\( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) Geometry from Fluxes

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Abstract

We provide a proof of the equivalence of \( \mathcal{N} = 1 \) dynamics obtained by deforming \( \mathcal{N} = 2 \) supersymmetric gauge theories by addition of certain superpotential terms, with that of type IIB superstring on Calabi-Yau threefold geometries with fluxes. In particular we show that minimization of the superpotential involving gaugino fields is equivalent to finding loci where Seiberg-Witten curve has certain factorization property. Moreover, by considering the limit of turning off of the superpotential we obtain the full low energy dynamics of \( \mathcal{N} = 2 \) gauge systems from Calabi-Yau geometries with fluxes.

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1. Introduction

It was conjectured in [1] that large $N$ dual of $U(N)\mathcal{N} = 2$ gauge theory deformed by certain superpotential terms is realized as type IIB string on Calabi-Yau threefolds with fluxes. The evidence for this conjecture was provided by checking that the low energy dynamics on both sides agree, at least up to the order checked. Namely the Calabi-Yau geometry led to a superpotential for the gaugino fields, whose extremization yielded information about the low energy dynamics. This was checked using the gauge theory analysis beginning with the exact $\mathcal{N} = 2$ answer and studying its deformation under the addition of superpotential. The two objects look rather different. On the gauge theory side one studies the Seiberg-Witten curve and its factorization locus, and on the geometry side one studies extremization of a superpotential. The agreement for the low energy dynamics (for example the tensions of the domain walls) was checked to some order in a series expansion. It is natural to ask how to prove this equivalence to all orders, which may also shed light on what it means to consider a factorization locus of the Seiberg-Witten curve from the $\mathcal{N} = 1$ perspective.

In this paper we find a proof of this equivalence. The idea is to relate the extremization of the superpotential to the existence of some meromorphic function on the Riemann surface with suitable divisors. This in turn is equivalent to specializing to the appropriate factorization locus of the Seiberg-Witten curve.

We also push this idea further and recover the full $\mathcal{N} = 2$ low energy dynamics for $U(N)$ gauge theory by considering a superpotential of degree $N + 1$ and considering the locus where $U(N)$ is broken down to $U(1)^N$. By turning off the superpotential we go back to a point on the Coulomb branch of the $\mathcal{N} = 2$ theory, and we are able to obtain the full low energy dynamics of the $\mathcal{N} = 2$ theory from the Calabi-Yau geometry with fluxes. It is quite interesting that in the limit of turning off the superpotential Calabi-Yau threefold becomes the product of an $A_1$ geometry with the complex plane, as is expected based on the enhanced supersymmetry. Nevertheless the information of the $\mathcal{N} = 2$ low energy dynamics survives in this limit. For example the gauge coupling constants are given by ratios of the periods of the Calabi-Yau threefold. Even though the periods vanish in this limit, the ratios are finite and yield the $\mathcal{N} = 2$ low energy gauge couplings.

The organization of this paper is as follows: In section 2 we discuss the gauge theory analysis. In section 3 we recall the geometric dual and present a proof of its equivalence with the gauge theory prediction. In section 4 we show how to recover the full $\mathcal{N} = 2$ geometry from this setup. Some technical aspects of the computation are discussed in the appendices A,B and C.
2. Field theory analysis

In this section we will review the analysis of [1] giving rise to the exact low energy superpotential of pure $\mathcal{N} = 2$ $U(N)$ Yang-Mills theory deformed to $\mathcal{N} = 1$ by a tree level superpotential for $\Phi$ given by,

$$W_{\text{tree}} = \sum_{i=1}^{n+1} g_i u_i$$  \hspace{1cm} (2.1)

where $u_i = \frac{1}{i} \text{Tr} \Phi^i$.

The solution of this model is achieved by using the Seiberg-Witten curve of the original $\mathcal{N} = 2$ theory and going to the points on the Coulomb branch where the susy vacua are not lifted by (2.1). As we will review below, this approach reduces the problem of finding the low energy superpotential $W_{\text{low}}$, that is only a function of $g_i$'s and $\Lambda$ (the scale of the $\mathcal{N} = 2$ theory), to a well posed factorization problem of a polynomial of degree $N$. Note that one disadvantage for this method is that there is no direct way to integrate in the gaugino superfields which are important in the low energy dynamics of the IR $\mathcal{N} = 1$ theory. This disadvantage is resolved in the geometric dual description that we will review in the next section.

Classically, the vacuum structure of the theory is very simple. Solutions to the $F$ and $D$-terms equations are given by $\Phi$ being diagonal with eigenvalues solutions of,

$$W'(x) = g_{n+1} x^n + \ldots + g_1 = g_{n+1} \prod_{i=1}^{n} (x - a_i) = 0.$$ 

The different vacua are given by the different choices of the number $N_i$ of eigenvalues of $\Phi$ equal to $a_i$. This is subject to the condition $\sum_{i=1}^{n} N_i = N$ and the gauge group $U(N)$ is broken down to $U(N_1) \times \ldots \times U(N_n)$. Thus in the IR we end up with pure $\mathcal{N} = 1$ Yang-Mills theory with the group $U(N_1) \times \ldots \times U(N_n)$.

In the Coulomb branch, the $\mathcal{N} = 2$ theory is described at low energies by an $U(1)^N$ effective theory. All the relevant quantum corrections in the IR are given in terms of an auxiliary Riemann surface and the periods of a particular meromorphic one form.

The SW curve for a pure $U(N)$ gauge theory is given by [2],

$$y^2 = P_N(x)^2 - 4\Lambda^{2N}$$

where $P_N(x, u_k) = \langle \det(xI - \Phi) \rangle$ and $u_k = \frac{1}{k} \text{Tr} \Phi^k$. 

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Once the tree level superpotential is introduced, all points in the Coulomb moduli space are lifted except those for which \(N-n\) mutually local magnetic monopoles become massless. The presence of the superpotential produces a condensate of monopoles and the Higgs mechanism in the magnetic theory gives the expected confinement of the electric \(\mathcal{N} = 1\) theory. Those points are where <\(u_k\)>’s are solution to,

\[
P_N(x)^2 - 4\Lambda^{2N} = F_{2n}(x)H_{N-n}^2(x)
\]

where \(F_{2n}(x)\) and \(H_{N-n}(x)\) are at this point arbitrary polynomials with simple zeroes of degrees \(2n\) and \(N-n\) respectively. The fact that \(H_{N-n}^2(x)\) appears in the above signifies the appearance of \(N-n\) mutually local massless magnetic monopoles. From the original \(U(1)^N\) only \(U(1)^n\) remains unbroken and the corresponding coupling constants are given by the period matrix of the reduced curve,

\[
y^2 = F_{2n}(x).
\]

These \(U(1)^n\) can also be thought of as \(U(1) \subset U(N_i)\) for \(i = 1, \ldots, n\) from the classically unbroken group. On the other hand the pure \(\mathcal{N} = 1\) \(SU(N_i)\) piece confines in the IR, has a mass gap and gaugino condensation, i.e, \(<\text{Tr}_{SU(N_i)}W^\alpha W_\alpha> \neq 0\).

For \(U(N)\) the Coulomb moduli space has dimension \(N\), parametrized for example by the roots of \(P_N(x)\). The condition (2.2) implies that \(N-n\) of those have to be tuned in order to produce the \(N-n\) double roots on the RHS. This implies that (2.2) is satisfied on a codimension \(N-n\) subspace of the Coulomb moduli space. Thus the factorization condition (2.2) does not lead to a unique answer and there is an \(n\) parameter family of such factorizations.

Thus, for this subspace <\(u_k\)>’s are functions of \(n\) parameters. Plugging this in the superpotential \(W_{\text{tree}}\) an effective superpotential is obtained for those \(n\) variables,

\[
W_{\text{eff}} = \sum_{k=1}^{n+1} g_k <u_k>.
\]

Using the field equations from varying (2.3) with respect to the \(n\) variables one could get all <\(u_k\)>’s as functions only of \(g_i\)’s and \(\Lambda\). Substituting back in \(W_{\text{eff}}\) one gets \(W_{\text{low}} = W_{\text{low}}(g_i, \Lambda)\).
However, it is possible to restate this latter extremization problem also in purely algebraic terms. In [1] it was shown that extremizing the effective superpotential is equivalent to imposing, (for a review of the proof see appendix A)

\[ g_{n+1}^2 F_{2n}(x) = W'(x)^2 + f_{n-1}(x) \]

where \( f_{n-1}(x) \) is a polynomial of degree \( n - 1 \) completely fixed by (2.2) as we will show.

Putting these two factorizations together we thus have a purely algebraic description of the low energy dynamics of the \( \mathcal{N} = 1 \) theory. The claim is that we end up with the following problem which is well posed and has a unique answer: Find \( P_N(x) \) such that,

\[ P_N^2(x) - 4\Lambda^{2N} = \frac{1}{g_{n+1}^2} (W'(x)^2 + f_{n-1}(x)) H_{2N-n}^2(x) \] (2.4)

where \( W'(x) = g_{n+1} \prod_{i=1}^{n}(x - a_i) \) is given, together with the following condition,

\[ P_N(x) \to \prod_{i=1}^{n}(x - a_i)^{N_i} \text{ as } \Lambda \to 0 \]

It is interesting to notice that these polynomials are a generalization of Chebyshev polynomials that are the solution to the problem for \( n = 1 \). The proof that the solution to (2.4) is unique is given in appendix B.

Once \( W_{\text{low}}(g_r, \Lambda) = \sum_{r=1}^{n+1} g_i < u_i > \) is obtained, the following information can be computed,

\[ \frac{\partial W_{\text{low}}}{\partial g_r} = < u_i > \quad \text{and} \quad \frac{\partial W_{\text{low}}}{\partial \log \Lambda^{2N}} = < S_1 + \ldots + S_n > \]

where \( S_i \equiv \text{Tr}_{SU(N_i)} W_\alpha W^\alpha \) are the glueball superfields of each \( SU(N_i) \) factor.

It is possible to show that \(-4g_{n+1} < S_1 + \ldots + S_n >\) is equal to the coefficient of the \( x^{n-1} \) monomial of \( f_{n-1}(x) \) in (2.4). This fact plays an important role in section 3 and its proof is given at the end of appendix A.

3. Geometric dual analysis

In [1] a geometric dual to the field theory in the previous section was given. The dual theory was conjectured to have all the IR holomorphic information of the original theory. More explicitly, the coupling constants of the \( U(1) \) factors and the effective superpotential for gaugino fields. These conjectures were tested in a semi-classical series expansion up to several orders.
In this section we will provide the proof that the gauge theoretic prediction for the low energy (holomorphic) dynamics is in exact agreement with the geometric prediction. The dual geometric description has the advantage of also providing the effective superpotential for gaugino fields.

First a review of the geometric construction is given in order to set the notation and then we show how the effective superpotential $W_{\text{eff}}(S_k)$ proposed in [1] gives equations whose solution is completely equivalent to solving the problem proposed in the previous section (2.4).

3.1. Review

The starting point is to geometrically engineer the $\mathcal{N} = 2$ $U(N)$ field theory deformed by the superpotential term (2.1) as the theory living on the world volume of D5 branes wrapping two cycles. We consider IIB string theory on a non-compact Calabi-Yau 3-fold. The 3-fold is a fibration of an $A_1$ ALE space over a complex plane with $D5$ branes wrapping the nontrivial $S^2$ in the blown up $A_1$ singularity. At $n$ isolated points the Calabi-Yau 3-fold thus constructed is singular and can be smoothed out by blowing up $S^2$'s or $S^3$'s. Let us discuss this geometry in more detail.

The geometry corresponding to the theory without superpotential, i.e., to the $\mathcal{N} = 2$ theory is a product space of a complex plane with coordinate $x$ and the $A_1$ ALE space,

$$uv + w^2 = 0.$$ 

In [3] it was shown that adding the tree level superpotential (2.1) to the field theory can be accounted for by allowing a nontrivial fibration given by,

$$uv + w^2 + W'(x)^2 = 0$$

(3.1)

where $W'(x) = g_{n+1} \prod_{i=1}^{n} (x - a_i)$. At each point $x = a_i$ there is a blown up $S^2$ and $N_i$ D5-branes wrapping around the $S^2$.

The dual theory proposed in [1] is obtained via a geometric transition (as a generalization of the $n = 1$ case in [4]). The transitions takes place when the $S^2$'s are blown down and $S^3$'s are blown up. The $N_i$ D5 branes wrapping the $S^2_i$ located at $x = a_i$ disappear and get replaced by $N_i$ units of $H_{RR}$ flux through the new non-trivial $S^3_i$.

The transition to $S^3$'s corresponds to a complex deformation of the geometry. The allowed deformations are computed by taking into account a normalizability condition.
The volume of a minimal lagrangian 3-cycle is given by the absolute value of the integral of the holomorphic 3-form over the cycle. In the non-compact geometry there are non-compact 3-cycles \( B_i \) whose volumes are infinite and need a large distance cut off \( \Lambda_0 \). The deformations that will correspond to dynamical fields are those for which the corresponding variation of the holomorphic form integrated over cycles will not depend on the cutoff \( \Lambda_0 \). This is needed for the mode to be localized. In other words, 

\[
\lim_{\Lambda_0 \to \infty} \frac{\partial}{\partial b_k} \int_{B_i} \Omega
\]

is finite, where \( b_k \)'s are the coefficients of the deformation. Actually we also allow log normalizable, i.e. allow divergence of the form \( \log \Lambda_0 \). This is deeply connected with asymptotic freedom of the underlying gauge theory. This condition fixes the form of the possible complex deformations of (3.1) to be, 

\[
uv + w^2 + W'(x)^2 + f_{n-1}(x) = 0
\]

where,

\[
f_{n-1}(x) = \sum_{j=0}^{n-1} b_j x^j.
\]

The variation of \( b_{n-1} \) term corresponds to log normalizable term. Type IIB on this geometry gives rise to an effective \( \mathcal{N} = 2 \ U(1)^n \) field theory in four dimensions. However, the presence of fluxes induces electric and magnetic FI terms in the effective action allowing for a spontaneous symmetry breaking to \( \mathcal{N} = 1 \).

The effective superpotential for Calabi-Yau 3-folds with fluxes was considered in \[5\] (see also the more recent work \[7\] ). This is given by

\[
W_{\text{eff}} = \int_{CY} H \wedge \Omega
\]

where \( H = H_{RR} - \tau_{IIB} H_{NS} \) and \( \Omega \) is the holomorphic three form of the CY 3-fold.

Let us choose a symplectic basis for three cycles \( A_i \) and \( B_i \), with \( A_i \) identified with the blown up \( S^3 \) and \( B_i \) with the dual non-compact cycle to \( S^3 \). In terms of this basis the superpotential corresponding to the classical vacuum\[8\] where \( N = \sum_{i=1}^{n} N_i \), is given by,

\[
W_{\text{eff}} = \sum_{i=1}^{n} \left( \int_{A_i} H \int_{B_i} \Omega - \int_{B_i} H \int_{A_i} \Omega \right).
\]

\[1\] We assume that \( N_i \)'s do not have a common factor
Using the fact that the D5-branes have been replaced by fluxes we get,

\[ \int_{A_i} H = N_i \quad \text{and} \quad \int_{B_i} H = \tau_{YM} \quad \text{for} \quad i, j = 1, \ldots, n. \tag{3.4} \]

The second condition implies that \( \int_{B_i} H \) is a constant independent of \( i \), and thus \( \int_{B_i - B_j} H = 0 \). Note that since \( B_i \) cycles are non-compact \( \int_{B_i} H \) is actually infinite. This IR divergence can be traced back to the original Yang-Mills UV divergence. This is dealt with by the introduction of a cut off \( \Lambda_0 \). Following the same steps we can identify the constant with \( \tau_{YM(\Lambda_0)} \), the bare Yang-Mills coupling as was done in [4].

Plugging this in the superpotential (3.3),

\[ W_{\text{eff}} = \sum_{i=1}^{n} N_i \Pi_i + \tau_{YM(\Lambda_0)} \sum_{i=1}^{n} S_i \tag{3.5} \]

where, \( S_i \equiv \int_{A_i} \Omega \) and \( \Pi_i \equiv \int_{B_i} \Omega \).

The \( S_i \) and \( \Pi_i \) period integrals can be shown to reduce to line integrals over the complex \( x \) plane of the following effective one form,

\[ \lambda_{\text{eff}} = \sqrt{W'(x)^2 + f_{n-1}(x)dx}. \tag{3.6} \]

There are \( 2n \) branch points on the \( x \)-plane with \( n \) branch cuts running between pairs as shown in Figure 1. \( S_i \)'s are integrals of \( \lambda_{\text{eff}} \) around the \( i \)-th branch cut, \( \alpha_i \). On the other hand, \( \Pi_i \)'s are integrals from \( x = \Lambda_0 \) on the lower sheet to \( x = \Lambda_0 \) on the upper sheet following \( C_i \)'s.

Adding the contours of all \( S_i \)'s and deforming it to enclose \( x = \infty \), it is easy to show that,

\[ \sum_{i=1}^{n} S_i = -\frac{1}{4g_{n+1}}b_{n-1} \]

by computing the residue of the pole at infinity.

Therefore, the superpotential can be written as,

\[ W_{\text{eff}} = \sum_{i=1}^{n} N_i \Pi_i - \tau_{YM(\Lambda_0)} \frac{1}{4g_{n+1}}b_{n-1}. \]

The effective superpotential is only a function of \( S_i \)'s for \( \Pi_i = \frac{\partial F}{\partial S_i} \), where \( F = F(S_1, \ldots, S_n) \) is the prepotential of the CY 3-fold. The field equations are given by,

\[ \frac{\partial W_{\text{eff}}(S_k)}{\partial S_i} = 0 \quad \text{for} \quad i = 1, \ldots, n. \]
However, it turns out to be more useful to use a change of variables from \( \{S_1, \ldots, S_n\} \) to \( \{b_{n-1}, \ldots, b_0\} \), which is generically non-singular.

With the change of variables, the field equations are given by,

\[
\sum_{i=1}^{n} N_i \frac{\partial \Pi_i}{\partial b_{n-1}} - \frac{\tau_{YM}(\Lambda_0)}{4g_{n+1}} = 0 \quad \text{and} \quad \sum_{i=1}^{n} N_i \frac{\partial \Pi_i}{\partial b_j} = 0 \quad \text{for} \quad j = 0, \ldots, n-2. \tag{3.7}
\]

![Diagram](image.png)

**Figure 1:** Contours of integration. The points \( P \) and \( Q \) represent \( \Lambda_0 \) on the upper and lower sheets of the Riemann surface.

### 3.2. Conjectures

Let us recall the conjectures made in [1]. Consider the original \( \mathcal{N} = 2 \) \( U(N) \) theory in the classical vacuum where \( U(N) \) is broken down to \( U(N_1) \times \ldots \times U(N_n) \). As mentioned in section 2, each factor \( U(N_i) = U(1) \times SU(N_i) \) in the IR is expected to give a free \( U(1) \) and gaugino condensate for the confining \( SU(N_i) \) piece, i.e., \( < \text{Tr}_{SU(N_i)} W^\alpha W_\alpha > \neq 0 \).

The holomorphic information, as mentioned before, is composed of the coupling constants \( \tau_{ij} \) of the \( U(1)^n \) factors and the effective superpotential for the glueball fields \( S_{gf}^i = \text{Tr}_{SU(N_i)} W^\alpha W_\alpha \), where \( \text{(gf)} \) stands for glueball field.

The duality map is the following: the \( \mathcal{N} = 2 \) \( U(1)^n \) vector superfields in the Calabi-Yau with blown up \( S^3 \) that can be decomposed in \( \mathcal{N} = 1 \) superfield notation as \( (W^i_\alpha, S_i) \), are identified with the \( U(1)^n \) \( W^i_\alpha \) and \( S_{gf}^i \) of the original theory respectively. Namely,
the lowest component of the glueball field $S_i^{\text{gf}}$ is the holomorphic volume of the $S_i^3$, i.e., 
$S_i = \int_{A_i} \Omega$.

With this identification two physical predictions follow which can be stated as mathematical conjectures, namely

**Conjecture 1:** The coupling constants $\tau_{ij}$ of the original $U(1)^n$ groups are given in the dual geometry by,

$$
\tau_{ij} = \left. \frac{\partial^2 \mathcal{F}}{\partial S_i \partial S_j} \right|_{S_k \to <S_k>}
$$

where $<S_k>$ are the expectation values of the massive $S_i$ fields. More precisely, in the original field theory, the overall $U(1) \subset U(N)$ decouples from the other $U(1)^{n-1}$'s. In an appropriate basis the couplings are given by,

$$
\tau_{ij} \quad \text{with} \quad i, j = 1, \ldots, n-1 \quad \tau_{i,n} = 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \tau_{nn} = \tau_{\text{YM}(\Lambda_0)}
$$

where $\tau_{ij}$ is the period matrix of the reduced SW curve, $y^2 = W'(x)^2 + f_{n-1}(x)$ solution to (2.4).

**Conjecture 2:** Solving the problem (2.4) to find $f_{n-1}(x)$ and $<\text{Tr}\Phi^k>$ for $k = 1, \ldots, n+1$ is equivalent to solving the field equations (3.7) arising from the dual effective superpotential (3.5). In particular, $f_{n-1}(x)$ appearing in the geometry is the same as that appearing in the field theory and

$$
W_{\text{eff}}(<S_i>) = W_{\text{low}}(g_i, \Lambda).
$$

In the next section we will give first the proof to conjecture 2 and then using the relation between the geometries conjecture 1 will be shown to follow.

3.3. Proof of conjectures

Consider the effective superpotential,

$$
W_{\text{eff}} = \int_{\text{CY}} H \wedge \Omega
$$

where the CY 3-fold is given by, $uw + w^2 + W'(x)^2 + f_{n-1}(x) = 0$. Recall that $f_{n-1}(x) = b_{n-1} x^{n-1} + \ldots + b_0$.

The field equations are obtained by varying with respect to the deformations $b_k$,

$$
\frac{\partial W_{\text{eff}}}{\partial b_k} = \int_{\text{CY}} H \wedge \frac{\partial \Omega}{\partial b_k} = 0.
$$
After integrating over the quadratic pieces in the geometry the integral over the CY-3-fold is reduced to an integral over a Riemann surface \( \Gamma \),

\[ y^2 = W'(x)^2 + f_{n-1}(x) \]  
(3.8)

There are two special points on \( \Gamma \) for our discussion, they are located at the two pre-images of \( \infty \) of \( x \). Let us denote them by \( P \) and \( Q \).

In section 3.1 we denoted the reduction of \( \Omega \) to a one form over \( \Gamma \) by,

\[ \lambda_{\text{eff}} = \sqrt{W'(x)^2 + f_{n-1}(x)} \, dx. \]

Let us also introduce a one form \( h \) for the reduction of \( H \),

\[ h = \int_{S^2} H. \]

Note that \( h \) is subject to constraints coming from (3.4), namely,

\[ \oint_{\alpha_i} h = N_i \quad \text{and} \quad \oint_{C_j} h = \tau_{\text{YM}} \to \oint_{C_i - C_j} h = 0 \]  
(3.9)

for all \( i \) and \( j \) in \( \{1, \ldots, n\} \). See figure 1 for the definition of \( \alpha_i \)'s and \( C_i \)'s.

Moreover, it is clear by adding up the \( \alpha_i \) contours that,

\[ \oint_P h = N \quad \text{and} \quad \oint_Q h = -N, \]  
(3.10)

where the integrals run over a path enclosing \( P \) and \( Q \) respectively. Therefore \( h \) should have precisely a pole of order 1 at \( P \) and \( Q \) with residue \( N \) and \( -N \) respectively.

On the Riemann surface the extremization of the superpotential gives

\[ \int \int_{\Gamma} h \wedge \frac{\partial \lambda_{\text{eff}}}{\partial b_k} = 0 \quad \text{for} \quad k = 0, \ldots, n-2, n-1. \]  
(3.11)

Notice that \( \frac{\partial \lambda_{\text{eff}}}{\partial b_k} \) for \( k = 0, \ldots, n-2 \) are holomorphic one forms on \( \Gamma \). These form a complete basis of holomorphic one forms. Therefore by the Riemann bilinear identities (3.11) is satisfied if and only if \( h \) is a holomorphic one form on \( \Gamma \setminus \{P, Q\} \). This will also make the equation for varying of \( b_{n-1} \) satisfied on \( \Gamma \setminus \{P, Q\} \). But for all \( b_k \) variations we also need to consider the potential contribution of the integral (3.11) from \( P, Q \). By using the Riemann bilinear identity this is equivalent to the contribution

\[ \oint_P h \oint_Q \frac{\partial \lambda_{\text{eff}}}{\partial b_k} - \oint_P \frac{\partial \lambda_{\text{eff}}}{\partial b_k} \oint_P h = 0. \]  
(3.12)
Using (3.10) and (3.9) we can write this as

$$N \int_P^Q \frac{\partial \lambda_{\text{eff}}}{\partial b_k} - \int_P^Q \frac{\partial \lambda_{\text{eff}}}{\partial b_k} \tau_{\text{YM}} = 0.$$ (3.13)

For $k = 1, \ldots, n - 2$ the second term vanishes because $\omega_k = \frac{\partial \lambda_{\text{eff}}}{\partial b_k}$ is a holomorphic one form. Thus we obtain

$$N \int_P^Q \omega_k = 0.$$ (3.14)

Note that this is well defined up to addition of periods, depending on which path one takes from $P$ to $Q$. This equation implies, according to Abel’s theorem that there must be a meromorphic function on $\Gamma$ with an $N$-th order zero on $P$ and an $N$-th order pole on $Q$. For $k = n - 1$, since $\omega_{n-1} = \frac{1}{g_{n+1} \partial \theta_{n-1}} \sim dx/x$ as $x \to \infty$, we have to introduce a cutoff $\Lambda_0$, as discussed before. We obtain

$$N \int_P^Q \omega_{n-1} - \tau_{\text{YM}} = 0$$

where the first term gives $2N \log[\Lambda/\Lambda_0]$ for some $\Lambda$ (depending on $b_i$) and we obtain

$$\int_P^Q h = \tau_{\text{YM}} = 2N \log[\Lambda/\Lambda_0]$$ (3.15)

Now that we have translated the field equations (3.11) into the existence of a holomorphic one form $h$ on $\Gamma$ with certain properties, and the existence of a meromorphic function with divisor $N[P - Q]$ the final step is to find $f_{n-1}(x)$ such that $\Gamma$ defined by (3.8) admits such an $h$ and such a meromorphic function.

We will now show that these exist if $f_{n-1}(x)$ is such that the following is true,

$$(W'^2(x) + f_{n-1}(x)) H_{N-n}^2(x) = g_{n+1}^2 (P_N(x)^2 - \gamma^2).$$ (3.16)

for some $H_{N-n}(x)$ and $P_N(x)$, where $P_N(x) \to \prod_{i=1}^n (x - a_i)^{N_i}$ as $\gamma \to 0$. The factor $g_{n+1}$ is introduced only to normalize the coefficient of $x^N$ in $P_N(x)$ to one.

Consider, the function $z$ on $\Gamma$, defined by,

$$z = P_N(x) - \frac{1}{g_{n+1}} y H_{N-n}(x).$$

Note that due to (3.16), $z$ satisfies the following equation on $\Gamma$:

$$z - 2P_N(x) + \frac{\gamma^2}{z} = 0.$$ (3.17)
$z$ has a zero of order $N$ at $P$ and a pole of order $N$ at $Q$. This is one condition we needed to satisfy. Moreover, $z$ does not have any zeros or poles in $\Gamma - \{P, Q\}$. This follows from (3.17). This implies that $\frac{1}{2\pi i} \frac{dz}{z}$ satisfies (3.10). We claim that,

$$h = \frac{1}{2\pi i} \frac{dz}{z}.$$ 

We need to check that the conditions (3.9) are satisfied. In order to compute the periods of $h = \frac{1}{2\pi i} \frac{dz}{z}$ over $\alpha_k$’s notice that the answer is independent of $\gamma$. This is because $z$ is a well defined function on $\Gamma - \{P, Q\}$, and its phases can change only by an integer multiple of $2\pi i$. This implies that the evaluation can be performed in the limit $\gamma \to 0$.

$$\oint_{\alpha_k} \frac{dz}{z} = \oint_{\alpha_k} d(\log z) = \oint_{\alpha_k} d(\log(2P_N(x)|_{\gamma \to 0})).$$

But $P_N(x)|_{\gamma \to 0} = \prod_{j=1}^{n} (x - a_j)^{N_j}$ and therefore,

$$\oint_{\alpha_k} h = \frac{1}{2\pi i} \oint_{\alpha_k} \frac{dz}{z} = N_k \frac{1}{2\pi i} \oint_{\alpha_k} d(\log(x - a_i)) = N_k.$$ 

We can also compute $\int_{C_i - C_j} \frac{dz}{z}$. This can be done using the same argument as before and realizing that going around the $C_i - C_j$ cycle we do not cross any branch cut of the $\log(x - a_i)$ functions. Hence,

$$\oint_{C_i - C_j} \frac{dz}{z} = 0.$$

Finally, we have to check (3.15) which relates $\gamma$ to the parameters of the original Yang-Mills theory. From the definition of $z$ we see that the $\gamma$ gets identified with $\gamma = \pm 2\Lambda^N$.

In order to complete the proof of the second conjecture we only have to show that

$$W_{\text{eff}}(\langle S_k \rangle) = W_{\text{low}}(b_k, \Lambda).$$

The final result in section 2, showed from field theory that,

$$\frac{\partial W_{\text{low}}}{\partial \log \Lambda^{2N}} = -\frac{1}{4g_{n+1}} b_{n-1}.$$ 

where $b_{n-1}$ is the coefficient of $x^{n-1}$ in $f_{n-1}(x)$ from field theory. However, we also have that,

$$\frac{\partial W_{\text{eff}}(\langle S_k \rangle)}{\partial \log \Lambda^{2N}} = -\frac{1}{4g_{n+1}} b_{n-1}$$

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where \( b_{n-1} \) is the coefficient of \( x^{n-1} \) in \( f_{n-1}(x) \) from the CY 3-fold. Given that we have shown that the two polynomials \( f_{n-1}(x) \) are equal, the following is true,

\[
\frac{\partial W_{\text{eff}}(<S_k>)}{\partial \log \Lambda^{2N}} = \frac{\partial W_{\text{low}}(b'_ks, \Lambda)}{\partial \log \Lambda^{2N}}.
\] (3.18)

Finally, showing that \( W_{\text{eff}}(<S_k>)|_{\Lambda \to 0} \) is equal to \( W_{\text{low}}(g_k, \Lambda \to 0) \) will complete the proof. From section 2, taking \( \Lambda \to 0 \) is the classical limit and

\[
W_{\text{low}}(g_k, \Lambda \to 0) = \sum_{i=1}^{n} N_i \sum_{k=1}^{n+1} \frac{1}{k} g_k a_i^k.
\]

On the other hand, from the geometry, setting \( \Lambda \) to zero gives \( f_{n-1}(x) = 0 \) and the effective one form simplifies \( \lambda_{\text{eff}} = ydx = W'(x)dx \). Taking into account that \( b_{n-1} \) goes to zero as a polynomial in \( \Lambda \) we get that the second term in the effective superpotential (3.3) given by \( \tau_{\text{YM}(\Lambda_0)} < S_1 + \ldots + S_n > \) goes to zero in the limit. Notice that we could add to our definition of \( W_{\text{eff}} \) (3.3) an arbitrary function of the form,

\[
(N_1 + N_2 + \ldots + N_n)G(g'_ks, \Lambda_0)
\]

which is \( \Lambda \) independent and does not affect the validity of (3.18). Such an addition does not have any effect on physical quantities. It does not affect the field equations because it is an additive constant to the superpotential. Having shown that such an addition is harmless, let us choose \( G(g'_ks, \Lambda_0) = W(\Lambda_0) \). Therefore taking the limit \( \Lambda \to 0 \) we get,

\[
W_{\text{eff}}(<S_k>) = -\sum_{i=1}^{n} N_i \int_{a_i}^{\Lambda_0} W'(x)dx + NW(\Lambda_0) = \sum_{i=1}^{n} N_i \sum_{k=1}^{n+1} \frac{1}{k} g_k a_i^k.
\]

This completes the proof of the second conjecture.

**Coupling constants for \( U(1)^n \)**

In order to establish conjecture 1 we only have to show that the couplings in the dual theory given by,

\[
\tau_{ij} = \frac{\partial^2 F}{\partial S_i \partial S_j}
\]

in some appropriate basis decompose into the period matrix of the auxiliary Riemann surface \( \Gamma \) and the coupling of original \( U(N) \) theory \( \tau_{\text{YM}(\Lambda_0)} \).
The change of basis is easy to guess if we look at the field equations from (3.5),
\[ \frac{\partial}{\partial S_j} \sum_{i=1}^{n} N_i \Pi_i + \tau_{YM(\Lambda_0)} = 0 \quad \text{for} \quad j = 1, \ldots, n \]
using that \( \Pi_i = \frac{\partial F}{\partial S_i} \) the equations can be written as,
\[ \frac{\partial}{\partial S_j} \sum_{i=1}^{n} N_i \frac{\partial}{\partial S_i} F = -\tau_{YM(\Lambda_0)}. \]

From this it is natural to define basis \( \{ S_{12}, S_{23}, \ldots, S_{n-1,n}, S_{+} \} \) such that,
\[ \frac{\partial}{\partial S_{+}} = \sum_{i=1}^{n} N_i \frac{\partial}{\partial S_i} \quad \text{and} \quad \frac{\partial}{\partial S_{i,i+1}} = \frac{\partial}{\partial S_i} - \frac{\partial}{\partial S_{i+1}}. \]

In this new basis the field equations become,
\[ \tau_{i,+} = \frac{\partial^2}{\partial S_{i,i+1} \partial S_{+}} F = 0 \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad \tau_{+ +} = \frac{\partial^2}{\partial S_{+}^2} F = -\frac{1}{N} \tau_{YM(\Lambda_0)}. \]

Finally, we only have to show that the remaining elements of \( \tau_{ij} \) give the period matrix of \( \Gamma \). Consider,
\[ \tau_{ij} = \frac{\partial^2}{\partial S_{i,i+1} \partial S_{j,j+1}} F = \frac{\partial}{\partial S_{i,i+1}} (\Pi_j - \Pi_{j+1}) \quad (3.19) \]
From figure 1 it is clear that \( \Pi_j - \Pi_{j+1} = \int_{C_j - C_{j+1}} y dx \) is an integral over a compact cycle. One more change of variables is needed. Let the new independent variables be \( \{ b_0, \ldots, b_{n-1} \} \). Using this, (3.19) becomes,
\[ \tau_{ij} = \sum_{k=0}^{n-2} \frac{\partial b_k}{\partial S_{i,i+1}} \frac{\partial}{\partial b_k} \left( \int_{C_j - C_{j+1}} y dx \right) + \frac{\partial b_{n-1}}{\partial S_{i,i+1}} \frac{\partial}{\partial b_{n-1}} \left( \int_{C_j - C_{j+1}} y dx \right) \quad (3.20) \]
However, recalling that \( b_{n-1} = -4(S_1 + \ldots + S_n) = -4S_{+} \) the second term drops out since \( \frac{\partial b_{n-1}}{\partial S_{i,i+1}} = 0. \)

Using that \( y^2 = W'(x)^2 + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \) it is easy to see that,
\[ \frac{\partial}{\partial b_k} (y dx) = \frac{1}{2} \frac{x^k}{y} dx \quad \text{for} \quad k = 0, \ldots, n-2 \]
forms a basis of holomorphic one forms over \( \Gamma \). Moreover, \( S_{i,i+1} = \int_{\gamma_i} y dx \), with \( \gamma_i \) integral linearly independent combinations of \( \alpha_j \)'s. Together, \( (\gamma_i, C_i - C_{i+1}) \) form a basis for \( H_1(\Gamma, \mathbf{Z}) \). Therefore, \( \tau_{ij} \) given in (3.20) is a period matrix of \( \Gamma \). This completes the proof of the conjectures.
4. Derivation of the Seiberg-Witten solution for $\mathcal{N} = 2$ $U(N)$ from Fluxes

The $\mathcal{N} = 1$ theories we have studied up to now are deformations of pure $\mathcal{N} = 2$ $U(N)$ Yang-Mills. It is natural to ask to what extent one can recover information of the original $\mathcal{N} = 2$ theory as the deformation is turned off. This is the main issue we want to address in this part of the paper.

The idea is to look for deformations $W_{\text{tree}}$ that will provide information about an arbitrary Coulomb point of the original $U(N)$ theory. This will turn out to be a potential of degree $N + 1$,

$$W_{\text{tree}} = \sum_{k=1}^{N+1} \frac{g_k}{k} \text{Tr} \Phi^k$$

and we consider the vacuum which breaks $U(N)$ to $U(1)^N$ generically. In this vacuum $\Phi$ is given by diag$(a_1, \ldots, a_N)$, where $a_i$’s are defined by,

$$W'(x) = g_{N+1}x^N + \ldots + g_1 = g_{N+1} \prod_{k=1}^{N} (x - a_i).$$

The limit that allows us to get back to the $\mathcal{N} = 2$ theory is $g_{N+1} \rightarrow 0$ while keeping $a_i$’s fixed. The $a_i$’s will correspond to a generic point in the Coulomb branch of the $\mathcal{N} = 2$ theory. It is natural to suggest that all $\mathcal{N} = 2$ information, if any, in the $\mathcal{N} = 1$ theory will have to come from quantities that do not depend on $g_{N+1}$. Moreover, intrinsically $\mathcal{N} = 1$ objects like gaugino vev’s will all vanish as $g_{N+1} \rightarrow 0$.

4.1. Seiberg-Witten Curve

We will see first how the $\mathcal{N} = 2$ curve arises as a solution to the field equations (3.7) of the effective superpotential (3.5) for gaugino fields $S_i$’s.

Let us rewrite (3.5) using $N_i = 1$ for $i = 1, \ldots, N$,

$$W_{\text{eff}}(S_1, \ldots, S_N) = \sum_{i=1}^{N} \Pi_i + \tau_{YM(\Lambda_0)} \sum_{i=1}^{N} S_i.$$

where $\Pi_i = \frac{\partial F}{\partial S_i}$ and $F$ is the prepotential of the CY 3-fold,

$$uv + w^2 + W'(x)^2 + f_{N-1}(x) = 0.$$

In this case the field equations arising from the superpotential are hard to solve. The main problem being the determination of the prepotential $F$. However, they can be solved for any $N_i$’s in a semi-classical expansion. See appendix C for examples.
Luckily, in this case the factorization problem (2.4) is trivial and in section 3 we gave a general proof of the equivalence of the two. So we can simply use (2.4) with \( n = N \) to get,

\[
P_N^2(x) - 4\Lambda^{2N} = \frac{1}{g_{N+1}^2} \left( W'(x)^2 + b_{N-1}(\Lambda)x^{N-1} + \ldots + b_0(\Lambda) \right).
\]

From this we get that \( b_k(\Lambda) = 0 \) for \( k = 1, \ldots, N-1 \) and \( b_0 = -4g_{N+1}\Lambda^{2N} \) is a solution. In appendix B we show that this is indeed the unique solution. Therefore, \( W'(x) = g_{N+1}P_N(x) = g_{N+1} < \det(xI - \Phi) > \).

This implies that the vev’s of the Casimirs \( u_k = \frac{1}{k} \text{Tr} \Phi^k \) are not modified quantum mechanically and \( < u_k > = (u_k)_{\text{class}} \). Let us check that this is indeed the case from the low energy superpotential of the dual theory, i.e. \( W_{\text{eff}}(< S_1 >, \ldots, < S_N >) \).

The effective superpotential after minimization procedure can be used to compute the quantum expectation value of the Casimir operator \( < u_k > \) as well as the expectation value of \( < S >= < S_1 + \ldots + S_N > \) as follows,

\[
\frac{\partial W_{\text{eff}}}{\partial g_k} = < u_k > \quad \text{and} \quad \frac{\partial W_{\text{eff}}}{\partial \text{Log} \Lambda^{2N}} = < S >
\]

But we know from the solution to the field equations that the expectation value of \( < S > \) is zero because it is proportional to \( b_{N-1} \). This implies that \( W_{\text{eff}}(< S_1 >, \ldots, < S_N >) \) is not a function of \( \Lambda \) and therefore it can be computed at any value, in particular, at \( \Lambda = 0 \). This implies that,

\[
W_{\text{eff}}(< S_1 >, \ldots, < S_N >) = W_{\text{class}}(g_l)
\]

Therefore,

\[
\frac{\partial W_{\text{eff}}(g_l)}{\partial g_k} = (u_k)_{\text{class}}
\]

as it should be consistent with the result from the curve. Now recall that the geometry of the Calabi-Yau 3 fold after the transitions is given by,

\[
uw + w^2 + W'(x)^2 + f_{N-1}(x) = 0.
\]

Using the result of minimizing the superpotential we get,

\[
uw + w^2 + g_{N+1}^2 \left( P_N(x)^2 - 4\Lambda^{2N} \right) = 0.
\]

There are several interesting observations to make from this: Notice that the auxiliary Riemann surface \( \Gamma \) used to compute periods is exactly equal to the Seiberg-Witten curve for pure \( \mathcal{N} = 2 \ U(N) \) after absorbing a factor of \( g_{N+1} \) in the definition of \( y \).
This is surprising given that we expect to recover the $\mathcal{N} = 2$ answer only when $g_{N+1}$ is taken to zero. However, the SW curve is the solution to the field equations for all $g_{N+1}$. Moreover, for $g_{N+1} \to 0$, the geometry of the CY 3-fold reduces to that of an $A_1$ singularity trivially fibered over the $x$-plane as expected from enhanced supersymmetry in this limit. This looks like the classical limit of the $\mathcal{N} = 2$ theory, and so one would like to see how the exact quantum $\mathcal{N} = 2$ answer is recovered.

Let us consider in more detail the way periods $S_i$'s and $\Pi_i$'s of the holomorphic three form over $A_i$'s and $B_i$'s cycles depend on $g_{N+1}$. As mentioned in section 3 the periods can be written as integrals of an effective one form (3.6) over the x-complex plane.

$$\lambda_{\text{eff}} = \sqrt{W'(x)^2 + f_{N-1}(x)dx} = g_{N+1}\sqrt{P(x)^2 - 4\Lambda^2 Ndx}$$

The contours of integration only depend on the position of the branching points $a_i$'s. This implies that, $\frac{1}{g_{N+1}}S_i$ and $\frac{1}{g_{N+1}}\Pi_i$ are independent of $g_{N+1}$. The $\mathcal{N} = 1$ fields $S_i$ and $\Pi_i$ go to zero in the limit $g_{N+1} \to 0$. Recall that the $U(1)^N$ couplings in the dual theory are given by,

$$\tau_{ij} = \frac{\partial}{\partial S_i} \Pi_j = \frac{\partial}{\partial \left( \frac{1}{g_{N+1}} S_i \right)} \left( \frac{1}{g_{N+1}} \Pi_j \right)$$

and therefore are trivially $g_{N+1}$ independent. Furthermore, as discussed in section 3, in a suitable basis $\{S_{12}, S_{23}, \ldots, S_{N-1,N}, S_+\}$ defined by,

$$\frac{\partial}{\partial S_{i,i+1}} = \frac{\partial}{\partial S_i} - \frac{\partial}{\partial S_j}$$

and $S_+ = S_1 + S_2 + \ldots + S_N$

the $U(1)$ coupling $\tau_{++} \equiv \frac{\partial^2 F}{\partial S_+^2} = -\frac{1}{N} \tau_{\text{YM}(\Lambda_0)}$ decouples, i.e, $\tau_{+i} = 0$ for $i = 1, \ldots, N-1$ and $\tau_{ij} \equiv \frac{\partial^2 F}{\partial S_{i,i+1} \partial S_{j,j+1}}$ is equal to the period matrix of $y^2 = P_N(x)^2 - 4\Lambda^2 N$.

We have thus recovered the $U(1)^N$ coupling constants of the $\mathcal{N} = 2$ theory that are originally given by $\tau_{++} = -\frac{1}{N} \tau_{\text{YM}(\Lambda_0)}$ and $\tau_{ij} = \frac{\partial D_{ij}}{\partial a_i}$.

4.2. $\mathcal{N} = 2$ dyons

The $\mathcal{N} = 2$ data also contains information about the mass of BPS particles. It is therefore interesting to see how this data comes out of our $\mathcal{N} = 1$ theory. At first sight this seems not to be possible given that in the $\mathcal{N} = 1$ theory dyons are not BPS states. However we will see that the key is to realize that the dual theory contains non zero fluxes of $H = H_{RR} + \tau_{\text{IIB}} H_{NS}$ through 3-cycles. This three form carries nontrivial
information because, as we will show, its integral over $S^2$ in the fiber is $g_{N+1}$ independent. Of course, the computation of the exact mass of the dyons is conceptually correct only when $g_{N+1} \rightarrow 0$.

Let us start by identifying the electric and magnetic objects of the $\mathcal{N} = 2$ system before and after the transition. Consider first the geometry before the transition with one $D5$ brane wrapping each $S^2_i$, where $S^2_i$ is the non-trivial element in $H_2$ of the $A_1$ fiber located at $x = a_i$. These are the points where the holomorphic volume $\alpha_i$ is zero i.e. the solutions to the classical field equations obtained from $W_{\text{tree}}$. Here we are considering the case where the kahler volume is also zero but the stringy volume is non zero due to the contribution of $B_{\text{NS}}$.

If $g_{N+1} \rightarrow 0$ we expect the fibration to become trivial. The geometry is just the product of the $x$-complex plane and the $A_1$ ALE space where the singularity is resolved only by $B_{\text{NS}}$ and $B_{\text{RR}}$. These fluxes are clearly constant. $D5$ branes are still wrapping the $S^2$ at the same locations in the x-plane, therefore we are at some point in the Coulomb branch of the classical $\mathcal{N} = 2$ $U(N)$ theory. At this generic point the gauge group is broken down to $U(1)^N$. The electric and magnetic particles can be easily identified as follows. $W$-bosons with charges $(1, -1)$ under $U(1)_i \times U(1)_j$ are identified with open strings stretching between $D5$ branes at $x = a_i$ and $x = a_j$. Given that the fibration is trivial, the mass of such a string is simply its tension times its length $m = |a_i - a_j|$. A magnetic object, on the other hand, can be identified with a $D3$ brane wrapping a 3-chain given by $S^2 \times I_{ij}$, where $I_{ij}$ is the interval in the x-plane from $a_i$ to $a_j$. Its mass is given by the volume of the 3-chain times the tension of the brane. The mass is therefore given by $m = |\int_{S^2}(\tau_{\text{IIB}}B_{\text{NS}} + B_{\text{RR}})| |a_i - a_j|$. Recall that the holomorphic coupling of the 4 dimensional field theory is given by $\tau_{\text{YM}} = \int_{S^2}(\tau_{\text{IIB}}B_{\text{NS}} + B_{\text{RR}})$, therefore $m = |\tau_{\text{YM}}(a_i - a_j)|$. This is also the result from field theory classically.

It is important to keep in mind that $\tau_{\text{YM}}$ is constant only in the classical theory. Quantum mechanically we expect $B$ to vary over $x$-plane. Thus the mass will be given by an integral over the path in the x-complex plane joining $a_i$ to $a_j$ times the $B$-field over each point. Let us now write the central charge $Z_m$ instead of the mass $m = \sqrt{2}|Z_m|$. Therefore,

$$Z_m = \int_{I_{ij}} \int_{S^2} B \wedge dx$$

(4.1)

where we have defined $B = \tau_{\text{IIB}}B_{\text{NS}} + B_{\text{RR}}$.  

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As $g_{N+1}$ is turned on, the IR physics is described by the geometry after the transition where D5 branes wrapping 2-cycles have been replaced by fluxes over the new 3-cycles.

As the $S^2_i$ and $S^2_j$ are blown down the 3-chain $S^2 \times I_{ij}$ on which the $D_3$ was wrapped becomes a 3-cycle given by $B_i - B_j$. In terms of the basic cycles in the auxiliary Riemann surface $\Gamma$ this is the same as $\beta_i - \beta_j$ and the integral (4.1) can be written as,

$$Z_m = \oint_{\beta_i} \int_{S^2} B \wedge dx - \oint_{\beta_j} \int_{S^2} B \wedge dx.$$ Integrating by parts in order to bring in $H = dB = H_{RR} + \tau H_{NS},$

$$Z_m = \int_{(\beta_i - \beta_j) \times S^2} x \ H. \quad (4.2)$$

This formula for the BPS mass is only valid in the limit $g_{N+1} \to 0$. The reason being that only when the geometry becomes a trivial fibration, the kahler volume of the $S^2$ and the complex volume of the interval $I_{ij}$ combine.

We are only left with the computation of the one form $h \equiv \int_{S^2} H$ in the dual geometry. But recall from section 3.3 that such a one form was found in the general case to be given by (3.17),

$$h = \frac{1}{2\pi i} \frac{dz}{z} \quad \text{with} \quad z - 2P_N(x) + \frac{4\Lambda^{2N}}{z} = 0. \quad (4.3)$$

This was derived from the constraints,

$$\int_{A_i} H = N_i = 1 \quad \text{and} \quad \int_{B_i} H = \tau_{YM}(\Lambda_0)$$

where $\tau_{YM}(\Lambda_0)$ is the bare Yang-Mills coupling of the original $\mathcal{N} = 2$ theory. It is clear from the definition of $z$ that $\frac{dz}{z}$ is independent of $g_{N+1}$ as required.

Finally, substituting (4.3) in (4.2) we get,

$$Z_m = \oint_{\beta_i} \frac{dz}{z} x - \oint_{\beta_j} \frac{dz}{z} x$$

from which it is possible to identify the Seiberg-Witten differential,

$$\lambda_{SW} = x \frac{dz}{z} = x \frac{P_N'(x)dx}{\sqrt{P_N^2(x) - 4\Lambda^{2N}}}$$

and the mass of the magnetic monopole as $m = |a_{D_i} - a_{D_j}|$. 

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The electric particle is harder to identify in the dual geometry. However, using the identification between $U(1)^N$ in the original theory and the $U(1)^N$ in the dual geometry, we can use the charges as a hint to identify the state. $W_{ij}$-boson are charged under $U(1)_i \times U(1)_j$ with charges $(1, -1)$. Therefore, it is natural to propose that the fundamental string stretched between the D5-brane at $x = a_i$ and the D5-brane at $x = a_j$ corresponds to a fundamental string stretched between a D3 brane wrapping $S^3_i$ and an anti-D3 brane wrapping $S^3_j$. The consistency of this argument relies heavily on the fact that there is one unit of $H_{RR}$ flux through each $S^3$, leading to a fundamental string charge once a D3 brane is wrapped over it. As for the BPS mass for electric states, this should agree with the gauge predictions, because we have already argued that $a_{Di}$ and $\tau_{ij}$ agree and we have the fundamental $\mathcal{N} = 2$ relation

$$\frac{\partial a_{Di}}{\partial a_j} = \tau_{ij}.$$  

4.3. Generalizations

A natural generalization of these ideas is to consider the case of the quiver theories studied in [3]. We leave the study of this large class of examples to the reader. It is quite satisfactory to see this merging of holomorphic techniques in studying vacuum structures for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauge systems via a geometric realization in string theory and it would be worthwhile studying more examples of how this works, which we leave to the interested reader.

Appendix A. Proof of $g^2_{n+1}F_{2n}(x) = W'(x)^2 + f_{n-1}(x)$

In this section we will review the proof given in [1] for the reformulation of the superpotential extremization in the gauge theory setup. The idea is to formulate the whole problem in terms of a superpotential with the conditions for massless monopoles imposed as constraints. Clearly the condition,

$$P_N(x)^2 - 4\Lambda^{2N} = F_{2n}(x)H_{N-n}^2(x)$$  \hspace{1cm} (A.1)

is equivalent to,

$$P_N(p_i) + \epsilon_i 2\Lambda^N = 0 \quad \text{and} \quad P'_N(p_i) = 0$$

for $H_{N-n}(x) = \prod_{i=1}^{N-n}(x - p_i)$ and $\epsilon_i = \pm 1$. 

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The total superpotential can then be written as,

$$W = \sum_{i=1}^{n} g_r u_r + \sum_{i=1}^{N-n} \left[ L_i(P_N(x)|_{x=p_i} - 2\epsilon_i \Lambda^N) + Q_i \frac{\partial}{\partial x} P_N(x)|_{x=p_i} \right]$$  \hspace{1cm} (A.2)

Notice that $l$ is arbitrary now but it will turn out to be $l \geq N - n$. The $L_i$, $Q_i$, and $p_i$ are treated as Lagrange multipliers.

The variation of (A.2) with respect to $p_i$ gives

$$Q_i \frac{\partial^2 P_N}{\partial x^2} \Big|_{x=p_i} = 0,$$  \hspace{1cm} (A.3)

where we used the $Q_i$ constraint to eliminate the term involving $L_i$. For generic $g_r$, the RHS of (A.1) has some double roots, but no triple or higher roots; therefore (A.3) implies that $\langle Q_i \rangle = 0$. Since the $\langle Q_i \rangle = 0$, the variation of (A.2) with respect to all $u_r$ is

$$g_r + \sum_{i=1}^{N-n} \sum_{j=0}^{N} L_i p_i^{N-j} \frac{\partial s_i}{\partial u_r},$$  \hspace{1cm} (A.4)

with the understanding that the $g_r = 0$ for $r > n + 1$. Using that $P_N(x) = \langle \det(x I - \Phi) \rangle$ and

$$\det(x I - \Phi) = x^N \exp \left( \text{tr} \log(I - \frac{1}{x} \Phi) \right) \bigg|_+ = x^N \exp \left( -\sum_{n=1}^{\infty} \frac{u_n}{x^n} \right) \bigg|_+ = \sum_{l=0}^{\infty} x^{N-l} s_l \bigg|_+$$

where $\sum_{k=-\infty}^{\infty} c_k x^k \bigg|_+ = \sum_{k=0}^{\infty} c_k x^k$, one can easily show that, $\frac{\partial s_j}{\partial u_k} = -s_{j-k}$. Therefore, (A.4) becomes

$$g_r = \sum_{i=1}^{N-n} \sum_{j=0}^{N} L_i p_i^{N-j} s_{j-r}.$$  \hspace{1cm} (A.5)

We should also impose the $L_i$ and $Q_i$ constraints in (A.2). These equations and (A.5) fix the $\langle u_r \rangle$, $\langle L_i \rangle$, $\langle p_i \rangle$, and $\langle Q_i \rangle$ as functions of the $g_r$ and $\Lambda$. The $\langle L_i \rangle$ are proportional to the expectation values $\langle q_i \tilde{q}_i \rangle$ of the $l \geq N - n$ condensed, mutually local, monopoles.
Following a similar argument in [9], we multiply (A.5) by $x^r$ and sum:

$$W'(x) = \sum_{r=1}^{N} g_r x^{r-1}$$

$$= \sum_{r=1}^{N} \sum_{i=1}^{l} \sum_{j=0}^{N} x^{r-1} p_i^{N-j} s_{j-r} L_i$$

$$= \sum_{r=-\infty}^{l} \sum_{i=1}^{N} \sum_{j=-\infty}^{0} x^{r-1} p_i^{N-j} s_{j-r} L_i - 2LA^N x^{-1} + O(x^{-2})$$

$$= \sum_{i=1}^{l} \frac{P_N(x; \langle u \rangle)}{x - p_i} L_i - 2LA^N x^{-1} + O(x^{-2}).$$

We define $L \equiv \sum_{i=1}^{l} L_i \epsilon_i$. Defining, as in [9], the order $l-1$ polynomial $B_{l-1}(x)$ by

$$\sum_{i=1}^{l} \frac{L_i}{x - p_i} = \frac{B_{l-1}(x)}{H_l(x)},$$

with $H_l(x)$ the polynomial appearing in (A.1), we thus have

$$W'(x) + 2LA^N x^{-1} = B_{l-1}(x) \sqrt{F_{2N-2l}(x) + \frac{4A^{2N}}{H_l(x)^2}} + O(x^{-2}).$$

(A.8)

Since the highest order term in $W'(x)$ is $g_{n+1} x^n$, we see that $B_{l-1}(x)$ should actually be order $n-N+l$. This shows that $l \geq N-n$ and, in particular, for $l = N-n$, $B_{N-n-1} = g_{n+1}$ is a constant. Squaring (A.8) gives

$$g_{n+1}^2 F_{2n} = W'(x)^2 + 4g_{n+1} L \Lambda^N x^{n-1} + O(x^{n-2}).$$

(A.9)

We have found, $g_{n+1}^2 F_{2n} = W'(x)^2 + f_{n-1}(x)$.

Notice that after varying with respect to all the Lagrange multipliers and solving the equations; $< L_i >$, $< Q_i >$, and $< p_i >$ will be functions of $g_i$ and $\Lambda$.

Let us now prove the statement made at the end of section 2. There it was claimed that $-4g_{n+1} < S_1 + \ldots + S_n >$ is equal to the coefficient of the $x^{n-1}$ term in $f_{n-1}(x)$.

Consider the term in the superpotential (A.2),

$$\sum_{i=1}^{N-n} (-2\epsilon_i L_i) \Lambda^N \equiv -2LA^N.$$
This tells us that after integrating out $p_i$’s, $Q_i$’s, $u_i$’s, and $L_i$’s, and $W$ becomes equal to $W_{\text{low}}$, then,

$$\frac{\partial W_{\text{low}}}{\partial \log \Lambda^{2N}} = \frac{\partial W}{\partial \log \Lambda^{2N}} = -\Lambda^N < L > .$$

From (2.5),

$$\frac{\partial W_{\text{low}}}{\partial \log \Lambda^{2N}} = < S_1 + \ldots + S_n >$$

we get,

$$\sum_{i=1}^{n} < S_i > = -\Lambda^N < L > .$$

Finally, using (A.9) we see that,

$$f_{n-1}(x) = 4g_{n+1}L\Lambda^N x^{n-1} + \mathcal{O}(x^{n-2}).$$

From which the statement we wanted to prove follows.

**Appendix B. Proof of Uniqueness**

We want to understand to what extent our answer for the curves is unique, let us assume that the following equation holds,

$$\tilde{W}'(x)^2 + \tilde{b}_{n-1}x^{n-1} + \ldots + \tilde{b}_0 = W'(x)^2 + b_{n-1}x^{n-1} + \ldots + b_0$$

where $W'(x) = x^n + s_1x^{n-1} + \ldots + s_n$ and $\tilde{W}'(x) = x^n + \tilde{s}_1x^{n-1} + \ldots + \tilde{s}_n$.

Consider the Riemann surface defined by,

$$y^2 = W'(x)^2 + b_{n-1}x^{n-1} + \ldots + b_0.$$  \hfill (B.1)

It is not difficult to show that if $C$ is a closed contour around $x = \infty$ on the upper sheet that does not contain any of the branching points, then,

$$s_k = \frac{1}{2\pi i} \oint_C x^{k-1-n} y dx.$$  \hfill (B.2)

This can be shown by expanding $y(x)$ around $x = \infty$ and reading the residue. From (B.2) we conclude that $s_k = \tilde{s}_k$.

It is also possible to see that,

$$\frac{1}{2} b_l = \frac{1}{2\pi i} \oint_C (x^{l-1} + s_1x^{l-2} + \ldots + s_{l-1}) y dx.$$  \hfill (B.3)

Therefore, using that $s_k = \tilde{s}_k$ we conclude that $b_l = \tilde{b}_l$.

We have shown that if the hyperelliptic curve can be written as (B.1) then the form of the curve is unique.
Appendix C. Calculability

We have shown how the superpotential equations are equivalent to finding a solution to problem stated in (2.4). However, we have not shown how this can be used to find the solution. In this appendix we will show that the equations from our effective superpotential are solvable in a systematic expansion in \( \Lambda \) around the semi-classical regime.

The superpotential (3.5),

\[
W_{\text{eff}} = \sum_{i=1}^{n} N_i \Pi_i + \tau_{\text{YM}(\Lambda_0)} \sum_{i=1}^{n} S_i
\]

is only a function of \( S_i \)'s. The periods over the non-compact cycles can be computed in terms of the prepotential of the CY 3-fold \( F = F(S_1, \ldots, S_n) \) by \( \Pi_i = \frac{\partial F(S_k)}{\partial S_i} \).

The main advantage of the geometric approach is that the prepotential does not depend on \( N_i \)'s and once it is found the problem is solved for any splitting \( N = \sum_{i=1}^{n} N_i \).

The semi-classical approximation in geometric language means that the deformation of \( W'(x)^2 = \Pi_{i=1}^{n} (x - a_i)^2 \) by \( W'(x)^2 + f_{n-1}(x) = \Pi_{i=1}^{n} (x - a_i^+) (x - a_i^-) \) is such that, \( |a_i - a_j| \gg |a_k^+ - a_k^-| \) for any \( \{i, j, k\} \). The first step is to rewrite the effective one form,

\[
\lambda_{\text{eff}} = \sqrt{W'(x)^2 + f_{n-1}(x)} = \prod_{i=1}^{n} \sqrt{(x - A_i)^2 - \delta_i^2}
\]

The compact and non-compact periods \( S_j \) and \( \Pi_j \) are computed by changing variables to \( y = x - A_j \) and writing,

\[
\lambda_{\text{eff}} = \sqrt{y^2 - \delta_j^2} \prod_{k \neq j} (y + A_j - A_k) \prod_{i \neq j} \sqrt{1 - \left( \frac{\delta_j}{y + A_j - A_i} \right)^2}.
\]

Expanding the square roots in the product one gets an infinite power series in \( \delta_i \)'s times \( \sqrt{y^2 - \delta_j^2} \). Integrals of the form,

\[
\int \sqrt{y^2 - \delta_j^2} \prod_{k \neq j} (y + A_j - A_k)^{l_k}
\]

where \( l_k \) are arbitrary integers, can be done in closed form.

The second step is to write the new variables \( \{A_1, \ldots, A_n, \delta_1, \ldots, \delta_n\} \) in terms of mixed ones \( \{a_1, \ldots, a_n, \delta_1, \ldots, \delta_n\} \). This is done by equating the \( x^{2n}, \ldots, x^n \) coefficients of \( W'(x)^2 + f_{n-1}(x) \) and of \( \prod_{i=1}^{n} ((x - A_i)^2 - \delta_i^2) \). This can be done order by order in \( \delta \)'s by solving linear systems of equations.
Finally, using $S_i$'s as functions of $\{a_1, \ldots, a_n, \delta_1, \ldots, \delta_n\}$ one can invert the relations to get $\delta_k$'s as functions of $\{a_1, \ldots, a_n, S_1, \ldots, S_n\}$. Substituting this in $\Pi_i = \Pi_i(a_1, \ldots, a_n, \delta_1, \ldots, \delta_n)$ one gets $\Pi_i = \Pi_i(a_1, \ldots, a_n, S_1, \ldots, S_n)$. The inversion process can also be done order by order by solving linear systems of equations.

In \cite{1} this procedure was carried out for $n = 2$ with the following result,

$$\Pi_1 = \ldots + S_1(\log \frac{S_1}{g\Delta} - 1) + 2S_2 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 +$$

$$+ g(\Delta)^3 \left[ \frac{1}{(g\Delta^3)^2} (2S_1^2 - 10S_1S_2 + 5S_2^2) + O \left( \frac{S_3^{3}}{(g\Delta^3)^3} \right) \right]$$

where the ellipses stand for terms independent of $S_i$'s, $\Delta = a_1 - a_2$, $g = g_3$ and $\Lambda_0$ is a large distance cut off. $\Pi_2$ can be found by replacing all 1's by 2's and vice versa.

In this case the parameters of the classical superpotential at each order only enter in an overall coefficient $(g\Delta^3)^{-n}$.

A much more interesting structure can be found for $n = 3$. In order to give the expression of the non-compact periods, a small change in notations has been introduced. (1, 2, 3) will be replaced by $(a, b, c)$ and $W'(x) = (x - a_1)(x - a_2)(x - a_3)$ will be replaced by $W'(x) = (x - \alpha)(x - \beta)(x - \gamma)$. In terms of the new notation we have,

$$\Pi_a = \ldots - S_a(1-\log S_a) + (2S_b - S_a) \log(\alpha - \beta) + (2S_c - S_a) \log(\alpha - \beta) - 2(S_a + S_b + S_c) \log \Lambda$$

$$+ h_{aa}S_a^2 + h_{bb}S_b^2 + h_{cc}S_c^2 + h_{ab}S_aS_b + h_{ac}S_aS_c + h_{bc}S_bS_c + O(S^3)$$

with,

$$h_{aa} = \frac{1}{2(\alpha - \beta)^2(\alpha - \gamma)^2} \left( 5 + 4\frac{(\alpha - \gamma)}{\alpha - \beta} + 4\frac{(\alpha - \beta)}{\alpha - \gamma} \right)$$

$$h_{bb} = -\frac{1}{(\beta - \alpha)^2(\beta - \gamma)^2} \left( 2 + 2\frac{(\gamma - \beta)}{\gamma - \alpha} + 5\frac{\beta - \gamma}{\beta - \alpha} \right)$$

$$h_{cc} = -\frac{1}{(\gamma - \alpha)^2(\gamma - \beta)^2} \left( 2 + 2\frac{\beta - \gamma}{\beta - \alpha} + 5\frac{\gamma - \beta}{\gamma - \alpha} \right)$$

$$h_{ab} = -\frac{2}{(\alpha - \beta)^2(\alpha - \gamma)(\beta - \gamma)} \left( -2 + 5\frac{\beta - \gamma}{\beta - \alpha} - 2\frac{\gamma - \beta}{\gamma - \alpha} \right)$$

$$h_{ac} = -\frac{2}{(\alpha - \gamma)^2(\alpha - \beta)(\gamma - \beta)} \left( 2 - 5\frac{\gamma - \beta}{\gamma - \alpha} + 2\frac{\beta - \gamma}{\beta - \alpha} \right)$$

$$h_{bc} = \frac{8}{(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)^2} \left( 1 - \frac{\beta - \gamma}{\beta - \alpha} - \frac{\gamma - \beta}{\gamma - \alpha} \right)$$

and ... represent the classical part $W_{\text{tree}}(\alpha)$ and the diverging pieces that are $S$-independent.
One can now solve the superpotential equations, \( \frac{\partial W_{\text{eff}}}{\partial S_i} = 0 \) for a given splitting \( N = \sum_{i=1}^{n} N_i \). This can be done order by order and gives \( \langle S_i \rangle = \langle S_i \rangle (a_1, \ldots, a_n, \Lambda) \). Using this result one can go back and compute \( b_k = b_k (a_1, \ldots, a_n, \Lambda) \) and \( W_{\text{low}} = W_{\text{eff}} (\langle S_i' \rangle) \).

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References

[1] F. Cachazo, K. Intriligator, and C. Vafa, “A Large N Duality via a Geometric Transition”, hep-th/0103067, Nucl.Phys. B603 (2001) 3-41.

[2] N. Seiberg and E. Witten, “Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory”, hep-th/9407087, Nucl.Phys. B426 (1994) 19-52; P.C. Argyres and A.E. Faraggi, “The Vacuum Structure and Spectrum of N=2 Supersymmetric SU(N) Gauge Theory”, hep-th/9411057, Phys.Rev.Lett. 74 (1995) 3931-3934; A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, “Simple Singularities and N=2 Supersymmetric Yang-Mills Theory”, hep-th/9411048, Phys.Lett. B344 (1995) 169-175

[3] F. Cachazo, S. Katz, and C. Vafa, “Geometric Transitions and N=1 Quiver Theories”, hep-th/0108120

[4] C. Vafa, “Superstrings and Topological Strings at Large N”, hep-th/0008142, J. Math. Phys. 42 (2001) 2798-2817

[5] T.R. Taylor and C. Vafa, “RR Flux on Calabi-Yau and Partial Supersymmetry Breaking”, hep-th/9912152, Phys.Lett. B474 (2000) 130-137

[6] P. Mayr, “On Supersymmetry Breaking in String Theory and its Realization in Brane Worlds”, hep-th/0003198, Nucl.Phys. B593 (2001) 99-126

[7] J. Louis and A. Micu, “Type II Theories Compactified on Calabi-Yau Threefolds in the Presence of Background Fluxes”, hep-th/0202168

[8] F. Cachazo, B. Fiol, K. Intriligator, S. Katz, and C. Vafa, “A Geometric Unification of Dualities”, hep-th/0110028, Nucl.Phys. B628 (2002) 3-78

[9] J. de Boer and Y. Oz, “Monopole Condensation and Confining Phase of N=1 Gauge Theories Via M Theory Fivebrane”, hep-th/9708044, Nucl.Phys. B511 (1998) 155-196