Universal factorization law in quantum information processing

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We identify a universal factorization law in quantum information processing protocols such as the quantum teleportation, remote state preparation, Bell-non-locality violation and particularly dynamics of geometric quantum correlation measures. This factorization law shows that when the system traverses the local quantum channel, various figure of merits for different protocols demonstrate a universal factorization decay behavior for dynamics. We find a family of quantum states and the corresponding quantum channels where the factorization law is satisfied. This factorization law simplifies the assessment of many quantum tasks.

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Introduction.— Quantum theory enables the processing of quantum information in a way that outperforms those of classical one in many aspects. The quantum correlations, quantified by various measures, among the computational systems were considered to be the origin of this superiority [1]. But when implementing the quantum information processing (QIP) tasks experimentally, one has to face the unavoidable interaction of the quantum devices with their environments, the process of which often induces decay of correlations [2]. Therefore, it is of practical significance to make clear the robustness of a correlation when it serves as a resource for carrying out certain QIP tasks.

To assess the time evolution of a given quantum correlation measure, one usually needs to derive first the time-dependence of the density matrix, and this is intractable. From a theoretical point of view, one needs to obtain the evolution of the system for every initial states, while from an experimental point of view, it requires the full reconstruction of the evolved state via quantum tomography [3]. This either needs much computational resources or is hard to do experimentally.

Seeking general dynamical law for various correlation measures can simplify these tasks. As an earlier proposed quantum correlation measure, entanglement has been the focus of researchers for many years [1]. Particularly, when one party of a two-qubit system passes through a noisy channel, the evolution equation of concurrence [4] was found to be governed by a factorization law (FL) for the initial pure states [3]. This simplifies the assessment of the evolution of concurrence as to evaluate a decay factor. Soon a universal curve describing the evolution of concurrence for general two-qubit states was revealed [6]. Since then, the evolution equations for many other entanglement measures [7–9] or their bounds [10–12] have been found.

Besides entanglement, the correlations in a system can also be characterized from other perspectives. The typical one is quantum discord [13], and its generalizations such as the geometric quantum discord (GQD) [14–19] and measurement-induced nonlocality (MIN) [20, 21], which are also resources for certain QIP tasks [22–24]. Moreover, many measures related to the explicit quantum tasks (see the following text) can also be considered as some kinds of correlations. The decay dynamics for these related measures under different environments have been extensively studied [25–31]. Yet, some general conclusions related specifically with the factorization law remain absent. Motivated by these, we study in this Letter the evolution equation for various geometric correlation measures when the system traverses a local quantum channel. We aim at revealing a general dynamical factorization law that governs the dynamics of these correlation measures.

Preliminaries.— We consider a general bipartite state ρ in the Hilbert space ℋAB. It can always be decomposed as

\[ ρ = \frac{1}{\sqrt{d_A d_B}} X_0 \otimes Y_0 + \sum_{k=1}^{d_A^2-1} x_k X_k \otimes Y_0 \]

\[ + X_0 \otimes \sum_{l=1}^{d_B^2-1} y_l Y_l + \sum_{k=1}^{d_A^2-1} \sum_{l=1}^{d_B^2-1} t_{kl} X_k \otimes Y_l, \tag{1} \]

where \( X_k \) (\( k = 0, 1, 2, \ldots, d^2_A - 1 \)) with \( d_A = \dim(ℋ_A) \) is the orthonormal Hermitian operator bases in ℋ_A that satisfy \( \text{tr} X_k X_{k'} = \delta_{kk'} \), with \( X_0 = \mathbb{1}_A/\sqrt{d_A} \) and \( \mathbb{1}_A \) the identity matrix in ℋ_A, and likewise for Y_l.

Two extensively studied cases in the literature are \( d_{AB} = 2 \) and 3, for which \( X_k \) (and Y_l) are given respectively by \( X_k = \sigma_k/\sqrt{2} \) (\( k = 1, 2, 3 \)), and \( X_k = \lambda_k/\sqrt{2} \) (\( k = 1, \ldots, 8 \)), with \( \sigma_k \) and \( \lambda_k \) being the Pauli and the Gell-Mann matrices [32]. They can describe the qubit, qutrit, and the hybrid qubit-qutrit systems, which are of central relevance to QIP.

The decomposed \( ρ \) in Eq. (1) enables the definition of correlation measures from a geometric perspective. We consider here the definition of the general form

\[ D^A_ρ(ρ) = \text{opt}_{Π^A ∈ ℳ} \| ρ - Π^A(ρ) \|_p, \tag{2} \]

where \( \| X \|_p = |\text{tr}(X^p)|^{1/p}/p \) is the Schatten p-norm, and opt represents the optimization over some class ℳ of the local measurements \( Π^A = \{ Π^A_k \} \) acting on party A.

The definition (2) covers a series of discord-like correlation measures being proposed recently. (i) If opt represents minimum and the class ℳ is that of the projection-valorued or the
positive operator valued measurements (POVM), one recovers the projection-based or the POVM-based GQD $D^2_A(\rho)$ when $p = 2$ \cite{14}, and the modified GQD $D^3_A(\rho)$ when $p = 1$ \cite{16}, which can remedy the noncontractivity problem occurred for $D^2_A(\rho)$. (ii) If $\rho$ is replaced by its square root $\sqrt{\rho}$, then one obtains the squared GQD $D_H(\rho)$ based on the Hellinger distance \cite{18,19}. (iii) If $\text{opt}$ represents maximum and $\mathcal{M}$ is confined to the locally invariant measurements that maintain $\rho_A = \text{tr}_B\rho$, Eq. (2) turns to be the MIN $N^3_A(\rho)$ when $p = 2$ \cite{20}, and its modified version $N^3_A(\rho)$ when $p = 1$ \cite{21}.

Besides various forms of $D^A(\rho)$ that are determined by the vectors $\vec{x} = (x_1, \ldots, x_{d_A^2-1})^t$, $\vec{y} = (y_1, \ldots, y_{d_A^2-1})^t$ (the superscript $t$ denotes transpose of vectors), and the correlation tensor $T = (t_{kl})$, there are many other measures related to explicit quantum tasks that are determined solely by $T$. For example, for quantum teleportation (QT) of the one-qubit state, the average fidelity was given by \cite{33}

$$F_{\text{qt}}(\rho) = \frac{1}{2} + \frac{1}{6} N_{\text{qt}}(\rho),$$

where $N_{\text{qt}}(\rho) = \text{tr}\sqrt{T^\dagger T}$. Moreover, for the remote state preparation (RSP) protocol of Ref. \cite{23}, the fidelity was

$$F_{\text{resp}}(\rho) = \frac{1}{2}(E_2 + E_3),$$

with $E_1 \geq E_2 \geq E_3$ being the eigenvalues of $T^\dagger T$. Thirdly, the Bell-nonlocality violation (BNV) of a two-qubit state can be detected by its violation of the inequality $|\langle B \rangle | \leq 2$, and the maximum of $|\langle B \rangle |$ was proved to be \cite{34}

$$B_{\text{max}}(\rho) = 2\sqrt{E_1 + E_2},$$

where $E_i (i = 1, 2, 3)$ are still the eigenvalues of $T^\dagger T$.

**General results.**— We restrict ourselves in the following to the projection-valued measurements $\Pi_A$, then it follows from Eqs. (1) and (2) that $D^A(\rho)$ is solely determined by the vector $\vec{x}$ and the correlation tensor $T$.

Now, we suppose the considered system $A$ passes through a quantum channel $\mathcal{S}$ such that $\vec{x}'$ and $T'$ for $\mathcal{S}(\rho)$ are given by

$$\vec{x}' = q\vec{x}, \quad T' = qT,$$

with $q$ being a time-dependent parameter that contains the information on $\mathcal{S}$’s structure and its coupling with $A$, then the definition (2) implies that the evolution equation of $D^A_p(\rho)$ obeys the following dynamical FL

$$D^A_p[\mathcal{S}(\rho)] = |q|^p D^A_p(\rho).$$

In the operator-sum representation, the evolved state of $A$ under the action of a local quantum channel $\mathcal{S}$ can be written compactly as $\mathcal{S}(\rho) = \sum_{\mu\nu} E_{\mu\nu} \rho E_{\mu\nu}$, with the Kraus operators $E_{\mu\nu} = E_\mu \otimes E_\nu$ satisfying $\sum_{\mu\nu} E^\dagger_{\mu\nu} E_{\mu\nu} = \mathbb{I}_{AB}$. Then, by turning to the Heisenberg picture (Fig. 1) to describe the map of the channel on $\mathcal{O}$ (an arbitrary operator) as

$$\mathcal{S}^H(\rho) = \sum_{\mu\nu} E^\dagger_{\mu\nu} \mathcal{O} E_{\mu\nu},$$

one can obtain that Eq. (6) can be satisfied by a broad class of $\mathcal{S}$ \cite{32}. To show this explicitly, we denote by $T_{K^S, L^S} (T_{K^S, L^S})$ for the $K^S$-th row ($L^S$-th column) of $T$, with $K^S$ being a proper subset of $K = \{1, \ldots, d_A\}$, $K^C = K - K^S$ the complementary set of $K^S$, and likewise for $L^S$ and $L^C$. Then we have:

(1) For quantum channels that give the map $\mathcal{S}^H(X_k) = qX_k$ or $\mathcal{S}^H(Y_l) = qY_l$ for all $k, l \neq 0$, the FL (7) holds for arbitrary bipartite state $\rho$ if the channel acts as $\mathcal{S} \otimes \mathbb{I}_B$, while it holds for the family of $\rho$ with $\vec{x} = 0$ ($\vec{x} = 0$ or $T = 0$) if the channel acts as $\mathcal{S} \otimes \mathcal{S}$ ($\mathcal{S} \otimes \mathcal{S}$).

(2) If the channels give $\mathcal{S}^H(X_k) = qX_k$ or $\mathcal{S}^H(Y_l) = qY_l$ only for partial $\{k, l\} \neq 0$, e.g., for the proper subset $K^S \subset K$ or $L^S \subset L$, then the one-sided channel $\mathcal{S} \otimes \mathcal{S}$ ($\mathcal{S} \otimes \mathcal{S}$), the FL (7) holds for the family of $\rho$ with $\vec{x} = 0$ ($\vec{x} = 0$ or $T = 0$) if the channel acts as $\mathcal{S} \otimes \mathcal{S}$ ($\mathcal{S} \otimes \mathcal{S}$), or with $x_{KC} = 0$ and $T_{KC} = 0$ ($\vec{x} = 0$ or $T_{KC} = 0$), while for the two-sided channel $\mathcal{S} \otimes \mathcal{S}$, it holds for $\rho f$ with $\vec{x} = 0$, $T_{KC} = 0$, and $T_{KC} = 0$, or with $x_{KC} = 0$ and $T = 0$.

Moreover, when restricted to the qubit system, the evolution equations of the measures for QT, RSP and BNV can also be described by some FL when $A$ and $B$ traverses a channel such that $T' = qT$ for $\mathcal{S}(\rho)$ is given by $T' = qT$. They are

$$N_{\text{qt}}[\mathcal{S}(\rho)] = |q|N_{\text{qt}}(\rho), \quad F_{\text{resp}}[\mathcal{S}(\rho)] = |q|^2 F_{\text{resp}}(\rho), \quad B_{\text{max}}[\mathcal{S}(\rho)] = |q|B_{\text{max}}(\rho),$$

The requirement $T' = qT$ can be satisfied by the local quantum channel which gives $\mathcal{S}^H(X_k) = qX_k$ or $\mathcal{S}^H(Y_l) = qY_l$ for the family of $\rho$ with nonzero coefficients $t_{kl}$.

Of course, the above conditions are sufficient but not necessary, namely, for certain specifically defined correlation measures, the FL may holds even if those requirements cannot be satisfied \cite{32,35}. The significance of the above conditions lie in that they provide a flexible way for identifying the family.
of $\rho$ that obeys the FL, and this is of special importance for assessing the robustness of certain quantum protocols.

**Depolarizing channel.**— We discuss now one paradigmatic type of the local quantum channel, i.e., the depolarizing channel, which represents the process in which a state $\rho$ is dynamically replaced by the maximally mixed one. It gives

$$S(\rho_S) = q\rho_S + (1-q)\frac{1}{d_{S}} \mathbb{I}_{S},$$

(10)

where $S = \{ A, B \}$. Eq. (10) corresponds to $S(X_k) = qX_k$ or $S(Y_l) = qY_l$ for all $X_k$ and $Y_l$ with nonzero $\{k,l\}$. Therefore, the FL (7) holds for arbitrary bipartite $\rho$ if the one-sided channel $S \otimes \mathbb{I}_B$ is applied. Moreover, it holds for the family of $\rho$ with $\vec{x} = 0$ ($\vec{x} = 0$ or $T = 0$) if the one-sided channel $\mathbb{I}_A \otimes S$ (the two-sided channel $S_1 \otimes S_2$) is applied.

The fact that the evolution equation of any $D^A_{\rho}(\rho)$ obeys a FL under the action of the depolarizing channel $S \otimes \mathbb{I}_B$ has by itself a practical significance, as one can infer the evolution of $D^A_{\rho}$ or acts locally on $\rho$ or $\rho^\dagger$. There, the entanglement measures obey a FL for arbitrary one-sided or two-sided.

For the qubit system, the possible actions of $S$ on a qubit can be characterized by at most four independent Kraus operators that are linear combinations of the identity and the Pauli matrices. Consider the following typical one

$$E_{0,i} = \frac{1}{2} \sqrt{1 + q_0 + 2q_0 \sigma_{0,i}}, \quad E_{j,k} = \frac{1}{2} \sqrt{1 - q_0 \sigma_{j,k}},$$

(13)

where $\sigma_0 = I_2$, and $\{i,j,k\}$ is a possible cyclic permutation of $\{1,2,3\}$. They give the map $S(\sigma_i) = q_0 \sigma_i$, and $S(\sigma_j \sigma_k) = q_0 \sigma_j \sigma_k$. Therefore, we have the following results:

First, the evolution equation of $D^A_{\rho}(\rho)$ obeys the FL (7) for the family of $\rho^f_1 (\rho^f_2)$ under the one-sided channel $S \otimes I_2 (I_2 \otimes S)$, and for the family of $\rho^f_3$ under the two-sided channel $S_1 \otimes S_2$, all with $K^{C} = \{i\}$ and $L^{C} = \{i\}$.

Second, the three measures related to QT, RSP, and BNV obey the FL (7) if $T_{i}^f$ (for $S \otimes I_2$), or $T_{i}^f$ (for $I_2 \otimes S$), or both $T_{i}^f$ and $T_{i}^f$ equal zero.

We note that the foregoing discussion covers many typical Pauli channels in the literature, which include the depolarizing channel ($q_0 = q$), and the bit flip ($i = 1$), bit-phase flip ($i = 2$), and phase flip ($i = 3$) channels when $q_0 = 1$.

Eq. (13) also enables us to identify the family of $\rho$ for which certain correlation measures are immune of decay, and this is appealing for QIP tasks. For example, we have $S(\sigma_3) = \sigma_3$, and $S(\sigma_{12}) = q_0 \sigma_{12}$ when $AB$ traverses the phase flip channel $S \otimes I_2$. Thus if $x_1 = 1$ or $T_{i}^f = 0$, then $D^A_{\rho}(\rho)$ and the measures related to QT, RSP, and BNV will do not decay in the whole time region. Similar results can be obtained for the bit flip and bit-phase flip channels.

The foregoing Pauli channels we considered all satisfy the unital condition $S(I_2) = I_2/2$. We now further consider a nonunital local channel, i.e., the generalized amplitude damping (GAD) channel, with the Kraus operators

$$E_{0,2} = \sqrt{\eta_0 (1 + q)(1 - q) \sigma_3},$$

$$E_{1,3} = \sqrt{\eta_3 (1 - q) \sigma_1},$$

(14)

for the two-qubit local channel, by replacing $\rho$ in Eq. (11) with ($I_2 \otimes S_2$)$\rho$ and then using Eq. (7), one can obtain

$$D^A_{\rho}[S(I_2 \otimes S_2)] \geq |q_A| D^A_{\rho}(\rho).$$

(12)

Similarly, for the 2-norm GQD, from the results of [15] one can show that it obeys the similar dynamical law as Eqs. (11) and (12), with however $|q_A|$ and $|q_B|$ being replaced by $|q_A|^2$ and $|q_B|^2$. Moreover, the FL (7) related to QT, RSP, and BNV holds for arbitrary two-qubit state $\rho$, whether the depolarizing channel is one-sided or two-sided.

**Pauli channels.**— We now turn to discuss some more general quantum channels, and show that one can construct many $S$ for which the FL is obeyed for certain families of $\rho$.

For the qubit system, the possible actions of $S$ on a qubit can be characterized by at most four independent Kraus operators that are linear combinations of the identity and the Pauli matrices. Consider the following typical one

$$E_{0,i} = \frac{1}{3} \sqrt{1 + 2q_3 \tau_3}, \quad E_{1,2} = \sqrt{\frac{1 - q}{2}} \lambda_{k_1,k_2},$$

(15)
which gives $S^1(\lambda_k) = \lambda_k$ for $k \in K^C$ with $K^C = \{k_1, k_2, k_3\}$ ($k_1 = 1, 2, 3$), and $S^q(\lambda_k) = q \lambda_k$ otherwise. Thus, for $\mathbb{F} \otimes I_3 \otimes \mathbb{F}$, and $S^1 \otimes S^2$, the FL (7) holds for the families of $\rho^{f_1}$, $\rho^{f_2}$, and $\rho^{f_3}$, respectively. Moreover, $D^q(p)$ is time-independent if $x_{K^S}$ and $T_{K^S}(K^S = K - K^C)$ equal zero.

The second class is described by the Kraus operators

$$E_0 = (2q - 1)I_3, \quad E_{1,2,3} = (1 - q)\lambda_{k_1,k_2,k_3},$$

(16)

where $k_1 \in \{1, 2, 3\}$, $k_2 \in \{4, 5\}$, and $k_3 \in \{6, 7\}$. They give $S^1(\lambda_k) = q \lambda_k$ for $k \in \{k_1, k_2, k_3\}$, and $S^q(\lambda_k) = (3q - 2)\lambda_k$ otherwise. Then, by replacing $\{k_1, 8\}$ with $\{k_1, k_2, k_3\}$ or its complementary set, one can obtain the families of $\rho$ obeying the FL (7).

Summary and discussion.— In summary, we have investigated the evolution equations for a series of geometric correlation measures. Moreover, by turning to the Heisenberg picture to describe the action of the local quantum channel, we also derive explicitly the sufficient conditions such that the FL holds, and identify several families of $\rho$ for which the evolution equations of the corresponding measures obey this FL under given noisy channels.

The existence of the FL is of central relevance for assessing the robustness of the related correlation measures. Those measures may be the resource for carrying out certain QIP tasks such as the input-output gate operation in sequential quantum computing and the experimental generation of quantum correlated resources in noisy environments. Additionally, the FL is potential for simplifying the detection of the structures of certain unknown reservoirs, for any $\rho$ belongs to the derived family can fulfill this task. Moreover, as $S(\rho)$ may represent the action of environment, of measures, or both on $\rho$, the significance of the FL (7) and (9) lies also in that they show unambiguously that all the $p$-norm based correlations may be enhanced by a sequential of operations such that the factor $|q|$ is increased with time. A deep exploration of the FL might be related with structure of entanglement spectrum in describing topology of band structures of many-body systems [37, 39]. We hope these results may shed some lights on understanding the essence of quantum correlations and their applications in QIP and condensed matter physics, especially from a geometric perspective.

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SUPPLEMENTAL MATERIAL

We provide in this supplemental material the proofs for the factorization law (FL) of certain geometric correlation measures based on the Schatten $p$-norm, as well as the details for some statements in the main text.

The map of local channel on $X_k$ and $Y_i$

From the decomposed form of $\rho$ in Eq. (1) of the main text, we know that $x_k = tr[A\rho(X_k \otimes Y_0)]$, $y_i = tr[A\rho(X_0 \otimes Y_i)]$, and $t_{kl} = tr[A\rho(X_k \otimes Y_l)]$, then if the system $AB$ traverses a local quantum channel $S_1 \otimes S_2$, we have

$$
t'_{kl} = tr[(S_1 \otimes S_2)\rho(X_k \otimes Y_l)]
= \sum_{\mu\nu} tr[E_{\mu\nu}\rho E_{\mu\nu}^T(X_k \otimes Y_l)]
= \sum_{\mu\nu} tr[\rho E_{\mu\nu}^T(X_k \otimes Y_l) E_{\mu\nu}]
= tr[\rho S_1^T(X_k) \otimes S_2^T(Y_l)],
$$

(A17)

where the Kraus operators $E_{\mu\nu} = E_{\mu} \otimes E_{\nu}$ satisfy the condition $\sum_{\mu\nu} E_{\mu\nu}^T E_{\mu\nu} = I_{AB}$, with $E_{\mu}$ and $E_{\nu}$ describing the effects of $S_1$ and $S_2$, respectively. Moreover, we denote by $S_1^T(X_k) = \sum_{\mu} E_{\mu}^T X_k E_{\mu}$ for the map of $S_1$ on $X_k$, and likewise for $S_2^T(Y_l)$.

Similarly, one can obtain

$$
x'_{k} = tr[\rho S_1^T(X_k) \otimes Y_0],
y'_{l} = tr[\rho X_0 \otimes S_2^T(Y_l)],
$$

(A18)

where we have used the facts that $S_1^T(X_0) = X_0$ and $S_2^T(Y_0) = Y_0$ for arbitrary $S_1$ and $S_2$.

Moreover, the above discussion also applies to the case of the one-sided channel acting solely on party $A$ ($B$) when $S_1 = I_A$ ($S_2 = I_B$).

Proofs of the FL for GQD

The 1-norm (trace norm) GQD for arbitrary two-qubit state $\rho$ had been derived as

$$
D_1^A(\rho) = \frac{1}{2}\sqrt{2\min_{\hat{c}} h(\hat{c})},
$$

(A19)

where $\hat{c}$ is a unit vector in $\mathbb{R}^3$, and

$$
h(\hat{c}) = a(\hat{c}) + \sqrt{a^2(\hat{c}) - b(\hat{c})},
$$

(A20)

where $a(\hat{c}) = Q + x_4^2$, and $b(\hat{c}) = 4(x_2^2 + y_2^2)$ (see Ref. [1] for the specific meanings of the four parameters $Q$, $x_{\perp}$, $x$, and $y$). Then, as $x' = \bar{x}$ and $T' = qT$ under the action of the one-sided depolarizing channel $\mathbb{I}_2 \otimes S$, we have

$$
a'(\hat{c}) = |q|^2 Q + x_4^2, b'(\hat{c}) = 4|q|^2(x_2^2 + |q|^2y_2^2),
$$

(A21)

After a complicated algebra we obtain $h'(\hat{c}) \geq |q|^2 h(\hat{c})$, and therefore

$$
D_1^A([\mathbb{I}_2 \otimes S]\rho) \geq |q|D_1^A(\rho).
$$

(A22)

Moreover, the 2-norm (Hilbert-Schmidt norm) GQD for arbitrary 2-qubit state $\rho$ was given by [2]

$$
D_2^A(\rho) = \frac{1}{4}[||\bar{x}||^2 + ||T||^2 - k_{\text{max}}(K)],
$$

(A23)

where $k_{\text{max}}(K)$ is the largest eigenvalue of $K = \bar{x}\bar{x}^T + TT^*$. Then, as $\bar{x}' = \bar{x}$ and $T' = qT$ when party $B$ of $AB$ traverses the depolarizing channel, we have

$$
D_2^A([\mathbb{I}_2 \otimes S]\rho) = \frac{1}{4}[||\bar{x}||^2 + ||T'||^2 - k_{\text{max}}(K')],
$$

(A24)

where $K' = \bar{x}\bar{x}^T + |q|^2TT^*$.

By rewriting $K'$ as $K' = |q|^2 K + (1 - |q|^2)\bar{x}\bar{x}^T$ and then using the Weyl’s theorem, we know that

$$
k_{\text{max}}(K') \leq |q|^2 k_{\text{max}}(K) + (1 - |q|^2)k_{\text{max}}(\bar{x}\bar{x}^T),
$$

$$
k_{\text{max}}(K') \geq |q|^2 k_{\text{min}}(K) + (1 - |q|^2)k_{\text{min}}(\bar{x}\bar{x}^T),
$$

(A25)

therefore

$$
D_2^A([\mathbb{I}_2 \otimes S]\rho) \geq \frac{1}{4}[||\bar{x}||^2 + |q|^2||T'||^2 - |q|^2 k_{\text{max}}(K)
- (1 - |q|^2)k_{\text{max}}(\bar{x}\bar{x}^T)]
= \frac{1}{4}[|q|^2[||\bar{x}||^2 + ||T'||^2 - k_{\text{max}}(K)]
+ (1 - |q|^2)[||\bar{x}||^2 - k_{\text{min}}(\bar{x}\bar{x}^T)]]
\geq |q|^2 D_2^A(\rho),
$$

(A26)

where the second inequality is due to $||\bar{x}||^2 - k_{\text{max}}(\bar{x}\bar{x}^T) \geq 0$. Similarly, one can derive

$$
D_2^A([\mathbb{I}_2 \otimes S]\rho) \leq |q|^2 D_2^A(\rho) + \frac{1}{4}(1 - |q|^2) \times ||\bar{x}||^2 - k_{\text{min}}(\bar{x}\bar{x}^T),
$$

(A27)
The Pauli and the Gell-Mann channels

Apart from the depolarizing channel, one can also construct many other quantum channels under which the geometric correlation measures obey a FL. As two examples, we consider here the general Pauli and Gell-Mann channels, which apply respectively to the qubit and the qutrit systems.

For the qubit system, we write the Pauli matrices as

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(A28)

Then, we consider the Pauli channel $\mathcal{S}$ with the Kraus operators

\[
E_0 = \sqrt{\epsilon_0 I_2}, \quad E_{1,2,3} = \sqrt{\epsilon_3 \sigma_{1,2,3}},
\]

(A29)

where $\sum_{\mu=0}^{3} E_\mu^\dagger E_\mu = I_2$. This yields

\[
\mathcal{S}^\dagger(\sigma_k) = (\epsilon_0 + \epsilon_k - \sum_{i \neq k} \epsilon_i) \sigma_k,
\]

(A30)

for $k = \{1, 2, 3\}$. Then, by supposing $x_k = q_k x_k$, we obtain

\[
\epsilon_0 = \frac{1}{4} (1 + q_1 + q_2 + q_3),
\]

\[
\epsilon_1 = \frac{1}{4} (1 + q_1 - q_2 - q_3),
\]

\[
\epsilon_2 = \frac{1}{4} (1 - q_1 + q_2 - q_3),
\]

\[
\epsilon_3 = \frac{1}{4} (1 - q_1 - q_2 + q_3).
\]

(A31)

This result enables us to construct a number of Pauli channels under the action of which the evolution equations of various geometric correlation measures obey a FL. For instance, the depolarizing channel acting on a qubit is in fact a special case of Eq. (A31) with $q_1 = q_2 = q_3 = q$. Moreover, the Pauli channels in Eq. (13) of the main text corresponds to the case of $q_i = q_j = q$ and $q_k = q_0$, where $\{i, j, k\}$ represents a possible cyclic permutation of $\{1, 2, 3\}$.

When considering the qutrit system, the explicit forms of the Gell-Mann matrices are of the following form:

\[
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

(A32)

The possible actions of a quantum channel $\mathcal{S}$ on a qutrit can be described by the Kraus operators that are linear combinations of the identity and the Gell-Mann matrices, and for convenience of representation, we call them Gell-Mann channels. Similar as the qubit system, we consider here the representative class of $\mathcal{S}$ with the Kraus operators described by

\[
E_0 = \sqrt{\epsilon_0 I_3}, \quad E_{1,\ldots,8} = \sqrt{\epsilon_\mu \lambda_{1,\ldots,8}},
\]

(A33)

where $\sum_{\mu=0}^{8} E_\mu^\dagger E_\mu = I_3$. Then, we suppose $\mathcal{S}^\dagger(\lambda_k) = q_k \lambda_k$ $(k = 1, \ldots, 8)$ for the purpose of finding the channels $\mathcal{S}$ under which $D_\mu^A(\rho)$ obeys the FL, and after a straightforward algebra, we obtain that the parameters $q_k$ must satisfy the following conditions

\[
q_1 + q_2 + q_3 = q_6 + q_7 + q_8,
\]

\[
q_4 + q_5 = q_6 + q_7,
\]

(A34)

under which we have

\[
\epsilon_0 = \frac{1}{9}(1 + 3q_6 + 3q_7 + 2q_8),
\]

\[
\epsilon_{1,2,3} = \frac{1}{12}(2 - 3q_6 - 3q_7 - 2q_8 + 6q_{1,2,3}),
\]

\[
\epsilon_{4,5} = \frac{1}{12}(2 + 3q_4 + 3q_5 - 2q_6),
\]

\[
\epsilon_{6,7} = \frac{1}{12}(2 + 3q_6 + 3q_7 - 2q_8),
\]

\[
\epsilon_8 = \frac{1}{12}(2 - 3q_6 - 3q_7 + 4q_8).
\]

(A35)

The depolarizing channel for a qutrit can be considered as a special case of Eq. (A35), which corresponds to $q_k = q$ for all $k = \{1, \ldots, 8\}$. Moreover, the other Gell-Mann channels discussed in the main text are also special cases of Eq. (A35). For instance, the Gell-Mann channels in Eq. (15) of the main text are those with $q_{k_1,8} = 1$ $(k_1 = 1, 2, 3)$, and $q_k = 2$ for $k \neq \{k_1, 8\}$, while Eq. (16) of the main text are those with $q_{k_1, k_2, k_3} = 2$ and $q_k = 3q - 2$ for $k \neq \{k_1, k_2, k_3\}$, where $k_1 = \{1, 2, 3\}$, $k_2 = \{4, 5\}$, and $k_3 = \{6, 7\}$.

In fact, one can also construct many other quantum channels under the action of which the geometric correlation measure $D_\mu^A(\rho)$ obeys the FL for certain families of the two-qubit states.

**Examples of other states obeying the FL**

We have pointed out in the main text that the conditions for the FL are sufficient but not necessary. We present here some explicit examples by considering the 1-norm GQD for certain families of the two-qubit states.

Consider first the family of the pure states

\[
|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle,
\]

(A36)

which gives to $D_1(|\Psi\rangle \langle \Psi|) = 2|\alpha\beta|$. When the system $AB$ passes through the local phase damping (phase flip) channel, one can obtain

\[
D_1(|S_1 \otimes S_2|\langle \Psi|) = 2|q_Aq_B|\alpha\beta|.
\]

(A37)
Therefore, the FL of Eq. (7) in the main text is obeyed even if $x_3 = \alpha^2 - \beta^2 \neq 0$. Note that Eq. (A37) covers also the one-sided phase damping channel acting solely on party $A$ ($B$) when $\mathbb{S}_1 = \mathbb{I}_2$ ($\mathbb{S}_2 = \mathbb{I}_2$).

The second family of quantum states we considered are the Bell-diagonal states

$$\rho^{BD} = \frac{1}{4}(\mathbb{I}_2 \otimes \mathbb{I}_2 + \sum_{i=1}^{3} c_i \sigma_i), \quad (A38)$$

for which $T' = \text{diag}\{q_1 q_2 c_1, q_1 q_2 c_2, q_1 q_2 c_3\}$ under the action of the local phase damping channel $\mathbb{S}_1 \otimes \mathbb{S}_2$. Clearly, it maintains the form of the Bell-diagonal states. As a result [3]

$$D_1^A[(\mathbb{S}_1 \otimes \mathbb{S}_2)\rho^{BD}] = \text{int}\{|q_A q_B c_1||q_A q_B c_2||c_3|\}, \quad (A39)$$

where ‘int’ in Eq. (A39) represents the process of taking the intermediate value. Then, a straightforward analysis about the magnitudes of $|c_1|$, $|c_2|$, and $|c_3|$ shows that

$$D_1^A[(\mathbb{S}_1 \otimes \mathbb{S}_2)\rho^{BD}] \geq |q_A q_B|D_1^A(\rho^{BD}), \quad (A40)$$

which implies that the FL of Eq. (7) in the main text is obeyed even if $c_3 \neq 0$. Moreover, this statement covers also the cases of the two different one-sided phase damping channels.

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