TWISTED MAZUR PATTERN SATELLITE KNOTS & BORDERED FLOER THEORY

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ABSTRACT. We use bordered Floer theory to study properties of twisted Mazur pattern satellite knots $Q_n(K)$. We prove that $Q_n(K)$ is not Floer homologically thin, with two exceptions. We calculate the 3-genus of $Q_n(K)$ in terms of the twisting parameter $n$ and the 3-genus of the companion $K$, and we determine when $Q_n(K)$ is fibered. As an application to our results on Floer thickness and 3-genus, we verify the Cosmetic Surgery Conjecture for many of these satellite knots.

1. INTRODUCTION

In its simplest form, knot Floer homology, introduced by Ozsváth-Szabó in [OS03b] and Rasmussen in [Ras03], assigns to a knot $K \subset S^3$ an abelian group $\widehat{HFK}(K)$ that is endowed with two $\mathbb{Z}$-gradings $M$ and $A$. We call $M$ the Maslov grading and $A$ the Alexander grading, and we denote their difference $M - A$ by $\delta$. Knot Floer homology has proven quite useful for studying knots in $S^3$. For example, it detects the 3-genus [OS04] and fiberedness [Ghi08, Ni07], and has a lot to say about knot concordance [OS03c, Hom14a, OSS17].

A knot $K \subset S^3$ is said to be knot Floer homologically thin ($\delta$-thin for short), if its knot Floer homology $\widehat{HFK}(K)$ takes a particularly simple form: all of its generators have the same $\delta$-grading. The class of $\delta$-thin knots includes alternating knots [OS03a], quasi-alternating knots [MO08], and some non-quasi-alternating knots [Gre10]. Recently, cable knots with nontrivial companions were shown to not be $\delta$-thin [Dey19]. It is natural to conjecture whether this is true for all satellite knots. Recall the satellite construction: Every framing $n \in \mathbb{Z}$ of a knot $K \subset S^3$ gives rise to an embedding of $S^1 \times D^2$ in $S^3$ as a tubular neighborhood of $K$, which is unique up to isotopy. We define the $n$-twisted satellite knot $P_n(K)$ of an $n$-framed companion knot $K$ with oriented pattern knot $P \subset S^1 \times D^2$ to be the image of $P$ under this embedding. Once we fix a generator of $H_1(S^1 \times D^2; \mathbb{Z})$, the winding number of $P$ is the integer $w$ for which $P$ represents $w$ times the generator.

In recent years, satellite knots with winding number $\pm 1$ have been instrumental in producing exotic structures on smooth 4-manifolds, see [Yas15, HMP19]. One of the more well-known winding number 1 patterns is the Mazur knot $Q$ in Figure 1. In [Maz61], Mazur used it to construct the first example of a contractible 4-manifold whose boundary is an integral homology sphere not equal to $S^3$. More recently, Levine in [Lev16] and Feller-Park-Ray in [FPR19] used 0-twisted Mazur pattern satellite knots to understand the structure of the smooth knot concordance group.

Key words and phrases. satellite knots, knot Floer homology, bordered Floer theory, 3-genus, fiberedness, Cosmetic Surgery Conjecture.
In this paper, we use bordered Floer homology to study some 3-dimensional properties of arbitrarily twisted Mazur pattern satellite knots $Q_n(K)$. We show that for all but two satellites, $Q_n(K)$ is not $\delta$-thin:

**Theorem 1.0.1.** $Q_n(K)$ is $\delta$-thick for all knots $K \subset S^3$ and integers $n$, except when $Q_n(K)$ is the trivial satellite $Q_0(U)$ or the $\mathbb{Z}_2$ satellite $Q_{-1}(U)$.

Since quasi-alternating knots are $\delta$-thin, Theorem 1.0.1 implies the following.

**Corollary 1.0.2.** $Q_n(K)$ is not quasi-alternating, except when $Q_n(K)$ is the trivial satellite $Q_0(U)$ or the $\mathbb{Z}_2$ satellite $Q_{-1}(U)$.

Given any knot $K \subset S^3$, the $\delta$-thickness of $K$, denoted $\text{th}(K)$, is defined as the difference between the maximum and minimum $\delta$-gradings in $\widehat{\text{HFK}}(K)$ [MO08]. We show that the $\delta$-thickness of $Q_n(K)$ increases without bound as we increase the number of twists:

**Theorem 1.0.3.** For any knot $K \subset S^3$, $\lim_{n \to \pm \infty} \text{th}(Q_n(K)) = \infty$.

We remark that Theorem 1.0.3 does not hold in general. For example, consider the pattern $P$ that is the $(2,1)$-cable in the solid torus, and any $\delta$-thin knot $K$ with $\tau(K) = 0$. One can verify $\text{th}(P_n(K)) = 2g(K)$ for any $n$.

In addition to the above results, in Section 6 we explicitly compute $\text{th}(Q_n(K))$ for $K$ a $\delta$-thin knot or an L-space knot.

By a classical theorem of Schubert [Sch53], the 3-genus $g(P_0(K))$ of a 0-twisted satellite knot $P_0(K)$, with nontrivial companion $K \subset S^3$ and pattern $P \subset S^1 \times D^2$, can be expressed in terms of the 3-genus $g(K)$ of $K$, the winding number $w$ of $P$, and a geometrically defined number $g(P)$ that depends only on $P$:

$$g(P_0(K)) = |w|g(K) + g(P).$$

We give an explicit formula for the 3-genus $g(Q_n(K))$ of an arbitrarily twisted Mazur pattern satellite $Q_n(K)$, in terms of the 3-genus $g(K)$ of the companion $K$ and the twisting $n$. Our result includes the case when the companion $K$ is trivial.

**Theorem 1.0.4.** For any nontrivial knot $K \subset S^3$,

$$g(Q_n(K)) = \begin{cases} g(K) - n & \text{if } n \leq -1 \\ g(K) + n + 1 & \text{if } n \geq 0. \end{cases}$$
When \( K \) is the unknot,

\[
g(Q_n(K)) = \begin{cases} 
-n & \text{if } n \leq 0 \\
n + 1 & \text{if } n \geq 1.
\end{cases}
\]

We remark that there is a 4-dimensional analogue of Theorem 1.0.4 due to Cochran-Ray in [CR16]. They showed that for certain companion knots \( K \), the 4-genus \( g_4(Q_n(K)) \) of \( Q_n(K) \) depends only on the 4-genus \( g_4(K) \) of the companion \( K \), and not on the framing \( n \).

We also fully determine when \( Q_n(K) \) is fibered. By a theorem of Hirasawa-Murasugi-Silver [HMS08], 0-twisted satellite knots \( P_0(K) \) with nontrivial companions \( K \) are fibered if and only if \( K \) is fibered and \( P \) is fibered in \( S^1 \times D^2 \). We show the following:

**Theorem 1.0.5.** If \( K \) is nontrivial, then \( Q_n(K) \) is fibered if and only if \( K \) is fibered and \( n \neq -1, 0 \). If \( K \) is trivial, then \( Q_n(K) \) is fibered if and only if \( n \neq -1 \).

Lastly, we consider a question about surgeries on satellite knots. Given a knot \( K \subset S^3 \), two surgeries \( S^3_r(K) \) and \( S^3_{r'}(K) \), with \( r \neq r' \), are said to be **truly cosmetic** if \( S^3_r(K) \) and \( S^3_{r'}(K) \) are homeomorphic as oriented manifolds. The Cosmetic Surgery Conjecture predicts that there are no truly cosmetic surgeries on nontrivial knots in \( S^3 \) [CG78]. The conjecture has been verified for several classes of knots, including genus 1 knots [Wan06], nontrivial cables [Tao19a], knots with genus at least 3 and \( \delta \)-thickness at most 5 [Han19], and most recently composite knots [Tao19b] and 3-braids [Var20]. One might ask whether Mazur pattern satellite knots also satisfy the conjecture. We give the following partial answer.

**Theorem 1.0.6.** Suppose \( K \) is an L-space knot or a \( \delta \)-thin knot. If \( K \) is an L-space knot, then all nontrivial satellites \( Q_n(K) \) satisfy the Cosmetic Surgery Conjecture. If \( K \) is a \( \delta \)-thin knot, then all nontrivial satellites \( Q_n(K) \) satisfy the Cosmetic Surgery Conjecture, unless one of the following holds:

- \( \{r, r'\} = \{\pm 2\}, n = -1, \text{ and } \Delta_K(t) = 2t - 5 + 2t^{-1} \)
- \( \{r, r'\} = \{\pm 1\}, n = -1, \text{ and } \Delta_K(t) = \begin{cases} 
2t - 5 + 2t^{-1}, & \text{or} \\
bt^2 - (4b + 2)t + (6b + 5) - (4b + 2)t^{-1} + bt^{-2} & \text{with } b \geq 1, \text{ or} \\
b^2 - (4b - 2)t + (6b - 5) - (4b - 2)t^{-1} + bt^{-2} & \text{with } b \geq 2, \text{ or} \\
(b + 1)t^2 - (4b + 6)t + (6b + 11) - (4b + 6)t^{-1} + (b + 1)t^{-2} & \text{with } b \geq 0.
\end{cases} \)
- \( \{r, r'\} = \{\pm 1\}, n = 0, \text{ and } \Delta_K(t) = \begin{cases} 
b^2 - 4bt + (6b - 1) - 4bt^{-1} + bt^{-2} & \text{with } b \geq 1 \text{ or} \\
b^2 - 4bt + (6b + 1) - 4bt^{-1} + bt^{-2} & \text{with } b \geq 1, \text{ and } \tau(K) = -1.
\end{cases} \)

**Organization.** We review the necessary bordered Floer homology background in Section 2. In Section 3, we use bordered Floer homology to study relevant properties of the knot Floer homology of \( Q_n(K) \). In Section 5, we prove Theorems 1.0.4 and 1.0.5. In Section 6, we prove Theorem 1.0.6.
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2. Preliminaries on bordered Floer theory

Bordered Floer homology is an extension of Heegaard Floer homology to manifolds with boundary [LOT08]. To a parametrized surface $F$, one associates a differential algebra $A(F)$, and to a manifold $Y$ whose boundary is identified with $F$, one associates a right $A_{\infty}$-module $\widehat{CFA}(Y)$ over $A(F)$, or a left type $D$ module $\widehat{CFD}(Y)$ over $A(F)$. These modules are invariants of the manifolds up to homotopy equivalence, and $\widehat{CFA}(Y_1) \otimes \widehat{CFD}(Y_2) \simeq \widehat{CF}(Y_1 \cup Y_2)$. Another variant of these structures is associated to knots in bordered 3-manifolds, and recovers $\overline{HFK}$ or $HFK^-$ after gluing. To define these structures, one uses bordered Heegaard diagrams. The algebra is graded by a certain nonabelian group $G$, domains on a bordered Heegaard diagram are graded by $\hat{G}$, and recovers $\overline{HFK}$ or $HFK^-$ after gluing. To define these structures, one uses bordered Heegaard diagrams. The algebra is graded by a certain nonabelian group $G$, domains on a bordered Heegaard diagram are graded by $G$ as well, and a right (resp. left) module associated to a Heegaard diagram is graded by a right (resp. left) coset in $G$ of the subgroup of gradings of periodic domains. The tensor product is then graded by double cosets in $G$, from where one could extract the usual Heegaard Floer grading.

Below, we recall relevant definitions in the case when the boundary $F$ is a torus. For more details, see [LOT08].

The algebra $A$ associated to the torus is generated over $\mathbb{F}_2$ by two idempotents denoted $\iota_0$ and $\iota_1$, and six nontrivial elements denoted $\rho_1$, $\rho_2$, $\rho_3$, $\rho_{12}$, $\rho_{23}$, and $\rho_{123}$. The differential is zero, the nonzero products are

$$\rho_1 \rho_2 = \rho_{12}, \quad \rho_2 \rho_3 = \rho_{23}, \quad \rho_1 \rho_{23} = \rho_{123}, \quad \rho_{12} \rho_3 = \rho_{123},$$

and the compatibility with the idempotents is given by

$$\rho_1 = \iota_0 \rho_1 \iota_1, \quad \rho_2 = \iota_1 \rho_2 \iota_0, \quad \rho_3 = \iota_0 \rho_3 \iota_1, \quad \rho_{12} = \iota_0 \rho_{12} \iota_0, \quad \rho_{23} = \iota_1 \rho_{23} \iota_1, \quad \rho_{123} = \iota_0 \rho_{123} \iota_1.$$

Let $X_{K,n}$ be the $n$-framed knot complement $X_K = S^3 \setminus \text{nhd}(K)$. One can compute $\widehat{CFD}(X_{K,n})$ from $\overline{CFK}^-(K)$ as follows.

There exist a pair of bases $\tilde{\eta} = \{\tilde{\eta}_0, \ldots, \tilde{\eta}_{2k}\}$ and $\tilde{\xi} = \{\tilde{\xi}_0, \ldots, \tilde{\xi}_{2k}\}$ for $\overline{CFK}^-(K)$ (over $\mathbb{F}_2[U]$) that are horizontally simplified and vertically simplified, respectively, indexed so that there is a horizontal arrow of length $l_i \geq 1$ from $\tilde{\eta}_{2i-1}$ to $\tilde{\eta}_{2i}$ and a vertical arrow of length $k_i \geq 1$ from $\tilde{\xi}_{2i-1}$ to $\tilde{\xi}_{2i}$. There are corresponding bases $\eta = \{\eta_0, \ldots, \eta_{2k}\}$ and $\xi = \{\xi_0, \ldots, \xi_{2k}\}$ for $\iota_0 \widehat{CFD}(X_{K,n})$, such that if $\xi_p = \sum_{i=0}^{2k} a_{ip} \tilde{\eta}_i$ and $\eta_p = \sum_{i=0}^{2k} b_{ip} \tilde{\xi}_i$, then $\xi_p = \sum_{i=0}^{2k} a_{ip} |U=0 \eta_i$ and $\eta_p = \sum_{i=0}^{2k} b_{ip} |U=0 \xi_i$. The summand $\iota_1 \widehat{CFD}(X_{K,n})$ has basis

$$\bigcup_{i=1}^{k} \{\kappa_1^i, \ldots, \kappa_{k_i}^i\} \cup \bigcup_{i=1}^{k} \{\lambda_1^i, \ldots, \lambda_{l_i}^i\} \cup \{\mu_1, \ldots, \mu_{2r(K,n)-l_i}\}.$$
and for each horizontal arrow $\tilde{\eta}_{2i-1} \rightarrow \tilde{\eta}_{2i}$ there are corresponding coefficient maps
\[
\eta_{2i-1} \xrightarrow{D_3} \lambda_1 \xrightarrow{D_{23}} \cdots \xrightarrow{D_3} \lambda_i \xrightarrow{D_2} \eta_{2i}.
\]
Depending on the framing $n$, there are additional coefficient maps
\[
\xi_0 \xrightarrow{D_{12}} \eta_0 \quad \text{if } n = 2\tau,
\]
\[
\xi_0 \xrightarrow{D_1} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_m \xrightarrow{D_3} \eta_0 \quad \text{if } n < 2\tau, \quad m = 2\tau - n,
\]
\[
\xi_0 \xrightarrow{D_{123}} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \cdots \xrightarrow{D_{23}} \mu_m \xrightarrow{D_2} \eta_0 \quad \text{if } n > 2\tau, \quad m = n - 2\tau.
\]
We refer to the above chains of coefficient maps as the vertical chains, the horizontal chains, and the unstable chain.

Given a doubly-pointed bordered Heegaard diagram $\mathcal{H}$ for a knot $Q$ in a solid torus $V$, one can compute the module $CFA^-(\mathcal{H})$ or $\overline{CFA}(\mathcal{H})$. There are homotopy equivalences $CFA^-(\mathcal{H}) \times \overline{CFA}(\mathcal{H}) \cong gCFK^- (Q_n (K))$ and $\overline{CFA}(\mathcal{H}) \times \overline{CFA}(\mathcal{H}) \cong gCFK^- (Q_n (K))$.

Given a right type $A$ structure $M$ and a left type $D$ structure $N$ with at least one of $M$ or $N$ bounded (an algebraic condition which our module $CFA^-(\mathcal{H})$ from Section 3.1 satisfies), their box tensor product is the chain complex $M \boxtimes N \simeq M \overline{\boxtimes} N$ defined as follows. As an $\mathbb{F}_2$ vector space, $M \boxtimes N$ is just $M \otimes_I N$. The differential $\partial^2 (x_1 \boxtimes y_1)$ has $x_2 \boxtimes y_2$ in the image whenever there is a sequence of coefficient maps $D_1, \ldots, D_{n}$ from $y_1$ to $y_2$ and a multiplication map $m_{n+1}(x_1, \rho_{n+1}, \ldots, \rho_{n})$ with $x_2$ in the image, both indexed the same way. Further, $\partial^2 (x_1 \boxtimes y)$ has $x_2 \boxtimes y$ in the image whenever $x_1$ is in the image of $m_1(x_2)$, and $\partial^2 (x \boxtimes y_1)$ has $x \boxtimes y_2$ in the image whenever there is a coefficient map with no label from $y_1$ to $y_2$. See [LOT08, Definition 2.26 and Equation (2.29)].

The algebra $A$ is graded by a group $G$ given by quadruples $(a; b, c; d)$ with $a, b, c \in \frac{1}{2} \mathbb{Z}$, $d \in \mathbb{Z}$, and $b + c \in \mathbb{Z}$ and group law
\[
(a_1; b_1, c_1; d_1) \cdot (a_2; b_2, c_2; d_2) = (a_1 + a_2 + \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}; b_1 + b_2, c_1 + c_2; d_1 + d_2).
\]
The grading function is the is defined by
\[
gr (\rho_1) = \left( -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; 0 \right) \\
gr (\rho_2) = \left( -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0 \right) \\
gr (\rho_3) = \left( -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; 0 \right)
\]
along with the rule that for homogeneous algebra elements $a, b$, we have $gr(ab) = gr(a)gr(b)$.

The type $D$ module $\overline{CFA}(X_{K,n})$ is graded by the coset space $G / (h_D)$, where $h_D = \left( -\frac{n}{2}, -\frac{1}{2}; -1, -n; 0 \right)$. A homogeneous generator $s$ of $\iota_0 \overline{CFA}(X_{K,n})$ has grading
\[
gr (s) = \left( M(\tilde{s}) - \frac{3}{2} A(\tilde{s}); 0, -A(\tilde{s}); 0 \right),
\]
where $M(\tilde{s})$ and $A(\tilde{s})$ are the Maslov grading and Alexander filtration of the corresponding generator $\tilde{s}$ in $CFK^- (K)$, respectively. In particular, we recall that
\[
A(\xi_0) = \tau(K) \quad M(\xi_0) = 0 \quad A(\eta_0) = -\tau(K) \quad M(\eta_0) = -2\tau(K).
\]
If $D_1$ is a coefficient map from $x$ to $y$ then the gradings of $x$ and $y$ are related by
\[ \text{gr}(y) = \lambda^{-1} \text{gr}(\rho_I)^{-1} \text{gr}(x) \in G/H_D, \] (3)
where $\lambda = (1; 0, 0; 0)$.

Given a doubly-pointed bordered Heegaard diagram $\mathcal{H}$ for a knot $Q$ in a solid torus $V$, the module $CFA^-(\mathcal{H})$ is graded by the coset $(h_A)\backslash G$, where $(h_A)$ is the subgroup of gradings of periodic domains. This subgroup depends on the knot $Q$; for the Mazur pattern, we find a generator $h_A$ in Section 3.1. For a multiplication map $m_{l+1}(x, \rho_{I_1}, \ldots, \rho_{I_l}) = U^iy$ we have the formula
\[ \text{gr}(y) = \text{gr}(x)\lambda_l^{-1} \text{gr}(\rho_{I_1}) \cdots \text{gr}(\rho_{I_l})(0; 0, 0; i) \in (h_A)\backslash G. \] (4)
When the underlying manifold is the solid torus $V$, this is sufficient to obtain a relative grading of all generators.

The tensor product $CFA^-(\mathcal{H}) \otimes \hat{\text{CFD}}(X_{K,n}) \simeq \hat{\text{CFK}}(Q_n(K))$ is graded by the double-coset space $(h_A)\backslash G/(h_D)$ via $\text{gr}(x \otimes y) = \text{gr}(x)\text{gr}(y)$. The double-coset space $(h_A)\backslash G/(h_D)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and for a homogeneous element $x \otimes y$, there is a unique grading representative of the form $(a; 0, 0; d)$ with $a, d \in \mathbb{Z}$. Up to an overall translation, $a$ agrees with the $z$-normalized grading $N$ of $x \otimes y$, and $d$ agrees with the Alexander grading $A$ of $x \otimes y$.

3. The complex $\hat{\text{CFA}}(\mathcal{H}) \otimes \hat{\text{CFD}}(X_{K,n}) \simeq \hat{\text{CFK}}(Q_n(K))$

In this section, we work out general grading formulas for the generators of $\hat{\text{CFA}}(\mathcal{H}) \otimes \hat{\text{CFD}}(X_{K,n}) \simeq \hat{\text{CFK}}(Q_n(K))$, where $\mathcal{H}$ is a doubly-pointed bordered Heegaard diagram for the Mazur pattern in the solid torus. We also make some useful observations about the differential on this complex.

3.1. $\hat{\text{CFA}}$ of the Mazur pattern in the solid torus. Let $V$ denote the solid torus $S^1 \times D^2$, and let $Q$ denote the Mazur pattern in $V$. Figure 2 is a doubly-pointed bordered Heegaard diagram $\mathcal{H}$ for $(V, Q)$, see also [Lev16, Figure 9].

![Figure 2](image-url)

**Figure 2.** A bordered Heegaard diagram $\mathcal{H}$ for the pair $(V, Q)$. 

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[Lev16] Levitt, S. (2016). *Geometric Topology of 3-Manifolds.* Cambridge University Press.

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Over $\mathbb{F}_2[U]$, the type $A$ structure $CFA^-(\mathcal{H})$ is generated by $x_0, x_2, x_4, y_2, y_4$ in idempotent $t_1$, and by $x_1, x_3, x_5, x_6, y_1, y_3, y_5, y_6$ in idempotent $t_2$. The multiplication maps are encoded by the labeled edges in Figure 3.4: an arrow from $v_1$ to $v_2$ with label $U^t a_1 \cdots a_n$ describes the multiplication map $m_{n+1}(v_1, a_1, \ldots, a_n) = U^t v_2$, while an arrow from $v_1$ to $v_2$ with label $U^t a_1 \cdots a_n + U^s b_1 \cdots b_p$ describes the multiplication maps $m_{n+1}(v_1, a_1, \ldots, a_n) = U^t v_2$ and $m_{p+1}(v_1, b_1, \ldots, b_p) = U^p v_2$.

Consider the periodic domain $B \in \pi_2(x_0, x_0)$ corresponding to traversing the loop

\[ x_0 \xrightarrow{U^2} y_1 \xrightarrow{U} x_1 \xrightarrow{\rho_2} x_0. \]

Using Equation 4, we compute the following relative gradings in $\langle h_A \rangle \backslash G$.

\[ \text{gr}(y_1) = \text{gr}(x_0) \text{gr}(\rho_3)(0; 0, 0, 1) \in \langle h_A \rangle \backslash G \]

\[ \text{gr}(y_1) = \text{gr}(x_1) \lambda^{-1}(0; 0, 0, 2) \in \langle h_A \rangle \backslash G \]

\[ \text{gr}(x_0) = \text{gr}(x_1) \text{gr}(\rho_2) \in \langle h_A \rangle \backslash G \]

The first equation is equivalent to

\[ \text{gr}(x_0) = \text{gr}(y_1)(0; 0, 0; -1) \text{gr}(\rho_3)^{-1}. \]

Substituting the left coset $\text{gr}(x_1) \lambda^{-1}(0; 0, 0, 2)$ for $\text{gr}(y_1)$, we get

\[ \text{gr}(x_0) = \text{gr}(x_1) \lambda^{-1}(0; 0, 0, 2)(0; 0, 0; -1) \text{gr}(\rho_3)^{-1} = \text{gr}(x_1) \text{gr}(\rho_3)^{-1}(-1; 0, 0, 1). \]

Further substituting $\text{gr}(x_1) = \text{gr}(x_0) \text{gr}(\rho_2)^{-1}$, we get

\[ \text{gr}(x_0) = \text{gr}(x_0) \text{gr}(\rho_2)^{-1} \text{gr}(\rho_3)^{-1}(-1; 0, 0, 1) = \text{gr}(x_0) \left( \frac{1}{2}; 0, -1; 1 \right). \]

So $\left( \frac{1}{2}; 0, -1; 1 \right) \in \langle h_A \rangle$, and since $\left( \frac{1}{2}; 0, -1; 1 \right)$ is not a positive multiple of another group element, it generates $\langle h_A \rangle$. From here on, we will use the generator

\[ h_A = (-\frac{1}{2}; 0, 1; -1) \]

of $\langle h_A \rangle$. 

**Figure 3.** The type $A$ structure $CFA^-(\mathcal{H})$. 
From here on, we abuse notation and denote cosets by their representatives. We normalize the grading by setting 
\[ \text{gr}(x_0) = (0; 0, 0, 0). \]
Since \( m_2(x_1, \rho_2) = x_0 \), we get 
\[ \text{gr}(x_0) = \text{gr}(x_1) \text{gr}(\rho_2), \]
so
\[ \text{gr}(x_1) = \text{gr}(x_0) \text{gr}(\rho_2)^{-1} = \left( \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}; 0 \right). \]
Since \( m_2(x_2, \rho_1) = x_1 \), we get 
\[ \text{gr}(x_1) = \text{gr}(x_2) \text{gr}(\rho_1), \]
so
\[ \text{gr}(x_2) = \text{gr}(x_1) \text{gr}(\rho_1)^{-1} = \left( \frac{1}{2}; 1, 0; 0 \right). \]
Continuing these computations along any spanning tree for the graph in Figure 3, we obtain the gradings of all generators. We summarize the result below.

\[
\begin{align*}
\text{gr}(x_0) &= (0; 0, 0, 0) \\
\text{gr}(x_1) &= \left( \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}; 0 \right) \\
\text{gr}(x_2) &= \left( \frac{1}{2}; -1, 0; 0 \right) \\
\text{gr}(x_3) &= \left( \frac{1}{2}; -\frac{3}{2}, -\frac{1}{2}; 1 \right) \\
\text{gr}(x_4) &= (0; 2, 0; 1) \\
\text{gr}(x_5) &= \left( -\frac{1}{2}; -\frac{1}{2}; 1 \right) \\
\text{gr}(x_6) &= \left( -\frac{3}{2}; -\frac{3}{2}; 1 \right)
\end{align*}
\]

We remind the reader that following a different path to a given generator may result in a different representative of the same coset.

3.2. The gradings on \( \widehat{\text{CFD}}(X_{K,n}) \). We begin with a discussion of the gradings in \( G/\langle h_D \rangle \) of the generators of \( \widehat{\text{CFD}}(X_{K,n}) \). Since the last component of the grading is always zero here, we omit it. Recall from Equation 1 that each homogeneous generator \( s \) of \( \iota_0 \widehat{\text{CFD}}(X_{K,n}) \) is graded by
\[ \text{gr}(s) = (M(\tilde{s}) - \frac{3}{2} A(\tilde{s}); 0, -A(\tilde{s})); \]
where \( M(\tilde{s}) \) and \( A(\tilde{s}) \) are the Maslov grading and Alexander filtration of the corresponding generator \( \tilde{s} \) in \( \text{CFK}^{-}(K) \), respectively.

Next consider the vertical chain
\[
\xi_{2i-1} \xrightarrow{D_1} \kappa_1^i \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \kappa_{k_i}^i \xrightarrow{D_{123}} \xi_{2i}.
\]
Using Equation 3, we see that
\[ \text{gr}(\kappa_1^i) = \lambda^{-1} \text{gr}(\rho_1)^{-1} \text{gr}(\xi_{2i-1}) \]
\[ = \lambda^{-1} \left( \frac{1}{2}; -\frac{1}{2}, \frac{1}{2} \right) \left( M(\tilde{\xi}_{2i-1}) - \frac{3}{2} A(\tilde{\xi}_{2i-1}); 0, -A(\tilde{\xi}_{2i-1}) \right) \]
\[ = (M(\tilde{\xi}_{2i-1}) - A(\tilde{\xi}_{2i-1}) - \frac{1}{2} ; -\frac{1}{2}, -A(\tilde{\xi}_{2i-1}) + \frac{1}{2}) . \]
Continuing along the chain, we obtain the general formula
\[ \text{gr}(\kappa_j^i) = \lambda^{-1} \text{gr}(\rho_{23})^{j-1} \text{gr}(\kappa_1^i) \]
\[ = \left( \frac{1}{2} - \frac{j}{2}; 0, j - 1 \right) \text{gr}(\kappa_1^i) \]
\[ = (M(\tilde{\xi}_{2i-1}) - A(\tilde{\xi}_{2i-1}) + j - \frac{3}{2}; -\frac{1}{2}, -A(\tilde{\xi}_{2i-1}) + j - \frac{1}{2}) . \]
Similarly, traversing the horizontal chain

\[
\eta_{2i-1} \xrightarrow{D_3} \lambda_1^i \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \lambda_{l_i}^i \xrightarrow{D_2} \eta_{2i},
\]

we get

\[
gr(\lambda_j^i) = (M(\eta_{2i-1}) - 2A(\eta_{2i-1}) - \frac{1}{2}; \frac{1}{2}, -A(\eta_{2i-1}) - i + \frac{1}{2}).
\]

Last, we traverse the unstable chain, starting from \(\xi_0\) and working towards \(\eta_0\). When \(n = 2\tau(K)\), there are no additional generators. When \(n < 2\tau(K)\), the unstable chain takes the form

\[
\xi_0 \xrightarrow{D_1} \mu_1 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \mu_{2\tau-\eta} \xleftarrow{D_3} \eta_0
\]

and we get

\[
gr(\mu_j) = \lambda^{j-2}gr(\rho_{23})^{j-1}gr(\rho_1)^{-1}gr(\xi_0) = (-\tau(K) + j - \frac{3}{2}; -\frac{1}{2}, -\tau(K) + j - \frac{1}{2}).
\]

When \(n > 2\tau(K)\), the unstable chain takes the form

\[
\xi_0 \xrightarrow{D_{123}} \mu_1 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \mu_{n-2\tau} \xrightarrow{D_3} \eta_0
\]

and we get

\[
gr(\mu_j) = \lambda^{-j}gr(\rho_{123})^{-1}gr(\rho_{23})^{1-j}gr(\xi_0) = (-\tau(K) - j + \frac{1}{2}; -\frac{1}{2}, -\tau(K) - j + \frac{1}{2}).
\]

3.3. The gradings on \(\widetilde{\text{CFA}}(\mathcal{H}) \boxtimes \widetilde{\text{CFD}}(X_{K,n})\). In this subsection, we compute grading representatives in the double-coset space \(h_A \backslash G / (h_D)\) of the form \((a; 0, 0; d)\) for all generators of \(\widetilde{\text{CFA}}(\mathcal{H}) \boxtimes \widetilde{\text{CFD}}(X_{K,n})\). Note that not all these generators survive in homology; the differential is discussed in the subsequent section. Recall that \(h_A = (-\frac{1}{2}; 0, 1; -1)\) and \(h_D = (-\frac{4}{2} - \frac{1}{2}; -1, -n; 0)\), where \(n\) is the framing of \(K\). Recall also that \(\widetilde{\text{CFA}}(\mathcal{H}) \boxtimes \widetilde{\text{CFD}}(X_{K,n}) \simeq g\widetilde{CFK}(Q_n(K))\). The procedure is as follows. Given a generator \(x_A \boxtimes x_D\), we multiply the coset grading representatives for \(x_A\) and \(x_D\) to obtain a double coset representative for \(x_A \boxtimes x_D\). Then we multiply the double coset representative by an appropriate power of \(h_A\) on the left, to obtain a representative with 0 in the second coordinate. Last, we multiply the new double coset representative by an appropriate power of \(h_D\) on the right, to obtain a double coset representative with 0 in the second and third coordinates. In the resulting representative \((a; 0, 0; d)\), \(a\) is the absolute \(z\)-normalized Maslov grading \(N\) of \(x_A \boxtimes x_D\) in \(g\widetilde{CFK}(Q_n(K))\), and \(d\) is the Alexander grading \(A\) of \(x_A \boxtimes x_D\) in \(g\widetilde{CFK}(Q_n(K))\), considered up to an overall translation.
For example,

\[
\text{gr}(x \boxtimes \kappa^j_i) = \text{gr}(x) \text{gr}(\kappa^j_i)
\]

\[
= (\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}; 0)(M(\xi_{2i-1}) - A(\xi_{2i-1}) + j - \frac{3}{2}; -\frac{1}{2}, -A(\xi_{2i-1}) + j + \frac{1}{2}; 0)
\]

\[
= (M(\xi_{2i-1}) - \frac{1}{2}A(\xi_{2i-1}) + \frac{1}{2}j - 1; -1, -A(\xi_{2i-1}) + j - 1; 0)
\]

\[
= (M(\xi_{2i-1}) - \frac{1}{2}A(\xi_{2i-1}) + \frac{1}{2}j - 1; -1, -A(\xi_{2i-1}) + j - 1; 0)h^1_D
\]

\[
= (M(\xi_{2i-1}) + \frac{1}{2}A(\xi_{2i-1}) - \frac{1}{2}j - \frac{n}{2} + \frac{1}{2}; 0, -A(\xi_{2i-1}) + j + n - 1; 0)
\]

\[
= h^A_D(\xi_{2i-1})^{-n+1}(M(\xi_{2i-1}) + \frac{1}{2}A(\xi_{2i-1}) - \frac{1}{2}j - \frac{n}{2} + \frac{1}{2}; 0, -A(\xi_{2i-1}) + j + n - 1; 0)
\]

\[
= (M(\xi_{2i-1}); 0; 0, -A(\xi_{2i-1}) + j + n - 1).
\]

Proceeding in this way, we find all remaining bigradings. We summarize the results in Table 1. We denote the Alexander grading up to overall translation by \( A_{\text{rel}} \). Since the \( w \)-normalized Maslov grading \( M \) is given by \( M = N + 2A \), we can then compute the \( \delta \)-grading \( \delta_{\text{rel}} \), also up to overall translation, as \( \delta_{\text{rel}} = N + A_{\text{rel}} \).

### 3.4. The differential on \( \widehat{CFA} (\mathcal{H}) \otimes \widehat{CFD} (X_{K,n}) \)

Setting \( U = 0 \) in \( CFA^- (\mathcal{H}) \), we obtain \( \widehat{CFA} (\mathcal{H}) \), see Figure 4.

![Figure 4](image-url)

**Figure 4.** The type A structure \( \widehat{CFA} (\mathcal{H}) \)

Since \( \widehat{CFA} (\mathcal{H}) \) is bounded, we can take the boxtensor with any type \( D \) structure, and so we use the model described in Section 2 without having to analyze its boundedness (in general, we may have to replace with an equivalent structure when \( \epsilon (K) = 0 \) and \( n = 0 \)). By [Hom14a, Lemmas 3.2-3.3], we may assume that our bases \( \eta = \{\eta_0, \ldots, \eta_{2k}\} \) and \( \xi = \{\xi_0, \ldots, \xi_{2k}\} \) are indexed so that when \( \epsilon (K) = 0 \) we have \( \eta_0 = \xi_0 \), when \( \epsilon (K) = 1 \) we have \( \xi_0 = \eta_2 \), and when \( \epsilon (K) = -1 \) we have \( \eta_0 = \xi_1 \).

To compute \( \partial^\beta \), we pair the multiplication maps represented by the arrows in Figure 4 with (sequences of) coefficient maps in \( \widehat{CFD} (X_{K,n}) \). From Figure 4, we see that we only need to consider the length one sequences \( D_1, D_2, D_{12} \), and the length two sequences \( D_2, D_1 \) and \( D_{12}, D_1 \).

The coefficient map \( D_1 \) is seen once from \( \xi_{2i-1} \) to \( \kappa^j_i \) in each vertical chain, and once from \( \xi_0 \) to \( \mu_1 \) in the unstable chain when \( n < 2\tau (K) \). The map \( D_2 \) is seen once from \( \lambda^j_i \) to \( \eta_{2i} \) in each horizontal chain, and once from \( \mu_m \) to \( \eta_0 \) in the unstable chain when \( n > 2\tau (K) \). The map \( D_{12} \) is only seen once, from \( \xi_0 \) to \( \eta_0 \) in the unstable chain, when \( n = 2\tau (K) \).

The sequence \( D_2, D_1 \) is seen from \( \lambda^j_i \) to \( \kappa^j_i \) whenever \( a_{ij} |_{U = 0} \neq 0 \), once when \( \epsilon (K) = 1 \) and \( n < 2\tau (K) \) (because we’ve assumed that \( \xi_0 = \eta_2 \), and once from \( \mu_m \) to \( \kappa^j_i \) when \( \epsilon (K) = -1 \).
| Generator | $N$ | $A_{rel}$ | $\delta_{rel}$ |
|-----------|-----|----------|---------------|
| $x_0 \otimes s$ | $M(s) - 2A(s)$ | $-A(s)$ | $M(s) - 3A(s)$ |
| $x_2 \otimes s$ | $M(s) + 1$ | $-A(s) + n$ | $M(s) - A(s) + n + 1$ |
| $3y_2 \otimes s$ | $M(s)$ | $-A(s) + n + 1$ | $M(s) - A(s) + n + 1$ |
| $4x_2 \otimes s$ | $M(s) + 2A(s) - 2n + 1$ | $-A(s) + 2n + 1$ | $M(s) + A(s) + 2$ |
| $4y_2 \otimes s$ | $M(s) + 2A(s) - 2n$ | $-A(s) + 2n + 2$ | $M(s) + A(s) + 2$ |
| $x_1 \otimes \lambda_i$ | $M(\eta_{2i-1}) - 2A(\eta_{2i-1})$ | $-A(\eta_{2i-1}) - j$ | $M(\eta_{2i-1}) - 3A(\eta_{2i-1}) - j$ |
| $y_1 \otimes \lambda_i$ | $M(\eta_{2i-1}) - 2A(\eta_{2i-1}) - 1$ | $-A(\eta_{2i-1}) - j + 2$ | $M(\eta_{2i-1}) - 3A(\eta_{2i-1}) - j + 1$ |
| $x_3 \otimes \mu_i$ | $M(\xi_{2i-1}) - 1$ | $-A(\xi_{2i-1}) - j + n + 1$ | $M(\xi_{2i-1}) - A(\xi_{2i-1}) + j + n - 1$ |
| $y_3 \otimes \mu_i$ | $M(\xi_{2i-1}) - 2A(\xi_{2i-1}) - 1$ | $-A(\xi_{2i-1}) - j - 2$ | $M(\xi_{2i-1}) - 3A(\xi_{2i-1}) - j - 1$ |
| $x_5 \otimes \lambda_i$ | $M(\eta_{2i-1}) - 2A(\eta_{2i-1}) - 1$ | $-A(\eta_{2i-1}) - j + 2$ | $M(\eta_{2i-1}) - 3A(\eta_{2i-1}) - j + 1$ |
| $y_5 \otimes \lambda_i$ | $M(\eta_{2i-1}) - 2A(\eta_{2i-1}) - 2$ | $-A(\eta_{2i-1}) - j + 3$ | $M(\eta_{2i-1}) - 3A(\eta_{2i-1}) - j + 1$ |
| $x_7 \otimes \kappa_i$ | $M(\xi_{2i-1}) - 1$ | $-A(\xi_{2i-1}) + j + n + 1$ | $M(\xi_{2i-1}) - A(\xi_{2i-1}) + j + n$ |
| $y_7 \otimes \kappa_i$ | $M(\xi_{2i-1}) - 2$ | $-A(\xi_{2i-1}) + j + n + 2$ | $M(\xi_{2i-1}) - A(\xi_{2i-1}) + j + n$ |
| $x_9 \otimes \lambda_i$ | $M(\eta_{2i-1}) + 2j - 3$ | $-A(\eta_{2i-1}) - j + n + 3$ | $M(\eta_{2i-1}) - A(\eta_{2i-1}) - j + n$ |
| $y_9 \otimes \lambda_i$ | $M(\eta_{2i-1}) + 2j - 4$ | $-A(\eta_{2i-1}) - j + n + 4$ | $M(\eta_{2i-1}) - A(\eta_{2i-1}) - j + n$ |
| $x_9 \otimes \kappa_i$ | $M(\xi_{2i-1}) + 2A(\xi_{2i-1}) - 2j - 2n - 1$ | $-A(\xi_{2i-1}) + j + 2n + 2$ | $M(\xi_{2i-1}) + A(\xi_{2i-1}) - j + 1$ |
| $y_9 \otimes \kappa_i$ | $M(\xi_{2i-1}) + 2A(\xi_{2i-1}) - 2j - 2n - 2$ | $-A(\xi_{2i-1}) + j + 2n + 3$ | $M(\xi_{2i-1}) + A(\xi_{2i-1}) - j + 1$ |

| Generator | $N$ | $A_{rel}$ | $\delta_{rel}$ |
|-----------|-----|----------|---------------|
| $x_1 \otimes \mu_j$ | $0$ | $-\tau(K) + j + n - 1$ | $-\tau(K) + j + n - 1$ |
| $y_1 \otimes \mu_j$ | $-1$ | $-\tau(K) + j + n + 1$ | $-\tau(K) + j + n + 1$ |
| $x_3 \otimes \mu_j$ | $2\tau(K) - 2j - 2n + 2$ | $-\tau(K) + j + 2n$ | $\tau(K) = j + 2n$ |
| $y_3 \otimes \mu_j$ | $2\tau(K) - 2j - 2n + 1$ | $-\tau(K) + j + 2n + 1$ | $\tau(K) = j + 2n + 1$ |
| $x_5 \otimes \mu_j$ | $-2j - 4$ | $-\tau(K) + j + n + 2$ | $-\tau(K) + j + n + 1$ |
| $y_5 \otimes \mu_j$ | $2\tau(K) - 2j - 2n - 1$ | $-\tau(K) + j + 2n + 2$ | $\tau(K) = j + 1$ |
| $y_6 \otimes \mu_j$ | $2\tau(K) - 2j - 2n - 2$ | $-\tau(K) + j + 2n + 3$ | $\tau(K) = j + 1$ |

| Generator | $N$ | $A_{rel}$ | $\delta_{rel}$ |
|-----------|-----|----------|---------------|
| $x_1 \otimes \mu_j$ | $1$ | $-\tau(K) - j + n$ | $-\tau(K) - j + n + 1$ |
| $y_1 \otimes \mu_j$ | $0$ | $-\tau(K) - j + n + 2$ | $-\tau(K) - j + n + 2$ |
| $x_3 \otimes \mu_j$ | $2\tau(K) + 2j - 2n + 1$ | $-\tau(K) - j + 2n + 1$ | $\tau(K) = j + 2n$ |
| $y_3 \otimes \mu_j$ | $2\tau(K) + 2j - 2n$ | $-\tau(K) - j + 2n + 2$ | $\tau(K) = j + 2n + 2$ |
| $x_5 \otimes \mu_j$ | $0$ | $-\tau(K) - j + n + 2$ | $-\tau(K) - j + n + 2$ |
| $y_5 \otimes \mu_j$ | $-1$ | $-\tau(K) - j + n + 1$ | $-\tau(K) - j + n + 1$ |
| $x_6 \otimes \mu_j$ | $2\tau(K) + 2j - 2n - 2$ | $-\tau(K) - j + 2n + 2$ | $\tau(K) = j + 1$ |
| $y_6 \otimes \mu_j$ | $2\tau(K) + 2j - 2n - 3$ | $-\tau(K) - j + 2n + 3$ | $\tau(K) = j + 1$ |

Table 1. The $z$-normalized Maslov gradings, relative Alexander gradings, and relative $\delta$-gradings on all generators of the complex $\widehat{\text{CFD}}(\mathcal{H}) \otimes \text{CFD}(X_{K,n})$.

and $n > 2\tau(K)$ (because we’ve assumed that $\eta_0 = \xi_1$). The sequence $D_{12}, D_1$ appears only once, from $\xi_0$ to $\kappa_1^j$, when $\epsilon = -1$ and $n < 2\tau(K)$ (because we’ve assumed that $\eta_0 = \xi_1$).
The following nontrivial differentials occur regardless of the value of $\epsilon(K)$ and the framing $n$.

$$\partial^\Theta(x_1 \otimes \lambda_i^i) = x_0 \otimes \eta_{2i} \quad i = 1, \ldots, k$$
$$\partial^\Theta(x_3 \otimes \lambda_i^i) = y_2 \otimes \eta_{2i} \quad i = 1, \ldots, k$$
$$\partial^\Theta(x_2 \otimes \xi_{2i-1}) = x_1 \otimes \kappa_i^1 \quad i = 1, \ldots, k$$
$$\partial^\Theta(x_4 \otimes \xi_{2i-1}) = x_3 \otimes \kappa_i^1 \quad i = 1, \ldots, k$$
$$\partial^\Theta(y_4 \otimes \xi_{2i-1}) = y_3 \otimes \kappa_i^1 \quad i = 1, \ldots, k$$
$$\partial^\Theta(y_3 \otimes \lambda_i^i) = y_1 \otimes \kappa_i^2$$

where the first two rows of differentials come from pairings for $D_2$, the next three rows come from pairings for $D_1$, and the last row comes from pairings for the sequence $D_2, D_1$.

There are also the following additional nontrivial differentials that depend on the framing $n$.

When $n < 2\tau(K)$, we have

$$\partial^\Theta(x_2 \otimes \xi_0) = x_1 \otimes \mu_1 \quad \partial^\Theta(x_4 \otimes \xi_0) = x_3 \otimes \mu_1 \quad \partial^\Theta(y_4 \otimes \xi_0) = y_3 \otimes \mu_1$$

where all three differentials come from pairings for $D_1$, as well as

$$\partial^\Theta(y_3 \otimes \lambda_i^i) = y_1 \otimes \mu_1$$

if $\epsilon(K) = 1$, coming from a pairing for the sequence $D_2, D_1$.

When $n = 2\tau(K)$, we have

$$\partial^\Theta(x_2 \otimes \xi_0) = x_0 \otimes \eta_0 \quad \partial^\Theta(x_4 \otimes \xi_0) = y_2 \otimes \eta_0$$

where both differentials come from pairings for $D_{12}$, as well as

$$\partial^\Theta(y_4 \otimes \xi_0) = y_1 \otimes \kappa_1^1$$

if $\epsilon(K) = -1$, coming from a pairing for the sequence $D_{12}, D_1$.

When $n > 2\tau(K)$, we have

$$\partial^\Theta(x_1 \otimes \mu_m) = x_0 \otimes \eta_0 \quad \partial^\Theta(x_3 \otimes \mu_m) = y_2 \otimes \eta_0$$

where both differentials come from pairings for $D_2$, as well as

$$\partial^\Theta(y_3 \otimes \mu_m) = y_1 \otimes \kappa_1^1$$

if $\epsilon(K) = -1$, coming from a pairing for the sequence $D_2, D_1$.

4. $\delta$-Thickness of $Q_n(K)$

We start by proving that $Q_n(K)$ is $\delta$-thick for all integers $n$ and knots $K$, except for two satellites obtained when $K$ is the unknot and $n$ is $-1$ or $0$.

**Proof of Theorem 1.0.1.** Recall that we have horizontally and vertically simplified bases $\tilde{\eta} = \{\tilde{\eta}_0, \ldots, \tilde{\eta}_{2k}\}$ and $\tilde{\xi} = \{\tilde{\xi}_0, \ldots, \tilde{\xi}_{2k}\}$, respectively, for $\text{CFK}^-(K)$ that induce bases $\eta = \{\eta_0, \ldots, \eta_{2k}\}$ and $\xi = \{\xi_0, \ldots, \xi_{2k}\}$ for the subspace $\iota_0\overline{CFD}(X_{K,n})$. Since $\tilde{\eta}$ and $\tilde{\xi}$
are simplified bases for $\text{CFK}^-(K)$, we can treat them as bases of $\widehat{\text{HFK}}(K)$ as well. In particular, this implies that $\text{rk} \, \widehat{\text{HFK}}(K) = 2k + 1$. We will make use of the following simple lemma.

**Lemma 4.0.1.** If $K$ is not the unknot or a trefoil, then there is some $\tilde{\eta}_{2i-1} \in \tilde{\eta}$ with $A(\tilde{\eta}_{2i-1}) < 0$.

*Proof.* Assume, to the contrary, $\tilde{\eta}_1, \tilde{\eta}_3, \ldots, \tilde{\eta}_{2k-1}$ all have nonnegative Alexander degree. Recall that the basis $\tilde{\eta}$ is indexed so that there is a horizontal arrow from $\tilde{\eta}_{2i-1}$ to $\tilde{\eta}_{2i}$ for each $i$. Since the horizontal arrows strictly increase the Alexander degree, it follows that $\tilde{\eta}_2, \tilde{\eta}_4, \ldots, \tilde{\eta}_{2k}$ all have positive Alexander degree. By symmetry of $\widehat{\text{HFK}}$, there must be at least $k$ generators in $\tilde{\eta}$ with negative Alexander degree. So $\text{rk} \, \widehat{\text{HFK}}(K) \geq 3k$.

On the other hand, since the dimension of $\widehat{\text{HFK}}$ is always odd and detects both the trefoil [HW18] and the unknot [OS04], we have $\text{rk} \, \widehat{\text{HFK}}(K) = 2k + 1 \geq 5$. This is a contradiction. \hfill $\square$

Now consider $\widehat{\text{CFD}}(X_{K,n})$, and consider the basis $\eta$ for $\iota_0 \widehat{\text{CFD}}(X_{K,n})$. At each odd-indexed element $\eta_{2i-1}$, we have the following arrows:

- An outgoing $D_3$-arrow to $\lambda_i^1$.
- An outgoing $D_1$-arrow to $\kappa_i^j$, whenever $\xi_{2j-1}$ appears with nonzero coefficient in $\eta_{2i-1}$.
- An outgoing $D_{123}$-arrow to $\kappa_{k_j}$, whenever $\xi_{2j}$ appears with nonzero coefficient in $\eta_{2i-1}$.
- An outgoing arrow labelled $D_1, D_{12},$ or $D_{123}$, depending on the framing, if $\xi_0$ appears with nonzero coefficient in $\eta_{2i-1}$.

Regardless of the framing and how the bases $\eta$ and $\xi$ are related, there are no incoming arrows at $\eta_{2i-1}$. Since there are no outgoing edges at the generators $x_0$ and $y_2$ of $\widehat{\text{CFA}}(\mathcal{H})$, the elements $x_0 \boxtimes \eta_{2i-1}$ and $y_2 \boxtimes \eta_{2i-1}$ survive in the homology of $\widehat{\text{CFA}}(\mathcal{H}) \boxtimes \widehat{\text{CFD}}(X_{K,n})$, i.e. they represent distinct generators of $\widehat{\text{HFK}}(Q_n(K))$.

Similarly, since the only outgoing arrows at each even-indexed element $\xi_{2i}$ are labeled $D_3$ or $D_{123}$, and there are no matching labels in our model for $\widehat{\text{CFA}}(V,Q)$, the elements $x_2 \boxtimes \xi_{2i}$ and $x_4 \boxtimes \xi_{2i}$ are all nonzero in $\widehat{\text{HFK}}(Q_n(K))$.

Next, we consider the relative $\delta$-gradings of the above generators of homology. From Table 1, we have

$$
\delta(x_0 \boxtimes \eta_{2i-1}) = M(\tilde{\eta}_{2i-1}) - 3A(\tilde{\eta}_{2i-1})
$$

$$
\delta(y_2 \boxtimes \eta_{2i-1}) = M(\tilde{\eta}_{2i-1}) - A(\tilde{\eta}_{2i-1}) + n + 1
$$

$$
\delta(x_4 \boxtimes \xi_{2i}) = M(\tilde{\xi}_{2i}) + A(\tilde{\xi}_{2i}) + 2.
$$

**Case 1:** Suppose there are two elements $\tilde{\eta}_{2i-1}$ and $\tilde{\eta}_{2s-1}$ with distinct $\delta$-degrees. Then $\delta(y_2 \boxtimes \eta_{2i-1}) \neq \delta(y_2 \boxtimes \eta_{2s-1})$, so $Q_n(K)$ is $\delta$-thick.

**Case 2:** Suppose all odd-indexed elements in $\tilde{\eta}$ are in the same $\delta$-degree.
Case 2.1: Suppose there exist two elements $\tilde{\eta}_{2t-1}$ and $\tilde{\eta}_{2s-1}$ in different Alexander degrees. Then
\[
\delta(x_0 \boxtimes \eta_{2t-1}) - \delta(x_0 \boxtimes \eta_{2s-1}) = M(\tilde{\eta}_{2t-1}) - 3A(\tilde{\eta}_{2t-1}) - M(\tilde{\eta}_{2s-1}) + 3A(\tilde{\eta}_{2s-1}) \\
= \delta(\tilde{\eta}_{2t-1}) - \delta(\tilde{\eta}_{2s-1}) - 2A(\tilde{\eta}_{2t-1}) + 2A(\tilde{\eta}_{2s-1}) \\
= -2A(\tilde{\eta}_{2t-1}) + 2A(\tilde{\eta}_{2s-1}) \neq 0,
\]
so $Q_n(K)$ is $\delta$-thick.

Case 2.2: Suppose all odd-indexed elements in $\tilde{\eta}$ have the same bidegree $(M, A)$. First, we consider a couple of special cases. If $K$ is the unknot, then $Q_n(K)$ is $\delta$-thick exactly when $n \not\in \{-1, 0\}$, by Proposition 6.1.1. If $K$ is a trefoil, then for all values of $n$, $Q_n(K)$ is $\delta$-thick, again by Proposition 6.1.1.

Now assume $K$ is any other knot. Lemma 4.0.1 implies that $A < 0$. By symmetry of $\HFK$, there are $k \geq 2$ generators in $\tilde{\eta}$ with bidegree $(M - 2A, -A)$, and the one remaining generator has Alexander degree zero.

Consider the basis $\tilde{\xi}$. It also has $k$ elements in Alexander degree $A$ and $k$ elements in Alexander degree $-A$. Recall that $k \geq 2$. Since $A < 0$ and the vertical arrows strictly decrease the Alexander grading, there is at least one element $\tilde{\xi}_{2t}$ in bidegree $(M, A)$ with $t \geq 1$. We see that $\delta(x_0 \boxtimes \eta_2) - \delta(x_4 \boxtimes \xi_{2t}) = -4A - 2$, which is nonzero, since $A$ is an integer. So $Q_n(K)$ is $\delta$-thick. □

Further, we show that as the number of twists on the Mazur pattern increases, the $\delta$-thickness increases without bound.

Proof of Theorem 1.0.3. Let $n < 2\tau(K)$. Observe that for any generator $\mu_i$ along the unstable chain of $\text{CFD}(X_{K,n})$, the tensor product $x_6 \boxtimes \mu_i$ survives in the homology of $\text{CFA}(\mathcal{H}) \boxtimes \text{CFD}(X_{K,n})$. Further, the generators $x_6 \boxtimes \mu_i$ all have distinct $\delta$-gradings. In particular,
\[
\text{th}(Q_n(K)) \geq \delta(x_6 \boxtimes \mu_1) - \delta(x_6 \boxtimes \mu_{2\tau(K) - n}) = 2\tau(K) - n - 1.
\]

Similarly, when $n > 2\tau(K)$, we have that every tensor product of the form $x_6 \boxtimes \mu_i$ survives in the homology of $\text{CFA}(\mathcal{H}) \boxtimes \text{CFD}(X_{K,n})$ and
\[
\text{th}(Q_n(K)) \geq \delta(x_6 \boxtimes \mu_{n - 2\tau(K)}) - \delta(x_6 \boxtimes \mu_1) = -2\tau(K) + n - 1.
\]
Thus,
\[
\lim_{n \to \pm \infty} \text{th}(Q_n(K)) = \infty. \quad \square
\]

5. 3-genus and fiberedness of $Q_n(K)$

In this section, we combine a bordered Floer homology computation with a couple of classical results to calculate the 3-genus of $Q_n(K)$ and to determine when $Q_n(K)$ is fibered, for all $n$ and $K$.

We first focus on the case where $K$ is the right-handed trefoil. To compute the 3-genus of $Q_n(K)$, it suffices to find the extremal Alexander degrees in $\HFK(Q_n(K))$ [OS04]. To determine whether $Q_n(K)$ is fibered, it suffices to compute the rank of $\HFK(Q_n(K))$ in the top Alexander degree [Ghi08, Ni07].
Recall that $\widehat{HFK}(Q_n(K)) \cong H_*(\widehat{\text{CFA}}(\mathcal{H}) \boxtimes \widehat{\text{CFD}}(X_{K,n}))$, where $\widehat{\text{CFD}}(X_{K,n})$ is as follows:

$$
\begin{array}{cccc}
\xi_0 & D_2 & D_3 & \eta_0 \\
\downarrow & \lambda_1 & \eta_1 & \uparrow \\
\kappa_1 & D_1 & \eta_0 & D_{123}
\end{array}
$$

**Figure 5.** $\widehat{\text{CFD}}(X_{K,n})$ for the right-handed trefoil $K$. The dotted arrow represents the unstable chain.

We use the values from Table 1, combined with the differential computed in Section 3.4, to find the extremal relative Alexander degrees in $\widehat{HFK}(Q_n(K))$ and the generators of $\widehat{HFK}(Q_n(K))$ in those degrees.

When $n < -1$, the nontrivial differentials on $\widehat{\text{CFA}}(\mathcal{H}) \boxtimes \widehat{\text{CFD}}(X_{K,n})$ are

$$
\begin{align*}
\partial^\text{rel}(x_2 \boxtimes \eta_1) &= x_1 \boxtimes \kappa_1, \\
\partial^\text{rel}(x_4 \boxtimes \eta_1) &= x_3 \boxtimes \kappa_1, \\
\partial^\text{rel}(y_4 \boxtimes \eta_1) &= y_3 \boxtimes \kappa_1, \\
\partial^\text{rel}(x_1 \boxtimes \lambda_1) &= x_0 \boxtimes \xi_0, \\
\partial^\text{rel}(x_3 \boxtimes \lambda_1) &= y_2 \boxtimes \xi_0, \\
\partial^\text{rel}(y_3 \boxtimes \lambda_1) &= y_1 \boxtimes \mu_1, \\
\partial^\text{rel}(x_2 \boxtimes \xi_0) &= x_1 \boxtimes \mu_1, \\
\partial^\text{rel}(x_4 \boxtimes \xi_0) &= x_3 \boxtimes \mu_1, \\
\partial^\text{rel}(y_4 \boxtimes \xi_0) &= y_3 \boxtimes \mu_1.
\end{align*}
$$

The generators of $\widehat{HFK}(Q_n(K))$, together with their relative Alexander degrees, are given by Table 2. One can easily verify that the minimum relative Alexander degree is $2n + 1$ realized only by generator $x_3 \boxtimes \mu_2$, and the maximum Alexander degree is $3$ realized only by generator $y_5 \boxtimes \mu_{2-n}$.

| Generator | $A_{\text{rel}}$ | Generator | $A_{\text{rel}}$ |
|-----------|----------------|-----------|----------------|
| $x_0 \boxtimes \eta_0$ | 1 | $x_5 \boxtimes \kappa_1$ | $n + 2$ |
| $x_2 \boxtimes \eta_0$ | $1 + n$ | $y_5 \boxtimes \kappa_1$ | $n + 3$ |
| $y_2 \boxtimes \eta_0$ | $2 + n$ | $x_6 \boxtimes \kappa_1$ | $2n + 3$ |
| $x_4 \boxtimes \eta_0$ | $2n + 2$ | $y_6 \boxtimes \kappa_1$ | $2n + 4$ |
| $y_4 \boxtimes \eta_0$ | $2n + 3$ | $x_1 \boxtimes \mu_j, j \in \{2, \ldots, 2 - n\}$ | $n + j - 2$ |
| $x_0 \boxtimes \eta_1$ | 0 | $y_1 \boxtimes \mu_j, j \in \{2, \ldots, 2 - n\}$ | $n + j$ |
| $y_2 \boxtimes \eta_1$ | $n + 1$ | $x_3 \boxtimes \mu_j, j \in \{2, \ldots, 2 - n\}$ | $2n + j - 1$ |
| $y_1 \boxtimes \lambda_1$ | 1 | $y_3 \boxtimes \mu_j, j \in \{2, \ldots, 2 - n\}$ | $2n + j$ |
| $x_5 \boxtimes \lambda_1$ | 1 | $x_5 \boxtimes \mu_j, j \in \{1, \ldots, 2 - n\}$ | $n + j$ |
| $y_5 \boxtimes \lambda_1$ | 2 | $y_5 \boxtimes \mu_j, j \in \{1, \ldots, 2 - n\}$ | $n + j + 1$ |
| $x_6 \boxtimes \lambda_1$ | $n + 2$ | $x_6 \boxtimes \mu_j, j \in \{1, \ldots, 2 - n\}$ | $2n + j + 1$ |
| $y_6 \boxtimes \lambda_1$ | $n + 3$ | $y_6 \boxtimes \mu_j, j \in \{1, \ldots, 2 - n\}$ | $2n + j + 2$ |
| $y_1 \boxtimes \kappa_1$ | $n + 2$ | $y_1 \boxtimes \mu_j, j \in \{1, \ldots, 2 - n\}$ | $2n + j + 2$ |

**Table 2.** The generators of $\widehat{HFK}(Q_n(K))$ and their relative Alexander degrees for the right-handed trefoil $K$ and framing $n < -1$.  

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**TWISTED MAZUR PATTERN SATELLITE KNOTS & BORDERED FLOER THEORY 15**
The cases when \( n \geq -1 \) are similar. We summarize the results in Table 3.

Now since the 3-genus is half the difference between the highest and the lowest Alexander degrees, Table 3 implies that

\[
g(Q_n(K)) = \begin{cases} 
1 - n & \text{if } n \leq -1, \\
n + 2 & \text{if } n \geq 0.
\end{cases}
\]

Furthermore, because a knot is fibered if and only if its knot Floer homology has rank 1 in the highest Alexander degree, we conclude that \( Q_n(K) \) is fibered if and only if \( n \) is not \(-1\) or \(0\).

Next we use our work for the right-handed trefoil to calculate the 3-genus of \( Q_n(K) \) and to determine when \( Q_n(K) \) is fibered, given any nontrivial knot \( K \) not equal to the right-handed trefoil and given any number of twists \( n \).

**Proof of Theorem 1.0.4** for nontrivial knots \( K \) not equal to the right-handed trefoil. We can think of \( Q_n(K) \) as the 0-twisted satellite knot \((Q_n)_0(K)\) with pattern the \(n\)-twisted Mazur knot \( Q_n \) and companion \( K \). Since the winding number of \( Q_n \) is 1, by a result attributed to Schubert [Sch53],

\[
g(Q_n(K)) = g((Q_n)_0(K)) = g(K) + g(Q_n),
\]

where \( g(Q_n) \) is a number that depends only on the pattern \( Q_n \). This means that we can determine the genus of \( Q_n(K) \) if we know the constant \( g(Q_n) \). The same equation also tells us that if we know the genus of some test companion \( K' \) and the genus of its corresponding satellite \( Q_n(K') \), then this constant \( g(Q_n) \) is just \( g(Q_n(K')) - g(K') \). Take \( K' \) to be the right-handed trefoil. From our work above,

\[
g(Q_n) = g(Q_n(K')) - g(K') = \begin{cases} 
-n & \text{if } n \leq -1 \\
n + 1 & \text{if } n \geq 0.
\end{cases}
\]

This completes the proof of Theorem 1.0.4 for nontrivial \( K \). \( \square \)

**Proof of Theorem 1.0.5** for nontrivial knots \( K \) not equal to the right-handed trefoil. Once again, we think of \( Q_n(K) \) as the 0-twisted satellite knot \((Q_n)_0(K)\) with pattern \( Q_n \) and companion \( K \). By a theorem of Hirasawa-Murasugi-Silver [HMS08, Theorem 2.1], \( Q_n(K) \) is fibered if and only if \( K \) is fibered and \( Q_n \) is fibered in the solid torus. Since the right-handed trefoil is fibered, the above computation shows that the pattern \( Q_n \) is fibered in the solid torus.
if and only if \( n \) is not \(-1\) or \(0\). Therefore, when \( n = -1,0 \), the satellite \( Q_n(K) \) is never fibered, and when \( n \neq -1,0 \), the satellite \( Q_n(K) \) is fibered if and only if \( K \) is fibered. \( \square \)

Lastly, we determine the 3-genus and fiberedness of \( Q_n(U) \). Again, we use the values from Table 1. The case analysis is similar to above.

When \( n < -1 \), the minimum and the maximum relative Alexander degrees are \(2n + 2\) and \(2\), realized by \(x_3 \boxtimes \mu_2\), and \(y_5 \boxtimes \mu_{-n}\), respectively. When \( n = -1 \), we have \( Q_{-1}(U) = 5_2 \), which is known to have genus 1 and not be fibered. When \( n = 0 \), we have \( Q_0(U) = U \) (genus zero, fibered). When \( n = 1 \), the minimum and the maximum relative Alexander degrees are 1 and 5, realized by \(x_2 \boxtimes \eta_0\), and \(y_6 \boxtimes \mu_1\), respectively. Last, when \( n > 1 \), the minimum and the maximum relative Alexander degrees are 1 and \(2n + 3\), realized by \(x_1 \boxtimes \mu_{n-1}\), and \(y_6 \boxtimes \mu_1\), respectively.

It follows that
\[
g(Q_n(U)) = \begin{cases} 
-n & \text{if } n \leq 0, \\
n + 1 & \text{if } n \geq 1,
\end{cases}
\]
and that \( Q_n(U) \) is fibered if and only if \( n \neq -1 \).

6. AN APPLICATION TO THE COSMETIC SURGERY CONJECTURE

In this section, we prove Theorem 1.0.6.

In [Han19, Theorem 2], Hanselman shows that if \( K \subset S^3 \) is a nontrivial knot and \( S^3_r(K) \cong S^3_{r'}(K) \), for \( r \neq r' \), then the pair of surgery slopes \( \{r, r'\} \) is either \( \{\pm 2\} \) or \( \{\pm \frac{1}{q}\} \) for some positive integer \( q \). Further, he shows that if \( \{r, r'\} = \{\pm 2\} \), then \( g(K) = 2 \), and if \( \{r, r'\} = \{\pm \frac{1}{q}\} \), then
\[
q \leq \frac{\text{th}(K) + 2g(K)}{2g(K)(g(K) - 1)}.
\]
In particular, if \( g(K) \geq 3 \), and
\[
\frac{\text{th}(K) + 2g(K)}{2g(K)(g(K) - 1)} < 1,
\]
the knot \( K \) automatically satisfies the cosmetic surgery conjecture. Define
\[
f(K) = 2(g(K))^2 - 4g(K) - \text{th}(K),
\]
and observe that Inequality 5 is equivalent to the inequality
\[
f(K) > 0.
\]
We will show that nontrivial satellites \( Q_n(K) \) satisfy the cosmetic surgery conjecture whenever \( K \) is a thin knot (with a small set of unverified exceptions) or an \( L \)-space knot. Except for a few special cases, which we analyze using other tools, we use Inequality 6, so we need to combine the genus values from Theorem 1.0.4 with a computation of the \( \delta \)-thickness \( \text{th}(Q_n(K)) \). There are two other tools that we will use for the special cases. The first is an obstruction of Boyer–Lines, which says that if \( \Delta_\delta'(1) \neq 0 \), then the knot \( J \) satisfies the cosmetic surgery conjecture; see [BL90, Proposition 5.1]. The second is an obstruction of Ni–Wu, which says that if \( \tau(J) \neq 0 \), then \( J \) satisfies the cosmetic surgery conjecture; see [NW15, Theorem 1.2].
6.1. Thin companions. In this subsection, we prove Theorem 1.0.6 in the case of thin companions.

We break the argument into cases that depend on \( n \) and \( K \). In each case, we begin by computing the \( \delta \)-thickness \( \text{th}(Q_n(K)) \). We then combine the thickness values with the genus values from Theorem 1.0.4, and check whether Inequality 6 holds. In the isolated cases where Inequality 6 does not hold, we use other methods to complete the proof.

6.1.1. The \( \delta \)-thickness of \( Q_n(K) \) when \( K \) is thin. In this subsection, we show that when the companion \( K \) is thin, the \( \delta \)-thickness of \( Q_n(K) \) is as follows.

**Proposition 6.1.1.** Suppose the companion \( K \) is thin. If \( K \) is the unknot, then

\[
\text{th}(Q_n(K)) = \begin{cases} 
-n - 1 & \text{if } n \leq -1 \\
n & \text{if } n \geq 0.
\end{cases}
\]

If \( K \) is the right-handed trefoil, then

\[
\text{th}(Q_n(K)) = \begin{cases} 
-n + 1 & \text{if } n \leq -1 \\
2 & \text{if } n = 0, 1 \\
n + 2 & \text{if } n \geq 2.
\end{cases}
\]

In all other cases,

\[
\text{th}(Q_n(K)) = \begin{cases} 
2g - n - 1 & \text{if } n \in (-\infty, -2g] \\
4g - 2 & \text{if } n \in [-2g + 1, 2g - 2] \\
2g + n & \text{if } n \in [2g - 1, \infty).
\end{cases}
\]

**Proof.** Recall from [Pet13, Lemma 7] that for a thin knot \( K \), the complex \( CFK^-(K) \), and hence the module \( \hat{CFD}(X_{K,n}) \), is particularly simple. More precisely, there exists a basis \( \tilde{\eta} = \{\tilde{\eta}_0, \ldots, \tilde{\eta}_{2k}\} \) for \( CFK^-(K) \) which is both horizontally and vertically simplified. With respect to that basis, the complex \( CFK^-(K) \) decomposes as a direct sum of “squares” and one “staircase” with length-one steps, as in Figure 6. If \( CFK^-(K) \) contains squares, then the number of squares with top right corner in any given Alexander degree \( a \) is the same as the number of squares with top right corner in Alexander degree \(-a\).

**Figure 6.** The three types of summands of \( CFK^-(K) \) for a thin knot \( K \).

Let \( Sq_a \) be a square summand of \( \hat{CFD}(X_{K,n}) \) with \( A(\tilde{\eta}_i) \) in degree \( a \) and generators labeled as in Figure 6.1.1. We will say that \( Sq_a \) is centered at \( a \). Using the differential
shows that \( 4 \) gives

\[
\eta_2 \xrightarrow{D_2} \lambda \xrightarrow{D_3} \eta_1
\]

\[
D_1 \quad \kappa' \quad D_2 \quad \lambda' \quad D_3 \quad \eta_3
\]

\[
D_{123} \quad \kappa' \quad D_{2} \quad \lambda' \quad D_{3} \quad \eta_3
\]

**Figure 7.** A summand \( S_{q,a} \) of \( \widehat{CFD}(X_{K,n}) \) corresponding to a square summand of \( CFK^-(K) \) with top right corner in Alexander degree \( a \).

computation from Section 3.4, we see that the differential on \( \widehat{CFA}(H) \boxtimes S_{q,a} \) is given by

\[
\partial^\frak{g}(x_1 \boxtimes \lambda) = x_0 \boxtimes \eta_2 \\
\partial^\frak{g}(x_2 \boxtimes \eta_1) = x_1 \boxtimes \kappa \\
\partial^\frak{g}(y_3 \boxtimes \lambda) = y_3 \boxtimes \kappa
\]

Using the values from Table 1, we compute the relative \( \delta \)-gradings of the generators of \( H_*(\widehat{CFA}(H) \boxtimes S_{q,a}) \) in terms of \( M(\overline{\eta}_1), A(\overline{\eta}_1), \tau(K), \) and \( n \). For example, the second row of Table 1 gives

\[
\delta_{\text{rel}}(x_2 \boxtimes s) = M(\overline{s}) - A(\overline{s}) + n + 1
\]

so we get

\[
\delta_{\text{rel}}(x_2 \boxtimes \eta_3) = M(\overline{\eta}_3) - A(\overline{\eta}_3) + n + 1 = (M(\overline{\eta}_1) - 1) - (A(\overline{\eta}_1) - 1) + n + 1.
\]

Since \( A(\overline{\eta}_1) = a \) and \( M(\overline{\eta}_1) = a + \tau(K) \), this simplifies to

\[
\delta_{\text{rel}}(x_2 \boxtimes \eta_3) = -\tau(K) + n + 1.
\]

We list all generators of \( H_*(\widehat{CFA}(H) \boxtimes S_{q,a}) \) and their \( \delta \) degrees in Table 4.

By symmetry, for every square \( S_q \) centered at \( a \), there is a square \( S_q' \) centered at \(-a\). Table 4 shows that \( S_q \) is supported in the following six or fewer (depending on \( \tau, n, \) and \( a \)) \( \delta \) degrees: \(-\tau + n + 1, \) \(-\tau + n + 2, \) \(-\tau - 2a, \) \(-\tau - 2a + 2, \) \(-\tau + 2a, \) and \(-\tau + 2a + 2\); changing \( a \) to \(-a\), we see that \( S_{q'} \) is supported in the same \( \delta \) degrees as \( S_q \). Hence, to analyze the thickness of \( Q_n(K) \), it is enough to consider the squares of \( CFK^-(K) \) centered at nonnegative degrees. Now, for \( a \geq 0 \), we have

\[
-\tau - 2a < -\tau - 2a + 2 \leq -\tau + 2a + 2,
\]

so the minimum \( \delta \) degree is in the set \( \{-\tau + n + 1, -\tau - 2a\} \), and the maximum is in the set \( \{-\tau + n + 2, -\tau + 2a + 2\} \). Further, if \( 0 \leq a' \leq a \), we have

\[
-\tau - 2a \leq -\tau - 2a',
\]

\[
-\tau + 2a' + 2 \leq -\tau + 2a + 2.
\]
so the δ degrees resulting from a square centered at \( a' \) are bounded by the minimum and maximum degrees resulting from a square centered at \( a \). Thus, to analyze the thickness of \( Q_n(K) \), it is in fact enough to only consider a highest-centered square of \( CFK^-(K) \). Let \( A \) be the highest Alexander degree at which a square is centered for our fixed thin knot \( K \). Table 5 summarizes the minimum and maximum relative δ degrees following from the above discussion, depending on the framing \( n \) relative to \( A \).

| \( n \leq -2A - 2 \) | \( n = -2A - 1 \) | \( n \in [-2A, 2A - 1] \) | \( n = 2A \) | \( n \geq 2A + 1 \) |
|---|---|---|---|---|
| Min \( \delta_{rel} \) | \( -\tau + n + 1 \) | \( -\tau + n + 1 = -\tau - 2A \) | \( -\tau - 2A \) | \( -\tau - 2A \) |
| Max \( \delta_{rel} \) | \( -\tau + 2A + 2 \) | \( -\tau + 2A + 2 = -\tau + n + 2 \) | \( -\tau + 2A + 2 \) | \( -\tau + n + 2 \) |

Table 5. Minimum and maximum relative δ degrees for the generators of \( H_*(\widehat{CFA}(\mathcal{H}) \boxtimes X_{K,n}) \cong \widehat{HFK}(Q_n(K)) \) coming from squares of \( CFK^-(K) \) when \( K \) is a thin knot with \( \tau(K) = \tau \), and \( A \) is the highest Alexander degree at which there is a square centered.

Minimum and maximum relative δ degrees for the generators of \( H_*(\widehat{CFA}(\mathcal{H}) \boxtimes X_{K,n}) \cong \widehat{HFK}(Q_n(K)) \) coming from the staircase summand of \( CFK^-(K) \) are obtained similarly, by a routine application of the differential formulas from Section 3.4, followed by a case analysis of the relative δ degrees computed in Section 3.3 applied to the surviving generators. We summarize the results in three tables below.

When \( 0 < |\tau| < g \), the highest Alexander degree for a staircase generator of \( CFK^-(K) \) is \( |\tau| \), and the highest overall Alexander degree of a generator of \( CFK^-(K) \) is \( g \). Hence, the highest Alexander degree is attained by a square generator, so there is at least one
square, and $A = g - 1$. The highest and lowest relative $\delta$ degrees of $\widehat{HFK}(Q_n(K))$ arising from tensoring $\widehat{CFA}(\mathcal{H})$ with squares and with the staircase are summarized in Table 6. For all values of $n$, the staircase degrees are bounded by the highest-square degrees, so the thickness of $Q_n(K)$ is the difference between the extremal square degrees; see the last row of Table 6 for $\text{th}(Q_n(K))$.

| $n \in$ | $(-\infty, -2g]$ | $[-2g + 1, -2|\tau|]$ | $[-2|\tau| + 1, 2|\tau| - 2]$ | $[2|\tau| - 1, 2g - 2]$ | $[2g - 1, \infty)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| Min $\delta_{\text{rel}}$ from squares | $-\tau + n + 1$ | $-\tau - 2g + 2$ | $-\tau - 2g + 2$ | $-\tau - 2g + 2$ | $-\tau - 2g + 2$ |
| Min $\delta_{\text{rel}}$ from staircase | $-\tau + n + 1$ | $-\tau + n + 1$ | $-\tau - 2|\tau| + 2$ | $-\tau - 2|\tau| + 2$ | $-\tau - 2|\tau| + 2$ |
| Max $\delta_{\text{rel}}$ from squares | $-\tau + 2g$ | $-\tau + 2g$ | $-\tau + 2g$ | $-\tau + 2g$ | $-\tau + n + 2$ |
| Max $\delta_{\text{rel}}$ from staircase | $-\tau + 2|\tau|$ | $-\tau + 2|\tau|$ | $-\tau + 2|\tau|$ | $-\tau + n + 2$ | $-\tau + n + 2$ |
| $\text{th}(Q_n(K))$ | $2g - n - 1$ | $4g - 2$ | $4g - 2$ | $4g - 2$ | $2g + n$ |

**Table 6.** Minimum and maximum relative $\delta$ degrees for the generators of $H_n(\widehat{CFA}(\mathcal{H}) \boxtimes X_{K,n}) \cong \widehat{HFK}(Q_n(K))$ when $K$ is thin with $\tau(K) = \tau \neq 0$ and $g(K) = g > |\tau|$, along with the resulting thickness $\text{th}(Q_n(K))$.

When $0 < |\tau| = g$, there may or may not be squares in $CFK^-(K)$. If there are squares, the highest one is centered at some $A \in \{0, \ldots, g - 1\}$. The highest and lowest relative $\delta$ degrees of $\widehat{HFK}(Q_n(K))$ arising from tensoring $\widehat{CFA}(\mathcal{H})$ with squares and with the staircase are summarized in Table 7. For all values of $n$, except when $\tau = 1$ and $n = 1$, the square degrees are bounded by the extremal staircase degrees, so the thickness of $Q_n(K)$ is the difference between the extremal staircase degrees. When $\tau = 1$ and $n = 1$, the maximum $\delta$ degree coming from the staircase is 1, and the maximum $\delta$ degree coming from squares, if there are any, is 2. The resulting thickness $\text{th}(Q_n(K))$ is then 2 if there are no squares, or 3 if there are squares. See the last row of Table 7 for $\text{th}(Q_n(K))$.

Last, we consider the case $\tau = 0$. The staircase of $CFK^-(K)$ consists of just one element $\tilde{\eta}_0$, and the corresponding summand of $\widehat{CFD}(X_{K,n})$ consists only of the unstable chain, which starts and ends at the corresponding element $\eta_0$. Analogous analysis to the above yields the extremal $\delta_{\text{rel}}$ degrees listed in Table 8. To compute the thickness of $Q_n(K)$, we need to also consider squares.

If $g \geq 2$ and $\tau = 0$, then there are squares, and the highest one is centered at $A = g - 1$. For all values of $n$, the staircase degrees are bounded by the highest-square degrees, so the thickness of $Q_n(K)$ is the difference between the extremal square degrees. See Table 8.

---

1 Except when $\tau = 1$ and $n = 1$ the maximum $\delta_{\text{rel}}$ degree coming from the staircase is 1, not 2.

2 Except when $\tau = 1$ and $n = 1$ the maximum $\delta_{\text{rel}}$ degree coming from the staircase is 1, not 2.

3 Except when $\tau = 1$, $n = 1$ and there are no squares, and hence by [HW18] $K$ is the right-handed trefoil, the value of $\text{th}(Q_n(K))$ is 2, not 3.
\[ n \in \begin{array}{cccccc} \infty, -2g & -2g + 1, -2A - 1 & -2A, 2A & 2A + 1, 2g - 2 & 2g - 1, \infty \\
\end{array} \]

\[ \begin{array}{cccccc} \text{Min } \delta_{\text{rel}} \text{ from squares} & -\tau + n + 1 & -\tau + n + 1 & -\tau - 2A & -\tau - 2A & -\tau - 2A \\
\text{Min } \delta_{\text{rel}} \text{ from staircase} & -\tau + n + 1 & -\tau - 2g + 2 & -\tau - 2g + 2 & -\tau - 2g + 2 & -\tau - 2g + 2 \\
\text{Max } \delta_{\text{rel}} \text{ from squares} & -\tau + 2A + 2 & -\tau + 2A + 2 & -\tau + 2A + 2 & -\tau + n + 2 & -\tau + n + 2 \\
\text{Max } \delta_{\text{rel}} \text{ from staircase} & -\tau + 2g & -\tau + 2g & -\tau + 2g & -\tau + 2g & -\tau + n + 2 \\
\text{th}(Q_n(K)) & 2g - n - 1 & 4g - 2 & 4g - 2 & 4g - 2 & 2g + n \\
\end{array} \]

Table 7. Minimum and maximum relative $\delta$ degrees for the generators of $H_n(CFA(H) \boxtimes X_{K,n}) \approx \widehat{HFK}(Q_n(K))$ when $K$ is thin with $\tau(K) = \tau \neq 0$ and $g(K) = g = |\tau|$, along with the resulting thickness $\text{th}(Q_n(K))$. 

\[ n \in \begin{array}{cccccc} \infty, -2g & -2g + 1, -2 & -1, 0 & 1, 2g - 2 & 2g - 1, \infty \\
\end{array} \]

\[ \begin{array}{cccccc} \text{Min } \delta_{\text{rel}} \text{ from squares} & n + 1 & -2g + 2 & -2g + 2 & -2g + 2 & -2g + 2 \\
\text{Min } \delta_{\text{rel}} \text{ from staircase} & n + 1 & n + 1 & n + 1 & 2 & 2 \\
\text{Max } \delta_{\text{rel}} \text{ from squares} & 2g & 2g & 2g & 2g & n + 2 \\
\text{Max } \delta_{\text{rel}} \text{ from staircase} & 0 & 0 & n + 1 & n + 2 & n + 2 \\
\text{th}(Q_n(K)) & 2g - n - 1 & 4g - 2 & 4g - 2 & 4g - 2 & 2g + n \\
\end{array} \]

Table 8. Minimum and maximum relative $\delta$ degrees for the generators of $H_n(CFA(H) \boxtimes X_{K,n}) \approx \widehat{HFK}(Q_n(K))$ when $K$ is thin with $\tau(K) = 0$ and $g(K) = g \geq 2$, along with the resulting thickness $\text{th}(Q_n(K))$. 

If $g = 1$ and $\tau = 0$, then there are squares, all centered at $A = 0$. Combining the staircase $\delta_{\text{rel}}$ values from Table 8 (which do not depend on any assumptions for $g(K)$) with the square values from Table 5, we see that the staircase degrees are bounded by the extremal square degrees. Then thickness of $Q_n(K)$ is the difference between the extremal square degrees. When $n \leq -1$, $\text{th}(Q_n(K)) = 1 - n$; when $n \geq 0$, $\text{th}(Q_n(K)) = n + 2$.

If $g = 0$, then $K$ is the unknot $U$. The staircase values from Table 8 are all we need to compute $\text{th}(Q_n(K))$. When $n \leq -2$, $\text{th}(Q_n(K)) = -n - 1$; when $n \in \{-1, 0\}$, $\text{th}(Q_n(K)) = 0$; when $n \geq 1$, $\text{th}(Q_n(K)) = n$.

This completes the proof of Proposition 6.1.1. \qed

6.1.2. The cosmetic surgery conjecture for $Q_n(K)$ when $K$ is thin. We are now ready to prove Theorem 1.0.6 in the case of thin companions.
Proof of Theorem 1.0.6 for thin companions. Apart from a few special cases, we will use Inequality 6.

Let $S = \{(0, -2), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (2, -1), (2, 0)\}$.

Case 1: Assume $(g, n) \notin S$. Theorem 1.0.4 implies that $g(Q_n(K)) \geq 3$. We will combine the thickness values from Proposition 6.1.1 with the genus values from Theorem 1.0.4 to show that Inequality 6 holds for all pairs $(g, n) \notin S$.

Case 1.1: Suppose $g = 0$, i.e. $K$ is the unknot.
- If $n \leq -3$, then $\text{th}(Q_n(K)) = -n - 1$, $g(Q_n(K)) = -n$, and $f(Q_n(K)) = 2(-n)^2 - 4(-n) - (-n - 1) > 0$.
- If $n \geq 2$, then $\text{th}(Q_n(K)) = n$, $g(Q_n(K)) = n + 1$, and $f(Q_n(K)) = 2(n + 1)^2 - 4(n + 1) - n > 0$.

Case 1.2: Suppose $g \geq 1$.
- If $n \leq -2g$, then $\text{th}(Q_n(K)) = 2g - n - 1$ and $g(Q_n(K)) = g - n$. One can verify (by hand, or plugging into a calculator) that $f(Q_n(K)) = 2(g-n)^2 - 4(g-n) - (2g-n-1)$ is always positive on the domain \{(g, n) \in \mathbb{Z} \times \mathbb{Z} | g \geq 1, n \leq -2g, (g, n) \notin S\}.
- If $n \in [-2g + 1, -1]$, then $\text{th}(Q_n(K)) = 4g - 2$ and $g(Q_n(K)) = g - n$. Again one sees that $f(Q_n(K)) = 2(g-n)^2 - 4(g-n) - (4g-2)$ is always positive.
- Suppose $n \in [0, 2g - 2]$. Then $\text{th}(Q_n(K)) = 4g - 2$ and $g(Q_n(K)) = g + n + 1$, so $f(Q_n(K)) = 2(g+n+1)^2 - 4(g+n+1) - (4g-2) > 0$.
- Suppose $n \geq 2g - 1$. If $K$ is the right handed trefoil, we have $\text{th}(Q_n(K)) = 2$ and $g(Q_n(K)) = 3$, so $f(Q_n(K)) = 4 > 0$. Otherwise, $\text{th}(Q_n(K)) = 2g + n$ and $g(Q_n(K)) = g + n + 1$, so $f(Q_n(K)) = 2(g+n+1)^2 - 4(g+n+1) - (2g+n) > 0$.

Thus, the satellite $Q_n(K)$ satisfies the cosmetic surgery conjecture whenever $K$ is thin and $(g, n) \notin S$.

Case 2: Assume $(g, n) \in S$. We cannot use Inequality 6 here, so we use [BL90, Proposition 5.1], [NW15, Theorem 1.2], and [Han19, Theorem 2].

Case 2.1: Suppose $g = 0$.
- If $(g, n) = (0, -2)$, then $Q_{-2}(K) = 12n121$, so $\Delta''_{12n121}(1) = 8 \neq 0$.
- If $(g, n) = (0, -1)$, then $Q_{-1}(K) = 52$, so $\Delta''_{Q_{-1}(K)}(1) = 4 \neq 0$.
- If $(g, n) = (0, 0)$, then $Q_0(K)$ is the trivial knot.
- If $(g, n) = (0, 1)$, then $Q_1(K) = 9_{42}$, so $\Delta''_{Q_1(K)}(1) = -4 \neq 0$.

Thus, all three nontrivial satellites above satisfy the cosmetic surgery conjecture.

Case 2.2: Suppose $g \geq 1$.
- Suppose $(g, n) = (1, -1)$. If $|\tau(K)| = 1$, the complex $\text{CFK}^-(K)$ consists of one 3-element staircase and possibly some squares centered at 0. Thus,

$$\Delta_K(t) = (s + 1)t - (2s + 1) + (s + 1)t^{-1},$$

where $s \geq 0$ is the number of squares. Since the $n$-twisted Mazur pattern has winding number 1 in the solid torus, we have

$$\Delta_{Q_n(K)}(t) = \Delta_{Q_n(U)}(t) \Delta_K(t).$$
Specifically, for \( n = -1 \) we have \( Q_{-1}(U) = 5_2 \), so
\[
\Delta_{Q_{-1}(K)}(t) = \Delta_{5_2}(t)\Delta_K(t) = (2t - 3 + 2t^{-1})((s + 1)t - (2s + 1) + (s + 1)t^{-1}).
\]
So \( \Delta''_{Q_{-1}(K)}(1) = 2s + 6 \), which is nonzero for any positive \( s \). By [BL90, Proposition 5.1], \( Q_{-1}(K) \) satisfies the cosmetic surgery conjecture.

If \( \tau(K) = 0 \), following the above reasoning, we see that \( Q_{-1}(K) \) satisfies the cosmetic surgery conjecture whenever \( \Delta_K(t) \neq 2t - 5 + 2t^{-1} \). We further obstruct possible cosmetic surgeries using [Han19, Theorem 2]. Since here \( \frac{\text{th}(K) + 2g(K)}{2g(K)(g(K) - 1)} = \frac{6}{4} < 2 \), it follows that \( S_3^3\left(\frac{Q_{-1}(K)}{3}\right) \cong S_2^3\left(\frac{Q_{-1}(K)}{3}\right) \) for \( q \geq 2 \). Hence, the only pairs of surgery manifolds we cannot distinguish are \( \{S_1^3\left(\frac{Q_{-1}(K)}{3}\right), S_1^3\left(\frac{Q_{-1}(K)}{3}\right)\} \) and \( \{S_2^3\left(\frac{Q_{-1}(K)}{3}\right), S_2^3\left(\frac{Q_{-1}(K)}{3}\right)\} \), where the companion \( K \) satisfies \( \Delta_K(t) = 2t - 5 + 2t^{-1} \), for example \( K = 6_1 \).

- Suppose \( (g, n) = (1, 0) \). Similar to the previous case, if \( |\tau(K)| = 1 \), we see that
\[
\Delta_K(t) = (s + 1)t - (2s + 1) + (s + 1)t^{-1},
\]
where \( s \geq 0 \). So
\[
\Delta_{Q_0(K)}(t) = \Delta_U(t)\Delta_K(t) = (s + 1)t - (2s + 1) + (s + 1)t^{-1},
\]
and we obtain \( \Delta''_{Q_0(K)}(1) = 2s + 2 \neq 0 \). If \( \tau(K) = 0 \), the complex \( CFK^-(K) \) consists of a one-element staircase and \( s \geq 1 \) squares centered at 0. Thus,
\[
\Delta_{Q_0(K)}(t) = st - (2s + 1) + st^{-1},
\]
so \( \Delta''_{Q_0(K)}(1) = 2s \neq 0 \). Thus, \( Q_0(K) \) satisfies the cosmetic surgery conjecture.

- Suppose \( (g, n) = (2, -1) \), and suppose \( S_3^3\left(\frac{Q_0(K)}{3}\right) \cong S_2^3\left(\frac{Q_0(K)}{3}\right) \). Since \( \text{th}(Q_n(K)) = 6 \), \( g(Q_n(K)) = 3 \), and \( \frac{\text{th}(K) + 2g(K)}{2g(K)(g(K) - 1)} = 1 \), we see that \( \{r, r'\} = \{\pm 1\} \). Further, by an argument analogous to the one when \( (g, n) = (1, -1) \), \( \Delta_K(t) \) must be of the form
\[
bt^2 - (4b + 2)t + (6b + 5) - (4b + 2)t^{-1} + bt^{-2} \quad \text{with } b \geq 1, \quad \text{or}
\]
\[
bt^2 - (4b - 2)t + (6b - 5) - (4b - 2)t^{-1} + bt^{-2} \quad \text{with } b \geq 2, \quad \text{or}
\]
\[
(b + 1)t^2 - (4b + 6)t + (6b + 11) - (4b + 6)t^{-1} + (b + 1)t^{-2} \quad \text{with } b \geq 0.
\]
An example is \( K = 8_6 \).

- Suppose \( (g, n) = (2, 0) \), and suppose \( S_3^3\left(\frac{Q_0(K)}{3}\right) \cong S_2^3\left(\frac{Q_0(K)}{3}\right) \). Since \( \text{th}(Q_n(K)) = 6 \), \( g(Q_n(K)) = 3 \), and \( \frac{\text{th}(K) + 2g(K)}{2g(K)(g(K) - 1)} = 1 \), we see that \( \{r, r'\} = \{\pm 1\} \). Further, combining [NW15, Theorem 1.2] with [Lev16, Theorem 1.4], we see that \( \tau(K) \in \{-1, 0\} \). Similar to above, we then compute that \( \Delta_K(t) \) is of the form
\[
bt^2 - 4bt + (6b - 1) - 4bt^{-1} + bt^{-2} \quad \text{with } b \geq 1 \text{ when } \tau(K) = -1 \quad \text{or}
\]
\[
bt^2 - 4bt + (6b + 1) - 4bt^{-1} + bt^{-2} \quad \text{with } b \geq 1 \text{ when } \tau(K) = -1.
\]
An example here is \( K = m8_{14} \).

This completes the proof of Theorem 1.0.6 for thin companions. \( \square \)
6.2. **L-space companions.** Recall that an *L-space* is a rational homology sphere $Y$ with the smallest possible Heegaard Floer homology in the sense that $\dim \widehat{HF}(Y) = |H_1(Y)|$. Knots that admit nontrivial L-space surgeries are referred to as *L-space knots*.

In this subsection, we prove the L-space portion of Theorem 1.0.6. We start by computing their $\delta$-thickness values.

6.2.1. **$\delta$-thickness values for L-space companions.** By [OS05, Corollary 1.3] and [HW18, Corollaries 8 and 9], the Alexander polynomial $\Delta_K(t)$ of any L-space knot $K$ takes the form

$$\Delta_K(t) = t^{-r_0} - t^{-r_1} + \ldots + (-1)^k t^{-r_k} + (-1)^{k+1} + (-1)^k t^r_k + \ldots - t^r_1 + t^{r_0},$$

for some integers $r_0, r_1, \ldots, r_k$ satisfying

- $r_0 = g$
- $0 < r_k < \ldots < r_1 < g$
- If $k = 0$, then $K$ is either the unknot or a trefoil knot
- If $k \geq 1$, then $r_1 = g - 1$.

Let $\ell_i = r_i - r_{i+1}$, for $i \in \{1, \ldots, k - 1\}$, and define

$$M = \begin{cases} \max(\ell_1, \ldots, \ell_{k-1}, r_k) & \text{if } k \geq 2 \\ r_1 & \text{if } k = 1 \\ 1 & \text{if } k = 0. \end{cases}$$

With the notation above, we have the following theorem:

**Proposition 6.2.1.** If $K$ is the unknot or a trefoil, then $\text{th}(Q_n(K))$ is given by Proposition 6.1.1. For all other L-space knots $K$,

$$\text{th}(Q_n(K)) = \begin{cases} 2g - n - 1 & \text{if } n \leq -2g \\ 4g - 2 & \text{if } n \in [-2g + 1, 2g - M - 1] \\ M + 2g + n - 1 & \text{if } n \geq 2g - M. \end{cases}$$

**Proof.** Let $K$ be neither the unknot, nor a trefoil.

Assume $K$ admits a positive L-space surgery. By [OS05, Theorem 1.2] and [Hom14b, Remark 6.6], there exists a basis $\{\xi_0, \omega_1, \ldots, \omega_k, \tilde{\omega}_k, \ldots, \tilde{\omega}_1, \tilde{\eta}_0\}$ for $\widehat{CFK}^{-}(K)$ with respect to which $\widehat{CFK}^{-}(K)$ looks like a right-handed staircase where the heights and widths of the steps are given by $1, \ell_i$, and $r_i$. See Figure 8. Since $\{\tilde{\xi}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_k, \tilde{\omega}_k, \ldots, \tilde{\omega}_1, \tilde{\eta}_0\}$ is horizontally and vertically simplified, the invariant $\widehat{CFD}(X_{K,n})$ is given by Figure 9. To compute $\text{th}(Q_n(K))$, we need the $\delta$-gradings of the generators of $\widehat{HFK}(Q_n(K)) \cong \widehat{H}_*(\widehat{CFA}(\mathcal{H}) \boxtimes \widehat{CFD}(X_{K,n}))$.

First observe that the basis elements $\tilde{\xi}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_k, \tilde{\omega}_k, \ldots, \tilde{\omega}_1, \tilde{\eta}_0$ have Alexander and Maslov gradings given by Table 9. The Alexander gradings come from the powers of the Alexander polynomial $\Delta_K(t)$. The Maslov gradings for $\tilde{\xi}_0$ and $\tilde{\eta}_0$ come from Equation 2. The rest of the Maslov gradings come from the fact that the differential in $\widehat{CFK}^{-}(K)$ decreases the Maslov grading by 1.

Next note that because $K$ is neither the unknot nor the right-handed trefoil, $k \geq 1$. For simplicity, we take $k$ to be odd, as the argument for the even case is similar. Then
\[\tilde{\xi}_0 \xleftarrow{1} \tilde{\omega}_1 \xrightarrow{\ell_1} \tilde{\omega}_2 \]
\[\tilde{\omega}_{k-1} \xleftarrow{\ell_{k-1}} \tilde{\omega}_k \]
\[\tilde{\theta}_{k-1} \xrightarrow{r_k} \tilde{\theta}_k \]
\[\tilde{\omega}_{k+1} \xleftarrow{\ell_k} \tilde{\omega}_k \]
\[\tilde{\theta}_1 \xrightarrow{\theta_1} \tilde{\eta}_0\]

**Figure 8.** \(\text{CFK}^- (K)\) for L-space knots \(K\) that admit positive L-space surgeries. The left staircase is for \(k\) odd, while the right staircase is for \(k\) even.

\[\tilde{\xi}_0 \xleftarrow{\rho_2} \lambda_1^{\omega_1} \xleftarrow{\rho_1} \omega_1 \]
\[\tilde{\omega}_2 \]
\[\tilde{\theta}_2 \xrightarrow{\rho_2} \lambda_{e_1}^{\theta_1} \ldots \lambda_{e_1}^{\theta_1} \xleftarrow{\rho_1} \theta_1 \]
\[\tilde{\theta}_1 \xrightarrow{\theta_1} \tilde{\eta}_0 \]

**Figure 9.** \(\widehat{\text{CFD}}(X_{K,n})\) for L-space knots \(K\) that admit positive L-space surgeries. The dotted arrow represents the unstable chain.

\(\text{CFK}^- (K)\) is given by the left staircase in Figure 8. We have several cases, depending on the framing \(n\) relative to \(2\tau(K) = 2g\). Recall that

\[M = \begin{cases} 
\max(\ell_1, \ldots, \ell_{k-1}, r_k) & \text{if } k \geq 2 \\
r_1 & \text{if } k = 1.
\end{cases}\]
| Basis element | $A$ | $M$ |
|---------------|-----|-----|
| $\xi_0$       | $r_0 = g$ | 0   |
| $\omega_i$, $i \in \{1, \ldots, k\}$ | $r_i$ | $r_i - g$ |
| $\omega_{k+1}$ | 0   | $-g$ |
| $\theta_i$, $i \in \{1, \ldots, k\}$ | $-r_i$ | $-r_i - g$ |
| $\eta_0$      | $-r_0 = -g$ | $-2g$ |

**Table 9.** The Alexander and Maslov gradings of the basis elements $\tilde{\xi}_0, \tilde{\omega}_i, \tilde{\theta}_i, \tilde{\eta}_0$.

- When $n = 2g$, the unstable chain in $\mathcal{CFD}(X_{K,n})$ takes the form

$$\xi_0 \xrightarrow{D_{2g}} \eta_0.$$ 

Table 10 gives the generators of $\widehat{HFK}(Q_n(K))$, together with their $\delta$-gradings. One can show that the minimal $\delta$-grading of all generators in $\widehat{HFK}(Q_n(K))$ is $-3g + 2$, and the maximal $\delta$-grading of all generators in $\widehat{HFK}(Q_n(K))$ is $M + g + 1$. Then $\text{th}(Q_n(K)) = M + 4g - 1$.

- When $n < 2g$, the unstable chain in $\mathcal{CFD}(X_{K,n})$ takes the form

$$\xi_0 \xrightarrow{D_1} \mu_1 \xleftarrow{D_{2g}} \cdots \xleftarrow{D_{2g}} \mu_{2g-n} \xleftarrow{D_1} \eta_0.$$ 

Table 11 gives the generators of $\widehat{HFK}(Q_n(K))$, together with their $\delta$-gradings. One can verify that

$$\min \{\delta_{rel}(u \boxtimes v) : u \boxtimes v \text{ generates } \widehat{HFK}(Q_n(K))\} = \begin{cases} n - g + 1 & \text{if } n \in (-\infty, -2g + 1] \\ -3g + 2 & \text{if } n \in [-2g + 1, 2g). \end{cases}$$

and

$$\max \{\delta_{rel}(u \boxtimes v) : u \boxtimes v \text{ generates } \widehat{HFK}(Q_n(K))\} = \begin{cases} g & \text{if } n \in (-\infty, 2g - M - 1] \\ M - g + n + 1 & \text{if } n \in [2g - M - 1, 2g). \end{cases}$$

Note that $-2g + 1 < 2g - M - 1$. Then we have that

$$\text{th}(Q_n(K)) = \begin{cases} 2g - n - 1 & \text{if } n \in (-\infty, -2g] \\ 4g - 2 & \text{if } n \in [-2g + 1, 2g - M - 1] \\ M + 2g + n - 1 & \text{if } n \in [2g - M, 2g). \end{cases}$$

- The $n > 2g$ case is similar to the $n < 2g$ case.

The case where $K$ admits a negative L-space surgery is analogous. This concludes the proof of Proposition 6.2.1.

6.2.2. **Cosmetic Surgery Conjecture for L-space companions.** In this subsection, we prove Theorem 1.0.6 for L-space companions. Our main technical tool will be Inequality 6. We use the same notation as in Section 6.2.1.
Proof of Theorem 1.0.6 for L-space companions. First suppose $K$ is neither the unknot nor a trefoil. Then $g = g(K) \geq 2$. By Theorem 1.0.4, $g(Q_n(K)) \geq 3$ for every $n$. This means that we can use Inequality 6 to test whether $Q_n(K)$ satisfies the cosmetic surgery conjecture. We consider several cases, depending on our values for $\text{th}(Q_n(K))$ from Proposition 6.2.1.

We begin with the case $n < -2g$. Then $\text{th}(Q_n(K)) = 2g - n - 1$ and $g(Q_n(K)) = g - n$. By the argument in Case 1.2 of Section 6.1.2, the satellites $Q_n(K)$ satisfy the cosmetic surgery conjecture.

Now suppose $n \in [-2g+1, 2g-2]$. Then $\text{th}(Q_n(K)) = 4g - 2$. We consider two subcases:

- Suppose $n \in [-2g+1, -1]$. Then $g(Q_n(K)) = g - n$. As seen in Section 6.2.1, for every $n \leq -1$ and $g \geq 2$, except for $n = -1$ and $g = 2$, $f(Q_n(K)) > 0$. Hence for every $n \leq -1$ and $g \geq 2$, except for $n = -1$ and $g = 2$, the satellites $Q_n(K)$ satisfy the cosmetic surgery conjecture. Now we resolve the remaining case where $n = -1$ and $g = 2$, using the Boyer-Lines obstruction in [BL90, Proposition 5.1].
First note that $\Delta_K(t) = t^{-2} - t^{-1} + 1 + t + t^2$ and $\Delta_{Q_{-1}(U)}(t) = \Delta_2(t) = 2t^{-1} - 3 + 2t$. Then $\Delta_{Q_{-1}(K)}(1) = \Delta''_{Q_{-1}(U)}(1) + \Delta''_{K}(1) = 4 + 6 = 10$. By [BL90, Proposition 5.1], $Q_{-1}(K)$ satisfies the cosmetic surgery conjecture.

- Suppose $n \in [0, 2g - 2]$. Then $g(Q_n(K)) = n + g + 1$. As seen in Section 6.2.1, for every $n \geq 0$ and $g \geq 2$, except for $n = 0$ and $g = 2$, $f(Q_n(K)) > 0$. Hence for every $n \geq 0$ and $g \geq 2$, except for $n = 0$ and $g = 2$, the satellites $Q_n(K)$ satisfy the cosmetic surgery conjecture. Now suppose $n = 0$ and $g = 2$. Then $\Delta_{Q_0(K)}(t) = \Delta_K(t) = t^{-2} - t^{-1} + 1 + t + t^2$, which implies that $\Delta''_{Q_0(K)}(1) = \Delta''_{K}(1) = 6$. By [BL90, Proposition 5.1], $Q_0(K)$ satisfies the cosmetic surgery conjecture.
Finally, we consider the case where \( n \geq 2g - M \). Then \( \text{th}(Q_n(K)) = M + 2g + n - 1 \) and \( g(Q_n(K)) = g + n + 1 \). For every \( n \geq 2 \) and \( g \geq 2 \), \( f(Q_n(K)) = 2n^2 + 4ng + 2g^2 - n - 2g - M - 1 > 0 \). Thus, when \( n \geq 2g - M \), the satellites \( Q_n(K) \) also satisfy the cosmetic surgery conjecture.

If \( K \) is the unknot or a trefoil, then by Section 6.2.1, all nontrivial \( Q_n(K) \) satisfy the cosmetic surgery conjecture. This concludes the proof of Theorem 1.0.6 for L-space companions.

\[ \square \]

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