Ground-state Stabilization of Open Quantum Systems by Dissipation

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Abstract

Control by dissipation, or environment engineering, constitutes an important methodology within quantum coherent control which was proposed to improve the robustness and scalability of quantum control systems. The system-environment coupling, often considered to be detrimental to quantum coherence, also provides the means to steer the system to desired states. This paper aims to develop the theory for engineering of the dissipation, based on a ground-state Lyapunov stability analysis of open quantum systems via a Heisenberg-picture approach. Algebraic conditions concerning the ground-state stability and scalability of quantum systems are obtained. In particular, Lyapunov stability conditions expressed as operator inequalities allow a purely algebraic treatment of the environment engineering problem, which facilitates the integration of quantum components into a large-scale quantum system and draws an explicit connection to the classical theory of vector Lyapunov functions and decomposition-aggregation methods for control of complex systems. The implications of the results in relation to dissipative quantum computing and state engineering are also discussed in this paper.

Key words: Open quantum systems; Lyapunov stability; Control by dissipation

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Control of quantum systems lies at the core of the quantum technology [41,61], while stability analysis provides the appropriate tool for the systematic development of quantum control theory. The stability analysis has been used in several quantum control synthesis problems [21,9,27,19,28,42]. The applications include measurement-based feedback control and coherent control for the generation of quantum states as well as the regulation of system performance. Among all the methods for stability analysis, the Lyapunov stability approach is the most fundamental, as the energy of a quantum system is well-defined for most of the physical systems and a Lyapunov function can be easily constructed [15,40,30,36,2]. Especially, as we will demonstrate in this paper, the Lyapunov method provides a means for engineering the dissipation to be used as coherent control.

Quantum computing often involves the execution of a sequence of unitary operations on quantum systems. However, the severe decoherence associated with the quantum systems presents a major obstacle to the scalability of this approach. For this reason, methods for robust realization of unitary operations are currently under discussion. The possible plans include topological quantum computing, adiabatic quantum computing and dissipative quantum computing. Among these schemes, the adiabatic quantum computing and dissipative quantum computing have direct relevance to the stability of quantum systems. For example, in dissipative quantum computing and state engineering, dissipation is introduced as a resource to coherently control the system [39]. The idea is to consider open quantum systems, and stabilize their quantum states by engineering the system-environment interaction. Under certain conditions, the dissipation will drive the system to a target steady state regardless of the initial state. This method can be used to generate highly entangled quantum states, and perform quantum computation by encoding the computation result to the steady state of the system. Since the target state is made to be a steady state of the dissipative system, it is quite robust to external perturbations and uncertainty in initial states. From an engineering point of view, this kind of coherent control approaches can be referred to as control by dissipation. Our goal in this paper is to formulate the method of control by dissipation within the framework of ground-state stability, and then propose approaches for the synthesis of dissipation that rely on the stability analysis methods.

Stability of quantum states has been the focus of many theoretical studies. Many of them have derived sufficient conditions for convergence of quantum Markov systems to a steady state [35,71,33,12,29,24]. In particular, the stability of quantum states in a dissipative setting has been considered in [37,32,33,41]. In these studies, the target state is often explicitly given and follows a Schrödinger-picture master equation. The dissipative couplings, com-
pensated by Hamiltonian control, can generate a Markov process that converges to the target states \[37,38\]. The implementation of the system-environment couplings with the practical resources has been investigated experimentally. Dissipative engineering of several types of quantum systems has been demonstrated in recent years \[3,14,10,17,31,33\].

In this paper, we adopt an alternative path to approach the stability theory within the Heisenberg picture, where we will deal with a Lyapunov operator instead. Previously in \[24\], we have obtained general results regarding the stability of Lyapunov operators, however these results do not readily fit into the model of control by dissipation. Generally speaking, within our approach the target states are encoded as the ground states of a Lyapunov operator, and the stability problem is transformed to the problem of stabilization of the ground states of the Lyapunov operator. The formalism of Lyapunov stability can thus be conveniently introduced to engineer the desired system dissipation within this framework. This allows to derive sufficient conditions expressed in term of operator inequalities, which can be used for the synthesis of the desired system-environment coupling.

An important advantage of the Heisenberg-picture approach developed here is that the target state does not need to be given in advance. In addition to the entangled-state engineering applications in which the Lyapunov operator is chosen based on the knowledge of the target state, there exists a large class of applications where the control goals are posed as minimization of an operator while the target state is not known. For example, the problems of sequential quantum computation and the quantum satisfiability problem (SAT) \[5,22\] involve operators which play the role of cost functions. In these problems, the target states are unknown and result from computation and/or control. Moreover, the target state in these applications may be not unique. This complicates the analysis based on the conventional Schrödinger-picture approach. Therefore, the Heisenberg-picture approach extends the applicability of the control by dissipation.

One of the main contributions of this paper is concerned with the scalability of the control by dissipation, when this control method is applied to large quantum systems comprised of multiple interacting subsystems coupled with the environment. The Heisenberg-picture Lyapunov approach has an advantage in that the problem can be treated in a way that resembles the decomposition-aggregation engineering \[4,34\] for complex classical systems. Namely, a large-scale quantum system is decomposed into subsystems which are governed by individual Lyapunov operators. This allows us to establish conditions, expressed in terms of the subsystems’ Lyapunov operators, under which the quantum system is guaranteed to converge to its ground state. Here we note a similarity with the classical connective stability conditions \[34\], which have proved to be useful in the synthesis of decentralized controllers for large-scale
A typical methodology for the synthesis of dissipations involves two problems, the calculation of the stabilizing system-environment couplings and the implementation of these couplings using the available physical resources. For example, it is possible to construct a coherent optical network to realize a linear coupling [23]. Therefore, in this paper we focus on the first problem of calculation of coupling operators that render the states of the ground energy asymptotically stable. Particularly, we can apply this method to check the feasibility of the solutions proposed in [39]. It is worth mentioning that the constraints on the system-environment couplings could be greatly relaxed if Hamiltonian control is available [37,38].

The paper is organised as follows. In Section 2, we introduce the notations and the model considered in this paper. In Section 3 we present the ground-state stability analysis of Lyapunov operators. Section 4 discusses the scalability problem, where a large quantum system may be governed by more than one Lyapunov operators. Section 5 concerns with the synthesis of the dissipation. More explicitly, this section concerns with the calculation of the correct coherent couplings for the ground-state stabilization when the Lyapunov operator is given. Conclusion is given in Section 6.

2 Notations and preliminaries

Control by dissipation is implemented by coupling the systems to a collection of environments. Consider the system defined on a Hilbert space $\mathcal{H}$, and the environment on a Fock space $\mathcal{H}_B$ over $L^2(\mathbb{R}_+, dt)$ corresponding to Boson field modes. An operator $X(t)$ of the system evolves as $X(t) = U(t)\dagger (X \otimes I) U(t)$ in the Heisenberg picture, where $U(t)$ is the unitary evolution operator of the combined system. The dynamical equation for $X(t)$ is given by [25]

$$dX(t) = (-i[X, H] + \mathcal{L}(X))dt + \sum_k [L_k^\dagger(t), X(t)]dB_k(t) + \sum_k [X(t), L_k(t)]dB_k^\dagger(t),$$

with

$$\mathcal{L}(X) = \sum_k L_k^\dagger X L_k - \frac{1}{2} L_k^\dagger L_k X - \frac{1}{2} X L_k^\dagger L_k.$$

Here $H$ is the Hamiltonian of the system, $L_k$ describes the coupling between the system and environment, $B_k(\cdot)$ and $B_k^\dagger(\cdot)$ are the annihilation and creation processes defined on $\mathcal{H}_B$. The generator of this Markov process is determined by

$$\mathcal{G}(X) = -i[X, H] + \mathcal{L}(X).$$

We could assume the total system is defined on the tensor product of Hilbert spaces $\mathcal{H} = \bigotimes_i \mathcal{H}_i$, while each subsystem is defined on one of the Hilbert
We only consider finite-dimensional systems throughout this paper. In other words, \( \mathcal{H} \) is finite-dimensional. \( \mathcal{B}(\mathcal{H}) \) is the space of bounded operators on \( \mathcal{H} \) and we let \( X \in \mathcal{B}(\mathcal{H}) \). \( X^T \) denotes the transpose of \( X \) and \( X^\dagger \) is the adjoint of \( X \). An operator \( X \) is called an observable if \( X^\dagger = X \). Given a bounded observable \( X \in \mathcal{B}(\mathcal{H}) \) and a trace class operator \( \rho \) on \( \mathcal{H} \), \( \langle X \rangle_\rho \) denotes the trace of \( X \rho \), \( \langle X \rangle_\rho = \text{Tr} X \rho \). \( \langle X \rangle_\rho \) is the mean value of \( X \) evaluated at the density state \( \rho \). In conjunction with the Heisenberg picture dynamics, the evolution of the density state \( \rho_t \) in the Schrödinger picture is given by

\[
\dot{\rho}_t = -i[H, \rho_t] + \sum_k L_k \rho_t L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_t - \frac{1}{2} \rho_t L_k^\dagger L_k.
\]

(4)

The notation \( X \geq 0 \) (\( X \leq 0 \)) means the operator \( X \) is a positive (negative) semidefinite operator. We write \( X > 0 \) if \( X \) is positive definite. Also, we will use the notation \( X \succeq 0 \) for positive semidefinite operators \( X \) whose smallest eigenvalue is equal to 0.

We recall the definition of the Lyapunov operator [24]:

**Definition 1** A quantum Lyapunov operator \( V \) is an observable (a self-adjoint operator) on a Hilbert space \( \mathcal{H} \) for which the following properties hold:

(i) \( V \succeq 0 \).

(ii) \( \mathcal{G}(V) \leq 0 \).

One natural choice of the Lyapunov operator is the energy operator of the system. For example, the Lyapunov operator can be defined by offsetting a system Hamiltonian \( H \) as \( V = H - d \succeq 0 \), where \( d \) is the smallest eigenvalue of \( H \).\(^1\)

In this paper, we neglect the unitary evolution in (3) by assuming

\[
[X, H] = 0
\]

throughout the paper, which is a common practice in dissipation engineering [39] if no additional Hamiltonian control is used [38]. This allows us to focus entirely on the analysis and synthesis of the effects associated with the environment. Under the assumption (5), the expression (3) for the system generator is simplified into

\[
\mathcal{G}(X) = \sum_k L_k^\dagger X L_k - \frac{1}{2} L_k^\dagger L_k X - \frac{1}{2} X L_k^\dagger L_k.
\]

(6)

\(^1\) In accordance with the common convention of quantum physics, the identity operator is omitted here and elsewhere, i.e., \( H - d \) should be understood as \( H - dI \).
We use $\rho_0$ to denote the initial density state of the system, and $\rho_t$ to denote the system state at time $t$. Following Meyer [20], any convergence of a trajectory in the form of $\rho_t \to \rho_\infty$ should be understood as convergence of probability distributions, i.e., $\rho_t \to \rho_\infty$ means $\text{Tr} \rho_t X \to \text{Tr} \rho_\infty X$ for all bounded $X$. This is a nice convergence property for which the Prokhorov theorem and tightness work. In this paper, since $\mathcal{H}$ is finite-dimensional, any trajectory $\rho_t$ is tight [20,24], which means $\rho_t$ always admits a subsequence converging to a limit point $\rho'$.

The ground-state stability of an operator $X$ is defined using the mean of the operator:

**Definition 2** Suppose the smallest eigenvalue of an observable $X$ is $d$. $X$ is said to be asymptotically ground-state stable if

$$\langle X(t) \rangle_{\rho_0} = \langle X \rangle_{\rho_t} \to d, \quad \text{as} \ t \to \infty. \quad (7)$$

Here $\langle X \rangle_{\rho_t}$ is an alternative representation of $\langle X(t) \rangle_{\rho_0}$ in terms of $\rho_t$.

Consequently, a Lyapunov operator $V$ is asymptotically ground-state stable if

$$\langle V(t) \rangle_{\rho_0} = \langle V \rangle_{\rho_t} \to 0. \quad (8)$$

**Definition 3** The state trajectory $\rho_t$ is said to converge to a set $S$ if the limit points of $\rho_t$ are all contained in $S$.

This definition is often used to characterize the convergence to an invariant set in invariance principle [21,40,24]. Denote $Z_X = \{ \rho : \langle X \rangle_{\rho} = d \}$ to be the set of the ground states of $X$.

**Proposition 4** The state trajectory $\rho_t$ is converging to $Z_X$ if and only if $X$ is asymptotically ground-state stable.

**Proof.** If $X$ is asymptotically ground-state stable, then $\langle X \rangle_{\rho_t} \to d$ by Definition 2. Since $\rho_t$ is tight, the limit point always exists. For any limit point $\rho'$ which is the limit of a converging subsequence $\rho_{t_k}$ satisfying $\rho_{t_k} \to \rho'$, we have $\langle X \rangle_{\rho'} = \lim_{k \to \infty} \langle X \rangle_{\rho_{t_k}} = \lim_{t \to \infty} \langle X \rangle_{\rho_t} = d$ and so $\rho'$ is contained in $Z_X$.

If $\rho_t$ is converging to $Z_X$, then for any converging subsequence $\rho_{t_k}$ of $\rho_t$ we have $\langle X \rangle_{\rho_{t_k}} \to \langle X \rangle_{\rho'} = d$, where $\rho'$ is the limit point. Therefore, $\lim_{t \to \infty} \langle X \rangle_{\rho_t}$ exists and equals $d$.

We will also exploit the notion of dissipation functional:
Definition 5 \cite{[18,7]} The dissipation functional of the Markov semigroup \cite{[11]} is defined as

\[ \mathcal{D}(X) = \mathcal{G}(X^\dagger X) - \mathcal{G}(X^\dagger)X - X^\dagger \mathcal{G}(X). \] (9)

The dissipation functional characterizes the dissipation of energy. With a single coupling operator \( L \), the dissipation functional is calculated to be

\[ \mathcal{D}(X) = [L^\dagger, X][X, L], \] (10)

and hence \( \mathcal{D}(X) \geq 0 \).

The first objective of this paper is to develop the ground-state stability theory using the notion of Lyapunov operator. We will consider Lyapunov operators which govern either a single quantum system, or a subsystem of the total system. The need in such theory can be illustrated by the following result from \cite{[24]}.

Proposition 6 \cite{[24], Theorem 8} Suppose \( V \) is a Lyapunov operator of the system. The state trajectory \( \rho_t \) will converge to \( Z_V \) if \( \langle \mathcal{D}(V) \rangle_\rho > 0 \) for \( \rho \notin Z_V \) and \( [\mathcal{G}(V), V] = 0 \).

As one may see, the conditions in Proposition 6 are not fully algebraic and also they are not easy to verify. This motivates us to revisit the ground-state stability theory for Lyapunov operators, in order to derive ground-state stability conditions expressed purely in terms of operator inequalities.

The second objective of this paper is to apply this theory to stabilization of large-scale quantum systems. We consider an operator \( W = \sum_\lambda X_\lambda, X_\lambda \succeq 0, \) as a candidate for the Lyapunov operator of a large-scale system, where each \( X_\lambda \succeq 0 \) acts nontrivially on a subsystem or on a collection of subsystems. Such operator sum representations naturally arise in many problems of control by dissipation, including the preparation of multipartite entangled states, hence the theory developed in this paper is aimed at these applications. For more information about the applications of control by dissipation please refer to \cite{[39, 26, 38]}.

In general, the fact that the individual observables \( X_\lambda \) have zero eigenvalue (as implied by the notation \( X_\lambda \succeq 0, \forall \lambda \)) does not guarantee that \( W = \sum_\lambda X_\lambda \) has a zero eigenvalue; in fact \( W \) can be positive definite. In the light of the definition of the Lyapunov operator, this means that the operator sum \( W = \sum_\lambda X_\lambda \) may result in an operator which does not satisfy formally all the properties of Definition 1 (recall that by definition, Lyapunov operators have a zero eigenvalue). For this reason, we use the notation \( W, X_\lambda \) instead of \( V \) in Section 4 because there is a possibility that \( W, X_\lambda \) cannot be made Lyapunov operators even if they are asymptotically ground-state stable. Similarly, for \( W \) to satisfy
the condition $G(W) \leq 0$, the condition $G(X_\lambda) \leq 0$ does not have to be satisfied for all $X_\lambda$; that is, $X_\lambda$ may not be a Lyapunov operator either (for $X_\lambda$ to be a Lyapunov operator, the condition $G(X_\lambda) \leq 0$ must be satisfied). We will show in Section 4 that the approaches to the engineering of the ground-state stability of $W$ can be quite different depending on whether or not $X_\lambda$ can be taken to be a Lyapunov operator of the subsystems.

It is worth mentioning that the issues discussed above are similar to those arising within the vector Lyapunov function approach [4]. In certain situations arising in the classical stability theory for large-scale systems, it is more convenient to use a vector Lyapunov function rather than a scalar function for a large-scale system [34]. Indeed, in general, scalar functions comprising the vector Lyapunov function of a stable large-scale system do not need to be Lyapunov functions individually. Particularly, the decomposition-aggregation method used in [34] to simplify the analysis by decomposing the large system into several subsystems made extensive use of the vector Lyapunov function machinery. When the subsystems are coupled together, a connective stability condition will ensure the total system is stable after the aggregation. In our case, $W$ is the quantum counterpart of the vector Lyapunov function, and $X_\lambda$ is the quantum counterpart of the scalar component of that function. Also, the scalability property discussed in this paper where the ground-state stability of the operator $W$ is derived from the ground-state stability properties of the addends $\{X_\lambda\}$, is parallel to the classical decomposition-aggregation approach mentioned above. Since each $X_\lambda$ may act on several subsystems and the couplings may act on more than one operator from the set $\{X_\lambda\}$, a scalability condition is needed to ensure the cross-couplings do not undermine the stability of $W$.

The following lemma summarizes the approaches to the stability of a large-scale quantum system.

**Lemma 7** Given a collection of observables $X_\lambda \succeq 0$, consider $W = \sum_\lambda X_\lambda$, whose the smallest eigenvalue is $d$.

1. Suppose $\langle X_\lambda \rangle_{\rho_t} \to 0$ for each $\lambda$. Then $\langle W \rangle_{\rho_t} \to 0$ and $d = 0$.
2. Conversely, suppose $\langle W \rangle_{\rho_t} \to 0$. Then $d = 0$ and each $X_\lambda$ is asymptotically ground-state stable. If $\langle W \rangle_{\rho_t} \to d > 0$, each $X_\lambda$ is not necessarily asymptotically ground-state stable.

**Proof.** To prove (i) we observe that if $\langle X_\lambda \rangle_{\rho_t} \to 0$, then $\langle W \rangle_{\rho_t} = \sum_\lambda \langle X_\lambda \rangle_{\rho_t} \to 0$. It remains to show that $d = 0$. Since $W$ is a finite-dimensional operator, it has finite number of eigenvalues. Suppose the smallest eigenvalue of $W$ is positive, i.e., $d > 0$. Then $W - d \succeq 0$ and $\langle (W - d) \rangle_{\rho_t} \geq 0$ for any state $\rho_t$. Thus, $\langle W \rangle_{\rho_t} \geq d > 0$ for any state $\rho_t$ (since $\text{Tr} \rho_t = 1$). This contradicts
Conversely, if \( \langle W \rangle_{\rho t} \to 0 \), then \( \langle X_{\lambda} \rangle_{\rho t} \to 0 \) for each \( \lambda \), which proves the ground-state stability of each \( X_{\lambda} \).

Lemma 7 suggests two different approaches to engineering of the ground-state stability of \( W \) (as we mentioned, investigation of such approaches is the main objective of this paper), namely, through engineering the ground-state stability of every \( X_{\lambda} \) or the ground-state stability of \( X_{\Lambda}' = \sum_{\lambda} X_{\lambda} \). Here \( \Lambda' \) is a subset of the set of all \( \Lambda \). The trivial case where \( \Lambda \) is divided into \( \{ \Lambda, \emptyset \} \) means that we engineer the ground-state stability of \( W \) directly. In this paper by engineering we mean the synthesis of coupling operators between the environment and the system.

We would like to mention that Lemma 7 has a connection with the notion of frustration-free Hamiltonian [39]. A Hamiltonian \( H \) in the form of

\[
H = \sum_{\lambda} H_{\lambda}
\]

is called frustration-free if the ground states of \( H \) are also the ground states of every \( H_{\lambda} \). Suppose \( H \geq 0 \), \( d \) is the smallest eigenvalue of \( H \) and \( \rho \) is one of the ground states of \( H \). Suppose \( H_{\lambda} \geq 0 \), which proves that \( H \) is frustration-free. Then, if \( d = 0 \), we have \( \langle H_{\lambda} \rangle_{\rho} = \sum_{\lambda} \langle H_{\lambda} \rangle_{\rho} = 0 \). and so \( \langle X_{\lambda} \rangle_{\rho t} \to 0 \), which proves that \( L \) is frustration-free. Therefore, if \( W \) and \( X_{\lambda} \) denote Hamiltonians in Lemma 7, the condition \( \langle X_{\lambda} \rangle_{\rho t} \to 0 \) of (i) and the condition \( \langle W \rangle_{\rho t} \to 0 \) of (ii) in fact imply the frustration-freeness of \( W \).

The property of the system observables to maintain their smallest eigenvalue to be \( d = 0 \) while adding the subsystem Hamiltonians \( H_{\lambda} \) and associated observables \( X_{\lambda} \) means that the system size can be increased without perturbing the ground energy. Such a scalability property is often desired in quantum engineering.

3 Lyapunov stability of the ground states

In this section, we consider the generator with one dissipation channel

\[
\mathcal{G}(X) = \mathcal{L}^\dagger X \mathcal{L} - \frac{1}{2} \mathcal{L}^\dagger \mathcal{L} X - \frac{1}{2} X \mathcal{L}^\dagger \mathcal{L}.
\]

(11)

Recall that a state \( \rho \) is an invariant state of the quantum system, if it satisfies the condition \( \langle X(t) \rangle_{\rho} = \langle X \rangle_{\rho} \) for any operator \( X \) [7,24]. Thus we have \( \langle \mathcal{G}(X) \rangle_{\rho I} = 0 \) for an invariant state \( \rho_I \).

The next statement gives the quantum version of the Lyapunov's second method for stability.
Lemma 8 Suppose $V$ is a Lyapunov operator of the system. If $\langle \mathcal{G}(V) \rangle_\rho < 0$ for any $\rho \notin Z_V$, then $V$ is asymptotically ground-state stable.

Proof. Since $\mathcal{G}(V) \leq 0$, $\langle V \rangle_{\rho_t} \leq \langle V \rangle_{\rho_0}$ and $\lim_{t \to \infty} \langle V \rangle_{\rho_t}$ exists. Recall that since $V$ is a Lyapunov operator, then $Z_V = \{ \rho : \langle V \rangle_\rho = 0 \}$. Hence $\forall \rho_0 \in Z_V$, $\langle V \rangle_{\rho_t} = \langle V \rangle_{\rho_0} = 0$; this implies that $Z_V$ is an invariant set. We only need to prove that $\rho_t$ will exit the domain $\{ \rho : \langle V \rangle_\rho \geq \epsilon \}$ for arbitrary $\epsilon > 0$. Now suppose the trajectory $\rho_t$ is restricted to a domain $\{ \rho : \langle V \rangle_\rho \geq \epsilon \}$ for some $\epsilon > 0$. There exists an invariant state $\rho_I$ which is the limit point of the tight sequence $\frac{1}{t} \int_0^t \rho_t^t dt'$. \[ \frac{1}{t} \int_0^t \rho_t^t dt' \] is the mean of $\rho_t$, and so $\rho_I$ is in $\{ \rho : \langle V \rangle_\rho \geq \epsilon \}$. By assumption, $\langle \mathcal{G}(V) \rangle_{\rho_I} < 0$.

Let the initial state be the invariant state $\rho_I$. Integrating $\mathcal{G}(V)$ over $[0, t]$ yields

$$\langle V \rangle_{\rho_I} - \langle V \rangle_{\rho_I} = 0 = \int_0^t \langle \mathcal{G}(V) \rangle_{\rho_I} dt'.$$

(12)

This leads to a contradiction as $\langle \mathcal{G}(V) \rangle_{\rho_I} < 0$. We thus conclude that for any $\epsilon > 0$, there exists $t(\epsilon)$ such that $\langle V \rangle_{\rho_t} \leq \langle V \rangle_{\rho_{t(\epsilon)}} < \epsilon$ $\forall t > t(\epsilon)$. That is, $V$ is asymptotically ground-state stable.

A special case of Lemma 8 is concerned with the generator satisfying the condition

$$\mathcal{G}(V) \leq -cV, \quad c > 0.$$ (13)

In this case, we can integrate (13) to obtain $\langle V(t) \rangle_{\rho_0} \leq e^{-ct} \langle V \rangle_{\rho_0}$. The system exponentially converges to the ground states of $V$.

The exponential convergence condition (13) does not describe all the dynamics that lead to the asymptotic stability of the ground states. Not all physical systems are exponentially stable. A more general treatment will involve dealing with the condition $\mathcal{G}(V) \leq 0$. To this end we will make use of the dissipation functional $\mathcal{D}(V)$.

Lemma 9 If $V$ is a Lyapunov operator of the system satisfying $\mathcal{D}(V) \geq cV^2$ for some $c > 0$, then $V$ is asymptotically ground-state stable.

Proof. Similar to the proof of Lemma 8 and the proof of Proposition 6 in [24], we only need to prove that for arbitrary $\epsilon > 0$, the domain $\{ \rho : \langle V \rangle_\rho \geq \epsilon \}$ does not contain invariant states $\rho_I$. Suppose this is not true and there is an invariant state $\rho_I$ in the domain $\{ \rho : \langle V \rangle_\rho \geq \epsilon \}$. Consider the positive operator $W = V^2$. The generator for $W$ is

$$\mathcal{G}(W) = V\mathcal{G}(V) + \mathcal{G}(V)V + \mathcal{D}(V).$$ (14)
Let the initial state be the invariant state \( \rho_I \). Integrating \( G(W) \) leads to

\[
\langle W \rangle_{\rho_I} - \langle W \rangle_{\rho_I} = \int_0^t \langle V G(V) + G(V) V + \mathfrak{D}(V) \rangle_{\rho_I} dt' = 0. \tag{15}
\]

To establish a contradiction, we use the following identity

\[
\frac{1}{c} V G(V) + \frac{1}{c} G(V) V + V^2 + \frac{1}{4c^2} G(V)^2 = [V + \frac{1}{2c} G(V)]^2. \tag{16}
\]

Note that by assumption, \( \langle V + \frac{1}{2c} G(V) \rangle_{\rho_I} = \langle V \rangle_{\rho_I} \geq \epsilon \), and hence \( \langle [V + \frac{1}{2c} G(V)]^2 \rangle_{\rho_I} \geq \epsilon^2 > 0 \) due to the positivity of the variance \( \langle X^2 \rangle_{\rho} - \langle X \rangle_{\rho}^2 \geq 0 \) for any Hermitian operator \( X \). Using \( \langle [V + \frac{1}{2c} G(V)]^2 \rangle_{\rho_I} > 0 \) we have

\[
\langle V G(V) + G(V) V + \mathfrak{D}(V) \rangle_{\rho_I} > \langle -cV^2 - \frac{1}{4c} G(V)^2 + \mathfrak{D}(V) \rangle_{\rho_I}. \tag{17}
\]

Next, choose a positive number \( d > 0 \) such that \( G(V) + d \geq 0 \), and then we have \( -G(V)(G(V) + d) \geq 0 \) since \( -G(V) \) and \( G(V) + d \) commute. The latter inequality can be written as \( -G(V)^2 \geq dG(V) \). This results in the following inequality

\[
\langle V G(V) + G(V) V + \mathfrak{D}(V) \rangle_{\rho_I} > \langle -cV^2 - \frac{1}{4c} G(V)^2 + \mathfrak{D}(V) \rangle_{\rho_I} \geq \langle -cV^2 + \mathfrak{D}(V) \rangle_{\rho_I} \geq 0. \tag{18}
\]

The last line of (18) is obtained using the assumption \( cV^2 \leq \mathfrak{D}(V) \). As a consequence, (15) is not consistent with (18). This contradiction shows that for arbitrary \( \epsilon > 0 \), the domain \( \{ \rho : \langle V \rangle_{\rho} \geq \epsilon \} \) does not contain invariant states \( \rho_I \), hence any trajectory \( \rho_t \) must exit the set \( \{ \rho : \langle V \rangle_{\rho} \geq \epsilon \} \). This conclusion results in the asymptotic ground-state stability of \( V \), which can be established using the same argument as in the proof of Lemma 8.

Particularly, we can make use of Lemma 9 to obtain the following result.

**Lemma 10** If \( V \) is a Lyapunov operator of the system satisfying \( cV \leq \mathfrak{D}(V) \) for some \( c > 0 \), then the state trajectory \( \rho_t \) will converge to \( Z_V \).

**Proof.** Choose a positive number \( d > 0 \) such that \( V - d \leq 0 \), from which we can conclude \( V(V - d) \leq 0 \) since \( V \) and \( (V - d) \) commute. This can be
rewritten as \( dV \geq V^2 \). Thus we have

\[
V^2 \leq dV \leq \frac{d}{c} \mathcal{D}(V). \tag{19}
\]

By Lemma 9, \( \rho_t \) converges to the ground states of \( V \).

With multiple dissipation channels, the generator of the Lyapunov operator \( V \) is expressed as \( (6) \) and the dissipation functional becomes

\[
\mathcal{D}(V) = \sum_k [L_k^\dagger, V] [V, L_k]. \tag{20}
\]

All the above stability results can be routinely extended to the multi-channel case. Therefore, the sufficient conditions for the convergence to the ground states of general quantum systems can be expressed by operator inequalities, as summarized in the following theorem:

**Theorem 11** If \( V \) is a Lyapunov operator of the system satisfying one of the following conditions:

\[
\mathcal{G}(V) \leq -cV, \quad c > 0, \tag{21}
\]

or

\[
\mathcal{G}(V) \leq 0, \\
cV \leq \mathcal{D}(V), \quad c > 0, \tag{22}
\]

then the state trajectory \( \rho_t \) will converge to \( Z_V \).

Below are two examples to illustrate the use of \( (21) \) and \( (22)-(23) \):

**Example 12** Consider a two-level quantum system. The single coupling operator \( L \) with complex entries is given by

\[
L = \begin{pmatrix}
l_{00} & l_{01} \\
l_{10} & l_{11}
\end{pmatrix}. \tag{24}
\]

Suppose we want to engineer the ground-state stability of the Lyapunov operator \( V \) which is defined as

\[
V = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}. \tag{25}
\]
By (23), we need to establish

\[ [L^\dagger, V][V, L] = \begin{pmatrix} |l_{10}|^2 & 0 \\ 0 & |l_{01}|^2 \end{pmatrix} \geq \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}. \]  

(26)

Letting \( l_{01} = 0 \) and choosing any \( l_{10} \neq 0 \) will satisfy (26). With these values, condition (22) becomes

\[
\begin{pmatrix}
-|l_{10}|^2 & -\frac{1}{2}l_{10}^*l_{11} \\
-\frac{1}{2}l_{11}^*l_{10} & 0
\end{pmatrix} \leq 0,
\]

(27)

and requires \(-\frac{1}{4}|l_{10}|^2|l_{11}|^2 \geq 0\) for it to hold. Therefore, we must let \( l_{11} = 0 \). As a result, coupling this system with the environment using any \( L \) of the form of

\[
\begin{pmatrix}
l_{00} & 0 \\
l_{10} & 0
\end{pmatrix}, \quad l_{10} \neq 0
\]

(28)

will stabilize \( V \) to its ground state.

In Example 12, the use of (22)-(23) also implies the satisfaction of (21), and the system is exponentially stable. However in general, (22)-(23) is a weaker condition compared to (21), as (22)-(23) does not necessarily lead to exponential convergence. This can be illustrated in the following example

**Example 13** Consider a three-level system. Suppose we want to engineer the ground-state stability of the Lyapunov operator

\[
V = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

(29)

To do this, the coupling operators are chosen as

\[
L_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad L_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

(30)
Using these values we compute

\[
  \mathcal{G}(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{D}(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]  

(31)

Obviously, \( \mathcal{G}(V) \leq 0 \). Additionally, we have \( \mathcal{D}(V) \geq \frac{1}{2}V \). As a result, the Lyapunov operator satisfies the sufficient conditions (22)-(23). However, we cannot conclude exponential convergence since (21) does not hold for any \( c > 0 \). In this case, the dissipation will still steer the system to the ground state, although the generator at the first-excited state is zero. In fact, since the dissipation strength at the first-excited state is two times the dissipation strength at the second-excited state, the system will be partly driven to the ground state from the first-excited state. However, there is also a possibility that the system will be re-excited to the second-excited state, which makes the calculation of the convergence speed difficult.

Example 13 provides the evidence that condition (21) may not always follow from (22)-(23). Indeed, the convergence speed is comparatively slow in Example 13, because in there the role of \( L_2 \) is mixing the higher energy states rather than steering them to the ground state.

4 Scalability of the Lyapunov methods

In this section, we address the question as to whether the Lyapunov stability of the subsystems can scale up when these subsystems interact and are coupled with environments. Associated with the subsystems, consider a collection of operators \( \{X_\lambda\} \), \( X_\lambda \succeq 0 \), \( \lambda \in \Lambda = \{1, 2, ..., N\} \). Each \( X_\lambda \), \( \lambda \in \Lambda \), may act on more than one subsystem. Coupling between the system and the environments is described by coupling operators \( \{L_k\} \), \( k \in \Delta = \{1, 2, ..., K\} \). Also, consider the operator in the following form

\[
  W = \sum_\lambda X_\lambda, \quad X_\lambda \succeq 0,
\]

(32)

with the generator and the dissipation functional of \( W \) calculated to be
\[ G(W) = \sum_{k,\lambda} L_k^\dagger X_\lambda L_k - \frac{1}{2} L_k^\dagger L_k X_\lambda - \frac{1}{2} X_\lambda L_k^\dagger L_k \]
\[ = \frac{1}{2} \sum_{k,\lambda} (L_k^\dagger [X_\lambda, L_k] + L_k^\dagger [X_\lambda, L_k]), \]  \( (33) \)
\[ D(W) = \sum_{k,\lambda,\lambda'} [L_k^\dagger, X_\lambda][X_{\lambda'}, L_k], \quad \lambda, \lambda' \in \Lambda, k \in \Delta. \]  \( (34) \)

The dissipation functional \((34)\) of the total system contains cross-coupling terms. In the reduced space of the subsystems on which \(X_\lambda\) acts, the generator and dissipation functional are

\[ G(X_\lambda) = \sum_k L_k^\dagger X_\lambda L_k - \frac{1}{2} L_k^\dagger L_k X_\lambda - \frac{1}{2} X_\lambda L_k^\dagger L_k, \]  \( (35) \)

and

\[ D(X_\lambda) = \sum_k [L_k^\dagger, X_\lambda][X_{\lambda'}, L_k], \quad \lambda \in \Lambda, k \in \Delta, \]  \( (36) \)

respectively. It follows from \((35)-(36)\) that

\[ G(W) = \sum_\lambda G(X_\lambda), \]  \( (37) \)

but in general

\[ D(W) \neq \sum_\lambda D(X_\lambda), \]  \( (38) \)

which indicates that the dissipation behaviour may be quite different between \(W\) and individual \(X_\lambda\).

If the couplings \(\{L_k\}\) are such that each operator \(X_\lambda\) satisfies conditions \((21)\) or \((22)-(23)\), i.e., \(X_\lambda\) is a Lyapunov operator and the system asymptotically converges to the set of ground states of each \(X_\lambda\), then the ground-state stability of \(W\) is guaranteed by Lemma \(7\). We say that in this case the stability of subsystems scales up. This case is discussed in Subsection \(4.1\).

Another way to approach the scalability of the subsystems is via studying the total system directly using conditions \((33)-(34)\), without imposing the ground-state stability requirement on individual subsystems and their corresponding operators \(X_\lambda\), which might be difficult if the system is complex. This case is discussed in Subsection \(4.2\).

### 4.1 Scalability of the ground-state stability of each \(X_\lambda\)

Combined with the results from Section \(3\), the first statement in Lemma \(7\) can be formulated in terms of the Lyapunov stability:
Theorem 14 If the subsystems satisfy the corresponding conditions (21) or (22)-(23), i.e., if one of the following conditions hold

\[ G(X_\lambda) \leq -c_\lambda X_\lambda, \quad c_\lambda > 0 \] (39)

or

\[ G(X_\lambda) \leq 0, \quad D(X_\lambda) \geq c_\lambda X_\lambda, \quad c_\lambda > 0 \] (40)

for each \( X_\lambda \succeq 0 \), then the system converges to the set of the ground states of \( W \) asymptotically. In addition, \( W \) is a Lyapunov operator.

**Proof.** Theorem 14 directly follows from Lemma 7.

We define the single-channel component of the generator \( G(X) \) as

\[ G(X)_{L_k} = L_k^\dagger XL_k - \frac{1}{2} L_k^\dagger L_k X - \frac{1}{2} X L_k^\dagger L_k. \] (41)

We first consider the ground-state stability of each \( X_\lambda \) separately. Suppose the ground states of \( X_\lambda \) are stabilized by the corresponding \( L_k \). After the aggregation, \( X_\lambda \) may be subjected to other coherent couplings \( \{L_{k'}, k' \neq k\} \). Therefore, we will ensure that the additional couplings associated with \( X_\lambda \) do not undermine the ground-state stability of \( X_\lambda \). Specifically, suppose we achieve

\[ G(X_\lambda)_{L_k} \leq -c_\lambda X_\lambda, \quad c_\lambda > 0. \] (42)

The ground states of \( X_\lambda \) are asymptotically stable under the action of \( L_k \). Then, to ensure (39) is satisfied in the presence of coupling with the channels other than \( k \), we need to impose a condition

\[ \sum_{k' \neq k} G(X_\lambda)_{L_{k'}} \leq 0. \] (43)

Clearly, (43) is a sufficient condition to guarantee that the satisfaction of condition (39) can be established from (42). For this reason, (43) will be referred to as scalability condition.

**Corollary 15** Suppose for each \( \lambda \in \Lambda \), there exists an \( L_k \) such that (42) holds, and (43) holds for all other couplings associated with \( X_\lambda \). Then, \( W \) is asymptotically ground-state stable.

**Proof.** Obvious from the previous discussions.

**Corollary 16** Suppose for each \( \lambda \in \Lambda \), there exists an \( L_k \) such that

\[ G(X_\lambda)_{L_k} \leq 0, \quad D(X_\lambda)_{L_k} \geq c_\lambda X_\lambda, \quad c_\lambda > 0, \] (44)
and (43) holds for all other couplings associated with \( X_\lambda \). Then, (40) is true and \( W \) is asymptotically ground-state stable.

**Proof.** In the light of the previous discussion, we have \( G(X_\lambda) \leq 0 \) by (43). Moreover, \( \mathfrak{D}(X_\lambda)_{L_{k'}} = [L_{k'}^\dagger, X_\lambda][X_\lambda, L_{k'}] \) is always non-negative for any \( L_{k'}, k' \neq k \), which yields the following relation

\[
\mathfrak{D}(X_\lambda) \geq \mathfrak{D}(X_\lambda)L_k \geq c_\lambda X_\lambda.
\]  

As Corollary 16 shows, we have dealt with the cross terms in \( \mathfrak{D}(W) \) by introducing a more conservative condition (43), which allowed us to engineer the condition (44) on each dissipation functional \( \mathfrak{D}(X_\lambda) \) individually. More explicitly, by stabilizing \( X_\lambda \) separately and imposing the scalability condition (43), we can guarantee the convergence to the set of the ground states of \( W \) without using the dissipation functional of the total system.

### 4.2 Ground-state stability of \( W \)

As said before, the other approach to the scalability problem is to engineer the ground-stability of the total system directly. One way to achieve this is by induction, by grouping \( X_\lambda, \lambda = 1, \ldots, n \), into \( \tilde{X}_n = \sum_{\lambda=1}^{n} X_\lambda \), and considering \( X_{n+1} \) as an additional observable. The question of interest is to achieve the ground-state stability of \( \tilde{X}_{n+1} = \sum_{\lambda=1}^{n+1} X_\lambda = \tilde{X}_n + X_{n+1} \), by synthesizing coupling operators \( \{L_k, k = M + 1, \ldots, K\} \) additional to the coupling operators \( \{L_k, k = 1, \ldots, M\} \) that ensure the ground-state stability of \( \tilde{X}_n \). Define \( d_n \) as the smallest eigenvalue of \( \tilde{X}_n \), then we have \( \tilde{X}_n - d_n \geq 0 \). Obviously, \( d_1 = 0 \).

The scalability conditions for integrating these systems are summarized in the following theorems:

**Theorem 17** Suppose \( G(\tilde{X}_n - d_n) \leq -c(\tilde{X}_n - d_n), \ c > 0, \) is achieved using a set of coupling operators \( \{L_k, k = 1, \ldots, M\} \). \( \tilde{X}_{n+1} \) is asymptotically ground-state stable if the additional coupling operators \( \{L_k, k = M+1, \ldots, K\} \) satisfy the Lyapunov condition

\[
G(X_{n+1}) + \sum_{k=M+1}^{K} G(\tilde{X}_n)L_k \leq -cX_{n+1} + c(d_{n+1} - d_n), \ c > 0.
\]  

(46)

**Theorem 18** Suppose the conditions \( G(\tilde{X}_n) \leq 0, \mathfrak{D}(\tilde{X}_n - d_n) \geq c(\tilde{X}_n - d_n), \ c > 0 \) are achieved using a set of coupling operators \( \{L_k, k = 1, \ldots, M\} \). The Lyapunov conditions to ensure the ground-state stability of \( \tilde{X}_{n+1} \) are
\[ \mathcal{G}(X_{n+1}) + \sum_{k=M+1}^{K} \mathcal{G}(\tilde{X}_n)_{L_k} \leq 0, \quad (47) \]
\[ \mathcal{D}(X_{n+1}) + 2 \sum_{k=1}^{K} \text{Re}([L_k^\dagger \tilde{X}_n][X_{n+1}, L_k]) \geq cX_{\lambda=n+1} - c(d_{n+1} - d_n), \quad c > 0. \]  

(48)

Here \( \{L_k, k = M + 1, \ldots, K\} \) denote additional coupling operators.

**Proof.** For simplicity, first we consider the integration of two operators represented by \( X_1 \) and \( X_2 \) with \( X_{1,2} \geq 0 \). \( W = X_1 + X_2 \) could be positive definite. Since our Lyapunov stability results are derived under the assumption that the candidate Lyapunove operator has zero eigenvalue, we circumvent this issue by considering the displaced operator \( W - d \geq 0 \), where \( d \geq 0 \) is the smallest eigenvalue of \( W \).

Suppose the Lyapunov condition \( \mathcal{G}(X_1)_{L_1} \leq -cX_1 \) has been established using the coherent coupling \( L_1 \). We are concerned with engineering an additional coupling between the environment and the part of the system characterized by the observable \( X_2 \) to achieve the following Lyapunov condition for the total system

\[ \mathcal{G}(W - d) = \mathcal{G}(X_1 + X_2) \leq -c(W - d) = -c(X_1 + X_2) + cd, \quad c > 0. \]  

(49)

Formally, this problem reduces to the that of the synthesis of an additional coupling operator \( L_2 \) which couples an environment to \( X_2 \) (and possibly acts non-trivially on \( X_1 \)). Decomposing (49) yields

\[ \mathcal{G}(X_1)_{L_1} + \mathcal{G}(X_1)_{L_2} + \sum_{k=1,2} \mathcal{G}(X_2)_{L_k} \leq -c(X_1 + X_2) + cd, \quad c > 0. \]  

(50)

Since \( \mathcal{G}(X_1)_{L_1} \leq -cX_1 \), a sufficient condition for (50) to hold is

\[ \mathcal{G}(X_1)_{L_2} + \mathcal{G}(X_2) \leq -cX_2 + cd. \]  

(51)

It follows from the above discussion that \( W \) is asymptotically ground-state stable if (51) holds. However, \( X_2 \) alone may not satisfy the Lyapunov condition (39) even if (51) is true. We can see that if \( \mathcal{G}(X_1)_{L_2} - cd \leq 0 \), then condition (51) implies (39), i.e., (51) implies the ground-state stability condition for \( X_2 \). But in general, the stability condition (39) on \( X_2 \) is no longer required.
Similar results can be obtained based on the assumption
\[ \mathcal{G}(X_1)_{L_1} \leq 0, \quad \mathcal{D}(X_1)_{L_1} \geq cX_1, \quad c > 0. \] (52)

In order to engineer the stability of the combined system achieving
\[
\begin{align*}
\mathcal{G}(W - d) &= \mathcal{G}(X_1 + X_2) \leq 0, \\
\mathcal{D}(W - d) &= \mathcal{D}(X_1 + X_2) \\
\geq c(W - d) &= c(X_1 + X_2) - cd, \quad c > 0,
\end{align*}
\] (53)
we exploit the following relation
\[
\begin{align*}
\mathcal{D}(\sum \lambda X_{\lambda}) &= \sum_{k, \lambda, \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k] \\
&= \sum_{\lambda} \mathcal{D}(X_{\lambda}) + \sum_{k, \lambda, \lambda' \neq \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k], \quad \lambda, \lambda' \in \Lambda
\end{align*}
\] (54)
to write the second inequality in (53) explicitly as
\[
\begin{align*}
\mathcal{D}(X_1) + \mathcal{D}(X_2) + \sum_{k; \lambda, \lambda' = 1, 2; \lambda \neq \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k] \\
\geq \mathcal{D}(X_1)_{L_1} + \mathcal{D}(X_2) + \sum_{k; \lambda, \lambda' = 1, 2; \lambda \neq \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k] \\
\geq c(X_1 + X_2) - cd.
\end{align*}
\] (55)

Accordingly, a sufficient condition to guarantee \( \mathcal{D}(W - d) \geq c(W - d) \) is
\[
\begin{align*}
\mathcal{D}(X_2) + \sum_{k; \lambda, \lambda' = 1, 2; \lambda \neq \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k] \geq cX_2 - cd.
\end{align*}
\] (56)

Compared with condition (40) for \( X_2 \), inequality (56) includes an extra cross-modulation term \( \sum_{k; \lambda, \lambda' = 1, 2; \lambda \neq \lambda'} [L_k^\dagger, X_{\lambda}][X_{\lambda'}, L_k] \).

The above methods can be readily extended to systems that involve an arbitrary number of subsystems and observables \( X_{\lambda} \), resulting in Theorem 17 and 18.

Interestingly, we can further obtain different sufficient conditions for Theorem 17 and Theorem 18 without knowing the value of \( \{d_n\} \). Note that \( X_{n+1} \geq 0 \) and \( d_n \) is the smallest eigenvalue of \( \tilde{X}_n \). So we have
\[
\begin{align*}
d_{n+1} &= \langle \tilde{X}_{n+1} \rangle_{\rho_{n+1}} = \langle \tilde{X}_n + X_{n+1} \rangle_{\rho_{n+1}} \\
&\geq \langle \tilde{X}_n \rangle_{\rho_{n+1}} \geq d_n,
\end{align*}
\] (57)
where $\rho_g^{n+1}$ denotes the ground state of $\tilde{X}_{n+1}$. Based on (57), we can obtain sufficient conditions which are not dependent on $\{d_n\}$.

**Corollary 19** Suppose the following conditions hold

\[
\mathcal{G}(X_{\lambda=n+1}) + \sum_{k=M+1}^{K} \mathcal{G}(\tilde{X}_n)_{L_k} \leq -cX_{\lambda=n+1},
\]

(58)

\[
\mathcal{D}(X_{\lambda=n+1}) + 2 \sum_{k=1}^{K} \text{Re}([L_k^\dagger,\tilde{X}_n][X_{\lambda=n+1}, L_k]) \geq cX_{\lambda=n+1}, \ c > 0.
\]

(59)

Then the conclusion of Theorems 17 and 18 holds.

**Proof.** Conditions (58) and (59) can imply (46) and (48) by using (57), respectively. The statement of the Corollary then follows from Theorems 17 and 18.

If the additional coupling operators satisfy $\{[L_k, \tilde{X}_n] = 0, \ k = M + 1, \ldots, K\}$, then the terms $\sum_{k=M+1}^{K} \mathcal{G}(\tilde{X}_n)_{L_k}$ in (58) and (47), and the cross-coupling terms $\sum_{k=1}^{K} \text{Re}([L_k^\dagger,\tilde{X}_n][X_{n+1}, L_k])$ in (59) all vanish. In this case, the conditions of Corollary 19 reduce to conditions (42) and (44).

In contrast to the scalability approach considered in Theorem 14 and Corollary 16, the cross-coupling terms appear in (48) and (59). These cross-coupling terms show that the condition on the dissipation functional of the subsystem and the condition on the dissipation functional of the total system do not necessarily imply each other. Therefore, the two scalable methods proposed in this section have different implications.

We would like to end this section with an illustration on how to scale up the systems using the Theorem 17 and Corollary 19:

**Example 20** Recall the two-level system where we engineer the ground-state stability of the operator $X_1 = \frac{1}{2}(1 + \sigma_z) \succeq 0$, which can be written as

\[
X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(60)

This operator has been studied in Example 12. The coupling operator $L_1 = \sigma_-$
defined as
\[ L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \] (61)
is found to satisfy the stability condition \( G(X_1)L_1 \leq -X_1 \). Now we consider the extended two-qubit system, on which the operator of interests is \( W = X_1 + X_2 \) with \( X_2 = \frac{1}{2}(1 + \sigma_z \sigma_z) \succeq 0 \). \( L_1 \) acts non-trivially on \( X_2 \). In this extended system, we denote \( X_1 = X_1 \otimes I_2 \) and \( L_1 = L_1 \otimes I_2 \). By (58), these operators should satisfy the following inequality of the form
\[ G(X_2)L_1 + G(X_2)L_2 + G(X_1)L_2 \leq -X_2. \] (62)

\( L_2 \) is the new coupling operator acting on the total system. (62) can be further simplified as
\[
G(X_2 + X_1)L_2 \leq -X_2 - G(X_2)L_1
\]
\[
= - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
\[
= - \begin{pmatrix} 0 & 0 \\ 0 & X'_3 \times 3 \end{pmatrix} \] (63)

Since \( X_1 + X_2 \) equals
\[
X_1 + X_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & X' \end{pmatrix}, \] (64)

we can define \( L_2 \) as
\[
L_2 = \begin{pmatrix} 1 & 0 \\ 0 & L_{3\times3} \end{pmatrix} \] (65)
to reduce the problem to solving the inequality
\[ G(X')L \leq -X', X' \succeq 0 \] (66)
within a subspace. The easiest way to solve (66) is to further decompose \( X' \).
as
\[ X' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \] (67)

and then use two additional coupling operators \( L_2, L_3 \) instead of a single \( L_2 \), which are readily computed to be
\[ L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (68)

In this example, since \( W \succeq 0 \), the individual \( X_{1,2} \) are also asymptotically ground-state stable. In addition, \( W \) is frustration-free.

5 Synthesis of the dissipation

In this section, we introduce the methods to find the correct coherent couplings between the system and environment that steer the system to the ground states of given candidate Lyapunov operators. Also, we will show how to calculate the system-environment couplings which satisfy the scalability conditions derived in Section 4.

5.1 Synthesis of single dissipation channel

In this first part, we use a single candidate Lyapunov operator \( V \succeq 0 \) and a single system-environment coupling operator \( L \) as the dissipation control. \( L \) can be calculated using the inequalities (21), or (22)-(23). Since (21), (22) and (23) are generally nonlinear matrix inequalities, solving these inequalities for large-scale systems is a challenging task.

We introduce a special class of dissipation controls that admit factorization \( L = UV \), where \( U \) is a unitary operator. In [39] the authors have suggested similar coupling operators \( L_{i,\lambda} = U_i H_\lambda \) for the ground-state engineering of a Hamiltonian \( H_\lambda \). They showed that this class of control could form a sufficient condition for convergence if \( \{U_i\} \) is a set of unitary operators which rotate part of the high-energy space with support in \( H_\lambda \) into the ground-state space,
according to [13]. However, it is not clear when this rotation exists, and how to solve for such unitary rotation.

In this section we characterize the unitary rotation $U$ required to establish the ground-state Lyapunov stability. Basically, we attempt to solve (21), or (22)-(23) for $U$. We have

$$G(V) = L^\dagger VL - \frac{1}{2} L^\dagger LV - \frac{1}{2} VL^\dagger L$$

$$= VU^\dagger VUV - \frac{1}{2} (VU^\dagger UV^2 + V^2 U^\dagger UV)$$

$$= VU^\dagger VUV - V^3$$

(69)

for single system-environment coupling $L$. We now consider several special choices for the operator $V$.

5.1.1 Special case 1. $V$ is a projection $(V^2 = V)$

In many cases, $V$ can be constructed as a projection, i.e. $V^2 = V$. With the aid of this property, the condition (21) can be rewritten as

$$VU^\dagger VUV \leq (1 - c)V, \quad 0 < c \leq 1,$$

(70)

which can be regarded as the mathematical formulation for the argument in [39]: $U$ should be designed to rotate part of the high-energy space into the zero-energy space. This shows that our stability results are consistent with the physical intuition.

Now we turn to conditions (22)-(23). With $L = UV$, they can be written as

$$VU^\dagger VUV \leq V,$$

(71)

and

$$\mathcal{D}(V) = [VU^\dagger, V][V, UV] = -VU^\dagger VUV + V \geq cV,\quad c > 0.$$  

(72)

Obviously, (72) implies (71). More importantly, (72) and (70) are the same conditions. As a result, the sufficient conditions (21) and (22)-(23) both reduce to the same expression (70) under the assumptions $L = UV$ and $V^2 = V$. In this case, (22)-(23) also leads to exponential convergence of $\langle V \rangle_{\rho_t}$ to 0.

A special case to (70) is $VUV = 0$, which corresponds to $c = 1$. In particular, a unitary rotation $U$ satisfying $VUV = 0$ always exists when stable states are engineered to be the ground states of $V$ [8,39].

In addition to the special case by letting $c = 1$, (70) can be solved by making
the substitution $P = VUV$ as
\[(1 - c)V - P^\dagger P \geq 0, \ 0 < c < 1.\] (73)

Since $V \succeq 0$, it can be decomposed as $V = Q^\dagger Q$. Therefore, $P = \sqrt{1 - c}Q$ is a solution to (73). The synthesis problem is transformed to solving
\[VUV = \sqrt{1 - c}Q, \ 0 < c \leq 1\] (74)

for a unitary $U$. (74) is equivalent to
\[(V^T \otimes V)\text{vec}(U) = \text{vec}(\sqrt{1 - c}Q),\] (75)

vec($U$) is the vectorization of an $n \times n$ matrix $U$ by stacking the columns of $U$ into a single column vector of dimension $n^2 \times 1$. The general solution to (75) is given by
\[
\text{vec}(U) = (V^T \otimes V)^{-1}\text{vec}(\sqrt{1 - c}Q) + (I_{n^2 \times n^2} - (V^T \otimes V)^{-1}(V^T \otimes V))x,
\]

where $x$ is an $n^2 \times 1$ vector of free parameters. $(V^T \otimes V)^{-1}$ denotes the unique Moore-Penrose pseudoinverse [16] of $V^T \otimes V$. For convenience, we adopt the notations
\[
I_{n^2 \times n^2} - (V^T \otimes V)^{-1}(V^T \otimes V) = (a_1^T a_2^T \ldots a_n^T)^T,
\]
\[
(V^T \otimes V)^{-1}\text{vec}(\sqrt{1 - c}Q) = (b_1^T b_2^T \ldots b_n^T)^T,
\]

where the elements $\{a_i, i = 1, \ldots, n\}$ are $n \times n^2$ matrices, and $\{b_i, i = 1, \ldots, n\}$ are $n \times 1$ vectors. According to (76), $U$ can be expressed as
\[U = (a_1x + b_1 \ a_2x + b_2 \ \ldots \ a_nx + b_n),\] (78)

which is an $n \times n$ matrix. The parameters $\{a_i\}$ and $\{b_i\}$ are already known because $V, Q$ are given, and $x$ is determined from the condition $U^\dagger U = I$. The latter condition can be explicitly written as
\[
\begin{pmatrix}
(a_1x + b_1)^\dagger \\
(a_1x + b_2)^\dagger \\
\vdots \\
(a_nx + b_n)^\dagger
\end{pmatrix}
\begin{pmatrix}
a_1x + b_1 & a_2x + b_2 & \ldots & a_nx + b_n
\end{pmatrix} = I.
\]

(79)

Equation (79) can be further organized as a set of bilinear equations:
\[ x^\dagger a_i^\dagger a_i x + x^\dagger a_i^\dagger b_i + b_i^\dagger a_i x + b_i^\dagger b_i = 1, \ i = 0, 1, \ldots, n, \]
\[ x^\dagger a_i^\dagger a_j x + x^\dagger a_i^\dagger b_j + b_j^\dagger a_j x + b_j^\dagger b_j = 0, \ i > j. \] (80)

The special case where \( VUV = 0 \ (c = 1) \) corresponds to \( b_1 = b_2 = \cdots = b_n = 0 \). In this case, (80) can be simplified as

\[ x^\dagger a_i^\dagger a_i x = 1, \ i = 1, \ldots, n, \]
\[ x^\dagger a_i^\dagger a_j x = 0, \ i > j. \] (81)

**Example 21** For the purpose of illustrating the difference between the cases \( c = 1 \) and \( c < 1 \), we again consider a quantum two-level system. Consider the problem of engineering the ground-state stability of the Lyapunov operator

\[ V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \] (82)

The Moore-Penrose pseudoinverse of \( V \) is calculated to be

\[ (V^T \otimes V)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (83)

First we solve (81) in the case where \( c = 1 \). \( \{a_i\} \) can be obtained using (83):

\[ a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \] (84)

With this, (81) can be written as
Parameterizing $x$ as $x = (x_1, x_2, x_3, x_4)^T$, we arrive at a set of bilinear equations

\[
|x_2|^2 = 1, \quad |x_3|^2 + |x_4|^2 = 1, \quad x_4^*x_2 = 0.
\] (86)

Particularly, we have $x_4 = 0$ by (86). Then the unitary rotation is

\[
U = (a_1x \quad a_2x) = \begin{pmatrix} 0 & x_3 \\ x_2 & 0 \end{pmatrix}, \quad |x_2|^2 = |x_3|^2 = 1,
\] (87)

and the desired system-environment coupling $L = UV$ is

\[
L = \begin{pmatrix} 0 & 0 \\ x_2 & 0 \end{pmatrix}.
\] (88)

Next we consider the case when $c < 1$. The decomposition $V = Q^*Q$ is not unique, however due to the particular form of (83), $(V^T \otimes V)^{-1}\text{vec}(\sqrt{1 - cQ})$ is nonzero only if the first entry of $Q$ is nonzero. For example, we can choose $Q$ as the square root of $V$

\[
Q = \sqrt{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\] (89)

which gives

\[
b_1 = \begin{pmatrix} \sqrt{1 - c} \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\] (90)

Equation (80) transforms to
Accordingly, the unitary rotation and coupling are

\[
U = \begin{pmatrix} \sqrt{1-c} & x_3 \\ x_2 & x_4 \end{pmatrix},
\]

\[L = \begin{pmatrix} \sqrt{1-c} & 0 \\ x_2 & 0 \end{pmatrix}, \quad |x_2|^2 = c. \tag{93}\]

Equation (93) gives the general form of the coupling operator \(L\) which satisfies (70) for \(c < 1\). Obviously, (93) does not incorporate the special case (88) for \(c = 1\) since \(|x_2| < 1\).

5.1.2 Special case 2. \(V^2 \geq V\)

In this case, the stability condition (21) or (22)-(23) is still satisfied if (70) holds because

\[\mathcal{G}(V) = VU^\dagger UV - V^3 \leq VU^\dagger VUV - V \leq -cV. \tag{94}\]

Additionally, we have

\[U^\dagger VU \leq U^\dagger V^2 U, \tag{95}\]
\[VU^\dagger VUV \leq VU^\dagger V^2 UV. \tag{96}\]

Now it is easy to see that if \(U\) satisfies (73), then it also satisfies (70), since

\[VU^\dagger VUV \leq VU^\dagger V^2 UV \leq (1-c)V, \quad c > 0. \tag{97}\]

As a result, the unitary solution \(U\) obtained from (73) could work for both cases \(V^2 = V\) and \(V^2 \geq V\).

5.2 Synthesis of multiple dissipation channels

In this section we extend the coupling synthesis approach considered in the previous section to construct multiple dissipation channels aimed at ground-state stabilization of a Lyapunov observable \(V\). We still assume \(V\) is a projection.
Letting \( L_k = U_k V \), we can re-express (21) as

\[
\sum_{k=1}^{K} V U_k^\dagger V U_k V = V (\sum_{k=1}^{K} U_k^\dagger V U_k ) V \leq (K - c) V, \quad c > 0.
\]  (98)

It is easy to verify that (22)-(23) is still equivalent to (21) if we assume the decomposition \( L_k = U_k V \) for each \( L_k \). This observation leads to the following “no-go theorem” concerning the validity of such decomposition.

**Corollary 22** Suppose the Lyapunov observable \( V \) is a projection. If it satisfies conditions (22)-(23) but the exponential stability condition (21) does not hold, then at least one of the coupling operators \( L_k \) does not admit decomposition of the form \( L_k = U_k V \) with a unitary \( U_k \). In the single channel case, the coupling operator \( L \) cannot be represented as form \( L = UV \) with a unitary operator \( U \).

To illustrate the above result, consider Example 13 where (22)-(23) is not equivalent to (21). We conclude that the matrix \( L_2 \) in Example 13 cannot be written as \( L_2 = U_2 V \) where \( U_2 \) is a unitary operator. If this decomposition was possible, we would have

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
U_{00} & U_{01} \\
U_{10} & U_{11}
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 2
\end{pmatrix},
\]  (99)

where \( U_{00} \) is a scalar. It is easy to see from (99) that

\[
U_{11} = \begin{pmatrix}
0 & \frac{1}{2} \\
1 & 0
\end{pmatrix}, \quad U_{01} = [0 \quad 0],
\]  (100)

which implies that \( U_2 \) cannot be a unitary operator.

The set of unitary operators \( \{U_k\} \) satisfying (98) can be calculated if a decomposition as

\[
(K - c) V = \sum_{k=1}^{K} Q_k^\dagger Q_k,
\]  (101)

is available. Then \( \{U_k\} \) are obtained by solving \( V U_k V = Q_k \), as did in (74).

### 5.3 Scalable dissipations

In this section we are concerned with a particular class of coupling operators of the form \( L_k = U_k X_\lambda, \quad X_\lambda \succeq 0 \) and \( U_k \) is a unitary operator. For this type
of coupling operators, we can re-express the condition \([L_{k'}, X_\lambda] = 0, k' \neq k\), which is one particular sufficient condition to guarantee satisfaction of the scalability condition [43], in terms of unitary operators \(U_k\). This leads to a sufficient condition for ground-state stability of the operator \(W = \sum_\lambda X_\lambda\), which follows from Corollaries [15] and [16].

**Corollary 23** Assume \([X_\lambda, X_{\lambda'}] = 0\) for \(\lambda \neq \lambda'\) and \([U_{k'}, X_\lambda] = 0\). If either (42) or (44) holds for each \(X_\lambda \succeq 0\), then \(W = \sum_\lambda X_\lambda\) is asymptotically ground-state stable.

**Proof.** The conclusion follows from

\[
[L_{k'}, X_\lambda] = [U_{k'}, X_{\lambda'}, X_\lambda] \\
= [U_{k'}, X_\lambda] X_{\lambda'} = 0, \quad \lambda' \neq \lambda.
\] (102)

Thus \(G(X_\lambda)_{L_{k'}} = 0\) and so the scalable condition [43] used in Corollaries [15] and [16] is satisfied.

If \(X_\lambda\) is a projection, the general sufficient condition [43] is then expressed as

\[
X_\lambda U_{k'}^\dagger X_\lambda U_{k'} X_\lambda \leq X_\lambda.
\] (103)

In the last are two examples of applying the scalable condition.

**Example 24** Consider a chain of qubits. The candidate Lyapunov operator for the system is defined as

\[
W = \sum_\lambda X_\lambda, \quad X_\lambda = \frac{1}{2}(\sigma_{z_{\lambda-1}} \sigma_{x_\lambda} \sigma_{z_{\lambda+1}} + 1), \quad X_\lambda \succeq 0.
\] (104)

\(\{X_\lambda\}\) are commuting due to

\[
[X_\lambda, X_{\lambda+1}] = \frac{1}{4}[\sigma_{z_{\lambda-1}} \sigma_{x_\lambda} \sigma_{z_{\lambda+1}}, \sigma_{z_\lambda} \sigma_{x_{\lambda+1}} \sigma_{z_{\lambda+2}}] \\
= \frac{1}{4}\sigma_{z_{\lambda-1}}[\sigma_{x_\lambda} \sigma_{z_{\lambda+1}}, \sigma_{z_\lambda} \sigma_{x_{\lambda+1}} \sigma_{z_{\lambda+2}}] = 0.
\] (105)

\(U_\lambda = \sigma_{z_\lambda}\) is a solution to \(X_\lambda U_\lambda X_\lambda = 0\) and (42). Furthermore, we have

\[
[U_{\lambda'}, V_\lambda] = \frac{1}{2}[\sigma_{z_{\lambda'}}, \sum_\lambda \sigma_{z_{\lambda-1}} \sigma_{x_\lambda} \sigma_{z_{\lambda+1}}] = 0
\] (106)
Fig. 1. The schematic representation of lattices on a torus with periodic boundary. The qubits are placed on the edges. The qubits 1, 2, 3, 4 are connected to a common vertex, and the qubits 3, 4, 5, 6 form a plaquette.

for $\lambda \neq \lambda'$. By Corollary 23, $W = \sum_{\lambda} X_{\lambda}$ is asymptotically ground-state stable. In particular, $W$ can be stabilized to its ground states by $\{L_{\lambda} = \sigma_{z\lambda}(\sum_i \sigma_{z\lambda-1}\sigma_{x\lambda}\sigma_{z\lambda+1} + 1)\}$.

**Example 25** The toric code [11,22] is defined on spin lattices. The qubits are placed on the edges, as shown in Figure 1. The toric code states can then be defined by the degenerate ground states of the stabilizer operators as

$$A = \prod_{i=1}^{4} \sigma_{x\lambda}, \quad B = \prod_{j=3}^{6} \sigma_{z\lambda}. \quad (107)$$

The code states are the ground states of both $A$ and $B$. $\{\sigma_{x\lambda}\}$ are four $X$-axis Pauli operators acting on the four qubits that connect to one vertex, and $\{\sigma_{z\lambda}\}$ are four Pauli operators acting on the four qubits that form one plaquette. Errors can be easily detected and corrected using these code states as the computation basis. Normally, the code states are defined using a large number of stabilizer operators, and so the qubits could cover a large area of the torus.

First, we consider two stabilizer operators as outlined in (107) and define the Lyapunov operator $V$ as

$$V = V_1 + V_2,$$

$$V_1 = -\frac{1}{2} \sigma_{x1} \sigma_{x2} \sigma_{x3} \sigma_{x4} + \frac{1}{2},$$

$$V_2 = -\frac{1}{2} \sigma_{z3} \sigma_{z4} \sigma_{z5} \sigma_{z6} + \frac{1}{2}. \quad (108)$$

Performing the similar analysis as in the last example, any of the four operators $U_i = \sigma_{z\lambda}, i \in \{1, 2, 3, 4\}$ can be shown to stabilize $V_1$. Particularly, $\sigma_{z\lambda}, i \in \{1, 2, 3, 4\}$ commutes with any of the four operators $\{\sigma_{z\lambda}, i \in \{3, 4, 5, 6\}\}$. In other words, $\sigma_{z\lambda}, i \in \{1, 2, 3, 4\}$ stabilizes $V_1$ without interfering with $V_2$. By
Corollary \[23\] \( V_1 \) and \( V_2 \) are scalable and \( V \) is asymptotically ground-state stable.

Suppose the code states are defined by the common ground states of three stabilizer operators, namely, \( A \), \( B \) and a third stabilizer operator as

\[
V_3 = \frac{1}{2}\sigma_x^1\sigma_x^7\sigma_x^8\sigma_x^9 + \frac{1}{2}.
\]  

(109)

\( V_1 \) and \( V_3 \) have one common edge. In Figure 1 the qubits 7, 8, 9 should be placed on the three edges connecting to the vertex on the left of qubit 1. If we use \( U = \sigma_z^1 \) to stabilize \( V_1 \), then we have \([\sigma_z^1, V_3] = [\sigma_z^1, -\frac{1}{2}\sigma_x^1\sigma_x^7\sigma_x^8\sigma_x^9] \neq 0\). The scalable condition in Corollary \[23\] does not hold and so \( \sigma_z^1 \) acts non-trivially on \( V_3 \). However, it can be easily seen that \( \sigma_z^1 \) indeed stabilizes \( V_3 \) as well, which can be considered as a special case where (43) holds.

6 Conclusion

We have developed the Lyapunov theory of the ground-state stability of quantum systems using a Heisenberg-picture approach. This theory is designed to serve as a foundation for a theory of stabilization by dissipation, which has significant applications in future quantum technologies. This theory allows us to engineer the systems by considering Lyapunov operators and manipulating Lyapunov inequalities, which is a common practice when engineering classical and quantum control systems.

Several issues should be taken into consideration concerning practical implementations of this theory. For example, the realization of the resulting coupling operators \( \{L_k\} \) with the available experimental resources is an important and interesting problem. Additional Hamiltonian control could be introduced if the choices of dissipation channels are limited, as was done in [37][38]. The convergence speed is also critical for a large-scale quantum system. We have shown that the Lyapunov condition \[22\]-\[23\] does not necessarily guarantee exponential convergence. Therefore, the scaling of the convergence speed as we build up the systems using weak Lyapunov conditions will need further investigation.

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