Optimum Placement of Post-1PN GW Chirp Templates
Made Simple at any Match Level
via Tanaka-Tagoshi Coordinates

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A simple recipe is given for constructing a maximally sparse regular lattice of spin-free post-1PN gravitational wave chirp templates under a given minimal-match constraint, using Tanaka-Tagoshi coordinates. Cardinal interpolation among the resulting maximally-sparse templates is discussed.

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I. INTRODUCTION

Gravitational wave *chirps* emitted by compact binary stars in their adiabatic inspiral phase will be primary targets for the early operation of broadband laser-interferometric detectors like TAMA300, GEO600, LIGO and VIRGO [1]. These signals have been thoroughly studied and accurately modeled [2].

The maximum-likelihood strategy [3] for detecting signals of known shape (except for a set of unknown parameters) in additive stationary colored gaussian noise consists in correlating the data with a set

\[ \{ g_i \} \]

of possible expected waveforms (*templates*), and using the largest correlator as a detection statistic [3]. The mentioned correlators are the (noise-weighted) scalar products:

\[ \langle a, g \rangle = 2 \left[ \int_{f_i}^{f_s} a(f) g^*(f) \frac{df}{\Pi(f)} + \text{c.c.} \right], \]

where \( (f_i, f_s) \) is the antenna spectral window, \( a(f) = h(f) + n(f) \) are the noise-corrupted (spectral) data, \( h(f) \) is a (possibly null) signal, \( n(f) \) is a realization of the antenna noise with (one-sided) power spectral density \( \Pi(f) \), \( g(f) \) is a template, and \text{c.c.} denotes complex-conjugation.

The issue of optimum placement of the templates in the waveform parameter-space is more or less obviously a crucial and still open one, and has been addressed by several Authors [4]-[8].

At fixed false-alarm probability, the detection probability is an increasing function of the *overlap*

\[ O(h, g) = \frac{|\langle h, g \rangle|}{||h|| \cdot ||g||} \leq 1 \]

between the signal \( h \) and the template \( g \), where the norm \( ||u|| = \langle u, u \rangle^{1/2} \). The template set \( \mathcal{L} = \{ g_i \} \) should be designed in such a way that, for any admissible signal \( h \), the overlap never drops below a prescribed value \( \Gamma \), which can be immediately related to the fraction \( (1 - \Gamma^3) \) of potentially observable sources which might be missed out [3].

The overlap can be readily maximized w.r.t. the (unknown but irrelevant) initial-phase and coalescence-time differences between the signal and the template [3]. The resulting partially-maximized overlap is called the *match*, is denoted by \( M(h, g) \), and is a function of the source and template *intrinsic* parameters only (e.g., at 2.5PN order, the binary companion-masses, spin-spin and spin-orbit parameters). The minimum-overlap condition rephrases obviously into the following minimal-match prescription:

\[ \forall h \in \mathcal{S}, \exists g \in \mathcal{L} : M(h, g) \geq \Gamma, \]

(1.3)

where \( \mathcal{S} \) is the set of admissible signals.

Discussion will be henceforth restricted to the 2D parameter-space of spin-free chirp-waveforms, in view of the widespread present consensus that these should be adequate for the early operation of interferometric detectors [3].

In the following we shall denote as \( G \) the (2D, spin-free) parameter-space point corresponding to waveform \( g \). We shall also denote as \( \gamma_T(G) \) the match contour-line, whose points represent the set of waveforms \( h \) such that \( M(h, g) = \Gamma \), and as \( S_T(G) \) the 2D-region bounded by \( \gamma_T(G) \), representing the set of waveforms \( h \) such that \( M(h, g) \geq \Gamma \).

The template set \( \Lambda = \{ G_i \} \) should fulfill the following basic requirements. First of all, (a) the minimal-match condition (1.3) should be met, which re-phrases into

\[ \bigcup_i S_T(G_i) \supseteq \Sigma, \]

(1.4)
where $\Sigma$ is the image of $S$ i.e., the subset of the waveform parameter-space corresponding to admissible sources. Pictorially, (1.4) means that the patches $S_I(G_i)$ should cover the whole set $\Sigma$ without leaving holes. At the same time, (b) the template set $\Lambda$ should be chosen as much sparse as possible, so as to minimize the overall number of templates, and hence the detection threshold-level, for any prescribed false-alarm probability $\mathcal{F}$. It might be further desirable (c) to deal with a set $\Lambda$ forming a regular (or piecewise-regular) grid, for computational ease.

A further template-placement requirement is also worth mentioning. When covering $\Sigma$ with (regularly spaced or patch-wise regularly spaced) templates, a certain amount of template spill-over beyond $\partial \Sigma$ is unavoidable. The post-1PN spin-free waveform parameter-space subset $\Sigma$ corresponding to admissible sources, for which $m_{\text{min}} \leq m_1 \leq m_2 \leq m_{\text{max}}$, is a three-vertex curved-side 2D domain. Spill-over across the $m_1 = m_2$ boundary line of $\Sigma$ (equal-mass boundary-line) is a true penalty, and thus (d) should be kept to a minimum. This is not the case for the other two boundary lines of $\Sigma$, where points beyond $\partial \Sigma$ could well correspond to possible (though unlikely) sources.

At large values of the minimal match, e.g. typically $M(h,g) \geq \Gamma \gtrsim 0.97$, as prescribed for a single-step search strategy $\mathcal{F}$, the match is very well approximated by a quadratic function $\mathcal{F}$ of the distance between $H$ and $G$ in parameter-space, and the (approximate) match contour-lines $G_\Gamma$ are ellipses. However when using post-1PN order waveforms (which will be required in a one-step search to achieve the prescribed large minimal-match levels $\mathcal{F}$), as $G$ moves throughout $\Sigma$, the above ellipses rotate and stretch as an effect of the intrinsic nonzero curvature of $\Sigma$.

An effective procedure for constructing a 2D spin-free post-1PN template set appropriate to this case has been formulated in $\mathcal{F}$, as a generalization of the 1PN strategy introduced in $\mathcal{F}$. The simplest rectangular-cell template-lattice is chosen, and the span of template $G_i$ (i.e., the set of points representing waveforms for which $M(h,g_j) > M(h,g_i), \forall j \neq i$) is accordingly taken as the largest rectangle inscribed in the minimal-match contour-line $G_\Gamma$.

In order to minimize spill-over across the equal-mass boundary-line of $\Sigma$, while optimizing template placement for equal-mass sources (which are credited to be most abundant $\mathcal{F}$), the algorithm is started by placing a sufficient number of templates along the equal-mass boundary line. The nearest-neighboring templates are placed in such a way that the boundaries of their rectangular spans touch without intersecting. As a result $\Sigma$ is covered with a set of patches, each of which is a stack of equal-width rectangles, whose height changes according to the shear/stretch of the elliptical $G_\Gamma$ contour-lines. The current release of GRASP $\mathcal{F}$ implements a similar procedure, resulting into a relatively straightforward template placement algorithm.

As already noted in $\mathcal{F}$ the above algorithm is not optimal, i.e., it does not necessarily satisfy requirement (b), in view of its a-priori restriction to rectangular cells $\mathcal{F}$.

The template placement issue becomes even more complicated at low minimal-match levels (e.g., typically $\Gamma > 0.7$) as prescribed in the first step(s) of hierarchical search strategies. In multi-step hierarchical searches a single template-lattice $\Lambda$ designed to achieve a large $\Gamma$ throughout $\Sigma$ is constructed. In the first step only the correlators corresponding to a suitably decimated subset of $\Lambda$ are computed, covering $\Sigma$ at a lower minimal-match level. In the second step, only a limited number of correlators, corresponding to a few non-decimated subsets of $\Lambda$ need to be actually computed, covering suitable neighbourhood(s) of the candidate signals discovered in the first step. Multi-step hierarchical search strategies should allow a sizeable reduction in the required computing power $\mathcal{F}$.

Mohanty $\mathcal{F}$ formulated a simple strategy for evaluating the lattice decimation-steps (taken as constant throughout extended patches of the lattice) for a post-1PN template lattice. Also in this case, however, requirement (b) will be most certainly violated. By itself, the template placement issue at low minimal-match levels is further complicated by the fact that the match contour-lines $G_\Gamma(G)$ besides varying with $G$ are no longer elliptical. For this reason, the procedure expounded in $\mathcal{F}$ can only be validated by extensive Monte-Carlo trials.

In this rapid communication we present a possible way to get rid of all the above difficulties and limitations, by capitalizing on the remarkable properties of the post-1PN waveform parametrization introduced by Tanaka and Tagoshi $\mathcal{F}$, and further discussed in $\mathcal{F}$. We accordingly introduce a simple and systematic procedure to set up an optimum template-lattice $\mathcal{F}$, which closely fulfills all conditions a), b) and c) above, at any prescribed minimal-match level $\mathcal{F}$.

The paper is organized as follows. In Sect. 2 the Tanaka-Tagoshi parametrization and its relevant properties, including the main features of the match contour-lines, are briefly introduced. In Sect. 3 our optimum template placement procedure is exposed in detail. In Sect. 4 the gain achievable by cardinal interpolation among these optimally-placed templates is discussed. Conclusions follow under Section 5.
II. TANAKA-TAGOSHI PARAMETRIZATION OF POST-1PN WAVEFORMS

The Tanaka-Tagoshi transformation \[ \Gamma = 0 \] is a linear one-to-one mapping \( F \in \Sigma \leftrightarrow \tilde{F} \in \mathcal{T} \) between the (2D, spin-free, post-1PN) admissible waveform parameter-space \( \Sigma \) and a globally flat manifold \( \mathcal{T} \) such that: i) the ratio

\[
\epsilon = \frac{|M(h, g) - M(\tilde{h}, g)|}{M(h, g)}
\] (2.1)

is negligibly small (typically of the order of \( 10^{-3} \)) and, ii) \( M(\tilde{h}, g) \) depends only on the distance between \( \tilde{H} \) and \( \tilde{G} \). The Tanaka-Tagoshi construction is effective for all first generation interferometers [21].

Clearly, property i) above allows to solve the optimum template placement problem in the globally flat manifold \( \mathcal{T} \), by finding a set \( \Lambda = \{ \tilde{G}_i \} \subset \mathcal{T} \) such that:

\[
\bigcup_i S_F(\tilde{G}_i) \supseteq \mathcal{T}.
\] (2.2)

On the other hand, property ii) implies that the sets \( S_F(\tilde{G}_i) \) are identical up to trivial translations. This means that \( \Lambda \subset \mathcal{T} \) is a globally uniform template lattice, featuring a regular grid of nodes, at any prescribed minimal-match level.

A. Constant Match Contour-Lines in Tanaka-Tagoshi Coordinates

The match contour-lines \( \gamma_F(\tilde{G}) \) in the (dimensionless) Tanaka-Tagoshi (spin-free) waveform parameter-space coordinates \((x_1, x_2)\) have different geometrical features, depending on the value of \( \Gamma \). In Fig. 1 the inverse of the curvature radius \( \rho_\gamma \) of \( \gamma_F(\tilde{G}) \) is displayed as a function of the polar angle \( \phi \) around \( \tilde{G} \) for several representative values of \( \Gamma \), for LIGO-I at 2.5PN order. The contour-lines \( \gamma_F \) are also shown in the insets, where \( \delta x_i = x_i(\tilde{G}) - x_i(\tilde{H}) \).

In the limit as \( \Gamma \to 1 \) the contour-lines \( \gamma_F \) are asymptotically circular and \( \rho_\gamma^{-1} \) is almost constant. See, e.g., the \( \Gamma = 0.99 \) contour-line in Fig. 1 top-left.

In a range \( \Gamma^* \leq \Gamma \leq 1 \), the contour-lines \( \gamma_F \) deviate from circular, but still remain globally convex. For LIGO-I, \( \Gamma^* \approx 0.9225 \). Correspondingly, \( \rho_\gamma^{-1} \) remains non negative throughout a full rotation around \( \tilde{G} \). See, e.g., the \( \Gamma = 0.93 \) contour-line in Fig. 1 top-right.

For \( \Gamma < \Gamma^* \), the \( \gamma_F \) contour-lines are no longer globally convex. In a range \( \Gamma^* \leq \Gamma \leq \Gamma^* \), they consist simply of two convex arcs and two concave ones joining smoothly, to form a "peanut" shape. See e.g. the \( \Gamma = 0.914 \) contour-line in Fig. 1 bottom-left. For LIGO-I, \( \Gamma^* \approx 0.9109 \). As \( \Gamma \) is decreased below \( \Gamma^* \), the contour-lines become more and more complicated, as more and more bumps and dents do appear. Correspondingly, the boundary curvature \( \rho_\gamma^{-1} \) exhibits more and more turning points throughout a full rotation around \( \tilde{G} \). See, e.g., the \( \Gamma = 0.875 \) contour-line in Fig. 1 bottom-right.

For all values of \( \Gamma \), the surface \( S_F(\tilde{G}) \) has always a centre of symmetry in \( \tilde{G} \).

III. OPTIMUM TEMPLATE PLACEMENT

In the following we shall discuss the optimum template placement strategy for each of the above mentioned minimal-match ranges. Optimum here and henceforth means that requirements (a), (b) and (c) discussed in Sect. 1 will be strictly fulfilled [19].

A. The Asymptotic \( \Gamma \to 1 \) Limit

In the asymptotic limit \( \Gamma \to 1 \), the match contour-lines are circular. Rotational symmetry of the match contour-lines implies that the templates should sit at the vertices of regular polygons in the waveform parameter-space. These regular (open) polygons should besides make up a (regular) tiling of the (Tanaka-Tagoshi) plane [22], and hence can only be triangles, squares or hexagons. Let \( U \) be one such polygon, and \( \{ \tilde{G}_i \} \) its vertices. The sparsest templates satisfying (2.2) are such that the circles \( \gamma_F(\tilde{G}_i) \) touch at a single point \( P \), which is obviously the center of symmetry of \( U \). This means that the curve \( \gamma_F(P) \) goes through all the \( \tilde{G}_i \)'s. Hence, the templates should be located at the vertices of a regular triangle, square or hexagon inscribed in a circle \( \gamma_F \).
The obvious question is which polygon is best in terms of the total number of templates needed to cover the admissible-waveform parameter-space $\mathcal{F}$ in the Tanaka-Tagoshi plane. Clearly, the best polygon will be the one for which the span of each (and any) template has the largest measure (area) \[23\].

The span (or, in more technical language, the Voronoi set \[24\]) of $\tilde{G}_i$ is the set of points $\tilde{H}$ for which $M(\tilde{h}, \tilde{g}_i) > M(\tilde{h}, \tilde{g}_j), \forall j \neq i$, which are the images in the Tanaka-Tagoshi plane of those waveforms $h$ which will be detected using template $g_i$. The Voronoi sets are shown shaded in Fig. 2, for the (regular) triangular, square and hexagonal tilings of the (Tanaka-Tagoshi) plane \[25\].

A choice can be made on the basis of a comparison among the measures (areas) of the pertinent Voronoi sets. Let $\mu(\cdot)$ denote the measure (area), $U^{(p)}$ the $p$-gonal tiling-cell, and $V^{(p)}$ the corresponding Voronoi set. It is readily seen that \[26\]:

\[
\frac{\mu[V^{(3)}]}{\mu[V^{(4)}]} = 1.299037, \quad \frac{\mu[V^{(6)}]}{\mu[V^{(4)}]} = 0.649519, \quad (3.1)
\]

The triangular tiling is thus seen to be the best choice, yielding a gain in terms of template-span of $\sim 30\%$, whereas the hexagonal one features a loss of $\sim 35\%$, w.r.t. the simple square tiling \[27\].

The triangular tiling yields a template lattice whose fundamental domain (lattice-cell) is a parallelogram \[18\]. Only three vertices of any lattice-cell belong to a single curve $\gamma_T$.

### B. The Range $\Gamma > \Gamma^*$

In the range $\Gamma > \Gamma^*$, the contour-lines $\gamma_T(\tilde{G})$ are no longer circles, but still globally-convex and center-symmetric. Circular symmetry being lost, we shall still seek the sparsest template collocation subject to \[2.2\] among the non-regular, but still center-symmetric (open) $p$-gons ($p = 3, 4, 6$) with all vertices on $\gamma_T$, which tile the Tanaka-Tagoshi plane.

It makes sense to compare for this case the template-span (Voronoi set) measures (areas) $\mu[V^{(p)}], p = 3, 4, 6$ to the template-span measure (area) $\mu[V_0^{(4)}]$ of the square tiling \[26\] pertinent to the naive (but usual) approximation of circular match contour-lines. The ratios

\[
r_p = \frac{\mu[V^{(p)}]}{\mu[V_0^{(4)}]}, \quad p = 3, 4, 6. \quad (3.2)
\]

are displayed in Fig. 3 as functions of $\Gamma$ for LIGO-I. Not unexpectedly, the triangular tiling is once more the best. This conclusion is valid for all first generation antennas, and the pertinent values of $r_3$ have been collected in Table-I.

| Antenna   | $r_3$ |
|-----------|-------|
| TAMA300   | 1.58  |
| LIGO-I    | 1.43  |
| GEO600    | 1.58  |
| VIRGO     | 1.38  |

Table I.- The ratio $r_3$, eq. (3.2), for first generation interferometers at $\Gamma = 0.97$.

### C. The Range $\Gamma < \Gamma^*$

By an argument of continuity (i.e., since the match contour-lines corresponding to $\Gamma < \Gamma^*$ can be obtained from those corresponding to $\Gamma > \Gamma^*$ by continuous transformations) we shall still seek the optimum tiling of Tanaka-Tagoshi plane using the (open) triangles with (all) vertices on $\gamma_T$. However, the match contour-lines being no longer globally convex, the largest-area triangle inscribed in $\gamma_T$ does not necessarily cope with eq. \(2.2\), as e.g. exemplified in Fig. 4.

One should accordingly seek three points $\tilde{G}_i$ on $\gamma_T$ yielding the largest-area triangle $\tilde{G}_0\tilde{G}_1\tilde{G}_2$, subject to eq. \(2.2\). The minimal-match condition \(2.2\) is equivalent to

\[
C - \bigcup_i \left[ C \cap S_T(\tilde{G}_i) \right] = \emptyset, \quad (3.3)
\]
where $C$ is the (parallelogram) fundamental-domain of the lattice corresponding to a given triangular tiling. Condition (3.3) is most easily checked, involving only four patches $S_P$.

Let $\tilde{P}, \tilde{P}', \tilde{Q}, \tilde{Q}'$ the points (ordered, e.g., counterclockwise) where $\gamma_\Gamma$ and its convex-hull detach, shown in Fig. 5. One can readily prove that two vertices of the sought optimum triangular-tile, say $\tilde{G}_0, \tilde{G}_1$, should be sought on the arc $\tilde{P}\tilde{P}'$ (or $\tilde{Q}\tilde{Q}'$) of $\gamma_\Gamma$, while the third should be sought on the arc $\tilde{Q}\tilde{Q}'$ (resp. $\tilde{P}\tilde{P}'$). It is also readily proved that the lattice vector $\tilde{G}_0\tilde{G}_1$, after a trivial translation making its midpoint coincident with $\tilde{G}$, should be contained in the butterfly-shaped region $(\tilde{P}\tilde{G}\tilde{Q}' \cup \tilde{Q}\tilde{G}\tilde{P}') \cap S_\Gamma(\tilde{G})$ depicted in Fig. 5.

The maximum-area triangular tiles subject to the constraint (2.2), are shown in Fig. 6, for several values of $\Gamma$ in the range $0.7 \leq \Gamma < 1$ (note that different scales are used in the the figure, for better readability). The lattice-cell measure corresponding to the optimum triangular-tile is displayed in Fig. 7 as a function of $\Gamma$, together with the number of templates needed to cover the companion-mass range $0.2M_\odot \leq m_1 \leq m_2 \leq 10M_\odot$, for a number of representative minimal-match values.

IV. CORRELATOR BANK ECONOMIZATION VIA CARDINAL INTERPOLATION

In recent papers it has been shown that a sensible reduction ($\approx 75\%$ for 1PN and higher-PN-order templates) in the number of correlators to be computed in order to achieve a prescribed minimal-match can be obtained using cardinal interpolation, thanks to the quasi-band-limited property of the match function $[29], [30]$. These results were obtained under the simplest assumption of a square-cell template-lattice.

Two obvious questions are now in order: i) whether/to what extent cardinal interpolation is still effective when using the optimum triangular-tiling discussed in the previous sections $[31]$, and ii) whether/to what extent cardinal interpolation is still useful at relatively low $\Gamma$ values, as needed in the early step(s) of hierarchical searches. Numerical simulations have been run to clarify both issues.

In Fig. 8 the template density reduction $[32]$ obtained by cardinal interpolation among optimally-placed templates is displayed as a function of the prescribed minimal-match $\Gamma$. In the inset of Fig. 8, the (boosted) minimal-match values obtained after cardinal interpolation among the optimally-placed templates corresponding to a number of representative values of $\Gamma$ are also listed.

At $\Gamma = 0.97$ a template density reduction $\approx 2.79$ is obtained. The apparent discrepancy between this value and the one reported in $[31]$ is readily explained. The template density required to achieve a prescribed minimal-match when using cardinal interpolation is essentially independent from the shape of the lattice fundamental-domain, depending only on its measure (area), according to the well-known Nyquist-rate condition $[33]$. Thus, the template density reduction $\approx 4$ in $[31], [17]$ when using the simplest square-cell lattice is seen to result from two factors: a factor $\approx 2.79$, expressing the template density reduction due to cardinal interpolation when using the optimum triangular-tiling, times a factor $\approx 1.43$ expressing the template density reduction implied in using the optimum triangular-tiling instead of the simplest square one.

V. CONCLUSIONS

We presented a simple and systematic procedure for constructing the sparsest lattice of templates subject to a given minimal-match constraint in the range between $\Gamma \sim 1$ down to $\Gamma \approx 0.7$ (and below) in the (spin-free) Tanaka-Tagoshi parameter-space of post-1PN gravitational wave chirps.

We also showed that cardinal interpolation can be effective for correlator-bank economization both in the late and early stage(s) of multi-step hierarchical searches.

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[1] B.F. Schutz, Class. Quant. Grav. 16, A131 (1999).
The template-density reduction factor is defined here as the ratio between the areas of the (optimal) triangular-tiles yielding the same minimal-match with and without cardinal interpolation.

The convex hull of a set $G \subseteq \mathbb{R}^n$, the Voronoi set or proximity locus of $G_k \in G$ is the set $V(G_k) = \{ x \in \mathbb{R}^n : \forall h \neq k, D(x, G_h) > D(x, G_k) \}$, where $D(\cdot, \cdot)$ is a suitable distance (metric).

It is seen from Fig. 2 that, but for trivial translations, $U^{(3)} = V^{(6)}$, $U^{(6)} = V^{(3)}$, and $U^{(4)} = V^{(4)}$, i.e., the triangular and hexagonal tiling are Voronoi-dual, while the square tiling is Voronoi self-dual.

We remind that for circular match contour-lines, the inscribed square-tile area is given by $\mu[V^{(4)}] = 2(1 - \Gamma)$.

Occasional confusion is made in the technical literature between the tiling-cell and tiling-cell’s span (Voronoi set) measures, leading e.g. to the often quoted statement that hexagonal tiling would be the most efficient.

The total number of templates needed to cover $T$ is approximately equal to the ratio between the measure (area) of $\Sigma$ and the measure (area) of the template span.

Given a set of points $G \subseteq \mathbb{R}^n$, the Voronoi set or proximity locus of $G_k \in G$ is the set $V(G_k) = \{ x \in \mathbb{R}^n : \forall h \neq k, D(x, G_h) > D(x, G_k) \}$, where $D(\cdot, \cdot)$ is a suitable distance (metric).

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The convex hull of a set $U$ is the smallest convex set containing $U$.

The standard 2D cardinal-interpolation formula applies to the case where the interpolation points form a rectangular grid. The cardinal-interpolation formula for the case where the interpolated points form a skew grid is more or less obviously obtained by first applying a trivial coordinate transformation which brings the grid into a rectangular one, then writing down the usual (rectangular-grid) cardinal-interpolation formula, and finally switching back to the original coordinates.

The template-density reduction factor is defined here as the ratio between the areas of the (optimal) triangular-tiles yielding the same minimal-match with and without cardinal interpolation.

See, e.g., J.R. Higgins, *Sampling Theory in Fourier and Signal Analysis* (Clarendon Press, Oxford, 1996), ch. 14.
CAPTIONS TO THE FIGURES

Fig 1 - Match contour-lines in Tanaka-Tagoshi coordinates. Curvature $\rho^{-1}_\gamma$ of $\gamma_\Gamma$ vs. polar angle $\phi$ (LIGO-I, 2.5PN).

Fig. 2 - Triangular, square and hexagonal tilings and pertinent Voronoi sets.

Fig. 3 - The ratios $r_p$, $p = 3, 4, 6$ relevant to Eq. (3.2) as functions of $\Gamma$ (LIGO-I, 2.5PN).

Fig. 4 - Inscribed triangle with largest area does not necessarily cope with minimal-match condition for $\Gamma < \Gamma^*$ (LIGO-I, 2.5PN, $\Gamma = 0.9$).

Fig. 5 - The butterfly-shaped set $(\tilde{P}\tilde{G}\tilde{Q}' \cup \tilde{Q}\tilde{G}\tilde{P}') \cap S_{\Gamma}(\tilde{G})$ (LIGO-I, 2.5PN, $\Gamma = 0.875$).

Fig. 6 - Optimum triangular-tilings and template-lattices for several values of $\Gamma$ (LIGO-I, 2.5PN).

Fig. 7 - Lattice-cell area corresponding to optimum triangular-tiling and number of templates needed to cover the range $0.2M_\odot \leq m_1 \leq m_2 \leq 10M_\odot$ vs. minimal-match (LIGO-I, 2.5PN).

Fig. 8 - Cardinal-interpolation gain (template density reduction factor and minimal-match boost) for optimum triangular-tiling (LIGO-I, 2.5PN).
Fig. 1 - Match contour-lines in Tanaka-Tagoshi coordinates. Curvature $\rho_{\gamma}^{-1}$ of $\gamma_r$ vs. polar angle $\phi$ (Ligo-I, 2.5 PN).
Fig. 2 - Triangular, square and hexagonal tilings and pertinent Voronoi sets.
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Fig. 4 - Inscribed triangle with largest area does not necessarily cope with minimal-match condition for $\Gamma < \Gamma^\circ$ (LIGO-I, 2.5 PN, $\Gamma = 0.9$).
Fig. 5 – The butterfly-shaped set \((\tilde{P}\tilde{G}\tilde{Q}' \cup \tilde{Q}\tilde{G}\tilde{P}') \cap S_f(\tilde{G})\) (LIGO-I, 2.5PN, \(\Gamma=.875\)).
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