Global existence of solutions of initial boundary value problem for nonlocal parabolic equation with nonlocal boundary condition

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1 | INTRODUCTION

We consider the following nonlinear nonlocal parabolic equation

$$u_t = \Delta u + a(x, t)u^r \int_\Omega u^p(y, t) \, dy - b(x, t)u^q, \quad x \in \Omega, \quad t > 0,$$

(1)

with nonlinear nonlocal boundary condition

$$u(x, t) = \int_\Omega k(x, y, t)u^l(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0,$$

(2)

and an initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

(3)

where $r$, $p$, $q$, $l$ are positive constants and $\Omega$ is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial \Omega$. 

We prove the global existence and blow-up of solutions of an initial boundary value problem for nonlinear nonlocal parabolic equation with nonlinear nonlocal boundary condition. Obtained results depend on the behavior of variable coefficients for large values of time.

KEYWORDS
blow-up, nonlinear parabolic equation, nonlocal boundary condition

MSC CLASSIFICATION
35B44; 35K20; 35K61
Throughout this paper, we suppose that \( a(x, t), \ b(x, t), \ k(x, y, t), \) and \( u_0(x) \) satisfy the following conditions:

\[
a(x, t), \ b(x, t) \in C_{\text{loc}}^\alpha (\tilde{\Omega} \times [0, \infty)), \ 0 < \alpha < 1, \ a(x, t) \geq 0, \ b(x, t) \geq 0;
\]

\[
k(x, y, t) \in C(\partial \Omega \times \tilde{\Omega} \times [0, \infty)), \ k(x, y, t) \geq 0;
\]

\[
u_0(x) \in C(\tilde{\Omega}), \ u_0(x) \geq 0, \ x \in \tilde{\Omega}, \ u_0(x) = \int_{\tilde{\Omega}} k(x, y, 0) u_0'(y) \, dy, \ x \in \partial \Omega.
\]

For the global existence and blow-up of solutions for parabolic equations with nonlocal boundary conditions, we refer to previous studies\(^{1-17}\) and the references therein. Initial-boundary value problems for nonlocal parabolic equations with nonlocal boundary conditions were considered in many papers also (see, for example, previous works\(^{18-24}\)). In particular, the blow-up problem for nonlocal parabolic equations with boundary condition (2) was investigated in literature.\(^{25-32}\) So, for example, Cui et al.\(^{25}\) studied (1)-(3) with \( b(x, t) \equiv 0, \ a(x, t) \equiv a(x) \) and \( k(x, y, t) \equiv k(x, y) \), and problem (1)-(3) with \( r = 0, \ a(x, t) \equiv 1, \ b(x, t) \equiv b > 0 \) and \( k(x, y, t) \equiv k(x, y) \) was considered in Mu et al.\(^{30}\) Gladkov and Guedda\(^{8}\) studied (1)-(3) with \( a(x, t) \equiv 0 \).

The existence of classical local solutions and the comparison principle for (1)-(3) were proved in Gladkov and Kavitova.\(^{33,34}\)

In this paper, we prove the global existence and blow-up of the solutions of (1)-(3). Obtained results depend on the behavior of the coefficients \( a(x, t), \ b(x, t) \) and \( k(x, y, t) \) as \( t \to \infty \).

This paper is organized as follows. The global existence of solutions for any initial data and blow-up in finite time of solutions for large initial data are proved in Section 2. In Section 3, we present finite time blow-up of all nontrivial solutions as well as the existence of global solutions for small initial data.

## 2 Global Existence and Blow-up of Solutions

Let \( Q_T = \Omega \times (0, T), \ S_T = \partial \Omega \times (0, T), \Gamma_T = S_T \cup \tilde{\Omega} \times \{0\}, \ T > 0. \)

**Definition 1.** We say that a nonnegative function \( u(x, t) \in C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T) \) is a supersolution of (1)-(3) in \( Q_T \) if

\[
u_t \geq \Delta u + a(x, t) u^r \int_{\Omega} u^q(y, t) \, dy - b(x, t) u^s, \ (x, t) \in Q_T, \tag{4}
\]

\[
u(x, t) \geq \int_{\Omega} k(x, y, t) u^l(y, t) \, dy, \ (x, t) \in S_T, \tag{5}
\]

\[
u(x, 0) \geq u_0(x), \ x \in \Omega, \tag{6}
\]

and \( u(x, t) \in C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T) \) is a subsolution of (1)-(3) in \( Q_T \) if \( u \geq 0 \) and it satisfies (4)-(6) in the reverse order. We say that \( u(x, t) \) is a solution of (1)-(3) in \( Q_T \) if \( u(x, t) \) is both a subsolution and a supersolution of (1)-(3) in \( Q_T \).

We will repeatedly use the following comparison principle (see Gladkov and Kavitova\(^{33,34}\)).

**Theorem 1.** Let \( \tilde{u}(x, t) \) and \( \bar{u}(x, t) \) be a subsolution and a supersolution of problem (1)-(3) in \( Q_T \), respectively. Suppose that \( \tilde{u}(x, t) > 0 \) or \( \bar{u}(x, t) > 0 \) in \( Q_T \cup \Gamma_T \) if \( \min(r, p, l) < 1 \). Then \( \bar{u}(x, t) \geq \tilde{u}(x, t) \) in \( Q_T \cup \Gamma_T \).

To prove the global existence of the solutions of (1)-(3), we suppose that

\[
b(x, t) > 0 \text{ for } x \in \tilde{\Omega} \text{ and } t \geq 0. \tag{7}
\]

**Theorem 2.** Let \( \max(r + p, l) \leq 1 \) or (7) hold and either \( l \leq 1, \ 1 < r + p < q \) or \( 1 < l < (q + 1)/2, \ \max(r + p, 2p + 1) < q. \) Then problem (1)-(3) has global solutions for any initial data.
Proof. Let $T$ be any positive constant and

$$M = \max \left( \sup_{Q_T} a(x, t), \sup_{\partial \Omega \times Q_T} k(x, y, t) \right). \quad (8)$$

In order to prove the global existence of solutions, we construct a suitable explicit supersolution of (1)-(3) in $Q_T$. Suppose at first that $\max(r + p, l) \leq 1$. Let $\lambda_1$ be the first eigenvalue of the following problem

$$\Delta \varphi(x) + \lambda \varphi(x) = 0, \quad x \in \Omega, \quad \varphi(x) = 0, \quad x \in \partial \Omega,$$

and $\varphi(x)$ be the corresponding nonnegative eigenfunction that is chosen to satisfy that for some $0 < \varepsilon < 1$,

$$M \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^r} \leq 1.$$

Then it is easy to check that

$$v(x, t) = \eta \exp(\mu t) / \varphi(x) + \varepsilon \quad (10)$$

is a supersolution of (1)-(3) in $Q_T$ if

$$\eta \geq \max \left( \sup_{\Omega} u_0(x) \sup_{\Omega} (\varphi(x) + \varepsilon), 1 \right),$$

$$\mu \geq \lambda_1 + \sup_{\Omega} 2|\nabla \varphi(x)|^2 / (\varphi(x) + \varepsilon]^2 + M \sup_{\Omega} (\varphi(x) + \varepsilon)^{1-r} \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^p}.$$

By Theorem 1, problem (1)-(3) has global solutions for any initial data.

From (7), we conclude that $b = \inf_{Q_T} b(x, t) > 0$. Let $l \leq 1, \quad r < p < q$. Then $v(x, t)$ in (10) is a supersolution of (1)-(3) in $Q_T$ if

$$\eta \geq \max \left( \sup_{\Omega} u_0(x) \sup_{\Omega} (\varphi(x) + \varepsilon), \left( \frac{M}{b} \sup_{\Omega} (\varphi(x) + \varepsilon)^{q-r} \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^p} \right)^{-1/r}, 1 \right),$$

$$\mu \geq \lambda_1 + \sup_{\Omega} 2|\nabla \varphi(x)|^2 / (\varphi(x) + \varepsilon)^2.$$

Let $1 < l < (q + 1)/2$, $\max(r + p, 2p + 1) < q$. To construct a supersolution, we use the change of variables in a neighborhood of $\partial \Omega$ as in Cortazar et al.\textsuperscript{35} Let $\bar{x} \in \partial \Omega$ and $\hat{n}(\bar{x})$ be the inner unit normal to $\partial \Omega$ at the point $\bar{x}$. Since $\partial \Omega$ is smooth, it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial \Omega \times [0, \delta] \to \mathbb{R}^n$ given by $\psi(\bar{x}, s) = \bar{x} + s \hat{n}(\bar{x})$ defines new coordinates $(\bar{x}, s)$ in a neighborhood of $\partial \Omega$ in $\bar{\Omega}$. A straightforward computation shows that, in these coordinates, $\Delta$ applied to a function $g(\bar{x}, s) = g(s)$, which is independent of the variable $\bar{x}$, evaluated at a point $(\bar{x}, s)$ is given by

$$\Delta g(\bar{x}, s) = \frac{\partial^2 \hat{g}}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \frac{\partial g}{\partial s}(\bar{x}, s), \quad (11)$$

where $H_j(\bar{x})$ for $j = 1, \ldots, n-1$, denotes the principal curvatures of $\partial \Omega$ at $\bar{x}$. For $0 \leq s \leq \delta$ and small $\delta$, we have

$$\left| \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \right| \leq \bar{c}. \quad (12)$$

Let $0 < \varepsilon < \omega < \min(\delta, 1), 2/(q - 1) - \beta < \min(1/p, 1/(l - 1)), 0 < \gamma < \beta/2, A = \sup_{\Omega} u_0(x)$. For points in $Q_{\delta, T} = \partial \Omega \times [0, \delta] \times [0, T]$ of coordinates $(\bar{x}, s, t)$ define

$$v(x, t) = v(\bar{x}, s, t) = \left[ (s + \varepsilon)^{-\gamma} - \omega^{-\gamma} \right]^\frac{\bar{c}}{\bar{c}} + A, \quad (13)$$
where \( s_{+} = \max(s, 0) \). For points in \( \overline{Q_T \setminus Q_{6,T}} \), we set \( v(x, t) = A \). We prove that \( v(x, t) \) is a supersolution of (1)-(3) in \( Q_T \).

It is not difficult to check that
\[
\frac{\partial v}{\partial s} \leq \beta \min \left( \left| [D(s)]^\alpha \right| \frac{2s+1}{\tau} \frac{(s+\epsilon)^{-\gamma} - \omega^{-\gamma}}{s + \epsilon)^{\beta+1}} \right),
\]
(14)

\[
\frac{\partial^2 v}{\partial s^2} \leq \beta(\beta + 1) \min \left( \left| [D(s)]^\alpha \right| \frac{2s+1}{\tau} \frac{(s+\epsilon)^{-\gamma} - \omega^{-\gamma}}{s + \epsilon)^{\beta+2}} \right),
\]
(15)

where
\[
D(s) = \frac{(s+\epsilon)^{-\gamma}}{(s+\epsilon)^{-\gamma} - \omega^{-\gamma}}.
\]

Then \( D'(s) > 0 \) and for any \( \bar{\epsilon} > 0 \),
\[
1 \leq D(s) \leq 1 + \bar{\epsilon}, \quad 0 < s \leq \bar{s},
\]
(16)

where \( \bar{s} = [\bar{\epsilon}/(1 + \bar{\epsilon})]^{1/\omega} - \epsilon, \quad \epsilon < [\bar{\epsilon}/(1 + \bar{\epsilon})]^{1/\omega} \). We denote
\[
Lv \equiv v_t - \Delta v - a(x, t)v^\gamma \int_\Omega v^p(y, t) \, dy + b(x, t)v^\beta
\]
(17)

and
\[
\bar{J} = \sup_{0<s<\bar{s}} \int_{\partial \Omega} |J(\bar{y}, s)| \, d\bar{y},
\]
(18)

where \( J(\bar{y}, s) \) is Jacobian of the change of variables in a neighborhood of \( \partial \Omega \). We use the inequality \((a + b)^p \leq 2^p(a^p + b^p), \quad a \geq 0, b \geq 0, p > 0 \) to estimate the integral in (17)
\[
\int_\Omega v^p(y, t) \, dy \leq 2^p \left( A^p|\Omega| + \int_0^{\bar{s}} \int_{\partial \Omega} J(\bar{y}, s) \left| (s+\epsilon)^{-\gamma} - \omega^{-\gamma} \right|^{\frac{2s+1}{\tau}} d\bar{y} \, ds \right)
\]
\[
\leq 2^p \left( A^p|\Omega| + \frac{1}{1 - \beta p} \right).
\]
(19)

Here, \(|\Omega|\) is Lebesgue measure of \( \Omega \). By (11)-(17) and (19), we can choose \( \bar{\epsilon} \) small and \( A \) large so that in \( Q_{5,T} \),
\[
Lv \geq b \left( \left| (s+\epsilon)^{-\gamma} - \omega^{-\gamma} \right|^{\frac{\beta}{\gamma}} + A \right)^q - \beta(\beta + 1)D(s) \left| [D(s)]^\alpha \right| \frac{2s+1}{\tau} \left[ (s+\epsilon)^{-\gamma} - \omega^{-\gamma} \right]^{\frac{2s+1}{\tau}}
\]
\[
- \beta \left| A^p|\Omega| + \frac{1}{1 - \beta p} \right| \geq 0.
\]

Let \( s \in [\bar{s}, \delta] \). From (11)-(15), we have
\[
|\Delta v| \leq \beta(\beta + 1) \omega^{-\gamma} \left( \frac{1 + \epsilon}{\epsilon} \right)^{\frac{\beta+1}{\gamma}} + \beta^2 \omega^{-\gamma} \left( \frac{1 + \epsilon}{\epsilon} \right)^{\frac{\beta+2}{\gamma}}
\]

and by (19), \( Lv \geq 0 \) for large \( A \). Obviously, in \( \overline{Q_T \setminus Q_{6,T}} \),
\[
Lv \geq -2^p MA^p \left( A^p|\Omega| + \frac{1}{1 - \beta p} \right) + bA^q \geq 0
\]

for large \( A \).
Now, we prove the following inequality

$$\nu((x_0,0),t) \geq \int_{\Omega} k(x,y,t)\nu(y,t) \, dy, \ (x,t) \in S_T$$

(20)

for a suitable choice of $\epsilon$. To do this, we use the change of variables in a neighborhood of $\partial \Omega$. Estimating the integral $I$ in (20), we get

$$I \leq 2^t M \int_0^{\omega-\epsilon} [(s+\epsilon)^{-\gamma} - \omega^{-\gamma}] \frac{d\omega}{\omega} \leq 2^t M \int_0^{\omega-\epsilon} [(s+\epsilon)^{-\gamma} - \omega^{-\gamma}] \, ds + 2^t M \omega^{1-\epsilon} \leq 2^t M \omega^{1-\epsilon}.$$

where

$$C(\epsilon) = \left\{ \begin{array}{ll} \epsilon^{-\beta l - 1}/(\beta l - 1), & \beta l > 1, \\ \omega^{1-\beta l}/(1 - \beta l), & \beta l < 1, \\ -\ln \epsilon, & \beta l = 1. \end{array} \right.$$ 

On the other hand, we have

$$\nu((x_0,0),t) = [\epsilon^{-\gamma} - \omega^{-\gamma}] \frac{d\omega}{\omega} + A.$$ 

Hence, (20) holds for small values of $\epsilon$, and by Theorem 1, $u(x,t) \leq \nu(x,t)$ in $\overline{Q_T}$. \hfill \Box

To prove finite time blow-up result, we need a lower bound for the solutions of (1)-(3) with large initial data.

**Lemma 1.** Let $u(x,t)$ be a solution of (1)-(3) in $\overline{Q_T}$. For any $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ and any positive constant $C$, there exists a positive constant $c$ such that if $u_0(x) \geq c$ in $\Omega_1$, then

$$u(x,t) \geq C \ \text{in} \ \overline{\Omega_0} \times [0, T].$$

(21)

**Proof.** Let $y(x,t)$ be a solution of the following problem

$$\begin{cases} 
  y_t - \Delta y, & x \in \Omega_1, \ 0 < t < T, \\
  y(x,t) = 0, & x \in \partial \Omega_1, \ 0 < t < T, \\
  y(x,0) = \chi(x), & x \in \Omega_1,
\end{cases}$$

(22)

where $\chi(x) \in C^\infty(\Omega_1)$, $\chi(x) = 1$ in $\Omega_0$ and $0 \leq \chi(x) \leq 1$. By the strong maximum principle,

$$\inf_{\Omega_0 \times (0,T)} y(x,t) > 0.$$ 

(23)

Suppose that $q \geq 1$. We put $m = \max \{ \sup_{Q_T} u(x,t), \sup_{Q_T} b(x,t) \}$ and define the function $\nu(x,t) = \exp(\rho t)u(x,t)$. For $\rho \geq m^q$, we have in $Q_T$

$$\nu_t - \Delta \nu = \exp(\rho t) \left( \rho u + a(x,t)u^r \int_{\Omega} u^q(y,t) \, dy - b(x,t)u^q \right) \geq \exp(\rho t) (u^q - b(x,t)u^q) \geq 0.$$ 

We assume $u_0(x) \geq c \chi(x)$ in $\Omega_1$, where the constant $c$ will be chosen below. Then by the comparison principle for (22), we get $\nu(x,t) \geq cy(x,t)$ in $\overline{\Omega_1} \times [0, T]$. Taking into account (23), we have (21) if $c = C \exp(\rho T)(\inf_{\Omega_0 \times (0,T)} y(x,t))^{-1}$.

Let $q < 1$. We set $w(x,t) = \exp(mt)(u(x,t) + 1)$. Since $u^q \leq u + 1$, we conclude that

$$w_t - \Delta w = \exp(mt) \left( m(u + 1) + a(x,t)u^r \int_{\Omega} u^q(y,t) \, dy - b(x,t)u^q \right) \geq 0$$

in $Q_T$. Arguing as in the previous case, we obtain

$$u(x,t) \geq c \exp(-mt) y(x,t) - 1 \ \text{in} \ \overline{\Omega_1} \times [0, T].$$
Choosing $c = (C + 1) \exp(mT)(\inf_{\Omega \times (0,T)} y(x, t))^{-1}$, we have (21).

Now we prove that problem (1)-(3) has finite time blow-up solutions if either $l > \max(1, (q + 1)/2)$ and

$$k(x, y, t) \geq k_0 > 0, \ x \in \partial \Omega, \ y \in \Omega, \ 0 < t < t_0,$$

for some positive constants $k_0$ and $t_0$ or $r + p > \max(q, 1)$ and

$$a(x, t) \geq a_0 > 0, \ x \in \Omega, \ 0 < t < t_1,$$

for some positive constants $a_0$ and $t_1$.

**Theorem 3.** There exist finite time blow-up solutions of (1)-(3) if either $l > \max(1, (q + 1)/2)$ and (24) holds or $r + p > \max(q, 1)$ and (25) holds.

**Proof.** We suppose at first that $l > \max(1, (q + 1)/2)$ and (24) holds. Let us consider the following problem

$$\begin{align*}
\frac{\partial w}{\partial t} &= \Delta w - b(x, t)w^q, \ x \in \Omega, \ t > 0, \\
w(x, t) &= \int_{\Omega} k(x, y, t)w^q(y, t) \ dy, \ x \in \partial \Omega, \ t > 0, \\
w(x, 0) &= u_0(x), \ x \in \Omega.
\end{align*}$$

(26)

As it is proved in Gladkov and Guedda, problem (26) has positive finite time blow-up solutions. We note that any solution of (26) is a subsolution of (1)-(3). Applying Theorem 1, we prove the theorem.

Now, we assume that $r + p > \max(q, 1)$ and (25) holds. We put $\bar{b} = \sup_{Q_1} b(x, t)$.

Let $r \geq q > 1$. We denote

$$J(t) = \exp(\lambda_1 t) \int_{\Omega} u(x, t)\phi(x) \ dx,$$

(27)

where $\phi(x)$ is the solution of (9) satisfying

$$\int_{\Omega} \phi(x) \ dx = 1.$$

(28)

Then using (1), (9), Green’s identity, and the inequality

$$\frac{\partial \phi(x)}{\partial n} \leq 0, \ x \in \partial \Omega,$$

(29)

where $n$ is the unit outward normal to $\partial \Omega$, we get for $t < t_1$,

$$J'(t) \geq \exp(\lambda_1 t) \int_{\Omega} \left( a(x, t)u^r \int_{\Omega} u^q(y, t) \ dy - b(x, t)u^q \right) \phi(x) \ dx$$

$$\geq \exp(\lambda_1 t) \int_{\Omega} u^p(y, t) \ dy \int_{\Omega} u^q(x, t)\phi(x) \ dx - \bar{b} \int_{\Omega} u^q(x, t)\phi(x) \ dx.$$ 

(30)

By Lemma 1,

$$a_0 \int_{\Omega} u^p(y, t) \ dy \left[ \int_{\Omega} u^q(x, t)\phi(x) \ dx \right]^{\frac{q-r}{r}} - \bar{b} \geq 1$$

(31)
for \( t < t_1 \) and large initial data. Taking into account (30), (31), and Hölder’s and Jensen’s inequalities, we have for \( t < t_1 \),

\[
J'(t) \geq \exp(\lambda_1 t) \int_{\Omega} u^q(x) \varphi(x) \, dx \left[ a_0 \int_{\Omega} u^p(y, t) \, dy \right] \left( \int_{\Omega} u^q(x) \varphi(x) \, dx \right)^{\frac{t - t_{1}}{q}} - \tilde{b}
\]

\[
\geq \exp(\lambda_1 (1 - q)t)J(t).
\]

Hence, \( J(t) \) blows up in finite time \( T (T < t_1) \) for large initial data.

Let \( r > 1 \geq q \). By Lemma 1,

\[
\int_{\Omega} u^p(y, t) \, dy \geq 1
\]

(32)

for \( t < t_1 \) and large initial data. Then using (30), (32), \( u^q \leq u + 1 \) and Jensen’s inequalities, we obtain for \( t < t_1 \),

\[
J'(t) \geq a_0 \exp(\lambda_1 (1 - r)t)J(t) - \tilde{b}J(t) - \tilde{b} \exp(\lambda_1 t)
\]

and again \( J(t) \) blows up in finite time \( T (T < t_1) \) for large initial data.

Let \( r < q \). Without loss of generality, we may assume that \( \Omega \) contains the origin. We introduce the designations

\[
G = \left\{ (x, t) : 0 \leq t < T_1, \ |x| \leq A\sqrt{T - t} \right\}, \quad G_r = G \cap \{ t = \tau \} \ (0 < \tau < T_1),
\]

where a positive constant \( A \) will be determined below, \( T_1 < T \) and \( T \) we choose in such a way that \( T < \min\{1, t_1\} \) and points \( x \) with \( |x| \leq A\sqrt{T} \) belong to \( \Omega \). Let \( u(x, t) \) be a positive in \( \Omega \) solution of (1)-(3) such that \( u_0(x) \geq (A^2 - |x|^2 / T)^+ \).

Obviously, \( u(x, t) \) is a supersolution of the auxiliary problem

\[
\begin{aligned}
&v_t = \Delta v + a(x, t)v^r \int_{\mathbb{G}_r} v^q(y, t) \, dy - b(x, t)v^q, \ (x, t) \in G, \\
v(x, t) = 0, \ |x| = A\sqrt{T - t}, \ 0 < t < T_1, \\
v(x, 0) = A^2 - |x|^2 / T, \ |x| < A\sqrt{T}.
\end{aligned}
\]

(33)

We construct a subsolution of (33) in the following form

\[
\overline{v}(x, t) = (T - t)^{-\gamma} V \left( \frac{|x|}{\sqrt{T - t}} \right),
\]

(34)

where \( V(\xi) = (A^2 - \xi^2)^+, \xi = |x| / \sqrt{T - t} \) and \( \gamma > 0 \) will be chosen below. It is easy to see \( \overline{v}(0, t) \to \infty \) as \( t \to T_1 \) and \( T_1 \to T \). We show that

\[
\Lambda \overline{v} \leq 0
\]

(35)

in \( G \), where

\[
\Lambda v = v_t - \Delta v - a(x, t)v^r \int_{\mathbb{G}_r} v^q(y, t) \, dy + b(x, t)v^q.
\]

Note that

\[
\int_{\mathbb{G}_r} V^p(\xi) \, d\xi = (T - t)^{n} \int_{|\xi| \leq A} (A^2 - |\xi|^2)^p_+ \, d\xi = C(A)(T - t)^{\frac{n}{2}}.
\]

(36)

By (36), we obtain

\[
\Lambda \overline{v} \leq \gamma(T - t)^{-\gamma - 1} V(\xi) - (T - t)^{-\gamma - 1} \xi^2 + 2n(T - t)^{-\gamma - 1}
\]

\[
- a_0 C(A)(T - t)^{\frac{n}{2} - \gamma - 1} V(\xi) + \tilde{b}(T - t)^{-\gamma} V^q(\xi)
\]

(37)

for points of \( G \). Further, we distinguish the two zones \( 0 \leq \xi < \theta A \) and \( \theta A \leq \xi < A \), where \( \theta \in (0, 1) \) will be chosen below.
For $\theta A \leq \xi < A$, we have
\[ V(\xi) \leq (1 - \theta^2)A^2. \] (38)

From (37) and (38), it follows that
\[ \Lambda u \leq \left( \gamma (1 - \theta^2)A^2 - \theta^2A^2 + 2n \right) (T - t)^{-\gamma - 1} - a_0 C(A)(T - t)^{5/2 - \gamma(r + p)}V^r(\xi) \]
\[ + \tilde{b}(T - t)^{-\gamma}V^q(\xi). \] (39)

We put
\[ A = 3\sqrt{n}, \theta^2 = \frac{\gamma + 1/2}{\gamma + 1} \] (40)
and estimate the first term on the right side of (39)
\[ \left( \gamma (1 - \theta^2)A^2 - \theta^2A^2 + 2n \right) (T - t)^{-\gamma - 1} = \frac{-5n}{2} (T - t)^{-\gamma - 1} < 0. \] (41)

By (38),
\[ a_0 C(A)(T - t)^{5/2 - \gamma(r + p)}V^r(\xi) \geq \tilde{b}(T - t)^{-\gamma}V^q(\xi) \] (42)
for small values of $T$ and $\gamma > n/[2(r + p - q)]$. From (39), (41), and (42), it follows (35) for $\xi \in [\theta A, A)$.

For $0 \leq \xi < \theta A$, we have
\[ V(\xi) \geq (1 - \theta^2)A^2 = \frac{9n}{2(\gamma + 1)}. \]

Then by (37), the inequality (35) still holds for $0 \leq \xi < \theta A$ if $T$ is small and
\[ \gamma > \max \left( \frac{n}{2(r + p - q)}, \frac{n + 2}{2(r + p - 1)} \right). \]

Applying the comparison principle for (33), we obtain $u(x, t) \geq v(x, t)$ in $G$. Hence, $u(x, t)$ blows up in finite time.

In the case $q \leq r \leq 1$, we have $\gamma q < \gamma + 1$. Then the function in (34) satisfies (35) for $0 \leq \xi < A$. Indeed, by virtue of (39)-(41), we have
\[ \Lambda u \leq \left( \frac{-5n}{2} (T - t)^{-\gamma - 1} + \tilde{b}(T - t)^{-\gamma}V^q(\xi) \right) - a_0 C(A)(T - t)^{5/2 - \gamma(r + p)}V^r(\xi) \leq 0 \]
for $\theta A \leq \xi < A$ and small values of $T$. For $0 \leq \xi < \theta A$, the inequality (35) holds if $\gamma > (n + 2)/[2(r + p - 1)]$ and $T$ is small. Arguing as in the previous case, we complete the proof. \qed

3 | BLOW-UP OF ALL NONTRIVIAL SOLUTIONS AND GLOBAL EXISTENCE OF SOLUTIONS FOR SMALL INITIAL DATA

In this section, we find the conditions that guarantee blow-up in finite time of all nontrivial solutions and prove the global existence of solutions for small initial data.

First, we show that for $q < \min(r + p, 1)$ under some conditions, problem (1)-(3) has nontrivial global solutions for any $a(x, t)$ and $k(x, y, t)$. Suppose that
\[ \inf_{\Omega} b(x, 0) > 0. \] (43)

**Theorem 4.** Let (43) hold and either $q < \min(r + p, 1)$, $l > 1$ or $q \leq r$, $(q + 1)/2 < l \leq 1$. Then problem (1)-(3) has global solutions for small initial data.

**Proof.** We put $b_0 = \inf_{\Omega} b(x, t)$ and choose $T$ so that $b_0 > 0$.

Suppose that $q < \min(r + p, 1)$, $l > 1$. A straightforward computation shows that for small $\beta, \epsilon$, and $u_0(x)$,
\[ g(t) = \beta [T - t]^{\frac{r}{q + 1} - 1} + \epsilon \]
is a supersolution of (1)-(3) in $Q_T$. Applying Theorem 1, we have $u(x, t) \leq g(t)$ in $Q_T$. Passing to the limit as $\varepsilon \to 0$, we obtain

$$u(x, t) \leq \beta(T - t)^{\frac{1}{\gamma}}_+, \ (x, t) \in Q_T.$$ 

Now, we put $u(x, t) \equiv 0$ for $t \geq T$.

For $r \geq q, (q + l)/2 < l \leq 1$ to construct a supersolution, we use the change of variables as in Theorem 2. For points of $Q_{\delta, l}$, define

$$v(x, t) = v((x, s), t) = (\delta - s - t)_+^l + \varepsilon,$$

where $\delta > 0$, $\varepsilon > 0$, $t_0 < \min(\delta, T)$, $2/(1 - q) < \gamma < 1/(1 - l)$ for $l < 1$ and $2/(1 - q) < \gamma$ for $l = 1$. In $Q_{\delta, l} \setminus Q_{\delta, \varepsilon}$, we put $v(x, t) = \varepsilon$. Then $v(x, t)$ is a supersolution of (1)-(3) in $Q_{\delta, l}$ if $u_0(x) \leq (\delta - s)_+^l$. Indeed, by (8), (11), (12), and (17)-(19) for small $\delta$ and $\varepsilon$, we have $Lv \geq 0$ in $Q_{\delta, \varepsilon}$ and

$$Lv \geq -\gamma(\delta - s - t)_+^{l-1} - \gamma(\gamma - 1)(\delta - s - t)_+^{l-2} - \gamma c(\delta - s - t)_+^{l-1}$$

$$-2\beta M((\delta - s - t)_+^l + \varepsilon)\gamma_{t}^p \left(\delta_{t}^{l+1} + \varepsilon_{t}^p |\Omega| \right) + b_0((\delta - s - t)_+^l + \varepsilon)^q \geq 0 \text{ in } Q_{\delta, l}.$$ 

Estimating the integral $I$ on the right side of (20), we obtain

$$I \leq M \left( \int_{\Omega} (\delta - s - t)_+^l \ dy + |\Omega| \varepsilon \right) \leq M \left( \int_{\Omega} (\delta - s - t)_+^{l+1} \gamma_{t}^l + |\Omega| \varepsilon \right).$$

On the other hand, we have $v((x, 0), t) = (\delta - t)_+^l + \varepsilon$ and (20) holds for

$$\delta < \left( \frac{\gamma_1 + 1}{2M} \right)^{\frac{1}{l+1}}, \ \varepsilon < \left\{ \frac{(\delta - t_0)^l}{2M|\Omega|} \right\}^{\frac{1}{l}}.$$

By Theorem 1, $u(x, t) \leq v(x, t)$ in $Q_{\delta, l}$, and passing to the limit as $t_0 \to \delta$ and $\varepsilon \to 0$, we deduce

$$u(x, t) \leq (\delta - s - t)_+^l \text{ in } Q_{\delta}.$$ 

We put $u(x, t) \equiv 0$ for $t \geq \delta$ and complete the proof. \hfill \Box

Now suppose that $q = 1$. We set*

$$\overline{a}(t) = \sup_{\Omega} a(x, t), \underline{a}(t) = \inf_{\Omega} a(x, t), \overline{b}(t) = \sup_{\Omega} b(x, t), \underline{b}(t) = \inf_{\Omega} b(x, t), \underline{k}(t) = \inf_{\partial\Omega \times \Omega} k(x, y, t). \quad (44)$$

Problem (1)-(3) has global solutions for small initial data if $q = 1$, $\min(r + p, l) > 1$ and

$$\int_{0}^{\infty} \overline{a}(t) \exp \left[ -(r + p - 1) \left( \sigma t + \int_{0}^{t} \overline{b}(r) \ dr \right) \right] dt < \infty, \ \sigma < \lambda_1, \quad (45)$$

$$\int_{\Omega} k(x, y, t) \ dy \leq K \exp \left( (l - 1) \left( \gamma t + \int_{0}^{t} \overline{b}(r) \ dr \right) \right), \ x \in \partial\Omega, \ t > 0, \ K > 0, \ \gamma < \lambda_1. \quad (46)$$

Conversely, problem (1)-(3) has no global nontrivial solutions if either $q = 1$, $\min(r, p) \geq 1$ and

$$\int_{0}^{\infty} \underline{a}(t) \exp \left[ -(r + p - 1) \left( \lambda_1 t + \int_{0}^{t} \overline{b}(r) \ dr \right) \right] dt = \infty, \quad (47)$$

*Correction added on 5 March 2020 after initial online publication. Equation 44 has been corrected in this version.
or \( q = 1, l > 1 \) and
\[
\int_0^\infty k(t) \exp \left[ - (l - 1) \left( \lambda_1 t + \int_0^l b(\tau) \, d\tau \right) \right] \, dt = \infty. \tag{48}
\]

**Theorem 5.** Let \( q = 1, \min(r + p, l) > 1 \) and (45), (46) hold. Then there exist global solutions of (1)-(3) for small initial data. If either \( q = 1, \min(r, p) \geq 1 \) and (47) holds or \( q = 1, l > 1 \) and (48) holds, then any nontrivial solution of (1)-(3) blows up in finite time.

**Proof.** Assume that \( T \) is any positive constant, \( q = 1, \min(r + p, l) > 1 \) and (45), (46) hold. We choose \( \lambda \) in such a way that
\[
\max(\sigma, \gamma) < \lambda < \lambda_1.
\]
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary such that \( \Omega \subset \subset \tilde{\Omega} \) and \( \lambda \) be the first eigenvalue of (9) in \( \tilde{\Omega} \). Then the corresponding eigenfunction \( \tilde{\varphi}(x) \) satisfies
\[
\sup_{\tilde{\Omega}} \tilde{\varphi}(x) < d
\]
for some \( d > 0 \). Choosing
\[
0 < \epsilon \leq \left( Kd^l \right)^{-\frac{1}{l}}, \quad \sup_{\Omega} \tilde{\varphi}(x) = d\epsilon,
\]
we have \( \inf_{\Omega} \tilde{\varphi}(x) > \epsilon \). We put \( N = \sup_{\Omega} \tilde{\varphi}^{-1}(x) \int_{\Omega} \tilde{\varphi}^p(y) \, dy \) and
\[
f(t) = \exp(-\lambda t) \left[ B - (r + p - 1)N \int_0^\infty \tilde{\varphi}(\tau) \exp \left[ -(r + p - 1) \left( \lambda \tau + \int_0^\tau b(s) \, ds \right) \right] \, d\tau \right]^\frac{1}{r+p-1},
\]
where
\[
B = 1 + (r + p - 1)N \int_0^\infty \tilde{\varphi}(\tau) \exp \left[ -(r + p - 1) \left( \lambda \tau + \int_0^\tau b(s) \, ds \right) \right] \, d\tau.
\]
It is easy to check that
\[
v(x, t) = \tilde{\varphi}(x) f(t) \exp \left( - \int_0^t b(\tau) \, d\tau \right)
\]
is a supersolution of (1)-(3) in \( Q_T \) for \( u_0(x) \leq B^{-\frac{1}{r+p-1}} \tilde{\varphi}(x) \). By Theorem 1, there exist global solutions of (1)-(3).

Now suppose that \( q = 1, \min(r, p) \geq 1 \) and (47) holds. Multiplying (1) by \( \varphi(x) \exp(\lambda_1 t) \), where \( \varphi(x) \) is defined in (9) and (28), and integrating the obtained equation over \( \Omega \), from (27), (29), Green’s identity, and Jensen’s inequality, we obtain
\[
J'(t) \geq \sup_{\Omega} \varphi(x) J(t) \exp[\lambda_1 (1 - r - p)t] J'' + b(t) J(t).
\]
Now, (47) guarantees blow-up of \( J(t) \) in finite time. The case \( q = 1, l > 1 \) is treated similarly.

**Remark 1.** The conclusion of Theorem 5 is not true if \( \sigma > \lambda_1 \) in (45), or \( \gamma = \lambda_1 \) in (46), or \( \lambda_1 \) is replaced by a smaller value in (47) or (48).

Further, we consider the case \( q > 1 \). To prove blow-up of all nontrivial solutions, we need an universal lower bound for solutions of (1)-(3). Assume that
\[
b(x, t) \leq \epsilon(t) \exp[\lambda_1 (q - 1)t], \quad x \in \Omega, \ t > 0, \tag{49}
\]
where
\[
\epsilon(t) \in C([0, \infty)), \ \epsilon(t) \geq 0, \ \int_0^\infty \epsilon(t) \, dt < \infty. \tag{50}
\]
Lemma 2. Let \( u(x, t) \) be a solution of (1)-(3) in \( Q_T, q > 1, u_0(x) \not\equiv 0 \) and (49), (50) hold. Then for any \( t_0 \in (0, T) \), there exists \( d > 0 \), which does not depend on \( T \), such that

\[
u(x, t) \geq d\varphi(x) \exp(-\lambda_1 t), \quad x \in \Omega, \quad t \in (t_0, T),
\]

where \( \varphi(x) \) is defined in (9) and (28).

Proof. For \( T_0 \in (0, T) \) denote \( m_0 = \max \left( \sup_{Q_{T_0}} \sup_{Q_{T_0}} u(x, t), \sup_{Q_{T_0}} b(x, t) \right) \). Let \( v(x, t) \) be a solution of the problem

\[
\begin{align*}
&v_t = \Delta v - b(x, t)v^q, \quad (x, t) \in Q_{T_0}, \\
&v(x, t) = 0, \quad (x, t) \in S_{T_0}, \\
&v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(52)

where \( v_0(x) \in C^\infty_0(\Omega), \quad 0 \leq v_0(x) \leq u_0(x) \) and \( v_0(x) \not\equiv 0 \). Obviously, \( u(x, t) \) is a supersolution of (52). By the comparison principle for (52), we obtain

\[
u(x, t) \geq v(x, t), \quad (x, t) \in Q_{T_0}.
\]

(53)

We put \( h(x, t) = \exp(\mu t)v(x, t) \), where \( \mu \geq m_0^q \). Then in \( Q_{T_0} \),

\[
h_t - \Delta h \geq \exp(\mu t)v(\mu - b(x, t)v^{q-1}) \geq 0.
\]

(54)

Since \( h(x, 0) = v_0(x) \) and \( v_0(x) \) is a nontrivial nonnegative function in \( \Omega \), by the strong maximum principle,

\[
h(x, t) > 0, \quad (x, t) \in Q_{T_0}.
\]

(55)

By virtue of Theorem 3.6 in Hu,\(^{36}\)

\[
\max_{\partial Q_{T_0}} \frac{\partial h(x, t_0)}{\partial n} < 0,
\]

(56)

where \( t_0 \in (0, T_0) \). From (54) and (55), it follows that

\[
v(x, t) > 0 \text{ in } Q_{T_0} \text{ and } \max_{\partial Q_{T_0}} \frac{\partial v(x, t_0)}{\partial n} < 0.
\]

(57)

Then there exists a positive constant \( d_0 \) such that

\[
v(x, t_0) \geq d_0\varphi(x) \exp(-\lambda_1 t_0), \quad x \in \Omega.
\]

(58)

By (53) and (56),

\[
u(x, t_0) \geq d_0\varphi(x) \exp(-\lambda_1 t_0), \quad x \in \Omega.
\]

(59)

A straightforward computation shows that for large \( f_0 \),

\[
v(x, t) = \varphi(x) \exp(-\lambda_1 t) \left\{ f_0 + (q - 1)[\sup_{\Omega} \varphi(x)]^{q-1} \int_{t_0}^{t} \varepsilon(\tau) d\tau \right\}^{-\frac{1}{q-1}}
\]

is a subsolution of (52) in \( Q_{T_0} \setminus Q_{t_0} \) with the initial datum \( v(x, t_0) = u(x, t_0) \). The application of the comparison principle for (52) completes the proof.

Next, we assume that

\[
\int_{\Omega} k(x, y, t) dy \leq A \exp(\sigma t), \quad x \in \partial \Omega, \quad t > 0, \quad A > 0, \quad \sigma < \lambda_1(l - 1)
\]

(60)

and

\[
b(x, t) \geq B a(x, t) \exp(-\omega t), \quad x \in \Omega, \quad t > 0, \quad B > 0, \quad \omega < \lambda_1(r + p - q)
\]

(61)
or $b(x, t)$ satisfies (49), (50), where
\[
\lim_{t \to \infty} \epsilon(t) = 0, \tag{59}
\]
and
\[
k(x, y, t) \geq D \exp[\lambda_1(t-1)t], \quad x \in \partial \Omega, \ y \in \Omega, \ D > 0 \tag{60}
\]
for large values of $t$.

**Theorem 6.** If $l > 1$, $1 < q < r + p$ and (57), (58) hold, then there exist global solutions of (1)-(3) for small initial data. If $l \geq q > 1$ and (49), (50), (59), and (60) hold, then any nontrivial solution of (1)-(3) blows up in finite time.

**Proof.** Let $l > 1$, $1 < q < r + p$ and (57), (58) hold. We choose $\lambda_1$ in the following way:
\[
\max (\omega/(r + p - q), \sigma/(l - 1)) < \lambda_1 < \lambda_1.
\]
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary such that $\Omega \subset \subset \tilde{\Omega}$ and $\lambda_1$ be the first eigenvalue of (9) in $\tilde{\Omega}$. The corresponding eigenfunction $\tilde{\varphi}(x)$ is chosen to satisfy that $\sup_{\Omega} \tilde{\varphi}(x) = 1$. Obviously, $\inf_{\Omega} \tilde{\varphi}(x) > d$ for some $d > 0$. Then $\nu(x, t) = \beta \exp(-\lambda_1 t)\tilde{\varphi}(x)$ is a supersolution of (1)-(3) in $\Omega_T$ for any $T > 0$ if
\[
\beta \leq \min \left( \frac{B}{\sup_{\tilde{\Omega}} \tilde{\varphi}^{-q}(x) \int_{\tilde{\Omega}} \tilde{\varphi}^q(y) dy}, \frac{d}{A} \right), \ u_0(x) \leq \beta \tilde{\varphi}(x).
\]
By Theorem 1, there exist global solutions of (1)-(3).
Let $u(x, t)$ be a nontrivial global solution of (1)-(3), $l \geq q > 1$ and (49), (50), (59), and (60) hold. Then by (2), (51), and (60), there exist positive constants $t_1$ and $d_1$ such that
\[
u(x, t) \geq d_1 \exp(-\lambda_1 t), \ x \in \partial \Omega, \ t \geq t_1. \tag{61}
\]
Let us consider the auxiliary problem
\[
\begin{cases}
\nu_t = \Delta \nu - b(x, t)\nu^q, \ x \in \Omega, \ t > t_2, \\
\nu(x, t) = u(x, t), \ x \in \partial \Omega, \ t > t_2, \\
\nu(x, t_2) = u(x, t_2), \ x \in \Omega,
\end{cases} \tag{62}
\]
where $t_2 \geq t_1$. Using (49), (59), and (61), we check that $\nu(x, t) = d_2 \exp(-\lambda_1 t)$ is a subsolution of (62) under a suitable choice of $t_2$ and $d_2 > 0$. The comparison principle for (62) gives
\[
u(x, t) \geq d_2 \exp(-\lambda_1 t), \ x \in \Omega, \ t \geq t_2. \tag{63}
\]
Let $\varphi(x)$ satisfy (9) and (28). Multiplying (1) by $\varphi(x) \exp(\lambda_1 t)$, integrating over $\Omega$ and using
\[
\int_{\partial \Omega} \frac{\partial \varphi(x)}{\partial n} \ ds = -\lambda_1,
\]
Green’s identity, Jensen’s inequality, and (27), (29), (49), (59)-(61), and (63), we obtain
\[
J'(t) \geq \int_{\Omega} \left( \lambda_1 [\sup_{\Omega} \varphi(x)]^{-1} D \exp[\lambda_1 lt] u^{r-q} - \epsilon(t) \exp[\lambda_1 qt] \right) u^q \varphi(x) \ dx \geq d_3 J_0(t)
\]
for some $d_3 > 0$ and large values of $t > 0$. Integrating the differential inequality, we prove the theorem. \qed
Remark 2. Theorem 6 does not hold if \( \sigma = \lambda_1(l - 1) \) in (57) or \( \lambda_1 \) is replaced by a smaller value in (60). Furthermore, we cannot replace by any positive constant in (49). Indeed, let \( a(x, t) \equiv 0, \ b(x, t) = b \exp[\lambda_1(q - 1)t], \ k(x, y, t) = k \exp[\lambda_1(l - 1)t], \) where \( b \) and \( k \) are positive constants. Then

\[
\overline{u}(x, t) = \left\{ \frac{\lambda_1}{b} \right\}^{\frac{1}{q-1}} \exp(-\lambda_1 t)
\]

is a supersolution of (1)-(3) if \( \min(q, l) > 1 \),

\[
k \leq \frac{1}{|\Omega|} \left\{ \frac{b}{\lambda_1} \right\}^{\frac{1}{q-1}} \text{ and } u_0(x) \leq \left\{ \frac{\lambda_1}{b} \right\}^{\frac{1}{q-1}}.
\]

By Theorem 1, there exist global solutions of (1)-(3).

To prove blow-up of all nontrivial solutions of (1)-(3) for \( \max(r, p) \geq q > 1 \), we assume that

\[
\overline{a}(t) = \gamma(t) \exp \left[ \lambda_1(r + p - q)t \right] \overline{b}(t),
\]

and

\[
\int_0^\infty \overline{a}(t) \exp \left[ -\lambda_1(r + p - 1)t \right] dt = \infty,
\]

where \( \lim_{t \to \infty} \gamma(t) = \infty, \overline{a}(t) \) and \( \overline{b}(t) \) are defined in (44).

**Theorem 7.** Let \( \max(r, p) \geq q > 1 \) and (49), (50), (64), and (65) hold. Then any nontrivial solution of (1)-(3) blows up in finite time.

**Proof.** Denote

\[
I(t) = \int_\Omega \left\{ \frac{1}{2} \overline{a}(t) u^r(x, t) \int_\Omega u^p(y, t) dy - \overline{b}(t) u^q(x, t) \right\} \varphi(x) dx,
\]

where \( \varphi(x) \) is defined in (9) and (28). Suppose at first that \( p \geq q \). By (51), (64), (66), and Hölder's inequality, it follows

\[
I(t) \geq \int_\Omega \left\{ \frac{1}{2} \overline{a}(t) u^r [\sup_\Omega \varphi(x)]^{-1} \int_\Omega u^p(y, t) \varphi(y) dy - \overline{b}(t) u^q \right\} \varphi(x) dx
\]

\[
\geq \overline{b}(t) \left[ \int_\Omega u^p \varphi dx \right] \gamma(t) \exp \left[ \lambda_1(p - q)t \right] \left( \int_\Omega \varphi^{p+1} dx \right)^{\frac{q}{p}} - 1 \right) \geq 0
\]

for large values of \( t \). Using Hölder's inequality again and (30), (51), and (67), we obtain

\[
J'(t) \geq \exp(\lambda_1 t) \int_\Omega \left\{ \overline{a}(t) u^r [\sup_\Omega \varphi(x)]^{-1} \int_\Omega u^p(y, t) \varphi(y) dy - \overline{b}(t) u^q \right\} \varphi(x) dx
\]
CONFLICT OF INTERESTS

This work does not have any conflicts of interest.

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CONFLICT OF INTERESTS

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REFERENCES

1. Amosov AA. Global solvability of a nonlinear nonstationary problem with a nonlocal boundary condition of radiative heat transfer type. Diff Equat. 2005;41:96-109.
2. Carl S, Lakshmikantham V. Generalized quasilinearization method for reaction-diffusion equation under nonlinear and nonlocal flux conditions. J Math Anal Appl. 2002;271:182-205.
3. Deng K. Comparison principle for some nonlocal problems. Quart Math. 1992;50:517-522.
4. Friedman A. Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions. Quart Appl Math. 1986;44:401-407.
5. Gao Y, Gao W. Existence and blow-up of solutions for a porous medium equation with nonlocal boundary condition. Appl Anal. 2011;90:799-809.
6. Gladkov A. Blow-up problem for semilinear heat equation with nonlinear nonlocal Neumann boundary condition. Commun Pure Appl Anal. 2017;16:2053-2068.
7. Gladkov A. Initial boundary value problem for a semilinear parabolic equation with absorption and nonlinear nonlocal boundary condition. Lith Math J. 2017;57:468-478.
8. Gladkov A, Guedda M. Blow-up problem for semilinear heat equation with absorption and a nonlocal boundary condition. Nonlinear Anal. 2011;74:4573-4580.
9. Gladkov A, Guedda M. Semilinear heat equation with absorption and a nonlocal boundary condition. Appl Anal. 2012;91:2267-2276.
10. Gladkov A, Kavitova T. Initial-boundary-value problem for a semilinear parabolic equation with nonlinear nonlocal boundary conditions. Ukr Math J. 2016;68:179-192.
11. Gladkov A, Kavitova T. Blow-up problem for semilinear heat equation with nonlinear nonlocal boundary condition. Appl Anal. 2016;95:1974-1988.
12. Gladkov A, Kim KI. Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition. J Math Anal Appl. 2008;338:264-273.
13. Kakumani BK, Tumuluri SK. Asymptotic behavior of the solution of a diffusion equation with nonlocal boundary conditions. Discrete Cont Dyn B. 2017;22:407-419.
14. Kozhanov AI. On the solvability of a boundary-value problem with a nonlocal boundary condition for linear parabolic equations. Vestnik Samara Gos Tekh Univ Ser Fiz,Mat Nauk. 2004;30:63-69.
15. Marras M, Vernier Piro S. Reaction-diffusion problems under non-local boundary conditions with blow-up solutions. J Inequal Appl. 2014;167:1-11.
16. Pao CV. Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions. J Comput Appl Math. 1998;88:225-238.
17. Wang Y, Mu C, Xiang Z. Blowup of solutions to a porous medium equation with nonlocal boundary condition. Appl Math Comput. 2007;192:579-585.
18. Cui Z, Yang Z. Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition. J Math Anal Appl. 2008;342:559-570.
19. Fang ZB, Zhang J. Influence of weight functions to a nonlinear p-Laplacian evolution equation with inner absorption and nonlocal boundary condition. J Inequal Appl. 2013;301:1-10.
20. Fang ZB, Zhang J. Global existence and blow-up properties of solutions for porous medium equation with nonlinear memory and weighted nonlocal boundary condition. Z Angew Math Phys. 2015;66:67-81.
21. Fang ZB, Zhang J, Yu SC. Roles of weight functions to a nonlocal porous medium equation with inner absorption and nonlocal boundary condition. Abstr Appl Anal. 2012;2012:1-16.
22. Lin Z, Liu Y. Uniform blowup profiles for diffusion equations with nonlocal source and nonlocal boundary. Acta Math Sci. 2004;24B:443-450.
23. Liu D. Blow-up for a degenerate and singular parabolic equation with nonlocal boundary condition. J Nonlinear Sci Appl. 2016;9:208-218.
24. Wang Y, Mu C, Xiang Z. Properties of positive solution for nonlocal reaction-diffusion equation with nonlocal boundary. Bound Value Probl. 2007;2007:1-12.
25. Cui Z, Yang Z, Zhang R. Blow-up of solutions for nonlinear parabolic equation with nonlocal source and nonlocal boundary condition. Appl Math Comput. 2013;224:1-8.
26. Fang ZB, Zhang J. Global and blow-up solutions for the nonlinear p-Laplacian evolution equation with weighted nonlinear nonlocal boundary condition. J Integral Equ Appl. 2014;26:171-196.
27. Li Y, Mi Y, Mu C. Properties of positive solutions for a nonlinear nonlinear diffusion equation with nonlocal nonlinear boundary condition. Acta Math Sci. 2014;34:748-758.
28. Liu D, Mu C. Blow-up analysis for a semilinear parabolic equation with nonlinear memory and nonlinear nonlinear boundary condition. Electron J Qual Theo. 2010;51:1-17.
29. Liu D, Mu C, Ahmed I. Blow-up for a semilinear parabolic equation with nonlinear memory and nonlocal nonlinear boundary. Taiwan J Math. 2013;17:1353-1370.
30. Mu C, Liu D, Zhou S. Properties of positive solutions for a nonlocal reaction-diffusion equation with nonlocal nonlinear boundary condition. *J Korean Math Soc*. 2010;47:1317-1328.

31. Zhong G, Tian L. Blow up problems for a degenerate parabolic equation with nonlocal source and nonlocal nonlinear boundary condition. *Bound Value Probl*. 2012;2012:1-14.

32. Zhou J, Yang D. Blowup for a degenerate and singular parabolic equation with nonlocal source and nonlocal boundary. *Appl Math Comput*. 2015;256:881-884.

33. Gladkov A, Kavitova T. On the initial-boundary value problem for a nonlocal parabolic equation with nonlocal boundary condition. *J Belarus State Univ Math Inform*. 2018;1:29-38.

34. Gladkov A, Kavitova T. Letter to the editors. *J Belarus State Univ Math Inform*. 2019;1:33-34.

35. Cortazar C, del Pino M, Elgueta M. On the short-time behaviour of the free boundary of a porous medium equation. *Duke J Math*. 1997;87:133-149.

36. Hu B. Blow-up theories for semilinear parabolic equations. *Lecture Note Math*. 2011;2018:1-127.

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