Adjacency Matrix and Co-occurrence Tensor of General Hypergraphs: Two Well Separated Notions

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Abstract

Adjacency and co-occurrence are two well separated notions: even if they are the same for graphs, they start to be two different notions for uniform hypergraphs. After having done the difference between the two notions, this paper contributes in the definition of a co-occurrence tensor reflecting the general hypergraph structure. It is a challenging issue that can have many applications if properly achieved, as it will allow the study of the spectra of such general hypergraph. In most of the applications only an hypermatrix associated to the tensor is needed. In this article, a novel way of building a symmetric co-occurrence hypermatrix is proposed that captures also the cardinality of the hyperedges and allows full separation of the different layers of the hypergraph.

1 Introduction

It has been several years since Berge and Minieka [1973] introduced the hypergraphs. Nonetheless recent improvements in tensor spectral theory and the limitations of graphs to capture $p$-adic relationship has made the research on the spectra of hypergraphs more relevant. For studying such spectra a proper definition of Laplacian tensor in hypergraph is needed and therefore the co-occurrence tensor must be properly defined.

In Pu [2013] a clear distinction is made between the pairwise relationship which is a binary relation and the co-occurrence relationship which is presented as the extension of the pairwise relationship to a $p$-adic relationship. The notion of co-occurrence is often used in linguistical data as the simultaneous apperance of linguistical units in a reference. The concept can be easily extended to nodes contained in a hyperedge.
Nonetheless it is more than an extension. Indeed graphs allow to connect nodes by pair through edges. Therefore the notion of adjacency is clearly a pairwise relationship in graphs. At the same time in an edge only two nodes are linked. Also given an edge only two nodes can co-occur. Thus the two notions are equivalent in graphs.

Extending the notion of adjacency to hypergraphs two nodes are said adjacent if it exists a hyperedge that connect them. Hence the adjacency notion still captures a binary relationship and can be modelized by an adjacency matrix.

Nevertheless co-occurrence is no more a pairwise relationship as a hyperedge being given more than two nodes can occur since a hyperedge contains \( p \geq 1 \) nodes. Therefore it is a \( p \)-adic relationship that has to be captured and to be modelised by tensor.

Consequently adjacency matrix of a hypergraph and co-occurrence tensor are two separated notions; nonetheless the co-occurrence tensor if often abusively named the adjacency tensor in the litterature.

This article contributes to a definition of a new co-occurrence tensor for general hypergraphs which not solely preserves all the structural informations of the hypergraph but also captures separately the information on the hyperedges held in the hypergraph.

After sketching the background and the related works on the adjacency and co-occurrence concepts for hypergraphs in Section 2, one proposal is made to build a new co-occurrence tensor which is built as unnormalized in Section 3. Section 4 tackles the particular case of graphs seen as 2-uniform hypergraphs. Future works and Conclusion are addressed in Section 5. A full example is given in Appendix A.

Notation

Exponents are indicated into parenthesis - for instance \( y^{(n)} \) - when they refer to the order of the corresponding tensor. Indices are written into parenthesis when they refer of a sequence of objects - for instance \( a_{(k)ij} \) is the elements at row \( i \) and column \( j \) of the matrix \( A_{(k)} \). The context should made it clear.

For the convenience of readability, it is written \( z_0 \) for \( z^1, \ldots, z^n \). Hence given a polynomial \( P \), \( P(z_0) \) has to be understood as \( P(z^1, \ldots, z^n) \).

Given additional variables \( y^1, \ldots, y^k \), it is written \( z_k \) for \( z^1, \ldots, z^n, y^1, \ldots, y^k \).

\( S_k \) is the set of permutations on the set \( \{ i : i \in \mathbb{N} \land 1 \leq i \leq k \} \).

2 Background and related works

Several definitions of hypergraphs exist and are reminded in Ouvrard et al. [2017]. Hypergraphs allow the preservation of the \( p \)-adic relationship in between nodes becoming the natural modeling of collaboration networks, co-author networks, chemical reactions, genome and all situations where the 2-adic relationship allowed by graphs is not sufficient and where the keeping of the grouping
information is important. Among the existing definitions the one of Bretto [2013] is reminded:

Definition 1. An (undirected) hypergraph $H = (V,E)$ on a finite set of $n$ vertices (or nodes) $V = \{v_1, v_2, \ldots, v_n\}$ is defined as a family of $p$ hyperedges $E = (e_1, e_2, \ldots, e_p)$ where each hyperedge is a non-empty subset of $V$.

Let $H = (V,E)$ be a hypergraph and $w$ a relation such that each hyperedge $e \in E$ is mapped to a real number $w(e)$. The hypergraph $H_w = (V,E,w)$ is said to be a weighted hypergraph.

The $2$-section of a hypergraph $H = (V,E)$ is the graph $[H]_2 = (V,E')$ such that:

$$\forall u \in V, \forall v \in V : (u,v) \in E' \Leftrightarrow \exists e \in E : u \in e \land v \in e$$

Let $k \in \mathbb{N}^*$. A hypergraph is said to be $k$-uniform if all its hyperedges have the same cardinality $k$.

A directed hypergraph $H = (V,E)$ on a finite set of $n$ vertices (or nodes) $V = \{v_1, v_2, \ldots, v_n\}$ is defined as a family of $p$ hyperedges $E = (e_1, e_2, \ldots, e_p)$ where each hyperedge contains exactly two non-empty subset of $V$, one which is called the source - written $e_s$ - and the other one which is the target - written $e_t$ -.

In this article only undirected hypergraphs will be considered. In a hypergraph a hyperedge links one or more vertices together. The role of the hyperedges in hypergraphs is playing the role of edges in graphs.

Definition 2. Let $H = (V,E)$ be a hypergraph.

The degree of a vertex is the number of hyperedges it belongs to. For a vertex $v_i$, it is written $d_i$ or $\deg(v_i)$. It holds: $d_i = |\{e : v_i \in e\}|$

Given a hypergraph the incident matrix of an undirected hypergraph is defined as follow:

Definition 3. The incidence matrix of a hypergraph is the rectangular matrix $H = [h_{kl}]_{1 \leq k \leq n}$ of $M_{n \times p}(\{0 ; 1\})$, where $h_{kl} = \begin{cases} 1 & \text{if } v_k \in e_l \\ 0 & \text{otherwise} \end{cases}$.

As seen in the introduction defining adjacency in a hypergraph has to be distinguished from the co-occurrence in a hyperedge of a hypergraph.

Definition 4. Let $H = (V,E)$ be a hypergraph. Let $u \in V$ and $v \in V$ be two vertices of this hypergraph.

$u$ and $v$ are said adjacent if it exists $e \in E$ such that $u \in e$ and $v \in e$.

Definition 5. Let $H = (V,E)$ be a hypergraph. Let $k \geq 1$ be an integer, $j \in \llbracket 1 ; k \rrbracket$, $i_j \in \llbracket 1 ; n \rrbracket$. For $j \in \llbracket 1 ; k \rrbracket$, let $u_{i_j} \in V$ be $k$ vertices.

Then $u_{i_1}, \ldots, u_{i_k}$ are said $k$-adjacent if it exists $e \in E$ such that for all $j \in \llbracket 1 ; k \rrbracket$, $u_{i_j} \in e$. 
With \( k = 2 \), the usual notion of adjacency is retrieved.  
If \( k \) vertices are \( k \)-adjacent then each subset of this \( k \) vertices of size \( l \leq k \) is \( l \)-adjacent.

**Definition 6.** Let \( \mathcal{H} = (V,E) \) be a hypergraph. Let \( e \in E \).

The nodes constituting \( e \) are said **co-occurrent** nodes.

If \( \mathcal{H} \) is \( k \)-uniform then the \( k \)-adjacency is equivalent to the co-occurrence of nodes in an edge.

For a general hypergraph, nodes that are \( k \)-adjacent with \( k < \max_{e \in E} |e| \) have to co-occur - potentially with other nodes - in one edge. In this case the two notions are actually distinct.

**Adjacency matrix**

The adjacency matrix of a hypergraph is related with the 2-adjacency. Several approaches have been made to define an adjacency matrix for hypergraphs.

In Bretto [2013] the adjacency matrix is defined as:

**Definition 7.** The **adjacency matrix** is the square matrix which rows and columns are indexed by the vertices of \( \mathcal{H} \) and where for all \( u,v \in V \), \( u \neq v \):

\[
a_{uv} = |\{ e \in E : u,v \in e \}| \quad \text{and} \quad a_{uu} = 0.
\]

The adjacency matrix is defined in Zhou et al. [2007] as follow:

**Definition 8.** Let \( \mathcal{H}_w = (V,E,w) \) be a weighted hypergraph.

The adjacency matrix of \( \mathcal{H}_w \) is the matrix \( A \) of size \( n \times n \) defined as

\[
A = HWHT - D_v
\]

where \( W \) is the diagonal matrix of size \( p \times p \) containing the weights of the hyperedges of \( \mathcal{H}_w \) and \( D_v \) is the diagonal matrix of size \( n \times n \) containing the degrees of the nodes of \( \mathcal{H}_w \), where \( d(v) = \sum_{e \in E : v \in e} w(e) \) for all \( v \in V \).

This last definition is equivalent to the one of Bretto for unweighted hypergraphs - ie weighted hypergraphs where the weight of all hyperedges is 1.

The problem of the matrix approach is that the multi-adic relationship is no longer kept as an adjacency matrix can link only pair of vertices. Somehow it doesn’t preserve the structure of the hypergraph: the hypergraph is extended in the 2-section of the hypergraph and the information is captured by this way.

Following a lemma cited in Dewar et al. [2016], it can be formulated:

**Lemma 1.** Let \( \mathcal{H} = (V,E) \) be a hypergraph and let \( u,v \in V \). If two vertices \( u \) and \( v \) are adjacent in \( \mathcal{H} \) then they are adjacent in the 2-section \( [\mathcal{H}]_2 \).

The reciprocal doesn’t hold as it would imply an isomorphism between \( \mathcal{H} \) and its 2-section \( [\mathcal{H}]_2 \).

Moving to the approach by co-occurrence will allow to keep the information on the structure that is held in the hypergraph.
Co-occurrence tensor

In Michael and Nachtergaele [2012] an unnormalized version of the \( k \)-adjacency tensor of a \( k \)-uniform hypergraph is given. This definition is also adopted in Ghoshdastidar and Dukkipati [2017].

**Definition 9.** The unnormalized (Author’s note: \( k \)-)adjacency tensor of a \( k \)-uniform hypergraph \( \mathcal{H} = (V,E) \) on a finite set of nodes \( V = \{v_1, v_2, ..., v_n\} \) and a family of hyperedges \( E = (e_1, e_2, ..., e_p) \) of equal cardinality \( k \) is the tensor \( A_{\text{raw}} = (a_{\text{raw}i_1...i_k})_{1 \leq i_1, ..., i_k \leq n} \) such that:

\[
a_{\text{raw}i_1...i_k} = \begin{cases} 
1 & \text{if } \{v_{i_1}, ..., v_{i_k}\} \in E \\
0 & \text{otherwise.}
\end{cases}
\]

In Cooper and Dutle [2012] a slightly different version exists for the definition of the adjacency tensor, called the degree normalized \( k \)-adjacency tensor

**Definition 10.** The (Author’s note: degree normalized \( k \)-)adjacency tensor of a \( k \)-uniform hypergraph \( \mathcal{H} = (V,E) \) on a finite set of nodes \( V = \{v_1, v_2, ..., v_n\} \) and a family of hyperedges \( E = (e_1, e_2, ..., e_p) \) of equal cardinality \( k \) is the tensor \( A = (a_{i_1...i_k})_{1 \leq i_1, ..., i_k \leq n} \) such that:

\[
a_{i_1...i_k} = \frac{1}{(k-1)!} \begin{cases} 
1 & \text{if } \{v_{i_1}, ..., v_{i_k}\} \in E \\
0 & \text{otherwise.}
\end{cases}
\]

This definition by introducing the coefficient \( \frac{1}{(k-1)!} \) allows to retrieve the degree of a vertex \( i \) summing the elements of index \( i \) on the first mode of the tensor. Also it will be called the degree normalized adjacency tensor.

**Proposition 1.** Let \( \mathcal{H} = (V,E) \) be a \( k \)-uniform hypergraph. Let \( v_i \in V \) be a vertex. It holds by considering the degree normalized \( k \)-adjacency tensor \( A = (a_{i_1...i_k})_{1 \leq i_1, ..., i_k \leq n} \):

\[
\deg(v_i) = \sum_{i_2, ..., i_k = 1}^{n} a_{i_1...i_k}.
\]

**Proof.** On the first mode of the degree normalized adjacency tensor, for a given node \( v_i \) that occurs in an hyperedge \( e = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\} \) the elements \( a_{i\sigma(i_2)...\sigma(i_k)} = \frac{1}{(k-1)!} \) where \( \sigma \in \mathcal{S}_{k-1} \) which exist in quantity \( (k-1)! \) in the first mode. Hence \( \sum_{\sigma \in \mathcal{S}_{k-1}} a_{i\sigma(i_2)...\sigma(i_k)} = 1 \).

Therefore doing it for all hyperedges where \( v_i \) is an element allows to retrieve the degree of \( v_i \).  

\( \square \)
This definition could be interpreted as the definition of the co-occurrence tensor for a uniform hypergraph since the notion of $k$-adjacency and co-occurrence are equivalent in a $k$-uniform hypergraph.

In Hu [2013] a full study of the spectra of an uniform hypergraph using the Laplacian tensor is given. The definition of the Laplacian tensor is linked to the existence and definition of the normalized ([Author’s note]: $k$)-adjacency tensor.

**Definition 11.** The ([Author’s note]: eigenvalues) normalized ([Author’s note]: $k$)-adjacency tensor of a $k$-uniform hypergraph $\mathcal{H} = (V, E)$ on a finite set of nodes $V = \{v_1, v_2, \ldots, v_n\}$ and a family of hyperedges $E = (e_1, e_2, \ldots, e_p)$ of equal cardinality $k$ is the tensor $A = (a_{i_1 \ldots i_k})_{1 \leq i_1, \ldots, i_k \leq n}$ such that:

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k - 1)!} \prod_{1 \leq j \leq k} \frac{1}{\sqrt{d_{i_j}}} & \text{if } \{v_{i_1}, \ldots, v_{i_k}\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The aim of the normalization is motivated by the bounding of the different eigenvalues of the tensor.

The normalized Laplacian tensor $L$ is given in the following definition.

**Definition 12.** The normalized Laplacian tensor of a $k$-uniform hypergraph $\mathcal{H} = (V, E)$ on a finite set of nodes $V = \{v_1, v_2, \ldots, v_n\}$ and a family of hyperedges $E = (e_1, e_2, \ldots, e_p)$ of equal cardinality $k$ is the tensor $L = I - A$ where $I$ is the $k$-th order $n$-dimensional diagonal tensor with the $j$-th diagonal element $i_{j \ldots j} = 1$ if $d_j > 0$ and 0 otherwise.

In Banerjee et al. [2017] the definition is extended to general hypergraph.

**Definition 13.** Let $\mathcal{H} = (V, E)$ on a finite set of nodes $V = \{v_1, v_2, \ldots, v_n\}$ and a family of hyperedges $E = (e_1, e_2, \ldots, e_p)$. Let $k_{\text{max}} = \max \{|e_i| : e_i \in E\}$ be the maximum cardinality of the family of hyperedges.

The adjacency hypermatrix of $\mathcal{H}$ written $A_H = (a_{i_1 \ldots i_{k_{\text{max}}}})_{1 \leq i_1, \ldots, i_{k_{\text{max}}} \leq n}$ is such that for a hyperedge: $e = \{v_{l_1}, \ldots, v_{l_s}\}$ of cardinality $s \leq k_{\text{max}}$.

$$a_{p_1 \ldots p_{k_{\text{max}}}} = \frac{s}{\alpha}, \text{ where } \alpha = \sum_{k_1, \ldots, k_s \geq 1} \frac{n!}{k_1! \ldots k_s!},$$

with $p_1, \ldots, p_{k_{\text{max}}}$ chosen in all possible way from $\{l_1, \ldots, l_s\}$ with at least once from each element of $\{l_1, \ldots, l_s\}$.

The other position of the hypermatrix are zero.

The first problem in this case is that the notion of $k$-adjacency as it has been mentioned earlier is not the most appropriated for a general hypergraph where the notion of co-occurrence is much stronger. The approach in Shao [2013] and Pearson and Zhang [2014] consists in the retrieval of the classical adjacency matrix for the case where the hypergraph is 2-uniform - ie is a graph.
3 Towards an unnormalized co-occurrence tensor

- by keeping their degree invariant: therefore the degree of each node can be retrieved on the first mode of the tensor by sum.

In Hu [2013] the focus is made on the spectra of the tensors obtained: the normalization is done to keep eigenvalues of the tensor bounded. Extending this approach for general hypergraph, Banerjee et al. [2017] spreads the information of lower cardinalities hyperedges inside the tensor. Even if it can be well explainable on its principle it might be hard to use practically as the elements to be filled in for just one hyperedge can explode not solely in number of occurrences but also fragmented in small values that could be hard to compute. It has the advantage of keeping the co-occurrence cubical hypermatrix of dimension $n$ and order $k_{\text{max}}$ at the price of splitting the elements in small fragments and mixing the different cardinalities in one hypermatrix forcing the nodes to occur in a $k_{\text{max}}$ manner without properly distinguishing the ones that are not and by decreasing the sparsity of the tensor. The focus is made only on the spectra to be obtained.

Nonetheless building the link with homogeneous polynomials as it is done in the current article, it seems there is no real explanation of the choice made to spread out the elements in the hypermatrix and that better way of building a co-occurrence hypermatrix can help to capture the co-occurrence properly and by the same time keeping the ability to hold the $k$-adjacency whatever the level it occurs.

In this article a new definition of a co-occurrence tensor is built that helps to preserve as much as possible the information of the original hypergraph meanwhile avoiding to explode the number of not null term in the tensor. The approach is based on the homogeneisation of sums of polynomials of different degrees and by considering a family of uniform hypergraphs. It is also motivated by the fact that the information on the cardinality of the hyperedges has to be kept in some ways and that the elements should not be mixed with the different layers of the hypergraph.

The next Section aims at building the unnormalized co-occurrence tensor.

3 Towards an unnormalized co-occurrence tensor

3.1 Family of tensors attached to a hypergraph

Let $\mathcal{H} = (V, E)$ be a hypergraph. A hypergraph can be decomposed in a family of uniform hypergraphs. To achieve it, let $\mathcal{R}$ be the equivalency relation: $e \mathcal{R} e' \iff |e| = |e'|$.

$E/\mathcal{R}$ is the set of classes of hyperedges of same cardinality. The elements of $E/\mathcal{R}$ are the sets: $E_k = \{ e \in E : |e| = k \}$.

Let $k_{\text{max}} = \max_{e \in E} |e|$, called the range of the hypergraph $\mathcal{H}$.

Considering $K = \{ k : E_k \in E/\mathcal{R} \}$, it is set $E_k = \emptyset$ for all $k \in [1 : k_{\text{max}}] \setminus K$.

Let consider the hypergraphs: $\mathcal{H}_k = (V, E_k)$ for all $k \in [1 : k_{\text{max}}]$ which are all $k$-uniform.
Figure 1: Illustration of a hypergraph decomposed in three layers of uniform hypergraphs

It holds: \( E = \bigcup_{k=1}^{k_{\text{max}}} E_k \) and \( E_j \cap E_k = \emptyset \) for all \( j \neq k \), hence \( (E_k)_{1 \leq k \leq k_{\text{max}}} \) formed a partition of \( E \) which is unique by the way it has been defined.

Before going forward the sum of two hypergraphs has to be defined:

**Definition 14.** Let \( \mathcal{H}_1 = (V_1, E_1) \) and \( \mathcal{H}_2 = (V_2, E_2) \) be two hypergraphs. The sum of this two hypergraphs is the hypergraph written \( \mathcal{H}_1 + \mathcal{H}_2 \) defined as:

\[
\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 \cup E_2).
\]

This sum is said direct if \( E_1 \cap E_2 = \emptyset \). In this case the sum is written \( \mathcal{H}_1 \oplus \mathcal{H}_2 \).

Hence:

\[
\mathcal{H} = \bigoplus_{k=1}^{k_{\text{max}}} \mathcal{H}_k.
\]

The hypergraph \( \mathcal{H} \) is said to be decomposed in a family of hypergraphs \( \{\mathcal{H}_k\}_{1 \leq k \leq k_{\text{max}}} \) where \( \mathcal{H}_k \) is \( k \)-uniform.

An illustration of the decomposition of a hypergraph in a family of uniform hypergraphs is shown in Figure 1. This family of uniform hypergraphs decomposes the original hypergraph in layers. A layer holds a \( k \)-uniform hypergraph \( (1 \leq k \leq k_{\text{max}}) \): therefore the layer is said to be of level \( k \).

Therefore, at each \( k \)-uniform hypergraph \( \mathcal{H}_k \) can be mapped a \((k\text{-adjacency})\) co-occurrence tensor \( A_k \) of order \( k \) which is hypercubic of dimension \( |V| = n \). This tensor can be unnormalized or normalized.

Choosing one type of tensor - normalized or unnormalized for the whole family of \( \mathcal{H}_k \) - the hypergraph \( \mathcal{H} \) is then fully described by the family of co-occurrence tensors \( A_{\mathcal{H}} = (A_k) \). In the case where all the \( A_k \) are chosen normalized this family is said pre-normalized. The final choice will be made further in Sub-Section 3.6 and explained to fulfill the expectations listed in the next Sub-Section.
3.2 Expectations for a co-occurrence tensor for a general hypergraph

The definition of the family of tensors attached to a general hypergraph has the advantage to open the way to the spectral theory for uniform hypergraphs which is quite advanced.

Nonetheless many problems remain in keeping a family of tensors of different orders: studying the spectra of the whole hypergraph could be hard to achieve by this means. Also it is necessary to get a co-occurrence tensor which covers the whole hypergraph and which retains the information on the whole structure.

The idea behind is to “fill” the hyperedges with sufficient elements such that the general hypergraph is transformed in an uniform hypergraph. A similar approach has been taken in Banerjee et al. [2017] where the filling elements are the vertices belonging to the hyperedge itself. In the next subsections the justification of the approach taken will be made via homogeneous polynomials.

Before getting to the construction, expected properties of such a tensor have to be listed.

Expectation 1. The tensor should be invariant to vertices permutation either globally or at least locally.

This expectation is motivated by the fact that in an hyperedge the nodes have no order. The fact that this expectation can be local remains in the fact that added special vertices will not have the same status that the one from the original hypergraph. Also the invariance by permutation is expected on the vertices of the original hypergraph.

Expectation 2. The co-occurrence tensor should allow the retrieval of the hypergraph it is originated from.

This expectation seems important to rebuild properly the original hypergraph from the co-occurrence tensor: all the necessary information to retrieve the original hyperedges has to be encoded in the tensor.

Expectation 3. Giving the choice of two representations the co-occurrence tensor the sparsest should be chosen.

Sparsity allows to compress the information and so to gain in place and complexity in calculus. Also sparsity is a desirable property for some statistical reasons as shown in Nikolova [2000] or expected in Bruckstein et al. [2009] for signal processing and image encoding.

Expectation 4. The co-occurrence tensor should allow the retrieval of the degrees of the nodes.

In the adjacency matrix of a graph the information on the degrees of the nodes is encoded directly. It is still the case, as it has been seen, with the \( k \)-adjacency degree normalised tensor that has been defined by Shao [2013] and Pearson and Zhang [2014].
3.3 Building an homogeneous polynomial attached to a general hypergraph

To get an homogeneous polynomial attached to a general hypergraph a mapping is going to be made between the family of co-occurrence tensors and homogeneous polynomials. This bound is used in Comon et al. [2015] where the author links symmetric tensors and homogeneous polynomials of degree $s$ to show that the problem of the CP decomposition of different symmetric tensors of different orders and the decoupled representation of multivariate polynomial maps are related.

Let $\mathbb{K}$ be a field. Here $\mathbb{K} = \mathbb{R}$.

Let $A_k \in L_k^0(\mathbb{K}^n)$ be a cubical tensor of order $k$ and dimension $n$ with values in $\mathbb{K}$.

**Definition 15.** Let define the *Segre outerproduct* $\otimes$ of $a = [a_i] \in \mathbb{K}^l$ and $b = [b_j] \in \mathbb{K}^m$ as:

$$a \otimes b = [a_ib_j]_{1 \leq i \leq l, 1 \leq j \leq m} \in \mathbb{K}^{l \times m}.$$ 

More generally as given in Comon et al. [2008] the outerproduct of $k$ vectors $u_{(1)} \in \mathbb{K}^{n_1}$, ..., $u_{(k)} \in \mathbb{K}^{n_k}$ is defined as:

$$k \otimes u_{(i)} = \left[ \prod_{i=1}^{k} u_{(i)j_i} \right]_{j_1 \leq \cdots \leq j_k \leq 1} \in \mathbb{K}^{n_1 \times \cdots \times n_k}.$$ 

Let $e_1$, ..., $e_n$ be the canonical base of $\mathbb{K}^n$.

$$(e_{i_1} \otimes \cdots \otimes e_{i_k})_{1 \leq i_1, \ldots, i_k \leq n}$$ is a basis of $L_k^0(\mathbb{K}^n)$.

Then $A_k$ can be written as:

$$A_k = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{(k)i_1...i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}$$

The notation $A_k$ will be used for the corresponding hypermatrix of coefficients $a_{(k)i_1...i_k}$ where $1 \leq i_1, \ldots, i_k \leq n$.

Let $z \in \mathbb{K}^n$, with $z = z_i e_i$ using the Einstein convention.

In Lim [2013] a multilinear matrix multiplication is defined as follow:

**Definition 16.** Let $A \in \mathbb{K}^{n_1 \times \cdots \times n_d}$ and $X_j = [x_{(j)kl}] \in \mathbb{K}^{m_j \times n_j}$ for $1 \leq j \leq d$.

$A' = (X_1, ..., X_d).A$ is the multilinear matrix multiplication and defined as the matrix of $\mathbb{K}^{m_1 \times \cdots \times m_d}$ of coefficients:

$$a'_{j_1...j_d} = \sum_{k_1, \ldots, k_d=1}^{n_1, \ldots, n_d} x_{(1)j_1k_1} \cdots x_{(d)j_dk_d} a_{k_1...k_d}$$

for $1 \leq j_i \leq m_i$ with $1 \leq i \leq d$.

Afterwards only vectors $z \in \mathbb{K}^n$ are needed and $A_k$ is cubical of order $k$ and dimension $n$. Writing $(z, ..., z) \in (\mathbb{K}^n)^k$, $(z)_{[k]}$
Therefore \((z)_{[k]} A_k\) contains only one element written \(P_k(z^1, \ldots, z^n) = P_k(z_0)\):

\[
P_k(z_0) = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{(k) i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}.
\]  

(1)

Therefore considering a hypergraph \(\mathcal{H}\) with its family of unnormalized tensor \(A_\mathcal{H} = (A_k)\), it can be also attached a family \(P_\mathcal{H} = (P_k)\) of homogenous polynomials with \(\deg(P_k) = k\).

The formulation of \(P_k\) can be reduced taking into account that \(A_k\) is symmetric for a uniform hypergraph:

\[
P_k(z_0) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} k! a_{(k) i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}.
\]  

(2)

Writing:

\[
\tilde{P}_k(z_0) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} a_{(k) i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}.
\]  

(3)

the reduced form of \(P_k\), it holds:

\[
P_k(z_0) = k! \tilde{P}_k(z_0).
\]

Writing for \(1 \leq i_1 \leq \ldots \leq i_k \leq n\):

\[
\alpha_{(k) i_1 \ldots i_k} = k! a_{(k) i_1 \ldots i_k}
\]

and \(\alpha_{(k) \sigma(i_1) \ldots \sigma(i_k)} = 0\) for \(\sigma \in S_k, \sigma \neq \text{Id}\)

It holds:

\[
P_k(z_0) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \alpha_{(k) i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}
\]

(4)

and:

\[
\tilde{P}_k(z_0) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \frac{\alpha_{(k) i_1 \ldots i_k}}{k!} z^{i_1} \ldots z^{i_k}.
\]  

(5)

**Reversibility of the process**

Reciprocally, given a homogenous polynomial of degree \(k\) that is with no restriction suppose reduced and ordered, a unique tensor can be built - at the order of the elements in each dimension - which is hypercubic and symmetric of order \(k\). It reflects uniquely a \(k\)-uniform hypergraph.

**Proposition 2.** Let \(P(k) = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}\) be a homogenous polynomial where:
• for $j \neq k$ : $z^j \neq z^k$
• for all $1 \leq j \leq n$: $\deg(z^j) = 1$
• and such that for all $\sigma \in S_k$: $a_{\sigma(i_1)...\sigma(i_k)} = a_{i_1...i_k}$.

Then $P$ is the homogeneous polynomial attached to a unique $k$-uniform hypergraph $\mathcal{H} = (V, E, w)$ - up to the indexing of vertices.

Proof. Considering the vertices $(v_i)_{1 \leq i \leq n}$ labelled by the elements of $[1, n]$.

If $a_{i_1...i_k} \neq 0$ then for all $\sigma \in S_k$: a unique hyperedge $e_j$ is attached corresponding to the vertices $v_{i_1}, ..., v_{i_k}$ and which has weight $w(e_j) = ka_{i_1...i_k}$.

Using a family of tensor will give less flexibility than using a single tensor. To build this new tensor different approaches can be foreseen: the first approach is to build iteratively a sequence of homogeneous polynomials which summarized the information included in the former polynomial built and the current $P_k$. To achieve it the homogenisation of the sum of two homogeneous polynomials of different order is needed. It can be achieved by creating $k_{\text{max}} - 1$ new variables.

The key idea behind is to fill the $k$-uniform hypergraph in such a way it becomes a $k + 1$-uniform hypergraph and to add it to the $k + 1$-uniform hypergraph of the above layer. It can be achieved by putting special vertices that differ from one layer to the other in the original hyperedges of the $k$-uniform hypergraph. Each of this special vertex will therefore be the representant of one of the additional variables.

Hence it transforms a $k$-uniform hypergraph into a $k + 1$-uniform almost bipartite hypergraph. Almost bipartite hypergraph is defined below extending the definition of bipartite hypergraph given in Annamalai [2015] of a $k$-uniform bipartite hypergraph.

**Definition 17.** Let $k \geq 1$ be an integer.

A hypergraph $\mathcal{H} = (W, E)$ is said **almost bipartite** if the vertex set $W$ can be partitioned into two sets $V$ and $V_s$ such that for every edge $e \in E$: $|e \cap V| \geq 1$ and $|e \cap V_s| = |e| - |e \cap V|$.

If for every edge $e \in E$: $|e \cap V_s| > 0$ then the hypergraph is said (strictly) **bipartite**.

The transformation of the $k$-uniform hypergraph - which can already be seen as an almost bipartite hypergraph - into a $k + 1$-uniform bipartite hypergraph where the set of vertices is composed of the vertices of the original hypergraph and the special vertex - which will be added to the special vertices if it comes from a $k$-uniform bipartite hypergraph - allows to sum it with an other $k + 1$-uniform hypergraph, making it an almost bipartite hypergraph $k + 1$-uniform.

From the final homogeneous polynomial built, a hypercubic symmetric tensor of order $k_{\text{max}}$ and dimension $n + k_{\text{max}} - 1$ is constructed. This is the first approach taken. This approach has the advantage of directly encoding the
TOWARDS AN UNNORMALIZED CO-OCCURRENCE TENSOR

A more straightforward approach is to build directly from the sum \( \sum_{k=1}^{k_{\text{max}}} P_k \) an homogeneous polynomial. It can be achieved by using only one additional variable that will keep the overall information on the cardinality of the hyperedges. The role of this additional variable can be linked to the hypergraph by saying that a special node is added to fill the hyperedge at the right level of uniformity, by duplicating this special vertex as many time as needed. Also a hypercubic symmetric tensor of order \( k_{\text{max}} \) and dimension \( n + 1 \) is built from the final homogeneous polynomial built.

A global overview of the approach is sketched in the figure 2.

**An iterative approach:** building an homogeneous family summarizing the different layers

A new family \( R_H = (R_k) \) of homogeneous polynomials of degree \( k \) is built from the family of unnormalized co-occurrence tensors \( A_H = (A_k) \) by following the subsequent steps. To achieve it properly real constants \( c_1, \ldots, c_{k_{\text{max}}} \) are introduced at each level that will allow to normalize the fact that the tensors are of different orders: they have to be considered in a first instance as purely technical. A deeper explanation will be given forward.

**Step 1:** \( R_1 (z_o) = c_1 P_1 (z_o) = c_1 \sum_{i=1}^{n} a_{(1)i} z^i. \)

This is the initialization step. The layer of level 1 of the hypergraph corresponds to a 1-uniform hypergraph which is represented by a order 1 symmetric tensor. This tensor is in correspondence with a degree 1 homogeneous polynomial.
Step 2: To build $R_2$ an homogeneization of the sum of $R_1$ and $c_2 P_2$ is needed. It holds:

$$R_1 (z_o) + P_2 (z_o) = c_1 \sum_{i=1}^{n} a_{(1)} i z^i + c_2 \sum_{i_1, i_2=1}^{n} a_{(2)} i_{1,2} z^{i_1} z^{i_2}$$

To achieve the homogeneization of $R_1 (z_o) + c_2 P_2 (z_o)$ a new variable $y^1$ is introduced.

It follows for $y^1 \neq 0$:

$$R_2 (z_1) = R_2 (z_0, y^1) = y^{1(2)} \left( R_1 \left( \frac{z_0}{y^1} \right) + c_2 P_2 \left( \frac{z_0}{y^{1(2)}} \right) \right) = c_1 \sum_{i=1}^{n} a_{(1)} i z^i + c_2 \sum_{i_1, i_2=1}^{n} a_{(2)} i_{1,2} z^{i_1} z^{i_2}.$$  

By continuous prolongation of $R_2$, it is set:

$$R_2 (z_o, 0) = c_2 \sum_{i_1, i_2=1}^{n} a_{(2)} i_{1,2} z^{i_1} z^{i_2}.$$  

In this step, the layer of level 1 is filled with one special vertex common to every hyperedge that transforms these hyperedges in 2-partite hyperedges. To achieve it the homogeneous polynomial $R_1$ of degree 1 is filled with one variable so it raises to an homogeneous polynomial of degree 2 able to be linked with a tensor of order 2. The merge with the level 2 of the original hypergraph is achieved by considering that an almost bipartite hypergraph has been constructed that can be modelized by a homogeneous polynomial of degree 2.

Step 3: Supposing that $R_k (z_{k-1})$ is an homogeneous polynomial of degree $k$ that can be written as:

$$R_k (z_{k-1}) = \sum_{j=1}^{k} c_j \sum_{i_1, \ldots, i_j=1}^{n} a_{(j)} i_1 \ldots i_j z^{i_1} \ldots z^{i_j} \prod_{l=j}^{k-1} y^{l},$$

with the convention that $\prod_{l=j}^{k-1} y^{l} = 1$ if $j > k-1$ and $\omega_{k-1} = z^1, \ldots, z^n, y^1, \ldots, y^{k-1}$

$R_{k+1}$ is built as an homogeneous polynomial from the sum of $R_k$ and $c_{k+1} P_{k+1}$ by adding a variable $y_k$ and factorizing by its $k + 1$-th power.

Therefore, for $y^{k-1} \neq 0$: 

$R_{k+1}(z_k) = y^k(k+1) \left( R_k \left( \frac{z_{k-1}}{y^k(k)} \right) + c_{k+1} P_{k+1} \left( \frac{z_0}{y^k(k+1)} \right) \right) = \left( \sum_{j=1}^{k} \sum_{i_1, \ldots, i_j=1}^{n} a_{(j) i_1 \ldots i_j} z_1^{i_1} \cdots z_j^{i_j} \prod_{l=j}^{k-1} y^l \right) y^k + c_{k+1} \sum_{i_1, \ldots, i_{k+1}=1}^{n} a_{(k+1) i_1 \ldots i_{k+1}} z_1^{i_1} \cdots z_{k+1}^{i_{k+1}}$

And for $y_k = 0$, it is set by continuous prolongation: $R_{k+1}(z_{k-1}, 0) = \left( \sum_{j=1}^{n} a_{(k+1) i_1 \ldots i_{k+1}} z_1^{i_1} \cdots z_{k+1}^{i_{k+1}} \right) = c_{k+1} \sum_{i_1, \ldots, i_{k+1}=1}^{n} a_{(k+1) i_1 \ldots i_{k+1}} z_1^{i_1} \cdots z_{k+1}^{i_{k+1}}$.

The fact that $P_{k+1}(z_n)$ can be null doesn’t prevent to do the step: the degree of $R_k$ will then be elevated of 1.

The interpretation of this step is similar to the one done at step 2.

Ending step: the process stops when $k = k_{max}$.

An other approach

An alternative approach would have been to create a family $Q_{\mathcal{H}} = (Q_k)$ of homogeneous polynomials by directly writing

$Q_k(z_0, y^k) = y^k(k) \sum_{j=1}^{k} P_j \left( \frac{z_0}{y^k} \right) = \sum_{j=1}^{k} \sum_{i_1, \ldots, i_j=1}^{n} a_{(j) i_1 \ldots i_j} z_1^{i_1} \cdots z_j^{i_j} y^{k-j}$

$Q_k$ is an homogeneous polynomial of order $k$.

$Q_{k_{max}}$ can be then seen as the homogeneous polynomial attached to the hypergraph.

It would have the great advantage of adding only one to the dimension of each axis of the tensor: this additional variable captures in somehow the cardinality of the hyperedge. Nonetheless this approach can be seen as a compressed form of the tensor of the iterative approach. Also it is harder to distinguish the different layers of the hypergraph through the homogeneous polynomial. It is why the iterative approach is preferred in this article.

3.4 Building an unnormalized symmetric tensor from this family of homogeneous polynomials

Based on $R_{\mathcal{H}}$

It is now valuable to interpret the built polynomials.

The notation $w_{(k)} = w_{(k)}^n, \ldots, w_{(k)}^{n+k-1}$ is used.
• The interpretation of $R_1$ is trivial as it holds the single element hyperedges of the hypergraph.

• $R_2$ is an homogeneous polynomial with $n + 1$ variables of order 2.

$$R_2(z_1) = c_1 \sum_{i=1}^{n} a_{(1)i} z_i y^1 + c_2 \sum_{i_1,i_2=1}^{n} a_{(2)i_1i_2} z_i^1 z_i^2$$

$$= c_1 \sum_{1 \leq i \leq n} a_{(1)i} z_i y^1 + c_2 \sum_{1 \leq i_1 \leq i_2 \leq n} a_{(2)i_1i_2} z_i^1 z_i^2$$

It can be rewritten:

$$R_2(w_{(2)}) = \sum_{i_1,i_2=1}^{n+1} r_{(2)i_1i_2} w_{i_1}^{(2)} w_{i_2}^{(2)}$$

where:

– for $1 \leq i \leq n$: $w_{i}^{(2)} = z^i$

– $w_{n+1}^{(2)} = y^1$

– for $1 \leq i_1 \leq i_2 \leq n$ and $\sigma \in S_2$:

$$r_{(2)\sigma(i_1)\sigma(i_2)} = \frac{c_2 \alpha_{(2)i_1i_2}}{2!} = c_2 a_{(2)i_1i_2}$$

– for $1 \leq i \leq n$ and $\sigma \in S_2$:

$$r_{(2)\sigma(i)\sigma(n+1)} = \frac{c_1 \alpha_{(1)i}}{2!} = c_1 a_{(1)i}$$

– the other coefficients: $r_{(2)i_1i_2}$ are null.

Also $R_2$ can be linked to a symmetric hypercubic tensor of order 2 and dimension $n + 1$.

• $R_k$ is an homogeneous polynomial with $n + k - 1$ variables of order $k$.

$$R_k(z_{k-1}) = \sum_{j=1}^{k} c_j \sum_{i_1,\ldots,i_j=1}^{n} a_{(j)i_1\ldots i_j} z_i^1 \ldots z_i^j \prod_{l=j}^{k-1} y_l$$

$$= \sum_{j=1}^{k} c_j \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n} \alpha_{(j)i_1\ldots i_j} z_i^1 \ldots z_i^j \prod_{l=j}^{k-1} y_l$$

with the convention that: $\prod_{l=j}^{k-1} y_l = 1$ if $j > k - 1$. 
It can be rewritten:

\[ R_k (w_{(k)}) = \sum_{i_1, \ldots, i_k=1}^{n+k-1} r_{(k)} (i_1 \ldots i_k) w_{(k)}^{i_1} \ldots w_{(k)}^{i_k} \]

where:
- for \(1 \leq i \leq n\): \(w_{(k)}^i = z^i \)
- for \(n+1 \leq i \leq n + k - 1\): \(w_{(k)}^i = y^{i-n} \)
- for \(1 \leq i_1 \leq \ldots \leq i_k \leq n\), for all \(1 \leq j \leq k - 1\), for all \(\sigma \in S_k\):
  \[* r_{(k)} (i_1) \ldots (i_k) \sigma (i_1) \ldots (i_k) = c_{k} \alpha (i_1) \ldots (i_k) \]
  \[* r_{(k)} (i_1) \ldots (i_j) \sigma (i_{n+j}) \ldots \sigma (i_{n+k-1}) = c_{j} \alpha (i_1) \ldots (i_j) \]
  \[ \frac{j!}{k!} = \frac{j!}{k!} \]
  \[ c_{j} \alpha (i_1) \ldots (i_j) \]
- the other elements \(r_{(k)} (i_1 \ldots i_k)\) are null.

Also \(R_k\) can be linked to a symmetric hypercubic tensor of order \(k\) and dimension \(n + k - 1\) written \(R_{k}\) whose elements are \(r_{(k)} (i_1 \ldots i_k)\).

The hypermatrix \(R_{k_{\text{max}}}\) is called the unnormalized tensor.

### 3.5 Interpretation and choice of the coefficients for the unnormalized tensor

There are different ways of setting the coefficients \(c_1, \ldots, c_{k_{\text{max}}}\) that are used. These coefficients can be seen as a way of normalizing the tensors of co-occurrence generated from the \(k\)-uniform hypergraphs.

A first way of choosing them is to set them all equal to 1. In this case no normalisation occurs. The impact on the co-occurrence tensor of the original hypergraph is that co-occurrence in hyperedges of size \(k\) have a weight of \(k\) times bigger than the co-occurrence in hyperedges of size 1.

A second way of choosing these coefficients is to consider that in a \(k\)-uniform hypergraph, each hyperedge holds \(k\) nodes and then contributes to \(k\) to the total degree. Representing this \(k\)-uniform hypergraph by the \(k\)-adjacency degree normalized tensor \(A_k = (a_{(k)} (i_1 \ldots i_k)) \sum_{1 \leq i_1, \ldots, i_k \leq n}\) it holds a revisited hand-shake lemma for \(k\)-uniform hypergraphs:

\[ \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{(k)} (i_1 \ldots i_k) = \sum_{i=1}^{n} d_{(k)} (i) = k |E_k| \]

where \(d_{(k)} (i)\) is the degree of the vertex \(v_i\) in \(H_k\).

If this formula is to be extended to general hypergraph:

\[ |E| = \sum_{k=1}^{k_{\text{max}}} |E_k| \]

\[ = \sum_{k=1}^{k_{\text{max}}} \frac{1}{k} \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{(k)} (i_1 \ldots i_k) \]
Also $c_k = \frac{1}{k}$ seems to be a good choice in this case.

The final choice will be taken in the next paragraph to answer to the required specifications on degrees. It will also fix the matrix chosen for the uniform hypergraphs.

### 3.6 Gages of the unnormalized co-occurrence tensor

**Gage 1.** The unnormalized co-occurrence tensor keeps the overall structure of the hypergraph.

*Proof.* It is inherent to the way the tensor has been built: the layer of level equal or under $j$ can be seen in the mode 1 at the $n + j$-th component of the mode. To have only elements of level $j$ one can project this mode so that it keeps only the first $n$ dimensions. $\square$

In the expectations of the built co-tensors listed in the paragraph 3.2, the co-occurrence tensor should allow the retrieval of the degree of the nodes. It implies to fix the choice of the $k$-adjacency tensors used to modelize each layer of the hypergraph as well as the normalizing coefficient.

Let consider:

$$I_{k,l,i} = \{(i_1, \ldots, i_l) : \exists j \in [1,l] : i_j = i \land \forall j \in [1,l] : 1 \leq i_j \leq n + k - 1\}$$

and its subset of ordered tuples

$$IO_{k,l,i} = \{(i_1, \ldots, i_l) : (i_1, \ldots, i_l) \in I_{k,l,i} \land \forall (j_1, j_2) \in [1,l]^2 : j_1 < j_2 \implies i_{j_1} < i_{j_2}\}.$$ 

Then:

$$\sum_{(i_1, \ldots, i_{k_{\text{max}}}) \in I_{k_{\text{max}}, k_{\text{max}}, l}} r_{i_1 \ldots i_{k_{\text{max}}}} = \sum_{(i_1, \ldots, i_{k_{\text{max}}}) \in IO_{k_{\text{max}}, k_{\text{max}}, l}} k_{\text{max}}! r_{i_1 \ldots i_{k_{\text{max}}}} = \sum_{j=1}^{k_{\text{max}}} \sum_{(i_1, \ldots, i_j) \in IO_{k_{\text{max}}, j, l}} j! c_j a_{(j)} i_1 \ldots i_j.$$ 

Hence, the expectation on the retrieval of degree imposes to set $c_j a_{(j)} i_1 \ldots i_j = \frac{1}{j!}$ for the elements of $A_{(j)}$ that are not null, which is coherent with the usage of the coefficient $c_j = \frac{1}{j}$ and of the degree-normalized tensor for $j$-uniform hypergraph where not null elements are equals to: $\frac{1}{(j-1)!}$.

This choice is then made for the rest of the article.

It follows immediately:

**Gage 2.** The unnormalized co-occurrence tensor allows the retrieval of the degree of the nodes of the hypergraph.
Proof. Defining for $1 \leq i \leq n$: $d_i = \deg(v_i)$.

From the previous choice, it follows that:

$$
\sum_{(i_1, \ldots, i_{k_{\text{max}}}) \in I_{k_{\text{max}}, k_{\text{max}}, i}} r_{i_1 \ldots i_{k_{\text{max}}}} = \sum_{j=1}^{k_{\text{max}}} \sum_{(i_1, \ldots, i_j) \in IO_{k_{\text{max}}, j, i}} j! c_j a(j)_{i_1 \ldots i_j} = \deg(v_i)
$$

as $j! c_j a(j)_{i_1 \ldots i_j} = 1$ only for hyperedges where $i$ is in it (and they are counted only once for each hyperedge).

\[ \square \]

Gage 3. The unnormalized co-occurrence tensor allows the retrieval of the cardinality of the hyperedges.

Proof. Defining $d_{n+i} = |\{e : |e| \leq i\}|$ for $1 \leq i \leq k_{\text{max}}$.

$$
d_{n+i} = \sum_{(i_1, \ldots, i_{k_{\text{max}}}) \in I_{k_{\text{max}}, k_{\text{max}}, n+i}} r_{i_1 \ldots i_{k_{\text{max}}}} = \sum_{j=1}^{k_{\text{max}}} \sum_{(i_1, \ldots, i_j) \in IO_{k_{\text{max}}, j, n+i}} j! c_j a(j)_{i_1 \ldots i_j}
$$

In this case $j! c_j a(j)_{i_1 \ldots i_j} = 1$ when the $i$-th special vertex is integrated into the hyperedge.

and setting: $d_{n+k_{\text{max}}} = |E|$.

Also $d_{n+j}$ allows to retrieve the number of hyperedges of cardinality equal or less than $j$.

Therefore:

- for $2 \leq j \leq k_{\text{max}}$: $|\{e : |e| = j\}| = d_{n+j} - d_{n+j-1}$
- for $j = 1$: $|\{e : |e| = 1\}| = d_{n+1}$

An other way of keeping directly the cardinality of the layer $k_{\text{max}}$ in the co-occurrence tensor would be to store it in an additional variable $y_{k_{\text{max}}}$.

\[ \square \]

Gage 4. The co-occurrence tensor is unique up to the labelling of the nodes for a given hypergraph.

Reciprocally, given the co-occurrence tensor and the number of nodes, the associated hypergraph is unique.

Proof. Given an hypergraph, the process of decomposition in layers is bijective as well as the formalisation by degree normalized $k$-adjacency tensor. Given the coefficients, the process of building the co-occurrence homogeneous polynomial is also unique and the reversion to a symmetric cubic tensor is unique.
Given the co-occurrence tensor and the number of nodes, as the co-occurrence tensor is symmetric, up to the labelling of the nodes, considering that the first \( n \) variables encoded in the co-occurrence tensor in each direction represents variables associated to nodes of the hypergraph and the last variables in each direction encode the information of cardinality. Therefore it is possible to retrieve each layer of the hypergraph uniquely and consequently the whole hypergraph.

3.7 Interpretation of the unnormalized co-occurrence tensor

The additional dimension in the co-occurrence tensor allows to retrieve the cardinality of the hyperedges. By decomposing a hypergraph in a set of uniform hypergraphs the hyperedges are quotiented depending on their cardinality.

The iterative approach consists in taking the set of hyperedges of level \( k \) regrouping them in one set and let them regrouped with the set of level \( k + 1 \) as illustrated in Figure 3. The packing set of level \( k \) includes the information that the elements inside it are at the former level.

The straightforward approach concept makes the regroupments all at a time the information is then in some way more hidden than in the first representation as it can be seen in Figure 4.

These two representations explain why in the first case \( k_{\text{max}} - 1 \) extra-variables are needed meanwhile only one is needed in the second case.

Viewed in an other way, co-occurrence hypermatrix of uniform hypergraph don’t need an extra dimension as the hyperedges are uniform, therefore there is no ambiguity. Adding an extra variable allows to capture the dimensionality of each hyperedge meanwhile preventing any ambiguity on the meaning of each element of the tensor.

4 The particular case of graphs

As a graph \( G = (V, E) \) with \(|V| = n\) can always be seen a 2-uniform hypergraph \( \mathcal{H}_G \), the approach given in this paragraph should allow to retrieve in a coherent way the spectral theory for normal graphs.

The hypergraph that contains the 2-uniform hypergraph is then composed of an empty level 1 layer and a level 2 layer that contains only \( \mathcal{H}_G \).

Let \( A \) be the adjacency matrix of \( G \). The co-occurrence tensor of the corresponding 2-uniform hypergraph is of order 2 and obtained from \( A \) by multiplying it by \( c_2 \) and adding one row and one column of zero. Therefore the co-occurrence tensor of the two level of the corresponding hypergraph is: 

\[
\mathcal{A} = \begin{pmatrix} c_2 A & 0 \\ 0 & 0 \end{pmatrix}
\]

Also as an eigenvalue \( \lambda \) of \( \mathcal{A} \) seen as a matrix is a solution of the characteristic polynomial \( \det (\mathcal{A} - \lambda I) = 0 \) \( \Leftrightarrow \) \( -\lambda \det (c_2 A - \lambda I) = 0 \) \( \Leftrightarrow \) \( -\lambda c_2 \det \left( A - \frac{\lambda}{c_2} I \right) = 0 \)
In the iterative approach the layers of level $n$ and $n + 1$ are merged together into the layer $n + 1$ by adding a filling vertex to the hyperedges of the layer $n$. On this example, during the first step the layer 1 and 2 are merged to form a 2-uniform hypergraph. In the second step, the 2-uniform hypergraph obtained in the first step is merged to the layer 3 to obtain a 3-uniform hypergraph.
0, the eigenvalues of $A$ are $c_2$ times the ones of $A$ and one additional 0 eigenvalue. This last eigenvalue is attached to the eigenvector $(0 \ldots 0 1)^T$. The other eigenvalues have same eigenvectors than $A$ with one additional $n + 1$ component which is 0.

**Proof.** Let consider $Y = \begin{pmatrix} X \\ y \end{pmatrix}$ with $X$ vector of dimension $n$. Let $\lambda$ be an eigenvalue of $A$ and $Y$ an eigenvector of $A$

$$AY = \lambda Y \iff A \begin{pmatrix} X \\ y \end{pmatrix} = \lambda \begin{pmatrix} X \\ y \end{pmatrix} \iff (c_2 A - \lambda I_n) X = 0 \land -\lambda y = 0 \iff X$$

eigenvalue of $A$ attached to $\frac{\lambda}{c_2}$, $y$ can be always taken equals to 0 to fit the second condition.

Therefore globally there is no change in the spectra: the eigenvectors hold, the eigenvalues of the initial graph are multiplied by the normalizing coefficient.

## 5 Future work and Conclusion

Having made a clear distinction between the different level of adjacency and the co-occurrence in a general hypergraph helps to refine the problem in a way it makes sense to solve it. Defining the co-occurrence tensor is a good step to allow future work on the impact on the spectra. The tensor obtained is so sparse and huge that it might not allow proper decomposition. Nonetheless as the cubical hypermatrix are made uniform then former works on the decomposition of such hypermatrices can be used.
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Appendix A

Example

Given the following hypergraph: $\mathcal{H} = (V, E)$ where: $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ with: $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_2, v_7\}$, $e_3 = \{v_6, v_7\}$, $e_4 = \{v_5\}$, $e_5 = \{v_4\}$, $e_6 = \{v_3, v_4\}$ and $e_7 = \{v_4, v_7\}$.

This hypergraph $\mathcal{H}$ is drawn in Figure 1.

The layers of $\mathcal{H}$ are:

- $\mathcal{H}_1 = (V, \{e_4, e_5\})$ with the associated unnormalized tensor:

$$A_{1\ raw} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and associated homogeneous polynomial:

$$P_1(z_0) = z_4 + z_5.$$
More generally, the version with a normalized tensor is:
\[ P_1(z_0) = a_{(1)4}z_4 + a_{(1)5}z_5 \]

- \( \mathcal{H}_2 = (V, \{e_3, e_6, e_7\}) \) with the associated unnormalized tensor:

\[
A_{2_{\text{raw}}} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

and associated homogeneous polynomial:
\[ P_2(z_0) = 2z_3z_4 + 2z_6z_7 + 2z_4z_7. \]

More generally, the version with a normalized tensor is:
\[ P_2(z_0) = 2!a_{(2)3}z_3z_4 + 2!a_{(2)6}z_6z_7 + 2!a_{(2)4}z_4z_7 \]

- \( \mathcal{H}_3 = (V, \{e_1, e_2\}) \) with the associated unnormalized tensor:

\[
A_{3_{\text{raw}}} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

and associated homogeneous polynomial:
\[ P_3(z_0) = 3!z_1z_2z_3 + 3!z_1z_2z_7. \]

More generally, the version with a normalized tensor is:
\[ P_3(z_0) = 3!a_{(3)1}z_1z_2z_3 + 3!a_{(3)2}z_1z_2z_7. \]

The iterative process of homogeneization is then the following using the degree-normalized adjacency tensor \( A_k = \frac{1}{(k-1)!}A_{k_{\text{raw}}} \) and the normalizing coefficients \( c_k = \frac{1}{k} \):

- \( R_1(z) = z_4 + z_5 \)
- \( R_2(z) = z_4y_1 + z_5y_1 + \frac{2!}{2} (z_3z_4 + z_6z_7 + z_4z_7) \)
- \( R_3(z) = z_4y_1y_2 + z_5y_1y_2 + z_4z_4y_2 + z_6z_7y_2 + z_4z_7y_2 + z_1z_2z_3 + z_1z_2z_7 \)
Therefore the co-occurrence tensor of $\mathcal{H}$ is obtained by writing the corresponding symmetric cubical tensor of order 3 and dimension 9, described by: $r_{489} = r_{589} = r_{349} = r_{679} = r_{479} = r_{123} = r_{127} = \frac{1}{3!}$. The other remaining non-null elements are obtained by permutation on the indices.

Finding the degree of one vertex from the tensor is easily achievable; for instance $\deg(v_4) = 3! (r_{489} + r_{349} + r_{479}) = 3$. 