Parameterized Algorithms for Conflict-free Colorings of Graphs

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Abstract. In this paper, we study the conflict-free coloring of graphs induced by neighborhoods. A coloring of a graph is conflict-free if every vertex has a uniquely colored vertex in its neighborhood. The conflict-free coloring problem is to color the vertices of a graph using the minimum number of colors such that the coloring is conflict-free. We consider both closed neighborhoods, where the neighborhood of a vertex includes itself, and open neighborhoods, where a vertex does not included in its neighborhood. We study the parameterized complexity of conflict-free closed neighborhood coloring and conflict-free open neighborhood coloring problems. We show that both problems are fixed-parameter tractable (FPT) when parameterized by the cluster vertex deletion number of the input graph. This generalizes the result of Gargano et al. (2015) that conflict-free coloring is fixed-parameter tractable parameterized by the vertex cover number. Also, we show that both problems admit an additive constant approximation algorithm when parameterized by the distance to threshold graphs.

We also study the complexity of the problem on special graph classes. We show that both problems can be solved in polynomial time on cographs. For split graphs, we give a polynomial time algorithm for closed neighborhood conflict-free coloring problem, whereas we show that open neighborhood conflict-free coloring is NP-complete. We show that interval graphs can be conflict-free colored using at most four colors.

1 Introduction

A hypergraph is a pair \( H = (V, E) \) where \( V \) is the set of vertices and \( E \) is a set of non-empty subsets of \( V \) called hyperedges. A proper \( k \)-coloring of \( H \) is an assignment of colors from \( \{1, 2, \cdots, k\} \) to every vertex of \( H \) such that every hyperedge contains at least two vertices of distinct colors. The minimum number of colors required to color the vertices of \( H \) is called the chromatic number of \( H \) and is denoted as \( \chi(H) \). The conflict-free coloring is a special case of coloring and it is defined as follows.

Definition 1. Let \( H = (V, E) \) be a hypergraph, a coloring \( C_H \) is called conflict-free coloring of \( H \) if for every \( e \in E \) there exists a vertex \( u \in e \) such that for all \( v \in e, u \neq v \) we have \( C_H(u) \neq C_H(v) \). The minimum number of colors needed to conflict-free color the vertices of a hypergraph \( H \) is called the conflict-free chromatic number of \( H \).
The conflict-free coloring problem was introduced by Even et al. \cite{Even10} to study the frequency assignment problem for cellular networks. These networks contain two types of nodes, base stations and clients. Fixed frequencies are assigned to base stations to allow connections to clients. Each client scans for the available base stations in this neighborhood and connects to one of the available base station. Suppose if two base stations are available to a client, which are assigned the same frequency then mutual interference occurs and the connection between the client and base stations can become noisy. Our aim is to reduce the disturbances occur in connections between base stations and clients. The frequency assignment problem on cellular networks is an assignment of frequencies to base stations such that for each client there exists a base station of unique frequency within his region. The goal here is to minimize the number of assigned frequencies, since available frequencies are limited and expensive.

We can model this problem using the hypergraphs. The vertices of hypergraph correspond to the base stations and the set of base stations available for each client is represented by a hyperedge. The problem reduces to assigning frequencies to vertices of hypergraph such that each hyperedge contains a vertex of unique frequency. Conflict-free coloring is well studied for hypergraphs induced by geometric objects like, intervals \cite{Chilakalapudi09}, rectangles \cite{Gupta11}, unit disks \cite{Nair15} etc. This problem also has applications in areas like radio frequency identification and robotics, VLSI design and many other fields.

In this paper, we study the conflict-free coloring of hypergraphs induced by graph neighborhoods. Let $G = (V, E)$ be a graph, for a vertex $v \in V(G)$, $N(v)$ denotes the set consisting of all vertices which are adjacent to $v$, called open neighborhood of $v$. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$. The conflict-free open neighborhood (CF-ON) coloring of a graph $G$ is defined as the conflict-free coloring of the hypergraph $H$ with

$$V(H) = V(G) \quad \text{and} \quad E(H) = \{N(v) : v \in V(G)\}$$

Similarly conflict-free closed neighborhood (CF-CN) coloring problem can be defined.

Alternatively we can also define both CF-CN coloring and CF-ON coloring problems as follows. Given a graph $G$ and a coloring $C_G$, we say that a subset $U \subseteq V(G)$ has a unique color with respect to $C_G$ if there exists a color $c$ such that $|\{u \in U \mid C_G(u) = c\}| = 1$.

**Definition 2.**
1. A coloring $C_G$ of a graph $G$ is called conflict-free closed neighborhood (CF-CN) coloring if for every vertex $v \in V(G)$, the set $N[v]$ has a unique color.
2. A coloring $C_G$ of a graph $G$ is called conflict-free open neighborhood (CF-ON) coloring if for every vertex $v \in V(G)$, the set $N(v)$ has a unique color.

The minimum value $k$ for which there is a CF-ON (resp. CF-CN) coloring of $G$ with $k$ colors is called the CF-ON (resp. CF-CN) chromatic number of $G$ and is denoted as $\chi_{cf}(G)$ (resp. $\chi_{cf}[G]$).
Fig. 1 A schematic showing the relation between the various parameters. An arrow from parameter $a$ to $b$ indicates that $a$ is larger than $b$. Parameters marked with * are studied in this paper.

Related work. Gargano et al. [12] studied the complexity of conflict-free colorings induced by the graph neighborhoods and showed that the CF-CN 2-coloring and CF-ON $k$-coloring are NP-complete. In the parameterized setting, both conflict-free closed and open neighborhood colorings are fixed-parameter tractable (FPT), when parameterized by the vertex cover number or the neighborhood diversity of the graph [12]. Ashok et al. [2] showed that maximizing the number of conflict-free colored edges in hypergraphs is FPT when parameterized by the number of conflict-free edges in solution.

Our contributions. In this paper, we give parameterized algorithms for CF-CN coloring and CF-ON coloring problems with respect to various structural parameters. Gargano et al. [12] showed that both problems are FPT parameterized by vertex cover number. Both problems are FPT parameterized by tree-width, which follows from an application of Courcelle’s Theorem [6] and the fact that the CF-CN and CF-ON problems can be expressed by a monadic second order (MSO) formula.

In this paper, we focus is on distance-to-triviality [13, 15] parameters. They measure how far a graph is from some class of graphs for which the problem is tractable. Then, it is natural to parameterize by the distance of a general instance to a tractable class. The main advantage of studying structural parameters is, if a problem is tractable on a class of graphs $\mathcal{F}$, then it is natural to expect the problem might be tractable on a class of graphs which are close to $\mathcal{F}$. Our notion of distance to a graph class is the vertex deletion distance. More precisely, for a class $\mathcal{F}$ of graphs we say that $X$ is a $\mathcal{F}$-modulator of a graph $G$ if there is a
subset $X \subseteq V(G)$ such that $G \setminus X \in \mathcal{F}$. If the size of the smallest modulator to $\mathcal{F}$ is $k$, we also say that the distance of $G$ to the class $\mathcal{F}$ is $k$.

We study the parameterized complexity of the conflict-free coloring problems with respect to the distance from following graph class: cluster graphs (disjoint union of cliques) and threshold graphs. Studying the parameterized complexity of conflict-free coloring problem with respect to these parameters improves our understanding about the tractable parameterizations. For instance, the parameterization by the distance to cluster graphs directly generalizes vertex cover and is not comparable with tree-width (see Fig. [1]). In particular we obtain the following results.

- We show that both variants of conflict-free coloring problems are FPT when parameterized by size of the modulator to cluster graphs (cluster vertex deletion number).
- We show that CF-CN (resp. CF-ON) coloring problem admits an additive 1-approximation (resp. 2-approximation) when parameterized by the size of the modulator to threshold graphs.
- We show that on split graphs CF-CN coloring admits a polynomial time algorithm, where as CF-ON coloring is \text{NP}-complete.
- We give polynomial time algorithms for both CF-CN and CF-ON coloring problems on cographs. We show that interval graphs can be conflict-free colored using at most four colors.

## 2 Preliminaries

In this section, we introduce the notation and the terminology that we will need to describe our algorithms. Most of our notation is standard. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. All graphs we consider in this paper are undirected, connected, finite and simple. For a graph $G = (V,E)$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. An edge in $E$ between vertices $x$ and $y$ is denoted as $xy$ for simplicity. For a subset $X \subseteq V(G)$, the graph $G[X]$ denotes the subgraph of $G$ induced by vertices of $X$. Also, we abuse notation and use $G \setminus X$ to refer to the graph obtained from $G$ after removing the vertex set $X$. For a vertex $v \in V(G)$, $N(v)$ denotes the set of vertices adjacent to $v$ and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of $v$. A vertex is called universal vertex if it is adjacent to every other vertex of the graph.

**Graph classes.** We now define the graph classes which are considered in this paper. A graph is a *split graph* if its vertices can be partitioned into a clique and an independent set. Split graphs are $(2K_2, C_4, C_5)$-free. The class of $P_3$-free graphs are called *cographs*. A graph is a *threshold graph* if it can be constructed recursively by adding an isolated vertex or a universal vertex. A cluster graph is a disjoint union of complete graphs. Cluster graphs are $P_3$-free graphs. A graph $G$ is called an *interval graph* if there exists a set $\{I_v \mid v \in V(G)\}$ of real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $(u,v) \in E(G)$.

It is easy to see that a graph that is both split and cograph is a threshold graph. We denote threshold graph (or a split graph) with $G = (C, I)$ where
\( C \) and \( I \) denotes the partition of \( G \) into a clique and an independent set. For any two vertices \( x, y \) in a threshold graph \( G \) we have either \( N(x) \subseteq N[y] \) or \( N(y) \subseteq N[x] \). For a class of graphs \( \mathcal{F} \), the distance to \( \mathcal{F} \) of a graph \( G \) is the minimum number of vertices to be deleted from \( G \) to get a graph in \( \mathcal{F} \). An \( \mathcal{F} \)-modulator is the smallest possible vertex subset \( X \) for which \( G \setminus X \in \mathcal{F} \). The size of a \( \mathcal{F} \)-modulator is a natural measure of closeness to \( \mathcal{F} \).

**Parameterized Complexity.** A parameterized problem denoted as \((I, k) \subseteq \Sigma^* \times \mathbb{N}\), where \( \Sigma \) is fixed alphabet and \( k \) is called the parameter. We say that the problem \((I, k)\) is fixed parameter tractable with respect to parameter \( k \) if there exists an algorithm which solves the problem in time \( f(k)|I|^{O(1)} \), where \( f \) is a computable function. A kernel for a parameterized problem \( \Pi \) is an algorithm which transforms an instance \((I, k)\) of \( \Pi \) to an equivalent instance \((I', k')\) in polynomial time such that \( k' \leq g(k) \) and \( |I'| \leq f(k) \) for some computable functions \( f \) and \( g \). It is known that a parameterized problem is fixed parameter tractable if and only if it has a kernel. For a detailed survey of the methods used in parameterized complexity, we refer the reader to the texts [7,9].

Since cluster graphs are \( P_3 \)-free and threshold graphs are \((P_4, C_4, 2K_2)\)-free, modulators can be computed in FPT time. Therefore we assume that a modulator to these graph classes is given as a part of the input. The problems we consider in this paper are formally defined as follows: We only define for closed neighborhood case and open neighborhood case can be defined similarly.

| Conflict-free closed neighborhood coloring (CF-CNC) |
|-----------------------------------------------|
| **Input:** A graph \( G \), a subset \( X \subseteq V(G) \) such that \( G \setminus X \in \mathcal{F} \) and an integer \( k \). |
| **Parameter:** The size \( d := |X| \) of the modulator to \( \mathcal{F} \). |
| **Question:** Does there exists a coloring \( C_G : V(G) \rightarrow [k] \) such that for each \( v \in V(G) \) the set \( N[v] \) has a unique color with respect to \( C_G \)? |

### 3 Parameterized Algorithms

In this section, we give parameterized algorithms for conflict-free open/closed neighborhood coloring problems parameterized by cluster vertex deletion number and distance to threshold graphs.

#### 3.1 Parameterized by Cluster Vertex Deletion Number

The cluster vertex deletion number (or distance to cluster) of a graph \( G \) is the minimum number of vertices that have to be deleted from \( G \) to get a disjoint union of complete graphs or cluster graph. Cluster vertex deletion number is an intermediate parameter between vertex cover number and clique-width/rank-width [8]. In this section we show that both variants of conflict free coloring
problems are FPT parameterized by cluster vertex deletion number. The following lemma gives an upper bound on number of colors needed in a conflict-free coloring.

**Lemma 1.** Let $X \subseteq V(G)$ of size $d$ such that $G \setminus X$ is a cluster graph then

$$\chi_{cf}[G] \leq d + 2 \quad \text{and} \quad \chi_{cf}(G) \leq 2d + 2$$

**Proof.** A CF-CN $(d + 2)$-coloring $C_G$ of $G$ can be obtained as follows.

1. For each clique $C \in G \setminus X$, assign $C_G(u) = 0$ for some vertex $u \in C$ and for all $v \in C \setminus \{u\}$, assign color $C_G(v) = 1$
2. For each $x \in X$ assign $C_G(x)$ a color from the set $\{2, \cdots, d+1\}$ that is not already used by $C_G$.

According to this coloring, the unique color in $N[x]$ is $C_G(x)$ if $x \in X$ and $0$ if $x \in G \setminus X$.

A CF-ON $(2d + 2)$-coloring $C_G$ of $G$ can be obtained as follows.

1. For each vertex $u \in G \setminus X$, assign the color $C_G(u) = 0$
2. For each $x \in X$ assign $C_G(x) \in \{1, \cdots, d\}$ that is not already used by $C_G$.
3. For a vertex $x \in X$ if $C_G(N(x)) = \{0\}$, then recolor any one vertex in $N(x)$ with a color from $\{d+1, \cdots, 2d\}$ which is not already used by $C_G$.
4. For each clique $C \in G \setminus X$, if $C_G(C) = \{0\}$, then recolor an arbitrary vertex $u$ in $C$ with color $2d + 1$.

It is easy to see that above coloring is a CF-ON coloring of $G$. \qed

**Theorem 1.** Both CF-CN and CF-ON coloring problems are fixed-parameter tractable when parameterized by the cluster vertex deletion number of the input graph.

**Proof.** Let $G$ be a graph and $X \subseteq V(G)$ of size $d$ such that $G \setminus X$ is a cluster graph.

**CF-CN coloring :** First we show that CF-CN coloring problem admits a kernel of size $O(d^{2d+1})$. Without loss of generality we assume that $k < d + 2$ otherwise from Lemma 1 we can obtain a CF-CN coloring of $G$. We partition the vertices of each clique $C$ in $G \setminus X$ based on their neighborhoods in $X$. For every subset $Y \subseteq X$, $T^C_Y := \{x \in C \mid N(x) \cap X = Y\}$. Notice that in this way we can partition vertices of a clique $C$ into at most $2^d$ subsets (called types), one for each $Y \subseteq X$. We represent each clique $C$ in $G \setminus X$ with a vector $T^C$ of length $2^d$, where each entry of $T^C$ corresponds to a type and its value equals to the number of vertices in that type.

**Reduction 1** For a clique $C \in G \setminus X$, if a type $T^C_Y$ has more than $k+1$ vertices for some $Y \subseteq X$, then removing all vertices except $k+1$ from $T^C_Y$ does not change $\chi_{cf}[G]$. 

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Proof. Let $G_1$ be the graph obtained after applying the reduction rule 1 on $G$. Given a CF-CN coloring $C_{G_1}$ of reduced instance $G_1$, we extend it to a CF-CN coloring $C_G$ of $G$. Let $C$ be a clique in $G \setminus X$ such that the number of vertices in $T_Y^C$ are more than $k + 1$ for some $Y \subseteq X$. We color the deleted vertices in $T_Y^C$ as follows. Since there are at least $k + 1$ vertices in $T_Y^C$ after applying reduction rule, there exists two vertices $u$ and $v$ in $T_Y^C$ such that $C_G(v) = C_G(u)$. For each deleted vertex of $T_Y^C$, we assign the color $C_G(u)$. Since $C_G(u)$ is not unique in $N[v]$ for any $v \in V(G)$, it is easy to see that $C_G$ is a CF-CN coloring of $G$.

After applying Reduction rule 1, each clique in $G_1 \setminus X$ has at most $k + 1$ vertices in each type. Now we partition the cliques in $G_1 \setminus X$ based on their type vector of length $2d$. For every subset $S \subseteq \{0, 1, \ldots, k+1\}^{2d}$, $T^{G_1}_S := \{C \in G_1 \setminus X | T_C^C = S\}$. Notice that in this way we can partition cliques of $G_1 \setminus X$ into at most $(k + 2)^{2d}$ subsets (called mega types), one for each $S \subseteq \{0, 1, \ldots, k+1\}^{2d}$.

Let $\tau$ be an arbitrary but fixed ordering on $V(G_1)$. For a vertex $v$ in the modulator $X$ and a clique $C$ in $G_1 \setminus X$, we say that $C$ is critical for $v$ with respect to a CF-CN coloring if the first uniquely colored vertex in the neighborhood of $v$ belongs to $C$, where the notion of the first vertex is with respect to the ordering $\tau$.

**Reduction 2** For a subset $S \subseteq \{0, 1, \ldots, k+1\}^{2d}$, If a mega type $T^{G_1}_S$ has more than $d + 1$ cliques then removing all cliques except $d + 1$ cliques from $T^{G_1}_S$ does not change the $\chi_{cf}(G_1)$.

Proof. Let $G_2$ be the graph obtained after applying the reduction rule 2 on $G_1$. Given a CF-CN coloring $C_{G_2}$ of reduced instance, we extend it to a CF-CN coloring $C_{G_1}$ of $G_1$. Let $T^{G_1}_S$ be a mega type with more than $d + 1$ cliques for some $S \subseteq \{0, 1, \ldots, k+1\}^{2d}$. We color the deleted cliques in $T^{G_1}_S$ as follows. First, for every vertex $v \in X$, we mark a clique $C$ in $T^{G_1}_S$ if it is critical for $v$ with respect to $C_{G_2}$. Since there are $d + 1$ cliques in $T^{G_1}_S$ after applying the reduction rule 2, at the end of the procedure above, at least one clique is not marked. Let this clique be $C$.

Note that reusing the colors of clique $C$ to color deleted cliques does not violate the uniqueness of a color in $N[x]$ for all $x \in X$. So we recolor all deleted cliques according to the coloring of $C$.

After applying the above reduction rules on input graph $G$, it is easy to see that the size of the reduced instance is at most $O((k + 2)^{2d}(d + 1)(k + 1))$. As $k < d + 2$, we get a kernel of size at most $O(d^{2d+2})$.

**CF-ON coloring** : The proof of CF-ON coloring is similar to CF-CN coloring except some minor changes in reduction rules.

**Reduction 1** For a clique $C \in G \setminus X$, if a type $T_Y^C$ has more than $2k + 1$ vertices for some $Y \subseteq X$, then removing all vertices except $2k + 1$ from $T_Y^C$ does not change $\chi_{cf}(G)$. 

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Proof. Let $G_1$ be the graph obtained after applying the reduction rule 1 on $G$. Given a CF-ON coloring $C_{G_1}$ of reduced instance $G_1$, we extend it to a CF-ON coloring $C_G$ of $G$. Let $C$ be a clique in $G \setminus X$ such that the number of vertices in $T_Y^C$ are more than $2k + 1$ for some $Y \subseteq X$. Given a CF-ON coloring $C_G$ of reduced instance, we can color the deleted vertices in $T_Y^C$ as follows. Since there are at least $2k + 1$ vertices in $T_Y^C$ after applying reduction rule, there exists three vertices $u, v$ and $w$ in $T_Y^C$ such that $C_G(v) = C_G(u) = C_G(w)$. For each deleted vertex of $T_Y^C$ we assign the color $C_G(u)$. Since $C_G(u)$ is not unique in $N(x)$ for any $x \in G$, it is easy to see that $C_G$ is a CF-ON coloring of $G$.

Reduction 2 For a subset $S \subseteq \{0, 1, \ldots, k + 1\}^{2d}$, if mega type $T_S^{G_1}$ has more than $d + 1$ cliques then removing all cliques except $d + 1$ cliques does not change $\chi_{cf}(G_1)$.

After applying the above two reduction rules on input graph $G$ we obtain a kernel of size at most $O(d^{2d+2})$ for CF-ON coloring problem. \qed

3.2 Parameterized by Distance to Threshold Graphs

In this section we obtain an additive one (resp. two) approximation algorithm for CF-CN (resp. CF-ON) coloring of a graph parameterized by distance to threshold graphs in FPT-time.

Theorem 2. Let $G$ be a graph and $X \subseteq V(G)$ of size $d$ such that $G \setminus X$ is a threshold graph. Then we can find a CF-CN $k$-coloring of $G$ such that $\chi_{cf}[G] \leq k \leq \chi_{cf}[G] + 1$ in FPT-time.

Proof. Let $G' = G[X \cup N(X)]$ be a subgraph of $G$ induced by $X$ and its neighbors. Let $H$ be a graph obtained from $G'$ by removing all edges between vertices of $N(X)$. The set $X$ is a vertex cover of $H$ of size $d$. We divide the problem into two subproblems. First, partial CF-CN coloring of $H$: color the vertices of $H$ using the minimum number of colors such that for each $x \in X$, $N[x]$ has a unique color. Second, color the vertices of $V(G) \setminus V(H)$ such that $N[x]$ has a unique color for each $x \in G \setminus X$. It is easy to see that $\chi_{cf}[G]$ is at least the number of colors needed in a partial CF-CN coloring of $H$.

Now, we give a procedure to find a partial CF-CN coloring of $H$. We partition the vertices of $N(X)$ based on their neighborhoods in $X$ into at most $2^d$ subsets (called types). For every subset $Y \subseteq X$, $T_Y^H := \{x \in N(X) \mid N(x) \cap X = Y\}$. Since every vertex cover of a graph is also a cluster vertex deletion set, therefore by following the proof of Theorem 1 we get a partial CF-CN coloring of $H$.

Color all vertices of $(G \setminus X) \setminus (N(X))$ with any arbitrary color used in $H$. Since these vertices are non-neighbors of $X$, this step does not disturb the existence of unique color in $N[x]$ for all $x \in X$. Recall that every connected threshold graph has a universal vertex. Recolor the universal vertex $u \in G \setminus X$ with a new color which is not used in $G$. This new color is unique color in $N[x]$ for all $x \in G \setminus X$. This completes the proof of CF-CN coloring. \qed
Corollary 1. Let \( G \) be a graph and \( X \subseteq V(G) \) of size \( d \) such that \( G \setminus X \) is a threshold graph. Then we can find a CF-ON \( k \)-coloring of \( G \) such that \( \chi_{cf}(G) \leq k \leq \chi_{cf}(G) + 2 \) in FPT-time.

Proof. The proof of CF-ON coloring is similar to Theorem 2 except at the end we need to recolor two vertices, universal vertex and some arbitrary vertex of \( G \setminus X \) with two new colors. Hence we get a CF-ON \( k \)-coloring of \( G \) such that \( \chi_{cf}(G) \leq k \leq \chi_{cf}(G) + 2 \).

4 Special Graph Classes

In this section we study the complexity of CF-CN and CF-ON coloring problems for bipartite graphs, split graphs, interval graphs and cographs. For a bipartite graph \( G = (A, B, E) \), the CF-CN coloring can be obtained by coloring all vertices of \( A \) with color 0 and \( B \) with 1. The CF-ON coloring for bipartite graphs is \( \text{NP} \)-complete (follows from Theorem 3 in [12]). We show that CF-CN coloring can be solved in polynomial time on split graphs, whereas CF-ON is \( \text{NP} \)-complete. For cographs we prove that both CF-CN and CF-ON coloring problems can be solved in polynomial time. For interval graphs we give upper bounds on the number of colors needed for both CF-CN and CF-ON colorings.

4.1 Split Graphs

We begin by showing that CF-CN coloring can be solved in polynomial time on split graphs.

Lemma 2. Let \( G \) be a split graph with at least one edge, then

\[ 2 \leq \chi_{cf}[G] \leq 3 \]

Proof. Let \( G = (C, I) \) be a split graph. We assume that \( |C| \geq 3 \) and \( |I| \geq 3 \). Clearly \( 2 \leq \chi_{cf}[G] \) and we obtain a CF-CN 3-coloring \( C_G \) of \( G \) as follows. For a vertex \( v \in C \), assign \( C_G(v) = 0 \) and for all \( u \in C \setminus \{v\} \) assign \( C_G(u) = 1 \). Color all vertices of independent set \( I \) with 2. It is easy to see that \( C_G \) is a CF-CN coloring of \( G \). Therefore we have \( 2 \leq \chi_{cf}[G] \leq 3 \). \( \square \)

In the following lemma we give a characterization of split graphs that admit a CF-CN coloring using two colors. If there is a universal vertex in the graph then coloring it with one color and the rest with second color gives a CF-CN coloring. We assume that graph does not contain universal vertices.

Lemma 3. Let \( G = (C, I) \) be a split graph without universal vertex. \( \chi_{cf}[G] = 2 \) if and only if for all \( v \in C, |N(v) \cap I| = 1 \).

Proof. For the reverse direction, assume for all \( v \in C, |N(v) \cap I| = 1 \) then color all vertices of \( C \) with 0 and \( I \) with 1 respectively to get a CF-CN coloring of \( G \) with two colors.
For the forward direction, assume \( \chi_{cf}(G) = 2 \). First, we show that all vertices in the clique get the same color. Suppose not, then at least one vertex \( v \) in the clique has another color. If \( N(v) \cap I = \emptyset \) and let \( v' \in C \) such that \( u' \in N(v') \cap I \) then color of \( u' \) have to be different from \( v' \), then \( v' \) don not have unique color in its neighborhood. If \( u \in N(v) \cap I \) then since \( v \) is not universal vertex there is a vertex \( u' \) in \( I \) not adjacent to \( v \). Let \( u' \) be the neighbor of \( u' \) in \( C \). Then it is easy to see that either \( u' \) or \( v' \) have no uniquely colored vertex in any coloring.

If all vertices in the clique have the same color, then all vertices in \( I \) must have a different color as otherwise their closed neighborhood would be in conflict. If \( C \) and \( I \) are completely colored as described above, then every vertex in \( C \) must have exactly one neighbor in \( I \) to make its neighborhood conflict-free. This completes the proof of the lemma.

**Theorem 3.** The CF-CN coloring problem is polynomial time solvable on the class of split graphs.

**Proof.** Follows from Lemma \( \text{2} \) and Lemma \( \text{3} \).

Now we show that CF-ON coloring problem is NP-complete for split graphs.

**Theorem 4.** The CF-ON coloring is NP-complete on the class of split graphs.

**Proof.** We give a reduction from the well-known NP-complete Graph Coloring problem. Given an instance \( (G, k) \) of graph coloring, define a graph \( G' \) with \( V(G') = V(G) \cup \{x, y\} \) and \( E(G') = E(G) \cup \{xy\} \cup \{xv, yv \mid v \in V(G)\} \). i.e., \( x \) and \( y \) are universal vertices in graph \( G' \). Construct a split graph \( H = (C, I) \) as follows.

\[
V(H) = V(G') \cup \{I_{uv} \mid uw \in E(G')\} \quad \text{and} \quad E(H) = \{uv \mid u, v \in V(G')\} \cup \{uI_{uv}, vI_{uv} \mid uw \in E(G')\}
\]

It is easy to see that the graph \( H \) is a split graph with clique \( C = H[V(G')] \) and independent set \( I = H[V(H) \setminus V(G')] \). The construction of graph \( H \) can be done in polynomial time. We show that for any \( k \geq 3 \), the graph \( G \) is \( k \)-colorable if and only if the split graph \( H \) has a CF-ON \( (k + 2) \)-coloring.

Let \( C_G \) be a \( k \)-coloring of \( G \). We construct a CF-ON \( (k + 2) \)-coloring \( C_H \) of \( H \) as follows. \( C_H(v) = C_G(v) \) for all \( v \in V(G) \), \( C_H(x) = k + 1 \), \( C_H(y) = k + 2 \) and color all independent set vertices of \( H \) with color \( k \). Now we show that \( C_H \) is a CF-ON \( (k + 2) \)-coloring of \( H \). According to the coloring \( C_H \), the unique color in \( N(v) \) is \( k + 2 \) if \( v \in V(G) \cup \{x\} \) and \( k + 1 \) if \( v = y \). For all \( I_{uv} \in V(H) \), \( N(I_{uv}) = \{u, v\} \) and we have \( C_H(u) = C_G(u) \neq C_G(v) = C_H(v) \) for all \( uv \in E(G') \), therefore \( N(I_{uv}) \) has a unique color.

For the reverse direction, Let \( C_H \) be a CF-ON \( (k + 2) \)-coloring of \( H \), we show that \( C_H \) when restricted to vertices of \( G \) gives a \( k \)-coloring of \( G \). By the
construction, for any \( I_{uv} \in V(H) \), we have \( N(I_{uv}) = \{u, v\} \), therefore \( C_H(u) \neq C_H(v) \) which implies \( C_H \) when restricted to vertices of \( G' \) gives a proper coloring of \( G' \) with \( k + 2 \) colors.

Now we show that the color of \( x \) (resp \( y \)) is unique in \( V(G') \). For every \( v \in V(G) \) we have \( N(I_{vx}) = \{v, x\} \), implies \( C_H(x) \neq C_H(v) \) for all \( v \in V(G) \). Similarly \( C_H(y) \neq C_H(v) \) for all \( v \in V(G) \) and \( C_H(x) \neq C_H(y) \) as \( N(I_{xy}) = \{x, y\} \). Therefore \( C_H \) when restricted to \( V(G) \) gives a \( k \)-coloring of \( G \). \( \square \)

### 4.2 Interval Graphs

In this section, we give upper bounds on the number of colors needed for both CF-CN and CF-ON coloring problems for interval graphs. Let \( G \) be an interval graph and \( I \) be an interval representation of \( G \), i.e., there is a mapping from \( V(G) \) to closed intervals on the real line such that for any two vertices \( u \) and \( v \), \( uv \in E(G) \) if and only if \( I_u \cap I_v \neq \emptyset \). For any interval graph, there exists an interval representation with all endpoints distinct. Such a representation is called a distinguishing interval representation and it can be computed starting from an arbitrary interval representation of the graph. Interval graphs can be recognized in linear time and an interval representation can be obtained in linear time. Let \( l(I_u) \) and \( r(I_u) \) denote the left and right endpoints of the interval corresponding to the vertex \( u \) respectively. We say that an interval \( I \in \mathcal{I} \) is rightmost interval if \( r(J) \leq r(I) \) for all \( J \in \mathcal{I} \).

**Theorem 5.** Let \( G \) be an interval graph with at least one edge, then

\[
2 \leq \chi_{cf}[G] \leq 4
\]

**Proof.** It is easy to see that for any graph with at least one edge, \( 2 \leq \chi_{cf}[G] \). The algorithm that computes the CF-CN coloring of \( G \) with at most four colors is given in Algorithm 1. Now we present details on the proof of correctness of the algorithm.

Let \( C_G \) be a CF-CN coloring of \( G \) obtained using Algorithm 1. For any \( u_i, u_j \in V(G) \), if \( C_G(u_i) = C_G(u_j) = 1 \) then \( u_i u_j \notin E(G) \). Suppose assume that \( u_i u_j \in E(G) \) and \( l(I_{u_i}) \leq l(I_{u_j}) \). If \( I_{u_j} \) is the rightmost interval in \( \mathcal{I} \) then \( C_G(u_j) = 2 \) [lines 7-8 in Algorithm 1], which is a contradiction to \( C_G(u_2) = 1 \). If \( I_{u_i} \) is not the rightmost interval in \( \mathcal{I} \), then there exists \( u_i \) adjacent to \( u_j \) such that \( r(I_{v_k}) \leq r(I_{v_i}) \) for all \( v_k \in N(u_i) \). So we have \( r(I_{u_i}) \leq r(I_{u_j}) \), and the algorithm colors \( u_i \) with 0, which is a contradiction as \( C_G(u_j) = 1 \). If \( I_{u_i} \) lies completely inside \( I_{u_j} \), then \( C_G(u_i) = 0 \), which is again a contradiction to \( C_G(u_j) = 1 \). Therefore \( u_i u_j \notin E(G) \). Similarly, we can show that any two vertices of color 2 or color 3 are also not adjacent. This shows that any vertex \( v \) colored with a non-zero color by Algorithm 1 have a unique color in the set \( N[v] \).

Now we show that if a vertex \( v \) is assigned a color 0 by Algorithm 1 then \( v \) has at least one non-zero unique color in \( N[v] \). It is easy to see that if a vertex is colored 0 by Algorithm 1 then it is adjacent to at least one vertex of non-zero color. Since \( C_G(v) = 0 \), \( v \) is not the rightmost interval in \( \mathcal{I} \). Suppose assume that
Theorem 6. Let $G$ be an interval graph with at least two edges, then

$$2 \leq \chi_{cf}(G) \leq 4$$

**Proof.** The algorithm that finds a CF-ON coloring of interval graphs with four colors is given in Algorithm 2. The correctness proof of Algorithm 2 is similar to proof of Theorem 4.

$\square$
Algorithm 2: Conflict-free open neighborhood coloring of an interval graph with at most four colors

Input: A connected interval graph $G$ along with its distinguishing interval representation $I$.

Output: A conflict-free open neighborhood coloring $C_G$ of $G$ with four colors.

1. for each interval $I_{v_i} \in I$ by increasing left end point do
   2. if $v_i$ is not colored then
      3. if $I_{v_i}$ is the rightmost interval in $I$ then
         4. if there is no vertex $v_{i'}$ such that $I_{v_{i'}} \subseteq I_{v_i}$ then
            5. $C_G(v_i) = 1$;
         6. else
            7. select an arbitrary vertex $v_{i'}$ in $N(v_i)$ with $l(I_{v_{i'}}) \geq l(I_{v_i})$;
            8. $C_G(v_i) = 1, C_G(v_{i'}) = 2$ and $C_G(v_j) = 0$ for all $v_j \in N(v_i)$ with $l(I_{v_j}) \geq l(I_{v_i})$;
      9. end if
   10. else
      11. find a $v_l \in N(v_i)$ such that $r(I_{v_l}) \leq r(I_{v_i})$ for all $v_j \in N[v_i]$;
      12. if $I_{v_l}$ is the rightmost interval in $I$ then
         13. $C_G(v_i) = 1, C_G(v_l) = 2$ and $C_G(v_j) = 0$ for all $v_j \in N(v_i \cup v_l)$ with $l(I_{v_j}) \geq l(I_{v_l})$;
      14. else
         15. find a $v_{l'} \in N(v_i)$ such that $r(I_{v_{l'}}) \leq r(I_{v_l})$ for all $v_j \in N[v_l]$;
         16. $C_G(v_i) = 1, C_G(v_l) = 2, C_G(v_{l'}) = 3$ and $C_G(v_j) = 0$ for all $v_j \in N(v_i \cup v_l \cup v_{l'})$ with $r(I_{v_j}) \leq r(I_{v_{l'}})$ and $l(I_{v_j}) \geq l(I_{v_l})$;
      17. end if
   18. end if
   19. end if
20. end for

4.3 Cographs

Theorem 7. The CF-CN (resp. CF-ON) coloring problem can be solved in polynomial time on cographs.

Proof. We use modular decomposition technique to solve conflict free coloring problem on cographs. First, we define the notion of modular decomposition. A set $M \subseteq V(G)$ is called module of $G$ if all vertices of $M$ have the same set of neighbors in $V(G) \setminus M$. The trivial modules are $V(G)$, and $\{v\}$ for all $v$. A prime graph is a graph in which all modules are trivial. The modular decomposition of a graph is one of the decomposition techniques which was introduced by Gallai [11]. The modular decomposition of a graph $G$ is a rooted tree $M_G$ that has the following properties:

1. The leaves of $M_G$ are the vertices of $G$.
2. For an internal node $h$ of $M_G$, let $M(h)$ be the set of vertices of $G$ that are leaves of the subtree of $M_G$ rooted at $h$. ($M(h)$ forms a module in $G$).
3. For each internal node $h$ of $M_G$ there is a graph $G_h$ (representative graph) with $V(G_h) = \{h_1, h_2, \ldots, h_r\}$, where $h_1, h_2, \ldots, h_r$ are the children of $h$ in $M_G$ and for $1 \leq i < j \leq r$, $h_i$ and $h_j$ are adjacent in $G_h$ iff there are vertices $u \in M(h_i)$ and $v \in M(h_j)$ that are adjacent in $G$.

4. $G_h$ is either a clique, an independent set, or a prime graph and $h$ is labeled Series if $G_h$ is clique, Parallel if $G_h$ is an independent set, and Prime otherwise.

Modular decomposition tree of cographs has only parallel and series nodes. Let $G$ be a cograph whose modular decomposition tree is $M_G$. Without loss of generality we assume that the root $r$ of tree $M_G$ is a series node, otherwise, $G$ is not connected and we can color each connected component independently. Let the children of $r$ be $x$ and $y$. Further, let the cographs corresponding to the subtrees at $x$ and $y$ be $G_x$ and $G_y$. First we consider CF-CN coloring problem on cographs. If $G$ has a universal vertex then color it with one color and the rest with a second color. If $G$ does not have a universal vertex, then color one vertex of $G_x$ with 0, one vertex of $G_y$ with 1 and all other remaining vertices of $G$ with 2.

For the CF-ON coloring, if $G$ contains only two vertices then color one vertex with color 0 and other one with color 1. If $G$ contains at least three vertices then color one vertex of $G_x$ with 0, one vertex of $G_y$ with 1 and all other remaining vertices of $G$ with 2.

5 Conclusion

In this paper, we have studied the parameterized complexity of conflict-free coloring problem with respect to open/closed neighborhoods. We have shown that both closed and open neighborhood conflict-free colorings are FPT parameterized by cluster vertex deletion number and also showed that both variants of the problem admit an additive constant approximation algorithm when parameterized by the distance to threshold graphs. We studied the complexity of the problem on special classes of graphs. We show that both closed and open neighborhood conflict-free colorings are polynomially solvable on cographs. On split graphs, closed neighborhood coloring can be solved in polynomial time, whereas open neighborhood coloring is NP-complete. For interval graphs, we give upper bounds on the number of colors needed for both open/closed conflict-free coloring problems.

The following problems remain open.

- Does conflict-free open/closed coloring admit a polynomial kernel when parameterized by (a) the size of a vertex cover (b) distance to clique?
- Is the CF-CN problem is FPT when parameterized by distance to cographs, distance to split graphs?
- What is the parameterized complexity of both the problems when parameterized by clique-width/rank-width?
- What is the complexity of CF-CN and CF-ON coloring problems on interval graphs?
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