DIVISIBILITY PROPERTIES OF THE FOURIER COEFFICIENTS OF
(MOCK) MODULAR FUNCTIONS AND RANANUJAN

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Dedicated to my teacher Bruce C. Berndt on his 80th Birthday.

Abstract. We survey divisibility properties of the Fourier coefficients of modular functions inspired by Ramanujan. Then using recent results of the generalized Hecke operator on harmonic Maass functions and known divisibility of Fourier coefficients of modular functions, we establish congruence relations that the Fourier coefficients of certain mock modular functions satisfy.

1. Introduction

Fourier coefficients of modular forms have rich arithmetic properties and give a wide range of applications. It is in the work of S. Ramanujan where we find important such examples and the rudiment of the arithmetic theory of modular forms.

In his paper [25, 26, 27], Ramanujan stated and proved his three famous congruences for the partition function $p(n)$, namely,

\begin{align*}
(1.1) & \quad p(5n + 4) \equiv 0 \pmod{5}, \\
(1.2) & \quad p(7n + 5) \equiv 0 \pmod{7}, \\
(1.3) & \quad p(11n + 6) \equiv 0 \pmod{11}, 
\end{align*}

where $n$ is any non-negative integer. The partition function is closely related with the \(\tau\)-function that was introduced by Ramanujan in his important paper [24] via

\[ \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^24 = \left( \prod_{n=1}^{\infty} (1 - q^n)^{25} \right) \sum_{n=0}^{\infty} p(n)q^{n+1}. \]

He proved a congruence for $\tau(n)$ as well:

\begin{equation}
(1.4) \quad \tau(n) \equiv \sigma_{11}(n) \pmod{691},
\end{equation}

where $\sigma_k(n)$ is the sum of $k$th powers of divisors of $n$. Ramanujan’s observations on $\tau(n)$ such as

\[ \tau(p^{\alpha+2}) = \tau(p)\tau(p^{\alpha+1}) - p^{11}\tau(p^{\alpha}) \]

for prime $p$ and $\alpha \geq 0$ led to the development of Hecke theory. The proofs of (1.1)–(1.3) in [27] employ Eisenstein series. In [6], B. C. Berndt also uses Eisenstein series and gives simple and unified proofs for (1.1)–(1.3). In fact, the proofs in [27] were extracted from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$ by G. H. Hardy after Ramanujan died in 1920.

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Berndt and K. Ono [7] edited this manuscript with an extensive commentary and references to the literature.

Ramanujan [25] conjectured that there are further congruences for $p(n)$ modulo powers of the primes 5, 7, and 11 and left a sketch of proofs of the conjecture for all powers of 5 and 7 in his manuscript [7]. G. N. Watson [28] completed the proofs. The case for all powers of 11 was proved by A. O. L. Atkin [3], who followed J. Lehner’s approach developed in [16, 19]. For more history and discoveries on congruences for $p(n)$ including two groundbreaking work by Atkin [4] and S. Ahlgren and Ono [1], see [5, 21, 22, 23].

It is too extensive to discuss all divisibility properties of Fourier coefficients of modular forms inspired by Ramanujan. We rather focus on the divisibility of those relating with modular functions. It was Lehner who started investigating the divisibility properties of the Fourier coefficients of a modular function, in particular, the modular $j$-invariant. For $\tau$ with $\text{Im}\tau > 0$ and $q = \exp(2\pi i \tau)$, the classical modular $j$-invariant $j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} b(n)q^n$ is a modular function on the full modular group, that is, a weight 0 weakly holomorphic modular form of level 1. The Fourier coefficients of this fundamental function in modular form theory and number theory are now known as the graded dimensions of the Monster module. Expanding the methods used in his proofs of Ramanujan’s partition congruences, Lehner showed in [17] that

$$b(5^{\alpha+1}n) \equiv 0 \pmod{5^\alpha},$$
$$b(7^{\alpha}n) \equiv 0 \pmod{7^\alpha},$$
$$b(11^2n) \equiv 0 \pmod{11^2},$$

for any positive integers $n$ and $\alpha$. In [3], Atkin generalized (1.7) to all positive powers $\alpha$:

$$b(11^\alpha n) \equiv 0 \pmod{11^\alpha}.$$

As he claimed in [3, p.22], congruence in (1.8) is best possible in the sense that $b(11^\alpha) \not\equiv 0 \pmod{11^{\alpha+1}}$. Lehner [18] further showed that the coefficients $b(n)$ are highly divisible by small primes dividing $n$ by proving that

$$b(2^{\alpha}3^\beta 5^\gamma 7^\delta n) \equiv 0 \pmod{2^{3\alpha+8}3^{2\beta+3}5^\gamma 7^\delta}$$

for any positive integers $\alpha$, $\beta$, $\gamma$, and $\delta$. He also claimed that the Fourier coefficients of certain modular functions of higher level also satisfy the same divisibility properties. Recently, these results have been generalized to a modular function of level of genus zero. M. Griffin [10] showed that similar congruences with (1.9) hold for all elements of the canonical basis $j_m(\tau) := q^{-m} + O(q)$ for the space of weakly holomorphic modular functions on the full modular group. N. Anderson and P. Jenkins [2] generalized Lehner’s claim on modular functions of higher level by proving the following theorem:

**Theorem 1.1.** [2, Theorem 2] Let $p \in \{2, 3, 5, 7\}$, and let

$$f_{p,m}(\tau) = q^{-m} + \sum_{n=0}^{\infty} b_p(m, n)q^n$$

be an element of the canonical basis for the space of level $p$ modular functions which are holomorphic at 0, with $m = p^\alpha m'$ and $(m', p) = 1$. Then, for $\beta > \alpha$,

$$b_2(2^{\alpha}m', 2^\beta n) \equiv 0 \pmod{2^{3(\beta-\alpha)+8}},$$
$$b_3(3^{\alpha}m', 3^\beta n) \equiv 0 \pmod{3^{2(\beta-\alpha)+3}},$$
Divisibility properties of Fourier coefficients of modular functions of level of genus zero are further investigated in the series of the papers by Jenkins and his collaborators in [11, 12, 13]. They showed that many of the Fourier coefficients of the basis elements for the space of modular functions of level \( N \) of genus zero are divisible by high powers of the prime dividing the level \( N \).

Ramanujan's mock theta functions are holomorphic parts of harmonic Maass forms. In this spirit and following Zagier, we call the holomorphic part of a harmonic Maass function (i.e., a weight zero harmonic Maass form) by a mock modular function. For each level \( N \), there are certain analytic continuations of Niebur-Poincaré series which form a basis for the space of harmonic weak Maass functions of level \( N \). We denote these basis elements by \( j_{N,m} \). These are considered to be generalizations of \( j_m \)'s, because \( j_{1,m}(\tau) = j_m(\tau) + 24\sigma_1(m) \) for any non-negative integer \( m \) [14, p.99]. The harmonic Maass function \( j_{N,m} \) is decomposed into ([20, Theorem 1], [8, Proposition 3.1])

\[
j_{N,m}(\tau) = j_{h_{N,m}}(\tau) + j_{nh_{N,m}}(\tau),
\]

where the Fourier expansion of the mock modular function \( j_{h_{N,m}} \) is given by

\[
j_{h_{N,m}}(\tau) = q^{-m} + c_N(m,0) + \sum_{n=1}^{\infty} c_N(m,n)q^n
\]

and its corresponding non-holomorphic part \( j_{nh_{N,m}} \) can be written as

\[
j_{nh_{N,m}}(\tau) = -q^m + \sum_{n=1}^{\infty} c_N(m,-n)q^n.
\]

The functions \( j_m \)'s form a Hecke system, that is, for the normalized Hecke operator \( T(m) \), they satisfy that

\[
j_m(\tau) = j_1|T(m)(\tau).
\]

In [15], the author with D. Jeon and C. H. Kim constructed a Hecke system for \( j_{N,m} \) with a suitable generalization of the Hecke operator in (1.13). Precisely, for any positive integers \( N \) and \( m \), we have

\[
j_{N,m}(\tau) = j_{N,1}|T(m)(\tau).
\]

Using this property and its applications, one can obtain several congruences for the Fourier coefficients of mock modular function \( j_{h_{N,m}} \).

**Theorem 1.2.** Let \( p \) be a prime and \( c_p(m,n) \) be the Fourier coefficients of \( j_{h_{p,m}}(\tau) \), the mock modular function of level \( p \) given in (1.12). Then

1. \( pc_p(m,pm) - c_p(pm,n) \in \mathbb{Z} \),
2. if \( p \nmid n \) and \( c_p(m,pm) \in \mathbb{Z} \), then \( c_p(pm,n) \equiv 0 \pmod{p} \).
Theorem 1.3. Let \( p \in \{2, 3, 5, 7\} \) and \( c_p(m, n) \) be the Fourier coefficients of \( j_{p,m}^h(\tau) \). If \((m', p) = 1\), \((n', p) = 1\) and \( \beta > \alpha + 1 \), then

\[
2c_2(2^\alpha m', 2^{\beta+1}n') - 2c_2(2^{\alpha+1}m', 2^\beta n') \equiv 0 \pmod{2^{3(\beta-\alpha)+5}},
\]

\[
3c_3(3^\alpha m', 3^{\beta+1}n') - 3c_3(3^{\alpha+1}m', 3^\beta n') \equiv 0 \pmod{3^{2(\beta-\alpha)+1}},
\]

\[
5c_5(5^\alpha m', 5^{\beta+1}n') - 5c_5(5^{\alpha+1}m', 5^\beta n') \equiv 0 \pmod{5^{(\beta-\alpha)}},
\]

\[
7c_7(7^\alpha m', 7^{\beta+1}n') - 7c_7(7^{\alpha+1}m', 7^\beta n') \equiv 0 \pmod{7^{(\beta-\alpha)-1}}.
\]

In the case when \( p = 3 \) above, we need more condition such that \( m'n' \equiv \pm 1 \pmod{3} \) and \( n'|\sigma_1(m')\sigma_1(n') \).

Unlike modular curves of level 2, 3, 5 and 7, the curve of level 11 is of genus 1 not of genus 0. Hence there is no Hauptmodul on this curve. In order to prove congruence relations relating with 11, Atkin and Lehner, each found a pair of two generators to construct a basis for the space of modular functions of level 11. In [15], reduced form bases for the spaces of the modular functions that are holomorphic away from the cusp at \( \imath\infty \) are constructed for all levels of genus 1. More specifically, each basis element function with level \( N \) of genus 1 is given by

\[
f_{N,m}(\tau) = q^{-m} + a_N(m, -1)q^{-1} + \sum_{n=1}^{\infty} a_N(m, n)q^n, \quad (m \geq 2)
\]

with integer coefficients. Using Atkin’s basis and its properties discovered in [3], we establish a divisibility property of the Fourier coefficients of \( f_{1,m}(\tau) \) modulo powers of 11:

Theorem 1.4. Let \( f_{11,m}(\tau) \) be an element of the basis for the space of level 11 modular functions which are holomorphic at 0 with the Fourier expansion in (1.15). If \( m = 11^\beta m' \geq 2 \) for a non-negative integer \( \beta \) and \((11, m') = 1\), then for any positive integers \( n \) and \( \alpha \),

\[
a_{11}(11^\beta m', 11^\alpha + \beta n) \equiv 0 \pmod{11^\alpha}.
\]

By the relation between \( f_{11,m}(\tau) \) and \( j_{11,m}(\tau) \) provided in (3.4), we find that for any positive integers \( n, \alpha \) and integers \( m \geq 2 \),

\[
c_{11}(m, 11^\alpha n) + a_{11}(m, -1)c_{11}(1, 11^\alpha n) \in \mathbb{Z}.
\]

Moreover, by Theorem 1.4, we have the following congruence:

Corollary 1.5. Let \( j_{11,m}^h(\tau) \) be the mock modular function in (1.12) and \( a_{11}(m, -1) \) denote the Fourier coefficient of \( q^{-1} \) in \( f_{11,m}(\tau) \) as in (1.15). If \( m = 11^\beta m' \geq 2 \) for a non-negative integer \( \beta \) and \((11, m') = 1\), then for any positive integers \( n \) and \( \alpha \),

\[
c_{11}(11^\beta m', 11^\alpha + \beta n) + a_{11}(11^\beta m', -1)c_{11}(1, 11^\alpha + \beta n) \equiv 0 \pmod{11^\alpha}.
\]

In [15], several congruences for the Fourier coefficients of \( f_{N,m}(\tau) \) are found. One of them leads to the following congruence:

Theorem 1.6. Let \( N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\} \) and \( j_{N,m}^h(\tau) \) be the mock modular function in (1.12). Also, let \( a_N(m, -1) \) denote the Fourier coefficient of \( q^{-1} \) in \( f_{N,m}(\tau) \) in (1.15). Then for any prime \( p \) and positive integers \( r, m \) and \( n \) with \( p \nmid m, p \nmid n \), we have

\[
c_N(mp^r, n) + a_N(m, -1)c_N(p^r, n) + (a_N(mp^r, -1) + a_N(m, -1)a_N(p^r, -1))c_N(1, n) \equiv 0 \pmod{p^r}.
\]
The rest of the paper is organized as follows. In Section 2, we introduce the \( p \)-lication formula for the harmonic Maass function \( j_{N,m} \) proved in [15] and congruence relations found in [10]. Theorems 1.2 and 1.3 follow immediately from them. In Section 3, we present the basis element \( f_{N,m} \) and its congruence property discussed in [15]. Proof of Theorem 1.6 then follows. In Section 4, utilizing the \( U \)-operator and applying Atkin’s results in [3], we prove Theorem 1.4 and Corollary 1.5. The last section includes some concluding remarks.

2. Proof of Theorems 1.2 and 1.3

For any positive integer \( N \), let \( M_k^!(N) \) (resp. \( H_k^!(N) \)) denote the space of weight \( k \) weakly holomorphic modular forms (resp. harmonic Maass forms) on \( \Gamma_0(N) \). Also, let \( M_k^!,\infty(N) \) (resp. \( M_k^!,0(N) \)) be its subspace of modular forms whose poles are supported only at infinity (resp. 0) and holomorphic at other cusps. For a prime \( p \), the \( U \)-operator that acts on a complex valued function on \( \mathbb{H} \) is defined by

\[
U_p f(\tau) = \frac{1}{p} \sum_{i=0}^{p-1} f\left(\frac{\tau + i}{p}\right)
\]

and \( U_p^{n+1} f(\tau) = U_p(U_p^m f(\tau)) \) for \( n \geq 1 \). Also for a non-negative integer \( i \), the \( p \)-lication of \( j_{N,m} \) is given by

\[
(j_{N,m})^{(p^i)} := j_{\frac{N}{(N,p^i)},m},
\]

where \( j_{N,m} \) is if \( p \nmid m \). In [15], the following \( p \)-lication formula is established, which was essential in constructing the generalized Hecke operator for \( j_{N,m} \) satisfying (1.14).

**Theorem 2.1.** [15, Theorem 1.5] For any prime \( p \), we have that

\[
p(U_p j_{N,m}(\tau) + j_{N,m}^{(p)}(p\tau)) = j_{N,pm}(\tau) + p j_{N,\frac{m}{p}}^{(p)}(\tau).
\]

Let \( N = p \) in (2.3) so that after rearrangement of terms, we have

\[
p(U_p j_{p,m}(\tau) - j_{p,pm}(\tau)) = p j_{1,\frac{m}{p}}(\tau) - j_{1,m}(p\tau).
\]

Note that the function on the right-hand side of (2.4) is weakly holomorphic, because \( j_{1,m} = j_m + 24\sigma_1(m) \). Hence we can decompose (2.4) into two identities:

\[
p(U_p j_{p,m}(\tau) - j_{p,pm}(\tau)) = p j_{1,\frac{m}{p}}(\tau) - j_{1,m}(p\tau)
\]

and

\[
p(U_p j_{p,m}(\tau) - j_{p,pm}(\tau)) = j_{p,pm}^{(p)}(\tau).
\]

By comparing the coefficients of \( q^n \) in both sides of (2.5), we find that

\[
p c_p(p, pm) - c_p(pm, n) = p c_1\left(\frac{m}{p}, n\right) - c_1(m, \frac{n}{p}).
\]

As the coefficients of \( j_{1,m}(\tau) \) are integers, so are the left-hand side of (2.7), the Fourier coefficients of the mock modular function \( p(U_p j_{p,m}(\tau) - j_{p,pm}(\tau)) \). Theorem 1.2 immediately follows from (2.7).
Applying known congruences of $c_1(m,n)$ into (2.7), one can obtain many congruences for $pc_p(m, pn) - c_p(m, n)$. In particular, we recall Griffin's congruences on the Fourier coefficients of $j_m(\tau)$, the most general congruences for $j_m(\tau)$ known. He used Zagier duality between the Fourier coefficients of weights $k$ and $2 - k$ weakly holomorphic modular forms and properties of Hecke operator.

**Theorem 2.2.** [10, Theorem 2.1] For each $p \in \{2, 3, 5, 7\}$, let $a_1, a_2 \geq 0$, $a = |a_1 - a_2|$, and $b_1, b_2 \not\equiv 0 \pmod{p}$. Then

For $p = 2$:

\[
c_1(2^{a_1} b_1, 2^{a_2} b_2) \equiv -2^{3a+8} 3^{a-1} b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^{3a+13}} \quad \text{if } a_2 > a_1,
\]

\[
\equiv -2^{3a+8} 3^{a-1} b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^{4a+13}} \quad \text{if } a_1 > a_2,
\]

\[
\equiv 20 b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^7} \quad \text{if } a = 0 \text{ and } b_1 b_2 \equiv 1 \pmod{8},
\]

\[
\equiv \frac{1}{2} b_1 \sigma_1(b_1) \sigma_1(b_2) \pmod{2^5} \quad \text{if } a = 0 \text{ and } b_1 b_2 \equiv 3 \pmod{8},
\]

\[
\equiv -12 b_1 \sigma_7(b_1) \sigma_7(b_2) \pmod{2^8} \quad \text{if } a = 0 \text{ and } b_1 b_2 \equiv 5 \pmod{8}.
\]

For $p = 3$:

\[
c_1(3^{a_1} b_1, 3^{a_2} b_2) \equiv +3^{3a+3} 10^{a-1} \sigma_1(b_1) \sigma_1(b_2) \pmod{2^{3a+6}} \quad \text{if } a_2 > a_1 \text{ and } b_1 b_2 \equiv \pm 1 \pmod{3},
\]

\[
\equiv +3^{3a+3} 10^{a-1} \sigma_1(b_1) \sigma_1(b_2) \pmod{3^{3a+6}} \quad \text{if } a_1 > a_2 \text{ and } b_1 b_2 \equiv \pm 1 \pmod{3},
\]

\[
\equiv 2 \cdot 3^3 \sigma_1(b_1) \sigma_1(b_2) \pmod{3^7} \quad \text{if } a = 0 \text{ and } b_1 b_2 \equiv \pm 1 \pmod{3}.
\]

For $p = 5$:

\[
c_1(5^{a_1} b_1, 5^{a_2} b_2) \equiv -5^{a+13} 3^{a-1} b_1^2 b_2 \sigma_1(b_1) \sigma_1(b_2) \pmod{5^{a+2}} \quad \text{if } a_2 > a_1,
\]

\[
\equiv -5^{2a+13} 3^{a-1} b_1^2 b_2 \sigma_1(b_1) \sigma_1(b_2) \pmod{5^{2a+2}} \quad \text{if } a_1 > a_2,
\]

\[
\equiv 10 b_1^2 b_2 \sigma_1(b_1) \sigma_1(b_2) \pmod{5^2} \quad \text{if } a = 0 \text{ and } \left(\frac{b_1 b_2}{5}\right) = -1.
\]

For $p = 7$:

\[
c_1(7^{a_1} b_1, 7^{a_2} b_2) \equiv 7^{2a+1} 5^{a-1} b_1^2 b_2 \sigma_3(b_1) \sigma_3(b_2) \pmod{7^{a+1}} \quad \text{if } a_2 > a_1,
\]

\[
\equiv 7^{2a} 5^{a-1} b_1^2 b_2 \sigma_3(b_1) \sigma_3(b_2) \pmod{7^{2a+1}} \quad \text{if } a_1 > a_2,
\]

\[
\equiv 2 b_1^2 b_2 \sigma_3(b_1) \sigma_3(b_2) \pmod{7} \quad \text{if } a = 0 \text{ and } \left(\frac{b_1 b_2}{7}\right) = 1.
\]

Utilizing each congruence of Theorem 2.2 into (2.7) for $p \in \{2, 3, 5, 7\}$ gives a congruence of $c_p(m, n)$ for a corresponding $p$. Specifically, congruences in Theorem 1.3 are results of employing congruences for the case when $a_2 > a_1$ in Theorem 2.2.

### 3. Proof of Theorem 1.6

Let $N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$. These are the values of $N$ precisely when the modular curves $X_0(N)$ are of genus 1. For each of these $N$, there are two canonical generators
We also define a function of the form \( q^{-2} + O(q^{-1}) \) and \( q^{-3} + O(q^{-1}) \), because the infinity point is not a Weierstrass point of the Riemann surface \( X_0(N) \). Using these, one can construct a basis for \( \mathcal{M}^1_0(N) \) which consists of unique modular functions with integral Fourier coefficients in the form of \( f_{N,m} = q^{-m} + a_N(m, -1)q^{-1} + \sum_{n=1}^{\infty} a_N(m, n)q^n \). Let \( f_{N,0}(\tau) = 1 \), \( f_{N,1}(\tau) = 0 \) and \( a_N(1, -1) = -1 \). For example, when \( N = 11 \), the smallest \( N \) for which \( X_0(11) \) is of genus 1, two generators of \( \mathbb{C}(X_0(11)) \) are found by Y. Yang [29]. They are:

\[
X = q^{-2} + 2q^{-1} + 4 + 5q + 8q^2 + 4q^3 + 7q^4 - 11q^5 + \cdots
\]

and

\[
Y = q^{-3} + 3q^{-2} + 7q^{-1} + 12 + 17q + 26q^2 + 10q^3 + 37q^4 - 15q^5 + \cdots
\]

We let \( f_{11,2} := X - 4 \) and \( f_{11,3} := Y - 3X \). One can continue to construct modular functions of the form \( f_{11,m} = q^{-m} + O(q^{-1}) = Q_m(X, Y) \), where \( Q_m(X, Y) \) is a polynomial in \( X \) and \( Y \) that eliminates the terms between \( q^{-m} \) and \( q^{-1} \) and also the constant term if there is any. The first few of \( f_{11,m}(\tau) \) are given by

\[
\begin{align*}
 f_{11,0} &= 1 \\
 f_{11,1} &= q^{-1} - q^{-1} + 0q + 0q^2 + 0q^3 + 0q^4 + 0q^5 + \cdots = 0 \\
 f_{11,2} &= q^{-2} + 2q^{-1} + 5q + 8q^2 + 4q^3 + 7q^4 - 11q^5 + \cdots = X - 4 \\
 f_{11,3} &= q^{-3} + q^{-1} + 2q + 2q^2 + 4q^3 + 7q^4 - 11q^5 + \cdots = X - 4 \\
 f_{11,4} &= q^{-4} - 2q^{-1} + 6q + 3q^2 + 2q^3 + 6q^4 + 16q^5 + \cdots = X^2 - 4Y - 4X - 36 \\
 f_{11,5} &= q^{-5} - q^{-1} - 14q - 16q^2 + 34q^3 + \cdots = XY - 2X^2 + X + 7Y + 60 \\
&\vdots
\end{align*}
\]

Using properties of the generalized Hecke operator appearing in (1.14), several congruences of the Fourier coefficients of \( f_{N,m}(\tau) \) are proved in [15]. The following theorem is one of them, which gives a strong divisibility property of Fourier coefficients of modular functions that holds for arbitrary prime powers, including powers of prime divisors of \( N \).

**Theorem 3.1.** [15, Theorem 1.6] Let \( N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\} \). Then for any prime \( p \) and positive integers \( r, m \) and \( n \) with \( p \nmid m, p \nmid n \), we have

\[
a_N(mp^r, n) + a_N(m, -1)a_N(p^r, n) \equiv 0 \pmod{p^r}.
\]

Let \( J_{N,m}(\tau) := j_{N,m}(\tau) - c_N(m, 0) \) and define \( J_{N,0}(\tau) := 1 \). As \( H^0_0(N) \) is generated by \( j_{N,m} \), each of \( f_{N,m} \) can be written as a linear combination of \( j_{N,m} \)'s. Specifically,

\[
f_{N,m} = J_{N,m} + a_N(m, -1)J_{N,1}.
\]

We also define \( f_{N,m} = 0 \) unless \( p \) divides \( m \). Theorem 1.6 is an immediate result of Theorem 3.1 and (3.4). For more congruences of the Fourier coefficients of \( f_{N,m}(\tau) \), see [15].

4. Proofs of Theorem 1.4 and Corollary 1.5

The method of proof given in this section is very similar to those in [2, 3, 17, 18]. In [3], Atkin constructed bases \( G_m \)'s for \( \mathcal{M}^1_0(11) \) and \( g_m \)'s for \( \mathcal{M}^1_0(11) \).

**Lemma 4.1.** [3, Lemma 3] For all integers \( m \geq 2 \), \( G_m(\tau) \) and \( g_m(\tau) \) satisfy the following properties:
(1) \( G_m(\tau) \in M^0_{\infty}(11) \) and \( g_m(\tau) \in M^0_{0}(11), \)
(2) \( G_m(-1/11\tau) = 11^{\theta(m)} g_m(\tau), \) where \( \theta(m) = 6k + 2, 3, 4, 6, 6 \) according as \( m = 5k + 2, 3, 4, 5, 6 \) \((k \geq 0), \)
(3) the Fourier coefficients of \( G_m(\tau) \) have integral coefficients with leading term \( q^{-m}, \)
(4) the Fourier coefficients of \( g_m(\tau) \) have integral coefficients with leading term \( q^{\psi(m)}, \) where \( \psi(m) = 5k + 1, 2, 3, 4, 5 \) \((k \geq 0). \)

Further, the function \( B(\tau) = G_2(\tau)g_2(\tau) - 12 \in M^0_0(11) \) has simple poles at \( \tau = 0 \) and \( \tau = i\infty \) such that \( B(-1/11\tau) = B(\tau) \) and the Fourier coefficients of \( B(\tau) \) have integral coefficients with leading term \( q^{-1}. \)

According to [3, Lemma 4], \( G_m(\tau), 11^{\theta(m)}g_m(\tau) \) and \( B(\tau) \) form a basis for \( M^0_0(11) \). Thus if \( F(\tau) \in M^0_0(11) \) has a pole of order \( M \) at \( \tau = 0 \) and a pole of order \( K \) at \( \tau = i\infty \), then

\[
(4.1) \quad F(\tau) = \sum_{r=2}^{K} \lambda_r G_r(\tau) + \lambda_{-1} B(\tau) + \lambda_0 + \sum_{r=2}^{M} \lambda_r 11^{\theta(r)} g_r(\tau)
\]

for some constants \( \lambda_r \) \((-K \leq r \leq M). \)

The Fourier expansions of \( G_m(\tau), g_m(\tau) \) and \( B(\tau) \) are found in [3, Table 1]. We note that \( G_2(\tau) = f_{11,2}(\tau) - 12 = X - 16 \) and \( G_3(\tau) = f_{11,3}(\tau) - 3f_{11,2}(\tau) + 24 = Y - 6X + 36. \) Also, note that \( g_2(\tau) = \frac{\eta^{12}(11\tau)}{\eta^{12}(\tau)} \) and \( G_5(\tau) = \frac{\eta^{12}(11\tau)}{\eta^{12}(\tau)} \) where \( \eta(\tau) \) is the Dedekind eta-function \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \) In general, if we write \( G_m(\tau) = q^{-m} + \sum_{\ell=m+1}^{\infty} a_m(\ell)q^\ell, \) then

\[
(4.2) \quad G_m(\tau) = f_{11,m}(\tau) + a_m(0) + \sum_{\ell=2}^{m-1} a_m(-\ell)f_{11,\ell}(\tau), \quad (m \geq 2).
\]

Conversely, we can also express the element \( f_{11,m}(\tau) \) of the reduced form basis of \( M^0_0(11) \) as a linear combination of \( G_2(\tau) \)'s with \( 2 \leq \ell \leq m. \) Thus the congruence (1.16) is equivalent to the same congruence of coefficients \( a_n(\ell) \) of \( G_m(\tau). \) More precisely, \( a_{11\beta m'}(11^{\alpha+\beta}m') \equiv 0 \) \((\text{mod } 11^\alpha). \)

In [3], Atkin showed the divisibility property of the Fourier coefficients of \( g_m \)'s. Using this and the fact \( U_{11}j(\tau) \in M^0_0(11) \) with a pole of order 121 at \( \tau = 0, \) he proved (1.8). From now till the end of the paper, we use the notation \( U := U_{11}. \)

**Lemma 4.2.** [16, Theorem 8 and (8.81)] Let \( f(\tau) \in M^0_0(11). \) Then

1. \( Uf(\tau) \in M^0_0(11), \)
2. \( 11(Uf)(-\frac{1}{11\tau}) = 11(Uf)(11\tau) + f(-\frac{1}{121\tau}) - f(\tau). \)

**Lemma 4.3.** [3, Corollary, p. 21] Define \( \xi(2) = 0, \xi(3) = 1 \) and \( \xi(n) = 5k + 1, 3, 3, 4, 5 \) according as \( n = 5k + 4, 5, 6, 7, 8 \) \((k \geq 0), \) We denote by \( \mathcal{S} \) the class of functions \( F(\tau) \) with \( \sum_{n=2}^{M} \lambda_n 11^{\xi(n)} g_n(\tau). \) Then whenever \( F(\tau) \in \mathcal{S}, \) we have \( 11^{-1}UF(\tau) \in \mathcal{S}. \)

We now let \( m = 11^\beta m' \) when \( (11, m') = 1 \) and want to prove the congruence \( a_{11}(m, 11^{\alpha+\beta}n) \equiv 0 \) \((\text{mod } 11^\alpha). \) Using induction on \( \beta. \) For brevity, we let \( f_m := f_{11,m}. \) Consider

\[
(4.3) \quad H_m(\tau) := 11(Uf_m)(\tau) - 11f_{m/11}(\tau).
\]
Here \( f_{m/11}(\tau) = 0 \) when \( 11 \nmid m \). By Lemma 4.2 (2), we have
\[
H_m\left(-\frac{1}{11\tau}\right) = 11(Uf_m(-\frac{1}{11\tau})) - 11f_{m/11}\left(-\frac{1}{11\tau}\right)
\]
(4.4)
\[
= 11(Uf_m(11\tau) + f_m(-\frac{1}{121\tau}) - f_m(\tau) - 11f_{m/11}\left(-\frac{1}{11\tau}\right))
= 11q^{-m} + O(1) + O(1) - q^{-m} - O(q^{-1}) - O(1)
= 10q^{-m} + O(q^{-1}).
\]
Apparently, when \( \beta = 0 \), \( H_m(\tau) \) is holomorphic at \( \tau = i\infty \) and has a pole of order \( m \) at \( \tau = 0 \). Hence it follows from (4.1) that
\[
Uf_m(\tau) = \lambda_0 + \sum_{r=2}^{m} \lambda_r 11^{\theta(r)-1} g_r(\tau)
\]
for some constants \( \lambda_0 \) and \( \lambda_r \) (\( 2 \leq r \leq m \)). Thus every coefficient of \( q^{11n} \) in \( f_m(\tau) \) is a multiple of 11. Moreover, by Lemma 4.3, after applying \( U \)-operator \( \alpha \) times, we find that
\[
a_{11}(m, 11^\alpha n) \equiv 0 \pmod{11^\alpha}
\]
for each positive integer \( \alpha \).

Nextly, if \( \beta = 1 \) and \( m' = 1 \), i.e., \( m = 11 \), \( H_{11}(\tau) \) has a simple pole at \( \tau = i\infty \) and a pole of order 11 at \( \tau = 0 \) so that we obtain from (4.1) that
\[
Uf_{11}(\tau) = \lambda_{-1} B(\tau) + \lambda_0 + \sum_{r=2}^{11} \lambda_r 11^{\theta(r)-1} g_r(\tau)
\]
(4.5)
for some constants \( \lambda_r \) (\( -1 \leq r \leq 11 \)). As \( UB(\tau) + 5 \in \mathcal{S} \) by [3, p. 22], every coefficient of \( q^{11n} \) in \( B(\tau) \) is a multiple of 11. We hence deduce from (4.5) that every coefficient of \( q^{121n} \) in \( f_{11}(\tau) \) is a multiple of 11. Again using Lemma 4.3 repeatedly, we find that
\[
a_{11}(11, 11^{\alpha+1} n) \equiv 0 \pmod{11^\alpha}.
\]

Now assume the congruence holds for all \( m \) of the form \( m = 11^k m' \) with \( b < \beta \). Then it remains to prove that it holds for \( m = 11^b m' \) as well. In this general case, \( H_m(\tau) \) has a pole of order \( m/11 \) at \( \tau = i\infty \) and a pole of order \( m \) at \( \tau = 0 \). Therefore, by (4.1)
\[
Uf_m(\tau) = \sum_{r=2}^{m/11} \lambda_{-r} G_r(\tau) + \lambda_{-1} B(\tau) + \lambda_0 + f_{m/11}(\tau) + \sum_{r=2}^{m} \lambda_r 11^{\theta(r)-1} g_r(\tau)
\]
for some constants \( \lambda_r \) (\( -m/11 \leq r \leq m \)). By induction hypothesis and the arguments so far, we see that every coefficient of \( q^{11^{\alpha+b-1}n} \) in \( Uf_m(\tau) \) is a multiple of \( 11^\alpha \), which implies the desired result
\[
a_{11}(m, 11^{\alpha+b} n) \equiv 0 \pmod{11^\alpha}.
\]

5. Concluding Remarks

For a harmonic weak Maass form \( f \), let us write its holomorphic part (or the mock modular form) as \( f^h(\tau) = \sum_{n=-\infty}^{\infty} c_f^+(n) q^n \). For a certain harmonic weak Maass form \( f \) with level of power of prime \( p \), it is shown in [9, Theorem 1.5 (2)] that almost every \( c_f^+(p^a n) \) is a multiple of \( p^b \), where \( a = \min\{d \geq 0 | c_f^+(p^d n) = 0 \text{ for all } n < 0\} \) and \( b \) is any positive integer. We have
proved that certain linear combinations of Fourier coefficients of a mock modular function \( j_{p,m} \) are integers and furthermore some linear combinations of Fourier coefficients of \( j_{11,m} \) are divisible by powers of 11. In addition, there are relations between Fourier coefficients of non-holomorphic parts of harmonic Maass functions such as (2.6). It seems that there are much more intriguing arithmetic properties to be further explored in the Fourier coefficients of a mock modular form and its corresponding non-holomorphic part.

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