The \( q \)-calculus for generic \( q \) is developed and related to the deformed oscillator of parameter \( q^{1/2} \). By passing with care to the limit in which \( q \) is a root of unity, one uncovers the full algebraic structure of \( \mathbb{Z}_n \)-graded fractional supersymmetry and its natural representation.

1 Introduction.

The \( q \)-calculus, in the generic case in which the fixed complex number \( q \) is not a root of unity but is otherwise arbitrary, involves a single non-commuting variable \( \theta \) and its left derivative operator \( \mathcal{D} = \mathcal{D}_L \) and is governed by the commutation relation

\[
\mathcal{D}\theta - q\theta\mathcal{D} = 1.
\]

It is closely related to the \( q \)-deformed oscillator \cite{1,2,3}, as is shown below.

The context in which \( q \) is a root of unity, \( q = \exp\left( \frac{2\pi i}{n} \right) \), is also of great interest. It involves \( \theta \) such that \( \theta^n = 0 \) and can be discussed by truncating the generic case so as to exclude powers of \( \theta \) higher than the \((n-1)\)-th. However, if we pass with care from the generic case to the limit in which \( q \) is a root of unity much more structure can be exposed. The algebraic structure in question is the full algebraic structure of fractional supersymmetry (FSUSY), not only the generalised Grassmann sector of this \( \mathbb{Z}_n \)-graded theory which is the part that where \( \theta \) enters but also its bosonic sector. The paper shows how both these ‘sectors’ emerge and discusses the representation of the theory in a product Hilbert space. This has an ordinary oscillator factor for the bosonic degree of freedom, and relates the generalised Grassmann sector to the \( q \)-deformed oscillator with deformation parameter \( q^{1/2} \), which is exactly what is needed to ensure proper hermiticity properties. We do not here make any extensive discussion of the interplay between the sectors. But some idea of the insights regarding this interplay can be obtained from \cite{4} which is devoted to the case of \( q = -1 \), which is that of ordinary (i.e., \( \mathbb{Z}_2 \)-graded) supersymmetry in zero space dimension. It seems worthwhile emphasising that the \( q \)-deformed oscillators at deformation parameter \( q^{1/2} \) emerge as those generalisations from \( n = 2 \) to higher
References to FSUSY, including many to the extensive work of others, can be found in our published \cite{5} and forthcoming \cite{6} work.

2 The $q$-calculus.

For any graded algebra, we define a graded bracket, initially for elements $A$ and $B$ of pure grade $g(A)$ and $g(B)$, by

$$[A, B]_{\gamma(A,B)} = AB - \gamma(A, B) BA, \quad \gamma(A, B) := q^{-g(A)g(B)} \quad (2)$$

This satisfies

$$[AB, C]_{\gamma(AB,C)} = A[B, C]_{\gamma(B,C)} + \gamma(B, C)[A, C]_{\gamma(A,C)} B \quad (3)$$

$$[A, BC]_{\gamma(A,BC)} = [A, B]_{\gamma(A,B)} C + \gamma(A, B)[A, C]_{\gamma(A,C)} B \quad (4)$$

wherein $g(AB) = g(A) + g(B)$ is implicit. The definition (2) extends by linearity to elements of the algebra not of pure grade.

In (2) and until section four, $q$ is ‘generic’ i.e., it is a fixed but arbitrary complex quantity that is not a root of 1. To define the $q$-calculus algebra, we employ a single non-commuting variable $\theta$ of grade 1, together with left and right derivatives $D_L$ and $D_R$ of grade $-1$. Since we shall not refer to $D_R$ here (cf. \cite{5,7}), we shall write $D_L \equiv D$. The action of $D$ upon powers and hence functions of $\theta$ is defined algebraically with the help of the graded bracket

$$1 = [D, \theta]_q := D\theta - q\theta D \quad (5)$$

so that, for any positive integer $m$, we have

$$[D, \theta^{(m)}]_q = \theta^{(m-1)} \quad (6)$$

where the bracketed exponent is defined by

$$B^{(m)} := B^m/[m]_q!, \quad [m]_q = (1 - q^m)/(1 - q) \quad (7)$$

The action extends obviously to $f(\theta) = \sum_{m=0}^{\infty} C_m \theta^{(m)}$, where the $C_m$ are complex numbers

$$\frac{df}{d\theta} \equiv [D, f(\theta)]_{\gamma} = \sum_{m=0}^{\infty} C_m [D, \theta^{(m)}]_{\gamma} = \sum_{m=1}^{\infty} C_m \theta^{(m-1)} \quad (8)$$

It extends further also to $f(\theta) = \sum_{m=0}^{\infty} \theta^{(m)} A_m$, where the $A_m$ are quantities independent of $\theta$ and of pure grade $g(A_m)$, so that $[\theta, A_m]_{\gamma}$ with $\gamma = q^{-g(A_m)}$.
The $q$-calculus for generic $q$ and $q$ a root of unity

giving

\[ \frac{df}{d\theta} = [\mathcal{D}, f(\theta)]_{\gamma} = \sum_{m=0}^{\infty} [\mathcal{D}, \theta^{(m)} A_m]_{\gamma}, \]

\[ = \sum_{m=0}^{\infty} ([\mathcal{D}, \theta^{(m)}] q^m A_m + q^m A_m [\mathcal{D}, A_m]_{\gamma}) = \sum_{m=1}^{\infty} \theta^{(m-1)} A_m, \quad (9) \]

provided that $[\mathcal{D}, A_m]_{\gamma} = 0$ where $\gamma = q^{A_m}$ (cf. 8), which can be seen to be compatible with the corresponding result assumed for $\theta$ and $A_m$. An illustration, featuring $\exp_q(\theta A) = \sum_{m=0}^{\infty} (\theta A)^{(m)}$ and wherein $A$ is of pure grade, gives rise to

\[ \frac{d}{d\theta} \exp_q(\theta A) = A \exp_q(\theta A). \quad (10) \]

Another notable result, involving a parameter $\varepsilon$ of grade 1 with $[\mathcal{D}, \varepsilon]_{q^{-1}} = 0$ and $[\varepsilon, \theta]_{q^{-1}} = 0$, shows that the quantity $G_L(\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^{(m)} \mathcal{D}^m$ generates the translation $\theta \mapsto \theta + \varepsilon$, i.e. $G_L(\varepsilon) f(\theta) G_L(\varepsilon)^{-1} = f(\theta + \varepsilon)$.

3 Number, creation and destruction operators.

We begin by constructing a number operator $N$ of grade zero with the properties

\[ [N, \theta] = \theta, \quad q^N \theta q^{-N} = \theta, \]

\[ [N, \mathcal{D}] = -\mathcal{D}, \quad q^N \mathcal{D} q^{-N} = q^{-1} \mathcal{D}. \quad (11) \]

Since $N$ is of grade zero, an expression for it in terms of $\theta$ and $\mathcal{D}$ may be expected to be of the form

\[ N = \sum_{m=0}^{\infty} C_m \theta^m \mathcal{D}^m, \quad (12) \]

and

\[ N = \sum_{m=1}^{\infty} \frac{(1-q)^m}{1-q^m} \theta^m \mathcal{D}^m, \quad (13) \]

satisfies both lines of (11). Likewise the right entries of (11) may be shown to be satisfied by

\[ q^N = \mathcal{D} \theta - \theta \mathcal{D} = 1 - (1-q)\theta \mathcal{D}, \quad (14) \]

where (8) has been used. Useful consequences of these results include

\[ \theta \mathcal{D} = [N]_q, \quad \theta^m \mathcal{D}^m = \frac{[N]_q^m}{[N-m]_q}, \quad N = \sum_{m=1}^{\infty} \frac{(1-q)^m}{1-q^m} \frac{[N]_q^m}{[N-m]_q}. \quad (15) \]

3
The last result in (15) is of interest as it gives an expression for \( N \) in terms of \( q^N \).

Acting on an eigenstate of \( N \) whose eigenvalue is a positive integer \( r \), this yields the identity

\[
r = \sum_{m=1}^{r} \frac{(1 - q)^m}{1 - q^m} \frac{[r]_q!}{[r-m]_q!}.
\]

(16)

If we now make the identification (to within a similarity transformation, discussion of which may be sought in [6])

\[
\theta = a^\dagger, \quad D = q^{N/2}a
\]

(17)

then (5) and (14) imply

\[
a a^\dagger - q^{\mp 1/2} a^\dagger a = q^{\pm N/2}.
\]

(18)

This important result indicates how the \( q \)-calculus is related to the \( q \)-deformed harmonic oscillator [1], [2] and [3]. If \( q \) is real, (18) admits representations in which \( a^\dagger \) is indeed the adjoint of \( a \) in a positive definite Hilbert space. Further, for \( q = \exp 2\pi i/n \) when \( n \) is an odd integer, the situation to be concentrated upon below, a similar statement also holds true because of the fact that the deformation parameter in (18) is \( q^{1/2} \). For simplicity the remaining sections of the Colloquium talk confined discussion to the indicated set of roots of unity. But the case \( q = -1 \) can also be treated in a similar spirit. It is of interest because it underlies an instructive view [4] of ordinary supersymmetry in much the same way as the present work does for fractional supersymmetry.

4 Lemmas for use at \( q \) a root of 1.

We now confine attention –as mentioned at the end of the previous section– to the \( q \)-values \( q = \exp \frac{2\pi i}{n} \) for odd integer \( n \). We use the shorthand \( \mathcal{L} \) to indicate the passage to the limit in which \( q \) takes on such \( q \)-values, \( \mathcal{L} := \lim_{q \to \exp(2\pi i/n)} \). We here deduce a sequence of lemmas to be used in subsequent sections to effect the systematic passage to the limit in question in the work of previous sections.

The following results can be proved in the order given:

\[
\mathcal{L} \frac{[rn]_q}{[n]_q} = r, \quad \text{for integer } r, \quad \mathcal{L} \frac{[rn]_q!}{[n]_q!((r-1)n)_q!} = r, \quad \mathcal{L} \frac{[rn]_q!}{[(n)_q]^r} = r!,
\]

(19)

In the next section, we shall retain \( q^{(m)} \), in the notation (6), for \( m = 1, 2, \ldots, (n-1) \) as \( \mathbb{Z}_n \)-graded variables, and explain the use of the case \( m = n \) to define a variable \( z \) of zero grade by setting \( z = \mathcal{L} q^{(n)} \). To handle powers \( m \) greater than \( n \), we require further lemmas to be deduced in order. Set \( m = rn + p \) for integer \( r \) and \( p = 1, 2, \ldots, (n-1) \). Then we have

\[
[rn + p]_q = [p]_q, \quad \mathcal{L} \frac{[rn + p]_q!}{[rn]_q!} = [p]_q!, \quad p = 1, \ldots, (n-1).
\]

(20)
The $q$-calculus for generic $q$ and $q$ a root of unity

\[ \mathcal{L}^{(r \theta^{n+p})} = \mathcal{L} \frac{\theta^{r \theta^{n+p}}}{([n]_q)^r r!} \mathcal{L}^{(r \theta^{n+p})} = \frac{\theta^p}{[p]_q} \frac{1}{r!} \mathcal{L} \left( \frac{\theta^n}{[n]_q} \right)^r = \frac{z^r}{r!} \theta^p . \]

(21)

An illustration of the use of these lemmas indicates what happens to $\exp_q(C\theta)$, $C$ a complex number, in the limit under study. We find

\[ \exp_q(C\theta) = \sum_{m=0}^{\infty} C^m \theta^{(m)} = \left( \sum_{r=0}^{\infty} \left( \frac{z C^m}{r!} \right) \left( \sum_{p=0}^{n-1} C^p \theta^{(p)} \right) \right) , \]

(22)

In other words

\[ \exp_q(C\theta) = \exp(z C^n) \times \text{truncated series} . \]

(23)

5 The $q$-calculus for $q = \exp 2 \pi i / n$ for odd integer $n$.

Now we consider what happens to the $q$-calculus for those values of $q$. We look first at an identity valid for generic complex values of $q$ and any positive integer $m$, namely

\[ [\mathcal{D}, \theta^{(m)}] = \theta^{(m-1)} , \]

(24)

where the notation \((\mathcal{L})\) is used. This makes sense for $m = n$ and $[n]_q = 0$ only if $\theta^n = 0$ at this $q$, and if $\mathcal{L} \theta^{(n)}$ attains a finite non-zero value. Indeed, we hereby define a new variable $z = \mathcal{L} \theta^{(n)}$ of grade zero, so that (24) assumes the form

\[ [\mathcal{D}, z] = \theta^{(n-1)} . \]

(25)

Also we see that the $q$-calculus involves the variables

\[ 1, \theta, \theta^{(2)}, \ldots, \theta^{(n-1)} \]

of grades $0, 1, 2, \ldots, n-1$ .

(26)

It is natural at this point to ask what happens to powers of the generalised Grassmann variable $\theta$ higher than the $n$-th. If they are simply discarded much insight into the nature of fractional supersymmetry \((\mathcal{L})\) (and likewise of ordinary supersymmetry \((\mathcal{L})\)) is lost. Actually lemma (21) of the previous section gives us directly an explicit non-trivial answer to the question. It follows that the generalised superfields of the context are linear combinations of the variables \((\mathcal{L})\) with coefficients that are functions of $z$ Thus $z$ plays the role for the present ($\mathbb{Z}_n$-graded fractional supersymmetry) context that $t$ plays in ordinary ($\mathbb{Z}_2$-graded) supersymmetric mechanics.

Next it is natural to ask about $\mathcal{D}^n$ and to ask how $\frac{\partial}{\partial z}$ enters the picture, plainly not unrelated matters. By looking at a suitable $n$-fold graded bracket involving $\theta$ and $\mathcal{D}$ each $n$ times, it is not hard to show that $\mathcal{D}^n$ must be a well defined quantity such that

\[ [\mathcal{D}^n, z] = 1 . \]

(27)

Thus we make the identification $\mathcal{D}^n = \frac{\partial}{\partial z}$.
R. S. Dunne, A. J. Macfarlane, J.A. de Azcárraga and J. C. Pérez Bueno

It is clear that we must adjust somewhat our view of the nature of the derivative operator $\mathcal{D}$. Presenting (24) in the form

$$[\mathcal{D}, z] = \theta^{(n-1)}\left(\frac{dz}{d\theta}\right),$$

suggests that we now must view $\mathcal{D}$ as a total derivative with respect to $\theta$ and write

$$\mathcal{D} = \frac{\partial}{\partial \theta} + \theta^{(n-1)}\frac{\partial}{\partial z},$$

which corresponds to the result

$$\left(\frac{df}{d\theta}\right) = \left(\frac{\partial f}{\partial \theta}\right) + \left(\frac{dz}{d\theta}\right)\frac{\partial f}{\partial z}.$$  

(30)

It follows from (29) that

$$1 = \left[\frac{\partial}{\partial \theta}, \theta\right]_{q}, \quad (\partial_{\theta})^{n} = 0.$$  

(31)

It might be judged from the form of (29) that $\mathcal{D}$ is closely related to the full supercharge of the $\mathbb{Z}_{n}$-graded fractional supersymmetry (FSUSY), and it can be seen in [6] (see [7] for $\mathbb{Z}_{3}$) that this is exactly correct. That $\mathcal{D}$ should therefore generate the full translational invariance of the theory is one aspect of this. We wish to exhibit how this emerges from the results at the end of section two where $\mathcal{D}$ is seen to generate translation of $\theta$ at generic $q$. First we note that $\theta \mapsto \theta + \varepsilon$ is compatible with $\theta^{n} = 0$ only if $\varepsilon^{n} = 0$, holds in addition, of course, to $\varepsilon\theta = q - 1 \varepsilon\theta$. Next, using lemmas from section four, we deduce

$$G_{L} = \mathcal{L} \sum_{m=0}^{\infty} \varepsilon^{(m)}D^{m} = \mathcal{L} \sum_{r=0}^{\infty} \sum_{p=0}^{n-1} \frac{\varepsilon^{p}D^{p}}{[p]_{q}!} \times \frac{\varepsilon^{r}n^{r}}{([n]_{q}!)^{r!}}.$$  

(32)

This makes it clear that we should define a grade zero parameter to associate with a translation of $z$ by means of

$$\mathcal{L}_{\varepsilon}^{(n)} = z\varepsilon.$$  

(33)

For then it follows that we may write (32) in the form

$$G_{L}(z\varepsilon, \epsilon) = \sum_{r=0}^{\infty} \sum_{p=0}^{n-1} \frac{z^{r}\partial^{r} \varepsilon^{(p)}D^{p}}{[p]_{q}!} \times \varepsilon^{(n)}D^{n} = \exp(z\varepsilon\partial_{z}) \sum_{p=0}^{n-1} \varepsilon^{(p)}D^{p}.$$  

(34)

The first factor – an ordinary exponential of zero grade quantities – generates $z \mapsto z + z\varepsilon$ and the second factor is exactly the one obtained in [6] as the generator of translations of $\theta$ in the FSUSY context. However, the key result, showing that the full non-trivial FSUSY transformation of $z$ is generated by $G_{L}(z\varepsilon, \epsilon)$, is

$$z \mapsto G_{L}zG_{L}^{-1} = z + z\varepsilon + \sum_{p=1}^{n-1} \varepsilon^{(p)}\theta^{(n-p)}.$$  

(35)

in agreement with [6].
6 Reduction of the Representation space.

It is rather obvious how we are to represent the algebra of $z, \partial_z, \theta$ and $\partial_\theta$. The first two describe a bosonic degree of freedom that commutes with the latter pair, one that describes in Bargmann style a harmonic oscillator Hilbert space $V_{HO}$. Also, with the evident analogue

$$\theta = a^\dagger, \quad \partial_\theta = q^{N/2} a,$$

of (14), we see that (18) still follows. Also $\theta^n = 0$ and $\partial_\theta^m = 0$ imply $a^n = 0, a^\dagger^m = 0$ so that the variables of non-zero grade are represented in a vector space $V_n$ of $n$ degrees of freedom. Crucially, since (18) involves the deformation parameter $q^{1/2}$, in the natural representation of $a$ and $a^\dagger$ in $V_n$ of positive definite metric, the latter operator is indeed the true adjoint of the former.

It is our purpose now to demonstrate how the structure just described emerges from the work of section three when one passes from the case of generic $q$ to $q = \exp(2\pi i/n)$ for odd integer $n$. A representation of $D$ and $\theta$ at generic $q$ in a space spanned by eigenkets of $N$, namely $|m\rangle$ for $m = 0, 1, 2, \ldots$, can be taken to within equivalence in the form

$$D|m\rangle = |m - 1\rangle, \quad D|0\rangle = 0, \quad \theta|m\rangle = |m + 1\rangle |m + 1\rangle.$$

This implies

$$\theta^{(n)}|m\rangle = ([m + n]_q! / [m]_q! [n]_q!) |m + n\rangle$$

is valid for generic $q$. Setting $m = rn + p$ as in section four and passing to the limit for $q$ a root of 1 with the aid of lemmas from section four gives $z|rn + p\rangle = (r + 1)(r + 1)n + p$. Also $\partial_z = D^n$ leads to $\partial_z|rn + p\rangle = |(r - 1)n + p\rangle$. Indeed we can see that the representation space at generic $q$ in the limit acquires a product structure. Setting $|rn + p\rangle \equiv |r, p\rangle \in V_{HO} \otimes V_n$, we may view $z, \partial_z$ as $z \otimes 1, \partial_z \otimes 1$ in the product space, so that

$$z|r\rangle = (r + 1)|r + 1\rangle, \quad \partial_z|r\rangle = |r - 1\rangle.$$

Likewise we may view $\theta, \ldots$, as $1 \otimes \theta$ and use

$$D = 1 \otimes \partial_\theta + \partial_z \otimes \theta^{(n-1)}.$$

to express $D$ in terms of creation and destruction operators. There is of course a similarity transformation involved in placing the representations considered here explicitly in equivalence with those in which

$$a|p\rangle = \left(\frac{q^{p/2} - q^{-p/2}}{q^{1/2} - q^{-1/2}}\right)^{1/2} |p - 1\rangle,$$

and in which the correct adjoint properties of $a^\dagger$ are evident. This is discussed in [6].
Acknowledgements

This paper describes research supported in part by E.P.S.R.C and P.P.A.R.C. (UK) and by the C.I.C.Y.T (Spain). J.C.P.B. wishes to acknowledge an FPI grant from the CSIC and the Spanish Ministry of Education and Science.

References

[1] M. Arik and D.D. Coon, J. Math. Phys. 17 (1976) 524-527.
[2] A.J. Macfarlane, J. Phys. A22 (1989) 4581-4588.
[3] L.C. Biedenharn, J. Phys. A22 (1989) L873-L878.
[4] R.S. Dunne, A.J. Macfarlane, J.A. de Azcárraga and J.C. Pérez Bueno, 
Supersymmetry from a braided point of view, DAMTP/96-52, FTUV/96-27 and IFIC/96-31, Phys. Lett. B, to be published.
[5] J.A. de Azcárraga and A.J. Macfarlane, J. Math. Phys. 37 (1996) 1115-1127.
[6] R.S. Dunne, A.J. Macfarlane, J.A. de Azcárraga and J.C. Pérez Bueno, Geometrical foundations of fractional supersymmetry, DAMTP/96-57, FTUV/96-49 and IFIC/96-47, forthcoming.
[7] J.A. de Azcárraga, R.S. Dunne, A.J. Macfarlane and J.C. Pérez Bueno, Braided structure of $Z_3$-fractional supersymmetry, these proceedings.