An approach for verification of finite-element analysis in nonlinear elasticity under large strains

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Abstract. An approach to verification of finite-element calculations of stress-strain state of nonlinear elastic bodies under large deformations is suggested. The problems that may be reduced to one-dimensional ones using a semi-inverse method are taken as test problems. An example of such a test problem is the Lame problem for a cylinder. Generally, this problem for compressible hyperelastic materials has no exact analytical solution, but it can be reduced to a boundary value problem for an ordinary second-order nonlinear differential equation, and in some cases - to the Cauchy problem. A numerical solution of this problem can be used as a test one for finite element calculations carried out in three-dimensional statement. Some results of such verification (finite element calculations were performed using the Fidesys CAE-system) are presented.

1. Introduction

In finite element simulation of the phenomena taking place in deformable solids, it is necessary to verify the results of calculations. The usual way to do such verification - is a comparison of results of calculations for specially selected model problems with the results obtained with the help of other methods. A comparison with the exact analytical solutions is the most preferred one. In the theory of elasticity there are a lot of exact analytical solutions, but in the framework of the nonlinear theory of elasticity at large strains the number of exact analytical solutions is small. Typically, such solutions are known either for incompressible materials [1,2] or for materials whose mechanical properties are described by potentials of a special form, e.g. John potential (harmonic) [3]. In this connection, a semi-inverse method for solution of nonlinear elasticity problems is of interest. The method consists in the following, that a certain class of motions of a continuous medium is set up in the coordinates of a particle in initial and final states containing unknown functions. By substituting these relationships into the equations and boundary conditions of the boundary value problem we obtain new equations and boundary conditions, which are less complicated than the initial ones. For example, the initial equations are the differential equations in...
partial derivatives, but after the substituting we obtain ordinary differential equations. The researchers aim is to find solutions to these new equations.

There are many publications dedicated to semi-inverse method, for example [4-11]. This method does not always give an opportunity to get an exact analytical solution of the problem. However, even in cases, where such a decision can not be found, the semi-inverse method can simplify the task greatly.

In this paper, the semi-inverse method is applied to solving of Lame problems for hollow cylinder at large deformations. Generally, this problem for compressible hyperelastic materials has no exact analytical solution, but it can be reduced to a boundary value problem for an ordinary second-order nonlinear differential equation, and in some cases - to the Cauchy problem. A numerical solution of this problem can be used as a test one for finite element calculations carried out in three-dimensional statement. Some results of such verification (finite element calculations were performed using the Fidesys CAE-system) are presented. The calculations were performed for the case where the mechanical properties of the cylinder material are described by defining relations for compressible Mooney-Rivlin material.

2. Problem statement
The static statement of nonlinear elasticity problem for finite deformations is formulated in the coordinates of the initial (non-deformed) state and includes the following equations.

Equilibrium equation (in the absence of mass forces)

\[ \nabla \cdot P = 0 \]  
(1)

Constitutive relations

\[ P = \frac{dW}{d\Psi} \]  
(2)

Kinematic relations

\[ \Psi = \nabla R \]  
(3)

In the equations (1)-(3) \( R \) – the radius vector of a particle in a deformed state; \( \Psi \) - deformation gradient; \( W \) - elastic potential (strain potential energy density); \( P \) - the first Piola-Kirchhoff stress tensor, which is related with the true stress tensor \( \sigma \) in the following way:

\[ \sigma = J^{-1} \Psi^T \cdot P \]  
(4)

here \( J = \det \Psi \); \( J - 1 \) is the relative volume change.

The elastic potential for a compressible Mooney-Rivlin material is defined as follows [12]:

\[ W = C_1 \bar{I}_1 + C_2 \bar{I}_2 + D(J - 1)^2 \]  
(5)

Here \( C_1 \), \( C_2 \), \( D \) are material constants; \( \bar{I}_1 = J^{-2/3} I_1 \); \( \bar{I}_2 = J^{-4/3} I_2 \); \( I_1 \), \( I_2 \) - invariants of the Green deformation tensor \( G = \Psi \cdot \Psi^T \).

Boundary conditions at a given pressure \( p \) at the boundary can be written as follows

\[ n \cdot P = -p/n \cdot (\Psi^{-1})^T \]  
(6)

A model problem of deformation of a hollow circular cylinder has been solved, a predetermined pressure was applied to the inner border of the cylinder, the outer boundary is free of loads and
cylinder bases are fixed, so that displacements in the direction of the axis of the cylinder are equal to zero.

3. Approaches to problem solving

Finite-element solution was prepared using the Fidesys CAE-system [13]. In this system, a finite element method based on the Galerkin method [14-17] is implemented. A system of nonlinear algebraic equations, obtained with the help of the finite element approximation, is solved using the Newton-Kantorovich method.

In solving the problem using semi-inverse method, a cylindrical coordinate system was used. Let's denote coordinates of the particle in the undeformed state in this coordinate system with \( r, \varphi, z \), and coordinates of the particle in the final state in the system - with \( R, Z, \Phi \). Considering the symmetry of the problem and fixation conditions, the solution can be found in the following form

\[
R = R(r), \quad Z = z, \quad \Phi = \varphi
\]  

Here \( R(r) \) - unknown function.

Substitution of expression (7) in the kinematic relations (3) gives

\[
\Psi_{rr} = R'(r), \quad \Psi_{\varphi\varphi} = R(r)/r, \quad \Psi_{zz} = 1
\]  

The off-diagonal components of the deformation gradient are equal to zero.

The expressions for the components of the first Piola stress tensor \( P \) are determined from (2) as follows:

\[
P_{rr} = \frac{\partial W}{\partial \Psi_{rr}}, \quad P_{\varphi\varphi} = \frac{\partial W}{\partial \Psi_{\varphi\varphi}}, \quad P_{zz} = \frac{\partial W}{\partial \Psi_{zz}}
\]  

The off-diagonal components of this tensor are equal to zero due to the isotropy of material. Considering (8), one can see that all the components of the tensor \( P \) in this problem will depend on the \( r \) only. Therefore, the vector equilibrium equation (1) reduces to a single scalar equation:

\[
\frac{dP_{rr}}{dr} + \frac{1}{r}(P_{rr} - P_{\varphi\varphi}) = 0
\]  

The other two equilibrium equations are completed identically.

Equation (10) in view of (9) and (8) is converted to an ordinary differential equation of the second order with respect to \( R(r) \). The boundary conditions for this equation are written on the basis of the boundary condition (6). By denoting \( a \) and \( b \) the radii of the outer and inner cylinder, respectively, these boundary conditions can be written in the form

\[
P_{rr}(a) = 0, \quad P_{rr}(b) = -P_{\varphi\varphi}(b)
\]  

Along with the boundary value problem for a differential equation obtained from (10) with the boundary conditions (11), we can consider the Cauchy problem for the same equation with the given values of \( R(a) \) and \( R'(a) \). From a mechanical point of view, this means that instead of setting up the pressure on the inner boundary, radius variation of the outer cylinder under the influence of this pressure should be given. The value of \( R'(a) \) is determined from the no-load conditions at the external
boundary. For this, the equation $P_r(a) = 0$ is solved with respect to $\Psi_{rr}(a) = R(a)/a$. The obtained value $\Psi_{rr}(a)$ according to (8) is equal to $R'(a)$.

Let's note that to solve the Cauchy problem than the boundary value problem. From the solution of the Cauchy problem one can define all the characteristics of the stress-strain state at $b \leq r \leq a$ and, in particular, to find the pressure $p$ at the inner boundary ($r = b$). By solving this problem for several values $R(a)$, one can draw a table of dependence of the pressure $p$ at the inner boundary against the outer cylinder radius after deformation. According to this table (for example, using interpolation) one can approximately find characteristics of the stress-strain state for a given pressure $p$. An accuracy depends on the pitch of the table: the smaller the step is, the more accurate solution will be obtained.

4. Calculation results

Finite element solution was obtained for a cylinder with a relation of the inner radius to the outer one of $b/a = 0.6$ with a height of $h = 2a$. Material parameters specification: $C_2 = 0$, the $D$ value is varied from $2C_1$ to $64C_1$. The calculations were carried out for three values of the internal pressure: $p/C_1 = 0.25$, 0.5 and 0.7. An unstructured mesh was used, the number of nodes - 8736, the number of elements - 1092.

Some results of the problem solution with the help of finite element method are shown in Table 1, and the results of solution of this problem using the semi-inverse method are shown in Table 2. The tables show the value of true hoop stress $\sigma_{\varphi\varphi}$ on the inner boundary of the cylinder for different values of the internal pressure $p$ and the material constant $D$. The stress is referred to the material constant $C_1$.

| $D/C_1$ | $p/C_1$ | 0.25 | 0.5 | 0.7 |
|---------|---------|------|-----|-----|
| 2       |         | 0.56 | 1.48| 3.29|
| 4       |         | 0.56 | 1.45| 2.93|
| 8       |         | 0.56 | 1.43| 2.79|
| 16      |         | 0.55 | 1.41| 2.71|
| 32      |         | 0.55 | 1.38| 2.62|
| 64      |         | 0.53 | 1.33| 2.49|

| $D/C_1$ | $p/C_1$ | 0.25 | 0.5 | 0.7 |
|---------|---------|------|-----|-----|
| 2       |         | 0.63 | 1.65| 3.67|
| 4       |         | 0.63 | 1.63| 3.30|
| 8       |         | 0.64 | 1.63| 3.18|
By comparing the results shown in Tables 1 and 2, it can be seen that the difference between the finite element solution and the solution obtained via semi-inverse method amounts to 10 - 20% of these solutions. Evidently, this difference is explained by the use of a large enough finite element mesh.

5. Conclusion

Thus, the article offered the option of applying of semi-inverse method to the solution of the Lame problem for large deformations. Numerical results are obtained for a compressible Mooney-Rivlin material. The results can be used for verification of finite element calculations, in particular the calculations carried out using the Fidesys CAE-system.

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