BF Topological Theories and Infinitely Reducible Systems

M. I. Caicedo and A. Restuccia

Universidad Simón Bolívar, Departamento de Física
Apartado postal 89000, Caracas 1080-A, Venezuela.
e-mail: mcaicedo@usb.ve, arestu@usb.ve

Abstract

We present a rigorous discussion for abelian $BF$ theories in which the base manifold of the $U(1)$ bundle is homeomorphic to a Hilbert space. The theory has an infinite number of stages of reducibility. We specify conditions on the base manifold under which the covariant quantization of the system can be performed unambiguously. Applications of the formulation to the superparticle and the superstring are also discussed.

In this paper we discuss the formulation of a $U(1)$ $BF$ topological theory formulated over an infinite dimensional base manifold ($\mathcal{M}_\infty$) which is locally homeomorphic to a Hilbert space ($\mathcal{H}$).

There are two main motivations for approaching this problem. The first of them is to study such theories as natural generalizations of $BF$ theories over finite dimensional manifolds in order to investigate their unknown structure of topological observables.

The second motivation for this work is directly related to the fact that $BF$ actions are infinitely reducible systems, this feature is shared by both the 10-
dimensional superparticle and the Type II Green-Schwartz’s Superstring. Indeed the authors of reference [2] have proposed a new approach for the quantization of the Superstring, in this formulation second class constraints are absent at the expense of introducing an infinite number of first class constraints and stages of reducibility. The covariant quantization of infinitely reducible systems needs the introduction of an infinite tower of ghosts for ghosts, a consistent treatment of this problem is still lacking and therefore one finds an obstruction to the covariant quantization of the superstring along these lines. Recently another approach to the superstring problem has appeared [3] which makes heavy use of the twistor approach but we shall not elaborate on such lines.

The main goal of this work is to show that under certain assumptions it is possible to give a rigorous treatment of the quantization of infinitely reducible $BF$ systems.

We consider a $U(1)$ principal bundle over a manifold $\mathcal{M}_\infty(\cong \mathbb{R} \times \Sigma)$, which is locally homeomorphic to a Hilbert space $\mathcal{H}$ and let $<\ldots>$ stand for integration over $\mathcal{M}_\infty$ with an appropriate measure.

Let $A^{(\infty)}$ be the $U(1)$ connection one form over $\mathcal{M}_\infty$ and $F^{(\infty)} = dA^{(\infty)}$ its associated curvature -we well use boldface for all the geometric objects in $\mathcal{M}_\infty$ and plain text for the geometric objects in $\Sigma$-. The $BF$ action is then given by:

$$S^{(\infty)} = < B \wedge F^{(\infty)} >$$

where $B$ belongs to the dual space of two forms. In terms of local indices the action can be written as

$$S^{(\infty)} = < B^{\mu\nu} F^{(\infty)}_{\mu\nu} >$$

$B^{\mu\nu}$ being antisymmetric. The convergence on the summation is guaranteed from the convergence of the expansions of $B$ and $F^{(\infty)}$ in their respective infinite basis.

The canonical analysis of constrained systems and the Dirac approach to the reduction to the physical submanifold can be done along the same lines followed in field theory. Consequently, the action may be rewritten in terms of the canonical fields:

$$(A_i^{(\infty)}, \pi^i = 2B^0_i)$$
as a singular action:

\[ S^{(\infty)} = \langle \pi^i A_i^{(\infty)} + A_0^{(\infty)} \partial_i \pi^i + B^{ij} F_{ij}^{(\infty)} \rangle \]  

(4)

where clearly \( A_0^{(\infty)} \) and \( B^{ij} \) are Lagrange multipliers associated to a couple of first class constraints whose Poisson bracket algebra is abelian.

As happens in the finite dimensional case, the Gauss constraint, namely: \( \partial_i \pi^i = 0 \) is irreducible and generates the standard gauge transformations on the principal bundle. The remaining constraint \( F_{ij}^{(\infty)} = 0 \) brings in the rich topological structure of BF theories and can be cast in a geometrical way by virtue of the structure of the basespace \( \mathcal{M}_\infty \). Indeed, this constraint is just the statement of flatness of the connection 1-form over \( \Sigma \) i.e. the vanishing of the curvature two form \( (F^{(\infty)} = 0) \). This constraint is reducible since the Bianchi identity is just:

\[ d_2 F^{(\infty)} = 0 \]  

(5)

where, \( d_2 \) is the exterior derivative operator acting on 2 forms over \( \Sigma \). Obviously, \( d_2 \) is a reducible operator since \( d_3 d_2 = 0 \), moreover the identity:

\[ d_{k+1} d_k \equiv 0 \]  

(6)

shows that the system has an infinite set of reducible operators given by the chain:

\[ d_1 \rightarrow d_2 \rightarrow \ldots \rightarrow d_N \rightarrow \ldots \]  

(7)

In the finite dimensional case \( \text{dim}(\mathbb{R} \times \Sigma) = 1 + (D-1) \) the \( d_{D-1} \) exterior derivative operator acting on \( D-1 \) forms over \( \Sigma \) is trivial, the above chain ends up after \( D-3 \) steps and consequently we are in presence of a system with \( D-3 \) stages of reducibility [4].

We can now formally construct the BRST charge \( (\Omega) \). Following the notation of [4] it is given by:

\[ \Omega = \langle C \partial_i \pi^i + C_1^{ij} F_{ij} + C_{11}^{ijk} \partial_k \mu_1^{ij} + \ldots \rangle \]  

(8)

an expression which is clearly geometrical:

\[ \Omega = \langle C \partial_i \pi^i + C_1 F + C_{11} d\mu^1 + C_{111} d\mu^{11} + \ldots \rangle \]  

(9)
From the geometrical point of view $\Omega$ can be thought of as the BRST charge corresponding to a gauge theory over the minimal sector of the extended (super)phase space subjected to the following extra set of first class constraints:

\[
\begin{align*}
d\mu^1 &= 0 \\
d\mu^{11} &= 0 \\
d\mu^{[p]} &= 0, \quad p = 3, \ldots
\end{align*}
\]  

(10)

where the bracketed superscript reflects the stage of reducibility to which a particular object belongs. These "extra" constraints generate gauge transformations on the $C$ fields through the appropriate Poisson brackets. For example:

\[
\delta_{\text{Gauge}} C_1(x) = \{ C_1(x), \langle d\mu^1 \varepsilon_1 \rangle \} = \langle d\delta_x \varepsilon_1 \rangle = \langle \delta_x \delta \varepsilon_1 \rangle = \delta \varepsilon_1(x)
\]  

(12)

where: $\delta_x$ is Dirac's delta, $\delta$ is the co-derivative and $\varepsilon_1$ is a 0-form -the parameter of the gauge transformation-. According to these definitions the expression (12) in components reads as follows:

\[
\delta_{\text{Gauge}} C_{ij}^{11}(x) = -\partial_k \varepsilon_1^{kij}
\]  

(13)

The higher level fields have gauge transformations whose parameters are higher order forms. We find

\[
\begin{align*}
\delta_{\text{Gauge}} C_{11}(x) &= \{ C_{11}, \langle d\mu^{11} \varepsilon_{11} \rangle \} = \delta \theta_{11}(x) \text{ or} \\
\delta_{\text{Gauge}} C_{ijk}^{11}(x) &= \partial_l \varepsilon_1^{lijk}(x) \\
\delta_{\text{Gauge}} C_{[p]}^{11}(x) &= \delta \varepsilon_{[p]}(x)
\end{align*}
\]  

(14)

At this point it is necessary to go a step backwards in order to briefly discuss some issues related with the quantization of finite dimensional $BF$ actions. As explained before, when the base manifold of the bundle $(\mathcal{M}_D)$
is finite dimensional the reducibility is up to $D - 3$ stages. In reference \[\text{5}\] we have shown the gauge fixing procedure that explicitly reduces the path integral of any (finitely) reducible theory to the physical modes. The process is as follows: one begins by defining a transverse-longitudinal ($T + L$) decomposition with respect to the highest stage reducibility operator. This decomposition is used in order to fix the the longitudinal part of the highest stage auxiliary fields. After this step one can recursively work backwards stage by stage and end up with an effective action and path integral formulated in terms of the transverse (physical) degrees of freedom only. The $T + L$ decomposition just described defines the set of admissible gauge fixing conditions for the auxiliary fields associated to reducible constraints through the fixing of the longitudinal part of such fields.

In the case under discussion ($BF$ theories) the $T + L$ decomposition does not break the $SO(D - 1)$ global invariance and at the end one may recover a covariant effective action. For the sake of completeness we will now present the procedure for a 4-dimensional $BF$ theory. In four dimensions the highest reducibility operator is given by the exterior derivative acting on three forms (or their duals the zero forms) this suggests the following $T + L$ decomposition for one forms:

\begin{equation}
\begin{aligned}
v_l &= \partial v + v_l^T \\
\partial_l v_l^T &= 0
\end{aligned}
\end{equation}

according to this formula, the gauge transformations for the ghost fields can be written as

\begin{equation}
\begin{aligned}
\delta_{\text{Gauge}} C_{i1}^j(x) &= -\epsilon^{jk} \partial T_{1k}(x) \\
\delta_{\text{Gauge}} C_{11}^{jk}(x) &= -\epsilon^{ijk} \partial_l \varepsilon_{11}(x)
\end{aligned}
\end{equation}

where $\varepsilon_1, \varepsilon_{11}...$ are appropriate forms.

It is convenient to introduce dual objects to the ghosts ($C_{i1}^{jk} \equiv \epsilon^{ijkl} K_l$ and $C_{11}^{ij} \equiv \epsilon^{ijkl} K_{kl}$) since in such terms the gauge transformations can be cast as follows

\begin{equation}
\begin{aligned}
\delta_{\text{Gauge}} K_l(x) &= \partial T_l \\
\delta_{\text{Gauge}} K_{kl}(x) &= \partial [k \varepsilon^T_{l}]
\end{aligned}
\end{equation}
showing that it is possible to completely fix the gauge for the longitudinal components of $K_l$ and $K_{kl}$ without breaking the explicit $SO(D-1)$ global invariance. With little extra effort it is possible to rearrange the full set of fields (including the tower of ghosts for ghosts) in order to end up with a manifestly covariant $SO(D)$ effective action $(-S_{\text{eff}}^{(D)})$ whose exponential we functionally integrate in all the fields -with unit Liouville measure $(D\mu)$- in order to find a covariant path integral:

$$I_{\text{Cov}}^{(D)} = \int D\mu e^{S_{\text{eff}}^{(D)}}$$ (21)

The covariant gauge fixing is admissible and as a consequence of this we may recall the $BFV$ theorem to show that the result of the last path integral coincides with another gauge fixed path integral within the admissible set.

After the above discussion the problem when treating the infinite dimensional case becomes obvious, there is not such a $T+L$ decomposition so it seems that the programme just outlined will not be applicable, nevertheless we shall see that by adequately complementing the conditions given for the base manifold one can rigorously treat the quantization of the infinite dimensional $BF$ theories.

Before embarking on such enterprise we will see how the light cone gauge technology can be used in order to build an effective action an path integral over the physical modes only. The main idea here consists in remembering that in order to build a correct quantum theory it is enough to find the right set of physical modes even at the expense of losing the manifest covariance. In the light cone gauge we can truncate the infinite tower of ghosts and their descendants at any stage of reducibility by completely breaking the gauge invariance. As an explicit example let us consider such breaking at the very lowest (zero) stage of reducibility. We begin by choosing appropriate coordinates i.e. $\{0, i\} \rightarrow \{-, +, I\}$ and use $x^-$ as ”time”, the gauge transformations for the first stage ghosts are then given by

$$\delta_{\text{Gauge}} C_{1}^{+I} = -\partial_K \varepsilon_1^{K+I}$$ (22)

$$\delta_{\text{Gauge}} C_{1}^{IJ} = -\partial_+ \varepsilon_1^{+IJ} - \partial_K \varepsilon_1^{KIJ}$$ (23)

The reducibility of the system manifests itself in the existence of the following residual gauge transformations
\[ \varepsilon_1^{KIJ} \rightarrow \varepsilon_1^{KIJ} + \partial_P \varepsilon_1^{PKIJ} \]  

which is nothing but the coordinate expression of \( d d \varepsilon_1 = 0 \).

If we assume the invertibility of \( \partial_+ \) - as is usually done when using the LCG - we can use the gauge fixing condition

\[ C_1^{IJ} = 0 \]  

leaving \( C_1^{ij} \) as independent fields, these last fields are invariant under the residual gauge. Moreover, the constraints that generate the residual gauge, namely:

\[ \partial_\mu \mu_1^{ij} = 0 \]  

can also be explicitly solved for \( \mu_1^{ij} \). Indeed, using the LCG we get

\[ \partial_+ \mu_1^{ij} + \partial_J \mu_1^{1+} + \partial_I \mu_1^{1} = 0 \]  

from which one obtains:

\[ \mu_1^{ij} = -\frac{1}{\partial_+} [\partial_J \mu_1^{1+} + \partial_I \mu_1^{1}] \]  

notice that this solution implies the identity:

\[ \partial_\mu \mu_1^{ij} = 0 \]

Since we have eliminated conjugate pairs we have shown an explicit canonical reduction to the finite degrees of freedom of the theory. We may now approach the construction of the light cone gauge path integral over the physical modes by functional integrating the exponential of the light cone action over the independent variables with unit measure \( (D\mu_{phys}) \) as usual:

\[ I_{LCG}^{(\infty)} = \int D\mu_{phys} \exp^{-S_{LCG}^{(\infty)}} \]  

We will now turn our attention back to the original problem: formulating a covariant effective action and path integral for the infinite dimensional \( BF \) theory whose classical action is \( S^{(\infty)} \). We will begin our detailed analysis by assuming the existence of a succession of finite \( (D\)-dimensional) manifolds \( M_D \) such that \( M_\infty \) is contractible to all \( M_D \) for big enough \( D \). Let us now
consider the succession of $U(1)$ $BF$ theories over $\mathcal{M}_D$ and their associated reducible constraints

$$F^{(D)} = 0 \tag{31}$$

The contractibility of $\mathcal{M}_\infty$ to $\mathcal{M}_D$ implies that the $U(1)$ principal bundles over such manifolds are equivalent. Since flat connections over one principal bundle have associated flat connections over the equivalent bundle there is a one to one correspondence between the space of solutions of $F^{(\infty)} = 0 \tag{32}$

We have then shown that under our assumptions both theories (finite dimensional and infinite dimensional) have the same number of degrees of freedom. We will take advantage of this fact in order to build the quantum theory for the infinite dimensional $BF$ theory.

To begin with we first notice that under the assumptions, the finite dimensional LCG effective action has a well defined limit given by:

$$\lim_{D \to \infty} S^{(D)}_{\text{eff}(\text{LCG})} = S^{(\infty)}_{\text{eff}(\text{LCG})} \tag{33}$$

on the other hand, and by virtue of the BFV theorem the finite dimensional LCG and covariant path integrals are equivalent i.e.

$$I^{(D)}_{\text{LCG}} \simeq I^{(D)}_{\text{Cov}} \tag{34}$$

This allows the following definition for the infinite dimensional covariant $BF$ path integral:

$$\lim_{D \to \infty} I^{(D)}_{\text{Cov}} \equiv I^{(\infty)}_{\text{Cov}} \tag{35}$$

The argument we have elaborated shows a way to solve the problem of infinitely reducible systems. The formulation of the problem needs additional structure contained in our assumption of contractibility of $\mathcal{M}_\infty$ to $\mathcal{M}_D$. If this stabilizing property can be added to a system with infinite stages of reducibility there is an unambiguous solution to the problem.

If we compare (5) and (6) with the reducibility problem found for the superparticle (4) or the superstring we find a close similarity. In fact, for the superparticle we have that for the reducible first class constraints
\[ \gamma^{\mu}p_\mu \psi = 0 \quad (36) \]
\[ (\gamma^{\mu}p_\mu)(\gamma^{\rho}p_\rho) = 0 \quad (37) \]

If we are able to introduce -in a consistent way- an odd weight for the operator \( \gamma^{\mu}p_\mu \) acting over odd elements of an infinite dimensional Grassman algebra \( G_\infty \) we may expect the set \( M_D \) to be related with a sucesion of finite dimensional Grassman algebras \( G_{(D)} \) approaching \( G_\infty \) when \( D \to \infty \).

**Acknowledgments**

This work has been partially supported by the "Decanato de Investigaciones de la Universidad Simón Bolívar" through the research fund number S10-CB-812.

**References**

[1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129.

[2] A. Restuccia and J. Stephany, Phys. Rev. D47 (1993) 3437.

[3] I. Bandos and A. A. Zheltukhin, Phys. Part. Nucl. 25 (1994), 453-477.

[4] M. I. Caicedo, R. Gianvittorio, A. Restuccia and J. Stephany, Phys. Lett. B 354 (1995) 292.

[5] M. I. Caicedo and A. Restuccia, Lectures given at the *First Venezuelan School in Relativity and Field Theory*, Mérida, Venezuela (October 1995)