KINETIC EQUATIONS FOR ULTRARELATIVISTIC PARTICLES IN A ROBERTSON-WALKER UNIVERSE AND ISOTROPIZATION OF RELICT RADIATION BY GRAVITATIONAL INTERACTIONS

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Abstract

Kinetic equations for ultrarelativistic particles with due account of gravitational interactions with massive particles in the Robertson-Walker universe are obtained. On the basis of an exact solution of the kinetic equations thus obtained, a conclusion is made as to the high degree of the uniformity of the relict radiation on scales with are less than $10^{10}$.

1 Introduction

Bisnovatyi-Kogan and Shykhman (1982) have shown that within the scope of a macro-scopically homogeneous isotropic cosmological model based on the Newtonian theory of gravitation we can form kinetic equations with an integral of gravitational collisions which converge at large distance of action. The kernel of the collision integral obtained by Bisnovatyi-Kogan (1982) is the same as that of the Landau collision integral in Landau (1937). However the ‘Coulomb’ logarithm turns out to be finite at any finite cosmological time. Kandrup (1982) substantiated this result, but the collision integral differs from the one obtained in Bisnovatyi-Kogan and Shykhman (1982) in so far it does not vanish on account of equilibrium distribution. The difference between the kernels of collision integrals in Bisnovatyi-Kogan and Shykhman (1982) and Kandrup (1982) seems to be caused by the following. A Robertson-Walker universe lacks homogeneity in time, which is why the energy of the particles is not an integral of motion. On the other hand, in consequence of the long-range character of gravitational interaction the act of gravitational collision is protracted. It is precisely the combination of the temporal non-locality of gravitational interaction and the absence of temporal homogeneity that leads to the collision integral in Kandrup (1982). In Zakharov (1984) an integral of Coulomb collisions is obtained for nonrelativistic charged particles in a Robertson-Walker universe filled with dust. This integral coincides with the collision integral obtained by Kandrup, if the same notations are used, although all calculations in Zakharov (1984) are made within the framework of Einstein’s theory of gravitation. It should be noted that the employment of Einstein’s theory of gravitation for the problem of a Coulomb interaction of non-relativistic particles is unwarranted, since non-relativistic particles interact by means of Coulomb’s field in an identical manner both in Einstein’s and in Newton’s treatment, whereas the Coulomb component of an electromagnetic field in an isotropic space, being bound up with the law of the conservation of a charge, does not differ from the traditional component. The are precisely these facts that account for the agreement between the results in Zakharov (1984) and Kandrup (1982).

Thus, the derivation of kinetic equations without regard to the radiation for non-relativistic particles in a Robertson-Walker universe can always be carried out within the scope of the Newtonian theory of gravitation and Newtonian mechanics. This assertion is based on two facts: (1) the adequacy of describing the effect of a gravitational field on non-relativistic particles

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1Original article is published in [1]. In the given version original typing errors are corrected.
within the scope of Einstein’s and Newton’s theories of gravitation, which makes it possible to employ Newtonian mechanics; (2) the smallness of the path of a non-relativistic particle as compared to the light horizon, which makes it possible to employ the Newtonian theory of gravitation.

There exists a fundamentally different condition in the case of ultrarelativistic particles. A description of the gravitational effect on these particles can only be carried out within the scope of a relativistic theory of gravitation. Indeed, on the one hand, the motion of an ultrarelativistic particle is affected by the components of a three-dimensional metric, which is why Newtonian mechanics cannot be employed to describe motion. On the other hand, an ultrarelativistic particle runs a distance comparable to the light horizon, which makes the Newtonian description of a gravitational field unacceptable. However, in the case of electromagnetic interaction the question of forming kinetic equations for ultrarelativistic particles in a Robertson-Walker universe is easily solved; in consequence of the conformal invariance of field equations and motion equations, these equations will be not different from those in a plane space (see Ignat’ev, 1982).

This brings to the fore the basic problem of formulating kinetic equations for ultrarelativistic particles in a Robertson-Walker universe with allowance for their gravitational interaction with non-relativistic ones. Suitable techniques for forming these equations are provided by Ignat’ev (1978, 1983). In the present paper a kinetic equation for ultrarelativistic particles is derived, with due account of their gravitational interaction with massless ones. This is carried out on the basis of averaging a collisionless kinetic equation over the local fluctuations of the Robertson-Walker metric caused by the gravitational fields of massive point particles. The solution of Einstein’s equations linearized about a Robertson-Walker solution shows that the local fields of point masses formed by a redistribution of Robertson-Walker matter, always remain within the sound horizon. In this connection the problem of the convergence of the collision integral at large distance of action becomes irrelevant. The kinetic equation thus obtained has a collision term whose structure is the same as that of the Belyaev-Budker collision integral (cf. Belayev and Budker, 1956), if one of the particles is taken to be non-relativistic in the latter. A solution of the kinetic equation thus obtained is presented. The solution describes the process of the isotropization of a homogeneous, but anisotropic distribution of massless relict particles. The estimates show that angular harmonics with a scale less than 10 angular minutes are strongly damped. All the notation, unless otherwise specified in the text, are according to those in Ignat’ev (1978, 1981, 1982, 1983), Belayev and Budker (1956), or Landau and Lifshitz (1972).

2 Massive Particles in a Robertson-Walker Universe

The gravitational field of a massive point particle in a Robertson-Walker universe is basically obtainable with the aid of the well-known Lifshitz solutions (see Landau and Lifshitz, 1972). However, the formulation of initial and boundary conditions in a problem with spherical symmetry in terms of plane waves loses its physical lucidity. Besides, the synchronous frame of reference by Landau and Lifshitz (1972) is unsuitable for our purpose. We shall write the energy-momentum tensor of a massive point particle (Ignat’ev, 1983) as \( \delta T^{ik} = \frac{m}{\hbar} \int u^i u^k D(x|x') dS \)

where \( D(x|x') \) is the invariant four-dimensional function of Dirac: i.e.

\[ \int D(x|x') d\Omega = 1 \]
(dΩ = \sqrt{-g}d^4x); integration in (2.1) is carried out along the entire world line of the particle \(x' = x'(S)\). As will be seen from the following, in a non-empty space the mass of a particle cannot remain constant; therefore, \(m = m(x)\) is a scalar function. The motion equations of a point particle with a variable mass is obtainable from an invariant Hamiltonian (Ignat’ev, 1982)

\[
H_m(x, P) = \frac{1}{m}g^{ik}P_iP_k - m = 0,
\]

from which we find using standard procedure (as in Ignat’ev, 1983)

\[
\frac{d\mathbf{u}^i}{dS} = (\ln m)_{,k}(g^{ik} - u^iu^k).
\]

(2.2)

If we assume the spherical symmetry of the problem we shall write the space-time metric in isotropic coordinates

\[
ds^2 = e^\nu dv^2 - e^\lambda(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)),
\]

where \(\nu = \nu(r, \eta), \lambda = \lambda = (r, \eta)\). Then the world line of the particle is the time line \(r = 0, m = m(\eta)\) being an arbitrary function in consequence of (2.2). Integrating (2.1) we get

\[
δT^i_k = δ_T^i_k \delta e^{3/2}\delta(r)
\]

where \(\delta(r)\) is Dirac’s three-dimensional \(\delta\) - function in plane space. We shall represent the total energy-momentum tensor \(T^{ik}\) as

\[
T^{ik} = T^{ik}_f + \delta T^{ik},
\]

where \(T^{ik}_f\) is the energy-momentum tensor of a fluid. In consequence of the Einstein equations, (2.1) and (2.2) the law of conservation is satisfied by

\[
T^{ik}_f \cdot k = -\nabla^i \int mD(x|x')dS.
\]

In particular, it is, in our case, of the form

\[
T^{ik}_f \cdot k = -m\varepsilon e^{-\mu+\lambda}\delta(r).
\]

Einstein’s non-trivial equations for the metric (2.3) take the form (see Landau and Lifshitz, 1972)

\[
\frac{1}{2}e^{-\lambda} \left[ \frac{\lambda'^2}{2} + \lambda' \nu' + \frac{2}{r}(\lambda' + \nu') \right] - e^{-\nu} \left( \dot{\lambda} - \frac{1}{2} \dot{\lambda} \nu + \frac{3}{4} \dot{\lambda}^2 \right) = 8\pi[p + \nu^2(\varepsilon + p)];
\]

\[
\frac{1}{4}e^{-\lambda} \left[ 2(\lambda'' + \nu'') + \nu'^2 + \frac{2}{r}(\lambda' + \nu') \right] - e^{-\nu} \left( \dot{\lambda} - \frac{1}{2} \dot{\lambda} \nu + \frac{3}{4} \dot{\lambda}^2 \right) = 8\pi p;
\]

\[
-e^{-\lambda} \left( \lambda'' + \frac{\lambda'^2}{4} + \frac{2}{r} \lambda' \right) + \frac{3}{4}e^{-\nu}\dot{\lambda}^2 = 8\pi \left[ me^{-\frac{1}{2}(\nu + p)}\delta(r) + \varepsilon + \nu^2(\varepsilon + p) \right];
\]

\[
\frac{1}{2}e^{-\lambda} (2\lambda' - \nu' \dot{\lambda}) = 8\pi(\varepsilon + p)e^{\frac{1}{2}(\nu + \lambda)}\sqrt{1 + \nu^2},
\]

where \(\nu = u^re^{\lambda/2}\) is the reference projection of the radial velocity of a fluid. If we subtract the second equation from the first, we get

\[
\frac{1}{2}e^{-\lambda} \left[ \frac{\lambda'^2}{2} + \lambda' \nu' + \frac{1}{2}(\lambda' + \nu') - (\lambda'' + \nu'') \right] = 8\pi(\varepsilon + p)\nu^2.
\]
We shall take \( m, v, \lambda', \nu' \) to be first-order infinitesimals. Then in a linear approximation the last equation is easily integrable, yielding

\[
\lambda + \nu = C_1(\eta) r^2 + C_2(\eta).
\]

By making a direct substitution we can verify that \( C_1 = 0 \). Admissible transformations of coordinates which preserve the form of the metric (2.3) are

\[
\eta = \eta(\tilde{\eta}); \quad r = k \tilde{r}; \quad (k = \text{Const}).
\]

In consequence, we can add to \( \nu \) an arbitrary time function, and to \( \lambda \) an arbitrary constant. Let us select this function so that

\[
\lambda = \ln a^2 + \xi(r, \eta), \quad \nu = \ln a^2 - \xi(r, \eta),
\]

where \( a = a(\eta), \xi \ll 1 \). Linearizing Einstein’s equations with respect to the smallness of \( m, v, \xi \), we obtain the system

\[
\frac{1}{a^2} \left( \frac{2 \ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) = -8 \pi p_0; \quad 3 \frac{\dot{a}^2}{a^2} = 8 \pi \varepsilon_0; \quad (2.5)
\]

\[
\dot{\xi} + 3 \frac{\dot{a}}{a} \left( \frac{2 \ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \xi = -8 \pi a^2 \frac{dp}{d\varepsilon} \delta \varepsilon; \quad (2.6)
\]

\[
-\frac{1}{a^2 r^2} \frac{\partial}{\partial r} (r^2 \xi') + 3 \frac{\dot{a}}{a^2} \frac{\partial}{\partial \eta} (a \xi) = 8 \pi \frac{m}{a^3} + 8 \pi \delta \varepsilon; \quad (2.7)
\]

\[
v = \frac{1}{8 \pi (\varepsilon_0 + p_0)} \frac{\partial}{\partial \eta} (a \xi'), \quad (2.8)
\]

where \( p_0 = p_0(\eta), \varepsilon_0 = \varepsilon_0(\eta) \), and we have put \( \delta p = (dp/d\varepsilon) \). Equations (2.5) describe the evolution of a flat-space Robertson-Walker universe (cf. Landau and Lifshitz, 1972); Equation (2.8) determines the radial velocity of a fluid. To solve the singular equations (2.6) and (2.7), let us put

\[
\xi = \frac{2}{ra} (m - \Psi), \quad (2.9)
\]

and \( m = m(\eta) \),

\[
\lim_{r \to 0} \Psi(r, \eta)/r < +\infty. \quad (2.10)
\]

If we substitute (2.9) into (2.7) and taking account of (2.10) we get

\[
4 \pi a^3 \delta \varepsilon = \frac{1}{r} \Psi'' + \frac{3 \dot{a}}{ar} \frac{\partial}{\partial \eta} (m - \Psi). \quad (2.11)
\]

By use of (2.11) in the right-hand side of Equation (2.6), we get closed equations for \( m(\eta) \) and \( \Psi(r, \eta) \):

\[
\ddot{m} + \frac{\dot{a}}{a} \dot{m} \left( 1 + 3 \frac{dp}{d\varepsilon} \right) + m \left( \frac{2 \ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) = 0; \quad (2.12)
\]

\[
\ddot{\Psi} + \frac{\dot{a}}{a} \dot{\Psi} \left( 1 + 3 \frac{dp}{d\varepsilon} \right) + \Psi \left( \frac{2 \ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) = \Psi \frac{dp}{d\varepsilon}; \quad (2.13)
\]
At the non-relativistic stage $p_0 = 0$ and, according to (2.5), $a \sim \eta^2$. Then (2.12) and (2.13) are easily integrable: yielding

$$m = \sigma \left( \frac{\eta_0}{\eta} \right)^3 + \mu \left( \frac{\eta}{\eta_0} \right)^2 ; \quad \Psi = W(r) \left( \frac{\eta_0}{\eta} \right)^3 + V(r) \left( \frac{\eta}{\eta_0} \right)^2,$$

(2.14)

where $\sigma, \mu$ are arbitrary constants and $W(r), V(r)$ are arbitrary functions. Thus, the mass of a point particle in a medium is not conserved, which is the consequence of its gravitational interaction with the fluid.

To determine the functions $W(r), V(r)$ it is necessary to consider the evolution of a metric at the ultrarelativistic stage of expansion ($\varepsilon_0 = 3p_0$). In this case Equations (2.12) and (2.13) take the form

$$\ddot{m} + \frac{2}{\eta} \dot{m} - \frac{2}{\eta^2} m = 0; \quad \ddot{\Psi} + \frac{2}{\eta} \dot{\Psi} - \frac{2}{\eta^2} \Psi = \frac{1}{3} \Psi''.$$

The first one is easily integrated, and yields

$$m = \sigma' \left( \frac{\eta_0}{\eta} \right)^2 + \mu' \frac{\eta}{\eta_0},$$

(2.15)

where $\sigma', \mu'$ are arbitrary constants; to solve the second equation we shall make a substitution $\eta = \sqrt{3} \tau$ and

$$\Psi = \frac{\partial}{\partial \tau} \frac{1}{\tau} \Phi(r, \tau).$$

(2.16)

On substituting (2.16) into the second equation and changing the order of differentiation we shall reduce it to the form

$$\frac{\partial}{\partial \tau} \frac{1}{\tau} (\Phi_{rr} - \Phi_{\tau r}),$$

(2.17)

whence

$$\Phi(r, \tau) = \Phi_+ (r + \tau) + \Phi_- (r, \tau).$$

(2.18)

To the solution obtained we can add $\tau F(r)$, where $F(r)$ is an arbitrary function. However, this addition, according to (2.16), does not alter the value of $\Psi$. A particular solution to (2.18) is $\Phi = 0$. Then, according to (2.9), we shall obtain a Newtonian potential caused by a point particle of variable mass $m(\eta)$ determined with the aid of (2.15). However, this solution is physically unacceptable for the description of the gravitational field of a point particle which arose as a result of fluctuation at the moment of time $t = 0$, since it is inconsistent with the principle of causality. Indeed, the Newtonian potential referred to make it possible to obtain information about a particle beyond its light horizon and even to determine the mass of the particle. Therefore, boundary conditions should be formulated in such a way that at least beyond the light horizon the potential $\xi$ should vanish together with its derivatives. Such boundary conditions are in accordance with the 'birth' of a particle as a result of the redistribution of Robertson-Walker matter. In fact, however, the horizon of the potential is not the light cone but the sound cone, since the change in the potential is bound up with the redistribution of matter, and the latter proceeds precisely at the speed of sound. Mathematically this is related to the characteristics of Equation (2.17) and it is precisely at the sound horizon $r = \tau = r/\sqrt{3}$ that boundary conditions take the simplest form.
For the mass of a particle to remain limited at \( t \to 0 \) it is necessary to put \( \sigma' = 0 \) in the solution (2.15). Boundary conditions at the light horizon which satisfy the principle of causality are of the form

\[
\xi(r, \tau)|_{r=\tau} = 0; \quad \xi'(r, \tau)|_{r=\tau} = 0; \quad v(r, \tau)|_{r=\tau} = 0.
\]

The latter condition is, however, automatically satisfied in consequence of the first two conditions. If we substitute \( \xi \) here from (2.9) we shall reduce the first two conditions to the form

\[
\Psi(\tau, \tau) = m(\tau) = \mu \tau_0; \quad \Psi(r, \tau)|_{r=\tau} = 0.
\] (2.19)

According to the definitions of (2.17) and (2.18) these conditions are equivalent to

\[
\Phi_+(2\tau) + \Phi_-(0) = 2\frac{\mu(\tau)}{\tau_0} r^3 + A\tau; \tag{2.20}
\]

\[
\Phi'_+(r + \tau)|_{r=\tau} - \Phi'_-(r - \tau)|_{r=\tau} = B\tau, \tag{2.21}
\]

where \( A \) and \( B \) are arbitrary constants. Besides, due to (2.10) another condition is to be satisfied: namely

\[
\Phi_+(\tau) + \Phi_-(\tau) = 0. \tag{2.22}
\]

From (2.20) and (2.22) we find that

\[
\Phi_+(x) = \frac{1}{4} \frac{\mu'}{\tau_0} x^3 + A x + \Phi_+(0),
\]

\[
\Phi_-(x) = -\frac{1}{4} \frac{\mu'}{\tau_0} x^3 - A x - \Phi_+(0).
\]

Differentiating these relations and substituting the results into (2.21), we get \( A = 0 \), \( = -\mu'/\tau_0 \).

If we substitute the obtained value of (, ) into (2.16), we finally get

\[
\Psi(r, \tau) = \begin{cases} \frac{1}{2} \frac{\mu'}{\tau_0} r \left( 3 - \frac{r^2}{\tau_0^2} \right), & r \leq \tau; \\ -\frac{\mu'}{\tau_0} r, & r > \tau. \end{cases} \tag{2.23}
\]

Let \( \eta_0 \) be a moment of 'time' when the ultrarelativistic stage is replaced by the non-relativistic. Suppressing in the solution of (2.14) terms which correspond to the dispersion of the mass and joining this solution to (2.23) at the moment \( \eta_0 \), we get an expression for the potential \( \xi(r, \tau) \) at the non-relativistic stage of expansion

\[
\xi(r, \tau) = \begin{cases} \frac{2\mu}{r} \left[ 1 - \frac{r}{2\tau_0} \left( 3 - \frac{r^2}{\tau_0^2} \right) \right], & r \leq \tau; \\ 0, & r > \tau. \end{cases} \tag{2.24}
\]

The potential obtained is time-independent. Therefore, if the distance \( \Delta r \) from a certain observer, synchronous in a Robertson-Walker metric, to a massive particle is more than the value \( \eta_0/\sqrt{3} \), the observer will never be affected by the gravitation of a particle. Thus, in a

\[\text{A solution to (2.23) is also obtainable as an auto-model one, putting } \Psi = \tau \Psi(r/\tau).\]
Robertson-Walker universe the effective range of the gravitational forces of point particles turns out to be finite. The ratio between this range \( l_g = a(\eta)\eta_0/\sqrt{3} \) and the distance to the light horizon \( l_c \) at the non-relativistic stage of expansion is less than unity and decreases as time goes on

\[
\frac{l_g}{l_c} = \frac{t_0}{t} \left( \frac{\eta_0}{t} \right)^{1/3}.
\]

At \( t_0 \sim 10^{13} \text{c}, t \sim 10^{18} \text{c}, l_g/l_c \sim 10^{-2} \). It is of interest to note that the dimension of gravitationally bound regions \( l_g \) in the present epoch turns out to be of the order of \( 100 \text{Mpc} \), which at the medium density \( \varrho = 10^{-29} \text{g} \cdot \text{cm}^{-3} \text{Mpc}^{-1} \) indicates a mass of the order of \( 10^{18} \text{M}_\odot \). It is also worth of noting that the mass \( m \sim m_{pl} \) per a Planck moment of time increases to values of the order of \( 10^{26} \text{g} \) in the present epoch.

If we substitute Equation (2.24) into (2.11), we shall obtain an expression for the perturbation of the energy density caused by a massive particle, of the form

\[
\frac{\delta \varepsilon}{\varepsilon_0} = \frac{-m(\eta)}{\frac{4}{3}\pi a^3 \varepsilon_0} + \xi(r) = \frac{m(\eta)\sqrt{3}}{2t_0} + \xi(r),
\]

from which follows an obvious fact that the perturbation of the energy density remains small as long as the mass of a particle is small as compared to the mass of the gravitationally bound region:

\[
m(\eta) \ll \frac{4}{3}\pi a^3 \varepsilon_0(\eta) = \frac{2t_0}{\sqrt{3}} = \overline{m}.
\]

At \( t_0 = 10^{13} \text{c} \) we have \( m = 10^{18} \text{M}_\odot \). According to (2.26), at the boundary of the gravitationally bound region \((r = r_0)\) there arises an abrupt change in density \( \delta \varepsilon(r_0, \eta)/\varepsilon_0(\eta) = -m(\eta)/\overline{m} \), which vanishes at \( \eta \to 0 \) and increases as time goes on. At the end the entire matter from within the sphere with the radius \( r_0 \) accretes by a massive particle and the finite gravitational field in an empty sphere will be described by a Schwarzschild metric with the gravitational mass \( \overline{m} \).

It is not difficult to obtain this finite metric by joining a Schwarzschild metric in an isotropic coordinate system to a Robertson-Walker metric (see, for instance, Landau and Lifshitz, 1972) on a sphere with the ‘radius’ \( r_0 = \text{const.} \) A smooth joining is possible precisely at \( r_g = 2\overline{m} \), where \( \overline{m} \) is described by Equation (2.27).

### 3 Local Fluctuations of the Metric and Their Averages

Let there be in a Robertson-Walker universe not one, but several massive identical particles with the coordinates \( r_a = \{x_a, y_a, z_a\} \). Then the total space-time metric can be approximately written in a form (linear approximation with respect to \( m \)):

\[
dS^2 = (g_{ik} + h_{ik}) dx^i dx^k,
\]

where \( g_{ik} \) is a Robertson-Walker metric,

\[
h_{ik} = -a^2 \delta_{ik} \sum_a \xi_a(|r - r_a|).
\]
Indeed, a contribution to the metric caused by the interaction of particles is of the order $m^2$; non-diagonal components of the metric tensor $g_{ab} \sim \xi_{ab} \sim m^2$ are also of the same order. Introduce a field of observers, macroscopic in the metric of (3.1), whose coordinate grid is on massive particles. Such observers are geodesic with respect to the Robertson-Walker metric. Let the coordinates of massive particles $r_a$ assume random equiprobable values throughout the entire three-dimensional space, with no correlation between the positions of these particles. Let $N = \text{Const}$, be the number of massive particles per volume $V = 4/3 \pi r_0^3$. Let us introduce an operation of averaging a certain field value $\varphi(x|r_1, r_2, \ldots)$, which is a function of the positions of the massive particles, on the scale of macroscopic observers: i.e.,

$$\langle \varphi(x) \rangle = \prod_a \frac{N}{V_a} \int d^3 r_a \varphi(x|x_1, x_2, \ldots)$$

(3.3)

where integration is carried out within the spheres with the radius $r_0$ with centres at the point; $V_a = V_{\xi}$. Then the average of the local fluctuations of the Robertson-Walker metric of (3.2), according to (2.24) and (3.3), is

$$\langle h_{ik} \rangle = -a^2 \delta_{ik} \frac{3\mu N}{5r_0} = a^2(\eta)\text{Const.}$$

(3.4)

If we follow the procedure of Ignat'ev (1978), we shall renormalize the macroscopic Robertson-Walker metric and the local fluctuations $h_{ik}$ so that the average of the latter should be equal to zero: i.e.,

$$g_{ik} \rightarrow g_{ik} + \langle h_{ik} \rangle; \quad h_{ik} \rightarrow h_{ik} - \langle h_{ik} \rangle; \quad \xi_a \rightarrow \xi_a - \frac{3\mu}{5r_0}.$$  

(3.5)

Due to (3.4) a renormalization of the macroscopic metric $g_{ik}$ reduces to multiplying $g_{44}$ and $g_{\alpha\beta}$ by constant numbers. Carrying out a further admissible infinitesimal scale transformation of the coordinates $\eta$ and $r$ with a constant scale coefficient we shall restore the former value of the Robertson-Walker metric

$$\langle dS^2 \rangle = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2).$$

(3.6)

By use of the value of the function of $\xi(r)$ redefined according to (3.5) we shall evaluate the change in the total mass $\delta M(r_0, \eta)$ within the radius $r_0$ caused by a massive particle. For this purpose, we shall transform the expression for the total mass $M(r)$ in the Schwarzschild coordinate system (Landau and Lifshitz, 1972), linearize it with respect to $\xi$ and employ the formula (2.24). As a result we get

$$\delta M(r_0, \eta) = 4\pi \frac{3\dot{a}}{a} \frac{d}{d\eta} \int_0^{r_0} \xi r^2 dr = 0$$

Thus, the solution we have obtained actually describes the gravitational field of a particle formed by a redistribution of Robertson-Walker matter.

If we calculate $\langle \xi^2(r) \rangle$ according to (3.3) and (3.5), we get

$$\langle \xi^2(r) \rangle = \sum_a \langle \xi_a^2 r_a \rangle = \frac{108}{175} N \left( \frac{2\mu}{r_0} \right)^2.$$  

(3.7)

In the formula (3.3) integration could be carried out over the entire space; however, this would yield nothing new, since according to (2.24) outside the sphere with the radius correlations vanish.
For the local and medium perturbations of the metric to be small, two conditions are to be satisfied: namely,

\[ |r_a - r| \gg 2\mu; \]  
(3.8)

\[ N \left( \frac{2\mu}{r_0} \right)^2 \ll 1. \]  
(3.9)

Since several massive particles can now find themselves within the sphere with the radius \( r_0 \), the condition of (2.27) must be replaced by

\[ \eta \frac{\mu}{r_0^2} \ll 1. \]  
(3.10)

By use of the properties of isotropy and homogeneity we can show the validity of the equalities

\[ \langle h_{ik} h_{lm} \rangle = \delta_{ik} \delta_{lm} a^4 \langle \xi^2 \rangle; \]
\[ \langle \partial_j (h_{ik}) h_{lm} \rangle = 0; \quad \langle \partial_k (h_{ik}) \partial_j (h_{lm}) \rangle = 0. \]  
(3.11)

Averages of the type \( \langle \partial_\alpha \xi \partial_\beta \xi \rangle \) diverge as \( r^{-1} \), the divergence of these values is connected with the ordinary divergence of the particle’s energy. To calculate these values we shall note that, due to the isotropy of space,

\[ \langle (\partial_\alpha \xi)(\partial_\beta \xi) \rangle = \frac{1}{3} \delta_{\alpha\beta} \langle (\partial_\gamma \xi)(\partial_\gamma \xi) \rangle = \frac{1}{3} \delta_{\alpha\beta} \left[ \langle \partial_\gamma (\xi \partial_\gamma \xi) \rangle - \langle \xi \partial_\gamma \xi \rangle \right]. \]

Thus,

\[ \langle (\partial_\alpha \xi)(\partial_\beta \xi) \rangle = -\frac{1}{3} \langle \xi \Delta \xi \rangle. \]  
(3.12)

By use of Equation (2.7) and the explicit form of the function \( \Delta \xi \) in (2.9), (2.24) to calculate \( \xi(r) \), we find

\[- \Delta \xi_a = 8\pi \mu \delta(r - r_a) - \frac{6\mu}{r_0^3}. \]

As a result we get

\[- \langle \xi \partial_\gamma \xi \rangle = \frac{8\pi \mu}{V} \int \delta(r) \xi_a(r) dV. \]

This integral diverges as \( r^{-1} \), which is caused by the divergence of the total energy of the particle. To calculate the integral we shall employ the procedure of renormalizing the mass, treating \( \delta(r)/r \) as \( -\delta'(r) \). Then, taking into account (3.12), we get eventually

\[- \langle \xi \Delta \xi \rangle = \frac{18\pi N}{r_0^2} \left( \frac{2\mu}{r_0} \right)^2; \]
\[ \langle (\partial_\alpha \xi)(\partial_\beta \xi) \rangle = \frac{6\pi N}{r_0^2} \left( \frac{2\mu}{r_0} \right)^2 \delta_{\alpha\beta}. \]
If we calculate the averages of corrections to the Einstein tensor caused by the fluctuations of the metric we shall find the corrections to the energy-momentum tensor of Robertson-Walker dust caused by the energy of the local gravitational fields:

\[ \delta T^{g}_{ij} = \frac{1}{8\pi} \langle \delta G_{ij} \rangle = \frac{g_{ij}}{8\pi a^2} (\xi \Delta \xi) = -\frac{gN}{4\pi a^2} \left( \frac{2\mu}{r_0} \right)^2 \frac{g_{ij}}{a^2}. \]

Thus, we shall obtain corrections to the energy density \( \delta \varepsilon_g \) and pressure \( \delta p_g \) of the form

\[ \delta \varepsilon_g = \delta p_g = -\frac{gN}{4\pi a^2} \left( \frac{2\mu}{r_0} \right)^2 \Rightarrow \delta \varepsilon_g + \delta p = 0. \]  

(3.13)

The allowance for the local fluctuations of the gravitational field in the Einstein equations is equivalent to adding to the Robertson-Walker dust a fluid with the equation of state \( \varepsilon = -p \).

Note that the ratio \( \delta \varepsilon_g / \varepsilon_0 \) increases in proportion to \( \eta^2 \), but remains small when the conditions of (3.8)-(3.10) are satisfied.

### 4 Derivation of the Kinetic Equation

Let \( \bar{f}(x, \bar{P}|x_1, x_2, \ldots) = \bar{f}(x, \bar{P}) \) be a macroscopic function of the distribution of massless particles; then the total number of particles recorded by the observers, associated with the velocity field \( u^i \) on hypersurface \( \Sigma \) orthogonal to this field (cf. Ignat’ev, 1983) is

\[ \bar{L} = \int \int \int_{\Sigma} \int d^4P u^i \bar{P}_i \delta(\bar{H}) \bar{f}(x, \bar{P}), \]  

(4.1)

where \( d^4 \bar{P} = d\bar{P}_1 d\bar{P}_2 d\bar{P}_3 d\sqrt{-g}; H(x, P) \) is the invariant Hamiltonian of massless particles.

We have used a bar to note the fact that all the values determined in a Robertson-Walker universe perturbed by local fluctuations. In accordance with our procedure we shall select as \( u^i \) a field of observers which is geodesic in a macroscopic Robertson-Walker universe; then \( u^i = \delta^4_i / \sqrt{g_{44}} \). Thus, taking into account (3.1), (3.2), (3.4), and (3.5), we get from (4.1)

\[ \bar{L} = \int \int \int_{\Sigma} \frac{d^3x}{\sqrt{g_{44}}} \int d^4\bar{P} \delta(\bar{H}) \bar{f}(x, \bar{P}). \]  

(4.2)

We shall take into account the fact that the Hamiltonian of massless particles is of the form

\[ \bar{H}(x, \bar{P}) = \frac{1}{2} g^{ik} \bar{P}_i \bar{P}_k = \frac{1}{2a^2} \left( \frac{\bar{P}_1^2}{1 - \xi} - \frac{\bar{P}_1^2}{1 + \xi} \right), \]  

(4.3)

where \( \bar{P}^2 = \bar{P}_1^2 + \bar{P}_2^2 + \bar{P}_3^2 \). Transform the formulae (4.2) and (4.3) to the new variables \( x^i, P_1^i \):

\[ P_\alpha = \bar{P}_\alpha; \quad P_4 = \bar{P}_4 \sqrt{\frac{1 + \xi}{1 - \xi}}. \]  

(4.4)

then

\[ \bar{H}(x, \bar{P}) = \frac{1}{1 + \xi} H(x, P), \]  

(4.5)

\[ ^{6}\text{In original article [1] because of negligence the error in a sign on density of energy } \delta \varepsilon_g \text{ is admitted. However, expression for } \delta T^{g}_{ij} \text{ is resulted the correct.} \]
where $H(x, P)$ is the Hamiltonian of massless particles in an unperturbed Robertson-Walker universe (3.6). Modifying the formula (4.2) taking into account the properties of $\delta(H)$, we get

$$\bar{L} = \int \int \int \frac{d^3x}{g^{44}} \int d^4P \; P_4 \delta(H) \bar{f}(x, \bar{P}),$$

which is precisely the same expression as the one in terms of an unperturbed metric (3.6). It should be noted that in (4.6) the fluctuation of the metric is present only in the function of the distribution of $f(x, P)$. Averaging (4.6) according to (3.3) we find that

$$\langle L \rangle \equiv L = \int \int \int \sigma \int d^4P \; u^i P_4 \delta(H) \bar{f}(x, \bar{P}),$$

(4.7)

where

$$f(x, P) \equiv \langle \bar{f}(x, \bar{P}) \rangle.$$ 

(4.8)

Thus, the average number of massless particles is evaluated with the aid of an ordinary formula, in a macroscopic Robertson-Walker universe, with respect to the distribution function averaged over the local fluctuations of the metric. This is the advantage of the frame of reference that we have selected. If we had selected, for instance, a macroscopically synchronous frame of reference, then instead of the simple relationship (4.8) we would have had a complex relationship which would include the derivatives of the distribution function $f(x, P)$ and the correlations of this function with the local fields. This is caused by the fact that a macroscopically local observer is subjected to the effect of the local gravitational fields, as a result of which the scales of clocks and rods on the micro – and macro – levels are different.

Now we shall proceed to average the microscopic collisionless kinetic equation (as in Ignatov, 1981)

$$\delta(H) \left[ \frac{\partial \bar{H}}{\partial \bar{P}_i} \frac{\partial \bar{f}}{\partial x^i} - \frac{\partial \bar{H}}{\partial x^i} \frac{\partial \bar{f}}{\partial \bar{P}_i} \right] = 0$$

(4.9)

If we carry out in (4.9) a preliminary transformation to the new variables $x^i, P_4$ according to (4.4), we get

$$\delta(H) \left\{ \frac{\partial H}{\partial P_4} \frac{\partial f}{\partial x^i} + \sqrt{1 + \xi - \xi} \frac{\partial f}{\partial P_4} \right\} = 0$$

(4.10)

In Equation (4.10) all the momentum variables $P_\alpha$ and $P_4$ are treated as independent. Expressing $P_4$ in terms of $P_\alpha$ with the aid of the mass shell equation $H(x, P) = 0$, we shall reduce (4.10) to a simpler form

$$P^\alpha \frac{\partial f}{\partial x^\alpha} + \sqrt{1 + \xi} \frac{P_\alpha \bar{f}}{P_4} - \frac{1}{2} \left( \partial_\alpha \ln \frac{1 + \xi}{1 - \xi} \right) \frac{P_4 \bar{f}}{P_4} = 0$$

(4.11)

($P_\alpha = -P_\alpha /a^2; P_4 = P_4 /a^2$). Our task is to obtain a kinetic equation for the macroscopic distribution function $f(x, P)$ to within an accuracy of terms quadratic with respect to the local fluctuations of the metric. Therefore, we shall expand (4.11) into a series with respect to the smallness of $\xi$ confining ourselves to the second-order terms:

$$P^\alpha \frac{\partial \bar{f}}{\partial x^\alpha} + \left( 1 + \xi + \frac{\xi^2}{2} \right) P_\alpha \frac{\partial \bar{f}}{\partial \eta} - (\partial_\alpha \xi) P_4 \frac{\partial \bar{f}}{\partial x^\alpha} = 0.$$ 

(4.12)
Taking into account the fact that correlations between the positions of massive particles are absent, we shall represent the macroscopic distribution function \( f(x, P) \) in the form
\[
\tilde{f}(x, P) = f(x, P) + g(x, P),
\]
where
\[
g(x, P|x_1, x_2, \ldots) = \sum_a g_a(x, P|x_a) \tag{4.13}
\]
and, in accordance with the definition (4.8),
\[
\langle g(x, P) \rangle = 0. \tag{4.14}
\]
If we substitute (4.13) into (4.12) and averaging the equation obtained, taking into account (4.14), we get
\[
P^\alpha \frac{\partial f}{\partial x^\alpha} + \left(1 + \frac{1}{2} \langle \xi^2 \rangle \right) P^4 \frac{\partial f}{\partial \eta} + P^4 \xi \frac{\partial g}{\partial \eta} - P_4 P^4 \frac{\partial}{\partial P^\alpha} \langle (\partial_\alpha \xi) g \rangle = 0. \tag{4.15}
\]
Averaging (4.12) over all the particles except the ath one, taking into account the results of Equation (4.15) in the equation obtained, and suppressing terms quadratic with respect to \( \xi \), we shall find an equation to determine the correlation function
\[
g_a(x, P|x_a) \equiv g_a, \tag{4.16}
\]
\[
P_i \frac{\partial g_a}{\partial x^i} = -\xi_a P^4 \frac{\partial f}{\partial \eta} + P_4 P^4 \frac{\partial f}{\partial P^\alpha} \partial_\alpha \xi_a. \tag{4.17}
\]
The integrals of Equation (4.16) are the integrals of geodesic lines in a Robertson-Walker universe: namely,
\[
P_\alpha = \text{Const}; \quad x^\alpha = x_0^\alpha + \pi^\alpha (\eta - \eta_0), \tag{4.18}
\]
where \( \pi^\alpha = P^\alpha / P^4 = -P_\alpha / P_4 = \text{Const} \). If we integrate (4.16) along the trajectories of (4.17), we find that
\[
g_a = -\int_{\eta_0}^{\eta} d\eta' \xi_a \left( \left| r'(\eta') - r_a \right| \right) \frac{\partial}{\partial \eta'} f[r'(\eta'), \eta', P_1] +
\]
\[
+P_4 \int_{\eta_0}^{\eta} d\eta' \partial_\alpha \xi_a \left( \left| r'(\eta') - r_a \right| \right) \frac{\partial}{\partial P^\alpha} f[r'(\eta), \eta', P_1] + \dot{g}_a (x^\alpha - \pi^\alpha (\eta - \eta_0), P_1). \tag{4.19}
\]
We shall expand in the integrands of (4.18) the function \( f[r'(\eta'), \eta', P_1] \)
\[
f[r'(\eta'), \eta', P_1] = f(r, \eta, P_1) - (\eta - \eta') \frac{P_i}{P_4} \frac{\partial f}{\partial x^i}.
\]
But according to (4.15) \( P^4 (\partial f / \partial x^i) = O(\xi^2) \); therefore, the distribution function to within an accuracy here required, can be factored outside the integral in (4.18). For the same reason the function \( g_a \) can be represented in the form of an arbitrary linear operator acting on \( f(x, P) \) selecting it so as to satisfy the condition of (4.14). Having noted this, we shall proceed to calculate the integrals in (4.18). A non-zero contribution to these integrals is only provided by the regions
\[
\left| r'(\eta') - r_a \right| \equiv \left| r - r_a + \pi(\eta - \eta') \right| \leqslant r_0. \tag{4.20}
\]
A non-zero contribution to the averages in Equation (4.15) is provided by the regions

\[ | \mathbf{r} - \mathbf{r}_a | \leq r_0. \] (4.21)

Therefore, we are interested in the values of correlation functions at the intersection of the regions (4.19) and (4.20).

To simplify the calculations we can put \( r_a = 0 \) and direct the velocity of the particle along the z-axis. The value of the correlation function obtained by integration is, generally speaking, time-depended; however, in the region (4.21)

\[ \eta - \eta_0 \geq 2r_0, \] (4.22)

\[ Q \equiv \int_{\eta_0}^{\eta} \xi d\eta' = 2\mu \left[ \ln \frac{r_0 + \sqrt{r_0^2 - \varrho^2}}{r_a + z_a} + z_a \left( \frac{9}{5r_0} - \frac{\varrho^2}{2r_0^3} \right) + \right. \]
\[ \left. \frac{z_a^3}{6r_0^3} - \left( \frac{9}{5r_0} - \frac{\varrho^2}{2r_0^3} \right) \sqrt{r_0^2 - \varrho^2} + \frac{1}{6r_0^3} (r_0^2 - \varrho^2)^{3/2} \right], \] (4.23)

\[ \Psi_a \equiv \int_{\eta_0}^{\eta} \partial_\alpha \xi d\eta' = \xi \pi^\alpha + 2\mu x^\beta (\delta^\alpha_\beta - \pi^\alpha \pi^\beta) \times \]
\[ \times \left[ -\frac{z_a}{\varrho^2 r_a} + \frac{\sqrt{r_0^2 - \varrho^2}}{\varrho^2 r_0} - \frac{1}{r_0^3} (z_a - \sqrt{r_0^2 - \varrho^2}) \right]. \]

It is necessary to generalize the expressions obtained, in the manner

\[ z_a \rightarrow (\vec{\pi}, \mathbf{r} - \mathbf{r}_a); \quad \varrho^2 = |\mathbf{r} - \mathbf{r}_a|^2 - (\vec{\pi}, \mathbf{r} - \mathbf{r}_a)^2; \]
\[ z_a \rightarrow |\mathbf{r} - \mathbf{r}_a|; \quad x^\beta_a \rightarrow x^\beta - x^\beta_a. \]

Since \( \langle \Psi_a \rangle = 0 \), the correlation function can be represented as

\[ g_a = -\frac{\partial f}{\partial \eta} (Q - \langle Q \rangle) + P_4 \frac{\partial f}{\partial P_\alpha} \Psi_\alpha. \] (4.24)

Calculating the averages in accordance with (4.23), we get

\[ \langle \xi (Q - \langle Q \rangle) \rangle = \langle \xi Q \rangle = \frac{3}{2} r_0 N \left( \frac{2\mu}{r_0} \right)^2 K; \]
\[ \langle \xi \Psi_\alpha \rangle = -\langle (\partial_\alpha \xi) Q \rangle = \pi^\alpha \xi^2; \] (4.25)
\[ \langle (\partial_\alpha \xi) \Psi_\beta \rangle = \frac{3}{2} N \left( \frac{2\mu}{r_0} \right)^2 (\delta_\alpha_\beta - \pi_\alpha \pi_\beta) \Lambda, \]

where

\[ K = \frac{17}{160} - \frac{7}{144} \ln 2 - \frac{29}{50} - \frac{31}{240} + \frac{13}{180} \left( \frac{19}{25} - \frac{5}{21} \right) \approx 0.11. \]

The last value of (4.24) is logarithmically divergent at the lower limit \( r \rightarrow 0 \), which is caused by the coincident illegitimacy of a linear approximation of the Einstein equations and a Born approximation. According to (3.8) at \( r \sim 2\mu \), the local fluctuations of the metric become
large, and at the same time the deflection angle of massless particle becomes large; therefore, integration in (4.24) is
\[
\Lambda = \ln \frac{r_0}{2\mu} - \ln 2 + \frac{1}{3}.
\]
Substitute (4.23), with the account of (4.24) into Equation (4.15)
\[
P^\alpha \frac{\partial f}{\partial x^\alpha} + \left(1 + \frac{3}{2} \langle \xi^2 \rangle \right) P^4 \frac{\partial f}{\partial \eta} - 3 \frac{r_0 N}{2 \mu} \left(\frac{2 \mu}{r_0}\right)^2 KP^4 \frac{\partial^2 f}{\partial \eta^2} - \langle \xi^2 \rangle P^4 P_\alpha \frac{\partial f}{\partial \eta} P_\alpha + \langle \xi^2 \rangle P_1 P^4 \frac{\partial f}{\partial \eta} - \frac{3 N}{2 r_0} \left(\frac{2 \mu}{r_0}\right)^2 \Lambda P^4 P_\alpha \frac{\partial f}{\partial \eta} P_\alpha \left[ P_4 \left(\delta_{\alpha\beta} - \frac{P_\alpha P_\beta}{P^2_4}\right) \frac{\partial f}{\partial P_\beta} \right] = 0.
\]

If we change the order of differentiation in the fourth and fifth term, respectively, suppressing the small term which is proportional to \(r_0/\eta\) and transposing the final term to the right-hand side, we eventually obtain kinetic equation for massless particles in the region (4.21)
\[
P^4 \left(1 + \frac{5}{2} \langle \xi^2 \rangle \right) \frac{\partial f}{\partial \eta} + P^\alpha \frac{\partial f}{\partial x^\alpha} = 3 \frac{N}{2 r_0} \left(\frac{2 \mu}{r_0}\right)^2 \Lambda P^4 P_\alpha \frac{\partial f}{\partial \eta} P_\alpha \left[ P_4 \left(\delta_{\alpha\beta} - \frac{P_\alpha P_\beta}{P^2_4}\right) \frac{\partial f}{\partial P_\beta} \right].
\]

The term on the right-hand side of Equation (4.25) is a collision 'integral', it describes the process of altering the momentum of massless particles by interaction with the local gravitational fields. The supplementary term on the left-hand side of (4.25), \(\frac{5}{2} \langle \xi^2 \rangle P_2 \frac{\partial f}{\partial \eta}\) describes the effective change in the velocity of a particle in local gravitational fields. Indeed, if we suppress the collision term in (4.25), then the equations of characteristics will be of the form
\[
P_\alpha = \text{Const}; \quad x^\alpha = \frac{\pi^\alpha (\eta - \eta_0)}{1 + \frac{5}{2} \langle \xi^2 \rangle} + x_0^\alpha.
\]
These equations describe the motion of a particle with the velocity
\[
v = c \left(1 - \frac{5}{2} \langle \xi^2 \rangle\right) < c.
\]

A local observer who measures the velocities of massless particles equal to the speed of light and assumes that medium velocity of these particles coincides with the speed of light will observe a local violation of the law of conservation of the particle number. Indeed, if the observer knows that a massless particle escapes from a certain point in space in his direction he has every reason to expect to register it within a completely definite time \(\Delta t = \Delta l/c\). If this fails to occur, the observer will record a local violation of the law of conservation of the particle number. However, since all massless particles travel at identical velocities (4.26), the observer can define this velocity as the speed of light. In agreement with this operation is the renormalization of the momentum in (4.25), in which the term on the left-hand side of the equation, quadratic with respect to \(\xi\), vanishes.

5 The Laws of Conservation and Generalized Kinetic Equations

Carrying out the foregoing renormalization of the momentum and changing over from a synchronous frame of reference, to an arbitrary one, we shall write Equation (4.25) in an invariant
form
\[ [H, f] = -\frac{\partial I_i}{\partial P_i}, \]  
(5.1)
where
\[ I_i = 3 \frac{N}{2r_0a} \left( \frac{2\mu}{r_0} \right)^2 \Lambda W_{ik} \frac{\partial f}{\partial P_k}, \]  
(5.2)
\[ W_{ik} = (u, P)[P_i P_k + g_{ik}(u, P)^2 - (u, P)(u_i P_k - u_k P_i)]. \]  
(5.3)
The coefficient in (5.2) can be given a clear physical meaning by taking account of the determination of the particle mass \( m = \mu a \) and the particle number \( N \). Let us introduce the medium mass density \( \rho_m = N m/4/3\pi(\alpha r_0)^3 \), which corresponds to massive particles. It should be noted that this value changes as \( \rho_m \sim t^{-4/3} \) at the linear stage of accretion as distinct from the total mass density \( \rho \sim t^{-2} \). Then (5.2) takes the form
\[ I_i = 8\pi\Lambda \rho_m W_{ik} \frac{\partial f}{\partial P_k}. \]  
(5.4)
The symmetric tensor \( W_{ik} \) has the following properties:
\[ W_{ik} P^k = 0; \quad W_{ik} u^k = 0. \]  
(5.5)
Let us integrate (5.1) over the space of momenta
\[ n^i = 0, \]
where
\[ n^i = \int P^i f \delta(H) d^4 P \]
is the density vector of the particle-number flux. Thus the local law of the conservation of particle number is also satisfied for a non-renormalized equation (4.25), which can be verified by integrating (4.25) over the space of momenta and the entire three-dimensional space. We shall multiply (5.1) by \( P^i \) and integrate over the space of momenta. If we integrate twice by parts on the right-hand side of the equation, we get, taking into account (5.4)
\[ T_{ik}^{\xi_k} = 32\Lambda \rho_m m(g^{ik} - u^iu^k)T_{kl}u^l, \]  
(5.6)
where
\[ T^{ik} = \int P^i P^k f \delta(H) d^4 P \]
is the energy-momentum tensor of massless particles. The energy of massless particles is conserved, which can be verified by comparing (5.5) with the time-like vector of conformal motion \( x^i = \delta^i_4 \):
\[ P^i = 0, \]
where \( P^i = T^{ik} \xi_k \). However, the three-dimensional momentum of the particle is not conserved, which is the consequence of the non-conservative nature of the system of massless particles, - a part of the momentum of massless particles is transferred to massive ones.
The kinetic equation (5.1) admits a natural generalization in the case where massive particles travel at arbitrary, but small velocities at which the rate of accretion is not altered significantly. Let $F(\eta, P')$ be a function of the distribution of massive particles with masses which are now arbitrary, so that

$$n_i^m = \int F(\eta, P') P'^i d^4 P'$$

is the density vector of the flux of massive particles. Then it is known that collision term of (5.2) ($P'_i = mu_i$) can be written as

$$I_i = 8\pi \int \Lambda m^2 F(P') W_{ik} \frac{\partial f}{\partial P'_k} d^4 P'.$$

To make sure of this it is sufficient to put $F(P') = m^{-1} \delta(P) \delta(H_m/2)$, i.e., to consider the distribution of fixed-mass particles at rest. The collision term of (5.6) takes account of the processes of transferring a momentum from massless particles to massive ones, but fails to take account of the reverse processes. To take into account the latter it is necessary to add to (5.6) a term which is antisymmetric respect to the transposition of particles

$$I_i = 8\pi \int d^4 P' \Lambda m^2 W_{ik} \left[ F(P') \frac{\partial f}{\partial P'_k} - f(P') \frac{\partial F(P')}{\partial P'_k} \right].$$

The kernel $W_{ik}$ of the collision integral thus obtained is the same as that of the Belaliev-Budker (1956) collision integral, if in the latter we let the momentum of one of the particles tend to infinity. The difference lies in the mutiplier $8\pi$ (instead of $2\pi$ in Belaliev and Budker (1956)), which is the consequence of the well-known effect, viz., the deflection angle of a photon in a gravitational field with the scope of Einstein’s theory is two times as great as its Newtonian value.

The collision integral of (5.7) as distinct from (5.1) now satisfies all the necessary laws of conservation. It can be used as the basis for showing that the total of energy and momentum within the system of ‘dust + massive particles + massless particles’ is conserved.

### 6 Isotropization of Homogeneous Distribution of Massless Particles

An exact solution of the kinetic equation (5.1) (as well as (4.25)) is an arbitrary isotropic distribution $f(P)$. If the initial distribution is anisotropic, however, it will be isotropized by gravitational interactions. Let the distribution of massless particles at the moment of time $\eta = \eta_1$ in a synchronous frame of reference be of the form

$$f(\eta_1, P_\alpha) = \sum_{l,m} f_{lm}(\eta_1, P) Y^l_m(\theta, \varphi),$$

where $\theta$ and $\varphi$ are the azimuthal and the polar angles in the momentum space, respectively. Representing the distribution of $f(\eta, P_\alpha)$ in a form analogous to (6.1), and on separating the variables, we obtain equations for the functions $f_{lm}(\eta, P)$,

$$\frac{\partial f_{lm}}{\partial \eta} = -(l+1)m\Lambda a f_{lm} \theta_m 8\pi,$$

$$f_{lm}(\eta, P) = f_{lm}(\eta_1, P) \exp \left[ -(l+1)8\pi \int_{\tau_1}^{t} \Lambda m \theta_m dt' \right].$$
In particular, for the dependence of \( m(t) \) and \( \rho_m(t) \) we have obtained, we shall find by integrating in (6.2):

\[
f_{lm}(\eta, P) = f_{lm}(\eta_1, P) \exp \left[ -l(l+1)24\pi \Lambda m \rho_m \left( 1 - \left( \frac{t}{t_1} \right)^{1/3} \right) \right],
\]

(6.3)

where \( m = m(t) \), \( \rho_m = \rho_m(t) \). At \( t \to \infty \) the expression in the exponent increases proportional to \( t^{1/3} \). Therefore, at \( t \to \infty \) the expression (6.1) retains only one harmonic with \( l = 0 \), i.e., the distribution is isotropized. The \( l \)-harmonic is presented by the angular scale \( \Delta \varphi = 2\pi(l+1) \). Therefore, at the given present-time values the particle masses \( m(t) \) and their medium density \( \rho_m(t) \) all harmonics with the angular dimension

\[
\Delta \varphi 2\pi < \sqrt{24\pi \Lambda m \rho_m t}
\]

(6.4)

vanish. At \( m \sim 10^{16} M_\odot \), \( \rho \sim 10^{30} \text{ g cm}^{-3} \) and \( t = 2 \times 10^{10} \text{ yr} \) we obtain from (6.4) \( \Delta \varphi < 10 \) angular minutes (Figures 1 and 2).

![Figure: 1. The dependence of the damping coefficient of the angular harmonic distribution \( \gamma = f_{lm}(t_0)/f_{lm}(t) \) from the angular scale \( \Delta \varphi = 2\pi(l+1) \). Curve 1: \( m = 10^{16} M_\odot \), \( \rho_m/\rho = 0.1 \). Curve 2: \( m = 10^{16} M_\odot \), \( \rho_m/\rho = 0.2 \). The angle \( \Delta \varphi \) is measured in angular minutes.](image)

Thus, any relict radiation must be highly homogeneous on scales less than 10 angular minutes. This effect can resolve the contradiction between the deductions of a standard adiabatic theory of galaxy formation (as in Zeldovich, 1983) and observation data which give evidence in favour of the absence of small-scale fluctuations of relict radiation.

Now let us assume that the accretion is completed at the moment of time \( t_1 \) i.e., \( m = \text{Const} \), and \( \rho_m \sim t^{-2} \). Then in place of (6.3) we obtain from (6.2),

\[
f_{lm}(\eta, P) = f_{lm}(\eta_1, P) \exp \left[ -l(l+1)8\pi \Lambda m \rho_m \frac{t^2}{t_1} \left( 1 - \frac{t_1}{t} \right) \right].
\]

(6.5)
In case, at $t \sim \infty$ the exponential index becomes large, $4\Lambda(l + l)m/3t_1$. This implies that harmonics with an angular scale of the order of tens of degrees can be damped, and isotropization takes place according to (6.5) at the earliest stages following the termination of accretion. Apparently, this cannot be the case. But it follows then that the masses of super aggregations should be either less than $10^{16} M_\odot$, or their formation should be completed at significantly later moments of time $t_1 > 10^{16}$ s.

Figure: 2. The dependence of the maximum resolution of $\Delta \varphi$ on the red shift $Z$: $m = 10^{15} M_\odot$; $\varrho_m / \varrho = 1$. The graph shows the values of $\Delta \varphi$ for which $\gamma = 1$.

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References

[1] Ignat’ev, Yu. G., Popov, A. A.: Astrophysics and Space Science, 163, 153-174, 1990.
[2] Belayev, S. G. and Budker, G. I.: 1956, Dokl Akad. Nauk USSR 107(6), 807.
[3] Bisnovatyi-Kogan, G. S. and Shykhman, I. G.: 1982, Zh. Experim. Teor. Fiz. 82(1),
[4] Ignat’ev, Yu. G.: 1978, in Gravitatsiya i teoriya otnositel’nosti (Gravitation and Relativity Theory), Kazan’ Univ. Press, Vol. 14, pp. 90-107.
[5] Ignat’ev, Yu. G.: 1981, Zh. Experim. Teor. Fiz. 81, 1(7), 3.
[6] Ignat’ev, Yu. G.: 1982, Izv. vyssh. ucheb. zaved., Fizika 4, 92.

[7] Ignat’ev, Yu. G.: 1983a, in Gravitatsiya i teoriya otnositel’nosti (Gravitation and Relativity), Kazan’ Univ. Press, Vol.20, pp. 50-109.

[8] Ignat’ev, Yu. G.: 1983b, Izv. vyssh. ucheb. zaved., Fizika 8, 15.

[9] Kandrup, H. E.: 1982, Astrophys. J. 259(1), 1.

[10] Landau, L. D.: 1937, Zh. Experim. Teor. Fiz. 7(2), 203.

[11] Landau, L. D. and Lifshitz, E. M.: 1972, Teoriya poly a (Classical Theory of Fields), Nauka, Moscow.

[12] Zakharov, A. V.: 1984, Zh. Experim. Teor. Fiz. 86(1), 3.

[13] Zeldovich, Ya.: 1983, in Itogi nauki i tekhniki (Results of Scientific and Technological Research), Astronomy 22, 4.