A New Class of Higher-order Hypergeometric Bernoulli Polynomials Associated with Hermite Polynomials

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ABSTRACT: In this paper, we introduce a new class of higher-order hypergeometric Hermite-Bernoulli numbers and polynomials. We shall provide several properties of higher-order hypergeometric Hermite-Bernoulli polynomials including summation formulae, sums of product identity, recurrence relations.

Key Words: Hermite polynomials, Higher-order hypergeometric Bernoulli polynomials, Higher-order hypergeometric Hermite-Bernoulli polynomials, Recurrence relations.

Contents

1 Introduction

2 Multiple hypergeometric Hermite-Bernoulli numbers and polynomials

3 Summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials

1. Introduction

The Bernoulli polynomials $B_n(x)$ are defined by the following generating function

\[ \left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \]  (1.1)

and $B_n = B_n(0)$ are named Bernoulli numbers. These numbers and polynomials have a long history, which arise from Bernoulli’s calculations of power sums in 1713, that is,

\[ \sum_{j=1}^{m} j^n = \frac{B_{n+1}(m + 1) - B_{n+1}}{n + 1}, \]

(see \cite[p.5, (2.2)]{19}). They have many applications in modern number theory, such as modular forms \cite{11} and Iwasawa theory \cite{9}. A recent book by Arakawa, Ibukiyama and Kaneko \cite{1} give a nice introduction of Bernoulli numbers and polynomials including their connections with zeta functions.

In 1924, Nörlund \cite{14} introduced and studied the generalized higher order Bernoulli polynomials defined by means of the following generating function

\[ \left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \]

(1.2)

We also have a similar expression of multiple power sums

\[ \sum_{l_1 \cdots l_n = 0}^{m-1} (t + l_1 + \cdots + l_n)^k, \]

in terms of higher order Bernoulli polynomials, (see \cite[Lemma 2.1]{12}).
Howard ([5], [6]) gave a generalization of Bernoulli polynomials by considering the following generating function
\[ \frac{t^2e^{xt}/2}{e^t - 1 - t} = \sum_{n=0}^{\infty} A_n^{(a)}(x) \frac{t^n}{n!}, \quad (1.3) \]
and more generally, for all positive integer \( N \)
\[ \frac{t^N}{e^t - T_{N-1}(t)} e^{xt} = \sum_{n=0}^{\infty} B_{N,n}(x) \frac{t^n}{n!}, \quad (1.4) \]
where \( T_{N-1}(t) \) is the Taylor polynomial of order \( N - 1 \) for the exponential function. For the case \( N = 1 \) and \( N = 2 \), (1.4) reduces to (1.1) and (1.3), respectively. We see that the polynomials \( B_{N,n}(x) \) have rational coefficients.

The polynomials \( B_{N,n}(x) \) are named hypergeometric Bernoulli polynomials, while the numbers \( B_{N,n} = B_{N,n}(0) \) are named hypergeometric Bernoulli numbers since the generating function \( f(t) = \frac{e^t - T_{N-1}(t)}{e^t} \)
can be expressed as \( _1F_1(1; N + 1; t) \), where the confluent hypergeometric function \( _1F_1(a; b; t) \) is defined by
\[ _1F_1(a; b; t) = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b)_n n!}, \quad (1.5) \]
and \((a)_n\) is the Pochhammer symbol, (see [20])
\[ (a)_0 := 1, \quad (a)_n = a(a + 1) \cdots (a + n - 1), \quad (n \in \mathbb{N}) := \{1, 2, 3, \cdots \}. \]

For \( N, r \in \mathbb{N} \), the higher-order hypergeometric Bernoulli polynomials \( B_{N,n}^{(r)}(x) \) are defined by means of the generating function, (see [2], [7], [10])
\[ \left( \frac{t^N}{e^t - T_{N-1}(t)} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x) \frac{t^n}{n!}. \quad (1.6) \]
For \( x = 0 \) in (1.6), \( B_{N,n}^{(r)} = B_{N,n}^{(r)}(0) \) are called the higher order hypergeometric Bernoulli numbers, (see [10], [13]). Again, on taking \( r = 1 \) in (1.6), \( B_{N,n}^{(1)}(x) = B_{N,n}(x) \) are called the hypergeometric Bernoulli polynomials and if we put \( x = 0 \) in (1.6), \( B_{N,n}^{(1)}(0) = B_{N,n} \) are called the hypergeometric Bernoulli numbers.

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) \( H_n(x,y) \) ([3], [4]) are defined as
\[ H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \quad (1.7) \]
It is easily seen that
\[ H_n(2x,-1) = H_n(x), \quad H_n(x, -\frac{1}{2}) = He_n(x), \]
where \( H_n(x) \) and \( He_n(x) \) are called the ordinary Hermite polynomials. Also
\[ H_n(x,0) = x^n. \]
The generating function for Hermite polynomial \( H_n(x,y) \) ([16]-[18]) are given by
\[ e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \quad (1.8) \]
The object of this paper is to present a systematic account of these families in a unified and generalized form. We develop some elementary properties and derive the implicit summation formulae for the higher-order hypergeometric Hermite-Bernoulli polynomials by using different analytical means on their respective generating functions. The approach given in recent papers of Pathan and Khan ([16]-[18]) has indeed allowed the derivation of implicit summation formulae in the two-variable higher-order hypergeometric Hermite-Bernoulli polynomials. In addition to this, some relevant connections between Hermite and higher-order hypergeometric Bernoulli polynomials and recurrence relations are given.

2. Multiple hypergeometric Hermite-Bernoulli numbers and polynomials

For every positive integer $N$ and $r$, the higher-order hypergeometric Hermite-Bernoulli numbers and polynomials $H_{N,n}^{(r)}(x,y)$ are defined by means of the following generating function defined in a suitable neighborhood of $t = 0$:

\[
F_{r,N}(x, y, t) = \frac{1}{1F_1(1; N + 1; t)} \left( \frac{t^N}{N!} \right)^r e^{xt+yt^2} = \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x,y) \frac{t^n}{n!}.
\]

For $x = y = 0$, $B_{N,n}^{(r)} = H_{N,n}^{(r)}(0,0)$ are called the higher-order hypergeometric Bernoulli numbers, (see [10, 13]). When $r = 1$, we obtain the hypergeometric Hermite-Bernoulli polynomials $H_{N,n}(x,y) = H_{N,n}^{(1)}(x,y)$ and $B_{N,n} = H_{N,n}^{(1)}(0,0)$ is the hypergeometric Bernoulli numbers, (see [8, 15]). If we put $N = 1$, the result reduces to the known result of Pathan and Khan, (see [16]).

Remark 2.1. On setting $y = 0$, (2.1) reduces to the known result of Aoki et al. [2] as follows:

\[
F_{r,N}(x, t) = \frac{1}{1F_1(1; N + 1; t)} e^{xt} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x) \frac{t^n}{n!}.
\]

In particular in terms of higher-order hypergeometric Bernoulli numbers $B_{N,n}^{(r)}$ and Hermite polynomials $H_{s}(x,y)$, the higher order Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x,y)$ are defined as

\[
H_{N,n}^{(r)}(x,y) = \sum_{s=0}^{n} \binom{n}{s} B_{N,n-s}^{(r)} H_{s}(x,y).
\]

Taking $r = N = 1$ and $x = 0$ in (2.1) gives the result

\[
\sum_{m=0}^{\infty} \binom{n}{2m} B_{n-2m} y^m = H_{1,n}^{(1)}(0,y).
\]

Using $e^{it} = \cos t + i \sin t$ and $N = 1$, the result reduces to

\[
\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n + 1),
\]

\[
(2.5)
\]
and 
\[
\left( \frac{it}{e^{it} - 1} \right)^r = \left( \frac{it(\cos t - 1 - i \sin t)}{(\cos t - 1 + i \sin t)(\cos t - 1 - i \sin t)} \right)^r = \left( \frac{it(\cos t - 1 - i \sin t)}{(\cos t - 1)^2 + (\sin t)^2} \right)^r
\]

where \( \Omega = (\cos t - 1)^2 + (\sin t)^2 \), together with the definition (2.1) and the result (2.5), we get (see Pathan and Khan [16]):

\[
e^{ixt-yt^2}\left( \frac{(t \sin t) + it(\cos t - 1)}{\Omega} \right)^r = \sum_{n=0}^{\infty} H B_{2n}^{(r)}(x, y) \frac{(-1)^n t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} H B_{2n+1}^{(r)}(x, y) \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \tag{2.6}
\]

where \( r \geq 1, \Omega = (\cos t - 1)^2 + (\sin t)^2 \).

On setting \( r = 1, x = y = 0 \) in the above results, we get the following well known classical results involving Bernoulli numbers, (see [16]):

\[
\sum_{n=0}^{\infty} H B_{2n}^{(r)}(x, y) \frac{(-1)^n t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} B_{2n} t^{2n}, \quad \sum_{n=0}^{\infty} H B_{2n+1}^{(r)}(x, y) \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} B_{2n+1} t^{2n+1}.
\]

**Theorem 2.2.** For \( n \geq 1 \), we have

\[
H_n(x, y) = n!(N)! \sum_{m=0}^{\infty} \sum_{i_1+\cdots+i_r=n-m} H B_{N,m}^{(r)}(x, y) \frac{t^n}{m!(N+i_1)\cdots(N+i_r)!}. \tag{2.7}
\]

**Proof.** From definition (2.1), we have

\[
\left( \frac{t}{(N)!} \right)^r e^{xt+yt^2} = \left( \frac{t+iN}{(i+N)!} \right)^r \left( \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \right)^r
\]

\[
= t^r N \left( \sum_{l=0}^{\infty} \sum_{i_1+\cdots+i_r=1} \frac{t^l}{l!(N+i_1)\cdots(N+i_r)!} \right) \left( \sum_{m=0}^{\infty} H B_{N,m}^{(r)}(x, y) \frac{t^m}{m!} \right)
\]

\[
= t^r N \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{i_1+\cdots+i_r=n-m} H B_{N,m}^{(r)}(x, y) \frac{t^n}{m!(N+i_1)\cdots(N+i_r)!}. \]

Comparing the coefficients of \( t^n \) on both sides, we get (2.7). \( \square \)

**Corollary 2.3.** For \( r = 1 \) in (2.7), we get

\[
H_n(x, y) = n!(N)! \sum_{m=0}^{\infty} \left( \frac{n+iN}{m} \right) H B_{N,m}(x, y). \tag{2.8}
\]

**Corollary 2.4.** For \( x = y = 0 \) in (2.7), the result reduces to the known result of Aoki et al. [2] as follows

\[
\sum_{m=0}^{n} \sum_{i_1+\cdots+i_r=n-m} \frac{B_{N,m}^{(r)}}{m!(N+i_1)\cdots(N+i_r)!} = 0. \tag{2.9}
\]

and \( r = 1 \) in (2.8), the result reduces to (see [7]):

\[
\sum_{m=0}^{n} \left( \begin{array}{c} n+iN \\ m \end{array} \right) B_{N,m}(x, y) = 0. \tag{2.10}
\]
Theorem 2.5. The following relationship holds true:

\[ H_n(x, y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{m! \Gamma(N+1)}{\Gamma(N+1+m)} H_{N-n-m}(x, y). \]  

(2.11)

Proof. Using equations (2.1), (1.5) and (1.8), we have

\[ \frac{1}{1F1(1; N + 1; t)} e^{xt+y^2} = \sum_{n=0}^{\infty} H_{N-n}(x, y) \frac{t^n}{n!} \]

\[ e^{xt+y^2} = \sum_{n=0}^{\infty} H_{N,n}(x, y) \frac{t^n}{n!} \]

\[ \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{(1)_m}{(N+1)_m m!} \sum_{n=0}^{\infty} H_{N,n}(x, y) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{m! \Gamma(N+1)}{\Gamma(N+1+m)} H_{N-n}(x, y) \frac{t^n}{n!} \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we arrive at the obtained result (2.11). □

Theorem 2.6. The following relationship holds true:

\[ \int_{0}^{1} (1-x)^{N-1} H_{N,n}^{(r)}(x, y) dx = (N-1)! \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(n-k)!}{(N+n-k)!} H_{N,k}^{(r)}(0, y). \]  

(2.12)

Proof. From (2.1), we have

\[ \frac{1}{1F1(1; N + 1; t)} e^{xt+y^2} = \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \]

\[ e^{xt} \sum_{n=0}^{\infty} H_{N,n}^{(r)}(0, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \]

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) H_{N,k}^{(r)}(0, y) x^{n-k} \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \]

Thus, we have

\[ H_{N,n}^{(r)}(x, y) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) H_{N,k}^{(r)}(0, y) x^{n-k}. \]  

(2.13)

Therefore, by integrating (2.13) with weight \( (1-x)^{N-1} \) and using the result ( [20], p.26(48) ), we obtain

\[ \int_{0}^{1} (1-x)^{N-1} H_{N,n}^{(r)}(x, y) dx = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) H_{N,k}^{(r)}(0, y) \int_{0}^{1} (1-x)^{N-1} x^{n-k} dx \]

\[ = (N-1)! \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(n-k)!}{(N+n-k)!} H_{N,k}^{(r)}(0, y), \]

which follows from (2.12). This completes the proof. □
Theorem 2.7. The following representation for higher-order hypergeometric Hermite-Bernoulli polynomials \( H^B_{N,n}(x, y) \) involving Hermite-Euler polynomials \( H^E_n(x, y) \) holds true:

\[
H^B_{N,n}(x, y) = \frac{1}{2} \left[ \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} \binom{m}{k} H^E_{n-m}(x, y) B^B_{N,m-k} + \sum_{m=0}^{n} \binom{n}{m} H^E_{n-m}(x, y) B^B_{N,m} \right]. \tag{2.14}
\]

Proof. Using generating function for Hermite-Euler polynomials as follows

\[
e^{xt+yt^2} = \frac{e^t + 1}{2} \sum_{n=0}^{\infty} H^E_n(x, y) \frac{t^n}{n!}, \quad \text{(see [18]).}
\]

Substituting this value of \( e^{xt+yt^2} \) in (2.1) gives

\[
\sum_{n=0}^{\infty} H^B_{N,n}(x, y) \frac{t^n}{n!} = \frac{1}{1 F_1(1; N + 1; t^r)} \frac{e^t + 1}{2} \sum_{n=0}^{\infty} H^E_n(x, y) \frac{t^n}{n!}
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} H^E_n(x, y) \frac{t^n}{n!} \sum_{m=0}^{\infty} \sum_{k=0}^{m} B^B_{N,m-k} \frac{t^m}{(m-k)!k!}
\]

\[
+ \sum_{n=0}^{\infty} \sum_{m=0}^{n} H^E_{n-m}(x, y) B^B_{N,m} \frac{t^n}{(n-m)!m!}
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{n}{m} \binom{m}{k} H^E_{n-m}(x, y) B^B_{N,m-k} + \sum_{m=0}^{n} \binom{n}{m} H^E_{n-m}(x, y) B^B_{N,m} \right] \frac{t^n}{n!}
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we required at the result (2.14).

\[\square\]

Theorem 2.8. For \( n \geq 0, p, q \in \mathbb{R} \), the following formula for higher-order hypergeometric Hermite-Bernoulli polynomials \( H^B_{N,n}(px, qy) \) holds true:

\[
H^B_{N,n}(px, qy) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} H^B_{N,n-k}(x, y) ((p-1)x)^{k-2j} ((q-1)y)^j \frac{t^n}{(n-k-2j)!j!k!}. \tag{2.15}
\]

Proof. Rewrite the generating function (2.1), we have

\[
\sum_{n=0}^{\infty} H^B_{N,n}(px, qy) \frac{t^n}{n!} = \frac{1}{1 F_1(1; N + 1; t^r)} e^{xt+yt^2} e^{(p-1)x} e^{(q-1)y} t^2.
\]

\[
= \left( \sum_{n=0}^{\infty} H^B_{N,n}(x, y) \frac{t^n}{n!} \right) \left( \sum_{k=0}^{\infty} ((p-1)x)^k \frac{k!}{k!} \right) \left( \sum_{j=0}^{\infty} ((q-1)y)^j \frac{t^2j}{j!} \right).
\]

\[
\left( \sum_{n=0}^{\infty} H^B_{N,n}(x, y) \frac{t^n}{n!} \right) \left( \sum_{k=0}^{\infty} ((p-1)x)^k \frac{k!}{k!} \right) \left( \sum_{j=0}^{\infty} ((q-1)y)^j \frac{t^2j}{j!} \right).
\]
\[
\sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) t^n/n!
\left(\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} ((p-1)x)^k((q-1)y)^j t^{k+2j}/n!k!j!\right).
\]

Replacing \(k\) by \(k-2j\) in above equation, we have
\[
\text{L.H.S.} = \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) t^n/n!
\left(\sum_{k=2j}^{\infty} ((p-1)x)^{k-2j}((q-1)y)^j t^k/(k-2j)!j!\right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{\lfloor n/k \rfloor} \left(\sum_{n=0}^{\infty} H B_{N,n-k}^{(r)}(x,y) t^n/n!\right) ((p-1)x)^{k-2j}((q-1)y)^j t^n/(n-k-2j)!j!k!.
\]

Again replacing \(n\) by \(n-k\) in the above equation, we have
\[
\text{L.H.S.} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{\lfloor n/k \rfloor} \left(\sum_{n=0}^{\infty} H B_{N,n-k}^{(r)}(x,y) t^n/n!\right) ((p-1)x)^{k-2j}((q-1)y)^j t^n/(n-k-2j)!j!k!.
\]

Finally, equating the coefficients of \(t^n\) on both sides, we acquire the result (2.15). \(\square\)

**Theorem 2.9.** For \(n \geq 0, p, q \in \mathbb{R}\) and \(x, y \in \mathbb{C}\), we have
\[
H B_{N,n}^{(r)}(px,qy) = \sum_{k=0}^{n} \binom{n}{k} H B_{N,n-k}^{(r)}(x,y) H_k((p-1)x, (q-1)y).
\] (2.17)

**Proof.** By using (2.16) and (1.8), we can easily derive (2.17). We omit the proof. \(\square\)

**3. Summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials**

In this section, we derive the summation formula, the sum of the product of identity and recurrence relations. First, we prove the following results involving higher-order hypergeometric Hermite-Bernoulli polynomials \(H B_{N,n}^{(r)}(x,y)\).

**Theorem 3.1.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials \(H B_{N,n}^{(r)}(x,y)\) holds true:
\[
H B_{N,k+l}^{(r)}(z,y) = \sum_{n,p=0}^{k,l} k!!(z-x)^n p!! H B_{N,k+l-p-n}^{(r)}(x,y)/(k-n)!(l-p)!n!p!.
\] (3.1)

**Proof.** We replace \(t\) by \(t + u\) and rewrite the generating function (2.1) as
\[
\frac{1}{1F_1(1; N+1; (t+u))^r} e^{(t+u)y} = e^{-x(t+u)^2} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x,y) t^k u^l/k! l!.
\] (3.2)

Replacing \(x\) by \(z\) in the above equation and equating the resulting equation to the above equation, we get
\[
e^{(z-x)(t+u)} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x,y) t^k u^l/k! l! = \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(z,y) t^k u^l/k! l!.
\] (3.3)
On expanding exponential function (3.3) gives
\[
\sum_{M=0}^{\infty} \frac{[(z - x)(t + u)]^M}{M!} \sum_{k,l=0}^{\infty} H_{N,k+l}^{(r)}(x,y) \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!}, \tag{3.4}
\]
which on using formula ([20], p.52(2))
\[
\sum_{M=0}^{\infty} f(M) \frac{(x + y)^M}{M!} = \sum_{n,m=0}^{\infty} f(n + m) \frac{x^n y^m}{n! m!}, \tag{3.5}
\]
in the left hand side becomes
\[
\sum_{n,p=0}^{\infty} \frac{(z - x)^{n+p}}{n!p!} \sum_{k,l=0}^{\infty} H_{N,k+l}^{(r)}(x,y) \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!}. \tag{3.6}
\]
Now replacing \(k\) by \(k - n\), \(l\) by \(l - p\) and using the lemma ([20], p.100(1)) in the left hand side of (3.6), we get
\[
\sum_{n,p=0}^{\infty} \frac{(z - x)^{n+p}}{n!p!} \sum_{k,l=0}^{\infty} \frac{H_{N,k+l}^{(r)}(x,y)}{(k-n)! (l-p)!} \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!}. \tag{3.7}
\]
Finally on equating the coefficients of the like powers of \(t\) and \(u\) in the above equation, we get the required result. \(\square\)

**Corollary 3.2.** On taking \(l = 0\) in Theorem 3.1, the result reduces to
\[
H_{N,k}^{(r)}(z,y) = \sum_{n=0}^{k} \binom{k}{n} (z - x)^n H_{N,k-n}^{(r)}(x,y). \tag{3.8}
\]

**Corollary 3.3.** On replacing \(z\) by \(z+x\) and setting \(y = 0\) in Theorem (3.1), we get the following result involving higher-order hypergeometric Hermite-Bernoulli polynomials of one variable:
\[
H_{N,k+l}^{(r)}(z+x) = \sum_{n,m=0}^{k,l} \frac{k!l! n^m H_{N,k+l-n-m}^{(r)}(x)}{(k-n)!(l-m)! n! m!}, \tag{3.9}
\]
whereas by setting \(z = 0\) in Theorem 3.1, we get another result involving hypergeometric Hermite-Bernoulli polynomials of one and two variables:
\[
H_{N,k+l}^{(r)}(y) = \sum_{n,m=0}^{k,l} \frac{k!l! (-x)^{n+m} H_{N,k+l-n-m}^{(r)}(x,y)}{(k-n)!(l-m)! n! m!}. \tag{3.10}
\]

**Theorem 3.4.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials \(H_{N,n}^{(r)}(x,y)\) holds true:
\[
H_{N,n}^{(r)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} B_{N,n-m}^{(r)}(x-z) H_{m}(z,y). \tag{3.11}
\]
**Proof.** By exploiting the generating function (2.1) and using (1.8), we can write equation (2.1) as
\[
\frac{1}{\Gamma_1(1; N+1; t)} e^{(x-z)t + y t^2} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x-z) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(z,y) \frac{t^m}{m!}.
\]
Replacing \( n \) by \( n - m \) in above equation and using lemma ([20], p.101(1)), we get
\[
\sum_{n=0}^{\infty} H_{N,m}^{(r)}(x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} H_{N,n-m}^{(r)}(x - z) H_m(z, y) \frac{t^n}{(n-m)!m!}.
\]
On equating the coefficients of the like powers of \( t \), we get (3.11). \( \square \)

**Corollary 3.5.** Letting \( z = x \) in Theorem 3.2 gives
\[
H_{N,n}^{(r)}(x, y) = \sum_{m=0}^{n} \binom{n}{m} H_{N,n-m}^{(r)} H_m(x, y).
\] (3.12)

**Theorem 3.6.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials \( H_{N,n}^{(r)}(x, y) \) holds true:
\[
H_{N,n}^{(r)}(x + 1, y) = \sum_{m=0}^{n} \binom{n}{m} H_{N,n-m}^{(r)}(x, y).
\] (3.13)

**Proof.** Using the generating function (2.1), we have
\[
\sum_{n=0}^{\infty} H_{N,n}^{(r)}(x + 1, y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} = \frac{1}{1F_1(1; N + 1; t)} (e^t - 1) e^{xt + yt^2}
\]
\[
= \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} - 1 \right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H_{N,n-m}^{(r)}(x, y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!}
\]
Finally equating the coefficients of the like powers of \( t \), we get (3.13). \( \square \)

**Theorem 3.7.** The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials \( H_{N,n}^{(r)}(x, y) \) holds true:
\[
H_{N,n}^{(r)}(x + z, y + u) = \sum_{m=0}^{n} \binom{n}{m} H_{N,n-m}^{(r)}(x, y) H_m(z, u).
\] (3.14)

**Proof.** We replace \( x \) by \( x + z \) and \( y \) by \( y + u \) in (2.1), use (1.2) and rewrite the generating function as
\[
\frac{1}{1F_1(1; N + 1; t)} e^{(x+y)t^2} e^{xt + yt^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x, y) \frac{t^m}{m!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n}^{(r)}(x, y) H_m(x, y) \frac{t^{n+m}}{n!m!}
\]
Replacing \( n \) by \( n - m \) in above equation, we have
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n-m}^{(r)}(x, y) H_m(x, y) \frac{t^n}{(n-m)!m!}
\]
Comparing the coefficients of \( t \) on both sides, we get the result (3.14). \( \square \)
Theorem 3.8. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x, y)$ holds true:

$$H_{N,n}^{(r)}(y, x) = \sum_{k=0}^{[\frac{n}{2}]} B_{N,n-2k}^{(r)}(y) \frac{x^k}{(n-2k)!k!}. \tag{3.15}$$

Proof. We replace $x$ by $y$ and $y$ by $x$ in equation (2.1) to get

$$\sum_{n=0}^{\infty} H_{N,n}^{(r)}(y, x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{N,n-2k}^{(r)}(y) \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{x^k t^{2k}}{k!}.$$

Now replacing $n$ by $n - 2k$ and comparing the coefficients of $t$, we get the result (3.15). \qed

Theorem 3.9. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x, y)$ holds true:

$$H_{N,n}^{(r)}(z, u) = \sum_{m=0}^{n} \binom{n}{m} H_{m}(\alpha - x + z, \beta - y + u) H_{N,n-m}^{(r)}(x - \alpha, y - \beta), \tag{3.16}$$

and

$$H_{N,n}^{(r)}(z - \alpha - x, u - \beta + y) = \sum_{m=0}^{n} \binom{n}{m} H_{m}(z, u) H_{N,n-m}^{(r)}(x - \alpha, y - \beta). \tag{3.17}$$

Proof. By exploiting the generating function (2.1), we can write

$$\begin{align*}
\sum_{n=0}^{\infty} H_{N,n}^{(r)}(z, u) \frac{t^n}{n!} &= \frac{1}{1 F_1(1; N + 1; t)^r} e^{zt + ut^2} \\
&= e^{-(x-z)t-(y-u)\beta t^2} e^{(x-\alpha)\beta t+(y-\beta)\alpha t^2} \frac{1}{1 F_1(1; N + 1; t)^r} \\
&= e^{-(x-z)t-(y-u)\beta t^2} \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x - \alpha, y - \beta) \frac{t^n}{n!},
\end{align*}$$

which yields

$$\sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} = \sum_{m=0}^{\infty} H_{m}(\alpha - x + z, \beta - y + u) \frac{t^m}{m!} \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x - \alpha, y - \beta) \frac{t^n}{n!}.$$

Replacing $n$ by $n - m$ in above equation and comparing the coefficients of $t$, we obtain (3.16). On replacing $z$ by $z - \alpha - x$ and $u$ by $u - \beta + y$ in (3.16), we get (3.17). \qed

Corollary 3.10. On setting $z = u = 0$ in (3.16), we have the following result for higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x, y)$ holds true:

$$B_{N,n}^{(r)} = \sum_{m=0}^{n} \binom{n}{m} H_{m}(\alpha - x, \beta - y) H_{N,n-m}^{(r)}(x - \alpha, y - \beta).$$
Theorem 3.11. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x,y)$ holds true:

$$
\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}(y)}{(n-2m)!m!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{N,n-2m}^{(r)}(x,y)B_{N,m}^{(r)}}{(n-2m)!m!},
$$

(3.18)

and

$$
\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}(y)}{(n-2m)!m!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{N,n-2m}^{(r)}(x,y)B_{N,m}^{(r)}}{(n-2m)!m!}.
$$

(3.19)

Proof. Consider the definition of (2.1), we have

$$
\sum_{n=0}^{\infty} B_{N,n}^{(r)}(y) \frac{t^{2n}}{n!} = \frac{1}{1F_1(1; N + 1; t^2)^r} e^{xt+y^2},
$$

(3.20)

where $x$ is replaced by $y$ and $t$ is replaced by $t^2$ in (2.1). On multiplying (2.1) and (3.20), we have

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}(y) \frac{t^{2n}}{(n-2m)!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n}^{(r)}(x,y) \frac{t^{2n}}{(n-2m)!m!}.
$$

(3.21)

Using the Cauchy product and comparing the coefficients of $t$, we obtain (3.18). Another way of defining the right hand side of equation (3.21) is suggested by replacing $e^{xt+y^2}$ by its series representation

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}(y) \frac{t^{2n}}{(n-2m)!m!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x,y) \frac{t^{2n}}{(n-2m)!m!}.
$$

Using the Cauchy product and comparing the coefficients of $t$, we get (3.19).

\square

Corollary 3.12. For $y=0$ in Theorem 3.7, we have the following result for higher-order hypergeometric Bernoulli polynomials $H_{N,n}^{(r)}(x,y)$ holds true:

$$
\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}}{(n-2m)!m!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{N,n-2m}^{(r)}(x,0)B_{N,m}^{(r)}}{(n-2m)!m!},
$$

and

$$
\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{N,n-2m}^{(r)}(x)B_{N,m}^{(r)}}{(n-2m)!m!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{N,n-2m}^{(r)}(x,y)B_{N,m}^{(r)}}{(n-2m)!m!}.
$$

Theorem 3.13. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x,y)$ holds true:

$$
\sum_{m=0}^{n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(x^r - y^r)^r}{r!} \frac{B_{N,m}^{(k)}(x,y)H_{N,n}^{(r)}(x,y)}{y^{n+m}m!r!(n-m-2r)!} = \sum_{m=0}^{n} \frac{B_{N,m}^{(k)}(x,y)}{x^{n+m}y^{n-k}(n-k)!}.
$$

(3.22)
Theorem 3.14. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H^{(r)}_{N,n}(x, y)$ holds true:

$$H^{(r)}_{N,n}(w, u)H^{(r)}_{N,m}(W, U) = \sum_{s,k=0}^{m,n} \binom{n}{s} \binom{m}{k} H_s(w - x - u - y)H^{(r)}_{N,n-s}(x, y) \times H_k(W - X, U - Y)H^{(r)}_{N,m-k}(X, Y).$$

Proof. Consider the product of higher-order hypergeometric Hermite-Bernoulli polynomials, equation (2.1) in the following form

$$\frac{1}{1F_1(1; N + 1; t)^r}e^{xt+yt^2} \frac{1}{1F_1(1; N + 1; T)^r}e^{XT+YT^2}$$

Replacing $x$ by $w$, $y$ by $u$, $X$ by $W$ and $Y$ by $U$ in (3.26) and equating the resultant equation to itself, we find

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(r)}_{N,n}(w, u)H^{(r)}_{N,m}(W, U) \frac{t^n}{n!} \frac{T^m}{m!}$$

$$= \exp \left( (w - x)t + (u - y)t^2 \right) \exp \left( (W - X)T + (U - Y)T^2 \right)$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(r)}_{N,n}(x, y) \frac{t^n}{n!} \frac{T^m}{m!}$$

$$= \exp \left( (w - x)t + (u - y)t^2 \right) \exp \left( (W - X)T + (U - Y)T^2 \right) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(r)}_{N,n}(x, y) \frac{t^n}{n!} \frac{T^m}{m!}.$$
A New Class of Higher-order Hypergeometric Bernoulli...

\[ \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} H_s(w-x, u-y) H_{N,n}(x,y) \frac{t^{n+s}}{n!s!} \times H_k(W-X, U-Y) H_{N,m}(X,Y) \frac{T^{m+k}}{m!k!}. \]

Finally, replacing \( n \) by \( n-s \) and \( m \) by \( m-k \) in the r.h.s. of the above equation and then equating the coefficients of like powers of \( t \) and \( T \), we get assertion (3.25) of Theorem (3.8).

Remark 3.15. Replacing \( u \) by \( y \) and \( U \) by \( Y \) in assertion (3.25) of Theorem (3.9), we deduce the following consequence of Theorem (3.9).

Corollary 3.16. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials \( H_{N,n}(x,y) \) holds true:

\[ H_{N,n}^{(r)}(w,y) H_{N,m}(W,Y) = \sum_{s,k=0}^{m,n} \binom{n}{s} \binom{m}{k} (w-x)^s H_{N,n-s}(x,y) \times (W-X)^k H_{N,m-k}(X,Y). \]

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