Research Article

Asymptotic Behavior of the Solutions of the Generalized Globally Modified Navier–Stokes Equations

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The paper is concerned with the existence and the asymptotic behavior of solutions to a class of generalized Navier–Stokes equations, which generalises the so-called globally modified Navier–Stokes equations. The existence and uniqueness of solutions are proved under different assumptions on the dissipation and modification factors. For the asymptotic behavior of solutions, we prove the existence of global attractors in proper spaces. The results generalize some results derived in our previous work Ann. Polon. Math. 122(2):101–128(2019).

1. Introduction

Well-posedness of 3D Navier–Stokes equation is one of the most challenging problems in modern mathematics [1, 2]. To understand or approximate Navier–Stokes equations, various kinds of modified models were introduced in different contexts, such as the Navier–Stokes-$\alpha$ models introduced by Chen et al. and Ilyin and Titi [3, 4], the Leray-$\alpha$, Clark-$\alpha$, and simplified Bardina models introduced by Titi et al. [5–7], and some other modified Navier–Stokes equations introduced and studied, respectively, by Caraballo and Kloeden, Constantin, Sohr, and Flandoli et al., see [8–12].

In [8], the authors proposed the global modified Navier–Stokes equations:

$$
\begin{cases}
\partial_t u - \nu \Delta u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = f, \\
\nabla \cdot u = 0,
\end{cases}
$$

(1)

where $F_N(r) = \min\{1, N/r\}$, $r \in \mathbb{R}^+$, $N \in \mathbb{R}^+$. Since the modifying factor $F_N(\|\Lambda^\beta u\|)$ decreases the singularity of the quadratic convection term $(u \cdot \nabla)u$, it allows the authors to derive the existence and uniqueness of global solutions [8]. Following [8], the existence results and the asymptotic behaviors of solutions to problem (1) were extensively studied in different contexts, see e.g., [13–21] and the review paper [22].

Recently, Dong and Song [23] studied the globally modified Navier–Stokes equations with fractional dissipation in the whole space $\mathbb{R}^3$:

$$
\begin{cases}
\partial_t u + \nu \Lambda^{2\alpha} u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = 0, \\
\nabla \cdot u = 0.
\end{cases}
$$

(2)

The existence and uniqueness of global solutions was obtained under the assumption $\alpha > (3/4)$, see also [24], for the existence and uniqueness results in a bounded domains. These results review that the modifying factor $F_N(\|\Lambda^\beta u\|)$ decreases the singularity of the term $(u \cdot \nabla)u$ “too much” so that one can control the nonlinear term by using only the fractional dissipation $(-\Delta)^\alpha u$, $\alpha < 1$ rather than $\Delta u$ in (1). This inspires us to weaken the modification term and to investigate that how the dissipation and modification terms interact with each other to determine the existence and uniqueness of the solutions. Precisely speaking, we shall consider the following modified Navier–Stokes equations in $\Omega = [0, L]^3$:

$$
\begin{cases}
\partial_t u - \nu \Delta u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = f, \\
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\begin{cases}
\partial_t u - \nu \Delta u + F_N(\|\nabla u\|)(u \cdot \nabla)u + \nabla P = f, \\
\nabla \cdot u = 0,
\end{cases}
$$

(3)
\[
\begin{aligned}
\bar{\partial}_t u + \nu \Delta^a u + F_N \left( \left\| \Lambda^s u \right\| \right) (u \cdot \nabla) u + \nabla P &= f, \\
\nabla \cdot u &= 0, \\
u \rho_0 &= u_0(x),
\end{aligned}
\] (3)

with periodic boundary conditions, where the constants \( \nu > 0 \) and \( \alpha, \beta \geq 0 \).

Assume that the initial data and the forcing term are mean-free functions, i.e.,
\[
\int_\Omega u_0 \, dx = 0.
\] (4)

Then, the solution is also a mean-free function, and the Poincaré inequality holds. We prove that system (3) admits an attractor and the upper bound of its fractal dimension.

Existence and uniqueness results of solutions, while in Section 3, we prove the existence of solutions.

We provide some preliminaries about the function spaces \([8, 23, 24]\) to more general settings.

\[ \text{Assume that the initial data and the forcing term are} \]
\[ \text{mean in} \ L^p_\Omega, \text{for any tempered distribution} \ f \text{,} \]
\[ \text{transform of} \ f, \text{where} \ \tilde{f}(\xi) \text{is the Fourier} \]
\[ \text{transform of} \ f(x). \text{Especially,} \ \Lambda = (-\Delta)^{1/2}. \text{Let} \ \mathcal{C}_p(\Omega) \text{be the space of restrictions of} \ \Omega \text{of infinitely differentiable functions that are} \ L \text{-periodic in each direction and} \]
\[ \text{and with zero mean in} \ \Omega. \text{For} \ \alpha \in \mathbb{R}, \text{we denote by} \ H^s(\Omega) \text{the closure of} \]
\[ \text{under the norm} \]
\[ \left\| f \right\|_{H^s} = \left\| \Lambda^s f \right\|_{L^2} = \left( \sum_{k \in \mathbb{Z}^d} |k|^{2s} \left| \tilde{f}(k) \right|^2 \right)^{1/2}, \] (6)

that is, the space of periodic functions with zero mean such that \( \| f \|_{H^s} < \infty \). It is obvious that \( H^s(\Omega) \to H^s(\Omega) \) (compact embedding), for any \( s_1 \leq s_2. \) Moreover, for \( p \in [1, \infty], \) we denote by \( H^{s,p}(\Omega) \) the space of periodic mean-free \( L^p(\Omega) \) functions \( \varphi, \) which can be written as \( \varphi = \Lambda^s \psi, \) with \( \psi \in L^p. \) This is normed by \( \| \varphi \|_{H^{s,p}} = \| \Lambda^s \varphi \|_{L^p}. \)

For \( s \in \mathbb{R}, \) we denote
\[
\mathbb{H}^s = \left\{ u \in H^s(\Omega)^3, \text{div} \ u = 0 \right\},
\]
\[
\mathbb{H}^{s,p} = \left\{ u \in H^{s,p}(\Omega)^3, \text{div} \ u = 0 \right\}.
\] (7)

Particularly, when \( s = 0, \) we denote \( \mathbb{H}^0 \) by \( \mathbb{H} \) for short.

In this study, for any Banach space \( X, \) we denote its norm as \( \| \cdot \|_X; \) particularly, \( \| \cdot \|_{L^2} \) will be abbreviated as \( \| \cdot \|. \)

Now, we recall the definitions of the global attractor and the fractal dimension, see \([26, 27]\).

Definition 1. Let \( \{ S(t) \}_{t \geq 0} \) be a semigroup on a Banach space \( X. \) A subset \( A \subset X \) is called a global attractor for the semigroup if \( A \) enjoys the following properties:

(i) \( A \) is compact in \( X. \)

(ii) \( A \) is invariant, i.e., \( S(t)A = A, \) for any \( t \geq 0. \)

(iii) \( A \) attracts every bounded subset of \( X, \) i.e., \( \forall B \subset X \text{bounded,} \lim_{t \to \infty} \text{dist}(S(t)B, A) = 0. \)

where dist is the Hausdorff semidistance between sets in \( X, \) defined as
\[
\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|_X, \quad \forall A, B \subset X. \] (8)

Definition 2. The fractal dimension of a compact set \( K \) in a Banach space \( X \) is defined as
\[
d_f(K) = \limsup_{\epsilon \to 0} \frac{\log \mathcal{N}_\epsilon(K)}{-\log \epsilon}, \] (9)

\[ \text{where} \mathcal{N}_\epsilon(K) \text{is the minimal number of balls of radius} \ \epsilon \text{in} \ X \text{needed to cover} \ K. \]

The following inequalities may be found in \([26, 28]\).

Lemma 1 (Young's inequality). For any positive constants \( a, b, \) and \( \epsilon \) and any \( 1 < p < \infty, \) it holds that
\[
ab \leq \frac{\epsilon}{p} + \frac{1}{p} \left( \frac{1}{p(p-1)} \right)^{b(p-1)}. \] (10)

Lemma 2 (Poincare's inequality). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \) and let \( p \) be a continuous seminorm on \( H^1(\Omega), \) which is a norm on the constants. Then, there exists a constant \( c \) depending only on \( \Omega \) such that
\[
\| u \|_{L^2(\Omega)} \leq C(\Omega) \| \nabla u \|_{L^2(\Omega)} + p(u), \quad \forall u \in H^1(\Omega). \] (11)
Lemma 3 (Gagliardo–Nirenberg inequality). Let $1 < p, q, r \leq \infty$, $0 \leq j < m$, $(j/m) \leq \lambda \leq 1$. For any $u \in W^{m-r}(\Omega) \cap L^q(\Omega)$, there exists a constant $C$ such that
\[
\|D^j u\|_{L^p(\Omega)} \leq C \|D^m u\|^\lambda_{L^p(\Omega)} \|u\|^{1-\lambda}_{L^q(\Omega)},
\]
where $p, q, r, n, m, j$, and $\lambda$ satisfy
\[
\frac{1}{r} - \frac{j}{n} = \lambda \left(\frac{1}{p} - \frac{m}{n}\right) + (1 - \lambda) \frac{1}{q}.
\]

The following product estimates play an essential role in our analysis (see [29]).

Lemma 4. Suppose that $f, g \in \mathcal{S}$ the Schwartz class. Then, for $s > 0, 1 < p < \infty$, there exist a positive constant $C$ such that
\[
\left\|\Lambda^s (fg)\right\|_{L^p(\Omega)} \leq C \left(\|f\|_{L^p(\Omega)} \|\Lambda^s g\|_{L^p(\Omega)} + \|\Lambda^s f\|_{L^p(\Omega)} \|g\|_{L^p(\Omega)}\right),
\]
with $q_1, q_2 \in (1, +\infty)$ satisfying
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.
\]

The following lemma will play an important role in the proof of our result. It was first proved by Romito in [30] for the case $\beta = 1$. The general case can be proved similarly.

Lemma 5. For every $u, v \in H^1$ and each $N > 0$, we have
\[
0 \leq F_N\left(\|\Lambda^N u\|, \|\Lambda^N v\|\right) \leq N,
\]
\[
\left|F_N\left(\|\Lambda^N u\|\right) - F_N\left(\|\Lambda^N v\|\right)\right| \leq \frac{F_N\left(\|\Lambda^N u\|\right) F_N\left(\|\Lambda^N v\|\right)}{N}
\]
\[
\|\Lambda^N u - \Lambda^N v\|.
\]

3. Existence and Uniqueness Results

We now give the definition of weak solutions to system (3).

Definition 3. Let $u_0, f \in \mathbb{H}$. A function $u$ is called a weak solution to system (3) if
\[
u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^D),
\]
\[
\|u\|_{L^2(0, T; \mathbb{H}^D)} \leq N,
\]
\[
\|u\|_{L^\infty(0, T; \mathbb{H})} \leq N,
\]
\[
\nu \in L^\infty(0, T; \mathbb{H}^D),
\]
\[
\|\nu\|_{L^\infty(0, T; \mathbb{H}^D)} \leq N.
\]

and for any function $\varphi \in \mathbb{H}$ and any $T > 0$, it holds that
\[
\int_0^T \int_\Omega (\nabla u \cdot \nabla \varphi) \, dx \, dt + \int_0^T \int_\Omega F_N\left(\|\Lambda u\|\right) (u \cdot \nabla)u \varphi \, dx \, dt
\]
\[
+ \nu \int_0^T \int_\Omega \Lambda \nu \Lambda \varphi \, dx \, dt
\]
\[
= \int_0^T \int_\Omega f \varphi \, dx \, dt + \int_0^T \int_\Omega \nu \varphi \, dx \, dt.
\]

Remark 1. Obviously, if $u(t)$ is a weak solution of system (3), then $u \in C([0, T]; \mathbb{H})$, see [26, 27].

Theorem 1. Let $\alpha$ and $\beta$ be two constants such that $4\alpha + 2\beta > 5$, $0 \leq \alpha < 5/4, 0 \leq \beta < 3/2$. (i) If $u_0, f \in \mathbb{H}$, there exists at least one weak solution $u(t)$ to problem (3) with
\[
u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^D)
\]
\[
\text{for all } T > 0.
\]

If in addition $4\alpha^2 - 5\alpha + 2\beta^2 \geq 0$ or $2\alpha + 4\beta > 5$, the weak solution is unique. (ii) On the other hand, if $u_0 \in \mathbb{H}^D, f \in \mathbb{H}^{m-a}$, and $s \geq \beta$, then problem (3) admits a unique global solution $u$ satisfying
\[
u(t) \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^\alpha)
\]
\[
\text{for all } T \geq 0.
\]

Remark 2. The standard existence result for the Navier–Stokes equations shows that system (3) possess a unique global solution, for all $\beta = 0$, when $\alpha > 5/4$, so we only consider the case $\alpha < 5/4$. However, when $\alpha \leq 1/2$, we cannot use the dissipation term of the equations to control the nonlinear term, and the existence results is difficult to prove in this case.

Proof. Let us divide the proof into several steps.

Step 1: we prove the existence of the weak solution by the Galerkin approximation method. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathbb{H}$ consisting of eigenfunctions of the Stokes operator $A$ and $\lambda_j$ are the corresponding eigenvalues which are increasing with $j$. Consider the following ordinary differential system:
\[
\frac{du_m}{dt} + \nu \Lambda^{2\alpha} u_m + P_m F_N\left(\|\Lambda^D u_m\|\right) (u_m \cdot \nabla)u_m = P_m f_m,
\]
\[
u_m(0) = P_m u_0.
\]
where $u_m = \sum_{j=1}^{m} \epsilon_j(t) \phi_j, \Lambda^2 u_m = \sum_{j=1}^{m} \lambda_j \epsilon_j(t) \phi_j, \Lambda^2 u_m = \sum_{j=1}^{m} \lambda_j^2 \epsilon_j(t) \phi_j$, and $P_m$ is the orthogonal projection form $H$ onto the space spanned by $\{\phi_1, \phi_2, \ldots, \phi_m\}$. By the standard existence theorem for ordinary differential equations, for each $m$, there exists a local solution $u_m$ to system (21) in the interval $[0, T_m]$.

Multiplying (21) by $u_m(t)$, using the Poincaré inequality $\lambda^2 \|u_m\| \leq \|\Lambda^2 u_m\|$, we can deduce that

$$\frac{d}{dt}\|u_m\|^2 + \nu \|\Lambda^2 u_m\|^2 \leq \frac{1}{\nu \lambda^2_1} \|f\|^2,$$

(22)

and integrating from 0 to $t$, we obtain

$$\|u_m(t)\|^2 + \nu \int_0^t \|\Lambda^2 u_m(s)\|^2 ds \leq \frac{t}{\nu \lambda^2_1} \|f\|^2$$

$$+ \|u_m(0)\|^2, \quad \forall t \geq 0. \quad (23)$$

Using Gronwall’s inequality, we obtain

$$\|u_m(t)\|^2 \leq \|u_m(0)\|^2 e^{-\nu \lambda^2_1 t} + \frac{\|f\|^2}{\nu \lambda^2_1} (1 - e^{-\nu \lambda^2_1 t})$$

(24)

which implies that

$$u_m \text{ is bounded in } L^\infty(0, T; H) \cap L^2(0, T; H^\alpha). \quad (25)$$

Let us perform the estimates for $\frac{\partial u_m}{\partial t}$. For any $\phi \in H^\alpha$, using H"older’s inequality, the product estimates (see Lemma 1), the Gagliardo–Nirenberg inequality, and Young’s inequality, we deduce that

$$\bigg| F_N \left( \|\Lambda^2 u_m\| \right) \int_\Omega (u_m \cdot \nabla u_m) \phi dx \bigg| \leq F_N \left( \|\Lambda^2 u_m\| \right) \|\Lambda^{-\alpha} (u_m \cdot \nabla u_m)\| \|\Lambda^\alpha \phi\|$$

$$\leq F_N \left( \|\Lambda^2 u_m\| \right) \|\Lambda^{-\alpha} u_m u_m\| \|\Lambda^\alpha \phi\|$$

$$\leq C F_N \left( \|\Lambda^2 u_m\| \right) \|u_m\|_{L^{2\alpha}) \|\Lambda^{-\alpha} u_m\|_{L^{\beta}} \|\Lambda^\alpha \phi\|$$

$$\leq C \|\Lambda^\alpha u_m\| \|\Lambda^\alpha \phi\|.$$  

(26)

The last inequality holds since

$$\frac{1}{2} - \frac{\beta}{3} = \frac{3 - 2\beta}{6}, \quad \frac{1}{2} - \frac{\alpha}{3} \leq \frac{\beta}{3} - \frac{1 - \alpha}{3}.$$

(27)

In view of (25), the sequence $\{F_N \left( \|\Lambda^2 u_m\| \right) (u_m \cdot \nabla) u_m\}$ is bounded in $L^2(0, T; H^{-\alpha})$. Obviously, $\{-\nu\Lambda^2 u_m\}$ and $\{P_m f\}$ are bounded in $L^2(0, T; H^{-\alpha})$. Hence, from (21), we conclude that

$$\left[ \frac{\partial u_m}{\partial t} \right] \text{ is bounded in } L^2(0, T; H^{-\alpha}). \quad (28)$$

Using the standard Aubin-Simon-type compactness results [26, 27], there exists an element

$$u \in L^2(0, T; H^\alpha) \cap L^\infty(0, T; H), \quad \text{for all } T > 0, \quad (29)$$

such that up to subsequences,

$$u_m \rightarrow u \text{ strongly in } L^2(0, T; H),$$

$$u_m \rightarrow u \text{ a.e. in } (0, T) \times \Omega,$$

$$u_m \rightarrow u \text{ weakly in } L^2(0, T; H^\alpha), \quad (30)$$

$$\frac{\partial u_m}{\partial t} \rightarrow \frac{\partial u}{\partial t} \text{ weakly }^* \text{ in } L^2(0, T; H^{-\alpha}).$$

Now, it remains to verify that $u$ is a weak solution to problem (3). We treat the case $\alpha > \beta$ and $\alpha \leq \beta$ separately.

Case 1. ($\alpha > \beta \geq 0$). By the Gagliardo–Nirenberg inequality and Hölder’s inequality [27], we deduce that
\[
\int_0^T \|A^\beta (u_m - u)\|^2 \, ds \leq C \int_0^T \|u_m - u\|^2 \|A^\alpha (u_m - u)\|^{2-2\theta_i} \, ds
\]
\[
\leq C \left( \int_0^T \|u_m - u\|^2 \, ds \right)^{\theta_i} \left( \int_0^T \|A^\alpha (u_m - u)\|^2 \, ds \right)^{1-\theta_i},
\]
(31)

where \( \theta_i = \alpha - \beta/\alpha \). This, together with (25) and (30), implies that
\[
u_m \to u \text{ strongly in } L^2 \left( 0, T; [0,\Omega^\beta] \right).
\]
(32)

Thus, up to subsequences,
\[
\|A^\beta u_m\| \to \|A^\beta u\| \text{ a.e. in } (0, T) \text{ for any } T > 0.
\]
(33)

And, hence,
\[
F_N \left( \|A^\beta u_m\| \right) \to F_N \left( \|A^\beta u\| \right) \text{ a.e. in } (0, T) \text{ for any } T > 0.
\]
(34)

Thanks to (30) and (34), taking \( \varphi \in \mathcal{H}^u \) as a test function in (21) and passing to the limits, we obtain that \( u \) is a weak solution to system (3). As the calculations are rather similar to those in [8], we omit the details for concision.

**Case 2.** (\( \beta > \alpha \)). Assume that there exist a positive integer \( N_0 \) such that \( N_0 \alpha < \beta < (N_0 + 1) \alpha \), without lose of generality, we set \( N_0 = 1 \). After multiplying equation (21) with \( A^{2\alpha} u_m \) and integrating, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|A^{\alpha} u_m\|^2 + \gamma \|A^{2\alpha} u_m\|^2 \leq F_N \left( \|A^\beta u_m\| \right) \|A (u_m + u)\| \|A^{2\alpha} u_m\|
\]
\[
\leq C F_N \left( \|A^\beta u_m\| \right) \|u_m\|_{L^{5/3}} \|A u_m\|_{L^{30}} \|A^{2\alpha} u_m\|
\]
\[
\leq C \|u_m\|_{5/3}^{\theta_i} \|A^{2\alpha} u_m\|^{2-\theta_i}
\]
\[
\leq \frac{\gamma}{4} \|A^{2\alpha} u_m\|^2 + C \|u_m\|^2,
\]
(35)

where \( \theta_i = \min\{1, (4\alpha + 2\beta - 5/4\alpha)\} \). Combining with (24), (35), and (36), we obtain
\[
\frac{d}{dt} \|A^{\alpha} u_m\|^2 + \gamma \|A^{2\alpha} u_m\|^2 \leq \frac{\gamma}{4} \|f\|^2 + C \|u_m\|^2
\]
\[
\leq \left( \frac{2}{\gamma} + \frac{C}{\gamma^\alpha \lambda^4} \right) \|f\|^2 + C \|u_0\|^2.
\]
(37)

For any \( t \geq \tau \geq 0 \), integrating (22) between \( t \) and \( t + \tau \) and using (24), we obtain
\[
y \int_t^{t+\tau} \|A^{\alpha} u_m\|^2 \, ds \leq \|u_0\|^2 + \frac{\|f\|^2}{\gamma^2 \lambda^4} \left( \tau + 1 + \frac{1}{\gamma \lambda^4} \right).
\]
(38)

Set
\[
a^2 = \frac{2}{\gamma \lambda^4} \gamma \left( \|u_0\|^2 + \frac{\|f\|^2}{\gamma^2 \lambda^4} \left( \tau + 1 + \frac{1}{\gamma \lambda^4} \right) \right),
\]
(39)

and denote by \( \Omega_m \) the Lebesgue measure of \( \Omega_m \). We have
\[
a^2 \Omega_m \leq \int_{\Omega_m} \|A^{\alpha} u_m\|^2 \, ds \leq \int_t^{t+\tau} \|A^{\alpha} u_m\|^2 \, ds
\]
\[
\leq \frac{1}{\gamma^2 \lambda^4} \gamma \left( \|u_0\|^2 + \|f\|^2 \left( \tau + 1 + \frac{1}{\gamma \lambda^4} \right) \right) = \frac{\alpha^2}{2},
\]
(40)

which implies that \( \|\Omega_m\| \leq \tau/2 \). Therefore, for any given \( \varepsilon > 0 \), there exist a \( t_0 \in (0, \varepsilon) \) such that
\[
\|A^{\alpha} u_m(t_0)\|^2 \leq \frac{2}{\gamma^2 \lambda^4} \gamma \left( \|u_0\|^2 + \|f\|^2 \left( \varepsilon + 1 + \frac{1}{\gamma \lambda^4} \right) \right).
\]
(41)

By using the Gronwall inequality, we obtain, for all \( t \geq \varepsilon \),
\[
\|A^{\alpha} u_m(t)\|^2 \leq \|A^{\alpha} u_m(t_0)\|^2 e^{-\frac{1}{\gamma \lambda^4} (t-t_0)} + \left( \frac{2}{\gamma \lambda^4} + \frac{C}{\alpha \lambda^{4}} \right) \|f\|^2
\]
\[
+ \frac{C}{\gamma \lambda^4} \|u_0\|^2.
\]
(42)

Integrating (37) from \( \varepsilon \) to \( T \) and taking (42) into consideration, we deduce that
\[
\|A^{\alpha} u_m(t)\|^2 \leq \int_0^T \|A^{2\alpha} u_m(t)\|^2 \, ds
\]
\[
\leq \int_0^T \left( \frac{1}{\gamma} + \frac{C}{\gamma^2 \lambda^4} \right) \|f\|^2 + C \|u_0\|^2 \, ds + \|A^{\alpha} u_m(t_0)\|^2
\]
\[
\leq \left( \frac{2}{\gamma} + \frac{C}{\gamma \lambda^4} \right) \|f\|^2 + C \|u_0\|^2 (T - \varepsilon)
\]
\[
+ \frac{2}{\gamma \lambda^4} \gamma \left( \|u_0\|^2 + \|f\|^2 \left( \varepsilon + 1 + \frac{1}{\gamma \lambda^4} \right) \right)
\]
\[
+ \frac{2}{\gamma \lambda^4} \gamma \left( C(N, C, \alpha, \beta) \left( \|u_0\|^2 + \|f\|^2 \left( \frac{\|f\|^2}{\gamma \lambda^4} \right) + \frac{2 \|f\|^2}{\gamma} \right) \right),
\]
(43)
for all \( \varepsilon \leq t \leq T \). Thus, we have

\[
\|u_m\| \text{ is bounded in } L^\infty(\varepsilon, T; \mathbb{H}^\alpha) \cap L^2(\varepsilon, T; \mathbb{H}^{2\alpha}), \text{ for any } T > \varepsilon. \tag{44}
\]

Taking \( \Lambda^\beta u_m, \Lambda^\alpha u_m, \ldots, \Lambda^{N_0+1} u_m \) as test functions and performing similar analysis, we may prove that

\[
\|u_m\| \text{ is bounded in } L^\infty(\varepsilon, T; \mathbb{H}^\alpha) \cap L^2(\varepsilon, T; \mathbb{H}^{(N_0+1)\alpha}).
\]

(45)

Denotes \( \alpha_1 = (N_0+1)\alpha \); since \( \beta < \alpha_1 \), we deduce that

\[
\int_\varepsilon^T \|\Lambda^\beta (u_m - u)\|^2 \, ds \leq C \int_\varepsilon^T \|u_m - u\|^{2\delta_1} \|\Lambda^{\alpha_1} (u_m - u)\|^{2 - 2\delta_1} \, ds
\]

\[
\leq C \left( \int_\varepsilon^T \|u_m - u\|^2 \, ds \right)^{\delta_1} \left( \int_\varepsilon^T \|\Lambda^{\alpha_1} (u_m - u)\|^2 \, ds \right)^{1 - \delta_1},
\]

where we have used the fact \( \int_0^T \nu \cdot \nabla u \, dx = 0. \) Since \( 4\alpha^2 - 5\alpha + 2\beta \geq 0 \) or \( 2\alpha + 4\beta > 5 \), the weak solution is unique. Let \( u \) and \( \nu \) be two solutions to system (3) corresponding to the initial condition \( u_0, \nu_0 \), respectively. Set \( w = u - \nu \) and let \( \mathcal{D} \) be the Helmholtz-Leray projection operator [27]. It is easy to check that \( w \) satisfies

\[
w_t + \nu \Lambda^{2\alpha} w + F_N(\|\Lambda^\beta u\|) \mathcal{D}(\nu \cdot \nabla) w + F_N(\|\Lambda^\beta u\|) \mathcal{D}(w \cdot \nabla) u
\]

\[
+ \left( F_N(\|\Lambda^\beta u\|) - F_N(\|\Lambda^\beta \nu\|) \right) \mathcal{D}(\nu \cdot \nabla) \nu = 0.
\]

(50)

Multiplying (50) by \( w \) and integrating, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \gamma \|\Lambda^\alpha w\|^2 \leq F_N(\|\Lambda^\beta u\|) \int_0^T w \cdot \nabla u \, dx
\]

\[
+ \left( F_N(\|\Lambda^\beta u\|) - F_N(\|\Lambda^\beta \nu\|) \right) \int_0^T \nu \cdot \nabla u \, dx,
\]

(51)

with \( \theta_1 = \alpha - \beta/\alpha \). When \( 2\alpha + 4\beta > 5 \), we can always find \( (p, q) = ((3/\beta), (6/3 - 2\beta)) \) such that \( (1/p) + (1/q) = (1/2) \) and

\[
\frac{1}{p} - \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{2}, \quad \frac{1}{q} - \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{2}.
\]

(54)

Thanks to the product estimates (see Lemma 1) and the fractional Sobolev inequality, we have

\[
\left| \int_0^T \nu \cdot \nabla u \, dx \right| \leq C \|\nabla u\| \|\Lambda^\alpha u\|^2 \|\Lambda^\beta \nu\| + \|\Lambda^\beta u\| \|\nabla \nu\| \|\Lambda^\beta \nu\|
\]

(55)

Combining (53) and (55) and using Young inequality, we obtain
\[
\left( F_N\left(\|\Lambda^\beta u\|\right) - F_N\left(\|\Lambda^\beta v\|\right)\right)\int_0^T v \cdot \nabla w dx
\]
\[
\leq C \frac{F_N\left(\|\Lambda^\beta u\|\right) F_N\left(\|\Lambda^\beta v\|\right)}{N} \|w\|^{\beta} \|\Lambda^\alpha u\|^{2-\beta} \left(\|\Lambda^\beta u\| \|\Lambda^\beta v\| + \|\Lambda^\beta v\| \|\Lambda^\beta u\|\right)
\]
\[
\leq C \|w\|^2 + \frac{\gamma}{4} \|\Lambda^\alpha u\|^2.
\]

Hence, if \(2\alpha + 4\beta \geq 5\), from (56), (52), and (51), we obtain that
\[
\frac{d}{dt} \|w\|^2 + \|\Lambda^\alpha u\|^2 \leq C \|w\|^2,
\]
for which the uniqueness result follows easily.

On the contrary, if \(2\alpha + 4\beta < 5\), setting \(p = (6/3 - 2\beta)\) and \(q = 3/\beta\), we have
\[
\|w(t)\|^2 \leq \|w(0)\|^2 \exp\left\{ C \int_0^t \|\Lambda^\alpha u\|^2 + \|\Lambda^\alpha v\|^2 + 1 ds \right\}.
\]

The uniqueness result follows easily.

Step 3: we now prove the second part of the theorem. If \(u_0 \in H^s, f \in H^{s-n}\), and \(s \geq \beta\), we multiply (21) by \(\Lambda^\gamma u_m\) to deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^\gamma u_m\|^2 + \gamma \|\Lambda^{\gamma+\gamma} u_m\|^2
\]
\[
\leq \|f\|_{H^{s-n}} \|\Lambda^{\gamma+\gamma} u_m\| + F_N\left(\|\Lambda^\beta u_m\|\right)\int_\Omega (u_m \cdot \nabla u_m) \Lambda^{2s} u_m dx
\]
\[
\leq \frac{1}{\gamma} \|f\|_{H^{s-n}} \|\Lambda^{\gamma+\gamma} u_m\|^2 + F_N\left(\|\Lambda^\beta u_m\|\right)
\]
\[
\|\Lambda^{s+1-a} (u_m u_m)\| \|\Lambda^{\gamma+\gamma} u_m\|.
\]

Using the product estimates and the Gagliardo–Nirenberg inequality, we deduce that
\[
F_N\left(\|\Lambda^\beta u_m\|, \|\Lambda^{\beta+1-a} (u_m, u_m)\|, \|\Lambda^{\alpha+a} u_m\|\right) \\
\leq CF_N\left(\|\Lambda^\beta u_m\|, \|u_m\|_{L^{65/3}} \|\Lambda^{\beta+1-a} u_m\|_{L^{65/3}} \|\Lambda^{\alpha+a} u_m\|\right) \\
\leq CN\|u_m\|^{\theta_2} \|\Lambda^{\alpha+a} u_m\|^{2-\theta_2} \\
\leq \frac{1}{2} C_0 (N, \nu, \alpha, \beta) \|u_m\|^2 + \frac{\nu}{4} \|\Lambda^{\alpha+a} u_m\|^2,
\]

where \( \theta_2 = 4N + 2\beta - 5/2N + 2\alpha \) and \( C_0 (N, \nu, \alpha, \beta) = (CN)^{1/6} (\nu/4 - 2\theta_2)^{65/2} \). This, combined with (63), yields that
\[
\frac{d}{dt}\|\Lambda^\alpha u_m\|^2 + \nu \|\Lambda^{\alpha+a} u_m\|^2 \leq C_0 (N, \nu, \alpha, \beta) \|u_m\|^2 + \frac{\nu}{\nu} \|f\|^2_{L^\nu}.
\]

Hence, for all \( t \geq \tau \geq 0 \),

\[
\|\Lambda^\alpha u (t)\|^2 + \nu \int_t^\tau \|\Lambda^{\alpha+a} u_m\|^2 \, d\zeta \\
\leq \|\Lambda^\alpha u (\tau)\|^2 + C_0 (N, \nu, \alpha, \beta) (t - \tau) \left( \|u_m\|^2 + \frac{\|f\|^2}{(\nu \lambda_N)^\alpha} \right) \\
+ \frac{2(t - \tau)}{\nu} \|f\|^2_{L^{\nu} (\tau)}.
\]

Thus, \( \|u_m\| \) is bounded in \( L^\infty (0, T; \mathbb{H}^\alpha) \cap L^2 (0, T; \mathbb{H}^{\alpha+a}) \), \( \forall T > 0 \). Passing to the limit, we obtain (20) immediately.

Next, we prove the uniqueness result. Let \( u \) and \( v \) be two solutions in \( L^\infty (0, T; \mathbb{H}^\alpha) \cap L^2 (0, T; \mathbb{H}^{\alpha+a}) \). Taking the inner product in (50) with \( \Lambda^{\alpha+a} u \), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^\alpha u\|^2 + \nu \|\Lambda^{\alpha+a} u\|^2 \leq F_N\left(\|\Lambda^\beta u\|, \|u\|_{L^{65/3}} \|\Lambda^{\alpha+a} v\|\right)
\]

\[
+ \frac{\nu}{\nu} \|\Lambda^{\alpha+a} u\|^2 \int_\Omega (u \cdot \nabla u) \Lambda^{\alpha+a} w \, dx \\
+ \frac{\nu}{\nu} \|\Lambda^{\alpha+a} u\|^2 \int_\Omega (v \cdot \nabla v) \Lambda^{\alpha+a} w \, dx \\
+ \frac{\nu}{\nu} \|\Lambda^{\alpha+a} u\|^2 \int_\Omega (v \cdot \nabla v) \Lambda^{\alpha+a} w \, dx \\
= I_1 + I_2 + I_3.
\]

For \( I_1 \), using Hölder’s inequality, the product estimates, Gagliardo–Nirenberg inequality, and Young’s inequality, we deduce that
\[
I_1 \leq F_N\left(\|\Lambda^\beta u\|, \|\Lambda^{\alpha+a} (u \cdot w)\|, \|\Lambda^{\alpha+a} w\|\right)
\]

\[
\leq CF_N\left(\|\Lambda^\beta u\|\right) \|u\|_{L^{60/3}} \|\Lambda^{\alpha+a} u\|_{L^{60/3}} \|\Lambda^{\alpha+a} w\| \\
+ \|w\|_{L^{60/3}} \|\Lambda^{\alpha+a} u\|_{L^{60/3}} \|\Lambda^{\alpha+a} w\| \\
\leq C\left(\|w\|_{L^{60/3}} \|\Lambda^{\alpha+a} u\|_{L^{60/3}} \|\Lambda^{\alpha+a} w\| \right)
\]

\[
\leq C\left(1 + \|w\|^2_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \|\Lambda^{\alpha+a} w\|^2 \right),
\]

where \( \theta_2 = \min\{1, 2\beta + 4\beta - 5/2\} \) (here, we may assume that \( s < 3/2 \). If \( s = 3/2 \), we can choose \( p, q \) satisfies \( (1/p) + (1/q) = (1/p) \) and \( 2 < q < 6/5 - 4\alpha \), and the Sobolev inequality implies that \( \|w\|_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \leq C \|w\|_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \). If \( s > 3/2 \), we may choose \( p, q \) to get \( \|w\|_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \leq C \|w\|_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \). Similarly, we have
\[
I_2 \leq C\left(1 + \|v\|^2_{L^{60}} \|\Lambda^{\alpha+a} u\|_{L^{60}} \|\Lambda^{\alpha+a} w\|^2 \right).
\]

Moreover,
\[
I_3 \leq F_N\left(\|\Lambda^\beta u\|\right) - F_N\left(\|\Lambda^\beta v\|\right) \|\Lambda^{\alpha+a} (u \cdot w)\| \|\Lambda^{\alpha+a} w\|
\]

\[
\leq C_2 - F_N\left(\|\Lambda^\beta u\|\right) \|\Lambda^\beta v\| \|\Lambda^{\alpha+a} u\|_{L^{65/3}} \|\Lambda^{\alpha+a} w\| \\
\leq C \|\Lambda^\beta u\| \|\Lambda^{\alpha+a} u\| \|\Lambda^{\alpha+a} w\| \\
\leq \nu \|\Lambda^{\alpha+a} u\|^2 + C \|\Lambda^{\alpha+a} u\|^2 \|\Lambda^\beta u\|^2.
\]

By (67)–(70), we obtain that
\[
\frac{d}{dt} \|\Lambda^\alpha u\|^2 \leq C \left(\|\Lambda^{\alpha+a} u\|^2 + \|\Lambda^{\alpha+a} v\|^2 + 1\right) \|\Lambda^\alpha u\|^2.
\]

Gronwall’s inequality then implies that
\[
\|\Lambda^\alpha u\|^2 \leq \exp \left[C \int_0^t \left(\|\Lambda^{\alpha+a} u\|^2 + \|\Lambda^{\alpha+a} v\|^2 + 1\right) \, dt\right] \|\Lambda^\alpha u(0)\|^2.
\]

from which the uniqueness of the solution and the continuity of the solution semigroup in \( \mathbb{H}^\alpha \) follow immediately.

Finally, let us verify that \( u \in C ([0, T]; \mathbb{H}^\alpha) \). Note that (66) implies that
\[ u \in L^2(0, T; H^{1+a}), \quad \forall \, T > 0, \text{ i.e., } \Lambda^\ast u \in L^2(0, T; H^a), \forall \, T > 0. \]  

(73)

Hence, according to the standard Sobolev embedding result [26, 27], we need only to show that

\[ \Lambda^\ast u \in L^2(0, T; \mathcal{H}^{-a}). \]  

(74)

Indeed, for any \( \varphi \in \mathcal{H}^a \), we have

\[
\langle \Lambda^\ast u, \varphi \rangle = -F_N\left(\|\Lambda^\beta u\|\right)\langle \Lambda^\ast (u \cdot \nabla u), \varphi \rangle - \langle \Lambda^{s+a} \lambda, u \rangle + \langle \Lambda^\ast f, \varphi \rangle. 
\]

(75)

Therefore,

\[
\|\Lambda^\ast u\|_{\mathcal{H}^{-a}} \leq F_N\left(\|\Lambda^\beta u\|\right)\|\Lambda^{s+a} (u \cdot \nabla u)\| + \|\Lambda^{s+a} u\| + \|\Lambda^{s-a} f\|, 
\]

(76)

which implies

\[
\|\Lambda^\ast u\|_{\mathcal{H}^{-a}} \leq F_N\left(\|\Lambda^\beta u\|\right)\|\Lambda^{s+a} (u \cdot \nabla u)\| + \|\Lambda^{s+a} u\| + \|\Lambda^{s-a} f\|. 
\]

(77)

Applying the product estimates and the imbedding of fractional Sobolev spaces, we have

\[
F_N\left(\|\Lambda^\beta u\|\right)\|\Lambda^{s+a} (u \cdot \nabla u)\| \leq CF_N\left(\|\Lambda^\beta u\|\right)\|u\|_{L^{2s+2}}, 
\]

\[
\|\Lambda^{s+a} u\|_{L^{2s}} \leq CN\|\Lambda^{s+a} u\|. 
\]

(78)

Therefore,

\[
\|\Lambda^\ast u\|_{\mathcal{H}^{-a}} \leq C\left(\|\Lambda^{s+a} u\| + \|\Lambda^{s-a} f\|\right). 
\]

(79)

Combining (73) and the assumption \( f \in \mathcal{H}^{0-a} \), we know that \( \Lambda^\ast u \in L^2(0, T; \mathcal{H}^{-a}) \). The proof is thus complete.

4. Long Time Behaviors

4.1. Attractor for Strong Solution. In this section, we prove the existence of a global attractor for system (3).

**Theorem 2.** Assume that \( 4\alpha + 2\beta > 5, 0 < \alpha < 5/4, 0 \leq \beta < 3/2 \), and \( f \in \mathcal{H}^{-a}, u_0 \in \mathcal{H}^a, s \geq \beta \). Then, system (3) generates a continuous semigroup \( \{S(t)\}_{t \geq 0} \) in \( \mathcal{H} \), and the semigroup possesses a global attractor \( \Lambda \), which is compact, invariant, and connected in \( \mathcal{H} \) and attracts all the bounded subsets of \( \mathcal{H} \) in the \( \mathcal{H}^{-a} \)-norm. Moreover, if \( s \geq \max(\beta, 1), f \in \mathcal{H}^{-1-a} \), the global attractor is bounded in \( \mathcal{H}^{-1-2a} \).

**Proof.** Thanks to Theorem 1, we know that the semigroup is continuous. It remains to prove the existence of an absorbing set and the compactness of the semigroup in \( \mathcal{H} \).

Absorbing set: let \( u(t) \) be the solution of system (1). Similar to (24), we have

\[
\|u(t)\|^2 \leq \|u(0)\|^2 e^{-\nu_1 t} + \frac{1}{\nu_1} \frac{\|f\|^2}{(\nu_1^2)^2}. 
\]

(80)

From the above inequality, we can deduce that there exists a \( T_0 = t(\|u_0\|) \) such that

\[
\|u(t)\|^2 \leq 2 \frac{\|f\|^2}{\nu_1^2}, \quad \forall \, t > T_0. 
\]

(81)

Multiplying (3) by \( \Lambda^\ast u \) and integrating, we have

\[
\frac{1}{2} \frac{d}{dt}\|\Lambda^\ast u\|^2 + \nu\|\Lambda^{s+a} u\|^2 \leq \|f\|^2 + \|\Lambda^{s+a} u\| + F_N\left(\|\Lambda^\beta u\|\right) 
\]

\[
\left| \int_\Omega (u \cdot \nabla u) \Lambda^{s+a} u \right|. 
\]

(82)

for all \( t > 0 \). Similar to (65), we deduce that

\[
\frac{d}{dt}\|\Lambda^\ast u\|^2 + \nu\|\Lambda^{s+a} u\|^2 \leq C_0(N, \nu, \alpha, \beta)\|u\|^2 
\]

\[
+ \frac{2}{\nu} \|f\|^2, \quad \forall \, t > 0, 
\]

where \( C_0(N, \nu, \alpha, \beta) = (CN)^{2\beta/3} (\nu/4 - 2\theta_2)^{\beta_2} \theta_2 = 4\alpha + 2\beta - 5/2 + 2\alpha \). Integrating the both sides from 0 to \( T_0 \) and taking (24) into consideration, we obtain

\[
\|\Lambda^\ast u(T_0)\|^2 \leq \|\Lambda^\ast u_0\|^2 + \int_0^{T_0} \left( C_0(N, \nu, \alpha, \beta)\|u\|^2 + \frac{2}{\nu} \|f\|^2 \right) \, dt 
\]

\[
\leq \|\Lambda^\ast u_0\|^2 + C_0(N, \nu, \alpha, \beta) T_0 \left(\|u_0\|^2 + \frac{\|f\|^2}{(\nu_1^2)^2}\right) 
\]

\[
+ \frac{2T_0^\nu}{\nu} \|f\|^2. 
\]

(84)

Using the Poincaré inequality and (81), we obtain from (83) that

\[
\frac{d}{dt}\|\Lambda^\ast u\|^2 + \nu\|\Lambda^{s+a} u\|^2 \leq C_0(N, \nu, \alpha, \beta)\|u\|^2 
\]

\[
+ \frac{1}{\nu_1^2} \left( C_0(N, \nu, \alpha, \beta) \|f\|^2 + \frac{2}{\nu}\|f\|^2 \right). 
\]

(85)

Gronwall’s inequality then implies that

\[
\|\Lambda^\ast u(t)\|^2 \leq \|\Lambda^\ast u(T_0)\|^2 e^{-\frac{\nu_1}{2}(t - T_0)} 
\]

\[
+ \frac{1}{\nu_1} \left( C_0(N, \nu, \alpha, \beta) \|f\|^2 + \frac{2}{\nu}\|f\|^2 \right). 
\]

(86)

Thanks to (96), we know that if

\[
t \geq \max\left\{ T_0, \frac{1}{\nu_1}, \frac{1}{\nu_1^2} \right\}, 
\]

(87)
\[
\|A^t u(t)\|^2 \leq \frac{1}{\nu_1 t} \left( C_0(N, \nu, \alpha, \beta) \|f\|^2 + \frac{2}{\nu} \|f\|_{H^0}^2 \right) + 1 = p_0^t.
\]

(88)

Therefore, there is an absorbing set \(B_1\) for the semigroup \(\{S(t)\}_{t \geq 0}\) in \(H^s\).

Compactness of the semigroup: we show that, for any bounded sequence \(\{v_0^t\}\) in \(H^s\) any \(t > 0\), the sequence \(\{S(t)v_0^t\} = \{v^t(\cdot)\}\) has a convergent subsequence in \(H^s\). Similar to (66) and (79), we can prove that

- \(A^t v^t(t)\) is bounded in \(L^2(0, 1; H^s)\),
- \(A^t v^t_1(t)\) is bounded in \(L^2(0, 1; H^{-s})\).

(89)

Using the Aubin-Simon type compactness results [26, 27], there exists an element \(v\) with

\[
\begin{align*}
A^s v &\in L^2(0, 1; H^s), \\
A^s v_1 &\in L^2(0, 1; H^{-s}),
\end{align*}
\]

such that up to subsequences,

\[
A^s v^t(t) \rightharpoonup A^s v(t) \text{ strongly in } L^2(0, 1; H),
\]

i.e.,

\[
v^t(t) \rightharpoonup v(t) \text{ strongly in } L^2(0, 1; H^s).
\]

(90)

In particular, there exists a \(\alpha \in (0, 1)\) such that

\[
v^t(\tau) \rightharpoonup v(\tau), \quad \text{in } H^s.
\]

(91)

Recall that the map \(S(t): H^s \rightarrow H^s\) is continuous, and we obtain that

\[
S(t)v_0^t = S(t - \tau)S(\tau)v_0^t = S(t - \tau)v^\alpha(\tau) \rightarrow S(t - \tau)v(\tau) \text{ in } H^s, \quad \text{for all } t \geq 1.
\]

(92)

Thus, the semigroup \(S(t)\) is compact, for any \(t \geq 1\). Thanks to the standard existence results on global attractors, we may obtain a global attractor in \(H^s\) for the solution semigroup \(S(t)\).

Regularity of the attractor: now, we prove that \(A\) is bounded in \(H^{s+\alpha}\) if \(f \in H^{s}, s_0 = s - 1 + \alpha\). Take the inner product of (3) with \(\Lambda^{s_0} u\), and we have

\[
\frac{d}{dt}\|\Lambda^{s_0} u\|^2 + \nu \|\Lambda^{s_0} u\|^2 \leq C_1(N, \nu, \alpha, \beta)\|u\|^2 + \frac{2}{\nu} \|f\|_{H^0}^2
\]

\[
\leq C_1(N, \nu, \alpha, \beta)\left(\|u_0\|^2 + \frac{2}{\nu} \|f\|_{H^0}^2\right) + \frac{2}{\nu} \|f\|_{H^0}^2, \quad \forall t > 0.
\]

(93)

If \(s_0 \leq s\), we have \(u_0 \in H^{s_0}\). By standard regularity result, we know that \(u(t)\) is bounded in \(L^\infty(0, 1; H^s)\cap L^2(0, 1; H^{s_0})\). Thus, there exists a time \(t_0 \in [0, 1]\) and a positive constant \(M_1\) such that \(\|\Lambda^{s_0} u(t_0)\|^2 < M_1\). By using Gronwall’s inequality, we deduce from (96) that

\[
\|\Lambda^{s_0} u(t)\|^2 \leq M_1 e^{-\nu t_0^2} + \frac{1}{\nu \lambda_1^2} C_0(N, \nu, \alpha, \beta)
\]

\[
\left(\|u_0\|^2 + \frac{2}{\nu} \|f\|_{H^0}^2\right) + \frac{2}{\nu} \|f\|_{H^0}^2, \quad \forall t > t_0.
\]

(94)

If \(s_0 > s\), since \(u_0 \in H^s\) and \(f \in H^{s_0}\), we know that \(u(t)\) is bounded in \(L^\infty(0, 1; H^s)\cap L^2(0, 1; H^{s_0})\). Since \(s_0 < s + \alpha\), we conclude that there exist a time \(t_1 \in [0, 1]\), and a positive constant \(M_2\) such that \(\|\Lambda^{s_0_0} u(t_1)\|^2 < M_2\). Let \(v(t) = S(t)v_0 = S(t)\chi_{H^s} u_0(\cdot)\). We know that \(v(t)\) is bounded in \(L^\infty(0, 1; H^s)\cap L^2(0, 1; H^{s_0})\). Then, there exists a time \(t_2 \in [0, 1]\) and a positive constant \(M_2\), which does not depend on \(v\), such that \(\|\Lambda^{s_0} v(t_2)\|^2 < M_2\), i.e., \(\|\Lambda^{s_0} u(t_2)\|^2 < M_2\). Denote \(t_3 = t_2 + t_1, t_3 \in (0, 2)\); then, \(u(t_3) \in H^{s_0}\). Similar to (97), we have

\[
\|\Lambda^{s_0} u(t)\|^2 \leq M_3 e^{-\nu t_3^2} + \frac{1}{\nu \lambda_1^2} C_0(N, \nu, \alpha, \beta)
\]

\[
\left(\|u_0\|^2 + \frac{2}{\nu} \|f\|_{H^0}^2\right) + \frac{2}{\nu} \|f\|_{H^0}^2, \quad \forall t > t_3.
\]

(95)

Since the attractor \(A\) is invariant, for any \(t > 0\) and any \(\chi \in A\), there exists a \(u_0 \in A\) such that \(S(t_0)u_0 = S(t - t_0)S(t_0)u_0 = u(t) = \chi\) (or \(S(t)u_0 = S(t - t_3)S(t_3)u_0 = u(t) = \chi\)). We assume \(t\) is large enough and takes (81) into consideration to obtain

\[
\|\Lambda^{s_0} \chi\|^2 = \|\Lambda^{s_0} u(t)\|^2
\]

\[
\leq \frac{2}{\nu \lambda_1^2} \left(3 C_4(N, \nu, \alpha, \beta) \frac{\|f\|_{H^0}^2}{\nu \lambda_1^2} + \frac{2}{\nu} \|f\|_{H^0}^2\right) = C_s^2.
\]

(96)

Therefore, \(A\) is bounded in \(H^{s_0+\alpha}\) by \(C_s\), which may depend on \(\|f\|, \|f\|_{H^{s_0+\alpha}}^2, N, \alpha, \beta, \nu\) and \(\nu\) and the bound in \(H^s\) of the attractor \(A\).
5. Finite Dimensionality of the Attractor

In this section, we provide the upper bound for the fractal dimension of the attractor derived in Section 4.

Theorem 3. Assume that $4\alpha + 2\beta > 5$, $0 < \alpha < 5/4$, $0 < \beta < 3/2$, $v_0 \in H^s$, and $f \in H^{s+1,r}$, $s \geq \max\{1, \beta\}$. Then, the fractal dimension of the global attractor $A$ derived in Theorem 2 is finite.

To prove Theorem 3, we use the following abstract results derived in [31–33].

Lemma 6. Let $H_0$ be a separable Hilbert space and let $M$ be a bounded closed set in $H_0$. Assume that there exists a mapping $S_0: M \rightarrow H_0$ such that $M \subseteq S_0(M)$:

(i) $S_0$ is Lipschitz on $M$, i.e., there exists $L > 0$ such that

$$\|S_0 v_1 - S_0 v_2\|_{H_0} \leq L \|v_1 - v_2\|_{H_0}, \quad \forall v_1, v_2 \in M.$$  

(ii) There exist finite dimension orthoprojectors $P_1$ and $P_2$ on $H_0$ such that

$$\|S_0 v_1 - S_0 v_2\|_{H_0} \leq \eta \|v_1 - v_2\|_{H_0} + K \left(\|P_1 (v_1 - v_2)\|_{H_0} + \|P_2 (v_1 - v_2)\|_{H_0}\right), \quad \forall v_1, v_2 \in M,$$  

where $0 < \eta < 1$ and $K > 0$ are constants.

Then,

$$\dim_M \leq \left(\dim P_1 + \dim P_2\right) \ln \left(1 + \frac{8 \sqrt{2} (1 + L)K}{1 - \eta}\right)^{-1}$$

Denote $Z_m = \text{span}\{e_j - k_j / |k| e^{ik \cdot x} : j = 1, 2, 3, |k| = \sqrt{|k_1|^2 + |k_2|^2 + |k_3|^2} \leq m\}$, where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3, k \neq 0$, and $e_1, e_2,$ and $e_3$ represent the canonical basis of $\mathbb{R}^3$. Let $P_m: L^2(\Omega) \rightarrow Z_m$ be the projection operator. Similar to Lemma 3.4 in [34] (see also Lemma 2.12 in [31]), we have the following lemma.

Lemma 7. Let $\eta \geq 0$ and $\theta > 0$. For any $\varepsilon > 0$, there exists a positive integer $m(\varepsilon)$ such that for $m \geq m(\varepsilon)$, and $|\phi(\varphi)|_{H^s} \leq \varepsilon \|\varphi\|_{H^s},$ and $\|P_m \phi\|_{H^s}, \quad \forall \varphi \in H^s,$  

where $m(\varepsilon) = \lfloor e^{-\varepsilon/|\theta|}\rfloor$, the integer part of number $e^{-\varepsilon/|\theta|}$.

Thanks to Lemma 8 in [24], we have the following.

Lemma 8. The projection operator $P_m: L^2(\Omega) \rightarrow Z_m$ has a finite range with

$$\dim P_m \leq 8(4m^3 + 6m^2 + 8m + 3).$$

Lemma 9. Assume that $4\alpha + 2\beta > 5$, $0 < \alpha < 5/4$, $0 < \beta < 3/2$, $v_0 \in H^s$, and $f \in H^{s+1,r}$, $s \geq \min\{1, \beta\}$. Let $\mathcal{A}$ be the global attractor of system (3) derived in Theorem 2 for the smooth solution. Let $u(t)$ and $v(t)$ be two solutions of system (3) corresponding to the initial data $u_0, v_0 \in \mathcal{A}$, respectively. Let $w(t) = u(t) - v(t)$, and let $\theta$ be a positive constant such that $\max |s, 3/2| < 1 - 2\alpha - \theta$. Then, for any $s_1 \in [s - 1 + 2\alpha - \theta, s - 1 + 2\alpha]$, we have

$$\left\|\Lambda^{s_1} w(t)\right\|^2 \leq \exp \left\{C(N, \alpha, \beta, \nu) \int_0^t \left(\left\|\Lambda^{s_1} u\right\|^2 + \left\|\Lambda^{s_1} v\right\|^2 + 1\right) \right\} \cdot \left\|\Lambda^{s_1} w(0)\right\|^2,$$

for some positive constant.

Proof. Take the inner product of (50) with $\Lambda^{s_1} w$, and we know that $w$ satisfies, for any $s - 1 + 2\alpha - \theta \leq s_1 \leq s - 1 + 2\alpha$:

$$\frac{1}{2} \frac{d}{dt}\left\|\Lambda^{s_1} u\right\|^2 + \left\|\Lambda^{s_1} v\right\|^2 \leq F_N\left(\left\|\Lambda^\beta u\right\|\right)$$

$$\int_{\Omega} (u \cdot \nabla w) \Lambda^{s_1} \omega dx$$

$$+ F_N\left(\left\|\Lambda^\beta u\right\|\right) \int_{\Omega} (w \cdot \nabla v) \Lambda^{s_1} \omega dx$$

$$\geq L_1 + L_2 + L_3.$$  

Since $A$ is bounded in $H^{s+\alpha}$, we have

$$\left\|\Lambda^{s+\alpha} u(t)\right\| \leq C_A, \left\|\Lambda^{s+\alpha} v(t)\right\| \leq C_A, \forall t \geq 0.$$  

For $L_1$, using Hölder’s inequality, the product estimates, Gagliardo–Nirenberg inequality, and Young’s inequality, we deduce that
Let $L_1 \leq F_N \left( \left\| \Lambda^{\beta} u \right\| \right) \left\| \Lambda^{\beta+1-a} (u \omega) \right\| \left\| \Lambda^{\alpha} u \right\|$

\[ \leq C \left\{ F_N \left( \left\| \Lambda^{\beta} u \right\| \right) \left( \| u \|_{L^\infty} \left\| \Lambda^{\beta+1-a} u \right\| + \| u \|_{L^\infty} \left\| \Lambda^{\beta+1-a} u \right\| \right) \left\| \Lambda^{\alpha} u \right\| \right\} \]

\[ \leq C \left\| \Lambda^{\beta+\alpha} u \right\| \left\| \Lambda^{\alpha} u \right\|^{\beta_5 - \delta_{5}} + C_2 \left\| \Lambda^{\alpha} u \right\| \left\| \Lambda^{\beta+\alpha} u \right\| \left\| \Lambda^{\alpha} u \right\| \]

\[ \leq C \left\| \Lambda^{\beta+\alpha} u \right\|^2 + C_1 \left( N, \alpha, \beta, \nu \right) \left( 1 + \left\| \Lambda^{\alpha} u \right\|^2 \right) \left\| \Lambda^{\alpha} u \right\|^2, \]

where $\theta_{6} = \min \{2 \alpha - 1, \alpha, 1 \}$. Similarly, we have

\[ L_2 \leq \frac{\nu}{6} \left\| \Lambda^{\alpha+\nu} u \right\|^2 + C_1 \left( N, \alpha, \beta, \nu \right) \left( 1 + \left\| \Lambda^{\alpha+\nu} u \right\|^2 \right) \left\| \Lambda^{\alpha+\nu} u \right\|^2. \]

(109)

Finally, for $L_3$, we use Hölder’s inequality, Lemma 2, product estimates, and imbedding of fractional Sobolev spaces to deduce that

\[ L_3 \leq F_N \left( \left\| \Lambda^{\beta} u \right\| \right) - F_N \left( \left\| \Lambda^{\beta} v \right\| \right) \left\| \Lambda^{\alpha+\nu} \left( \nu \nu \nu \right) \right\| \left\| \Lambda^{\alpha+\nu} u \right\|

\[ \leq C_2 \left( \left\| \Lambda^{\beta} u \right\| \right) \left\| \Lambda^{\alpha+\nu} \left( \nu \nu \nu \right) \right\| \left\| \Lambda^{\alpha+\nu} u \right\|

\[ \leq \frac{C_2}{N} \left\| \Lambda^{\alpha+\nu} \left( \nu \nu \nu \right) \right\| \left\| \Lambda^{\alpha+\nu} u \right\|

\[ \leq \frac{C_2^2}{N} \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \frac{3 C_2^2}{2 N^2 \nu} \left\| \Lambda^{\alpha+\nu} u \right\|^2 \]

(110)

\[ \left\| \Lambda^{\alpha+\nu} (t) \right\|^2 \leq \exp \left\{ C \left( N, \alpha, \beta, \nu \right) \int_{0}^{t} \left( \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \left\| \Lambda^{\alpha+\nu} v \right\|^2 + 1 \right) dt \right\} \left\| \Lambda^{\alpha+\nu} (0) \right\|^2 = \tilde{C} (t) \left\| \Lambda^{\alpha+\nu} (0) \right\|^2, \]

where $\epsilon_{0} = 1/2 \min \{2 \alpha - 1, 2 \alpha, 1 \}$, $\gamma = s + 1 + 2 \alpha$, $\tilde{C} (t)$ is from (113), and $C_{\alpha}$ is from (99).

Lemma 10. Assume that $4 \alpha + 2 \beta > 5, \alpha > 1/2, \beta \geq 0,$

\[ u_0 \in H^s, \text{ and } f \in H^s, \text{ s \geq max } \{1, \beta \}. \]

Let $u$ and $v$ be two solutions of system (3) with initial condition $u_0, v_0 \in \mathfrak{A},$ respectively. Setting $w = u - v,$ then there exists a positive constant $C$ such that $\forall t \geq 0$:

\[ \| w (t) \|_{1/2} \leq C e^{-x_{0} / 2} \| w (0) \|_{1/2} \]

\[ + \nu^{\epsilon_0} C \left( C_{\alpha} + 2 C_{\alpha} \right) \tilde{C} (t) \| w (0) \|_{1/2-s}, \]

(114)

\[ \text{Combining (106)--(110), we obtain that} \]

\[ \frac{d}{dt} \left\| \Lambda^{\alpha} u \right\|^2 \leq C \left( N, \alpha, \beta, \nu \right) \left( \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \left\| \Lambda^{\alpha+\nu} v \right\|^2 + 1 \right) \left\| \Lambda^{\alpha} u \right\|^2, \]

(111)

with

\[ C \left( N, \alpha, \beta, \nu \right) = \max \left\{ C_1 C_{a} \right\}^{\nu} \left( \frac{3}{12 - 6 \beta} \right) \frac{3 C_2^2}{2 N^2 \nu} \frac{3 C_2^2}{2 N^2 \nu} \}

(112)

Gronwall’s inequality then implies (105):

\[ \left\| \Lambda^{\alpha+\nu} (t) \right\|^2 \leq \exp \left\{ C \left( N, \alpha, \beta, \nu \right) \int_{0}^{t} \left( \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \left\| \Lambda^{\alpha+\nu} v \right\|^2 + 1 \right) dt \right\} \left\| \Lambda^{\alpha+\nu} (0) \right\|^2 = \tilde{C} (t) \left\| \Lambda^{\alpha+\nu} (0) \right\|^2, \]

(113)

\[ \text{Obviously, } \tilde{C} (t) \text{ is a monotone function with respect to } t, \]

and it is finite for finite time $t$. \hfill \Box

\[ \text{Lemma 10. Assume that } 4 \alpha + 2 \beta > 5, \alpha > 1/2, \beta \geq 0, \]

\[ u_0 \in H^s, \text{ and } f \in H^s, \text{ s \geq max } \{1, \beta \}. \]

Let $u$ and $v$ be two solutions of system (3) with initial condition $u_0, v_0 \in \mathfrak{A},$ respectively. Setting $w = u - v,$ then there exists a positive constant $C$ such that $\forall t \geq 0$:

\[ \| w (t) \|_{1/2} \leq C e^{-x_{0} / 2} \| w (0) \|_{1/2} \]

\[ + \nu^{\epsilon_0} C \left( C_{\alpha} + 2 C_{\alpha} \right) \tilde{C} (t) \| w (0) \|_{1/2-s}, \]

(114)

\[ \text{Combining (106)--(110), we obtain that} \]

\[ \frac{d}{dt} \left\| \Lambda^{\alpha} u \right\|^2 \leq C \left( N, \alpha, \beta, \nu \right) \left( \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \left\| \Lambda^{\alpha+\nu} v \right\|^2 + 1 \right) \left\| \Lambda^{\alpha} u \right\|^2, \]

(111)

with

\[ C \left( N, \alpha, \beta, \nu \right) = \max \left\{ C_1 C_{a} \right\}^{\nu} \left( \frac{3}{12 - 6 \beta} \right) \frac{3 C_2^2}{2 N^2 \nu} \frac{3 C_2^2}{2 N^2 \nu} \}

(112)

Gronwall’s inequality then implies (105):

\[ \left\| \Lambda^{\alpha+\nu} (t) \right\|^2 \leq \exp \left\{ C \left( N, \alpha, \beta, \nu \right) \int_{0}^{t} \left( \left\| \Lambda^{\alpha+\nu} u \right\|^2 + \left\| \Lambda^{\alpha+\nu} v \right\|^2 + 1 \right) dt \right\} \left\| \Lambda^{\alpha+\nu} (0) \right\|^2 = \tilde{C} (t) \left\| \Lambda^{\alpha+\nu} (0) \right\|^2, \]

(113)

where $\epsilon_{0} = 1/2 \min \{2 \alpha - 1, 2 \alpha, 1 \}$, $\gamma = s + 1 + 2 \alpha$, $\tilde{C} (t)$ is from (113), and $C_{\alpha}$ is from (99).

\[ \text{Proof. Similar to (50), we know that } w \text{ satisfies} \]

\[ w_t + \nu \Lambda^{\alpha+\nu} w + F_N \left( \| \Lambda^{\beta} u \| \right) \rho (u, v) w + F_N \left( \| \Lambda^{\beta} u \| \right) \rho (w, v) v \]

\[ + \left\{ F_N \left( \| \Lambda^{\beta} u \| \right) - F_N \left( \| \Lambda^{\beta} v \| \right) \right\} \rho (v, v) v = 0. \]

(115)

By using the Duhamel principle [35], the solution of (114) can be given by
\[ w(t) = e^{-v(t - t)}w(0) - \int_0^t e^{-v(t - r)} \left\{ F_N \left( \left\| \Lambda^u \right\| \right) \mathbf{P} (u \cdot \nabla) w + F_N \left( \left\| \Lambda^v \right\| \right) \mathbf{P} (v \cdot \nabla) v \right. \\
\left. + \left( F_N \left( \left\| \Lambda^u \right\| \right) - F_N \left( \left\| \Lambda^v \right\| \right) \right) \mathbf{P} (v \cdot \nabla) v \right\} \, dr. \] (116)

And moreover, we have the following estimate of the semigroup \( e^{-v(t-t)} \):
\[ \left\| e^{-v(t-t)} \right\|_{\mathcal{L}^{1/p}(\mathcal{H})} \leq C e^{-\pi/2} \left\| w(0) \right\|_{\mathcal{H}^s}, \] (117)
and then,
\[ \left\| w(t) \right\|_{\mathcal{H}^s} \leq C e^{-v(t-t)/2} \left\| w(0) \right\|_{\mathcal{H}^s} + C \int_0^t e^{-\pi/2} (t-r) \left( t-r \right)^{-1 + s} \mathbf{P} (v) \, dr, \] (119)
and hence,
\[ \left\| w(t) \right\|_{\mathcal{H}^s} \leq C e^{-\pi/2} \left\| w(0) \right\|_{\mathcal{H}^s} + C \int_0^t e^{-\pi/2} (t-r) \left( t-r \right)^{-1 + s} \mathbf{P} (v) \, dr, \] (120)

Next, we estimate \( S_1, S_2, \) and \( S_3 \) one by one. By the product estimates of Sobolev spaces, we have
\[ S_1 \leq C f_N \left( \left\| \Lambda^u \right\| \right) \left( \left\| w \right\|_{\mathcal{H}^{s+1}} + \left\| u \right\|_{\mathcal{H}^{s+1}} \left\| w \right\|_{\mathcal{H}^{s+1}} \right), \] (121)

for positive integers \( p, q_i \) satisfying \( \left( 1/p_1 \right) + \left( 1/q_i \right) = \left( 1/2 \right), i = 1, 2 \). Let \( \epsilon \leq 2 \alpha - 1/2 + 1 \); since \( 4 \alpha + 2 \beta > 5, \alpha > 1/2, \) and \( s \geq \max \{ 1, \beta \} \), we have
\[ \frac{2y + 4a - 5}{4a + 2} \geq \frac{2(\beta + 1/2a) + 4a - 5}{2a + 1} > \frac{2\alpha - 1}{2a + 1} > \epsilon > 0. \] (122)

Hence, it is easy to check that there exists a pair of positive integers \( (p_1, q_1) \) such that \( \left( 1/p_1 \right) + \left( 1/q_1 \right) = \left( 1/2 \right), \) and
\[ \frac{1}{p_1} \geq \frac{1 - y}{2}, \]
\[ \frac{1}{q_1} \geq \frac{1 - y - \sigma - 1 - \epsilon}{3}. \] (123)

\[ \sigma + 1 \leq y - \epsilon. \]

By the Sobolev inequality, we have
\[ \left\| u \right\|_{\mathcal{H}^{p_1/2}} \left\| w \right\|_{\mathcal{H}^{s+1}} \leq C \left\| u \right\|_{\mathcal{H}^{p_1/2}} \left\| w \right\|_{\mathcal{H}^{s+1}}. \] (129)

Combining (121)–(129), we obtain
\[ S_1 \leq C \left\| u \right\|_{\mathcal{H}^{p_1/2}} \left\| w \right\|_{\mathcal{H}^{s+1}}, \] (130)

when \( \epsilon \in (0, 2\alpha - 1/2 + 1) \).

Similarly, for \( S_2 \), we have
\[ S_2 \leq C \left\| w \right\|_{\mathcal{H}^{s+1}} \left\| w \right\|_{\mathcal{H}^{s+1}}. \] (131)

Since \( \epsilon \in (0, 2\alpha - 1/2 + 1), s \geq \max \{ 1, \beta \}, \) we have that \( \beta < y - \epsilon \). For \( S_3, \) using Lemma 2, the product estimate, and the Sobolev inequality, it is easy to deduce that
\[
S_3 \leq C \left( \frac{F_N \left( \|A^N u\| \right)}{N} \right) \left( \frac{F_N \left( \|A^N v\| \right)}{N} \right) \|A^N u\| \|v\|_{\ell^p} \|v\|_{\ell^p} \ (132)
\]

Therefore, combining (99) and (130)–(132), we obtain

\[
\|w(t)\|_{\ell^p} \leq C \exp \left\{ \frac{\gamma \lambda_1 t}{2} \right\} \|w(0)\|_{\ell^p} + C(C_A + 2C_A) \int_0^t \exp \left\{ -\frac{\gamma \lambda_1 (t - \tau)}{2} \right\} (t - \tau)^{-1/\epsilon_0} \|w(\tau)\|_{\ell^p} d\tau
\]

\[
\leq C \exp \left\{ \frac{\gamma \lambda_1 t}{2} \right\} \|w(0)\|_{\ell^p} + C(C_A + 2C_A) \bar{C}(t) \|w(0)\|_{\ell^p} \ (134)
\]

where \( \Gamma(\cdot) \) is the Gamma function. This completes the proof of Lemma 10.

**Proof.** of Theorem 3. Combining with Lemmas 6–10, we can give an upper bound on the fractal dimension of the attractor derived in Theorem 2 as follows. Let \( w \) be a solution of (115). It follows from Lemma 7 and Lemma 10 that

\[
\|w(t)\|_{\ell^p} \leq C \exp \left\{ \frac{\gamma \lambda_1 t}{2} \right\} \|w(0)\|_{\ell^p} + C_v v \left( C_A^2 + 2C_A \right) \bar{C}(t) \|w(0)\|_{\ell^p} \ (135)
\]

with \( m \geq m(\varepsilon) = [\varepsilon^{-1/\alpha_0}] \) and \( \varepsilon_0 = 1/2 \min\{2\alpha - 1/2\alpha + 1, 2\beta\} \). After some elementary calculations, we can choose

\[
t_0 = \frac{2 \ln 4C}{\gamma \lambda_1},
\]

\[
\varepsilon = \frac{\sqrt{2}}{4} \left( C_A^2 + 2C_A \right)^{-1/2} \bar{C}(t_0)^{-1},
\]

such that

\[
C \exp \left\{ \frac{\gamma \lambda_1 t_0}{2} \right\} = \frac{1}{4}, \ (136)
\]

\[
\varepsilon v \left( C_A^2 + 2C_A \right) \bar{C}(t_0) = \frac{1}{4}, \ (137)
\]

with

\[
\left( S_1 + S_2 + S_3 \right) \leq C \left( \|u\|_{\ell^p} + \|v\|_{\ell^p} + \|v\|_{\ell^p} \right) \|w\|_{\ell^p} \ (133)
\]

Now, set \( \varepsilon_0 = 1/2 \min\{2\alpha - 1/2\alpha + 1, 2\beta\} \). Taking (132) into (119) and using (99) and (105) (note that \( s - 1 + 2\alpha - \theta \leq s - 1 + 2\alpha \), we obtain that

\[
\|w(t)\|_{\ell^p} \leq \bar{C}(t_0) \left( \frac{C_A^2 + 2C_A}{4} \right) \|w(0)\|_{\ell^p} \ (138)
\]

Combining with (135) and (137), we have

\[
\|w(t)\|_{\ell^p} \leq \frac{1}{2} \|w_0\|_{\ell^p} + \varepsilon v \left( C_A^2 + 2C_A \right) \bar{C}(t_0) \|P_m w(0)\|_{\ell^p} \ (139)
\]

Meanwhile, it follows from Lemma 9 that

\[
\left( \Lambda^{-1} w(t_0) \right)^2 \leq \exp \left\{ C(N, \alpha, \beta, \nu) \int_0^t \left( \left\| \Lambda^{\nu+1} u \right\|^2 + \left\| \Lambda^{\nu+1} v \right\|^2 + 1 \right) \right\}
\]

\[
\|w(t_0)\|_{\ell^p} \leq \bar{C}(t_0) \left( \frac{C_A^2 + 2C_A}{4} \right) \|w(0)\|_{\ell^p} \ (140)
\]
Define $S_0 = S(t_0)$: $w(0) \mapsto w(t_0)$, and we can check that $S_0$ satisfies conditions (i) and (ii) of Lemma 6, and

$$M = A,$$

$$H_0 = H^F = \mathcal{H}^N \cap 2a + 1,$$

$$S_0 = S(t_0): w(0) \mapsto w(t_0),$$

$$t_0 = \frac{2 \ln 4C}{\nu \lambda_1},$$

$$\eta = \frac{1}{2},$$

$$K = \nu^{-\varepsilon_0}C(C^2 A^2 + 2C)\overline{C}(t_0),$$

$$L = \overline{C}(t_0),$$

$$m = \left[ C \nu^{-\varepsilon_0} \left( C^2 A^2 + 2C \right) \right]^{1/\varepsilon_0} \overline{C}(t_0)^{1/\varepsilon_0},$$

$$\varepsilon_0 = \frac{1}{2} \min \left\{ \frac{2a - 1}{2a + 1}, 2 \right\},$$

$$\dim(P_2) = 0,$$

$$P_1 \leq P_m,$$

$$\dim(P_m) \leq 8\left( 4m^3 + 6m^2 + 8m + 3 \right),$$

where $C_A$ is from (99), $\overline{C}(t_0)$ is from (140), $\delta$ is the constant in the Lemma 9, $C$ is a universal constant, and $\lfloor x \rfloor$ denotes the integer part of the number $x$.

Thus, we have

$$\dim(A) \leq 8\left( 4m^3 + 6m^2 + 8m + 3 \right)$$

$$\ln \left\{ 1 + 16\sqrt{2} \left( 1 + \overline{C}(t_0) \right) \nu^{-\varepsilon_0} C \left( C^2 A^2 + 2C \right) \overline{C}(t_0) \right\} \left( \frac{4}{\nu} \right)^{-1}. $$

(142)

6. Conclusions

The authors established the existence of weak solutions under proper assumptions on $\alpha$ and $\beta$. The existence of finite-dimensional global attractors is also obtained. It is possible to study the global modified MHD equations by using similar ideas herein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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