LOCAL WELL-POSEDNESS FOR THIRD ORDER BENJAMIN-ONO TYPE EQUATIONS ON THE TORUS

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ABSTRACT. We consider the Cauchy problem of third order Benjamin-Ono type equations on the torus. Nonlinear terms may yield derivative losses, which prevents us from using the classical energy method. In order to overcome that difficulty, we add a correction term into the energy. We also use the Bona-Smith type argument to show the continuous dependence.

1. Introduction

We consider the Cauthy problem of the following third order Benjamin-Ono type equations on the torus $T := \mathbb{R}/2\pi \mathbb{Z}$:

$$\begin{align*}
\partial_t u - \partial_x^3 u + u^2 \partial_x u + c_1 \partial_x (u \mathcal{H} \partial_x u) + c_2 \mathcal{H} \partial_x (u \partial_x u) &= 0, \\
u(0, x) &= \varphi(x),
\end{align*}$$

where the initial data $\varphi$ and the unknown function $u$ are real valued, and $c_1, c_2 \in \mathbb{R}$. $\mathcal{H}$ is the Hilbert transform on the torus defined by

$$\hat{\mathcal{H}} f(0) = 0 \quad \text{and} \quad \hat{\mathcal{H}} f(k) = -i \text{sgn}(k) \hat{f}(k), \quad k \in \mathbb{Z} \setminus \{0\},$$

where $\hat{f}$ is the Fourier transform of $f$: $\hat{f}(k) = \mathcal{F} f(k) = (2\pi)^{-1/2} \int_T f(x) e^{-ixk} dx$. The well-known Benjamin-Ono equation

$$\partial_t u + \mathcal{H} \partial_x^2 u + 2u \partial_x u = 0$$

describes the behavior of long internal waves in deep stratified fluids. The equation (1.3) also has infinitely many conservation laws, which generates a hierarchy of Hamiltonian equations of order $j$. The equation (1.1) with $c_1 = c_2 = \sqrt{3}/2$ is the second equation in the Benjamin-Ono hierarchy [1].

There are a lot of literature on the Cauchy problem on (1.3). On the real line case, Ionescu-Kenig [8] showed the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 0$ (see also [7] for another proof and [25] for the local well-posedness with small complex valued data). On the periodic case, Molinet [9, 10] showed the local well-posedness.

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in $H^s(\mathbb{T})$ for $s \geq 0$ and that this result was sharp. See [17, 18, 19, 20, 22, 23, 24] for former results.

On the Cauchy problem of (1.1) with $c_1 = c_2 = \sqrt{3}/2$ on the real line, Feng-Han [11] proved the unique existence in $H^s(\mathbb{R})$ for $4 \leq s \in \mathbb{N}$ by using the theory of complete integrability. They also used the energy method with a correction term in order to show the uniqueness. Feng [12] modified the energy method used in [11] and used an a priori bound of solutions in $H^s(\mathbb{R})$ to show the “weak” continuous dependence in the following sense:

$$\varphi_n \to \varphi \text{ in } H^{s-2}(\mathbb{R}) \text{ as } n \to \infty \Rightarrow u_n \to u \text{ in } C([0,T]; H^{s-2}(\mathbb{R})) \text{ as } n \to \infty,$$

(1.4)

for $\varphi, \varphi_n \in H^s(\mathbb{R})$ and $6 \leq s \in \mathbb{N}$. Here, $u_n$ (resp. $u$) denotes the corresponding solution of (1.1) with $c_1 = c_2 = \sqrt{3}/2$ and the initial data $\varphi_n$ for $n \in \mathbb{N}$ (resp. $\varphi$). Note that the topology of the convergence is weaker than $H^s$. Linares-Pilod-Ponce [13] and Molinet-Pilod [14] succeed in proving the local well-posedness in $H^s(\mathbb{R})$ of the following equation

$$\partial_t u + d_1 \partial_x^3 u - d_2 \mathcal{H} \partial_x^2 u = d_3 u \partial_x u - d_4 \partial_x (u \mathcal{H} \partial_x u + \mathcal{H} (u \partial_x u)),$$

for $s \geq 2$ and $s \geq 1$, respectively. Here, coefficients satisfy $d_1 \in \mathbb{R}$, $d_1 \neq 0$ and $d_j > 0$ for $j = 2, 3, 4$. Their proof involves the gauge transform and the Kato type smoothing estimate.

On the periodic case, as far as the author knows, there are no well-posedness results for the Cauchy problem of (1.1) available in the literature. Although proofs in Feng-Han [11] and Feng [12] above works, and we cannot obtain the local well-posedness, that is, the resultant continuous dependence (1.4) is weak. And their proofs heavily depend on the complete integrability. In particular, it is very important to have $c_1 = c_2$ in their proofs. It should also be pointed out that in the periodic case, we do not have the Kato type smoothing estimate, which implies that the local well-posedness is far from trivial.

Therefore, in this article, we are interested in establishing the local well-posedness of (1.1) in $H^s(\mathbb{T})$ for $s$ less than 4 without using the theory of complete integrability. In particular, we improve the “weak” continuous dependence (1.4) shown in [12] in order to fulfill conditions of the local well-posedness. Moreover, thanks to Lemma 2.5, we can show the local well-posedness of the non-integrable case (1.1).

The main result is the following:

**Theorem 1.1.** Let $s \geq s_0 > 5/2$. For any $\varphi \in H^s(\mathbb{T})$, there exist $T = T(\|\varphi\|_{H^{s_0}}) > 0$ and the unique solution $u \in C([-T,T]; H^s(\mathbb{T}))$ to the IVP (1.1)–(1.2) on $[-T,T]$. 
Moreover, for any $R > 0$, the solution map $\varphi \mapsto u(t)$ is continuous from the ball
$\{ \varphi \in H^s(\mathbb{T}); \| \varphi \|_{H^s} \leq R \}$ to $C([-T,T]; H^s(\mathbb{T}))$.

Now, we mention the idea of the proof of Theorem 1.1. The standard energy
method gives us the local well-posedness of (1.3) in $H^s(\mathbb{T})$ for $s > 3/2$. On the other
hand, nonlinear terms $\partial_x(uH\partial_x u)$ and $H\partial_x(w\partial_x u)$ in (1.1)
have two derivatives, and the energy estimate gives only the following:

$$
\frac{d}{dt}\|\partial_x^k u(t)\|_{L^2}^2 \lesssim (1 + \|\partial_x^2 u\|_{L^\infty})^2 \|\partial_x^k u(t)\|_{L^2}^2 + \left| \int \partial_x u(\mathcal{H}\partial_x^{k+1} u)\partial_x^k u dx \right|.
$$

(1.5)

It is difficult to handle the last term in the right hand side by $\|u\|_{H^k}$, which is the
main difficulty in this problem. To overcome that difficulty, we add a correction
term into the energy (see Definition 2):

$$
E_s(u) := \|u\|_{L^2}^2 + \|D^s u\|_{L^2}^2 + a_s \|u\|_{L^2}^{4s+2} + b_s \int u(\mathcal{H}D^s u)D^{s-2}\partial_x u dx,
$$

where $D := \mathcal{F}^{-1}|\xi|\mathcal{F}$, following the idea from Kwon [4], who studied the local well-
posedness of the fifth order KdV equation (see also Segata [5], Kenig-Pilod [16] and
Tsugawa [15]). The correction term allows us to cancel out the worst term in (1.5),
which makes it possible to evaluate the $H^s$-norm of the solution by that of the initial
data. It is worth pointing out that our proof refines the idea in [12]. Indeed, Feng
introduced the following energy estimate in order to show the “weak” continuous
dependence (1.4):

$$
\frac{d}{dt}\left(\|\partial_x^{k-2} w\|_{L^2}^2 + \frac{2k-3}{4} \int_{\mathbb{R}} (u + v)\partial_x^{k-3} w\mathcal{H}\partial_x^{k-2} w dx \right)
\leq C(T, \|\varphi\|_{H^k}, \|\psi\|_{H^k})\|w(t)\|_{H^{k-2}}^2,
$$
on $[0,T]$, where $w = u - v$ and $u, v \in C([0,T]; H^k(\mathbb{R}))$ satisty (1.1) with $c_1 = c_2 = \sqrt{3}/2$ and initial data $\varphi, \psi \in H^k(\mathbb{R})$, respectively. Here, we would like to have the
estimate for $\|w\|_{H^k}$. If we simply replace $k - 2$ with $k$ in the above estimate, the
constant in the right hand side depends on $\|\varphi\|_{H^{k+2}}$ (resp. $\|\psi\|_{H^{k+2}}$), which cannot
be handled by $\|\varphi\|_{H^k}$ (resp. $\|\psi\|_{H^k}$). Therefore, we need to find a different correction
term (see Definition 2) and estimate the difference between two solutions in $H^k(\mathbb{T})$
more carefully (see the proof of Proposition 4.4) so as to complete the continuous
dependence.

Another difficulty is the presence of the Hilbert transform $\mathcal{H}$, which restricts the
possibility of using the integration by parts for some terms. Recall that for real
valued functions $f, g$, we have

$$
|\langle fD^s g, D^s\partial_x g \rangle_{L^2}| \leq \frac{1}{2}\|\partial_x f\|_{\infty}\|D^s g\|_{L^2}^2.
$$
However, in our problem we cannot apply the integration by parts to
\[ \langle \partial_x f \mathcal{H} \partial_x g, D^s g \rangle_{L^2}, \]
which is nothing but the term which we cancel out by introducing a correction term.

We notice that the $L^2$-norm is conserved by solutions of equations (1.1) with $c_1 = c_2$ thanks to the following equality:
\[ \langle \mathcal{H} \partial_x (u \partial_x u), u \rangle_{L^2} + \langle \partial_x (u \mathcal{H} \partial_x u), u \rangle_{L^2} = 0, \]
which helps us to handle nonlinear terms. In the case $c_1 \neq c_2$, we use Lemma 2.5 originally proved in [21].

Subsequently, using the conservation law corresponding to the $H^3$-norm of the solution, we can obtain the following result:

**Corollary 1.2.** The Cauchy problem (1.1)–(1.2) with $c_1 = c_2 = \sqrt{3}/2$ is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 3$.

This paper is organized as follows. In Section 2, we fix some notations and state a number of estimates. We also obtain a solution of the regularized equation associated to (1.1). In Section 3, we give an a priori estimate for the solution to (1.1). In Section 4, we show the existence of the solution, uniqueness, the persistence, and the continuous dependence.

**2. Notations, Preliminaries and Parabolic Regularization**

In this section, we give some notations and collect a number of estimates which will be used throughout this paper. We denote the norm in $L^p(\mathbb{T})$ by $\| \cdot \|_p$. In particular, we simply write $\| \cdot \| := \| \cdot \|_2$. We denote $\| f \|_{H^s} := 2^{-1/2}(\| f \|^2 + \| D^s f \|^2)^{1/2}$ for a function $f$ and $s \geq 0$, where $D = \mathcal{F}^{-1}|\xi|\mathcal{F}$. Let $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2}$. We also use the same symbol for $\langle \cdot \rangle := (1 + | \cdot |^2)^{1/2}$. Let $[A, B] = AB - BA$.

We use the following Gagliardo-Nirenberg inequality on the torus:

**Lemma 2.1.** Assume that $l \in \mathbb{N} \cup \{0\}$ and $s \geq 1$ satisfy $l \leq s - 1$ and a real number $p$ satisfies $2 \leq p \leq \infty$. Put $\alpha = (l + 1/2 - 1/p)/s$. Then, we have
\[ \| \partial_x^l f \|_p \lesssim \begin{cases} \| f \|^{1-\alpha} \| D^s f \|^\alpha & \text{(when } 1 \leq l \leq s - 1), \\ \| f \|^{1-\alpha} \| D^s f \|^\alpha + \| f \| & \text{(when } l = 0), \end{cases} \]
for any $f \in H^s(\mathbb{T})$.

**Proof.** In the case $s$ is an integer, see Section 2 in [6]. The general case follows from the integer case and the Hölder inequality. \[\square\]
The following inequality is helpful when we estimate the difference between two solutions in $L^2$.

**Lemma 2.2.** Let $k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds true:

$$\|\mathcal{H}\partial_x^k f + (D)^{-1}\partial_x^{k+1} f\| \leq \|f\|_{H^{k-1}}$$

for any $f \in H^{k-1}(\mathbb{T})$.

*Proof.* We have $|\text{sgn}(\xi) - \xi(\xi)^{-1}| \leq (\xi)^{-1}$ for any $\xi \in \mathbb{Z}$, which shows that

$$\|\mathcal{H}\partial_x^k f + (D)^{-1}\partial_x^{k+1} f\| = \| (\text{sgn}(\xi) - \xi(\xi)^{-1})\xi^k \hat{f}(\xi)\| \leq \|f\|_{H^{k-1}}$$

as desired. \qed

**Definition 1.** For $s \geq 0$ and functions $u, v$ defined on $\mathbb{T}$, we define

$$P_s(f, g) := D^s\partial_x f\partial_x g - D^s\partial_x f\partial_x g - (s + 1)\partial_x f\partial_x^2 g,$$

$$Q_s(f, g) := \mathcal{H}D^s\partial_x f\partial_x g - (\mathcal{H}D^s\partial_x f)\partial_x g - \partial_x f\mathcal{H}D^s\partial_x^2 g - (s + 1)\partial_x f\mathcal{H}D^s\partial_x g.$$

We introduce several commutator estimates. For general theory on the real line, see [26]. We shall use extensively the following commutator estimate.

**Lemma 2.3.** Let $s \geq 1$ and $s_0 > 5/2$. Then there exists $C = C(s, s_0) > 0$ such that for any $f, g \in H^s(\mathbb{T}) \cap H^{s_0}(\mathbb{T}),$

$$\|P_s(f, g)\|, \|Q_s(f, g)\| \leq C(\|f\|_{H^{s_0}}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{H^{s_0}}).$$

*Proof.* We show only the inequality for $P_s(f, g)$ with $s > 1$. The case $s = 1$ follows from Lemma 2.5. Another one follows from a similar argument since $D = \mathcal{H}\partial_x$. It suffices to show that there exists $C = C(s)$ such that

$$\|\xi^s\eta - |\xi - \eta|^s(\xi - \eta)\eta - |\eta|^s\eta^2 - (s + 1)(\xi - \eta)|\eta|^s\eta\| \leq C(|\xi - \eta|^s|\eta|^2 + |\xi - \eta|^2|\eta|^s)$$

(2.1)

for any $\xi, \eta \in \mathbb{Z}$. We split the summation region into three regions: $R_1 = \{3|\eta| \leq |\xi - \eta|\}$, $R_2 = \{|\eta| \geq 3|\xi - \eta|\}$ and $R_3 = \{|\xi - \eta|/4 \leq |\eta| \leq 4|\xi - \eta|\}$. On $R_1$, the mean value theorem shows that (2.1) holds. On $R_2$, note that $|\xi| \sim |\eta|$. It immediately follows that $|\xi - \eta|^s(\xi - \eta)\eta \lesssim |\xi - \eta|^s|\eta|^2$. Set $\sigma(x) = x|x|^s$ for $x \in \mathbb{R}$. Note that $\sigma \in C^2(\mathbb{R})$. The Taylor theorem shows that there exist $\tilde{\eta} \in (\xi, \eta)$ or $\tilde{\eta} \in (\eta, \xi)$ such that

$$\sigma(\xi) = \sigma(\eta) + \sigma'(\eta)(\xi - \eta) + \frac{\sigma''(\tilde{\eta})}{2}(\xi - \eta)^2.$$

This together with the fact that $|\tilde{\eta}| \sim |\xi| \sim |\eta|$ implies that (2.1) holds. On $R_3$, it is obvious. \qed
Lemma 2.4. Let $s \geq 1$, $s_0 > 1/2$ and $\Lambda_s = D^s$ or $D^{s-1}\partial_x$. Then we have the following:

(i) There exists $C(s, s_0) > 0$ such that for any $f, g \in H^{s_0+1}(\mathbb{T}) \cap H^s(\mathbb{T})$,

$$
\| [\Lambda_s, f] \partial_x g \| \lesssim \| f \|_{H^{s_0+1}} \| g \|_{H^s} + \| f \|_{H^s} \| g \|_{H^{s_0+1}}.
$$

(ii) There exists $C(s_0) > 0$ such that for any $f \in H^{s_0+1}(\mathbb{T})$ and $g \in L^2(\mathbb{T})$,

$$
\| \langle D \rangle^{-1} \Lambda_s f \| \lesssim \| f \|_{H^{s_0+1}} \| g \|.
$$

Proof. We omit the proof of the (i) since it is identical with that of the previous Lemma. We show the case (ii) with $\Lambda_2 = \partial_x^2$ only. The other case follows from a similar argument. It suffices to show that $|\xi^2 \langle \xi \rangle^{-1} - \eta^2 \langle \eta \rangle^{-1}| \lesssim |\xi - \eta|$ for any $\xi, \eta \in \mathbb{Z}$. Set $\sigma(x) = -x^2 \langle x \rangle^{-1}$ for $x \in \mathbb{R}$. Note that $\sigma \in C^1(\mathbb{R})$ and that $\sigma'(x) = -(x^2 + 2x) \langle x \rangle^{-3}$. It then follows that there exists $C > 0$ such that $|\sigma'(x)| \leq C$ for any $x \in \mathbb{R}$. This together with the mean value theorem implies that we have

$$
|\sigma(\xi) - \sigma(\eta)| \leq C|\xi - \eta|,
$$

which completes the proof. \hfill \square

The following estimate is essential for our analysis in the case $c_1 \neq c_2$ in (1.1). For $L^p$ cases on the real line, see [21].

Lemma 2.5. Let $s_0 > 1/2$ and $k \in \mathbb{N}$. Then, there exists $C = C(s_0) > 0$ such that for any $f \in H^{s_0}(\mathbb{T})$ and $g \in L^2(\mathbb{T})$

$$
\| [\mathcal{H}, f] \partial_x^k g \| \leq C \| f \|_{H^{s_0+k}} \| g \|.
$$

Proof. It suffices to show that

$$
| \text{sgn}(\xi) - \text{sgn}(\eta)| |\eta|^k \lesssim |\xi - \eta|^k
$$

(2.2)

for any $\xi, \eta \in \mathbb{Z}$. We split the summation region into three regions: $R_1 = \{3|\eta| \leq |\xi| \}, R_2 = \{|\eta| \geq 3|\xi| \}$ and $R_3 = \{|\xi|/4 \leq |\eta| \leq 4|\xi| \}$. It is clear that (2.2) holds on $R_1$ and $R_2$. It is also clear that (2.2) holds when $\xi \eta > 0$. Therefore, we consider the region $R_3 \cap \{\xi \eta \leq 0 \}$. We first assume that $\xi \geq 0$ and $\eta \leq 0$. Note that $|\xi - \eta| \geq |\xi| \geq |\eta|/4$. Similarly, in the case $\xi \leq 0$ and $\eta \geq 0$ we have $|\xi - \eta| \geq |\eta|$. Therefore, we have (2.2), which concludes the proof. \hfill \square

Lemma 2.6. Let $s_0 > 1/2$ and $u, v$ be sufficiently smooth function defined on $\mathbb{T}$. Then there exists $C = C(s_0) > 0$ such that

$$
| \langle v \mathcal{H} \partial_x^2 u + \partial_x v \mathcal{H} \partial_x u, u \rangle | \leq C \| v \|_{H^{s_0+2}} \| u \|^2.
$$
Proof. This follows from the equality
\[ 2\langle \mathcal{H} \partial_x^2 u + \partial_x v \mathcal{H} \partial_x u, u \rangle = -([\mathcal{H}, v] \partial_x^2 u, u) - \langle \partial_x^2 v \mathcal{H} u, u \rangle \]
together with Lemma 2.5. \qed

We shall also use extensively the following estimate.

Lemma 2.7. Let \( s_0 > 1/2 \). Then, there exists a constant \( C = C(s_0) > 0 \) such that for any \( f \in H^{s_0+1}(\mathbb{T}) \) and \( g \in H^1(\mathbb{T}) \)
\[ |\langle f \partial_x g, g \rangle| \leq C \| f \|_{H^{s_0+1}} \| g \|^2. \]

Proof. This follows from the density argument and the integration by parts. \qed

The following lemma helps us calculate a correction term.

Lemma 2.8. For sufficiently smooth functions \( f, g \) and \( h \) defined on \( \mathbb{T} \), it holds that
\[ \langle \partial^2_x fg, h \rangle + \langle f \partial^2_x g, h \rangle + \langle fg, \partial^2_x h \rangle = 3\langle \partial_x f \partial_x g, \partial_x h \rangle. \]

Proof. See Lemma 2.2 in [16]. \qed

We shall repeatedly use estimates of the following type:

Lemma 2.9. Let \( s_0 > 5/2 \).

(i) Let \( s \geq 1 \). There exists a constant \( C(s, s_0) > 0 \) such that for any \( f_1 \in H^s(\mathbb{T}) \cap H^{s_0}(\mathbb{T}) \) and \( f_2 \in H^{s+1}(\mathbb{T}) \cap H^{s_0}(\mathbb{T}) \),
\[ |\langle f_1 \mathcal{H} D^s f_2, \mathcal{H} D^s (f_1 \partial_x f_2) \rangle| \leq C(\|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s}). \]

(ii) Let \( s \geq 2 \). There exists a constant \( C(s, s_0) > 0 \) such that for any \( f_1 \in H^{s+1}(\mathbb{T}) \cap H^{s_0}(\mathbb{T}) \) and \( f_2 \in H^{s+2}(\mathbb{T}) \cap H^{s_0}(\mathbb{T}) \),
\[ |\langle f_1 \mathcal{H} D^s \partial_x (f_1 \mathcal{H} \partial_x f_2), D^{s-2} \partial_x f_2 \rangle| \leq C(\|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s}). \]

Proof. First we show (i). Note that
\[ |\langle f_1 \mathcal{H} D^s f_2, \mathcal{H} D^s (f_1 \partial_x f_2) \rangle| \leq |\langle f_1 \mathcal{H} D^s f_2, [\mathcal{H} D^s, f_1] \partial_x f_2 \rangle| + |\langle f_1^2 \mathcal{H} D^s f_2, \mathcal{H} D^s \partial_x f_2 \rangle|. \]
Lemma 2.7 together with (i) of Lemma 2.4 shows (i). Next we show (ii). Lemma 2.3 shows that
\[ |\langle D^{s+1} (f_1 \mathcal{H} \partial_x f_2), f_1 D^{s-2} \partial_x f_2 \rangle - R_1 - R_2 - R_3| \lesssim \|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2 + \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s}, \]
where $R_1 = \langle D^{s+1} f_1 \mathcal{H} \partial_x f_2, f_1 D^{s-2} \partial_x f_2 \rangle$, $R_2 = \langle f_1 \mathcal{H} D^{s+1} \partial_x f_2, f_1 D^{s-2} \partial_x f_2 \rangle$ and $R_3 = (s + 1) \langle \partial_x f_1 \mathcal{H} D^{s+1} f_2, f_1 D^{s-2} \partial_x f_2 \rangle$. It is easy to see that

$$|R_1| \lesssim \|f_1\|_{H^{s_0}} \|f_1\|_{H^s} \|f_2\|_{H^{s_0}} \|f_2\|_{H^s} \quad \text{and} \quad |R_3| \lesssim \|f_1\|_{H^{s_0}}^2 \|f_2\|_{H^s}^2.$$

For $R_2$, we have

$$R_2 = -\langle f_1^2 D^s \partial_x f_2, D^{s-2} \partial_x f_2 \rangle - \langle f_1^2 D^s \partial_x f_2, D^s f \rangle = -2 \langle \partial_x (f \partial_x f_1 D^{s-2} \partial_x f_2), D^s f_2 \rangle + \langle f_1 \partial_x f_1, (D^s f_2) \rangle,$$

which can be bounded by $\lesssim \|f_1\|^2_{H^{s_0}} \|f_2\|^2_{H^s}$. This concludes the proof. \qed

**Lemma 2.10.** For any $s \geq 1$ and $s_0 > 5/2$, there exists $C(s, s_0) > 0$ such that for any $u, v \in H^{s+2}(\mathbb{T}) \cap H^{s_0}(\mathbb{T})$,

$$\left| \langle D^s \partial_x (u \mathcal{H} \partial_x u - v \mathcal{H} \partial_x v), D^s w \rangle \right| - s \langle \partial_x u \mathcal{H} D^s \partial_x w, D^s w \rangle \right| \leq C \|w\|_{H^s} \left( \|u\|_{H^{s_0}} + \|v\|_{H^{s_0}} \right) \|w\|_{H^s} + \|u\|_{H^s} + \|v\|_{H^s} \|w\|_{H^{s_0}} + \|w\|_{H^{s_0-2}} \|v\|_{H^{s+2}} + \|w\|_{H^{s_0-1}} \|v\|_{H^{s+1}},$$

where $w = u - v$.

**Proof.** Adding and subtracting terms, we obtain

$$\left| \langle D^s \partial_x (u \mathcal{H} \partial_x w + w \mathcal{H} \partial_x v), D^s w \rangle \right| - s \langle \partial_x u \mathcal{H} D^s \partial_x w, D^s w \rangle \right| \leq \left| \langle P_s(u, \mathcal{H} w) + P_s(w, \mathcal{H} v), D^s w \rangle \right| + \left| \langle D^s \partial_x u \mathcal{H} \partial_x w, D^s w \rangle \right| + \frac{1}{2} \|\mathcal{H} \partial_x^2 w, (D^s w)^2 \| + \left| \langle w \mathcal{H} D^s \partial_x^2 w + \partial_x u \mathcal{H} D^s \partial_x w, D^s w \rangle \right| + (s + 1) \left| \langle \partial_x w \mathcal{H} D^s \partial_x v, D^s w \rangle \right|$$

since we have

$$\langle \partial_x w \mathcal{H} D^s \partial_x u + \partial_x v \mathcal{H} D^s \partial_x w, D^s w \rangle = \langle \partial_x u \mathcal{H} D^s \partial_x w + \partial_x w \mathcal{H} D^s \partial_x v, D^s w \rangle.$$

Note that

$$\left| \langle D^s \partial_x u \mathcal{H} \partial_x w, D^s w \rangle \right| \leq \|w\|_{H^{s_0}} \|w\|_{H^s} + \|w\|_{H^s} \|w\|_{H^{s_0-1}} \|v\|_{H^{s+1}}$$
by Lemma 2.11. This together with Lemma 2.3 and 2.6 gives the desired inequality, which completes the proof. □

**Definition 2.** Let $s \geq 2$ and $a, b, c \geq 0$. Set $\lambda(s') = -2((c_1 + c_2)s' + c_2)/3$ for $s' \geq 0$. For $f, g \in H^s(\mathbb{T})$ we define

$$E_s(f, g; a) := a\|f - g\|^2 + \|D^s(f - g)\|^2 + \lambda(s) \int_\mathbb{T} f(\mathcal{H}D^s(f - g))D^{s-2}\partial_x(f - g)dx,$$

$$E_s(f; b) := E_s(f, f; 0) + b\|f\|^{4s+2}.$$

For $f, g \in L^2(\mathbb{T})$ we define

$$\tilde{E}(f, g; c) := c\|f - g\|^2_{H^{-1}} + \|f - g\|^2 - \lambda(0) \int_\mathbb{T} f(\langle D \rangle^{-1}(f - g))(f - g)dx.$$

**Lemma 2.11.** Let $s \geq s_0 > 5/2$ and $K > 0$. Then

(i) If $f, g \in H^s(\mathbb{T})$ and $f$ satisfies $\|f\| \leq K$, then there exist $C = C(s, K)$ and $a = a(s, K)$ such that

$$\|f - g\|^2_{H^s} \leq E_s(f, g; a) \leq C\|f - g\|^2_{H^s}. \tag{2.3}$$

(ii) If $f \in H^s(\mathbb{T})$, there exist $C = C(s)$ and $b = b(s)$ such that

$$\|f\|^2_{H^s} \leq E_s(f; b) \leq C(1 + \|f\|^{4s})\|f\|^2_{H^s}. \tag{2.4}$$

(iii) If $f, g \in L^2(\mathbb{T})$ and $f$ satisfies $\|f\| \leq K$, then there exist $c = c(K)$ and $C = C(K)$ such that

$$\frac{1}{2}\|f - g\|^2 \leq \tilde{E}(f, g; c) \leq C\|f - g\|^2. \tag{2.5}$$

**Proof.** We see from Lemma 2.11 and the Young inequality that

$$\int_\mathbb{T} |f(\mathcal{H}D^s(f - g))D^{s-2}\partial_x(f - g)|dx \leq \|f\|\|D^s(f - g)\|\|D^{s-2}\partial_x(f - g)\|_{\infty} \leq C\|f - g\|^{1/2}\|D^s(f - g)\|^{2-1/2}\leq C\|f - g\|^2 + \frac{1}{2}\|D^s(f - g)\|^2.$$

Choosing $a > 0$ so that $a - C \geq 1/2$, we obtain the left hand side of (2.3). The right hand side of (2.3) follows immediately, which shows (i).

Next we prove (2.4). A similar argument to the proof of (2.3) yields that

$$\int_\mathbb{T} |f(\mathcal{H}D^sf)D^{s-2}\partial_x f|dx \leq C\|f\|^{4s+2} + \frac{1}{2}\|D^sf\|^2.$$

Choosing $b > 0$ so that $b - C > 1/2$, we obtain (2.4). The proof of (iii) is identical with that of (i). □
In what follows, we simply write \( E_s(f, g) := E_s(f, g; a) \), \( E_s(f) := E_s(f; b) \) and \( \tilde{E}_s(f, g) := \tilde{E}_s(f, g; c) \), where \( a, b \) and \( c \) are defined by Lemma 2.11.

**Definition 3.** Let \( s \geq 0 \), \( f \in H^s(\mathbb{T}) \) and \( \gamma \in (0, 1) \). And let \( \rho \in C_0^\infty(\mathbb{R}) \) satisfy \( \text{supp}\, \rho \subset [-2, 2] \), \( 0 \leq \rho \leq 1 \) on \( \mathbb{R} \) and \( \rho \equiv 1 \) on \([-1, 1]\). We put

\[
\tilde{J}_s f(k) := \rho(\gamma k) \hat{f}(k).
\]

For the proof of the following lemma, see Remark 3.5 in [2].

**Lemma 2.12.** Let \( s \geq 0 \), \( \alpha \geq 0 \), \( \gamma \in (0, 1) \) and \( f \in H^s(\mathbb{T}) \). Then, \( J_s f \in H^\infty(\mathbb{T}) \) satisfies

\[
\|J_s f - f\|_{H^s} \to 0 \quad (\gamma \to 0), \quad \|J_s f - f\|_{H^{s+\alpha}} \lesssim \gamma^{-\alpha} \|f\|_{H^s},
\]

\[
\|J_s f\|_{H^{s+\alpha}} \leq \|f\|_{H^{s+\alpha}}, \quad \|J_s f\|_{H^{s+\alpha}} \lesssim \gamma^{-\alpha} \|f\|_{H^s}.
\]

We employ the parabolic regularization on the problem (1.1)-(1.2):

\[
\partial_t u - \partial_x^3 u + u^2 \partial_x u + c_1 \partial_x (u \mathcal{H} \partial_x u) + c_2 \mathcal{H} \partial_x (u \partial_x u) = -\gamma D^{5/2} u,
\]

\[
u(0, x) = \varphi(x),
\]

where \((t, x) \in [0, \infty) \times \mathbb{T}\) and \( \gamma \in (0, 1) \). In what follows, we only consider \( t \geq 0 \). In the case \( t \leq 0 \), we only need to replace \(-\gamma D^{5/2} u\) with \( \gamma D^{5/2} u \) in (2.6).

**Proposition 2.13.** Let \( s \geq 2 \) and \( \gamma \in (0, 1) \). For any \( \varphi \in H^s(\mathbb{T}) \), there exist \( T_\gamma \in (0, \infty) \) and the unique solution \( u \in C([0, T_\gamma), H^s(\mathbb{T})) \) to the IVP (2.6)-(2.7) on \([0, T_\gamma)\) such that (i) \( \liminf_{t \to T_\gamma} \|u(t)\|_{H^2} = \infty \) or (ii) \( T_\gamma = \infty \) holds. Moreover, \( u \) satisfies

\[
u \in C((0, T_\gamma), H^{s+\alpha}(\mathbb{T})), \quad \forall \alpha > 0.
\]

**Proof.** This follows from the standard argument, for example, see Proposition 2.8 in [13], but we reproduce the proof here for the sake of completeness. First we consider the case \( s = 2 \). For simplicity, set \( F(u) = -u^2 \partial_x u - c_1 \partial_x (u \mathcal{H} \partial_x u) - c_2 \mathcal{H} \partial_x (u \partial_x u) \). Let \( U_\gamma(t) \) be the linear propagator of the linear part of (2.6), i.e.,

\[
U_\gamma(t) \varphi = \mathcal{F}^{-1}[e^{-\xi^4 t - \gamma |\xi|^{5/2} t} \hat{\varphi}]
\]

for a function \( \varphi \). Note that

\[
\|D^\alpha U_\gamma(t) \varphi\| \leq \frac{C(\alpha)}{(\gamma t)^{2\alpha/5}} \|\varphi\| \quad \text{and} \quad \|U_\gamma(t) \varphi\|_{H^\alpha} \leq C(\alpha)(1 + (\gamma t)^{-2\alpha/5}) \|\varphi\| \quad \text{for} \ t > 0 \text{ and} \ \alpha > 0.
\]

We show the map

\[
\Gamma(u(t)) = U_\gamma(t) \varphi + \int_0^t U_\gamma(t - \tau) F(u) d\tau
\]
is a contraction on the ball

\[ B_r = \left\{ u \in C([0,T];H^2(\mathbb{T})); \|u\|_X := \sup_{t \in [0,T]} \|u(t)\|_{H^2} \leq r \right\}, \]

where \( r > 0 \) and \( T \) will be chosen later (which is sufficiently small and depends only on \( \|\varphi\|_{H^2} \) and \( \gamma \)). Set \( r = 2\|\varphi\|_{H^2} \). We show that \( \Gamma \) maps from \( B_r \) to \( B_r \). Let \( u \in B_r \). Obviously,

\[ \|\Gamma(u(t))\|_{H^2} \leq \|\varphi\|_{H^2} + \int_0^t \|U_\gamma(t-t')F(u)\|_{H^2} dt'. \]

The Plancherel theorem implies that

\[ \|U_\gamma(t-t')\partial_x u^3\|_{H^2} = \|\langle \xi \rangle^2 |\xi|^{-\gamma(t-t')}|\xi|^{5/2} \mathcal{F} u^3\|_2 \]

\[ \lesssim \gamma^{-2/5}(t-t')^{-2/5}\|u^3\|_{H^2} \leq \gamma^{-2/5}(t-t')^{-2/5}\|\varphi\|_{H^2}^3. \]

Similarly, we have

\[ \|U_\gamma(t-t')\mathcal{H}_\partial_x(u\partial_x u)\|_{H^2} \lesssim \gamma^{-2/5}(t-t')^{-2/5}\|\varphi\|_{H^2}^2. \]

On the other hand,

\[ \|U_\gamma(t-t')\partial_x(u\mathcal{H}_\partial_x u)\|_{H^2} \lesssim (1 + \gamma^{-4/5}(t-t')^{-4/5})\|\varphi\|_{H^2}^2. \]

It then follows that

\[ \sup_{t \in [0,T]} \|\Gamma(u(t))\|_{H^2} \]

\[ \leq \|\varphi\|_{H^2} + C\{\|\varphi\|_{H^2}^2 \gamma^{-2/5}T^{3/5} + \|\varphi\|_{H^2}(T + \gamma^{-4/5}T^{1/5})\} \|\varphi\|_{H^2} \leq 2\|\varphi\|_{H^2} \]

for sufficiently small \( T = T(\|\varphi\|_{H^2}, \gamma) > 0 \) and any \( u \in B_r \). By a similar argument, we can show that \( \|\Gamma(u) - \Gamma(v)\|_X \leq 2^{-1}\|u-v\|_X \) when \( u, v \in B_r \). Therefore, \( \Gamma \) is a contraction map from \( B_r \) to \( B_r \), which implies that there exists \( u \in B_r \) such that \( u = \Gamma(u) \) on \([0,T] \). Since \( \|u(T)\|_{H^2} \) is finite, we can repeat the argument above with initial data \( u(T) \) to obtain the solution on \([T, T+T'] \). Iterating this process, we can extend the solution on \([0, T_\gamma) \) where \( T_\gamma = \infty \) or \( \lim \inf_{t \to T_\gamma} \|u(t)\|_{H^2} = \infty \) holds.

Next, we consider the case \( s > 2 \). The solution obtained by the argument above satisfies

\[ u(t) = U_\gamma(t)\varphi + \int_0^t U_\gamma(t-t')F(u)dt'. \quad (2.10) \]

Note that

\[ \|U_\gamma(t-t')\partial_x u^3\|_{H^2} \lesssim \gamma^{-2/5}(t-t')^{-2/5}\|u^3\|_{H^2} \lesssim \gamma^{-2/5}(t-t')^{-2/5}\|\varphi\|_{H^2}^2. \]
We can estimate the other nonlinear terms in the same manner as above. It then follows that

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \leq \|\varphi\|_{H^s} + C\{\|\varphi\|_{H^2}^2 \gamma^{-2/5} T^{3/5} + \|\varphi\|_{H^2}(T + \gamma^{-4/5} T^{1/5})\}\|\varphi\|_{H^s} \leq 2\|\varphi\|_{H^s}$$

for sufficiently small $T = T(\|\varphi\|_{H^2}, \gamma) > 0$. By using (2.10), we also obtain $u \in C([0,T]; H^s(\mathbb{T}))$. Since $\|u(T)\|_{H^s}$ is finite, we can repeat the argument above with initial data $u(T)$ to obtain $u \in C([T,T + T']; H^s(\mathbb{T}))$. We can iterate this process as far as $\|u(t)\|_{H^2} < \infty$. Therefore, we obtain $u \in C([0,T\gamma); H^s(\mathbb{T}))$. We omit the proof of the uniqueness since it follows from a standard argument. Let $0 < \delta < T\gamma/2$. We see from (2.9) and (2.10) that $u \in C([\delta, T\gamma); H^{s+1/4}(\mathbb{T}))$. The same argument as above with the initial data $u(\delta) \in H^{s+1/4}(\mathbb{T})$ shows that $u \in C([\delta + \delta/2, T\gamma); H^{s+1/2}(\mathbb{T}))$. Iterating this procedure, we obtain (2.8) since $\delta$ is arbitrary, which completes the proof. \(\square\)

3. ENERGY ESTIMATE

In this section, we obtain an a priori estimate of the solution of (1.1), which is important to have the time $T$ independent of $\gamma$.

**Proposition 3.1.** Let $s \geq s_0 > 5/2$, $\gamma \in (0,1)$, $\varphi \in H^s(\mathbb{T})$. Let $T\gamma > 0$ and let $u \in C([0,T\gamma), H^s(\mathbb{T})) \cap C((0,T\gamma); H^{s+3}(\mathbb{T}))$ be the solution to (2.6)–(2.7), both of which are obtained by Proposition 2.13. Then, there exist $T = T(s_0, \|\varphi\|_{H^{s_0}}) > 0$ and $C = C(s, s_0, \|\varphi\|_{H^{s_0}}) > 0$ such that

$$T\gamma \geq T, \quad \sup_{t \in [0,T]} E_s(u(t)) \leq CE_s(\varphi), \quad \frac{d}{dt} E_s(u(t)) \leq CE_s(u(t)) \quad (3.1)$$

on $[0,T]$, where $T$ (resp. $C$) is monotone decreasing (resp. increasing) with $\|\varphi\|_{H^{s_0}}$.

Before proving Proposition 3.1, we give the following lemma.

**Lemma 3.2.** Let $s \geq s_0 > 5/2$, $\gamma \in [0,1)$, $T > 0$, $u \in C([0,T], H^s(\mathbb{T})) \cap C((0,T]; H^{s+3}(\mathbb{T}))$ satisfy (2.4) on $[0,T] \times \mathbb{T}$ and $\sup_{t \in [0,T]} E_{s_0}(u(t)) \leq K$ for $K > 0$. Then, there exists $C = C(s, s_0, K) > 0$ such that

$$\frac{d}{dt} E_s(u(t)) \leq CE_s(u(t))$$

on $[0,T]$. 
Proof. First observe that
\[ \frac{d}{dt} \|u(t)\|^2 = 2\langle \partial_x^3 u - u^2\partial_x u - c_1 \partial_x(uH\partial_x u) - c_2 H\partial_x(u\partial_x u), u \rangle \]
\[ \lesssim \|u(t)\|^2_{H^1} \leq \|u(t)\|^2_{H^s}. \]

We can estimate the time derivative of \( \|u(t)\|^{4s+2} \) in a similar manner. Next we consider
\[ \frac{d}{dt} \|D^s u\|^2 = 2\langle D^s\partial_x^2 u, D^s u \rangle - 2\langle D^s(u^2\partial_x u), D^s u \rangle - 2c_1 \langle D^s\partial_x(uH\partial_x u), D^s u \rangle \]
\[ - 2c_2 \langle H D^s\partial_x(u\partial_x u), D^s u \rangle - 2\gamma \langle D^{s+5/2} u, D^s u \rangle \]
\[ =: R_1 + R_2 + R_3 + R_4 + R_5. \]

It is clear that \( R_1 = 0 \). We have
\[ |R_2| \leq 2|\langle [D^s, u^2]\partial_x u, D^s u \rangle| + 2|\langle u^2 D^s\partial_x u, D^s u \rangle| \lesssim \|u\|_{H^s} \]
by (i) of Lemma 2.4 and Lemma 2.7. Lemma 2.10 with \( v = 0 \) shows that
\[ |R_3 + 2c_1 s\langle \partial_x uH D^s\partial_x u, D^s u \rangle| + |R_4 + 2c_2 (s + 1)\langle \partial_x uH D^s\partial_x u, D^s u \rangle| \lesssim \|u\|_{H^s} \]
Finally, we have \( R_5 = -2\gamma \|D^{s+5/4} u\|^2 \). Therefore, we have
\[ \frac{d}{dt} \|D^s u\|^2 \leq C \|u\|_{H^s}^2 + 3\lambda(s) \int \partial_x u(\partial_x uD^s u)D^s u dx - 2\gamma \|D^{s+5/4} u\|^2, \quad (3.2) \]
where \( \lambda(s) \) is defined in Definition 2. Next we evaluate the correction term. We put
\[ \frac{d}{dt} \langle uH D^s u, D^{s-2}\partial_x u \rangle \]
\[ = \langle \partial_x uH D^s u, D^{s-2}\partial_x u \rangle + \langle uH D^s\partial_x u, D^{s-2}\partial_x u \rangle + \langle uH D^s u, D^{s-2}\partial_x \partial_t u \rangle \]
\[ =: R_6 + R_7 + R_8. \]
Moreover, we set
\[ R_6 = \langle \partial_x^3 uH D^s u, D^{s-2}\partial_x u \rangle - \langle u^2\partial_x uH D^s u, D^{s-2}\partial_x u \rangle \]
\[ - c_1 \langle uH\partial_x uH D^s u, D^{s-2}\partial_x u \rangle - c_2 \langle (H\partial_x(u\partial_x u))H D^s u, D^{s-2}\partial_x u \rangle \]
\[ - \gamma D^{s+2} uH D^s u, D^{s-2}\partial_x u \rangle =: R_{61} + R_{62} + R_{63} + R_{64} + R_{65}. \]
And we set
\[ R_7 = \langle uH D^s\partial_x^3 u, D^{s-2}\partial_x u \rangle - \langle uH D^s(u^2\partial_x u), D^{s-2}\partial_x u \rangle \]
\[ - c_1 \langle uH D^s\partial_x(uH\partial_x u), D^{s-2}\partial_x u \rangle + c_2 \langle uD^s\partial_x(u\partial_x u), D^{s-2}\partial_x u \rangle \]
\[ - \gamma \langle uH D^{s+5/2} u, D^{s-2}\partial_x u \rangle =: R_{71} + R_{72} + R_{73} + R_{74} + R_{75}. \]
Finally, we set
\[ R_8 = \langle u \mathcal{H} D^s u, D^{s-2} \partial_x^4 u \rangle - \langle u \mathcal{H} D^s u, D^{s-2} \partial_x (u^2 \partial_x u) \rangle + c_1 \langle u \mathcal{H} D^s u, D^s (u \mathcal{H} \partial_x u) \rangle + c_2 \langle u \mathcal{H} D^s u, \mathcal{H} D^s (u \partial_x u) \rangle - \gamma \langle u \mathcal{H} D^s u, D^{s+1/2} \partial_x u \rangle =: R_{s1} + R_{s2} + R_{s3} + R_{s4} + R_{s5}. \]

Lemma 2.8 shows that
\[ R_{s1} + R_{s2} + R_{s3} = 3 \langle \partial_x u \mathcal{H} D^s \partial_x u, D^{s-2} \partial_x^2 u \rangle = -3 \langle \partial_x u \mathcal{H} D^s \partial_x u, D^s u \rangle, \]
which cancels out the second term in the right hand side in (3.2) by multiplying \( \lambda(s) \). It is easy to see that \( |R_{s2}| + |R_{s3}| + |R_{s4}| \lesssim \|u\|_{H^s}^2 \). By (i) of Lemma 2.4, we have \( |R_{s5}| + |R_{s6}| \lesssim \|u\|_{H^s}^2 \). We see from (ii) of Lemma 2.9 that \( |R_{s7}| \lesssim \|u\|_{H^s}^2 \). Lemma 2.7 and (i) of Lemma 2.4 give \( |R_{s8}| \lesssim \|u\|_{H^s}^2 \). For \( R_{s9} \), it follows from (i) of Lemma 2.9 that \( |R_{s9}| \lesssim \|u\|_{H^s}^2 \). Finally, we estimate \( R_{s6}, R_{s7} \) and \( R_{s5} \). Lemma 2.1 implies that
\[ \|D^{s-2} \partial_x u\|_\infty \leq C \|D^{s-2} u\|^{1/4} \|D^s u\|^{3/4} \leq C \|u\|^{1-(4s-2)/(4s+5)} \|D^{s+5/4} u\|^{(4s-2)/(4s+5)}. \]

Then we have
\[ |R_{s5}| \leq \gamma \|D^{s-2} \partial_x u\| \|D^s u\| \|D^{s-2} \partial_x u\|_\infty \]
\[ \leq \gamma C \|u\|^{1+2/(4s+5)} \|D^{s+5/4} u\|^{2-2/(4s+5)} \leq C \|u\|^{4s+7} + \frac{\gamma^{1+1/4(s+1)}}{3} \|D^{s+5/4} u\|^2. \]

A similar argument yields
\[ |R_{s7}| + |R_{s8}| \leq C \|u\|^{4s+7} + C \|u\|^{2s+9/2} + \frac{2\gamma^{1+1/4(s+1)}}{3} \|D^{s+5/4} u\|^2. \]

Therefore, the fact that \( \gamma \in [0,1] \) shows that
\[ \frac{d}{dt} E_s(u(t)) \leq C \|u(t)\|_{H^s}^2 \leq C E_s(u(t)) \]
on \([0,T]\). Note that the implicit constant does not depend on \( \gamma \). This completes the proof. \( \square \)

Now, we are ready to prove Proposition 3.1

**Proof of Proposition 3.1.** Assume that the set \( F = \{ t \geq 0; E_{s_0}(u(t)) > 2 E_{s_0}(\varphi) \} \) is not empty. Set \( T^*_\gamma = \inf F \). Note that \( 0 < T^*_\gamma \leq T_\gamma \) and \( E_{s_0}(u(t)) \leq 2 E_{s_0}(\varphi) \) on \([0,T^*_\gamma]\). Assume that there exists \( t' \in [0,T^*_\gamma] \) such that \( E_{s_0}(u(t')) > 2 E_{s_0}(\varphi) \). This implies that \( t' \geq T^*_\gamma \) by the definition of \( T^*_\gamma \). Then we have \( t' = T^*_\gamma \). Thus,
sup_{t \in [0,T^*_\gamma]} E_{s_0}(u(t)) \leq C(\|\varphi\|_{H^{s_0}}) \text{ by (ii) of Lemma 2.11.} By Proposition 3.2, there exists \( C'_s = C(s,s_0,\|\varphi\|_{H^{s_0}}) \) such that
\[
\frac{d}{dt} E_s(u(t)) \leq C'_s E(u(t))
\]
on \([0,T^*_\gamma]\). The Gronwall inequality gives that
\[
E_s(u(t)) \leq E_s(\varphi) \exp(C'_s t)
\]
on \([0,T^*_\gamma]\). Here, we put \( T = \min\{ (2C'_s)^{-1}, T^*_\gamma \} \). Then (3.3) with \( s = s_0 \) shows that
\[
E_{s_0}(u(t)) \leq E_{s_0}(\varphi) \exp(2^{-1}) < 2E_{s_0}(\varphi),
\]
on \([0,T]\). By the definition of \( T^*_\gamma \) and the continuity of \( E_{s_0}(u(t)) \), we obtain
\[0 < T = (2C'_s)^{-1} < T^*_\gamma \leq T_\gamma. \]
If \( F \) is empty, then we have \( T^*_\gamma = T_\gamma = \infty \). In particular, we can take \( T = (2C'_s)^{-1} < \infty \), which concludes the proof.

4. UNIQUENESS, PERSISTENCE AND CONTINUOUS DEPENDENCE

In this section, we prove Theorem 1.1. We first show the existence of the solution of (1.1) by the limiting procedure. We also prove the uniqueness and the persistence property \( u \in C([0,T];H^s(\mathbb{T})) \). Then we estimate the difference between two solutions of (4.3)–(4.4) in \( H^s(\mathbb{T}) \), which is essential to show the continuous dependence.

Lemma 4.1. Let \( s \geq s_0 > 5/2, \gamma_j \in (0,1), T > 0, u_j \in C([0,T];H^s(\mathbb{T})) \cap C((0,T];H^{s+1}(\mathbb{T})) \) satisfy (2.9) with \( \gamma = \gamma_j \) on \([0,T] \times \mathbb{T}\) and \( \sup_{t \in [0,T]} \| u_j(t) \|_{H^{s_0}} \leq K \) for \( K > 0, j = 1,2 \). Then there exists \( C = C(K,s) \) such that
\[
\frac{d}{dt} \tilde{E}(u_1,u_2) \leq C(\tilde{E}(u_1,u_2) + \max\{\gamma_1^2,\gamma_2^2\})
\]
on \([0,T]\).

Proof. Set \( w := u_1 - u_2 \) so that \( w \) satisfies the following equation:
\[
\partial_t w - \partial^3_x w + \frac{1}{3} \partial_x \{(u_1^2 + u_1 u_2 + u_2^2)w\}
+ \frac{c_1}{2} \partial_x(wH\partial_x z) + \frac{c_1}{2} \partial_x(zH\partial_x w) + \frac{c_2}{2} H\partial_x(w\partial_x z) + \frac{c_2}{2} H\partial_x(z\partial_x w)
= -\gamma_1 D^{5/2} w - (\gamma_1 - \gamma_2) D^{5/2} u_2,
\]
where \( z = u_1 + u_2 \). By the presence of the operator \( \langle D \rangle^{-1} \), we can easily obtain
\[
\frac{d}{dt} \| \langle D \rangle^{-1} w \|^2 \lesssim \| w \|^2 + \max\{\gamma_1^2,\gamma_2^2\}.
\]
Next, we estimate the $L^2$-norm of $w$. Set

$$\frac{d}{dt}\|w\|^2 = 2\langle \partial_x^3 w, w \rangle - \frac{2}{3}\langle \partial_x \{ (u_1^2 + u_1 u_2 + u_2^2) w \}, w \rangle - c_1 \langle \partial_x (w H \partial_x z), w \rangle$$

$$- c_1 \langle \partial_x (z H \partial_x w), w \rangle - c_2 \langle H \partial_x (w \partial_x z), w \rangle - c_2 \langle \mathcal{H} \partial_x (z \partial_x w), w \rangle$$

$$- 2\gamma_1 \langle D^{5/2} w, w \rangle - 2(\gamma_1 - \gamma_2) \langle D^{5/2} u_2, w \rangle$$

$$=: R_9 + R_{10} + R_{11} + R_{12} + R_{13} + R_{14} + R_{15} + R_{16}.$$ 

Again, it is clear that $R_9 = 0$. By Lemma 2.7 we have $|R_{10}| + |R_{11}| \lesssim \|w\|^2$. Note that

$$\langle [\mathcal{H}, \partial_x z] \partial_x w, w \rangle + \langle [\mathcal{H}, z] \partial_x^2 w, w \rangle$$

$$= \langle \mathcal{H}(\partial_x z \partial_x w), w \rangle - \langle \partial_x z \mathcal{H} \partial_x w, w \rangle + \langle \mathcal{H}(z \partial_x^2 w), w \rangle - \langle z \mathcal{H} \partial_x^2 w, w \rangle$$

$$= \langle \partial_x (\partial_x z \mathcal{H} w), w \rangle - \langle \partial_x z \mathcal{H} \partial_x w, w \rangle - \langle \partial_x^2 (z \mathcal{H} w), w \rangle - \langle z \mathcal{H} \partial_x^2 w, w \rangle$$

$$= -2\langle \partial_x (z \mathcal{H} \partial_x w), w \rangle.$$ 

Then Lemma 2.5 shows that $|R_{12}| + |R_{14}| \lesssim \|w\|^2$. We can reduce $R_{13}$ to

$$R_{13} = -2c_2 \langle \partial_x u_1 \mathcal{H} \partial_x z, w \rangle - c_2 \langle \partial_x w \mathcal{H} \partial_x z, w \rangle$$

since $z = 2u_1 - w$. The last term in the right hand side can be bounded by $\lesssim \|w\|^2$ by using Lemma 2.7. Observe that $R_{15} = -\gamma_1 \|D^{5/4} w\|^2 \leq 0$ and that $|R_{16}| \lesssim \|w\|^2 + \max\{\gamma_1^2, \gamma_2^2\}$. Therefore, we have

$$\frac{d}{dt}\|w\|^2 \leq C\|w\|^2 + 3\lambda(0) \int T \partial_x u_1 (\mathcal{H} \partial_x w) w dx + \max\{\gamma_1^2, \gamma_2^2\}.$$ 

The correction term in $\tilde{E}$ cannot exactly cancel out the second term, but Lemma 2.2 shows that the difference is harmless. Set

$$\frac{d}{dt} \langle u_1 \langle D \rangle^{-1} w, w \rangle = \langle \partial_t u_1 \langle D \rangle^{-1} w, w \rangle + \langle u_1 \langle D \rangle^{-1} \partial_x w, w \rangle + \langle u_1 \langle D \rangle^{-1} w, \partial_x w \rangle$$

$$=: R_{17} + R_{18} + R_{19}.$$ 

Moreover, we set $R_{171} = \langle \partial_x^2 u_1 \langle D \rangle^{-1} w, w \rangle$ and set

$$R_{18} = \langle u_1 \langle D \rangle^{-1} \partial_x^2 w, w \rangle - \frac{1}{3} \langle u_1 \langle D \rangle^{-1} \partial_x \{ (u_1^2 + u_1 u_2 + u_2^2) w \}, w \rangle$$

$$- c_1 \langle u_1 \langle D \rangle^{-1} \partial_x (w H \partial_x z), w \rangle - c_1 \langle u_1 \langle D \rangle^{-1} \partial_x (z H \partial_x w), w \rangle$$

$$- c_2 \langle u_1 \langle D \rangle^{-1} H \partial_x (w \partial_x z), w \rangle - c_2 \langle u_1 \langle D \rangle^{-1} H \partial_x (z \partial_x w), w \rangle$$

$$- \gamma_1 \langle u_1 \langle D \rangle^{-1} D^{5/2} w, w \rangle - (\gamma_1 - \gamma_2) \langle u_1 \langle D \rangle^{-1} D^{5/2} u_2, w \rangle$$

$$=: R_{181} + R_{182} + R_{183} + R_{184} + R_{185} + R_{186} + R_{187} + R_{188}.$$
We set $R_{19k}$ for $k = 1, \ldots, 8$ in the same manner as above. Lemma 2.8 shows that
\[ R_{171} + R_{181} + R_{191} = -3\langle \partial_x u_1(D)^{-1} \partial^2_x w, w \rangle - 3\langle \partial^2_x u_1(D)^{-1} \partial_x w, w \rangle, \]
which together with Lemma 2.2 shows that $|R_{13} - \lambda(0)(R_{171} + R_{181} + R_{191})| \lesssim \|w\|^2$. It is easy to see that
\[ |\langle u^2 \partial_x u_1 + c_1 \partial_x (u_1 H \partial_x u_1) + c_2 H \partial_x (u_1 \partial_x u_1) + \gamma_1 D^{5/2} u_1 \rangle(D)^{-1} w, w \rangle| \lesssim \|w\|^2. \]
We have $|R_{182} + |R_{183}| + |R_{185}| + |R_{192}| + |R_{193}| + |R_{195}| \lesssim \|w\|^2$ because of the presence of the operator $\langle D \rangle^{-1}$. In order to handle $R_{184}, R_{186}, R_{194}$ and $R_{196}$, we see from Lemma 2.2 and (i) of Lemma 2.4 that
\[ |R_{196}| = \left| \frac{c_2}{2} \langle u_1(D)^{-1} w, H \partial^2_x (zw) \rangle + \frac{c_2}{2} \langle u_1(D)^{-1} w, H \partial_x (zw) \rangle \right| \lesssim |\langle u_1(D)^{-1} \partial_x w, (H \partial_x + D^{-1} \partial_x^2)(zw) \rangle| + |\langle u_1(D)^{-1} \partial_x w, (D)^{-1} \partial_x^2(zw) \rangle| + \|w\|^2 \lesssim |\langle u_1(D)^{-1} \partial_x w, (D)^{-1} \partial^2_x(zw) \rangle| + \|w\|^2 \lesssim \|w\|^2. \]
We can obtain $|R_{184}| + |R_{186}| + |R_{194}| \lesssim \|w\|^2$ from a similar argument. Finally, it is easy to see that $|R_{187} + |R_{188} + |R_{197}| + |R_{198}| \lesssim \|w\|^2 + \max\{\gamma_1^2, \gamma_2^2\}$. Summing these estimates above and applying (iii) of Lemma 2.11 we obtain (4.1), which concludes the proof.

Now we obtain the solution to \((1.1) - (1.2)\). Let $\varphi \in H^*(\mathbb{T})$ and let $\gamma_1, \gamma_2 \in (0, 1)$. Let $u_{\gamma_j}$ be the solution to \((2.6) - (2.7)\) with $\gamma = \gamma_j$ for $j = 1, 2$, obtained by Proposition 2.13. Note that $E(u_{\gamma_1}(0), u_{\gamma_2}(0)) = E(\varphi, \varphi) = 0$. Proposition 3.1 shows that there exists $T = T(s_0, \|\varphi\|_{H^s})$ such that \((3.1)\) holds. We see from (iii) of Lemma 2.11 and Lemma 4.1 that
\[ \sup_{t \in [0, T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|^2 \leq \sup_{t \in [0, T]} E(u_{\gamma_1}(t), u_{\gamma_2}(t)) \leq C \max\{\gamma_1^2, \gamma_2^2\} \to 0 \]
as $\gamma_1, \gamma_2 \to +0$. This implies that there exists $u \in C([0, T]; L^2(\mathbb{T}))$ such that
\[ u_{\gamma} \to u \quad \text{in} \quad C([0, T]; L^2(\mathbb{T})) \quad \text{as} \quad \gamma \to 0. \]
The above convergence can be verified in $C([0, T]; H^r(\mathbb{T}))$ for any $r < s$ by interpolating with $L^\infty([0, T]; H^s(\mathbb{T}))$. It is clear that $u$ satisfies \((1.1) - (1.2)\) on $[0, T]$.

For the proof of the following uniqueness result, see Thorem 6.22 in [3].

**Lemma 4.2** (Uniqueness). Let $\delta > 0$ and $\varepsilon > 0$, $u_j \in L^\infty([0, \delta]; H^{5/2+\varepsilon}(\mathbb{T}))$ satisfy \((1.1)\) on $[0, \delta]$ with $u_1(0) = u_2(0)$ and satisfy
\[ u_j \in C([0, \delta]; H^2(\mathbb{T})) \cap C^4([0, \delta]; H^{-1}(\mathbb{T})) \]
for $j = 1, 2$. Then $u_1 \equiv u_2$ on $[0, \delta]$. 

It remains to show the persistent property, i.e., \( u \in C([0, T]; H^s(T)) \) and the continuous dependence. In what follows, we employ the Bona-Smith approximation argument. We consider the following initial value problem:
\[
\partial_t u - \partial_x^2 u + u^2 \partial_x u + c_1 \partial_x (u \partial_x u) + c_2 \partial_x (u \partial_x u) = 0, \quad x \in T, \tag{4.3}
\]
\[
u(0, x) = J_\varphi(x), \tag{4.4}
\]
where \( J_\varphi \) is defined in Definition 3. Let \( s \geq s_0 > 5/2, \varphi \in H^s(T) \) and \( \epsilon > 0 \). Lemma 2.12 shows that \( J_\varphi \in H^\infty(T) \). Let \( u_\gamma \in C([0, T_\gamma); H^{s+3+\epsilon}(T)) \) be the solution (2.6) with the initial data \( J_\varphi \) obtained by Proposition 2.13. Lemma 2.12 and Proposition 3.1 imply that there exists \( T = T(s_0, \|\varphi\|_{H^{s_0}}) (\leq T' = T(s_0, \|\varphi\|_{H^{s_0}})) \) such that (3.1) holds for \( s + 3 + \epsilon \). Lemma 4.1 and the above argument show that there exists \( \tilde{u} \in C([0, T]; H^{s+3}(T)) \) such that \( \tilde{u} \) solves (4.3)–(4.4). Therefore, we have the following corollary:

**Corollary 4.3.** Let \( s \geq s_0 > 5/2, T > 0, u_j \in C([0, T]; H^{s+1}(T)) \) satisfy (4.3) on \([0, T] \times T \) and \( \sup_{t \in [0, T]} \|u_j(t)\|_{H^s} \leq K \) for \( K > 0, j = 1, 2 \). Then there exists \( C = C(K, s_0, s) \) such that
\[
\frac{d}{dt} \tilde{E}(u_1(t), u_2(t)) \leq C \tilde{E}(u_1(t), u_2(t)) \tag{4.5}
\]
on \([0, T] \).

**Proposition 4.4.** Let \( s \geq s_0 > 5/2, T > 0, u_j \in C([0, T]; H^{s+3}(T)) \) satisfy (4.3) on \([0, T] \times T \) and \( \sup_{t \in [0, T]} \|u_j(t)\|_{H^s} \leq K \) for \( K > 0, j = 1, 2 \). Then there exists \( C = C(s, s_0, K) \) such that
\[
\frac{d}{dt} E_s(u_1(t), u_2(t)) \leq C\left(\|u_1(t) - u_2(t)\|_{H^s}^2 + \|u_1(t) - u_2(t)\|_{H^{s_0-1}}^2 \|u_2\|_{H^{s+1}}^2 \right.
\]
\[
+ \left. \|u_1(t) - u_2(t)\|_{H^{s_0-2}}^2 \|u_2\|_{H^{s+2}}^2 \right) \tag{4.6}
\]
on \([0, T] \).

**Proof.** Set \( w = u_1 - u_2 \) and \( z = u_1 + u_2 \). It is easy to see that
\[
\frac{d}{dt} \|w\|^2 \lesssim \|w\|_{H^1}^2 \leq \|w\|_{H^s}^2.
\]
Set
\[
\frac{d}{dt} \|D^s w\|^2 = 2\langle D^s \partial_x^2 w, D^s w \rangle - 2\langle D^s (u_1^2 \partial_x w), D^s w \rangle - 2\langle D^s (zw \partial_x u_2), D^s w \rangle
\]
\[
- 2c_1 \langle D^s \partial_x (u_1 \partial_x u_1 - u_2 \partial_x u_2), D^s w \rangle
\]
\[
- 2c_2 \langle D^s \partial_x (u_1 \partial_x u_1 - u_2 \partial_x u_2), D^s w \rangle =: R_1 + R_2 + R_3 + R_4 + R_5.
\]
It is easy to see that $R_1 = 0$ and $|R_2| \lesssim \|w\|_{H^s}^2$ by (i) of Lemma 2.4. For $R_3$, we have $|R_3| \lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s-1}}^2 \|u_2\|_{H^{s+1}}^2$. Lemma 2.10 shows that

$$|R_4 + R_5 - 3\lambda(s) \langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^s w \rangle| \lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s-1}}^2 \|u_2\|_{H^{s+1}}^2 + \|w\|_{H^{s-1}} \|u_2\|_{H^{s+1}}^2.$$ 

Therefore, the time derivative of $\|D^s w\|^2$ yields

$$\frac{d}{dt} \|D^s w\|^2 \leq C \|w\|_{H^s}^2 + C \|w\|_{H^{s-1}}^2 \|u_2\|_{H^{s+1}}^2 + C \|w\|_{H^{s-1}} \|u_2\|_{H^{s+1}}^2 + 3\lambda(s) \int_T \partial_x u_1 (\mathcal{H} D^s \partial_x w) D^s w dx.$$ 

(4.7)

Next, we evaluate the time derivative of the correction term. Lemma 2.8 with $f = u_1$, $g = \mathcal{H} D^s w$ and $h = D^{s-2} \partial_x w$ shows that

$$\langle \partial_x^2 u_1 \mathcal{H} D^s w, D^{s-2} \partial_x w \rangle + \langle u_1 \mathcal{H} D^s \partial_x^2 w, D^{s-2} \partial_x w \rangle + \langle u_1 \mathcal{H} D^s w, D^{s-2} \partial_x^4 w \rangle = 3 \langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^{s-2} \partial_x w \rangle - 3 \langle \partial_x u_1 \mathcal{H} D^s \partial_x w, D^s w \rangle.$$ 

Multiplying by $\lambda(s)$, we can cancel out the last term in the right hand side in (4.7). On the other hand, it is easy to see that

$$\langle \partial_x u_1 - \partial_x^3 u_1 \rangle \mathcal{H} D^s w, D^{s-2} \partial_x w \rangle \lesssim \|w\|_{H^s}^2.$$ 

We set

$$\langle u_1 \mathcal{H} D^s (\partial_x w - \partial_x^3 w), D^{s-2} \partial_x w \rangle$$

$$= - \frac{1}{3} \langle u_1 \mathcal{H} D^s \partial_x \{ (u_1^2 + u_1 u_2 + u_2^2) w \}, D^{s-2} \partial_x w \rangle$$

$$- c_1 \langle u_1 \mathcal{H} D^s \partial_x (u_1 \mathcal{H} \partial_x w), D^{s-2} \partial_x w \rangle + c_2 \langle u_1 D^s \partial_x (u_1 \partial_x w), D^{s-2} \partial_x w \rangle$$

$$- c_1 \langle u_1 \mathcal{H} D^s \partial_x (w \mathcal{H} \partial_x u_2), D^{s-2} \partial_x w \rangle + c_2 \langle u_1 D^s \partial_x (w \partial_x u_2), D^{s-2} \partial_x w \rangle$$

$$= : R_9 + R_{10} + R_{11} + R_{12} + R_{13}$$

and

$$\langle u_1 \mathcal{H} D^s w, D^{s-2} \partial_x (\partial_x w - \partial_x^3 w) \rangle$$

$$= \frac{1}{3} \langle u_1 \mathcal{H} D^s w, D^s \{ (u_1^2 + u_1 u_2 + u_2^2) w \} \rangle + c_1 \langle u_1 \mathcal{H} D^s w, D^s (u_1 \mathcal{H} \partial_x w) \rangle$$

$$+ c_2 \langle u_1 \mathcal{H} D^s w, \mathcal{H} D^s (u_1 \partial_x w) \rangle + c_1 \langle u_1 \mathcal{H} D^s w, D^s (w \mathcal{H} \partial_x u_2) \rangle$$

$$+ c_2 \langle u_1 \mathcal{H} D^s w, \mathcal{H} D^s (w \partial_x u_2) \rangle = : R_{14} + R_{15} + R_{16} + R_{17} + R_{18}.$$ 

By (i) of Lemma 2.4, we have $|R_9| + |R_{14}| \lesssim \|w\|_{H^s}^2$. We see from (ii) of Lemma 2.9 that $|R_{10}| \lesssim \|w\|_{H^s}^2$. We also have $|R_{16}| \lesssim \|w\|_{H^s}^2$ by (i) of Lemma 2.9. Similarly, we can obtain $|R_{11}| + |R_{15}| \lesssim \|w\|_{H^s}^2$. On the other hand, by (i) of Lemma 2.4, we have
\[ |R_{12}| + |R_{13}| + |R_{17}| + |R_{18}| \lesssim \|w\|_{H^s}^2 + \|w\|_{H^{s-2}}^2 \|u_2\|_{H^{s+1}}^2. \]

Summing these estimates above, we obtain (4.6) on \([0, T]\), which concludes the proof. \(\square\)

Now, we can show the persistent property and the continuous dependence.

**Proof of Theorem 1.1.** In what follows, without loss of generality, we may assume that \(s_0\) is strictly smaller than \(s\) since the assumption \(\|\varphi\|_{H^{s_0}} \leq K\) is weaker than \(\|\varphi\|_{H^{s_0}} \leq K\) when \(s_0 < s_0'\). First we prove the persistence property. Let \(0 < \gamma_1 < \gamma_2 < 1\). Let \(u_{\gamma_j} \in C([0, T]; H^{s+3}(\mathbb{T}))\) be the solution to (1.1)–(1.4) with the initial data \(J_{\gamma}\varphi\) for \(\varphi \in H^s(\mathbb{T})\) and \(j = 1, 2\). Corollary 4.3 with the Gronwall inequality shows that

\[
\sup_{t \in [0, T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|^2 \leq C \tilde{E}(u_{\gamma_1}(0), u_{\gamma_2}(0)) \leq C \|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|^2 \leq C\gamma_2^{2s}
\]

since \(\gamma_1 < \gamma_2\). This together with the interpolation implies that

\[
\sup_{t \in [0, T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|_{H^{s-\alpha}}^2 \leq C\gamma_2^{2(s-\alpha)}
\]

for any \(0 \leq \alpha < s\). On the other hand, Lemma 2.12 and 3.2 show that

\[
\sup_{t \in [0, t]} \|u_{\gamma_2}(t)\|_{H^{s+3}}^2 \leq C \|J_{\gamma_2}\varphi\|_{H^{s+3}}^2 \leq C\gamma_2^{-2\alpha} \|\varphi\|_{H^{s}}^2
\]

for \(\alpha \geq 0\). This together with the Gronwall inequality and Proposition 4.4 implies that

\[
\sup_{t \in [0, T]} \|u_{\gamma_1}(t) - u_{\gamma_2}(t)\|_{H^{s}}^2 \lesssim \|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|_{H^{s}}^2 + \gamma_2^{2(s-s_0)} \to 0
\]

as \(\gamma_2, \gamma_1 \to 0\) since \(\|J_{\gamma_1}\varphi - J_{\gamma_2}\varphi\|_{H^{s}} \to 0\) as \(\gamma_1, \gamma_2 \to 0\). Then, there exists \(\tilde{u} \in C([0, T]; H^s(\mathbb{T}))\) such that

\[ u_{\gamma} \to \tilde{u} \text{ in } C([0, T]; H^s(\mathbb{T})) \text{ as } \gamma \to 0. \]

It is clear that the function \(\tilde{u}\) coincides with our solution \(u \in C([0, T]; H^r(\mathbb{T}))\) for \(r < s\) to (1.1)–(1.2), which shows the persistence property.

Finally, we prove the continuous dependence, which is the only thing left to prove. We will claim that

\[
\forall \varphi \in H^s(\mathbb{T}), \forall \epsilon > 0, \exists \delta > 0, \forall \psi \in H^s(\mathbb{T}) : \quad \left[ \|\varphi - \psi\|_{H^s} < \delta \Rightarrow \sup_{t \in [0, T/2]} \|u(t) - v(t)\|_{H^s} < \epsilon \right], \tag{4.8}
\]

where \(u, v\) represent the solution to (1.1) with initial data \(\varphi, \psi \in H^s(\mathbb{T})\), respectively, which are obtained by the above argument. In (4.8) we take the interval \([0, T/2]\) with \(T\) as defined by Proposition 3.1 to guarantee that if \(\|\varphi - \psi\|_{H^s} < \delta\), then the
solution \( v(t) \) is defined in the time interval \([0, T/2]\). Fix \( \varphi \in H^s(\mathbb{T}) \) and \( \epsilon > 0 \). Let \( 0 < \gamma_1 < \gamma_2 < 1 \). Assume that \( \|\varphi - \psi\|_{H^r} < \delta \), where \( \delta > 0 \) will be chosen later. Note that by the triangle inequality we have

\[
\sup_{t \in [0,T/2]} \|u(t) - v(t)\|_{H^s} 
\leq \sup_{t \in [0,T/2]} \|u(t) - u^{\gamma_2}(t)\|_{H^s} + \sup_{t \in [0,T/2]} \|u^{\gamma_2}(t) - v^{\gamma_1}(t)\|_{H^s} + \sup_{t \in [0,T/2]} \|v^{\gamma_1}(t) - v(t)\|_{H^s},
\]

(4.9)

where \( u^{\gamma_2} \) and \( v^{\gamma_1} \) represent the solution to the IVP (1.1) with the initial data \( J_{\gamma_2} \varphi \) and \( J_{\gamma_1} \psi \), respectively. First we handle the second term in the right hand side in (4.9). Again, the triangle inequality shows that

\[
\|J_{\gamma_2} \varphi - J_{\gamma_1} \psi\|_{H^r} \leq \|J_{\gamma_2} \varphi - \varphi\|_{H^r} + \|\varphi - \psi\|_{H^r} + \|\psi - J_{\gamma_1} \psi\|_{H^r}
\]

for \( r \leq s \). Proposition 4.4 with \( u_1 = v^{\gamma_1} \) and \( u_2 = u^{\gamma_2} \) gives that

\[
\sup_{t \in [0,T/2]} \|u^{\gamma_2}(t) - v^{\gamma_1}(t)\|_{H^s} 
\leq C_{J_{\gamma_2} \varphi} + C\|\psi - J_{\gamma_1} \psi\|_{H^s} + C\gamma_2^{s-s_0} + C\gamma_2^{-1}\delta^{1+1/s-s_0/s} + C\gamma_2^{-1}\psi - J_{\gamma_1} \psi\|^{1+1/s-s_0/s} + C\gamma_2^{s-s_0} + C\gamma_2^{-2}\delta^{1+2/s-s_0/s} + C\gamma_2^{-2}\|\psi - J_{\gamma_1} \psi\|^{1+2/s-s_0/s}.
\]

Therefore, we choose \( \gamma_2 > 0 \) so that

\[
\sup_{t \in [0,T/2]} \|u(t) - u^{\gamma_2}(t)\|_{H^s} + C\|J_{\gamma_2} \varphi - \varphi\|_{H^s} + C\gamma_2^{s-s_0} < \frac{\epsilon}{3}.
\]

Then we take \( \delta > 0 \) such that

\[
C(\delta + \gamma_2^{-1}\delta^{1+1/s-s_0/s} + \gamma_2^{-2}\delta^{1+2/s-s_0/s}) < \frac{\epsilon}{3}
\]

and finally for each \( \psi \in H^s(\mathbb{T}) \) satisfying \( \|\varphi - \psi\|_{H^s} < \delta \) we take \( \gamma_1 \in (0, \gamma_2) \) such that

\[
\sup_{t \in [0,T/2]} \|v^{\gamma_1}(t) - v(t)\|_{H^s} + C\|\psi - J_{\gamma_1} \psi\|_{H^s} 
\leq C\gamma_2^{-1}\|\psi - J_{\gamma_1} \psi\|^{1+1/s-s_0/s} + C\gamma_2^{-2}\|\psi - J_{\gamma_1} \psi\|^{1+2/s-s_0/s} < \frac{\epsilon}{3},
\]

which completes the proof of (4.8).

\[\square\]

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