ON THE STRONG CONTINUITY OF GENERALISED SEMIGROPS

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Abstract. It is well known that weakly continuous semigroups defined over \( \mathbb{R}_+ \) are automatically strongly continuous. We extend this result to more generally defined semigroups, including multiparameter semigroups.

1. Introduction

By a well-known result, under certain basic conditions, semigroups over Banach spaces are automatically continuous wrt. the strong operator topology (sot). Engel und Nagel proved this in [2, Theorem 5.8] under the assumption of continuity wrt. the weak operator topology (wot). In that reference and here, semigroups are ordinarily defined over \( \mathbb{R}_+ \). Specifically, one considers operator-valued functions,

\[ T : \mathbb{R}_+ \to \mathcal{L}(E), \]

where \( E \) is a Banach space, and \( T \) satisfies \( T(0) = I \) and \( T(s + t) = T(s)T(t) \) for all \( s, t \in \mathbb{R}_+ \). In other words, semigroups are nothing other than morphisms between the monoids \( (\mathbb{R}_+, +, 0) \) and \( (\mathcal{L}(E), \circ, I) \).

Now, for our purposes, there is no particular reason to focus on semigroups over \( \mathbb{R}_+ \), also known as one-parameter semigroups. A natural abstraction is to define semigroups over topological monoids. In this note, we shall define a broad class of semigroups, including ones defined over \( \mathbb{R}^d_+ \) for \( d \geq 1 \), i.e. multiparameter semigroups, and to which we generalise the automatic continuity proof in [2].

Our generalisation may also be of interest to other fields. For example, multiparameter semigroups occur in the study of diffusion equations in space-time dynamics (see e.g. [10]) and the approximation of periodic functions in multiple variables (see e.g. [9]).

2. Definitions

Note that no assumptions about commutativity shall be made, and hence monoids and groups shall be expressed multiplicatively. We define generalised semigroups as follows.

Definition 2.1 A semigroup over a Banach space, \( E \), defined over a monoid, \( M \), shall mean any operator-valued function, \( T : M \to \mathcal{L}(E) \), satisfying \( T(1) = I \) and \( T(st) = T(s)T(t) \) for \( s, t \in M \).

We now define a large class of monoids to which the classical continuity result shall be generalised.

Definition 2.2 Let \( M \) be a locally compact Hausdorff topological monoid. We shall call \( M \) extendible, if there exists a locally compact Hausdorff topological group, \( G \), such that \( M \) is topologically and algebraically isomorphic to a closed subset of \( G \).

If \( M \) is extendible to \( G \) via the above definition, then one can assume without loss of generality that \( M \subseteq G \).

Definition 2.3 Let \( G \) be a locally compact Hausdorff group. We shall call a subset \( A \subseteq G \) positive in the identity, if for all neighbourhoods, \( U \subseteq G \), of the group identity, \( U \cap A \) has non-empty interior within \( G \).

Example 2.4 (The non-negative reals). Consider \( M := \mathbb{R}_+ \) viewed under addition. Since \( M \subseteq \mathbb{R} \) is closed, we have that \( M \) is an extendible locally compact Hausdorff monoid. For
any open neighbourhood, \( U \subseteq \mathbb{R} \), of the identity, there exists an \( \varepsilon > 0 \), such that \( (-\varepsilon, \varepsilon) \subseteq U \) and thus \( U \cap M \supseteq (0, \varepsilon) \neq \emptyset \). Hence \( M \) is positive in the identity.

**Example 2.5 (The \( p \)-adic integers).** Consider \( M := \mathbb{Z}_p \) with \( p \in \mathbb{P} \), viewed under addition with the topology generated by the \( p \)-adic norm. Since \( M \subseteq \mathbb{Q}_p \) is clopen, we have that \( M \) is an extendible locally compact Hausdorff monoid. Since \( M \) is clopen, it is clearly positive in the identity.

**Example 2.6 (Discrete cases).** Let \( G \) be a discrete group, and let \( M \subseteq G \) contain the identity and be closed under group multiplication. Clearly, \( M \) is a locally compact Hausdorff monoid, extendible to \( G \) and positive in the identity. For example one can take the free-group \( \mathbb{F}_2 \) with generators \( \{a, b\} \), and \( M \) to be the closure of \( \{1, a, b\} \) under multiplication.

**Example 2.7 (Non-discrete, non-commutative cases).** Let \( d \in \mathbb{N} \) with \( d > 1 \) and consider the space, \( X \), of \( \mathbb{R} \)-valued \( d \times d \) matrices. Topologised with any matrix norm (equivalently the strong or the weak operator topologies), this space is homeomorphic to \( \mathbb{R}^{d^2} \) and thus locally compact Hausdorff. Since the determinant map \( X \ni T \mapsto \det(T) \in \mathbb{R} \) is continuous, the subspace of invertible matrices \( \{T \in X \mid \det(T) \neq 0\} \) is open and thus a locally compact Hausdorff topological group. Now the subspace, \( G \), of upper triangular matrices with positive diagonal entries, is a closed subgroup and thus locally compact Hausdorff. Letting

\[
G_0 := \{T \in G \mid \det(T) = 1\},
\]

\[
G_+ := \{T \in G \mid \det(T) > 1\},
\]

\[
G_- := \{T \in G \mid \det(T) < 1\},
\]

it is easy to see that \( M := G_0 \cup G_+ \) is a topologically closed subspace containing the identity and is closed under multiplication. Moreover \( M \) is a proper monoid, since the inverses of the elements in \( G_+ \) are clearly in \( G \setminus M \). Consider now an open neighbourhood, \( U \subseteq G \), of the identity. Since inversion is continuous, \( U^{-1} \) is also an open neighbourhood of the identity. Since, as a locally compact Hausdorff space, \( G \) satisfies the Baire category theorem, and since \( G_+ \cup G_- \) is clearly dense (and open) in \( G \), and thus comeagre, we clearly have \( (U \cap U^{-1}) \cap (G_+ \cup G_-) \neq \emptyset \). So either \( U \cap G_+ \neq \emptyset \) or else \( U^{-1} \cap G_- \neq \emptyset \), from which it follows that \( U \cap G_+ = (U^{-1} \cap G_-)^{-1} \neq \emptyset \). Hence in each case \( U \cap M \) contains a non-empty open subset, viz. \( U \cap G_+ \). So \( M \) is extendible to \( G \) and positive in the identity.

Next, consider the subgroup, \( G_h \subseteq G \), consisting of matrices of the form \( T = I + E \) where \( E \) is a strictly upper triangular matrix with at most non-zero entries on the top row and right hand column. That is, \( G_h \) is the continuous Heisenberg group, \( H_{2d-3}(\mathbb{R}) \), of order \( 2d - 3 \). The elements of the Heisenberg group occur in the study of Kirillov’s orbit method (cf. [4]) and have important applications in physics (see e.g. [5]). Clearly, \( G_h \) is topologically closed within \( G \) and thus locally compact Hausdorff. Now consider the subspace,

\[
M_h := \{T \in G_h \mid \forall i, j \in \{1, 2, \ldots, d\} : T_{ij} \geq 0\},
\]

of matrices with only non-negative entries. This is clearly a topologically closed subspace of \( G_h \) containing the identity and closed under multiplication. Moreover, if \( S, T \in M_h \setminus \{I\} \) we clearly have

\[
ST = I + ((S - I) + (T - I) + (S - I)(T - I)) \in M_h \setminus \{I\},
\]

which implies that no non-trivial element in \( M_h \) has its inverse in \( M_h \), making \( M_h \) a proper monoid. Consider now an open neighbourhood, \( U \subseteq G_h \), of the identity. Since \( G_h \) is homeomorphic to \( \mathbb{R}^{2d-3} \), there exists some \( \varepsilon > 0 \), such that

\[
U = \{T \in G_h \mid \forall (i, j) \in \mathcal{I} : T_{ij} \in (-\varepsilon, \varepsilon)\},
\]

where \( \mathcal{I} := \{(1, 2), (1, 3), \ldots, (1, d), (2, d), \ldots, (d - 1, d)\} \). Hence

\[
U \cap M_h \supseteq \{T \in G_h \mid \forall (i, j) \in \mathcal{I} : T_{ij} \in (0, \varepsilon)\} =: V,
\]

where \( V \) is clearly a non-empty open subset of \( G_h \), since the \( 2d - 3 \) entries in the matrices can be freely and independently chosen. Thus \( M_h \) is extendible to \( G_h \) and positive in the identity.

Finally, we may consider the subgroup, \( G_u := U_{\mathbb{T}}(d) \), of upper triangular matrices over \( \mathbb{R} \) with unit diagonal. The elements of \( U_{\mathbb{T}}(d) \) have important applications in image analysis (see e.g. [5] and [6, §5.5.2]) and representations of the group have been studied in [8, Chapter 6].
Proof. First note that the principle of uniform boundedness applied twice to the
space coincide (cf. [1, Theorem 5.98]), it follows that the convex hull, $co(D)$, is strongly dense in $E$.

Now, to prove the sot-continuity of $T$, we need to show that
\[ t \in M \mapsto T(t)x \in E \tag{3.2} \]
is strongly continuous for all $x \in E$. Since $M$ is locally compact and $T$ is norm-bounded on compact subsets of $M$, the set of $x \in E$ such that (3.2) is strongly continuous, is itself a strongly closed convex subset of $E$. So, since $co(D)$ is strongly dense in $E$, it suffices to prove the strong continuity of (3.2) for each $x \in D$. 

3. MAIN RESULT

We can now state and prove the generalisation. Our argumentation builds on [2, Theorem 5.8].

**Theorem 3.1** Let $M$ be a locally compact Hausdorff monoid and $E$ a Banach space. Assume that $M$ is extendible to a locally compact Hausdorff group, $G$, and that $M$ is positive in the identity. Then all wot-continuous semigroups, $T : M \to \Sigma(E)$, are automatically sot-continuous.

**Proof.** First note that the principle of uniform boundedness applied twice to the wot-continuous function, $T$, ensures that $T$ is norm-bounded on all compact subsets of $M$. Fix now a left-invariant Haar measure, $\lambda$, on $G$ and set
\[ S := \{ F \subseteq G \mid F \text{ a compact neighbourhood of the identity} \}. \]

Consider arbitrary $F \subseteq S$ and $x \in E$. By the closure of $M$ in $G$ as well as positivity in the identity, $M \cap F$ is compact and contains a non-empty open subset of $G$. It follows that $0 < \lambda(M \cap F) < \infty$. The wot-continuity of $T$, the compactness (and thus measurability) of $M \cap F$, and the norm-boundedness of $T$ on compact subsets ensure that
\[ \langle x_F, \varphi \rangle := \frac{1}{\lambda(M \cap F)} \int_{t \in M \cap F} \langle T(t)x, \varphi \rangle \, dt, \quad \text{for } \varphi \in E' \tag{3.1} \]
describes a well-defined element $x_F \in E''$. Exactly as in [2, Theorem 5.8], one may now argue by the wot-continuity of $T$ and compactness of $M \cap F$ that in fact $x_F \in E$ for each $x \in E$ and $F \in S$. Moreover, since $M$ is locally compact, one can readily see that each $x \in E$ can be weakly approximated by the net, $(x_F)_{F \in S}$, ordered by inverse inclusion. So
\[ D := \{ x_F \mid x \in E, F \in S \} \]
is weakly dense in $E$. Since the weak and strong closures of any convex subset in a Banach space coincide (cf. [1, Theorem 5.98]), it follows that the convex hull, $co(D)$, is strongly dense in $E$. 

Now, to prove the sot-continuity of $T$, we need to show that
\[ t \in M \mapsto T(t)x \in E \tag{3.2} \]
is strongly continuous for all $x \in E$. Since $M$ is locally compact and $T$ is norm-bounded on compact subsets of $M$, the set of $x \in E$ such that (3.2) is strongly continuous, is itself a strongly closed convex subset of $E$. So, since $co(D)$ is strongly dense in $E$, it suffices to prove the strong continuity of (3.2) for each $x \in D$. 

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To this end, fix arbitrary $x \in E$, $F \in S$ and $t \in M$. We need to show that $T(t')x_F \rightarrow T(t)x_F$ strongly for $t' \in M$ as $t' \rightarrow t$.

First recall, that by basic harmonic analysis, the canonical left-shift, $L : G \rightarrow \mathcal{L}(L^1(G))$, defined via $(L_t f)(s) = f(t^{-1}s)$ for $s, t \in G$ and $f \in L^1(G)$, is an sot-continuous morphism (cf. [7, Proposition 3.5.6 ($\lambda_1 - \lambda_4$)]. Now, by compactness, $f := 1_{M \cap F} \in L^1(G)$ and it is easy to see that $\|L_t f - L_{t'} f\|_1 = \lambda(t'(M \cap F) \Delta t(M \cap F))$ for $t' \in M$. The sot-continuity of $L$ thus yields

$$\lambda(t'(M \cap F) \Delta t(M \cap F)) \rightarrow 0 \quad (3.3)$$

for $t' \in M$ as $t' \rightarrow t$.

Fix now a compact neighbourhood, $K \subseteq G$, of $t$. For $t' \in M \cap K$ and $\varphi \in E'$ one obtains

$$|\langle T(t')x_F - T(t)x_F, \varphi \rangle| = \left| \langle x_F, (T(t')^* \varphi) - \langle x_F, T(t)^* \varphi \rangle \right|$$

$$= \frac{1}{\lambda(M \cap F)} \left| \int_{s \in M \cap F} \langle T(s)x, T(t')^* \varphi \rangle \, ds - \int_{s \in M \cap F} \langle T(s)x, T(t)^* \varphi \rangle \, ds \right|$$

by construction of $x_F$ in (3.1)

$$= \frac{1}{\lambda(M \cap F)} \left| \int_{s \in M \cap F} \langle T(t's)x, \varphi \rangle \, ds - \int_{s \in M \cap F} \langle T(ts)x, \varphi \rangle \, ds \right|$$

since $T$ is a semigroup

$$\leq \frac{1}{\lambda(M \cap F)} \left| \int_{s \in M \cap F} \langle T(s)x, \varphi \rangle \, ds \right|$$

by left-invariance

$$\leq \frac{1}{\lambda(M \cap F)} \sup_{s \in (M \cap K)(M \cap F)} \|T(s)\| \cdot \|x\| \cdot \|\varphi\| \cdot \lambda(t'(M \cap F) \Delta t(M \cap F))$$

since $t, t' \in M \cap K$.

Since $K' := (M \cap K)(M \cap F)$ is compact, and $T$ is uniformly bounded on compact sets (see above), it holds that $C := \sup_{s \in K'} \|T(s)\| < \infty$. The above calculation thus yields

$$\|T(t')x_F - T(t)x_F\| = \sup_{\varphi \in E', \|\varphi\| \leq 1} |\langle T(s)x_F - T(t)x_F, \varphi \rangle| \leq \frac{1}{\lambda(M \cap F)} \cdot C \cdot \|x\| \cdot \lambda(t'(M \cap F) \Delta t(M \cap F)) \quad (3.4)$$

for all $t' \in M$ sufficiently close to $t$.

By (3.3), the right-hand side of (3.4) converges to 0 and hence $T(t')x_F \rightarrow T(t)x_F$ strongly as $t' \rightarrow t$. This completes the proof.

Theorem 3.1 applied to Corollary 2.9 immediately yields:

**Corollary 3.2** Let $d \in \mathbb{N}$ and let $E$ be a Banach space. Then all wot-continuous semigroups, $T : \mathbb{R}_+^d \rightarrow \mathcal{L}(E)$, are automatically sot-continuous.

**Remark 3.3** In the proof of Theorem 3.1, weak continuity only played a role in obtaining the boundedness of $T$ on compact sets, as well as the well-definedness of the elements in $D$. Now, another proof of the classical result exists under weaker conditions, viz. weak measurability, provided the semigroups are almost separably valued (cf. [3, Theorem 9.3.1 and Theorem 10.2.1–3]). It remains open, whether the approach in [3] can be adapted to our context, to yield the result under weaker assumptions.

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**References**

[1] Aliprantis, C. D., and Border, K. C. *Infinite Dimensional Analysis, a Hitchhiker’s Guide*, 3rd ed. Springer-Verlag, 2005.
[2] Engel, K.-J., and Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics. Springer-Verlag, 1999.

[3] Hille, E., and Phillips, R. S. *Functional analysis and semi-groups*, vol. 31. American Mathematical Society, 2008.

[4] Kirillov, A. A. Unitary representations of nilpotent Lie groups. *Russian Mathematical Surveys* 17, 4 (1962), 53–104.

[5] Kirillov, A. A. Two more variations on the triangular theme. In *The Orbit Method in Geometry and Physics*. Birkhäuser Boston, 2003, pp. 243–258.

[6] Penne, X., and Lorenzi, M. Beyond Riemannian geometry: The affine connection setting for transformation groups. In *Riemannian Geometric Statistics in Medical Image Analysis*, X. Pennec, S. Sommer, and T. Fletcher, Eds. Academic Press, 2020, pp. 169–229.

[7] Reiter, H., and Stegeman, J. D. *Classical Harmonic Analysis and Locally Compact Groups*, 2nd ed. Oxford University Press, 2000.

[8] Samoilenko, Y. S. *Spectral Theory of Families of Self-Adjoint Operators*. Springer Netherlands, Dordrecht, 1991, pp. 124–144.

[9] Terehin, A. P. A multiparameter semigroup of operators, mixed moduli and approximation. *Mathematics of the USSR-Izvestiya* 9, 4 (1975), 887–910.

[10] Zelik, S. Multiparameter semigroups and attractors of reaction-diffusion equations in $\mathbb{R}^n$. *Transactions of the Moscow Mathematical Society* 65 (2004), 105–160.

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