Critical Sets in Bipartite Graphs

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G) (\Omega(G))$ we mean the set of all (maximum) independent sets of $G$, while $\alpha(G) = |S|$ for $S \in \Omega(G)$ [6].

The neighborhood of $A \subseteq V$ is denoted by $N(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$, where $N(v)$ is the neighborhood of the vertex $v$. The number $d(X) = |X| - |N(X)|$ is the difference of the set $X \subseteq V$, and $d_c(G) = \max\{d(I) : I \in \text{Ind}(G)\}$ is called the critical difference of $G$. A set $X$ is critical if $d(X) = d_c(G)$ [4].

For a graph $G$ we define $\ker(G) = \cap \{S : S$ is a critical independent set$\}$, while $\text{diadem}(G) = \cup \{S : S$ is a critical independent set$\}$.

For a bipartite graph $G = (A, B, E)$, with bipartition $\{A, B\}$, Ore [11] defined $\delta(X) = d(X)$ for every $X \subseteq A$, while $\delta_0(A) = \max\{\delta(X) : X \subseteq A\}$. Similarly is defined $\delta_0(B)$.

In this paper we prove that for every bipartite graph $G = (A, B, E)$ the following assertions hold:

- $d_*(G) = \delta_0(A) + \delta_0(B)$;
- $\ker(G) = \text{core}(G)$;
- $|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G)$.

Keywords: maximum independent set, maximum matching, critical set, critical difference, Kőnig-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean either the subgraph $G[V - W]$, if $W \subseteq V(G)$, or the partial subgraph $H = (V, E - W)$ of $G$, for
denote a bipartite graph having \( \{A, B\} \) as a bipartition and we assume that \( A \neq \emptyset \neq B \).

The neighborhood of a vertex \( v \in V \) is the set \( N(v) = \{w : w \in V \text{ and } vw \in E\} \), while the neighborhood of \( A \subseteq V \) is denoted by \( N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\} \), and \( N[A] = N(A) \cup A \).

A matching is a set of non-incident edges of \( G \); a matching of maximum cardinality \( \mu(G) \) is a maximum matching, and a perfect matching is a matching covering all the vertices of \( G \). If \( M \) is a matching, then \( M(v) \) means the mate of the vertex \( v \) by \( M \), and \( M(X) = \{M(v) : v \in X\} \) for \( X \subseteq V(G) \).

A set \( S \subseteq V(G) \) is independent (or stable) if no two vertices from \( S \) are adjacent, and by \( \text{Ind}(G) \) we denote the set of all independent sets of \( G \). An independent set of maximum size will be referred to as a maximum independent set of \( G \), and the independence number of \( G \) is \( \alpha(G) = \max\{|S| : S \in \text{Ind}(G)\} \). Let \( \Omega(G) \) be the family of all maximum independent sets of \( G \), and \( \text{core}(G) = \cap\{S : S \in \Omega(G)\} \) \[6\].

Recall from \cite{14} the following definitions for a graph \( G = (V, E) \):

- \( d(X) = |X| - |N(X)| \), \( X \subseteq V \) is the difference of the set \( X \);
- \( d_c(G) = \max\{d(X) : X \subseteq V\} \) is the critical difference of \( G \);
- a set \( U \subseteq V \) is \( d \)-critical if \( d(U) = d_c(G) \);
- \( \text{id}_c(G) = \max\{d(I) : I \in \text{Ind}(G)\} \) is the critical independence difference of \( G \);
- if \( A \subseteq V \) is independent and \( d(A) = \text{id}_c(G) \), then \( A \) is critical independent.

For a graph \( G \) let us denote

\[
\text{ker}(G) = \cap\{S : S \subseteq V \text{ is a critical independent set}\}, \quad \text{diadem}(G) = \cup\{S : S \subseteq V \text{ is a critical independent set}\}.
\]

For instance, the graph \( G_1 \) from Figure 1 has \( X = \{x, y, z, u, v\} \) as a critical set, because \( N(X) = \{a, b, u, v\} \) and \( d(X) = 1 = d_c(G_1) \). In addition, let us notice that \( \text{ker}(G_1) = \{x, y\} \subseteq \text{core}(G_1) \), and \( \text{diadem}(G_1) = \{x, y, z\} \). The graph \( G_2 \) from Figure 1 has \( d_c(G_1) = d(\{v_1, v_2\}) = |\{v_1, v_2\}| - |N(\{v_1, v_2\})| = 1 \). It is easy to see that \( \text{core}(G_1) \) is a critical set, while \( \text{core}(G_2) \) is not a critical set, but \( \text{ker}(G_2) = \{v_1, v_2\} \subseteq \text{core}(G_2) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig.png}
\caption{core\((G_1) = \{x, y\}, \text{ while core}(G_2) = \{v_1, v_2, v_7, v_{11}\}.} \end{figure}

The following results will be used in the sequel.
Theorem 1.1 Let $G$ be a graph. Then the following assertions are true:

(i) $\{d_c(G) = id_c(G)$. 
(ii) $\{d_c(G)$ is a matching from $N(S)$ into $S$, for every critical independent set $S$. 
(iii) $\{d_c(G)$ each bipartite graph enjoys this property [3], [4]. 
(iv) $\{d_c(G)$ the function $d$ is supermodular, i.e.,
\[ d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y) \text{ for every } X, Y \subseteq V(G); \]
(v) $\{d_c(G)$ if $S_1, S_2$ are $d$-critical sets, then $S_1 \cap S_2, S_1 \cup S_2$ are $d$-critical as well;
(vi) $\{d_c(G)$ there is a unique minimal $d$-critical set, namely, $\ker(G)$.
(vii) $\{d_c(G)$ the function $d$ is supermodular, i.e.,
\[ d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y) \text{ for every } X, Y \subseteq V(G); \]

If $\alpha(G) + \mu(G) = |V(G)|$, then $G$ is called a König-Egerváry graph. [2], [13]. It is well-known that each bipartite graph enjoys this property [3], [4].

Theorem 1.2 If $G = (V, E)$ is a König-Egerváry graph, $M$ is a maximum matching, and $S \in \Omega(G)$, then:
(i) $\{M$ matches $V - S$ into $S$, and $N(\text{core}(G))$ into $\text{core}(G)$;
(ii) $\{S$ is $d$-critical, and $d_c(G) = \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|$. 

Following Ore [11], [12], the number
\[ \delta(X) = d(X) = |X| - |N(X)| \]
is called the deficiency of $X$, where $X \subseteq A$ or $X \subseteq B$ and $G = (A, B, E)$ is a bipartite graph. Let
\[ \delta_0(A) = \max\{\delta(X) : X \subseteq A\}, \quad \delta_0(B) = \max\{\delta(Y) : Y \subseteq B\}. \]

A subset $X \subseteq A$ having $\delta(X) = \delta_0(A)$ is called $A$-critical, while $Y \subseteq B$ having $\delta(Y) = \delta_0(B)$ is called $B$-critical. For a bipartite graph $G = (A, B, E)$ let us denote $\ker_A(G) = \cap\{S : S \text{ is } A\text{-critical}\}$ and $\text{diadem}_A(G) = \cup\{S : S \text{ is } A\text{-critical}\}$. Similarly, $\ker_B(G) = \cap\{S : S \text{ is } B\text{-critical}\}$ and $\text{diadem}_B(G) = \cup\{S : S \text{ is } B\text{-critical}\}$.

It is convenient to define $d(\emptyset) = \delta(\emptyset) = 0$.

![G](image)

Figure 2: $G$ is a bipartite graph without perfect matchings.

For instance, the graph $G = (A, B, E)$ from Figure 2 has: $X = \{a_1, a_2, a_3, a_4\}$ as an $A$-critical set, $\ker_A(G) = \{a_1, a_2\}$, $\text{diadem}_A(G) = \{a_i : i = 1, ..., 5\}$ and $\delta_0(A) = 1$, while $Y = \{b_i : i = 4, 5, 6, 7\}$ is a $B$-critical set, $\ker_B(G) = \{b_4, b_5, b_6\}$, $\text{diadem}_B(G) = \{b_i : i = 2, ..., 7\}$ and $\delta_0(B) = 2$. 

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Theorem 1.3 \cite{11} Let \( G = (A, B, E) \). Then the following are true:

(i) the function \( \delta \) is supermodular, i.e., \( \delta(X \cup Y) + \delta(X \cap Y) \geq \delta(X) + \delta(Y) \) for every \( X, Y \subseteq A \) (or \( X, Y \subseteq B \)).

(ii) there is a unique minimal \( A \)-critical set, namely, \( \ker_A(G) \), and there is a unique maximal \( A \)-critical set, namely, \( \text{diadem}_A(G) \); similarly, for \( \ker_B(G) \) and \( \text{diadem}_B(G) \);

(iii) \( \mu(G) = |A| - \delta_0(A) = |B| - \delta_0(B) \).

In this paper we define two new graph parameters, namely, \( \ker \) and \( \text{diadem} \). Further, we analyze their relationships with two other parameters, \( \text{core} \) and \( \text{corona} \), for bipartite graphs.

2 Preliminaries

Theorem 2.1 Let \( G = (A, B, E) \) be a bipartite graph. Then the following assertions are true: 

(i) \( d_c(G) = \delta_0(A) + \delta_0(B) \);

(ii) \( \alpha(G) = |A| + \delta_0(B) = |B| + \delta_0(A) = \mu(G) + \delta_0(A) + \delta_0(B) = \mu(G) + d_c(G) \);

(iii) if \( X \) is an \( A \)-critical set and \( Y \) is a \( B \)-critical set, then \( X \cup Y \) is a \( d \)-critical set; 

(iv) if \( Z \) is a \( d \)-critical independent set, then \( Z \cap A \) is an \( A \)-critical set and \( Z \cap B \) is a \( B \)-critical set;

(v) if \( X \) is either an \( A \)-critical set or a \( B \)-critical set, then there is a matching from \( N(X) \) into \( X \).

Proof. (i) By Theorems \cite{12},(iii) and \cite{11},(ii) we get 
\[
d_c(G) = \alpha(G) - \mu(G) = |A| + |B| - 2\mu(G) = |A| + |B| - (|A| - \delta_0(A)) - (|B| - \delta_0(B)) = \delta_0(A) + \delta_0(B).
\]

(ii) Using Theorem \cite{12},(iii), we infer that 
\[
\alpha(G) = |A \cup B| - \mu(G) = |A \cup B| - |A| + \delta_0(A) = |B| + \delta_0(A).
\]

Similarly, one can find \( \alpha(G) = |A| + \delta_0(B) \).

According to part (i), we obtain 
\[
\mu(G) + d_c(G) = \mu(G) + \delta_0(A) + \delta_0(B) = |A| - \delta_0(A) + \delta_0(A) + \delta_0(B) = |A| + \delta_0(B) = \alpha(G).
\]

(iii) By supermodularity of the function \( d \) (Theorem \cite{12},(iv)) and part (i), we have 
\[
d_c(G) \geq d(X \cup Y) = d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y) = \delta(X) + \delta(Y) = \delta_0(A) + \delta_0(B) = d_c(G).
\]

(iv) Since \( Z = (Z \cap A) \cup (Z \cap B) \) and \( N(Z \cap A) \cap (Z \cap B) = \emptyset = N(Z \cap B) \cap (Z \cap A) \), then 
\[
d(Z \cap A) + d(Z \cap B) = |Z \cap A| - |N(Z \cap A)| - |Z \cap B| + |N(Z \cap B)| = |Z \cap A| + |Z \cap B| - |N(Z \cap A)| - |N(Z \cap B)| = |Z| - |N(Z)| = d(Z).
\]
Using the fact that $d(Z) = d_c(G) = \delta_0(A) + \delta_0(B)$, it follows that $d(Z \cap A) = \delta_0(A)$ and $d(Z \cap B) = \delta_0(B)$.

(v) Let $X$ be an $A$-critical set. Suppose to the contrary that there is no matching from $N(X)$ into $X$. By Hall’s Theorem it means that there exists $U \subseteq N(X)$ such that $|N(U) \cap X| < |U|$. Consequently, we obtain

\[
\delta(X - N(U)) = |X - N(U)| - |N(X - N(U))| = |X| - |X \cap N(U)| - (|N(X)| - |U|) = |X| - |N(X)| + (|U| - |X \cap N(U)|) = \delta_0(A) + (|U| - |X \cap N(U)|) > \delta_0(A),
\]

which contradicts the fact that $X$ is an $A$-critical set. ■

It is known that a bipartite graph $G$ has a perfect matching if and only if $\alpha(G) = \mu(G)$. Hence using Theorem 2.1(ii), we deduce the following.

**Corollary 2.2** \textbf{[11]} A bipartite graph $G = (A, B, E)$ has a perfect matching if and only if $\delta_0(A) = 0 = \delta_0(B)$.

**Lemma 2.3** Let $G = (A, B, E)$ be a bipartite graph. If $X$ is an $A$-critical set and $Y$ is a $B$-critical set, then $|X \cap N(Y)| = |N(X) \cap Y|$. Moreover, there is a perfect matching between $X \cap N(Y)$ and $N(X) \cap Y$.

**Proof.** By Theorem 2.1(v), there is a matching $M_1$ from $N(X)$ into $X$, and a matching $M_2$ from $N(Y)$ into $Y$. For each $b \in N(X) \cap Y$, it follows that $M_1(b) \in N(b) \subseteq N(Y)$ and $M_1(b) \in X$. Hence $M_1(b) \in X \cap N(Y)$, which implies $M_1(N(X) \cap Y) \subseteq X \cap N(Y)$, and further

\[
|N(X) \cap Y| = |M_1(N(X) \cap Y)| \leq |X \cap N(Y)|.
\]

Similarly, we have

\[
|X \cap N(Y)| = |M_2(X \cap N(Y))| \leq |N(X) \cap Y|.
\]

Consequently, we deduce that $|X \cap N(Y)| = |N(X) \cap Y|$ and the restriction of $M_1$ to $N(X) \cap Y$ is a perfect matching from $N(X) \cap Y$ onto $X \cap N(Y)$. ■

**Corollary 2.4** Let $G = (A, B, E)$ be a bipartite graph.

(i) \textbf{[12]} If $X = \ker_A(G)$ and $Y$ is a $B$-critical set, then $X \cap N(Y) = N(X) \cap Y = \emptyset$;

(ii) \textbf{[13]} $\ker_A(G) \cap N(\ker_B(G)) = N(\ker_A(G)) \cap \ker_B(G) = \emptyset$.

**Proof.** (i) Assume, to the contrary, that $X \cap N(Y) \neq \emptyset$. By Lemma 2.3, we have $|X \cap N(Y)| = |N(X) \cap Y|$.

If $x \in X - X \cap N(Y)$ has $N(x) \cap Y \neq \emptyset$, then $x \in N(y) \subseteq N(Y)$, which is impossible. Hence $N(X - X \cap N(Y)) \subseteq N(X) - N(X) \cap Y$, and further, we get

\[
|X - X \cap N(Y)| - |N(X - X \cap N(Y))| \geq |X - X \cap N(Y)| - |N(X) - N(X) \cap Y| = |X| - |X \cap N(Y)| - |N(X)| = |N(X) \cap Y| = \delta(X) = \delta_0(A),
\]

and this contradicts the minimality of $X$.

(ii) It immediately follows from part (i), when $Y = \ker_B(G)$. ■
3 Ker and Core

**Theorem 3.1** Let $X$ be a critical independent set in a graph $G$. Then the following statements are equivalent:

(i) $X = \ker(G)$;

(ii) there is no set $Y \subseteq N(X)$, $Y \neq \emptyset$ such that $|N(Y) \cap X| = |Y|$;

(iii) for each $v \in X$ there exists a matching from $N(X)$ into $X - v$.

**Proof.** (i) $\implies$ (ii) By Theorem 1.1, there is a matching, say $M$, from $N(\ker(G))$ into $\ker(G)$. Suppose, to the contrary, that there exists some non-empty set $Y \subseteq N(\ker(G))$ such that $|M(Y)| = |N(Y) \cap \ker(G)| = |Y|$. It contradicts the minimality of the set $\ker(G)$, because

$$d(\ker(G) - N(Y)) = d(\ker(G)),$$

while $\ker(G) - N(Y) \not\subseteq \ker(G)$.

(ii) $\implies$ (i) Suppose $X - \ker(G) \neq \emptyset$. By Theorem 1.1, there is a matching, say $M$, from $N(X)$ into $X$. Since there are no edges connecting vertices from $\ker(G)$ with vertices of $N(X) - N(\ker(G))$, we obtain that $M(N(X) - N(\ker(G))) \subseteq X - \ker(G)$. Moreover, we have that $|N(X) - N(\ker(G))| = |X - \ker(G)|$, otherwise

$$|X| - |N(X)| = (|\ker(G)| - |N(\ker(G))|) + (|X - \ker(G)| - |N(X) - \ker(G)|) > (|\ker(G)| - |N(\ker(G))|) = d_e(G).$$

It means that the set $N(X) - N(\ker(G))$ contradicts the hypothesis of (ii), because

$$|N(X) - N(\ker(G))| = |X - \ker(G)| = |N(X) - \ker(G)) \cap X|.$$  

Consequently, the assertion is true.

(ii) $\implies$ (iii) By Theorem 1.1, there is a matching, say $M$, from $N(X)$ into $X$. Suppose, to the contrary, that there is no matching from $N(X)$ into $X - v$. By Hall’s Theorem, it implies the existence of a set $Y \subseteq N(X)$ such that $|N(Y) \cap X| = |Y|$, which contradicts the hypothesis of (ii).

(iii) $\implies$ (ii) Suppose, to the contrary, there is a non-empty subset $Y$ of $N(X)$ such that $|N(Y) \cap X| = |Y|$. Let $v \in N(Y) \cap X$. Hence, we get $|N(Y) \cap X - v| < |Y|$. Then, by Hall’s Theorem, it is impossible to find a matching from $N(X)$ into $X - v$, which contradicts the hypothesis of (iii). \qed

**Lemma 3.2** If $G = (A, B, E)$ is a bipartite graph with a perfect matching, say $M$, $S \in \Omega(G)$, $X \in \text{Ind}(G)$, $X \subseteq V(G) - S$, and $G[X \cup M(X)]$ is connected, then

$$X^1 = X \cup M((N(X) \cap S) - M(X))$$

is an independent set, and $G[X^1 \cup M(X^1)]$ is connected.

**Proof.** Let us show that the set $M((N(X) \cap S) - M(X))$ is independent. Suppose, to the contrary, that there exist $v_1, v_2 \in M((N(X) \cap S) - M(X))$ such that $v_1v_2 \in E(G)$. Hence $M(v_1), M(v_2) \in (N(X) \cap S) - M(X)$.
If \( M(v_1) \) and \( M(v_2) \) have a common neighbor \( w \in X \), then \( \{v_1, v_2, M(v_2), w, M(v_1)\} \) spans \( C_5 \), which is forbidden for bipartite graphs.

Otherwise, let \( w_1, w_2 \in X \) be neighbors of \( M(v_1) \) and \( M(v_2) \), respectively. Since \( G[X \cup M(X)] \) is connected, there is a path with even number of edges connecting \( w_1 \) and \( w_2 \). Together with \( \{w_1, M(v_1), v_1, v_2, M(v_2), w_2\} \) this path produces a cycle of odd length in contradiction with the hypothesis on \( G \) being a bipartite graph.

To complete the proof of independence of the set

\[
X^1 = X \cup M((N(X) \cap S) - M(X))
\]

it is enough to demonstrate that there are no edges connecting vertices of \( X \) and \( M((N(X) \cap S) - M(X)) \).

![Figure 3: S ∈ Ω(G), Y = (N(X) ∩ S) − M(X) and X^1 = X ∪ M(Y).](image)

Assume, to the contrary, that there is \( vw \in E \), such that \( v \in M((N(X) \cap S) - M(X)) \) and \( w \in X \). Since \( M(v) \in (N(X) \cap S) - M(X) \) and \( G[X \cup M(X)] \) is connected, it follows that there exists a path with an odd number of edges connecting \( M(v) \) to \( w \).

This path together with the edges \( vw \) and \( vM(v) \) produces cycle of odd length, in contradiction with the bipartiteness of \( G \).

Finally, since \( G[X \cup M(X)] \) is connected, \( G[X^1 \cup M(X^1)] \) is connected as well, by definitions of set functions \( N \) and \( M \).

Theorem 1.1(vii) claims that \( \ker(G) \subseteq \text{core}(G) \) for every graph.

**Theorem 3.3** If \( G \) is a bipartite graph, then \( \ker(G) = \text{core}(G) \).

**Proof.** The assertions are clearly true, whenever \( \text{core}(G) = \emptyset \), i.e., for \( G \) having a perfect matching. Assume that \( \text{core}(G) \neq \emptyset \).

Let \( S \in \Omega(G) \) and \( M \) be a maximum matching. By Theorem 1.2(i), \( M \) matches \( V(G) - S \) into \( S \), and \( N(\text{core}(G)) \) into \( \text{core}(G) \).

According to Theorem 1.1 it is sufficient to show that there is no set \( Z \subseteq N(\text{core}(G)) \), \( Z \neq \emptyset \), such that \( |N(Z) \cap \text{core}(G)| = |Z| \).

Suppose, to the contrary, that there exists a non-empty set \( Z \subseteq N(\text{core}(G)) \) such that \( |N(Z) \cap \text{core}(G)| = |Z| \). Let \( Z_0 \) be a minimal non-empty subset of \( N(\text{core}(G)) \) enjoying this equality.

Clearly, \( H = G[Z_0 \cup M(Z_0)] \) is bipartite, because it is a subgraph of a bipartite graph. Moreover, the restriction of \( M \) on \( H \) is a perfect matching.
**Claim 1.** $Z_0$ is independent.

Since $H$ is a bipartite graph with a perfect matching it has two maximum independent sets at least. Hence there exists $W \in \Omega (H)$ different from $M (Z_0)$. Thus $W \cap Z_0 \neq \emptyset$. Therefore, $N (W \cap Z_0) \cap \text{core}(G) = M (W \cap Z_0)$. Consequently,

$$|N (W \cap Z_0) \cap \text{core}(G)| = |M (W \cap Z_0)| = |W \cap Z_0|.$$ 

Finally, $W \cap Z_0 = Z_0$, because $Z_0$ has been chosen as a minimal subset of $N (\text{core}(G))$ such that $|N (Z_0) \cap \text{core}(G)| = |Z_0|$. Since $|Z_0| = \alpha(H) = |W|$ we conclude with $W = Z_0$, which means, in particular, that $Z_0$ is independent.

**Claim 2.** $H$ is a connected graph.

Otherwise, for any connected component of $H$, say $\tilde{H}$, the set $V(\tilde{H}) \cap Z_0$ contradicts the minimality property of $Z_0$.

**Claim 3.** $Z_0 \cup (\text{core}(G) - M (Z_0))$ is independent.

By Claim 1 $Z_0$ is independent. The equality $|N (Z_0) \cap \text{core}(G)| = |Z_0|$ implies $N (Z_0) \cap \text{core}(G) = M (Z_0)$, which means that there are no edges connecting $Z_0$ and $\text{core}(G) - M (Z_0)$. Consequently, $Z_0 \cup (\text{core}(G) - M (Z_0))$ is independent.

**Claim 4.** $Z_0 \cup (\text{core}(G) - M (Z_0))$ is included in a maximum independent set.

Let $Z_i = M ((N (Z_{i-1}) \cap S) - M (Z_{i-1}))$, $1 \leq i < \infty$. By Lemma 3.2 all the sets $Z^i = \bigcup_{0 \leq j \leq i} Z_j$, $1 \leq i < \infty$ are independent. Define

$$Z^\infty = \bigcup_{0 \leq i \leq \infty} Z_i,$$

which is, actually, the largest set in the sequence $\{Z^i, 1 \leq i < \infty\}$.

![Diagram](image)

Figure 4: $S \in \Omega(G)$, $Q = \text{core}(G) - M (Z_0)$, $Y_0 = M (Z_0)$, $Y_1 = (N (Z_0) - M (Z_0)) \cap S$, $Y_2 = \ldots$ and $Z_i = M (Y_i)$, $i = 1, 2, \ldots$.

The inclusion

$$Z_0 \cup (\text{core}(G) - M (Z_0)) \subseteq (S - M (Z^\infty)) \cup Z^\infty$$

is justified by the definition of $Z^\infty$. 

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Since $|M(Z^\infty)| = |Z^\infty|$ we obtain $|(S - M(Z^\infty)) \cup Z^\infty| = |S|$. According to the definition of $Z^\infty$ the set 
$$(N(Z^\infty) \cap S) - M(Z^\infty)$$
is empty. In other words, the set $(S - M(Z^\infty)) \cup Z^\infty$ is independent. Therefore, we arrive at 
$$(S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$
Consequently, $(S - M(Z^\infty)) \cup Z^\infty$ is a desired enlargement of $Z_0 \cup (\text{core}(G) - M(Z_0))$.

Claim 5. $\text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) = \text{core}(G) - M(Z_0)$.
The only part of $(S - M(Z^\infty)) \cup Z^\infty$ that interacts with $\text{core}(G)$ is the subset 
$$Z_0 \cup (\text{core}(G) - M(Z_0)).$$
Hence we obtain
$$\text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) =$$ 
$$= \text{core}(G) \cap (Z_0 \cup (\text{core}(G) - M(Z_0))) = \text{core}(G) - M(Z_0).$$
Since $Z_0$ is non-empty, by Claim 5 we arrive at the following contradiction
$$\text{core}(G) \nsubseteq (S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$
Finally, we conclude with the fact there is no set $Z \subseteq N(\text{core}(G)), Z \neq \emptyset$ such that $|N(Z) \cap \text{core}(G)| = |Z|$, which, by Theorem 3.1, means that $\text{core}(G)$ and $\text{ker}(G)$ coincide. $\blacksquare$

Notice that there are non-bipartite graphs enjoying the equality $\text{ker}(G) = \text{core}(G)$; e.g., the graphs from Figure 5. Notice that only $G_1$ is a König–Egerváry graph.

![Figure 5: $\text{core}(G_1) = \text{ker}(G_1) = \{x, y\}$ and $\text{core}(G_2) = \text{ker}(G_2) = \{a, b\}$.](image)

There is a non-bipartite König-Egerváry graph $G$, such that $\text{ker}(G) \neq \text{core}(G)$. For instance, the graph $G_1$ from Figure 6 has $\text{ker}(G_1) = \{x, y\}$, while $\text{core}(G_1) = \{x, y, u, v\}$. The graph $G_2$ from Figure 6 has $\text{ker}(G_2) = \emptyset$, while $\text{core}(G_2) = \{w\}$.

![Figure 6: Both $G_1$ and $G_2$ are König-Egerváry graphs. Only $G_2$ has a perfect matching.](image)
4 Ker and Diadem

Proposition 4.1 If $G$ is a König–Egerváry graph, then
$$N(\text{core}(G)) = \cap \{V(G) - S : S \in \{G\}\}, \text{ i.e., } N(\text{core}(G)) = V(G) - \text{corona}(G).$$

There is a non-König-Egerváry graph $G$ with $V(G) = N(\text{core}(G)) \cup \text{corona}(G)$; e.g., the graph $G$ from Figure 7.

Theorem 4.2 If $G$ is a König–Egerváry graph, then

(i) $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$;

(ii) $\text{diadem}(G) = \text{corona}(G)$, while $\text{diadem}(G) \subseteq \text{corona}(G)$ is true for every graph;

(iii) $|\text{ker}(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$.

Proof. (i) Using Theorem 1.2 (ii) and Proposition 4.1 we infer that
$$|\text{corona}(G)| + |\text{core}(G)| = |\text{corona}(G)| + |N(\text{core}(G))| + |\text{core}(G)| - |N(\text{core}(G))| =$$
$$= |V(G)| + d_c(G) = \alpha(G) + \mu(G) + d_c(G) = 2\alpha(G).$$
as claimed.

(ii) Every $S \in \Omega(G)$ is $d$-critical, by Theorem 1.2 (ii). Further, Theorem 1.2 (ii) ensures that $\text{corona}(G) \subseteq \text{diadem}(G)$. On the other hand, each critical independent set is included in a maximum independent set, according to Theorem 1.1 (iii). Thus, we have $\text{diadem}(G) \subseteq \text{corona}(G)$. Consequently, the equality $\text{diadem}(G) = \text{corona}(G)$ holds.

(iii) It follows by combining parts (i), (ii) and Theorem 1.1 (vii). ■

Notice that the graph from Figure 7 has $|\text{corona}(G)| + |\text{core}(G)| = 13 > 12 = 2\alpha(G)$.

For a König–Egerváry graph with $|\text{ker}(G)| + |\text{diadem}(G)| < 2\alpha(G)$ see Figure 6.

Figure 7 shows that it is possible for a graph to have $\text{diadem}(G) \subset \text{corona}(G)$ and $\text{ker}(G) \subset \text{core}(G)$.

Figure 8: $G_1$ is a non-bipartite König–Egerváry graph, such that $\text{ker}(G_1) = \text{core}(G_1)$ and $\text{diadem}(G_1) = \text{corona}(G_1)$; $G_2$ is a non-König–Egerváry graph, such that $\text{ker}(G) = \text{core}(G) = \{x, y\}$; $\text{diadem}(G) \cup \{z, t, v, w\} = \text{corona}(G)$.

The combination of $\text{diadem}(G) \subset \text{corona}(G)$ and $\text{ker}(G) = \text{core}(G)$ is realized in Figure 8.

Now we are ready to describe both ker and diadem of a bipartite graph in terms of its bipartition.
Theorem 4.3 Let $G = (A, B, E)$ be a bipartite graph. Then the following assertions are true:

1. $\ker_{A}(G) \cup \ker_{B}(G) = \ker(G)$;
2. $|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G)$;
3. $|\ker_{A}(G)| + |\text{diadem}_{B}(G)| = |\ker_{B}(G)| + |\text{diadem}_{A}(G)| = \alpha(G)$;
4. $\text{diadem}_{A}(G) \cup \text{diadem}_{B}(G) = \text{diadem}(G)$.

Proof. (i) By Theorem 2.1(iii), $\ker_{A}(G) \cup \ker_{B}(G)$ is $d$-critical in $G$. Moreover, $\ker_{A}(G) \cup \ker_{B}(G)$ is independent in accordance with Corollary 2.4. Assume that $\ker_{A}(G) \cup \ker_{B}(G)$ is not minimal. Hence the unique minimal $d$-critical set of $G$, say $Z$, is a proper subset of $\ker_{A}(G) \cup \ker_{B}(G)$, by Theorem 1.1(iv). According to Theorem 2.1(iv), $Z_{A} = Z \cap A$ is an $A$-critical set, which implies $\ker_{A}(G) \subseteq Z_{A}$, and similarly, $\ker_{B}(G) \subseteq Z_{B}$. Consequently, we get that $\ker_{A}(G) \cup \ker_{B}(G) \subseteq Z$, in contradiction with the fact that $\ker_{A}(G) \cup \ker_{B}(G) \neq Z \subseteq \ker_{A}(G) \cup \ker_{B}(G)$.

(ii), (iii), (iv) By Corollary 2.2 we have

$$|\ker_{A}(G)| - \delta_{0}(A) + |\text{diadem}_{B}(G)| = |N(\ker_{A}(G))| + |\text{diadem}_{B}(G)| \leq |B|.$$ 

Hence, according to Theorem 2.1(ii), it follows that

$$|\ker_{A}(G)| + |\text{diadem}_{B}(G)| \leq |B| + \delta_{0}(A) = \alpha(G).$$

Changing the roles of $A$ and $B$, we obtain

$$|\ker_{B}(G)| + |\text{diadem}_{A}(G)| \leq \alpha(G).$$

By Theorem 2.1(iv), $\text{diadem}(G) \cap A$ is $A$-critical and $\text{diadem}(G) \cap B$ is $B$-critical. Hence $\text{diadem}(G) \cap A \subseteq \text{diadem}_{A}(G)$ and $\text{diadem}(G) \cap B \subseteq \text{diadem}_{B}(G)$. It implies both the inclusion $\text{diadem}(G) \subseteq \text{diadem}_{A}(G) \cup \text{diadem}_{B}(G)$, and the inequality

$$|\text{diadem}(G)| \leq |\text{diadem}_{A}(G)| + |\text{diadem}_{B}(G)|.$$ 

Combining Theorem 3.3, Theorem 4.2(ii) and part (i) with the above inequalities, we deduce

$$2\alpha(G) \geq |\ker_{A}(G)| + |\ker_{B}(G)| + |\text{diadem}_{A}(G)| + |\text{diadem}_{B}(G)| \geq$$

$$|\ker(G)| + |\text{diadem}(G)| = |\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G).$$

Consequently, we infer that

$$|\text{diadem}_{A}(G)| + |\text{diadem}_{B}(G)| = |\text{diadem}(G)|,$$

$$|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G),$$

$$|\ker_{A}(G)| + |\text{diadem}_{B}(G)| = |\ker_{B}(G)| + |\text{diadem}_{A}(G)| = \alpha(G).$$

Since $\text{diadem}(G) \subseteq \text{diadem}_{A}(G) \cup \text{diadem}_{B}(G)$ and $\text{diadem}_{A}(G) \cap \text{diadem}_{B}(G) = \emptyset$, we finally obtain that

$$\text{diadem}_{A}(G) \cup \text{diadem}_{B}(G) = \text{diadem}(G),$$

as claimed. ■
5 Conclusions

In this paper we focus on interconnections between \( \text{ker} \), \( \text{core} \), \( \text{diadem} \), and \( \text{corona} \) for König-Egerváry graphs, in general, and bipartite graphs, in particular.

In [9] we showed that \( 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)| \) is true for every graph. By Theorem 4.2 (i), this equality is true whenever \( G \) is a König-Egerváry graph.

According to Theorem 1.1 (vii), \( \text{ker}(G) \subseteq \text{core}(G) \) for every graph. On the other hand, Theorem 1.1 (iii) implies the inclusion \( \text{diadem}(G) \subseteq \text{corona}(G) \). Hence

\[
|\text{ker}(G)| + |\text{diadem}(G)| \leq |\text{core}(G)| + |\text{corona}(G)|
\]

for each graph \( G \). These remarks together with Theorem 4.2 (iii) motivate the following.

**Conjecture 5.1** \( |\text{ker}(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \) is true for every graph \( G \).

When it is proved one can conclude that the following inequalities:

\[
|\text{ker}(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|
\]

hold for every graph \( G \).

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