Abstract—This article addresses the problem of optimal scheduling of an aggregated power profile (during a coordinated discharging or charging operation) by means of a heterogeneous fleet of storage devices subject to availability constraints. The devices have heterogeneous initial levels of energy, power ratings, and efficiency; moreover, the fleet operates without cross-charging of the units. An explicit feedback policy is proposed to compute a feasible schedule whenever one exists, and scalable design procedures to achieve maximum time to failure or minimal uncharged energy in the case of unfeasible aggregated demand profiles. Finally, a time-domain characterization of the set of feasible demand profiles using aggregate constraints is proposed, suitable for optimization problems where the aggregate population behavior is of interest.

Index Terms—Distributed optimal control, flexible demand, optimal storage management, storage fleet aggregation.

I. INTRODUCTION

POWER networks, in the coming years, are likely to significantly rely on distributed storage assets in order to ease the task of balancing demand and supply, both during normal operation or in case of power outages. This is expected, on one hand, because of increased penetration of renewable technologies and the volatility of supply it entails; on the other, because of widespread decarbonization of the transport sector and consequent adoption of electric vehicles (EVs). The flexibility afforded by a considerable amount of storage capacity connected permanently or intermittently to the network has a lot of potential for limiting peak demands and related costs, and delivering balancing services to the grid. Recent research has systematically classified a significant amount of literature that deals with optimal energy management for storage devices [14], [15], [16].

Specifically, the work [14] introduces an adaptive charging algorithm with the objective of peakload management. Ma et al. [15] proposed a decentralized strategy for numerous identical EVs under the noncooperative games to minimize the charging cost. Cooperative control of network control theory is developed in [16] to ensure the satisfaction of both energy balance and fair utilization among dispersed energy storage systems.

Under such future scenarios, one issue becomes fundamental: How to schedule power profiles of a multitude of storage devices while respecting individual rated power and energy constraints and, at the same time, fulfilling a preassigned aggregate power profile for the fleet? This question arises both during a hypothetical “discharging operation,” viz. when the fleet is acting as a “service provider” and compensates for a lack of power due to outages or fluctuations in availability of renewable generation, or during the “charging phase,” when storage devices are recharged to meet individual energy needs and the aggregate profile is designed so as to possibly reduce peaks in demand or to minimize conventional generation costs.

Such questions have recently received considerable attention from the community (see [1] for a recent survey and reasoned literature classification based on keywords, such as energy storage and optimal policy/strategy/management). For instance, Sioshansi et al. [2] presented an application of dynamic programming in the estimation of the capacity value of storage devices. A dynamic model to approximate the power/energy capacity of aggregations of EVs is developed in [19]. The aggregate flexibility provided by a collection of thermostatically controlled loads (TCLs) is characterized by using a stochastic battery model [23]. Fortenbacher et al. [18] adopted model predictive control (MPC) to control multiple battery sets to track an aggregated set-point trajectory while minimizing battery degradation, battery system, and network losses. Dall’Anese et al. [22] proposed an algorithmic framework which controls the dispatchable distributed energy resources (DERs) to the power demand request from transmission systems at the feeder substation. Demand dispatch for the regulation of the power grid is considered in [17], based on randomized local control algorithms for homogeneous load in a mean field control setting. More recently, the control architectures for a fleet of diverse DERs using the packetized energy management dispatch paradigm have been developed in [21].

Notions of controlled invariant sets have been proposed in [3] to achieve optimality preserving aggregation of fleets. In [20], invariant sets are also used to ensure safe coordination of systems with both local and global constraints, while a population of homogeneous air conditioners tracks a power trajectory. Zhu and Zhang et al. [4] introduced an optimal coordination policy for fleets seeking to fulfill a preassigned reference signal (subject to
penalty costs for unmet demand) and achieving profit maximization taking into account the service and recharging phases. Our approach complements the existing literature in two respects: 1) it considers fully heterogeneous fleets, allowing different power ratings, different initial and target energy levels, different nonunity efficiency, and different availability windows and 2) it provides guaranteed and scalable optimal solutions which may help an aggregator monitor its flexibility provision in unidirectional power transfer operations, viz. neglecting the recovery phase, by computing exact optimal dispatch profiles in real time.

In this context, we adopt and further develop the approach proposed in [5], [6], [7], and [8]. In [5], an optimal causal dispatch policy is introduced for heterogeneous storage fleets unable to cross-charge and seeking to deliver a preassigned aggregate demand profile, while maximizing future flexibility, viz. the ability of meeting future power requests. Remarkably, the same policy was first introduced in [9], in the context of hydro-reservoirs. Further optimality properties, including the ability to minimize time to failure or unserved energy were highlighted in subsequent publications [6], [8], while an explicit and remarkably effective time-domain characterization of the set of feasible power profile demands is provided in [7], using the notion of load duration curves. An in depth compendium of the theory, with new insights and subsequent interpretations, can be found in [10]. While such theory has been developed by taking into account heterogeneous fleets of devices, it neglects the possibility of partial availability, viz. the fact that devices may be connected to the grid during different time intervals over the considered prediction horizon, as would typically be expected of, i.e., EVs.

This article extends the approach of [10] to deal with the case of storage devices with different availability windows (or, more generally, availability sets), heterogeneous power ratings, and initial energy values. Its contribution is manifold as follows.

1) On one hand, it shows how to convert the dispatch design problem for a fleet subject to availability constraints (under a no cross-charging assumption) into the dispatch design of an augmented demand signal for a similar fleet without availability constraints and with possible complemented initial energy levels; hence, it broadens applicability of the previous results to the current setup.

2) It shows by means of an example, that no single causal policy exists in this case and provides a characterization of the set of aggregate power profiles that a heterogeneous fleet of given initial energy, power ratings, and availability sets is able to deliver. Counter-examples show why simpler necessary conditions are unable to capture the full complexity of the feasible set of power profiles.

3) It highlights how to leverage the new proposed policies in order to derive optimal power schedules with minimum unserved energy or maximum time to failure and by considering novel medium and large scale examples, where such approaches are illustrated.

II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

A. System Description and Objectives

Let $\mathcal{N}$ denote a finite collection of batteries (of cardinality $N$). Each battery $i \in \mathcal{N}$ is constrained by a rated power $\bar{P}_i$, quantifying its maximum discharging rate. Our aim is to analyze and design dispatch policies for the fleet over a given bounded time interval $\mathcal{T} \subset [0, +\infty)$. To this end, let $E_i(t)$ denote the state of charge of the $i$th device at time $t \in \mathcal{T}$, viz. the amount of externally measured energy stored in the device. Notice that measuring energy externally allows to factor in possibly heterogeneous and nonunity efficiencies, in particular, by defining $\bar{E}_i(t) = \eta_i \bar{E}_i(t)$, where $\bar{E}_i$ denotes internally measured energy and $\eta_i$ the round-trip efficiency coefficient. The differential equations given as follows describe the time-evolution of $E_i(t)$:

$$\dot{E}_i(t) = -u_i(t)$$

with initial configuration of energy levels, $E_i(0), i \in \mathcal{N}$. The variable $u_i(t)$ is the instantaneous power delivered by the $i$th battery, which needs to fulfill

$$u_i(t) \in [0, \bar{P}_i] \quad \forall t \in \mathcal{T}. \quad (2)$$

Together with constraints (2), we consider the additional possibility of devices operating within a preassigned availability window (or more in general availability set) $A_i \subset \mathcal{T}$, so that

$$u_i(t) = 0 \quad \forall t \in \mathcal{T} \setminus A_i. \quad (3)$$

Our first objective is to ascertain, for any given power profile $d : \mathcal{T} \rightarrow [0, +\infty)$, if there exists a control action $u_i(\cdot), i \in \mathcal{N}$, fulfilling constraints (2) and (3), while at the same time

$$\sum_{i \in \mathcal{N}} u_i(t) = d(t) \quad \forall t \in \mathcal{T} \quad (4)$$

and the associated solution of (1) fulfills $E_i(t) \geq 0 \forall t \in \mathcal{T}$.

**Assumption 1:** Without loss of generality, we assume the following inequality:

$$d(t) \leq \sum_{i \in \mathcal{N} : t \in A_i} \bar{P}_i. \quad (5)$$

Since any power in excess of this bound could never be delivered, the scheduling problem can be reformulated by considering a modified demand signal saturated at this maximum level.

For ease of notation, we arrange energy values in a vector $E(t) := [E_1(t), E_2(t), \ldots, E_N(t)]$. Moreover, for any given $E(0)$, we define the set of feasible power profiles

$$\mathcal{F}(E(0)) := \left\{ d(\cdot) : \mathcal{T} \rightarrow [0, +\infty) : \exists \{u_i : \mathcal{T} \rightarrow [0, \bar{P}_i]\}_{i \in \mathcal{N}} : u_i(t) = 0 \quad \forall t \notin A_i \forall i \in \mathcal{N} \\
E_i(0) \geq \int_\mathcal{T} u_i(\tau) d\tau \forall i \in \mathcal{N} \\
d(t) = \sum_{i \in \mathcal{N}} u_i(t) \forall t \in \mathcal{T} \right\}. \quad (6)$$

Hence, our preliminary task is to find out if $d(\cdot) \in \mathcal{F}(E(0))$ and, if so, what is a suitable dispatch policy. In practice, this question arises whenever a fleet of storage devices is required to coordinate in delivering energy (without cross-charging) to jointly fulfill a given power reference signal $d(t)$. Previous literature has answered such questions for the case of full availability, viz. windows coinciding with $\mathcal{T}$ [5], [8]. We approach the problem by suitably modifying the scheduling proposed in [5] [later denoted Greedy-Greatest-Discharge-Duration First policy (GGDDF)] to address the issue of availability windows.

Our main contribution is a constructive design algorithm for a feasible policy and a supporting theoretical analysis, showing
that the problem can be equivalently framed as one of delivery of an auxiliary (increased) power profile for a fleet with suitably augmented initial energy levels, identical power ratings, and full availability over the considered interval.

As in [5], we introduce a new set of coordinates, the so-called time-to-discharge variables, defined as

$$x_i(t) = \frac{E_i(t)}{\bar{P}_i},$$

(7)

Accordingly, the state evolution is governed by

$$\dot{x}_i(t) = -u_i(t) / \bar{P}_i, \quad x_i(0) = \frac{E_i(0)}{\bar{P}_i}. \quad (8)$$

Some additional notations are useful to formulate the proposed feedback policy. More closely, for any discharge duration $\tau$, we denote the set of agents $\mathcal{N}_\tau(x) := \{ i \in \mathcal{N} : x_i = \tau \}$, where $x \in \mathbb{R}^N$ is the stacked state vector of time to discharge over all devices. Clearly, $\mathcal{N} = \bigcup_{\tau \geq 0} \mathcal{N}_\tau(x)$, and this partitions $\mathcal{N}$ as $\mathcal{N}_{\tau_1} \cap \mathcal{N}_{\tau_2} = \emptyset$ for $\tau_1 \neq \tau_2$. Overall, only a finite number of $\mathcal{N}_\tau$ are nonempty, at each time $t$, and we order the corresponding discharge time as, $\tau_1(t) > \tau_2(t) > \tau_3(t) > \ldots > \tau_{\mathcal{G}(t)}(t)$, with $\mathcal{G}(t) \leq N$.

**B. Greedy-Greatest-Discharge-Duration First Policy**

It is useful to first introduce the GGDDF policy without any reference to availability sets

$$u_i(t) = \begin{cases} \bar{P}_i, & \text{if } i \in \mathcal{N}_{\tau_1} \text{ and } \sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j \leq d(t) \\ \bar{r}(t) \bar{P}_i, & \text{if } i \in \mathcal{N}_{\tau_2} \text{ and } \sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j \leq d(t) \\ d(t) < \sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j, & \text{otherwise} \end{cases} \quad (9)$$

where the value of $\bar{r}(t) \in [0, 1]$ is determined according to

$$\bar{r}(t) = \frac{d(t) - \sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j}{\sum_{j \in \mathcal{N}_h} \bar{P}_j},$$

where $k$ is such that $\sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j \leq d(t) < \sum_{h < k} \sum_{j \in \mathcal{N}_h} \bar{P}_j$. Accordingly, the amount of power extracted from all devices (regardless of availability) equals the instantaneous demand $d(t)$, viz.

$$\sum_{i \in \mathcal{N}} u_i(t) = d(t). \quad (10)$$

For later use, we denote this feedback policy explicitly as

$$u(t) = K(x(t), d(t)) \quad (11)$$

with the associated system of equations

$$\dot{x}(t) = -\Omega K(x(t), d(t)). \quad (12)$$

where $\Omega = \text{diag}[P_1^{-1}, P_2^{-1}, \ldots, P_N^{-1}]$. The solution of (12) is denoted as $\hat{x}(t, x(0), d(\cdot))$.

While taking into account the availability sets, the explicit feedback policy to dispatch $d(t)$ at time $t$ is expressed as

$$u_i(t) = \begin{cases} \bar{P}_i, & \text{if } i \in \mathcal{N}_{\tau_1} \text{ and } \sum_{h < k} \sum_{j \in \mathcal{N}_h,i \in A_j} \bar{P}_j \leq d(t) \\ r(t) \bar{P}_i, & \text{if } i \in \mathcal{N}_{\tau_2} \text{ and } \sum_{h < k} \sum_{j \in \mathcal{N}_h,i \in A_j} \bar{P}_j \leq d(t) \\ d(t) < \sum_{h < k} \sum_{j \in \mathcal{N}_h,i \in A_j} \bar{P}_j, & \text{otherwise} \end{cases} \quad (13)$$

where the value of $r(t) \in [0, 1]$ is determined according to

$$r(t) = \frac{d(t) - \sum_{h < k} \sum_{j \in \mathcal{N}_h,i \in A_j} \bar{P}_j}{\sum_{j \in \mathcal{N}_h,i \in A_j} \bar{P}_j} \quad (14)$$

which is referred to as the GGDDF policy with availability sets. We remark that time-dependence is made explicit in (15) since the availability of devices at any time $t$ directly influences the amount of power extracted from each battery. With this in mind, the fleet’s closed-loop equations read

$$\dot{x}(t) = -\Omega K(t, x(t), d(t)). \quad (16)$$

Of course, the schedule (15) might induce solutions of (16) violating the constraint that $x_i(t) \geq 0$ due to insufficient initial energy; we ignore this, for the time being, as our main goal is to compute how much energy the feedback policy extracts outside of the availability window for the given power profile $d(t)$. Notice also that the constraint $u_i(t) = 0$ for $t \notin \mathcal{A}_i$ is, for the time-being, not enforced. Specifically, if a device has a sufficiently high time to discharge, it will be discharged by (15) regardless of its availability window. Denote by $\varphi(t, x(0), d(\cdot))$, the solution of (16) at time $t$ with initial condition $x(0)$. Following the similar steps, as in [11], one can show the properties 1) to 4) stated as follows.

1. For all initial conditions $x(0) \in \mathbb{R}^N$, there exists a unique Filippov solution [24] in (16) defined for $t \geq 0$.
2. The relative ordering of time to discharge is preserved along solutions. Namely, for all $t \geq \tau \geq 0$, it holds

$$x_i(\tau) \geq x_j(\tau) \Rightarrow \varphi_i(t, x(0), d(\cdot)) \geq \varphi_j(t, x(0), d(\cdot)).$$

3. A corollary of the previous implication is

$$x_i(\tau) = x_j(\tau) \Rightarrow \varphi_i(t, x(0), d(\cdot)) = \varphi_j(t, x(0), d(\cdot)).$$

4. Furthermore, sets of agents with the same discharge duration are monotonically nondecreasing in time (and, in fact, strictly increasing whenever two or more sets merge with each other as the corresponding discharge times equalize).

5. $\varphi(t, x(0), d(\cdot))$ is a cooperative (monotone) system [12]; $x_1 \geq x_2 \Rightarrow \varphi(t, x_1, d(\cdot)) \geq \varphi(t, x_2, d(\cdot)) \forall t \geq 0$, where $x_1$ and $x_2$ are temporarily used to denote two different vectors of time to discharge, $\geq$ componentwise inequalities between vectors.

6. $\varphi$ fulfills translation invariance, viz. $\varphi(t, x(0) + \delta \mathbf{1}, \cdot) = \varphi(t, x(0), \cdot) + \delta \mathbf{1}$, where $\delta \in \mathbb{R}$ is arbitrary and $\mathbf{1}$ denotes the vector of all ones of dimension $N$.

7. The flow is weakly contracting with respect to the infinity norm: $\|\varphi(t, x_1, d(\cdot)) - \varphi(t, x_2, d(\cdot))\|_{\infty} \leq \|x_1 - x_2\|_{\infty}$.

We prove properties 5)–7) as follows.
5) Monotonicity of $\varphi$: Notice that the feedback $K$ is such that $K_j(t, x, d)$ is nonincreasing with respect to $x_j$ for all $j \neq i$ (increasing $x_j$ might trigger an increase in its priority so that more energy will be taken from $x_j$ and, consequently, possibly less will be extracted from $x_i$). Moreover, while $K$ is discontinuous (in fact, piecewise constant in $x$), similar steps, as in [11, Proof of Lemma 3.2.1], show that there exists a unique (Filippov’s or Carathéodory) solution that fulfills $\dot{x}(t) = -\Omega K(t, x(t), d(t))$ for almost all $t$. By uniqueness of solutions, combined with nondecreasingness of $\hat{x}$ with respect to $x_i$ for all $i \neq j$, we conclude that system (16) is cooperative ([12]). Hence, denoting by $\geq_{\varphi}$ componentwise inequalities between vectors, we see that $x_1 \geq x_2 \Rightarrow \varphi(t, x_1(d)) \geq \varphi(t, x_2(d)) \forall t \geq 0$.

6) Translation Invariance: Let $1$ denote the vector of all $1$’s of dimension $N$. For all $x \in \mathbb{R}^N$, all $d \geq 0$ and any $\delta \in \mathbb{R}$ it holds $K(t, x + \delta 1, d) = K(t, x, d)$, thus follows, since the relative ordering of time to discharge is unaffected by a simultaneous $\delta$ increase (or decrease) affecting all batteries. Hence, considering any solution $\varphi(t, x(d))$ we see that $\frac{d}{dt} \varphi(t, x(d)) + \delta 1 = -\hat{\Omega} K(t, \varphi(t, x(d)), d(t)) = -\Omega K(t, \varphi(t, x(d)) + \delta 1, d(t))$. This proves that $\varphi(t, x(d)) + \delta 1$ is a solution of system (16), with initial condition $x + \delta 1$. In other words, $\varphi(t, x + \delta 1, d) = \varphi(t, x, d) + \delta 1$.

7) Weak Contraction: To prove weak contraction, let $x_1, x_2$ be arbitrary in $\mathbb{R}^N$. Let the vectors $\bar{x}$ and $\bar{x}$ be defined as $\bar{x} := \min \{x_1, x_2\}$ and $\bar{x} := \max \{x_1, x_2\}$, where min and max are meant componentwise. Clearly $\bar{x} \geq x_1 \geq \bar{x} \geq x_2 \geq \bar{x}$. Exploiting monotonicity we see that, for any $d \geq 0$ and any $\delta \geq 0$, $\varphi(t, \bar{x}, d) \geq \varphi(t, x_1, d) \geq \varphi(t, x_2, d)$. Rearranging the previous inequalities we can show that

$$\varphi(t, \bar{x}, d) - \varphi(t, x_1, d) \geq \varphi(t, x_1, d) - \varphi(t, x_2, d) \geq -\varphi(t, \bar{x}, d) + \varphi(t, x_2, d).$$

Denoting componentwise absolute values of a vector as $|\cdot|$, the previous inequality can equivalently be written as

$$|\varphi(t, x_1, d) - \varphi(t, x_2, d)| \leq |\varphi(t, \bar{x}, d) - \varphi(t, x_2, d)|.$$

Define next $\delta := \max_{x \in \mathbb{R}^N} |x_1 - x_2|$. Clearly, $\bar{x} \leq \bar{x} + \delta 1$, hence, by monotonicity and translation invariance

$$\varphi(t, \bar{x}, d) - \varphi(t, \bar{x}, d) \leq \varphi(t, \bar{x} + \delta 1, d) - \varphi(t, x_2, d) = \delta 1.$$

Combining (17) with (18) yields $|\varphi(t, x_1, d) - \varphi(t, x_2, d)| \leq \delta 1$, and ultimately $\max_{x \in \mathbb{R}^N} |\varphi(t, x_1, d) - \varphi(t, x_2, d)| \leq \delta 1$. Denoting by $\|\cdot\|_\infty$ infinity norms, the previous inequality reads $\|\varphi(t, x_1, d) - \varphi(t, x_2, d)\|_\infty \leq \|x_1 - x_2\|_\infty$, which proves weak contractivity of $\varphi$. □

C. Equivalent Feasible Power Dispatching

Let us introduce a vector $\lambda = [\lambda_1]_{i \in \mathbb{N}} \in [0, 1]^N$ which we use to define the auxiliary time-to-discharge vector $\tilde{x}_i = x_i + \lambda_i \mu(T \setminus A_i)$, where $\mu$ denotes the Lebesgue measure in $\mathbb{R}$. Informally $\lambda_i$ modulates between a minimum value of 0 (no energy) and 1 (energy corresponding to discharging at full rate) the auxiliary energy needed by agent $i$ to account for discharging happening outside of its availability window. For each $\tilde{x}(0)$ we define the corresponding solution according to our feedback policy, viz. $\tilde{x}(t) := \varphi(t, \tilde{x}(0), d(t))$. Then, for each battery $i \in \mathbb{N}$, we integrate the amount of energy delivered outside of its availability set

$$\Delta_i = \int_{T \setminus A_i} K(t, \varphi(t, \tilde{x}(0), d(t)), d(t)) dt.$$

Notice that $\Delta_i$ is a function of $\lambda$, as $\tilde{x}(0)$ is such. In particular, we focus on the map

$$\Lambda(\lambda) := [\Delta_i(\tilde{P}_i \cdot \mu(T \setminus A_i))]_{i \in \mathbb{N}}.$$

The following properties of $\Lambda$ are of interest.

1) $\Lambda([0, 1]^N) \subset [0, 1]^N$.
2) $\Lambda : [0, 1]^N \rightarrow [0, 1]^N$ is a continuous function.

Proof: Continuity of $\Lambda$: We prove the continuity of $\Lambda$ under the assumption that each availability set $A_i$ is at most the union of a finite number of (disjoint) intervals. Because of this, the same is true of the complement

$$T \setminus A_i = [t_1, t_2] \cup [t_3, t_4] \cup \ldots \cup [t_{M-1}, t_M]$$

for some even integer $M$. Let $\mu \equiv \mu(T \setminus A_i) \in \mathbb{N}$. Then, for each $\lambda$, we define $\hat{x}(0) = \text{diag}(\lambda_1 \lambda_2)$. Define the associated map as: $\Lambda(\lambda) = 1/P_j \int_{T \setminus A_i} K_j(t, \varphi(t, \hat{x}(0), d(t)), d(t)) dt$. Recalling that $K_j/P_j$ is the derivative of $-x_j$ with respect to time, and exploiting (21) we see that

$$\Lambda(\lambda) = \varphi_j(t_1, \hat{x}(0), d(\cdot)) - \varphi_j(t_2, \hat{x}(0), d(\cdot)) + \ldots + \varphi_j(t_{M-1}, \hat{x}(0), d(\cdot)) - \varphi_j(t_M, \hat{x}(0), d(\cdot)).$$

In order to assess the variation of $\Lambda$ with respect to $\lambda_1$ and $\lambda_2$, we consider, $\hat{x}_1(0) = \text{diag}(\mu) \lambda_1$ and $\hat{x}_2(0) = \text{diag}(\mu) \lambda_2$. By the previous equation, then

$$\Lambda(\lambda_1) - \Lambda(\lambda_2)$$

$$= \varphi_j(t_1, \hat{x}_1(0), d(\cdot)) - \varphi_j(t_2, \hat{x}_1(0), d(\cdot)) + \ldots + \varphi_j(t_{M-1}, \hat{x}_1(0), d(\cdot)) - \varphi_j(t_M, \hat{x}_1(0), d(\cdot)) - \varphi_j(t_1, \hat{x}_2(0), d(\cdot)) + \varphi_j(t_2, \hat{x}_2(0), d(\cdot)) + \ldots - \varphi_j(t_{M-1}, \hat{x}_2(0), d(\cdot)) + \varphi_j(t_M, \hat{x}_2(0), d(\cdot))$$

$$= \sum_{m=1}^M (1)^{m-1} \varphi_j(t_m, \hat{x}_1(0), d(\cdot)) - \varphi_j(t_m, \hat{x}_2(0), d(\cdot))$$

$$\leq M\|\hat{x}_1(0) - \hat{x}_2(0)\|_\infty = \|\text{diag}(\mu)\|_1 - \|\lambda_1 - \lambda_2\|_\infty.$$

Notice that a similar inequality holds for $\Lambda_1(\lambda_2) - \Lambda_1(\lambda_1)$. Hence, continuity of $\lambda$ follows, since $\|\Lambda_{\lambda_1} - \Lambda_{\lambda_2}\|_\infty \leq M\|\lambda_1 - \lambda_2\|_\infty$, where $M$ is the maximum of $M$ over the fleet of devices.

Notice that, thanks to property 1., 2., and by virtue of the Brouwer’s fixed point theorem, $\Lambda$ admits (at least) one fixed point, namely, a value $\lambda \in [0, 1]^N$ such that $\Lambda(\lambda) = \lambda$. Finally, we present our first main result as follows, and later provide details of its proof.

Theorem 1: Consider a fleet of devices $N$ with initial time to discharge, power ratings, and availability windows $x_j(0)$, $P_j$, and $A_j$, respectively, for $j \in \mathbb{N}$. Let $d(\cdot) : T \rightarrow \mathbb{R}_{\geq 0}$ be the aggregated power demand signal. Define, for convenience, the corresponding energy vector $E_j(0) = x_j(0) P_j$. Let $\lambda$ be a fixed point of the associated map $\Lambda$ and $\tilde{x}(0)$ be the corresponding vector of time to discharge, viz. $\tilde{x}(0) = x_j(0) + \lambda_j \mu(T \setminus A_j)$. The following facts are equivalent.
1) The signal \( \tilde{d}(\cdot) \) belongs to \( \mathcal{F}(E(0)) \) (viz. it is feasible for the fleet \( \mathcal{N} \) subject to availability constraints).

2) The signal

\[
\tilde{d}(t) = \sum_{j \in \mathcal{N}} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) \quad (22)
\]

is feasible for the fleet \( \mathcal{N} \) without availability constraints from the initial condition \( \tilde{x}(0) \).

**Remark 1:** Notice that \( \Lambda(\cdot) \) computes the average power absorbed outside of availability windows (for each device normalized by its own rated power), while \( \lambda(\cdot) \) is the (normalized) vector of additional energy provided to each device. Hence, a fixed-point of \( \Lambda(\cdot) \) corresponds to the situation where the extra power provided exactly matches absorption outside of availability windows.

Recalling that the policy \( \bar{K} \) in (12) differs from the feedback policy \( K \) in (16) by the power aggregation equations (10) and (14). In addition, the feedback policy \( K \) is time-invariant (unlike \( \bar{K} \)) as its formulation neglects availability windows. In order to prove our main result Theorem 1, it is useful to establish a mathematical relation between policies \( K \) and \( \bar{K} \) introduced in (11) and (15), respectively.

**Lemma 1:** For any \( x \in \mathbb{R}^N \), any \( d \geq 0 \) and any \( t \in \mathcal{T} \), it holds

\[
K(t, x, d) = \bar{K}(x, d + \sum_{j_t \notin A_j} K_j(t, x, d(\cdot)), d(t)).
\]

For the sake of readability, we defer the proof of the Lemma to the Appendix A. A useful consequence of Lemma 1 is the following alternative expression for solutions of (12).

**Lemma 2:** Let \( x \in \mathbb{R}^N \) be arbitrary, and \( d : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \) denote a given power profile. Let \( \tilde{d}(t) = d(t) + \sum_{j_t \notin A_j} K_j(t, x, d(\cdot)), d(t)) \). Then, for any \( t \in \mathcal{T} \), it holds

\[
\varphi(t, x, d(\cdot)) = \tilde{\varphi}(t, x, d(\cdot)).
\]

**Proof:** Let \( x(t) := \varphi(t, x, d(\cdot)) \). Clearly \( x(0) = x \), moreover, taking derivatives with respect to time yields

\[
\dot{x}(t) = -\Omega K(t, x(t), d(t)) = -\tilde{\varphi}(t, x, d(\cdot)).
\]

Hence, \( x(t) \) is also the solution of (12) with initial condition \( x(t) \) and input signal \( \tilde{d}(t) \). This proves Lemma 2.

We are now ready to prove Theorem 1.

**Proof:** We show first the implication 1 \( \Rightarrow \) 2. Let \( d : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \) be a feasible power profile with respect to availability sets \( A_j \) (\( j \in \mathcal{N} \)) and with initial time-to-discharge distribution \( \tilde{x}(0) \). Let \( \tilde{x}(0) \) be a fixed point of \( \Lambda(\cdot) \) and \( \tilde{x}(0) \) defined according to \( \tilde{x}(0) = x(t) + \lambda(t) \mu(T \setminus A_j) \). Then, there exist \( u_j(\cdot) : \mathcal{T} \rightarrow [0, \bar{P}_j] \), such that

\[
1) \sum_{j \in \mathcal{N}} u_j(t) = d(t);
2) u_j(t) = 0 \quad \text{for all } j \text{ and all } t \notin A_j;
3) \int_T u_j(t) dt \leq x_j(0) \bar{P}_j.
\]

Consider the following auxiliary input signals:

\[
\tilde{u}_j(t) = \begin{cases} u_j(t), & \text{if } t \in A_j \\ K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)), & \text{if } t \notin A_j, \end{cases}
\]

We claim that \( \tilde{u}_j(t) \) are feasible input for demand profile \( \tilde{d}(t) \) without availability constraints and for initial time-to-discharge distribution \( \tilde{x}(0) \). To this end, notice that

\[
\sum_{j \in \mathcal{N}} \tilde{u}_j(t) = \sum_{j_t \notin A_j} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) = d(t) + \sum_{j_t \notin A_j} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t))
\]

Moreover

\[
\int_T \tilde{u}_j(t) dt = \int_{A_j} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) dt \leq x_j(0) \bar{P}_j + \lambda_j \mu(T \setminus A_j) \bar{P}_j = \tilde{x}_j(0) \bar{P}_j,
\]

where the last inequality follows by recalling that \( \lambda \) is a fixed point of \( \Lambda(\cdot) \). This completes the proof of our claim.

We show next the implication 2 \( \Rightarrow \) 1. Let us assume \( \tilde{d}(\cdot) \) be feasible for initial condition \( \tilde{x}(0) \) and disregarding availability windows. Then, by the main result in [5], it can be dispatched through the GGDDF policy, and in particular: \( \hat{\varphi}(t, \tilde{x}(0), \bar{P}) \geq 0 \) for all \( t \in \mathcal{T} \). By Lemma 2, it follows that \( \varphi(t, \tilde{x}(0), d(t)) = \tilde{\varphi}(t, \tilde{x}(0), \bar{P}) \geq 0 \) for all \( t \in \mathcal{T} \). Equivalently, for all \( j \in \mathcal{N} \)

\[
\bar{P}_j \tilde{x}_j(0) - \int_T K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) dt \geq 0.
\]

We claim that the dispatch policy \( u_j(t) \) defined as follows, proves feasibility of \( d(t) \) for the fleet with initial condition \( x(t) \) and availability windows \( A_j \):

\[
u_j(t) = \begin{cases} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)), & \text{if } t \in A_j \\ 0, & \text{if } t \notin A_j, \end{cases}
\]

Indeed, by definition of \( K \), \( u_j(t) \in [0, \bar{P}_j] \) for all \( t \in \mathcal{T} \). Moreover, according to (24), \( u_j(t) = 0 \) for \( t \notin A_j \). In addition

\[
\int_T u_j(t) dt = \int_{A_j} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) dt \geq \int_{T \setminus A_j} K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) dt \leq \bar{P}_j \tilde{x}_j(0) - \lambda_j \mu(T \setminus A_j) \bar{P}_j = \tilde{P}_j \tilde{x}_j(0),
\]

where the last inequality follows by (23) and (19), while the subsequent derivations are from (20) and recalling that \( \lambda \) is a fixed point of \( \Lambda(\cdot) \). This completes the proof of the implication.

**III. DISCUSSION AND INTERPRETATIONS**

**A. Construction of the Feasible Policy**

Our first result, Theorem 1, provides a characterization of feasible aggregate demand profiles for storage fleets in the presence...
of availability constraints, by converting it to the simpler (and previously addressed) problem of dispatch for a fleet without availability constraints. These are the steps that can be followed to compute the schedule.

1) Run a numerical or analytical simulation of the GGDDF policy with availability windows within the time-interval \( T \) to compute \( \varphi(t, x_0, d(\cdot)) \) and to integrate the amount of energy delivered by each agents outside their availability window, according to (19).

2) For any \( \lambda \)-dependent initialization \( \tilde{x}(0) \), define the map \( \Lambda(\cdot) \), as specified in (20).

3) Compute fixed point \( \tilde{\lambda} \) of \( \Lambda(\cdot) \) by using a numerical iterative scheme.

Notice that, by continuity of \( \Lambda \), and forward invariance of \([0,1]^{2N} \), we may apply the Brouwer’s fixed point theorem, [13], to conclude that such \( \tilde{\lambda} \) always exists.

Extensive simulations have shown that the limit \( \lim_{k \to +\infty} \Lambda^k(\tilde{x}(0)) \) exists for any choice of \( \tilde{x}(0) \) (though this was not proved formally) leading to conjecture that a fixed point can simply be computed by iterating the map \( \Lambda \). As shown in (24), the GGDDF policy initialized with \( \tilde{x}(0) \) computed from \( \tilde{\lambda} \) allows to use \( K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(t)) \) restricted to the availability set \( A_j \) as a feasible dispatch input for device \( j \).

This dispatch policy matches a GGDDF policy without availability windows for a suitably inflated demand signal, but unlike the case of full availability, it is not causal, as it requires prior computation of the fixed point \( \tilde{\lambda} \), which, implicitly, takes into account demand over the whole horizon. As shown in the following, this is not a weakness of the approach, but rather a consequence of the considered setup. Indeed, considering a two batteries fleet, is enough to show that no causal policy exists in general.

### B. Impossibility of Causal Dispatch

Consider a fleet \(\mathcal{N} = \{1,2\} \), with rated powers \( P_1 = P_2 = 1 \) and assume an initial discharge time \( x_1(0) = 3 \) and \( x_2(0) = 6 \), respectively. The availability windows \( A_1 = [0,5] \) and \( A_2 = [0,12] = T \) are assigned. We propose the next two feasible aggregated demand signals, \( d_1(t) \) and \( d_2(t) \), as shown in Fig. 1. Notice that \( d_1(t) \) and \( d_2(t) \) coincide over the initial interval \([0,2] \). However, it is their respective behaviors for \( t \geq 2 \) which determines who is supposed to deliver the first two power units. Specifically, \( d_1(t) \) can only be fulfilled with device 1 delivering power in the interval \([0,3] \) and device 2 delivering on the interval \([5,11] \). On the contrary, \( d_2(t) \) can only be met if device 1 delivers power over the interval \([2,5] \) and device 2 in the interval \([0,6] \). This proves that, even in elementary situations, causal dispatch policies do not exist.

In the light of our previous result Theorem 1, it is worth noticing that the dispatch for \( d_2(t) \) coincides with a fixed point \( \tilde{\lambda} = [0,0] \). Indeed, the GGDDF policy (without availability windows) would initially allocate all the power to the second device (because of its higher discharge time), and additionally trigger battery 1 in the interval \([2,5] \). The profile \( d_1(t) \) corresponds instead to \( \tilde{\lambda} = [6/7,0] \). Notice that, with such choice of \( \tilde{\lambda} \) we get \( \tilde{x}(0) = [3 + 7 \cdot 6/7,6] = [9,6] \). Hence, in \([0,3] \) only device 1 is discharging, due to its highest discharge time, while at time 3 the following equality \( \tilde{x}_1(3) = \tilde{x}_2(3) = 6 \) is achieved, so that devices will then discharge together until time 11, when they will both be empty. The energy delivered outside the availability window by device 1 is therefore 6, which corresponds to \( \lambda_1 = 6/\mu(T, A_1) = 6/7 \). Instead \( A_2 = T \) and therefore, \( \lambda_2 = 0 \).

### IV. Time-Domain Characterization of Feasible Set

In this section, we present necessary and sufficient conditions on the aggregate demand signal \( d(t) \) for feasibility with respect to a fleet \(\mathcal{N} \) with given initial conditions, power ratings, and availability sets. Our main result states that a signal \( d(t) \) is feasible if and only if it fulfills a set of linear constraints that can be explicitly computed as functions of initial conditions and availability windows.

**Theorem 2:** Consider a fleet \(\mathcal{N} \) of batteries with maximum power ratings \( P_j \), availability windows \( A_j \subset T \) and initial time-to-discharge \( x_j(0) \), respectively, for all \( j \in \mathcal{N} \). A signal \( d : T \to [0,\infty) \) is feasible for \(\mathcal{N} \) if and only if, for all \( W \subset T \) the following inequality holds:

\[
\int_{W} d(t) \, dt \leq \sum_{j \in \mathcal{N}} \min\{\mu(A_j \cap W), x_j(0)\} P_j.
\]

From a physical point of view, (25) requires that the energy request over any time window \( W \) be less than what the fleet can deliver, over the same time window \( W \) by operating at rated power (for devices who have enough energy) or at any other rate that yields a depletion of the battery within the considered time window, for those who do not have. It is worth pointing out that the above conditions, in the case of \( A_j = [0, +\infty) \) for all \( j \in \mathcal{N} \) boil down to existing characterizations of feasibility. Indeed, (25) reads

\[
\int_{W} d(t) \, dt \leq \sum_{j \in \mathcal{N}} \mu(W) P_j - \int_{0}^{\mu(W)} 1(t) - 1(t - x_j(0)) \, dt
\]

with \( 1(\cdot) \) denoting the Heaviside function. Specifically, for nonincreasing demand signals \( d(\cdot) \), the maximum integral value for a given set \( W \) of assigned measure \( \mu(W) \), is achieved for \( W = [0, \mu(W)] \), viz., \( \int_{W} d(t) \, dt \leq \int_{0}^{\mu(W)} d(t) \, dt \). Hence, for all \( W \) of measure \( \mu(W) = T \) condition (26) is fulfilled if and only if it holds on the interval \([0, T] \). Hence, (26) can equivalently be stated (in the case of nonincreasing signals) as

\[
\int_{0}^{T} d(t) \, dt \leq \int_{0}^{T} \sum_{j \in \mathcal{N}} P_j [1(t) - 1(t - x_j(0))] \, dt
\]

for all \( T \in T \) which is in agreement with the result in [7].

An important Corollary of Theorem 2 is for demand signals \( d(t) \) which are piecewise constant on finitely many equally spaced time intervals, and for availability sets which are the union of such equally spaced intervals. This corresponds to \( d(t) \) of the form \( d(t) = \sum_{k=0}^{T-1} d_k [1(t - k) - 1(t - (k + 1))] \),
where \(1(\cdot)\) is the Heaviside function. From a practical point of view this corresponds to a discrete time formulation of the dispatch problem. Let us denote the time-horizon as \(\mathcal{T} = \{1, 2, \ldots, T\}\) and the availability windows \(A_j \subseteq \mathcal{T}\).

**Corollary 1:** Consider a fleet \(\mathcal{N}\) of batteries with maximum power ratings \(P_j\), availability windows \(A_j \subset \mathcal{T}\) and initial time-to-discharge \(x_j(0)\), respectively, for all \(j \in \mathcal{N}\). A signal \(d: \mathcal{T} \to [0, +\infty)\) is feasible for \(\mathcal{N}\) if and only if, for all \(W \subset \mathcal{T}\) the following inequality holds:

\[
\sum_{k \in W} d(k) \leq \sum_{j \in \mathcal{N}} \min\{\text{card} (A_j \cap W), x_j(0)\} P_j. \tag{27}
\]

It is worth pointing out that inequalities (27) completely characterize the set of feasible demand profiles for any fleet \(\mathcal{N}\) with arbitrary availability sets. For a time-horizon of \(T\) sampling intervals, \(2^T\) constraints are enough to characterize the polytope of feasible aggregated demand profiles, regardless of the size of the fleet.

**V. ON OPTIMALITY OF DISPATCH POLICIES**

Throughout this section, we highlight important optimality properties of the dispatch policies previously introduced. We defined already the set of feasible demand profiles for a fleet subject to availability constraints, according to (6). We define a similar notion, for fleets without full availability.

The notion of unserved energy is an important measure of reliability of supply. From the mathematical point of view, in the context of this article, we may define, for each policy \(u(t)\) defined over \(\mathcal{T}\) the following functional:

\[
\mathcal{U}(u(\cdot)) := \int_{\mathcal{T}} \max \left\{ d(t) - \sum_{j \in \mathcal{N}} u_j(t), 0 \right\} dt. \tag{29}
\]

When a demand signal is feasible, one might find \(u(\cdot)\) such that the corresponding unserved energy is zero (and this is what the discussed policies allow to do); however, for unfeasible demand signals, it is still desirable to minimize \(\mathcal{U}\). The problem can be formulated as follows:

\[
\begin{aligned}
\min_{u(\cdot), E(t)} & \mathcal{U}(u(\cdot)) \\
\text{s.t.} & \dot{E}(t) = -u(t) \quad \forall t \in \mathcal{T} \\
& E(0) = \Omega^{-1} x(0) \\
& 0 \leq u_j(t) \leq P_j \quad \forall t \in \mathcal{T} \forall j \in \mathcal{N} \\
& E(t) \geq 0 \quad \forall t \in \mathcal{T} \\
& u_j(t) = 0 \quad \forall t \notin A_j \forall j \in \mathcal{N}. \tag{30}
\end{aligned}
\]

The solution to problem (30) can be found by considering an augmented demand signal \(\bar{d}\) for a fleet with full availability and suitably augmented initial time-to-discharge \(\bar{x}(0)\). Consider the modified cost functional

\[
\mathcal{U}(\bar{u}(\cdot)) := \int_{\mathcal{T}} \max \left\{ \bar{d}(t) - \sum_{j \in \mathcal{N}} \bar{u}_j(t), 0 \right\} dt \tag{31}
\]

where \(\bar{d}\) is defined according to (22). Its minimization disregarding partial availability constraints can be formulated as

\[
\begin{aligned}
\min_{\bar{u}(\cdot), \bar{E}(\cdot)} & \mathcal{U}(\bar{u}(\cdot)) \\
\text{s.t.} & \dot{\bar{E}}(t) = -\bar{u}(t) \quad \forall t \in \mathcal{T} \\
& \bar{E}(0) = \Omega^{-1} \bar{x}(0) \\
& 0 \leq \bar{u}_j(t) \leq P_j \quad \forall t \in \mathcal{T} \forall j \in \mathcal{N} \\
& \bar{E}(t) \geq 0 \quad \forall t \in \mathcal{T}. \tag{32}
\end{aligned}
\]

Previous results (see [6]) show that problem (32) can be solved through a GGDDDF policy. Our main result in this respect is to connect the optimal solution of (30) to that of (32).
Theorem 4: Consider the optimization problems in (30) and (32), where the augmented demand \( \tilde{d} \) is defined according to (22) and the initial condition \( \tilde{x}(0) \) fulfills \( \tilde{x}(0) = x_j(0) + \tilde{\lambda}_j u(T \setminus A_j) \) for some fixed point \( \tilde{\lambda} \) of \( \Lambda() \). Then, the minimum value of unserved energy for (30) equals the minimum of (32).

Proof: As a first step it is useful to remark that minimizing unserved energy can be equivalently formulated as

\[
\min_{u(),E()} \sum_{j \in N} \min \{ E(\tilde{x}), 0 \}
\]

s.t. \( \dot{E}(t) = -u(t) \quad \forall t \in \mathcal{T} \)
\[E(0) = \Omega^{-1}x(0)\]
\[0 \leq u_j(t) \leq \tilde{P}_j \quad \forall t \in \mathcal{T} \quad \forall j \in N\]
\[u_j(t) = 0 \quad \forall t \notin A_j, j \in N\]
\[d(t) = \sum_{j \in N} u_j(t) \tag{33}\]

where \([0, \bar{\tau}] = \mathcal{T}\). To see this, notice that any feasible \( u() \), and associated \( E() \) of (33) can be modified as
\[\tilde{u}_j(t) = \begin{cases} u_j(t), & \text{if } E_j(t) \geq 0 \\ 0, & \text{if } E_j(t) < 0. \end{cases}\]

Hence, the corresponding \( \dot{E}_j(t) \) fulfills, \( \dot{E}_j(t) \geq 0 \) for all \( t \in \mathcal{T} \). Moreover
\[\mathcal{U}(\tilde{u}) = \int_{\mathcal{T}} \max \{ d(t) - \sum_{j \in N} \tilde{u}_j(t), 0 \} \, dt \]
\[= \int_{\mathcal{T}} d(t) - \sum_{j \in N} u_j(t) + [\tilde{u}_j(t) - u_j(t)] \, dt \]
\[= \sum_{j \in N} \int_{\{t \in \mathcal{T} : E_j(t) < 0\}} -u_j(t) \, dt \]
\[= \sum_{j \in N} \int_{\{t \in \mathcal{T} : E_j(t) < 0\}} \dot{E}_j(t) \, dt \]
\[= -\sum_{j \in N} \min \{ E_j(\bar{\tau}), 0 \}.\]

Hence, any feasible pair \( u(), E() \) of (33) maps to a feasible pair of (30), whose unserved energy coincides with the cost of (33). Hence, the minimum of (30) is less or equal than the optimum of (33). Conversely, under Assumption 1, we see that any optimal \( u^*(\cdot) \) of (30) can be made into a feasible \( \tilde{u} \) of (33) by injecting some complementary power \( \Delta u \) so as to achieve for the modified signal \( \tilde{u} = u^* + \Delta u \), \( d(t) = \sum_j u_j(t) \). Notice that, for the corresponding optimal solution \( E^* \) it holds necessarily \( E_j^*(\bar{\tau}) = 0 \) for all devices \( j \) such that there exists \( t \in \mathcal{T} \) with \( d(t) > \sum_{j \in N} u_j^*(t) \). Because of this latter constraint, the corresponding \( E \) signal fulfills
\[-\sum_{j \in N} \min \{ \dot{E}_j(\bar{\tau}), 0 \} \leq -\sum_{j \in N} \dot{E}_j(\bar{\tau}) - E_j(\bar{\tau}) \]
\[= \sum_{j \in N} \int_{\mathcal{T}} \Delta u_j^*(t) \, dt \]
\[= \int_{\mathcal{T}} d(t) - \sum_{j \in N} u_j^*(t) \, dt = \mathcal{U}(u^*).\]

Hence, by feasibility of \( \tilde{u} \) and \( \dot{E} \) for problem (33) the optimal cost of (33) is less than or equal to the optimal cost of (30). Similarly, for fleets without partial availability constraints, problem (32) is equivalently written as
\[\min_{\tilde{u}(),\dot{E}()} \sum_{j \in N} \min \{ \dot{E}_j(\bar{\tau}), 0 \}
\]
\[\text{s.t. } \dot{E}(t) = -\tilde{u}(t) \quad \forall t \in \mathcal{T} \]
\[\dot{E}(0) = \Omega^{-1}\tilde{x}(0)\]
\[0 \leq \tilde{u}_j(t) \leq \tilde{P}_j \quad \forall t \in \mathcal{T} \quad \forall j \in N\]
\[\tilde{u}_j(t) = 0 \quad \forall t \notin A_j, j \in N\]
\[\dot{d}(t) = \sum_{j \in N} \tilde{u}_j(t).\tag{34}\]

Notice that (33) and (34) relax the positivity constraint on \( E \) and \( \dot{E} \), respectively. In its place, an equality constraint is added, forcing the total power delivered to meet aggregate demand. In this way, input signals that push charge levels to negative values are still regarded as feasible, but their impact is accounted for in the cost functional. The result is proved through a series of inequality. Consider the subset of signals \( \tilde{u} \) achieved through the following construction:
\[\tilde{u}_j(t) = \begin{cases} u_j(t), & \text{if } t \in A_j \\ K_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(\cdot)), & \text{if } t \notin A_j \end{cases}\]
\[\tilde{u}_j(t) = \begin{cases} u_j(t), & \text{if } t \in A_j \\ \dot{K}_j(t, \varphi(t, \tilde{x}(0), d(\cdot)), d(\cdot)), & \text{if } t \notin A_j \end{cases}\]
\[\dot{d}(t) = \sum_{j \in N} \tilde{u}_j(t).\tag{35}\]

When avoiding energy curtailment is to be prioritized, another functional of interest is the so called time to failure. For each dispatch policy \( u \) of aggregate demand \( d(\cdot) \) this is defined as the first time some battery state of charge becomes negative. Formally, \( T_f(x(\cdot)) := \inf \{ t \in \mathcal{T} : \min_{j \in N} x_j(t) < 0 \} \).

Our aim is to solve the following optimization problem:
\[\tau^* := \max_{u(),x(\cdot)} T_f(x(\cdot))\]
\[\text{s.t. } x(0) = x_0\]
\[x(t) = -\Omega u(t) \quad \forall t \in \mathcal{T}\]
\[0 \leq u_j(t) \leq \tilde{P}_j \quad \forall t \in \mathcal{T} \quad \forall j \in N\]
\[u_j(t) = 0 \quad \forall t \notin A_j, j \in N\]
\[d(t) = \sum_{j \in N} u_j(t).\tag{36}\]
TABLE I
COMPUTATIONAL COMPLEXITY OF SOLVING THE FIXED λ.

| Device number | 10 | 50 | 100 | 250 | 500 |
|---------------|----|----|-----|-----|-----|
| Computation time (min) | $1 \times 10^{-4}$ | 0.002 | 0.05 | 1.51 | 30.13 |

For fleets without availability constraints this can be maximized through the use of a GGDDF policy. However, it is shown in the following section that the naive application of this same policy does not achieve maximization of time to failure. We propose an iterative procedure to compute the optimal time to failure and the associated dispatch.

1) Let $T = [0, \tau_0 := \bar{\tau}]$; let $k = 0$.
2) Repeat:
   a) compute $\lambda_k$, fixed point of $\Lambda$, over window $[0, \tau_k]$;
   b) apply GGDDF policy from $\bar{x}(0)$. Let $\bar{x}_k$ be corresponding state evolution;
   c) $\tau_{k+1} := T_j(\bar{x}_k)$; increase $k$.

Our main result is the following.

**Theorem 5:** The iteration defined above converges to the optimal time to failure, i.e., $\lim_{k \to +\infty} \tau_k = \tau^*$, as in (36).

VI. NUMERICAL EXAMPLES

The proposed dispatch algorithm for heterogeneous devices is applied to numerical case studies. The computational tasks were implemented using MATLAB R2019a and solved by the routine `fsolve` or `fmincon`, on a computer with 2-core 3.50 GHz Intel(R) Xeon(R) E5-1650 processor and 32 GB RAM.

A. Feedback Policy

Next, the proposed feedback policy is tested on a fleet of 500 devices. A time interval of 24 h, from 12 : 00 h to 12 : 00 h of the next day, is considered. For every device $j \in \mathcal{N} = \{1, 2, \ldots, 500\}$, the rated power $P_j$ is set to the same value $P_j = 1$ kW. The initial time-to-discharge $x_j(0)$ (initial energy $E_j(0)$) follows a normal distribution with mean $\mu_E = 8$ kWh and standard deviation $\sigma_E = 1.5$ kWh. It is assumed that each device can discharge only within a continuous time interval $[t_j, t_j + d_j]$, where $t_j$ and $d_j$ follow normal distributions, with the following mean and standard deviation: $\mu_t = 18 : 00$ h, $\sigma_t = 1$ h, $\mu_d = 10$ h, $\sigma_d = 2$ h.

To provide a clearer demonstration, the boundaries of availability windows are chosen as integers and we consider piecewise constant aggregate demand signals with integer switching time instants. The calculation of fixed $\lambda$ has been completed in about 30 min. Table I presents how the computational time for $\lambda$ changes with respect to the number of devices in the fleet. It is observed that the time to obtain the fixed $\lambda$ is growing approximately as $N^4$, thus exhibiting good scalability properties. When the proposed approach is adopted into the receding horizon framework, the convergence of $\lambda$ will be faster because a warm start is available if there is no significant mismatch between the realization and prediction. Moreover, in real-world applications, more powerful machines can further reduce the computational time and easily perform the algorithm on larger number of devices.

Fig. 2(a) shows the aggregate availability, viz. $\sum_{j \in \mathcal{A}} P_j$ and required demand profile. This demand is feasible since the auxiliary time to discharge of all devices, in Fig. 2(b) and (c), are nonnegative. Some representative examples of individual power dispatch signals are shown in Fig. 3. Their availability is displayed with a green area and individual discharging power profiles are decided according to the ranking of auxiliary time to discharge which is presented in Fig. 2(b) and (c). When the demand $d(t) = 0$ before 16 : 00 h and after 11 : 00 h, all devices have a constant time to discharge. If the demand is positive, higher time-to-discharge devices are prioritized for discharging with $P$, medium time-to-discharge devices are discharging at a fraction of rated power, and lower time-to-discharge devices are preserving energy for later use.

B. Minimum Unserved Energy versus Maximum Time to Failure

We consider a smaller number of devices $N = 20$ with parameters of energy and partial availability constraints following the same probability density functions, as in Section VI-A. The maximum aggregate availability of these devices and the required demand profile are shown in Fig. 4. Numerically, the aggregate initial energy is $\sum_{j \in \mathcal{N}} E_j(0) = 163.25$ kWh and the total energy by the demand profile is $\int_T d(t)dt = 168.05$ kWh. As a result, this demand profile is unfeasible due to insufficient total energy, which leads to the consequence that some devices finish the discharging task with negative time to discharge according to the policy (15).

Fig. 5 demonstrates the auxiliary and actual time to discharge using two dispatch policies. Over the 24 h full time window,
the GGDDF policy serves total energy 151.40 kWh, hence, the minimum unserved energy is \( U(\cdot) = 16.65 \) kWh. Nevertheless, this policy has a relatively short time to failure, since one device crosses into negative time to discharge at 24 : 30 h on the left-bottom subplot of Fig. 5.

We compare the above scheduling with the policy resulting from the iteration algorithm corresponding to Theorem 5. This identifies the discharging schedules for all devices up to around 4 : 15 h, which is the maximum time to failure. This is significantly larger (\( \approx 3 : 45 h \)) than what is achieved by the previous dispatch. Notice that the total energy served before this time instant is 149.69 kWh. The considered example exhibits a significant gap between the time to failure of the original minimum unserved energy schedule and the maximum time to failure achievable. This is in contrast to the case of full availability, where the GGDDF policy achieve both maximum time to failure and minimum unserved energy.

**VII. Conclusion**

This article solves the optimal dispatch problem for heterogeneous fleets of storage devices, subject to partial availability constraints and without cross-charging. This significantly extends previous known results which were limited to fleets with full availability, [5], [6], [7], [8]. The approach transforms the problem to one of dispatch of a fleet over the same time-horizon but in the absence of availability constraints, for some auxiliary and increased demand signal and correspondingly increased initial energies (or discharge time). Admissible dispatch policies are provided whenever the aggregate demand signal is feasible and policies maximizing time to failure or minimizing unserved energy are formulated and discussed, for the case of unfeasible demand.

A characterization of the feasible set of aggregate demand signals is presented, which may serve as an effective computational approach to embed flexible demand or coordinated fleet operation during outages in large scale optimization problems, by exactly capturing the degrees of freedom afforded by the fleet without explicit mention of individual power schedules. Finally, examples of application of the techniques are provided for fleets of medium and large size, to demonstrate the effectiveness and scalability of the approach. Several open questions remain in this area, particularly related to more realistic battery models, multiarea dispatch problems or bidirectional power transfers.

**APPENDIX A**

**Proof of Lemma 1**

We prove the lemma by separately considering all the cases involved in the policy definition.

1) Let \( i \in \mathcal{N}_{\tau_h} \), with \( \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}}: t \in A_j \) \( \bar{P}_j \) \( \leq d \). Then \( K_i(t, x, d) = \bar{P}_i \), moreover, \( K_j(t, x, d) = \bar{P}_j \) for all \( j \in \mathcal{N}_{\tau_h} \) with \( h \leq k \). Hence

\[
\sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}} \bar{P}_j = \sum_{h \leq k} \left( \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j + \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} K_j(t, x, d) \right) \leq d + \sum_{j \notin A_j} K_j(t, x, d).
\]

By definition of \( \tilde{K} \) this implies \( \tilde{K}_i(x, d + \sum_{j \notin A_j} K_j(t, x, d)) = \tilde{P}_i = K_i(t, x, d) \).

2) Let \( i \in \mathcal{N}_{\tau_h} \) with \( \sum_{h < k} \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j \leq d < \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j \). Then \( K_j(t, x, d) = r(t) \bar{P}_j \) for all \( j \in \mathcal{N}_{\tau_h} \) and \( K_j(t, x, d) = \bar{P}_j \) for all \( j \in \mathcal{N}_{\tau_h} \) with \( h < k \). In addition, \( K_j(t, x, d) = 0 \) for all \( j \in \mathcal{N}_{\tau_h} \) with \( h > k \). As a consequence

\[
\sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}} \bar{P}_j = \sum_{h \leq k} \left( \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j + \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} K_j(t, x, d) \right) \leq d + \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} K_j(t, x, d) = d + \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} K_j(t, x, d) \]

Moreover

\[
d \sum_{j \notin A_j} K_j(t, x, d) < \left( \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j \right) + \sum_{j \notin A_j} K_j(t, x, d) = \left( \sum_{h \leq k} \sum_{j \in \mathcal{N}_{\tau_h}, t \in A_j} \bar{P}_j \right) + \left( \sum_{h < k} \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} \bar{P}_j \right) + \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} r(t) \bar{P}_j = \left( \sum_{h < k} \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} \bar{P}_j \right) + \sum_{j \in \mathcal{N}_{\tau_h}, t \notin A_j} r(t) \bar{P}_j.
\]
\[ \left( \sum_{h \leq k \in \mathcal{N}_{N_h}} \tilde{P}_j \right) \geq \left( \sum_{h \leq k \in \mathcal{N}_{N_h}} P_j \right). \]

Hence, provided we can show \( r(t) = \tilde{r}(t) \), we see that, \( \hat{K}_i(x, d + \sum_{j \in \mathcal{A}_j} K_j(t, x, d)) = \tilde{r}(t) \tilde{P}_i = r(t) \tilde{P}_i = K_i(t, x, d). \)

To complete the proof of this case, notice that

\[
\begin{align*}
\tilde{r}(t) \left( \sum_{j \in \mathcal{N}_{N_k}} \tilde{P}_j \right) &= r(t) \sum_{j \in \mathcal{N}_{N_k}} P_j + \sum_{j \in \mathcal{N}_{N_k}} r(t) \tilde{P}_j \\
&= r(t) \left( \sum_{j \in \mathcal{N}_{N_k}} P_j \right) + \sum_{j \in \mathcal{N}_{N_k}} K_j(t, x, d) \\
&= d \sum_{j \in \mathcal{N}_{N_k}} P_j + \sum_{j \in \mathcal{N}_{N_k}} K_j(t, x, d) \\
&= \tilde{r}(t) \sum_{j \in \mathcal{N}_{N_k}} \tilde{P}_j.
\end{align*}
\]

3) Finally, when \( i \in \mathcal{N}_{N_h} \) and \( d < \sum_{h < k} \sum_{j \in \mathcal{N}_{N_h}} P_j \), we have \( K_j(t, x, d) = 0 \) for all \( j \in \mathcal{N}_{N_h} \) for all \( h \geq k \). As a consequence

\[
\begin{align*}
d + \sum_{j \in \mathcal{A}_j} K_j(t, x, d) &< \left( \sum_{h < k \in \mathcal{N}_{N_h}} \sum_{j \in \mathcal{A}_j} \tilde{P}_j \right) + \sum_{j \in \mathcal{A}_j} K_j(t, x, d) \\
&\leq \left( \sum_{h < k \in \mathcal{N}_{N_h}} \sum_{j \in \mathcal{A}_j} \tilde{P}_j \right) + \sum_{h < k} \sum_{j \in \mathcal{N}_{N_h}} \tilde{P}_j \\
&= \sum_{j \in \mathcal{N}_{N_h}} \tilde{P}_j.
\end{align*}
\]

This proves that \( \hat{K}_i(x, d + \sum_{j \in \mathcal{A}_j} K_j(t, x, d)) = 0 = K_i(t, x, d). \)

This concludes the proof of the Lemma.

**APPENDIX B
PROOF OF THEOREM 2**

We show first necessity of condition (25). Let \( d(\cdot) \) be a feasible demand signal. Then there exists \( u_j(\cdot) \) such that

1) total demand constraint: \( \sum_{j} u_j(t) = d(t) \) for all \( t \geq 0 \);
2) power constraint: \( 0 \leq u_j(t) \leq \bar{P}_j \) for all \( t \geq 0 \);
3) availability constraints: \( u_j(t) = 0 \) for all \( t \notin \mathcal{A}_j \);
4) energy constraint: \( \int_0^\infty u_j(t) \, dt \leq \bar{P}_j x_j(0) \).

Hence, for every \( \mathcal{W} \subset \{0, +\infty\} \) we see the following:

\[
\int_{\mathcal{W}} d(t) \, dt = \int_{\mathcal{W}} \sum_{j \in \mathcal{N}} u_j(t) \, dt = \sum_{j \in \mathcal{N}} \int_{\mathcal{W}} u_j(t) \, dt
\]

This completes the necessity proof.

Conversely, let \( d(\cdot) \) be unfeasible. Consider the associated map \( \lambda \), as defined in (20) and (19). Let \( \tilde{\lambda} \) be a fixed point of the map (which always exists by the Brower’s fixed point theorem) and \( \tilde{x}(0) \) the associated initial condition. Since \( d \) is unfeasible, the set defined below is nonempty

\[ \mathcal{N}_e = \{ j \in \mathcal{N} : \varphi_j(\sup \mathcal{T}, \tilde{x}(0), \tilde{d}(\cdot)) < 0 \} \]

viz. of batteries which have negative energy at the end of the considered time-horizon, when the signal \( d \) is dispatched. Equivalently, by virtue of Lemma 2, \( \mathcal{N}_e \) can be expressed as \( \mathcal{N}_e = \{ j \in \mathcal{N} : \varphi_j(\sup \mathcal{T}, \tilde{x}(0), d(\cdot)) < 0 \} \).

Let \( u_j(t) \) be defined as

\[
\begin{cases}
K(t, \varphi(t, \tilde{x}(0), d(\cdot), d(t))), & \text{if } t \in \mathcal{A}_j \\
0, & \text{if } t \notin \mathcal{A}_j
\end{cases}
\]

The set \( \mathcal{N}_e \) can equivalently be expressed as: \( \mathcal{N}_e = \{ j \in \mathcal{N} : \int_{\mathcal{W}} u_j(t) \, dt > \tilde{P}_j x_j(0) \} \), thanks to the fact that \( \tilde{\lambda} \) is a fixed point of \( \lambda \). The following implication is a consequence of the order-preserving property of the maps \( \varphi \) and \( \varphi \) of the definition of \( K \):

\[ \exists i \in \mathcal{N}_e : u_i(t) > 0 \Rightarrow u_j(t) = \tilde{P}_j \forall j \notin \mathcal{N}_e : t \in \mathcal{A}_j. \]

Let \( \mathcal{W} \) denote the set \( \mathcal{W} = \bigcup_{j \notin \mathcal{N}_e} \supp(u_j) \). Hence we might proceed to the following manipulations:

\[
\int_{\mathcal{W}} d(t) \, dt = \int_{\mathcal{W}} \sum_{j \in \mathcal{N}_e} u_j(t) \, dt
\]

This shows that (25) is violated and concludes the proof.

**APPENDIX C
PROOF OF THEOREM 5**

We first show by induction that \( \tau^* \in [0, \tau_k] \) for all \( k \in \mathbb{N} \).

The claim is trivially true for \( k = 0 \), given the initialization \( \tau_0 \).

Assume next that \( \tau^* \in [0, \tau_k] \). We will show that \( \tau^* \in [0, \tau_{k+1}] \).

Let \( \tilde{\lambda}_k \) be the fixed point of \( \lambda \) at the \( k \)th iteration of the algorithm.

Consider \( \tilde{x}_k(0) \), the corresponding value of time to discharge with augmented energy proportional to \( \tilde{\lambda}_k \). Denote by \( \tilde{x}_k(\cdot) \) the solution corresponding to a GGDDF policy for the fleet without availability constraints. It is known that this solution

\[ = \sum_{j \in \mathcal{N} \setminus \mathcal{W} \cap \mathcal{A}_j} u_j(t) \, dt \leq \min \{ \mu(\mathcal{W} \cap \mathcal{A}_j), x_j(0) \} \tilde{P}_j. \]
maximizes time to failure, viz.
\[
T_f(\tilde{x}_k) = \max_{u(\cdot), \bar{x}(\cdot)} \int_{t(0)}^{t(T)} \mathcal{J}(\bar{x}(\cdot))
\]
subject to
\[
\begin{align*}
\dot{x}(t) &= -\Omega \tilde{u}(t) & \forall t \in [0, \tau_k] \\
0 &\leq u_j(t) \leq \tilde{P}_j & \forall t \in [0, \tau_k] \forall j \in \mathcal{N} \\
\tilde{d}_k(t) &= \sum_{j \in \mathcal{N}} \tilde{u}_j(t).
\end{align*}
\]

Moreover, this also equals the maximum time to failure for demand signal \( \tilde{d} \) over a restricted class of input policies, viz.
\[
T_f(\tilde{x}_k) = \max_{u(\cdot), \bar{x}(\cdot)} \int_{t(0)}^{t(T)} \mathcal{J}(\bar{x}(\cdot))
\]
subject to
\[
\begin{align*}
\dot{x}(t) &= 0 & \forall t \in [0, \tau_k] \\
\dot{\tilde{u}}_j(t) &= \left( K_j(t, \varphi(t, \tilde{x}_k(0), \tilde{d}_k(t)), \tilde{d}_k(t)) \right. & t \in \mathcal{A}_j \\
\dot{\tilde{x}}(t) &= -\Omega \tilde{u}(t) & \forall t \in [0, \tau_k] \\
0 &\leq u_j(t) \leq \tilde{P}_j & \forall t \in [0, \tau_k] \forall j \in \mathcal{N} \\
\tilde{d}_k(t) &= \sum_{j \in \mathcal{N}} \tilde{u}_j(t).
\end{align*}
\]

Since the energy delivered by each device outside \( \mathcal{A}_j \) in the above optimization is equal to the additional energy provided at time 0, we see that \( \tilde{x}(t) \geq x(t) \), where \( x_j \) denotes the state evolution for input \( u_j \) (which matches \( \tilde{u}_j \) in \( \mathcal{A}_j \) but is zero otherwise). Hence, the following inequality holds:
\[
T_f(\tilde{x}_k) \geq \max_{u(\cdot), x(\cdot)} \int_{t(0)}^{t(T)} \mathcal{J}(x(\cdot))
\]
subject to
\[
\begin{align*}
x(0) &= \Omega E(0) \\
\dot{x}(t) &= -\Omega u(t) & \forall t \in [0, \tau_k] \\
0 &\leq u_j(t) \leq \tilde{P}_j & \forall t \in [0, \tau_k] \forall j \in \mathcal{N} \\
u_j(0) &= 0 & \forall t \notin \mathcal{A}_j \forall j \in \mathcal{N} \\
d(t) &= \sum_{j \in \mathcal{N}} u_j(t).
\end{align*}
\]

The maximization problem in (37), however, yields \( \tau^* \) due to the induction hypothesis: \( \tau^* \in [0, \tau_k] \). Hence, \( \tau_{k+1} = T_f(\tilde{x}_k(\cdot)) \geq \tau^* \). Notice that \( \tau_k \) is, by construction, a nonincreasing and lower bounded sequence. Hence it admits a limit \( \bar{\tau} \). By the previous inequality, we see that \( \bar{\tau} = \lim_{k \to +\infty} \tau_k \geq \tau^* \). We need to show that the equality holds.

By definition of \( \tilde{x}_k \), we have \( \tilde{x}_k(\tau_k+1) = 0 \forall t \in [0, \tau_k+1] \).

Let \( k_n \) be any divergent sequence such that \( \tilde{x}_{k_n}(0) \) converges to some initial condition \( x_{\infty}(0) \) as \( n \to +\infty \). Accordingly, \( \tilde{x}_{k_n}(\cdot) \) will converge to signal \( \varphi(t, x_{\infty}(0), d(\cdot)) \) and \( \tilde{d}_{k_n}(\cdot) \) will converge to some limit \( d_{\infty}(\cdot) \) fulfilling definition
\[
\tilde{d}_{k_n}(t) = d(t) + \sum_{j \in \mathcal{A}_j} K_j(t, \varphi(t, x_{\infty}(0), d(\cdot)), d(t)).
\]

In particular, taking limits along subsequence \( k_n \) yields \( \tilde{x}_{\infty}(\cdot) \geq 0 \) for all \( t \in [0, \tau] \). Hence \( d_{\infty}(\cdot) \) is feasible over \( [0, \tau] \). Clearly \( \Delta(\tilde{\lambda}_{\infty}) = \tilde{\lambda}_{\infty} \), and therefore, by Theorem 1, \( d \) is feasible for the fleet with partial availability constraints over the interval \([0, \tau] \). Hence \( \tau^* \geq \tau \). Since we already proved the opposite inequality, \( \tau^* = \tau \) which completes the proof.