Sampling from a couple of negatively correlated gamma variates

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Abstract
We propose two algorithms for sampling from two gamma variates possessing a negative correlation. The case of positive correlation is easily solved, so we just mention it. The main problem is the lowest value of the correlation coefficient that can be reached. The starting point of both algorithms is generation from a bivariate density with uniform negatively correlated marginals. Actually the first method uses a degenerate bivariate density since it considers two uniforms related by a linear relationship. Then we resort essentially to the inverse transform method. For both algorithms we stress restrictions on the parameters and rigidities.

1 Introduction
To fix notation, if $X$ follows a gamma distribution with parameters $\lambda$ and $\alpha$, that is $X \sim \mathcal{G}(\lambda, \alpha)$, then the density is

$$f_X(x) = \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha - 1}}{\Gamma(\alpha)}, \quad x > 0.$$  

Johnson and Kotz [2, 1972] define a multivariate gamma distribution in the following way. We have $m + 1$ independent standard (that is, each with $\lambda = 1$), gamma variates (in general with different $\alpha$’s) $X_0, X_1, \ldots, X_m$. Define

$$Y_j = X_0 + X_j, \quad j = 1, \ldots, m$$
Then the $Y_j$, $j = 1, \ldots, m$ are distributed according a $m$-variate gamma variable. We can see that each marginal $Y_j$ is a standard gamma variable with $\alpha = \alpha_0 + \alpha_j$ and

$$\text{Cov}(Y_j, Y_{j'}) = \text{Var}(X_0)$$
$$= \alpha_0,$$

and so all the marginals are positively correlated.

We follow this suggestion, with $m = 2$, but we exploit the preserving monotonicity property of the inverse transform method, see Fishman [1, 1996] to generate a couple of gamma variates, $X_1$ and $X_2$, with a rigid amount of negative correlation and then generate $X_0$ to allow for some flexibility.

### 2 First Method

Let $X_1 \sim \mathcal{G}(1, r)$ and $X_2 \sim \mathcal{G}(1, s)$, with $r$ and $s$ integers. Assume, without loss of generality, $r < s$. Let $U_i$ be $s$ independent uniforms on the unit interval, that is $U_i \sim \mathcal{U}(0, 1)$, $i = 1, \ldots, s$. Then we can write, using properties of the gamma variate and the inverse transform method,

$$X_1 = -\sum_{i=1}^{r} \ln U_i,$$
$$X_2 = -\sum_{i=1}^{s} \ln(1 - U_i).$$

It follows that

$$\text{Cov}(X_1, X_2) = \text{Cov}\left( \sum_{i=1}^{r} \ln U_i, \sum_{i=1}^{s} \ln(1 - U_i) \right)$$
$$= \sum_{i=1}^{r} \text{Cov} (\ln U_i, \ln(1 - U_i))$$
$$= \sum_{i=1}^{r} \left[ \left( 2 - \frac{\pi^2}{6} \right) - 1 \right]$$
$$= r \left( 1 - \frac{\pi^2}{6} \right).$$

Now define, with $X_0 \sim \mathcal{G}(1, \alpha_0)$ independent of $X_1$ and $X_2$,

$$Y_1 = X_0 + X_1.$$
\[ Y_2 = X_0 + X_2. \]

It follows \( Y_1 \sim \mathcal{G}(1, \alpha_0 + r) \) and \( Y_2 \sim \mathcal{G}(1, \alpha_0 + s) \). Furthermore

\[
\text{Cov}(Y_1, Y_2) = \text{Var}(X_0) + \text{Cov}(X_1, X_2) = \alpha_0 + r \left(1 - \frac{\pi^2}{6}\right).
\]

Because

\[ c = 1 - \frac{\pi^2}{6} = -0.644934, \]

we can see that varying \( r \) and \( \alpha_0 \) we have some freedom in modelling a negative covariance.

The correlation coefficient between \( Y_1 \) and \( Y_2 \) is given by

\[
\rho(Y_1, Y_2) = \frac{\alpha_0 + rc}{\sqrt{(\alpha_0 + r)(\alpha_0 + s)}}. \tag{1}
\]

For \( \alpha_0 + rc < 0 \), which guarantees a negative correlation, we see that

\[
\frac{\partial \rho(Y_1, Y_2)}{\partial \alpha_0} > 0,
\]

and so when the correlation is negative, it is an increasing function of \( \alpha_0 \).

The lower bound for \( \rho \) is given by

\[
\frac{rc}{\sqrt{rs}} = c\sqrt{s}. 
\]

When \( r = s \) and assuming that \( \alpha_0 = 0 \) means that \( X_0 = 0 \) with probability 1 we reach the most negative correlation possible between two gamma variates.

If we are given \( \alpha_0 + r = m, \alpha_0 + s = n \) and \( \rho = \rho_0 \) then from Equation (1) we get

\[
\rho_0 = \frac{m + r(c - 1)}{\sqrt{mn}}, \tag{2}
\]

which can be solved for \( r \) and subsequently obtaining \( \alpha_0 \) and \( s \). The drawback is that \( r \) and \( s \) are required to be integers, so we cannot arbitrarily choose \( m, n \) and \( \rho \). To give an idea of the situation we present some tables, assuming for simplicity that both \( m \) and \( n \) are integers. It follows that also \( \alpha_0 \) has to be an integer. Because we want a negative correlation and because \( \alpha_0 \geq 0 \) we have the restriction

\[ r \leq m < r(1 - c). \]
Using Equation 2 we obtain

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 2 | 2 | 3 | .5266 | 2 | 2 | 5 | .4078 |
| 2 | 3 | 4 | .0837 | 2 | 3 | 6 | .0683 |
| 5 | 5 | 6 | .5887 | 5 | 5 | 8 | .5098 |
| 5 | 6 | 7 | .3432 | 5 | 6 | 9 | .3027 |
| 5 | 7 | 8 | .1636 | 5 | 7 | 10 | .1463 |
| 5 | 8 | 9 | .0264 | 5 | 8 | 11 | .0239 |
| 8 | 8 | 9 | .6080 | 8 | 8 | 11 | .5500 |
| 8 | 9 | 10 | .4384 | 8 | 9 | 12 | .4002 |
| 8 | 10 | 11 | .3012 | 8 | 10 | 13 | .2771 |
| 8 | 11 | 12 | .1879 | 8 | 11 | 14 | .1740 |
| 8 | 12 | 13 | .0928 | 8 | 12 | 15 | .0864 |
| 8 | 13 | 14 | .0118 | 8 | 13 | 16 | .0110 |
| 12 | 12 | 13 | .6196 | 12 | 12 | 15 | .5768 |
| 12 | 13 | 14 | .4995 | 12 | 13 | 16 | .4672 |
| 12 | 14 | 15 | .3960 | 12 | 14 | 17 | .3720 |
| 12 | 15 | 16 | .3059 | 12 | 15 | 18 | .2884 |
| 12 | 16 | 17 | .2267 | 12 | 16 | 19 | .2144 |
| 12 | 17 | 18 | .1565 | 12 | 17 | 20 | .1485 |
| 12 | 18 | 19 | .0940 | 12 | 18 | 21 | .0894 |
| 12 | 19 | 20 | .0379 | 12 | 19 | 22 | .0361 |

Nothing essentially changes if we require gamma's with the first parameter different from 1: it is enough to multiply \( Y_1 \) and \( Y_2 \) by, say, a constant \( \beta \). The correlation coefficient is of course not affected.

### 3 Joint Density

Obtaining the joint density of \( Y_1 \) and \( Y_2 \) in the general case is cumbersome. To give an idea we consider the simplest case, \( r = s = 1 \), and we set \( \alpha < -c \) that is \( \alpha < 0.644934 \), to guarantee a negative correlation. So we have

\[
Y_1 = X_0 - \ln U, \quad (3)
\]

\[
Y_2 = X_0 - \ln(1 - U). \quad (4)
\]
with independence of $X_0$ and $U$. It turns out that marginally $Y_1$ and $Y_2$ are both $G(1, 1 + \alpha_0)$. Setting
\[
\begin{align*}
y_1 &= x_0 - \ln u \\
y_2 &= x_0 - \ln(1 - u),
\end{align*}
\]
we get
\[
\begin{align*}
u &= \frac{1}{1 + e^{y_1 - y_2}} \\
x_0 &= y_1 - \ln \left(1 + e^{y_1 - y_2}\right).
\end{align*}
\]
Because $X_0 \geq 0$ we have the inequality
\[
y_1 - \ln \left(1 + e^{y_1 - y_2}\right) \geq 0.
\]
The solution to the equation
\[
y_1 - \ln \left(1 + e^{y_1 - y_2}\right) = 0
\]
is given by
\[
y_2 = y_1 - \ln \left(e^{y_1} - 1\right).
\]
It follows that we have $x_0 \geq 0$ if
\[
y_2 \geq y_1 - \ln \left(e^{y_1} - 1\right).
\]
For the Jacobian of this transformation we have
\[
J = \frac{e^{y_1 - y_2}}{(1 + e^{y_1 - y_2})^2}.
\]
The joint density of $X_0$ and $U$ is given by
\[
f_{X_0,U}(x_0, u) = e^{-x_0} x_0^{a-1}, \quad \begin{cases} 0 \leq u \leq 1 \\ 0 \leq x_0. \end{cases}
\]
The quantity $e^{-x_0}J$ simplifies to
\[
\frac{1}{e^{y_1} + e^{y_2}}
\]
and consequently we get for the joint density of $Y_1$ and $Y_2$
\[
f_{Y_1,Y_2}(y_1, y_2) = \frac{[y_1 - \ln \left(1 + e^{y_1 - y_2}\right)]^{a-1}}{e^{y_1} + e^{y_2}}, \quad \begin{cases} y_1 > 0 \\ y_2 > y_1 - \ln \left(e^{y_1} - 1\right). \end{cases}
\]
We see that for $\alpha = 1$ this simplifies to
\[ f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{e^{y_1} + e^{y_2}}, \quad \begin{cases} y_1 > 0 \\ y_2 > y_1 - \ln(e^{y_1} - 1) \end{cases}, \]
but in this case the correlation is positive:
\[ \rho(Y_1, Y_2) = \frac{1 + c}{2} = 0.177533. \]
In this last situation the marginals are $G(1, 2)$. This can be checked using the fact that
\[ \int_{b}^{+\infty} \frac{1}{a + e^{x}} \, dx = \frac{-b + \ln(a + \sqrt{a})}{a}. \]

4 Second Method

Again looking for flexibility we analyze another method where we start generating $s$ samples $(U_1, U_2)$ from a bivariate distribution with density
\[ f(u_1, u_2) = 1 + \theta(1 - 2u_1)(1 - 2u_2), \quad 0 \leq u_1, u_2 \leq 1. \quad (5) \]
This density is of the form studied by Long and Krzysztofowicz [5, 1995]. For $-1 \leq \theta \leq 1$ this function is a proper density. It turns out that marginally $U_1$ and $U_2$ are uniforms over the unit interval and the correlation coefficient is given by
\[ \rho(U_1, U_2) = \frac{\theta}{3}. \]
Now define
\[ X_1 = -\sum_{i=1}^{r} \ln U_{1i}, \]
\[ X_2 = -\sum_{i=1}^{s} \ln U_{2i}. \]
Now
\[ \text{Cov}(\ln U_1, \ln U_2) = \int_{0}^{1} \int_{0}^{1} \ln x \ln y (1 + \theta(1 - 2x)(1 - 2y)) \, dx \, dy - 1 \]
\[ = 1 + \frac{\theta}{4} - 1 \]
\[ = \frac{\theta}{4}. \]
It follows that
\[ \text{Cov}(X_1, X_2) = \frac{r\theta}{4}. \]

Proceeding as in the other method we define, with \( X_0 \sim G(1, \alpha_0) \) independent of \( X_1 \) and \( X_2 \),
\[ Y_1 = X_0 + X_1, \]
\[ Y_2 = X_0 + X_2, \]
and we get
\[ \rho(Y_1, Y_2) = \frac{\alpha_0 + \frac{r\theta}{4}}{\sqrt{\alpha_0 + r}(\alpha_0 + s)}, \quad (6) \]

Now we have a negative correlation if \( 4\alpha_0 + r\theta < 0 \). Because of this inequality the admissible range for \( \theta \) is \(-1 \leq \theta < 0\). The lower bound for \( \rho \) is
\[ \frac{\theta}{4\sqrt{s}}, \]
so we cannot hope to do better than
\[ \rho > -0.25. \]

Repeating the same reasoning that led us to Equation 3 now we have
\[ \rho_0 = \frac{4m - r(4 - \theta)}{4\sqrt{mn}}. \]

The restrictions are now
\[ r \leq m < \frac{r(4 - \theta)}{4}. \quad (7) \]

Solving for \( r \) we get
\[ r = \frac{4m - 4\rho_0\sqrt{mn}}{4 - \theta}. \quad (8) \]

Other restrictions on the minimum value admissible for \( \rho_0 \) arise from Equation 8 and the fact that \( r \) must be integer. The minimum value of \( r \) as given by Equation 8 is obtained for \( \theta = -1 \). Then
\[ \frac{4m - 4\rho_0\sqrt{mn}}{5} \leq m - 1, \]
which implies
\[ \rho_0 \geq -\frac{m - 5}{4\sqrt{mn}}. \quad (9) \]
Under the condition $m \geq 6$ this attainable lower bound is an increasing function of $n$ and because $n \geq m$ it is a decreasing function of $m$: in particular if $m = n$ the limit for $m$ going to infinity is $-0.25$.

Once observed the conditions on $m$ and $\rho$ we have the following algorithm.

1. Set $y = 4m - 4\rho_0\sqrt{mn}$.
2. Obtain $a = \frac{y}{\rho}$.
3. Set $r = [a]$, that is the lowest integer not lower than $a$. Because of the construction such an $r$ exists and $r < m$.
4. Obtain $\theta = 4 - y/r$.

As an example, imagine $m = 7$, $n = 10$. The attainable lower bound is $-0.0597$. Set for example $\rho_0 = -0.05$. Then $\frac{y}{\rho} = 5.2653$. Take $r = 6$, so that $\theta = 4 - 4.38778 = -0.38778$.

Once $\theta$ is obtained, the next step is to evaluate $\alpha_0 = m - r$ and $s = n - \alpha_0$. Then we generate $s$ samples from the density given in Equation 5; this can be done for instance with the acceptance-rejection method. And then we follow the final construction to obtain $Y_1$ and $Y_2$.

We can note that in this second method the attainable lower bound for $\rho$ is sensibly greater than in the first method, but for the admissible values we have a complete flexibility. The other negative point is that we have to generate samples from a bivariate density, whose covariance scalar $\theta$ (as termed by Long and Krzysztofowicz) has to be preliminarily obtained, instead of generations from univariate densities.

References

[1] G.S. Fishman (1996), *Monte Carlo. Concepts, Algorithms, and Applications*, Springer, New York.

[2] N.L. Johnson and S. Kotz (1972), *Distributions in Statistics: Continuous Multivariate Distributions*, Wiley, New York.

[3] D. Long and R. Krzysztofowicz (1995), A Family of Bivariate Densities Constructed from Marginals, *Journal of the American Statistical Association*, 90: 739-748.