On some class of nonlinear mean random values

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Abstract. In this paper we study the properties of one class of nonlinear means, including a geometric mean, a harmonic mean, and also a power mean of random variables. The considered random variables are assumed to be positive and have finite mathematical expectations. The asymptotics and extremal properties, as well as algebraic properties and inequalities between means are also studied below.

1. Introduction
Means are widely used in statistical research. Mathematical expectation and median are the most usable among average random quantities. However, sometimes the other characteristics are useful too. Below the properties of one class of nonlinear mean random variables are investigated, including the geometric mean, the harmonic mean and the power mean. These means are assumed to be positive and have finite mathematical expectations.

Let $X$ be positive random variable and $EX$ — its mathematical expectation. Consider some types of nonlinear mean nonlinear random variables. As is known (see [1] Chapt. 5, p. 5.6.2) the geometric mean $G(X)$ of a random variable $X$ ($X > 0$) is given by the formula

$$G(X) = e^{E(lnX)}.$$  (1)

Harmonic mean $H(X)$, ($X > 0$) — by the formula

$$H(X) = 1/(E(1/X)).$$  (2)

Note that the geometric mean value is used in studying processes whose growth is proportional to the level already achieved (population growth, gross product, etc.), as well as in calculating price indices.

The harmonic mean $H(X)$ is used in economics in index calculations.

Along with the geometric means and harmonic means, we will also consider the power means. As is known, the power mean of $p$-th order ($p > 0$), ($X > 0$) is defined by the formula

$$E_p(X) = (E(X^p))^{1/p}.$$  (3)

The power mean of $p$-th order is the $p$-th initial moment of a random variable $X$ of $1/p$ degree. In the case of $p = 1$ the power mean turns into mathematical expectation.
Let $X$ be a sample of the volume $n$, obtained from several independent observations of random variable $X$. By definition, all random variables $X_1, X_2, ..., X_n$ are independent and distributed equally, as well as the random variable $X$.

As an estimate of the average (7) $\hat{E}_\phi(X)$ consider

$$
\hat{E}_\phi(X_1, X_2, ..., X_n) = \phi^{-1}(\frac{1}{n}\sum_{i=1}^{n} \phi(X_i)).
$$

Note that (8) — is a widely known formula for the mean common species (see e.g. [2] Chapt. III, § 3.1, [3], Chapt. I, § 5 and [4]).

Assuming in formula (8) $\phi(x) = \ln x$, we get the geometric mean

$$
\hat{G}(X_1, X_2, ..., X_n) = \sqrt[n]{X_1 X_2 ... X_n},
$$

i.e., respectively, the geometric mean, the harmonic mean and the power mean of the set of numbers $x_1, ..., x_n$. The "cap" sign above the mean here and below denotes the discrete (empirical) mean.

If, in this example, $P(X = x_i) = p_i$, we obtain, respectively, the weighted geometric mean, the weighted harmonic mean and the weighted power mean with weights $p_i$.

Denote by $R_+$ real half-axis $(0, \infty)$. The above formulas (1)–(3) for nonlinear means can be combined into one.

Let $\phi : R_+ \rightarrow R$ be the real-valued strictly monotonic continuous function. Because of assumptions, there is a unique inverse function $\phi^{-1}$. Define a nonlinear mean $\hat{E}_\phi(X)$ of a random variable $X > 0$ by the expression

$$
\hat{E}_\phi(X) = \phi^{-1}(E\phi(X)),
$$

assuming that there is a finite mathematical expectation $E\phi(X)$.

The function $\phi$ that appears in (7) will be called the defining function. In the case $\phi(x) = \ln x$, $\phi^{-1}(y) = e^y$ we have $\hat{E}_\phi(X) = G(X)$ — the geometric mean. The corresponding defining function we denote $\phi_G$. In the case $\phi(x) = x^p$, $\phi^{-1}(y) = y^{1/p}$ we have $\hat{E}_\phi(X) = E_p(X)$ — the power mean. The corresponding defining function we denote $\phi_p$. In the case $\phi(x) = 1/x$, $\phi^{-1}(y) = y^{-1}$ we have $\hat{E}_\phi(X) = H(X)$ — the harmonic mean. The corresponding defining function we denote $\phi_H$.

The asymptotic and extreme properties of the class of nonlinear mean random variables defined by the formula (7) are discussed below, as well as algebraic properties and inequalities between means of one random variable.

2. Asymptotic and extreme properties

Let $X_1, X_2, ..., X_n$ be a sample of the volume $n$, obtained from several independent observations of random variable $X$. By definition, all random variables $X_1, X_2, ..., X_n$ are independent and distributed equally, as well as the random variable $X$.

As an estimate of the average (7) $\hat{E}_\phi(X)$ consider

$$
\hat{E}_\phi(X_1, X_2, ..., X_n) = \phi^{-1}(\frac{1}{n}\sum_{i=1}^{n} \phi(X_i)).
$$

Example 1. Consider the discrete random variable $X$, having the uniform distribution. Let it takes various positive values $x_1 < x_2 < ... < x_n$ with probabilities $p_i = P(X = x_i) = 1/n$, $i = 1, 2, ..., n$. Then

$$
G(X) = \hat{G}(x_1, x_2, ..., x_n) = \sqrt[n]{x_1 x_2 ... x_n},
$$

$$
H(X) = \hat{H}(x_1, x_2, ..., x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1},
$$

$$
E_p(X) = \hat{E}_p(x_1, x_2, ..., x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p\right)^{1/p},
$$

and the weighted harmonic mean and the weighted power mean with weights $p_i$.
Assuming $\phi(x) = 1/x$ — the geometric mean

$$\hat{H}(X_1, X_2, ..., X_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i} \right)^{-1}, \quad (10)$$

assuming $\phi(x) = x^p$ — the power mean

$$\hat{E}_p(X_1, X_2, ..., X_n) = \left( \frac{1}{n} \sum_{i=1}^{n} X_i^p \right)^{1/p}. \quad (11)$$

In a particular implementation $x_1, x_2, ..., x_n$ of the sample $X_1, X_2, ..., X_n$ the formulas (9)–(11) are the same as (4)–(6). Other mean such as (8) are also found in statistics, for example, trigonometric ([3], Chapt. I, § 7).

Note the relationship between formulae (7) and (8). In (7) the argument of $\phi^{-1}$ from (8) is replaced by the mathematical expectation $E\phi(X)$.

The equivalent of the non-displacement property for the estimate (8) is

**Statement 1.** Let the continuous function $\phi(x)$ be defined and strictly monotoneous on $(0, \infty)$, $X$ — be a positive random value and there is a finite mathematical expectation $E\phi(X)$ of the random value $\phi(X)$. Then

$$E\left(\frac{1}{n} \sum_{i=1}^{n} \phi(X_i)\right) = E\phi(X).$$

Indeed, because of the continuity of the function $\phi$ the random values $\phi(X_i)$ are independent, and because of the strict monotony of $\phi$ they are equally distributed (such as $\phi(X)$). So $E\phi(X_i) = E\phi(X) \ (i = 1, 2, ..., n)$. Then $E\left(\frac{1}{n} \sum_{i=1}^{n} \phi(X_i)\right) = \frac{1}{n} \sum_{i=1}^{n} E\phi(X_i) = E\phi(X)$.

From the statement 1, because of the strict monotonocity of the function $\phi^{-1}$, we have

**Corollary 1.** Under the conditions of statement 1 there is a formula

$$\phi^{-1}(E\left(\frac{1}{n} \sum_{i=1}^{n} \phi(X_i)\right)) = \hat{E}\phi(X).$$

The estimate (8) is consistent, namely the following statement holds

**Statement 2.** $\phi^{-1}(E\left(\frac{1}{n} \sum_{i=1}^{n} \phi(X_i)\right)) \to \hat{E}\phi(X)$ when $n \to \infty$ with probability 1.

The rationale for this fact is based on the following sentence proved in [5].

**Lemma 1.** Let the function $\phi(x)$ continuous and strictly increases (or decreases) on $(0, \infty).$ Let $\xi_1, \xi_2, ..., \xi_n$ - a family of equally distributed, pairwise independent random values. And there is a finite mathematical expectation $E\phi(\xi_1) = b$. Then $\phi^{-1}(\frac{1}{n} \sum_{i=1}^{n} \phi(\xi_i)) \to \phi^{-1}(b)$ when $n \to \infty$ with probability 1.

Note that statement 2 entails the consistency of estimates (9) — (11).

Lemma 1 is a kind of so-called continuity theorems (see e.g., [6] Chapt. I, § 5)

With regard to the effectiveness of evaluation (8) (estimates (9) to (11), respectively), we will give the following:

**Statement 3.** The estimate $\hat{Y}_n = \frac{1}{n} \sum_{i=1}^{n} \alpha_i \phi(X_i)$ is effective in the class $L_n$ of all linear estimates of type $Y_n = \sum_{i=1}^{n} \alpha_i \phi(X_i)$ with real-valued coefficients $\alpha_i$, such as $EY_n = E\phi(X)$. Namely,

$$E(\hat{Y}_n - E\phi(X))^2 \leq E(Y_n - E\phi(X))^2 \ (\forall Y_n \in L_n). \quad (12)$$
Proof relies on the fact of pairwise independence of random values \( \phi(X_i) \) and equality
\[
E\phi(X_i) = E\phi(X) \quad (i = 1, 2, ..., n).
\]
This fact has limited the problem to the search of minimum of function \( \sum_{i=1}^{n} \alpha_i^2 \) with condition \( \sum_{i=1}^{n} \alpha_i = 1 \). Solving this problem gives the required result (see e.g., [7] Chapt. I, § 6).

On the given numerical function \( \phi(x) \) we will define scalar function \( f(x, y) \) — a measure of difference between \( x, y \in R \) by the formula: \( f(x, y) = |\phi(x) - \phi(y)|^2 \). Let \( X, Y \) be a random value. We call the (standard) distance between them the value of \( d_\phi(X, Y) = (E|\phi(X) - \phi(Y)|^2)^{1/2} \). It is easy to see that for \( d_\phi \) all properties of the metric are fulfilled. In a particular case \( \phi(x) = x \) is the square distance mean between random values of \( X \) and \( Y \).

In these terms, statement 3 entails

**Corollary 2.** The estimate \( Z_n = \phi^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \phi(X_i)\right) \) is effective on the distance \( d_\phi \) in the class of all estimates such as \( Z_n = \phi^{-1}(Y_n) \) when \( Y_n \in L_n \). Namely,
\[
d_\phi(\bar{Z}_n, \bar{E}_\phi(X)) \leq d_\phi(\phi^{-1}(Y_n), \bar{E}_\phi(X)) \quad (\forall Y_n \in L_n).
\]

Consider the extreme properties of nonlinear mean characteristics of the form (7).

**Statement 4.** For a given random value of \( X > 0 \), the value of \( a_0 = \phi E_\phi(X) \) minimizes the mean squared error of \( E(\phi(X) - a)^2 \) at \( a \in R \).

Indeed, according to the definition \( a_0 = \phi E_\phi(X) = E\phi(X) \). Then, according to the extreme property of mathematical expectations (see, for example, [8] chapt. I, § 4), applied to a random value \( \phi(X) \), we get
\[
\min_a E(\phi(X) - a)^2 = E(\phi(X) - E\phi(X))^2.
\]

Which is what the statement says.

**Corollary 3.** The mean squared deviation of a natural logarithm of a positive random value \( (\ln X) \) from a natural logarithm of a geometric mean \( (\ln G(X)) \) is not greater than the mean squared deviation of \( \ln X \) from any other real constant. The mean squared deviation of \( p \)-th degree of random value \( X \) \( (X^p) \) from the initial moment \( (EX^p) \) does not exceed the mean square deviation of \( X^p \) from any real constant. The mean square deviation of \( X^{-1} \) from the inverse of the harmonic mean \( H(X)^{-1} \) is not more than the mean square deviation of \( X^{-1} \) from any other real constant.

For discrete analogues (4)–(6) the corresponding properties are marked, for example, in [3] chapt. I, § 5.

The result about extreme properties can be interpreted within the framework of the theory of mean random variables relative to some measure of difference. Let some function \( \phi(x) \), a random variable \( X \) and a numerical parameter \( y \) be given. Consider the extreme problem: to find \( \min_{y \in R} E|\phi(X) - \phi(y)|^2 \).

The solution of this problem \( E(X, f) \) is called the mean of the random value \( X \) relative to the measure of difference \( f(x, y) = |\phi(x) - \phi(y)|^2 \) (see [9], Chapt. 5, § 5.5).

According to statement 4, the non-linear average of \( E_\phi(X) \) is the mean of a random variable \( X \) relative to the measure of difference \( f(x, y) = |\phi(x) - \phi(y)|^2 \). In particular, for the defining function \( \phi_G(x) = \ln x \), the comparison function \( f_G(x, y) = |\ln x - \ln y|^2 \) and the mean \( E(X, f_G) \) coincide with the geometric mean \( G(X) \) as compared.

Dwell on the question about estimation of the given (predicted) random value \( Y \) using the function of another (predicted) random value \( Z \). As is known ([8] Chapt. II, § 8), the best estimate of \( Z \) in the RMSE sense is a function of regression \( E(Y|Z) \), i.e., a conditional expectation. Consider the question about the best estimate concerning the measure of difference \( f(x, y) = |\phi(x) - \phi(y)|^2 \) for the given defining function \( \phi \).
Let $X$ and $Z$ be random values. Let’s consider a random value $\tilde{E}_\phi(X|Z) := \phi^{-1}(E((\phi(X))/Z))$. It can be considered as a function of $Z$.

**Statement 5.** The function $\tilde{E}_\phi(X|Z)$ is the optimal predictive function of a random value $X$ in terms of the squared mean distance $d_\phi$ as the difference between $f(x, y) = |\phi(x) - \phi(y)|^2$.

Namely, for any Borel function $\psi(Z)$ the following inequality is correct

$$d_\phi(X, \tilde{E}_\phi(X|Z)) \leq d_\phi(X, \psi(Z)).$$

In fact, let $X$ and $Z$ be random values. Put $Y = \phi(X)$. Then $\phi(\tilde{E}_\phi(X|Z)) = E(Y|Z)$.

Let $\psi(Z)$ be an arbitrary Borel function of a random value $Z$. By the regression property we have $E(Y - E(Y|Z))^2 \leq E(Y - \phi(\psi(Z))^2$. According to the definition of $Y$, we have (13) fulfilled.

**Corollary 4.** The ratio (13) is correct for geometric mean, power mean and harmonic mean in case of defining functions $\phi_G$, $\phi_p$ and $\phi_H$.

**Example 2.** Consider two independent discrete random variables $X$ and $Y$, taking values $x_i$ ($i = 1,...,n$) with probabilities of $p_i$ and, accordingly, values of $y_j$ ($j = 1,...,m$) with probabilities of $q_j$. The product $XY$ is a random variable that takes all the values $x_iy_j$ ($i = 1,...,n$) ($j = 1,...,m$) with probabilities of $p_iq_j$. Therefore, for any $\alpha > 0$ we have

$$E(XY)^\alpha = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_iy_j)^\alpha p_iq_j =$$

$$= (\sum_{i=1}^{n} x_i^\alpha p_i)(\sum_{j=1}^{m} y_j^\alpha q_j) = (EX^\alpha)(EY^\alpha).$$

From above-mentioned we have

$$E_\alpha(XY) = E_\alpha(X)E_\alpha(Y).$$

**Remark 1.** As is known, for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ the Helder’s inequality is fulfilled (See, [7] Chapt. II, §6). In our designations it takes the form $E(XY) \leq E_p(X)E_q(Y)$.

For the harmonic mean it leads to

$$H(XY) = E\left(\frac{1}{X/Y}\right)^{-1} \geq (E_p\left(\frac{1}{X}\right))^{-1}(E_q\left(\frac{1}{Y}\right))^{-1} = (H(X^p))^{1/p}(H(Y^q))^{1/q}.$$

### 3. Algebraic properties and inequalities

Below we will list the properties of nonlinear characteristics (1)–(3) close to the properties of mathematical expectations and point out the differences.

**Property 1.** The value of the mean (7) from the positive constant — there is this constant itself.

Indeed, let $C > 0$ be some constant. The stated property follows from the formula (7) by the calculation

$$\tilde{E}_\phi(C) = \phi^{-1}(E\phi(C)) = \phi^{-1}(\phi(C)) = C.$$

Specifically, $G(C) = C$, $H(C) = C$ and $E_p(C) = C$.

**Property 2 (homogeneity) $G(X)$, $H(X)$ and $E_p(X)$ are positive homogeneous.**

Give a reasonings for the mean harmonic. Let $X$ be a positive random value. For any $k > 0$ we can write

$$H(kX) = \frac{1}{E(1/kX)} = \frac{1}{E(1/kX)} = \frac{1}{kE(1/X)} = kH(X).$$
For the mean geometric this statement will follow from the multiplicity property. For the power mean — from its definition.

**Property 3 (multiplicity).** For arbitrary positive random variables \( X \) and \( Y \), with the condition of existence of the mathematical expectations of all above random values, there is equality \( G(XY) = G(X)G(Y) \). With additional assumption of independence of random values \( X \) and \( Y \), the equations \( H(XY) = H(X)H(Y) \) and \( E_p(XY) = E_p(X)E_p(Y) \) are hold.

In fact, for the geometric mean we get

\[
G(XY) = e^{E(\ln(XY))} = e^{E(\ln X + \ln Y)} = e^{E\ln X + E\ln Y} = e^{E\ln X} \cdot e^{E\ln Y} = G(X)G(Y).
\]

In particular, if \( Y = C \) is constant, then \( G(XC) = G(X) \cdot G(C) = G(X) \cdot C \).

Let’s consider this property for the harmonic mean. Let the random values \( X > 0 \) and \( Y > 0 \) be independent.

Then

\[
H(XY) = E\left(\frac{1}{XY}\right)^{-1} = (E\left(\frac{1}{X} \cdot \frac{1}{Y}\right))^{-1} = ((E\frac{1}{X})(E\frac{1}{Y}))^{-1} = H(X) \cdot H(Y).
\]

Here we use the independence \( \frac{1}{X} \) and \( \frac{1}{Y} \), following from the independence of \( X \) and \( Y \).

The same calculation is valid for the power mean.

Property 3 shows that the geometric mean is convenient to use in multiplicative models.

Note that of the discrete analogues, property 3 is true only for the geometric mean (4).

Explain the essentiality of the independence condition in property 3 by the example of the power mean.

**Property 4. (monotonous growth)** Let the random values \( X \) and \( Y \) be such that \( X > Y > 0 \), and there are finite mathematical expectations \( \phi(X) \) and \( \phi(Y) \). Then the inequality \( \tilde{E}_\phi(X) > \tilde{E}_\phi(Y) \) is fulfilled for the corresponding means (7).

Indeed, at first assume that the defining function \( \phi \) grows monotonically.

Since \( X > Y \), then due to the monotonous growth of the function \( \phi \), we have \( \phi(X) > \phi(Y) \).

And following the property of mathematical expectations \( E(\phi(X)) > E(\phi(Y)) \). Then, due to the monotonous growth of \( \phi^{-1} \) we get the required result.

Analogous case is considered when the function \( \phi \) decreases monotonically: if \( X > Y \), then \( \phi(X) < \phi(Y) \). Then \( E(\phi(X)) < E(\phi(Y)) \) and due to the monotonous decrease of the function \( \phi^{-1} \) we get \( \phi^{-1}(E(\phi(X))) > \phi^{-1}(E(\phi(Y))) \). Q.E.D.

**Corollary 5.** The values \( G(X) \), \( H(X) \) and \( E_p(X) \) monotonically increase in the class of positive random variables.

Note that the above properties 1 – 4 are similar to the properties of mathematical expectations.

**Corollary 6.** Let \( X \) and \( Y \) be positive random variables. Then

\[
2\tilde{E}_\phi(\min\{X,Y\}) \leq \tilde{E}_\phi(X + Y) \leq 2\tilde{E}_\phi(\max\{X,Y\})
\]

assuming that all the necessary finite mathematical expectations are exist.

**Corollary 7 (the equivalent of Cauchy inequality).** Let \( X \) and \( Y \) be positive random variables. For the geometric mean, as well as for the harmonic and power mean, with an additional assumption of independence of \( X \) and \( Y \), the inequality \( \tilde{E}_\phi(X + Y) \geq 2\tilde{E}_\phi(\sqrt{X})\tilde{E}_\phi(\sqrt{Y}) \) holds for the corresponding defining functions \( \phi_G, \phi_H, \phi_p \).

This assumes the existence of all the necessary mathematical expectations.
Consider the case of the geometric mean. Since \( X + Y \geq 2\sqrt{XY} \) and following the property 4 we have \( G(X + Y) \geq G(2\sqrt{XY}) \). So

\[
G(2\sqrt{XY}) = e^{E\ln(2\sqrt{XY})} = e^{E(\ln 2 + \ln \sqrt{X} + \ln \sqrt{Y})} = e^{E\ln 2} \cdot e^{E\ln \sqrt{X}} \cdot e^{E\ln \sqrt{Y}} = 2G(\sqrt{X})G(\sqrt{Y}),
\]

which leads to the statement made in the case of the geometric mean.

Let’s suppose additionally that random values \( X \) and \( Y \) are independent. Show these reasonings for the power mean. Since \( X + Y \geq 2\sqrt{XY} \), then taking into account property 4 we get \( E_p(X + Y) \geq E_p(2\sqrt{XY}) = 2E_p(\sqrt{X})E_p(\sqrt{Y}) \). The last equality is valid by virtue of property 3. It is similar for the harmonic mean. Since \( X \) and \( Y \), which leads to the statement made in the case of the geometric mean.

Remark 2. For the power mean at \( p > 1 \) the inequality of Minkowski holds ([8] Chapt. II, § 6) \( E_p(X + Y) \leq E_p(X) + E_p(Y) \). At additional assumption about independence of \( X \) and \( Y \) corollary 7 for power mean gives an estimation of \( E_p(X + Y) \) from below.

Consider the inequalities between different nonlinear means of the same random value.

First of all, we will give general statements about the ratio of non-linear means (7) and mathematical expectation \( EX \). Note that the defining functions \( \phi_G, \phi_H \) and \( \phi_p \), except for the properties of strict monotony, have certain properties of convexity. In this connection, the Jensen’s inequality is used here (see, for example, [8], Chapt. II, § 6). Namely: if \( \phi(x) \) is convex up (concave) Borel function, then for any random value \( X \), for which the mathematical expectation exists, the inequality \( E(\phi(X)) \leq \phi(EX) \) is executed; if \( g(x) \) is convex down Borel function, then the opposite inequality \( E(g(X)) \geq g(EX) \) takes place.

Statement 5. Let the defining function \( \phi \) in (7) be continuous, convex up and strictly increases monotonically on \((0, +\infty)\). Then for any positive random value of \( X \) there is inequality \( E_p(X) \leq EX \). Let the defining function convex down and increase strictly monotonically on \((0, +\infty)\). Then \( \hat{E}_p(X) \geq EX \). It is assumed that all the necessary mathematical expectations exist.

Let’s do the argumentation for one of the variants. For example, let the function \( \phi \) be upward and increase strictly monotonically on \((0, +\infty)\). Let \( X \) be an arbitrary positive random value and mathematical expectations \( EX \) and \( E(\phi(X)) \) are exist. By assumption \( EX > 0 \), and by Jensen’s inequality \( E(\phi(X)) \leq \phi(EX) \). Then, due to the strictly monotonous decrease of the function \( \phi^{-1} \), we get \( \hat{E}_p(X) \leq EX \).

Corollary 8. Assuming that all the necessary mathematical expectations exist for a positive random value of \( X \), we have \( G(X) \leq EX, E_p(X) \leq EX \) at \( p < 1 \), \( H(X) \leq EX \) and \( E_p(X) \geq EX \) at \( p > 1 \).

The general property of the majority of the means is known (8). Namely, for the specified defining functions \( \phi(x) \) and \( \psi(x) \) there is inequality between the sample averages of the form (8) \( \hat{E}_\phi \) and \( \hat{E}_\psi \)

\[
\hat{E}_\phi(X_1, ..., X_n) > \hat{E}_\psi(X_1, ..., X_n),
\]

if the major condition is satisfied (see e.g. [9], section 2, p. 30).

\[
\frac{\phi''(x)}{\phi'(x)} > \frac{\psi''(x)}{\psi'(x)} \quad (\forall x > 0).
\]  

(14)

Statement 6. Let the condition (14) for the defining functions \( \phi(x) \) and \( \psi(x) \) be fulfilled, and \( X \) be a positive random value. Then, for nonlinear means (7), the following inequality is valid

\[
\hat{E}_\phi(X) \geq \hat{E}_\psi(X).
\]  

(15)
Indeed, in the context of the law of large numbers (statement 2), inequality (15) follows from (14) by means of passage to the limit.

**Corollary 9.** Let $X$ be a positive random value. For non-linear means (1)–(3), assuming that all the necessary mathematical expectations exist, the ratio

$$H(X) \leq G(X) \leq E(X) \leq E_p(X)$$

with $p > 1$.

The ratio (16) is analogous to the famous inequality of

$$\hat{H}(X) \leq \hat{G}(X) \leq \hat{E}(X) \leq \hat{E}_p(X)$$

between the empirical means (4)–(6) (see e.g. [2] Chapt. II, § 2.5, [3] Chapt. I, § 7). Here $\hat{E}(X)$ is the power mean with $p = 1$, i.e. the mathematical expectation.

The limit relationship between the means of $E_p(X)$ and $G(X)$ describes

**Statement 7.** For an arbitrary positive random value $X$, for which the mathematical expectations $E(X)$ and $E(\phi_G(X))$ are finite, the ratio of $\lim_{p \to 0} E_p(X) = G(X)$ is valid.

Give a scheme of proof. Note that because of the inequality of moments from the existence of $E(X)$ follows the existence of $E(X^p)$ at $p < 1$. Consider $\ln E_p(X) = \frac{1}{p} \ln E(X^p)$. Calculating the limit from the right side at $p \to 0$, using the Lopital rule, we get

$$\lim_{p \to 0} \frac{\ln(E(X^p))}{p} = E \ln X.$$

Then

$$\lim_{p \to 0} E_p(X) = \lim_{p \to 0} e^{\ln E_p(X)} = \lim_{p \to 0} \ln E_p(X) = \ln X = G(X).$$

This result is similar to well-known for discrete means (6) and (4) (see e.g. [3] Chapt. I, § 2).

4. Conclusion

In the present paper, the general class of nonlinear random variables of the form (7) is selected and studied. In particular, new properties of mean geometric and mean harmonic random variables have been established. These results can be useful for statistical analysis of economic processes. It is important to identify the connection between the class of nonlinear means introduced by us and the well-known class of associative means (8). Note that the study of the mean of the form (8) actively continues (see, for example, [11, 12]). Other aspects of the theory of nonlinear means are presented in [13] and [14]. The results of this work allow the development of nonlinear mean fuzzy random variables (cf. [15]).

5. References

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