On a Babuška Paradox for Polyharmonic Operators: Spectral Stability and Boundary Homogenization for Intermediate Problems

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Abstract. We analyse the spectral convergence of high order elliptic differential operators subject to singular domain perturbations and homogeneous boundary conditions of intermediate type. We identify sharp assumptions on the domain perturbations improving, in the case of polyharmonic operators of higher order, conditions known to be sharp in the case of fourth order operators. The optimality is proved by analysing in detail a boundary homogenization problem, which provides a smooth version of a polyharmonic Babuška paradox.

Keywords. Spectral analysis, Polyharmonic operators, Boundary homogenization.

1. Introduction

A recurrent topic in the Analysis of Partial Differential Equations, in Spectral Theory, and their applications is the study of the variation of the solutions to elliptic boundary value problems on domains subject to boundary perturbation, with contributions rooting back in the works of Courant and Hilbert [27], and Keldysh [37]. The mathematical interest in this type of problems is also given by the possible appearance of an unexpected asymptotic behaviour of the solutions, which can be understood as a spectral instability phenomenon. Probably the most famous example in elasticity theory is the celebrated Babuška paradox which concerns the approximation of a thin hinged circular plate by means of an invading sequence of convex polygons. This problem was considered by Babuška in [10] and was further discussed by Maz’ya and Nazarov in [38] where among various results they present a variant of the Babuška paradox consisting in the approximation a thin hinged circular plate by means of an invading sequence of non-convex, indented polygons (see [33, § 1.4], for a recent discussion on this subject and for more details concerning the related results of Sapondžyan [44]). We find convenient to briefly recall the formulation of the paradox.
Given a circle $\Omega$ in $\mathbb{R}^2$ and a datum $f \in L^2(\Omega)$, consider the following boundary value problem

\[
\begin{cases}
\Delta^2 u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, \\
\partial^2 u / \partial n^2 = 0, & \text{on } \partial \Omega, 
\end{cases}
\]

(1.1)
in the unknown real-valued function $u$. Note that here and in the sequel, boundary value problems will be understood in the weak sense. Thus, problem (1.1) consists in finding $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ such that

\[
\int_\Omega D^2 u : D^2 \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \text{for all } \varphi \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega),
\]

where $D^2 u : D^2 \varphi = \sum_{i,j=1}^N u_{x_i x_j} \varphi_{x_i x_j}$ is the Frobenius product of the two Hessian matrices of $u$ and $\varphi$. In the theory of elastic plates, $u$ represents the deflection of a hinged thin plate with midplane $\Omega$ and normal load $f$.

Define inside $\Omega$ an invading sequence of indented polygons $\Omega_n$ obtained by modifying an inscribed convex polygon with $n$ vertexes $p^n_j$, $j = 1, \ldots, n$, and replacing its contour line in a neighbourhood of each $p^n_j$ by a $V$-shaped line as in Fig. 1. The small curvilinear triangles appearing have height equal to $h^n_j$ and base of length $\eta^n_j$, while the length of the nearby chord (the side of the polygon) is denoted by $\zeta^n_j$. Consider now the same boundary value problem in $\Omega_n$

\[
\begin{cases}
\Delta^2 u_n = f, & \text{in } \Omega_n, \\
u_n = 0, & \text{on } \partial \Omega_n, \\
\partial^2 u_n / \partial n^2 = 0, & \text{on } \partial \Omega_n, 
\end{cases}
\]

(1.2)
in the unknown $u_n \in W^{2,2}(\Omega_n) \cap W^{1,2}_0(\Omega_n)$. The paradox lies in the fact that if

\[
\max_{1 \leq j \leq n} \frac{|\zeta^n_j|}{|\eta^n_j|} = O(1), \quad \max_{1 \leq j \leq n} \frac{|\eta^n_j|}{|h^n_j|^{2/3}} = o(1),
\]

as $n \to \infty$, then the solution $u_n \in W^{2,2}(\Omega_n) \cap W^{1,2}_0(\Omega_n)$ of (1.2) does not converge to the solution $u$ of (1.1), but to the solution $v$ of the boundary value problem

\[
\begin{cases}
\Delta^2 v = f, & \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega, \\
\partial v / \partial n = 0, & \text{on } \partial \Omega. 
\end{cases}
\]

(1.3)

Here $v$ represents the deflection of a clamped thin plate. Note that it is possible to choose $|\zeta^n_j| = 0$ for all $j$ and $n$ in order to obtain the wild looking set $\Omega_n$ in Fig. 2.

In [7,8] the authors considered a smooth version of this paradox. Given a sufficiently regular bounded domain $W$ in $\mathbb{R}^{N-1}$, $N \geq 2$, they define a family of domains $(\Omega_\epsilon)_{0 < \epsilon < \epsilon_0}$ by setting

\[
\Omega = W \times (-1,0), \quad \Omega_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < \epsilon \alpha b(\bar{x}/\epsilon)\}
\]

(1.4)
Figure 1. Indented polygon

Figure 2. Degenerate indented polygon

where \( \bar{x} = (x_1, \ldots, x_{N-1}) \), and \( b \) is a non-constant, smooth, positive, periodic function of period \( Y = [-1/2, 1/2]^{N-1} \). The geometry of this perturbation is described in Fig. 3 below.

By comparing Figs. 3a and 2, one realizes that the perturbations look similar locally at the boundary. This analogy goes further if we define \( h^n_j = \epsilon^\alpha \) and \( \eta^n_j = \epsilon \), with \( \epsilon = 1/n \). Indeed, in [8] it was proved that if

\[
\frac{|\eta^n_j|}{|h^n_j|^{2/3}} = \frac{\epsilon}{\epsilon^{2/3} \alpha} = o(1),
\]

as \( \epsilon \to 0 \), that is if \( \alpha < 3/2 \), then the same Babuška-type paradox appears. Moreover, it was also proved that if \( \alpha > 3/2 \) then no Babuška paradox appears and there is spectral stability. The threshold \( \alpha = 3/2 \) is then critical and represents a typical case of study for homogenization theory; in fact, it was proved in [8] that the limiting problem contains a ‘strange term’ which could be interpreted as a ‘strange curvature’.

It is then natural to wonder whether Babuška-type paradoxes may be detected in the case of polyharmonic operators \((-\Delta)^m, \ m > 2\) subject to
intermediate boundary conditions. The answer is not as straightforward as it may appear, and it is necessary to clarify first what are the possible boundary conditions for those operators. Indeed, there exists a whole family of boundary value problems depending on a parameter $k = 0, 1, \ldots, m$, the formulation of which reads as follows: given a bounded domain (i.e., a connected open set) $\Omega$ in $\mathbb{R}^N$ with sufficiently smooth boundary, $m \in \mathbb{N}$, and $f \in L^2(\Omega)$, find $u \in W^{m,2}(\Omega) \cap W^{k,2}_0(\Omega)$ such that

$$\int_{\Omega} D^m u : D^m \varphi dx + \int_{\Omega} u \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in W^{m,2}(\Omega) \cap W^{k,2}_0(\Omega). \quad (1.5)$$

Here we denote by $W^{m,2}(\Omega)$ the standard Sobolev space of functions in $L^2(\Omega)$ with weak derivatives up to order $m$ in $L^2(\Omega)$ and by $W^{k,2}_0(\Omega)$ the closure in $W^{k,2}(\Omega)$ of the $C^\infty$-functions with compact support in $\Omega$. Note that for $k = m$ one obtains the Dirichlet problem

$$\begin{cases}
(-\Delta)^m u + u = f, & \text{in } \Omega, \\
\frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega, \quad \text{for all } 0 \leq l \leq m - 1,
\end{cases} \quad (1.6)$$

while for $k = m - 1$ one gets the significantly different problem

$$\begin{cases}
(-\Delta)^m u + u = f, & \text{in } \Omega, \\
\frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega, \quad \text{for all } 0 \leq l \leq m - 2, \\
\frac{\partial^m u}{\partial n^m} = 0, & \text{on } \partial\Omega.
\end{cases} \quad (1.7)$$

Finally, for $k = 0$ one gets the problem with natural boundary conditions, also known as Neumann problem, and this explains why problem (1.7) is called intermediate. Actually, in this paper we refer to problem (1.7) as to the strong intermediate problem to emphasise the fact that (1.7) is the intermediate
problem with the largest $k$ and to distinguish it from the other cases where $0 < k < m - 1$ which are called here weak intermediate problems. According to these considerations, one is led to ask the following:

**Question.** Are there Babuška-type paradoxes for polyharmonic operators $(-\Delta)^m$, $m > 2$ satisfying intermediate boundary conditions, and which are the natural assumptions which prevent the appearance of this paradox?

We are able to answer to this question in the geometric setting given by (1.4). Since when $m = 2$ problem (1.7) coincides with the hinged plate (1.1), the Babuška paradox will be discussed for polyharmonic operators with strong intermediate boundary conditions (in short, (SIBC)), being the natural higher order version of the intermediate boundary conditions for the biharmonic operator.

Let us describe one of the two main results of this paper. Let $\Omega_\epsilon$ and $\Omega$ be as in (1.4), $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon)$. For every $\epsilon > 0$, let $u_\epsilon \in V(\Omega_\epsilon)$ be the solution of
\[
\int_{\Omega_\epsilon} D^m u_\epsilon : D^m \varphi + u_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} f \varphi \, dx, \quad \text{for all } \varphi \in V(\Omega_\epsilon). \tag{1.8}
\]
Recall that this is the weak formulation of the Poisson problem for $(-\Delta)^m + I$ with (SIBC). For $u \in W^{m,2}(\Omega)$, define $T_\epsilon u = u \circ \Phi_\epsilon$ where $\Phi_\epsilon$ is a smooth diffeomorphism mapping $\Omega_\epsilon$ into $\Omega$ that coincides with the identity on a large part $K_\epsilon$ of $\Omega$, with $|\Omega \setminus K_\epsilon| \to 0$ as $\epsilon \to 0$, see (3.5). Let $u$ be such that $\|u_\epsilon - T_\epsilon u\|_{L^2(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$.

Theorem 7 states that the limit $u$ solves different differential problems according to the values of the parameter $\alpha$. More precisely, we have the following trichotomy:

(i) (Stability) If $\alpha > 3/2$, then $u$ solves (1.8) in $\Omega$, that is, $u$ satisfies $(-\Delta)^m u + u = f$ in $\Omega$ and (SIBC) on $\partial \Omega$;

(ii) (Degeneration) If $\alpha < 3/2$, then $u$ satisfies $(-\Delta)^m u + u = f$ in $\Omega$, with Dirichlet boundary conditions on $W \times \{0\}$, that is
\[
\frac{\partial^l u}{\partial n^l} = 0, \quad \text{for all } 0 \leq l \leq m - 1,
\]
and (SIBC) on the rest of the boundary of $\Omega$;

(iii) (Strange term) If $\alpha = 3/2$, then $u$ satisfies $(-\Delta)^m u + u = f$ in $\Omega$ with the following boundary conditions on $W \times \{0\}$
\[
\begin{aligned}
D^l u = 0, & \quad \text{for all } 0 \leq l \leq m - 2, \\
\frac{\partial^m u}{\partial n^m} + K \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0, & \quad \text{for } m > 2,
\end{aligned}
\]
and (SIBC) on the rest of the boundary of $\Omega$. Here $K$ is a certain positive constant that can be characterized as the energy of a suitable $m$-harmonic function in $Y \times (-\infty, 0)$.

It follows that if $\alpha < 3/2$ a polyharmonic Babuška paradox appears. It is interesting to observe that the critical value $3/2$ is the same for all the polyharmonic operators with (SIBC).
The techniques used to prove Theorem 7 vary drastically depending on the case (i)–(iii) considered. Theorem 7(i) is a consequence of Theorem 2, which is the second main result of the paper and provides a general stability criterion for self-adjoint elliptic differential operators of order $2m$ with non-constant coefficients and compact resolvents (or, more precisely, for their realization in the space $W^{m,2}(Ω) \cap W^{k,2}_0(Ω), 0 < k < m$) on varying domains featuring a fast oscillating boundary.

Theorem 2 is an improvement of a previous result (see [8, Lemma 6.2]) and can be summarized and simplified in the following way. Let $Ω$ and $Ω_\epsilon$ be bounded domains in $\mathbb{R}^N$ defined as follows:

$$Ω = \{(\bar{x}, x_N) \in W \times (a, b) : \bar{x} \in W, a < g(\bar{x}) < b\},$$

$$Ω_\epsilon = \{(\bar{x}, x_N) \in W \times (a, b) : \bar{x} \in W, a < g_\epsilon(\bar{x}) < b\},$$

where $W \subset \mathbb{R}^{n-1}$ is as above, $a + \rho < g, g_\epsilon < b - \rho$, $a, b \in \mathbb{R}$, and $g, g_\epsilon \in C^m(\bar{W})$. If $\|g - g_\epsilon\|_{\infty}$ converges to zero as $\epsilon$ goes to zero and, for all $|\beta| = m$, $\|D^\beta(g - g_\epsilon)\|_{\infty}$ converges to zero or diverges to infinity with a suitable rate expressed in terms of a power of $\|g - g_\epsilon\|_{\infty}$, then the spectrum of the realization of a self-adjoint elliptic differential operator in $W^{m,2}(Ω_\epsilon) \cap W^{k,2}_0(Ω_\epsilon), 1 \leq k \leq m - 1$ is stable as $\epsilon \to 0$. We note that [8, Lemma 6.2] is sharp in the case $m = 2$ and $k = 1$. In Theorem 2 we allow a rate of convergence or divergence for $\|D^\beta(g - g_\epsilon)\|_{\infty}$ which is much better when $k > 1$. For example, going back to Theorem 7(i), we note the following fact: upon considering profile functions $g_\epsilon$ of the type $g_\epsilon(\bar{x}) = \epsilon^\alpha b(\frac{\bar{x}}{\epsilon})$, where $b$ is a non-constant periodic function, we could apply [8, Lemma 6.2] to the polyharmonic problem in a straightforward way; however, this would only guarantee the spectral stability for $\alpha > m - 1/2$. Our improved stability Theorem 2 guarantees the spectral stability for the better range $\alpha > m - k + 1/2$.

The proof of Theorem 7(ii) is based on a consequence of a degeneration argument that was introduced in [21], and which was already exploited in [8].

The reader may wonder if it is possible to push the arguments contained in the proof of Theorem 7 in order to discuss the general case of weak intermediate problems for polyharmonic operators. The main issue is that the degeneration argument in Theorem 7(ii) is restricted to the case of $(SIBC)$. Hence, a detailed analysis of the various possible situations seems to us much more involved and almost prohibitive for arbitrary values of $m$ and $k$. We mention that the case $m = 3, k = 1$ will be the object of a forthcoming paper and we refer to [30] for a number of results in this direction.

We remark that our main results, in particular Theorem 2 and Theorem 7, are based on the notion of $\mathcal{E}$-convergence in the sense of Vainikko [46] which is related to Stummel’s discrete convergence and to Anselone and Palmer’s collective compactness, see [45] and [2] respectively. For a recent survey on these topics and further generalisations, we refer to [11].

Finally, we mention that, in the case of second-order operators, counterexamples to the spectral stability with respect to domain perturbation are well-known, see for example the classical [27, Chp. VI, 2.6]. Related problems for the Neumann Laplace operator and for the Schrödinger operator with Neumann boundary conditions have been considered in [6,22] and [3].
respectively. Regarding higher order elliptic operators on variable domains, several contributions can be found in [4,12–14,16–18,32]. In particular, for a possible approach to these topics via asymptotic analysis, we refer to the articles [19,26,34] and to the monographs [39,40]. We refer also to the monograph [33] and the articles [31,43] where polyharmonic operators are considered. For a wider discussion about perturbation theory for linear operators we mention the monographs [35,36,41].

This paper is organised as follows. Section 2 is devoted to preliminaries and notation, in particular to the definition of the class of operators and open sets under consideration. Section 3 contains a general discussion concerning the spectral stability of elliptic operators, and the proof of Theorem 2 and its corollaries, see in particular Theorem 4. In Sect. 4 we prove a Polyharmonic Green Formula which is used in the sequel and has its own interest. Section 5 is devoted to the analysis of strong intermediate boundary conditions and to the proof of Theorem 7. In “Appendix” we prove a technical lemma used in the proof of Theorem 7(iii).

2. Preliminaries and Notation

In the sequel, we will use the following basic notation:

- $\mathbb{N}$ denotes the set of positive integers. Moreover, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- Given a normed space $X$, $\mathcal{L}(X)$ is the space of bounded linear operators on $X$;
- If not otherwise specified, $m \in \mathbb{N}$ will always be greater or equal to 2;
- $\Omega, \Omega_\epsilon, \epsilon_0 \geq \epsilon > 0$ will always denote bounded domains (i.e., open connected open sets in $\mathbb{R}^N$);
- The standard Sobolev spaces with summability order 2 and smoothness order $m$ are denoted by $W^{m,2}_0(\Omega)$ and $W^{m,2}(\Omega)$.
- The notation $V(\Omega), V(\Omega_\epsilon)$ will often be used for subspaces of $W^{m,2}(\Omega)$ (resp. $W^{m,2}_0(\Omega_\epsilon)$), containing $W^{m,2}_0(\Omega)$ (resp. $W^{m,2}_0(\Omega_\epsilon)$).

2.1. Classes of Operators

Let $M$ be the number of multiindices $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ with length $|\alpha| = \alpha_1 + \cdots + \alpha_N = m$. For all $\alpha, \beta \in \mathbb{N}_0^N$ such that $|\alpha| = |\beta| = m$, let $A_{\alpha\beta}$ be bounded measurable real-valued functions defined on $\mathbb{R}^N$ satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq 0,$$

(2.1)

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$. For all open subsets $\Omega$ of $\mathbb{R}^N$ we define

$$Q_\Omega(u,v) = \sum_{|\alpha|=|\beta|=m} \int_{\Omega} A_{\alpha\beta} D^\alpha u D^\beta v \, dx + \int_{\Omega} uv \, dx,$$

(2.2)

for all $u,v \in W^{m,2}(\Omega)$ and we set $Q_\Omega(u) = Q_\Omega(u,u)$. Note that by (2.1) $Q_\Omega$ is a positive quadratic form, densely defined in the Hilbert space $L^2(\Omega)$. Hence, $Q_\Omega(\cdot, \cdot)$ defines a scalar product in $W^{m,2}(\Omega)$. 

Let $V(\Omega)$ be a linear subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$. By standard Spectral Theory, if $V(\Omega)$ is complete with respect to the norm $Q_{\Omega}^{1/2}$, then there exists a uniquely determined non-negative self-adjoint operator $H_{V(\Omega)}$ such that $\mathcal{D}(H_{V(\Omega)}^{1/2}) = V(\Omega)$ and

$$Q_{\Omega}(u, v) = \left(H_{V(\Omega)}^{1/2}u, H_{V(\Omega)}^{1/2}v\right)_{L^2(\Omega)}, \quad \text{for all } u, v \in V(\Omega).$$ (2.3)

By [29, Lemma 4.4.1] it follows that the domain $\mathcal{D}(H_{V(\Omega)})$ of $H_{V(\Omega)}$ is the subset of $W^{m,2}(\Omega)$ containing all the functions $u \in V(\Omega)$ for which there exists $f \in L^2(\Omega)$ such that

$$Q_{\Omega}(u, v) = (f, v)_{L^2(\Omega)}, \quad \text{for all } v \in V(\Omega),$$ (2.4)

in which case $H_{V(\Omega)}u = f$. If $u$ is a smooth function satisfying identity (2.4) and the coefficients $A_{\alpha\beta}$ are smooth, by integration by parts it is immediate to verify that (2.4) is the weak formulation of problem $Lu = f$ in $\Omega$, where $L$ is the operator defined by

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(A_{\alpha\beta}D^\beta u) + u,$$

and the unknown $u$ is subject to suitable boundary conditions depending on the choice of $V(\Omega)$.

If the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, then the operator $H_{V(\Omega)}$ has compact resolvent. Consequently, its spectrum is discrete, and it consists of a sequence of isolated eigenvalues $\lambda_n[V(\Omega)]$ of finite multiplicity diverging to $+\infty$. By [29, Theorem 4.5.3] the eigenvalues $\lambda_n[V(\Omega)]$ are determined by the following Min-Max principle:

$$\lambda_n[V(\Omega)] = \min_{E \subset V(\Omega)} \max_{u \in E \atop \dim E = n \atop u \neq 0} \frac{Q_{\Omega}(u)}{\|u\|^2_{L^2(\Omega)}},$$

for all $n \geq 1$. Furthermore, there exists an orthonormal basis in $L^2(\Omega)$ of eigenfunctions $\varphi_n[V(\Omega)]$ associated with the eigenvalues $\lambda_n[V(\Omega)]$.

We remark that in our assumptions there exist two positive constants $c, C \in \mathbb{R}$ independent of $u$ such that

$$c \|u\|_{W^{m,2}(\Omega)} \leq Q_{\Omega}^{1/2}(u) \leq C \|u\|_{W^{m,2}(\Omega)},$$

which means that the two norms $Q_{\Omega}^{1/2}$ and $\|\cdot\|_{W^{m,2}(\Omega)}$ are equivalent on $V(\Omega)$. Note that in general the constant $c$ may depend on $\Omega$. However, if the coefficients $A_{\alpha\beta}$ satisfy the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta \sum_{|\alpha|=m} |\xi_\alpha|^2,$$ (2.5)

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ and for some $\theta > 0$, then $c$ can be chosen independent of $\Omega$. 
2.2. Classes of Open Sets

We recall the following definition from [16, Definition 2.4] where for any given set \( V \in \mathbb{R}^N \) and \( \delta > 0 \), \( V_\delta \) is the set \( \{ x \in \mathbb{R}^N : d(x, \partial \Omega) > \delta \} \), and by a cuboid we mean any rotation of a rectangular parallelepiped in \( \mathbb{R}^N \).

**Definition 1.** Let \( \rho > 0 \), \( s, s' \in \mathbb{N} \) with \( s' < s \). Let also \( \{ V_j \}_{j=1}^s \) be a family of bounded open cuboids and \( \{ r_j \}_{j=1}^s \) be a family of rotations in \( \mathbb{R}^N \). We say that \( \mathcal{A} = (\rho, s, s', \{ V_j \}_{j=1}^s, \{ r_j \}_{j=1}^s) \) is an atlas in \( \mathbb{R}^N \) with parameters \( \rho, s, s' \), \( \{ V_j \}_{j=1}^s \), \( \{ r_j \}_{j=1}^s \), briefly an atlas in \( \mathbb{R}^N \). Moreover, we consider the family of all open sets \( \Omega \subset \mathbb{R}^N \) satisfying the following:

(i) \( \Omega \subset \cup_{j=1}^s (V_j)_{\rho} \) and \( (V_j)_{\rho} \cap \Omega \neq \emptyset \)

(ii) \( V_j \cap \partial \Omega \neq \emptyset \) for \( j = 1, \ldots, s' \) and \( V_j \cap \partial \Omega = \emptyset \) for \( s' < j \leq s \)

(iii) for \( j = 1, \ldots, s \) we have

\[
 r_j(V_j) = \{ x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \ldots, N \}, \quad j = 1, \ldots, s
\]

\[
 r_j(V_j \cap \Omega) = \{ x \in \mathbb{R}^N : a_{Nj} < x_N < g_j(\bar{x}) \}, \quad j = 1, \ldots, s'
\]

where \( \bar{x} = (x_1, \ldots, x_{N-1}) \), \( W_j = \{ x \in \mathbb{R}^{N-1} : a_{ij} < x_i < b_{ij}, i = 1, \ldots, N-1 \} \) and \( g_j \in C^{k,\gamma}(W_j) \) for \( j = 1, \ldots, s' \), with \( k \in \mathbb{N}_0 \) and \( 0 \leq \gamma \leq 1 \). Moreover, for \( j = 1, \ldots, s' \) we have \( a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho \), for all \( \bar{x} \in W_j \).

We say that an open set \( \Omega \) is of class \( C^{k,\gamma}_M(\mathcal{A}) \) if all the functions \( g_j, j = 1, \ldots, s' \) defined above are of class \( C^{k,\gamma}(W_j) \) and \( \|g_j\|_{C^{k,\gamma}(W_j)} \leq M \). We say that an open set \( \Omega \) is of class \( C^{k,\gamma}(\mathcal{A}) \) if it is of class \( C^{k,\gamma}_M(\mathcal{A}) \) for some \( M > 0 \). Also, we say that an open set \( \Omega \) is of class \( C^{k,\gamma} \) if it is of class \( C^{k,\gamma}_M(\mathcal{A}) \) for some atlas \( \mathcal{A} \) and some \( M > 0 \). Finally, we denote by \( C^k \) the class \( C^{k,0} \) for \( k \in \mathbb{N} \cup \{0\} \).

It is important to note that if \( \Omega \) is a \( C^0 \) bounded open set then the Sobolev space \( W^{m,2}(\Omega) \) (and consequently all the spaces \( W^{m,2}(\Omega) \cap W^{k,2}_0(\Omega) \), \( 1 \leq k \leq m \)) is compactly embedded in \( L^2(\Omega) \), see e.g., Burkenov [15]. Moreover, by using a common atlas as in Definition 1, it is possible to define a distance.

**Definition 2.** (Atlas distance) Let \( \mathcal{A} = (\rho, s, s', \{ V_j \}_{j=1}^s, \{ r_j \}_{j=1}^s) \) be an atlas in \( \mathbb{R}^N \). For all \( \Omega_1, \Omega_2 \in C^m(\mathcal{A}) \) and for all \( h = 0, \ldots, m \) we set

\[
d^{(h)}_{\mathcal{A}}(\Omega_1, \Omega_2) = \max_{j=1, \ldots, s'} \sup_{0 \leq |\beta| \leq h} \sup_{(x, x_N) \in r_j(V_j)} \left| D^\beta g_{1j}(\bar{x}) - D^\beta g_{2j}(\bar{x}) \right|,
\]

where \( g_{1j}, g_{2j} \) respectively, are the functions describing the boundaries of \( \Omega_1, \Omega_2 \) respectively, as in Definition 1. Moreover, we set \( d_{\mathcal{A}} = d^{(0)}_{\mathcal{A}} \) and we call \( d_{\mathcal{A}} \) ‘atlas distance’.

2.3. Formulae for Higher Order Derivatives of Composite Functions

We recall here few well-known multidimensional formulae for the derivatives of composite functions. We will use the following notation: by \( \mathcal{P}(A) \) we denote the set of all subsets of a given finite non-empty set \( A \) and by \( \text{Part}(A) \) we denote the set of all possible partitions of \( A \). Namely, \( \pi \in \text{Part}(A) \) is a set the
elements of which are pairwise disjoint subsets of $A$ whose union is $A$. Given $n \in \mathbb{N}$, we often write $\text{Part}(n)$ in place of $\text{Part}([1, \ldots, n])$ and $\mathcal{P}(n)$ in place of $\mathcal{P}([1, \ldots, n])$. Moreover we use the symbol $|A|$ to denote the cardinality of $A$; hence, for example $|\pi|$ with $\pi \in \text{Part}(A)$ is the number of subsets of $A$ in the partition $\pi$. Let $\Omega$ be an open set in $\mathbb{R}^N$. If $I$ is an open set in $\mathbb{R}$ and $f$ is a $C^m$ function from $I$ to $\mathbb{R}$ and $\Phi$ is a $C^n$ function from $\Omega$ to $I$, then the Faà di Bruno formula reads
\[
\frac{\partial^m f(\Phi(x))}{\partial x_1 \cdots \partial x_m} = \sum_{\pi \in \text{Part}(n)} f(|\pi|) \left( \Phi(x) \right)^{\pi} \prod_{S \in \pi} \frac{\partial |S| \Phi(x)}{\partial x_1}.
\] (2.6)
Moreover, the Leibnitz formula for the derivatives of the product of two functions $u, v$ of class $C^m(\Omega)$ can be written as follows
\[
\frac{\partial^n (uv)}{\partial x_1 \cdots \partial x_n} = \sum_{S \in \mathcal{P}(n)} \prod_{j \in S} \frac{\partial \prod_{i \in j} f(x)}{\partial x_j} \prod_{i \notin S} \frac{\partial (n-|S|) v}{\partial x_i},
\] (2.7)
where $j \notin S$ means that $j$ lies in the complement of $S$ in $\{1, \ldots, n\}$. We recall that in general, if $\Phi$ is a $C^m$ function from an open subset $U$ of $\mathbb{R}^N$ to an open subset $V$ of $\mathbb{R}^r$, and $f$ is a function in $W^{m,1}_r(V)$ then the Faà di Bruno formula reads
\[
\frac{\partial^n f(\Phi(x))}{\partial x_1 \cdots \partial x_n} = \sum_{\pi \in \text{Part}(n)} \sum_{j_1, \ldots, j_n \in \{1, \ldots, r\}} \prod_{k=1}^{\pi} \frac{\partial |\pi| f(\Phi(x))}{\partial x_{j_k}} \prod_{k=1}^{\pi} \frac{\partial |S_k| \Phi(j_k)}{\partial x_{i_{\pi}}}
\] (2.8)

3. Higher Order Operators on Domains with Perturbed Boundaries

Let $m \in \mathbb{N}$, $m \geq 2$ and let $\epsilon > 0$. Let $V(\Omega), V(\Omega_{\epsilon})$ be subspaces of $W^{m,2}(\Omega), W^{m,2}(\Omega_{\epsilon})$ respectively, containing $W_0^{m,2}(\Omega), W_0^{m,2}(\Omega_{\epsilon})$ respectively. Moreover, let $H_V(\Omega), H_V(\Omega_{\epsilon}), Q_\Omega, Q_{\Omega_{\epsilon}}$ be as in (2.3). A fundamental part of our analysis will be based on the following:

**Definition 3.** [8, Definition 3.1] Given open sets $\Omega_{\epsilon}, \epsilon > 0$ and $\Omega \in \mathbb{R}^N$ with corresponding elliptic operators $H_V(\Omega_{\epsilon}), H_V(\Omega)$ defined on $\Omega_{\epsilon}, \Omega$ respectively, we say that condition $\mathcal{C}$ is satisfied if there exists open sets $K_{\epsilon} \subset \Omega \cap \Omega_{\epsilon}$ such that
\[
\lim_{\epsilon \to 0} |\Omega \setminus K_{\epsilon}| = 0,
\] (3.1)
and the following conditions are satisfied:

(C1) If $v_\epsilon \in V(\Omega_{\epsilon})$ and $\sup_{\epsilon > 0} Q_{\Omega_{\epsilon}}(v_\epsilon) < \infty$ then $\lim_{\epsilon \to 0} \|v_\epsilon\|_{L^2(\Omega \setminus K_{\epsilon})} = 0$.

(C2) For each $\epsilon > 0$ there exists an operator $T_\epsilon$ from $V(\Omega)$ to $V(\Omega_{\epsilon})$ such that for all fixed $\varphi \in V(\Omega)$

(i) $\lim_{\epsilon \to 0} Q_{K_{\epsilon}}(T_\epsilon \varphi - \varphi) = 0$;

(ii) $\lim_{\epsilon \to 0} Q_{\Omega_{\epsilon}\setminus K_{\epsilon}}(T_\epsilon \varphi) = 0$;

(iii) $\lim_{\epsilon \to 0} \|T_\epsilon \varphi\|_{L^2(\Omega_{\epsilon})} < \infty$. 


(C3) For each \( \epsilon > 0 \) there exists an operator \( E_\epsilon \) from \( V(\Omega_\epsilon) \) to \( W^{m,2}(\Omega) \) such that the set \( E_\epsilon(V(\Omega_\epsilon)) \) is compactly embedded in \( L^2(\Omega) \) and such that

(i) If \( v_\epsilon \in V(\Omega_\epsilon) \) is a sequence such that \( \sup_{\epsilon > 0} Q_{V(\Omega_\epsilon)}(v_\epsilon) < \infty \), then

\[
\lim_{\epsilon \to 0} Q_{K_\epsilon}(E_\epsilon v_\epsilon - v) = 0;
\]

(ii) \[
\sup_{\epsilon > 0} \sup_{v \in V(\Omega_\epsilon) \setminus \{0\}} \frac{\|E_\epsilon v\|_{W^{m,2}(\Omega)}}{Q_{\Omega_\epsilon}^1(v)} < \infty;
\]

(iii) If \( v_\epsilon \in V(\Omega_\epsilon) \) for all \( \epsilon > 0 \), \( \sup_{\epsilon > 0} Q_{\Omega_\epsilon}(v_\epsilon) < \infty \) and there exists \( v \in L^2(\Omega) \) such that, up to a subsequence, we have \( \|E_\epsilon v_\epsilon - v\|_{L^2(\Omega)} \to 0 \), then \( v \in V(\Omega) \).

It is proved in [8, Theorem 3.5] that Condition (C) guarantees the spectral convergence of the operators \( H_{V(\Omega_\epsilon)} \) to the operator \( H_{V(\Omega)} \) as \( \epsilon \to 0 \).

The convergence of the operators is understood in the sense of the compact convergence, as defined in [46]. Let us briefly recall the setting. Let \( \mathcal{E} \) be the extension-by-zero operator, mapping any given real-valued function \( u \) defined on some subset \( A \) of \( \mathbb{R}^N \), to the function \( \mathcal{E}u \) such that \( \mathcal{E}u = u \) a.e. in \( A \) and \( \mathcal{E}u = 0 \) a.e. in \( \mathbb{R}^N \setminus A \). By using \( \mathcal{E} \) we can map functions in \( L^2(\Omega) \) to the space \( L^2(\Omega_\epsilon) \), for every \( \epsilon > 0 \), so that \( \mathcal{E} \) defines a “connecting system” between \( L^2(\Omega) \) and the family of spaces \( (L^2(\Omega_\epsilon))_{\epsilon > 0} \). We then say that:

- \( v_\epsilon \in L^2(\Omega_\epsilon) \) \( \mathcal{E} \)-converges to \( v \in L^2(\Omega) \) if \( \|v_\epsilon - \mathcal{E}v\|_{L^2(\Omega_\epsilon)} \to 0 \) as \( \epsilon \to 0 \);
- a family of bounded linear operators \( B_\epsilon \in \mathcal{L}(L^2(\Omega_\epsilon)) \) \( \mathcal{E} \)-converges to \( B \in \mathcal{L}(L^2(\Omega)) \) if \( B_\epsilon \mathcal{E}v_\epsilon \to Bv \) whenever \( v_\epsilon \) \( \mathcal{E} \)-converges to \( v \);
- a family of bounded, compact linear operators \( B_\epsilon \in \mathcal{L}(L^2(\Omega_\epsilon)) \) is said to \( \mathcal{E} \)-compact converges to \( B \in \mathcal{L}(L^2(\Omega)) \) if \( B_\epsilon \mathcal{E}v_\epsilon \to Bv \) and for any family of functions \( v_\epsilon \in L^2(\Omega_\epsilon) \) with \( \|v_\epsilon\|_{L^2(\Omega_\epsilon)} \leq 1 \) there exists a subsequence, denoted by \( v_\epsilon \) again, and a function \( w \in L^2(\Omega) \) such that \( B_\epsilon v_\epsilon \to w \).

We refer to [8, Section 2.2], for further information on this type of convergence. Importantly, in our assumptions on the operators \( H_{V(\Omega_\epsilon)}, H_{V(\Omega)} \), the compact convergence of the resolvent operators is a sufficient condition for the spectral convergence. In particular, we have the following

**Theorem 1.** Let \( \Omega_\epsilon, \epsilon > 0 \) and \( \Omega \) be open sets in \( \mathbb{R}^N \). Let \( H_{V(\Omega_\epsilon)}, H_{V(\Omega)} \) be operators with compact resolvents, associated with \( V(\Omega_\epsilon), V(\Omega) \), respectively, as in (2.3), such that condition (C) is satisfied. Let \( \lambda_k, \lambda^\epsilon_k \) be the \( k \)-th eigenvalue of \( H_{V(\Omega)}, H_{V(\Omega_\epsilon)} \), respectively. Then \( H_{V(\Omega_\epsilon)}^{-1} \) \( \mathcal{E} \)-compact converges to \( H_{V(\Omega)}^{-1} \) as \( \epsilon \to 0 \). Moreover,

(i) \( \lambda^\epsilon_n \to \lambda_n \) as \( \epsilon \to 0 \), for all \( n \in \mathbb{N} \).

(ii) If \( \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+h-1} \) is an eigenvalue of multiplicity \( h \) and \( \varphi^\epsilon_n, \varphi^\epsilon_{n+1}, \ldots, \varphi^\epsilon_{n+h-1} \) is an orthonormal set in \( L^2(\Omega_\epsilon) \) of eigenfunctions associated with the corresponding eigenvalues \( \lambda^\epsilon_n, \lambda^\epsilon_{n+1}, \ldots, \lambda^\epsilon_{n+h-1} \) then there exists an orthonormal set \( \varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+h-1} \) in \( L^2(\Omega) \) of eigenfunctions associated with the eigenvalues \( (\lambda_{n+t-1})^h_{t=1} \) such that,
possibly passing to a suitable subsequence, $\varphi_{n+i-1}^{\epsilon}$ $\mathcal{E}$-converges to $\varphi_{n+i-1}^{\epsilon}$ as $\epsilon \to 0$ for all $i = 1, \ldots, h$.

(iii) If $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+h-1}$ is an eigenvalue of multiplicity $h$ and $\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+h-1}$ is an orthonormal set $L^2(\Omega)$ of eigenfunctions associated with $(\lambda_{n+i-1})_t^{h}$ then for every $\epsilon > 0$ there exists an orthonormal set in $L^2(\Omega)$ of eigenfunctions $\varphi_{n}^{\epsilon}, \varphi_{n+1}^{\epsilon}, \ldots, \varphi_{n+h-1}^{\epsilon}$ associated with the corresponding eigenvalues $\lambda_{n}^{\epsilon}, \lambda_{n+1}^{\epsilon}, \ldots, \lambda_{n+h-1}^{\epsilon}$ such that $\varphi_{n+i-1}^{\epsilon}$ $\mathcal{E}$-converges to $\varphi_{n+i-1}$ as $\epsilon \to 0$ for all $i = 1, \ldots, h$.

When the claims (i)–(ii)–(iii) of the previous theorem are verified, we say that $H_{V(\Omega_{\epsilon})}$ spectrally converges to $H_{V(\Omega)}$ as $\epsilon \to 0$.

3.1. An Explicit Condition for the Spectral Stability

We consider now the following geometric setting:

(G1) There exists a cuboid $V$ of the form $W \times (a, b)$, where $W \subset \mathbb{R}^{N-1}$ is an open, connected and bounded set of class $C^{m}$, and $g, g_{\epsilon} \in C^{m}(W)$ such that

$$\Omega \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g(\bar{x})\},$$

$$\Omega_{\epsilon} \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g_{\epsilon}(\bar{x})\}.$$  \hspace{1cm} (3.2) \hspace{1cm} (3.3)

Assume that $\Omega \setminus V = \Omega_{\epsilon} \setminus V$ for all $\epsilon > 0$.

It is convenient to set $\Omega_{0} = \Omega$. According to Definition 1, if $\Omega_{\epsilon} \in C^{m}(A)$ for all $\epsilon \geq 0$, then we can assume (G1) without loss of generality. For all $\epsilon \geq 0$, let us consider the quadratic forms $Q_{\Omega_{\epsilon}}$ on $\Omega_{\epsilon}$ defined as in (2.2), where the coefficients $A_{\alpha\beta}$ are independent of $\epsilon \geq 0$ and satisfy the uniform ellipticity condition (2.5). Then we consider the non-negative self-adjoint operators $H_{V(\Omega_{\epsilon})}$ defined by (2.3) with $V(\Omega)$ replaced by $V(\Omega_{\epsilon}) = W_{m,2}(\Omega_{\epsilon}) \cap W^{k,2}_{0}(\Omega_{\epsilon})$ for some $1 \leq k < m$. Since $\Omega_{\epsilon}$ is of class $C^{m}$, $V(\Omega_{\epsilon})$ is compactly embedded in $L^{2}(\Omega_{\epsilon})$ hence $H_{V(\Omega_{\epsilon})}$ has compact resolvent.

We now state our first result, concerning an explicit condition sufficient to guarantee the spectral convergence of the operators $H_{V(\Omega_{\epsilon})}$. This theorem is a generalisation of [8, Lemma 6.2].

**Theorem 2.** Let $\Omega_{\epsilon}, \epsilon \geq 0$ satisfy assumption (G1). Suppose that for some $k \in \mathbb{N}$, with $1 \leq k < m$, $V(\Omega_{\epsilon}) = W_{m,2}(\Omega_{\epsilon}) \cap W^{k,2}_{0}(\Omega_{\epsilon})$ for all $\epsilon \geq 0$. If for all $\epsilon > 0$ there exists $\kappa_{\epsilon} > 0$ such that

(i) $\kappa_{\epsilon} > \|g_{\epsilon} - g\|_{\infty}, \ \forall \epsilon > 0, \ \lim_{\epsilon \to 0} \kappa_{\epsilon} = 0,

(ii) \lim_{\epsilon \to 0} \frac{\|D^\beta(g_{\epsilon} - g)\|_{m-|\beta|-k+1/2}}{\kappa_{\epsilon}^{m-|\beta|-k+1/2}} = 0, \forall \beta \in \mathbb{N}_0^N \text{ with } |\beta| \leq m,$

then $H_{V(\Omega_{\epsilon})}^{-1} \mathcal{E}$-compact converges to $H_{V(\Omega)}^{-1}$ as $\epsilon \to 0$. In particular, $H_{V(\Omega_{\epsilon})}$ spectrally converges to $H_{V(\Omega)}$ as $\epsilon \to 0$.

**Proof.** We first observe that the last statement is a direct consequence of Theorem 1. The case $k = 1$ is proved in [8, Lemma 6.2]. Thus, we suppose $k > 1$. It is possible to assume directly that $\Omega = \Omega \cap V$ and $\Omega_{\epsilon} = \Omega_{\epsilon} \cap V$ in (3.2) and (3.3) respectively. Define $k_{\epsilon} = M \kappa_{\epsilon}$ for a suitable constant $M > 2m$. Let $\tilde{g}_{\epsilon} = g_{\epsilon} - k_{\epsilon}$ and

$$K_{\epsilon} = \{(\bar{x}, x_N) \in W \times [a, b] : a < x_N < \tilde{g}_{\epsilon}(\bar{x})\}.$$
Note that with this definition of $K_{\epsilon}$ (3.1) is satisfied. By the standard one dimensional estimate

$$
\|f\|_{L^\infty(a,b)} \leq C \|f\|_{W^{1,2}(a,b)},
$$

(3.4)

and Tonelli Theorem it follows that condition (C1) is satisfied.

We now define a suitable family of diffeomorphisms $\Phi_{\epsilon} : \Omega \rightarrow \Omega$ by setting

$$
\Phi_{\epsilon}(\bar{x}, x_N) = (\bar{x}, x_N - h_{\epsilon}(\bar{x}, x_N)),
$$

for all $(\bar{x}, x_N) \in \Omega_{\epsilon}$, where

$$
h_{\epsilon}(\bar{x}, x_N) = \begin{cases} 
0, & \text{if } a \leq x_N \leq \tilde{g}_{\epsilon}(\bar{x}), \\
(g_{\epsilon}(\bar{x}) - g(\bar{x})) \left( \frac{x_N - \tilde{g}_{\epsilon}(\bar{x})}{g_{\epsilon}(\bar{x}) - \tilde{g}_{\epsilon}(\bar{x})} \right)^{m+1}, & \text{if } \tilde{g}_{\epsilon}(\bar{x}) < x_N \leq g_{\epsilon}(\bar{x}).
\end{cases}
$$

Then consider the map $T_{\epsilon}$ from $V(\Omega)$ to $V(\Omega_{\epsilon})$ defined by

$$
T_{\epsilon}\varphi = \varphi \circ \Phi_{\epsilon},
$$

(3.5)

for all $\varphi \in V(\Omega)$. One can check that $T_{\epsilon}$ is well-defined and that condition (C2)(i) is satisfied. We now want to prove that conditions (C2)(ii), (iii) are satisfied. We need to estimate the derivatives of $\varphi \circ \Phi_{\epsilon}$. Here we can improve the estimate given in [8, Lemma 6.2] by taking advantage of the decay of $D^\gamma \varphi$ in a neighbourhood of $\partial \Omega$, for $|\gamma| \leq k - 1$. We divide the proof in two steps.

**Step 1** We aim at proving a decay estimate for the $L^2$-norms of the derivatives of $\varphi$ near the boundary, namely estimate (3.12). First, note that

$$
\Phi_{\epsilon}(\Omega_{\epsilon} \setminus K_{\epsilon}) = \Omega \setminus K_{\epsilon} = \{(\bar{x}, x_N) \in \Omega : \bar{x} \in W, g_{\epsilon}(\bar{x}) - k_{\epsilon} \leq x_N \leq g(\bar{x})\},
$$

for any $\epsilon > 0$. Fix $x \in \Phi_{\epsilon}(\Omega_{\epsilon} \setminus K_{\epsilon})$ and $\beta \in \mathbb{N}_0^N$, $|\beta| \leq k - 1$. Suppose for the moment $\varphi \in C^m(\overline{\Omega})$. By the Taylor’s formula with remainder in integral form, we get that

$$
D^\beta \varphi(x) = \sum_{l=0}^{k-1-|\beta|} \frac{1}{l!} \frac{\partial^l (D^\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} (x_N - g(\bar{x}))^l + R(\beta, x),
$$

where

$$
R(\beta, x) := \frac{(x_N - g(\bar{x}))^{k-|\beta|}}{(k - |\beta| - 1)!} \times 
\int_0^1 (1-t)^{k-1-|\beta|} \frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}} D^\beta \varphi(\bar{x}, g(\bar{x}) + t(x_N - g(\bar{x}))) dt.
$$

Note that $-2k_{\epsilon} \leq g_{\epsilon}(\bar{x}) - g(\bar{x}) - k_{\epsilon} \leq x_N - g(\bar{x}) \leq 0$. By Jensen’s inequality,

$$
|R(\beta, x)|^2 \leq (2k_{\epsilon})^{2(k-|\beta|)} \int_0^1 \left| \frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}} D^\beta \varphi(\bar{x}, g(\bar{x}) + t(x_N - g(\bar{x}))) \right|^2 dt.
$$

(3.6)
An integration in the variable $x_N$ in (3.6) and inequality (3.4) applied to the interval $(a, g(\bar{x}))$ yield
\[
\int_{g(x)-k_c}^{g(x)} |R(\beta, x)|^2 \, dx_N \leq Ck_\epsilon^{2(k-|\beta|)+1} \left\| \frac{\partial^{k-|\beta|+1}}{\partial x_N^{k-|\beta|+1}} D_\beta \varphi(\bar{x}, \cdot) \right\|_{W^{2,2}(a, g(\bar{x}))}^2 \tag{3.8}
\]
By integrating both sides of (3.7) with respect to $\bar{x} \in W$, we finally get
\[
\int_{\Phi_c(\Omega_c \setminus K_c)} |R(\beta, x)|^2 \, dx \leq Ck_\epsilon^{2(k-|\beta|)+1} \| \varphi \|_{W^{m,2}(\Omega)}^2 \tag{3.9}
\]
for sufficiently small $\epsilon$, for all $|\beta| \leq k - 1$. Thus, by (3.1) we get
\[
\int_{\Phi_c(\Omega_c \setminus K_c)} |D_\beta \varphi(x)|^2 \, dx \leq Ck_\epsilon^{2(k-|\beta|)+1} \| \varphi \|_{W^{m,2}(\Omega)}^2 \\
+ C \int_W \int_{g(x)-k_c}^{g(x)} \left| \sum_{l=0}^{k-1-|\beta|} \frac{\partial^l(D_\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right| |x_N - g(\bar{x})|^2 \, d\bar{x} \, dx_N ,
\]
(3.10)
for all sufficiently small $\epsilon$, and $|\beta| \leq k - 1$. We now estimate the last integral in the right-hand side of (3.9) in the following way
\[
\sum_{l=0}^{k-1-|\beta|} \int_W \int_{g(x)-k_c}^{g(x)} \left| \frac{\partial^l(D_\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right| |x_N - g(\bar{x})|^2 \, d\bar{x} \, dx_N
\]
\[
\leq \sum_{l=0}^{k-1-|\beta|} k_\epsilon^{2l+1} \int_W \left| \frac{\partial^l(D_\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right|^2 \, d\bar{x}
\]
\[
= \sum_{l=0}^{k-1-|\beta|} Ck_\epsilon^{2l+1} \left\| \frac{\partial^l(D_\beta \varphi)}{\partial x_N^l} \right\|_{L^2(\Gamma)}^2,
\]
(3.10)
where $\Gamma := \{ (\bar{x}, g(\bar{x})) : \bar{x} \in W \}$. Thus, by (3.9), (3.10) we obtain
\[
\int_{\Phi_c(\Omega_c \setminus K_c)} |D_\beta \varphi(x)|^2 \, dx \\
\leq \sum_{l=0}^{k-1-|\beta|} Ck_\epsilon^{2l+1} \left\| \frac{\partial^l(D_\beta \varphi)}{\partial x_N^l} \right\|_{L^2(\Gamma)}^2 + Ck_\epsilon^{2(k-|\beta|)+1} \| \varphi \|_{W^{m,2}(\Omega)}^2 .
\]
(3.11)
Inequality (3.11) holds for smooth functions. If $\varphi \in W^{m,2}(\Omega) \cap W^{k-2}_{0,2}(\Omega)$, then we can choose a sequence $(\psi_n)_{n \geq 1} \subset C^\infty(\overline{\Omega})$ such that $\psi_n \to \varphi$ in $W^{m,2}(\Omega)$ (this is possible because $\partial \Omega$ is Lipschitz continuous). We then use (3.11) for $\psi_n$, and we pass to the limit as $n \to \infty$ by using the continuity of the trace operator and standard estimates on the intermediate derivatives of Sobolev functions (see e.g., [15, §4.4]). We deduce that
\[
\int_{\Phi_c(\Omega_c \setminus K_c)} |D_\beta \varphi(x)|^2 \, dx \leq Ck_\epsilon^{2(k-|\beta|)+1} \| \varphi \|_{W^{m,2}(\Omega)}^2 ,
\]
(3.12)
for all sufficiently small $\epsilon$. Actually, inequality (3.12) holds also for $|\beta| = k$ (possibly modifying the constant in the right hand side). Indeed, $D^\beta \varphi \in W^{2,2}(\Omega)$, for any $|\beta| = k$, hence by standard boundedness of Sobolev functions on almost all vertical lines [see (3.4)] we find that

$$
\int_W \int_{g_{\epsilon}-k_{\epsilon}} |D^\beta \varphi(x)|^2 \, dx_N \, d\bar{x} \leq 2k_{\epsilon} \int_W \|D^\beta \varphi(\bar{x}, \cdot)\|_\infty^2 \, d\bar{x} \leq 2Ck_{\epsilon} \|\varphi\|_{W^{m,2}(\Omega)}^2.
$$

This concludes Step 1.

Step 2 We claim that Condition (C2)(ii) holds. Let $\varphi \in V(\Omega)$ and let $\alpha$ be a fixed multiindex such that $|\alpha| = m$. We write

$$D^\alpha \varphi(\Phi_\epsilon(x)) = \sum_{1 \leq |\beta| \leq m} D^\beta \varphi(\Phi_\epsilon(x))p_{m,\beta}^\alpha(\Phi_\epsilon)(x),$$

where $p_{m,\beta}^\alpha(\Phi_\epsilon)$ is a homogeneous polynomial of degree $|\beta|$ in derivatives of $\Phi_\epsilon$ of order not exceeding $m - |\beta| + 1$. Note that the polynomial $p_{m,\beta}^\alpha(\Phi_\epsilon)$ appearing in (3.13) is the sum of several terms $\Theta$ in the following form

$$\Theta = D^{k_1}(\delta_{j_1,N} - \frac{\partial h_{\epsilon}}{\partial x_{j_1}}) \cdots D^{k_n}(\delta_{j_n,N} - \frac{\partial h_{\epsilon}}{\partial x_{j_n}}) \frac{\partial \Phi^{i_{n+1}}}{\partial x_{i_{n+1}}} \cdots \frac{\partial \Phi^{i_{|\beta|}}}{\partial x_{i_{|\beta|}}},$$

where $1 \leq n \leq |\beta|$, $1 \leq j_i \leq N$ for all $i = 1, \ldots, n$, $i_{n+1}, \ldots, i_{|\beta|}$ are in \{1, $\ldots$, $N$ - 1\}, and $k_1, \ldots, k_n$ are multiindexes satisfying $|k_1| + \cdots + |k_n| = m - |\beta|$. Moreover, $\Theta$ is a sum of terms of the type $D^{L_1} h_{\epsilon} \cdots D^{L_l} h_{\epsilon}$, for all $1 \leq l \leq n$, for suitable multiindexes $L_1, \ldots, L_l$ satisfying

$$|L_1| + \cdots + |L_l| = m - |\beta| + l.$$  \hspace{1cm} (3.14)

Now by [8, Inequality (6.7)] and hypothesis (iii) we have

$$\|D^{L_1} h_{\epsilon} \cdots D^{L_l} h_{\epsilon}\|_\infty \leq C \left( \sum_{|\gamma_1| \leq |L_1|} \frac{\|D^{\gamma_1}(g_{\epsilon} - g)\|_{\kappa_{\epsilon}^{L_1}}}{K_{\epsilon}^{1-|\gamma_1|}} \right) \cdots \left( \sum_{|\gamma_l| \leq |L_l|} \frac{\|D^{\gamma_l}(g_{\epsilon} - g)\|_{\kappa_{\epsilon}^{L_l}}}{K_{\epsilon}^{1-|\gamma_l|}} \right) \leq o(1) \left( \sum_{|\gamma_1| \leq |L_1|} \frac{\kappa_{\epsilon}^{m-|\gamma_1|-k+1/2}}{K_{\epsilon}^{1-|\gamma_1|}} \right) \cdots \left( \sum_{|\gamma_l| \leq |L_l|} \frac{\kappa_{\epsilon}^{m-|\gamma_l|-k+1/2}}{K_{\epsilon}^{1-|\gamma_l|}} \right) \leq o(1) \kappa_{\epsilon}^{l(m-k+1/2)} \sum_i |L_i| \leq o(1) \kappa_{\epsilon}^{l(m-k+1/2)} - \sum_i |L_i| - |\beta| + k + 1/2 \geq 0.
$$

By (3.14), we have to check that $l(m-k+1/2) - (m - |\beta| + l) - |\beta| + k + 1/2 \geq 0$, which is verified if and only if $l(m - k - 1/2) \geq m - k - 1/2$, and this holds

\footnote{Here it is understood that for $|\beta| = 1$ the terms $\frac{\partial \Phi^{i_{n+1}}}{\partial x_{i_{n+1}}} \cdots \frac{\partial \Phi^{i_{|\beta|}}}{\partial x_{i_{|\beta|}}}$ are not present; recall that $m \geq 2$.}
true because \( m - k - 1/2 > 0 \) and \( l \geq 1 \). Hence we have proved that

\[
\| p^{\alpha, \beta}_m(\Phi_\epsilon) \|_\infty \leq o(1) \kappa_\epsilon^{(|\beta|-k-1/2)}.
\]  

(3.15)

By inequalities (3.12) and (3.15), we deduce that

\[
Q_{\Omega \setminus K_\epsilon} (T_\epsilon \varphi) \leq \int_{\Omega \setminus K_\epsilon} |\varphi(\Phi_\epsilon)|^2 \, dx + C \sum_{|\alpha|=m} \int_{\Omega \setminus K_\epsilon} \| D^\alpha \varphi(\Phi_\epsilon) \|^2 \, dx
\]

\[
\leq C \int_{\Phi_\epsilon(\Omega \setminus K_\epsilon)} |\varphi|^2 \, dx + C \sum_{1 \leq |\beta| \leq m} \| p^{\alpha}_{m, \beta}(\Phi_\epsilon) \|_\infty^2 \int_{\Omega \setminus K_\epsilon} \| D^\beta \varphi(\Phi_\epsilon(x)) \|^2 \, dx
\]

\[
+ C \sum_{k < |\beta| \leq m} \| p^{\alpha}_{m, \beta}(\Phi_\epsilon) \|_\infty^2 \int_{\Omega \setminus K_\epsilon} \| D^\beta \varphi(\Phi_\epsilon(x)) \|^2 \, dx
\]

\[
\leq C \| \varphi \|^2_{L^2(\Omega \setminus K_\epsilon)} + o(1) \kappa_\epsilon^{2(|\beta|-k-1/2)} \kappa_\epsilon^{2(k-|\beta|)+1} + o(1) \| \varphi \|^2_{W^{m,2}(\Omega \setminus K_\epsilon)}
\]  

(3.16)

for all \( \epsilon > 0 \) sufficiently small. Since the right-hand side of (3.16) vanishes as \( \epsilon \to 0 \) we conclude that condition (C2)(ii) is satisfied.

It remains to prove condition (C3). To prove that conditions (C3)(i), (C3)(ii) are satisfied it is sufficient to set \( E_\epsilon u = (\text{Ext}_\Omega, u)_{|\Omega} \) for all \( u \in V(\Omega_\epsilon) \), where \( \text{Ext}_\Omega \) is the standard Sobolev extension operator mapping \( W^{m,2}(\Omega_\epsilon) \) to \( W^{m,2}(\mathbb{R}^N) \). Finally, in order to prove condition (C3)(iii) it is sufficient to prove that the weak limit \( v \) of the uniformly bounded sequence \( v_\epsilon \) (appearing in the statement of condition (C3)(iii)) lies in \( W^{k,2}_0(\Omega) \). This is easily achieved by considering the extension-by-zero of the functions \( v_\epsilon \) outside \( \Omega_\epsilon \), passing to the limit and recalling that the limit set \( \Omega \) has Lipschitz boundary. \( \Box \)

Theorem 2 can be actually applied to open sets \( \Omega \) in the atlas class \( C^m(\mathcal{A}) \) by requiring that the assumptions of Lemma 2 are satisfied by all the profile functions \( g_j \) describing their boundaries. Then we can prove the following

**Theorem 3.** Let \( \mathcal{A} \) be an atlas in \( \mathbb{R}^N \), \( M > 0 \), \( m \in \mathbb{N} \), \( m \geq 2 \). For all \( \epsilon \geq 0 \), let \( \Omega_\epsilon \in C^m_M(\mathcal{A}) \). Let \( k \in \mathbb{N} \) with \( 1 \leq k < m \) and define, for all \( \epsilon \geq 0 \), \( V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W^{k,2}_0(\Omega_\epsilon) \). If

\[
\lim_{\epsilon \to 0} d_A^{(m-k)}(\Omega_\epsilon, \Omega) = 0,
\]

then condition (C) is satisfied, hence \( H^{-1}_{V(\Omega_\epsilon), E} \)-compact converges to \( H^{-1}_{V(\Omega)} \) as \( \epsilon \to 0 \).

**Proof.** By using a standard partition of unity argument, it suffices to prove that the assumptions of Theorem 2 are satisfied by all the profile functions \( g_{j,\epsilon}, g_j \) describing the boundaries of \( \Omega_\epsilon, \Omega \), respectively, and this follows by choosing \( \kappa_\epsilon = (d_A^{(m-k)}(\Omega_\epsilon, \Omega))^\frac{1}{m} \). \( \Box \)

In order to prove that the assumptions of Lemma 2 are sharp, we now consider the following geometric setting:
(G2) Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $b \in C^\infty(\mathbb{W})$ a positive, non-constant periodic function, with periodicity cell given by $Y = ] - 1/2, 1/2[^{N-1}$. Let us set

$$g_\epsilon(\vec{x}) = \epsilon^\alpha b\left(\frac{\vec{x}}{\epsilon}\right), \quad g(\vec{x}) = 0,$$

for all $\vec{x} \in \mathbb{W}$. For simplicity, we set $g_0 = g$ and for all $\epsilon \geq 0$ we consider the open sets

$$\Omega_\epsilon = \{(\vec{x}, x_N) \in \mathbb{R}^N : \vec{x} \in \mathbb{W}, -1 < x_N < g_\epsilon(\vec{x})\}$$

Then we have the following

**Theorem 4.** Let $\Omega_\epsilon$, $\epsilon \geq 0$ be as in (G2) and let $k \in \mathbb{N}$ satisfy $1 \leq k \leq m - 1$. Let $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_0^{k,2}(\Omega_\epsilon)$ for all $\epsilon \geq 0$. If $\alpha > m - k + \frac{1}{2}$, then $H_{V(\Omega)}^{-1}$ $\mathcal{E}$-compact converges to $H_{V(\Omega)}^{-1}$ as $\epsilon \to 0$.

**Proof.** We aim at applying Theorem 2 with $\kappa_\epsilon = \epsilon^\alpha \|b\|_\infty$, for some $\theta \in (0,1)$ to be specified. By the classical Gagliardo-Nirenberg interpolation inequality

$$\|D^{\beta}f\|_\infty \leq C(\sum_{|\alpha|=m} \|D^\alpha f\|_\infty)^{\beta/m} \|f\|^{1-\beta/m}_\infty,$$

for all $f \in W^{m,\infty}(\Omega)$ (see e.g., [42, p.125]), in order to verify condition (iii) in Theorem 2 it is sufficient to verify it for $|\beta| = 0$ and $|\beta| = m$ (see also [8, Proposition 6.17]). When $|\beta| = 0$ we have

$$\lim_{\epsilon \to 0} \frac{\|g_\epsilon - g\|_\infty}{\kappa_\epsilon^{m-k+1/2}} = c \lim_{\epsilon \to 0} \frac{\epsilon^\alpha}{\epsilon^{\alpha\theta(m-k+1/2)}} = c \lim_{\epsilon \to 0} \epsilon^{\alpha(1-\theta(m-k-1/2))},$$

where $c$ is a constant depending only on $\|b\|_\infty$. The right-hand side clearly tends to 0 as soon as $\theta < \frac{1}{m-k+1/2}$.

When $|\beta| = m$, we must check that

$$\lim_{\epsilon \to 0} \frac{\|D^{\beta}g_\epsilon\|_\infty}{\kappa_\epsilon^{-k+1/2}} = c \frac{\epsilon^{\alpha-m}}{\epsilon^{\alpha\theta(-k+1/2)}} = c^{\alpha(1-\theta(-k+1/2))-m},$$

and the right hand side tends to zero if and only if

$$\alpha \left(1 + \theta \left(k - \frac{1}{2}\right)\right) - m > 0. \quad (3.17)$$

By letting $\theta \to \frac{1}{m-k+1/2}$ in (3.17) we obtain that inequality (3.17) is satisfied when $\alpha > m - k + 1/2$, true by assumption. By Lemma 2 we deduce the validity of Theorem 4.

**Remark 1.** When $k = m - 1$, Theorem 4 states that if $\alpha > \frac{3}{2}$, $H_{V(\Omega_\epsilon)}^{-1} \overset{c}{\to} H_{V(\Omega)}^{-1}$ as $\epsilon \to 0$, independently on $m \geq 2$. Actually, it is possible to prove that $\alpha = 3/2$ in this case is the critical exponent, in the sense that when $\alpha \leq 3/2$ the operator $H_{V(\Omega_\epsilon)}^{-1}$ does not converge to $H_{V(\Omega)}^{-1}$. We refer to Theorem 7 for a complete discussion about the spectral convergence of $H_{V(\Omega_\epsilon)}$ depending on $\alpha$. 

\[\square\]
4. A Polyharmonic Green Formula

In this section we provide a formula which turns out to be useful in recognising the possible natural boundary conditions for polyharmonic operators of any order. Let us begin by stating an easy integration-by-parts formula.

**Proposition 1.** Let $\Omega$ be a bounded domain of class $C^{0,1}$ in $\mathbb{R}^N$. Let $m \in \mathbb{N}$ and let $f \in C^{m+1}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$. Then

$$
\int_\Omega D^m f : D^m \varphi \, dx = - \int_\Omega D^{m-1}(\Delta f) : D^{m-1} \varphi \, dx
+ \int_{\partial \Omega} D^m f : (n \otimes D^{m-1} \varphi) \, dS,
$$

(4.1)

where the symbol $: \varphi$ stands for the Frobenius product, $n$ is the unit outer normal to $\partial \Omega$, and $\otimes$ is the tensor product, defined by $(n \otimes D^{m-1} \varphi)_{i,j,\cdots,j_{m-1}} = n_i \frac{\partial^{m-1} \varphi}{\partial x_{j_1} \cdots \partial x_{j_{m-1}}}$ for all $i,j_1,\cdots,j_{m-1} \in \{1,\cdots,N\}$.

**Proof.** The proof is a simple integration by parts. Indeed, dropping the summation symbols we get

$$
\int_\Omega D^m f : D^m \varphi \, dx = \int_\Omega \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} \, dx
- \int_\Omega \frac{\partial^{m+1} f}{\partial x^2_{j_1} \cdots \partial x_{j_m}} \frac{\partial^{m-1} \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} \, dx + \int_{\partial \Omega} \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} \frac{\partial^{m-1} \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} n_{j_i} \, dS
= - \int_\Omega D^{m-1}(\Delta f) : D^{m-1} \varphi \, dx + \int_{\partial \Omega} (D^m f) : (n \otimes D^{m-1} \varphi) \, dS.
$$

By applying $m$ times the integration by parts argument used in the proof of formula (4.1), we deduce the validity of the following

**Corollary 1.** Let $m \in \mathbb{N}$. Let $f \in C^{2m}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$. Then

$$
\int_\Omega D^m f : D^m \varphi \, dx = (-1)^m \int_\Omega \Delta^m f \varphi \, dx
+ \sum_{k=0}^{m-1} (-1)^k \int_{\partial \Omega} (D^{m-k}(\Delta^k f)) : (n \otimes D^{m-k-1} \varphi) \, dS.
$$

(4.2)

**Theorem 5.** (Polyharmonic Green Formula - Flat case). Let $H$ be the half-space $H = \{(\bar{x},x_N) \in \mathbb{R}^N : x_N < 0\}$. Let $m \in \mathbb{N}$. Let $f \in C^{2m}(\overline{H})$, $\varphi \in C^m(\overline{H})$ with compact support in $\overline{H}$. Then,

$$
\int_H D^m f : D^m \varphi \, dx = (-1)^m \int_H \Delta^m f \varphi \, dx + \sum_{t=0}^{m-1} \int_{\mathbb{R}^{N-1}} B_t(f) \frac{\partial^t \varphi}{\partial x_N} \, d\bar{x},
$$

(4.3)

where $B_t : C^{2m}(\partial H) \to C^{t+1}(\partial H)$ is defined by

$$
B_t(f) = \sum_{l=t}^{m-1} (-1)^{m-t-1} \binom{m}{t} \Delta_{N-1}^{l-t} \left( \frac{\partial^{t+1}}{\partial x_N^{t+1}} (\Delta^{m-l-1} f) \right),
$$

(4.4)

and $\Delta_{N-1}$ is the Laplace operator in the first $N-1$ variables.
Proof. Let \( r = m - k - 1 \). First note that we can write
\[
\int_{\mathbb{R}^{N-1}} \left( D^r \left( \Delta^k \left( \frac{\partial f}{\partial x_N} \right) \right) \right) : D^r \varphi \, d\bar{x}
\]
\[
= \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \left( D^t_{\bar{x}} \left( \Delta^k \left( \frac{\partial^{t+1} f}{\partial x_{t+1}^N} \right) \right) \right) : \left( D^r-t \left( \frac{\partial^t \varphi}{\partial x_N^t} \right) \right) \, d\bar{x}.
\]
Then, by using (4.5) in the last integral in the right-hand side of (4.2) we get the following as boundary term
\[
\sum_{k=0}^{m-1} (-1)^k \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left( \frac{\partial^{t+1} (\Delta^k f)}{\partial x_{t+1}^N} \right) \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left( \frac{\partial^t \varphi}{\partial x_N^t} \right) \, d\bar{x},
\]
(4.6)
By dropping the summation symbols, the integrand in (4.6) becomes
\[
\int_{\mathbb{R}^{N-1}} \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left( \frac{\partial^{t+1} (\Delta^k f)}{\partial x_{t+1}^N} \right) \frac{\partial^{r-t}}{\partial x_{i_1} \cdots \partial x_{i_{r-t}}} \left( \frac{\partial^t \varphi}{\partial x_N^t} \right) \, d\bar{x},
\]
(4.7)
where the indexes \( i_j \) run on the first \( N - 1 \) coordinates. By integrating by parts \( r - t \) times in \( i_1, \ldots, i_{r-t} \) in (4.7) we deduce that (4.6) equals
\[
\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \frac{\partial^{2(r-t)}}{\partial^2 x_{i_1} \cdots \partial^2 x_{i_{r-t}}} \left( \frac{\partial^{t+1} (\Delta^k f)}{\partial x_{t+1}^N} \right) \frac{\partial^t \varphi}{\partial x_N^t} \, d\bar{x},
\]
where we have no other boundary terms because \( \varphi \) has compact support. We rewrite the last expression as
\[
\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \Delta^{r-t}_{N-1} \left( \frac{\partial^{t+1} (\Delta^k f)}{\partial x_{t+1}^N} \right) \frac{\partial^t \varphi}{\partial x_N^t} \, d\bar{x}.
\]
(4.8)
We now apply the change of summation index \( r = m - k - 1 \) in the first sum of (4.8). We deduce that (4.8) equals
\[
\sum_{r=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{r} \binom{r}{t} \int_{\mathbb{R}^{N-1}} \Delta^{r-t}_{N-1} \left( \frac{\partial^{t+1} (\Delta^{m-r} f)}{\partial x_{t+1}^N} \right) \frac{\partial^t \varphi}{\partial x_N^t} \, d\bar{x}.
\]
(4.9)
By exchanging the two sums in (4.9) we get (4.3). \( \square \)

Remark 2. If \( m = 2 \), then (4.3) reads
\[
\int_H D^2 f : D^2 \varphi \, dx = \int_H \Delta^2 f \varphi \, dx + \int_{\mathbb{R}^{N-1}} \frac{\partial^2 f}{\partial x_N^2} \frac{\partial \varphi}{\partial x_N} \, d\bar{x}
\]
\[
- \int_{\mathbb{R}^{N-1}} \left( \Delta_{N-1} \left( \frac{\partial f}{\partial x_N} \right) + \Delta \left( \frac{\partial f}{\partial x_N} \right) \right) \varphi \, d\bar{x},
\]
which is consistent with the formula provided in [8, Lemma 8.56]. Indeed, if the domain is a hyperplane, the boundary integral \( \int_{\partial H} (\operatorname{div}_{\partial H} (D^2 f \cdot n))_{\partial \Omega} \varphi \, dS \) appearing in [8, Lemma 8.56] coincides with \( \int_{\mathbb{R}^{N-1}} \Delta_{N-1} \left( \frac{\partial f}{\partial x_N} \right) \varphi \, d\bar{x} \).
Theorem 6. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ of class $C^{0,1}$, $m \in \mathbb{N}$, $m \geq 2$. Let $f \in W^{2m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$ and $\varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$. Then
\[
\int_{\Omega} D^m f : D^m \varphi \, dx = (-1)^m \int_{\Omega} \Delta^m f \varphi \, dx + \int_{\partial \Omega} \frac{\partial^m f}{\partial n^{m-1}} \frac{\partial^{m-1} \varphi}{\partial n^{m-1}} \, dS. \tag{4.10}
\]

Proof. By (4.2) it is easy to see that
\[
\int_{\Omega} D^m f : D^m \varphi \, dx = (-1)^m \int_{\Omega} \Delta^m f \varphi \, dx + \int_{\partial \Omega} D^m f : (n \otimes D^{m-1} \varphi) \, dS, \tag{4.11}
\]
for all $\varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$, since $D^l \varphi = 0$ on $\partial \Omega$ for all $l \leq m-2$. We note that $D^m f : (n \otimes D^{m-1} \varphi) = (n^T D^m f) : D^m \varphi$. Moreover we claim that $D^{m-1} \varphi = \frac{\partial^{m-1} \varphi}{\partial n^{m-1}} \otimes \frac{1}{n}$ on $\partial \Omega$ and we prove it by induction. If $m = 2$ the claim is a direct consequence of the gradient decomposition $\nabla|_{\partial \Omega} = \nabla|_{\partial \Omega} + \frac{\partial}{\partial n} n$. Now we assume that $m > 2$ and that the claim holds for $m-1$. Then, by using the fact that $D^{m-2} \varphi|_{\partial \Omega} = 0$, for all $\varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$, we get
\[
D^{m-1} \varphi|_{\partial \Omega} = D(D^{m-2} \varphi)|_{\partial \Omega} = \left( D \left( \frac{\partial^{m-2} \varphi}{\partial n^{m-2}} \otimes_{i=1}^{m-1} n \right) \right) \otimes n = \frac{\partial^{m-1} \varphi}{\partial n^{m-1}} \otimes_{i=1}^{m-1} n,
\]
for all $\varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega)$. This proves the claim. Then we can rewrite (4.11) as
\[
\int_{\Omega} D^m f : D^m \varphi \, dx = (-1)^m \int_{\Omega} \Delta^m f \varphi \, dx + \int_{\partial \Omega} \frac{\partial^{m-1} \varphi}{\partial n^{m-1}} (n^T D^m f) : \left( \otimes_{i=1}^{m-1} n \right) \, dS, \tag{4.12}
\]
and since $(n^T D^m f) : \left( \otimes_{i=1}^{m-1} n \right) = D^m f : \left( \otimes_{i=1}^{m-1} n \right) = \frac{\partial^m f}{\partial n^m}$ we deduce (4.10). \hfill \Box

5. Polyharmonic Operators with Strong Intermediate Boundary Conditions

Let $\Omega_\epsilon$, $\epsilon \geq 0$ be as in (G2). Consider the polyharmonic operators $(-\Delta)^m + I$ subject to strong intermediate boundary conditions, corresponding to the energy space $V(\Omega_\epsilon) := W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$. More precisely, let $H_{\Omega_\epsilon,S}$ be the non-negative self-adjoint operator such that
\[
(H_{\Omega_\epsilon,S} u, v)_{L^2(\Omega_\epsilon)} = (H_{\Omega_\epsilon,S}^{1/2} u, H_{\Omega_\epsilon,S}^{1/2} v)_{L^2(\Omega_\epsilon)} = Q_{\Omega_\epsilon}(u, v), \tag{5.1}
\]
for all functions $u, v \in W^{m,2}(\Omega_\epsilon) \cap W^{m-1,2}_0(\Omega_\epsilon)$, where $Q_{\Omega_\epsilon}(u, v) := \int_{\Omega_\epsilon} D^m u : D^m v + uv \, dx$, is the quadratic form canonically associated with $H_{\Omega_\epsilon,S}$. As it is explained in Sect. 2 the equation $H_{\Omega_\epsilon,S} u = f$ with datum $f \in L^2(\Omega_\epsilon)$, corresponds exactly to the weak Poisson problem (1.8).
Let $H_{\Omega,D}$ be the polyharmonic operator satisfying strong intermediate boundary conditions on $\partial \Omega \setminus \overline{W}$ and Dirichlet boundary conditions on $W$, whose associated boundary value problem reads

$$
\begin{cases}
(-\Delta)^m u + u = f, & \text{in } \Omega, \\
\frac{\partial^l u}{\partial n^l} = 0, & \text{on } W, \text{ for all } 0 \leq l \leq m - 1, \\
\frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial \Omega \setminus \overline{W}, \text{ for all } 0 \leq l \leq m - 2, \\
\frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial \Omega \setminus \overline{W}.
\end{cases}
$$

(5.2)

Note that we are identifying $W$ with $W \times \{0\}$. Then the following theorem holds.

**Theorem 7.** Let $m \in \mathbb{N}$, $m \geq 2$, $\Omega_\epsilon$ as in (G2), $H_{\Omega_\epsilon}$ as in (5.1), for all $\epsilon > 0$. Then the following statements hold true.

(i) [Spectral stability] If $\alpha > 3/2$, then $H_{\Omega_\epsilon,S}^{-1} \subset H_{\Omega,S}^{-1}$ as $\epsilon \to 0$.

(ii) [Instability] If $\alpha < 3/2$, then $H_{\Omega_\epsilon,S}^{-1} \subset H_{\Omega,D}^{-1}$ as $\epsilon \to 0$, where $H_{\Omega,D}$ is defined in (5.2).

(iii) [Strange term] If $\alpha = 3/2$, then $H_{\Omega_\epsilon,I}^{-1} \subset H_{\Omega}^{-1}$ as $\epsilon \to 0$, where $H_{\Omega}$ is the operator $(-\Delta)^{m} + I$ with strong intermediate boundary conditions on $\partial \Omega \setminus \overline{W}$ and the following boundary conditions on $W$: $D^l u = 0$, for all $l \leq m - 2$, $\partial_{x_N}^m u + K \partial_{x_N}^{m-1} u = 0$, where the factor $K$ is given by

$$
K = \int_Y |D^m V|^2 \, dy
$$

$$
= - \int_Y \left( \frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m - 1) \Delta_{N-1} \left( \frac{\partial^{m-1}V}{\partial x_N^{m-1}} \right) \right) b(\bar{y}) \, d\bar{y},
$$

and the function $V$ is $Y$-periodic in the variable $\bar{y}$ and satisfies the following microscopic problem

$$
\begin{cases}
(-\Delta)^m V = 0, & \text{in } Y \times (-\infty,0), \\
\frac{\partial^l V}{\partial y_N^{l}}(\bar{y},0) = 0, & \text{on } Y, \text{ for all } 0 \leq l \leq m - 3, \\
\frac{\partial^{m-2}V}{\partial y_N^{2}}(\bar{y},0) = b(\bar{y}), & \text{on } Y, \\
\frac{\partial^{m-1}V}{\partial y_N^{1}}(\bar{y},0) = 0, & \text{on } Y.
\end{cases}
$$

**Proof.** Statement (i) is a straightforward application of Theorem 4 with $k = m - 1$. To prove (ii) we check that Condition (C) in Definition 3 is satisfied with $V(\Omega) = W_{0,W}^{m,2}(\Omega) \cap W_{0}^{m-1,2}(\Omega)$, and $V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W_{0}^{m-1,2}(\Omega_\epsilon)$.

Here $W_{0,W}^{m,2}(\Omega)$ is the closure in $W^{m,2}(\Omega)$ of the space of functions vanishing in a neighborhood of $W$. Let $K_\epsilon = \Omega$ for all $\epsilon > 0$. Then we see immediately that condition (3.1) and condition (C1) are satisfied. We define now $T_\epsilon$ as the extension by zero operator from $W_{0,W}^{m,2}(\Omega)$ to $W^{m,2}(W \times (-1,\infty))$ and $E_\epsilon$ as the restriction operator to $\Omega$. With these definitions it is not difficult to prove that conditions (C2) and (C3)(i),(ii) are satisfied. It remains to prove that condition (C3)(iii) holds. Let $v_\epsilon \in W^{m,2}(\Omega_\epsilon) \cap W_{0}^{m-1,2}(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq C$ for all $\epsilon > 0$. Possibly passing to a subsequence
there exists a function \( v \in W^{m-1,2}(\Omega) \) such that \( v|\Omega \to v \) in \( W^{m,2}(\Omega) \) and \( v|\Omega \to v \) in \( W^{m-1,2}(\Omega) \). By considering the sequence of functions \( T_\epsilon(v|\Omega) \) it is not difficult to prove that \( v \in W_0^{m-1,2}(\Omega) \). It remains to check that

\[
\frac{\partial^{m-1}v}{\partial x_N^{m-1}} = 0 \quad \text{on} \quad W \times \{0\}.
\]

This is proven exactly as in [8, Theorem 7.3] by applying Lemma 4.3 from [20] to the vector field \( V^i_\epsilon \) defined by

\[
V^i_\epsilon = \left(0, \ldots, 0, -\frac{\partial^{m-1}v}{\partial x_N^{m-1}}, 0, \ldots, 0, \frac{\partial^{m-1}v}{\partial x_N^{m-2}}\partial x_i\right),
\]

for all \( i = 1, \ldots, N-1 \), where the only non-zero entries are the \( i \)-th and the \( N \)-th ones. We remark that it is possible to apply Lemma 4.3 from [20] because by Theorem 4 the critical threshold for all the polyharmonics operator with strong intermediate boundary conditions is \( \alpha = 3/2 \), which coincides with the critical value in [20]. We then deduce that

\[
\frac{\partial^{m-1}v(\bar{x},0)}{\partial x_N^{m-1}} \frac{\partial b(y)}{\partial y} = 0, \quad \text{a.e.} \quad W \times Y.
\]

Since \( b \) is a non-constant smooth function we must have

\[
\frac{\partial^{m-1}v(\bar{x},0)}{\partial x_N^{m-1}} = 0 \quad \text{a.e.} \quad W.
\]

This concludes the proof of condition \((C3)(iii)\).

We provide a proof of (iii) in Sections 5.1 and 5.2.

Remark 3. We take the chance to point out a misprint in [5, Theorem 1, (ii)] where the condition

\[
\frac{\partial^{m-1}u}{\partial x_N^{m-1}} + Ku = 0
\]

appears for \( m = 3 \) with \(-K\) instead of \(+K\) as it should be.

5.1. Critical Case: Macroscopic Problem

In this section we prove Theorem 7(iii). Let us define a diffeomorphism \( \Phi_\epsilon \) from \( \Omega_\epsilon \) to \( \Omega \) by

\[
\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)), \quad \text{for all} \quad x = (\bar{x}, x_N) \in \Omega_\epsilon,
\]

where \( h_\epsilon \) is defined by

\[
h_\epsilon(\bar{x}, x_N) = \begin{cases} 
0, & \text{if } -1 \leq x_N \leq -\epsilon, \\
g_\epsilon(\bar{x})\left(\frac{x_N + \epsilon}{g_\epsilon(\bar{x}) + \epsilon}\right)^{m+1}, & \text{if } -\epsilon \leq x_N \leq g_\epsilon(\bar{x}).
\end{cases}
\]

By standard calculus one can prove the following

**Lemma 1.** The map \( \Phi_\epsilon \) is a diffeomorphism of class \( C^m \) and there exists a constant \( c > 0 \) independent of \( \epsilon \) such that \( \|h_\epsilon\| \leq c\epsilon^\alpha \) and \( \|\partial^l h_\epsilon\| \leq c\epsilon^{\alpha-l} \), for all \( l = 1, \ldots, m \), \( \epsilon > 0 \) sufficiently small.

As in [8, Section 8.1], we introduce the pullback operator \( T_\epsilon \) from \( L^2(\Omega) \) to \( L^2(\Omega_\epsilon) \) given by \( T_\epsilon u = u \circ \Phi_\epsilon \) for all \( u \in L^2(\Omega) \).

In order to proceed we find convenient to recall some notation and results in homogenization theory regarding the unfolding operator. We refer to [1,23,24,28] for the proof of the main properties of the operator, and we mention that recent developments can be found in the article [9].
For any \( k \in \mathbb{Z}^{N-1} \) and \( \epsilon > 0 \) we define
\[
\begin{align*}
C^k_\epsilon &= \epsilon k + \epsilon Y, \\
I_{W,\epsilon} &= \{ k \in \mathbb{Z}^{N-1} : C^k_\epsilon \subset W \}, \\
\hat{W}_\epsilon &= \bigcup_{k \in I_{W,\epsilon}} C^k_\epsilon.
\end{align*}
\tag{5.3}
\]
Then we give the following

**Definition 4.** Let \( u \) be a real-valued function defined in \( \Omega \). For any \( \epsilon > 0 \) sufficiently small the unfolding \( \hat{u} \) of \( u \) is the real-valued function defined on \( \hat{W}_\epsilon \times Y \times (-1/\epsilon,0) \) by
\[
\hat{u}(\bar{x}, \bar{y}, y_N) = u\left( \epsilon \left[ \frac{\bar{x}}{\epsilon} \right] + \epsilon \bar{y}, \epsilon y_N \right),
\]
for almost all \((\bar{x}, \bar{y}, y_N) \in \hat{W}_\epsilon \times Y \times (-1/\epsilon,0)\), where \( \left[ \frac{\bar{x}}{\epsilon} \right] \) denotes the integer part of the vector \( \bar{x}^{-1} \) with respect to \( Y \), i.e., \( \left[ \bar{x}^{-1} \right] = k \) if and only if \( \bar{x} \in C^k_\epsilon \).

The following lemma will be often used in the sequel. For a proof we refer to [25, Proposition 2.5(i)].

**Lemma 2.** Let \( a \in [-1,0[ \) be fixed. Then
\[
\int_{\hat{W}_\epsilon \times (a,0)} u(x) dx = \epsilon \int_{\hat{W}_\epsilon \times Y \times (a/\epsilon,0)} \hat{u}(\bar{x}, y) d\bar{x} dy \tag{5.4}
\]
for all \( u \in L^1(\Omega) \) and \( \epsilon > 0 \) sufficiently small. Moreover
\[
\int_{\hat{W}_\epsilon \times (a,0)} \left| \frac{\partial^l u(x)}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^2 dx = \epsilon^{1-2l} \int_{\hat{W}_\epsilon \times Y \times (a/\epsilon,0)} \left| \frac{\partial^l \hat{u}}{\partial y_{i_1} \cdots \partial y_{i_l}}(\bar{x}, y) dx \right|^2 dy,
\]
for all \( l \leq m \), \( u \in W^{m,2}(\Omega) \) and \( \epsilon > 0 \) sufficiently small.

Let \( W_{\text{Pery}, \text{loc}}^{m,2}(Y \times (-\infty,0)) \) be the subspace of \( W_{\text{loc}}^{m,2}(\mathbb{R}^{N-1} \times (-\infty,0)) \) containing \( Y \)-periodic functions in the first \((N-1)\) variables \( \bar{y} \). We then define \( W_{\text{loc}}^{m,2}(Y \times (-\infty,0)) \) to be the space of functions in \( W_{\text{Pery}, \text{loc}}^{m,2}(Y \times (-\infty,0)) \) restricted to \( Y \times (-\infty,0) \). Finally we set
\[
w_{\text{Pery}}^{m,2}(Y \times (-\infty,0)) := \{ u \in W_{\text{Pery}, \text{loc}}^{m,2}(Y \times (-\infty,0)) : \| D^\gamma u \|_{L^2(Y \times (-\infty,0))} < \infty, \forall|\gamma| = m \}.
\tag{5.5}
\]

For any \( d < 0 \), let \( P^l_{\text{hom},Y}(Y \times (d,0)) \) be the space of homogeneous polynomials of degree at most \( l \) restricted to the domain \((Y \times (d,0))\). Let \( \epsilon > 0 \) be fixed. We define the projectors \( P_l \) from \( L^2(\tilde{W}_\epsilon, W_{\text{hom},Y}^{m,2}(Y \times (-1/\epsilon,0))) \) to \( L^2(\hat{W}_\epsilon, P^l_{\text{hom},Y}(-1/\epsilon,0)) \) by setting
\[
P_l(\psi) = \sum_{|\gamma| = l} \int_Y D^\gamma \psi(\bar{x}, \zeta, 0) d\zeta \eta^{\gamma} \gamma!.
\]
for all \( i = 0, \ldots, m-1 \). We now set \( Q_{m-1} = P_{m-1} \), \( Q_{m-2} = P_{m-2}(I - Q_{m-1}) \), etc., up to \( Q_0 = P_0(I - \sum_{j=1}^{m-1} Q_j) \). Note that \( Q_{m-j}, j = 1, \ldots, m \) is a projection on the space of homogeneous polynomials of degree \( m - j \), with the property that \( Q_{m-k}(p) = 0 \) for all polynomials \( p \) of degree \( m - k \) with \( k \neq j \). We finally set

\[
P = Q_0 + Q_1 + \cdots + Q_{m-1},
\]

which is a projector on the space of polynomials in \( y \) of degree at most \( m-1 \). Note that \( D_y^\beta P(\psi(\bar{x}, \bar{y}, 0)) = \int_Y D_y^\beta \psi(\bar{x}, \bar{y}, 0) d\bar{y} \) for all \( |\beta| = 0, \ldots, m-1 \). In particular, it follows that \( \int_Y (D_y^\beta \psi(\bar{x}, \bar{y}, 0) - D_y^\beta P(\psi)(\bar{x}, \bar{y}, 0)) d\bar{y} = 0 \) for almost all \( \bar{x} \) in \( \hat{W}_\epsilon \), for all \( |\beta| = 0, \ldots, m-1 \).

**Lemma 3.** Let \( m \in \mathbb{N}, m \geq 2 \) be fixed. The following statements hold:

(i) Let \( v_\epsilon \in W^{m,2}(\Omega) \) with \( \|v_\epsilon\|_{W^{m,2}(\Omega)} \leq M \), for all \( \epsilon > 0 \). Let \( V_\epsilon \) be defined by

\[
V_\epsilon(\bar{x}, y) = \hat{v}_\epsilon(\bar{x}, y) - P(v_\epsilon)(\bar{x}, y),
\]

for \((\bar{x}, y) \in \hat{W}_\epsilon \times Y \times (-1/\epsilon, 0)\), where \( P \) is defined by (5.6). Then, possibly passing to a subsequence, there exists a function \( \hat{v} \in L^2(\hat{W}, w_{m,2}^{\epsilon, \forall, Y}((X \times (-\infty, 0))) \) such that for every \( d < 0 \)

(a) \( \frac{D^\gamma \hat{v}}{m-1} \to D_y^\gamma \hat{v} \) in \( L^2(W \times Y \times (d, 0)) \) as \( \epsilon \to 0 \), for any \( \gamma \in \mathbb{N}^N \), \( |\gamma| \leq m - 1 \).

(b) \( \frac{D^\gamma \hat{v}}{m-1} \to D_y^\gamma \hat{v} \) in \( L^2(W \times Y \times (-\infty, 0)) \) as \( \epsilon \to 0 \), for any \( \gamma \in \mathbb{N}^N \), \( |\gamma| = m \).

where it is understood that the functions \( V_\epsilon, D^\gamma V_\epsilon \) are extended by zero to the whole of \( W \times Y \times (-\infty, 0) \) outside their natural domain of definition \( \hat{W}_\epsilon \times Y \times (-1/\epsilon, 0) \).

(ii) If \( \psi \in W^{1,2}(\Omega) \), then \( \lim_{\epsilon \to 0} (T_\epsilon \psi)|_{\Omega} = \psi(\bar{x}, 0) \) in \( L^2(W \times Y \times (-1, 0)) \).

**Proof.** The proof follows as in the proof [8, Lemma 8.9] by noting that \( P \) is a projector on the space of polynomials of degree at most \( m - 1 \), so that a Poincaré-Wirtinger-type inequality still holds. \( \square \)

Let \( f_\epsilon \in L^2(\Omega_\epsilon) \) and \( f \in L^2(\Omega) \) be such that \( f_\epsilon \to f \) in \( L^2(\mathbb{R}^N) \) as \( \epsilon \to 0 \), with the understanding that the functions are extended by zero outside their natural domains. Let \( v_\epsilon \in V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon) \cap W^{-m-1,2}_0(\Omega_\epsilon) \) be such that for all \( \epsilon > 0 \) small enough

\[
H_{\Omega_\epsilon, S}v_\epsilon = f_\epsilon.
\]  

Then \( \|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq M \) for all \( \epsilon > 0 \) sufficiently small, hence, possibly passing to a subsequence there exists \( v \in W^{m,2}(\Omega) \cap W^{-m-1,2}_0(\Omega) \) such that \( v_\epsilon \to v \) in \( W^{m,2}(\Omega) \) and \( v_\epsilon \to v \) in \( L^2(\mathbb{R}^N) \).

Let \( \varphi \in V(\Omega) = W^{m,2}(\Omega) \cap W^{-m-1,2}_0(\Omega) \) be fixed. Since \( T_\epsilon \varphi \in V(\Omega_\epsilon) \), by (5.7) we have

\[
\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, dx + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, dx,
\]

(5.8)
and passing to the limit as $\epsilon \to 0$ we get $\int_{\Omega} v_{\epsilon} T_{\epsilon} \varphi \, dx \to \int_{\Omega} v \varphi \, dx$ and $\int_{\Omega} f_{\epsilon} T_{\epsilon} \varphi \, dx \to \int_{\Omega} f \varphi \, dx$.

Now consider the first integral in the right hand-side of (5.8). Set $K_{\epsilon} = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega \setminus \Omega$, $\Omega \setminus K_{\epsilon}$ and $K_{\epsilon}$ and by arguing as in [8, Section 8.3] one can show that

$$\int_{K_{\epsilon}} D^{m} v_{\epsilon} : D^{m} \varphi \, dx \to \int_{\Omega} D^{m} v : D^{m} \varphi \, dx$$

and $\int_{\Omega \setminus \Omega} D^{m} v_{\epsilon} : D^{m} T_{\epsilon} \varphi \, dx \to 0$, as $\epsilon \to 0$. Let us define $Q_{\epsilon}$ by

$$Q_{\epsilon} = \hat{W}_{\epsilon} \times (-\epsilon, 0).$$

We split again the remaining integral in two summands as follows:

$$\int_{\Omega \setminus K_{\epsilon}} D^{m} v_{\epsilon} : D^{m} T_{\epsilon} \varphi \, dx$$

$$= \int_{\Omega \setminus (K_{\epsilon} \cup Q_{\epsilon})} D^{m} v_{\epsilon} : D^{m} T_{\epsilon} \varphi \, dx + \int_{Q_{\epsilon}} D^{m} v_{\epsilon} : D^{m} T_{\epsilon} \varphi \, dx. \tag{5.9}$$

As in [8, Section 8.3], $\int_{\Omega \setminus (K_{\epsilon} \cup Q_{\epsilon})} D^{m} v_{\epsilon} : D^{m} T_{\epsilon} \varphi \, dx \to 0$, as $\epsilon \to 0$. It remains to analyse the limit as $\epsilon \to 0$ of the last summand in the right-hand side of (5.9). To do so, we also need the following lemma in the proof of which we use notation and rules of calculus recalled in Sect. 2.

**Lemma 4.** Let $l \in \mathbb{N}$, $l \leq m$, and let $i_{1}, \ldots, i_{l} \in \{1, \ldots, N\}$. The functions $\hat{h}_{\epsilon}(x, y)$, $\frac{\partial^{l} h_{\epsilon}}{\partial x_{i_{1}} \cdots \partial x_{i_{l}}}(x, y)$ defined for $y \in Y \times (-1, 0)$, are independent of $\hat{x}$.

Moreover, $\left\| \frac{\partial^{l} h_{\epsilon}}{\partial x_{i_{1}} \cdots \partial x_{i_{l}}}(x, y) \right\|_{L^{\infty}} = O(\epsilon^{3/2})$ and $\left\| \frac{\partial^{l} h_{\epsilon}}{\partial x_{i_{1}} \cdots \partial x_{i_{l}}}(x, y) \right\|_{L^{\infty}} = O(\epsilon^{3/2})$ as $\epsilon \to 0$.

If $l \geq 2$ we have $\epsilon^{l-3/2} \frac{\partial^{l} h_{\epsilon}}{\partial x_{i_{1}} \cdots \partial x_{i_{l}}}(x, y) \to \frac{\partial^{l} b(y)(y_{N+1})^{m+1}}{\partial y_{i_{1}} \cdots \partial y_{i_{l}}}$ as $\epsilon \to 0$, uniformly in $y \in Y \times (-1, 0)$.

**Proof.** First, note that the part of the statement involving the asymptotic behaviour of $\hat{h}_{\epsilon}$ as $\epsilon \to 0$ follows directly from Lemma 1 and Definition 4. Assume now that $l \geq 2$. By applying formula (2.7) we have that

$$\frac{\partial^{l} h_{\epsilon}}{\partial x_{i_{1}} \cdots \partial x_{i_{l}}}(\bar{x}, y) = \sum_{S \in \mathcal{P}(l)} \frac{\epsilon^{\alpha}}{\epsilon^{\left| S \right|}} \frac{\partial^{|S|} b(\bar{y})}{\partial y_{i_{j}}} \frac{\partial^{l-\left| S \right|}}{\partial x_{i_{j}}} \left( \frac{x_{N} + \epsilon}{g_{\epsilon}(\bar{x}) + \epsilon} \right)^{m+1} \tag{5.10}$$

Standard Calculus computations based on Formulas (2.6) and (2.7) give

$$\frac{\partial^{l-\left| S \right|}}{\partial y_{i_{j}}} \left( \frac{x_{N} + \epsilon}{g_{\epsilon}(\bar{x}) + \epsilon} \right)^{m+1} = C(\left| S \right|) \frac{\epsilon^{l-\left| S \right|}}{(\epsilon^{1-l} b(\bar{y}) + 1)^{m+1}} \prod_{j \notin S} \delta_{i_{j} N} \left( y_{N} + 1 \right)^{m+1-l-\left| S \right|} \left( \frac{m + |\pi|)!}{m! \prod_{j \notin S}} \prod_{i \notin \pi} \delta_{i_{k} N} \prod_{B \in \pi} \frac{\partial^{B} b(\bar{y})}{\partial y_{i_{k}}}, \tag{5.11}$$

where $\Lambda \in \mathcal{P}(SC)$, $\pi \in \text{Part}(\Lambda)$, and $\Lambda \neq \emptyset$.
where \( C(|S|) = \frac{(m+1)!}{(m+l-|S|)!} \). By (5.10) and (5.11) we deduce that
\[
\epsilon^{l-\alpha} \frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}(\bar{x}, y) = \epsilon^{l-\alpha} \sum_{S \in P(l)} \epsilon^{-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \in S} \partial y_{i_j}} C(|S|) \epsilon^{-l+|S|} \frac{(y_N + 1)^{m+1-l+|S|}}{\left(\epsilon^{\alpha-1} b(\bar{y}) + 1\right)^{m+1}} \prod_{j \notin S} \delta_{i_j N} \\
+ \epsilon^{l-\alpha} \sum_{S \in P(l)} \epsilon^{-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \notin S} \partial y_{i_j}} \sum_{\Lambda \in P(S^c \setminus \Lambda)} \sum_{\Lambda \neq \emptyset} \epsilon^{\Lambda \setminus |\Lambda| - |l| - |S|} \left(-1\right)^{|\Lambda|} \frac{(m+|\pi|)!}{m!} \\
\cdot C(|S \cup \Lambda|) \frac{(y_N + 1)^{m+1-l+|S \cup \Lambda|}}{\left(\epsilon^{\alpha-1} + 1\right)^{m+1+|\pi|}} \prod_{k \in (S^c \setminus \Lambda)} \delta_{i_k N} \prod_{B \in \pi} \epsilon^{-|B|} \frac{\partial^{|B|} b(\bar{y})}{\prod_{l \in B} \partial y_{i_l}}.
\]
(5.12)

It is possible to prove by direct computation that all the summands appearing in the second line in the right-hand side of (5.12) are vanishing as \( \epsilon \to 0 \). By letting \( \epsilon \to 0 \) in (5.12) we see that
\[
\lim_{\epsilon \to 0} \epsilon^{l-\alpha} \frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}(\bar{x}, y) = \sum_{S \in P(l)} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j \notin S} \partial y_{i_j}} C(|S|) (y_N + 1)^{m+1-l+|S|} \prod_{j \notin S} \delta_{i_j N} \\
= \frac{\partial^l}{\partial y_{i_1} \cdots \partial y_{i_l}} (b(\bar{y})(y_N + 1)^{m+1}),
\]
concluding the proof. \( \square \)

Finally, we are ready to prove the following

**Proposition 2.** Let \( v_\epsilon \in V(\Omega_\epsilon) \) be such that \( \|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq M \) for all \( \epsilon > 0 \). Let \( \bar{Y} = Y \times (-1, 0) \) and \( g(y) = b(\bar{y})(1 + y_N)^{m+1} \) for all \( y \in \bar{Y} \). Moreover, let \( \hat{v} \in L^2(W, w_{Perv}^{m,2}(Y \times (-\infty, 0))) \) be as in Lemma 3. Then
\[
\int_{Q_\epsilon} D^m v_\epsilon : D^m(T_\epsilon \varphi) \, dx \\
- \sum_{l=1}^{m-1} \left( \begin{array}{c} m \\ l+1 \end{array} \right) \int_W \int_Y \frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left( \frac{\partial^{m-l-1} \hat{v}(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) \cdot D_y^{l+1} g(y) dy \frac{\partial^{m-l-1} \varphi(\bar{x}, 0)}{\partial x_N^{m-l-1}} \, dx,
\]
for all \( \varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \), as \( \epsilon \to 0 \).

**Proof.** We set
\[
P_1(t) = \{ \pi = (S_1, \ldots, S_t) \in \text{Part}(\{1, \ldots, m\}) : \exists! S_k \text{ with } |S_k| > 1 \},
\]
\[
P_2(t) = \{ \pi \in \text{Part}(\{1, \ldots, m\}) : |\pi| = t, \pi \notin P_1(t) \}.
\]
We note that in the definition of \( P_1(t) \) we may assume without loss of generality that the only element \( S_k \) with cardinality strictly bigger than 1 is \( S_1 \). In the sequel, we always assume that a given partition \( \pi \) of cardinality \( t \) is represented by \( \pi = \{S_1, \ldots, S_t\} \). In the following calculations, we use the index notation and we drop the summation symbols \( \sum_{j_1, \ldots, j_m=1}^N \) and \( \sum_{i_1, \ldots, i_m=1}^N \). With the help of (2.8) we compute
\[
\int_{Q_\epsilon} D^m v_\epsilon \cdot D^m (T_\epsilon \varphi) \, dx = \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^m (\varphi \circ \Phi_\epsilon)}{\partial x_{i_1} \cdots \partial x_{i_m}} \, dx \\
= \sum_{\pi \in \text{Part}\{1, \ldots, m\}} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^{|\pi|} \varphi}{\partial x_{j_k} \prod_{k=1}^{|\pi|} \partial x_{j_k}} (\Phi_\epsilon(x)) \prod_{k=1}^{|\pi|} \frac{\partial |S_k| \Phi_{(j_k)}}{\partial x_{i_l}} \, dx \\
= \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} (\Phi_\epsilon(x)) \frac{\partial \Phi_{(j_1)}}{\partial x_{i_1}} \cdots \frac{\partial \Phi_{(j_m)}}{\partial x_{i_m}} \, dx \\
+ \sum_{t=1}^{m-1} \sum_{\pi \in \Pi_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_{j_k} \prod_{k=1}^t \partial x_{j_k}} (\Phi_\epsilon(x)) \prod_{k=1}^t \frac{\partial |S_k| \Phi_{(j_k)}}{\partial x_{i_l}} \, dx \\
+ \sum_{t=2}^{m-2} F_t(v_\epsilon, \varphi, \Phi_\epsilon),
\]

where \( F_t(v_\epsilon, \varphi, \Phi_\epsilon) \) is defined by

\[
F_t(v_\epsilon, \varphi, \Phi_\epsilon) = \sum_{\pi \in \Pi_2(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_{j_k} \prod_{k=1}^t \partial x_{j_k}} \prod_{k=1}^t \frac{\partial |S_k| \Phi_{(j_k)}}{\partial x_{i_l}} \, dx.
\]

We consider separately the three summands in the right hand side of (5.13). Let us remark for future use that

\[
\frac{\partial \Phi_{(k)}}{\partial x_i} = \begin{cases} 
\delta_{ki}, & \text{if } k \neq N, \\
\frac{\partial \Phi_{(k)}}{\partial x_i}, & \text{if } k = N,
\end{cases} \quad \frac{\partial^l \Phi_{(k)}}{\partial x_i} = \begin{cases} 
0, & \text{if } k \neq N, \\
-\frac{\partial^l \Phi_{(k)}}{\partial x_i}, & \text{if } k = N.
\end{cases}
\]

for all \(2 \leq l \leq m\). Consider now the first term in the right hand side of (5.13). We unfold it by taking into account (5.4) in order to obtain

\[
\epsilon \int_{W_\epsilon} \int_{\tilde{Y}} \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \cdots \partial y_{i_m}} \frac{\partial^m \varphi}{\partial y_{j_1} \cdots \partial y_{j_m}} (\hat{\Phi}_\epsilon(y)) \frac{\partial \hat{\Phi}_{(j_1)}}{\partial y_{i_1}} \cdots \frac{\partial \hat{\Phi}_{(j_m)}}{\partial y_{i_m}} \, dyd\tilde{x} \\
= \epsilon^{-2m+1} \left| \int_{W_\epsilon} \int_{\tilde{Y}} \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \cdots \partial y_{i_m}} \frac{\partial^m \varphi}{\partial y_{j_1} \cdots \partial y_{j_m}} (\hat{\Phi}_\epsilon(y)) \frac{\partial \hat{\Phi}_{(j_1)}}{\partial y_{i_1}} \cdots \frac{\partial \hat{\Phi}_{(j_m)}}{\partial y_{i_m}} \, dyd\tilde{x} \right|
\]

\[
\leq C \epsilon^{-m+1} \left| \int_{W_\epsilon} \int_{\tilde{Y}} \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \cdots \partial y_{i_m}} \frac{\partial^m \varphi}{\partial y_{j_1} \cdots \partial y_{j_m}} (\hat{\Phi}_\epsilon(y)) \, dyd\tilde{x} \right| \\
\leq C \epsilon^{1/2} \left\| \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \cdots \partial y_{i_m}} \right\|_{L^2(W_\epsilon, \tilde{Y})} \left\| \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} (\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W_\epsilon, \tilde{Y})} \\
\leq C \epsilon^{1/2} \left\| \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} (\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W_\epsilon, \tilde{Y})} \leq C \left\| \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} \right\|_{L^2(\Phi(Q_\epsilon))},
\]

which vanishes as \( \epsilon \to 0 \). In the first inequality we have used the fact that

\[
\left| \frac{\partial \Phi_{(k)}}{\partial y_i} \right| \leq C \epsilon, \text{ for sufficiently small } \epsilon > 0.
\]

Let now \(1 \leq t \leq m-1\) be fixed and consider
\[ \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_{j_k}} (\Phi_\epsilon(x)) \prod_{k=1}^t \frac{\partial j_{S_k} \Phi^{(j_k)}}{\partial x_{i_k}} \, dx \]

where to shorten the notation we have identified \( S_2, \ldots, S_t \) with the only element they contain. Note that if \( j_1 \neq N \) then the integral in (5.14) is zero. Thus, without loss of generality we set \( j_1 = N \). Note that we have \( \frac{\partial \Phi^{(N)}}{\partial x_{i_t}} = \delta_{N,i_t} + \frac{\partial h_i}{\partial x_{i_t}} \) and \( |\frac{\partial h_i}{\partial x_{i_t}}| \leq C\epsilon^{1/2} \) as \( \epsilon \to 0 \). In order to simplify the expressions we will not write down the higher order terms in \( \epsilon \). Hence, by setting \( j_1 = N \) in (5.14) we deduce that the lower order terms in \( \epsilon \) are given by

\[ \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_{N S_2} \cdots \partial x_{S_1}} (\Phi_\epsilon) \prod_{l \in S_1} \frac{\partial^{m-t+1} \Phi^{(N)}}{\partial x_{i_1} \cdots \partial x_{i_{S_2} \cdots \partial x_{i_{S_1}}} \partial x_{i_{S_1}}} \, dx \]

where in the last equality in (5.15) we have used the fact that each of the summands

\[ \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_{N S_2} \cdots \partial x_{S_1}} (\Phi_\epsilon) \prod_{l \in S_1} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{S_2} \cdots \partial x_{i_{S_1}}} \partial x_{i_{S_1}}} \, dx \]

equals

\[ \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_{N S_2} \cdots \partial x_{S_1}} (\Phi_\epsilon) D^{m-t+1} \left( \frac{\partial^{m-t+1} \Phi^{(N)}}{\partial x_{i_{S_2} \cdots \partial x_{i_{S_1}}} \partial x_{i_{S_1}}} \right) \, dx \]

and in particular they do not depend on the choice of \( \pi \) (note that the cardinality of \( P_1(t) \) equals \( \binom{m}{t-1} \)). By unfolding the right-hand side of (5.15) and using the fact that \( m - t + 2 \geq 2 \) we have that

\[ \begin{align*}
&= \left( \frac{m}{t-1} \right) \int_{W_\epsilon} \int_{\hat{\gamma}} \frac{\partial^t \varphi}{\partial x_{N S_2} \cdots \partial x_{S_1}} (\Phi_\epsilon(y)) \prod_{l \in S_1} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{S_2} \cdots \partial x_{i_{S_1}}} \partial x_{i_{S_1}}} \, dy d\hat{x} \\
& \quad + \left( \frac{m}{t+1} \right) \int_{W_\epsilon} \int_{\hat{\gamma}} \frac{\partial^t \varphi}{\partial x_{N S_2} \cdots \partial x_{S_1}} (\Phi_\epsilon(y)) \prod_{l \in S_1} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{S_2} \cdots \partial x_{i_{S_1}}} \partial x_{i_{S_1}}} \, dy d\hat{x} \end{align*} \]

It is easy to see that the final expression appearing in the right-hand side of (5.16) can be written as
\[-\left(\frac{m}{t+1}\right) \int_{W_\epsilon} \int_{\tilde{Y}} \left[ \epsilon^{-m+1/2} \prod_{l \in S_1} \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_{S_1}}} \right] \left[ \prod_{l \in S_1} \frac{\partial^{m-1} \hat{\varphi}}{\partial x_{i_l}} \right] \left[ \epsilon^{m-t+1} \frac{\partial^{m-t+1} \hat{h}_\epsilon}{\partial x_{i_l}} \right] dy d\bar{x}.\]

(5.17)

Now

\[\epsilon^{-m+1/2} \prod_{l \in S_1} \frac{\partial^m \hat{v}_\epsilon}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_{S_1}}} \rightarrow \prod_{l \in S_1} \frac{\partial^m \hat{v}}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_{S_1}}},\]

weakly in \(L^2(W_\epsilon \times Y \times (-1,0))\) as \(\epsilon \to 0\), by Lemma 3, and

\[\epsilon^{m-t+1-3/2} \frac{\partial^{m-t+1} \hat{h}_\epsilon}{\partial x_{i_l}} \rightarrow \frac{\partial^{m-t+1} \hat{h}}{\partial x_{i_l}},\]

in \(L^\infty(W_\epsilon \times Y \times (-1,0))\) as \(\epsilon \to 0\), by Lemma 4. Moreover, by Lemma 6 in

“Appendix” it follows that

\[\frac{1}{\epsilon^{m-t-1}} \frac{\partial^t \varphi}{\partial x_N^{t}}(\hat{\Phi}_\epsilon(y)) \rightarrow \frac{y_{N}^{m-t-1}}{(m-t-1)!} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x}, 0),\]

and

\[\frac{1}{\epsilon^{m-t-1}} \frac{\partial^t \varphi}{\partial x_N \partial x_{i_2} \cdots \partial x_{i_{S_1}}} \rightarrow 0,\]

strongly in \(L^2(W \times Y \times (-1,0))\) as \(\epsilon \to 0\), if at least one of the indexes \(i_{S_2}, \ldots, i_{S_N}\) is not equal to \(N\). Hence (5.17) tends to

\[-\left(\frac{m}{t+1}\right) \int_{W_\epsilon} \int_{Y \times (-1,0)} \frac{y_{N}^{m-t-1}}{(m-t-1)!} D_y^{m-t+1} \left(\frac{\partial^{t-1} \hat{v}}{\partial y_{N}^{t-1}}\right) : \left[D_y^{m-t+1} \left(b(\tilde{y})(1 + y_N)^{m+1}\right) dy \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x}, 0) d\bar{x}.\]

By setting \(m - t = l\) we recover the limiting expression in the statement. Then, in order to conclude the proof it is sufficient to prove that the integrals in \(F_i(v_\epsilon, \varphi, \Phi_\epsilon)\) vanish as \(\epsilon \to 0\). We will show this by comparing each integral appearing in the definition of \(F_t(v_\epsilon, \varphi, \Phi_\epsilon)\) with the corresponding integral of the form (5.14), which is convergent as \(\epsilon \to 0\), hence it is uniformly bounded in \(\epsilon\). Note that by Lemma 4

\[\frac{\partial^{m-t+1} \hat{\Phi}_\epsilon^{(j_1)}}{\partial y_{i_1}} \frac{\partial^{m-t+1} \hat{\Phi}_\epsilon^{(j_2)}}{\partial y_{i_2}} \cdots \frac{\partial^{m-t+1} \hat{\Phi}_\epsilon^{(j_1)}}{\partial y_{i_{S_1}}} = O(\epsilon^{3/2+t-1}) = O(\epsilon^{1/2+t}),\]

for all \(\pi \in P_1(t)\), whereas if we consider \(\pi' = (S_1', \ldots, S_t') \in P_2(t)\) with \(|S'_1| = m - t < m - t + 1\) there must exists \(S'_k\), \(k > 1\) with \(|S'_k| = 2\). Let us assume that \(k = 2\). Then we have

\[\frac{\partial^{m-t} \hat{\Phi}_\epsilon^{(j_1)}}{\partial y_{i_1}} \frac{\partial^{m-t} \hat{\Phi}_\epsilon^{(j_2)}}{\partial y_{i_2}} \cdots \frac{\partial^{m-t} \hat{\Phi}_\epsilon^{(j_1)}}{\partial y_{i_{S_1}}} = O(\epsilon^{3/2+t} \epsilon^{3/2-2}) = O(\epsilon^{1+t}),\]
and since \( \epsilon^{1+t} = o(\epsilon^{1/2+t}) \) as \( \epsilon \to 0 \) and the integral (5.14) is bounded, we deduce that the integral in \( F_t(v_\epsilon, \varphi, \Phi_\epsilon) \) involving

\[
\frac{\partial^m v_\epsilon}{\partial x_1 \cdots \partial x_m} \cdot \frac{\partial^t \varphi}{\partial x_{k_1} \partial y_{l_1}} \cdot \frac{\partial^{m-t} \Phi^{(ji)}_\epsilon}{\partial y_{l_2} \partial x_{k_2}} \cdot \frac{\partial^2 \Phi^{(jz)}_\epsilon}{\partial y_{l_3} \partial x_{k_3}} \cdot \frac{\partial \Phi^{(ji)}_\epsilon}{\partial y_{l_4}} \cdot \frac{\partial \Phi_\epsilon}{\partial y_{l_5}} \cdot \ldots \cdot \frac{\partial \Phi_\epsilon}{\partial y_{l_5}}
\]

for all \( \pi' \in P_2(t) \) defined above, vanishes as \( \epsilon \to 0 \). By arguing in a similar way for all the terms in \( F_t(v_\epsilon, \varphi, \Phi_\epsilon) \) we deduce the validity of the statement. 

We summarise the previous discussion in the following

**Theorem 8.** Let \( f_\epsilon \in L^2(\Omega_\epsilon) \), \( f \in L^2(\Omega) \) be such that \( f_\epsilon \rightharpoonup f \) in \( L^2(\Omega) \). Let \( g(y) = b(\bar{y})(1 + y_N)^{m+1} \) for all \( y \in Y \times (-1,0) \). Moreover, let us assume that \( v_\epsilon \in W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon) \) is the solution to \( H_{\Omega_\epsilon, \epsilon} v_\epsilon = f_\epsilon \) for all \( \epsilon > 0 \). Then there exist \( v \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \) and a function \( \hat{v} \) in the space \( L^2(W, w^2_{P_r}(Y \times (-\infty,0))) \) such that, possibly passing to a subsequence, \( v_\epsilon \rightharpoonup v \) in \( W^{m,2}(\Omega) \), \( v_\epsilon \rightharpoonup v \) in \( L^2(\mathbb{R}^N) \), and statements (a) and (b) in Lemma 3 hold. Moreover, the following integral equality holds

\[
-\sum_{l=1}^{m-1} \left( \frac{m}{l+1} \right) \int_W \int_{Y \times (-1,0)} \left[ \frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left( \frac{\partial^{m-l-1} \hat{v}(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) \right] \cdot D_y^{l+1} g(y) \, dy \right] dx + \int_\Omega D^m v : D^m \varphi + u \varphi \, dx = \int_\Omega f \varphi \, dx.
\]

(5.18)

for all \( \varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \).

**Notation.** We will use the following notation:

\[
q_\varphi(f, g) := \sum_{l=1}^{m-1} \left( \frac{m}{l+1} \right) \int_{Y \times (-1,0)} \left[ \frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left( \frac{\partial^{m-l-1} f(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) \right] \cdot D_y^{l+1} g(y) \, dy
\]

for all \( f \in L^2(W, w^2_{P_r}(Y \times (-\infty,0))) \), \( g \in C^m_{P_r}(Y \times (-1,0)) \). We refer to

\[
- \int_W q_\varphi(\hat{v}, g) \frac{\partial^{m-l-1} \varphi(\bar{x}, 0)}{\partial x_N^{m-l-1}} \, d\bar{x}
\]

(5.19)

as the strange term appearing in the homogenization.

### 5.2. Critical Case: Microscopic Problem

The aim of this section is to characterize the strange term (5.19) as the energy of a suitable polyharmonic function and in particular to conclude that it is different from zero. We will use periodically oscillating test functions matching the intrinsic \( \epsilon \)-scaling of the problem.

Let then \( \psi \in C^\infty(\overline{W \times Y} \times (-\infty,0]) \) be such that \( \text{supp} \, \psi \subset C \times Y \times [d,0] \) for some compact set \( C \subset W \) and for some \( d \in (-\infty,0) \). Moreover, assume that \( \psi(\bar{x}, \bar{y}, 0) = D^l \psi(\bar{x}, \bar{y}, 0) = 0 \) for all \( (\bar{x}, \bar{y}) \in W \times Y \), for all \( 1 \leq l \leq m-2 \). Let also \( \psi \) be \( Y \)-periodic in the variable \( \bar{y} \). We set

\[
\psi_\epsilon(x) = \epsilon^{m-\frac{1}{2}} \psi\left( \frac{\bar{x}}{\epsilon}, \frac{x_N}{\epsilon} \right),
\]
for all $\epsilon > 0$, $x \in W \times [-\infty, 0]$. Then $T_\epsilon \psi_\epsilon \in V(\Omega_\epsilon)$ for all sufficiently small $\epsilon$, hence we can use it as a test function in the weak formulation of the problem in $\Omega_\epsilon$, getting
\[
\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, dx + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \psi_\epsilon \, dx = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \psi_\epsilon \, dx.
\]
It is not difficult to prove that
\[
\int_{\Omega_\epsilon} v_\epsilon T_\epsilon \psi_\epsilon \, dx \to 0, \quad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \psi_\epsilon \, dx \to 0
\] as $\epsilon \to 0$. By arguing as in [8, §8.4], it is also possible to prove that
\[
\int_{\Omega \setminus \Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, dx \to 0,
\] as $\epsilon \to 0$. Moreover, a suitable modification of [8, Lemma 8.47] yields
\[
\int_{\Omega} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, dx \to \int_{W \times Y \times (-\infty, 0)} D^m \hat{v}(\bar{x}, y) : D^m \psi(\bar{x}, y) \, d\bar{x} dy.
\] (5.22)

**Theorem 9.** Let $\hat{v} \in L^2(W, w^{m,2}_{\text{Per}_Y}(Y \times (\infty, 0)))$ be the function from Theorem 8. Then
\[
\int_{W \times Y \times (-\infty, 0)} D^m \hat{v}(\bar{x}, y) : D^m \psi(\bar{x}, y) \, d\bar{x} dy = 0,
\] (5.23)
for all $\psi \in L^2(W, w^{m,2}_{\text{Per}_Y}(Y \times (\infty, 0)))$ such that $\psi(\bar{x}, \bar{y}, 0) = D^l_y \psi(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$, for all $1 \leq l \leq m - 2$. Moreover, for any $j = 1, \ldots, N - 1$, we have
\[
\frac{\partial^{m-1} \hat{v}}{\partial y_j \partial y_N^{m-2}}(\bar{x}, \bar{y}, 0) = - \frac{\partial b}{\partial y_j} (\bar{y}) \frac{\partial^{m-1} \psi}{\partial x_N^{m-1}}(\bar{x}, 0), \quad \text{on } W \times Y,
\] (5.24)
and
\[
\frac{\partial^{m-1} \hat{v}}{\partial y_{i_1} \cdots \partial y_{i_m-1}}(\bar{x}, \bar{y}, 0) = 0, \quad \text{on } W \times Y,
\] (5.25)
for all $i_1, \ldots, i_{m-1} = 1, \ldots, N - 1$.

**Proof.** The first part of the statement follows from (5.20), (5.21) and (5.22) by arguing as in [8, Theorem 8.53]. In order to prove formulas (5.24) and (5.25) we note that, since $D^{m-2} v_\epsilon(\bar{x}, g_\epsilon(\bar{x})) = 0$ for all $\bar{x} \in W$, we have
\[
\frac{\partial^{m-2} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m-2}}(\bar{x}, g_\epsilon(\bar{x})) = 0, \quad \text{for all } i_1, \ldots, i_{m-2} = 1, \ldots, N, \bar{x} \in W.
\]
Differentiating with respect to $x_j, j \in \{1, \ldots, N - 1\}$ yields
\[
\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m-2}}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m-2} \partial x_N}(\bar{x}, g_\epsilon(\bar{x})) \frac{\partial g_\epsilon(\bar{x})}{\partial x_j} = 0,
\] for all $\bar{x} \in W$. Hence, by setting
\[
V_\epsilon^j = \left(0, \ldots, 0, -\frac{\partial^{m-1} v_\epsilon}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-2}}}, 0, \ldots, 0, \frac{\partial^{m-1} v_\epsilon}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_{m-2}}}, \right),
\]
for all \( i_1, \ldots, i_{m-2} = 1, \ldots, N, \) \( j = 1, \ldots, N - 1, \) where the only non-zero entries are the \( j \)-th and the \( N \)-th, we obtain that \( V^j \cdot n_\epsilon = 0, \) on \( \Gamma_\epsilon, \) where \( n_\epsilon \) is the outer normal to \( \Gamma_\epsilon \equiv \{(\bar{x}, g_\epsilon(\bar{x})) : \bar{x} \in W\}. \) By using Lemma 3
\[
\frac{\partial^{m-1} \hat{v}}{\partial y_1 \cdots \partial y_{i_{m-2}} \partial y_j} (\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_j} (\bar{y}) \frac{\partial^{m-1} v}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-2}}} (\bar{x}, 0),
\]
in \( L^2(W \times Y \times [d, 0]) \) for any \( d < 0. \) This combined with [20, Lemma 4.3] (see also [8, Lemma 8.56]) yields
\[
\int_{Y \times (-\infty, 0)} D^m V : D^l \psi \, dy = 0,
\]
for all \( \psi \in w_{Per}^{m,2}(Y \times (-\infty, 0)) \) such that \( D^l \psi(\bar{y}, 0) = 0 \) on \( Y, \) for all \( 0 \leq l \leq m - 2, \) and the boundary conditions
\[
\begin{cases}
\frac{\partial^l V}{\partial y_N} (\bar{y}, 0) = 0, & \text{for all } l = 0, \ldots, m - 3, \text{ on } Y, \\
\frac{\partial^{m-2} V}{\partial y_{i_{m-2}}} (\bar{y}, 0) = b(\bar{y}), & \text{on } Y.
\end{cases}
\]
The function \( V \) is unique up to the sum of a monomial in \( y_N \) of degree \( m - 1 \) of the type \( a y_N^{m-1} \) with \( a \in \mathbb{R}. \) Moreover \( V \in W_{Per}^{2m,2}(Y \times (d, 0)) \) for any \( d < 0 \) and it satisfies the equation
\[
(-\Delta)^m V = 0, \quad \text{in } Y \times (d, 0),
\]
subject to the boundary conditions
\[
\begin{cases}
\frac{\partial^l V}{\partial y_N} (\bar{y}, 0) = 0, & \text{on } Y, \text{ for all } 0 \leq l \leq m - 3, \\
\frac{\partial^{m-2} V}{\partial y_{i_{m-2}}}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\
\frac{\partial^m V}{\partial y_N} (\bar{y}, 0) = 0, & \text{on } Y.
\end{cases}
\]

**Proof.** Similar to the proof of [8, Lemma 8.60], we just note that in order to deduce the classical formulation of problem (5.26) it is sufficient to choose test functions \( \psi \) as in the statement with bounded support in the \( y_N \) direction. By using the Polyharmonic Green Formula (4.3) we then deduce that
\[
\int_{Y \times (-\infty, 0)} D^m V : D^m \psi \, dy = (-1)^m \int_{Y \times (-\infty, 0)} \Delta^m V \psi \, dy + \int_Y \frac{\partial^m V}{\partial y_N^m} \frac{\partial^{m-1} \psi}{\partial y_{N-1}} \, dy.
\]
By the arbitrariness of \( \psi \) it is then easy to conclude the proof. \( \square \)
Theorem 10. Let \( V \) be as in Lemma 5 and \( g(y) = b(\bar{y})(1 + y_N)^{m+1} \), for all \( y \in Y \times (-1, 0) \). Then
\[
q_Y(V, g) = \int_{Y \times (-\infty, 0)} |D^m V|^2 \, dy. 
\]
Furthermore
\[
\int_{Y \times (-\infty, 0)} |D^m V|^2 \, dy 
= - \int_Y \left( \frac{\partial^{m-1} V}{\partial x_N^{m-1}} + (m-1) \Delta_{N-1} \left( \frac{\partial^{m-1} V}{\partial x_N^{m-1}} \right) \right) b(\bar{y}) \, d\bar{y}. 
\]

Proof. Let \( \phi \) be the real-valued function defined on \( Y \times (-\infty, 0) \) by \( \phi(y) = \frac{y_{N-2}}{(m-2)!} g(y) \) if \(-1 \leq y_N \leq 0 \) and \( \phi(y) = 0 \) if \( y_N < -1 \). Then \( \phi \in W^{m,2}(Y \times (-\infty, 0)) \), \( \frac{\partial^l \phi}{\partial y^l} (\bar{y}, 0) = 0 \) for all \( 0 \leq l \leq m - 3 \), and
\[
\frac{\partial^{m-2} \phi}{\partial y_{m-2}^2} (\bar{y}, 0) = b(\bar{y}), \quad \text{for all } y \in Y. 
\]
Now note that the function \( \psi = V - \phi \) is a suitable test-function in equation (5.26); by plugging it in (5.26) we deduce that
\[
\int_{Y \times (-\infty, 0)} |D^m V|^2 \, dy = \int_{Y \times (-1, 0)} D^m V : D^m \phi \, dy.
\]
By the Leibnitz rule we have that
\[
\int_{Y \times (-1, 0)} D^m V : D^m \phi \, dy 
= \int_{Y \times (-1, 0)} \frac{\partial^m V}{\partial x_{j_1} \cdots \partial x_{j_m}} \sum_{S \in P(S)} \frac{1}{(m-2)!} \prod_{j \in S} \frac{\partial |S| y_{N-2}^{-2} \partial (n-|S|) g}{\partial x_j} \, dy.
\]
Using the obvious fact that
\[
\frac{\partial^{m-k} y_{N-2}^{-2} \partial x_{j_1} \cdots \partial x_{j_{m-k}}}{\partial x_{i_1} \cdots \partial x_{i_{m-k}}} = \begin{cases} 0, & \text{if } k = 0, 1; \\ y_{N-2}^{k-2} \delta_{i_1} \cdots \delta_{i_{m-k}} N, & \text{for } k \geq 2. \end{cases}
\]
we can rewrite the right-hand side of (5.30) as follows
\[
\sum_{k=2}^m \binom{m}{k} \int_{Y \times (-1, 0)} D^k \left( \frac{\partial^{m-k} V(y)}{\partial y_{N-2}^{m-k}} \right) : \left( \frac{y_{N-2}^{k-2}}{(k-2)!} \partial^k g(y) \right) \, dy 
= \sum_{k=2}^m \binom{m}{k} \int_{Y \times (-1, 0)} \frac{y_{N-2}^{k-2}}{(k-2)!} \partial^k \left( \frac{\partial^{m-k} V(y)}{\partial y_{N-2}^{m-k}} \right) : \partial^k g(y) \, dy,
\]
which coincides with the left-hand side of (5.27) up to the change of summation index defined by \( k = l + 1 \). Finally, (5.28) follows by applying the polyharmonic Green formula (4.3) on \( \int_{Y \times (-1, 0)} D^m V : D^m \phi \, dy \). Indeed, we note that the boundary integrals on \( \partial Y \times (-1, 0) \) are zero, due to the periodicity of \( V \) and \( b \). Moreover the boundary integral on \( \partial Y \times \{ -1 \} \) is zero since \( \phi \) vanishes there together with all its derivatives. Then, the only non-trivial boundary integral is supported on \( Y \times \{ 0 \} \). More precisely, we have
Theorem 11. Let $m \in \mathbb{N}$, $m \geq 2$. Let $V$ be as in Lemma 5. Let $v, \hat{v}$ be the functions defined in Theorem 8. Let also $g(y) = b(y)(1 + y_N)^{m+1}$ for all $y \in Y \times (-1,0)$. Then

$$\hat{v}(x,y) = -V(y) \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) + a(x)y^{m-1},$$

for some $a(\bar{x}) \in L^2(W)$. Moreover, the strange term (5.19) is given by

$$- \int_W q_y(\hat{v},g) \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) d\bar{x}$$

$$= \int_{Y \times (-\infty,0)} |D^m V|^2 dy \int_W \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) d\bar{x}$$

$$= - \int_Y \left( \frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta N-1 \left( \frac{\partial^{m-1} V}{\partial x_N^{m-1}} \right) \right) b(y) d\bar{y}$$

$$\cdot \int_W \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) d\bar{x}.$$
and by the Green Formula (4.10) for all \( \varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \)
\[
\int_{\Omega} D^m v : D^m \varphi \, dx = (-1)^m \int_{\Omega} \Delta^m v \varphi + \int_{\partial \Omega} \frac{\partial^{m-1} v}{\partial n^{m-1}} \varphi \, dS. \tag{5.33}
\]
Hence, in the weak formulation of the limiting problem we find the following boundary integral
\[
\int_{\Omega} \frac{\partial^{m-1} v}{\partial x_N^{m-1}} \varphi \, \Phi(\epsilon) \, d\bar{x} + \left( \int_{Y \times (-\infty,0)} |D^m V|^2 \, dy \right) \frac{\partial^{m-1} v}{\partial x_N^{m-1}}(\bar{x},0) \, \Phi(\epsilon) \, d\bar{x},
\]
for all \( \varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \). By (5.32), (5.33), (5.34) and the arbitrari-
ness of \( \varphi \) we deduce the statement of Theorem 7, part (iii).

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6. Appendix

In this section we prove the following technical result used in the proof of Proposition 2.

**Lemma 6.** Let \( l, m \in \mathbb{N}, m \geq 2, 1 \leq l \leq m - 1, i_1, \ldots, i_{m-l-1} \in \{1, \ldots, N\} \). Then for all \( \varphi \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega) \) we have
\[
\frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l} \varphi}{\partial x_N^{m-l}}(\Phi(\epsilon)(y)) \to \frac{y_N^{m-l}}{(l-1)!} \frac{\partial^{m-l} \varphi}{\partial x_N^{m-l}}(\bar{x},0),
\]
in \( L^2(W \times Y \times (-1,0)) \) as \( \epsilon \to 0 \) and if at least one of the indexes \( i_1, \ldots, i_{m-l-1} \) does not coincide with \( N \) we also have
\[
\frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l} \varphi}{\partial x_N\partial x_{i_1} \cdots \partial x_{i_{m-l-1}}}(\Phi(\epsilon)(y)) \to 0
\]
in \( L^2(W \times Y \times (-1,0)) \) as \( \epsilon \to 0 \).
Proof. Note that for \( l = 1 \) the claim follows by Lemma 3. Then assume \( l > 1 \). Fix \( \varphi \in W^{m,2}(\Omega) \cap W^{m-1,2}_0(\Omega) \cap C^\infty(\Omega) \). Then

\[
\int_{W_c \times Y \times (-1,0)} \left| \frac{1}{e^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\Phi_c(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) \right|^2 \, d\bar{x}dy
\]

\[
= \int_{-1}^0 \sum_{k \in I_{W,c}} \int_{C_k^c} \int_{C_k^c} \left| \frac{1}{e^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} \left( \epsilon \left[ \frac{x}{\epsilon} \right] + \epsilon y_N - h_c \left( \epsilon \left[ \frac{x}{\epsilon} \right] + \epsilon y_N \right) \right) \right|^2 \, d\bar{x}dyN
\]

\[
= \int_{-1}^0 \sum_{k \in I_{W,c}} \int_{C_k^c} \int_{C_k^c} \left| \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) \right|^2 \, d\bar{x} \, d\bar{z} \, dyN. \quad (6.1)
\]

Now, let \( \bar{z} \in C^k_\epsilon \) be fixed. By expanding \( \varphi \) in Taylor’s series with remainder in Lagrange form we deduce that

\[
\frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, y_N - h_c(\bar{z}, y_N)) = \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \left( y_N - h_c(\bar{z}, y_N) \right)^{l-1},
\]

for some \( \xi \in (0, y_N - h_c(\bar{z}, y_N)) \). We then deduce that the term appearing in the right-hand side of (6.1) can be rewritten as

\[
\int_{-1}^0 \sum_{k \in I_{W,c}} \int_{C_k^c} \int_{C_k^c} \left| \frac{1}{e^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \left( y_N - h_c(\bar{z}, y_N) \right)^{l-1} \right|^2 \, d\bar{x} \, d\bar{z} \, dyN. \quad (6.2)
\]

We then estimate (6.2) from above. Note that

\[
\int_{-1}^0 \sum_{k \in I_{W,c}} \int_{C_k^c} \int_{C_k^c} \left| \frac{1}{e^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \left( y_N - h_c(\bar{z}, y_N) \right)^{l-1} \right|^2 \, d\bar{x} \, d\bar{z} \, dyN
\]

\[
\leq \int_{-1}^0 \sum_{k \in I_{W,c}} \int_{C_k^c} \int_{C_k^c} \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) - \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{x},0) \right|^2 \, d\bar{x} \, d\bar{z} \, dyN
\]

\[
+ \sum_{s=1}^{l-1} \left( \frac{l-1}{s} \right) \frac{1}{e^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \left( y_N - h_c(\bar{z}, y_N) \right)^{l-1-s} \left( -h_c(\bar{z}, y_N) \right)^s \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}}(\bar{z}, \xi) \right|^2 \, d\bar{x} \, d\bar{z} \, dyN, \quad (6.3)
\]

and the right-hand side of (6.3) is estimated from above by
Now consider separately the three integrals on the right-hand side of (6.4). The first integral can be estimated in the following way

\[
\int_{-1}^{0} \sum_{k \leq 0} \int_{C_k} \int_{C_k} \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{z}, \xi) - \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{x}, 0) \right|^2 \frac{d\bar{x}}{\epsilon} \frac{d\bar{z}}{\epsilon^N} dy_N
\]

\[
= C \int_{-1}^{0} \sum_{k \leq 0} \int_{C_k} \int_{C_k} \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{z}, \xi) - \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{x}, 0) \right| \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{x}, 0) \right| \frac{d\bar{x}}{\epsilon} \frac{d\bar{z}}{\epsilon^N} dy_N
\]

\[
\leq C \epsilon \left\| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} \right\|_{L^2(W \times (-\epsilon, 0))}^2.
\]

Now consider the second integral in (6.4). We have the following estimate

\[
\int_{-1}^{0} \sum_{k \leq 0} \int_{C_k} \int_{C_k} \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{z}, 0) - \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{x}, 0) \right|^2 \frac{d\bar{x}}{\epsilon N-1} d\bar{z} dy_N
\]

\[
= \sum_{k \leq 0} \int_{C_k} \int_{C_k} \left| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{z}, 0) - \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} (\bar{x}, 0) \right|^2 \frac{d\bar{x}}{\epsilon^N} \frac{d\bar{z}}{\epsilon^N} dy_N
\]

\[
\leq C \epsilon \left\| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} \right\|_{B^{1/2}_2(W)}^2 \leq C \epsilon \left\| \frac{\partial^{m-1} \varphi}{\partial x_N^{m-1}} \right\|_{W^{2,2}(\Omega)}^2,
\]

where we have used the classical Trace Theorem and the standard Besov space $B^{1/2}_2(W)$ of exponents 2, 1/2. Finally we consider the third integral in (6.4), which is easily estimated by using Lemma 1 as follows:
By using Lemma 2, a Trace Theorem, Poincaré inequality and a typical diagonal argument, it is not difficult to see that right hand-side of (6.9) tends to zero as \( \epsilon \to 0 \), concluding the proof of the first part of the statement.

The second part of the second statement can be proved as follows. By assumption, at least one of the indexes \( i_j \) is different from \( N \). This implies that the function \( \frac{\partial^{m-1} \varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_m}} \) is not only in \( W^{l,2}(\Omega) \cap W_0^{l-1,2}(\Omega) \) but also in \( W_{0,W}^{l,2}(\Omega) \). Thus, formula (5.4) and an iterated application of the Poincaré inequality in the \( x_N \) direction, \( l - 1 \) times, yield
\[
\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-1} \varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}} (\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W \times Y \times (-1,0))} \leq C \left\| \frac{\partial^{m-1} \varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}} (\hat{\Phi}_\epsilon(y)) \right\|_{L^2(W \times Y \times (-1,0))}
\]

which allows to conclude since the right-hand side of the previous inequality tends to zero as \( \epsilon \to 0 \) in virtue of Lemma 3(ii) and of the vanishing of the trace of \( \frac{\partial^{m-1} \varphi}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-l-1}}} \) on \( W \).

\[\square\]

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