ON THE COLLAPSING RATE OF THE KÄHLER-RICCI FLOW
WITH FINITE-TIME SINGULARITY

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Abstract. We study the collapsing behavior of the Kähler-Ricci flow on a
compact Kähler manifold $X$ admitting a holomorphic submersion $X \to \Sigma$
where $\Sigma$ is a Kähler manifold with dim $\Sigma < \dim X$. We give cohomological
and curvature conditions under which the fibers $\pi^{-1}(z), z \in \Sigma$ collapse at the
optimal rate $\text{diam}_t(\pi^{-1}(z)) \sim (T - t)^{1/2}$.

1. Introduction

Let $X$ be a compact connected Kähler manifold with $\dim X = n$. Suppose $(\Sigma, \omega_\Sigma)$
is a Kähler manifold with $\dim \Sigma = n - r < n$ and $X \to \Sigma$ is a surjective holomorphic submersion. For each $z \in \Sigma$, we call $\pi^{-1}(z)$ a fiber based at $z$, which is a
complex submanifold of $X$ with $\dim \pi^{-1}(z) = r$. By the classical Ehresmann’s fibration theorem [Eh], $X$ is a smooth fiber bundle over $\Sigma$ and in particular $\pi^{-1}(z)$'s are diffeomorphic. Nonetheless, the induced complex structure on each $\pi^{-1}(z)$ may vary. In the case where all fibers $\pi^{-1}(z)$'s are biholomorphic, a classical theorem due to Fischer-Grauert [FG] asserts that $X$ is a holomorphic fiber bundle over $\Sigma$.

In this short note, we study the Kähler-Ricci flow on $X$ defined by
\begin{equation}
\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t), \quad \omega|_{t=0} = \omega_0.
\end{equation}
The Kähler class $[\omega_t]$ at time $t$ is precisely given by $[\omega_0] - t c_1(X)$. The maximal existence time $T$ of (1.1) is uniquely determined by a result of Tian-Zhang in [TZ], namely
\[ T = \sup \{ t : [\omega_0] - t c_1(X) > 0 \}. \]
In particular, if $c_1(X) : [\pi^{-1}(z)] > 0$ for some $z \in \Sigma$, we must have $T < \infty$. In this article, we only focus on the case where (1.1) encounters finite-time singularity at $T < \infty$. We will study the collapsing behavior of the Kähler-Ricci flow (1.1) starting with an initial Kähler class $[\omega_0]$ such that as $t \to T$ the Kähler class $[\omega_t]$ limits to:
\begin{equation}
[\omega_0] - T c_1(X) = [\pi^* \omega_\Sigma].
\end{equation}
We will prove the following result:

Theorem 1.1. Let $X \to \Sigma$ be a surjective holomorphic submersion. Suppose the
Kähler-Ricci flow (1.1) on $X$ encounters finite-time singularity at $T < \infty$ and the initial Kähler class $[\omega_0]$ satisfies (1.2), we have
\[ \text{Ric}(\omega_t) \leq B \omega_0 \quad \text{for some uniform constant } B > 0 \]
\[ \Rightarrow \quad C^{-1} (T - t)^{1/2} \leq \text{diam}_t(\pi^{-1}(z)) \leq C (T - t)^{1/2} \quad \text{for any } t \in [0, T), z \in \Sigma, \]
where $C$ is a uniform constant depending only on $n, r, \omega_0, \omega_\Sigma$ and $B$. 

In the case where $X$ is a $\mathbb{P}^r$-bundle over $\Sigma$ and $X \xrightarrow{\pi} \Sigma$ is the bundle map, the fiber-collapsing behavior of the Kähler-Ricci flow was studied in [SW1, SSW] and by the author in [F1]. In [SW1], Song-Weinkove studied the case where $r = 1$ and $(\Sigma, \omega_\Sigma) = (\mathbb{P}^{n-1}, \omega_{FS})$ and the initial metric $\omega_0$ on $X$ is constructed by Calabi’s Ansatz (a cohomogeneity-1 symmetry). In this case, the Kähler class $[\omega_0]$ can be expressed as $-a_0[\Sigma_0] + b_0[\Sigma_\infty]$ with $0 < a_0 < b_0$, where $[\Sigma_0]$ and $[\Sigma_\infty]$ denote the Poincaré duals of the zero and infinity sections respectively, and (1.2) can be achieved by a suitable choice of $a_0$ and $b_0$. It was shown that under the condition (1.2) the Kähler-Ricci flow collapses the $\mathbb{P}^1$-fiber. In [F1], the author extended this result to allow $\Sigma$ to be any Kähler-Einstein manifold and proved that the singularity must be of Type I modelled on $\mathbb{C}^{n-1} \times \mathbb{P}^1$ when the initial metric has Calabi symmetry. In [SSW], Song-Székelyhidi-Weinkove generalized this fiber-collapsing result to $\mathbb{P}^r$-bundles over a smooth projective variety $\Sigma$ and removed the symmetry assumption.

In [SSW], it was proved that the diameters of fibers decay at the rate of at most $(T - t)^{1/3}$ which is sufficient in their proof of Gromov-Hausdorff convergence towards the base manifold. It is more desirable for the diameters of fibers decay at a rate $\sim (T - t)^{1/2}$ as far as rescaling analysis (as in [F1]) is concerned. Rescaling analysis has been fundamental in the study of singularity formations of the Ricci flow. The Cheeger-Gromov limit obtained from a suitably rescaled and dilated sequence of the Ricci flow encodes crucial geometric information of the singularity region near the singular time. In case of Type I singularity, the rescaling factor of the metric tensor is uniformly equivalent to $(T - t)^{-1}$. If a fiber $\pi^{-1}(z)$ shrinks at a rate such that $C^{-1}(T - t)^{1/2} \leq \text{diam}_{\omega_t}(\pi^{-1}(z)) \leq C(T - t)^{1/2}$, then after the $(T - t)^{-1}$-rescaling the diameter of the fiber $F$ remains bounded away from 0 and $\infty$. It is conjectured in [SSW, SW2] that the diameter decay can be improved to $O((T - t)^{1/2})$. In this article, we give a partially affirmative answer to this conjecture under the assumption that the Ricci curvature stays bounded along the flow.

It is interesting to note that in the very special case where $\Sigma$ is a point, the only “fiber” is simply the whole manifold $X$ and (1.2) says that $\omega_0$ is in the canonical class, i.e. $[\omega_0] = Tc_1(X)$. It is well-known by an unpublished result of Perelman (see [ScT]) that the diameter stays bounded along the normalized Kähler-Ricci flow, and equivalently, decays at a rate of at most $(T - t)^{1/2}$ along the unnormalized flow (1.1). Note that the Ricci curvature bound is not needed in Perelman’s work. The proof involves the use of the monotonicity of the $W$-functional introduced in [P].

We mainly focus on finite-time singularity in this article. The collapsing behavior of the Kähler-Ricci flow with long-time existence was studied by, for instance, Song-Tian in [ST] when $X$ is a minimal elliptic surface. When the base Riemann surface has genus greater than 1, it was proved in [ST] that the nonsingular Calabi-Yau fibers collapse at the rate $\sim e^{-t}$ under the normalized Kähler-Ricci flow $\partial_t \tilde{\omega}_t = -\text{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t$. 

2. Some estimates

In this section, we give the estimates necessary to establish Theorem 1.1. The approach to establish the main result is motivated by the techniques adopted in [ST, To, Z1] etc.

We rewrite the Kähler-Ricci flow (1.1) as a parabolic complex Monge-Ampère equation in the same way as in [ST, SW1, SSW] etc. We define a family of reference metrics \( \hat{\omega}_t \) in the same Kähler class as \( \omega_t \) by:

\[
\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t\pi^*\omega_\Sigma).
\]

One can argue that \( [\omega_t] = [\hat{\omega}_t] \) by observing that \( [\omega_t] = [\omega_0] - tc_1(X) \) which is a linear path connecting \( [\omega_0] \) and \( [\pi^*\omega_\Sigma] \) as \( t \) goes from 0 to \( T \) by our assumption (1.2). By the \( \partial\bar{\partial} \)-lemma, there exists a family of smooth functions \( \phi_t \) such that \( \omega_t = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi_t \). Let \( \Omega \) be a volume form on \( X \) such that

\[
\sqrt{-1}\partial\bar{\partial}\log \Omega = \frac{\partial}{\partial t} \hat{\omega}_t = \frac{1}{T}(\pi^*\omega_\Sigma - \omega_0).
\]

Then it is easy to check that the Kähler-Ricci flow (1.1) is equivalent to the following complex Monge-Ampère equation:

\[
\frac{\partial \phi_t}{\partial t} = \log \left( \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi_t)^n}{(T-t)^r\Omega} \right).
\]

In all the estimates below, we will denote \( C > 0 \) to be a uniform constant which depends only on \( n, r, \omega_0, \omega_\Sigma \) and \( B \), and may change from line to line. We first prove the following:

**Lemma 2.1.** Given that \( \text{Ric}(\omega_t) \leq B\omega_0 \) for some uniform constant \( B > 0 \), then along (2.3) there exists a uniform constant \( C = C(n, r, \omega_0, \omega_\Sigma, B) \) such that

\[
\frac{\partial \phi_t}{\partial t} \leq \frac{1}{[\omega_0]^n} \int_X \log \frac{\omega_0^n}{(T-t)^r\Omega} \omega_0^n + C.
\]

**Proof.** We let \( G(x, y) : X \times X - \{ x = y \} \rightarrow \mathbb{R} \) be a nonnegative Green’s function with respect to \( \omega_0 \). Then we have for any \( x \in X \),

\[
\frac{\partial \phi_t}{\partial t}(x) - \frac{1}{[\omega_0]^n} \int_X \frac{\partial \phi_t}{\partial t} \omega_0^n = \frac{1}{[\omega_0]^n} \int_X G(x, \cdot) \Delta_{\omega_0} \left( \frac{\partial \phi_t}{\partial t} \right) \omega_0^n
\]

\[
= -\frac{1}{[\omega_0]^n} \int_X G(x, \cdot) \Delta_{\omega_0} \log \frac{\omega_0^n}{(T-t)^r\Omega} \omega_0^n
\]

\[
= \int_X G(x, \cdot) \text{Tr}_{\omega_0} \left( \text{Ric}(\omega_t) + \frac{\partial \omega_t}{\partial t} \right) \omega_0^n
\]

\[
= \int_X G(x, \cdot) \left( \text{Tr}_{\omega_0} \text{Ric}(\omega_t) + \frac{1}{T} \text{Tr}_{\omega_0} (\pi^*\omega_\Sigma - \omega_0) \right) \omega_0^n,
\]

where we have used (2.2) in the third and fourth steps. By our assumption \( \text{Ric}(\omega_t) \leq B\omega_0 \), it is easy to see that

\[
\text{Tr}_{\omega_0} \text{Ric}(\omega_t) + \frac{1}{T} \text{Tr}_{\omega_0} (\pi^*\omega_\Sigma - \omega_0) \leq C.
\]

for some uniform constant \( C = C(n, r, \omega_0, \omega_\Sigma, B) > 0 \).
Therefore, we have for $x \in X$,
\[
\frac{\partial \varphi_t}{\partial t}(x) - \frac{1}{[\omega_0]_n} \int_X \frac{\partial \varphi_t}{\partial t} \omega_0^n \leq C \int_X G(x, y) \omega_0^n(y).
\]

Note that $\int_X G(x, y) \omega_0^n(y) \equiv \text{constant}$. It completes the proof of the lemma. \(\square\)

Next we derive a uniform upper bound for $\frac{\partial \varphi_t}{\partial t}$. By the previous lemma, it suffices to bound the total volume of $(X, \omega_t)$, which depends only on $[\omega_t]$. Let us consider the reference metric $\hat{\omega} = \frac{1}{t}(T-t)\omega_0 + t\pi^*\omega_\Sigma)$. We have
\[
\hat{\omega}_t^n = \frac{1}{T^n} \sum_{k=0}^n \binom{n}{k} (T-t)^k t^{n-k} \omega_0^k \wedge (\pi^*\omega_\Sigma)^{n-k}.
\]
Since $\Sigma$ has complex dimension $n - r$, we have $(\pi^*\omega_\Sigma)^{n-k} = 0$ for any $k < r$. Therefore, we have
\[
\hat{\omega}_t^n = \frac{1}{T^n} \sum_{k=r}^n \binom{n}{k} (T-t)^k t^{n-r} \omega_0^k \wedge (\pi^*\omega_\Sigma)^{n-r} + \cdots + (T-t)^{n-r} \omega_0^n.
\]
It is not difficult to see that there exists a uniform constant $C = C(n, r, \omega_0, \omega_\Sigma) > 0$ such that for any $t \in [0, T)$, we have
\[
\frac{1}{C} (T-t)^r \omega_0^n \leq \hat{\omega}_t^n \leq C(T-t)^r \omega_0^n.
\]
By Jensen’s Inequality, we have
\[
\int_X \log \frac{\omega_t^n}{(T-t)^r \Omega [\omega_0]^n} \leq \log \left( \int_X \frac{\omega_t^n}{(T-t)^r \Omega [\omega_0]^n} \frac{\omega_0^n}{[\omega_0]^n} \right)
\leq \log \left( \frac{C}{[\omega_0]^n} \int_X (T-t)^r \right)
\leq \log C \quad \text{using (2.6)}.
\]
Combining with Lemma 2.1, we have established the following:

**Lemma 2.2.** There exists a uniform constant $C = C(n, r, \omega_0, \omega_\Sigma, B) > 0$ such that for any $t \in [0, T)$, we have
\[
\frac{\hat{\omega}_t^n}{(T-t)^r \Omega} = e^{\frac{\partial \varphi_t}{\partial t}} \leq C.
\]

Lemma 2.2 gives a pointwise bound for the volume form, which will be used in the next lemma to show the Kähler potential $\varphi_t$ is decay at a rate of $O(T-t)$ after a suitable normalization.

For each $z \in \Sigma$ and $t \in [0, T)$, we denote $\omega_{t, z}$ to be the restriction of $\omega_t$ on the fiber $\pi^{-1}(z)$. For each $t \in [0, T)$, we define a function $\Phi_t : \Sigma \rightarrow \mathbb{R}$ by
\[
\Phi_t(z) = \frac{1}{\text{Vol}_{\omega_{t, z}}(\pi^{-1}(z))} \int_{\pi^{-1}(z)} \varphi_t \omega_{t, z}^n,
\]
which is the average value of $\varphi_t$ over each fiber $\pi^{-1}(z)$. The pull-back $\pi^*\Phi_t$ is then a function defined on $X$. For simplicity, we also denote $\pi^*\Phi_t$ by $\Phi_t$. 

Lemma 2.3. There exists a uniform constant $C = C(n, r, \omega, \omega, B)$ such that for any $t \in [0, T)$, we have

\begin{equation}
\frac{|\hat{\varphi}_t - \Phi_t|}{T - t} \leq C
\end{equation}

Proof. Denote $\hat{\varphi}_t = \frac{\varphi_t - \Phi_t}{T - t}$. For each $z \in \Sigma$, we have $\dot{\omega}_t, z = \frac{T - t}{T} \omega_0, z$, and so

$$\omega_t, z = \frac{T - t}{T} \omega_0, z + \sqrt{-1} d \bar{\partial} \varphi_1 |_{\pi^{-1}(z)}.$$ 

Since $\Phi_t$ depends only on $z \in \Sigma$, we have $\sqrt{-1} d \bar{\partial} \Phi_1 |_{\pi^{-1}(z)} = 0$. By rearranging, we have

\begin{equation}
\frac{1}{T - t} \omega_t, z = \frac{1}{T} \omega_0, z + \sqrt{-1} d \bar{\partial} \hat{\varphi}_1 |_{\pi^{-1}(z)}.
\end{equation}

Regard (2.9) to be a metric equation on the manifold $\pi^{-1}(z)$, then we have

\begin{equation}
\left( \frac{1}{T} \omega_0, z + \sqrt{-1} d \bar{\partial} \hat{\varphi}_1 |_{\pi^{-1}(z)} \right)^r = \left( \frac{1}{T - t} \omega_t, z \right)^r.
\end{equation}

Using Lemma 2.2, we can see

\begin{equation}
\frac{\omega_t, z}{\omega_0, z} = \frac{\omega_t, z \wedge (\pi^* \omega)}{\omega_0, z \wedge (\pi^* \omega)} \\
\leq \frac{\omega_t, z \wedge (\pi^* \omega)}{\omega_t \wedge (\pi^* \omega)} \cdot \frac{\omega_0, z \wedge (\pi^* \omega)}{\omega_0, z \wedge (\pi^* \omega)} \\
\leq C(Tr_{\omega^*} \pi^* \omega)^r \cdot (T - t)^r.
\end{equation}

It is well-known that $Tr_{\omega^*} \pi^* \omega \leq C$ by the cohomological condition (1.2) and a maximum principle argument (see e.g. [SW1, SSW]). Combining this with (2.11), we see than (2.10) can be restated as

\begin{equation}
\left( \frac{1}{T} \omega_0, z + \sqrt{-1} d \bar{\partial} \hat{\varphi}_1 |_{\pi^{-1}(z)} \right)^r = F_z(\xi, t) \left( \frac{1}{T} \omega_0, z \right)^r
\end{equation}

where $F_z(\xi, t) : \pi^{-1}(z) \times [0, T) \to \mathbb{R}_{>0}$ is uniformly bounded.

Since $\int_{\pi^{-1}(z)} \hat{\varphi}_1 \omega_0, z = 0$, by applying Yau’s $L^\infty$-estimate (see [Y] on (2.9), we then have

\begin{equation}
\sup_{\pi^{-1}(z) \times [0, T)} |\hat{\varphi}_t| \leq C_z,
\end{equation}

where $C_z$ depends on $n, r, \omega_0, \omega, B, sup_{\pi^{-1}(z) \times [0, T)} F_z, Vol_{\omega_0, z} (\pi^{-1}(z))$, the Sobolev and Poincaré constants of $\pi^{-1}(z)$ with respect to metric $\omega_0, z$, all of which can be bounded uniformly independent of $z$. It completes the proof of the lemma. $\square$

Remark 2.4. In our setting, the uniform boundedness of Sobolev and Poincaré constants of $\pi^{-1}(z), \omega_0, z$ follow from the compactness of $\Sigma$ and the absence of singular fibers. It is also possible to give such uniform bounds by noting that $\pi^{-1}(z)$’s are minimal submanifolds of $X$ and hence they can be embedded into some Euclidean space $\mathbb{R}^N$ with bounded mean curvature. Combining the classical results in [MS, Ch1, LY] one can obtain uniform bounds on the Sobolev and Poincaré constants. A detail discussion in this regard can be found in [T] which also dealt with singular fibers.
3. Proof of Theorem \[1.1\]

Proof of Theorem \[1.1\]. We apply maximum principle to the following quantity

\[ Q := \log((T - t)\text{Tr}_{\omega_0}) - \frac{A}{T - t}(\varphi_t - \Phi_t), \]

where \( A \) is a positive constant to be chosen. Denote \( \Box_{\omega_t} = \partial_t - \Delta_{\omega_t} \), we have

\[ \Box_{\omega_t} \log((T - t)\text{Tr}_{\omega_t}) \leq -\frac{1}{T - t} + \frac{1}{\text{Tr}_{\omega_t}\omega_0}g^{ij}g^{kl}\text{Rm}(g_0)_{ijkl}, \]

where \( \tilde{C} \) depends only on the curvature of \( g_0 \).

\[ \Box_{\omega_t} \frac{A}{T - t}(\varphi_t - \Phi_t) = \frac{A}{T - t}\left(\frac{\partial \varphi_t}{\partial t} - \frac{\partial \Phi_t}{\partial t}\right) - \frac{A}{(T - t)^2}(\varphi_t - \Phi_t) \]

\[ - \frac{A}{T - t}(\Delta_{\omega_t} \varphi_t - \Delta_{\omega_t} \Phi_t) \]

\[ \geq \frac{A}{T - t}\left(\log \frac{\omega_t^n}{(T - t)\Omega} - \int_{\pi^{-1}(z)} \frac{\partial \varphi_t}{\partial t} \omega_{t,0,z}^r\right) - CA \frac{A}{T - t} \]

\[ - \frac{A}{T - t}(n - \text{Tr}_{\omega_t} \hat{\omega}_t - \Delta_{\omega_t} \Phi_t) \]

Note that

\[ \text{Tr}_{\omega_t} \hat{\omega}_t \geq n \left(\frac{\omega_t^n}{\omega_t^r}\right)^{1/n} \geq cn \left(\frac{(T - t)\Omega}{\omega_t^n}\right)^{1/n}. \]

Since \( x \mapsto -\log x + \frac{1}{2}cnx^{1/n} \) is uniformly bounded from below, we have

\[ \Box_{\omega_t} \frac{A}{T - t}(\varphi_t - \Phi_t) \geq -\frac{AC}{T - t} \]

\[ + \frac{A}{T - t} \left(\Delta_{\omega_t} \Phi_t - \int_{\pi^{-1}(z)} \frac{\partial \varphi_t}{\partial t} \omega_{t,0,z}^r\right) \]

\[ + \frac{A}{2(T - t)\text{Tr}_{\omega_t} \hat{\omega}_t}. \]

Combining (3.1) and (3.3), we have

\[ \Box_{\omega_t} Q \leq \frac{AC}{T - t} + \tilde{C}\text{Tr}_{\omega_t}\omega_0 - \frac{A}{2(T - t)\text{Tr}_{\omega_t}} \left(\frac{T - t}{T}\omega_0 + \frac{t}{T} \pi^*\omega_\Sigma\right) \]

\[ - \frac{A}{T - t} \left(\Delta_{\omega_t} \Phi_t - \int_{\pi^{-1}(z)} \frac{\partial \varphi_t}{\partial t} \omega_{t,0,z}^r\right) \]

\[ \leq \frac{AC}{T - t} + \left(\tilde{C} - \frac{A}{2T}\right)\text{Tr}_{\omega_t}\omega_0 \]

\[ - \frac{A}{T - t} \left(\Delta_{\omega_t} \Phi_t - \int_{\pi^{-1}(z)} \frac{\partial \varphi_t}{\partial t} \omega_{t,0,z}^r\right). \]
By Lemma 2.2, we have \( \frac{\partial \varphi_t}{\partial t} \) for some uniform constant \( C \). It follows that
\[
\int_{\pi^{-1}(z)} \frac{\partial \varphi_t}{\partial t} \omega^0_{t, z} \leq C \text{Vol}_{\omega^0_{t, z}}(\pi^{-1}(z)) \leq C'.
\]
Note that \( \text{Vol}_{\omega^0_{t, z}}(\pi^{-1}(z)) \) is independent of \( z \). For the Laplacian term of \( \Phi_t \), we have
\[
\Delta_{\omega_t} \int_{\pi^{-1}(z)} \varphi_t \omega^0_{t, z} = \text{Tr}_{\omega_t} \int_{\pi^{-1}(z)} \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \omega^0_{t, z}
= \text{Tr}_{\omega_t} \int_{\pi^{-1}(z)} (\omega_t - \hat{\omega}_t) \wedge \omega^0_{t, z}
\geq -\text{Tr}_{\omega_t} \int_{\pi^{-1}(z)} \hat{\omega}_t \wedge \omega^0_{t, z}
\geq -\text{Tr}_{\omega_t} \int_{\pi^{-1}(z)} (\omega_t \wedge \omega^0_{t, z} + \pi^* \omega_{\Sigma} \wedge \omega^0_{t, z}).
\]
By the fact that \( \text{Tr}_{\omega_t} \pi^* \omega_{\Sigma} \leq C \) and \( \int_{\pi^{-1}(z)} (\omega_t \wedge \omega^0_{t, z} + \pi^* \omega_{\Sigma} \wedge \omega^0_{t, z}) \) is a \((1,1)\)-form on \( \Sigma \) independent of \( t \), we have
\[
\Delta_{\omega_t} \int_{\pi^{-1}(z)} \varphi_t \omega^0_{t, z} \geq -C.
\]
for some uniform constant \( C \). Back to (3.5), we have
\[
\square_{\omega_t} Q \leq \frac{AC}{T - t} + \left( \hat{C} - \frac{A}{2T} \right) \text{Tr}_{\omega_t} \omega_0 \leq \frac{AC}{T - t} - \text{Tr}_{\omega_t} \omega_0
\]
if we choose \( A \) sufficiently large such that \( \hat{C} - \frac{A}{2T} < -1 \).

Hence, for any \( \varepsilon > 0 \), at the point where \( Q \) achieves its maximum over \( X \times [0, T - \varepsilon] \), we have \( \text{Tr}_{\omega_t} (T - t) \omega_0 \leq C \) for some uniform constant \( C \) independent of \( \varepsilon \). Together with Lemma 2.3, it shows that for any \( t \in [0, T) \) we have,
\[
C^{-1} (T - t) \omega_0 \leq \hat{\omega}_t.
\]
Combining this with the fact that \( \omega_t \geq C^{-1} \pi^* \omega_{\Sigma} \), we have
\[
C^{-1} \hat{\omega}_t \leq \omega_t.
\]
Together with (2.7) and (2.11), we also have \( \omega_t \leq C \hat{\omega}_t \) for any \( t \in [0, T) \). It completes the proof of the theorem since one can check that for the reference metric \( \hat{\omega}_t \), we have
\[
C^{-1} (T - t)^{1/2} \leq \text{diam}_{\hat{\omega}_t} (\pi^{-1}(z)) \leq C (T - t)^{1/2}.
\]
\( \square \)

4. Remarks on curvatures

We would like to end this article by a discussion of the implications of Theorem 1.1 on the blow-up rate of curvatures.

There are “folklore conjectures” concerning the blow-up of curvatures along the Kähler-Ricci flow (1.1) with finite-time singularity (see Section 7 in SW2). It is known by Hamilton [H], Sesum [Se] and Zhang [Z2] that \( \sup ||\text{Rm}||_{g(t)}, \sup ||\text{Ric}||_{g(t)} \) and the scalar curvature \( \sup R(g(t)) \) must blow-up to \( +\infty \) as \( t \to T \) when \( T < \infty \) is the singular time. However, it is still open whether they blow-up at the rate of \( O((T - t)^{-1}) \) for the Kähler-Ricci flow (1.1).
In the case where \([\omega_0] = c_1(X) > 0\), it was established by Perelman (see \[SeT\]) that \(R(g(t)) = O((T - t)^{-1})\) along \((1.1)\). For the normalized Kähler-Ricci flow \(\partial_t \hat{g}_t = -\text{Ric}(\hat{g}_t) - \hat{g}_t\) with finite-time singularity, Zhang established in \[Z2\] that \(R(\hat{g}(t)) = O((T - t)^{-2})\) under a cohomological assumption analogous to \([1.2]\). In the special case where \(X\) is a \(\mathbb{P}^1\)-bundle over a compact Kähler-Einstein manifold, it is proved in \[F1\] that when the \(\mathbb{P}^1\)-fibers collapse the Kähler-Ricci flow \((1.1)\) must develop Type I singularity (i.e. \(\|Rm\|_{g(t)} = O((T - t)^{-1})\)) assuming that the initial metric has Calabi symmetry.

Under the cohomological setting in this article, the implications of the boundedness of \(\text{Tr}_{\omega_0} \text{Ric}(\omega_t)\) on the curvature blow-up rates are as follows:

**Corollary 4.1.** Under the same assumptions as in Theorem \([Z2]\), we have

\[
\text{Ric}(\omega_t) \leq B\omega_0 \quad \text{for some uniform constant } B > 0
\]

\[
\Rightarrow \quad R(\omega_t) = O((T - t)^{-1}), \quad \text{and } \|Rm\|_{\omega_t} = O((T - t)^{-2}).
\]

**Proof.** The \(O((T - t)^{-1})\) blow-up rate of the scalar curvature \(R(\omega_t) = \text{Tr}_{\omega_t} \text{Ric}(\omega_t)\) follows trivially from \(\text{Ric}(\omega_t) \leq B\omega_0\) and \([1.0]\).

For the blow-up of the Riemann curvature tensor, we use the result in the proof of Theorem \([Z2]\) that there exists a uniform constant \(C > 0\) such that for any \(t \in [0, T)\), we have

\[
C^{-1}(T - t)\omega_0 \leq \omega_t \leq C\omega_0.
\]

Using the estimates in \[PSS, ShW\] one can establish that

\[
\|Rm\|_{\omega_t} = O((T - t)^{-2}).
\]

Note that in \[ShW\] the authors assumed \(N\omega_0 \leq \omega_t \leq \frac{1}{N}\omega_0\) and asserted that \(\|Rm\|_{\omega_t} = O(N^{-4})\). One can easily check this result can be extended without much difficulty so that \(N\) is any positive non-increasing function \(N(t)\) defined on \([0, T)\). Furthermore, if the metric upper bound \(\frac{1}{N}\omega_0\) is replaced by \(C\omega_0\) for some uniform constant \(C > 0\), the result can be generalized to \(\|Rm\|_{\omega_t} = O(N^{-2})\) with an almost identical proof. \(\square\)

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