Abstract. In bounded, polygonal domains $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial \Omega$ consisting of a finite number of Jordan curves admitting analytic parametrizations, we analyze $hp$-FEM discretizations of linear, second order, singularly perturbed reaction diffusion equations on so-called geometric boundary layer meshes. We prove, under suitable analyticity assumptions on the data, that these $hp$-FEM afford exponential convergence in the natural “energy” norm of the problem, as long as the geometric boundary layer mesh can resolve the smallest length scale present in the problem. Numerical experiments confirm the robust exponential convergence of the proposed $hp$-FEM.

Key words. anisotropic $hp$–refinement, geometric corner refinement, exponential convergence.

AMS subject classifications. 65N12, 65N30.

1. Introduction. The need for accurate numerical approximations of solutions to singularly perturbed partial differential equations in nonsmooth domains arises in a wide range of applications. Higher order numerical methods must cope with the appearance of boundary layers and their interaction with geometric corner and edge singularities. They are due to length scales introduced into weak solutions by small or large parameters in the differential operator. Accordingly, a large body of numerical analysis research has been developed during the past decades on their efficient numerical resolution; we mention only the texts [25, 28] and the references there for reaction-advection-diffusion problems, and [33] for viscous, incompressible flow. The discretization methods presented and surveyed in [28] are of fixed order and of Finite Difference or Finite Element type. They account specifically for the appearance of boundary and interior layers in solutions of the singularly perturbed boundary value problems. Being of fixed order, the corresponding discretization methods can afford at best fixed, algebraic orders of convergence whose convergence is, however, robust: the constants implicit in a priori error bounds are independent of the singular perturbation parameters and, hence, of the physical length scales in solutions that are implied by the singular perturbation of the governing equations.

As it is well-known, elliptic boundary value problems in domains $\Omega$ with piecewise analytic boundary for differential operators with analytic in $\Omega$ coefficients and forcing terms admit exponential convergence rates by Galerkin approximations with local mesh refinement and concurrent, judicious increase of the polynomial degree. This so-called $hp$-Finite Element approach has been analyzed in a series of papers, see [5, 35] and, more recently, in [34, 31, 32], and the references there, for regular elliptic boundary value problems.

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The study of parametric regularity and the proof of robust, algebraic convergence rates of discretizations for singularly perturbed, elliptic boundary value problems on polygons seems to have been initiated by G.I. Shishkin in the 1980s [39].

For singular perturbation problems, corresponding results on robust exponential convergence rates and corresponding analytic regularity estimates for solutions have been obtained in a series of papers in the 1990s, see [36, 23, 19, 20, 22] and the references there. The results from [23, 19, 20] were restricted to domains $\Omega$ with smooth (analytic) boundary $\partial \Omega$. In [24] analytic regularity results for elliptic reaction-diffusion problems in two space dimensions that are uniform with respect to the perturbation parameter were obtained. These results allowed us in [23] to prove robust exponential convergence of $hp$-FEM with so-called two-element and geometric boundary layer meshes, in domains $\Omega$ with a smooth (analytic) boundary $\partial \Omega$. In [22], an analytic regularity theory for solutions of linear, singularly perturbed elliptic reaction-diffusion problems in polygonal domains $\Omega$ as depicted in Fig. 1.1 was developed. In these works, elliptic singular perturbations with exactly one small length scale were considered. Examples of elliptic singular perturbation problems where several characteristic length scales appear simultaneously comprise in particular dimensionally reduced models of curved thin solids (“shells”) in so-called “bending-dominated” states (see, e.g., [10] and the references there for a detailed discussion of possible length scales), and linear, elliptic reaction-diffusion boundary value problems that result from implicit time-discretizations of parabolic evolution equations and, more recently, from discretizations of fractional powers of elliptic operators (see, e.g., [6, 21] and the references there). Further applications of the presently developed results arise in electromagnetics in so-called eddy-current models (where the small parameter is a complex number) (see, e.g., [9] and the references there), and in $hp$-FEM for advection-diffusion problems where stability, in addition to consistency, is a major issue (see, e.g., [20, 12]).

The present work extends in the reaction-diffusion case the $hp$-error analysis of Part I [23] to polygons and addresses the question how to approximate problems with multiple scales. The meshes presented here are also appropriate in situations where the precise length scales are unavailable, as is the case for example in problems from computational mechanics, see, e.g., [10].

In the following Section 1.1 we present a model singularly perturbed linear elliptic diffusion problem, where the perturbation parameter $0 < \varepsilon \leq 1$ dictates the single boundary length scale $\varepsilon$. We hasten to add that the ensuing $hp$-approximation results hold, independent of the particular model problem, and apply to a wider range of singular perturbations, as plate and shell models with possibly multiple length scales. See, e.g., [2, 14, 10, 15] and the references there.

1.1. Model reaction-diffusion problem. In a bounded domain $\Omega \subset \mathbb{R}^2$, which is assumed to be scaled to unit size, and for a parameter $0 < \varepsilon \leq 1$, we consider the $hp$-FE approximation of the model reaction-diffusion Dirichlet problem

\begin{equation}
-\varepsilon^2 \nabla \cdot (A(x)\nabla u_\varepsilon) + c(x)u_\varepsilon = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial \Omega.
\end{equation}

We assume

\begin{equation}
A, \ c, \ f \text{ analytic on } \overline{\Omega}, \text{ independent of } \varepsilon, \\
A \text{ symmetric, positive definite uniformly in } \overline{\Omega}, \ c \geq c_0 > 0 \text{ on } \overline{\Omega}.
\end{equation}

To design $H^1_0(\Omega)$-conforming, $hp$-FE approximations of the solutions $\{u_\varepsilon : 0 < \varepsilon \leq 1\}$
of (1.1) under the analyticity assumptions (1.2) which converge exponentially in the $\varepsilon$-dependent energy norm $\|\cdot\|_{\varepsilon, \Omega}$ given by

$$\|v\|_{\varepsilon, \Omega}^2 := \varepsilon^2 \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} , \quad v \in H^1(\Omega),$$

is the purpose of the present paper. We prove in particular in Theorem 4.1 a robust exponential convergence error bound, i.e., all constants in the exponential convergence bound do not depend on $\varepsilon > 0$.

We again emphasize that we consider (1.1) for illustration. The scope of the robust exponential $hp$ convergence rate bounds below extends well beyond (1.1) to more complex, singularly perturbed PDE such as those in [2, 14, 10].

### 1.2. Geometric Preliminaries.

In (1.2), the domain $\Omega \subset \mathbb{R}^2$ is a curvilinear polygonal domain, schematically depicted in Fig. 1.1. Specifically, the boundary $\partial \Omega$ is assumed to consist of $J \in \mathbb{N}$ closed curves $\Gamma(i)$. Each curve $\Gamma(i)$ in turn is assumed to be partitioned into finitely many open, disjoint, analytic arcs $\Gamma(i)_j$, in the sense that there are numbers $J_i \in \mathbb{N}$ such that

$$\Gamma(i) = \bigcup_{j=1}^{J_i} \Gamma(i)_j , \quad i = 1, \ldots, J .$$

Here, analytic arcs $\Gamma(i)_j$ admit nondegenerate, analytic parameterizations, i.e.,

$$\Gamma(i)_j = \left\{ x_j(i)(\theta) | \theta \in (0, 1) \right\} , \quad i = 1, \ldots, J, \quad j = 1, \ldots, J_i .$$

with the coordinate functions $x_j(i)$, $y_j(i)$ of $x_j(i)(\theta) = (x_j(i)(\theta), y_j(i)(\theta))$ assumed to be (real) analytic functions of $\theta \in [0, 1]$ and such that

$$\min_{\theta \in [0,1]} \left\{ \left| \frac{d}{d\theta} x_j(i)(\theta) \right|^2 + \left| \frac{d}{d\theta} y_j(i)(\theta) \right|^2 \right\} > 0 \quad j = 1, \ldots, J_i , \quad i = 1, \ldots, J .$$

We denote $\partial \Gamma(i)_j = \{ A_{j-1}^{(i)}, A_j^{(i)} \}$ where $A_{j-1}^{(i)} = x_j^{(i)}(0)$ and $A_j^{(i)} = x_j^{(i)}(1)$. For each boundary component $\Gamma(i)$, we enumerate $\{ A_j^{(i)} \}_{j=1}^{J_i}$ cyclically, counterclockwise by indexing with $j$ modulo $J_i$, thereby identifying in particular $A_j^{(i)} := A_{j+J_i}^{(i)}$. The
interior angle at $A^{(i)}$ is denoted $\omega^{(i)}_j \in (0, 2\pi)$. For notational simplicity, we assume henceforth that $J = 1$, i.e., $\partial \Omega$ consists of a single component of connectedness. We write $A_j = A^{(1)}_j$, $\Gamma_j$ for $\Gamma^{(1)}_j$, $x_j = x^{(1)}_j$, $y_j = y^{(1)}_j$. Then, in a vicinity of any point $x \in \partial \Omega$, a curvilinear polygon $\Omega$ is analytically diffeomorphic to either a half-space, or to a plane sector with vertex situated at the origin.

1.3. Contributions. The principal contribution of the present paper is robust, exponential convergence of a class of $hp$-FEM approximations of the singular perturbation problem (1.1) under the analyticity assumptions (1.2), in curvilinear polygons $\Omega$ as described in Section 1.2. The convergence proof in Sections 3 and 4 is done under a scale resolution condition that corresponds, roughly speaking, to the $hp$-FE partitions resolving the shortest length scale that occurs in the solution $u_\varepsilon$, and it is strongly based on parameter-explicit, analytic regularity results for the parametric solution family $\{u_\varepsilon : 0 < \varepsilon \leq 1\} \subset H^1_0(\Omega)$ of (1.1), which were obtained by one of the authors in [22]. Importantly, and distinct from earlier work on robust exponential $hp$-FE convergence for (1.1), a patch-based convergence proof is developed which also enables an algorithmic, patchwise-structured anisotropic mesh specification, described in Section 2, which is applicable in domains $\Omega$ of the generality admitted in Section 1.2. As we show in numerical experiments in Section 5, the recently developed, automatic mesh generator Netgen [29] does produce automatically, i.e., without “expert pruning”, anisotropic, geometric meshes in $\Omega$ with the required boundary and corner refinement that satisfy the requirement of the present robust, exponential convergence bounds.

1.4. Outline of this paper. In Section 2, we introduce the geometric mesh families in $\Omega$ that underlie our robust exponential convergence results. The meshes require concurrent anisotropic geometric partitions of $\Omega$ towards the boundary $\partial \Omega$ and isotropic geometric refinement towards the corners $A_j$. We define these meshes in a macro-element fashion based on an initial, coarse regular partition of the physical domain $\Omega$ into a macro-triangulation consisting of a regular, finite, and fixed partition of the physical domain $\Omega$ that is described in Section 2.1. Its elements will be referred to as (macro) patches and are assumed to be images of a finite number of quadrilateral reference patches under analytic patch maps. The reference patches are key in ensuring robust exponential convergence rate bounds of our $hp$-FEM approximation. Following earlier work [19, 20, 22], we consider so-called geometric boundary layer meshes, denoted by $T_{\text{geo}}^{L,n}$. We introduce these in Def. 2.3. Unlike the so-called “two-element” meshes considered earlier in [38, 37], which are designed to approximate only a single small scale in a robust way, the presently considered geometric boundary layer meshes afford robust exponential convergence rates of $hp$-FEM also in the presence of multiple physical length scales. This situation arises in a number of applications (e.g., [10, 6]). Section 2.3 introduces the geometric boundary layer mesh, first on the reference patches, and then in Section 2.4 in curvilinear polygons.

Section 3 is devoted to the polynomial approximation of functions on geometric boundary layer meshes. The approximation is based on Gauss-Lobatto interpolation operators in the reference triangle, indicated by $\triangle$, and in the reference square, indicated by $\square$, in Section 3.1. These are then assembled into (nodal) patch approximation operators on the geometric boundary layer patches in Section 3.2. Then, the robust $hp$-approximation of corner singularities and boundary layer functions is proved. These functions are the key solution components of singularly perturbed problems in polygons such as the model problem (1.1).

Section 4 assembles the patch $hp$-approximation results and interpolants into a
global approximation operator, and presents the main result of this paper: robust exponential convergence rate bounds for the global \(hp\)-interpolation of the solution of (1.1) assembled from the patch approximations.

Section 5 presents several illustrative numerical experiments in polygons, which underline the theoretical results. They are based on \(hp\)-meshes that are furnished by the automated mesh generation procedure Netgen [29] to make the point that the somewhat technical construction of geometric boundary layer meshes is, in principle, available and feasible automatically. Appendix A contains proofs of auxiliary results on analytic regularity estimates under analytic changes of variables. Appendix B collects (mostly known) results on univariate polynomial approximation for convenient reference in the main text.

1.5. Notation. We employ standard notation for Sobolev spaces. Constants \(C, \gamma, b > 0\) may be different in different instances. However, they will be independent of parameters of interest such as \(\varepsilon \in (0,1]\) and the polynomial degree \(q\).

The notation \(\nabla^n u\) stands for the collection of all partial derivatives of order \(n\) and \(|\nabla^n u|^2 = \sum_{|\alpha|=n} \frac{n!}{\alpha!} |D^{\alpha} u|^2\). Points in \(\mathbb{R}^2\) will be denoted depending on the context as either \(x = (x, y)\) (physical domain) or \(\tilde{x} = (\tilde{x}, \tilde{y})\) (patch domains) or \(\hat{x} = (\hat{x}, \hat{y})\) (reference domains). We abbreviate \(\{\tilde{x} = 0\}, \{\tilde{y} = 0\}, \{\hat{x} = \hat{y}\}\) for the line segments \(\{\tilde{x} = (0, \tilde{y}) | 0 < \tilde{y} < 1\}, \{\tilde{x} = (\tilde{x}, 0) | 0 < \tilde{x} < 1\}, \{\hat{x} = (\hat{x}, \hat{y}) | 0 < \hat{x} = \hat{y} < 1\}\), respectively. We write \(\{\tilde{y} \leq \tilde{x}\}\) for \(\{\tilde{x} = (\tilde{x}, \tilde{y}) | 0 < \tilde{y} \leq \tilde{x} < 1\}\). The origin will be denoted \(0 = (0,0)\). The reference square and triangle are \(\hat{S} := (0,1)^2\) and \(\hat{T} := \{(\hat{x}, \hat{y}) | 0 < \hat{x} < 1, 0 < \hat{y} < \hat{x}\}\). The region covered by the reference patch will be denoted \(\hat{S} := \hat{S} = (0,1)^2\). It will be convenient to introduce \(\bar{T} := \hat{T}\) and set \(T_{\text{hp}} := \hat{S} \setminus \bar{T}\). We denote the space of polynomials of total degree \(q\) by \(P_q = \text{span}\{x^iy^j | 0 \leq i + j \leq q\}\); the tensor product space \(Q_q\) is \(Q_q = \text{span}\{x^iy^j | 0 \leq i, j \leq q\}\).

2. Macro triangulation. Geometric boundary layer mesh. Our robust exponentially convergent \(hp\) approximation is based on so-called geometric boundary layer meshes, denoted by \(T_{\text{geo},\sigma}\). To facilitate our error analysis, the \(T_{\text{geo},\sigma}\) are generated as push-forwards of a small number of so-called reference patches, which are partitions of \(\hat{S}\), under the patch maps. The images of \(\hat{S}\) under the patch maps form a (coarse) macro triangulation of \(\Omega\) satisfying some minimal conditions, which are described in Section 2.1. This concept was also used in the context of \(hp\)-FEM for singular perturbations in [22, Sec. 3.3.3] and in [18, 13].

2.1. Macro triangulations. We assume given a fixed macro-triangulation \(T^M = \{K^M | K^M \in T^M\}\) of \(\Omega\) consisting of curvilinear quadrilaterals \(K^M\) with bijective element maps \(F_{K^M} : \hat{S} \to K^M\) that are analytic in \(\hat{S}\) and that in addition satisfy the usual compatibility conditions. I.e., the partition \(T^M\) does not have hanging nodes and, for any two distinct elements \(K_1^M, K_2^M \in T^M\) that share an edge \(e\), their respective element maps induce compatible parametrizations of \(e\) (cf., e.g., [22, Def. 2.4.1] for the precise conditions).

Each element of the fixed macro-triangulation \(T^M\) is further subdivided according to one of the refinement patterns in Definition 2.1 (see also [22, Sec. 3.3.3] or [13]). The actual triangulation is then obtained by transplanting refinement patterns on the square reference patch \(\hat{S}\) into the physical domain \(\Omega\) by the element maps \(F_{K^M}\) of the macro-triangulation resulting in the physical triangulation \(T\). For any element \(K \in T\), the element maps \(F_K : \hat{K} \to K\) are then concatenations of affine maps
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\[ \tilde{T}^{\text{BL}, L} \]

\[ \tilde{T}^{\text{geo}, \sigma} \]

\[ \tilde{T}^{\text{C}, n} \]

\[ \tilde{T}^{\text{T}, L, n} \]

\[ \tilde{T}^{\text{M}, L, n} \]

\[ \tilde{T}^{\text{C}, \text{half}, L, n} \]

\[ \tilde{T}^{\text{C}, \text{half}, \text{flip}, L, n} \]

\[ \tilde{T}^{\text{half}, L, n} \]

\[ \tilde{T}^{\text{geo}, \sigma} \]

\[ S_1 = (0, \sigma L)^2 \]

\[ \hat{\tilde{T}}_1 := \{ 0 < \hat{x} < \sigma, 0 < \hat{y} < \hat{z} \} \]

\[ \hat{\tilde{T}}^{\text{M}, \text{half}, L, n} \]

\[ \hat{\tilde{T}}^{\text{C}, \text{half}, L, n} \]

\[ \hat{\tilde{T}}^{\text{C}, \text{half}, \text{flip}, L, n} \]

\[ \hat{\tilde{T}}^{\text{half}, L, n} \]

\[ \hat{\tilde{T}}^{\text{geo}, \sigma} \]

\[ A_K : \hat{K} \rightarrow \tilde{K}, \text{ which realize the mapping from } \hat{K} \in \{ \hat{S}, \hat{T} \} \text{ to the elements in the patch refinement pattern, and the analytic patch maps } F_{K,M}. \text{ That is, the element maps have the form } F_K = F_{K,M} \circ A_K \text{ for an affine } A_K. \text{ Throughout the article, we will denote by } \hat{K} \in \{ \hat{S}, \hat{T} \} \text{ the reference element corresponding to an element } K \text{ of a triangulation, and we will denote by } \tilde{K} \text{ the elements of the triangulation of the reference patterns. Points in the reference patch } \hat{S} \text{ are denoted } \hat{x} = (\hat{x}, \hat{y}) \in \hat{S}; \text{ variables } (x, y) \text{ are employed to indicate points in } \Omega, \text{ and } \hat{x} = (\hat{x}, \hat{y}) \text{ are used for points of the reference square } \hat{S} \text{ and reference triangle } \hat{T}. \]

**2.2. Refinement patterns in the reference configuration (patch catalog \( \mathfrak{P} \)).** The admissible patch refinement patterns are collected in a catalog \( \mathfrak{P} \) and are depicted in Fig. 2.1. They are based on geometric refinement towards a vertex and/or an edge; the parameter \( L \) controls the number of layers of refinement towards an edge whereas the natural number \( n \geq L \) measures the number of geometric refinements.
towards vertices.

**Definition 2.1 (catalog $\mathcal{P}$ of refinement patterns).** Given $\sigma \in (0, 1)$, $L, n \in \mathbb{N}_0$ with $n \geq L$ the catalog $\mathcal{P}$ of admissible refinement patterns consists of the following patches:

1. The trivial patch: The reference square $\tilde{S}$ is not further refined. The corresponding triangulation of $S$ consists of the single element: $\mathcal{T} = \{\tilde{S}\}$.
2. The geometric boundary layer patch $\mathcal{T}_{\text{geo}}^{\text{BL}, L, \sigma}$: $S$ is refined anisotropically towards $\{\tilde{y} = 0\}$ into $L$ elements as depicted in Fig. 2.1 (top left). The mesh $\mathcal{T}_{\text{geo}}^{\text{BL}, L, \sigma}$ is characterized by the nodes $(0, 0), (0, \sigma^i), (1, \sigma^i), i = 0, \ldots, L,$ and the corresponding rectangular elements generated by these nodes.
3. The geometric corner patch $\mathcal{T}_{\text{geo}, \sigma}^{\text{C}, L, n}$: $S$ is refined isotropically towards $(0, 0)$ as depicted in Fig. 2.1 (top middle). Specifically, the reference geometric corner patch mesh $\mathcal{T}_{\text{geo}, \sigma}^{\text{C}, L, n}$ in $S$ with geometric refinement towards $(0, 0)$ and $n$ layers is given by triangles and based on the nodes $(0, 0), (0, \sigma^i), (\sigma^i, 0), (\sigma^j, \sigma^i), i = 0, 1, \ldots, n$.
4. The tensor product patch $\mathcal{T}_{\text{geo}, \sigma}^{\text{T}, L, n, \tau}$: $S$ is triangulated in $\tilde{S}_1 := (0, \sigma^L)^2$ and $\tilde{S}_2 := \tilde{S} \setminus \tilde{S}_1$ separately as depicted in Fig. 2.1 (bottom left). The triangulation of $\tilde{S}_1$ is a scaled version of $\mathcal{T}_{\text{geo}, \sigma}^{\text{C}, L, n, \tau}$ and based on the nodes $(0, \sigma^i), (\sigma^i, 0), i = L, \ldots, n$. The triangulation of $\tilde{S}_2$ is based on the nodes $(\sigma^i, \sigma^j), i, j = 0, \ldots, L$.
5. The mixed patches $\mathcal{T}_{\text{geo}, \sigma}^{\text{M}, L, n, \tau}$: The triangulation consists of both anisotropic elements and isotropic elements as depicted in Fig. 2.1 (bottom right) and is obtained by triangulating the regions $\tilde{S}_1 := (0, \sigma^L)^2, \tilde{S}_2 := (\tilde{S} \setminus \tilde{S}_1) \cap \{\tilde{y} \leq \tilde{x}\}, \tilde{S}_3 := \tilde{S} \setminus (\tilde{S}_1 \cup \tilde{S}_2)$ separately. The set $\tilde{S}_1$ is a scaled version of $\mathcal{T}_{\text{geo}, \sigma}^{\text{C}, L, n, \tau}$ based on the nodes $(0, \sigma^i), (\sigma^i, 0), i = L, \ldots, n$. The triangulation of $\tilde{S}_2$ is based on the nodes $(\sigma^i, 0), (\sigma^i, \sigma^j), 0 \leq i \leq L, i \leq j \leq L$ and consists of rectangles and triangles, and only the triangles abut on the diagonal $\{\tilde{x} = \tilde{y}\}$. The triangulation of $\tilde{S}_3$ consists of triangles only and is based on the nodes $(0, \sigma^i), (\sigma^i, \sigma^j), i = 0, \ldots, L$.

**Remark 2.2.** We kept the catalog $\mathcal{P}$ of admissible patch refinement patterns in Definition 2.1 small in order to reduce the number of cases to be discussed for the hp-FE error bounds. A larger number of refinement patterns provides greater flexibility.
in the mesh generation. In particular, the reference patch meshes of Def. 2.1 do not contain general quadrilaterals but only rectangles; this restriction is not essential but simplifies the hp-FE error analysis. Also certain types of anisotropic triangles (e.g., splitting anisotropic rectangles along the diagonal), which are altogether excluded in the present analysis, could be accommodated at the expense of additional technicalities.

The addition of the diagonal line in the reference corner, tensor, and mixed patches is done to be able to apply the regularity theory of [22]. It is likely not necessary in actual computations. We also mention that with additional constraints on the macro triangulation \( T^M \) the diagonal line could be dispensed with in certain situations as is illustrated in Section 2.4.

2.3. Geometric boundary layer mesh. The following definition of the geometric boundary layer mesh \( T_{\text{geo}, \sigma}^{L,n} \) formalizes the patchwise construction of meshes on \( \Omega \) based on transplanting meshes of the reference configurations to \( \Omega \) via the patch maps \( F_{K,M} \).

**Definition 2.3** (geometric boundary layer mesh \( T_{\text{geo}, \sigma}^{L,n} \) in \( \Omega \)). Let \( T^M \) be a fixed macro-triangulation consisting of quadrilaterals with analytic element maps that satisfy [22, Def. 2.4.1].

Given \( \sigma \in (0,1) \), \( L, n \in \mathbb{N}_0 \) with \( n \geq L \), a regular mesh \( T_{\text{geo}, \sigma}^{L,n} \) in \( \Omega \) is called a geometric boundary layer mesh if the following conditions are satisfied:
1. \( T_{\text{geo}, \sigma}^{L,n} \) is obtained by refining each element \( K^M \in T^M \) according to one of the refinement patterns given in Definition 2.1 using the given parameters \( \sigma \), \( L \), and \( n \).
2. The resulting mesh \( T_{\text{geo}, \sigma}^{L,n} \) is a regular triangulation of \( \Omega \), i.e., it does not have hanging nodes. Since the element maps for the refinement patterns are assumed to be affine, this requirement ensures that the resulting triangulation satisfies [22, Def. 2.4.1].

For each macro-patch \( K^M \in T^M \), exactly one of the following cases is possible:
3. \( K^M \cap \partial \Omega = \emptyset \). Then the trivial patch is selected as the reference patch.
4. \( K^M \cap \partial \Omega \) is a single point. Then two cases can occur:
   (a) \( K^M \cap \partial \Omega = \{A_j\} \) for a vertex \( A_j \) of \( \Omega \). Then the corresponding reference patch is the corner patch \( T_{\text{geo}, \sigma}^{C,n} \) with \( n \) layers of refinement towards \( 0 \). Additionally, \( F_{K,M}(0) = A_j \).
   (b) \( K^M \cap \partial \Omega = \{P\} \), where \( P \) is not a vertex of \( \Omega \). Then the refinement pattern is the corner patch \( T_{\text{geo}, \sigma}^{C,L} \) with \( L \) layers of geometric mesh refinement towards \( 0 \). Additionally, it is assumed that \( F_{K,M}(0) = P \in \partial \Omega \).
5. \( K^M \cap \partial \Omega = \tau \) for an edge \( e \) of \( K^M \) and neither endpoint of \( e \) is a vertex of \( \Omega \). Then the refinement pattern is the boundary layer patch \( T_{\text{geo}, \sigma}^{BL,L} \) and additionally \( F_{K,M}(\{\tilde{y} = 0\}) \subset \partial \Omega \).
6. \( K^M \cap \partial \Omega = \tau \) for an edge \( e \) of \( K^M \) and exactly one endpoint of \( e \) is a vertex \( A_j \) of \( \Omega \). Then the refinement pattern is the mixed layer patch \( T_{\text{geo}, \sigma}^{M,L,n} \) and additionally \( F_{K,M}(\{\tilde{y} = 0\}) \subset \partial \Omega \) as well as \( F_{K,M}(0) = A_j \).
7. Exactly two edges of a macro-element \( K^M \) are situated on \( \partial \Omega \). Then the refinement pattern is the tensor patch \( T_{\text{geo}, \sigma}^{T,L,n} \). Additionally, it is assumed that \( F_{K,M}(\{\tilde{y} = 0\}) \subset \partial \Omega \), \( F_{K,M}(\{x = 0\}) \subset \partial \Omega \), and \( F_{K,M}(0) = A_j \) for a vertex \( A_j \) of \( \Omega \).

Finally, the following technical condition ensures the existence of certain meshlines:
Fig. 2.4. Generating a boundary layer mesh from a regular triangulation $T^0$: solid lines are the triangulation $T^0$, dashed lines connect edge midpoints with element barycenters to create a mesh consisting of quadrilaterals. $B$ stands for boundary layer, $M$ for mixed, $C^k$ for corner layer patches $T^k_{\text{geo},\sigma}$, $C^n$ for corner layer patches $T^n_{\text{geo},\sigma}$, empty quadrilaterals are trivial patches.

8. For each vertex $A_j$ of $\Omega$, introduce a set of lines

\[ \ell = \bigcup_{K^M: A_j \in K^M} \{ F_{K^M}([\bar{y} = 0]), F_{K^M}([\bar{x} = 0]), F_{K^M}([\bar{x} = \bar{y}]) \}. \]

Let $\Gamma_j, \Gamma_{j+1}$ be the two boundary arcs of $\Omega$ that meet at $A_j$. Then there exists a line $e \in \ell$ such that the interior angles $\angle(e, \Gamma_j)$ and $\angle(e, \Gamma_{j+1})$ are both less than $\pi$.

Remark 2.4. The last condition, requirement 8. in Definition 2.3, is merely a technical condition that results from our applying the regularity theory for singular perturbations of [22]. Very likely, it could be dropped.

The condition that $F_{K,M}(0) \in \partial \Omega$ or that $F_{K,M}(\{ \bar{y} = 0 \}) \subset \partial \Omega$ are not conditions on the patch geometry but on the maps $F_{K,M}$. They are not essential but introduced for notational simplicity. They could be enforced by suitably concatenating the maps $F_{K,M}$ with an orthogonal transformation.

Remark 2.5. The meshes $T^{L,n}_{\text{geo},\sigma}$ are refined towards both vertices and edges of $\Omega$. The parameter $L \in \mathbb{N}_0$ measures the number of layers of geometric refinement towards $\partial \Omega$ whereas the parameter $n \in \mathbb{N}$ characterizes the number of layers of geometric refinement towards the vertices. For $L = 0$ (or, more generally, $L$ fixed), the meshes $T^{L,n}_{\text{geo},\sigma}$, $n = 1, 2, \ldots$, realize the “geometric meshes” introduced in [3, 4] (see also [35, Sec. 4.4.1]) for the hp-FEM applied to elliptic boundary value problems with piecewise analytic data.

Example 2.6. Fig. 2.3 (left and middle) shows an example of an $L$-shaped domain with macro triangulation and suitable refinement patterns.

2.4. Geometric boundary layer meshes in curvilinear polygons. Geometric boundary layer meshes can be constructed in various ways. A first approach, which is in line with the illustration in Fig. 2.3, is to create one layer of quadrilateral elements that partition a tubular neighborhood $T_{\text{geo}}$ of $\partial \Omega$. Each quadrilateral $K$ should fall into one of the following 3 categories: a) $K \cap \partial \Omega$ is an edge of $K$; b) $K \cap \partial \Omega$ consists of two contiguous edges and the shared vertex is a vertex of $\Omega$; c) $K \cap \partial \Omega$ is a vertex of $\Omega$. In the second step, refinement patterns from Definition 2.3 are applied to each quadrilateral. In the final step, $\Omega' := \Omega \setminus T_{\text{geo}}$ is triangulated under the constraint that the boundary nodes of the triangulation of $\Omega'$ on $\partial \Omega'$ coincide with the nodes of the triangulation of $T_{\text{geo}}$ that also lie on $\partial \Omega'$. This triangulation of $\Omega'$ could be chosen to consist of triangles (and/or quadrilaterals). All elements of that triangulation will
be denoted “trivial patches”; we mention without proof that the approximation result holds also if we include “trivial” triangles in the list of refinement patterns.

Geometric boundary layer meshes can also be constructed for general (curvilinear) polygons \( \Omega \) starting from any regular initial triangulation \( T^0 \) of \( \Omega \). This triangulation \( T^0 \) is assumed to consist of (curvilinear) triangles with analytic element maps and satisfying the “usual” conditions for triangulations as spelled out in [22, Def. 2.4.1].

Then, the geometric boundary layer mesh is generated in 3 steps (cf. Fig. 2.4):

1. (Ensure condition 8 of Def. 2.3) For each vertex \( A_j \) of \( \Omega \) verify if an edge \( e \) of \( T^0 \) splits the interior angle at \( A_j \) into two angles each less than \( \pi \). If not, then suitably split an appropriate triangle abutting on \( A_j \) into two triangles (so that the newly introduced edge will satisfy this condition) and remove the newly introduced hanging node by a mesh closure. The resulting triangulation has again analytic element maps and satisfies [22, Def. 2.4.1]; it is again denoted \( T^0 \).

2. (Create a macro triangulation \( T^M \) consisting of quadrilaterals only.) Split each triangle \( K \in T^0 \) into 3 quadrilaterals as follows: split the reference triangle \( \hat{T} \) into 3 quadrilaterals \( \hat{K}_i \), \( i = 1, 2, 3 \), characterized by the vertices of \( \hat{T} \), its barycenter, and by the 3 midpoints of the edges of \( \hat{T} \). The element maps of the 3 quadrilaterals \( F_K(\hat{K}_i) \), \( i = 1, 2, 3 \), are obtained by concatenating the bilinear bijections \( F_{\hat{K}_i} : \hat{S} \to \hat{K}_i \) with \( F_K \). The triangulation \( T^M \) of \( \Omega \) obtained in this way realizes a decomposition of \( \Omega \) into (curvilinear) quadrilaterals, and the element maps satisfy [22, Def. 2.4.1].

3. (Generate the geometric boundary layer mesh.) The refinement pattern for each \( K \in T^M \) is determined since \( K \) falls into exactly one of the categories 3—7 of Definition 2.3 as can be seen by the following observations: a) At most 2 edges of \( K \) are on \( \partial \Omega \) (since the two edges that meet in the barycenter of the parent triangle cannot be on \( \partial \Omega \)). b) If two edges of \( K \) are situated on \( \partial \Omega \), then they have to be subsets of the two edges of the parent triangle with common vertex \( V \); since \( T^0 \) is a regular triangulation, the common vertex \( V \) has to be a vertex of \( \Omega \). Additionally, if necessary, the assumptions on where the reference element vertex \( 0 \) and/or the edges \( \{ \hat{y} = 0 \}, \{ \hat{x} = 0 \} \) are mapped can be ensured by suitably adjusting the element map with the aid of an orthogonal transformation of \( \hat{S} \). Finally, condition 8 of Def. 2.3 is satisfied by step 1.

It remains to see that after selecting the refinement patterns the resulting triangulation satisfies [22, Def. 2.4.1]. This follows from the fact that the parameters \( \sigma \), \( L \), \( n \) are the same for all macro elements and the structure of the refinement patterns: If an edge \( e \) of the macro triangulation inherits a further refinement from a refinement pattern, then the edge either lies on \( \partial \Omega \) (which is immaterial for the question of satisfying [22, Def. 2.4.1]) or it is in \( \Omega \) and exactly one of its endpoints \( V \) lies on \( \partial \Omega \). This edge \( e \) is shared by two macro elements. If \( V \) is a vertex of \( \Omega \), then the refinement patterns are such that the induced 1D-mesh on \( e \) is the same geometric mesh with \( n \) layers for both macro elements. If \( V \in \partial \Omega \) is not a vertex of \( \Omega \), then the induced 1D-mesh on \( e \) is the same geometric mesh with \( L \) layers for both macro elements. Hence, the resulting mesh satisfies [22, Def. 2.4.1].

2.5. Properties of the mesh patches. We note that parts of the mixed patch, the tensor patch, and the corner patch are identical or at least structurally similar. For the analysis of the approximation properties of \( hp \)-FEM on geometric boundary layer meshes it is therefore convenient to single out these meshes:
DEFINITION 2.7 (half-patches, cf. Fig. 2.2). The mixed half-patch \( \tilde{T}^{M,\text{half},L,n}_{\text{geo,}\sigma} \) and the corner half-patch \( \tilde{T}^{C,\text{half},n}_{\text{geo,}\sigma} \) on \( \tilde{T} = \{(x, y) | 0 < \tilde{x} < 1, 0 < \tilde{y} < 1\} \) are obtained by restricting \( \tilde{T}^{M,\text{half},L,n}_{\text{geo,}\sigma} \) and \( \tilde{T}^{C,n}_{\text{geo,}\sigma} \) to \( \tilde{T} \). The flipped corner half-patch \( \tilde{T}^{C,\text{half,flip},n}_{\text{geo,}\sigma} \) on \( T^{\text{flip}} := \{(x, y) | 0 < \tilde{x} < 1, \tilde{x} < \tilde{y} < 1\} \) is obtained by reflecting \( \tilde{T}^{C,\text{half},L,n}_{\text{geo,}\sigma} \) at the diagonal \( \{(\tilde{x}, \tilde{x}) | \tilde{x} \in (0, 1)\} \) of \( \tilde{S} \).

We will approximate functions on boundary layer meshes \( T^{L,n}_{\text{geo,}\sigma} \) with the aid of an elementwise defined operator \( \Pi_q \). To estimate the total error in \( L^2 \)-based norms, the elemental error contributions are summed up on each mesh patch separately. The following Lemma 2.8 provides tools to conveniently do that. In order to formulate Lemma 2.8, we introduce some additional notation, which represents the pull-back of the parts of the boundary of the reference patch that is mapped to \( \partial \Omega \) and is marked by bold lines or dots in Figs. 2.1 and 2.2:

\[
\begin{align*}
(2.1a) & \quad \Gamma^C := \Gamma^{C,\text{half}} := \Gamma^{C,\text{half,flip}} := \{0\}, \\
(2.1b) & \quad \Gamma^{BL} := \Gamma^M := \Gamma^{M,\text{half}} := \{\tilde{y} = 0\}, \\
(2.1c) & \quad \Gamma^T := \{\tilde{y} = 0\} \cup \{\tilde{x} = 0\} \cup \{0\}.
\end{align*}
\]

LEMMA 2.8 (properties of mesh patches). The reference patches (cf. Def. 2.1) and half patches (cf. Def. 2.7) have the following properties:

(i) The triangular elements \( K \) of the reference patches are shape regular with shape regularity constant depending solely on \( \sigma \). For the rectangular elements \( \tilde{K} \) of the reference patches, the element maps \( A^{'}_{\tilde{K}} \) are affine with

\[
A^{'}_{\tilde{K}} = \begin{pmatrix} h_{\tilde{K},\tilde{x}} & h_{\tilde{K},\tilde{y}} \end{pmatrix},
\]

where \( h_{\tilde{K},\tilde{x}} \) and \( h_{\tilde{K},\tilde{y}} \leq 1 \) are the side lengths (in \( \tilde{y} \) and \( \tilde{x} \)-direction) of \( \tilde{K} \). We denote

\[
(2.2) \quad h_{\tilde{K},\text{min}} := \min\{h_{\tilde{K},\tilde{x}}, h_{\tilde{K},\tilde{y}}\}, \quad h_{\tilde{K},\text{max}} := \max\{h_{\tilde{K},\tilde{x}}, h_{\tilde{K},\tilde{y}}\}.
\]

(ii) There is \( c_{\text{dist}} > 0 \) depending only on \( \sigma \) such that for all triangular elements \( \tilde{K} \) of a reference patch \( \tilde{T} \) or a half-patch \( \tilde{T} \) the following dichotomy holds:

\[
\text{either } \tilde{K} \cap \Gamma \neq \emptyset \quad \text{or} \quad \text{dist}(\tilde{K}, \Gamma) \geq c_{\text{dist}}h_{\tilde{K}},
\]

where \( h_{\tilde{K}} = \text{diam}(\tilde{K}) \) and \( \Gamma \in \{\Gamma^C, \Gamma^{C,\text{half}}, \Gamma^{C,\text{half,flip}}, \Gamma^T, \Gamma^M, \Gamma^{M,\text{half}}, \Gamma^{BL}\} \) for \( \tilde{T} \in \{\tilde{T}^{C,n}_{\text{geo,}\sigma}, \tilde{T}^{C,\text{half},n}_{\text{geo,}\sigma}, \tilde{T}^{C,\text{half,flip},n}_{\text{geo,}\sigma}, \tilde{T}^{T,n}_{\text{geo,}\sigma}, \tilde{T}^{M,n}_{\text{geo,}\sigma}, \tilde{T}^{M,\text{half},n}_{\text{geo,}\sigma}, \tilde{T}^{BL,n}_{\text{geo,}\sigma}\} \), respectively.

(iii) There is \( c_{\text{dist}} > 0 \) depending only on \( \sigma \) such that for all rectangular elements \( \tilde{K} \) of a reference patch \( \tilde{T} \) or half-patch \( \tilde{T} \), the following dichotomy holds:

\[
\text{Either } \tilde{K} \cap \Gamma = \emptyset \quad \text{or} \quad \text{dist}(\tilde{K}, \Gamma) \geq c_{\text{dist}}h_{\tilde{K},\text{min}},
\]

where \( \Gamma \in \{\Gamma^T, \Gamma^M, \Gamma^{M,\text{half}}, \Gamma^{BL}\} \) for \( \tilde{T} \in \{\tilde{T}^{T,n}_{\text{geo,}\sigma}, \tilde{T}^{M,n}_{\text{geo,}\sigma}, \tilde{T}^{M,\text{half},n}_{\text{geo,}\sigma}, \tilde{T}^{BL,n}_{\text{geo,}\sigma}\} \), respectively.
(iv) There is $c_{\text{dist}} > 0$ depending only on $\sigma$ such that for all rectangular elements $\tilde{K}$ of a mixed patch, a mixed half-patch, or a tensor patch there holds $\text{dist}(\tilde{K}, 0) \geq c_{\text{dist}} h_{\tilde{K}, \max}$.

(v) There is $C > 0$ depending only on $\sigma$ such that for all elements $\tilde{K}$ of a reference patch or half-patch there holds $\text{dist}(\tilde{K}, 0) \leq C \text{diam} \tilde{K}$.

(vi) Let $\delta > 0$ and consider a reference patch or half-patch. Let $\tilde{T}^\triangle$ be the collection of triangles of that reference patch or half-patch that do not abut on the vertex $0$. Then, there exists a constant $C > 0$ depending solely on $\delta$ and $\sigma$ such that

$$\sum_{K \in \tilde{T}^{\triangle}} h_{\tilde{K}}^\delta \leq C.$$ 

(vii) Let $\delta > 0$ and consider a reference mixed patch, tensor patch, mixed half-patch or corner half-patch. Let $\tilde{T}^{\square}$ be the collection of rectangles of that reference patch. Then there exists a constant $C > 0$ depending solely on $\delta$ and the parameter $\sigma$ such that

$$\sum_{K \in \tilde{T}^{\square}} \frac{h_{\tilde{K}, \min}}{h_{\tilde{K}, \max}} h_{\tilde{K}}^\delta \leq C.$$ 

(viii) Let $\delta \in (0, 1]$, $\alpha > 0$, and consider a reference patch or half-patch. Let $\tilde{T}^{\triangle}$ be the collection of triangles of that reference patch or half-patch that do not abut on the vertex $0$. Then, there holds, with a $C > 0$ depending solely on $\delta$, $\alpha$, and $\sigma$,

$$\forall \varepsilon \in (0, 1]: \sum_{K \in \tilde{T}^{\triangle}} (h_{\tilde{K}}/\varepsilon)^\delta e^{-\alpha h_{\tilde{K}}/\varepsilon} \leq C.$$ 

(ix) Let $\delta \in (0, 1]$, $\alpha > 0$, and consider a reference mixed patch, mixed half-patch, or a reference tensor patch. Let $\tilde{T}^{\square}$ be the collection of rectangles of that reference patch. Then there exists a constant $C > 0$ depending solely on $\delta$, $\alpha$, and $\sigma$ such that

$$\forall \varepsilon \in (0, 1]: \sum_{K \in \tilde{T}^{\square}} \frac{h_{\tilde{K}, \min}}{h_{\tilde{K}, \max}} (h_{\tilde{K}, \max}/\varepsilon)^\delta e^{-\alpha h_{\tilde{K}, \max}/\varepsilon} \leq C.$$ 

**Proof.** Items (i)–(v) follow by construction. Since items (vi), (vii) are shown by similar arguments, we only prove the case of (vii) for the specific case of the mixed patch as shown in Fig. 2.1, bottom right panel. Inspection of that panel shows that for each $\tilde{K} \in \tilde{T}^{\square}$ we have $h_{\tilde{K}, \min} = h_{\tilde{K}, y}$ and $h_{\tilde{K}, \max} = h_{\tilde{K}, x}$. Additionally, the elements can be enumerated as $K_{i,j}$, $i = 1, \ldots, L$, $j = 1, \ldots, i$ with $h_{K_{i,j}, x} \sim \sigma^{L-i}$, $h_{K_{i,j}, y} \sim \sigma^{L-j}$. Hence,

$$\sum_{K \in \tilde{T}^{\square}} \frac{h_{\tilde{K}, \min}}{h_{\tilde{K}, \max}} h_{\tilde{K}}^\delta \lesssim \sum_{i=1}^{L} \sum_{j=1}^{i} \sigma^{L-j} \sigma^{L-i} \lesssim \sum_{i=1}^{L} \sigma^{2L-i} \lesssim 1.$$ 

The proof of items (viii), (ix) is also done in similar ways. Therefore, we will only show (ix). The key observation is that by comparing sums with integrals, there is a constant $C > 0$ depending solely on $\delta$, $\alpha$, and $\sigma$ such that

$$\forall \varepsilon \in (0, 1]: \sum_{i=0}^{\infty} (\sigma^i/\varepsilon)^\delta e^{-\alpha \sigma^i/\varepsilon} \leq C.$$ 

The proof of (ix) now follows by a reasoning similar to that in (2.3).
3. Approximation on the reference elements and on the reference configurations. In Sec. 3.1 we construct polynomial approximation operators on the reference square and triangle that coincide with the Gauss-Lobatto interpolant on the edges, which affords convenient $H^1$-conforming approximations. Sec. 3.2 studies the approximation properties of spaces of piecewise polynomials on the reference patches. It is shown that functions of boundary layer or corner layer type can be approximated at exponential rates, robustly in the parameter $\varepsilon$ that characterizes the strength of the layer.

3.1. Polynomial approximation operators on the reference element. We introduce polynomial approximation operators on the reference triangle $\hat{T}$ in Lemma 3.1 and the reference square $\hat{S}$ in Lemma 3.2. Before actually doing so, we highlight a technical detail: the triangular elements (on the reference patches) are shape-regular so that isotropic scaling arguments can be brought to bear; only the rectangles (of the reference patches) may be anisotropic, for which tensor product polynomial approximation operators (specifically, the Gauss-Lobatto interpolation operator) are used for their favorable anisotropic scaling properties.

Lemma 3.1 (element-by-element approximation on triangles). Let $\hat{T}$ be the reference triangle. Then for every $q \in \mathbb{N}$, there exists a linear operator $\hat{\Pi}_q^\Delta : C(\hat{T}) \to \mathbb{P}_q$ with the following properties:

(i) For each edge $e$ of $\hat{T}$, $(\hat{\Pi}_q^\Delta u)|_e$ coincides with the Gauss-Lobatto interpolant $\iota_q(u)|_e$ of degree $q$ on edge $e$.

(ii) (projection property) $\hat{\Pi}_q^\Delta v = v$ for all $v \in \mathbb{P}_q$.

(iii) (stability) There exists a constant $C > 0$ such that for every $q \in \mathbb{N}$ there holds

$$\forall u \in W^{1,\infty}(\hat{T}) : \quad \|u - \hat{\Pi}_q^\Delta u\|_{W^{1,\infty}(\hat{T})} \leq C q^4 \|\nabla u\|_{L^\infty(\hat{T})},$$

$$\forall u \in C(\hat{T}) : \quad \|u - \hat{\Pi}_q^\Delta u\|_{L^\infty(\hat{T})} \leq C q^2 \|u\|_{L^\infty(\hat{T})}.$$

(iv) Let $A$ be one of the vertices of $\hat{T}$ and $\beta \in [0,1)$. Then there is $C > 0$ (depending only on $\beta$) such that, provided the right-hand side is finite,

$$\|u - \hat{\Pi}_q^\Delta u\|_{L^\infty(\hat{T})} + \|u - \hat{\Pi}_q^\Delta u\|_{H^1(\hat{T})} \leq C q^4 \|\text{dist}(\cdot,A)^\beta \nabla^2 u\|_{L^2(\hat{T})}.$$

(v) Let $u \in C^\infty(\hat{T})$ satisfy, for some $C_u$, $\gamma > 0$ and for some $h$, $\varepsilon \in (0,1]$,

$$\forall n \in \mathbb{N}_0 : \quad \|\nabla^n u\|_{L^\infty(\hat{T})} \leq C_u \gamma^n h^n \max\{n + 1, \varepsilon^{-1}\}^n.$$

Then there are $\delta, C, \eta, b > 0$ depending solely on $\gamma$ such that, under the provision that the scale resolution condition

$$h \leq \frac{q}{\varepsilon} \leq \delta$$

is satisfied, there holds (with the constant hidden in $\lesssim$ independent of $u$, $h$, $q$ and $\varepsilon$)

$$\|u - \hat{\Pi}_q^\Delta u\|_{W^{1,\infty}(\hat{T})} \lesssim C_u \left( \left( \frac{h}{h + \eta} \right)^{q+1} + \left( \frac{h}{q\varepsilon\eta} \right)^{q+1} \right) \lesssim C_u e^{-bq \min\{1, h/\varepsilon\}}.$$
Proof. The operator $\hat{\Pi}_q$ is taken as the one defined in [22, Thm. 3.2.20], where items (i)–(iii) are shown (the $W^{1,\infty}$-estimate follows with an additional polynomial inverse estimate). Item (iv) is taken from [22, Prop. 3.2.21]. For Item (v), we note that the projection property of (ii) and the stability assertions (iii) reduce the error estimate to a best approximation problem, which can be taken from [17, Lemma C.2].

**Lemma 3.2** (approximation properties of the Gauss-Lobatto interpolant). Let $\hat{S}$ be the reference square. For each $q \in \mathbb{N}$ the tensor-product Gauss-Lobatto interpolation operator $\hat{\Pi}_q^n : C(\hat{S}) \to \mathbb{Q}_q$ satisfies the following:

(i) **(projection property)** $\hat{\Pi}_q^n v = v$ for all $v \in \mathbb{Q}_q$.

(ii) For each edge $e$, the restriction $(\hat{\Pi}_q^n u)_e$ coincides with the univariate Gauss-Lobatto interpolant $i_q(u_e)$ on $e$.

(iii) **(stability)**

\[
\forall u \in C(\hat{S}) : \quad \| u - \hat{\Pi}_q^n u \|_{L^\infty(\hat{S})} \leq C_q \| u \|_{L^\infty(\hat{S})},
\]

\[
\forall u \in C^1(\hat{S}) : \quad \| \partial_x (u - \hat{\Pi}_q^n u) \|_{L^\infty(\hat{S})} \leq C_q^4 \| \partial_x u \|_{L^\infty(\hat{S})},
\]

\[
\forall u \in C^1(\hat{S}) : \quad \| \partial_y (u - \hat{\Pi}_q^n u) \|_{L^\infty(\hat{S})} \leq C_q^4 \| \partial_y u \|_{L^\infty(\hat{S})}.
\]

(iv) Let $u \in C^\infty(\hat{S})$ satisfy for some $C_u > 0$, $\epsilon_x, \epsilon_y, h_x, h_y \in (0,1]$ and all $(n,m) \in \mathbb{N}_0^2$

\[
\| \partial_x^m \partial_y^n u \|_{L^\infty(\hat{S})} \leq C_u \gamma^{n+m} h_x^m h_y^n \max\{n + 1, \epsilon_y^{-1}\} \max\{m + 1, \epsilon_x^{-1}\}.
\]

Then there are constants $\delta, C, \eta, b > 0$ depending solely on $\gamma$ such that under the scale-resolution condition

\[
\frac{h_x}{\epsilon_x} + \frac{h_y}{\epsilon_y} \leq \delta
\]

there holds

\[
\| \partial_x (u - \hat{\Pi}_q^n u) \|_{L^\infty(\hat{S})} \leq CC_u \epsilon_x \left( \frac{h_x}{h_x + \eta} \right)^q + \left( \frac{h_x}{\epsilon_x q \eta} \right)^q + \left( \frac{h_y}{h_y + \eta} \right)^{q+1} + \left( \frac{h_y}{\epsilon_y q \eta} \right)^{q+1},
\]

\[
\| \partial_y (u - \hat{\Pi}_q^n u) \|_{L^\infty(\hat{S})} \leq CC_u \epsilon_y \left( \frac{h_x}{h_x + \eta} \right)^{q+1} + \left( \frac{h_x}{\epsilon_x q \eta} \right)^{q+1} + \epsilon_y \left( \frac{h_y}{h_y + \eta} \right)^q + \left( \frac{h_y}{\epsilon_y q \eta} \right)^q,
\]

\[
\| u - \hat{\Pi}_q^n u \|_{L^\infty(\hat{S})} \leq CC_u \left[ \left( \frac{h_x}{h_x + \eta} \right)^{q+1} + \left( \frac{h_x}{\epsilon_x q \eta} \right)^{q+1} + \left( \frac{h_y}{h_y + \eta} \right)^{q+1} + \left( \frac{h_y}{\epsilon_y q \eta} \right)^{q+1} \right].
\]

Proof. Items (i), (ii) are well-known. We let $\Lambda_q$ denote the Lebesgue constant of the univariate Gauss-Lobatto interpolation operator of polynomial degree $q \in \mathbb{N}$ (cf. Lemma B.2). The $L^\infty$-stability in (iii) follows from tensor product arguments, viz.

\[
\| \hat{\Pi}_q^n u \|_{L^\infty(\hat{S})} \leq \Lambda_q^2 \| u \|_{L^\infty(\hat{S})} \text{ and the (generous) bound } \Lambda_q^2 \leq C_q \text{ for } q \geq 1.
\]

For the remaining estimates, we introduce the tensor-product Gauss-Lobatto interpolation
operator $\tilde{\Pi}_q^\square = i_q \otimes i_q = i_q^\square \otimes i_q^\square$, where we use the superscripts $\tilde{x}$ and $\tilde{y}$ to emphasize the variable with respect to which the univariate Gauss-Lobatto interpolant acts. From
\[
\hat{u} - i_q^\square \otimes i_q^\square u = u - (I \otimes i_q^\square) u + I \otimes i_q^\square (u - i_q^\square \otimes I u)
\]
we get in view of the univariate stability bound Lemma B.2
\[
\|\partial_2 (\hat{u} - i_q^\square \otimes i_q^\square u)\|_{L^\infty(S)} \lesssim A_q \sup_{x \in (0,1)} \inf_{\tilde{v} \in P_q} \|\partial_2 \tilde{u}(x, \cdot) - v\|_{L^\infty(0,1)}
\]
\[
+ q^2 A_q^2 \sup_{\tilde{v} \in P_q} \inf_{v \in P_q} \|\partial_2 (\tilde{u}(\cdot, \tilde{y}) - v)\|_{L^\infty(0,1)};
\]
an analogous estimate holds for $\partial_y (\hat{u} - i_q^\square \otimes i_q^\square u)$. The estimate (3.4) gives the stability estimates in $W^{1,\infty}$ of (iii) by selecting $v = 0$ in the infima. The estimate (3.4) reduces the question of approximation on $\hat{S}$ to questions of univariate polynomial approximation. The pertinent approximation results to prove item (iv) are given in Lemma B.1.

### 3.2. Approximation on the reference patches.
In this section, we study the approximation of functions on the reference patches (or the half-patches) described in Defs. 2.1, 2.7. The non-trivial reference patches consist of meshes that are anisotropically refined towards the edge $\{\tilde{y} = 0\}$, which can resolve algebraic singularities at $\{\tilde{y} = 0\}$ or boundary layers. We show exponential approximability of functions that have algebraic singularities at $0$ or boundary layers at $\{\tilde{y} = 0\}$.

Throughout this section, we will use the notation
\[
\tilde{r}(\cdot) := \text{dist}(0, \cdot).
\]
In this section, we present piecewise polynomial approximations on reference patches using the following elementwise defined interpolation operator:
\[
(\tilde{\Pi}_q)^K u := \begin{cases} 
\tilde{\Pi}_q^\triangle (u \circ A_{\tilde{K}}) & \text{if } \tilde{K} \text{ is a triangle } \triangle \\
\tilde{\Pi}_q^\square (u \circ A_{\tilde{K}}) & \text{if } \tilde{K} \text{ is a rectangle } \square,
\end{cases}
\]
where $A_{\tilde{K}} : \tilde{K} \to \tilde{K} = A_{\tilde{K}}(\tilde{K}) \subset \tilde{S}$ is the affine bijection between the reference element and the corresponding element on the reference patch. The edge-traces of the interpolators $\tilde{\Pi}_q^\triangle$ and $\tilde{\Pi}_q^\square$ coincide with the univariate Gauss-Lobatto interpolation operator on the edges of $\tilde{K}$. Hence, $H^1$-conformity of the elementwise defined operator $\tilde{\Pi}_q$ is ensured. We will frequently use the stability estimates
\[
\|\tilde{\Pi}_q u\|_{L^\infty(\tilde{K})} \leq C q^2 \|u\|_{L^\infty(\tilde{K})}, \quad \|\nabla \tilde{\Pi}_q u\|_{L^\infty(\tilde{K})} \leq C q^4 \|\nabla u\|_{L^\infty(\tilde{K})};
\]
these estimates are easily seen to hold for triangles with the isotropic scaling property and Lemma 3.1, (iii). The anisotropic nature of the rectangles is accounted for by separately scaling the bounds for the partial derivatives in Lemma 3.2, (iii).

#### 3.2.1. $hp$-FE approximation of corner singularity functions.

**Lemma 3.3** (approximation of corner singularity functions).
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(i) Let \( \tilde{T} \in \{ \tilde{T}^{M, \text{half}, L, n}, \tilde{T}^{C, \text{half}, n}, \tilde{T}^{C, \text{half, flip}, n}, \tilde{T}^{L, n}, \tilde{T}^{\text{geo}, \sigma}, \tilde{T}^{\text{geo}, \sigma} \} \). Let \( \mathcal{O} \) be the region covered by the elements of \( \tilde{T} \), i.e., let \( \mathcal{O} = \tilde{S} \) if \( \tilde{T} \) is a reference patch, \( \mathcal{O} = \tilde{T} \) if \( \tilde{T} \in \{ \tilde{T}^{C, \text{half}, n}, \tilde{T}^{M, \text{half}, L, n} \} \) is a reference half-patch, and \( \mathcal{O} = T^{\text{flip}} \) if \( \tilde{T} = \tilde{T}^{\text{flip}} \). Let \( \tilde{u} \) be analytic on \( \mathcal{O} \) and assume there exist constants \( \varepsilon \in (0, 1], \beta \in [0, 1), \gamma, C_u > 0 \) such that for all \( p \in \mathbb{N}_0 \) and all \( \tilde{x} \in \tilde{S} \)

\[
\| \nabla^p (\tilde{u}(\tilde{x}) - \tilde{u}(0)) \| \leq C_u \varepsilon^{-1-\gamma} p(\bar{\tau}(\tilde{x})/\varepsilon)^{1-\beta} \bar{r}(\tilde{x})^{-p} \max\{ p + 1, \bar{r}(\tilde{x})/\varepsilon \}^{p+1}. 
\]

Then, there are constants \( C, b, \kappa > 0 \) depending only on \( \gamma, \sigma, \) and \( \beta \) (in particular, independent of \( \varepsilon, n, L \)) such that under the scale resolution condition

\[
q \varepsilon \geq \kappa
\]

there holds

\[
\| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{L^\infty(\mathcal{O})} + \| \nabla (\tilde{u} - \tilde{\Pi}_q \tilde{u}) \|_{L^2(\mathcal{O})} \leq CC_u \left( q^d \sigma^{n(1-\beta)} + e^{-bq} \right).
\]

(ii) Let \( \tilde{T} \in \{ \tilde{T}^{L, n}, \tilde{S} \} \). Let \( \tilde{u} \) be analytic on \( \tilde{S} \) and assume that there are constants \( C_u, \gamma > 0 \) such that for all \( p \in \mathbb{N}_0 \)

\[
\| \nabla^p \tilde{u} \|_{L^\infty(\tilde{S})} \leq C_u \gamma^p \max\{ p + 1, \varepsilon^{-1} \}^{p+1}.
\]

Then there are constants \( C, b, \kappa > 0 \) depending only on \( \gamma \) and \( \sigma \) (in particular, they are independent of \( \varepsilon \) and \( L \)) such that under the constraint (3.9) there holds

\[
\| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{W^{1, \infty}(\tilde{S})} \leq CC_u e^{-bq}.
\]

Proof. Proof of (i): Step 1: Elements abutting on 0: Only triangles \( \triangle \) may abut on 0. Let \( \tilde{K} \) be such a triangle. From (3.8) and estimating (generously) \( \max\{1, \bar{r}(\cdot)/\varepsilon\}^3 \lesssim \varepsilon^{-3} \), we get the existence of \( C > 0 \) independent of \( \varepsilon \in (0, 1] \) with

\[
\forall \tilde{x} \in \tilde{K} : \quad |\nabla^2 \tilde{u}(\tilde{x})| \leq C(\bar{r}(\tilde{x}))^{-1-\beta} \varepsilon^{-5+\beta}.
\]

By scaling this bound and invoking Lemma 3.1, (iv), we get with \( h_{\tilde{K}} := \text{diam } \tilde{K} \) for any fixed \( \tilde{\beta} \in (\beta, 1) \)

\[
\| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{L^\infty(\tilde{K})} + \| \nabla (\tilde{u} - \tilde{\Pi}_q \tilde{u}) \|_{L^2(\tilde{K})} \leq C q^d h_{\tilde{K}}^{1-\tilde{\beta}} \| \nabla^2 \tilde{u} \|_{L^2(\tilde{K})}.
\]

\[
(3.13)
\leq C q^d h_{\tilde{K}}^{1-\tilde{\beta}} \varepsilon^{-5+\beta} \lesssim C q^{4+5-\beta} h_{\tilde{K}}^{1-\beta} \lesssim q^{-3+\beta} \sigma^{n(1-\beta)}.
\]

Step 2: Elements not abutting on 0: From Lemma 2.8, (v) we get \( \bar{r}(\cdot) \lesssim h_{\tilde{K}} = \text{diam } \tilde{K} \) on \( \tilde{K} \). The regularity assumption (3.8) then implies that there exist (suitably adjusted) constants \( C, \gamma \) such that for all \( p \in \mathbb{N}_0 \)

\[
\| \nabla^p (\tilde{u} - \tilde{u}(0)) \|_{L^\infty(\tilde{K})} \leq C q^p \varepsilon^{-3+\beta} h_{\tilde{K}}^{1-\beta} \bar{r}^{-p} \max\{ p + 1, \varepsilon^{-1} \}^p.
\]

We now consider the approximation on triangles and rectangles separately.

Step 2.1: \( \tilde{K} \) is a triangle \( \triangle \). Lemma 2.8, (ii) implies in particular that \( h_{\tilde{K}} \lesssim \bar{r}(\cdot) \) on \( \tilde{K} \). Scaling the bounds (3.15) to the reference element \( \tilde{K} = \tilde{T} \) therefore gives for
\( \hat{u} := \overline{u} \circ A_{\hat{K}} \), where \( A_{\hat{K}} : \hat{K} \rightarrow \tilde{K} \) is the affine element map for \( \tilde{K} \), the existence of constants \( C, \gamma > 0 \) such that

\[
(3.16) \quad \forall p \in \mathbb{N}_0: \| \nabla^p (\hat{u} - \overline{u}(0)) \|_{L^\infty(\tilde{K})} \leq C \gamma^p \varepsilon^{-3+\gamma} h_{\tilde{K}}^{1-\beta} \max\{p+1, \varepsilon^{-1}\}^p.
\]

In order to be able to apply the approximation properties of Lemma 3.1, we note

\[
\max\{p+1, \varepsilon^{-1}\}^p = \max\{(p+1)^p, \varepsilon^{-p}(p+1)^{-p}(p+1)^p\} = (p+1)^p \max\{1, \varepsilon^{-p}(p+1)^{-p}\} \leq (p+1)^p \max\left\{1, \left(\frac{1}{\varepsilon}\right)^p\right\}
\]

\[
(3.17) \quad \leq (p+1)^p e^{1/\varepsilon} \varepsilon^{\eta \varepsilon \kappa} \leq (p+1)^p e^{1/\kappa}.
\]

Inserting (3.17) into (3.16) yields that there are constants \( C > 0, \gamma > 0 \) such that

\[
(3.18) \quad \forall p \in \mathbb{N}_0: \| \nabla^p (\hat{u} - \overline{u}(0)) \|_{L^\infty(\tilde{K})} \leq C e^{\eta / \kappa} \varepsilon^{-3+\beta} h_{\tilde{K}}^{1-\beta} (p+1)^p.
\]

We are in a position to apply Lemma 3.1. The parameter \( \delta \) in (3.1) is determined by \( \gamma \). In view of \( q \varepsilon \geq \kappa \), we can ensure condition (3.1) by selecting \( \kappa \) sufficiently large to obtain from Lemma 3.1 with a \( b > 0 \) depending only on \( \gamma \)

\[
\| \hat{u} - \hat{u}^{h^q} \|_{W^{1,\infty}(\tilde{K})} \leq C h_{\tilde{K}}^{1-\beta} e^{\eta / \kappa} \varepsilon^{-3+\beta} e^{-bq},
\]

We may assume that \( \kappa \) is so large that \( 1/\kappa - b \leq -b/2 \) to absorb \( e^{\eta / \kappa} \). Finally, in view of \( q \varepsilon \geq \kappa \), we can also absorb the factor \( \varepsilon^{-3+\beta} \leq q^{3-\beta} \) in the exponentially decaying one by adjusting \( b \). Upon scaling from \( \hat{K} \) to \( \tilde{K} \) we get the existence of constants \( b, C > 0 \) such that

\[
(3.19) \quad \forall q \in \mathbb{N}: \| \hat{u} - \hat{u}^{h^q} \|_{L^\infty(\hat{K})} + \| \nabla (\hat{u} - \hat{u}^{h^q}) \|_{L^2(\hat{K})} \leq C h_{\hat{K}}^{1-\beta} e^{-bq}.
\]

Step 2.2: \( \tilde{K} \) is a rectangle \( \square \). We argue as in the case of a triangle in Step 2.1. Starting point is again the regularity assertion (3.8). The rectangle \( \tilde{K} \) has side lengths \( h_{\tilde{K},y} \leq h_{\tilde{K},x} \leq 1 \). From Lemma 2.8, (iv) we have \( h_{\tilde{K},y} \leq h_{\tilde{K},x} \leq C \varepsilon^{1/\kappa} \) on \( \tilde{K} \). Hence, the (anisotropic) scaling to the reference square \( \hat{S} \) of the estimates (3.8) yields, for all \( (n, m) \in \mathbb{N}_0^2 \), in view of (3.17)

\[
\| \partial_x^n \partial_y^m (\hat{u} - \overline{u}(0)) \|_{L^\infty(\hat{S})} \leq C \varepsilon^{-3+\beta} h_{\tilde{K},x}^{1-\beta} n^m h_{\tilde{K},y}^m h_{\tilde{K},y}^{-(n+m)} e^{\eta / \kappa} (n+m)^{\gamma n+m} + m!,
\]

\[
(3.20) \quad \leq C \varepsilon^{-3+\beta} h_{\tilde{K},x}^{1-\beta} n^m e^{\eta / \kappa} n! m!,
\]

where we again suitably adjusted the value of \( \gamma \). Lemma 3.2 (with \( h_y = h_{\tilde{K},y} / h_{\tilde{K},x} \leq 1 \) and \( \varepsilon_y = \varepsilon_y = 1 \) there) yields with the regularity estimates (3.20) the existence of constants \( C, b > 0 \) such that for all \( q \in \mathbb{N}_0 \)

\[
\| \hat{u} - \hat{u}^{h^q} \|_{L^\infty(\hat{K})} + \| \partial_x (\hat{u} - \hat{u}^{h^q}) \|_{L^\infty(\hat{K})} \leq C \varepsilon^{-3+\beta} h_{\tilde{K},x}^{1-\beta} e^{-bq},
\]

\[
\| \partial_y (\hat{u} - \hat{u}^{h^q}) \|_{L^\infty(\hat{K})} \leq C \varepsilon^{-3+\beta} h_{\tilde{K},y}^{1-\beta} h_{\tilde{K},x} e^{-bq},
\]
where the factor $e^{2q/\kappa}$ was absorbed again in the exponentially decaying term by taking $\kappa$ sufficiently large. We obtain on $\overline{S}$
\begin{equation}
\|\partial^l_\nu (\bar{u} - \Pi_q \bar{u})\|_{L^2(\overline{S})} \leq C e^{-3+\beta} \sqrt{h_{K,\beta} h_{\tilde{K},\beta}^{-1} h_{\tilde{K},\overline{x}}} e^{-bq} \tag{3.21a}
\end{equation}
\begin{equation}
\|\partial^l_\gamma (\bar{u} - \Pi_q \bar{u})\|_{L^2(\overline{S})} \leq C e^\kappa \sqrt{h_{K,\beta} h_{\tilde{K},\beta}^{-1} h_{\tilde{K},\overline{x}}} e^{-bq}, \tag{3.21b}
\end{equation}
\begin{equation}
\|\bar{u} - \Pi_q \bar{u}\|_{L^\infty(\overline{S})} \leq C h_{K,\beta}^{-\gamma} e^{-bq}, \tag{3.21c}
\end{equation}
where again we adjusted the values of the constants $b$, $C$ in both estimates to absorb algebraic factors in $q$.

Step 3: Summation of the elemental errors: We note that the element size $h_{\tilde{K}}$ of the elements abutting on $0$ is $h_{\tilde{K}} \sim \sigma^n$. For the finitely many contributions from the (triangular) elements $\tilde{K}$ touching $0$ we have by (3.14) the existence of $C > 0$ such that for every $q \geq 1$
\begin{equation}
\sum_{\tilde{K} : 0 \in \overline{\tilde{K}}} \|\bar{u} - \Pi_q \bar{u}\|_{H^1(\overline{\tilde{K}})}^2 \leq C q^{18} \sigma^{2n(1-\beta)}. \tag{3.14}
\end{equation}
The sum of squared error contributions over all triangular elements not touching $0$ is also bounded by $e^{-2bq}$ by combining (3.19) and Lemma 2.8, (vi). Likewise, the sum over all rectangular elements is bounded by $e^{-2bq}$ by combining (3.21) and Lemma 2.8, (vii).

Proof of (ii): The proof is similar to the proof of case (i) and can be obtained from it by formally setting $\beta = 0$ and $\tilde{E} \equiv 1$ and dropping the error contribution $q^0 \sigma^{n(1-\beta)}$ that is due to the small elements touching $0$.

3.2.2. $hp$-FE approximation of boundary layer functions.

Lemma 3.4 (approximation of boundary layer functions). Fix $c_1 > 0$.

(i) Let $\tilde{T} \in \{\tilde{T}_{g,e,0}^{M,\text{half}}, \tilde{T}_{g,e,0}^{B,\text{L}}\}$. Let $\tilde{T}' \subset \tilde{T}$ and let $O := \text{interior}(\cup\{\tilde{K} | \tilde{K} \in \tilde{T}'\})$ be the union of the elements of $\tilde{T}'$. Let $\bar{u}$ be analytic on $O$ and satisfy for some $C_u, \gamma, \alpha > 0, \varepsilon \in (0,1]$ and for all $(m,n) \in N_0^2$ and all $\tilde{x} = (\tilde{x}, \tilde{y}) \in O$
\begin{equation}
|\partial^l_\nu \partial^m_\gamma \bar{u}(\tilde{x})| \leq C_u \gamma^{n+m} m! \max\{n, \varepsilon^{-1}\}^n e^{-\alpha |\tilde{y}|/\varepsilon}. \tag{3.22}
\end{equation}
Assume that $L$ is such that the scale resolution condition
\begin{equation}
\sigma^L \leq c_1 \varepsilon \tag{3.23}
\end{equation}
is satisfied. Then there are constants $C$, $b > 0$ depending only on $\gamma$, $\alpha$, $c_1$, $\sigma$ such that
\begin{equation}
\forall q \in \mathbb{N}: \|\bar{u} - \Pi_q \bar{u}\|_{L^\infty(O)} + \varepsilon \|\nabla (\bar{u} - \Pi_q \bar{u})\|_{L^\infty(O)} \leq CC_u e^{-bq}. \tag{3.24}
\end{equation}

(ii) Let $\tilde{T}'' \subset \tilde{T}_{g,e,0}^{C,\text{half},n}$ or $\tilde{T}'' \subset \tilde{T}_{g,e,0}^{C,\text{half},\text{flip},n}$. Let $O := \text{interior}(\cup\{\tilde{K} | \tilde{K} \in \tilde{T}''\})$ be the union of the elements of $\tilde{T}''$. Let $\bar{u}$ be analytic on $O$ and satisfy for some $C_u, \gamma, \alpha > 0, \varepsilon \in (0,1]$ \begin{equation}
\forall p \in N_0 \forall \tilde{x} \in O: \|\nabla^p \bar{u}(\tilde{x})\| \leq C_u \gamma^p \max\{p, \varepsilon^{-1}\}^p e^{-\alpha |\tilde{x}|/\varepsilon}. \tag{3.25}
\end{equation}
Assume that $n$ is such that $c_1 > 0$, $n \in \mathbb{N}$ satisfies the scale scale resolution condition
\begin{equation}
(3.26) \quad \sigma^n \leq c_1 \varepsilon
\end{equation}
is satisfied. Then, there are constants $C, b > 0$ (depending only on $\gamma, \alpha, c_1, \sigma$) such that
\begin{equation}
(3.27) \quad \forall q \in \mathbb{N}: \|\overline{u} - \overline{\Pi}_q \overline{u}\|_{L^\infty(Q)} + \varepsilon\|\nabla(\overline{u} - \overline{\Pi}_q \overline{u})\|_{L^\infty(Q)} \leq Cq e^{-bq}.
\end{equation}

Proof. Proof of (ii): We only consider the case $\mathcal{T}'' \subset \mathcal{T}_{geo,\sigma}^{C,\text{half},n}$ as the case $\mathcal{T}'' \subset \mathcal{T}_{geo,\sigma}^{C,\text{half},\text{flip},n}$ is handled similarly. We note that the patch $\mathcal{T}_{geo,\sigma}^{C,\text{half},n}$ consists of triangles only, which are all shape-regular. Let $\tilde{K} \subset Q$ be a triangle and let $h_{\tilde{K}} = \text{diam}\tilde{K}$. In the case that $\tilde{K}$ touches $0$, the condition (3.26) implies that $h_{\tilde{K}} \lesssim \sigma^n \lesssim c_1 \varepsilon$ so that
\begin{equation}
(3.28) \quad \frac{h_{\tilde{K}}}{q^\varepsilon} \lesssim \frac{1}{q}.
\end{equation}
Hence, for every fixed choice of the constant $c_1$ there exists $q_0 = q_0(c_1) \in \mathbb{N}$ (independent of $\varepsilon$) such that for every $q \geq q_0$ one has the scale resolution condition (3.1). Then, Lemma 3.1, (v) implies for suitable $b > 0$ (independent of $\varepsilon$)
\begin{equation}
(3.29) \quad \|\overline{u} - \overline{\Pi}_q \overline{u}\|_{L^\infty(\tilde{K})} + \varepsilon\|\nabla(\overline{u} - \overline{\Pi}_q \overline{u})\|_{L^\infty(\tilde{K})} \lesssim e^{-bq}.
\end{equation}

If $\tilde{K}$ does not touch $0$, we distinguish between two further cases. In the first case, we assume that $h_{\tilde{K}}/(q^\varepsilon) \leq \delta$ and proceed as above: The scale resolution condition (3.1) is satisfied, and we arrive again at (3.29). In the case $h_{\tilde{K}}/(q^\varepsilon) \geq \delta$, we note that Lemma 2.8, (ii) implies $\text{dist}(\tilde{K}, 0) \geq c_2 h_{\tilde{K}} \geq c_2 \delta q^\varepsilon$. Hence, by the decay properties of $\overline{u}$ in (3.25) we have
\begin{equation}
(3.30) \quad \|\overline{u}\|_{L^\infty(\tilde{K})} + \varepsilon\|\nabla \overline{u}\|_{L^\infty(\tilde{K})} \leq Ce^{-\alpha c_2 h_{\tilde{K}}/\varepsilon} \leq C e^{-\alpha c_2 q^\varepsilon}.
\end{equation}
In view of the stability properties (3.7), we conclude
\begin{equation}
(3.31) \quad \|\overline{u} - \overline{\Pi}_q \overline{u}\|_{L^\infty(\tilde{K})} + \varepsilon\|\nabla(\overline{u} - \overline{\Pi}_q \overline{u})\|_{L^\infty(\tilde{K})} \lesssim e^{-bq}.
\end{equation}

Proof of (i): We distinguish between triangular and rectangular elements.

Approximation of $\overline{u}$ on triangular elements $\tilde{K}$: Triangular elements do not appear in boundary layer patches $\mathcal{T}_{geo,\sigma}^{BL,L}$ but only in $\mathcal{T}_{geo,\sigma}^{M,\text{half},L,n}$. For patches $\mathcal{T}_{geo,\sigma}^{M,\text{half},L,n}$ inspection (cf. Fig. 2.2) shows that two types of triangles occur: the first type are the triangles $\tilde{K}$ in $\tilde{T} \setminus \tilde{T}_1$ on which one has $\tilde{r}(\tilde{x}, \tilde{y}) \sim \tilde{y}$ (uniformly in $L, n$). The second type are the triangles in $\tilde{T}_1$. For the first type, we have from (3.22) the regularity assertion (with suitably adjusted $C_u, \gamma, \alpha$ independent of $\varepsilon$)
\begin{equation*}
\forall \tilde{x} \in \tilde{K} \forall n \in \mathbb{N}_0: \quad |\nabla^n \overline{u}(\tilde{x})| \leq C_u\gamma^n \max\{n, \varepsilon^{-1}\} \varepsilon^{-\alpha(\tilde{K})/\varepsilon}.
\end{equation*}
This is the same regularity assumption that underlies the proof of part (ii) of the lemma so that the same arguments can be brought to bear as in the case of part (ii). For the second type of triangles, i.e., $\tilde{K} \subset \tilde{T}_1 \subset (0, \sigma L)^2$, the resolution assumption (3.23) implies for the element size $h_{\tilde{K}} \lesssim \sigma L \lesssim \varepsilon$. Hence, again Lemma 3.1, (v) is applicable and yields the desired exponential approximation.
Approximation of \( \tilde{u} \) on rectangular elements \( \tilde{K} \): The case of rectangular elements \( \tilde{K} \) with side lengths \( h_{\tilde{K},x}, h_{\tilde{K},y} \) is similar to the case of triangles. We note that the patches \( \tilde{T}_{\text{geo},\sigma} \) and \( \tilde{T}_{\text{half},\sigma}^{M} \) are such that \( h_{\tilde{K},\tilde{y}} \leq h_{\tilde{K},\tilde{x}} \leq 1 \). The anisotropic scaling from \( K \) to \( \tilde{K} \) and the regularity assumption (3.22) show that the pull-back \( \tilde{u} \) to \( \tilde{K} \) satisfies for all \( (m,n) \in \mathbb{N}_0^2 \)

\[
\| \partial_x^n \partial_y^m \tilde{u} \|_{L^\infty(\tilde{K})} \leq Ce^{-\alpha \text{dist}(\tilde{K},(\tilde{y}=0))/\varepsilon} h_{\tilde{K},x}^m h_{\tilde{K},y}^n + \max\{n+1,\varepsilon^{-1}\}^n.
\]

That is, \( \tilde{u} \) satisfies the analytic regularity condition (3.2) with \( \varepsilon_y = \varepsilon, \varepsilon_x = 1, h_x = h_{\tilde{K},\tilde{x}}, h_y = h_{\tilde{K},\tilde{y}} \) and \( C_u = Ce^{-\alpha \text{dist}(\tilde{K},(\tilde{y}=0))/\varepsilon} \). We observe that the resolution condition (3.3) can be achieved if \( \tilde{K} \) touches the line \( \{ \tilde{y} = 0 \} \) in view of (3.23) provided that \( q \geq q_0 \geq 1 \) for suitable \( q_0 \) (depending on \( c_1, \sigma, \gamma \)). If \( \tilde{K} \) does not touch the line \( \{ \tilde{y} = 0 \} \), then two cases may occur: If the resolution condition (3.3) is still satisfied then we obtain again exponential convergence. If not, we note that \( h_{\tilde{K},\tilde{y}} \leq h_{\tilde{K},\tilde{x}} \) and that we may assume \( h_{\tilde{K},\tilde{x}}/q \leq \delta/2 \) by assuming \( q \geq q_0 \geq 1 \) (note: trivially, \( h_{\tilde{K},\tilde{x}} \leq 1 \) so that \( q_0 \geq 2/\delta \) will work). Furthermore, Lemma 2.8, (iii) reveals again that \( \text{dist}(\tilde{K},(\tilde{y}=0)) \geq c_3 h_{\tilde{K},\tilde{y}} \) since \( h_{\tilde{K},\tilde{y}}/(\varepsilon q) \geq \delta/2 \) we get \( \text{dist}(\tilde{K},(\tilde{y}=0))/\varepsilon \geq c_3 q_0 \delta/2 \). Hence, \( \exp(-\alpha \text{dist}(\tilde{K},(\tilde{y}=0))/\varepsilon) \leq \exp(-q_0 c_3 \delta/2) \) and we may argue as in the case of triangles that \( \tilde{u} \) is exponentially (in \( q \)) small on \( \tilde{K} \). The stability of \( \tilde{u} \) given in (3.7) then concludes the argument.

\[ \Box \]

### 3.2.3. \( hp \)-FE approximation of corner layer functions.

**Lemma 3.5** (approximation of corner layer functions). Fix \( c_1 > 0 \). Let \( \tilde{T} \in \{ \tilde{T}_{\text{geo},\sigma}, \tilde{T}_{\text{half},\sigma} \} \). Let \( \tilde{T}' \subset \tilde{T} \) and let \( \bar{O} := \text{interior}(\mathbb{K} | K \in \tilde{T}' \} \). Let \( \tilde{u} \) be analytic on \( \bar{O} \) and satisfy for some \( \beta \in [0,1], \varepsilon \in (0,1), \sigma > 0 \)

\[
(3.32) \quad \forall \tilde{x} \in \bar{O} \quad \forall \tilde{p} \in \mathbb{N}_0 : \quad |\nabla^\beta \tilde{u}(\tilde{x})| \leq C_u \varepsilon^{\beta-1} p(\tilde{x})^{1-\beta} r p e^{-\sigma(\tilde{x})/\varepsilon}.
\]

Assume that \( n \in \mathbb{N} \) is such that the scale resolution condition

\[
(3.33) \quad \sigma^n \leq c_1 \varepsilon.
\]

is satisfied. Then there are constants \( C, b > 0 \) depending only on \( \gamma, \alpha, c_1, \sigma, \) and \( \beta \) (in particular, independent of \( \varepsilon, q, n, L \)) such that for all \( q \in \mathbb{N} \)

\[
(3.34) \quad \| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{L^\infty(\bar{O})} \leq CC_u \left( e^{-bq} + q^4 \varepsilon^{\beta-1} \sigma^n(1-\beta) \right),
\]

\[
(3.35) \quad \| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{L^2(\bar{O})} + \varepsilon \| \nabla(\tilde{u} - \tilde{\Pi}_q \tilde{u}) \|_{L^2(\bar{O})} \leq CC_u \varepsilon \left( e^{-bq} + q^4 \varepsilon^{\beta-1} \sigma^n(1-\beta) \right).
\]

In the estimates (3.34), (3.35) the term \( q^4 \varepsilon^{\beta-1} \sigma^n(1-\beta) \) can be dropped if \( \bar{O} \) does not touch \( \tilde{K} \).

**Proof.** The approximation of functions of corner layer type \( \tilde{u} \) proceeds structurally along the same lines as in the case of the singularity functions in Lemma 3.3. We distinguish between the elements touching \( \tilde{K} \) and the remaining ones.

**K touches \( \tilde{K} \):** Selecting \( \beta \in (\beta, 1) \) we obtain by arguing as in (3.14)

\[
(3.36) \quad \| \tilde{u} - \tilde{\Pi}_q \tilde{u} \|_{L^\infty(\tilde{K})} + \| \nabla(\tilde{u} - \tilde{\Pi}_q \tilde{u}) \|_{L^2(\tilde{K})} \lesssim h_{\tilde{K}}^{1-\beta} q^4 \| \tilde{\beta}^2 \tilde{u} \|_{L^2(\tilde{K})} \lesssim q^4 (h_{\tilde{K}}/\varepsilon)^{1-\beta}.
\]
Since $h_K \lesssim \sigma^n$ for elements $\widehat{K}$ touching $0$, their contributions lead to the term $q^4 \varepsilon^{\beta-1} \sigma^{n(1-\beta)}$.

**K does not touch 0:** We distinguish between triangular and rectangular elements.

**Step 1:** $K$ is a triangular element: As in the case of the approximation in Lemma 3.3, we get from Lemma 3.1, (v) and scaling that (for suitably adjusted $C, \alpha$)

$$\left(3.37\right) \quad \|\hat{u} - \hat{\Pi}_q \hat{u}\|_{L^\infty(\widehat{K})} + \|\nabla(\hat{u} - \hat{\Pi}_q \hat{u})\|_{L^2(\widehat{K})} \leq C(h_{\widehat{K}}/\varepsilon)^{1-\beta} e^{-bq} e^{-\alpha h_{\widehat{K}}/\varepsilon}.$$

**Step 2:** $K$ is a rectangular element: We recall $h_{\widehat{K},\widehat{y}} \leq h_{\widehat{K},\widehat{x}} \leq 1$. By Lemma 2.8, (iv) we have $\tilde{r}(\cdot) \sim h_{\widehat{K},\widehat{x}}$ on $\widehat{K}$. As in the case of Lemma 3.3 we observe for the pull-back to the reference element $\widehat{K}$

$$\forall (m, n) \in \mathbb{N}_0^2: \quad \|\partial^m q^n \hat{u}\|_{L^\infty(\widehat{K})} \leq C(h_{\widehat{K},\widehat{x}}/\varepsilon)^{1-\beta} e^{-bq} e^{-\alpha h_{\widehat{K},\widehat{x}}/\varepsilon}.$$

Using Lemma 3.2, (iv) (with $\varepsilon = \varepsilon_y = 1$ and $h_y = h_{\widehat{K},\widehat{y}} / h_{\widehat{K},\widehat{x}}$, $h_x = 1$ there), we arrive at

$$\left(3.38\right) \quad \|\partial_x (\hat{u} - \hat{\Pi}_q \hat{u})\|_{L^\infty(\widehat{K})} \leq C(h_{\widehat{K}}/\varepsilon)^{1-\beta} e^{-bq} e^{-\alpha h_{\widehat{K},\widehat{x}}/\varepsilon},$$

$$\left(3.39\right) \quad \|\partial_y (\hat{u} - \hat{\Pi}_q \hat{u})\|_{L^\infty(\widehat{K})} \leq C(h_{\widehat{K}}/\varepsilon)^{1-\beta} e^{-bq} e^{-\alpha h_{\widehat{K},\widehat{x}}/\varepsilon},$$

$$\left(3.40\right) \quad \|\partial_y (\hat{u} - \hat{\Pi}_q \hat{u})\|_{L^\infty(\widehat{K})} \leq C(h_{\widehat{K}}/\varepsilon)^{1-\beta} e^{-bq} e^{-\alpha h_{\widehat{K},\widehat{x}}/\varepsilon}.$$

**Step 3 ($L^\infty$-bound):** Since $\sup_{t>0} t^{1-\beta} e^{-t} < \infty$, the $L^\infty$-estimates follow easily from (3.36), (3.37) (3.38).

**Step 4 (energy norm estimate):** Proceeding as in Step 3 of the proof of Lemma 3.3 we set $e_K := \hat{u} - \hat{\Pi}_q \hat{u}$ and get, using $h_K \lesssim \varepsilon$ for the elements abutting on 0:

$$\sum_{\widehat{K}: \widehat{K} \text{ abuts on } 0} \|e_K\|^2_{L^2(\widehat{K})} + \varepsilon^2 \|
abla e_K\|^2_{L^2(\widehat{K})} \lesssim q^8 \sum_{\widehat{K}: \widehat{K} \text{ abuts on } 0} (h_{\widehat{K}}^2 + \varepsilon^2)(h_{\widehat{K}}/\varepsilon)^{(1-\beta)} \lesssim q^8 \varepsilon^{2\beta} \sigma^{2n(1-\beta)}.$$

For the remaining elements, we consider the triangular elements and the rectangular ones. In both cases, we employ the simple observation

$$\left(3.41\right) \quad \|e_K\|_{L^2(\widehat{K})} \lesssim h_{\widehat{K}} \|e_K\|_{L^\infty(\widehat{K})} = \varepsilon h_{\widehat{K}} \|e_K\|_{L^\infty(\widehat{K})}.$$

The sum over all triangles, collected in $\tilde{T}^\Delta$, yields by combining (3.37) and (3.41) with Lemma 2.8, (viii)

$$\sum_{K \in \tilde{T}^\Delta} \|e_K\|^2_{L^2(\widehat{K})} + \varepsilon^2 \|e_K\|^2_{H^1(\widehat{K})} \lesssim \varepsilon^2 e^{-2bq}.$$

Likewise, the sum over all rectangular elements, collected in $\tilde{T}^\Box$, yields by combining Lemma 2.8, (ix) with (3.38), (3.41) for the $L^2$-part and with (3.39), (3.40) for the $H^1$-part

$$\sum_{K \in \tilde{T}^\Box} \|e_K\|^2_{L^2(\widehat{K})} + \varepsilon^2 \|e_K\|^2_{H^1(\widehat{K})} \lesssim \varepsilon^2 e^{-2bq}.$$

This concludes the proof. \(\square\)
4. *hp*-FE approximation of singularly perturbed problems on geometric boundary layer meshes. The principal result of the present paper is a robust, exponential approximation result for solutions of the singular perturbation problem (1.1), (1.2) in curvilinear polygonal domains from spaces based on geometric boundary layer meshes that are able to resolve the length scales present in the problem. The meshes are independent of $\varepsilon$ but subject to the (weak) scale resolution condition (4.1).

**Theorem 4.1.** Let the Lipschitz domain $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon with $J$ vertices as described in Section 1.2. Let $A$, $c$, $f$ satisfy (1.2). Fix $c_1 > 0$. Let $T_{geo,\sigma}^{L,n}$ be a geometric boundary layer mesh in sense of Definition 2.3.

Then there are constants $C$, $b > 0$, $\beta \in [0,1)$ depending solely on the data $A$, $c$, $f$, $\Omega$, on the parameter $c_1$, on the (fixed) macro-triangulation $T^M$, and on $\sigma \in (0,1)$ such that the following holds: If $\varepsilon \in (0,1]$ and $L$ satisfy the scale resolution condition

\begin{equation}
\frac{\sigma^L}{\varepsilon} \leq c_1,
\end{equation}

then for every $q$, $n \in \mathbb{N}$ the solution $u_{\varepsilon} \in H^1_0(\Omega)$ of (1.1) can be approximated from $S^q_0(\Omega, T_{geo,\sigma}^{L,n})$ such that

\begin{equation}
\inf_{v \in S^q_0(\Omega, T_{geo,\sigma}^{L,n})} \|u_{\varepsilon} - v\|_{\varepsilon,\Omega} \leq C q^\beta \sigma^{(1-\beta)n} + \varepsilon^{-bq},
\end{equation}

\begin{equation}
N := \dim S^q_0(\Omega, T_{geo,\sigma}^{L,n}) \leq C(L^2 q^2 \text{card } T^M + nq^2 J).
\end{equation}
Proof. Before proving the result, let us comment on the scale resolution (4.1) and
its relation to previous scale resolution conditions (3.9), (3.33), and (3.23). Condition
(4.1) ensures that $L$ layers of anisotropic refinement towards the boundary are
performed, which is the condition (3.23) needed to resolve functions of boundary layer
type. Since $n \geq L$ (by Def. 2.3), condition (4.1) also enforces the condition (3.33),
which provides the approximation of corner layer functions. Finally, the situation
(3.9) is of a different nature as in that case, the polynomial degree is so large that
already very coarse meshes can resolve the boundary layers.

We employ the analytic, parametric regularity theory for the solution $u_\varepsilon$ presented
in [22, Thms. 2.3.1, 2.3.4]. The infimum in (4.2) is estimated with the aid of the
interpolation operator $\Pi_q$ that is defined elementwise by

$$(\Pi_q u)|_K \circ F_K := \begin{cases} \hat{\Pi}^\triangle_q (u \circ F_K) & \text{if } K \text{ is a triangle} \\ \hat{\Pi}^\square_q (u \circ F_K) & \text{if } K \text{ is a rectangle}. \end{cases}$$

Here, the operator $\hat{\Pi}^\triangle_q$ is defined in Lemma 3.1 and the operator $\hat{\Pi}^\square_q$ in Lemma 3.2.
Since $\hat{\Pi}^\triangle_q$ and $\hat{\Pi}^\square_q$ reduce to the Gauss-Lobatto interpolation operator on the edges
of the reference element, the operator $\Pi_q$ indeed maps into $S^q_0(\Omega, T_{geo,\sigma})$. We recall
that the element maps $F_K$ have the form

$$F_K = F_{K,M} \circ A_K,$$

where $A_K : \tilde{K} \to \tilde{K} := A_K(\tilde{K}) = F_{K,M}^{-1}(K) \subset \tilde{S}$ is an affine bijection. Indeed, for
triangular elements it is clear that $A_K$ is affine and for rectangular elements, this
follows from the special form of the reference patches (cf. also Lemma 2.8, (i)).

The notation $\tilde{u}$ denotes the pull-back of $u$ to the reference element, i.e., $\tilde{u} := u|_K \circ F_K$ whereas $\bar{u} := (u \circ F_{K,M})|_{\tilde{K}} = \tilde{u} \circ A_{K,1}$ is the corresponding function on $K$.
We recall the notation $\hat{\Pi}_q$ from (3.6) and note that on a macro-element $K^M$ we have

$$(\Pi_q u)|_K \circ F_K = \hat{\Pi}_q \bar{u}.$$ 

For $k \in \mathbb{N}_0$ we have for all elements $K \subset K^M$ with $\tilde{K} = F_{K,M}^{-1}(K)$

$$\forall v \in H^k(K): \|v \circ F_{K,M}\|_{H^k(\tilde{K})} \sim \|v\|_{H^k(K)},$$

(4.4a) $$\forall v \in W^k,\infty(K): \|v \circ F_{K,M}\|_{W^k,\infty(\tilde{K})} \sim \|v\|_{W^k,\infty(K)},$$

(4.4b)

where in both cases the constants implied in $\sim$ depend solely on $k$ and the macro-
element $K^M$. The equivalences (4.4) show that the approximation error $v - \Pi_q u$ on
$K$ is equivalent to the corresponding error $\tilde{v} - \hat{\Pi}_q \bar{v}$ on $\tilde{K}$.

The approximation theory distinguishes between the “asymptotic case” $\varepsilon \geq \kappa$ of
large polynomial degree $q$ and the “preasymptotic case” $\varepsilon \leq \kappa$, where the parameter
$\kappa > 0$ (depending only on $A, c, f, \Omega$, the macro-triangulation, and $\sigma$) is of size $O(1)$
and will be determined in the course of the analysis of the “asymptotic case” in Step I.

Step I: Asymptotic case $\varepsilon \geq \kappa$. We consider mesh patches $K^M$ that abut on a
vertex $A_j$ and those with a positive distance from the vertices separately in Steps I.1
and I.2.

Step I.1: $K^M$ abuts on a vertex $A_j$: The regularity of [22, Thm. 2.3.1] asserts
the existence of $C, \gamma > 0, \beta_j \in [0, 1)$ such that with $r_j(\cdot) := \text{dist}(\cdot, A_j)$, there holds
for every $p \in \mathbb{N}_0$ and for every $0 < \varepsilon \leq 1$

$$|\nabla^p (u_\varepsilon(\cdot) - u_\varepsilon(A_j))| \leq C\gamma^p \varepsilon^{-1} \min\{1, r_j(\cdot)/\varepsilon\}^{1-\beta_j} (r_j(\cdot))^{-p} \max\{p+1, r_j(\cdot)/\varepsilon\}^{p+1}.$$
Recall from (3.5) that \( \bar{r}(\cdot) = \text{dist}(\cdot, \mathbf{0}) \). Set \( \bar{u}_\varepsilon := u_\varepsilon \circ F_{K^M} \). Note \( F_{K^M}(\mathbf{0}) = A_j \) and \( \bar{r}(\bar{x}) \sim r_j(F_{K^M}(\bar{x})) \). The analyticity of \( F_{K^M} \) and Lemma A.2 imply, for suitably modified constants \( C, \gamma \) independent of \( \varepsilon \in (0, 1] \), for every \( p \in \mathbb{N}_0 \) holds on \( \bar{K} \)

\[
|\nabla^p(\bar{u}_\varepsilon(\cdot) - \bar{u}(\mathbf{0}))| \leq C\gamma^p\varepsilon^{-1} \min\{1, \bar{r}/\varepsilon\}^{1-\beta_j}\bar{r}(\cdot)^{-p} \max\{p + 1, \bar{r}(\cdot)/\varepsilon\}^{p+1}.
\]

Lemma 3.3 then yields

\[
\|\bar{u}_\varepsilon - \bar{\Pi}_q\bar{u}_\varepsilon\|_{L^\infty(K^M)} + \|\nabla(\bar{u}_\varepsilon - \bar{\Pi}_q\bar{u}_\varepsilon)\|_{L^\infty(K^M)} \leq C\varepsilon^q \left( \sigma(1-\beta_j)^n + \varepsilon^{-bq} \right)
\]

provided that \( \kappa \) is chosen sufficiently large (depending on \( \gamma \)).

**Step I.2:** \( K^M \) does not abut on a vertex \( A_j \): [22, Thm. 2.3.1] asserts

\[
\forall x \in K^M, \forall p \in \mathbb{N}_0: |\nabla^p u(x)| \leq C\gamma^p \max\{p + 1, \varepsilon^{-1}\}^{p+2}
\]

for constants \( C, \gamma > 0 \) independent of \( \varepsilon \in (0, 1] \). Since \( K^M \) is a trivial patch, it consists of a single (curvilinear) quadrilateral. The analyticity of \( F_{K^M} = F_K \) and Lemma A.2 imply, for suitably modified \( C, \gamma \) independent of \( \varepsilon \in (0, 1] \), that

\[
\forall \bar{x} \in \bar{S}, \forall p \in \mathbb{N}_0: |\nabla^p \bar{u}(\bar{x})| \leq C\gamma^p \max\{p + 1, \varepsilon^{-1}\}^{p+2}.
\]

Lemma 3.2 then implies that there are \( C, b > 0 \) such that for sufficiently large, fixed \( \kappa \) and for every \( \varepsilon \in (0, 1] \) and every \( q \in \mathbb{N} \) holds

\[
\|\bar{u} - \bar{\Pi}_q\bar{u}\|_{L^\infty(K^M)} + \|\nabla(\bar{u} - \bar{\Pi}_q\bar{u})\|_{L^\infty(\bar{S})} \leq Ce^{-bq}.
\]

**Step I.3:** Combining the approximation results of Steps I.1, I.2 for the finitely many patches leads to the desired estimate (4.2).

**Step II:** Preasymptotic case \( q\varepsilon \leq \kappa \). The parameter \( \kappa \) has been fixed in Step I through the appeal to Lemmas 3.3 and 3.2. In the regime \( q\varepsilon \leq \kappa \), we employ the regularity theory of [22, Thm. 2.3.4], which furnishes the decomposition \( u_\varepsilon = w_\varepsilon + \chi^{\text{BL}}w^{\text{BL}}_\varepsilon + \chi^{\text{CL}}w^{\text{CL}}_\varepsilon + r_\varepsilon \) into a smooth part \( w_\varepsilon \), a boundary layer part \( w^{\text{BL}}_\varepsilon \), a corner layer part \( w^{\text{CL}}_\varepsilon \), and a small remainder \( r_\varepsilon \); the functions \( \chi^{\text{BL}}, \chi^{\text{CL}} \) are suitable localizations near the boundary and the vertices of \( \Omega \). We approximate each of these four contributions in turn.

**Step II.1:** Approximation of \( w_\varepsilon \). By [22, Thm. 2.3.4] the smooth part \( w_\varepsilon \) is analytic on \( \bar{\Omega} \) with constants independent of \( \varepsilon \). Therefore, one can show \( \|w_\varepsilon - \Pi_qw_\varepsilon\|_{W^{1,\infty}(\Omega)} \leq Ce^{-bq} \) using similar techniques as in the asymptotic case above (essentially, setting \( \varepsilon = 1 \) there and ignoring the special treatment of the elements abutting on the vertices of \( \Omega \)).

**Step II.2:** Approximation of \( \chi^{\text{BL}}u^{\text{BL}}_\varepsilon \).

**Step II.2.a:** Regularity of \( u^{\text{BL}}_\varepsilon \): The regularity of \( u^{\text{BL}}_\varepsilon \) in [22, Thm. 2.3.4] is described in terms of boundary fitted coordinates (cf. Fig. 4.1). Associated with each edge \( \Gamma_j \) are fitted coordinates \((\rho_j, \theta_j)\), where \( \rho_j \) is the distance from the analytic continuation \( \Gamma_j \) of the boundary arc \( \Gamma_j \), and \( \theta_j \) is a parametrization of \( \Gamma_j \). The map \( \psi_j : (\rho_j, \theta_j) \to (x, y) \in \Omega \) is analytic with an analytic inverse. An analytic arc \( \Gamma'_j \) emanates from each vertex \( A_j \), which can be chosen arbitrarily but is assumed to be such that the angles between \( \Gamma_j \) and \( \Gamma'_j \) and between \( \Gamma_{j+1} \) and \( \Gamma'_j \) are both less than \( \pi \). Condition 8 of Definition 2.3 ensures that \( \Gamma'_j \) can be chosen to be a meshline of a boundary layer mesh since it can be chosen as the image of an edge of \( \bar{S} \) or a diagonal of \( \bar{S} \) under a patch map.
The regions $\Omega_j \subset \{ x \in \Omega \mid \text{dist}(x, \Gamma) < \delta \}$ for a sufficiently small $\delta$ are defined by the lines $\Gamma_j$, $\Gamma_j'$, and $\Gamma_j - 1$, as shown in Fig. 4.2. By [22, Thm. 2.3.4], the function $u^e_\varepsilon$ is analytic on each $\Omega_j$ and satisfies there, for constants $C, \gamma, \alpha > 0$ independent of $\varepsilon \in (0, 1]$ and all $(m, n) \in \mathbb{N}_0$,

$$\begin{align*}
|\partial^m_{\rho_j} \partial^n_{\theta_j} u^e_\varepsilon \circ \psi_j(\rho_j, \theta_j)| & \leq C \varepsilon^{-n} \gamma^{n+m} m! e^{-\alpha_\rho_j / \varepsilon} \\
(4.9) & \leq C \gamma^{n+m} \max\{n + m, \varepsilon^{-1}\}^{n+m} m! e^{-\alpha_\rho_j / \varepsilon}.
\end{align*}$$

Finally, the cut-off function $\chi^B$ is supported by $\bigcup_j \overline{\Omega_j}$ and is identically 1 near $\partial \Omega$.

**Step II.2.b:** Approximation of $\chi^B u^BL$ far from $\partial \Omega$: In the interest of simplicity of notation, we make the assumption that patches $K^M$ touching $\partial \Omega$ are fully contained in the tubular neighborhood $\bigcup_j \overline{\Omega_j}$ of $\partial \Omega$. Since patches $K^M$ not touching $\partial \Omega$ have a positive distance from $\partial \Omega$, the function $\chi^B u^BL$ is exponentially small (in $1/\varepsilon$) there; in view of the stability (3.7) (and thus the stability of $\Pi_q$) $\|\chi^B u^BL - \Pi_q(\chi^B u^BL)\|_{W^{1,\infty}(K)} \leq C \varepsilon^{-b/\varepsilon}$ for $K \in K^M$. Since $q/\varepsilon \leq 1/\varepsilon$ the error contribution of these patches is controlled in the desired fashion.

**Step II.2.c:** Approximation of $\chi^B u^BL$ near $\partial \Omega$: Let $K^M$ be a patch touching $\partial \Omega$. Consider, for a fixed $j$ the pull-back $F^M_1(K^M \cap \Omega_j)$. By the assumptions on the boundary layer mesh (Def. 2.3) this pull back is either empty, the full square $\overline{S}$, half the square $\overline{T} = \{ \overline{x} = (\overline{x}, \overline{y}) \mid 0 < \overline{x} < 1, 0 < \overline{y} < \overline{x} \}$, or the other half $\overline{T}^{\text{flip}} = \{ \overline{x} = (\overline{x}, \overline{y}) \mid 0 < \overline{x} < 1, \overline{x} < \overline{y} < 1 \}$. To fix ideas, let us assume that $K^M$ is a mixed patch. The reference mixed patch restricted to $\overline{T}$ is the half-patch $T^{\text{mixed},L,n}$ and its restriction to $T^{\text{flip}}$ is $\overline{T}^{\text{mixed,fitted,n}}$. We approximate $(\chi^B u^BL) \circ F_{K^M}$ on these two parts separately, starting with the approximation on $\overline{T}$. The assumptions on boundary layer meshes (Def. 2.3) allow us to assume that $F_{K^M}(\overline{T}) \subset \Omega_j$ for some $j$. We recall that $F_{K^M}$ maps the edge $\{ \overline{y} = 0 \}$ of $\overline{T}$ to a subset of $\partial \Omega$, which corresponds to $\rho_j = 0$ in the boundary fitted coordinates. The shape-regularity of $F_{K^M}$ implies that $\psi_j^{-1} \circ F_{K^M}$ has the form

$$\begin{align*}
\overline{T} \ni (\overline{x}, \overline{y}) & \mapsto (\rho_j, \theta_j) = (\overline{y} \rho(\overline{x}, \overline{y}), \theta(\overline{x}, \overline{y}))
\end{align*}$$

for a pair of functions $\rho, \theta$ with $\rho \geq \rho_0 > 0$. The analyticity of $\psi_j^{-1}$ and $F_{K^M}$ implies that $\rho$ and $\theta$ are in fact analytic on $\overline{T}$. Hence, the transformed function

$$\begin{align*}
\overline{u}^BL_\varepsilon := u^BL_\varepsilon \circ F_{K^M} = u^BL_\varepsilon \circ \psi_j^{-1} \circ (\psi_j^{-1} \circ F_{K^M})
\end{align*}$$

admits by Lemma A.1 and (4.9) the analytic regularity

$$\begin{align*}
\forall (m, n) \in \mathbb{N}_0^2 \quad \forall (\overline{x}, \overline{y}) \in \overline{T}: \quad |\partial^m_{\overline{\rho_j}} \partial^n_{\overline{\theta_j}} \overline{u}^BL_\varepsilon(\overline{x}, \overline{y})| & \leq C \gamma^{m+n} m! \max\{n + 1, \varepsilon^{-1}\}^{n+m} e^{-b\overline{y}/\varepsilon},
\end{align*}$$

where the constants $C, \gamma, b > 0$ are independent of $\varepsilon \in (0, 1]$. We decompose the set of elements $T^{\text{mixed},L,n}$ into two sets $\overline{T}_1 := \{ \overline{K} \in T^{\text{mixed},L,n} \mid \chi^B(\overline{K}) \equiv 1 \}$ and $\overline{T}_2 := T^{\text{mixed},L,n}_{\text{geo,}\sigma} \setminus \overline{T}_1$. For the elements of $\overline{T}_1$, Lemma 3.4, (i) and (4.12) give that there are $C, b$ such that for every $q \in \mathbb{N}$ and every $\overline{K} \in \overline{T}_1$

$$\begin{align*}
\|\chi^B \overline{u}^B_\varepsilon - \overline{\Pi}_q(\chi^B \overline{u}^B_\varepsilon)\|_{L^\infty(\overline{K})} & + \varepsilon \|\nabla(\chi^B \overline{u}^B_\varepsilon - \overline{\Pi}_q(\chi^B \overline{u}^B_\varepsilon))\|_{L^\infty(\overline{K})} \leq C e^{-bq}.
\end{align*}$$

For the elements of the set $\overline{T}_2$, we use (4.10) to see that $\overline{K} \in \overline{T}_2$ implies $\text{dist}(\overline{K}, \{ \overline{y} = 0 \}) > c$ for some $c > 0$ that depends solely on $F_{K^M}$ and $\psi_j$. Hence, the smoothness
of $\chi^{\text{BL}}$ and (4.12) provide $\|\chi^{\text{BL}}u_\varepsilon\|_{W^{1,1}(\tilde{K})} \leq Ce^{-b/\varepsilon}$ for suitable $C, b > 0$ and every $\tilde{K} \in \tilde{T}_2$. Hence, the stability properties of $\tilde{\Pi}_q$ provided in (3.7) and $q/k \leq 1/\varepsilon$ imply for all $\tilde{K} \in \tilde{T}_2$

\begin{equation}
(4.14) \quad \|\chi^{\text{BL}}u_\varepsilon - \tilde{\Pi}_q(\chi^{\text{BL}}u_\varepsilon)\|_{L^\infty(\tilde{K})} + \varepsilon\|\nabla(\chi^{\text{BL}}u_\varepsilon - \tilde{\Pi}_q(\chi^{\text{BL}}u_\varepsilon))\|_{L^\infty(\tilde{K})} \leq Ce^{-bq}.
\end{equation}

Let us now sketch the arguments for the approximation of $\chi^{\text{BL}}u_\varepsilon$ on $T^{\text{flip}}$. For notational simplicity, assume that $F_{KM}(T^{\text{flip}}) \subset \Omega_j$. (If $F_{KM}(T^{\text{flip}}) \subset \Omega_j'$ for some different $j'$, then replace $j$ with $j'$ in what follows.) The regularity assertion (4.12) is still valid. Next, one observes that on $T^{\text{flip}}$, one has $\tilde{y} \sim \tilde{r}(\tilde{x}, \tilde{y}) = \text{dist}(\tilde{x}, \tilde{y}), 0)$. Hence, recalling (4.12), $u_\varepsilon^{\text{BL}}$ satisfies, for suitable $C, b > 0$ and for all $p \in \mathbb{N}_0$,

\begin{equation}
(4.15) \quad |\nabla^p u_\varepsilon^{\text{BL}}(\cdot)| \leq C\max\{p + 1, \varepsilon^{-1}\}& e^{-b\tilde{r}(\cdot)/\varepsilon} \text{ on } T^{\text{flip}}.
\end{equation}

Replacing the appeal to Lemma 3.4, (i) with a reference to Lemma 3.4, (ii), we may argue as above to obtain

\begin{equation}
\|\chi^{\text{BL}}u_\varepsilon - \tilde{\Pi}_q\chi^{\text{BL}}u_\varepsilon\|_{L^\infty(T^{\text{flip}})} + \varepsilon\|\nabla(\chi^{\text{BL}}u_\varepsilon - \tilde{\Pi}_q\chi^{\text{BL}}u_\varepsilon)\|_{L^\infty(T^{\text{flip}})} \leq Ce^{-bq}.
\end{equation}

This concludes the arguments for the approximation of $u_\varepsilon^{\text{BL}}$ on a mixed patch $\tilde{T}^{M,M,.L,.n}$. The approximation on corner patches $\tilde{T}^{C,n}_{\text{geo,} \sigma}$, tensor patches $\tilde{T}^{T,n}_{\text{geo,} \sigma}$, or boundary layer patches $\tilde{T}^{\text{BL},L,n}_{\text{geo,} \sigma}$ is similar.

**Step II.3: Approximation of $\chi^{\text{CL}}u_\varepsilon^{\text{CL}}$:** Structurally, the proof is similar to the procedure in Step II.2. From [22, Thm. 2.3.4] we have in a neighborhood $B_j$ of vertex $A_j$ that $u_\varepsilon^{\text{CL}}$ satisfies on $(B_j \cap \Omega_j) \cup (B_j \cap \Omega_j + 1)$ with $r_j(\cdot) = \text{dist}(\cdot, A_j)$

\begin{equation}
(4.16) \quad \forall p \in \mathbb{N}_0: \quad |\nabla^p u_\varepsilon^{\text{CL}}(\cdot)| \leq C\gamma^p p!\varepsilon^\beta \cdot r_j(\cdot)^{1-p-\beta}e^{-\alpha r_j(\cdot)/\varepsilon},
\end{equation}

where $C, \alpha > 0$ and $\beta \in [0, 1)$ are independent of $\varepsilon \in (0, 1]$. Let $K^M$ be a patch abutting on $A_j$. Such a patch has to be either a corner patch or a mixed patch. Then $K^M \cap \Omega_j$ (and similarly $K^M \cap \Omega_{j+1}$) consists of one or two half-patches that are push-forwards of $T' \in \{\tilde{T}^{M,\text{half},L,.n}_{\text{geo,} \sigma}, \tilde{T}^{C,\text{half},n}_{\text{geo,} \sigma}, \tilde{T}^{C,\text{half},\text{flip},n}_{\text{geo,} \sigma}\}$. For simplicity of exposition, assume that $K^M \subset B_j$. By the analyticity of the patch-map $F_{KM}$, the shape regularity of $F_{KM}$ together with $F_{KM}(0) = A_j$, and by Lemma A.2, we get that $u_\varepsilon^{\text{CL}} := u_\varepsilon^{\text{CL}} \circ F_{KM}$ satisfies on $\mathcal{O} := F_{KM}^{-1}(K^M \cap \Omega_j)$

\begin{equation}
(4.17) \quad \forall p \in \mathbb{N}_0: \quad |\nabla^p u_\varepsilon^{\text{CL}}(\cdot)| \leq C\gamma^p p!\varepsilon^\beta \cdot \tilde{r}(\cdot)^{1-p-\beta}e^{-\alpha \tilde{r}(\cdot)/\varepsilon},
\end{equation}

with possibly adjusted values for $C, \gamma, \alpha > 0$. We also note that the pull-back $\chi^{\text{CL}}$ is smooth and identically 1 near 0. Hence, using Lemma 3.5 we obtain

\begin{equation}
\|\chi^{\text{BL}}u_\varepsilon^{\text{CL}} - \tilde{\Pi}_q(\chi^{\text{BL}}u_\varepsilon^{\text{CL}})\|_{L^2(\mathcal{O})} + \varepsilon\|\nabla(\chi^{\text{BL}}u_\varepsilon^{\text{CL}} - \tilde{\Pi}_q(\chi^{\text{BL}}u_\varepsilon^{\text{CL}}))\|_{L^2(\mathcal{O})} \\
\leq C\left(\varepsilon e^{-bq} + \varepsilon^\beta q^4\alpha^{\alpha(1-\beta)}\right).
\end{equation}

**Step II.4: Approximation of $r_\varepsilon$:** We approximate $r_\varepsilon$ by zero. We note that [22, Thm. 2.3.4] asserts that $r_\varepsilon|_{\partial \Omega} = 0$ and that $\|r_\varepsilon\|_{H^1(\Omega)} \leq Ce^{-b/\varepsilon}$ for suitable $C, b > 0$ independent of $\varepsilon \in (0, 1]$.

**Corollary 4.2.** Assume the hypotheses on $\Omega$ and the data $A, c, f$ as in Theorem 4.1. Let $\tilde{T}^{L,n}_{\text{geo,} \sigma}$ be a geometric boundary layer mesh as in Definition 2.3.
Then, for every fixed $0 < \sigma < 1$ and $c_1 > 0$ there exist constants $C$, $b > 0$ such that, for every $0 < \varepsilon \leq 1$, with the choices $q \simeq n \geq L \geq c_1 \log \varepsilon$, the solution $u_\varepsilon \in H^1_0(\Omega)$ of (1.1) can be approximated from $S^q_0(\Omega, T_{geo,\sigma}^{L,n})$ at an exponential rate:

$$\inf_{v \in S^q_0(\Omega, T_{geo,\sigma}^{L,n})} \|u_\varepsilon - v\|_{H^1(\Omega)} \leq C \exp\left(-b\sqrt[4]{N}\right), \quad N = \dim(S^q_0(\Omega, T_{geo,\sigma}^{L,n})).$$

**Remark 4.3.** In addition to the approximation in the energy norm in Cor. 4.2, exponential approximation results in the so-called “balanced norm” $\|v\|_{\sqrt{\varepsilon}} := \varepsilon \|\nabla v\|_{L^2(\Omega)}$, $\|v\|_{L^2(\Omega)}^2$ (or even in $H^1(\Omega)$) are possible under slightly stronger conditions: for sufficiently large $C_1$, the constraint $q \simeq n \geq L \geq C_1 \log \varepsilon$ yields

$$\inf_{v \in S^q_0(\Omega, T_{geo,\sigma}^{L,n})} \|u_\varepsilon - v\|_{\sqrt{\varepsilon}H^1(\Omega)} \leq C \exp(-b'\sqrt[4]{N})$$

for suitable $b' > 0$ since a factor $\varepsilon^{-1/2}$ can be compensated by the exponentially decaying terms $e^{-b\sqrt{N}}$ in Cor. 4.2.

Theorem 4.1 is restricted to $\varepsilon \in (0, 1]$. For $\varepsilon \geq 1$, (1.1) is a regularly perturbed elliptic boundary value problem and exponential convergence of $hp$-FEM with mere geometric corner refinement follows by standard results [35, 22].

**Proposition 4.4.** Assume the hypotheses on $\Omega$ and the data $A, c, f$ as in Theorem 4.1. Let $T_{geo,\sigma}^{L,n}$ be a geometric boundary layer mesh. Then, there are constants $C, b > 0, \beta \in (0, 1)$ depending solely on $A, c, f$, the analyticity properties of the patch maps for the macro-triangulation, and $\sigma$ such that for any $\varepsilon \geq 1$, the solution $u_\varepsilon$ of (1.1) satisfies for every $n, L, q \in \mathbb{N}$

$$\inf_{v \in S^q_0(\Omega, T_{geo,\sigma}^{L,n})} \|u_\varepsilon - v\|_{H^1(\Omega)} \leq C \varepsilon^{-2} \left(q^0 \sigma (1-\beta)n + e^{-bq}\right). \quad (4.18)$$

**Proof.** The solution $u_\varepsilon \in H^1_0(\Omega)$ satisfies

$$-\nabla \cdot (A\nabla u_\varepsilon) + \varepsilon^{-2} c u_\varepsilon = \varepsilon^{-2} f \quad \text{in } H^{-1}(\Omega). \quad (4.19)$$

For $\varepsilon \geq 1$, the term $\varepsilon^{-2}c$ represents a regular perturbation and the analytic regularity theory for linear, second order elliptic boundary value problems (e.g. [5] and the references there) is applicable. The resulting regularity assertions are then those employed in the “asymptotic case” in the proof of Theorem 4.1 with $\varepsilon = 1$ there. The factor $\varepsilon^{-2}$ in (4.18) is a reflection of the fact that the right-hand side of (4.19) include the factor $\varepsilon^{-2}$. \qed

**5. Numerical experiments.** For $0 < \varepsilon \leq 1$ and $f \equiv 1$ we consider the Dirichlet problem: find $u_\varepsilon \in H^1_0(\Omega)$ such that

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } H^{-1}(\Omega).$$

Here, the domain $\Omega$ is either the unit square $\Omega_1 = (0,1)^2$, the so-called “L-shaped, polygonal domain” $\Omega_2 \subset \mathbb{R}^2$ determined by the vertices $\{(0,0), (1,0), (1,1), (-1,1), (-1,-1), (0,-1)\}$, or the square domain with a slit $\Omega_3 = (-1,1)^2 \setminus (-1,0) \times \{0\}$.

In Figures 5.1–5.3 we show examples of the meshes used in our computations on the three domains. These are constructed using the NGSolve/Netgen package [29]. For the square domain $\Omega = \Omega_1$ the resulting mesh is the geometric boundary layer mesh $T_{geo,\sigma}^{L,L}$ with $L = 4$ and $\sigma = 0.25$. The same parameters are used in NGSolve/Netgen.
to construct the meshes for the other two domains, with the resulting meshes differing slightly from the strict definition of $T_{L,L,geo,σ}$ near the re-entrant corners. Nevertheless, we denote these meshes also by $T_{L,L,geo,σ}$ and make use of the finite element spaces $S_p^0(Ω, T_{geo,σ})$. We also mention that in accordance with Remark 2.4, the meshes shown in Figs. 5.1–5.3 do not satisfy requirement 8 of Definition 2.3.

For each $p = 1, 2, 3, \ldots$, we use the finite element space $S_p^0(Ω, T_{geo,σ})$ with uniform polynomial order $q = p$ and with $L = p$ refinement levels towards boundaries and corners with refinement factor $σ = 0.25$. We denote by $u_h^ε ∈ S_p^0(Ω, T_{geo,σ})$ the corresponding finite element solution. We measure the error in energy norm

$$\text{error} = \left( ε^2 \left\| \nabla (u_ε - u_h^ε) \right\|^2_{L^2(Ω)} + \left\| u_ε - u_h^ε \right\|^2_{L^2(Ω)} \right)^{1/2},$$

where $u_h^ε$ denotes the discrete solution. In place of the (unknown, for the considered examples) exact solution $u_ε$ we use a numerical approximation on a sufficiently fine mesh. The plots of the estimated numerical errors for the three domains are depicted in Figures 5.1–5.3. Evidently, exponential convergence occurs. In agreement with the theoretical analysis, the experimentally observed exponential convergence has two regimes: (i) an asymptotic regime in which the scale resolution condition $σ^p ≲ ε$ is satisfied and (ii) a pre-asymptotic regime with $σ^p ≳ ε$.

The observed exponential convergence in the presymptotic regime (not rigorously shown in Theorem 4.1) is plausible for the following reason: the approximation error for boundary layer functions is dominated by the error on the elements touching the boundary and is of size $O(σ^{p/2})$ for every $p ∈ \mathbb{N}$. The approximation error of the corner layer functions is likewise dominated by the error on the elements abutting on the vertices of $Ω$ and is of size $O(σ^{p(1-β)})$ for every $p ∈ \mathbb{N}$ and some fixed $β ∈ [0, 1)$.

6. Conclusions. We established robust exponential convergence of $hp$-FEM for solutions of elliptic singular perturbation problems in polygons. These solutions contain, usually, boundary layers, corner singularities and combinations of the two. We admitted possibly multiple length scales, and built the $hp$-FE approximations on (patches of) geometric boundary layer meshes as described in Section 2, and depicted in Fig. 2.1. The $hp$-FEM on this class of partitions is capable to resolve exponential boundary- and corner-layers with multiple physical length scales under a scale resolution condition that incorporates the smallest physical length scale. The number of
geometric mesh refinements to achieve this grows only logarithmically with respect to the smallest length scale. The proposed, spectral numerical boundary layer resolution by \( hp \)-FEM is based on boundary fitted, structured mesh-patches in the physical domain \( \Omega \). They are pushforwards from a finite catalog \( \Phi \) of canonical, highly structured, anisotropic reference mesh patterns. Such pushforwards are readily available in industrial CAD / CAM environments such as, e.g., NgSolve [29, 30]. For deployment, it only requires (a lower bound on) the smallest physical length-scale \( \varepsilon \). This can often be deduced from heuristic, physical considerations, e.g. scaling or dimensional analysis. The methodology should be contrasted with so-called “augmented/enriched spectral discretizations” proposed even recently in [15, 11] and in references there. For this numerical approach, explicit, functional forms of boundary and corner layer components of the solution are required. The analytic derivation of closed forms for such solution components in general geometries for possibly nonlinear PDEs requires an elaborate asymptotic analysis, which is unnecessary in the present approach.

As we explained in the numerical experiments section patch-structured meshes as required here can be generated, in general geometries, by specialized mesh generators.

**Fig. 5.2.** Right panel: Convergence in the energy norm (5.1) for the L-shaped domain for different values of \( \varepsilon \) and \( q = L = n = p \). Left panel: a Netgen-generated mesh used for the computations.

**Fig. 5.3.** Right panel: Convergence in the energy norm (5.1) for the slit domain for different values of \( \varepsilon \) and \( q = L = n = p \). Left panel: a Netgen-generated mesh used for the computations.
such as Netgen [29]. We hasten to add, however, that our analysis can readily be extended to cover more general partitions, such as geometric boundary layer meshes that also contain anisotropic triangles.

The focus of the present work was on robust exponential convergence rate bounds for singular perturbation problems in nonsmooth domains by \( hp \) finite element methods. We proved that they afford robust, exponential convergence on patchwise structured meshes with possibly anisotropic, geometric refinement towards the “support set” (i.e., the subset of \( \Omega \) off which the layer components decay exponentially), of the boundary and corner layers. As a rule, robust exponential convergence requires genuine \( hp \)-FE capabilities, i.e., simultaneous mesh refinement and polynomial degree increase, as featured in the \( hp \)-FE spaces \( \{S_0^p(\Omega, T_{\text{geo},\sigma})\}_{p \geq 1} \) used in our numerical experiments. The corresponding, patchwise structured triangulations can be automatically generated by specialized mesh generators, in domains of engineering interest (see, e.g., [29]). Although we mainly considered the model linear, second order elliptic singular perturbation problem (1.1), corresponding solution families are known to arise for several common models in solid and fluid mechanics, see, e.g. [2, 14, 10, 15] and the references there.

The present \( hp \)-error analysis implies exponential upper bounds on Kolmogorov \( N \)-widths of solution sets \( \{u_\varepsilon : 0 < \varepsilon \leq 1\} \) of (1.1). We recall that, for a normed linear space \( X \) (with norm \( \| \cdot \|_X \) ) and for a subset \( B \subset X \), the \( N \)-width is given by

\[
d_N(B, X) = \inf_{E_N} \sup_{f \in B} \inf_{g \in E_N} \|f - g\|_X,
\]

where the first infimum is taken over all subspaces \( E_N \) of \( X \) of dimension \( N \in \mathbb{N} \). Subspace sequences \( \{E_N\}_{N \geq 1} \) which attain the rates of \( d_N(B, X) \) in (6.1) can be realized numerically by non-polynomial, so-called reduced bases (see [26]).

In (6.1) we choose \( (X, \| \cdot \|_X) = (H^1_0(\Omega), \| \cdot \|_{H^1(\Omega)}) \) with the energy norm \( \| \cdot \|_{H^1(\Omega)} \) of (1.3). Given a complex neighborhood \( G \subset \mathbb{C}^2 \) of \( \Omega \) we take \( B \subset X \) as the set of solutions of (1.1) corresponding right-hand sides \( f \) that admit a holomorphic extension to \( G \) with \( \|f\|_{L^\infty(G)} \leq 1 \). From Corollary 4.2, with \( E_N = S_0^0(\Omega, T_{\text{geo},\sigma}) \) we obtain \( d_N(B, X) \leq \exp(-b\sqrt{N}) \) with a continuous, piecewise polynomial interpolant to bound the inner infimum in (6.1). Remark that [16, Theorem 3.2] stipulates for (1.1) the (sharp) majorization \( d_N(A, X) \leq \exp(-b'\sqrt{N}) \) with \( b' > 0 \) possibly different from \( b \) and with nonpolynomial \( E_N \), but with \( b' \) and the constant hidden in \( \leq \) independent of \( \varepsilon \). For analytic \( \partial \Omega \) and a single, known boundary layer length scale \( \varepsilon \in (0, 1) \), this rate is attained by \( hp \)-FEM on so-called minimal boundary layer meshes (e.g. [38, 23]) which are \( \varepsilon \)-dependent, however.

The underlying concept of using patchwise structured meshes to approximate parametric solution families to linear, elliptic singularly perturbed boundary value problems extends also to \( h \)-version FEM. Here, in patches abutting on the boundary analogs of so-called “Shishkin meshes”, see, e.g., [39], [28, Sec. 3.5.2], could be employed to achieve robust, algebraic rates of convergence under weaker, finite order differentiability assumptions on the data \( A, c, \) and \( f \) than the presently assumed analyticity in \( \overline{\Omega} \) of these data. The present results will constitute a foundation for proving exponential convergence of several \( hp \) discretizations of (spectral) fractional diffusion problems as presented in [6] in curvilinear polygonal domains \( \Omega \). Details will be developed in [7].

The model problem (1.1) considers homogeneous Dirichlet boundary conditions. The approximation result Theorem 4.1 relies on the regularity results of [22], which decomposes the solution (1.1) into boundary and corner layer components. Similar
Appendix A. Analytic changes of variables. The following lemma shows how boundary layer functions are transformed under the patch maps if the edge \( \{ y = 0 \} \) of \( \bar{S} \) is mapped to a subset of \( \partial \Omega \):

**Lemma A.1.** Let \( G_x \subset \mathbb{R} \times \mathbb{R}^+ \) be a domain. Let the map \( M : (\bar{x}, \bar{y}) \mapsto (\theta, \rho) \) be of the form \( M(\bar{x}, \bar{y}) = (\hat{\theta}(\bar{x}, \bar{y}), \hat{\rho}(\bar{x}, \bar{y})) \) for some functions \( \hat{\theta}, \hat{\rho} \geq \rho_0 > 0 \) that are analytic on closure \( (G_x) \), i.e., there are constants \( C_M, \gamma_M > 0 \) such that \( \| \nabla^n \hat{\theta} \|_{L^\infty(G_x)} \leq C_M \gamma_M^m m! \) for all \( n \in \mathbb{N}_0 \). Let \( \Omega_x \subset G_x \) be open and let \( \Omega \) be an open neighborhood of \( M(\Omega_x) \). Let \( u \) be analytic on \( \Omega \) and assume that, for some function \( C_u \) and some constants \( b > 0 \), \( \gamma > 0 \), there holds

\[
\forall (m, n) \in \mathbb{N}_0^2 \quad (\rho, \theta) \in \Omega : \quad |\partial^m \partial^n u(\rho, \theta)| \leq C_u(\rho, \theta) e^{-b \rho^2 / \gamma} \gamma^{m+n} \max\{n, 1\} m!
\]

Then there are constants \( b', \gamma > 0 \) (depending only on \( b, \gamma \), and \( M \)) such that the function \( \hat{u} := u \circ M \) satisfies with the notation \( \rho(\bar{x}, \bar{y}) = M(\bar{x}, \bar{y}) \)

\[
\forall (m, n) \in \mathbb{N}_0^2 \quad \forall (\bar{x}, \bar{y}) \in \Omega_x : \quad |\partial^m \partial^n \hat{u}(\bar{x}, \bar{y})| \leq C_u(\rho, \theta) e^{-b' \rho^2 / \gamma} \gamma^{m+n} \max\{n, 1\} m!
\]

**Proof.** The proof uses arguments employed in [22, Sec. 4.3]. Consider a fixed \( (\bar{x}, \bar{y}) \in \Omega_x \) and set \( (\theta', \rho') = M(\bar{x}, \bar{y}) \). Then \( (\theta, \rho) \mapsto u(\theta, \rho) \) is holomorphic on the polydisc

\[
B_{1/\gamma}(\theta') \times B_{1/\gamma}(\rho') \subset \mathbb{C}^2
\]

with the bound

\[
|u(\theta' + \zeta_1, \rho' + \zeta_2)| \leq C_u(\rho', \theta') e^{-b' \rho^2 / \gamma} \gamma^{m+n} \max\{m, 1\} m!
\]

Since the functions \( \hat{\theta}, \hat{\rho} \) are holomorphic on the closure of \( M(G_x) \), there are \( C_1, \delta > 0 \) (independent of \( (\bar{x}, \bar{y}) \)) such that for \( \zeta_1, \zeta_2 \in B_\delta(0) \subset \mathbb{C} \) there holds

\[
|\hat{\theta}(\bar{x} + \zeta_1, \bar{y} + \zeta_2) - \hat{\theta}(\bar{x}, \bar{y})| \leq C_1 |\zeta_1| \|\zeta_2\|,
\]

\[
|\bar{y}(\bar{x} + \zeta_1, \bar{y} + \zeta_2) - \bar{y}(\bar{x}, \bar{y})| \leq C_1 |\zeta_1| \|\zeta_2\|,
\]

and we may assume that \( \delta > 0 \) is such that for \( \zeta_1, \zeta_2 \in B_\delta(0) \) we have \( M(\bar{x} + \zeta_1, \bar{y} + \zeta_2) \in B_{1/(2\gamma)}(\theta') \times B_{1/(2\gamma)}(\rho') \). This implies in view of (A.1) the bounds

\[
|\hat{u}(\bar{x} + \zeta_1, \bar{y} + \zeta_2)| = |u(M(\bar{x} + \zeta_1, \bar{y} + \zeta_2))| \leq C_Cu(\theta', \rho') e^{-b' \rho^2 / \gamma} \gamma^{m+n} \max\{m, 1\} m! \exp(C_1 \gamma|\zeta_1| / \gamma) \exp(C_1 \gamma|\zeta_2| / \gamma).
\]

For \( \delta_1, \delta_2 < \delta \) Cauchy’s integral formula for derivatives gives

\[
\frac{\partial^{a_1}_{\bar{x}} \partial^{a_2}_{\bar{y}} \hat{u}(\bar{x}, \bar{y})}{4\pi^2} \int_{\Omega_x} \frac{1}{\int_{\partial B_{1/\delta}(0)} \int_{\partial B_{1/\delta}(0)}} \frac{\bar{u}(\bar{x} + \zeta_1, \bar{y} + \zeta_2)}{(-\zeta_1)^{a_1+1}(-\zeta_2)^{a_2+1}} d\zeta_1 d\zeta_2
\]

so that

\[
|\partial^{a_1}_{\bar{x}} \partial^{a_2}_{\bar{y}} \hat{u}(\bar{x}, \bar{y})| \leq C_Cu(\rho', \theta') e^{-b' \rho^2 / \gamma} \gamma^{m+n} \max\{m, 1\} m! \exp(C_1 \gamma|\zeta_1| / \gamma) \exp(C_1 \gamma|\zeta_2| / \gamma) \exp(C_1 \gamma|\delta_1 + \delta_2| / \gamma).
\]

Selecting \( \delta_1 = \delta := b \rho_0/(4C_1) \) and \( \delta_2 = \min\{(|\zeta_2| + 1)\epsilon, \delta\} \) yields the desired result with \( b' = b/2 \) since \( C_1 \gamma \epsilon \delta \leq 2\delta C_1 \gamma \epsilon \leq 2\delta C_1 \rho' / \rho_0 = b/2 \).
The following lemma shows how functions that may have a singular behavior are transformed under analytic changes of variables:

**Lemma A.2** ([22, Lemma 4.3.3]). Let $\tilde{G} \subset \mathbb{R}^2$ be a domain and $M: \tilde{G} \to \mathbb{R}^2$ be analytic on $\text{closure}(\tilde{G})$. Let $\tilde{\Omega} \subset \tilde{G}$ be open and $\Omega$ be an open neighborhood of $M(\tilde{\Omega})$. Let $u$ be analytic on $\Omega$ and assume that for some (positive) function $\Lambda$, $r: \Omega \to \mathbb{R}$ and some $\gamma \geq 0$ there holds

(A.4) \quad \forall n \in \mathbb{N}_0 \quad \forall x \in \Omega : \quad |\nabla^n u(x)| \leq \Lambda(x)\gamma^n \max\{(n+1)/r(x), \varepsilon^{-1}\}^n.

Then the function $\tilde{u} := u \circ M$ is analytic on $\tilde{\Omega}$ and there are constants $C, \tilde{\gamma} > 0$ depending solely on $M$ and $\gamma$ such that for each $\tilde{x} \in \tilde{\Omega}$ there holds with the notation $x = M(\tilde{x})$

$$\forall n \in \mathbb{N}_0 : \quad |\nabla^n \tilde{u}(\tilde{x})| \leq C \Lambda(\tilde{x})\tilde{\gamma}^n \max\{(n+1)/r(x), \varepsilon^{-1}\}^n.$$ 

**Proof.** The statement is taken from [22, Lemma 4.3.3] except that we explicitly allow $r$ to be a function of $x$. The proof is similar to that of Lemma A.1. We fix $\tilde{x} \in \tilde{G}$ and set $x = M(\tilde{x})$. The assumption (A.4) implies that $u$ has a holomorphic extension to $B_{cr}(x) \subset \mathbb{C}^2$ with $c > 0$ depending solely on $\gamma$. Additionally, we have the bound for $z \in B_{cr}(x) \subset \mathbb{C}^2$ (we write $r = r(x)$)

$$|u(x + z)| \leq \Lambda(x) \sum_{n=0}^{\infty} \frac{1}{n!} |z|^n |\nabla^n u(x)| \leq C \left[ \frac{1}{1 - |z|/(cr)} + \exp(C'/|z|/\varepsilon) \right].$$

for suitable $C, C'$. The analyticity of $M$ on $\text{closure}(\tilde{G})$ implies the existence of $\delta > 0$ (independent of $\tilde{x} = (\tilde{x}, \tilde{y}) \in \tilde{G}$) such that

$$M(\tilde{x} + B_{\delta r}(0), \tilde{y} + B_{\delta r}(0)) \subset B_{1/2cr}(x).$$

For $\alpha \in \mathbb{N}_0^2$ let $\theta := \min\{\delta r(x), (|\alpha| + 1)\varepsilon\}$. The Cauchy integral theorem for derivatives gives

$$\partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{u}(\tilde{x}, \tilde{y}) = \frac{\alpha_1!\alpha_2!}{4\pi^2} \int_{z_1 \in \partial B_\theta(0)} \int_{z_2 \in \partial B_\theta(0)} \frac{\tilde{u}(M(\tilde{x} + (z_1, z_2)))}{(z_1 - \theta)^{\alpha_1+1}(z_2 - \theta)^{\alpha_2+1}},$$

so that we get

$$|\partial_x^{\alpha_1} \partial_y^{\alpha_2} \tilde{u}(\tilde{x}, \tilde{y})| \lesssim \alpha_1!\alpha_2! \theta^{-|\alpha|} [1 + \exp(C'\theta/\varepsilon)]$$

$$\lesssim |\alpha|! \max\{(\delta r(x))^{-1}, (|\alpha| + 1)^{-1}\varepsilon^{-1}\} |\alpha|! [1 + \exp(C'|\alpha|)],$$

which proves the asserted estimate. \□

**Appendix B. Univariate Approximation.**

**Lemma B.1.** Let $I = (-1, 1)$ and $u \in C^\infty(I)$ satisfy, for some constants $C_u$, $\gamma_u > 0$, for some $h \in (0, 1]$, $\varepsilon \in (0, 1]$ the bound

(B.1) \quad \forall n \in \mathbb{N}_0 : \quad \|D^n u\|_{L^\infty(I)} \leq C_u (\gamma_u h)^n \max\{n, \varepsilon^{-1}\}^n.

Then there are constants $C, \eta, \delta > 0$ depending solely on $\gamma_u$ such that under the constraint

(B.2) \quad \frac{h}{\varepsilon q} \leq \delta
there holds

\[ \forall q \in \mathbb{N}: \inf_{v \in \mathbb{P}_q} \|u - v\|_{W^{1,\infty}(I)} \leq CC_u \left( \frac{h}{h + \eta} \right)^{q+1} + \left( \frac{h}{\eta e q} \right)^{q+1}. \]

**Proof.** We start with the observation that Taylor’s theorem yields for \( x > 0 \)

\[ \sum_{n \geq q+1} \frac{1}{n!} x^n = e^x - \sum_{n=0}^{q} \frac{x^n}{n!} = \frac{1}{q!} \int_{0}^{x} (x-t)^q e^t \, dt \leq \frac{x^{q+1}}{q!} e^x. \]

**Case 1:** Let \( e \gamma_u h < 1/2 \). Then the Taylor series of \( u \) about \( x_0 = 0 \) converges in \( I \) and the Taylor polynomials \( T_q \in \mathbb{P}_q \) satisfy the error bounds

\[ \|u - T_q\|_{L^\infty(I)} \leq \sum_{n=q+1}^{\infty} \frac{|D^n u(0)|}{n!} \leq C_u \sum_{n=q+1}^{\infty} (\gamma_u h)^n + \frac{(\gamma_u h/\varepsilon)^n}{n!} \]

\[ \leq C_u \left( \frac{h \gamma_u h/\varepsilon}{1 - (h \gamma_u h/\varepsilon)} + \frac{(h \gamma_u h/\varepsilon)^q}{q!} e^{\gamma_u h/\varepsilon} \right) \]

\[ \leq C_u \left( 2(h \gamma_u h)^q + C(e^{\gamma_u h/\varepsilon})^{q+1} e^{\gamma_u h/\varepsilon} \right), \]

for suitable \( \gamma' > \gamma_u \). The assumption (B.2) allows us to estimate \( e^{\gamma_u h/\varepsilon} \leq e^{\gamma_u \delta q} \) and the desired result follows for the \( L^\infty \)-estimate. An analogous argument applies for the \( W^{1,\infty} \)-estimate.

**Case 2:** Let \( 1 \geq h > 1/(2e \gamma_u) \). Introduce for \( \rho > 1 \) the ellipse \( \mathcal{E}_\rho := \{ z \in \mathbb{C} \mid |z-1|+|z+1| < \rho+1/\rho \} \) and set \( G_\kappa(I) := \cup_{x \in I} B_\kappa(x) \). By geometric considerations (e.g., with the aid of [8, Lemma 3.14]) one has \( \mathcal{E}_{1+\kappa} \subset G_\kappa(I) \). Taylor’s theorem gives that \( u \) is holomorphic on \( G_{1/(\gamma_u h)}(I) \) and for every \( \kappa < 1/(\gamma_u h) \) we have

\[ \|u\|_{L^\infty(G_\kappa)} \leq C_u \sum_{n=0}^{\infty} \frac{1}{n!} (h \gamma_u \kappa)^n \max\{n, \varepsilon^{-1}\} \leq C_u \left[ \frac{1}{1 - e^{\gamma_u h/\varepsilon}} + \exp(\kappa \gamma_u h/\varepsilon) \right]. \]

Well-established polynomial approximation results (see, e.g., [1, Thm. 6]) then yield for fixed \( \kappa > 0 \) the existence of \( \rho_1 = \rho_1(\kappa) > 1 \) such that

\[ \inf_{v \in \mathbb{P}_q} \|u - v\|_{W^{1,\infty}(I)} \leq CC_u \rho_1^{-q} \|u\|_{L^\infty(G_\kappa)} \leq CC_u \rho_1^{-q} e^{\kappa \gamma_u h/\varepsilon} \leq CC_u \rho_1^{-q} e^{\kappa \gamma_u \delta q}. \]

Fix \( 1 < \rho_2 < \rho_1 \). Then we may select \( \delta > 0 \) sufficiently small so that there exists a constant \( C > 0 \) such that

\[ \forall q \in \mathbb{N} : \inf_{v \in \mathbb{P}_q} \|u - v\|_{W^{1,\infty}(I)} \leq CC_u \rho_2^{-q}. \]

Using \( h \geq 1/(2e \gamma_u) \) and suitably choosing \( \eta \), we can estimate

\[ \rho_2^{-q} \leq \left( \frac{h}{h + \eta} \right)^q. \]

**Lemma B.2** (stability of the 1d-Gauss-Lobatto (GL) interpolant). Let \( I = [-1,1] \). There exists a constant \( C > 0 \) such that for any \( q \in \mathbb{N} \), the Gauss-Lobatto interpolation operator \( i_q : C(I) \to \mathbb{P}_q \) satisfies:

\[ \|u - i_q u\|_{L^\infty(I)} \leq (1 + A_q) \inf_{v \in \mathbb{P}_q} \|u - v\|_{L^\infty(I)}, \quad A_q = C \ln(q + 1), \]

\[ \|(u - i_q u)^\prime\|_{L^\infty(I)} \leq C(1 + q^2 A_q) \inf_{v \in \mathbb{P}_q} \|(u - v)^\prime\|_{L^\infty(I)}. \]
Proof. The bound (B.6) follows from the projection property of the Gauss-Lobatto interpolation; the logarithmic growth of the Lebesgue constant $\Lambda_q$ is shown in [40].

For (B.7), we estimate for arbitrary $v \in \mathbb{P}_q$

$$
\|(u - i_qv')_Q\|_{L^\infty(I)} \leq \|(u - v')_Q\|_{L^\infty(I)} + \|(i_q(u - v))'\|_{L^\infty(I)}
\lesssim \|(u - v')_Q\|_{L^\infty(I)} + q^2\|i_q(u - v)\|_{L^\infty(I)} \lesssim \|(u - v')_Q\|_{L^\infty(I)} + q^2\Lambda_q\|u - v\|_{L^\infty(I)}.
$$

Constraining $v$ to satisfy $v(-1) = u(-1)$ the result follows from a Poincaré inequality. □

Finally, we recall two inequalities of Stirling’s type.

(B.8) $\forall n \in \mathbb{N}: \sqrt{2\pi n^{n+1/2}}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}$, (B.9) $\forall n \in \mathbb{N}_0 \forall \alpha \in \mathbb{N}_0: \alpha!n! \geq 2^{-(\alpha+n)}(\alpha+n)! \geq (2e)^{-(\alpha+n)}(\alpha+n)^{\alpha+n}$.

(B.8) follows from [27]. In B.9, the first bound follows from the binomial formula

$$
\sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu = (1 + x)^n
$$

and the second bound follows from (B.8).

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