MULTIDIMENSIONAL CONSENSUS MODEL
ON A BARABÁSI-ALBERT NETWORK

DIRK JACOBMEIER
Institute for Theoretical Physics, Cologne University
50923 Köln, Germany
E-mail: dj@thp.uni-koeln.de

A Consensus Model according to Deffuant on a directed Barabási-Albert network was simulated. Agents have opinions on different subjects. A multi-component subject vector was used. The opinions are discrete. The analysis regards distribution and clusters of agents which are on agreement in the opinions of the subjects. Remarkable results are on the one hand, that there mostly exists no absolute consens. It determines depending on the ratio of number of agents to the number of subjects, whether the communication ends in a consens or a pluralism. Mostly a second robust cluster remains, in its size depending on the number of subjects. Two agents agree either in (nearly) all or (nearly) no subject.

The operative parameter of the consens-formating-process is the tolerance in change of views of the group-members.

Keywords: Opinion Dynamics; Deffuant-Model; Sociophysics; Monte-Carlo Simulation.

1. Introduction

"Winwood Reade is good upon the subject," said Holmes. "He remarks that, while the individual man is an insoluble puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will do, but you can say with precision what an average number will be up to. Individuals vary, but percentages remain constant. So says the statistician. ..."

The Sign of the Four; Arthur Conan Doyle, 1890

The enigma of man shouldn’t be solved here. It is our aim to build a model, which imitates the behaviour of a group of humans. Therefore from the behaviour of people rules have been deduced which were placed into models 1,2,3.

Consensus models recreate the opinion forming process of a group of agents. One starts from a random distribution of opinions. After a simulation of the communication of agents with each other, the resulting distribution of opinions is regarded. There have been developed and simulated some consensus models in which the choice of connections of the agents among each other is multifaceted. Models differ in the used topologies, in their way of communication, the relationship of agents to a subject, the dimensionality of geometry, subject, and opinion, etc. 4,5,6,7,8,9.

The present model continues this tradition, and offers a consensus model according
to Deffuant\(^7\) on a directed Barabási-Albert network\(^10\) with discrete opinions and several subjects (= questions, themes, ... ). Every agent \(i\) \((i = 1, 2, ..., N)\) has on each subject \(S_k\) \((k = 1, 2, ..., S)\) an opinion \(O^k_i\). The discrete opinion spectrum comprises natural numbers from 1 to \(O\).

Simulations of a consensus model á la Deffuant on a directed Barabási-Albert network with one subject and discrete opinions have been made by Stauffer et al.\(^{11}\).

2. Model

2.1. Network assembly

At the beginning one knot of \(m\) agents, each connected with all others, is built. Every newly added agent \(i\) connects itself with \(m\) already existing agents in the network. The connection takes place stochastically. With it the probability of connecting with a already existing agent is proportional to the total number of its connections (“The rich get richer”).

Besides the connection is directed, i.e., the agents communicate along the \(m\) connections, which they assemble themselves. The connections, with whom they get in touch when new agents are added, can not be chosen by themselves.

2.2. Communication

The communication takes place along the connections. The agents become the active communicator \(i\) in the order they have been bound into the network. The partner for communication \(j\) will be chosen randomly from the \(m\) with those to whom \(i\) has connected itself. Then the over-all distance \(\delta\) to the partner of communication will be calculated. This \(\delta\) results from the absolute value of the distance of all opinions to each other

\[\delta = \sum_{k=1}^{S} |O^k_i - O^k_j|,\]

and is the indicator for the start of a communication:

- If \(\delta\) greater than a given \(\Delta = OS \varepsilon\) a communication will be impossible and ends for this agent, where \(\varepsilon\) with \(0 < \varepsilon < 1\) is an input parameter. Then it is the next agents’ turn.

- If \(\delta\) is lower or equal the given \(\Delta\) then a communication will start.

The outcomes of a simulation with \(\Delta = OS \varepsilon\) don’t differ in the substance compared with the outcomes of a simulation with \(\Delta = (O - 1)S \varepsilon\). The typical appearances of the model remain preserved.

2.2.1. Rules for Simulating the Communication:

Now agents \(i\) and \(j\) look randomly for a subject \(S_k\) on which they will communicate.
• If the difference of opinions \((O^k_i - O^k_j)\) of both partners of communication on the subject \(k\) results in zero, then they agreeing and the communication ends.

• If the difference of opinions equals one, one communicant will adopt randomly the opinion from the other.

• If the difference of opinions is larger than one, both communicants approach each other about \(d\), with rounding the opinion.

With \(d = \sqrt{\frac{1}{10} (O^k_i - O^k_j)}\), it will be \(O^k_i := O^k_i - d\) and \(O^k_j := O^k_j + d\). After that it is the next agents turn.

The simulation ends, when during \(n\) iterations over all agents no change of opinion in one of the communications takes place.

### 2.3. Parameter
The parameters of the model, which have been modified, are \(N\): Number of agents; \(S\): Number of subjects; \(O\): Number of opinions per subject; \(\varepsilon\): tolerance, \(\Delta = O S \varepsilon, 0 \leq \varepsilon \leq 1\); \(n\): stop criterion; the simulation stops if during \(n\) consecutive iterations over all agents no opinion was changed.

### 2.4. Methods of Evaluation

a) **Distribution**
   The distribution specifies how many persons in how many subjects share the same opinion. The quota will be plotted as a fraction of the total number of all agents.

   At the beginning the shape of distribution resembles a random distribution.

b) **“Center of Mass”**
   The “Center of Mass” of the distribution is the average number of subjects on which one agents agree with one another and results from the weighted distribution (cp. 1: For the “Center of Mass” the shaded area of the figure is considered).

   The ”Center of Mass” is analogous to the order parameter of Klemm et al. 8.

c) **Cluster**
   A cluster contains all agents in the network, who share the same opinion on any of the \(S\) subjects.

### 3. Simulation

#### 3.1. Description
The tolerance \(\varepsilon\) was varied. At very small \(\varepsilon\) the distribution (Fig.1) is on the left part a combination of random distributions. The number of agents on the left side
Fig. 1. Distribution for $S = 10$: Quota of pairs of agents agreeing on the plotted number of subjects. With growing tolerance $\varepsilon$, the number of maximal agreeing agents grows. The places of the distribution with an intermediate number of agreements remain vacant (resp. very sparsely populated). The opinion of agents matching in all subjects lies in the centrist opinion of the opinion spectrum $O$ (here with $O = 10$ on the opinion 5 or 6 each) in every subject.

decreases with growing $\varepsilon$. The number of agents on the right side (agreeing in (nearly) all subjects) increases to the same extent. Two peculiarities are to take note of. On the one hand, no agents are found, which agree in an intermediate number of subjects. The places of the distribution with an intermediate number of agreements remain vacant. The extremal distribution of accordance and dissent are favoured. On the other hand, there is no “absolute consensus” (an “absolute consensus” is an accordance of all agents in all subjects). There remain for $\varepsilon > \varepsilon_2$ ($\varepsilon_2$ is the $\varepsilon$ above which the average number of clusters remains stable) several (typically 2) clusters.
The other $\varepsilon$ below which the number of clusters is constant (see inset Fig. 2), is called $\varepsilon_1$.

3.2. Analysis

$N$, $S$ and $O$ have been varied, as outlined.
3.2.1. $N$

With increasing $N$, $\varepsilon_1$ decreases and the number of clusters as well as $\varepsilon_2$ increase, Figure 3.

In a social group of many people they sooner find each other for a discussion, and thereby starting a process of agreement. As $\varepsilon_2$ is weakly depending on the number of agents, its change with $N$ for small groups is higher (Fig. 3).

3.2.2. $S$

$\varepsilon_1$ and $\varepsilon_2$ (Fig. 4a, b) also depend on the number of the discussed subjects $S$.

A large $S$ needs a larger $\varepsilon_1$ to start discussion, compared with small $S$ (Fig. 4b). I.e., if many subjects are available the communicants need larger $\varepsilon_1$ to start discussion, compared with small $S$.

The less subjects are available for selection the smaller is the $\varepsilon_1$ where a talk begins. But then it needs a larger tolerance of the people speaking together, before an agreement will be found.
3.2.3. $O$

The opinion spectrum $O$ has only an influence when it is small. Large spectra saturate very fast. $O$ has less influence on $\varepsilon_1$ (Fig. 5b) (Inset) and a little more influence on $\varepsilon_2$. Increasing $O$ increases both $\varepsilon_1$ and $\varepsilon_2$. Thus diversity of opinion complicates the process of agreement.

4. No absolute Consensus

Almost always several clusters remain of different size (= number of agents within one cluster) in the area $> \varepsilon_2$ where the number of clusters in the average stays stable small.
4.1. Remaining Cluster

The number of remaining clusters depends on the ratio $N/S$ and the absolute value of $S$ (Fig. 6).

- At $S=1$ exists an absolute consensus (remaining cluster = 1).

- If the ratio $N/S$ is large (small $S$) on average $\approx 2$ clusters remain in the area $> \varepsilon_2$. The difference of the clusters is $\delta = 1$. One cluster dominates in its number of members, the other depends in its size on $S$. The latter I call “2nd Cluster”.

- When $S$ becomes larger and the ratio $N/S$ smaller, but still $N/S > 1$, a huge increase of the number of remaining clusters is seen. At the beginning there
Fig. 5. The inset shows the left end more detailed.
a) shows that large opinion spectra saturate early and so only small $O$ have an influence on $\varepsilon_2$.
b) shows the same for $\varepsilon_1$.

is still a large central cluster and some few small clusters. With increasing $S$ an increasing number of differently sized clusters exists.

- At very large $S$ ($N/S \leq 1$) a great many number (maximum $N$) of very small clusters exists. The number of clusters is for $N/S < 1$ the maximum ($N$).

If only one subject will be discussed, it gives an absolute consensus. If several subjects are discussed, no absolute consensus will be given, but in general a majority opinion.
If the ratio of the number of agents to the number of subjects is small, but $S < N$, then a fragmentation into many clusters is seen.
Multidimensional Consensus Model on a Barabási-Albert Network

Fig. 6. Dependence of the number of clusters on the fraction $N/S$. Over a wide range about 2 clusters occur. One dominating with a very high occupation and a 2nd minor, with a low occupation dependent on $S$. On the left a region with a rapid growing number of clusters is attached; many little clusters exist; there is no all-dominant large cluster. For $N/S < 1$ the results are the same as for $N/S = 1$.

4.2. Number of Occupants of the 2nd Cluster

The case of two remaining cluster is prevalent over a wide range of $N/S$. Therefore I have considered particularly the occupation of the 2nd cluster.

The size of the 2nd cluster varies proportional to $S$ (Fig. 7) for intermediate $S$. Even a rise of $n$ to 200 iterations does not erase the existence of the 2nd cluster (Fig. 8).

4.3. Explanation

I have looked for an explanation of the effects described above.

A variation of this model has been simulated, in which one subject after the other is simulated. The whole population discussed the same subject until the stop criterion for this subject is reached. In this modified model there are always both, an absolute consensus and also occupied places of the distribution with an intermediate number of agreements ($:= \text{PDIM}$, cp. Fig. 1). Consequently the existence of a 2nd (or several) remaining cluster(s) arises from the random selection of the subject to discuss.

Therewith I explain the remaining clusters and their dependence of $S$ as follows:
Fig. 7. The maximum in the number of the size of the 2nd cluster is reached in the area where the number of cluster increases with increasing $S$. In the extreme areas of many existing clusters and absolute consensus respectively, the number of occupants decreases to 1 resp. 0.

A dynamical barrier may be develop between two agents $i$ and $j$.

For explanation I take the state that $i$ and $j$ differ only about one opinion unit ($\delta = 1$) in subject $S_c$ and agree in all other subjects. At a discussion between $i$ and $j$ a subject of agreement is chosen with a probability $1 - 1/S$. Due to the present congruence of opinions the discussion stops immediately. With probability $1/S$ the subject $S_c$ will be chosen for discussion. If a discussion will start then the probability is $1/2$ that a takeover of a opinion takes place (Second rule for simulating the communication). The agents $i$ and $j$ obtain an agreement in subject $S_c$ with probability $1/2S$.

Now it is assumed that a consensus of all agents in the network is given with only two agents $i$ and $j$ differing in their opinion on subject $S_c$ by one opinion unit. The difference in opinion vanishes with a probability $1/2S$, leading to total agreement. Therefore the difference in opinions persists with a probability $(1 - 1/2S)$. Depending on the number $k$ of connections of agent $i$ within the network the probability is

$$\left(\frac{k}{m}\right)e \left(1 - \frac{1}{2S}\right)^e \left(\frac{1}{2S}\right)^{m-e}, \quad (2)$$

therefore the subject $S_c$ is either not addressed or the partner for discussion takes over the opinion of $i$ on the subject $S_c$. Let $e$ be the number, out of $k/m$ for agent
Fig. 8. The existence of the 2nd cluster stays highly robust. Even after \( n = 200 \) consecutive iterations of communication there still remain occupants.

\( i \) with \( k \) neighbours, of events where the opinion stays at \( S_c \). \( m \) is the number of agents to talk to, chosen during the assembly of the network. Then the probability for agent \( i \) not to change opinion is (2). Shall the opinion in \( S_c \) be preserved then \( e \) has to be as large as all \( k/m \) requests for a discussion. I.e., at no call for discussion may the subject \( S_c \) be addressed or change the opinion. With \( e = \frac{k}{m} \) the probability not to change opinion

\[
\frac{k}{m} \left( \frac{1}{2S} \right)^{\frac{k}{m}} \left( \frac{1}{2S} \right)^{\frac{k}{m}} = \left( 1 - \frac{1}{2S} \right)^{\frac{k}{m}},
\]

(3) thus is the difference of opinion to stay, and with it

\[
1 - \left( 1 - \frac{1}{2S} \right)^{\frac{k}{m}},
\]

(4) for the difference to vanish.

In this way single nodes could become quasi inactive in relation to \( S_c \). With an existing difference in opinion by \( i \) and \( j \) at subject \( S_c \) this difference stays with a probability (3). If this probability (3) is very small I call it “dynamic barrier”. If the stop criterion intervenes before the dynamic barrier is negotiated the network owns a node, inactive up to this moment in relation to a subject.
Fig. 9. The Probability for an existing difference in opinions to stay depends on the number of subjects $S$ and on how often the agent with the opinion at issue will be spoken to. This depends on the number of its connections $k$ within the network. Many connections result in many addresses and therefore in a low possibility of surviving for a opinion at issue. This graph shows the equation 3 for different $k = 3, 100, 300, 1000$ and once the equation 4 for $k = 100$ (full diamonds).

Only the same subject can be discussed by two agents. So we can imagine, that the same subject together owns a separate network. With it, we have $S$ networks. Consequently an inactive node in relation to a subject is a inactive node of a network. Albert et al.\cite{12} describe that a not performing node in a Barabási-Albert Network leads to clustering. Thereby a large cluster remains and some small clusters with very small number of members (1-16 agents with $m = 3$ and $N = 10000$). But this only takes place if no node with many connections $k$ (center agent) is unperforming. If a center agent (in\cite{12} call Hub) is inactive, then the network disintegrates in many clusters of different sizes.

The probability (3) of a barrier for an agent depending on its $k$ connections can be seen in Figure 9. A center agent with many connections has a high probability for a barrier to vanish.

### 4.4. Not occupied PDIM’s

As mentioned before (start of section 4.3), in the distribution (Fig. 1) the occupation of the places with an average value of agreed subjects depends on the random choice
of subjects during the simulation. The non-occupation of these intermediate states required that nearly all agents which can discuss with agent $i$ at a particular $\varepsilon$, will discussed until consensus is reached, before the simulation stops. If it is not discussed up to a high level of agreement there would be given agents which agree in an intermediate number of subjects.

A discussion up to a near consensus can only happen, if the number of iterations during the communication is large enough. The random (as opposed to sequential at start of section 4.3) choice of subject let the number of iterations raise (Fig. 10) in the network. The number of iteration seems to be high enough, that every agent, which can and has begun to discuss, will discuss up to the end (discuss till a near consensus).

4.5. General Remarks

The agents that occupy the extremes clusters at first are the center agents. Because of their high number of connections $k$, they often get in touch with agents which hold many different opinions in their subjects and from there are forced into a centrist opinion where they stay. The stabilised center agents pull the agents they
are connected to into their state of opinion by their links. Center agents are leaders which are formed and stabilised by public opinion.

5. Conclusion

The behaviour of this model with its multi-components subjects on a scale-free network is not analogous to hitherto simulated one subject-models or models on vectors, lattices or higher dimensional-geometries. The essential difference to these is the missing of a total consensus at more than one subject $(S > 1)$. In the present models the number of remaining cluster is $\propto S/N$ (cp. Fig. 6).

Acknowledgement

I would like to thank D. Stauffer that he has given me the possibility to work on this subject. And I thanks very much N. Klietsch for stimulating discussions.

References

1. Schelling, T.S., *J. Mathematical Sociology* 1, 143 (1971).
2. Galam, S., Gefen, Y. and Shapir, Y., *J. Mathematical Sociology* 9, 1 (1982).
3. Callen, E. and Shapero, D., *Physics Today*, July 1974, 23.
4. Hegselmann, R. and Krause, M., *Journal of Artificial Societies and Social Simulation* 5, issue 3, paper 2 (jasss.soc.surrey.ac.uk) (2002).
5. Sznajd-Weron, K. and Sznajd, J., *Int. J. Mod. Phys. C* 11, 1157 (2000).
6. Stauffer, D., *Journal of Artificial Societies and Social Simulation* 5, issue 1, paper 4 (jasss.soc.surrey.ac.uk) (2002).
7. Deffuant, G., Amblard, F., Weisbuch, G. and Faure, T., *Journal of Artificial Societies and Social Simulation* 5, issue 4, paper 1 (jasss.soc.surrey.ac.uk) (2002).
8. Klemm, K., Eguiluz, V. M., Toral, R. and San Miguel, S., *Physica A* 327, 1 (2003).
9. Axelrod, R., *J. Conflict Resolut.* 41, 203 (1997).
10. Barabási, A.-L. and Albert, R., *Science* 286, 509 (1999).
11. Stauffer, D., Sousa, A. and Schulze, C., *Journal of Artificial Societies and Social Simulation* 7, issue 3, paper 7 (jasss.soc.surrey.ac.uk) (2004).
12. Albert, R., Jeong, H. and Barabási, A.-L., *Nature* 406, 378-382 (2000).