Thermodynamics, topology and dimension of initial real tunneling manifolds

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Abstract

Based on the Non-Boundary proposal in quantum cosmology, we develop the argument that initial real tunneling in quantum gravity might be contemplated as a thermodynamical analogous to a black hole condensate in equilibrium with Hawking’s radiation in a box. The total entropy is always maximized in the Lorentzian sector of the theory, and, in this sense, tunneling is predicted. The maximum relative increase of the entropy is achieved if the Euclidean geometry has topology \( S^2 \times M^2 \) (\( M^2 \) so far an arbitrary bidimensional manifold) and the total spacetime dimension is four. We propose that a tunneling from nothing configuration in quantum cosmology be contemplated as an initial condition for the Universe.

1. Introduction.

Real tunneling manifolds are a very restrictive set of solutions of Euclidean Einstein’s equations with positive definite metrics, compact topologies and finite actions being the ”complexified” spacetimes of some suitable real Lorentzian signature metrics solving the field equations with a new time coordinate \( t = i \tau \) (\( \tau \) standing for the \( x^0 \) coordinate in the Euclidean geometry). These have been described as gravitational instantons. On the other hand, a general positive metric will not have a section on which the metric is real and Lorentzian. The importance of these metrics comes from the fact that they must dominate in the path integral defining the wave function of a closed Universe. Moreover, a real tunneling solution describes transitions from a purely ”Euclidean” metric to a purely Lorentzian (i.e., a ”tunneling from nothing” configuration). In this scenario, the creation of the Universe ”ex nihilo” is due to the presence of an effective cosmological constant and the result may also be considered as a sort of classical change of signature or a bounce in spacetime (i.e., the transition between two solutions with the same boundary conditions having different actions). In quantum cosmology, a tunneling solution of the Wheeler-DeWitt equation in minisuperspace with a positive cosmological constant would represent the quantum rate of production corresponding to a spacetime of the type of a round Euclidean sphere joined on an equator to the Lorentzian space at its radius of maximum contraction.

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Based on the Non-Boundary proposal we can also visualize gravitational instantons as having thermodynamical properties rather like a black hole. Though, it is not obvious from the behaviour of the wave function in the saddle points approximation, $\Psi \sim \exp[-I]$ (here, $I$ stands for Einstein’s Euclidean action depending on the topology). On the other hand, semiclassically, one can assign a probability measure to the compact instanton simply as the square of the wave amplitude for the Universe, i.e., $P = |\Psi|^2 \sim \exp\{-2\text{Re}[I]\}$. It leads to computing the logarithm of the number of available (microcanonical) states as in the case of a thermal system with an entropy rather like $S = \log(P) \sim -2\text{Re}[I]$, which is finite and non vanishing due to the fact that the topology of the instanton is not trivial, a feature which also holds in the black hole case (a quite direct and interesting example of this is observed in four dimensions when there exists a positive cosmological constant, $\lambda$, so that $I \sim -\kappa\chi/(16\pi G\lambda)$ – $\chi$ stands for the Euler-Poincaré characteristic of the instanton and $\kappa \leq 12\pi^2$ is some numerical constant).

For a compact manifold there exists obviously no real time, so there should not either be necessarily an ordinary space. For instance, we might expect the available dimensions of the global manifold to be an arbitrary, yet undetermined, number $n$. Moreover, that part of the compact instanton describing an effective change of signature having a Lorentzian sector will probably consist on a $S^d$ ($d$, so far, also some free parameter). This is because we expect to extend the analogy with black holes to the existence of thermal Green functions on the complexified spacetime having some real period equating the inverse of the physical temperature. It imposes a periodic topology $S^d$. If the geometry is connected the instanton would consist on a generic topological product, $\mathcal{M}^n \approx S^d \times \mathcal{M}^{n-d}$. Nonetheless, as we will see, thermodynamical consistence will strongly restrict the topology of the instanton.

For $n$-dimensional gravitational instantons (when $n > 3$) as well as for black hole thermodynamics one finds a negative specific heat. This is in conflict with the positivity of $(\delta U)^2$ for the fluctuations of the thermal energy. It means that both systems are thermally unstable. As such this is not surprising since instability is typical for gravitational phenomena, a feature which in turns seems to appear even in Newtonian theory as showed by Jeans. Moreover, one can use the the microcanonical ensemble to obtain an equilibrium configuration. This was done previously by Gibbons and Perry who considered a black hole immersed in a bath of radiation with fixed volume: they obtained that, at a sufficiently high energy density, a black hole nucleates from pure radiation in a way analogously to a liquid drop can condense out of saturated vapour. From the similarities existing between black holes and gravitational instantons, the previous heuristic picture may also be seen as useful in order to understand the physical phenomenon which underlies in a generic tunneling configuration even in the cosmological case (so that, the phenomenon of creation of the Universe from nothing be compared to that of a condensed liquid drop from saturated vapour).

Quantum gravity is the available technology of the fabric of spacetime and, indeed, as every technology does, it is constrained by thermodynamics; to us, it remains to obtain the answer to the question: what is the topology of the Euclidean instanton from which a stable thermal Universe may have been nucleated using this technology? This problem is admittedly elusive, a fully satisfactory solution would require a more convincing theory of quantum gravity, here, however, we may try to deal with some simplified arguments, in the spirit of the semiclassical approximations, so that the question be affordable.
The paper is organized as follows, in section 2 we review the thermodynamical properties of those gravitational instantons corresponding to ”tunneling from nothing configurations”. In section 3 we develop the analogy between thermal equilibrium of a black hole made of pure radiation inside a box and the nucleation of a initial tunneling manifold. Section 4 is devoted to obtaining the ”entropically favoured states”, that is, the topology and dimension corresponding to the maximization of a relative entropy function, already defined in section 3, upon considering the Boltzmann thermal equilibrium of the nucleating manifold with the radiation in an arbitrary space-like cavity. Our final conclusions are written in last section.

2. Topology and thermodynamics

A real tunneling manifold \( \mathcal{M}^n \) might be also contemplated as a gravitational instanton in an arbitrary dimensional ”complexified” spacetime. In this case, it is described by means of its classical action and it has a non zero entropy

\[
S = -2I(\mathcal{M}^n/2),
\]

where \( I(\mathcal{M}^n/2) \) is the classical Euclidean action computed for half of the compact manifold joined to the Lorentzian geometry at the spacelike hypersurface of vanishing extrinsic curvature.

The corresponding temperature of the instanton is given by the usual thermodynamical formula

\[
T^{-1} = \frac{\partial S}{\partial U},
\]

\( U \) standing for the thermal energy of the instanton. On the other hand, the temperature is the inverse of the period of the complex time coordinate of a d-dimensional sphere of radius \( r_d \) immersed into the Euclidean n-dimensional geometry \( \mathcal{M}^n \) in the form of an arbitrary topological product \( \mathcal{M}^n \approx S^d[r_d] \times \mathcal{M}^{n-d} \), i.e.,

\[
\beta = T^{-1} = 2\pi r_d.
\]

In order to compute the action we require the expression for the Ricci tensor of \( S^d[r_d] \), it is given by,

\[
R_{ab}[S^d] = \frac{(d-1)}{r_d^2} g_{ab}[S^d],
\]

and,

\[
R[S^d] = \frac{(d-1)d}{r_d^2}.
\]

On the other hand, Einstein’s equations are given in terms of an effective cosmological constant

\[
R_{\mu\nu}[\mathcal{M}^n] = \frac{2\lambda}{n-2} g_{\mu\nu}[\mathcal{M}^n].
\]

Equations (4)-(6) directly impose

\[
r_d^2 = \frac{(n-2)(d-1)}{2\lambda},
\]
that is,
\[
\beta = 2\pi \left( \frac{(n-2)(d-1)}{2} \right)^{1/2}\lambda^{-1/2}
\]  

(8)

As a matter of fact, we can not obtain the general expression for the entropy corresponding to the nucleating manifold (i.e., the d-sphere, which undertakes an effective change of signature in the \(x^0\) coordinate when analytically continuing the Euclidean metric into the Lorentzian sector). This is because, the Euclidean action also depends on the value of Newton’s constant in \(n\)-dimensions \(G_n = \left(\frac{k_n}{m_p}\right)^{n-2}\), and the \(k_n\) are arbitrary so far but in the standard four dimensional case \((k_4 = 1)\),

\[
S = -2I(\mathcal{M}^n/2) = \frac{2}{16\pi G_n} \int_{d/2} \{g[S^d]\}^{1/2} d^d x \int_{\mathcal{M}'} (R - 2\lambda)\{g[M']\}^{1/2} d^{n-d} x = \frac{2}{16\pi G_n} \frac{4}{n-2} \lambda^{1/2} \mathcal{V}_d \mathcal{V}_{M'} = \zeta(n, d) \frac{\lambda^{-(n-2)/2}}{G_n}.
\]  

(9)

We used the fact that the compactified internal space \(\mathcal{M}'^{n-d}\) has a volume \(\mathcal{V}_{M'} \sim \lambda^{-(n-d)/2}\); it will remain compact after the effective change of signature. The function \(\zeta(n, d)\) depends on the topology of the global manifold. In the special case that \(d = n\), we get

\[
\zeta(n, n) = \frac{v_{n-2}}{4} \frac{(n-1)(n-2)}{2} \left(\frac{n-2}{2}\right)^{2},
\]  

(10)

where,

\[
v_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)},
\]  

(11)

henceforth we obtain the result

\[
S = \frac{A}{4G_n},
\]  

(12)

\(A\) standing for the surface area of the horizon at which the effective change of signature takes place. It is exactly the same expression than the one obtained for black holes. It suggests a deep connection between compact instanton thermodynamics and that from black holes, though, the physical analogy is problematic since, in the black hole case, the area law corresponds to the presence of an horizon in a non compact Lorentzian spacetime.

On the other hand, from Eqs. (8)-(9) we can write for the entropy

\[
S = \gamma(n, d) \frac{\beta^{n-2}}{G_n},
\]  

(13)

for \(\gamma(n, d)\) a known function of the parameters of the topology of the compact manifold. Now the thermal energy is simply,

\[
U = F + TS = \int \beta^{-2} S d\beta + \beta^{-1} S = \gamma(n, d) \frac{\beta^{n-3}}{G_n} \frac{n-2}{n-3},
\]  

(14)

and,

\[
S = U \beta^{n-3} = \frac{n-2}{n-2} = 2\pi r_d U^{n-2} < 2\pi r_d U;
\]  

(15)
which is Bekenstein’s bound\[13\]. Let us write the above expressions in the form
\[S = C(d, n, k_n) \left( \frac{U}{m_{Pl}} \right)^{(n-2)/(n-3)}.\] (16)
Here, \(C(d, n, k_n)\) is a function of the total dimension and the topology which also depends on the physically undertermined quantities \(k_n\); these values should only be restricted by the fact that, recalling a classical analogy with statistical mechanics, we expect \(S\) to increase with the total dimension \(n\) in a way that should mimic a comparative increase of the volume of phase space with dimensionality.

Heat capacity can not be defined positive for a gravitational instanton, therefore, thermal energy fluctuations become imaginary, a signal that the system is thermally unstable; this is also the case for a generic black hole. In the later case, however, one can consider the canonical ensemble for a hole inside a cavity full of radiation in Boltzmann equilibrium with it\[14\] it leads to a thermodynamically stable description making possible to understand the nucleation of black holes from hot flat space\[15\]. The latter suggests to extend the themodynamical analogy of compact instantons with black holes to the Lorentzian sector of the tunneling manifold where there would also exist radiation and \((d - 1\ dimensional)\) volume terms.

More simply, let us study the microcanonical ensemble concerning the maximization of the total entropy of the system with radiation in a \textit{fixed} volume at a given \textit{constant} total energy.

3. Thermal equilibrium with radiation

Correspondingly to the non-vanishing temperature, we should consider the black body radiation inside an arbitrary \(d - 1\) dimensional cavity of the nucleated space having a volume \(V_{d-1}\). Boltzmann thermal equilibrium is reached for a given volume and fixed total energy configuration that maximizes the total entropy of the system including the radiation terms,
\[S_T = C(U/m_{Pl})^{(n-2)/(n-3)} + (1 + \frac{1}{d-1})T^{d-1}V_{d-1}\sigma(d)\] (17)
\[E = U + \sigma(d)V_{d-1}T^d\] (18)
Here, \(\sigma(d)\) is Stefan’s constant in \(d\) dimensions and we assume the standard expressions for a black body gas in a \(d - 1\) dimensional cavity. This consideration is in complete analogy with that from the equilibrium of a black hole with Hawking’s radiation in a box discussed earlier by Gibbons and Perry\[10\].

Now, upon eliminating \(T = [(E - U)/(\sigma_dV_{d-1})]^{1/d}\) from the above equations and defining
\[A(E, V_{d-1}) = \frac{d}{d-1}(\sigma_dV_{d-1})^{(d-1)/d}E^{1/d};\] (19)
\[g(E) = C(U/m_{Pl})^{(n-2)/(n-3)};\] (20)
\[\Omega = S_T/g(E);\] (21)
\[\omega = A(E, V_{d-1})/g(E);\] (22)
\[u = U/E,\] (23)
\[\]
we can express the total entropy as a function of the previous adimensional variables,

\[ \Omega(u, \omega) = u^{(n-2)/(n-3)} + \omega(1 - u)^{(d-1)/d}, \]  

(24)

\( \Omega \) represents the relative total entropy weighted by the entropy of an instanton of total thermal energy. It does not depend on the exact topology of the compactified space \( \mathcal{M}^{n-d} \) neither on the value of the unknown constants \( k_n \). On the other hand, we propose that the energy variable \( 0 \leq u \leq 1 \) had the following heuristic meaning: a configuration such that \( u < 1 \) would represent a tunneling configuration with some amount of radiation in thermal equilibrium (at some temperature) with the gravitational field (i.e., the horizon would absorb exactly the same energy than it emits). It is in full analogy with black hole nucleation from hot flat space \( \mathbb{R}^4 \). The radiation would have, therefore, an energy which is a portion of the total one, \( E_{rad} = E(1 - u) \). Of course, the solution to \( \Omega(u, \omega) = 1 \) is also \( u = 1 \) for all values of \( \omega \) (the instanton configuration).

Thus, in the above scenario, we are led to a picture which is the cosmological analog to a black hole condensate in thermodynamical equilibrium with pure radiation \( \mathbb{R}^4 \). Here, however, the existence of radiation is a consequence of the condensation of a tunneling configuration in its Lorentzian sector (since, previously to the creation of the Universe from nothing it is impossible to define radiation in a space-like cavity). This is conversely to the black hole case in which radiation is previous to the existence of the hole. Thus \( \Omega(1, \omega) = 1 \) would be the total relative entropy in the special case that there were no volume terms. Nucleation is entropically favoured if and only if there exist a value \( u' < 1 \) such that \( \Omega(u', w) > 1 \). Therefore, tunneling would be allowed if the total entropy had increased in the Lorentzian sector with respect to the entropy defined by means of the topological properties of the compact instanton. In the black hole case, nucleation will imply, in general, a spontaneous increase of the free energy \( F \) (since hot flat space in four dimensions have negative \( F \) whereas black hole Helmholtz’s free energy is necessarily positive). That difficulty, however, does not apply to the cosmological case since, as we just mentioned, the standard picture evolution is formally reversed.

In order to find the value \( u' \) that maximizes the relative entropy in Eq. (24), we should obtain \( \omega = \omega' \) satisfying the constraint

\[ \frac{\partial \Omega}{\partial u}|_{u'} = 0 \]  

(25)

or,

\[ u' = \left\{ \frac{(n-2)}{(d-1)(n-3)} + 1 + \frac{1}{n-3} \right\}(1 - u')^{1/d}u'^{1/(n-3)} \]  

(26)

On the other hand, the global maximum of \( \Omega \) is reached at \( u = 0 \), unless there were a different value \( \Omega(u') \) such that

\[ \Omega(0) = \omega' = u'^{(n-2)/(n-3)} + \omega'(1 - u')^{(d-1)/d} = \Omega(u'), \]  

(27)

solving the above equation for \( \omega' \) we obtain,

\[ \omega' = \frac{u'^{(n-2)/(n-3)}}{1 - (1 - u')^{(d-1)/d}}. \]  

(28)
now, from Eq. (26) and Eq. (28), we directly get

\[ 1 - u' = \left( \frac{d(u' - n + 2) + u'(n - 3)}{d(2 - n)} \right)^d, \] (29)

or, upon doing the definitions

\[ a(n, d) = \frac{d - 3 + n}{n - 2}; \] \hspace{1cm} (30)

\[ x = u' a(n, d) \] \hspace{1cm} (31)

and substituting them in Eq. (29) we finally get:

\[ 1 - \frac{x}{a(n, d)} = \left[ 1 - \frac{x}{d} \right]^d. \] (32)

The above equation has solutions \( x \) different from zero if and only if \( a > 1 \), that is \( d > 1 \), this is also physically consistent since \( S^1 \) could not represent a tunneling manifold. For \( n = d = 4 \), we obtain, \( a[S^4] = 5/2 \), then, \( u' = x/a[S^4] \approx 0.97702 \) is the fraction of the energy corresponding to the nucleation of deSitter spacetime from nothing. Notice that it coincides with Gibbons and Perry heuristic estimates in the case of a neutral black hole condensate from pure radiation. The latter amounts to the well known analogy between deSitter’s spacetime and Schwarzschild’s black hole thermodynamics.

Yet, since \( \Omega(u') \) should also be the local maximum, it satisfies,

\[ \frac{\partial^2 \Omega}{\partial u^2} \bigg|_{u',u} < 0. \] (33)

This directly requires -using Eq.(26)

\[ \frac{d}{d + n - 3} < u' < 1, \] (34)

that is \( n > 3 \). Therefore, \textit{thermodynamics requires that the minimum number of total dimensions for a tunneling manifold be four!}

Moreover, it is a simple task to prove that the equilibrium temperature coincides with the instanton temperature. This can be easily done, for instance, for the special case that \( n = d \) (see Appendix A).

4. Topology and dimension

Since \( u = U/E \sim r_d^{n-3} \sim \lambda^{-(n-3)/2} \), a finite value of \( u = u' \) which maximizes the total entropy also represents a finite value of the effective cosmological constant whose dynamical meaning is that of a large potential for the scalar matter field (inflaton). Its value would remain approximately constant if we could neglect back reaction of radiation on the gravitational field. In this case \( E_{\text{rad}} \ll E \) and \( u' \sim 1 \). This fact could be considered as a physical requirement of consistence in the quantum cosmological tunneling scenario. Moreover, more straightforwardly, a solution such that \( u' \approx 0 \) amounts to \( \lambda/m_{Pl}^2 \gg 1 \) and
this clearly breaks down the semiclassical approximations on which the thermal description was based.

On the other hand, the solutions \( u' = ax \) of Eq. (32) have the following behaviour for large \( n \) and finite \( d \),

\[
u'_{n \gg 1, d \sim O(1)} \sim \frac{2d}{n^{3}}[1 - \frac{4(d - 2)}{3n} + O(n^{-2})] \to 0. \quad (35)
\]

Which seems to be in contradiction with the thermal description after the previous heuristic reasoning. It means that, at least in the semiclassical domain of energies, we must consider \( d \sim O(n) \), in particular, \( d = n(1 - \alpha) \), for some \( \alpha \leq \alpha_{\text{max}} < 1 \) and \( n \leq d/(1 - \alpha_{\text{max}}) \).

Now, in order to find the solution of Eq. (32), we need to know the range of \( a(n, d) \) for the allowed tunneling configurations. First, since \( n \geq 4 \), we have from Eq. (30), and that \( d = n(1 - \alpha) \),

\[
\frac{1}{n} = \frac{1}{2} \left[ 1 - \frac{1 - 2\alpha}{2a - 3} \right] \leq \frac{1}{4}, \quad (36)
\]

that is,

\[
1 < a \leq \frac{5}{2} - 2\alpha = \frac{1}{2} + \frac{2d(n)}{n}. \quad (37)
\]

The topology of \( M^n \) is encoded within the variable \( a \), for instance, the case \( a = 3/2 \) may correspond to the topological product of identical spheres \( S^{d} \times \cdots \times S^{d} \) whereas \( a = 5/2 \) denotes the 4-sphere \( S^{4} \). We can assume that \( a \) takes all possible rational values in its interval and so, it is a dense variable. Yet, \( a > 1 \) and the inequality above also suggests that \( d \) must be some suitable function of \( n \) (otherwise \( \lim_{n \gg 1} a \leq 1/2 \), contrary to the assumption that there exist non zero solutions of Eq. (32)).

Since we would like to find the entropically favoured topologies, we should evaluate the absolute maximum of \( \Omega[u'] = \omega' \) at the solutions of Eq. (32), i.e., \( u' = x(a, d)/a \). It is easy to show (see Appendix B) that, in the semiclassical regime, i.e., when \( u' \sim 1 \), \( \omega' \) in Eq. (28) is approximated by,

\[
\omega' \sim \exp\left\{ \frac{1}{d\alpha^3} \right\} + O((1 - u')^2), \quad (38)
\]

Moreover, the simpler approximation to the solution of Eq. (32) is to take \( x = au' \sim a \) independent of \( d \); a much better approximation is (see also Appendix B)

\[
x \sim a \exp\left\{ -\frac{1}{a^3} \left[ 1 - \frac{a}{d} + O((1/d)^2) \right] \right\}, \quad (39)
\]

Since \( S^1 \) is not a tunneling manifold, we have \( d \geq 2 \) and recalling Eq. (30),

\[
d = n(a - 1) - 2a + 3 \geq 2 \quad (40)
\]

namely,

\[
a \geq \frac{n - 1}{n - 2}, \quad (41)
\]

but, using Eq. (37), also recalling that thermal (semiclassical) description requires \( d = d(n) \sim O(n) \), we get

\[
\frac{n - 1}{n - 2} \leq \frac{1}{2} + \frac{2d(n)}{n}, \quad (42)
\]

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obtaining,

\[ d(n) \geq \frac{n^2}{4(n - 2)}. \]  

Eq. (43) could be considered as an additional physical constraint for the effective \( d \)-dimensional signature change spacetime geometry. Thus, for instance, the maximum total dimension of the Euclidean manifold having a Lorentzian 4-dimensional sector could be estimated to be \( n_{\text{max}} = 13 \). Yet, this is in accordance with Embacher’s related previous calculations for the available Euclidean geometries in quantum gravity, using the procedure of minimizing Einstein’s action for the Euclidean manifold corresponding to different dimensions having various topologies \[1\].

Feeding \( d = n(1 - \alpha) \) in Eq. (43) we obtain

\[ \alpha_{\text{max}} < \frac{3}{4}. \]  

In order to find \( \alpha_{\text{max}} \), we may consider the following physical conderation: if a generic \( n-d \) dimensionalsional space remains compactified during cosmological evolution, there would exist, associated to it, an inverse temperature whose maximum possible (non equilibrium) value is in terms of the radius of a maximal \( n - d \)-sphere, \( S^{n-d} \)

\[ \beta_c = 2\pi r_{n-d}, \]  

where \( r_{n-d} \) is,

\[ r_{n-d}^2 = \frac{(n - 2)(n - d - 1)}{2\lambda} \]  

The previous simplification does not seem to be relevant to evaluate \( \Omega \) for recall that \( \Omega \) does not depend on the exact topology of \( \mathcal{M}^{n-d} \).

Yet, the cosmological horizon holds an inverse temperature given by Eq. (3) and Eq. (7) and, in order to prevent it from evaporating out (so that the horizon radiates less energy than it absorbs from extra dimensions) it is required that the compactified space had a highest possible associated inverse temperature satisfying,

\[ \beta_c \leq \beta_{\text{Horizon}} = \beta(r_d), \]  

the latter directly amounts to \( n - d \leq d \), i.e., \( \alpha_{\text{max}} = 1/2 \). It is compatible with the constraint in Eq. (44). Moreover, the compactified space will remain "small" during cosmological evolution until it eventually reaches some Planckian size whereas cosmological horizon inflates.

We have finally reached the state on which we would be able to give an answer to the question which is the aim of this paper: in the semiclassical inflationary regime corresponding to the Lorentzian sector of a generic initial real tunneling manifold, the only available Euclidean topologies compatible with a stable thermal regime are \( \mathcal{M}^{n} \approx S^d \times \mathcal{M}^{n-d} \), where \( \dim[\mathcal{M}] \leq 2d \), and \( d \) is such that there exist an absolute maximum for \( \omega' \) written in Eq (38). The exact topology of \( \mathcal{M}^{n-d} \) remains in principle arbitrary.
5. Conclusions

A maximal increase of the relative entropy is obtained when the variable \( a \) reaches its minimum value, i.e., for \( a = 3/2 \). The latter encodes that the Euclidean topology of the tunneling manifold should be \( S^d \times M' \). The absolute maximum is now for the minimal available dimension \( d = 2 \) of the effective signature change geometry so that the topology of the compact instanton is \( S^2 \times M'^2 \). The total dimension of spacetime should be four, at least for the range of energies compatible with the semiclassical thermal approximations. The previous beautiful, although admittedly heuristic, result indicates that a typical physical "a priori" such as the dimension of spacetime should not be assumed but derived from more developed theories of gravity such as string theory but that, since Einstein’s gravity should be recovered at low energies, the compactification to four dimensional spacetime have to take place by consistence of the semiclassical theory as we have stablished here.

On the other hand, Eq. (38) states that a relative increase of the total entropy is always achieved in the Lorentzian sector of a tunneling manifold. This suggest that a "tunneling from nothing" configuration should be considered as an initial condition for the Universe in the sense of thermodynamics, i.e., that there should exist an \( \text{arrow of time} \) in the direction of increased entropy. It also means that tunneling solutions can be thought as initial states in cosmology. Conversely, a "collapse into nothing" configuration seems to be strongly thermodynamically prohibited by these (semiclassical) estimates.

The latter might be seen as an argument against the elaborated comments of Kiefer and Zeh about the non existence of a generic arrow of time in cosmology. This is not the case. The author agrees with the former in their line of thought concerning the partial effectiveness of cosmological decoherence in order to define an arrow of time. The latter is true in particular for the solutions of Wheeler-DeWitt equation which could not (in principle) be used to describe a formal evolution in time, but, on the other hand, the existence of a non trivial entropy seems to be strongly related on peaking up some suitable boundary condition for the wave function of the Universe (in this case the Hartle-Hawking state). That is why, indirectly, a thermal picture, being not independent on the selection of the boundary conditions, arises. In order to derive consistence one can device simple \textit{Gedankenexperimente} as the one we have suggested here. In general we should never ignore the very existence of some, perhaps, absolute (i.e., topological) entropy in the discussions, a fact which should be in the core of quantum gravity.

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Apendix A

Let us proove that the equilibrium temperature is exactly that of the instanton in the special simpler case that the topology of the compact manifold is \( S^n \). In this case we have
to maximize $S_T$, at fixed total energy (we set $E = 1$ for convenience) and fixed volume $V_{n-1}$.

$$S_T = \frac{1}{4} v_{n-2} r_n^{-2} + \frac{d}{d-1} \sigma_n V_{n-1} T^{n-1}$$

$$E = 1 = \frac{n^3 (n-2) v_{n-2}}{8\pi (n-3)} + \sigma_n V_{n-1} T^n.$$

We now define the quantities $A = n/(n-1)(\sigma_n V_{n-1})^{1/d}$, $C = v_{n-2}/4$, $B = v_{n-2}(n-2)/(8\pi(n-3))$ and $u = B r_n^{n-3}$; so that the entropy of a generic tunneling configuration be written as

$$S(1) = \frac{C}{B(n-2)/(n-3)},$$

$$S_T = S(1) u^{(n-2)/(n-3)} + A (1-u)^{(n-1)/n}. \tag{A.1}$$

Yet, the equilibrium temperature is from $A.1$,

$$T_0 = \left\{ \frac{1-u}{\sigma_n V_{n-1}} \right\}^{1/d} = \frac{n (1-u)^{1/n}}{(n-1)A}, \tag{A.2}$$

but, from $A.2$, the constraint $\partial S/\partial u = 0$ implies

$$A = (1-u)^{1/n} \frac{2\pi n}{n-1} \left( \frac{u}{B} \right)^{1/(n-3)}. \tag{A.3}$$

Now, using $u = B r_n^{n-3}$, we get, from $A.3$, and $A.4$,

$$T_0 = (2\pi r_n)^{-1}, \tag{A.4}$$

which coincides with the temperature of the compact instanton.

**Appendix B**

If we take into account Eq. (30), i.e., $1/n = (a-1)/(d+2a-3)$, Eq. (28) becomes,

$$\omega' = \frac{u^{(d-1)/(d-a)}}{1 - (1-u')^{(d-1)/d}}, \tag{B.1}$$
if we define $\gamma \equiv [1 - x/d]^d$, then $u' \approx \exp[-\gamma]$ and $B$. 1 may be conveniently approximated by (assuming $\gamma \ll 1$),

$$\omega' \approx \exp[-\gamma \frac{d-1}{d-a} + \gamma^{(d-1)/d}],$$

B. 2

that is, since semiclassically $a \sim x$, neglecting terms which are $O[\gamma^2] = O[(u' - 1)^2]$, we get

$$\omega' \sim \exp[\frac{1}{d}[1 - x/d]^{d-1}] + O[(u' - 1)^2].$$

B. 3

On the other hand, in order to obtain Eq. (42) we use, as a first approximation, the solutions of Eq. (32) for $d = 2$, i.e., $x(a, d) \sim x(a, 2) = 4(a - 1)/a$, then

$$u' \approx \exp[-1 - \frac{4(a - 1)}{da}],$$

B. 4

at $d = 4$ it provides $u' \approx \exp[-1/a^4]$ which is correct up to an approximation of 1%. Now, we expand Eq. B. 4 in a series of $(1/d)^k$ near $d = 4$, to obtain

$$u' \sim \exp\{\frac{1}{d^3}[1 - \frac{x(a, d)}{d}]^{d-1} + O[(1/d)^2]\},$$

B. 5

here, we have replaced $x(a, 2)$ by $x(a, d)$. Eq. (42) is obtained if we further approximate $x(a, d) \sim a$. Since, on the other hand, $u' \approx \exp[-\gamma]$, Eq. B. 5 allows us to write the following useful expression:

$$[1 - \frac{x(a, d)}{d}]^{d-1} \sim \frac{1}{a^3} + O[(1/d)^2].$$

B. 6

Finally, Eqs. B. 3 and B. 6, lead to Eq. (41)

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