Components of the Stack of Torsion-Free Sheaves of Rank 2 on Ruled Surfaces

Charles H. Walter*
Laboratoire de Mathématiques
Université de Nice
F-06108 Nice Cedex 02 FRANCE
walter@math.unice.fr

Abstract

Let $S$ be a ruled surface without sections of negative self-intersection. We classify the irreducible components of the moduli stack of torsion-free sheaves of rank 2 sheaves on $S$.

We also classify the irreducible components of the Brill-Noether loci in $\text{Hilb}^N(P^1 \times P^1)$ given by $W^0_N(D) = \{ [X] \mid h^1(I_X(D)) \geq 1 \}$ for $D$ an effective divisor class. Our methods are also applicable to $P^2$ giving new proofs of theorems of Strømme (slightly extended) and Coppo.

Let $\pi: S = \mathbf{P}(\mathcal{A}) \to C$ be a ruled surface with tautological line bundle $\mathcal{O}(1) := \mathcal{O}_{\mathbf{P}(\mathcal{A})}(1)$. The current classification of isomorphism classes of rank 2 vector bundles $\mathcal{E}$ on $S$ ([B] [Br] [HS] [Ho]) proceeds by stratifying the moduli functor (or stack) and then classifying the sheaves in each stratum independently. The numerical data used to distinguish the strata are usually (i) the splitting type $\mathcal{O}_{\mathbf{P}_1}(a) \oplus \mathcal{O}_{\mathbf{P}_1}(b)$ of the generic fiber of $\pi$ (with $a \geq b$), and (ii) the degree of the locally free sheaf $\pi_*(\mathcal{E}(-a))$ on $C$. On each stratum, $\mathcal{U} := \pi^*(\pi_*(\mathcal{E}(-a)))(a)$ is naturally a subsheaf of $\mathcal{E}$, and the possible quotient sheaves $\mathcal{E}/\mathcal{U}$ and extension classes $\text{Ext}^1(\mathcal{E}/\mathcal{U}, \mathcal{U})$ have been classified.

To the author’s knowledge, rank 2 torsion-free sheaves on $S$ have not been given a similar classification, but one could clearly adapt the ideas used for vector bundles.

What this approach has usually not described is the relationship between the strata particularly for the strata parametrizing only unstable sheaves. In this paper we give a first result along these lines by describing which strata are generic, i.e. which are open in the (reduced) moduli stack. Thus we are really classifying the irreducible components of the moduli stack of rank 2 torsion-free sheaves on $S$. We use a method developed by Strømme [S] for rank 2 vector bundles on $P^2$ modified by deformation theory techniques which originate in [DL].

We will divide our irreducible components into two types. The first type we call prioritary because the general member of a component of this type is a prioritary sheaf in the sense that we used in [W1]. That is, if for each $p \in C$ we write $f_p := \pi^{-1}(p)$ for the corresponding fiber,

*Supported in part by NSA research grant MDA904-92-H-3009.
then a coherent sheaf \(E\) on \(S\) is prioritary if it is torsion-free and satisfies \(\text{Ext}^2(E, E(-f_p))\) for all \(p\). We showed in \([W1]\) Lemma 7, that if one polarizes \(S\) by an ample divisor \(H\) such that \(H \cdot (K_S + f_p) < 0\), then \(H\)-semistable sheaves are prioritary. Thus the prioritary components should be viewed as playing a role one might otherwise assign to semistable components. But the condition of priority is simpler to use than semistability because it does not depend on the choice of a polarization, and moreover the existence problem has a simpler solution (particularly in higher rank).

The second type of components are nonprioritary ones.

Our main result is the following. We use the convention that if \(D \in \text{NS}(S)\), then \(O_S(D)\) is the line bundle corresponding to the generic point of the corresponding component \(\text{Pic}^D(S)\) of the Picard scheme. This is well-defined on all surfaces for which numerical and algebraic equivalence coincide, including all of ours.

**Theorem 0.1.** Let \(S\) be a ruled surface without curves of negative self-intersection, and let \(f\) be the numerical class of a fiber of \(S\). Let \(c_1 \in \text{NS}(S)\) and \(c_2 \in \mathbb{Z}\). The irreducible components of the stack \(\text{TF}_S(2, c_1, c_2)\) of torsion-free sheaves on \(S\) of rank 2 and Chern classes \(c_1\) and \(c_2\) are the following:

(i) A unique prioritary component if \(c_1 f\) is even and \(c_2 \geq \frac{1}{4} c_1^2\), or if \(c_1 f\) is odd and \(c_2 \in \mathbb{Z}\). This component is generically smooth of dimension \(-\chi(E, E)\), and the general sheaf in the component is locally free.

(ii) For every pair \((D, n) \in \text{NS}(S) \times \mathbb{Z}\) such that \(D f \leq -1 + \frac{1}{4} c_1 f\) and \(0 \leq n \leq c_2 + D(D - c_1) \leq \chi(O_S(-c_1)) + D(2D - 2c_1 - K_S)\) a unique nonprioritary component whose general member is a general extension

\[0 \to I_{Z_1}(c_1 - D) \to E \to I_{Z_2}(D) \to 0\]

where \(Z_1\) (resp. \(Z_2\)) is a general set of \(n\) (resp. \(n' := c_2 + D(D - c_1) - n\) points on \(S\). These components have dimensions \(-\chi(E, E) + \chi(O_S(-c_1)) + D(D - c_1 - K_S) - c_2\) but have generic embedding codimension \(n' + h^1(O_S(2D - c_1))\). The general sheaf in the component is locally free except at \(Z_1\).

For \(\mathbb{P}^2\) the components of \(\text{TF}_{\mathbb{P}^2}(2, c_1, c_2)\) containing locally free sheaves were classified by Strømme using a similar method (\(S\) Theorem 3.9). We wish to add to his classification the components of \(\text{TF}_{\mathbb{P}^2}(2, c_1, c_2)\) whose general member is not locally free. We recall from \([\text{III}]\) that a prioritary sheaf \(E\) on \(\mathbb{P}^2\) is one that is torsion-free and satisfies \(\text{Ext}^2(E, E(-1)) = 0\).

**Theorem 0.2.** Let \(S\) be \(\mathbb{P}^2\) and let \(f \in \text{NS}(S)\) be the class of a line. Let \((c_1, c_2) \in \mathbb{Z}^2\). Then the irreducible components of \(\text{TF}_{\mathbb{P}^2}(2, c_1, c_2)\) have the same classification as in Theorem 0.1 except that the prioritary component exists if and only if \(c_2 \geq \frac{1}{4} c_1^2 - \frac{1}{4}\).

The uniqueness of the prioritary components was proven for ruled surfaces (resp. for \(\mathbb{P}^2\)) in \([W1]\) (resp. \([\text{III}]\)) although of course there were many earlier results by many authors concerning semistable components on \(\mathbb{P}^2\) and on various ruled surfaces.

The classification of the irreducible components of the stacks of torsion-free sheaves has an interesting application to Brill-Noether problems. Let \(S\) be a smooth projective algebraic surface, \(E\) an effective divisor class on \(S\), and \(N\) a positive integer such that \(N \leq h^0(O_S(E))\).
For simplicity we will assume that $H^1(\mathcal{O}_S) = H^1(\mathcal{O}_S(E)) = 0$. We consider the Brill-Noether loci in $\text{Hilb}^N S$ defined as

$$W^i_N(E) = \{[X] \in \text{Hilb}^N S \mid h^1(\mathcal{I}_X(E)) \geq i + 1\}.$$  

Thus $W^i_N(E)$ parametrizes those 0-schemes of length $N$ which impose at least $i + 1$ redundant conditions on divisors in $|E|$. What we wish to consider is:

**The Brill-Noether Problem.** Classify the irreducible components of the $W^i_N(E)$ and compute their dimensions.

It is known from general principles that each component has codimension at most $(\chi + 1) + i$ in $\text{Hilb}^N S$ where $\chi = h^0(\mathcal{O}_S(E)) - N \geq 0$, but there can be many components of various smaller codimensions.

The Brill-Noether problem is related to the problem of classifying irreducible components of the stack of torsion-free sheaves on $S$ as follows. By an elementary argument (cf. [3] p. 732) the general $X$ in any component of $W^i_N(E)$ has $h^1(\mathcal{I}_X(E)) = i + 1$. One then uses Serre duality $H^1(\mathcal{I}_X(E))^* \cong \text{Ext}^1(\mathcal{I}_X(E - K_S), \mathcal{O}_S)$ to get an extension

$$0 \to \mathcal{O}_S^{|E| + 1} \to \mathcal{E} \to \mathcal{I}_X(E - K_S) \to 0.$$  

So we get a Serre correspondence between $X \in W^i_N(E) - W^{i-1}_N(E)$ and pairs $(\mathcal{E}, V)$ where $\mathcal{E}$ is torsion-free of rank $i + 2$ with $c_1(\mathcal{E}) = E - K_S$, $c_2(\mathcal{E}) = N$, $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$, and for which there exists an $(i + 1)$-dimensional subspace $V \subset H^0(\mathcal{E})$ such that the natural map $V \otimes \mathcal{O}_S \to \mathcal{E}$ is injective with torsion-free quotient. Note that these properties are all open conditions on $\mathcal{E}$ within the stack of torsion-free sheaves on $S$. So Theorems 0.1 and [1,2] yields a classification of the irreducible components of the $W^0_N(E)$ for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

This classification has been previously obtained by Coppo for $\mathbb{P}^2$ by a different method ([3] Théorème 3.2.1) but seems new for $\mathbb{P}^1 \times \mathbb{P}^1$.

**Theorem 0.3.** Let $S$ be $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$), let $E$ be an effective divisor of degree $e$ (resp. of bidegree $(e_1, e_2)$), and let $N$ be an integer such that $0 < N \leq \chi(\mathcal{O}_S(E))$. Then the irreducible components of the Brill-Noether locus $W^0_N(E)$ are the following:

(i) For every pair $(D, n) \in \text{NS}(S) \times \mathbb{Z}$ such that $D$ is an effective and irreducible divisor class of degree $d$ on $\mathbb{P}^2$ (resp. of bidegree $(d_1, d_2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$) such that $d \leq \frac{1}{2}(e + 1)$ (resp. $d_2 \leq \frac{1}{2}e_2$), $D(E - D) \leq \chi(\mathcal{O}_S(E)) - N$, $n \geq 0$, and $0 \leq N - D(E - D - K_S) - n \leq \chi(\mathcal{O}_S(D + K_S))$, there exists a unique irreducible component of codimension $D(E - D) + 1$ in $\text{Hilb}^N(S)$ whose general member is the union of $n$ general points of $S$ and $N - n$ points on a curve in $|D|$.

(ii) If $S$ is $\mathbb{P}^2$ (resp. if $S$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_2$ is even, resp. if $S$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_2$ is odd), then there exists one additional component of codimension $\chi(\mathcal{O}_S(E)) - N + 1$ in $\text{Hilb}^N(S)$ if $N \geq \frac{1}{2}(e + 2)(e + 4)$ (resp. $N \geq \frac{1}{2}(e_1 + 2)(e_2 + 2)$, resp. $N \geq \frac{1}{2}(e_1 + 2)(e_2 + 1) + 1$).

If $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $(e_1, e_2, N) = (e_1, 1, e_1 + 2)$ there is also one additional component of codimension $\chi(\mathcal{O}_S(E)) - N + 1$.

In part (i) the $N - n$ points on the curve $C \in |D|$ have the property that their union is a divisor on $C$ belonging to a linear system of the form $|\Gamma + E|_{|C} - K_C|$ with $\Gamma$ an effective
divisor satisfying \( h^0(O_C(\Gamma)) = 1 \). The necessary condition \( 0 \leq \deg(\Gamma) \leq g(C) \) is exactly the condition \( 0 \leq N - D(E - D - K_S) - n \leq \chi(O_S(D + K_S)) \).

The main tool which we use to obtain our results is interesting in its own right. We use the notation \( \chi(F,G) = \sum (-1)^i \dim \text{Ext}^i(F,G) \).

**Proposition 0.4.** Let \( S \) be a projective surface, and \( E \) a coherent sheaf on \( S \) with a filtration \( 0 = F_0(E) \subset F_1(E) \subset \cdots \subset F_r(E) = E \). Suppose that the graded pieces \( \text{gr}_i(E) := F_i(E)/F_{i-1}(E) \) satisfy \( \text{Ext}^2(\text{gr}_i(E), \text{gr}_j(E)) \) for \( i \geq j \). Then

(i) the deformations of \( E \) as a filtered sheaf are unobstructed,

(ii) if \( E \) is a generic filtered sheaf, then the \( \text{gr}_i(E) \) are generic, and

(iii) if \( E \) is generic as an unfiltered sheaf, then also \( \chi(\text{gr}_i(E), \text{gr}_{i+1}(E)) \geq 0 \) for \( i = 1, \ldots, r - 1 \).

The outline of the paper is as follows. In the first section we review some necessary facts about algebraic stacks and their dimensions. In the second section we prove our technical tool Proposition 0.4 and describe some situations where it applies. In the third section we classify the prioritary components of the TF \( S(2, c_1, c_2) \) and the \( W^0_N(E) \). In the fourth section we classify the nonprioritary components. In the short final section we complete the proofs of the main theorems.

This paper was written in the context of the group on vector bundles on surfaces of Europroj. The author would like to thank A. Hirschowitz and M.-A. Coppo for some useful conversations.

### 1 Algebraic Stacks

In this paper we use stacks because in that context there exist natural universal families of coherent (or torsion-free) sheaves. The paper should be manageable even to the reader unfamiliar with algebraic stacks if he accepts them as some sort of generalization of schemes where there are decent moduli for unstable sheaves. For the reader who wishes to learn about algebraic stacks we suggest [LM]. Alternative universal families of coherent sheaves which stay within the category of schemes would be certain standard open subschemes of Quot schemes. This is the approach taken in [S]. But the language of algebraic stacks is the natural one for problems which involve moduli of unstable sheaves.

Stacks differ from schemes in the way their **dimensions** are calculated. For the general definition of the dimension of an algebraic stack at one of its points the reader should consult [LM] §5. But the dimension of the algebraic stack \( \text{Coh}_S \) of coherent sheaves on \( S \) (or of any open substack of \( \text{Coh}_S \) such as a TF \( S(r, c_1, c_2) \)) at a point corresponding to a sheaf \( E \) is the dimension of the Kuranishi formal moduli for deformations of \( E \) (i.e. the fiber of the obstruction map \( (\text{Ext}^1(E,E), 0)^{\wedge} \rightarrow (\text{Ext}^2(E,E), 0)^{\wedge} \)) minus the dimension of the automorphism group of \( E \). Thus if we write \( e_1 = \dim \text{Ext}^1(E,E) \), then \( -e_0 + e_1 - e_2 \leq \dim [E] \text{Coh}_S \leq -e_0 + e_1 \). If \( S \) is a surface, this means

\[
-\chi(E,E) \leq \dim [E] \text{Coh}_S \leq -\chi(E,E) + e_2.
\]

If \( E \) is a stable sheaf, then \( \dim [E] \text{Coh}_S \) is one less than the dimension of the moduli scheme at \([E]\) because \( E \) has a one-dimensional family of automorphisms, the homotheties.
Generally speaking, the dimension of an algebraic stack are well-behaved. It is constant on an irreducible component away from its intersection with other components; the dimension of a locally closed substack is smaller than the dimension of the stack; etc. But stacks can have negative dimensions.

2 When is a Filtered Sheaf Generic?

In this section we prove our main technical tool Proposition [4.4] and then give two corollaries applying the proposition to birationally ruled surfaces.

Proof of Proposition [4.4]. We begin by recalling some of the deformation theory of [DL]. We consider the abelian category of sheaves with filtrations of a fixed length $r$:

$$0 = F_0(E) \subset F_1(E) \subset \cdots \subset F_r(E) = E.$$ 

On this category we can define functors

$$\text{Hom}_-(E,F) = \{ \phi \in \text{Hom}(E,F) \mid \phi(F_i(E)) \subseteq F_i(F) \text{ for all } i \},$$

$$\text{Hom}_{neg}(E,F) = \{ \phi \in \text{Hom}(E,F) \mid \phi(F_i(E)) \subseteq F_{i-1}(F) \text{ for all } i \}.$$ 

These have right-derived functors denoted $\text{Ext}_-^p$ and $\text{Ext}_{neg}^p$ which may be computed by the spectral sequences ([DL] Proposition 1.3)

$$E_1^{pq} = \begin{cases} 
\prod_i \text{Ext}_-^{p+q}(\text{gr}_i(E),\text{gr}_{i-p}(E)) & \text{if } p \geq 0 \\
0 & \text{if } p \leq -1 
\end{cases} \Rightarrow \text{Ext}_-^{p+q}(E,E), \quad (2)$$

$$E_1^{pq} = \begin{cases} 
\prod_i \text{Ext}_{neg}^{p+q}(\text{gr}_i(E),\text{gr}_{i-p}(E)) & \text{if } p \geq 1 \\
0 & \text{if } p \leq 0 
\end{cases} \Rightarrow \text{Ext}_{neg}^{p+q}(E,E). \quad (3)$$

There is also a long exact sequence

$$\cdots \to \text{Ext}_{neg}^p(E,E) \to \text{Ext}_-^p(E,E) \to \prod_i \text{Ext}^p(\text{gr}_i(E),\text{gr}_i(E)) \to \text{Ext}_{neg}^{p+1}(E,E) \to \cdots. \quad (4)$$

(i) The tangent space for the deformations of $E$ as a filtered sheaf is $\text{Ext}_-^1(E,E)$ and the obstruction space is $\text{Ext}_-^2(E,E)$. The latter vanishes because of the spectral sequence (2).

(ii) From (3) and (4) we see that the map $\text{Ext}_-^1(E,E) \to \prod_i \text{Ext}^1(\text{gr}_i(E),\text{gr}_i(E))$ is surjective. Thus any first-order infinitesimal deformation of the $\text{gr}_i(E)$ can be induced from a first-order infinitesimal deformation of $E$ as a filtered sheaf. But because of (i) any first-order infinitesimal deformation of the filtered sheaf $E$ is induced by a noninfinitesimal deformation of $E$. So if $E$ is generic, then the $\text{gr}_i(E)$ must also be generic in their respective stacks.

(iii) We consider $E$ with two filtrations: the original filtration and its subfiltration obtained by suppressing the term $F_i(E)$. We write $\text{Ext}_-^p$ (resp. $\text{Ext}_{-\text{sub}}^p$) for the $\text{Ext}_-^p$ associated to these two filtrations. We have a long exact sequence

$$\cdots \to \text{Ext}_-^p(E,E) \to \text{Ext}_{-\text{sub}}^p(E,E) \to \text{Ext}^p(\text{gr}_i(E),\text{gr}_{i+1}(E)) \to \text{Ext}_{-\text{sub}}^{p+1}(E,E) \to \cdots.$$ 

Also $\text{Ext}_-^2(E,E) = 0$ by (i). So the formal moduli for the deformations of $E$ as a filtered sheaf for the full filtration is of dimension $\dim \text{Ext}_-^1(E,E)$. The formal moduli for the deformations of $E$ as a filtered sheaf for the subfiltration is by general principles of dimension at least

$$\dim \text{Ext}_{-\text{sub}}^1(E,E) - \dim \text{Ext}_{-\text{sub}}^2(E,E) \geq \dim \text{Ext}_-^1(E,E) - \chi(\text{gr}_i(E),\text{gr}_{i+1}(E)).$$
So if \( \chi(\text{gr}_i(\mathcal{E}), \text{gr}_{i+1}(\mathcal{E})) < 0 \), then the natural morphism from the formal moduli for the deformations of \( \mathcal{E} \) with the full filtration to the formal moduli for the deformations of \( \mathcal{E} \) with the subfiltration could not be surjective. So there would be finite deformations of \( \mathcal{E} \) which preserve the subfiltration but not the full filtration. This would contradict the genericity of \( \mathcal{E} \) as an unfiltered sheaf.

There are several situations in which there are filtrations to which Proposition 0.4 applies. For the first situation, let \( S \) be a smooth projective surface and \( H \) an ample divisor on \( S \). Recall that the \( H \)-slope of a torsion-free sheaf \( \mathcal{F} \) on \( S \) is \( \mu_H(\mathcal{F}) := (Hc_1(\mathcal{F}))/\text{rk}(\mathcal{F}) \). We write \( \mu_{H,\max}(\mathcal{F}) \) is the maximum \( H \)-slope of a nonzero subsheaf of \( \mathcal{F} \), and \( \mu_{H,\min}(\mathcal{F}) \) is the minimum slope of a nonzero torsion-free quotient sheaf of \( \mathcal{F} \).

Lemma 2.1. Let \( S \) be a smooth projective surface and \( H \) an ample divisor on \( S \) such that \( HK_S < 0 \). Let \( \mathcal{F} \) and \( \mathcal{G} \) be torsion-free sheaves on \( S \) such that \( \mu_{H,\max}(\mathcal{F}) + HK_S < \mu_{H,\min}(\mathcal{G}) \). Then \( \text{Ext}^2(\mathcal{F}, \mathcal{G}) = 0 \).

Proof. By Serre duality we have \( \text{Ext}^2(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{G}, \mathcal{F}(K_S))^* \). If there were a nonzero \( \phi \in \text{Hom}(\mathcal{G}, \mathcal{F}(K_S)) \), then we would have

\[
\mu_{H,\max}(\mathcal{F}) + HK_S = \mu_{H,\max}(\mathcal{F}(K_S)) \geq \mu(\text{im}(\phi)) \geq \mu_{H,\min}(\mathcal{G}),
\]

a contradiction. \( \square \)

It follows that if \((S, \mathcal{O}_S(H))\) is a polarized surface such that \( HK_S < 0 \), then Proposition 0.4 applies to the Harder-Narasimhan filtration of a torsion-free sheaf \( \mathcal{E} \) on \( S \). It also applies to the weak Harder-Narasimhan filtration for torsion-free sheaves on \( \mathbb{P}^2 \) described in [W2].

The other situation in which Proposition 0.4 applies is the relative Harder-Narasimhan filtration of a torsion-free sheaf \( \mathcal{E} \) on a ruled surface \( \pi: S \to C \). To describe this let \( f_q \) be the generic fiber of \( \pi \). Write \( \mathcal{E}|_{f_q} \cong \bigoplus_{i=1}^n \mathcal{O}_{f_q}(e_i)^{n_i} \) with \( e_1 > e_2 > \cdots > e_s \) and the \( n_i > 0 \). There exists a unique filtration \( 0 = F_0(\mathcal{E}) \subset F_1(\mathcal{E}) \subset \cdots \subset F_s(\mathcal{E}) = \mathcal{E} \) such that the graded pieces \( \text{gr}_i(\mathcal{E}) \) are torsion-free and satisfy \( \text{gr}_i(\mathcal{E})|_{f_q} \cong \mathcal{O}_{f_q}(e_i)^{n_i} \). The \( F_i(\mathcal{E}) \) may be obtained as the inverse image in \( \mathcal{E} \) of the torsion subsheaf of \( \mathcal{E}/E_i \) where \( E_i \) is the image of the natural map \( \pi^*(\mathcal{O}(e_i))|_{f_q} \to \mathcal{E} \). Proposition 0.4 applies to this relative Harder-Narasimhan filtration because of

Lemma 2.2. Let \( \pi: S \to C \) be a ruled surface, and let \( \mathcal{E} \) and \( \mathcal{G} \) be torsion-free sheaves on \( S \). Suppose that the restrictions of \( \mathcal{E} \) and \( \mathcal{G} \) to a general fiber \( F \) of \( \pi \) are of the forms \( \mathcal{E}|_F \cong \bigoplus_i \mathcal{O}_F(e_i) \) and \( \mathcal{G}|_F \cong \bigoplus_j \mathcal{O}_F(g_j) \) with \( \max\{e_i\} - 2 < \min\{g_j\} \). Then \( \text{Ext}^2(\mathcal{E}, \mathcal{G}) = 0 \). In particular, if \( \max\{e_i\} - \min\{e_j\} < 2 \), then \( \mathcal{E} \) is priority.

Proof. Again Serre duality gives \( \text{Ext}^2(\mathcal{E}, \mathcal{G}) \cong \text{Hom}(\mathcal{G}, \mathcal{E}(K_S))^* \). If there were a nonzero \( \phi \in \text{Hom}(\mathcal{G}, \mathcal{E}(K_S)) \), then there would be a nonzero \( \phi|_F \in \bigoplus_{i,j} \mathcal{H}^0(\mathcal{O}_F(e_i - 2 - g_j)) \). This is impossible since \( f_i - 2 - g_j < 0 \) for all \( i \) and \( j \).

If \( \max\{e_i\} - \min\{e_i\} < 2 \), then we may set \( G = E(-f_p) \) for any fiber \( f_p = \pi^{-1}(p) \) to get \( \text{Ext}^2(\mathcal{E}, E(-f_p)) = 0 \) for all \( p \in C \). Thus \( \mathcal{E} \) is priority. \( \square \)
3 Prioritary Components

In this section we prove the necessary lemmas for classifying the principal components of the $\text{TF}_S(2, c_1, c_2)$ and $W^0_N(E)$. We use [W1] and [HL] as our basic sources for existence and uniqueness results because these use our preferred language of prioritary sheaves. But existence and uniqueness results for only marginally different classes of sheaves on $\mathbb{P}^2$ and of rank 2 sheaves on ruled surfaces had already been proven in [Ba] [BS] [Br] [DL] [E] [ES] [HS], [Ho], [Hu1], [Hu2] (and perhaps elsewhere).

Proposition 3.1. Let $\pi: S \to C$ be a ruled surface, and let $f \in \text{NS}(S)$ be the numerical class of a fiber of $\pi$. Then $\text{TF}_S(r, c_1, c_2)$ has a unique prioritary component if $r$ divides $c_1 f$ and $2rc_2 \geq (r-1)c_1^2$, or if $r$ does not divide $c_1 f$. Otherwise it has no prioritary components. The prioritary component is smooth of dimension $-\chi(E, E)$.

Proof. The uniqueness and smoothness of the prioritary component was proven in [W1] Proposition 2. Since by definition a prioritary sheaf $E$ satisfies $\text{Ext}^2(E, E(-f_p)) = 0$ for all $p \in C$, it also satisfies $\text{Ext}^2(E, E) = 0$. So the prioritary component has dimension $-\chi(E, E)$ according to $[H]$. For existence of a prioritary sheaf $E$, note that $2rc_2 - (r-1)c_1^2$ is invariant under twist as is the residue of $c_1 f$ modulo $r$. Then by replacing $E$ by a twist $E(n)$ if necessary, we may assume that $d := -c_1 f$ satisfies $0 \leq d < r$. In the proof of [W1] Proposition 2 it was shown that a general prioritary sheaf with such a $c_1$ fits into an exact sequence

$$0 \to \pi^*(K) \to E \to \pi^*(L) \otimes \Omega_S(1) \to 0 \quad (5)$$

where $K$ is a vector bundle on $C$ of rank $r - d$ and $L$ a coherent sheaf on $C$ of rank $d$. Let $k = \text{deg}(K)$ and $l = \text{deg}(L)$. Write $h = c_1(O(1))$ so that $\{h, f\}$ is a basis of $\text{NS}(S)$. Then $E$ has rank $r$ and Chern classes $c_1 = (k + l)f - dh$ and $c_2 = \frac{1}{2}d(d - 1)h^2 - (k + l)d + l$. So to finish the proof of the lemma we need to show that if $0 < d < r$, then there exist prioritary sheaves of the form $(5)$ for all $k$ and $l$, while if $d = 0$, then there exist prioritary sheaves of that form if and only if $(k, l)$ satisfies $l \geq 0$.

If $0 < d < r$, then for any $k$ and $l$ and any locally free sheaves $K$ (resp. $L$) on $C$ of rank $r - d$ and degree $k$ (resp. rank $d$ and degree $l$), the sheaf $F := \pi^*(K) \oplus [\pi^*(L) \otimes \Omega_S(1)]$ has splitting type $O_{\mathbb{P}^1}^{-d} \oplus O_{\mathbb{P}^1}(-1)^d$ on all fibers and hence is prioritary by Lemma 2.2.

If $d = 0$, then $L$ has rank 0. So its degree $l$ must be nonnegative. Conversely if $k$ is any integer and $l \geq 0$, then an $E$ as in $(5)$ can be constructed for any locally free sheaf $K$ of rank $r$ and degree $k$ on $C$ as an elementary transform of $\pi^*(K)$ along $l$ fibers of $\pi$. Such an $E$ is prioritary by Lemma 2.2 because its restriction to the general fiber of $\pi$ is trivial. Thus for $d = 0$ there exists prioritary sheaf $E$ of the form $(5)$ for and only for those $(k, l)$ satisfying $l \geq 0$. This completes the proof of the lemma. □

Proposition 3.2. (Hirschowitz-Laszlo) The stack $\text{TF}_{\mathbb{P}^2}(r, c_1, c_2)$ has a unique prioritary component if $2rc_2 - (r-1)c_1^2 \geq -d(r-d)$ where $c_1 \equiv -d \pmod{r}$ and $0 \leq d < r$. Otherwise it has no prioritary components. The prioritary component is smooth of dimension $-\chi(E, E)$.
Proof. Let \( c_1 = mr - d \). Let \( \mu = c_1/r \) be the slope and \( \Delta = (2rc_2 - (r - 1)c_1^2)/2r^2 \) the discriminant of \( \mathcal{E} \). Then in [11] Chap. I, Propositions 1.3 and 1.5 and Théorème 3.1, it is shown that \( \text{TF}_{\mathbb{P}^2}(r, c_1, c_2) \) has a priority component if and only if the Hilbert polynomial
\[
P(n) = r \left( \frac{1}{2}(\mu + n + 2)(\mu + n + 1) - \Delta \right)
\]
is nonpositive for some integer \( n \), and that in that case the priority component is unique and smooth. The dimension of such a component is again \( -\chi(\mathcal{E}, \mathcal{E}) \) by [1] because the priority condition implies \( \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 0 \).

We show that the Hilbert polynomial criterion of [11] is equivalent to the criterion asserted by the lemma. But \( P(n) - P(n - 1) = \mu + n + 1 = m - \frac{4}{3} + n + 1 \) is nonnegative if and only if \( n \geq -m \). So \( \min_{n \in \mathbb{Z}} P(n) = P(-m - 1) = r \left( \frac{1}{2}(1 - \frac{4}{3})(-\frac{4}{3}) - \Delta \right) \), and this is nonpositive if and only if \( 2r^2\Delta \geq -d(r - d) \).

We recall the Riemann-Roch formula for a coherent sheaf \( \mathcal{E} \) of rank \( r \) and Chern classes \( c_1 \) and \( c_2 \) on a surfaces \( S \):
\[
\chi(\mathcal{E}) = r\chi(\mathcal{O}_S) + \frac{1}{2} c_1(c_1 - K_S) - c_2.
\] (6)

**Lemma 3.3.** Let \( \pi: S \to C \) be a ruled surface or let \( S \) be \( \mathbb{P}^2 \). Suppose \( \mathcal{E} \) is a priority sheaf on \( S \) of rank \( r \geq 2 \) such that \( H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0 \). Let \( H \) be a very ample divisor on \( S \).

(i) If \( \mathcal{F} \) is a general priority sheaf of rank \( r \) and Chern classes \( c_1 = c_1(\mathcal{E}) \) and \( c_2 \geq c_2(\mathcal{E}) \) such that \( \chi(\mathcal{F}) \geq 0 \), then \( H^1(\mathcal{F}) = H^2(\mathcal{F}) = 0 \).

(ii) If in addition \( H^1(\mathcal{E}(H)) = H^2(\mathcal{E}(H)) = 0 \) and \( \chi(\mathcal{E}(H)) \geq \chi(\mathcal{E}) \), then for all \( n \geq 2 \) the sheaf \( \mathcal{F}(nH) \) is generated by global sections and its general section has degeneracy locus of codimension 2.

**Proof.** (i) By semicontinuity it is enough to exhibit one such \( \mathcal{F} \). We go by induction on \( c_2 \). If \( c_2 = c_2(\mathcal{E}) \), we may take \( \mathcal{F} = \mathcal{E} \). If \( c_2 > c_2(\mathcal{E}) \), let \( \mathcal{G} \) be a priority sheaf of rank \( r \) and Chern classes \( c_1 \) and \( c_2 - 1 \) with \( H^1(\mathcal{G}) = H^2(\mathcal{G}) = 0 \). By [8] we have \( \chi(\mathcal{G}) = \chi(\mathcal{F}) + 1 > 0 \). So \( \mathcal{G} \) must have a nonzero global section \( s \). If \( x \in S \) is a general point of \( S \) and \( \mathcal{G} \otimes k(x) \to k(x) \) a general one-dimensional quotient of the fiber of \( \mathcal{G} \) at \( x \), then the image of \( s \) in \( k(x) \) is nonzero. So if \( \mathcal{F} \) is the kernel
\[
0 \to \mathcal{F} \to \mathcal{G} \to k(x) \to 0,
\]
then \( h^0(\mathcal{F}) = h^0(\mathcal{G}) - 1 \) and \( H^1(\mathcal{F}) = H^2(\mathcal{F}) = 0 \).

(ii) Under the added hypotheses the general \( \mathcal{F} \) also satisfies \( H^1(\mathcal{F}(H)) = H^2(\mathcal{F}(H)) = 0 \). But \( H^1(\mathcal{F}(H)) = H^2(\mathcal{F}) = 0 \) implies that \( \mathcal{F}(nH) \) is generated by global sections for all \( n \geq 2 \) by the Castelnuovo-Mumford lemma. The general section of \( \mathcal{F}(nH) \) will drop rank in codimension 2 by Bertini’s theorem.

**Lemma 3.4.** Let \( \mathcal{F} \) be a generic priority sheaf of rank 2 and Chern classes \( c_1 \geq -4 \) and \( c_2 \) on \( \mathbb{P}^2 \). The two conditions (a) \( H^1(\mathcal{F}) = H^2(\mathcal{F}) = 0 \) and (b) \( \mathcal{F}(3) \) has a section with degeneracy locus of codimension 2 hold if and only if \( \chi(\mathcal{F}) \geq 0 \).
such that the sheaf of the given rank and Chern classes is of the form
\[ S \]
Moreover, the extension is uniquely determined by \( E \) either of a fiber of \( S \) with respect to a suitable polarization of the surface.

The next two lemmas show that if \( S \) is the gener ic sheaf of an irreducible component of the stack of torsion-free rank 2 sheaves on \( S \).

**Lemma 3.5.** Let \( F \) be a generic primary sheaf of rank 2 and Chern classes \( c_1 = (a_1, a_2) \) and \( c_2 \) on \( P^1 \times P^1 \). Suppose the \( a_i \geq -2 \).

(i) If \( a_2 \) is even, then the two conditions (a) \( H^1(F) = H^2(F) = 0 \) and (b) \( F(2, 2) \) has a section with degeneracy locus of codimension 2 hold if and only if \( \chi(F) \geq 0 \).

(ii) If \( a_2 \) is odd, then (a) and (b) hold if and only if one has both \( \chi(F) \geq 0 \) and either \( c_2 \geq \frac{1}{2}a_1(a_2 - 1) - 1 \) or \( (a_1, a_2, c_2) = (a_1, -1, -a_1 - 2) \).

Proof. (i) We apply Lemma 3.3 with \( E \) either \( O(\frac{a_1}{2}, \frac{a_2}{2}) \) or \( O(\frac{a_1-1}{2}, \frac{a_2}{2}) \) or \( O(\frac{a_1+1}{2}, \frac{a_2}{2}) \).

(ii) Before beginning, recall that if \( a_2 = 2b - 1 \) is odd, then by (3) the general primary sheaf of the given rank and Chern classes is of the form
\[ 0 \to O(a_1 - p, b) \to F \to O(p, b - 1) \to 0 \]
with \( p \) determined by \( c_2 = (a_1 - p)(b - 1) + pb = \frac{1}{2}a_1(a_2 - 1) + p \).

Now suppose that (a) and (b) hold. Then clearly \( \chi(F) = h^0(F) \geq 0 \). If \( p \geq -1 \), then \( c_2 \geq \frac{1}{2}a_1(a_2 - 1) - 1 \) as desired. If on the other hand \( p \leq -2 \), then the sequence splits. So if (a) and (b) hold, then \( H^1(O(p, b - 1)) = 0 \) while \( O(p + 2, b + 1) \) is generated by global sections. These are possible simultaneously only if \( p = -2 \) and \( b = 0 \).

Conversely, if \( \chi(F) \geq 0 \) and \( c_2 \geq \frac{1}{2}a_1(a_2 - 1) - 1 \), then (a) and (b) hold by Lemma 3.3 using \( E = O(a_1 + 1, b) \oplus O(-1, b - 1) \). If \( (a_1, a_2, c_2) = (a_1, -1, -a_1 - 2) \), then one may pick \( F = O(a_1 + 2, 0) \oplus O(-2, -1) \). \( \square \)

**4 Nonprimary Components**

In this section we study nonprimary components of \( TF_S(r, c_1, c_2) \) and of \( W^0_N(E) \). According to Proposition 3.4 on a ruled surface \( \pi: S \to C \) or on \( P^2 \) with \( f \) denoting the numerical class either of a fiber of \( \pi \) or of a line in \( P^2 \), the general member of any nonprimary component of \( TF_S(r, c_1, c_2) \) a nonprimary generic extension of twisted ideal sheaves, i.e. an extension
\[ 0 \to I_{Z_1}(L_1) \to E \to I_{Z_2}(L_2) \to 0 \]
such that the \( O_S(L_i) \) are generic line bundles having \( L_1f > L_2f + 1 \) and the \( Z_i \) are generic sets of \( n_i \) points in \( S \). In addition, the proposition says that
\[ \chi(I_{Z_1}(L_1), I_{Z_2}(L_2)) = \chi(O(L_2 - L_1)) - n_1 - n_2 \geq 0. \]}

Moreover, the extension is uniquely determined by \( E \) since it defines the Harder-Narasimhan filtration of \( E \) with respect to a suitable polarization of the surface.

The next two lemmas show that if \( S \) is a nonprimary extension of twisted ideal sheaves satisfying (8) is the generic sheaf of an irreducible component of the stack of torsion-free rank 2 sheaves on \( S \).
Lemma 4.1. Suppose either that \( \pi : S \to C \) is a birationally ruled surface or \( S \) is \( \mathbb{P}^2 \). If a nonprioritary generic extension of twisted ideal sheaves \( \mathcal{E} \) as in (7) specializes to another nonprioritary generic extension of twisted ideal sheaves \( 0 \to \mathcal{I}_{Z_1}(L_1') \to \mathcal{E}' \to \mathcal{I}_{Z_2}(L_2') \to 0 \), then

(i) \( \chi(\mathcal{I}_{Z_1}(L_1'), \mathcal{I}_{Z_2}(L_2')) < \chi(\mathcal{I}_{Z_1}(L_1), \mathcal{I}_{Z_2}(L_2)) \) and

(ii) there exists an effective divisor \( \Gamma \) on \( S \) such that \( -\Gamma \cdot \Gamma > (L_1 - L_2 + K_S) \cdot \Gamma \).

Proof. (i) Let \( \text{FiltCoh}_S \) be the stack parametrizing filtered coherent sheaves \( \mathcal{F}_1 \subset \mathcal{F} \). Because the tangent space for automorphisms of \( \mathcal{F}_1 \subset \mathcal{F} \) (resp. the tangent space for deformations of \( \mathcal{F}_1 \subset \mathcal{F} \), resp. the obstruction space for deformations of \( \mathcal{F}_1 \subset \mathcal{F} \)) is \( \text{Ext}^i(\mathcal{F}, \mathcal{F}) \) for \( i = 0 \) (resp. \( i = 1 \), resp. \( i = 2 \)), one has

\[
-\chi_-(\mathcal{F}, \mathcal{F}) \leq \dim_{[\mathcal{F}_1 \subset \mathcal{F}]} \text{FiltCoh}_S \leq -\chi_-(\mathcal{F}, \mathcal{F}) + \dim \text{Ext}^2(\mathcal{F}, \mathcal{F})
\]

where \( \chi_-(\mathcal{F}, \mathcal{F}) = \sum (-1)^i \dim \text{Ext}^i(\mathcal{F}, \mathcal{F}) \). The forgetful functor \( \text{FiltCoh}_S \to \text{Coh}_S \) defined by \( [\mathcal{F}_1 \subset \mathcal{F}] \mapsto [\mathcal{F}] \) induces maps on infinitesimal automorphism, tangent, and obstruction spaces \( \text{Ext}^i(\mathcal{F}, \mathcal{F}) \to \text{Ext}^i(\mathcal{F}, \mathcal{F}) \). So if \( \text{Hom}_+(\mathcal{F}, \mathcal{F}) := \text{Hom}(\mathcal{F}_1, \mathcal{F}/\mathcal{F}_1) = 0 \), then the morphism \( \text{FiltCoh}_S \to \text{Coh}_S \) is unramified at \( [\mathcal{F}_1 \subset \mathcal{F}] \), and \( \text{FiltCoh}_S \) can be viewed as more or less a locally closed substack of \( \text{Coh}_S \) in a neighborhood of \([\mathcal{F}]\). In our case the substack \( \mathcal{F}_1 = \mathcal{I}_{Z_1}(L_1) \) is unique, so \( \text{FiltCoh}_S \) is a locally closed substack of \( \text{Coh}_S \) in a neighborhood of \([\mathcal{F}]\).

Thus the dimension of the locally closed substack of torsion-free sheaves numerically equivalent to \( \mathcal{E} \) which admit a filtration with the subsheaf numerically equivalent to \( \mathcal{I}_{Z_1}(L_1) \) (resp. to \( \mathcal{I}_{Z_1}(L_1') \)) and with \( \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 0 \) is

\[
-\chi_-(\mathcal{E}, \mathcal{E}) = -\chi(\mathcal{E}, \mathcal{E}) + \chi(\mathcal{I}_{Z_1}(L_1), \mathcal{I}_{Z_2}(L_2))
\]

(resp. \( -\chi(\mathcal{E}, \mathcal{E}) + \chi(\mathcal{I}_{Z_1}(L_1'), \mathcal{I}_{Z_2}(L_2')) \)). If the former substack contains the latter in its closure, its dimension must be larger.

(ii) As \( \mathcal{E} \) specializes to \( \mathcal{E}' \), its subsheaf \( \mathcal{I}_{Z_1}(L_1) \) specializes to a subsheaf of \( \mathcal{E}' \). Because this subsheaf destabilizes \( \mathcal{E}' \), it must be contained in \( \mathcal{I}_{Z_1}(L_1') \). Hence \( \mathcal{O}_S(L_1) \) specializes to a line bundle of the form \( \mathcal{O}_S(L_1' - \Gamma) \) with \( \Gamma \) an effective divisor, and \( \mathcal{O}_S(L_2) \) specializes to \( \mathcal{O}_S(L_2' + \Gamma) \).

Since \( L_2' - L_1' \equiv L_2 - L_1 - 2\Gamma \), the Riemann-Roch formula leads to

\[
\chi(\mathcal{O}(L_2' - L_1')) = \chi(\mathcal{O}(L_2 - L_1)) + (2(L_1 - L_2 + \Gamma) + K_S) \cdot \Gamma.
\]

We also have

\[
n_1 + n_2 + (L_1 \cdot L_2) = c_2(\mathcal{E}) = c_2(\mathcal{E}') = n_1' + n_2' + (L_1' \cdot L_2'),
\]

from which we see that

\[
n_1' + n_2' = n_1 + n_2 + (L_1 - L_2 + \Gamma) \cdot \Gamma.
\]

Thus

\[
\chi(\mathcal{I}_{Z_1}(L_1), \mathcal{I}_{Z_2}(L_2)) = \chi(\mathcal{I}_{Z_1}(L_1), \mathcal{I}_{Z_2}(L_2)) + (L_1 - L_2 + \Gamma + K_S) \cdot \Gamma.
\]

Because of (i) this now implies the lemma. \( \square \)
Lemma 4.2. Suppose either that $\pi: S \to C$ is a ruled surface without curves of negative self-intersection or that $S$ is $P^2$. If a nonprioritary generic extension of twisted ideal sheaves $E$ as in (4) specializes to another generic extension of twisted ideal sheaves $E'$, then $\chi(O(L_2-L_1)) \leq 0$.

Proof. Because $E$ is not prioritary, the restriction of $O(L_2-L_1)$ to a general fiber of $\pi$ is a general line of $P^2$ of degree $0$. Hence the relative Harder-Narasimhan filtration to a generic line of $P^2$ of $E$ is the sheaf corresponding to a generic point of a locally closed substack of $TF_S(2,c_1,c_2)$ has a unique component whose general member $E$ is a nonprioritary generic extension of twisted ideal sheaves

$$0 \to I_{Z_1}(c_1-D) \to E \to I_{Z_2}(D) \to 0$$

(9)

with $\deg(Z_i) = n_i$ if and only if $Df \leq -1 + \frac{1}{2}c_1f$, and the $n_i$ are nonnegative and satisfy $n_1+n_2 = c_2-D(D-c_1) \leq \chi(O_S(2D-c_1))$. Such a component of $TF_S(2,c_1,c_2)$ has dimension $-\chi(E,E) + \chi(O_S(2D-c_1)) - n_1-n_2$ and generic embedding codimension $n_2+h^1(O_S(2D-c_1))$.

Proof. If $E$ is a generic nonprioritary sheaf, then its restriction to a general fiber $F$ of $\pi$ (resp. to a generic line of $P^2$) must be of the form $E|_F \cong O_F(a) \oplus O_F(b)$ with $a \geq b \geq 2$ by Lemma 2.2 (resp. by Chap. I, Proposition 1.2). Hence the relative Harder-Narasimhan filtration $E$ which was described before Lemma 2.2 (resp. the Harder-Narasimhan filtration of $E$ on $P^2$) must be of the form $0 \subset I_{Z_1}(c_1-D) \subset E$ with $E/I_{Z_1}(c_1-D) \cong I_{Z_2}(D)$ for some divisor $D$ on $S$ and some 0-dimensional subschemes $Z_i \subset S$ such that $(c_1-D)f \geq Df + 2$, or $Df \leq -1 + \frac{1}{2}c_1f$. Clearly one has $n_i := \deg(Z_i) \geq 0$ and $c_2 = D(c_1-D) + n_1 + n_2$. Lemma 2.2 shows that Proposition 4.1 is applicable to the filtered sheaf $0 \subset I_{Z_1}(c_1-D) \subset E$. So $\chi(I_{Z_1}(c_1-D),I_{Z_2}(D)) = \chi(O_S(2D-c_1)) - n_1 - n_2 \geq 0$. Thus to any nonprioritary irreducible component of $TF_S(2,c_1,c_2)$ there is an associated triple $(D,n_1,n_2)$ satisfying the asserted numerical conditions.

Conversely suppose $(D,n_1,n_2)$ satisfy all the numerical conditions. Let $Z_i$ be a general set of $n_i$ points on $S$ and let $E$ be a generic extension as in (4). Then $E$ cannot be a specialization of another nonprioritary generic extension $0 \to I_{Z_1}(c_1-D) \to E' \to I_{Z_2}(D') \to 0$ because in that case Lemma 4.2(i) would imply

$$\chi(O_S(2D'-c_1)) - n_1' - n_2' > \chi(O_S(2D-c_1)) - n_1 - n_2 \geq 0$$

contradicting Lemma 4.2. Nor can $E$ be a specialization of a generic prioritary sheaf because it is the sheaf corresponding to a generic point of a locally closed substack of $TF_S(2,c_1,c_2)$ whose dimension was calculated in the proof of Lemma 4.2(i) as $-\chi(E,E) + \chi(O_S(2D-c_1)) - n_1 - n_2$. This is at least $-\chi(E,E)$, the dimension of the prioritary component. So $E$ is the generic sheaf of an irreducible component of $TF_S(2,c_1,c_2)$ of dimension $-\chi(E,E) + \chi(O_S(2D-c_1)) - n_1 - n_2$. 

The embedding codimension is the dimension of the cokernel of the map $\alpha$ between the tangent spaces of the stack of filtered sheaves and the stack of unfiltered sheaves which is given by

$$\text{Ext}^1_\alpha(\mathcal{E}, \mathcal{E}) \xrightarrow{\alpha} \text{Ext}^1(\mathcal{I}_{Z_1}(c_1 - D), \mathcal{I}_{Z_2}(D)) \to 0.$$ 

But since $\text{Hom}(\mathcal{I}_{Z_1}(c_1 - D), \mathcal{I}_{Z_2}(D)) = 0$, the dimension of $\text{cok}(\alpha)$ is the difference between the two numbers

$$\dim \text{Ext}^2(\mathcal{I}_{Z_1}(c_1 - D), \mathcal{I}_{Z_2}(D)) = h^0(\mathcal{I}_{Z_1}(c_1 - 2D + K_S)) = [h^2(\mathcal{O}(2D - c_1)) - n_1]_+,$$

$$\chi(\mathcal{I}_{Z_1}(c_1 - D), \mathcal{I}_{Z_2}(D)) = [h^2(\mathcal{O}(2D - c_1)) - n_1] - [h^1(\mathcal{O}(2D - c_1)) + n_2].$$

Because the $\chi$ is nonnegative, we see that $h^2(\mathcal{O}(2D - c_1)) - n_1 \geq 0$, and that therefore the difference between the two numbers is $n_2 + h^1(\mathcal{O}(2D - c_1))$. \(\square\)

**Remark 4.4.** The components of $\text{TF}_{\mathbb{P}^2}(2, c_1, c_2)$ containing locally free sheaves were already classified by Strømme in \(\square\) Theorem 3.9, but he made one minor error with the embedding codimensions. The prioritary components of $\text{TF}_{\mathbb{P}^2}(2, c_1, \frac{1}{3}c_1^2 + 1)$ are generically smooth like all prioritary components. But they appear in Strømme’s classification in \(\square\) Theorem 3.9 as the component with $(d, c_1, c_2) = (0, 0, 1)$ which was said to be nonreduced with generic embedding codimension 1. The computation of the $h^i(\mathcal{E}nd(\mathcal{E}))$ in \(\square\) Proposition 1.4 is wrong in that single case.

We now consider what the classification of generic rank 2 sheaves entails for Brill-Noether loci. For the sake of simplicity, we will restrict ourselves to those surfaces covered by Lemma \(\square\) which also have vanishing irregularity, thus $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, so that we do not need to analyze nongeneric line bundles which might have more cohomology than the corresponding generic line bundles.

**Lemma 4.5.** Let $S$ be $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{P}^2$) and let $f \in \text{NS}(S)$ be the class of a fiber of $\text{pr}_1$ (resp. a line). Let $\mathcal{F}$ be a nonprioritary generic extension of twisted ideal sheaves

$$0 \to \mathcal{I}_{Z_1}(c_1 - D) \to \mathcal{F} \to \mathcal{I}_{Z_2}(D) \to 0$$

(10)

with $c_1 - K_S$ effective, $Df \leq -1 + \frac{1}{2}c_1 f$, and $\chi(\mathcal{O}_S(2D - c_1)) \geq 0$. Let $n_i := \text{deg}(Z_i)$. Then the two conditions (a) $H^1(\mathcal{F}) = H^2(\mathcal{F}) = 0$ and (b) $\mathcal{F}(-K_S)$ has a section with degeneracy locus of codimension 2 hold if and only if the three conditions hold: (i) $\chi(\mathcal{F}) \geq 0$, (ii) $D - K_S$ is an effective and irreducible divisor class, and (iii) $n_2 \leq h^0(\mathcal{O}_S(D))$.

**Proof.** We will prove the lemma for $\mathbb{P}^1 \times \mathbb{P}^1$ only. The proof for $\mathbb{P}^2$ is similar and actually simpler.

Let $(a_1, a_2)$ be the bidegree of $c_1 - D$ and $(b_1, b_2)$ the bidegree of $D$. We claim that the hypotheses of the lemma imply that $a_1 \geq 0$ and $a_2 \geq 0$. To see this first note that the effectiveness of $c_1 - K_S$ is equivalent to $a_1 + b_1 \geq -2$ and $a_2 + b_2 \geq -2$. The condition $Df \leq -1 + \frac{1}{2}c_1 f$ is equivalent to $b_2 \leq -1 + \frac{1}{2}(a_2 + b_2)$, or $a_2 - b_2 \geq 2$. Adding gives $a_2 \geq 0$ as claimed. Also we have

$$0 \leq \chi(\mathcal{O}_S(2D - c_1)) = (b_1 - a_1 + 1)(b_2 - a_2 + 1).$$
Since $b_2 - a_2 + 1 < 0$, this gives $b_1 - a_1 + 1 \leq 0$. Thus $a_1 > b_1$. Adding this to $a_1 + b_1 \geq -2$ now gives $a_1 \geq 0$ as claimed.

The fact that $c_1 - D$ has bidegree $(a_1, a_2)$ with $a_1 \geq 0$ and $a_2 \geq 0$ implies that $H^i(O_S(c_1 - D)) = 0$ for $i = 1, 2$ and that $O_S(c_1 - D + K_S) \not\cong O_S$.

Now suppose that (a) and (b) hold. Then $\chi(F) = h^0(F) \geq 0$, whence (i). To prove conditions (ii) and (iii), note first that $D \cdot h^0(F)$ holds for $F$ by a simple application of the Riemann-Roch formula for a line bundle on $D$ since according to (ii) $F$ and $b$ is very ample, or that $D$ is effective. Thus (a) holds in the special case where $F = K_S$ is base-point-free, so $D - K_S$ is of bidegree $(0, 1)$, and if also $D - K_S$ has bidegree $(b_1 + 2, b_2 + 2)$ is very ample, or that $(b_1, b_2)$ is $(−1, d)$ or $(d, −1)$ with $d \leq −2$. But if $(b_1, b_2)$ had one of these last two forms, and if also $d \leq −3$, then $O_S(D - K_S) = O_S(b_1 + 2, b_2 + 2)$ would not have any global sections. Hence all sections of $F = K_S$ would lie in $I_{Z_2}(c_1 - D - K_S)$. But we have shown that the line bundle $O_S(c_1 - D - K_S)$ is always nontrivial. So all global sections of $F = K_S$ would degenerate along a nontrivial curve, contradicting (b). Hence the only possible cases where (a) and (b) hold with $D - K_S$ not very ample are the cases where $D - K_S$ is of bidegree $(0, 1)$ or $(1, 0)$, whence (ii). Thus (a) and (b) imply (i), (ii) and (iii).

Conversely, suppose (i), (ii) and (iii) hold for $F$. We begin by proving (a) in the special case where $n_1 = 0$. Condition (ii) implies that either both $b_i \geq −1$ or one $b_i = −1$. Therefore $H^1(O_S(D)) = H^2(O_S(D)) = H^2(I_{Z_2}(D)) = 0$. Because $Z_2$ consists of $n_2 \leq h^0(O_S(D))$ generic points of $S$ by condition (iii), we have $H^1(I_{Z_2}(D)) = 0$ also. And we have already shown that $H^i(O_S(c_1 - D)) = 0$ for $i = 1, 2$. It now follows by (i) that $H^1(F) = H^2(F) = 0$.

If $n_1 > 0$, then we may prove (a) by induction on $n_1$ using the same method as in the proof of Lemma 3.3(i).

For (b) let $H$ be a divisor of bidegree $(1, 1)$. Then

$$\chi(F(H)) = \chi(F) + (c_1 + 2H)H + 2 > \chi(F) \geq 0$$

since $c_1 + 2H = c_1 - K_S$ is effective. So (i) holds for $F(H)$. Since (ii) holds for $F$, the divisor $D - K_S$ is base-point-free, so $D + H - K_S$ is very ample. Hence (ii) holds for $F(H)$. And

$$h^0(O_S(D + H)) = h^0(O_S(D)) + (D + H)H + 1 \geq h^0(O_S(D)) \geq n_2$$

since according to (ii) $D + H$ is either effective or of bidegree $(0, −1)$ or $(−1, 0)$. So (iii) also holds for $F(H)$. By what we have already verified, the fact that (i), (ii) and (iii) all hold for $F$ and $F(H)$ implies that (a) also holds for $F$ and $F(H)$. Hence $H^1(F(H)) = H^2(F) = 0$. So $F(2H) = F = K_S$ is generated by global sections by the Castelnuovo-Mumford lemma. Condition (b) now follows from Bertini’s theorem.

5 Proofs of the Theorems

Proof of Theorems 0.1 and 0.2. Theorem 0.1 follows from the classification of the prioritarity components of $T_F(S_2, c_1, c_2)$ in Lemma 3.1 and the classification of the nonprioritary components of $T_F(S_2, c_1, c_2)$ in Lemma 3.3. Note that the expression $\chi(O_S(c_1 - D) + D(2D - 2c_1 - K_S)$ appearing the Theorem 0.1 is equal to the expression $\chi(O_S(2D - c_1))$ appearing in Lemma 3.3 by a simple application of the Riemann-Roch formula for a line bundle on $S$. 

□
Theorem 0.2 follows from Lemmas 3.2 and 4.3 in the same manner.

Proof of Theorem 0.3. According to the argument given before the statement of Theorem 0.3, there is a correspondence between irreducible components of $W_N^0(E)$ correspond to the irreducible components of $TF_S(2, E - K_S, N)$ whose general member $\mathcal{E}$ satisfies $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$ and has a section with zero locus of codimension 2. These irreducible components of $TF_S(2, E - K_S, N)$ may either be nonprioritary or prioritary. According to Theorems 0.1 and 0.2, the nonprioritary components of $TF_S(2, E - K_S, N)$ correspond to pairs $(D, n) \in NS(S) \times \mathbb{Z}$ such that $Df \leq -1 + \frac{1}{2}((E - K_S)f)$ and $0 \leq n \leq N + D(D - E + K_S)$ and $N \leq \chi(\mathcal{O}_S(-E + K_S)) + D(D - E) = \chi(\mathcal{O}_S(E)) - D(E - D)$. According to Lemma 1.3, the general member of such an irreducible component has $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$ and a section with zero locus of codimension 2 if and only if (i) $\chi(\mathcal{E}(K_S)) \geq 0$, (ii) $D$ is an effective and irreducible divisor class, and (iii) $n_2 = N + D(D - E + K_S) - n \leq \chi(\mathcal{O}_S(D + K_S))$. Since $\chi(\mathcal{E}(K_S)) = \chi(\mathcal{I}_X(E)) + 1 = \chi(\mathcal{O}_S(E)) - N + 1 > 0$, we see that the irreducible components of $W_N^0(E)$ with nonprioritary $\mathcal{E}$ are precisely the components described in part (i) of Theorem 0.3. Moreover, the geometry of $X$ can be recovered from $\mathcal{E}$, $D$ and $n$ via the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_S & \rightarrow & \mathcal{O}_S \\
\downarrow & & \downarrow \\
0 \rightarrow \mathcal{I}_{Z_1}(E - K_S - D) & \rightarrow & \mathcal{E} & \rightarrow & \mathcal{I}_{Z_2}(D) & \rightarrow & 0 \\
\parallel & & & & \downarrow & & \downarrow \\
0 \rightarrow \mathcal{I}_{Z_1}(E - K_S - D) & \rightarrow & \mathcal{I}_X(E - K_S) & \rightarrow & \mathcal{K} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

The bottom row must be a twist of the residual exact sequence for $\mathcal{I}_X(E - K_S)$ with respect to a curve $C \in |D|$. So $K = \mathcal{I}_X \cap C/(E - K_S)$. Thus $X = Z_1 \cup (X \cap C)$ with $Z_1$ a generic set of $n$ points of $S$ and $X \cap C$ a set of $N - n$ points on $C$. Thus the irreducible components of $W_N^0(E)$ such that $\mathcal{E}$ is nonprioritary are exactly those described in part (i) of Theorem 0.3. In addition, $TF_S(2, E - K_S, N)$ may have a unique prioritary component. For $\mathbb{P}^2$ this component exists if and only if $N \geq 3(e + 4)(e + 2)$ by Theorem 0.2. Its general member $\mathcal{E}$ always satisfies $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$ and has a section with zero locus of codimension 2 according to Lemma 3.4 because $\chi(\mathcal{E}(K_S)) = \chi(\mathcal{O}_S(E)) - N + 1 > 0$. If $S$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_2$ is even, the prioritary component exists if and only if $N \geq 3(e_1 + 2)(e_2 + 2)$ according to Theorem 0.1 and its general member always satisfies $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$ and has a section with zero locus of codimension 2 according to Lemma 3.3 because $\chi(\mathcal{E}(K_S)) > 0$. If $S$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $e_2$ is odd, then the prioritary component exists for all $N$ according to Theorem 1.1. But according to Lemma 3.3, its general member $\mathcal{E}$ satisfies $H^1(\mathcal{E}(K_S)) = H^2(\mathcal{E}(K_S)) = 0$ and has a section with zero locus of codimension 2 only if either $N \geq \frac{3}{2}(e_1 + 2)(e_2 + 1) + 1$ or $(e_1, e_2, N) = (1, 1, e_1 + 2)$. This gives all the components described in part (ii) of Theorem 0.3.

The dimension of a component of $W_N^0(E)$ is the dimension of the corresponding component of $TF_S(2, E - K_S, N)$ plus $h^0(\mathcal{E}) - 1$ (for the choice of a section of $\mathcal{E}$ modulo $k^\times$) plus 1 (to cancel the negative contribution of $\dim \text{Aut}(\mathcal{I}_X(E))$ in the stack computations). Hence the
prioritary components have dimension $-\chi(E, E) + \chi(E)$ which a straightforward Riemann-Roch computation shows is $2N - (\chi(O_S(E) - N + 1)$. Since $\dim \text{Hilb}^N(S) = 2N$, this is the asserted codimension $\chi(O_S(E) - N + 1)$. The nonprioritary components of $\text{TF}_S(2, E - K_S, N)$ have dimensions greater by $\chi(O_S(E)) + D(D - E) - N$. So the nonprioritary components of $W^0_N(E)$ have codimensions $D(E - D) + 1$.

\[\square\]

References

[Ba] Barth, W.: Moduli of vector bundles on the projective plane. Invent. math. 42, 63-91 (1977).

[BS] Brünzănescu, V., Stoia, M.: Topologically Trivial Vector Bundles on Ruled Surfaces II. Springer Lect. Notes Math. 1056, 34-46 (1984).

[Br] Brosius, J. E.: Rank-2 Vector Bundles on a Ruled Surface I, II. Math. Ann. 265, 155-168 (1983), and 266, 199-214 (1983).

[C] Coppo, M.-A.: Familles maximales de systèmes de points surabondants dans $\mathbb{P}^2$. Math. Ann. 291, 725-735 (1991).

[DL] Drezet, J.-M., Le Potier, J.: Fibrés stables et fibrés exceptionnels sur $\mathbb{P}^2$. Ann. scient. Ec. Norm. Sup. 18, 193-244 (1985).

[E] Ellingsrud, G.: Sur l’irréductibilité du module de fibrés stables sur $\mathbb{P}^2$. Math. Z. 182, 189-192 (1983).

[ES] Ellingsrud, G., Strømme, S. A.: On the moduli space for stable rank 2 vector bundles on $\mathbb{P}^2$. preprint, Oslo 1979.

[Ha] Hartshorne, R.: Algebraic Geometry (Graduate Texts in Mathematics vol. 52) New York Heidelberg Berlin: Springer-Verlag 1977.

[HL] Hirschowitz, A., Laszlo, Y.: Fibrés génériques sur le plan projectif. Math. Ann. 297, 85-102 (1993).

[HS] Hoppe, H. J., Spindler, H.: Modulräume stabiler 2-Bündel auf Regelflächen. Math. Ann. 249, 127-140 (1980).

[Ho] Hoppe, H. J.: Modulräume stabiler Vektorraumbündel vom Rang 2 auf rationalen Regelflächen. Math. Ann. 264, 227-239 (1983).

[Hu1] Hulek, K.: Stable rank 2 vector bundles on $\mathbb{P}^2$ with $c_1$ odd. Math. Ann. 242, 241-266 (1979).

[Hu2] Hulek, K.: On the classification of stable rank $k$ vector bundles over the projective plane. In: Hirschowitz, A. (ed.): Vector Bundles and Differential Equations (Progress in Mathematics vol. 7, pp. 113-144) Basel Stuttgart Boston: Birkhäuser 1980.

[LM] Laumon, K., Moret-Bailly, L.: Champs algébriques. Preprint, Orsay 1992.
[S] Stromme, S. A.: Deforming Vector Bundles on the Projective Plane. Math. Ann. 263, 385-397 (1983).

[W1] Walter, C.: Irreducibility of Moduli Spaces of Vector Bundles on Birationally Ruled Surfaces. Preprint, Nice 1993.

[W2] Walter, C.: On the Harder-Narasimhan Filtration for Torsion-Free Sheaves on $\mathbb{P}^2$: I. Preprint, Nice 1993.