Universal behaviour of interfaces in 2d and dimensional reduction of Nambu-Goto strings

M. Billó, M. Caselle, L. Ferro

Dipartimento di Fisica Teorica, Università di Torino
and Istituto Nazionale di Fisica Nucleare - sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy

ABSTRACT: We propose a simple effective model for the description of interfaces in 2d statistical models, based on the first-order treatment of an action corresponding to the length of the interface. The universal prediction of this model for the interface free energy agrees with the result of an exact calculation in the case of the 2d Ising model. This model appears as a dimensional reduction of the Nambu-Goto stringy description of interfaces in 3d, i.e., of the capillary wave model.

KEYWORDS: Lattice Gauge Field Theories, Interfaces, Nambu-Goto String.

*Work partially supported by the European Community’s Human Potential Programme under contract MRTN-CT-2004-005104 “Constituents, Fundamental Forces and Symmetries of the Universe” and by the European Commission TMR programme HPRN-CT-2002-00325 (EUCLID) and by the Italian M.I.U.R under contract PRIN-2005023102 “Strings, D-branes and Gauge Theories”.
1. Introduction

The appearance of interfaces in statistical systems under particular conditions has always raised much interest in various fields of research, ranging from condensed matter to high energy physics. Recently, there have been remarkable theoretical and computational improvements in the study of this phenomenon.

Numerical investigations of interfaces in statistical systems (and of their analogue in Lattice Gauge Theories) have attained a great level of accuracy and reliability. In particular, the interface free energy in the Ising 3d model has been thoroughly studied by means of Monte Carlo simulations. Indeed, spin models provide a simple realization of interfaces since in their broken symmetry phase an interface separating coexisting vacua of different magnetization can be easily induced in the system by suitably choosing the boundary conditions.

On the theoretical side, different effective models can be used to describe the behaviour of interfaces in 3d systems and in particular to evaluate their free energy. The most popular is the capillary wave model (CWM)[1, 2] which is based on the assumption of an action proportional to the area of the surface swept by the interface (for a review, see for instance [3]). This model is tantamount to consider the interfaces as bosonic strings embedded in three dimensions, with a Nambu-Goto action [4, 5]. Although this effective string approach neglects the conformal anomaly which appears for target space dimensions $d \neq 26$, the effects of this approximation appear to be subleading [6] for large worldsheets. This description has exactly the same nature of the effective string description of certain observables, such as Polyakov and Wilson loops, in LGT (see for instance [7, 8] and references therein).

The standard procedure to treat the effective Nambu-Goto string [9] in the last twenty years was to fix the so called ”physical gauge” and expand perturbatively the partition
function around the classical solution (the surface of minimal area) in terms of the inverse product of the string tension $\sigma$ with the area of the surface.

The increasing precision of recent numerical simulations allowed in the last few years to test this expansion beyond the leading order [10, 11]. Resorting to the first order formulation à la Polyakov of the NG model, it has been possible to re-sum the loop expansion obtained in the physical gauge-fixing, in particular in the case of Polyakov correlators [12] and of interfaces [13]. This treatment of the NG model yields a very good agreement with Monte Carlo results, at least for sufficiently large world-sheets; from this point of view, the comparison with MC data sets establishes a lower bound on the world-sheet area, below which one cannot neglect the effects of the conformal anomaly.

Our paper lays in this context. We focus on the universal properties of periodic interfaces arising in general 2d models. Following the philosophy of the CWM we make the simple assumption that the weight of the interface is proportional to its length (which indeed corresponds to a 2d version of the CWM) and treat at the quantum level this action by means of its first-order formulation. In this way we show in section 2 that the dominant term in the partition function acquires an universal form, proportional to $L m K_1(m R)$, where $m$ is the inverse of the correlation length, $R$ is the lattice size in the direction of the interface, $L$ the lattice size orthogonal to the interface and $K_1$ is the modified Bessel function of the first order.

In section 3 we consider an explicit statistical model, the 2d Ising model. Upon a suitable choice of boundary conditions, this system allows the formation of interfaces. The 2d Ising model is under full analytic control, and we can derive directly the form of the dominant term in the interface partition function. We find agreement with the universal expression predicted by our 2d CWM.

In section 4, we consider the exact expression of the free energy for interfaces in 3d obtained in [12], and we perform a dimensional reduction in one of the directions along the interface. We retrieve in this way the 2d expression proportional to $K_1(m R)$ which we propose in this paper. This dimensional reduction is the exact analogue for the interface boundary conditions of the dimensional reduction from Polyakov loop correlators to spin-spin correlators studied in [14, 15]. A non trivial consistency test of this correspondence is that the relation which links the 2d mass $m$ with the 3d string tension $\sigma$ is the same in the case of interfaces and of Polyakov loops.

Recently, a new set of high precision Monte Carlo data for the free energy of interfaces in the 3d Ising model became available [16]. These data include asymmetric geometries in which one of the sides of the interface becomes much smaller than the other; in this situation one expects that the dimensionally reduced expression can describe the data accurately. In the last section, we test this expectation comparing these data with the predictions of the full 3d Nambu-Goto treatment and of the 2d simple model described here.

2. A simple model for 2d interfaces

Consider a physical system, defined on a two-dimensional space with periodic boundary conditions in both directions, in which two different phases, separated by a one-dimensional
We will describe this situation in a very simple way, analogous to the so-called capillary wave model [1, 2] for two-dimensional interfaces in 3d systems. We make the assumption that the effective action is just given by the length of the interface $\Gamma$:

$$S[x] = m \int_{\Gamma} ds = m \int_{0}^{1} d\tau \sqrt{(\dot{x})^2},$$  

(2.1)

where $\tau$ parametrizes the curve $\Gamma$, the dots denotes $\tau$ derivatives, $(\dot{x})^2 \equiv \dot{x}_i \dot{x}^i$ and $m$ is a constant\(^1\) with the dimension of an inverse length. This is just the analogue of the conjecture which, in three dimensions, leads to the CWM [1, 2].

The corresponding partition function is given by the functional integral

$$Z = \int Dx e^{- S[x]},$$  

(2.2)

and it should correspond to the interface free energy, in analogy to what happens in the stringy description of 3d interfaces [13].

The action eq. (2.1), as well known, admits a first-order reformulation:

$$S[x, e] = \frac{m}{2} \int_{0}^{1} d\tau \left( \frac{(\dot{x})^2}{e} + e \right),$$  

(2.3)

where $e(\tau)$ is the einbein\(^2\), so that the partition function can be written as

$$Z = \int De De^{- S[x,e]}.$$  

(2.4)

The action eq. (2.3) is invariant under reparametrizations of $\Gamma$:

$$\delta \tau = \epsilon,$$

$$\delta e = \dot{\epsilon} e + \epsilon \ddot{e}.$$  

(2.5)

\(^1\)We can view $m$ as the “mass” of a relativistic particle in an Euclidean target space (the torus) whose world-line is the interface.

\(^2\)Namely one has $ds^2 = e^2(\tau) d\tau^2$
We can gauge-fix this invariance\(^3\) by choosing \(e(\tau) = \lambda = \text{const.}\), so that the action becomes

\[
S[x, e] = \frac{m \lambda}{2} + \frac{m}{2 \lambda} \int_0^1 d\tau (\dot{x})^2 .
\]

(2.6)

The constant \(\lambda\) represents the length of the path and is a Teichmüller parameter: it cannot be changed using the reparametrizations in eq. (2.5). As such, it must be integrated over in the partition function, which now reads

\[
Z = \int_0^\infty d\lambda e^{-\frac{m \lambda}{2}} Z_x Z_{\text{gh}} .
\]

(2.7)

Here

\[
Z_x = \int D x \exp \left\{ -\frac{m}{2 \lambda} \int_0^1 d\tau (\dot{x})^2 \right\}
\]

(2.8)

and we took into account the Faddeev-Popov ghosts \(b, c\) for the chosen gauge-fixing of the invariances in eq. (2.5). For them we have, by standard treatment,

\[
Z_{\text{gh}} = \int D b D c \exp \left\{ -\lambda \int_0^1 d\tau b \partial_\tau c \right\} .
\]

(2.9)

With the chosen periodic boundary conditions on \(\tau\), it turns out that

\[
Z_{\text{gh}} = \frac{1}{\lambda} .
\]

(2.10)

The partition function \(Z_x\) is simply that of a non-relativistic particle of mass \(\mu = m/\lambda\), whose Hamiltonian is

\[
H = \frac{p^2}{2\mu} .
\]

(2.11)

Since the particle moves on a two-dimensional torus of sides \(L_1, L_2\), the components of the momentum are quantized:

\[
p^i = \frac{2\pi n^i}{L_i} ,
\]

(2.12)

for \(i = 1, 2\), with \(n^i \in \mathbb{Z}\). The partition function of this system is

\[
\sum_{\{n_i\}} \exp \left\{ -\frac{4\pi^2}{2\mu} \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \right) \right\} = \frac{L_1 L_2 \mu}{2\pi} \sum_{\{m_i\}} \exp \left\{ -\frac{\mu^2}{2} \left( L_1^2 n_1^2 + L_2^2 n_2^2 \right) \right\} .
\]

(2.13)

In the second step above we performed a Poisson re-summation, which reorganizes the sum in contributions of sectors describing quantum fluctuations around classical solutions of wrapping numbers \(m_i\).

To describe an interface such as the one depicted in Fig. 1, we set

\[
m_1 = 0 , \quad m_2 = 1 ,
\]

(2.14)

\(^3\)Notice that in the first-order formulation we have to take into account the constraints which follow from the variation of \(S\) w.r.t. the einbein \(e\), which (in the chosen gauge) are simply \((\dot{x})^2 = \lambda\).
getting therefore (if we remember that $\mu = m/\lambda$)

$$Z_x = L_1 L_2 \frac{m}{2\pi \lambda} \exp \left\{ -\frac{m}{2\lambda} L_2^2 \right\}.$$ \hspace{1cm} (2.15)

Inserting eq. (2.10) and eq. (2.15) into the expression eq. (2.7) of the total partition function we obtain finally

$$Z = L_1 L_2 \frac{m}{2\pi} \int_0^\infty d\lambda \frac{1}{\lambda^2} \exp \left\{ -\frac{m}{2\lambda} \lambda L_2^2 \right\} = \frac{L_1 m}{\pi} K_1(mL_2),$$ \hspace{1cm} (2.17)

where $K_1$ is the modified Bessel function of first order.

In the next section, we will consider a concrete physical system where the interface free energy can be computed directly, namely the 2d Ising model, and we will find that our proposed general expression eq. (2.17) does indeed capture its dominant behaviour.

3. Interfaces in the 2d Ising model

Interfaces can be realized in the context of spin models, since under peculiar boundary conditions their broken symmetry phase allows separated but coexisting vacua of different magnetization. Being interested in interfaces in 2d systems, it is natural to consider the 2d Ising model, where an explicit analytical treatment is viable.

To evaluate the interface free energy of the 2d Ising model in a simple way, we can take advantage of the fact that the continuum limit of the Ising model is described by the Quantum Field Theory of a free fermionic field of mass $m$. This framework allows the calculation of the partition function with any type of boundary conditions. Being interested in estimating the interface free energy, we will impose periodic boundary conditions in one direction and antiperiodic ones in the other direction. Explicit expressions of various partition functions can be found in [17] and (using the $\zeta$ function regularization) in [18] \(^5\).

It’s worth pointing out that the solution proposed in [17] is nothing else than the continuum limit of Kaufmann solution [19]. Such continuum limit was performed by Ferdinand and Fischer in [20] at the critical point and the result of [17] is essentially the extension of [20] outside the critical point.

Using the notations of [17], eq.s (91)-(105), the partition function of a fermion on a rectangular torus of sizes $L_1$ and $L_2$, which for notational simplicity we shall henceforth denote as $L$ and $R$, is given by the so-called “massive fermion determinant”:

$$D_{\alpha,\beta}(m|L, R) = e^{-\delta_{\beta}\pi L \zeta(\beta)/6R} \prod_{n \in \mathbb{Z} + \beta} (1 - \delta_{\alpha} e^{-L \zeta(\beta)/R}).$$ \hspace{1cm} (3.1)

Here $\alpha, \beta$ can take the values 0, 1/2 and label the boundary conditions in the $L$ and $R$ directions respectively: the value 0 corresponds to periodic b.c.s, 1/2 to antiperiodic ones.

\(^4\)We use the integral

$$\int_0^\infty \frac{d\lambda}{\lambda^a} e^{-A^2 \lambda - \frac{\gamma^2}{\lambda}} = 2 \left( \frac{A}{B} \right)^{a-1} K_{a-1}(2AB).$$ \hspace{1cm} (2.16)

\(^5\)Notice however that in [18] there is a sign mistake which was later corrected in [17].
Moreover, \( \delta_\alpha = e^{2\pi i \alpha} \) and
\[
\frac{c_n(r)}{R} = \sqrt{m^2 + \left(\frac{2\pi n}{R}\right)^2}.
\] (3.2)

The adimensional variable \( r \equiv mR \) sets the scale of the theory.

Depending on the boundary conditions, the coefficient \( c_\beta(r) \) can assume the following values:
\[
c_\frac{1}{2}(r) = \frac{6r}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} K_1(kr)
\] (3.3)
or
\[
c_0(r) = \frac{6r}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k} K_1(kr).
\] (3.4)

The partition function of the Ising model with periodic boundary conditions in both directions can be written as follows:
\[
Z_{\text{Ising}}(m|L, R) = \frac{1}{2} \left( D_{\frac{1}{2}, \frac{1}{2}} + D_{0, \frac{1}{2}} + D_{\frac{1}{2}, 0} - D_{0, 0} \right).
\] (3.5)

If the \( L \) direction is antiperiodic, the partition function becomes
\[
Z_{\text{Ising, ap}}(m|L, R) = \frac{1}{2} \left( D_{\frac{1}{2}, \frac{1}{2}} + D_{0, \frac{1}{2}} - D_{\frac{1}{2}, 0} - D_{0, 0} \right).
\] (3.6)

This is the situation which generates an odd number of interfaces along the \( R \) direction. The interface free energy will be given by the following expression:
\[
e^{-F_{\text{interface}}} \equiv Z_{\text{interface}} = \frac{Z_{\text{Ising, ap}}}{Z_{\text{Ising}}}.
\] (3.7)

Notice, as a side remark, that taking the limit \( r \to 0 \), one moves to the critical point and all the expressions written above flow toward the conformal invariant ones and agree with the standard CFT results and with those reported in the pioneering work of Ferdinand and Fisher [20].

We are now interested in the large \( mL \) and \( mR \) limit of eq.(3.7). The large \( mL \) limit allows us to neglect the infinite product in eq.(3.1) while in the large \( mR \) we approximate
\[
c_\frac{1}{2}(r) \sim c_0(r) \sim \frac{6r}{\pi^2} K_1(r).
\] (3.8)

The partition functions in eq.(3.5) and (3.6) simplify and we end up with
\[
Z_{\text{Ising}} \sim 1,
\]
\[
Z_{\text{Ising, ap}} \sim \frac{\pi L}{6R} \left( c_\frac{1}{2}(r) + c_0(r) \right),
\] (3.9)

from which we find
\[
Z_{\text{interface}} \sim \frac{\pi L}{6R} \left( c_\frac{1}{2}(r) + c_0(r) \right) \sim \frac{2Lr}{\pi R} K_1(r) = \frac{2Lm}{\pi} K_1(mR).
\] (3.10)

By comparing this result to eq. (2.17), we see that the behaviour of the interface free energy in the 2d Ising model for large scales is perfectly captured by our simple effective model.

\[\text{In this regime, one must keep in mind that the parameter } \tau \text{ of [20] corresponds to } r/2.\]
4. Dimensional reduction of the Nambu-Goto effective description of 3d interfaces

In this section we shall discuss the behaviour of the Nambu-Goto effective string description of fluctuating interfaces when dimensional reduction occurs along one of the two lattice sizes which define the interface.

As we discussed above, interfaces can be realized in spin models. In this realization, the physical ideas underlying the dimensional reduction process are perhaps best discussed. Focusing on the 3d Ising model, let us recall that it is mapped by duality into the 3d Ising gauge model. Under this mapping, the interface free energy becomes a close relative of the Wilson loop expectation value of the 3d gauge Ising model, the only difference being in the boundary conditions of the two observable: fixed in the Wilson loop case and periodic in the interface case.

In lattice gauge theories, dimensional reduction has a very important physical meaning. The size of the lattice direction which is reduced is interpreted as the inverse temperature of the system and the point in which dimensional reduction occurs coincides with the deconfinement phase transition. The behaviour of the model in the vicinity of the deconfinement transition is well described, according to the Svetitsky-Yaffe conjecture [21], by the 2d Ising spin model. More generally, for other gauge groups with continuous deconfinement phase transitions, it is described by the 2d spin model with global symmetry group the center of the original gauge group.

As a consequence of this observation, we expect that the interface free energy of the 3d Ising model should smoothly map into the 2d one when the dimensional reduction scale is approached from above, while it should vanish below it, since in the dual model this is the deconfined phase. Any reliable effective description of the free energy should therefore be compatible with these two requirements. This is a rather non trivial test for the effective string models which are expected to effectively describe both the Wilson and Polyakov loop expectation value and the interface free energy.

In this section, we start from the expression for the interface free energy in 3d obtained in [13] and consider its behaviour under dimensional reduction; we find that it does indeed smoothly reduce to the 2d expression given here in eq. (2.17). This represents a test of our effective approach to the interface free energy both in three and in two dimensions.

In the next section, we will refine this test by comparing both the 3d and the 2d theoretical expressions to Monte Carlo data for asymmetric interfaces: the more asymmetric are the interfaces, the more we expect the prediction of our 3d and 2d expression to become compatible and reliable.

Let us consider a three-dimensional torus of sides $L_{1,2,3}$, and assume that the interface lies in the plane orthogonal to $L_1$. Our starting point is the expression for the 3d interface partition function in the Nambu-Goto approximation evaluated in [13] (eq.s (2.23, 2.24)):

$$\mathcal{Z}^{(3)} = 2 \left( \frac{\sigma}{2\pi} \right)^{\frac{1}{2}} L_1 \sqrt{\sigma A u} \sum_{k,k'=0}^\infty c_k c_{k'} \left( \frac{E_u}{u} \right) K_1 (\sigma A E) ,$$  \hspace{1cm} (4.1)
where
\[ A = L_3 L_2 \, , \, \frac{L_2}{L_3} \] (4.2)
and
\[ E = \sqrt{1 + \frac{4\pi u}{\sigma A} (k + k' - \frac{1}{12}) + \frac{4\pi^2 u^2 (k - k')^2}{(\sigma A)^2}} \] (4.3)
is a function of the occupation numbers \( k \) and \( k' \). The coefficients \( c_k \) and \( c_{k'} \) are the number of partitions of \( k \) and \( k' \).

The dimensional reduction of eq. (4.1) can be most easily discussed by writing the interface free energy as a function of \( L_3 \). Following the notations of [13], let us introduce the “energy levels” \( E_{k,k'} \equiv \sigma L_3 \mathcal{E} \). With this choice the argument of the Bessel functions in eq.(4.1) becomes \( L_2^2 E_{k,k'} \) and we have:

\[ E_{k,k'} = \sqrt{\sigma^2 L_3^2 + 4\pi\sigma (k + k' - \frac{1}{12}) + \frac{4\pi^2 (k - k')^2}{L_3^2}}. \] (4.4)

From this expression it is evident that, as the ratio \( L_2/L_3 \) increases, the separation between consecutive levels in the spectrum becomes larger and larger and, in particular, the gap between the lowest state \( k = k' = 0 \) and the second one increases.

In the dimensional reduction limit \( L_3 \ll L_2 \) we can thus truncate the sum in eq.(4.1) to the first term. In this limit, the argument of the Bessel function in eq.(4.1) becomes large, allowing us to use the asymptotic expansion

\[ K_j(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + O \left( \frac{1}{z} \right) \right\}, \] (4.5)

from which we see that higher states are exponentially suppressed with respect to the \( k = k' = 0 \) one. Setting \( k = k' = 0 \), we get

\[ \mathcal{I}^{(3)}_{(0,0)} = 2 \left( \frac{\sigma}{2\pi} \right)^{1/2} L_1 \sqrt{\sigma A u} \left( \frac{\mathcal{E}}{u} \right) K_1 (\sigma A \mathcal{E}) , \] (4.6)

where now
\[ E = \sqrt{1 - \frac{\pi}{3\sigma A} u} = \sqrt{1 - \frac{\pi}{3\sigma L_3^2}} . \] (4.7)

Equation (4.6) can therefore be rewritten as:

\[ \mathcal{I}^{(3)}_{(0,0)} = \left( \frac{2}{\pi} \right)^{1/2} L_1 \sigma L_3 \sqrt{1 - \frac{\pi}{3\sigma L_3^2}} K_1 \left( \sigma L_3 L_2 \sqrt{1 - \frac{\pi}{3\sigma L_3^2}} \right) . \] (4.8)

If we now define
\[ m_{\text{eff}} = \sigma L_3 \sqrt{1 - \frac{\pi}{3\sigma L_3^2}} \] (4.9)
we find
\[ \mathcal{I}^{(3)}_{(0,0)} = \left( \frac{2}{\pi} \right)^{1/2} L_1 m_{\text{eff}} K_1 (m_{\text{eff}} L_2) , \] (4.10)
which is in perfect agreement with the partition function given by (2.17) a part from the normalization constant.

It is clear from eq. (4.7) that the above discussion is consistent only for

$$L_3 \geq \sqrt{\frac{\pi}{3\sigma}}.$$  \hspace{1cm} (4.11)

In the string language, this bound represents the tachyonic singularity of the bosonic string: for $L_3 \leq \sqrt{\frac{\pi}{3\sigma}}$ the lowest state has a negative energy. In the 3d Ising model framework, the critical value

$$L_{3,c} = \sqrt{\frac{\pi}{3\sigma}} = 1.023... \times \frac{1}{\sqrt{\sigma}}$$  \hspace{1cm} (4.12)

at which the mass of the lowest state vanishes can be considered as the effective string prediction for the dimensional reduction scale which we were looking for. We thus see that this scale is naturally embedded in the NG formulation of the interface free energy. The same argument in the dual case of the Polyakov loop correlator leads to the identification of this scale with deconfinement temperature [22]. The value obtained in this way for the dimensional reduction scale (or, equivalently, for the deconfinement temperature in the dual model) is in rather good agreement with the Monte Carlo estimates. For the 3d gauge Ising model the disagreement is only of about 20%, as the Monte Carlo estimate turns out to be $\sqrt{\sigma}L_c = 0.8186(16)$ [23] instead of 1.023... and it further decreases if one looks at $SU(N)$ pure Yang Mills theories.

It is important to stress that the mapping between 3d and 2d observables which emerges from this comparison is exactly the same which is found considering the finite temperature behaviour of the Polyakov loop correlators in the vicinity of the deconfinement transition in the 3d gauge Ising model. In particular, the relation between the mass scale of the 2d model and the string tension (in the present case the interface tension) of the 3d one reported in eq. (4.9) is the same which one finds in the Polyakov loop case. This is a rather non trivial consistency check of the whole dimensional reduction picture.

As a last remark let us stress that the smooth flow of the 3d Nambu-Goto result towards the 2d ones under dimensional reduction has various important implications on our understanding of the Nambu-Goto effective action. Recall that the treatment leading to eq. (4.1) neglected the conformal anomaly, i.e., equivalently, did not take into account the Liouville field which does not decouple in the Polyakov formulation of the string when $d \neq 26$. Most probably the smooth flow toward the 2d result occurs exactly because we neglected the Liouville field contribution. It is thus interesting to compare simultaneously the Nambu-Goto expression (without Liouville field), the 2d result presented here and the output of the Monte Carlo simulations for the free energy of interfaces on lattices with a value of $L_3$ small enough to make the 2d approximation a reliable one. We shall devote the next section to this comparison.

5. Comparison with the numerical data

Following the above discussion we compare in this section the expressions for the 2d interface free energy given in eq. (2.17) and for the 3d free energy [13], reported in eq. (4.1),
Table 1: Numerical results of the free energy in the 3d Ising model at $\beta = 0.223101$ are compared to various theoretical estimates. See the main text for detailed explanations.

| $L_3$ | $L_2$ | $L_1$ | $F_{\text{num}}$ | $F$ | $F_{(0,0)}$ | $F_{\text{1st}}$ | $F_{1\text{-loop}}$ | $F_{2\text{-loop}}$ |
|-------|-------|-------|-------------------|-----|-------------|-----------------|-----------------|-----------------|
| 24    | 64    | 96    | 6.8855(20)        | 6.7495 | 6.7495      | 6.9974         | 6.8347         |
| 28    | 64    | 96    | 7.6929(21)       | 7.6537 | 7.6537      | 7.7875         | 7.6821         |
| 32    | 64    | 96    | 8.4626(20)       | 8.4518 | 8.4518      | 8.5380         | 8.4632         |
| 36    | 64    | 96    | 9.1996(21)       | 9.2012 | 9.2012      | 9.2632         | 9.2062         |
| 40    | 64    | 96    | 9.9227(23)       | 9.9231 | 9.9231      | 9.9713         | 9.9253         |
| 44    | 64    | 96    | 10.6203(23)      | 10.6278 | 10.6278     | 10.6674        | 10.6288        |
| 48    | 64    | 96    | 11.3138(25)      | 11.3209 | 11.3209     | 11.3548        | 11.3213        |

Table 2: Numerical results of the free energy in the 3d Ising model at $\beta = 0.226102$ are compared to various theoretical estimates. See the main text for detailed explanations.

| $L_3$ | $L_2$ | $L_1$ | $F_{\text{num}}$ | $F$ | $F_{(0,0)}$ | $F_{\text{1st}}$ | $F_{1\text{-loop}}$ | $F_{2\text{-loop}}$ |
|-------|-------|-------|-------------------|-----|-------------|-----------------|-----------------|-----------------|
| 24    | 64    | 96    | 18.4131(26)      | 18.4121 | 18.4121     | 18.4555        | 18.4152        |
| 28    | 64    | 96    | 21.2414(27)      | 21.2450 | 21.2450     | 21.2724        | 21.2463        |
| 32    | 64    | 96    | 24.0310(27)      | 24.0306 | 24.0306     | 24.0497        | 24.0312        |
| 36    | 64    | 96    | 26.7859(28)      | 26.7873 | 26.7873     | 26.8017        | 26.7875        |
| 40    | 64    | 96    | 29.5271(30)      | 29.5250 | 29.5250     | 29.5365        | 29.5251        |
| 44    | 64    | 96    | 32.2449(31)      | 32.2498 | 32.2498     | 32.2594        | 32.2498        |
| 48    | 64    | 96    | 34.9623(33)      | 34.9653 | 34.9653     | 34.9736        | 34.9653        |

In particular we isolated two sets of data. The first one is reported in Tab. 1 and displays the values of the free energies obtained in the 3d Ising model at $\beta = 0.223101$. The interface string tension for this value of $\beta$ is $\sigma = 0.0026083$ and the dimensional reduction scale is exactly 16 lattice spacings [23]. The data correspond to values of the $L_3$ size ranging from a minimum value of $3/2$ the dimensional reduction scale up to 3 times.

In the table we report, from left to right, the lattice sizes $L_{1,2,3}$, the numerical estimate $F_{\text{num}}$ for the free energy (with its statistical error), the theoretical estimate $F$ according to eq. (4.1), the estimate $F_{(0,0)}$ obtained in the full dimensional reduction limit by truncating eq. (4.1) to $k = k' = 0$, which corresponds to our 2d formula eq. (2.17), and the estimate $F_{1\text{st}}$ in which also the first excited states, $k + k' = 1$, are kept into account. Finally, as a comparison, in the last two columns we report the one loop and two loop perturbative approximations to the whole NG action discussed, for instance, in [9, 10].

In Tab. 2 we report the same set of data evaluated at $\beta = 0.226102$, corresponding to $\sigma = 0.0105254$ and to a dimensional reduction scale of 8 lattice spacings [23].

Looking at these tables one can see that the 2d expression $F_{(0,0)}$ always gives a very
good approximation of the whole NG result \( F \): the difference between the two is always smaller than the statistical error on the numerical data. One can also notice that the 2d approximation is more reliable when the dimensional reduction scale \( \sqrt{\pi/(3\sigma)} \) is smaller, i.e. in Tab. 2, and that within each data set it becomes slightly worse as \( L_3 \) increases. Looking at the seventh column we see that the deviation from the full NG result is completely due to the first excited state.

In fact it is easy to evaluate the gap \( \Delta \) between the lowest state and the first excitation which appears at the exponent of the asymptotic expansion of the Bessel function in eq. (4.1). In the large \( \sigma L_3^2 \) limit, the first excited states correspond to \( k = 1, k' = 0 \) or \( k = 0, k' = 1 \) and \( \Delta \) takes the value \( 2 \pi L_2 / L_3 \). When \( \pi / 3 \leq \sigma L_3^2 \leq \pi \) the first excited state is instead the one with \( k = 1, k' = 1 \) and the variation of the gap is

\[
\sqrt{\frac{8}{3}} \frac{\pi L_2}{L_3} \geq \Delta \geq \frac{\pi L_2}{L_3}.
\]

(5.1)

We see that the dominating effect is due to the asymmetry ratio \( L_2 / L_3 \) (which in our data ranges from 8/3 to 4/3) and that a subdominant role is played by the \( \sigma L_3^2 \) combination, in complete agreement with the results reported in Table 1 and 2.

Given this agreement between 2d and whole NG results it is very interesting to compare them both to the Montecarlo data. The agreement is rather good for large values of \( F \), see in particular the results reported in Tab. 2. As \( F \) decreases, however, the NG predictions compare less favorably to the data, see Table 1. As a matter of fact, in this regime the Monte Carlo data seem to better agree with the two loop perturbative expansion than with the whole NG result (and its 2d approximation). This fact has already been discussed in [16] and is most probably due to the fact that the two loop result is actually consistent also at the quantum level, in the sense that at this order the contribution due to the Liouville field vanishes [24, 25, 26, 27].

As expected, these problems disappear if one works exactly in two dimensions, and in fact the direct calculation of the interface partition function in the 2d Ising model reported in sec. 3 shows that the expression eq. (2.17) obtained in section 2 should be valid to all orders.

6. Conclusions

In this paper, we addressed the description of some universal properties of interfaces in two-dimensional physical systems. Using a simple model in which one assumes an action proportional to the length of the interface, in analogy with the 3d capillary wave model, we proposed a general expression for the interface free energy. This expression agrees with an exact calculation in the case of the 2d Ising model. We also show that our 2d result represents the dimensional reduction of the interface 3d free energy obtained using the capillary wave model, which is tantamount to the Nambu-Goto effective string description, when the effects of the conformal anomaly are neglected. Finally, we compared the 3d and 2d theoretical predictions to Monte Carlo numerical estimates for the interface free energy in the case of the 3d Ising model. In the range of values that we studied the
difference between the 3d and 2d estimates is always smaller than the statistical error of
the numerical data. Moreover we were able to show that this difference is completely due
to the first excited state of the Nambu-Goto action.

Acknowledgments

We warmly thank F. Bastianelli, O. Corradini, F. Gliozzi, M.Hasenbusch and M. Panero
for useful discussions.

References

[1] F. P. Buff, R. A. Lovett and F. H. Stillinger Jr., Phys. Rev. Lett. 15 (1965) 621.
[2] J. Rowlinson and S. Widom, *Molecular theory of capillarity*, Clarendon Press, 1982.
[3] V. Privman, Int. J. Mod. Phys. C 3 (1992) 857 [arXiv:cond-mat/9207003].
[4] T. Goto, Prog. Theor. Phys. 46, 1560 (1971).
[5] Y. Nambu, Phys. Rev. D 10, 4262 (1974).
[6] P. Olesen, Phys. Lett. B 160 (1985) 144.
[7] M. Caselle, M. Hasenbusch and M. Panero, JHEP 0503 (2005) 026 [arXiv:hep-lat/0501027].
[8] J. Kuti, PoS LAT2005 (2005) 001 [arXiv:hep-lat/0511023].
[9] K. Dietz and T. Filk, Phys. Rev. D 27 (1983) 2944.
[10] M. Caselle, R. Fiore, F. Gliozzi, M. Hasenbusch, K. Pinn and S. Vinti, Nucl. Phys. B 432
    (1994) 590 [arXiv:hep-lat/9407002].
[11] M. Caselle, M. Hasenbusch and M. Panero, JHEP 0603 (2006) 084 [arXiv:hep-lat/0601023].
[12] M. Billo and M. Caselle, JHEP 0507 (2005) 038 [arXiv:hep-th/0505201].
[13] M. Billo, M. Caselle and L. Ferro, JHEP 0602 (2006) 070 [arXiv:hep-th/0601191].
[14] M. Caselle, G. Delfino, P. Grinza, O. Jahn and N. Magnoli, J. Stat. Mech. 0603 (2006) P008
    [arXiv:hep-th/0511168].
[15] M. Caselle, P. Grinza and N. Magnoli, J. Stat. Mech. 0611 (2006) P003
    [arXiv:hep-th/0607014].
[16] M. Caselle, M. Hasenbusch and M. Panero, arXiv:0707.0055 [hep-lat].
[17] T. R. Klassen and E. Melzer, Nucl. Phys. B 350 (1991) 635.
[18] C. Itzykson and H. Saleur, J. Stat. Phys. 48, 449 (1987).
[19] B. Kaufman, Phys. Rev. 76, 1232 (1949).
[20] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185, 832 (1969).
[21] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B 210 (1982) 423.
[22] P. Olesen, Phys. Lett. B160 (1985) 408.
[23] M. Caselle and M. Hasenbusch, Nucl. Phys. B 470, 435 (1996) [arXiv:hep-lat/9511015].
[24] J. Polchinski and A. Strominger, Phys. Rev. Lett. 67 (1991) 1681.
[25] M. Lüscher and P. Weisz, JHEP 0407 (2004) 014 [arXiv:hep-th/0406205].

[26] J. M. Drummond, arXiv:hep-th/0411017.

[27] N. D. Hari Dass and P. Matlock, arXiv:hep-th/0606265.