( E_\infty \)-ring spectra and we give different expressions for it: via the augmentation ideal, via a stabilization process, i.e. as a sequential colimit, and via indecomposables.

Before going to \( E_\infty \)-ring spectra, let us say a word on the general definition. The relative cotangent complex according to Lurie is a suspension spectrum, in the following sense:

**Definition 3.1.** Let \( \mathcal{C} \) be a presentable \( \infty \)-category and \( f : A \to B \) in \( \mathcal{C} \). Consider the suspension spectrum functor

\[ \mathcal{C}_{A/B} \xrightarrow{\Sigma^\infty_{+}} \text{Sp}(\mathcal{C}_{A/B}). \]

The **relative cotangent complex** of \( f \) is the image of \( A \xrightarrow{f} B \xrightarrow{\text{id}_B} B \) by this functor, and it is denoted \( L_{B/A} \). If \( A \) is an initial object of \( \mathcal{C} \), then \( L_{B/A} \) is also denoted \( L_B \) and it is called the **absolute cotangent complex** of \( B \).

**Remark 3.2.** Let us say a word about the \( \Sigma^\infty_{+} \) functor above. Since \( \text{id} : B \to B \) is the terminal object of \( \mathcal{C}/B \), then

\[ \mathcal{C}_{B/B} = (\mathcal{C}/B)_{\text{id}_B} \simeq (\mathcal{C}/B)_{+}. \]

Note as well that \( \mathcal{C}_{A/B} = (\mathcal{C}/B)/f \simeq (\mathcal{C}/A)/f \). On the other hand, by [Lur17, 7.3.3.9], we have \((\mathcal{C}_{A/B})_{+} \simeq \mathcal{C}_{B/B} \). Therefore, \( \Sigma^\infty_{+} \) factors as

\[ \mathcal{C}_{A/B} \xrightarrow{-\cup_{A/B}} \mathcal{C}_{B/B} \xrightarrow{\Sigma^\infty_{+}} \text{Sp}(\mathcal{C}_{B/B}). \]

In order to address the issue of functoriality of the cotangent complex, Lurie uses the **tangent bundle** of \( \mathcal{C}_{A/B} \). We shall not be needing this, so for the sake of simplicity we will not introduce it.

Let us now concentrate on the case \( \mathcal{C} = \text{CAlg} \).

3.1. **The cotangent complex via the augmentation ideal.** When \( \mathcal{C} = \text{CAlg} \), we may identify \( \text{Sp}(\mathcal{C}_{B/B}) \) with a more familiar \( \infty \)-category, namely \( \text{Mod}_B \), as we shall now see. Note that \( \text{CAlg}_{B/B} \) is the \( \infty \)-category of **augmented** \( E_\infty \)-algebras: its objects are \( E_\infty \)-algebras \( C \) with a map \( C \to B \) of \( E_\infty \)-algebras.

**Definition 3.3.** Let \( B \in \text{CAlg} \). The **augmentation ideal** functor

\[ I : \text{CAlg}_{B/B} \to \text{Mod}_B \]
takes $C$ to the fiber in $\text{Mod}_B$ of the augmentation, i.e. to the pullback

$$
\begin{array}{c}
I(C) \rightarrow C \\
\downarrow \downarrow \\
0 \rightarrow B
\end{array}
$$

in $\text{Mod}_B$, where 0 denotes a zero object of $\text{Mod}_B$.

**Remark 3.4.** The functor $I$ is right adjoint to the free $E_\infty$-$B$-algebra functor $M \mapsto \bigvee_{n \geq 0} (M^{\wedge n})_{\Sigma_n}$ where $(-)_{\Sigma_n}$ denotes the (homotopy) orbits for the $\Sigma_n$-action [Lur17, 3.1.3.14], which is augmented over $B$ via the projection to the 0-th summand [Lur17, 7.3.4.5]. Note that $I$ is reduced, as it takes $B$ to the zero module.

**Notation 3.5.** We let $\text{NUCA}_B$ denote the $\infty$-category of non-unital $E_\infty$-$B$-algebras, which we call $\text{nucas}$ [Lur17, 5.4.4.1].

**Remark 3.6.** The augmentation ideal functor factors through $\text{NUCA}_B$ as follows:

$$
\begin{array}{ccc}
\text{CAlg}_{B//B} & \overset{I}{\longrightarrow} & \text{Mod}_B \\
\downarrow l_0 & & \downarrow U \\
\text{NUCA}_B & \overset{l}{\cong} & \text{Mod}_B
\end{array}
$$

where $U$ is the forgetful functor. Indeed, one can take the pullback defining $I$ in Definition 3.3 in the $\infty$-category $\text{NUCA}_B$ instead of in $\text{Mod}_B$, which defines $l_0$. The functor $l_0$ is a right adjoint to the functor $N \mapsto B \vee N$, and it is in fact an equivalence [Lur17, 5.4.4.10], [Bas99, 2.2]. Note as well that $I$ commutes with sifted colimits, since $U$ does [Lur17, 3.2.3.1]. In particular, $I$ commutes with filtered or sequential colimits.

**Theorem 3.7.** [Lur17, 7.3.4.7/14] The functor

$$
\text{Sp(CAlg}_{B//B}) \overset{\partial I}{\longrightarrow} \text{Sp(Mod}_B)
$$

is an equivalence of $\infty$-categories; in particular, $\text{Sp(CAlg}_{B//B}) \overset{\partial I}{\longrightarrow} \text{Sp(Mod}_B) \overset{\Omega^\infty}{\longrightarrow} \text{Mod}_B$ is an equivalence as well.

**Remark 3.8.** A model-categorical precedent can be found as Theorem 3 of [BM05]. There, the functor fitting in the place of $\partial I$ is defined as follows. First of all, in their framework a spectrum in a model category $M$ is a sequence of objects $\{X_n\}_{n \geq 0}$ of $M$ with maps $\Sigma X_n \rightarrow X_{n+1}$. Spectra in $M$ have a model structure whose fibrant objects are the $\Omega$-spectra. Thus, any topological left Quillen functor $F$ between model categories enriched over based spaces induces a left Quillen functor $\mathcal{F}$ between the corresponding model categories of spectra: the arrows $\Sigma X_n \rightarrow X_{n+1}$ get sent to $\Sigma F(X_n) \simeq F(\Sigma X_n) \rightarrow F(X_{n+1})$. After passing to the underlying $\infty$-categories, $I$ gives a functor equivalent to $\partial I$.

**Remark 3.9.** Recall from 2.(4) that $\Omega^\infty_{\text{Mod}_B} \circ \partial I \simeq I \circ \Omega^\infty_{\text{CAlg}_{B//B}}$, i.e. $\partial I$ commutes with $\Omega^\infty$. On the other hand, note that $\partial I$ typically does not commute with $\Sigma^\infty$. If it did, then

$$
\Omega^\infty_{\text{Mod}_B} \circ \partial I \circ \Sigma^\infty_{\text{CAlg}_{B//B}} \simeq \Omega^\infty_{\text{Mod}_B} \circ \Sigma^\infty_{\text{Mod}_B} \circ I \simeq I,
$$

but on the other hand this is equivalent to $I \circ \Omega^\infty_{\text{CAlg}_{B//B}} \circ \Sigma^\infty_{\text{CAlg}_{B//B}}$, which is the excisive approximation of $I$ by 2.(8). Therefore, $I$ would be excisive. Since $\text{Mod}_B$ is stable, this would mean that $I$ preserves pushouts; since $I$ is also reduced, then $I$ would be right exact. But this is typically false. For example, $I$ does not commute with coproducts: if we take $B = S$, the

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$^3$The “c” in “nuca” stands for commutative.
coproduct of $S[S^1]$ with itself in $\text{CAlg}_{S//S}$ is $S[S^1 \times S^1]$. Its augmentation ideal is $\Sigma^\infty (S^1 \times S^1) \simeq \Sigma^\infty S^1 \vee \Sigma S^1 \vee \Sigma^\infty S^2$, which is not equivalent to the coproduct of $I(S[S^1]) \simeq \Sigma^\infty S^1$ with itself.

If $f : A \to B$ is in $\text{CAlg}$, then $L_{B/A} \in \text{Sp}(\text{CAlg}_{B//B})$ by definition. Given Theorem 3.7, in this situation we redefine $L_{B/A}$ to mean the image of $A \xrightarrow{f} B \xrightarrow{id} B$ under the composition

$$CAlg_{A//B} \xrightarrow{B \wedge_A -} CAlg_{B//B} \xrightarrow{\Sigma^\infty} \text{Sp}(\text{CAlg}_{B//B}) \xrightarrow{\Omega^\infty \circ \partial I} \text{Mod}_B.$$  

(3.10)

Therefore, by 2.6, $L_{B/A}$ is equivalently the value of an excisive approximation to $I : CAlg_{B//B} \to \text{Mod}_B$ evaluated in $B \wedge_A B$. In symbols,

$$L_{B/A} \simeq (P_1I)(B \wedge_A B).$$

(3.11)

3.2. The cotangent complex as a colimit. Let $A$ be an $E_\infty$-ring spectrum. The general definition of a cotangent complex also applies to an $E_k$-$A$-algebra $B$. In [Lur17, 7.3.5] Lurie analyzes this particular case. He denotes by $L_{B/A}^{(k)}$ the resulting $E_k$-cotangent complex.

Forgetting structure, every $E_\infty$-$A$-algebra $B$ is an $E_1$-$A$-algebra for every $k \geq 0$. Lurie observes in [Lur17, 7.3.5.6] that since the $E_\infty$-operad is the colimit of the $E_k$-operads, these $E_k$-cotangent complexes recover the cotangent complex as follows:

$$L_{B/A} \simeq \colim( L_{B/A}^{(1)} \longrightarrow L_{B/A}^{(2)} \longrightarrow L_{B/A}^{(3)} \longrightarrow \cdots ).$$

These $E_k$-cotangent complexes admit a different expression which is sometimes computable, as we shall see in this section. That is what we shall use in Section 4 to compute the cotangent complex of Thom $E_\infty$-algebras.

Let $B \in \text{CAlg}$ and $C \in \text{CAlg}_{B//B}$. Since $I$ is left exact (it is a right adjoint) and commutes with sequential colimits (Remark 3.6), then by 2.3,

$$P_1I(C) \simeq \colim( I(C) \xrightarrow{e_0} \Omega I(\Sigma C) \xrightarrow{\Omega e_1} \Omega^2 I(\Sigma^2 C) \xrightarrow{\Omega^2 e_2} \cdots )$$

(3.12)

Here $e_0 : I(\Sigma^n C) \to \Omega(\Sigma^{n+1} C)$ is the natural map obtained as in [Lur17, 1.4.2.12].

Let $f : A \to B$ be a morphism in $\text{CAlg}$. Applying (3.12) to $C = B \wedge_A B$ we get a quite explicit colimit formula for $L_{B/A}$. But we can be more explicit: we are going to recast $I(\Sigma^n (B \wedge_A B))$ in other terms.

Any presentable $\infty$-category $\mathcal{C}$ is tensored over spaces: there is a functor $- \otimes - : S \times \mathcal{C} \to \mathcal{C}$ which preserves colimits separately in each variable. If $X \in S$ and $c \in \mathcal{C}$, then

$$X \otimes c \simeq \colim(X \xrightarrow{\{c\}} \mathcal{C})$$

where $\{c\}$ denotes the constant functor with value $c$. If $\mathcal{C}$ is moreover pointed, then it is tensored over pointed spaces: there is a functor $- \otimes - : S_+ \times \mathcal{C} \to \mathcal{C}$ which preserves colimits separately in each variable. If $(X, x_0) \in S_+$ and $c \in \mathcal{C}$, then

$$X \otimes c \simeq \text{cofib}_c(c \simeq c \otimes c x_0 \otimes \id \longrightarrow X \otimes c).$$

(3.13)

See [RSV19, Section 2] for more details.

Remark 3.14. The suspension $\Sigma A$ of an object $A$ in a pointed presentable $\infty$-category $\mathcal{C}$ can be expressed as $S^1 \otimes A$. Indeed, write $S^1 = \colim(\ast \leftarrow S^0 \to \ast)$ and apply the colimit-preserving functor $- \otimes A$. By induction, $\Sigma^n A \simeq S^n \otimes A$ for all $n \geq 0$.

Notation 3.15. Let $\otimes_B$ denote the tensor of $\text{CAlg}_{B//B}$ over $S_+$. Let $\otimes_A$ denote the tensor of $\text{CAlg}_A$ over $S$. 
Remark 3.16. If \( f : A \to B \) in \( \text{CAlg} \) and \((X,x_0) \in S_\ast\), then \( X \otimes_A B \in \text{CAlg}_{B//B'} \) with unit and augmentation given by

\[
B \simeq \ast \otimes_A B \xrightarrow{x_0 \otimes \text{id}} X \otimes_A B \xrightarrow{\ast \otimes \text{id}} \ast \otimes_A B \simeq B.
\]

In the following lemma, we shall prove a couple of results about this construction.

Lemma 3.17. (1) There is a functor

\[
- \otimes_A : S_\ast \times \text{CAlg}_{B//B} \to \text{CAlg}_{B//B}
\]

which preserves colimits separately in each variable and extends \(- \otimes_A : S \times \text{CAlg}_A \to \text{CAlg}_A\), in the sense that it makes the following diagram commute:

\[
\begin{array}{ccc}
S_\ast \times \text{CAlg}_{B//B} & \xrightarrow{- \otimes_A} & \text{CAlg}_{B//B} \\
\downarrow & & \downarrow \\
S \times \text{CAlg}_A & \xrightarrow{- \otimes_A} & \text{CAlg}_A
\end{array}
\]

Here the two vertical maps are forgetful functors.

(2) The resulting functor \(- \otimes_A B : S_\ast \to \text{CAlg}_{B//B}\) is equivalent to \(- \otimes_B (B \wedge_A B)\), where \(B \wedge_A B\) denotes the object \(B \to B \wedge_A B \xrightarrow{\mu} B\).

Proof. (1) Since the tensor is a colimit and colimits in overcategories are created in the original \(\infty\)-category \([Lur09, 1.2.13.8]\), the functor \(- \otimes_A : S \times \text{CAlg}_A \to \text{CAlg}_A\) begets a functor

\[
(\text{CAlg}_A)_{/B} \simeq (\text{CAlg}_{A_f})_{/f} \simeq \text{CAlg}_{A//B}
\]

which extends the original one and preserves colimits separately in each variable. Now, note as in Remark 3.2 that

\[
(\text{CAlg}_{A_f})_{/B} \simeq (\text{CAlg}_{A_f})_{/f} \simeq \text{CAlg}_{A//B}.
\]

Therefore, (3.18) induces a functor on \(\infty\)-categories of arrows which preserves colimits separately in each variable,

\[
- \otimes_A : S^{A^1} \times (\text{CAlg}_{A//B})^{A^1} \to (\text{CAlg}_{A//B})^{A^1}.
\]

It is defined on objects as follows:

\[
(\text{Y} \xrightarrow{g} \text{X}) \otimes_A \text{D} = \text{Y} \otimes_A \text{X} \otimes_A \text{C}
\]

where the equivalences \(A \simeq \varnothing \otimes_A A\) and \(B \simeq \ast \otimes_A B\) are understood.

Restricting to the full subcategories where the domain of the arrow is equivalent to the terminal object gives us a functor which preserves colimits separately in each variable,

\[
- \otimes_A : S_\ast \times (\text{CAlg}_{A//B})_{\ast} \to (\text{CAlg}_{A//B})_{\ast}.
\]

\footnote{The first arrow can be taken to be the unit in the first or in the second factor: up to composing with the symmetry \(B \wedge_A B \simeq B \wedge_A B\), the two choices are equivalent.}
and acts on objects as follows:

\[
\begin{align*}
& \xymatrix{
A & \ar[l]_f & \mathcal{X} 
\ar[r]^u & \mathcal{Y} 
\ar[l]_h & B 
\ar[d]_{\text{id}} & \ar[u]^c & \ar[l]_{\text{id}}^0 & \ar[d]^\text{id} & \mathcal{X} 
\ar[r]^u & \mathcal{Y} 
\ar[l]_h & B
}
\end{align*}
\]

Now recall from Remark 3.2 that \((\text{CAlg}_{A/B})^* \simeq \text{CAlg}_{B/B}^*\). This equivalence just disregards the redundant upper part of the squares, and gives us the functor we were looking for.

(2) To prove \(- \otimes_A B\) and \(- \otimes_B (B \wedge_A B)\) are equivalent, it suffices to see that they send \(S^0\) to equivalent objects. Indeed, they are colimit-preserving functors from \(S_*\) to a pointed presentable \(\infty\)-category, but \(S_*\) is freely generated under colimits by \(S^0\) in pointed presentable \(\infty\)-categories [RSV19, 2.29].

Now, indeed both functors send \(S^0\) to \(B \wedge_A B\), and the proof is finished. \(\square\)

**Lemma 3.19.** Let \(B \xrightarrow{u} C \xrightarrow{c} B\) be an object in \(\text{CAlg}_{B/B}^*\). Then \(I(C)\) is equivalent to the cofiber of \(u\) in \(\text{Mod}_B\), i.e.

\[
I(C) = \text{fib}_{\text{Mod}_B}(c : C \to B) \simeq \text{cofib}_{\text{Mod}_B}(u : B \to C) =: J(C).
\]

**Proof.** Consider the following commutative diagram in \(\text{Mod}_B\):

\[
\begin{array}{ccc}
0 & \xrightarrow{id} & 0 \\
\downarrow & & \downarrow \\
I(C) & \xrightarrow{c} & J(C) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\text{id}} & B.
\end{array}
\]

The bottom left square and the big rectangle formed by the two squares on the left are pullbacks, so the square on the top left is a pullback [Lur09, 4.4.2.1]. It is therefore a pushout, by stability of \(\text{Mod}_B\). The square on the top right is also a pushout, hence the big rectangle formed by the two squares on top is a pushout [Lur09, Dual of 4.4.2.1], proving the result. \(\square\)

We adopt the notation of [Kri93] or [Sch11, Page 164]:

**Notation 3.20.** Let \(X \in S_*\) and \(f : A \to B\) in \(\text{CAlg}\). By the previous lemma, there is an equivalence \(I(X \otimes_A B) \simeq J(X \otimes_A B)\) in \(\text{Mod}_B\): we denote the common value by \(X \tilde{\otimes}_A B\).

From the two lemmas above, we deduce:

**Corollary 3.21.** Let \(f : A \to B\) in \(\text{CAlg}\) and \(X \in S_*\). There is an equivalence of \(B\)-modules

\[
I(X \otimes_B (B \wedge_A B)) \simeq X \tilde{\otimes}_A B.
\]

Letting \(C = B \wedge_A B\) and using Remark 3.14, we can now recast (3.12) in a different form:

**Proposition 3.22.** Let \(f : A \to B\) in \(\text{CAlg}\). There is an equivalence of \(B\)-modules

\[
L_{B/A} \simeq \text{colim}_{\text{Mod}_B}(S^0 \tilde{\otimes}_A B \xrightarrow{\Omega(S^1 \tilde{\otimes}_A B)} \Omega^2(S^2 \tilde{\otimes}_A B) \xrightarrow{\Omega^3} \cdots)
\]

where \(\Omega\) denotes the loop functor in \(\text{Mod}_B\).
Remark 3.23. The previous proposition was known to the experts (it is mentioned e.g. in [Sch11, Page 164]), but we do not think a complete derivation had been spelled out in the literature before.

Finally, let us make the connection between the $\tilde{\otimes}$ construction and the $E_k$-cotangent complex mentioned above.

Remark 3.24. Let $B$ be an $E_\infty$-$A$-algebra. Forgetting structure, we may consider $B$ as an $E_k$-$A$-algebra, for all $k \geq 0$, and thus we may form its $E_k$-cotangent complex $L_B^{(k)}$ [Lur17, 7.3.5]. Lurie [Lur17, 7.3.5.1/3] proved that for each $k \geq 1$ there is a fiber sequence of $B$-modules

$$L_B^{(k)} \rightarrow \Omega^{k-1}(S^{k-1} \otimes_A B) \xrightarrow{\Omega^{k-1}(\ast \otimes \text{id})} \Omega^{k-1} B,$$

where $\ast : S^{k-1} \rightarrow *$ denotes the unique map. Since $\ast \otimes \text{id} : S^{k-1} \otimes_A B \rightarrow B$ is the augmentation of $S^{k-1} \otimes_A B$ and loops commute with pullbacks, this identifies $L_B^{(k)}$ with $\Omega^{k-1}(S^{k-1} \otimes_A B)$, so by Proposition 3.22 we obtain an equivalence of $B$-modules

$$L_B^{(k)} \simeq \text{colim}( L_B^{(1)} \rightarrow L_B^{(2)} \rightarrow L_B^{(3)} \rightarrow \cdots )$$

recovering [Lur17, 7.3.5.6].

3.3. The cotangent complex via indecomposables. The first published definition of the cotangent complex was established in the context of model categories using the indecomposables functor [Bas99]. The goal of this subsection is to prove that this definition of the cotangent complex is equivalent to the definition adopted in (3.10). We are not aware of a discussion of the approach using indecomposables in the $\infty$-categorical setting.

The content of this subsection will not be used in the sequel: the reader interested in Thom spectra should feel free to jump ahead to Section 4.

Definition 3.25. Let $B \in \text{CAlg}$. We denote by

$$Q : \text{NUCA}_B \rightarrow \text{Mod}_B$$

the indecomposables functor that takes $N$ to the cofiber in $\text{Mod}_B$ of the multiplication, i.e. to the pushout

$$\begin{array}{ccc}
N \wedge_B N & \xrightarrow{\mu} & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & Q(N)
\end{array}$$

in $\text{Mod}_B$.

The functor $Q$ is left adjoint to the functor which takes a module $M$ and endows it with a zero multiplication map. More precisely:

Lemma 3.26. (1) The functor $Q$ is left adjoint to a functor $Z : \text{Mod}_B \rightarrow \text{NUCA}_B$ such that $UZ \simeq \text{id}_{\text{Mod}_B}$, and for each $M \in \text{Mod}_B$ the multiplication map $ZM \wedge_B ZM \rightarrow ZM$ is zero.

(2) $Q \circ F \simeq \text{id}_{\text{Mod}_B}$, where $F : \text{Mod}_B \rightarrow \text{NUCA}_B$ is the free functor.

(3) There exists a unique functor $Z : \text{Mod}_B \rightarrow \text{NUCA}_B$ such that $UZ \simeq \text{id}_{\text{Mod}_B}$, up to equivalence.

Here $U : \text{NUCA}_B \rightarrow \text{Mod}_B$ denotes the forgetful functor.
Proof. (1) We will use a criterion for adjointness from [GK17, 2.3]: $Q$ admits a right adjoint if and only if for every $M \in \text{Mod}_B$, the comma $\infty$-category $(Q \downarrow M)$, defined as the pullback

$$(Q \downarrow M) \longrightarrow \text{NUCA}_B$$

$$\downarrow\quad \downarrow$$

$$(\text{Mod}_B)/_M \longrightarrow \text{Mod}_B$$

has a terminal object. Fix $M \in \text{Mod}_B$. It suffices to see that there exists a $B$-nuca $ZM$ such that $M \simeq QZM$, for then $QZM \simeq M \overset{\text{id}}{\rightarrow} M$ is a terminal object of $(Q \downarrow M)$.

To see this, first consider the trivial square-zero extension $B \oplus M$ [Lur17, 7.3.4.16]. This is an augmented $E_\infty$-$B$-algebra such that the multiplication map of $I_0(B \oplus M)$ is zero. Define $ZM$ to be $I_0(B \oplus M)$. By definition, $QZM$ is the cofiber in $\text{Mod}_B$ of the multiplication map $ZM \wedge_B ZM \rightarrow ZM$ which is zero, so $QZM \simeq M$ as required.

(2) $Q \circ F$ is the left adjoint to $U \circ Z \simeq \text{id}_{\text{Mod}_B}$, so it is equivalent to the identity.

(3) The existence of such a $Z$ has just been proven. Now suppose we have a functor $Z' : \text{Mod}_B \rightarrow \text{NUCA}_B$ such that $UZ' \simeq \text{id}_{\text{Mod}_B}$. Let $M, M' \in \text{Mod}_B$. We have natural equivalences of spaces

$$\text{Map}_{\text{NUCA}_B}(FM', Z'M) \simeq \text{Map}_{\text{Mod}_B}(M', UZ'M) \simeq \text{Map}_{\text{Mod}_B}(QFM', M).$$

Let $N \in \text{NUCA}_B$. By [Lur17, 4.7.3.14/15], there exists a simplicial object $N_\bullet$ in $\text{NUCA}_B$ which depends functorially on $N$ and satisfies that $\text{colim}(N_\bullet) \simeq N$, $\text{colim}(N_\bullet)$ is given by free nucas, and $\text{colim}(QN_\bullet) \simeq QN$ [Lur17, 4.7.2.4]. Using the above, one obtains a natural equivalence of spaces

$$\text{Map}_{\text{NUCA}_B}(N, Z'M) \simeq \text{Map}_{\text{Mod}_B}(QN, M).$$

By [Cis19, 6.1.23], this proves that $Z'$ is a right adjoint to $Q$, but then by uniqueness of adjoints ([Lur09, 5.2.6.2] or [Cis19, 6.1.9]), we deduce that $Z' \simeq Z$. \hfill $\square$

Remark 3.27. Let $K$ denote the composition

$$\text{Mod}_B \xrightarrow{\sim} \text{Sp}(\text{CAlg}_{B//B}) \xrightarrow{\Omega^\infty} \text{CAlg}_{A//B};$$

here, the first arrow is an inverse to $\Omega^\infty \circ \partial I : \text{Sp}(\text{CAlg}_{B//B}) \xrightarrow{\sim} \text{Mod}_B$. Note that $K(M)$ is the trivial square-zero extension $B \oplus M$ [Lur17, 7.3.4.16]. By definition of $K$ and $L_{B//A}$, we get the following equivalence for every $B$-module $M$,

$$\text{Map}_{\text{Mod}_B}(L_{B//A}, M) \simeq \text{Map}_{\text{CAlg}_{A//B}}(B, B \oplus M).$$

These spaces can be interpreted as the spaces of $A$-linear derivations from $B$ into $M$ [Lur17, 7.3].

Now, observe that $K$ is equivalent to the composition

$$\text{Mod}_B \xrightarrow{Z} \text{NUCA}_B \xrightarrow{B-\text{nuca}} \text{CAlg}_{B//B} \rightarrow \text{CAlg}_{A//B},$$

the last functor being a forgetful functor. Indeed, since $B \vee -$ is an equivalence with inverse given by $I_0$, we may equivalently verify that $I_0 \circ K \simeq Z$. But this follows from Part (2) of the previous lemma.

In the following theorem, we prove how to get the cotangent complex via indecomposables. A model categorical precedent of the result can be found in [BM05, Theorem 4].
Theorem 3.28. Let $B \in \mathbf{CAlg}$. The following diagram commutes:

\[
\begin{array}{ccc}
\mathbf{CAlg}_{/B} & \xrightarrow{\Sigma^\infty} & \mathbf{Sp}(\mathbf{CAlg}_{/B}) \\
\downarrow^{i_0} & & \downarrow^{\Omega^\infty \partial I} \\
\mathbf{NUCA}_B & \xrightarrow{Q} & \mathbf{Mod}_B.
\end{array}
\]

Proof. Recall from 2.(6) that $P_1 I$, the excisive approximation to $I$, is equivalent to $\Omega^\infty \circ \partial I \circ \Sigma^\infty$. Thus, we have to prove that $Q \circ i_0$ is equivalent to $P_1 I$. Recall that $I \simeq U \circ i_0$. By [Lur17, 6.1.1.30], we have $P_1 I \simeq P_1 U \circ i_0$. We will now prove that $P_1 U \simeq Q$, finishing the proof.

By 2.(6), $P_1 U$ is equivalent to $\Omega^\infty_{\mathbf{Mod}_B} \circ \partial U \circ \Sigma^\infty_{\mathbf{NUCA}_B}$. Note that $Q$ is excisive, since it preserves pushouts and $\mathbf{Mod}_B$ is stable, so $Q \simeq P_1 Q$. Therefore, to prove that $P_1 U \simeq Q$ it suffices to prove that $\partial U \simeq \partial Q$, by 2.(6) once more.

We have the free–forgetful adjunction $\mathbf{Mod}_B \xrightarrow{F} \mathbf{NUCA}_B$, and taking derivatives gives an adjunction $\mathbf{Sp}(\mathbf{Mod}_B) \xrightarrow{\partial F/\partial U} \mathbf{Sp}(\mathbf{NUCA}_B)$ by [Lur17, 6.2.2.15]. By Lemma 3.26.(2), $Q \circ F \simeq \text{id}_{\mathbf{Mod}_B}$, so by [Lur17, 6.2.1.4/24] we get $\partial Q \circ \partial F \simeq \text{id}_{\mathbf{Sp}(\mathbf{Mod}_B)}$. To prove that $\partial Q \simeq \partial U$, it suffices to prove that $\partial U$ is an equivalence. Indeed, if it is, then $\partial F \circ \partial U \simeq \text{id}_{\mathbf{Sp}(\mathbf{NUCA}_B)}$, so $\partial Q \simeq \partial Q \circ \partial F \circ \partial U \simeq \partial U$.

To prove that $\partial U$ is an equivalence, we proceed similarly as in [Lur17, Proof of 7.3.4.7]: namely, since $U$ is a monadic functor, then by [Lur17, 6.2.2.17] it suffices to prove that the unit $\text{id}_{\mathbf{Mod}_B} \Rightarrow U \circ F$ induces an equivalence of derivatives. The proof of this is very similar to that of [Lur17, 7.3.4.10], only simpler.

Note that $U \circ F : \mathbf{Mod}_B \to \mathbf{Mod}_B$ is given on objects by $M \mapsto M \vee \bigvee_{n \geq 1} (M^{\wedge g^n})_{\Sigma_n}$ and the unit of the $(F, U)$ adjunction includes $M$ into the separate $M$ factor, so by [Lur17, 7.3.4.8] it suffices to see that the derivative of the functor $\text{Sym}^n : \mathbf{Mod}_B \to \mathbf{Mod}_B, M \mapsto (M^{\wedge g^n})_{\Sigma_n}$ is nullhomotopic. Now note that $\text{Sym}^n$ is the colimit of the functor $B \Sigma_{g^n} \to \text{Fun}(\mathbf{Mod}_B, \mathbf{Mod}_B)$ which considers the functor $(-)^{\wedge g^n} \circ \text{diag} : M \mapsto M^{\wedge g^n}$ together with its $\Sigma_n$-action. By [Lur17, 7.3.4.8], it suffices to see that the derivative of $(-)^{\wedge g^n} \circ \text{diag} : \mathbf{Mod}_B \to \mathbf{Mod}_B$ is nullhomotopic for $n \geq 2$. This follows from [Lur17, 6.1.3.12].

Remark 3.29. The key aspect of the previous proof is the fact that $\partial (U \circ F) \simeq \text{id}_{\mathbf{Sp}(\mathbf{Mod}_B)}$, which is proven in the last paragraph. Notice this is equivalent to

$$P_1 (U \circ F) \simeq \text{id}_{\mathbf{Mod}_B},$$

as $P_1 (U \circ F) \simeq \Sigma^\infty \circ \partial (U \circ F) \circ \Omega^\infty$ by 2.(6), and $\Sigma^\infty, \Omega^\infty$ are equivalences since $\mathbf{Mod}_B$ is stable. Using the analogy of Goodwillie calculus with classical calculus, we can gain some intuition for this result. Under this analogy, functors correspond to smooth functions of the real line, so the functor $U \circ F$ which maps $M \in \mathbf{Mod}_B$ to $\bigvee_{n \geq 1} (M^{\wedge g^n})_{\Sigma_n}$ corresponds to the power series $f(x) = \sum_{n=1}^{\infty} x^n$. The linear approximation at $0$ of this function is the identity map $x \mapsto x$. Continuing with the analogy, linear approximations of functions correspond to 1-excisive approximations of functors, which provides some intuition for the equivalence $P_1 (U \circ F) \simeq \text{id}_{\mathbf{Mod}_B}$.

From Theorem 3.28 we immediately deduce:

Corollary 3.30. Let $f : A \to B$ be a map of $E_\infty$-ring spectra. There is an equivalence of $B$-modules

$$L_{B/ A} \simeq Q i_0 (B \wedge_A B).$$

This is analogous to what Basterra [Bas99] adopted as a definition for $L_{B/ A}$ in a model-categorical setting. That was the first published definition. The approach from (3.10) had been
used in the preprint [Kri93], albeit formulated in a different language. Only in [BM05] were the two approaches first proven to be equivalent.

4. THE COTANGENT COMPLEX OF THOM SPECTRA

In this section, we will prove the main result, Theorem 4.3, giving an expression for the cotangent complex of Thom $E_\infty$-algebras.

An $E_\infty$-monoid is a commutative algebra object in $\mathcal{S}$ in the sense of [Lur17, 2.1.3.1], or, equivalently, a commutative monoid object in $\mathcal{S}$ in the sense of [Lur17, 2.4.2.1] (i.e. a special $\Gamma$-space). They form an $\infty$-category $\text{Mon}_{E_\infty}(\mathcal{S})$. If an $E_\infty$-monoid $M$ is grouplike, i.e. if the monoid $\pi_0(M)$ is a group, we say $M$ is an $E_\infty$-group. These form an $\infty$-category denoted by $\text{Grp}_{E_\infty}(\mathcal{S})$.

Let $R$ be an $E_\infty$-ring spectrum. An $R$-module $M$ is invertible if there exists an $R$-module $N$ such that $M \wedge_R N \simeq R$. We let $\text{Pic}(R)$ be the Picard space of $R$: this is the core (i.e. the maximal subspace) of the full subcategory of $\text{Mod}_R$ on the invertible $R$-modules.

If $Z$ is a space and $f : Z \to \text{Pic}(R)$ is a map of spaces, the Thom $R$-module of $f$ is defined as

$$Mf := \text{colim}( Z \xrightarrow{f} \text{Pic}(R) \xrightarrow{c} \text{Mod}_R ).$$

This defines a functor $M : S_{/\text{Pic}(R)} \to \text{Mod}_R$. As noted in [ABG18, 7.7], [ACB19, Section 3], Pic($R$) is an $E_\infty$-group. If $G$ is an $E_\infty$-monoid and $f$ is an $E_\infty$-map, then $Mf$ becomes an $E_\infty$-$R$-algebra [ABG18, 8.1], [ACB19, 3.2]. When $f = \{R\}$ is the constant map at $R \in \text{Pic}(R)$, then $Mf \simeq R \wedge_S [G]$ as $E_\infty$-$R$-algebras.

4.1. The main result. We will need the following proposition.

**Proposition 4.1.** Consider $\Sigma^\infty \Omega^\infty : \text{Sp} \to \text{Sp}$. The counit natural transformation

$$\Sigma^\infty \Omega^\infty \Rightarrow \text{id}_{\text{Sp}}$$

exhibits $\text{id}_{\text{Sp}}$ as the excisive approximation to $\Sigma^\infty \Omega^\infty$.

In particular, for a spectrum $X$, there is an equivalence

$$(4.2) \quad X \simeq \text{colim}_{\text{Sp}}(\Sigma^\infty \Omega^\infty X \to \Omega^\infty \Sigma^\infty X \to \Omega^2 \Sigma^\infty \Omega^\infty \Sigma^2 X \to \cdots)$$

natural in $X$.

**Proof.** We will first establish the natural equivalence (4.2). We will then observe that this equivalence is obtained from the counit $\Sigma^\infty \Omega^\infty X \to X$, in such a way that the main assertion will have been proven, by 2.(3).

Let $X$ be a spectrum. First, we prove there is a natural equivalence of functors

$$\text{Map}_{\text{Sp}}(X, -) \xrightarrow{\sim} \lim_{n \in \mathbb{N}} \text{Map}_{S_*}(\Omega^\infty \Sigma^n X, \Omega^\infty \Sigma^n -)$$

where the arrows in the sequential limit are loop functors. We do this carefully, taking care of naturality: indeed, it is easier to see that the two functors are objectwise equivalent using that $\text{Sp} \simeq \lim_{\to} \Omega_S \xrightarrow{\Omega_S} \Omega_S$ [Lur17, 1.4.2.24], writing a mapping space as a pullback and commuting pullbacks with limits.

By [Lur09, 2.2.1.2], it suffices to establish an equivalence of the corresponding left fibrations. The left hand side corresponds to the left fibration $\text{Sp}_{X/} \to \text{Sp}$. For the right hand side, first observe that for $n \in \mathbb{N}$, the functor $\text{Map}_{S_*}(\Omega^\infty \Sigma^n X, -)$ corresponds to the left fibration $(S_n)_{\Omega^\infty \Sigma^n X/} \to S_*$. Pulling back this left fibration along the functor $\Omega^\infty \Sigma^n : \text{Sp} \to S_*$,
We want to prove this map is an equivalence. Note we can recover the right hand side directly which by the Yoneda lemma is induced by a map \( \operatorname{colim} \). This gives us the desired result.

We prove that

\[
\pi_2 \circ I : \text{Sp}_X/ \to \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/.
\]

We want to prove this map is an equivalence. Note we can recover the right hand side directly as the following pullback:

\[
\begin{array}{ccc}
\text{Sp} \times \text{lim}_N S_* & \longrightarrow & (S_*)_{\Omega^\infty \Sigma^n X}/ \\
\downarrow & & \downarrow \\
\text{Sp} & \longrightarrow & S_*
\end{array}
\]

gives us the left fibration that classifies the functor \( \operatorname{Map}_{S_*}(\Omega^\infty \Sigma^n X, \Omega^\infty \Sigma^n -) \). Taking limits, it follows that the functor \( \lim_n \operatorname{Map}_{S_*}(\Omega^\infty \Sigma^n X, \Omega^\infty \Sigma^n -) \) corresponds to the left fibration \( \lim_n (\text{Sp} \times S_*)_{\Omega^\infty \Sigma^n X}/ \simeq \text{Sp} \times \text{lim}_NS_* \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/. \)

By the Yoneda lemma, a map of left fibrations \( \text{Sp}_X/ \to \text{Sp} \times \text{lim}_NS_* \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/ \) is uniquely determined by a choice of object in the fiber of \( \text{Sp} \times \text{lim}_NS_* \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/ \to \text{Sp} \) over \( X \), which is given by the space \( \{X\} \times \lim_n \operatorname{Map}_{S_*}(\Omega^\infty \Sigma^n X, \Omega^\infty \Sigma^n X). \) Thus the object \( (X, (\operatorname{id}_{\Omega^\infty \Sigma^n X})_n \in \mathbb{N}) \) in the fiber induces a map of left fibrations over \( \text{Sp} \)

\[ I : \text{Sp}_X/ \to \text{Sp} \times \text{lim}_NS_* \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/. \]

The bottom equivalence is \( \text{Sp} \simeq \text{lim}(\cdots \Omega \rightarrow S_* \rightarrow S_*) = \text{lim}_NS_* \), from [Lur17, 1.4.2.24]. This implies that the top horizontal map is an equivalence as well [Lur09, 3.3.1.3]. Thus, in order to prove that \( I \) is an equivalence, by 2-out-of-3 it suffices to show that

\[ \pi_2 \circ I : \text{Sp}_X/ \to \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/ \]

is an equivalence.

First, we will construct an equivalence \( \text{Sp}_X/ \xrightarrow{\simeq} \lim_n (S_*)_{\Omega^\infty \Sigma^n X}/. \) Then we will observe it is indeed \( \pi_2 \circ I \). We have

\[
\text{Sp}_X/ \simeq \text{Sp}A^1 \times \text{Sp}A^0 \xrightarrow{\simeq} (\text{lim}_N S_*)^{A^1} \times \text{lim}_NS_* A^0.
\]

Using that limits commute with pullbacks and exponentials \((-)^{A^1},\)

\[
\text{lim}_N (S_*)^{A^1} \times (S_* A^0) \simeq \text{lim}_n (S_*)_{\Omega^\infty \Sigma^n X}/
\]

gives us an equivalence. Notice this equivalence takes the object \( \operatorname{id}_X \) in \( \text{Sp}_X/ \) to \( (\text{id}_{\Omega^\infty \Sigma^n X})_n \in \mathbb{N} \) and thus, by the Yoneda lemma, is equivalent to \( \pi_2 \circ I \), as they both take \( \operatorname{id}_X \) to the same object. This gives us the desired result.

Next, we have a natural equivalence

\[
\text{lim}_{n} \operatorname{Map}_{S_*}(\Omega^\infty \Sigma^n X, \Omega^\infty \Sigma^n -) \xrightarrow{\simeq} \text{lim}_{n} \operatorname{Map}_{\text{Sp}}(\Omega^\infty \Sigma^n \Omega^\infty \Sigma^n X, -)
\]

induced by the adjunction \( (\Sigma^n, \Omega^n) \). Finally, we have a natural equivalence

\[
\text{lim}_{n} \operatorname{Map}_{\text{Sp}}(\Omega^\infty \Sigma^n \Omega^\infty \Sigma^n X, -) \xrightarrow{\simeq} \operatorname{Map}_{\text{Sp}}(\text{colim}_{n} \Omega^\infty \Omega^\infty \Omega^\infty \Sigma^n X, -).
\]

Combining all three gives us a natural equivalence

\[
\operatorname{Map}_{\text{Sp}}(X, -) \xrightarrow{\simeq} \operatorname{Map}_{\text{Sp}}(\text{colim}_{n} \Omega^\infty \Sigma^n \Omega^\infty \Sigma^n X, -)
\]

which by the Yoneda lemma is induced by a map \( \text{colim}_{n} \Omega^\infty \Sigma^n \Omega^\infty \Sigma^n X \rightarrow X \) and is explicitly given by the image of the identity map \( \operatorname{id}_X \) under this natural equivalence. By inspection, the map \( \text{colim}_{n} \Omega^\infty \Omega^\infty \Omega^\infty \Sigma^n X \rightarrow X \) can be characterized as the unique map that comes from the
cocone given by \( \Omega^n \Sigma^n \Omega^\infty \Sigma^n X \to \Omega^n \Sigma^n X \xrightarrow{\sim} X \) where the first arrow is induced by the counit of the \((\Sigma^\infty, \Omega^\infty)\) adjunction. \(\square\)

Let \( B^\infty : \text{Grp}_{E_2}(S) \to \text{Sp}^\infty \) denote the standard equivalence between the \(\infty\)-categories of \(E_\infty\)-groups and of connective spectra [Lur17, 5.2.6.26].

**Theorem 4.3.** Let \( R \) be an \( E_\infty \)-ring spectrum. Let \( f : G \to \text{Pic}(R) \) be a map of \( E_\infty \)-groups. There is an equivalence of \( Mf \)-modules

\[
L_{Mf/R} \simeq Mf \wedge B^\infty G.
\]

**Proof.** By [RSV19, 4.11], \( S^n \otimes_R Mf \simeq Mf \wedge S[B^n G] \) in \( \text{CAlg}_{Mf//Mf} \). Now, note that

\[
(4.4)
\]

as \( Mf \)-modules. Indeed, since \( S^0 \simeq * \xrightarrow{(x_0)_{+}} (B^n G)_{+} \longrightarrow B^n G \) is a cofiber sequence in \( S_x \) where \( x_0 \) denotes the basepoint of \( B^n G \), then applying \( Mf \wedge \Sigma^\infty(-) \) we get a cofiber sequence in \( \text{Mod}_{Mf} \)

\[
Mf \simeq Mf \wedge \Sigma^\infty(+) \longrightarrow Mf \wedge \Sigma^\infty B^n G \longrightarrow Mf \wedge \Sigma^\infty B^n G,
\]

whereas by the definition of \( J \) in Lemma 3.19 we get the equivalence (4.4).

By Proposition 3.22 we get equivalences of \( Mf \)-modules

\[
L_{Mf/R} \simeq \text{colim}_{\text{Mod}_{Mf}}( Mf \wedge \Sigma^\infty G \longrightarrow \Omega(Mf \wedge \Sigma^\infty BG) \longrightarrow \Omega^2(Mf \wedge \Sigma^\infty B^2G) \longrightarrow \cdots)
\]

(4.5) \( \simeq Mf \wedge \text{colim}_{\text{Sp}}( \Sigma^\infty G \longrightarrow \Omega\Sigma^\infty BG \longrightarrow \Omega^2\Sigma^\infty B^2G \longrightarrow \cdots) \)

where in the second line we have used the stability of \( \text{Sp} \) and \( \text{Mod}_{Mf} \) together with the fact that \( Mf \wedge - : \text{Sp} \to \text{Mod}_{Mf} \) commutes with colimits; note that \( \Omega \) now denotes the loop functor in \( \text{Sp} \).

We will now prove that \( B^\infty G \) is equivalent to the colimit in (4.5), which we can rewrite more formally as

\[
(4.6)
\]

where \( U_n : \text{Grp}_{E_n}(S) \to \text{Grp}_{E_{n+1}}(S) \), \( E_0 \)-groups are pointed spaces, and \( i_n : S_x^{\geq n} \to S_x \) is the inclusion functor of \((n-1)\)-connected pointed spaces. Now note that \( B^\infty \) is constructed as the limit of the \( B^n \) as follows [Lur17, 5.2.6.26]

\[
\begin{array}{ccc}
\text{Sp}^\infty & \simeq & \text{lim}(\cdots \xrightarrow{\Omega} S_x^{\geq n+1} \xrightarrow{\Omega} S_x^{\geq n} \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} S_x^{\geq 1} \xrightarrow{\Omega} S_x) \\
\text{Grp}_{E_n}(S) & \simeq & \text{lim}(\cdots \xrightarrow{\text{Grp}_{E_{n+1}}(S)} \xrightarrow{\text{Grp}_{E_n}(S)} \cdots \xrightarrow{\text{Grp}_{E_1}(S)} S_x),
\end{array}
\]

so in particular we have a commutative diagram

\[
\begin{array}{ccc}
\text{Sp}^\infty & \xrightarrow{B^\infty} & S_x^{\geq n} \\
\text{Grp}_{E_n}(S) & \xrightarrow{U_n} & \text{Grp}_{E_n}(S)
\end{array}
\]

where the top horizontal functor is the projection to the corresponding term of the limit. After composing it with \( i_n \) this functor becomes \( \Omega^{\infty-n} = \Omega^{\infty-n} : \text{Sp}^\infty \to S_x \), the \( n \)-th space functor. Therefore, (4.6) is equivalent to

\[
\text{colim}_{\text{Sp}}(\Sigma^\infty \Omega^\infty B^\infty G \longrightarrow \Omega^\infty \Omega^\infty \Sigma^\infty B^\infty G \longrightarrow \Omega^2 \Sigma^\infty \Omega^\infty \Sigma^2 B^\infty G \longrightarrow \cdots).
\]

This is equivalent to \( B^\infty G \) by Proposition 4.1. \(\square\)
Remark 4.7. A model-categorical version of Theorem 4.3 first appeared in [BM05, Corollary, Page 907]. Their result, however, only applies to Thom spectra of maps to $BGL_1(S)$, whereas ours applies to maps to $Pic(R)$ where $R$ is any $E_\infty$-ring spectrum. Already considering maps into $Pic(S)$ leads to interesting, non-connective examples, as we shall now see.

Example 4.8. Let $MUP$ denote the complex cobordism spectrum, which is the Thom $E_\infty$-ring spectrum of the complex $J$-homomorphism $BU \to Pic(S)$. Then

$\text{L}_{MUP} \simeq MUP \wedge ku$

as $MUP$-modules, where $ku \simeq B^\infty(BU \times \mathbb{Z})$ is the connective complex $K$-theory spectrum.

4.2. The cotangent complex as a Thom spectrum. If $G$ is not only an $E_\infty$-group but also an $E_\infty$-ring space in the sense of [GGN15, 7.1], then $B^\infty G$ is a connective $E_\infty$-ring spectrum. Many interesting examples of $E_\infty$-ring spaces arise as group completions of $E_\infty$-rig spaces (similar to $E_\infty$-ring spectra but now the underlying additive structure does not necessarily admit inverses). For example, $\bigcup_{n \geq 0} \Sigma_n, \bigcup_{n \geq 0} BU(n)$ and $\bigcup_{n \geq 0} BGL_n(A)$ where $A$ is a commutative ring are examples of $E_\infty$-rig spaces: their corresponding connective $E_\infty$-ring spectra are $S$, $ku$ and the algebraic $K$-theory $K(A)$ respectively. See [GGN15, Sections 7/8] for more details.

Proposition 4.9. Let $R$ be an $E_\infty$-ring spectrum, let $G$ be an $E_\infty$-ring space and let $f : G \to Pic(R)$ be an $E_\infty$-map (with respect to the $E_\infty$-group structure of $G$). Then the $R$-module $L_{Mf/R}$ underlies a Thom $E_\infty$-$(R \wedge B^\infty G)$-algebra.

Proof. First, note that if $R \to T$ is a map of $E_\infty$-ring spectra, then extension of scalars induces an $E_\infty$-map $- \wedge_R T : Pic(R) \to Pic(T)$ making the following diagram commute:

$$
\begin{array}{ccc}
Pic(R) & \longrightarrow & \text{Mod}_R \\
\downarrow -\wedge_RT & & \downarrow -\wedge_ST \\
Pic(T) & \longrightarrow & \text{Mod}_T.
\end{array}
$$

Indeed, the functor $- \wedge_R T : \text{Mod}_R \to \text{Mod}_T$ takes invertible $R$-modules to invertible $T$-modules.

Now recall that, as an $R$-module, $Mf \simeq \text{colim}(G \xrightarrow{f} Pic(R) \to \text{Mod}_R)$. Let $T = R \wedge B^\infty G$. Since $- \wedge_R T : \text{Mod}_R \to \text{Mod}_T$ preserves colimits, we get equivalences of $T$-modules

$$
Mf \wedge B^\infty G \simeq Mf \wedge_R T
= \text{colim}(G \xrightarrow{f} Pic(R) \xrightarrow{-\wedge_ST} \text{Mod}_T)
$$

Since we proved in Theorem 4.3 that $Mf \wedge B^\infty G \simeq L_{Mf/R}$ and $(- \wedge_R T) \circ f$ is an $E_\infty$-map, this proves the result.

Remark 4.10. This echoes with the result that the factorization homology of a Thom $E_\infty$-algebra is a Thom spectrum [Kla18, 4.2], or with the related result that the tensor of a Thom $E_\infty$-algebra with a space is again a Thom $E_\infty$-algebra [RSV19, 4.10].
Example 4.11. Continuing the example of MUP from Example 4.8, if we take \( f \) to be the complex \( J \)-homomorphism \( BU \times \mathbb{Z} \to \text{Pic}(S) \), then \( L_{\text{MUP}} \simeq MUP \wedge ku \) underlies a Thom \( E_\infty \)-\( ku \)-algebra, even if \( ku \) is not the Thom spectrum of any \( E_3 \)-map from an \( E_3 \)-group [AHL09].

In Proposition 4.9, we proved that \( L_{MF/R} \) is the Thom module of an \( E_\infty \)-map provided \( G \) is an \( E_\infty \)-ring space. The hypothesis was not superfluous: let us now give an example of a cotangent complex of a Thom \( E_\infty \)-ring spectrum which cannot be the Thom module of an \( E_\infty \)-map, simply because it would then be an \( E_\infty \)-ring spectrum and this cannot happen in this example:

Example 4.12. Let \( G \) be an abelian group of infinite order such that every element has finite order. For example, one could take \( \mathbb{Q}/\mathbb{Z} \) or an infinite direct sum of cyclic groups of arbitrarily large order.

Let \( f : G \to \text{Pic}(S) \) be the constant map at \( S \). Let us prove that \( L_{MF} \) is not the underlying spectrum of an \( E_\infty \)-ring spectrum. Such a structure would induce a ring structure on \( \pi_0(L_{MF}) \), so it suffices to prove that the latter has no ring structure extending its abelian group structure.

Before we prove this, let us use Theorem 4.3 to compute \( L_{MF} \). We have equivalences of spectra

\[
L_{MF} \simeq MF \wedge B^\infty G \simeq \Sigma^\infty_+ G \wedge HG \simeq \bigvee_G S \wedge HG \simeq \bigvee_G HG
\]

where we used the fact that \( B^\infty G \simeq HG \) is the Eilenberg–Mac Lane spectrum of \( G \), and that since \( G \) is a discrete group then \( \Sigma^\infty_+ G \) is a wedge of sphere spectra. In particular, \( \pi_0(L_{MF}) = \bigoplus_G G \).

Now, note that \( G' = \bigoplus_G G \) is again an abelian group of infinite order such that every element has finite order. This implies that it is not the underlying abelian group of a ring. If it were, the multiplicative unit would have finite order, and this number would bound the order of all the other elements, contradicting that \( G' \) has infinite order.

Remark 4.13. As we just observed, in general \( L_{MF/R} \) is not the Thom module of an \( E_\infty \)-map. It is, however, the colimit of iterated loops of Thom modules of \( E_\infty \)-maps by (4.5), since

\[
MF \wedge \Sigma^\infty_+ B^n G \simeq M(G \times B^n G \xrightarrow{\pi_1} G \xrightarrow{f} \text{Pic}(R))
\]

and so

\[
L_{MF/R} \simeq \text{colim}_{\text{Mod}_{MF}} (\Omega^n M(G \times B^n G \xrightarrow{\pi_1} G \xrightarrow{f} \text{Pic}(R))).
\]

5. Étale extensions

We will now extend the results of the previous section to compute cotangent complexes of étale extensions of Thom \( E_\infty \)-algebras.

Following [Lur17, 7.5.0.1/2/4], a map of (ordinary) commutative rings \( A \to B \) is étale if \( B \) is finitely presented as an \( A \)-algebra, \( B \) is flat as an \( A \)-module, and there exists an idempotent element \( e \in B \otimes_A B \) such that the multiplication map \( B \otimes_A B \to B \) induces an isomorphism \( (B \otimes_A B)[e^{-1}] \cong B \). If \( A \to B \) is a map of \( E_\infty \)-ring spectra, it is étale if \( \pi_0(A) \to \pi_0(B) \) is étale and \( B \) is flat as an \( A \)-module, i.e. the natural map

\[
\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)
\]

is an isomorphism.

Proposition 5.1. If \( A \to B \) is an étale map of \( E_\infty \)-ring spectra, then \( L_{B/A} \) vanishes. Therefore, if \( R \) is an \( E_\infty \)-ring spectrum and \( A \to B \) is a map of \( E_\infty \)-\( R \)-algebras which is étale, there is an equivalence of \( B \)-modules

\[
L_{B/R} \simeq B \wedge_A L_{A/R}.
\]
Proof. The first statement is [Lur17, 7.5.4.5]. Note that Lurie adds a connectivity hypothesis, but it is not used in the proof. The second statement now follows from the transitivity cofiber sequence [Lur17, 7.3.3.6]

\[ B \wedge_A L_{A/R} \to L_{B/R} \to L_{B/A}. \]

\[ \text{Corollary 5.2. Let } R \text{ be an } E_{\infty} \text{-ring spectrum. Let } f : G \to \text{Pic}(R) \text{ be a map of } E_{\infty} \text{-groups. Let } Mf \to B \text{ be a map of } E_{\infty} \text{-R-algebras which is étale. There is an equivalence of } B \text{-modules} \]

\[ L_{B/R} \simeq B \wedge B^\infty G. \]

Proof. By Proposition 5.1, \( L_{B/R} \simeq B \wedge Mf \) \( L_{Mf/R} \) as \( B \)-modules, and by Theorem 4.3 we have an equivalence of \( Mf \)-modules \( L_{Mf/R} \simeq Mf \wedge B^\infty G \). Putting these together finishes the proof. □

Example 5.3. Let \( R \) be an \( E_{\infty} \)-ring spectrum and \( x \in \pi_*(R) \). By [Lur17, 7.5.0.6/7], [Lur18a, 4.3.17], there is an étale map of \( E_{\infty} \)-ring spectra \( R \to R[x^{-1}] \) which universally inverts the homotopy element \( x \). We deduce from Proposition 5.1 that there is an equivalence of \( R[x^{-1}] \)-modules

\[ L_{R[x^{-1}]} \simeq (L_R)[x^{-1}]. \]

Example 5.4. Let \( KU \) denote the periodic complex topological \( K \)-theory \( E_{\infty} \)-ring spectrum. Snaith [Sna79], [Sna81] proved that \( KU \simeq S[K(Z, 2)][x^{-1}] \) as homotopy commutative ring spectra (i.e. commutative monoids in the homotopy category of spectra), where \( x \in \pi_2 S[K(Z, 2)] \) is induced by the fundamental class in \( K(Z, 2) \). See [Lur18a, 6.5.1] for one improvement of such an equivalence to an equivalence of \( E_{\infty} \)-ring spectra.

Since \( S[K(Z, 2)] \simeq M(K(Z, 2) \xrightarrow{[5]} \text{Pic}(S)) \), we can apply Corollary 5.2 to deduce that \( L_{KU} \simeq KU \wedge \Sigma^2 HZ \). Recall that the inclusion \( Z \to Q \) induces an equivalence \( KU \wedge HQ \simeq KU \wedge HQ \) [Swi75, 16.25]. Combining this result with Bott periodicity, we obtain:

\[ L_{KU} \simeq KU \wedge HQ, \]

the rationalization of \( KU \), a result first gotten in [Sto19, 8.4] in a model-categorical context.

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