Spacetime Structures and Physical Theories

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Abstract

General relativity is applied to the strong interaction; the nexus between the two being arrived at by constructing a line element having the Yukawa form, which is used to describe geometrically the classical dynamics of a particle moving under the influence of the short-range strong interaction. It is shown that, with reasonable assumptions, the theory of general relativity can be made compatible with quantum mechanics by using the general relativistic field equations to construct a Robertson-Walker metric for a quantum particle. The resulting line element of the particle can be transformed entirely to that of a Minkowski spacetime, and the spacetime dynamics of the particle described by a Minkowski observer takes the form of quantum mechanics. It is also discussed the physical aspects of the affine connection in general relativity and its relationship with the field strength of the electromagnetic field and strong interaction. A heuristic geometric formulation of the electromagnetic field as an independent spacetime structure is presented.
1 Introduction

The geometrical character of physical reality has been discussed since the time of the Greek philosophers. However, a serious geometrical formulation of physical theories only started with the advent of the general theory of relativity, formulated by Einstein earlier this century [1]. Despite the fact that it is a relativistic generalisation of Newton’s theory of gravitation, general relativity is a theory about curved spacetime structures in which gravity is described geometrically as a manifestation of the curvature of a Riemannian spacetime manifold. The dynamical description of physics then becomes force-free in the sense that spacetime is also considered as a dynamical entity, whose metrical structure is determined by matter fields through a system of nonlinear field equations. The concept of geometrisation of physics has led to successful theories in fundamental interactions in contemporary physics, especially within the framework of gauge theories of elementary particles. In this work we first consider the possibility of formulating the strong interaction in terms of general relativity, focussing attention on the case where the strong force between nucleons is described by the Yukawa potential. Although general relativity was developed to describe the gravitational field, its general formulation allows the formalism to be applied to other physical fields, as long as the requirement of mathematical consistency is satisfied. We apply the formalism of general relativity to the ‘strong’ interaction whose force carriers are the Yukawa virtual pions. In this situation the strong interaction is attractive and charge-independent, although the matter that produces the attractive nucleonic force may not have the same characteristics as the matter that produces the gravitational field. It should be emphasised that gravitating matter cannot be specified self-consistently (i.e., geometrically) within general relativity, but can only be incorporated into the theory through the energy-momentum tensor. General relativity may thus be viewed as having the status of Newton’s second law in classical mechanics, which is regarded as a definition of force in terms of mass and acceleration. This conceptual framework then allows for the expression of physical laws, such as Newton’s law of gravitation and the Lorentz force law of electromagnetism, to be formulated [2, 3]. It is noted that the energy-momentum tensor used to define the field equations of general relativity contains only the general concept of force, and so it does not specify a priori any particular physical law in nature. We verify the applicability of the formalism of general relativity to the strong interaction by showing that the field equations of general relativity admit a line element that contains the Yukawa potential of the strong force as an exact solution. This result reveals a possible relationship between the strong interaction described in terms the Yukawa potential and the mathematical formalism of general relativity.

The most interesting situation arises when we apply the general relativistic formulation to quantum particles. When quantum particles are described as spacetime structures, it is possible to make general relativity compatible with quantum mechanics by constructing a spacetime structure for a quantum particle, so that the dynamics of the spacetime structure when viewed by a Minkowski observer will produce quantum mechanics. In this case, space and time of the observer are directly connected to, and determined by, the structure of space and time of the quantum particle. The spacetime structure is determined by the particle’s own characteristics, such as energy density, and it is not possible for an observer to specify in a deterministic way the physical observables of the particle in the Minkowski spacetime. Furthermore, we show that the geometrical and topological structures of a quantum particle may be entirely different from that normally considered within the present framework of quantum physics. For example, quantum particles may exist as timeless four-dimensional objects.
Such speculations on quantum particles as hyperspheres have been discussed previously from a philosophical viewpoint (see, e.g., [8, 9]).

2 General relativity and the strong interaction

2.1 Introductory remarks

In this section we discuss a relationship between general relativity and the strong interaction\(^1\) which is assumed to be described by the Yukawa potential. We consider this simple model because the form of the Yukawa potential can be incorporated into a line element that satisfies the field equations of general relativity. Since the strong interaction has been investigated almost entirely within the context of a quantum theory, e.g., quantum chromodynamics [6], problems associated with the classical dynamics of a particle under the influence of the strong force and the concomitant classical laws of the strong force have not received much attention. Obviously, a fundamental question is whether the strong interaction also has a classical analogue? In classical physics, however, since a particle is characterised only by its mass and charge, the dynamics of a particle is assumed to follow either the laws of classical electrodynamics or the laws of general relativity. Therefore, if the strong force does not have any kind of relationship with the charge of a particle, then it is conjectured that the dynamics of a particle under the influence of the strong force would also follow the laws of general relativity. This suggestion is based on the fact that the strong interaction is attractive and charge-independent; most importantly, the formalism of general relativity does not specify \textit{a priori} the nature of matter that produces the effect of attraction. Furthermore, the role of physical principles, such as the principle of equivalence and the principle of covariance, incorporated into the general relativistic formulation of the gravitational field can be argued to be inessential (see, e.g., [5]). Although the covariance principle does make assertions about the mathematical formulation of the theory, it does not have a direct physical significance in the sense that it is possible to formulate covariantly any physical theory [4]. The status of the equivalence principle has also been subject to controversy. The problem here is related to whether it is physically significant to specify a coordinate transformation to a local reference frame so that the gravitational field is be eliminated [7]. In the following, we assume the applicability of the general relativistic formulation to the strong interaction, and investigate the consequences that emerge from the resulting formulation. Similar to the Newtonian gravitational field, that forms a line element satisfying the field equations of general relativity, we can construct a line element that contains an appropriate potential characterising the strong force, and which satisfies the general relativistic field equations. The relevant potential that we consider is the Yukawa potential. This simple assumption and concomitant formulation, therefore, does not take into account the quark structure of the nucleons as in quantum chromodynamics (see, e.g., [11]). However, this approach reveals a possible way to make general relativity compatible with quantum mechanics, within the formulation of general relativity itself.

\(^1\)By the strong interaction we are referring to the nucleon-nucleon interaction mediating by Yukawa bosons (i.e., pions).
2.2 A line element of the Yukawa potential

In order to see whether the strong force can be describes by the laws of general relativity, it is first necessary to consider whether the Yukawa potential can form a line element that satisfies the field equations of general relativity, \( R_{\mu\nu} \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \). Assuming a centrally symmetric field, the spacetime metric can be written as

\[
ds^2 = e^\mu c^2 dt^2 - e^\nu dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\] (1)

With this line element, the vacuum solutions satisfy the system of equations

\[
\frac{\partial \mu}{\partial r} + \frac{1}{r} \frac{e^\nu}{r} = 0, \tag{2}
\]

\[
\frac{\partial \nu}{\partial r} - \frac{1}{r} \frac{e^\nu}{r} = 0, \tag{3}
\]

\[
\frac{\partial \nu}{\partial t} = 0, \tag{4}
\]

\[
2 \frac{\partial^2 \mu}{\partial r^2} + \left( \frac{\partial \mu}{\partial r} \right)^2 + \frac{2}{r} \left( \frac{\partial \mu}{\partial r} - \frac{\partial \nu}{\partial r} \right) - \frac{\partial \mu}{\partial r} \frac{\partial \nu}{\partial r} - e^{\nu-\mu} \left[ 2 \frac{\partial^2 \nu}{\partial t^2} + \left( \frac{\partial \nu}{\partial t} \right)^2 - \frac{\partial \nu}{\partial t} \frac{\partial \mu}{\partial t} \right] = 0. \tag{5}
\]

These equations are not independent, since it can be verified that the last equation follows from the first three equations. Furthermore, the first two equations give

\[
\frac{\partial \nu}{\partial r} + \frac{\partial \mu}{\partial r} = 0,
\]

which leads to \( \nu + \mu = 0 \), due to the possibility of an arbitrary transformation of the time coordinate.

If we assume a line element in the form of Yukawa potential, then it can be written in the following simple form (see, e.g., [12])

\[
e^{-\nu} = 1 - \frac{\alpha}{r} e^{-\beta r}, \tag{6}
\]

where the quantity \( \alpha \) plays the role of the charge of a particle in electromagnetism, and the quantity \( \beta = 1/R \), with \( R = \hbar/mc \), specifies the range of the strong force (which is assumed to be described by the Yukawa potential). The quantity \( m \) is the rest mass of Yukawa quanta, (i.e., the virtual pions), whose continuous transfer between two nucleons is assumed to give rise to the strong interaction. Since the quantity \( \nu \) is now time-independent, by differentiating Eq.(6), it is found that

\[
\frac{d\nu}{dr} = -\alpha \frac{e^{-\beta r}}{r} \frac{1 + \beta r}{r - \alpha e^{-\beta r}}. \tag{7}
\]

On the other hand, the quantity \( d\nu/dr \) that follows from equations (3) and (6) implies that

\[
\frac{d\nu}{dr} = -\alpha \frac{e^{-\beta r}}{r} \frac{1}{r - \alpha e^{-\beta r}}. \tag{8}
\]

Equation (7) reduces to equation (8) if the condition \( \beta r \ll 1 \), or \( r \ll R \), is satisfied. In general, the quantity \( R \) specifies a range, so that the line element of the Yukawa form (6) can be approximated as a solution to the field equations of general relativity for the region \( r \ll R \). It is seen that for the maximal possible range, where \( R \to \infty \), the metric of the Yukawa form reduces to the familiar Schwarzschild metric, which is used to describe a spherically symmetric gravitational field. In the case of short range nuclear forces, the quantity \( R \) can be assigned a
value in terms of the fundamental constants $\hbar$ and $c$, and the rest mass of the Yukawa quanta. Consequently, for the short range of the strong force, the field equations of general relativity admit a line element that takes the form of a Yukawa potential. This leads to the conclusion that by specifying an appropriate matter source that characterises the strong interaction, it is possible to consider the strong interaction as a manifestation of general relativity at short range.

The strong interaction, however, has massive exchange quanta, unlike the assumed massless force carriers of the gravitational field. The masslessness of the latter quanta is consistent with the vacuum solution to the field equations of general relativity. Therefore, any solutions to the field equations of general relativity that are used to describe the strong field should be non-vacuum solutions. In order to find an appropriate solution to describe the strong force, an energy-momentum tensor must be specified. Obviously, at present nuclear physics does not allow us to specify a precise form for the strong energy-momentum tensor. In this situation it is appropriate to construct an energy-momentum tensor for the strong force, so that it not only gives rise to the desired metric of the Yukawa form as an exact solution to the field equations of general relativity, but also satisfies the conservation law $T_{\mu,\nu} = 0$. In what follows we discuss a particular form of the strong energy-momentum tensor, which admits a line element of the Yukawa potential as an exact solution to the field equations of general relativity. We consider a strong energy-momentum tensor of the form

$$T_{\mu} = \begin{pmatrix}
-\frac{\alpha \beta e^{-\beta r}}{r^2} & 0 & 0 & 0 \\
0 & -\frac{\alpha \beta e^{-\beta r}}{r^2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha \beta^2 e^{-\beta r}}{2r}
\end{pmatrix}$$

(9)

With this energy momentum tensor, the field equations of general relativity reduce to the system of equations

$$e^{-\nu} \left( \frac{\partial \nu}{\partial r} - \frac{1}{r} \right) + \frac{1}{r} = -\frac{\alpha \beta e^{-\beta r}}{r},$$

(10)

$$-e^{-\nu} \left( \frac{\partial \mu}{\partial r} + \frac{1}{r} \right) + \frac{1}{r} = -\frac{\alpha \beta e^{-\beta r}}{r},$$

(11)

$$\frac{\partial \nu}{\partial t} = 0,$$

(12)

$$-e^{-\nu} \left[ 2 \frac{\partial^2 \mu}{\partial r^2} + \left( \frac{\partial \mu}{\partial r} \right)^2 + \frac{2}{r} \left( \frac{\partial \mu}{\partial r} - \frac{\partial \nu}{\partial r} \right) - \frac{\partial \mu \partial \nu}{\partial r \partial r} \right] + e^{-\mu} \left[ 2 \frac{\partial^2 \nu}{\partial t^2} + \left( \frac{\partial \nu}{\partial t} \right)^2 - \frac{\partial \nu \partial \mu}{\partial t \partial t} \right] = \frac{\alpha \beta^2 e^{-\beta r}}{r}.$$  

(13)

As in the case of vacuum solutions, it can be verified that the first two equations give $\partial \nu/\partial r + \partial \mu/\partial r = 0$, which leads to $\nu + \mu = 0$; while equation (13) follows from the first three equations. However, this system of equations when integrated gives a metric of the following form

$$e^{-\nu} = 1 - \alpha e^{-\beta r} + \frac{Q}{r},$$

(14)

where $Q$ is a constant of integration. The term $Q/r$ can be interpreted as a Coulomb repulsive force which arises for charged particles, such as two protons. This term, however, may be set to zero for strong interactions that involve neutral particles, such as two neutrons.
The form of the energy-momentum tensor (9) is mathematically valid since it can be shown to satisfy the conservation law

\[ \nabla_{\nu} T_{\mu}^{\nu} = \frac{1}{\sqrt{-g}} \partial T_{\mu}^{\nu} \sqrt{-g} - \frac{1}{2} g^{\lambda\sigma} T_{\lambda\sigma} = 0. \]  

(15)

An important feature emerges from the above model that relates to the nature of the quantity \( \alpha \) in the line element and the energy momentum tensor. That is, since the quantities \( \kappa \) and \( \beta \) are positive, the energy component \( T_{00} \) and the quantity \( \alpha \) always have opposite signs. Therefore, since \( g^{00} \) is positive, if the energy component \( T_{00} \) is considered to be positive, then the energy component \( T_{00} = g^{00} T_{00} \) must also be positive, and in this case, the quantity \( \alpha \) must be negative. Since the quantity \( \alpha \) in the line element should be defined in terms of a matter source, the matter source that produces the strong interaction is negative if the quantity \( \alpha \) is negative. This property of matter will be discussed further later. Furthermore, it is also noted from the metric of the Yukawa potential that if the quantity \( \alpha \) is negative, then there would be no Schwarzschild-like singularity when the constant of integration \( Q \) is set to zero.

2.3 Classical dynamics in the Yukawa strong force field

If we assume that the motion of a particle in the Yukawa strong force field is also governed by the geodesic equation

\[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0, \]  

(16)

then using the metric with the Yukawa potential, i.e.,

\[ ds^2 = \left( 1 - \alpha e^{-\beta r} \right) c^2 dt^2 - \left( 1 - \alpha e^{-\beta r} \right)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]  

(17)

the equations for the geodesics can be written explicitly as

\[ \left( 1 - \alpha e^{-\beta r} \right)^{-1} \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\theta}{d\tau} \right)^2 + r^2 \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 - c^2 \left( 1 - \alpha e^{-\beta r} \right) \left( \frac{dt}{d\tau} \right)^2 = -c^2, \]  

(18)

\[ \frac{d}{d\tau} \left( r^2 \frac{d\theta}{d\tau} \right) - r^2 \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \]  

(19)

\[ \frac{d}{d\tau} \left( r^2 \sin^2 \theta \frac{d\phi}{d\tau} \right) = 0, \]  

(20)

\[ \frac{d}{d\tau} \left[ \left( 1 - \alpha e^{-\beta r} \right) \frac{dt}{d\tau} \right] = 0. \]  

(21)

By choosing spherical polar coordinates and considering motion in the plane \( \theta = \pi/2 \), the equations (20) and (21) reduce to

\[ \frac{d\phi}{d\tau} = l r^2, \quad \frac{dt}{d\tau} = kr r - \alpha e^{-\beta r}, \]  

(22)
where \( l \) and \( k \) are constants of integration. With these relations, the equation for the orbit can be obtained from the equation (18) as

\[
\left( \frac{l}{r^2} \frac{dr}{d\phi} \right)^2 + \frac{l^2}{r^2} = c^2(k^2 - 1) + \frac{\alpha c^2}{r} e^{-\beta r} + \frac{\alpha l^2}{r^3} e^{-\beta r}.
\]

In the case, when the condition \( \beta r \ll 1 \) is satisfied, and using the approximation \( e^{-\beta r} = 1 - \beta r \), the equation for the orbit becomes

\[
\left( \frac{l}{r^2} \frac{dr}{d\phi} \right)^2 + \frac{l^2(1 + \alpha \beta)}{r^2} = c^2(k^2 - (1 + \alpha \beta)) + \frac{\alpha c^2}{r} + \frac{\alpha l^2}{r^3}.
\]

By letting \( u = 1/r \) and differentiating the resulting equation with respect to the variable \( \phi \), it is found that

\[
\frac{d^2u}{d\phi^2} + (1 + \alpha \beta)u = \frac{\alpha c^2}{2l^2} + \frac{3\alpha l^2}{2} u^2.
\]

Hence, the classical dynamics of a particle under the influence of strong force of Yukawa potential is similar to that of a particle in the Schwarzschild gravitational field. In the next section we discuss the quantum dynamics of a particle in terms of general relativity, by showing that the mathematical formulation of general relativity may contain within it the quantum theory!

### 3 General relativity and quantum mechanics

It is often stated that general relativity may not be compatible with quantum theory, because the former is formulated in terms of curved spacetimes while the latter is formulated from the viewpoint of an observer in flat Minkowski spacetime; the quantum dynamics of a particle is then described in terms of a Hilbert space of physical states (see, e.g., [14]). Nevertheless, various attempts have been made to unify the theory of general relativity with quantum theory by employing different quantisation procedures [16]. For example, the path integral approach to quantum gravity has emerged as a promising quantisation procedure [17]. However, by approaching the problem in a more intuitive manner, we conjecture that general relativity and quantum mechanics may be reconciled if the curved spacetime of a quantum particle is constructed in such manner that it can be transformed to Minkowski spacetime. As we discussed previously it is possible to apply directly the general relativistic formulation to the strong interaction so that a strong force exhibit a geometrical character.

In the present section we are concerned with the possibility of deducing the quantum dynamics of a particle from the mathematical formulation of general relativity. First we note that despite the energy-momentum tensor (9) gives an exact solution to the field equations of general relativity, which, although mathematically acceptable, may not be physically realistic. Nonetheless, the form of the energy-momentum tensor (9) reveals the possibility that at the quantum level the energy density may vary only as the inverse square of the distance, and that the pressure may be ignored compared to the energy density. With these assumptions, and for the purpose of describing the spacetime dynamics of a single quantum particle, let us consider a general relativistic spacetime model for a quantum particle based on the Robertson-Walker metric (see, e.g., [18, 19])

\[
ds^2 = c^2 dt^2 - S^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right)
\]
and the energy-momentum tensor $T_{\mu\nu}$ of the form

$$T_{\mu\nu} = \frac{A}{S^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (27)$$

where $A$ is a constant. The quantity $S(t)$ is considered as the radius of curvature and in general is a function of time which will be determined by the field equations of general relativity. The parameter $k = -1, 0, 1$. The sign of $k$ will be discussed shortly. Using the expression for the Ricci tensor in the form

$$R_{\mu\nu} = \frac{\partial^2 \ln \sqrt{-g}}{\partial x^\mu \partial x^\nu} - \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\sigma} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} \frac{\partial \ln \sqrt{-g}}{\partial x^\sigma}, \quad (28)$$

the field equations of general relativity then reduce to

$$\frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - \frac{\Lambda c^2}{3} = \frac{\kappa c^2 A}{3 S^2} \quad (29)$$

$$2 \frac{\ddot{S}}{S} + \frac{S^2 + kc^2}{S^2} - \Lambda c^2 = 0. \quad (30)$$

This system of equations has a static solution

$$S_0^2 = \frac{kc^4}{4\pi G \epsilon} \quad (31)$$

which is similar to the Einstein static model with an energy density $\epsilon$ which may be very large. However, the possibility of negative energy density should not be ruled out because at the quantum level a particle may have a curved spacetime with negative curvature as will be discussed presently. Since we are discussing curved spacetimes at the quantum level, the quantity $\Lambda = k/S_0^2$ will change drastically for a small variation in $S_0$. Hence, the quantity $\Lambda$ may also be considered as an inverse square function of $S$, i.e., $\Lambda = B/S^2$, where $B$ is constant. The system of field equations (29) and (30) is then modified to the following equations

$$\frac{\dot{S}^2}{S^2} + \frac{kc^2}{S^2} - \frac{c^2 B}{3 S^2} = \frac{\kappa c^2 A}{3 S^2} \quad (32)$$

$$2 \frac{\ddot{S}}{S} + \frac{S^2 + kc^2}{S^2} - \frac{c^2 B}{S^2} = 0. \quad (33)$$

This system of equations has a solution of the form

$$S = act, \quad \text{where} \quad a = \sqrt{\frac{\kappa A}{2} - k} \quad (34)$$

Let us first consider the case $k = 1$. It is seen that in this case a real solution requires the spacetime of a quantum particle to have a very large positive energy density in its own reference frame. The curved spacetime of the particle in this case cannot be transformed to

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2Actually the constant $B$ could be set to zero in the discussion that follows.
the Minkowski spacetime of a quantum observer. However, results from nucleon scattering experiments (see, e.g., [20]) have shown that such a large energy density is not appropriate for the surrounding spacetime of quantum particles like protons and neutrons. Therefore, if we assume a reasonable value for the energy density so that $\kappa A/2 \ll 1$, then we are forced to quantize the spacetime structure of the particle by introducing the imaginary number $i$, hence $a \approx i$ or $S \approx ict$. The reason for the introduction of this quantisation is that the Robertson-Walker metric can be transformed to a manifestly Minkowski metric when $S = ict$. Actually, this kind of quantisation turns the pseudo-Riemannian curved spacetime of the particle into a Riemannian spacetime. This means that we assume the particle to be able to measure its temporal distance in exactly the same way as its spatial distances. In other words, the particle is viewed as an object existing as a timeless four-dimensional Riemannian space. Speculations on elementary particles being tiny hyperspheres are normally regarded as rather philosophical in nature. However, it is interesting to quote [8], ‘... Another case would arise if space were four-(or more) dimensional in its smallest elements, but three-dimensional as a whole. This situation would correspond to the case of a thin layer of grains of sand which, although each is three-dimensional if taken individually, taken as a whole forms essentially a two-dimensional space. Similarly, atoms which individually are higher-dimensional could cluster into three-dimensional structures. In such a world, a macroscopic structure would have only the degrees of freedom of the three dimensions of space, while an atom would have many more degrees of freedom. Sense perceptions in such a world would not be noticeably different from those of our ordinary world; and conversely, it is in principle possible to infer from our ordinary experiences the higher-dimensional character of the microscopic world. Incidentally, it is not impossible that quantum mechanics will lead to such results.’

Quantisation is realisable only when the Robertson-Walker metric (with $S = ict$) of the particle can be transformed to a Minkowski metric. This means that the curved spacetime structure of the particle can be viewed in Minkowski spacetime. This is in fact the case, for if we apply the coordinate transformations [18]

$$iR = cR, \quad cT = ct\sqrt{1 - v^2}$$  \hspace{1cm} (35)$$

or

$$r = \left(1 - \frac{c^2T^2}{R^2}\right)^{-1/2}, \quad ct = iR\left(1 - \frac{c^2T^2}{R^2}\right)^{1/2}$$  \hspace{1cm} (36)$$

then, as can be readily verified, using the formula for the transformation of the metric tensor

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta},$$  \hspace{1cm} (37)$$

the coordinate transformations (35) reduce the Robertson-Walker metric (26) of the quantum particle (with $S = ict$) to a manifestly Minkowski metric of the form

$$ds^2 = c^2dT^2 - dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (38)$$

It is seen that the spacetime dynamics of the quantum particle can now be investigated by an observer using a Minkowski metric. The description can be achieved by writing the quantity $S$ in terms of the coordinates $(R, cT)$ in the form of an action integral

$$S = -i\sqrt{c^2T^2 - R^2} = -i \int ds = -ic \int \sqrt{1 - \frac{v^2}{c^2}} dT,$$  \hspace{1cm} (39)$$
where $ds$ is the usual Minkowski spacetime interval and $v = R/T$. It should be emphasised that because the above coordinate transformations involve the imaginary number $i$, in order to get real observables of the quantum particle, the velocity should be defined as $v = iR/T$. This definition is consistent with the definition of the canonical momentum operator, $p = -i\hbar\nabla$, in quantum mechanics. In order to relate this result with the path integral formulation of quantum mechanics we introduce a new quantity $\Psi$, defined by the relation $S = K \ln \Psi$, where $K$ is a dimensional constant, which will be identified shortly. We then obtain

$$\Psi = e^{i\frac{1}{K} \int ds}.$$  

(40)

With this form, we can recover standard quantum mechanics, at least for free particles in the nonrelativistic limit, by applying the Feynman path integral method [21]. Perhaps, the most important point that should be emphasised is that the Minkowski coordinates in this case depend entirely on the metric structure of the quantum particle. So observers in Minkowski spacetime cannot perform measurements of physical observables of the particle using their own choice of standard spatial and temporal gauges. This may be a reason for the unpredictable behaviour of quantum particles formulated in Minkowski spacetime. This leads to the conclusion that the action integral (39) does not have a deterministic character, which is implied in its formulation in classical physics. The intrinsic relationship between the spacetime of a quantum particle and that of a Minkowski observer, as specified by the transformations (26), also justifies the mathematical formulation of the Feynman random path integral formulation of quantum mechanics.

In Minkowski spacetime the quantity $S$ has an action integral form; in this case a quantum description can be constructed by following Schrödinger’s original method [24, 25]. We begin by noting that the quantity $S$ satisfies the relation

$$-\frac{1}{c^2} \left( \frac{\partial S}{\partial T} \right)^2 + \left( \frac{\partial S}{\partial R} \right)^2 - 1 = 0.$$  

(41)

Using the relation $S = K \ln \Psi$, the quantity $\Psi$ then satisfies the following equation

$$-\frac{1}{c^2} \left( \frac{\partial \Psi}{\partial T} \right)^2 + \left( \frac{\partial \Psi}{\partial R} \right)^2 - \frac{K^2}{2} \Psi^2 = 0.$$  

(42)

Since $\partial \Psi / \partial R = \nabla \Psi \cdot (\partial \mathbf{R} / \partial R) = |\nabla \Psi| \cos \alpha$, using the variational principle, after averaging the above equation with $\langle \cos^2 \alpha \rangle = 1/2$, we obtain a Klein-Gordon-like wave equation

$$-\frac{1}{c_a^2} \frac{\partial^2 \Psi}{\partial T^2} + \nabla^2 \Psi - \frac{1}{K_a^2} \Psi = 0,$$  

(43)

where $c_a = c / \sqrt{2}$ and $K_a = K / \sqrt{2}$. If this equation is compared with the Klein-Gordon equation in relativistic quantum mechanics (26), then it is seen that equation (43) describes the quantum dynamics of a particle with an average velocity $c_a$ which is less than the velocity of light $c$. This may give a reason why the force carriers in the strong and weak interactions are massive. The comparison between Eq.(43) and the Klein-Gordon equation also gives $K_a = \hbar/mc_a$. We remark that the treatment of a quantum particle as a spacetime manifold possessing a Robertson-Walker metric is consistent with our previous static solution in which the Yukawa potential was used to construct a line element for the strong interaction. This
is because the Yukawa potential is actually derived from the Klein-Gordon wave equation in relativistic quantum mechanics.

Now let us consider the case $k = -1$. In this case we have $a \approx 1$ or $S \approx ct$, again assuming $\kappa A/2 \ll 1$. This is in fact the well known Milne model that arises from Milne’s work on kinematic relativity [22]. The coordinate transformations of the form [18]

$$R = c\tau, \quad cT = ct\sqrt{1 + r^2} \quad (44)$$

or

$$r = \left(\frac{c^2T^2}{R^2} - 1\right)^{-1/2}, \quad ct = R\left(\frac{c^2T^2}{R^2} - 1\right)^{1/2} \quad (45)$$

also reduce the Robertson-Walker line element (26) of a quantum particle, with $S = ct$, to that of Minkowski spacetime. The quantity $S$ written in terms of the coordinates $(R, cT)$ as a classical action integral now takes the form

$$S = \sqrt{c^2T^2 - R^2} = \int ds = c\int \sqrt{1 - \frac{v^2}{c^2}}dT \quad (46)$$

and satisfies the relation

$$-\frac{1}{c^2} \left(\frac{\partial S}{\partial T}\right)^2 + \left(\frac{\partial S}{\partial R}\right)^2 + 1 = 0. \quad (47)$$

The quantity $\Psi$, defined by the relation $S = K\ln\Psi$, now becomes

$$\Psi = e^{\frac{1}{K}\int ds}. \quad (48)$$

Using the variational principle, it can be shown that $\Psi$ satisfies the equation

$$-\frac{1}{c^2} \frac{\partial^2\Psi}{\partial T^2} + \nabla^2\Psi + \frac{1}{K^2}\Psi = 0. \quad (49)$$

Equation (49), however, differs from the Klein-Gordon equation by the appearance of a plus sign in the last term. This results from the fact that the quantum particle in this case has negative curvature. This kind of particle structure is not usually found in quantum mechanics. However, it is seen that if the energy density is negative, then our formulation is real-valued and compatible with the usual relativistic description with timelike intervals. If the spacetime structure of a quantum particle in this case is also quantised, by turning equation (49) into the Klein-Gordon equation, so that the ‘conventional’ quantum mechanics is used to describe the quantum dynamics of the particle, then we replace $K \rightarrow iK$. This process of quantisation is equivalent to turning the spacetime structure of a particle with negative curvature into that with positive curvature, and specifying a positive energy density for the particle in Minkowski spacetime.

Finally, in the case when $k = 0$, a real solution is obtained for any positive energy density, $S = act$. The spatial part of the Robertson-Walker line element simply becomes the Euclidean metric scaled by the factor $S$. If we apply the coordinate transformations

$$R = actr, \quad cT = ct \quad (50)$$

then $dR = act dr + acrdT$. It is seen that when the term $acrdT \ll 1$, the spacetime structure of a particle can be reduced to that of the Minkowski spacetime. For a particle with large energy
density, i.e., a large, its curved spacetime metric can only be transformed to a Minkowski metric for a short time $dT$.

We conclude this section with some remarks on the energy-momentum conservation laws in general relativity. For the simple models that have been discussed the solution $S = S_0$ satisfies strictly the conservation laws required by general relativity. However, when the theory is applied to quantum physics, the conservation laws should not be expected to be satisfied strictly at the quantum level; the uncertainty principle would allow violations. Furthermore, it seems that at the quantum level constraints such as positivity of energy density also become relative and coordinate-dependent, and this may affect the way in which a Minkowski observer describes a physical process. These fundamental problems are as yet unresolved and require further investigation.

4 Further discussions and speculations

4.1 A heuristic geometric formulation of electromagnetism

We first remark that all attempts at the unification of the electromagnetic field with the gravitational field have focused on incorporating electromagnetism into gravitation by modifying a Riemannian metric structure; this metric structure essentially describes the gravitational field. Since all such attempts have not been successful, a natural question arises as to whether electromagnetism should be regarded as more fundamental than gravitation, so that electromagnetism should be described geometrically as an independent affine structure of a spacetime manifold, without referring to the existence of a Riemannian metric structure that represents the gravitational field. This radical alternative approach to the geometrisation of physics may allow gravity to arise as an additional structure to the spacetime structure of electromagnetism. In this section we discuss how a heuristic geometrical formulation of electromagnetism can be realised in terms of non-Riemannian geometry. In this formulation, a simple quantisation procedure can be introduced into an electromagnetic structure of the spacetime manifold to make it compatible with the quantum theory. This quantisation is carried out by considering complex-valued changes to a vector under an infinitesimal parallel displacement. In this case, there are similarities between the formulation of quantum mechanics in an electromagnetic spacetime and the theory of gravitation using the Riemannian spacetime. The free particle Schrödinger wave equation in a Euclidean space must be modified to the free particle Schrödinger wave equation in an electromagnetic spacetime. The latter wave equation is identical to the wave equation of a charged particle moving in an electromagnetic field in the background Euclidean space.

We adopt the view that electromagnetism should be described in its own right, as an independent affine structure of a spacetime manifold without reference to any other possible metric structures, such as the Riemannian metric structure used to describe the gravitational field. This approach is equivalent to postulating independent physical fields in field theory where, with the same background Minkowski spacetime, a particular field can be formulated by the introduction of a particular mathematical structure. It is assumed that a spacetime manifold can be endowed with a geometric structure from which a particular, affine or metric, structure can be chosen to describe a physical field. Unless it is equipped with a geometric structure, the background spacetime manifold cannot describe physical laws.

In order to describe electrodynamics geometrically, an affine connection is introduced into the differentiable manifold of spacetime. The introduction of such a connection can be carried
out by adopting a heuristic approach modelled on parallel transport of a vector field and its covariant derivative. Instead of the form given to the connection used to formulate general relativity, the change $\delta V^\mu$ in the components of a vector $V^\mu$ under an infinitesimal parallel displacement in the present situation is assumed to be of the form

$$\delta V^\mu = -\beta \Lambda_\nu V^\mu dx^\nu,$$

in which case the covariant derivatives are defined as

$$\nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \beta \Lambda_\nu V^\mu,$$  \hspace{1cm} (52)

$$\nabla_\nu V_\mu = \frac{\partial V_\mu}{\partial x^\nu} - \beta \Lambda_\nu V_\mu.$$  \hspace{1cm} (53)

These are required to transform like a tensor under general coordinate transformations. Here the quantity $\Lambda_\nu$ is an affine connection of the spacetime manifold, which will be identified with the electromagnetic four-vector potential. The quantity $\beta$ is an arbitrary dimensional constant. The transformation law for the affine connection $\Lambda_\mu$ can be deduced from the transformation properties of the covariant derivative. Under a general coordinate transformation $x'^\mu = x'_\nu(x'^\tau)$, the connection $\Lambda_\mu$ transforms as

$$\beta \Lambda'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} (\beta \Lambda_\nu) + \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x'^\sigma} \frac{\partial x'^\sigma}{\partial x^\nu}.$$  \hspace{1cm} (54)

The generalisation of Eqs.(52) and (53) can be obtained from the definition of a covariant derivative, i.e.,

$$\nabla_\sigma A^{\mu_1 \cdots \mu_m} = \frac{\partial A^{\mu_1 \cdots \mu_m}}{\partial x^\sigma} + m \beta \Lambda_\sigma A^{\mu_1 \cdots \mu_m},$$  \hspace{1cm} (55)

$$\nabla_\sigma A_{\nu_1 \cdots \nu_n} = \frac{\partial A_{\nu_1 \cdots \nu_n}}{\partial x^\sigma} - n \beta \Lambda_\sigma A_{\nu_1 \cdots \nu_n},$$  \hspace{1cm} (56)

$$\nabla_\sigma A^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_n} = \frac{\partial A^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_n}}{\partial x^\sigma} + (m - n) \beta \Lambda_\sigma A^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_n}.$$  \hspace{1cm} (57)

It is interesting to note that the covariant derivative of a mixed tensor having equal number of superscripts and subscripts is identical to its ordinary derivative.

It should be reiterated that the electromagnetic structure is assumed to be an independent structure, which defines a curved spacetime. This must not be considered as an additional structure arising from the postulate of gauge invariance, as in Weyl’s theory, which assumes a change $\delta l = -l \phi_\mu dx^\mu$ of the length $l = g_{\mu\nu} \xi^\mu \xi^\nu$ of a vector $\xi^\mu$ under parallel transport. In this latter case $g_{\mu\nu}$ represent a Riemannian spacetime structure of gravitation and $\phi_\mu$ is a four-vector function which is identified with the four-vector potential of an electromagnetic field. The electromagnetic ‘spacetime’ in our case is assumed to exist by itself, independent of any other spacetime structures, such as the gravitational field. The purpose of the introduction of the connection $\Lambda_\mu$ is to construct a non-Riemannian spacetime manifold which can be used to represent electromagnetism alone. In this way that an appropriate topological structure of the manifold can be related to the quantum dynamics of a particle in spacetime.

The quantities $\Lambda_\mu$ in general are arbitrary functions of the coordinate variables and they do not form a tensor under general coordinate transformations, but they do form a tensor under the group of linear transformations. This result is compatible with the usual formulation of electromagnetism as a physical field in a Minkowski spacetime, which is invariant
under Lorentz transformations. In a particular coordinate system, if the connection is identified with the electromagnetic potentials, then, since the four-potential should transform as a vector, the effect caused by the extra term in the transformed connection (54) is not an electromagnetic effect. It is related purely to coordinate transformations, i.e., it represents an inertial effect. Therefore, if the potentials are significant then our geometrical formulation of electromagnetism is covariant only under the group of linear transformations. This is reasonable since an inertial effect caused by non-linear coordinate transformations has not been related to any kind of electromagnetic properties. The formulation of a physical theory is normally required to be covariant only under some particular group of transformations, except for the general theory of relativity in which the formalism is based on the requirement of general covariance [1].

However, in the present geometrical formulation of electromagnetism, the geometrical object which plays the role of the Riemannian curvature tensor is covariant under general coordinate transformations. This object is expected to take the familiar form of the electromagnetic field tensor \( F_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu} \). This curvature can be derived by considering the change \( \Delta V_\mu = \oint \delta V_\mu \) of a vector \( V_\mu \) parallel transported around an infinitesimal closed path. To first order, an infinitesimal closed path permits the components of the vector \( V_\mu \) at points inside the path to be uniquely determined by their values on the path. By Stokes theorem, it is found that

\[
\Delta V_\sigma = \oint_\Gamma \Lambda_\nu V_\sigma dx^\nu = \frac{1}{2} \left( \frac{\partial \Lambda_\mu}{\partial x^\nu} - \frac{\partial \Lambda_\nu}{\partial x^\mu} \right) V_\sigma \Delta f^{\mu\nu},
\]

(58)

where \( \Delta f^{\mu\nu} \) is the area enclosed by the closed path \( \Gamma \). Since \( V_\mu \) is a vector and \( \Delta f^{\mu\nu} \) is a tensor, and since \( \Delta V_\mu \) is also a vector, because it is the difference between the values of vectors at the same point after parallel displacement, the tensor character of the curvature, defined by the relation \( F_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \) is determined from the quotient theorem in tensor calculus. The quantities \( F_{\mu\nu} \) therefore form a tensor under general coordinate transformations. This result shows that if only the field strength of the electromagnetic field is considered significant, then the present geometrical formulation of electromagnetism, like the general relativistic formalism of gravitation, is also covariant with respect to the general group of transformations. However, as we shall discuss in detail later, when the electromagnetic field strength is defined in terms of the potentials, the existence of the electromagnetic field strength requires restrictions on the analytic properties of the four-potential. We shall also demonstrate that the present formulation of electromagnetism may lead to the possibility of introducing an asymmetric connection, in which the combined electromagnetic effects of two electromagnetic fields on a charged particle provide a geometrical framework for describing the dynamics of the particle. Remarkably, we can show that when the combined effects on a charged particle are electromagnetically neutral\(^3\), they can be identified with gravity within a general relativistic framework.

The curvature \( F_{\mu\nu} \) automatically satisfies the homogeneous equations of classical electrodynamics, \( \partial_\alpha F_{\mu\nu} + \delta_\mu F_{\nu\alpha} + \delta_\nu F_{\alpha\mu} = 0 \). The result shows that the homogeneous equations of electrodynamics are geometrical rather than dynamical when the connection \( \Lambda_\mu \) is considered

\(^3\)Electromagnetic neutrality is defined within the context of the Lorentz force law, where opposing electromagnetic fields lead to the possibility of the net force on the particle being zero.
as being a purely geometrical object. As usual, to determine the dynamics of the electromagnetic spacetime manifold, an action, that may or may not relate geometrical properties of the manifold to matter or charge, must be specified. If such an action is defined by the form 

\[ S = - \int (F_{\mu\nu} F^{\mu\nu} - \kappa \Lambda_{\mu} j^\mu) \, dx^4, \]

where \( \kappa \) is an arbitrary dimensional constant, then the variation of the action \( S \) with respect to the connection \( \Lambda_{\mu} \) leads to the inhomogeneous equations of classical electrodynamics, i.e., \( \partial_\mu F^{\mu\nu} + \kappa j^\nu = 0. \) The external current density, \( j^\mu \), whose geometrical character is unknown, plays the role of the stress tensor in the field equations of general relativity. In the case when there is no external current, the field equations of electromagnetism \( \partial_\mu F^{\mu\nu} = 0 \) describe a vacuum spacetime structure, similar to the source free gravitational field equations, \( R_{\mu\nu} = 0 \), in general relativity.

The spacetime structure of electromagnetism that has been described using the connection \( \Lambda_{\mu} \) and the curvature \( F_{\mu\nu} \) is entirely affine. An affine structure is not capable of providing a dynamical description of the motion of a particle in the field. This is exactly the case in classical electrodynamics where the Lorentz force must be added to the Maxwell field equations for a dynamical description of a charged particle. With a geometrical formulation of the physical field, the dynamics can be provided by introducing a metric tensor \( g_{\mu\nu} \) onto the spacetime manifold through the defining relation \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \) When the spacetime manifold is endowed with a metric, a relationship between the metric and the connection can be obtained by demanding that the metric be covariantly constant, in the sense that the inner product of two vectors remains constant under parallel transport along a curve. This requirement leads to the condition

\[ \nabla_\sigma g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - 2\beta \Lambda_\sigma g_{\mu\nu} = 0. \]  

(59)

Eq. (59) can be rewritten in the form

\[ \Lambda_\mu = g^{\lambda\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} = \frac{1}{2\beta g} \frac{\partial g}{\partial x^\mu} = \frac{1}{\beta} \frac{\partial \ln \sqrt{-g}}{\partial x^\mu}. \]

(60)

where \( g = \det(g_{\mu\nu}). \) Under the usual gauge transformation

\[ N'_\mu = \Lambda_\mu + \frac{\partial \chi}{\partial x^\mu}, \]

(61)

with \( \chi = (\ln \sqrt{\sigma})/\beta, \) we obtain

\[ N'_\mu = \frac{1}{\beta} \frac{\partial \ln \sqrt{-\sigma g}}{\partial x^\mu} = \frac{1}{2\beta (\sigma g)} \frac{\partial (\sigma g)}{\partial x^\mu}. \]

(62)

Here \( \chi, \) and hence \( \sigma, \) is an arbitrary function of the coordinate variables.

If the potentials are considered to have a physical significance and covariance of the theory is imposed, the transformation group is restricted to the linear group. In this case it is not possible to introduce a metric tensor that satisfies the requirement of being covariantly constant \( \nabla_\sigma g_{\mu\nu} = 0, \) since linear coordinate transformations imply \( g = \det(g_{\mu\nu}) = constant \) and the affine connection \( \Lambda_{\mu} \) defined in terms of the metric tensor in Eq. (60) vanishes. This result leads to the conclusion that electromagnetism is a non-metrical spacetime structure. Although the conclusion seems to contradict a basic requirement of classical physics, namely, that the local validity of physical laws requires a metric, the non-metric character of electromagnetism
seems to comply with the principles of quantum mechanics whose non-deterministic formulation implies the impossibility of using a metric for investigating the dynamics of a physical system. In other words, the process of physical measurement in a deterministic manner is not possible for the electromagnetic field, which is covariant only under the linear group of coordinate transformations.

On the other hand, if the field strength is considered to be the relevant physically significant object, as in classical electrodynamics, and the effects caused by non-linear coordinate transformations are treated as inertial effects, then a metric tensor can be introduced so that the electromagnetic potential can be defined through it. However, the relationship between the connection, the curvature and the metric tensor means that the electromagnetic field strength can exist only at spacetime points where the condition of integrability is not satisfied. In spacetime regions where the determinant of the metric tensor is a smooth function, or at least twice differentiable, there will only be a four-potential without a concomitant electromagnetic field strength. It is the analytical behaviour of the metric tensor that determines the existence of the electromagnetic field strength. This result leads to the conclusion that for metric spacetime structures describing electromagnetism, for which the electromagnetic potentials are considered significant, it is not possible to measure the electromagnetic field strength of the system smoothly (i.e., the field strength is not differentiable). This clearly contradicts the usual assumption that physical laws can be formulated locally in classical physics. Whether this situation can be incorporated consistently into the present formulation of quantum mechanics is a problem that requires further investigation. For example, it is known that quantum mechanics utilises potentials to describe the Aharonov-Bohm effect.

The equation of motion of a charged particle in an electromagnetic spacetime manifold can also be obtained from the requirement that the path of a particle is a geodesic, i.e.,

$$\frac{d^2x^\mu}{ds^2} + \beta \Lambda_\mu \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} = 0. \quad (63)$$

where the parameter $s$ is identified with the arc-length only when a metric exists. Since the affine connection $\Lambda_\mu$ is entirely geometrical, the equation of motion in this form does not have the dynamical character required of a physical process, where some physical quantity is required to characterise the physical state of a particle. Therefore, it is necessary to introduce some kind of relationship between the geometrical objects and the physical quantities in order to provide a possible dynamical description of the system consisting of particle and field. For example, if the following relationship is assumed

$$\beta \Lambda_\mu \frac{dx^\mu}{ds} = -\frac{q}{m} F^\mu_\nu, \quad (64)$$

then the familiar form of Lorentz force law for the motion of a charged particle in an electromagnetic field is regained, i.e.,

$$\frac{d^2x^\mu}{ds^2} = \frac{q}{m} F^\mu_\nu \frac{dx^\nu}{ds}. \quad (65)$$

However, since there is no physical basis for its introduction, the relation (64) should be considered as an intrinsic relationship between the field and the experimentally defined physical quantities that characterise the mass and the charge of a particle.

In the case when a metric is introduced onto the electromagnetic manifold, the relation (60) between the metric tensor and the connection can be rewritten in the form

$$g = g_0 \exp \left(2\beta \int \Lambda_\mu dx^\mu \right). \quad (66)$$
In the non-relativistic limit, the determinant $g$ can be reduced further to the form

$$g = g_0 \exp \left( 2\alpha \frac{ct}{r} \right), \quad (67)$$

where $g_0$ and $\alpha$ are constants. Utilising this form of the determinant of the metric tensor, with $\Lambda_\mu = (\phi, -\mathbf{A})$, the four-vector potential is obtained

$$\phi = \frac{\alpha}{\beta} \frac{1}{r}, \quad \mathbf{A} = \frac{\alpha c t}{\beta} \frac{\mathbf{r}}{r^3}. \quad (68)$$

With this form of the four-potential, it is noted that at finite time $t$, except at the origin $r = 0$, the electromagnetic field strength defined in terms of the four-potential vanishes everywhere. Any possible influence on a charged particle in this electromagnetic spacetime can only be expressed in terms of the potentials themselves. This result is similar to the situation in general relativity where the effect of a gravitational field on a particle can only be expressed in terms of the potential, which forms a line element for the field.

The non-relativistic equation of motion takes the form

$$\frac{d^2 \mathbf{r}}{dt^2} + \frac{\alpha}{r} \frac{d\mathbf{r}}{dt} = 0, \quad (69)$$

where the scalar potential $\phi = \alpha/\beta r$ has been used. It can be shown that a semi-classical relation can be derived from differential geometry which relates the momentum and the de Broglie wavelength of a particle, i.e.,

$$\frac{d\mathbf{r}}{dt} = \frac{\hbar}{m} \frac{\mathbf{r}}{r^2}. \quad (70)$$

The equation of motion then takes the familiar form of Newton’s equation of motion for a charged test particle moving in a spherically symmetric electrostatic field

$$m \frac{d^2 \mathbf{r}}{dt^2} = -e \frac{\mathbf{r}}{r^3}. \quad (71)$$

Here we have set $\alpha = e^2/\hbar c$, with $e$ denoting the fundamental electronic charge. The constant $\alpha$ is the fine structure constant, and the constant $\beta$ is equal to $1/\hbar c$. In this case the electric field strength is determined experimentally, since classically the vector potential $\mathbf{A}$ is considered merely as a mathematical convenience, with no direct physical significance, and so would have no physical effect on the motion of a charged particle (see, e.g., [23]).

Let us now consider a simple procedure for quantising the electromagnetic spacetime structure. It is known that the canonical quantisation of a classical field requires the introduction of the imaginary number $i$ into the mathematical structure used to formulate the theory in an appropriate way. Specifically, quantisation is achieved by replacing the classical Poisson bracket by the quantum Poisson bracket; namely, $[q, p] \rightarrow i\hbar[q, p]$. Alternative quantisation procedures, such as the path integral formulation of quantum mechanics, introduce a complex-valued transition amplitude; however, this method still retains the classical notion of an action integral. We discussed previously a possible relationship between the theory of general relativity and quantum theory, in which it is necessary to employ the quantity $i$ in order to obtain a coordinate transformation from the spacetime structure of a quantum particle to that of a Minkowski observer. In so doing the spacetime dynamics of a quantum
particle manifests itself in Minkowski spacetime as the quantum dynamics described by a relativistic wave equation. We now examine the quantisation procedure within the present geometric formulation of the electromagnetic field. It has already been noted that the determinant of a metric tensor on an electromagnetic spacetime manifold has the property that it generates an electromagnetic field through its analytical behaviour. This property is reflected in the gauge transformation (62). Each quantity \( g \) corresponds to an infinite number of possible spacetime metrics, with the same electromagnetic field generated by all analytically equivalent determinants of the form \( \sigma g \). For example, there is an infinite number of metric structures corresponding to the potentials given in (68). Within the context of classical electrodynamics, the vector potential \( A = (\alpha ct)/(\beta r^3) \) in this case produces a zero magnetic field, i.e., \( B = \nabla \times A \equiv 0 \). On the other hand, the scalar potential \( \alpha/(\beta r) \) can be identified with the Coulomb potential used in the quantum dynamical description of a hydrogen atom. Therefore, a dynamical equation, such as the Schrödinger wave equation, that contains only the potentials (68) cannot be used to determine the metric structure of a physical system. This implies the impossibility of determining the path of a particle. Let us now consider a purely imaginary change of a vector under an infinitesimal displacement

\[
\delta A^\mu = -i\beta \Lambda_\nu A^\mu dx^\nu. \tag{72}
\]

The covariant derivative will take the form

\[
\nabla_\nu A^\mu = \frac{\partial A^\mu}{\partial x^\nu} + i\beta \Lambda_\nu A^\mu. \tag{73}
\]

It is known that the gauge invariant formulation of quantum mechanics requires the free particle Schrödinger wave equation to be modified when analysing a charged particle moving in an electromagnetic field. In both cases the free particle equation and the modified equation take the Euclidean space as their background space. We can demonstrate that this situation may be described geometrically within the present formulation of electromagnetism. First we recall that general relativity describes the gravitational field in terms of geometry, without the concept of force; namely, the free particle Newtonian equation, \( \eta_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds) = 0 \), in a Euclidean space must be modified to the geodesic equation, \( d^2 x^\mu/ds^2 + \Gamma^{\mu}_{\sigma\lambda}(dx^\sigma/ds)(dx^\lambda/ds) = 0 \), in a Riemannian manifold to describe the dynamics of a particle in a gravitational field. In the present case, however, if we take the Schrödinger wave equation as the fundamental dynamical equation, then the free particle Schrödinger wave equation, \( i\hbar \partial_t \psi + (\hbar^2/2m)\nabla^2 \psi = 0 \), in a Euclidean space must be replaced by the wave equation, \( i\hbar \nabla_0 \psi + (\hbar^2/2m)\nabla_\mu \nabla^\mu \psi = 0 \), in a curved electromagnetic manifold for the dynamical description of a charged particle in an electromagnetic field. However, the problem that remains here is whether it is possible to derive the Schrödinger wave equation from the field equations of the electromagnetic spacetime manifold, as is the case in general relativity where the geodesic equation for a particle in a gravitational field can be derived from the Einstein field equations. However, as we already showed, it is indeed possible to obtain the Schrödinger wave equation from the field equations of general relativity by a coordinate transformation.

### 4.2 Physical aspects of the affine connection

General relativity is a physical theory that describes particle dynamics in terms of the geometry of a spacetime manifold. In this interpretation, gravity is a manifestation of the curvature of a Riemannian metric spacetime structure under the influence of matter. In Einstein’s
theory of gravitation, the metric tensor plays the role of the gravitational potential, while the metric connection, expressed in terms of the metric tensor, only plays an intermediate role and has no direct physical interpretation. However, the more fundamental object in the differential geometric formulation of spacetime manifolds is the affine connection. The question arises as to whether it is possible to assign a direct physical meaning to the connection, in the sense that the connection will play the fundamental role of a physical field which actually determines the dynamics of a particle. This point of view can be justified in part by noting that the geodesic equation of motion is directly governed by the connection itself. Historically, in an attempt to describe gravitation and electromagnetism within the context of a single spacetime structure, Einstein tried to formulate a ‘unified’ theory using a nonsymmetric affine connection rather than the metric tensor \[1\].

Unlike other equations in physics, which are usually deduced from experiments, and formulated as relations between physical quantities, the field equations of general relativity, \( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \), were postulated via an examination of tensor properties of the Einstein tensor, and the stress tensor. The former is a purely geometrical object, while the latter is a physical quantity that has no direct mathematical interpretation as a geometrical object. The question has been raised as to whether it is possible to construct a purely geometrical model for the theory of general relativity by looking for a geometrical interpretation of the stress tensor \[27\]. However, an alternative viewpoint can be adopted in which we look instead for a physical construction of the Einstein tensor in terms of known physical entities, so that the field equations of general relativity adopt the status of physical equations, describing a general relationship between purely physical quantities. This alternative viewpoint provides the basis for a physical interpretation of the affine connection in the theory of general relativity. In what follow we discuss the possibility of constructing the affine connection in terms of the potentials and the field strengths of two coupled electromagnetic fields.

In classical physics, electromagnetism and gravitation are considered to be two different physical structures that exist independently of each other in the spacetime manifold. In such a situation a gravitational field is not considered to have a direct relationship with the charge of a particle in the sense that there is no interaction between them. On the other hand, the mass of the particle is not related to the electromagnetic properties of an electromagnetic field. Only when the dynamics of the particle is studied using the laws of motion are these dynamical aspects connected. Consider a charged particle in a region of spacetime in which there is an electromagnetic field \( \Phi_{\nu} = (\phi, -A) \), where \( \phi \) and \( A \) are the scalar and vector potential, respectively. If we assume that in this same region another electromagnetic field could be set up so that at every point in the region the magnitudes of the two fields are always equal but the two fields are in opposite directions, then, according to the classical Lorentz force law, the charged particle would not be affected electromagnetically by the fields because their effects on the particle cancel. In this case the existence of the electromagnetic fields can be ignored within the context of classical electrodynamics. In other words, the spacetime surrounding region is ‘neutralised’ by the presence of the second field. The spacetime region surrounding an hydrogen atom, or the spacetime region surrounding the earth, obviously satisfies this requirement. For example, if a particle \( A \) with charge \( q \) and a particle \( B \) with charge \( -q \) are located at the origin of a coordinate system, then the field strengths and the potentials of the particles are

\[
E_A = \frac{kq}{r^2} \mathbf{r}, \quad \phi_A = -\frac{kq}{r} \tag{74}
\]
\[ E_B = -\frac{kq}{r^3}r, \quad \phi_B = \frac{kq}{r}, \quad (75) \]

and Newton’s law of motion \( m \mathbf{a} = \mathbf{F} \) would result in a net zero effect on a charged particle in the surrounding spacetime region, since in classical physics, the potentials play no direct role in determining the dynamics of a particle but are merely a mathematical convenience \[23\]. However, this situation changes if quantum effects are taken into account, since it is known that in quantum mechanics the potentials themselves may be significant and can determine the dynamics of a charged particle in a spacetime region where the fields vanish (see, e.g., \[28, 29\]). In such a situation it is possible that the potential and the field strength of one electromagnetic field might couple to the field and the potential of the other, so that the coupling would give rise to some kind of observable effects on the particle. Although dynamical effects on the charged particle by the opposing electromagnetic fields are cancelled, the presence of potentials may produce significant results, since one could consider products like \( \phi E \) as physical observables. If such terms produce observable effects on a charged particle, then in the above example, both particles \( A \) and \( B \) would give rise to the same dynamical effects on a charged particle. In the next section it is shown that we can construct such a model in terms of differential geometry, and that the resulting electromagnetic effects can be identified with the gravitational force.

4.3 Asymmetric connection of the form \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu \)

We now consider an asymmetric connection of the form \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu \). The motivation for this form of asymmetric connection is that it allows a construction of an affine connection in terms of two electromagnetic fields. The quantity \( \Phi_\mu \) will then be identified with the four-vector potential of one electromagnetic field and the quantity \( \Lambda^\mu_\nu \) with the field strength of the second opposing field. Since the affine connection does not behave like a tensor under general coordinate transformations, neither does \( \Phi_\mu \) nor \( \Lambda^\mu_\nu \). However, they do form tensors under the group of linear transformations, as should be the case in electrodynamics. Therefore we must address the problem of how to couple two electromagnetic fields to produce a spacetime structure in such a way that the gravitational field can be identified as a manifestation of that geometry. When the affine connection is formed from two electromagnetic fields, its possible effects on the motion of a particle are genuine physical effects if the electromagnetic field is viewed as a physical field determined from experiment rather than a geometrical structure, as discussed above. However, due to the asymmetry of the connection, these effects cannot be identified with gravity, since the theory of general relativity requires a symmetric connection. To meet this requirement, it is necessary to reduce the physical Ricci tensor, which is formed by two electromagnetic fields, to a symmetric form.

The affine connection of the particular form \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu \) reduces the curvature tensor \( R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\lambda_{\alpha\beta} \Gamma^\alpha_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\beta\lambda} \) to the simpler form

\[ R^\alpha_{\beta\mu\nu} = \frac{\partial}{\partial x^\mu} \left( \Lambda^\alpha_{\beta} \Phi_\nu \right) - \frac{\partial}{\partial x^\nu} \left( \Lambda^\alpha_{\mu} \Phi_\beta \right). \quad (76) \]

The Ricci tensor becomes

\[ R_{\mu\nu} = \frac{\partial}{\partial x^\sigma} \left( \Lambda^\sigma_{\mu} \Phi_\nu \right) - \frac{\partial}{\partial x^\nu} \left( \Lambda^\sigma_{\nu} \Phi_\mu \right) = \left( \frac{\partial \Phi_\nu}{\partial x^\sigma} - \frac{\partial \Phi_\sigma}{\partial x^\nu} \right) \Lambda^\sigma_\mu + \Phi_\nu \frac{\partial \Lambda^\sigma_\mu}{\partial x^\sigma} - \Phi_\sigma \frac{\partial \Lambda^\sigma_\mu}{\partial x^\nu}. \quad (77) \]
The Ricci tensor in this form can be reduced to a symmetric form if the quantities \( \Lambda^\sigma_\mu \) satisfy, for example, the relation

\[
\frac{\partial \Lambda_\sigma^\mu}{\partial x^\nu} = \eta_{\mu\nu\sigma} \Phi^\nu \Lambda^\sigma_\mu, \tag{78}
\]

where \( \eta_{\mu\nu\sigma} \) are arbitrary functions of the coordinate variables. The indices \( \mu, \nu \) and \( \sigma \) indicate that the functions \( \eta_{\mu\nu\sigma} \) are specified independently for each term of the quantity \( \Lambda^\sigma_\mu \). For the trivial case of a constant field, i.e., \( \Lambda^\sigma_\mu = \text{constant} \), we have \( \eta_{\mu\nu\sigma} \equiv 0 \). A more detailed discussion will be given via another example shortly. The Ricci tensor then becomes

\[
R_{\mu\nu} = \Lambda^\sigma_\mu F_{\sigma\nu}, \tag{79}
\]

where we have defined \( F_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu \). It is seen that the reduced form (79) of the Ricci tensor is symmetric if the quantity \( \Lambda^\sigma_\mu \) is assumed to be the transpose of the quantity \( F_{\mu\nu} \).

In a new coordinate system, \( x''^\mu = x^\mu(x'^\nu) \), the transformed affine connection is still defined as a product \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu \). Since the relation (78) retains the same form in the new coordinate system, due to the arbitrariness of the function \( \eta \), it is apparent that the relation (79) is form-invariant under general coordinate transformations; although it behaves like a tensor only under the linear group of coordinate transformations. Because the affine connection \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu \) behaves like a tensor only under the linear group, the quantities \( \Lambda^\sigma_\mu \) and \( \Phi_\nu \) behave like a tensor and a vector, respectively, only under linear coordinate transformations. However, since the tensor properties of the quantities \( \Lambda^\sigma_\mu \) and \( \Phi_\nu \) are not essential in the following discussion, it is sufficient to assume that their combination transforms like an affine connection under general coordinate transformations, without further postulating any particular properties of transformations for each of them separately.

### 4.4 Gravity as a coupling of two electromagnetic fields

The reduced form of the Ricci tensor suggests that in order to incorporate it into electromagnetism, the quantity \( \Phi_\mu \) should be identified with the four-vector potential and the quantity \( F_{\mu\nu} \) with the field strength of an electromagnetic field; the quantity \( F_{\mu\nu} \) has been defined in terms of the quantity \( \Phi_\mu \) by the familiar relation in electrodynamics, namely, \( F_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu \). In this case, if the reduced form of the Ricci tensor is required to be symmetric, the quantity \( \Lambda^\sigma_\mu \) can be identified as the field opposite to the field \( F_{\mu\nu} \). The reduced form of the Ricci tensor therefore may be used as a counterpart of the energy-momentum tensor to form field equations for gravitation. It should be emphasised again that the Ricci tensor in this case is a physical quantity which is formed by two electromagnetic fields of equal magnitude but opposite direction. Its effect on the motion of a particle in the fields are physical effects which require physical laws to describe them. These physical laws will be identified with the field equations of general relativity, because according to the classical Lorentz force law two opposing electromagnetic fields of equal magnitude are considered to have no classical electromagnetic effects on a charged particle moving in the coupled fields. In terms of the field strengths, the reduced form (2.46) of the Ricci tensor takes the explicit form

\[
R_{\mu\nu} = \begin{pmatrix}
E_1^2 + E_2^2 + E_3^2 & E_3 B_2 - E_2 B_3 & E_1 B_3 - E_3 B_1 & E_2 B_1 - E_1 B_2 \\
E_3 B_2 - E_2 B_3 & E_1^2 + B_3^2 + B_2^2 & E_1 E_2 - B_1 B_2 & E_1 E_3 - B_1 B_3 \\
E_1 B_3 - E_3 B_1 & E_1 E_2 - B_1 B_2 & E_2^2 + B_1^2 + B_2^2 & E_2 E_3 - B_2 B_3 \\
E_2 B_1 - E_1 B_2 & E_1 E_3 - B_1 B_3 & E_2 E_3 - B_2 B_3 & E_3^2 + B_1^2 + B_2^2
\end{pmatrix}. \tag{80}
\]
The quantity $R_{\mu\nu}$ is a physical quantity which is covariant only with respect to the group of linear transformations, and this reflects the covariant properties of the electromagnetic field under the linear group. Hence, if a symmetric Ricci tensor is required to describe a gravitational field then only effects caused by the reduced form (80) of the Ricci tensor are considered, and therefore possible spacetime structures are restricted to those determined by it. We assume that the dynamics of a charged particle in a region of spacetime, whose structure is determined by the connection $\Lambda^\sigma_\mu \Phi_\nu$ formed by two electromagnetic fields, is governed by the reduced form of the Ricci tensor. This is reasonable since the dynamics of a charged particle is not effected classically by the two opposing electromagnetic fields of equal magnitude. In order to define lengths of the paths of the particle, and hence to determine dynamical aspects of the particle in the spirit of general relativity, a new symmetrical metric tensor $g_{\mu\nu}$ is introduced according to the defining relation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. With the introduction of this symmetrical metric tensor into the restricted spacetime structure, determined by the reduced form of the Ricci tensor, it is now possible to construct field equations which have a similar form to the Einstein field equations of gravitation, $R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor. The Ricci tensor in these field equations is constructed from the new symmetrical metric tensor $g_{\mu\nu}$, and is a geometrical object which is used to describe geometrically the physical effect caused by the quantity $R_{\mu\nu}$ in (80). Because the field equations of general relativity do not require a physical basis for the affine connection, any physical effect on a particle moving in two coupled electromagnetic fields will manifest itself through the form of the energy-momentum tensor rather than through the geometrical Ricci tensor. Describing physical effects by a geometrical description is fundamental to the theory of general relativity, where the Newtonian force is described by a curved spacetime manifold.

It is interesting to note that if we assume that the effect of the coupling of two opposing electromagnetic fields can be interpreted as a gravitational field, then the connection $\Gamma^\sigma_\mu \Phi_\nu$, gives rise to a geodesic equation of the form

$$\frac{d^2x^\mu}{ds^2} + \left(\Phi_\sigma \frac{dx^\nu}{ds}\right) \Lambda^\mu_\nu \frac{dx^\nu}{ds} = 0.$$  \hspace{1cm} (81)

It is known that this equation admits a linear first integral of the form

$$\Phi_\mu \frac{dx^\mu}{ds} = -\frac{q}{m},$$  \hspace{1cm} (82)

provided the quantities $\Phi_\mu$ satisfy the condition

$$\nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu = 0.$$  \hspace{1cm} (83)

Here we have set the constant in the first integral equal to $-q/m$ for convenience. This condition identifies $\Phi_\mu$ as a Killing vector field, which defines a direction of symmetry along which the motion leaves the spacetime geometry unchanged. The geodesic equation then has the form of the Lorentz force law

$$\frac{d^2x^\mu}{ds^2} = \frac{q}{m} \Lambda^\mu_\nu \frac{dx^\nu}{ds}.$$  \hspace{1cm} (84)

We can interpret this result in the following way. When one of the electromagnetic fields drives the charged particle according to the laws of classical electrodynamics, the opposite
field resists such motion of the particle and the \textit{resistance} manifests itself as the mass of the particle via the linear first integral (82).

As an illustration, let us consider the simple situation of two opposite electric fields associated with particles \( A \) and \( B \) (see Eqs. (74) and (75)). First, we must examine whether this system satisfies the relation (78), i.e., \( \partial_{\nu} \Lambda_{\mu}^0 = \eta_{\mu\nu\sigma} \Phi_{\nu} \Lambda_{\sigma}^0 \). If the potentials and the field strengths of the particles \( A \) and \( B \) are specified only by the relations (74) and (75), then this system of fields does not satisfy the relation (78). By rewriting \( \Phi_{\mu} \) and \( \Lambda_{\nu}^0 \) explicitly in matrix form

\[
\Phi_{\mu} = (\phi, 0, 0, 0) \quad \text{and} \quad \Lambda_{\nu}^0 = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & 0 & 0 \\
E_y & 0 & 0 & 0 \\
E_z & 0 & 0 & 0
\end{pmatrix},
\]

(85)

it is seen that \( \eta_{\mu\nu\sigma} \Phi_{\nu} \Lambda_{\sigma}^0 \equiv 0 \) but \( \partial_{\nu} \Lambda_{\mu}^0 \neq 0 \) for \( \nu \neq 0 \). This results from the fact that in classical electrodynamics a stationary charged particle does not produce a magnetic field; a pure gauge vector potential is insignificant because it does not have any effect (in the context of the classical electrodynamics) on the dynamics of a charged particle described by the Lorentz force law. However, in the present situation, if we take the pure gauge vector potential into account then the difficulty can be resolved. Instead of the relations (74) and (75), we now consider the potentials and the field strengths

\[
\begin{align*}
E_A &= \frac{\kappa q}{r^3} r, \quad \phi_A = -\frac{kq}{r}, \quad B_A = 0, \quad A_A = \nabla \chi \\
E_B &= -\frac{kq}{r^3} r, \quad \phi_B = \frac{kq}{r}, \quad B_B = 0, \quad A_B = -\nabla \chi,
\end{align*}
\]

(86)

(87)

where \( \chi \) is an arbitrary function of the coordinate variables. In this case since \( \Phi_{\mu} \neq 0 \) for all \( \mu \), the relation \( \partial_{\nu} \Lambda_{\mu}^0 = \eta_{\mu\nu\sigma} \Phi_{\nu} \Lambda_{\sigma}^0 \) can be satisfied by a suitable specification of the function \( \eta_{\mu\nu\sigma} \). The reduced form (80) of the Ricci tensor is written explicitly in terms of the field strengths of the system as

\[
R_{\mu\nu} = \frac{k^2 q^2}{r^6} \begin{pmatrix}
r^2 & 0 & 0 & 0 \\
0 & x^2 & xy & xz \\
0 & xy & y^2 & yz \\
0 & xz & yz & z^2
\end{pmatrix},
\]

(88)

If a charged particle is located in the coupled fields of particles \( A \) and \( B \) then we assume that its dynamics is influenced by this reduced form of the Ricci tensor. In terms of the electromagnetic field strengths of the two opposing fields, the dynamical effect on the charged particle caused by the quantity (88) is regarded as a non-electromagnetic effect, since as we discussed earlier, dynamical effects on a charged particle due to the opposing electromagnetic fields are cancelled. The problem now is to look for a dynamical equation to describe the effect of the quantity (88) on the motion of the charged particle. Since in classical physics there are no other known forces, besides the gravitational force, we identify this effect with gravity and formulate the problem in terms of the field equations of general relativity. The procedure is to geometrise the physical object (88) and to specify its physical effect by defining an energy-momentum tensor in which the concept of the mass of a particle is introduced. This is similar to the case where the charge of a particle is introduced into physics for the purpose of formulating the electrodynamics of a charged particle (see, e.g., [23]). We note
from (88) that the effect is dominated by the term \( R_{00} \) for regions far from the origin at which the charged particle is located. This result is analogous to the weak field approximation in general relativity in which Newton’s law of gravitation is recovered [31].

Having asserted that gravity can be described geometrically in terms of two coupled electromagnetic fields, it seems natural to pose the question as to whether the connection \( \Lambda^\sigma_{\mu\nu} \) itself also produces physical effects. It is observed that the Ricci tensor (77) can be put in the form

\[
R_{\mu\nu} = \frac{\partial J^\sigma_{\mu\nu}}{\partial x^\sigma},
\]

where the quantities \( J^\sigma_{\mu\nu} \) are defined by

\[
J^\sigma_{\mu\nu} = \Lambda^\sigma_{\mu\nu} - \delta^\sigma_{\nu} \Lambda^\lambda_{\mu} \Phi^\lambda.
\]

In this case the equation \( R_{\mu\nu} = 0 \) results in conservation laws. For example, the energy density component \( R_{00} \) leads to an equation of continuity formed by the potential and the field strength, i.e.,

\[
\frac{\partial (\mathbf{A} \cdot \mathbf{E})}{\partial t} + \nabla \cdot (\phi \mathbf{E}) = 0,
\]

where \( \Phi_{\nu} = (\phi, -\mathbf{A}) \), and \( \mathbf{E} \) is the corresponding electromagnetic field strength of the opposing field.

To conclude this section we remark that in the case where two electromagnetic fields do not cancel out completely, it is possible to resolve the fields at each point in a region of spacetime into a pair of two opposing electromagnetic fields of equal magnitude, and a ‘residual’ field which acts as a single ‘conventional’ electromagnetic field; whence, we have the situation where a charged particle is considered to move under the influence of a combined gravitational and electromagnetic field. If we denote by \( \Lambda_{\mu}^{\nu} \), the connection formed by the opposing fields, where \( \Omega_{\nu} \) represents the four-potential of one of the two opposing fields, and \( \Phi_{\mu} \) is the four-potential of the single electromagnetic field, then the change \( \delta V^\mu \) in the components of vector \( V^\mu \) under an infinitesimal parallel displacement is given by

\[
\delta V^\mu = -(\Lambda_{\mu}^{\nu} \Omega_{\nu} + \beta \Phi_{\nu} V^\mu) dx^\nu.
\]

It is seen that the total connection is identified with the quantities \( \Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_{\mu\nu} + \beta \delta^\sigma_{\nu} \Phi_{\nu} \).

The covariant derivative of a tensor now takes the form

\[
\nabla_\sigma A^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} = \frac{\partial A^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}}{\partial x^\sigma} + \Gamma^\mu_{\lambda\sigma} A^{\mu_1 \ldots \lambda \ldots \mu_m}_{\nu_1 \ldots \nu_n} - \Gamma^\lambda_{\nu\sigma} A^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \lambda \ldots \nu_n} + (m - n) \beta \Phi_{\sigma} A^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n}.
\]

The curvature tensor can now be written as

\[
R^\sigma_{\beta\mu\nu} = \frac{\partial (\Lambda^\sigma_{\beta\nu})}{\partial x^\mu} - \frac{\partial (\Lambda^\sigma_{\beta\mu})}{\partial x^\nu} + \beta \left( \frac{\partial (\delta^\sigma_{\beta} \Phi_{\nu})}{\partial x^\mu} - \frac{\partial (\delta^\sigma_{\beta} \Phi_{\mu})}{\partial x^\nu} \right),
\]

and the Ricci tensor reduces to a simple form

\[
R_{\mu\nu} = \frac{\partial (\Lambda^0_{\mu\nu})}{\partial x^\mu} - \frac{\partial (\Lambda^0_{\nu\mu})}{\partial x^\nu} + \beta \left( \frac{\partial \Phi_{\nu}}{\partial x^\mu} - \frac{\partial \Phi_{\mu}}{\partial x^\nu} \right).
\]

If the parts formed by the first and second terms on the right of this Ricci tensor is reduced to a symmetric form, for example, by utilising the relation \( \partial_\nu \Phi^0_{\mu} = \eta_{\mu\nu} \partial_\nu \Lambda^0_{\mu} \), then it can be used to describe both the gravitational and electromagnetic field, whence gravitation can be incorporated consistently into the electromagnetic spacetime structure.
4.5 Strong interaction as a coupling of two strong fields

At short range we have demonstrated that it is possible to describe the strong interaction by the field equations of general relativity. We now look for a mechanism responsible for this interaction. We conjecture that it is a spacetime structure that can be formulated in terms of differential geometry. As in the case of gravity, which may be thought of as the coupling of two electromagnetic fields, we postulate an affine connection of the form \( \Gamma_{\mu}^{\sigma} = \Lambda_{\sigma}^{\sigma} \Phi_{\nu} \).

However, the relation between the quantities \( \Lambda_{\mu}^{\sigma} \) and \( \Phi_{\nu} \) is now given by

\[
\Phi_{\sigma} \frac{\partial \Lambda_{\mu}^{\sigma}}{\partial x^{\nu}} = -C_{\nu\sigma}^{\alpha\beta} \Phi_{\alpha} \Lambda_{\mu}^{\alpha},
\]

(96)

where the quantities \( C_{\nu\sigma}^{\alpha\beta} \) are arbitrary, but are assumed antisymmetric with respect to the subscript indices. The quantity \( \Phi_{\mu} \) is identified with the potential of the strong interaction, and the quantity \( \Lambda_{\nu}^{\mu} \) with the field strength. We further assume antisymmetry of the quantities \( \partial_{\nu} \Lambda_{\mu}^{\sigma} \), which is obtained by imposing the condition \( \partial_{\nu} \Lambda_{\mu}^{\sigma} + \partial_{\sigma} \Lambda_{\mu}^{\nu} = 0 \). With these assumptions, the Ricci tensor (77) then takes the form

\[
R_{\mu\nu} = \left( \frac{\partial \Phi_{\nu}}{\partial x^{\sigma}} - \frac{\partial \Phi_{\sigma}}{\partial x^{\nu}} + C_{\nu\sigma}^{\alpha\beta} \Phi_{\alpha} \Phi_{\beta} \right) \Lambda_{\mu}^{\sigma}
= F_{\sigma\nu} \Lambda_{\mu}^{\sigma},
\]

(97)

where the quantity \( F_{\mu\nu} \) is defined by the relation

\[
F_{\mu\nu} = \frac{\partial \Phi_{\nu}}{\partial x^{\mu}} - \frac{\partial \Phi_{\mu}}{\partial x^{\nu}} + C_{\mu\nu}^{\alpha\beta} \Phi_{\alpha} \Phi_{\beta}.
\]

(98)

In this form the quantities \( F_{\mu\nu} \) can be regarded as a tensor field for the strong interaction. As with the gravitational field, all strong sources produce attractive strong fields, whence it is possible to identify the quantities \( \Lambda_{\mu}^{\sigma} \) with the strong field \( F_{\mu\sigma} \). Since the product of two antisymmetric tensors results in a symmetric tensor, the Ricci tensor has been reduced to a symmetric form, so that the field equations of general relativity can be applied. As with the case of the gravitational field, the reduced form (97) of the Ricci tensor behaves like a tensor only under the linear group of coordinate transformations. Similarly, the reduced form (97) of the Ricci tensor (77) must be geometrised in order to investigate the dynamics of a particle under its influence. This can be carried out by introducing a metric onto the spacetime manifold, as in the case of a charged particle under the influence of two coupled electromagnetic fields. The field equations that govern the metric tensor are assumed to take the form of the field equations of general relativity, since it has already been shown that the field equations of general relativity admit a line element of a Yukawa potential. However, there is a fundamental difference between the present situation and that discussed for two opposing electromagnetic fields. In the present case, since the field \( \Lambda_{\mu}^{\nu} \) is considered to be identical to the field \( F_{\mu\nu} \), the effect of coupling of the two fields \( \Lambda_{\mu}^{\nu} \) and \( F_{\mu\nu} \), which gives rise to the strong interaction is additive rather than subtractive as in the case of two opposing electromagnetic fields.

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