Ward identity for loop corrected soft photon theorem for massless QED coupled to gravity.

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**Abstract**

We study the loop level asymptotic conservation law proposed by Campiglia and Laddha [1] for massless scalar QED in presence of dynamical gravity. We show that the equivalence between the loop level Ward identity and the Sahoo-Sen soft photon theorem [2] continues to hold. In presence of gravity, the new feature is that photons also acquire a dressing due to long range gravitational force and contribute to the charge. Similar to the massive case, the terms in the asymptotic charge are directly related to the dressing of scalars and photons due to long range forces.
1 Introduction and Result

Asymptotic symmetries strongly constrain low energy physics of gauge theories [3–5]. Leading soft theorems are manifestations of asymptotic symmetries. Soft theorems are statements about universal properties of amplitudes in the limit when energy of some of the interacting massless particles is taken to be small [7–10].\(^1\) The equivalence between the two was first demonstrated for gravitational theories in the seminal paper [11]. The asymptotic group for perturbative gravity is the so called (extended) BMS group [12, 13] which is an infinite dimensional subgroup of diffeomorphisms. In [4], it was conjectured that BMS is an exact symmetry of gravitational S-matrix. Further in [11], the authors proved that the Ward identity corresponding to the BMS supertranslation charges is exactly equivalent to Weinberg’s soft graviton theorem. Similar analysis was carried out

\(^1\)Interested readers can look up the references of [2] and [37] for recent literature on Soft theorems.
for QED in [5, 6]; it was shown that leading soft photon theorem is equivalent to the Ward identity of corresponding asymptotic symmetries that are called the large gauge transformations and constitute an infinite dimensional subgroup of U(1) gauge transformations. The leading soft theorems and the corresponding Ward identities for photons and gravitons are true to all loop orders and hence are exact quantum statements.

Analogous investigations have been carried out to understand the possible symmetry origins of tree level subleading soft theorems. In [15, 16], the authors had proposed an extension of original BMS group to include local conformal transformations on the celestial sphere (i.e. a Virasoro algebra). In [14], it was shown that subleading soft graviton theorem indeed implies existence of a Virasoro symmetry and was proposed to be related to above mentioned extension of BMS group. In [17–19], the authors proposed a different extension of BMS group that includes local diffeomorphisms of $S^2$ and showed that the corresponding Ward identities are exactly equivalent to tree level subleading soft graviton theorem.

Ward identity corresponding to Low’s subleading photon theorem has been similarly studied in [20–23]. The symmetry underlying this Ward identity or its relation to U(1) gauge group is not clear. In [39], the authors proved an infinite hierarchy of conservation laws for classical electromagnetism and showed that quantum version of one of these laws is equivalent to Low’s subleading soft photon theorem. The authors also provide evidence that suggests that this entire hierarchy is equivalent to the infinite hierarchy of tree level soft theorems proved in [24]. Thus, subleading soft theorems in QED can be related to asymptotic conservation laws though the question of existence of a well defined underlying symmetry still persists. The story is similar for gravity beyond subleading order: Ward identities for tree level graviton soft theorems beyond subleading order were studied in [25, 26] and the authors pointed out similar questions about the corresponding symmetry.

Beyond the leading order, soft theorems receive non-trivial loop corrections in four spacetime dimensions as shown in [27–29]. A part of these loop corrections are divergent. In [30], the authors studied the effect of these divergent terms on the tree level Virasoro Ward identity and showed that the divergence can absorbed by renormalising the stress tensor. In the seminal paper [2], the authors extended the regulating technique introduced in [32] and used it to show that loop effects lead to a new logarithmic term in soft expansion of amplitudes in four spacetime dimensions. They explicitly obtained the coefficient of the log term through Feynmann diagram calculations and showed that it is universal for both gravity and QED. In [42], the authors discussed the direct experimental consequence of above loop corrected soft graviton theorem. Interestingly, they have predicted a new tail to the well known memory effect [34–36].

In [1], the authors proposed a new loop level conservation law and reproduced above Sahoo-Sen soft theorem for QED coupled to massive scalars. This is quite a remarkable result given the fact that the loop level soft factor has a very complicated structure [2]. The authors also showed that this loop correction is a direct effect of long range electromagnetic
force on massive fields and established a one to one correspondence between the loop-level charge and the Fadeev-Kulish dressing of massive particles [31]. Like the case for tree level subleading charge, the symmetry associated to the loop level asymptotic charge is not clear.

In this paper, our aim is to construct the analogue of this charge for massless scalar QED in presence of gravitational couplings and show that the Ward identity reproduces the soft theorem. Let us state the Sahoo-Sen soft photon theorem in presence of gravitational couplings and massless scalars [2]:

\[ M_{n+1}(p_i, k) = \frac{S_0}{\omega} M_n(p_i) + S_{\log} \log \omega \ M_n(p_i) + \ldots, \]

here, \( S_0 = \sum_i e_i \frac{\varepsilon_{p_i}}{p_i \cdot q} \) is the leading soft factor,

\[
S_{\log} = -\frac{ig}{4\pi} \sum_i e_i \frac{\varepsilon_{p_i}}{p_i \cdot k} \sum_{\eta_j=1} k \cdot p_j + \frac{ig}{4\pi} \sum_{i \neq j} \frac{e_i \varepsilon_{p_i}}{p_i \cdot k} \left[ p_j^\mu p_i^\mu - p_j^\mu p_i^\mu \right] \\
- \frac{1}{4\pi^2} \sum_{i \neq j} e_i \varepsilon_{p_i} \sum_j k \cdot p_j \log p_j \cdot q,
\]

here, \( \varepsilon \) is the polarisation vector for the soft photon and \( k = \omega q \) is the soft momentum. The indices \( i, j \) take values from 1 to \( n \), where \( n \) is the number of hard particles. The momenta are defined including \( \eta \) factors such that \( \eta_i = 1(-1) \) for outgoing (incoming) particles. In above expression, we have introduced \( g \) to keep track of gravitational terms. (We later set \( g = 8\pi G=1 \).)

**Discussion of the result:**

The soft theorem can be rewritten as:

\[
\lim_{\omega \to 0} \omega \partial^2_{\omega} \omega \ M_{n+1}(p_i, k) = S_{\log}.
\]

This paper shows that above relation is equivalent to the Ward identity:

\[
[Q, S] = 0,
\]

where \( Q \) is the loop level charge [1]. To simplify calculations, we consider an amplitude such that all the hard particles are scalars. The charge receives contribution from both electromagnetic and gravitational dressing of massless scalars. Thus, this part of the charge is directly related to the late time acceleration of scalars under the long range forces. The dressing of massless scalar is given by (31):

\[
\phi(x) = -\frac{ie^{iA_0}(\hat{x}) \log r}{8\pi^2 r} \int d\omega \left[ b(\omega, \hat{x}) e^{-i\omega u} e^{i\omega \log r \frac{k_{L}(\hat{x})}{2}} - d^l(\omega, \hat{x}) e^{i\omega u} e^{-i\omega \log r \frac{k_{L}(\hat{x})}{2}} \right].
\]
\( A^1_{\sigma}, h^{1}_{\sigma} \) have been defined in (10) and (11) respectively. Though the precise structure of dressing of massless scalars is different from that of massive scalars [1], the contribution to the charge has very similar structure. Photons also acquire gravitational dressing (58):

\[
A_\sigma(x) = -\frac{i}{8\pi^2 r} \int d\omega \left[ a_\sigma(\omega, \hat{x}) e^{-i\omega u} e^{i\omega \log(r \omega) h^{1}_{\sigma}(\hat{x})} - a_\sigma^\dagger(\omega, \hat{x}) e^{-i\omega u} e^{-i\omega \log(r \omega) h^{1}_{\sigma}(\hat{x})} \right].
\]

The leading order \( \log r \) gravitational dressing factor of photons and massless scalars is similar. Photons acquire additional \( \log \omega \) dressing and this additional dressing term contributes to the charge. This is consistent with the proposals of [2, 41]. This part of the charge corresponds to the late time acceleration of the soft photon under long range gravitational force.

The two terms in the first line of (1) are present in the classical radiative field (in soft limit) [41]; the last two lines are absent in soft classical radiation. We see that the classical terms are purely gravitational, thus an interesting feature for QED coupled to massless scalars (in absence of gravity) is that there is no classical \( \log \omega \) term. At the level of dressing too, we see that the classical part of electromagnetic dressing factor becomes trivial (59) and there is no corresponding contribution to charge. In [1], the authors pointed out that there is a purely quantum mode in photon field and derived the 'quantum' part of \( \log \omega \) coefficient from this mode. Analogue of this purely quantum mode exists for metric field as well and the last two lines of (1) can be obtained from these quantum modes of photon and graviton fields. It is interesting to note that these quantum contributions have diverging pieces (coming from collinear configurations) that cancel out due to conservation of momentum and electric charge respectively.

The organisation of this paper is as follows: In section 2, we discuss some preliminaries and review conservation equations. In sections 3 and 4, we discuss dressing of massless scalars and photons respectively and also write down the contribution of the dressings to the asymptotic charge. In section 5, we discuss the coulombic and radiative modes in photons and gravitons that contribute to the charge. In section 6, we finally write down the Ward identity for the asymptotic charge and show its equivalence to the Sen-Sahoo soft theorem.

## 2 Preliminaries

We consider a theory with U(1) gauge field \( A_\mu \) minimally coupled to massless scalar \( \phi \) and gravitational field \( g_{\mu\nu} \). So, our system is described by the action:

\[
S = -\int d^4x \sqrt{-g} \left[ \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) + R \right],
\]

where \( D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi \).

We are interested in asymptotic dynamics of above system. Massless particles end up at future null infinity \((t, r \to \infty)\) which is represented as \( I^+ \). To describe late time
dynamics of massless fields, we need to use retarded co-ordinate system. The flat metric takes following form in this co-ordinate system ($u = t - r$):

$$ds^2 = -du^2 - 2dudr + r^2 \gamma_{zz} \ dzd\bar{z}; \quad \gamma_{zz} = \frac{2}{(1 + z\bar{z})^2}.$$ We use $\hat{x}$ or $(z, \bar{z})$ interchangeably to describe points on $S^2$. A useful parametrisation of a 4 dimensional spacetime point is given by (Greek indices will be used to denote 4d cartesian components):

$$x^\mu = rq^\mu + ut^\mu, \quad q^\mu = (1, \hat{x}), \quad t^\mu = (1, \vec{0}).$$

Here, $q^\mu$ is a null vector that can be parameterised in terms of $(z, \bar{z})$ as

$$q = \frac{1}{1 + z\bar{z}} \{1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}\}.$$

Dynamics of scalar is given by:

$$g^{\mu\nu}D_\mu D_\nu \phi(x) = 0. \tag{5}$$

Solution to this equation can be expanded around future null infinity. Using stationary phase approximation, we can obtain the leading order coefficient in asymptotic expansion for massless scalars [5]:

$$\phi(u, r, \hat{x}) = \frac{1}{r} \phi^1(u, \hat{x}) + ... \tag{6},$$

where

$$\phi^1(u, \hat{x}) = -i \frac{8}{\pi^2} \int d\omega \left[b(\omega, \hat{x}) \ e^{-i\omega u} - d^\dagger(\omega, \hat{x}) \ e^{i\omega u}\right]. \tag{7}$$

Next we turn to the gauge field. Choosing Lorentz gauge $\partial_\mu A^\mu = 0$, Maxwell’s equations reduce to

$$\Box A_\mu = -j_\mu, \quad j_\mu = i e (\phi D_\mu \phi^* - \phi^* D_\mu \phi). \tag{8}$$

This is in absence of gravitaional couplings. The effect of gravity will be analysed in 4. With fall off in (6) for massless scalars, the currents admit following asymptotic fall offs:

$$j_\mu = \frac{j^2_\mu(u, \hat{x})}{r^2} + ..., \quad j_A = \frac{j^2_A(u, \hat{x})}{r^2} + ..., \quad j_r = \frac{j^2_r(u, \hat{x})}{r^4} + ..., \quad (A = z, \bar{z}). \tag{9}$$

Above and henceforth, we denote the vector components on $S^2$ by capital latin alphabets. Hence, the asymptotic behaviour of gauge field components is given by:

$$A_r = A^1_r(\hat{x}) + A^2_r(u, \hat{x}) \frac{\log r}{r^2} + ... ; \quad A_u = A^1_u(u, \hat{x}) \frac{\log r}{r} + A^2_u(u, \hat{x}) + ... ; \quad A_A = A^1_A(u, \hat{x}) + A^2_A(u, \hat{x}) \frac{\log r}{r} + ... . \tag{10}$$
We will work in the perturbative linear gravity regime where gravitational dynamics is confined to perturbations around flat space-time: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the de Donder gauge $\partial_{\mu}h^{\mu} = 0$, the metric field satisfies $\Box h_{\mu\nu} = -2T_{\mu\nu}$. Thus, the metric field admits following expansion: \(^2\)

\[
\begin{align*}
    h_{rr} &= \frac{h_{rr}^1(\hat{x})}{r} + h_{rr}^2(u, \hat{x}) \frac{\log r}{r^2} + \ldots, \\
    h_{ur} &= \frac{h_{ur}^1(u, \hat{x})}{r} + h_{ur}^2(u, \hat{x}) \frac{\log r}{r^2} + \ldots, \\
    h_{uu} &= h_{uu}^1(u, \hat{x}) \frac{\log r}{r} + h_{uu}^2(u, \hat{x}) \frac{\log r}{r^2} + \ldots, \\
    h_{rA} &= h_{rA}^0(u, \hat{x}) + h_{rA}^1(u, \hat{x}) \frac{\log r}{r} + \ldots, \\
    h_{uA} &= h_{uA}^0(u, \hat{x}) + h_{uA}^1(u, \hat{x}) \frac{\log r}{r} + \ldots, \\
    h_{AB} &= r h_{AB}^1(u, \hat{x}) + \log r h_{AB}^2(u, \hat{x}) + \ldots.
\end{align*}
\]

There exists similar asymptotic expansion of fields at past null infinity ($-t, r \to \infty$) represented by $\mathcal{I}^-$. 

2.1 Asymptotic conservation laws

Classical equations of motion can be used to derive conservation laws of the form: \(^3\)

\[ Q^+[\epsilon^+] \mid \mathcal{I}^+ = Q^-[\epsilon^-] \mid \mathcal{I}^- . \]

The quantum version of above statements can be related to soft theorems. In this section, we will review these conservation laws for QED. The Maxwell’s equations:

\[ \nabla^\nu F_{\sigma\nu} = j_\sigma, \]

can be expanded order by order at large $r$. Currents in (9) lead to following fall offs for the field strength:

\[
\begin{align*}
F_{rA} &= \frac{F_{rA}^2(u, \hat{x})}{r^2} + \ldots, \\
F_{ru} &= \frac{F_{ru}^2(u, \hat{x})}{r^2} + \ldots, \\
F_{uA} &= F_{uA}^0(u, \hat{x}) + \ldots, \\
F_{AB} &= F_{AB}^0(u, \hat{x}) + \ldots,
\end{align*}
\]

with the coefficients satisfying:

\[
\begin{align*}
    \partial_u F_{ru} + \partial_u B^B A^0_B &= j_u^2, \\
    \partial_u F_{rA} - \frac{1}{2} \partial_A F_{ru} + \frac{1}{2} D^B F_{AB} &= \frac{1}{2} j_A^2.
\end{align*}
\]

\(^2\)Some of the coefficients are independent of $u$, this follows from the de Donder gauge condition itself. \(^3\) $\mathcal{I}^+$ is the $u \to -\infty$ sphere of $\mathcal{I}^+$. Similarly, $\mathcal{I}^-$ is the $v \to \infty$ sphere of $\mathcal{I}^-$. 

7
Above equations can be used to study the u-fall offs of the field strength components $F^{2}_{ru}$ and $F^{2}_{rA}$ for given profile of free radiative data $A_{A}^{0}$. Using following behaviour for radiative data:

$$A_{A}^{0} = e^{\pm}(\hat{x})u^{0} + \mathcal{O}\left(\frac{1}{u^{\infty}}\right) \quad u \rightarrow \pm \infty,$$

$$A_{A}^{0} = e^{\pm}(\hat{x})u^{0} + \mathcal{O}\left(\frac{1}{u^{\infty}}\right) \quad u \rightarrow \pm \infty,$$

(16)

Around $u \rightarrow -\infty$, the field strength components admit following behaviour:

$$F_{ru} = \frac{1}{r^{2}} \left[ u^{0} F^{2,0}_{ru}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

$$F_{ru} = \frac{1}{r^{2}} \left[ u^{0} F^{2,0}_{ru}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

(17)

$$F_{rA} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + u^{0} F^{2,0}_{rA}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

$$F_{rA} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + u^{0} F^{2,0}_{rA}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

(18)

In [37], for specific physical processes, the authors showed that following relation holds between asymptotic values of the fields:

$$\frac{\partial}{\partial x}^{2,0} F_{ru}(\hat{x}) \mid_{x^{+}} = \frac{\partial}{\partial x}^{2,0} F_{ru}(\hat{\hat{x}}) \mid_{x^{-}}.$$

$$\frac{\partial}{\partial x}^{2,0} F_{ru}(\hat{x}) \mid_{x^{+}} = \frac{\partial}{\partial x}^{2,0} F_{ru}(\hat{\hat{x}}) \mid_{x^{-}}.$$

(19)

Above statement can be rewritten as conservation law for charges parameterized by a scalar function:

$$Q^{+}[\epsilon^{+}] \mid_{x^{+}} = Q^{-}[\epsilon^{-}] \mid_{x^{-}}.$$

$$Q^{+}[\epsilon^{+}] \mid_{x^{+}} = Q^{-}[\epsilon^{-}] \mid_{x^{-}}.$$

(20)

$Q^{+}[\epsilon^{+}] = \int d^{2}z \, \epsilon^{+}(\hat{x}) \, F^{2,0}_{ru}(\hat{x})$. $Q^{-}$ is defined analogously. And $\epsilon^{+}(\hat{x}) = \epsilon^{-}(\hat{\hat{x})}$. Above conservation law was proved for generic processes in [38]. The Ward identity for above charges is exactly equivalent to the leading soft photon theorem [37].

Similarly, in [39], the authors proved following conservation equation and showed that it is equivalent to the tree level subleading soft photon theorem.

$$\frac{\partial}{\partial x}^{2,0} F_{rA}(\hat{x}) \mid_{x^{+}} = \frac{\partial}{\partial x}^{2,0} F_{rA}(\hat{\hat{x}}) \mid_{x^{-}}.$$

$$\frac{\partial}{\partial x}^{2,0} F_{rA}(\hat{x}) \mid_{x^{+}} = \frac{\partial}{\partial x}^{2,0} F_{rA}(\hat{\hat{x}}) \mid_{x^{-}}.$$

(21)

In [1], the authors studied the leading order correction due to long range electromagnetic interaction on massive scalars (electromagnetic dressing) and showed that there are new logarithmic fall-off’s in the field strength. At tree level, in presence of massive fields the field strength components at future null infinity admit fall offs similar to (18) i.e.

$$F_{rA}|_{u \rightarrow -\infty} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + u^{0} F^{2,0}_{rA}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

$$F_{rA}|_{u \rightarrow -\infty} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + u^{0} F^{2,0}_{rA}(\hat{x}) + u^{-\infty} \right] + \ldots,$$

(22)

Including the long range electromagnetic effects this gets modified to:

$$F_{rA}|_{u \rightarrow -\infty} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + \log u F^{2,0}_{rA}(\hat{x}) + \ldots \right] + \ldots.$$

$$F_{rA}|_{u \rightarrow -\infty} = \frac{1}{r^{2}} \left[ u F^{2,0}_{rA}(\hat{x}) + \log u F^{2,0}_{rA}(\hat{x}) + \ldots \right] + \ldots.$$

(23)
Similarly, expansion around the past null infinity is given by:

\[ F_{rA}|_{v \to \infty} = \frac{\log r}{r^2} [v^0 F_{rA}^{0}(\hat{x}) + \ldots] + \frac{1}{r^2} [v F_{rA}^{2,0}(\hat{x}) + v^0 F_{rA}^{2,0}(\hat{x}) + v^{-\infty}] + \ldots . \]  

(24)

In [1], the authors proposed following new loop level conservation equation and showed that it is equivalent to the loop corrected soft photon theorem.

\[ \frac{2 \log F_{rA}(\hat{x})}{z^+} = \frac{\log_0 F_{rA}(\hat{x})}{z^-}. \]  

(25)

Now, we want to study the same conservation law in presence of massless scalars and dynamical gravity. So, we will first identify logarithmic fall-offs that arise due to dressing of massless scalars and photons\(^4\) and contribute to the conservation equation. The loop level soft theorem can then be derived from this conservation equation. From (15), we have,

\[ \partial_u^2 F_{rA} + \frac{1}{2} \partial_u \partial_A D^B A^0_B + \frac{1}{2} \partial_u D^B F^0_{AB} = \frac{1}{2} \partial_u j^2_A. \]  

(26)

So, for \( A = z \) component, we get:

\[ \partial_u^2 F_{rz}^2 = -D_z D^z \partial_u A^0_z + \frac{1}{2} \partial_u j_z^2. \]  

(27)

Above equation relates \( 1/u \) term in \( A^0_z \) to \( \log u \) term in \( F_{rz}^2 \) as noted in [1]. Thus, the tree level radiative fall-offs in (16) get modified due to long range forces to:

\[ A^0_A \sim e^{\pm u^0} + d^{\pm} \frac{1}{u} + \frac{1}{u^\infty} \ldots \quad u \to \pm \infty. \]  

(28)

In [42], the authors discuss the gravitational analogue of above \( 1/u \) term and show that it leads to a tail to the well known memory effect [34–36]. Electromagnetic memory effect due to \( 1/u \) term in photon data \( A^0_A \) has been discussed in [43]. In the next section, we study dressing of massless scalar and identify the resultant \( 1/u \) term in \( A^0_A \) that eventually contributes to (25).

3 Dressing of massless scalar field

In absence of long range forces, particles are free asymptotically. For massive fields the effect of long range forces can be obtained perturbatively by studying asymptotic potential order by order around \( t \to \infty \). This leads to the well known Fadeev-Kulish dressing of massive scalars [31]. For massless scalars, the asymptotic states live at null infinity. So,

\(^4\) Dressing of gravitons does not affect the Ward identity for photons.
we will study the corrections to the free equation of motion at null infinity. Massless scalars satisfy following equation:

\[ g^{\mu\nu} D_\mu D_\nu \phi(x) = 0. \tag{29} \]

Let us expand above equation around future null infinity. Using the fall-offs given in (10) and (11), we find that the leading order equation is (at \( \mathcal{O}(\frac{1}{r^2}) \)):

\[ -2\partial_u \partial_r \phi - \frac{2}{r} \partial_u \phi = \frac{h^1_{rr}(\hat{x})}{r} \partial^2_u \phi - 2ie \frac{A^1_r(\hat{x})}{r} \partial_u \phi. \tag{30} \]

Thus, the leading order effect of longe range forces on the massless field is given by \( h^1_{rr} \) and \( A^1_r \). The solution of above equation is given by:

\[ \phi(x) = -\frac{ie^{iA^1_r(\hat{x})} \log r}{8\pi r^2} \int d\omega \left[ b(\omega, \hat{x}) e^{-i\omega u} e^{i\omega \log r \frac{h^1_{rr}(\hat{x})}{r^2}} - d^\dagger(\omega, \hat{x}) e^{i\omega u} e^{-i\omega \log r \frac{h^1_{rr}(\hat{x})}{r^2}} \right], \tag{31} \]

where, \( b \) is the annihilation operator for free particles while \( d \) annihilates free antiparticles (see (7)). Analogous to the Fadeev-Kulish dressing for a massless scalar particle, the dressing factor associates a cloud of photons and gravitons to the free scalar particle. Next we find the correction to the U(1) current. Dressing of scalar field leads to a new logarithmic fall off in the current (9):

\[ j_A = j_A^{\log} \frac{\log r}{r^2} + \frac{2}{r^2} j_A^u + ... , \tag{32} \]

where

\[ j_A^{\log} = -\frac{1}{2} \partial_A h^1_{rr} j_u^2 + 2e^2 \partial_A A^1_r |\phi|^2. \tag{33} \]

Similarly, using the logarithmic fall off of the gauge field : \( A_r = A^1_r + A^{\log}_r \frac{\log r}{r^2} + ... \) in the expression of U(1) current we get a logarithmic term in the r-component of the current:

\[ j_r = j_r^{\log} \frac{\log r}{r^4} + \frac{j_r^4}{r^4} + ... . \]

Such that,

\[ j_r^{\log} = -2e^2 A^{\log}_r |\phi|^2. \tag{34} \]

Also,

\[ j_u = j_u^2 + j_u^{\log} \frac{\log r}{r^3} + ... . \]
Let us find the corrections to the gauge field due to these new logarithmic fall-offs in the current. In Lorentz gauge, we have $\Box A_\mu = -j_\mu$. (This equation admits corrections due to gravity; gravitational corrections will be analysed in section 4). So,

$$A_\sigma(x) = \frac{1}{2\pi} \int d^4x' \delta_{\text{ret}}( (x-x')^2) j_\sigma(x').$$ (35)

The cartesian components of the U(1) current admit following fall-offs:

$$j_\mu = j_{\mu}^2 + j_{\mu}^\log \frac{\log r}{r^3} + \ldots.$$ (36)

Let us take the limit $r \to \infty$ keeping $u$ finite:

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi r} \int du' dr' d^2z' \left[ \frac{2}{-q.q'} j_\sigma(u', z') + \frac{\log(u - u')}{u} + j_{\sigma}^{\log}(u', z') \frac{1}{u} \right] + \mathcal{O}(\frac{1}{r^2}).$$ (37)

We are interested in studying the $u$-behaviour in $u \to \infty$ limit. In (37), the most dominant fall-off in $u \to \infty$ limit is $\frac{\log u}{u}$. It comes from the region $u' \ll u$. First we rewrite the coefficient in retarded co-ordinates (Recalling that $q^\mu = (1, \hat{x})$) :

$$\log j_\sigma = -q_\sigma j_u^\log + \gamma^{AB} \partial_B q_\sigma^\log A_\sigma.$$ (38)

Now, $j_u^\log$ can be eliminated using the conservation equation of current :

$$\partial_u j_\sigma^\log = \partial_u j_r - D^A j_A^\log.$$ (39)

Substituting in the expression for $j_\sigma^\log$ :

$$\log j_\sigma = -q_\sigma \partial_u j_r + D^A [q_\sigma \log j_A].$$ (40)

Thus, $j_\sigma^\log$ is a total derivative. The second term vanishes trivially due to sphere integral. Using (34) let us study the behaviour of $j_r^\log$ as $|u| \to \infty$. Following the logic of [20], we know that $\phi \sim \frac{1}{u^{1+\epsilon}}$ as $|u| \to \infty$. Now, let us find find the $u$-fall off of $A_r^\log$. Using the gauge condition we have : $\partial_\sigma A_r^\log = -A_u^\log$. Then $A_u^\log$ can be related to the current by Maxwell’s equation : $2 \partial_u A_u^\log = j_{u}^2$. Hence, $A_r^\log$ can have a $\mathcal{O}(u)$ term as $|u| \to \infty$. Using these $u$-fall offs in the expression (34) we get $j_r^\log \to 0$ as $|u| \to \infty$. Thus, the first term in (40) is also 0. Hence the coefficient of $\frac{\log u}{u}$ vanishes.
The next term falls off as $1/u$ and this is the term relevant for loop level charge. Let us rewrite the $1/u$-term in a nice form. To start with, we have:

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi u} \int du' d^2 z' \left[ \frac{3}{4} j_\sigma(z') \log(-q.q') \right].$$

We manipulate $j^3_\sigma$ in similar fashion:

$$j^3_\sigma = -q_\sigma j^3_u + \gamma^{AB} \partial_B q_\sigma j^4_A = -q_\sigma \partial_u j^4_r - q_\sigma j^4_u + D^A[q_\sigma j^2_A],$$

and we get:

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi u} \int du' d^2 z' \left[ \frac{q_\sigma}{u} \log(-q.q') \left[ q_\sigma \partial_u j^4_r + D^A[q_\sigma j^2_A] \log(-q.q') \right] \right].$$

We again substitute for $j^3_\mu$ using (40). Upto total derivative terms above expression can be rewritten as:

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi u} \int du' d^2 z' \left[ q_\sigma \log(-q.q') \left[ q_\sigma \partial_u j^4_r + D^A[q_\sigma j^2_A] \log(-q.q') \right] \right].$$

The last term drops out as $j^4_\sigma \to 0$ as $|u| \to \infty$ (proved in [20]) and we have already checked that $j^4_r \to 0$ as $|u| \to \infty$. We can rewrite the first term as

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi u} \int du' d^2 z' \left[ q_\sigma \log(-q.q') \left[ q_\sigma \partial_u j^4_r + D^A[q_\sigma j^2_A] \log(-q.q') \right] \right].$$

Finally we perform a coordinate transformation (using (4)):

$$A_\sigma(u, r, \hat{x}) = \frac{1}{4\pi u} \int du' d^2 z' \left[ q_\sigma \log(-q.q') \left[ q_\sigma \partial_u j^4_r + D^A[q_\sigma j^2_A] \log(-q.q') \right] \right].$$

In above expression we have used the following basis for polarization vectors [11]:

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \bar{z}} [(1 + z\bar{z}) q^\mu], \quad \epsilon_+^\mu = \frac{1}{\sqrt{2}} \frac{\partial}{\partial z} [(1 + z\bar{z}) q^\mu].$$

The expression for $A_\sigma$ can be obtained from the expression for $A_\bar{z}$ by replacing $\epsilon_-$ by $\epsilon_+$. So, to summarise we have studied the $\log r$ dressing of massless scalars and showed that it leads to a $1/u$ term in $A_A$. This expression is exactly analogous to its counterpart for massive scalars case [1]. A point to be noted is that the dressing of scalars (31) does not change the $u$-fall offs of the currents, in particular there is no $1/u$-term in the modified currents. This fact will play a role in definition of hard charge.
We can use (27) to relate the $1/u$ term in (43) to the future charge (25). The precise relation is:

$$2 \log F_{rz} = -\gamma^z z D_z^0 A_z.$$  \hspace{0.5cm} (45)

Next, we find the contribution of scalar dressing to the charge at past. We recall that the charge at past is defined in terms of following mode (25):

$$\log r r^2 F r A(\hat{x})|_{\mathcal{I}^-}.$$  \hspace{0.5cm}

To study this mode first we will expand Maxwell's equations in large $r$ at finite $v$ and take $v \to \infty$ limit in the solution. Around $\mathcal{I}^-$, the gauge field equation $\Box A_\mu = -j_\mu$ is:

$$[2\partial_v \partial_r + 2r \partial_v + \partial_r^2 + \frac{D^2}{r^2}] A_\sigma = -j_\sigma.$$  \hspace{0.5cm}

Using the asymptotic expansion for current:

$$j_\mu = j_\mu \frac{1}{r^2} + j_\mu \frac{\log r}{r^3} + ..., \hspace{0.5cm}$$

we get:

$$A_\mu = A^{\log 1}_\mu \frac{\log r}{r} + A^{\log 2}_\mu \frac{1}{r^2} + \cdots.$$  \hspace{0.5cm}

The coefficients satisfy:

$$2\partial_v A^{\log 1}_\sigma = -j^{\log 1}_\sigma,$$

$$-2\partial_v A^{\log 2}_\sigma + (D^2 + 2) A^{\log 1}_\sigma = -j^{\log 2}_\sigma.$$  \hspace{0.5cm}

The $\log r/r^2$ term in $F_{rA}$ comes from $A^{\log 2}_\sigma$. $A^{\log 1}_\sigma$ is $O(e)$ term and contributes an $O(v)$ term at $v \to -\infty$. So, we ignore it henceforth:

$$A^{\log 2}_\sigma(x) = \frac{1}{2} \int v^0 \log j_\sigma(v', z) + ....$$  \hspace{0.5cm} (46)

In above solution, we have chosen the integration constant such that $A^{\log 2}_\sigma \to 0$ as $v \to -\infty$. With a co-ordinate transformation, we get (index i denotes cartesian spatial components):

$$F_{rz}|_{\mathcal{I}^+} = -\frac{\log r}{2r^2} \partial_z q^i \int d' \frac{\log}{\log z} j_i(v', z).$$  \hspace{0.5cm} (47)

Substituting for $j_\sigma$ we get:

$$F_{rz}|_{\mathcal{I}^+} = \frac{\log r}{2r^2} \partial_z q^i \int d' D^A[q_i j_A](v', z).$$  \hspace{0.5cm} (48)
This can be rewritten as:

\[
F_{rz}|_{z^+} = -\log \frac{r}{2r^2} \int d'v' j_z(v', z) = -\log \frac{r}{2r^2} \int d'v' \int d^2z' \delta(z' - z) j_z(v', z'),
\]

\[
= -\frac{1}{4\pi} \int d^2z' D_z' \left[ \sqrt{2} \frac{\epsilon^\mu_\nu q^\nu}{1 + z z'} q q' L^A_{\mu\nu} j_A(z') \right],
\]

(49)

here, \( L^A_{\mu\nu} = q_\mu D^A q_\nu - q_\nu D^A q_\mu \). Thus, the charge at past can be also be written as:

\[
\log F_{rz}|_{z^+} = -\gamma z \bar{z} D_z'^2 \log B.
\]

(50)

We can compare above equation to (45) and also note that \( B \) is exactly the same operator as \( A_z^{01} \) (43).

4 Dressing of gauge field

In this section, we will study the effect of long range gravitational force on gauge fields. Choosing Lorentz gauge \( \nabla_\mu A^\mu = 0 \), Maxwell’s equations reduce to:

\[
\nabla^2 A_\mu = -j_\mu + R_\mu^\nu A_\nu.
\]

(51)

\( R_{\mu\nu} \) is the Ricci tensor. Above equation can be written as: (Ignoring the U(1) current here.)

\[
\Box A_\sigma = j_\sigma^{grav},
\]

(52)

where we have defined:

\[
j_\sigma^{grav} = h^\mu\nu \partial_\mu A_\sigma + \eta^\mu\nu \Gamma^\rho_{\mu\nu} \partial_\rho A_\sigma + 2\eta^\mu\nu \Gamma^\rho_{\mu\nu} \partial_\nu A_\rho + \eta^\mu\nu A_\lambda \partial_\mu \Gamma^\lambda_{\nu\sigma} + [\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma}] A^\nu + O(G^2).
\]

Since we are working pertubatively, \( j_\sigma^{grav} \) can treated as a source i.e. evaluated on the zeroth order solution \( A^0_\sigma = A^0_\sigma(u, \hat{x}) + A^1_\sigma(u, \hat{x}) + ... \) which is the solution in absence of gravity. Using (11), we see that the source has following behaviour around future null infinity:

\[
j_\sigma^{grav}(x') = \frac{1}{r^2} h^1_{rr} \partial_u^2 A^1_\sigma + O(\frac{1}{r^{3}}). \]

(53)

The \( O(\frac{1}{r^3}) \) terms in \( j_\sigma^{grav}(x') \) produce subleading corrections, hence are not relevant for our analysis. Analogous to the massless scalar equation (30), we get:

\[
-2\partial_u \partial_v A_\sigma - \frac{2}{r} \partial_u A_\sigma = \frac{h^1_{rr}(\hat{x})}{r} \partial_u^2 A_\sigma.
\]

(54)
The solution to above equation is given by:

\[ A_\sigma(u, \hat{x}) = -\frac{i}{8\pi^2} \int d\omega \left[ a_\sigma(\omega, \hat{x}) e^{-i\omega u} e^{i\omega \log r \frac{h_{1r}(\omega)}{2}} - a_\sigma^*(\omega, \hat{x}) e^{i\omega u} e^{-i\omega \log r \frac{h_{1r}(\omega)}{2}} \right]. \] (55)

Thus, the log \( r \) dressing of photons is exactly similar to the log \( r \) dressing of massless scalars. This dressing does not contribute to the loop level charge. The contribution to the loop level charge comes from 1/scalars. This dressing does not contribute to the loop level charge. The contribution to the symmetry to align \( \hat{r} \) \( u \) in \( \hat{A}_\sigma \).

The solution to above equation is given by:

\[ A_\sigma(x) = -\frac{1}{2\pi} \int d^4x' \delta_+( (x - x')^2) h_{rr}(z') \partial^2_{u'} A_\sigma^1(u', z'). \]

Taking the limit \( r \to \infty \) with \( u < r \): (For dressing of incoming photons analogous calculation needs to be done around \( \mathcal{I}^- \).)

\[ A_\sigma(u, r, \hat{x}) = -\frac{1}{4\pi r} \int du' dr' d^2z' \delta_+(u' + r' - u - \vec{x}.\hat{x}) h_{rr}(z') \partial^2_{u'} A_\sigma^1(u', z') + O(\frac{1}{r^2}), \]

\[ = -\frac{1}{4\pi r} \partial_{u'} \left[ \int du' dr' d^2z' \delta(u' + r' - u - \vec{x}.\hat{x}) h_{rr}(z') \partial_{u'} A_\sigma^1(u', z') \right]. \]

\( \partial_{u'} A_\sigma^0 \) vanishes for \( |u'| > u_0 \) as to the zeroth order the particles are free for \( |u'| > u_0 \), where, \( u_0 \) is some time scale that is set by short range interactions. We can use rotational symmetry to align \( \hat{x} \) along \( z'\)-axis:

\[ A_\sigma(u, r, \hat{x}) = -\frac{1}{2r} \partial_u \left[ \int_{-u_0}^{u_0} du' \int dr' d\cos \theta' \frac{1}{r'} \delta(\cos \theta' - 1 + \frac{u - u'}{r'}) h_{rr}(\theta') \partial_{u'} A_\sigma^1(u', \theta') \right]. \]

We will use the delta function to do the \( \theta' \) integral. \( \cos \theta' \in [-1, 1] \) leads to a bound on other integration variables. There are two allowed ranges: \( u > u', \ 2r' > u - u'; \ u > u, \ 2r' < -(u' - u) \). The second range is inadmissible as \( r' \) needs to be positive. Also, \( r'\)-integral needs to be regulated with some IR cutoff.

\[ A_\sigma(u, r, \hat{x}) = -\frac{1}{2r} \partial_u \left[ \int_{-u_0}^{u_0} du' \int_{u - u'}^{R} \frac{dr'}{r'} h_{rr}(\theta') \partial_{u'} A_\sigma^1(u', \theta') \bigg|_{\cos \theta' = 1 - \frac{u - u'}{r'}} \right]. \]

Taylor expanding the integrand around \( \cos \theta' = 1 \), we get the leading order contribution in \( u \to \infty \) limit to be:

\[ A_\sigma(u, r, \hat{x}) = \frac{1}{2r} \left[ \int_{-u_0}^{u_0} du' \frac{1}{u - u'} h_{rr}(\theta') \partial_{u'} A_\sigma^1(u', \theta') \bigg|_{\cos \theta' = 1} \right]. \] (56)
Above expression can be readily related to insertion of leading soft mode:

\[ A_\sigma(u, r, \mathring{x})_{u \to \infty} = \frac{1}{2r u} h_{rr}^1(\mathring{x}) \int_{-\infty}^{\infty} du' \partial_\omega A_\sigma^1(u', \mathring{x}), \]

\[ = -\frac{1}{16\pi r u} h_{rr}^1(\mathring{x}) \lim_{\omega \to 0} \omega \left[ a_\sigma(\omega, \mathring{x}) - a_\sigma^\dagger(-\omega, \mathring{x}) \right]. \quad (57) \]

This term is due to acceleration of outgoing photons under gravitational field \( h_{rr}^1 \). Combining above expression with (55), we obtain dressing that is consistent with the proposal given in [2, 41]:

\[ A_\sigma(u, r, \mathring{x}) = -\frac{i}{8\pi^2 r} \int d\omega \left[ a_\sigma(\omega, \mathring{x}) e^{-i\omega u} e^{i\omega \log(r\omega) \frac{h_{rr}^1(\mathring{x})}{2}} - a_\sigma^\dagger(\omega, \mathring{x}) e^{i\omega u} e^{-i\omega \log(r\omega) \frac{h_{rr}^1(\mathring{x})}{2}} \right]. \quad (58) \]

Thus, we have obtained the dressing of photon due to the presence of long range gravitational force. Similar to massless scalars, the dressing depends only on \( h_{rr}^1 \).

5 Expressions for \( h_{rr}^1 \) and \( A_r^1 \)

So far, we have identified the entire 1/u term of \( A_0^A \). We see that it depends on \( h_{rr}^1 \) and \( A_r^1 \). In this section we write down the expressions for \( h_{rr}^1 \) and \( A_r^1 \). First we write down the inhomogenous part of the expressions. Then we study the homogeneous mode that contributes to above terms. This homogeneous mode is purely quantum. Thus, the classical contribution comes entirely from the inhomogenous terms.

5.1 Inhomogenous part

We know that the solution for gauge field in Lorentz gauge is given by:

\[ A_\mu(x^\mu) = \frac{1}{2\pi} \int d^4x' \delta((x-x')^2) \Theta(t-t') j_\mu(x'). \]

The leading order term at large r is given by:

\[ A_\mu(u, r, \mathring{x}) = -\frac{1}{4\pi r} \int du'd^2z' \frac{j_\mu^2(z', u')}{q.q'}. \]

Above expression is consistent with the fall-offs mentioned in (10). In particular we have:

\[ A_r^1(\mathring{x}) = \frac{1}{4\pi r} \int du'd^2z' j_\sigma^2(z', u'). \quad (59) \]

The inhomogenous part of \( A_r^1(x) \) is just a constant (i.e. independent of x) hence does not contribute to the hard charge. Thus, the classical electromagnetic dressing is trivial which
is consistent with the absence of classical log $\omega$ term in soft electromagnetic radiation (in absence of gravitation coupling) [2].

Similarly, in De-Donder gauge, the mertric perturbations satisfy $\Box \bar{h}_{\mu\nu} = -2T_{\mu\nu}$ with the solution given by:

$$\bar{h}_{\mu\nu}(x') = \frac{1}{\pi} \int d^4x' \delta((x - x')^2) \Theta(t - t') T_{\mu\nu}(x').$$

The leading order solution around future null infinity is given by:

$$\bar{h}_{\mu\nu}(u, r, \hat{x}) = -\frac{1}{2\pi r} \int d\omega' d^2q' \frac{T_{\mu\nu}(z', u')}{q.q'}. $$

Above expression is consistent with the fall-offs mentioned in (11). A key point about the perturbations is that $\partial_\omega \bar{h}^1_{\mu\nu} = 0$. This kills off lot of terms that would have otherwise been present in (30). Finally we have:

$$h^1_{rr}(\hat{x}) = -\frac{1}{2\pi r} \int d\omega' d^2q' T_{uu}(z', u').$$

5.2 Homogenous part

We discuss a purely quantum mode present in both metric and gauge fields that will eventually contribute to the charge. In [1], authors pointed out that discontinuity in $\omega \tilde{A}_A$ as $\omega \to 0$ has a non trivial consequence at quantum mechanical level. The discontinuity leads to a log $u$ term in $A_A$ that vanishes classically. First let us discuss the gravitational analogue of this purely quantum log $u$ mode.

We denote gravitational free data by $C_{AB}$. ($C_{AB} = \lim_{r \to \infty} \frac{h_{AB}}{r}$). Let us consider $C^+_{zz}(u)$ that has only positive frequencies. We know that around $\omega \sim 0$, the behaviour the radiative data is given by $\tilde{C}^+_{zz} = \frac{1}{\omega} \tilde{C}^{+0}_{zz} + ...$, hence:

$$C^+_{zz}(u) = \frac{1}{2\pi} \int_0^\infty d\omega \left[ \frac{1}{\omega} \tilde{C}^{+0}_{zz} + ... \right] e^{-i\omega u},$$

$$= \frac{1}{2\pi} \log(u^{-1}) \tilde{C}^{+0}_{zz} + ... .$$

Similarly for negative frequencies, we have:

$$C^-_{zz}(u) = -\frac{1}{2\pi} \log(u^{-1}) \tilde{C}^{-0}_{zz} + ... .$$

Collecting the positive and negative frequency terms we get:

$$C_{zz}(u) = -\frac{1}{2\pi} \left[ \tilde{C}^{+0}_{zz} - \tilde{C}^{-0}_{zz} \right] \log |u| + ... ,$$

$$= -\frac{1}{2\pi} \lim_{\omega \to 0^+} \left[ \omega \tilde{C}^+_{zz}(\omega) + \omega \tilde{C}^-_{zz}(-\omega) \right] \log |u| + ... .$$
If $C_{zz}$ were continuous at $\omega = 0$, the log $u$ term would have cancelled as it does classically. But quantum mechanically positive frequencies involve annihilation operator while negative frequencies involve creation operator:

$$C_{zz}(\omega, z) = \frac{-ic_+(\omega, z)}{2\pi(1 + z^2)} \quad \omega > 0, \quad C_{zz}(\omega, z) = \frac{i\epsilon_+(-\omega, z)}{2\pi(1 + z^2)} \quad \omega < 0. \quad (64)$$

Thus, we get:

$$C_{zz}(u, \hat{x}) = \frac{1}{4\pi^2 (1 + z^2)^2} \lim_{\omega \to 0} \omega [c_+(\omega, \hat{x}) + c_+^*(-\omega, \hat{x})] \log |u| + \ldots. \quad (65)$$

Similarly, for $C_{zz}$ we have,

$$\log C_{zz}(\hat{x}) = \frac{1}{4\pi^2 (1 + z^2)^2} \lim_{\omega \to 0} \omega [c_-(\omega, \hat{x}) + c_+^*(-\omega, \hat{x})]. \quad (66)$$

Next, we want to find the correction to $h_{rr}^1$ due to the log $u$ term. For that we will use Herdegen-like representation of $h_{\mu\nu}$. Herdegen representation [33] for photon is a way to write a generic homogenous solution for gauge field in Lorentz gauge in terms of free data $A^0_A$. Similarly, here we write a generic homogenous solution for metric field in De Donder gauge in terms of free data $C_{AB}$ (See Appendix B for details):

$$h^\text{hom}_{\mu\nu}(x) = -\frac{1}{4\pi} \int d^2 z' (1 + z'z)^2 \left[ \varepsilon^- \varepsilon^- C_{zz}(u = -x \cdot q', \hat{q}') + \varepsilon^+ \varepsilon^+ C_{zz}(u = -x \cdot q', \hat{q}') \right], \quad (67)$$

$q'$ is defined according to (4). From above expression it can be seen that $C_{zz} \sim \log u$ gives rise to a $\frac{1}{r}$ term in $h_{\mu\nu}$. By co-ordinate transformation, we get $(x^\mu = rq^\mu + \ldots)$:

$$h^\text{hom}_{rr}^1(x) = \frac{1}{4\pi} \int d^2 z' (1 + z'z)^2 \frac{1}{q'q} \left[ \log [\varepsilon^- \varepsilon^- u \cdot \varepsilon^- \varepsilon^- q C_{zz}(\hat{x}') + \varepsilon^+ \varepsilon^+ q \varepsilon^+ \varepsilon^+ C_{zz}(\hat{x}')] \right]. \quad (68)$$

Now we will use (65) and (66) in above expression and then use the leading soft theorem to evaluate action of (65) and (66). So, action of $h^1_{rr}$ on a generic state is given by:

$$<\text{out}|h^\text{hom}_{rr}^1(x) = i <\text{out}| \int \frac{d^2 z'}{16\pi^3} \left[ \frac{\varepsilon^- \cdot q \varepsilon^- \cdot q}{q'q} \sum_j \frac{\varepsilon^+ \cdot p_j \varepsilon^+ \cdot p_j}{q'p_j} + \frac{\varepsilon^+ \cdot q \varepsilon^+ \cdot q}{q'q} \sum_j \frac{\varepsilon^- \cdot p_j \varepsilon^- \cdot p_j}{q'p_j} \right]. \quad (69)$$

Using completeness relations for polarisation tensors:

$$\frac{\varepsilon^- \cdot q \varepsilon^- \cdot q}{q'q} \sum_j \frac{\varepsilon^+ \cdot p_j \varepsilon^+ \cdot p_j}{q'p_j} + \frac{\varepsilon^+ \cdot q \varepsilon^+ \cdot q}{q'q} \sum_j \frac{\varepsilon^- \cdot p_j \varepsilon^- \cdot p_j}{q'p_j} = \sum_j \frac{2(q \cdot p_j)^2}{q'q \cdot q'p_j}, \quad (70)$$
Thus, in (69), we need to do following integral :

\[
I = \int d^2 z' \frac{1}{q.q'} q'.p_j = \int d^2 z' \int_0^1 dx \frac{1}{[q'.(x q + (1 - x) \omega_j q_j) - x - (1 - x) \omega_j]^2}. \tag{71}
\]

But \( I \) is divergent. These are collinear divergences that appear as we dealing with massless particles. We will see that the diverging terms cancel and the charge is finite. Regulating the integral using a small \( \epsilon \) parameter :

\[
I = 2\pi \int_0^1 dx \frac{1}{[x(1 - x) \omega_j(1 - q_j q_j)]^{1-\epsilon}}, \quad \epsilon \Gamma(1 + 2\epsilon) \frac{1}{[q.p_j]^{1-\epsilon}}. \tag{72}
\]

The infinite piece is as follows :

\[
h_{rr}^1(\hat{x})|_{inf} = -\frac{i}{2\pi^2 \epsilon} \sum_j (q.p_j). \tag{73}
\]

Above term vanishes due to conservation of momentum. The finite piece gives us :

\[
h_{rr}^1(\hat{x}) = -\frac{i}{2\pi^2} \sum_j (q.q_j) \log(q.p_j). \tag{74}
\]

This is the term that contributes to the charge. Now, we repeat the calculation for gauge field. To start with, we have [1] :

\[
hom A_\mu(x) = \frac{1}{2\pi} \int d^2 z' \sqrt{2} \frac{1}{q'.x} \left[ \varepsilon^- A_z + \varepsilon^+ A_\bar{z} \right], \tag{75}
\]

where,

\[
\log A_z(\hat{x}') = \frac{i}{8\pi^2 (1 + |z'|^2)} \lim_{\omega \to 0} \omega[a_-(\omega, \hat{x}') + a^+_\omega(-\omega, \hat{x}')],
\]

\[
\log A_\bar{z}(\hat{x}') = \frac{i}{8\pi^2 (1 + |z'|^2)} \lim_{\omega \to 0} \omega[a_+(\omega, \hat{x}') + a^+\omega(-\omega, \hat{x}')]. \tag{76}
\]

We extract out the 1/r-term :

\[
hom A_1^1(\hat{x}) = \frac{1}{2\pi} q^\mu \int d^2 z' \sqrt{2} \frac{1}{q'.q} \left[ \varepsilon^- A_z + \varepsilon^+ A_\bar{z} \right]. \tag{77}
\]
The action of $A_r^1$ can be evaluated on a generic out state:

$$< \text{out}| A_r^1(\hat{x}) S = i < \text{out}| \int \frac{d^2z'}{16\pi^3} \left[ \varepsilon^- \cdot q \sum_j e_j \frac{\varepsilon^+ \cdot p_j}{q' \cdot p_j} + \frac{\varepsilon^+ \cdot q}{q' \cdot p_j} \sum_j e_j \frac{\varepsilon^- \cdot p_j}{q' \cdot p_j} \right] S,$$

$$= \frac{i}{16\pi^3} \int d^2z' \sum_j e_j \frac{p_j \cdot q}{q' \cdot p_j} S.$$  \hspace{1cm} (78)

Above integral can be calculated similar to the earlier one. The infinite piece is a constant and vanishes by charge conservation. We have:

$$h_{\text{om}}^1 A_r^1(\hat{x}) = -\frac{i}{4\pi^2} \sum_j e_j \log(q \cdot p_j).$$ \hspace{1cm} (79)

Finally, we have the complete expressions for $h_{rr}^1$ and $A_r^1$ which will be needed to evaluate the action of hard charge in the next section.

### 6 Asymptotic charge and Ward identity

Finally we have all the ingredients to write down the loop level Ward identity. We begin by reviewing the definition of the loop level asymptotic charge from the conservation equation (25). The charge at future infinity is defined as (where $V_A^2$ is an arbitrary parameter):

$$Q_+ [V] = - \int d^2z \ V_A^2 \log F_{rA} \big| \mathcal{I}_+^{+},$$

$$= u^2 \partial_u^2 \int d^2z \ V_A^2 F_{rA}^2 \big| u \to -\infty.$$  

The $u$-operator isolates the log coefficient of $F_{rA}^2$. Next we follow the usual steps: write the term as an integral over entire future null infinity minus the term at $\mathcal{I}_+^{+}$.

$$Q_+ [V] = - \int du \ d^2z \ V_A \partial_u^2 \left[ u^2 \partial_u^2 F_{rA}^2 \right] - \int d^2z \ V_A \log F_{rA} \big| \mathcal{I}_+^{+},$$

$$:= Q_+^{\text{soft}}[V] + Q_+^{\text{hard}}[V].$$ \hspace{1cm} (80)

This defines the soft and hard parts of asymptotic charge. The expression at Past null infinity is:

$$Q_- [V] = - \int d^2z \ V_A \log F_{rA} \big| \mathcal{I}_-^{+}. $$
We know from (47) that $F_{rA}^{\log}$ depends only on particle currents i.e. it has no contribution from radiation. Thus, at past the charge is entirely made of hard modes.

$$Q_-[V] = -\int d^2 z \ V^A F_{rA}^{2\log} \big|_{I^+} := Q_-^{\text{hard}}[V].$$

Thus, the conservation law that we have started with in (25), reproduces outgoing soft theorem. An analogous conservation law that relates $\log v$ mode at $I^-$ to log $r$ (a purely hard mode) at $I^+$ will reproduce the incoming soft theorem.

We can simplify the soft charge expression further. Using Maxwell’s equation (106) for $\partial_u F_{rA}$, we get:

$$Q_-^{\text{soft}} = -\frac{1}{2} \int du \ d^2 z' \ V^A \partial_u \left[ u^2 \partial_u [\partial_A F_{ru} - D^B F_{AB}^0 + j_A^2] \right] + ..., \quad (81)$$

here, ‘...’ represent the gravity corrections, we explicitly check in Appendix A that these corrections vanish. As we noted in section 3, $j_A^2$ does not have a $1/u$-term, so $j_A^2$ also drops out and we get:

$$Q_-^{\text{soft}} = \int du \ d^2 z' \left[ V^z(\hat{x}') \partial_u \left[ u^2 \partial_u [\partial_A F_{ru} - D^B F_{AB}^0 + j_A^2] \right] + z' \leftrightarrow \bar{z}' \right],$$

where

$$Q_-^{\text{soft}} = \int du \ d^2 z' \left[ D^2 V^z \gamma^{zz} \partial_u \left[ u^2 \partial_u [\partial_A F_{ru} - D^B F_{AB}^0 + j_A^2] \right] + z' \leftrightarrow \bar{z}' \right]. \quad (82)$$

The last line was derived using integration by parts. Next it is instructive to go to the frequency space:

$$Q_-^{\text{soft}} = \int d^2 z' \left[ D^2 V^z \gamma^{zz} \lim_{\omega \to 0} \omega \partial_\omega^2 \omega \ A_0^0(\omega, \hat{x}') + z' \leftrightarrow \bar{z}' \right]. \quad (84)$$

The gauge field can be expressed in terms of Fock operators as:

$$\tilde{A}_z^0(\omega, z) = -i \sqrt{2} \frac{a_-(\omega, z)}{4\pi(1 + z\bar{z})} \ ... \ \omega > 0, \quad \tilde{A}_0^0(\omega, z) = i \sqrt{2} \frac{a_+^\dagger(-\omega, z)}{4\pi(1 + z\bar{z})} \ ... \ \omega < 0. \quad (83)$$

Since, we have only outgoing radiation, we will define soft limit as:

$$Q_-^{\text{soft}} = -\frac{i}{4\pi} \int d^2 z' \left[ D^2 V^z \sqrt{\gamma^{zz}} \lim_{\omega \to 0} \omega \partial_\omega^2 \omega \ a_-(\omega, \hat{x}') + z' \leftrightarrow \bar{z}' \right]. \quad (84)$$

Thus, this operator is related to zero energy photon modes. The $\omega$-derivatives in particular isolate the coefficient of soft log $\omega$ mode.

Now, let us turn to the hard charge expression:

$$Q_+^{\text{hard}} = -\int d^2 z' \ V^A F_{rA}^{2\log}(\hat{x}').$$
Substituting above relations in the expression for hard charge and get:

\[
Q_{\text{hard}}^+ = \int d^2 z' V^z \gamma^{zz} D_{zz}^0 A_z(\hat{x}') + \int d^2 z' V^z \gamma^{zz} D_{zz}^0 A_z(\hat{x}')
\]

(85)

To avoid unnecessary clattering of equations we will work with \(V_{\bar{z}} = 0\). Integrating by parts:

\[
Q_{\text{hard}}^+ = \int d^2 z' D_{zz}^2 V^z \gamma^{zz} \bar{A}_z(\hat{x}')
\]

(86)

The charge at past can also be cast as (49):

\[
Q_{\text{hard}}^- = \int d^2 z' D_{zz}^2 V^z \gamma^{zz} \log B(\hat{x}'),
\]

(87)

The Ward identity for S matrix for above asymptotic charge can written down:

\[
\left[ Q, S \right] = 0,
\]

\[
\Rightarrow \left( Q_{\text{soft}}^+ S - S Q_{\text{soft}}^+ \right) = -\left( Q_{\text{hard}}^+ S - S Q_{\text{hard}}^- \right),
\]

\[
= -\int d^2 z' D_{zz}^2 V^z \gamma^{zz} \left( A_z(\hat{x}') \right) \log B(\hat{x}').
\]

(88)

Next we need to evaluate the action of hard charge. We will work with the usual Fock states. For simplicity, we consider all hard particles to be scalars.

### 6.1 Hard charge

Let us first write down the \(1/u\) term due to dressing of massless scalars (43):

\[
0.1 A_z(z)|_{\text{scalar}} = \frac{v_{z_{zz}}}{4\pi} \int du' d^2 z' \frac{q'^\mu q^\nu}{q.q'} L_{\sigma\mu} \left[ -\frac{1}{2} h_{rr}^1(z') j_u^2(z') + 2e^2 A_r^1(z') \phi^1(z') \right],
\]

(89)

where, \(L_{\sigma\mu} = q_{\alpha} \partial q^\sigma - q_{\mu} \partial q^\sigma\). Expression for \(B^\log\) is exactly analogous (50). The action of (89) can be easily evaluated.

\[
< \text{out} | \left[ Q_{\text{hard}}^+, S \right] |\text{scalar} \ | \text{in} >
\]

\[
= \int d^2 z' D_{zz}^2 V^z(z, z') \frac{v_{z_{zz}}}{4\pi} \frac{q'^\mu q^\nu}{q'.q_i} \sum_i L_{i\sigma\mu} \left[ \frac{e_i}{2} h_{rr}^1(z_i) + \frac{e_i^2 A_r^1(z_i)}{\omega_i} \right] < \text{out} | S | \text{in} > .
\]

(90)
As noted from (59), the classical part of \( A^1_t \) is trivial and does not contribute. Using (60) for \( h^1_{rr} \), the RHS of (90) becomes:

\[
- \int d^2 z' D_z^2 V^z(z, z') \frac{\sqrt{\gamma^{zz}}}{16\pi^2} \sum_{\eta, \eta_j = 1} e_i^\eta q^\eta_{ij} e_\eta p_{j, q_i} L_{i \sigma \mu} M_n = - \sum_{\eta, \eta_j = 1} e_i U^{\sigma \mu}(q_i) (p_{j, q_{i\sigma}} - p_{j, q_{i\mu}}) M_n. \tag{91}
\]

where we have defined

\[
U^{\sigma \mu}(q_i) = \int d^2 z' D_z^2 V^z(z, z') \frac{\sqrt{\gamma^{zz}}}{16\pi^2} \frac{e_\sigma q^\mu}{q', q_i}, \tag{92}
\]

to make the expressions compact. \( M_n = <\text{out} | S | \text{in}> \).

Now, we will substitute the quantum part of \( h^1_{rr} \) and \( A^1_t \) from (74) and (79) in (90), the RHS becomes:

\[
- i \int d^2 z' D_z^2 V^z(z, z') \frac{\sqrt{\gamma^{zz}}}{16\pi^2} \sum_{\eta, \eta_j \neq j} e_i^\eta q^\eta_{ij} e_\eta p_{j, q_i} L_{i \sigma \mu} [p_{j, q_i} \log(p_{j, p_i}) + \frac{e_i e_j}{\omega_i} \log(p_{j, p_i})] M_n = - \frac{i}{\pi} \sum_{\eta, \eta_j \neq j} e_i U^{\sigma \mu}(q_i) \left[ \frac{e_i e_j}{q_j q_i} (q_{i\mu} q_{i\sigma} - q_{j\sigma} q_{j\mu}) + (p_{j, q_{i\sigma}} - p_{j, q_{i\mu}}) [1 + \log(p_{j, p_i})] \right] M_n \tag{93}
\]

Next we turn to the similar contribution coming from photon dressing.

The \( 1/u \) term due to dressing of photon dressing is given in (57). Since we are interested in outgoing photon theorem, we will define the soft limit from positive side:

\[
0.1 A^1_z(z)|_{\text{photon}} = - \frac{\sqrt{\gamma^{zz}}}{8\pi} h^1_{rr}(\hat{x}) \lim_{\omega \to 0} \omega a_-(\omega, \hat{x}). \tag{94}
\]

We substitute above term in the expression for hard charge (86) and get:

\[
<\text{out} | \left[ Q^{\text{hard}}_t, S \right] |_{\text{photon}} | \text{in}> = - \int d^2 z' D_z^2 V^z(z') \frac{\sqrt{\gamma^{zz}}}{8\pi} \sum_{i} \frac{e_i \varepsilon_- \cdot p_i}{q_i q_i} \left< \text{out} | h^1_{rr}(z') S | \text{in} > \right. \tag{95}
\]

Using the classical part of \( h^1_{rr} \) (60) in (95), the RHS of (95) becomes:

\[
\int d^2 z' D_z^2 V^z(z') \frac{\sqrt{\gamma^{zz}}}{16\pi^2} \sum_{i} \frac{e_i \varepsilon_- \cdot p_i}{q_i q_i} \sum_{\eta_j = 1} p_{j, q_i} M_n,
\]

\[
= \sum_{i} e_i q_{i\sigma} U^{\sigma \mu} \sum_{\eta_j = 1} p_{j, q_i} M_n. \tag{96}
\]
$U^{\mu\nu}$ has been defined in (92). Now, we substitute the quantum part of $h^1_{rr}$ and $A^1_r$ from (74), the RHS of (95) is:

$$i \int d^2 z' D_\bar{z}^2 V^z(z') \frac{\sqrt{\gamma_{z\bar{z}}}}{16\pi^3} \sum_i \frac{e_i e^- p_i}{q' p_i} \sum_j q.p_j \log(q.p_j) \mathcal{M}_n,$$

$$= \frac{i}{\pi} \sum_i e_i \epsilon_\sigma U^{\sigma 0} \sum_j q.p_j \log(q.p_j) \mathcal{M}_n. \quad (97)$$

Collecting together (91), (93), (96) and (97) we finally get the complete action of the hard charge and we can write down the Ward identity.

Thus, the S-matrix needs to satisfy following Ward identity for a generic vector field $V^A$ that lives on a sphere:

$$[Q_{soft}(V), S] = - C(p_i, e_i, V) S. \quad (98)$$

$Q_{soft}(V)$ defined in (84), inserts soft modes of photon. Dependence on $V^A$ is via $U^{\mu\nu}$ defined in (92) and

$$C(p_i, e_i, V) = \sum_i e_i \epsilon_\sigma U^{\sigma 0}(q_i) \sum_{\eta_j} p_j \cdot q - \sum_{\eta_i,\eta_j} e_i U^{\sigma \mu}(q_i) (p_{j\mu} q_{i\sigma} - p_{j\sigma} q_{i\mu})$$

$$- \frac{i}{\pi} \sum_{i \neq j} e_i U^{\sigma \mu}(q_i) \left[ \frac{e_i e_j}{q_j \cdot q_i} (q_{j\mu} q_{i\sigma} - q_{j\sigma} q_{i\mu}) + (p_{j\mu} q_{i\sigma} - p_{j\sigma} q_{i\mu}) [1 + \log(-p_j \cdot p_i)] \right]$$

$$+ \frac{i}{\pi} \sum_i e_i \epsilon_\sigma U^{\sigma 0} \sum_j q.p_j \log(-q.p_j) \quad (99)$$

### 6.2 The Sahoo-Sen soft theorem

Let us derive the Sahoo-Sen soft theorem from above Ward identity. To derive negative helicity soft theorem we choose [1]:

$$V^z(z, z') = \sqrt{2} (1 + z' \bar{z}') \frac{z - z'}{\bar{z} - \bar{z}'} , \quad V^\bar{z} = 0. \quad (100)$$

Performing the sphere $(z', \bar{z}')$ integral in (84), we get:

$$Q_{soft}^+ = -i \lim_{\omega \to 0} \omega \partial^2_{\omega} \omega a_-(\omega). \quad (101)$$
Next we will use \((100)\) in the expression for hard charge \((99)\). The sphere integral in the expression for \(U\) \((92)\) can be done easily. We get:

\[
C(p_i, e_i) = \frac{1}{4\pi} \sum_i e_i \frac{\epsilon \cdot p_i}{p_i \cdot k} \sum_{\eta_i=1} k \cdot p_j - \frac{1}{4\pi} \sum_{i \neq j} e_i \frac{\epsilon \cdot k}{p_i \cdot k} (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu)
\]

\[-\frac{i}{4\pi^2} \sum_{i \neq j} e_i \frac{\epsilon \cdot k}{p_i \cdot k} \left[ [e_i e_j + p_i \cdot p_j] (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu) + (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu) \log[p_i \cdot p_j]\right]
\]

\[+ \frac{i}{4\pi^2} \sum_i e_i \frac{\epsilon \cdot p_i}{p_i \cdot k} \sum_j k \cdot p_j \log p_j \cdot q \tag{102}\]

So, the Ward identity can be recast as:

\[
\lim_{\omega \to 0} \omega \frac{\partial^2}{\partial \omega^2} \omega a_-(\omega)
\]

\[= -\frac{i}{4\pi} \sum_i e_i \frac{\epsilon \cdot p_i}{p_i \cdot k} \sum_{\eta_i=1} k \cdot p_j + \frac{i}{4\pi} \sum_{i \neq j} e_i \frac{\epsilon \cdot k}{p_i \cdot k} (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu)
\]

\[-\frac{1}{4\pi^2} \sum_{i \neq j} e_i \frac{\epsilon \cdot k}{p_i \cdot k} \left[ [e_i e_j + p_i \cdot p_j] (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu) + (p_j^\mu p_i^\mu - p_i^\mu p_j^\mu) \log[p_i \cdot p_j]\right]
\]

\[+ \frac{1}{4\pi^2} \sum_i e_i \frac{\epsilon \cdot p_i}{p_i \cdot k} \sum_j k \cdot p_j \log p_j \cdot q. \tag{103}\]

This is exactly the Sahoo-Sen soft theorem.

So, we have derived the soft theorem from the Ward identity. The Ward identity (with \(V^z = 0\)) can be derived from the soft theorem by multiplying both sides of the statement of soft theorem with \(\int d^2 z \int d^2 z V^z(z) \frac{1}{16\pi^2} \). Thus, we can conclude that the Ward identity \((98)\) is exactly equivalent to the Sahoo-Sen soft photon theorem \((1)\). Thus, the equivalence between the conservation law proposed in \([1]\) and the log \(\omega\)-soft photon theorem \([2]\) holds for massless scalar QED coupled to gravity as well. The effect of dynamical gravity on massive scalar QED is being studied in \([40]\).

**Acknowledgments:**

I am deeply thankful to Nabamita Banerjee, Miguel Campiglia and Alok Laddha for many helpful discussions and suggestions. I thank Chennai Mathematical Institute where part of this work was done. I am grateful to the people of India for their support to theoretical sciences.
A Maxwell’s equations in presence of gravity

In this section we write down Maxwell’s equations in presence of gravitational fluctuations given by (11).
Let us study the \( \nabla \mu F_{\mu \nu} = j_\nu \) equation. Expanding the equation around \( r \to \infty \), at \( O(\frac{1}{r^2}) \) we get:

\[
\partial_u F^2_{ru} + \partial_u D^B A_0^B - j_u^2 = -\gamma^{CB} h_{Cr}^0 \partial_u F_{uB}^0. \tag{104}
\]

In the equation \( \nabla \mu F_{A\mu} = 0 \), there appears a gravity correction even at \( O(\frac{1}{r}) \):

\[
\partial_u F_{rA}^1 - \gamma_{AB} h_{Cr}^0 \partial_u F_{Au}^0 = 0. \tag{105}
\]

This implies log \( r \) dressing of \( A_A \) that also been derived in (55). \( \nabla \mu F_{A\mu} = 0 \) at \( O(\frac{1}{r^2}) \) gives:

\[
2\partial_u F^2_{rA} - \partial_A F^2_{ru} + D^B F^0_{AB} - j_A^2 = h_{rr}^1 \partial_u F_{Au} + h_{rr}^0 \partial_u F_{Au} - \gamma^{CB} h_{Cr}^0 \partial_u F_{AB}^0 - \gamma^{CB} h_{Cr}^0 D_B F_{Au}^0 - \gamma^{BC} h_{AB}^{-1} F_{uC}^0 \\
+ \partial_u h_{rr}^0 F_{Au} + \frac{1}{2} h_{rr}^1 F_{Au} - \frac{1}{2} \gamma^{BC} h_{BC}^{-1} F_{Au}^0 - \gamma^{BC} D_B h_{Au}^0 - \gamma^{BC} h_{Au}^0 D_B F_{Au}^0 - 2 h_{ur}^1 F_{Au}^0. \tag{106}
\]

We use above equation to substitute for \( \partial_u F_{rA}^2 \) in (80) i.e. in

\[
Q_{+}^{\text{soft}} = -\int du \ d^2 z' \ V^A \partial_u \left[ u^2 \partial_u F_{rA} \right], \tag{107}
\]

and we get (81) i.e.

\[
Q_{+}^{\text{soft}} = -\frac{1}{2} \int du \ d^2 z' \ V^A \partial_u \left[ u^2 \partial_u [\partial_A F_{ru} - D^B F_{AB}^0 + j_A^2] \right] + \ldots, \tag{108}
\]

where ”...” refers to the gravity corrections that come from RHS of (106) and (104). We will analyse them one by one. Out of the metric components appearing in Maxwell’s equations only \( h_{rr}^1 \) and \( h_{AB}^{-1} \) depend on \( u \), rest of them are \( u \)-independent. This simplifies the analysis for most of the terms.

**Term** \( h_{rr}^1 \partial_u F_{Au}^1 \)

\[
Q_{1}^{\text{cor}} = -\frac{1}{2} \int du \ d^2 z' \ V^z \partial_u \left[ u^2 \partial_u [h_{rr}^1 \partial_u F_{uz}] \right] + z \leftrightarrow \bar{z}. \tag{109}
\]

Using Bianchi identities we can simplify above expression to:

\[
Q_{1}^{\text{cor}} = -\frac{1}{2} \int d^2 z' \ V^z \int du \ h_{rr}^1 \partial_u \left[ u^2 \partial_u^2 D_z A_z^0 \right] + z \leftrightarrow \bar{z}. \tag{109}
\]

26
The operator picks out difference between boundary values of log \( u \) piece of \( A_A \) which is 0.

**Term** \( h^2_{rr} \, \partial_u F^0_{uA} \)

\[
Q^\text{cor}_2 = -\frac{1}{2} \int d\nu d^2 z \, V^A \partial_u \left[ u^2 \partial_u [h^2_{rr} \, \partial_u F^0_{uA}] \right]
\]  

(110)

\( h^2_{rr} \) has atmost a \( \mathcal{O}(u) \) term. Using the \( u \)-behaviour of \( F^0_{uA} \) we see that this term is also 0.

**Term** \( \gamma^{BC} h^{-1}_{BA} \, F^0_{uC} \)

\[
Q^\text{cor}_3 = \frac{1}{2} \int d\nu d^2 z \, V^A \partial_u \left[ u^2 \partial_u [\gamma^{BC} h^{-1}_{BA} \, F^0_{uC}] \right]
\]

This term vanishes trivially for classical fall-offs of \( F^0_{uC} \). For the quantum log \( u \) fall offs we get 2 terms for \( A=\bar{z} \) (the analysis is similar for \( A= z \) ) :

\[
Q^\text{cor}_3 = -\frac{1}{2} \int d\nu d^2 z \, V^z \gamma^{\bar{z}z} \partial_u h^{-1}_{ZZ} A^0_{\bar{z}} \]  

(111)

Upto unimportant overall terms that are common to both terms, the first integrand is :
\[ \lim_{\omega \to 0} \omega [c_+ (\omega) + c_+^\dagger (\omega)] \]  
\[ \lim_{\omega \to 0} \omega [a_- (\omega) - a_-^\dagger (\omega)] \]  
Similarly the second integrand is :
\[ \lim_{\omega \to 0} \omega [c_+ (\omega) - c_+^\dagger (\omega)] \]  
\[ \lim_{\omega \to 0} \omega [a_- (\omega) + a_-^\dagger (\omega)] \]  
Thus, \( Q^\text{cor}_3 = 0 \).

**Term** \( h^0_{rC} \, \partial_u F^0_{AB} \)

\[
Q^\text{cor}_4 = \frac{1}{2} \int d\nu d^2 z \, V^A \partial_u \left[ u^2 \partial_u [\gamma^{CB} h^0_{rC} \, \partial_u F^0_{AB}] \right]
\]

\[ = \frac{1}{2} \int d\nu d^2 z \, \gamma^{CB} h^0_{rC} \, V^A \partial_u \left[ u^2 \partial_u \partial_u F^0_{AB} \right] \]

(112)

This is similar to (109) and vanishes by same logic. The analysis for rest of the terms is exactly similar.

**B Herdegen like representation for graviton**

The usual momentum space expression for free metric field is :

\[
h_{\mu\nu}(x) = \sum_{r=+,-} \frac{1}{2(2\pi)^3} \int_0^\infty d\omega d^2 q \left[ e^{-i\omega(u+r-q,\bar{q})} c^r_{\mu\nu}(\omega, q) - e^{-i\omega(u+r-q,\bar{q})} c^{tr}_{\mu\nu}(-\omega, q) \right]
\]

(113)
The angular integral can be performed using stationary phase approximation at large $r$, we can obtain following well known expressions [11]:

$$h_{zz}(u, q) = \frac{r}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega u} \tilde{C}_{zz}(\omega)$$  \hspace{1cm} (114)$$

where:

$$\tilde{C}_{zz}(\omega) = \frac{c_{+}(\omega, q)}{2\pi i (1 + |z|^2)^2} \ldots \omega > 0, \quad \tilde{C}_{zz}(\omega) = \frac{-c_{+}(-\omega, q)}{2\pi i (1 + |z|^2)^2} \ldots \omega < 0. \hspace{1cm} (115)$$

And

$$h_{\bar{z}\bar{z}}(u, q) = \frac{r}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega u} \tilde{C}_{\bar{z}\bar{z}}(\omega)$$  \hspace{1cm} (116)$$

$$\tilde{C}_{\bar{z}\bar{z}}(\omega) = \frac{c_{-}(\omega, q)}{2\pi i (1 + |z|^2)^2} \ldots \omega > 0, \quad \tilde{C}_{\bar{z}\bar{z}}(\omega) = \frac{-c_{-}(-\omega, q)}{2\pi i (1 + |z|^2)^2} \ldots \omega < 0. \hspace{1cm} (117)$$

Thus (113) can be rewritten as:

$$h_{\mu\nu}(x) = \frac{i}{2(2\pi)^2} \int_{-\infty}^{\infty} \omega d\omega \, d^2 q \, (1 + |z|^2)^2 \left[ \epsilon_{\mu\nu} \tilde{C}_{zz} + \epsilon^+_{\mu\nu} \tilde{C}_{\bar{z}\bar{z}} \right] e^{-i\omega(u+r \cdot \hat{q} \cdot \hat{x})}$$

$$= \frac{i}{2(2\pi)^2} \int_{-\infty}^{\infty} \omega d\omega \, d^2 q \, (1 + |z|^2)^2 \epsilon_{\mu\nu} \left[ \int_{-\infty}^{\infty} du' \, e^{i\omega u'} C_{zz} \right] e^{-i\omega(u+r \cdot \hat{q} \cdot \hat{x})}$$

$$+ \frac{i}{2(2\pi)^2} \int_{-\infty}^{\infty} \omega d\omega \, d^2 q \, (1 + |z|^2)^2 \epsilon^+_{\mu\nu} \left[ \int_{-\infty}^{\infty} du' \, e^{i\omega u'} C_{\bar{z}\bar{z}} \right] e^{-i\omega(u+r \cdot \hat{q} \cdot \hat{x})}$$

$$= -\frac{1}{4\pi} \int d^2 q \, (1 + |z|^2)^2 \left[ \epsilon_{\mu\nu} \tilde{C}_{zz}(u = -x \cdot q, \hat{q}) + \epsilon^+_{\mu\nu} \tilde{C}_{\bar{z}\bar{z}}(u = -x \cdot q, \hat{q}) \right], \hspace{1cm} (118)$$

here, $C_{AB} = \lim_{r \to \infty} \frac{1}{r} h_{AB}$. Above expression is analogous to the expression for gauge field obtained by Herdegen [33].

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