Oscillating Population Models

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Abstract
Oscillating population model realistic situations in different contexts. We examine this situation with reasonable mathematical models and come to interesting conclusions, such as for example, that the population at most points of the cycle approximately equals half the maximum attainable population.

1 Introduction

The Logistic equation
\[
\frac{d(P(t))}{dt} = r(M - P(t))P(t), \ r > 0
\] (1)

where \(P(t)\) is the population at given time \(t\) and \(M\) is the maximum sustainable population was formulated to refine the well-known Malthusian Model of population growth.\[1, 2\] This model takes into consideration that the resources available are limited. The logistic form is a very useful mathematical tool in describing the phenomena of growth. Experiments have shown a fairly satisfactory fit between empirical observations and theoretical calculations based on this logistic form of Verlhurst. This logistic model gives a theory of growth in the simplest stage — a struggle for existence. The Logistic equation can be treated as an equation of the Bernoulli type. By using a partial fraction decomposition it becomes
\[
\frac{1}{M}\left(\frac{1}{P} + \frac{1}{M - P}\right)\frac{dP}{dt} = r
\] (2)
On integration this yields

\[ \frac{P}{M - P} = Ae^{rMT} \]  

(3)

A, the constant of integration is chosen from the initial condition

\[ P = P_0 \text{ at } t = t_0 \]

Thus from (3) we get

\[ P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-rM(t-t_0)}} \]  

(4)

We now consider equation (1) in the following context:

The maximum attainable population would be different at different periods. For example this would be the case when seasonal factors come into play as in the case of insect populations like mosquitoes which manifest highs and lows at different times of the year due to temperature dependent factors.

Secondly the well-known 10 year cycle in lynx and snow-shoe hare populations of the boreal forests of Canada is perhaps controlled by the interaction which can be best represented as follows [3]:

vegetation \(\rightarrow\) eaten by hares \(\rightarrow\) eaten by lynx

Thus the lynx exhibits “driven oscillations” in its population.

It maybe mentioned that oscillating populations have also been studied in certain mammals in a predator-prey model [4, 5]

2 Oscillating Populations

We now model the above considerations in two simple ways.

2.1

First we consider the case where M assumes two different values \(M_1\) and \(M_2\),

\[ M = M_1 \quad 0 \leq t \leq h/2 \]

\[ M = M_2 \quad h/2 \leq t \leq h \]
We now write equation (1) as
\[
\frac{1}{r} \frac{dP}{dt} = MP - P^2 = -\left( P - \frac{M}{2} \right)^2 + \frac{M^2}{4}
\]
Putting
\[
P - \frac{M}{2} = W
\]
we get the Riccati equation,
\[
\frac{1}{r} \frac{dW}{dT} = -W^2 + \frac{M^2}{4} - \frac{1}{2} \frac{dM}{dt}
\]
(5)
We observe that equation (5) shows that P has a periodic solution with the same period as M. This would also be expected from (1) itself.

Let \( M_1 \) and \( M_2 \) be the maximum sustainable populations during different time periods. Then as indicated above equation (4) would have two different forms:

\[
P_1(t) = \frac{M_1 P_0}{P_0 + (M_1 - P_0) e^{-rM_1 h/2}} \quad 0 \leq t \leq h/2
\]
(6)
\[
P_2(t) = \frac{M_2 P_1}{P_1 + (M_2 - P_1) e^{-rM_2 h/2}} \quad h/2 \leq t \leq h
\]
(7)
Equations (6) and (7) share the interlinkage of populations in the two intervals — this of course is quite expected.

For large positive values of \( M_1 \) and \( M_2 \), \( M_1 \gg P_0, M_2 \gg P_1 \) in equations (6) and (7), we further get,

\[
P_1(t) \approx M_1, \quad P_2 \approx M_2
\]

This shows that the maximal values are soon attained in the two intervals. If we now impose the condition

\[
\frac{M_1 + M_2}{2} \approx \langle P \rangle,
\]
that is, that on the average, the population in the interval \( 0 < t < h \) is the mean of the two maximum sustainable populations in the two sub-intervals, then we deduce that

\[
P_1 \approx P_2
\]
2.2

We now consider the case when \( M \) is periodic, with period \( h \) which models oscillating sustainability. As already remarked after (5) or as can be seen from (1) there exists a solution \( P \) which is periodic with the same period as \( M \). In other words the population could follow the same periodic pattern as \( M \). Let us first scale equation (1) by dividing both sides with a large number \( N^2 \) so the maximum attainable population is normalized to 1.

Thus we have

\[
\frac{1}{N^2} \frac{dP}{dt} = \frac{MP}{N^2} - \frac{P^2}{N^2}
\]

Put

\[
\frac{M}{N} \equiv \mathcal{M}, \quad \frac{P}{N} \equiv \mathcal{P},
\]

and integrating both sides from 0 to \( h \), we get,

\[
\int_0^h \frac{1}{N} d\mathcal{P} = \int_0^h (\mathcal{MP} - \mathcal{P}^2) dt
\]

The left hand side = 0, since \( \mathcal{P} \) is taken to be periodic. The right hand side can be written as

\[
- \int_0^h [(\mathcal{M}/2 - \mathcal{P})^2 - \mathcal{M}^2/4] dt
\]

whence we get

\[
\int_0^h (\mathcal{M} - \mathcal{P}/2)^2 dt = \int_0^h \mathcal{M}^2/4 dt
\]

As can be seen, for small \( \mathcal{M} \), \( \mathcal{M}^2 \) can be neglected except in a small interval near its maximum, whence

\[
\int (\mathcal{M}/2 - \mathcal{P})^2 dt \simeq 0,
\]

\[
\mathcal{M}/2 \simeq \mathcal{P}
\]

This shows that the maximum sustainable population which continuously and periodically changes with time is twice the actual population at that time at any point in the cycle.
2.3

We explicitly solve the equation (1) to get a graphical feel of the above considerations. Putting

\[ z = \frac{1}{P}, \quad z' = -\frac{P''}{P^2}, \]

we get

\[ -z' - rMz + r = 0 \]  \hspace{1cm} (9)

The integrating factor for (9) is \( e^{\int_0^t M(t) dt} \equiv Q \).

Solving we get

\[ z = \frac{r}{Q} \int_T^t Q dt \]

and hence

\[ P = \frac{1}{z} = \frac{Q}{r \int_T^t Q dt} \]  \hspace{1cm} (10)

\[ Q = e^{r \cos t_0} e^{-r \cos t} \]

(10) gives the population at time \( t \). We now graphically consider (10) for a simple special case:

\[ M(t) = \sin t \]

\[ y = \frac{r}{e^{-r \cos t}} \int_T^t e^{r \cos t} dt \]  \hspace{1cm} (11)

3 Remarks

We remark that equation (1) maybe considered with multiple parameters \[ 6, 7 \]. On the other hand, the behaviour of the solution is dependent on the parameters. \[ 8 \] the value of \( P \) typically increases and is proportional to its initial value, but at large times it \( \rightarrow M \), here treated as a constant, this being independent of the initial value. On the other hand the discrete version of the logistic equation, equation (1) is even more complicated in that as small \( r \) increases, first one, then two and subsequently more and more number of solutions appear. \( P \) oscillates between them and ultimately becomes totally random.
Acknowledgement

I am thankful to Dr. B. G. Sidharth for useful discussions.

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