Integration measure and extended BRST covariant quantization

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Abstract

We propose an extended BRST invariant Lagrangian quantization scheme of general gauge theories based on explicit realization of the ”modified triplectic algebra” that was announced in our previous investigation (hep-th/0104189). The algebra includes, besides the specific odd operators $V^a$ appearing in the triplectic formalism, also the odd operators $U^a$ introduced within modified triplectic quantization and the second-order odd operators $\Delta^a$. While the operators $V^a$ can be viewed as anti-Hamiltonian vector fields generated by a second-rank irreducible $Sp(2)$ tensor, the operators $U^a$ are the anti-Hamiltonian vector fields generated by a $Sp(2)$ scalar. We show that some even supersymplectic structure defined on the space of fields and antifields, provides the extended BRST path integral with a well-defined integration measure. All the known Lagrangian quantization schemes based on the extended BRST symmetry are obtained by specifying the (free) parameters of that method.

1 Introduction

The Batalin-Vilkovisky (BV) formalism \cite{1} of Lagrangian quantization of general gauge theories, since its introduction, attracts permanent interest due to its covariance, universality and mathematical elegance. The BV formalism is outstanding also from a purely mathematical point of view because it is formulated in terms of seemingly exotic objects: the antibracket (odd Poisson bracket) and the (related) second-order operator $\Delta$. The study of the formal geometrical structure of the BV formalism, performed during the last ten years, allowed to introduce its interpretation in terms of more traditional mathematical objects as well as to find the unusual behaviour of the antibracket with respect to integration theory \cite{2}.

On the other hand, there exists an extended, $Sp(2)$ symmetric (BRST/antiBRST invariant) version of the BV formalism \cite{3}, and its geometrized version known as “triplectic formalism” \cite{4, 5} (see also \cite{3, 4, 6}).

The main ingredients of triplectic quantization are a pair of antibrackets $(\cdot, \cdot)^{\alpha}$, a pair of operators $\Delta^{\alpha}$, and a pair of odd vector fields $V^{\alpha}$, with $\alpha = 1, 2$ being related to BRST and antiBRST symmetry. However, the degeneracy of the antibrackets appearing in the $Sp(2)$ symmetric quantization schemes, and the structure of $V^{\alpha}$ fields, lead to difficulties in establishing the links between the various quantization schemes as well as in the geometrical analysis of their structure \cite{7}. For example, within the “canonical triplectic quantization” \cite{4, 5} the initial version of the $Sp(2)$ symmetric Lagrangian quantization
is not contained. However, within the “modified triplectic quantization” which is specified by the presence of an additional pair of odd operators $U^a$ the initial $Sp(2)$ symmetric formalism is contained as a special limit. Thereby, the separation of appropriate variables [in our context to be identified with $x = \{\phi, \bar{\phi}\}$] can be considered as essential means for a consistent quantization procedure allowing for the formulation of a suitable boundary condition for quantum action and for the definition of an appropriate Lagrangian manifold on which a natural volume form is defined ensuring the integral of the vacuum functional well-defined. The pair of operators $U^a$ appearing in the modified triplectic quantization may serve for these purposes.

An interesting observation on the structure of $Sp(2)$ invariant formalism was made in Ref. [5]: there, it was found that the “triplectic algebra” generates a Poisson bracket on an appropriate subspace of the whole space of fields and antifields. However, this Poisson bracket did not appear in the triplectic formalism. It seemed, that it plays an auxiliary role, giving the means for the geometrical formulation of the gauge-fixing conditions, and the compatibility conditions for the antibrackets and $V^a$ fields as well [7].

In our recent paper [9], starting from this observation, we proposed a covariant realization of the triplectic algebra by making use of this Poisson bracket and of a flat connection which respects it. We have shown, that the operators $V^a$ can be viewed as the anti-Hamiltonian vector fields generated by a second-rank irreducible $Sp(2)$ tensor, $S_{ab}$, while the operators $U^a$ are the anti-Hamiltonian vector fields generated by some $Sp(2)$ scalar, $S_0$. There, it was also found that the whole space of the triplectic formalism can be equipped with an even symplectic structure. Note, that for the first time a flat symplectic connection was used by I. A. Batalin and I. V. Tyutin for the covariant formulation of deformation quantization in Ref. [10].

Now, in this paper we define the extended BRST invariant path integral whose integration measure corresponds to that symplectic structure, and we formulate the master equations for the quantum action $W$ and gauge-fixing functional $X$. Furthermore, we show that the various known Lagrangian quantization schemes based on the extended BRST symmetry [3, 4, 8] are included in the former one as special cases. Hence, the above-mentioned Poisson bracket plays the basic role in the $Sp(2)$ symmetric Lagrangian quantization schemes: It provides them with a well-defined integration measure and allows for a reparametrization-invariant realization of triplectic algebra.

The paper is arranged as follows.

In Section 2 we give a brief description of the general statements and explicit realizations of the triplectic quantization scheme.

In Section 3, for the reason of completeness, the basic ingredients of quantization scheme including the $\Delta^a$—operators, the antibrackets $(\cdot, \cdot)^a$, the $V^a$- and $U^a$- fields are presented. These objects were proposed in [3] and given in reparametrized invariant form with the help of a flat symmetric connection which respects the Poisson bracket on the space of fields.

In Section 4 we complete these structures by the integration measure defined by the use of an even symplectic structure which is constructed on the whole space of fields and antifields. Then we construct a quantization scheme based on the modified triplectic algebra and we show that all the known approaches to triplectic quantizations can be considered as specific ones within the proposed method.

Finally in the Appendix we collect various relations among these objects which we need for the construction of the triplectic algebra.
2 Extended BRST symmetric formalisms

Let us present the basic ingredients of extended BRST invariant Lagrangian quantization schemes, thereby, following the original papers [3, 4, 5].

In these schemes one requires the existence of a pair of odd (differential) operators $\Delta^a$ and of odd vector fields $V^a$ ($a = 1, 2$) satisfying the following consistency conditions:

\[
\Delta^a \Delta^b = 0, \quad \epsilon(\Delta^a) = 1, \\
V^a V^b = 0, \quad \epsilon(V^a) = 1, \\
\Delta^a V^b + V^a \Delta^b = 0.
\] (2.1)

Here, and in the following, the curly bracket denotes symmetrization with respect to the enclosed indices $a$ and $b$. The operators $\Delta^a$, due to their obstruction of the Leibniz rule, generate a pair of antibrackets,

\[
(-1)^{\epsilon(f)}(f, g)^a = \Delta^a (fg) - (\Delta^a f)g - (-1)^{\epsilon(f)} f(\Delta^a g),
\] (2.4)

which obey the properties of graded antisymmetry,

\[
(f, g)^a = -(-1)^{\epsilon(f)+\epsilon(g)+1} (g, f)^a,
\] (2.5)

of the Leibniz rule,

\[
(f, gh)^a = (f, g)^a h + (-1)^{\epsilon(h)} g(\Delta^a h)^a,
\] (2.6)

and of the consistency conditions:

\[
\Delta^a (f, g)^b = (\Delta^a f, g)^b + (-1)^{\epsilon(f)+1}(f, \Delta^a g)^b,
\] (2.7)

\[
V^a (f, g)^b = (V^a f, g)^b + (-1)^{\epsilon(f)+1}(f, V^a g)^b.
\] (2.8)

The partition function in the triplectic formalism is defined by the expression

\[
Z = \int dz \, d\lambda \exp \left\{ \frac{i}{\hbar} [W + X] \right\},
\] (2.10)

where $z$ denotes the whole set of fields and antifields, $W = W(z)$ is viewed as the quantum action of the theory, and $X = X(z, \lambda)$ is considered as the gauge fixing term. Obviously, this division of the gauge fixed action into two pieces is to a large extent arbitrary, but we shall follow the conventions of Refs. [4, 5]. The gauge-fixing function $X$ restricts the partition function to the “space of effective fields” being necessary to describe the quantum dynamics, and $\lambda$ are a some additional parametric field variables which simply become Lagrangian multipliers of gauge constraints when $X$ depends on them only linearly.

The partition function (2.10) is gauge independent if the following quantum master equations hold:

\[
(\Delta^a + \frac{i}{\hbar} V^a) \exp \left\{ \frac{i}{\hbar} W \right\} = 0, \quad (\Delta^a - \frac{i}{\hbar} V^a + \ldots) \exp \left\{ \frac{i}{\hbar} X \right\} = 0.
\] (2.11)

In the last equation the dots indicate those extra terms which are required due to the variation of the field $\lambda$. The explicit form of the second master equation will be specified in the following. The extended BRST transformations are generated by the operators $\delta^a$,

\[
\delta^a = (X - W, \cdot)^a + 2V^a + i\hbar \Delta^a.
\] (2.12)

For convenience let us now give the explicit realizations of the triplectic algebra and the corresponding quantization procedure in the various versions of the extended BRST invariant Lagrangian formalism being under consideration in the literature.
2.1 $Sp(2)$—symmetric version \[3\]

In the initial version of the $Sp(2)$ invariant formalism the triplectic algebra is realised on the “physical fields” $\phi^A$ (including the original fields, ghosts, antighosts and Lagrangian multipliers), the (supplementary) fields $\bar{\phi}_A$ of the same grading $\epsilon(\bar{\phi}_A) = \epsilon(\phi^A) \equiv \epsilon_A$ and the pair of antifields $\phi^*_A$ of the opposite grading $\epsilon(\phi^*_A) = \epsilon_A + 1$.

On these fields the triplectic algebra is realized as follows $(\phi^*_A, \phi^*_B)^a = \delta^a_c \delta^c_B$:

$$\Delta^a = \Delta^a_0 \equiv (-1)^{\epsilon_A} \partial_l \frac{\partial}{\partial\phi^*_A}, \quad V^a = \hat{V}^a_0 \equiv \epsilon^{ab} \phi^*_b \frac{\partial}{\partial\phi^*_A},$$

(2.13)

where, according to Eq. (2.4), the corresponding antibrackets are defined by the relations

$$\pi^a(\phi^A, \phi^*_b) = \delta^a_B \delta^B_A.$$  

(2.14)

In order to be able to formulate the gauge fixing conditions in Ref. \[3\], via introduction of additional fields $\pi^{Aa}$, $\epsilon(\pi^{Aa}) = \epsilon_A + 1$, and $\lambda^A$, $\epsilon(\lambda^A) = \epsilon_A$, a further extension of the space of fields has been performed. In these terms the partition function can be presented in the form

$$Z = \int dz \, d\lambda \exp \left\{ \frac{i}{\hbar} (W + X + S_0) \right\}$$

(2.15)

In this version the quantum action $W$ is supposed to obey the master equation (2.11) together with the boundary condition to coincide with the initial classical action when $\hbar = \phi^* = \phi = 0$; no condition is required for $X$.

2.2 Triplectic version \[4\]

Another version of the $Sp(2)$ formalism, suggested by Batalin and Marnelius \[4\], considers the auxiliary fields $\pi^{Aa}$ as canonically conjugate with respect to the fields $\Phi_A$. The triplectic algebra is realized in the following way:

$$\Delta^a = \Delta^a_0 + (-1)^{\epsilon_A + 1} \epsilon^{ab} \frac{\partial}{\partial\pi^{Ab}} \frac{\partial}{\partial\phi^*_A}, \quad V^a = \frac{1}{2} \left( \hat{V}^a_0 + (-1)^{\epsilon_A + 1} \pi^{Aa} \frac{\partial}{\partial\phi^*_A} \right),$$

(2.18)

with $\Delta^a_0$ and $\hat{V}^a_0$ being defined through Eqs. (2.13); then, the corresponding antibrackets are defined by the relations

$$\pi^a(\phi^A, \phi^*_b) = \delta^a_B \delta^B_A, \quad \pi^a(\bar{\phi}_A, \pi^{B}) = \delta^a_B \delta^B_A,$$

(2.19)

thus making obvious the triplet structure of the formulation. For later convenience let us write the $V^a$—operators as

$$V^a = \frac{1}{2} (\hat{V}^a_0 + \hat{\bar{V}}^a_0) \quad \text{with} \quad \hat{\bar{V}}^a_0 \equiv (-1)^{\epsilon_A + 1} \pi^{Aa} \frac{\partial}{\partial\phi^*_A}.$$  

(2.20)
While the quantum action $W = W(z)$ is required to satisfy (2.11), the gauge fixing functional $X = X(z, \lambda)$, is required to satisfy the following master equation:

$$\left(\Delta^a - \frac{i}{\hbar} V^a\right) \exp\{(i/\hbar)X\} = 0.$$  \hspace{1cm} (2.21)

Notice that by the explicit realization of the anti-Hamiltonian vector fields $V^a$ in the form (2.18), cf., also Eq. (3.8), and of the above introduced antibrackets the stronger relations

$$\Delta^a V^b + V^a \Delta^b = 0,$$  \hspace{1cm} (2.22)

$$V^a(f, g)^b = (V^a f, g)^b - (-1)^{c(f)}(f, V^a g)^b$$  \hspace{1cm} (2.23)

hold, i.e., without symmetrization of the indices $a$ and $b$.

### 2.3 Modified triplectic version [8]

The essential point of the original triplectic quantization [4] consists in dividing the task of constructing the partition function $Z$ into two parts: first, in the construction of the quantum action $W$, and second, in the construction of a suitable gauge fixing functional $X$. Either problem is solved by means of the appropriate master equations (2.11), (2.21).

Unfortunately, the explicit realization of this attractive idea within original version meets the problem of the boundary condition for $W$. Due to the special structure of the operators $V^a$, Eq. (2.18), it is not possible, in contrast to all previously known schemes of Lagrangian quantization, to consider the initial classical action as a natural boundary condition to the solution of the quantum master equations. This was the main reason in Ref. [8] to reformulate the original triplectic scheme by a modified one.

Remaining in the same configuration space of fields and antifields $z = (\phi, \phi^*; \bar{\phi}, \pi)$, and accepting the idea of a separate treatment of the two above mentioned actions $W$ and $X$, it was proposed to change from the beginning the triplectic algebra and the generating master equations by introducing an additional set of $Sp(2)$ doublets of operators $U^a$, $\epsilon(U^a) = 1$, by requiring

$$\Delta^{[a} \Delta^{b]} = 0, \quad V^{[a} V^{b]} = 0, \quad \Delta^{[a} V^{b]} + V^{[a} \Delta^{b]} = 0, \quad (2.24)$$

$$U^{[a} U^{b]} = 0, \quad \Delta^{[a} U^{b]} + U^{[a} \Delta^{b]} = 0, \quad U^{[a} V^{b]} + V^{[a} U^{b]} = 0. \quad (2.25)$$

This algebra can be considered as an extension of triplectic algebra (2.1), (2.2) and (2.3) and is referred to as the ”modified triplectic algebra”. An explicit realization of the operators may be given by $\Delta^a$ as introduced in Eqs. (2.18), $V^a \equiv V^a_0$ and $U^a \equiv U^a_0$, as introduced in Eqs. (2.13) and (2.20), respectively.

Again, the partition function is given by the functional integral (2.15),

$$Z = \int \! dz \, d\lambda \, \exp \{(i/\hbar)(W + X + S_0)\}, \quad (2.26)$$

but now $W = W(z)$ and $X = X(z, \lambda)$ are required to satisfy the following quantum master equations:

$$\left(\Delta^a + \frac{i}{\hbar} V^a\right) \exp\{(i/\hbar)W\} = 0, \quad \left(\Delta^a - \frac{i}{\hbar} U^a\right) \exp\{(i/\hbar)X\} = 0,$$  \hspace{1cm} (2.27)

where the functional $S_0$ was defined in (2.17). Now one can use the standard boundary condition for $W$ in the form of the initial classical action when $\hbar = \phi^* = \bar{\phi} = 0$. Note that the difference of the operators $V^a - U^a \equiv \hat{V}^a_0 - \hat{U}^a_0$ can be represented as anti-Hamiltonian vector fields:

$$V^a - U^a = (S_0, \cdot)^a.$$  \hspace{1cm} (2.28)
3 Triplectic algebra, modified triplectic algebra and Poisson brackets

The basic ingredients of the triplectic formalism and the quantization in arbitrary coordinates were formally constructed and analyzed in terms of an “anti-triplectic non-degenerate metric” [5].

An important feature of triplectic algebra which was observed in Ref. [5] is the existence of some Lagrangian subspace \( \mathcal{M}_0 \) on which a \( Sp(2) \) invariant even Poisson bracket can be defined, viz.,

\[
\{ u, w \}_0 = \epsilon_{ab}(u, V^a w)^b .
\]

This subspace \( \mathcal{M}_0 \) is specified by requiring for any functions \( u, v, w \) on \( \mathcal{M}_0 \) to hold:

\[
(u, v)^a = 0, \quad (u, V^{(a} v)^b ), w^c = 0.
\]

In the canonical and primary triplectic versions [3, 4] of the \( Sp(2) \) covariant Lagrangian quantization schemes this construction yields a canonical Poisson bracket (in Darboux coordinates \( x^i = \{ \phi, \bar{\phi} \} \)):

\[
\{ u(x)^i, v(x)^j \}_0 = \partial_i u(x) \partial_j v(x) - \partial_i v(x) \partial_j u(x) .
\]

Now, we try to find an explicit realization of the triplectic algebra on a Poisson (super)space where the Poisson bracket \( \{ \cdot, \cdot \}_0 \) in local coordinates \( x^i \) ("fields") is given by the expression

\[
\{ u(x)^i, v(x)^j \}_0 = \partial_i u(x) \omega^{ij}(x) \partial_j v(x),
\]

where

\[
\omega^{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega^{ij}, \quad (-1)^{\epsilon_i \epsilon_k} \omega^{kl} \partial_i \omega^{ij} + \text{cycl. perm } (ijk) = 0.
\]

Then we consider the superspace \( \mathcal{M} \) parametrized by the coordinates \( z^\alpha = (x^i, \theta^{ia}) \), \( (\epsilon(\theta^{ia}) = \epsilon_i + 1) \), where \( \theta^{ia} \) ("antifields") are transformed as \( \partial_i = \partial/\partial x^i \) under reparametrizations of \( \mathcal{M}_0 \). This superspace can be equipped by a pair of “canonical” antibrackets \( \{ \cdot, \cdot \}_0^a \)

\[
(f, g)^a_0 = \partial_f \partial g + (-1)^{\epsilon(f)} \partial_f g \partial x^i .
\]

On that superspace \( \mathcal{M} \) there exists a naturally defined object being a \( Sp(2) \) irreducible second rank tensor, namely

\[
S_{ab} = \frac{1}{6} \theta^{ia} \omega^{ij}(x) \theta_{jb}, \quad S_{ab} = S_{ba} .
\]

This tensor obeys the condition

\[
(S_a b, S_c d)^d_0 = 0 ,
\]

which is nothing but the Jacobi identity for the Poisson bracket (3.4). With the help of this tensor we introduce the vector fields \( V^a \) in an explicitly \( Sp(2) \) symmetric way,

\[
V_a = (S_{ab}, \cdot )^b_0 .
\]
with the following results:

\[
V^a = \frac{1}{2} \omega^{ij} \theta_j^a \partial_i + \frac{1}{6} \partial_i \omega^{jk} \theta^a \theta_{kb} \frac{\partial}{\partial \theta_{ib}}, \quad \theta^{ia} = \varepsilon^{ab} \theta_b^i.
\]  

(3.9)

In accordance with (3.1) these vector fields \( V^a \) generate on \( M_0 \) the even Poisson bracket.

It should be noticed that it is not necessary to use the algebraic properties of \( V^a \) which are dictated by the triplectic algebra in order to find the relation between antibrackets and Poisson bracket. Namely, from the definition (3.9) we obtain the following result

\[
V_{\{a}V_{b\}} = (S_{c\{a}, (S_{b}d), \cdot\}^d_{\text{can}})\text{can} =
\]

\[
= \frac{1}{12} \omega^{ij} \partial_i \omega^{mn} \theta_{ma} \theta_{nb} \partial_j + \frac{1}{6} \left( \frac{1}{6} \partial_i \omega^{mk} \partial_j \omega^{in} + \frac{1}{6} \omega^{km} \partial_i \partial_j \omega^{in} + \right.
\]

\[
+ \frac{1}{3} \left( \partial_i \omega^{mn} \partial_j \omega^{ik} + \partial_i \omega^{nk} \partial_j \omega^{im} + \partial_i \omega^{km} \partial_j \omega^{in} \right) \theta_{mb} \theta_{ka} \theta_{nc} \frac{\partial}{\partial \theta_{jc}}.
\]

(3.10)

Obviously, the conditions (2.2) are failed in general. It can be easily checked that the conditions (2.9) are failed as well. Hence, the existence of a \( Sp(2) \) invariant Poisson bracket defined by the expression (3.1) does not require the conditions (2.2), (2.9).

To obtain the explicit realization of the triplectic algebra generating the Poisson structure (3.3), let us equip \( M_0 \) by a symmetric connection which respects the Poisson bracket

\[
\Gamma^k_{ij} = \Gamma^k_{ji}, \quad \partial_i \omega^{kj} + \omega^{kl} \Gamma^j_{il} + \omega^{ij} \Gamma^k_{il} = 0.
\]  

(3.11)

Notice, that from the symmetry of the Christoffel symbols the Jacobi identity for the Poisson bracket follows. When the Poisson bracket is non-degenerate the symmetric Poisson connection coincides with the symmetric symplectic connection which is known as “Fedosov connection” [4] because of Fedosov’s work on globally defined deformation quantization [12]. It was recently found to be the natural object in the Hamiltonian BRST quantization as well [13]. In what follows we consider the case when the Poisson bracket is non-degenerate.

Using this connection, we can define on the superspace \( M \) the differential operators corresponding to the covariant derivative on \( M_0 \):

\[
\nabla_i = \partial_i + \Gamma^k_{ij} \theta_{ka} \frac{\partial}{\partial \theta_{ja}}, \quad [\nabla_i, \nabla_j] = R^k_{mij} \theta_{ka} \frac{\partial}{\partial \theta_{ma}},
\]

(3.12)

where \( R^k_{mij} \) are the components of the curvature tensor of the Fedosov connection,

\[
R^l_{ijk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ij} + \Gamma^m_{ki} \Gamma^l_{jm} - \Gamma^m_{ij} \Gamma^l_{km}, \quad R^l_{ij} = -R^l_{ikj}.
\]

(3.13)

To get the realization of the triplectic algebra in general coordinates, one can consider a minimal generalization by replacing within the antibrackets (3.5) the usual derivative, \( \partial_i \), by the covariant one, \( \nabla_i \), namely

\[
(f, g)^a = (\nabla_i f) \frac{\partial g}{\partial \theta_{ia}} + (-1)^{\epsilon(f)} \frac{\partial f}{\partial \theta_{ia}} (\nabla_i g)
\]

(3.14)

and in the definition of the operators \( \Delta^a \),

\[
\Delta^a = \nabla_i \frac{\partial}{\partial \theta_{ia}} + \frac{1}{2} (\rho(x), \cdot)^a
\]

(3.15)
where \( \rho \) is some functional on \( \mathcal{M}_0 \) which will be restricted even more in the Appendix. These operators obviously generate through (2.4) a pair of bilinear operations (3.14) acting on \( \mathcal{M} \) and obeying evidently the properties (2.5) and (2.7).

Taking into account that the following equation holds:

\[
\nabla_i S_{ab} = 0, \tag{3.16}
\]

we obtain that the vector fields \( V^a \), defined through (3.14), are of the form

\[
V_a = (S_{ab}, \cdot)^b = \frac{1}{2} \theta_i a \omega^{ij} \nabla_j. \tag{3.17}
\]

For these objects to be antibrackets, these operations (3.14) must satisfy the Jacobi identities (2.7). On the other hand, the Jacobi identities hold, if the commutators of the \( \Delta^a \)-operators are first order differential operators. However, the explicit calculation yields the result (see Appendix)

\[
\Delta^i (\Delta^j) = R^k_{\ ij} \theta^c \frac{\partial}{\partial \theta^l} \theta^{ja} \theta^{lc} \nabla^i. \tag{3.18}
\]

Hence, we should require \( \Gamma^k_{ij} \) to be a flat connection:

\[
R^k_{\ ijm} = 0. \tag{3.19}
\]

The existence of such flat connections directly follows from the Darboux theorem, since in Darboux coordinates one can choose the trivial connection which is obviously flat. Note that every non-linear canonical transformation transfers the trivial connection into a non-zero one. Such a flat connection, respecting the Poisson bracket, was used in Ref. [10] for the formulation of a coordinate-free scheme of deformation quantization.

Straightforward calculations (see Appendix) of the algebra of operators \( \Delta^a \) (3.15), vector fields \( V^a \) (3.17) and operations \( (\cdot, \cdot)^a \) yield the conclusion that the remaining relations of the triplectic algebra (2.1), (2.2), (2.4), (2.7), (2.8), (2.23) and (2.22) in its strong version are fulfilled when \( R^k_{\ ijm} = 0 \). Therefore we arrive at an explicit realization of the triplectic algebra on \( \mathcal{M} \) with an arbitrary flat Poisson space \( \mathcal{M}_0 \).

Now we extend the triplectic algebra to the modified one by the introduction of additional vector fields \( U^a \) as follows: Let us equip \( \mathcal{M}_0 \) by a symmetric tensor \( g_{ij}(x) = g_{ji}(x) \). Then we are able to construct a \( Sp(2) \) scalar function \( S_0 \) being defined on the superspace \( \mathcal{M} \),

\[
S_0 = \frac{1}{2} \theta_i a g^{ij} \theta_j b \epsilon^{ab}, \quad \epsilon(S_0) = 0, \quad g_{ij} = \omega^{im} g_{mn} \omega^{nj}. \tag{3.20}
\]

It generates the anti-Hamiltonian vector fields \( U^a \)

\[
U^a = (S_0, \cdot)^a = \frac{1}{2} g_{ji} \theta_j \theta_l \frac{\partial}{\partial \theta^l} + g^{ij} \theta^a \nabla_i, \tag{3.21}
\]

where

\[
g_{ij} = \partial_i g^{jl} + \Gamma^j_{in} g^{nl} + \Gamma^l_{in} g^{jn}
\]

denotes the covariant derivative of tensor \( g^{ij} \). The requirement of the conditions (2.25) yields the following equations for \( S_0 \):

\[
(S_0, S_0)^a = 0, \quad V^a S_0 = 0, \quad \Delta^a S_0 = 0. \tag{3.22}
\]
Let us define the traceless matrix \( I^i_k \) by the relations \( I^i_k = \omega^{ij}g_{jk} \). In terms of the operator \( \hat{I} \) the relations (3.22) read
\[
I^i_k - I^k_i = 0, \quad N^k_{ij} \equiv I^l_i I^k_{jl} - I^l_j I^k_{il} - I^l_l (I^l_i - I^l_j) = 0,
\]
where \( N^k_{ij} \) is the Nijenhuis tensor. When these equations are fulfilled the set of operators \( \Delta^a, \) Eq. (3.15), \( V^a, \) Eq. (3.17), and \( U^a, \) Eq. (3.21), form the modified triplectic algebra (2.25).

Notice that in Darboux coordinates the matrix \( \omega^{ij} \) is in the form
\[
\omega^{ij} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix},
\]
where ‘id’ is the unit matrix. Then the vector fields \( V^a \) (3.17) are exactly reduced to the operators used in the original triplectic quantization [4],
\[
V^a = \frac{1}{2}(\hat{V}^a_0 + \hat{U}^a_0),
\]
where \( \hat{V}^a_0 \) and \( \hat{U}^a_0 \) were introduced in (2.13) and (2.20) respectively.

Choosing for the matrix \( g^{ij} \) the representation
\[
g^{ij} = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix},
\]
we obtain for the vector fields \( U^a \) (3.21) the expressions
\[
U^a = \hat{V}^a_0 - \hat{U}^a_0.
\]
The vector fields used in formulation of modified triplectic quantization [8] are linear combinations of the vector fields \( V^a \) (3.24) and \( U^a \) (3.25).

4 Quantization procedure

The superspace \( \mathcal{M} \) can be equipped with an even symplectic structure. Indeed we can construct the even closed two-form, which is non-degenerated on antifields (cf., Ref. [14]):
\[
\Omega_2 = d(\theta_{ia} \omega^{ij} D\theta^a_j) = \frac{1}{2} R_{ijkl} \theta^{al} \theta^i_k dx^i \wedge dx^j + \omega^{ij} D\theta_{ia} \wedge D\theta^a_j,
\]
where \( D\theta_{ia} = d\theta_{ia} - \Gamma^k_{ij} \theta_{ka} dx^j, \) and \( \Gamma^k_{ij} \) is some Fedosov connection (not necessarily flat). Hence, requiring the connection to be flat, we can equip the superspace \( \mathcal{M} \), in addition to the triplectic algebra, with an even symplectic structure
\[
\Omega = dz^\mu \Omega_{\mu\nu} dz^\nu = \omega_{ij} dx^i \wedge dx^j + \kappa^{-1} \Omega_2 = \omega_{ij} dx^i \wedge dx^j + \kappa^{-1} \omega^{ij} D\theta_{ia} \wedge D\theta^a_j,
\]
where \( \kappa \) is an arbitrary constant. The corresponding non-degenerate Poisson bracket reads
\[
\{ f(z), g(z) \} = (\nabla_i f) \omega^{ij} (\nabla_j g) + \kappa \frac{\partial f}{\partial \theta^a_i} \omega_{ij} \frac{\partial g}{\partial \theta_j^a}.
\]
Using the even symplectic structure we can define on \( \mathcal{M} \) the analogue of the Liouville measure [13],
\[
\mathcal{D}_0 = \sqrt{\text{Ber} \, \Omega_{\mu\nu} = \kappa (\det \omega_{ij})^{3/2}},
\]
which is invariant under supercanonical transformations of the Poisson bracket (4.2).

Now we construct the vacuum functional and the generating master equations for a quantum action $W = W(z)$ and a gauge fixing functional $X = X(z, \lambda)$ using basic operators $\Delta^a, V^a, U^a$ and the function $S_0$ introduced above. Define the vacuum functional $Z$ as the following path integral

$$Z = \int dz \, d\lambda \, D_0 \exp \{ (i/\hbar)[W + X + \alpha S_0]\}$$

(4.5)

where $D_0$ is the integration measure (4.4) and $\alpha$ is a constant. We suggest the following form of generating master equations for $W$ and $X$:

$$\frac{1}{2}(W, W)^a + V^a W = i\hbar \Delta^a W,$$

(4.6)

$$\frac{1}{2}(X, X)^a + U^a X = i\hbar \Delta^a X,$$

(4.7)

where $V^a$ and $U^a$ are first order differential operators constructed from $V^a$ and $U^a$.

Let us consider the more general transformations of coordinates $z$ generated by antibrackets $(\cdot, \cdot)^a$ and operators $V^a, U^a$

$$\delta z = (z, F)^a \mu_a + \beta \mu_a V^a z + \gamma \mu_a U^a z$$

(4.8)

where $\beta, \gamma$ are some constants, $\mu_a$ is a $Sp(2)$ doublet of anticommuting constant parameters and $F = F(z, \lambda)$ is a bosonic functional. The transformations (4.8) lead to the variation

$$\delta(W + X + \alpha S_0) = \mu_a [ (W + X, F)^a - \alpha U^a F + \beta V^a (W + X) + \gamma U^a (W + X) ] .$$

The corresponding Jacobian $J$ has the form

$$J = \exp\{2\Delta^a F \mu_a\}.$$

Choosing $F = X - W$ and taking into account the generating master equations (4.6), (4.7), the requirement of invariance of the integrand in (4.5) takes the form

$$[2V^a W - 2U^a X - (\alpha U^a + \beta V^a + \gamma U^a) W + (\alpha U^a - \beta V^a - \gamma U^a) X ] \mu_a = 0.$$

These equations are satisfied if we define the operators $V^a, U^a$ by the relations

$$V^a = \frac{1}{2}(\alpha U^a + \beta V^a + \gamma U^a), \quad U^a = \frac{1}{2}(\alpha U^a - \beta V^a - \gamma U^a).$$

(4.9)

Evidently for arbitrary constants $\alpha, \beta, \gamma$ the operators $V^a, U^a$ obey the properties

$$V^{(a} V^{b)} = 0, \quad U^{(a} U^{b)} = 0, \quad V^{(a} U^{b)} + U^{(a} V^{b)} = 0.$$

(4.10)

Therefore, the operators $\Delta^a$, Eq. (3.13), and $V^a, U^a$, Eqs. (4.3) realize the modified triplectic algebra. Obviously, the integrand of vacuum functional (4.5) is invariant under the BRST and antiBRST transformations of the coordinates $z$ defined by the generators

$$\delta^a = (X - W, \cdot)^a + V^a - U^a.$$

(4.11)

Note that the same result concerning the integration measure $D_0$ in the functional integral (4.5) can be obtained by requiring the extended BRST symmetry principle.

Finally, we arrive at the conclusion, that the given construction includes all the $Sp(2)-$covariant quantization schemes listed in Section 2:
• $\alpha = 0$:
  The operators $V^a$, $U^a$ are linearly dependent $U^a = -V^a$. In this case there exists only one set of the vector fields $V^a$ under consideration and the modified triplectic algebra is reduced to the triplectic one.

• $\alpha = 0, \beta = 2, \gamma = 0$:
  In this case one obtains $V^a = V^a$ in fact representation of vector fields used in constructing the triplectic quantization in general coordinates [5]. In Darboux coordinates the vector fields $V^a = \frac{1}{2}(\hat{V}_0^a + \hat{U}_0^a)$, $U^a = -V^a$ coincide with ones used in original version of the triplectic quantization [3].

• $\alpha = 1, \beta = 2, \gamma = 0$:
  Using Darboux coordinates in the operators (3.24), (3.25) we reproduce the vector field $V^a = \hat{V}_0^a$, $U^a = -\hat{U}_0^a$ adopted in [3] as well as in the modified triplectic quantization [8]. In this case $\rho = const$, $S_0 = \hat{\phi}_a^A \pi^A$ and the vacuum functional (4.5) coincides with that within the modified triplectic quantization (2.26).

In conclusion, we developed the extended BRST invariant quantization scheme in Lagrangian formalism which includes all the ingredients appeared in realization of the modified triplectic algebra. The formulation exactly uses the integration measure extracted from the even symplectic structure on the whole space of fields and antifields. It allows to consider all existing covariant quantization schemes with extended BRST symmetry as special limits.

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Appendix

A Algebra of operators $\Delta^a$, $V^a$ and $(\ , \ )^a$

Let us consider algebra of operators $\Delta^a$ (3.15), $V^a$ (3.17) and the operations $(\ , \ )^a$ (3.14). Straightforward calculations yield the following results

$$\Delta^{(a}\Delta^{b)} = \hat{R}^{ab}, \quad \hat{R}^{ab} = R^k_{lij} \theta^a_{kc} \frac{\partial}{\partial \theta^a_{lc}} \frac{\partial}{\partial \theta^a_{jb}}, \quad (A.1)$$

$$V^{(a} V^{b)} = \frac{1}{4} R_{mij}^k \theta^a_{mc} \theta^b_{kc} \frac{\partial}{\partial \theta^a_{mc}}, \quad (A.2)$$

$$2(\Delta^a V^b + V^b \Delta^a) = \varepsilon^{ab} \omega_{ij} R_{lij}^k \theta^a_{kc} \frac{\partial}{\partial \theta^a_{lc}} + R^k_{lij} \theta^b_{kc} \frac{\partial}{\partial \theta^b_{ja}} \frac{\partial}{\partial \theta^b_{lc}} +$$

$$+ \varepsilon^{ab} (\partial_i \omega_{ij} - \frac{1}{2} \omega_{ij} \frac{\partial \rho}{\partial x^i}) \nabla_j + (\partial_i \Gamma_{jl}^i - \Gamma_{jm}^i \Gamma_{il}^m + \frac{1}{2} \frac{\partial \rho}{\partial x^i} \Gamma_{jl}^i - \frac{1}{2} \frac{\partial^2 \rho}{\partial x^i \partial x^j}) \theta^{jb} \frac{\partial}{\partial \theta^a_{la}}. \quad (A.3)$$
The function $\rho$ satisfies the relations
\[
\partial_i \omega^{ij} = \frac{1}{2} \omega^{ji} \frac{\partial \rho}{\partial x^i}.
\] (A.4)

In particular this means that
\[
\frac{1}{2} \frac{\partial \rho}{\partial x^i} = \Gamma_{ij}^i.
\] (A.5)

It is not difficult to solve the equation (A.4) for $\rho$. Indeed, it holds,
\[
\frac{\partial \rho}{\partial x^k} = 2 \omega_{kj} \frac{\partial}{\partial x^i} \omega^{ij} = \left( \omega_{kj} \frac{\partial}{\partial x^i} \omega^{ij} + \omega_{ik} \frac{\partial}{\partial x^j} \omega^{ij} \right) = -\omega_{ji} \frac{\partial \omega^{ij}}{\partial x^k},
\] (A.6)

and therefore
\[
\rho = -\log \det \omega^{ij} = \log \det \omega_{ij}.
\] (A.7)

where the Jacobi identities in the form $\omega_{kj} \frac{\partial}{\partial x^i} \omega^{ij} + \omega_{ik} \frac{\partial}{\partial x^j} \omega^{ij} + \omega_{ji} \frac{\partial}{\partial x^k} \omega^{ij} = 0$ were used.

From (3.13) the final expression for the anticommutator (A.3) follows:
\[
\Delta^a V^b + V^b \Delta^a = \frac{1}{2} \left( \epsilon^{ab} \omega^{ij} R_{lijk} \theta_{kc} \frac{\partial}{\partial \theta_{lc}} + R_{lijk} \theta_{kc} \theta^{ib} \frac{\partial}{\partial \theta_{ja}} \frac{\partial}{\partial \theta_{lc}} + R_{lijk} \theta^{ib} \frac{\partial}{\partial \theta_{la}} \right).
\] (A.8)

In its turn the actions of $\Delta^a$ and $V^a$ on the antibrackets $(\ , \ )^a$ are given by the rule
\[
\Delta^a (f, g)^b - (\Delta^a f, g)^b + (-1)^{\epsilon(f)} (f, \Delta^a g)^b = (-1)^{\epsilon(f)} \left( \hat{R}_{ab}^c (fg) - (\hat{R}_{ab}^c)g - f(\hat{R}_{ab}^c) \right),
\] (A.9)

\[
V^a (f, g)^b - (V^a f, g)^b + (-1)^{\epsilon(f)} (f, V^a g)^b = \frac{1}{2} R_{lijk} \theta^{ka} \theta_{kc} \left( \frac{\partial f}{\partial \theta_{ja}} \frac{\partial g}{\partial \theta_{lb}} - \frac{\partial f}{\partial \theta_{ja}} \frac{\partial g}{\partial \theta_{lb}} \right).
\] (A.10)

The properties of compatibility for $(\ , \ )^a$ have the form
\[
(-1)^{\epsilon(f)+\epsilon(h)+1} (f, (g, h)^{(a)})^b + \text{cycl. perm. } (f, g, h) = \\
= \left( \hat{R}_{ab}^c (fg) - f \hat{R}_{ab}^c (gh) - \hat{R}_{ab}^c (fg)h - \hat{R}_{ab}^c (fh)g(-1)^{\epsilon(h)}(g) + \\
(\hat{R}_{ab}^c)gh + f(\hat{R}_{ab}^c)g + f(\hat{R}_{ab}^c)h \right) (-1)^{\epsilon(f)+\epsilon(h)+\epsilon(f)+\epsilon(g)+\epsilon(h)}.
\] (A.11)

Therefore, on a flat Fedosov manifold, $R_{lijk}^k = 0$, an explicit realization of the strong triplectic algebra is constructed.
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