EQUILIBRIUM STATES OF GENERALISED SINGULAR VALUE POTENTIALS AND APPLICATIONS TO AFFINE ITERATED FUNCTION SYSTEMS

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Abstract. We completely describe the equilibrium states of a class of potentials over the full shift which includes Falconer’s singular value function for affine iterated function systems with invertible affinities. We show that the number of distinct ergodic equilibrium states of such a potential is bounded by a number depending only on the dimension, answering a question of A. Käenmäki. We prove that all such equilibrium states are fully supported and satisfy a Gibbs inequality with respect to a suitable subadditive potential. We apply these results to demonstrate that the affinity dimension of an iterated function system with invertible affinities is always strictly reduced when any one of the maps is removed, resolving a folklore open problem in the dimension theory of self-affine fractals. We deduce a natural criterion under which the Hausdorff dimension of the attractor has the same strict reduction property.

1 Introduction and context

If $T_1, \ldots, T_N : \mathbb{R}^d \to \mathbb{R}^d$ are contractions it is well-known that there exists a unique nonempty compact set $X \subset \mathbb{R}^d$ which solves the equation $X = \bigcup_{i=1}^N T_iX$. In this article we will refer to the tuple $(T_1, \ldots, T_N)$ as an iterated function system and the set $X$ as its attractor. When the transformations $T_i$ are all similarity transformations the set $X$ is called self-similar; when they are assumed only to be affine transformations $X$ is called self-affine. Subject to suitable hypotheses which guarantee that the distinct images $T_iX$ do not substantially overlap, the dimension theory of self-similar sets has been well understood since the work of Hutchinson [H81]. (Investigation of the overlapping case remains an active and challenging research topic: see e.g. [FH15, H14, Hoc, SS16].) The dimension theory of self-affine sets, by contrast, has been a source of stubborn open problems since its initiation in the 1980s by Bedford [B84], McMullen [M84] and Falconer [F88], and affine iterated function systems remain the focus of substantial research interest (see e.g. [B07, BK17, BR18, BF13, DS17, FS14, F16, JPS07, M16, MS, Rap]). A persistent feature of
the literature on affine iterated function systems has been the requirement for additional hypotheses on the linear parts of the affinities in order to obtain results: they may be required to be positive or dominated [B15,BR18,FK17,FK,HL95], to be of low dimension [B07,B15,BR18,B84,FK,F12,HL95,KM,M84,MS], to have a simple algebraic structure [B07,B84,DS17,FM07,F12,M84] or to induce invariant measures on projective space which themselves satisfy suitable dimension hypotheses [B15,FK,MS,Rap]. In this article we contribute to the still very small literature of results on affine iterated function systems which require no hypotheses whatsoever on the affinities other than that they be contracting and invertible.

We recall that an iterated function system \((T_1,\ldots,T_N)\) satisfies the open set condition if there exists a nonempty open set \(U \subseteq \mathbb{R}^d\) such that \(T_i U \subseteq U\) for every \(i = 1,\ldots,N\) and \(T_i U \cap T_j U = \emptyset\) when \(i \neq j\). If \(T_1,\ldots,T_N\) are similarities satisfying the open set condition with \(T_i\) having contraction ratio \(r_i \in (0,1)\), a well-known theorem of Hutchinson [H81] asserts that the Hausdorff dimension \(s\) of the attractor satisfies the equation \(\sum_{i=1}^{N} r_i^s = 1\). Let us note three trivial consequences of this formula: firstly, if the maps \(T_i\) are perturbed within this class then the value of the Hausdorff dimension predicted by the formula varies continuously with the perturbation; secondly, the value of \(s\) may clearly be computed to within any prescribed accuracy in a finite amount of time when the contraction ratios \(r_i\) are known; thirdly, if one of the maps \(T_i\) is deleted then the value of the dimension predicted by the formula is strictly decreased. The extent of the difficulties presented by affine iterated function systems may perhaps be appreciated by observing that in the affine context an analogue of the first property was not established until 2014 by Feng and Shmerkin [FS14] and an analogue of the second property was unknown until established by the second named author in the recent article [M16]. Prior to the present article the third property was known in the affine context only in dimensions three and lower [KM] or when the affinity dimension (defined below) is a rational number [KL17]. As a corollary of the main result of this article we will establish the third of these three properties unconditionally for invertible affine iterated function systems of arbitrary dimension.

Let us describe the appropriate generalisation of Hutchinson’s formula to affine iterated function systems. When \(T_1\) is a similarity transformation all of the essential information about the \(s\)-dimensional volume of the image of the unit ball is captured by its contraction ratio \(r_i\), but when \(T_i\) is an affine transformation more detailed information is required. If \(A\) is a linear transformation of \(\mathbb{R}^d\) we recall that the singular values of \(A\) are defined to be the non-negative square roots of the eigenvalues of the positive semidefinite linear map \(A^\top A\). We will write the singular values as \(\alpha_1(A) \geq \alpha_2(A) \geq \cdots \geq \alpha_d(A)\), allowing repetition in the case of multiple eigenvalues. The existence of the singular value decomposition of \(A\) implies that the image of the unit ball under \(A\) is an ellipsoid with the lengths of the semiaxes equal to the singular values of \(A\). Given a real number \(s > 0\) and linear transformation \(A\) of \(\mathbb{R}^d\) we define the singular value function \(\varphi^s(A)\) by