A high order time discretization of the solution of the non-linear filtering problem

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Abstract

The solution of the continuous time filtering problem can be represented as a ratio of two expectations of certain functionals of the signal process that are parametrized by the observation path. We introduce a class of discretization schemes of these functionals of arbitrary order. The result generalizes the classical work of Picard, who introduced first order discretizations to the filtering functionals. For a given time interval partition, we construct discretization schemes with convergence rates that are proportional with the $m$-power of the mesh of the partition for arbitrary $m \in \mathbb{N}$. The result paves the way for constructing high order numerical approximation for the solution of the filtering problem.

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Key words: Non-linear filtering, Kallianpur-Striebel’s formula, high order time discretization.

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1 Introduction

Partially observed dynamical systems are ubiquitous in a multitude of real-life phenomena. The dynamical system is typically modelled by a continuous time stochastic process called the signal process $X$. The signal process cannot be measured directly, but only via a related process $Y$, called the observation process. The filtering problem is that of estimating the current state of the dynamical system at the current time given the observation data accumulated up to that time. Mathematically the problem entails computing the conditional distribution of the signal process $X_t$, denoted by $\pi_t$, given $\mathcal{Y}_t$, the $\sigma$-algebra generated by $Y$.

In a few special cases, $\pi_t$ can be expressed in closed form as a functional of the observation path. For example, the celebrated Kalman-Bucy filter does this in the linear case. In general, an explicit formula for $\pi_t$ is not available and inferences can only be made by numerical approximations of $\pi_t$. As expected the problem has attracted a lot of attention in the last fifty years (see Chapter 8 of [2] for a survey of existing numerical methods for approximating $\pi_t$).

The basis of this class of numerical methods is the representation of $\pi_t$ given by the Kallianpur–Striebel formula (see (2.2) below). In the case when the signal process is modelled by the solution of a stochastic differential equation (SDE) and the observation process is a function of the signal perturbed by white noise (see Section 2 below for further details), the formula entails the computation of expectations of functionals of the solution of the signal SDE that are parametrized by the observation path. The numerical approximation of $\pi_t$ requires three procedures:

- the discretization of the functionals (corresponding to a partition of the interval $[0, t]$).
- the approximation of the law of the signal with a discrete measure.
- the control of the computational effort.

The first step is typically achieved by the discretization scheme introduced by Picard in [15]. This offers a first order approximation for the functionals appearing in formula (2.2). More precisely, the $L^1$-rate of convergence of the approximation is proportional with the mesh of the partition of the time interval $[0, t]$ (see Theorem 21.5 in [4]). The second and the third step are achieved by a combination of an Euler approximation of the solution of the SDE, a Monte Carlo step that gives a sample from the law of the Euler approximation and a re-sampling step that acts as a variance reduction method and keeps the computational effort in control. There are a variety of algorithms that follow this template. Further details can be found, for instance, in Part VII of [5]. It is worth pointing out that once the functional discretization and the Euler approximation have been applied, the problem can be reduced to one where the signal evolves and is observed in discrete time. The discrete version of the filtering problem is popular both with practitioners and with theoreticians. The majority of the existing theoretical results and the numerical algorithms are constructed and analyzed in the discrete framework. For more details, the interested reader can consult the comprehensive theoretical monograph [7] and the reference therein and the equally comprehensive methodological volume [8] and the references therein with some updates in Part VII of [5].

The first order discretization introduced by Picard creates a bottleneck: There exist higher order schemes for approximating the law of the signal that can be used, but which won’t bring any substantial improvements because of this. For example, in the recent paper [6], the authors employ high order cubature methods to approximate the law of the signal with only minimal improvements due to the low order discretization of the required functionals. The aim of this paper is to address this issue. More precisely, we introduce
a class of high order discretizations of the functionals. As we shall see, we prove that the $L^1$-rate of convergence of the approximations is proportional with the $m$-power of the mesh of the partition of the time interval $[0, t]$. For details, see Theorem 2 below. In a work in progress, this discretization procedure is employed to produce a second order particle filter. It is hoped that this discretization will be used in conjunction with other high order approximations of the law of the signal, in particular with cubature methods. We are not aware of any other similar high order discretization schemes.

The paper is organized as follows: In Section 2 we introduce some basic definitions and state the main result of the paper, Theorem 2. Section 3 is devoted to prove our main result. We start by proving several auxiliary results on iterated stochastic integrals and on the integrability of the likelihood function and its discretizations. These lead to the two main results of the section, Proposition 15 and Proposition 16, from which we will deduce our main result. In Section 4 we address the most technical aspects of the paper. We first introduce some technical tools on Malliavin calculus (subsection 4.1), the Stroock-Taylor formula (subsection 4.2) and backward martingales (subsection 4.3). Then, with the aid of the theses tools, we prove in subsection 4.4 the estimates on the conditional expectation with respect to $\mathcal{Y}_t$ that are essential in proving Proposition 15.

2 Basic framework and statement of the main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space together with a filtration $(\mathcal{F}_t)_{t \geq 0}$ which satisfies the usual conditions. On $(\Omega, \mathcal{F}, P)$ we consider a $d_X \times d_Y$-dimensional partially observed system $(X, Y)$ satisfying

$$X_t = X_0 + \int_0^t f(X_s) \, ds + \int_0^t \sigma(X_s) \, dV_s,$$

$$Y_t = \int_0^t h(X_s) \, ds + W_t,$$

where $V$ is a standard $\mathcal{F}_t$-adapted $d_Y$-dimensional Brownian motion and and $W$ is a a standard $\mathcal{F}_t$-adapted $d_Y$-dimensional Brownian motion, independent of each other. We also assume that $X_0$ is a random variable independent of $V$ and $W$ and denote by $\pi_0$ its law. We assume that $f = (f_i)_{i=1,\ldots,d_X} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_X}$ and $\sigma = (\sigma_{i,j})_{i=1,\ldots,d_X,j=1,\ldots,d_Y} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_X \times d_Y}$ are globally Lipschitz continuous. In addition, we assume that $h = (h_i)_{i=1,\ldots,d_Y} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_Y}$ is measurable and has linear growth.

Let $\mathcal{Y} = \{\mathcal{Y}_t\}_{t \geq 0}$ be the usual augmentation of the filtration generated by the process $Y$, that is, $\mathcal{Y}_t = \sigma(Y_s, s \in [0, t]) \vee \mathcal{N}$, where $\mathcal{N}$ are all the $P$-null sets of $(\Omega, \mathcal{F}, P)$. We are interested in determining $\pi_t$, the conditional law of the signal $X$ at time $t$ given the information accumulated from observing $Y$ in the interval $[0, t]$. More precisely, for any Borel measurable and bounded function $\varphi$, we want to compute $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t]$. By an application of Girsanov’s theorem (see, for example, Chapter 3 in [2]) one can construct a new probability measure $\tilde{P}$, absolutely continuous with respect to $P$, under which $Y$ becomes a Brownian motion independent of $X$ and the law of $X$ remains unchanged. The Radon-Nikodym derivative of $\tilde{P}$ with respect to $P$ is given by the process $Z(X,Y) = (Z_t(X,Y))_{t \geq 0}$ given by

$$Z_t(X,Y) = \exp \left( \sum_{i=1}^{d_Y} \int_0^t h_i^{(X_s)} dY_s^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_0^t h_i^2(X_s) \, ds \right), \quad t \geq 0,$$  \hspace{1cm} (2.1)
which is an $\mathcal{F}_t$-adapted martingale under $\tilde{P}$ under the assumptions introduced above. We will denote by $\tilde{E}$ to be the expectation with respect to $\tilde{P}$. In the following we will make use of the measure valued process $\rho = (\rho_t)_{t \geq 0}$, defined by the formula $\rho_t(\varphi) = \tilde{E}[\varphi(X_t)Z_t|\mathcal{Y}_t]$, for any bounded Borel measurable function $\varphi$. The processes $\pi$ and $\rho$ are connected through the Kallianpur-Striebel’s formula:

$$
\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} = \frac{\tilde{E}[\varphi(X_t)\exp\left(\sum_{i=1}^{dY} \int_0^t h_i(X_s)dY_s^i - \frac{1}{2} \sum_{i=1}^{dY} \int_0^t h_i^2(X_s)ds\right)|\mathcal{Y}_t]}{\tilde{E}[\exp\left(\sum_{i=1}^{dY} \int_0^t h_i(X_s)dY_s^i - \frac{1}{2} \sum_{i=1}^{dY} \int_0^t h_i^2(X_s)ds\right)|\mathcal{Y}_t]}, \quad (2.2)
$$

$P$-a.s., where $1$ is the constant function. As a result, $\rho$ is called the unnormalized conditional distribution of the signal. For further details on the filtering framework, see [2].

It follows from (2.2) that $\pi_t(\varphi)$ is a ratio of two conditional expectations of functionals of the signal that depend on the stochastic integrals with respect to the process $Y$. In the following we will introduce a class of time discretization schemes for these conditional expectations which, in turn, will generate time discretisation schemes $\pi_t$ (of any order).

This is the main result of the paper and is stated Theorem 2 below.

We first introduce some useful notation and definitions. We denote by:

- $B_b$ the space of bounded Borel-measurable functions.
- $B_P$ the space of Borel-measurable functions with polynomial growth.
- $C_k^p$ the space of continuously differentiable functions up to order $k \in \mathbb{Z}_+$ with bounded derivatives of order greater or equal to one.
- $C_P^k$ the space of continuously differentiable functions up to order $k \in \mathbb{Z}_+$ such that the function and its derivatives have at most polynomial growth.
- $L^p(\Omega, \mathcal{F}, \tilde{P})$ the space of $p$-integrable random variables (with respect to $\tilde{P}$) and denote by $||\cdot||_p$ the corresponding norm on $L^p(\Omega, \mathcal{F}, \tilde{P})$, i.e., for $\xi \in L^p(\Omega, \mathcal{F}, \tilde{P})$, $||\xi||_p = \tilde{E}[||\xi||^p]^{1/p}$.

In the following, we will use the notation introduced in Section 5.4 in Kloeden and Platen [10]. More precisely, let $S$ be a subset of $\mathbb{Z}_+$ and denote by $\mathcal{M}^*(S)$ the set of all multi-indices with values in $S$. In addition, define $\mathcal{M}(S) = \mathcal{M}^*(S) \cup \{v\}$, where $v$ denotes the multi-index of length zero. For $\alpha = (\alpha_1, ..., \alpha_k) \in \mathcal{M}(S)$ define the following operations

$$
|\alpha| \triangleq k,
|\alpha|_0 \triangleq \# \{j : \alpha_j = 0, j = 1, ..., k\},
\alpha_- \triangleq (\alpha_1, ..., \alpha_{k-1}),
-\alpha \triangleq (\alpha_2, ..., \alpha_k),
$$

where $|v| = 0, -v = v- = v$. Given two multi-indices $\alpha, \beta \in \mathcal{M}(S)$ we denote its concatenation by $\alpha * \beta$. Itô-Taylor expansions are usually done with a particular subsets of multi-indices, the so called hierarchical sets. We call a subset $\mathcal{A} \subset \mathcal{M}(S)$ a hierarchical set if $\mathcal{A}$ is nonempty, $\sup_{\alpha \in \mathcal{A}} |\alpha| < \infty$, and

$$
-\alpha \in \mathcal{A} \quad \text{if} \quad \alpha \in \mathcal{A} \setminus \{v\}.
$$

For any given hierarchical set $\mathcal{A}$ we define the remainder set $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ by

$$
\mathcal{R}(\mathcal{A}) \triangleq \{\alpha \in \mathcal{M}(S) \setminus \mathcal{A} : -\alpha \in \mathcal{A}\}.
$$
We will consider the hierarchical set $\mathcal{M}_m(S)$ and its associated remainder set $\mathcal{R}(\mathcal{M}_m(S))$, that is,

$$\mathcal{M}_m(S) \triangleq \{ \alpha \in \mathcal{M}(S) : |\alpha| \leq m \},$$

and

$$\mathcal{R}(\mathcal{M}_m(S)) \triangleq \{ \alpha \in \mathcal{M}(S) : |\alpha| = m + 1 \}.$$

Observe that $\mathcal{R}(\mathcal{M}_m(S)) = \mathcal{M}_{m+1}(S) \setminus \mathcal{M}_m(S)$. We shall use the sets of multi-indices with values in the sets $S_0 = \{0, 1, \ldots, d_V\}$ and $S_1 = \{1, \ldots, d_V\}$. Note also that the set $\mathcal{R}(\mathcal{M}_m(S_0))$ can be partitioned in the following way

$$\mathcal{R}(\mathcal{M}_m(S_0)) = \bigcup_{k=0}^{m+1} \mathcal{R}(\mathcal{M}_m(S_0))_k,$$

where $\mathcal{R}(\mathcal{M}_m(S_0))_k = \{ \alpha \in \mathcal{R}(\mathcal{M}_m(S_0)) : |\alpha|_0 = k \}, k = 0, \ldots, m+1$, that is, $\mathcal{R}(\mathcal{M}_m(S_0))_k$ is the set of multi-indices of length $m+1$ with values in $S_0$ which contains $k$ zeros.

To simplify the notation, it is convenient to add an additional component to the Brownian motion $V$. Let $V^0_s = s$, for all $s \geq 0$ and consider the $(d_V + 1)$-dimensional process $V = (V^i)^{d_V}_{t=0}$. We will consider the filtration $\mathbb{F}^{0,V} = \{ \mathcal{F}^{0,V}_s \}_{s \geq 0}$ defined to be the usual augmentation of the filtration generated by the process $V$ and initially enlarged with the random variable $X_0$. Moreover, for fixed $t$, we will also consider the filtration $\mathbb{F}^t = \{ \mathcal{H}^t_s \triangleq \mathcal{F}^{0,V}_s \vee \mathcal{Y}_t \}_{s \geq 0}$. For $\alpha \in \mathcal{M}(S_0)$, denote by $I_\alpha(h)_{s,t}$ the following Itô iterated integral

$$I_\alpha(h)_{s,t} = \begin{cases} h_t & \text{if } \alpha = v \\ \int_s^t I_\alpha(h)_{u,t} dV^\alpha_u & \text{if } |\alpha| \geq 1 \end{cases},$$

where $h = \{h_s\}_{s \geq 0}$ is an $\mathbb{H}^t$-adapted process (satisfying appropriate integrability conditions). We introduce the differential operators $L^0, L^r, r = 1, \ldots, d_V$ defined by

$$L^0 g(x) \triangleq \sum_{k=1}^{d_X} f^k(x) \frac{\partial g}{\partial x^k}(x) + \frac{1}{2} \sum_{k,l=1}^{d_X} \sum_{r=1}^{d_V} \sigma_{k,r}(x) \sigma_{l,r}(x) \frac{\partial^2 g}{\partial x^k \partial x^l}(x),$$

$$L^r g(x) \triangleq \sum_{k=1}^{d_X} \sigma_{k,r}(x) \frac{\partial g}{\partial x^k}(x), \quad r = 1, \ldots, d_V,$$

where $g : \mathbb{R}^{d_X} \to \mathbb{R}$ belongs to $C^2_{\mathbb{F}}(\mathbb{R}^{d_X}; \mathbb{R})$. For $\alpha \in \mathcal{M}(S_0)$, with $\alpha = (\alpha_1, \ldots, \alpha_k)$, and the differential operator $L^\alpha$ is defined by

$$L^\alpha g = L^{\alpha_1} \circ L^{\alpha_2} \circ \cdots \circ L^{\alpha_k} g,$$

and, by convention $L^0 g = g$. Finally, let $\tau \triangleq \{0 = t_0 < \cdots < t_i < \cdots < t_n = t\}$ be a partition of $[0, t]$. Associated to $\tau$ we define the following elements

$$\delta_i \triangleq t_i - t_{i-1}, \quad i = 1, \ldots, n,$$

$$\delta \triangleq \max_{i=1,\ldots,n} \delta_i,$$

$$\delta_{\min} \triangleq \min_{i=1,\ldots,n} \delta_i,$$

$$\tau(s) \triangleq t_{i-1}, \quad s \in [t_{i-1}, t_i), i = 1, \ldots, n,$$

$$\eta(s) \triangleq t_i, \quad s \in [t_{i-1}, t_i), i = 1, \ldots, n.$$
We will only consider partitions satisfying the following condition

$$\delta \leq C \delta_{\text{min}},$$  \hspace{1cm} (2.3)

for some finite constant $C \geq 1$. We denote by $\Pi(t)$ the set of all partitions of $[0, t]$ satisfying (2.3) and such that $\delta$ converges to zero when $n$ tends to infinity. We denote by $\Pi(t, \delta_0)$ the set of all partitions of $[0, t]$ satisfying (2.3), such that $\delta$ converges to zero when $n$ tends to infinity and $\delta < \delta_0$.

**Remark 1.** Under the assumption (2.3) one has that

$$n \leq t \delta_{\text{min}}^{-1} \leq C t \delta^{-1}. \hspace{1cm} (2.4)$$

To simplify the notation, we will add an additional component to the Brownian motion $Y$. Let $Y^0$ be the process $Y^0_s = s$, for all $s \geq 0$ and consider the $(d_{\nu} + 1)$-dimensional process $Y = (Y^i)_{i=0}^{d_{\nu}}$. Then the martingale $Z = (Z_t)_{t \geq 0}$ defined in (2.1) can be written as $Z_t = \exp(\xi_t)$, $t \geq 0$, where

$$\xi_t = \frac{d_{\nu}}{2} \sum_{i=0}^{d_{\nu}} \int_0^t h_i(X_s) dY^i_s, \quad t \geq 0,$$

and $h_0 = -\frac{1}{2} \sum_{i=1}^{d_{\nu}} h_i^2$. For $\tau \in \Pi(t)$ and $m \in \mathbb{N}$ we consider the processes

$$\xi^{\tau, m}_t \triangleq \sum_{j=0}^{n-1} \xi^{\tau, m}_t(j) \triangleq \sum_{j=0}^{n-1} \sum_{i=0}^{d_{\nu}} \sum_{\alpha \in \mathcal{M}_{m-1}(S_0)} L^\alpha h_{t}(X_{t_j}) \int_{t_j}^{t_{j+1}} I_{\alpha}(1)_{t_j, s} dY^i_s \bigg|_{s = \tau(j)}$$

$$= \sum_{i=0}^{d_{\nu}} \int_0^t \left\{ \sum_{\alpha \in \mathcal{M}_{m-1}(S_0)} L^\alpha h_{t}(X_{t_s}) I_{\alpha}(1)_{t_s, \tau(s)} \right\} dY^i_s.$$

For $m > 2$, we can write

$$\xi^{\tau, m}_t = \xi^{\tau, 2}_t + \sum_{j=0}^{n-1} \mu^{\tau, m}_t(j),$$

where

$$\mu^{\tau, m}_t(j) \triangleq \sum_{i=0}^{d_{\nu}} \sum_{\alpha \in \mathcal{M}_{1, m-1}(S_0)} L^\alpha h_t(X_{t_j}) \int_{t_j}^{t_{j+1}} I_{\alpha}(1)_{t_j, s} dY^i_s,$$

and

$$\mathcal{M}_{1, m-1}(S_0) \triangleq \mathcal{M}_{m-1}(S_0) \setminus \mathcal{M}_1(S_0)$$

$$= \{ \alpha : |\alpha| \in [2, m - 1], \alpha_k \in \{0, ..., d_{\nu}\}, k = 1, ..., |\alpha| \}.$$

The processes $\xi^{\tau, m}_t$ are obtained by replacing $h_i(X_s)$ in the formula for the process $\xi$ with the truncation of degree $(m - 1)$ of the corresponding stochastic Taylor expansion of $h_i(X_s)$. They are used to produce discretization schemes of order 1 and 2 for $\pi_t(\varphi)$. They *cannot* be used to produce discretization schemes of order $m > 2$ as they don’t have finite exponential moments (required to define the discretization schemes). More precisely, the quantities $\mu^{\tau, m}_t(j)$ do not have finite exponential moments because of the high order iterated integral.
involved. For this, we need to introduce a truncation of $\mu_{\tau,m}(j)$ resulting in a (partial) taming procedure to the stochastic Taylor expansion of $h_i(X)$. We define the processes

$$\xi_{t}^{\tau,m} \triangleq \sum_{j=0}^{n-1} \xi_{t}^{\tau,m}(j),$$

where

$$\xi_{t}^{\tau,i}(j) = \begin{cases} \xi_{t}^{\tau,i}(j) & \text{if } i = 1, 2 \\ \xi_{t}^{\tau,2}(j) + \Gamma_{m-\frac{1}{2} \delta_{j}}(\mu_{\tau,m}(j)) & \text{if } i > 2 \\ \end{cases}$$

for $j = 0, ..., n - 1$ with the truncation function $\Gamma$ being defined as

$$\Gamma(q, \delta) = \frac{z}{1 + (z/\delta)^2}, \quad z \in \mathbb{R}$$

for some $\delta > 0$ and $q \in \mathbb{N}$. Finally, for $\tau \in \Pi(t)$ and $m \in \mathbb{N}$ consider the processes $Z^{\tau,m} = (Z_{t}^{\tau,m})_{t \geq 0}$ given by

$$Z_{t}^{\tau,m} = \exp(\xi_{t}^{\tau,m}).$$

For any Borel measurable function $\varphi$ such that $\varphi(X_t) Z_{t}^{\tau,m} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ define the $m$-th order discretizations

$$\rho_{t}^{\tau,m}(\varphi) \triangleq \mathbb{E}[\varphi(X_t) Z_{t}^{\tau,m} | \mathcal{Y}_t],$$

and

$$\pi_{t}^{\tau,m}(\varphi) \triangleq \rho_{t}^{\tau,2}(\varphi)/\rho_{t}^{\tau,m}(1),$$

of $\rho_{t}$ and $\pi_{t}$, respectively.

Let $m \in \mathbb{N}$, our main assumption is the following:

**Assumption ($H(m)$).** We have that:

- $f = (f_i)_{i=1, \ldots, d_X} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_X} \in \mathcal{B}_b \cap C_{2+2(2m-1)}$,
- $\sigma = (\sigma_{i,j})_{i=1, \ldots, d_X, j=1, \ldots, d_Y} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_X \times d_Y} \in \mathcal{B}_b \cap C_{2m}$,
- $h = (h_i)_{i=0, \ldots, d_Y} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_Y + 1} \in \mathcal{B}_b \cap C_{2m+1}$,
- $X_0$ has moments of all orders.

Note that if assumption $H(m)$ holds for some $m \in \mathbb{N}$, then it also holds for any $n \leq m$.

**Theorem 2.** Let assumption $H(m)$ be satisfied. Then, there exists constants $C_0, C > 0$ not depending on the choice of the partition $\tau \in \Pi(t, \delta_0)$, such that

$$\|\rho_{t}(\varphi) - \rho_{t}^{\tau,m}(\varphi)\|_2 \leq C\delta^m,$$

for $\varphi \in C_{P}^{m+1}$. Moreover, if $\sup_{\tau \in \Pi(t, \delta_0)} \|\pi_{t}^{\tau,m}(\varphi)\|_{2+\varepsilon} < \infty$, for some $\varepsilon > 0$, then

$$\mathbb{E}\left[\|\pi_{t}(\varphi) - \pi_{t}^{\tau,m}(\varphi)\|\right] \leq \tilde{C}\delta^m,$$

where $\tilde{C}$ is another constant independent of $\tau \in \Pi(t, \delta_0)$.

**Remark 3.** The assumption $\sup_{\tau \in \Pi(t, \delta_0)} \mathbb{E}\left[|\pi_{t}^{\tau,m}(\varphi)|^{2+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$ is satisfied if $\varphi$ is bounded. If $\varphi$ is unbounded, note that by using Jensen’s inequality one has

$$\mathbb{E}\left[|\pi_{t}^{\tau,m}(\varphi)|^{2+\varepsilon}\right] = \mathbb{E}\left[\mathbb{E}\left[|\varphi(X_t) Z_{t}^{\tau,m}| \mathcal{Y}_t\right]^{2+\varepsilon}\right]$$

$$\leq \mathbb{E}\left[|\varphi(X_t)|^{2+\varepsilon} \exp((2+\varepsilon)(\xi_{t}^{\tau,m} - \mathbb{E}[\xi_{t}^{\tau,m} | \mathcal{Y}_t]))\right] .$$

Hence, one can reason as in Lemma 13 to justify that $\sup_{\tau \in \Pi(t, \delta_0)} \mathbb{E}\left[|\pi_{t}^{\tau,m}(\varphi)|^{2+\varepsilon}\right] < \infty$. 

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Remark 4.
i. In the case \( m = 1 \) we can consider any partition \( \tau \in \Pi(t) \). For \( m \geq 2 \), we must consider partitions \( \tau \) with mesh \( \delta \) smaller than
\[
\delta_0 = \frac{1}{2 \| Lh \|_\infty \sqrt{dy \cdot dy}},
\]
where
\[
\| Lh \|_\infty \triangleq \max_{i=1,\ldots, dY} \| L^r h^i \|_\infty.
\]

ii. The functional discretization given in (2.5) is recursive. More precisely, if \( \tau' \in \Pi(t + s) \) is a partition that includes \( t \) as an intermediate point, for example \( \tau' \triangleq \{ 0 = t_0 < \cdots < t_k = t < t_{k+1} \cdots < t_n = t + s \} \) with \( 0 < k < n \), then
\[
Z_{t+s}^{\tau',m} = Z_{t}^{\tau,m} \prod_{j=k}^{n-1} \exp \left( \xi_{t}^{\tau,m}(j) \right).
\]

This property is essential for implementation purposes as at every discretization time we only need to use the previous functional discretization and the term corresponding to the next interval to obtain the new functional discretization.

iii. The discretization introduced by Picard in [15] corresponds to the case \( m = 1 \). In this case, \( \rho_t^{\tau,m} \) can be explicitly written as
\[
\rho_t^{\tau,m}(\varphi) \triangleq \mathbb{E} \left[ \varphi(X_t) \exp \left( \sum_{j=0}^{n-1} \sum_{i=0}^{dy} h_i(X_{t_j}) \left( Y_{t_{j+1}}^i - Y_{t_j}^i \right) \right) \right],
\]
(2.7)

This discretization scheme leads to a wealth of numerical methods that can be used to approximate \( \pi_t \). Among them, particle methods\(^1\) are algorithms which approximate \( \pi_t \) with discrete random measures of the form \( \sum_i a_i(t) \delta_{v_i(t)} \), in other words with empirical distributions associated with sets of randomly located particles of stochastic masses \( a_1(t), a_2(t), \ldots \), which have stochastic positions \( v_1(t), v_2(t), \ldots \). These methods are currently among the most successful and versatile for numerically solving the filtering problem. Based on (2.7), the “garden variety” particle filter uses particles that evolve according to the signal equation (or, rather, the Euler approximation of the signal) and carry exponential weights. These weights are proportional with
\[
\exp \left( \sum_{i=0}^{dy} h_i(v_{t_n}^i) \left( Y_{t_{n+1}}^i - Y_{t_n}^i \right) \right),
\]
where \( v^i \) is the process modelling the trajectory of the particle and \( t_n \) is the update time. The method also involves a variance reduction procedure (for further details, see for example Chapter 9 in [2]). Alternatively one can use a cubature method to approximate the law of the signal, see [6]. In both cases, higher order approximations of the signal can be used, but this would not improve the rate of convergence of the method as Picard’s discretisation has an error of order 1. The remedy is to exploit the result in this paper and use a higher order discretisation. The second author is working on a particle filter that uses the second order discretisation presented in this paper.

\(^1\)Also known as particle filters or sequential Monte Carlo methods.
3 Proof of the main result

We start by recalling and introducing some basic results on iterated integrals and martingale representations. Throughout the rest of the paper we will be assuming that $H(m)$ holds, without recalling it in each result statement. Moreover, $C$ will denote a constant that usually depends on $d_Y, d_X, d_Y, f, \sigma, h$ and possibly other parameters but NOT on the partition $\tau$. As we are interested in showing a rate of convergence for our approximations, the particular form of dependence of $C$ with respect to these parameters is not relevant and, hence, omitted. Of course, the choice of the constant $C$ may change from line to line.

Remark 5. Some immediate consequences of assumption $H(m)$ are the following:

1. The signal process $X$ has moments of all orders and for any $p \geq 1$, we have
   \[
   \mathbb{E} \left[ \sup_{s \in [0,t]} |X_s|^p \right] < \infty,
   \]
   for all $i \in \{1, \ldots, d_X\}$.

2. If $\Upsilon : \mathbb{R}^{d_X} \to \mathbb{R}$ is a function with polynomial growth we have
   \[
   \mathbb{E} \left[ \sup_{s \in [0,t]} |\Upsilon(X_s)|^p \right] < \infty,
   \]
   in particular,
   \[
   \mathbb{E} \left[ \sup_{s \in [0,t]} |L^\alpha h_i(X_s)|^p \right] < \infty,
   \]
   for $i = 0, \ldots, d_Y$ and $\alpha \in \mathcal{M}_m(S_0) = \mathcal{M}_{m-1}(S_0) \mathcal{R} (\mathcal{M}_{m-1}(S_0))$.

3. The processes $\xi_t$ and $\xi_{t,m}^\tau$, $m \in \mathbb{N}$, as defined above have finite moments of all orders.

Remark 6. Consider the truncation function $\Gamma_{q,\delta}(z) = \frac{z}{1 + (z/\delta)^{2q}}$ defined as above corresponding to the real parameters $q \geq 1$ and $\delta > 0$.

1. For any $z \in \mathbb{R}$, $|\Gamma_{q,\delta}(z)| \leq \delta$. To check this observe that if $|z| \leq \delta$ we have that $1 + (z/\delta)^{2q} \geq 1$ and then
   \[
   |\Gamma_{q,\delta}(z)| = \frac{|z|}{1 + (z/\delta)^{2q}} \leq |z| \leq \delta.
   \]
   On the other hand, if $|z| > \delta$ we have that $|z/\delta|^{-1} + |z/\delta|^{2q-1} > 1$ and then
   \[
   |\Gamma_{q,\delta}(z)| = \frac{|z|}{1 + |z|^{2q} \delta^{-2q}} = \frac{1}{|z|^{-1} + |z|^{2q-1} \delta^{-2q}} = \frac{\delta}{|z/\delta|^{-1} + |z/\delta|^{2q-1} \delta^{-2q}} \leq \delta.
   \]

2. Moreover, if we define
   \[
   \mathcal{E}_{q,\delta}(z) \triangleq \Gamma_{q,\delta}(z) - z,
   \]
   we get that
   \[
   |\mathcal{E}_{q,\delta}(z)| = |\Gamma_{q,\delta}(z) - z| = \left| \frac{z}{1 + (z/\delta)^{2q}} - z \right| = \frac{|z|^{2q+1} \delta^{-2q}}{1 + (z/\delta)^{2q}} = \frac{|z|^{2q+1}}{\delta^{2q} + z^{2q}} \leq \delta^{-2q} |z|^{2q+1}, \quad \forall z \in \mathbb{R}.
   \]

(3.1)
3. Finally, note that for \( m \geq 3 \) we can write
\[
\bar{\xi}_{t,m} = \xi_{t,2} + \sum_{j=0}^{n-1} \Gamma_{m-\frac{1}{2},\delta_j} (\mu_{t,m}^{\gamma, \theta})(j)
\]
Equation 3.2

\[= \xi_{t,m} + \sum_{j=0}^{n-1} \mathcal{E}_{m-\frac{1}{2},\delta_j} (\mu_{t,m}^{\gamma, \theta})(j)\]  \(\text{(3.2)}\)

3.1 Iterated integrals

The following two results are well known and can be found in Kloeden and Platen [10], Theorem 5.5.1 and Lemma 5.6.5, respectively.

**Theorem 7.** Let \( \rho_1 \) and \( \rho_2 \) be two stopping times with \( 0 \leq \rho_1 \leq \rho_2 \leq t \), a.s., let \( \mathcal{A} \subset \mathcal{M}(S_0) \) be a hierarchical set and \( g : \mathbb{R}^d \rightarrow \mathbb{R} \). Then, the Itô-Taylor expansion
\[
g(X_{\rho_2}) = \sum_{\alpha \in \mathcal{A}} L^\alpha g(X_{\rho_1}) I_\alpha(1)_{\rho_1,\rho_2} + \sum_{\alpha \in \mathcal{R}(\mathcal{A})} I_\alpha(L^\alpha g(X))_{\rho_1,\rho_2}, \] \(\text{(3.3)}\)

holds, provided all of the derivatives of \( g, f \) and \( \sigma \) and all of the iterated Itô integrals appearing in (3.3) exist.

**Lemma 8.** Let \( \alpha \in \mathcal{M}(S_0) \), let \( \theta = \{ \theta_s \}_{s \in [0,t]} \) be an \( \mathbb{F}^t \)-adapted process, let \( p \geq 1 \) and let \( \rho_1 \) and \( \rho_2 \) be two stopping times with \( 0 \leq \rho_1 \leq \rho_2 \leq t \) and \( \rho_2 \) being \( \mathcal{H}_{\rho_1}^t \)-measurable. Then,
\[
\hat{\mathbb{E}} \left[ \left| I_\alpha(\theta)_{\rho_1,\rho_2} \right|^{2p} | \mathcal{H}_{\rho_1}^t \right] \leq C R(\theta)(\rho_2 - \rho_1)^p|\alpha| + |\alpha_0|,
\]
where
\[
R(\theta) = \hat{\mathbb{E}} \left[ \sup_{\rho_1 \leq s \leq \rho_2} |\theta_s|^{2p} | \mathcal{H}_{\rho_1}^t \right].
\]

The following lemma gives a basic estimate on the difference between the log likelihood functional \( \xi_t \) and its \( m \)-th order discretization. Its proof relies on Theorem 7 and Lemma 8.

**Lemma 9.** We have that
\[
\xi_t - \xi_{t,m} = \sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0))} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY^i_s, \] \(\text{(3.4)}\)

and
\[
\hat{\mathbb{E}} \left[ |\xi_t - \xi_{t,m}|^{2p} \right] \leq C \delta^{pm}.
\]

**Proof.** By Remark 5, we can apply Theorem 7 and get equation 3.4. Applying the Itô isometry and Jensen’s inequality (or Jensen’s inequality directly if \( i = 0 \)), we obtain the following bound
\[
\hat{\mathbb{E}} \left[ |\xi_t - \xi_{t,m}|^{2p} \right] = \hat{\mathbb{E}} \left[ \sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0))} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY^i_s \right]^{2p}
\]
\[
\leq C \sum_{\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0))} \int_0^t \hat{\mathbb{E}} \left[ I_\alpha(L^\alpha h_i(X))_{\tau(s),s}^{2p} \right] ds.
\]
Let $\alpha \in \mathcal{R} (\mathcal{M}_{m-1}(S_0))$, by Lemma 8 and Remark 5 we get that

$$
\mathbb{E} \left[ |I_\alpha(L^\alpha h_i(X_s))_{\tau(s),s}|^{2p} \right] \leq C \mathbb{E} \left[ \sup_{\tau(s) \leq u \leq s} |L^\alpha h_i(X_u)|^{2p} \right] (s - \tau(s))^{p|\alpha|+|\alpha|_0} \leq C \delta^{\rho m},
$$

where in the last inequality we have used that $|\alpha| + |\alpha|_0 \geq m$ for $\alpha \in \mathcal{R} (\mathcal{M}_{m-1}(S_0))$. From the previous inequality the result follows easily.

**Lemma 10.** Let $p, q \geq 1$ and $m \geq 3$. Then, we have that

$$
\mathbb{E} \left[ \sum_{j=0}^{n-1} \mathcal{E}_{q,\delta_j} (\mu^r,m) (j) \right]^{2p} \leq C (t, d_Y, p, m) \delta^{(2q+1)}.
$$

**Proof.** We have that

$$
\mathbb{E} \left[ \sum_{j=0}^{n-1} \mathcal{E}_{q,\delta_j} (\mu^r,m) (j) \right]^{2p} \leq C (p) n^{2p-1} \sum_{j=0}^{n-1} \mathbb{E} \left[ \mathcal{E}_{q,\delta_j} (\mu^r,m) (j) \right]^{2p}.
$$

Moreover, using similar arguments as in Lemma 9, for any $r \geq 1$, we have that

$$
\mathbb{E} \left[ |\mu^r,m (j)|^r \right] \leq C (d_Y, m) \delta^{(1+|\alpha|+|\alpha|_0)} \leq C (d_Y, m) \delta^{2r},
$$

because as $\alpha \in \mathcal{M}_{1,m-1}(S_0), m \geq 3$ then $|\alpha| \in [2, m - 1]$ and $|\alpha|_0 \in [0, m - 1]$. Then, using equation (3.1) and Remark 1 we get that

$$
n^{2p-1} \sum_{j=0}^{n-1} \mathbb{E} \left[ \mathcal{E}_{q,\delta_j} (\mu^r,m) (j) \right]^{2p} \leq n^{2p-1} \sum_{j=0}^{n-1} \delta^{-4pq} \mathbb{E} \left[ |\mu^r,m (j)|^{2p(2q+1)} \right] \leq C (t, d_Y, m) n \delta^{-(2p-1)\delta^{-4pq}\delta^{(2q+1)}} \leq C (t, d_Y, m) \delta^{(2q+1)}.
$$

**Lemma 11.** Let $\theta = \{\theta_s\}_{s \in [0,t]}$ and $\Psi = \{\Psi_s\}_{s \in [0,t]}$ be two $\mathbb{H}^{t'}$-adapted process. Then:

1. For $\alpha \in \mathcal{M}(S_0)$ and $0 \leq s_1 \leq s_2 \leq s_3 \leq t$, we have that

$$
\mathbb{E} \left[ I_\alpha (\theta)_{s_2,s_3} | \mathcal{H}^{t}_{s_1} \right] = 1_{\{|\alpha| = |\alpha|_0\}} I_\alpha \left( \mathbb{E}[\theta | \mathcal{H}^{t}_{s_1}] \right)_{s_2,s_3}. 
$$

2. For $\alpha \in \mathcal{M}(S_0)$ with $|\alpha| \neq |\alpha|_0, r \in \{1, \ldots, d_Y\}, 0 \leq s_1 \leq s_2 \leq t$ and $0 \leq s_3 \leq s_4 \leq t$ we have that

$$
\mathbb{E} \left[ \left( \int_{s_3}^{s_4} \Psi_s dV_s \right) I_\alpha (\theta)_{s_1,s_2} | \mathcal{Y}_t \right] = 1_{\{|\alpha|_0 = 0\}} \int_{s_1}^{s_2} \mathbb{E} \left[ \left( \int_{s_3}^{s_4} \Psi_s dV_s \right) I_\alpha - (\theta)_{s_1,u} | \mathcal{Y}_t \right] du
$$

$$
+ 1_{\{|\alpha|_0 = r\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_4} \mathbb{E} \left[ \Psi_u I_\alpha - (\theta)_{s_1,u} | \mathcal{Y}_t \right] du.
$$
3. For $\alpha \in \mathcal{M}(S_0)$ with $|\alpha| \geq 2, \alpha_{|\alpha|} \neq 0, \alpha_{|\alpha|-1} \neq 0, r_1, r_2 \in \{1, ..., d_V\}, 0 \leq s_1 \leq s_2 \leq t$ and $0 \leq s_3 \leq s_4 \leq s_5 \leq s_6 \leq t$ we have that

$$
\mathbb{E} \left[ \left( \int_{s_1}^{s_6} \int_{s_3}^{s_5} \Psi s_{s_4}dV^r_{s_4}dV^r_{s_5} \right) I_{\alpha}(\theta)_{s_1,s_2}\left| Y_1 \right. \right] = 1_{\{\alpha_{|\alpha|} = r_2\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_6} \mathbb{E} \left[ \left( \int_{s_3}^{s_5} \Psi s_{s_4}dV^r_{s_4} \right) I_{\alpha_-(\theta)}_{s_1,u}\left| Y_1 \right. \right] du
$$

and

$$
= 1_{\{\alpha_{|\alpha|} = r_2, \alpha_{|\alpha|-1} = r_1\}} \int_{s_1 \vee s_3}^{u} \int_{s_1 \vee s_3}^{u} \mathbb{E} \left[ \Psi I_{\alpha_-(\theta)}_{s_1,v}\left| Y_1 \right. \right] dvdu
$$

Proof.
1. If $|\alpha| \neq |\alpha|_0$, then the iterated integral $I_{\alpha}(\theta)_{s_2,s_3}$ contains a Brownian differential $dV^r$ and it vanishes when we take the conditional expectation with respect to $\mathcal{H}_{s_1}^t$. If $|\alpha| = |\alpha|_0$, all the differentials in the iterated integral $I_{\alpha}(\theta)_{s_2,s_3}$ are Lebesgue differentials and we can write the conditional expectation inside the inner integral.

2. Note that if $\alpha_{|\alpha|} = 0$ we can write

$$
\mathbb{E} \left[ \left( \int_{s_3}^{s_4} \Psi s_{s_4}dV^r_{s_4} \right) \int_{s_2}^{s_3} I_{\alpha_-(\theta)}_{s_1,u}\left| Y_1 \right. \right] = \mathbb{E} \left[ \int_{s_2}^{s_3} \left( \int_{s_3}^{s_4} \Psi s_{s_4}dV^r_{s_4} \right) I_{\alpha_-(\theta)}_{s_2,u}\left| Y_1 \right. \right],
$$

because we can push the random variable $\left( \int_{s_3}^{s_4} \Psi s_{s_4}dV^r_{s_4} \right)$ inside the Lebesgue integral. If $\alpha_{|\alpha|} \neq 0$ we can write

$$
\mathbb{E} \left[ \left( \int_{s_3}^{s_4} \Psi s_{s_4}dV^r_{s_4} \right) \int_{s_1}^{s_2} I_{\alpha_-(\theta)}_{s_1,u,dV_{u}^{\alpha_{|\alpha|}}}\left| Y_1 \right. \right]
$$

$$
= \mathbb{E} \left[ \mathbb{E} \left[ \left( \int_{s_3}^{s_4} \Psi s_{s_4}dV^r_{s_4} \right) \int_{s_1}^{s_2} I_{\alpha_-(\theta)}_{s_1,u,dV_{u}^{\alpha_{|\alpha|}}}\left| \mathcal{H}_0 \right. \right] \left| Y_1 \right. \right]
$$

$$
= 1_{\{\alpha_{|\alpha|} = r\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_4} \mathbb{E} \left[ \Psi I_{\alpha_-(\theta)}_{s_1,u}\left| Y_1 \right. \right] du
$$

where we have just applied the $\mathcal{H}_{s_1}^t$-semimartingale covariation formula.

3. Same reasoning as for statement 2. \hfill \square

### 3.2 Integrability of the likelihood functional and its discretizations

In this section we state some integrability results for the likelihood functional and its discretizations. The first result is on the integrability of the likelihood functional. It follows from the basic fact that any Gaussian distribution has exponential moments of all orders.

**Lemma 12.** Assume that $\mathbf{H}(1)$ holds. Let $p \geq 1$ and $\tau$ be any partition. Then, one has that

$$
\mathbb{E} \left[ |Z_t|^p \right] = \mathbb{E} \left[ \exp(p|\xi_t|) \right] \leq \mathbb{E} \left[ \exp(p |\xi_t|) \right] < \infty,
$$

and

$$
\mathbb{E} \left[ |Z_t^{\tau,1}|^p \right] = \mathbb{E} \left[ \exp(p|\xi_t^{\tau,1}|) \right] < \infty.
$$
Proof. We have that
\[ \mathbb{E} \left[ \exp(p \mid \xi_t \mid) \right] = \mathbb{E} \left[ \exp \left( p \sum_{i=1}^{d_Y} \left( \int_0^t h^i(X_s) dY_s^i \right) - \frac{p}{2} \sum_{i=1}^{d_Y} \left( \int_0^t h^i(X_s)^2 ds \right) \right) \right] \]
\[ \leq \exp \left( \frac{p}{2} d_Y \|h\|_\infty^2 t \right) \mathbb{E} \left[ \exp \left( p \sum_{i=1}^{d_Y} \left( \int_0^t h^i(X_s) dY_s^i \right) \right) \right] . \]

Recall that if \( Z \sim \mathcal{N}(0, \sigma^2) \) under \( \hat{P} \), then
\[ \mathbb{E} \left[ e^{\mid Z \mid} \right] = 2e^{\frac{\sigma^2}{2} \phi(p\sigma)} , \]
where \( \phi \) is the cumulative distribution function of a standard normal random variable. As \( Y \) is a Brownian motion independent of \( X \) under \( \hat{P} \), we have that
\[ \mathbb{E} \left[ \exp \left( p \sum_{i=1}^{d_Y} \left( \int_0^t h^i(X_s) dY_s^i \right) \right) \right] = \mathbb{E} \left[ \exp \left( p \sum_{i=1}^{d_Y} \left( \int_0^t h^i(X_s) dY_s^i \right) \mid \mathcal{F}_t^{0,V} \right) \right] \]
\[ = 2 \exp \left( \frac{p^2}{2} \sum_{i=1}^{d_Y} \int_0^t h^i(X_s)^2 ds \right) \phi \left( p \left( \sum_{i=1}^{d_Y} \int_0^t h^i(X_s)^2 ds \right)^{1/2} \right) \]
\[ \leq 2 \exp \left( \frac{p^2}{2} d_Y \|h\|_\infty^2 t \right) , \]
and we can conclude that \( \mathbb{E} \left[ \exp(p \mid \xi_t \mid) \right] < \infty \). The proof that \( \mathbb{E} \left[ \mid Z_t^{\tau,2} \mid^p \right] < \infty \) follows by similar arguments.

The following lemma ensures the \( L^p(\Omega) \) integrability of the second order discretization of the likelihood function, provided the discretization is done on a sufficiently fine partition. We give a bound on the mesh of the partition in terms of \( p \), the uniform bounds on the sensor function \( h \) and its derivatives and the dimensions of the noise driving the signal and the observation process. The proof is based on the fact that the square of a centered Gaussian random variable has finite exponential moment of order sufficiently small.

**Lemma 13.** Assume that \( \textbf{H}(2) \) holds. Let \( p \geq 1 \) and \( \tau \) be a partition with mesh size
\[ \delta < \left( p \|Lh\|_\infty \sqrt{d_Y d_V} \right)^{-1} , \]
where
\[ \|Lh\|_\infty \triangleq \max_{i=1,\ldots,d_Y} \|L^r h^i\|_\infty . \]
Then, one has that
\[ \mathbb{E} \left[ \mid Z_t^{\tau,2} \mid^p \right] = \mathbb{E} \left[ \exp(p \xi^{\tau,2}) \right] < \infty . \]

**Proof.** We can write \( \exp(p \xi^{\tau,2}) \triangleq \prod_{i=1}^{d_Y} \left( K_t^{\tau,2,i} \right)^p \), where
\[ K_t^{\tau,2,i} \triangleq \exp \left( \sum_{i=1}^{d_Y} \sum_{r=1}^{d_Y} \int_0^t L^r h^i(X_{\tau(s)}) (V_{s} - V_{\tau(s)}) dV_s^i \right) , \]
We get exponential moments of any order. The law of \( F \) have finite moments of all orders. In what follows, let \( \varepsilon > 0 \) because

Let \( \varepsilon > 0 \), then, by Hölder inequality, we have

\[
\mathbb{E} \left[ \exp \left( p \xi_{t}^{2} \right) \right] \leq \mathbb{E} \left[ \left| K_{t}^{\tau,1.1} \right|^{p(1+\varepsilon)} \right]^{\frac{1}{1+\varepsilon}} \mathbb{E} \left[ \prod_{i=2}^{4} \left| K_{t}^{\tau,1.1} \right|^{p(1+\varepsilon)} \right]^{\frac{1}{1+\varepsilon}}.
\]

Hence, the result follows by showing that \( K_{t}^{\tau,1.1} \) has finite \( p(1+\varepsilon) \)-moment and

\[
\mathbb{E} \left[ \prod_{i=2}^{4} \left| K_{t}^{\tau,1.1} \right|^{p(1+\varepsilon)} \right] < \infty.
\]

(3.5)

Applying Hölder inequality twice, condition (3.5) follows by showing that \( K_{t}^{\tau,2.1}, i = 2, \ldots, 4 \) have finite moments of all orders. In what follows, let \( q \geq 1 \) be a fixed real constant. We start by the easiest term, \( K_{t}^{\tau,2.3} \). We have that

\[
\mathbb{E} \left[ \left| K_{t}^{\tau,2.3} \right|^{q} \right] \leq \exp \left( \frac{qdy}{2} t \left( \| h \|_{\infty}^{2} + \delta \| L_{0}h_{2} \|_{\infty} \right) \right) < \infty,
\]

because \( \| h \|_{\infty} \) and \( \| L_{0}h_{2} \|_{\infty} = \max_{i=1,\ldots,dy} \| L_{0}(h_{2}^{i}) \|_{\infty} \) are finite due to the assumptions on \( f, \sigma \) and \( h \). For the term \( K_{t}^{\tau,2.4} \), we can write

\[
\mathbb{E} \left[ \left| K_{t}^{\tau,2.4} \right|^{q} \right] \leq \mathbb{E} \left[ \exp \left( \frac{qdy}{2} t \left( \| L(h_{2}) \|_{\infty} \int_{0}^{t} \left| V_{s}^{1} - V_{\tau(s)}^{1} \right| ds \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{qdy}{2} t \left( \| L(h_{2}) \|_{\infty} \left( \int_{0}^{t} (s - \tau(s)) ds \right) \left| V_{1}^{1} \right| \right) \right] \\
\leq \mathbb{E} \left[ \exp \left( \frac{qdy}{2} t \left( \| L(h_{2}) \|_{\infty} \sqrt{\delta} \left| V_{1}^{1} \right| \right) \right] < \infty,
\]

because \( \| L(h_{2}) \|_{\infty} = \max_{i=1,\ldots,dy} \| L^{*}(h_{2}^{i}) \|_{\infty} \) is finite, the law of \( V_{s}^{1} - V_{\tau(s)}^{1} \) coincides with the law of \( (s - \tau(s))^{1/2}V_{1}^{1} \) by the scaling properties of the Brownian motion and \( |V_{1}^{1}| \) has exponential moments of any order.

For the term \( K_{t}^{\tau,2.2} \), we first condition with respect to \( \mathcal{F}_{s}^{V} = \sigma(V_{s}, 0 \leq s \leq t) \) and use the fact that, conditionally to \( \mathcal{F}_{t}^{V} \), the stochastic integrals with respect to \( Y \) are Gaussian. We get

\[
\mathbb{E} \left[ \left| K_{t}^{\tau,2.2} \right|^{q} \right] = \mathbb{E} \left[ \exp \left( q \sum_{i=1}^{dy} \int_{0}^{t} \left\{ h_{1}^{i}(X_{\tau(s)}) + L_{0}h_{1}^{i}(X_{\tau(s)})(s - \tau(s)) \right\} dY_{s}^{i} \right) \left| \mathcal{F}_{t}^{V} \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{q}{2} \sum_{i=1}^{dy} \int_{0}^{t} \left\{ h_{1}^{i}(X_{\tau(s)}) + L_{0}h_{1}^{i}(X_{\tau(s)})(s - \tau(s)) \right\}^{2} ds \right) \right]
\]

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\[ = \exp(q^2d_Y t \{ \| h \|_\infty^2 + \| L^0 h \|_\infty^2 \}) < \infty. \]

Finally, the term \( K_t^{r,2,1} \) is more delicate because, in order to show that has finite \((p + \varepsilon)\)-moment, a relationship between the mesh of the partition \( \delta \) and \( p + \varepsilon \) is needed. Proceeding as with the term \( K_t^{r,2,2} \), we obtain

\[
\mathbb{E} \left[ |K_t^{r,2,1}|^{p(1 + \varepsilon)} \right] = \mathbb{E} \left[ \exp \left( p(1 + \varepsilon) \sum_{i=1}^{d_r} \int_0^t L^r h^i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) dY_s^i \right) \right] \\
= \mathbb{E} \left[ \prod_{i=1}^{d_r} \mathbb{E} \left[ \exp \left( \int_0^t p(1 + \varepsilon) \sum_{i=1}^{d_r} L^r h^i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) dY_s^i \right) \bigg| \mathcal{F}_t^V \right] \right].
\]

Now, conditionally to \( \mathcal{F}_t^V \), the terms in the exponential are centered Gaussian random variables and we get that

\[
\mathbb{E} \left[ |K_t^{r,2,1}|^{p(1 + \varepsilon)} \right] = \mathbb{E} \left[ \prod_{i=1}^{d_r} \exp \left( \frac{p^2(1 + \varepsilon)^2}{2} \int_0^t \left( \sum_{i=1}^{d_r} L^r h^i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) \right)^2 ds \right) \right] \\
\leq \mathbb{E} \left[ \prod_{i=1}^{d_r} \exp \left( \frac{p^2(1 + \varepsilon)^2}{2} \int_0^t \left( \sum_{i=1}^{d_r} L^r h^i(X_{\tau(s)}) \right)^2 (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{p^2(1 + \varepsilon)^2}{2} \int_0^t (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right] \\
= \mathbb{E} \left[ \exp \left( \frac{p^2(1 + \varepsilon)^2}{2} \int_0^t (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right]^{d_r}.
\]

So we need to find conditions on \( \beta > 0 \), such that \( \mathbb{E} \left[ \exp \left( \beta \int_0^t (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right] < \infty \). We can write

\[
\mathbb{E} \left[ \exp \left( \beta \int_0^t (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right] = \mathbb{E} \left[ \exp \left( \beta \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (V_s^r - V_{t_{j-1}}^r)^2 ds \right) \right] \\
= \prod_{j=1}^n \mathbb{E} \left[ \exp \left( \beta \int_{t_{j-1}}^{t_j} (V_s^r - V_{t_{j-1}}^r)^2 ds \right) \right] \\
\triangleq \prod_{j=1}^n \Theta (\beta, \delta_j).
\]

Denote by \( M_t \triangleq \sup_{0 \leq s \leq t} V_s^1 \) and recall that the density of \( M_t \) is given by

\[
f_{M_t} (x) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} 1_{(0,\infty)},
\]

see Karatzas and Shreve [9], page 96. Moreover, note that for any \( A > 0 \),

\[
\frac{2}{\sqrt{2\pi \sigma^2}} \int_0^\infty \exp \left\{ -A \frac{x^2}{2\sigma^2} \right\} dx = A^{-1/2}.
\]

Then, we have that

\[
\Theta (\beta, \delta_j) \leq \mathbb{E}[\exp(\beta \delta_j M_t^2)] = \int_0^\infty \frac{2}{\sqrt{2\pi \delta_j}} \exp \left\{ \beta \delta_j x^2 - \frac{x^2}{2\delta_j} \right\} dx = A^{-1/2}.
\]
\[
\int_0^\infty \frac{2}{\sqrt{2\pi \delta_j}} \exp \left\{ - \left(1 - 2\beta \delta_j^2 \right) \frac{x^2}{2\delta_j} \right\} = \left(1 - 2\beta \delta_j^2 \right)^{-1/2} < \infty,
\]
as long as \(1 - 2\beta \delta_j^2 > 0\). On the other hand,
\[
(1 - 2\beta \delta_j^2)^{-1} = \sum_{k=0}^{\infty} (2\beta \delta_j^2)^k = 1 + 2\beta \delta_j^2 \left( \sum_{k=0}^{\infty} (2\beta \delta_j^2)^k \right) \\
\leq 1 + 2\beta \delta_j^2 \left( \sum_{k=0}^{\infty} (2\beta \delta_j^2)^k \right) = 1 + \frac{2\beta \delta_j^2}{1 - 2\beta \delta_j^2} \\
\leq \exp \left( \frac{2\beta \delta_j^2}{1 - 2\beta \delta_j^2} \right),
\]
and, therefore,
\[
\prod_{j=1}^n \Theta (\beta, \delta_j) \leq \prod_{j=1}^n \exp \left( \frac{\beta \delta_j^2}{1 - 2\beta \delta_j^2} \right) \leq \exp \left( \frac{\beta \sum_{j=1}^n \delta_j^2}{1 - 2\beta \delta_j^2} \right) \\
\leq \exp \left( \frac{\beta \delta}{1 - 2\beta \delta_j^2} \right) < \infty.
\]

As \(\beta = \frac{p^2(1+\varepsilon)^2 dy dv \|Lh\|^2_2}{2}\) and \(\varepsilon > 0\) can be made arbitrary small we get the following condition for the partition mesh \(\delta < \left( p \|Lh\|_\infty \sqrt{dy dv} \right)^{-1} \).

We complete the section with an application of the previous two lemmas. Note that, in order to control the high order discretizations of the likelihood function, we reduce the problem to the control of the second order discretization via the truncation procedure as described in Remark 6.

**Corollary 14.** Let \(\varphi \in B_p\). One has that:

1. If \(\textbf{H} (1)\) holds, then there exists \(\varepsilon > 0\) such that
\[
\tilde{E} \left[ |\varphi(X_t)e^{\xi_t}|^{2+\varepsilon} \right] < \infty,
\]
and
\[
\sup_{\tau \in \Pi(t)} \tilde{E} \left[ |\varphi(X_t)e^{\xi_{t-1}}|^{2+\varepsilon} \right] < \infty.
\]

2. If \(\textbf{H} (2)\) holds, then there exists \(\varepsilon > 0\) and \(\delta_0 = \delta_0 (h, f, \sigma, ) > 0\) such that
\[
\sup_{\tau \in \Pi(t, \delta_0)} \tilde{E} \left[ |\varphi(X_t)e^{\xi_{t-2}}|^{2+\varepsilon} \right] < \infty.
\]

3. If \(\textbf{H} (m)\) with \(m \geq 3\) holds, then there exists \(\varepsilon > 0\) and \(\delta_0 > 0\) such that
\[
\sup_{\tau \in \Pi(t, \delta_0)} \tilde{E} \left[ |\varphi(X_t)e^{\xi_{t-m}}|^{2+\varepsilon} \right] < \infty.
\]
Proof. Combining Lemmas 12 and 13 with Hölder inequality and Remark 5 we obtain (3.6), (3.7) and (3.8). Moreover, for \( m \geq 3 \), note that

\[
\bar{\xi}_t^{\tau,m} = \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \Gamma_{m-\frac{1}{2},\delta_j} (\mu^{\tau,m}(j)) \\
\leq \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \Gamma_{m-\frac{1}{2},\delta_j} (\mu^{\tau,m}(j)) \\
\leq \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \delta_j = \xi_t^{\tau,2} + t,
\]

and (3.9) follows from (3.8). \( \square \)

### 3.3 Proof of the Theorem 2

In this section we prove the main theorem of the paper. We start by stating and proving two main propositions.

**Proposition 15.** Let \( m \in \mathbb{N} \) and assume that condition \( H(m) \) holds and \( \varphi \in C^m_{\mathbb{P}} \). Then, there exists a constant \( C \) independent of the partition \( \pi \in \Pi \) such that

\[
\mathbb{E} \left[ \mathbb{E} \left[ (\xi_t - \xi_t^{\tau,m}) \varphi(X_t)e^{\xi_t} | \mathcal{Y}_t \right] \right]^2 \leq C \delta^{2m}.
\]

Proof. By Lemma 9 we can write

\[
(\xi_t - \xi_t^{\tau,m}) \varphi(X_t)e^{\xi_t} = \varphi(X_t)e^{\xi_t} \left( \sum_{i=0}^{dY_i} \int_0^t \left\{ \sum_{\alpha \in R(M_{m-1}(S_0))} I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} \right\} dY_s^i \right).
\]

For \( i = 0 \), the result follows from Lemmas 34 and 35. Recall that

\[
\mathcal{R}(M_{m-1}(S_0))_k = \{ \alpha \in \mathcal{R}(M_{m-1}(S_0)) : |\alpha|_0 = k \}, \quad k = 0, \ldots, m,
\]

that is, \( \mathcal{R}(M_{m-1}(S_0))_k \) is the set of multi-indices in \( \mathcal{R}(M_{m-1}(S_0)) \) that contain \( k \) zeros. This collection of sets are obviously a disjoint partition of \( \mathcal{R}(M_{m-1}(S_0)) \), that is,

\[
\mathcal{R}(M_{m-1}(S_0)) = \bigcup_{k=0}^m \mathcal{R}(M_{m-1}(S_0))_k.
\]

For \( i \neq 0 \), we will divide the proof of the theorem in cases, depending on \( \alpha \) belonging to one of these subsets. The cases for \( m \in \{1,2\} \) are:

- \( m = 1, \alpha \in \mathcal{R}(M_0(S_0))_0 \): Lemma 37.
- \( m = 1, \alpha \in \mathcal{R}(M_0(S_0))_1 \): Lemma 34.
- \( m = 2, \alpha \in \mathcal{R}(M_1(S_0))_0 \): Lemma 38.
- \( m = 2, \alpha \in \mathcal{R}(M_1(S_0))_1 \): Lemma 39.
- \( m = 2, \alpha \in \mathcal{R}(M_1(S_0))_2 \): Lemma 34.

For arbitrary \( m > 2 \), the proof follows the same ideas as for \( m \in \{1,2\} \). In the case that \( \alpha \in \mathcal{R}(M_{m-1}(S_0))_m \) the result follows from applying Lemma 34. For \( \mathcal{R}(M_{m-1}(S_0))_k \) with \( k \in \{0, m-1\} \), first one needs to use the truncated Stroock-Taylor formula of order \( k \) to express \( \varphi(X_t)e^{\xi_t} \) as a sum of iterated integrals with respect to the Brownian motion. The goal is to use the covariance between the iterated integrals in the Stroock-Taylor expansion of \( \varphi(X_t)e^{\xi_t} \) and \( I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} \) in order to generate the right order of convergence in
\( \delta \). However, this is not straightforward due to the presence of the stochastic integral with respect to \( Y \). The process \( Y \) as an integrator makes impossible to use directly an integration by parts formula because the two iterated integrals are semimartingales with respect to different filtrations. To overcome this difficulty, the idea is to compute this covariance along a partition. We use an integration by parts formula, in each subinterval and only to the integral with respect to \( Y \), to obtain

\[
\int_{t_j}^{t_{j+1}} I_{\alpha}(L^\alpha h_t(X))_{r(s),s} dY^i_s = \int_{t_j}^{t_{j+1}} \left( Y^i_{t_{j+1}} - Y^i_{t_j} \right) I_{\alpha}(L^\alpha h_t(X))_{r(s),s} dV^{\alpha,|i|}_s.
\]

The term on the right hand side in the last expression is an \( \mathbb{H}^q \)-semimartingale and we can compute its covariation with the terms in the Stroock-Taylor expansion of \( \varphi(X_t)e^{\xi_t} \), see Lemmas 37, 38 and 39.

**Proposition 16.** Let \( m \in \mathbb{N} \) and assume that condition \( H(m) \) holds and \( \varphi \in \mathcal{B}_p \). Then, there exist \( \delta_0 > 0 \) and constant \( C \) independent of any partition \( \pi \in \Pi(t, \delta_0) \) such that

\[
\mathbb{E} \left[ \left| \varphi(X_t) \right| \left( e^{\xi_t} + e^{\tilde{\xi}_t} \right) (\xi_t - \tilde{\xi}_t)^2 \right] \leq C\delta^{2m}.
\]

**Proof.** As \( H(m) \) holds, let \( \varepsilon > 0 \) such as in Corollary 14. By Hölder inequality we have that

\[
\mathbb{E} \left[ \left| \varphi(X_t) \right| \left( e^{\xi_t} + e^{\tilde{\xi}_t} \right) (\xi_t - \tilde{\xi}_t)^2 \right] \\
\leq \mathbb{E} \left[ \left| \varphi(X_t) \right| \left( e^{\xi_t} + e^{\tilde{\xi}_t} \right) \right]^{2+\varepsilon} 2^{1-2\varepsilon} \mathbb{E} \left[ (\xi_t - \tilde{\xi}_t)^4 \right]^{1+\varepsilon/2} 2^{\varepsilon/2}.
\]

Corollary 14 yields that there exists \( \delta_0 > 0 \) such that

\[
\sup_{\tau \in \Pi(t, \delta_0)} \mathbb{E} \left[ \left| \varphi(X_t) \right| \left( e^{\xi_t} + e^{\tilde{\xi}_t} \right) \right]^{2+\varepsilon} < \infty.
\]

On the other hand, by equation (3.2), for any \( p \geq 1 \) we get that

\[
|\xi_t - \tilde{\xi}_t|^2 \leq C \left\{ |\xi_t - \tilde{\xi}_t|^2 + \sum_{j=0}^{n-1} \mathcal{E}_{m-\frac{1}{2}}(\mu_t) \right\}.
\]

By Lemma 9, we obtain

\[
\mathbb{E} \left[ |\xi_t - \tilde{\xi}_t|^{2p} \right] \leq C\delta^{pm},
\]

and by Lemma 10 with \( q = m - \frac{1}{2} \) we have that

\[
\mathbb{E} \left[ \sum_{j=0}^{n-1} \mathcal{E}_{m-\frac{1}{2}}(\mu_t) \right]^{2p} \leq C(t, \delta, p, m) \delta^{2pm}
\]

Hence, setting \( p = 2(2+\varepsilon)/\varepsilon \), we obtain

\[
\mathbb{E} \left[ |\xi_t - \tilde{\xi}_t|^{4(2+\varepsilon)/\varepsilon} \right]^{\varepsilon/(2+\varepsilon)} \leq C\delta^{2m}.
\]

\( \square \)
We are finally ready to put everything together and deduce Theorem 2.

**Proof of Theorem 2.** To get the desired rate of convergence for the unnormalised conditional distribution $\rho_t^{\tau,m}$, we can write

\[
\rho_t(\varphi) - \rho_t^{\tau,m}(\varphi) = \mathbb{E}[\varphi(X_t)(\xi_t - \tilde{\xi}_t^{\tau,m})e^{\xi_t}|\mathcal{Y}_t]
\]

\[
+ \hat{\mathbb{E}}\left[\varphi(X_t)e^{\xi_t} - \varphi(X_t)e^{\tilde{\xi}_t^{\tau,m}} - \varphi(X_t)(\xi_t - \tilde{\xi}_t^{\tau,m})e^{\xi_t}|\mathcal{Y}_t\right].
\]

Using the inequality

\[
|e^x - e^y - (x - y)e^x| \leq \frac{e^x + e^y}{2}(x - y)^2,
\]

we get that

\[
\hat{\mathbb{E}}\left[|\rho_t(\varphi) - \rho_t^{\tau,m}(\varphi)|^2\right]
\]

\[
\leq C\left\{\hat{\mathbb{E}}\left[\mathbb{E}\left[|(\xi_t - \tilde{\xi}_t^{\tau,m})\varphi(X_t)e^{\xi_t}|\mathcal{Y}_t\right]|^2\right] + \hat{\mathbb{E}}\left[|\varphi(X_t)|\frac{e^{\xi_t} + e^{\tilde{\xi}_t^{\tau,m}}}{2}(\xi_t - \tilde{\xi}_t^{\tau,m})^2\right]\right\}.
\]

Now, Propositions 15 and 16 yield

\[
\hat{\mathbb{E}}\left[|\rho_t(\varphi) - \rho_t^{\tau,m}(\varphi)|^2\right] \leq C\delta^{2m}.
\]

To prove the rate for the normalised conditional distribution observe that we can write

\[
\pi_t^{\tau,m}(\varphi) - \pi_t(\varphi) = \frac{1}{\rho_t(1)}\frac{\rho_t^{\tau,m}(\varphi)}{\rho_t^{\tau,m}(1)}(\rho_t(1) - \rho_t^{\tau,m}(1)) + \frac{1}{\rho_t(1)}(\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)).
\]

Hence,

\[
\mathbb{E}[|\pi_t(\varphi) - \pi_t^{\tau,m}(\varphi)|]
\]

\[
\leq C\hat{\mathbb{E}}\left[\frac{Z_t}{|\rho_t(1)|}\left||\pi_t^{\tau,m}(\varphi)||\rho_t(1) - \rho_t^{\tau,m}(1)| + |\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|\right||\frac{Z_t|\mathcal{Y}_t}{\rho_t(1)}\right]
\]

\[
= C\hat{\mathbb{E}}\left[\frac{Z_t}{|\rho_t(1)|}\left||\pi_t^{\tau,m}(\varphi)||\rho_t(1) - \rho_t^{\tau,m}(1)| + |\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|\right||\frac{Z_t|\mathcal{Y}_t}{\rho_t(1)}\right]
\]

\[
\leq C\left\{\hat{\mathbb{E}}\left[|\pi_t^{\tau,m}(\varphi)||\rho_t(1) - \rho_t^{\tau,m}(1)| + \hat{\mathbb{E}}[|\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|]\right]\right\}.
\]

\[
\leq C\left\{\hat{\mathbb{E}}\left[|\pi_t^{\tau,m}(\varphi)|^2\right]^{1/2}\hat{\mathbb{E}}\left[|\rho_t(1) - \rho_t^{\tau,m}(1)|^2\right]^{1/2} + \hat{\mathbb{E}}\left[|\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|^2\right]^{1/2}\right\},
\]

where in the last inequality we have applied Hölder inequality. Combining the bounds for the unnormalised distribution and the hypothesis on $\pi_t^{\tau,m}(\varphi)$ we can conclude. \[\square\]

### 4 Technical Lemmas

We collate in this section the technical lemmas required to prove the main results. We begin with some limited background material on Malliavin Calculus (and partial Malliavin Calculus) with a view to deduce the necessary properties of the functionals to be discretised.
4.1 Malliavin calculus

Let $B = \{B_t\}_{t \in [0,T]}$ be a $d$-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $S$ denote the class of smooth random variables such that a random variable $F \in S$ has the form

$$F = f (B_{t_1}; \ldots; B_{t_n}),$$

where the function $f \left( x^{i_1}, \ldots, x^{i_n}; \ldots; x^{i_1}, \ldots, x^{i_n} \right)$ belongs to $C^\infty_b (\mathbb{R}^{dn})$ and $t_1, \ldots, t_n \in [0, T]$. The Malliavin derivative of a smooth functional $F$ can be defined as the $d$-dimensional stochastic processes given by

$$(DF)_t^j = \sum_{i=1}^n \frac{\partial f}{\partial x^{ji}} (B_{t_1}; \ldots; B_{t_n}) 1_{[0,t_i]} (t),$$

for $t \in [0, T]$ and $j = 1, \ldots, d$. The derivative $DF$ can be regarded as a random variable taking values in the Hilbert space $H = L^2 ([0, T]; \mathbb{R}^d)$. Noting the isometry $L^2 (\Omega \times [0, T]; \mathbb{R}^d) \simeq L^2 (\Omega; H)$ we can identify $(DF)_t^j$ as the value at time $t$ of the $j$th component of and $\mathbb{R}^d$-valued stochastic process. We will also the notation $D^1_{\mathbb{D}} F$ for $(DF)_t^j$. One can see that the operator $D$ is closable from $L^p (\Omega)$ to $L^p (\Omega; H)$, $p \geq 1$ and we will denote the domain of $D$ in $L^p (\Omega)$ by $\mathbb{D}^{1,p}$. That is, meaning that $\mathbb{D}^{1,p}$ is the closure of smooth random variables $S$ with respect to the norm

$$\|F\|_{\mathbb{D}^{1,p}} = (\mathbb{E} [\|F\|^p] + \mathbb{E} [\|DF\|^p])^{1/p}.$$

We define the $k$-th derivative of $F$, $D^k F$, as the $H^{\otimes k}$-valued random variable

$$(D^k F)_{s_1, \ldots, s_k} = \sum_{i_1, \ldots, i_k=1}^n \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}} (B_{t_1}; \ldots; B_{t_n}) 1_{[0,t_{i_1}]} (s_1) \cdots 1_{[0,t_{i_k}]} (s_k),$$

where $s_1, \ldots, s_k \in [0, T]$ and $i_1, \ldots, i_k = 1, \ldots, d$. We will also write $D^1_{s_1, \ldots, s_k} F$ for $(D^k F)_{s_1, \ldots, s_k}$ and notice that it coincides with the iterated derivative $D^1_{t_1} \cdots D^k_{t_k} F$. For any integer $k \geq 1$ and any real number $p > 1$ we introduce the norm on $S$ given by

$$\|F\|_{k,p} = \left( (\mathbb{E} [\|F\|^p] + \sum_{j=1}^k \mathbb{E} [\|D^j F\|_{H^{\otimes j}}^p])^{1/p} \right),$$

where

$$\|D^k F\|_{H^{\otimes k}} = \left( \sum_{j_1 \ldots j_k=1}^d \int_{[0,T]^k} |D^j_{s_1, \ldots, s_k} F|^2 ds_1 \cdots ds_k \right)^{1/2}.$$ 

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of random variables $S$ with respect to the norm $\|\cdot\|_{k,p}$. We also define the space $\mathbb{D}^{k,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{k,p}$. We have the following chain rule formula for the Malliavin derivative.

**Proposition 17.** Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be of class $C^1_b (\mathbb{R}^m)$. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to $\mathbb{D}^{1,\infty}$. Then, $\varphi (F) \in \mathbb{D}^{1,\infty}$ and

$$D^j_{t} \varphi (F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x^{i}} (F) D^j_{t} F^{i},$$

where $t \in [0, T]$ and $j = 1, \ldots, d$. 

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Proof. The proof follows the same ideas as the proof of Proposition 1.2.3 in Nualart [12], where is proved for \( \varphi \in C^1_b (\mathbb{R}^m) \) and \( F \in D^{1,p} \). One can extend the result to \( \varphi \in C^1_P (\mathbb{R}^m) \) by requiring \( F \in D^{1,\infty} \) and using Hölder inequality. □

As a corollary of Proposition 17 one obtains that the product rule and the binomial formula holds for the Malliavin derivative of products of random variables in \( D^{1,\infty} \). However, Proposition 17 does not apply to the exponential function. In order to show that the likelihood functional \( e^{\xi t} \) is smooth in the Malliavin sense we need the following lemma.

**Lemma 18.** Let \( F \in D^{1,\infty} \) and such that

\[
\mathbb{E} [\exp (p |F|)] < \infty, \tag{4.1}
\]

for all \( p \geq 1 \). Then \( G = e^F \in D^{1,\infty} \) and

\[
D^j_t G = G D^j_t F, \tag{4.2}
\]

where \( t \in [0,T] \) and \( j = 1, \ldots, d \).

**Proof.** Define \( G_n = \sum_{k=0}^n \frac{F^k}{k!} \). As \( F \in D^{1,\infty} \), Proposition 17 yields that \( G_n \in D^{1,\infty} \) and

\[
DG_n = \sum_{k=1}^n F^{k-1} \frac{DF}{k!} = G_{n-1} DF.
\]

In order to prove that \( G \in D^{1,\infty} \) and that the identity (4.2) is satisfied, it suffices to show that for all \( p \geq 1 \) one has that \( G_n \) converges to \( G \) in \( L^p (\Omega) \) and

\[
\mathbb{E} [\|DG - DG_n\|_{L^p}^p] \rightarrow 0,
\]

when \( n \) tends to infinity. Note that

\[
\mathbb{E} [\|DG - DG_n\|_{L^p}^p] = \mathbb{E} \left[ \left( G - G_{n-1} \right)^p \int_0^T |D_t F|^2 \, dt \right]^{p/2} \leq \mathbb{E} \left[ \left( G - G_{n-1} \right)^{2p} \right]^{1/2} \mathbb{E} \left[ \|DF\|_{L^2([0,T])}^{2p} \right]^{1/2}.
\]

Hence, the problem is reduced to show that \( G_n \) converges to \( G \) in \( L^p \) for all \( p \geq 1 \). Equivalently, defining

\[
G_n^c := G - G_n = \sum_{k=n+1}^\infty \frac{F^k}{k!},
\]

it suffices to prove that \( G_n^c \) converges to 0 in \( L^p \) for all \( p \geq 1 \). Clearly, \( G_n^c \) converges to 0 almost surely and, thanks to assumption (4.1), the dominated convergence theorem yields that \( G_n^c \) also converges to 0 in \( L^p (\Omega) \) for all \( p \geq 1 \). □

We also have the following relationship between the conditional expectation and the Malliavin derivative.

**Lemma 19.** Let \( F \in D^{1,2} \) and \( \mathcal{F}_t = \{ \mathcal{F}_t \}_{t \in [0,T]} \) be the \( P \)-augmented natural filtration generated by \( B \). Then \( \mathbb{E} [F | \mathcal{F}_t] \in D^{1,2} \) and \( D^j_s \mathbb{E} [F | \mathcal{F}_t] = \mathbb{E} [D^j_s F | \mathcal{F}_t] 1_{[0,t]} (s), j = 1, \ldots, d \).

**Proof.** The lemma is a particular case of Proposition 1.2.8 in Nualart [12]. □
The following is an important result regarding the Malliavin differentiability of the solution of a stochastic differential equation.

**Lemma 20.** If $X_t \in \mathbb{R}^n$ is the solution to

$$X_t = x + \int_0^t V_0(X_s) \, ds + \int_0^t V(X_s) \, dB_s,$$

where the components of $V_0$ and $V$ are $m$-times continuously differentiable with bounded derivatives of order greater or equal than one and $B_t = (B^1_t, \ldots, B^d_t)$ is a $d$-dimensional Brownian motion. Then, $X^i_t \in \mathbb{D}^{m, \infty}, t \in [0, T], i = 1, \ldots, n$. Furthermore, for any $p \geq 1$ one has that

$$\sup_{r_1, r_2, \ldots, r_k \in [0, T]} \mathbb{E} \left[ \sup_{r_1 \leq t \leq r_k} \left| D^{|\alpha|}_{r_1, \ldots, r_k} X^i_t \right|^p \right] < \infty,$$

for all $p \geq 1, i = 1, \ldots, n, j_k \in \{1, \ldots, d\}$ and $1 \leq k \leq m$.

**Proof.** See Nualart [12], Theorem 2.2.1. and 2.2.2. \qed

**Remark 21.** We will be using a variation of the classical Malliavin calculus known as partial Malliavin calculus. This calculus was introduced in Kusuoka and Stroock [11] and Nualart and Zakai [13] with a view towards its application to the stochastic filtering problem, see also Tanaka [17]. The idea is to consider only the Malliavin derivative operator with respect some of the components of the Brownian motion $B$. In our setting $B = (V, Y)$ is a $d_Y$-dimensional Brownian motion under $\mathbb{P}$ and the Malliavin differentiation will be only with respect to the Brownian motion $V$. The main consequence of this approach is that the Malliavin derivative with respect to $V$ commutes with the stochastic integral with respect to $Y$.

**Lemma 22.** Let $m \in \mathbb{N}$ and assume that $H(m)$ holds and $\varphi \in C^{m+1}_P$. Then, the random variable $\varphi(X_t)e^{\xi_t}$ belongs to $\mathbb{D}^{m+1, \infty}$. Moreover,

$$\sup_{r_1, \ldots, r_m \in [0, T]} \mathbb{E} \left[ \left| D^{|\alpha|}_{r_1, \ldots, r_m} (\varphi(X_t)e^{\xi_t}) \right|^p \right] < \infty,$$

for all $p \geq 1$ and $\alpha \in \mathcal{M}_{m+1}(S_t)$.

**Proof.** To ease the notation we are only going to give the proof for $d_Y = d_V = d_X = 1$.

We will also use the notation $D^k_{r_1, \ldots, r_k} F = D^k_{r_1, \ldots, r_k} F$. Lemma 20 yields that $X_t \in \mathbb{D}^{1, \infty}$. Applying iteratively Proposition 17 we obtain that $\varphi(X_t) \in \mathbb{D}^{m+1, \infty}$ and $h(X_t) \in \mathbb{D}^{m+1, \infty}$. Taking into account Remark 21, we have that $\xi_t \in \mathbb{D}^{m+1, \infty}$. Moreover, thanks to Lemma 12, we can apply iteratively Lemma 19 and conclude that $e^{\xi_t} \in \mathbb{D}^{m+1, \infty}$. For any $\alpha \in \mathcal{M}_{m+1}(S_t)$, by Leibniz’s rule, we can write

$$D^{|\alpha|}_{r_1, \ldots, r_m} (\varphi(X_t)e^{\xi_t}) = D^{|\alpha|}_{r_1, \ldots, r_m} (\varphi(X_t)) e^{\xi_t}$$

$$= \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} (D^k_{r_1, \ldots, r_k} \varphi(X_t)) (D^{|\alpha|-k}_{r_1, \ldots, r_m-k} e^{\xi_t}),$$

and applying Schwartz’s inequality one has that

$$\mathbb{E} \left[ \left| D^{|\alpha|}_{r_1, \ldots, r_m} (\varphi(X_t)e^{\xi_t}) \right|^p \right] \leq C \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} \mathbb{E} \left[ \left| (D^k_{r_1, \ldots, r_k} \varphi(X_t)) (D^{|\alpha|-k}_{r_1, \ldots, r_m-k} e^{\xi_t}) \right|^p \right],$$

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Lemma 31, we can bound

\[ \leq C \sum_{k=0}^{\lvert \alpha \rvert} \binom{\lvert \alpha \rvert}{k} \mathbb{E} \left[ \left\lvert D_{r_1, \ldots, r_k}^k \varphi (X_t) \right\rvert^{2p} \right]^{1/2} \mathbb{E} \left[ \left\lvert D_{r_1, \ldots, r_{\lvert \alpha \rvert - k}}^{\lvert \alpha \rvert - k} e^{\xi_t} \right\rvert^{2p} \right]^{1/2}. \]

Hence, the result follows if we show that

\[ \sup_{r_1, \ldots, r_k \in [0, t]} \mathbb{E} \left[ \left\lvert D_{r_1, \ldots, r_k}^k \varphi (X_t) \right\rvert^p \right] < \infty, \quad 0 \leq k \leq \lvert \alpha \rvert, \quad (4.3) \]

\[ \sup_{r_1, \ldots, r_k \in [0, t]} \mathbb{E} \left[ \left\lvert D_{r_1, \ldots, r_k}^k e^{\xi_t} \right\rvert^p \right] < \infty, \quad 0 \leq k \leq \lvert \alpha \rvert, \quad (4.4) \]

for any \( p \geq 1. \)

\( \triangleright \) **Proof of (4.3):**

If \( k = 0, \) using that \( \mathbf{H} (m) \) holds and \( \varphi \in C_{\mathcal{P}}^{m+1}, \) we have that \( \mathbb{E} \left[ \lvert \varphi (X_t) \rvert^p \right] < \infty, \) by Remark 5. If \( 1 \leq k \leq \lvert \alpha \rvert, \) we use Faà di Bruno’s formula to obtain an expression for \( D_{r_1, \ldots, r_k}^k \varphi (X_t) \) in terms of the so called partial Bell polynomials, which are given by

\[ B_{k,a}(x_1, \ldots, x_k) = \sum_{(j_1, \ldots, j_k) \in \Lambda (k, a)} \frac{k!}{j_1! (1!)^{j_1} j_2! (2!)^{j_2} \cdots j_k! (k!)^{j_k}} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}, \]

where \( 1 \leq a \leq k \) and

\[ \Lambda (k, a) = \{(j_1, \ldots, j_k) \in \mathbb{Z}_+^k : j_1 + 2j_2 + \cdots + kj_k = k, j_1 + j_2 + \cdots + j_k = a\}. \]

In particular, we have that

\[ D_{r_1, \ldots, r_k}^k \varphi (X_t) = \sum_{a=1}^{k} \varphi^{(a)} (X_t) B_{k,a}(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \ldots, D_{r_1, \ldots, r_k}^k X_t). \]

Hence, for any \( p \geq 1, \) applying Cauchy-Schwarz inequality we get

\[
\mathbb{E} \left[ \left\lvert D_{r_1, \ldots, r_k}^k \varphi (X_t) \right\rvert^p \right] 
\leq C \sum_{a=1}^{k} \mathbb{E} \left[ \lvert \varphi^{(a)} (X_t) \rvert B_{k,a}(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \ldots, D_{r_1, \ldots, r_k}^k X_t) \rvert^p \right] 
\leq C \sum_{a=1}^{k} \mathbb{E} \left[ \lvert \varphi^{(a)} (X_t) \rvert^{2p} \right]^{1/2} \mathbb{E} \left[ \left\lvert B_{k,a}(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \ldots, D_{r_1, \ldots, r_k}^k X_t) \right\rvert^{2p} \right]^{1/2}. 
\]

The terms \( \mathbb{E} \left[ \lvert \varphi^{(a)} (X_t) \rvert^{2p} \right] < \infty, a = 1, \ldots, k, \) due to Remark 5 combined with that \( \mathbf{H} (m) \) holds and \( \varphi \in C_{\mathcal{P}}^{m+1}. \) On the other hand, using the generalized version of Hölder’s inequality, Lemma 31, we can bound

\[ \mathbb{E} \left[ \lvert B_{k,a}(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \ldots, D_{r_1, \ldots, r_k}^k X_t) \rvert^{2p} \right], \quad 1 \leq a \leq k, \]

by a sum of products of expectations of powers of Malliavin derivatives of \( X \) of different orders. Combining this bound with Lemma 20 we get that the integrability condition (4.3) is satisfied.

\( \triangleright \) **Proof of (4.4):**

If \( k = 0, \) we have that \( \mathbb{E} \left[ \lvert e^{\xi_t} \rvert^p \right] < \infty \) due to Lemma 12. If \( 1 \leq k \leq \lvert \alpha \rvert, \) using again Faà di Bruno’s formula we get

\[ D_{r_1, \ldots, r_k}^k e^{\xi_t} = \sum_{a=1}^{k} \frac{d^a}{dx^a} \bigg|_{x=\xi_t} B_{k,a}(D_{r_1}^1 \xi_t, D_{r_1, r_2}^2 \xi_t, \ldots, D_{r_1, \ldots, r_k}^k \xi_t). \]
\[
\sum_{a=1}^{k} \exp(\xi_t) B_{k,a}(D_{r_1,\ldots,r_a}^1, D_{r_1,\ldots,r_a}^2, \ldots, D_{r_1,\ldots,r_a}^k, \xi_t).
\]

We can repeat exactly the same arguments as in the proof of (4.3), due to the fact that by Lemma 12 \(e^\xi_t\) has moment of all orders, provided we can show that

\[
\sup_{r_1,\ldots,r_a \in [0,t]} \mathbb{E}[|D_{r_1,\ldots,r_a}^a \xi_t|^p] < \infty, \quad 1 \leq a \leq k, \quad (4.5)
\]

for any \(p \geq 1\). As noted in Remark 21, the Malliavin derivative commute with the stochastic integral with respect to \(Y\) and we can write

\[
D_{r_1,\ldots,r_a}^a \xi_t = D_{r_1,\ldots,r_a}^a \left( \int_0^t h(X_s) dY_s - \frac{1}{2} \int_0^t h^2(X_s) ds \right)
= \int_0^t D_{r_1,\ldots,r_a}^a h(X_s) dY_s - \frac{1}{2} \int_0^t D_{r_1,\ldots,r_a}^a (h^2(X_s)) ds.
\]

Hence, by Burkholder-Davis-Gundy inequality and Jensen’s inequality, we get for any \(p \geq 1\) that

\[
\mathbb{E}[|D_{r_1,\ldots,r_a}^a \xi_t|^{2p}] \
\leq C \left\{ \mathbb{E} \left[ \left( \int_0^t D_{r_1,\ldots,r_a}^a h(X_s) dY_s \right)^{2p} \right] + \mathbb{E} \left[ \left( \int_0^t D_{r_1,\ldots,r_a}^a (h^2(X_s)) ds \right)^{2p} \right] \right\}
\leq C \left\{ \mathbb{E} \left[ \left( \int_0^t D_{r_1,\ldots,r_a}^a h(X_s) \right)^{2p} ds \right] + \mathbb{E} \left[ \left( \int_0^t D_{r_1,\ldots,r_a}^a (h^2(X_s))^{2p} ds \right) \right] \right\}
\leq C \{ A_1 + A_2 \}.
\]

Applying Faà di Bruno formula we can write

\[
A_1 \leq C \sum_{l=1}^{a} \int_0^t \mathbb{E}[|h^{(l)}(X_s) B_{a,l}(D_{r_1}^1 X_s, D_{r_1,r_2}^2 X_s, \ldots, D_{r_1,\ldots,r_a}^a X_s)|^{2p}] ds
\leq C \|h\|_{\infty,a}^q \sum_{l=1}^{a} \int_0^t \mathbb{E}[|B_{a,l}(D_{r_1}^1 X_s, D_{r_1,r_2}^2 X_s, \ldots, D_{r_1,\ldots,r_a}^a X_s)|^{2p}] ds,
\]

where

\[
\|h\|_{\infty,a}^q \triangleq \sum_{i=0}^{dy} \sum_{l=0}^{a} \|h_i^{(l)}\|_{\infty} < \infty;
\]

because \(H(m)\) holds. Therefore, using the generalized version of Hölder inequality, Lemma 31, and Lemma 20 we get \(A_1 < \infty\). We can repeat the same argument for \(A_2\) and obtain (4.5).

\[\square\]

4.2 Martingale representations and Clark-Ocone formula

In this section we recall the Clark-Ocone formula. This formula relates the kernels in the Itô martingale representation of Malliavin differentiable functionals with the Malliavin derivatives of such functionals. We present a truncated version of the well known Stroock-Taylor formula, see Stroock [16], that can be seen as an extension of the Clark-Ocone formula and it will be essential in deducing several conditional expectation estimates (see Section
4.4). We also show that, if the coefficients $f$ and $\sigma$ of the SDE modeling the signal, the sensor function $h$ and the test function $\varphi$ are regular enough with bounded derivatives then, the kernels in the truncated Stroock-Taylor formula for $\varphi(X_t) e^{\xi_t}$ satisfy a uniform integrability property. Finally, we show that those kernels also satisfy a Hölder continuity property.

**Theorem 23.** Let $F \in L^2(\Omega, \mathcal{H}_t^1, \tilde{P})$. Then, $F$ admits the following martingale representation

$$F = \mathbb{E}[F|\mathcal{H}_0^t] + \sum_{r=1}^{d_V} \int_0^t J_s^r dV_s^r,$$

where $J^r = \{J^r_s, s \in [0, t]\}, r = 1, ..., d_V$ are $\mathcal{H}_s^t$-progressively measurable processes such that

$$\mathbb{E} \left[ \int_0^t |J^r_s|^2 ds \right] < \infty, \quad r = 1, ..., d_V.$$

Moreover, if $F \in \mathbb{D}^{1,2}$ then

$$J^r_s = \mathbb{E} [D_s^r F|\mathcal{H}_s^t], \quad s \in [0, t],$$

which is known as the Clark-Ocone formula.

**Proof.** The proof is similar to that of Lemma 17 in Crisan [4] and the proof of the Clark-Ocone formula can be found in Nualart [12], Proposition 1.3.14. \(\square\)

By applying Theorem 23 to the kernels $J^r, r = 1, ..., d_V$ one can get the following result.

**Theorem 24** (Stroock-Taylor formula of order $m$). Assume that $F \in L^2(\Omega, \mathcal{H}_t^1, \tilde{P})$. Then, for $m \in \mathbb{N}$ we can write

$$F = \sum_{\beta \in \mathcal{M}_{m-1}(S_1)} I_\beta \left( \mathbb{E} \left[ J^\beta_{s_1, ..., s_{|\beta|}} |\mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{R}(\mathcal{M}_{m-1}(S_1))} I_\beta \left( J^\beta_{s_1, ..., s_{|\beta|}} \right)_{0,t},$$

where the kernels $J^\beta_{s_1, ..., s_{|\beta|}}$ for $\beta \in \mathcal{M}_m(S_1)$ are obtained from the martingale representation of $J^\beta_{s_2, ..., s_{|\beta|}}$, that is, they satisfy the following relationship

$$J^\nu \triangleq F,$$

$$J^{-\beta}_{s_2, ..., s_{|\beta|}} = \mathbb{E} \left[ J^{-\beta}_{s_2, ..., s_{|\beta|}} |\mathcal{H}_0^t \right] + \sum_{\beta_1=1}^{d_V} \int_0^{s_2} J^{\beta_{1, s_{|\beta|}}}_{s_1, ..., s_{|\beta|}} dV_{s_1}.$$

Moreover, if $\varphi(X_t) e^{\xi_t} \in \mathbb{D}^{m,2}$ then

$$J^{\beta}_{s_1, ..., s_{|\beta|}} = \mathbb{E} \left[ D^{\beta}_{s_1, ..., s_{|\beta|}} F|\mathcal{H}_s^t \right], \quad \beta \in \mathcal{M}_m(S_1).$$

**Proof.** We prove the result by induction. For $m = 1$, the result is precisely Theorem 23. We assume that the result holds for $m-1 \geq 0$ and prove that this implies that it also holds for $m$. By the induction hypothesis we have that

$$F = \sum_{\beta \in \mathcal{M}_{m-1}(S_1)} I_\beta \left( \mathbb{E} \left[ J^\beta_{s_1, ..., s_{|\beta|}} |\mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{R}(\mathcal{M}_{m-2}(S_1))} I_\beta \left( J^\beta_{s_1, ..., s_{|\beta|}} \right)_{0,t}. $$
Applying Theorem 23 to $J_{s_1,\ldots,s_{|\beta|}}^\beta$, $\beta \in \mathcal{R}(\mathcal{M}_{m-2}(S_1))$ we get

$$F = \sum_{\beta \in \mathcal{M}_{m-2}(S_1)} I_\beta \left( \hat{\mathbb{E}} \left[ J_{s_1,\ldots,s_{|\beta|}}^\beta | \mathcal{H}_0 \right] \right)_{0,t} + \sum_{\beta \in \mathcal{R}(\mathcal{M}_{m-2}(S_1))} I_\beta \left( \hat{\mathbb{E}} \left[ J_{s_1,\ldots,s_{|\beta|}}^\beta | \mathcal{H}_0 \right] \right)_{0,t}$$

$$+ \sum_{\beta \in \mathcal{R}(\mathcal{M}_{m-2}(S_1))} \sum_{r=1}^{d_V} I_{rS_2} \left( J_{s_1,\ldots,s_{|\beta|}}^{rS_2} \right)_{0,t}$$

$$= \sum_{\beta \in \mathcal{M}_{m-1}(S_1)} I_\beta \left( \hat{\mathbb{E}} \left[ J_{s_1,\ldots,s_{|\beta|}}^\beta | \mathcal{H}_0 \right] \right)_{0,t} + \sum_{\beta \in \mathcal{R}(\mathcal{M}_{m-1}(S_1))} I_\beta \left( J_{s_1,\ldots,s_{|\beta|}}^\beta \right)_{0,t},$$

where in the last equality we have used that

$$\mathcal{M}_{m-1}(S_1) = \mathcal{M}_{m-2}(S_1) \cup \mathcal{R}(\mathcal{M}_{m-2}(S_1)),$$

the definitions of $\mathcal{R}(\mathcal{M}_{m-1}(S_1))$ and the concatenation of multi-indices.

The Clark-Ocone representation of the kernels also follows from a straightforward induction. \hfill \Box

**Proposition 25.** Let $m \in \mathbb{N}$ and assume that $H(m)$ holds and $\varphi \in C_{p}^{m+1}$. Then, the kernels $J_{s_1,\ldots,s_{|\beta|}}^\beta$, $\beta \in \mathcal{M}_{m+1}(S_1)$ appearing in the Stricker-Taylor formula of order $m+1$ for $\varphi(X_t)e^{\xi_t}$ satisfy

$$\sup_{0 \leq s_1 < \cdots < s_{|\beta|} \leq t} \hat{\mathbb{E}} \left[ | J_{s_1,\ldots,s_{|\beta|}}^\beta |^p \right] < \infty,$$

for $p \geq 1$

**Proof.** It is a straightforward combination of Theorem 24, Jensen’s inequality for conditional expectations and Lemma 22. \hfill \Box

**Lemma 26.** Assume that $H(1)$ holds and $\varphi \in C_{p}^2$. Then, the kernels $J_r = \{ J_s^r, s \in [0,t] \}, r = 1,\ldots,d_V$ in the martingale representation of $\varphi(X_t)e^{\xi_t}$ satisfy the following H"{o}lder continuity property:

$$\hat{\mathbb{E}} \left[ | J_s^r - J_u^r |^{2p} \right] \leq C (s - u)^p, \quad 0 \leq u \leq s \leq t,$$

for $p \geq 1$.

**Proof.** The idea is to use the Clark-Ocone formula, Theorem 23. That is, one has the following representation

$$J_s^r = \hat{\mathbb{E}} \left[ D_s^r \{ \varphi(X_t)e^{\xi_t} \} | \mathcal{H}_s \right], \quad 0 \leq s \leq t,$$

where $D_s^r$ denotes the Malliavin derivative with respect to $V^r$. Hence, we can write

$$J_s^r - J_u^r = \hat{\mathbb{E}} \left[ D_s^r \{ \varphi(X_t)e^{\xi_t} \} - D_u^r \{ \varphi(X_t)e^{\xi_t} \} | \mathcal{H}_s \right]$$

$$+ \hat{\mathbb{E}} \left[ D_u^r \{ \varphi(X_t)e^{\xi_t} \} | \mathcal{H}_s \right] - \hat{\mathbb{E}} \left[ D_u^r \{ \varphi(X_t)e^{\xi_t} \} | \mathcal{H}_u \right]$$

$$\triangleq A_1 + A_2.$$
In this section we start reviewing some basic concepts of backward Itô integration that can be found, for instance, in Pardoux and Protter [14], Bensoussan [3] and Applebaum [1].

### 4.3 Backward martingales estimates

For the term $A_2$ the result follows from the martingale representation theorem, Theorem 23, applied to the random variable $D_u \{ \varphi (X_t) e^{\xi_t} \} \in L^2 \left( \Omega, \mathcal{H}_t, \tilde{P} \right)$ which yields

$$
D_u \{ \varphi (X_t) e^{\xi_t} \} = \tilde{E} \left[ D_u \{ \varphi (X_t) e^{\xi_t} \} | \mathcal{H}_0 \right] + \sum_{r_1=1}^{d_v} \int_0^t G_{r_v}^{r_1} dV_{r_v}^{r_1},
$$

and, hence, $A_2 = \sum_{r_1=1}^{d_v} \int_u^s G_{r_v}^{r_1} dV_{r_v}^{r_1}$ and

$$
\tilde{E} \left[ |A_2|^p \right] \leq C \sum_{r_1=1}^{d_v} \tilde{E} \left[ \left( \int_u^s |G_{r_v}^{r_1}|^2 \right)^p \right] \leq C (s-u)^{p-1} \sum_{r_1=1}^{d_v} \int_u^s \tilde{E} \left[ |G_{r_v}^{r_1}|^2 \right] dv \leq C (s-u)^p \sum_{r_1=1}^{d_v} \sup_{0 \leq v \leq t} \tilde{E} \left[ |G_{r_v}^{r_1}|^2 \right] \leq C (s-u)^p,
$$

where in the last inequality we have used Proposition 25.

**Remark 27.** If $H(m)$ holds and $\varphi \in C_{m+1}^m$, using the same reasonings as in Lemma 26, one can show that the kernels $J^\beta, \beta \in M_m (S_1)$ in the Stroock-Taylor formula for $\varphi (X_t) e^{\xi_t}$ satisfy the following Hölder continuity property:

$$
\tilde{E} \left[ \left| J_{s_1,\ldots,s_i}^{\beta} - J_{s_1,\ldots,s_{i-1},u,s_{i+1},\ldots,s_{[\beta]}}^{\beta} \right|^{2p} \right] \leq C |s-u|^p,
$$

for $p \geq 1$, $s_{i-1} \leq u \leq s \leq s_{i+1}$, $i = 2, \ldots, m - 1$.

### 4.3 Backward martingales estimates

In this section we start reviewing some basic concepts of backward Itô integration that can be found, for instance, in Pardoux and Protter [14], Bensoussan [3] and Applebaum [1].
Then we compute some technical estimates related to products of backward Itô integrals and backward stochastic exponentials that will be useful in the next section.

We know that under \( \bar{P} \) the observation process \( Y \) is a Brownian motion with respect to the filtration \( \bar{Y} \). For fixed \( t \geq 0 \), we can consider the process \( \bar{Y}^t = \{ \bar{Y}^t_s = Y^t_s - Y^t_t \}_{0 \leq s \leq t} \) which is a Brownian motion with respect to the backward filtration

\[
\bar{Y}^t = \left\{ Y^t_s \triangleq \sigma \left( \bar{Y}_u, s \leq u \leq t \right) \vee \mathcal{N} \right\}_{0 \leq s \leq t},
\]

where \( \mathcal{N} \) are all the \( P \)-null sets of \((\Omega, \mathcal{F}, P)\). We can also consider the filtration \( \bar{Y}^{0,V} = \{ \bar{Y}^{0,V}_s \triangleq \mathcal{F}^{0,V}_s \vee \mathcal{Y}_s \}_{0 \leq s \leq t} \) and the backward filtration \( \bar{Y}^{0,V,t} = \{ \bar{Y}^{0,V,t}_s \triangleq \mathcal{F}^{0,V}_s \vee \mathcal{Y}^t_s \}_{0 \leq s \leq t} \). As \( X_0 \) and \( V \) are independent of \( Y \) under \( \bar{P} \), we also have that \( Y \) is a \( \bar{Y}^{0,V} \)-Brownian motion and \( \bar{Y} \) is \( \bar{Y}^{0,V,t} \)-Brownian motion.

If \( \eta = \{ \eta^i_s, \cdots, \eta_d^i \}_{0 \leq s \leq t} \) is a square integrable measurable process adapted to \( \bar{Y}^{0,V,t} \) we can define the backward Itô integral of \( \eta \) with respect to \( \bar{Y} \) by

\[
\int_s^t \eta_u d\bar{Y}_u \triangleq \sum_{i=1}^{dY} \sum_{j=1}^{n-1} \eta_{i,j+1} \left( \bar{Y}_{i,j+1}^t - \bar{Y}_{i,j}^t \right),
\]

\[
= L^2(\bar{P}) - \lim_{\tau \in \Pi(t),|\tau| \to 0} \sum_{i=1}^{dY} \sum_{j=0}^{n-1} \eta_{i,j+1} \left( \bar{Y}_{i,j+1}^t - \bar{Y}_{i,j}^t \right),
\]

**Remark 28.** If a square integrable process \( \theta = \{ \theta_u \}_{0 \leq u \leq t} \) is simultaneously adapted to \( \bar{Y}^{0,V} \) and \( \bar{Y}^{0,V,t} \) both, the Itô and the backward Itô integrals, can be defined over the same interval but, in general, they will be different. However, if \( \theta_u \) is measurable with respect to \( \bar{Y}^{0,V} \) and \( \bar{Y}^{0,V,t} \), then both integrals coincide. In fact, they coincide with the Stratonovich integral, see Pardoux and Protter [14]. This means that in the statement of Lemma 32 we can change all backward Itô integrals by Itô integrals and the estimates will hold true. However, in the proof of Lemma 32 we use the properties of the backward integral and for that reason we keep the notation of backward integration.

The backward Itô integral is analogous to the Itô integral. In particular, the backward Itô integral has zero expectation and it is a backward martingale with respect to \( \bar{Y}^{0,V,t} \), that is

\[
\mathbb{E} \left[ \int_s^t \eta_u d\bar{Y}_u \bigg\vert \bar{Y}^{0,V,t}_{s_2} \right] = \int_s^{s_2} \eta_u d\bar{Y}_u, \quad 0 \leq s_1 < s_2 \leq t.
\]

A backward Itô process is a process of the following form

\[
Z_s = Z_t + \int_s^t \eta_u d\bar{Y}_u, \quad 0 \leq s \leq t,
\]

where \( \nu \) and \( \eta \) are two square integrable, measurable and \( \bar{Y}^{0,V,t} \)-adapted processes of the appropriate dimensions. For backward Itô processes and \( f \in C^{1,2}((0, t) \times \mathbb{R}^d; \mathbb{R}) \) we have the following Itô formula, see Bensoussan [3],

\[
f(s, Z_s) = f(t, Z_t) + \int_s^t \left\{ -\partial_t f(u, Z_u) + \nu_u D f(u, Z_u) + \frac{1}{2} \text{tr} \left( D^2 f(u, Z_u) \eta_u \eta_u^T \right) \right\} du
\]
Proof. Let $Y$ with respect to $\theta = \{\theta_u\}_{0 \leq u \leq t}$ be the process
\[ \Psi \equiv \left( \int_s^t \eta_u \, d\tilde{Y}_u \right) \left( \int_s^t \eta_u \, d\tilde{Y}_u \right) \eta_u \, d\tilde{Y}_u \]

Moreover, by the same reasoning as in Lemma 12, we have for all $s \leq t$
\[ \mathbb{E} \left[ M^t \left( \phi \right) \right] < \infty. \] 

Lemma 29. Let $0 \leq s_1 \leq s_2 \leq s_3 \leq t$, $\Psi$ be $\mathcal{Y}^{0,V,t}$-measurable random variable and $\theta = \{\theta_u\}_{0 \leq u \leq t}$ a square integrable and measurable process such that $\theta_u$ is measurable with respect to $\mathcal{Y}_{s_3}^{0,V,t}$ for all $s_2 \leq u \leq s_3$. Then,
\[ \mathbb{E} \left[ \Psi M_{s_3}^t \left( \phi \right) \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right] = \mathbb{E} \left[ \Psi M_{s_3}^t \left( \phi \right) \int_{s_2}^{s_3} \phi_i (X_u) \, \theta_u \, d\tilde{Y}_u \right]. \]

Proof. By the backward martingale properties of $M^t$ and equation (4.8) we can write
\[ \mathbb{E} \left[ \Psi M_{s_3}^t \left( \phi \right) \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right] = \mathbb{E} \left[ \Psi \mathbb{E} \left[ M_{s_3}^t \left( \phi \right) \mid \mathcal{Y}_{s_2}^{0,V,t} \right] \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right] = \mathbb{E} \left[ \Psi M_{s_2}^t \left( \phi \right) \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right] = \mathbb{E} \left[ \Psi M_{s_3}^t \left( \phi \right) \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right] = \mathbb{E} \left[ \Psi \left( \int_{s_2}^{s_3} M_{s_2}^t \left( \phi \right) \phi_{i_1} (X_u) \, d\tilde{Y}_{i_1}^u \right) \left( \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right) \right] + \sum_{i=1}^{d_y} \mathbb{E} \left[ \Psi \left( \int_{s_2}^{s_3} M_{s_2}^t \left( \phi \right) \phi_{i_1} (X_u) \, d\tilde{Y}_{i_1}^u \right) \left( \int_{s_2}^{s_3} \theta_u \, d\tilde{Y}_u \right) \right].
Next, note that

\[ \mathbb{E}\left[ M_{s_3}^t (\phi) \int_{s_2}^{s_3} \theta_u dY_u^{\ast} \right] = \mathbb{E}\left[ M_{s_3}^t (\phi) \mathbb{E}\left[ \int_{s_2}^{s_3} \theta_u dY_u^{\ast} | \mathcal{Y}_{s_3}^{0,V,t} \right] \right] = 0, \]

and using equation (4.7) we have

\[
\begin{align*}
\mathbb{E}\left[ \Psi \left( \int_{s_2}^{s_3} M_u^t (\phi) \phi_{i_1} (X_u) dY_u^{\ast} \right) \left( \int_{s_2}^{s_3} \theta_u dY_u^{\ast} \right) \right] \\
= 1_{\{i_1=i\}} \mathbb{E}\left[ \Psi \left( \int_{s_2}^{s_3} M_u^t (\phi) \phi_{i_1} (X_u) \theta_u du \right) \right] \\
= 1_{\{i_1=i\}} \mathbb{E}\left[ \Psi \left( \int_{s_2}^{s_3} \mathbb{E}\left[ M_u^t (\phi) | \mathcal{Y}_{s_3}^{0,V,t} \right] \phi_{i_1} (X_u) \theta_u du \right) \right] \\
= 1_{\{i_1=i\}} \mathbb{E}\left[ \Psi M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \phi_{i_1} (X_u) \theta_u du \right) \right].
\end{align*}
\]

Hence, the result follows. \(\square\)

**Lemma 30.** Let \(0 \leq s_1 \leq s_2 \leq s_3 \leq t\), \(\Psi\) be \(\mathcal{Y}_{s_3}^{0,V,t}\)-measurable random variable and \(\theta^1 = \{\theta_u^1\}_{0 \leq u \leq t}\) and \(\theta^2 = \{\theta_u^2\}_{0 \leq u \leq t}\) be two square integrable measurable processes such that \(\theta_u^1\) and \(\theta_u^2\) are also measurable with respect to \(\mathcal{Y}_{s_3}^{0,V,t}\) for all \(s_2 \leq u \leq s_3\). Then,

\[
\begin{align*}
\mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \left( \int_{s_2}^{s_3} \theta_u^1 dY_u^{\ast} \right) \left( \int_{s_2}^{s_3} \theta_u^2 dY_u^{\ast} \right) \right] \\
= \mathbb{E}\left[ \Psi M_{s_3}^t (\phi) \int_{s_2}^{s_3} \phi_{i_1} (X_u) \theta^1_u \left( \int_{s_2}^{s_3} \phi_{i_2} (X_v) \theta^2_v dv \right) du \right] \\
+ \mathbb{E}\left[ \Psi M_{s_3}^t (\phi) \int_{s_2}^{s_3} \phi_{i_2} (X_u) \theta^2_u \left( \int_{s_2}^{s_3} \phi_{i_1} (X_v) \theta^1_v dv \right) du \right] \\
+ 1_{\{i_1=i_2\}} \mathbb{E}\left[ \Psi M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \theta^1_u \theta^2_u du \right) \right].
\end{align*}
\]

**Proof.** Using the integration by parts formula (4.6) we can write

\[
\begin{align*}
\mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \left( \int_{s_2}^{s_3} \theta_u^1 dY_u^{\ast} \right) \left( \int_{s_2}^{s_3} \theta_u^2 dY_u^{\ast} \right) \right] \\
= \mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \int_{s_2}^{s_3} \theta_u^1 \left( \int_{s_2}^{s_3} \theta_v^2 dY_v^{\ast} \right) dY_u^{\ast} \right] \\
+ \mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \int_{s_2}^{s_3} \theta_u^2 \left( \int_{s_2}^{s_3} \theta_v^1 dY_v^{\ast} \right) dY_u^{\ast} \right] \\
+ 1_{\{i_1=i_2\}} \mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \left( \int_{s_2}^{s_3} \theta^1_u \theta^2_u du \right) \right].
\end{align*}
\]

Then, using the same reasonings as in Lemma 29 we get that

\[
\begin{align*}
\mathbb{E}\left[ \Psi M_{s_1}^t (\phi) \int_{s_2}^{s_3} \theta_u^1 \left( \int_{s_2}^{s_3} \theta_v^2 dY_v^{\ast} \right) dY_u^{\ast} \right] \\
= \mathbb{E}\left[ \Psi \int_{s_2}^{s_3} M_u^t (\phi) \phi_{i_1} (X_u) \theta_u^1 \left( \int_{s_2}^{s_3} \theta_v^2 dY_v^{\ast} \right) du \right]
\end{align*}
\]

Next, by Fubini's theorem, Lemma 29 and Fubini's theorem again we obtain

\[
\begin{align*}
\mathbb{E}\left[ \Psi \int_{s_2}^{s_3} M_u^t (\phi) \phi_{i_1} (X_u) \theta_u^1 \left( \int_{s_2}^{s_3} \theta_v^2 dY_v^{\ast} \right) du \right]
\end{align*}
\]

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that for the last term we only need to take conditional expectation with respect to

$\Phi_{t_1} (X_u) \theta_{11} M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \phi_{12} (X_u) \theta_{22} dv \right) du$

$= \int_{s_2}^{s_3} [\Phi_{t_1} (X_u) \theta_{11} M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \phi_{12} (X_u) \theta_{22} dv \right) du$

$= \int_{s_2}^{s_3} [\Phi_{t_1} (X_u) \theta_{11} M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \phi_{12} (X_u) \theta_{22} dv \right) du$

By symmetry we get an analogous expression for the term $\int_{s_2}^{s_3} \theta_{22} \left( \int_{s_2}^{s_3} \theta_{11} \Phi_{t_1} (X_u) \theta_{11} M_{s_3}^t (\phi) \left( \int_{s_2}^{s_3} \phi_{12} (X_u) \theta_{22} dv \right) du$

The next lemma is a well known generalization of Hölder’s inequality.

**Lemma 31** (Generalized Hölder’s inequality). Let $p_i > 1$, $i = 1, ..., m$ such that $\sum_{i=1}^{m} \frac{1}{p_i} = 1$, and $X_i \in L^{p_i} (\Omega, \mathcal{F}, \hat{P})$, $i = 1, ..., m$. Then,

$$\mathbb{E} \left[ \prod_{i=1}^{m} X_i \right] \leq \prod_{i=1}^{m} \mathbb{E} \left[ \left| X_i \right|^{p_i} \right]^{1/p_i} < \infty.$$

**Lemma 32.** Let $\tau = \{0 = t_0 < t_1 \leq \cdots < t_n = \tau\}$ be a partition of $[0, \tau]$, $\phi \in \mathcal{B}_\beta (\mathbb{R}^d \times \mathbb{R}^d)$, $\mathcal{T} \in L^{p} (\Omega, \mathcal{F}_{[0,\tau]} \hat{P})$ for any $p \geq 1$, $\beta$, be a deterministic processes satisfying

$$|\beta_s 1_{[t_j, t_{j+1}[} (s)| \leq \delta^m \quad (4.10)$$

for some $m \in \mathbb{N}$ and $\theta^1, \theta^2, \kappa^1, \kappa^2$, be stochastic processes measurable with respect to $\mathcal{F}_{[0,\tau]}$, such that

$$\sup_{0 \leq s \leq \tau} \mathbb{E} \left[ \left| \theta^j_s \right|^p \right] < \infty, \quad j = 1, 2, \quad (4.11)$$

for any $p \geq 1$. Then:

1. For $i \in \{1, ..., d_Y\}$, we have that

$$\left| \mathbb{E} \left[ \mathcal{T} M_{d_Y}^i (\phi) \left( \int_{t_j}^{t_{j+1}} \beta_s dY^i_s \right) \left( \int_{t_k}^{t_{k+1}} \beta_s dY^i_s \right) \right] \right| \leq C \left\{ 1_{(j \neq k)} \delta^{2m+2} + 1_{(j = k)} \delta^{2m+1} \right\}.$$

2. For $i, i_1 \in \{1, ..., d_Y\}$, we have that

$$\left| \mathbb{E} \left[ \mathcal{T} M_{d_Y}^i (\phi) \left( \int_{t_j}^{t} \theta_s dY^i_s \right) \left( \int_{t_k}^{t_{k+1}} \beta_s dY^i_s \right) \left( \int_{t_j}^{t_{k+1}} \beta_s dY^i_s \right) \right] \right| \leq C \left\{ 1_{(j \neq k)} \delta^{2m+2} + 1_{(j = k)} \delta^{2m+1} \right\}.$$

3. For $i, i_1, a_1 \in \{1, ..., d_Y\}$, we have that

$$\left| \mathbb{E} \left[ \mathcal{T} M_{d_Y}^i (\phi) \left( \int_{t_j}^{t} \theta_s dY^i_s \right) \left( \int_{t_k}^{t} \kappa_s dY^i_s \right) \left( \int_{t_j}^{t_{k+1}} \beta_s dY^i_s \right) \left( \int_{t_j}^{t_{k+1}} \beta_s dY^i_s \right) \right] \right| \leq C \left\{ 1_{(j \neq k)} \delta^{2m+2} + 1_{(j = k)} \delta^{2m+1} \right\}.$$
4. For \( i, i_1, i_2 a_1, a_2 \in \{1, ..., d_Y \} \), we have that

\[
\left| \mathbb{E} \left[ \prod_{l=1}^{2} \left( \int_{t_j}^{t_{j+1}} \theta_s^l d\bar{Y}^i_s \right) \left( \int_{t_j}^{t_{j+1}} \beta_s d\bar{Y}^i_s \right) \right] \right| \leq C \left\{ 1_{\{j \neq k\}} \delta^{2m+2} + 1_{\{j = k\}} \delta^{2m+1} \right\}.
\]

**Proof.** The full proof of the lemma is lengthy and depends on applying Lemmas 29 and 30 repeatedly. We do not present it in full, but only write in detail the proof of the statement (4), the others being similar and easier. Note that by the assumptions on \( \Upsilon \), the expectations in the statement of the lemma are finite. We start with some preliminary estimations. In what follows \( C(t), C(\phi) \) and \( C(t, \phi) \) will denote constants that only depends on \( t \), on \( \phi \) and on \( t \) and \( \phi \), respectively. For any \( p \geq 1 \) we have

- Let \( 0 \leq s \leq t \), then

\[
\tilde{\mathbb{E}} \left[ |M^t_s (\phi)|^p \right] = \mathbb{E} \left[ M^t_s (p\phi) \exp \left( \frac{p^2 - p}{2} \sum_{i=1}^{d_Y} \int_{s}^{t} \phi_i^2 (X_u) du \right) \right] \leq \exp \left( d_Y \|\phi\|_\infty^2 \frac{p^2 - p}{2} (t - s) \right),
\]

where we have used that for \( \phi \in \mathcal{B}_b (\mathbb{R}^{d_x}; \mathbb{R}^{d_Y}) \) one has that \( M^t_s (\phi) \) is a backward martingale with expectation equal to one. The previous estimate yields that

\[
\sup_{0 \leq s \leq t} \tilde{\mathbb{E}} \left[ |M^t_s (\phi)|^p \right] \leq \exp \left( d_Y \|\phi\|_\infty^2 \frac{p^2 - p}{2} t \right) \leq C (t, \phi).
\]

- Let \( 0 \leq s_1 \leq s_2 \leq t \), then

\[
\tilde{\mathbb{E}} \left[ \int_{s_1}^{s_2} \theta_s Y^i_s \right] \leq \tilde{\mathbb{E}} \left[ \left( \int_{s_1}^{s_2} |\theta_s|^2 du \right)^{p/2} \right]\]

\[
\leq \tilde{\mathbb{E}} \left[ (s_2 - s_1)^{p/2-1} \left( \int_{s_1}^{s_2} |\theta_s|^p du \right) \right] \leq \sup_{0 \leq s \leq t} \tilde{\mathbb{E}} \left[ |\theta_s|^p (s_2 - s_1)^{p/2} \right] \leq C (s_2 - s_1)^{p/2},
\]

where we have used the Burkholder-Davis-Gundy inequality for backward martingales, Jensen’s inequality and Fubini’s theorem. The previous estimate yields that

\[
\sup_{0 \leq s_1 \leq s_2 \leq t} \tilde{\mathbb{E}} \left[ \left| \int_{s_1}^{s_2} \theta_s Y^i_s \right|^p \right] \leq \sup_{0 \leq s \leq t} \tilde{\mathbb{E}} \left[ |\theta_s|^p \right] t^{p/2} \leq C (t).
\]

Moreover, using Jensen’s inequality and Fubini’s theorem we have that

\[
\tilde{\mathbb{E}} \left[ \left| \int_{s_1}^{s_2} \phi_i (X_s) \theta_s du \right|^p \right] \leq \|\phi\|_\infty^p \tilde{\mathbb{E}} \left[ \left| \int_{s_1}^{s_2} |\theta_s| du \right|^p \right] \leq \|\phi\|_\infty^p \tilde{\mathbb{E}} \left[ (s_2 - s_1)^{p-1} \int_{s_1}^{s_2} |\theta_s| du \right].
\]
\begin{align*}
&\leq \|\phi\|_\infty^p \sup_{0 \leq s \leq t} \mathbb{E} [\|\theta_s\|^p] (s_2 - s_1)^p. \\
\end{align*}

The previous estimate yields that
\begin{align*}
\mathbb{E} \left[ \left| \int_{s_1}^{s_2} \phi_i (X_s) \theta ds \right|^p \right] &\leq \|\phi\|_\infty^p \sup_{0 \leq s \leq t} \mathbb{E} [\|\theta_s\|^p] t^{p/2} \leq C (t, \phi). & (4.15)
\end{align*}

- Let $0 \leq s_1 \leq s_2 \leq t$, then similar reasonings as in the previous point and hypothesis (4.10) give
\begin{align*}
\mathbb{E} \left[ \left| \int_{s_1}^{s_2} \beta_u Y_u^q \right|^p \right] &\leq \mathbb{E} \left[ \left( \int_{s_1}^{s_2} |\beta_u|^2 du \right)^{p/2} \right] \\
&\leq (s_2 - s_1)^{p/2 - 1} \int_{s_1}^{s_2} |\beta_u|^p du \\
&\leq t^{p/2 - 1} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\beta_u|^p du \\
&\leq t^{p/2} \delta^mp \leq C (t) & (4.16)
\end{align*}

Moreover, if $s_1 = t_j$ and $s_2 = t_{j+1}$ for some $j \in \{0, ..., n-1\}$ we can conclude using hypothesis (4.10) that
\begin{align*}
\mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right|^p \right] &\leq \|\phi\|_\infty^p \delta^{(m+1)p} = C (\phi) \delta^{(m+1)p} & (4.17)
\end{align*}

\begin{itemize}
\item \textbf{Case} $j = k$: \\
Using Cauchy-Schwarz inequality, Lemma 31 and inequalities (4.12), (4.14) and (4.17) we can write
\end{itemize}
\begin{align*}
\mathbb{E} \left[ Y_{\delta_{1}} \left( \phi_0 \right) \right] &\leq \mathbb{E} \left[ Y_{\delta_{1}} \left( \phi_0 \right) \right] \\
&\leq \left( \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right] \mathbb{E} \left[ \left| Y_{\delta_{1}} \right|^{12} \right] \right)^{1/12} \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right]^{1/12} \\
&\leq \left( \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right] \mathbb{E} \left[ \left| Y_{\delta_{1}} \right|^{12} \right] \right)^{1/12} \delta^{\frac{1}{2} (m+1) 4} \\
&\leq C (t) \delta^{2m+1}
\end{align*}

\begin{itemize}
\item \textbf{Case} $j < k$: \\
Using Lemma 29, we can write
\end{itemize}
\begin{align*}
\mathbb{E} \left[ Y_{\delta_{1}} \left( \phi_0 \right) \right] &\leq \mathbb{E} \left[ Y_{\delta_{1}} \left( \phi_0 \right) \right] \\
&\leq \left( \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right] \mathbb{E} \left[ \left| Y_{\delta_{1}} \right|^{12} \right] \right)^{1/12} \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right]^{1/12} \\
&\leq \left( \mathbb{E} \left[ \left| \phi_0 \right|^{12} \right] \mathbb{E} \left[ \left| Y_{\delta_{1}} \right|^{12} \right] \right)^{1/12} \delta^{\frac{1}{2} (m+1) 4} \\
&\leq C (t) \delta^{2m+1}
\end{align*}
\[
= \mathbb{E} \left[ Y \left( \int_{t_k}^{t_{k+1}} \beta_s dY_s \right) \prod_{l=1}^{2} \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \kappa_s^l dY_s^{l_2} \right) \right] \\
\times M_{t_{lj+1}} (\phi) \left( \int_{t_{lj}}^{t_{lj+1}} \phi_l (X_s) \beta_s ds \right) \\
= \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \prod_{l=1}^{2} \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \right]
\]
\[
\sum_{i=1}^{9} A_i,
\]

Where
\[
Y_1 = Y \left( \int_{t_k}^{t_{k+1}} \beta_s dY_s \right) \left( \int_{t_{lj}}^{t_{lj+1}} \phi_l (X_s) \beta_s ds \right) \prod_{l=1}^{2} \left( \int_{t_{kj+1}}^{t_l} \kappa_s^l dY_s^{l_2} \right),
\]

and

\[
A_1 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_2 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_3 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_4 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_5 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_6 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_7 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_8 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[
A_9 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]

The treatment of some of the terms is completely analogous. We distinguish four subcases:

\[\triangleright \triangleright \textbf{Subcase 1:} \text{ This subcase covers term } A_9. \text{ We apply Lemma 30 to write}\]

\[A_9 = \mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_1} \right) \left( \int_{t_{lj+1}}^{t_l} \theta_s^l dY_s^{l_2} \right) \right]
\]
\[\mathbb{E} \left[ Y_1 M_{t_{lj+1}} (\phi) \left( \int_{t_{lj+1}}^{t_l} \phi_s (X_s) \beta_s ds \right) \left( \int_{t_{lj+1}}^{t_l} \phi_s (X_s) \beta_s ds \right) \right],\]
where
\[ \Gamma_1 \triangleq \mathcal{Y} \prod_{l=1}^{2} \left( \int_{t_{k+1}}^{t} \kappa_{s}^{l} d\tilde{Y}_{s}^{l} \right) \left( \int_{t_{k+1}}^{t} \theta_{s}^{l} d\tilde{Y}_{s}^{l} \right). \]

Hence, using Lemma 31 and inequality (4.14) we have, for \( p \geq 1 \), that
\[ \mathbb{E} \left[ |\Gamma_1|^p \right] \leq C \left( t \right), \]
and using Lemma 31 and inequalities (4.12) and (4.18) we obtain
\[ |A_9| \leq \mathbb{E} \left[ |\Gamma_1|^4 \right]^{1/4} \mathbb{E} \left[ \left| M_{t_{k+1}}^{l} (\phi) \right|^4 \right]^{1/4} \mathbb{E} \left[ \left| \int_{t_{j+1}}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right|^4 \right]^{1/4} \times \mathbb{E} \left[ \left| \int_{t_{k}}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right|^4 \right]^{1/4} \leq C \left( t, \phi \right) \delta^{2m+2}. \]

\> \> **Subcase 2:** The terms \( A_2, A_4, A_5, A_6 \) and \( A_8 \) are treated analogously. We will write the proof for \( A_2 \). By Lemma 30, we can write
\[ A_2 = \mathbb{E} \left[ \sum_{l=1}^{2} \left( \int_{t_{j+1}}^{t_{j+1}} \theta_{s}^{l} d\tilde{Y}_{s}^{l} \right) \left( \int_{t_{k}}^{t_{k+1}} \theta_{s}^{2} d\tilde{Y}_{s}^{2} \right) \right]. \]
where
\[ \Gamma_2 \triangleq \mathcal{Y} \left( \int_{t_{j+1}}^{t_{j+1}} \theta_{s}^{1} d\tilde{Y}_{s}^{1} \right) \left( \int_{t_{k}}^{t_{k+1}} \theta_{s}^{2} d\tilde{Y}_{s}^{2} \right) \prod_{l=1}^{2} \left( \int_{t_{k+1}}^{t} \kappa_{s}^{l} d\tilde{Y}_{s}^{l} \right). \]

Hence, using Lemma 31 and inequality (4.14) we have, for \( p \geq 1 \), that
\[ \mathbb{E} \left[ |\Gamma_2|^p \right] \leq C \left( t \right), \]
and using Lemma 31 and inequalities (4.12),(4.18), (4.17) and (4.13) we obtain
\[ |A_2| \leq \mathbb{E} \left[ |\Gamma_2|^5 \right]^{1/5} \mathbb{E} \left[ \left| M_{t_{j+1}}^{l} (\phi) \right|^5 \right]^{1/5} \mathbb{E} \left[ \left| \int_{t_{j+1}}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right|^5 \right]^{1/5} \times \mathbb{E} \left[ \left| \int_{t_{k}}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right|^5 \right]^{1/5} \leq C \left( t, \phi \right) \delta^{(m+1)} \delta^{(m+2)} \delta^{1/2} = C \left( t, \phi \right) \delta^{2m+2}. \]

\> \> **Subcase 3:** The terms \( A_3 \) and \( A_7 \) are treated analogously. We will write the proof for \( A_3 \). We apply Lemma 30 twice to write
\[ A_3 = \mathbb{E} \left[ \sum_{l=1}^{2} \left( \int_{t_{j+1}}^{t_{j+1}} \theta_{s}^{1} d\tilde{Y}_{s}^{1} \right) \left( \int_{t_{k+1}}^{t} \theta_{s}^{2} d\tilde{Y}_{s}^{2} \right) \right]. \]
\[
\begin{align*}
&= \mathbb{E} \left[ \Gamma_3 \left( \int_{t_k}^{t_{k+1}} \beta_s d\tilde{Y}^{s}_{12} \right) M^t_{t_k} (\phi) \left( \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_s) \theta^1_s ds \right) \left( \int_{t_j}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right) \right] \\
&= \mathbb{E} \left[ \Gamma_3 \left( \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_s) \theta^1_s ds \right) M^t_{t_{k+1}} (\phi) \left( \int_{t_k}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right) \right] \\
&= \mathbb{E} \left[ \Gamma_3 \left( \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_s) \theta^1_s ds \right) \left( \int_{t_k}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right) \right]
\end{align*}
\]

where

\[\Gamma_3 = \Upsilon \left( \int_{t_k}^{t_{k+1}} \phi^2_s d\tilde{Y}^s_{12} \right) \prod_{l=1}^2 \left( \int_{t_k}^{t_{k+1}} \kappa^l_s d\tilde{Y}^{l}_{12} \right).\]

Using Lemma 31 and inequalities (4.14) and (4.15) we have, for \( p \geq 1 \), that

\[\mathbb{E} \left[ \left| \Gamma_3 \left( \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_s) \theta^1_s ds \right) \right|^p \right] \leq C (t, \phi),\]

and using Lemma 31 and inequalities (4.12),(4.18), (4.17) and (4.13)

\[|A_3| \leq \mathbb{E} \left[ \left| \Gamma_3 \left( \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_s) \theta^1_s ds \right) \right|^4 \right]^{1/4} \mathbb{E} \left[ \left| M^t_{t_{k+1}} (\phi) \right|^5 \right]^{1/5}
\]
\[\times \mathbb{E} \left[ \left| \int_{t_k}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right|^4 \right]^{1/4} \mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right|^4 \right]^{1/4}
\]
\[\leq C (t, \phi) \delta^{m+1} \delta^{m+1} = C \delta^{2m+1}.
\]

\[\triangleright \triangleright \text{Subcase 4: This subcase corresponds to the term } A_1. \text{ Applying Lemma 30 and Lemma 29 we can write }
\]
\[A_1 = \mathbb{E} \left[ \Upsilon_1 M^t_{t_{j+1}} (\phi) \left( \int_{t_{j+1}}^{t_k} \theta^1_s d\tilde{Y}^s_{12} \right) \left( \int_{t_{j+1}}^{t_k} \theta^2_s d\tilde{Y}^{s}_{12} \right) \right]
\]
\[= \mathbb{E} \left[ \Gamma_4 \Gamma_5 M^t_{t_k} (\phi) \left( \int_{t_k}^{t_{k+1}} \beta_s d\tilde{Y}^s_{12} \right) \left( \int_{t_j}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right) \right]
\]
\[= \mathbb{E} \left[ \Gamma_4 \Gamma_5 M^t_{t_{k+1}} (\phi) \left( \int_{t_k}^{t_{k+1}} \phi_i (X_s) \beta_s ds \right) \left( \int_{t_j}^{t_{j+1}} \phi_i (X_s) \beta_s ds \right) \right]
\]

where

\[\Gamma_4 = \Upsilon \prod_{l=1}^2 \left( \int_{t_{k+1}}^{t_k} \kappa^l_s d\tilde{Y}^{l}_{12} \right)
\]
\[\Gamma_5 = 1_{\{i_1 = i_2\}} \int_{t_{j+1}}^{t_k} \theta^1_s \theta^2_s du + \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_u) \theta^1_u \left( \int_{t_{j+1}}^{t_k} \phi_{i_2} (X_v) \theta^2_v dv \right) du + \int_{t_{j+1}}^{t_k} \phi_{i_1} (X_u) \theta^2_u \left( \int_{t_{j+1}}^{t_k} \phi_{i_2} (X_v) \theta^1_v dv \right) du
\]

To finish the proof one follows the same reasonings in the previous subcase taking into account that \( \Gamma_4 \) and \( \Gamma_5 \) have moments of all orders that only depend on \( t \) and \( \phi \).
In this subsection we will show the main estimates for conditional expectations with respect to \( \kappa' \)s.

\[ 4.4 \text{ Conditional expectation estimates} \]

Remark 33. Under the assumptions of Lemma 32 one can prove that for any \( q \in \{1, \ldots, m\} \) and \( i, i_1, \ldots, i_q, a, \ldots, a_q \in \{1, \ldots, d_Y\} \), we have that

\[
\hat{E} \left[ \prod_{\lambda=1}^{q} \left( \int_0^t \theta_{i_\lambda} d\gamma_{i_\lambda} \right) \right] \leq C \left\{ \delta_{2m+2} + \delta_{2m+1} \right\}.
\]

Case \( j > k \):

This is completely symmetric to the previous case by swapping the role of the \( \theta' \)s by the \( \kappa' \)s.

\[ \square \]

4.4 Conditional expectation estimates

In this subsection we will show the main estimates for conditional expectations with respect to the observation filtration that will allow the proof of our result. Throughout this section we assume that \( \varphi \in \mathcal{B}_p \), which ensures that Corollary 14 holds, and \( m \in \mathbb{N} \).

Lemma 34. Assume that \( H(m) \) holds. For \( \alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0)) \) with \( |\alpha|_0 = m \) and \( i \in \{0, 1, \ldots, d_Y\} \) we have

\[
\hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} dY_s^i | Y_t \right)^2 \right] \leq C \delta^{2m}.
\]

Proof. Using Jensen inequality, Hölder inequality and Burkholder-Davis-Gundy inequality, if \( i \neq 0 \), or Jensen inequality, if \( i = 0 \), we get

\[
\hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} dY_s^i | Y_t \right)^2 \right] \leq \hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} dY_s^i \right)^{2+\varepsilon} \right] \Delta^{\varepsilon} \hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} dY_s^i \right)^{\frac{2+\varepsilon}{\varepsilon}} \right] \]

\[
\leq C \left\| \varphi(X_t) e^{\xi t} \right\|_{2+\varepsilon} \hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} dY_s^i \right)^{\frac{2+\varepsilon}{\varepsilon}} \right].
\]

Next, using Jensen inequality again, Fubini’s Theorem, Assumption \( H(m) \), Lemma 8 and that \( |\alpha| + |\alpha|_0 = 2m \) we get

\[
\hat{E} \left[ \left( \int_0^t I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} \right)^2 ds \right] \leq C \hat{E} \left[ \left( I_\alpha(L^\alpha h_\alpha(X)) \tau_{(s),s} \right)^{2+\varepsilon} \right] ds
\]

\[
\leq C \int_0^t \hat{E} \left[ \sup_{\tau(s) \leq u \leq s} \left| L^\alpha h_\alpha(X_u) \right|^{2+\varepsilon} \right] \left( s - \tau(s) \right)^{\varepsilon (|\alpha| + |\alpha|_0)} ds
\]

\[
\leq C \delta^{2m+\frac{2+\varepsilon}{\varepsilon}},
\]

from which follows the result. \( \square \)
Lemma 35. Assume that $H(m)$ holds. For $\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0))$ with $|\alpha|_0 \neq m$

\[
\hat{E} \left[ \hat{E} \left[ \varphi(X_t) e^{\xi \int_0^t I_\alpha(L^\alpha h_0(X_\cdot))_{\tau(s),s} ds |Y_t} \right] \right]^2 \leq C \delta^{2m}
\]

Proof. We will give only the proof for the case $m \in \{1, 2\}$. The proof for $m > 2$ follows the same ideas but it is tedious to write down and we leave it to the reader. We split the proof depending on $|\alpha|_0$, the number of zeros in $\alpha$. If $m = 1$, $|\alpha|_0 \in \{0\}$ and if $m = 2$, $|\alpha|_0 \in \{0, 1\}$. We group the three cases into two: $|\alpha|_0 = m - 1$ and $|\alpha|_0 = 0$, (of course the two overlap when $m = 1$).

Assume that $|\alpha|_0 = m - 1$. Then, using Theorem 23 we can write

\[
\hat{E} \left[ \varphi(X_t) e^{\xi \int_0^t I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} ds |Y_t} \right]
\]

\[
= \int_0^t \hat{E} \left[ \varphi(X_t) e^{\xi \int_0^t I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} ds |Y_t} \right] ds
\]

\[
= \int_0^t \hat{E} \left[ \hat{E} \left[ \varphi(X_t) e^{\xi \int_0^t I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} ds |Y_t} \right] \right] ds
\]

\[
+ \sum_{r=1}^{d\nu} \int_0^t \hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] ds
\]

\[
= \int_0^t \hat{E} \left[ \varphi(X_t) e^{\xi \int_0^t I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} ds |Y_t} \right] \hat{E} \left[ I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |H_0^t \right] |Y_t} \right] ds
\]

Moreover, by Lemma 11 (1), we get $\hat{E} \left[ I_\alpha(L^\alpha h_0(X_\cdot))_{\tau(s),s} |H_0^t \right] = 0$ and, by Lemma 11 (2), for $r = 1, ... , d\nu$ we have

\[
\hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] \hat{E} \left[ I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |H_0^t \right] |Y_t} \right] ds
\]

Next, using Jensen's inequality, Cauchy-Schwartz inequality, Itô isometry, Lemma 8 and Remark 5 we get

\[
\hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] \right] du
\]

\[
\leq C \delta \int_0^t \hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] \right] du
\]

\[
\leq C \delta \int_0^t \hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) |Y_t} \right] \right] \hat{E} \left[ I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] du
\]

\[
\leq C \delta \int_0^t \hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) |Y_t} \right] \right] \hat{E} \left[ I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] du
\]

\[
\leq C \delta \int_0^t \hat{E} \left[ \left( \int_0^t J_u^r dV_u^r \right) |Y_t} \right] \right] \hat{E} \left[ I_\alpha(L^\alpha h_i(X_\cdot))_{\tau(s),s} |Y_t} \right] du
\]

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and using similar reasonings we get

\[
\mathbb{E} \left[ \left( \int_{\tau(s)}^t \mathbb{E} \left[ J_u^r (L^o h_i (X_s)) \right] du \right)^2 \right] \leq C \delta^{m-1+|\alpha|-l_0+2} = C \delta^{2m},
\]

and the result for the case \(|\alpha|_0 = m - 1\) follows.

The last case is \(|\alpha|_0 = 0\) and \(m = 2\). Applying Theorem 23 we can write

\[
\mathbb{E} \left[ \mathbb{E} \left[ \varphi (X_t) e^{\xi t} \right] \int_0^t I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \right] = \int_0^t \mathbb{E} \left[ \mathbb{E} \left[ \varphi (X_t) e^{\xi t} \right] I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \right] ds
\]

\[
+ \sum_{r=1}^{d_r} \int_0^t \mathbb{E} \left[ \mathbb{E} \left[ J_u^r | \mathcal{H}_0^r \right] I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \right] ds
\]

\[
+ \sum_{r_1, r_2 = 1}^{d_r} \int_0^t \mathbb{E} \left[ \mathbb{E} \left[ J_u^{r_1, r_2} | \mathcal{H}_0^{r_1, r_2} \right] I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \right] ds.
\]

\[\triangleq A_1 + \sum_{r=1}^{d_r} A_2 (r) + \sum_{r_1, r_2 = 1}^{d_r} A_3 (r_1, r_2).\]

Applying Lemma 11 (1), we see that the term \(A_1\) vanishes. Applying Lemma 11 (2) and, then, Lemma 11 (1), for \(r = 1, ..., d_r\), we can write

\[A_2 (r) = 1_{\{a_2 = r\}} \int_0^t \int_{\tau(s)}^s \mathbb{E} \left[ J_u^r | \mathcal{H}_0^r \right] I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \] duds

\[= 1_{\{a_2 = r\}} \int_0^t \int_{\tau(s)}^s \mathbb{E} \left[ J_u^r | \mathcal{H}_0^r \right] \mathbb{E} \left[ I_\alpha (L^o h_i (X_s)) \tau(s), s | \mathcal{H}_0^r \right] | \mathcal{Y}_t] duds
\]

\[= 1_{\{a_2 = r, |\alpha|_1 = |\alpha|_0\}} \int_0^t \int_{\tau(s)}^s \mathbb{E} \left[ J_u^r | \mathcal{H}_0^r \right] \times I_\alpha \left( \mathbb{E} \left[ L^o h_i (X_s) | \mathcal{H}_0^r \right] \right) \tau(s), s | \mathcal{Y}_t] duds
\]

which is equal to zero because \(1 = |\alpha|_1 \neq |\alpha|_0 = 0\). Applying Lemma 11 (3), for \(r_1, r_2 = 1, ..., d_r\) we can write

\[A_3 (r_1, r_2) = 1_{\{a_2 = r_2, a_1 = r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^{u_{r_2}} \mathbb{E} \left[ J_{v,u}^{r_1, r_2} I_{(\alpha)} (L^o h_i (X_s)) \tau(s), s | \mathcal{Y}_t \right] duds
\]

\[= 1_{\{a_2 = r_2, a_1 = r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^{u_{r_2}} \mathbb{E} \left[ J_{v,u}^{r_1, r_2} L^o h_i (X_s) | \mathcal{Y}_t \right] duds
\]

\[\leq 1_{\{a_2 = r_2, a_1 = r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^{u_{r_2}} \mathbb{E} \left[ |J_{v,u}^{r_1, r_2}|^2 | \mathcal{H}_0^r \right]^{1/2}
\]

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Hence, using Jensen’s inequality, Cauchy-Schwartz inequality and Remark 5 we have

\[
\mathbb{E} \left[ |A_3(r_1, r_2)|^2 \right] \leq C (\delta^2) \int_0^t \int_{\tau(s)}^{\tau(u)} \mathbb{E} \left[ |J_{r,u}^r| \right] \, dvdu,
\]

and we can conclude. \(\square\)

**Lemma 36.** Let \(m \in \{1, 2\}\) and assume that \(H(m)\) holds. For \(\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0)), |\alpha|_0 \neq |\alpha|\) and \(i \neq 0\), we can write

\[
\mathbb{E} \left[ \varphi(X_t)e^{\xi t} \int_0^t I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s)dY_s^i \big| \mathcal{Y}_t \right] = \sum_{r=1}^{d} \sum_{j=0}^{n-1} \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} J_{r,s}^i dV_s^i \right) \left( \int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s)dY_s^i \right) \right] \mathcal{Y}_t,
\]

and

\[
\mathbb{E} \left[ \varphi(X_t)e^{\xi t} \int_0^t I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s)dY_s^i \big| \mathcal{Y}_t \right] = \sum_{r=1}^{d} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} J_{r,s}^i dV_s^i \right) \left( \int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s)dY_s^i \right) \right] \mathcal{Y}_t.
\]

**Proof.** Note that, as \(|\alpha|_0 \neq |\alpha|\), by Lemma 11 (1) we have that if \(0 \leq u \leq v \leq w \leq t\) then

\[
\mathbb{E} \left[ I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s) | \mathcal{Y}_u \right] = 0.
\]

Using Theorem 23 we can write

\[
\varphi(X_t)e^{\xi t} \int_0^t I_{\alpha}(\lambda \cdot h_i(X_s))\tau(s)dY_s^i
\]

\[
= \varphi(X_t)e^{\xi t} \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} I_{\alpha}(\lambda \cdot h_i(X_s))t_{j+1}dY_s^i
\]

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Next, for \( j \geq 0 \) we get, using equation (4.21) that

\[
\tilde{E} \left[ \varphi(X_t) e^{\xi t} \left| \mathcal{H}_0^t \right. \right] = \mathbb{E} \left[ \varphi(X_t) e^{\xi t} \left| \mathcal{H}_0^t \right. \right] \int_{t_{j+1}}^{t_{j+1}} I_{\alpha}(L^\alpha h_i(X))_{t_{j},s} dY_s^i
\]

Moreover, for \( r = 1, \ldots, d_r \), if \( j_2 > j_1 \) we get that

\[
\tilde{E} \left[ \left( \int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \left( \int_{t_{j_1}}^{t_{j_1+1}} I_{\alpha}(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right) \left| \mathcal{H}_{t_{j_2}}^i \right. \right] = 0,
\]

and if \( j_2 < j_1 \) we get that

\[
\tilde{E} \left[ \left( \int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \left( \int_{t_{j_1}}^{t_{j_1+1}} I_{\alpha}(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right) \left| \mathcal{H}_{t_{j_1}}^i \right. \right] = 0.
\]

Hence, using the tower property of the conditional expectation we can write

\[
\tilde{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \left| \mathcal{Y}_t \right. \right] = \sum_{r=1}^{d_r} \tilde{E} \left[ \sum_{j=0}^{n-1} \left( \int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \right) \left| \mathcal{Y}_t \right. \right].
\]

By integration by parts formula for \( F_{s}^{Y_{t_0}} \cap \mathcal{Y}_s \)-semimartingales we have

\[
\int_{t_j}^{t_{j+1}} I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} dY_s^i = \left( Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_{\alpha}(L^\alpha h_i(X))_{t_{j},t_{j+1}} - \int_{t_j}^{t_{j+1}} \left( Y_{s}^i - Y_{t_j}^i \right) I_{\alpha} - (L^\alpha h_i(X))_{\tau(s),s} dV_s^\alpha.
\]

Moreover, we can rewrite the right hand side of the previous equality as a well defined \( \mathcal{H}_s \)-iterated integral and obtain

\[
\int_{t_j}^{t_{j+1}} I_{\alpha}(L^\alpha h_i(X))_{\tau(s),s} dY_s^i = \int_{t_j}^{t_{j+1}} \left( Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_{\alpha} - (L^\alpha h_i(X))_{\tau(s),s} dV_s^\alpha,
\]

which combined with equation (4.22) gives equation (4.19). Finally, using Theorem 24 with \( k=1 \) and repeating the same reasonings as before we get equation (4.20). \( \square \)
Lemma 37. Assume that $H(1)$ holds and $\varphi \in C^2_\mathbb{P}$. For $\alpha \in \mathcal{R}(\mathcal{M}_0(S_0))$ with $|\alpha|_0 \neq 1$ and $i \neq 0$ we have that

$$\mathbb{E} \left[ \mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t \alpha(L^\alpha h_i(X_s)) \tau(s) dY_s^i | \mathcal{Y}_t \right]^2 \right] \leq C\delta^2.$$  

Proof. We divide the proof into several steps.

Step 1. First we will find a more convenient expression for

$$\mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t \alpha(L^\alpha h_i(X_s)) \tau(s) dY_s^i | \mathcal{Y}_t \right].$$

Recall that $\alpha \in \mathcal{R}(\mathcal{M}_0(S_0))$ with $|\alpha|_0 \neq 1$ concides with the set of multiindices $\alpha = (\alpha_1)$ with $\alpha_1 \in \{1, \ldots, d_\nu\}$. Using Lemma 36, equation (4.19), and taking into account that $I_{\alpha}(L^\alpha h_i(X_s)) \tau(s) = L^{\alpha_1} h_i(X_s)$, we can write

$$\mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t \alpha(L^\alpha h_i(X_s)) \tau(s) dY_s^i | \mathcal{Y}_t \right] = \sum_{r=1}^{d_\nu} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} J^r_s dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} (Y_s^i - Y_{t_j}^i) L^{\alpha_1} h_i(X_s) dV_s^{\alpha_1} \right) | \mathcal{Y}_t \right].$$

Next, by Lemma 11 (2) we get that

$$\mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t \alpha(L^\alpha h_i(X_s)) \tau(s) dY_s^i | \mathcal{Y}_t \right] = \sum_{r=1}^{d_\nu} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} J^r_s (Y_s^i - Y_{t_j}^i) L^{\alpha_1} h_i(X_s) ds | \mathcal{Y}_t \right] = \sum_{r=1}^{d_\nu} (B_1(r) + B_2(r) + B_3(r)), $$

where

$$B_1(r) \triangleq \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} (J^r_s - J^r_{\tau(s)}) (Y_s^i - Y_{t_j}^i) L^{\alpha_1} h_i(X_s) ds | \mathcal{Y}_t \right],$$

$$B_2(r) \triangleq \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} J^r_{\tau(s)} (Y_s^i - Y_{t_j}^i) (L^{\alpha_1} h_i(X_s) - L^{\alpha_1} h_i(X_{\tau(s)})) ds | \mathcal{Y}_t \right],$$

$$B_3(r) \triangleq \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} J^r_{\tau(s)} L^{\alpha_1} h_i(X_s) (Y_s^i - Y_{t_j}^i) ds | \mathcal{Y}_t \right].$$

Step 2. Next, we prove the result for $B_1(r)$. Applying Jensen inequality, Cauchy-Schwarz inequality, Hölder inequality, Remark 5, that $Y^i$ is a Brownian motion under $\tilde{P}$ and Lemma 26 we have that

$$\mathbb{E} \left[ |B_1(r)|^2 \right] \leq C(t) \int_0^t \mathbb{E} \left[ \left( J^r_s - J^r_{\tau(s)} \right)^2 (Y_s^i - Y_{t_j}^i)^2 | \mathcal{Y}_t \right] \mathbb{E} \left[ \left( L^{\alpha_1} h_i(X_s) \right)^2 | \mathcal{Y}_t \right] ds.$$
Applying Hölder inequality and Proposition 25 we can conclude that

\[
\begin{align*}
&\leq C(t) \int_0^t \mathbb{E} \left[ (J^r_{s} - J^r_{\eta(s)})^2 (Y^i_{\eta(s)} - Y^i_{s})^2 \right] ds \\
&\leq C(t) \int_0^t \mathbb{E} \left[ (J^r_{s} - J^r_{\eta(s)})^{2+\varepsilon} 2(2+\varepsilon) \mathbb{E} \left[ (Y^i_{\eta(s)} - Y^i_{s})^{2+\varepsilon/(2+\varepsilon)} \right] ds \\
&\leq C(t) \delta \int_0^t \mathbb{E} \left[ (J^r_{s} - J^r_{\eta(s)})^{2+\varepsilon} \right] ds \\
&\leq C(t) \delta^2.
\end{align*}
\]

**Step 3.** Here, we prove the result for \( B_2 (r) \). Applying Jensen inequality and Cauchy-Schwarz inequality we get

\[
\begin{align*}
\mathbb{E} \left[ |B_2 (r)|^2 \right] &\leq C(t) \int_0^t \mathbb{E} \left[ |J^r_{\eta(s)} (Y^i_{\eta(s)} - Y^i_{s})|^2 |\mathcal{Y}_t\right] ds \\
&\leq C(t) \int_0^t \mathbb{E} \left[ |J^r_{\eta(s)} (Y^i_{\eta(s)} - Y^i_{s})|^2 \mathbb{E} \left[ |L^r h_i (X_s) - L^r h_i (X_{\tau(s)})|^2 |\mathcal{Y}_t\right] ds.
\end{align*}
\]

Applying Hölder inequality and Proposition 25 we can conclude that

\[
\mathbb{E} \left[ |J^r_{\eta(s)} (Y^i_{\eta(s)} - Y^i_{s})|^2 \right] \leq \mathbb{E} \left[ |J^r_{\eta(s)}|^{2+\varepsilon} \mathbb{E} \left[ |(Y^i_{\eta(s)} - Y^i_{s})|^{2+\varepsilon/(2+\varepsilon)} \right]^{\frac{2+\varepsilon}{\varepsilon/(2+\varepsilon)}} \right] \leq \delta \sup_{0 \leq s \leq t} \mathbb{E} \left[ |J^r_{s}|^{2+\varepsilon} \right]^{\frac{2+\varepsilon}{\varepsilon/(2+\varepsilon)}} \leq C\delta.
\]

On the other hand, we can write

\[
L^r h_i (X_s) - L^r h_i (X_{\tau(s)}) = \int_{\tau(s)}^s L^{(0,r)} h_i (X_u) du + \sum_{r_1 = 1}^{d_V} \int_{\tau(s)}^s L^{(r_1,r)} h_i (X_u) dV_u^{r_1}.
\]

As the worst rate is achieved by the terms with the stochastic integral, it suffices to show that

\[
\mathbb{E} \left[ \left( \int_{\tau(s)}^s L^{(r_1,r)} h_i (X_u) dV_u^{r_1} \right)^2 \right] \leq C\delta,
\]

which easily follows by Itô isometry and Remark 5.

**Step 4.** Finally, we prove the result for \( B_3 (r) \). We can write

\[
\begin{align*}
B_3 (r) &= \sum_{j=0}^{n-1} \mathbb{E} \left[ \int_{l_j}^{l_{j+1}} J^r_{l_{j+1}} L^r h_i (X_{l_j}) \left( Y^i_{l_{j+1}} - Y^i_{l_j} \right) ds \right] |\mathcal{Y}_l |
\\
&= \sum_{j=0}^{n-1} \mathbb{E} \left[ J^r_{l_{j+1}} L^r h_i (X_{l_j}) \int_{l_j}^{l_{j+1}} \left( Y^i_{l_{j+1}} - Y^i_{l_j} \right) ds \right] |\mathcal{Y}_l |
\\
&= \sum_{j=0}^{n-1} \mathbb{E} \left[ J^r_{l_{j+1}} L^r h_i (X_{l_j}) \int_{l_j}^{l_{j+1}} (s - t_j) dY^i_s \right] |\mathcal{Y}_l |
\\
&\triangleq \sum_{j=0}^{n-1} \mathbb{E} \left[ J^r_{l_{j+1}} L^r h_i (X_{l_j}) \int_{l_j}^{l_{j+1}} \beta^i_s dY^i_s \right] |\mathcal{Y}_l |
\end{align*}
\]

Moreover,

\[
J^r_{l_{j+1}} = \mathbb{E} \left[ D^r_{l_{j+1}} [\varphi (X_{l_j}) e^{\xi}] |\mathcal{H}^\xi_{l_{j+1}} \right],
\]

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by the Clark-Ocone formula. Using the product formula for the Malliavin derivative, we get
\[ D_{t_{j+1}}^r \left[ \phi \left( X_t \right) e^{\xi t} \right] = e^{\xi t} D_{t_{j+1}}^r \phi \left( X_t \right) + \phi \left( X_t \right) D_{t_{j+1}}^r e^{\xi t} \]

Therefore, using the tower property of the conditional expectation and the previous expression for the Malliavin derivative, we have
\[
\begin{align*}
\tilde{E} \left[ J_{t_{j+1}}^r L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i | \mathcal{F}_t \right] \\
= \tilde{E} \left[ \tilde{E} \left[ D_{t_{j+1}}^r \left[ \phi \left( X_t \right) e^{\xi t} \right] | \mathcal{H}_{t_{j+1}}^r \right] L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i | \mathcal{F}_t \right] \\
= \tilde{E} \left[ e^{\xi t} D_{t_{j+1}}^r \phi \left( X_t \right) L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i | \mathcal{F}_t \right] \\
+ \tilde{E} \left[ \phi \left( X_t \right) D_{t_{j+1}}^r e^{\xi t} L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i | \mathcal{F}_t \right].
\end{align*}
\]

Then,
\[
\tilde{E} \left[ |B_3 \left( r \right)|^2 \right] \leq \tilde{E} \left[ \left( \sum_{j=0}^{n-1} \tilde{E} \left[ J_{t_{j+1}}^r L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i | \mathcal{F}_t \right] \right)^2 \right] \\
\leq 2 \tilde{E} \left[ \sum_{j=0}^{n-1} e^{\xi t} D_{t_{j+1}}^r \phi \left( X_t \right) L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right]^2 \\
+ 2 \tilde{E} \left[ \sum_{j=0}^{n-1} \phi \left( X_t \right) D_{t_{j+1}}^r e^{\xi t} L^r h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right]^2 \\
= 2 A_1 \left( r \right) + 2 A_2 \left( r \right).
\]

Next, note that
\[
D_{t_{j+1}}^r e^{\xi t} = e^{\xi t} \left( \sum_{k=1}^{d_y} \int_0^t D_{t_{j+1}}^r h_k (X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_0^t D_{t_{j+1}}^r \left[ h_k (X_s)^2 \right] ds \right) \\
= e^{\xi t} \left( \sum_{k=1}^{d_y} \int_{t_{j+1}}^t D_{t_{j+1}}^r h_k (X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^t D_{t_{j+1}}^r \left[ h_k (X_s)^2 \right] ds \right) \\
= e^{\xi t} \left( \sum_{k=1}^{d_y} \int_{t_{j+1}}^t \alpha_s^{j,k,1} dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^t \alpha_s^{j,k,2} ds \right),
\]

where we have used that \( D_{t_s}^r h_k (X_s) = 0, \) \( s < u < t. \) In addition, note that
\[
e^{2 \xi t} = M'_0 \left( 2h \right) \exp \left( \sum_{i=1}^{d_y} \int_0^t h_i^2 \left( X_u \right) du \right),
\]
where
\[
M'_0 \left( h \right) = \exp \left( \sum_{i=1}^{d_y} \int_s^t h_i (X_u) dY_u^i - \frac{1}{2} \sum_{i=1}^{d_y} \int_s^t h_i^2 \left( X_u \right) du \right),
\]

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is an exponential martingale. Defining

\[
\Gamma(j_1, j_2) \equiv D_{j_1+1}^r \varphi(X_t) D_{j_2+1}^r \varphi(X_t) L h_i(X_{t_{j_1}}) L h_i(X_{t_{j_2}}) \exp \left( \sum_{i=1}^{d_y} \int_0^t h^2_i(X_u) \, du \right),
\]

and

\[
\Lambda(j_1, j_2) \equiv \varphi(X_t)^2 L h_i(X_{t_{j_1}}) L h_i(X_{t_{j_2}}) \exp \left( \sum_{i=1}^{d_y} \int_0^t h^2_i(X_u) \, du \right),
\]

we can write

\[
A_1(r) = \sum_{j_1, j_2=0}^{n-1} \mathbb{\hat{E}} \left[ \Gamma(j_1, j_2) M_0^r (2h) \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^1 dY^s \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^2 dY^s \right],
\]

and

\[
A_2(r) = \sum_{k_1, k_2=1}^{d_y} A_{2,1} (r, k_1, k_2) - \frac{1}{2} A_{2,2} (r, k_1, k_2) - \frac{1}{2} A_{2,3} (r, k_1, k_2) + \frac{1}{4} A_{2,4} (r, k_1, k_2),
\]

where

\[
A_{2,1} (r, k_1, k_2) \equiv \sum_{j_1, j_2=0}^{n-1} \mathbb{\hat{E}} \left[ \Lambda(j_1, j_2) M_0^r (2h) \int_{t_{j_1}}^{t_{j_1+1}} \alpha_s^{j_1, k_1} dY^s \int_{t_{j_2}}^{t_{j_2+1}} \alpha_s^{j_2, k_2} dY^s \right],
\]

\[
A_{2,2} (r, k_1, k_2) \equiv \sum_{j_1, j_2=0}^{n-1} \mathbb{\hat{E}} \left[ \Lambda(j_1, j_2) M_0^r (2h) \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^1 dY^s \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^2 dY^s \right],
\]

\[
A_{2,3} (r, k_1, k_2) \equiv \sum_{j_1, j_2=0}^{n-1} \mathbb{\hat{E}} \left[ \Lambda(j_1, j_2) M_0^r (2h) \int_{t_{j_1}}^{t_{j_1+1}} \alpha_s^{j_1, k_1} ds \int_{t_{j_2}}^{t_{j_2+1}} \alpha_s^{j_2, k_2} dY^s \right],
\]

\[
A_{2,4} (r, k_1, k_2) \equiv \sum_{j_1, j_2=0}^{n-1} \mathbb{\hat{E}} \left[ \Lambda(j_1, j_2) M_0^r (2h) \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^1 dY^s \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^2 dY^s \right].
\]

The result follows by applying Lemma 32, taking into account Remark 28, to the terms \( A_1, A_{2,1}, A_{2,2}, A_{2,3} \) and \( A_{2,4} \). \( \square \)

**Lemma 38.** Assume that \( H(2) \) holds and \( \varphi \in C^2_p \). For \( \alpha \in \mathcal{R} (M_1(S_0)) \) with \( |\alpha|_0 = 1 \) and \( i \neq 0 \) we have that

\[
\mathbb{\hat{E}} \left[ \mathbb{\hat{E}} \left[ \varphi(X_t) e^{\xi t} \int_0^t I_\alpha (L^\alpha h_i(X_s)) \tau_s, s \, dY^s \, | Y_t \right]^2 \right] \leq C \delta^4.
\]
Proof. The proof of this lemma is analogous to the proof of Lemma 37. Using Lemma 36, we can write
\[
\mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t I_\alpha(L^\alpha h_\tau(X))_\tau(s),s dY_s^\alpha | \mathcal{Y}_t \right]
\]
\[
= \sum_{r=1}^{d_V} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-} (L^\alpha h_\tau(X))_\tau(s),s dV_s^\alpha | \mathcal{Y}_t \right) \right]
\]
\[
\triangleq \sum_{r=1}^{d_V} A(r).
\]
Therefore, by Lemma 11 (2), we have that
\[
A(r) = \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} L^\alpha h_\tau(X_u) dV_u^\alpha \right) | \mathcal{Y}_t \right]
\]
\[
= 1_{\{\alpha_1 = 0, \alpha_2 = r\}} \mathbb{E} \left[ \left( \int_0^t (Y_{\eta(s)}^i - Y_s^i) J_s^r \left( \int_{\tau(s)}^s L^\alpha h_\tau(X_u) du \right) ds \right) | \mathcal{Y}_t \right]
\]
\[
+ 1_{\{\alpha_1 = r, \alpha_2 = 0\}} \mathbb{E} \left[ \left( \int_0^t (Y_{\eta(s)}^i - Y_s^i) \left( \int_{\tau(s)}^s J_s^r L^\alpha h_\tau(X_u) du \right) ds \right) | \mathcal{Y}_t \right]
\]
\[
\triangleq 1_{\{\alpha_1 = 0, \alpha_2 = r\}} A_1(r) + 1_{\{\alpha_1 = r, \alpha_2 = 0\}} A_2(r).
\]
Next, the proof follows by similar reasonings as in Lemma 37.

\[\square\]

Lemma 39. Assume that $H(2)$ holds and $\varphi \in C^2_F$. For $\alpha \in \mathcal{R}(\mathcal{M}_1(S_0))$ with $|\alpha|_0 = 0$ and $i \neq 0$ we have that
\[
\mathbb{E} \left[ \mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t I_\alpha(L^\alpha h_\tau(X))_\tau(s),s dY_s^\alpha | \mathcal{Y}_t \right]^2 \right] \leq C \delta^4.
\]

Proof. We divide the proof into several steps.

Step 1. Using Lemma 36, we can write
\[
\mathbb{E} \left[ \varphi(X_t) e^{\xi t} \int_0^t I_\alpha(L^\alpha h_\tau(X))_\tau(s),s dY_s^\alpha | \mathcal{Y}_t \right]
\]
\[
= \sum_{r_1=1}^{d_V} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-} (L^\alpha h_\tau(X))_\tau(s),s dV_s^\alpha | \mathcal{Y}_t \right) \right]
\]
\[
\times \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_{\alpha-} (L^\alpha h_\tau(X))_\tau(s),s dY_s^\alpha | \mathcal{Y}_t \right)
\]
\[
+ \sum_{r_1, r_2=1}^{d_V} \sum_{j=0}^{n-1} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left( \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} J_s^{r_1} dV_s^{r_1} dV_s^{r_2} \right) \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_{\alpha-} (L^\alpha h_\tau(X))_\tau(s),s dY_s^\alpha | \mathcal{Y}_t \right) \right]
\]
\[
\triangleq \sum_{r_1=1}^{d_V} \sum_{j=0}^{n-1} A(r_1, j) + \sum_{r_1, r_2=1}^{d_V} \sum_{j=0}^{n-1} A(r_1, r_2, j).
\]
Therefore, by Lemma 11 (2), we have that

$$A(r_1, j)$$

$$= \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} \tilde{E} \left[ J_{s}^2 | \mathcal{H}_t^0 \right] dV_s^1 \right) \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_j+1}^i - Y_s^i \right) \left( \int_{t_j}^{s} L^a h_i (X_u) dV_u^a \right) dV_s^a \right) | \mathcal{Y}_t \right]$$

$$= 1_{\{a_2=r_1\}} \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} \tilde{E} \left[ J_{s}^2 | \mathcal{H}_t^0 \right] \left( Y_{t_j+1}^i - Y_s^i \right) \left( \int_{t_j}^{s} L^a h_i (X_u) dV_u^a \right) dV_s^a \right) | \mathcal{Y}_t \right]$$

$$= 1_{\{a_2=r_1\}} \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_j+1}^i - Y_s^i \right) \left( \int_{t_j}^{s} L^a h_i (X_u) dV_u^a \right) dV_s^a \right) | \mathcal{Y}_t \right]$$

$$= 0.$$ 

and, by Lemma 11 (3) and Lemma 11 (2), we obtain

$$A(r_1, r_2, j)$$

$$= \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} \int_0^{s_2} J_{s_1,s_2}^1 dV_{s_1}^1 dV_{s_2}^1 \right) \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_j+1}^i - Y_s^i \right) \left( \int_{t_j}^{s} L^a h_i (X_u) dV_u^a \right) dV_s^a \right) | \mathcal{Y}_t \right]$$

$$= 1_{\{a_2=r_1\}} \tilde{E} \left[ \left( \int_{t_j}^{t_{j+1}} \left( Y_{t_j+1}^i - Y_s^i \right) \left( \int_{t_j}^{s} J_{u,s}^1 L^a h_i (X_u) du \right) dV_s^a \right) | \mathcal{Y}_t \right]$$

Hence, we can write

$$\tilde{E} \left[ \varphi (X_t) e^{\xi t} \int_0^t I_a (L^a h_i (X)) \tau (s) ds dY_s^a | \mathcal{Y}_t \right]$$

$$= \sum_{r_1, r_2=1}^{d_v} 1_{\{a_2=r_2, a_1=r_1\}} \tilde{E} \left[ \left( \int_0^t \left( Y_{\eta(s)}^i - Y_s^i \right) \left( \int_{\tau(s)}^{s} J_{u,s}^1 L^a h_i (X_u) du \right) ds \right) | \mathcal{Y}_t \right]$$

$$= \sum_{r_1, r_2=1}^{d_v} 1_{\{a_2=r_2, a_1=r_1\}} (B_1 (r_1, r_2) + B_2 (r_1, r_2) + B_3 (r_1, r_2) + B_4 (r_1, r_2),$$

where

$$B_1 (r_1, r_2) = \tilde{E} \left[ \left( \int_0^t \left( Y_{\eta(s)}^i - Y_s^i \right) \left( \int_{\tau(s)}^{s} \left( J_{u,s}^1 - J_{a,s}^1 \right) L^a h_i (X_u) du \right) ds \right) | \mathcal{Y}_t \right],$$

$$B_2 (r_1, r_2) = \tilde{E} \left[ \left( \int_0^t \left( Y_{\eta(s)}^i - Y_s^i \right) \left( J_{a,s}^1 L^a h_i (X_u) - L^a h_i (X_{\tau(s)}) \right) du \right) ds \right) | \mathcal{Y}_t \right]$$

$$B_3 (r_1, r_2) = \tilde{E} \left[ \left( \int_0^t \left( Y_{\eta(s)}^i - Y_s^i \right) \left( J_{a,s}^1 L^a h_i (X_u) - L^a h_i (X_{\tau(s)}) \right) \left( \int_{\tau(s)}^{s} L^a h_i (X_{\tau(s)}) \right) du \right) ds \right) | \mathcal{Y}_t \right]$$

$$B_4 (r_1, r_2) = \tilde{E} \left[ \left( \int_0^t \left( Y_{\eta(s)}^i - Y_s^i \right) \left( J_{a,s}^1 L^a h_i (X_{\tau(s)}) \right) \left( \int_{\tau(s)}^{s} du \right) ds \right) | \mathcal{Y}_t \right]$$

**Step 2.** That the terms $B_1 (r_1, r_2), B_2 (r_1, r_2)$ and $B_3 (r_1, r_2)$ have the right order is deduced analogously to the **Steps 2** and 3 in Lemma 37.
Step 3. Finally, we prove the result for \( B_4 (r_1, r_2) \). We can write

\[
B_4 (r_1, r_2) = \sum_{j=0}^{n-1} \tilde{E} \left[ \int_{t_j}^{t_{j+1}} J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i (X_{t_j}) \left( \int_{t_j}^{s} \frac{1}{2} (s-t_j)^2 dY^s | \mathcal{Y}_t \right) \right]
\]

where \( |\beta_s^j| \leq \delta^2 \). Moreover,

\[
J_{t_{j+1}, t_{j+1}}^{r_1, r_2} = \tilde{E} \left[ D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \{ \varphi (X_t) e^{\xi_t} \} | \mathcal{H}_{t_j} \right],
\]

by the Clark-Ocone formula. Using the definition of the iterated Malliavin derivative and the product formula, we get

\[
D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \{ \varphi (X_t) e^{\xi_t} \} = D_{t_{j+1}}^{r_1} \left( D_{t_{j+1}}^{r_2} \{ \varphi (X_t) e^{\xi_t} \} \right)
\]

Reasoning as in Step 4 of Lemma 37, we get that

\[
\tilde{E} \left[ |B_4 (r_1, r_2)|^2 \right] \leq \tilde{E} \left[ \sum_{j=0}^{n-1} \tilde{E} \left[ J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY^s | \mathcal{Y}_t \right] \right]^2
\]

\[
\leq C \tilde{E} \left[ \sum_{j=0}^{n-1} D_{t_{j+1}}^{r_1} e^{\xi_t} D_{t_{j+1}}^{r_2} \varphi (X_t) L^{(r_1, r_2)} h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY^s \right]^2
\]

\[
+ C \tilde{E} \left[ \sum_{j=0}^{n-1} e^{\xi_t} D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \varphi (X_t) L^{(r_1, r_2)} h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY^s \right]^2
\]

\[
+ C \tilde{E} \left[ \sum_{j=0}^{n-1} D_{t_{j+1}}^{r_1} \varphi (X_t) D_{t_{j+1}}^{r_2} e^{\xi_t} L^{(r_1, r_2)} h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY^s \right]^2
\]

\[
+ C \tilde{E} \left[ \sum_{j=0}^{n-1} \varphi (X_t) D_{t_{j+1}, t_{j+1}}^{r_1, r_2} e^{\xi_t} L^{(r_1, r_2)} h_i (X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY^s \right]^2
\]

\[
\triangleq C \left\{ F_1 (r_1, r_2) + F_2 (r_1, r_2) + F_3 (r_1, r_2) + F_4 (r_1, r_2) \right\}
\]

The term \( F_2 (r_1, r_2) \) is analogous to the term \( A_1 (r) \) in Lemma 37 and the terms \( F_2 (r_1, r_2) \) and \( F_3 (r_1, r_2) \) are analogous to the term \( A_2 (r) \) in Lemma 37. For the term \( F_4 (r_1, r_2) \) we
have that

\[
D_{t_{j+1}, t_{j+1}}^{r_1, r_2} e^{\xi t} = D_{t_{j+1}}^{r_1} \left\{ e^{\xi t} \left\{ \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_2} \left[ h_k(X_s)^2 \right] ds \right\} \right\} = e^{\xi t} \left\{ \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_1} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_1} \left[ h_k(X_s)^2 \right] ds \right\} \times \left\{ \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_2} \left[ h_k(X_s)^2 \right] ds \right\} + e^{\xi t} \left\{ \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}, t_{j+1}}^{r_1, r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_y} \int_{t_{j+1}}^{t} D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \left[ h_k(X_s)^2 \right] ds \right\}
\]

All the terms obtained in the previous expression can be dealt analogously to the terms in Lemma 37 except the terms

\[
G(j, r_1, r_2) \triangleq e^{\xi t} \sum_{k_1, k_2=1}^{d_y} \left( \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_1} h_{k_1}(X_s) dY_s^{k_1} \right) \left( \int_{t_{j+1}}^{t} D_{t_{j+1}}^{r_2} h_{k_2}(X_s) dY_s^{k_2} \right).
\]

Let

\[
H \triangleq \hat{E} \left[ \sum_{j=0}^{n-1} \varphi(X_t) G(j, r_1, r_2) L^{(r_1, r_2)} h_i(X_t) \int_{t_j}^{t_{j+1}} \beta^i_s dY_s^i \right]^2.
\]

Defining

\[
\Lambda(j_1, j_2) \triangleq \varphi(X_t)^2 L^{(r_1, r_2)} h_i(X_{t_{j_1}}) L^{(r_1, r_2)} h_i(X_{t_{j_2}}) \exp \left( \sum_{i=1}^{d_y} \int_{0}^{t} h_i^2(X_u) du \right),
\]

we can write

\[
H = \sum_{k_1, \ldots, k_4=1}^{d_y} \sum_{j_1, j_2=0}^{n-1} \hat{E} \left[ \Lambda(j_1, j_2) M_4^i (2h) \right. \times \left( \int_{t_{j_1+1}}^{t} D_{t_{j_1+1}}^{r_1} h_{k_1}(X_s) dY_s^{k_1} \right) \left( \int_{t_{j_1+1}}^{t} D_{t_{j_1+1}}^{r_2} h_{k_2}(X_s) dY_s^{k_2} \right) \left. \times \left( \int_{t_{j_2+1}}^{t} D_{t_{j_2+1}}^{r_1} h_{k_3}(X_s) dY_s^{k_3} \right) \left( \int_{t_{j_2+1}}^{t} D_{t_{j_2+1}}^{r_2} h_{k_4}(X_s) dY_s^{k_4} \right) \times \left( \int_{t_{j_1}}^{t_{j_1+1}} \beta^i_s dY_s \right) \left( \int_{t_{j_2}}^{t_{j_2+1}} \beta^i_{s} dY_s \right) \right],
\]

and the result follows from Lemma 32 and Remark 28.

\[\square\]

Remark 40. Following Remarks 27 and 33, the results in Lemmas 38 and 39 can be extended analogously to \(m > 2\) and \(\alpha \in \mathcal{R}(\mathcal{M}_{m-1}(S_0))\) with \(|\alpha|_0 \in \{0, \ldots, m - 1\}\) without any additional difficulties.
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