Galerkin finite element methods for the numerical solution of two classical-Boussinesq type systems over variable bottom topography

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Abstract

We consider two ‘Classical’ Boussinesq type systems modelling two-way propagation of long surface waves in a finite channel with variable bottom topography. Both systems are derived from the 1-d Serre-Green-Naghdi (SGN) system; one of them is valid for stronger bottom variations, and coincides with Peregrine’s system, and the other is valid for smaller bottom variations. We discretize in the spatial variable simple initial-boundary-value problems (ibvp’s) for both systems using standard Galerkin-finite element methods and prove $L^2$ error estimates for the ensuing semidiscrete approximations. We couple the schemes with the 4th order-accurate, explicit, classical Runge-Kutta time-stepping procedure and use the resulting fully discrete methods in numerical simulations of dispersive wave propagation over variable bottoms with several kinds of boundary conditions, including absorbing ones. We describe in detail the changes that solitary waves undergo when evolving under each system over a variety of variable-bottom environments. We assess the efficacy of both systems in approximating these flows by comparing the results of their simulations with each other, with simulations of the SGN-system, and with available experimental data from the literature.

Keywords: Boussinesq systems, surface dispersive long-wave propagation, variable bottom topography, Galerkin finite element methods, Error estimates, solitary waves

2020 MSC: 65M60, 65M12

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1 Introduction

The ‘Classical’ Boussinesq system, [1], in one spatial dimension is the nonlinear, dispersive system of pde’s

\[
\begin{align*}
\zeta_t + u_x + \varepsilon(\zeta u)_x &= 0, \\
u_t + \zeta_x + \varepsilon uu_x - \frac{\mu}{3} u_{xxx} &= 0.
\end{align*}
\] (CB)

It has been derived, cf. e.g. [1], as an approximation of the two-dimensional Euler equations of water-wave theory, and models two-way propagation of long waves of small amplitude on the surface of an ideal fluid (say, water) in a horizontal channel of finite depth. The variables in (CB) are nondimensional and scaled: \(x\) and \(t\) are proportional to length along the channel and time, respectively and the function \(\varepsilon \zeta(x,t)\) represents the free surface elevation of the water above a level of rest at \(z = 0\). (Here \(z\) is proportional to the depth variable and is taken positive upwards). The function \(u = u(x,t)\) is the depth-averaged horizontal velocity of the fluid. The scaling parameters \(\varepsilon, \mu\) are defined as \(\varepsilon = \frac{A}{h_0}\), where \(A\) is a typical amplitude of the surface wave and \(h_0\) is the depth of the channel, and as \(\mu = \frac{\lambda^2}{h_0^2}\), where \(\lambda\) is a typical wavelength of the waves. The assumptions behind the derivation of (CB) are that \(\varepsilon \ll 1\), \(\mu \ll 1\), and that \(\varepsilon\) and \(\mu\) are related so that \(\varepsilon = O(\mu)\), i.e. are in the so-called Boussinesq scaling regime.

The first pde in (CB) is exact while the second is an \(O(\varepsilon^2)\) approximation to a relation obtained from the Euler equations. It is to be noted that in the variables of (CB), the horizontal bottom lies at \(z = -1\).

The initial-value problem for (CB) with initial data \(\zeta(x,0) = \zeta_0(x), u(x,0) = u_0(x)\) on the real line has been studied by Schonbek [2] and Amick [3], who established global existence and uniqueness of smooth solutions under the assumption that \(1 + \varepsilon \inf_x \zeta_0(x) > 0\). One conclusion of this theory is that for all \(t \geq 0\), \(1 + \varepsilon \inf_x \zeta(x,t) > 0\), i.e. that there is always water in the channel. Existence-uniqueness of solutions globally in time in Sobolev spaces were established in [1]. The initial-boundary-value problem (ibvp) for (CB) posed on a finite interval, say \([0,1]\), with zero boundary conditions for \(u\) at \(x = 0\) and \(x = 1\), and no boundary conditions for \(\zeta\), was proved in [5] to possess global weak (distributional) solutions.

The system (CB) has been used and solved numerically extensively in the engineering literature. We will refer here just to [6] and [7] for error estimates of Galerkin-finite element methods for the ibvp for (CB) mentioned above and a computational study of the properties of the solitary-wave solutions of the system. For the numerical analysis of the periodic ivp we refer to [8].

In this paper we will be interested in the numerical solution of extensions of (CB) valid in channels of variable-bottom topography. Several such extensions have been derived in the literature. Here we will follow [9] and consider two specific such variable-bottom models that may be derived from the Serre-Green-Naghdi (SGN) system of equations, [10, 11, 12]; for their derivation and theory of their validity we refer to [9] and [13] and their references.
In order to describe the topography of the bottom, in addition to \( \varepsilon \) and \( \mu \) we consider the scaling parameter \( \beta \) defined by \( \beta = \frac{B}{h_0} \), where \( B \) is a typical bottom topography variation and \( h_0 \) is now a reference depth. In scaled nondimensional variables consistent with those in (CB) the Serre-Green-Naghdi equations are written as

\[
\begin{align*}
\zeta_t + (\eta_u)_x &= 0, \\
\left(1 + \frac{\mu}{\eta} \mathcal{T}[\eta, \beta b]\right) u_t + \zeta_x + \varepsilon u u_x \\
&+ \mu \varepsilon \left\{-\frac{1}{3\eta} (\eta^3 (w u_{xx} - (u_x)^2)_x + Q[\eta, \beta b] u)\right\} = 0,
\end{align*}
\] 

(SGN)

where the operators \( \mathcal{T}[\eta, \beta b] \), \( Q[\eta, \beta b] \) are defined by

\[
\begin{align*}
\mathcal{T}[\eta, \beta b] w &= -\frac{1}{3} (\eta^3 w_x)_x + \frac{\beta}{2} [(\eta^2 b')_x - \eta^2 b' w_x] + \beta^2 \eta (b')^2 w, \\
Q[\eta, \beta b] w &= \frac{\beta}{2\eta} \left\{ (\eta^2 w w_x b')_x + (\eta^2 b'' w^2)_x - \eta^2 [w w_{xx} - w_x^2] b' \right\} \\
&+ \beta^2 b' b'' w^2 + \beta^2 w w_x (b')^2.
\end{align*}
\]

In these variables the bottom topography is given by \( z = -\eta_b(x, t) \), where \( \eta_b(x) = 1 - \beta b(x) \) and \( b \) is assumed to be a \( C^2 \) function. Since the free surface is at \( z = \varepsilon \zeta(x, t) \), cf. Figure 1, the water depth \( \eta \) in (SGN) is given by \( \eta = \varepsilon \zeta + \eta_b \).

The assumptions under which (SGN) is a valid approximation to the 2d-Euler equations are, cf. [9],

\[
\mu \ll 1, \quad \varepsilon = \mathcal{O}(1), \quad \beta = \mathcal{O}(1).
\] 

(1.1)

It may then be seen that the second pde in (SGN) is formally an \( \mathcal{O}(\mu^2) \) approximation of an analogous expression for the Euler equations. (The first pde is exact.) It will be also assumed that the bottom never reaches the undisturbed surface i.e. that \( \eta_b(x) = 1 - \beta b(x) > 0 \), for all \( x \). It will also be assumed that at \( t = 0 \) the water depth \( \eta \) is positive. Part of the theory of existence-uniqueness
of solutions of the initial-value problem for (SGN) is to prove that the data is such that $\eta$ remains positive for the duration of existence of solutions. This is what is proved locally in time and in some generality in $[13]$. Also, in the case of the 1d (SGN) as given above, a local in time theory of existence and uniqueness of solutions of the ivp with energy methods has been given by Israwi, $[14]$.

The model (SGN) has been used in many computational studies of long-surface wave propagation over uneven bottoms. We refer, for example, to $[15]$, $[16]$, and $[17]$ and their references for computations with finite differences and finite volume methods, and to $[18]$ for a finite element scheme. An error analysis of the Galerkin-finite element method in the case of a horizontal bottom (i.e. when $\beta = 0$), appears in $[19]$ in the case of the periodic ivp.

As was mentioned previously, our aim in this paper is to consider two simplifications of (SGN) that are variable-bottom extensions of (CB). The derivation of the first of those systems, in addition to (1.1), is made under the Boussinesq scaling hypothesis that $\varepsilon = O(\mu)$ (we usually take $\varepsilon = \mu$), and allows arbitrary bottom topography, i.e. $\beta = O(1)$, cf. $[9]$. When we take $\varepsilon = \mu$ in the second pde in (SGN) and ignore $O(\mu^2)$ terms (thus retaining the formal accuracy of (SGN) as approximation of the Euler equations), it is not hard to see that $\eta = \eta_b + O(\mu)$ and that $\frac{\eta}{\eta} = \frac{\eta}{\eta} + O(\mu^2)$. Therefore $\frac{\eta}{\eta} T[\eta, \beta b] w = \frac{\eta}{\eta} T[\eta, \beta b] w + O(\mu^2)$. So, since $\mu \varepsilon = O(\mu^2)$, if we ignore $O(\mu^2)$ terms the second pde in (SGN) becomes

$$\left(1 + \frac{\mu}{\eta} T[\eta, \beta b]\right) u_t + \zeta_x + \varepsilon uu_x = 0.$$  

Together with the first pde in (SGN) we obtain therefore a simplified system of equations that incorporates the hypothesis $\varepsilon = O(\mu)$ but allows $\beta = O(1)$. This system will be called ‘Classical’ Boussinesq system with strongly varying bottom topography and abbreviated as (CBs). It is given by the pde’s

$$\zeta_t + (\eta u)_x = 0,$$

$$\left(1 + \frac{\mu}{\eta} T[\eta_b, \beta b]\right) u_t + \zeta_x + \varepsilon uu_x = 0,$$  

where $\eta = \eta_b + \varepsilon \zeta > 0$, $\eta_b = 1 - \beta b > 0$, $\varepsilon = O(\mu) \ll 1$, and $T[\eta, \beta b] w$ is given by its expression in (SGN) when we replace $\eta$ by $\eta_b$.

This system, as a little algebra shows, coincides with the system that was first derived from the Euler equations by Peregrine in $[20]$; it is usually called the ‘Peregrine system’ in the literature and has been used widely in practice in coastal dynamics computations. We will refer to several computational studies with (CBs) in Section 3 of the present paper. If we now assume in (CBs) following $[9]$ that $\beta = O(\varepsilon)$, i.e. that the variation of bottom is small and specifically of the order $\varepsilon$ of the nonlinear and dispersion terms in (CBs), we obtain a system that we will call here the ‘Classical’ Boussinesq system with weakly varying bottom topography, (CBw). It is straightforward to see that if $\beta = O(\varepsilon)$ the first equation in (CBs) remains intact and that the second equation, up to $O(\varepsilon^2)$ terms that we neglect, coincides with the second equation

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in \((\text{CB})\). Thus we have the system
\[
\begin{align*}
\zeta_t + (\eta u)_x &= 0, \\
u_t + \zeta_x + \varepsilon uu_x - \frac{\mu}{3} u_{xxx} &= 0,
\end{align*}
\]
where of course we still assume that \(\varepsilon = O(\mu), \mu \ll 1\). The dependence on the bottom topography occurs now explicitly (but weakly) through the first pde only, since \(\eta = \eta_b + \varepsilon \zeta = 1 - \beta b + \varepsilon \zeta\) with \(\beta = O(\varepsilon)\). This system has also been used widely in computations in the engineering literature, and coincides with the system derived in [21]. It should be noted that another rigorous derivation of the two variable-bottom ‘Classical’ Boussinesq systems and various other associated models has been given in [22].

The theory of existence and uniqueness of solutions, at least locally in time, for the ivp for \((\text{CBs})\) may be easily inferred from the analogous theory of \((\text{SGN})\), cf. e.g. [14], while that of \((\text{CBw})\) is practically the same as the one for \((\text{CB})\) plus a ‘source’-type linear term of the form \(-\beta (bu)_x\) in the left-hand side of the first pde.

In this paper we will discretize in space ibvp’s for the systems \((\text{CBs})\) and \((\text{CBw})\), with zero b.c. for \(u\) at the endpoints of \([0, 1]\) and no b.c. for \(\zeta\), by the standard Galerkin-finite element method on a quasiuniform mesh and prove \(L^2\)-error estimates in Section 2 for the resulting semidiscretizations. Under certain standard assumptions on the finite element spaces we will prove error estimates of the form
\[
\|\zeta - \zeta_h\| + \|u - u_h\| \leq Ch^{r-1},
\]
where \(\zeta_h, u_h\) are the semidiscrete approximations of \(\zeta\) and \(u\), respectively, \(h = \max_i h_i\), and \(r - 1 \geq 2\) is the degree of the piecewise polynomials in the finite element space. (\(\|\cdot\|\) and \(\|\cdot\|_1\) denote, respectively the \(L^2\) and \(H^1\) norms of functions on \([0, 1]\).) This type of error estimate is of the same type as the one proved in [6] for the analogous ibvp for \((\text{CB})\) in the case of a quasiuniform mesh.

In Section 3 we show the results of several numerical experiments that we performed with both systems using a fully discrete scheme with the above spatial discretization and with temporal discretization effected by the classical, \(4^{th}\) order, 4-stage Rugne-Kutta method. The resulting schemes are stable under a mild Courant number restriction and highly accurate. In Section 3.1 we check that the schemes also work for piecewise linear continuous functions (i.e. for \(r = 2\) and are of optimal order in \(L^2\) for both \(u\) and \(\zeta\) in the case of uniform mesh. In Section 3.2 we discuss the application of simple, approximate, absorbing boundary conditions for the systems as an alternative to the reflection b.c. \(u = 0\) at the endpoints. In Section 3.3 we perform a series of numerical experiments aimed at describing in detail the changes that solitary waves undergo when evolving under \((\text{CBs})\) or \((\text{CBw})\) in a variety of variable-bottom environments. We assess the efficacy of these systems in approximating these flows by comparing them with each other and with the \((\text{SGN})\) system and available experimental data. In the Ph.D. thesis of the first listed author, [23], one may find more details on the theory behind, and more numerical experiments with
these systems, as well as with related models of surface water wave propagation over variable bottom.

In the sequel, we denote, for integer \( k \geq 0 \), \( C^k = C^k[0,1] \) the spaces of \( k \)-times continuously differentiable functions on \([0,1]\) and by \( H^k = H^k(0,1) \) the usual \( L^2 \)-based Sobolev spaces of functions on \((0,1)\). \( \tilde{H}^1 \) will denote the elements of \( H^1 \) which vanish at \( x = 0 \) and \( x = 1 \). The inner product in \( L^2 = L^2(0,1) \) will be denoted by \((\cdot,\cdot)\), its norm by \( \| \cdot \| \), and the norm on \( H^k \) by \( \| \cdot \|_k \). The norms on \( W^k_\infty \) and \( L^\infty \) on \((0,1)\) are denoted by \( \| \cdot \|_{k,\infty} \) and \( \| \cdot \|_\infty \), respectively. \( P_r \) are the polynomials of degree at most \( r \).

2 Error analysis of the Galerkin semidiscretization

2.1 The finite element spaces

Let \( 0 \leq x_1 < x_2 < \ldots < x_{N+1} = 1 \) be a quasiuniform partition of \([0,1]\) with \( h := \max(x_{i+1} - x_i) \). For integers \( r \geq 2 \) and \( 0 \leq k \leq r - 2 \) we consider the finite element space \( S_h = \{ \phi \in C^k : \phi \mid_{[x_i,x_{i+1}]} \in \mathbb{P}_{r-1} \} \) and \( S_{h,0} = \{ \phi \in S_h : \phi(0) = \phi(1) = 0 \} \). It is well known that if \( w \in H^r \) there exists \( \chi \in S_h \) such that

\[
\| w - \chi \| + h \| w' - \chi' \| \leq C h^r \| w \|_r \tag{2.1}
\]

for some constant \( C \) independent of \( h \) and \( w \), and that a similar property holds in \( S_{h,0} \). In addition, if \( P \) is the \( L^2 \) projection operator onto \( S_h \), then it holds, cf. [24], that

\[
\| P v \|_{\infty} \leq C \| v \|_{\infty}, \quad \forall v \in L^\infty, \tag{2.2a}
\]

\[
\| P v - v \|_{\infty} \leq C h^r \| v \|_{r,\infty}, \quad \forall v \in C^r. \tag{2.2b}
\]

Due to the quasiuniformity of the mesh, the inverse inequalities

\[
\| \chi \|_1 \leq C h^{-1} \| \chi \|, \quad \| \chi \|_\infty \leq C h^{-1/2} \| \chi \| \tag{2.3}
\]

are valid for \( \chi \in S_h \) (or \( \chi \in S_{h,0} \)).

2.2 Semidiscretization in the case of a strongly varying bottom

Using the notation of the Introduction we consider the following initial-boundary-value problem (ibvp) for [C8]. For \( T > 0 \) we seek \( \zeta = \zeta(x,t) \), \( u = u(x,t) \), for \((x,t) \in [0,1] \times [0,T] \), such that

\[
\zeta_t + (\eta u)_x = 0, \\
\left(1 + \frac{\mu}{\eta b} T[\eta b, \beta b]\right) u_t + \zeta_x + \varepsilon uu_x = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\
\zeta(x,0) = \zeta_0(x), \quad u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \\
u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq T, \tag{2.4}
\]

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where \( \zeta_0, u_0 \) are given functions of \([0, 1]\) and
\[
\eta = \varepsilon \zeta + \eta_b > 0, \quad \eta_b(x) = 1 - \beta b(x) > 0,
\]
\( \varepsilon, \mu, \beta, \) are positive constants with \( \varepsilon = O(\mu), \mu \ll 1, \beta = O(1), \) and \( b \in C^2[0, 1]. \)

The operator \( T[\eta_b, \beta b] \) is defined as in Section [1] by
\[
T[\eta_b, \beta b]w = -\frac{1}{3}(\eta_b^3 w_x)_x + \frac{\beta}{2}[(\eta_b^2 b')' + \eta_b^2 b' w_x] + \beta^2 \eta_b(b')^2 w.
\]

All the variables above are nondimensional and scaled. We will assume that there are positive constants \( \varepsilon, \beta, \mu_1 \) such that
\[
\varepsilon, \beta, \mu_1 > 0 \quad \text{and} \quad \beta \eta_b(b') \ll 1, \text{ for } x \in [0, 1], \tag{2.5}
\]
and that \( \varepsilon \eta_b(b') = -\beta b', \) we have
\[
T[\eta_b, \beta b]w = -\frac{1}{3}(\eta_b^3 w_x)_x - \frac{1}{2}\eta_b^2 \eta_b'' w. \tag{2.5}
\]

Using in first pde of (2.4) the definition of \( \eta, \) multiplying the second pde by \( \eta_b \), and taking into account (2.5), we rewrite the ibvp (2.4) in the form
\[
\zeta_t + \varepsilon (\zeta u)_x + (\eta_b u)_x = 0,
\]
\[
(\eta_b - \frac{\mu}{2} \eta_b'' u) u_t - \frac{\mu}{3} (\eta_b^3 u u_x)_x + \eta_b \zeta_x + \varepsilon \eta_b u u_x = 0, \quad (x, t) \in [0, 1] \times [0, T],
\]
\[
\zeta(x, 0) = \zeta_0(x), \quad u(x, 0) = u_0(x), \quad x \in [0, 1],
\]
\[
u(0, t) = u(1, t) = 0, \quad t \in [0, T]. \tag{2.6}
\]

We assume that there are positive constants \( c_1 \) and \( c_2 \) such that
\[
\eta_b(x) \geq c_1, \tag{2.7a}
\]
\[
\eta_b(x) - \frac{\mu}{2} \eta_b''(x) \eta_b''(x) \geq c_2, \tag{2.7b}
\]
for all \( x \in [0, 1]. \) Since \( \eta_b \) and its derivatives are \( O(1), \) (2.7b) holds for \( \mu \) sufficiently small. We also consider the bilinear form \( A : H^1_0 \times H^1_0 \to \mathbb{R} \) defined by
\[
A(v, w) = \left( (\eta_b - \frac{\mu}{2} \eta_b'' v) v, u \right) + \frac{\mu}{3} \eta_b^3 v', \tag{2.8}
\]
which is symmetric, bounded on \( H^1 \times H^1, \) and, because of (2.7), coercive, with
\[
A(v, v) \geq c_2 \|v\|^2 + \frac{\mu c_3}{3} \|v'\|^2 \geq c_4 \|v\|^2, \quad \forall v \in H^1, \tag{2.9}
\]
where \( c_4 = \min(c_2, \mu c_3/3). \) Consider now a weighted \( H^1 \) (‘elliptic’) projection associated with the bilinear form (2.9) as the map \( R_h : H^1 \to S_{1,0} \) defined by
\[
A(R_h v, \chi) = A(v, \chi), \quad \forall \chi \in S_{1,0}. \tag{2.10}
\]
for which, cf. e.g. [24], it holds that

\[ \| R_h v - v \| + h R_h v - v \| \leq C h^r \| v \|, \quad \text{if } v \in H^r \cap H^1_0, \]

\[ \| R_h v - v \| \leq C h^r \| v \|_{r, \infty}, \quad \text{if } v \in W^{r, \infty} \cap H^1_0. \]

We now define the standard Galerkin finite element semidiscretization of the
ibvp (2.6). We seek \( \zeta_h : [0, T] \rightarrow S_h, u_h : [0, T] \rightarrow S_{h,0} \) such that

\[ (\zeta_{ht}, \phi) + \varepsilon ((\zeta_h u_h, \phi)_x, \phi) + ((\eta_h u_h)_x, \phi) = 0, \quad \forall \phi \in S_h, \]

\[ A(u_{ht}, \chi) + (\eta_h u_h u_{hx}, \chi) = 0, \quad \forall \chi \in S_{h,0}, \]

with initial conditions

\[ \zeta_h(0) = P \zeta_0, \quad u_h(0) = R_h u_0. \]

The ode ivp given by (2.13)–(2.15) has a unique solution locally in time. As
part of Theorem 2.1 below we will prove that for sufficiently small \( h \), its solution
may be extended up to \( t = T \).

**Theorem 2.1.** Suppose that the solution \((\zeta, u)\) of (2.6) is sufficiently smooth
and that the conditions (2.7) hold. Then, if \( h \) sufficiently small, there exists a constant \( C \) independent of \( h \) such that the semidiscrete problem (2.13)–(2.15)
has a unique solution \((\zeta_h, u_h)\) for \( 0 \leq t \leq T \), that satisfies

\[ \max_{0 \leq t \leq T} \| (\zeta(t) - \zeta_h(t)) \| + \| u(t) - u_h(t) \|_1 \leq C h^{r-1}. \]

**Proof.** Let \( P = \zeta - P \zeta, \theta = P \zeta - \zeta_h, \sigma = u - R_h u, \xi = R_h u - u_h. \) From (2.6) and (2.13)–(2.15) we get

\[ (\theta_t, \phi) + \varepsilon ((\zeta u - \zeta_h u)_{x}, \phi) + ((\eta u + \eta_h \xi)_x, \phi) = 0, \quad \forall \phi \in S_h, \]

\[ A(\xi_t, \chi) + (\eta u \xi + \eta_h \xi) = 0, \quad \forall \chi \in S_{h,0}, \]

that are valid while the semidiscrete problem has a unique solution. For the nonlinear terms we have

\[ \zeta u - \zeta_h u_h = \zeta(\sigma + \xi) + u(\rho + \theta) - (\rho + \theta)(\sigma + \xi), \]

\[ u u_x - u_h u_{hx} = (u \sigma)_x + (u \xi)_x - (\sigma \xi) - \sigma \sigma_x - \xi \xi_x. \]

Let now \( t_h \in (0, T] \) be the maximal temporal instance for which the solution of
(2.6) exists and it holds that \( \| \theta(t) \|_\infty + \| \xi(t) \|_\infty \leq 1 \), for \( t \leq t_h \). Putting \( \phi = \theta \)
in (2.17), using (2.1), (2.2b), (2.11), (2.12), (2.3), and integrating by parts we
have for $t \leq t_h$

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 = -\varepsilon((\zeta \sigma, \theta) - \varepsilon((\zeta \xi, \theta) - \varepsilon((u \sigma) x, \theta) - \varepsilon((u \xi) x, \theta) + \varepsilon((\rho \sigma) x, \theta) + \varepsilon((\rho \xi) x, \theta) + \varepsilon((\theta \sigma) x, \theta) + \varepsilon((\theta \xi) x, \theta) - ((\eta \sigma) x, \theta) - ((\eta \xi) x, \theta) \leq \varepsilon(\|\xi\|_\infty \|\sigma\| + \|\xi\|_\infty \|\xi\|) \|\theta\| + \varepsilon(\|\zeta\|_\infty \|\xi\| + \|\zeta\|_\infty \|\xi\|) \|\theta\| + \varepsilon(\|u\|_\infty \|\rho\| + \|u\|_\infty \|\rho\|) \|\theta\| + \frac{1}{2} \|\xi\|_\infty \|\theta\|^2 + \varepsilon(\|\sigma\|_\infty \|\rho\| + \|\sigma\|_\infty \|\sigma\|) \|\theta\| + \varepsilon(\|\xi\|_\infty \|\rho\| + \|\rho\|_\infty \|\xi\|) \|\theta\| + \frac{1}{2} \|\sigma\|_\infty \|\xi\| \|\theta\| + \frac{1}{2} \|\xi\|_\infty \|\xi\| \|\theta\| + ((\|\eta\|_\infty \|\xi\| + \|\eta\|_\infty \|\xi\|) \|\theta\| \leq C(h^{r-1} + \|\xi\|_1 + \|\theta\|) \|\theta\|, \quad (2.19)$$

for some constant $C$ independent of $h$.

In addition, with $\lambda = \xi$ in (2.18) we obtain for $t \leq t_h$

$$\frac{1}{2} \frac{d}{dt} A(\xi, \xi) = -(\eta \rho \xi + \eta \theta \xi, \xi) - \varepsilon(\eta \sigma \xi, \xi) + \varepsilon(\eta \xi, \xi) + \varepsilon(\eta \xi, \xi)$$

$$= \varepsilon(\|\xi\|_\infty \|\xi\| + \|\xi\|_\infty \|\xi\|) \|\xi\| + \varepsilon(\|\xi\|_\infty \|\xi\|) \|\xi\| + \varepsilon(\|\xi\|_\infty \|\xi\|) \|\xi\| + \frac{1}{2} \|\xi\|_\infty \|\xi\| \|\xi\| + \frac{1}{2} \|\xi\|_\infty \|\xi\| \|\xi\| + \frac{1}{2} \|\xi\|_\infty \|\xi\| \|\xi\| + \frac{1}{2} \|\xi\|_\infty \|\xi\| \|\xi\| \quad (2.20)$$

With estimates analogous to those used in (2.19) we get

$$\frac{1}{2} \frac{d}{dt} A(\xi, \xi) \leq (\|\eta\|_\infty \|\xi\| + \|\eta\|_\infty \|\xi\|) \|\xi\| + \|\eta\|_\infty \|\xi\| \|\xi\| + \|\eta\|_\infty \|\xi\| \|\xi\| + \|\eta\|_\infty \|\xi\| \|\xi\| \quad (2.20)$$

where $C$ is independent of $h$. From (2.19) and (2.20) we see that

$$\frac{d}{dt} (\|\theta\|^2 + A(\xi, \xi)) \leq C_1 h^{2 r - 2} + C_2 (\|\theta\|^2 + \|\xi\|^2) \quad (2.21)$$

where $C_1$, $C_2$ are independent of $h$. From this inequality and (2.9) it follows that

$$\frac{d}{dt} (\|\theta\|^2 + A(\xi, \xi)) \leq C_1 h^{2 r - 2} + C_\mu (\|\theta\|^2 + A(\xi, \xi)) \quad (2.21)$$

for $t \leq t_h$, where $C_\mu = C_2 \max(1, 1/c_\mu)$. Using Gronwall’s lemma in the above we obtain for $t \leq t_h$,

$$\|\theta(t)\|^2 + A(\xi(t), \xi(t)) \leq e^{C_r T} (\|\theta(0)\|^2 + A(\xi(0), \xi(0))) + C_1 e^{C_r T} h^{2 r - 2},$$

from which, in view of (2.9) and since $\theta(0) = \xi(0) = 0$, we see that

$$\|\theta(t)\| + \|\xi(t)\|_1 \leq \left(\frac{2 C_1}{C_\mu C_r T} e^{C_r T} h^{2 r - 2}\right)^{1/2} h^{r - 1}.$$
for \( t \leq t_h \), where \( \widetilde{C}_\mu = \min(1, c_\mu) \). Now, since \([2.3]\) gives \( ||\theta||_\infty \leq Ch^{-1/2}||\theta|| \) and \( ||\xi||_\infty \leq ||\xi||_1 \), if \( h \) is taken sufficiently small, we have that \( ||\theta||_\infty + ||\xi||_\infty < 1 \) for \( 0 \leq t \leq t_h \), and therefore we may take \( t_h = T \). The result follows from \([2.20]\) and the approximation properties of the finite element spaces.

As suggested by numerical experiments for the \((\text{CB})\) on a horizontal bottom, shown in \([6]\), the convergence rates in the error estimate \([2.16]\) are sharp in the case of a horizontal bottom; they are sharp in the case of variable-bottom models as well. The \( H^1 \) convergence rate of the error of \( u_h \) is optimal, while the \( L^2 \) rate for \( \eta_h \) suboptimal, as expected, since the first pde in \([2.4]\) is of hyperbolic type and we are using the standard Galerkin method on a nonuniform mesh. (For \( r = 2 \) the numerical experiments in \([6]\) also suggest the improved estimate \( ||u - u_h|| = O(h^2) \).) In the case of uniform mesh, better results were proved in \([6]\) in the case of horizontal bottom. The numerical experiments in Section 3 of the paper at hand suggest that such improved rates of convergence for uniform mesh persist in the presence of a variable bottom as well.

### 2.3 Semidiscretization in the case of a weakly varying bottom

In the case of a weakly varying bottom, following the remarks in Section 1, we consider the following ibvp for the system \((\text{CBw})\). For \( T > 0 \) we seek \( \zeta = \zeta(x,t), \quad u = u(x,t), \) for \((x,t) \in [0,1] \times [0,T] \), such that

\[
\begin{align*}
\zeta_t + (\eta u)_x &= 0, \\
u_t + \zeta_x + \varepsilon uu_x - \frac{\mu}{3} u_{xxx} &= 0, & 0 \leq x \leq 1, & 0 \leq t \leq T, \\
\zeta(x,0) &= \zeta_0(x), & u(x,0) &= u_0(x), & 0 \leq x \leq 1, \\
u(0,t) &= u(1,t), & 0 \leq t \leq T,
\end{align*}
\]

\((2.22)\)

where

\[ \eta = \varepsilon \zeta + \eta_b > 0, \quad \eta_b(x) = 1 - \beta b(x) > 0, \]

and \( \varepsilon, \mu, \beta \), are positive constants with \( \varepsilon = O(\mu), \beta = O(\mu), \mu \ll 1, \) and \( b = C^2[0,1] \). All the variables above are nondimensional and scaled. We assume that \((2.22)\) has a unique solution, smooth enough for the purposes of the error estimate below.

Let \( a : H^1_0 \times H^1_0 \to \mathbb{R} \) denote the weighted \( H^1 \)-inner product defined by \( a(v,w) = (v, w) + \frac{2}{3}(v', w') \) and consider the weighted \( H^1 \) (‘elliptic’) projection associated with \( a(\cdot,\cdot) \), defined as the map \( \tilde{R}_h : \tilde{H}^1 \to S_{h,0} \) such that

\[
a(\tilde{R}_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_h.
\]

\((2.23)\)

Obviously, \( \tilde{R}_h \) satisfies the properties \((2.11)\) and \((2.12)\).
The standard Galerkin finite element semidiscretization of the ibvp \(2.22\) is the following. We seek \(\zeta_h : [0, T] \rightarrow S_h, \ u_h : [0, T] \rightarrow S_{h,0}\), such that
\[
\begin{align*}
(\zeta_{ht}, \phi) + \varepsilon ((\zeta_h u_h)_x, \phi) + (\eta_b u_h)_x, \phi) &= 0, \quad \forall \phi \in S_h, \quad (2.24) \\
a(u_{ht}, \chi) + (\zeta_{hx}, \chi) + \varepsilon (u_h u_{hx}, \chi) &= 0, \quad \forall \chi \in S_{h,0}, \quad (2.25)
\end{align*}
\]
with initial conditions
\[
\begin{align*}
\zeta_h(0) &= P \zeta_0, \quad u_h(0) = \tilde{R}_h u_0. \quad (2.26)
\end{align*}
\]
In analogy with Theorem 2.1, the following error estimate holds for the semidiscrete scheme \(2.24\)–\(2.26\). (We omit the proof since it is very similar to that of Theorem 2.1, \textit{mutatis mutandis}.)

**Theorem 2.2.** Suppose that the solution \((\zeta, u)\) of \(2.22\) is sufficiently smooth. Then, if \(h\) is sufficiently small, there exists a constant \(C\) independent of \(h\) such that the semidiscrete problem \(2.24\)–\(2.26\) has a unique solution \((\zeta_h, u_h)\) for \(0 \leq t \leq T\), that satisfies
\[
\max_{0 \leq t \leq T} (\|\zeta(t) - \zeta_h(t)\| + \|u(t) - u_h(t)\|_1) \leq C h^{r-1}. \quad (2.27)
\]

### 3 Numerical experiments

In this section we present results of numerical experiments that we performed using the two models \(\boxed{\text{CBs}}\) and \(\boxed{\text{CBw}}\) of the classical Bousinesq system with variable bottom. We discretized the two systems in space using the Galerkin finite element method analyzed in the previous section. For the temporal discretization we used the ‘classical’, explicit, 4-stage, 4th order Runge-Kutta scheme (RK4). The convergence of this fully discrete scheme was analyzed, in the case of the ibvp for the \(\boxed{\text{CB}}\) with horizontal bottom and \(u = 0\) at the endpoints in \([6]\), where it was shown that under a Courant number stability restriction of the form \(\frac{k}{\pi} \leq \alpha\) the scheme is stable, is fourth-order accurate in time, and preserves the spatial order of convergence of the semidiscrete problem; here \(k\) denotes the (uniform) time step.

#### 3.1 Convergence rates

The spatial convergence rates proved in Theorems 2.1 and 2.2 in the case of a general quasiuniform mesh are sharp as is suggested by numerical experiments (not shown here). In the case of a uniform spatial mesh better convergence rates may be achieved. This was proved in \([6]\) for the \(\boxed{\text{CB}}\) (horizontal bottom and \(u = 0\) at the endpoints of the spatial interval) in the case of piecewise linear continuous functions \((r = 2)\) and cubic splines \((r = 4)\). The numerical results to be presented in the sequel suggest that the improved rates persist in the case of a variable bottom as well for both CB models. (We do not show the optimal-order results for the piecewise linear case \((r = 2)\) but concentrate instead in the case of cubic splines \((r = 4)\).)
The exact solution of the test problem used for the error rate computations is 
\[ \zeta(x, t) = e^{2t}(\cos(\pi x) + x + 2), \]
\[ u(x, t) = e^{xt}(\sin(\pi x) + x^3 - x^2) \] for \((x, t) \in [0, 1] \times [0, 1/4]\); the bottom topography was given by the function \(\eta_b(x) = 1 - \beta \sin \pi x\).

The scaling parameters (not important for the convergence rate computations) were taken as \(\varepsilon = 1, \mu = 1/10, \beta = 1/10\). Appropriate right-hand sides and initial conditions were found from the above data. We solved numerically the ibvp's (2.20) and (2.22) with the above exact solution and bottom profile using the spatial discretizations (2.13)–(2.15) and (2.24)–(2.26), respectively, with cubic splines with uniform mesh of meshlength \(h = 1/N\). The temporal discretization was effected by the RK4 scheme with stability restriction \(k h \leq 1/4\); the resulting time steps were small enough so that the temporal errors were much smaller than the spatial ones. We used 3-point Gauss quadrature to evaluate the finite element integrals in every mesh interval. (Since we wished to obtain detailed information about the spatial convergence rates, we computed throughout with quadruple precision and evaluated the \(L^2\)-errors using 5-point Gauss quadrature and the \(L^\infty\) errors by taking the maximum value of the error on all these quadrature points.)

In Table 1 we show the \(L^2\), \(L^\infty\), and \(H^1\) (seminorm) spatial errors and convergence rates in the case of the (CBw) model. The numerical results suggest strongly that the \(L^2\) rates for \(\zeta\) and \(u\) are equal to 3.5 and 4, respectively, the \(L^\infty\) rates equal to 3 and 4, while the \(H^1\) ones 2.5 and 3, respectively. The same

| \(N\) | \(L_2\) error | rate | \(L_\infty\) error | rate | \(H_1\) semi-nrm | rate |
|-------|---------------|------|-------------------|------|------------------|------|
| 128   | 6.0526e-09    | -    | 5.6333e-08        | -    | 3.5964e-06       | -    |
| 256   | 5.3006e-10    | 3.5133 | 6.9294e-09        | 3.0232 | 6.2857e-07       | 2.5164 |
| 512   | 4.6605e-11    | 3.5076 | 8.5918e-10        | 3.0117 | 1.1046e-06       | 2.5086 |
| 1024  | 4.1074e-12    | 3.5042 | 1.0696e-10        | 3.0059 | 1.9466e-08       | 2.5045 |
| 2048  | 3.6250e-13    | 3.5022 | 1.3343e-11        | 3.0029 | 3.4357e-09       | 2.5023 |
| 4096  | 3.2016e-14    | 3.5011 | 1.6862e-12        | 3.0015 | 6.0686e-10       | 2.5012 |
| 8192  | 2.8287e-15    | 3.5006 | 2.0816e-13        | 3.0007 | 1.0724e-10       | 2.5006 |

(a) \(\zeta\)

| \(N\) | \(L_2\) error | rate | \(L_\infty\) error | rate | \(H_1\) semi-nrm | rate |
|-------|---------------|------|-------------------|------|------------------|------|
| 128   | 2.9812e-10    | -    | 6.0043e-10        | -    | 2.4080e-07       | -    |
| 256   | 1.8618e-11    | 4.0011 | 3.7468e-11        | 4.0023 | 3.0098e-08       | 3.0001 |
| 512   | 1.1632e-12    | 4.0005 | 2.3407e-12        | 4.0006 | 3.7623e-09       | 3.0000 |
| 1024  | 7.2689e-14    | 4.0002 | 1.4628e-13        | 4.0002 | 4.7029e-09       | 3.0000 |
| 2048  | 4.5427e-15    | 4.0001 | 9.1419e-15        | 4.0001 | 5.8786e-11       | 3.0000 |
| 4096  | 2.8391e-16    | 4.0001 | 5.7136e-16        | 4.0000 | 7.3483e-12       | 3.0000 |
| 8192  | 1.7744e-17    | 4.0000 | 3.5710e-17        | 4.0000 | 9.1853e-13       | 3.0000 |

(b) \(u\)

Table 1: Spatial errors and rates of convergence, \(t = 1/4\), (CBw), cubic splines on uniform mesh, \(h = 1/N\). (a) \(\zeta\), (b) \(u\).

rates are observed (cf. Table 2) in the numerical integration by the same method.
of the analogous ibvp for the CBs model.

As a remark of theoretical interest we point out that in the case of the analogous ibvp for CB on a horizontal bottom two of the authors proved in [6]

| N   | L₂ error rate | L∞ error rate | H₁ semi-nrm rate |
|-----|---------------|---------------|------------------|
| 128 | 6.0165e-09    | 5.5101e-08    | 3.5538e-06       |
| 256 | 5.2848e-10    | 6.8528e-09    | 3.0037           |
| 512 | 4.6535e-11    | 8.5440e-10    | 1.1012e-07       |
| 1024| 4.1044e-12    | 1.0666e-10    | 1.9436e-08       |
| 2048| 3.6236e-13    | 1.3324e-11    | 3.4331e-09       |
| 4096| 3.2010e-15    | 1.6650e-12    | 6.0663e-10       |
| 8192| 2.8284e-14    | 1.3324e-12    | 3.0073           |

(a) \( \zeta \)

| N   | L₂ error rate | L∞ error rate | H₁ semi-nrm rate |
|-----|---------------|---------------|------------------|
| 128 | 2.9818e-10    | 6.0086e-10    | 3.0099e-08       |
| 256 | 1.8621e-11    | 3.7476e-11    | 3.7623e-09       |
| 512 | 1.1634e-12    | 2.3411e-12    | 3.7623e-09       |
| 1024| 7.2699e-14    | 1.4630e-13    | 4.7029e-10       |
| 2048| 4.5433e-15    | 9.1432e-15    | 5.8786e-11       |
| 4096| 2.8395e-16    | 5.7144e-16    | 7.3483e-12       |
| 8192| 1.7746e-17    | 3.5714e-17    | 9.1853e-13       |

(b) \( u \)

Table 2: Spatial errors and rates of convergence, \( t = 1/4 \), CBs, cubic splines on uniform mesh, \( h = 1/N \). (a) \( \zeta \), (b) \( u \).

\[ L^2 \text{ error estimates } \| \zeta - \zeta_h \| \leq C h^{3.5} \sqrt{\ln h}, \| u - u_h \| \leq C h^{4} \sqrt{\ln h}, \] for the semidiscretization with cubic splines on a uniform mesh. The increased accuracy of our present code affords investigating computationally if the logarithmic factors are actually present in these estimates. To this end we considered the ibvp for CB with the exact solution given previously, but now in the case of the horizontal bottom \( \eta_b = 1 \), and found that the rates \( \frac{\| \zeta - \zeta_h \|}{h^{3.5}} \) stabilized to the value 0.124 (the values of \( h \) used were less than 1/1024 and the errors \( \| \zeta - \zeta_h \| \) were of \( O(10^{-12}) \) or smaller,) while the ratio \( \frac{\| \zeta - \zeta_h \|}{h^{3.5} \sqrt{\ln h}} \) did not stabilize for the same range of \( h \)’s. Similar observations were made for the \( u \) component of the error. Therefore these increased accuracy experiments suggest that the error estimates in [6] are not sharp.

### 3.2 Approximate absorbing boundary conditions

In the case of the shallow water (SW) equations on a horizontal bottom, obtained if we set \( \mu = 0 \) in the CB system, i.e. for the equations

\[
\begin{align*}
\zeta_t + u_x + \varepsilon (\zeta u)_x &= 0, \\
u_t + \zeta_x + \varepsilon uu_x &= 0,
\end{align*}
\]

(SW)
(written here in nondimensional, scaled variables, and where it is assumed that $1 + \varepsilon \zeta > 0$), it is well known that using Riemann invariants and the theory of characteristics, one may derive transparent, characteristic boundary conditions at the endpoints of a finite spatial interval, say $[0, 1]$. These boundary conditions allow an initial pulse that is generated in the interior of $(0, 1)$ and travels in both directions to exit the interval cleanly. In the case of a subcritical flow, which will be of interest here, i.e. when the solution of (SW) satisfies $u^2 < (1 + \varepsilon \zeta)/\varepsilon^2$, the characteristic boundary conditions are of the form

$$
\varepsilon u(0, t) + 2\sqrt{1 + \varepsilon \zeta(0, t)} = \varepsilon u_0 + 2\sqrt{1 + \varepsilon \zeta_0},
\varepsilon u(1, t) - 2\sqrt{1 + \varepsilon \zeta(1, t)} = \varepsilon u_0 - 2\sqrt{1 + \varepsilon \zeta_0}.
$$

(3.1)

Here it is assumed that outside the interval $[0, 1]$ the flow is uniform and satisfies $\zeta(x, t) = \zeta_0, u(x, t) = u_0$, where $u_0, \zeta_0$ are constants such that $u^2_0 < (1 + \varepsilon \zeta_0)/\varepsilon^2$.

In addition, the initial conditions $\zeta(x, 0), u(x, 0)$ of (SW) should satisfy the subcriticality conditions and be compatible at $x = 0$ and $x = 1$ with the uniform flow outside $[0, 1]$. In [25] two of the authors analyzed the space discretization of (SW) with characteristic boundary conditions (both in the subcritical and supercritical case) using Galerkin finite element methods. Analytical and computational evidence in [25] suggests that the discretized characteristic boundary conditions, although not exactly transparent, are nevertheless highly absorbent.

We note that the same type of characteristic absorbing conditions may be used for the (SW) over a variable bottom, at least in the case where the bottom is locally horizontal at the endpoints cf. e.g. [26] and its references.

Finding (exact) transparent boundary conditions for the (CB) is not easy, as a nonlocal problem should be solved for this nonlinear system. In practice, for small $\mu$, it is reasonable to assume that the Riemann invariants do not change much over short distances along the characteristics, and, consequently, to pose the b.c. [3.1] as approximate, absorbing b.c.’s for (CB) as well. This has been widely done in practice, for example in numerical simulations of the Serre equations cf. e.g. [16], [17]; in [27] the related problem of deriving one-way approximations of the Serre equations is discussed. Our aim in this subsection is to assess, by numerical experiment, the accuracy of (3.1) as approximate absorbing boundary conditions for the (CB), paying special attentions to their efficacy in simulating outgoing solitary-wave solutions of the (CB).

In order to derive (classical) solitary-wave solutions of (CB) on the real line, we let $\zeta = \zeta_s(x - c_s t), u = u_s(x - c_s t)$, where $c_s$ is the speed of the solitary wave and $\zeta_s(\xi), u_s(\xi)$ are smooth functions that tend to zero, along with their derivatives, as $|\xi| \to \infty$. Inserting these expressions in (CB) and integrating we see that the equations for $\eta_s$ and $u_s$ decouple and give

$$
\zeta_s = \frac{u_s}{c_s - \varepsilon u_s}, \quad \frac{c_s \mu}{3} u_s'' + \frac{\varepsilon}{2} u_s^2 - c_s u_s + \frac{u_s}{c_s - \varepsilon u_s} = 0.
$$

(3.2)

A further integration yields that $u_s$ satisfies the ode

$$
\frac{c_s \mu}{6} (u_s')^2 + \frac{\varepsilon}{6} u_s^3 - \frac{c_s}{2} u_s^2 - \frac{1}{\varepsilon} u_s - \frac{c_s}{\varepsilon^2} \ln \frac{c_s - \varepsilon u_s}{c_s} = 0.
$$

(3.3)
It is straightforward to see that $\zeta_s$ and $u_s$ have a single positive maximum at some point $\xi_0$ (we assume that $\xi_0 = 0$). Denoting $A = \max \zeta_s$, $B = \max u_s$, we get

$$A = \frac{B}{c_s - \varepsilon B}, \quad \varepsilon B^3 - \frac{c_s^2}{2} B^2 = \frac{1}{\varepsilon} B - \frac{c_s}{\varepsilon^2} \ln \left( \frac{c_s - \varepsilon B}{c_s} \right) = 0, \quad (3.4)$$

from which one may compute the speed-amplitude relation

$$c_s = \frac{\sqrt{6}(1 + \varepsilon A) \sqrt{(1 + \varepsilon A) \ln(1 + \varepsilon A) - \varepsilon A}}{\varepsilon A}. \quad (3.5)$$

For fixed $\varepsilon$, $c_s$ is monotonically increasing with $A_s$ but stays below the straight line $c_s = 1 + \frac{\varepsilon A}{2}$, which is the speed-amplitude relation of the solitary waves of the Serre equations. (The formulas (3.2)–(3.5) were derived in [7] in the case of the unscaled (CB). Note that there are some typographical errors in [7]: In equation (1.58) of [7] the last term in the left-hand side of the equation should have the sign $+$, while in the equation preceding (3.2) in [7] the third term in the left-hand side should have the sign $+$ and the last term the sign $-$. However formulae (3.1) and (3.2) of [7], which are the analogues of (3.5) and (3.4) above, are correct.)

When $\varepsilon A_s$ is not large, i.e. when (CB) is a valid model for surface waves, it may be seen by (3.5) and also by numerical simulations that the solitary-wave solutions of (CB) satisfy the subcriticality condition. (Since there is no closed-form formula for the solitary waves we generate them numerically by solving for given $c_s$ the nonlinear o.d.e. (3.2) that $u_s$ satisfies, taking zero boundary conditions for $u_s$ and $u'_s$ at the endpoints of a large enough spatial interval using the routine bvp4c of [28].)

In the numerical experiments to be described in the sequel we solved the (SW) and the (CB), unless otherwise specified, on the spatial interval [0, 50] using cubic splines on a uniform mesh with $h = 0.025$, coupled with RK4 time stepping with time step satisfying $\frac{k}{h} = \frac{1}{2}$, up to $T = 50$.

We set the stage by solving numerically the (SW) with $\varepsilon = 1$ with the b.c. (3.1), posed now at the endpoints of $x = 0$ and $x = 50$. As initial condition we take the solitary wave of (CB) with $\mu = \varepsilon = 1$ of speed $c_s = 1.18112$, centered at $x = 25$, which we multiply by a factor 0.1 (thus it is no longer a solitary wave), so that no discontinuities develop in its evolution under (SW) for the duration of the experiment. As expected, the initial single-hump wave is split in two pulses: a larger one of amplitude of about 0.04 traveling to the right with a speed of about 1.057 and which starts exiting the computational interval at $x = 50$ at about $t = 22.5$, (the exit is completed by about $t = 30$), and a smaller one of amplitude of about 0.0035 that travels to the left with speed 1.005 and exits the interval at $x = 0$ at about $t = 24.5$.

In Figure 2 we present some graphs that are relevant for assessing the accuracy of the absorbing b.c.’s for this example. (All graphs refer to $\zeta$.) In Fig. 2(a) we observe the temporal variation of the wavefield at $x = 40$. The pulse that travels to the right passes this gauge and exits the interval. What
Figure 2: Accuracy of the numerical characteristic b.c.’s for the $\text{SW}$, $\varepsilon = 1$, (a) $\zeta(40, t)$ with magnification underneath, (b) $\max_x \zeta(x, t)$ with magnification underneath, (c) Magnification of $\zeta(x, 50)$
remains after $t \simeq 30$ is a small residual consisting of small-amplitude oscillations reflected from the boundary due to the inexactness of the discretized b.c.’s and shown in the magnification of $2(a)$ to be of $O(10^{-9})$. In $2(b)$ we show the maximum amplitude of $\zeta$ with respect to $x$ over the whole interval as a function of $t$, while $2(c)$ shows the small oscillations still present in the computational interval at the end of the experiment ($t = 50$). The are all of magnitude at most $10^{-9}$ and consist of a main wavepacket of high frequency and amplitude of about $4 \times 10^{-10}$ centered at about $x = 40$ and moving to the right, and three larger amplitude ‘thin’ wavetrains of small support centered at about $x = 5$ (moving to the right), $x = 20$ (moving to the left) and $x = 37.5$ (moving to the left), respectively. The main oscillatory wavepacket is produced when the right-traveling pulse exits the boundary at $x = 50$. This wavepacket moves to the left with speed equal to about 7 and has undergone three reflections at the boundary by $T = 50$. The thinner wavetrains (of speed about 1) are generated by the interaction of this wavepacket with the boundaries (The left-traveling pulse produced by the splitting of the initial condition produces, when it hits the boundary at $x = 0$, artificial reflections with amplitude well below $10^{-10}$).

In Figure 3, resp. 4 we show analogous graphs in the case of the (CB) system in the cases $\varepsilon = \mu = 0.1$, resp. $\varepsilon = \mu = 0.01$. As initial condition we took now the exact solitary-wave profile of (CB) for these values of $\varepsilon$, $\mu$, and of speed $c_s = 1.18112$. As a consequence, the wave moves to the right without changing its shape. The fact that the characteristic b.c.’s are no longer exactly transparent for the continuous system is manifested by the larger magnitudes of the residual artificial oscillations, which are now of $O(10^{-3}), O(10^{-4})$. (Note their dispersive character in the larger $\mu$ case, Fig. 2(c)). The main pulse in graph (c) of Figures 3 and 4 is due to the modelling, i.e. the approximate character of the characteristic b.c.’s, while the superimposed noise in Fig. 4(c) disappears as $h$ is decreased. The amplitude of the residual was equal to about $2.1 \times 10^{-3}$ for $\varepsilon = \mu = 0.1$ and fell to $3.2 \times 10^{-4}$ for $\varepsilon = \mu = 0.01$, and to $3.3 \times 10^{-5}$ for $\varepsilon = \mu = 0.001$ (figure not shown). We thus observe that it decreases linearly with $\mu$ when $\varepsilon = \mu$. As expected, for fixed $\varepsilon$ we observed that this amplitude decreased with $\mu$. For example, for $\varepsilon = 0.01$ and $\mu = 10^{-3}$ it was equal to about $3.6 \times 10^{-5}$, for $\mu = 10^{-4}$ it was of $O(10^{-6})$.

Our conclusion is that for small $\varepsilon = \mu$, i.e. when the (CB) is a valid model, the (approximate) characteristic b.c.’s for the (CB) are satisfactorily absorbing. We extended these b.c.’s in the case of the variable bottom models (CBw) and (CBS) and used them in numerical experiments with these systems that will be reported in the next subsection.

### 3.3 Propagation of solitary waves over a variable bottom

In this subsection we present the results of several numerical experiments we performed with the variable-bottom models (CBw) and (CBS) in order to validate the numerical methods used for their solution, compare the two models, and compare the results of (CBS) with those obtained by the Serre-Green-Naghdi system and with experimental measurements. We mainly use test problems al-
Figure 3: Accuracy of the numerical characteristic b.c.'s for the (CB), $\varepsilon = \mu = 0.1$, (a) $\zeta(40, t)$ with magnification underneath, (b) $\max_x \zeta(x, t)$ with magnification underneath, (c) Magnification of $\zeta(x, 50)$
Figure 4: Accuracy of the numerical characteristic b.c.’s for the (CB), $\varepsilon = \mu = 0.01$. (a) $\zeta(40, t)$ with magnification underneath, (b) $\max_x \zeta(x, t)$ with magnification underneath, (c) Magnification of $\zeta(x, 50)$
ready considered in the literature, whose main theme is the study of the changes that solitary-wave pulses undergo when propagating over an uneven bottom.

3.3.1 Solitary waves on a sloping beach

We first consider the problem of a solitary wave climbing a sloping beach of mild slope that was studied by Peregrine in his pioneering study [20], in which he derived the (CBs) system and solved it numerically by a finite difference scheme. In our experiments we used the (CBs) in unscaled, nondimensional variables (i.e. setting $\varepsilon = \mu = 1$) and solved it with our fully discrete scheme using cubic splines on a uniform mesh with $N = 2000$ spatial intervals and $M = 2N$ temporal steps. Following [20] we consider, using our notation, a bottom of uniform slope $\alpha > 0$ given by $\eta_b(x) = \alpha x$ on a spatial interval of the form $[0, L_\alpha]$. As initial condition we take as in [20] a solitary wave of the form

$$\zeta_0(x) = a_0 \text{sech}^2 \left[ \frac{1}{2} \sqrt{3a_0} (x - x_0) \right], \tag{3.6}$$

where $x_0 = 1/\alpha$. This is a solitary wave of the KdV type equation $\zeta_t + \zeta_x + \frac{3}{2} \zeta \zeta_x + \frac{1}{6} \zeta_{xxx} = 0$ with speed $c_s = 1 + a_0/2$. The KdV equation in this form is obtained as a one-way approximation of the (CB) with $\varepsilon = \mu = 1$ in the standard manner, cf. [1]. The particular solitary wave (3.6) is centered at $x_0 = 1/\alpha$, where the (undisturbed) water depth is equal to one. The initial velocity of the pulse, found by inserting (3.6) in the continuity equation, is given by

$$u_0(x) = -\frac{1 + \frac{1}{2} a_0}{\alpha x + \zeta_0(x)} \zeta_0(x). \tag{3.7}$$

Thus the initial condition (3.6), (3.7) is not an exact solitary-wave solution of (CB) but a close approximation thereof. We took an interval of length $L_\alpha = 1/\alpha + 20$ to ensure that the support of the initial pulse was well within the spatial interval of integration. At $x = 0$ we used the b.c. $u = 0$ (which produced no reflections as the wave did not reach the left boundary within the temporal range of the experiment), posed absorbing (characteristic) boundary conditions at $x = L_\alpha$, and ran the experiment up to $t = 25$.

During this temporal interval the wave moves to the left, steepens (wave 'shoaling') and grows in amplitude; its evolution resembles that of Fig. 1 of [20], which corresponds to $\alpha = 1/30$, $a_0 = 0.1$. We compared our numerical results with those of the finite-difference scheme of Peregrine (given in the Appendix of [20]) that we implemented. (Note that there is a misprint in the last equation of this scheme in [20]. In the discretization of the term $\eta_b u_x$, the denominator should be $4\Delta x$.) We observed that the maximum discrepancy in the amplitude of $\zeta$ approximated by the two methods occurred at $t = 25$ where the values were 0.14100 for our scheme and 0.13634 for the scheme of [20] (implemented with $\Delta x = \Delta t = 0.1$), which corresponds to a difference of about 3.4% (Fig. 1 of [20] shows a $\zeta$-amplitude of about 0.15 at $t = 25$ which does not correspond to the actual numerical results that the scheme of [20] gives and is probably due to some inaccuracy in the graphics.)
We also repeated with our scheme the numerical experiments leading to Fig. 2 of [20] that depicts the change of amplitude of the solitary wave with depth for various values of $a_0$ in the case of a beach of slope $\alpha = 1/20$. There was good agreement for low values of $a_0$; however the values given in [20] for $a_0 = 0.2$ seem too high as the depth approaches 0.4. (All the amplitudes computed by our scheme stay below the curve of Green’s law for depths larger than 0.5.)

As the solitary wave climbs the sloping beach a small-amplitude flat wave of elevation is reflected backwards due to the presence of the sloping bed. The results of our computations agree with those of Fig. 3 in [20]. Peregrine, op. cit., derives an approximate expression for the amplitude of the reflected wave of the form

$$\zeta_{\text{max, refl}} \simeq \frac{1}{2} \alpha \left( \frac{1}{3} a_0 \right)^{\frac{1}{2}},$$

using characteristic variables for the linearized shallow water equations. We found quite a good agreement between our numerical results and the values computed by (3.8). For example, for $\alpha = 1/40$, $a_0 = 0.1$, our computations gave $\zeta_{\text{max, refl}} = 0.0023$, while (3.8) gives 0.0025. We will return to the reflections due to the uneven bottom in subsection 3.3.3 in the sequel.

As was previously mentioned, we used the approximate characteristic boundary conditions discussed in subsection 3.2 at the right-hand boundary $x = L_\alpha$. We found that the b.c. also works for a sloping bottom provided the length of the domain is taken sufficiently large so that the artificial oscillations created at the boundary do not interfere as they travel to the left with the reflected wave due to the slope. As an example we consider a beach of slope $\alpha = 1/40$ on the spatial interval $[0, 70]$. As initial condition we took $\zeta(x, 0) = \zeta_0(x)$ given by (3.6) with $x_0 = 40$, $a_0 = 0.1$, and $u(x, 0) = 0$, i.e. a ‘heap’ of water, so that sizeable pulses are generated and propagate in both directions. Figure 5 shows a magnified profile of the surface elevation $\zeta$ as a function of $x$ in the interval $[20, 70]$ at $t = 50$, by which time the right-travelling pulse has left the domain. In the interval $[20, 45]$ we observe the small-amplitude (of height approximately $5 \times 10^{-4}$) reflection due to interaction of the left-travelling pulse with the sloping bottom. In the interval $[45, 60]$ we observe the artificial oscillations reflected...
from the right-hand boundary at $x = 70$ due to the approximate absorbing b.c. after the exit of the main right-travelling pulse. The ratio of the amplitude of the artificial reflection to that of the main pulse is about 4%. Finally, one may also observe on the extreme right the dispersive-tail oscillations that follow the main right-travelling pulse as they exit the domain.

### 3.3.2 Transformation of a solitary wave propagating onto a shelf

We next consider in detail an example of the transformation that a solitary wave undergoes as it propagates over a bottom of shelf type like the one shown in Figure 6. This test problem was considered by Madsen and Mei in [29]. In this subsection we work in dimensionless, unscaled variables with $\varepsilon = \mu = 1$.

![Figure 6: Solitary wave propagating onto a shelf](image)

The initial elevation of the solitary wave is given again by (3.6), in which $x_0$ is taken far enough from the toe of the sloping part of the bottom at $x = x_B$, so that $\zeta_0(x_B)/a_0 \ll 1$. The initial velocity is found again from the continuity equation but is now computed for a bottom of constant depth $h_0 = 1$, i.e. as

$$u_0(x) = \frac{1 + \frac{1}{2}a_0}{1 + \zeta_0(x)} \zeta_0(x). \quad (3.9)$$

The solitary wave travels to the right, changes in amplitude and shape as it climbs the slope, and resolves itself into a sequence of solitary-wave pulses as it travels on the shelf of uniform depth $h_1 < 1$, cf. Figure 9.

In [29] the pde model used was a Boussinesq system of KdV-BBM type with variable-bottom terms originally derived in [30], and which, in the case of horizontal bottom, is locally well-posed, cf. [4]. The initial-value problem was integrated with a type of a method of ‘characteristics’. In order to form some idea of the proximity of the model used in [29] to [CBs] we integrated both systems using our fully discrete scheme with cubic splines and RK4 time stepping over a variable bottom domain like that of Figure 6 with $0 \leq x \leq 150$, $x_B = 60$, $h_1 = 0.5$, $\alpha = 1/20$. As initial values we took solitary waves of the respective systems of the same amplitude $a_0 = 0.12$ and centered at $x_0 = 30$. 

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(Their speeds are very close but the wavelength of the solitary wave of the system of [29] was about 22% larger. The difference of the two-solitary waves in $L^2$ was about $4.37 \times 10^{-2}$.) At the end of the computational domain at $t = 22.5$, when both waves had climbed well onto the shelf and resolved themselves into two solitary waves plus dispersive tail, the two wavetrains had an $L^2$ distance of $5.3 \times 10^{-2}$, while the leading solitary waves had a difference in amplitude of about $3 \times 10^{-3}$ and a phase difference (distance of positions of the crest) of 0.15. We conclude that in the time scales of this and similar experiments typical solutions of the two systems stay close to each other, so that it is fair to compare in a general way the results of numerical experiments in [29] with similar ones that we ran with (CBs) to be described in the sequel.

We first make some quantitative remarks on the transformation of the solitary wave as it climbs on the sloping part of the bottom in Figure 6. As observed in subsection 3.3.1, the amplitude of the solitary wave increases as the depth of the water decreases. In order to quantify this increase in the case of (CBs) and our numerical method, and motivated by analogous experiments in [29], we took $h_1 = 0.1$, $\alpha = 1/20$, $0 \leq x \leq 150$, $x_B = 60$, and computed with cubic splines, $N = 3000$, $M = 2N$, the evolution (according to (CBs)) of a solitary wave of (CB) centered at $x = 30$. We recorded the variation of the normalized amplitude $\zeta_{\text{max}}/a_0$ of the solitary wave as a function of the water depth $\eta_b$ for various values of the initial amplitude $a_0$. In Figure 7 we show the outcome of these numerical experiments corresponding to solitary waves of initial amplitudes $a_0 = 0.1, 0.15$ and 0.2. (The graph starts when the crest of the solitary wave is at $x = x_B$. At that point $\zeta_b = 1$, but the forward point of the solitary wave is already travelling on the sloping bed; hence, the corresponding value of $\zeta_{\text{max}}/a_0$ is about 1.034 and not 1. For $\eta_b$ larger than about 0.6 the three curves

![Figure 7: Amplitude variation with depth for beach slope $\alpha = 1/20$ for $a_0 = 0.2, 0.15, 0.1$. Computation stopping criteria: solid lines, $\max_x(\zeta(x,t)/\eta_b(x)) < 0.4$; dotted lines, $\max_x(\zeta(x,t)/\eta_b(x)) < 0.6$.](image)

of these numerical experiments corresponding to solitary waves of initial amplitudes $a_0 = 0.1, 0.15$ and 0.2. (The graph starts when the crest of the solitary wave is at $x = x_B$. At that point $\zeta_b = 1$, but the forward point of the solitary wave is already travelling on the sloping bed; hence, the corresponding value of $\zeta_{\text{max}}/a_0$ is about 1.034 and not 1. For $\eta_b$ larger than about 0.6 the three curves
corresponding to the three amplitudes chosen are quite close to each other with the lowest initial amplitude \( a_0 = 0.10 \) giving the highest values of \( \zeta_{\text{max}}/a_0 \). For \( \eta_b \) smaller than about 0.5 the sequence is reversed with the highest \( a_0 = 0.2 \) giving the highest \( \zeta_{\text{max}}/a_0 \) values. The initial solid-line part of the three curves represents the values of \( \zeta_{\text{max}}/a_0 \) up to the point where \( \max_x \left( \frac{\zeta(x,t)}{\eta_b(x)} \right) = 0.4 \), which is probably a large upper bound of the range of validity of \( \text{(CBs)} \), while the dotted-line extensions of the curves go up to \( \max_x \left( \frac{\zeta(x,t)}{\eta_b(x)} \right) = 0.6 \), which is probably beyond that range. We also show the curve of Green’s law given by \( \zeta_{\text{max}}/a_0 \equiv \eta^{-1/4} \) for comparison purposes. It is to be noted that our results are in satisfactory agreement with those of the corresponding Fig. 3 of [29] for values of \( \eta_b \) in the range 1 to 0.75.

These results are supplemented by those of Figure 8 in which we record the variation of \( \zeta_{\text{max}}/a_0 \) as a function of \( \eta_b \) for a solitary wave of fixed \( a_0 = 0.1 \) and slopes equal to 0.023, 0.05, and 0.065. For \( \eta_b \) larger than about 0.65 all curves are fairly close to each other with the steeper slopes giving slightly higher values of \( \zeta_{\text{max}}/a_0 \). For values of \( \eta_b \) less than about 0.65 the smaller slope gives the highest ratio \( \zeta_{\text{max}}/a_0 \) while the two other curves remain close together (stopping criteria as in Figure 6). A qualitatively similar behavior is observed in the analogous Figure 4 of [29].

The distortion the solitary wave suffers as it travels upslope causes the wave, when it reenters a horizontal-bottom region reaching the shelf, to resolve itself into a sequence of solitary waves followed by dispersive oscillations. This phenomenon was noticed in [29] for the model used in that paper, and is also present in our case of the \( \text{(CBs)} \) system as well. In Figure 9 we show this phenomenon, which may be viewed as a manifestation of the stability of solitary waves of
Figure 9: Transformation of a CB solitary wave \((a_0 = 0.12)\) propagating up a slope of \(\alpha = 1/20\), onto a shelf of smaller depth, \(h_1 = 0.5h_0\). We took a spatial interval \([0, 150]\), \(h_1 = 0.5, x_B = 60, \alpha = 1/20, \) and considered the evolution of a solitary wave of initial amplitude \(a_0 = 0.12\). The graphs in Figure 9 show the temporal evolution every 25 temporal units (“seconds”). The solitary wave distorts as it climbs the sloping part of the bottom (depicted in the lower part of the graph), increases in amplitude, and by \(t = 125\) it has resolved itself into two solitary waves (a third is also possibly forming) plus a dispersive tail. The first solitary wave has an amplitude of about 0.2099 and travels at a speed of about 0.84. (We checked that it is indeed a CB-solitary wave.) This wavetrain is followed by the usual for upsloping environments flat reflection wave that travels to the left. The results of a similar experiment in [29] are qualitatively the same.

3.3.3 Reflection and dispersion from various types of variable bottom

As already mentioned in subsection 3.3.1 when a solitary wave propagates up a sloping bottom, a small-amplitude, flat wave of elevation is generated by reflection from the uneven bottom and travels in the opposite direction. This phenomenon has been shown e.g. in Figs 5 and 9. (In this subsection we work again in dimensionless, unscaled variables with \(\varepsilon = \mu = 1.\) Using characteristic variables theory for the linearized shallow water equations, in addition to the approximate formula (3.8) for the reflected wave, Peregrine predicted in [20] that the reflected wave will have a wavelength of about \(2L\) if the slope occurs over a horizontal interval of length \(L\). In order to check these results we integrated the CBs over the variable bottom shown in the lower graph of Figure 9 with an initial solitary wave of CB, varying the slope and the initial amplitude \(a_0\) of the wave; we present the results in Table 3 that shows the amplitudes and
wavelengths of the reflected wave predicted in [20] and the numerical results given by our code. (Due to the shape of the reflected wave we measured its length by the formula \( \frac{1}{\pi} \int_I \zeta \, dx \), where \( I = \{ x : \zeta > 0.8 \zeta_{\text{max}} \} \), at a short time after the full reflected wave had formed. In the case \( \alpha = 1/40 \), \( a_0 = 0.18 \), we took \( I = \{ x : \zeta > 0.6 \zeta_{\text{max}} \} \).) We conclude that the predictions of [20]

\[
\begin{array}{cccccc}
\alpha & a_0 & L & \text{refl. ampl. by (3.8)} & \text{reflected amplitude} & \text{reflected wavelength} \\
1/20 & .12 & 10 & 5.000e-3 & 5.578e-3 & 22.35 \\
1/40 & .12 & 20 & 2.500e-3 & 2.875e-3 & 43.00 \\
1/20 & .18 & 10 & 6.124e-3 & 6.880e-3 & 21.25 \\
1/40 & .18 & 20 & 3.062e-3 & 3.451e-3 & 41.65 \\
\end{array}
\]

Table 3: Predicted and numerical values of amplitude and wavelength of reflected wave.

underestimate by a small amount the actual numerical results.

In [20] Peregrine also made some qualitative comments about the type of reflected waves generated by various kinds of uneven bottoms. We verified his general statements by performing various numerical experiments, the results of some of which appear in Figure 6. In each case an initial wave, originally on a horizontal bottom, is let to evolve under (CBs) and travel over uneven bottoms of various simple topographies shown in the lower graphs in Figure 10. Fig. 10(a) shows a CB solitary wave of amplitude \( a_0 = 0.12 \) passing into shallower water. The resulting reflected wave is a wave of depression; this solitary wave seems to be dispersing as a result of its interaction with the bottom. In the case of a hump (Fig. 10(b)) the same initial wave gives rise first to a reflected wave of elevation followed by a reflected wave of depression as one would expect. This particular perturbation due to this bottom topography seems to lead to a solitary wave very close to the initial one plus a trailing dispersive tail. Finally, an initial wave of depression climbing upslope gives rise to a reflected wave of depression and large-amplitude dispersive oscillations as it travels on the shelf.

3.3.4 Comparison of (CBs) and (CBw) as the variation of the bottom increases

As was mentioned in the Introduction (CBs) is valid as a model for bottoms where topography, described by \( \eta_b(x) = 1 - \beta b(x) \), may vary arbitrarily (so that \( \eta_b > 0 \) of course), i.e. where the parameter \( \beta \) can be taken as an \( O(1) \) quantity, while (CBw) was derived under the assumption that \( \beta = O(\varepsilon) \). In this subsection we suppose that the systems are written in scaled, dimensionless variables with \( \mu = \varepsilon \) and we compare computationally the behavior of an initial CB solitary wave as it evolves according to each of the two systems travelling over a bottom of smooth topography with a fixed shelf-like function \( b(x) \) and a parameter \( \beta \) that varies from \( O(\varepsilon) \) to \( O(1) \), so that the bottom becomes steeper.
Figure 10: Reflection due to sloping bottom, various topographies. $\zeta(x,t)$ as a function of $x$ at various $t$. (a) solitary wave travelling into deeper water, (b) solitary wave passing over a hump, (c) wave of depression travelling into shallower water.
For this purpose we solve both systems with our fully discrete scheme using cubic splines with uniform mesh, \( N = 2000 \) and the RK4 with \( M = 2N \) on a spatial interval of \([0, 140]\) with a CB solitary wave of amplitude 0.5 as initial condition. (We experimented with several values of \( \varepsilon = \mu \) but the results were qualitatively similar, so we show in Figure 11 below only the case \( \varepsilon = \mu = 0.05 \).)

As \( b(x) \) we took a fixed profile given by

\[
b(x) = \begin{cases} 
0, & x \in \left[0, L - \frac{3}{2}\right], \\
\frac{1}{2} (1 + \sin \left( \frac{\pi}{3} (x - L) \right)), & x \in \left[L - \frac{3}{2}, L + \frac{3}{2}\right], \\
1, & x \in \left[L + \frac{3}{2}, 140\right],
\end{cases}
\]

with \( L = 70 \). Thus \( b \) is a \( C^1 \) nonnegative function that bridges 0 and 1 over an interval of length 3. As a result, the undisturbed water depth \( \eta_b \) will vary from 1 to a shelf of depth \( 1 - \beta \) smoothly over this interval. We consider three cases: \( \beta = \varepsilon = 0.05, \beta = 0.4, \beta = 0.6, \) and present the results of the evolution for \( 0 \leq t \leq 89 \) in Figure 11. In Fig. 11(a), where \( \beta = \varepsilon = 0.05, \) there is, as expected, practically no difference between the two solitary waves that suffer only a very small perturbation due to the bottom. But for \( \beta = O(1), \) i.e. when the bottom is steeper, we observe in Figure 11(b) \( (\beta = 0.4) \) and 11(c) \( (\beta = 0.6) \) large differences in the solutions of the two systems. As it travels on the shelf the solitary wave evolves under \( \text{(CBs)} \) into a sequence of solitary waves as expected, whilst no such resolution is discernible in the case of the evolution under \( \text{(CBw)} \) at least for the time frame of this experiment. Both systems produce the same small-amplitude reflection waves. Our conclusion is that for \( \beta = O(1) \) \( \text{(CBw)} \) does not seem to give the correct longer-time behaviour of solutions in the case of strongly varying bottoms.

### 3.3.5 Comparison of (CBs) with the Serre-Green-Naghdi system

Finally, we compare by means of numerical experiment, the evolution of an initial solitary wave as it climbs a sloping bed, and as it is reflected by a vertical wall at the end of a slope. Recall from the Introduction that the system of Serre-Green-Naghdi (SGN) equations models two-way propagation of long dispersive waves (i.e. for which \( \mu \ll 1 \)) without the assumption of small amplitude, i.e. with no restriction of \( \varepsilon, \) and that \( \text{(CBs)} \) is obtained from the (SGN) system with variable bottom under the Boussinesq scaling \( \varepsilon = O(\mu), \) [9]. The SGN system has been used in many computations, cf. e.g. [16], [17], [18], and their references, that agree quite well with experimental results of long-wave propagation over variable bottoms. In [19], two of the authors of the paper at hand, together with D. Mitsotakis, have analyzed Galerkin-finite element methods for \( \text{SGN} \) on a horizontal bottom (i.e. for the Serre equations) and shown optimal-order, \( L^2 \)-error estimates in the case of periodic splines \( (r \geq 3) \) on uniform meshes.

Our aim in this subsection is to compare the results of numerical simulations of two test problems with \( \text{(CBs)} \), computed with our code, with numerical results for \( \text{SGN} \) obtained by Mitsotakis et al. in [18]. The spatial semidiscretization used in [18] is based on a modified Galerkin finite element scheme that uses
Figure 11: Comparison of evolution of a solitary wave under (CBw) and (CBs) over a bottom of varying steepness: $\eta_b = 1 - \beta b(x)$, $b(x)$ given by (3.10), $\varepsilon = \mu = 0.05$. 

(a) $\beta = \varepsilon = 0.05$

(b) $\beta = 0.4$

(c) $\beta = 0.6$
a projection of a term containing a second-order derivative in SGN so that the scheme is also well defined for piecewise linear continuous elements (i.e. for $r = 2$) as well. In what follows we will solve numerically (CBs) using cubic splines on a uniform mesh with $N = 2000$ and RK4 time stepping with $M = 2N$. All variables for this experiment are nondimensional and unscaled with $\varepsilon = \mu = 1$.

In the first experiment (shoaling of a solitary wave) we consider the variable-bottom example in §4.1 of [18]. The geometry, in our notation, consists of a channel in the interval $[0, 84]$. The bottom is horizontal at a depth equal to $-1$ for $0 \leq x \leq x_B = 50$, and upsloping with slope $\alpha = 1/35$ up to $x = 84$ where the water depth is equal to $1/35$. The initial condition is a solitary wave of the form (3.6), (3.9) of amplitude $a_0 = 0.2$ with crest at $x_0 = 29.8829$. The evolution of the numerical solution is monitored at ten gauges numbered 0, 1, . . . , 9, and located, respectively, at $x = 45$, 70.96, 72.55, 73.68, 74.68, and 76.91. In this experiment the variables are dimensionless and unscaled with $\varepsilon = \mu = 1$. In the experimental data and the (SGN) computations $g$ was equal to 1. In Figure 12 we show the elevation of the wave at gauge 0 (at $x = x_B - 5 = 45$, i.e. on the left of the toe of the slope), as a function of $t$. The three graphs shown correspond to the numerical solutions of (CBs) and (SGN), and to experimental data for this problem due to Grilli et al. [31], and are all in satisfactory agreement. Figure 13 shows the corresponding graphs of the elevation of the wave as a function of time recorded at gauges 1, 3, 5, 7, and 9 on the sloping bed. The numerical solution of (SGN) is in good agreement with the experimental data of [31]. As the wave climbs up the slope the (CBs) solution grows to a higher amplitude, whose ratio to the amplitude of the (SGN) wave increases monotonically from 1.14 for gauge 1 to 1.49 for gauge 9.

![Figure 12: Elevation of wave at $x = x_B - 5 = 45$ as a function of time. Markers show the experimental data, [31], dotted lines the numerical solution of (SGN), [18], and solid lines the numerical solution of (CBs) system.](image)

For the second numerical experiment (shoaling and reflection of a solitary
Figure 13: Elevation of wave at various gauges as a function of time for the shoaling on a beach of slope 1 : 35 of a solitary wave with $a_0 = 0.12$. Markers show experimental data, [31], dotted lines the numerical solution of \((SGN)\), [18], and solid lines the numerical solution of \((CBs)\) system.

wave from a vertical wall at the end of the sloping beach), we consider a benchmark problem, cf. e.g. [18], [32], [33], [16], [17], among other, that we solve numerically with our code of \((CBs)\) and compare the results with those found by the numerical integration of \((SGN)\) in Section 4.3 of [18], and with experimental data due to Dodd, [33]. The setup consists of a channel of length \([0, 70]\), initially horizontal at a depth of $h_0 = 0.7$, a sloping bed of uniform slope 1 : 50 that starts rising at $x_B = 50$ and ends at $x = 70$, where a vertical wall is placed. (This is shown in the lower graph of Figure 14.) We consider two solitary waves of the form \((3.6), (3.9)\) (suitably modified so that the horizontal part of the waveguide has now a depth of $h_0 = 0.7$) with amplitudes 0.07 and 0.12 and crest initially located at $x = 20$. We solve the problem numerically with our code for \((CBs)\) with a boundary condition $u = 0$ using cubic splines, $N = 2000$, $M = 2N$. All variables for this experiment are dimensional, $x$ and $\eta$ are measured in meters and $t$ in seconds. The parameters $\epsilon$ and $\mu$ are equal to 1. The value of the gravitational acceleration constant is $g = 9.80665 \text{m/s}^2$ (standard gravity).

In Figure 14 we show snapshots every 3 secs of the \((CBs)\)-free surface elevation as a function of $x$ as the wave (of initial amplitude $a_0 = 0.07$) climbs up the slope and is reflected by the wall at $x = 70$ between $t = 15$ and $t = 18$. The reflected pulse apparently consists of a leading pulse followed by a dispersive tail. This wave travels downslope, and by $t = 30$ the leading pulse is located well within the horizontal-bottom region. The maximum runup at the wall was recorded to be equal to .1899.

In the (related) Figure 15 we show the temporal histories of the wave elevation $\zeta(x, t)$, generated by the solitary wave of amplitude $a_0 = 0.07$, at three gauges $g_1, g_2, g_3$, located at $x = 50$, $x = 66.25$, and $x = 67.75$ (very close to the
Figure 14: Evolution of the solitary wave of amplitude $a_0 = 0.07$ according to \( \text{CBs} \) on a beach of slope $1:50$, reflected on a vertical wall at $x = 70$. Vertical lines depict the location of gauges 1, 2 and 3.

We observe that there is quite a good agreement between the three curves. The maximum amplitude of the reflected wave at gauge $g_3$ is found to be equal to 0.11080 for \( \text{CBs} \) and to 0.10280 for \( \text{SGN} \), giving a ratio of about 1.08.

Figure 16 depicts the analogous graphs in the case of the initial solitary wave of amplitude $a_0 = 0.12$. (Note the different scale of the $\zeta$-axis.) This wave becomes steeper as it climbs up the slope; the reflected wave is of higher amplitude as well. The incident waves computed by the two models are quite close to each other and to the experimental data but the short-time behavior of the reflected pulse is somewhat different. For example, at $g_3$ the amplitude of the reflected \( \text{CBs} \) pulse is now equal to 0.2285 while the amplitude of the \( \text{SGN} \) reflected pulse is 0.1838 (giving a ratio of about 1.24), and there are phase and amplitude differences in the leading trailing oscillations. When the reflected wave has returned to the horizontal part of the channel (i.e. at $g_1$ in Figure 16 for $t \geq 25$) the agreement is much better and the ratio is now 0.98. The leading reflected pulse of the \( \text{SGN} \) solution is in satisfactory agreement with the data at all three gauges. The maximum runup at the wall of \( \text{CBs} \) for this amplitude was equal to 0.4012.

Our conclusion from the two numerical experiments in this subsection is that when the elevation wave steepens either while climbing up a sloping beach or after reflection from a vertical wall and close to the wall, the \( \text{CBs} \) solution overestimates that of the \( \text{SGN} \); the latter stays quite close to the available experimental data in the cases that we tried.
Figure 15: Reflection at a vertical wall located at $x = 70$ of a shoaling wave over a beach of slope 1:50, with toe at $x_B = 50$. Initial solitary wave amplitude $a_0 = 0.07$.

Figure 16: Reflection at a vertical wall located at $x = 70$ of a shoaling wave over a beach of slope 1:50, with toe at $x_B = 50$. Initial solitary wave amplitude $a_0 = 0.12$.  

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Acknowledgements

This research was partially supported by IACM-FORTH by the grant “Innovative Actions in Environmental Research and Development (PÉrAn)” (MIS 5002358), implemented under the “Action for the strategic development of the Research and Technological sector” funded by the Operational Program “Competitiveness, and Innovation” (NSRF 2014-2020) and cofinanced by Greece and the EU (European Regional Development Fund). G. Kounadis also acknowledges scholarship support in the initial stages of the project from the Stavros Niarchos Foundation ‘Archers’ grant to FORTH. The authors also express their thanks to Dr. D. E. Mitsotakis for making available to them the numerical data for [SGN] of [18] quoted in the last two experiments.

References

[1] G. B. Whitham, Linear and Nonlinear Waves, Wiley, 1974.

[2] M. E. Schonbek, Existence of solutions for the boussinesq system of equations, Journal of Differential Equations 42 (3) (1981) 325–352. doi:10.1016/0022-0396(81)90108-X

[3] C. J. Amick, Regularity and uniqueness of solutions to the boussinesq system of equations, Journal of Differential Equations 54 (2) (1984) 231–247. doi:10.1016/0022-0396(84)90160-8

[4] J. L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. the nonlinear theory, Nonlinearity 17 (3) (2004) 925–952. doi:10.1088/0951-7715/17/3/010

[5] K. Adany, Existence of solutions for a boussinesq system on the half line and on a finite interval, Discrete & Continuous Dynamical Systems - A 29 (1) (2011) 25–49. doi:10.3934/dcds.2011.29.25

[6] D. C. Antonopoulos, V. A. Dougalis, Error estimates for galerkin approximations of the classical boussinesq system, Mathematics of Computation 82 (282) (2013) 689–717. doi:10.1090/S0025-5718-2012-02663-9

[7] D. C. Antonopoulos, V. A. Dougalis, Numerical solution of the classical boussinesq system, Mathematics and Computers in Simulation 82 (6) (2012) 984–1007, nonlinear Waves: Computation and Theory-IX, WAVES 2009. doi:10.1016/j.matcom.2011.09.006

[8] D. C. Antonopoulos, V. A. Dougalis, D. E. Mitsotakis, Galerkin approximations of periodic solutions of boussinesq systems, Bull. Greek Math. Soc 57 (2010) 13–30.
[9] D. Lannes, P. Bonneton, Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation, Physics of Fluids 21 (1) (2009) 016601. doi:10.1063/1.3053183

[10] F. Serre, Contribution à l’étude des écoulements permanents et variables dans les canaux, La Houille Blanche (3) (1953) 374–388. doi:10.1051/lhb/1953034

[11] F. Serre, Contribution à l’étude des écoulements permanents et variables dans les canaux, La Houille Blanche (6) (1953) 830–872. doi:10.1051/lhb/1953058

[12] A. E. Green, P. M. Naghdi, A derivation of equations for wave propagation in water of variable depth, Journal of Fluid Mechanics 78 (2) (1976) 237–246. doi:10.1017/s0022112076002425

[13] D. Lannes, The Water Waves Problem: Mathematical Analysis and Asymptotics, Vol. 188, American Mathematical Society, Providence, RI, 2013. doi:10.1090/surv/188

[14] S. Israwi, Large time existence for 1d green-naghdi equations, Nonlinear Analysis: Theory, Methods & Applications 74 (1) (2011) 81–93. doi:10.1016/j.na.2010.08.019

[15] E. Barthélemy, Nonlinear shallow water theories for coastal waves, Surveys in Geophysics 25 (3-4) (2004) 315–337. doi:10.1007/s10712-003-1281-7

[16] R. Cienfuegos, E. Barthélemy, P. Bonneton, A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive boussinesq-type equations. part ii: boundary conditions and validation, International Journal for Numerical Methods in Fluids 53 (9) (2007) 1423–1455. doi:10.1002/fld.1359

[17] P. Bonneton, F. Chazel, D. Lannes, F. Marche, M. Tissier, A splitting approach for the fully nonlinear and weakly dispersive green–naghdi model, Journal of Computational Physics 230 (4) (2011) 1479–1498. doi:10.1016/j.jcp.2010.11.015

[18] D. E. Mitsotakis, C. Synolakis, M. McGuinness, A modified galerkin/finite element method for the numerical solution of the serre-green-naghdi system, International Journal for Numerical Methods in Fluids 83 (10) (2017) 755–778. doi:10.1002/fld.4293

[19] D. C. Antonopoulos, V. A. Dougalis, D. E. Mitsotakis, Error estimates for galerkin approximations of the serre equations, SIAM Journal on Numerical Analysis 55 (2) (2017) 841–868. doi:10.1137/16M1078355

[20] D. H. Peregrine, Long waves on a beach, Journal of Fluid Mechanics 27 (4) (1967) 815–827. doi:10.1017/S0022112067002605
[21] M. Chen, Equations for bi-directional waves over an uneven bottom, Mathematics and Computers in Simulation 62 (1) (2003) 3–9. doi:10.1016/S0378-4754(02)00193-3

[22] F. Chazel, Influence of bottom topography on long water waves, ESAIM: Mathematical Modelling and Numerical Analysis 41 (4) (2007) 771–799. doi:10.1051/m2an:2007041

[23] G. Kounadis, Numerical methods for shallow water equations, Ph.D. thesis, National and Kapodistrian University of Athens (2020).

[24] J. Douglas, T. Dupont, L. Wahlbin, Optimal $l_\infty$ error estimates for galerkin approximations to solutions of two-point boundary value problems, Mathematics of Computation 29 (130) (1975) 475–483. doi:10.1090/S0025-5718-1975-0371077-0

[25] D. C. Antonopoulos, V. A. Dougalis, Galerkin-finite element methods for the shallow water equations with characteristic boundary conditions, IMA Journal of Numerical Analysis 37 (1) (2017) 266–295. doi:10.1093/imanum/drw017.

[26] G. Kounadis, V. A. Dougalis, “galerkin finite element methods for the shallow water equations over variable bottom”, (to appear in J. Comput. Appl. Math) (2019). arXiv:1901.04230

[27] F. Dias, P. Milewski, On the fully-nonlinear shallow-water generalized serre equations, Physics Letters A 374 (8) (2010) 1049–1053. doi:10.1016/j.physleta.2009.12.043

[28] MATLAB, version 9.4.0 (R2018a), The MathWorks Inc., Natick, Massachusetts, 2018.

[29] O. S. Madsen, C. C. Mei, The transformation of a solitary wave over an uneven bottom, Journal of Fluid Mechanics 39 (4) (1969) 781–791. doi:10.1017/S0022112069002461

[30] C. C. Mei, B. Le Méhauté, Note on the equations of long waves over an uneven bottom, Journal of Geophysical Research 71 (2) (1966) 393–400. doi:10.1029/JZ071i002p00393

[31] S. Grilli, R. Subramanya, I. Svendsen, J. Veeramony, Shoaling of solitary waves on plane beaches, Journal of Waterway, Port, Coastal, and Ocean Engineering 120 (6) (1994) 609–628. doi:10.1061/(ASCE)0733-950X(1994)120:6(609)

[32] M. Walkley, M. Berzins, A finite element method for the one-dimensional extended boussinesq equations, International Journal for Numerical Methods in Fluids 29 (2) (1999) 143–157. doi:10.1002/(SICI)1097-0363(19990130)29:2<143::AID-FLD779>3.0.CO;2-S
[33] N. Dodd, Numerical model of wave run-up, overtopping, and regeneration, Journal of Waterway, Port, Coastal, and Ocean Engineering 124 (2) (1998) 73–81. doi:10.1061/(ASCE)0733-950X(1998)124:2(73)