Deformation Quantization of Nambu Mechanics

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Abstract. Phase Space is the framework best suited for quantizing superintegrable systems—systems with more conserved quantities than degrees of freedom. In this quantization method, the symmetry algebras of the hamiltonian invariants are preserved most naturally, as illustrated on nonlinear $\sigma$-models, specifically for Chiral Models and de Sitter $N$-spheres. Classically, the dynamics of superintegrable models such as these is automatically also described by Nambu Brackets involving the extra symmetry invariants of them. The phase-space quantization worked out then leads to the quantization of the corresponding Nambu Brackets, validating Nambu’s original proposal, despite excessive fears of inconsistency which have arisen over the years. This is a pedagogical talk based on [1, 2], stressing points of interpretation and care needed in appreciating the consistency of Quantum Nambu Brackets in phase space. For a parallel discussion in Hilbert space, see T Curtright’s contribution in these Proceedings, [hep-th/0303088].

INTRODUCTION

Highly symmetric quantum systems are often integrable, and, in special cases, superintegrable and exactly solvable [3]. A superintegrable system of $N$ degrees of freedom has more than $N$ independent invariants, and a maximally superintegrable one has $2N - 1$ invariants. The classical evolution of all functions in phase space for such systems is alternatively specified through Nambu Brackets (NB) [4, 5, 6, 7]. However, quantization of NBs has been considered problematic ever since their inception. We find that it need not be.

In the case of velocity-dependent potentials, when quantization of a classical system presents operator ordering ambiguities involving $x$ and $p$, the general consensus has long been [8, 9, 10, 11] to select those orderings in the quantum hamiltonian which maximally preserve the symmetries present in the corresponding classical hamiltonian. Even for simple systems, such as $\sigma$-models considered here, such constructions may become involved and needlessly technical.

There is a quantization procedure ideally suited to this problem of selecting the quantum hamiltonian which maximally preserves integrability. In contrast to conventional operator quantization, this problem is addressed most cogently in Moyal’s phase-space quantization formulation [12, 13], reviewed in [14]. The reason is that the variables involved in it (“phase-space kernels” or “Weyl-Wigner inverse transforms of operators”) are c-number functions, like those of the classical phase-space theory, and have the same interpretation, although they involve $\hbar$-corrections (“deformations”), in general—so $\hbar \to 0$ reduces to the classical expression. It is only the detailed algebraic struc-
ture of their respective brackets and composition rules which contrast with those for
the variables of the classical theory. This complete formulation is based on the Wigner
Function (WF), which is a quasi-probability distribution function in phase-space, and
comprises the kernel function of the density matrix. Observables and transition am-
plitudes are phase-space integrals of kernel functions weighted by the WF, in analogy
to statistical mechanics. Kernel functions, however, unlike ordinary classical functions,
multiply through the $\star$-product, a noncommutative, associative, pseudodifferential oper-
ation, which encodes the entire quantum mechanical action and whose antisymmetriza-
tion (commutator) is the Moyal Bracket (MB) \cite{12, 13, 14}, the quantum analog of the
Poisson Bracket (PB).

Groenewold’s correspondence principle theorem \cite{15} (to which van Hove’s extension
is routinely attached \cite{16}) evinces that there is no invertible linear map from
all functions of phase space $f(x, p), g(x, p), \ldots$, to hermitean operators in Hilbert space $\Omega(f),
\Omega(g), \ldots$, such that the PB structure is preserved consistently,

$$\Omega(\{f, g\}) = \frac{1}{i\hbar} \left[ \Omega(f), \Omega(g) \right].$$
\hspace{1cm} (1)

This cannot be achieved, in general.

Instead, the Weyl correspondence map \cite{17, 14} from functions to ordered operators,

$$\mathcal{M}(f) \equiv \frac{1}{(2\pi)^2} \int d\tau d\sigma dxdp \ f(x, p) \exp(i\tau(\rho - p) + i\sigma(\tau - x)), $$

\hspace{1cm} (2)
specifies an associative (and non-commutative) $\star$-product \cite{15, 14},

$$\star \equiv \exp \left( \frac{i\hbar}{2} (\leftarrow \partial_x \rightarrow \partial_p - \leftarrow \partial_p \rightarrow \partial_x) \right),$$

\hspace{1cm} (3)

so that $\mathcal{M}(f \star g) = \mathcal{M}(f) \mathcal{M}(g)$, and thus

$$\mathcal{M}(\{\{f, g\}\}) = \frac{1}{i\hbar} \left[ \mathcal{M}(f), \mathcal{M}(g) \right].$$

\hspace{1cm} (4)

Here, the Moyal Bracket is defined as

$$\{\{f, g\}\} \equiv \frac{f \star g - g \star f}{i\hbar},$$

\hspace{1cm} (5)

and as $\hbar \to 0$, MB $\to$ PB. That is, it is the MB instead of the PB which maps invertibly to
the quantum commutator, completing Dirac’s original proposal. This relation underlies
the foundation of phase-space quantization \cite{15, 14}.

Conversely, given an arbitrary operator $\mathcal{F}(x, p)$ consisting of operators $x$ and $p$, one
might imagine rearranging it by use of Heisenberg commutations to a canonical com-
pletely symmetrized (Weyl-ordered) form, in general with $O(\hbar)$ terms generated in the
process. It might then be mapped uniquely to its Weyl-correspondent c-number kernel
function $f$ in phase space $x \mapsto x$, and $p \mapsto p$, $\mathcal{M}^{-1}(\mathcal{F}) = f(x, p, \hbar)$. (In practice, there is
a more direct inverse transform formula \cite{15, 14} which bypasses a need to rearrange to a
canonical Weyl ordered form explicitly.) Clearly, operators differing from each other by different orderings of their $rs$ and $ps$ correspond to kernel functions $f$ coinciding with each other at $O(\hbar^0)$, but different in $O(\hbar)$, in general. Thus, a survey of all alternate operator orderings in a problem with such ambiguities amounts, in deformation quantization, to a survey of the “quantum correction” $O(\hbar)$ pieces of the respective kernel functions, i.e., the inverse Weyl transforms of those operators, and their study is greatly systematized and expedited. Choice-of-ordering problems then reduce to purely $\star$-product algebraic ones, as the resulting preferred orderings are specified through particular deformations in the $c$-number kernel expressions resulting from the particular solution in phase space.

Hietarinta [18] has investigated in this phase-space quantization language the simplest integrable systems of velocity-dependent potentials. In each system, he has promoted the vanishing of the Poisson Bracket (PB) of the (one) classical invariant $I$ (conserved integral) with the Hamiltonian, $\{H,I\} = 0$, to the vanishing of its (quantum) Moyal Bracket (MB) with the Hamiltonian, $\{\{H_{qm},I_{qm}\}\} = 0$. This dictates quantum corrections, addressed perturbatively in $\hbar$, specifically $O(\hbar^2)$ corrections to the $Is$ and $H$ ($V$), needed for quantum symmetry. The expressions found are quite simple, as the systems chosen are such that the polynomial characteristic of the $ps$, or suitable balanced combinations of $ps$ and $qs$, ensure collapse or subleading termination of the MBs. The specification of the symmetric Hamiltonian then is complete, since, as indicated, the quantum Hamiltonian in terms of classical phase-space variables corresponds uniquely to the Weyl-ordered expression for these variables in operator language.

Here, nonlinear $\sigma$-models (with explicit illustrations on $N$-spheres and chiral models) are utilized to argue for the general principles of power and convenience in isometry-preserving quantization in phase space, for large numbers of invariants, in principle (as many as the isometries of the relevant manifold). In the cases illustrated, the number of algebraically independent invariants matches or exceeds the dimension of the manifold, leading to superintegrability [3], whose impact is best surveyed through Classical Nambu Brackets (CNBs). The procedure of determining the proper symmetric quantum Hamiltonian then yields remarkably compact and elegant expressions.

Briefly, we find that the symmetry generator invariants are undeformed by quantization, but the Casimir invariants of their MB algebras are deformed, in accord with the Groenewold-van Hove theorem. Hence, the Hamiltonians are also deformed by terms $O(\hbar^2)$, as they consist of quadratic Casimir invariants. Their spectra are then read off through group theory, suitably adapted to phase space [1]. The basic principles are illustrated for the simplest curved manifold, the two-sphere, and are then generalized to larger classes of symmetric manifolds such as Chiral Models and $N$-spheres.

Given the elegant quantization of maximally superintegrable systems in phase space, as worked out here, and further given their classical dynamics specified by CNBs, one might wonder why quantization of NBs had been deemed to be beset with inconsistencies. Direct comparison of Nambu’s early quantization prescription [4] to the conventional quantum answers now at hand actually reveals that there are no insurmountable inconsistencies. Comparison to the standard Moyal deformation quantization vindicates Nambu’s quantization prescription which yields identical results, and thus also identical to standard Hilbert space quantization, for all systems discussed here. We find that, in exceptional cases, the algebra of invariants ensures that the Quantum Nambu Brackets
(QNBs) are derivations. However, in the more generic case, we find that, even though QNBs are not derivations, they still involve derivations entwined in symmetrized (Jordan) products with invariants, and are fully consistent.

**SPHERES AND CHIRAL MODELS**

Consider a particle on a curved manifold, in integrable one-dimensional $\sigma$-models,

$$L(q, \dot{q}) = \frac{1}{2} g_{ab}(q) \dot{q}^a \dot{q}^b, \quad (6)$$

so that

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = g_{ab} \dot{q}^b, \quad \dot{q}^a = g^{ab} p_b. \quad (7)$$

Thus,

$$H(p, q) = \frac{1}{2} g^{ab} p_a p_b = (= L). \quad (8)$$

The isometries of the manifold generate the conserved integrals of the motion [19]. The classical equations of motion are

$$\dot{p}_a = -\frac{g_{bc}}{2} p_b p_c = \frac{g_{bc,a}}{2} \dot{q}^b \dot{q}^c. \quad (9)$$

As the simplest possible nontrivial illustration, consider a particle on a 2-sphere of unit radius, $S^2$. In Cartesian coordinates (after the elimination of $z$, so $q^1 = x, q^2 = y$), one has, for $a, b = 1, 2$:

$$g_{ab} = \delta_{ab} + \frac{q^aq^b}{u}, \quad g^{ab} = \delta_{ab} - q^a q^b, \quad \det g_{ab} = \frac{1}{u}, \quad u = 1 - x^2 - y^2. \quad (10)$$

($u$ is the sine-squared of the latitude, since this represents the orthogonal projection of the globe on its equatorial plane). The momenta are then

$$p_a = \dot{q}^a + q^a \frac{h}{u} = \dot{q}^a + q^a (q \cdot p), \quad h \equiv -\dot{u}/2 = xx + yy. \quad (11)$$

The classical equations of motion here amount to

$$\dot{p}_a = p_a q \cdot p, \quad \text{i.e.} \quad \dot{q}^a + q^a \left(\frac{h}{u} + \frac{h^2}{u^2}\right) = 0. \quad (12)$$

It is then easy to find the three classical invariants, the components of the conserved angular momentum in this nonlinear realization,

$$L_z = xp_y - yp_x, \quad L_y = \sqrt{u} \ p_x, \quad L_x = -\sqrt{u} \ p_y. \quad (13)$$
The last two are the de Sitter “momenta”, or non-linearly realized “axial charges” corresponding to the “pions” $x, y$ of the $\sigma$-model: linear momenta are, of course, not conserved. Their PBs close into $so(3)$,

$$\{L_x, L_y\} = L_z, \quad \{L_y, L_z\} = L_x, \quad \{L_z, L_x\} = L_y.$$  \hfill (14)

Thus, it follows algebraically that their PBs with the Casimir invariant $L \cdot L$ vanish. Naturally, since $H = L \cdot L/2$, they are manifested to be time-invariant,

$$\dot{L} = \{L, H\} = 0.$$  \hfill (15)

In quantizing this system, operator ordering issues arise, given the effective velocity(momentum)-dependent potential. In phase-space quantization, one may insert $\star$-products in strategic points and orderings of the variables of (8), to maintain integrability. That is, the classical invariance expressions (PB commutativity),

$$\{I, H\} = 0,$$  \hfill (16)

are to be promoted to quantum invariances (MB commutativity),

$$\{\{I_{qm}, H_{qm}\} \equiv \frac{I_{qm} \star H_{qm} - H_{qm} \star I_{qm}}{i\hbar} = 0 .$$  \hfill (17)

Here, this argues for a (c-number kernel function) hamiltonian of the form

$$H_{qm} = \frac{1}{2}(L_x \star L_x + L_y \star L_y + L_z \star L_z).$$  \hfill (18)

The reason is that, in this realization, the algebra (14) is promoted to the corresponding MB expression \textit{without any modification}, since all of its MBs collapse to PBs by the linearity in momenta of the arguments: all corrections $O(\hbar)$ vanish. Consequently, these particular invariants are undeformed by quantization, $L = L_{qm}$. As a result, given associativity for $\star$, the corresponding quantum quadratic Casimir invariant $L \cdot L$ has vanishing MBs with $L$ (but not vanishing PBs\(^1\)), and automatically serves as a symmetry-preserving hamiltonian. The specification of the maximally symmetric quantum hamiltonian is thus complete.

The $\star$-product in this hamiltonian trivially evaluates to yield the quantum correction to (8),

$$H_{qm} = H + \frac{\hbar^2}{8}(\det g - 3).$$  \hfill (19)

One might wish to contrast this quantum correction to the free-space angular-momentum quantum correction for zero curvature in the underlying manifold familiar from atomic physics [20]: that one too is $O(\hbar^2)$, but it is a constant, reflecting the vanishing curvature.

\(^1\) Likewise, in comportance with the Groenewold-van Hove theorem, the above classical hamiltonian $H$ does not MB-commute with the invariants.
Predictably, on the North pole above, \( u = 1 \), and these expressions coincide. This difference and pole coincidence carries over for all dimensions, as evident in the quantum correction (35) for \( S^n \) derived below.

In phase-space quantization \([12, 13, 14]\), the WF (the kernel function of the density matrix) evolves according to Moyal’s equation \([12]\),

\[
\frac{\partial f}{\partial t} = \{\{H_{qm}, f\}\}.
\]

But, in addition, for pure stationary states, the spectrum is specified by the fundamental \(*\)-genvalue equations \([21, 13]\) satisfied by the respective WFs,

\[
H_{qm}(x, p) \ast f(x, p) = f(x, p) \ast H_{qm}(x, p)
\]

\[
= H_{qm}\left(x + \frac{i\hbar}{2} \partial_p \, , \, p - \frac{i\hbar}{2} \partial_x\right) f(x, p) = E f(x, p).
\]

The spectrum of this specific hamiltonian (18), then, is proportional to the \( \hbar^2 l(l+1) \) spectrum of the \( SO(3) \) Casimir invariant \( L \cdot \ast L = L_x \ast L_+ + L_\ast \ast L_- - \hbar L_z \), for integer \( l \) \([22]\). It can be produced algebraically by the standard recursive ladder operations in \(*\) space which obtain in the operator formalism Fock space \([1]\). Similar \(*\)-ladder arguments and inequalities apply directly in phase space to all Lie algebras.

The treatment of the 3-sphere \( S^3 \) is very similar, with some significant differences, since it also accords to the standard chiral model technology. The metric and eqns of motion, etc, are identical in form to those above, except now \( u \equiv 1 - x^2 - y^2 - z^2 = 1/\det g \), \( h \equiv -\dot{u}/2 = x\dot{x} + y\dot{y} + z\dot{z} \), and \( a, b = 1, 2, 3 \). However, the description simplifies upon utilization of Vielbeine, \( g_{ab} = \delta_{ij} V_i^a V_j^b \) and \( g^{ab} V_i^a V_j^b = \delta^{ij} \).

Specifically, the Dreibeine, are either left-invariant, or right invariant \([23]\):

\[
(\pm)V_i^a = \varepsilon^{iab} q^b \pm \sqrt{u} g_{ai}, \quad (\pm)V^a = \varepsilon^{iab} q^b \pm \sqrt{u} \delta^{ai}.
\]

The corresponding right and left conserved charges (left- and right-invariant, respectively) then are

\[
R^i = (+)V^i_a q^a = (+)V^a p_a, \quad L^i = (-)V^i_a q^a = (-)V^a p_a.
\]

More intuitive than those for \( S^2 \) are the linear combinations into Axial and Isospin charges (again linear in the momenta),

\[
\frac{R - L}{2} = \sqrt{u} p \equiv A, \quad \frac{R + L}{2} = q \times p \equiv I.
\]

It can be seen that the \( L \)s and the \( R \)s have PBs closing into standard \( su(2) \otimes su(2) \), ie, \( su(2) \) relations within each set, and vanishing between the two sets. Thus they are seen to be constant, since the hamiltonian (and the lagrangian) can, in fact, be written in terms of either quadratic Casimir invariant,

\[
H = \frac{1}{2} L \cdot L = \frac{1}{2} R \cdot R = L.
\]
Quantization consistent with integrability thus proceeds as above for the 2-sphere, since the MB algebra collapses to PBs again, and so the quantum invariants \( L \) and \( R \) again coincide with the classical ones, without deformation (quantum corrections). The \(*\)-product is now the obvious generalization to 6-dimensional phase-space. The eigenvalues of the relevant Casimir invariant are now \( j(j+1) \), for half-integer \( j \). However, this being a chiral model \((G \otimes G)\), the symmetric quantum hamiltonian is simpler than the previous one, since it can now also be written geometrically as

\[
H_{qm} = \frac{1}{2} (p_a V^{ai}) (V^{bi} p_b) = \frac{1}{2} \left( g^{ab} p_a p_b + \frac{\hbar^2}{4} \partial_a V^{bi} \partial_b V^{ai} \right). \tag{26}
\]

The Dreibeine throughout this formula can be either \( +V^i_a \) or \( -V^i_a \), corresponding to either the right, or the left-acting quadratic Casimir invariant. The quantum correction then amounts to

\[
H_{qm} - H = \frac{\hbar^2}{8} \left( \det g - 7 \right). \tag{27}
\]

In general, the above discussion also applies to all chiral models, with \( G \otimes G \) replacing \( su(2) \otimes su(2) \) above. I.e, the Vielbein-momenta combinations \( V^a_j \) represent algebra generator invariants, whose quadratic Casimir group invariants yield the respective hamiltonians, and whence the properly \(*\)-ordered quantum hamiltonians as above. (We follow the conventions of [19], taking the generators of \( G \) in the defining representation to be \( T_j \).)

That is to say, for

\[
i U^{-1} \frac{d}{dt} U = (+) V^a_j T_j q^a = (+) V^a_j p_a T_j, \quad i U \frac{d}{dt} U^{-1} = (-) V^a_j p_a T_j, \tag{28}
\]

it follows that the PBs of the left- and right-invariant charges \( (\pm) V^a_j p_a = \frac{1}{2} Tr T_j U^{\pm1} \frac{d}{dt} U^{\pm1} \) close to the identical Lie algebras,

\[
\{ (\pm) V^a_j p_a, (\pm) V^{bk} p_b \} = -2 f^{jkn} (\pm) V^{an} p_a, \tag{29}
\]

and PB commute with each other,

\[
\{ (+) V^a_j p_a, (-) V^{bk} p_b \} = 0. \tag{30}
\]

These two statements are proven directly in [1].

MBs collapse to PBs by linearity in momenta as before, and the hamiltonian is identical in form to (26). The quantum correction in (26) to amounts to

\[
H_{qm} - H = \frac{\hbar^2}{8} \left( \Gamma^b_{ac} g^{cd} \Gamma^a_{bd} - f_{ijk} f_{ijk} \right), \tag{31}
\]

(reducing to (27) for \( S^3 \)). The spectra are given by the Casimir eigenvalues for the relevant algebras and representations.
For the generic sphere models, $S^N$, the maximally symmetric Hamiltonians are the quadratic Casimir invariants of $so(N+1)$,

$$H = \frac{1}{2} P_a P_a + \frac{1}{4} L_{ab} L_{ab},$$  \hspace{1cm} (32)

where

$$P_a = \sqrt{u} p_a, \hspace{1cm} L_{ab} = q^a p_b - q^b p_a,$$ \hspace{1cm} (33)

for $a = 1, \cdots, N$, the de Sitter momenta and angular momenta of $so(N+1)/so(N)$. All of these $N(N+1)/2$ sphere translations and rotations are symmetries of the classical Hamiltonian.

Quantization proceeds as in $S^2$, maintaining conservation of all $P_a$ and $L_{ab}$,

$$H_{qm} = \frac{1}{2} P_a \times P_a + \frac{1}{4} L_{ab} \times L_{ab},$$  \hspace{1cm} (34)

and hence the quantum correction is

$$H_{qm} - H = \hbar^2 8 \left( \frac{1}{u} - 1 - N(N - 1) \right).$$  \hspace{1cm} (35)

The spectra are proportional to the Casimir eigenvalues $l(l + N - 1)$ for integer $l$ [22]. For $N = 3$ of the previous section, this form is reconciled with the Casimir expression for (25) as $l = 2j$, and agrees with [9, 10, 11].

**MAXIMAL SUPERINTEGRABILITY AND THE NAMBU BRACKET**

All the models considered above have extra invariants beyond the number of conserved quantities in involution (mutually commuting) required for integrability in the Liouville sense. The most systematic way of accounting for such additional invariants, and placing them all on a more equal footing, even when they do not all simultaneously commute, is the NB formalism.

For example, the classical mechanics of a particle on an N-sphere as discussed above may be summarized elegantly through Nambu Mechanics in phase space [4, 6]. Specifically, [5, 7], in an $N$-dimensional space, and thus $2N$-dimensional phase space, motion is confined on the constant surfaces specified by the algebraically independent integrals of the motion (eg, $L_x, L_y, L_z$ for $S^2$ above.) Consequently, the phase-space velocity $\mathbf{v} = (q, p)$ is always perpendicular to the $2N$-dimensional phase-space gradients $\nabla = (\partial q, \partial p)$ of all these integrals of the motion, and $\nabla \cdot \mathbf{v} = 0$.

As a consequence, if there are $2N - 1$ algebraically independent such integrals $L_i$, possibly including the Hamiltonian, (ie, the system is maximally superintegrable [3]), the phase-space velocity must be proportional [5] to the cross-product of all those gradients,
and hence the motion is fully specified for any phase-space function \( k(q,p) \) by a phase-space Jacobian which amounts to the Nambu Bracket:

\[
\frac{dk}{dt} = \nabla k \cdot v \propto \partial_{i_1} k \, \varepsilon^{i_1 i_2 \cdots i_{2N}} \partial_{i_2} L_{1_1} \cdots \partial_{i_{2N}} L_{2N-1} = \frac{\partial(k, L_1, \ldots, L_{2N-1})}{\partial(p_1, q_1, q_2, p_2, \ldots, q_N, p_N)} = \{k, L_1, \ldots, L_{2N-1}\}, \tag{36}
\]

as an alternative to Hamiltonian mechanics. For instance, for the above \( S^2 \),

\[
\frac{dk}{dt} = \frac{\partial(k, L_x, L_y, L_z)}{\partial(x, p_x, y, p_y)} = \{k, L_x\} \{L_y, L_z\} - \{k, L_y\} \{L_x, L_z\} + \{k, L_z\} \{L_x, L_y\}. \tag{37}
\]

For the more general \( S^N \), one now has a choice of \( 2N - 1 \) of the \( N(N+1)/2 \) invariants of \( so(N+1) \); one of several possible expressions is

\[
\frac{dk}{dt} = \frac{(-1)^{(N-1)}}{P_2 P_3 \cdots P_{N-1}} \left. \frac{\partial(k, P_1, L_{12}, P_2, L_{23}, P_3, \ldots, P_{N-1}, L_{N-1}, L_{N-1}N, P_N)}{\partial(x_1, p_1, x_2, p_2, \ldots, x_N, p_N)} \right|_{\mathcal{L}}, \tag{38}
\]

where \( P_a = \sqrt{\mu_a} p_a \), for \( a = 1, \ldots, N \), and \( L_{a,a+1} = q^a p_{a+1} - q^{a+1} p_a \), for \( a = 1, \ldots, N - 1 \).

In general \([25, 6]\), NBs, being Jacobian determinants, possess all antisymmetries of such; being linear in all derivatives, they also obey the Leibniz rule of partial differentiation,

\[
\{k(L, M), f_1, f_2, \ldots\} = \frac{\partial k}{\partial L} \{L, f_1, f_2, \ldots\} + \frac{\partial k}{\partial M} \{M, f_1, f_2, \ldots\}. \tag{39}
\]

Thus, an entry in the NB algebraically dependent on the remaining entries leads to a vanishing bracket. For example, it is seen directly from above that the hamiltonian is constant,

\[
\frac{dH}{dt} = \left\{ \frac{L \cdot L}{2}, \ldots \right\} = 0, \tag{40}
\]

since each term of this NB vanishes. Naturally, this also applies to all explicit examples discussed here, as they are all maximally superintegrable.

Finally, the impossibility to antisymmetrize more than \( 2N \) indices in \( 2N \)-dimensional phase space,

\[
\varepsilon^{a \ldots c [i} \varepsilon_{j_1 j_2 \ldots j_{2N}]} = 0, \tag{41}
\]

leads to the so-called Fundamental Identity, \([25, 6]\), slightly generalized here \([1, 2]\),

\[
\{V \{A_1, \ldots, A_{m-1}, A_m\}; A_{m+1}, \ldots, A_{2m-1}\} + \{A_m, V \{A_1, \ldots, A_{m-1}, A_{m+1}\}, A_{m+2}, \ldots, A_{2m-1}\} + \cdots + \{A_m, \ldots, A_{2m-2}, V \{A_1, \ldots, A_{m-1}, A_{2m-1}\}\} = \{A_1, \ldots, A_{m-1}, V \{A_m, A_{m+1}, \ldots, A_{2m-1}\}\}. \tag{42}
\]
Its name is inapposite, however, to the extent that it merely reflects the fact that CNBs, being linear in derivatives, obey Leibniz’s rule (they are derivations [25, 2]); even though it has sometimes been analogized to the Jacobi Identity, unlike the Jacobi identity, it is not fully antisymmetrized in all of its arguments, nor does it reflect associativity in the abstract [2, 26, 27].

The proportionality function $V$ in (36),

$$\frac{dk}{dt} = V\{k, L_1, ..., L_{2N-1}\},$$

has to be a time-invariant [7] if it has no explicit time dependence. This is seen from consistency of (43), application of which to

$$\frac{d}{dt}(V\{A_1, ..., A_{2N}\}) = \dot{V}\{A_1, ..., A_{2N}\} + V\{\dot{A}_1, ..., A_{2N}\} + ... + V\{A_1, ..., \dot{A}_{2N}\},$$

yields

$$V\{V\{A_1, ..., A_{2N}\}, L_1, ..., L_{2N-1}\} = \dot{V}\{A_1, ..., A_{2N}\}$$

$$+ V\{V\{A_1, L_1, ..., L_{2N-1}\}, ..., A_{2N}\} + ... + V\{A_1, ..., V\{A_{2N}, L_1, ..., L_{2N-1}\}\},$$

and, by virtue of (42), $\dot{V} = 0$ follows.

Actually, PBs result from a maximal reduction of NBs, by inserting $2N - 2$ phase-space coordinates and summing over them, thereby taking symplectic traces,

$$\{L, M\} = \frac{1}{(N-1)!}\{L, M, x_{i_1}, p_{i_1}, ..., x_{i_{N-1}}, p_{i_{N-1}}\},$$

where summation over all $N - 1$ pairs of repeated indices is understood. Fewer traces lead to relations between NBs of maximal rank, $2N$, and those of lesser rank, $2k$,

$$\{L_1, ..., L_{2k}\} = \frac{1}{(N-k)!}\{L_1, ..., L_{2k}, x_{i_1}, p_{i_1}, ..., x_{i_{N-k}}, p_{i_{N-k}}\}$$

(which is one way to define the lower rank NBs for $k \neq 1$), or between two lesser rank NBs [2]. Note that $\{L_1, ..., L_{2k}\}$ acts like a Dirac Bracket (DB) up to a normalization, $\{L_1, L_2\}_{DB}$, where the fixed additional entries $L_3, ..., L_{2k}$ in the NB play the role of the constraints in the DB [24]. (In effect, this has been previously observed, eg, [7], for the extreme case $N = k$, without symplectic traces.)

As a simplest illustration, consider $N = k = 2$ for the system (37), but now taking $L_x, L_y$ as second-class constraints:

$$\{f, g, L_x, L_y\} = \{f, g\}\{L_x, L_y\} + \{f, L_x\}\{L_y, g\} - \{f, L_y\}\{L_x, g\} \equiv \{L_x, L_y\}\{f, g\}_{DB}. \quad (48)$$

That is,

$$\{f, g\}_{DB} = \left(\{L_x, L_y\}\right)^{-1}\{f, g, L_x, L_y\}, \quad (49)$$

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so that, from (42) with \( V = (\{L_x, L_y\})^{-1} = 1/L_z \) (also see [7]), it follows that the Dirac Brackets satisfy the Jacobi identity,

\[
\{\{f, g\}_DB, h\}_DB + \{g, h\}_DB, f\}_DB + \{h, f\}_DB, g\}_DB = 0 ,
\]

(50)
a property usually established by explicit calculation [24], in contrast to this derivation. Naturally, \( \{f, L_x, L_y, L_z\} = \{f, H\}_DB \).

## NAMBU QUANTIZATION

The quantization method of NBs considered by Nambu in an operator context [4], when applied to the phase-space formalism, motivates defining QNBs as fully antisymmetrized associative \(*\)-product sequences,

\[
[A, B]_* \equiv A * B - B * A = i\hbar \{[A, B]\},  \\
[A, B, C]_* \equiv A * B * C - A * C * B + B * C * A - B * A * C + C * A * B - C * B * A,  \\
[A, B, C, D]_* \equiv A *[B, C, D]_* - B *[C, D, A]_* + C *[D, A, B]_* - D *[A, B, C]_*  \\
= [A, B]_* *[C, D]_* + [A, C]_* *[D, B]_* + [A, D]_* *[B, C]_* + [C, D]_* *[A, B]_* + [D, B]_* *[A, C]_* + [B, C]_* *[A, D]_*,
\]

(51)
etc. These antisymmetrized \(*\)-products are used in the quantum theory instead of the previous jacobians. As for the above 4-QNB, the resolution of all even-QNBs into symmetrized \(*\)-products of 2-brackets (MBs) reflecting the full antisymmetry of the structure is a general useful result [2].

These QNBs, consisting of associative strings of \(*\)-products, satisfy Generalized Jacobi Identities [26, 27, 2], but not, in general [28], the Leibniz property (39) and the consequent Fundamental Identity (42). The loss of the latter two properties is a subjective shortcoming, contingent on the specific application context. Objectively, this approach is in agreement with the \(*\)-product quantization of all the examples given above, the N-spheres, Chiral Models, and, in addition, \(n\)-dimensional isotropic oscillator systems [2].

Specifically, for \(S^2\), it follows directly from (51) and (14) (with MBs supplanting PBs, \(\{L_x, L_y\} = L_z, \{L_y, L_z\} = L_x, \{L_z, L_x\} = L_y\) that the Moyal Bracket with the hamiltonian (18) equals Nambu’s QNB, for an arbitrary function \(k\) of phase space,

\[
[k, L_x, L_y, L_z]_* = i\hbar \ [k, L_x \cdot L_y]_* = -2\hbar^2 \{[k, H_{qm}]\},
\]

(52)
so that for \(k\) with no explicit time dependence,

\[
\frac{dk}{dt} = -\frac{1}{2\hbar^2} [k, L_x, L_y, L_z]_* .
\]

(53)
This provides a good quantization for (43), for this particular system (37). For \(\hbar \to 0\), it naturally goes to (37).
As a derivation, it ensures that consistency requirements (39) and (42) are satisfied, with the suitable insertion of $\star$-multiplication in the proper locations to ensure full combinatoric analogy,

$$[A \star B, L_x, L_y, L_z] = A \star [B, L_x, L_y, L_z] + [A, L_x, L_y, L_z] \star B, \quad (54)$$

and

$$[[L_x, L_y, L_z, D], E, F, G] + [D, [L_x, L_y, L_z, E], F, G] + [D, E, [L_x, L_y, L_z, F], G] = [D, E, [L_x, L_y, L_z, G], F, G] \quad (55)$$

The reader might also wish to note from (51) that, for constant $A$ (independent of phase-space variables), thus $dA/dt = 0$,

$$[A, B, C, D] = 0 \quad (56)$$

holds identically, in contrast to the 3-argument QNB [4]. Thus, no debilitating constraint among the arguments $B, C, D$ is imposed; the inconsistency identified in ref [4] is a feature of odd-argument QNBs and does not restrict the even-argument QNBs of phase space considered here.

As indicated, in the generic case, the QNB (which provide the correct quantization rule for all the systems considered here) need not satisfy the Leibniz property and FI for consistency, as they are not necessarily derivations. However, they entwine derivations within symmetrized Jordan products of invariants: it turns out that the proportionality function $V$ of (43) is not irrelevant$^2$.

For instance, for $S^3$, to quantize (38) for $N = 3$,

$$P_2 \frac{dk}{dt} = \{k, P_1, L_{12}, P_2, L_{23}, P_3\}, \quad (57)$$

note that

$$[k, P_1, L_{12}, P_2, L_{23}, P_3] = 3i\hbar^3 \left( P_2 \star \{\{k, H_{qm}\}\} + \{\{k, H_{qm}\} \star P_2 \right) + \mathcal{Q}, \quad (58)$$

where $\mathcal{Q}$ is an $O(\hbar^5)$ rotation, a sum of triple commutators of $k$ with invariants. Consequently, the proper quantization of (57) is

$$[k, P_1, L_{12}, P_2, L_{23}, P_3] = 3i\hbar^3 \frac{d}{dt} \left( \frac{P_2 \star k + k \star P_2}{2} \right) + \mathcal{Q}, \quad (59)$$

and again reduces to (57) in the $\hbar \to 0$ limit, as $\mathcal{Q}$ is subdominant in $\hbar$ to the time derivative term. The right hand side not being an unadorned derivation on $k$, it does not

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$^2$ If, for some reason related to specific application contexts, one still insisted on a bracket structure which is a derivation, with some effort, the relevant structures might be unentwined through infinite series solutions of (58), requiring special attention to convergence issues [2].
impose a Leibniz rule analogous to (54) on the left hand side, and so it fails the FI, at no compromise to its validity, however. The $N > 3$ case and Chiral Models parallel the above through use of fully symmetrized products.

More elaborate isometries of general manifolds in such models are expected to yield to analysis similar to what has been illustrated for the prototypes considered here. An empirical methodology suggested by a plethora of examples [2] argues for, first, manipulating the CNB entries to simplify $V$, and to select the entries such that they combine into the hamiltonian in the PB resolution of the CNB. The corresponding QNB would then be expected to yield entwined structures as illustrated above, upon working out its MB resolution, through analogous combinatoric operations; with the hamiltonian appearing in a MB with $k$ (hence its time derivative), now entwined with invariants. The resulting intriguing dynamical laws are further discussed in T Curtright’s talk [these Proceedings, hep-th/0303088].

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