Dither computing: a hybrid deterministic-stochastic computing framework

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Abstract—Stochastic computing has a long history as an alternative method of performing arithmetic on a computer. While it can be considered an unbiased estimator of real numbers, it has a variance and MSE on the order of $\Omega(\frac{1}{N^2})$. On the other hand, deterministic variants of stochastic computing remove the stochastic aspect, but cannot approximate arbitrary real numbers with arbitrary precision and are biased estimators. However, they have an asymptotically superior MSE on the order of $O(\frac{1}{N^2})$. Recent results in deep learning with stochastic rounding suggest that the bias in the rounding can degrade performance. We proposed an alternative framework, called dither computing, that combines aspects of stochastic computing and its deterministic variants and that can perform computing with similar efficiency, is unbiased, and with a variance and MSE also on the optimal order of $\Theta(\frac{1}{N^2})$. We also show that it can be beneficial in stochastic rounding applications as well. We provide implementation details and give experimental results to comparatively show the benefits of the proposed scheme.

Index Terms—computer arithmetic; stochastic computing; stochastic rounding; deep learning;

I. INTRODUCTION

Stochastic computing [1]–[4] has a long history and is an alternative framework for performing computer arithmetic using stochastic pulses. While not as efficient as binary encoding in representing numbers, it does provide a simpler implementation for arithmetic units and can tolerate errors, making it a suitable choice for computational substrates that are inherently noisy such as analog computers and quantum systems. Stochastic computing can approximate real numbers and perform arithmetic on them to reach the correct value in expectation, but the stochastic nature means that the result is not precise each time. Recently, Ref. [5] suggests that deterministic variants of stochastic computing can be just as efficient, and does not have the random errors introduced by the random nature of the pulses. Nevertheless in such deterministic variants the finiteness of the scheme implies that it cannot approximate general real numbers with arbitrary accuracy. This paper proposes a framework that combines these two approaches to get the best of both worlds, and inherit some of the best properties of both schemes. In the process, we also provide a more complete probabilistic analysis of these schemes. In addition to considering both the first moment of the approximation error (e.g. average error) and the variance of the representation, we also consider the set of real numbers that are represented and processed to be drawn from an independent distribution as well. This allows us to provide a more complete picture of the tradeoffs in the bias, variance of the approximation and the number of pulses along with the prior distribution of the data.

II. REPRESENTATION OF REAL NUMBERS VIA SEQUENCES

For a real number $x$, $\text{round}(x)$ has the usual definition of $\text{round}(x) = \lfloor x + 0.5 \rfloor$. We consider two independent random variables $X, Y$ with support in the unit interval $[0,1]$. A common assumption is that $X$ and $Y$ are uniformly distributed. The interpretation is that $X$ and $Y$ generate the real numbers that we want to perform arithmetic on. In order to represent a sample $x \in [0,1]$ from $X$ the main idea of stochastic computing (and other representations such as unary coding [6]) is to use a sequence of $N$ binary pulses. In particular, $x$ is represented by a sequence of independent $N$ Bernoulli trials $X_i$. We estimate $x$ via $X_s = \frac{1}{N} \sum_{i=1}^N X_i$. Our standing assumption is that $X, Y, X_s$ and $Y_s$ are all independent. We are interested in how well $X_s$ approximates a sample $x$ in $X$. In particular, we define $L_x = E((X_s - x)^2|X = x)$ and are interested in the expected mean squared error (EMSE) defined as $L = E_X(L_x)$. Note that $L_x$ consists of two components, bias and variance, and the bias-variance decomposition [7] is given by $L_x = \text{Bias}(X_s, x)^2 + \text{Var}(X_s)$ where $\text{Bias}(X_s, x) = E(X_s) - x$. The following result gives a lower bound on the EMSE:

Theorem 2.1: $L \geq \frac{1}{N^2} \int_0^1 p_X(x)(Nx - \text{round}(Nx))^2 dx$.

Proof: Follows from the fact that $X_s$ is a rational number with denominator $N$ and thus $|X_s - x| \geq \frac{1}{N}|Nx - \text{round}(Nx)|$. □

Given the standard assumption that $X$ is uniformly distributed, this implies that $L \geq 2N \int_0^1 x^2 dx = \frac{1}{12N^2}$, i.e. the EMSE can decrease at a rate of at most $\Omega(\frac{1}{N^2})$.

In the next sections, we analyze how well $X_s$ approximates samples in $X$ asymptotically as $N \to \infty$ by analyzing the error $L$ for various variants of stochastic computing.

A. Stochastic computing

A detailed survey of stochastic computing can be found in Ref. [2]. We give here a short description of the unipolar format. Using the notation above, $X_i$ are chosen to be iid Bernoulli trials with $p(X_i = 1) = x$. Then $E(X_i) = x$ and $X_s$ is an unbiased estimator of $x$. Since $\text{Bias}(X_s, x) = 0$ and $\text{Var}(X_s) = \frac{1}{N^2} x(1-x)$, i.e. $\text{Var}(X_s) = \Omega(\frac{1}{N^2})$, we have...
\[ L_x = \Omega\left(\frac{1}{N}\right) \text{ for } x \in (0, 1). \] More specifically, if \( X \) has a uniform distribution on \([0, 1]\), then \( L = \frac{1}{N} \int_0^1 x(1-x)dx = \frac{1}{2N} \).

### B. A deterministic variant of stochastic computing

In [5] deterministic variants of stochastic computing [4] are proposed. Several approaches such as clock dividing, and relative prime encoding are introduced and studied. One of the benefits of a deterministic algorithm is the lack of randomness, i.e., the representation of \( x \) via \( X_i \) does not change and \( Var(X_s) = 0 \). However, the bias term \( \text{Bias}(X_s, x) \) can be nonzero. Because \( x \) is represented by counting the number of 1’s in \( X \), it can only represent fractions with denominator \( N \).

For \( x = \frac{m}{2N} \) where \( m \) is odd, the error is \( X_s - x = \frac{1}{2N} \). This means that such values of \( x \), \( L_x = \text{Bias}(X_s, x)^2 + \text{Var}(X_s) = \frac{1}{2N} + \text{Var}(X_s) = O\left(\frac{1}{N^2}\right) \). If \( X \) is a discrete random variable with support only on the rational points \( \frac{m}{2N} \) for integer \( 0 \leq m \leq N \), then \( L = 0 \). However, in practice, we want to represent the arbitrary real numbers in \([0, 1]\). Assume that \( X \) is uniformly distributed in \([0, 1] \). By symmetry, we only need to analyze the \( x \in \left[0, \frac{2}{2N}\right] \). Then \( X_s - x = x \) and \( L_x = x^2 \).

It follows that \( L = 2N \int_0^{\frac{2}{2N}} x^2 dx = \frac{1}{12N^2} \).

### C. Stochastic rounding

As the deterministic variant (Sect. II-B) has a better asymptotic EMSE than stochastic computing (Sect. II-A), one might wonder why is stochastic computing useful. It is instructive to consider a special case: 1-bit stochastic rounding [8], in which rounding a number \( x \in [0, 1] \) is given as a Bernoulli trial \( X_i \) with \( P(X_i = 1) = x \). This type of rounding is equivalent to the special case \( N = 1 \) of the stochastic computing mechanism in Sect. II-A. In deterministic rounding, \( X_i = \text{round}(x) \) and the corresponding EMSE is \( \tilde{L} = \frac{1}{12} \).

For stochastic rounding, \( X_i \) has a Bernoulli distribution. If \( P(X_i = 1) = p \), then \( L_x = \text{Bias}(X_s, x)^2 + \text{Var}(X_s) = (p - x)^2 + p(1-p) - p(1-2x) + x^2 \).

Since \( \frac{\partial L}{\partial x} = 1 - 2x \), it follows that for \( x \in \left[0, \frac{1}{2}\right] \), \( L_x \) is minimized when \( p = 0 \) and for \( x \in \left[\frac{1}{2}, 1\right] \), \( L_x \) is minimized when \( p = 1 \), i.e. \( L_x \) is minimized when \( p = \text{round}(x) \). Thus \( L_x \geq L_x \) with equality exactly when \( p = \text{round}(x) \).

This shows that the EMSE for deterministic rounding is minimal among all stochastic rounding schemes. Thus at first glance, deterministic rounding is preferred over stochastic rounding. While deterministic rounding has a lower EMSE than stochastic rounding, it is a biased estimator. This is problematic for application such as reduced precision deep learning where an unbiased estimator such as stochastic rounding has been shown to provide improved performance over a biased estimator such as deterministic rounding. As indicated in [9], part of the reason is that subsequent values that are rounded deterministically are correlated and in this case the stochastic rounding prevents stagnation.

### D. Dither computing: A hybrid deterministic-stochastic computing framework

The main goal of this paper is to introduce dither computing, a hybrid deterministic-stochastic computing framework that combines the benefits of stochastic computing (Sec. II-A) and its deterministic (Sec. II-B) variants and eliminates the bias component while preserving the optimal \( O\left(\frac{1}{N}\right) \) asymptotic rate for the EMSE \( L \). Furthermore, it converges to the zero bias faster than stochastic computing. The main idea here is to approximate deterministically as close as possible to the desired quantity and to stochastically approximate the remaining difference. The encoding is constructed as follows. Let \( \sigma \) be a permutation of \([1, 2, \ldots, N]\).

For \( x \in \left[0, \frac{1}{2}\right] \), let \( n = \lfloor Nx \rfloor \leq \frac{N}{2} \) and \( 0 \leq r = x - \frac{n}{N} \leq \frac{1}{2N} \). Then we pick the \( N \) Bernoulli trials with \( P(X_{\sigma(i)} = 1) = 1 \) for \( 1 \leq i \leq n \) and \( P(X_{\sigma(i)} = 1) = \delta \) for \( n+1 \leq i \leq N \) with \( \delta = \frac{N+r}{N} \). Then \( E(X_s) = \frac{1}{N} (n + \delta(N-n)) = x \).

In addition, since \( n \leq \frac{N}{2} \) and \( rN \leq 1 \), this implies that \( \delta \leq \frac{2}{N} \). \( Var(X_s) = \frac{1}{N} (N-n) (1-\delta) \leq \frac{2}{N} \).

Thus the bias is 0 and the EMSE is of the order \( O\left(\frac{1}{N^2}\right) \). It is clear that this remains true if \( \sigma \) is either a deterministic or a random permutation as \( X_s \) does not depend on \( \sigma \).

For \( x \in \left[\frac{1}{2}, 1\right] \), let \( n = \lfloor Nx \rfloor \geq \frac{N}{2} \) and \( 0 \leq r = x - \frac{n}{N} \leq \frac{1}{2N} \). We pick the \( N \) Bernoulli trials with \( P(X_{\sigma(i)} = 1) = 1-\delta \) for \( 1 \leq i \leq n \) and \( P(X_{\sigma(i)} = 1) = 0 \) for \( n+1 \leq i \leq N \) with \( \delta = \frac{N-r}{N} \). \( E(X_s) = \frac{n(1-\delta)}{N} = x \).

In addition, since \( n \geq \frac{N}{2} \) and \( rN \leq 1 \), this implies that \( \delta \leq \frac{2}{N} \). \( Var(X_s) = \frac{1}{N} (1-\delta) (N-\delta) \leq \frac{2}{N} \).

Thus again the bias is 0 and the EMSE is of the order \( O\left(\frac{1}{N^2}\right) \).

The above analysis shows that dither computing offers better EMSE error than stochastic computing while preserving the zero bias property. In order for such representations to be useful in building computing machinery, we need to show that this advantage persists under arithmetic operations such as multiplication and (scaled) addition.

### III. Multiplication of values

In this section, we consider whether this advantage is maintained for these schemes for multiplication of sequences via bitwise AND. The sequence corresponding to the product of \( X_i \) and \( Y_i \) is given by \( Z_i = X_i Y_i \) and the product \( z = xy \) is estimated via \( Z_s = \frac{1}{N} \sum_{i} Z_i \).

#### A. Stochastic computing

In this case we want to compute the product \( z = xy \). Let \( X_i \) and \( Y_i \) be independent with \( P(X_i = 1) = x \) and \( P(Y_i = 1) = y \). Then for \( Z_i = X_i Y_i \), \( Z_i \) are Bernoulli with \( P(Z_i = 1) = xy \) and \( E(Z_i) = xy = z \). \( Var(Z_i) = z(1-z) \) and \( Var(Z_s) = \frac{1}{N} z(1-z) = \Omega\left(\frac{1}{N}\right) \). Thus \( \text{Bias}(Z_s, z) = 0 \) and the variance and the MSE of the product maintains the suboptimal \( \Omega\left(\frac{1}{N}\right) \) asymptotic rate.

#### B. Deterministic variant of stochastic computing

For numbers \( x, y \in [0, 1] \), we consider a unary encoding for \( x \), i.e., \( P(X_i = 1) = 1 \) for \( 1 \leq i \leq R \) and \( P(X_i = 1) = 0 \) otherwise, where \( R = \text{round}(Nxy) \). For \( y \) we have \( P(Y_i) = 1 \).
shown that the bias is possible. In particular, let 

\[ y_i \]

Bernoulli trials with \( X \) between the two sequences by defining \( W \) on average and \( E \). Then

\[ E(x,y) \in x,y \text{ independent from } X \]

is a uniformly distributed random variable on \([0,1]\). The control sequence \( W_i \) is defined as \( N \) independent Bernoulli trials with \( P(W_i = 1) = \frac{1}{2} \). It is assumed that \( W_i, X_i \) and \( Y_i \) are independent. Then \( E(U_s) = \frac{1}{2} E(X_s) + \frac{1}{2} E(Y_s) = \frac{1}{4} (x+y) = u \), i.e., \( \text{Bias}(U_s, u) = 0 \). \( \text{Var}(U_s) = \frac{1}{N} \left( x(1-\frac{1}{2}x) + y(1-\frac{1}{2}y) \right) = \Omega\left(\frac{1}{N}\right) \). Thus \( L = \Omega\left(\frac{1}{N}\right) \).

### B. Deterministic variant of stochastic computing

For this case \( W_i \) are deterministic and we define \( W_i = 1 \) if \( i \) is even and \( W_i = 0 \) otherwise. Let \( N_e = \lceil \frac{N}{2} \rceil \) and \( N_o = N - N_e \) be the number of even and odd numbers in \( \{1, \ldots, N\} \) respectively. Then \( \text{Var}(E_s) = 0 \) and \( E(U_s) = \frac{1}{2} E(X_s) + \frac{1}{2} E(Y_s) \). If \( N \) is even, \( E(U_s) = \frac{1}{2} (E(X_s) + E(Y_s)) \). If \( N \) is odd, \( \frac{1}{N} - \frac{1}{2} \). In either case, \( |E(U_s) - u| = O\left(\frac{1}{N}\right) \) and \( \text{Bias}(U_s, u) = O\left(\frac{1}{N^2}\right) \), \( L = O\left(\frac{1}{N^2}\right) \).

### C. Dither computing

We set \( \sigma_x \) and \( \sigma_y \) both equal to the identity permutation and define sequence \( \{s_i\} \) with \( s_i = 1 \) for \( i \) odd and 0 otherwise. With probability \( \frac{1}{2} \), \( W_i = s_i \) for all \( i \) and \( W_i = 1 - s_i \) for all \( i \) otherwise. Thus the 2 sequences \( \{s_i\} \) and \( \{1 - s_i\} \) are each chosen with probability \( \frac{1}{2} \). Note that \( W_i \) and \( W_j \) are correlated, \( E(W_i) = \frac{1}{2} \) and \( \text{Var}(W_i) = \frac{1}{4} \). This means that \( E(U_s) = u \) and the bias is 0. The 2 sequences for \( W_i \) selects 2 disjoint sets of random variables \( X_i \) and \( Y_i \), the average of which has variance \( O\left(\frac{1}{N^2}\right) \). \( U_s \) can be written as \( U_s = W(A_s + (1 - W)B_s) = B_s + W(A_s - B_s) \), where \( W \) is a Bernoulli trial with \( p = \frac{1}{2} \), \( A_s = \frac{1}{N} \sum_{i=1}^{N} s_i X_i + (1 - s_i) Y_i \), and \( B_s = \frac{1}{N} \sum_{i=1}^{N} s_i Y_i + (1 - s_i) X_i \). Since \( A_s \) and \( B_s \) takes about half of all the \( X_i \) and the \( Y_i \) alternating over the indices \( i \), it is easy to show that \( E(A_s - B_s) = O\left(\frac{1}{N}\right) \). Furthermore \( \text{Var}(A_s) = O\left(\frac{1}{N^2}\right) \), \( \text{Var}(B_s) = O\left(\frac{1}{N^2}\right) \) and the formula for the variance of products of independent r.v.’s shows that \( \text{Var}(U_s) = O\left(\frac{1}{N^2}\right) \) and thus \( L = O\left(\frac{1}{N^2}\right) \).

### V. Numerical results

In Figs. **I**6 we show the EMSE \( L \) and the bias for the computing schemes above by generating 1000 independent pairs \( (x, y) \) from a uniform distribution of \( X \) and \( Y \). For each pair \( (x, y) \), 1000 trials of dither computing and stochastic computing are used to represent them and compute the product \( z \) and average \( u \).

![Fig. 1: Sample estimate of EMSE L to represent x for various values of N.](image1)

![Fig. 2: Sample estimate of |Bias| to represent x for various values of N.](image2)
Representing $z = xy$

**Fig. 3:** Sample estimate of EMSE $L$ to represent $z = xy$ for various values of $N$.

Representing $z = xy$

**Fig. 4:** Sample estimate of $|\text{Bias}|$ to represent $z = xy$ for various values of $N$.

Representing $u = (x + y)/2$

**Fig. 5:** Sample estimate of EMSE $L$ to represent $u = \frac{x + y}{2}$ for various values of $N$.

**Fig. 6:** Sample estimate of $|\text{Bias}|$ to represent $u = \frac{x + y}{2}$ for various values of $N$.

**TABLE I:** Asymptotic behavior of Bias, Variance and EMSE for stochastic computing, deterministic variant and dither computing to represent a number, to multiply 2 numbers and to perform the average (scaled addition) operation.

|                  | Stoch. Comp. | Determin. Variant | Dither Comp. |
|------------------|--------------|-------------------|--------------|
| Bias (repr.)     | 0            | $\Theta(\frac{1}{N})$ | 0            |
| Variance (repr.) | $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |
| EMSE $L$ (repr.) | $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |
| Bias (mult.)     | 0            | $\Theta(\frac{1}{N})$ | 0            |
| Variance (mult.) | $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |
| EMSE $L$ (mult.) | $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |
| Bias (average)   | 0            | $\Theta(\frac{1}{N})$ | 0            |
| Variance (average)| $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |
| EMSE $L$ (average)| $\Omega(\frac{1}{N})$ | 0 | $\Theta(\frac{1}{N^{2}})$ |

**VI. ASYMMETRY IN OPERANDS**

In the dither computing scheme (and in the deterministic variant of the stochastic computing as well), the encoding of the two operands $x$ and $y$ are different. For instance $x$ is encoded as a unary number (denoted as Format 1) and $y$ has its 1-bits spread out as much as possible (denoted as
for multiplication while both $x$ and $y$ are encoded as unary numbers for scaled addition. For multilevel arithmetic operations, this asymmetry requires additional logic to convert the output of multiplication and scaled addition into these 2 formats depending on which operand and which operation the next arithmetical operation is. On the other hand, there are several applications where the need for this additional step is reduced. For instance,

1) In memristive crossbar arrays \cite{10}, the sequence of pulses in the product is integrated and converted to digital via an A/D converter and thus the product sequence of pulses is not used in subsequent computations.

2) Using stochastic computing to implement the matrix-vector multiply-and-add in neural networks \cite{11}, one of the operands is always a weight or a bias and thus fixed throughout the inference operation. Thus the weight can be precoded in Format 2 for multiplication and the bias value is precoded in Format 1 for addition, whereas the data to be operated on is always in Format 1 and the result recoded to Format 1 for the next operation.

VII. DITHER Rounding: Stochastic Rounding Revisited

In order to reduce power while increasing throughput, there has been much research activities in using reduced precision hardware, in particular in deep learning both for training and inference \cite{12, 13}. In these applications, the data and operations are performed with far lower precision than traditional fixed point or floating point arithmetic, going to as low as a single bit \cite{14}. To address the loss of precision in such reduced precision arithmetic units, stochastic rounding has emerged as an alternative mechanism to deterministic rounding for using reduced precision hardware in applications such as solving differential equations \cite{15} and deep learning \cite{16}. As mentioned in Sec. II-C, 1-bit stochastic rounding can be considered as the special case of stochastic computing with $N = 1$. For $k$-bit stochastic rounding, the situation is similar as only the least significant bit is stochastic. Another alternative interpretation is that stochastic computing is stochastic rounding in time, i.e. $X_i$, $i = 1, \cdots, N$ can be considered as applying stochastic rounding $N$ times. Since the standard error of the mean of dither computing is asymptotically superior to stochastic computing, we expect this advantage to persist for rounding as well when applied over time.

Thus we introduce dither rounding as follows. We assume $\alpha \geq 0$ as the case $\alpha < 0$ can be handled similarly. We define dither rounding of a real number $\alpha \geq 0$ as $d(\alpha, i) = \lfloor \alpha \rfloor + X_i$ where $\{X_i\}$ is the dither computing representation of $x = \alpha - \lfloor \alpha \rfloor$ as defined in Sect. II-D and $\alpha - \lfloor \alpha \rfloor$ is the fractional part of $\alpha$. Note that there is an index $i$ in the definition of $d(\cdot, \cdot)$ which is an integer $0 \leq i < N$. In practice we will compute $\mathcal{I} = \sigma(i, \mod N)$, where $i$ counts how many times the dither rounding operation has been applied so far and $\sigma$ is a fixed permutation, one for the left operand and one for the right operand of the scalar multiplier. The implementation complexity of dither rounding is similar to stochastic rounding, except that we need to keep track of the index $i$, whereas in stochastic rounding, the rounding of the elements are done independently.

To illustrate the performance of these different rounding schemes, consider the problem of matrix-matrix multiplication, a workhorse of computational science and deep learning algorithms. Let $A$ and $B$ be $p \times q$ and $q \times r$ matrices with elements in $[0, 1]$. We denote the $(i, j)$-th element of $A$ as $A_{ij}$. The goal is to compute the matrix $C = AB$. A straightforward algorithm for computing $C$ requires $pqr$ (scalar) multiplications. Although there are asymptotically more optimal algorithms for square matrices such as Strassen \cite{17}, Coppersmith-Winograd \cite{18} and Alman-Williams \cite{19} they are not widely used in practice. Let us assume that we have at our disposal only $k$-bit fixed point digital multipliers and thus floating point real numbers are rounded to $k$-bits before using the multiplier. We use the following standard definition of a $k$-bit quantizer. For simplicity, we will only deal with nonnegative numbers. The quantized value is simply $q(x) = \text{round}(x)$ for $x \in [0, 2^k - 1]$. If $x < 0$, then $q(x) = 0$ (underflow) and if $x > 2^k - 1$, then $q(x) = 2^k - 1$ (overflow).

Fig. 7: Dither rounding to compute the partial result $A_{ij}B_{jk}$. $X_u$ and $Y_v$ are the dither computing representations of $A_{ij} - \lfloor A_{ij} \rfloor$ and $B_{jk} - \lfloor B_{jk} \rfloor$.

In Sect. II-C it was shown that deterministic rounding has the lowest EMSE, but Ref. \cite{9} argues that correlation in subsequent data makes an unbiased estimator such as stochastic rounding a better choice for deep learning. We
propose another reason why dither or stochastic rounding can be superior to deterministic rounding in deep learning. If the prior distribution of the data is not uniform, the output levels of the quantizer are not obtained uniformly and some of the output levels may be wasted by rarely being used. As an extreme example, if the input is in the range $[0, \frac{1}{2})$, the output of a $k$-bit quantizer in deterministic rounding will always be 0 and all information about the input is lost. On the other hand, the output from a dither or stochastic rounding will take on both values 0 and 1.

If we assume that the range of the matrix elements is narrow when compared to the quantization interval, then we expect dither rounding (and stochastic rounding) to outperform traditional rounding. For example, take the special case of $A = \alpha J$ and $B = \beta J$, where $J$ is the square matrix of all 1’s and $\alpha, \beta \in [0, 1]$. When we use traditional rounding to round the elements of $A$ and $B$, the corresponding $C$ is $\gamma J^2$, where $\gamma = \text{round}((2^k - 1)\alpha) \cdot \text{round}((2^k - 1)\beta)/(2^k - 1)^2$ which is $\neq \alpha \beta$ in general. The analysis in Section III shows that for both dither rounding and stochastic rounding the resulting $C$ satisfies $E(C) = \alpha \beta J^2 = AB$, with $E(e_f) = \Theta(\frac{1}{N})$ for dither rounding and $E(e_f) = \Theta(\frac{1}{\sqrt{N}})$ for stochastic rounding.

To ensure there is no overflow or underflow in the quantization process, in practical applications such as deep learning training, the numbers are conservatively scaled to lie well within the range of the quantizer so the above assumption is reasonable. As an another example, we generate 100 pairs of 100 by 100 matrices $A$ and $B$ where elements of $A$ and $B$ are randomly chosen from the range $[0, \frac{1}{2})$ and choose $N = 100$. The average $e_f$ for traditional rounding, stochastic computing and dither computing are shown in Fig. 8. We see that dither rounding has smaller $e_f$ than stochastic rounding and that for small $k$ both dither computing and stochastic rounding has significant lower error in computing $AB$ than traditional rounding. There is a threshold $\tilde{k}$ where traditional rounding outperforms dither or stochastic rounding for $k \geq \tilde{k}$, and we expect this threshold to increase when $N, p, q, r$ increase.

VIII. OTHER DITHER Rounding SCHEMES FOR MATRIX MULTIPLICATION

In the above matrix multiplication scheme, the dither (or stochastic) rounding operation is performed on each of the $pqr$ partial products and 2 rounding operations per partial product (Fig. 7), resulting in $2pqr$ rounding operations. We next consider other variants for dither rounding schemes for matrix multiplication. For instance, one can compute the partial product $A_{ij}B_{jk}$ where $A_{ij}$ is rounded once for each $(i, j)$ and applied to each $k$ whereas $B_{jk}$ is rounded for each partial product. For the MNIST task this corresponds to the input being only quantized once. We find that this dither rounding variant results in slighter better performance than stochastic rounding as shown in Figs. 11 and 12. The number of rounding operations is $pq + pqr = pq(r + 1)$.

Fig. 9: Comparison of deterministic, dither and stochastic rounding for the MNIST handwritten digits classification task. The mean accuracy is computed over 1000 trials.

Fig. 8: Comparison of various rounding methods for multiplying two 100 by 100 matrices with entries in $[0, 0.5)$.

Next, we apply this approach to a simple neural network

3Note that for traditional rounding and $k = 1$, $A$ and $B$ are both rounded to the zero matrix, and $e_f = \|AB\|_F$ in this case.

solving the MNIST handwritten digits recognition task [20]. The input images are $28 \times 28$ grayscale images with pixel values in the range $[0, 1]$. A single layer neural network with a softmax function can obtain an accuracy of 92.4% on the 10000 sample test set, which we denote as the baseline accuracy. The neural network has a single weight matrix and a single bias vector and inference includes a single matrix-matrix multiplication of the input data matrix and the weight matrix. We scaled the weight matrix to the range $[-1, 1]$. In order to use a $k$-bit fixed point multiplier, we rescale both the weights and the input from $[-1, 1]$ to $[0, 2^k - 1]$ and apply the standard $k$-bit quantizer. Note that since the input is restricted to the range $[0, 1]$, it did not fully utilize the full range of the quantizer. We apply the 3 rounding methods discussed earlier to compute the partial products in the matrix multiplication (Fig. 9). We see in Fig. 9 that dither rounding has similar classification accuracy as stochastic rounding and both of them are significantly better than deterministic rounding for small $k > 1$. Note that the rounding method sometimes has slightly better accuracy than using full precision. Furthermore, dither rounding has less variance on the classification accuracy compared with stochastic rounding (Fig. 10).

Fig. 9: Comparison of deterministic, dither and stochastic rounding for the MNIST handwritten digits classification task. The mean accuracy is computed over 1000 trials.
Another variant is to apply the rounding to $A$ and $B$ separately and then perform matrix multiplication on the rounded matrices. This will only require $pq + qr = (p + r)q$ rounding operations which can be much less than $2pqr$. We show that this variant (for both dither and stochastic rounding) can still provide benefits when compared with deterministic rounding. The results are shown in Figs. 13-14. For stochastic and dither rounding, the mean accuracy over 1000 trials are plotted. We see that the results are similar to Figs. 9-10.

Fig. 13: Comparison of deterministic, dither and stochastic rounding for MNIST by quantizing the matrices separately. The mean accuracy is computed over 1000 trials.

Fig. 14: Classification accuracy variance in MNIST classification by quantizing the matrices separately. The sample variance is computed over 1000 trials.

We also performed the same experiment for the Fashion MNIST clothing image recognition task [21]. Since this is a harder task than MNIST, we use a 3-layer MLP (multi-layer perceptron) network with ReLu activation functions between layers and a softmax output function. Each of the 3 weight matrices, the input data matrix and the intermediate result matrices are rounded separately before applying the matrix multiplication operations. The results are shown in Figs. 15-16 which illustrate similar trends for this task as well, although
the range of $k$ where dither and stochastic rounding are beneficial is much narrower ($3 \leq k \leq 4$). We plan to provide further results on various variants of dither rounding and explore the various tradeoffs in a future paper.

![Fashion-MNIST classification](image)

Fig. 15: Comparison of deterministic, dither and stochastic rounding for the Fashion MNIST task using a 3-layer MLP. The mean accuracy is shown over 1000 trials.

![Fashion-MNIST classification](image)

Fig. 16: Classification accuracy variance in Fashion MNIST classification using a 3-layer MLP. The sample variance is computed over 1000 trials.

IX. CONCLUSIONS

We present a hybrid stochastic-deterministic scheme that encompasses the best features of stochastic computing and its deterministic variants by achieving the optimal $\Theta(\frac{1}{k^2})$ asymptotic rate for the EMSE of the deterministic variant while inheriting the zero bias property of stochastic computing schemes with faster convergence to the zero bias. We also show how it can be beneficial in stochastic rounding applications as well, where dither rounding has similar mean performance as stochastic rounding, but with lower variance and is superior to deterministic rounding especially in cases where the range of the data is smaller than the full range of the quantizer.

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