A Representation for the Anyon Integral Function

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Abstract

The Fermi-Dirac and Bose-Einstein particles satisfy corresponding statistical distributions. In the phenomena of charge fractionalization and the fractional quantum Hall effect it is found that particles behave as if they are neither fermions nor bosons. Such particles are called anyons. The integral functions for bosons and fermions are available in the literature. However, there is no anyon integral function available. In this note we propose a pair of functions that interpolate, in some sense, between the gamma function and the zeta function, which we call “gamma-zeta” functions. It is pointed out that this pair of functions very naturally provides a representation of the anyon integral function.

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1 Introduction

There are two aspects to this note: (1) the presentation of a new special function; and (2) its physical application. It is necessary to briefly review the background for both aspects, so that readers familiar with one area can follow the motivation and reasoning for the other. In this section, we will first recall the special function aspect and then go on to explain the physical aspect.

Most of the so-called “special functions” arise as solutions of commonly occurring first or second order linear differential equations. However, two of the most significant special functions do not come from differential equations, namely the gamma function and the family of zeta functions. In fact both arose from the study of numbers. (For a discussion of the relevant special functions, see for example [1].) Considering their relevance and utility, and the fact that they are related in the above sense, it seemed worthwhile to try to put them together so that they appear as limiting cases of either one of a pair of functions, which we call the gamma-zeta function [2]. It turns out that in doing so we find that we can “interpolate” between the Fermi-Dirac and Bose-Einstein integral functions in a way that follows the physical needs for doing so, as will be explained.

The gamma function was defined by Euler to extend the domain of the factorial function from the integers and then the complex numbers. It has the integral representation

\[ \Gamma(\alpha) := \int_0^\infty e^{-t}t^{\alpha-1}dt, \]  

which is singular for negative integer values of \( \alpha \), but is well defined everywhere else. The zeta function, defined over the reals, was first studied by Euler who also gave a product formula for it known as the Euler product,

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}. \]  

Riemann while studying the distribution of prime numbers rediscovered the zeta function and studied it as a function defined over complex numbers. He also established the amazing link between the complex zeroes of zeta function and the distribution of prime numbers. The Riemann zeta function, as it is now known, has the integral representation

\[ \zeta(\alpha) := \frac{1}{C(\alpha)} \int_0^\infty \frac{t^{\alpha-1}dt}{e^t + 1} \quad (\alpha = \sigma + i\tau, \sigma > 0), \]  

where

\[ C(\alpha) := \Gamma(\alpha)(1 - 2^{1-\alpha}). \]  

The function has the analytic continuation

\[ \zeta(\alpha) := \frac{1}{\Gamma(\alpha)(e^{2\pi i\alpha} - 1)}I(\alpha) \]  

where

\[ I(\alpha) := \frac{e^{-i\pi\alpha} \Gamma(1 - \alpha)}{2\pi i}I(\alpha), \]
where

\[ I(\alpha) := \int_C \frac{z^{\alpha-1}dz}{e^z - 1} \]  \hspace{1cm} (6)

is the loop integral. The contour \( C \) consists of the real axis from \( \infty \) to \( \rho \) (\( 0 < \rho < 2\pi \)), the circle \( |z| = \rho \), and the axis from \( \rho \) to \( \infty \). The integral \( I(\alpha) \) is uniformly convergent in any finite region of the complex plane and so defines an integral function of \( \alpha \). The representations (3) and (5) provide an analytic continuation of \( \zeta(\alpha) \) over the whole complex plane. Since \( I(1) = 2\pi i \) and \( I(n) = 0 \) \( (n = 2, 3, \ldots) \), the poles of the gamma function \( \Gamma(1 - \alpha) \) at \( \alpha = 2, 3, 4, \ldots \) cancel the zeros of \( I(\alpha) \). It follows from (5) that \( \zeta(\alpha) \) has only a simple pole, due to \( \Gamma(1 - \alpha) \), at \( \alpha = 1 \) with residue 1.

2 Bose-Einstein and Fermi-Dirac Particles

In quantum mechanics particles (for example see [4]) are represented by wave functions that satisfy the Schrodinger equation. Making the theory relativistic requires that one shift over to quantum field theory (for example see [5]). Following the Dirac quantization procedure leads to the Klein-Gordon equation, which has problems of interpretation due to the fact that it has a second derivative relative to the time appearing in it. To avoid these problems Dirac “took the square root of the Klein-Gordon equation” to obtain a first order equation in space and time variables. It turned out that the Klein-Gordon equation represents spin-less particles while the Dirac equation applies to particles with spin. The theory further requires that the spin go up in half-integer multiples of the quantity, \( \hbar \), defined to be \( h/2\pi \), where \( h \) is Planck’s constant. The wave function for identical particles with integer spin is symmetric while for half-integer spin it is anti-symmetric under the exchange of particles. On account of this property, particles with half-integer spin can not have the same quantum numbers, while those with integer spin can. The former are said to be subject to the Pauli exclusion principle.

Statistical mechanics (for example see [6]) had been developed to deal with ensembles of classical particles using the Maxwell distribution:

\[ f(\varepsilon) = \frac{e^{(\mu - \varepsilon)/\tau}}{e^{(\mu - \varepsilon)/\tau} - 1}, \]  \hspace{1cm} (1)

where \( \tau \) stands for thermal energy and is hence related to the temperature, \( \varepsilon \) stands for the kinetic energy of the given particle and \( \mu \) for the chemical potential. The \( f \) gives the probability of the particle having the given kinetic energy. It turned out that this distribution did not apply precisely to the quantum particles but could be used as an approximation for a mixture of the two types. Also, it was a good approximation for the ensemble at high temperatures. For lower temperatures of systems of particles of one type other distributions were required. For the integer spin particles it was propounded by Bose and is called the Bose-Einstein distribution:

\[ f_B(\varepsilon) = \frac{1}{e^{(\mu - \varepsilon)/\tau} - 1}; \]  \hspace{1cm} (2)

while for the particles with half-integer spin it was propounded by Fermi and is called the Fermi-Dirac distribution and is given by:

\[ f_F(\varepsilon) = \frac{1}{e^{(\mu - \varepsilon)/\tau} + 1}. \]  \hspace{1cm} (3)
Particles of the former type are called *bosons* and of the latter type *fermions*.

The cumulative probabilities for bosons and fermions are given by the Bose-Einstein and the Fermi-Dirac integral functions:

\[
B_q(x) := \frac{1}{\Gamma(q + 1)} \int_0^\infty \frac{t^q}{e^{t-x} - 1} dt \quad (q > 0); \tag{4}
\]

\[
F_q(x) := \frac{1}{\Gamma(q + 1)} \int_0^\infty \frac{t^q}{e^{t-x} + 1} dt \quad (q > -1). \tag{5}
\]

### 3 Anyons

It appeared that all elementary particles would belong to one of these classes. However, more recently, in the phenomenon of the fractional quantum Hall effect [7] it was found that under certain conditions electron, a fermion, can behave as if it is made of yet more fundamental particles with spin a fractional multiple of \(\frac{1}{2}\hbar\). Such particles were dubbed "anyons" [8]. (For a review see [9]). Particles such as anyons with spin/statistics interpolating between bosons and fermions can only exits in two dimensions. There are both topological and group theoretic reasons for their non-existence in higher dimensions. As mentioned in the last section, the wave function of bosons is symmetric under the exchange of two particles whereas the wave function of the fermions is antisymmetric. This has to do with the fact that in three and higher dimensions the symmetry group is the permutation group and it has only two one dimensional representations. The trivial representation corresponds to the bosons and the non-trivial representation gives fermions. In two dimensions, however, the situation is much more interesting. The symmetry group is larger than the permutation group, it is the braid group\(^4\). The braid group has a one dimensional representation for every real number \(\nu\). Thus a two particle wave function under exchange of two particles behaves in the following way:

\[
\psi(b, a) = \begin{cases} 
(+1) \psi(a, b), & \text{Bosons,} \\
(-1) \psi(a, b), & \text{Fermions,} \\
(-1)^\nu \psi(a, b), & \text{Anyons.}
\end{cases} \tag{1}
\]

There is no functional representation for anyons available in the literature corresponding to that for bosons or fermions. The gamma-zeta functions connect across the two in that for real \(\nu\) they give an interpolation between them. As such, they should be considered as candidates for representing the anyon integral function. Note that \(\Phi\) and \(\Psi\) are sufficiently directly related that either (or a linear combination of the two) could be used. The choice would be guided by examining how convenient the function is in the resulting formulae.

It was found [10] that there are phenomena in which it appeared that there were fractional charges appearing. This was regarded as exciting because it seemed that these may be the "quarks" that had been proposed in high energy physics [11]. However, the theory did not

\(^4\)Braid group, \(B_N\), is generated by transpositions \(\{\phi_1, \ldots, \phi_N\}\) such that \(\phi_i\phi_j = \phi_j\phi_i\) for \(|i - j| > 1\) and \(\phi_i\phi_{i+1}\phi_i = \phi_{i+1}\phi_i\phi_{i+1}\). If we further impose the relation \(\phi_i^2 = 1\) then we get the permutation group. It is the absence of this last relation in the braid group which allows the possibility of anyons.
allow free quarks [12] and it was found that the fractions did not necessarily correspond to
those required for quarks. This phenomenon was later explained as arising from the statistics
not being well defined, namely that the particles behave like neither fermions nor bosons but
something in between. The explanation proposed can be understood in terms of an example.
Consider fermions of spin 1/2 (in units of the Planck action $\hbar$) in an infinite linear array.
The axial component of the spin can then either be +1/2 or -1/2. The energy is minimized
by having them alternate in the axial component of the spin. Had they been bosons there
would be invariance under shift by a lattice length. For fermions we would have to reverse
the sign on the shift. This can be achieved by shifting and then “rotating the particle”
through an angle $\pi$, i.e. changing the phase of the particle wave function. This way we can
have the fermion behave like a boson in some sense. This is relevant for our purposes. More
generally the transformation may be through some other fraction of $2\pi$.

We should be able to follow the same procedure for the integral functions, of shifting the
phase by $\pi$, or more generally, by some other fraction of $2\pi$. Our proposed new “gamma-
zeta” function has the Bose-Einstein and Fermi-Dirac integral functions as limits. As such,
itis able to achieve just this. As will be seen, the procedure adopted corresponds to
the physical argument just mentioned.

4 The Gamma-Zeta Functions

Our pair of new functions, which we call gamma-zeta functions, is obtained by using the
Mellin and Weyl transforms. The Mellin transform of a function $\varphi(t)$, if it exists, is defined by [13]

$$
\Phi_M(\alpha) := M[\varphi; \alpha] = \int_0^\infty t^{\alpha-1} \varphi(t) dt \quad (\alpha = \sigma + i\theta).
$$

(1)

It is inverted by

$$
\varphi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi_M(z) t^{-z} dz.
$$

(2)

A convolution-like product of two functions $\varphi$ and $\psi$, $\varphi \circ \psi$, can be defined by [14]

$$
(\varphi \circ \psi)(t) := \int_0^\infty \varphi(xt) \psi(x) dx.
$$

(3)

Note that the operation “$\circ$” is not commutative and

$$
M[\varphi \circ \psi; \alpha] = \Phi_M(\alpha) \Psi_M(1 - \alpha),
$$

(4)

where

$$
\Phi_M(\alpha) := M[\varphi; \alpha]
$$

(5)

and

$$
\Psi_M(\alpha) := M[\psi; \alpha].
$$

(6)

It is to be noted that if $\varphi \in L^1_{\text{loc}}[0, \infty)$ is such that

$$
\varphi(t) = \begin{cases} 
O(t^{-\alpha_1}) & (t \to 0^+) \\
O(t^{-\alpha_2}) & (t \to \infty)
\end{cases}
$$

(7)
then $\Phi_M(\alpha)$ exists for $\sigma_1 < \sigma < \sigma_2$. In particular, if $\varphi$ is continuous on $[0, \infty)$ and has \textit{rapid decay} at infinity, the Mellin transform (4.1) will converge absolutely for $\sigma > 0$.

Let $X_M(\sigma_1, \sigma_2)$ be the space of all $\varphi \in L^1_{loc}[0, \infty)$ such that the corresponding integral (4.1) converges uniformly and absolutely in the strip $\sigma_1 < \sigma < \sigma_2$. Then, for each $\varphi \in X_M(\sigma_1, \sigma_2)$, $\Phi_M(\alpha)$ is analytic in the interior of the strip $\sigma_1 \leq \sigma \leq \sigma_2$, [13].

Similarly the Weyl transform of $\varphi \in X_M[0, \infty)$ is given by [14]

$$\Phi_W(\alpha; x) := W^{-\alpha}[\varphi](x) = \frac{1}{\Gamma(\alpha)} M[\varphi(x + t); \alpha] \quad (\alpha = \sigma + i\tau, \sigma > 0),$$

where

$$\Phi_W(0, x) := \varphi(x),$$

and

$$\Phi_W(-\alpha; x) := D^m(\Phi_W(\alpha; x)) \quad (\alpha = \sigma + i\tau, \sigma > 0),$$

where $m$ is the smallest integer greater than $\sigma = \text{Re}(\alpha)$ and

$$D^n := (-1)^n \frac{d^n}{dx^n} \quad (n = 0, 1, 2, \ldots).$$

In particular we have

$$\Phi_W(-n; x) = (-1)^n \varphi^{(n)}(x) \quad (n = 0, 1, 2, \ldots).$$

For our purposes we need to consider some sufficiently well-behaved functions. The class $\Sigma(\sigma_0)$ of “good” functions is defined as follows: A function $\varphi \in \Sigma(\sigma_0)$ if

(P.1) $\varphi \in X_M(0, \sigma_0)$,

(P.2) $\Phi_W(\alpha, x)$ is analytic in $x$ in the region $\text{Re}(x) \geq 0$ for $0 \leq \sigma < \sigma_0$.

It is to be noted that the class $\Sigma(\sigma_0)$ is nonempty. To see this consider any polynomial $P(t)$. If $a > 0$, the function $P(t)e^{-at}$ is a member of the class $\Sigma(\sigma_0) \forall \sigma > 0$. Also, as $\Phi_W(0, x)$ is analytic in the region $\text{Re}(x) \geq 0$, each of $\varphi \in \Sigma(\sigma_0)$ is a $C^{\infty}[0, \infty)$ function as well.

Let us now define

$$\varphi_\nu(t) := \frac{e^{-\nu t}}{e^t - 1}$$

and

$$\psi_\nu(t) := \frac{e^{-\nu t}}{e^t + 1}, \quad (\nu \geq 0, t > 0),$$

and consider their Weyl transforms:

$$\Phi_\nu(\alpha; x) := W^{-\alpha}[\varphi_\nu](x)$$

and

$$\Psi_\nu(\alpha; x) := W^{-\alpha}[\psi_\nu](x).$$
We call the functions $\Phi_\nu(\alpha; x)$ and $\Psi_\nu(\alpha; x)$ the *gamma-zeta functions*. Note that

$$
\Phi_\nu(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \varphi(t + x) dt
$$

$$
= e^{-\nu x} \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\nu t} dt
$$

$$
= e^{-(\nu+1)x} \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-\nu t}}{e^t - e^{-x}} dt.
$$

Similarly, we have

$$
\Psi_\nu(\alpha; x) = e^{-(\nu+1)x} \Phi(e^{-x}, \alpha, \nu + 1)
$$

From (17) and (18) we find that the gamma-zeta function $\Phi_\nu(\alpha; x)$ is related to the general Hurwitz-Lerch zeta function \[15\] $\Phi(z, s, \nu)$ via

$$
\Phi_\nu(\alpha; x) = e^{-(\nu+1)x} \Phi(e^{x}, \alpha, \nu + 1)
$$

which leads to the series representation

$$
\Phi_\nu(\alpha; x) = e^{-(\nu+1)x} \sum_{n=0}^\infty \frac{e^{-nx}}{(n + \nu + 1)^\alpha}
$$

$$
(\nu \neq -1, -2, -3, \ldots; \ Re(x) > 0; x = 0, \sigma > 1). \tag{20}
$$

Similarly, we have the relation

$$
\Psi_\nu(\alpha; x) = e^{-(\nu+1)x} \Phi(-e^{-x}, \alpha, \nu + 1)
$$

that leads to the series representation

$$
\Psi_\nu(\alpha; x) = e^{-(\nu+1)x} \sum_{n=0}^\infty \frac{(-1)^n e^{-nx}}{(n + \nu + 1)^\alpha}
$$

$$
(\nu \neq -1, -2, -3, \ldots; \ Re(x) > 0; x = 0, \sigma > 0). \tag{22}
$$

**Theorem 1** The functions $\Phi_\nu(\alpha; x)$ and $\Psi_\nu(\alpha; x)$ are related via

$$
\Psi_\nu(\alpha; x) = (-1)^{(\nu+1)} \Phi_\nu(\alpha; x + i\pi).
$$

**Proof.** Replace $x$ by $x + i\pi$ in (20) and compare with (22) to obtain the result. \[\square\]

Another useful relation, also proved elsewhere [2] is:

**Theorem 2** The functions $\Phi_\nu(\alpha; x)$ and $\Psi_\nu(\alpha; x)$ are related via

$$
\Psi_{2\nu}(\alpha; x) = \Phi_{2\nu}(\alpha; x) - 2^{1-\alpha} \Phi_\nu(\alpha; 2x). \tag{24}
$$
Proof:

\[
e^{(2\nu+1)x}(\Psi_{2\nu}(\alpha : x) - \Phi_{2\nu}(\alpha; x)) = \sum_{n \geq 0} e^{-nx} - (-1)^n e^{-nx} (n + 1 + 2\nu)^\alpha
\]
\[
= \sum_{k \geq 0} 2e^{-(2k+1)x} \frac{e^{-2kx}}{(2k + 2 + 2\nu)^\alpha}
\]
\[
= 2^{1-\alpha} e^{-x} \sum_{k \geq 0} \frac{e^{-2kx}}{(n + 1 + \nu)^\alpha}
\]
\[
= 2^{1-\alpha} e^{-x} \Phi_\nu(\alpha; 2x)e^{(\nu+1)2x}
\]

Note that our gamma-zeta functions are dual to each other in the sense that the above relation can be easily inverted, so that each is similarly related to the other. This fact is of special relevance for us.

5 The Anyon Integral Functions

From the integral representations (17) of \(\Phi_\nu(\alpha; x)\) and (2.4) of the Bose-Einstein integral function \(B_\alpha(x)\), we find that

\[
B_{\alpha-1}(-x) = \Phi_0(\alpha; x),
\]
leading to the fact that the gamma-zeta function \(\Phi_\nu(\alpha; x)\) is a natural extension of the Bose-Einstein integral function. Similarly, from (18) and (2.5), we find that

\[
F_{\alpha-1}(-x) = \Psi_0(\alpha; x),
\]
which shows that the second gamma-zeta function \(\Psi_\nu(\alpha; x)\) naturally extends the Fermi-Dirac integral function.

Putting \(\nu = 0\) and replacing \(x\) by \(-x\) and \(\alpha\) by \(\alpha + 1\) in (24) we find that the Fermi-Dirac and Bose-Einstein integrals are related via

\[
F_\alpha(x) = B_\alpha(x) - 2^{1-\alpha} B_\alpha(2x).
\]
The relation (5.3) does not seem to have been realized earlier.

Using the fractional Weyl transform \(W^{-\alpha}\) on \(\Phi_\nu(\beta; t)\) we obtain

\[
\Phi_\nu(\alpha + \beta; x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \Phi_\nu(\beta; t + x).
\]

Putting \(\nu = 0\) above we find the interesting integral representation

\[
B_{\alpha+\beta-1}(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} B_{\beta-1}(x - t)dt
\]
for the Bose-Einstein integral. Similarly, we obtain

\[
F_{\alpha+\beta-1}(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} F_{\beta-1}(x - t)dt.
\]
Putting $\alpha = 1$ in (5.5) and (5.6) we obtain

$$B_\beta(x) = \int_0^\infty B_{\beta-1}(x-t)dt,$$

(7)

and

$$F_\beta(x) = \int_0^\infty F_{\beta-1}(x-t)dt.$$  

(8)

These representations for the Bose-Einstein and Fermi-Dirac integral functions may prove useful.

We can interpolate between the Bose-Einstein and Fermi-Dirac integral functions using gamma-zeta function. To see this consider $G_\nu(\alpha; x)$,

$$G_\nu(\alpha; x) = a(\nu)\Phi_\nu(\alpha; x) + b(\nu)\Psi_{\nu-1}(\alpha; x).$$  

(9)

If we appropriately chose $a(\nu), b(\nu)$ satisfying the conditions $a(0) = 1 - b(0) = 1$ and $a(1) = b(1) - 1 = 0$ then $G_\nu(\alpha; x)$ interpolates between the Bose-Einstein and Fermi-Dirac integral functions.

6 Concluding Remarks

The family of zeta functions including the Riemann, Hurwitz, Epstein, Lerch, Selberg and their generalizations, constantly find new applications in different areas of mathematics (number theory, analysis, numerical methods, etc.) and physics (quantum field theory, string theory, cosmology, etc.) A useful generalization of the family is expected to have wide applications in all these areas as well. On the mathematical side, the gamma-zeta functions $\Phi_\nu(\alpha; x)$ and $\Psi_{\nu}(\alpha; x)$ discussed in this paper provide a unified approach to the study of the zeta family that is remarkably simple. This is discussed in more detail in a separate paper [2].

For our present purposes of the physical applications, the first pair of gamma-zeta functions also provide a representation of an anyon integral function. There was no such representation available in the literature. They also led to a new relation between the Bose-Einstein and Fermi-Dirac integral functions that reflect the nature of the two functions. The “mixing” of fermions and bosons displayed in this relationship demonstrates how the fractional spin behaviour arises.

One might have hoped that the gamma-zeta functions could be applied to supersymmetry [16], as they “unify” bosons and fermions in some sense. Despite the “duality” manifested by them, they cannot be so used. The reason is that supersymmetry does not interpolate between the two types of particles. They remain quantized with spin whole or half integer with nothing in between. The utility of the gamma-zeta functions is that they do provide such an interpolation. This lack of intermediate particles is in sharp contradistinction to the situation for grand unified theories [17], in which quarks and leptons are unified. In this case there are intermediate particles, called lepto-quarks that are neither quarks nor leptons but both. Such particles are required for spontaneous symmetry breaking of the unified symmetry [18]. This highlights the problem with supersymmetry, that it does not have a spontaneous symmetry breaking mechanism and has to use “soft symmetry breaking”.
Again, one sees a strong relationship between the physical and mathematical requirements in the context of our gamma-zeta functions.

It is expected that the operational properties of these functions can and will be exploited further in solving difficult problems in mathematics and physics.

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