Geometry of extended Bianchi-Cartan-Vranceanu spaces

Angel Ferrández¹⁺, Antonio M. Naveira² and Ana D. Tarrío³

¹Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. E-mail address: aferr@um.es
²Departamento de Matemáticas, Universidad de Valencia (Estudi General), Campus de Burjassot, 46100 Burjassot, Spain. E-mail address: naveira@uv.es
³Departamento de Matemáticas, Universidade da Coruña, Campus A Zapateira, 15001 A Coruña, Spain. E-mail address: madorana@udc.es

Abstract

The differential geometry of 3-dimensional Bianchi, Cartan and Vranceanu (BCV) spaces is well known. We introduce the extended Bianchi, Cartan and Vranceanu (EBCV) spaces as a natural seven dimensional generalization of BCV spaces and study some of their main geometric properties, such as the Levi-Civita connection, Ricci curvatures, Killing fields and geodesics.

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1 Introduction

Let us denote by $H^{2n+1}$ the $(2n + 1)$-dimensional complex Heisenberg group in $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ with coordinates $(z, t) = (x, y, t) = (x_1, y_1, \ldots, x_n, y_n, t)$ whose group law writes

$$(x, y, t) \cdot (x', y', t') = (z, t) \cdot (z', t') = (z + z', t + t' + \frac{1}{2} \text{Im} \sum_{j=1}^{n} z_j \bar{z}_j).$$

Set $g_0 = (z_0, t) \in H^{2n+1}$ and let $l_{g_0}g = g_0g$ be the left translation by $g_0$. We can then easily see that the left invariant vector fields write down

$$X_\alpha = \frac{\partial}{\partial x_\alpha} + \frac{1}{2} y_\alpha \frac{\partial}{\partial t}, \quad Y_\alpha = \frac{\partial}{\partial y_\alpha} - \frac{1}{2} x_\alpha \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

(*) Corresponding author A. Ferrández.
and \( \{X_\alpha, Y_\alpha, \alpha = 1, \ldots, n \} \) form an orthonormal basis of a distribution \( D \) with respect to the sub-Riemannian metric \( ds^2 = \sum_{\alpha=1}^{n} (dx_\alpha^2 + dy_\alpha^2) \) and satisfy the following bracket relations:

\[
[X_\alpha, Y_\alpha] = T, \quad [X_\alpha, T] = 0, \quad [Y_\alpha, T] = 0, \quad \alpha = 1, \ldots, n.
\]

Taking \( n = 1 \) and \( t \in S^1 \) we obtain the complex Heisenberg group, which is a manifold equipped with a contact structure.

The quaternionic Heisenberg group serves as a flat model of quaternionic contact manifolds. We can consider the following model \( \mathbb{Q}^n \times \text{Im}\mathbb{Q} \) with the group law

\[
(q, p) = (q_1, p_1) \circ (q_2, p_2) = (q_1 + q_2, p_1 + p_2 + \frac{1}{2} \text{Im}(p_1 \bar{p}_2)),
\]

where \( q_1, q_2 \in \mathbb{Q}^n, p_1, p_2 \in \text{Im}\mathbb{Q} \), and \( \mathbb{Q} \) stands for the quaternionic field \( \mathbb{Q} = \{ q = w + xI + yJ + zK, (w, x, y, z) \in \mathbb{R}^4 \} \) and \( \text{Im}\mathbb{Q} = \{ p = ri + sj + tk, (r, s, t) \in \mathbb{R}^3 \} \) with the Pauli matrices

\[
I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

Bearing in mind these elementary algebraic computations, it is easily understood the definitions of the Bianchi-Cartan-Vranceanu spaces (BCV spaces for short) as well as their natural extensions, as we will do in next sections.

## 2 The Bianchi-Cartan-Vranceanu (BCV) spaces (see [4])

It was Cartan ([5]) who obtained the families of today known as BCV-spaces by classifying three-dimensional Riemannian manifolds with four-dimensional isometry group. They also appeared in the work of L. Bianchi ([2, 3]), and G. Vranceanu ([15]). These kind of spaces have been extensively studied and classified (see for instance [10, 14]). In theoretical cosmology they are known as Bianchi-Kantowski-Saks spaces, which are used to construct some homogeneous spacetimes ([5]).

For real numbers \( m \) and \( l \), consider the set

\[
\text{BCV}(m, l) = \{(x, y, z) \in \mathbb{R}^3 : 1 + m(x^2 + y^2) > 0\}
\]

equipped with the metric

\[
ds_{m,l}^2 = \frac{dx^2 + dy^2}{(1 + m(x^2 + y^2))^2} + \left( dr + \frac{l}{2} \frac{x dy - y dx}{1 + m(x^2 + y^2)} \right)^2.
\]

Observe that this metric is obtained as a conformal deformation of the planar Euclidean metric by adding the imaginary part of \( z \, d\bar{z} \), for a complex number \( z \).

The complete classification of BCV spaces is as follows:

(i) If \( m = l = 0 \), then \( \text{BCV}(m, l) \cong \mathbb{R}^3 \);

(ii) If \( m = \frac{1}{4} \), then \( \text{BCV}(m, l) \cong (\mathbb{S}^3(m) - \{\infty\}) \);

(iii) If \( m > 0 \) and \( l = 0 \), then \( \text{BCV}(m, l) \cong (\mathbb{S}^2(4m) - \{\infty\}) \times \mathbb{R} \);
(iv) If $m < 0$ and $l = 0$, then $BCV(m, l) \cong (\mathbb{H}^2(4m) - \{\infty\}) \times \mathbb{R}$;
(v) If $m > 0$ and $l \neq 0$, then $BCV(m, l) \cong SU(2) - \{\infty\}$;
(vi) If $m < 0$ and $l \neq 0$, then $BCV(m, l) \cong \widetilde{SL}(2, \mathbb{R})$;
(vii) If $m = 0$ and $l \neq 0$, then $BCV(m, l) \cong Nil_3$.

The following vector fields form an orthonormal frame of $BCV(m, l)$:

$$E_1 = (1 + m(x^2 + y^2)) \partial_x - \frac{l}{2}y \partial_z, \quad E_2 = (1 + m(x^2 + y^2)) \partial_y + \frac{l}{2}x \partial_z, \quad E_3 = \partial_z.$$  

Let $\mathcal{D}$ be the distribution generated by $\{E_1, E_2\}$, then the manifold $(BCV(m, l), \mathcal{D}, ds_{m,l}^2)$ is an example of sub-riemannian geometry (see [3, 12]) and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

### 3 Extended Bianchi-Cartan-Vranceanu spaces

#### 3.1 Set up

Observe that letting $z = x + iy$, we see that $\text{Im}(z \, d\bar{z}) = ydx - xdy$, which reminds us the map $\mathbb{C} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}$ given by $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2(z_1 \bar{z}_2))$, that easily leads to the classical Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, where coordinates in $S^2$ are given by $(|z_1|^2 - |z_2|^2, 2\text{Re}(z_1 \bar{z}_2), 2\text{Im}(z_1 \bar{z}_2))$.

In the same line, using quaternions $\mathbb{H}$ instead of complex numbers, we get the fibration $S^3 \hookrightarrow S^7 \to S^4$. Quaternions are usually presented with the imaginary units $i, j, k$ in the form $q = x_0 + x_1i + x_2j + x_3k$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$ with $i^2 = j^2 = k^2 = ijk = -1$. They can also be defined equivalently, using the complex numbers $c_1 = x_0 + x_1i$ and $c_2 = x_2 + x_3i$, in the form $q = c_1 + c_2j$. Then for a point $(q_1 = \alpha + \beta j, q_2 = \gamma + \delta j) \in S^7$, we get the following coordinate expressions $(|q_1|^2 - |q_2|^2, 2\text{Re}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\text{Im}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\text{Re}(\alpha\delta - \beta\gamma), 2\text{Im}(\alpha\delta - \beta\gamma))$.

For any $q = w + xi + yj + zk \in \mathbb{H}$ we find that $qd\bar{q} = wdw + xdx + ydy + zdz + (xdw - wdx + zdy - ydz)i + (ydw - wdy + xdz - zdx)j + (zdw - wdz + ydx - xdy)k$. As the quaternionic contact group $\mathbb{H} \times \text{Im}\mathbb{H}$, with coordinates $(w, x, y, z, r, s, t)$ can be equipped with the metric

$$ds^2 = (dw^2 + dx^2 + dy^2 + dz^2) + \left(dr + \frac{1}{2}(xdw - wdx + zdy - ydz)\right)^2 + \left(ds + \frac{1}{2}(ydw - wdy + xdz - zdx)\right)^2 + \left(dt + \frac{1}{2}(zdw - wdz + ydx - xdy)\right)^2.$$

Then, by extending this metric, it seems natural to find a 7-dimensional generalization of the 3-dimensional $BCV$ spaces endowed with the two-parameter family of metrics.
\[ \text{ds}_{m,l}^2 = \frac{dw^2 + dx^2 + dy^2 + dz^2}{K^2} + \left( \frac{dr + \frac{1}{2}wdx - xdw + ydz - zdy}{K} \right)^2 \]
\[ + \left( ds + \frac{1}{2}wdy - ydw + zdx - xdz \right)^2 + \left( \frac{dt + \frac{1}{2}wdz - zdw + xdy - ydx}{K} \right)^2, \]

where \( l, m \) are real numbers and \( K = 1 + m(w^2 + x^2 + y^2 + z^2) > 0 \).

Then \((EBCV, \text{ds}_{m,l}^2)\) will be called extended BCV spaces \((EBCV\) for short).

That metric is obtained as a conformal deformation of the Euclidean metric of \( \mathbb{R}^4 \) by adding three suitable terms which depend on \( l \) and \( m \) concerning the imaginary part of \( q\overline{q} \), for a quaternion \( q \). When \( m = 0 \) we get a one-parameter of Riemannian metrics depending on \( l \). Furthermore, if \( l = 1 \), we find the 7-dimensional quaternionic Heisenberg group (see [7] and [16]). The manifold \( EBCV \) provides another example of sub-riemannian geometry and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

Observe that when \( m = l = 0 \), \( EBCV \) is nothing but \( \mathbb{R}^7 \); when \( m > 0, l = 0 \), \( EBCV \cong S^4(4m) \times \mathbb{R}^3 \) and when \( m < 0, l = 0 \), \( EBCV \cong H^4(4m) \times \mathbb{R}^3 \).

The metric \( \text{ds}_{m,l}^2 \) can also be written as
\[ \text{ds}_{m,l}^2 = \sum_{\alpha=1}^{7} \omega^a \otimes \omega^a, \]

where
\[
\begin{align*}
\omega^1 &= dr + \frac{l}{2K} (wdx - xdw + ydz - zdy), & \omega^4 &= \frac{1}{K} dw, \\
\omega^2 &= ds + \frac{l}{2K} (wdy - ydw + zdx - xdz), & \omega^5 &= \frac{1}{K} dx, \\
\omega^3 &= dt + \frac{l}{2K} (wdz - zdw + xdy - ydx), & \omega^6 &= \frac{1}{K} dy, \\
\omega^7 &= dz.
\end{align*}
\]

with the corresponding dual orthonormal frame
\[
\begin{align*}
X_1 &= \partial_r, & X_2 &= \partial_s, & X_3 &= \partial_t, \\
X_4 &= K \partial_w + \frac{lx}{2} \partial_r + \frac{ly}{2} \partial_s + \frac{lz}{2} \partial_t, & X_5 &= K \partial_x - \frac{lw}{2} \partial_r - \frac{lx}{2} \partial_s + \frac{ly}{2} \partial_t, \\
X_6 &= K \partial_y + \frac{lx}{2} \partial_r - \frac{lw}{2} \partial_s - \frac{lx}{2} \partial_t, & X_7 &= K \partial_z - \frac{ly}{2} \partial_r + \frac{lx}{2} \partial_s - \frac{lw}{2} \partial_t.
\end{align*}
\]

Writing \( 1 \leq i, j \leq 3, 4 \leq a \leq 7 \), we find that
\[ [X_i, X_j] = 0; \quad [X_i, X_a] = 0, \]
as well as

\[ [X_4, X_5] = -l\{1 + m(x^2 + y^2)\}X_1 + ml(wx + xy)X_2 - ml(wy - xz)X_3 - 2mxX_4 + 2mwX_5, \]

and so on (see Appendix).

For later use, when \( m = 0 \) brackets reduce to

\[ [X_4, X_5] = -lX_1, \quad [X_4, X_6] = -lX_2, \quad [X_4, X_7] = -lX_3, \]

\[ [X_5, X_6] = -lX_3, \quad [X_5, X_7] = lX_2, \quad [X_6, X_7] = -lX_1. \]

**Remark 1** When \( l = 1 \), we have the brackets of the quaternionic contact manifold.

As for the Levi-Civita connection we find out

\[ \nabla_{X_i}X_j = 0, \quad \nabla_{X_i}X_a = \nabla_{X_a}X_i, \]

and

\[ \nabla_{X_i}X_4 = \frac{l}{2}\{1 + m(y^2 + z^2)\}X_5 + \frac{ml}{2}(wx - yz)X_6 - \frac{ml}{2}(wy + xz)X_7, \]

\[ \nabla_{X_i}X_5 = -\frac{l}{2}\{1 + m(y^2 + z^2)\}X_4 + \frac{ml}{2}(wy + xz)X_6 + \frac{ml}{2}(wx - yz)X_7, \]

\[ \nabla_{X_i}X_6 = -\frac{ml}{2}(wx - yz)X_4 - \frac{ml}{2}(wy + xz)X_5 + \frac{l}{2}(1 - m(y^2 + z^2))X_7, \]

\[ \nabla_{X_i}X_7 = \frac{ml}{2}(wy + xz)X_4 - \frac{ml}{2}(wx - yz)X_5 - \frac{l}{2}(1 + m(w^2 + x^2))X_6, \]

and so on (see Appendix).

When \( m = 0 \), the Levi-Civita connection reduces to

\[ \nabla_{X_1}X_4 = \frac{l}{2}X_5, \quad \nabla_{X_1}X_5 = -\frac{l}{2}X_4, \quad \nabla_{X_1}X_6 = \frac{l}{2}X_7, \quad \nabla_{X_1}X_7 = -\frac{l}{2}X_6, \]

\[ \nabla_{X_2}X_4 = \frac{l}{2}X_6, \quad \nabla_{X_2}X_5 = -\frac{l}{2}X_7, \quad \nabla_{X_2}X_6 = \frac{l}{2}X_5, \quad \nabla_{X_2}X_7 = -\frac{l}{2}X_4, \]

\[ \nabla_{X_3}X_4 = \frac{l}{2}X_7, \quad \nabla_{X_3}X_5 = \frac{l}{2}X_6, \quad \nabla_{X_3}X_6 = \frac{l}{2}X_5, \quad \nabla_{X_3}X_7 = -\frac{l}{2}X_4, \]

\[ \nabla_{X_4}X_4 = \frac{l}{2}X_1, \quad \nabla_{X_4}X_5 = \frac{l}{2}X_7, \quad \nabla_{X_4}X_6 = \frac{l}{2}X_5, \quad \nabla_{X_4}X_7 = \frac{l}{2}X_6, \]

\[ \nabla_{X_5}X_4 = \frac{l}{2}X_1, \quad \nabla_{X_5}X_5 = \frac{l}{2}X_7, \quad \nabla_{X_5}X_6 = \frac{l}{2}X_5, \quad \nabla_{X_5}X_7 = -\frac{l}{2}X_4, \]

\[ \nabla_{X_6}X_4 = \frac{l}{2}X_7, \quad \nabla_{X_6}X_5 = \frac{l}{2}X_6, \quad \nabla_{X_6}X_6 = \frac{l}{2}X_1, \quad \nabla_{X_6}X_7 = \frac{l}{2}X_5, \]

**Remark 2** When \( l = 1 \), we find the Levi-Civita connection of the quaternionic contact manifold.

As for the curvature tensor \( R \) we have

\[ R_{X_1X_4X_1X_4} = R_{X_1X_5X_1X_5} = \frac{l^2}{4}\{1 + m(K + 1)(y^2 + z^2)\}, \]

\[ R_{X_1X_6X_1X_6} = R_{X_1X_7X_1X_7} = \frac{l^2}{4}\{1 + m(K + 1)(w^2 + x^2)\}, \]

and so on (see Appendix).

**Remark 3** When \( m = 0 \), the curvature of the quaternionic contact manifold reduces to

\[ R_{X_1X_4X_1X_4} = \frac{l^2}{4}, \quad \ldots \quad R_{X_6X_6X_6X_6} = \frac{3l^2}{4}. \]
3.2 The Ricci tensor

**Proposition 4** The matrix representing the Ricci tensor is given by

\[
\begin{pmatrix}
\frac{l^2}{T}(K^2 + 1) & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{l^2}{T}(K^2 + 1) & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{l^2}{T}(K^2 + 1) & 0 & 0 & 0 \\
-mlx(K+2) & -mly(K+2) & -mlz(K+2) & -mlw(K+2) & mlz(K+2) & mlw(K+2) \\
mlw(K+2) & mlz(K+2) & -mly(K+2) & mlx(K+2) & -mlz(K+2) & mlw(K+2) \\
-mlz(K+2) & mlw(K+2) & mlx(K+2) & -mlz(K+2) & mlw(K+2) & mlx(K+2) \\
mlx(K+2) & mlw(K+2) & mlz(K+2) & mlx(K+2) & mlw(K+2) & mlz(K+2) \\
\end{pmatrix}
\]

where

\[
A = -l^2(K+1) \quad \text{and} \quad B = 12m - 3/2l^2.
\]

Some particular cases could be interesting, for instance we get the following Ricci matrix when \(K = 1\) (or \(m = 0\))

\[
\text{Ric}_1 = \begin{pmatrix}
l^2 & 0 & 0 & 0 & 0 & 0 \\
0 & l^2 & 0 & 0 & 0 & 0 \\
0 & 0 & l^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -3/2l^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -3/2l^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -3/2l^2 \\
\end{pmatrix}
\]

**Remark 5** When \(l = 1\), we find the Ricci curvature of the quaternionic contact manifold.

An easy computation leads to

**Corollary 6** The \(EBCV\) manifold has constant scalar curvature \(S = 48m\).
4 The homogeneous structure

In [1] W. Ambrose and I. M. Singer proved that a connected, complete and simply-connected Riemannian manifold \((M, g)\) is homogeneous if and only if there exists a \((1,2)\) tensor field \(T\) such that

\[
\begin{align*}
(i) & \quad g(T_X Y, Z) + g(Y, T_X Z) = 0, \\
(ii) & \quad (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{TXYZ} - R_{Y TXZ}, \\
(iii) & \quad (\nabla_X T)_Y = [T_X, T_Y] - T_{TXY},
\end{align*}
\]

for \(X, Y, Z \in \mathfrak{X}(M)\), where \(\nabla\) stands for the Levi-Civita connection and \(R\) is the Riemann curvature tensor of \(M\) (see [13]). As a consequence, Tricerri and Vanhecke define a homogeneous Riemannian structure on \((M, g)\) as a \((1,2)\) tensor field \(T\) which is a solution of the above three equations. Instead of taking \((1,2)\) tensors it is preferred to work with \((0,3)\) tensors via the isomorphism \(T_{uvw} = g(T_u v, w)\), for \(u, v, w \in T_p M\) and \(p \in M\).

Then they consider the vector space \(\mathfrak{F}\) of \((0,3)\) tensors having the same symmetries as a homogeneous structure, i. e., \(\mathfrak{F} = \{T : T_{uvw} = -T_{uwv}, u, v, w \in T_p M\}\). The natural action of the orthogonal group \(O(T_p M)\) on \(\mathfrak{F}\) gives us the decomposition into eight irreducible invariant components. The main building blocks are defined as follows:

\[
\begin{align*}
\mathfrak{F}_1 & = \{T \in \mathfrak{F} : T_{uvw} = g(u,v)\alpha(w) - g(u,w)\alpha(v), \alpha \in T^*_p(M)\}, \\
\mathfrak{F}_2 & = \{T \in \mathfrak{F} : T_{uvw} + T_{uwv} + T_{vuw} = 0, c_{12}(T) = 0\}, \\
\mathfrak{F}_3 & = \{T \in \mathfrak{F} : T_{uvw} + T_{vuw} = 0\},
\end{align*}
\]

where \(u, v, w \in T_p M\) and \(c_{12}(T)(w) = \sum T_{e_i e_i w},\) for any orthonormal basis \(\{e_i\}\) of \(T_p M\).

We consider in \(EBCV\) the characteristic connection \(D\) defined by (see [11]):

\[
D_A B = \nabla_A B + \frac{P}{2}(\nabla_A P)B,
\]

where \(P\) is the natural almost product structure given by \(P = \mathcal{V} - \mathcal{H}, Id = \mathcal{V} + \mathcal{H}\). Let us remember that the vertical distribution in \(EBCV\) is spanned by \(X_1, X_2, X_3\) and the horizontal distribution by \(X_4, X_5, X_6, X_7\). Then we have

\[
\begin{align*}
D_{X_i} X_j & = \mathcal{V}(\nabla_{X_i} X_j), \quad i, j = 1, 2, 3, \\
D_{X_a} X_j & = \mathcal{V}(\nabla_{X_a} X_j), \quad a = 4 \ldots, 7; \quad j = 1, 2, 3, \\
D_{X_i} X_b & = \mathcal{H}(\nabla_{X_i} X_b), \quad i = 1, 2, 3; \quad b = 4 \ldots, 7, \\
D_{X_a} X_b & = \mathcal{H}(\nabla_{X_a} X_b), \quad a, b = 4 \ldots, 7.
\end{align*}
\]

This is a metric connection which makes parallel both the curvature and the torsion tensors and can be completely obtained by using the table giving the Levi-Civita connection.

By denoting \(T^D\) the torsion tensor of \(D\), that is,

\[
T^D_L M \equiv T^D(L, M) = D_L M - D_M L - [L, M],
\]

where \(L, M \in \mathfrak{X}(M)\).
or equivalently

\[ T^D(L, M) = \frac{P}{2} ((\nabla_L P) M - (\nabla_M P) L), \]

we find out

\[ T^D(X_i, X_j) = T^D(X_i, X_a) = 0, \quad i, j = 1, 2, 3; \quad a = 4, \ldots, 7, \]

\[ T^D(X_4, X_4) = T^D(X_5, X_5) = T^D(X_6, X_6) = T^D(X_7, X_7) = 0, \]

as well as

\[ T^D(X_4, X_5) = l\{ (1 + m(y^2 + z^2))X_1 - m(wx + yz)X_2 + m(wz + xy)X_3 \}, \]

\[ T^D(X_4, X_6) = l\{ m(wz - xy)X_1 + (1 + m(x^2 + z^2))X_2 - m(wx + yz)X_3 \}, \]

\[ T^D(X_4, X_7) = l\{-m(wz + xy)X_1 + (m(wx - yz))X_2 + (1 + m(x^2 + y^2))X_3 \}, \]

\[ T^D(X_5, X_6) = l\{ m(wy + xz)X_1 - (m(wx - yz)X_2 + (1 + m(w^2 + z^2))X_3 \}, \]

\[ T^D(X_5, X_7) = l\{ m(wz - xy)X_1 - (1 + m(w^2 + y^2))X_2 - m(wx + yz)X_3 \}, \]

\[ T^D(X_6, X_7) = l\{ (1 + m(w^2 + x^2))X_1 + m(wx + xy)X_2 - m(wy - xz)X_3 \}. \]

Then \( T^D \) defines a homogeneous structure on \( EBCV \) in the sense of Tricerri-Vanhecke (see [13], pags I and 15-16). Furthermore, it is easy to see that

\[ c_{12}(T) = \sum T_{X_r}X_r = 0, \quad r = 1, \ldots, 7, \]

so that \( T^D \) defines a homogeneous structure which is lying in the class \( \mathfrak{F}_2 \oplus \mathfrak{F}_3 \).

However, \( T^D \) does not belong to \( \mathfrak{F}_2 \), since, for instance,

\[ T^D_{X_1X_4X_5} + T^D_{X_5X_1X_4} + T^D_{X_4X_5X_1} = (T^D_{X_1X_4X_5}) + (T^D_{X_5X_1X_4}) + (T^D_{X_4X_5X_1}) = l\{ 1 + m(y^2 + z^2) \} \neq 0. \]

Finally, from the definition of \( T^D \) we have that

\[ T^D_{XYZ} = (T^D_X Y, Z) = - (T^D_Y X, Z), \]

that is, \( T^D_{XYZ} + T^D_{YXZ} = 0 \), and therefore \( T^D \) is lying in \( \mathfrak{F}_3 \).
5 Killing vector fields in $EBCV$

Remember that a Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are continuous isometries of the manifold. Specifically, a vector field $X$ is a Killing vector field if the Lie derivative with respect to $X$ of the metric $g$ vanishes: $\mathcal{L}_X g = 0$ or equivalently

$$\mathcal{L}_X ds^2_{i,m} = (\mathcal{L}_X \omega^a) \otimes \omega^a = 0,$$

where

$$\mathcal{L}_X \omega^a = \iota_X d\omega^a + d(\iota_X \omega^a).$$

In terms of the Levi-Civita connection, Killing’s condition is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0.$$  

(2)

It is easy to prove that

**Proposition 7** $\mathcal{L}_X g(Y, Z) = 0$ if and only if $\mathcal{L}_X g(X_i, X_j) = 0$ for basic vector fields $X_i, X_j$.

We know that the dimension of the Lie algebra of the Killing vector fields is $m \leq n(n + 1)/2$ and the maximum is reached on constant curvature manifolds ([6], p. 238, Vol. II), then for our manifold $m < 28$. Then obviously

**Proposition 8** The basic vertical vector fields $X_1, X_2, X_3$ are Killing fields.

From (2) it is easy to prove that the horizontal basic vector fields $X_4, \cdots, X_7$ are not Killing vector fields.

In her thesis, Profir [10] proved that the Lie algebra of Killing vector fields of $BCV$ spaces is 4-dimensional. Our problem now is to determine the space of Killing vector fields in $EBCV$.

5.1 The Killing equations

In the usual coordinate system $(r, s, t, w, x, y, z)$ on $EBCV$, a vector field $X = \sum_{a=1}^{7} f_a X_a$ will be a Killing field if and only if the real functions $f_i$ satisfy the following system of 28-partial differential equations:

$$\mathcal{L}_X ds^2_{i,m} = (\mathcal{L}_X \omega^a) \otimes \omega^a = 0,$$

where

$$\mathcal{L}_X \omega^a = \iota_X d\omega^a + d(\iota_X \omega^a).$$

In terms of the Levi-Civita connection, Killing’s condition is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0.$$  

(2)
\[ \partial_r(f_1) = 0, \]
\[ \partial_s(f_2) = 0, \]
\[ \partial_t(f_3) = 0, \]
\[ \partial_r(f_2) + \partial_s(f_1) = 0, \]
\[ \partial_r(f_3) + \partial_t(f_1) = 0, \]
\[ \partial_s(f_3) + \partial_t(f_2) = 0, \]
\[ \partial_r(f_4) + K\partial_w(f_1) + \frac{\partial}{\partial x}\partial_r(f_1) + \frac{\partial}{\partial y}\partial_r(f_1) - l\{1 + m(y^2 + z^2)\}f_5 - ml(wz - xy)f_6 + ml(wy + xz)f_7 = 0, \]
\[ \partial_r(f_5) + K\partial_x(f_1) - \frac{\partial}{\partial x}\partial_r(f_1) + \frac{\partial}{\partial y}\partial_r(f_1) + l\{1 + m(y^2 + z^2)\}f_4 - ml(wy + xz)f_6 - ml(wz - xy)f_7 = 0, \]
\[ \partial_r(f_6) + K\partial_y(f_1) - \frac{\partial}{\partial x}\partial_r(f_1) + \frac{\partial}{\partial y}\partial_r(f_1) + ml(wz - xy)f_4 + ml(wy + xz)f_5 - l\{1 + m(w^2 + x^2)\}f_7 = 0, \]
\[ \partial_r(f_7) + K\partial_z(f_1) + \frac{\partial}{\partial x}\partial_r(f_1) - \frac{\partial}{\partial y}\partial_r(f_1) - ml(wy + xz)f_4 + ml(wz - xy)f_5 + \{1 + m(w^2 + x^2)\}f_6 = 0, \]
\[ \partial_s(f_4) + K\partial_w(f_2) + \frac{\partial}{\partial x}\partial_s(f_2) + \frac{\partial}{\partial y}\partial_s(f_2) + ml(wz + xy)f_5 - l\{1 + m(x^2 + z^2)\}f_6 - ml(wx - yz)f_7 = 0, \]
\[ \partial_s(f_5) + K\partial_x(f_2) - \frac{\partial}{\partial x}\partial_s(f_2) + \frac{\partial}{\partial y}\partial_s(f_2) - ml(wz + xy)f_4 + ml(wx - yz)f_6 + l\{1 + m(w^2 + y^2)\}f_7 = 0, \]
\[ \partial_s(f_6) + K\partial_y(f_2) - \frac{\partial}{\partial x}\partial_s(f_2) - \frac{\partial}{\partial y}\partial_s(f_2) + ml(wx - yz)f_4 - ml(wz + xy)f_5 - ml(wx + yz)f_7 = 0, \]
\[ \partial_s(f_7) + K\partial_z(f_2) - \frac{\partial}{\partial x}\partial_s(f_2) - \frac{\partial}{\partial y}\partial_s(f_2) + ml(wx + yz)f_4 + l\{1 + m(w^2 + z^2)\}f_5 + ml(wx - yz)f_7 = 0, \]
\[ \partial_t(f_4) + K\partial_w(f_3) + \frac{\partial}{\partial x}\partial_t(f_3) + \frac{\partial}{\partial y}\partial_t(f_3) + l\{1 + m(x^2 + y^2)\}f_4 - ml(wx + yz)f_5 - ml(wx - yz)f_6 = 0, \]
\[ K\partial_w(f_4) + \frac{\partial}{\partial x}\partial_t(f_4) + \frac{\partial}{\partial y}\partial_t(f_4) + 2mzf_4 - 2myf_6 - 2mzf_7 = 0, \]
\[ K\partial_w(f_5) + \frac{\partial}{\partial x}\partial_t(f_5) + \frac{\partial}{\partial y}\partial_t(f_5) + l\{1 + m(x^2 + z^2)\}f_5 = 0, \]
\[ K\partial_w(f_6) + \frac{\partial}{\partial x}\partial_t(f_6) + \frac{\partial}{\partial y}\partial_t(f_6) + 2mzf_4 + 2myf_5 = 0, \]
\[ K\partial_w(f_7) + \frac{\partial}{\partial x}\partial_t(f_7) + \frac{\partial}{\partial y}\partial_t(f_7) + 2mzf_5 = 0, \]
\[ K\partial_s(f_4) - \frac{\partial}{\partial x}\partial_s(f_4) - \frac{\partial}{\partial y}\partial_s(f_4) = 0, \]
\[ K\partial_s(f_5) - \frac{\partial}{\partial x}\partial_s(f_5) - \frac{\partial}{\partial y}\partial_s(f_5) + 2mzf_4 = 0, \]
\[ K\partial_s(f_6) - \frac{\partial}{\partial x}\partial_s(f_6) - \frac{\partial}{\partial y}\partial_s(f_6) + 2mzf_5 = 0, \]
\[ K\partial_s(f_7) - \frac{\partial}{\partial x}\partial_s(f_7) - \frac{\partial}{\partial y}\partial_s(f_7) + 2mzf_6 = 0, \]
\[ K\partial_t(f_4) - \frac{\partial}{\partial x}\partial_t(f_4) - \frac{\partial}{\partial y}\partial_t(f_4) - 2mzf_4 - 2myf_6 - 2mzf_7 = 0, \]
\[ K\partial_t(f_5) - \frac{\partial}{\partial x}\partial_t(f_5) - \frac{\partial}{\partial y}\partial_t(f_5) - 2mzf_4 - 2myf_5 = 0, \]
\[ K\partial_t(f_6) - \frac{\partial}{\partial x}\partial_t(f_6) - \frac{\partial}{\partial y}\partial_t(f_6) - 2mzf_5 = 0, \]
\[ K\partial_t(f_7) - \frac{\partial}{\partial x}\partial_t(f_7) - \frac{\partial}{\partial y}\partial_t(f_7) - 2mzf_6 = 0, \]
It seems that the solution of the system is very difficult, so that we focus on solving the system for \( m = 0 \), that is:

\[
\begin{align*}
\partial_r(f_1) &= 0, \\
\partial_s(f_2) &= 0, \\
\partial_t(f_3) &= 0, \\
\partial_r(f_2) + \partial_s(f_1) &= 0, \\
\partial_r(f_3) + \partial_t(f_1) &= 0, \\
\partial_s(f_3) + \partial_t(f_2) &= 0, \\
\partial_r(f_4) + \partial_w(f_1) + \frac{lw}{2} \partial_s(f_1) + \frac{l}{2} \partial_t(f_1) - lf_5 &= 0, \\
\partial_r(f_5) + \partial_x(f_1) - \frac{lx}{2} \partial_s(f_1) + \frac{l}{2} \partial_t(f_1) + lf_4 &= 0, \\
\partial_r(f_6) + \partial_y(f_1) - \frac{ly}{2} \partial_s(f_1) - \frac{l}{2} \partial_t(f_1) - lf_7 &= 0, \\
\partial_r(f_7) + \partial_z(f_1) + \frac{l}{2} \partial_s(f_1) - \frac{lz}{2} \partial_t(f_1) + lf_6 &= 0, \\
\partial_s(f_4) + \partial_u(f_2) + \frac{lu}{2} \partial_r(f_2) + \frac{l}{2} \partial_t(f_2) - lf_6 &= 0, \\
\partial_s(f_5) + \partial_x(f_2) - \frac{lx}{2} \partial_r(f_2) + \frac{l}{2} \partial_t(f_2) + lf_7 &= 0, \\
\partial_s(f_6) + \partial_y(f_2) + \frac{ly}{2} \partial_r(f_2) - \frac{l}{2} \partial_t(f_2) + lf_4 &= 0, \\
\partial_s(f_7) + \partial_z(f_2) - \frac{lz}{2} \partial_r(f_2) - \frac{l}{2} \partial_t(f_2) - lf_5 &= 0, \\
\partial_t(f_4) + \partial_w(f_3) + \frac{lw}{2} \partial_r(f_3) + \frac{l}{2} \partial_s(f_3) - lf_7 &= 0, \\
\partial_t(f_5) + \partial_x(f_3) - \frac{lx}{2} \partial_r(f_3) - \frac{l}{2} \partial_s(f_3) - lf_6 &= 0, \\
\partial_t(f_6) + \partial_y(f_3) + \frac{ly}{2} \partial_r(f_3) - \frac{l}{2} \partial_s(f_3) + lf_5 &= 0, \\
\partial_t(f_7) + \partial_z(f_3) - \frac{lz}{2} \partial_r(f_3) + \frac{l}{2} \partial_s(f_3) + lf_1 &= 0, \\
\partial_w(f_4) + \frac{lw}{2} \partial_r(f_4) + \frac{l}{2} \partial_s(f_4) + \frac{l}{2} \partial_t(f_4) &= 0, \\
\partial_w(f_5) + \frac{lw}{2} \partial_r(f_5) + \frac{l}{2} \partial_s(f_5) + \frac{l}{2} \partial_t(f_5) + \partial_x(f_4) - \frac{lx}{2} \partial_s(f_4) - \frac{l}{2} \partial_t(f_4) &= 0, \\
\partial_w(f_6) + \frac{lw}{2} \partial_r(f_6) + \frac{l}{2} \partial_s(f_6) + \frac{l}{2} \partial_t(f_6) + \partial_y(f_4) + \frac{ly}{2} \partial_r(f_4) - \frac{l}{2} \partial_s(f_4) - \frac{l}{2} \partial_t(f_4) &= 0, \\
\partial_w(f_7) + \frac{lw}{2} \partial_r(f_7) + \frac{l}{2} \partial_s(f_7) + \frac{l}{2} \partial_t(f_7) + \partial_z(f_4) - \frac{lz}{2} \partial_s(f_4) + \frac{l}{2} \partial_t(f_4) &= 0, \\
\partial_x(f_5) - \frac{lx}{2} \partial_r(f_5) - \frac{l}{2} \partial_s(f_5) + \frac{l}{2} \partial_t(f_5) &= 0, \\
\partial_x(f_6) - \frac{lx}{2} \partial_r(f_6) - \frac{l}{2} \partial_s(f_6) + \frac{l}{2} \partial_t(f_6) + \partial_y(f_5) + \frac{ly}{2} \partial_r(f_5) - \frac{l}{2} \partial_s(f_5) - \frac{l}{2} \partial_t(f_5) &= 0, \\
\partial_x(f_7) - \frac{lx}{2} \partial_r(f_7) - \frac{l}{2} \partial_s(f_7) + \frac{l}{2} \partial_t(f_7) + \partial_z(f_5) - \frac{lz}{2} \partial_s(f_5) + \frac{l}{2} \partial_t(f_5) &= 0, \\
\partial_y(f_6) + \frac{ly}{2} \partial_r(f_6) - \frac{l}{2} \partial_s(f_6) - \frac{l}{2} \partial_t(f_6) &= 0, \\
\partial_y(f_7) + \frac{ly}{2} \partial_r(f_7) - \frac{l}{2} \partial_s(f_7) - \frac{l}{2} \partial_t(f_7) + \partial_z(f_6) - \frac{lz}{2} \partial_s(f_6) + \frac{l}{2} \partial_t(f_6) &= 0, \\
\partial_x(f_7) - \frac{lx}{2} \partial_r(f_7) + \frac{l}{2} \partial_s(f_7) - \frac{l}{2} \partial_t(f_7) &= 0.
\end{align*}
\]
whose solution is given by

\[ f_1(r, s, t, w, x, y, z) = (P + R)s + (S - N)t + \frac{l}{2}(-M(w^2 + x^2) - U(y^2 + z^2) + (R - P)(wy + xz)
+ (N + S)(wz - xy) + 2Tw - 2Qx + 2Wy - 2Vz) + C_1, \]

\[ f_2(r, s, t, w, x, y, z) = -(P + R)r + (M + U)t - \frac{l}{2}\{N(w^2 + y^2) - S(x^2 + z^2) + (R - P)(wx - yz)
+ (M - U)(wz + xy) - 2Vw + 2Wx + 2Qy - 2Tz\} + C_2, \]

\[ f_3(r, s, t, w, x, y, z) = -(S - N)r - (M + U)s - \frac{l}{2}\{P(w^2 + z^2) + R(x^2 + y^2) + (N + S)(wx + yz)
+ (U - M)(wy - xz) - 2Ww - 2Vx + 2Ty + 2Qz\} + C_3, \]

\[ f_4(r, s, t, w, x, y, z) = Mx + Ny + Pz + Q, \]

\[ f_5(r, s, t, w, x, y, z) = -Mw + Ry + Sz + T, \]

\[ f_6(r, s, t, w, x, y, z) = -Nw - Rx + Uz + V, \]

\[ f_7(r, s, t, w, x, y, z) = -Pw - Sx - Uy + W, \]

where \( M, N, P, Q, R, S, T, U, V, C_1, C_2, C_3 \in \mathbb{R} \).

As a consequence, when \( m = 0 \), we obtain

**Proposition 9** The Lie algebra of Killing vector fields is 13-dimensional.

### 6 Computing horizontal geodesics of the quaternionic Heisenberg group (see [9])

A Riemannian metric on a manifold \( M \) is defined by a covariant two-tensor, which is to say, a section of the bundle \( S^2(T^*M) \). There is no such object in subriemannian geometry. Instead, a subriemannian metric can be encoded as a contravariant symmetric two-tensor, which is a section of \( S^2(TM) \). This two-tensor has rank \( k < n \), where \( k \) is the rank of the distribution, so it cannot be inverted to obtain a Riemannian metric. We call this contravariant tensor the **cometric**.

**Definition 10** A cometric is a section of the bundle \( S^2(TM) \subset TM \otimes TM \) of symmetric bilinear forms on the cotangent bundle of \( M \).

Since \( TM \) and \( T^*M \) are dual, any cometric defines a fiber-bilinear form \((\cdot, \cdot) : T^*M \otimes T^*M \to \mathbb{R}\), i.e. a kind of inner product on covectors. This form in turn defines a symmetric bundle map \( \beta : T^*M \to TM \) by \( p(\beta_q(\mu)) = ((p, \mu))_q \), for \( p, \mu \in T^*_qM \) and \( q \in M \). Thus \( \beta_q(\mu) \in T_qM \). The adjective symmetric means that \( \beta \) equals its adjoint \( \beta^* : T^*M \to T^{**}M = TM \).

The cometric \( \beta \) for a subriemannian geometry is uniquely defined by the following conditions:
(1) \( \text{im}(\beta_q) = \mathcal{H}_q \);

(2) \( p(v) = \langle \beta_q(p), v \rangle \), for \( v \in \mathcal{H}_q \), \( p \in T_qM \),

where \( \langle \beta_q(p), v \rangle_q \) is the subriemannian inner product on \( \mathcal{H}_q \). Conversely, any cometric of constant rank defines a subriemannian geometry whose underlying distribution has that rank.

**Definition 11** The fiber-quadratic function \( H(q, p) = \frac{1}{2} (p, p)_q \), where \( (\cdot, \cdot)_q \) is the subriemannian inner product on \( \mathcal{H}_q \), is called the subriemannian Hamiltonian, or the kinetic energy.

The Hamiltonian \( H \) is related to length and energy as follows. Suppose that \( \gamma \) is a horizontal curve. Then, \( \dot{\gamma}(t) = \beta_{\gamma(t)}(p) \), for same covector \( p \in T_{\gamma(t)}^*M \), and

\[
\frac{1}{2} ||\dot{\gamma}||^2 = H(q, p).
\]

\( H \) uniquely determines \( \beta \) by polarization, and \( \beta \) uniquely determines the subriemannian structure. This proves the following proposition:

**Proposition 12** The subriemannian structure is uniquely determined by its Hamiltonian. Conversely, any nonnegative fiber-quadratic Hamiltonian of constant fiber rank \( k \) gives rise to a subriemannian structure whose underlying distribution has rank \( k \).

To compute the subriemannian Hamiltonian we can start with a local frame \( \{X_a\}, a = 1, \ldots, k \), of vector fields for \( \mathcal{H} \). Thinking of the \( X_a \) as fiber-linear functions on the cotangent bundle, we rename them \( P_a \) so that

\[
P_a(q, p) = p(X_a(q)), \quad q \in M, p \in T_q^*M.
\]

**Definition 13** Let \( X \) be a vector field on the manifold \( M \). The fiber-linear function on the cotangent bundle \( P_X : T^*M \to \mathbb{R} \), defined by \( P_X(q, p) = p(X(q)) \) is called the momentum function for \( X \).

Thus the \( P_a = P_{X_a} \) are the momentum functions for our horizontal frame. If \( X_a = \sum X_a^\alpha(x) \frac{\partial}{\partial x^\alpha} \) is the expression for \( X_a \) relative to coordinates \( x^\alpha \), then \( P_{X_a}(x, p) = \sum X_a^\alpha(x)p_\alpha \), where \( p_\alpha = P_{\frac{\partial}{\partial x^\alpha}} \) are the momentum functions for the coordinate vector fields. The \( x^\alpha \) and \( p_\alpha \) together form a coordinate system on \( T^*M \). They are called canonical coordinates.

Let \( g_{ab}(q) = \langle X_a(q), X_b(q) \rangle_q \) be the matrix of inner products defined by our horizontal frame. Let \( g^{ab}(q) \) be its inverse matrix. Then \( g^{ab} \) is a \( k \times k \) matrix-valued function defined in some open set of \( M \).

**Proposition 14** Let \( P_a \) and \( g^{ab} \) be the functions on \( T^*M \) that are induced by a local horizontal frame \( X_a \) as just described. Then

\[
H(q, p) = \frac{1}{2} \sum g^{ab}(q) P_a(q, p) P_b(q, p).
\]

13
Indeed,

$$H(q, p) = \frac{1}{2}(p, p)_q = \frac{1}{2}\left(\sum p_a dx^a, \sum p_b dx^b\right) = \frac{1}{2} \sum g^{ab}(q)(p_a, p_b)$$

$$= \frac{1}{2} \sum g^{ab}(q)(p(X_a)(q), p(X_b)(q)) = \frac{1}{2} \sum g^{ab}(q)P_a(q, p)P_b(q, p).$$

Note, in particular, that if the $X_a$ are an orthonormal frame for $H$ relative to the subriemannian inner product, then $H = \frac{1}{2}P^2_a$.

**Normal geodesics.** Like any smooth function on the cotangent bundle, our function $H$ generates a system of Hamiltonian differential equations. In terms of canonical coordinates $(x^a, p_a)$, these differential equations are

$$\dot{x}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial x^a}. \quad (4)$$

**Definition 15** The Hamiltonian differential equations (4) are called the normal geodesic equations.

Riemannian geometry can be viewed as a special case of subriemannian geometry, one in which the distribution is the entire tangent bundle. The cometric is the usual inverse metric, written $g^{ij}$ in coordinates. The normal geodesic equations in the Riemannian case are simply the standard geodesic equations, rewritten on the cotangent bundle.

### 6.1 One remark about the Hamiltonian

We follow word for word the computations in [9]. The vector fields

$$W = \partial_w + \frac{1}{2}(x\partial_r + y\partial_s + z\partial_t),$$

$$X = \partial_x - \frac{1}{2}(w\partial_r + z\partial_s - y\partial_t),$$

$$Y = \partial_y + \frac{1}{2}(z\partial_r - w\partial_s - x\partial_t),$$

$$Z = \partial_z - \frac{1}{2}(y\partial_r - x\partial_s - w\partial_t),$$

which are the old $X_4, \ldots, X_7$ ones, provided $m = 0, l = 1$, along with $\{\partial_r, \partial_s, \partial_t\}$, form an orthonormal frame for the quaternionic contact manifold $\mathbb{H} \times \text{Im}\mathbb{H}$. This means that $\{W, X, Y, Z\}$ form the fourth plane $\mathcal{H}$ and they are orthonormal with respect to the inner product $ds^2 = (dw^2 + dx^2 + dy^2 + dz^2)|_\mathcal{H}$ on the distribution. According to the above discussion, the subriemannian Hamiltonian is

$$H = \frac{1}{2}(P^2_W + P^2_X + P^2_Y + P^2_Z). \quad (5)$$
where \( P_W, P_X, P_Y, P_Z \) are the momentum functions of the vector fields \( W, X, Y, Z \), respectively. Thus

\[
\begin{align*}
P_W &= p_w + \frac{1}{2}(xp_r + yp_s + zp_t), \\
P_X &= p_x - \frac{1}{2}(wp_r + zp_s - yp_t), \\
P_Y &= p_y + \frac{1}{2}(zp_r - wp_s - xp_t), \\
P_Z &= p_z - \frac{1}{2}(yp_r - xp_s + wp_t),
\end{align*}
\]

where \( p_w, p_x, p_y, p_z, p_r, p_s, p_t \) are the fiber coordinates on the cotangent bundle of \( \mathbb{R}^7 \) corresponding to the cartesian coordinates \( w, x, y, z, r, s, t \) on \( \mathbb{R}^7 \). Again, these fiber coordinates are defined by writing a covector as \( w, x, y, z, r, s, t \) in the more common form of Hamilton’s equations. By letting the functions \( \{ \cdot, \cdot \} : \mathbb{R}^7 \rightarrow \mathbb{R} \) be any smooth function on the cotangent bundle, we will need the Poisson bracket. The Poisson bracket on the cotangent bundle \( T^*\mathbb{R}^7 \) of a manifold \( \mathbb{R}^7 \) is a canonical Lie algebra structure defined on the vector space \( \mathbb{C}^\infty(T^*\mathbb{R}^7) \) of smooth functions on \( T^*\mathbb{R}^7 \). The Poisson bracket is denoted \( \{ \cdot, \cdot \} : \mathbb{C}^\infty \times \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty \), where \( \mathbb{C}^\infty = \mathbb{C}^\infty(T^*\mathbb{R}^7) \), and can be defined by the coordinate formula

\[
\{ f, g \} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p_i}.
\]

This formula is valid in any canonical coordinate system, and can be shown to be coordinate independent. The Poisson bracket satisfies the Leibniz identity

\[
\{ f, gh \} = g\{ f, h \} + h\{ f, g \},
\]

which means that the operation \( \{ \cdot, \cdot \} \) defines a vector field \( X_H \), called the Hamiltonian vector field. By letting the functions \( f \) vary over the collection of coordinate functions \( x^i \) and we get the more common form of Hamilton’s equations

\[
\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}.
\]

Indeed, for the first one we take \( f = w \) and \( g = H \). Then \( \{ w, H \} = \frac{\partial w}{\partial x} \frac{\partial H}{\partial p_r} - \frac{\partial H}{\partial x} \frac{\partial w}{\partial p_r} \) if and only if \( \dot{w} = \frac{\partial H}{\partial p_w} \). Also we have

\[
\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}.
\]
These equations are in turn equivalent to the above formulation \((\ref{eq:7})\), which is more convenient to use, because the momentum function \(W \mapsto P_W\) is a Lie algebra anti-homomorphism from the Lie algebra of all smooth vector fields on \(\mathbb{R}^7\) to \(C(T^*\mathbb{R}^7)\) with the Poisson brackets:

\[
\{P_W, P_X\} = -P_{[W,X]}, \quad \{P_W, P_Y\} = -P_{[W,Y]}, \quad \{P_W, P_Z\} = -P_{[W,Z]}, \\
\{P_X, P_Y\} = -P_{[X,Y]}, \quad \{P_X, P_Z\} = -P_{[X,Z]}, \quad \{P_Y, P_Z\} = -P_{[Y,Z]}.
\]

Since all calculations are similar, we only prove the first one:

\[
\{P_W, P_X\} = \{p_w + \frac{x}{2}p_r + \frac{y}{2}p_s + \frac{z}{2}p_t, p_x - \frac{w}{2}p_r - \frac{z}{2}p_s + \frac{y}{2}p_t\} = p_r = -P_{[W,X]}.
\]

For the quaternionic contact group, with our choose of \(W, X, Y, Z\) as a frame for \(\mathcal{H}\), we compute

\[
[W, X] = -\partial_r, \quad [W, Y] = -\partial_s, \quad [W, Z] = -\partial_t,
\]

\[
[X, Y] = -\partial_t, \quad [X, Z] = \partial_s, \quad [Y, Z] = -\partial_t,
\]

\[
[W, \partial_r] = [W, \partial_r s] = [W, \partial_t] = [X, \partial_r] = [X, \partial_s] = [X, \partial_t] = 0,
\]

\[
[Y, \partial_r] = [Y, \partial_s] = [Y, \partial_t] = [Z, \partial_r] = [Z, \partial_s] = [Z, \partial_t] = 0.
\]

Thus

\[
\{P_W, P_X\} = \partial_r := P_r, \quad \{P_W, P_Y\} = \partial_s := P_s, \quad \{P_W, P_Z\} = \partial_t := P_t,
\]

\[
\{P_X, P_Y\} = P_t, \quad \{P_X, P_Z\} = -P_s = -P_s, \quad \{P_Y, P_Z\} = P_r = P_r.
\]

We can prove that

\[
\{P_W, P_r\} = \{P_W, P_s\} = \{P_W, P_t\} = \{P_X, P_r\} = \{P_X, P_s\} = \{P_X, P_t\} = 0,
\]

\[
\{P_Y, P_r\} = \{P_Y, P_s\} = \{P_Y, P_t\} = \{P_Z, P_r\} = \{P_Z, P_s\} = \{P_Z, P_t\} = 0.
\]

These relations can also easily be computed by hand, from our formulae for \(P_W, P_X, P_Y, P_Z\) and the bracket in terms of \(w, x, y, z, r, s, t\).

**Lemma 16** By letting \(f\) vary over the functions \(w, x, y, z, r, s, t\), using the bracket relations and equation \((\ref{eq:7})\), we find that Hamilton’s equations are equivalent to the system

\[
\dot{w} = P_W, \quad \dot{x} = P_X, \quad \dot{y} = P_Y, \quad \dot{z} = P_Z, \\
\dot{r} = \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z), \quad \dot{s} = \frac{1}{2}(yP_W - zP_X + xP_Y - wP_Z), \\
\dot{t} = \frac{1}{2}(zP_W + yP_X - xP_Y - wP_Z), \quad \dot{P}_W = p_rP_X + p_sP_Y + p_tP_Z, \\
\dot{P}_X = -p_rP_W - p_sP_Z + p_tP_Y, \quad \dot{P}_Y = p_rP_Z - p_sP_W - p_tP_X, \\
\dot{P}_Z = -p_rP_Y + p_sP_X - p_tP_W, \quad \dot{P}_r = 0, \quad \dot{P}_s = 0, \quad \dot{P}_t = 0.
\]
To see it, remember that $H = \frac{1}{2}(P_W^2 + P_X^2 + P_Y^2 + P_Z^2)$. Then

\[
\dot{w} = \{w, H\} = P_w \frac{\partial P_W}{\partial p_w} = P_W,
\]

\[
\dot{x} = \{x, H\} = P_X \frac{\partial P_X}{\partial p_x} = P_X,
\]

\[
\dot{y} = P_Y,
\]

\[
\dot{z} = P_Z.
\]

Also, considering that:

\[
\frac{\partial P_W}{\partial p_r} = \frac{x}{2}, \quad \frac{\partial P_W}{\partial p_s} = \frac{y}{2}, \quad \frac{\partial P_W}{\partial p_t} = \frac{z}{2},
\]

\[
\frac{\partial P_X}{\partial p_r} = -\frac{w}{2}, \quad \frac{\partial P_X}{\partial p_s} = -\frac{z}{2}, \quad \frac{\partial P_X}{\partial p_t} = \frac{y}{2},
\]

\[
\frac{\partial P_Y}{\partial p_r} = \frac{z}{2}, \quad \frac{\partial P_Y}{\partial p_s} = -\frac{w}{2}, \quad \frac{\partial P_Y}{\partial p_t} = -\frac{x}{2},
\]

\[
\frac{\partial P_Z}{\partial p_r} = -\frac{y}{2}, \quad \frac{\partial P_Z}{\partial p_s} = \frac{x}{2}, \quad \frac{\partial P_Z}{\partial p_t} = -\frac{w}{2},
\]

we have

\[
\dot{r} = \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z).
\]

Indeed,

\[
\dot{r} = \{r, H\} = P_W \frac{\partial P_W}{\partial p_r} + P_X \frac{\partial P_X}{\partial p_r} + P_Y \frac{\partial P_Y}{\partial p_r} + P_Z \frac{\partial P_Z}{\partial p_r}
\]

\[
= \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z)
\]

\[
\dot{s} = \{s, H\} = P_W \frac{\partial P_W}{\partial p_s} + P_X \frac{\partial P_X}{\partial p_s} + P_Y \frac{\partial P_Y}{\partial p_s} + P_Z \frac{\partial P_Z}{\partial p_s}
\]

\[
= \frac{1}{2}(yP_W - zP_X + xP_Y - wP_Z)
\]

\[
\dot{t} = \{t, H\} = P_W \frac{\partial P_W}{\partial p_t} + P_X \frac{\partial P_X}{\partial p_t} + P_Y \frac{\partial P_Y}{\partial p_t} + P_Z \frac{\partial P_Z}{\partial p_t}
\]

\[
= \frac{1}{2}(zP_W + yP_X - xP_Y - wP_Z).
\]

Working as above we obtain

\[
\dot{P}_W = \{P_W, H\} = p_r P_X + p_s P_Y + p_t P_Z,
\]

\[
\dot{P}_X = \{P_X, H\} = -p_r P_W - p_s P_Z + p_t P_Y,
\]

\[
\dot{P}_Y = \{P_Y, H\} = p_r P_Z - p_s P_W - p_t P_X,
\]

\[
\dot{P}_Z = \{P_Z, H\} = -p_r P_Y + p_s P_X - p_t P_W.
\]
Then we are ready to show the following

**Theorem 17** The horizontal geodesics of the quaternionic Heisenberg group are exactly the horizontal lifts of arcs of circles, including line segments as a degenerate case.

**Proof.** It is not difficult to see that \( \dot{P}_r = \dot{P}_s = \dot{P}_t = 0 \). These equations assert that \( P_r = p_r \), \( P_s = p_s \) and \( P_t = p_t \) are constant. The variables \( r, s, t \) appears nowhere in the right-hand sides of these equations. It follows that the variables \( w, x, y, z, P_W, P_X, P_Y, P_Z \) evolve independently of \( r, s, t \), and so we can view the system as defining a one-parameter family of dynamical systems on \( \mathbb{R}^8 \) parameterized by the constant value of \( P_r, P_s, P_t \).

Combine \( w, x, y, z \) into a single quaternionic variable \( \omega = w + ix + jy + kz \) and taking into account the fourteen equations one has

\[
\frac{d\omega}{du} = P_W + iP_X + jP_Y + kP_Z
\]

The \( u \)-derivative of \( P_W + iP_X + jP_Y + kP_Z \) is \( -(ip_r + jp_s + kp_t)(P_W + iP_X + jP_Y + kP_Z) \). Then we have \( \frac{d^2\omega}{du^2} = -(ip_r + jp_s + kp_t)\frac{d\omega}{du} \), where \( p_r, p_s \) and \( p_t \) are constant.

By integrating the above expression we get

\[
P_W + iP_X + jP_Y + kP_Z = P(0)\exp(-((ip_r + jp_s + kp_t)t),
\]

where \( P(0) = P_W(0) + iP_X(0) + jP_Y(0) + kP_Z(0) \).

A second integration yields the general form of the geodesics on the quaternionic contact group:

\[
\omega(u) = w(u) + ix(u) + jy(u) + kz(u) = \frac{P(0)}{ip_r + jp_s + kp_t}(\exp(-((ip_r + jp_s + kp_t)t) - 1) + w(0) + ix(0) + jy(0) + kz(0)),
\]

\[
r(u) = r(0) + \frac{1}{2} \int_0^t \text{Im}_I(\bar{\omega} d\omega),
\]

\[
s(u) = s(0) + \frac{1}{2} \int_0^t \text{Im}_J(\bar{\omega} d\omega),
\]

\[
r(u) = t(0) + \frac{1}{2} \int_0^t \text{Im}_K(\bar{\omega} d\omega).
\]
7 Appendix

The brackets:

\[ [X_4, X_5] = -l \{1 + m(x^2 + y^2)\}X_1 + ml(wz + xy)X_2 - ml(wy - xz)X_3 - 2mxX_4 + 2mwX_5, \]
\[ [X_4, X_6] = -ml(wx - yz)X_1 - \{1 + m(x^2 + z^2)\}X_2 + ml(wx + yz)X_3 - 2myX_4 + 2mwX_6, \]
\[ [X_4, X_7] = ml(wy + xz)X_1 - ml(wx - yz)X_2 - l\{1 + (x^2 + y^2)\}X_3 - 2mzX_4 + 2mwX_7, \]
\[ [X_5, X_6] = -ml(wx + yz)X_1 + ml(wx - yz)X_2 - l\{1 + m(w^2 + z^2)\}X_3 - 2myX_5 + 2mxX_6, \]
\[ [X_5, X_7] = ml(xy - wz)X_1 + l\{1 + m(w^2 + y^2)\}X_2 + ml(wx + yz)X_3 - 2mzX_5 + 2mxX_7, \]
\[ [X_6, X_7] = -l\{1 + m(w^2 + x^2)\}X_1 - ml(wz + xy)X_2 - ml(wx - yz)X_3 - 2mzX_6 + 2myX_7. \]

The Levi-Civita connection:

\[ \nabla_{X_1}X_4 = \frac{l}{2}(1 + m(y^2 + z^2))X_5 + \frac{ml}{2}(wz - xy)X_6 - \frac{ml}{2}(wy + xz)X_7, \]
\[ \nabla_{X_1}X_5 = \frac{l}{2}(1 + m(y^2 + z^2))X_4 + \frac{ml}{2}(wy + xz)X_6 + \frac{ml}{2}(wz - xy)X_7, \]
\[ \nabla_{X_1}X_6 = -\frac{ml}{2}(wz - xy)X_4 - \frac{ml}{2}(wy + xz)X_5 + \frac{l}{2}(1 + m(w^2 + x^2))X_7, \]
\[ \nabla_{X_1}X_7 = \frac{ml}{2}(wz + xy)X_5 - \frac{ml}{2}(wz - xy)X_7 - \frac{l}{2}(1 + m(w^2 + x^2))X_6, \]
\[ \nabla_{X_2}X_4 = -\frac{ml}{2}(wz + xy)X_5 + \frac{l}{2}(1 + m(x^2 + z^2))X_6 + \frac{ml}{2}(wz - yz)X_7, \]
\[ \nabla_{X_2}X_5 = \frac{ml}{2}(wz + xy)X_4 + \frac{l}{2}(1 + m(x^2 + y^2))X_5 - \frac{ml}{2}(wx + yz)X_6, \]
\[ \nabla_{X_2}X_6 = \frac{ml}{2}(wx - yz)X_5 - \frac{ml}{2}(wx + yz)X_6 + \frac{l}{2}(1 + m(x^2 + y^2))X_7, \]
\[ \nabla_{X_2}X_7 = -\frac{ml}{2}(wx + yz)X_4 - \frac{ml}{2}(wx - yz)X_5 + \frac{l}{2}(1 + m(x^2 + z^2))X_7, \]
\[ \nabla_{X_3}X_4 = \frac{ml}{2}(wy - xz)X_5 - \frac{ml}{2}(wx - yz)X_6 + \frac{l}{2}(1 + m(x^2 + y^2))X_7, \]
\[ \nabla_{X_3}X_5 = \frac{ml}{2}(wy - xz)X_4 + \frac{l}{2}(1 + m(w^2 + z^2))X_6 - \frac{ml}{2}(wx + yz)X_7, \]
\[ \nabla_{X_3}X_6 = \frac{ml}{2}(wx + yz)X_4 - \frac{l}{2}(1 + m(w^2 + z^2))X_5 + \frac{ml}{2}(wy - xz)X_7, \]
\[ \nabla_{X_3}X_7 = -\frac{l}{2}(1 + m(x^2 + y^2))X_4 + \frac{ml}{2}(wx + yz)X_5 - \frac{ml}{2}(wy - xz)X_6, \]
\[ \nabla_{X_4}X_1 = 2m(wx + yX_5 + yX_6 + zX_7), \]
\[ \nabla_{X_4}X_5 = -\frac{l}{2}(1 + m(y^2 + z^2))X_1 + \frac{ml}{2}(wz + xy)X_2 - \frac{ml}{2}(wy - xz)X_3 - 2mxX_4, \]
\[ \nabla_{X_4}X_6 = -\frac{ml}{2}(wz - xy)X_1 - \frac{l}{2}(1 + m(x^2 + z^2))X_2 + \frac{ml}{2}(wx + yz)X_3 - 2myX_4, \]
\[ \nabla_{X_4}X_7 = \frac{ml}{2}(wy + xz)X_1 - \frac{ml}{2}(wx - yz)X_2 - \frac{l}{2}(1 + m(x^2 + y^2))X_3 - 2mzX_4, \]
\[ \nabla_{X_5}X_1 = \frac{l}{2}(1 + m(y^2 + z^2))X_4 - \frac{ml}{2}(wz + xy)X_2 + \frac{ml}{2}(wx - yz)X_3 + 2mwX_5, \]
\[ \nabla_{X_5}X_2 = 2m(wX_4 + yX_6 + zX_7), \]
\[ \nabla_{X_5}X_6 = -\frac{ml}{2}(wy + xz)X_1 + \frac{l}{2}(wx - yz)X_2 - \frac{l}{2}(1 + m(w^2 + z^2))X_3 - 2myX_5, \]
\[ \nabla_{X_5}X_7 = -\frac{ml}{2}(wz - xy)X_1 + \frac{l}{2}(1 + m(w^2 + y^2))X_2 + \frac{ml}{2}(wx + yz)X_3 - 2mzX_5, \]
\[ \nabla_{X_6} X_4 = \frac{ml}{2}(wz - xy)X_1 + \frac{l}{2}(1 + m(x^2 + z^2))X_2 - \frac{ml}{2}(wx + yz)X_3 - 2mwX_6, \]
\[ \nabla_{X_6} X_5 = \frac{ml}{2}(wy + xz)X_1 - \frac{ml}{2}(wx - yz)X_2 + \frac{l}{2}(1 + m(w^2 + z^2))X_3 - 2mxX_6, \]
\[ \nabla_{X_6} X_6 = 2m(wx_4 + xX_5 + zX_7), \]
\[ \nabla_{X_6} X_7 = -\frac{l}{2}(1 + m(w^2 + z^2))X_1 - \frac{ml}{2}(wx + yz)X_2 + \frac{ml}{2}(wy - xz)X_3 - 2mzX_6, \]
\[ \nabla_{X_7} X_4 = -\frac{ml}{2}(wy + xz)X_1 + \frac{ml}{2}(wx - yz)X_2 + \frac{l}{2}(1 + m(x^2 + y^2))X_3 - 2mwX_7, \]
\[ \nabla_{X_7} X_5 = \frac{ml}{2}(wz - xy)X_1 - \frac{l}{2}(1 + m(w^2 + y^2))X_2 - \frac{ml}{2}(wx + yz)X_3 - 2mxX_7, \]
\[ \nabla_{X_7} X_6 = \frac{l}{2}(1 + m(w^2 + x^2))X_1 + \frac{ml}{2}(wz + xy)X_2 - \frac{ml}{2}(wy - xz)X_3 - 2myX_7, \]
\[ \nabla_{X_7} X_7 = 2m(wx_4 + xX_5 + yX_6). \]

The curvature tensor:

\[ R_{X_1X_4X_1X_4} = R_{X_1X_5X_1X_5} = \frac{l^2}{4}\{1 + m(K + 1)(y^2 + z^2)\}, \]
\[ R_{X_1X_6X_1X_6} = R_{X_1X_7X_1X_7} = \frac{l^2}{4}\{1 + m(K + 1)(w^2 + x^2)\}, \]
\[ R_{X_2X_4X_2X_4} = R_{X_2X_6X_2X_6} = \frac{l^2}{4}\{1 + m(K + 1)(x^2 + z^2)\}, \]
\[ R_{X_2X_5X_2X_5} = R_{X_2X_7X_2X_7} = \frac{l^2}{4}\{1 + m(K + 1)(w^2 + y^2)\}, \]
\[ R_{X_3X_4X_3X_4} = R_{X_3X_7X_3X_7} = \frac{l^2}{4}\{1 + m(K + 1)(x^2 + y^2)\}, \]
\[ R_{X_3X_5X_3X_5} = R_{X_3X_6X_3X_6} = \frac{l^2}{4}\{1 + m(K + 1)(w^2 + z^2)\}, \]
\[ R_{X_4X_5X_4X_5} = 4m - 3R_{X_1X_4X_1X_4}, \]
\[ R_{X_4X_6X_4X_6} = 4m - 3R_{X_2X_4X_2X_4}, \]
\[ R_{X_4X_7X_4X_7} = 4m - 3R_{X_3X_4X_3X_3}, \]
\[ R_{X_5X_6X_5X_6} = 4m - 3R_{X_3X_5X_3X_5}, \]
\[ R_{X_5X_7X_5X_7} = 4m - 3R_{X_2X_5X_2X_5}, \]
\[ R_{X_6X_7X_6X_7} = 4m - 3R_{X_1X_6X_1X_6}. \]

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