Calculus on Dual Real Numbers

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Abstract

We present the basic theory of calculus on dual real numbers, and prove the counterpart of the ordinary fundamental theorem of calculus in the context of dual real numbers.

The purpose of this paper is to study calculus on dual real numbers. Unlike the multi-variables calculus on the Cartesian product of finite many copies of the real number field and the complex analysis on the complex number field, the generalizations of the order relation on the real number field plays a central role in the theory of calculus on dual real numbers. Hence, calculus on dual real numbers seemd to be closer to the well-known single variable calculus than both multi-variables calculus and complex analysis.

The main result of this paper is to explain how to develop the basic theory of calculus on dual real numbers. In section 1, we make the dual real number algebra into a normed algebra and introduce two generalizations of the order relation on the real number field. In section 2, we define the differentiability in dual real numbers, and characterize the differentiability by using the real-valued component functions of a dual real number-valued function. In section 3, we introduce two types of integrals based on the two generalized order relations and prove the counterpart of the ordinary fundamental theorem of calculus in the context of dual real numbers.

1 Two generalized order relations

We begin this section by recalling some facts about dual real numbers. For any two elements \((x_1, x_2)\) and \((y_1, y_2)\) from the 2-dimensional real vector space \(\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}\), we define their product according to the following rule called dual number multiplication:

\[(x_1, x_2) \cdot (y_1, y_2) := (x_1y_1, x_1y_2 + x_2y_1).\]
The vector space \( \mathcal{R}^2 \) with respect to the dual number multiplication is a real associative algebra, which is called the **dual real number algebra** and denoted by \( \mathcal{R}^{(2)} \). An element of \( \mathcal{R}^2 \) is called a dual real number. Clearly, the dual real number algebra \( \mathcal{R}^{(2)} \) is both unital and commutative. We denote the multiplication identity \((1, 0)\) by 1, and the element \((0, 1)\) by \(1^\#\). Then every dual real number \((x_1, x_2)\) of \( \mathcal{R}^{(2)} \) can be expressed in a unique way as a linear combination of 1 and \(1^\#\):

\[
x = (x_1, x_2) = x_1 + x_2 \cdot 1^\# \quad \text{for } x_1, x_2 \in \mathcal{R}^{(2)},
\]

where \( \text{Re } x := x_1 \) and \( \text{Ze } x := x_2 \) are called the **real part** and the **zero-divisor part** of \( x \), respectively.

The dual real number algebra \( \mathcal{R}^{(2)} \) is not a field and has many zero-divisors. In fact, if \( 0 \neq x \in \mathcal{R}^2 \), then \( x \) is a zero-divisor if and only if \( \text{Re } x = 0 \) and \( x \) is invertible if and only if \( \text{Re } x \neq 0 \). Moreover if \( x \) is invertible, then the inverse \( x^{-1} \) of \( x \) is given by \( x^{-1} = \frac{1}{\text{Re } x} - \frac{\text{Ze } x}{(\text{Re } x)^2} \cdot 1^\# \).

**Definition 1.1** The real-valued function \( || \cdot || : \mathcal{R}^{(2)} \rightarrow \mathcal{R} \) defined by

\[
||x|| := \sqrt{2(\text{Re } x)^2 + (\text{Ze } x)^2} \quad \text{for } x \in \mathcal{R}^{(2)}
\]

is called the **norm** in \( \mathcal{R}^{(2)} \).

**Proposition 1.1** Let \( x, y \in \mathcal{R}^{(2)} \) and \( a \in \mathcal{R} \).

(i) \( ||x|| \geq 0 \), with equality only when \( x = 0 \).

(ii) \( ||ax|| = |a||x|| \), where \( |a| \) denotes the absolute value of the real number \( a \).

(iii) \( ||x + y|| \leq ||x|| + ||y|| \).

(iv) \( ||xy|| \leq ||x|| \cdot ||y|| \).

**Proof** a direct computation. \( \square \)

Unlike the Cartesian product of finite many copies of the real number field and the complex field, there are two generalized order relations on \( \mathcal{R}^{(2)} \) which are compatible with the multiplication in \( \mathcal{R}^{(2)} \).

**Definition 1.2** Let \( x \) and \( y \) be two elements of \( \mathcal{R}^{(2)} \).
We say that $x$ is type 1 greater than $y$ (or $y$ is type 1 less than $x$) and we write $x \overset{1}{>} y$ (or $y \overset{1}{<} x$) if

$$\text{either } \begin{cases} \text{Re } x \overset{1}{>} \text{Re } y \\ \text{Ze } x \overset{1}{\geq} \text{Ze } y \end{cases} \text{ or } \begin{cases} \text{Re } x = \text{Re } y \\ \text{Ze } x \overset{1}{>} \text{Ze } y \end{cases}$$

(ii) We say that $x$ is type 2 greater than $y$ (or $y$ is type 2 less than $x$) and we write $x \overset{2}{>} y$ (or $y \overset{2}{<} x$) if

$$\text{either } \begin{cases} \text{Re } x \overset{2}{>} \text{Re } y \\ \text{Ze } y \overset{2}{\geq} \text{Ze } x \end{cases} \text{ or } \begin{cases} \text{Re } x = \text{Re } y \\ \text{Ze } y \overset{2}{>} \text{Ze } x \end{cases}$$

We use $x \overset{\theta}{\geq} y$ when $x \overset{\theta}{>} y$ or $x = y$ for $\theta = 1, 2$. By Definition 1.1, if $\text{Re } x = \text{Re } y$, then $x \overset{\theta}{>} y \iff y \overset{\theta}{<} x$; if $\text{Ze } x = \text{Ze } y$, then $x \overset{\theta}{>} y \iff x \overset{\theta}{<} y$.

The following proposition gives the basic properties of the two generalized order relations.

**Proposition 1.2** Let $x, y$ and $z$ be elements of $\mathbb{R}^{(2)}$ and $\theta = 1, 2$.

(i) One of the following holds:

$$x \overset{1}{>} y, \quad y \overset{1}{>} x, \quad x \overset{2}{=} y, \quad x \overset{2}{>} y, \quad y \overset{2}{>} x.$$

(ii) If $x \overset{\theta}{>} y$ and $y \overset{\theta}{>} z$, then $x \overset{\theta}{>} z$.

(iii) If $x \overset{\theta}{>} y$, then $x + z \overset{\theta}{>} y + z$.

(iv) If $x \overset{\theta}{>} 0$ and $y \overset{\theta}{>} 0$, then $xy \overset{\theta}{\geq} 0$.

(v) If $x \overset{\theta}{>} y$, then $-x \overset{\theta}{<} -y$.

**Proof** Clear. □

### 2 Differentiation

By Proposition [11], the dual real number algebra $\mathbb{R}^{(2)}$ is a metric space with the distance function $|| \cdot ||$. If $c \in \mathbb{R}^{(2)}$ and $\epsilon \in \mathbb{R}$, we use $N(c; \epsilon)$ and $N^*(c; \epsilon)$ to denote the ordinary $\epsilon$-neighborhood and deleted $\epsilon$-neighborhood of $c$, respectively, i.e.,

$$N(c; \epsilon) = \{ x \in \mathbb{R}^{(2)} \mid ||x - c|| < \epsilon \} \quad \text{and} \quad N^*(c; \epsilon) := N(c; \epsilon) \setminus \{ c \}.$$
For $\theta \in \{1, 2\}$, the set $N_\theta(c; \epsilon) := \{x \in N(c; \epsilon) \mid x^\theta \geq c \text{ or } x^\theta \leq c\}$ is called the type $\theta$ $\epsilon$- neighborhood of $c$ and the set $N^*_\theta(c; \epsilon) := N_\theta(c; \epsilon) \setminus \{c\}$ is called the deleted type $\theta$ $\epsilon$- neighborhood of $c$. By Proposition 1.2 (i), we have

$$N(c; \epsilon) = N^*_1(c; \epsilon) \cup N^*_2(c; \epsilon).$$

We now introduce the differentiability in the following

**Definition 2.1** Let $D$ be an open subset of $\mathbb{R}^{(2)}$ and let $c \in D$.

(i) We say that $f : D \to \mathbb{R}^{(2)}$ is type $\theta$ differentiable at $c$ with $\theta \in \{1, 2\}$ if for each positive real number $\epsilon > 0$ there exist a positive real number $\delta > 0$ and a dual real number $f'_\theta(c) \in \mathbb{R}^{(2)}$ such that

$$x \in N^*_\theta(c; \delta) \subseteq D \Rightarrow \frac{f(x) - f(c) - f'_\theta(c)(x - c)}{\|x - c\|} \in N(0; \epsilon).$$

The dual real number $f'_\theta(c)$ is called the type $\theta$ derivative of $f$ at $c$, which is also denoted by $\frac{df}{d_{\theta}x}(c)$.

(ii) We say that $f : D \to \mathbb{R}^{(2)}$ is differentiable at $c$ if for each positive real number $\epsilon > 0$ there exist a positive real number $\delta > 0$ and a dual real number $f'(c) \in \mathbb{R}^{(2)}$ such that

$$x \in N^*(c; \delta) \subseteq D \Rightarrow \frac{f(x) - f(c) - f'(c)(x - c)}{\|x - c\|} \in N(0; \epsilon).$$

The dual real number $f'(c)$ is called the derivative of $f$ at $c$, which is also denoted by $\frac{df}{dx}(c)$. If $f$ is differentiable at each point of the open subset $D$, then $f$ is said to be differentiable on $D$.

It is easy to check that if a dual real number-valued $f$ is type $\theta$ differentiable at $c \in \mathbb{R}^{(2)}$, then the type $\theta$ derivative of $f$ at $c$ is unique for $\theta \in \{1, 2\}$.

Let $S$ be a subset of $\mathbb{R}^{(2)}$. A function $f : S \to \mathbb{R}^{(2)}$ can be expressed as

$$f(x) = u(x) + v(x) 1^\# \quad \text{for } x = x_1 + x_2 1^\# \in \mathbb{R}^{(2)},$$

where $u(x) := u(x_1, x_2)$ and $v(x) := v(x_1, x_2)$ are two real-valued functions of two real variables $x_1$ and $x_2$, which are called the real component and the zero-divisor component of $f$, respectively. The following proposition provides an useful characterization of differentiability for dual real-valued functions in terms of their real and zero-divisor components.
Proposition 2.1 Let \( f : D \to \mathbb{R}^{(2)} \) be a dual real-valued function given by
\[
f(x) = u(x_1, x_2) + v(x_1, x_2) 1^\# \quad \text{for} \ x = x_1 + x_2 1^\# \in \mathbb{R}^{(2)},
\]
where \( D \) is an open subset of \( \mathbb{R}^{(2)} \), \( u(x_1, x_2) \) and \( v(x_1, x_2) \) are the real component and the zero-divisor component of \( f \), respectively. Let \( c = c_1 + c_2 1^\# \in D \) with \( c_1, c_2 \in \mathbb{R} \).

(i) If the first-order partial derivatives \( u_{x_1}, u_{x_2}, v_{x_1} \) and \( v_{x_2} \) exist at \( (c_1, c_2) \) and are continuous at \( (c_1, c_2) \), and the following equations
\[
\begin{align*}
  u_{x_1} = v_{x_2}, & \quad u_{x_2} = 0 \\
\end{align*}
\]
hold at \( (c_1, c_2) \), then \( f \) is differentiable at \( c \) and the derivative \( f'(c) \) of \( f \) at \( c \) is given by
\[
f'(c) = u_{x_1}(c) + v_{x_1}(c) 1^\#.
\]

(ii) If \( f \) is differentiable at \( c \), then the equations in (2) hold at \( c = c_1 + c_2 1^\# \).
In this case, the derivative \( f'(c) \) of \( f \) at \( c \) is given by (3).

**Proof** The proof of Proposition 2.1 is similar to the proof of the famous fact which characterizes the complex differentiability by using Cauchy-Riemann equations.

For example, let us prove (ii). If \( f \) is differentiable at \( c \), then exists \( L = L_1 + L_2 1^\# \) with \( L_1, L_2 \in \mathbb{R} \) such that for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( N(c; \delta) \subseteq D \) and
\[
x \in N^*(c; \delta) \Rightarrow \frac{\|f(x) - f(c) - L(x - c)\|}{\|x - c\|} < \varepsilon. \tag{4}
\]

By dual number multiplication, we have
\[
\begin{align*}
f(x) - f(c) - L(x - c) &= u(x_1, x_2) - u(c_1, c_2) - L_1(x_1 - c_1) + \\
&\quad + [v(x_1, x_2) - v(c_1, c_2) - L_1(x_2 - c_2) - L_2(x_1 - c_1)] 1^\#.
\end{align*} \tag{5}
\]

Let \( x_2 = c_2 \) and choose \( x_1 \) such that \( 0 < |x_1 - c_1| < \frac{\delta}{\sqrt{2}} \). Then
\[
0 < \|x - c\| = \|x_1 - c_1\| = \sqrt{2}|x_1 - c_1| < \delta,
\]
which implies that \( x = x_1 + c_2 1^\# \in N^*(c; \delta) \). By (4) and (5), we get
\[
\frac{\|f(c_1 + x_2 1^#) - f(c) - L(c_1 + x_2 1^# - c)\|}{\|c_1 + x_2 1^# - c\|} > \varepsilon
\]
\[
= \frac{1}{\sqrt{2}} \left\{ \frac{\|u(x_1, c_2) - u(c_1, c_2)\|}{x_1 - c_1} - L_1 \right\}^2 + \\
+ \left\{ \frac{\|v(x_1, c_2) - v(c_1, c_2)\|}{x_1 - c_1} - L_2 \right\}^2 \right)^{\frac{1}{2}}. \tag{6}
\]
It follows from (6) that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
\left| \frac{u(x_1, c_2) - u(c_1, c_2)}{x_1 - c_1} - L_1 \right| < \epsilon \quad \text{and} \quad \left| \frac{v(x_1, c_2) - v(c_1, c_2)}{x_1 - c_1} - L_2 \right| < \epsilon \sqrt{2}
\]
whenever \( 0 < |x_1 - c_1| < \frac{\delta}{\sqrt{2}} \). This proves that
\[
u_{x_1}(c_1, c_2) = L_1 \quad \text{and} \quad v_{x_1}(c_1, c_2) = L_2. \tag{7}
\]
Similarly, let \( x_1 = c_1 \) and choose \( x_2 \) such that \( 0 < |x_2 - c_2| < \delta \). Then

\[
0 < ||x - c|| = \|(x_2 - c_2)1\#\| = |x_2 - c_2| < \delta,
\]
which implies that \( x = c_1 + x_21\# \in N^*(c; \delta) \). By (4) and (5), we get
\[
\epsilon > \frac{\| f(c_1 + x_21\#) - f(c) - L(c_1 + x_21\# - c) \|}{\| c_1 + x_21\# - c \|}
= \left\{ 2 \left[ \frac{u(c_1, x_2) - u(c_1, c_2)}{x_2 - c_2} \right]^2 + \left[ \frac{v(c_1, x_2) - v(c_1, c_2)}{x_2 - c_2} - L_1 \right]^2 \right\}^{\frac{1}{2}}.
\]
which implies that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
\left| \frac{u(c_1, x_2) - u(c_1, c_2)}{x_2 - c_2} \right| < \frac{\epsilon}{\sqrt{2}} \quad \text{and} \quad \left| \frac{v(c_1, x_2) - v(c_1, c_2)}{x_2 - c_2} - L_1 \right| < \epsilon
\]
whenever \( 0 < |x_2 - c_2| < \delta \). This proves that
\[
u_{x_2}(c_1, c_2) = 0 \quad \text{and} \quad v_{x_2}(c_1, c_2) = L_1. \tag{8}
\]
By (7) and (8), (ii) holds.

\[ \square \]

### 3 Type \( \theta \) Integrals

In the remaining of this paper, \( \theta \) always denote an element in the set \( \{1, 2\} \). Let \( f : S \to R^{(2)} \) be a function on a subset \( S \) of \( R^{(2)} \). For convenience, we will use \( f_{Re} \) and \( f_{Ze} \) to denote the real component and the zero-divisor component of a function \( f : S \to R^{(2)} \), respectively. Thus, we have

\[
f(x) = f_{Re}(x) + f_{Ze}(x)1\# \quad \text{for} \quad x = x_1 + x_21\# \in R^{(2)}.
\]
We say that the function \( f : S \to R^{(2)} \) is \textbf{bounded on} \( S \) if both \( f_{Re} \) and \( f_{Ze} \) are bounded on \( S \) \(( \subseteq R^2 = R \times R) \).

Let \( a, b \in R^{(2)} \) and \( a \leq b \). The \textbf{type \( \theta \) closed interval} \([a, b]_\theta\) is defined by

\[
[a, b]_\theta := \{ x \in R^{(2)} | a \leq x \leq b \}.
\]
Definition 3.1 Let \([a, b]_\theta\) be a type \(\theta\) closed integral. A partition \(P\) of \([a, b]_\theta\) is a finite set of points \(\{x^{(0)}, x^{(1)}, \ldots, x^{(n)}\}\) in \([a, b]_\theta\) such that
\[
a = x^{(0)} < x^{(1)} < \ldots < x^{(n)} = b. \tag{9}\]
If \(P\) and \(Q\) are two partitions of \([a, b]_\theta\) with \(P \subseteq Q\), then \(Q\) is called a refinement of \(P\).

Let \([a, b]_\theta\) be a type \(\theta\) closed interval. Suppose that \(f : [a, b]_\theta \to \mathcal{R}^{(2)}\) is bounded and \(P = \{x^{(0)}, x^{(1)}, \ldots, x^{(n)}\}\) is a partition of \([a, b]_\theta\). For \(1 \leq i \leq n\), the length \(\Delta x^{(i)}\) of the \(i\)-th type \(\theta\) subinterval \([x^{(i-1)}, x^{(i)}]_\theta\) is defined by \(\Delta x^{(i)} := x^{(i)} - x^{(i-1)}\). Clearly, \(\Delta x^{(i)} > 0\), and \(\Delta x^{(i)}\) is a real number if and only if \(\mathcal{Z}_\theta(x^{(i)}) = \mathcal{Z}_\theta(x^{(i-1)})\) for \(1 \leq i \leq n\). Since both \(f_{Re}\) and \(f_{Ze}\) are bounded on \([a, b]_\theta\), both \(f_{Re}\) and \(f_{Ze}\) are bounded on \([x^{(i-1)}, x^{(i)}]_\theta\) for \(1 \leq i \leq n\). Hence, both \(\sup f_\bullet := \sup \{f_\bullet(x) \mid x \in [x^{(i-1)}, x^{(i)}]\}\) and \(\inf f_\bullet := \inf \{f_\bullet(x) \mid x \in [x^{(i-1)}, x^{(i)}]\}\) exist as real numbers for \(\bullet \in \{Re, Ze\}\) and \(1 \leq i \leq n\). Based on these facts, we define the type \(\theta\) upper sum \(U_\theta(P, f)\) of \(f\) with respect to the partition \(P\) to be
\[
U_\theta(P, f) = \begin{cases} 
\sum_{i=1}^{n} (\sup f_{Re} + 1^\# \sup f_{Ze}) \Delta x^{(i)} & \theta = 1; \\
\sum_{i=1}^{n} (\sup f_{Re} + 1^\# \inf f_{Ze}) \Delta x^{(i)} & \theta = 2
\end{cases}
\]
and the type \(\theta\) lower sum \(L_\theta(P, f)\) of \(f\) with respect to the partition \(P\) to be
\[
L_\theta(P, f) = \begin{cases} 
\sum_{i=1}^{n} (\inf f_{Re} + 1^\# \inf f_{Ze}) \Delta x^{(i)} & \theta = 1; \\
\sum_{i=1}^{n} (\inf f_{Re} + 1^\# \sup f_{Ze}) \Delta x^{(i)} & \theta = 2
\end{cases}
\]
Then the following four sets
\[
\{\bullet U_\theta(P, f) \mid P \in \mathcal{P}_\theta\}, \quad \{\bullet L_\theta(P, f) \mid P \in \mathcal{P}_\theta\} \quad \text{with} \quad \bullet \in \{Re, Ze\} \quad \text{(10)}
\]
are bounded subsets of the real number field \(\mathcal{R}\), where \(\mathcal{P}_\theta\) is the set of all partitions of \([a, b]_\theta\), i.e. \(\mathcal{P}_\theta := \{P \mid P\ \text{is a partition of } [a, b]_\theta\}\). Hence, the supremums and infimums of the four sets in (10) exist. Using these facts, we introduce the type \(\theta\) lower integral \(\int_a^b f(x)dx_\theta\) and the type \(\theta\) upper integral \(\int_a^b f(x)dx_\theta\) of \(f(x)\) on \([a, b]_\theta\) in the following way:
\[
\int_a^b f(x)dx_\theta = \sup \{Re L_1(P, f) \mid P \in \mathcal{P}_1\} + 1^\# \sup \{Ze L_1(P, f) \mid P \in \mathcal{P}_1\}.
\]
\[
\int_a^b f(x) \, dx = \text{sup} \{ \text{Re} L_2(P, f) \mid P \in \mathcal{P}_2 \} + 1 \# \text{inf} \{ \text{Ze} L_2(P, f) \mid P \in \mathcal{P}_2 \},
\]

\[
\int_a^b f(x) \, dx = \text{inf} \{ \text{Re} U_1(P, f) \mid P \in \mathcal{P}_1 \} + 1 \# \text{inf} \{ \text{Ze} U_1(P, f) \mid P \in \mathcal{P}_1 \}
\]

and
\[
\int_a^b f(x) \, dx = \text{inf} \{ \text{Re} U_2(P, f) \mid P \in \mathcal{P}_2 \} + 1 \# \text{sup} \{ \text{Ze} U_2(P, f) \mid P \in \mathcal{P}_2 \}.
\]

If the type \(\theta\) lower integral and the type \(\theta\) upper integral of \(f(x)\) on \([a, b]_\theta\) are equal, i.e., if
\[
\int_a^b f(x) \, dx = \text{inf} \{ \text{Re} U_2(P, f) \mid P \in \mathcal{P}_2 \} + 1 \# \text{sup} \{ \text{Ze} U_2(P, f) \mid P \in \mathcal{P}_2 \},
\]

then we say that \(f\) is \textbf{type \(\theta\) integrable} on \([a, b]_\theta\), we denote their common value by \(\int_a^b f(x) \, dx\) which is called the \textbf{type \(\theta\) integral} of \(f\) on \([a, b]_\theta\).

**Proposition 3.1** Let \(f : [a, b]_\theta \to \mathbb{R}^{(2)}\) is a bounded function, where \(\theta \in \{1, 2\}\).

(i) If \(P\) and \(P^*\) are partitions of \([a, b]_\theta\) and \(P^*\) is a refinement of \(P\), then
\[
\text{L}_\theta(P, f)_{\theta} \leq \text{L}_\theta(P^*, f)_{\theta} \leq \text{U}_\theta(P^*, f)_{\theta} \leq \text{U}_\theta(P, f)_{\theta}.
\]

(ii) \(\int_a^b f(x) \, dx_{\theta} \leq \int_a^b f(x) \, dx_{\theta}.
\]

(iii) \(f\) is type \(\theta\) integrable if for each \(\varepsilon \in \mathbb{R}^{(2)}\) with \(\varepsilon > 0\) and \((\text{Re} \varepsilon)(\text{Ze} \varepsilon) \neq 0\) there exists a partition \(P\) of \([a, b]_\theta\) such that
\[
\text{U}_\theta(P, f)_{\theta} - \text{L}_\theta(P, f)_{\theta} < \varepsilon.
\]

**Proof** The proof of Proposition 3.1 follows from the definitions above and the properties of the supremums and infimums. For example, let us prove (i) for \(\theta = 1\), i.e.,
\[
\text{L}_1(P, f)_{\frac{1}{1}} \leq \text{L}_1(P^*, f)_{\frac{1}{1}} \leq \text{U}_1(P^*, f)_{\frac{1}{1}} \leq \text{U}_1(P, f)_{\frac{1}{1}}.
\]

The middle inequality in (11) follows directly from the definitions of type 1 upper and lower sums. Suppose that \(P = \{x^{(0)}, x^{(1)}, \ldots, x^{(n)}\}\) and consider the partition \(P^*\) formed by joining just one point \(x^*\) to \(P\), where \(x^{(k-1)} \leq x^* \leq x^{(k)}\) for some \(k\) with \(1 \leq k \leq n\). Let
\[
\alpha_{1}(f^{*}) := \text{inf} \{ f^{*}(x) \mid x \in [x^{(k-1)}, x^*]_{1} \},
\]
\[
\alpha_2(f_{\clubsuit}) := \inf \{ f_{\clubsuit}(x) \mid x \in [x^*, x^{(k)}]_1 \},
\]
where \(\clubsuit \in \{Re, Ze\}\). The terms in \(L_1(P^*, f)\) and \(L_1(P, f)\) are all the same except those over the subinterval \([x^{(k-1)}, x^{(k)}]_1\). Thus we have

\[
L_1(P^*, f) - L_1(P, f) = \left[ (\alpha_1(f_{Re}) - \inf_k f_{Re}) + 1^#(\alpha_1(f_{Ze}) - \inf_k f_{Ze}) \right] (x^* - x^{(k-1)}) + \\
+ \left[ (\alpha_2(f_{Re}) - \inf_k f_{Re}) + 1^#(\alpha_2(f_{Ze}) - \inf_k f_{Ze}) \right] (x^{(k)} - x^*).
\] (12)

Since \(\alpha_j(f_{\clubsuit}) \geq \inf_k f_{\clubsuit}\) for \(j \in \{1, 2\}\) and \(\clubsuit \in \{Re, Ze\}\), we get

\[
(\alpha_j(f_{Re}) - \inf_k f_{Re}) + 1^#(\alpha_j(f_{Ze}) - \inf_k f_{Ze}) \geq 0 \quad \text{for } j \in \{1, 2\}.
\] (13)

Using (13) and the facts: \(x^* - x^{(k-1)} > 0\) and \(x^{(k)} - x^* > 0\), we get from (12) that \(L_1(P^*, f) - L_1(P, f) \geq 0\) or \(L_1(P^*, f) \geq L_1(P, f)\).

Similarly, let

\[
\beta_1(f_{\clubsuit}) := \sup \{ f_{\clubsuit}(x) \mid x \in [x^{(k-1)}, x^*]_1 \},
\]
\[
\beta_2(f_{\clubsuit}) := \sup \{ f_{\clubsuit}(x) \mid x \in [x^*, x^{(k)}]_1 \},
\]
where \(\clubsuit \in \{Re, Ze\}\). Since the terms in \(U_1(P^*, f)\) and \(U_1(P, f)\) are all the same except those over the subinterval \([x^{(k-1)}, x^{(k)}]_1\), we have

\[
U_1(P^*, f) - U_1(P, f) = \left[ (\sup_k f_{Re} - \beta_2(f_{Re})) + 1^#(\sup_k f_{Ze} - \beta_2(f_{Ze})) \right] (x^{(k)} - x^*) + \\
+ \left[ (\sup_k f_{Re} - \beta_1(f_{Re})) + 1^#(\sup_k f_{Ze} - \beta_1(f_{Ze})) \right] (x^* - x^{(k-1)}).
\] (14)

Since \(\sup_k f_{\clubsuit} \geq \beta_j(f_{\clubsuit})\) for \(j \in \{1, 2\}\) and \(\clubsuit \in \{Re, Ze\}\), we have

\[
(\sup_k f_{Re} - \beta_j(f_{Re})) + 1^#(\sup_k f_{Ze} - \beta_j(f_{Ze})) \geq 0 \quad \text{for } j \in \{1, 2\}.
\] (15)

It follows from (14) and (15) that \(U_1(P, f) - U_1(P^*, f) \geq 0\) or \(U_1(P, f) \geq U_1(P^*, f)\).

This proves that (11) holds.
As a corollary of Proposition 3.1 (iii), we have that if \( f = f_{\Re} + 1^\# f_{\Ze} \) is a function on a type \( \theta \) closed interval \([a, b]_\theta\) such that real-valued functions \( f_{\Re} \) and \( f_{\Ze} \) are continuous on the rectangle \([\Re a, \Re b] \times [\Ze a, \Ze b] \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}\), then \( f \) is type \( \theta \) integrable on \([a, b]_\theta\), where \( \theta = 1 \) or \( 2 \).

The algebraic properties of the ordinary integral are still true for the type \( \theta \) integrals.

**Proposition 3.2**  Let \( \theta = 1 \) or \( 2 \) and let \( k \in \mathbb{R}^{(2)} \) be a dual real number.

(i) If \( f \) and \( g \) are type \( \theta \) integrable on \([a, b]_\theta\), then \( f + g \) and \( kf \) are type \( \theta \) integrable on \([a, b]_\theta\) and

\[
\int_a^b (f + g) d\theta x = \int_a^b f d\theta x + \int_a^b g d\theta x, \quad \int_a^b kf d\theta x = k \int_a^b f d\theta x.
\]

(ii) If \( f \) is type \( \theta \) integrable on both \([a, c]_\theta\) and \([c, b]_\theta\), where \( a \lt c \lt b \), then \( f \) is type \( \theta \) integrable on both \([a, b]_\theta\) and

\[
\int_a^b f d\theta x = \int_a^c f d\theta x + \int_c^b f d\theta x.
\]

(iii) If \( f, g : [a, b]_\theta \to \mathbb{R}^{(2)} \) are type \( \theta \) integrable and \( f(x) \theta \geq g(x) \) for all \( x \in [a, b]_\theta\), then

\[
\int_a^b f(x) d\theta x \theta \geq \int_a^b g(x) d\theta x.
\]

**Proof** Both (i) and (ii) are proved by using Proposition 3.1 and (iii) is proved by using the definitions of type \( \theta \) integrals and the properties of the two generalized order relations on the dual real number algebra.

We finish this paper with the following counterpart of the ordinary fundamental theorem of calculus in the context of dual real numbers.

**Proposition 3.3**  Let \( a, b \in \mathbb{R}^{(2)} \) and \( a \lt b \), where \( \theta = 1 \) or \( 2 \).

(i) If \( f : [a, b]_\theta \to \mathbb{R}^{(2)} \) is a function such that the real-valued functions \( f_{\Re} \) and \( f_{\Ze} \) are continuous on the rectangle \([\Re a, \Re b] \times [\Ze a, \Ze b] \subseteq \mathbb{R}^2\), then the function \( F(x) : [a, b]_\theta \to \mathbb{R}^{(2)} \) defined by

\[
F(x) := \int_a^x f(t) d\theta t \quad \text{for} \quad x \in [a, b]_\theta
\]

is type \( \theta \) differential at each \( c \in [a, b]_\theta \) and \( F'_\theta(c) = f(c) \).
(ii) If \( f(x) : [a, b]_\theta \to \mathcal{R}^{(2)} \) is differential on \([a, b]_\theta\) and the derivative \( f'(x) \) of \( f(x) \) is integrable on \([a, b]_\theta\), then

\[
\int_a^b f'(x) \, d_\theta x = f(b) - f(a).
\]

**Proof** The way of proving Proposition 3.3 comes from the application of the algebraic properties of type \( \theta \) integrals in Proposition 3.2. Let us prove (i) to explain the way of doing the proofs.

By the definitions of type \( \theta \) integrals, we have

\[
a \leq b \implies \int_a^b \, d_\theta x = b - a \quad \text{for} \quad \theta \in \{1, 2\}. \tag{16}
\]

Clearly, (i) holds if we can prove that for each positive real number \( \epsilon > 0 \) there exists a positive real number \( \delta > 0 \) such that

\[
x \in N_\theta^*(c; \delta) \cap [a, b]_\theta \implies \frac{\|F(x) - F(c) - f(c)(x - c)\|}{\|x - c\|} < \epsilon, \tag{17}
\]

where \( \theta \in \{1, 2\} \). The proofs of (17) for \( \theta = 1 \) and \( \theta = 2 \) are similar, so we prove (17) for \( \theta = 1 \). First, we choose two positive real numbers \( \varepsilon_{Re} \) and \( \varepsilon_{Ze} \)

\[
0 < \varepsilon_{Re} < \frac{\epsilon}{3} \quad \text{and} \quad 0 < \varepsilon_{Ze} < \frac{\epsilon - 3\varepsilon_{Re}}{\sqrt{2}}. \tag{18}
\]

Next, since both real-valued functions \( f_{Re} \) and \( f_{Ze} \) are continuous on the rectangle \([Re \ a, Re \ b] \times [Ze \ a, Ze \ b] \subseteq \mathcal{R}^2\), both \( f_{Re} \) and \( f_{Ze} \) are uniformly continuous on the rectangle \([Re \ a, Re \ b] \times [Ze \ a, Ze \ b] \). Hence, there exist a positive real number \( \delta > 0 \) such that

\[
\sqrt{(t_1 - s_1)^2 + (t_2 - s_2)^2} < \delta \implies |f_{\bullet}(t_1, t_2) - f_{\bullet}(s_1, s_2)| < \varepsilon_{\bullet} \tag{19}
\]

for all \((t_1, t_2), (s_1, s_2) \in [a_1, b_1] \times [a_2, b_2]\) and \( \bullet \in \{Re, Ze\} \).

Let \( x \in N_1^*(c; \delta) \cap [a, b]_\theta \). Then \( x \geq c \) and \( x \leq c \).

**Case 1:** \( x \geq c \), in which case, by Proposition 3.2 and (16), we have

\[
F(x) - F(c) - f(c)(x - c) = \int_c^x f(t) \, dt - \int_c^c f(t) \, dt - f(c)(x - c) = \int_c^x f(t) \, dt - f(c) \int_c^x \, dt = \int_c^x f(t) \, dt + \int_c^x (-f(c)) \, dt = \int_c^x [f(t) - f(c)] \, dt. \tag{20}
\]
For $c = c_1 + c_2 1^#$ and $x = x_1 + x_2 1^#$, where $c_1, c_2, t_1, t_2, x_1$ and $x_2 \in \mathcal{R}$, we have
\[
\sqrt{(t_1 - c_1)^2 + (t_2 - c_2)^2} \leq \sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2}
\]
\[
\leq \sqrt{2(x_1 - c_1)^2 + (x_2 - c_2)^2} = ||x - c|| < \delta
\]
\[
\Rightarrow |f_\bullet(t_1, t_2) - f_\bullet(c_1, c_2)| < \varepsilon_\bullet \quad \text{for} \quad \bullet \in \{Re, Ze\},
\]
which implies
\[
-\varepsilon = -\varepsilon_{Re} - \varepsilon_{Ze} 1^# \leq f(t) - f(c) \leq \varepsilon_{Re} + \varepsilon_{Ze} 1^# = \varepsilon. \tag{21}
\]
It follows from (10), (21) and Proposition 3.2 (iii) that
\[
-\varepsilon(x - c) = \int_c^x (-\varepsilon) \, dt \leq \int_c^x [f(t) - f(c)] \, dt \leq \int_c^x \varepsilon \, dt = \varepsilon(x - c),
\]
which gives
\[
\left| f_\bullet \left( \int_c^x [f(t) - f(c)] \, dt \right) \right| \leq |f_\bullet(\varepsilon(x - c))| \quad \text{for} \quad \bullet \in \{Re, Ze\}. \tag{22}
\]
Since
\[
\varepsilon(x - c) = (\varepsilon_{Re} + \varepsilon_{Ze} 1^#)((x_1 - c_1) + (x_2 - c_2) 1^#)
\]
\[
= \varepsilon_{Re}(x_1 - c_1) + [\varepsilon_{Re}(x_2 - c_2) + \varepsilon_{Ze}(x_1 - c_1)]1^#,
\]
we have
\[
Re(\varepsilon(x - c)) = \varepsilon_{Re}(x_1 - c_1) \tag{23}
\]
and
\[
Ze(\varepsilon(x - c)) = \varepsilon_{Re}(x_2 - c_2) + \varepsilon_{Ze}(x_1 - c_1). \tag{24}
\]
By (22), (23) and (24), we have
\[
\left\| \left( \int_c^x [f(t) - f(c)] \, dt \right) \right\|
\]
\[
= \sqrt{2 \left[ Re \left( \int_c^x [f(t) - f(c)] \, dt \right) \right]^2 + Ze \left( \int_c^x [f(t) - f(c)] \, dt \right)^2}
\]
\[
\leq \sqrt{2} \left| Re \left( \int_c^x [f(t) - f(c)] \, dt \right) \right| + \left| Ze \left( \int_c^x [f(t) - f(c)] \, dt \right) \right|
\]
\[
\leq \sqrt{2} \left| \varepsilon_{Re}(x_1 - c_1) + \varepsilon_{Re}(x_2 - c_2) + \varepsilon_{Ze}(x_1 - c_1) \right|
\]
\[
\leq \sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} |x_2 - c_2| + \varepsilon_{Ze} |x_1 - c_1| \tag{25}
\]
It follows from (20) and (25) that

\[
\frac{\|F(x) - F(c) - f(c)(x-c)\|}{\|x-c\|} = \frac{\left\| \int_c^x [f(t) - f(c)] \, dt \right\|}{\|x-c\|} \leq \frac{\sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} |x_2 - c_2| + \varepsilon_{Zc} |x_1 - c_1|}{\sqrt{2(x_1 - c_1)^2 + (x_2 - c_2)^2}}
\]

\[= \sqrt{2} \varepsilon_{Re} + \varepsilon_{Re} \frac{|x_2 - c_2|}{|x_1 - c_1|} + \varepsilon_{Zc} \frac{x_1 - c_1}{x_1 - c_1} < \sqrt{2} \varepsilon_{Re} + \varepsilon_{Re} \frac{x_2 - c_2}{x_1 - c_1} + \varepsilon_{Zc} = \frac{3}{\sqrt{2}} \varepsilon_{Re} + \varepsilon_{Zc} < \frac{\varepsilon}{\sqrt{2}} < \varepsilon.
\]

Case 11: \(x_1 = c_1\), in which case, by (18) and (26), we get

\[
\frac{\|F(x) - F(c) - f(c)(x-c)\|}{\|x-c\|} \leq \frac{\varepsilon_{Re} |x_2 - c_2|}{\sqrt{(x_2 - c_2)^2}} = \varepsilon_{Re} < \frac{\varepsilon}{3} < \varepsilon.
\]

Case 12: \(x_1 \neq c_1\) and \(\frac{|x_2 - c_2|}{|x_1 - c_1|} \leq \frac{1}{\sqrt{2}}\), in which case, using (18) and (20) again, we get

\[
\frac{\|F(x) - F(c) - f(c)(x-c)\|}{\|x-c\|} \leq \sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} |x_2 - c_2| + \varepsilon_{Zc} |x_1 - c_1| \leq \sqrt{2} \varepsilon_{Re} + \varepsilon_{Re} \frac{|x_2 - c_2|}{|x_1 - c_1|} + \varepsilon_{Zc} \leq \frac{3}{\sqrt{2}} \varepsilon_{Re} + \varepsilon_{Zc} < \frac{\varepsilon}{\sqrt{2}} < \varepsilon.
\]

Case 13: \(x_1 \neq c_1\) and \(\frac{|x_2 - c_2|}{|x_1 - c_1|} > \frac{1}{\sqrt{2}}\). In this case, we have \(\frac{|x_1 - c_1|}{|x_2 - c_2|} \leq \sqrt{2}\) and

\[
\frac{\|F(x) - F(c) - f(c)(x-c)\|}{\|x-c\|} \leq \sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} |x_2 - c_2| + \varepsilon_{Zc} |x_1 - c_1| \leq \sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} + \varepsilon_{Zc} \frac{x_1 - c_1}{x_2 - c_2} < \frac{3}{\sqrt{2}} \varepsilon_{Re} + \varepsilon_{Zc} < \varepsilon.
\]

\[\leq \sqrt{2} \varepsilon_{Re} |x_1 - c_1| + \varepsilon_{Re} + \varepsilon_{Zc} |x_1 - c_1| \leq \sqrt{2} \varepsilon_{Re} \sqrt{2} + \varepsilon_{Re} + \varepsilon_{Zc} \sqrt{2} = 3\varepsilon_{Re} + \varepsilon_{Zc} \sqrt{2} < \varepsilon.
\]

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It follows from (27), (28) and (29) that (17) holds in Case 1.

**Case 2:** $x \leq c$, in which case, a similar computation shows that (17) holds in Case 2.

This proves that (17) holds for each $x \in N^*_1(c; \delta) \cap [a, b]_1$. \qed

**References**

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[2] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.