HEEKAARD DIAGRAMS OF CERTAIN 3-MANIFOLDS

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ABSTRACT

This paper is an exercise in topology of 3-manifolds. We present three examples of genus 2 Heegaard splittings of manifolds with different geometries diagramatically.

1 Introduction

1.1 Brief history of classification of 3-manifolds

The history of manifolds began in 1851, when Bernhard Riemann used the word for the first time. It was the beginning of low-dimensional topology as we know it today. Mathematicians soon started working on the classification of manifolds. Trivial classification of 0-dimensional and 1-dimensional manifolds, and then the uniformization theorem for 2-dimensional manifolds, painted a promising picture for classifying manifolds of higher dimensions. However, the simplest question in classification of 3-manifolds, the Poincaré conjecture, was only settled after almost eighty years of work. Of course there was much progress in general theory of 3-manifolds, to just mention the equivalence of smooth, piecewise linear and topological categories (which we will encounter below via the Moise theorem). This allowed for the development of new methods and new paths of investigation.

1.1.1 Prime decomposition, JSJ decomposition and Geometrization Conjecture

One of the first approaches to the classification of 3-manifolds was to cut them into smaller and hopefully simpler pieces. Around 1930 Hellmuth Kneser proved the existence of the prime decomposition (Theorem 1.5 in [10]), which was based on cutting 3-manifolds along the spheres. He showed that every compact orientable 3-manifold factors as a connected sum of prime manifolds, i.e. that cannot be expressed as a connected sum where none of the summands is a 3-sphere. This theorem also says that this decomposition is unique up to insertion or deletion of \( S^3 \) summands, which may give hope to reduce the problem of classification just to classification of the prime manifolds. However, the prime manifolds remained elusive and despite much insight, this venue of investigation ultimately proved inferior to (and was only settled by) the geometrization methods described below.

In 1979 William Jaco, cooperating with Peter Shalen, and Klaus Johannson, who worked independently, came up with a way to cut an irreducible ([10], page 4) orientable closed 3-manifold along tori to obtain either Seifert-fibered ([10], page 13) or atoroidal ([10], page 12) pieces. It is called the JSJ decomposition or sometimes the toral decomposition and was an important shift in the outlook on 3-manifolds. It paved the way for geometrization, as an additional (geometric) structure was needed to fully understand the atoroidal pieces (Seifert manifolds being already classified in the 1930’s, cf. [10], Theorem 2.3).

Three years later, in 1982, William Thurston published the Geometrization Conjecture (described in [1]), which built upon his work on 3-manifolds. He had already proved this conjecture for Haken ([21], page 71) manifolds and recognized the most difficult cases. His ideas are similar to the JSJ decomposition – they are based on cutting oriented prime closed 3-manifold along tori in such a way that the interior of every obtained piece has a geometric structure with

1 We will not indulge in up to which category’s isomorphisms the classification is considered – mainly because it would obfuscate the exposition, but also because it is a well known fact that in dimension 3 the distinction is irrelevant.
finite volume, modeled on one of the eight geometries possible in the dimension 3 ([1], Chapter 2). The Geometrization Conjecture was proved in the beginning of the 21st century by Grigorij Perelman by means of the celebrated Ricci flow and it is now considered a full classification of 3-manifolds.

1.1.2 Heegaard decomposition

A completely different approach was proposed by Poul Heegaard: splitting a 3-manifold not into many pieces along tori or spheres, but into two pieces glued along a high genus surface. While the theory was introduced 1898, it did not gain much attention. There was some response in the 1960s, for example by Wolfgang Haken ([8]), although it was not until 1987 that it was revived in the work of Andrew Casson and Cameron Gordon ([3]).

As stated, the Heegaard splitting is the decomposition of a compact oriented 3-manifold into two handlebodies with the same genus. We will interchangeably decompose a manifold into pieces or work in reverse, gluing two handlebodies with the same genus and study the obtained manifold. For examining the Heegaard decomposition of a specific 3-manifold it is crucial to know the genus of these two handlebodies and the homeomorphism that defines how exactly one should glue the handlebodies together.

The fundamental theorem about Heegaard splittings is that every closed, orientable 3-manifold admits one. While the theorem gives hope to classify 3-manifolds this way, there is a (still wide open) problem of finding the minimal Heegaard genus, and classifying all decompositions of a fixed genus of a fixed manifold. There are well understood families of 3-manifolds with completely classified Heegaard splittings, like the three-sphere or lens spaces ([23], page 894). For the reference later on, we note here the following.

Remark. If a manifold has a Heegaard splitting of genus 0 or 1, then it is either a sphere, a lens space or \( S^2 \times S^1 \).

Using that Remark, we know that the decompositions we present below are indeed minimal.

Recently, the classification was obtained for the Heegaard splittings for some 3-manifolds with the specific Thurston geometry (for example it is known for the manifolds with Sol geometry ([5]) or hyperbolic manifolds that are two-bridge knot complements ([13])). There is also a wide array of computer programs that can help to deal with Heegaard splittings, such as SnapPy, Regina, or Heegaard ([20]). However, there is still a lot of manifolds with unknown Heegaard splitting or with incomplete understanding of all possible splittings, and they do not fit well with the geometric framework now prevalent in low dimensional topology (as discussed below). Mostly because the picture is so incomplete, it is also rather difficult to find sources for splittings of specific 3-manifolds.

1.2 Three 3-manifolds

To exhibit non-trivial examples of Heegaard decompositions, we chose three remarkable 3-manifolds. They are chosen to have different Thurston’s geometries, but will turn out to have the same Heegaard genus. The manifolds are also noteworthy for the reasons described below.

1.2.1 Poincaré sphere

Henri Poincaré defined the first topological invariants in his series of papers ([18]). He conjectured that if a 3-manifold has the homology groups of the 3-sphere, then it is homeomorphic to the 3-sphere. He himself found a counterexample in the last paper from this series: a manifold which homologically is a 3-sphere but has a non-trivial fundamental group. This manifold is called the Poincaré homology sphere and led Poincaré to pose the correct version of the Poincaré conjecture that shaped much of the 20th century mathematics. The article written by Klaus Volkert ([24]) goes further into the history and various constructions of this manifold.

The Poincaré sphere has the spherical Thurston geometry and is Seifert fibered.

1.2.2 Heisenberg manifold

In 1976 Thurston used a well known complex surface to exhibit a compact symplectic but not Kähler 4-manifold. It was known to Kodaira (and present in his classification of complex surfaces), thus it is now known as Kodaira-Thurston manifold, and can be presented either as a torus bundle over a torus, or a (trivial) circle bundle over a 3-manifold. This 3-manifold is a quotient of a group of upper triangular \( 3 \times 3 \)-matrices (with 1s on the diagonal). We will present it as our second example and call it the Heisenberg manifold.

The Heisenberg manifold has the Nil Thurston geometry and is Seifert fibered.
1.2.3 Weeks manifold

A hyperbolic 3-manifold is a closed, connected manifold with the complete Riemannian metric of constant negative curvature \(-1\). Equivalently (by the Killing-Hopf theorem, [26], Corollary 2.4.10), every such manifold can be written as \(\mathbb{H}^3/\Gamma\), where \(\Gamma\) is a discrete group of isometries of the hyperbolic 3-space. Orientable hyperbolic manifolds \(\mathbb{H}^3/\Gamma\) have a finite hyperbolic volume, which is a topological invariant (by the Mostow rigidity theorem, [21], Theorem 5.7.1 and 5.7.2). The Thurston-Jørgensen theorem ([21], Theorem 5.12.1) says that the volumes of hyperbolic 3-manifolds constitute a well-ordered subset of type \(\omega^2\) in the real line, hence there is the smallest hyperbolic 3-manifold. In 2007 it was proved that the Weeks manifold, obtained by the (5, 2) and (5, 1) Dehn surgery on the Whitehead link, is the smallest volume example ([7], where also the surgery in question is explained).

The Weeks manifold has (by definition) the hyperbolic Thurston geometry, thus is not Seifert fibered ([1], Proposition 31).

1.3 Organization of the paper

This paper focuses on presenting the Heegaard decompositions of the minimal genus for the three manifolds above, without delving into all technical details surrounding the vast subject of 3-dimensional topology. As above, we use very powerful and complicated theorems as tools, referring the Reader to the sources for proofs or even statements, as merely citing the relevant definitions for each 3-manifolds methods of decomposition would make this thesis swell uncontrollably. We chose not to do this, wanting to exhibit Heegaard diagrams of three manifolds that would each require a separate geometric treatment if the text was to be self-contained. This is the most obvious in Section 5 as the Weeks manifold is given somewhat implicitly. However, we spend time to describe how the Heegaard splittings arise (in Section 5). We hope the Reader will forgive us for the size of the Bibliography section and the practice of citing separate pages of some books.

In Sections 3, 4 and 5 we give a model of each manifold in question, describe its geometry and discuss the minimal Heegaard splitting, sometimes preceding with discussion of non-minimal (but more readily visible) cases. We also discuss the a priori estimates on Heegaard genus in each case.

We must note that the pictures and decompositions provided here are somewhat hard to find in literature.

2 Heegaard decomposition and Heegaard diagram of a 3-manifold

In this section we define precisely how Heegaard splittings were expected to classify 3-manifolds. The main (however oversimplified) corollary of the discussion in the present paper (as certified by the history of the field in 20th and 21st century) should be that the Heegaard decomposition may be not well-suited for this task, since the three very different manifolds presented in this paper have decompositions of the same genus. The in-depth discussion of this point would amount to the investigation of the homeomorphisms groups of the surfaces, a notoriously hard subject.

In the first two subsections we will follow the reasoning from [12], which is based on [19]. Most of the proofs will be presented only as sketches, all details can be found in this book.

2.1 Handlebodies and Heegaard splittings

Firstly, we define a handlebody.

**Definition 2.1** (Definition 1.7 in [12]). Let \(B_1, \ldots, B_n\) be a collection of closed 3-balls and let \(D_1, \ldots, D_m, D'_1, \ldots, D'_m\) be a collection of pairwise disjoint 2-discs in \(\bigcup B_i\), where \(i = 1, \ldots, n\). For each \(i \leq m\), let \(\phi_i: D_i \to D'_i\) be an orientation reversing homeomorphism. Let \(H\) be a result of gluing along \(\phi_1\), then along \(\phi_2\), and so on. After the final gluing, if \(H\) is connected, then \(H\) is a handlebody.

Obviously, the definition doesn’t depend on the order of gluing and on the relative position of the discs on the boundary of each ball.

**Remark.** If a handlebody is connected, then \(m \geq n - 1\).

An example of a handlebody is shown in the Figure 1.
Definition 2.2. The genus of a handlebody $H$ is the genus of its boundary $\partial H$.

Remark. A handlebody with genus $g$ is homeomorphic to a solid torus with the same genus $g$. We note that both the 3-manifold with boundary $H$ and its boundary $\partial H$ are oriented due to the homeomorphisms in Definition 2.1 reversing the orientation.

We can compute the genus of a handlebody using the Euler characteristic. With simple computations (which can be found on page 7 of [12]) we get that the genus equals the difference between number of pairs of discs $m$ and number of balls $n$ incremented by one, so $m - n + 1$.

In the next subsection, it will be convenient to use the following lemma (again, the full proof can be found in [12]), but first let’s recall the relevant definition.

Definition 2.3 (page 30 in [11]). We call $A$ a neat submanifold of $M$ if $\partial A = A \cap \partial M$ and $A$ is covered by charts $(\phi, U)$ into $V = \mathbb{R}^{\text{dim}M}$ in the interior or $V = \mathbb{R}_{>0} \times \mathbb{R}^{\text{dim}M-1}$ on the boundary such that $A \cap U = \phi^{-1}(\mathbb{R}^{\text{dim}A})$. A neat embedding is one whose image is a neat submanifold.

Lemma 2.1 (Lemma 1.13 in [12], reverse construction of a handlebody). Let $M$ be a manifold with boundary and let $D_1, \ldots, D_m$ be a collection of disks neatly embedded in $M$. If $N$ is the interior of $M$ and $N \setminus \bigcup D_i$ is a collection of $n$ open balls, then $M$ is a handlebody with genus $m - n + 1$.

Now we define a Heegaard splitting which is based on gluing two handlebodies of the same genus.

Definition 2.4 (Definition 1.14 in [12]). A Heegaard splitting of a 3-manifold $M$ is an ordered triple $(\Sigma, H_1, H_2)$, where $\Sigma$ is a closed surface embedded in $M$ and $H_1$ and $H_2$ are handlebodies embedded in $M$ such that

$$\partial H_1 = \Sigma = \partial H_2 = H_1 \cap H_2 \text{ and } H_1 \cup H_2 = M.$$ 

The surface $\Sigma$ is called a Heegaard surface.

Definition 2.5. The Heegaard genus of $M$ is the smallest possible genus of a Heegaard surface of a Heegaard decomposition of $M$.

2.2 Existence of decomposition

The most important theorem about Heegaard splittings is the following:

Theorem 2.2 (Theorem 1.15 in [12]). Every compact, closed, connected, orientable 3-manifold has a Heegaard splitting.

Because of this Theorem, Definition 2.5 is well-posed. To prove Theorem 2.2, we need to use the result of Moise.

Theorem 2.3 (Moise, [16], 1952). Every compact 3-manifold is homeomorphic to a 3-dimensional simplicial complex ([12], Definition 1.1).

We will now prove Theorem 2.2 by constructing a Heegaard splitting from a triangulation. Firstly, we need the following lemma that allows us to see the connection between the closure of a regular neighborhood (an appropriate analogue of a tubular neighborhood in [11], page 109) of 1-skeleton and a handlebody.

Lemma 2.4 (Lemma 1.16 in [12]). Let $M$ be an orientable 3-manifold and let $K$ be a (piecewise linear) graph embedded in $M$. If $K$ has $n$ vertices and $m$ edges then the closure of a regular neighborhood of $K$ is homeomorphic to a handlebody with genus $m - n + 1$. 4
**Sketch of the proof.** We can take $K$ as an image in $M$ of 1-skeleton of a triangulation of $M$. Taking a regular neighborhood of this 1-skeleton and discs along midpoints of every edge, we get precisely the data we need to use the reverse construction of a handlebody (Lemma 2.1), and thus we obtain that $N$ is a $m - n + 1$ genus handlebody.

Figure 2: [12], page 11. A regular neighborhood of a graph embedded in a 3-manifold is a handlebody. The number of balls in the handlebody responds to the number of vertices in the graph, the number of discs to the number of edges.

**Sketch of the proof of Theorem 2.2.** Let $M$ be a compact, closed, connected, orientable 3-manifold. By Theorem 2.3, there is a simplicial complex $K$ homeomorphic to $M$. Let $N$ be a sufficiently small regular neighborhood of the 1-skeleton of $K$ (Figure 3). Let $H_1$ be the closure of $N$. We know by Lemma 2.4 that $H_1$ is homeomorphic to a handlebody.

Then let $H_2$ be the complement of $N$. It is closed, because $N$ is open. Let $\Sigma = \partial H_1 = \partial H_2$. To show that the triple $(\Sigma, H_1, H_2)$ is a Heegaard splitting, we have to show that $H_2$ is a handlebody.

To find a collection of discs properly embedded in $H_2$ such that their complement is a collection of open balls, we need to observe that every tetrahedron of the simplicial complex is an embedded ball with a boundary. When we remove from this tetrahedron a sufficiently small regular neighborhood of the 1-skeleton, we will get a manifold still homeomorphic to a ball. The intersection of the original boundary of the tetrahedron with this complement of the small neighborhood of the 1-skeleton has four connected components. Each connected component is homeomorphic to a disc, so in this way we constructed the discs between the balls from the Reverse Construction of a Handlebody (Lemma 2.1), and that finishes the proof.

Figure 3: [12], page 12. A triangulation of a manifold $M$ suggests a Heegaard splitting of $M$. The Heegaard surface is the boundary of a regular neighborhood of the 1-skeleton.

### 2.3 Heegaard diagrams

In order to obtain a manifold that admits a prescribed Heegaard decomposition, we need to describe the gluing of the handlebodies. Let $H_1$ and $H_2$ be two handlebodies with the same genus, $H_1 \sqcup H_2$ be their disjoint union, let $h: \partial H_1 \to \partial H_2$ be a homeomorphism, and let $M$ be a manifold obtained by gluing $H_1$ and $H_2$ along $h$, $M = H_1 \cup_h H_2$. Then of course $\langle i_1(\partial H_1) = i_2(\partial H_2), H_1, H_2 \rangle$ is a Heegaard decomposition of $M$, where $i_1$ and $i_2$ are respective inclusions in $M$.  

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Conversely, if we start with a manifold $M$ with a Heegaard splitting $(\Sigma, H_1, H_2)$, then the composition of the inclusion maps $\partial H_1 \to \Sigma \to \partial H_2$ is our gluing map $h: \partial H_1 \to \partial H_2$, such that $M$ is homeomorphic to $H_1 \cup_h H_2$. It remains to describe $h$ diagrammatically.

To that end, if we take the collection of the discs $D_1, \ldots, D_m \in H_1$ from the first handlebody as in the definition 2.1 (the discs $D_1, \ldots, D_m, D'_1, \ldots, D'_m$ after gluing them by the homeomorphism $\phi$), then the image of their boundaries by the homeomorphism $h$ is a collection of loops in the second handlebody, $h(\partial D_i) = l_i \in \partial H_2$. Conversely, any such collection of $g$ loops defines a homeomorphism of a genus $g$ surface, which allows us to give the following definition, uniquely describing a Heegaard decomposition.

**Definition 2.6** (Definition 2.25 in [12]). A **Heegaard diagram** is an ordered pair $(H, \{l_1, \ldots, l_g\})$, where $H$ is a handlebody from a Heegaard splitting, $g$ is a genus of $H$ and $\{l_i\}$ is a collection of disjoint, embedded, simple closed curves in $H$, such that $H \setminus \bigcup\{l_i\}$ is a collection of punctured spheres.

The simplest way to describe the Heegaard splitting as gluing two handlebodies together is to draw the Heegaard diagram. We can take a closer look into the example of the Poincaré sphere. The picture in Figure 4 shows a sphere with two handles: identifying $+A$ with $-A$ yields the first handle and $+B$ with $-B$ yields the second handle. Identified circles $A$ and $B$ are the glued discs $D_1, D_2$ from 2.1, the definition of a handlebody.

![Figure 4](image1.png)

**Figure 4**: [18], page 494. Poincaré’s Heegaard diagram for the Poincaré’s homology sphere.

![Figure 5](image2.png)

**Figure 5**: Streamlined Heegaard diagram for the Poincaré’s homology sphere. Meaning of the relations on the arrows is explained in Section 2.4.

![Figure 6](image3.png)

**Figure 6**: Toral Heegaard diagram for the Poincaré’s homology sphere

Two kinds of curves with arrows in the picture represent the way of attaching two discs from the second handlebody. These curves can be also depicted in other kinds of diagrams. Figure 5 portraits the same Heegaard diagram as the one
from the Poincaré’s paper in Figure 4. There is no substantial difference between these pictures but the second one can be clearer and is the most popular way of drawing Heegaard diagrams.

The third diagram in Figure 6 is the most cluttered, however it doesn’t demand imagining gluing the discs because we already have it depicted as a torus. This kind of diagram is not common, because it can be hard to present more complicated relations in a clear way. For example, Heegaard and Dehn published a paper in 1907 about the Poincaré sphere with a toral diagram (Figure 7). The picture is incorrect because it presents a different fundamental group, of the form \( \langle a, b | a^3b^2 = a^2b^{-1} = 1 \rangle \). Dehn noticed their mistake and published an erratum soon afterwards.

**2.4 Heegaard splittings and fundamental groups**

To see the connection between Heegaard splittings and fundamental groups, we need to observe that a handlebody is homotopy equivalent to its core which is a wedge sum of 1-spheres \( \bigvee S^1 \). With this fact, we can see that the fundamental group of a handlebody is the free group \( \langle x_1, \ldots, x_g \rangle \) on \( g \) generators, where we may think of \( x_1, \ldots, x_g \) as counted by discs in the definition of a handlebody (i.e., \( A \) and \( B \) in the figure above). The loops from the Heegaard diagram give the relations \( r_1, \ldots, r_g \) (full and dashed in the streamlined version or orange and blue in the toral version): since the generators are counted by the discs, each segment travelling from one circle in the picture to another adds an appropriate letter to the relation. Then it can be shown using the Seifert-van Kampen theorem (Theorem 1.20 in [9]) that the fundamental group of the manifold from the Heegaard diagram is \( \langle x_1, \ldots, x_g | r_1, \ldots, r_g \rangle \) (the explicit proof can be found in [12], Lemma 3.10).

We know from this that rank of the fundamental group of a manifold is less or equal then the Heegaard genus of the manifold.

**Corollary 2.4.1.** \( \text{rank}(\pi_1(M)) \leq g(M) \).

We note that the ad hoc application of Seifert-van Kampen would only give us \( \text{rank}(\pi_1(M)) \leq 2g(M) \), the whole diagram is needed to get this better lower bound on the Heegaard genus of a manifold. We also note that there was an outstanding conjecture that the inequality in the Corollary above is actually equality, but this was disproved with a hyperbolic counterexample ([2]). Thus it is not a priori sufficient to find the minimal presentation of the fundamental group to exactly compute the Heegaard genus of a manifold.

An offshoot of these considerations is that if we want to find a Heegaard decomposition of a manifold, we can start with a diagram with appropriate generators and relations for its fundamental group. However, this will construct a 3-manifold with the same fundamental group, but further insight is needed to prove that we obtain exactly the manifold we wanted. In the following sections we will use quite a heavy machinery to conclude just that.

**3 Genus 2 decomposition of the Poincaré sphere**

One of the most popular construction of the Poincaré sphere is by taking a dodecahedron with a 3-cell inside and identifying the opposite faces. In this way from a dodecahedron, which has 12 faces, 30 edges and 20 vertices, we obtain a spherical dodecahedron space (homeomorphic to the original Poincaré homology sphere), which has 6 faces, 10 edges and 5 vertices. In the Figure, we can see how gluing the faces identifies the vertices and edges. Every vertex...
belongs to three faces so it is identified with three others vertices. For example the vertex $A$ is a vertex of the face $ABCDE$ (so is identified with the vertex $C'$), but it is also a vertex of the face $ABKH'F$ (so is identified with the vertex $H$), and a vertex of the face $AFJ'GE$ (so is identified with the vertex $G'$). This process is simply depicted by marking these four vertices with pink color. Every vertex belongs to three faces so four vertices become one. This construction (and much more about this manifold) is discussed in detail in [17].

![Dodecahedron with identified faces](image1.jpg)

Figure 8: [17] Dodecahedron with the identified faces.

We recall this construction of the Poincaré sphere since it is more intuitive, gives an easy to visualise Heegaard decomposition by taking the regular neighborhood of the 1-skeleton (although of genus 6, which we know from Section 2.3 not to be minimal), and also a concrete presentation of the fundamental group of the Poincaré sphere: as the icosahedral group (isomorphic to the alternating group $A_5$) acting on the 3-sphere.

However, we have already seen the genus 2 decomposition of the Poincaré sphere, or so we claimed. Consider the diagram

![Heegaard diagram for Poincaré sphere](image2.jpg)

Figure 9: A Heegaard diagram for the Poincaré sphere.
If we read the fundamental group of the manifold arising from this diagram, we see the presentation of $A_5$,

$$\pi_1(P) = \langle a, b \mid a^4 ba^{-1} b = b^{-2} a^{-1} ba^{-1} = 1 \rangle.$$ 

The first relation is depicted as the full curves and the second relation as the dashed curves.

As it was mentioned in Section 2.3, this diagram is the same as the diagram drawn by Poincaré, who defined the Poincaré sphere by it. In order to see this fact, one should imagine the diagrams on a solid 2-sphere, where the circles on the Poincaré’s diagram and the hexes on ours are the discs we identify to get a handlebody with genus 2. The only difference between these diagrams is that the Poincaré’s $A_+$ circle is our $A_-$ hex and his $A_-$ circle is our $A_+$ hex, which is just a notation we choose. On a 2-sphere we can move the big $A_+$ circle in the Poincaré’s diagram to get the $A_-$ hex in our diagram by homeomorphism without changing the order of points on these discs or the relations (arrows) between them. Manifolds obtained from homeomorphic Heegaard diagrams are homeomorphic (Lemma 2.26 in [12]). If we were not in such a convenient situation and had only our diagram and the construction of the Poincaré sphere from a dodecahedron, we could prove that they are homeomorphic by Thurston’s elliptization conjecture ([21], page 28), which states that every 3-manifold with finite fundamental group has an elliptic structure, hence the manifold obtained from our diagram is spherical so it is of the form $S^3/\Gamma$, where $\Gamma$ is a fundamental group from our diagram. The group is a representation of $A_5$ acting by isometries. The dodecahedral construction is also a quotient of $S^3$ by a representation of $A_5$ also acting by isometries and we know from the classification of finite subgroups of $SO(4)$ that these two representations are conjugated, hence the manifolds obtained from the diagram and the dodecahedral construction are diffeomorphic. However, it is a well-known fact that the dodecahedral construction gives the same manifold that can be obtained from the Poincaré’s Heegaard diagram. The dodecahedral construction was introduced by W. Threlfall and H. Seifert in 1931. They stated a question if the obtained manifold is homeomorphic to the Poincaré sphere and the proof of the positive answer can be found in the work of C. Weber and H. Seifert published in 1933 ([25], page 244).

4 Genus 2 decomposition of the Heisenberg manifold

**Definition 4.1.** The Heisenberg group is the group $H_3(\mathbb{R})$ of real $3 \times 3$ upper triangular matrices with the operation of matrix multiplication of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$ 

The Heisenberg group is homeomorphic with $\mathbb{R}^3$.

The Heisenberg group contains a discrete lattice $H_3(\mathbb{Z})$ of matrices with integer entries. By the well known theorems of Cartan on quotients of Lie groups, we know that the quotient of the Heisenberg group by this lattice is a manifold, and we will call it the Heisenberg manifold $H$.

Since the universal cover of the Heisenberg manifold is $\mathbb{R}^3$, we know that the fundamental group of the Heisenberg manifold is the discrete Heisenberg group $\pi_1(H) = H_3(\mathbb{Z})$.

This group has a natural presentation as $\pi_1(H) = H_3(\mathbb{Z}) = \langle a, b, c \mid aba^{-1}b^{-1} = bab^{-1}a^{-1}b^{-1}a = 1 \rangle$, where

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

which we can simplify to get the presentation with just two generators,

$$\pi_1(H) = \langle a, b \mid aba^{-1}b^{-1}a^{-1}b^{-1}a = 1 \rangle.$$
In the Figure 10 we see a Heegaard diagram for the Heisenberg manifold. The full arrows represent the first relation and the dashed arrows represent the second relation.

To show that the manifold arising from the quotient is indeed homeomorphic to the one constructed from the diagram, note that the fundamental group of both is virtually nilpotent but not virtually abelian, which means that the geometric structure on this manifold is Nil geometry ([22], Theorem 4.7.8). This means that the universal covering of both manifolds is contractible, so both are acyclic 3-manifolds with the same fundamental group. By the Borel conjecture (which holds in dimension 3, [14], Remark 4.5), they are homeomorphic.

5 Genus 2 decomposition of the Weeks manifold

The Weeks manifold is defined as (5,2) and (5,1) Dehn surgeries on the Whitehead link and was shown to be the closed, oriented, hyperbolic 3-manifold with the minimal volume in [7]. In 1996 Vesnin proved ([23]) that the Heegaard genus of the Weeks manifold is 2. It was proved in a non-constructive way, using Dehn surgery and deep results about branched coverings and 3-link presentation.

On the page 24 of [4] we can find that the fundamental group of the Weeks manifold is

\[ \pi_1(W) = \langle a, b | a^2b^2a^{-1}ba^{-1}ab^{-1} = 1 \rangle. \]

It can be computed by using the surgery theory with the program SnapPea.

Using this group presentation, we can draw the Heegaard diagram for the Weeks manifold (Figure 11). The full arrows represent the first relation, the dashed arrows represent the second relation.
Just as in the case of the Heisenberg manifold, the question of geometry of Weeks manifold boils down to the properties of its fundamental group. Since the fundamental group does not split, both Weeks manifold and the manifold arising from the diagram are prime, and for prime manifolds there is an algorithm ([15]) that decides if the fundamental group is hyperbolic (in the group-theoretic sense) and decides if the manifold is hyperbolic (in the usual sense). Since we know that Weeks manifold is hyperbolic, so too must be the manifold from the diagram, and by Mostow rigidity (or again by the Borel conjecture) they must be homeomorphic.

We note that if a hyperbolic metric was described simply from the Heegaard diagram in some natural way, one would not need to consider the hyperbolicity of fundamental group but we would show the two manifold to be homeomorphic from Mostow rigidity alone. This however seems to be challenging.

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