An optimal entropic uncertainty relation in a two-dimensional Hilbert space*

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Abstract

We derive an optimal entropic uncertainty relation for an arbitrary pair of observables in a two-dimensional Hilbert space. Such a result, for the simple case we are considering, definitively improves all the entropic uncertainty relations which have appeared in the literature.

Key words: Entropy; Uncertainty relation; Complementary observables.

PACS: 0.3.65.Bz

1 Introduction

The uncertainty principle of quantum mechanics is expressed by the well-known Robertson relation [1]:

$$\Delta_{\psi} A \Delta_{\psi} B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|$$

where $\Delta_{\psi} A$ and $\Delta_{\psi} B$ represent the variances of the observables $A$ and $B$ when the state of the system is $|\psi\rangle$. This inequality, when the expectation value of the commutator $[A, B]$ does not vanish, expresses the intrinsic quantum mechanical limitation on the possibility of preparing

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homogeneous quantum ensembles with arbitrarily narrow variances for the involved observables and gives, in general, a physically useful information about the considered observables for the pure case associated to the state $|\psi\rangle$. In a Letter of various years ago, Deutsch [2] moved a compelling criticism about the inadequacy of such inequality on the ground that a true indeterminacy relation should not exhibit a dependence on the state vector on its right-hand side, as it happens with Eq. (1.1). Indeed, according to the author, such an hypothetical relation should be an inequality whose left-hand side quantifies, in a way to be defined appropriately, the uncertainty in the results of measurement processes of a pair of observables, while the right-hand side should contain a fixed and irreducible lower bound. This kind of mathematical expression cannot be generally obtained when dealing with a relation displaying the form of Eq. (1.1). In fact, given a pair of non-commuting observables $A$ and $B$, belonging to an arbitrary Hilbert space $\mathcal{H}$, it is easy to show that the quantity $\Delta_\psi A \Delta_\psi B$ can either vanish or become arbitrarily close to zero if at least one of the two observables is bounded. The proof of this goes as follows: suppose $B$ is the bounded observable and suppose $A$ possesses a discrete eigenvalue. In this case the variance of $A$ becomes null in correspondence of the proper eigenvector associated to the discrete eigenvalue and the indeterminacy relation assumes the desired, but absolutely trivial, form

$$\Delta_\psi A \Delta_\psi B \geq 0 \quad \forall |\psi\rangle \in \mathcal{H},$$

since $\Delta_\psi B$ is always finite for bounded $B$. A similar result holds when the observable $A$ has a purely continuous spectrum: also in this case the variance $\Delta_\psi A$ can be made arbitrarily close to zero (\forall \epsilon > 0, \exists |\psi\rangle \in \mathcal{H} : \Delta_\psi A \leq \epsilon), while the other variance remains bounded. On the contrary, a non-trivial uncertainty relation (of the kind we are searching for) can be written whenever the commutator of $A$ and $B$ is equal to a multiple of the identity, a case implying that both operators are unbounded. The paradigmatic example is represented by the pair of position $Q$ and momentum $P$ operators for which $[Q, P] = i\hbar$: in this case the usual uncertainty relation $\Delta_\psi Q \Delta_\psi P \geq \hbar/2$ is obtained. This formula is significant since it exhibits a non-zero irreducible lower bound $\hbar/2$ constraining the possible values of the two variances.

So, if one pretends that the right-hand side of an indeterminacy relation of the type (1.1) does not depend on the chosen state $|\psi\rangle$, one unavoidably ends up with the trivial result (1.2), for every pair of observables in any Hilbert space when at least one of them is bounded.

In order to overcome this problem Deutsch proposed Shannon entropy as an optimal measure of the amount of uncertainty which should be naturally connected with the measurement process of a pair of observables. Given an observable $A$ of a Hilbert space $\mathcal{H}$ with purely discrete spectrum $\{a_i\}$, the Shannon entropy of $A$ in the state $|\psi\rangle$ is defined as the quantity

$$S_\psi(A) \equiv - \sum_i p_i \log p_i$$

in terms of the probabilities $p_i$ of getting the eigenvalue $a_i$ in a measurement of the observable $A$ in the given state $|\psi\rangle$.

Now, in the particular case in which the couple of observables $A$ and $B$ of $\mathcal{H}$ have a non-degenerate spectrum, calling $|a_i\rangle$ and $|b_j\rangle$ the (unique) eigenvectors associated to the eigenvalues $a_i$ and $b_j$ respectively, the following relevant entropic uncertainty relation holds:

$$S_\psi(A) + S_\psi(B) \geq 2 \log \left( \frac{2}{1 + \sup_{ij} |\langle a_i | b_j \rangle|} \right) \quad \forall |\psi\rangle \in \mathcal{H}.$$
This inequality, originally derived by Deutsch [2] under rather restrictive assumptions, has been shown later to be a particular case of a more general formula involving totally arbitrary observables [3] and has the appealing advantage of displaying a right-hand side which is independent of the state $|\psi\rangle$. Moreover, it gives a non-trivial information, i.e., a strictly positive lower bound, concerning the sum of the uncertainties associated to measurement outcomes when the observables $A$ and $B$ do not have any common eigenvector. A further improvement of the previous formula, in the case of a finite-dimensional Hilbert space, has been subsequently obtained in Ref. [4], where it has been proved that

$$S_\psi(A) + S_\psi(B) \geq -2 \log \left( \max_{ij} |\langle a_i | b_j \rangle| \right) \quad \forall |\psi\rangle \in \mathcal{H}$$

However, the quantities appearing at the right-hand side of Eqs. (1.4) and (1.5) are in general not optimal. The optimal lower bound for two given observables $A$ and $B$ can only be found by calculating explicitly the minimum of $S_\psi(A) + S_\psi(B)$ over all the normalized state vectors $|\psi\rangle \in \mathcal{H}$.

In this Letter we show how to determine exactly such an (optimal) lower bound in the particular case of a two-dimensional Hilbert space (i.e., when $n=2$). Such a value significantly improves the lower bounds given by Eqs. (1.4) and (1.5).

2 Optimal entropic uncertainty relation when $n = 2$

Let us consider two arbitrary Hermitian operators $A = (\alpha_1 I + \beta_1 \vec{\sigma} \cdot \vec{m})$ and $B = (\alpha_2 I + \beta_2 \vec{\sigma} \cdot \vec{n})$ of $\mathcal{H} = \mathbb{C}^2$, where $(\alpha_i, \beta_i)$ are real numbers, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and $(\vec{m}, \vec{n})$ are two unit vectors in the three-dimensional Euclidean space. According to what has been said previously, our purpose consists in determining the following quantity

$$\min_{|\psi\rangle} \left[ S_\psi(A) + S_\psi(B) \right], \quad (2.1)$$

where the minimum is calculated over the set of all normalized states $|\psi\rangle \in \mathbb{C}^2$, for fixed $\alpha_i, \beta_i, \vec{m}, \vec{n}$.

In order to simplify the calculations of the quantities $S_\psi(A)$ and $S_\psi(B)$, we begin by reducing the number of the involved parameters. To this purpose we note that it is possible to restrict our attention to the class of Hermitian operators of the form $A = \vec{\sigma} \cdot \vec{m}$ and $B = \vec{\sigma} \cdot \vec{n}$ only. In fact, the Shannon entropy $S_\psi(A)$ (an equivalent consideration holds for $S_\psi(B)$) does not depend on the eigenvalues of the involved observable but only on the scalar products of its eigenstates $\{|\uparrow \vec{m}\rangle, |\downarrow \vec{m}\rangle\}$ with the state $|\psi\rangle$. Since the eigenstates of $A = \alpha_1 I + \beta_1 \vec{\sigma} \cdot \vec{m}$ are the same as those of the simpler operator $A = \vec{\sigma} \cdot \vec{m}$, our simplification does not affect the final result.

Accordingly, the entropy $S_\psi(A)$ of Eq. (1.3) takes the form

$$S_\psi(A) = -|\langle \vec{m} \uparrow |\psi\rangle|^2 \log |\langle \vec{m} \uparrow |\psi\rangle|^2 - |\langle \vec{m} \downarrow |\psi\rangle|^2 \log |\langle \vec{m} \downarrow |\psi\rangle|^2 \quad (2.2)$$

and the corresponding formula for $S_\psi(B)$ is obtained by replacing $\vec{m}$ with $\vec{n}$.

The two-dimensional Hilbert space we are dealing with exhibits two remarkable geometrical properties which will be of help in simplifying the expression (2.2): i) every normalized state $|\psi\rangle \in \mathbb{C}^2$ can be associated to a unit vector $\vec{k}$ in the three-dimensional Euclidean space $\mathbb{R}^3$ (all
these vectors forming the so-called Bloch sphere) by requiring that $|\psi\rangle$ is the eigenstate of the observable $\vec{\sigma} \cdot \vec{k}$ pertaining to the eigenvalue $+1$; ii) scalar products of state vectors belonging to $\mathbb{C}^2$ are simply related to the Euclidean scalar product of their corresponding three-dimensional vectors, according to $|\langle \vec{m} \uparrow \mid \vec{k} \uparrow \rangle|^2 = \frac{1}{2}(1 + \vec{m} \cdot \vec{k})$ and $|\langle \vec{m} \downarrow \mid \vec{k} \uparrow \rangle|^2 = \frac{1}{2}(1 - \vec{m} \cdot \vec{k})$.

Owing to these nice features, the sum $S_\psi(A) + S_\psi(B)$ displays the following appealing form:

$$S_\psi(A) + S_\psi(B) = -\frac{1}{2}(1 - \vec{m} \cdot \vec{k}) \log \frac{1}{2}(1 - \vec{m} \cdot \vec{k}) - \frac{1}{2}(1 + \vec{m} \cdot \vec{k}) \log \frac{1}{2}(1 + \vec{m} \cdot \vec{k})$$

$$-\frac{1}{2}(1 - \vec{n} \cdot \vec{k}) \log \frac{1}{2}(1 - \vec{n} \cdot \vec{k}) - \frac{1}{2}(1 + \vec{n} \cdot \vec{k}) \log \frac{1}{2}(1 + \vec{n} \cdot \vec{k}).$$  \hspace{1cm} (2.3)

Our problem consists in finding the minimum value of the above expression over all possible orientations of the (three-dimensional) unit vector $\vec{k}$ for fixed $\vec{m}$ and $\vec{n}$. So, the quest for the minimum of the sum of the Shannon entropies of two arbitrary observables $A$ and $B$ has been turned into the geometrical problem of finding the Euclidean vector $\vec{k}$ which minimizes the quantity (2.3) for the two fixed spatial directions.

The problem can be further simplified by observing that the minimum of (2.3) is attained when $\vec{k}$ lies on the plane $\tau$, determined by the vectors $\vec{m}$ and $\vec{n}$. In order to prove that let us consider an arbitrary plane $\tau_\perp$ which is orthogonal to $\tau$, and let us take into account the function, appearing in Eq. (2.3), $H_0(x) = -\frac{1}{2}(1 - x) \log \frac{1}{2}(1 - x) - \frac{1}{2}(1 + x) \log \frac{1}{2}(1 + x)$ for $x \in [0,1]$. Such a function is monotonically decreasing from its maximum value $\log 2$ to 0. Accordingly $H_0(x_0) \leq H_0(x)$ for $x_0 \geq x$. Let us identify now the variable $x$ with $|\vec{m} \cdot \vec{k}|$ for $\vec{k}$ any unit vector belonging to $\tau_\perp$. If we choose as $\vec{k}_0$ one of the two unit vectors of $\tau_\perp$ belonging to $\tau$, we obviously have that $|\vec{m} \cdot \vec{k}| \leq |\vec{m} \cdot \vec{k}_0| \quad \forall \vec{k} \in \tau_\perp$, the equality sign holding when $\vec{k} = \pm \vec{k}_0$. There follows that $H_0(|\vec{m} \cdot \vec{k}_0|) \leq H_0(|\vec{m} \cdot \vec{k}|)$ (an analogous consideration holds when $\vec{m}$ is replaced with $\vec{n}$) which implies that to find the minimum of (2.3) we can confine our considerations to unit vectors belonging to the plane $\tau$.

Having restricted our attention to an expression involving only vectors lying on the plane identified by the vectors $\vec{m}$ and $\vec{n}$, we introduce the (fixed) angle $\alpha$ between these two vectors and the (freely variable) angle $\theta$ formed by $\vec{m}$ and $\vec{k}$. Due to symmetry considerations the angle $\alpha$ can be chosen to belong to the interval $[0, \pi)$.

In terms of these angles, we can now tackle the problem of finding the minimum of the quantity of interest

$$S_\psi(A) + S_\psi(B) = -\frac{1}{2}(1 - \cos \theta) \log \frac{1}{2}(1 - \cos \theta) - \frac{1}{2}(1 + \cos \theta) \log \frac{1}{2}(1 + \cos \theta)$$

$$-\frac{1}{2}(1 - \cos(\alpha - \theta)) \log \frac{1}{2}(1 - \cos(\alpha - \theta)) - \frac{1}{2}(1 + \cos(\alpha - \theta)) \log \frac{1}{2}(1 + \cos(\alpha - \theta)).$$  \hspace{1cm} (2.4)

when $\theta$ runs over the interval $[0,2\pi)$, for a fixed $\alpha \in [0, \pi)$. The above expression depends smoothly on $\theta$ (and it does not display any singularity) and the number of its minima depends on the value of $\alpha$. In fact, from numerical plots of (2.4), one easily notices that there is a critical value of $\alpha$, which we denote as $\bar{\alpha}$, in correspondence of which the number of absolute minima of $S_\psi(A) + S_\psi(B)$ within $[0,2\pi)$, changes from two to four (we will denote this phenomenon as parametric bifurcation).
More precisely an analytical expression for the minimum value of (2.4), in terms of the angle \( \alpha \), can be given when \( \alpha \in [0, \bar{\alpha}] \cup [\pi - \bar{\alpha}, \pi) \), such a minimum being attained for two well-defined directions. On the contrary, when \( \alpha \in (\bar{\alpha}, \pi - \bar{\alpha}) \) there are four minima and the minimum value cannot be given in terms of elementary functions of \( \alpha \) but can only be determined by means of numerical calculations. Let us now analyze in detail and separately all possible cases.

For \( \alpha \in [0, \bar{\alpha}] \) the vectors \( \vec{\bar{k}} \) which minimize (2.4) lay half-way between the directions \( \vec{\bar{m}} \) and \( \vec{\bar{n}} \), that is the minima are located at \( \theta = \alpha/2 \) and \( \theta = \pi + \alpha/2 \). Such a result is obtained by calculating the first and second order derivatives of (2.4) with respect to \( \theta \): only when \( \theta = \alpha/2 \) and \( \theta = \pi + \alpha/2 \) the first derivative vanishes while the second turns out to be always positive.

In this case we can easily obtain an analytical expression for the optimal entropic uncertainty relation, depending only on the angle \( \alpha \) specified by the two observables:

\[
S_\psi(A) + S_\psi(B) \geq -(1 - \cos \frac{\alpha}{2}) \log \frac{1}{2} (1 - \cos \frac{\alpha}{2}) - (1 + \cos \frac{\alpha}{2}) \log \frac{1}{2} (1 + \cos \frac{\alpha}{2}) \quad \forall |\psi\rangle \in \mathcal{H}. \tag{2.5}
\]

For \( \alpha \in (\bar{\alpha}, \pi/2) \) the bifurcation phenomenon occurs and four minima appear. As \( \alpha \) grows, two minima spring symmetrically out of \( \theta = \alpha/2 \) and move towards the vectors \( \vec{\bar{m}} \) and \( \vec{\bar{n}} \), while the other two are symmetrically placed with respect to the opposite direction \( \theta = \pi + \alpha/2 \) and move towards \( -\vec{\bar{m}} \) and \( -\vec{\bar{n}} \). In this case an analytical expression for the entropic uncertainty principle in terms of elementary functions cannot be given and we need to resort to direct numerical calculations for obtaining the value of the minimum for every fixed value of \( \alpha \) (see Fig. I).

When \( \alpha = \pi/2 \) we are considering a couple of complementary observables \([5]\). In a two-dimensional Hilbert space, complementary observables are represented by spin components along orthogonal axis, and their entropic uncertainty relation reduces to:

\[
S_\psi(A) + S_\psi(B) \geq \log 2 \quad \forall |\psi\rangle \in \mathcal{H}. \tag{2.6}
\]

For \( \alpha \in (\pi/2, \pi - \bar{\alpha}) \) there exist again four vectors \( \vec{\bar{k}} \) which minimize (2.4): two of them are located symmetrically around the direction \( \theta = (\pi + \alpha)/2 \) and move towards it from \( \vec{\bar{m}} \) and \( -\vec{\bar{m}} \); the other two are located around \( \theta = (3\pi + \alpha)/2 \) moving towards it from \( -\vec{\bar{n}} \) and \( \vec{\bar{n}} \). In this case also, we have to resort to numerical calculations for obtaining an exact value of the minimum of \( S_\psi(A) + S_\psi(B) \) and its values are plotted in Fig. I.

Finally, when \( \alpha \in (\pi - \bar{\alpha}, \pi) \), the bifurcation disappears and only two minima remain corresponding to vectors \( \vec{\bar{k}} \) directed along \( \theta = (\pi + \alpha)/2 \) and \( \theta = (3\pi + \alpha)/2 \). An analytical expression for the entropic uncertainty relation can be given in terms of the angle \( \alpha \) in this situation too:

\[
S_\psi(A) + S_\psi(B) \geq -(1 + \sin \frac{\alpha}{2}) \log \frac{1}{2} (1 + \sin \frac{\alpha}{2}) - (1 - \sin \frac{\alpha}{2}) \log \frac{1}{2} (1 - \sin \frac{\alpha}{2}) \quad \forall |\psi\rangle \in \mathcal{H}. \tag{2.7}
\]

An exact plot displaying the minimum value of \( S_\psi(A) + S_\psi(B) \) with respect to the variable \( \alpha \) can be finally obtained by plotting Eqs. (2.5) and (2.7) together with the numerical values obtained when \( \alpha \in (\bar{\alpha}, \pi - \bar{\alpha}) \).

The dashed curves of Fig. I represent the unotopimal estimates for the entropic uncertainty relations \([14]\) and \([15]\): the plot clearly shows how much our result, represented by the curve in solid line, outperforms the currently known lower bounds, when \( \alpha \neq \pi/2 \). Such curves, when
n = 2, can be easily proved to be equal to $S_\psi(A) + S_\psi(B) \geq -2 \log \frac{1}{2}(1 + c)$ and $S_\psi(A) + S_\psi(B) \geq -2 \log c$ respectively, where $c = \cos \frac{\alpha}{2}$ when $\alpha \in [0, \pi/2)$ and $c = \sin \frac{\alpha}{2}$ when $\alpha \in [\pi/2, \pi)$.

Fig. 2 shows instead the angular position $\theta \in [0, 2\pi)$ (on the vertical axis) of all the possible minima of $S_\psi(A) + S_\psi(B)$ with respect to the variable $\alpha \in [0, \pi)$ (on the horizontal axis): the peculiar phenomenon of the bifurcation is clearly visible in correspondence of the critical points $\bar{\alpha}$ and $\pi - \bar{\alpha}$ (in dashed line).

Figure 2: Angular position of the minima of $S_\psi(A) + S_\psi(B)$ with respect to angle $\alpha \in [0, \pi)$.

It remains to explain how to determine the angular value $\bar{\alpha}$ for which the bifurcation phenomenon appears. From an analysis of (2.4) when $\alpha \in [0, \pi/2]$, we notice that the angle $\theta = \alpha/2$ is always an extremal point for the sum of the two entropies (since the first derivative of (2.4) vanishes in it), passing from a (relative) minimum to a (relative) maximum point in correspondence of the critical value $\bar{\alpha}$. Therefore, if we take the second derivative of (2.4) with respect to the variable $\theta$ and we evaluate it for $\theta = \alpha/2$, the function we obtain must change sign exactly for the critical value $\alpha = \bar{\alpha}$. Therefore the desired critical angle $\bar{\alpha}$ turns out to be the (unique) solution of the following equation within the range $[0, \pi/2]$:

$$\left.\frac{\partial^2}{\partial \theta^2}[S_\psi(A) + S_\psi(B)]\right|_{\theta=\alpha/2} = -\cos \frac{\alpha}{2} \cdot \log \left[\left(\tan \frac{\alpha}{4}\right)^2\right] - 2 = 0. \quad (2.8)$$

From a numerical analysis, we have obtained the approximated value $\bar{\alpha} \simeq 1.17056$ for the critical angle.
3 Conclusions

The entropic uncertainty relations for a pair of observables in a finite-dimensional Hilbert space constitute an appealing measure of the degree of uncertainty for measurement outcomes. In the particular case of a two-dimensional Hilbert space, we have been able to determine (in part analytically and in part numerically) an optimal entropic uncertainty relation for spin-observables which improves the known lower bounds.

Our procedure works easily in this case, since we have been able to reduce our problem to a simple geometrical one by resorting to the well-known correspondence between state vectors and points of a unit sphere within the three-dimensional Euclidean space. An analogue result for higher dimensional spaces is still lacking.

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