Virasoro blocks at large exchange dimension

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Abstract

In this paper, we analyze Virasoro conformal blocks in the limit when the operator exchange dimension is taking to be large in comparison with the other parameters dependence of the block. We do this by using Zamolodchikov’s recursion relations. We found a dramatically simplified solution at leading order in an inverse power expansion in large exchange conformal dimension which appear to be related to a quasi-modular form in an Eisenstein series representation.

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1 Introduction

Conformal field theories in two dimensions play a major role in fundamental physics and are ubiquitous in modern theoretical high energy physics. They find applications across a wide range of topics and space-time dimensions from realistic phenomena to purely mathematically motivated models. Some examples worth mentioning are: condense matter systems at criticality, such as the Ising model and models within its universality class, fractional quantum Hall effect, deep inelastic scattering, string theory as well as AdS/CFT correspondence. Within them we find the most successful application so far of the nowadays mainstream bootstrap program, in particular because this approach lead to the discovery of an important large family of solvable theories known as minimal models [1]. They have become important in the study of partition functions in supersymmetric gauge theories through the so-called AGT correspondence [2, 3] and more recently, appear to be related to the asymptotic symmetries of quantum gravity in four dimensions [4, 5, 6]. They have served multiple times as an inspiration for developments in pure mathematics and find applications, in for example, the theory of Riemann surfaces.

The main observables in a conformally invariant theory are correlation functions, which admit an operator product expansion decomposition in terms of conformal blocks. In the two dimensional case, the symmetry algebra is infinitely large and is known as the Virasoro algebra. The representations of this algebra can in turn be decomposed as a direct sum of the so-called Verma modules, each of which contains the whole family of descendants states created from a given highest weight state. The contribution to a correlation function from an entire Verma module is contained in a corresponding Virasoro block [1] and therefore they are completely determined by the conformal symmetry. By knowing them, we can therefore methodically isolate the symmetry constrains on a generic correlation function and we are then left with the computation of the expansion coefficients, or OPE coefficients, in order to completely solve a particular theory. Despite the importance of the Virasoro blocks for the study and use of conformal field theories, a close, generic, complete and useful expression for them has elude us so far [1].

Some few Virasoro blocks has been computed exactly for particular theories and correlation functions. Minimal models are solvable theories where this blocks simplify drastically into Gauss hypergeometric functions [1]. For a particular correlator between four operators of the same weight and particular central charge \(c = 1\) and \(c = 25\), an exact expression is known from [9] [2]. Some few particular cases were obtained by relating Virasoro blocks to Painlevé VI equation [11, 12, 13, 14]. Some combinatorial formulas for the coefficients in a series expansion on cross ratios have also been obtained [15, 16]. A particular vacuum block in a semiclassical limit (large \(-c\)) has been computed, and extensively studied in a series of papers [17, 18, 19, 20, 21, 22, 23, 24, 25].

Remarkably, crossing symmetry has been solved with not need of an explicit representation of the Virasoro blocks [1, 5].

Another nice example of a theory with \(c = 1\) has been discussed in [10].
One could compute Virasoro blocks from first principles by using OPE expansions and the Virasoro algebra, computing level by level the contributions to the Virasoro blocks as is usually taught in textbooks [26,27], but this becomes cumbersome very quickly due to the highly combinatorial nature of this approach, so it would be helpful to have alternative ways to treat them. In a couple of beautiful papers, Zamolodchikov developed an alternative approach by deriving a couple of recursion relations for Virasoro blocks, by viewing them as an expansion in poles in the central charge [28], and as an expansion in poles in exchange dimension [29]. This recursions have been later generalized to other cases, like to torus one-point functions [30,31,32,33], and the so-called heavy-light semiclassical limit [20]. Among the applications of this recursion relations we can mention a few where it has been used in determining combinatorial expression for the coefficients in cross-ratio expansions [16] and in performing numerics in the context of AdS/CFT correspondence in [34].

Motivated by the simplified expression for the Virasoro blocks from [19,20,23,24] obtained by a perturbative semiclassical analysis in $c^{-1}$, we expect similar simplifications in a perturbative treatment in a large exchange conformal dimension $1/h$. It is the goal of this paper to provide such an example by solving (one of) Zamolodchikov’s recursion relation at leading order in a $1/h$ expansion for large $-h$. We find that the leading order correction, is given in terms of a Eisenstein series quasi-modular form, which is in agreement with expected results from the study of Virasoro blocks over elliptic curves.

This limit is important because, among other particularities, it controls a universal sector of all non-rational two dimensional conformal field theories, which in turn behaves Liouville-like as has been recently proved in [35]. This can be checked in a traditional bootstrap computation [36] and some aspects of it has been considered from the point of view of the modular bootstrap in [37,38]. In particular, for conformal field theories with holographic description, the Liouville-like universal behavior of two dimensional conformal field theories in the large exchange region has been argued some little ago [39].

Therefore, we believe our results in this paper will find applications in the study of this universal sector.

## 2 Virasoro conformal blocks

The spectrum of states for two dimensional conformal field theories fall into representations of the Virasoro algebra generated by the Laurent modes of infinitesimal conformal transformations. Due to the fact that in two dimensions, local conformal transformations corresponds to arbitrary analytic functions, the generic Laurent expansion of such functions will be in principle expanded by an infinite number of modes, and henceforth the number of generators in two dimensions is
infinitely large. The Virasoro algebra is explicitly given,
\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}n(n^2 - 1)\delta_{m+n,0} , \]
for the holomorphic sector and a similar algebra for the anti-holomorphic one. Modes with negative integer label corresponds to raising operators whereas positive label modes corresponds to lowering operators. The spectrum is build upon acting with raising operators over eigenstates of the zero mode, corresponding to local primary operator by means of the state-operator correspondence, defined by,
\[ L_0 O_h |0\rangle \equiv L_0 |O_h\rangle = h |O_h\rangle . \]
States created out of the primary states by the application of raising operators are known as descendants and the set of descendants from a given primary is known as a Verma module. The spectrum is then classified as a direct sum of all Verma modules.

Let us now consider a four-point correlation function of primary operators \( \langle \prod_{i=1}^{4} O_{h_i}(z_i) \rangle \).
Conformal invariance greatly constrains the form of this observable, as being proportional to a function of the cross ratios only,
\[ z = \frac{z_{12}z_{34}}{z_{13}z_{24}} , \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} , \]
where we have used \( z_{ij} \equiv z_i - z_j \). Explicitly, it is constrained to be of the form,
\[ \langle \prod_{i=1}^{4} O_{h_i}(z_i) \rangle = G(z, \bar{z}) \prod_{i<j}^{4} \frac{\sum_{i} h_i - h_j}{z_{ij}^3} \frac{\sum_{\bar{i}} \bar{h}_i - \bar{h}_j}{\bar{z}_{ij}^3} . \]
we can additionally use global conformal invariance to fix three of the primary positions, namely \( z_1 = \infty, z_2 = 1, z_4 = 0 \), which lead us to \( z_3 = z \), and similarly for the bared coordinates.

By taking the limit \( (z, \bar{z}) \rightarrow (0, 0) \), the function \( G(z, \bar{z}) \) can be expanded as,
\[ G(z, \bar{z}) = \sum_{h,\bar{h}} C_{12, h, \bar{h}} C_{34}^{-h, \bar{h}} |F(c, h_i, h, z)|^2 , \]
where the “basis” functions \( F(c, h_i, h, z) \) (and \( F(c, \bar{h}_i, \bar{h}, \bar{z}) \)) are known as Virasoro blocks, and encode the contributions to the four-point function from the whole Verma module corresponding to the primary with conformal weight \( (h, \bar{h}) \). They can be expanded in representations of the global conformal group, which would allow us to make direct contact with the analogous higher dimensional conformal block expansion, but in this paper we are more interested in study the expansion in the “elliptic basis”, obtained from mapping the branched sphere into an elliptic curve.
2.1 Elliptic expansion

In this section we want to present the expansion of conformal blocks on a elliptic curve which is equivalent to a double cover of a branched sphere.

From general considerations, the original blocks $F(c, h_i, h, z)$ have branches at the positions of the primary fields, namely at $\{\infty, 1, z, 0\}$. The map

$$q = e^{\pi i \tau}, \quad \tau = i \frac{K(1 - z)}{K(z)},$$

(2.6)

provides a conformal transformation into the upper-half plane $\text{Im}(\tau) > 0$ in such a way that the conformal block is a single valued function there, and therefore a series expansion in $q$ converges uniformly in $\text{Im}(\tau) > 0$ [29].

The variable $\tau$ corresponds to the modulus of a torus given by a double-cover of the Riemann sphere branched at $\{\infty, 1, z, 0\}$. This torus in turns can be described as an elliptic curve with the same branches as,

$$y^2 = x(z - x)(1 - x).$$

(2.7)

We can compute CFT correlators over the torus in the usual way, by taking expectations values with a partition function given by,

$$\text{Tr} \left( e^{\pi \tau (L_0 - \frac{c}{12})} e^{-\pi \bar{\tau} (\bar{L}_0 - \frac{c}{12})} \right) = \text{Tr} \left( q^{L_0 - \frac{c}{24}} q^{\bar{L}_0 - \frac{c}{24}} \right).$$

(2.8)

Finally, we can then write an expansion for $G(z, \bar{z})$ on the elliptic curve as,

$$G(z, \bar{z}) = \Lambda(z) \Lambda(\bar{z}) g(q, \bar{q}),$$

(2.9)

where

$$\Lambda(z) = z^{\frac{c}{24} - h_1 - h_2} (1 - z) \frac{c}{24} - h_3 \left[ \theta_3(q) \right]^{\frac{c}{24} - 4 \sum_{i=1}^{4} h_i},$$

(2.10)

and $\theta_3(q)$ is a Jacobi Theta function. $g(q, \bar{q})$ is now expanded in a basis of blocks on the torus as,

$$g(q, \bar{q}) = \sum_{h, \bar{h}} C_{12, h, \bar{h}} C_{h, \bar{h}}^h \nu_{h, h, c}(q) \nu_{\bar{h}, \bar{h}, c}(\bar{q}),$$

(2.11)

with the Virasoro blocks over the elliptic curve given by,

$$\nu_{h, h, c}(q) = \frac{(16q)^{\frac{c-1}{24}}}{\eta(q^2)} H(c, h, h, q),$$

(2.12)

being $\eta(q)$ the Dedekind eta function.

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3For details, look at the analysis in section 7 of [40].
This modified blocks can be computed by projecting the correlator into contributions to the trace from each Verma module, schematically,

\[ V_{h,h,c}(q) \sim \sum_{N} \sum_{\{\sum_{n_i,l_i}\}}^{\infty} \frac{\langle \mathcal{O}_h \prod_{n_i,l_i} L_{n_i}^{l_i} \prod_{j=1}^4 \mathcal{O}_{h_j}(z_j) \prod_{n_i,l_i} L_{-n_i}^{l_i} | \mathcal{O}_h \rangle}{\langle \mathcal{O}_h \prod_{n_i,l_i} L_{n_i}^{l_i} \prod_{n_i,l_i} L_{-n_i}^{l_i} | \mathcal{O}_h \rangle} q^{N}, \]  

(2.13)

however this procedure is very cumbersome. A more convenient way to deal with the blocks is through Zamolodchikov’s recursion relations [28, 29] which we proceed to describe in the next section.

3 Zamolodchikov’s recursion relations

In two beautiful papers [28, 29] Zamolodchikov realized that generic blocks are strongly constrained by the existence of degenerate representations of the Virasoro algebra. In particular, the function \( H(c,h_i,h,q) \) should have poles as a function of \( h \) at points \( h_{m,n} \) (defined below) corresponding to degenerate representations,

\[ V_{h,h_i,c}(q) = \sum_{m,n} \frac{S_{m,n}}{h - h_{m,n}}, \]  

(3.1)

The residue \( S_{m,n} \) of the pole at \( h_{m,n} \) will be proportional to the block \( V_{h_{m+n,m,n},h_i,c}(q) \) whose intermediate operator is evaluated on the degenerate representation with dimension \( h_{m,n} + mn \). These residues will have higher powers \( q^{mn} \) producing a series expansion in \( q \). All in all, this leads to the following recursion relation for \( H \),

\[ H(b,h_i,h,q) = 1 + \sum_{m,n \geq 1} q^{mn} \frac{R_{m,n}}{h - h_{m,n}} H(b,h_i,h_{m,n} + mn,q), \]  

(3.2)

borrowing the parametrization from Liouville theory, the central charge \( c \), external operator dimensions \( h_i \) and the degenerate operator dimensions \( h_{mn} \) are written as,

\[ Q = \left( b + \frac{1}{b} \right), \quad c = 1 + 6Q^2, \quad h_i = \frac{Q^2}{4} - \lambda_i^2, \quad h_{m,n} = \frac{Q^2}{4} - \lambda_{m,n}^2, \]  

(3.3)

with

\[ \lambda_{m,n} = \frac{1}{2} \left( \frac{m}{b} + nb \right), \]  

(3.4)

and adopting the notation from [34], \( R_{m,n} \) is given by

\[ R_{m,n} = 2 \prod_{p,q} (\lambda_1 + \lambda_2 - \lambda_{p,q}) (\lambda_1 - \lambda_2 - \lambda_{p,q}) (\lambda_3 + \lambda_4 - \lambda_{p,q}) (\lambda_3 - \lambda_4 - \lambda_{p,q}) \frac{\prod_{k,l} \lambda_{k,l}}{\prod_{k,l} \lambda_{k,l}}, \]  

(3.5)
and the ranges of $p, q, k,$ and $l$ are:

\begin{align*}
p &= -m + 1, -m + 3, \cdots, m - 3, m - 1, \\
q &= -n + 1, -n + 3, \cdots, n - 3, n - 1, \\
k &= -m + 1, -m + 2, \cdots, m, \\
l &= -n + 1, -n + 2, \cdots, n.
\end{align*}

with the prime in the denominator implying that $(k, l) = \{(0, 0), (m, n)\}$ are excluded from the product. The factor $R_{m,n}$ calls for a bit of faith and although it has passed several checks, mostly numerical (see for example [29]), we are unaware of a definite prove of it yet. It is constructed in such way that it has zeros when an operator in the correlator belongs to any of the operators in the spectrum within the fusion rules of other two external operators.

Similarly to the above, Zamolodchikov derived an equivalent recursion relation by expanding the Virasoro blocks as sum over poles in the central charge $c$ instead of intermediate state dimension $h$ [29].

### 3.1 Method of computing

In order to simplify the analysis we will consider the simplest case with all external operator dimension equal to each other, i.e. $h_i = h_1$ for $i = 1 \cdots 4$. At this configuration $R_{m,n} = 0$ whenever $mn$ is odd. This means that every $H_{m,n}$ with odd $mn$ is also zero, as every term contributing to it contains at least one $R_{m_i,n_i}$ with odd $m_i n_i$. Therefore only even powers of $q$ will appear and we can forget about odd’s $mn$.

Let us start by noticing that by evaluating the left hand side of (3.2) on the degenerate values $h = h_{mn} + mn$ we get a recursion for the coefficients,

\begin{equation}
H(b, h_1, h_{mn} + mn, q) = 1 + \sum_{r,s \geq 1} q^{rs} R_{r,s} h_{mn} + mn - h_{r,s} H(b, h_1, h_{r,s} + rs, q). \tag{3.6}
\end{equation}

Now let us expand the function $H(b, h_1, h, q)$ and its coefficients \(^4\) in a series expansion in $q$,

\begin{equation}
H(b, h_1, h, q) = \sum_{k \in \mathbb{N}} H^{(k)}(b, h_1, h) q^k, \quad H(b, h_1, h_{mn}, q) = \sum_{k \in \mathbb{N}} H^{(k)}_{mn}(b, h_1, h_{mn}) q^k. \tag{3.7}
\end{equation}

Plugging this expansion in (3.2) and (3.6) respectively we find

\begin{equation}
H^{(k)}(b, h_1, h) = \sum_{i \in \mathbb{N}} \sum_{m_i n_i = i} R_{m_i,n_i} H^{(k-i)}_{m_i,n_i}(b, h_1, h_{m_i,n_i}), \tag{3.8}
\end{equation}

\(^{4}\) $H(b, h_1, h_{mn}, q)$ are coefficients as functions of $h.$
and

\[ H_{m,n}^{(k)}(b,h_1,h_{mn}) = \sum_{i \in 2\mathbb{N}} \sum_{\{m_i n_i = i\}} R_{m_i n_i} \frac{H_{m_i n_i}^{(k-i)}(b,h_1,h_{mn})}{h_{m,n} + mn - h_{m_i n_i}} H_{m_i n_i}^{(k-i)}(b,h_1,h_{mn}), \]  

(3.9)

where the notation \( \{m_i n_i = i\} \) means the set of all pair of integers \( \{m_i n_i\} \) whose product equals \( i \).

From the two recursions above we see the strategy, for a given \( k \), we expand (3.8) in terms of the coefficients \( H_{m_i n_i}^{(k-i)} \) and then we use (3.9) to write them in terms of lower order coefficients \( H_{m_i n_i}^{(k' < k-i)} \). Unfortunately, this becomes tedious very quickly, but is friendly enough to allow being implemented in a computer algebra algorithm (see [34] for a nice implementation), although it gives not very illuminating expression at moderate higher orders.

4 Large-\( h \) analysis

Notice that due to the fact

\[ \lim_{h \to \infty} H(b,h_1,h,q) = 1, \]  

(4.1)

the prefactor in front of \( H(b,h_1,h,q) \) in (2.12) corresponds to the \( h \to \infty \) limit of \( \mathcal{V}_h \). Based on this observation, it is natural to believe that the recursion relation might simplify in a series expansion around \( \frac{1}{h} \). Is the goal of this section to provide evidence that at leading order in \( \frac{1}{h} \), the recursion relation can be solved.

Let us go back to the expansion (4.2),

\[ H(b,h_1,h,q) = 1 + \sum_{m,n \geq 1} q^{mn} \frac{R_{m,n}}{h - h_{m,n}} H(b,h_1,h_{m,n} + mn, q), \]  

(4.2)

the key observation here is that the residues, and in particular the coefficients \( H(b,h_1,h_{m,n} + mn, q) \), does not depend on \( h \), henceforth assuming that \( h \gg h_{m,n} \) for any \( (m,n) \)\(^5\), such as we can approximate the expansion above by,

\[ H(b,h_1,h,q) = 1 + \frac{1}{h} \sum_{m,n \geq 1} q^{mn} R_{m,n} \left( 1 + \frac{h_{mn}}{h} + \left( \frac{h_{mn}}{h} \right)^2 + \cdots \right) H(b,h_1,h_{m,n} + mn, q), \]  

(4.3)

we are interested into the leading term only, in other words, the expansion we want to deal with has the simplified form,

\[ H(b,h_1,h,q) = 1 + \frac{1}{h} \sum_{m,n \geq 1} q^{mn} R_{m,n} H(b,h_1,h_{m,n} + mn, q) + O \left( \frac{1}{h^2} \right), \]  

(4.4)

\(^5\)Notice that we are taking \( c \) to be finite and then \( h_{mn} \) will be small respect to \( h \) as long as \( (m,n) \ll O(h) \).
4.1 Leading order

Unfortunately the recursion for the coefficients \(3.9\) does not get affected by the limit and we still need to deal with the cumbersome long expression coming from several iterations. However, after evaluating some few terms with some involved algebra, the first few orders of \(H(b, h_1, h, q)\) simplify more than we could have wished for. For a given order at the \(q\)–expansion \(4.4\) the corresponding contribution from all the contributing coefficients become proportional to a single global function defined as,

\[
H_1(h_1, c) = \frac{1}{16}((c + 1) - 32h_1)((c + 5) - 32h_1).
\]

(4.5)

By using an accordingly modified version of the algorithm developed in [34] we were able to compute a few low order terms up to \(q^{18}\),

\[
H(b, h_1, h, q) = 1 + \frac{H_1(h_1, c)}{4h} (q^2 + 3q^4 + 4q^6 + 7q^8 + 6q^{10} + 12q^{12} + 8q^{14} + 15q^{16} + 13q^{18} \cdots ) + O\left(\frac{1}{h^2}\right).
\]

(4.6)

The sequence accompanying the \(q\) expansion can be quickly recognized as generated by the sigma divisor function of order one, \(\sigma_1(k)\), which gives the sum of all divisors of an integer \(k\).\(^6\) Assuming that the pattern holds at higher orders, we can write the solution to the Virasoro block at leading order in a large \(-h\) expansion as,

\[
H(b, h_1, h, q) = 1 + \frac{H_1(h_1, c)}{4h} \sum_{k=1}^{\infty} q^{2k} \sigma_1(k) + O\left(\frac{1}{h^2}\right).
\]

(4.7)

As a sanity check, we can immediately recognize that the only two solutions to the condition \(H_1(h_1, c) = 0\) are \(\{c=1, h_1=1/16\}\) and \(\{c=25, h_1=15/16\}\) which agrees with the results from [9], and the Virasoro blocks in that case are given by

\[
F\left(1, \frac{1}{16}, h, z\right) = \frac{(16q)^h}{\eta(q^2)}(z(1-z))^{-\frac{1}{8}}\theta_3(q)^{-1},
\]

(4.8)

with a similar expression for the second solution.

The solution (4.7) can be checked numerically to a reasonably high order by giving numerical values to \(h_1\) and \(c\) in order to compute a numerical solution to \(H(b, h_1, h, q)\) as a function of \(h\) from the numerical code [34]. Then expanding in \(1/h\) and dividing each coefficient at a given order in the \(q\)–expansion by the numerical value of the global function (4.5), we can check that the left-over coefficients are indeed generated by the sigma divisor function up to order \(q^{500}\).

\(^6\)see table 24.6 [42]

\(^7\)We thank Shouvik Datta for performing this numerical check
We can write the solution (4.7) perhaps in a more illuminating form in terms of an Eisenstein series defined as,

$$G_{2n}(\tau) = \sum_{m,n \in \mathbb{Z}^2(0,0)} \frac{1}{(m+n\tau)^{2n}}$$

(4.9)

which can be represented in terms of sigma divisors by the relation,

$$G_{2n}(\tau) = 2\zeta(2n) \left(1 + c_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k)q^{2k}\right),$$

(4.10)

here it is important to note that as a modular form the function $G_{2n}(\tau)$ only makes sense for $n > 1$, otherwise the $SL(2,\mathbb{Z})$-invariance would not hold. Nevertheless, an analogue of the holomorphic Eisenstein series can be defined even for $n = 1$, but it would give us a quasimodular form instead.

By allowing $n = 1$, we can take

$$\sum_{k=1}^{\infty} \sigma_1(k)q^{2k} = \left(\zeta(2) / (2\pi)^2 - G_2(\tau) / 8\pi^2\right),$$

(4.11)

and therefore rewrite the solution (4.7) as,

$$H(b, h_1, h, q) = 1 - \frac{H_1(h_1, c)}{4h} \left(\frac{G_2(\tau)}{8\pi^2} - \frac{1}{24}\right) + \mathcal{O}\left(\frac{1}{h^2}\right).$$

(4.12)

In this form, the leading order contribution seems to agree with what is expected from intuition. Due to the fact that we have written the Virasoro block on a elliptic curve, it has to be covariant under the fractional linear transformations preserving the lattice structure from the torus, i.e.

$$H \left(\frac{az+b}{cz+d}\right) = (cz+d)^w H(z),$$

(4.13)

this invariance in particular implies that the Virasoro blocks over the torus should be periodic $H(z+1) = H(z)$ and therefore have a Fourier expansion of the form

$$H(z) = \sum_{k \in \mathbb{Z}} H^{(k)}q^{2k},$$

(4.14)

with $q = e^{\pi i \tau}$ and which is equivalently a consequence of the recursion relation itself. Finally, automorphic forms on $SL(2,\mathbb{R})$ that are invariant under the action of the subgroup $SL(2,\mathbb{Z})$, can

Another possibly useful representation of this solution might be in the form of a Lamber series

$$\sum_{k=1}^{\infty} q^{2k} \sigma_1(k) = \sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}}.$$

8
be written in terms of Eisenstein series. It is however puzzling for us, that the solution is given by a quasi-modular form instead of a modular form. A possible naive explanation for it, is that even though we expect the exact answer at finite $h$ to be given by a modular form, we can thought the $1/h$ expansion as a Taylor series, however derivatives of modular forms are quasimodular. Even more, quasimodular forms are closed under derivation, so it might be not surprising to get also quasimodular forms at higher orders in $1/h$. This issue, certainly is worth of a deeper study.

4.2 Some application

Within the bootstrap framework in higher dimensions, it is well known that the light cone limit for the OPE expansion of the vacuum, is controlled by exchange operators at large spin \cite{20,43}. This has lead to the development of a large spin perturbation theory \cite{44,45} whose asymptotic series can be resummed from a brilliant inversion formula \cite{46,47}. We believe the results from this paper can be use to perform equivalent analysis in two dimensional conformal field theories, as those done for example in the computation of anomalous dimension at large spin in \cite{49,48,50}. This program has already been started very recently in \cite{51,35} by using the Fusion kernel instead of conformal blocks, and therefore it would be interesting to understand those new results from the perspective of Virasoro blocks at large dimension and large spin.

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