ON RIGIDITY OF NICHOLS ALGEBRAS

IVÁN ANGIONO*, MIKHAIL KOCHETOV**, AND MITJA MASTNACK**

Abstract. We study deformations of graded braided bialgebras using cohomological methods. In particular, we show that many examples of Nichols algebras, including the finite-dimensional ones arising in the Andruskiewitsch-Schneider program of classification of pointed Hopf algebras, are rigid. This result can be interpreted as nonexistence of “braided Lie algebras” with nontrivial bracket.

1. Introduction

Let $k$ be a field of characteristic 0 and $V$ a $k$-vector space. The symmetric algebra $S(V) = \bigoplus_{n \geq 0} S^n(V)$ is a graded bialgebra by declaring the elements of $V$ primitive, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in V$, and extending to a morphism of (unital) algebras $\Delta: S(V) \to S(V) \otimes S(V)$. Then Lie brackets on $V$ are in one-to-one correspondence with graded deformations of $S(V)$ as a bialgebra (or just as an augmented algebra).

We are interested in graded deformations of bialgebras generalizing $S(V)$, namely, the Nichols algebras of braided vector spaces, which have become prominent in the theory of Hopf algebras (see the survey [1] and references therein). Recall that a braided vector space is a vector space $V$ equipped with a linear isomorphism $c: V \otimes V \to V \otimes V$ that satisfies the braid equation

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c),$$

where $\text{id} = \text{id}_V$. The Nichols algebra of $(V, c)$, denoted by $B(V, c)$ or just $B(V)$ if the braiding is clear from the context, is the unique (up to isomorphism) graded braided bialgebra $B = \bigoplus_{n \geq 0} B_n$ with $B_0 = k, B_1 = V$ such that the restriction of the braiding of $B$ to $V$ is $c$, $B$ is generated by $V$ as an algebra, and $V$ coincides with the space $P(B)$ of primitive elements of $B$.

In the case of symmetric braiding, i.e., $c^2 = \text{id}$, the concept of braided Lie algebra is well understood [18, 8, 20, 23, 21]. This includes the usual Lie algebras (when $c$ is the flip $v \otimes w \mapsto w \otimes v$), Lie superalgebras (when $V$ is graded by $\mathbb{Z}_2$ and $c$ is the signed flip $v \otimes w \mapsto (-1)^{p(v)p(w)}w \otimes v$ where $p$ denotes parity) and color Lie superalgebras. It follows from Kharchenko’s version of PBW Theorem [20, Theorem 7.1] that such Lie structures on $(V, c)$ are in one-to-one correspondence with graded deformations of $B(V, c)$ as a braided bialgebra with a fixed braiding (see Section 3).

2010 Mathematics Subject Classification. Primary 17B37; Secondary 16E40, 16S80, 16T20.

Key words and phrases. Braided algebra, Nichols algebra, graded deformation, bialgebra cohomology, generalized Lie structures.

* Partially supported by CONICET, FONCyT-ANPCyT, Secyt (UNC).

**Supported by Discovery Grants of the Natural Sciences and Engineering Research Council (NSERC) of Canada.
It is an important and difficult question for what finite-dimensional braided vector spaces the Nichols algebra is also finite-dimensional. This condition puts severe restrictions on \( c \). For example, in the case of signed flip, this happens if and only if the even part of \( V \) is zero, in which case the Nichols algebra is the exterior algebra \( \Lambda(V) \) and there are no nontrivial graded deformations.

We believe that such rigidity is typical for finite-dimensional Nichols algebras. We establish it for a wide class of symmetric braidings (Theorem 3.3) using the description of finite-dimensional triangular Hopf algebras by Etingof and Gelaki \([11,15,12]\). We also establish a sufficient condition of rigidity (Theorem 5.3) using cohomological techniques, and verify that it is satisfied for finite-dimensional Nichols algebras in the Yetter-Drinfeld category \( k\Gamma\text{YD} \) over an abelian group \( \Gamma \) (Theorem 6.3) using a description of these Nichols algebras in terms of generators and relations \([4]\). It follows that any finite-dimensional Nichols algebra arising from a diagonal braiding, i.e., a braiding of the form \( c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \) where \( \{x_1, \ldots, x_\theta\} \) is a basis of \( V \) and \( q_{ij} \in k^\times \), does not admit nontrivial graded deformations (Theorem 6.4).

It should be mentioned that the so-called bosonizations of these Nichols algebras often admit nontrivial graded deformations (or “liftings”), as has been shown by Andruskiewitsch and Schneider in the course of their program of classification of pointed Hopf algebras \([3]\).

Our sufficient condition also applies to some interesting infinite-dimensional Nichols algebras (see Section 7) and other braided bialgebras close to Nichols algebras (Theorem 7.1). This may explain the difficulty of constructing new examples in \([7]\).

2. Preliminaries

2.1. Braided tensor categories. It is often more convenient to work in a category rather than with a stand-alone braided vector space. By a tensor category we always mean a strict monoidal \( k \)-linear category, see e.g. \([24]\) for details. We are mostly interested in categories of \( k \)-vector spaces endowed with some additional structure. To simplify notation, we omit associativity isomorphisms and parentheses in tensor products. In particular, we denote the tensor powers of an object \( V \) by \( V^\otimes n \) for all \( n \geq 0 \), where \( V^\otimes 0 \) is the unit object.

A braided tensor category is a tensor category \( \mathcal{V} \) with a braiding, i.e. a natural family of isomorphisms \( c_{U,V,W} : V \otimes W \to W \otimes V \) in \( \mathcal{V} \) satisfying the so-called hexagon axioms

\[
c_{U,V,W} = (\text{id}_V \otimes c_{U,W}) (c_{U,V} \otimes \text{id}_W), \quad c_{U \otimes V,W} = (c_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes c_{V,W}),
\]

for all \( U,V,W \) in \( \mathcal{V} \). The category is symmetric if \( c_{W,V} c_{V,W} = \text{id}_{V \otimes W} \) for all \( V, W \) in \( \mathcal{V} \).

The most well known braided tensor categories are the category of (co)modules over a (co)quasitriangular bialgebra and the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode. We will now briefly recall the relevant definitions and fix notation; details can be found in textbooks such as \([27,22]\). We use the standard Sweedler notation for coalgebras and comodules.
A coquasitriangular (CQT) bialgebra is a pair \((H, \beta)\) where \(H\) is a bialgebra and \(\beta\) is a bilinear form \(H \times H \to k\) that is invertible with respect to convolution and satisfies
\[
\beta(h_{(1)}, k_{(1)})h_{(2)}k_{(2)} = \beta(h_{(2)}, k_{(2)})h_{(1)},
\]
\[
\beta(hk, \ell) = \beta(h, \ell_{(1)})\beta(k, \ell_{(2)}),
\]
\[
\beta(\ell, hk) = \beta(\ell_{(2)}, h)\beta(\ell_{(1)}, k),
\]
for all \(h, k, \ell \in H\). The category of right comodules \(\mathcal{M}^H\) is braided as follows:
\[
c_{V,W}(v \otimes w) = \beta(v_{(-1)}w_{(0)} \otimes v_{(0)}), \quad \text{for all } v \in V, w \in W.
\]
Similarly, the category of left comodules \(\mathcal{M}^H\) is braided by
\[
c_{V,W}(v \otimes w) = \beta(w_{(-1)}v_{(0)} \otimes v_{(0)}), \quad \text{for all } v \in V, w \in W.
\]
If \(G\) is a group then the Hopf algebra \(H = kG\) admits a CQT structure \(\beta\) if and only if \(G\) is abelian. In this case the possible maps \(\beta\) are just linear extensions of bicharacters \(G \times G \to k^\times\). Right \(H\)-comodules are just \(G\)-graded vector spaces, \(V = \bigoplus_{g \in G} V_g\), and the braiding is given by \(v \otimes w \mapsto \beta(g,h)w \otimes v\) for all \(v \in V_g, w \in W_h, g, h \in G\).

An object \(V\) of the Yetter-Drinfeld category \(\mathcal{YD}^H\) is simultaneously a left module and a left comodule such that the following compatibility condition holds:
\[
h_{(1)}v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = (h_{(1)} \cdot v)_{(-1)}h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)} \quad \text{for all } v \in V, h \in H.
\]
A morphism is a linear map preserving both action and coaction. The braiding is given by
\[
c_{V,W} : v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}.
\]
The category of right Yetter-Drinfeld modules \(\mathcal{YD}^H\) is defined in a similar manner. If \(\Gamma\) is a group and \(H = k\Gamma\) then an object in \(\mathcal{YD}^H\) is just a \(\Gamma\)-graded vector space with a left action of \(\Gamma\) such that \(g \cdot V_h = V_{gh^{-1}}\), for all \(g, h \in \Gamma\). The braiding is given by \(v \otimes w \mapsto g \cdot w \otimes v\), for all \(v \in V_g, w \in W_h\). In particular, if \(\Gamma\) is abelian then the semisimple objects in \(\mathcal{YD}^H\) are vector spaces graded by the direct product \(\Gamma \times \hat{\Gamma}\) where \(\hat{\Gamma}\) is the character group of \(\Gamma\). For a vector space \(V\) with such a grading, we will denote the homogeneous component of degree \((g, \chi)\) by \(V^\chi_g\). The braiding becomes \(v \otimes w \mapsto \psi(g)w \otimes v\), for all \(v \in V_g^\chi\) and \(w \in W_h^\psi\).

If a CQT bialgebra \((H, \beta)\) is a Hopf algebra then its antipode is bijective. Moreover \(\mathcal{M}^H\) can be regarded as a full subcategory of the Yetter-Drinfeld category \(\mathcal{YD}^H\) if we define the right action of \(H\) on a right comodule \(V\) by means of the usual left action of \(H^*\) and the homomorphism of algebras \(H^{op} \to H^* : h \mapsto \beta(\cdot, h)\), i.e., \(v \cdot h = \sum \beta(v_{(1)}, h)v_{(0)}\), for all \(v \in V, h \in H\). Similarly, \(\mathcal{M}^H\) can be regarded as a full subcategory of \(\mathcal{YD}^H\).

If \((U, c)\) is a finite-dimensional braided vector space then the FRT construction \([22, 29]\) yields a CQT bialgebra \((H, \beta)\) such that \(U \in \mathcal{M}^H\) and \(c = c_{U,U}\) where \(c_{U,U}\) is given by \([1]\). Moreover, for any \(V, W \in \mathcal{M}^H\) and a linear map \(f : V \to W\) that commutes with the braiding with \(U\) in the sense that \((f \otimes \text{id})c_{U,V} = c_{U,W}(\text{id} \otimes f)\) and \((\text{id} \otimes f)c_{V,U} = c_{W,U}(f \otimes \text{id})\), there exists a biideal \(I\) of \(H\) contained in the left and right kernels of the bilinear form \(\beta\) such that \(f\) is a morphism in \(\mathcal{M}^H/I\) \([29, \text{Corollary 1.9}]\). Hence, replacing \((H, \beta)\) by \((H, \beta)\), where \(H\) is the quotient of \(H\) by the largest biideal contained in the left and right kernels of \(\beta\) and where \(\beta\) is induced by \(\beta\), we obtain a braided category, \(\mathcal{M}^H\), that contains \((U, c)\) and all linear maps that commute with the braiding with \(U\).
There is a Hopf algebra version of the above construction — see e.g. [29] and references therein — for braided vector spaces satisfying a certain condition, called rigidity in [29], which allows us to define the braiding operators \(c_{U,U^*}, c_{U^*,U}\) and \(c_{U^*,U^*}\), where \(U^*\) is the dual space. Namely, there exists a CQT Hopf algebra \((H,\beta)\) such that \(U \in \mathcal{M}^H\) and \(c = c_{U,U}\). Again, any linear map that commutes with the braiding with \(U\) can be included in the category \(\mathcal{M}^{H/I}\) where \(I\) is a Hopf ideal contained in the left and right kernels of \(\beta\), see the proof of [29 Proposition 5.4]. Since the largest biideal contained in the kernels of \(\beta\) is automatically a Hopf ideal, we obtain a CQT Hopf algebra \(\overline{H}\) such that \(\mathcal{M}^{H}\) includes \((U,c)\) and all linear maps that commute with the braiding with \(U\).

We are especially interested in the case of diagonal braiding: \(c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i\) where \(\{x_1,\ldots,x_\theta\}\) is a basis of \(U\) and \(q_{ij} \in \mathbb{K}^\times\). Here we can take \(H = \mathbb{K}G\), where \(G\) is the free abelian group \(\mathbb{Z}^\theta\), and define the bicharacter \(\beta\) by setting \(\beta(e_i,e_j) = q_{ij}\), where \(\{e_1,\ldots,e_\theta\}\) is the standard basis of \(\mathbb{Z}^\theta\). If we make \(U\) a \(G\)-graded vector space by declaring \(x_i \in U_{e_i}\) then we get \(c = c_{U,U}\) in \(\mathcal{M}^H\). Alternatively, we can make \(U\) an object of \(\mathbb{K}^{\Gamma\mathcal{YD}}\) for each abelian group \(\Gamma\) containing elements \(g_1,\ldots,g_\theta\) such that there exist characters \(\chi_1,\ldots,\chi_\theta \in \widehat{\Gamma}\) satisfying \(\chi_j(g_i) = q_{ij}\); then we declare \(x_i \in U_{\chi_i}\) and get \(c = c_{U,U}\) in \(\mathbb{K}^{\Gamma\mathcal{YD}}\). We can choose the group \(\Gamma\) so that it is generated by \(g_1,\ldots,g_\theta\) and the characters \(\chi_1,\ldots,\chi_\theta\) separate points of \(\Gamma\). It is easy to see that in this case a linear map \(f: V \rightarrow W\) commutes with the braiding with \(U\) if and only if \(f\) is a morphism in \(\mathbb{K}^{\Gamma\mathcal{YD}}\).

### 2.2. Braided bialgebras

A bialgebra in a braided tensor category \(\mathcal{V}\) with unit object \(\mathbb{1}\) is an object \(\mathcal{B}\) with four morphisms: multiplication \(m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}\), unit \(u: \mathbb{1} \rightarrow \mathcal{B}\), comultiplication \(\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}\) and counit \(\varepsilon: \mathcal{B} \rightarrow \mathbb{1}\) such that \((\mathcal{B},m,u)\) is a unital algebra, \((\mathcal{B},\Delta,\varepsilon)\) is a counital coalgebra, and the following compatibility conditions hold:

\[
\Delta m = (m \otimes m)(\text{id}_\mathcal{G} \otimes c_{\mathcal{B},\mathcal{B}} \otimes \text{id}_\mathcal{B})(\Delta \otimes \Delta), \quad \varepsilon u = \text{id}_1, \quad \varepsilon m = \varepsilon \otimes \varepsilon, \quad \Delta u = u \otimes u.
\]

Note that the braiding appears only in the compatibility condition involving \(m\) and \(\Delta\).

One can define a braided bialgebra without reference to any categories [29]: it is a braided vector space \((\mathcal{B},c)\) with four linear maps, \(m: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}\), \(u: k \rightarrow \mathcal{B}\), \(\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}\) and \(\varepsilon: \mathcal{B} \rightarrow k\), that commute with the braiding induced by \(c\) among the tensor powers of \(\mathcal{B}\) and satisfy the following conditions: \((\mathcal{B},m,u)\) is a unital algebra, \((\mathcal{B},\Delta,\varepsilon)\) is a counital coalgebra, \(u\) is a comultiplication on \(\mathcal{B}\), \(c\) is a unital algebra map, and finally \(\Delta m = (m \otimes m)(\text{id}_\mathcal{B} \otimes c \otimes \text{id}_\mathcal{B})(\Delta \otimes \Delta)\).

Obviously, a bialgebra \(\mathcal{B}\) in a braided tensor category consisting of vector spaces and linear maps (such as \(\mathcal{M}^H\) or \(\mathcal{M}^{H/\mathcal{YD}}\)) satisfies the definition of braided bialgebra with \(c = c_{\mathcal{B},\mathcal{B}}\). Conversely, it is shown in [29] that any finite-dimensional braided bialgebra \((\mathcal{B},m,u,\Delta,\varepsilon,c)\) can be included in the category \(\mathcal{M}^H\) over a suitable CQT bialgebra (Hopf algebra if \(c\) is rigid) \(H\) such that \(m,u,\Delta,\varepsilon\) are morphisms in \(\mathcal{M}^H\) and \(c = c_{\mathcal{B},\mathcal{B}}\) in \(\mathcal{M}^H\).

We are mainly interested in the case of the Nichols algebra \(\mathcal{B}(V)\) of a finite-dimensional vector space \(V\) with a rigid braiding \(c\), which is a graded braided Hopf algebra, not necessarily finite-dimensional. It can be constructed as the quotient of the tensor algebra \(T(V)\) by a graded biideal \(Z(V)\) [1 Proposition 2.2], which is determined by the grading \(c\); indeed the homogeneous components of \(Z(V)\) are the kernels of the so-called quantum symmetrizers on the tensor powers of \(V\) [1 Proposition 2.11]. This construction can be carried out
either with the stand-alone braided vector space \((V, c)\) or in a suitable braided category of comodules or Yetter-Drinfeld modules.

2.3. **Graded deformations and liftings.** We review the theory of formal graded deformations and liftings from [25], but in a slightly more general setting. The theory of formal bialgebra deformations was introduced by Gerstenhaber and Schack [16], while the graded version and its connection to liftings was considered by Du, Chen and Ye [10].

Let \((B, m, u, \Delta, \varepsilon)\) be a graded bialgebra in a braided tensor category \(V\) consisting of vector spaces and linear maps. A (formal) graded deformation of \(B\) is a \(k[t]-\)linear graded structure \((m_t, \Delta_t)\) on \(B[t]\) with \(t\) being of degree 1, such that \(B[t]\) is again a graded bialgebra. We say that \((B[t], m_t, \Delta_t)\) and \((B'[t], m'_t, \Delta'_t)\) are equivalent if there exists a \(k[t]-\)linear graded bialgebra isomorphism \(f: (B[t], m_t, \Delta_t) \to (B'[t], m'_t, \Delta'_t)\).

A lifting \((U, \pi)\) of \(B\) consists of a filtered bialgebra \(U\) and a filtered vector space isomorphism \(\pi: U \to B\) such that \(\text{gr}\ \pi: \text{gr} U \to \text{gr} B = B\) is an isomorphism of graded bialgebras. An equivalence between liftings \((U, \pi)\) and \((U', \pi')\) is a filtered bialgebra isomorphism \(f: U \to U'\) such that \(\text{gr}\ \pi \circ gr f = gr \pi'\).

A graded deformation is given by a sequence of pairs of maps \((m_i, \Delta_i)\), \(i \geq 0\), of degree \(-i\) such that \(m_{i|B=B} = m + \sum_{i \geq 1} m_i t^i\) and \(\Delta_{i|B} = \Delta + \sum_{i \geq 1} \Delta_i t^i\). We also denote \((m_0, \Delta_0) = (m, \Delta)\). A graded deformation \((B[t], m_t, \Delta_t)\) defines a lifting \((U, \pi)\), where \(U\) is \(B\) as a filtered vector space, \(\pi\) is identity, and \((m_0, \Delta_0) = (m_t, \Delta_t)|_{t=1}\).

If \((U, \pi)\) is a lifting, then the linear maps \(\tilde{m}: B \otimes B \xrightarrow{\pi^{-1} \otimes \pi^{-1}} U \otimes U\) and \(\tilde{\Delta}: B \xrightarrow{\pi^{-1}} U \xrightarrow{\Delta_{\mathcal{B}}} U \otimes U\) decompose into direct sums of homogeneous components \(m_i, \Delta_i\) of degrees \(-i\) for \(i \geq 0\), and the structure maps \((m_t, \Delta_t) = (\sum_i m_i t^i, \sum_i \Delta_i t^i)\) on \(B[t]\) define a formal graded deformation of \(B\).

Up to equivalence, these correspondences are inverses of each other.

2.4. **Graded bialgebra cohomology.** Let \(B\) be a bialgebra in \(V\). Consider the bisimplicial complex \(B^{p,q} = \text{Hom}(B^p \otimes q, B^{\otimes q})\).

The left and right diagonal actions and coactions of \(B\) on \(B^{\otimes n}\) will be denoted by \(\lambda_l, \lambda_r, \rho_l, \rho_r\), respectively. Note that they involve the braiding. The horizontal faces

\[\partial_i^h: \text{Hom}(B^p \otimes B^q) \to \text{Hom}(B^{p+1} \otimes B^q)\]

and degeneracies

\[\sigma_i^h: \text{Hom}(B^{p+1} \otimes B^q) \to \text{Hom}(B^p \otimes B^q)\]

are those for computing Hochschild cohomology:

\[\partial_i^h f = \lambda_i(id \otimes f),\]

\[\partial_i^h f = f(id \otimes \ldots \otimes m \otimes \ldots \otimes id),\ \ 1 \leq i \leq p,\]

\[\partial_{p+1}^h f = \lambda_r(f \otimes id),\]

\[\sigma_i^h f = f(id \otimes \ldots \otimes u \otimes \ldots \otimes id);\]

the vertical faces

\[\partial_j^v: \text{Hom}(B^p \otimes B^q) \to \text{Hom}(B^{p+1} \otimes B^{q+1})\]

and degeneracies

\[\sigma_j^v: \text{Hom}(B^{p+1} \otimes B^{q+1}) \to \text{Hom}(B^p \otimes B^q)\]
and degeneracies

$$\sigma^\varepsilon: \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes (q+1)}) \to \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^q)$$

are those for computing coalgebra (Cartier) cohomology:

$$\delta^p f = (\text{id} \otimes f) \rho_p, \quad \delta^q f = (f \otimes \ldots \otimes \Delta \otimes \ldots \otimes \text{id}) f, \quad 1 \leq j \leq q,$$

$$\partial^0 f = (f \otimes \text{id}) \rho_q, \quad \sigma^\varepsilon f = (\text{id} \otimes \ldots \otimes \varepsilon \otimes \ldots \otimes \text{id}) f.$$

The vertical and horizontal differentials are given by the usual alternating sums

$$\partial^h = \sum (-1)^i \partial^h_i, \quad \partial^c = \sum (-1)^j \partial^c_j.$$

By abuse of notation we identify a cosimplicial bicomplex with its associated cochain bicomplex. The *bialgebra cohomology* of $\mathcal{B}$ is then defined as

$$\Pi^*_b(\mathcal{B}) = H^*(\text{Tot} \mathcal{B}).$$

where

$$\text{Tot} \mathcal{B} = \mathcal{B}^{0,0} \to \mathcal{B}^{1,0} \oplus \mathcal{B}^{0,1} \to \ldots \to \bigoplus_{p+q=n} \mathcal{B}^{p,q} \to \ldots$$

and $\partial^h$ is given by the sign trick (i.e., $\delta^p|_{\mathcal{B}^{p,q}} = \partial^h \otimes (-1)^p \partial^c : \mathcal{B}^{p,q} \to \mathcal{B}^{p+1,q} \oplus \mathcal{B}^{p,q+1}$).

Let $\mathcal{B}_0$ denote the bicomplex obtained from $\mathcal{B}$ by replacing the edges by zeroes, i.e., $\mathcal{B}_0^{p,q} = 0 = \mathcal{B}^{0,q}$ for all $p, q$. The *truncated bialgebra cohomology* is

$$\hat{\Pi}^*_b(\mathcal{B}) = H^{*+1}(\text{Tot} \mathcal{B}_0).$$

For computations, it is convenient to use the normalized bicomplex $\mathcal{B}^+$, which is obtained from the cochain bicomplex $\mathcal{B}$ by replacing $\mathcal{B}^{p,q} = \text{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})$ with the intersection of degeneracies

$$(\mathcal{B}^+)^{p,q} = (\cap \ker \sigma^h) \cap (\cap \ker \sigma^c) \simeq \text{Hom}((\mathcal{B}^+)^{\otimes p}, (\mathcal{B}^+)^{\otimes q}),$$

where $\mathcal{B}^+ = \ker(\varepsilon)$. This change does not affect the cohomology.

We can define the first two cohomology groups as follows:

$$\hat{\Pi}^1_b(\mathcal{B}) = \{ f : \mathcal{B}^+ \to \mathcal{B}^+ \mid f(ab) = af(b) + f(a)b, \Delta f(a) = a_{(1)} \otimes f(a_{(2)}) + f(a_{(1)}) \otimes a_{(2)} \}$$

and

$$\hat{\Pi}^2_b(\mathcal{B}) = \hat{Z}^2_b(\mathcal{B})/\hat{B}^2_b(\mathcal{B}),$$

where

$$\hat{Z}^2_b(\mathcal{B}) = \{ (f, g) \mid f : \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+, \quad g : \mathcal{B}^+ \to \mathcal{B}^+ \otimes \mathcal{B}^+ \}$$

and

$$\hat{B}^2_b(\mathcal{B}) = \{ (f, g) : \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+ \},$$

$$\hat{Z}^2_b(\mathcal{B}) = \{ (f, g) \mid f : \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+, \quad g : \mathcal{B}^+ \to \mathcal{B}^+ \otimes \mathcal{B}^+ \}$$

(2) $af(b, c) + f(a, bc) = f(ab, c) + f(a, b)c,$

(3) $c_{(1)} \otimes g(c_{(2)}) + (\text{id} \otimes \Delta)g(c) = (\Delta \otimes \text{id})g(c) + g(c_{(1)}) \otimes c_{(2)},$

(4) $(f \otimes m)\Delta(a \otimes b) - \Delta f(a, b) + (m \otimes f)\Delta(a \otimes b) = (\Delta a)g(b) + g(ab) - g(a)(\Delta b)$
and

\[ \widehat{B}^2_b(B) = \{ (f,g) \mid \exists h : B^+ \to B^+, f(a,b) = ah(b) - h(ab) + h(a)b, \]
\[ g(c) = -c(1) \otimes h(c(2)) + \Delta h(c) - h(c(1)) \otimes c(2) \}. \]

All maps above are assumed to be morphisms in \( \mathcal{V} \). By \( \Delta(a \otimes b) \) we mean the braided coproduct in \( B \otimes B \), namely, \((\text{id} \otimes c \otimes \text{id})(a(1) \otimes a(2) \otimes b(1) \otimes b(2))\), and we write \( f(-,-) \) instead of \( f(\ -\,-) \). In the resulting deformation (see the next subsection), Equation (2) will correspond to associativity, Equation (3) to coassociativity and Equation (4) to compatibility.

Now assume that \( B \) is \( \mathbb{Z} \)-graded and let \( B_\ell \) denote the subcomplex of \( B \) consisting of homogeneous maps of degree \( \ell \), i.e.,

\[ B^{p,q}_\ell = \text{Hom}(B^{\otimes p}, B^{\otimes q})_\ell = \{ f : B^{\otimes p} \to B^{\otimes q} \mid f \text{ is homogeneous of degree } \ell \}. \]

Complexes \( (B_0)_\ell, B^+_\ell \) and \( (B_0^+)_\ell \) are defined analogously. The graded bialgebra and truncated graded bialgebra cohomologies are then defined by:

\[ H^*_b(B)_\ell = H^*(\text{Tot } B_\ell) = H^*(\text{Tot } B^+_\ell), \]
\[ H^*_b(B)_\ell = H^{*+1}(\text{Tot } (B_0)_\ell) = H^{*+1}(\text{Tot } (B_0^+)_\ell). \]

Note that if the support of the grading is finite, in particular if \( B \) is finite-dimensional, then

\[ H^*_b(B) = \bigoplus_{\ell \in \mathbb{Z}} H^*_b(B)_\ell \quad \text{and} \quad \widehat{H}_b(B) = \bigoplus_{\ell \in \mathbb{Z}} \widehat{H}_b(B)_\ell. \]

### 2.5. Cohomological aspects of graded deformations.

Given a graded deformation of \( B \), let \( r \) be the smallest positive integer for which \( (m_r, \Delta_r) \neq (0, 0) \) (if such an \( r \) exists). Then \( (m_r, \Delta_r) \) is a 2-cocycle in \( \widehat{H}^2_b(B)_- \). Every nontrivial deformation is equivalent to one for which the corresponding \( (m_r, \Delta_r) \) represents a nontrivial cohomology class \([16, 10]\). Hence, if \( \widehat{H}^2_b(B)_\ell = 0 \) for all \( \ell \leq 0 \), then \( B \) is rigid, i.e., has no nontrivial graded deformations.

Conversely, given a positive integer \( r \) and a 2-cocycle \((m', \Delta')\) in \( \widehat{H}^2_b(B)_- \), the maps \( m + t^r m' \) and \( \Delta + t^r \Delta' \) define a bialgebra structure on \( B[t]/(t^{r+1}) \) over \( k[t]/(t^{r+1}) \). There may or may not exist \( (m_r+k, \Delta_r+k), k \geq 1 \), for which \( m_t = m + t^r m' + \sum_{k \geq 1} t^{r+k} m_{r+k} \) and \( \Delta_t = \Delta + t^r \Delta' + \sum_{k \geq 1} t^{r+k} \Delta_{r+k} \) make \( B[t] \) into a bialgebra over \( k[t] \).

An \( r \)-deformation of \( B \) is a graded deformation of \( B \) over \( k[t]/(t^{r+1}) \), i.e., a pair \((m'_r, \Delta'_r)\) defining a bialgebra structure on \( B[t]/(t^{r+1}) \) over \( k[t]/(t^{r+1}) \) such that \((m'_0, \Delta'_0)|_{t=0} = (m, \Delta) \). For any 2-cocycle \((m', \Delta')\) in \( \widehat{H}^2_b(B)_- \), there exists an \( r \)-deformation, given by \((m + t^r m', \Delta + t^r \Delta')\).

If a given \((r-1)\)-deformation can be extended to an \( r \)-deformation, then all ways of doing so are parametrized by \( \widehat{H}^2_b(B)_- \). More precisely, suppose that \((B[t]/(t^r), m'^{-1}_r, \Delta'^{-1}_r)\) is an \((r-1)\)-deformation, where

\[ m'^{-1}_t = m + tm_1 + \ldots + t^{r-1}m_{r-1}, \quad \Delta'^{-1}_t = \Delta + t \Delta_1 + \ldots + t^{r-1} \Delta_{r-1}. \]

If

\[ D = (B[t]/(t^{r+1}), m'^{-1}_r + t^r m_r, \Delta'^{-1}_r + t^r \Delta_r) \]

then

\[ D = (B[t]/(t^{r+1}), m'^{-1}_r + t^r m_r, \Delta'^{-1}_r + t^r \Delta_r) \]
is an $r$-deformation, then
\[ D' = (\mathcal{B}[t]/(t^{r+1}), m^{r}_{t} - t^{r}m_{t}, \Delta^{r}_{t} - t^{r}\Delta_{t}) \]
is an $r$-deformation if and only if $(m^{r}_{t} - m_{t}, \Delta^{r}_{t} - \Delta_{t}) \in \tilde{Z}^{3}_{r}(\mathcal{B})$. Note also that if $(m^{r}_{t} - m_{t}, \Delta^{r}_{t} - \Delta_{t}) \in \tilde{Z}^{3}_{r}(\mathcal{B})$, then deformations $D$ and $D'$ are equivalent.

The obstruction to extend $r$-deformations to $(r + 1)$-deformations lies in $\tilde{H}^{3}_{r}(\mathcal{B})$.

3. The case of symmetric braiding

Let $(V, c)$ be a braided vector space with $c^{2} = \text{id}$. Then $\mathcal{B}(V)$ is a quadratic algebra: it is the quotient of $T(V)$ by the ideal generated by the elements $x \otimes y - c(x \otimes y)$, for $x, y \in V$. If $c$ is the flip (respectively, signed flip) then $\mathcal{B}(V) = S(V)$ (respectively, $S(V_{0}) \otimes A(V_{1})$) and the graded deformations of $\mathcal{B}(V)$ are in one-to-one correspondence with brackets $[,] : V \otimes V \to V$ making $V$ a Lie algebra (respectively, superalgebra). For arbitrary $c$, we need the following generalization of Lie algebra introduced by Gurevich [18] under the name “Lie $c$-algebra”.

**Definition 3.1.** Let $L$ be a vector space, $c : L \otimes L \to L \otimes L$ a symmetric braiding, and $[,] : L \otimes L \to L$ a linear map. Then $(L, [ , , c )$ is a braided Lie algebra if
\[
\begin{align*}
[c([, \otimes \text{id}_{L})] = (\text{id}_{L} \otimes [ , ])(c \otimes \text{id}_{L})(\text{id}_{L} \otimes c) \quad & \text{(compatibility),} \\
[ , ](\text{id}_{L} \otimes [ , ] + c) = 0 \quad & \text{(anticommutativity)} \\
[ , ][(, \otimes \text{id}_{L})(\text{id}_{L} \otimes L + c)(c \otimes \text{id}_{L})(\text{id}_{L} \otimes c) + (c \otimes \text{id}_{L})(\text{id}_{L} \otimes c)] = 0 \quad & \text{(Jacobi identity).}
\end{align*}
\]

Note that the compatibility condition (together with $c^{2} = \text{id}$) simply means that the bracket commutes with $c$, and the above Jacobi identity implies a similar identity for $[,](\text{id}_{L} \otimes [ , ])$ instead of $[,]([ , ] \otimes \text{id}_{L})$. It is straightforward to check that if a vector space $A$ is equipped with a symmetric braiding $c$ and an associative product $m : A \otimes A \to A$ that commutes with $c$ then $(A, [, , c)$ is a braided Lie algebra, where $[,]c$ is the braided commutator $m(\text{id}_{A} \otimes A - c)$.

Braided Lie algebras naturally arise as Lie algebras in a symmetric tensor category $\mathcal{V}$. A Lie algebra in $\mathcal{V}$ is an object $L$ endowed with a morphism $[,] : L \otimes L \to L$ such that the anticommutativity and Jacobi identity hold for $c = c_{L,L}$. If $(H, \beta)$ is a cotriangular bialgebra (i.e., a CQT bialgebra satisfying $\beta^{-1}(h, k) = \beta(k, h)$ for all $h, k \in H$) then the category $M^{H}$ is symmetric; Lie algebras in this category were introduced and studied in [8, 9] under the name $(H, \beta)$-Lie algebras. By an argument similar to [29] (see Subsection 2 above), any finite-dimensional braided Lie algebra can be regarded as an $(H, \beta)$-Lie algebra for a suitable cotriangular bialgebra (Hopf algebra if the braiding is rigid).

Given a braided Lie algebra $(L, [, , c)$, the universal enveloping algebra, which we will denote $\mathcal{U}_{c}(L)$, is the quotient of the tensor algebra $T(L)$ by the ideal generated by the degree 2 elements $x \otimes y - c(x \otimes y) - [x, y]$ where $x, y \in L$. The usual increasing filtration of $T(L)$ gives rise to the standard filtration of $\mathcal{U}_{c}(L)$. As one would expect, $\mathcal{U}_{c}(L)$ becomes a braided bialgebra if we declare the elements of $L$ primitive. It is not true in general that, given an ordered basis of $L$, the corresponding PBW monomials form a basis of $\mathcal{U}_{c}(L)$. However, the following version of PBW Theorem holds.
Theorem 3.2. [20, Theorem 7.1] The graded algebra $\text{gr} \mathcal{U}(L)$ associated to the standard filtration of $\mathcal{U}(L)$ is naturally isomorphic to $\mathcal{U}(L^0)$ where $L^0$ denotes the braided Lie algebra with the same underlying braided vector space as $L$ but with zero bracket. □

The standard filtration of $\mathcal{U}(L)$ coincides with its coradical filtration. Also $\mathcal{U}(L^0) = \mathcal{B}(L,c)$.

It follows that graded deformations of $\mathcal{B}(V,c)$ as a braided augmented algebra or as a braided bialgebra (with a fixed braiding) are in one-to-one correspondence with brackets on $V$ making it a braided Lie algebra. Here the “graded deformations” and “braided Lie algebras” can be understood in the sense of a stand-alone object or an object in $\mathcal{M}^H$ for a suitable cotriangular bialgebra $(H,\beta)$.

For $H = kG$, where $G$ is an abelian group, the cotriangular structures on $H$ are linear extensions of skew-symmetric bicharacters $\beta : G \times G \to k^\times$. In this case the $(H,\beta)$-Lie algebras are known as the color Lie superalgebras with grading group $G$ and commutation factor $\beta$. Note that the braiding is diagonal and, conversely, any braided Lie algebra with a diagonal braiding can be regarded as a color Lie superalgebra for some $G$ and $\beta$.

By a trick going back to Scheunert [28], color Lie superalgebras can be twisted to become ordinary Lie superalgebras. This procedure works in the same way for all color Lie superalgebras with given $G$ and $\beta$, and is associated to a suitable cocycle twist of $(kG,\beta)$ as a CQT bialgebra. Recall that a right 2-cocycle on a bialgebra $H$ is a convolution-invertible map $\sigma : H \otimes H \to k$ satisfying the following equations for all $h,k,\ell \in H$:

$$\sigma(h,k_1\ell_1)\sigma(k_2,\ell_2) = \sigma(h_1k_1,\ell)\sigma(h_2,k_2), \quad \sigma(h,1) = \sigma(1,h) = \varepsilon(h).$$

Also recall that if $(H,\beta)$ is a cotriangular (more generally, CQT) bialgebra then $(H_\sigma,\beta_\sigma)$ is again a cotriangular (respectively, CQT) bialgebra, see e.g. [22]; here $H_\sigma = H$ as a coalgebra, the multiplication of $H_\sigma$ is given by

$$h \cdot_\sigma k = \sigma^{-1}(h_1k_1)h_2k_2\sigma(h_3,k_3),$$

and

$$\beta_\sigma(h,k) = \sigma^{-1}(h_1,k_1)\beta(h_2k_2)\sigma(h_3,k_3).$$

Moreover, $\sigma$ yields an equivalence of braided tensor categories $\mathcal{M}^H$ and $\mathcal{M}^{H_\sigma}$, which is the identity on objects and morphisms and only transforms the tensor product. If $A$ is an algebra (not necessarily associative) in $\mathcal{M}^H$ with multiplication $m : A \otimes A \to A$, then the corresponding algebra in $\mathcal{M}^{H_\sigma}$ is $A$ as an $H$-comodule but with new multiplication:

$$m_\sigma(a \otimes b) = \sigma(a_1,b_1)m(a_0 \otimes b_0).$$

We denote this new algebra by $A_\sigma$ and call it the $\sigma$-twist of $A$. It is shown in [24] that multilinear polynomial identities of $A$ are preserved under $\sigma$-twist if we interpret them in each of the categories $\mathcal{M}^H$ and $\mathcal{M}^{H_\sigma}$ in terms of the appropriate action of symmetric groups on tensor powers of $A$. In particular, associative algebras remain associative and $(H,\beta)$-Lie algebras become $(H_\sigma,\beta_\sigma)$-Lie algebras.

If $H$ is cocommutative then $H_\sigma = H$ but $\beta$ is twisted. If $H = kG$, with $G$ an abelian group, then there exists a 2-cocycle $\sigma : G \times G \to k^\times$ such that $\beta_\sigma$ is a “sign bicharacter”:

$$\beta_\sigma(g,h) = \begin{cases} -1 & \text{if } g,h \in G_-, \\ 1 & \text{otherwise}; \end{cases}$$
where $G_- = G \setminus G_+$ and $G_+$ is a subgroup of index $\leq 2$. It follows that $\sigma$ twists any color Lie superalgebra $L$ with commutation factor $\beta$ into a Lie superalgebra with even part $L_+$ and odd part $L_-$, where $L_{\pm} = \bigoplus_{g \in G_{\pm}} L_g$.

Etingof and Gelaki \cite{11} showed that, under a certain condition on the antipode called \textit{pseudo-involutivity}, a cotriangular Hopf algebra $(H, \beta)$ can be twisted by a suitable cocycle to become the algebra of regular functions on a pro-algebraic group $G$ such that $\beta_\sigma = \frac{1}{2}(\epsilon \otimes \epsilon + \epsilon \otimes a + a \otimes \epsilon - a \otimes a)$ for some central element $a \in G$ with $a^2 = 1$. It immediately follows \cite[Theorem 4.3]{23} that the same cocycle twists $(H, \beta)$-Lie algebras to Lie superalgebras equipped with a $G$-action. Here the even and odd components are just the eigenspaces with respect to the action of $a$, with eigenvalues 1 and $-1$ respectively.

If $H$ is finite-dimensional then pseudo-involutivity of the antipode is equivalent to involutivity and hence to semisimplicity of $H$. Later, Etingof and Gelaki \cite[12, 15]{12} described all finite-dimensional cotriangular Hopf algebras by showing that $(H, \beta)$ can be twisted in such a way that its dual triangular Hopf algebra becomes a “modified supergroup algebra”. As a corollary, any $(H, \beta)$-Lie algebra is twisted to a Lie superalgebra equipped with a supergroup action \cite[Theorem 4.6]{23}.

One can use the twisting procedure to transfer known properties of Lie superalgebras to $(H, \beta)$-Lie algebras in the above cases. Let $U_\beta(L)$ be the universal enveloping algebra of an $(H, \beta)$-Lie algebra $L$, i.e., $U_\beta(L)$ for $c = c_{L,L}$ determined by $\beta$. It is straightforward to verify that $U_{\beta_\sigma}(L_\sigma)$ is naturally isomorphic to $(U_{\beta_\sigma}(L_\sigma))_\sigma$. In particular, for $V$ in $\mathcal{M}^H$ and $c = c_{V,V}$ induced by $\beta$, the $\sigma$-twist of the Nichols algebra $B(V, c)$ is naturally isomorphic to $B(V, c')$ where $c'$ is the braiding on $V$ induced by $\beta_\sigma$. This gives an alternative proof of PBW Theorem for $(H, \beta)$-Lie algebras \cite{23}.

\textbf{Theorem 3.3.} Let $(H, \beta)$ be a cotriangular Hopf algebra that is either pseudo-involutive or finite-dimensional. Let $V$ be a finite-dimensional $H$-comodule with the corresponding braiding $c$. If the Nichols algebra $B(V, c)$ is finite-dimensional then it does not admit nontrivial graded deformations as an augmented algebra or bialgebra in $\mathcal{M}^H$.

\textbf{Proof.} By our assumption on $(H, \beta)$, there exists a cocycle $\sigma$ such that $(H_{\sigma}, \beta_{\sigma})$ is as described by Etingof and Gelaki, Then the braiding $c'$ induced by $\beta_{\sigma}$ on $V$ is just the signed flip associated to a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$, so $B(V, c') = S(V_0) \otimes \Lambda(V_1)$, which is finite-dimensional only if $V_0 = 0$. But in this case $V$ does not admit nontrivial Lie superalgebra structures. It follows that $V$ does not admit nontrivial $(H, \beta)$-Lie algebra structures and hence $B(V, c)$ is rigid in $\mathcal{M}^H$. \hfill $\square$

\textbf{Corollary 3.4.} Let $(V, c)$ be a finite-dimensional braided vector space such that $c$ can be obtained from a coaction by a finite-dimensional cotriangular Hopf algebra. If $B(V, c)$ is finite-dimensional then it does not admit nontrivial graded deformations as a braided augmented algebra or bialgebra.

\textbf{Proof.} By assumption, $V$ can be regarded as an object in $\mathcal{M}^H$ for some finite-dimensional cotriangular Hopf algebra $(H, \beta)$ such that $c = c_{V,V}$. Any graded deformation of $B(V, c)$ can be realized in $\mathcal{M}^H$ for some quotient $(\bar{H}, \bar{\beta})$ of the cotriangular Hopf algebra $(H, \beta)$, so it must be trivial by the above theorem. \hfill $\square$
4. The vanishing of second algebra cohomology for a class of augmented algebras in a braided category

Let \( \mathcal{V} \) be a braided tensor category consisting of vector spaces and linear maps. Let \((\mathcal{B}, \varepsilon)\) be an augmented algebra in \( \mathcal{V} \) acting trivially (i.e., via \( \varepsilon \)) on some \( U \) in \( \mathcal{V} \).

\( \diamond \) A map \( f: \mathcal{B} \otimes \mathcal{B} \to U \) in \( \mathcal{V} \) is an \( \varepsilon \)-cocycle if \( f(1, a) = 0 = f(a, 1) \) and \( f(xy, z) = f(x, yz) \) for all \( a \in \mathcal{B} \) and all \( x, y, z \in \mathcal{B}^+ \). The space of all \( \varepsilon \)-cocycles is denoted by \( Z^2_\varepsilon(\mathcal{B}, U) \).

\( \diamond \) An \( \varepsilon \)-cocycle is an \( \varepsilon \)-coboundary if there exists a map \( t: \mathcal{B} \to U \) such that \( t(1) = 0 \) and \( f(x, y) = t(xy) \) for all \( x, y \in \mathcal{B}^+ \). The space of all \( \varepsilon \)-coboundaries is denoted by \( B^2_\varepsilon(\mathcal{B}, U) \).

\( \diamond \) The quotient of \( \varepsilon \)-cocycles by \( \varepsilon \)-coboundaries is denoted by \( H^2_\varepsilon(\mathcal{B}, U) = Z^2_\varepsilon(\mathcal{B}, U)/B^2_\varepsilon(\mathcal{B}, U) \).

In what follows \((\mathcal{B}^+)\) denotes the range of the multiplication \( \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+ \), i.e., \((\mathcal{B}^+) = \text{span}\{xy | x, y \in \mathcal{B}^+\} \).

**Lemma 4.1** (cf. [25] Subsection 4.1). Let \( \mathcal{B} \) be an augmented algebra in \( \mathcal{V} \) and let \( M = \ker \left( \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B} \right) \). If the map \( \mathcal{B}^+ \otimes \mathcal{B}^+ \to (\mathcal{B}^+) \) splits in \( \mathcal{V} \), then for every space \( U \in \mathcal{V} \), we have \( H^2_\varepsilon(\mathcal{B}, U) = \text{Hom}(M, U) \).

**Proof.** Let \( \varphi: (\mathcal{B}^+) \to \mathcal{B}^+ \otimes \mathcal{B}^+ \) be a splitting of \( M \) and let \( p: \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+ \otimes \mathcal{B}^+ \) be the canonical projection. We define a map \( \Phi: \text{Hom}(M, U) \to H^2_\varepsilon(\mathcal{B}, U) \) as follows: if \( f: M \to U \), then the cocycle \( \Phi(f): \mathcal{B}^+ \otimes \mathcal{B}^+ \to U \) is \( \Phi(f) = f(p - \varphi m) \). The inverse \( \Psi \) of \( \Phi \) is defined as follows: if \( g: \mathcal{B}^+ \otimes \mathcal{B}^+ \to U \) is a cocycle, then \( \Psi(g): M \to U \) is the unique map such that \( \Psi(g)p = g \). Now observe that maps \( \Phi \) and \( \Psi \) are well defined: \( \Phi(f) \) is always a cocycle and \( \Psi(g) = 0 \) whenever \( g \) is a coboundary. Note also that \( \Psi \Phi = \text{id} \) and that the range of \( \Phi \) consists of coboundaries. \( \square \)

Let \( V \) be an object in \( \mathcal{V} \), \( T(V) \) its tensor algebra and \( I \) an ideal generated by homogeneous elements of degree at least two. Let \( \mathcal{B} = T(V)/I \) and let \( \pi: T(V) \to \mathcal{B} \) be the canonical projection. We also abbreviate \( T(V)^+ = \bigoplus_{n \geq 1} V^\otimes n \) and \( T(V)_2 = \bigoplus_{n \geq 2} V^\otimes n \).

**Lemma 4.2.** The following is a commutative diagram:

\[
\begin{array}{ccc}
I \otimes T(V)^+ + T(V)^+ \otimes I & \xrightarrow{m} & I \\
\downarrow & & \downarrow \\
T(V)^+ \otimes T(V) \otimes T(V)^+ & \xrightarrow{\text{id} \otimes m - m \otimes \text{id}} & T(V)^+ \otimes T(V)^+ \\
\downarrow & & \downarrow \\
\mathcal{B}^+ \otimes \mathcal{B} \otimes \mathcal{B}^+ & \xrightarrow{\text{id} \otimes m - m \otimes \text{id}} & \mathcal{B}^+ \otimes \mathcal{B}^+ \\
\downarrow & & \downarrow \\
(B^+) & \xrightarrow{m} & (B^+) \\
\end{array}
\]

where the maps \( \overline{m} \) and \( \pi \otimes \pi \) are the universal maps arising from fact (1) below. Moreover, we have the following facts:

(1) The second and third rows of the diagram are cokernel diagrams.

(2) The second column of the diagram is exact at \( T(V)^+ \otimes T(V)^+ \).
(3) The composition \( T(V)_{(2)} \xrightarrow{\sim \otimes \pi} B^+ \otimes_B B^+ \xrightarrow{\bar{m}} (B^+)^2 \) is equal to the restriction of \( \pi \) to \( T(V)_{(2)} \).

(4) The map \( \bar{\pi} \otimes \bar{\pi} \) is surjective.

(5) If \( \varphi: T(V)_{(2)} \rightarrow T(V)^+ \otimes T(V)^+ \) is any splitting of multiplication (e.g., the composition \( T(V)_{(2)} \xrightarrow{\sim} V \otimes T(V)^+ \rightarrow T(V)^+ \otimes T(V)^+ \) is such a splitting), then \( \bar{\pi} \otimes \bar{\pi} = p(\pi \otimes \pi) \varphi \).

**Proof.** Clearly, each of the squares of the diagram commutes. We prove the remaining claims below:

(1) The third row is a cokernel diagram by definition. The second row is a cokernel diagram due to the fact that \( T(V)^+ = V \otimes T(V) \) as a right \( T(V) \)-module (with the obvious action on the second tensor factor), hence \( T(V)^+ \otimes_{T(V)} T(V)^+ = V \otimes T(V)^+ \), and \( V \otimes T(V)^+ \xrightarrow{m} T(V)_{(2)} \) is an isomorphism.

(2) Clear.

(3) As \( \pi \) is an algebra map, we have \( m(\pi \otimes \pi) m = \pi m \). Hence \( \bar{m}(\bar{\pi} \otimes \bar{\pi}) m = \pi m \). By the universal property of cokernels this means that \( \bar{m}(\bar{\pi} \otimes \bar{\pi}) = \pi \).

(4) Follows from the fact that maps \( p \) and \( \pi \otimes \pi \) are surjective.

(5) Follows from the universal property of cokernels.

\[ \square \]

**Corollary 4.3.** The following sequence is exact:

\[ 0 \rightarrow T(V)^+ I + IT(V)^+ \rightarrow I \xrightarrow{\bar{\pi} \otimes \pi} B^+ \otimes_B B^+ \xrightarrow{\bar{m}} (B^+)^2 \rightarrow 0 \]

Therefore, \( I/(T(V)^+ I + IT(V)^+) \simeq \ker (B^+ \otimes_B B^+ \rightarrow (B^+)^2) \).

**Proof.** To avoid ambiguity, we denote the restriction of \( \bar{\pi} \otimes \bar{\pi} \) to \( I \) by \( \tau \). We first prove that \( \ker(\tau) = T(V)^+ I + IT(V)^+ \). The inclusion \( T(V)^+ I + IT(V)^+ \subseteq \ker(\bar{\pi} \otimes \bar{\pi}) \) follows from \( \bar{\pi} \otimes \bar{\pi}(T(V)^+ I + IT(V)^+) = (\bar{\pi} \otimes \bar{\pi})(m(T(V)^+ \otimes I + I \otimes T(V)^+) = p(\pi \otimes \pi)(T(V)^+ \otimes I + I \otimes T(V)^+) = 0 \).

Let \( x \in \ker(\tau) \). Since \( m(T(V)^+ \otimes T(V)^+) = T(V)_{(2)} \), there exists \( y \in T(V)^+ \otimes T(V)^+ \) such that \( m(y) = x \). Now \( 0 = (\bar{\pi} \otimes \bar{\pi}) m(y) = p(\pi \otimes \pi)(y) \), and hence \( (\pi \otimes \pi)y = (\text{id} \otimes m - m \otimes \text{id})z \) for some \( z \in B^+ \otimes B \otimes B^+ \). Let \( w \in T(V)^+ \otimes T(V) \otimes T(V)^+ \) be such that \( (\pi \otimes \pi \otimes \pi)(w) = z \). Define \( y' = y - (\text{id} \otimes m - m \otimes \text{id})w \). As \( (\pi \otimes \pi)y' = 0 \) we have that \( y' \in I \otimes T(V)^+ + T(V)^+ \otimes I \) and hence \( x = m(y) = m(y') \in IT(V)^+ + T(V)^+ I \).

We now prove that \( \bar{\pi} \otimes \bar{\pi}(I) = \ker(B^+ \otimes_B B^+ \xrightarrow{\bar{m}} (B^+)^2) \). The inclusion \( \subseteq \) follows from part (3) of the lemma above: \( \bar{m}(\pi \otimes \pi)(I) = \pi(I) = 0 \). The inclusion \( \supseteq \) follows from the fact that \( \bar{\pi} \otimes \bar{\pi} \) is surjective.

\[ \square \]

**Corollary 4.4.** If \( I \) is generated by a subobject \( R \), then the induced morphism

\[ R \rightarrow \ker (B^+ \otimes_B B^+ \rightarrow (B^+)^2) \]

is surjective.

\[ \square \]

We summarize the above results in a theorem which will be needed in the next section to establish rigidity of certain graded bialgebras in \( \mathcal{V} \).
Theorem 4.5. Let $V$ be an object in $\mathcal{V}$ and $T(V)$ its tensor algebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in $\mathcal{V}$. Consider the augmented algebra $\mathcal{B} = T(V)/\langle R \rangle$ and an object $U$ in $\mathcal{V}$ on which $\mathcal{B}$ acts trivially (i.e., via $\varepsilon$). If the multiplication map $\mathcal{B}^+ \otimes \mathcal{B}^+ \xrightarrow{m} (\mathcal{B}^+)^2$ splits in $\mathcal{V}$, then there is an injection $H^2_\varepsilon(\mathcal{B}, U) \rightarrow \text{Hom}(R, U)$.

In particular, if $f$ is an $\varepsilon$-cocycle such that for every $u \in \mathcal{B} \otimes \mathcal{B}$ in the range of the composition $R \rightarrow V \otimes T(V)^+ \rightarrow \mathcal{B} \otimes \mathcal{B}$ we have $f(u) = 0$, then $f$ is an $\varepsilon$-coboundary. □

5. A sufficient condition for rigidity of graded bialgebras in a braided category

Let $\mathcal{B}$ be a graded bialgebra in $\mathcal{V}$. For a homogeneous map $f : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ of degree $\ell$ and a nonnegative integer $r$ we define $f_r : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ by $f_r|_{(\mathcal{B} \otimes \mathcal{B})_r} = f$ and $f_r|_{(\mathcal{B} \otimes \mathcal{B})_{s}} = 0$ for $s \neq r$. For $g : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$, we define $g_r$ analogously. We also define $f_{\leq r}$ by $f_{\leq r} = \sum_{i=0}^r f_i$, and $f_{< r}, g_{\leq r}, g_{< r}$ in a similar fashion.

Lemma 5.1 (cf. [25] Lemma 2.3.6). Let $\mathcal{B}$ be a graded bialgebra in $\mathcal{V}$ such that $\mathcal{B}_0 = k$ and $\mathcal{B}$ is generated as an algebra by $\mathcal{B}_1$.

(1) If $(f, g) \in Z^2_\varepsilon(\mathcal{B})$, $r > 1$, $f_{\leq r} = 0$, and $g_{< r} = 0$, then $g_r = 0$.

(2) If $(f, g) \in Z^2_\varepsilon(\mathcal{B})$, $\ell < 0$, and $f_{\leq r} = 0$, then $g_{\leq r} = 0$.

(3) If $(0, g) \in Z^2_\varepsilon(\mathcal{B})$, $\ell < 0$, then $g = 0$.

Proof. The proof in [25] carries over word for word. First note that for every $(f, g) \in Z^2_\varepsilon(\mathcal{B})$ we have $f_{\leq 1} = 0$ and $g_{\leq 2} = 0$, due to the fact that $(\mathcal{B}^+ \otimes \mathcal{B}^+)_0 = 0 = (\mathcal{B}^+ \otimes \mathcal{B}^+)_1$. Hence (1) easily yields (2) and (3).

For (1) recall that $\partial f = -\partial^h g$ by Equation (4). If $r > 1$, $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_{r-1}$, then

$$(\partial^c f)(a, b) = 0 = -\partial^h g(a, b) = -(\Delta a) g(b) + g(ab) - g(a)(\Delta b) = g(ab).$$

As $\mathcal{B}_r$ is spanned by such products $ab$, we have that $g(\mathcal{B}_r) = 0$. □

Lemma 5.2 (cf. [25] Lemma 2.3.5). Let $\mathcal{B}$ be a connected graded bialgebra in $\mathcal{V}$, let $r \in \mathbb{N}$, and let $f : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ be a homogeneous unital Hochschild cocycle in $\mathcal{V}$ (with respect to left and right regular actions of $\mathcal{B}$ on itself). If $f_{< r} = 0$, then $f_r : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ is an $\varepsilon$-cocycle.

Proof. This follows directly from $\partial^h f = 0$, see Equation (2). □

Theorem 5.3 (cf. [25] Lemma 4.2.2). Let $V$ be an object in $\mathcal{V}$ and $T(V)$ its (braided) tensor bialgebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in $\mathcal{V}$ and generates a biideal in $T(V)$. Consider the quotient $\mathcal{B} = T(V)/\langle R \rangle$, which is a graded bialgebra in $\mathcal{V}$, and assume that the multiplication map $m : \mathcal{B}^+ \otimes \mathcal{B}^+ \rightarrow (\mathcal{B}^+)^2$ splits in $\mathcal{V}$. If for some negative $\ell$ we have that $\text{Hom}(R, P(\mathcal{B}))_{\ell} = 0$, then $\widehat{H}^2_\varepsilon(\mathcal{B})_{\ell} = 0$.

In particular, if $\text{Hom}(R, P(\mathcal{B}))_{\ell} = 0$ for all negative $\ell$, then $\mathcal{B}$ is rigid.

Proof. Let $(f, g) \in Z^2_\varepsilon(\mathcal{B})$. We will find a map $s = \sum_{r=0}^{\infty} s_r : \mathcal{B} \rightarrow \mathcal{B}$ such that for every nonnegative $r$, $(f, g)_r = (\partial^h s_r, -\partial^c s_r)$, from where the result trivially follows since $(f, g) = \partial^h \sum_{r=0}^{\infty} s_r$. Here the sum $s = \sum_{r=0}^{\infty} s_r$ is potentially infinite but locally finite. The cases $r = 0, 1$ are clear. Suppose that $s_0, \ldots, s_{r-1}$ have been found. Let $(f', g') = (f, g) - \partial^h s_{< r} = (f, g) - \sum_{i=0}^{r-1} (\partial^h s_i, -\partial^c s_i)$. Note that, by assumption, $f'|_{< r} = 0$ and hence, by Lemma 5.1 also $g'|_{< r} = 0$. Let $u \in (\mathcal{B} \otimes \mathcal{B})$ be in the range of the composition $R \rightarrow V \otimes T(V)^+ \rightarrow \mathcal{B} \otimes \mathcal{B}$.
6. Nichols algebras of diagonal type

In what follows $(V, c)$ will denote a braided vector space of diagonal type, $\dim V = \theta$, such that the associated Nichols algebra $B(V)$ has a finite root system $\Delta^+_V$ in the sense of [19], i.e., $\Delta^+_V$ is the set of $\mathfrak{h}^0_\theta$-degrees of generators of a PBW basis. In particular, this is the case if $B(V)$ is finite-dimensional. Let

$$-c_{ij}^V := \min \{ n \in \mathbb{N}_0 \mid (n + 1)q_{ii}(1 - q_{ii}q_{jj}) = 0 \} , \quad j \neq i.$$  

Now we fix

- a basis $\{x_1, \ldots, x_\theta\}$ of $V$ and $q_{ij} \in \mathbb{k}^\times$ such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$,
- elements $x_\alpha \in B(V)$ of degree $\alpha, \alpha \in \Delta^+_V$, which generate a PBW basis, see [4].

We use the following notation:

- $\tilde{q}_{ij} := q_{ij}q_{ji}$ for all $i \neq j$.
- $\chi: \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbb{k}^\times$ is the bicharacter such that $\chi(\alpha_i, \alpha_j) = q_{ij}, 1 \leq i, j \leq \theta$, where $\{\alpha_1, \ldots, \alpha_\theta\}$ is the canonical basis of $\mathbb{Z}^\theta$.
- $N_\alpha$ is the order of $q_{\alpha} := \chi(\alpha, \alpha), \alpha \in \Delta^+_V$.
- $G_N$ is the group of roots of unity of order $N$ and $G'_N$ is the subset of primitive roots of unity of order $N, N \in \mathbb{N}$.

$O(V)$ is the set of Cartan roots of $V$, i.e., the orbit of Cartan vertices under the action of the Weyl groupoid. Recall that $i \in \{1, \ldots, \theta\}$ is a Cartan vertex of $V$ if

$$\tilde{q}_{ij} = q_{i}^{c_{ij}^V} \text{ for all } j \neq i \text{ [4 Definition 2.6].}$$

We recall the following result, which gives a presentation by generators and relations for any Nichols algebra of diagonal type with finite root system.

**Theorem 6.1.** [4] $B(V)$ is presented by generators $x_1, \ldots, x_\theta$ and relations:

$$x_\alpha^{N_\alpha}, \quad \alpha \in O(V);$$

$$\langle \text{ad}_c x_i \rangle^{1-c_{ij}^V} x_j, \quad q_{ii}^{1-c_{ij}^V} \neq 1;$$

$$x_i^{N_i}, \quad i \text{ is not a Cartan vertex};$$

- if $i, j \in \{1, \ldots, \theta\}$ satisfy $q_{ii} = \tilde{q}_{ij} = q_{jj} = -1$, and there exists $k \neq i, j$ such that $\tilde{q}_{ik}^2 \neq 1$ or $\tilde{q}_{jk}^2 \neq 1$,

$$x_{ij}^2;$$

- if $i, j, k \in \{1, \ldots, \theta\}$ satisfy $q_{jj} = -1, \tilde{q}_{ik} = \tilde{q}_{ij}\tilde{q}_{ki} = 1, \tilde{q}_{ij} \neq -1$,

$$[x_{ijk}, x_j]_c;$$

- if $i, j \in \{1, \ldots, \theta\}$ satisfy $q_{jj} = -1, q_{ii}\tilde{q}_{ij} \in G'_3, \tilde{q}_{ij} \neq -1$, and also $q_{ii} \in G'_3$ or $-c_{ij}^V \geq 3$,

$$[x_{iij}, x_{ij}]_c;$$

Since $m(u) = 0$ we have from Equation (4) that $f_r(u) \in P(B)$. Therefore, the composition $R \to V \otimes T(V)^+ \to B \otimes B \xrightarrow{f} B$ has range in $P(B)$ and must be the zero map. By Theorem 4.5 we get a map $t: B \to B$ such that $f_r = t \cdot m$. Now define $s_r = t_r$ and observe that $f_{\leq r} = f_r = \partial^h s_r$. Hence, by Lemma 5.1 we also have $g_r' = -\partial^h s_r$. $\square$
\[ q_{ij} = \pm \tilde{q}_{ij} \in \mathbb{C}_q, \quad \tilde{q}_{ik} = 1, \text{ and also } -q_{jj} = \tilde{q}_{ij}\tilde{q}_{jk} = 1 \text{ or } q_{jj}^{-1} = \tilde{q}_{ij} = \tilde{q}_{jk} \neq -1, \]

\[ x_{ijk} = \frac{1 - \tilde{q}_{jk}}{q_{kj}(1 - \tilde{q}_{ik})} \left[ x_{ik}, x_{ij} \right]_c - q_{ij}(1 - \tilde{q}_{jk}) x_j x_k; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]

\[ [x_{ij}, [x_{ij}, x_{ijk}]_c]_c; \]
\[ (26) \]
\[ (27) \]
\[ (28) \]
\[ (1 + q_{ij})(1 - q_{ij})x_{ij}^2; \]
\[ (29) \]
\[ (30) \]
\[ (31) \]
\[ (32) \]
\[ (33) \]
\[ (34) \]
\[ x_{4\alpha_i + 3\alpha_j} = [x_{3\alpha_i + 2\alpha_j}, x_{ij}]; \]
\[ \text{where } \zeta = q_{ij}, a = (1 - \zeta)(1 - q_{ii}^3), b = (1 - \zeta)(1 - q_{ii}^5) - a q_{ii} \zeta. \]
We fix a realization of \((V, c)\) as a Yetter-Drinfeld module over an abelian group \(\Gamma\), i.e., there exist \(g_i \in \Gamma\), \(\chi_i \in \hat{\Gamma}\) such that \(\chi_j(g_i) = q_{ij}\) and we make \(V\) an object of \(\text{h}^\Gamma\text{YD}\) by declaring \(x_i \in V_{g_i}\). Let \(\mathcal{R}_V\) be the set of relations defining \(B(V)\) according to the previous theorem. Note that \(k\mathcal{R}_V\) is a Yetter-Drinfeld submodule of \(T(V)\), because each relation is \(\mathbb{Z}^\theta\)-homogeneous. For each \(R \in \mathcal{R}_V\) of degree \((a_1, \ldots, a_\theta) \in \mathbb{Z}^\theta\), set
\[
(35) \quad g_R := g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_R := \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \quad \text{so } R \in T(V)_{g_R} \chi_R.
\]
The support of \(R \in \mathcal{R}_V\) is the set \(\text{supp } R := \{i \mid a_i \neq 0\}\), i.e., the set of indices of letters \(x_i\) appearing in \(R\).

**Proposition 6.2.** For every \(R \in \mathcal{R}_V\) and \(t \in \{1, 2, \ldots, \theta\}\), we have \((g_R, \chi_R) \neq (g_t, \chi_t)\).

**Proof.** We prove this for each defining relation. For (1), see [6] Proposition 3.1; the proof does not use that the brading is of standard type.

We discard easily the cases \((9), (8), (14), (15), (16), (17), (18), (25)\) because \(\chi_R(g_R) = 1\). For the remaining cases, note that the propositions in [11] Section 3 show that \((g_R, \chi_R) \neq (g_t, \chi_t)\) for each \(t \notin \text{supp } R\). Therefore, we have to consider only the case \(t \in \text{supp } R\).

For each remaining relation \(R\), we compute \(\chi_R(g_R)\) and/or \(\{\chi_R(g_i)\chi_i(g_R) \mid t \in \text{supp } R\}\).

\[
\begin{align*}
\text{(10):} \quad &\text{we have } \chi_R(g_R) = q_{ii} q_{kk} \neq q_{ii}, q_{kk}. \text{ Suppose that } g_R = g_j, \chi_R = \chi_j. \text{ Then } \tilde{q}_{ij} = \chi_R(g_i) \chi_i(g_R) = (q_{ii} \tilde{q}_{ij})^2 \text{ and } \tilde{q}_{kj} = \chi_R(g_k) \chi_k(g_R) = (q_{kk} \tilde{q}_{kj})^2, \text{ so } q_{ij} q_{ij} = q_{kk} q_{kj} = 1. \\
\text{(11):} \quad &\text{now } \chi_R(g_R) = q_{ii}^3 \neq q_{ii}, \tilde{q}_{ij} \neq \chi_R(g_i) \chi_i(g_R) = \tilde{q}_{ij}^2, \text{ so } (g_R, \chi_R) \neq (g_i, \chi_i), (g_j, \chi_j). \\
\text{(12):} \quad &\text{for both sets of conditions, } \tilde{q}_{ij} q_{jj} \tilde{q}_{jk} = 1 \text{ so } \chi_R(g_R) = q_{ii} q_{kk} \neq q_{ii}, q_{kk}. \text{ Suppose that } g_R = g_j, \chi_R = \chi_j. \text{ But } \tilde{q}_{ij} \neq \chi_R(g_i) \chi_i(g_R) = \tilde{q}_{ij}^2, \text{ a contradiction.} \\
\text{(13):} \quad &\text{recall that } i q_{ij} q_{kk} q_{jk} = 1. \text{ Suppose that } g_R = g_i, \chi_R = \chi_i. \text{ Then } q_{ii} = \chi_R(g_i) = q_{ii} q_{ij} q_{kk}, \text{ so } q_{ij} q_{kk} = 1. \text{ Also } q_{ij} q_{ki} = 1, \text{ so } q_{ij} q_{ik} = 1 \text{ and then } q_{jk} = 1, \text{ a contradiction.} \\
\text{(14):} \quad (\text{i}) \quad &\text{simply note that } \chi_R(g_R) = -q_{kk} \neq -1, q_{kk}. \\
\text{(14):} \quad (\text{ii}) \quad &\text{as } \chi_R(g_R) = q_{ii} q_{kk} \neq q_{ii}, q_{kk}, \text{ the remaining case is } t = j. \text{ But also } \tilde{q}_{ij} = -1 \neq \chi_R(g_i) \chi_i(g_R) = -q_{ii}. \\
\text{(14):} \quad (\text{iii}) \quad &\text{it follows since } \chi_R(g_R) = -q_{ii} \neq -1, q_{ii}. \\
\text{(14):} \quad (\text{iv}) \quad &\text{again } \chi_R(g_R) = q_{ii} q_{kk} \neq q_{ii}, q_{kk}, \text{ so the remaining case is } t = j. \text{ Suppose that } g_R = g_j, \chi_R = \chi_j, \text{ so } 1 = q_{jj}^2 = \chi_R(g_j) \chi_j(g_R) = \tilde{q}_{jj}^2 q_{kk} = -q_{ii}, \text{ a contradiction.} \\
\text{(15):} \quad &\text{it follows since } \chi_R(g_R) = -q_{kk} \neq -1, q_{kk}. \\
\text{(16):} \quad &\text{again } \chi_R(g_R) = q_{ii} q_{kk} \neq q_{ii}, q_{kk}. \text{ Suppose that } g_R = g_j, \chi_R = \chi_j, \text{ so } q_{jj} = q_{ii} q_{kk}, \quad 1 = \tilde{q}_{ij} q_{jk} = \chi_R(g_i) \chi_i(g_R) \chi_R(g_k) \chi_k(g_R) = q_{ij} q_{kk}^2 = q_{jj}^2, \\
\text{which is a contradiction.} \\
\text{(17):} \quad &\text{it follows from } \chi_R(g_R) = q_{jj}^2 \neq q_{ii}, q_{jj}, q_{kk}. \\
\text{(18):} \quad &\text{it follows from } \chi_R(g_R) = q_{jj} \neq q_{ii}, q_{kk}, \text{ and } \chi_R(g_i) \chi_i(g_R) = 1 \neq \tilde{q}_{ij}. \\
\text{(19):} \quad &\text{the proof is analogous to the one for (16).} \\
\text{(20):} \quad &\text{As } \chi_R(g_R) = q_{jj}^2 q_{kk} \text{ and } q_{jj} \neq \pm 1, \text{ we discard the case } t = k. \text{ The case } t = j \text{ is also discarded because } 1 = \chi_R(g_i) \chi_i(g_R) \neq \tilde{q}_{ij}. \text{ Finally suppose that } \chi_R = \chi_i, g_R = g_i, \text{ so }
\end{align*}
\]
But this case corresponds to a diagram which is not in [19], a contradiction.

21: Note that $\chi_R(g_R) = q_{ij}^2 \neq q_{ij}$, $-1 = q_{ii} = q_{kk}$ because $q_{ij}^2 = q_{ij}^2 \in G'_3$.

22: simply $\chi_R(g_R) = -q_{ii} \neq q_{ij}, q_{jk}, q_{kl}$ in all the possible cases.

23: for $t = l$ we have that $\chi_R(g_R) = q_{ij}^3 q_{ii} \neq q_{ii}$, and for $t = i, k$ we have $\chi_R(g_j) \chi_j(g_R) = 1 \neq q_{ij}, q_{kj}$. Suppose that $\chi_R = \chi_j$ and $g_R = g_j$. Then $q_{ij} = \chi_R(g_i) \chi_i(g_R) = q_{ij}^3$, which is a contradiction because $q_{ij} \neq \pm 1$.

24: now, $\chi_R(g_i) \chi_i(g_R) = \chi_R(g_j) \chi_j(g_R) = 1 \neq q_{ij}, q_{kj}$, so we discard the cases $t = i, j, k$.

Now $q_{kl} = q_{kk}^1 \neq \chi_R(g_k) \chi_k(g_R) = q_{kk}$ so also $(\chi_R, g_R) \neq (\chi_i, g_i)$.

25(i): again $\chi_R(g_i) \chi_i(g_R) = \chi_R(g_j) \chi_j(g_R) = 1 \neq q_{ij}, q_{kj}$, and the cases $t = i, j, k$ are solved. As $q_{kl} = q_{kk}^3 \neq \chi_R(g_k) \chi_k(g_R) = q_{kk}$, we conclude that $(\chi_R, g_R) \neq (\chi_i, g_i)$.

26: for $t = j, k$ note that $\chi_R(g_i) \chi_i(g_R) = q_{ij} q_{ik} \neq q_{ij}, q_{ik}$. For $(\chi_R, g_R) = (\chi_i, g_i)$, $q_{ii} = \chi_R(g_R) = -q_{ij} q_{kk}$, $q_{ij} = \chi_R(g_j) \chi_j(g_R) = q_{ij}^3 q_{jj}^2$, $q_{ik} = \chi_R(g_k) \chi_k(g_R) = q_{ij}^3 q_{kk}^2$, so $q_{jj} = -q_{kk} = \pm q_{ii}^2$, but this diagram is not in [19], a contradiction.

28: we look for the possible generalized Dynkin diagrams for which we need $R$.

29: consider each possible generalized Dynkin diagram.

30: again consider each possible generalized Dynkin diagram.
\[\begin{align*}
\sigma^6 \zeta^0 &\ -1, \quad \zeta \in \mathbb{C}_{24}^7: \chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}. \\
\sigma^{-5} \zeta^{-12} \sigma^5 &\ -1, \quad \zeta \in \mathbb{C}_{15}^1: \chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}.
\end{align*}\]

\((31)\): the unique diagram is \[\zeta^8 \zeta^{-8} \sigma^1 \sigma^{-1}, \quad \zeta \in \mathbb{C}_9^9, \text{ and } \chi_R(g_R) = -\zeta^6 \neq q_{ii}, q_{jj}.\]

\((32)\): we consider each possible generalized Dynkin diagram.

\[\begin{align*}
\sigma^2 &\ -1, \quad \zeta \in \mathbb{C}_3^3: \chi_R(g_R) = 1 \neq q_{ii}, q_{jj}. \\
\sigma^7 &\ -1, \quad \zeta \in \mathbb{C}_20^5: \chi_R(g_R) = \zeta^5 \neq q_{ii}, q_{jj}. \\
\sigma^7 &\ -1, \quad \zeta \in \mathbb{C}_20': \chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}. \\
\sigma^3 &\ -\zeta^4 \sigma^{-4} \sigma^11, \quad \zeta \in \mathbb{C}_{15}^1: \chi_R(g_R) = \zeta \neq q_{ii}, q_{jj}. \\
\sigma^3 &\ -\zeta^13 \sigma^{-1}, \quad \zeta \in \mathbb{C}_{15}^1: \chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}.
\end{align*}\]

\((33)\): the unique diagram is \[\zeta^3 \zeta^2 \sigma^1 \sigma^{-1}, \quad \zeta \in \mathbb{C}_9^9, \text{ and } \chi_R(g_R) = \zeta^9 \neq q_{ii}, q_{jj}.\]

**Theorem 6.3.** Suppose \(V\) is an object in \(k\Gamma\mathcal{Y}D\) such that its Nichols algebra has a finite root system. Then \(\text{Hom}_{k\Gamma}^\mathcal{Y}(kR_V, V) = 0\).

**Proof.** If \(f \in \text{Hom}_{k\Gamma}^\mathcal{Y}(kR_V, V)\) and \(R \in \mathcal{R}_V\), then \(f(R) \in V_{gR}^R\). By Proposition 6.2, \(V_{gR}^R = 0\) for each \(R \in \mathcal{R}_V\), so \(f = 0\).

**Theorem 6.4.** If \(B(V)\) is a Nichols algebra of diagonal type with finite root system then \(B(V)\) does not admit nontrivial graded deformations as a braided bialgebra.

**Proof.** We fix a realization of \((V, c)\) in \(k\Gamma\mathcal{Y}D\) where \(\Gamma\) is an abelian group. Without loss of generality, we may assume that the \(g_i\)'s generate \(\Gamma\) and the \(\chi_i\)'s generate \(\hat{\Gamma}\). By Theorem 6.3, the conditions needed to invoke Theorem 5.3 are satisfied, so \(B(V)\) does not admit nontrivial graded deformations in \(k\Gamma\mathcal{Y}D\). But our choice of realization ensures that any graded deformation of \(B(V)\) is in \(k\Gamma\mathcal{Y}D\) and hence must be trivial.

### 7. Examples

**7.1. Positive parts of quantum groups.** It is well known that, in the generic case, the positive part of a quantized enveloping algebra is a Nichols algebra of diagonal type. By Theorem 6.4, these positive parts are rigid. More generally, this applies to the “diagram” of the pointed Hopf algebra \(U(D)\) associated to a generic datum \(D\) of finite Cartan type — see [2], where it is shown that any pointed Hopf algebra whose group-like elements form a finitely generated abelian group is isomorphic to some \(U(D)\) if it is a domain with finite Gelfand-Kirillov dimension and its infinitesimal braiding is positive.

**7.2. Distinguished pre-Nichols algebras.** These are infinite-dimensional braided Hopf algebras projecting onto the corresponding finite-dimensional Nichols algebras. They were formally defined in [5, Definition 3.1] generalizing the situation with quantum groups at roots of unity and the corresponding small quantum groups. Let \(V\) be a braided vector
space of diagonal type such that \( \mathcal{B}(V) \) is finite-dimensional. Then the distinguished pre-Nichols algebra \( \tilde{\mathcal{B}}(V) \) is the quotient of \( T(V) \) by the relations in Theorem 6.1 except the powers of root vectors (6). As a consequence of Theorem 6.3, we have:

**Theorem 7.1.** Let \((V,c)\) be a braided vector space of diagonal type such that \( \mathcal{B}(V) \) is finite-dimensional. Then \( \tilde{\mathcal{B}}(V) \) does not admit nontrivial graded deformations as a braided bialgebra.

### 7.3. Nichols algebras over dihedral groups.

Let \( D_m \) denote the dihedral group of order \( 2m \). For odd \( m \), it is not known whether the category of Yetter-Drinfeld modules over \( D_m \) has any finite-dimension Nichols algebras. For even \( m \geq 4 \), the only known finite-dimensional Nichols algebras have a symmetric braiding [13], so Theorem 3.3 applies.

### 7.4. Nichols algebras over symmetric groups.

Let \( n \geq 3 \). The quadratic algebra \( \mathcal{FK}_n \), introduced by Fomin and Kirillov [14], is presented by generators \( x_{(ij)} \), \( 1 \leq i < j \leq n \), and relations

\[
\begin{align*}
  x_{(ij)}^2 &= 0, & 1 \leq i < j \leq n, \\
  x_{(ij)}x_{(jk)} &= x_{(j)k}x_{(ik)} + x_{(ik)}x_{(ij)}, & 1 \leq i < j < k \leq n, \\
  x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, & 1 \leq i < j < k \leq n, \\
  x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, & \#\{i, j, k, l\} = 4.
\end{align*}
\]

Milinski and Schneider [26] showed how to make \( \mathcal{FK}_n \) a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group \( S_n \). As an algebra, it is generated by the vector space \( V_n \) with basis \( \{x_{(ij)} \mid 1 \leq i < j \leq n\} \). Identifying \((ij)\) with the corresponding transposition in \( S_n \), we can make \( V_n \) a Yetter-Drinfeld module where the coaction is defined by declaring \( x_{(\sigma)} \) a homogeneous element of degree \( \sigma \), and the action is the conjugation twisted by the sign. The corresponding braiding on \( V_n \) is given by

\[
c(x_{(\sigma)} \otimes x_{(\tau)}) = \chi(\sigma, \tau)x_{(\sigma \tau)^{-1}} \otimes x_{(\sigma)}, \quad \chi(\sigma, \tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij), i < j, \\ -1 & \text{otherwise,} \end{cases}
\]

where \( \sigma \) and \( \tau \) are transpositions. Then the above relations generate a biideal in the (braided) tensor bialgebra \( T(V_n) \).

It is easy to see that \( \mathcal{FK}_n \) projects onto the Nichols algebra \( \mathcal{B}(V_n) \). For \( n = 3, 4, 5 \), it is known that \( \mathcal{FK}_n = \mathcal{B}(V_n) \) and has dimension, respectively, 12, 576 and 8294400 (see [26] for \( n = 3, 4 \) and [17] for \( n = 5 \)). Milinski and Schneider conjectured that \( \mathcal{FK}_n \) coincides with \( \mathcal{B}(V_n) \) for all \( n \). Moreover, it has been conjectured that \( \dim \mathcal{FK}_n = \infty \) for \( n \geq 6 \) [14].

**Theorem 7.2.** Let \( n \geq 3 \). Then \( \mathcal{FK}_n \) does not admit nontrivial graded deformations as a braided bialgebra.

**Proof.** All relations are in degree 2 and cannot have coaction given by transposition. As the only primitives in degrees smaller than 2 are in degree 1 and have coaction given by transpositions, the assumption of Theorem 5.3 is satisfied and these algebras are rigid. \( \square \)
ON RIGIDITY OF NICHOLS ALGEBRAS

REFERENCES

[1] N. Andruskiewitsch and H.-J. Schneider, Pointed Hopf algebras, “New directions in Hopf algebras”, MSRI series Cambridge Univ. Press; 1–68 (2002).
[2] N. Andruskiewitsch, H-J. Schneider, A characterization of quantum groups, J. Reine Angew. Math. 577 (2004), 81–104.
[3] N. Andruskiewitsch, H-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. of Math. (2) 171 (2010), no. 1, 375–417.
[4] I. Angiono, On Nichols algebras of diagonal type, J. Reine Angew. Math. 683 (2013), 189–251.
[5] I. Angiono, Distinguished Pre-Nichols algebras, arXiv:1405.6681.
[6] I. Angiono, A. García Iglesias, Pointed Hopf algebras with standard braiding are generated in degree one, Contemp. Math. 537 (2011), 57–70.
[7] A. Ardizzoni, Universal enveloping algebras of PBW type, Glasg. Math. J. 54 (2012), no. 1, 9–26.
[8] Y. Bahturin, D. Fischman, S. Montgomery, On the generalized Lie structure of associative algebra, Israel J. Math. 96 (1996), 27–48.
[9] Y. Bahturin, D. Fischman, S. Montgomery, Bicharacters, twistings, and Scheunert’s Theorem for Hopf algebras, J. Algebra 236 (2001), 246–276.
[10] Y. Du, X. Chen, Y. Ye, On graded bialgebra deformations, Algebra Colloq. 14 (2007), no. 2, 301–312.
[11] P. Etingof, S. Gelaki, On cotriangular Hopf algebras, Amer. J. Math. 123 (2001), 699–713.
[12] P. Etingof, S. Gelaki, The classification of finite-dimensional triangular Hopf algebras over an algebraically closed field of characteristic 0, Mosc. Math. J. 3 (2003), no. 1, 37–43, 258.
[13] F. Fantino, G. A. García, On pointed Hopf algebras over dihedral groups, Pacific J. Math., Vol. 252 (2011), no. 1, 69–91.
[14] S. Fomin, A. N. Kirillov, Quadratic algebras, Dunkl elements and Schubert calculus, Progr. Math. 172 (1999), 146–182.
[15] S. Gelaki, On the classification of finite-dimensional triangular Hopf algebras, In “New directions in Hopf algebras”, MSRI Publications, Vol. 43, Cambridge University Press, Cambridge, 2002, 69–116.
[16] M. Gerstenhaber, S. D. Schack, Bialgebra cohomology, deformations, and quantum groups, Proc. Nat. Acad. Sci. U.S.A. 87 (1990), no. 1, 478–481.
[17] M. Graña, Zoo of finite-dimensional Nichols algebras of non-abelian group type, available at http://mate.dm.uba.ar/ matiasg/zoo.html.
[18] D. Gurevich, The Yang–Baxter equation and a generalization of formal Lie theory, Soviet Math. Dokl. 33 (1986), 758–762.
[19] I. Heckenberger, Classification of arithmetic root systems, Adv. Math. 220 (2009) 59–124.
[20] V. Kharchenko, Connected braided Hopf algebras, J. Algebra 307 (2007), no. 1, 24–48.
[21] V. Kharchenko, I. Shestakov, Generalizations of Lie algebras, Adv. Appl. Clifford Algebr. 22 (2012), no. 3, 721–743.
[22] A. Kliment, K. Schmudgen, Quantum groups and their representations, Texts and Monographs in Physics, Springer–Verlag, Berlin, 1997.
[23] M. Kochetov, Generalized Lie algebras and cocycle twists, Comm. Algebra 36 (2008), no. 11, 4032–4051.
[24] S. Mac Lane, Categories for the working mathematician, Second edition, Graduate Texts in Mathematics, Vol. 5, Springer–Verlag, New York, 1998.
[25] M. Mastnak, S. Witherspoon, Bialgebra cohomology, pointed Hopf algebras, and deformations, J. Pure Appl. Algebra 213 (2009), no. 7, 1399–1417.
[26] A. Milinski, H. J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, Contemp. Math. 267 (2000), 215–236.
[27] S. Montgomery, Hopf Algebras and their Actions on Rings, CBMS Lectures, Vol. 82, AMS, Providence, RI, 1993.
[28] M. Scheunert, Generalized Lie algebras, J. Math. Physics 20 (1979), 712–720.
[29] M. Takeuchi, Survey of braided Hopf algebras, New trends in Hopf algebra theory (La Falda, 1999), 301–323, Contemp. Math., 267, Amer. Math. Soc., Providence, RI, 2000.
