On the causal characterization of singularities in spherically symmetric spacetimes

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Abstract

The causal character of the zero-areal-radius (\(R = 0\)) singularity in spherically symmetric spacetimes is studied. By using the techniques of the qualitative behaviour of dynamic systems, we are able to present the most comprehensive scheme so far to try to find out their causal characterization, taking into account and analysing, the possible limitations of the approach. We show that, with this approach, the knowledge of the scalar invariant \(m \equiv \frac{R(1 - g^{\mu\nu}\partial_{\mu}R\partial_{\nu}R)}{2}\) suffices to characterize the singularity. We apply our results to the study of the outcome of black hole evaporation and show different possibilities. In this way, we find that a persistent naked singularity could develop in the final stages of the evaporation and show its distinctive features. Likewise, we study the options for the generation of naked singularities in the collapse of an object (such as a star) as a means of violating the cosmic censorship conjecture.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Singularities are not part of the spacetime since they are related to diverging curvature invariants, to the incompleteness of curves and/or to the lack of tangent vectors. At most, it would seem reasonable to say that singularities are situated in the boundary of the spacetime, provided that a suitable definition of boundary is given. In fact, one such definition was first introduced by Penrose in 1963 [1]. His idea was to embed the spacetime under study with metric \(g\) into another Lorentzian manifold (the unphysical spacetime) with metric \(\bar{g}\) conformally,

\[
g = \Omega^{2}\bar{g},
\]

so that the causal properties are trivially kept. In this way, the boundary acquires causal properties itself which are obtained by its mere examination in the unphysical spacetime. Specifically, it now becomes meaningful to give a singularity attributes such as spacelike, timelike or lightlike. Furthermore, in spherically symmetric spacetimes, where the \(SO(3)\) group...
orbits form a spacelike 2-surface (the 2-spheres), it is possible to perform just the conformal compactification of the two-dimensional surface orthogonal to the 2-spheres retaining all the important information. This is so because, by means of a coordinate change, the induced Lorentzian metric or first fundamental form of the two-dimensional surface can always be brought into a conformally flat form

$$\text{d}s^2 = \Omega^2(x_0, x_1) (\text{d}x_0^2 + \text{d}x_1^2).$$

In this way, it can be naturally embedded in an unphysical two-dimensional Minkowskian spacetime (see, for instance, [2]). This allows us to draw simple two-dimensional diagrams, called Penrose diagrams, from which one can find out the properties of the boundary at a glance. In case the boundary is a $C^1$ curve at a given point in the Minkowskian spacetime, one could causally characterize the point according to the tangent vector to the curve as usual. However, as the cases analysed in the literature show [3], one usually finds boundaries that are just piecewise $C^1$ and which provide us with ‘piecewise causal characterizations’.

In order to find out the main features of the local causal character of a singular boundary in a spherically symmetric spacetime, it is not necessary to follow completely the conformal procedure explained above. This is interesting because an analytic conformal compactification can only be found for certain particular cases. The alternative procedure is based on the general fact that the concept of a null geodesic is a conformally invariant one, so that for every null geodesic in the physical two-dimensional surface, there is a corresponding null geodesic in the unphysical two-dimensional Minkowskian spacetime [4]. It seems reasonable, as we will try to show, that the study of the behaviour of the null geodesics in the physical spacetime in the neighbourhood of a singularity will provide us with information about the behaviour of the geodesics around its corresponding conformal boundary and, as a consequence, on its causal character.

On the other hand, we will mainly deal with probably the most interesting type of singularities in spherically symmetric spacetimes: the zero-areal-radius scalar curvature singularities [5]. In order to define the concept, we will use the areal radius $R$ such that the area of a 2-sphere is $4\pi R^2$. Then, we say that there is a zero-areal-radius scalar curvature singularity at a point $p$ in $R = 0$ if any scalar-invariant polynomial in the Riemann tensor diverges when approaching it along any incomplete curve.

The difficulties to apply the above procedure will lie not in the identification of this type of singularities, but in the study of the behaviour of the radial null geodesics. In order to carry out this study, we will analyse the system of differential equations that describe the null geodesics by means of the standard qualitative theory of dynamic systems [6–10]. (Note also that we have written appendices A and B with the main results on this subject for the less usual cases.) The information supplied by this theory about the behaviour of the radial null geodesics will allow us to mathematically classify the neighbourhood of $p$ in the cases where the theory can be applied. As we will see, this together with the correct interpretation of the results for the $R \geq 0$ region will provide us with a piecewise characterization of the singularity around the chosen point.

But before putting this plan into practice, let us comment on the previous results and the relevance of this analysis. In fact, the study of $R = 0$ singularities has been carried out for important particular solutions. In some cases, the conformal boundaries have been obtained by using a conformal compactification (for example, Schwarzschild’s and Reissner–Nordström’s

Note that many authors call this singularity ‘central’, especially when studying the collapse of massive objects, regardless of its causal characterization. However, this terminology is hard to justify in general. For instance, the big-bang singularity in a Robertson–Walker model is simply a particular case of an $R = 0$ singularity, but it would hardly be called central.
solutions—see, for instance, [11]—and Vaidya’s solution when there is a linear mass function [12, 13]), while in other cases, there is no analytical compactification and the alternative method of studying the radial null geodesics has been used in order to get the local causal characterization (for example, Vaidya’s solution in the general imploding case [14]4 or so many different collapsing stellar models, like those found in [15–17]).

In addition to the analysis of particular cases, some general approaches for studying zero-areal-radius singularities have also been carried out by analysing the properties of the radial null geodesics. In particular, it has been shown [18] that an $R = 0$ singularity is spacelike (and trapped) at a point $p$ in $R = 0$ if $m^p_p > 0$ and timelike (and untrapped) if $m^p_p < 0$, where $m \equiv R(1 - g^\mu\nu \partial_\mu R \partial_\nu R)/2$. Nevertheless, the case $m^p_p = 0$, in which there could be either a regular centre or a (spacelike, lightlike or timelike) singularity at $p$, must be analysed in detail for every particular case. The trouble is that this is precisely the most interesting case in many different physical situations. For example, what is the outcome of the evaporation of a black hole (BH) if it gets rid of all of its mass? What transformations can the BH’s singularity undergo in the process? And, since only particular cases have been treated in the current literature [13, 19, 21–23], have all the possibilities been considered? On the other hand, if a singularity-free star collapses, can it generate a massless singularity at the evaporation event due to the focusing of the different shells that constitute its interior? And, in the affirmative case, what can its causal character be?

In order to clarify the importance of these questions, let us recall that the singularity theorems [5, 11] show that, given a few reasonable assumptions, a collapse can terminate in a gravitational singularity. However, the theorems do not inform us about many properties of the singularities [5]. Among others, there is lack of information about the type of singularity, the divergence of the energy density of matter fields and whether the singularity is hidden from outside view by the formation of a BH. With regard to the last point, we must emphasize that the theorems do allow for the possibility that the singularity could be seen by observers close enough to it (in which case, we say that there is a locally naked singularity) or from the future null infinity (globally naked singularity). In any case, there would be a hypersurface (the Cauchy horizon) beyond which general relativity loses its predictability. The question on whether general relativity contains a built-in safety feature that precludes the formation of naked singularities in generic gravitational collapses was put forward by Penrose in 1969 [24] and gave rise to what is known as the cosmic censorship conjecture (CCC). Clearly, the causal character of the singularity is central in the resolution of this conjecture since, by construction, there are always null geodesics which are past incomplete whenever the spacetime possesses timelike or past null singularities (see, for example, section 2). Therefore, timelike and past null singularities are always naked.

Some counterexamples to the cosmic censorship conjecture have been proposed. An outstanding collapsing and radiating model can be found in the work by Demianski and Lasota [25]. Even if it was not first proposed to be such a counterexample, but as an evaporating model, it was later shown [26] that it possesses an instantaneous naked scalar curvature singularity at the evaporating event. Another counterexample of historical importance was discussed in [12, 13, 27] considering the collapse of null dust modelled by using Vaidya’s solution. It was shown that it suffices that pure radiation (or null dust) with a sufficiently weak wave travelling into an initially flat spacetime focuses in $R = 0$ in order to create a null singularity which is at least locally visible. We will not intend now to exhaust all the different counterexamples to the

4 Incidentally, the reader can analyse this case, in which a singularity develops from a regular $R = 0$ with $m = 0$, to verify that the approach of studying the causal character of $n = dR$ (or $g^{\mu\nu}$, provided that the metric is given in suitable coordinates) in order to characterize an $R = 0$ singularity on the boundary of the spacetime (!) is not a reliable method.
cosmic censorship conjecture that can be found in the literature, but just to mention some of them we point out that naked singularities in spherically symmetric models are also possible in the collapse of dust [15, 16, 28, 30], perfect fluids [31], general fluids [32], massless scalar fields [33] and even in higher dimensions [34]. On the other hand, more general studies on the formation of naked singularities in spherically symmetric spacetimes based on the study of the radial null geodesics can be found in [36, 37] and, by using the ad hoc devised procedure, in [38].

Notwithstanding the above (incomplete) list of proposed counterexamples to the CCC, the subject is still open. This is so because, on the one hand, any specific example is unlikely to be considered generic in some appropriate sense and specific examples satisfying the CCC exist for the different matter fields above (see [39, 40] and references therein). Moreover, in [41] the weak version of the CCC [42] has been shown to hold for a wide variety of spherically symmetric coupled Einstein-matter systems. On the other hand, some examples possessing naked singularities have been shown to have some kind of instability. However, it is in general unclear whether the instabilities can really hide the singularity (see [39, 43–45] and references therein).

Our aim in this paper is to apply the techniques of the qualitative behaviour of dynamic systems to the radial null geodesics around every \( R = 0 \) singular point in order to study the causal characterization of the \( R = 0 \) singularities. This will allow us to ascertain the relevant quantities (as well as their associated values) that determine whether a singularity is essentially spacelike, lightlike or timelike (even if the singularity is piecewise timelike or piecewise spacelike, for the case of piecewise \( C^1 \) boundaries). We will also find out and explicitly state the limits for the applicability of our results coming from our specific approach. We would like to remark that ours is a geometrical approach requiring only the existence of a spacetime, but not the fulfilment of Einstein’s equations. Thus, we just try to discover the possibilities allowed by this geometrical approach which includes the classical as well as the semiclassical framework. With the obtained information we will be able to study different possibilities for the final outcome of BH evaporation and the different options for the generation of naked singularities. We will emphasize new models and possibilities that have not been taken into account so far.

The paper has been divided as follows. In section 2, we study the relationship between the null geodesics in the physical and the unphysical spacetimes and we revise how to extract information about the causal characterization from them. In section 3, we establish a general spherically symmetric spacetime and the conditions required for it to have an \( R = 0 \) scalar curvature singularity as well as the equations governing its radial null geodesics. Section 4 is an application for the trivial and well-known case of a non-zero function \( m \) as \( R \) tends to zero. In sections 5 and 6, we use the theory of qualitative behaviour of dynamic systems to deal with the analysis of isolated \( m(R \to 0) = 0 \) points, since, as we will see, they turn out to be the isolated critical points of the system of differential equations describing the radial null geodesics. Specifically, the hyperbolic and the non-hyperbolic cases are treated in sections 5 and 6, respectively. In order to exhaust all the different possibilities, the non-isolated \( m(R \to 0) = 0 \) points are treated in section 7. The last section is devoted to the consequences and applications of our results. In particular, the general cases of BH evaporation and the generation of naked singularities are analysed.

2. Null geodesics and causal characterization

Let us assume that we are given an oriented two-dimensional Lorentzian manifold and that we embed it into an unphysical two-dimensional Minkowskian spacetime, as explained in
the introduction. Provided that the singular boundary in the Minkowskian spacetime is $C^1$, at a point $p$ we will be able to compute the tangent vector to the singular boundary at $p$. Then, by definition, the causal character of the singular boundary at $p$ coincides with the causal character (spacelike, timelike or lightlike) of its tangent vector. Furthermore, inspired by the definitions appearing in [46], we will also specify that there is a past spacelike (or past lightlike) singularity at $p$ if only past-directed causal curves end up at a spacelike (or lightlike) singularity at $p$ (see, for example, figures 1(i) and (iv), respectively). Likewise, we say that there is a future spacelike (or future lightlike) singularity at $p$ if only future-directed causal curves end up at a spacelike (or lightlike) singularity at $p$ (see, for example, figures 1(ii) and (v), respectively).

Let us now consider a sufficiently small simply connected open set $U_M$ in the two-dimensional Minkowskian spacetime with corresponding points in the physical spacetime (specifically, in an open set $U$—defined in section 3) and such that a part of its boundary consists of an open interval of the singular boundary containing $p$. Two nonvanishing null vector fields may be defined on $U_M$ such that they are linearly independent at each point of $U_M$ [47]. The integral curves of the two null vector fields provide us with two families ($\mathcal{F}_1$ and $\mathcal{F}_2$) of (non-necessarily affine parametrized) null geodesics.

2.1. Relating the causal character of singular points to null geodesics

If we parametrize the $C^1$ curve describing the singular boundary with a parameter $\lambda$, then the value of the norm of its tangent vector will be given by a continuous function $N(\lambda)$. The continuity of $N(\lambda)$ implies that if it is bigger than zero at a point $p$ corresponding to a certain $\lambda = \lambda_0$, then it will be bigger than zero for an open interval $\lambda_- < \lambda_0 < \lambda_+$. So that there is an open interval around $p$ on the singular boundary where the boundary is necessarily spacelike. On the other hand, in this situation there are only two possibilities for the behaviour of the lightlike geodesics at $p$ that traverse $U_M$.

- There is a past (or future) spacelike singularity at $p$. A null geodesic of every family leaves (resp. reaches) $p$. (See figure 1(i) (resp. figure 1(ii)), where we have drawn the singularity horizontally$^5$).

5 The situation does not change if we draw the singularity with an inclination bigger than $-45^\circ$ but less than $45^\circ$ around $p$, as is guaranteed by $N(\lambda_- < \lambda_0 < \lambda_+) > 0$. 

\[\text{Figure 1. The behaviour of null geodesics (arrows) at } p \text{ in an open } C^1 \text{ interval of the singular boundary (wavy line) is shown for (i) a past spacelike singularity at } p, (ii) \text{ a future spacelike singularity at } p \text{ and (iii) a timelike singularity at } p. \text{ When only a null geodesic leaves or reaches } p, \text{ there are two possibilities: (iv) } p \text{ is part of a past lightlike singularity or (v) } p \text{ is part of a future lightlike singularity. (vi) There is a lightlike singularity at } p \text{ while the rest of the interval is timelike. Thus, it is an example of a piecewise timelike interval. (vii) There is a lightlike singularity at } p \text{ while the rest of the interval is spacelike. This is an example of a piecewise past spacelike interval.} \]
Similarly, if $N(\lambda_0) < 0$, then $N$ will be less than zero for an open interval $\lambda_- < \lambda_0 < \lambda_+$ and there is only a possibility for the behaviour of the lightlike geodesics at $p$ that traverse $\mathcal{U}_M$.

- There is a timelike singularity at $p$. A null geodesic of one family reaches $p$ and a null geodesic of the other family leaves it. (See figure 1(iii).)

It is useful to note that if one, and only one, null geodesic leaves or reaches a point $p$ in the boundary, then, by the process of elimination, the singularity must be lightlike at $p$. Specifically, there are two cases.

- If only a null geodesic leaves (or reaches) $p$, then there is a past (resp. future) lightlike singularity at $p$. (An example of this situation is shown in figure 1(iv) (resp. figure 1(v)).)

2.2. Intervals of piecewise constant causal characterization

Let us now classify the intervals in the singular boundary according to the behaviour of the null geodesics. (We enumerate them according to the number of geodesics that leaves or reaches every of their points.)

1$^+$ If intervals where, for every one of their points, only a null geodesic leaves (resp. reaches) it. Then, according to our previous subsection, the interval is a past (resp. future) lightlike singularity.

2$^+$ Intervals where, for every one of their points, a null geodesic of one family reaches the point and a null geodesic of the other family leaves it. Then we have seen in the previous subsection that the interval can contain points where it would be timelike ($N(\lambda_0) < 0$) and, as a result, subintervals around these points would be timelike.

The possibility of having a point in this singular interval where the singularity is spacelike is trivially discarded since the behaviour of the null geodesics does not correspond with the expected one according to the first two items in the previous subsection. The possibility of having a lightlike subinterval is also discarded since, if we could draw the subinterval, it would have an inclination of $\pm 45^\circ$, so that its points could not be reached by null geodesics from both families as demanded in this case. However, there is still the possibility for the interval to have lightlike points only if every neighbourhood of the lightlike point contains timelike subintervals (an example is shown in figure 1(vi)).

In this case, we will say that the interval is piecewise timelike.

Following the reasoning in the previous item, there is only the additional possibility of having lightlike points if every neighbourhood of the lightlike point in the interval contains spacelike subintervals (an example is shown in figure 1(vii) (resp. its time reversal)). In this case, we will say that the interval is a piecewise past (resp. future) spacelike singularity.

Here finishes our list of possible $C^1$ intervals according to the behaviour of the two families of null geodesics traversing $\mathcal{U}_M$. (Note that there cannot be $C^1$ intervals such that every point is reached or left by a total of three or four null geodesics traversing $\mathcal{U}_M$.)

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6 Note that, in general, the singularity does not have to be lightlike all around $p$.

7 In fact, it can be argued that the distinction between strictly timelike and piecewise timelike intervals is not very relevant from a physical point of view. For example, if we are worried about the formation of a naked singularity in a particular model, we will have one in both cases.
Figure 2. We show, to the left, a proper spacelike open interval of the singular boundary and, to the right, a piecewise spacelike singularity (since the angular boundary is not $C^1$ at $p$). Both possibilities are indistinguishable if we only take into account that a null geodesic of every family leaves every point in the boundary.

As some exact solutions found in the literature show, we could also deal with just piecewise $C^1$ boundaries. In these cases, there would be points where it will not be possible to define a tangent vector to the singular boundary and, strictly speaking, no matter the chosen approach to ascertain the causal character of the boundary, we should resign ourselves to deal with piecewise causal characterizations. An additional difficulty appears when one analyses the null geodesics for these piecewise $C^1$ boundaries in order to get its causal characterization. Consider, for example, an open interval in the singular boundary where two null geodesics leave the boundary at every of its points. Then we cannot tell with just this scrutiny of null geodesics whether there is some point on the interval where the singular boundary is not $C^1$ (see figure 2). Likewise, a point in the singular boundary where the boundary is not $C^1$ can pass unnoticed either in intervals where two null geodesics leave the boundary at every of its points (imagine the time reversal of figure 2) or in intervals that are reached and left by one null geodesic of every family on every point. However, intervals where, at every point, only a null geodesic leaves it (past lightlike singularities) or reaches it (future lightlike singularities) are clearly $C^1$ (they are defined by $\pm 45^\circ$-inclined straight lines) and their causal character is therefore strictly determined.

Let us now call transition points those points in the singular boundary where two different intervals from the ones defined above join (for example, a $2^0$ interval is followed by a $2^-$ interval in the singular boundary). This definition implies either that the singular boundary is lightlike at a transition point $p$ or that no tangent vector to the boundary can be defined at $p$. Since every piecewise $C^1$ boundary will be composed of intervals with a (piecewise) single causal character joined through transition points, we will be able to sketch the causal character of the singular boundary by identifying the causal character of every (piecewise)-single-causal-character interval and overlooking whether the transition points really are lightlike. In order to exemplify this, in figure 3, we have sketched a spacelike open interval joined to a timelike open interval through a transition point $q$. In this spirit, from now on we will omit the adverb piecewise in single-causal-character intervals.

2.3. Translation into a physical spherically symmetric spacetime

Let us assume now that we are working with a spherically symmetric spacetime and that we have identified the physical two-dimensional Lorentzian surface orthogonal to the 2-spheres. Assuming that the invariant areal radius $R$ can be used as a coordinate in a local chart of the Lorentzian surface, we will work in local coordinates \{\(R, u\)\}, where $u$ is a lightlike coordinate. Note that $u = \text{constant}$ has a clear geometrical meaning, even if no further constraint is imposed on $u$, since it corresponds to an invariantly defined radial null geodesic. Moreover, according to our project, with this choice of coordinates a family of radial null geodesics is given ($u = \text{constant}$) and we need only to compute the other family of radial null geodesics.
in order to obtain the sketched causal characterization of any $R = 0$ singularity. This is so because, as explained in the introduction, we know that for every null geodesic in this physical two-dimensional Minkowskian spacetime, there is a corresponding null geodesic in the unphysical two-dimensional Minkowskian spacetime and vice versa. The local translation from the physical ($\mathcal{U}$) to the unphysical ($\mathcal{U}_M$) spacetime is rather straightforward. We have listed the possibilities for single-causal-character open intervals in figure 4. The only noteworthy case appears when an infinite number of null geodesics from the second family tend to (or depart from) a singularity at $\{ R = 0, u = u_0 \}$, which corresponds to the existence of a lightlike singularity, as can be seen in figures 4(vi) and (vii).

### 3. General spherically symmetric metric

Let us consider an oriented four-dimensional spherically symmetric spacetime $\mathcal{V}$. In order to study the local causal behaviour around $R = 0$, we will consider a local chart endowed with coordinates $\{ x^\mu \} = \{ u, R, \theta, \phi \}$ ($\mu = 0, 1, 2, 3$) and an open set $\mathcal{U} \equiv \{ (u, R) \} - \delta u < u < \delta u, 0 < R < \delta R \}$. The line element can be expressed in this local chart as

\[
\text{d}s^2 = -e^{4\beta} \text{d}u^2 + 2 \varepsilon e^{2\beta} \text{d}u \text{d}R + R^2 \text{d}\Omega^2.
\]

where $R$ is the areal radius, $e^2 = 1$, $\beta$ and $\chi$ are assumed to be at least $C^2$ functions\(^8\) on $\{ u, R \}$ in the local chart, $\beta$ is also assumed to be bounded as it approaches $R = 0$ and $\text{d}\Omega^2 \equiv \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2$.

In spherical symmetry, one can define the scalar invariant

\[
m \equiv \frac{R}{2} (1 - g^{uR} \partial_u R \partial_R R)
\]

(see [48] and also [18], [49] and references therein). If one computes $m$ for metric (1), it is easily checked that

\[
m = \frac{R}{2} (1 - \chi).
\]

\(^8\) This is a minimum requirement which guarantees the existence and uniqueness of null geodesics in $\mathcal{U}$ [11]. It also enables to use Einstein’s equations (in case one works in the framework of GR).
Figure 4. Translation table between the chart in the physical spacetime and the sketches in the unphysical spacetime for single causal character open intervals. In cases (i)–(iii) at every point in a whole open interval of \( u \) with \( R = 0 \), a null geodesic of every family tends to (or departs from) the point. (We depict only those tending to (or departing from) a point \( p \{ R = 0, u = u_0 \} \) in the interval.) In cases (iv) and (v), only a family of null geodesics tend to (or depart from) the points in a whole open interval of \( u \) with \( R = 0 \). In the last possibilities, (vi) and (vii), an infinite number of null geodesics (in the grey zone of the \( R-u \) diagram) tend to a single point \( \{ R = 0, u = u_0 \} \), which must be interpreted as a lightlike singularity in the unphysical spacetime.

On the other hand, if we do not want \( R = 0 \) scalar curvature singularities, the scalar invariant polynomials in the Riemann tensor must remain finite at \( R = 0 \). It is well known that there are only four algebraically independent scalar invariants associated with a general spherically symmetric metric [50]. We can take, for example, [51–53]

\[
\begin{align*}
\mathcal{R} &\equiv g^{\gamma \rho} g^{\delta \beta} R_{\alpha \beta \gamma \delta}, \\
r_1 &\equiv S_\alpha^\beta S_\beta^\alpha, \\
r_2 &\equiv S_\alpha^\beta S_\beta^\gamma S_\gamma^\alpha, \\
w_2 &\equiv C_{\alpha \beta \gamma \delta} \tilde{C}_{\mu \nu \lambda}^\gamma \tilde{C}_{\mu \nu \lambda}^\beta,
\end{align*}
\]

where \( S_\alpha^\beta \equiv R_\alpha^\beta - \delta_\alpha^\beta R/4 \), with \( R_\alpha^\beta \) being the Ricci tensor and \( \mathcal{R} \) the curvature scalar, \( C_{\alpha \beta \gamma \delta} \) the Weyl tensor and \( \tilde{C}_{\alpha \beta \gamma \delta} \equiv (C_{\alpha \beta \gamma \delta} + i * C_{\alpha \beta \gamma \delta})/2 \) the complex conjugate of the self-dual Weyl tensor, with \(*C_{\alpha \beta \gamma \delta} \equiv \epsilon_{\alpha \beta \mu \nu} C_{\gamma \delta}^{\mu \nu} / 2 \) being the dual of the Weyl tensor. If one evaluates these invariants for (1) using (2), one arrives at the following statement [53, 54]: all scalar invariant polynomials in the Riemann tensor will be finite at \( R = 0 \), preventing the existence of scalar curvature singularities.
of scalar curvature singularities if, and only if,
\[
\lim_{R \to 0} m = \lim_{R \to 0} \frac{m}{R^2} = \lim_{R \to 0} \frac{m}{R^3} = \lim_{R \to 0} \frac{\beta - \beta_0}{R} = 0,
\]
\[
\lim_{R \to 0} \frac{\beta - \beta_0}{R^2} = \beta_2(u), \quad \lim_{R \to 0} \frac{m}{R^3} = m_3(u),
\]
where \( \beta_0(u) \equiv \lim_{R \to 0} \beta(u, R) \) and both \( \beta_2(u) \) and \( m_3(u) \) are finite functions.

Moreover, if the set of scalar invariants is finite at \( R = 0 \), there will only be two algebraically independent scalar invariants at \( R = 0 \), say \( R \) and \( r_1 \), since \( w_2 = 0 \) and \( 343r_1^6 = 3087r_2^4 \).

Let us finally remark that one of the interesting properties of the scalar curvature singularities is that their existence cannot be an artefact of the coordinate system used. We now know that if conditions (3) are not fulfilled, then for any curve approaching \( R = 0 \) there will be at least one scalar invariant that will grow without limit along the curve as it approaches the \( R = 0 \) singularity. It is obvious that, if one uses different local coordinates, the value of the scalar invariant at every point of the curve must coincide, when computed with the new local coordinates, with the values obtained at the same points with the coordinate system used in this paper. Therefore, no matter what local coordinates are used, the same scalar invariant will grow without limit along the curve, which unmistakably identifies an \( R = 0 \) scalar curvature singularity.

3.1. Radial null geodesics

We choose \( u \) growing to the future. Then
\[
l = \frac{d}{d\ell} = -\frac{\varepsilon}{2R} \frac{\partial}{\partial R}
\]
is a future-directed radial null vector tangent to the null geodesics \( u = \text{constant} \) (namely the \( F_1 \) family) and \( \ell \) is a future-directed parameter. Since the expansion [11] of these null geodesics is given by
\[
\theta_l = -\frac{\varepsilon}{R^2},
\]
if \( \varepsilon = -1 \) (or +1), the expansion is positive (negative, respectively) and they are outgoing (ingoing, respectively) radial null geodesics directed towards increasing \( R \)s (decreasing \( R \)s, respectively) according to (4). Therefore, the behaviour of the \( F_1 \) family, with regard to whether the null geodesics are coming or are directed towards the \( R = 0 \) singularity, is absolutely defined by the sign of \( \varepsilon \), a fact that we will use throughout this paper. On the other hand,
\[
k = \frac{d}{d\kappa} = 2Re^{-2\beta} \frac{\partial}{\partial u} + \varepsilon(R - 2m) \frac{\partial}{\partial R}
\]
is a future-directed radial null vector such that \( l \cdot k = -1 \) and \( \kappa \) is a future-directed parameter. For later purposes, let us write explicitly the equations governing the geodesics that have \( k \) as its tangent vector field (the \( F_2 \) family):
\[
\begin{align*}
\frac{dR}{d\kappa} &= \varepsilon[R - 2m(u, R)] \\
\frac{du}{d\kappa} &= 2Re^{-2\beta}.
\end{align*}
\]
In this case, the expansion is given by
\[
\theta_k = \frac{2\varepsilon}{R}(R - 2m).
\]
If at a given 2-sphere $\chi > 0$ ($\Rightarrow R > 2m$) and $\varepsilon = -1$ (or $+1$), $k$ is tangent to a family of null geodesics with negative (positive, respectively) expansion, the areal coordinate $R$ decreases (increases, respectively) along them according to (6) and, therefore, these radial null geodesics are ingoing (outgoing, respectively) in the considered 2-sphere. However, it is interesting to note that if, in a given 2-sphere, $\chi < 0$ and $\varepsilon = -1$ ($\varepsilon = +1$), then the two radial null vectors have both positive (negative) expansion which means that the 2-sphere is a closed surface trapped to its past (future, respectively).

4. Characterization of the singularity: case $\lim_{\kappa \to \kappa_0} m(u, R) \neq 0$

Let us consider, in $\mathcal{U}$, the set of radial null geodesics belonging to the family $\mathcal{F}_2$ that approaches $R = 0$ (either towards their past or their future). For every such geodesic, we define $\kappa_0$ to be the value of its parameter such that $\lim_{\kappa \to \kappa_0} (\mathcal{R}(\kappa), u(\kappa)) = (0, u_0)$, where the precise value $u_0 \in (-\delta u, \delta u)$ depends on every geodesic. Now, if $\lim_{\kappa \to \kappa_0} m(u, R) \neq 0$ along every of these null geodesics, according to (6),

\[
\begin{align*}
\lim_{\kappa \to \kappa_0} \frac{dR}{d\kappa} &= -2\varepsilon \lim_{\kappa \to \kappa_0} m(u, R) \\
\lim_{\kappa \to \kappa_0} \frac{du}{d\kappa} &= 0,
\end{align*}
\]

which implies that

- if $\lim_{\kappa \to \kappa_0} m(u, R) > 0$ in $\mathcal{U}$, then the two different families of radial null geodesics are ingoing ($\varepsilon = -1$) or outgoing ($\varepsilon = +1$) in $\mathcal{U}$ (see section 3.1 for the $u$ constant family) and, therefore, the singularity is spacelike at $R = 0$;
- if $\lim_{\kappa \to \kappa_0} m(u, R) < 0$ in $\mathcal{U}$, then one radial null geodesic is outgoing, while the other is ingoing and, therefore, the singularity is timelike at $R = 0$.

As a corollary we have the following more workable and well-known [18] result: if $\lim_{(u \to m_0, R \to 0)} m(u, R)$ exits and it is positive for all $u_0$ in $\mathcal{U}$, then the singularity is spacelike at $R = 0$ and if $\lim_{(u \to 0, R \to 0)} m(u, R)$ exists and it is negative for all $u_0$ in $\mathcal{U}$, then the singularity is timelike at $R = 0$.

5. Characterization of the singularity: case $\tilde{m}(u = 0, R = 0) = 0$ and $\tilde{m}_{u0} (u = 0, R = 0) \neq 0$

In this section, we assume, as usual, that $m$ and $\beta$ are $C^{2-}$ in $\mathcal{U}$ and also that $C^1$ auxiliary extensions (devoid of any physical meaning) $\tilde{m}$ and $\tilde{\beta}$, respectively, exist in $\mathcal{U} \equiv \{(u, R), -\delta u < u < \delta u, -\delta R < R < \delta R\}$ such that, formally, $\tilde{m} = m$ and $\tilde{\beta} = \beta$ for $(-\delta u < u < \delta u, 0 < R < \delta R)$.

In the case considered in this section ($\tilde{m}(u = 0, R = 0) = 0$ and $\tilde{m}_{u0} (u = 0, R = 0) \neq 0$), there is a scalar curvature singularity at $R = u = 0$ since the regularity conditions (3) are violated.

Without loss of generality, we will consider from now on $\tilde{\beta}(u, 0) = 0$ since, if it were not, we could always perform a coordinate change $u \rightarrow u'$ such that the new coordinate $u'$ were defined by $du' = e^{\varepsilon \tilde{\beta}(u, 0)} du$.

Now $u = R = 0$ is a critical point of the system (6). In order to analyse the qualitative behaviour of the radial null geodesics, we try a linear approximation at the critical point:

9 Note that in the literature one usually finds the same functions $m$ and $\beta$ used for $R \leq 0$, which could be considered as the natural extension.
If we define $f^R \equiv \varepsilon[R - 2\bar{m}(u,R)]$ and $f^u \equiv 2R e^{-2\bar{\beta}}$, then the linearization matrix is

$$A_{\bar{\beta}}^x \equiv f^u_{,\bar{\beta}} (u = 0, R = 0)$$

$$A = \begin{pmatrix} \varepsilon (1 - 2\bar{m},R (0,0)) & -2\varepsilon \bar{m},u (0,0) \\ \frac{2}{\bar{\epsilon}} & 0 \end{pmatrix}.$$ \hfill (8)

The characteristic roots for this matrix are

$$\lambda_{\pm} = \frac{\pm \varepsilon (1 - 2\bar{m},R (0,0)) \pm \sqrt{(1 - 2\bar{m},R (0,0))^2 - 16\varepsilon \bar{m},u (0,0)}}{2}.$$ \hfill (9)

The critical point $u = R = 0$ will be hyperbolic [6] if none of the characteristic roots have zero real part. In order to analyse this for our case, let us define $\Delta \equiv (1 - 2\bar{m},R (0,0))^2 - 16\varepsilon \bar{m},u (0,0).$ Then $u = R = 0$ will be hyperbolic if

$$\bar{m},u (0,0) \neq 0$$ \hfill (10)

and, in case $\Delta \leq 0$, if the extra condition $\bar{m},R (0,0) \neq 1/2$ is satisfied. Under the assumption that $\bar{m}$ and $\bar{\beta}$ are $C^1$, we can apply the Hartman–Grobman theorem [6]. According to this theorem, if the critical point is hyperbolic, then the behaviour of the nonlinear system near the critical point is qualitatively determined by the behaviour of the linear system $\dot{\bar{x}} = A_{\bar{\beta}}^{\bar{x}}$. Furthermore, the theorem implies that the qualitative behaviour is independent of the extension chosen for $m$ and $\bar{\beta}$ when $[-\delta u < u < \delta u, -\delta R < R \leq 0]$ as long as it is a $C^1$ extension. In other words, and taking into account (8), the results will only depend on the partial derivatives of $m$ as we approach $R = u = 0$. Note that in the hyperbolic case near the critical point, the curves defined by $f^R(u,R) = 0$ and $f^u(u,R) = 0$ behave approximately as straight lines in the $[u,R]$ plane intersecting at the point $u = R = 0$, which is an isolated critical point [9].

5.1. Different possibilities

- If $\Delta < 0$ and $\bar{m},R (0,0) \neq 1/2$, the characteristic roots are complex conjugates of each other with nonzero real part. This means that the critical point $u = R = 0$ is a focus [6, 8]. In this way, the future-directed radial null geodesics in a neighbourhood of the critical point must start at $R = 0$, $u < 0$ and end at $R = 0$, $u > 0$ after going around the critical point. Therefore,

  (a) If $\varepsilon = -1$, then the $R = 0$ singularity must be spacelike near the critical point for $u < 0$, while it must be timelike near the critical point for $u > 0$. See figure 5.

  (b) If $\varepsilon = +1$, then the singularity must be timelike near the critical point for $u < 0$, while it must be spacelike near the critical point for $u > 0$.

- If $\Delta < 0$ and $\bar{m},R (0,0) = 1/2$, the point is not hyperbolic. However, even if the linearization does not suffice to distinguish the exact qualitative behaviour, it can be guaranteed that $u = R = 0$ will be either a focus or a centre [6, 7]. Therefore, the same characterization as in the above item applies.

- If $\Delta > 0$ and $\varepsilon \bar{m},u (0,0) > 0$, then the characteristic roots are real, with the same sign and distinct so that the critical point is a node [6, 8]. The slope of the radial null geodesics ending or starting at $u = R = 0$ is $\xi_0 \equiv \lim_{k \to \xi_0} u(k)/R(k)$. Using (6), we find $\xi_0 = 2\varepsilon/(1 - 2\bar{m},R (0,0) + \varepsilon \bar{m},u (0,0))$. The two real roots $\xi_{0,\pm}$ of this quadratic equation are

$$\xi_{0,\pm} = \frac{(1 - 2\bar{m},R (0,0)) \pm \sqrt{(1 - 2\bar{m},R (0,0))^2 - 16\varepsilon \bar{m},u (0,0)}}{4\bar{m},u (0,0)}.$$ \hfill (11)

10 Throughout the paper we will use the fact that the radial null geodesics are future directed in order to interpret in correct physical terms the results provided by the qualitative theory of dynamic systems.
If $T = \varepsilon (1 - 2 \bar{m}, u \left( 0, 0 \right)) > 0$, then $\lambda_{\pm} > 0$, $\xi_{0\pm} > 0$ and the node is unstable. Then it can be shown [7] that around $u = R = 0$, all, except for one, radial null geodesics must start at this point with a definite slope which is $\xi_{0+}$ for $\varepsilon = +1$ and $\xi_{0-}$ for $\varepsilon = -1$. The exception is just one null geodesic starting with the slope $\xi_{0-}$ for $\varepsilon = +1$ and $\xi_{0+}$ for $\varepsilon = -1$. The behaviour of the second family of null geodesics ($F_2$) for the $\varepsilon = -1$ case together with the corresponding sketched Penrose’s diagram is shown in figure 6.

- If $T < 0$, then $\lambda_{\pm} < 0$, $\xi_{0\pm} < 0$ and the node is stable. Therefore, the result is similar to figure 7 with equal critical directions.

- If $\Delta = 0$ and $\varepsilon \bar{m}, u \left( 0, 0 \right) \neq 0$ (no matter what the value of $\varepsilon (1 - 2 \bar{m}, u \left( 0, 0 \right))$ is), the characteristics roots are real with opposite sign and the critical point is a saddle [6, 8]. Likewise, the two critical directions also have a opposite sign since $\text{sign}(\xi_{0-}) = \varepsilon = \text{sign}(\xi_{0+})$. The behaviour of the second family of radial null geodesics for the $\varepsilon = -1$ case together with the corresponding sketched Penrose’s diagram is shown in figure 8.

We have collected the results for $\bar{m}, u \left( u = 0, R = 0 \right) \neq 0$ in figure 9.
In the left of this figure, we sketch the unstable node behaviour of the second family of radial null geodesics in the $\epsilon = -1$ case near $u = R = 0$ when $\det A > 0$, $T > 0$ and $\Delta > 0$. The two straight lines crossing at $u = R = 0$ indicate the critical directions of the radial null geodesics starting at $u = R = 0$. From these geodesics only, one leaves the critical point with the direction $\xi_0^+$, while all the others leave it with the direction $\xi_0^-$. In the right, we show the corresponding Penrose diagram around $R = u = 0$ with its characteristic $R = u = 0$ lightlike singularity. Here we have pointed out the single radial null geodesic that leaves the critical point in the direction $\xi_0^+$.

In the left of this figure, we sketch the stable node behaviour of the second family of radial null geodesics in the $\epsilon = -1$ case near $u = R = 0$ when $\det A > 0$, $T < 0$ and $\Delta > 0$. In the right, we show the corresponding Penrose diagram around $R = u = 0$ with its characteristic $R = u = 0$ lightlike singularity.

6. Characterization of the singularity: case $\bar{m}(u = 0, R = 0) = 0$ and $\bar{m}_{su}(u = 0, R = 0) = 0$

In this case, $u = R = 0$ is an isolated critical point of system (6), but, as we showed in the previous section, since $\bar{m}_{su}(u = 0, R = 0) = 0$, the critical point is not hyperbolic. In fact, if we demand $\bar{m}_{su}(0, 0) \neq 1/2$, the critical point will be semi-hyperbolic [10] and we
Figure 8. In the left of this figure, we sketch the saddle behaviour of the second family of radial null geodesics in the $\epsilon = -1$ case near $u = R = 0$ when $\det A < 0$ and $\Delta > 0$. In the right, we show the corresponding sketched Penrose diagram around $u = R = 0$ with its characteristic timelike–spacelike for the $\epsilon = -1$ case.

will have to use a different approach. (The theory regarding the qualitative behaviour of these points can be found in \cite{9, 10}. The reader can also find a summary of the main results in appendix A.) In order to apply the theory for semi-hyperbolic points, we must now demand, first, the existence of a natural number $n \geq 2$ such that $\lim_{(u \to 0, R \to 0)} \partial_{u n} (u, R) \neq 0$, while $\lim_{(u \to 0, R \to 0)} \partial_{u i} (u, R) = 0$ for $i = 1, \ldots, n - 1$, and, second, of $C^n$ extensions\textsuperscript{11} $\bar{m}$ and $\bar{\beta}$.

First we define

$$x = -\frac{2\epsilon}{1 - 2\bar{m}_R (0, 0)} R + u \quad (12)$$

$$y = R \quad (13)$$

$$t = \epsilon (1 - 2\bar{m}_R (0, 0)) \kappa. \quad (14)$$

Then the system (6) can be rewritten in the normal form

$$\begin{align*}
\frac{dx}{dt} &= P_2 (x, y), \\
\frac{dy}{dt} &= y + Q_2 (x, y),
\end{align*}$$

where $P_2 (x, y) \equiv \tilde{P}_2 (u(x, y), R(y))$, $Q_2 (x, y) \equiv \tilde{Q}_2 (u(x, y), R(y))$,

$$\tilde{P}_2 (u, R) = \frac{2\epsilon}{1 - 2\bar{m}_R (0, 0)} \left( R e^{-2\beta(u, R)} - \frac{R - 2\bar{m}(u, R)}{1 - 2\bar{m}_R (0, 0)} \right),$$

$$\tilde{Q}_2 (u, R) = 2 \frac{\bar{m}_R (0, 0) R - \bar{m}(u, R)}{1 - 2\bar{m}_R (0, 0)}$$

and they satisfy $P_2 (0, 0) = Q_2 (0, 0) = P_2 (0, 0) = Q_2 (0, 0) = P_2 (0, 0) = Q_2 (0, 0) = 0$. The assumed degree of differentiability together with the implicit function theorem guarantees that the equation $y + Q_2 (x, y) = 0$ has a solution $y = \varphi (x)$ which can

\textsuperscript{11} Private communication with the authors of \cite{10}, where $C^\infty$ is assumed independently of the value of the finite $n$. 
Figure 9. Characterization of the singularity when $\tilde{m}(u = 0, R = 0) = 0$ and $\tilde{m}_{\alpha\beta}(u = 0, R = 0) \neq 0$. 

The theory of the qualitative behaviour of dynamical systems tells us that we need only to know whether $n$ is even or odd and the sign of $\Delta_n$ to tell the qualitative behaviour of the dynamical systems.

Note that the requirement on the degree of differentiability for $\tilde{m}$ and $\tilde{\beta}$—taking into account Taylor’s theorem—allows us to write some functions derived from $\tilde{m}$ and $\tilde{\beta}$ as a Taylor polynomial plus a remainder term. From now on we will only write the first non-zero term of the Taylor polynomial (that is only guaranteed to exist thanks to our assumptions) and suspension points—as in (15).
geodesic curves around the critical point which can now be a saddle node, a topological saddle or a topological node. In fact, the saddle node behaviour is the only possibility that we have not treated yet. While all saddle nodes are the union of one parabolic and two hyperbolic sectors [9], we still have to study the geodesics only in $\mathcal{U}$, which implies $R \geq 0$. There appear four different possibilities that we show in the self-explanatory figures 10–13 for the case $\epsilon = -1$.

We have collected all the possibilities for $\tilde{m}_u (u = 0, R = 0) = 0$ and $\tilde{m}_R (0, 0) \neq 1/2$ in figure 14.
Figure 12. In the left of this figure, we sketch the behaviour of the second family of radial null geodesics in the \( \epsilon = -1 \) case near \( u = R = 0 \) when \( \tilde{m}_u(0,0) = 0 \), \( \Delta_u > 0 \) with \( n \) even and \( \tilde{m}_{,R}(0,0) > 1/2 \). The two hyperbolic sectors (H) and the parabolic sector (P) are partially visible for \( R \geq 0 \). We also show by means of a straight line the finite critical direction \( \xi_0 \). In the right, we show the corresponding Penrose diagram around \( u = R = 0 \), which is a lightlike singularity.

Figure 13. In the left of this figure, we sketch the behaviour of the second family of radial null geodesics in the \( \epsilon = -1 \) case near \( u = R = 0 \) when \( \tilde{m}_u(0,0) = 0 \), \( \Delta_u > 0 \) with \( n \) even and \( \tilde{m}_{,R}(0,0) < 1/2 \). We also show by means of a straight line the regular finite critical direction \( \xi_0 \) of the curve starting at \( u = R = 0 \) that separates a hyperbolic sector (H) from the parabolic sector (P) for \( R \geq 0 \). In the right, we show the corresponding Penrose diagram around \( u = R = 0 \) with its timelike singularity.

Finally, let us analyse the case with \( \tilde{m}_u(0,0) = 0 \) and \( \tilde{m}_{,R}(u = 0, R = 0) = 1/2 \). In this case, the critical point will be nilpotent [10]. Again we refer the reader to [9, 10] for the theory of the qualitative behaviour of these points and to appendix B for a summary. In order to apply the theory of nilpotent points, we must demand the existence of, first, a natural number \( n \geq 2 \) defined by \( n = \text{Max}\{i, k\} \), where \( i \) is the lowest value of \( j \) such that \( \lim_{(u \to 0, R \to 0)} \partial m/\partial u^j(u, R) \neq 0 \) and \( k = l + 1 \), where \( l \) is the lowest value such that \( \lim_{(u \to 0, R \to 0)} (\partial m/\partial R)/\partial u^i \neq 0 \) and, second, \( C^n \) extensions\(^{13} \) \( \tilde{m} \) and \( \tilde{\beta} \).

\(^{13}\) Private communication with the authors of [10], where \( C^\infty \) is assumed independently of the value of the finite \( n \).
Figure 14. Characterization of the singularity when $\bar{m}(u = 0, R = 0) = 0, \bar{m}_u (u = 0, R = 0) = 0$ and $\bar{m}_R (0, 0) \neq 1/2$. 

| $n$ | $\Lambda_n$ | $\varepsilon$ | $\partial m/\partial u |_{u=0}$ | $\partial m/\partial R |_{R=0}$ |
|-----|-------------|---------------|----------------|-----------------|
| even | $> 0$ | $> 1/2$ | |
| even | $> 0$ | $< 1/2$ | |
| even | $< 0$ | $> 1/2$ | |
| even | $< 0$ | $< 1/2$ | |
| odd | $> 0$ | $> 1/2$ | |
| odd | $> 0$ | $< 1/2$ | |
| odd | $< 0$ | $\neq 1/2$ | |
In this case, we simply define \( x = u/2, y = R, t = \kappa \) in order to rewrite the system (6) in the 

\[ \begin{align*}
\frac{dx}{dt} &= y + P_2(x, y) \\
\frac{dy}{dt} &= Q_2(x, y),
\end{align*} \]

where

\[ P_2(x, y) \equiv \hat{P}_2(u(x), R(y)), \quad \hat{P}_2(u, R) = R(e^{-2\beta(u,R)} - 1), \]
\[ Q_2(x, y) \equiv \hat{Q}_2(u(x), R(y)), \quad \hat{Q}_2(u, R) = \varepsilon(R - 2m(u, R)), \]

and they satisfy \( P_2(0, 0) = Q_2(0, 0) = P_{2,\varepsilon}(0, 0) = Q_{2,\varepsilon}(0, 0) = P_{2,\varepsilon}(0, 0) = Q_{2,\varepsilon}(0, 0) = 0 \). The solution of the equation \( y + P_2(x, y) = 0 \), which by the implicit function theorem in general takes the form \( y = \psi(x) \), in this case is simply \( y = 0 \). Following the general procedure, if we define the function \( \psi(x) = Q_2(x, \varepsilon(x)) \), then the assumed degree of differentiability guarantees that its truncated series expansion will have the form

\[ \psi(x) = a_k x^k + \cdots, \quad (18) \]

where \( k \geq 2 \) is assumed to be finite and, considering the Taylor polynomial for \( \tilde{m} \), one obtains

\[ a_k = -\frac{2^{k+1}}{k!} \frac{\partial^k \tilde{m}}{\partial u^k}(0, 0). \quad (19) \]

On the other hand, if we define \( \sigma(x) = P_{2,\varepsilon}(x, \varepsilon(x)) + Q_{2,\varepsilon}(x, \varepsilon(x)) \), then either its truncated series expansion can be written as

\[ \sigma(x) = b_n x^n + \cdots, \quad (20) \]

where

\[ b_n = -\varepsilon \frac{2^{n+1}}{n!} \frac{\partial^n \tilde{m}}{\partial u^n} \left( \frac{\partial \tilde{m}}{\partial R} \right)(0, 0). \quad (21) \]

if \( b_n \neq 0 \) or, alternatively, \( \sigma(x) \equiv 0 \) in which case \( b_n = 0 \forall n \).

The theory of the qualitative behaviour of dynamic systems tells us that we need only to know whether \( n \) is even or odd and the sign of \( a_k \) and \( b_n \) to tell the qualitative behaviour of the null curves around the critical point which can now be a saddle node, a topological saddle, a topological node, a cusp, a focus-centre or an elliptic region. In fact, only the equilibrium state with elliptic region behaviour provides us with a behaviour for the singularity that we have not encountered so far. We show this possibility in figure 15.

We have collected all the possibilities for \( \tilde{m}, \sigma(u = 0, R = 0) = 0 \) and \( \tilde{m}, \varepsilon(0, 0) = 1/2 \) in figures 16 and 17 with supplementary details in their corresponding captions.

The cases analysed so far cover all the possibilities whenever \( u = R = 0 \) is an isolated critical point.

7. Characterization when \( \tilde{m}(-\delta u < u < \delta u, R = 0) = 0 \)

In this case, the curves \( f_R(u, R) = \varepsilon[R - 2\tilde{m}(u, R)] = 0 \) and \( f_u(u, R) = 2R e^{-2\beta} = 0 \) must cross in an interval made up of points which are non-isolated critical points for the system of differential equations (6). Assuming the existence of the \( C^1 \) extensions \( \tilde{m} \) and \( \tilde{P} \), the equation for the radial null geodesics (6) can be easily written on the interval, up to first order, as

\[ \varepsilon(u) \equiv \frac{dR}{du}(u, R = 0) = \frac{\varepsilon(1 - 2\tilde{m},(u, 0))}{2}, \quad (22) \]
where we have used the fact that $\tilde{m}(-\delta u < u < \delta u, R = 0) = 0$ implies that $\tilde{m}_{,u} = 0$ in the interval. Let us consider the characterization of the singularity for the case $\varepsilon = -1$ (the case $\varepsilon = +1$ can be easily obtained later on as its time reversal). Then the $u =$ constant radial null geodesics are outgoing while

- If $\tilde{m}_{,R}(u, R = 0) < 1/2$ in $\mathcal{U}$, the second family of radial null geodesics satisfies $\zeta < 0$ and they are ingoing. Therefore, the singularity there is timelike.
- If $\tilde{m}_{,R}(u, R = 0) > 1/2$ in $\mathcal{U}$, the second family of radial null geodesics satisfies $\zeta > 0$ and they are outgoing. Therefore, the singularity there is spacelike.
- If $\tilde{m}_{,R}(u, R = 0) = 1/2$ in $\mathcal{U}$, then $\zeta = 0$.

In this way, the trajectories of the second family of radial null geodesics tend to be parallel to the $R = 0$ interval the closer they are to the singularity, and thus, the singularity is lightlike.

We have sketched the three situations in figure 18 for $\varepsilon = -1$. (It is not necessary to sketch the case $\varepsilon = +1$ since it produces the time-reversal diagrams for every case.)

8. Concluding remarks and some applications

Let us assume that the reader finds a specific singular model and wants to check the causal character of its $R = 0$ singularity by means of the results provided in this paper. In order to clarify the applicability of our approach and to make the reader’s task easier, we would now like to summarize the assumptions that have been made along the paper and the path that the reader should follow. On the one hand, we have assumed that

14 In case there is a singularity, since this case includes the regular one (3).
Figure 16. Characterization of the singularity when $\overline{m}(u = 0, R = 0) = 0$, $\overline{m}_u (u = 0, R = 0) = 0$, $\overline{m}_R (0, 0) = 1/2$ and $k$ is odd. In the extra conditions we have used $p = (k - 1)/2$ and $\lambda = b_n^2 + 2(k + 1)a_n$. The reader can check (see appendix B or [9, 10]) that the first row corresponds to a topological saddle, the second to a focus-centre, the third and fourth to a topological node and the fifth and sixth to an elliptic region. Note that the topological node and the elliptic region have two possibilities depending on the sectors appearing in $R > 0$, which depends on the sign of $b_n$. 

| $\frac{\partial^2 \overline{m}}{\partial u^2} \bigg|_{u=0}$ | $\frac{\partial^2 \overline{m}}{\partial R^2} \bigg|_{R=0}$ | Extra-conditions | $\varepsilon = -1$ | $\varepsilon = +1$ |
|---------------------------------|---------------------------------|------------------|----------------|----------------|
| $> 0$ | Any | No | ![Image](image1.png) | ![Image](image2.png) |
| $< 0$ | $= 0$ | No | ![Image](image3.png) | ![Image](image4.png) |
| | $\neq 0$ | $n > p$ | ![Image](image5.png) | ![Image](image6.png) |
| | | $n = p \cup \lambda < 0$ | | |
| $< 0$ | $> 0$ | $n$ even $\cup$ n$p$ | ![Image](image7.png) | ![Image](image8.png) |
| | | $n$ even $\cup$ n$p \cup \lambda > 0$ | | |
| $< 0$ | $< 0$ | $n$ even $\cup$ n$p$ | ![Image](image9.png) | ![Image](image10.png) |
| | | $n$ even $\cup$ n$p \cup \lambda > 0$ | | |
| $< 0$ | $> 0$ | $n$ odd $\cup$ n$p$ | ![Image](image11.png) | ![Image](image12.png) |
| | | $n$ odd $\cup$ n$p \cup \lambda > 0$ | | |
| $< 0$ | $< 0$ | $n$ odd $\cup$ n$p$ | ![Image](image13.png) | ![Image](image14.png) |
| | | $n$ odd $\cup$ n$p \cup \lambda > 0$ | | |
Figure 17. Characterization of the singularity when $\bar{m}(u=0, R=0) = 0, \bar{m}_u (u=0, R=0) = 0, \bar{m}_R (0,0) = 1/2$ and $k$ is even. In the extra conditions, we have used $p = k/2$. The reader can check (see appendix B or [9, 10]) that the first and second rows correspond to cusps and the rest to saddle nodes. Note that both the cusp and the saddle node have different possibilities depending on the sectors appearing in $R > 0$, which now depends on the sign of $a_k$ (and also of $b_p$ in the saddle node case).
Figure 18. We have sketched above the $u-R$ graphics for the three different cases when there is a non-isolated critical point and $\varepsilon = -1$. The order is, from the left to the right, $\bar{m}(u, R = 0) < 1/2$, $\bar{m}(u, R = 0) > 1/2$ and $\bar{m}(u, R = 0) = 1/2$. Below these graphics we have drawn the corresponding sketched Penrose diagrams.

- We are working with a time-orientable spherically symmetric spacetime possessing a local chart endowed with coordinates $\{u, R, \theta, \varphi\}$ (see section 3). In this way, we assume that the invariant areal radius $R$ can be used as a coordinate in our local chart.

- The metric, written as in (1), depends on the functions $\beta$ and $\chi$ which are at least $C^2$ in the local chart (this is a minimum requirement in order to guarantee the existence and uniqueness of null geodesics. Note however that the degree of differentiability is usually required to be higher). $\beta$ is also assumed to be bounded.

- The local study is carried out for finite values of $u$ and, specifically, we have chosen to work around $u = 0$ (if necessary, it suffices a simple coordinate change $u = \bar{u} + \text{constant}$). We assume that, at least, (future or past directed) radial null geodesics of the $F_1$ family ($u = \text{constant}$) reach $R = 0$ in an interval around $u = 0$ ($-\delta u < u < \delta u$).

- The spacetime has a singular boundary that is (in the unphysical spacetime), at least, piecewise $C^1$.

- In the cases where an extension ($\bar{m}$) of the invariant $m$ is required, then we have not considered working around non-isolated critical points of (6) that are accumulation points of the set $\bar{m}(R = 0, u) = 0$.

On the other hand, the reader should be aware that, for every case, our use of the qualitative theory of dynamic systems implies that the complete path could only be followed if some extra requirements on the differentiability of the functions $m$ and $\beta$ are satisfied. Let us then summarize the aforesaid path along with every differentiability requirements.

- If $\lim_{(u \to u_0, R \to 0)} m(u, R) \neq 0$ (\(\forall u_0 \in \mathcal{U}\), i.e. for an interval of $u$s), there is no assumption required here\(^{15}\). The limit can be either finite or infinite and the characterization depends

\(^{15}\) Except for the obvious minimum degree of differentiability ($C^2$–) for $m$ and $\beta$ in $\mathcal{U}$.
only on whether it is positive (spacelike singularity) or negative (timelike singularity) (final corollary in section 4).

Note that if \( \lim_{u \to u_0, R \to 0} m(u, R) \) does not exist, then we can still evaluate \( m \) along the radial null geodesics of the second family: \( \lim_{u \to u_0} m(u, R) \) (see section 4). If it is not zero, no matter if it is finite or infinite, the characterization depends only on whether it is positive (piecewise spacelike singularity) or negative (piecewise timelike singularity). On the other hand, if only the directional limit exists and it is zero, the method is not conclusive.

- If \( \lim_{(u \to u_0, R \to 0)} m(u, R) = 0 \) (\( \forall u_0 \in \mathcal{U} \), i.e. for an interval of \( u \)), then we demand the existence of \( C^1 \) extensions for \( m \) and \( \beta \). If \( \lim_{(u \to u_0, R \to 0)} m_{\bar{R}}(u, R) \) is less than 1/2 in \( \mathcal{U} \), then there is a timelike singularity; if the limit equals 1/2 in \( \mathcal{U} \), then there is a lightlike singularity; and if it is greater than 1/2, then there is a spacelike singularity (section 7).

- If \( \lim_{(u \to 0, R \to 0)} m(u, R) = 0 \) and \( \lim_{(u \to 0, R \to 0)} m_{\bar{u}}(u, R) \neq 0 \), then we demand the existence of a \( C^1 \) extension for \( m \) and \( \beta \) (section 5). In this case, the results can be found in figure 9.

- If \( \lim_{(u \to 0, R \to 0)} m(u, R) = 0 \) (and only for \( u \to 0 \)), \( \lim_{(u \to 0, R \to 0)} m_{\bar{u}}(u, R) = 0 \) and \( \lim_{(u \to 0, R \to 0)} m_{\bar{R}}(u, R) \neq 1/2 \), then we demand, first, the existence of a natural number \( n \geq 2 \) such that \( \lim_{(u \to 0, R \to 0)} \partial m/\partial u^i(u, R) \neq 0 \), while \( \lim_{(u \to 0, R \to 0)} \partial m/\partial u^i(u, R) = 0 \) for \( i = 1, \ldots, n = 1 \), and, second, \( C^n \) extensions \( \bar{m} \) and \( \bar{\beta} \) (section 6). In this case, the results can be found in figure 14.

- If \( \lim_{(u \to 0, R \to 0)} m(u, R) = 0 \) (and only for \( u \to 0 \)), \( \lim_{(u \to 0, R \to 0)} m_{\bar{u}}(u, R) = 0 \) and \( \lim_{(u \to 0, R \to 0)} m_{\bar{R}}(u, R) = 1/2 \), then we demand the existence of, first, a natural number \( n \geq 2 \) defined by \( n = \max(i, k) \), where \( i \) is the lowest value of \( j \) such that \( \lim_{(u \to 0, R \to 0)} \partial j/\partial u^i(u, R) \neq 0 \) and \( k = l + 1 \), where \( l \) is the lowest value such that \( \lim_{(u \to 0, R \to 0)} \partial l/\partial u^i(u, R) \neq 0 \). In this case, the results can be found in figures 16 and 17.

Assuming that we use a single local chart and that the required \( C^n \) extensions exist for every case, an interesting question appears: Are there any forbidden possibilities for the causal behaviour of the singular boundary? If one inspects the figures in the paper, it is easy to see that, indeed, some possibilities do not appear. Specifically, there are four possibilities which do not appear due to the degree of differentiability required. We have collected them under the name of ‘135’ singularities (see figures 19(1–4)). This is expected from the qualitative theory of dynamic systems since the theorems involved only allow for specific combinations of sectors. On the other hand, the use of a local chart does not allow us to work around transition points that can be reached by two radial null geodesics and left by other two (pointed singularities, see figures 19(5) and (6)) since, as explained in subsection 3.1, the \( u = \) constant radial null geodesics must always be outgoing or ingoing in a single local chart.

8.1. The role of the scalar invariant \( m \) in the causal characterization

As we have seen, despite that the function \( \beta \) is involved in the behaviour of the radial null geodesics (6), the causal characterization of the zero-areal-radius singularity around \( u = R = 0 \) for all the different possibilities depends on quantities (\( \Delta, T, \det A, \Delta_R, \Delta_a, b_R \)) that are just obtained from the knowledge of the lowest-order non-zero derivatives of \( m(u, R) \) as \( u \) and \( R \) tend to zero. Then we can state the following result.

**Proposition 8.1.** Provided that the listed assumptions are satisfied, the knowledge of the scalar invariant \( m \) is all that is required in order to obtain the sketched causal characterization of the \( R = 0 \) singularity in a spherically symmetric spacetime.
Figure 19. The forbidden possibilities if one works with a single local chart and assuming the existence of the required $C^r$ extension. From (1) to (4) they are particular transitions from timelike to lightlike singularities and from spacelike to lightlike singularities forbidden by the requirement of the degree of differentiability for the extension. Specifically, the forbidden possibilities are those that in our sketches represented forming an angle of $135^\circ$ in the spacetime. From (5) to (6), we show the pointed singularities that cannot be described around $p$ with a single local chart in our coordinates. We explicit a possibility showing that the $u = \text{constant}$ geodesics cannot be always ingoing or outgoing as required (in both cases $u_1$ is ingoing, while $u_2$ is outgoing).

We can refine this result taking into account that the knowledge of $m$ is not a necessary, but a sufficient condition in order to characterize the singularity in this approach. By inspecting the quantities defining the character of the singularity, we see that the knowledge of the lowest non-zero $\lim_{R \to 0} \partial^{\alpha} m / \partial u^\alpha (k > 0)$ and $\lim_{R \to 0} \partial^{\alpha} m(R, 0) / \partial u^\alpha (0, 0) (n > 0)$ suffices to characterize the singularity. Then, assuming the existence of the extension for $m$ with its corresponding degree of differentiability and using Taylor’s theorem up to that degree, we see that a rough knowledge of $\bar{m}(u, R = 0)$ and $\bar{m}_R (u, R = 0)$ (note that they are evaluated just at $R = 0$) suffices to characterize any $R = 0$ singularity in $U$.

8.2. Lightlike singularities and shell-focussing nakedness

When dealing with naked singularities that form due to the gravitational collapse, one usually looks for the generation of timelike or lightlike singularities. Provided that our assumptions are satisfied, the timelike case is straightforward; it requires either $\lim_{R \to 0} m < 0$ in an interval of $u$ in $U$\(^{16}\) (section 4) or $\lim_{R \to 0} m = 0$ (if the extra condition $\lim_{R \to 0} m(R, u) < 1/2$ is satisfied for some proper interval of $u$ in $U$) (section 7). The lightlike case, however, requires a careful study to which we will devote this subsection. For example, assuming the existence of the corresponding extension for $m$ and by inspecting the hyperbolic and the semi-hyperbolic cases (figures 9 and 14, respectively), we arrive at the following.

**Proposition 8.2.** Assuming the existence of, first, a natural number $n \geq 1$ such that $\lim_{u \to 0, R \to 0} \partial^{\alpha} m / \partial u^\alpha (u, R) \neq 0$ (while, in case $n \neq 1$, $\lim_{u \to 0, R \to 0} \partial^{\alpha} m / \partial u^\alpha (u, R) = 0$ for $i = 1, \ldots, n - 1$) and, second, $C^n$ extensions $\bar{m}$ and $\bar{\beta}$, if an $R = 0$ singularity satisfies

---

\(^{16}\) Think, for instance, of the timelike singularity in the Reissner–Nordström solution, where a negative limit is justified due to the presence of electrical charge.
\[ \ddot{m}(0) = 0, \]
\[ \varepsilon \frac{\partial^n \ddot{m}}{\partial u^n}(0, 0) [1 - 2\ddot{m},_R(0, 0)]^{n+1} > 0, \]  
\[ [1 - 2\ddot{m},_R(0, 0)]^2 - 16\varepsilon \ddot{m},_u(0, 0) \geq 0; \]
then it has a lightlike singularity at \( R = u = 0 \).

Furthermore, if \( \varepsilon (1 - 2\ddot{m},_R(0, 0)) > 0 \), there is a past lightlike singularity, whereas if \( \varepsilon (1 - 2\ddot{m},_R(0, 0)) < 0 \), there is a future lightlike singularity.

As a particular application, let us consider the case of Vaidya’s radiating solution. Since \( C^0 \) and the nonlinear \( C^1 \) cases.

Corollary 8.1. Any Vaidya metric admitting, first, a natural number \( n \geq 1 \) such that \( \lim_{u \to 0} d^m(u)/du^i \neq 0 \) (while, in case \( n \neq 1 \), \( \lim_{u \to 0} d^m/du^i = 0 \) for \( i = 1, \ldots, n-1 \)) and, second, a \( C^n \) extension \( \ddot{m} \), will develop a lightlike singularity at \( R = u = 0 \) if

\[ \ddot{m}(u = 0) = 0, \]
\[ \varepsilon \frac{\partial \ddot{m}}{\partial u}(u = 0) \leq \frac{1}{16}, \]
\[ \varepsilon \frac{\partial^2 \ddot{m}}{\partial u^2}(u = 0) > 0. \]

This corollary generalizes previous results on Vaidya’s metric with \( \varepsilon = +1 \) for the linear \([13, 27]\) and the nonlinear \([14]\) cases.

Among the lightlike singularities, some of them are persistent naked singularities. By ‘persistent’ we denote a naked singularity such that a whole family of future-directed lightlike radial null geodesics emerges from it. It has been pointed out that these naked singularities can appear as a consequence of shell-focusing \([13, 15–17, 27, 55]\) during the collapse and that they are, therefore, relevant to the cosmic censorship conjecture. If we want to avoid that the 2-spheres close enough to the \( R = 0 \) singularity become past-trapped closed surfaces (usually only future-trapped 2-surfaces are considered as possible byproducts of collapse), we should also demand \( \varepsilon = +1 \) (see subsection 3.1). By filtering the lightlike singularities with these properties in the above proposition, we arrive at the following.

Proposition 8.3. A general spherically symmetric spacetime with metric (1) admitting, first, a natural number \( n \geq 1 \) such that \( \lim_{u \to 0, R \to 0} d^n m/du^n(u, R) \neq 0 \) (while, in case \( n \neq 1 \), \( \lim_{u \to 0} d^n m/du^n(u, R) = 0 \) for \( i = 1, \ldots, n-1 \)) and, second, \( C^n \) extensions \( \ddot{m} \) and \( \ddot{\bar{m}} \), will develop a persistent lightlike naked singularity as a consequence of gravitational collapse if \( \varepsilon = +1 \), \( \ddot{m}(0, 0) = 0, \ddot{m},_R(0, 0) < 1/2, \)
\[ [1 - 2\ddot{m},_R(0, 0)]^2 - 16\ddot{m},_u(0, 0) \geq 0, \]
\[ \frac{\partial^n \ddot{m}}{\partial u^n}(0, 0) > 0. \]

Note that persistent lightlike naked singularities are also possible in the special case \( \ddot{m},_R(0, 0) = 1/2 \). The reader can write explicitly the conditions for them directly from figures 16–18.

So far we have been working with a single local chart in \( \mathcal{U} \) and assuming that certain extensions of \( m \) exist and possess a certain degree of differentiability. Therefore, the cases in which these requirements are not fulfilled have been excluded up to now. However, in
some cases, the method can be generalized to include different local charts or more general
differentiability requirements. For example, particular models for collapsing stars with an
initially regular centre \( \bar{m}(u < 0, R = 0) = 0 \) in which the shell focussing eventually
generates a persistent lightlike naked singularity \( \bar{m}(u = 0) \) could be constructed with the usual
matching techniques (see [13, 27] for particular cases), which allow the avoidance of the
differentiability requirements and the limitations of the use of a single local chart. It is only
required that there is a regular spacetime modelling the initially regular interior that there is
a singular spacetime whose scalar invariant \( m \) satisfies the conditions in proposition 8.3 and
that the two spacetimes can be matched through a non-spacelike hypersurface reaching \( R = 0 \)
at \( u = 0 \).

In the next subsection, we will also go beyond the limitations of dealing with a single
local chart and with the required differentiability assumptions by working with a particular
interesting novel application: evaporating BHs that can develop persistent naked singularities.
With these treatments, our results could be used to cover most of the spherically symmetric
solutions found in the literature. Nevertheless, let us remark that one can still find particular
cases where even the generalized treatment would fail. An example of this would be the model
in [25] which possesses a function \( m \) that is not even well defined at \( u = R = 0 \).

Figure 20. Typical Penrose’s diagram for a collapsing star generating a BH that evaporates due to
the emission of Hawking radiation. The grey zone corresponds with the interior of the star. The
exterior of the star will usually contain radiation coming out from the star (the wavy arrows) and
particles and radiation due to the Hawking effect.
8.3. Beyond the restrictions: evaporating BHs developing persistent naked singularities

Let us consider a future spacelike singularity that reaches a point, say at \( u = 0 \), where \( \lim_{u \to R \to 0} m = 0 \) and then it is followed by a regular \( R = 0 \)—see \((3)\)^{17}—for \( u > 0 \). If this happens, we will say that the singularity **evaporates**. In fact, this is the usually expected behaviour for the singularity of an evaporating black hole (EBH) [56]. (We have illustrated it with a particular complete model in figure 20.)

In this case there will be a future-directed radial null geodesic starting at \( u = R = 0 \) in the singularity. In this way, the singularity must be naked and this geodesic defines a Cauchy horizon (CH) of the model. Sometimes this singularity is called an **instantaneous** naked singularity [31] since an observer crossing the CH could only detect a single flash of null radiation coming from the singularity.

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^{17} Note that the singularity disappears for \( u > 0 \) provided that \( \beta \) also satisfies its own regularity conditions \((3)\) as it approaches \( R = 0 \) for \( u > 0 \).
Now we would like to show that the requirement of regularity for \( u > 0 \) in \( \mathcal{I} \) allows the interesting possibility of the development of a \textit{persistent} naked singularity at \( u = 0 \). Specifically the results in section 6 imply the following.

**Proposition 8.4.** A singular spherically symmetric spacetime with metric (1) admitting a natural number \( n \geq 2 \) such that \( \lim_{(u \to 0, R \to 0)} \partial m/\partial u^i (u, R) \neq 0 \), while \( \lim_{(u \to 0, R \to 0)} \partial m/\partial u^i (u, R) = 0 \) for \( i = 1, \ldots, n - 1 \), admitting \( C^\infty \) extensions \( \widehat{m} \) and \( \widehat{\beta} \) and with an \( R = 0 \) future spacelike singularity will develop a persistent lightlike naked singularity at \( R = u = 0 \) if \( \varepsilon = +1 \),

\[
\begin{align*}
\widehat{m}(0, 0) &= 0, \\
\widehat{m}_{,u}(0, 0) &= 0, \\
\frac{\partial^n \widehat{m}(0, 0)}{\partial u^n} &> 0,
\end{align*}
\]

provided that \( n \) is even, and one of these two options is satisfied:
Figure A1. (H) Hyperbolic sector, (P) parabolic sector and (E) elliptic sector.

(a) \[ \frac{\partial \tilde{m}}{\partial R} (0, 0) < \frac{1}{2}, \]  (25)

(b) \[ \frac{\partial \tilde{m}}{\partial R} (0, 0) = \frac{1}{2} \quad \text{and} \quad \frac{\partial^i}{\partial \tilde{u}^i} \left( \frac{\partial \tilde{m}}{\partial R} \right) (0, 0) < 0, \]  (26)

where \( i \) is the lowest number such that the partial derivative is non-null provided that \( i < n/2 \).

Note that the requirement \( n \geq 2 \) indicates that the generation of such lightlike singularities is related to the presence of a scalar invariant \( m \) reaching its zero value slowly enough. In figure 21, we have sketched a procedure for constructing a future spacelike singularity developing a persistent lightlike naked singularity and followed by a regular centre using the usual matching technique.

On the other hand, in figure 22, we show a complete spacetime with a persistent lightlike globally naked singularity and a regular \( R = 0 \) centre for \( u > 0 \).

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Appendix A. Semi-hyperbolic critical points

Let us assume that the origin is an isolated critical point of a planar system. Then, if the linearization matrix \( A \) satisfies \( \det A = 0 \) and its trace \( T \neq 0 \), then the critical point will be semi-hyperbolic [10]. The details on the theory regarding the qualitative behaviour of these points can be found in [9, 10]. We will summarize here only the main results.

The neighbourhood of the origin can be divided into open regions called sectors which can be hyperbolic, parabolic or elliptic according to their topological equivalence with figures A1(H), A1(P) or A1(E), respectively. The trajectories which lie on the boundary of a hyperbolic sector are called separatrices.

Given a system possessing an isolated semi-hyperbolic critical point, there is a suitable linear transformation that allows us to write it as

\[
\begin{align*}
\frac{dx}{d\tilde{u}} &= P_2(x, y) \\
\frac{dy}{d\tilde{u}} &= y + Q_2(x, y),
\end{align*}
\]
where $P_2$ and $Q_2$ are functions satisfying $P_2(0,0) = Q_2(0,0) = P_{2,x}(0,0) = Q_{2,x}(0,0) = P_{2,y}(0,0) = Q_{2,y}(0,0) = 0$. By the implicit function theorem, the equation $y + Q_2(x,y) = 0$ has a solution $y = \varphi(x)$ in the neighbourhood of $O$. If we define the function $\psi(x) = P_2(x, \varphi(x))$, then its truncated series expansion will have the form

$$\psi(x) = \Delta_n x^n + \cdots,$$

where $n \geq 2$ and $\Delta_n \neq 0$. Then it can be shown ([9], theorem 65, p 340) that

- If $n$ is odd and $\Delta_n > 0$, $(0,0)$ is a topological node, i.e. there is a trivial sectorial decomposition consisting in only one parabolic sector.
- If $n$ is odd and $\Delta_n < 0$, $(0,0)$ is a topological saddle, i.e. there are four hyperbolic sectors separated by four separatrices. Two of these separatrices tend to $(0,0)$ in the directions 0 and $\pi$, the other two in the directions $\pi/2$ and $3\pi/2$.
- If $n$ is even, then $(0,0)$ is a saddle node, i.e. one parabolic and two hyperbolic sectors separated by three separatrices.

1. If $\Delta_n < 0$, the hyperbolic sectors contain a segment of the positive $x$-axis.
2. If $\Delta_n > 0$, the hyperbolic sectors contain a segment of the negative $x$-axis.

Appendix B. Nilpotent critical points

Let us assume that the origin is an isolated critical point of a planar system and that $A$ (its linearization matrix) is not the zero matrix, but $\det A = 0$ and its trace satisfies $T = 0$; then the critical point will be nilpotent [10]. Again we refer the reader to [9, 10] for details on the theory of the qualitative behaviour of these points. We will summarize here only the main results.

Given a system possessing a nilpotent critical point, there is a suitable linear transformation that allows us to write it as

\[\begin{align*}
\frac{dx}{dt} &= y + P_2(x,y) \\
\frac{dy}{dt} &= Q_2(x,y),
\end{align*}\]

where $P_2$ and $Q_2$ are functions satisfying $P_2(0,0) = Q_2(0,0) = P_{2,x}(0,0) = Q_{2,x}(0,0) = P_{2,y}(0,0) = Q_{2,y}(0,0) = 0$. By the implicit function theorem, the equation $y + P_2(x,y) = 0$ has a solution $y = \varphi(x)$ in a neighbourhood of $O$. If we define the function $\psi(x) = Q_2(x, \varphi(x))$, then its truncated series expansion will have the form

$$\psi(x) = a_k x^k + \cdots,$$

where $k \geq 2$ and $a_k \neq 0$. On the other hand, if we define $\sigma(x) = P_{2,x}(x, \varphi(x)) + Q_{2,y}(x, \varphi(x))$, then either its truncated series expansion can be written as

$$\sigma(x) = b_n x^n + \cdots,$$

where $b_n \neq 0$ or $\sigma(x) \equiv 0$ in which case $b_n = 0 \forall n$. Then it can be shown that

- If $k$ is odd ([9], theorem 66, p 357), then we define $p \equiv (k-1)/2$ and $\lambda \equiv b_n^2 + 2(k+1)a_k$.

  1. If $a_k > 0$, then $O$ is a topological saddle point.
  2. If $a_k < 0$, then $O$ is a
(a) focus or centre if either (1) $b_n = 0$, (2) $b_n \neq 0$ and $n > p$ or (3) $b_n \neq 0$, $n = p$ and $\lambda < 0$; (b) topological node if either (1) $n$ is even, $b_n \neq 0$ and $n < p$ or (2) $n$ is even, $b_n \neq 0$, $n = p$ and $\lambda \geq 0$; (c) equilibrium state with an elliptic region, i.e. an elliptic sector and a hyperbolic sector separated by two separatrices, if either (1) $n$ is odd, $b_n \neq 0$ and $n < p$ or (2) $n$ is odd, $b_n \neq 0$ and $n = p$ and $\lambda \geq 0$.

If $k$ is even (9), theorem 67, p 362, then we define $p \equiv k/2$ and the critical point is a (1) cusp (i.e. two hyperbolic sectors separated by two separatrices) if either $b_n = 0$ or $b_n \neq 0$ and $n \geq p$; (2) saddle node if $b_n \neq 0$ and $n < p$.

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