On Self–Similar Global Textures in an expanding Universe

Stefan Åminneborg and Lars Bergström

Department of Physics
University of Stockholm
Vanadisvägen 9
S–113 46 Stockholm, Sweden

Abstract

We discuss self-similar solutions to $O(4)$ textures in Minkowski space and in flat Friedmann-Robertson-Walker backgrounds. We show that in the Minkowski case there exist no solutions with winding number greater than unity. However, we find besides the known solution with unit winding number also previously unknown solutions corresponding to winding number less than one. The applicability of the non-linear sigma model approximation is discussed. We point out that no spherically symmetric exactly self-similar solutions exist for radiation or matter dominated FRW cosmologies, but we find a way to relax the assumptions of self-similarity that give us approximative solutions valid on intermediate scales.

1E-mail: stefan@vana.physto.se
2E-mail: lbe@vana.physto.se
1 Introduction

Since some time, it has been realized [1] that defects (textures) associated with the non-trivial winding of massless scalar fields may be of interest even though they are intrinsically unstable if the winding number is large enough [2, 3, 4]. Indeed, it is the fact that textures continually enter the horizon during the evolution of the universe that makes the spectrum of density fluctuations near scale-invariant (although not Gaussian) and makes textures promising candidates for large scale structure formation even in light of the COBE results [5, 6].

A simple theory that admits global textures is given by the lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial _\mu \Phi \cdot \partial ^\mu \Phi - \frac{\lambda}{4} (\Phi ^2 - \phi _0^2)^2 \]

(1)

where \( \Phi \) is a 4 component real scalar field, Here \( \phi _0 \) is the symmetry breaking scale and \( \lambda \) is a dimensionless coupling constant. For convenience we do the rescaling \( \Phi \rightarrow \phi _0 \Phi \), so the action for this theory will be

\[ S[\Phi] = \phi _0^2 \int d^4 x \sqrt{-g} \left( \frac{1}{2} \partial _\mu \Phi \cdot \partial ^\mu \Phi - \frac{w}{4} (\Phi ^2 - 1)^2 \right), \]

(2)

were \( w = \lambda \phi _0^2 \) and \( g \) is the determinant of the space-time metric.

Upon quantization we have in this theory one massive Higgs particle with mass \( m_H = \sqrt{w}/2 \) and \( N - 1 \) massless (Goldstone) bosons. We will however not be interested in the particle spectrum, instead we will only consider the classical equations of motion, for the field \( \Phi \),

\[ \partial _\mu (\sqrt{-g} \partial ^\mu \Phi) = -\sqrt{-g} w (\Phi ^2 - 1) \Phi. \]

(3)

For cosmological applications, it is important that the Goldstone modes remain massless, creating long-range correlations and field dynamics over cosmologically relevant length scales. Arguments, based on quantum gravity effects, have been given [7] which seem to make the survival of exact global symmetries questionable. This statement is, however, based on unknown physics at the Planck scale. More specifically, mechanisms present, e.g. in string models may well protect the potential of the Goldstone modes of the texture scalar fields. (For a recent discussion of such mechanisms, see [14].)

Here we will not enter into this discussion but simply assume that textures can exist and study the properties of their evolution when the dynamics is given by the action (2).

We will parameterize the field using hyper-spherical coordinates \( \rho, \chi, \tilde{\theta} \) and \( \tilde{\varphi} \), in the following way,

\[ \phi (r, t) = \rho (\cos \chi \cos \tilde{\theta}, \sin \chi \sin \tilde{\theta} \cos \tilde{\varphi}, \sin \chi \sin \tilde{\theta} \sin \tilde{\varphi}). \]

(4)
We will look at the “spherically symmetric” (or hedgehog) ansatz, where we let the coordinate functions depend on time \( t \) and the spatial spherical coordinates \( r, \theta \) and \( \varphi \) as \( \rho = \rho(r, t), \chi = \chi(r, t), \bar{\theta} = \theta \) and \( \bar{\varphi} = \varphi \).

It is common to consider the field as a stiff source, which means that one is assuming that the self-coupling of the field is much stronger than the self-gravitational coupling. Thus only the background metric is required in the equation of motion for the field. The perturbation of the background metric can then be calculated from Einstein’s equations with the stress-energy tensor of the texture field added. The applicability of the stiff approximation in the self-similar case is discussed in \[13\].

When one studies the formation of large scale structure in the early universe, the background metric is taken to be Friedmann-Robertson-Walker (FRW). We will discuss how one can find solutions valid at medium large scales using the equation for the Minkowski background.

## 2 Minkowski background

For a Minkowski background the equations of motion (3) in terms of the hyper-spherical coordinates become

\[
\begin{align*}
    r^2(\rho^2 \dot{\chi}) &= (\rho^2 r^2 \dot{\chi})' - \rho^2 \sin 2\chi \\
    r^2 w(\rho^2 - 1) &= r^2(\dot{\chi}^2 - \chi'^2) - 2 \sin^2 \chi + ((r^2 \rho')' - r^2 \ddot{\rho})/\rho.
\end{align*}
\]

(5)

For length scales larger than the inverse mass of the radial “Higgs” mode \( m_i^{-1} = (\lambda \phi_i^2)^{-1/2} \) the dynamics of the field can be described by a nonlinear \( \sigma \) model (NLSM) \[1\]. The NLSM is characterized by that the field is exactly on the vacuum-manifold everywhere, thus \( \rho = 1 \). In this case the first equation of (5) admits a self-similar ansatz \( y = r/t \) and becomes

\[
(1 - y^2) (y^2 \chi_{yy} + 2y \chi_y) = \sin 2\chi(y).
\]

(6)

This equation has a singular behavior at \( y = 0 \) and \( y = \pm 1 \), the conditions for regular solutions are \( \sin 2\chi(0) = 0 \) and \( \sin 2\chi(\pm 1) = 0 \).

This equation has the well known solutions found by Turok and Spergel \[9\]:

\[
\chi(y) = m\pi \pm 2 \arctan(\pm y),
\]

(7)

where \( m \) is an integer.

These solutions are indeed very special, coming from the spherically symmetric self-similar ansatz to the non-linear sigma model approximation. An
important feature is, however, that these symmetric scaling solutions appears in the numerical simulations as attractors [2]. In fact, many of the calculations of the effects of textures, e.g. on the microwave background radiation, rely on the use of these simple analytic solutions [1, 10].

In order to see when the NLSM approximation is applicable we go a step beyond it. Instead of insisting on $\rho \equiv 1$, we will assume only that the derivatives of $\rho$ are negligible in the equations of motion. We will then recover the NLSM equation for $\chi$, so for $\chi$ we will use the selfsimilar NLSM solution (7). The second of the eqs. (5) $\rho$ becomes

$$
\rho^2 = 1 + (\chi^2 - \chi'^2 - \frac{2}{r^2} \sin^2 \chi)/w,
$$

which gives

$$
\rho^2 = 1 - \frac{12t^2 - 4r^2}{w(r^2 + t^2)^2},
$$

upon insertion of our self-similar NLSM solution for $\chi$. It can be checked that $\rho'$ and $\dot{\rho}$ can be neglected in the first of the eqs. (5) if $r^2 + t^2 >> 1/w$. Thus we conclude that there exists a $r_0 >> 1/\sqrt{w}$ such that the selfsimilar solution (7) is valid for all $r^2 + t^2 > r_0$.

If we have a field that initially for $t < 0$, $t^2 >> 1/w$ is described by $\chi(r,t) = 2 \arctan(-r/t)$ we will have an unwinding event for $r^2 + t^2 < r_0^2$ where the field is forced to leave the vacuum manifold. The solution (7) is valid right to the time $t = r - r_0$, when the information from the unwinding event reaches $r$. We thus can match the solution at $t = 0$, $\chi(r,0) = \pi$, $\dot{\chi}(r,0) = 2/r$, with $\chi(r,t) = 2\pi - 2 \arctan(r/t)$, valid for $0 < t < r - r_0$.

How the field behaves for $t > r - r_0$ depends on the details of the unwinding event and must be decided by making a numerical simulation of the full field equations [2]. One thus find that the field after the unwinding goes asymptotically to the NLSM solution $\chi(r,t) = \pi + 2 \arctan(r/t)$ for $t > r$. This solution describes an expanding shell of goldstone bosons, the winding number is zero.

Before we continue discussing Minkowskian self-similar solutions, we study the equations for the FRW background metric.

3 FRW background

We will discuss the evolution of spherical textures in a flat FRW background

$$
\text{ds}^2 = a^2(\eta)(d\eta^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)).
$$

4
Here $\eta$ is the conformal time. The time dependence for $a$ is $a(\eta) \propto \eta^\alpha$, where $\alpha = 1$ corresponds to a radiation dominated universe and $\alpha = 2$ corresponds to a matter dominated universe.

The equation of motion for the NLSM is now

$$(r^2\chi_r)_r - r^2(\chi_{\eta\eta} + 2\frac{\alpha}{\eta}\chi_\eta) = \sin 2\chi \quad (11)$$

We are interested in solutions where unwinding can occur so we try the selfsimilar ansatz $\chi(r, t) = \chi(r/t)$, where $t = \eta - \eta_*$ and $\eta_*$ is the time for the unwinding. The equations with $\alpha \neq 0$ admit this ansatz only if $\eta_* = 0$ which coincides with the time for the big bang singularity. One would nevertheless hope to have some use of this ansatz if we are interested in expanding textures that unwinded very early.

The self-similar ansatz $y = r/\eta$ gives

$$y^2(1 - y^2)\chi_{yy} + 2y(1 + y^2(\alpha - 1))\chi_y = \sin 2\chi(y). \quad (12)$$

However, we will show that with $\alpha = 1$ or 2 there does not exist any non-trivial solutions to (12) passing through $y = 1$ with finite derivative.

To show this we use the regularity conditions that we get by letting $y \to 1$ in (12) and the first and second derivative of that equation. We thus have at $y = 1$

$$2\alpha\chi_y = \sin 2\chi,$$

$$(\alpha - 1)\chi_{yy} + (3\alpha - 2 - \cos 2\chi)\chi_y = 0,$$

$$(\alpha - 2)\chi_{yyy} + (6\alpha - 9 - \cos 2\chi)\chi_{yy} + (6(\alpha - 1) + 2\cos 2\chi)\chi_y = 0. \quad (13)$$

For $\alpha = 1$ we find $(\cos 2\chi(1) - 1)\chi_y(1) = 0$. This gives $\chi_y(1) = 0$ since $\cos 2\chi(1) = 1$ implies $\chi_y(1) = 0$. For $\alpha = 2$ we find

$$(\cos^2 2\chi(1) - \frac{14}{3} \cos 2\chi(1) + \frac{11}{3})\chi_y(1) = 0,$$

the two solutions for $\cos 2\chi(1)$ are $\cos 2\chi(1) = \frac{7 \pm 4}{3}$ so the only real solution is also here $\chi_y(1) = 0$.

We see that in both cases for regular solutions we must have $\chi_y(1) = 0$. Since $\chi(y) = n\pi/2$, integer $n$ are solutions to (12) with $\chi_y(1) = 0$ and $\sin 2\chi(1) = 0$ we conclude that there does not exist any non-trivial regular solutions.
Instead we go over to discuss the validity of using the equation for the Minkowski background as the limiting case when we look at small scales. If we substitute \( \eta = \eta_* + t \) in (11) and assume that \( t << \eta_* \) we get

\[
(r^2 \chi_r)_r - r^2 (\chi_{rt} + \frac{2 \alpha}{\eta*} \chi_t) = \sin 2 \chi.
\]

(14)

Now if we can neglect the term linear in \( \chi_t \) compared to \( \chi_{tt} \) we recover the Minkowski equation. Inserting the solution \( \chi(r/t) = 2 \arctan(r/t) \) we see that this approximation is consistent only when \( |t| \eta_* >> \alpha (r^2 + t^2) \). Thus we have to try a different approximation in order to get something valid for \( t \approx 0 \).

We have found a way to get rid of the term linear in the time-derivative by a certain transformation. In the case \( \alpha = 1 \) when the equation of motion is

\[
(r^2 \chi_r)_r - r^2 (\chi_{rt} + \frac{2}{\eta} \chi_t) = \sin 2 \chi
\]

(15)

we can make the substitution \( \chi(r, \eta) = \frac{\eta}{\eta_*} \psi(r, \eta) \) and get

\[
(r^2 \psi_r)_r - r^2 \psi_{rt} = \frac{\eta}{\eta_*} \sin 2 \frac{\eta}{\eta_*.} \psi,
\]

(16)

which become the Minkowskian equation after substituting \( \eta = \eta_* + t \) and neglecting \( t \) compared with \( \eta_* \). Thus we find for \( \alpha = 1 \) the solutions

\[
\chi(r, \eta) = \frac{\eta_*}{\eta_* + t} \psi_M(r/t)
\]

(17)

valid for \( t << \eta_* \), where \( \psi_M(r/t) \) is any solution to the Minkowskian equation.

The same trick can be done in the case \( \alpha = 2 \), but first we have to change to the coordinates \( u = 3 \eta_*^2 r \) and \( \tau = \eta^3 \) in eq. (11) giving

\[
(u^2 \chi_u)_u - (\tau/\tau_*)^{4/3} u^2 (\chi_{\tau\tau} + \frac{2}{\tau} \chi_{\tau}) = \sin 2 \chi,
\]

(18)

which has the solutions

\[
\chi(u, \tau) = \frac{\tau_*}{\tau_* + t} \psi_M(u/t)
\]

(19)

valid for \( t << \tau_* \).

In the linear approximation of Einstein’s eqs. one can calculate the Newtonian gravitational acceleration from the unwinding texture solution (7), the result is

\[
\vec{g} = \varepsilon \frac{r}{r^2 + t^2} \hat{r},
\]

(20)

6
where \( \varepsilon = 8\pi G \phi_0^2 \), G the gravitational constant and \( \hat{r} \) is the radial unit vector. This acceleration give rise to a velocity kick inwards of surrounding homogenous dust of the amount \( \pi \varepsilon \).

For the solution (19) the same calculation gives to first order in \( t/\tau* \);

\[
\vec{g} = -\varepsilon \frac{r}{r^2 + t^2} (1 - t/\tau*)\hat{r}.
\]  

We notice that the acceleration (21) is enhanced at \( t < 0 \) compared with the ordinary (20), and vice versa for \( t > 0 \), which can be of importance for the form of the resulting matter perturbations. The velocity kick of the dust over the time interval \( \{-t_0, t_0\} \), \( t_0 \ll \tau* \) is \( 2\varepsilon \arctan(t_0/r) \), so at \( r \ll t_0 \) it is still \( \pi \varepsilon \).

4 New solutions

We now examine the self-similar solutions of the Minkowskian equation of motion in greater detail.

The solutions (19) have a winding charge \( Q = \pm 1 \). One would perhaps believe that there exist solutions with higher \( |Q| \) than 1 as has been claimed in [3], but this is not the case.

For a self-similar spherical texture we have \( |Q| \leq 1 \). This is implied by the theorem we prove in the appendix that if \( \chi(y) \) is a regular solution to (6) with \( \chi(0) = 0 \) then \( 0 < |\chi(y)| < \pi \), for all finite \( y \), and we have \( 0 < |\chi(\infty)| \leq \pi \).

The argumentation of reference [3] concerning solutions with \( |Q| > 1 \) is based on the erroneous assumption that there exist solutions satisfying the boundary conditions \( \chi(0) = 0, \chi(1) = n\pi/2 \), with \( n > 1 \), (see the corollary of Lemma 1).

We also want to emphasize that a boundary value problem such as (6) with \( \chi(0) = 0 \) and \( \chi(1) = \pi/2 \), does not necessarily possess a unique solution.

Actually we have by numerical means been able to demonstrate the existence of whatseems to be a countably infinite set of additional solutions with total winding charge \( Q \) less than unity. These solutions are characterized by the number of oscillations around the value \( \chi = \pi/2 \), and have rapidly increasing derivatives at the origin.

We want to demonstrate the existence of these new solutions by accurate numerical techniques [11] with some modifications necessary to handle the singular points of the equation. We want to solve the boundary value problem using the shooting technique. The most straightforward strategy would then be to consider the initial value problem \( \chi(0) = 0, \chi_y(0) = \beta, \)
and numerically integrate this to the point $y = 1$, we denote the solutions by $\chi(y, \beta)$. The equation $\chi(1, \beta) = \pi/2$ may now be solved for $\beta$ by trial. However, because of the singularities it becomes numerically impossible to start the integration from $y = 0$, so we have to modify our method. We must start the integration at a small distance away from $y = 0$ and use a series expansion in order to get an appropriate initial condition.

By making a series expansion of $\chi(y)$ close to the origin of the form:

$$
\chi(y) = \sum_{k=0}^{\infty} a_{2k+1} y^{2k+1}
$$

and inserting this into the differential equation Eq. (1), one can find the coefficients $a_3, a_5, \ldots$ in terms of $a_1 = \beta$. One finds, e.g.,

$$
a_3 = \frac{2\beta - 4\beta^3}{10}
$$

and

$$
a_5 = \frac{3\beta - 3\beta^3 + \beta^5}{35}.
$$

Let us now consider the initial value problem,

$$
\chi(\varepsilon) = \beta \varepsilon + a_3 \varepsilon^3 + a_5 \varepsilon^5, \quad \chi_y(\varepsilon) = \beta,
$$

and integrate only up to $y = 1 - \varepsilon$. For each $\varepsilon$ we may find a $\beta$ such that

$$
\chi(1 - \varepsilon, \beta) + \varepsilon \chi_y(1 - \varepsilon, \beta) = \pi/2,
$$

we then must check that the value of $\beta$ converges when we choose $\varepsilon$ smaller and smaller. It is also possible to use a more accurate extrapolation formula near $y=1$, one may again use the form of the original differential equation to write

$$
\chi(1 - z) = \frac{\pi}{2} - \gamma z - \frac{\gamma^2}{2} z^2 - \frac{\gamma^3}{6} z^3 + \frac{\gamma(1 - \gamma^2)}{18} z^4 + \ldots
$$

where $\gamma = \chi_y(1)$. This expression can also be used to continue the solution past the singular point $y=1$.

With this careful treatment of the singular points of the differential equation, its solution is otherwise straightforward. For the numerical solution we used a Runge-Kutta method with adaptive size control. We have employed this technique and thus discovered a set of such different $\beta$’s. In Fig. 1 the first four of the solutions corresponding to these $\beta$’s are displayed, we number them with the mode number $n$ starting with $n = 0$ for the analytical solution.
These solutions appear to be very robust according to various stability checks we have made of our numerical algorithm, so we are confident in the belief that the presence of the solutions is not a numerical artefact. We have also checked that when we vary $\alpha$ in (12) around 0 we still find solutions which approach our solutions in a continuous way when $\alpha \to 0$. From the conspicuously regular pattern of the first solutions shown in Fig. 1, we conjecture that the number of solutions is countably infinite.

That self-similar solutions with winding number less than unity exist is potentially of great importance, since as shown in numerical simulations [2] and backed by analytical arguments [3], configurations with $Q > 1/2$ collapse and contribute to structure formation. We expect that well inside the horizon where spacetime is approximately Minkowski our new scaling solutions could play a dynamical role in structure formation. These solutions need a very high resolution numerical code to appear in the simulations since the derivatives at $y = 0$ are very high. We plan to investigate these questions as well as the attractor nature of the solutions in future work. The applicability of the NLSM for the solutions with winding number less than one can be examined in the same way as for the analytical solution. The NLSM is valid for $r^2 + t^2 > r_0^2$ and we find that we get a factor of around hundred extra in $r_0$ for each mode, the condition reads $r_0 >> 100^n/\sqrt{w}$, where $n$ is the mode number. We can use the solutions for times $t < r - r_0$ as for the analytical solution, but they can not be matched at $t = r$ with any selfsimilar solution, thus for $t > r$ the selfsimilarity will necessarily be lost.

To conclude, we have investigated in quite some detail the nature and validity of the self-similar ansatz to the texture equations of motion. We have analyzed possible modifications when one goes beyond the non-linear sigma model approximation, the Minkowski background approximation, and the ”ground state” arctangent solution. In future work, the effects caused by including the self-gravitational coupling will be investigated.

We are grateful to P. Ernström for useful discussions. The work of L.B. was supported by the Swedish Natural Science Research Council (NFR) and EEC-SCIENCE contract no. SC1*-CT91-0650.
Appendix

In this appendix, we prove the following theorem:

**Theorem** If \( \chi(y) \) is a regular solution to (3) with \( \chi(0) = 0 \) then \( 0 < |\chi(y)| < \pi \), for all finite \( y \), and we have \( 0 < |\chi(\infty)| \leq \pi \).

For the proof we need some lemmas:

**Lemma 1** If \( \chi(y) \) is a regular solution to (3) with \( \chi(0) = 0 \) then \( 0 < |\chi(y)| < \pi \) for \( 0 < y \leq 1 \).

The proof is similar to one used in [8] concerning the boundary conditions for the static ansatz \( \chi(r, t) = f(r) \):

We make the variable substitution \( x = 1/y \) in (3) and get,

\[
(x^2 - 1)\chi_{xx} = \sin 2\chi(x), \quad \chi(\infty) = 0.
\]  
(25)

We multiply this with \( \chi_x \) and integrate from \( x \) to \( \infty \),

\[
\left[(x^2 - 1)\chi_x^2\right]_x^\infty - 2\int_x^\infty dx x\chi_x^2 = -\left[\cos 2\chi(x)\right]_x^\infty.
\]  
(26)

Since \( \chi_x(x) = -\frac{1}{x^2}(\chi_y(0) + O(1/x)) \) for large \( x \), (26) reduces to

\[
-(x^2 - 1)\chi_x^2(x) - 2\int_x^\infty dx x\chi_x^2 = \cos 2\chi(x) - 1.
\]  
(27)

The left-hand side is always negative for \( 1 \leq x < \infty \) so we must have \( 0 < |\chi(x)| < \pi \) for \( 1 \leq x < \infty \).

**Corollary** If \( \chi(y) \) is a regular solution to (3) with \( \chi(0) = 0 \) and \( \chi_y(0) > 0 \) then \( \chi(1) = \pi/2 \), if \( \chi_y(0) < 0 \) then \( \chi(1) = -\pi/2 \).

This follows immediately from lemma 1 and the regularity condition \( \chi(1) = n\pi/2 \), integer \( n \).

**Lemma 2** If \( \chi(y) \) is a solution to (3) with \( \chi(1) = \pi/2 \) and \( \chi_y(1) > 1 \) then there exists a \( 0 < y_0 < 1 \) such that \( \chi(y_0) = 0 \), if \( \chi_y(1) < -1 \) then there exists a \( 0 < y_0 < 1 \) such that \( \chi(y_0) = \pi \).

A brief outline of the proof:

We make the substitution \( y = \tan \theta \) which gives the equation

\[
\cos 2\theta(\sin^2 \theta \chi_{\theta\theta} + \sin 2\theta \chi_\theta) = \sin 2\chi(\theta).
\]  
(28)

After differentiation of this equation one can get some inequalities on the third derivative of \( \chi \), these can then be used in order to show that if \( \chi_\theta(\pi/4) > 2 \) then \( \chi_\theta(\theta) > \chi_\theta(\pi/4) \), \( 0 \leq \theta < \pi/4 \). (Note that \( \chi_\theta(\pi/4) = 2\chi_y(1) \).) This leads to the existence of a \( 0 < \theta_0 < \pi/4 \) such that \( \chi(\theta_0) = 0 \).

A similar argument shows that if \( \chi_\theta(\pi/4) < -2 \) then there exists a \( 0 < \theta_0 < \pi/4 \) such that \( \chi(\theta_0) = \pi \).
**Lemma 3** If $\chi(y)$ is a solution to (6) with $\chi(1) = \pi/2$ and $|\chi_y(1)| < 1$ then $0 < \chi(y) < \pi$, $y \geq 1$.

An outline of the proof:

We denote the known solutions with $\chi(1) = \pi/2$ by $\chi^a(x) = \pi/2 \pm (2 \arctan x - \pi/2)$. Using the equation we get if we multiply (25) with $\chi_x$ and integrate from $x$ to 1, we can show that if $|\chi_x(1)| < 1$ then $|\chi_x(x)| < |\chi^a_x(x)|$ for $0 \leq x \leq 1$. Since $0 \leq \chi^a(x) \leq \pi$ we thus have $0 < \chi(x) < \pi$, $0 \leq x \leq 1$.

**Proof of the theorem** :

If $\chi(y)$ is a regular solution to (6) with $\chi(0) = 0$ and $\chi_y(0) > 0$ then the corollary tells us that $\chi(1) = \pi/2$. From lemma 1 together with lemma 2 it follows that we cannot have $|\chi_y(1)| > 1$. Together with the existence of the solution $\chi(y) = 2 \arctan y$ (which has $\chi_y(1) = 1$ and $\chi(\infty) = \pi$) we thus conclude that $-1 < \chi_y(1) \leq 1$. From lemma 3 it now follows that $0 < \chi(y) < \pi$, $y \geq 1$, this together with Lemma 1 thus tells us that $0 < \chi(y) < \pi$, for all finite $y$. A similar reasoning gives $-\pi < \chi(y) < 0$ if $\chi_y(0) < 0$, which completes the proof.

**References**

[1] N. Turok, Phys. Rev. Lett. 63 (1989) 2625.

[2] D. Spergel, N. Turok, W. Press and B. Ryden, Phys. Rev. D43 (1990) 1038.

[3] J. Borrill, E. Copeland and A. Liddle, Phys. Rev. D46 (1992) 524.

[4] S. Åminneborg, Nucl. Phys. B388 (1992) 521.

[5] G.F. Smoot et al., Astrophys. J. 396 (1992) L1.

[6] N. Turok, Proc. “Trends in Astroparticle Physics”, Stockholm, Sep. 1994, eds L. Bergström, P. Carlson, P.O. Hulth and H. Snellman, Nucl. Phys. B. Proc. Suppl., in press.

[7] M. Kamionkowski and J. March-Russell, Phys. Lett. B282 (1992) 137. S.M. Barr and D. Seckel, Phys. Rev. D46 (1992) 539; R. Holman et al., NSF-ITP-92-04 (1992).

[8] M. Iwasaki, H. Ohyama, Phys. Rev. D40 (1989) 3125.

[9] N. Turok and D. Spergel, Phys. Rev. Lett. 64 (1990) 2736.
[10] R. Durrer, M. Heusler, P. Jetzer and N. Straumann, Nucl. Phys. B368 (1992) 527.

[11] W. Press, B. Flannery, S. Teukolsky and W. Vettering, *Numerical Recipes*, Cambridge Univ. Press, 1986.

[12] J. Barriola, Phys. Rev. D48 (1993) 5576.

[13] S. Åminneborg, in preparation.

[14] R. Kallosh, A. Linde, D. Linde, L. Susskind, SU-ITP-95-2 (hep-th/9502069), 1995.
Figure Caption

Fig. 1 The previously known self-similar solution to the non-linear sigma model (solid line) and the first four of the new class of solutions found in this paper. Here $\chi(y)$ is the radial function in the spherically symmetric ansatz, and $y = r/t$ is the self-similarity variable. The values of the derivative at the origin are for each solution, $\chi_y^0 = 2$, $\chi_y^1 = 21.757$, $\chi_y^2 = 234.50$, $\chi_y^3 = 2521.3$ and $\chi_y^4 = 27102$. 
