The Marshall-Olkin Flexible Weibull Extension Distribution

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Abstract

This paper introduces a new generalization of the flexible Weibull distribution with three parameters this model called the Marshall-Olkin flexible Weibull extension (MO-FWE) distribution which exhibits bathtub-shaped hazard rate. We studied its statistical properties include, quantile function skewness and kurtosis, the mode, \( r \)th moments and moment generating function and order statistics. We used the method of maximum likelihood for estimating the model parameters and the observed Fisher’s information matrix is derived. We illustrate the usefulness of the proposed model by applications to real data.

Keywords: Weibull distribution; flexible Weibull extension distribution; Marshall-Olkin flexible Weibull; maximum likelihood estimation.

1 Introduction

The Weibull distribution (WD) introduced by Weibull [20], is a popular distribution for modeling lifetime data where the hazard rate function is monotone. Recently appeared new classes of distributions were based on modifications of the Weibull distribution (WD) to provide a good fit to data set with bathtub hazard failure rate Xie and Lai [18]. Among of these, modified Weibull distribution (MWD), Lai et al. [8] and Sarhan and Zaindin [14], moreover the beta-Weibull distribution (BWD) has been derived by Famoye et al. [6], beta modified Weibull distribution (BMWD), Silva et al. [17] and Nadarajah et al. [12]. Recently, there are many generalization of the WD like a Kumaraswamy Weibull distribution (KWD), Cordeiro et al. [5], generalized modified Weibull distribution (GMWD), Carrasco et al. [4] and exponentiated modified Weibull extension distribution (EMWED), Sarhan and Apaloo [15].

The flexible Weibull distribution (FWED), Bebbington et al. [3] has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. The origin and other aspects of this distribution can be found in [3].

A random variable \( X \) is said to have the flexible Weibull Extension distribution with parameters \( \alpha, \beta > 0 \) if its probability density function (pdf) is given by

\[
g(x) = \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp \left\{ -e^{\alpha x - \frac{\beta}{x}} \right\}, \quad x > 0, \tag{1.1}
\]

while the cumulative distribution function (cdf) is given by

[1]
\[ G(x) = 1 - \exp\left\{ -e^{\alpha x - \frac{\beta}{x}} \right\}, \quad x > 0. \] (1.2)

The survival function is given by the equation
\[ S(x) = \exp\left\{ -e^{\alpha x - \frac{\beta}{x}} \right\}, \quad x > 0, \] (1.3)
the hazard rate function is
\[ h(x) = \left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}}, \] (1.4)
and the reversed hazard rate function is
\[ r(x) = \frac{\left( \alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} \exp\left\{ -e^{\alpha x - \frac{\beta}{x}} \right\}}{1 - \exp\left\{ -e^{\alpha x - \frac{\beta}{x}} \right\}}. \] (1.5)

In this paper we present a new generalization of the flexible Weibull extension distribution called the Marshall-Olkin flexible Weibull extension distribution (MO-FWED). By using the Marshall and Olkin’s method for adding a new parameter to an existing distribution, this new generalized referred to as the Marshall-Olkin flexible Weibull extension distribution. Marshall and Olkin proposed a new family of distributions called the Marshall-Olkin extended (MOE) family by adding a new parameter to the baseline distribution. They defined a new survival function \( S_{MO}(x) \) by introducing the additional shape parameter \( \theta \) such that \( \theta > 0 \) and \( \bar{\theta} = 1 - \theta \). Marshall and Olkin called the parameter \( \theta \) the tilt parameter and they interpreted \( \theta \) in terms of the behavior of the hazard rate function of \( S_{MO}(x) \). Their ratio is increasing in \( x \) for \( \theta \geq 1 \) and decreasing in \( x \) for \( 0 < \theta < 1 \). They consider for any arbitrary continuous distribution called baseline distribution having cumulative distribution function \( G(x, \varphi) \) with the related probability density function pdf \( g(x, \varphi) \), then the cumulative distribution function of the Marshall Olkin (MO) family of distribution is given by
\[ F_{MO}(x) = \frac{G(x, \varphi)}{1 - \theta S(x, \varphi)}, \quad -\infty < x < \infty, \] (1.6)
where \( \theta > 0 \) and \( \bar{\theta} = 1 - \theta \).

The probability density function corresponding to Eq.(1.6) becomes
\[ f_{MO}(x) = \frac{\theta g(x, \varphi)}{[1 - \theta S(x, \varphi)]^2}, \quad -\infty < x < \infty. \] (1.7)

The survival function, hazard rate function, reversed hazard rate function and cumulative hazard rate function of the Marshall-Olkin (MO) family of a probability distribution are given by
\[ S_{MO}(x) = \frac{\theta S(x, \varphi)}{1 - \theta S(x, \varphi)}, \] (1.8a)
\[ h_{MO}(x) = \frac{h(x, \varphi)}{1 - \theta S(x, \varphi)}, \] (1.8b)
\[ r_{MO}(x) = \frac{\theta r(x, \varphi)}{1 - \theta S(x, \varphi)}, \] (1.8c)
\[ H_{MO}(x) = -\log (S_{MO}(x)) = -\log \left( \frac{\theta S(x, \varphi)}{1 - \theta S(x, \varphi)} \right), \] (1.8d)
respectively, where $\theta > 0$, $\bar{\theta} = 1 - \theta$.

This paper is organized as follows, we define the cumulative distribution, probability density and hazard functions of the Marshall-Olkin flexible Weibull extension distribution (MO-FWED) in Section 2. In Sections 3 and 4, we introduced the statistical properties include, quantile function, the mode, skewness and kurtosis, $r$th moments and moment generating function. The distribution of the order statistics is expressed in Section 5. The maximum likelihood estimation of the parameters is determined in Section 6. Real data sets are analyzed in Section 7 and the results are compared with existing distributions. The conclusions are introduced in Section 8.

2 Marshall-Olkin Flexible Weibull Extension Distribution

In this section, we studied the three parameters Marshall-Olkin flexible Weibull extension distribution. Substituting from Eqs. (1.2) and (1.3) into Eq. (1.6), the cumulative distribution function of the Marshall-Olkin flexible Weibull extension distribution (MO-FWE) is given by

$$F(x; \alpha, \beta, \theta) = \frac{1 - e^{-e^{\alpha x - \frac{\beta}{x^\theta}}} - e^{\alpha x - \frac{\beta}{x^{1-\theta}}}}{1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}, \quad x > 0, \quad \alpha, \beta, \theta > 0. \quad (2.1)$$

Substituting from Eqs. (1.1) and (1.3) in Eq. (1.7), the pdf corresponding to Eq. (2.1) is given by

$$f(x; \alpha, \beta, \theta) = \frac{\theta (\alpha + \frac{\beta}{x^\theta}) e^{\alpha x - \frac{\beta}{x^\theta}} e^{-e^{\alpha x - \frac{\beta}{x^{1-\theta}}}}}{1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}^2, \quad x > 0, \quad \alpha, \beta, \theta > 0. \quad (2.2)$$

The survival function, hazard rate function, reversed-hazard rate function and cumulative hazard rate function of $X \sim \text{MO-FWED}(\alpha, \beta, \theta)$ are given by

$$S(x; \alpha, \beta, \theta) = \frac{\theta e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}{1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}, \quad (2.3a)$$

$$h(x; \alpha, \beta, \theta) = \frac{\left(\alpha + \frac{\beta}{x^\theta}\right) e^{\alpha x - \frac{\beta}{x^\theta}}}{1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}, \quad (2.3b)$$

$$r(x; \alpha, \beta, \theta) = \frac{\left(\alpha + \frac{\beta}{x^\theta}\right) e^{\alpha x - \frac{\beta}{x^\theta}} e^{-e^{\alpha x - \frac{\beta}{x^{1-\theta}}}}}{1 - e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}} \left[1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}ight], \quad (2.3c)$$

$$H(x; \alpha, \beta, \theta) = -\log \left(\frac{\theta e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}{1 - (1 - \theta)e^{-e^{\alpha x - \frac{\beta}{x^\theta}}}}\right), \quad (2.3d)$$

respectively, $x > 0$ and $\alpha, \beta, \theta > 0$.

Figures (1–6) display the cdf, pdf, survival function, hazard rate function, reversed hazard rate function and cumulative hazard rate function of the MO-FWED($\alpha, \beta, \theta$) for some parameter values.
Figure 1: The cdf of MO-FWED for different values of parameters.

Figure 2: The pdf of MO-FWED for different values of parameters.

Figure 3: The survival function of MO-FWED for different values of parameters.
Figure 4: The hazard rate function of MO-FWED for different values of parameters.

Figure 5: The reversed hazard rate function of MO-FWED for different values of parameters.

Figure 6: The cumulative hazard rate function of MO-FWED for different values of parameters.
3 Statistical Properties

In this section, we study the statistical properties for the MO-FWE distribution, specially quantile and simulation, median, skewness, kurtosis and moments.

3.1 Quantile and simulation

The quantile $x_q$ of the MO-FWE($\alpha, \beta, \theta$) random variable is given by

$$F(x_q; \alpha, \beta, \theta) = q, \quad 0 < q < 1.$$  \hspace{1cm} (3.1)

Using the cumulative distribution function of the MO-FWE distribution, from (2.1) in Eq. (3.1), we have

$$\alpha x_q^2 - k(q)x_q - \beta = 0,$$  \hspace{1cm} (3.2)

where

$$k(q) = \ln \left[ - \ln \left( \frac{1 - q}{1 - (1 - \theta)q} \right) \right].$$  \hspace{1cm} (3.3)

So, the simulation of the MO-FWE random variable is straightforward. Let $U$ be a uniform variate on the unit interval $(0,1)$, thus, by means of the inverse transformation method, we consider the random variable $X$ given by

$$X = \frac{k(u) \pm \sqrt{k(u)^2 + 4\alpha \beta}}{2\alpha}. \hspace{1cm} (3.4)$$

Since the median is 50% quantile then by setting $q = 0.5$ in Eq. (3.2), the median $M$ of the MO-FWED can be obtained the median.

3.2 The Mode of MO-FWE

In this subsection, we will derive the mode of the MO-FWED($\alpha, \beta, \theta$) by deriving its pdf with respect to $x$ and equal it to zero thus the mode of the MO-FWED($\alpha, \beta, \theta$) can be obtained as a nonnegative solution of the following nonlinear equation

$$\left[1 - (1 - \theta)e^{-\alpha x - \frac{\beta}{x}}\right] \left[-2\beta x + (\alpha x^2 + \beta)^2\right] - (\alpha x^2 + \beta)^2 \left[1 + (1 - \theta)e^{-\alpha x - \frac{\beta}{x}}\right] e^{\alpha x - \frac{\beta}{x}} = 0.$$  \hspace{1cm} (3.5)

From Figure 2, the pdf for MO-FWED has only one peak, It is a unimodal distribution, so the above equation has only one solution. It is not possible to get an explicit solution of Eq.3.5 in the general case. Numerical methods should be used such as bisection or fixed-point method to solve it.

3.3 The Skewness and Kurtosis

The analysis of the variability Skewness and Kurtosis on the shape parameters $\alpha, \beta, \theta$ can be investigated based on quantile measures. The short comings of the classical Kurtosis measure are well-known. The Bowely’s skewness based on quartiles is given by, Kenney and Keeping [7],

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}, \hspace{1cm} (3.6)$$
and the Moors Kurtosis based on quantiles, Moors [10],

\[ K_u = \frac{q(0.875) - q(0.625) - q(0.375) + q(0.125)}{q(0.75) - q(0.25)}, \] (3.7)

where \( q(x) \) represents quantile function.

### 3.4 The Moments

Now in this subsection, we derive the \( r \)th moment for MO-FWED. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

**Theorem 3.1.** If \( X \) has MO-FWED \((\alpha, \beta, \theta)\), then the \( r \)th moments of random variable \( X \), is given by

\[
\mu_r' = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}(k+1)^{j+1} \beta^j \theta^k (1-\theta)^k}{i! j! (j+1)^2 \alpha^{r-i-1}} \left[ \frac{\Gamma(r-i+1)}{\alpha(j+1)^2} + \beta \Gamma(r-i-1) \right].
\] (3.8)

**Proof.** The \( r \)th moment of the random variable \( X \) with probability density function \( f(x) \) is given by

\[
\mu_r' = \int_{0}^{\infty} x^r f(x; \alpha, \beta, \theta) dx.
\] (3.9)

Substituting from Eq. (2.2) into Eq. (3.9) we get

\[
\mu_r' = \int_{0}^{\infty} x^r \left( \alpha + \beta x^2 \right)^{\alpha x - \frac{\beta}{\alpha}} e^{-\frac{\beta}{\alpha}} \left[ 1 - (1-\theta) e^{-\frac{\beta}{\alpha}} \right]^{-2} dx.
\]

Since \( 0 < (1-\theta) e^{-\frac{\beta}{\alpha}} < 1 \) for \( x > 0 \) we can use the binomial series expansion of

\[
\left[ 1 - (1-\theta) e^{-\frac{\beta}{\alpha}} \right]^{-2} = \sum_{k=0}^{\infty} (k+1)(1-\theta)^k e^{-k e^{-\frac{\beta}{\alpha}}},
\]

then we get

\[
\mu_r' = \sum_{k=0}^{\infty} (k+1) \theta (1-\theta)^k \int_{0}^{\infty} x^r \left( \alpha + \beta x^2 \right)^{\alpha x - \frac{\beta}{\alpha}} e^{-\frac{(k+1)\alpha x - \beta}{\alpha}} dx,
\]

using series expansion of \( e^{-\frac{(k+1)\alpha x - \beta}{\alpha}} \),

\[
e^{-\frac{(k+1)\alpha x - \beta}{\alpha}} = \sum_{j=0}^{\infty} \frac{(-1)^j (k+1)^j}{j!} e^{j(\alpha x - \frac{\beta}{\alpha})},
\]

we obtain

\[
\mu_r' = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (k+1)^{j+1} \theta (1-\theta)^k}{j!} \int_{0}^{\infty} x^r \left( \alpha + \beta x^2 \right)^{\alpha x - \frac{\beta}{\alpha}} e^{-j(\alpha x - \frac{\beta}{\alpha})} dx,
\]
Finally, we obtain the \( r \)th moment of MO-FWE distribution in the form

\[
\mu'_r = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i j(k + 1)^{j+1} (j + 1)^i \beta^i \theta (1 - \theta)^k}{i! j!} \int_0^\infty x^{r-i} (\alpha + \beta x^{-2}) e^{(j+1)\alpha x} dx,
\]

we obtain

\[
\mu'_r = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i j(k + 1)^{j+1} (j + 1)^i \beta^i \theta (1 - \theta)^k}{i! j!} \int_0^\infty x^{r-i} (\alpha + \beta x^{-2}) e^{(j+1)\alpha x} dx,
\]

by using the definition of gamma function (Zwillinger [19], in the form,

\[
\Gamma(z) = x^z \int_0^\infty t^{z-1} e^{xt} dt, \quad z, x > 0.
\]

Finally, we obtain the \( r \)th moment of MO-FWE distribution in the form

\[
\mu'_r = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i j(k + 1)^{j+1} (j + 1)^i \beta^i \theta (1 - \theta)^k}{i! j!} \times \int_0^\infty x^{r-i} (\alpha + \beta x^{-2}) e^{(j+1)\alpha x} dx.
\]

This completes the proof.

\[\square\]

4 The Moment Generating Function

The moment generating function (mgf), \( M_X(t) \), of a random variable \( X \) provides the basis of an alternative route to analytic results compared with working directly with the pdf and cdf of \( X \).

**Theorem 4.1.** The moment generating function (mgf) of MO-FWE distribution is given by

\[
M_X(t) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^i j(k + 1)^{j+1} \beta^i \theta (1 - \theta)^k t^r}{r! i! j! (j + 1)^{r-2i-1} \alpha^{r-i-1}} \left[ \Gamma(r-i+1) + \beta \Gamma(r-i-1) \right].
\]

**Proof.** The moment generating function of the random variable \( X \) with probability density function \( f(x) \) is given by

\[
M_X(t) = \int_0^\infty e^{tx} f(x) dx,
\]
using series expansion of $e^{tx}$, we obtain

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{0}^{\infty} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r.' \quad (4.3)$$

Substituting from Eq. (3.8) into Eq. (4.3), we obtain the moment generating function of MO-FWED in the following form

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}(k+1)^{i+j+1}\beta(1-\theta)^k t^r}{r!j!(j+1)^{r-1}\alpha^{r-i-1}} \left[ \frac{\Gamma(r-i+1)}{\alpha(j+1)^2} + \beta \Gamma(r-i-1) \right].$$

This completes the proof. \hfill \Box

5 Order Statistics

In this section, we derive closed form expressions for the probability density function of the $r$th order statistic of the MO-FWED. Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which taken from a continuous population with cumulative distribution function $F(x; \varphi)$ and probability density function $f(x; \varphi)$, then the probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x; \varphi) = \frac{1}{B(r, n-r+1)} [F(x; \varphi)]^{r-1} [1-F(x; \varphi)]^{n-r} f(x; \varphi), \quad (5.1)$$

where $f(x; \varphi), F(x; \varphi)$ are the pdf and cdf of MO-FWED $(\alpha, \beta, \theta)$ given by Eqs. (2.2) and (2.1) respectively, $\varphi = (\alpha, \beta, \theta)$ and $B(., .)$ is the beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$, and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since $0 < F(x; \varphi) < 1$ for $x > 0$, we can use the binomial expansion of $[1-F(x; \varphi)]^{n-r}$ given as follows.

$$[1-F(x; \varphi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^i. \quad (5.2)$$

Substituting from Eq. (5.2) into Eq. (5.1), we obtain

$$f_{r:n}(x; \varphi) = \frac{1}{B(r, n-r+1)} f(x; \varphi) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^{i+r-1} \quad (5.3)$$

Substituting from Eqs. (2.1) and (3.2) into Eq. (5.3), we obtain pdf of $r$th order statistics for MOFWED$(\alpha, \beta, \theta)$. Relation (5.3) shows that $f_{r:n}(x; \varphi)$ is the weighted average of the Marshall Olkin flexible Weibull extension MO-FWED with different shape parameters.

6 Parameters Estimation

In this section, point and interval estimation of the unknown parameters of the MO-FWED are derived by using the method of maximum likelihood based on a complete sample.
6.1 Maximum likelihood estimation

Let \(x_1, x_2, \cdots, x_n\) denote a random sample of complete data from the MO-FWED. The likelihood function is given as

\[
L = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta),
\]

(6.1)

substituting from Eq. (2.2) into Eq. (6.1), we have

\[
L = \prod_{i=1}^{n} \theta (\alpha + \frac{\beta}{x_i^2}) e^{\alpha x_i - \frac{\beta}{x_i}} e^{-e^{\alpha x_i - \frac{\beta}{x_i}}} \left[1 - (1 - \theta) e^{-e^{\alpha x_i - \frac{\beta}{x_i}}} \right]^{-2}.
\]

The log-likelihood function is

\[
\mathcal{L} = n \ln(\theta) + \sum_{i=1}^{n} \ln(\alpha + \frac{\beta}{x_i}) + \sum_{i=1}^{n} \left(\alpha x_i - \frac{\beta}{x_i}\right) - \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - 2 \sum_{i=1}^{n} \ln \left[1 - (1 - \theta) e^{-e^{\alpha x_i - \frac{\beta}{x_i}}} \right].
\]

(6.2)

The maximum likelihood estimation of the parameters are obtained by differentiating the log-likelihood function, \(\mathcal{L}\), with respect to the parameters \(\alpha, \beta\) and \(\theta\) and setting the result to zero, we have the following normal equations.

\[
\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{x_i^2}{\beta + \alpha x_i^2} + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} + 2 \sum_{i=1}^{n} \left(1 - \theta\right)x_i e^{\alpha x_i - \frac{\beta}{x_i}} = 0,
\]

(6.3)

\[
\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\beta + \alpha x_i^2} - \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} - 2 \sum_{i=1}^{n} x_i \left[\frac{(1 - \theta)e^{\alpha x_i - \frac{\beta}{x_i}}}{1 - \theta - e^{\alpha x_i - \frac{\beta}{x_i}}}\right] = 0,
\]

(6.4)

\[
\frac{\partial \mathcal{L}}{\partial \theta} = \frac{n}{\beta} + 2 \sum_{i=1}^{n} \frac{1}{1 - \theta - e^{\alpha x_i - \frac{\beta}{x_i}}} = 0.
\]

(6.5)

The MLEs can be obtained by solving the nonlinear equations previous, (6.3)–(6.5), numerically for \(\alpha, \beta\) and \(\theta\).

6.2 Asymptotic confidence bounds

In this section, we derive the asymptotic confidence intervals when \(\alpha, \beta > 0\) and \(\theta > 0\) as the MLEs of the unknown parameters \(\alpha, \beta > 0\) and \(\theta > 0\) can not be obtained in closed forms, by using variance covariance matrix \(I^{-1}\) see Lawless [9], where \(I^{-1}\) is the inverse of the observed information matrix which defined as follows.

\[
I^{-1} = \begin{pmatrix}
\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \theta} \\
-\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} & \frac{\partial^2 \mathcal{L}}{\partial \beta^2} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \theta} \\
-\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \theta} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \theta} & \frac{\partial^2 \mathcal{L}}{\partial \theta^2}
\end{pmatrix}^{-1}
= \begin{pmatrix}
\text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\beta}) & \text{cov}(\hat{\alpha}, \hat{\theta}) \\
\text{cov}(\hat{\beta}, \hat{\alpha}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta}, \hat{\theta}) \\
\text{cov}(\hat{\theta}, \hat{\alpha}) & \text{cov}(\hat{\theta}, \hat{\beta}) & \text{var}(\hat{\theta})
\end{pmatrix}.
\]

(6.6)
where

\[ \frac{\partial^2 L}{\partial \alpha^2} = -n \sum_{i=1}^{n} \frac{x_i^4}{(\beta + \alpha x_i^2)^2} - n \sum_{i=1}^{n} x_i^2 e^{\alpha x_i - \frac{\beta}{x_i}} + 2(1 - \theta) \sum_{i=1}^{n} x_i^2 B_i \] (6.7)

\[ \frac{\partial^2 L}{\partial \alpha \partial \beta} = -n \sum_{i=1}^{n} \frac{x_i^2}{(\beta + \alpha x_i^2)^2} + n \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - 2(1 - \theta) \sum_{i=1}^{n} B_i \] (6.8)

\[ \frac{\partial^2 L}{\partial \alpha \partial \theta} = 2n \sum_{i=1}^{n} x_i A_i e^{\alpha x_i - \frac{\beta}{x_i}} \] (6.9)

\[ \frac{\partial^2 L}{\partial \beta^2} = -n \sum_{i=1}^{n} \frac{1}{(\beta + \alpha x_i^2)^2} - n \sum_{i=1}^{n} \frac{1}{x_i^2} e^{\alpha x_i - \frac{\beta}{x_i}} + 2(1 - \theta) \sum_{i=1}^{n} B_i \] (6.10)

\[ \frac{\partial^2 L}{\partial \beta \partial \theta} = -2n \sum_{i=1}^{n} \frac{1}{x_i} A_i e^{\alpha x_i - \frac{\beta}{x_i}} \] (6.11)

\[ \frac{\partial^2 L}{\partial \theta^2} = -n \frac{\theta}{\theta^2} - 2n \sum_{i=1}^{n} \left[ 1 - \theta - e^{\alpha x_i - \frac{\beta}{x_i}} \right]^{-2} \] (6.12)

where

\[ A_i = e^{\alpha x_i - \frac{\beta}{x_i}} \left[ 1 - \theta - e^{\alpha x_i - \frac{\beta}{x_i}} \right]^{-2} \] and

\[ B_i = A_i \left[ 1 - \theta - e^{\alpha x_i - \frac{\beta}{x_i}} (1 - e^{\alpha x_i - \frac{\beta}{x_i}}) \right]. \]

We can derive the \((1 - \delta)100\%\) confidence intervals of the parameters \(\alpha, \beta\) and \(\theta\), by using variance matrix as in the following forms

\[ \hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\beta})}, \quad \hat{\theta} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\theta})}, \]

where \(Z_{\frac{\delta}{2}}\) is the upper \((\frac{\delta}{2})\)-th percentile of the standard normal distribution.

7 Application

In this section, we present the analysis of two examples for a real data sets using the MO-FWE \((\alpha, \beta, \theta)\) model and compare it with the other fitted models like a flexible Weibull extension (FWE), Weibull (W), linear failure rate (LFR), exponentiated Weibull (EW), generalized linear failure rate (GLFR), exponentiated flexible Weibull (EFW), modified Weibull (MW), reduced additive Weibull (RAW) and Extended Weibull (EW) distributions using Kolmogorov Smirnov (K-S) statistic, as well as Akaike Information Criterion (AIC), \[2\], Akaike Information Citerion with correction (AICC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC) \[16\] values.

Example 7.1. Consider the data have been obtained from Aarset \[11\], and widely reported in many literatures. It represents the lifetimes of 50 devices, and also, possess a bathtub-shaped failure rate property, Table 1.

| Table 1: Lifetime of 50 devices, see Aarset \[11\]. |
|-----------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
|                | 0.1          | 0.2          | 1            | 1            | 1            | 1            | 1            | 1            | 2            |
|                | 3            | 6            | 7            | 11           | 12           | 18           | 18           | 18           | 18           | 18           | 21           | 32           |
|                | 36           | 40           | 45           | 46           | 47           | 50           | 55           | 60           | 63           | 63           | 67           | 67           | 67           | 72           | 75           | 79           | 82           | 82           | 83           |
|                | 84           | 84           | 85           | 85           | 85           | 85           | 85           | 85           | 86           | 86           | 87           | 87           | 87           | 87           | 87           | 87           | 87           | 87           | 87           | 87           |

We can derive the \((1 - \delta)100\%\) confidence intervals of the parameters \(\alpha, \beta\) and \(\theta\), by using variance matrix as in the following forms

\[ \hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\beta})}, \quad \hat{\theta} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\theta})}, \]

where \(Z_{\frac{\delta}{2}}\) is the upper \((\frac{\delta}{2})\)-th percentile of the standard normal distribution.
Table 2 gives MLEs of parameters of the MO-FWED and K–S Statistics. The values of the log-likelihood functions, AIC, AICC, BIC and HQIC are presented in Table 3.

| Model                          | MLE of the parameters | K-S    | P-value       |
|--------------------------------|-----------------------|--------|---------------|
| Flexible Weibull               | \( \hat{\alpha} = 0.0122, \hat{\beta} = 0.7002 \) | 0.4386 | \( 4.29 \times 10^{-9} \) |
| Weibull                        | \( \hat{\alpha} = 0.0223, \hat{\beta} = 0.949 \) | 0.2397 | 0.0052        |
| Linear Failure rate            | \( \hat{a} = 0.014, \hat{b} = 2.4 \times 10^{-4} \) | 0.1955 | 0.0370        |
| Exponentiated Weibull          | \( \hat{\alpha} = 0.0109, \hat{\beta} = 4.69, \hat{\gamma} = 0.164 \) | 0.1841 | 0.0590        |
| Generalized Linear Failure rate| \( \hat{a} = 0.0038, \hat{b} = 3.04 \times 10^{-4}, \hat{c} = 0.533 \) | 0.1620 | 0.1293        |
| Exponentiated Flexible Weibull | \( \hat{\alpha} = 0.0147, \hat{\beta} = 0.133, \hat{\theta} = 4.22 \) | 0.1433 | 0.2617        |
| MO-FWE(\( \alpha, \beta, \theta \)) | \( \hat{\alpha} = 0.017, \hat{\beta} = 0.401, \hat{\theta} = 9.043 \) | 0.1269 | 0.3756        |

We find that the MO-FWE distribution with three parameters provides a better fit than the previous models flexible Weibull (FW), Weibull (W), linear failure rate (LFR), exponentiated Weibull (EW), generalized linear failure rate (GLFR) and exponentiated flexible Weibull (EFW). It has the largest likelihood, and the smallest K-S, AIC, AICC, BIC and HQIC values among those considered in this paper.

Substituting the MLE’s of the unknown parameters \( \alpha, \beta \) and \( \theta \) into (6.6), we get estimation of the variance covariance matrix as the following

\[
I^{-1} = \begin{pmatrix}
1.523 \times 10^{-6} & -1.782 \times 10^{-5} & 2.177 \times 10^{-3} \\
-1.782 \times 10^{-5} & 0.022 & -0.061 \\
2.177 \times 10^{-3} & -0.061 & 8.458
\end{pmatrix}
\]

The approximate 95% two sided confidence intervals of the unknown parameters \( \alpha, \beta \) and \( \theta \) are \([0.015, 0.019]\), \([0.108, 0.694]\) and \([3.343, 14.743]\), respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of \( \alpha, \beta \) and \( \theta \) in Figures 7, 8.
Figure 7: The profile of the log-likelihood function of $\alpha$, $\beta$ for the Aarset data.

Figure 8: The profile of the log-likelihood function of $\theta$ for the Aarset data.

The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be MO-FWE, FW, W, LFR, EW, GLFR and EFW are computed and plotted in Figure 9.
Figure 9: The Kaplan-Meier estimate of the survival function for the Aarset(1987) data.

Figures 10 and 11 give the form of the hazard rate and cdf for the MO-FWE, FW, W, LFR, EW, GLFR and EFW which are used to fit the Aarset(1987) data after replacing the unknown parameters included in each distribution by their MLE.

Example 7.2. The data have been obtained from [13], it is for the time between failures (thousands of hours) of secondary reactor pumps, Table 4.

Table 4: Time between failures (thousands of hours) of secondary reactor pumps [13]

|        | 2.160 | 0.746 | 0.402 | 0.954 | 0.491 | 6.560 | 4.992 | 0.347 |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
|        | 0.150 | 0.358 | 0.101 | 1.359 | 3.465 | 1.060 | 0.614 | 1.921 |
|        | 4.082 | 0.199 | 0.605 | 0.273 | 0.070 | 0.062 | 5.320 |
Table 5 gives MLEs of parameters of the MO-FWE distribution and K-S Statistics. The values of the log-likelihood functions, AIC, AICC, BIC and HQIC are in Table 6.

| Model           | $\hat{\alpha}$ | $\hat{\beta}$ | $\theta$ | K-S |
|-----------------|-----------------|----------------|----------|-----|
| Flexible Weibull| 0.0207          | 2.5875         | –        | 0.1342 |
| Weibull         | 0.8077          | 13.9148        | –        | 0.1173 |
| Modified Weibull| 0.1213          | 0.7924         | 0.0009   | 0.1188 |
| Reduced Additive Weibull | 0.0070      | 1.7292         | 0.0452   | 0.1619 |
| Extended Weibull| 0.4189          | 1.0212         | 10.2778  | 0.1057 |
| MO-FWE          | 0.2160          | 0.2350         | 1.2960   | 0.0793 |

Table 6: Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted.

| Model             | $-2\mathcal{L}$ | AIC    | AICC   | BIC    | HQIC   |
|-------------------|-----------------|--------|--------|--------|--------|
| Flexible Weibull  | -83.3424        | 166.6848 | 170.6848 | 171.2848 | 172.95579 | 171.2559 |
| Weibull           | -85.4734        | 170.9468 | 174.9468 | 175.5468 | 177.21779 | 175.5179 |
| Modified Weibull  | -85.4677        | 170.9354 | 176.9354 | 178.1986 | 180.34188 | 177.7921 |
| Reduced Additive Weibull | -86.0728 | 172.1456 | 178.1456 | 179.4088 | 181.55208 | 179.0023 |
| Extended Weibull  | -86.6343        | 173.2686 | 179.2686 | 180.5318 | 182.67508 | 180.1253 |
| MO-FWE            | -30.2110        | 60.4220 | 66.4220 | 67.6852 | 69.8285 | 67.2787 |

We find that the MO-FWE distribution with the three-number of parameters provides a better fit than the previous new modified Weibull distributions like a flexible Weibull (FW), Weibull (W), modified Weibull (MW), reduced additive Weibull (RAW) and extended Weibull (EW) distributions. It has the largest likelihood, and the smallest K-S, AIC, AICC, BIC and HQIC values among those considered in this paper.

Substituting the MLE’s of the unknown parameters $\alpha$, $\beta$ and $\theta$ into (6.6), we get estimation of the variance

\[ I_0^{-1} = \begin{pmatrix}
1.996 \times 10^{-3} & -7.744 \times 10^{-4} & 8.987 \times 10^{-3} \\
-7.744 \times 10^{-4} & 5.487 \times 10^{-3} & -0.022 \\
8.987 \times 10^{-3} & -0.022 & 0.326
\end{pmatrix} \]

The approximate 95% two sided confidence intervals of the unknown parameters $\alpha$, $\beta$ and $\theta$ are [0.128, 0.304], [0.09, 0.38] and [0.177, 2.415], respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of $\alpha$, $\beta$ and $\theta$ in Figures 12 and 13.
The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be MO-FWE, FW, W, MW, RAW and EW are computed and plotted in Figure 14.
Figure 15 gives the form of the CDF for the MO-FWE, FW, W, MW, RAW and EW which are used to fit the data after replacing the unknown parameters included in each distribution by their MLE.

![Figure 15: The Fitted cumulative distribution function for the data.](image)

### 8 Conclusions

A new distribution MO-FWE, it’s generalized of the flexible Weibull extension distribution based on the Marshall and Olkin’s method, has been proposed and its properties are studied. The idea is to add parameter to a flexible Weibull extension distribution, so that the hazard function is either increasing or more importantly, bathtub shaped. Using Marshall and Olkin extended family by adding a new parameter to the baseline distribution, the distribution has flexibility to model the second peak in a distribution. We have shown that the Marshall Olkin flexible Weibull extension distribution fits certain well-known data sets better than existing modifications.

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