LIKELIHOOD EQUATIONS AND SCATTERING AMPLITUDES

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We relate scattering amplitudes in particle physics to maximum likelihood estimation for discrete models in algebraic statistics. The scattering potential plays the role of the log-likelihood function, and its critical points are solutions to rational function equations. We study the ML degree of low-rank tensor models in statistics, and we revisit physical theories proposed by Arkani-Hamed, Cachazo and their collaborators. Recent advances in numerical algebraic geometry are employed to compute and certify critical points. We also discuss positive models and how to compute their amplitudes.

1. Introduction

Likelihood equations are equations among rational functions that arise in various contexts, notably in high energy physics [2; 3] and in algebraic statistics [24; 28]. We establish a new link between these two fields. This is interesting for both sides, and may lead to unexpected advances in nonlinear algebra [23]. Specifically, we develop the connection between maximum likelihood estimation [13; 18] and the geometric theory of scattering amplitudes [9; 11]. Our goal is the practical solution of likelihood equations with certified numerical methods [7; 8].

On the statistics side, a discrete model is a subvariety $X$ of the real projective space $\mathbb{P}^n$, which is assumed to intersect the simplex $\Delta_n$ of positive points. The homogeneous coordinates $p = (p_0 : p_1 : \cdots : p_n)$ are interpreted as unknown probabilities for the $n+1$ states, subject to the constraint that $p$ lies in the model $X$. When collecting data, we write $s_i$ for the number of times the $i$-th state was observed. The data vector $s = (s_0, s_1, \ldots, s_n)$ is also viewed modulo scaling, i.e., $s$ lies in $\Delta_n \subset \mathbb{P}^n$. We are interested in the log-likelihood function

$$s_0 \cdot \log(p_0) + s_1 \cdot \log(p_1) + \cdots + s_n \cdot \log(p_n) - (s_0 + s_1 + \cdots + s_n) \cdot \log(p_0 + p_1 + \cdots + p_n).$$

This is a well-defined function on $\Delta_n \subset \mathbb{P}^n$. The aim of likelihood inference in data analysis is to maximize (1) over all points $p$ in the model $X \cap \Delta_n$. In algebraic statistics, we care about all complex critical points. Their number, for generic $s$, is the maximum likelihood (ML) degree of the model $X$. If $X$ is smooth then the ML degree equals the Euler characteristic of the open variety $X^o$, which is the complement of the divisor in $X$ defined by $p_0 p_1 \cdots p_n (\sum_{i=0}^n p_i) = 0$. Computing ML degrees and identifying critical points is an active area of research [25].

The situation is similar in the study of potentials and associated amplitudes in physics. Here the role of the data vector $s$ is played by the vector of Mandelstam invariants, which is constrained to lie in the
kinematic space. This mirrors the constraint that the coefficients in (1) sum to zero. In recent physical theories [4; 9; 11], the variety $X^o$ is the configuration space of $m$ points in general position in $\mathbb{P}^{k-1}$, up to projective transformations. This is modeled by the Grassmannian $\text{Gr}(k, m) \subset \mathbb{P}^{(n\choose m)-1}$, modulo the action of the torus $(\mathbb{C}^*)^m$. Let $\text{Gr}(k, m)^o$ be the open Grassmannian where all Plücker coordinates are nonzero. We work in the $(k-1)(m-k-1)$-dimensional manifold $X^o = \text{Gr}(k, m)^o / (\mathbb{C}^*)^m$. The ML degree of $X^o$ is the number of critical points on $X^o$ of the potential function, for generic $s$. A scattering amplitude is the sum of a certain rational function over all critical points. This is a global residue [14; 15], so it evaluates to a rational function in the Mandelstam invariants $s$.

The present article is organized as follows. Section 2 develops the promised connection for the Grassmannian of lines ($k = 2$). Here $X^o$ is the moduli space $\mathcal{M}_{0,m}$ of $m$ marked points in $\mathbb{P}^1$. This prominent space is here recast as a statistical model. The ML degree of that model is $(m - 3)!$ and all critical points are real, thanks to Varchenko’s theorem [24, Theorem 1.5]. Our computational results for $m \leq 13$ are found in Table 1. This is extended to arbitrary linear statistical models in Section 3. We show in that setting how the software HomotopyContinuation.jl [7; 8] is used to find and certify all critical points of (1). A key idea is to refrain from clearing denominators and work with rational functions directly.

Section 4 concerns higher Grassmannians ($k \geq 3$) and their associated likelihood equations. We focus on the case $m = 8$, $k = 3$, where the amplitudes literature [11; 12] reports the ML degree 188112. We interpret this CEGM theory as a nonlinear statistical model with $n = 47$. Our method computes and certifies all 188112 critical points in a few minutes, for random Mandelstam invariants with $s \geq 0$ in (1), and we show that most of them are real.

In Section 5 we apply our approach to a class of models that is important in statistics, namely conditional independence of identically distributed random variables. This corresponds to symmetric tensors of low rank, so here $X$ is a Veronese secant variety. We determine the ML degree in several new cases, well beyond the degree 12 for tossing coins in the running example of [17]. This opens up a new chapter in likelihood inference for tensors.

In Section 6 we finally turn to amplitudes. We build on the theory of stringy canonical forms due to Arkani-Hamed, He and Lam [5]. Definition 12 introduces a statistical version of positive geometries [4; 6]. The biadjoint theory amplitudes in [5] are limits of marginal likelihood integrals. They can be computed combinatorially from Newton polytopes, or as global residues, by summing the reciprocal toric Hessian of the function (1) over its critical points.

2. Points on the Line

We begin with a first direct connection between algebraic statistics and particle physics. The $m$-particle CHY scattering equations [9] will be presented as likelihood equations for a linear statistical model on the moduli space $\mathcal{M}_{0,m}$. We introduce these rational function equations, and we solve them using state-of-the-art tools from numerical algebraic geometry [7; 8; 27].

We consider $m \geq 4$ points in $\mathbb{P}^1$ whose homogeneous coordinates are the columns of

\[
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & 0 & x_1 & x_2 & \cdots & x_{m-4} & x_{m-3} & 1
\end{bmatrix}
\]
We write $q_{ij}$ for the $2 \times 2$ minor given by the $i$-th and the $j$-th column of this $2 \times m$-matrix. The moduli space $\mathcal{M}_{0,m} = \text{Gr}(2,m)^o/(\mathbb{C}^*)^m$ is the set of points for which these minors are nonzero. This is the complement of a hyperplane arrangement in $\mathbb{C}^{m-3}$. The corresponding real arrangement in $\mathbb{R}^{m-3}$ has $(m-3)!$ bounded regions, given by the possible orderings of $x_1, x_2, \ldots, x_{m-3}$ in $[0,1]$. These regions are simplices and they define a triangulation of the cube $[0,1]^{m-3}$. One of them is the positive region

$$\mathcal{M}_{0,m}^+ = \{0 < x_1 < x_2 < \cdots < x_{m-3} < 1\}.$$ 

We now define a statistical model $X$ on $n + 1 = m(m - 3)/2$ states. The states are the pairs $(i,j)$ where $2 \leq i < j \leq m$ and $(i,j) \neq (2,m)$. The parameter vector $(x_1, \ldots, x_{m-3})$ is assumed to lie in $\mathcal{M}_{0,m}^+$. The probability of observing the state $(i,j)$ is $p_{ij} = \alpha_{ij}q_{ij}$, where

$$\alpha_{im} = \frac{1}{m-3}, \quad \alpha_{ij} = \frac{1}{(m-3)^2} \quad \text{and} \quad \alpha_{2j} = \frac{2m-2j-1}{(m-3)^2} \quad \text{for } 3 \leq i < j \leq m-1. \quad (3)$$

These positive constants are chosen so that the sum of the $n + 1$ linear expressions $p_{ij}$ equals 1.

Suppose we collect data. For each of the $n + 1$ states $(i,j)$ as above, we record the number $s_{ij}$ of observations of that state. The aim of statistical inference is to find the point $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{m-3})$ in the parameter space $\mathcal{M}_{0,m}^+$ that best explains the data. Adopting the classical frequentist framework, this is done by maximizing the log-likelihood function

$$L(x) = \sum_{(i,j)} s_{ij} \log(p_{ij}(x)) = \sum_{(i,j)} s_{ij} \log(q_{ij}(x)) + \text{const.} \quad (4)$$

We write $\text{Crit}(L)$ for the set of critical points of $L$, i.e., the solutions of the likelihood equations

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \cdots = \frac{\partial L}{\partial x_{m-3}} = 0. \quad (5)$$

This is a system of $m-3$ rational function equations in the $m-3$ unknowns $x_1, \ldots, x_{m-3}$.

**Proposition 1.** If all $s_{ij}$ are positive then (5) has precisely $(m-3)!$ complex solutions. All solutions are real, and there is one solution for each of the orderings of the $m-3$ coordinates.

**Proof.** This result is known in the physics literature due to Cachazo, Mizera and Zhang [10]. We here derive it from Varchenko’s theorem in algebraic statistics [13, Theorem 13]. That theorem states that the likelihood equations of a linear space $X$ have only real solutions, and there is one solution in each bounded region of the arrangement in the real linear space $X_\mathbb{R}$ defined by the $n + 1$ hyperplanes $\{p_i = 0\}$. For the CHY model, we identify $X_\mathbb{R}$ with the parameter space $\mathbb{R}^{m-3}$, where the hyperplanes are $\{x_i = 0\}, \{x_i = 1\}$ and $\{x_i = x_j\}$. Every point with $x_i < 0$ or $x_i > 1$ for some $i$ can be moved to infinity without crossing a hyperplane. This implies that the bounded regions are the simplices $\{0 < x_{\pi_1} < \cdots < x_{\pi_{m-3}} < 1\}$, where $\pi$ runs over all $(m-3)!$ permutations of the set $\{1, 2, \ldots, m-3\}$. \qed
Example 2 $(m = 6, n = 8)$. We consider a linear model $X$ on nine states $23, 24, \ldots, 56$. Their probabilities, which sum to 1, are linear functions of three model parameters $x_1, x_2, x_3$:

$$
\begin{align*}
p_{23} &= 5x_1/9, & p_{24} &= x_2/3, & p_{25} &= x_3/9, \\
p_{34} &= (x_2 - x_1)/9, & p_{35} &= (x_3 - x_1)/9, & p_{45} &= (x_3 - x_2)/9, \\
p_{36} &= (1 - x_1)/3, & p_{46} &= (1 - x_2)/3, & p_{56} &= (1 - x_3)/3.
\end{align*}
$$

(6)

This maps the tetrahedron $M_{0,6}^+ = \{ 0 < x_1 < x_2 < x_3 < 1 \}$ into the probability simplex $\Delta_8$. Suppose we collect data with sample size 170, and the resulting data vector has coordinates

$$
\begin{align*}
s_{23} &= 25, & s_{24} &= 23, & s_{25} &= 16, & s_{34} &= 12, & s_{35} &= 22, & s_{45} &= 16, & s_{36} &= 14, & s_{46} &= 15, & s_{56} &= 27.
\end{align*}
$$

(7)

We must solve an optimization problem on $M_{0,6}^+$, namely to maximize the function

$$
L = s_{23} \log(p_{23}) + s_{24} \log(p_{24}) + s_{25} \log(p_{25}) + s_{34} \log(p_{34}) + s_{35} \log(p_{35}) + s_{45} \log(p_{45}) \\
+ s_{36} \log(p_{36}) + s_{46} \log(p_{46}) + s_{56} \log(p_{56}).
$$

(8)

The set Crit$(L)$ has one point in each bounded region of the arrangement of nine planes $\{p_{ij} = 0\}$ in $\mathbb{R}^3$. The six bounded regions lie in the cube $[0, 1]^3$. They correspond to the orderings of the values $x_1, x_2, x_3$. For instance, for the data in (7), the six critical points are

$$
\begin{align*}
\hat{x}_1 &= 0.240043275929170, & \hat{x}_2 &= 0.508172206739870, & \hat{x}_3 &= 0.777005866817260; \\
x_1 &= 0.223437550855307, & x_2 &= 0.843543048681696, & x_3 &= 0.518706389808326; \\
x_1 &= 0.48196772641097, & x_2 &= 0.235545240880672, & x_3 &= 0.781115679885971; \\
x_1 &= 0.618277926209287, & x_2 &= 0.851974456945199, & x_3 &= 0.155992558374125; \\
x_1 &= 0.861996060709608, & x_2 &= 0.217605043343923, & x_3 &= 0.453238947004789; \\
x_1 &= 0.863192417250353, & x_2 &= 0.578669456252017, & x_3 &= 0.157960116395912.
\end{align*}
$$

The first triple is the maximum likelihood estimate. The learned distribution in the model is

$$
\begin{align*}
\hat{p}_{23} &= 0.13336, & \hat{p}_{24} &= 0.16939, & \hat{p}_{25} &= 0.08633, & \hat{p}_{34} &= 0.02979, & \hat{p}_{35} &= 0.05966, \\
\hat{p}_{36} &= 0.25332, & \hat{p}_{45} &= 0.02987, & \hat{p}_{46} &= 0.16394, & \hat{p}_{56} &= 0.07433.
\end{align*}
$$

(9)

We shall see that this computation can be done for much larger values of $m$ and $n$.

We now turn to physics. In quantum field theory, the $s_{ij}$ are known as Mandelstam invariants. One writes them in a symmetric $m \times m$-matrix with zeros on the diagonal, so we have $s_{ii} = 0$ and $s_{ij} = s_{ji}$. Momentum conservation means that the row sums are zero, i.e., $\sum_{j=1}^{m} s_{ij} = 0$ for $i = 1, \ldots, m$. These equations define the kinematic space, which has dimension $n + 1 = \binom{n}{2} - m$. On that space, the $m$ Mandelstam invariants $s_{12}, s_{13}, \ldots, s_{1m}$ and $s_{2m}$ can be written uniquely in terms of our counts $s_{ij}$ in the statistical model above. For instance, for $m = 6$, the kinematic space is parametrized by the nine counts
\( m + 1 \) \( (m-3)! \) \( t_C \) \( t_R \) \( t_{\text{cert}} \)

| \( m \) | \( n + 1 \) | \( (m-3)! \) | \( t_C \) | \( t_R \) | \( t_{\text{cert}} \) |
|-------|---------|---------|-------|-------|-------|
| 10    | 35      | 5040    | 0.75  | 0.28  | 0.5   |
| 11    | 44      | 40320   | 13.4  | 3.4   | 4.0   |
| 12    | 54      | 362880  | 124.6 | 43.7  | 45.0  |
| 13    | 65      | 3628800 | 2141.5| 578.2 | 1178.0|

**Table 1.** Computing and certifying solutions to CHY scattering equations with the method in Section 3. Here \( t_C \), \( t_R \), \( t_{\text{cert}} \) denote timings (in seconds) that are explained in Example 3.

\[
s_{12} = s_{34} + s_{35} + s_{36} + s_{45} + s_{46} + s_{56}, \quad s_{13} = -s_{23} - s_{34} - s_{35} - s_{36},
\]
\[
s_{16} = s_{23} + s_{34} + s_{35} + s_{24} + s_{45} + s_{25}, \quad s_{14} = -s_{24} - s_{34} - s_{45} - s_{46},
\]
\[
s_{26} = -s_{23} - s_{34} - s_{35} - s_{24} - s_{45} - s_{25} - s_{56}, \quad s_{15} = -s_{25} - s_{35} - s_{45} - s_{56}.
\]

The scattering potential in the CHY model coincides with the log-likelihood function \( L \), up to the additive constant in (4). Hence the scattering equations are the likelihood equations.

We now come to the punchline of this section: *current off-the-shelf software from numerical algebraic geometry is highly efficient and reliable in solving our equations.* For our computations we used the julia package HomotopyContinuation.jl, due to Breiding and Timme [7], including the recent certification feature [8] which rests on interval arithmetic.

In Table 1 we present the timings we obtained for solving the scattering equations (5) when the number of particles is \( m = 10, 11, 12, 13 \). Recall that the solutions are the critical points of \( L \) in the moduli space \( \mathcal{M}_{0,m} \). In later sections we apply these methods for solving likelihood equations coming from other statistical models, including higher Grassmannians.

The first two columns in Table 1 show the number \( n + 1 = m(m-3)/2 \) of states in the statistical model and the ML degree \( (m-3)! \). The last three columns show computation times. The most relevant among these is \( t_R \). This is the time in seconds for computing all \( (m-3)! \) real critical points for a given system of Mandelstam invariants \( s_{ij} > 0 \). For instance, for \( m = 12 \), it takes less than one minute to compute all \((12-3)! = 362880 \) solutions.

### 3. Linear models and how to compute

We here explain our methodology for solving the likelihood equations. For ease of illustration we consider linear statistical models, with the understanding that the computations are analogous for nonlinear models. The scope of that becomes visible in the next two sections.

Fix affine-linear polynomials \( p_0(x), p_1(x), \ldots, p_n(x) \) with real coefficients in \( d \) unknowns \( x = (x_1, x_2, \ldots, x_d) \). We assume that \( p_0(x) + p_1(x) + \cdots + p_n(x) = 1 \) and that the convex polytope

\[
\Theta = \{x \in \mathbb{R}^d : p_i(x) \geq 0\}
\]
has dimension $d$. The model is the $d$-dimensional linear space $X$ in $\mathbb{P}^n$ parametrized by

$$x \mapsto (p_0(x) : \cdots : p_n(x)).$$

Given any positive real data vector $s = (s_0, s_1, \ldots, s_n)$, we wish to find all critical points of the log-likelihood function $L$ in (1).

By Varchenko’s theorem, all complex critical points are real, and there is one critical point in each bounded region of the arrangement of $n+1$ hyperplanes $\{p_i(x) = 0\}$ in $\mathbb{R}^d$. One of these bounded regions is the polytope $\Theta$, so this contains a unique critical point $\hat{x}$. Its image $\hat{p} = p(\hat{x})$ in $\Delta_n$ is the distribution in the model $X$ that best explains the data $s$.

The software HomotopyContinuation.jl [7] is very user-friendly. We will show how to compute all critical points with version 2.3.1. We start by generating a random linear model:

```julia
@var x[1:d]
c = rand(n+1); c = c/sum(c)
p = [randn(d)’*x + c[i] for i = 1:n]
p = push!(p,1-sum(p))
The array $p$ contains $n+1$ affine polynomials in the unknowns $x$. Their constant terms are the positive reals in $c$ that sum to 1. The polytope $\Theta$ has dimension $d$ since $0 \in \text{int}(\Theta)$. The next step is to construct the log-likelihood function and compute its derivatives. Using the logarithm function in HomotopyContinuation.jl, this can be done in two lines of code:

```julia
@var s[0:n]
L = sum([s[i]*log(p[i]) for i = 1:n+1])
F = System(differentiate(L,x), parameters = s)
Here $F$ represents the rational map

$$F : \mathbb{C}^d \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^d, \quad (x, s) \mapsto \left( \frac{\partial L}{\partial x_1}, \ldots, \frac{\partial L}{\partial x_d} \right).$$

We choose a random complex data vector $s^* \in \mathbb{C}^{n+1}$, and we solve the system $F(x; s^*) = 0$ as follows:

```julia
monodromy_result = monodromy_solve(F)
s_star = parameters(monodromy_result)
```

This uses the monodromy method for solving a generic instance of a parametrized family [16]. We stress that we do not turn rational functions into polynomials by clearing denominators. Working directly with the rational functions allows for cheaper evaluation of $F$ and it avoids spurious solutions in the hyperplanes $\{p_i(x) = 0\}$. Once we have the solutions for $s^* \in \mathbb{C}^{n+1}$, we can find the solutions for any data vector $(s^*)_t \in \mathbb{R}^{n+1}$ via a (straight line) coefficient parameter homotopy. Here the vector $s$ moves from $s^*$ to $(s^*)_t$ along a straight line in $\mathbb{C}^{n+1}$.

As the start parameter values $s^*$ move to the target parameter values $(s^*)_t$, the solutions of $F(x; s^*) = 0$ move towards the solutions of $F(x; (s^*)_t) = 0$. We can track them numerically. For details, see [27,
Chapter 7]. The coefficient parameter homotopy is implemented in the solve function. The following code solves \( F(x; (s^*)') = 0 \) for random \((s^*)' \in R_{>0}^{n+1}\) :

\[
\text{startsols} = \text{solutions(monodromy_result)}
\]

\[
s_{\text{star prime}} = \text{rand(length(s))}
\]

\[
\text{cp_result} = \text{solve}(F, \text{startsols}; \text{start_parameters} = s_{\text{star}}, \text{target_parameters} = s_{\text{star prime}})
\]

The solutions computed via monodromy are stored in \( \text{startsols} \). These can be used as starting points in the coefficient parameter homotopy for solving any new instance \( F(x; (s^*)') = 0 \) of our equations. Hence, the monodromy computation happens only once for a given model.

Finally, we certify the solutions found by the coefficient parameter homotopy using the certification technique described recently in [8]. Each solution that has been certified is guaranteed to be an approximate solution, in a suitable sense, to our system of equations:

\[
\text{cert} = \text{certify}(F, \text{solutions(cp_result)}, s_{\text{star prime}})
\]

In our discussion we described a workflow consisting of three steps: monodromy, coefficient parameter homotopy, and certification. These steps are easy to run, and they can be applied to any statistical model and hence to any system of scattering equations in physics. A nice feature of linear models, like CHY in Section 2, is that the method can solve the likelihood equations using real arithmetic only. This allows us to reduce the computation time.

We now explain the real arithmetic idea. In general, one uses complex start values \( s^* \) to avoid the discriminant locus of the family \( F(x; s) = 0 \). For linear models, this discriminant is an affine transformation of the entropic discriminant [26]. It is known from [21, Theorem 6.2] that the entropic discriminant is a sum of squares. This implies that the real locus of our discriminant has codimension \( \geq 2 \). One also finds that this locus is disjoint from \( R_{>0}^{n+1} \).

As the data vector \( s \) varies continuously in \( R_{>0}^{n+1} \), the solutions to \( F(x; s) = 0 \) move in distinct bounded regions in \( R^d \). Therefore, once we have solved \( F(x; s^*) = 0 \) for some \( s^* \in R_{>0}^{n+1} \), we can solve \( F(x; (s^*)') = 0 \) for any \( (s^*)' \in R_{>0}^{n+1} \) via a straight line coefficient parameter homotopy that uses only real arithmetic. In particular, for computing the MLE, we only need to track one solution, namely that in the distinguished polytope \( \Theta \).

**Example 3** (scattering equations on \( \mathcal{M}_{0,m} \)). Section 2 addressed a linear model from physics [9; 11] with \( d = m - 3, n = m(m - 3)/2 - 1 \) and ML degree \( (m - 3)! \). Our computations for Table 1 used the workflow described above. The columns \( t_C \) and \( t_{\text{cert}} \) show the computation times (in seconds) for the coefficient parameter homotopy from \( s^* \in C^{n+1} \) to \( (s^*)' \in R_{>0}^{n+1} \) and for the certification respectively. The column \( t_R \) shows the time for path tracking over the reals, from \( s^* \in R_{>0}^{n+1} \) to \( (s^*)' \in R_{>0}^{n+1} \). In each run, all \( (m - 3)! \) solutions were certified. The time for the monodromy step is not reported, as it is an off-line step which happens only once. For instance, for \( m = 12 \), the off-line step takes about 14 minutes. All computations were run on a 16 GB MacBook Pro with an Intel Core i7 processor working at 2.6 GHz.

**Example 4** (random linear models). We examined random models for various \((n, d)\) unlike in Section 2, the \( p_i(x) \) are now dense. The number of bounded regions equals \( \binom{n}{d} \). This is the ML degree; see
Table 2. Solving the likelihood equations for random linear models. Here \( \binom{n}{d} \) is the ML degree, \( t_C \) and \( t_R \) are the timings for solving, and \( t_{\text{cert}} \) is the timing for certifying the solutions.

| \( (n, d) \) | \( \binom{n}{d} \) | \( t_C \) | \( t_R \) | \( t_{\text{cert}} \) |
|-------------|----------------|--------|--------|----------------|
| (12, 6)     | 924            | 0.25   | 0.09   | 0.15           |
| (13, 6)     | 1716           | 0.46   | 0.13   | 0.27           |
| (14, 7)     | 3432           | 1.34   | 0.44   | 0.87           |
| (15, 7)     | 6435           | 2.12   | 0.87   | 1.46           |
| (16, 8)     | 12870          | 5.06   | 2.00   | 2.91           |
| (17, 8)     | 24310          | 6.25   | 3.60   | 7.25           |

[17, equation (8)]. Using the same computer as in Example 3, we obtained the results in Table 2, for various central binomial coefficients. Again, we do not report the timings for the off-line step, which happens once per pair \( (n, d) \). All models in Table 2 were solved easily using the default settings in HomotopyContinuation.jl. Larger values of \( (n, d) \) are more challenging. The straightforward approach we presented above ran into numerical difficulties. The monodromy loop sometimes failed to find a full set of starting solutions, and a few paths got lost in the coefficient parameter homotopy. Solving larger problems reliably will require a more clever approach or more conservative settings.

Examples 3 and 4 lead to the following conclusion. The special combinatorial structure of the CHY scattering equations allows us to solve large instances with a fairly naive method. Things are different for generic linear models. We encountered numerical issues for the default settings when the ML degree exceeds 20000. The same dichotomy occurs for the models studied in the next two sections. Low degree and sparsity render the equations from physics especially suitable for reliable and certified computations with HomotopyContinuation.jl.

4. Higher Grassmannians

Let \( \text{Gr}(k, m) \) denote the Grassmannian in its Plücker embedding in \( \mathbb{P}^{\binom{m}{k}-1} \), with Plücker coordinates \( p_I \) indexed by increasing sequences \( I = (1 \leq i_1 < i_2 < \cdots < i_k \leq m) \). We write \( \text{Gr}(k, m)^o \) for the open part where all \( p_I \) are nonzero and \( X^o \) for its quotient modulo \( (\mathbb{C}^*)^m \). We represent each point in \( X^o \) by a \( k \times m \) matrix that has been normalized and contains \( (k-1)(m-k-1) = \dim(X^o) \) unknowns. There are different conventions for setting this up. For \( k = 3 \), we place \( 2m-8 \) unknowns \( x_1, \ldots, x_{m-4} \) and \( y_1, \ldots, y_{m-4} \) in the matrix as follows:

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\
0 & -1 & 0 & 1 & x_1 & x_2 & x_3 & \cdots & x_{m-4} \\
1 & 0 & 0 & 1 & y_1 & y_2 & y_3 & \cdots & y_{m-4}
\end{bmatrix}.
\]

This ensures that \( m \) special minors \( p_I \) are equal to 1. These are the minors indexed by

\[ I = 123, 124, \ldots, 12m, 134, 234. \]
Table 3. Computation times for solving the CEGM scattering equations.

| m  | n + 1 | ML degree | tC  | t_cert |
|----|-------|-----------|-----|--------|
| 6  | 14    | 26        | 0.02| 0.01   |
| 7  | 28    | 1272      | 0.35| 0.19   |
| 8  | 48    | 188112    | 70.03| 47.71  |

The following result is known in the literature on scattering amplitudes; see [5, Section 7.1; 12, Section 3; 11, Appendix C]. Our computations furnish an independent verification.

**Proposition 5.** The ML degree of the models $X^o$ for $m = 6, 7, 8$ equals 26, 1272 and 188112.

**Sketch of proof.** The certification with HomotopyContinuation.jl furnishes a solid proof of the lower bound. The proof is an identity in interval arithmetic [8]. The upper bound requires more work. We can either use the degenerations known as soft limits [12], or Thomas Lam’s approach (mentioned in [12, Section 1]) that rests on finite fields and the Weil conjectures, or the trace test method in numerical algebraic geometry. It would be desirable to find a general formula and theoretical understanding for the Euler characteristic of $X^o = \text{Gr}(k, m)^o$. □

In the development of algebraic statistics there was an earlier attempt to view the Grassmannian Gr($k, m$) as a discrete statistical model. It has dimension $k(m - k)$, it has $n + 1 = \binom{m}{k}$ states, and the Plücker coordinates are the probabilities. We refer to [17, Section 5] where the numbers 4 and 22 were reported for the ML degrees of Gr(2, 4) and Gr(2, 5). That model is different from the one studied here, where the dimension is $(k - 1)(m - k - 1)$, the number of states is $n + 1 = \binom{m}{k} - m$, and Gr(2, m) has ML degree $(m - 3)!$. In light of the ubiquity and importance of the moduli space $\mathcal{M}_{0,m}$, we have concluded that the physical model $X^o$ above is the better way to think about the Grassmannian in the setting of algebraic statistics.

In what follows we work in the set-up for $k = 3$ as in [11; 12]. The task is to compute the set $\text{Crit}(L)$ of critical points of the scattering potential $L = \sum I s_I \log(p_I)$. We assign positive reals to the $\binom{m}{3} - m$ Mandelstam invariants $s_I$ where $I$ is any triple not listed in (12). The $m$ remaining Mandelstam invariants $s_I$ from (12) are determined from the kinematic relations

$$\sum_{jk} s_{ijk} = 0 \quad \text{for } i = 1, \ldots, m.$$  

Here $(s_{ijk})$ is a symmetric tensor with $s_{ijk} = 0$ unless $i, j, k$ are distinct; see [11, equation (1.6)]. Rewriting the kinematic equations, we obtain formulas that are analogous to (10). However, the $m$ Mandelstam invariants $s_I$ from (12) do not matter for us, since $\log(p_I) = 0$, so they do not appear in the scattering potential $L$. For the other $n + 1$ indices $I$, the polynomials $p_I$ are bilinear in the unknowns $x_i, y_i$. In conclusion, our task is to solve a system of $2m - 8$ rational function equations in $2m - 8$ unknowns, namely $\partial L / \partial x_i = \partial L / \partial y_i = 0$ for $i = 1, \ldots, m - 4$.

We use the techniques from Section 3 to solve these equations for $m = 6, 7, 8$. The results are reported in Table 3 using the same notation as in the previous sections. For $m = 6$, we confirmed that all 26
solutions are real (see [11, Appendix C]). In the case \( m = 7 \), all 1272 solutions are computed in a fraction of a second. For concreteness, let us consider the data

\[
s_{135} = 45, \quad s_{235} = 597, \quad s_{145} = 473, \quad s_{245} = 745, \quad s_{345} = 29, \quad s_{136} = 296, \quad s_{236} = 503, \\
s_{146} = 725, \quad s_{246} = 402, \quad s_{346} = 132, \quad s_{156} = 557, \quad s_{256} = 649, \quad s_{356} = 461, \quad s_{456} = 246, \\
s_{137} = 636, \quad s_{237} = 662, \quad s_{147} = 37, \quad s_{247} = 945, \quad s_{347} = 87, \quad s_{157} = 613, \quad s_{257} = 819, \\
s_{357} = 889, \quad s_{457} = 473, \quad s_{167} = 665, \quad s_{267} = 57, \quad s_{367} = 340, \quad s_{467} = 621, \quad s_{567} = 562. \\
\tag{13}
\]

These are the \( n + 1 = 28 \) Mandelstam invariants not in (12). For these data, we found 1272 solutions in 0.35 seconds, and we certified them in 0.19 seconds. Precisely 1210 of the solutions are real. To the best of our knowledge, no complete set of solutions to the scattering equations for \( m = 7 \) with general \( s_{ijk} \) has been reported in the literature so far.

Using HomotopyContinuation.jl we can also solve the likelihood equations for \( m = 8 \). This works in the order of minutes. But there are challenges for this large nonlinear model. While our earlier models showed the power of \texttt{solve} as a blackbox routine, here the situation is more delicate. It may happen that not all 188112 paths are tracked successfully in the coefficient parameter homotopy. For an example, fix the \( n + 1 = 48 \) Mandelstam invariants

\[
s_{135} = 632, \quad s_{235} = 5076, \quad s_{145} = 6368, \quad s_{245} = 619, \quad s_{345} = 8083, \quad s_{136} = 5762, \\
s_{236} = 2099, \quad s_{146} = 7767, \quad s_{246} = 9208, \quad s_{346} = 4889, \quad s_{156} = 4412, \quad s_{256} = 1024, \\
s_{356} = 5988, \quad s_{456} = 924, \quad s_{137} = 3430, \quad s_{237} = 1017, \quad s_{147} = 6235, \quad s_{247} = 8010, \\
s_{347} = 9867, \quad s_{157} = 2364, \quad s_{257} = 9661, \quad s_{357} = 7008, \quad s_{457} = 4706, \quad s_{167} = 2892, \\
s_{267} = 7670, \quad s_{367} = 5769, \quad s_{467} = 3188, \quad s_{567} = 9696, \quad s_{138} = 6264, \quad s_{238} = 5878, \\
s_{148} = 1442, \quad s_{248} = 1501, \quad s_{348} = 4225, \quad s_{158} = 579, \quad s_{258} = 7524, \quad s_{358} = 394, \\
s_{458} = 878, \quad s_{168} = 7684, \quad s_{268} = 5985, \quad s_{368} = 9306, \quad s_{468} = 8429, \quad s_{568} = 648, \\
s_{178} = 697, \quad s_{278} = 8414, \quad s_{378} = 3151, \quad s_{478} = 369, \quad s_{578} = 3176, \quad s_{678} = 8649. \\
\tag{14}
\]

Starting with the output \texttt{startsols} from the off-line phase, the command \texttt{solve} finds 188109 distinct solutions in 70 seconds. The remaining three solutions are found by a few extra minutes of monodromy loops. The 188109 earlier solutions in \texttt{cp\_result} serve as seeds:

\[ R = \text{monodromy\_solve}(F,\text{solutions}(\text{cp\_result}),s\_\text{star\_prime}) \]

When running the off-line step for any new statistical model, it is very helpful to know the ML degree ahead of time. In our situation, with knowledge of Proposition 5, we can use the option \texttt{target\_solutions\_count = 188112} in the command \texttt{monodromy\_solve}, both for off-line and for on-line. This interrupts the monodromy loop when all solutions are found.

All in all, the on-line phase for a given vector of Mandelstam invariants takes no more than a few minutes. This includes the coefficient parameter homotopy, the on-line monodromy phase described above, and the certification step that furnishes the proof of correctness.
Remark 6. A notable feature of the \( k = 2 \) model in Section 2 is that all critical points of the log-likelihood function are real (Proposition 1). The same holds for \( k = 3, \ m = 6 \). For \( k = 3, \ m \geq 7 \), this is no longer true. In fact, in these cases, it is not known whether all solutions can be real. However, we observed experimentally that most of the solutions are real. For instance, for \( k = 3, \ m = 8 \), with the data in (14), precisely 149408 out of 188112 critical points are real. We do not know whether the 48 Mandelstam invariants \( s_{ijk} \) can be chosen so that all 188112 complex solutions are real.

Remark 7. It would be interesting to investigate the likelihood geometry of positroid cells in \( \text{Gr}(k, m) \), taken modulo the \((\mathbb{C}^*)^m\) action as in [6]. The software HomotopyContinuation.jl will be useful for finding the ML degrees of such models. For these computations, one replaces the matrices in (2) and (11) with the network parametrization of positroid cells [6; 29].

5. Low rank tensors

In this section we return to algebraic statistics. We apply our methods to the model of conditional independence for identically distributed random variables. This corresponds to symmetric tensors of low rank. We here study their ML degree and likelihood equations.

We consider symmetric tensors of format \( m \times m \times \cdots \times m \) where the number of factors is \( \ell \). Our model \( X \) is the variety of symmetric tensors of rank \( \leq k \), or equivalently, the \( \ell \)-th secant variety of the \( \ell \)-th Veronese embedding of \( \mathbb{P}^{m-1} \). The dimension of the model equals \( \dim(X) = km - 1 \). We follow the set-up in (1), with the number of states \( n + 1 = \binom{m+\ell-1}{\ell} \). The state space is the set \( \Omega_{m,\ell} \) of sequences \( I = (i_1, i_2, \ldots, i_m) \in \mathbb{N}^m \) with \( i_1 + i_2 + \cdots + i_m = \ell \).

The parameter space for our statistical model is the polytope \( \Theta = (\Delta_{m-1})^k \times \Delta_{k-1} \), where the points \( x_i \) in the \( i \)-th simplex \( \Delta_{m-1} \) are distributions on the \( i \)-th random variable with \( m \) states, and points \( y \) in the simplex \( \Delta_{k-1} \) specify the mixture parameters. Hence \( x = (x_{i,j}) \) is a nonnegative \( k \times m \) matrix whose rows sum to 1, and \( y \) is a nonnegative vector in \( \mathbb{R}^k \) whose entries sum to 1. The probability of observing the state \( I = (i_1, i_2, \ldots, i_m) \) equals

\[
p_I(x, y) = \frac{\ell!}{i_1!i_2!\cdots i_m!} \sum_{j=1}^k y_j x_{j,1}^{i_1} x_{j,2}^{i_2} \cdots x_{j,m}^{i_m},
\]

(15)

The resulting natural parametrization of the conditional independence model is the map

\[
\Theta \to \Delta_n, \quad (x, y) \mapsto (p_I(x, y))_{I \in \Omega_{m,\ell}}.
\]

(16)

This polynomial map is \( k! \)-to-1, due to label swapping, which amounts to permuting rows of \( x \) and entries of \( y \). The variety \( X \) is the image in \( \mathbb{P}^n \) of the complexification of the map (16).

Fix counts \( s_I \in \mathbb{N} \) for \( I \in \Omega_{m,\ell} \). Statisticians aim to maximize the log-likelihood function

\[
L = \sum_{I \in \Omega_{m,\ell}} s_I \cdot \log(p_I(x, y)).
\]

In this formula we incorporate the substitutions \( x_{j,m} = 1 - \sum_{i=1}^{m-1} x_{j,i} \) and \( y_k = 1 - \sum_{j=1}^{k-1} y_j \).
We shall compute all complex critical points of $L$ by solving the likelihood equations
\[ \frac{\partial L}{\partial x_{i,j}} = \frac{\partial L}{\partial y_{i,j}} = 0 \quad \text{for} \quad i = 1, 2, \ldots, k \quad \text{and} \quad j = 1, 2, \ldots, m - 1. \] (17)

This is a system of $km - 1$ rational function equations in $km - 1$ unknowns. We denote the corresponding rational map by $F(x, y, s) : \mathbb{C}^{km-1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{km-1}$. The ML degree of the model $X$ is the number of complex solutions to the system (17) divided by $k! = 1 \cdot 2 \cdots k$. The maximum likelihood parameter $(\hat{x}, \hat{y})$ is one of the real solutions in the polytope $\Theta$.

**Example 8.** Two small instances were studied in [17]. The case $k = m = 2$, $\ell = 4$ is featured in [17, Section 1] where a gambler tosses one of two biased coins four times, and $X \subset \mathbb{P}^4$ is the hypersurface given by a $3 \times 3$ Hankel determinant. This has ML degree 12, so (17) has 24 solutions. A data vector $s$ with three local maxima in $\Delta_4$ is listed in [17, Example 10]. In the table at the end of [17, Section 5] we learn that the model with $k = m = 2$, $\ell = 5$ has ML degree 39, so (17) has 78 solutions. At that time, over 15 years ago, symbolic computing with *Singular* was the method of choice, and finding 78 solutions was not that easy.

Using the numerical methods presented in Section 3, we solved the likelihood equations for $m = 2$, 3 and $k = 2, 3$. For various $\ell$, we ran many iterations of the monodromy loop\(^1\) to count the number of solutions to (17). Dividing that number by $k!$ gives an integer, and that is the ML degree for the model. A subsequent run of the certification feature in *HomotopyContinuation.jl* furnishes a proof that the proposed number is a lower bound on the ML degree. However, our method does not give a proof that this is also an upper bound.

**Remark 9** (a view from nonlinear algebra). Points in the ambient space $\mathbb{P}^n$ for our models in Table 4 correspond to binary forms and ternary forms. For example, the entry 111 on the left is the ML degree for the $4 \times 4$ Hankel determinant which defines binary sextics of rank 3. The entry 646 on the right concerns plane cubic curves of rank 3. This is the hypersurface in $\mathbb{P}^9$ defined by the *Aronhold invariant*, shown in equation (9.15) and Example 11.12 in [23]. We solved the likelihood equations (17) in the naive way, by computing all $646 \times 3! = 3876$ zeros of the rational functions. Further computational progress is surely possible. But, just like in Example 4, this will require exploiting the special structure of the problem at hand.

\(^{1}\)The optional argument group_action of monodromy_solve can be used to speed up the computations.
A next goal is the likelihood geometry of $4 \times 4 \times 4$ tensors. For a geometer, these are cubic surfaces in $\mathbb{P}^3$, with parameters $m = 4$, $\ell = 3$, $n = 19$. In the book cover of [24], this means that DiaNA now juggles three dice, each labeled A, C, G, T. We studied this model for cubic surfaces of rank $k = 2$. Our computations suggest that the ML degree equals 6483.

We next present an explicit numerical example, for the model of plane cubics of rank 2.

**Example 10** ($m = \ell = 3$, $k = 2$, $n = 9$). Consider the data vector $s \in \mathbb{N}^{10}$ with coordinates

$s_{300} = 8263$, $s_{210} = 4935$, $s_{201} = 8990$, $s_{120} = 7238$, $s_{111} = 5034$,
$s_{102} = 5106$, $s_{030} = 5181$, $s_{021} = 6843$, $s_{012} = 5282$, $s_{003} = 9501$.

The log-likelihood function $L$ has 242 complex critical points, so there are 121 critical points in the secant variety $X \subset \mathbb{P}^9$. Precisely eight of them lie in the actual model $X \cap \Delta_9$. These come from 16 critical points in $\Theta = \Delta_2 \times \Delta_2 \times \Delta_1$. The maximum likelihood estimate equals

$\hat{p}_{300} = 0.0661$, $\hat{p}_{210} = 0.1585$, $\hat{p}_{201} = 0.0937$, $\hat{p}_{120} = 0.1269$, $\hat{p}_{111} = 0.1542$,
$\hat{p}_{102} = 0.0711$, $\hat{p}_{030} = 0.0340$, $\hat{p}_{021} = 0.0658$, $\hat{p}_{012} = 0.0883$, $\hat{p}_{003} = 0.1414$.

The 121 critical points in $X$ are $3 \times 3 \times 3$ tensors of complex rank 2. Among these 121 tensors, 47 are real. We found that 20 have real rank 2, so each has two real preimages in $\mathbb{R}^5$. The other 27 have real rank 3. They come from complex conjugate pairs of parameters $(x, y)$.

We now offer some pertinent remarks on numerical algebraic geometry. Our object of interest is the rational map $F : \mathbb{C}^{km-1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{km-1}$ defined by the gradient of $L$. To find all solutions of $F(x, y; s^*) = 0$ for general complex data $s^* \in \mathbb{C}^{n+1}$, it is necessary that the monodromy action on $F(x, y; s^{*})^{-1}(0)$ is transitive. This happens if and only if the incidence variety $\{(x, y, s) : F(x, y, s) = 0\}$ in the total space $\mathbb{C}^{km-1} \times \mathbb{C}^{n+1}$ is irreducible [16, Section 2]. However, for our parametrized tensor models, this incidence variety is reducible.

**Example 11** ($m = k = 2$, $\ell = 4$). For any $s \in \mathbb{C}^5$, we consider the solutions to the critical equations $F = (\partial L/\partial x_{11}, \partial L/\partial x_{21}, \partial L/\partial y_1) = 0$ in the open subset where the denominators are nonzero. The incidence variety $Y$ is the closure of this set in $\mathbb{C}^3 \times \mathbb{C}^5$. In a Gröbner basis approach, this would be computed by clearing denominators in $F$ and then saturating the denominators. To appreciate the complexity of this, note that the three numerators have degree 25, with 2025 terms, 2418 terms and 2439 terms respectively. This is why we do not clear denominators.

We see that $Y$ is reducible because all terms of $\partial L/\partial x_{11}$ are multiples of $y_1$. Points in the locus $\{y_1 = 0\}$ parametrize tensors of rank one. This gives an extraneous component of $Y$. Interestingly, $Y$ has dimension 6, because a rank 1 tensor arises from a line of parameter values, given by $x_{11} = x_{21}$ and $y_1$ arbitrary. The fibers of the map $Y \rightarrow \mathbb{C}^5$ contain a line and 24 isolated points that represent $24/2! = 12$ rank-2 tensors. These are the critical points we are interested in. The corresponding 5-dimensional component of $Y$ parametrizes the likelihood correspondence, i.e., the irreducible variety in $\mathbb{P}^4 \times \mathbb{P}^4$ from [18, Definition 1.5].
In summary, one drawback of our approach in this paper is the presence of extraneous components in the incidence variety. From a numerical point of view, this makes the monodromy procedure more challenging. The phenomenon of \textit{path jumping} may bring us to other components, leading to the computation of spurious solutions. For computing the ML degree of our tensor models, we are only interested in critical points in the regular locus of \( X \). These are tensors of complex rank exactly \( k \). They live on a component of \( Y = \{ F(x, y, s) = 0 \} \subset \mathbb{C}^{km-1} \times \mathbb{C}^{n+1} \), called the \textit{dominant component} in [16, Remark 2.2]. We can compute all solutions on that component by making sure that our seed lies on it.

6. Positive models and their amplitudes

The physical theory of scattering amplitudes is concerned with evaluating certain integrals of rational functions. In our statistical setting, these correspond to \textit{marginal likelihood integrals}

\[
\int_\Theta p_0(x)^{s_0}p_1(x)^{s_1} \cdots p_n(x)^{s_n} \mu(x) \, dx.
\]  

(18)

Such integrals arise in Bayesian statistics. In that paradigm one integrates the likelihood function over the parameter space \( \Theta \) where the kernel is given by a measure \( \mu(x) \), known as the \textit{prior belief}. In general, it is a difficult problem to evaluate the integral (18) exactly and reliably. See [22] for an approach in the context of conditional independence as in Section 5.

It is a classical theme in mathematical statistics to connect Bayesian inference with the optimization problem (MLE) we explored in the previous sections. In this section we present new ideas for advancing that theme. These are inspired by positive geometries from Feynman diagrams and scattering amplitudes. We build on the theory of stringy canonical forms [5].

\textbf{Definition 12.} A discrete statistical model \( X \) is called \textit{positive} if it has a parametrization by positive rational functions \( p_i(x) \) that sum to 1, where the parameter space is the orthant \( \Theta = \mathbb{R}^d_{>0} \). A positive rational function is the ratio of two polynomials with positive coefficients.

Many familiar models in statistics are positive. To begin with, the probability simplex \( \Delta_n \) of all distributions on \( n+1 \) states is a positive model, thanks to the parametrization

\[
p : \mathbb{R}_{>0}^n \to \Delta_n, \quad x \mapsto \frac{1}{1+x_1+x_2+\cdots+x_n} (1, x_1, x_2, \ldots, x_n).
\]  

(19)

Next are the two families in [24, Section 1.2]. In a \textit{toric model}, \( p_i(x) \) is a monomial with a positive coefficient divided by the sum of these \( n+1 \) monomials [1; 18]. Every \textit{linear model} \( X \) is a positive model, since \( X \cap \Delta_n \) is a polytope whose vertices have nonnegative coordinates. For instance, a positive parametrization \( y \mapsto p(y) \) for Example 2 is found by replacing

\[
x_1 = \frac{y_1}{1 + y_1 + y_2 + y_3}, \quad x_2 = \frac{y_1 + y_2}{1 + y_1 + y_2 + y_3} \quad \text{and} \quad x_3 = \frac{y_1 + y_2 + y_3}{1 + y_1 + y_2 + y_3}.
\]  

(20)

Every model \( X \) with ML degree one is a positive model, by the parametrization in [18, Corollary 3.12]. Mixtures of positive models are positive models, by using (19) for the mixture parameters. In particular, all discrete conditional independence models [28, Chapter 4] are positive models. In the setting of Section 5,
we use (19) to positively parametrize the factors in \( \Theta = (\Delta_{m-1})^k \times \Delta_{k-1} \), and we then compose this with the positive polynomials in (15).

Fix a positive model \( X \). We factor the numerator and denominator of each \( p_i(x) \) into positive polynomials, we write \( q_1(x), \ldots, q_e(x) \) for all the factors that occur, and we augment this list by \( x_1, \ldots, x_d \). We now rewrite the marginal likelihood integral (18) in the form seen in [5, (1.3)]. To this end, we set \( \varepsilon = 1 \) and \( \mu(x) = 1 \) for now. Then the integral (18) becomes

\[
\varepsilon^d \int_{\mathbb{R}_{>0}^d} \left[ x_1^{u_1} \cdots x_d^{u_d} q_1(x)^{-v_1} q_2(x)^{-v_2} \cdots q_e(x)^{-v_e} \right] \varepsilon \frac{dx_1}{x_1} \cdots \frac{dx_d}{x_d},
\]

where \( u_i, v_j \) are certain \( \mathbb{Z} \)-linear combinations of \( s_0, \ldots, s_n \). The log-likelihood function equals

\[
L = \sum_{i=1}^d u_i \log(x_i) - \sum_{j=1}^e v_j \log(q_j(x)).
\]

The set \( \text{Crit}(L) \subset \mathbb{C}^d \) of all critical points of \( L \) can be computed reliably using the methods in this paper. We define the toric Hessian of \( L \) to be the symmetric \( d \times d \)-matrix \( H_L(x) \) whose entries are the rational functions \( \theta_i \theta_j L \), where \( \theta_i = x_i \partial_{x_i} \) is the \( i \)-th Euler operator.

In their recent work [5], Arkani-Hamed, He and Lam define the amplitude of \( L \) to be the limit of the integral (21) as \( \varepsilon \) tends to zero. This corresponds to taking the field theory limit of integrals arising in string theory, where \( \varepsilon = 1/\alpha' \) represents the inverse of the string tension. Given data \( s \) such that all \( v_j \) are positive, they consider the polytope \( P = \sum_{j=1}^e v_j \text{New}(q_i) \) and assume that \( u = (u_1, \ldots, u_d) \) lies in \( P \).

**Theorem 13.** The amplitude of a positive model \( X \) is a rational function in the data \( s_0, s_1, \ldots, s_n \). It equals the volume of the dual polytope \( (P - u)^* \), and it can be computed as

\[
\text{amplitude}(X) = (-1)^d \sum_{\xi \in \text{Crit}(L)} \det(H_L(\xi))^{-1}.
\]

**Sketch of proof.** This is our interpretation of the results in [5]. The amplitude depends only on the Newton polytopes \( \text{New}(q_j) \) and not on the specific positive coefficients of \( q_j(x) \). The hypothesis that \( u \) is in the interior of \( P \) ensures that (21) converges [5, Section 4.1]. The volume formula appears in [5, (2.5)] for \( s = 1 \) and in [5, (4.15)] for \( s \geq 2 \). The critical equations of \( L \) are the saddle point equations for the marginal likelihood integral (21) when \( \varepsilon \to \infty \). These equations appear in [5, Section 7.1]. They encode the pushforward formula for canonical forms of positive geometries. The toric Hessian is a convenient tool for writing the Jacobian of the system [5, (7.3)], and hence for computing the integral in [5, (7.5)]. \( \square \)

**Example 14** \((d = n = 2)\). For the model \( X = \mathbb{P}^2 \), parametrized by (19), the integral (21) is

\[
\varepsilon^2 \int_0^\infty \int_0^\infty \left[ \frac{x_1^{s_1} x_2^{s_2}}{(1 + x_1 + x_2)^{s_0 + s_1 + s_2}} \right] \varepsilon \frac{dx_1}{x_1} \frac{dx_2}{x_2}.
\]

The log-likelihood function \( L = s_1 \log(x_1) + s_2 \log(x_2) - (s_0 + s_1 + s_2) \log(1 + x_1 + x_2) \) has only one critical point, namely \((\hat{x}_1, \hat{x}_2) = (1/s_0)(s_1, s_2)\). Substituting this into \( 1/\det(H_L(x)) \), we get

\[
\text{amplitude}(X) = \frac{1}{s_0 s_1} + \frac{1}{s_0 s_2} + \frac{1}{s_1 s_2} = \text{area}((P - (s_1, s_2))^*).
\]
Here, $P$ is the unit triangle $\text{conv}\{(0, 0), (0, 1), (1, 0)\}$ scaled by the sample size $s_0 + s_1 + s_2$.

The special case $\epsilon = 1$ in Theorem 13 corresponds to the class of toric models in statistics; see [18, Section 3; 24, Section 1.2]. Any polynomial $q(x) = \sum_{j=0}^{n} c_j x^a_j$ with positive coefficients $c_j > 0$ defines a toric model $X$, by setting $p_j(x) = c_j x^{a_j} / q(x)$ for $j = 0, \ldots, n$. The ML degree of $X$ depends in subtle ways on the coefficients $c_j$. This was observed in [5, Section 7.1] and studied in detail in [1]. Both sources contain many open problems. For instance, it is conjectured in [5, Section 7.1] that the number $(m - 3)!$ from Section 2 is the minimal ML degree among all toric models supported on the associahedron. The diffeomorphism referred to in [5, Claim 4] is the familiar toric moment map [23, Theorem 8.24].

Up to a nonzero constant factor, the amplitude of the toric model $X$ equals the defining polynomial of the adjoint hypersurface of $P$, in the sense of Wachspress geometry [20]. Here we also need to divide by the product of the linear forms given by the facets of $P$. We learned this from unpublished lecture notes by Christian Gaetz which connect [3] with [20].

**Example 15** (measuring the dual of a square). The toric model for $q(x) = 1 + x_1 + x_2 + x_1 x_2$ is the independence model for two binary random variables, with data $s = (s_{ij})_{0 \leq i, j \leq 1}$. Here $n = 3$, $d = 2$, and $X$ is the Segre quadric in $\mathbb{P}^3$. The marginal likelihood integral in (21) is

$$
\varepsilon^2 \int_0^\infty \int_0^\infty \left[ \frac{x_1^{s_{10}+s_{11}} x_2^{s_{01}+s_{11}}}{\prod_{i=0}^{1} \left( (1+x_1)(1+x_2) \right)^{s_{00}+s_{01}+s_{10}+s_{11}}} \right]^\varepsilon \frac{dx_1}{x_1} \frac{dx_2}{x_2}.
$$

The limit for $\varepsilon \to 0$ is the amplitude. Its denominator is the product of the row and column sums of the contingency table $s$. The adjoint is the square of the sample size. Hence,

$$
\text{amplitude}(X) = \frac{(s_{00} + s_{01} + s_{10} + s_{11})^2}{(s_{00} + s_{01})(s_{10} + s_{11})(s_{00} + s_{10})(s_{01} + s_{11})}.
$$

Here $P$ is the square $[0, 1]^2$ times the sample size. This is translated by $u = (s_{10} + s_{11}, s_{01} + s_{11})$. The normalized area of the dual quadrilateral $(P - u)^*$ equals the amplitude. Note that the assumption $u \in P$ from Theorem 13 is naturally satisfied in the statistical setting.

In earlier sections we showed that \texttt{HomotopyContinuation.jl} is fast for computing the critical set $\text{Crit}(L)$ of the log-likelihood function $L$. And it comes with certification. We use this to compute the sum (22) and hence to evaluate amplitudes for positive models. While the meaning of these amplitudes for Bayesian statistics is not clear yet, there is considerable interest in such computations among particle physicists. We next illustrate this for the CHY and CEGM models in Sections 2 and 4. We follow the set-up in [5, Section 6.2].

Let us begin with the $k = 2$ model in Section 3, with positive reparametrization as in (20).

**Example 16** ($k = 2$, $m = 6$). We compute the amplitude for the CHY model in Example 2. In terms of the positive parameters $y_1, y_2, y_3$ from (20), the log-likelihood function in (8) is

$$
L = s_{23} \log(y_1) + s_{34} \log(y_2) + s_{45} \log(y_3) + s_{24} \log(y_1 + y_2) + s_{25} \log(y_1 + y_2 + y_3) + s_{35} \log(y_2 + y_3) + s_{36} \log(1 + y_2 + y_3) + s_{46} \log(1 + y_3) - \left( \sum_{(i,j)} s_{ij} \right) \log(1 + y_1 + y_2 + y_3).
$$
The toric Hessian $H_L(y)$ is a symmetric $3 \times 3$-matrix whose entries are rational functions. The sum of the values of $-\det(H_L(y))^{-1}$ at the six critical points of $L$ is the amplitude

$$
\frac{1}{s_{12} s_{34} s_{56}} + \frac{1}{s_{12} s_{56} s_{123}} + \frac{1}{s_{23} s_{56} s_{123}} + \frac{1}{s_{23} s_{56} s_{234}} + \frac{1}{s_{34} s_{56} s_{234}} + \frac{1}{s_{16} s_{23} s_{45}} + \frac{1}{s_{12} s_{34} s_{345}} + \frac{1}{s_{12} s_{45} s_{123}} + \frac{1}{s_{16} s_{24} s_{345}} + \frac{1}{s_{16} s_{24} s_{234}} + \frac{1}{s_{16} s_{34} s_{345}} + \frac{1}{s_{16} s_{34} s_{234}} + \frac{1}{s_{23} s_{45} s_{123}} + \frac{1}{s_{23} s_{45} s_{456}}.
$$

Here we abbreviate $s_{ijk} = s_{ij} + s_{ik} + s_{jk}$. The 14 terms in this sum correspond to the planar trivalent trees with six labeled leaves, and hence to the vertices of the associahedron in $\mathbb{R}^3$.

For a numerical example take the data in (7) and (10). The unique positive critical point $(\hat{y}_1, \hat{y}_2, \hat{y}_3) = (1.076\ldots, 1.202\ldots, 1.205\ldots)$ maps to the MLE in (9). The amplitude (23) equals

$$
\frac{16074421}{56770632000} = 0.00028314676856\ldots
$$

Using the abbreviation $y_{i,j} = \sum_{i \leq \ell \leq j} y_\ell$, the associated integral (21) has the form

$$
\varepsilon^3 \int_{\mathbb{R}_{>0}^3} \left[ \frac{y_{1,2}^{s_{23}} y_{1,3}^{s_{45}} y_{2,3}^{s_{45}}}{y_{1,2}^{s_{24}} y_{1,3}^{s_{25}} y_{2,3}^{s_{25}} (1 + y_{2,3})^{-s_{46}} (1 + y_{1,3})^{-s_{46}} (1 + y_{1,3})^{s_{i,j}} y_i^{s_{ij}} y_j^{s_{ij}}} \right] \varepsilon \, dy_1 \, dy_2 \, dy_3.
$$

The theory in [5] requires the hypotheses

$$
\begin{align*}
s_{23} &\geq 0, \\
s_{34} &\geq 0, \\
s_{45} &\geq 0, \\
s_{24} &\leq 0, \\
s_{25} &\leq 0, \\
s_{35} &\leq 0, \\
s_{36} &\leq 0, \\
s_{46} &\leq 0, \\
\sum_{(i,j)} s_{ij} &\geq 0.
\end{align*}
$$

If this holds then the leading order ($\varepsilon \to 0$) of the integral equals the volume of $(P - (s_{23}, s_{34}, s_{45}))^*$, where $P$ is the associahedron

$$
c_{24} \text{New}(y_{1,2}) + c_{25} \text{New}(y_{1,3}) + c_{35} \text{New}(y_{2,3}) + c_{36} \text{New}(1 + y_{2,3}) + c_{46} \text{New}(1 + y_{1,3}) + \sum_{(i,j)} s_{ij} \text{New}(1 + y_{1,3}).
$$

Here $c_{ij} = -s_{ij}$. The hypothesis fails for (7), but summing over Crit($L$) always works.

We now reiterate the punchline from Section 2 for amplitudes: current off-the-shelf software from numerical algebraic geometry is highly efficient and reliable for computing amplitudes by evaluating the sum (22). For our computations we used HomotopyContinuation.jl [7; 8]. We carried this out for models with $k = 2$ and $k = 3$. If $v_1, \ldots, v_e > 0$ then the amplitude measures the volume of the dual polytope in Theorem 13.

For $k = 2$, our computations validate known formulas involving planar trees like (23). For $k = 3$, $m \leq 7$, Cachazo et al. [11] describe formulas in terms of rays of the positive tropical Grassmannian, but in general there is still plenty of room for further discovery.

**Example 17** ($k = 2$). We used the positive parametrization [5, (1.5)] of $M_{0,m}^+$ to verify (22) for the CHY model. Fix integer values for the Mandelstam invariants such that the hypotheses on $u$ and $v$ in Theorem 13 are satisfied. We compute the volume of $(P - u)^*$ in two ways. First the exact rational number is obtained using Polymake.jl [19]. Secondly, summing over the computed critical points as in (22) gives a floating point approximation. The cases we checked are $m = 5, 6, \ldots, 10$. Using double
precision arithmetic, the numerical evaluation of (22) agrees with the volume up to at least 12 significant digits in all cases. Computing the Hessian determinant and summing over the 5040 solutions for \( m = 10 \) takes about 20 seconds. The computation time for finding these solutions appears in Table 1.

**Example 18** \((k = 3)\). For \( m = 7 \), we compute the amplitude of the CEGM model for the data in (13). Our code finds the numerical value \(3.5930250842 \cdot 10^{-19}\). This equals

\[
338162987564411101121443846899477268242222303915555493151128494904729904959110279
\]

\[
9411651751278943751720147762872529781445322700567349835265330333349828096628760094174245472501760000
\]

This rational number is computed with a formula from [11, Section 4] which was kindly shared with us by Nick Early. In our study of amplitudes for CEGM models, we used the positive parametrization obtained from (11) by recursively setting \( x_0 = y_0 = 1 \) and

\[
x_\ell = x_{\ell-1} + z_\ell, \quad y_\ell = y_{\ell-1} + z_\ell(1 + w_1 + \cdots + w_\ell), \quad \ell = 1, \ldots, m-4.
\]  

(24)

Since this parametrization augments the degree of the equations, it is better to first solve the scattering equations using the formulation (11) and then compute the \((z, w)\) coordinates of the solutions \((x, y)\) via (24). Computing the sum (22) over the 1272 solutions takes about 11 seconds. Like Example 16, this illustrates the validity of (22) when the assumptions on \( u, v \) in Theorem 13 are violated. For \( m = 8 \), we obtain the numerical approximation \(1.3609103649662523 \cdot 10^{-34}\) for the amplitude of the CEGM model with data (14).

We conclude with a summary of what has been accomplished in this paper. A connection has been made between algebraic statistics and the study of scattering amplitudes in physics. Positive models play the role of positive geometries. We showed how to solve the likelihood equations with certified numerical methods, and how to use this for evaluating amplitudes. Our case study offers a new tool kit for statistics and physics, based on nonlinear algebra.

Here is what we did not do: we did not prove new theorems in pure mathematics. We did not achieve notable methodological progress in statistics or theoretical advances in physics. The contribution of this work lies in building a bridge. Others may now cross that bridge, and use our tool kit to gain insights on the numerous fascinating problems that remain open.

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