POD-based study of structure and dynamics in turbulent plane Poiseuille flow: comparing quasi-linear simulations to DNS

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Turbulence supported in the quasi-linear restricted nonlinear (RNL) dynamics is analyzed and compared with DNS of Poiseuille turbulence at Reynolds number $R = 1650$. The turbulence structure is obtained by POD analysis of the two components of the flow partition used in formulating the RNL dynamics: the streamwise-mean flow and the associated fluctuations. POD analysis of the streamwise-mean flow indicates that the dominant structure in both RNL and DNS is a coherent roll-streak in which the roll is collocated with the streak in a manner configured to reinforce the streak by the lift-up process. POD analysis of fluctuations from the streamwise-mean streak reveals that similar complex structures, consisting in part of oblique waves collocated with the streak, characterize these fluctuations in both the RNL simulations and DNS. The origin of these structures is identified by their correspondence to POD modes obtained by stochastically forcing the DNS and RNL streaks white in energy and solving for the POD structure predicted by the associated stochastic turbulence model (STM), which elicits a response determined primarily by optimally growing structures on the streak. The mechanism sustaining turbulence in RNL comprises rolls forced by Reynolds stress torques arising from fluctuations in the form of Lyapunov vectors that maintain the streak by lift-up. This close correspondence between the roll-streak structure and the fluctuations in RNL, DNS and the STM implies that the self-sustaining mechanism in DNS is the same as that in RNL, which has the advantage that its mechanism and components have been analytically characterized.

Key words:

1. Introduction

The Proper Orthogonal Decomposition (POD) analysis of a time-dependent velocity field proceeds by first obtaining a time independent mean flow and then forming the average spatial covariance of the components of the velocity fluctuations about this mean flow. The eigenfunctions of this covariance are the POD modes of the flow (Lumley 1967; Aubry \textit{et al.} 1988; Moin & Moser 1989; Berkooz \textit{et al.} 1993; Sirovich \textit{et al.} 1990; Moehlis \textdagger

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et al. 2002; Hellström et al. 2011; Hellström & Smits 2017). The Eckart-Young-Mirsky theorem (Eckart & Young 1936; Mirsky 1960) assures that the POD modes constitute an optimal representation of the fluctuation covariance, eigenanalysis of which provides an orthogonal basis ordered in contribution to the fluctuation variance. The POD modes therefore provide an optimally compact basis for representing fluctuation variance which can be used to compare simulations to other simulations or to observations. POD modes have also been proposed as a means to identify coherent structures in simulations and observations. However caution is needed in the use of POD modes for purposes other than the optimally compact representation of a covariance obtained from a data set, which is the purpose validated by the Eckart-Young-Mirsky theorem. One reason for using caution when interpreting POD modes is that there is arbitrariness in the structures that produce a given covariance. As Cantwell (1981) points out there is no unique relationship between the covariance obtained from a data set and the states that produce it. Indeed, the most general class of states that produce the same covariance is that of a unitary linear combination of the POD modes (Schroedinger 1936; Farrell & Ioannou 2002). It follows that while the POD modes provide a basis for optimally representing the fluctuation variance there is no reason to expect the members of this basis to resemble structures appearing in the flow. Ancillary information is required to connect the POD modes to structure. A trivial example of ancillary information would be a rank one covariance in which a single POD mode identifies the only structure that appears in the flow. This example is perhaps not as trivial as it seems because often a single POD mode in a flow does dominate the variance, as revealed by its eigenvalue being substantially larger than the others, in which case one can expect this dominant mode to be prominently seen when observing the flow. Unfortunately, a second structure can not in general be identified with the POD mode having the next largest eigenvalue. This is because, except in the special case of the Langevin system mentioned in the next paragraph, the structure of the second most prominent contributor to variance appearing in the flow will not in general be orthogonal to the first and so the structure educed from it by POD mode analysis will be largely determined by the requirement that it be projected onto the subspace orthogonal to the first, and so on for all other POD modes. More generally what is needed as ancillary information to obtain structure identification is the relative displacement among the POD modes so that their superposition is accounted for in forming a coherent structure. From this viewpoint the POD modes are regarded as a compact basis but the structure of the individual POD mode is not regarded as providing complete structure information which requires obtaining the relative phases of the POD modes. For a flow that is homogeneous in a given direction, the POD modes comprise the Fourier basis and displacement information is encoded in the relative phases of the harmonics. If one makes the random phase assumption then the fluctuation structure becomes minimally localized in that coordinate with the form of a spatially red noise process, while if one makes the zero phase assumption a maximally compact structure is obtained in which the harmonics add coherently at the origin.

A case for which explicit interpretation of the POD modes can be made is that of a normal linear dynamical system of Langevin form forced white in space and time. In this case the POD modes identify the eigenmodes of the system and the real parts of their eigenvalues (North 1984) and so provide sufficient information to identify the system. This example has led to the inference that the POD modes can be used to infer information about both structure and dynamics in turbulent flows. This inference, even when few POD modes explain all the variance, is misleading except under the highly restricting assumptions mentioned. Not only are the individual POD modes not necessarily structures that straightforwardly correspond to prominent physical entities that appear in the flow,
but neither do they provide sufficient information to characterize the flow dynamics. In particular, as the dynamics in wall-bounded flows is highly non-normal, the highly growing structures that give rise to the POD modes are very different from these energy bearing structures themselves and optimally representing dynamics requires including the growing structures in addition to the POD modes in the basis supporting the dynamics (Farrell & Ioannou 1993b, 2001; Rowley 2005).

POD analysis was originally advanced as a method for identifying coherent structures in wall turbulence and investigating their dynamics (Lumley 1967; Berkooz et al. 1993). It was presumed that as the coherent structures represent a large fraction of the variance, the POD modes would therefore correspond to these structures. However, this project of associating the POD modes with coherent structures faced the difficulties mentioned above and in addition issues specific to the wall-turbulence problem. The model problems addressed in studies of wall-turbulence are assumed to be homogeneous in the streamwise and spanwise direction. Over time, the structures in these turbulent flows explore all spanwise and streamwise locations equally, resulting in a time mean flow and covariance that are asymptotically homogeneous in the spanwise and streamwise directions. The mean flow then depends only on the cross-stream direction and the covariance depends only on the relative separation of points in the spanwise and streamwise directions. This implies that the POD modes are Fourier modes in the spanwise and streamwise coordinates and eigenanalysis of the covariance can only identify the amplitude of these harmonic POD modes together with their associated cross-stream structure, the cross-stream being the only inhomogeneous direction, but leaves their phase and associated structure in the spanwise and streamwise directions undetermined. This absence of information about the phase of the POD modes in the spanwise and streamwise renders the POD modes incapable of identifying coherent structures which, ironically, was the original motivation for studying them. This was recognized by Lumley (1981) who proposed to obtain relative phase information in the homogeneous coordinates from higher order statistics and in this way to complete the identification of the coherent structures using POD analysis. In the pursuit of this goal, of particular interest are the results and methods of Moin & Moser (1989) who used statistical methods for estimating the POD modes in a turbulent channel flow by which they identified a dominant coherent structure consisting of a compact streamwise elongated low-speed streak flanked by a pair of compact rolls, which they associated with the coherent structure that arises in the bursting process; Jiménez (2018) contains a recent review of these methods. The results of these attempts to obtain the phases of the POD modes by statistical means is to elicit structure similar to that predicted by the minimum entropy assumption of choosing the phase between the modes to produce a maximally compact structure. A related problem of identifying traveling coherent structures using POD analysis was addressed using slicing and centering methods (Rowley & Marsden 2000; Froehlich & Cvitanović 2012; Willis et al. 2013). An analogous procedure is employed in this paper to isolate the low-speed streak and its associated fluctuation field.

We have reviewed the conceptual basis for using POD-based analyses to study structure and dynamics in wall-turbulence as well as the limitations of POD-based analyses. Our objective in this work is to exploit POD-based analyses to study the dynamical mechanism supporting turbulence. We do this by appropriately modifying POD analysis methods to compare structure and dynamics between the restricted nonlinear (RNL) quasi-linear system and the associated DNS. The modifications to POD analysis used are designed to overcome limitations of POD-based analysis in e.g. identifying coherent structure in coordinates for which the problem is homogeneous. The motivation for doing such a comparison is that RNL is obtained directly from the Navier-Stokes equations with only
the omission of the nonlinear interaction of the streamwise varying components of the flow. Important for our study is that RNL sustains a realistic turbulence with highly simplified dynamics subject to analytic characterization, as will be described (Farrell et al. 2016). It follows that if a convincing case can be made for the essential similarity in the dynamics underlying turbulence in RNL and DNS, then the simplicity of the dynamics of RNL turbulence can be exploited to provide insight into the mechanism of wall-turbulence.

We proceed by briefly reviewing the formulation of RNL as a quasi-linear approximation of the Navier-Stokes (N-S) equations, the simplifications that result from this approximation and the implications this approximation provides for understanding the mechanism of wall-turbulence (Farrell & Ioannou 2012; Farrell et al. 2017a). To obtain the RNL approximation, the N-S equations are first decomposed into equations governing the streamwise-mean flow and the fluctuations from the streamwise-mean. At this point no approximations have been made to the Navier-Stokes equations. The RNL approximation consists of neglecting in the fluctuation equations the fluctuation-fluctuation interactions. It follows that RNL dynamics comprises the quasi-linear interaction between the time-dependent streamwise-mean flow and the fluctuations from the streamwise-mean. At this point no approximations have been made to the Navier-Stokes equations. The RNL approximation consists of neglecting in the fluctuation equations the fluctuation-fluctuation interactions. It follows that RNL dynamics comprises the quasi-linear interaction between the time-dependent streamwise-mean flow and the fluctuation field. This time-dependent interaction with the mean flow provides periods of fluctuation growth and decay. Given that the fluctuation field is bounded, its time-mean growth must be exactly zero or equivalently the top Lyapunov exponent of fluctuations growing on the time-varying streamwise-mean flow must be exactly the real number zero while the time-varying mean flow is regulated to neutral Lyapunov stability by the Reynolds stresses of the fluctuations. This implies that the fluctuation field of RNL turbulence lies in the subspace of the Lyapunov vectors of the time-varying streamwise mean flow that have zero Lyapunov exponent and the mean flow is regulated by feedback from the Reynolds stresses of these Lyapunov vectors to neutral Lyapunov stability. This simplification of the turbulence to a subset of fluctuations supported by as few as a single streamwise-varying harmonic occurs spontaneously in the RNL system. The fact that RNL is supported on the small set of Lyapunov vectors with precisely zero Lyapunov exponent and that the time mean state is feedback regulated to exact Lyapunov neutrality provides analytic characterization of the fluctuations and the regulation of the statistical mean state of RNL turbulence.

It is interesting to note that this quasilinear adjustment to neutral stability constitutes a solution for the statistical state of the turbulence to second order that vindicates the program of Malkus (1956) to obtain a quasi-linear equilibration identifying the statistical mean state of turbulence in shear flow - it only being required to recognize that it is not the inflectional or the viscous instability of the time-mean turbulent profile that is neutralized, as proposed by Malkus (1956) and discussed by Reynolds & Tiederman (1967), but rather the parametric instability of the time-dependent streamwise mean flow, which includes also the time varying roll-streak structure (R-S).

An intriguing consequence of RNL turbulence is that given the absence of fluctuation-fluctuation interactions in the fluctuation dynamics of the RNL system is that RNL turbulence is not sustained through non-normal amplification of fluctuations arising from fluctuation-fluctuation nonlinearity as proposed by Trefethen et al. (1993); Farrell & Ioannou (1993b); Gebhardt & Grossmann (1994). This nonlinear scattering into the growing subspace mechanism for sustaining turbulence is implicit in regeneration mechanisms in which optimal perturbations recycled from the turbulent debris, typically ascribed to a streak breakdown process, through their growth sustain the turbulence (Jiménez & Moin 1991; Jiménez 2018). Further example of the regeneration mechanism sustaining turbulence include the baroclinic turbulence of the midlatitude atmosphere (DelSole 2007; Farrell & Ioannou 2009). It was recently shown by Lozano-Durán et al. (2021) that non-normal amplification of fluctuations regenerated through fluctuation-
fluctuation interactions in an externally maintained stable time-independent mean flow can sustain a turbulent fluctuation field. We conclude that the nonlinear scattering regeneration mechanism is available to support turbulence. However, RNL turbulence makes a radical departure from regeneration theories by sustaining the turbulent field without any input from fluctuation-fluctuation nonlinearity.

An important additional insight inherent in the RNL formulation is isolation of the primary coherent structure, which is the R-S, in the mean equation. This partition of structure in the turbulence into mean and fluctuation with the R-S spontaneously forming in the mean equation poses a fundamental conceptual challenge to mechanistic theories of wall-turbulence. The necessary inference is that the R-S in RNL is maintained by fluctuation Reynolds stresses and not by a fluctuation-fluctuation scattering regeneration mechanism. A primary goal of this work is to use POD-based diagnostics to construct a compelling argument from data diagnostics that DNS and RNL turbulence both maintain the R-S by this same mechanism of Reynolds stress torque in contrast to the implications of the regeneration by fluctuation-fluctuation scattering mechanism. The mechanism sustaining turbulence in RNL is directly derived from the Navier-Stokes equations and is consistent mechanistically with the SSP mechanism advanced in Hamilton et al. (1995) and illustrated by the model based studies of Waleffe (1997). The streak in RNL is supported by roll induced lift-up with the roll in turn being maintained by continuously exerted torques from the Reynolds stresses of the fluctuation field. This mechanism for sustaining the R-S, is predicted to arise spontaneously by RNL dynamics and is fundamentally different from the mechanism proposed by e.g. Jiménez (2013a, b, 2022) in which streaks arise as scars left in the streamwise velocity from the linear growth of episodically excited optimal perturbations (Encinar & Jiménez 2020).

The goal of this work is to validate the essential dynamical similarity of DNS and RNL using POD-based analyses. We wish to emphasize that it is not our intent to advance the RNL as an LES to replace the DNS. To the contrary we have maximized the simplicity of these systems in order to isolate the essential underlying dynamics governing wall-bounded turbulence without the intent of obtaining identity between them. We first compare the POD modes of the streamwise mean flow predicted under the assumption of spanwise homogeneity in DNS and RNL. We then obtain a coherent R-S by collocating the observed streaks in both systems and verify that the POD amplitudes are consistent with the Fourier amplitudes of the R-S obtained by collocation, verifying that the collocation process identifies the phases of the POD modes. Having obtained the coherent R-S we compare their structures in DNS and RNL, which are found to be remarkably similar. This result verifies the spontaneous symmetry breaking in the spanwise direction by the emergence of the R-S instability as predicted by the S3T SSD (Farrell & Ioannou 2012; Farrell et al. 2017b). Having obtained the R-S structure we then perform POD analysis of the fluctuations about the mean R-S in both DNS and RNL and verify that the fluctuation fields are consistent with the prediction of oblique waves as required to maintain a coherent streamwise roll in the SSP (Farrell et al. 2022). The fluctuation POD modes are also found to be consistent with predictions for optimally growing structures over typical temporal correlation times by comparing them with the average structure of stochastically excited evolving fluctuations over 30 advective time units (which correspond to 4 shear time units in the wall region). Given that the SSP is analytically characterized in RNL this identification of structures chosen to reveal the dynamics maintaining the R-S in RNL and DNS constitutes a compelling argument that the turbulence in this wall-bounded shear flow is mediated by the SSP analytically obtained from RNL dynamics.
2. DNS and its RNL approximation

We study a pressure driven constant mass-flux plane Poiseuille flow in a channel which is doubly periodic in the streamwise, \( x \), and spanwise, \( z \), direction. The incompressible non-dimensional Navier-Stokes equations governing the channel flow are decomposed into equations for the streamwise mean flow, \( \mathbf{U} = (U, V, W) \), and the fluctuations, \( \mathbf{u} = (u, v, w) \), as follows:

\[
\begin{align*}
\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} - G(t)\hat{x} + \nabla P - R^{-1} \Delta \mathbf{U} &= -\overline{\mathbf{u} \cdot \nabla \mathbf{u}}, \\
\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \nabla p - R^{-1} \Delta \mathbf{u} &= - (\mathbf{u} \cdot \nabla \mathbf{u} - \overline{\mathbf{u} \cdot \nabla \mathbf{u}}). \\
\nabla \cdot \mathbf{U} &= 0, \quad \nabla \cdot \mathbf{u} = 0.
\end{align*}
\]  

(2.1a)\( \quad \) (2.1b)\( \quad \) (2.1c)

No-slip impermeable boundaries are placed at \( y = 0 \) and \( y = 2 \), in the wall-normal variable. The pressure gradient \( G(t)\hat{x} \) is adjusted in time to maintain constant mass flux, \( \hat{x} \) is the unit vector in the streamwise direction. An overline, e.g. \( \overline{\mathbf{u} \cdot \nabla \mathbf{u}} \), denotes averaging in \( x \). Capital letters indicate streamwise averaged quantities. Lengths have been made non-dimensional by \( h \), the channel’s half-width, velocities by \( \langle \mathbf{U} \rangle_c \), the center velocity of the time-mean flow, and time by \( h/\langle \mathbf{U} \rangle_c \). The Reynolds number is \( R = \langle \mathbf{U} \rangle_c h/\nu \), with \( \nu \) the kinematic viscosity.

The corresponding RNL equations are obtained by suppressing nonlinear interactions among streamwise-varying flow components in the fluctuation equations resulting in the right hand side of (2.1b) being neglected. The RNL equations are:

\[
\begin{align*}
\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} - G(t)\hat{x} + \nabla P - R^{-1} \Delta \mathbf{U} &= -\overline{\mathbf{u} \cdot \nabla \mathbf{u}}, \\
\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \nabla p - R^{-1} \Delta \mathbf{u} &= 0. \\
\nabla \cdot \mathbf{U} &= 0, \quad \nabla \cdot \mathbf{u} = 0.
\end{align*}
\]

(2.2a)\( \quad \) (2.2b)\( \quad \) (2.2c)

Under this quasi-linear restriction, the fluctuation field interacts non-linearly only with the mean, \( \mathbf{U} \), flow and not with itself. This quasi-linear restriction of the dynamics results in the spontaneous collapse of the support of the fluctuation dynamics to a small subset of streamwise Fourier components. It is important to recognize that this restriction in the support of RNL turbulence to a small subset of streamwise Fourier components is not imposed but rather is a property of the dynamics with significant implication. The components that are retained spontaneously by the RNL dynamics identify the streamwise harmonics that are dynamically active in the sense that this subset of streamwise harmonics participate in the parametric instability that sustains the fluctuation component of the turbulent state (Farrell & Ioannou 2012; Thomas et al. 2014, 2015; Farrell & Ioannou 2017; Nikolaidis & Ioannou 2022).

The data were obtained from a DNS of Eq. (2.1) and from simulation of the associated RNL governed by Eq. (2.2). The Reynolds number \( R = \langle \mathbf{U} \rangle_c h/\nu = 1650 \) is imposed in both the DNS and the RNL simulations. A summary of the parameters of the simulations is given in Table 1. The time averaged streamwise-mean flow \( \langle \mathbf{U} \rangle = \langle \mathbf{U} \rangle \hat{x} \) and its associated shear in the DNS and RNL simulation are shown in Fig. 1 and the time-averaged rms profiles of the fluctuations from the mean flow \( \langle \mathbf{U} \rangle \), \( \mathbf{u}' = \mathbf{u} - \langle \mathbf{U} \rangle \hat{x} \), are shown in Fig. 2. The RNL simulation reported here is supported by only three streamwise components with wavelengths \( \lambda_x/h = 4\pi, 2\pi, 4\pi/3 \), which correspond to the three largest streamwise Fourier components of the channel, \( n_x = 1, 2, 3 \). These streamwise Fourier components sustained in RNL are not imposed but rather the RNL spontaneously selects the streamwise Fourier components that are retained in the turbulent state. We have included 16 streamwise wavenumbers in the integration of the RNL in order to allow freedom for the RNL system to select the streamwise wavenumbers that it sustains.
Table 1: Simulation parameters. \([L_x, L_y, L_z]/h\), where \(h\) is the channel half-width, is the domain size in the streamwise, wall-normal and spanwise direction. Similarly, \([L_x^+, L_y^+, L_z^+]\), indicates the domain size in wall-units. \(N_x, N_z\) are the number of Fourier components after dealiasing and \(N_y\) is the number of Chebyshev components. \(R_\tau = u_\tau h/\nu\) is the Reynolds number of the simulation based on the friction velocity \(u_\tau = \sqrt{\nu \frac{d\langle U\rangle}{dy}|_w}\), where \(d\langle U\rangle/dy|_w\) is the shear at the wall.

Figure 1: Left panel: The mean velocity profile of the DNS (red) and RNL simulations (blue) normalized to the average centerline velocity \(\langle U\rangle_c\). Right panel: The corresponding normalized mean shear in the two simulations.

For the numerical integration the dynamics were expressed in the form of evolution equations for the wall-normal vorticity and the Laplacian of the wall-normal velocity, with spatial discretization and Fourier dealiasing in the two wall-parallel directions and Chebychev polynomials in the wall-normal direction (Kim et al. 1987). Time stepping was implemented using the third-order semi-implicit Runge-Kutta method.

3. Analysis method used in obtaining the POD modes

POD analysis requires the two-point same-time spatial covariance of the flow variables. The perspective on turbulence dynamics adopted in this work is that of the S3T statistical state dynamics (SSD) closed at second order (Farrell & Ioannou 2012) and its RNL approximation (Thomas et al. 2014). The insights on turbulence dynamics obtained by using this SSD proceed from its formulation which is based on using the streamwise-mean and associated fluctuations to express the dynamics. The choice of the streamwise-mean
in the cumulant expansion of this SSD serves to isolate the dynamics of the dominant coherent structure supporting turbulence, which is the R-S. If the R-S were not supported by the Reynolds stress torque mechanism it would not appear in the mean equation. In order to further isolate the R-S structure the $k_x = 0$ POD analysis is confined to deviations of the streamwise mean flow from its spanwise mean. Adopting the notation $\langle \cdot \rangle$ for the time average, $[\cdot]$ for the spanwise average, and $(\cdot)^T$ for transposition, the covariance of deviations of the streamwise-mean velocity field from its spanwise-mean is:

$$C = \langle \mathbf{U} \mathbf{U}^T \rangle,$$

in which

$$\mathbf{U} = [U_s, V_s, W_s]^T,$$

is the column vector comprising deviations of the three streamwise mean components from their spanwise-mean, $([U](y, t), [V](y, t), [W](y, t))$, i.e. $U_s = U - [U]$, $V_s = V - [V]$, and $W_s = W - [W]$. A requirement for $C$ to be a covariance is that $\langle U_s \rangle = 0$, $\langle V_s \rangle = 0$ and $\langle W_s \rangle = 0$, which demands that $\langle [U] \rangle = \langle U \rangle$, $\langle [V] \rangle = \langle V \rangle$ and $\langle [W] \rangle = \langle W \rangle$. This condition places a requirement of homogeneity on the velocity components in $z$, which was verified.

The dominant structures in the fluctuation field are localized about the streamwise streak. In order to isolate these structures the fluctuations are obtained by first collocating the dominant streak together with its associated fluctuation field in the flow prior to

Figure 2: Wall-normal profiles of the rms of velocity fluctuations (a,b,c) and the tangential Reynolds stress (d) for the DNS (red) and RNL (blue) simulations.
extracting the fluctuations from the dominant streak. These fluctuations are used to form
the covariance on which the POD analysis of fluctuations from the streamwise mean R-S
structure is done, as described in section 5. The covariance of the fluctuation flow field is
expressed as:
\[ c = \langle U'U'^T \rangle, \]  
(3.3)
with
\[ U' = [u, v, w]^T, \]  
(3.4)
which are the column vector of the three velocity components of the streamwise varying
flow, i.e. the components of the velocity deviations from the average streak structure in
the flow.

The POD modes for the mean flow fluctuations and for the fluctuations from the
dominant streak are obtained by eigenanalysis of the two-point covariances, $C$ and $c$. The
resulting orthonormal set of eigenvectors is then ordered descending in eigenvalue. The
eigenvalue of each POD mode is its time-averaged contribution to the variance of the
velocity.

To obtain a sufficiently converged covariance to identify the primary POD modes for the
streamwise mean flow requires a long time series of the turbulent flow field. Convergence is
facilitated by taking into account the statistical symmetries of the flow fields: homogeneity
in the $x$ and $z$ direction, mirror symmetry in $y$ about the $x-z$ plane at the channel
center, and mirror symmetry in $z$ about the $y-x$ plane at the channel center. We further
average the POD modes from the upper and lower half channel about the $x-z$ plane at
$y=0$ under the assumption that the POD modes in the upper and lower half-planes are
independently occurring structures. Details of the implementation of the symmetries for
the calculation of the POD modes are given in Appendix A.

Statistics of flow quantities have been verified to approach asymptotically these
symmetries, as the averaging time increases. These statistical symmetries are not necessary
consequences of the translation and mirror symmetry of the NS equations in a periodic
channel because the turbulent flow field may undergo symmetry breaking. For example
stability analysis of the S3T SSD of wall-bounded flows in periodic domains predicts
symmetry breaking of spanwise homogeneity before the turbulent state is established,
and an imperfect manifestation of this symmetry breaking is clearly seen in the related
DNS (Farrell & Ioannou 2012; Farrell et al. 2017b). Casting the Navier-Stokes equations
in SSD form permits identification of the instability underlying this symmetry breaking,
an instability without counterpart in the Navier-Stokes equations written in traditional
velocity-pressure component state variables (Farrell & Ioannou 2019).

If an underlying symmetry breaking instability exists in a turbulent system but
stochastic fluctuations cause the modes of this instability to random walk in a homogeneous
coordinate so that in the limit of long time the phase information localizing the mode
is lost, rendering the phase random, then one approach to identifying the symmetry
breaking mode is to obtain an approximation to the covariance over short enough times
that the phase randomization is not complete while another is to collocate the symmetry
breaking structures in the flow so the effect of the random walk is removed - we employ
the latter method in this work, which is equivalent to the centering or slicing method for
unveiling coherent structure in data in dynamics with continuous symmetries (Rowley &
Marsden 2000; Froehlich & Cvitanović 2012).
4. POD modes of the DNS and RNL streamwise mean flow

The POD basis for the $k_x = 0$ component of the deviations from the time and spanwise mean velocity in the DNS and the RNL simulation will first be described under the assumption of spanwise statistical homogeneity. Accepting the assumption of statistical homogeneity in $z$ implies that the eigenvectors of $C$, which are the POD modes, are single Fourier harmonics in the spanwise direction (Berkooz et al. 1993). Under this assumption the POD mode, $n_z$, with spanwise wavenumber $k_z = 2\pi n_z/L_z$ is given by:

$$\Phi_{k_z} = \begin{pmatrix} A_{k_z}(y) \\ B_{k_z}(y) \\ I_{k_z}(y) \end{pmatrix} e^{ik_z z}, \tag{4.1}$$

where $A_{k_z}(y)$ is the streamwise component of the velocity field associated with the POD, $B_{k_z}(y)$ the wall-normal and $I_{k_z}(y)$ the spanwise component. All these components are specified as $N_y$ dimensional column vectors, with $N_y$ the number of discretization points in $y$. At each sampling time the $3N_y$ column vector of a $k_z \neq 0$ Fourier component, $U_{k_z}(t)$, of the flow field $U$ is obtained and used to form $N_{k_z}$ average covariances:

$$C_{k_z} = \left\langle U_{k_z}(t)U_{k_z}^\dagger(t) \right\rangle, \tag{4.2}$$

where $N_{k_z}$ is the number of $k_z \neq 0$ Fourier components retained in the simulation and $\dagger$ is the Hermitian transpose. Eigenanalysis of these covariances determines $3N_y \times N_{k_z}$ eigenvectors comprising the POD orthonormal basis of the $k_x = 0$ flow field taking into account the restriction to deviations from the spanwise mean mentioned above. These POD modes are ordered decreasing in eigenvalue which corresponds to variance.

As discussed above, because of the statistical homogeneity of the flow in the $z$ direction, the $k_z = 0$ POD modes come in sin$(k_z z)$ and cos$(k_z z)$ pairs. The first three spanwise harmonics of the POD modes of both the DNS and RNL simulation account for 75% of the $k_x = 0$ variance are shown in Fig. 3. Shown are both the streak velocity and vectors of the corresponding roll velocity field for each POD mode. Note that the POD modes obtained from the DNS and the RNL simulation exhibit a similar structure of a streamwise velocity collocated with a supporting roll. The variances explained by the first three POD modes are similar and the structures of the modes are also similar, as shown in Fig. 4, although the variance accounted for by the individual modes is not identical. The result of importance is the structural similarity of the modes which is indicative of the dynamics.

Of dynamical significance is the systematic correlation between the wall-normal velocity $V_s$ of the roll and the corresponding streak velocity in these POD modes: positive $V_s$ is correlated with low speed streaks (defects in the streamwise average flow) and vice versa in all the POD modes. That all the POD modes exhibit this correlation is consistent with the interpretation that the rolls and the streaks form a coherent structure in which the lift-up mechanism arising from the roll is acting to maintain the streak. Consistently, previous work has revealed that the Reynolds stress resulting from streak-induced organization of turbulence in S3T and RNL results in a lift-up process supporting roll-streaks with the same structure as these POD modes (Farrell & Ioannou 2012; Farrell et al. 2016). Note that the first 3 DNS POD modes with $k_x = 0$ have roll-streak structure nearly identical to those in the RNL model as shown in Fig. 3. The wall-normal velocities of the roll component of the POD modes in both RNL and DNS are about 1/10 the streak velocity which, assuming an average non-dimensional mean flow shear of magnitude 2 (cf. Fig. 1) is consistent with the emergence of the associated streak through the lift-up mechanism over 5 time units.
Figure 3: Left panel: The first 3 POD modes of the streamwise-mean flow from a 310000 advective time units DNS. Right panel: The corresponding modes of the streamwise-mean flow from a 83000 advective time units simulation of RNL. The contours show levels of the streamwise $U_s$ velocity and the arrows show the cross stream-spanwise velocity vector $(V_s, W_s)$. The ratio $U_s/V_s$ is in all cases about 10. Notice that in DNS the POD mode with the largest contribution to the variance is the $n_z = 2$ mode, while in RNL it is the $n_z = 1$ mode. The contour level is 0.2 in all panels.

This similarity of the RNL and DNS streamwise-mean POD modes suggests that the same dynamics is operating. In the case of RNL this dynamics is known to be that the streaks organize the fluctuations so that their associated Reynolds stresses produce streamwise torque configured to force rolls collocated with the streak in such a manner as to reinforce the preexisting streak by the lift-up process. This reinforcement mechanism is persuasively manifest in the idealized problem of the instability of a background of spanwise homogeneous turbulence to the formation of streamwise streaks. Statistical state dynamics calculations closed at second order identify this instability, which is the fundamental instability underlying the dynamics of turbulence in shear flow, by showing that the R-S is the streamwise mean component of unstable eigenfunction in the SSD. Moreover, this unstable eigenfunction has the same form as the POD modes we have identified in RNL and DNS. Furthermore, these eigenfunctions have the property of destabilizing the roll-streak structure at all scales, indicating that the mechanism of streak-roll formation is intrinsically scale independent (Farrell & Ioannou 2012; Farrell et al. 2017b).

5. POD modes of the streamwise-varying fluctuations from the dominant streak occurring in flow realizations

A fundamental dynamical property of turbulence in wall-bounded flows is the spontaneous breaking of the spanwise symmetry by the formation of the roll-streak structure. Although there is no instability associated with this symmetry breaking in the traditional formulation of the NS using velocity components for the state, this symmetry is broken...
Figure 4: Left panel: Percentage variance of the streamwise-mean ($k_x = 0$) flow explained by each POD mode doublet in the DNS and RNL simulation. Right panel: The cumulative variance accounted for by the POD modes in the DNS and RNL simulation as a function of the number of POD modes included in the sum. In DNS the first POD mode has spanwise wavenumber $n_z = 2$, the second POD mode has $n_z = 1$. In RNL the first POD mode has spanwise wavenumber $n_z = 1$ and the second POD mode has $n_z = 2$.

by the most unstable mode of the simplest nontrivial SSD which is a cumulant expansion closed at second order using streamwise-mean velocity and fluctuation covariance for the state variables (Farrell & Ioannou 2012). While the underlying roll-streak symmetry breaking instability is analytic in pre-transitional flow analyses made using the S3T SSD (Farrell et al. 2017b), the manifestation of this symmetry breaking instability is made imperfect by time dependence both in the pre-transitional and post-transitional DNS and RNL solutions so that the roll-streak structure, while prominent, is randomly spatially displaced rather than persisting at a fixed spanwise location. Nevertheless, the existence of the underlying symmetry break in the spanwise by the roll-streak S3T instability is clearly manifest in the substantial spatial extent in the steamwise direction and persistence in time of the roll-streak structure in RNL and DNS, indicative of the analytical underlying bifurcation. Informed by the existence of an analytic bifurcation to a time and space independent roll-streak structure in pre-transitional flow, we wish to isolate structures underlying this fundamental mechanism of roll-streak maintenance from the secondary property of random variation of the streak location in the spanwise direction. By this simplification we are able to concentrate on the interaction of the roll-streak with streamwise fluctuations, which is widely recognized to be associated with the maintenance of turbulence, although the dynamics of this interaction remains controversial (cf. Jiménez & Moin (1991); Hamilton et al. (1995); Waleffe (1997); Jiménez & Pinelli (1999); Schoppa & Hussain (2002); Farrell & Ioannou (2012, 2017); Farrell et al. (2017a); Lozano-Durán et al. (2021)). A point of agreement of the above studies is that the streak and fluctuations are collocated to form a dynamical structure. Thus, an accurate statistical description of
Figure 5: (a) Top panel: A snapshot of the streamwise velocity $u$ at $t[U_c]/h = 177768$ from the NL100 simulation at the wall-normal plane $y/h = 0.21$. Bottom panel: The $U_s$ component of the above snapshot. The white dashed line in both figures indicates the spanwise location of the $U_s$ minimum. (b) Same as (a) for a snapshot of the RNL100 at $t[U_c]/h = 76827$. (c) Top panel: A temporal sequence of $U_s$ snapshots for which the streak minima have been aligned at the channel half-width $z/h = \pi/2$. The total flow snapshot is also subjected to the same shift. Bottom panel: The ensemble average $U_s$ converges to a negative central streak region with weak positive regions on its flanks, whereas the remaining flow is almost spanwise homogeneous. (d) Same as (c) for the ensemble average $U_s$ of the RNL100 simulation.
Figure 6: Contours of the time average collocated $U_s$ and vectors of the roll $(W_s, V_s)$ velocity on the $(z, y)$ plane for the NS100 (panel(a)) and RNL100 (panel (b)). The contour level step is 0.025 in both panels. Panel (a): max($|U_s|$) = 0.21, max($V_s$) = 0.024. Panel (b): max($|U_s|$) = 0.32, max($V_s$) = 0.03.

the $k_x \neq 0$ structures will be sought by performing at every instant of time a spanwise translation of the entire flow field data so that the dominant streak together with its associated fluctuations is at the center of the channel, $z/h = \pi/2$.

A reliable indicator of the streak location is the spanwise $z/h$ coordinate of the min($U_s$) associated with the dominant low speed streak structure (Figs. 5a and 5b). We proceed to identify this location in the flow realizations by finding the $z$ coordinate of this minimum velocity at a fixed distance from the wall, $y/h = 0.21$, where the $|\text{min}(U_s)|$ attains it’s largest values, and translate the total flow field so that the $U_s$ minima occur at the same spanwise location at the center of the channel at $z/h = \pi/2$, while retaining the time order. The effect of this operation on the streamwise average $U_s$ velocity is shown in the top panel of Figs. 5c and 5d for NL100 and RNL100 respectively. The modified time-series of the $U_s$ produce an aligned slow speed region at $z/h = \pi/2$ in both cases, while further away from this core region the uncorrelated high and low speed streaks cancel out. The streamwise-mean streak, $U_s$, on the plane $y/h = 0.21$ resulting from this procedure is shown in the bottom panel of Figs. 5c and 5d. The structure in the $y - z$ plane of the roll-streak for NL100 and RNL100 is shown in Figs. 6a,b using contours for $U_s$ and vectors for $(W_s, V_s)$. 
Figure 7: The percentage variance accounted for by the first Fourier spanwise components of the mean streaks in Fig. 6. Dashed red line: for the mean streak of the DNS, dashed blue line: for the mean streak of the RNL. The solid lines with the corresponding colors show the percentage variance of the corresponding POD modes with spanwise wavenumbers $n_z = 1, 2, 3$.

Figure 8: Contours of the $U_s$ and vectors of the roll $(W_s, V_s)$ velocity on the $(z, y)$ plane of the first three spanwise Fourier components of the mean streaks of Fig. 6. Left column of the DNS, right column of the RNL. The contour level step is 0.2 in all panels.

5.1. Relating POD modes to roll-streak structures in the flow

There are alternative explanations for the striking appearance of POD modes for $k_x = 0$ which differ in spanwise wavenumber while exhibiting roll-streak structure (cf. Fig. 3). One interpretation of these structures is that they correspond to stable linear S3T modes that are maintained by fluctuations in the homogeneous background of turbulence. Due to the
scale insensitivity of the roll-streak formation process, a spectral hierarchy of self-similar roll-streak structures are supported as modes by the turbulence as revealed by S3T (Farrell & Ioannou 2012). These roll-streak modes have different scales and damping rates and are therefore expected to be excited at different amplitudes. This interpretation of the POD modes is appropriate in the case of the roll-streaks that emerge in pre-transitional flows as discussed in Farrell et al. (2017b). Also in beta-plane turbulence one observes intermittent emergence of jets with structure corresponding to stochastically excited S3T modes, manifestations of which are referred to in observations as latent jets (Constantinou et al. 2014; Farrell & Ioannou 2019). In this interpretation of the POD modes with various spanwise wavenumber, the POD modes are identifying structures that are regarded in the traditional manner as being independent harmonic structures in $z$, as is appropriate for a homogeneous coordinate in the flow.

However, there is an alternative interpretation, which is that the dominant structure in the flow is the finite amplitude localized low-speed streak of Fig. 6. In this interpretation the POD modes reflect the amplitude of the spanwise Fourier components collocated to comprise the Fourier synthesis of this structure rather than corresponding to structures with independent existence. The assumption underlying this interpretation is that a mode has arisen in the flow that results in a spontaneous symmetry breaking; in the example under analysis here the S3T R-S instability is implicated.

In order to determine which of these alternative explanations for the POD structure at $k_x = 0$ is correct we compare in Fig. 7 the amplitudes of the Fourier components of the low-speed streak with the amplitude of the POD with corresponding spanwise Fourier component and in Fig. 8 we plot the structure of the first three Fourier components obtained from spanwise Fourier analysis of the velocity components of the mean streak. The Fourier modes of the collocated R-S structure for DNS and RNL shown in Fig. 8 are strikingly similar to the POD modes shown in Fig. 3. The mode projections shown in Fig. 7 are ordered the same way and show nearly identical distribution with the RNL and the DNS falling off with Fourier mode number. The agreement is the more remarkable given the amplitudes of the Fourier components in Fig. 8 were derived from analysis of only the mean low-speed streak and its accompanying roll circulation, while the variance accounted for by the POD modes includes the variance associated with the high-speed streaks. The agreement shown in these figures leads us to conclude that the second of these explanations, that the POD spectrum arises primarily from Fourier decomposition of the low-speed R-S, is correct. In summary we conclude that while POD analysis is consistent with identification of independent R-S structures, the alternative interpretation that the POD modes rather identify the individual components comprising the Fourier synthesis of a nonlinearly equilibrated localized coherent structure with complex R-S form requiring many Fourier modes in its representation is clearly preferable.

5.2. Determining the $k_x \neq 0$ POD modes associated with the colocated low-speed streak flows

Having isolated the streamwise mean R-S structure in RNL and DNS and identified the $k_x = 0$ POD modes as consistent with Fourier synthesis of this coherent structure, we turn now to exploiting POD analysis to obtain and interpret dynamically the fluctuations about the mean flow containing the streak structures of Fig. 6. We first Fourier decompose the fluctuation velocity $U' = [u(x,t), v(x,t), w(x,t)]^T$ in $x$ so that $U' = U'_k(y,z)e^{ik_xx}$ The POD modes are obtained from eigenanalysis of the two point spatial covariance:

$$C_{k_x}(y_i, z_i, y_j, z_j) = \langle U'_{k_x}(y_i, z_i)U'_{k_x}^*(y_j, z_j) \rangle,$$  \hspace{1cm} (5.1)
Figure 9: Percentage variance accounted for by the POD modes as a function of the order of the mode; (a) in NL100 and (b) in RNL100. POD modes with streamwise Fourier component \( n_x = 1 \) are in blue; those with streamwise Fourier component \( n_x = 2 \) are in red; and those with streamwise Fourier component \( n_x = 3 \) in green. The sinuous modes are indicated with an S, the varicose with a V. The corresponding streamwise wavenumber is \( k_x = 2\pi n_x / L_x \).

where \( \langle \cdot \rangle \) denotes the time mean, and \( \dagger \) indicates the Hermitian transpose. Each POD mode is of the form \( \begin{bmatrix} \alpha_{k_x}(y,z), \beta_{k_x}(y,z), \gamma_{k_x}(y,z) \end{bmatrix}^T e^{ik_x x} \), with \( \alpha_{k_x}(y,z) \), \( \beta_{k_x}(y,z) \) and \( \gamma_{k_x}(y,z) \) determining the \( (y,z) \) spatial structure of the velocity components of the POD mode. Note that the \( n_x \) component of the velocity field has streamwise wavenumber \( k_x = n_x \alpha \).

The flow fields shown in Fig. 6 reveal spanwise localized R-S structures symmetric about \( z/h = \pi/2 \). In order to isolate the streamwise-varying POD modes associated with the localized low-speed streaks while avoiding contamination by the far-field we weight the data used to calculate the covariances \( C_{k_x} \) with a spatial filter that suppresses the variance in the far-field. We have chosen a Tukey filter in the interval \( z = [0, \pi] \) with equation:

\[
f(z) = \begin{cases} 
1/2 \left[ 1 + \cos \left( \frac{\pi}{\delta} \left( \frac{\pi - 2z}{\pi} + (1 - \delta) \right) \right) \right], & (\pi - 2z)/\pi < \delta - 1, \\
1, & |(\pi - 2z)/\pi| \leq 1 - \delta, \\
1/2 \left[ 1 + \cos \left( \frac{\pi}{\delta} \left( \frac{\pi - 2z}{\pi} - (1 - \delta) \right) \right) \right], & (\pi - 2z)/\pi > 1 - \delta. 
\end{cases}
\] (5.2)

The parameter \( \delta \) dictates the width of the filter and is chosen to sample the fluctuation field in the vicinity of the streak. The values \( \delta = 0.7 \) and \( \delta = 0.55 \) are selected for NL100 and RNL100, respectively, since the RNL100 streak covers a wider area of the spanwise flow.

5.3. Results for the POD modes with \( k_x \neq 0 \) associated with the collocated low-speed streak flows

The energy density accounted for by the first 10 POD modes for each of the first three \( n_x \) wavenumbers is shown in Fig. 9a for NL100 and in Fig. 9b for RNL100. Note that the dominant POD modes in both DNS and RNL are characterized by a similar
intricate complex three dimensional structure that reflects the complexity of the underlying dynamics.

In the $y-z$ cross sections (top panels) the POD modes exhibit streamwise streaks produced by lift-up which are seen to be coincident with supporting roll circulations. Similar roll circulation and associated streak structures were deduced from analysis of DNS data to arise in association with sweep and ejection events by Lozano-Durán et al. (2012) (cf. their Fig. 12d) and Encinar & Jiménez (2020) (cf. their Fig. 12). Note that these streaks and associated rolls are located at the flanks of the central streamwise streak and are harmonic in the streamwise direction and should not be confused with the entirely different roll-streak structure that arise from lift-up induced by the Reynolds stresses of the fluctuations and which in contrast force the central streak with $k_x = 0$.

In the $x-y$ cross sections (middle panels) the POD modes exhibit the tilted structure indicative of linear amplification by the Orr mechanism. This characteristic Orr structure has been associated with sweep and ejection events in wall-turbulence by Encinar & Jiménez (2020) (cf. their Fig. 1) and evolution of these structures was found to accord with the linear evolution of optimal perturbations with Orr form on the cross-stream shear (Jiménez 2013a; Encinar & Jiménez 2020).

In the $x-z$ cross sections (bottom panels) the POD modes exhibit orientation with the streak indicative of an energy extracting sinuous oblique wave collocated with the streak. Sinuous structure of streak fluctuations has been associated with inflectional instability (Waleffe 1997) and with optimally growing perturbations (Schoppa & Hussain 2002).

The dominance of the top sinuous structure variance, shown in Fig 9a, indicates that it is preferentially expressed relative to the other components of the fluctuation field in both DNS and RNL. The second in variance POD mode is usually varicose and it also exhibits a common universal structure (not shown). The sinuous and varicose ordering of the POD modes is indicated in Figures 9a and 9b.

Qualitative agreement in structure between the top POD modes of the DNS and RNL is apparent. Despite differences such as the greater extent in $y$ of the streak structure of the RNL POD modes (cf. top panels of Figures 10, 11 and 12) the compelling similarity in structural features and the phasing between fluctuation fields revealed by the DNS and RNL POD modes argues for the operation of a parallel dynamics in these two systems. The exact statistical steady state, including the spatial extent of the similar structures, is determined by the non-linear feedback control between the mean and fluctuation fields (Farrell & Ioannou 2012). In particular the feedback control produces an RNL equilibrium with a 50% greater spatial extent in the fluctuation streak shown in the upper panels which is consistent with a similarly greater $V$ velocity in the RNL (cf. Fig. 6). While similarity of the R-S structure between RNL and DNS is manifest, there is no simple argument for the exact amplitude of the associated mean velocity components that such a non-linear regulator settles on.

5.4. Relation of the POD modes of the fluctuations in DNS to the POD modes of the fluctuations in a linear stochastic turbulence model

As remarked earlier, the striking structural similarity in the POD modes of fluctuations on the streamwise-mean streak in DNS and RNL suggests a common dynamical origin for these modes. Given that the streak is modally stable the default explanation would be excitation of transiently growing fluctuations to the streamwise streak by the turbulent background velocity field. Transient growth of optimal perturbations on a cross-stream shear has recently been shown to reasonably accord with the short time evolution of the structures that arise during sweep-ejection events (Encinar & Jiménez 2020). These structures when they are fully evolved assume the characteristic tilt of the POD modes
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Figure 10: The first sinuous POD mode with streamwise Fourier component $n_x = 1$ in NL100 (panels on the left) and in RNL100 (panels on the right). Top panels: contours of the $u$ velocity of the POD mode in the $z - y$ plane at $x = 0$ and vectors of $(w,v)$ velocity on this plane. Middle panels: contours of the $w$ velocity in the $x - y$ plane at the center of streak where the $u$ and $v$ velocities vanish. Bottom panels: contours of the $v$ velocity in the $x - z$ plane at the center of streak and vectors of $(u,w)$ velocity on this plane. The mean flow structure is indicated by the solid back line. The black contours in the top panels show the streak contours in the interval $[-0.35, -0.1]$ at contour intervals of 0.05. All other quantities have been normalized to 1 and the contour levels is 0.2. The first sinuous DNS POD mode accounts for 9.8% of the total variance of the streamwise varying velocity fluctuations of the flow (which includes all $k_x \neq 0$), while the first sinuous RNL POD mode accounts for 21.6% of the total fluctuation variance (cf. Fig. 9).

shown in the middle panels of Figures 10, 11 and 12. Within the context of POD analysis the appropriate extension of the optimal transient growth analysis of Encinar & Jiménez (2020) would be calculation of the POD modes arising from the ensemble mean fluctuation covariance excited by the entirety of the background turbulence.

This is ordinarily implemented by calculating the covariance using the stochastic
Figure 11: As in Fig. 10 for the first sinuous POD mode with streamwise Fourier component $n_x = 2$. A single streamwise wavelength of the POD mode has been plotted. The first sinuous DNS POD mode (which is the first in variance POD) accounts for 5.7% of the total variance of the streamwise varying velocity fluctuations of the flow, while the first sinuous RNL POD mode (which is also the first in variance POD) accounts for 7.6% of the total fluctuation variance.

The equations are the fluctuation equations of the DNS (2.1b) in which the fluctuation-fluctuation nonlinearity in this decomposition has been replaced by a state independent forcing, $f$, of Langevin form, white is space and time. This simplest parameterization imposes the least influence on the dynamics and is adequate for our purpose. The time dependent streamwise-mean flow $\mathbf{U}$ of the DNS has been replaced by the time-independent turbulence model (STM) governed by the equations:

$$
\partial_t \mathbf{u} + \mathbf{U} \partial_x \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{U} \hat{x} + \nabla p - R^{-1} \Delta \mathbf{u} = f, \quad \nabla \cdot \mathbf{u} = 0,
$$

with no slip boundary conditions at the channel walls and periodic boundary conditions in $x$ and $z$, where $U(y, z)\hat{x}$ is the equilibrium low-speed streak of the DNS (or the RNL).
Figure 12: As in Fig. 10 for the first sinuous POD mode with streamwise Fourier component $n_x = 3$. A single streamwise wavelength of the POD mode has been plotted. The first sinuous DNS POD mode (which is the first in variance POD) accounts for 5.7% of the total variance of the streamwise varying velocity fluctuations of the flow, while the first sinuous RNL POD mode (which is also the first in variance POD) accounts for 1.2% of the total fluctuation variance.

flow $U(y,z)\bar{x}$ obtained by collocating and averaging the low-speed streak. This mean streak is shown in Fig. 6.

The mean flows shown in Fig. 6 are stable. This assures the asymptotic approach of the covariance of the fluctuations governed by (5.3) to a statistical equilibrium covariance, $C_\infty$, satisfying in matrix form the Lyapunov equation:

$$AC_\infty + C_\infty A^\dagger = -I,$$

(5.4)

where $A$ is the operator, in matrix form, governing the linear dynamics associated with Eq. (5.3), $A^\dagger$ is the hermitian transpose of $A$, and $I$ is the spatial covariance $\langle f f^\dagger \rangle$. 

Figure 13: Comparison of the first sinuous POD mode in NL100 with streamwise Fourier component $n_x = 1$ (panels on the left) with the first sinuous and first POD mode of the STM with $T_d = 30$ on the DNS mean low-speed streak shown in Fig. 6a (panels on the right). The velocity fields are as in Fig. 10. The POD with the largest variance is the sinuous mode in both DNS and STM.

appropriate for the white in space stochastic excitation, $f$, with the implication that equal energy input is imparted to each degree of freedom (cf. Farrell & Ioannou (1993b, 1996)).

However, in turbulence the mean flow is time-dependent and the time invariant formulation of the STM producing the infinite horizon fluctuation covariance $C_\infty$ is not an appropriate model for the covariance arising in a time-dependent mean-flow. In DNS and RNL the coherence time for fluctuation growth is limited by the temporal coherence of the mean flow. Typical temporal coherence of the mean flow is of the order $T_d = O(1 \ h/\tau_v)$ (Lozano-Durán et al. 2021), which corresponds in our simulation to $T_d = O(20 \ h/U)$, and it is therefore appropriate to restrict the fluctuation development to extend over a time interval, $T_d$, commensurate with this coherence time. In relating the POD mode structure to the STM this coherence time is appropriate for both DNS and the RNL. However, while in the DNS the excitation term $f$ has traditionally been related to the fluctuation-fluctuation nonlinearity, in the case of the RNL a similar excitation
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Figure 14: As in Fig. 13 for \(n_x = 2\) fluctuations. A single streamwise wavelength of the POD mode has been plotted.

arises from the effective nonlinear scattering of the fluctuations by their interaction with the time-dependent mean flow. This excitation can be similarly parameterized by \(f\). It is interesting to note at this point that the fact that RNL maintains a full field of fluctuations at each retained streamwise wavenumber in the absence of fluctuation-fluctuation nonlinearity calls into question the role of fluctuations-fluctuation nonlinearity in DNS as well.

It can be shown (cf. (Farrell & Ioannou 1998)) that the covariance of the dynamics governed by Eq. (5.3) when restricted in time to \(T_d\) is

\[
C_{T_d} = e^{A T_d} e^{A^\dagger T_d} + \int_0^{T_d} ds \, e^{A s} e^{A^\dagger s} C_{\infty} - e^{A T_d} C_{\infty} e^{A^\dagger T_d}.
\]

An alternative to limiting the temporal extent over which fluctuations develop is the inclusion in Eq. (5.3) of an appropriate eddy viscosity (del Álamo & Jiménez 2006). Either
Figure 15: As in Fig. 13 for $n_x = 3$ fluctuations. A single streamwise wavelength of the POD mode has been plotted.

intervention in the linear dynamics of Eq. (5.3) has been shown to result in dynamical structures and spectra commensurate with those observed (Butler & Farrell 1993; Farrell & Ioannou 1998; del Álamo & Jiménez 2006; Hwang & Cossu 2010; Madhusudanan et al. 2019).

Eigenanalysis of the covariance $C_{T_d}$ determines the POD modes of the STM evolving from an initial state having covariance, $I$, and also with excitation covariance, $I$. We have obtained best agreement with the observed POD modes in both DNS and RNL when we choose a disruption time of $T_d = 30$ in the STM. The covariance obtained from Eq. (5.5) reflects both the influence of the transient growth of an unbiased initial state as well as the accumulated transient growth of an unbiased excitation over the coherence interval $T_d$. We find that in this problem for $T_d = 30$ the initial condition does not appreciably influence the covariance.

Because the streamwise mean streak is mirror symmetric in the spanwise direction the POD modes of the STM will be either sinuous or varicose. We find that in both DNS and RNL, as well as in the STM, the top POD modes are sinuous. This result is consistent with the optimally growing perturbation to a low-speed streak being of sinuous form and moreover extending this consistency is the fact that the Reynolds stresses of the sinuous
fluctuations are favorably configured to amplify the low-speed streak through the lift-up process (Farrell et al. 2022).

The top POD modes with streamwise wavenumber $n_x = 1, 2, 3$ of the STM are shown in Figures 13, 14 and 15 next to the corresponding POD modes of the DNS. The STM POD modes obtained from the RNL mean flow are similar to those obtained from the DNS mean flow and are not shown. This is expected because the mean low-speed streaks in RNL and DNS have similar structure and differ primarily only in amplitude, as seen in Figs. 6 and 7.

The dominant POD modes of the STM (cf. Figures 13, 14 and 15) exhibit complex three dimensional velocity fields with striking similarity in structure and phasing to those of DNS and RNL (cf. Figures 10, 11 and 12) indicative of a parallel mechanism underlying their dynamics. This coincidence in the dynamical structure of the POD modes among STM, DNS and RNL identifies the dynamical origin of the fluctuation variance in DNS and RNL: it is the growth of linearly evolving optimal perturbations arising in the mean flow streak. This identification explains the universality in structure of the POD modes in DNS and RNL as being reflective of the universality of optimal perturbations arising in similar background flows (Farrell & Ioannou 1993a).

6. Discussion and Conclusion

POD analysis was carried out on a DNS of turbulent Poiseuille flow at $R = 1650$ and the corresponding quasi-linear RNL simulation. The RNL system was chosen for this comparison because it is dynamically similar to S3T so that the S3T→RNL→DNS sequence of dynamical systems form a conceptual bridge connecting the analytically comprehensive characterization of turbulence in S3T to DNS turbulence, which lacks a similarly complete analytic characterization. The motivation for this work is to exploit this conceptual bridge to extend the comprehensive understanding of S3T dynamics to obtain a similar comprehensive understanding of the dynamics of DNS.

The POD modes analyzed were chosen to correspond to the two variables of the S3T SSD which are the first and second cumulants, these being the streamwise-mean flow and the covariance of fluctuations from it. In general this SSD is closed by parameterizing the third cumulant using stochastic excitation. In the present case this stochastic excitation has been set to zero. In RNL the equivalent to the covariance in S3T is the covariance of the Lyapunov vectors with zero Lyapunov exponent of the time-varying linear operator linearized about the fluctuating streamwise-mean flow. The Lyapunov vectors that are spontaneously emergent by the dynamics are analogous to neutral eigenmodes supporting a time independent mean state. The structure of the first cumulant is given by the known equilibrium structure forced by the boundaries subject to the deviations arising from the Reynolds stresses of the analytically known structures of the second cumulant. And finally, the statistical state of the turbulence is regulated by feedback from the second cumulant to bring the time-varying streamwise-mean flow to neutral stability, in the sense that the characteristic Lyapunov exponent of the linear operator governing the second cumulant is exactly zero. As an illustrative example of the power and utility of being in possession of an analytic theory for the dynamics of wall-turbulence consider the problem of understanding the mechanism determining the statistical mean state of the turbulence. Of all the possible mechanisms that one might hypothesize, this mechanism is identified analytically in S3T-RNL to be modification of the time-dependent streamwise mean state by Reynolds stress feedback from the fluctuations specifically to bring the characteristic Lyapunov exponent of the linear fluctuation equation to the real number zero. Extensive study of DNS has verified that the characteristic exponent of the DNS streamwise mean
state corresponds to this parametric growth stabilization mechanism (Nikolaidis et al. 2018). It is worth noting that this mechanism of regulating turbulence to its statistical mean state feedback between the fluctuations and the mean state such as to stabilize the mean state to linear instability is similar to that hypothesis by Malkus (1956) which posited that the statistical state of turbulence is determined by feedback regulation to neutrality of the mean-state’s inflectional modes. This hypothesis was not verified for the case of wall-bounded turbulence (Reynolds & Tiederman 1967), but needed as we have seen above only substitution of parametric neutrality of the time and spanwise varying streamwise-mean flow for the posited inflection mode neutrality of the temporal mean flow.

In our study of the POD modes we considered first the streamwise-mean component of the turbulence. We found striking resemblance between the POD modes of the DNS and RNL fields. An initial interpretation of this similarity suggested that the scale invariant R-S formation mechanism analytically identified in the S3T-RNL SSD is also operating in DNS. Although scale invariant R-S dynamics provides a possible explanation for the POD modes found in both DNS and RNL, the random phase assumption, which is traditionally taken to characterize the POD modes in directions in which solutions to the equations are statistically homogeneous, is not necessarily valid when a mechanism of symmetry breaking is active. In the present case an instability process occurs to break the spanwise symmetry resulting in R-S structures (Farrell & Ioannou 2012; Farrell et al. 2017b). If this process is active it would imply that the random phase assumption of the POD \( k_x = 0 \) modes is not valid. In order to examine this possibility we aligned the most prominent low-speed streak of the flow to obtain a spatially coherent time-averaged low speed streak and determined the spanwise Fourier components of this coherent streak. We then verified that the Fourier components of this coherent streak corresponded to the structure and the amplitudes of the POD modes that were obtained making the random phase assumption. Furthermore, we have verified that removing the random phase assumption among the POD modes of harmonic form by aligning them to zero phase difference revealed that the POD modes constituted the Fourier components of the coherent non-harmonic R-S structure educed by aligning the low-speed streak in the turbulence simulation.

After having determined the temporal mean of the R-S that was obtained by collocation, it remained to identify the \( k_x \neq 0 \) POD modes associated with this R-S. In order to isolate the POD modes of the \( k_x \neq 0 \) fluctuations that are associated with the R-S structure, the fluctuation velocity fields were also translated to be aligned with the collocated low speed streak. POD analysis of the aligned velocity fields revealed close correspondence between the \( k_x \neq 0 \) POD modes in DNS and RNL. The dominant POD modes in both DNS and RNL reveal a prominent component of streak-localized sinuous oblique form. Sinuous oblique waves had previously been identified in analysis of the S3T SSD (Farrell & Ioannou 2012). In those studies sinuous oblique waves were shown to give rise to Reynolds stresses properly collocated with the streak to force roll circulation that amplify the streak through the lift-up process. Moreover, these sinuous oblique waves are exactly the structures that are predicted to arise from a turbulent background flow field because they are the optimally growing perturbations. This was confirmed using a stochastic turbulence model (STM) that showed that the POD modes of the STM closely correspond to the POD modes of both the DNS and RNL.

In this work we have provided evidence using POD analysis that the roll-streak SSP operating in DNS turbulence is the same as that operating in RNL. Because the mechanism maintaining turbulence in the S3T-RNL system has been comprehensively characterized (Farrell & Ioannou 2012; Thomas et al. 2014, 2015; Farrell et al. 2016, 2017a), this close correspondence between the dominant structures in RNL and DNS and the fact that the
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RNL has the same dynamical restriction as the S3T SSD and is similarly characterized in its dynamics argues that the dynamics identified in S3T-RNL identifies the dynamics of turbulence in DNS.

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Declaration of interest

The authors report no conflict of interest

Appendix A. Construction of the covariances with symmetry restrictions

Homogeneity in the streamwise and spanwise directions allows the decomposition of velocity field snapshots into sums of plane waves with Fourier coefficients that depend on the wall-normal direction. Application of mirror symmetries in $y$ and $z$ incorporates the 2-point statistics from the total flow field into a single covariance for each $|k_x|,|k_z|$ wavenumber pair. Convergence towards these statistical symmetries is slow. For example in Fig. 16 we demonstrate the slow convergence of the statistics to the asymptotic mirror symmetric state about the wall-normal plane at the center of the channel.

For a single $k_z, k_x$ pair the three components of the velocity field is comprised by two independent plane waves

$$
\Phi_{k_x} e^{i k_z x} = \begin{pmatrix}
A_{k_x, k_z}(y) \\
B_{k_x, k_z}(y) \\
\Gamma_{k_x, k_z}(y)
\end{pmatrix} e^{i (k_x x + k_z z)} + \begin{pmatrix}
A_{k_x, -k_z}(y) \\
B_{k_x, -k_z}(y) \\
\Gamma_{k_x, -k_z}(y)
\end{pmatrix} e^{i (k_x x - k_z z)}. \tag{A 1}
$$

With $A$ we denote the streamwise component of the velocity field, $B$ the wall-normal and with $\Gamma$ the spanwise component. A special case is the $k_x = 0$ component for which the coefficients of $k_z$ and $-k_z$ will be complex conjugates. The two symmetries we consider are mirror symmetry in $y$ with respect to the half-channel $x - z$ plane at $y = 1$ and in $z$ with respect to the half width plane $x - y$ at $z = \pi/2$. Those produce a fourfold increase in the amount of data that will be included in the covariance matrix.

First we consider the spanwise mirroring operation. This will transform $z$ to $\pi - z$ and change sign in the spanwise velocity component.

$$
\hat{S}_z \Phi_{k_x} e^{i k_z x} = \begin{pmatrix}
A_{k_x, -k_z}(y) \\
B_{k_x, -k_z}(y) \\
-\Gamma_{k_x, -k_z}(y)
\end{pmatrix} e^{i (k_x x + k_z (z - \pi))} + \begin{pmatrix}
A_{k_x, k_z}(y) \\
B_{k_x, k_z}(y) \\
-\Gamma_{k_x, k_z}(y)
\end{pmatrix} e^{i (k_x x - k_z (z - \pi))}. \tag{A 2}
$$

The $-i k_z \pi$ phase that appears in the plane wave will cancel out when the covariance is formed.
Figure 16: The 1-norm of the difference $C_{kz} - \hat{S}_y C_{kz}$ between the covariance matrix $C_{kz}$ (4.2), and the covariance of the reflected flow about the $x-z$ plane at the center of the flow ($y = 1$) as a function of the averaging time, $T_{av}$, for $hk_z = 2, 4, 6, 8$ for a DNS of NL100. $\hat{S}_y$ is defined in Appendix A, Eq. (A 8). This plot verifies that reflection symmetry about the centerline is a statistical symmetry of the flow and that this symmetry is approached at the rate $1/T_{av}$ consistent with the law of large numbers for quadratic statistics. Time is non-dimensionalized by $h/U$.

In the wall-normal mirroring the effect is to transform $y$ to $2 - y$ and change sign in the wall-normal velocity component,

$$
\hat{S}_y \Phi_{kz} e^{i k_x x} = \begin{pmatrix}
A_{k_x,k_z}(2-y) \\
-B_{k_x,k_z}(2-y) \\
\Gamma_{k_x,k_z}(2-y)
\end{pmatrix} e^{i(k_x x+k_z z)} + \begin{pmatrix}
A_{-k_x,-k_z}(2-y) \\
-B_{k_x,-k_z}(2-y) \\
\Gamma_{k_x,-k_z}(2-y)
\end{pmatrix} e^{i(k_x x-k_z z)}
$$

(A 3)

What the $2 - y$ coordinate implies is that the wall-normal structure will be inverted for each component. Summarizing the above operations, the total covariance will be comprised by the individual covariances obtained for each of the 4 components below

$$
\Phi_{kz} = \begin{pmatrix}
A_{k_z}(y) \\
B_{k_z}(y) \\
\Gamma_{k_z}(y)
\end{pmatrix} e^{i k_z z}, \quad \hat{S}_z \Phi_{kz} = \begin{pmatrix}
A_{-k_z}(y) \\
B_{-k_z}(y) \\
\Gamma_{-k_z}(y)
\end{pmatrix} e^{i k_z z},
$$

(A 4)
where the $k_z$ subscript has been omitted.

We form the covariance obtained from the initial wave. To highlight the inner structure of this covariance due to the different velocity components the following representation is chosen

$$
C_{k_z} = \begin{pmatrix}
C_{uu}^{k_z} & C_{uv}^{k_z} & C_{uw}^{k_z} \\
C_{vu}^{k_z} & C_{vv}^{k_z} & C_{vw}^{k_z} \\
C_{wu}^{k_z} & C_{uw}^{k_z} & C_{ww}^{k_z}
\end{pmatrix},
$$

(A 5)

with $C_{k_z}^{u_i u_j} = (C_{k_z}^{u_j u_i})^\dagger$. In the following the $k_z$ subscript will be omitted where possible and instead of $u_i u_j$ the superscript $ij$ will be used. So the covariance can be written as:

$$
C = \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}.
$$

(A 6)

Statistical symmetry in reflections of the velocities in $z$ merge the covariance of the $-k_z$ component with the $k_z$. The negative $k_z$ covariance will be modified to account for this symmetry

$$
\hat{S}_z C_{-k_z} = \begin{pmatrix}
(C_{11}) & -C_{12} & -C_{13} \\
(C_{21}) & (C_{22}) & -C_{23} \\
-(C_{31}) & -(C_{32}) & (C_{33})
\end{pmatrix}.
$$

(A 7)

Reflections in $y$ require to reverse the order of the row and column indexes in each individual covariance and if this operation is noted as $\hat{S}_y C_{ij}^R = C_{ij}^R$ we have:

$$
\hat{S}_y C = \begin{pmatrix}
-C_{11}^R & C_{12}^R & C_{13}^R \\
-C_{21}^R & -C_{22}^R & -C_{23}^R \\
C_{31}^R & C_{32}^R & C_{33}^R
\end{pmatrix}.
$$

(A 8)

The total covariance will be comprised by the following components

$$
C_{k_z}^t = (C_{k_z} + \hat{S}_y C_{k_z} + \hat{S}_z C_{-k_z} + \hat{S}_y \hat{S}_z C_{-k_z})/4
$$

(A 9)

To account correctly for the relative energy between $k_z = 0$ and $k_z \neq 0$ components the eigenvalues of covariances with $k_z \neq 0$ are doubled in the ordering process.

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