On Distributionally Robust Multistage Convex Optimization:
New Algorithms and Complexity Analysis

Shixuan Zhang$^1$ and Xu Andy Sun$^2$

$^1$Wm Michael Barnes ’64 Department of Industrial & Systems Engineering, Texas A&M University, College Station, TX 77843, USA  shixuan.zhang@tamu.edu

$^2$Operations Research and Statistics, Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02139,  sunx@mit.edu

Abstract

This paper presents an algorithmic study and complexity analysis for solving distributionally robust multistage convex optimization (DR-MCO). We generalize the usual consecutive dual dynamic programming (DDP) algorithm to DR-MCO and propose a new nonconsecutive DDP algorithm that explores the stages in an adaptive fashion. We introduce dual bounds in the DDP recursions to prevent the growth of Lipschitz constants of the dual approximations caused by recursive cutting plane methods. We then provide a thorough subproblem-oracle-based complexity analysis of the proposed algorithms, proving both upper complexity bounds and a matching lower bound.

To the best of our knowledge, this is the first nonasymptotic complexity result for DDP-type algorithms on DR-MCO, which reveals that in some practical settings the new nonconsecutive DDP algorithm scales linearly with respect to the number of stages. Numerical examples are given to show the effectiveness of the proposed nonconsecutive DDP algorithm and the dual-bounding technique, including the reduction of the computation time or the number of subproblem oracle evaluations, and the capability to solve problems without relatively complete recourse.

Keywords: distributionally robust convex optimization, multistage optimization, dual dynamic programming algorithm, complexity analysis, cutting plane method

1 Introduction

Distributionally robust multistage convex optimization (DR-MCO) is a sequential decision making problem with convex objective functions and constraints, where the exact probability distribution of the uncertain parameters is unknown and decisions need to be made considering a family of distributions. DR-MCO provides a unified framework for studying decision-making under uncertainty. It includes as special cases both multistage stochastic convex optimization (MSCO), where some distribution of the uncertain parameters is known and the expected cost is to be minimized, and multistage robust convex optimization (MRCO), where the uncertainty is described by a set and the worst-case cost is to be minimized. DR-MCO as a general decision framework finds ubiquitous applications in energy system planning, supply chain and inventory control, portfolio optimization, finance, and many other areas (see e.g. [39, 6]).

DR-MCO is in general challenging to solve, due to the fast growth of the number of decisions with respect to the number of decision stages [36, 15, 17]. Meanwhile, real-world problems are often endowed with special structures in the uncertainty. In particular, the uncertainty may exhibit stagewise independence (SI), i.e., the uncertainty in different stages are independent from each other. Many uncertainty structures, such as autoregressive stochastic models, can be reformulated to satisfy SI [41]. This versatile modeling capability of SI has great implications on computation. It allows recursive
formulations of cost-to-go functions in each stage of a DR-MCO to be independent of the outcomes in its previous stages, thus making efficient approximations of the cost-to-go functions possible. Indeed, SI has been successfully exploited by various algorithms in solving MSCO and MRCO [37, 32, 17, 30, 3, 4, 1, 43, 38].

Among algorithms for DR-MCO, dual dynamic programming (DDP) is a prevalent class of recursive cutting plane algorithms that originate from nested Benders decomposition for multistage stochastic linear optimization [8, 28]. The earliest form of DDP for MSCO using stochastic sampling method was proposed in [29], known as stochastic dual dynamic programming (SDDP), where in each iteration the scenarios are sampled randomly and solved sequentially before updating the cost-to-go functions recursively. The algorithm has since been widely adopted in areas such as energy systems scheduling [16, 11, 40]. The deterministic sampling version of DDP was later proposed, which uses both over- and under-approximations for sampling and termination [3, 4]. DDP has also been extended to multistage stochastic nonconvex problems [43, 44, 1, 42]. Robust dual dynamic programming (RDDP) is proposed for multistage robust linear optimization [17]. Due to its intrinsic difficulty, the uncertainty sets are assumed to be polytopes such that the subproblem in each stage can be solved via a vertex enumeration technique over the uncertainty set. Similar to the deterministic DDP, RDDP constructs both over- and under-approximations to select the worst-case outcome. Moreover, it has the advantage of being able to terminate the algorithm with a guaranteed optimal first stage solution, in contrast to the commonly used decision rules [24, 7, 21]. Recently, DDP has been further extended to DR-MCO with promising out-of-sample performance [2, 22, 31, 14]. In particular, [31] uses an ambiguity set defined by a \( \chi^2 \)-distance neighborhood. In [14], the ambiguity sets are taken to be Wasserstein metric balls with finite support and centered at the empirical distributions, and the algorithm is shown to converge asymptotically with stochastic sampling methods. We comment that all of the above variants of DDP algorithms rely on the assumption of relatively complete recourse (RCR), while it is indeed possible to have MSCO that do not satisfy such an assumption [27].

The convergence analysis of DDP begins with multistage linear optimization [32, 37, 26, 10], where an almost sure finite convergence is established based on polyhedral structures. In [18], an asymptotic convergence is proved for MSCO problems. Due to the multistage structure, a main complexity question concerning DDP is the dependence of its iteration complexity on the number of decision stages, which is recently answered for MSCO, independently in [25, 42]. In particular, [18, 25] assume RCR and, moreover, the value functions are all Lipschitz continuous. A weaker regularity assumption concerning exact penalization is made instead in [42] that allows the DDP algorithm to work without RCR. However, it is not yet known whether the DDP algorithm, together with the complexity analysis, works for DR-MCO. The present paper introduces new algorithmic techniques to DDP and answers this question in the positive for the first time.

While the DDP algorithms have achieved great computational success for MSCO problems, one major issue of their generalization to the DR-MCO problems is the lack of proper termination criterion. Due to the distributional uncertainty in the model, the commonly used statistical upper bound for the policy evaluation in the MSCO literature (e.g., [37, 43]) is no longer valid for the DR-MCO problems. As a result, computational experiments in [31] and [14] terminate at a fixed number of iterations or cuts, without a good guarantee of the solution quality. To overcome the lack of statistical upper bound, we follow the idea of deterministic upper bounds [33, 3] and focus on dynamically constructing both under- and over-approximations to certify the optimality of the solutions, similar to the RDDP algorithm [17]. Our paper makes the following contributions.

1. We provide a unified framework for studying DR-MCO under SI assumption. We find that the traditional cutting plane methods can cause the Lipschitz constants of the under-approximations to grow with respect to the number of stages. Motivated by this phenomenon, we introduce bounds on the dual variables in the DDP recursion, which could effectively control the growth of Lipschitz constants and can dispense with the RCR assumption.

2. We extend the usual consecutive DDP (CDDP) to DR-MCO with consecutive exploration of the single stage subproblems, and also propose a novel nonconsecutive variant (NDDP), which adaptively explores the stages in a nonconsecutive fashion. We provide subproblem-oracle-based
complexity upper bounds for both algorithms. In particular, the proposed NDDP has a linear complexity growth with respect to the number of stages \( T \), assuming that the target optimality gap is allowed to scale with \( T \).

3. We construct a class of multistage robust convex problems to obtain a complexity lower bound for the new algorithms for the first time, which shows the complexity upper bounds are essentially tight. The complexity bounds can be applied to more general DR-MCO problems with continuous distributions.

4. Numerical results on a multi-commodity inventory problem and a hydro-thermal power planning problem are given to illustrate the effectiveness of our algorithms, including the capability to solve problems without RCR and the reduction in the computation time and number of subproblem oracle evaluations.

The rest of the paper is organized as follows. Section 2 contains the formulations of the problems and the discussion on the dual bounds in the DDP recursions. In Section 3, we introduce the single stage subproblem oracles and present the CDDP and NDDP algorithms with complexity bounds. In Section 4, we present two classes of numerical examples that demonstrate the effectiveness of the algorithms. Section 5 concludes the paper.

## 2 Formulations and Recursive Approximation

In this section, we introduce formulations of distributionally robust multistage convex optimization (DR-MCO). In the case where the DR-MCO has finite support, we build approximations of the value functions using recursions. We then discuss the dual-bounding technique and its exactness that is used for complexity analysis in Section 3.

### 2.1 Problem Formulations

We present a general definition of DR-MCO under the assumption of stagewise independence (SI). The SI assumption is necessary for efficient algorithmic development and we refer any interested reader to [39] for more general settings. Then we show that most MSCO and MRRO in the existing literature (such as [18] and [17]) can be encompassed within this DR-MCO framework even with only finitely many uncertainty outcomes in each stage.

Let \( \mathcal{T} := \{1, 2, \ldots, T\} \) denote the set of stage indices and \( \mathcal{T'} := \mathcal{T} \setminus \{1\} \). For each \( t \in \mathcal{T} \), the decision variable in stage \( t \) is denoted as \( x_t \) and constrained in a compact convex set \( X_t \subset \mathbb{R}^{d_t} \) with dimension \( d_t \in \mathbb{Z}_{\geq 0} \). The uncertainty in stage \( t \) is modeled as a random vector \( \xi_t \) with some possibly unknown probability distribution and its convex support set denoted as \( \Xi_t \). For simplicity, we use \( x_0 \) and \( \xi_1 \) to denote deterministic parameters, known as the initial conditions. The cost incurred by the decisions and the uncertainty outcomes in stage \( t \) is modeled by a nonnegative lower semicontinuous function \( f_t(x_{t-1}, x_t; \xi_t) \), that is convex in \( (x_{t-1}, x_t) \) for each \( \xi_t \in \Xi_t \) and allowed to take \( +\infty \) for infeasibility. We show by the following example that the usual constraints can be modeled as part of the cost functions \( f_t \).

**Example 1.** Let \( \mathcal{Y}_t \) denote a compact convex decision set of auxiliary variables \( y_t \) in stage \( t \in \mathcal{T} \). Given a nonnegative, convex, continuous real-valued cost function \( f_t^{\leq \infty}(x_{t-1}, y_t, x_t; \xi_t) \) and any convex continuous functional feasibility constraints \( g_t(x_{t-1}, y_t, x_t, \xi_t) \leq 0 \), we can define \( \mathcal{F}_t := \{(x_{t-1}, x_t, \xi_t) \in \mathcal{X}_{t-1} \times X_t \times \Xi_t : \exists y_t \in \mathcal{Y}_t \text{ s.t. } g_t(x_{t-1}, y_t, x_t, \xi_t) \leq 0 \} \) and set the extended real-valued cost function as

\[
    f_t(x_{t-1}, x_t; \xi_t) = \begin{cases} \min_{y_t \in \mathcal{Y}_t} f_t^{\leq \infty}(x_{t-1}, y_t, x_t; \xi_t), & \text{if } (x_{t-1}, x_t, \xi_t) \in \mathcal{F}_t, \\ +\infty, & \text{otherwise.} \end{cases}
\]

It can be checked that the cost function \( f_t \) in this example is indeed nonnegative, lower semicontinuous, convex in \( (x_{t-1}, x_t) \), and proper if \( \mathcal{F}_t \neq \emptyset \).

3
Let $\mathcal{P}_t$ denote a set of Borel probability measures supported on the uncertainty set $\Xi_t$, which is known as the ambiguity set, for each stage $t \in T'$. Now we can define our DR-MCO problem as follows.

\[
\min_{x_1 \in X_1} f_1(x_0, x_1; \xi_1) + \sup_{p_2 \in \mathcal{P}_2} \mathbb{E}_{\xi_2 \sim p_2} \min_{x_2 \in X_2} \left[ f_2(x_1, x_2; \xi_2) + \sup_{p_3 \in \mathcal{P}_3} \mathbb{E}_{\xi_3 \sim p_3} \min_{x_3 \in X_3} \left[ f_3(x_2, x_3; \xi_3) + \cdots + \sup_{p_T \in \mathcal{P}_T} \mathbb{E}_{\xi_T \sim p_T} \min_{x_T \in X_T} f_T(x_{t-1}, x_T; \xi_T) \right] \right].
\]

(1)

Here, each expectation $\mathbb{E}_{\xi_i \sim p_i}$ is taken with respect to some given probability measure $p_i \in \mathcal{P}_t$. Note that our problem (1) relies on the SI assumption in the sense that the ambiguity sets and the expectations are independent across stages. Consequently, we can define the (worst-case expected) cost-to-go functions recursively from $t = T$ to $t = 2$ as:

\[
Q_{t-1}(x_{t-1}) := \sup_{p_i \in \mathcal{P}_i} \mathbb{E}_{\xi_i \sim p_i} \min_{x_i \in X_i} f_i(x_{t-1}, x_i; \xi_i) + Q_t(x_t),
\]

(2)

with $Q_T(x_T) \equiv 0$ for any $x_T \in X_T$. To simplify the notation, we also define the value function as

\[
Q_t(x_{t-1}; \xi_t) := \min_{x_i \in X_i} f_i(x_{t-1}, x_i; \xi_i) + Q_t(x_t),
\]

(3)

such that $Q_{t-1}(x_{t-1}) = \sup_{p_i \in \mathcal{P}_i} \mathbb{E}_{\xi_i \sim p_i} Q_t(x_{t-1}; \xi_t)$ for all $t \in T'$. Our algorithmic goal in this paper is to find a near-optimal first stage solution

\[
x_1^* \in \arg\min_{x_i \in X_i} f_1(x_0, x_1; \xi_1) + Q_1(x_1),
\]

(4)

as a multistage solution may already take exponentially many operations in $T$ to be written down, e.g., $2^{T-1}$ if we have an MSCO with $|\Xi_t| = 2$ for $t = 2, \ldots, T$. We say that $x_1^*$ is an $\varepsilon$-optimal first stage solution if

\[
f_1(x_0, x_1^*; \xi_1) + Q_1(x_1^*) \leq \min_{x_1 \in X_1} f_1(x_0, x_1; \xi_1) + Q_1(x_1).
\]

(5)

It is well-known that if the ambiguity set $\mathcal{P}_t$ is a singleton set, then the supremum is superficial so the DR-MCO (1) reduces to an MSCO. If the ambiguity set $\mathcal{P}_t$ contains all Dirac atomic measures (i.e., measures $\delta_\xi$ for all $\xi \in \Xi_t$ such that $\mathbb{E}_{\xi \sim \delta_\xi} g(\xi) = g(\xi)$ for any Borel measurable function $g$), then the supremum

\[
\sup_{p_i \in \mathcal{P}_i} \mathbb{E}_{\xi_i \sim p_i} Q_t(x_{t-1}; \xi_t) = \sup_{\xi_t \in \Xi_t} Q_t(x_{t-1}; \xi_t)
\]

and thus the DR-MCO (1) reduces to an MRCO. The following proposition checks that minimizations in DR-MCO (1) (and thus also in MSCO and MRCO) are convex and well-defined.

**Proposition 1.** The cost-to-go functions $Q_t(x_t)$ are lower semicontinuous (lsc) and convex for all $t \in T$.

**Proof.** We prove by recursion from $t = T$ to $t = 1$. By definition, $Q_T(x_T) \equiv 0$ is lsc and convex. Now assume $Q_t$ is lsc and convex for some $t \in T$. Then the sum $f_t(x_{t-1}, x_t; \xi_t) + Q_t(x_t)$ is also lsc and convex in $(x_{t-1}, x_t)$ for any $\xi_t \in \Xi_t$ due to the assumption of $f_t$. Since $X_t$ is compact, we have $Q_t(x_{t-1}; \xi_t) := \min_{x_t \in X_t} f_t(x_{t-1}, x_t; \xi_t) + Q_t(x_t)$ is lsc (see e.g., Lemma 1.30 in [5]) and convex. Now fix any Borel probability measure $p_t \in \mathcal{P}_t$ and take any sequence $\{x^i\} \subset X_t$ with $\lim_{i \to \infty} x^i = x_{t-1}$. Note that $Q_t$ is nonnegative by definition, so by Fatou’s lemma (see e.g., Lemma 1.28 in [35]) we have

\[
\liminf_{i \to \infty} \mathbb{E}_{\xi_t \sim p_t} Q_t(x^i; \xi_t) \geq \int_{\Xi_t} \liminf_{i \to \infty} Q_t(x^i; \xi_t) \, dp_t(\xi_t) \geq \int_{\Xi_t} Q_t(x_{t-1}; \xi_t) \, dp_t(\xi_t).
\]
The expectation and the integrals are well-defined since \( Q_t \) is lsc, hence Borel measurable. This inequality shows that the function \( \mathbb{E}_{\xi_t \sim p_t} Q_t(x_{t-1}; \xi_t) \) is lsc in \( x_{t-1} \). It is also convex in \( x_{t-1} \) by the linearity and monotonicity of expectations (see e.g., Theorem 7.46 in [39]). Finally, the epigraph of \( Q_{t-1}(\cdot) \) is the intersection of epigraphs of \( \mathbb{E}_{\xi_t \sim p_t} Q_t(\cdot; \xi_t) \) for all \( p_t \in \mathcal{P}_t \), which shows that \( Q_{t-1}(\cdot) \) is lsc and convex.

### 2.1.1 Finitely Supported Uncertainty Sets

If the uncertainty set \( \Xi_t = \{ \hat{\xi}_t, 1, \ldots, \hat{\xi}_t, n_t \} \) is finite, then the ambiguity set is a subset of a \( n_t \)-dimensional simplex \( \mathcal{P}_t \subseteq \Delta^n := \{ p_t \in \mathbb{R}_{\geq 0}^n : \sum_{k=1}^n p_{t,k} = 1 \} \). In this case, the expectations in the definitions (1) and (2) become finite weighted summations. For MSCO, finite supports often arise from sample average approximations [37]. It turns out that for MRCO, we can also restrict our attention to a finite support set when the problem is polyhedral (cf. [17]).

**Proposition 2.** Let \( Q_t \) denote the cost-to-go functions of an MRCO, i.e., a DR-MCO (1) with all Dirac atomic measures included in \( \mathcal{P}_t \) for all \( t \in \mathcal{T} \). If the cost functions \( f_t \) are jointly convex in \((x_t, \xi_t)\) for each \( x_t \in \mathcal{X}_t \) and the uncertainty sets \( \Xi_t \) are polytopes for all \( t \in \mathcal{T} \), then

\[
Q_{t-1}(x_{t-1}) = \max_{p_t \in \Delta^n} \sum_{k=1}^{n_t} p_{t,k} \cdot Q_t(x_{t-1}; \hat{\xi}_t,k),
\]

where \( \{ \hat{\xi}_t, 1, \ldots, \hat{\xi}_t, n_t \} \) is the finite set of extreme points of \( \Xi_t \).

**Proof.** By Proposition 1, each worst-case cost-to-go function \( Q_t \) is convex for any \( t \in \mathcal{T} \). From the assumption, the value function \( Q_t(x_{t-1}; \xi_t) \) is thus also convex in \( \xi_t \) for each \( x_{t-1} \in \mathcal{X}_{t-1} \). Consequently, we have

\[
Q_{t-1}(x_{t-1}) = \sup_{\xi_t \in \Xi_t} Q_t(x_{t-1}; \xi_t) = \max_{p_t \in \Delta^n} \sum_{k=1}^{n_t} p_{t,k} Q_t(x_{t-1}; \hat{\xi}_t,k),
\]

due to the linearity of the maximization. We point out that the above equation holds even when \( \sup_{\xi_t \in \Xi_t} Q_t(x_{t-1}; \xi_t) = +\infty \) because in this case we must have \( Q_t(x_{t-1}; \hat{\xi}_t,k) = +\infty \) for some \( k = 1, \ldots, n_t \) by its convexity.

In view of Proposition 2, we decide to focus on finitely supported uncertainty sets when we discuss numerical implementations in this paper (such as in Sections 3.1.1 and 4), although the algorithms and the complexity questions are intended for a more general setting of DR-MCO problems.

### 2.2 Approximation of Recursions

We now discuss the approximation of functions \( Q_{t-1} \) and \( Q_t \) in the recursions (2) and (3). The following lemma relates the Lipschitz continuity of the value functions \( Q_t(\cdot; \xi_t) \) for \( \xi_t \in \Xi_t \) and that of the cost-to-go function \( Q_t \).

**Lemma 1.** If \( Q_t(\cdot; \xi_t) \) is \( L_t \)-Lipschitz continuous on \( \mathcal{X}_t \) for any \( \xi_t \in \Xi_t \) and for some \( L_t > 0 \), then so is \( Q_t(\cdot) \).

**Proof.** Take any two points \( x^1, x^2 \in \mathcal{X}_t \). For any \( \varepsilon > 0 \), take \( p^1_t \in \mathcal{P}_t \) such that

\[
Q_{t-1}(x^1) - \varepsilon \leq \mathbb{E}_{\xi_t \sim p^1_t} Q_t(x^1; \xi_t) \leq Q_{t-1}(x^1).
\]

Then we have

\[
Q_{t-1}(x^1) - Q_{t-1}(x^2) \leq \varepsilon + \mathbb{E}_{\xi_t \sim p^1_t} Q_t(x^1; \xi_t) - \sup_{p_t \in \mathcal{P}_t} \mathbb{E}_{\xi_t \sim p_t} Q_t(x^2; \xi_t)
\]

\[
\leq \varepsilon + \sup_{p_t \in \mathcal{P}_t} \mathbb{E}_{\xi_t \sim p_t} \left[ Q_t(x^1; \xi_t) - Q_t(x^2; \xi_t) \right]
\]

\[
\leq \varepsilon + \mathbb{E}_{\xi_t \sim p^1_t} L_t \| x^1 - x^2 \| = \varepsilon + L_t \| x^1 - x^2 \|.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have \( Q_{t-1}(x^1) - Q_{t-1}(x^2) \leq L_t \|x^1 - x^2\| \). The proof is then completed by exchanging \( x^1 \), \( x^2 \) and repeating the argument. \( \square \)

Combining Lemma 1 and Proposition 1, we know that if the value functions are convex and Lipschitz continuous, then so are the cost-to-go functions. In such a case, we can use cutting plane methods to build an under-approximation of the cost-to-go functions. To be precise, for every \( \xi_t \in \Xi_t \), let \( V_t(\cdot; \xi_t) \) denote an affine function such that \( Q_t(x_{t-1}; \xi_t) \geq V_t(x_{t-1}; \xi_t) \) for all \( x_{t-1} \in \mathcal{X}_{t-1} \). Such an affine function is referred to as a linear valid inequality or a linear cut for the value function, which is generated in the following way. Let \( \mathcal{Q}_t \) denote an under-approximation of the cost-to-go function \( Q_t \) and \( \hat{x}_{t-1} \in \mathcal{X}_{t-1} \) a feasible state. For any fixed \( \xi_t \in \Xi_t \), we introduce a redundant variable \( z_t \in \mathbb{R}^{d_t-1} \) with the duplicating constraint \( z_t = \hat{x}_{t-1} \). Then the Lagrangian dual problem

\[
\sup_{\lambda_t \in \mathbb{R}^{d_t-1}} \inf_{x_t \in \mathcal{X}_t, \ z_t \in \mathbb{R}^{d_t-1}} f_t(z_t, x_t; \xi_t) + \mathcal{Q}_t(x_t) + \langle \lambda_t, \hat{x}_{t-1} - z_t \rangle
\]

(6)

gives an affine function \( V_t(x_{t-1}; \xi_t) := \mathcal{V}_t(\xi_t) + \langle \hat{\lambda}_t(\xi_t), x_{t-1} - \hat{x}_{t-1} \rangle \), where we assume that an optimal dual solution \( \hat{\lambda}_t(\xi_t) \) of (6) and the associated infimum value \( \mathcal{V}_t(\xi_t) > -\infty \) are found. Then, by definition (3) and weak duality, we have for every \( x_{t-1} \in \mathcal{X}_{t-1} \)\n
\[
Q_t(x_{t-1}; \xi_t) \geq \sup_{\lambda_t \in \mathbb{R}^{d_t-1}} \inf_{x_t \in \mathcal{X}_t, \ z_t \in \mathbb{R}^{d_t-1}} f_t(z_t, x_t; \xi_t) + \mathcal{Q}_t(x_t) + \langle \lambda_t, x_{t-1} - z_t \rangle
\]

(7)

\[
\geq \inf_{x_t \in \mathcal{X}_t, \ z_t \in \mathbb{R}^{d_t-1}} f_t(z_t, x_t; \xi_t) + \mathcal{Q}_t(x_t) + \langle \hat{\lambda}_t(\xi_t), \hat{x}_{t-1} - z_t \rangle + \langle \hat{\lambda}_t, x_{t-1} - \hat{x}_{t-1} \rangle
\]

\[
= \mathcal{V}_t(\xi_t) + \langle \hat{\lambda}_t(\xi_t), x_{t-1} - \hat{x}_{t-1} \rangle.
\]

Therefore, \( V_t(\cdot; \xi_t) \) is a valid linear cut for the value function \( Q_t(\cdot; \xi_t) \). Finally, we can aggregate the valid linear cuts to get an affine function

\[
V_{t-1}(x_{t-1}) := \mathbb{E}_{\xi_t \sim \hat{\rho}_t} V_t(x_{t-1}; \xi_t),
\]

(8)

where \( \hat{\rho}_t \in \mathcal{P}_t \) can be arbitrarily chosen. The affine function \( V_{t-1}(\cdot) \) is a valid linear cut for the cost-to-go function \( Q_{t-1}(\cdot) \) because for any \( x_{t-1} \in \mathcal{X}_{t-1} \),

\[
Q_{t-1}(x_{t-1}) - V_{t-1}(x_{t-1}) \geq \mathbb{E}_{\xi_t \sim \hat{\rho}_t} \left[ Q_t(x_{t-1}; \xi_t) - V_t(x_{t-1}; \xi_t) \right] \geq 0.
\]

(9)

The above procedure of generating valid linear cuts requires finding the supremum in the Lagrangian dual problem (6). If the problem (1) does not have relatively complete recourse (RCR), the supremum in (6) may not be attained. Moreover, even with RCR, as we use some under-approximation \( \mathcal{Q}_t \) in the dual problem (6), the maximum may only be attained outside a ball with its radius growing with \( T \), as illustrated by the following example.

**Example 2.** Consider a deterministic problem (i.e., \( |\Xi_t| = 1 \) for all \( t \in \mathcal{T} \)) defined by the recursions for \( t = T, \ldots, 2 \)

\[
Q_{t-1}(x_{t-1}) := \min_{y_t, x_t} 1 + y_t + Q_t(x_t)
\]

s.t. \( y_t \geq 0, \ y_t \geq 1 - 2x_{t-1}, \ x_t \leq x_{t-1} + \frac{1}{2}, \ 0 \leq x_t \leq 1 \).

Note that for each stage \( t \in \mathcal{T} \), since \( x_{t-1} \in [0, 1] \), we have a feasible solution \( x_t = 1/2 \) and \( y_t = 0 \), which implies that \( Q_t(x_t) = T - t + \max\{0, 1 - 2x_t\} \) for \( t \leq T - 1 \). However, if we start our approximation
with points $x_t^0 = 0$ for all stages $t \in \mathcal{T}$, then the linear cut $V_{t-1}(x_{t-1}) = V_t(x_{t-1}) := v_t + \langle \lambda_t, x_t - x_t^0 \rangle$ can be generated from the following dual problem at stage $t \in \mathcal{T}'$:

$$\max_{\lambda_t \in \mathbb{R}} \inf_{z_t, y_t, x_t} 1 + y_t + \lambda_t(0 - z_t) + Q_t^0(x_t)$$

s.t. $y_t \geq 0$, $y_t \geq 1 - 2z_t$, $x_t \leq z_t + \frac{1}{2}$, $0 \leq x_t \leq 1$.

The optimal dual solution is $\hat{\lambda}_T = -2$ in stage $T$. Thus the under-approximation of $Q_{T-1}(x_{T-1})$ after adding the linear cut is $Q_t^0(x_{T-1}) = V_T(x_{T-1}) = 2 - 2x_{T-1}$. The dual problem in stage $t = T - 1$ now becomes

$$\max_{\lambda_t \in \mathbb{R}} \inf_{z_t, y_t, x_t} 1 + y_t - \lambda_t z_t + (T - t + 1) - c_t x_t$$

s.t. $y_t \geq 0$, $y_t \geq 1 - 2z_t$, $x_t \leq z_t + \frac{1}{2}$, $0 \leq x_t \leq 1$.

with $c_{T-1} = 2$. Note that if $\lambda_t > -c_t - 2$, the inner minimum value would be strictly smaller than $T - t + 3 - c_t/2$ (e.g., taking $z_t = 1/2$, $y_t = 0$, and $x_t = 1$). Thus any optimal dual solution must satisfy $\lambda_t \leq -c_t - 2$ as $z_t = 0$, $y_t = 1$, and $x_t = 1/2$ gives an inner minimum value $T - t + 3 - c_t/2$. By repeating the argument recursively, it can be checked that the under-approximation $Q_t^0(x_t) = V_t(x_t) = T - t + 1 - c_t x_t$ with $c_t = 2(T - t)$. This implies that any optimal dual solution in stage $t$ would lie outside the open ball of radius $2(T - t + 1)$, which is greater than 2, the Lipschitz constant of $Q_t$ on $X_t$. We illustrate the example with Figure 1 using $T = 3$.

![Figure 1: Cost-to-go Functions and First Iteration Linear Cuts for $T = 3$](image)

2.3 Exactness of Dual Bounds

Example 2 suggests that the norm of optimal dual solutions for linear cut generation cannot be bounded by the Lipschitz constants of the value functions. Intuitively speaking, larger norms of dual variables lead to steeper linear cuts (6), which makes the under-approximation less useful at points away from the previously visited ones. For this reason, we propose to enforce bounds on the dual variables for recursive approximations and investigate the exactness of these artificial dual bounds. To begin with, let us fix algorithmic parameters $M_t > 0$ for stages $t \in \mathcal{T}'$ and define alternative value function recursively

$$Q_t^R(x_{t-1} : \xi_t) := \max_{\| \lambda_t \|_t \leq M_t} \inf_{x_t \in X_t, \xi_t} f_t(x_{t-1}, x_t; \xi_t) + Q_{t+1}^R(x_t) + \langle \lambda_t, x_{t-1} - z_t \rangle,$$  

(10)
with \( Q^R_T(x_T) \equiv 0 \) on \( X_T \) and

\[
Q^R_{t-1}(x_{t-1}) = \sup_{p_t \in P_t} E_{\xi_t \sim p_t} Q^R_T(x_{t-1}; \xi_t)
\]

for \( t = T, T-1, \ldots, 2 \). We call them (Lipschitz) regularized value functions and (Lipschitz) regularized cost-to-go functions, due to the following observation.

**Proposition 3.** For any \( t \in T^t \) and \( \xi_t \in \Xi_t \), the function \( Q^R_t(:; \xi_t) \) is \( M_t \)-Lipschitz continuous, and \( Q^R_t(x_{t-1}; \xi_t) \leq Q_t(x_{t-1}; \xi_t) \) for all \( x_{t-1} \in X_{t-1} \). Moreover, we can write \( Q^R_t(x_{t-1}; \xi_t) \) as a value function

\[
Q^R_t(x_{t-1}; \xi_t) = \min_{x_t \in X_t, z_t \in \mathbb{R}^{d_t-1}} f_t(z_t, x_t; \xi_t) + Q^R_t(x_t) + M_t \| x_{t-1} - z_t \|.
\]

**Proof.** We begin our proof for the equation with the following observation

\[
\max_{\| \lambda_t \| \leq M_t} \inf_{x_t \in X_t, z_t \in \mathbb{R}^{d_t-1}} f_t(x_{t-1}, x_t; \xi_t) + Q^R_t(x_t) + (\lambda_t, x_{t-1} - z_t)
\]

\[
= \inf_{x_t \in X_t, \| \lambda_t \| \leq M_t} \max_{z_t \in \mathbb{R}^{d_t-1}} f_t(x_{t-1}, x_t; \xi_t) + Q^R_t(x_t) + (\lambda_t, x_{t-1} - z_t)
\]

\[
= \min_{x_t \in X_t, z_t \in \mathbb{R}^{d_t-1}} f_t(x_{t-1}, x_t; \xi_t) + Q^R_t(x_t) + M_t \| x_{t-1} - z_t \|.
\]

Here, the first equality is due to Sion’s minimax theorem [23], where the maximization is taken over a compact ball of radius \( M_t \). The second equality follows from the dual representation of the norm functions [9, A.1.6]. The infimum is attained since the sum \( f_t(x_{t-1}, x_t; \xi_t) + Q^R_t(x_t) + M_t \| x_{t-1} - z_t \| \) is bounded from below by our nonnegativity assumption of \( f_t \). Now it can be checked recursively for \( t = T, \ldots, 2 \) using the definition of \( Q^R_t \) that \( Q^R_t(x_{t-1}; \xi_t) \leq Q_t(x_{t-1}; \xi_t) \) for any \( x_{t-1} \in X_{t-1} \) because \( z_t = x_{t-1} \) is a feasible solution to the minimization.

It remains to show the \( M_t \)-Lipschitz continuity of \( Q^R_t(:, \xi_t) \). Pick any \( x^1, x^2 \in X_{t-1} \) and let \( x^i_t, z^i_t \) denote the solutions in the minimization associated with \( x^i \) for \( i = 1 \) and \( 2 \), respectively. Then, as \( x^2_t, z^2_t \) are also feasible solutions for the minimization associated with \( x^1 \), we have

\[
Q^R_t(x^1; \xi_t) - Q^R_t(x^2; \xi_t) \\
\leq f_t(z^2_t, x^2_t; \xi_t) + Q^R_t(x^2_t) + M_t \| x^1 - z^2_t \| \\
- f_t(z^1_t, x^1_t; \xi_t) - Q^R_t(x^1_t) - M_t \| x^2 - z^1_t \| \\
= M_t (\| x^1 - z^2_t \| - \| x^2 - z^1_t \|) \leq M_t \| x^1 - x^2 \|.
\]

Now we can repeat the argument with exchanged indices 1 and 2 and show that \( Q^R_t(x^2; \xi_t) - Q^R_t(x^1; \xi_t) \leq M_t \| x^1 - x^2 \| \), which completes the proof.

We can now generate a linear cut \( V_t(:, \xi_t) \) using a dual solution \( \hat{\lambda}_t \) and its associated infimum value \( \nu_t > -\infty \) of the following dual problem (cf. (6))

\[
\max_{\| \lambda_t \| \leq M_t} \inf_{x_t \in X_t, z_t \in \mathbb{R}^{d_t-1}} f_t(z_t, x_t; \xi_t) + Q_t(x_t) + (\lambda_t, x_{t-1} - z_t).
\]

(12)

Whenever \( Q_t \) is an under-approximation of the regularized cost-to-go function \( Q^R_t \), we can use the same argument as in (7) to see the validness of \( V_t(:, \xi_t) \) as a linear cut for \( Q^R_t(:, \xi_t) \), for all \( \xi_t \in \Xi_t \), and thus also the validness of \( V_{t-1}(:, \cdot) \) as an aggregate linear cut for \( Q^R_{t-1}(:, \cdot) \). The Lipschitz constants of these linear cuts \( V_t(:, \xi_t) \) and \( V_{t-1}(:, \cdot) \) are bounded by \( M_t \).

By solving the bounded dual problems (12) for linear cut generation, by Proposition 3, we are essentially replacing the value functions \( Q_t \) and the cost-to-go functions \( Q_t \) with the regularized ones, \( Q^R_t \) and \( Q^R_t \), respectively. We want to discuss that such replacement does not compromise any feasibility or optimality, which requires a technical lemma.
Lemma 2. Let \( \mathcal{X} \subseteq \mathbb{R}^d \) be a full-dimensional convex set and \( f : \mathbb{R}^d \to \mathbb{R} \) be a convex function. If the restriction \( f|_\mathcal{X} \) is \( M \)-Lipschitz continuous, then we have \( f(x) = \inf_{z \in \mathcal{X}} \{ f(z) + M\|z - x\| \} \) for any \( x \in \mathcal{X} \).

Proof. Assume for contradiction that for some \( x \in \mathcal{X} \), there exists \( z \in \mathbb{R}^d \) and \( \varepsilon > 0 \) such that \( f(x) > f(z) + (1 + \varepsilon)M\|z - x\| \). If \( x \) lies in the interior \( \text{int}\mathcal{X} \), then we can find \( z' = x + \delta(x - z) \in \mathcal{X} \) for some \( \delta > 0 \). Thus by convexity, \( (f(z') - f(x))/\|z' - x\| \geq (f(x) - f(z))/\|x - z\| > (1 + \varepsilon)M \), which contradicts the \( M \)-Lipschitz continuity of \( f|_\mathcal{X} \).

Otherwise if \( x \notin \text{int}\mathcal{X} \), since \( \mathcal{X} \) is full-dimensional, we can find \( x' \in \text{int}\mathcal{X} \) with \( \|x' - x\| \leq \|z - x\|/\varepsilon/2 \), so \( \|x' - z\| \leq (1 + \varepsilon/2)\|z - x\| \). Besides, by the \( M \)-Lipschitz continuity of \( f|_\mathcal{X} \), we have \( f(x') \geq f(x) - M\|x - x'\| > f(z) + (1 + \varepsilon/2)M\|z - x\| \). Therefore,

\[
\frac{f(x') - f(z)}{\|x' - z\|} > \frac{(1 + \varepsilon/2)M\|z - x\|}{(1 + \varepsilon/2)\|z - x\|} = M,
\]

which shows contradiction by the argument above since \( x' \in \text{int}\mathcal{X} \).

Proposition 4. Suppose that the state spaces \( \mathcal{X}_t \) are full-dimensional. If the value function \( Q_t(\cdot; \xi_t) \) is \( M_t \)-Lipschitz continuous for any \( \xi_t \in \mathcal{X}_t \) and \( t \in T \), then we have \( Q_t^R(x_{t-1}; \xi_t) = Q_t(x_{t-1}; \xi_t) \) for all \( x_{t-1} \in \mathcal{X}_{t-1} \) and \( t \in T \).

Proof. We prove the assertion by applying Lemma 2 recursively for \( t = T, T - 1, \ldots, 1 \). Suppose that \( Q_t = Q_t^R \) on \( \mathcal{X}_t \) for some \( t \in T \), which holds trivially for \( t = T \). We see that \( Q_t(\cdot; \xi_t) = Q_t^R(\cdot; \xi_t) \) everywhere on \( \mathcal{X}_{t-1} \) for any \( \xi_t \in \mathcal{X}_t \) by Proposition 3. Thus by definition (11), \( Q_{t-1} = Q_{t-1}^R \) on \( \mathcal{X}_{t-1} \).

Proposition 4 ensures the exactness of the bounded dual recursion (10) and (11), provided that the value functions \( Q_t(\cdot; \xi_t) \) are \( M_t \)-Lipschitz continuous for some known values \( M_t > 0 \). We remark that the idea of bounding the dual variables, or equivalently adding a regularization term in the dual problems, could work more generally for multistage stochastic problems without RCR, and refer any interested readers to the paper [42].

3 Algorithms and Complexity Analysis

In this section, we first define single stage subproblem oracles (SSSO) for DR-MCO (1), based on which we define the notion of complexity of the algorithms. A simple implementation of the SSSO is then discussed for the finitely supported problems. We generalize the consecutive DDP (CDDP) algorithm to DR-MCO and more importantly, we introduce a new nonconsecutive DDP algorithm (NDDP). The complexity upper bound for each algorithm is then presented, and finally we provide a complexity lower bound for both algorithms, which shows the upper complexity bounds are nearly tight.

3.1 Single Stage Subproblem Oracles

A subproblem oracle is an oracle that gives a solution to the subproblem given its own information as well as the data generated by the algorithm. The single stage subproblem oracles (SSSO) used in this paper solve an approximation of the problem defined by (10) and (11) for some stage \( t \in T \).

Definition 1 (Initial stage subproblem oracle). Let \( \underline{Q}_1, \overline{Q}_1 : \mathcal{X}_1 \to \mathbb{R} \cup \{+\infty\} \) denote two lsc, convex functions, with \( \underline{Q}_1(x_1) \leq Q_1^R(x_1) \leq \overline{Q}_1(x_1) \) for any \( x_1 \in \mathcal{X}_1 \). Consider the following subproblem for the first stage \( t = 1 \),

\[
\min_{x_1 \in \mathcal{X}_1} f_1(x_0, x_1; \xi_1) + \underline{Q}_1(x_1). \tag{I}
\]

The initial stage subproblem oracle provides an optimal solution \( x_1 \) to (I) and calculates the approximation gap \( \gamma_1 := \overline{Q}_1(x_1) - \underline{Q}_1(x_1) \). We thus define the subproblem oracle formally as a map \( \mathcal{O}_1 : (\underline{Q}_1, \overline{Q}_1) \mapsto (x_1; \gamma_1) \).
**Definition 2** (Noninitial stage subproblem oracle). For any \( t \in \mathcal{T} \), let \( Q_t, \overline{Q}_t : \mathcal{X}_t \to \mathbb{R} \cup \{+\infty\} \) denote two lsc, convex functions, with \( Q_t(x_t) \leq Q_t^R(x_t) \leq \overline{Q}_t(x_t) \) for all \( x_t \in \mathcal{X}_t \). Then given a feasible state \( x_{t-1} \in \mathcal{X}_{t-1} \), the stage-\( t \) subproblem oracle provides a feasible state \( x_t \in \mathcal{X}_t \), an \( M_t \)-Lipschitz continuous linear cut \( V_{t-1}(\cdot) \), and an over-approximate value \( v_{t-1} \) such that

- they are valid, i.e., \( v_{t-1}(\cdot) \leq Q_t^R(x_t) \) on \( \mathcal{X}_{t-1} \) and \( v_{t-1} \geq Q_t^R(x_{t-1}) \);
- the gap is controlled, i.e., \( v_{t-1}(x_{t-1}) - V_{t-1}(x_{t-1}) \leq \gamma_t := \overline{Q}_t(x) - Q_t^R(x) \).

We thus define the subproblem oracle formally as a map \( \Theta_t : (x_{t-1}, Q_t, \overline{Q}_t) \mapsto (v_{t-1}, v_{t-1}, x_t; \gamma_t) \).

The noninitial stage subproblem oracles are different from the initial stage subproblem oracle, in the sense that it does not necessarily provide any optimal solution to some optimization problem. Instead, it provides some feasible state, which could be used for exploration of the following stages, a linear cut and an estimate value for updating the approximation in the previous stages. As implementations for Definition 2 may be less intuitive than those for Definition 1, we provide some illustration using finitely supported DR-MCO below.

### 3.1.1 SSSO Implementation for Finitely Supported DR-MCO

We now propose a possible realization of the noninitial stage subproblem oracle based on the linear cut generation procedure discussed in Section 2.2. Suppose \( \mathcal{X}_t = \{\tilde{x}_1, \ldots, \tilde{x}_{n_t}\} \) for each \( t \in \mathcal{T} \). By solving the bounded Lagrangian dual problem (12) for each \( \tilde{x}_{t,k} \), we can get primal solutions \( \hat{x}_{t,k}, \hat{z}_{t,k} \), and linear cuts \( V_t(x_{t-1}; \tilde{x}_{t,k}) \) for \( k = 1, \ldots, n_t \) that are all \( M_t \)-Lipschitz continuous. We then aggregate them using \( v_{t-1}(\cdot) := \sum_{k=1}^{n_t} \hat{p}_{t,k} V_t(x_{t-1}; \tilde{x}_{t,k}) \), where \( \hat{p}_t \in \arg \max_{p_t \in \mathcal{P}_t} \sum_{k=1}^{n_t} \hat{p}_{t,k} V_t(x_{t-1}; \tilde{x}_{t,k}) \) is a maximizer at the given state \( x_{t-1} \). It is clear that \( v_{t-1}(\cdot) \) is also \( M_t \)-Lipschitz continuous. Next we solve the maximization problem

\[
v_{t-1} := \max_{p_t \in \mathcal{P}_t} \sum_{k=1}^{n_t} \hat{p}_{t,k} \cdot \left( f_t(x_{t-1}, \hat{x}_{t,k}; \hat{z}_{t,k}) + \overline{Q}_t(\hat{z}_{t,k}) + M_t \|x_{t-1} - \hat{z}_{t,k}\| \right).
\]  

(13)

Note that by assumption \( \overline{Q}_t(x_t) \geq Q_t^R(x_t) \) for all \( x_t \in \mathcal{X}_t \), and \( (\hat{x}_{t,k}, \hat{z}_{t,k}) \) is a feasible solution to the problem

\[
\min_{x_t \in \mathcal{X}_t} f_t(z_t, x_t; \hat{z}_{t,k}) + \overline{Q}_t(x_t) + M_t \|x_{t-1} - \hat{z}_{t,k}\| \geq Q_t^R(x_{t-1}; \tilde{x}_{t,k}).
\]

Thus the value \( v_{t-1} \geq \sum_{k=1}^{n_t} \hat{p}_{t,k} Q_t^R(x_{t-1}; \tilde{x}_{t,k}) = Q_{t-1}^R(x_{t-1}) \) is a valid overestimation.

It remains to determine the feasible state \( x_t \) and calculate the gap \( \gamma_t \) that satisfy the second requirement in Definition 2. Let \( \gamma_{t,k} := \overline{Q}_t(\hat{x}_{t,k}) - Q_t^R(\hat{z}_{t,k}) \) for each \( k = 1, \ldots, n_t \). We pick the index \( k^* \) corresponding to the largest gap \( \gamma_{t,k^*} \), and set \( x_t = \hat{x}_{t,k^*}, \gamma_t = \gamma_{t,k^*} \). Consequently, we have

\[
v_{t-1} = \max_{p_t \in \mathcal{P}_t} \sum_{k=1}^{n_t} \hat{p}_{t,k} \left( f_t(\hat{x}_{t,k}, \hat{z}_{t,k}; \hat{z}_{t,k}) + \overline{Q}_t(\hat{x}_{t,k}) + M_t \|x_{t-1} - \hat{z}_{t,k}\| \right)
= \max_{p_t \in \mathcal{P}_t} \sum_{k=1}^{n_t} \hat{p}_{t,k} \left( f_t(\hat{x}_{t,k}, \hat{x}_{t,k}; \hat{z}_{t,k}) + M_t \|x_{t-1} - \hat{z}_{t,k}\| \right)
\leq \gamma_t + \max_{p_t \in \mathcal{P}_t} \sum_{k=1}^{n_t} \hat{p}_{t,k} \left( f_t(\hat{x}_{t,k}, \hat{x}_{t,k}; \hat{z}_{t,k}) + \overline{Q}_t(\hat{x}_{t,k}) + M_t \|x_{t-1} - \hat{z}_{t,k}\| \right)
= \gamma_t + V_{t-1}(x_{t-1}).
\]

We summarize the above realization of the noninitial stage subproblem oracles in Algorithm 1.

We remark that Algorithm 1 is not the only way to realize the SSSO in Definition 2. For example, it is discussed in [17] that a polyhedral single stage subproblem of MRCO can be reformulated as mixed-integer linear optimization, which may then be solved by branch-and-bound type algorithms.
To ease the notation, we use \( \text{conv} \{ h_1, h_2 \} \) to denote the function corresponding to the closed convex hull of the epigraphs of functions \( h_1 \) and \( h_2 \). More precisely, we define

\[
\text{conv} \{ h_1, h_2 \}(x) := \sup_{\lambda} \inf_z \left\{ \min_{\mu_1, \mu_2} \{ \mu_1(z_1), \mu_2(z_2) \} + \langle \lambda, x - z \rangle \right\}.
\]

Note that if \( h_1, h_2 \) are further polyhedral, then by linear optimization strong duality, the function \( \text{conv} \{ h_1, h_2 \} \) (assuming it is proper) can be written as

\[
\text{conv} \{ h_1, h_2 \}(x) = \min \left\{ v \in \mathbb{R} : (z, v) = \mu_1(z_1, v_1) + \mu_2(z_2, v_2), v_i \geq h_i(z_i), \mu_i \in \mathbb{R}_{\geq 0}, i = 1, 2, \mu_1 + \mu_2 = 1 \right\}.
\]

For each iteration \( i \in \mathbb{N} \), the main loop of CDDP (Algorithm 2) consists of three steps. The forward step uses the state \( x_{t-1}^i \) in the previous stage and the approximations \( \text{Q}^i_{t-1} \) and \( \text{Q}^i_{t-1} \) to produce a new state \( x_t^i \). Then the backward step at stage \( t \) uses the cut \( \text{V}^i_{t-1}(x) \) and the value \( v_{t-1}^i \) to update the approximations \( \text{Q}^i_{t-1}, \overline{\text{Q}}^i_{t-1} \) in its precedent stage \( t-1 \). The initial stage step produces a first stage solution \( x_1^* \) and updates the lower and upper bounds.

We next discuss the correctness of Algorithm 2, i.e., the returned solution \( x_1^* \) is \( \varepsilon \)-optimal as defined in (5), while leaving the finiteness proof to Section 3.3. From the termination of the while-loop, it suffices to show that the approximations are valid \( \text{Q}^i_t(x) \leq \text{Q}^R_t(x) \leq \overline{\text{Q}}^i_t(x) \) for each \( t \in \mathcal{T} \) and \( i \in \mathbb{N} \). The first inequality follows from the validness of linear cuts \( \text{V}^i_t \) (Definition 2). The second inequality is due to the \( M_t \)-Lipschitz continuity of \( \text{Q}^R \) (Proposition 3). In particular, the
We describe the NDDP algorithm in Algorithm 3. To start the algorithm, it requires an additionally more efficient approximation updates. Alternatively, we propose a nonconsecutive version, NDDP, algorithm that could conduct the information in the previous iteration for deciding the next stage to be solved: the precedent stage $t - 1$ or the subsequent one $t + 1$. If the algorithm decides to proceed to the subsequent stage $t + 1$, then the current state $x_t$ is used; otherwise the generated linear cut $\mathcal{V}_{t-1}$ and over-estimate value $v_{t-1}$ are generated using only the information in the previous iteration $i - 1$. In fact, the subproblem oracles can be re-evaluated in the backward steps to produce tighter approximations. We simply keep the CDDP algorithm in its current form because it is already sufficient for us to prove its complexity bounds, and hence its convergence. Alternatively, we propose a nonconsecutive version, NDDP, algorithm that could conduct more efficient approximation updates.

### 3.2.2 Nonconsecutive Dual Dynamic Programming

We describe the NDDP algorithm in Algorithm 3. To start the algorithm, it requires an additionally chosen vector of approximation gaps $\delta := (\delta_t)_{t=1}^T$ such that $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$, compared with the CDDP algorithm. These predetermined approximation gaps serve as criteria at stage $t$ for deciding the next stage to be solved: the precedent stage $t - 1$ or the subsequent one $t + 1$. If the algorithm decides to proceed to the subsequent stage $t + 1$, then the current state $x_t$ is used; otherwise the generated linear cut $\mathcal{V}_{t-1}$ and over-estimate value $v_{t-1}$ are used for updating the approximations. The above argument of validness of approximations imply that $\mathcal{Q}_t^i(x) \leq \mathcal{Q}_t^R(x) \leq \mathcal{Q}_t^i(x)$ for all $x \in \mathcal{X}_t$ holds for any stage $t \in T$ and any index $i_t \in \mathbb{N}$. Therefore, when NDDP terminates, the returned solution $x^*_t$ is indeed $\varepsilon$-optimal.

---

**Algorithm 2 Consecutive Dual Dynamic Programming Algorithm**

**Require:** subproblem oracles $\mathcal{O}_t$ for $t \in T$, target optimality gap $\varepsilon > 0$

**Ensure:** an $\varepsilon$-optimal first stage solution $x^*_1$

1. initialize: $\mathcal{Q}_0^t \leftarrow 0, \mathcal{O}_0^t \leftarrow +\infty, t \in T \setminus \{T\}$; $\mathcal{Q}_t^0, \mathcal{O}_t^0 \leftarrow 0, j \in \mathbb{N}$; $i \leftarrow 1$
2. evaluate $(x^*_1; \gamma^*_1) \leftarrow \mathcal{O}_1^0(\mathcal{Q}_0^1, \mathcal{O}_1^0)$
3. set LOWERBOUND $\leftarrow f_1(x_0, x_1^1; \xi_t)$, UPPERBOUND $\leftarrow +\infty$
4. while $\text{UPPERBOUND} - \text{LOWERBOUND} > \varepsilon$ do
5. for $t = 2, \ldots, T$ do
6. evaluate $(\mathcal{V}_{t-1}^i, v_{t-1}^i, x_{t-1}^i; \gamma_{t-1}^i) ) = \mathcal{O}_i(x_{t-1}^i, \mathcal{Q}_{t-1}^{i-1}, \mathcal{O}_{t-1}^{i-1})$  $	riangleright$ forward step
7. end for
8. for $t = T, \ldots, 2$ do
9. update $\mathcal{Q}_{t-1}^i(x) \leftarrow \max\{\mathcal{Q}_{t-1}^{i-1}(x), \mathcal{V}_{t-1}^i(x)\}$  $	riangleright$ backward step
10. update $\mathcal{Q}_{t-1}^i(x) \leftarrow \text{conv}\{\mathcal{Q}_{t-1}^{i-1}(x), v_{t-1}^i + M_t ||x - x_{t-1}^i||\}$
11. end for
12. evaluate $(x_{t+1}^i, \gamma_{t+1}^i) \leftarrow \mathcal{O}_1^1(\mathcal{Q}_1^1, \mathcal{O}_1^1)$  $	riangleright$ initial stage step
13. update LOWERBOUND $\leftarrow f_1(x_0, x_{t+1}^1; \xi_t) + \mathcal{Q}_1^1(x_{t+1}^1)$
14. update UPPERBOUND $\prime \leftarrow f_1(x_0, x_{t+1}^1; \xi_t) + \mathcal{Q}_1^1(x_{t+1}^1)$
15. if $\text{UPPERBOUND} < \text{UPPERBOUND} $ then
16. set $x_t^* \leftarrow x_{t+1}^i$. UPPERBOUND $\leftarrow \text{UPPERBOUND} $’
17. end if
18. update $i \leftarrow i + 1$
19. end while

Inequality $v^i_{t-1} \geq \mathcal{Q}_{t-1}^R(x_{t-1}^i)$ implies $v^i_{t-1} + M_t ||x - x_{t-1}^i|| \geq \mathcal{Q}_{t-1}^R(x)$ for all $x \in \mathcal{X}_{t-1}$. Given that $\mathcal{Q}_{t-1}^{i-1}(x) \geq \mathcal{Q}_{t-1}^R(x)$ for $x \in \mathcal{X}_{t-1}$, which is obviously true for $i = 1$, we conclude that

$$\min\{\mathcal{Q}_{t-1}^{i-1}(x), v^i_{t-1} + M_t ||x - x_{t-1}^i||\} \geq \mathcal{Q}_{t-1}^R(x), \quad \forall x \in \mathcal{X}_{t-1}.$$

By taking the closed convex hull of the epigraphs on both sides, we have shown that $\mathcal{Q}_{t-1}^i(x) \geq \mathcal{Q}_{t-1}^R(x)$ for all $x \in \mathcal{X}_{t-1}$. The above argument shows inductively that for all $i \in \mathbb{N}$, the approximations are valid, which then implies the correctness of the algorithm.

We comment that the linear cut $\mathcal{V}_{t-1}^i$ and the over-estimate value $v_{t-1}^i$ are generated using only the information in the previous iteration $i - 1$. In fact, the subproblem oracles can be re-evaluated in the backward steps to produce tighter approximations. We simply keep the CDDP algorithm in its current form because it is already sufficient for us to prove its complexity bounds, and hence its convergence. Alternatively, we propose a nonconsecutive version, NDDP, algorithm that could conduct more efficient approximation updates.
3.3 Complexity Upper Bounds

In this section, we provide a complexity analysis for the proposed CDDP and NDDP algorithms, which implies that both algorithms terminate in finite time for any finite $M_2, \ldots, M_T > 0$ and $\varepsilon > 0$. Our goal is to derive an upper bound on the total number of subproblem oracle evaluations before the termination of the algorithm. To begin with, let $\mathcal{J}_t, t > 1$, denote the set of pairs of indices $(i_{t-1}, i_t)$ such that the noninitial stage subproblem oracle is evaluated at the $i_{t-1}$-th time at the state $x_t^{i_{t-1}}$, i.e., $(x_t^{i_{t-1}}, x_t^{i_t}; \gamma_t^{i_t}) = \mathcal{O}_t(x_t^{i_{t-1}}, Q_t^{i_t}, \overline{Q}_t^{i_t})$. For the CDDP algorithm, all stages share the same iteration index $i_t = i$, so $\mathcal{J}_t = \{(i, i) : i \in \mathbb{N}\}$ for all $t > 1$. We define the following sets of indices for each $t \in \mathcal{T} \setminus \{T\}$:

$$
\mathcal{I}_t(\delta) := \left\{ i_t \in \mathbb{N} : \gamma_t^{i_t} > \delta_t \text{ and } \gamma_{t+1}^{i_{t+1}} \leq \delta_{t+1}, (i_t, i_{t+1}) \in \mathcal{J}_{t+1} \right\}.
$$

Here, for NDDP algorithm, $\delta$ is the given approximation gap vector, while for CDDP algorithm, $\delta = (\delta_1, \ldots, \delta_T)$ can be any vector satisfying $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$ for the purpose of analysis, since it is not required for the CDDP algorithm. We adopt the convention that the gap for the last stage $\gamma_T^{i_T} \equiv 0$ such that $i_{T-1} \in \mathcal{I}_{T-1}(\delta)$ if and only if $\gamma_{T-1}^{i_{T-1}} > \delta_{T-1}$ and $(i_{T-1}, i_T) \in \mathcal{J}_T$. In plain words, $\mathcal{I}_t(\delta)$ contains the pairs of indices where the state $x_t^{i_t}$ has an approximation gap greater than $\delta_t$ at $i_t$-th
evaluation, but at the iteration the stage oracle finds a solution with approximation gap smaller than \( \delta_{i+1} \). An important observation is that all these index sets are finite (before algorithm termination) \(| \mathcal{I}_i | < \infty \), which is more precisely stated in the following lemma.

**Lemma 3.** For stage any \( t < T \), suppose the state space \( \mathcal{X}_t \subset \mathbb{R}^{d_t} \) is contained in a ball with diameter \( D_t > 0 \). Then,

\[
| \mathcal{I}_t(\delta) | \leq \left( 1 + \frac{2M_{t+1}D_t}{\delta_t - \delta_{t+1}} \right)^{d_t}.
\]

**Proof.** We claim that for any \( j, k \in \mathcal{I}_t \), \( j \neq k \), it holds that \( ||x_t^j - x_t^k|| < (\delta_t - \delta_{t+1})/(2M_{t+1}) \). Assume for contradiction that \( ||x_t^j - x_t^k|| \geq (\delta_t - \delta_{t+1})/(2M_{t+1}) \) for some \( j < k \), \( j, k \in \mathcal{I}_t(\delta) \). By definition of \( \mathcal{I}_t(\delta) \), the \((t+1)\)-th subproblem oracle is evaluated at the state \( x_t^j \), and in both the CDDP and the NDDP algorithms, the approximations \( \mathcal{Q}^j_t \) and \( \mathcal{Q}^k_t \) are updated since \( \gamma_{t+1}^{j+1} \leq \delta_{t+1} \) for some \( i_{t+1} \in \mathbb{N} \). Then by Definition 2 of the noninitial stage subproblem oracle, we have \( \mathcal{Q}^{j+1}_t(x_t^j) - \mathcal{Q}^{j+1}_t(x_t^k) \leq \delta_{t+1} \). Noting that for any point \( x \in \mathcal{X}_t \) with \( ||x - x_t^j|| \leq (\delta_t - \delta_{t+1})/(2M_{t+1}) \), we have \( \mathcal{Q}^{j+1}_t(x) - \mathcal{Q}^{j+1}_t(x_t^k) \leq \delta_t \) because of the \( M_{t+1} \)-Lipschitz continuity of the approximations. Since \( \mathcal{Q}^k_t(x) \leq \mathcal{Q}^{j+1}_t(x) \) and \( \mathcal{Q}^k_t(x) \geq \mathcal{Q}^{j+1}_t(x) \), by setting \( x = x_t^k \), we see a contradiction with the assumption that \( k \in \mathcal{I}_t(\delta) \), which proves the claim.

To ease the notation, let \( r_t := (\delta_t - \delta_{t+1})/(2M_{t+1}) \) denote the radius of the \( d_t \)-dimensional balls \( \mathcal{B}^{d_t}(x_t^j; r_t) \) centered at \( x_t^j \) for \( j \in \mathcal{I}_t(\delta) \), and let \( \mathcal{B}_t \subset \mathcal{X}_t \) denote a ball with diameter \( D_t \). From the above claim, we know that \( x_t^j \notin \mathcal{B}^{d_t}(x_t^j; r_t) \) for any \( j, k \in \mathcal{I}_t(\delta) \) with \( j < k \). In other words, the smaller balls \( \mathcal{B}^{d_t}(x_t^j; r_t/2) \) are disjoint. Meanwhile, note that each of these smaller balls satisfies \( \mathcal{B}^{d_t}(x_t^j; r_t/2) \subset \mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2) \) (the Minkowski sum in the Euclidean space \( \mathbb{R}^{d_t} \)). Therefore, the volumes satisfy the relation

\[
\text{Vol} \left( \bigcup_{j \in \mathcal{I}_t(\delta)} \mathcal{B}^{d_t}(x_t^j; r_t/2) \right) = | \mathcal{I}_t(\delta) | \cdot \text{Vol} \mathcal{B}^{d_t}(0; r_t/2) \leq \text{Vol} \left( \mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2) \right),
\]

which implies that

\[
| \mathcal{I}_t(\delta) | \leq \frac{\text{Vol}(\mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2))}{\text{Vol} \mathcal{B}^{d_t}(0; r_t/2)} = \left( \frac{D_t/2 + r_t/2}{r_t/2} \right)^{d_t} = \left( 1 + \frac{2M_{t+1}D_t}{\delta_t - \delta_{t+1}} \right)^{d_t}.
\]

\[\square\]

We prove the following complexity upper bounds for the CDDP algorithm (Theorem 1) and the NDDP algorithm (Theorem 2).

**Theorem 1.** Suppose the state spaces \( \mathcal{X}_t \subset \mathbb{R}^{d_t} \) are contained in balls, each with diameter \( D_t > 0 \). Then for the CDDP algorithm (Algorithm 2), the total number of subproblem oracle evaluations

\[
\#\text{Eval}_{\text{CDDP}} \text{ before termination is bounded by}
\]

\[
\#\text{Eval}_{\text{CDDP}} \leq 1 + T \cdot \inf_{\delta} \left\{ \sum_{i=1}^{T-1} \left( 1 + \frac{2M_{i+1}D_i}{\delta_i - \delta_{i+1}} \right)^{d_i} : \varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0 \right\}.
\]

**Proof.** We prove by showing that for any approximation gap vector \( \delta \) satisfying \( \varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0 \), the largest iteration index \( i \) is bounded by

\[
i \leq \sum_{i=1}^{T-1} \left( 1 + \frac{2M_{i+1}D_i}{\delta_i - \delta_{i+1}} \right)^{d_i}.
\]

We claim that each iteration \( i \in \mathbb{N} \) must lie in either of the following two cases:

1. the initial stage step has \( \gamma_1^i \leq \varepsilon \); or
2. the \(i\)-th forward step is in the index set \(i \in \mathbb{I}_t(\delta)\) for some stage \(t < T\).

To see the claim, suppose that the iteration \(i \in \mathbb{N}\) is not in the first case. Then we have \(\gamma^*_t \leq \varepsilon\) and by convention \(\gamma^*_T = 0 \leq \delta_T\). Therefore, there exists a stage \(t < T\) such that \(\gamma^*_t > \delta_t\) and \(\gamma^*_{t+1} \leq \delta_{t+1}\), which is the second case. Note that when the first case happens, we have \(\text{UpperBound} - \text{LowerBound} \leq \gamma^*_1 \leq \varepsilon\) and thus the CDDP algorithm terminates. By Lemma 3, the second case can only happen at most \(\sum_{t=1}^{T-1} |\mathbb{I}_t(\delta)|\) times, proving the bound (16). The theorem then follows from the fact that in each CDDP iteration, the subproblem oracle is evaluated \(T + 1\) times with the additional evaluation of the initial stage subproblem oracle for checking the termination criterion.

\[\text{Remark 1.}\] When \(d_t = d\), \(M_t = M\), and \(D_t = D\) for all \(t = 1, \ldots, T - 1\), the bound in Theorem 1 can be simplified nicely. Note that the function

\[
\sum_{t=1}^{T-1} \left(1 + \frac{2MD}{\varepsilon \cdot \sigma_t}\right)^d
\]

is strictly convex and symmetric under permutation on the simplex \(\Delta^{T-1} := \{\sigma = (\sigma_1, \ldots, \sigma_{T-1}) \in \mathbb{R}_{T-1} : \sum_{t=1}^{T-1} \sigma_t = 1\}\). Therefore, it has a unique optimal solution \(\delta = (\frac{1}{T-1}, \ldots, \frac{1}{T-1})\), which implies the infimum in (16) is attained by \(\delta_t = \frac{1}{T-1} \varepsilon\), \(t \in T\). In this case, the CDDP complexity upper bound in Theorem 1 becomes

\[i \leq 1 + T(T - 1) \left(1 + \frac{2MD(T - 1)}{\varepsilon}\right)^d.
\]

\[\text{Theorem 2.}\] Suppose the state spaces \(X_t \subset \mathbb{R}^{d_t}\) are contained in balls, each with diameter \(D_t > 0\). Then, for the NDDP algorithm (Algorithm 3) with the predetermined approximation gap vector \((\delta_t)_{t=1}^{T-1}\) satisfying \(\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0\), the total number of subproblem oracle evaluations \#Eval_{NDDP} before termination is bounded by

\[\#\text{Eval}_{NDDP} \leq 1 + 2 \sum_{t=1}^{T-1} \left(1 + \frac{2M_{t+1}D_t}{\delta_t - \delta_{t+1}}\right)^d_t.
\]

\[\text{Proof.}\] For the NDDP algorithm, each time when it decides to go back to the precedent stage \(t \leftarrow t - 1\), we must have \(\gamma^*_t \leq \delta_t\) while \(\gamma^*_t \geq \delta_t\) for some \((i_{t-1}, i_t) \in \mathbb{I}_t\). In this case, we have by definition that \(i_{t-1} \in \mathbb{I}_{t-1}(\delta)\). By Lemma 3, such “going back” steps can only happen at most \(\sum_{t=1}^{T-1} |\mathbb{I}_t(\delta)|\) times. The total number of evaluations is then bounded by two times the number of “going back” steps due to the corresponding “going forward” steps, plus an additional evaluation of the initial stage oracle for checking the termination criterion.

Let us compare the complexity bounds of the two algorithms. If we fix the approximation gap vector \(\delta\) in Theorem 1 to be the same in the NDDP algorithm, then the complexity bound of CDDP grows faster than that of NDDP as \(T \geq 2\). However, since an optimal choice of the gap vector \(\delta\) is usually not known, CDDP has the advantage of not requiring estimates of these factors for the algorithm to be executed.

In many practical problems, it is reasonable to allow the target optimality gap to grow with \(T\). We say \(x^*_t \in X_t\) is an \(\alpha\)-relative optimal solution if

\[f_t(x_0, x^*_1; \xi_t) + Q_t(x^*_t) \leq (1 + \alpha) \left(\min_{x_t \in X_t} f_t(x_0, x_t; \xi_t) + Q_t(x_t)\right).
\]

An \(\varepsilon\)-optimal first stage solution is \(\alpha\)-relative optimal if

\[\varepsilon \leq \alpha \left(\min_{x_t \in X_t} f_t(x_0, x_t; \xi_t) + Q_t(x_t)\right).
\]

We provide below an important simplification of the above complexity bounds when considering relative optimality gaps.
Corollary 1. Suppose that all the state spaces have the same dimension \( d_t = d \) and are bounded by a common diameter \( D_t \leq D \), and let \( M := \max \{ M_t : t \in T \} \). If for each stage \( t \in T \), the local cost functions are uniformly bounded from below by \( \alpha C > 0 \) for all feasible solutions \( x_{t-1}, x_t \) and uncertainty outcomes \( \xi_t \), i.e., \( f_t(x_{t-1}, x_t; \xi_t) \geq C \), then the total number of subproblem oracle evaluations before achieving an \( \alpha \)-relative optimal solution \( x_t^* \) for CDDP and NDDP are upper bounded respectively by

\[
\#\text{Eval}_{\text{CDDP}} \leq 1 + T(T - 1) \left( 1 + \frac{2MD}{\alpha C} \right)^d,
\]

\[
\#\text{Eval}_{\text{NDDP}} \leq 1 + 2(T - 1) \left( 1 + \frac{2MD}{\alpha C} \right)^d.
\]

Proof. By assumption, \( \min_{x_t \in X_t} f_t(x_0, x_1; \xi_1) + Q_1(\xi_1) \geq CT \). Thus the assertion is proved if \( x_t^* \) is \( \varepsilon \)-optimal with \( \varepsilon = \alpha C(T - 1) < \alpha CT \). We can apply Theorems 1 and 2 by setting \( \delta_t = (T - t)\alpha C \). \( \square \)

Corollary 1 shows that for problems that have strictly positive cost in each stage, the proposed complexity bounds for an \( \alpha \)-relative optimal solution grow at most quadratically for CDDP and linearly for NDDP with respect to the number of stages \( T \). At the same time, they both grow exponentially with the dimension \( d \). In what follows, we show that this exponential dependence on \( d \) is nearly tight by providing a matching lower bound.

3.4 Complexity Lower Bound

Note that if we take \( \delta_t = \varepsilon(T - t)/(T - 1) \) for \( t \in T \), then the complexity upper bounds in Theorems 1 and 2 depend on terms \((T - 1)^{d_t + 1}\) or \((T - 1)^{d_t + 2}\), where \( d_t \) is the state space dimension of stage \( t < T \). It is natural to ask whether it is possible to achieve an \( \varepsilon \)-optimal solution without the exponential dependence on the state space dimensions as in Corollary 1. We construct a class of convex problems to show that this is indeed impossible.

Given a \( d \)-sphere \( S^d(r) = \{ x \in \mathbb{R}^{d+1} : \| x \|_2 = r \} \) with radius \( r > 0 \), a spherical cap with depth \( \theta > 0 \) centered at a point \( x \in S^d(r) \) is the set \( S^d_\theta(r, x) := \{ y \in S^d(r) : \| y - x \| \geq \theta r \} \). Our construction is based on the following technical lemma, where we use \( \Gamma(\cdot) \) to denote the gamma function.

Lemma 4 ([42, Lemmas 8 and 9]). Given a \( d \)-sphere \( S^d(r), d \geq 2 \) and depth \( \theta < (1 - \frac{\sqrt{2}}{2})r \), there exists a finite set of points \( W \) with

\[
|W| \geq \frac{(d^2 - 1)\sqrt{\pi}}{d} \Gamma(d/2 + 1) \Gamma(d/2 + 3/2) \left( \frac{r}{2\theta} \right)^{(d-1)/2},
\]

such that for any \( w \in W, S^d_\theta(r, w) \cap W = \{ w \} \). Moreover, given a positive constant \( l > 0 \), any function \( F^W : W \to (\frac{1}{2}l\theta, l\theta) \) can be extended to an \( l \)-Lipschitz continuous convex function \( F : B^d_\theta(r) \to \mathbb{R} \), such that \( F \) is differentiable at any point of \( W \), and the approximate functions

\[
F_w(x) := \max \{ F(v) + \langle \nabla F(v), x - v \rangle : v \in W, v \neq w \},
\]

\[
\overline{F}_w(x) := \text{conv} \{ F(v) + l\| x - v \| : v \in W, v \neq w \},
\]

satisfy \( F_w(w) < 0 \) and \( \overline{F}_w(w) > \frac{3}{2}l\theta \).

We next construct a class of deterministic DR-MCO problems using such convex functions, with the following parameters: \( T \geq 2 \) as the number of stages, \( L > 0 \) as an upper bound on Lipschitz constants, \( d \geq 3 \) as the state space dimension, \( D = 2r > 0 \) as the state space diameter, and \( \varepsilon > 0 \) as the target optimality gap. Set \( \varepsilon_t = \varepsilon/(T - 1) \). Construct sets of points \( W_t := \{ w_{t,k} \}_{k=1}^{K_t} \), where \( \theta_t = \varepsilon_t/L \) for \( t \in T \). Let \( F_t(x) = F_{\hat{W}_t}(x) \) be an \( L \)-Lipschitz continuous convex function on \( B^d_\theta(r) \) constructed in Lemma 4 with \( F_t(w_{t,k}) \in (\varepsilon_t/2, \varepsilon_t) \) for each \( w_{t,k} \in W_t \). And we denote \( F_t(x) \equiv 0.\)
Our DR-MCO problems for the complexity lower bound are defined by the following recursions for $t = T, \ldots, 2$

$$Q_t(x_t-1) = \min_{x_{t} \in \mathcal{B}^t(r)} \left\{ F_t(x_{t-1}) + Q_t(x_t) \right\},$$

(17)

where $Q_T(x) \equiv 0$, and the first stage problem is defined as $\min_{x_1 \in \mathcal{B}^1(r)} Q_1(x_1)$. We assume that the dual bounds satisfy $M_t \geq L$, $t \in T'$ for the exactness guaranteed by Proposition 4. To describe the asymptotic behavior of a lower bound, we use the $\Omega$-notation, i.e., for functions $a, b : \mathbb{Z} \to \mathbb{R}$, $a(t) = \Omega(b(t))$ if $\lim \inf_{t \to \infty} \frac{a(t)}{b(t)} > 0$.

**Theorem 3.** For the problem (17), there exist single stage subproblem oracles such that the number of evaluations $\#\text{Eval}$ for either of Algorithms 2 and 3 before termination has the following lower bound

$$\#\text{Eval} \geq \frac{d}{d-1} \sqrt{\frac{\pi}{2}} (d^2 - 4) \left( \frac{DL(T - 1)}{2 \varepsilon} \right)^{d/2 - 1} = \Omega(T^{d/2 - 1}) \text{ as } T \to \infty.$$

**Proof.** We assume that our SSSO at stage $t \in T$ returns points in $W_t$, $t \in K_t$ evaluations without repetition. More specifically, for $t \in T'$ and $i_t \leq K_t$, given $Q_t^{i_t}, \overline{Q}_t^{i_t}$ and $x_{t-1}^{i_t}$ at some $(i_{t-1}, i_t) \in J_t$, the SSSO returns:

- a state $x_t^{i_t} \in W_t$ with $x_t^{i_t} \neq x_t^{i_t'}$ for all $i_t' < i_t$;
- a linear cut $V_t^{i_t}(x) := V_t^{i_t}(x) + Q_t^{i_t}(x_t^{i_t})$;
- an overestimate $v_t^{i_t} := F_t(x_{t-1}^{i_t}) + \overline{Q}_t^{i_t}(x_t^{i_t})$.

where $V_t^{i_t}(x) := F_t(x_{t-1}^{i_t}) + \langle \nabla F_t(x_{t-1}^{i_t}), x - x_{t-1}^{i_t} \rangle$ for any $(i_{t-1}, i_t) \in J_t$. Thus our under- and over-approximations of $Q_t$ can be written as

$$Q_{i_t-1}^{i_t}(x) = \max \left\{ 0, \max \left\{ Q_t^{i_t}(x_t^{i_t}) + V_t^{i_t}(x) : (i_{t-1}, i_t') \in J_t, i_{t-1}' \leq i_{t-1} \right\} \right\},$$

$$\overline{Q}_{i_t-1}^{i_t}(x) = \text{conv}\left( Q_t^{i_t}(x_t^{i_t}) + F_t(x_{t-1}^{i_t}) + M_t \| x - x_{t-1}^{i_t} \| : (i_{t-1}, i_t') \in J_t, i_{t-1}' \leq i_{t-1} \right\}.$$

To see that the above assumption on the SSSO does not conflict with Definition 2, it is straightforward to check that the state $x_t^{i_t}$ is feasible, the overestimate $v_t^{i_t} = F_t(x_{t-1}^{i_t}) + \overline{Q}_t^{i_t}(x_t^{i_t}) \geq Q_t-1(x_{t-1}^{i_t})$, and the gap is controlled $v_t^{i_t} - V_t^{i_t}(x_{t-1}^{i_t}) = \overline{Q}_t^{i_t}(x_t^{i_t}) - Q_t^{i_t}(x_t^{i_t})$. For the validity of the linear cut $V_t^{i_t}$, note that by Lemma 4, $V_t^{i_t}(x) < 0$ for all $x \in W_t \setminus \{ x_{t-1}^{i_t} \}$. Thus by a recursive argument from $t = T$ to 2, we know that $Q_{i_t-1}^{i_t}(x) \equiv 0$ when $i_t \leq K_t$ for all $t \in T'$. Similarly, one can verify that $\overline{Q}_{i_t-1}(x_{t-1}^{i_t}) \geq \frac{3}{2} \sum_{t=2}^{T} \varepsilon_t'$. Therefore, $\text{UPPERBOUND} - \text{LOWERBOUND} = \overline{Q}_{i_t}^{i_t}(x_t^{i_t}) - Q_{i_t}^{i_t}(x_t^{i_t}) > \frac{3}{2} \sum_{t=2}^{T} \varepsilon_t > \varepsilon$. In other words, when the algorithms terminate, we must have $i_t \geq K_t$ for some $t \in T'$, which implies

$$\#\text{Eval} \geq K_t \geq \frac{(d - 1)^2 - 1}{d - 1} \sqrt{\frac{\pi}{2}} \frac{\Gamma((d - 1)/2 + 1)}{\Gamma((d - 1)/2 + 3/2)} \left( \frac{rL}{\varepsilon_t} \right)^{(d-2)/2} \geq \frac{d(d - 2)\sqrt{\pi} \Gamma(d/2 + 1/2)}{d - 1} \left( \frac{DL(T - 1)}{2 \varepsilon} \right)^{(d-2)/2} \geq \frac{d}{d-1} \sqrt{\frac{\pi}{2}} (d^2 - 4) \left( \frac{DL(T - 1)}{2 \varepsilon} \right)^{d/2 - 1}.$$

Here, the last inequality is due to Wendel’s bound on the ratio of two gamma functions [34]. \hfill \Box

**Remark 2.** The CDDP and NDDP algorithms (Algorithms 2 and 3) and their complexity analyses depend only on the SSSO (Definitions 1 and 2). Therefore, while we only discuss SSSO implementations for finitely supported DR-MCO in Section 3.1.1, the CDDP and NDDP algorithms work for more general DR-MCO problems. Moreover, the complexity upper bounds (Theorems 1 and 2, Corollary 1) and the lower bound (Theorem 3) remain valid for them as well.
4 Numerical Experiments

In this section, we numerically test the proposed CDDP and NDDP algorithms. The first test problem is a robust multi-commodity inventory problem with customer demand uncertainty. The second test problem is a distributionally robust hydro-thermal power planning problem with stochastic energy inflows. The computation budget consists of 10 3.0-GHz CPU cores and a total of 70 GBytes of RAM.

The algorithms are implemented using JuMP package ([13], v1.14) in Julia language (v1.9) with Gurobi 9.5 [20] as its underlying LP solver.

4.1 Multi-Commodity Inventory Problem

We consider an MRCO of multi-commodity inventory problem with uncertain customer demands and deterministic holding and backlogging costs, following the description in [17]. This problem can be described through the local cost functions $f_t$ and the uncertainty sets $\Xi_t$. Let $\mathcal{J} := \{1, 2, \ldots, J\}$ denote the set of product indices. We first describe the variables in each stage $t \in \mathcal{T}$. We use $x_{t,j}^l$ to denote the inventory level, $y_{t,j}^e$ (resp. $x_{t,j}^b$) to denote the amount of express (resp. standard) order fulfilled in the current (resp. subsequent) stage, of some product $j \in \mathcal{J}$. Let $\xi_t \in \Xi_t$ denote the uncertainty vector controlling the customer demands in stage $t$. The state variables consist of the collection $x_t := (x_{t,1}, \ldots, x_{t,j}, x_{t,j+1}, \ldots, x_{t,J})$, while the internal variables are simply the express order amounts $y_t = (y_{t,1}^e, \ldots, y_{t,j}^e)$. In stage $t \in \mathcal{T}$, the local cost function $f_t$ and the state space $\mathcal{X}_t$ are defined by

$$f_t(x_{t-1}, x_t; \xi_t) := \min_{y_t} C^F + \sum_{j \in \mathcal{J}} (C_j^{a} y_{t,j}^a + C_j^{b} x_{t,j}^b + C_j^{H} [x_{t,j}^l]^+ + C_j^{B} [x_{t,j}^l]^-)$$

subject to

$$y_{t,j}^a \leq B^e,$$

$$x_{t,j}^l - y_{t,j}^a - x_{t-1,j}^l = x_{t,j}^l - D_{t,j}(\xi_t), \quad \forall j \in \mathcal{J},$$

$$y_{t,j}^a \in [0, B^a_j], \quad \forall j \in \mathcal{J}.$$ (18)

and

$$\mathcal{X}_t := \{ x_t \in \mathbb{R}^{2J} : 0 \leq x_{t,j}^l \leq B^b_j, -B^l_j \leq x_{t,j}^l \leq B^l_j, \forall j \in \mathcal{J} \}.$$ (19)

Here in the formulation, $C_j^a$ (resp. $C_j^b$) denotes the express (resp. standard) order unit cost, $C_j^H$ (resp. $C_j^B$) the inventory holding (resp. backlogging) unit cost, $B^a_j$ (resp. $B^b_j$) the bound on the express (resp. standard) order, and $B^l_j$ the inventory level bound, for the product $j \in \mathcal{J}$, respectively. The first constraint in (18) is a cumulative bound $B^e$ on the express orders, the second constraint characterizes the change in the inventory level, and the last of (18) and the state space (19) represent bounds on the decision variables with respect to each product. We also put $C^F > 0$ as a fixed cost to ensure the cost function is strictly positive (cf. Corollary 1). We use $[x]^+ := \max\{x, 0\}$ and $[x]^- := -\min\{0, x\}$ to denote the positive and negative part of a real number $x$. The initial state $x_0$ is given by $x_{0,j}^l = x_{0,j}^b = 0$ for all $j \in \mathcal{J}$. The uncertainty set $\Xi_t$ is an $E$-dimensional box $[-1, 1]^E$, and the customer demand is predicted by the following factor model:

$$D_{t,j}(\xi_t) = \begin{cases} 
2 + \sin \left(\frac{(t-1)\pi}{5}\right) + \Phi_{t,j}^T \xi_t, & j \leq J/2, \\
2 + \cos \left(\frac{(t-1)\pi}{5}\right) + \Phi_{t,j}^T \xi_t, & j > J/2,
\end{cases}$$

(20)

where $\Phi_{t,j}$ is an $E$-dimensional vector where each entry is chosen uniformly at random from $[-1/E, 1/E]$. Thus the value $\Phi_{t,j}^T \xi_t \in [-1, 1]$ and $D_{t,j}(\xi_t) \geq 0$ for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$.

For the following numerical test, we set the number of products $J = 5$, the number of uncertainty factors $E = 4$, $B^a_j = B^b_j = B^l_j = 10$, $C^a_j = 1$ for all $j \in \mathcal{J}$, $C^F = 1$, and $B^e = 0.3J = 1.5$. The costs are
generated uniformly at random within $C^g_j \in [1, 3]$, $C^H_j, c^B_j \in [0, 2]$ for all $j \in J$. Due to lack of RCR of the problem (18), we first test both CDDP (Algorithm 2) and NDDP (Algorithm 3) with dual bounds $M_t = 1.0 \times 10^2$ for all $t \in T$. The optimality gap set to be relative $\alpha = 1\%$ and approximation gaps set dynamically by $\delta^t_i = \text{LOWERBOUND} \cdot \alpha (T - t) / (T - 1)$ for $t \in T$. Over 3 independently generated test cases, we plot the growth of computational time and number of oracle evaluations with respect to the number of stages in Fig. 2, which shows that NDDP performs increasingly better than CDDP as the number of stages increases. In particular, NDDP can save up to 47.8% computation time and up to 44.2% subproblem oracle evaluations comparing to CDDP for these 40-stage problems.

As a comparison, we implement the NDDP algorithm without dual bounds. Instead, it generates linear feasibility cuts for approximation of the feasible sets (see definition of feasibility cuts in, e.g., [19]). We limit the computational budget to $2000 \cdot T$ subproblem oracle calls. For 5 independently generated test cases, we have obtained the following results (Table 1).

Table 1: Comparison of NDDP Performances with or without dual bounds

| Stage | With Dual Bounds | Without Dual Bounds |
|-------|------------------|---------------------|
|       | LB   | UB   | Time (s) | #Eval | LB   | UB   | Time (s) | #Eval |
| 10    | 154.70 | 156.02 | 30.61 | 1554 | 154.70 | 155.30 | 33.34 | 1497 |
| 15    | 232.46 | 233.85 | 63.84 | 2776 | 232.13 | inf | 440.07 | 30037 |
| 20    | 291.78 | 293.94 | 196.57 | 6637 | 286.88 | inf | 615.68 | 40054 |
| 25    | 369.62 | 372.54 | 257.33 | 8984 | 369.66 | 371.10 | 102.86 | 7334 |
| 30    | 428.81 | 432.41 | 545.98 | 15212 | 429.11 | 429.69 | 165.55 | 10217 |
|       | 429.34 | 432.08 | 414.91 | 12639 | 429.77 | 430.27 | 141.67 | 9549 |
|       | 383.74 | 387.01 | 72.00 | 11820 | 384.22 | 384.22 | 73.06 | 11368 |
|       | 386.54 | 390.41 | 277.57 | 9430 | 386.56 | 386.58 | 151.97 | 9824 |
|       | 379.46 | 382.84 | 649.69 | 16730 | 379.27 | 379.27 | 1230.98 | 60016 |

In Table 1, the inf indicates values of infinity or numerically infinity values (i.e., values greater than $10^8$) upon termination. As we see from the table, the NDDP algorithm together with feasibility cuts fails to solve 2 out of 5 cases even when there are only 15 stages, showing the instability of the performance of feasibility cuts. In contrast, the algorithm with the dual-bounding technique solves all of the cases within a reasonable computation time and number of subproblem oracle evaluations, without any optimality gap on those cases that both formulations are able to solve. This demonstrates the ability of the our proposed DDP algorithms handling problems without RCR. It is worth mentioning that for cases where the NDDP algorithm converges without dual bounds, the computation time used is usually smaller than the regularized problem, which can be explained by better numerical conditions.
Figure 2: Growth of computation expenses for CDDP and NDDP
of feasibility cuts and their effect on reducing the effective volumes of the state space.

4.2 Hydro-Thermal Power Planning Problem

We next consider the Brazilian interconnected power system described in [12]. By assuming the stagewise independence and taking sample average approximation in the underlying stochastic energy inflow, we formulate the problem below as a finitely supported DR-MCO (1). Let $\mathcal{J} = \{1, \ldots, J\}$ denote the indices of regions in the system ($J = 4$ in our data), and $\mathcal{L} = \bigcup_{j \in \mathcal{J}} \mathcal{L}_j$ the indices of thermal power plants, where each of the disjoint subsets $\mathcal{L}_j$ is associated with the region $j \in \mathcal{J}$. We first describe the decision variables in each stage $t \in \mathcal{T}$. Let $\Xi_t = \{\xi_{t,1}, \ldots, \xi_{t,n_t}\}$ denote the finite uncertainty set of energy inflow in stage $t \in \mathcal{T}$. We use $x^t_{i,j}$ to denote the stored energy level, $y^h_{t,j}$ to denote the hydro power generation, and $y^s_{t,j}$ to denote the energy spillage, of some region $j \in \mathcal{J}$; and $y^q_{t,l}$ to denote the thermal power generation for some thermal power plant $l \in \mathcal{L}$. For two different regions $j \neq j' \in \mathcal{J}$, we use $y^e_{t,j,j'}$ to denote the energy exchange from region $j$ to region $j'$, and $y^e_{t,j,j',j''}$ to denote the deficit account for region $j$ in region $j'$. Let $x_t := (x^t_{i,1}, \ldots, x^t_{i,j})$ be the state variables and $y_t$ denote the internal variables consisting of $y^h_{t,j}, y^s_{t,j}, y^q_{t,l}, y^e_{t,j,j'},$ and $y^e_{t,j,j',j''}$ for any $j, j' \in \mathcal{J}$ and $l \in \mathcal{L}$. Then in stage $t \in \mathcal{T}$, the state space is simply $X_t := \prod_{l=1}^t [0, B^t_l]$, and the local cost function $f_t$ can be defined as

$$f_t(x_{t-1}, x_t; \xi_t) := \min_{y_t} \sum_{j \in \mathcal{J}} \left( C^h_{t,j} y^h_{t,j} + \sum_{l \in \mathcal{L}_j} C^q_{t,l} y^q_{t,l} + \sum_{j' \in \mathcal{J}} \left( C^{e}_{t,j,j'} y^e_{t,j,j'} + C^{s}_{t,j,j'} y^s_{t,j,j'} \right) \right)$$

s.t. $x^t_{i,j} + y^h_{t,j} + y^s_{t,j} = x^t_{i-1,j} + \xi_{t,j}$, $\forall j \in \mathcal{J}$,

$y^h_{t,j} + \sum_{l \in \mathcal{L}_j} y^q_{t,l} + \sum_{j' \in \mathcal{J}} (y^e_{t,j,j'} - y^e_{t,j',j'} + y^e_{t,j',j''}) = D_{t,j}$, $\forall j \in \mathcal{J}$,

$y^h_{t,j} \in [0, B^h_t]$, $\forall j \in \mathcal{J}$,

$y^q_{t,l} \in [B^{q-}_l, B^{q+}_l]$, $\forall l \in \mathcal{L}$,

$y^e_{t,j,j',j''} \in [0, B^{e}_j]$, $y^e_{t,j,j'} \in [0, B^{e}_{j'}]$, $\forall j, j', j'' \in \mathcal{J}$.

Here in the formulation, $C^h_t$ denotes the unit penalty on energy spillage, $C^q_t$ the unit cost of thermal power generation of plant $l$, $C^{e}_{t,j,j'}$ the unit cost of power exchange from region $j$ to region $j'$, $C^{s}_{t,j,j'}$ the unit cost on the energy deficit account for region $j$ in region $j'$, $D_{t,j}$ the deterministic power demand in stage $t$ and region $j$, $B^h_t$ the bound on the storage level in region $j$, $B^{q-}_t, B^{q+}_t$ the lower and upper bounds of thermal power generation in plant $l$, $B^{e}_{j'}$ the bound on the deficit account for region $j$ in region $j'$, and $B^{e}_{j',j''}$ the bound on the energy exchange from region $j$ to region $j'$. The first constraint in (21) characterizes the change of energy storage levels in each region $j$; the second constraint imposes the power generation-demand balance for each region $j$, and the rest are bounds on the decision variables. The initial state $x_0$ and $\xi_1$ are given by data.

The energy inflow outcomes are sampled from multivariate lognormal distributions that are inter-stage independent. Then the distributional ambiguity set is constructed using Wasserstein metric to reduce the effect of overtraining with the sampled outcome, according to [14]. To be precise, suppose $\hat{p}_t = (1/n_t, \ldots, 1/n_t) \in \Delta^{n_t}$ is an empirical distribution of outcomes $\Xi_t = \{\xi_{t,1}, \ldots, \xi_{t,n_t}\}$. Then, the distributional ambiguity set $\mathcal{P}_t$ is described by

$$\mathcal{P}_t := \{p_t \in \Delta^{n_t} : W_t(p_t, \hat{p}_t) \leq \rho_t\},$$

for some radius $\rho_t \geq 0$, where the Wasserstein metric $W_t$ for finitely supported distributions is defined
by

\[ W_t(p_t, \hat{p}_t) := \min_{\pi_{k,k'} \geq 0} \sum_{k,k'=1}^{n_t} \| \hat{\xi}_{t,k} - \hat{\xi}_{t,k'} \| \pi_{k,k'} \]  \hspace{1cm} (23)

s.t. \[ \sum_{k=1}^{n_t} \pi_{k,k'} = p_{t,k}, \quad \forall k' = 1, \ldots, n_t, \]
\[ \sum_{k'=1}^{n_t} \pi_{k,k'} = \hat{p}_{t,k} = \frac{1}{n_t}, \quad \forall k = 1, \ldots, n_t. \]

Note when the radius \( \rho_t = 0 \), the ambiguity set \( \mathcal{P}_t \) becomes a singleton. In our numerical tests, we choose the radius to be relative to the total distances, i.e., \( \rho_t = \beta \cdot \sum_{k,k'=1}^{n_t} \| \hat{\xi}_{t,k} - \hat{\xi}_{t,k'} \| \) for some \( \beta \geq 0 \).

At the same time, we use uniform dual bounds for the tests, i.e., \( M_t = M > 0 \) for all \( t \in \mathcal{T}' \). When the relative optimality gap \( \alpha \) is smaller than the threshold 5\%, we check whether all the active cuts in the recent iterations are strictly smaller than the regularization factor. If they are, then the algorithm is terminated, and otherwise the dual bound \( M \) is increased by a factor of \( \sqrt{10} \approx 3.1623 \) with all the over-approximations reset to \( Q_i(x) \leftarrow +\infty, t \in \mathcal{T} \). Five scenarios are sampled independently in each stage for the nominal problem \( n_t = 5 \) before the distributional robust counterpart is constructed by (22).

For the 24-stage problem that we consider, the samples already give a total \( 5^{24} \approx 5 \times 10^{16} \) scenario paths, which is already impossible to solve via an extensive formulation. We have then obtained the following results (Table 2).

In Table 2, the lower bound (LB), the upper bound (UB) at termination, the computation time (Med. Time) and the number of subproblem oracles (Med. #Eval) shown are the median of the five test cases. The logarithmic regularization factors \( \log_{10}(M) \) listed in the table correspond to the initial regularization factors. We see that for different choices of the relative radii \( \beta \), the median computation time and number of subproblem oracle evaluations are usually smaller when \( \log_{10}(M) = 3 \), without compromising the quality of upper and lower bounds. This can be explained by the better numerical conditions for the smaller regularization factors (the cuts have smaller Lipschitz constants), leading to shorter subproblem oracle evaluation times (cf. Algorithm 1). We thus conclude that the regularization technique could lead to smaller number of subproblem oracle evaluations, as well as shorter computation time for a given DR-MCO problem.

### 5 Concluding Remarks

In this work, we extend CDDP algorithms to a broad class of DR-MCO problems, and propose a novel NDDP algorithm that decides its stage exploration strategy dynamically based on the cost-to-go function approximation qualities. We enforce bounds on the DDP recursion dual variables to effectively control the growth of Lipschitz constants in the approximation. We provide a comprehensive complexity analysis of both CDDP and NDDP algorithms, proving both upper complexity bounds and a matching lower bound, which reveal, in a precise way, the dependence of the complexity of the DDP-type algorithms on the number of stages, the dimension of the decision space, and various regularity characteristics of DR-MCO. In particular, our NDDP algorithm scales linearly with \( T \), assuming that the target optimality gap scales with \( T \). This is the first complexity analysis of DDP-type algorithms in such a general setting, and we believe it provides key insights for further developing efficient computational tools for the very many applications of sequential decision making under uncertainty.

We also provide numerical examples to show the capability of the DDP-type algorithms method to solve problems without RCR, and reduction in computation time and number of subproblem oracle evaluations, due to the dual-bounding technique.
Table 2: CDDP on hydro-thermal power planning with different radii

| $\beta$ | $\log_{10}(M)$ | LB ($10^\alpha$) | UB ($10^\alpha$) | Med. Time (s) | Med. #Eval |
|---------|----------------|------------------|------------------|--------------|------------|
| 0.00    | 2              | 4.60             | 4.84             | 784.45       | 33075      |
|         | 3              | 4.59             | 4.83             | 301.06       | 15345      |
|         | 4              | 4.58             | 4.82             | 374.88       | 17505      |
|         | 5              | 4.58             | 4.82             | 378.80       | 17595      |
|         | 6              | 4.58             | 4.82             | 386.86       | 17910      |
| 0.02    | 2              | 4.84             | 5.09             | 439.28       | 22095      |
|         | 3              | 4.84             | 5.09             | 263.26       | 13725      |
|         | 4              | 4.82             | 5.07             | 365.72       | 17100      |
|         | 5              | 4.82             | 5.07             | 358.08       | 16830      |
|         | 6              | 4.82             | 5.07             | 405.42       | 18090      |
| 0.04    | 2              | 5.10             | 5.36             | 921.99       | 36720      |
|         | 3              | 5.06             | 5.32             | 237.34       | 13095      |
|         | 4              | 5.06             | 5.33             | 327.38       | 15975      |
|         | 5              | 5.06             | 5.33             | 325.62       | 15930      |
|         | 6              | 5.06             | 5.33             | 350.65       | 16650      |
| 0.06    | 2              | 5.34             | 5.62             | 606.61       | 27810      |
|         | 3              | 5.32             | 5.60             | 284.40       | 14445      |
|         | 4              | 5.31             | 5.59             | 277.21       | 14310      |
|         | 5              | 5.30             | 5.58             | 271.26       | 14130      |
|         | 6              | 5.31             | 5.59             | 285.30       | 14445      |
| 0.08    | 2              | 5.57             | 5.86             | 519.87       | 25380      |
|         | 3              | 5.55             | 5.84             | 204.36       | 11790      |
|         | 4              | 5.54             | 5.83             | 290.96       | 14715      |
|         | 5              | 5.54             | 5.83             | 223.92       | 12510      |
|         | 6              | 5.54             | 5.83             | 257.26       | 13500      |
| 0.10    | 2              | 5.81             | 6.12             | 446.94       | 22140      |
|         | 3              | 5.78             | 6.08             | 194.85       | 10980      |
|         | 4              | 5.78             | 6.09             | 262.42       | 12915      |
|         | 5              | 5.78             | 6.09             | 215.10       | 11790      |
|         | 6              | 5.79             | 6.09             | 237.22       | 12420      |
References

[1] S Ahmed, F G Cabral, and B F Paulo da Costa. Stochastic lipschitz dynamic programming, 2019.
[2] E J Anderson and A B Philpott. Improving sample average approximation using distributional robustness. Optimization Online, 2019.
[3] R Baucke, A Downward, and G Zakeri. A deterministic algorithm for solving multistage stochastic programming problems. Optimization Online, 2017.
[4] R Baucke, A Downward, and G Zakeri. A deterministic algorithm for solving stochastic minimax dynamic programmes. Optimization Online, 2018.
[5] Heinz H Bauschke, Patrick I. Combettes, et al. Convex analysis and monotone operator theory in Hilbert spaces, volume 408. Springer, 2011.
[6] A Ben-Tal, L El Ghaoui, and A Nemirovski. Robust optimization, volume 28. Princeton University Press, 2009.
[7] D Bertsimas and A Georghiou. Design of near optimal decision rules in multistage adaptive mixed-integer optimization. Operations Research, 63(3):610–627, 2015.
[8] John R. Birge. Decomposition and Partitioning Methods for Multistage Stochastic Linear Programs. Operations Research, 33(5):989–1007, October 1985. ISSN 0030-364X, 1526-5463. doi: 10.1287/opre.33.5.989.
[9] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.
[10] Z L Chen and W B Powell. Convergent Cutting-Plane and Partial-Sampling Algorithm for Multistage Stochastic Linear Programs with Recourse. Journal of Optimization Theory and Applications, 102(3):497–524, September 1999. ISSN 0022-3239, 1573-2878. doi: 10.1023/A:1022641805263.
[11] Vitor L. de Matos, Andy B. Philpott, and Erlon C. Finardi. Improving the performance of Stochastic Dual Dynamic Programming. Journal of Computational and Applied Mathematics, 290:196–208, December 2015. ISSN 03770427. doi: 10.1016/j.cam.2015.04.048.
[12] L Ding, S Ahmed, and A Shapiro. A python package for multi-stage stochastic programming. Optimization Online, pages 1–41, 2019.
[13] I Dunning, J Huchette, and M Lubin. Jump: A modeling language for mathematical optimization. SIAM Review, 59(2):295–320, 2017. doi: 10.1137/15M1020575.
[14] D Duque and D P Morton. Distributionally robust stochastic dual dynamic programming. Optimization Online, 2019.
[15] Martin Dyer and Leen Stougie. Computational complexity of stochastic programming problems. Mathematical Programming, 106(3):423–432, May 2006. ISSN 0025-5610, 1436-4646. doi: 10.1007/s10107-006-0597-0.
[16] B.C. Flach, L.A. Barroso, and M.V.F. Pereira. Long-term optimal allocation of hydro generation for a price-maker company in a competitive market: Latest developments and a stochastic dual dynamic programming approach. IET Generation, Transmission & Distribution, 4(2):299, 2010. ISSN 17518687. doi: 10.1049/iet-gtd.2009.0107.
[17] A Georghiou, A Tsoukalas, and W Wiesemann. Robust dual dynamic programming. Operations Research, 67(3):813–830, 2019.
[18] P. Girardeau, V. Leclere, and A. B. Philpott. On the Convergence of Decomposition Methods for Multi-stage Stochastic Convex Programs. Mathematics of Operations Research, 40(1):130–145, February 2015. ISSN 0364-765X, 1526-5471. doi: 10.1287/moor.2014.0664.
[19] A Grothey, S Leyffer, and KIM McKinnon. A note on feasibility in benders decomposition. Numerical Analysis Report NA/188, Dundee University, 1999.
[20] Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual, 2023. URL https://www.gurobi.com.
[21] M J Hadjiyiannis, P J Goulart, and D Kuhn. A scenario approach for estimating the suboptimality of linear decision rules in two-stage robust optimization. In 2011 50th IEEE Conference on Decision and Control and European Control Conference, pages 7386–7391. IEEE, 2011.
[22] J Huang, K Zhou, and Y Guan. A Study of Distributionally Robust Multistage Stochastic Optimization. arXiv:1708.07930 [math], August 2017.
