Kinks in two-phase lipid bilayer membranes

MICHAEL HELMERS
Institute for Applied Mathematics, University of Bonn
Endenicher Allee 60, 53115 Bonn, Germany
Email: helmers@iam.uni-bonn.de

Abstract. Common models for two-phase lipid bilayer membranes are based on an energy that consists of an elastic term for each lipid phase and a line energy at interfaces. Although such an energy controls only the length of interfaces, the membrane surface is usually assumed to be at least $C^1$ across phase boundaries. We consider the spontaneous curvature model for closed rotationally symmetric two-phase membranes without excluding tangent discontinuities at interfaces a priori. We introduce a family of energies for smooth surfaces and phase fields for the lipid phases and derive a sharp interface limit that coincides with the Γ-limit on all reasonable membranes and extends the classical model by assigning a bending energy also to tangent discontinuities. The theoretical result is illustrated by numerical examples.

Keywords: Γ-convergence, phase field model, lipid bilayer, two-component membrane.

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1 Introduction

Lipid bilayer membranes are the building block of numerous biological systems and appear in a rich variety of shapes. In particular, membranes consisting of two or more lipid phases display a complex morphology, which is affected by their elastic properties as well as phase separation [21, 9, 3].

A well-established model for the shape of two-phase vesicles is the spontaneous curvature model, where equilibrium shapes are described as surfaces minimising the energy

$$\sum_{j=\pm} \int_{M_j} k_j^l (H - H_j^s)^2 + k_j^G K d\mu + \sigma \mathcal{H}^1(\partial M^+)$$

among all closed surfaces $M = M^+ \cup M^- \cup \partial M^+$, $M^+ \cap M^- = \emptyset$ with prescribed areas for $M^\pm$ (and prescribed enclosed volume) [7, 12, 13, 16, 22]. Here $H$ and $K$ are the mean and the Gauss curvature of the membrane surface $M$, and $\mu$ is its area measure. The bending rigidities $k_j^l > 0$ and the Gauss rigidities $k_j^G$ are elastic material parameters, and $H_j^s$, the so-called spontaneous or preferred curvatures, are supposed to reflect an asymmetry in the membrane. In the simplest case, the rigidities and spontaneous curvatures are constant within each lipid phase but different between the two phases.

Apart from the length $\mathcal{H}^1(\partial M^+)$ of phase boundaries multiplied by a constant line tension $\sigma$, (1.1) does not control the membrane surface at the interface $\partial M^+ = \partial M^-$. Studies of two-phase membranes, however, commonly include an a priori smoothness assumption. Jülicher and Lipowsky [16] consider the Euler-Lagrange equations of (1.1) for axially symmetric membranes that have exactly one interface between the two lipid phases and are $C^1$ across this interface. Du, Wang [28] and Lowengrub, Rätz, Voigt [17] perform numerical simulations using a phase field for both the membrane and the lipid phases; Elliot and Stinner [10, 11] consider a surface phase field model. Convergence to the sharp interface limit in these approaches is obtained by asymptotic expansion and under strong smoothness assumptions on the limit surface; in particular, the membrane is again assumed to be at
least $C^1$ across interfaces. In [14] we prove that for rotationally symmetric membranes this regularity need not be assumed, but is included in the $\Gamma$-limit of an appropriate surface phase field approximation.

Mathematically, however, the natural setting for the energy (1.1) does not contain $C^1$ regularity across interfaces. Moreover, the numerical simulations in [28, 10] show that equilibrium shapes of models including $C^1$ regularity have rather ample neck regions, see also Figure 3.3 on the left, while shapes observed in experiments do not [3]. In this paper we study a diffuse interface approximation for the lipid phases of rotationally symmetric membranes whose sharp interface limit allows tangent discontinuities or kinks at interfaces, thus infinitesimally small neck regions. More precisely, for a closed surface $M_\gamma$ obtained by rotating a curve $\gamma$ about the $x$-axis and an associated rotationally symmetric phase field $u: M_\gamma \to \mathbb{R}$, we consider an approximate energy of the form

$$
\int_{M_\gamma} u^2 k(u)(H - H_s(u))^2 + u^2 k_G(u)K \, d\mu + \int_{M_\gamma} \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu + \int_{M_\gamma} \varepsilon|B|^2 \, d\mu. \quad (1.2)
$$

Apart from the surface setting, the second integral in (1.2) is the usual Modica-Mortola approximation of the interface energy. With a standard double well potential such as $W(u) = (1 - u^2)^2$, the phase field is forced to $\pm 1$ as $\varepsilon \to 0$ and the first integral in (1.2) resembles the curvature energy in (1.1), provided that $k$, $k_G$, and $H_s$ are extensions of $k^\pm = k(\pm 1)$, $k_G^\pm = k_G(\pm 1)$, and $H^\pm_s = H_s(\pm 1)$. The third integral, where $B$ denotes the second fundamental form of $M_\gamma$, is the on the one hand necessary for compactness of energy-bounded sequences as $\varepsilon \to 0$. On the other hand, it assigns a curvature to kinks in the limit by penalising their size, thereby providing a meaningful extension of (1.1).

The preferred curvature extension $H_s: \mathbb{R} \to \mathbb{R}$ can be any continuous and bounded functions such that $k(\pm 1) = k^\pm$, $k_G(\pm) = k_G^\pm$,

$$
\inf_{\mathbb{R}} k > 0, \quad \sup_{\mathbb{R}} k_G < 0, \quad \text{and} \quad \inf_{\mathbb{R}} (k + k_G/2) > 0. \quad (1.3)
$$

The preferred curvature extension $H_s: \mathbb{R} \to \mathbb{R}$ can be any continuous and bounded function such that $H_s(\pm 1) = H_s^\pm$. Under these conditions there is a $\delta > 0$ such that $(1 - \delta)k > -k_G/2$, and Young’s inequality together with $|B|^2 = H^2 - 2K$ implies

$$
u^2 k(H - H_s)^2 + u^2 k_G K
\leq -u^2 \frac{k_G}{2} |B|^2 + u^2 \left(k + \frac{k_G}{2}\right) H^2 + u^2 k_H s^2 - 2u^2 k H H_s
\geq -u^2 \frac{k_G}{2} |B|^2 + u^2 \left((1 - \delta)k + \frac{k_G}{2}\right) H^2 - \frac{1}{\delta} \|k H^2_s\|_{\infty} \|u\|_{\infty}^2
\geq -C\|u\|_{\infty}^2,
$$

where $C > 0$ depends only on $k$, $k_G$, and $H_s$. By (1.4) the energy (1.2) provides weighted $L^2$-bounds for $B$ and $H$, which are used to establish equi-coercivity and a lower bound inequality. These bounds degenerate for $u \approx 0$, and similarly to the Ambrosio-Tortorelli approximation for free discontinuity problems [2], this degeneracy allows curvatures to become large and yield kinks in the limit. Interestingly, conditions such as (1.3) also appear in [4], where the authors obtain a partial $\Gamma$-convergence result for a diffuse interface approximation of the membrane surface of single-phase vesicles.
Under the above restrictions on the parameters, we prove that a limit of (1.2) is given by
\[
\int_{M_{\gamma}\cap\{y>0\}\setminus S} k(u)(H - H_s(u))^2 + k_C(u)K d\mu + 2\pi \sum_{s \in S} (\sigma + \hat{\sigma}|\gamma'(s)|)y(s) + 2\pi \hat{\sigma}L_{\gamma}(\{y = 0\})
\]
(1.5)
for membranes \((\gamma, u), \gamma = (x, y)\) with membrane surface \(M_{\gamma}\) and lipid phases \(u \in \{\pm 1\}\). Here \(S\) denotes the set of interfaces, that is, the set of jumps of \(u\), and of tangent discontinuities of \(\gamma\). An interface is penalised essentially by its length \(2\pi y(s) = H^1(M_{\gamma}(\{s\}))\), while a kink carries an additional “bending” energy \(2\pi |\gamma'(s)|y(s)\), where \(|\gamma'(s)|\) is the modulus of the angle enclosed by the two one-sided tangent vectors at \(s\) modulo \(2\pi\). The constants \(\sigma\) and \(\hat{\sigma}\) are given by
\[
\sigma = \int_{-1}^{1} 2\sqrt{W(u)} du \quad \text{and} \quad \hat{\sigma} = 2\sqrt{W(0)}.
\]
(1.6)
The set \(M_{\gamma}(\{y > 0\}\setminus S)\) is the part of \(M_{\gamma}\) that is obtained by rotating the restriction of \(\gamma\) to \(\{y > 0\}\setminus S\), and \(L_{\gamma}(\{y = 0\})\) denotes the length of the segment \(\gamma(\{y = 0\})\). Here and in the following \(\{y = 0\}\) is the set where \(\gamma\) lies on the axis of revolution and \(\{y > 0\}\) the set where it does not. A limit membrane \((\gamma, u)\) for which (1.5) is finite may touch the axis of revolution not only at the end points of \(\gamma\), but also in regions in the interior, see Figure 3.1 for a non membrane-like example. Our limit equals the \(\Gamma\)-limit of \(\mathcal{F}_\varepsilon\) on “good” membranes where \(|\gamma'| = \text{const}\), and exactly regions on the axis of revolution prevent us from obtaining full \(\Gamma\)-convergence. However, a “good” limit is for instance a membrane that touches the axis of revolution only at countably many points, hence our result covers reasonable biological membranes.

The paper is organised as follows. Section 2 is a brief recapitulation of surfaces of revolution. In Section 3 we state our approximate setting, the limit, and the convergence result; we also present some numerical examples and compare our model to one without kinks. The proof of the convergence theorem is presented in Section 4, and in Section 5 we consider some generalisations of our result, including a brief discussion of the full \(\Gamma\)-limit.

2 Surfaces of revolution

Let \(I \subset \mathbb{R}\) be an open and bounded interval and \(\gamma = (x, y): I \to \mathbb{R} \times \mathbb{R}_{\geq 0}\) a Lipschitz parametrised curve in the upper half of the \(xy\)-plane. We denote by \(M_{\gamma}\) the surface in \(\mathbb{R}^3\) obtained by rotating \(\gamma\) about the \(x\)-axis, that is, \(M_{\gamma}\) is the image of \(T \times [0, 2\pi)\) under the Lipschitz continuous map
\[
\Phi: (t, \theta) \mapsto (x(t), y(t) \cos \theta, y(t) \sin \theta);
\]
\(\gamma\) is called generating curve of \(M_{\gamma}\). Since \(\gamma\) and \(\Phi\) are Lipschitz, they are weakly and almost everywhere differentiable with bounded derivatives. The length of \(\gamma\), the area measure of \(M_{\gamma}\), and the area of \(M_{\gamma}\) are well-defined and given by
\[
L_{\gamma} = \int_I |\gamma'(t)| dt, \quad d\mu = |\gamma'|y dt d\theta, \quad \text{and} \quad A_{\gamma} = 2\pi \int_I |\gamma'|y dt,
\]
respectively. If \(J\) is a measurable subset of \(I\), we denote by \(M_{\gamma}(J)\) the part of \(M_{\gamma}\) that is obtained by rotating the curve segment \(\gamma(J)\) and write \(L_{\gamma}(J)\) and \(A_{\gamma}(J)\) for the corresponding length and area.
After removing at most countably many constancy intervals, pulling holes together and reparametrising, we may assume that $\gamma$ is parametrised with constant speed $|\gamma'| \equiv L_\gamma/|I| =: q_\gamma > 0$ almost everywhere in $I$ [6, Lemma 5.23]. Then the tangent space $\mathcal{T}_{(t,\theta)}M_\gamma$, which exists for almost every $(t,\theta) \in I \times (0,2\pi)$, is spanned by the orthonormal vectors

$$\xi_1 = \frac{\partial \Phi}{\partial \theta} \frac{1}{|\gamma'|} (x', y' \cos \theta, y' \sin \theta) \quad \text{and} \quad \xi_2 = \frac{\partial \Phi}{\partial \theta} (0, -\sin \theta, \cos \theta),$$

and a unit normal is given by

$$\nu = \frac{\partial \Phi \wedge \partial \Phi}{|\partial \Phi \wedge \partial \Phi|} \frac{1}{|\gamma'|} (-y', x' \cos \theta, x' \sin \theta). \quad (2.1)$$

Since $M_\gamma$ is not necessarily embedded, tangent space, normal, and the quantities defined below are associated to the parameter $(t,\theta)$ and not to the point $\Phi(t,\theta)$ on the surface $M_\gamma$.

We consider a function $f : M_\gamma \to \mathbb{R}^k$ to be a function $F : I \times [0,2\pi) \to \mathbb{R}^k$ of the parameters; on embedded parts of $M_\gamma$ this amounts to $f(\Phi(t,\theta)) = F(t,\theta)$. Given a tangent vector $\xi$ at $(t_0, \theta_0) \in I \times (0,2\pi)$, the directional derivative of $f$ in direction $\xi$ is defined as

$$D\xi f(t_0, \theta_0) = \left. \frac{d}{ds} F(\eta(s)) \right|_{s=0},$$

where $\eta : (-\delta, \delta) \to I \times [0,2\pi)$ is a $C^1$-curve satisfying $\eta(0) = (t_0, \theta_0)$ and $\left. \frac{d}{ds} \Phi(\eta(s)) \right|_{s=0} = \xi$. The tangential gradient of $f : M_\gamma \to \mathbb{R}$ is the vector

$$\nabla_M f = (D\xi_1 f) \xi_1 + (D\xi_2 f) \xi_2,$$

and since $D\xi_1 f = |\gamma'|^{-1} \partial_\theta F$, we obtain for a rotationally symmetric $f$, which does not depend on $\theta$, that

$$\nabla_M f(t, \theta) = \frac{F'(t)}{|\gamma'(t)|} \xi_1(t, \theta) \quad \text{and} \quad |\nabla_M f(t, \theta)| = \frac{|F'(t)|}{|\gamma'(t)|},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$.

For the rest of this section let $\gamma \in W^{2,1}_{loc}(I; \mathbb{R}^2)$ be twice weakly differentiable, thus twice differentiable almost everywhere, and assume that $y > 0$ in $I$. Then the normal $\nu$ is weakly differentiable, thus the shape operator $S : \mathcal{T}_{(t_0,\theta_0)}M \to \mathcal{T}_{(t_0,\theta_0)}M$, $\zeta \mapsto D\zeta \nu$ and the second fundamental form $\mathcal{B} : \mathcal{T}_{(t_0,\theta_0)}M \times \mathcal{T}_{(t_0,\theta_0)}M \to \mathbb{R}$, $(\zeta, \xi) \mapsto \xi \cdot S \zeta$ are well-defined for almost every $(t_0, \theta_0)$. The matrix representation with respect to the basis $\{\xi_1, \xi_2\}$ of both is diag$(\kappa_1, \kappa_2)$ where

$$\kappa_1 = \frac{-y'' x' + y' x''}{|\gamma'|^2} \quad \text{and} \quad \kappa_2 = \frac{x'}{y|\gamma'|-1}.$$

The eigenvalues $\kappa_1, \kappa_2$ of $S$ are the principal curvatures of $M_\gamma$, and $\kappa_1$ is just the signed curvature of $\gamma$ with respect to the normal $\frac{1}{|\gamma'|}(y', -x')$. The mean curvature $H$ and the Gauss curvature $K$ of $M_\gamma$ are

$$H = \text{trace } S = \kappa_1 + \kappa_2 \quad \text{and} \quad K = \det S = \kappa_1 \kappa_2.$$

By $|B|^2 = |S|^2 = \kappa_1^2 + \kappa_2^2$ we denote the squared Frobenius norm of $S$ and $B$. The signs of the principal curvatures and the mean curvature depend on the choice of the normal. In the above setting a unit ball has outer unit normal $\nu$ as in (2.1) and curvatures $\kappa_1 = \kappa_2 = +1$ when it is parametrised “from left to right” such that $x' \geq 0$, for instance by $\gamma(t) = (-\cos t, \sin t)$, $t \in [0, \pi]$. 
Let $\varphi: I \to \mathbb{R}$ be an angle function for $\gamma$, that is, let $\varphi(t)$ be the angle between the positive $x$-axis and the tangent vector $\gamma'(t)$. Since $W^{2,1}_{\text{loc}}(I)$ embeds into $C^1_{\text{loc}}(I)$, the angle $\varphi$ can be chosen continuously in $I$ and is then uniquely determined up to adding multiples of $2\pi$. In terms of $\varphi$, the curve $\gamma$ is characterised by fixing one point and

$$x' = |\gamma'| \cos \varphi, \quad y' = |\gamma'| \sin \varphi.$$  

(2.3)

The principal curvatures take the form

$$\kappa_1 = -\frac{\varphi'}{|\gamma'|}, \quad \kappa_2 = \frac{\cos \varphi}{|\gamma'|},$$

(2.4)

and we have

$$K = -\frac{\varphi' \cos \varphi}{|\gamma'|} = -\frac{(\sin \varphi)'}{|\gamma'|} = -\frac{(y'/|\gamma'|)'}{|\gamma'|}. $$

(2.5)

If $\gamma$ is parametrised with constant speed $q_\gamma > 0$, we see from (2.5) that

$$\int_{M_{\gamma}(J)} |K| \, d\mu = \frac{2\pi}{q_\gamma} \int_J |y''| \, dt$$

(2.6)

is the $L^1$-norm of $y''$ up to a constant factor. Moreover, for such $\gamma$ we have $\varphi'^2 q_\gamma^2 = |\gamma''|^2$, and therefore

$$\int_{M_{\gamma}(J)} \kappa_1^2 \, d\mu = \frac{2\pi}{q_\gamma} \int_J |\varphi'|^2 y \, dt = \frac{2\pi}{q_\gamma^2} \int_J |\gamma''|^2 y \, dt$$

(2.7)

is a weighted $L^2$-norm of $\varphi'$ and $\gamma''$.

For a more detailed discussion of surfaces and basic geometric analysis we refer to [8] or [24, §7].

3 The models

3.1 Approximate setting

We study the energy (1.2) with continuous bounded functions $H_s, k, k_G$. The precise values of $k$ and $k_G$ do not enter our arguments as long as (1.3) is satisfied, so we assume $k = k = 1 = -k_G \equiv -k_G$ for simplicity of notation. Thus, our approximate energy is given by

$$F_\varepsilon(\gamma, u) = \mathcal{H}_\varepsilon(\gamma, u) + \mathcal{I}_\varepsilon(\gamma, u),$$

(3.1)

where

$$\mathcal{H}_\varepsilon(\gamma, u) = \int_{M_{\gamma}} u^2 (H - H_s(u))^2 - u^2 K \, d\mu$$

is the Helfrich energy of the membrane $(\gamma, u)$ and

$$\mathcal{I}_\varepsilon(\gamma, u) = \int_{M_{\gamma}} \varepsilon |\nabla M_{\gamma}, u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu + \varepsilon \int_{M_{\gamma}} |B|^2 \, d\mu$$

the interface energy. For $J \subset I$ we denote by $F_\varepsilon(\gamma, u, J), \mathcal{H}_\varepsilon(\gamma, u, J)$, and $\mathcal{I}_\varepsilon(\gamma, u, J)$ the corresponding integrals restricted to the set $M_{\gamma}(J)$. The double well potential $W: \mathbb{R} \to [0, \infty)$ is a continuous function that vanishes only in $\pm 1$ and is $C^2$ around these points; for simplicity of notation we assume that $W$ is symmetric.

We study (3.1) for rotationally symmetric membranes $(\gamma, u) \in C \times \mathcal{P}$, where

$$C := \left\{ \gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) \cap W^{2,1}_{\text{loc}}(I; \mathbb{R}^2) : \right. \\
|\gamma'| = \text{const}, \ y(\partial I) = \{0\}, \ y(I) \subset (0, \infty), \ x' \geq 0, \int_{M_{\gamma}} |B|^2 \, d\mu < \infty, \ A_\gamma = A_0 \left. \right\}.$$
and

\[ P := \{ u \in W_{\text{loc}}^{1,1}(I) : \int_{M_{\gamma}} |\nabla u| d\mu < \infty, \| u \|_{\infty} \leq C_0, \int_{M_{\gamma}} u d\mu = mA_0 \} \]

The first three conditions in the definition of \( C \) ensure that the constant speed curve \( \gamma \) generates a closed surface \( M_{\gamma} \). The requirement \( x' \geq 0 \) fixes the orientation and, since by embedding \( \gamma \in C \) belongs to \( C_{\text{loc}}^1(I; \mathbb{R}^2) \), it guarantees that \( M_{\gamma} \) is embedded; its main purpose is to exclude some very non membrane-like behaviour such as infinitely many self-intersections or zigzagging of curves in the limit. The \( L^2 \)-bound on \( B \) and the first two conditions on the phase fields ensure that (3.1) is well-defined for \((\gamma, u) \in C \times P\). The uniform bound \( \| u \|_{\infty} \leq C_0 \) with a constant \( C_0 \gg 1 \) is more restrictive than necessary, and in many places in the proof it can be replaced by weaker conditions such as integral bounds on \( u \). We impose the \( L^\infty \) bound for convenience though, and as one expects phase fields with small energy to be roughly between \(+1\) and \(-1\) for small \( \varepsilon \), this is not a strong restriction.

The area constraints for the two lipid phases are incorporated by prescribing the area of \( M_{\gamma} \) for all \((\gamma, u) \in C \times P \). Moreover, we have the individual bounds \( J \) for any \((\gamma, u) \in C \times P \). The \( C \times P \) speed parametrisation belongs to \( C \times P \) requirements of \( \partial I \) perpendicular to the axis of revolution at \( \partial I \); and these regularity properties cannot be improved [14, Section 2.2]. The energy (3.1) is invariant under reparametrisations that preserve the orientation of \( \gamma \) and the regularity properties of \((\gamma, u) \). In particular, if \((\gamma, u) \) satisfies all requirements of \( C \times P \) but only \( |\gamma'| \neq 0 \) instead of \( |\gamma'| = \text{const} \), the corresponding constant speed parametrisation belongs to \( C \times P \) and has the same energy. Hence, considering only \( |\gamma'| = \text{const} \) is no geometric restriction.

By our assumptions (1.3) on the bending parameters, the calculations in (1.4) yield

\[
H_\varepsilon(\gamma, u, J) \geq \int_{M_{\gamma}(J)} \frac{1}{2} u^2 |B|^2 - u^2 H_s(u)^2 d\mu
\]

\[
\geq \int_{M_{\gamma}(J)} \frac{1}{2} u^2 |B|^2 d\mu - \| H_s \|_\infty^2 \| u \|_{\infty}^2 A_\gamma
\]  

(3.2)

for any \( J \subset I \). Since (3.2) provides a lower bound for \( H_\varepsilon \) on \( C \times P \), also \( F_\varepsilon \) is bounded from below. Moreover, we have the individual bounds

\[
|F_\varepsilon(\gamma, u)| \leq C (F_\varepsilon(\gamma, u) + \| H_s \|_\infty^2 C_0^2 A_0), \quad (3.3)
\]

\[
I_\varepsilon(\gamma, u) \leq F_\varepsilon(\gamma, u) + \| H_s \|_\infty^2 C_0^2 A_0, \quad (3.4)
\]

\[
\int_{M_{\gamma}} u^2 |B|^2 d\mu + \varepsilon \int_{M_{\gamma}} |B|^2 d\mu \leq C (F_\varepsilon(\gamma, u) + \| H_s \|_\infty^2 C_0^2 A_0) \quad (3.5)
\]

for all \((\gamma, u) \in C \times P \), where \( C > 0 \) is a generic constant independent of \((\gamma, u) \). From (3.4) and (3.5) we derive a bound on the first variation of \( M_{\gamma} \), that is, on the first variation of the area of \( M_{\gamma} \).
Lemma 3.1. There is a constant $C > 0$ such that
\[
\frac{1}{\sqrt{2}} \int_{M_\gamma} |H| \, d\mu \leq \int_{M_\gamma} |B| \, d\mu \leq C(F_\varepsilon(\gamma, u) + 1)
\]
for all $(\gamma, u) \in C \times P$.

Proof. Splitting $M_\gamma$ into two pieces where the phase field is small and large, respectively, and applying Hölder’s inequality, we get
\[
\int_{M_\gamma} |B| \, d\mu \leq \int_{M_\gamma \cup \{\{|u| \leq 1/2\}\}} |B| \, d\mu + \int_{M_\gamma \cap \{\{|u| > 1/2\}\}} |B| \, d\mu
\]
\[
\leq \left( \frac{1}{\varepsilon} A_\gamma(\{|u| \leq 1/2\}) \right)^{1/2} \left( \int_{M_\gamma} \varepsilon |B|^2 \, d\mu \right)^{1/2} + 2 \sqrt{A_0} \left( \int_{M_\gamma} u^2 |B|^2 \, d\mu \right)^{1/2} .
\]

From the interface energy we obtain the estimate
\[
\mathcal{I}_\varepsilon(\gamma, u) \geq \int_{M_\gamma \cup \{\{|u| \leq 1/2\}\}} \frac{1}{\varepsilon} W(u) \, d\mu \geq \left( \inf_{\{|u| \leq 1/2\}} W(u) \right) \frac{A_\gamma(\{|u| \leq 1/2\})}{\varepsilon},
\]
and since $W$ has a positive minimum on $[-1/2, 1/2]$, we find
\[
\int_{M_\gamma} |B| \, d\mu \leq C \left( \mathcal{I}_\varepsilon(\gamma, u)^{1/2} + 1 \right) \left[ \left( \int_{M_\gamma} \varepsilon |B|^2 \, d\mu \right)^{1/2} + \left( \int_{M_\gamma} u^2 |B|^2 \, d\mu \right)^{1/2} \right].
\]

The conclusion now follows from (3.4), (3.5), and the elementary inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \leq \sqrt{2(\sqrt{a} + \sqrt{b})}$ for $a, b \geq 0$. □

To establish compactness, we use that the first variation of $M_\gamma$ bounds the length of the generating curve $\gamma$. Such a bound is for instance deduced from the well-known bound on the intrinsic diameter from [26] or for surfaces of revolution easily proved by an integration by parts as in [14, Section 2.3].

Lemma 3.2. Let $\gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) \cap W^{2,1}_{\text{loc}}(I; \mathbb{R}^2)$ be a curve such that $y(I) \subset (0, \infty)$, $y(\partial I) = \{0\}$. Then
\[
\int_{M_\gamma} |H| \, d\mu \geq 2\pi L_\gamma.
\]

Remark. Combining Lemmas 3.1 and 3.2, we see that any sequence $(\gamma_\varepsilon, u_\varepsilon) \in C \times P$ with uniformly bounded energy has uniformly bounded length. For this conclusion we could have argued with $\int u^2 H^2 + \varepsilon H^2 \, d\mu$ directly, instead of using the second fundamental form in the proof of Lemma 3.1. The following example shows that an additional energy term like $\varepsilon \int H^2 \, d\mu$ or $\varepsilon \int |B|^2 \, d\mu$ is necessary to obtain the length bound. Let $M_\varepsilon$ be a sequence of “dumbbells” that consist of two spheres, which are smoothly connected by a cylinder of length $l_\varepsilon$ and diameter $h_\varepsilon$, and let the phase field $u_\varepsilon$ be 0 on the cylinder and $+1$ and $-1$ on the spheres with exactly one transition with gradient of order $\varepsilon^{-1}$ at each connection. Then $H_\varepsilon \sim 0$ on the cylinder and $H_\varepsilon$ is bounded independently of $\varepsilon$ on the spheres. The contribution of $u_\varepsilon \sim 0$ on the cylinder and of the two phase transitions stems from $\int \varepsilon \nabla u_\varepsilon^2 + \frac{1}{2} W(u_\varepsilon) \, d\mu$, and is of order $\frac{1}{\varepsilon} l_\varepsilon h_\varepsilon + h_\varepsilon$, and the smoothing of the connections between the cylinder and the spheres can be done where $u_\varepsilon \sim 0$; see Section 4.3 for the details of the construction of a recovery sequence. Thus, if $l_\varepsilon \to \infty$ and $h_\varepsilon \to 0$ such that $l_\varepsilon h_\varepsilon \sim \varepsilon$, the energy without $\varepsilon \int |B|^2 \, d\mu$ is bounded, but the length of the generating curve is unbounded as $\varepsilon \to 0$; $(\gamma_\varepsilon, u_\varepsilon)$ can easily be made admissible for some $A_0$ and $m$. 

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since the area and phase constraint, which are disturbed by the vanishing cylinder, can be recovered by slightly perturbing the spheres. On the other hand, \( \epsilon \int |B|^2 \, d\mu \sim \epsilon l_\varepsilon/h_\varepsilon \) on the cylinder, and therefore \( l_\varepsilon \to \infty \) is excluded by a uniform bound on \( F_\varepsilon \).

The scaling of \( \varepsilon \) in the stabilising term is critical. If the energy contains \( \varepsilon^p \int |B|^2 \, d\mu \) with \( p > 1 \), the above example still works and there is no length bound. If \( p < 1 \), tangent discontinuities in the limit are excluded, since an argument similar to the proof of Lemma 3.1 yields an \( L^q \)-bound for some \( q > 1 \) on the second fundamental form and thus on \( \kappa_1 \); compare the equi-coercivity arguments in Section 4.1.

3.2 Limit setting

The major technical difficulties in the limit of (3.1) as \( \varepsilon \to 0 \) stem from the appearance of kinks and from the axis of revolution. In particular, at the axis the compactness result for sequences \((\gamma_n, u_n)\) and the regularity properties of the limit are weaker than elsewhere. Limit curves will have parametrisations in

\[
D := \left\{ \gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) : 
\begin{align*}
|\gamma'| \equiv q_\gamma & = \text{const in } \{y > 0\}, 
|\gamma'| = x' \leq q_\gamma \text{ in } \{y = 0\}, 
y(\partial I) = \{0\}, \quad y \geq 0, \quad x' \geq 0 
\end{align*}
\text{there is } S_\gamma \subset \{y > 0\} \text{ s.t. } H^1(M_\gamma(S_\gamma)) < \infty \text{ and }
\right\}.
\]

A curve \( \gamma \in D \) is globally Lipschitz with Lipschitz constant \( q_\gamma \), and its restriction to \( \{y > 0\} \) is a constant speed parametrisation. Since \( \{y > 0\} \subset \mathbb{R} \) is open, it is the union of its countably many connected components, which are disjoint intervals. In a slight abuse of language we refer to a component \( \omega \) of \( \{y > 0\} \) also as component of \( \gamma \) and call \( M_\gamma(\omega) \) a component of \( M_\gamma \). Thus, \( M_\gamma \) consists of at most countably many components, which are connected through the axis of revolution.

Due to \( H^1(M_\gamma(S_\gamma)) < \infty \) the set \( S_\gamma \cap J \) is finite for any \( J \in \{y > 0\} \), and since \( S \) can be written as countable union of such sets it is countable. The bound on the second fundamental form yields \( \gamma \in W^{2,2}(J \setminus S_\gamma; \mathbb{R}^2) \). By embedding into \( C^1(J; \mathbb{R}^2) \) the tangent vector \( \gamma' \) is continuous from either side at any \( s \in S_\gamma \), that is, \( S_\gamma \) indeed contains the tangent discontinuities of \( \gamma \) in \( \{y > 0\} \).

In contrast to \( C \), a component \( M_\gamma(\omega) \) of \( M_\gamma \), \( \gamma \in D \) is embedded only between adjacent kinks, but in general not globally. Moreover, if kinks accumulate at \( a \in \partial \omega \), the limit of \( \gamma'(t) \) as \( t \to a, t \in \omega \) need not exist; \( \gamma' \) is perpendicular to the axis of revolution in the following weak sense.

**Lemma 3.3.** Let \( \omega = (a, b) \) be a component of \( \gamma = (x, y) \in D \) and assume that \( (s_j) \subset S_\gamma \cap \omega \) is a decreasing sequence such that \( s_j \to a \) as \( j \to \infty \) and \( \gamma \in W^{2,2}(s_{j+1}, s_j) \) for all \( j \in \mathbb{N} \). Then the one-sided approximate limit of \( x' \) vanishes at \( a \), that is

\[
\lim_{\rho \searrow 0} \frac{1}{\rho} \int_a^{a+\rho} x' \, dt = 0,
\]

and \( |y'| \) has one-sided approximate limit \( q_\gamma \). Moreover, \( \gamma \) is almost piecewise straight near \( a \) in the sense

\[
\lim_{j \to \infty} \text{osc}_{(a, b)}(s_j) \gamma' = \lim_{j \to \infty} \sup_{t, s \in (s_j, s_{j+1})} |\gamma'(t) - \gamma'(s)| = 0.
\]
Figure 3.1: Example of a curve $\gamma \in \mathcal{D}$. The component on the left is regular except for one kink. In the right component kinks accumulate at both ends, where at the left end the limit tangent exist, but at the right end it does not. In the centre there are countably many self-similar components $\omega_k$ decreasing from right to left such that one can easily find a scaling of $\gamma(\omega_k)$ and $L_\gamma(\omega_k)$ that leaves $B$ bounded in $L^2$.

**Proof.** Lipschitz continuity of $y$ implies $y(t) \leq y(a) + q_\gamma(t - a) \leq q_\gamma \rho$ in $(a, a + \rho)$, hence using (2.2) we conclude

$$\frac{1}{\rho} \int_a^{a+\rho} x'^2 \, dt \leq q_\gamma^2 \int_a^{a+\rho} \frac{x'^2}{q_\gamma y} \, dt \leq \frac{q_\gamma^2}{2\pi} \int_{M_\gamma((a,a+\rho) \setminus S)} \kappa_2^2 \, d\mu.$$  

The right hand side tends to zero as $\rho \to 0$, because the $L^2$-norm of the second fundamental form of $M_\gamma(\omega \setminus S)$ is finite. The approximate limit $q_\gamma$ of $|y'|$ follows from $y'^2 = q_\gamma^2 - x'^2$ almost everywhere in $\omega$.

For the straightness recall (2.6) and $|B|^2 \geq |K|$ almost everywhere in $\omega$, thus

$$\sum_{j=1}^{\infty} \int_{s_j}^{s_{j+1}} |y''| \, dt \leq \int_{M_\gamma(\omega \setminus S)} |B|^2 \, d\mu < \infty.$$  

Consequently,

$$\sup_{t, s \in (s_{j+1}, s_j)} |y'(t) - y'(s)| \leq \int_{s_j}^{s_{j+1}} |y''| \, dt \to 0 \quad \text{as } j \to \infty,$$

and likewise for $x'$ due to $x' \geq 0$ and

$$|x'(t) - x'(s)|^2 \leq |x'^2(t) - x'^2(s)| = |y'^2(t) - y'^2(s)| \leq 2q_\gamma |y'(t) - y'(s)|. \quad \square$$

According to Lemma 3.3, $\gamma$ consists roughly of straight line segments when approaching the component boundary, but the directions of these segments may vary as long as the approximate limit is perpendicular to the axis of revolution. If kinks do not accumulate near a component boundary, then, as for $\gamma \in \mathcal{C}$, the classical limit tangent exists and is perpendicular to the axis of revolution. An example for a curve in $\mathcal{D}$ is given in Figure 3.1.

To $\gamma \in \mathcal{D}$ we associate a phase field $u$ in

$$\mathcal{Q} := \left\{ u ; I \to [-C_0, C_0] : u \in \{ \pm 1 \} \text{ piecewise constant in } \{ y > 0 \}, \int_{M_\gamma} u \, d\mu = mA_0, \mathcal{H}^1(M_\gamma(S_u)) < \infty \right\}.$$
Here $S_u \subset \{y > 0\}$ denotes the jump set of $u$, and we call $s \in S_u$ and the corresponding circle $M_\gamma(\{s\})$ an interface of $(\gamma, u)$. The set $Q$ resembles the set of special functions of bounded variation SBV with values in $\{\pm 1\}$, weighted with the height $y$ of the generating curve $\gamma = (x, y) \in D$. Indeed, for $u \in Q$ and any $J \Subset \{y > 0\}$ we have $u \in SBV(J; \{\pm 1\})$, but jumps of height 2 may accumulate near the axis of revolution and $u$ is not specified in $\{y = 0\}$. We emphasise that in our notation $S_u$ and $S_\gamma$ are subsets of $\{y > 0\}$, because kinks and interfaces on the axis of revolution do not contribute to the limit energy defined below. Moreover, kinks are not restricted to interfaces, that is, there may be points $s \in S_\gamma \setminus S_u$. We call such points ghost interfaces, as opposed to proper interfaces, since their contribution to the limit energy is concentrated on lines as for interfaces and contains the interface energy of the latter.

For $(\gamma, u) \in D \times Q$ we consider the energy $\mathcal{F} = \mathcal{H} + \mathcal{I}$ with Helfrich energy

$$\mathcal{H}(\gamma, u) = \int_{M_\gamma(\{y > 0\}\setminus S_\gamma)} (H - H_s(u))^2 - K d\mu$$

and interface energy

$$\mathcal{I}(\gamma, u) = 2\pi \sum_{s \in S_\gamma \cup S_u} (\sigma + \hat{\sigma} |[\gamma](s)|) y(s) + 2\pi \hat{\sigma} \mathcal{L}_\gamma(\{y = 0\});$$

as before, $\mathcal{F}(\cdot, J)$, $\mathcal{H}(\cdot, J)$, and $\mathcal{I}(\cdot, J)$ denote the restrictions to $J \subset I$. Recall that $\sigma, \hat{\sigma}$ are given by (1.6) and that $|[\gamma](s)|$ denotes the modulus of the angle enclosed by the two one-sided tangent vectors at $s$ modulo $2\pi$, that is, the jump of the tangent vector because its length is fixed. The size of $\{y = 0\}$ appears in $\mathcal{I}$, because it stems from the second fundamental form in $\mathcal{I}_x$, and $\{y = 0\}$ might be interpreted as a (ghost) interface between components of $M_\gamma$. As for $\mathcal{F}_\varepsilon$, we find

$$\mathcal{H}(\gamma, u) \geq \frac{1}{2} \int_{M_\gamma(\{y > 0\}\setminus S_\gamma)} |B|^2 d\mu - \|H_s\|_\infty^2 A_\gamma$$

and bounds corresponding to (3.3)–(3.5). Moreover, also $\mathcal{F}$ is invariant under reparametrisations that preserve orientation and regularity properties.

### 3.3 Convergence theorem

We extend $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ to $C^0(I; \mathbb{R}^2) \times L^1(I)$ by setting $\mathcal{F}_\varepsilon(\gamma, u) = \mathcal{F}(\gamma, u) = \infty$ whenever $(\gamma, u)$ does not belong to $C \times P$ or $D \times Q$, respectively. The main result of this paper is the following theorem.

**Theorem 3.4.** The energies $\mathcal{F}_\varepsilon$ are equi-coercive, that is, any sequence $((\gamma_\varepsilon, u_\varepsilon)) \subset C \times P$ with uniformly bounded energy $\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon)$ admits a subsequence that converges in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ to some $(\gamma, u) \in D \times Q$. Furthermore, $\mathcal{F}_\varepsilon$ converges to $\mathcal{F}$ in the following sense:

1. **any sequence $((\gamma_\varepsilon, u_\varepsilon)) \subset C^0(I; \mathbb{R}^2) \times L^1(I)$ that converges to $(\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ satisfies the lower bound inequality**

   $$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{F}(\gamma, u);$$

2. **for any $(\gamma, u) \in D \times Q$ such that $\gamma$ is parametrised with constant speed almost everywhere in $I$ there exists a recovery sequence $((\gamma_\varepsilon, u_\varepsilon)) \subset C \times P$ that converges to $(\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ and satisfies the upper bound inequality**

   $$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \leq \mathcal{F}(\gamma, u).$$
Theorem 3.4 differs from Γ-convergence in two aspects. First, the underlying convergence of the phase fields \( u_\varepsilon \) in \( L^1(\{y > 0\}) \) depends on the limit curve \( \gamma = (x, y) \), because there is in general insufficent control on \( u_\varepsilon \) in \( \{y = 0\} \). Of course, due to \( \|u_\varepsilon\|_\infty \leq C_0 \) we could extract a weakly-* convergent subsequence, but since the \( L^\infty \)-bound is artificial and the value of the limit \( u \) in \( \{y = 0\} \) is not used by \( \mathcal{F} \), we prefer the above setting, where the limit phase field is essentially undefined in \( \{y = 0\} \). Second, in the upper bound inequality we construct a recovery sequence only for limits \((\gamma, u)\) with constant speed \(|\gamma'|\) in all of \( I \). Nevertheless, as for Γ-convergence it is true that almost minimising sequences for \( \mathcal{F}_\varepsilon \) cluster only in minimisers of \( \mathcal{F} \).

**Corollary 3.5.** Let \((\gamma_\varepsilon, u_\varepsilon) \in C \times \mathcal{P} \) converge to \((\gamma, u) \in D \times Q \) in \( C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\}) \) such that \( \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) = \inf \mathcal{F}_\varepsilon + o(1)_{\varepsilon \to 0} \). Then \((\gamma, u)\) minimises \( \mathcal{F} \) in \( D \times Q \).

**Proof.** Given an arbitrary \((\tilde{\eta}, \tilde{w}) \in D \times Q \), a constant speed parametrisation of the membrane represented by \((\tilde{\eta}, \tilde{w})\) is found as in Section 2 and the first remark in Section 3.1: First, removing constancy intervals of \( \tilde{\eta} \) in \( \{y = 0\} \) does not change the membrane, its area, energy \( \mathcal{F} \), or phase integral. Then, if \( \tilde{\eta} \) has no constancy intervals and \( I = (a, b) \), the function \( \psi(t) = a + \frac{b - a}{L_{\tilde{\eta}}} \int_a^t |\tilde{\eta}'(s)| \, ds \) is strictly increasing, and the parametrisation \((\eta, w) = (\tilde{\eta} \circ \psi^{-1}, \tilde{w} \circ \psi^{-1})\) has constant speed \( L_{\tilde{\eta}}/(b - a) \). Since \( \psi \) is affine in each component of \( \tilde{\eta} \), the pair \((\eta, w)\) inherits its differentiability properties and bounds in \( \{y_\eta > 0\} \) from \((\tilde{\eta}, \tilde{w})\) in \( \{y_{\tilde{\eta}} > 0\} \). Again, energy, area and phase integral are unchanged.

With a recovery sequence \((\eta_\varepsilon, w_\varepsilon)\) for \((\eta, w)\) we now obtain

\[
\mathcal{F}(\gamma, u) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) = \liminf_{\varepsilon \to 0} (\inf \mathcal{F}_\varepsilon) \leq \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\eta_\varepsilon, w_\varepsilon) \leq \mathcal{F}(\eta, w) = \mathcal{F}(\tilde{\eta}, \tilde{w}),
\]

and by arbitrariness of \((\tilde{\eta}, \tilde{w})\) we conclude that \((\gamma, u)\) has minimal energy \( \mathcal{F} \).

The relation between \( \mathcal{F} \) and Γ-lim \( \mathcal{F}_\varepsilon \) is discussed further in Section 5.

### 3.4 Numerical examples

Since it is defined for relatively smooth membranes, \( \mathcal{F}_\varepsilon \) is better suited for numerical simulation than \( \mathcal{F} \) and can be compared to an approximation of a limit without kinks given by

\[
\mathcal{E}_\varepsilon(\gamma, u) = \int_{M_{\gamma}} (H - H_s(u))^2 - K \, d\mu + \int_{M_{\gamma}} \varepsilon|\nabla_M u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu \quad (3.6)
\]

for \((\gamma, u) \in C \times \mathcal{P} \). The energy (3.6) has been studied in numerical simulations and by means of formal asymptotic expansion for arbitrary smooth surfaces, for instance in [10, 11]. For rotationally symmetric membranes the Γ-limit of (3.6) is given by

\[
\mathcal{E}(\gamma, u) = \int_{M_{\gamma}} (H - H_s(u))^2 - K \, d\mu + \sigma \mathcal{H}^1(M_{\gamma}(S_u)) \quad (3.7)
\]

on membranes \((\gamma, u)\) such that \( M_{\gamma} \) is a topological sphere [14].

For numerical illustrations we consider a gradient flow type evolution for \( \mathcal{E}_\varepsilon \) and \( \mathcal{F}_\varepsilon \) that consists of an \( L^2 \) flow for the surface and a weighted \( L^2 \) flow for the phase field; the constraints are incorporated by Lagrange multipliers. This flow has to our knowledge first been studied in [10, 11], where the derivation of the flow equations is presented in full detail.
Figure 3.2: Initial data for the numerical examples in Section 3.4, cross section of the surface on the left, phase field over arc length on the right. The marks on the horizontal axis indicate the interface and the connection of spherical caps and cylinder.

The numerical results below were obtained by incorporating the phase field into the scheme for rotational symmetric surfaces flows from [18].

The initial data for the simulations below is shown in Figure 3.2. The surface is a cylinder of length 3 and radius $1/2$ with spherical caps; it is centred in the origin so that the $x$-coordinate ranges from $-2$ to 2. The initial phase field is

$$u(x,y) = \begin{cases} 
-1 & \text{if } x \leq -\frac{5}{4}, \\
\frac{4}{3}x + \frac{2}{3} & \text{if } -\frac{5}{4} < x < \frac{1}{4}, \\
+1 & \text{if } \frac{1}{4} \leq x.
\end{cases}$$

The spontaneous curvature $H_s(u)$ is the fifth-order polynomial interpolation of $H_s(1) = 2$, $H_s(-1) = 1$, $H'_s(\pm1) = H''_s(\pm1) = 0$ in $[-1,1]$ and extended constantly to the whole real line.

Figure 3.3 shows the numerically stationary membranes and the angle between their generating curves and the positive $x$-axis for $E_\varepsilon$ and $F_\varepsilon$ with $\varepsilon = 0.05$. Obviously, while there is a smooth and rather ample neck region for $E_\varepsilon$, the curve for $F_\varepsilon$ makes a sharp turn: the angle almost jumps from about $-0.5$ to $1.05$, and the neck region is limited to a small neighbourhood of the approximate kink, which compares well with the experimental observations in [3]. Also, the light phase $u_\varepsilon = 1$ of the membrane is closer to a round sphere for $F_\varepsilon$ than for $E_\varepsilon$.

A different behaviour can be seen in Figure 3.4, which shows the numerically stationary membrane for the energy $\tilde{F}_\varepsilon$ that differs from $F_\varepsilon$ in that no Gauss curvature is present; we will discuss in Section 5.1 that our theorem can be adapted to this case. The stationary shape for the corresponding energy $\tilde{E}_\varepsilon$ is the same as for $E_\varepsilon$ in Figure 3.3, because the Gauss curvature integral in $E_\varepsilon$ is a topological invariant. One can see that the neck for $\tilde{F}_\varepsilon$ has smaller diameter than for $\tilde{E}_\varepsilon$, there is, however, no kink, and the neck region is as ample as for $\tilde{E}_\varepsilon$.

4 Proof of Theorem 3.4

The proof of Theorem 3.4 follows the ideas of [15], where we studied a one-dimensional analogue of two-phase membranes, and is split into the three steps equi-coercivity, lower bound, and upper bound inequality. In the following we write $M_\varepsilon$ instead of $M_{\gamma_\varepsilon}$ and so forth when considering sequences of membranes. If convenient for clarification, we also add an index $\gamma$ or $\varepsilon$ to other quantities such as $\mu$, $H$, and so on.
4.1 Equi-coercivity

Lemma 4.1. Let \((\gamma_\varepsilon, u_\varepsilon) \subset C \times P\) be a sequence with uniformly bounded energy \(F_\varepsilon(\gamma_\varepsilon, u_\varepsilon)\). Then there exist \((\gamma, u) \in D \times \mathcal{Q}\), \(\gamma = (x, y)\), a countable set \(S \subset \{y > 0\}\) with \(S_\gamma \cup S_u \subset S\) and \(S \cap J\) finite for any \(J \Subset \{y > 0\}\), and a subsequence, not relabelled, such that

- \(\gamma_\varepsilon \rightharpoonup \gamma\) in \(W^{1,\infty}(I; \mathbb{R}^2)\);
- \(u_\varepsilon \rightarrow u\) in \(L^p(\{y > 0\})\) for any \(p \in [1, \infty)\);
- \(\gamma_\varepsilon \rightharpoonup \gamma\) in \(W^{2,2}_{\text{loc}}(\{y > 0\} \setminus S; \mathbb{R}^2)\);
- in any \(J \Subset \{y > 0\} \setminus S\) there holds \(|u_\varepsilon| \geq 1/2\) for all sufficiently small \(\varepsilon\).

Proof. Let \(\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)\) and \(|\gamma'_\varepsilon| = q_\varepsilon\). With Lemma 3.1, Lemma 3.2, and Hölder’s inequality we find

\[
2\pi q_\varepsilon |I| = 2\pi L_\varepsilon \leq \left( A_\varepsilon \int_{M_\varepsilon} H^2 d\mu \right)^{1/2},
\]

thus the sequence \((q_\varepsilon)\) is uniformly bounded from above. Since translations in \(x\)-direction do not change the energy, we may assume that all \(\gamma_\varepsilon\) have a common end point. Hence, \((\gamma_\varepsilon)\) is bounded in \(W^{1,\infty}(I; \mathbb{R}^2)\) and we may extract a subsequence such that \(q_\varepsilon \rightharpoonup q\) and \(\gamma_\varepsilon \rightharpoonup (x, y)\) in \(W^{1,\infty}(I; \mathbb{R}^2)\) and \(C^{0,1}(I; \mathbb{R})\). In particular, \(y \geq 0\), \(y(\partial I) = \{0\}\), and the convergence of \((\gamma_\varepsilon)\) is uniform in \(I\). Since the set of non-negative functions is closed under
weak-$\star$ convergence in $L^\infty(I)$, $\gamma$ satisfies $x' \geq 0$. From

$$A_0 = A_\varepsilon = 2\pi q_\varepsilon \int_I y_\varepsilon \, dt \to 2\pi q \int_I y \, dt$$

we conclude that neither $q = 0$ nor $y \equiv 0$ in $I$. Without loss of generality, we assume $q = 1$, thus $|\gamma'| \leq 1$ almost everywhere in $I$.

Uniform convergence implies that for any $J \Subset \{y > 0\}$ there is a constant $c_J > 0$ such that $y_\varepsilon \geq c_J$ in $J$ for all sufficiently small $\varepsilon$. Therefore,

$$\frac{1}{2\pi} \int_{M_\varepsilon(J)} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon \geq c_J \int_J \frac{1}{q_\varepsilon} |u'_\varepsilon|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \, dt$$  \hspace{1cm} (4.1)

and the well-known arguments of Modica and Mortola [19, 20] apply in $J$, see in particular [5, Lemma 6.2 and Remark 6.3] for a proof in one dimension. The outcome is a finite set of points $S_J \subset J$ and a piecewise constant function $u: J \to \{\pm 1\}$ whose jump set is contained in $S_J$ such that a subsequence of $u_\varepsilon$ converges to $u$ in measure and almost everywhere in $J \setminus S_J$. Since $(u_\varepsilon)$ is uniformly bounded in $L^\infty(I)$, convergence in $L^p(J)$ for any $p \in [1, \infty)$ follows. Moreover, in the one-dimensional setting we obtain that in any set compactly contained in $J \setminus S_J$ we have $|u_\varepsilon| \geq 1/2$ for all sufficiently small $\varepsilon$.

Exhausting $\{y > 0\}$ by a sequence of increasing sets such as $J_k = \{y > 1/k\}$ as $k \to \infty$ and taking a diagonal sequence, we find an at most countable set $S \subset \{y > 0\}$ with $S \cap J$ finite for any $J \Subset \{y > 0\}$, a function $u: \{y > 0\} \to \{\pm 1\}$ with $S_u \subset S$, and subsequence of $(u_\varepsilon)$ that converges to $u$ in measure and almost everywhere in $\{y > 0\}$ and that satisfies $|u_\varepsilon| \geq 1/2$ in any $J \Subset \{y > 0\} \setminus S$ for all sufficiently small $\varepsilon > 0$ depending on $J$. From the uniform $L^\infty$-bound on $u_\varepsilon$ we infer convergence in $L^p(\{y > 0\})$ for any $p \in [1, \infty)$.

Along this subsequence we establish further compactness of the curves. Given $J \Subset \{y > 0\} \setminus S$, there holds $q_\varepsilon \leq 2$, $|u_\varepsilon| \geq 1/2$, and $y_\varepsilon \geq c_J$ for all sufficiently small $\varepsilon > 0$. Therefore, using (2.7) we find that

$$\int_{M_\varepsilon} u^2_\varepsilon |B_\varepsilon|^2 \, d\mu_\varepsilon \geq \frac{1}{4} \int_{M_\varepsilon(J)} \kappa^2_\varepsilon \, d\mu_\varepsilon \geq \frac{\pi c_J}{16} \int_{\gamma'_\varepsilon} |\gamma''_\varepsilon|^2 \, dt$$

is uniformly bounded for all sufficiently small $\varepsilon$, and a subsequence of $\gamma''_\varepsilon$ converges weakly to some $\gamma''_J$ in $L^2(J; \mathbb{R}^2)$. From $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2)$ we infer that $\gamma''_J$ is the weak derivative of $\gamma'$ in $J$ and that the whole sequence converges weakly in $W^{2,\infty}(J; \mathbb{R}^2)$. This shows $\gamma_\varepsilon \rightharpoonup \gamma$ in $W^{2,\infty}(\{y > 0\} \setminus S; \mathbb{R}^2)$ and by embedding $\gamma_\varepsilon \rightharpoonup \gamma$ in $C^1_{\text{loc}}(\{y > 0\} \setminus S; \mathbb{R}^2)$ and $\gamma'_\varepsilon \rightharpoonup \gamma'$ pointwise in $\{y > 0\} \setminus S$. Hence, we obtain $S_\gamma \subset S$, $1 = \lim q^2_\varepsilon = \lim |\gamma''_\varepsilon|^2 = |\gamma'|^2$ in $\{y > 0\} \setminus S$, and

$$A_0 = A_\varepsilon = 2\pi \int_{\{y > 0\}} q_\varepsilon y_\varepsilon \, dt + 2\pi \int_{\{y = 0\}} q_\varepsilon y_\varepsilon \, dt \to 2\pi \int_{\{y > 0\}} y \, dt = A_\gamma$$

as well as

$$mA_0 = \int_{M_\varepsilon} u_\varepsilon \, d\mu_\varepsilon \to \int_{M_\gamma} u \, d\mu.$$
Since the right hand side is bounded independently of $J$, we obtain
\[
\int_{M_r(\{y>0\}\setminus S)} |B|^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \int_{M_{\varepsilon}} u_{\varepsilon}^2 |B_{\varepsilon}|^2 \, d\mu_{\varepsilon} < \infty
\]
by exhausting $\{y > 0\} \setminus S$. The inequality $H^1(M_r(S_u \cup S_\gamma)) < \infty$ follows from (4.1) and the fact that each kink or interface $s \in S_u \cup S_\gamma$ carries at least an energy of $2\pi \sigma y(s)$ in the limit $\varepsilon \to 0$. The details are given in the lower bound section and thus are omitted here.

The following corollary renders the convergence around possible kinks more precise and will be used to establish the lower bound.

**Corollary 4.2.** For any subsequence as in Lemma 4.1 there are angle functions $\varphi_{\varepsilon} \in L^\infty(I) \cap W^{1,2}_{\text{loc}}(I)$ of $\gamma_{\varepsilon}$ that converge weakly in $BV_{\text{loc}}(\{y > 0\})$ to an angle function $\varphi$ of $\gamma$ in $\{y > 0\}$. Moreover, $\varphi \in W^{1,2}(J \setminus S)$ for any $J \Subset \{y > 0\}$.

**Proof.** Without loss of generality let $I = (0, L_\gamma)$ and $q_\gamma = 1$. Since $\gamma_{\varepsilon} \in W^{2,2}_{\text{loc}}(I; \mathbb{R}^2)$ and $x_{\varepsilon}' \geq 0$ by definition of $C$, there are angle functions $\varphi_{\varepsilon} \in W^{1,2}_{\text{loc}}(I; [-\pi/2, \pi/2])$ of $\gamma_{\varepsilon}$. Recalling $\varphi_{\varepsilon}' = -\kappa_{1,\varepsilon} q_{\varepsilon}$, uniform convergence of $y_{\varepsilon}$, and Lemma 3.1, we fix $J \Subset \{y > 0\}$ and obtain that
\[
\int_J |\varphi_{\varepsilon}'| \, dt = \int_J |\kappa_{1,\varepsilon}| q_{\varepsilon} \, dt \leq \frac{1}{2\pi c J} \int_{M_{\varepsilon}} |B_{\varepsilon}| \, d\mu_{\varepsilon}
\]
is uniformly bounded by Lemma 3.1 for all sufficiently small $\varepsilon > 0$. Hence, $\varphi_{\varepsilon}$ is uniformly bounded in $W^{1,1}(J)$ and there exists a subsequence that converges weakly in $BV(J)$ to some $\varphi$, that is, $\varphi_{\varepsilon} \rightharpoonup \varphi$ in $L^1(J)$ and $\kappa_{1,\varepsilon} \, dt$ restricted to $J$ converges weakly to the measure $d\varphi'$. Consequently, $\gamma_{\varepsilon}' = q_{\varepsilon} (\cos \varphi_{\varepsilon}, \sin \varphi_{\varepsilon}) \to (\cos \varphi, \sin \varphi) = \gamma'$ in $L^p(J; \mathbb{R}^2)$, that is, $\varphi$ is an angle function of $\gamma$. Since this argument can be applied to any subsequence, convergence of the whole sequence $(\varphi_{\varepsilon})$ in $BV(J)$ follows, and $\varphi$ is defined almost everywhere in $\{y > 0\}$ by exhaustion. Arguing as for $\gamma_{\varepsilon}''$ in Lemma 4.1, we obtain $\varphi_{\varepsilon} \to \varphi$ in $W^{1,2}(J)$ for any $J \Subset \{y > 0\} \setminus S$ and
\[
\int_J |\varphi'|^2 \, dt \leq \frac{4}{\pi c J} \liminf_{\varepsilon \to 0} \int_{M_{\varepsilon}} u_{\varepsilon}^2 \kappa_{1,\varepsilon}^2 \, d\mu_{\varepsilon} < \infty.
\]
Exhausting $J \Subset \{y > 0\}$ by $\tilde{J} \Subset J \setminus S$ yields $\varphi' \in L^2(J \setminus S)$ and $\varphi \in W^{1,2}(J \setminus S)$. 

\section{4.2 Lower bound}

To prove the lower bound
\[
\liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon}(\gamma_{\varepsilon}, u_{\varepsilon}) \geq \mathcal{F}(\gamma, u)
\]
whenever $(\gamma_{\varepsilon}, u_{\varepsilon})$ converges to $(\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$, it suffices to consider sequences such that the left hand side of (4.2) is finite and the limit inferior is attained. Then by definition of $\mathcal{F}_{\varepsilon}$ we have $(\gamma_{\varepsilon}, u_{\varepsilon}) \in \mathcal{C} \times \mathcal{P}$, and the equi-coercivity result yields $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ and the convergence properties listed in Lemma 4.1 and Corollary 4.2. In the following we consider the bulk energy $\mathcal{H}$, kinks and interfaces, and the axis of revolution separately.

\subsection{4.2.1 Bulk lower bound}

**Lemma 4.3.** There holds
\[
\liminf_{\varepsilon \to 0} \mathcal{H}_{\varepsilon}(\gamma_{\varepsilon}, u_{\varepsilon}, \{y > 0\}) \geq \mathcal{H}(\gamma, u).
\]
Proof. Let \( J \in \{ y > 0 \} \setminus S \). From \( \gamma_\varepsilon \to \gamma \) in \( W^{2,2}(J; \mathbb{R}^2) \), \( u_\varepsilon \to u \in \{ \pm 1 \} \) in \( L^2(J) \), and \( \sup_\varepsilon \| u_\varepsilon \|_{L^\infty(J)} < \infty \) we infer that
\[
\varepsilon \rightarrow 0, \quad \varepsilon H_\varepsilon \sqrt{|\gamma_\varepsilon'|y_\varepsilon} \to uH \sqrt{|\gamma'|y} \quad \text{in } L^2(J)
\]
and, using (2.5) and \( |\gamma_\varepsilon'| = q_\varepsilon \), that
\[
\varepsilon^2 K \gamma_\varepsilon |\gamma_\varepsilon'|y_\varepsilon = -\varepsilon^2 y''_\varepsilon \frac{q_\varepsilon}{q} = \varepsilon^2 K |\gamma'|y_\varepsilon \quad \text{in } L^1(J).
\]
Moreover, we have
\[
u_\varepsilon H_\varepsilon(u_\varepsilon)\sqrt{|\gamma_\varepsilon'|y_\varepsilon} \to uH_\varepsilon(u)\sqrt{|\gamma'|y} \quad \text{in } L^2(\{ y > 0 \}). \quad (4.3)
\]
Hence the inequality
\[
\mathcal{H}(\gamma, u, J) + \int_{M_\varepsilon(J)} H_\varepsilon(u)^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \left( \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) + \int_{M_\varepsilon(J)} u_\varepsilon^2 H_\varepsilon(u_\varepsilon)^2 \, d\mu_\varepsilon \right) \quad (4.4)
\]
holds. As seen in (3.2), the integrand on the right hand side of (4.4) is non-negative, so we estimate the integral from above by extending its domain to \( M_\varepsilon(\{ y > 0 \}) \). The right hand side is then independent of \( J \in \{ y > 0 \} \setminus S \), and by exhausting \( \{ y > 0 \} \setminus S \) we obtain
\[
\mathcal{H}(\gamma, u) + \int_{M_\varepsilon(\{ y > 0 \} \setminus S)} H_\varepsilon(u)^2 \, d\mu \leq \liminf_{\varepsilon \to 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{ y > 0 \}) + \limsup_{\varepsilon \to 0} \int_{M_\varepsilon(\{ y > 0 \})} u_\varepsilon^2 H_\varepsilon(u_\varepsilon)^2 \, d\mu_\varepsilon.
\]
The claim now follows from the convergence (4.3). \( \square \)

### 4.2.2 Kinks and interfaces

Next we consider the interface energies \( \mathcal{I}_\gamma \) and \( \mathcal{I} \) in \( \{ y > 0 \} \). Points in \( S \setminus (S_u \cup S_v) \) do not contribute to the limit energy \( \mathcal{I} \), so it suffices to examine \( s \in S_u \cup S_v \). In the following let \( J \in \{ y > 0 \} \) be an interval around \( s \) such that \( \overline{J} \setminus S = \{ s \} \), which exists because \( S \cap \{ y > y(s)/2 \} \) is finite. Again, we assume without loss of generality that \( q = 1 \).

If \( s \in S_u \setminus S_v \) is an interface without kink, we estimate the curvature term in \( \mathcal{I}_\varepsilon \) from below by zero and use a standard argument for the other terms as in [5, Chapter 6]. That is, from the convergence of \( u_\varepsilon \) we deduce that there are points \( a_\varepsilon, b_\varepsilon \in J \) such that \( a_\varepsilon \to s, b_\varepsilon \to s, u_\varepsilon(a_\varepsilon) \to -1, u_\varepsilon(b_\varepsilon) \to +1 \), and without loss of generality \( a_\varepsilon < s < b_\varepsilon \). By Young’s inequality and a change of variables we obtain
\[
\frac{1}{2\pi} \int_{M_\varepsilon(a_\varepsilon, b_\varepsilon)} \varepsilon |\nabla M_\varepsilon u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon \geq \left( \inf_{(a_\varepsilon, b_\varepsilon)} y_\varepsilon \right) \int_{a_\varepsilon}^{b_\varepsilon} 2\sqrt{W(u_\varepsilon)} |u_\varepsilon'| \, dt \\
\geq \left( \inf_{(a_\varepsilon, b_\varepsilon)} y_\varepsilon \right) \int_{a_\varepsilon}^{b_\varepsilon} 2\sqrt{W(u)} \, du,
\]
and taking the lower limit as \( \varepsilon \to 0 \) yields
\[
\liminf_{\varepsilon \to 0} \int_{M_\varepsilon(a_\varepsilon, b_\varepsilon)} \varepsilon |\nabla M_\varepsilon u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon \geq 2\pi y(s) \sigma. \quad (4.5)
\]
If \( s \in S_\epsilon \cap S_\gamma \) is a kink and a proper interface, let \( (\varphi_\epsilon) \) be angle functions of \( (\gamma_\epsilon) \) in \( J \) that converge weakly in \( BV(J) \) to an angle function \( \varphi \) of \( \gamma \). We then have

\[
\int_J \varphi'_\epsilon y_\epsilon \, dt \to [\varphi](s)y(s) - \int_{J \setminus \{s\}} \kappa_1 y \, dt,
\]

where \( \kappa_1 = -\varphi' \in L^2(J \setminus \{s\}) \) is the curvature of \( \gamma \) in \( J \setminus \{s\} \) and \([\varphi](s)\) the jump of the angle \( \varphi \) at \( s \). Since \( x_\epsilon' \geq 0 \), we may assume \( \varphi_\epsilon \in [-\pi/2, \pi/2] \), so that \([\varphi'] = [\varphi] \in [-\pi, \pi] \). The key step for the lower bound in \( J \) is to formalise the intuition that \( \gamma_\epsilon \) approaches the kink where \( u_\epsilon \) is close to zero.

**Lemma 4.4.** For sufficiently small \( \delta > 0 \) let \( J_{\epsilon,\delta} = \{ t \in I : |u_\epsilon| \leq \delta \} \). Then

\[
\liminf_{\epsilon \to 0} \left| \int_{J \setminus J_{\epsilon,\delta}} \varphi'_\epsilon y_\epsilon \, dt \right| \geq y(s)[[\varphi](s)].
\]

**Proof.** We show that the complement of \( J_{\epsilon,\delta} \) in \( J \) contains only the absolutely continuous part of \( d\varphi' \). Let \( \beta > 0 \) be arbitrary but fixed, and let \( U_\beta = [s - \beta, s + \beta] \). As \( J \setminus U_\beta \subseteq \{ y > 0 \} \setminus S \), we have \( |u_\epsilon| \geq 2\delta \) in \( J \setminus U_\beta \) for all sufficiently small \( \epsilon \) according to Lemma 4.1, and therefore \( J \cap J_{\epsilon,\delta} \subset U_\beta \). Writing \( w_\epsilon = \varphi'_\epsilon y_\epsilon + \kappa_1 y_\epsilon \), we have

\[
\left| \int_{J \setminus J_{\epsilon,\delta}} w_\epsilon \, dt \right| \leq \int_{J \setminus U_\beta} w_\epsilon \, dt + \int_{(J \setminus J_{\epsilon,\delta}) \cap U_\beta} |w_\epsilon| \, dt
\]

for all sufficiently small \( \epsilon \). The first term on the right hand side converges to \( 0 \) by weak convergence of \( w_\epsilon \) in \( J \setminus U_\beta \), and the second integral is bounded by a constant times \( \sqrt{\beta} \) due to Hölder’s inequality and the uniform bound on the second fundamental forms of \( M_\epsilon \) in \( I \setminus J_{\epsilon,\delta} \). As \( \beta > 0 \) is arbitrary, we obtain

\[
\limsup_{\epsilon \to 0} \left| \int_{J \setminus J_{\epsilon,\delta}} w_\epsilon \, dt \right| = 0,
\]

and taking the lower limit in the inequality

\[
\left| \int_{J \cap J_{\epsilon,\delta}} w_\epsilon \, dt \right| \geq \left| \int_J w_\epsilon \, dt \right| - \left| \int_{J \setminus J_{\epsilon,\delta}} w_\epsilon \, dt \right|
\]

yields the claim because \( |J \cap J_{\epsilon,\delta}| \to 0 \) as \( \epsilon \to 0 \) due to the uniform bound on

\[
\int_{M_\epsilon(J \cap J_{\epsilon,\delta})} \frac{1}{\epsilon} W(u_\epsilon) \, d\mu_\epsilon \geq \left( \inf_{[-\delta,\delta]} W \right) \frac{|J \cap J_{\epsilon,\delta}|}{\epsilon}
\]

and \( y\kappa_1 \in L^2(J \setminus \{s\}) \). \( \square \)

Using the above splitting of \( J \) into \( J \cap J_{\epsilon,\delta} \) and \( J \setminus J_{\epsilon,\delta} \), we prove the lower bound inequality.

**Lemma 4.5.** There holds

\[
\liminf_{\epsilon \to 0} \mathcal{I}_\epsilon(\gamma_\epsilon, u_\epsilon, J) \geq 2\pi \left( \hat{\sigma} \|\gamma'\|_1 + \sigma \right) y(s).
\]
Proof. With the notation of the Lemma 4.4 we have
\[
\frac{\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J)}{2\pi} \geq \int_{J \cap J_{\varepsilon, \delta}} \left( \frac{\varepsilon}{q_\varepsilon} |\varphi_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt + \int_{J \setminus J_{\varepsilon, \delta}} \left( \frac{\varepsilon}{q_\varepsilon} |u_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt.
\]
Estimating the first term on the right hand side with Young’s inequality we obtain
\[
\int_{J \cap J_{\varepsilon, \delta}} \left( \frac{\varepsilon}{q_\varepsilon} |\varphi_\varepsilon'|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) \right) y_\varepsilon \, dt \geq \int_{J \cap J_{\varepsilon, \delta}} 2\sqrt{W(u_\varepsilon)} |\varphi_\varepsilon'| y_\varepsilon \, dt
\geq 2 \inf_{u \in [-\delta, \delta]} \sqrt{W(u)} \left| \int_{J \cap J_{\varepsilon, \delta}} \varphi_\varepsilon y_\varepsilon \, dt \right|.
\]
With the second integral we deal as in (4.5); the only difference is that we now find an interval \((a_\varepsilon, b_\varepsilon) \subset J \setminus J_{\varepsilon, \delta}\) such that \(u_\varepsilon(a_\varepsilon) \to \delta, u_\varepsilon(b_\varepsilon) \to 1\) on one side of \(s\), and the same with \(-\delta\) and \(-1\) on the other side. Combining both estimates and taking the lower limit as \(\varepsilon \to 0\) yields
\[
\frac{1}{2\pi} \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \geq 2y(s)[\varphi](s) \inf_{[-\delta, \delta]} \sqrt{W(u)}
+ 2y(s) \int_{-\delta}^{\delta} \sqrt{W(u)} \, du + 2y(s) \int_{-1}^{-\delta} \sqrt{W(u)} \, du.
\]
Taking the supremum over \(\delta > 0\) finishes the proof.

Finally, if \(s \in S_\gamma \setminus S_u\) is a ghost interface, then the phase field \(u\) is constant in \(J\), say \(u \equiv 1\). The argument with the splitting of \(J\) into \(J \cap J_{\varepsilon, \delta}\) and \(J \setminus J_{\varepsilon, \delta}\) is as in Lemma 4.5, but now there is an interval \((a_\varepsilon, b_\varepsilon) \subset J \setminus J_{\varepsilon, \delta}\) such that \(u_\varepsilon(a_\varepsilon) \to \delta, u_\varepsilon(b_\varepsilon) \to 1\) on either side of \(s\). Hence, we conclude
\[
\frac{1}{2\pi} \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) \geq \sigma|\gamma'(s)|y(s) + 4y(s) \int_{0}^{1} \sqrt{W(u)} \, du,
\]
and the right hand side is equal to \(\hat{\sigma}|\gamma'(s)|y(s) + \sigma y(s)\) due to the symmetry of \(W\). The same argument holds when \(u \equiv -1\) near \(s\).

The above reasoning extends to any finite subset \(S\) of \(S_\gamma \cup S_u\), and we obtain
\[
\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{y > 0\}) \geq 2\pi \sum_{s \in S} \left( \sigma|\gamma'(s)| + \sigma \right) y(s).
\]
Since the left hand side is independent of \(S\), we conclude the lower bound for kinks and interfaces
\[
\liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, \{y > 0\}) \geq 2\pi \sum_{s \in S_\gamma \cup S_u} \left( \sigma + \hat{\sigma}|\gamma'(s)| \right) y(s) = \mathcal{I}(\gamma, u, \{y > 0\}).
\]

### 4.2.3 Axis of revolution

To motivate the lower bound estimate at the axis of revolution, we first consider the simple example that \(\gamma_\varepsilon(t) = (qt, y_\varepsilon), y_\varepsilon \in \mathbb{R}\) is a straight horizontal line segment in \(R = \{y = 0\}\) such that \(y_\varepsilon \to 0\) as \(\varepsilon \to 0\). Then \(\kappa_{1,\varepsilon} = 0\), while \(\kappa_{2,\varepsilon} = 1/y_\varepsilon\) blows up and contributes to the limit of \(\mathcal{F}_\varepsilon\). From the uniform bound on
\[
\int_{M_\varepsilon(R)} u_\varepsilon^2 \kappa_2^2 \, d\mu_\varepsilon \geq \frac{2\pi}{q_\varepsilon \sup_R y_\varepsilon} \int_{R} u_\varepsilon^2 x_\varepsilon^2 \, dt
\]
and \( x'_\varepsilon = q_\varepsilon \rightarrow q = x' \) in \( R \) we get \( u_\varepsilon \rightarrow 0 \) in \( R \). Therefore, the potential term contributes to the limit energy. On the other hand, there is no reason for \( u_\varepsilon \) to have a large gradient in \( R \), and it is reasonable to assume that \( u_\varepsilon \) tends to zero sufficiently fast so that there is no contribution of \( \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \) in the limit. Then

\[
\mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \sim \int_{M_\varepsilon(R)} \frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon \kappa_{2,\varepsilon}^2 \, d\mu_\varepsilon \sim \int_R \frac{y_\varepsilon}{\varepsilon} + \varepsilon \, dy_\varepsilon \, dt
\]

is bounded as \( \varepsilon \rightarrow 0 \) if and only if \( y_\varepsilon \sim \varepsilon \), and in this case \( \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \sim \mathcal{H}^1(R) \).

To extend this reasoning to general \((\gamma_\varepsilon, u_\varepsilon)\), when in particular the behaviour of \( u_\varepsilon \) is not known, recall that \( A_\varepsilon(R) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) and \( \|u_\varepsilon\|_\infty \leq C_0 \). These properties imply

\[
\int_{M_\varepsilon(R)} u_\varepsilon^2 H_\varepsilon(u_\varepsilon)^2 \, d\mu_\varepsilon \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,
\]

and with (3.2) we conclude

\[
\liminf_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) = \liminf_{\varepsilon \rightarrow 0} \left( \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) + \int_{M_\varepsilon(R)} u_\varepsilon^2 H_\varepsilon(u_\varepsilon)^2 \, d\mu_\varepsilon \right) \geq 0.
\]

For the interface energy we consider again \( J_{\varepsilon,\delta} = \{|u_\varepsilon| \leq \delta\} \). Similar to the proof of Lemma 4.4, Hölder’s inequality yields

\[
\left( \frac{\delta}{\varepsilon} \int_{M_\varepsilon(R-J_{\varepsilon,\delta})} |B_\varepsilon| \, d\mu_\varepsilon \right)^2 \leq A_\varepsilon(R) \int_{M_\varepsilon} u_\varepsilon^2 |B_\varepsilon|^2 \, d\mu_\varepsilon,
\]

and the right hand side of (4.6) vanishes in the limit \( \varepsilon \rightarrow 0 \). Thus, using Young’s inequality and \( x'_\varepsilon \xrightarrow{\ast} x' \) in \( L^\infty(I) \), we find

\[
\liminf_{\varepsilon \rightarrow 0} J_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \geq \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(R \cap J_{\varepsilon,\delta})} \varepsilon |B_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon
\]

\[
\geq 2 \left( \inf_{u \in [-\delta,\delta]} \sqrt{W(u)} \right) \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(R \cap J_{\varepsilon,\delta})} |B_\varepsilon| \, d\mu_\varepsilon
\]

\[
= 2 \left( \inf_{u \in [-\delta,\delta]} \sqrt{W(u)} \right) \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(R)} |B_\varepsilon| \, d\mu_\varepsilon
\]

\[
\geq 4\pi \left( \inf_{u \in [-\delta,\delta]} \sqrt{W(u)} \right) \int_R x' \, dt.
\]

Taking the supremum over \( \delta > 0 \) and combining with the estimate for \( \mathcal{H}_\varepsilon \) yields

\[
\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \geq \tilde{\sigma} \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(R)} |B_\varepsilon| \, d\mu_\varepsilon \geq 2\pi \tilde{\sigma} \int_R x' \, dt.
\]

Due to \( x' \geq 0 \) and \( y' = 0 \) in \( R \), we have

\[
2\pi \int_R x' \, dt = 2\pi \int_R |y'| \, dt = 2\pi \mathcal{L}_{\gamma}(R) = \mathcal{H}^1(M_\gamma(R)),
\]

and this concludes the proof of the lower bound (4.2).
4.3 Upper bound inequality

This section is devoted to the upper bound inequality
\[
\lim_{\epsilon \to 0} \sup \mathcal{F}_\epsilon(\gamma_\epsilon, u_\epsilon) \leq \mathcal{F}(\gamma, u)
\]
whenever \((\gamma, u) \in \mathcal{D} \times \mathcal{Q}\) has finite energy and \(\gamma\) is parametrised with constant speed. We first approximate \((\gamma, u)\) by a sequence of simple membranes in \(\mathcal{D} \times \mathcal{Q}\) that have a finite number of components and finitely many (ghost) interfaces. We construct a recovery sequence for such a simple membrane by employing essentially local changes to curve and phase field. A diagonal sequence then recovers \((\gamma, u)\).

Throughout this section we assume that \((\gamma, u) \in \mathcal{D} \times \mathcal{Q}\) has finite energy \(\mathcal{F}(\gamma, u)\) and constant speed \(|\gamma'| \equiv q\) in \(I\), without loss of generality \(q = 1\). Since the value of \(u\) in \(\{y = 0\}\) does not enter the energy or our arguments, we assume \(u = 0\) in \(\{y = 0\}\).

4.3.1 Approximation by simple configurations

Lemma 4.6. Assume that \(M_\gamma\) has infinitely many components. Then there is a sequence \((\gamma_\delta, u_\delta) \in \mathcal{D} \times \mathcal{Q}\) that converges to \((\gamma, u)\) in \(C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})\) as \(\delta \to 0\) such that \(\mathcal{F}(\gamma_\delta, u_\delta) \to \mathcal{F}(\gamma, u)\) and each \(M_\delta\) has finitely many components.

Proof. Since \(\mathcal{L}_\gamma, \mathcal{A}_\gamma,\) and \(|\mathcal{F}(\gamma, u)|\) are finite, approximations \((\gamma_\delta, u_\delta)\) can be constructed by replacing all components of \((\gamma, u)\), whose curve length is less than \(\delta\), with a horizontal on the axis of revolution and phase field equal to zero. Convergence of curves and phase fields as \(\delta \to 0\) are easily checked. The energy difference consists of the total energy of the removed components and the interface energy of the new horizontals on the axis of revolution. Thus we have
\[
|\mathcal{F}(\gamma, u) - \mathcal{F}(\gamma_\delta, u_\delta)| \leq \sum_{\omega:\mathcal{L}_\gamma(\omega) \leq \delta} \left(|\mathcal{F}(\gamma, u, \omega)| + 2\pi \delta \mathcal{L}_\gamma(\omega)\right),
\]
and both terms on the right hand side converge to 0 as \(\delta \to 0\). The area difference satisfies
\[
\mathcal{A}_\gamma - \mathcal{A}_{\gamma_\delta} = 2\pi \sum_{\omega:\mathcal{L}_\gamma(\omega) \leq \delta} \int_\omega |\gamma'|y dt \leq 2\pi \delta \sum_{\omega:\mathcal{L}_\gamma(\omega) \leq \delta} \mathcal{L}_\gamma(\omega) = o(\delta),
\]
and it remains to recover the constraints exactly so that \((\gamma_\delta, u_\delta)\) is admissible.

First, if there is an interval \(J \in \{y > 0\} \setminus S_\gamma \cup S_u\) such that \(x' > 0\) in \(J\), then the corresponding component belongs to \((\gamma_\delta, u_\delta)\) for all sufficiently small \(\delta > 0\) and we can add a perturbation that is compactly supported in \(J\), tends to zero in \(W^{2,2}(J; \mathbb{R}^2)\) as \(\delta \to 0\), and recovers the area; if necessary, a reparametrisation fixes the constant speed requirement. If there is no such interval \(J\), then \(\gamma\) consists only of vertical line segments interrupted by kinks and the area constraint is easily established for \(M_{\gamma_\delta}\) by adapting the length of two adjacent line segments.

Second, if there is at least one proper interface without kink in \((\gamma, u)\) then this interface also belongs to \((\gamma_\delta, u_\delta)\) for all sufficiently small \(\delta\). It can be moved by an order less than \(\delta\) and with change in energy of the same order to recover the phase integral constraint. If \((\gamma, u)\) contains no proper interface, introducing one or a finite number of new interfaces at a height of order less than \(\sqrt{\delta}\) above the axis of revolution and flipping the sign of \(u_\delta\) below these new interfaces recovers the constraint. The change in energy contributed by each new interface is proportional to its height above the axis of revolution and thus vanishes in the limit \(\delta \to 0\).
Lemma 4.7. Assume that \( M_\gamma \) has finitely many components. Then there is a sequence \((\gamma_\delta, u_\delta) \in \mathcal{D} \times \mathcal{Q}\) that converges to \((\gamma, u)\) in \( C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})\) as \( \delta \to 0 \) such that \( \mathcal{F}(\gamma_\delta, u_\delta) \to \mathcal{F}(\gamma, u) \) and \( \mathcal{H}^0(S_\gamma \cup S_u) < \infty \). Every component \( \omega = (a, b) \) of \( M_\gamma \) meets the axis of revolution in a line perpendicular to it, that is \( \gamma_\delta = (0, 1) \) near \( a \) in \( \omega \) and \( \gamma_\delta' = (0, -1) \) near \( b \). Two adjacent components are connected by a horizontal segment on the axis of revolution.

Proof. The approximations are constructed by changing \((\gamma, u)\) in segments of length of order \( \delta \) around component boundaries. More precisely, let \( \omega = (a, b) \) be a component of \( M_\gamma \) and \( \delta > 0 \) sufficiently small so that \( b_\delta = b - \delta \in \omega \). In \((b_\delta, b)\) we replace \( \gamma \) by

\[
\gamma_\delta(t) = \begin{cases} (x(b_\delta), y(b_\delta) - t + b_\delta) & \text{if } t \in (b_\delta, \hat{b}_\delta), \\ (x(b_\delta) + t - \hat{b}_\delta, 0) & \text{if } t \in (\hat{b}_\delta, b), 
\end{cases}
\]

that is, we move vertically down until we reach the axis of revolution at \( \hat{b}_\delta = y(b_\delta) + b_\delta \) and fill the remaining interval \((\hat{b}_\delta, b)\) by moving to the right. At the other component boundary \( a \) we do the same but with the horizontal to the left.

Making this replacement for every component, shifting remaining segments of \( \gamma \) slightly in \( x \)-direction to glue all parts together continuously and setting the phase field to, say, \( 1 \) on the new verticals and \( 0 \) on the horizontals we obtain \((\gamma_\delta, u_\delta)\) such that \( \gamma_\delta \to \gamma \) in \( C^0(I; \mathbb{R}^2) \) and \( u_\delta \to u \) in \( L^1(I) \). Denoting by \( M_{\text{orig}} \) all parts of \( M_\gamma \) that have been removed and by \( M_{\text{hor}} \) and \( M_{\text{ver}} \) the introduced horizontals and verticals, the Helfrich energy difference is bounded by

\[
|\mathcal{H}(\gamma, u) - \mathcal{H}(\gamma_\delta, u_\delta)| \leq \int_{M_{\text{orig}}} |H_\gamma - H_s(u)|^2 + |K_\gamma| \, d\mu + \int_{M_{\text{ver}}} H_s(u_\delta)^2 \, d\mu_\delta.
\]

The second term is bounded by \( \|H_s\|_{\infty}^2 \mu_\delta(M_{\text{ver}}) \to 0 \) as \( \delta \to 0 \), and the first tends to 0 as \( \delta \to 0 \) due to \( \mu(M_{\text{orig}}) \to 0 \) and uniform continuity of the integral. The difference in the interface energy consists of original (ghost) interfaces that are omitted in \((\gamma_\delta, u_\delta)\), the two probably introduced kinks for each component, and the new pieces on the axis of revolution. Therefore, we obtain

\[
\frac{1}{2\pi} |\mathcal{I}(\gamma, u) - \mathcal{I}(\gamma_\delta, u_\delta)| \leq \sum_{s \in S_\gamma \cup S_u, y(s) \leq \delta} (\sigma + \hat{\sigma} |(\gamma'|(s)))y(s) + 2N_\gamma(\sigma + \hat{\sigma} \pi)\delta + 2N_\gamma \hat{\sigma} \delta,
\]

where \( N_\gamma \) denotes the number of components of \( \gamma \). The first term converges to 0 as \( \delta \to 0 \) because the sum over all (ghost) interfaces is finite, thus the energy difference vanishes in the limit \( \delta \to 0 \). Since \( y_\delta \leq y \leq \delta \) where \( \gamma \) has been replaced, one easily finds that \( |A_\gamma - A_\delta| \) is of order \( \delta^2 \); thus the constraints can be recovered as in Lemma 4.6.

If necessary, additional minor changes such as adding horizontal segments between adjacent components or removing horizontals at \( \partial I \) can be applied. \( \square \)

From now on we assume that \((\gamma, u)\) has the form of the approximations constructed in Lemma 4.7. As an example, Figure 4.1 shows an approximation of the curve in Figure 3.1.

### 4.3.2 Kinks and interfaces

Let \( s \in S_\gamma \) be a kink, \( S = S_\gamma \cup S_u \), and \( J \subseteq \{y > 0\} \) with \( \overline{J} \cap S = \{s\} \). For simplicity of notation we formulate the following arguments for curves and phase fields given in an interval \( J \) around \( s = 0 \); recall that \( |\gamma'| = 1 \) in \( J \). First we smooth out kinks by a linear interpolation of the tangent angle of \( \gamma \) around \( s = 0 \). This local procedure disturbs the constraints and disrupts the curve, so that we have to add some corrections, one of which is a global shift in \( x \)-direction because of the requirement \( x' \geq 0 \).
Figure 4.1: Approximation of the curve in Figure 3.1, original segments in black, new parts in grey. Small components are removed and the ends of the remaining components are replaced by line segments; between two adjacent components there is always a small horizontal segment on the axis of revolution. The number of (ghost) interfaces is finite.

**Lemma 4.8.** Let $J = (-a, a)$. For all sufficiently small $\varepsilon > 0$ there is $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon) \in W^{2,2}(J; \mathbb{R}^2)$ such that

- $\gamma_\varepsilon$ satisfies $\inf_J y_\varepsilon > 0$ and $x_\varepsilon' \geq 0$;
- $\gamma_\varepsilon$ fits almost into $\gamma$, that is, at the end points of $\gamma_\varepsilon(J)$ we have $\gamma_\varepsilon(-a) = \gamma(-a)$, $\gamma_\varepsilon(-a) = \gamma'(-a)$, $\gamma_\varepsilon(a) = (x(a) + o(1), y(a))$, $\gamma'_\varepsilon(a) = \gamma'(a)$;
- $\gamma_\varepsilon \to \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for any $p \in [1, \infty)$ as $\varepsilon \to 0$;
- $A_\varepsilon(J) = A_\gamma(J) + O(\varepsilon)$ and $\int_{M_\varepsilon(J)} u \, d\mu_\varepsilon = \int_{M_\gamma(J)} u \, d\mu + O(\varepsilon)$ as $\varepsilon \to 0$; and
- with $J_\varepsilon = (-\delta_\varepsilon, \delta_\varepsilon)$, where $\delta_\varepsilon = \frac{||\gamma'||}{\sigma} \varepsilon$, there holds
  \[ \lim_{\varepsilon \to 0} I_\varepsilon(\gamma_\varepsilon, 0, J_\varepsilon) = 2\pi \sigma ||\gamma'||(0) y(0). \]

Moreover, $\gamma'_\varepsilon = \gamma' + r_\varepsilon$ in $J \setminus J_\varepsilon$ where $\text{spt } r_\varepsilon \Subset J \setminus J_\varepsilon$ is independent of $\varepsilon$ and $r_\varepsilon \to 0$ in $W^{1,2}(J; \mathbb{R}^2)$ as $\varepsilon \to 0$.

**Proof.** Let $\varphi$ be an angle function for $\gamma$ in $J$ that is uniformly continuous on either side of $s = 0$ and satisfies $|\varphi| \leq \pi/2$. Denote by $\varphi^+$ and $\varphi^-$ the one-sided limit of $\varphi$ at $s = 0$ from the right and the left, respectively; then the kink carries the “bending energy” $2\pi \sigma y(s)|\varphi^+ - \varphi^-|$ and we have $\delta_\varepsilon = |\varphi^+ - \varphi^-| / \sigma$.

In the simple case that $\gamma$ consists of two straight lines in $J$, the linear interpolation $\varphi_\varepsilon \in W^{1,p}(J)$ of $\varphi^\pm$ in $J_\varepsilon = (-\delta_\varepsilon, \delta_\varepsilon) \subset J$ is given by

\[ \varphi_\varepsilon(t) = \begin{cases} \varphi^- & \text{if } t < -\delta_\varepsilon, \\ \frac{\varphi^+ - \varphi^-}{2\delta_\varepsilon} t + \frac{\varphi^+ + \varphi^-}{2} & \text{if } -\delta_\varepsilon \leq t < \delta_\varepsilon, \\ \varphi^+ & \text{if } \delta_\varepsilon \leq t. \end{cases} \]

The curve $\gamma_\varepsilon$, defined by $\gamma'_\varepsilon = (\cos \varphi_\varepsilon, \sin \varphi_\varepsilon)$ and $\gamma_\varepsilon(-\delta_\varepsilon) = \gamma(-\delta_\varepsilon)$, converges in $W^{1,p}(J; \mathbb{R}^2)$
Figure 4.2: Linear interpolation of the tangent angle on the left and the corresponding curve on the right. Original angle and curve are black, interpolations grey.

To $\gamma$ as $\varepsilon \to 0$. Using Young’s inequality one easily verifies

$$
\frac{1}{2\pi} I_\varepsilon(\gamma, 0, J_\varepsilon) = \left( \frac{2\delta_\varepsilon}{\varepsilon} W(0) + \frac{\varepsilon}{2\delta_\varepsilon} (\varphi^+ - \varphi^-)^2 \right) \frac{1}{2\delta_\varepsilon} \int_{-\delta_\varepsilon}^{\delta_\varepsilon} y\varepsilon dt + \varepsilon \int_{-\delta_\varepsilon}^{\delta_\varepsilon} \kappa^2_{2,\varepsilon} y\varepsilon dt
$$

and by our choice of $\delta_\varepsilon$ we have equality in (4.8). The first integral in (4.8) divided by $2\delta_\varepsilon$ converges to $y(0)$ as $\varepsilon \to 0$, and the second term vanishes because the integral of $\kappa^2_{2,\varepsilon} y\varepsilon$ is bounded. Thus $I_\varepsilon(\gamma, 0, J_\varepsilon) \to 2\pi \hat{\theta} ||\gamma'|| y(0)$ as desired.

For a general angle function $\varphi$ the interpolation is

$$
\varphi_\varepsilon(t) = \begin{cases} 
\varphi(t) & \text{if } |t| > \delta_\varepsilon, \\
\frac{(\varphi(\delta_\varepsilon) - \varphi(-\delta_\varepsilon)) t + (\varphi(\delta_\varepsilon) + \varphi(-\delta_\varepsilon))}{2} & \text{if } |t| \leq \delta_\varepsilon,
\end{cases}
$$

see Figure 4.2, and similarly as above we get

$$
\frac{1}{2\pi} I_\varepsilon(\gamma, 0, J_\varepsilon) = \sqrt{W(0)} \left( ||\varphi|| + \frac{|\varphi(\delta_\varepsilon) - \varphi(-\delta_\varepsilon)|^2}{||\varphi||} \right) \frac{1}{2\delta_\varepsilon} \int_{-\delta_\varepsilon}^{\delta_\varepsilon} y\varepsilon dt + \varepsilon \int_{-\delta_\varepsilon}^{\delta_\varepsilon} \kappa^2_{2,\varepsilon} y\varepsilon dt
$$

$$
\to \hat{\theta} ||\gamma'|| y(0).
$$

By construction, $\varphi_\varepsilon \in [-\pi/2, \pi/2]$, that is $x_\varepsilon' \geq 0$, and $|\gamma_\varepsilon'| \equiv 1$ in $J$. Also, $\gamma_\varepsilon \to \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ because $\varphi_\varepsilon \to \varphi$ in $L^p(J)$ for any $p \in [1, \infty)$. Therefore, $y\varepsilon \geq \inf_J y/2 > 0$ for all sufficiently small $\varepsilon$.

It remains to correct the $y$-coordinate of the right end point of $\gamma_\varepsilon(J)$ and to calculate the error in the area and phase integral constraint. We fix $\tilde{J} \Subset J \setminus J_\varepsilon$ independently of all sufficiently small $\varepsilon$ and $f \in C^\infty_c(\tilde{J})$ such that $f \geq 0$ and $\int_{\tilde{J}} f dt = 1$. The perturbed curve $\tilde{\gamma}_\varepsilon = (x_\varepsilon, y_\varepsilon + \alpha_\varepsilon F)$, where $F(t) = \int_0^t f(s) ds$, has the desired end point $y$-coordinate for

$$
\alpha_\varepsilon = y(a) - y_\varepsilon(a).
$$

Since

$$
|\gamma_\varepsilon(t) - \gamma(t)| \leq \int_{-\delta_\varepsilon}^{\delta_\varepsilon} |\gamma_\varepsilon' - \gamma'| ds \leq 4\delta_\varepsilon = O(\varepsilon),
$$

also $\alpha_\varepsilon$, $\|\tilde{\gamma}_\varepsilon - \gamma\|_\infty$, and $\|\gamma_\varepsilon' - \gamma'|_{L^\infty(J \setminus J_\varepsilon)}$ are at most of order $\varepsilon$. The claims for area and phase constraint follow, and $r_\varepsilon = (0, \alpha_\varepsilon f)$. \qed
Next we construct a recovery sequence for the phase field in $J$ which is in line with $\gamma_\epsilon$ of Lemma 4.8. It is well known, see for instance [1], that in the classical one-dimensional Modica-Mortola setting the optimal $\epsilon$-energy profile for a transition of $u_\epsilon$ from $-1$ to $+1$ is obtained by minimising

$$G_\epsilon(u) = \int_{\mathbb{R}} \epsilon |u'|^2 + \frac{1}{\epsilon} W(u) \, dt$$

among functions $u$ that satisfy $u(0) = 0$ and $u(\pm \infty) = \pm 1$. Indeed, setting $u_\epsilon(t) = u(t/\epsilon)$, we find $G_\epsilon(u_\epsilon) = G_1(u) \geq 2 \int_{\mathbb{R}} \sqrt{W(u)} u' \, dt = 2 \int_{\mathbb{R}} \sqrt{W(u)} \, du = \sigma$.

Equality holds if and only if

$$u' = \sqrt{W(u)},$$

(4.9)

which admits a local solution $p$ with initial condition $p(0) = 0$ because $\sqrt{W(\cdot)}$ is continuous. Obviously, the constants $+1$ and $-1$ are a global super- and sub-solution of (4.9), hence $p$ can be extended to the whole real line. Since $W(p) > 0$ for $p \in (-1, +1)$, $p(t)$ converges to $\pm 1$ as $t \to \pm \infty$. Thus $p(t/\epsilon)$ minimises $G_\epsilon$, and due to the symmetry of $W$ we may assume $-p(-t) = p(t)$.

The building block $p_\epsilon$ of our phase field recovery is given by

$$p_\epsilon(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \delta_\epsilon, \\
p \left( \frac{1 - \delta_\epsilon}{\epsilon} \right) & \text{if } \delta_\epsilon < t \leq \delta_\epsilon + \sqrt{\epsilon}, \\
p(1/\sqrt{\epsilon}) + \frac{1}{\epsilon} (t - \delta_\epsilon - \sqrt{\epsilon}) & \text{if } \delta_\epsilon + \sqrt{\epsilon} < t \leq \delta_\epsilon + \sqrt{\epsilon} + \epsilon (1 - p(1/\sqrt{\epsilon})) , \\
1 & \text{if } \delta_\epsilon + \sqrt{\epsilon} + \epsilon (1 - p(1/\sqrt{\epsilon})) < t, 
\end{cases}$$

which connects $p_\epsilon = 0$ and $p_\epsilon = 1$ by an appropriately scaled optimal profile and a linear segment. In addition, there is a “plateau” \{p_\epsilon = 0\} to smooth out the kink, see Figure 4.3.

In the following lemma we estimate the interface energy of $\gamma_\epsilon$ combined with a suitable phase field $u_\epsilon$ based on $p_\epsilon$.

**Lemma 4.9.** Let $\gamma_\epsilon$ be as in Lemma 4.8. Then there exists $u_\epsilon \in W^{1,p}(J)$ such that $\|u_\epsilon\|_{L^\infty(J)} \leq C_0$, $u_\epsilon = u$ on $\partial J$, $u_\epsilon \to u$ in $L^p(J)$ for any $p \in [1, \infty)$, $\int_{M_\epsilon(J)} u_\epsilon \, d\mu_\epsilon = \int_{M_\epsilon(J)} u \, d\mu + o(\sqrt{\epsilon})$, and

$$\limsup_{\epsilon \to 0} I_\epsilon(\gamma_\epsilon, u_\epsilon, J) \leq 2\pi (\sigma + \delta |[\gamma'](0)|) y(0),$$

$$\limsup_{\epsilon \to 0} H_\epsilon(\gamma_\epsilon, u_\epsilon, J) \leq H(\gamma, u, J).$$

---

**Figure 4.3:** Construction of $p_\epsilon$ consisting of a “plateau” for the curve recovery, the optimal profile, and the connection to 1.
Proof. If $s = 0$ is a proper interface let

$$u_\varepsilon(t) = \begin{cases} p_\varepsilon(t) & \text{if } 0 \leq t, \\ -p_\varepsilon(t) & \text{if } t > 0, \end{cases}$$

if $u(t) = \text{sign } t$ in $J$, and the negative of it, if $u(t) = -\text{sign } t$; for a ghost interface take the combination of $p_\varepsilon(t)$ and $p_\varepsilon(-t)$ or its negative. Obviously, $u_\varepsilon \rightharpoonup u$ in $L^p(J)$, $|u_\varepsilon| \leq C_0$ in $J$, and $u_\varepsilon = u$ on $\partial J$. For the energy estimates we assume $u(t) = \text{sign } t$, the proof of the other cases works with the obvious changes.

Due to $u_\varepsilon \equiv 0$ in $J_\varepsilon = (\varepsilon - \delta_\varepsilon, \delta_\varepsilon)$, Lemma 4.8 provides

$$\limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_\varepsilon) \leq 2\pi \sigma |\gamma'(0)|y(0).$$

Since $\gamma \in W^{2,2}(J \setminus \{0\}; \mathbb{R}^2)$ and $\gamma_\varepsilon = \gamma + o(1)$ in $W^{2,2}(J \setminus J_\varepsilon; \mathbb{R}^2)$, the curvature term in $\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J \setminus J_\varepsilon)$ vanishes in the limit $\varepsilon \to 0$. The other terms are easily estimated by

$$\frac{1}{2\pi} \int_{M_\varepsilon(J \cap \{t > \delta_\varepsilon\})} \varepsilon |\nabla_M p_\varepsilon|^2 + \frac{1}{\varepsilon} W(p_\varepsilon) \, d\mu_\varepsilon \leq \left( \sup_{|\delta_\varepsilon, \delta_\varepsilon + \varepsilon|} y_\varepsilon \right) \int_0^{1/\varepsilon} |p'(t)|^2 + W(p(t)) \, dt + \|y_\varepsilon\|_{\infty} \left( 1 - p \left( 1/\varepsilon \right) \right) (1 + \sup W)$$

on the positive side of $s = 0$, and similarly on the other side. The second term on the right hand side vanishes in the limit $\varepsilon \to 0$ because $p(1/\varepsilon) \to 1$, while in the first term the integral is bounded by $\sigma/2$ and the supremum converges to $y(0)$. Hence, taking the limit superior as $\varepsilon \to 0$ proves the upper bound for $\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J)$. The estimate for the Helfrich energy follows from $\mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J) = \mathcal{H}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J \setminus J_\varepsilon)$, using convergence of $u_\varepsilon$ and $\gamma_\varepsilon \chi_{\varepsilon \cap J_\varepsilon}$.

Finally, one easily sees that

$$\left| \int_{M_\varepsilon(J)} u \, d\mu_\varepsilon - \int_{M_\varepsilon(J)} u_\varepsilon \, d\mu_\varepsilon \right| \leq \delta_\varepsilon + \int_{\delta_\varepsilon}^{\delta_\varepsilon + \varepsilon} \left( 1 - p_\varepsilon(t) \right) \, dt + \varepsilon \left( 1 - p(1/\varepsilon) \right),$$

and since

$$\int_{\delta_\varepsilon}^{\delta_\varepsilon + \varepsilon} \left( 1 - p_\varepsilon(t) \right) \, dt = \sqrt{\varepsilon} \int_0^1 1 - p(t/\varepsilon) \, dt = o(\sqrt{\varepsilon}),$$

the phase integral difference is also of order $\sqrt{\varepsilon}$. \hfill \Box

### 4.3.3 Axis of revolution

Let $J_0 \subset \{y = 0\}$ be an interval that is enclosed by two intervals $J_l$, $J_r$ such that $(J_l \cup J_0 \cup J_r) \cap S = \emptyset$,

$$\gamma'(t) = \begin{cases} (0, -1) & \text{in } J_l, \\ (1, 0) & \text{in } J_0, \\ (0, 1) & \text{in } J_r, \end{cases}$$

and $|u| = 1$ in $J_l \cup J_r$. The limit energy in $J_0$ is $\mathcal{F}(\gamma, u, J_0) = 2\pi \sigma \mathcal{H}^1(J_0)$, and this is easily recovered by setting $u_\varepsilon = 0$ and $\gamma_\varepsilon = \gamma + (0, 2\varepsilon/\delta)$ in $J_0$ because

$$\mathcal{I}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, J_0) = 2\pi \int_{J_0} \left( \frac{1}{\varepsilon} W(0) + \varepsilon^2 \kappa^2_\varepsilon \right) y_\varepsilon \, dt = 2\pi \int_{J_0} \frac{2}{\delta} W(0) + \frac{\delta}{2} \, dt = 2\pi \sigma \mathcal{H}^1(J_0).$$
In $J_l$ and $J_r$ we use the same construction as for kinks. If for simplicity of notation $J_r = (0, a)$, $\gamma(0) = (0, 0)$, and $u \equiv 1$ in $J_r$, we consider the approximate curve $\gamma_\varepsilon$ given by $\gamma_\varepsilon(0) = (0, 2\varepsilon/\hat{\sigma})$ and the angle function
\[
\varphi_\varepsilon(t) = \begin{cases} \frac{\pi}{2} & \text{if } 0 < t < \alpha\varepsilon, \\ \frac{\pi}{2} & \text{if } t > \alpha\varepsilon. \end{cases}
\]

The purpose of $\alpha = 2\pi/(\hat{\sigma}(\pi - 2))$ is to ensure $y_\varepsilon(t) = t = y(t)$ for $t \geq \alpha\varepsilon$. Thanks to $y_\varepsilon \geq 2\varepsilon/\hat{\sigma}$ in $J_r$, we have
\[
\frac{1}{2\pi} \int_{M_e(J_r)} \varepsilon \kappa^2 \, d\mu_\varepsilon = \varepsilon \int_0^{\alpha\varepsilon} \frac{x^2}{y_\varepsilon} \, dt \leq \frac{\hat{\sigma}}{2} \int_0^{\alpha\varepsilon} x^2 \, dt \leq \frac{\hat{\sigma}}{2} \alpha\varepsilon,
\]
and since the computations for all other terms of $I_\varepsilon$ from Section 4.3.2 still apply, we conclude $I_\varepsilon(\gamma_\varepsilon, p_\varepsilon, J_r) \to 0 = I(\gamma, u, J_r)$. For the Helfrich energy we use that $\gamma_\varepsilon$ is a vertical line where $p_\varepsilon \neq 0$ in $J_r$ and obtain
\[
\mathcal{H}_\varepsilon(\gamma_\varepsilon, p_\varepsilon, J_r) = \mathcal{H}_\varepsilon((x_\varepsilon, t), p_\varepsilon, (\alpha\varepsilon, a)) = \int_{M_e(\alpha\varepsilon, a)} p_\varepsilon^2 H_s(p_\varepsilon)^2 \, d\mu_\varepsilon
\]
\[
\to \int_{M(J_r)} H_s(u)^2 \, d\mu = \mathcal{H}(\gamma, u, J_r),
\]
where $x_\varepsilon \equiv \int_0^{\alpha\varepsilon} \cos p_\varepsilon \, dt = 2\alpha\varepsilon/\pi$. The change in area when replacing $\gamma$ by $\gamma_\varepsilon$ is of order $\varepsilon$, and the difference in the phase integral is of order $o(\sqrt{\varepsilon})$ as in Lemma 4.9. A similar construction applies to $u = -1$ and in $J_l$.

### 4.3.4 Recovery of simple configurations

**Corollary 4.10.** Let $(\gamma, u) \in D \times Q$, $|\gamma'| = \text{const}$ in $I$, be a simple membrane as constructed in Lemma 4.7. Then there exists a sequence $(\gamma_\varepsilon, u_\varepsilon) \in C \times P$ such that $(\gamma_\varepsilon, u_\varepsilon) \to (\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ and $\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \leq \mathcal{F}(\gamma, u)$.

**Proof.** We obtain a sequence $(\gamma_\varepsilon, u_\varepsilon)$ that converges in energy and approximates $(\gamma, u)$ in $C^0(I; \mathbb{R}^2) \times L^1(\{y > 0\})$ by combining the local approximations for kinks, interfaces, and the axis of revolution with the unchanged parts of $(\gamma, u)$, taking into account possible $x$-shifts to join segments continuously. This sequence satisfies $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon + O(\varepsilon) = A_0 + O(\varepsilon)$ and $\int_{M_\varepsilon} u_\varepsilon \, d\mu_\varepsilon = mA_0 + o(\sqrt{\varepsilon})$, and the area constraint is recovered as in Lemma 4.6. For the phase integral let $h: J \to \mathbb{R}$ be smooth, have compact support in an interval $J$, where $(\gamma, u)$ is unchanged except for an $x$-shift, and satisfy $\int_{M_\varepsilon} h \, d\mu_\varepsilon = 1$. Then $u_\varepsilon + \alpha\varepsilon h$ satisfies the constraint if
\[
\alpha\varepsilon = \int_{M_\varepsilon} u \, d\mu - \int_{M_\varepsilon} u_\varepsilon \, d\mu_\varepsilon = mA_0 - \int_{M_\varepsilon} u_\varepsilon \, d\mu_\varepsilon.
\]

Convergence of $u_\varepsilon + \alpha\varepsilon h \to u$ in $L^p(J)$ as $\varepsilon \to 0$ and of the Helfrich energy are obvious. Since $\alpha\varepsilon$ is of order $o(\sqrt{\varepsilon})$, also the interface energy $I(\gamma_\varepsilon, u_\varepsilon + \alpha\varepsilon h)$ still converges to $I(\gamma, u)$, thanks to
\[
\frac{1}{\varepsilon} W(\pm 1 + \alpha\varepsilon h) = \frac{1}{\varepsilon} \left(W(\pm 1) + \alpha\varepsilon h W'(\pm 1) + O(\alpha^2)\right) = o(1). \quad \square
\]

### 5 Generalisations and open problems

Finally, we discuss some extensions of Theorem 3.4 and open problems. First of all, the proof is easily adapted to non-symmetric potentials $W$. In this case one splits $\sigma$ into two
constants
\[ \sigma^+ = \int_0^1 2\sqrt{W(u)} \, du \quad \text{and} \quad \sigma^- = \int_{-1}^0 2\sqrt{W(u)} \, du \]
and distinguishes proper interfaces and ghost interfaces in the different phases \( u = \pm 1 \) by the line tensions \( \sigma^+ = \sigma^- = 2\sigma^+ \), or \( 2\sigma^- \) instead of \( \sigma \) in the limit energy. One may also consider potentials as \( W(u) = (1-u)^2 \) and drop the phase integral constraint. Then there is only one lipid phase, and \( \nu_\varepsilon \) is merely an auxiliary variable that allows curvature induced kinks in the limit. As already stated, rigidities other than \( k^\pm = -k^\pm_G = 1 \) can be considered as long as the conditions (1.3) hold. Also, the \( u^2 \) in \( H_\varepsilon \) can be replaced by other continuous functions that are equal to 0 for \( u = 0 \) and 1 for \( u = \pm 1 \).

Without change of the proof, the constraint of prescribed area for the approximate setting can be relaxed to
\[ 0 < \inf_{\gamma \in \mathcal{C}_\varepsilon} \mathcal{A}_\gamma \leq \sup_{\gamma \in \mathcal{C}_\varepsilon} \mathcal{A}_\gamma < \infty, \]
and thus incorporated as penalty term in the energy. The same is true for the phase integral. Other constraints that change continuously under the convergence proved in Lemma 4.1 can also be imposed, for instance on the enclosed volume \( V_\gamma = \pi \int_{M_\gamma} x^2 y^2 \, dt \).

The arguments can be adapted to open surfaces of revolution generated by curves \( \gamma = (x, y) : I \to \mathbb{R} \times \mathbb{R}_{>0} \) with prescribed boundary conditions. Alternatively, a uniform bound on \( \nu_\varepsilon \) derived from an energy like \( \mathcal{F}_\varepsilon + \mathcal{G} \), where
\[ \mathcal{G}(\gamma) = \int_{M_\gamma(\partial I)} dH^1 = 2\pi \sum_{s \in \partial I} y(s), \]
is sufficient, as it still ensures a bound on the curve length [14]; the corresponding limit \( \mathcal{F} + \mathcal{G} \) models open lipid bilayers with kinks, see for instance [27]. If boundary conditions for \( \gamma' \) are prescribed, then kinks may appear at the boundary in the sense that the tangent vector of the limit curve differs from the prescribed one and contributes to the limit energy like a ghost interface.

5.1 Gauss curvature, axis of revolution, and full \( \Gamma \)-limit of \( \mathcal{F}_\varepsilon \)

In the study of membranes it is often assumed that \( k^+_G = k^-_G \); then the Gauss curvature integral in (1.1) is a topological invariant and omitted, see for instance [16]. Therefore it is desirable to drop the Gauss curvature in \( \mathcal{F}_\varepsilon \) and to consider
\[ \tilde{\mathcal{F}}_\varepsilon(\gamma, u) = \int_{M_\varepsilon} u^2 (H - H_\varepsilon(u))^2 \, d\mu_\varepsilon + \mathcal{I}_\varepsilon(\gamma, u). \]
Since \( \tilde{\mathcal{F}}_\varepsilon \) still bounds the first variation of \( M_\gamma \), see Lemma 3.1 and the remark after Lemma 3.2, the arguments for equi-coercivity and the lower bound in bulk and at (ghost) interfaces still apply. At the axis of revolution we obtain the estimate
\[ \liminf_{\varepsilon \to 0} \tilde{\mathcal{F}}_\varepsilon(\gamma_\varepsilon, u_\varepsilon, R) \geq \hat{\sigma} \liminf_{\varepsilon \to 0} \int_{M_\varepsilon(R)} |H_\varepsilon| \, d\mu_\varepsilon \quad (5.1) \]
in place of (4.7). A subsequence of the measure \( \nu_\varepsilon = |H_\varepsilon|\mu_\varepsilon \) converges to some \( \nu \) in the sense of finite Radon measures, thus the right hand side of (5.1) is bounded from below by \( \hat{\sigma} \nu(R) \). We would like to connect \( \nu(R) \) or (5.1) to the limit curve \( \gamma \), but due to the lack of good bounds on \( (\nu_\varepsilon) \) in \( R \), we are able to do this only in special situations. If for instance \( J \subset R \) is an interval, \( |\gamma_\varepsilon| \equiv q_\varepsilon, x_\varepsilon' \geq 0 \), and \( \varphi_\varepsilon \) an angle function for \( \gamma_\varepsilon \) we can use the angle formulas (2.3), (2.4) and integrate by parts to find
\[ \frac{1}{2\pi} \int_{M_\varepsilon(J)} H_\varepsilon \, d\mu_\varepsilon = q_\varepsilon \int_J \varphi_\varepsilon \sin \varphi_\varepsilon + \cos \varphi_\varepsilon \, dt - \varphi_\varepsilon y_\varepsilon |\partial J| \geq \mathcal{L}_\varepsilon(J) - \varphi_\varepsilon y_\varepsilon |\partial J|, \quad (5.2) \]
where the last inequality holds due to $\varphi_\varepsilon \sin \varphi_\varepsilon + \cos \varphi_\varepsilon \geq 1$ for $\varphi_\varepsilon \in [-\pi/2, \pi/2]$. Exhausting int $R$ by such intervals $J$, we conclude

$$\liminf_{\varepsilon \to 0} \int_{M_\varepsilon(\text{int } R)} |H_\varepsilon| \, d\mu_\varepsilon \geq 2\pi \liminf_{\varepsilon \to 0} L_\varepsilon(\text{int } R) \geq 2\pi L_{\gamma}(\text{int } R).$$

However, $R$ is a closed set, and in general we only know $\liminf \hat{\mathcal{F}}_k(\cdot, \cdot, R) \geq 0$. Our limit functional is thus

$$\hat{\mathcal{F}}(\gamma, u) = \int_{M_\gamma} (H - H_u(u))^2 \, d\mu + 2\pi \sum_{s \in S_\gamma \cup S_u} (\sigma + \hat{\sigma}|\gamma'(s)|) y(s),$$

which does not provide any information about the axis of revolution at all and, as seen before, cannot be recovered in general.

The following example satisfies $R = \partial R$ and shows that finding the lower bound in int $R$ is not sufficient. Moreover, it highlights the difference between our limit energies and the full $\Gamma$-limit of $\mathcal{F}_\varepsilon$ and $\hat{\mathcal{F}}_\varepsilon$. Let $\rho = (x_\rho, y_\rho)$ be a periodic rectangular signal and

$$\gamma_{\varepsilon_k}(t) = (x_k(t), y_k(t)) = \begin{pmatrix} 0 \\ \varepsilon_k \end{pmatrix} + \begin{pmatrix} x_\rho(kt)/k^2 \\ y_\rho(kt)/k \end{pmatrix}$$

for $t$ in some interval $J$, where $\varepsilon_k = c/k$, $0 < c \ll 1$; see Figure 5.1. Using the constructions from Section 4.3, it is straightforward to check that $(\gamma_{\varepsilon_k}, u_{\varepsilon_k})$ with $u_{\varepsilon_k} \equiv 0$ can be made admissible by smoothing the kinks, reparametrising for constant speed, and extending $\gamma_{\varepsilon_k}(J)$ smoothly to obtain closed surfaces of revolution of prescribed area. Then $\hat{\mathcal{F}}_{\varepsilon_k}(\gamma_{\varepsilon_k}, u_{\varepsilon_k})$ and $\mathcal{F}_{\varepsilon_k}(\gamma_{\varepsilon_k}, u_{\varepsilon_k})$ are uniformly bounded, and in the limit membrane $(\gamma, u)$ the segment $\gamma_{\varepsilon_k}(J)$ collapses to a single point. Hence, if the other segments of $\gamma$ do not touch the axis of revolution in the interior, we have $R = \partial R$. The length of $\gamma_{\varepsilon_k}(J)$ is $2 + 1/k$, thus we find $\liminf \mathcal{F}_{\varepsilon_k}(\gamma_{\varepsilon_k}, u_{\varepsilon_k}, J) \geq 4\pi$, but $\mathcal{F}(\gamma, u, J) = L_{\gamma}(J) = 0$. This suggests that $\Gamma$-lim $\mathcal{F}_\varepsilon$ is not geometric, that is, it is not invariant under reparametrisations.

Recall, on the other hand, that the lower limit of $\mathcal{F}_\varepsilon$ at the axis of revolution is non-negative. Since changes of $(\gamma, u) \in \mathcal{D} \times \mathcal{Q}$ at the axis of revolution do not affect area or phase integral, removing segments of $(\gamma, u)$ at the axis is admissible and only reduces the limit energies. Minimisers of $\mathcal{F}$ and $\Gamma$-lim $\mathcal{F}_\varepsilon$ should thus have no energy at the axis of revolution at all, and for such membranes the two energies agree.

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