An exit measure construction of the total local time of super-Brownian motion

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Abstract

We use a renormalization of the total mass of the exit measure from the complement of a small ball centered at $x \in \mathbb{R}^d$ for $d \leq 3$ to give a new construction of the total local time $L^x$ of super-Brownian motion at $x$.

Keywords: super-Brownian motion; local time; exit measure.

MSC2020 subject classifications: 60J55; 60G57; 60J68; 35J75.

Submitted to ECP on November 9, 2020, final version accepted on June 20, 2021.

Supersedes arXiv:2001.07269.

1 Introduction and main results

The local time of super-Brownian motion (SBM) has been well studied by many authors, e.g., Adler and Lewin [1], Barlow, Evans and Perkins [2], Krone [9], Sugitani [14], etc. It may be formally defined as the density function of the occupation measure of super-Brownian motion. Let $M_F = M_F(\mathbb{R}^d)$ be the space of finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ equipped with the topology of weak convergence of measures. A super-Brownian motion $X = (X_t, t \geq 0)$ starting at $\mu \in M_F$ is a continuous $M_F$-valued strong Markov process defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $X_0 = \mu$ a.s. Write $\mu(\phi) = \int \phi(x) \mu(dx)$ for any measure $\mu$. It is well known that super-Brownian motion is the solution to the following martingale problem (see [13], II.5): For any $\phi \in C^2_b(\mathbb{R}^d)$,

$$X_t(\phi) = X_0(\phi) + M_t(\phi) + \int_0^t X_s(\frac{\Delta}{2} \phi)ds,$$  \hspace{1cm} (1.1)

where $(M_t(\phi))_{t \geq 0}$ is a continuous $\mathcal{F}_t$-martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$[M(\phi)]_t = \int_0^t X_s(\phi^2)ds.$$  

Here $C^2_b(\mathbb{R}^d)$ is the space of bounded functions which are twice continuously differentiable. The above martingale problem uniquely characterizes the law $P_{X_0}$ of super-Brownian motion $X$, starting from $X_0 \in M_F$, on $C([0, \infty), M_F)$, the space of continuous functions from $[0, \infty)$ to $M_F$ furnished with the compact-open topology.

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For any $0 \leq t \leq \infty$, the occupation measure of super-Brownian motion $X$ up to time $t$ is the random measure defined by

$$I_t(A) = \int_0^t X_s(A) ds.$$ (1.2)

In dimensions $d \leq 3$, the occupation measure $I_t$ has a density, $L^x_t$, called the local time of $X$, which satisfies

$$I_t(f) = \int_0^t X_s(f) ds = \int_{\mathbb{R}^d} f(x)L^x_t dx$$

for all continuous $f$ with compact support. (1.3)

Moreover, Theorems 2 and 3 of Sugitani [14] imply that $(t, x) \mapsto L^x_t$ is continuous on $[0, \infty) \times S(X_0)'$, where $S(\mu) = \text{Supp}(\mu)$ denotes the closed support of a measure $\mu$. The extinction time of $X$ is a.s. finite (see, e.g., Chp IV.5 in [13]) and so we set $L^x_t = L^x_\infty$ to be the (total) local time of $X$. We define the range, $\mathcal{R}$, of $X$ to be $\mathcal{R} = \text{Supp}(I_\infty)$.

Now consider SBM under the canonical measure $\mathbb{N}_{x_0}$, which is a $\sigma$-finite measure on $C([0, \infty), M_F)$. If $\Xi = \sum_{i \in I} \delta_{\nu^i}$ is a Poisson point process on $C([0, \infty), M_F)$ with intensity $\mathbb{N}_{x_0}(dv) = \int \mathbb{N}_x(dv)X_0(dx)$, then

$$X_t = \sum_{i \in I} \nu^i_t = \int \nu_t \Xi(dv), \quad t > 0,$$ (1.4)

has the law, $\mathbb{P}_{x_0}$, of a super-Brownian motion $X$ starting from $X_0$. We refer the readers to Theorem II.7.3(c) of [13] for more details. The global continuity of the total local time $L^x$ under $\mathbb{N}_{x_0}$ is given in [6] (see, e.g., Theorem 1.2 of the same reference). By (1.4) we may decompose the total local time $L^x$ under $\mathbb{P}_{X_0}$ as

$$L^x = \sum_{i \in I} L^x(\nu^i) = \int L^x(\nu)\Xi(dv).$$ (1.5)

Intuitively the total local time $L^x$ measures the amount of mass distributed by super-Brownian motion on the singleton $x$. This mechanism is pretty similar to the exit measure from the complement of a small ball centered at $x$. To define the exit measure in an appropriate way, we first recall Le Gall’s Brownian snake.

Let $\mathcal{W} = \bigcup_{s \geq 0} C([0, s], \mathbb{R}^d)$ be equipped with the natural metric (see, e.g., Chp. IV.1 of Le Gall [11]). For any $w \in \mathcal{W}$, we write $\zeta(w) = s$ if $w \in C([0, s], \mathbb{R}^d)$. We call $\zeta(w)$ the lifetime of $w$. The Brownian snake $W = (W_t, t \geq 0)$ is a $\mathcal{W}$-valued continuous strong Markov process. Let $\zeta_t = \zeta(W_t)$ and use $W_t(\zeta_t)$ to denote the tip of the snake at time $t$. Recall the canonical measure $\mathbb{N}_x$ of super-Brownian motion from above. By slightly abusing the notation, we let $\mathbb{N}_x$ denote the excursion measure of the snake, on $C([0, \infty), \mathcal{W})$, starting from the trivial path at $x \in \mathbb{R}^d$ with zero lifetime. Then we may use the Brownian snake $W$ to construct a measure-valued process $X(W) = (X_t(W), t \geq 0)$ under $\mathbb{N}_x$ such that the law of $X(W)$ under $\mathbb{N}_x$ is equal to that of a super-Brownian motion under the canonical measure $\mathbb{N}_x$, thus justifying our abusive notation. We use $X_t(W)$ to denote the super-Brownian motion associated with the snake $W$ instead of the integral with respect to $X_t$. This should be clear if one recalls that $W$ is not a function on $\mathbb{R}^d$ but the snake. The construction of the super-Brownian motion $X(W)$ by the snake $W$ is not important for our discussion here, and so we refer the interested readers to Theorem IV.4 of [11] for more information. If $\Xi = \sum_{j \in J} \delta_{W_j}$ is a Poisson point process on $\mathcal{W}$ with intensity $\mathbb{N}_{X_0}(dW) = \int \mathbb{N}_x(dW)X_0(dx)$, then it follows from (1.4) that

$$X_t = \sum_{j \in J} X_t(W_j) = \int X_t(W)\Xi(dW) \text{ for } t > 0$$ (1.6)
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has the law, \( P_{X_0} \), of a super-Brownian motion \( X \) starting from \( X_0 \). It also follows from (1.5) that the total local time \( L^x \) under \( P_{X_0} \) may be decomposed as

\[
L^x = \sum_{j \in J} L^x(W_j) = \int L^x(W) \Xi(dW). \tag{1.7}
\]

Now we turn to the exit measure. The exit measure from an open set \( G \), under \( P_{X_0} \) or \( \mathbb{N}_{X_0} \), is a random finite measure supported on \( \partial G \) and is denoted by \( X_G \) (see Chp. V of [11] for the construction of the exit measure). Intuitively \( X_G \) represents the mass started at \( X_0 \) which is stopped at the instant it leaves \( G \). We note [11] also suffices as a reference for the properties of \( X_G \) described below. Let \( B(x_0, \varepsilon) = B_\varepsilon(x_0) = \{ x : |x - x_0| < \varepsilon \} \) denote an open ball centered at \( x_0 \in \mathbb{R}^d \) with radius \( \varepsilon > 0 \). Define the complement of a closed ball centered at \( x_0 \) with radius \( \varepsilon > 0 \) to be

\[
G_{\varepsilon^+} = G_\varepsilon(x_0) = \{ x : |x - x_0| > \varepsilon \} \quad \text{and let } G_\varepsilon = G_\varepsilon(0). \tag{1.8}
\]

For any \( K_1, K_2 \) non-empty, set

\[
d(K_1, K_2) = \inf\{|x - y| : x \in K_1, y \in K_2\}.
\]

We assume that \( x_0 \in \mathbb{R}^d \) and \( \varepsilon > 0 \) satisfy \( d(B_\varepsilon(x_0), S(X_0)) > 0 \). In what follows we will only be considering exit measures \( X_G \) for \( G = G_{\varepsilon^+} \) with \( x_0 \in \mathbb{R}^d \) and \( \varepsilon > 0 \) as above. Under \( \mathbb{N}_x \) we have the range \( \mathcal{R} \) of super-Brownian motion \( X = X(W) \), defined by \( \mathcal{R} = S(I_\infty) \) with \( I_\infty \) as in (1.2), may also be written as (see, e.g., equation (8) in the proof of Theorem IV.7(iii) of [11])

\[
\mathcal{R} = \{ \hat{W}(s) : s \in [0, \sigma] \}, \tag{1.9}
\]

where \( \sigma = \sigma(W) = \inf\{ t > 0 : G_t = 0 \} > 0 \) is the length of the excursion path. For any \( x \in G_\varepsilon \), under \( \mathbb{N}_x \), we may use the definition of exit measure in Chp. V of [11] to get (see also (2.3) of [8])

\[
X_G \quad \text{is a finite random measure supported on } \partial G \cap \mathcal{R} \quad \text{a.e.} \tag{1.10}
\]

The extension of (1.10) to \( \mathbb{N}_{X_0} \) is immediate as \( \mathbb{N}_{X_0}(dW) = \int \mathbb{N}_x(dW) X_0(dx) \). It also works under \( P_{X_0} \) as we may, equivalently, set (see, e.g., (2.23) of [12])

\[
X_G = \sum_{j \in J} X_G(W_j) = \int X_G(W) \Xi(dW), \tag{1.11}
\]

where \( \Xi \) is a Poisson point process on \( W \) with intensity \( \mathbb{N}_{X_0} \).

Let \( d(x, K) = \inf\{|x - y| : y \in K\} \). It has been shown in Proposition 6.2(b) of [8] that for any \( x \in S(X_0)^c \), under \( \mathbb{N}_{X_0} \) or \( P_{X_0} \), the family \( \{ X_{G_{\varepsilon^+}}(1), 0 \leq r < r_0 \} \) with \( r_0 = d(x, S(X_0))/2 \) has a càdlàg version which is a supermartingale if \( d = 3 \), a martingale if \( d = 2 \). Throughout the rest of the paper, we will always work with this càdlàg version. For any \( \varepsilon > 0 \), set

\[
\psi_0(\varepsilon) = \begin{cases} 
\frac{1}{2} \log^+(1/\varepsilon), & \text{in } d = 2, \\
\frac{1}{2\pi} \frac{1}{\varepsilon}, & \text{in } d = 3. 
\end{cases} \tag{1.12}
\]

The following result gives a new construction of the total local time \( L^x \) in terms of the local asymptotic behavior of the exit measures at \( x \). This result is also useful in the construction of a boundary local time measure whose support is the topological boundary of the range of super-Brownian motion in \( d = 2 \) and \( d = 3 \) (see [7]).

**Notation.** For a collection of random variables \( \{ \xi_t, t \in T \} \), we say \( \xi_t \) converges in measure to \( \xi_{t_0} \) under \( \mathbb{N}_{X_0} \), as \( t \to t_0 \) if for any \( \eta > 0 \), \( \mathbb{N}_{X_0}(|\xi_t - \xi_{t_0}| > \eta) \to 0 \) as \( t \to t_0 \). The same definition applies under \( P_{X_0} \).
Theorem 1.1. Let $d = 2$ or $d = 3$ and $X_0 \in M_F(\mathbb{R}^4)$. For any $x \in S(X_0)^c$, we have

$$X_G; (1) \psi_0(\varepsilon) \text{ converges in measure to } L^x \text{ under } \mathbb{N}_{X_0} \text{ or } \mathbb{P}_{X_0} \text{ as } \varepsilon \downarrow 0,$$

(1.13)

where $\psi_0$ is as in (1.12). Moreover, in $d = 3$ the convergence holds $\mathbb{N}_{X_0}$-a.e. or $\mathbb{P}_{X_0}$-a.s.

Remark 1.2. In $d = 3$, the family $\mathcal{A} := \{X_{G^{G_{-r}}}(1) \psi_0(r_0 - r), 0 \leq r < r_0\}$ with $r_0 = d(x, S(X_0))/2$ is indeed a martingale (see the proof of the above theorem in Section 3). This allows us to use martingale convergence to conclude a.s. convergence in $d = 3$. In $d = 2$, we already know from Proposition 6.2(b) of [8] that the family $\{X_{G^{G_{-r}}}(1), 0 \leq r < r_0\}$ is a martingale, and so one can check that $\mathcal{A}$ will be a submartingale in $d = 2$. Whether or not a.s. convergence holds in $d = 2$ remains unresolved.

2 The special Markov property

We will state the special Markov property for the Brownian snake from [10] that plays an essential role in our proof. We first deal with $\mathbb{N}_{X_0}$. Recall that we are working with exit measures $X_G$ for $G = G^x_\varepsilon$ with $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ satisfying $d(B_t(x_0), S(X_0)) > 0$.

Define

$$S_G(W_u) = \inf\{t \leq \xi_u : W_u(t) \notin G\} \quad (\inf \emptyset = \infty),$$

$$\eta^G_u(W) = \inf\{t : \int_0^t 1(\xi_u \leq S_G(W_u)) du > s\},$$

$$\mathcal{E}_G = \sigma(W_0^G, s \geq 0) \vee \{\mathbb{N}_{X_0} - \text{null sets}\},$$

(2.1)

where $s \rightarrow W^G_0$ is continuous (see p. 401 of [10]). Intuitively one may think of $\mathcal{E}_G$ as the $\sigma$-field generated by the excursions of $W$ inside $G$. Write the open set $\{u : S_G(W_u) < \xi_u\}$ as countable union of disjoint open intervals, $\cup_{i \in I(a_i, b_i)}$. Then for all $u \in [a_i, b_i]$, one notices $S_G(W_u) = S^i_G < \infty$ where $S^i_G = S_G(W_u) > 0$, and we may define $W^i_t(t) = W_{(a_i, a_i) \wedge b_i}^i(S^i_G + t)$ for $0 \leq t \leq \xi_{(a_i, a_i) \wedge b_i} - S^i_G$.

In this way, we have $W^i$ are the excursions of $W$ outside $G$ for each $i \in I$. Proposition 2.3 of [10] implies that $X_G$ is $\mathcal{E}_G$-measurable and Corollary 2.8 of the same reference gives the following special Markov property:

$$\left\{ \begin{array}{l}
\text{Conditional on } \mathcal{E}_G, \text{ the point measure } \sum_{i \in I} \delta_{W^i} \text{ is a Poisson point measure with intensity } \mathbb{N}_{X_0}.
\end{array} \right.$$

(2.2)

Here $\mathbb{N}_{X_0}(dW) = \int \mathbb{N}_t(dW) X_G(dx)$ is a (random) intensity measure on the space of the snake, i.e. $C([0, \infty), W)$. Consider $G = G^{x_\varepsilon}_\varepsilon$ and $D = G^{x_\varepsilon}_\varepsilon$, with $\varepsilon_1 > \varepsilon_2 > 0$. We can define the exit measure $X_D(W^i)$ for each $W^i$ following the construction of exit measure in Chapter V.1 of [11]. As in (2.6) of [8], one may conclude

$$X_D = \sum_{i \in I} X_D(W^i).$$

(2.3)

If $U$ is an open subset of $S(X_0)^c$, then $L^U$, the restriction of the total local time $L^x$ to $U$, is in $C(U, \mathbb{R})$ which is the set of continuous functions on $U$. Here are some consequences of (2.2) that are already proved in Proposition 2.2(a) of [8].

Proposition 2.1. For any $X_0 \in M_F(\mathbb{R}^d)$, fix some $x \in S(X_0)^c$. Define $G_1 = G^{x_\varepsilon_1}$ and $G_2 = G^{x_\varepsilon_2}$, with $0 < \varepsilon_2 < \varepsilon_1 < d(x, S(X_0))$.

(i) If $\psi : C(\overline{G_1}, \mathbb{R}) \rightarrow [0, \infty)$ is Borel measurable, then

$$\mathbb{N}_{X_0}(\psi_1(L^G_{\overline{G_1}}) \mid \mathcal{E}_{G_1}) = \mathbb{E}_{X_{G_1}}(\psi_1(L^G_{\overline{G_1}})).$$

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(ii) If $\psi_2 : M_F(\mathbb{R}^d) \to [0, \infty)$ is Borel measurable, then

$$\mathbb{N}_{X_0}(\psi_2(X_{G_2})) = E_{X_{G_1}}(\psi_2(X_{G_2})).$$

The $\sigma$-finiteness of $\mathbb{N}_{X_0}$ is not an issue here as we may define the above conditional expectation by, e.g., using Radon-Nikodym derivative.

We will need a version of the above under $\mathbb{P}_{X_0}$ as well, which follows immediately from Proposition 2.3 of [8].

**Proposition 2.2.** For any $X_0 \in M_F(\mathbb{R}^d)$, fix some $x \in S(X_0)^c$. Define $G_1 = G_{x_1}^x$ and $G_2 = G_{x_2}^x$ with $0 < x_2 < x_1 < d(x, S(X_0))$.

(i) If $\phi_1 : C(G_{x_1}^x, \mathbb{R}) \to [0, \infty)$ is Borel measurable, then

$$E_{X_0}(\phi_1(L_{G_{x_1}^x})) = E_{X_0} \left( E_{X_{G_1}}(\phi_1(L_{G_{x_1}^x})) \right).$$

(ii) If $\phi_2 : M_F(\mathbb{R}^d) \to [0, \infty)$ is Borel measurable, then

$$E_{X_0}(\phi_2(X_{G_2})) = E_{X_0} \left( E_{X_{G_1}}(\phi_2(X_{G_2})) \right).$$

### 3 Construction of the total local time by exit measure

In this section we will give the proof of Theorem 1.1. We assume throughout this section that $d = 2$ or $d = 3$. The Laplace transform of $L^x$ derived in Lemma 2.2 of [12] is given by

$$E_{X_0}(\exp(-\lambda L^x)) = \exp \left( - \int_{\mathbb{R}^d} V^\lambda(x - y) X_0(dy) \right), \quad (3.1)$$

where $V^\lambda$ is the unique solution to

$$\frac{\Delta V^\lambda}{2} = \frac{(V^\lambda)^2}{2} - \lambda \delta_0, \quad V^\lambda > 0 \text{ on } \mathbb{R}^d. \quad (3.2)$$

Here $\delta_0$ is the Dirac delta function and the above differential equation is interpreted in a distributional sense. One can check that $V^\lambda$ is radially symmetric and we may write $V^\lambda(|x|)$ for $V^\lambda(x)$. Recall $\psi_0$ from (1.12). It is known that (see, e.g., p. 187 of [4]) $V^\lambda$ is smooth in $\mathbb{R}^d \setminus \{0\}$, and near the origin, Lemma 8 of [3] gives that

$$\lim_{x \to 0} V^\lambda(x) = \frac{\lambda}{\psi_0(|x|)} \quad \text{as } x \to 0. \quad (3.3)$$

**Proof of Theorem 1.1.** The outline for the proof is as follows: First we get some $L^2$ convergence, associated with $X_{G_1}$ and $L^x$, using the Laplace transforms. Then we show that this implies the convergence in measure. When $d = 3$, we prove there is an a.s. limit by the martingale arguments. It is then immediate that $L^x$, as the limit of convergence in measure, is in fact the a.s. limit, thus completing the proof.

We first consider the $\mathbb{N}_{X_0}$ case. Fix any $x \in S(X_0)^c$ and fix $0 < \varepsilon < \delta/2$, we have

$$I := \mathbb{N}_{X_0} \left( \left( \exp(-\lambda X_{G_2}^x(1)\psi_0(\varepsilon)) - \exp(-\lambda L^x) \right)^2 \right)$$

$$= \mathbb{N}_{X_0} \left( \exp(-2\lambda X_{G_2}^x(1)\psi_0(\varepsilon)) + \exp(-2\lambda L^x) - 2 \exp(-\lambda X_{G_2}^x(1)\psi_0(\varepsilon)) \exp(-\lambda L^x) \right)$$

$$= \mathbb{N}_{X_0} \left( \exp(-2\lambda X_{G_2}^x(1)\psi_0(\varepsilon)) + E_{X_{G_2}^x} \left( \exp(-2\lambda L^x) \right) - 2 \exp(-\lambda X_{G_2}^x(1)\psi_0(\varepsilon)) E_{X_{G_2}^x} \left( \exp(-\lambda L^x) \right) \right), \quad (3.4)$$

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where we have used Proposition 2.1 (i) in the last equality. Apply (3.1) with \( ECP \) from Proposition VI.2 of [11]. Hence (3.8) becomes

\[
0 < \varepsilon < \frac{\lambda}{V}.
\]

In the second equality we have used the fact that the exit measure \( X_G \) is supported on \( \partial G^\varepsilon \) by (1.10) and then apply the radial symmetry of \( V^\lambda \) to get \( V^\lambda(x - y) = V^\lambda(|x - y|) = V^\lambda(\varepsilon) \) for any \( y \in \partial G^\varepsilon \). The above still holds true if we replace \( \lambda \) with 2\( \lambda \) in (3.5). Use the above in (3.4) to arrive at

\[
I = E \left[ \exp \left( -2\lambda X_G (1) \psi_0 (\varepsilon) \right) + \exp \left( -X_G (1) V^\lambda (\varepsilon) \right) - 2 \exp (\lambda X_G (1) \psi_0 (\varepsilon)) \right].
\]

We first deal with \( I_1 \).

\[
|I_1| \leq E \left[ \exp \left( -2\lambda X_G (1) \psi_0 (\varepsilon) \right) - \exp \left( -\lambda \frac{V^\lambda (\varepsilon)}{\psi_0 (\varepsilon)} X_G (1) \psi_0 (\varepsilon) \right) \right] = E \left[ \exp \left( -\lambda' (\varepsilon) X_G (1) \psi_0 (\varepsilon) \right) \left( 2\lambda - (\lambda + \frac{V^\lambda (\varepsilon)}{\psi_0 (\varepsilon)}) \right) \right] \leq \lambda - \frac{V^\lambda (\varepsilon)}{\psi_0 (\varepsilon)} \cdot E \left[ X_G (1) \psi_0 (\varepsilon) \exp \left( -\lambda' (\varepsilon) X_G (1) \psi_0 (\varepsilon) \right) \right],
\]

where the second line is by the mean value theorem with \( \lambda' (\varepsilon) (\omega) \) chosen between \( 2\lambda \) and \( \lambda + V^\lambda (\varepsilon)/\psi_0 (\varepsilon) \). When \( \varepsilon > 0 \) is small, (3.3) implies \( V^\lambda (\varepsilon)/\psi_0 (\varepsilon) > \lambda/2 \), and so \( E \) \( X_G \) \( \psi_0 \) \( \lambda \) a.e. we have \( \lambda' (\varepsilon) \geq \min (2\lambda, \lambda + V^\lambda (\varepsilon)/\psi_0 (\varepsilon)) \) > \( 3\lambda/2 > \lambda \). Hence (3.7) becomes

\[
|I_1| \leq \lambda - \frac{V^\lambda (\varepsilon)}{\psi_0 (\varepsilon)} \cdot E \left[ X_G (1) \psi_0 (\varepsilon) \exp \left( -\lambda X_G (1) \psi_0 (\varepsilon) \right) \right].
\]

Recall \( \delta = d(x, S(X_\theta)) \). Define \( S(X_\theta)^{\delta/4} = \{ y : d(y, S(X_\theta)) > \delta/4 \} \) so that for any \( 0 < \varepsilon < \delta/2 \), we have \( \partial G^\varepsilon \subset S(X_\theta)^{\delta/4} \). Recall \( R \) from (1.9). Apply (1.10) to see for all \( 0 < \varepsilon < \delta/2 \), we have

\[
\mathcal{R} \cap S(X_\theta)^{\delta/4} = \emptyset \text{ implies } X_G (1) = 0, \quad \text{N}_X \text{-a.e.}\]  

Use the above to get

\[
E \left[ \exp \left( -\lambda X_G (1) \psi_0 (\varepsilon) \right) \right] = E \left[ \exp \left( -\lambda X_G (1) \psi_0 (\varepsilon) \right) 1(\mathcal{R} \cap S(X_\theta)^{\delta/4} \neq \emptyset) \right] \leq \lambda^{-1} e^{-\lambda} E \left[ X_G (1) \psi_0 (\varepsilon) 1(\mathcal{R} \cap S(X_\theta)^{\delta/4} \neq \emptyset) \right] := \lambda^{-1} e^{-\lambda} C(X_\theta, \delta) < \infty,
\]

where the first inequality is by \( x e^{-\lambda x} \leq \lambda^{-1} e^{-\lambda}, \forall x \geq 0 \). The finiteness of \( C(X_\theta, \delta) \) follows from Proposition VI.2 of [11]. Hence (3.8) becomes

\[
|I_1| \leq \lambda - \frac{V^\lambda (\varepsilon)}{\psi_0 (\varepsilon)} \cdot \lambda^{-1} e^{-\lambda} C(X_\theta, \delta) \to 0 \text{ as } \varepsilon \downarrow 0,
\]
where the convergence to 0 follows from (3.3).

Turning to $I_2$, we have

\[
|I_2| \leq \mathbb{E}_{X_0} \left( \exp \left( - \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} X_{G^2}(1) \psi_0(\varepsilon) \right) - \exp \left( - \frac{\lambda + V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} X_{G^2}(1) \psi_0(\varepsilon) \right) \right)
\]

\[
= \mathbb{E}_{X_0} \left( X_{G^2}(1) \psi_0(\varepsilon) \exp \left( - \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} (\frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} - (\lambda + \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}) \right) \right)
\]

\[
\leq \left| \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right| - \lambda - \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \cdot \mathbb{E}_{X_0} \left( X_{G^2}(1) \psi_0(\varepsilon) \exp \left( - \frac{\lambda}{\psi_0(\varepsilon)} X_{G^2}(1) \psi_0(\varepsilon) \right) \right),
\]

(3.12)

where in the second line we have used the mean value theorem with $\hat{\lambda}(\varepsilon)(\omega)$ chosen between $V_{2\lambda}(\varepsilon)/\psi_0(\varepsilon)$ and $\lambda + V_{\lambda}(\varepsilon)/\psi_0(\varepsilon)$. When $\varepsilon > 0$ is small, (3.3) implies $V_{2\lambda}(\varepsilon)/\psi_0(\varepsilon) > 3\lambda/2$ and $V_{\lambda}(\varepsilon)/\psi_0(\varepsilon) > \lambda/2$. So $\mathbb{E}_{X_0}$-a.e. we have

\[
\hat{\lambda}(\varepsilon) \geq \min \left\{ \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}, \lambda + \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right\} > \frac{3\lambda}{2} > \lambda.
\]

(3.13)

Use the above to see that (3.12) becomes

\[
|I_2| \leq \left| \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right| - \lambda - \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \cdot \mathbb{E}_{X_0} \left( X_{G^2}(1) \psi_0(\varepsilon) \exp \left( - \frac{\lambda}{\psi_0(\varepsilon)} X_{G^2}(1) \psi_0(\varepsilon) \right) \right).
\]

Apply (3.10) to see that

\[
|I_2| \leq \left| \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right| - \lambda - \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \cdot \lambda^{-1} e^{-1} C(X_0, \delta)
\]

\[
\leq \left( \left| \frac{V_{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right| - 2\lambda + \left| \lambda - \frac{V_{\lambda}(\varepsilon)}{\psi_0(\varepsilon)} \right| \right) \cdot \lambda^{-1} e^{-1} C(X_0, \delta) \rightarrow 0 \text{ as } \varepsilon \downarrow 0,
\]

(3.14)

where the convergence to 0 follows from (3.3).

Recall $I$ from (3.4). We may conclude from (3.11) and (3.14) that $I \rightarrow 0$ as $\varepsilon \downarrow 0$, thus giving the $L^2$ convergence of $\exp(-\lambda X_{G^2}(1) \psi_0(\varepsilon))$ to $\exp(-\lambda L^x)$ under $\mathbb{E}_{X_0}$. By Corollary 2.32 of Folland [5], for any sequence $\varepsilon_n \downarrow 0$, we may pick a subsequence $\varepsilon_{nk} \downarrow 0$ so that

\[
\lim_{\varepsilon_{nk} \downarrow 0} \exp(-\lambda X_{G^2_{\varepsilon_{nk}}}(1) \psi_0(\varepsilon_{nk})) = \exp(-\lambda L^x), \quad \mathbb{E}_{X_0}$-a.e.
\]

(3.15)

We note the arguments in Folland [5] remain valid for our setting with the $L^2$ convergence under the $\sigma$-finite measure $\mathbb{E}_{X_0}$. It is immediate from (3.15) that

\[
\lim_{\varepsilon_{nk} \downarrow 0} X_{G^2_{\varepsilon_{nk}}}(1) \psi_0(\varepsilon_{nk}) = L^x, \quad \mathbb{E}_{X_0}$-a.e.
\]

(3.16)

At this stage, we may not conclude the convergence in measure due to the $\sigma$-finiteness of $\mathbb{E}_{X_0}$. This issue could be solved by noticing that the event $\{X_{G^2} \neq 0 \text{ or } L^x \neq 0\}$ has only finite measure under $\mathbb{E}_{X_0}$. By using Proposition 2.1 (i), we get for any $0 < \varepsilon < \delta/2$,

\[
\mathbb{E}_{X_0}(\{L^x > 0\} \cap \{X_{G^2}(1) = 0\}) = \mathbb{E}_{X_0}(1\{X_{G^2}(1) = 0\} \mathbb{E}_{X_0}(1\{L^x > 0\} | \mathcal{G}_\varepsilon))
\]

\[
= \mathbb{E}_{X_0}(1\{X_{G^2}(1) = 0\} \mathbb{E}_{X_{G^2}}(L^x > 0)) = 0,
\]

thus giving $\mathbb{E}_{X_0}$-a.e. $X_{G^2}(1) = 0$ implies $L^x = 0$. Together with (3.9), we get for any $0 < \varepsilon < \delta/2$,

\[
\mathcal{R} \cap S(X_0)^{\delta/4} = \emptyset \text{ implies } L^x = 0 \text{ and } X_{G^2}(1) = 0, \quad \mathbb{E}_{X_0}$-a.e.
\]

(3.17)
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Therefore it follows that for any $\eta > 0$,

$$\mathbb{N}_{X_0}\left(|X_{G_1}(1)\psi_0(\epsilon) - L^x| > \eta \right) = \mathbb{N}_{X_0}\left(|X_{G_{\varepsilon_{nk}}}(1)\psi_0(\varepsilon) - L^x| > \eta \right) \cap \{ \mathcal{R} \cap S(X_0)^{\delta/4} \neq \emptyset \},$$

(3.18)

and so we may work with the finite measure $\mathbb{N}_{X_0}(\cdot \cap \{ \mathcal{R} \cap S(X_0)^{\delta/4} \neq \emptyset \})$ when considering the convergence in measure under $\mathbb{N}_{X_0}$. Apply Dominated Convergence Theorem with (3.16) and (3.18) to get

$$\lim_{\varepsilon_{nk} \downarrow 0} \mathbb{N}_{X_0}\left(|X_{G_{r_{nk}}}(1)\psi_0(\varepsilon_{nk}) - L^x| > \eta \right) = 0.$$  (3.19)

Hence for any sequence $\varepsilon_n \downarrow 0$, there is a subsequence $\varepsilon_{nk} \downarrow 0$ such that (3.19) holds, thus completing the proof of convergence in measure under $\mathbb{P}_{X_0}$. For the $\mathbb{P}_{X_0}$ case, the above arguments work in a similar and even easier way, and so we omit the details.

Now we turn to the a.s. convergence in $d = 3$. For any $x \in S(X_0)^c$, set $r_0 = \delta/2$ where $\delta = d(x, S(X_0)) > 0$. In $d = 3$, by (6.10) of [8], for any $0 < \varepsilon < r_0$ we have

$$E_{X_0}(X_{G_{\varepsilon}}(1)) = \mathbb{N}_{X_0}(X_{G_{\varepsilon}}(1)) = \int \frac{\varepsilon}{|x - x_0|} dx_0(x_0).$$

(3.20)

Hence for $0 < \varepsilon_2 < \varepsilon_1 < r_0$, we may apply Proposition 2.1(ii) to get

$$\mathbb{N}_{X_0}\left(\frac{X_{G_{\varepsilon_2}}(1)}{\varepsilon_2} |_{E_{G_{\varepsilon_1}}} \right) = E_{X_{G_{\varepsilon_1}}}(\frac{X_{G_{\varepsilon_2}}(1)}{\varepsilon_2}) = \frac{X_{G_{\varepsilon_1}}(1)}{\varepsilon_1},$$

(3.21)

where the last equality follows by applying (3.20) with $X_0 = X_{G_{\varepsilon_1}}$ and by using the fact that the exit measure $X_{G_{\varepsilon_1}}$ is supported on $\partial G_{\varepsilon_1}$ by (1.10). Recall that in $d = 3$ we have $\psi_0(\varepsilon) = 1/(2\pi\varepsilon)$. Use (3.21) to conclude

$$\mathbb{N}_{X_0}\left(\frac{X_{G_{\varepsilon_2}}(1)\psi_0(\varepsilon_2)}{\varepsilon_2} |_{E_{G_{\varepsilon_1}}} \right) = X_{G_{\varepsilon_1}}(1)\psi_0(\varepsilon_1),$$

(3.22)

which implies $\{X_{G_{r_{nk}}}(1)\psi_0(r_0 - r), 0 \leq r < r_0\}$ is a nonnegative martingale. Note that we always work with the càdlàg version of $X_{G_{r_{nk}}}(1)$ on $0 \leq r < r_0$. Now we may apply the martingale convergence theorem to get $\mathbb{N}_{X_0}$-a.e. $\lim_{r \to r_0} X_{G_{r_{nk}}}(1)\psi_0(r_0 - r)$ exists. Since we already have $X_{G_{\varepsilon}}(1)\psi_0(\varepsilon)$ converges to $L^x$ in measure under $\mathbb{N}_{X_0}$ (see also (3.16)), we conclude that $\mathbb{N}_{X_0}$-a.e. $\lim_{\varepsilon \downarrow 0} X_{G_{\varepsilon}}(1)\psi_0(\varepsilon) = L^x$. The case for $\mathbb{P}_{X_0}$ follows in a similar way.

References

[1] R. Adler and M. Lewin. Local time and Tanaka formulae for super-Brownian motion and super stable processes. Stochastic Process. Appl., 41: 45–67, (1992). MR-1162718
[2] M. Barlow, S. Evans and E. Perkins. Collision local times and measure-valued diffusions. Can. J. Math., 43: 897-938, (1991). MR-1138572
[3] H. Brezis and L. Oswald. Singular solutions for some semilinear elliptic equations, Archive Rational Mech. Anal. 99, 249-259, (1987). MR-0888452
[4] H. Brezis, L. Peletier and D. Terman. A very singular solution of the heat equation with absorption, Archive Rational Mech. Anal. 95 (1986) pp. 185-209. MR-0853963
[5] G. Folland. Real analysis: Modern techniques and their applications. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley&Sons, Inc., New York, (1999). MR-1681462

ECP 26 (2021), paper 40.  https://www.imstat.org/ecp

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Exit measure construction of the local time

[6] J. Hong. Renormalization of local times of super-Brownian motion. *Electron. J. Probab.*, 23: no. 109, 1–45, (2018). MR-3878134

[7] J. Hong. On the boundary local time measure of super-Brownian motion. *Electron. J. Probab.*, 25: no. 106, 66 pp, (2020). MR-4147519

[8] J. Hong, L. Mytnik and E. Perkins. On the topological boundary of the range of super-Brownian motion. *Ann. Probab.*, 48: no. 3, 1168–1201, (2020). MR-4112711

[9] S. Krone. Local times for superdiffusions. *Ann. Probab.*, 21 (b): 1599-1623, (1993). MR-1235431

[10] J.F. Le Gall. The Brownian snake and solutions of \( \Delta u = u^2 \) in a domain. *Probab. Theory Relat. Fields*, 102: 393–432, (1995). MR-1339740

[11] J.F. Le Gall. Spatial Branching Processes, Random Snakes and Partial Differential Equations. Lectures in Mathematics, ETH, Zurich. Birkhäuser, Basel (1999). MR-1714707

[12] L. Mytnik and E. Perkins. The dimension of the boundary of super-Brownian motion. *Prob. Th. Rel Fields* 174: 821–885, (2019). MR-3980306

[13] E.A. Perkins. Dawson-Watanabe Superprocesses and Measure-valued Diffusions. *Lectures on Probability Theory and Statistics*, no. 1781, *Ecole d’Été de Probabilités de Saint Flour 1999*. Springer, Berlin (2002). MR-1915445

[14] S. Sugitani. Some properties for the measure-valued branching diffusion processes. *J. Math. Soc. Japan*, 41:437–462, (1989). MR-0999507

**Acknowledgments.** This work was done as part of the author’s graduate studies at the University of British Columbia. I would like to thank my supervisor, Professor Edwin Perkins, for suggesting this problem and for the helpful discussions and suggestions throughout this work. I also thank two anonymous referees for their comments and suggestions which help to improve the readability of the manuscript.
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