Black Hole fragmentation and holography

M. Cadoni

Dipartimento di Fisica, Università di Cagliari, and INFN sezione di Cagliari,
Cittadella Universitaria 09042 Monserrato, ITALY

We discuss the entropy change due to fragmentation for black hole solutions in various dimensions. We find three different types of behavior. The entropy may decrease, increase or have a mixed behavior, characterized by the presence of a threshold mass. For two-dimensional (2D) black holes we give a complete characterization of the entropy behavior under fragmentation, in the form of sufficient conditions imposed on the function \( J \), which defines the 2D gravitational model. We compare the behavior of the gravitational solutions with that of free field theories in \( d \) dimensions. This excludes the possibility of finding a gravity/field theory realization of the holographic principle for a broad class of solutions, including asymptotically flat black holes. We find that the most natural candidates for holographic duals of the black hole solutions with mixed behavior are field theories with a mass gap. We also discuss the possibility of formulating entropy bounds that make reference only to the energy of a system.

I. INTRODUCTION

One of the most striking novelties in the research on gravitational physics is the possibility that gravity in \( d \) dimensions could be described by a local field theory in \( d - 1 \) dimensions [1, 2, 3]. The theoretical evidence for such holographic description of gravity is mounting. Indications that holography could be a fundamental feature of the gravitational interaction come from different directions: string theory, black hole physics, cosmology [4, 5, 6] (For a recent review see [7]). A particularly interesting output of these investigations has been the formulation of stringent holographic bounds for the entropy of a system occupying a given region of space [3, 8].

An explicit realization of the holographic principle has been found only in particular cases, essentially for anti-de Sitter (AdS) (and de Sitter) gravity, the so-called anti de Sitter/conformal field theory (AdS/CFT) correspondence [1, 2, 3]. A general way to explicit realize the holographic principle for generic gravitational systems, in particular for gravity in asymptotically flat spacetimes, is still lacking. In particular, it is not clear if the realizations of the holographic principle take always the form a correspondence between \( d \)-dimensional gravity and a field theory in \( d - 1 \) dimensions, or if there could be some alternative, still unknown, realization of it. If the holographic principle has to be considered a genuine feature of every quantum theory of gravity, one could explain our lacking in understanding the holographic principle as a lacking in understanding quantum gravity. However, there are strong indications that holography is a feature of gravity that appears and therefore should be explained, already at the semiclassical level. The Bekenstein-Hawking area law for the black hole entropy is the most striking example of holographic behavior of a gravitational system that has to be explained already at the semiclassical level.

An alternative strategy one can use in this context, is to explore the similarities and the differences between gravity and local field theories in order to check at a fundamental level the possibility of finding correspondences between the two classes of theories. This approach can be very powerful. One nice example, discussed in almost every introducing paper on the holographic principle, is the scaling behavior of the entropy as a function of the volume of the system for a local field theory compared to that of a black hole. For a local field theory the entropy is an extensive quantity, it scales as the volume of the space. On the other hand the entropy of a black hole scales as the area of the horizon. This simple fact enables one to conclude that the correspondence between gravity and field theory, if it exists, must be holographic.

In this paper we will focus on another aspect of the relationship between gravity and local field theory, namely on the dependence for composite systems of the entropy from the energy. Working in the microcanonical ensemble the entropy/energy relation for a free field theory in \( d \)-dimensions is given by

\[ S \propto E^{(d-1)/d}, \]

which considering two arbitrary excitations with energy \( E_1, E_2 \) satisfies the inequality

\[ S(E_1 + E_2) < S(E_1) + S(E_2), \]

*Electronic address: mariano.cadoni@ca.infn.it*
Conversely asymptotically flat black holes of masses $M_1$, $M_2$ in $d \geq 4$ dimensions satisfy

$$S(M_1 + M_2) > S(M_1) + S(M_2),$$

(3)

i.e fragmentation of a black hole of mass $M_1 + M_2$ into two smaller black holes of masses $M_1$ and $M_2$ is entropically not preferred. Assuming the existence of a gravity/field theory correspondence, one has to identify the black hole masses $M_1$, $M_2$ as excitations $E_1$, $E_2$ of the field theory. It follows that Eq. (2) contradicts Eq. (3) and consequently that the assumed correspondence gravity/field theory cannot be true. Also skipping this problem, i.e assuming the existence of black hole solutions satisfying inequality (2) rather then (3), one is faced by another problem. At first sight Eq. (2) seems incompatible with every entropy bound because a black hole could always increase its entropy by fragmenting into smaller black holes.

In this paper we will analyze in detail for various classes of black holes in various dimensions the validity of the relation (3). We will show that Eq. (3) holds true only for asymptotically flat black holes. Black holes with different asymptotic behavior (for instance AdS black holes) may satisfy Eq. (2). For the two-dimensional (2D) case we will be able to give a complete characterization of the entropy for composite black holes in terms of the asymptotic behavior (for instance AdS black holes) may satisfy Eq. (2). For the two-dimensional (2D) case we will be able to give a complete characterization of the entropy for composite black holes in terms of the asymptotic behavior (for instance AdS black holes) may satisfy Eq. (2).

II. ASYMPTOTICALLY FLAT BLACK HOLES

Let us first consider the $d$-dimensional Schwarzschild black hole ($d \geq 4$)

$$ds^2 = - \left(1 - \frac{k_d GM}{r^{d-3}}\right) dt^2 + \left(1 - \frac{k_d GM}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2,$$

(4)

where $M$ is mass and $k_d = 16\pi/(d - 2)\Omega_{d-2}$, $\Omega_{d-2}$ being the volume of the unit $S^{d-2}$ transverse sphere. The black hole entropy is given by the area law

$$S = \frac{A}{4G} = \frac{\Omega_{d-2}}{4} (k_d L_p M)^{(d-2)/(d-3)},$$

(5)

where $L_p = G^{1/(d-2)}$ is the Planck length.

In order to discuss the inequality (3) on a physical ground, we will make here and throughout the all paper the following assumptions: 1) The theory admits, at least in some approximation, multi black hole solutions and 2) The gravitational potential energy of the multi black hole configuration can be neglected with respect to the black hole masses. Assumption 1) is necessary in order to give to a multi black hole configuration a precise meaning whereas assumption 2) assures us that the multi black hole configuration can be treated as a composite system with zero binding energy, whose mass and entropy is simply the sum of that of the elementary constituents.

Using Eq. (5) one easily find that Eq. (3) is satisfied. As expected, for Schwarzschild black holes the entropy is maximized by the single black hole configuration with mass $M_1 + M_2$. The same holds true for asymptotically flat charged black holes. Considering for instance the four-dimensional Reissner-Nordström solution one finds for the entropy

$$S = \left(ML_p + \sqrt{M^2 L_p^2 - Q^2}\right)^2,$$

(6)

where $Q$ is the electric charge of the black hole. Again, from Eq. (6) it follows $S(M_1 + M_2) > S(M_1) + S(M_2)$. There is one simple argument that can be used to argue that inequality (3) holds in general for asymptotically flat black holes at least when $ML_p$ is much bigger then the (eventually present) black hole charges. The area law gives $S \propto (r_+/L_p)^{d-2}$, where $r_+$ is the horizon radius. For asymptotically flat black holes and when $ML_p >> Q_i$, where $Q_i$ are the charges associated with the black hole, the gravitational potential is dominated by the Newtonian term so that we have $r_+ \propto L_p^{(d-2)/(d-3)} M^{1/(d-3)}$. It follows $S \propto (L_p M)^{(d-2)/(d-3)}$.

III. ADS BLACK HOLES

The argument presented at the end of the previous section does not apply to non-asymptotically flat black holes. For 2D black holes a general discussion we will presented in the next section. Here we will discuss only the most interesting
case, namely asymptotically AdS black holes. The entropy-mass relation for the $d$-dimensional Schwarzschild-anti de Sitter black hole ($d \geq 4$),

$$ds^2 = -(1 + \lambda^2 r^2 - \frac{k_d G M}{r^{d-3}}) dt^2 + \left(1 + \lambda^2 r^2 - \frac{k_d G M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2,$$  \hspace{1cm} (7)

is rather complicated. However a simple formula can be found for black holes with $ML^d_p \lambda^{d-3} \gg 1$. In this latter case we have

$$S \propto \left(\frac{M}{L_p^2 \lambda^2}\right)^{(d-2)/(d-1)}. \hspace{1cm} (8)$$

Differently from the asymptotically flat case, we see that now inequality 2 is satisfied. For AdS black holes (at least for those with large enough mass) fragmentation is entropically preferred. This fact is perfectly consistent with the existence of a AdS/CFT correspondence between $d$-dimensional AdS gravity and $d-1$-dimensional conformal field theory. The $d$-dimensional black hole entropy reproduces correctly, after the $E = M$ identification, the entropy for a free field theory in $d-1$-dimensions.

Let us now discuss in some detail the three-dimensional (3D) case, the Banados-Teitelboim-Zanelli (BTZ) black hole 3. This case is very instructive not only because it is possible to solve exactly the inequality 2 but also because it clarifies the role played by the black hole ground state in our considerations. The BTZ black hole solution with zero angular momentum is

$$ds^2 = -(\lambda^2 r^2 - 8GM) dt^2 + \frac{(\lambda^2 r^2 - 8GM)^{-1} dr^2 + r^2 d\phi^2}{(1 + \lambda^2 r^2)^2}.$$  \hspace{1cm} (9)

where the black hole mass $M$ is defined with reference to the $M = 0$ black hole ground state, $ds^2 = -\lambda^2 r^2 dt^2 + \lambda^{-2} r^{-2} dr^2 + r^2 d\phi^2$. The entropy is given by $S = 2\pi r_+ / 4G = (\pi/\lambda) \sqrt{2M/G}$ so that Eq. 2 is satisfied for every value of the mass $M$. On the other hand, if the black hole mass is defined with reference to the full, geodetically complete AdS spacetime $ds^2 = -(1 + \lambda^2 r^2) dt^2 + (1 + \lambda^2 r^2)^{-1} dr^2 + r^2 d\phi^2$, the mass spectrum looks rather different. The $M = 0$ ground state is separated from the continuous part of the spectrum with $M \geq 1/8G$ by a mass gap. For $M \geq 1/8G$ the entropy is given by $S = (\pi/2\lambda G) \sqrt{8GM - 1}$. Taking for simplicity $M_1 = M_2 = M$ we see that Eq. 2 is satisfied only for $M \geq 3/16G$, whereas for $1/8G \leq M \leq 3/16G$ we have $S(M_1 + M_2) \geq S(M_1) + S(M_2)$. We will come back to this point in Sect. V, where will argue that this behavior is typical for a spectrum with a mass gap.

IV. TWO-DIMENSIONAL BLACK HOLES

In $d \geq 3$ spacetime dimensions it is very difficult to formulate general criteria that enable us to decide if a black hole satisfies either Eq. 2 or Eq. 3. These criteria can be found for 2D black holes. The 2D case is interesting for several reasons. Two-dimensional gravity supports a realization of the AdS/CFT correspondence 10. In two spacetime dimensions we can formulate entropy bounds 11. Moreover, 2D black holes can be used to describe black holes in higher dimensions. Two dimensional black holes arise as effective description of the near-horizon, near-extremal behavior of $d$-dimensional charged black hole and branes. Every $d$-dimensional spherically symmetric solution can be described, after dimensional reduction, by a 2D dilaton gravity model.

The generic 2D dilaton gravity model (For a recent review see 12) can be completely characterized by a dilaton potential $V(\Phi)$, the action for the model being given by $A = (1/2) \int d^2x \sqrt{-g}(\Phi R + \lambda^2 V(\Phi))$. The general 2D black hole solution takes the form

$$ds^2 = -(J(\Phi) - \frac{2M}{\lambda}) dt^2 + \left(\frac{J(\Phi) - \frac{2M}{\lambda}}{\lambda}\right)^{-1} dr^2, \hspace{1cm} \Phi = \lambda r,$$  \hspace{1cm} (10)

where $J(\Phi) = \int V(\Phi)$ and $M$ is the black hole mass. To be sure that Eq. 10 describes a black hole we will take $M \geq 0$, $\Phi > 0$ and $J(\Phi) > 0$ a strictly increasing function of $\Phi$, (we will therefore have $J(\infty) = \infty$). The temperature and entropy associated with the black hole are given by

$$S = 2\pi \Phi_h, \hspace{1cm} T = 4\pi \Phi(\Phi_h), \hspace{1cm} (11)$$

where $\Phi_h = J^{-1}(M)$ is the value of the dilaton at the black hole horizon.

Using Eq. 10 one easily finds

$$S = 2\pi J^{-1}(M). \hspace{1cm} (12)$$
One can now find sufficient conditions to be imposed on the function \( J(\Phi) \) such that Eq. (2) is satisfied \( \forall M_1, M_2 \). Let us first show that if both conditions (the prime denotes derivation with respect to \( M \))

\[
\begin{align*}
(J^{-1})''(M) &< 0, \quad \forall M, \\
J^{-1}(M = 0) &> 0,
\end{align*}
\]

are satisfied then

\[
J^{-1}(M_1 + M_2) < J^{-1}(M_1) + J^{-1}(M_2),
\]

which, owing to Eq. 12 implies that the inequality (2) for the entropy is also satisfied. The convexity condition (13) implies for every positive \( \hat{M}, M_1 \), \( (J^{-1})'(\hat{M} + M_1) < (J^{-1})'(\hat{M}) \), from which it follows \( J^{-1}(\hat{M} + M_1 + M_2) + J^{-1}(\hat{M}) < J^{-1}(\hat{M} + M_1) + J^{-1}(\hat{M} + M_2) \). Evaluating the previous inequality at \( \hat{M} = 0 \) and using condition 14 one easily recovers Eq. 15.

Analogously, we can show that if conditions

\[
\begin{align*}
(J^{-1})''(M) &> 0, \quad \forall M, \\
J^{-1}(M = 0) &< 0,
\end{align*}
\]

hold then the inequality 3 is satisfied for every \( M_1, M_2 \). If \( (J^{-1})'' = 0 \), identically, then the black hole solution of the model will satisfy \( S(M_1 + M_2) = S(M_1) + S(M_2) \). If \( (J^{-1})'' \) changes sign, we cannot make any definite statement about inequalities 2 or 3.

Any function \( J^{-1}(M) \) which is everywhere convex (concave) and strictly growing must diverge for \( M \to \infty \) lesser (faster) then \( J^{-1} = M \). It follows that \( J(\Phi) \) must diverge for \( \Phi \to \infty \) faster (lesser) then \( J = \Phi \). We have reached an important result: for models satisfying the required criteria about the derivatives of \( J^{-1} \) and \( J(0) = 0 \), fragmentation of 2D black holes can be or not be entropically preferred depending on the asymptotic behavior of the function \( J \). The black solutions of the 2D dilaton gravity model, characterized by a function \( J \), which asymptotically diverges faster (lesser) then \( \Phi \), will satisfy Eq. 2 (Eq. 3).

When \( J^{-1}(0) \neq 0 \) but condition 13 still holds we will have a mixed behavior. The black hole will satisfies Eq. 2 for \( M > M_0 \) and Eq. 3 for \( M < M_0 \), where \( M_0 \) is some threshold mass. Let us first notice that only the case \( J^{-1}(0) = \Phi_0 < 0 \) has physical relevance. Being \( J^{-1} \) a strictly growing function, \( \Phi_0 > 0 \) implies \( J^{-1}(M_0) = 0 \) with \( M_0 \) negative, i.e the presence of negative masses in the spectrum. Conversely, \( \Phi_0 < 0 \) implies

\[
J^{-1}(M_0) = 0,
\]

with \( M_0 \) positive, i.e the black hole spectrum is limited from below by the extremal, non vanishing value of the mass \( M_0 \). In general the extremal mass \( M_0 \) is simply related with the threshold mass \( M_0 \). Using Eqs. 12 and 15 one easily realizes that the entropy of the extremal state is zero, \( S(M_0) = 0 \). Depending on the behavior of the dilaton potential at \( M_0 \) we can have two cases. 1) \( J^{-1}(M_0) = 0 \) and \( V(M_0) = 0 \). The extremal state has zero entropy and from Eq. 14 also zero temperature. 2) \( J^{-1}(M_0) = 0 \) and \( V(M_0) \neq 0 \). The extremal state has zero entropy but nonvanishing temperature. In the next section we shall see that in both cases this behavior can be explained with the presence of a mass gap.

Let us now give some examples to illustrate our general results. The simplest model satisfying both conditions 15 is the Jackiw-teitelboim model, \( V = 2\Phi \). We have \( J^{-1}(M) = \sqrt{2M/\lambda} \), which satisfies both \( (J^{-1})'' < 0 \) and \( J^{-1}(0) = 0 \). For the entropy we get \( S \propto \sqrt{M} \), from which Eq. 2 follows for every value of the mass. A more general model is given by 12

\[
V = (h+1)\Phi^h, \quad h > -1.
\]

For \( h > 0 \) the model satisfies the condition 13 and 14. The black hole entropy is given by

\[
S \propto (M/\lambda)^{1/(h+1)},
\]

which as expected satisfies Eq. 2. This class of models contains, as particular cases, 2D gravity models arising as near-horizon limit of dilatonic zero-branes 12, black 3-branes 12 and heterotic string black holes 12. It is interesting to note that the entropy-energy relation 21 becomes that of a free field theory given by Eq. 1 identifying \( h = 1/(d - 1) \). This fact gives an hint about the possibility of finding a correspondence between these 2D dilaton gravity models and a free field theory.

For \( -1 < h < 0 \) the model satisfies the conditions 16 and 17. The black hole entropy now satisfies Eq. 3. An important particular case is given by \( h = -1/2 \), which describes the spherical dimensional reduction of the
Schwarzschild black hole. For \( h = 0 \) we have the CGHS model \( \text{[17]} \). Eq. \( \text{[19]} \) gives \((J^{-1})'' = 0\) identically. The entropy depends linearly on the mass, so that, as expected, \( S(M_1 + M_2) = S(M_1) + S(M_2) \).

As an example of a model fulfilling condition \( \text{[13]} \), but not \( \text{[14]} \), let us first consider the exponential potential \( V = \beta \exp(\beta\Phi) \), with \( \beta > 0 \). The black hole horizon is located at \( \lambda \Phi = J^{-1}(M) = (1/\beta) \ln(2M/\lambda) \). \( J^{-1} \) diverges asymptotically lesser than \( M \) but we have \( J^{-1}(0) \neq 0 \). We therefore expect the presence of a threshold mass \( \tilde{M}_0 \) separating the two regions of the spectrum where Eq. \( \text{[2]} \), respectively Eq. \( \text{[4]} \) hold.

The black hole mass has an extremal value, \( M \geq M_0 = \lambda/2 \). However, \( J^{-1}(M_0) = 0 \) and \( V(M_0) \neq 0 \). The entropy and temperature of the black hole are

\[
S = \frac{2\pi}{\beta} \ln \frac{2M}{\lambda}, \quad T = \frac{\beta}{2\pi} M. \tag{21}
\]

The ground state has zero entropy but finite temperature \( T(M_0) = (\beta/4\pi)\lambda \). Using Eq. \( \text{[21]} \) an taking for simplicity \( M_1 = M_2 = M \), one can easily find that the inequality \( \text{[2]} \) is satisfied only for \( M > \tilde{M}_0 = \lambda \), whereas for \( \lambda/2 < M < \lambda \) Eq. \( \text{[3]} \) holds. Black hole fragmentation becomes entropically preferred for \( M \) bigger then the threshold mass \( \tilde{M}_0 \). Notice that the threshold mass \( \tilde{M}_0 \) although of the same order do not coincide with the extremal mass \( M_0 \), we have \( M_0 = 2\tilde{M}_0 \).

As a second example of models with \( J^{-1}(0) \neq 0 \) let us consider \( J(\Phi) = \Phi^2 + 1 \). This is the 2D analogue of the 3D BTZ black hole discussed at the end of the previous section. Also in this case the black hole mass has an extremal value \( M_0 = \lambda/2 \). However, now \( S \propto T \propto \sqrt{(2M/\lambda) - 1} \), so that the extremal state has zero entropy and temperature, \( S(M_0) = T(M_0) = 0 \). The threshold mass at which black hole fragmentation becomes entropically preferred is given by \( \tilde{M}_0 = (3/2)M_0 = (3/4)\lambda \).

Until now we have considered a system in which the total energy is constant, i.e fragmentation of a black hole of mass \( M \) into two smaller black holes of masses \( M_1 + M_2 = M \). Let us now consider systems at constant volume. We will show that the energy is minimized when the conditions which maximize the entropy are satisfied.

We have to consider the black hole mass \( M \) as a function of its radius \( R \) and to solve the inequality

\[
M(R_1 + R_2) > M(R_1) + M(R_2). \tag{22}
\]

Using Eq. \( \text{[10]} \) one finds \( M(R) = (\lambda/2)J(\Phi_R) = (\lambda/2)J(R) \). Eq. \( \text{[22]} \) becomes \( J(R_1 + R_2) > J(R_1) + J(R_2) \). Because \((J^{-1})'' < 0 \) and \( J^{-1}(0) = 0 \) imply, respectively, \( J'' > 0 \) and \( J(0) = 0 \) it follows that conditions \( \text{[15]} \), \( \text{[14]} \) for the function \((J^{-1})\) are equivalent to conditions \( \text{[16]} \), \( \text{[17]} \) for the function \( J \). As a consequence, whenever conditions \( \text{[15]} \) and \( \text{[14]} \) are satisfied we have not only \( S(M_1 + M_2) < S(M_1) + S(M_2) \) but also \( M(R_1 + R_2) > M(R_1) + M(R_2) \). The process of fragmentation of a black hole maximizes the entropy if the total mass is constant and minimizes the mass if the total volume is constant. Conversely, if conditions \( \text{[16]} \) and \( \text{[17]} \) hold we have \( S(M_1 + M_2) > S(M_1) + S(M_2) \) but also \( M(R_1 + R_2) < M(R_1) + M(R_2) \). The configuration with maximal entropy and minimal mass is now given by the single black hole.

V. MASS GAP

It is evident that for what concerns the entropy of composite solutions 2D dilaton gravity models satisfying the conditions \( \text{[19]} \) and \( \text{[14]} \) of the previous section are very similar to free field theories. The mass of the gravitational solutions are naturally identified with the energy of the excitation in the field theory. The same is true for generic \( d \)-dimensional AdS solutions when the mass of the gravitational solution is much bigger then \( L_{d-2}^d \lambda^{d-3} \). We can ask ourselves if the 2D dilaton gravity models satisfying condition \( \text{[15]} \) but not \( \text{[14]} \) have a field theory counterpart. More in general, one would like to find field theoretical counterparts of \( d \)-dimensional black hole solutions whose spectrum exhibits extremal \( M \neq 0 \) solutions. The most natural candidates are field theories with mass gaps. This is rather obvious for the 2D dilaton gravity models discussed at the end of Sect. IV (for instance the model with exponential dilaton potential), which are characterized by a ground state of mass \( M_0 \) with zero entropy and nonvanishing temperature. The only way to have a nondegenerate ground state at finite temperature is the presence of a mass gap in the spectrum. The energy gap \( M_0 \) must be of the order of the temperature. In fact for the model with exponential potential discussed in Sect. IV we have \( E_{\text{gap}} = M_0 \propto T(M_0) \propto \lambda \).

The relationship between black hole solutions and field theories with mass gaps seems to be more general. Crucial for the existence of such correspondence are both the asymptotic behavior of the metric and the presence of an extremal solution with \( M_0 \neq 0 \). To illustrate this correspondence let us consider a generic field theory whose spectrum has a mass gap of energy \( E_0 \) separating the \( E = 0 \) state from the continuos part of the spectrum \( E \geq E_0 \). We will also assume that for \( E >> E_0 \) the spectrum is that of a generic free field theory and that the entropy of the \( E = E_0 \) state vanishes. For \( E >> E_0 \) the entropy/energy relation will be given by Eq. \( \text{[1]} \) which satisfies
$S(E_1 + E_2) < S(E_1) + S(E_2)$. On the other hand for $E \approx E_0$ we have $S = S(E - E_0)$ with $S(E_0) = 0$. Thus, the inequality $S(E_1 + E_2 - E_0) > S(E_1 - E_0) + S(E_2 - E_0)$ will be satisfied at least for $E_1 = E_2 = E_0$. Because for $E >> E_0$ the opposite inequality holds, this implies the existence of a threshold energy $E_0$ separating the two regimes in complete analogy with what happens for black hole solutions. This behavior is rather intuitive. For small excitations near $E_0$ the single state of energy $E = E_1 + E_2$ has more degeneracy then the two states of energies $E_1$ and $E_2$ because the states below the gap do not contribute. For excitations of energy $E >> E_0$ the contribution of the gap is irrelevant and the entropy is dominated by the contribution coming from the continuos part of the spectrum.

VI. ENTROPY BOUNDS

The holographic principle in its usual formulation puts an upper bound to the entropy, hence to the amount of information, which can be stored in given region of space, $S \leq A/4G$, where $A$ is the area of the surface enclosing the region. At fixed volume the entropy cannot increase by splitting a system into parts, i.e $S(V_1 + V_2) \geq S(V_1) + S(V_2)$. This inequality is satisfied not only for extensive systems, when the entropy scales as the volume of the system, but also when the entropy scales as the area of the surface enclosing the system. An holographic bound cannot be violated by fragmentation. Hence the holographic principle is compatible with both Eqs. (2) and (3). This is true for the holographic bound but it is no necessarily true for entropy bounds of different type, which can make reference not only to the volume the system (or to the area enclosing it) but also to its energy. For instance the Bekenstein bound $S \leq 2\pi ER$ refers not only to the linear size $R$ but also to the energy $E$ of the system.

In principle one could also try to formulate entropy bounds, which make reference to nothing but the energy of system. This could be done following the same line of reasoning one uses to formulate the Bekenstein-Hawking bound. In this latter case one takes a system in a given region of space and finds that its entropy is bounded by the area of the surface enclosing the region. More in detail, when one pumps energy in the system keeping its spatial extension constant, the entropy increases until the energy equals the mass pertaining to a black hole fitting in the region. A black hole forms and the bound $S \leq A/4G$ is saturated.

Instead, one could take a system with a constant amount of energy $M$ and look for a bound on the associated entropy. Spreading the system over regions of space with ever decreasing volumes the entropy will increase until the corresponding Schwarzschild radius is reached. A black hole forms and a bound $S \leq \pi GM^2$ is found. The problem is that this bound cannot be universal. Although, the Schwarzschild black hole cannot increase its entropy by fragmentating, we have seen that there exist many black hole solution that can increase their entropy trough fragmentation. For this simple reason an entropy bound which refers only to the energy of the system cannot be universal.

VII. CONCLUSIONS

In this paper we have discussed the holographic principle and entropy bounds for composite gravitational systems. We have shown that with respect to fragmentation black hole solutions exhibit three different types of behavior. The entropy may decrease, increase or have a mixed behavior. In the latter case there will exist a threshold mass separating the regime where fragmentation is entropically preferred from the regime where it is not. Moreover, for 2D black holes we have been able to find a complete characterization of the entropy behavior, in the form of sufficient conditions imposed on the function $J$, which defines the 2D gravitational model.

Our results have a strong impact on the way one can realize the holographic principle. In fact only for gravity theories satisfying relation (2) one can hope to find a realization of the holographic principle in terms of a correspondence gravity/field theory. For gravity theories satisfying (3) this correspondence cannot be realized as a gravity/field theory duality and must necessarily have an alternative, still unknown, form. Because asymptotically flat black holes, and in particular the Schwarzschild black hole, belongs to this latter case we can exclude the existence of a correspondence between asymptotically flat black holes and a field theory.

On the other hand we have seen that the class of gravity theories satisfying Eq. (2) is, at least in the 2D case, rather broad. It contains not only AdS gravity, for which the gravity/field theory duality is well established, but also other models, for instance those with a exponential dilaton potential. The possibility of finding a gravity/field theory duality for these theories is an open question, which deserves further investigations.

An other point of interest is the existence of black holes with mixed behavior, i.e exhibiting a transition from a regime where Eq. (2) holds to a regime where instead Eq. (3) is satisfied. We have argued that the most natural candidates for holographic duals of this black holes are field theories with a mass gap. Finally, our results seem to exclude the possibility of formulating an entropy bound only in terms of the energy of a system: the existence of black holes satisfying Eq. (2) will always allow to violate the bound by black hole fragmentation.
Acknowledgments

We thank S. Mignemi for discussions and valuable comments.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] arXiv:hep-th/9711200.
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428 (1998) 105 arXiv:hep-th/9802109.
[3] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253 arXiv:hep-th/9802150.
[4] G. ’t Hooft, arXiv:gr-qc/9310026.
[5] L. Susskind, J. Math. Phys. 36 (1995) 6377 arXiv:hep-th/9409089.
[6] E. Verlinde, arXiv:hep-th/0008140.
[7] R. Bousso, Rev. Mod. Phys. 74 (2002) 825 arXiv:hep-th/0203101.
[8] R. Bousso, JHEP 9907 (1999) 004 arXiv:hep-th/9905177.
[9] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849 arXiv:hep-th/9204099.
[10] M. Cadoni and S. Mignemi, Phys. Rev. D 59 (1999) 081501 arXiv:hep-th/9810251.
[11] S. Mignemi, arXiv:hep-th/0307205.
[12] D. Grumiller, W. Kummer and D. V. Vassilevich, Phys. Rept. 369 (2002) 327 arXiv:hep-th/0204253.
[13] S. Mignemi, Annals Phys. 245 (1996) 23 arXiv:hep-th/9411153.
[14] M. Cadoni, P. Carta, M. Cavaglia and S. Mignemi, Phys. Rev. D 65 (2002) 024002 arXiv:hep-th/0105113.
[15] M. Cadoni, arXiv:hep-th/0306069.
[16] M. Cadoni, Phys. Rev. D 60 (1999) 084016 arXiv:hep-th/9904011.
[17] C. G., Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D 45 (1992) 1005 arXiv:hep-th/9111056.
[18] J. D. Bekenstein, Phys. Rev. D 23 (1981) 287.