Integrable structure of the low-energy string gravity equations in $D = 4$ space-times with two commuting isometries

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Abstract

The generalized Einstein - Maxwell field equations which arise from a truncated bosonic part of the low-energy string gravity effective action in four dimensions (the so called Einstein-Maxwell - axion - dilaton theory) are considered. The integrable structure of these field equations for $D = 4$ space-times with two commuting isometries is elucidated. We express the dynamical part of the reduced equations as integrability conditions of some overdetermined $4 \times 4$-matrix linear system with a spectral parameter. The remaining part of the field equations are expressed as the conditions of existence for this linear system of two $4 \times 4$-matrix integrals of special structures. This provides a convenient base for a generalization to these equations of various solution generating methods developed earlier in General Relativity.

1 Introduction

The generalized Einstein - Maxwell field equations considered below describes a specifically coupled to gravity massless fields (axion, dilaton and one $U(1)$ gauge ("electromagnetic") fields). The corresponding action (representing a truncated

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bosonic sectors of some low energy string gravity models in four dimensions) takes in Planck units the well known form:

\[ S = -\frac{1}{16\pi} \int \left\{ R^{(4)} - 2\nabla_k \phi \nabla^k \phi + e^{-2\phi} F_{ik} F^{ik} - \frac{1}{12} e^{-4\phi} H_{ijk} H^{ijk} \right\} \sqrt{-g} d^4 x \quad (1) \]

where the indices \( i, j, k, \ldots \) are 4-dimensional, the metric \( g_{ik} \) possess the “most negative” Lorentz signature, \( \phi \) is the dilaton field and the Maxwell two-form \( F_{ik} \) and a three-form \( H_{ijk} \) are related to the gauge field \( A_k \) and the antisymmetric tensor \( B_{ik} \) by \( F_{ik} = dA_k \) and \( H_{ijk} = dB_{ik} - 2A_k \wedge F_{ik} \). In \( D = 4 \) space-time, \( F_{ik} \) and \( H_{ijk} \) can be characterized also by their dual fields \( \tilde{F}_{ik} \) and \( \tilde{a}^i \) respectively, and the field equations for the action (1) imply the existence of the “magnetic” vector \( B_i \) and the scalar \( a \) potentials for these dual fields (\( \varepsilon^{ijkl} \) is the 4-dimensional Levi-Civita tensor):

\[
H_{ijk} = \varepsilon_{ijkl} a^l, \quad F_{ik} = -\frac{1}{2} \varepsilon_{ijkl} \tilde{F}^{kl},
\]

\[
a_i = e^{4\phi} \nabla_i a, \quad \tilde{F}_{ik} = e^{2\phi} \left[ \partial_i B_k - \partial_k B_i + a \left( \partial_i A_k - \partial_k A_i \right) \right].
\]

Construction of exact solutions for the gravity model (1) and investigation of their physical and geometrical properties has drawn much attention in the studies of the last decade. These studies revealed a nice internal symmetry structure of these field equations for space-times with two commuting isometries. In [1] it was shown that in the particular case of vanishing electromagnetic field these equations are integrable. In this case, the dynamical variables and corresponding reduced equations split into two independent sectors (pure “gravitational” sector and the axion-dilatonic one) each governed by the Ernst \( \sigma \)-model equations. In the presence of \( U(1) \) gauge field, the structure of the reduced dynamical equations can be expressed in the form of specific complex symmetric \( 2 \times 2 \)-matrix generalization of the known vacuum Ernst equation in General Relativity. This type of \( 2 \times 2 \)-matrix generalization of of the Ernst equation had been derived in [2]. The symmetries of the dynamical equations for various effective string gravity models with isometries have been studied extensively (see, for example, [3], [4] and the references therein).

A number of attempts were made for the analysis of integrability and applications of some known solution generating methods for these equations. In particular, in [5] it was shown that a Belinskii - Zakharov - like spectral problem can be associated with the reduced field equations corresponding to the action (1). Another kind of associated linear system has been suggested recently in [6]. Some applications of the linear problem [5] for construction of particular form of soliton solutions have been considered in [7], [8]. Another way, based on the so called “monodromy transform” approach, was outlined in [9] (see also the references therein). One can notice, however, that in these studies the integrable structure of the equations under consideration has not been elaborated in all necessary details and any nontrivial solution generating methods have not been realized in these approaches.
For a construction of some effective solution generating methods one usually needs to reformulate the dynamical part of the field equations as some equivalent spectral problem, i.e. as integrability conditions of some associated linear system with spectral parameter, supplied with all necessary constraint conditions (such as, for example, reality of some variables, given canonical structures, symmetric or coset structures of auxiliary matrix functions, etc.) also expressed more or less explicitly as the constraints on the solutions of this linear system. Such reformulation of the reduced dynamical equations for the action (1) for space-times with two commuting isometries, following the approach [9], is the purpose of the present paper.

2 Space-times with isometries

We restrict our consideration of the gravity model (1) to the field configurations whose components and potentials all are functions of two of the four space-time coordinates only. These includes two physically important cases: the fields depending on the time and only one spatial coordinate (the hyperbolic case) or on two spatial coordinates only (the elliptic case). Both of these cases are considered below in the same manner and the only sign symbol \( \epsilon \) (\( \epsilon = 1 \) and \( \epsilon = -1 \) for the hyperbolic and the elliptic case respectively) will recall us about the difference between them.

2.1 Structure of the metric components

The structure of the action (1) together with our space-time symmetry conjecture imply that the space-time metric can be considered in the block-diagonal form

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{ab}dx^a dx^b
\]

where \( \mu, \nu = 1, 2, a, b, \ldots = 3, 4 \) and the metric components \( g_{\mu\nu} \) and \( g_{ab} \) depend on the coordinates \( x^\mu \) and not on \( x^a \).\(^1\) An appropriate choice of the coordinates \( x^\mu \) allows to present the metric components \( g_{\mu\nu} \) in a conformally flat form:

\[
g_{\mu\nu} = f \eta_{\mu\nu}, \quad \eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad \epsilon_1 = \pm 1, \quad \epsilon_2 = \pm 1
\]

where the conformal factor \( f(x^\mu) > 0 \). For \( g_{ab} \) we use a parametrization

\[
g_{ab} = \epsilon_0 \begin{pmatrix} H & H\Omega \\ H\Omega & H\Omega^2 + \epsilon \alpha^2 \end{pmatrix}, \quad \epsilon_0 = \pm 1, \quad \det \|g_{ab}\| \equiv \epsilon \alpha^2, \quad \epsilon \equiv -\epsilon_1 \epsilon_2
\]

\(^1\)We do not specify here whether the time-like coordinate is among the coordinates \( x^a \), or it is one of the “ignorable” coordinates \( x^a \).
where the metric functions $\alpha(x^\mu) > 0$, $H(x^\mu) > 0$ and $\Omega(x^\mu)$ are introduced and the value of the sign symbol $\epsilon_0$ as well as the relation between the sign symbols $\epsilon_1$, $\epsilon_2$ and $\epsilon$ should provide the Lorentz signature of (2).

### 2.2 Geometrically defined coordinates

It is easy to observe that in accordance with the field equations for the action (1), the function $\alpha$ should satisfy the linear two-dimensional equation of the form $\eta_{\mu\nu}\partial_\mu\partial_\nu\alpha = 0$ where the matrix $\eta_{\mu\nu}$ is inverse to $\eta_{\mu\nu}$. This is the d’Alambert (for $\epsilon = 1$) or Laplace (for $\epsilon = -1$) equation. It allows to introduce the function $\beta(x^\mu)$ “harmonically” conjugated to $\alpha$ using the relation: $\partial_\mu\beta = -\epsilon\epsilon_{\mu\nu}\partial_\nu\alpha$ where $\epsilon_{\mu\nu} = \eta_{\mu\gamma}\epsilon^{\gamma\nu}$, $\epsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These geometrically defined functions can be used for construction of a convenient pair of real null coordinates in the hyperbolic case or complex conjugated to each other coordinates in the elliptic case:

- $\xi = \beta + j\alpha$
- $\eta = \beta - j\alpha$

$\jmath = \begin{cases} 1, & \epsilon = 1 \quad \text{-- the hyperbolic case,} \\ i, & \epsilon = -1 \quad \text{-- the elliptic case.} \end{cases}$

### 2.3 Matter fields and their potentials

The space-time symmetry described above implies that the following gauge conditions can be imposed on the potentials $A_i$, $B_i$ and $B_{ik}$:

- $A_\mu = 0$, $B_\mu = 0$, $B_{\mu\nu} = 0$, $B_{\mu a} = 0$

In this case, the fields $F_{ik}$ and $H_{ijk}$ are represented by their non-vanishing components $A_a$, $B_{ab}$ depending on the coordinates $x^\mu$ only. The antisymmetric matrix $B_{ab}$ can be expressed in the form $B_{ab} = \Theta\epsilon_{ab}$, where $\epsilon_{ab}$ is the antisymmetric Levi - Civita symbol. The components $B_a$ and the scalar function $\Theta$ can be used also instead of the components $A_a$ and the axion field $a$. Thus, any of the field configuration under our consideration is described by a set of dynamical variables

$$g_{ab}, \ A_a \ (or \ B_a), \ a \ (or \ \Theta), \ \phi$$

representing the components of metric, gauge, axion and dilaton fields respectively.

### 3 Symmetry reduced field equations

The dynamical part of the field equations of the string gravity with the action (1), similarly to vacuum Einstein equations ($2 \times 2$-matrices) and electrovacuum Einstein
- Maxwell field equations (3 × 3-matrices) in General Relativity, can be written in some complex 4 × 4-matrix form:

\[
U_\eta + V_\xi + \frac{1}{2\iota j\alpha}[U, V] = 0, \quad U_\eta - V_\xi = 0,
\]  

where \( \alpha \equiv (\xi - \eta)/2j \), and the complex 4 × 4 matrices \( U \) and \( V \) should possess a universal (viz. solution independent) canonical Jordan forms \( U_0 \) and \( V_0 \):

\[
U_0 = \text{diag} \{ i, i, 0, 0 \}, \quad V_0 = \text{diag} \{ i, i, 0, 0 \}
\]

Besides that, the matrices \( U \) and \( V \) should satisfy additional constraints. To describe these, we define at first a Hermitian matrix \( G \) associated with every pair of matrices \( U \) and \( V \):

\[
2dG = \Omega dU - dU^\dagger \Omega, \quad dU \equiv U d\xi + V d\eta,
\]

where \( ^\dagger \) stands for a Hermitian conjugation of matrices (and matrix-valued 1-forms) and \( \Omega \) is a constant matrix defined just below. Then the corresponding algebraic constraints on \( U \) and \( V \) can be expressed by the relations:

\[
GU = ij\alpha \Omega U, \quad \Omega U + U^T \Omega = i\Omega, \\
G V = -ij\alpha \Omega V, \quad \Omega V + V^T \Omega = i\Omega, \\
\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

where \( ^T \) means a matrix transposition and \( I \) denotes a 2 × 2 unit matrix. After an appropriate identification of the components of \( U, V \) and \( G \) a direct calculations show that (4) - (7) are equivalent to the reduced field equations for (1).

4 Generalized (matrix) Ernst equation

Similarly to the case of vacuum Einstein equations in General Relativity, the equations (4) - (7) can be reduced to one complex matrix equation for a symmetric complex 2 × 2-matrix function, generalizing the well known vacuum Ernst equation for a complex scalar Ernst potential and coinciding with such equation found in [2]. For this, using the upper right 2 × 2-blocks \( U_{(1)}^{(2)} \) and \( V_{(1)}^{(2)} \) of the 4 × 4-matrices \( U \) and \( V \) we can define a generalized matrix Ernst potential by the differential relation \( dE = -U_{(1)}^{(2)} d\xi - V_{(1)}^{(2)} d\eta \). In the most general form which includes both the hyperbolic and the elliptic cases this matrix Ernst equation takes the form

\[
\begin{cases}
\eta^{\mu\nu} \left( \partial_\mu + \frac{\partial_\mu \alpha}{\alpha} \right) \partial_\nu E - \eta^{\mu\nu} \partial_\mu E \cdot (\text{Re } E)^{-1} \cdot \partial_\nu E = 0, \\
\eta^{\mu\nu} \partial_\mu \partial_\nu \alpha = 0
\end{cases}
\]
In the geometrically defined coordinates \((\xi, \eta)\) introduced in the previous section, this equation takes a more simple form \((\alpha \equiv (\xi - \eta)/2j)\):

\[
E_{\xi\eta} - \frac{1}{4j\alpha}(E_{\xi} - E_{\eta}) - \frac{1}{2}(E_{\xi} \cdot (\text{Re} E)^{-1} \cdot E_{\eta} + E_{\eta} \cdot (\text{Re} E)^{-1} \cdot E_{\xi}) = 0
\]

### 5 Equivalent spectral problem

Generalizing the similar constructions for vacuum Einstein equations and electrovacuum Einstein-Maxwell field equations \([10]-[12]\), we consider the following spectral matrix problem for the four \(4 \times 4\)-matrix functions \(\Psi(\xi, \eta, w), U(\xi, \eta), V(\xi, \eta)\) and \(W(\xi, \eta, w)\) (\(w\) is a complex “spectral” parameter), which should satisfy the following three groups of conditions.

- The overdetermined linear system for \(\Psi\) whose matrix coefficients \(U, V\) should possess a universal (i.e. solution independent) canonical forms:
  \[
  \begin{align*}
  2i(w - \xi)\partial_\xi \Psi &= U \Psi, \\
  2i(w - \eta)\partial_\eta \Psi &= V \Psi
  \end{align*}
  \]
  \[
  \begin{align*}
  U &= F_+ U_0 F_+^{-1}, \\
  V &= F_- V_0 F_-^{-1}
  \end{align*}
  \]
  with some transformation matrices \(F_\pm\) depending on the field variables.

- This system should admit a Hermitian matrix integral \(K(w)\):
  \[
  \Psi^\dagger W \Psi = K(w), \quad K^\dagger(w) = K(w) \quad \frac{\partial W}{\partial w} = i\Omega
  \]
  where \(K(w)\) is coordinate independent and \(W\) depends on the field variables.

- This system should admit also an antisymmetric matrix integral \(L(w)\)
  \[
  \Sigma \cdot \Psi^T \Omega \Psi = L(w), \quad L^T(w) = -L(w), \quad \Sigma^2 = (w - \xi)(w - \eta).
  \]
  where \(\Sigma\) is an auxiliary scalar function.

It can be shown directly (see, for the method details [11]), that solution of this spectral problem is equivalent to solution of \([4]-[7]\), and therefore, to solution of the symmetry reduced field equations for the string gravity model \([1]\).

### 6 Calculation of the field components

In this concluding section we describe a relation between the reduced field equations and the spectral problem formulated above. For this we present here a general explicit expression of the components of the matrix \(W\) in terms of the field components: \(W \equiv i(w - \beta)\Omega + G\) with
\[ G = (F^{-1})^\dagger \left( \begin{array}{cccc}
-e_0 H^2 - e_0 \frac{\epsilon \alpha^2}{H} & e_0 H \Omega & 0 & 0 \\
e_0 H \Omega & -e_0 H & 0 & 0 \\
0 & 0 & e^{-2\phi} \Theta^2 + \epsilon \alpha^2 e^{2\phi} & -e^{-2\phi} \Theta \\
0 & 0 & -e^{-2\phi} \Theta & e^{-2\phi}
\end{array} \right) F^{-1} \]

where we see two diagonal $2 \times 2$ blocks which possess pure gravitational and axion-dilatonic nature respectively, while the $F$-multipliers are of pure gauge nature:

\[ F = \left( \begin{array}{cccc}
1 & 0 & \sqrt{2}A & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & \sqrt{2}A & 0 \\
\sqrt{2}A & -\sqrt{2}A & 0 & 1
\end{array} \right). \]

We denote here $A_a = \{ A, \tilde{A} \}$. It is useful to notice also that the spectral problem presented above possess an obvious gauge symmetry of the form

\[ \Psi \rightarrow A \cdot \Psi \quad U \rightarrow A \cdot U \cdot A^{-1} \]
\[ W \rightarrow A^\dagger \cdot W \cdot A^{-1} \quad V \rightarrow A \cdot V \cdot A^{-1} \]

where the constant matrix $A$ is real and satisfies the constraint $A^T \cdot \Omega \cdot A = \Omega$. However, this transformation possesses a nontrivial nature. In particular, it can mix the mentioned above sectors and generate solutions with nonzero gauge fields $A_i$ from pure vacuum solutions.

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