Maps That Take Lines to Circles, in Dimension 4

V. Timorin

Abstract. We list all analytic diffeomorphisms between an open subset of the 4-dimensional projective space and an open subset of the 4-dimensional sphere that take all line segments to arcs of round circles. These are the following: restrictions of the quaternionic Hopf fibrations and projections from a hyperplane to a sphere from some point. We prove this by finding the exact solutions of the corresponding system of partial differential equations.

1 Introduction

Let $U$ be an open subset of the 4-dimensional real projective space $\mathbb{R}P^4$ and $V$ an open subset of the 4-dimensional sphere $S^4$. We study diffeomorphisms $f : U \to V$ that take all line segments lying in $U$ to arcs of round circles lying in $V$. For the sake of brevity we will always say in the sequel that $f$ takes all lines to circles. The purpose of this article is to give the complete list of such analytic diffeomorphisms.

Remark. Given a diffeomorphism $f : U \to V$ that takes lines to circles, we can compose it with a projective transformation in the preimage (which takes lines to lines) and a conformal transformation in the image (which takes circles to circles). The result will be another diffeomorphism taking lines to circles.

Example 1. For example, suppose that $S^4$ is embedded in $\mathbb{R}^5$ as a Euclidean sphere and take an arbitrary hyperplane and an arbitrary point in $\mathbb{R}^5$. Obviously, the projection of the hyperplane to $S^4$ form the given point takes all lines to circles. The restriction of this projection to some open subset of the hyperplane is a diffeomorphism between this subset and its image. Diffeomorphisms obtained in this way will be called classical projections. Of course, classical projections live in all dimensions, not only 4.

Example 2. Another example comes from the quaternionic Hopf fibration. Let us recall the definition. Consider the standard projection from the left (resp., right) quaternionic 2-dimensional vector space $\mathbb{H}^2$ to the left (resp., right) quaternionic projective line $\mathbb{H}P^1$. The latter is identified with $S^4$, a conformal quaternionic coordinate being the ratio of the homogeneous coordinates (in a specified order). Clearly, this projection descends to $\mathbb{R}P^7$ — the real projectivization of $\mathbb{H}^2$. Thus we obtain a map from $\mathbb{R}P^7$ to $S^4$. This is the quaternionic Hopf fibration. It is known (see e.g. [1]) that the quaternionic Hopf fibration takes all lines to circles. Therefore, the same is true for the restriction of it to any 4-dimensional projective subspace of $\mathbb{R}P^7$. 

Our main result is the following

**Theorem 1** Suppose that an analytic diffeomorphism $f$ between an open set in $\mathbb{RP}^4$ and an open set in $S^4$ takes all lines to circles. Then it is either a restriction of a classical projection or a restriction of a (left or right) quaternionic Hopf fibration.

**Remark.** When we say that a map $f : U \subseteq \mathbb{RP}^4 \to S^4$ is a restriction of a classical projection, we mean the following. There is a projective map $i$ from $\mathbb{RP}^4$ to $\mathbb{R}^5$ defined everywhere on $U$, a conformal identification $j$ of $S^4$ with a Euclidean sphere in $\mathbb{R}^5$ and a central projection $\pi$ to this Euclidean sphere from some point such that $f = j^{-1} \circ \pi \circ i$. Analogously, the statement "$f$ is a restriction of a quaternionic Hopf fibration" has the following meaning. There is a projective map $i$ from $\mathbb{RP}^4$ to $\mathbb{RP}^7$, a (right or left) quaternionic Hopf fibration $\pi : \mathbb{RP}^7 \to S^4$ and a conformal identification $j$ of the sphere in the image of $f$ with the sphere in the image of $\pi$ such that $f = j^{-1} \circ \pi \circ i$. Note in particular, that Theorem 1 is insensitive to a projective transformation in the preimage and a conformal transformation in the image.

Actually, it is not hard to see that we can always assume $j$ to be fixed from the very beginning.

**History of the problem.** The problem of finding maps that take lines to circles came from nomography (for an introduction to nomography see [2]). G.S. Khovanskii in 1970-s posed the following question: find all diffeomorphisms between open subsets of $\mathbb{R}^2$ that take lines to circles, or, in the language of nomography, that transform nomograms with aligned points to circular nomograms. Circular nomograms are more convenient in practice whereas nomograms with aligned points are theoretically easier to deal with. This problem was solved by A.G. Khovanskii [3] who proved that all such diffeomorphisms come from projections of planes in $\mathbb{R}^3$ to Euclidean spheres. Actually, his result was stated in a different form: up to a projective transformation in the preimage and a Möbius transformation in the image there are only 3 such diffeomorphisms, and they correspond to classical geometries. Izadi in [4] extended the results of Khovanskii to the 3-dimensional case.

It turned out (see [5]) that Khovanskii's theorem does not extend to dimension 4. The simplest counterexample is a complex projective transformation which takes lines to circles but does not come from a projection of a hyperplane to a sphere. This is a particular case of example 2 above. In [5] it was proved that all ample enough rectifiable bundles of circles passing through a point in $\mathbb{R}^4$ are obtained by means of example 2 (those obtained from example 1 can be obtained from example 2 as well). This may be regarded as a first step in proving Theorem 1. This article completes the proof.

**Outline of the proof of Theorem 1** Suppose that $f : U \to V$ is a diffeomorphism taking all lines to circles. First choose a projective identification of $U$ with a region in $\mathbb{R}^4$ and a conformal identification of $V$ with a region in $\mathbb{H}$, the skew-field of quaternions. Then $f$ can be regarded as a quaternion-valued function on $U$.

For any constant vector field $\alpha$ denote by $\partial_\alpha$ the Lie derivative along $\alpha$. Put $A_\alpha = \partial_\alpha f$. The quaternions $A_\alpha$ are best to be understood as components of a quaternion-
valued differential 1-form $A$ on $U$, namely, the differential of $f$. This form is closed: $dA = 0$ or, in components, $\partial_\alpha A_\beta = \partial_\beta A_\alpha$ for any pair of constant vector fields $\alpha$ and $\beta$ on $U$.

By Lemma 5.2 from [5], at every point of $U$ there exists a quaternion $B_\alpha$ that depends linearly on $\alpha$ and satisfies one of the equations

$$\partial_\alpha A_\alpha = B_\alpha A_\alpha \quad \text{or} \quad \partial_\alpha A_\alpha = A_\alpha B_\alpha.$$ 

The products in the right-hand sides are in the sense of quaternions. Assume that the first equation holds everywhere on $U$ (this is not very restrictive; see the end of Section 4). We show in Section 2 that the integrability condition for this equation is

$$\partial_\alpha B_\beta = \frac{1}{2}B_\alpha B_\beta + \frac{1}{3}C(\overline{A_\alpha A_\beta} + \overline{A_\beta A_\alpha} + A_\alpha A_\beta).$$

where $C$ is a quaternion-valued function. We call it the \textit{first integrability condition}. Then we use once again that $f$ takes lines to circles, now to deduce that $C$ is real-valued. If $C = 0$ identically, then the first integrability condition is integrable, and it gives us the maps from example 2. This will be proved in Section 3.

Suppose now that $C \neq 0$. Then we need to write down an integrability condition for the first integrability condition — the \textit{second integrability condition}. It will be obtained in Section 4. The second integrability condition turns out to be a differential relation between $A$, $B$ and $C$ only, with no additional parameters. It expresses all first logarithmic derivatives of $C$ in terms of the values of $A$ and $B$. The fact that $C$ is real imposes a restriction on possible values of $A$ and $B$. Namely, $B_\alpha = p(A_\alpha) + A_\alpha q$ where $p$ is a real-valued 1-form and $q$ is a quaternion. Now fix some admissible values of $A$, $B$ and $C$ at some point of $U$. We will show in Section 5 that there exists a map from example 1 with the same values of $A$, $B$ and $C$ at the given point. On the other hand, from the second integrability condition it follows that such analytic map is unique. Thus our map $f$ is from example 1.

## 2 First integrability condition

In this section, we obtain an integrability condition for the equation

$$\partial_\alpha A_\alpha = B_\alpha A_\alpha. \quad (1)$$

Since $A_\alpha$ are derivatives, we have $\partial_\alpha A_\beta = \partial_\beta A_\alpha$. From this and from equation (1) it follows immediately that

$$\partial_\alpha A_\beta = \frac{1}{2}(B_\alpha A_\beta + B_\beta A_\alpha). \quad (2)$$

An integrability condition for equation (1) is obtained from the equality $\partial_\alpha \partial_\beta A_\gamma = \partial_\gamma \partial_\alpha A_\gamma$ by expressing derivatives of $A$ with the help of (2). If we introduce a tensor $C$ given in components by

$$C_{\alpha\beta} = \partial_\alpha B_\beta - \frac{1}{2}B_\alpha B_\beta,$$
then the integrability condition reads as follows:

\[ C_{\alpha\beta}A_{\gamma} + C_{\alpha\gamma}A_{\beta} = C_{\gamma\alpha}A_{\beta} + C_{\gamma\beta}A_{\alpha}. \]  
(3)

If we apply to equation (3) the transposition of indices \( \beta \leftrightarrow \gamma \), then we have:

\[ C_{\alpha\beta}A_{\gamma} + C_{\alpha\gamma}A_{\beta} = C_{\gamma\alpha}A_{\beta} + C_{\gamma\beta}A_{\alpha}. \]  
(3')

Take the sum of equations (3) and (3'). It can be written as follows:

\[ (C_{\beta\gamma} + C_{\gamma\beta})A_{\alpha} = (2C_{\alpha\gamma} - C_{\gamma\alpha})A_{\beta} + (2C_{\alpha\beta} - C_{\beta\alpha})A_{\gamma}. \]  
(3'')

Put \( \beta = \gamma \) in equation (3). We obtain that

\[ (2C_{\alpha\beta} - C_{\beta\alpha})A_{\beta} = C_{\beta\beta}A_{\alpha}. \]  
(4)

or, replacing \( \beta \) with \( \gamma \),

\[ (2C_{\alpha\gamma} - C_{\gamma\alpha})A_{\gamma} = C_{\gamma\gamma}A_{\alpha}. \]  
(4')

Replace \( 2C_{\alpha\beta} - C_{\beta\alpha} \) and \( 2C_{\alpha\gamma} - C_{\gamma\alpha} \) in (3'') by their expressions obtained from (4) and (4'), respectively. We obtain

\[ C_{\beta\gamma} + C_{\gamma\beta} = C_{\beta\beta}A_{\alpha}(A_{\beta}^{-1}A_{\gamma}^{-1})A_{\alpha}^{-1} + C_{\gamma\gamma}A_{\alpha}(A_{\gamma}^{-1}A_{\beta})A_{\alpha}^{-1}. \]  
(5)

Take a point in \( U \). In equation (5) written at this point, \( \alpha, \beta \) and \( \gamma \) may be arbitrary vectors. Fix \( \beta \) and \( \gamma \), and let \( \alpha \) vary. Then \( A_{\alpha} \) runs over all quaternions, since \( f \) is a diffeomorphism. We need the following

**Lemma 2** Let \( a \) and \( b \) be arbitrary quaternions, and \( x \) an imaginary quaternion (i.e. not a real number). Suppose that the number \( ay + by^{-1} \) stays the same for all quaternions \( y \) obtained from \( x \) by inner conjugations (i.e. \( y = qxq^{-1} \) for an arbitrary quaternion \( q \)). Then \( b = a|x|^2 \).

**Proof.** Denote by \( x_0 \) the real part of \( x \). Then it is easy to see that

\[ ay + by^{-1} = \left( a - \frac{b}{|x|^2} \right) y + \frac{2bx_0}{|x|^2}. \]

The second term in the right-hand side is independent of \( y \). Therefore, the first term should be also independent of \( y \) which is possible only if the coefficient \( a - b/|x|^2 \) vanishes. \( \square \)

We can apply this lemma to equation (5) where we put \( a = C_{\beta\beta} \), \( b = C_{\gamma\gamma} \) and \( x = A_{\beta}^{-1}A_{\gamma} \). If \( \beta \) is not parallel to \( \gamma \), then \( x \) is imaginary. Thus by the lemma we have

\[ \frac{C_{\beta\beta}}{|A_\beta|^2} = \frac{C_{\gamma\gamma}}{|A_\gamma|^2}. \]

It follows that the left-hand side is independent of \( \beta \). Hence

\[ C_{\beta\beta} = C|A_\beta|^2 \]
(6)
where $C$ is a quaternion which does not depend on the vector $\beta$, but it may depend on a point — so $C$ is a quaternion-valued function on $U$.

Plug in (6) to (5):

$$C_{\beta\gamma} + C_{\gamma\beta} = C(\overline{A_\beta}A_\gamma + \overline{A_\gamma}A_\beta). \quad (6')$$

Comparing equations (6') (with $\beta$ changed to $\alpha$ and $\gamma$ changed to $\beta$) and (4) we can conclude that

$$C_{\alpha \beta} = \frac{1}{3}C(\overline{A_\alpha}A_\beta + \overline{A_\beta}A_\alpha + A_\alpha \overline{A_\beta}). \quad (6'')$$

This is the first integrability condition.

3 Examples

Let us now take a closer look on examples 1 and 2 from Section 1.

Example 1. Consider the left quaternionic Hopf fibration $\pi : \mathbb{RP}^7 \to S^4$. A point in the preimage is represented by a pair of quaternions $(y, z)$ up to multiplication of both $y$ and $z$ by a common real factor. The image of this point under $\pi$ is an element of the left projective quaternionic line with homogeneous coordinates $y$ and $z$. We can take $y^{-1}z$ as an affine conformal quaternionic coordinate in the image.

Suppose that $f$ is obtained as the composition of $\pi$ with some projective embedding of (a part of) $\mathbb{R}^4$ to $\mathbb{RP}^7$. This projective embedding sends a point $x \in \mathbb{R}^4$ to a point in $\mathbb{RP}^7$ with coordinates

$$y = L(x), \quad z = M(x),$$

where $L$ and $M$ are some affine maps from $\mathbb{R}^4$ to $\mathbb{R}^4$. Thus in the given coordinates $f$ looks as follows:

$$f : x \mapsto L(x)^{-1}M(x) \quad (E_1)$$

where the multiplication and the inverse are in the sense of quaternions.

Denote by $\mathcal{L}$ and $\mathcal{M}$ the linear parts of $L$ and $M$, respectively. This means that $\mathcal{L}$ (resp., $\mathcal{M}$) is a linear operator such that $L$ (resp., $M$) is a composition of $\mathcal{L}$ (resp., $\mathcal{M}$) and a translation. In other words, $\mathcal{L}$ and $\mathcal{M}$ are differentials of $L$ and $M$ at any point.

Compute $A_\alpha = \partial_\alpha f$ for our map $f$:

$$A_\alpha(x) = -L^{-1}(x)\mathcal{L}(\alpha)L^{-1}(x)M(x) + L^{-1}(x)\mathcal{M}(\alpha).$$

Differentiating once again along $\alpha$ we obtain:

$$B_\alpha = -2L^{-1}(x)\mathcal{L}(\alpha).$$

We can see from here that $\partial_\alpha B_\beta = \frac{1}{2}B_\alpha B_\beta$, so in this case $C = 0$. 

5
Case $C = 0$. Let us work out the case when $C$ vanishes everywhere in $U$. In this case the first integrability condition (6′′) reads as follows:

$$\partial_\alpha B_\beta = \frac{1}{2} B_\alpha B_\beta.$$  

Consider this equation together with equation (2). This is a system of partial differential equations of first order which expresses all first derivatives of $A$ and $B$ through the values of $A$ and $B$. Denote this system by $S$. Fix any point $x \in U$ and initial values of $1$-forms $A$ and $B$ at $x$. Thus for any vector $\alpha$ at $x$ we know $A_\alpha$ and $B_\alpha$. Note that if a solution of $S$ with given initial values exists, then it is unique. Indeed, all higher derivatives of $A$ and $B$ can be expressed through the initial values by means of the system $S$. The existence of a solution follows from example 1. In this example, at the given point $x$ linear maps $A : \alpha \mapsto A_\alpha$ and $B : \alpha \mapsto B_\alpha$ can be arbitrary.

We have just proved the following

**Proposition 3** If $C = 0$ identically in $U$, then the general solution of equations (2) and (6′′) is given by $(E_1)$.

**Example 2.** Consider a Euclidean sphere $S$ in $\mathbb{R}^5$. We can choose this sphere to be centered at the origin and to have radius 1. Introduce a conformal coordinate system on $S$ by means of the stereographic projection $j$ to the equatorial hyperplane with the center at the North pole. To this end we need to choose an orthogonal splitting $\mathbb{R}^5 = \mathbb{R}^4 \times \mathbb{R}$ the latter factor being the line between the North and South poles. Thus any point in $\mathbb{R}^5$ is represented as $(y, z)$ where $y \in \mathbb{R}^4$ and $z \in \mathbb{R}$. The point on $S$ corresponding to $(y, 0)$ by means of the stereographic projection is

$$j^{-1}(y) = \left( \frac{2y}{1 + |y|^2}, \frac{|y|^2 - 1}{1 + |y|^2} \right).$$

Suppose that $f$ is the central projection from a horizontal hyperplane $z = z_1$ to $S$ with the center $(0, z_0)$. Under this projection, a point $(x, z_1)$ corresponds to a point $j^{-1}(y)$ on $S$ if and only if

$$x = \frac{2y(z_0 - z_1)}{(z_0 + 1) + (z_0 - 1)|y|^2}. \ \ (E_2.1)$$

We can now rescale $y$ and choose specific values of $z_0$ and $z_1$ so that

$$x = \frac{y}{1 + |y|^2}. \ \ (E_2.1')$$

This can be considered as an implicit equation defining $f : x \mapsto y$. Recall that $f$ was defined as a classical projection. Another interpretation of $f$ is that it establishes a correspondence between the Klein and the Poincaré models of the hyperbolic (Lobachevsky) geometry.

Multiply both sides of equation $(E_2.1)$ by the denominator to get $y = (1 + |y|^2)x$ and then differentiate both sides along $\alpha$:

$$A_\alpha = \frac{2(y, A_\alpha)y}{1 + |y|^2} + (1 + |y|^2)\alpha. \ \ (E_2.2)$$
Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product corresponding to the given quaternionic structure. Namely, for 2 vectors $\xi$ and $\eta$ we have $\langle \xi, \eta \rangle = \text{Re}(\bar{\xi}\eta)$ where in the right-hand side $\xi$ and $\eta$ are considered as quaternions. To express $A_\alpha$ it terms of $y$ only we need to find an expression for $\langle y, A_\alpha \rangle$. For that, take the inner product of both sides of equation (E$_2$.2) with $y$. This leads to a linear equation for $\langle y, A_\alpha \rangle$. Solving this equation, we obtain

$$\langle y, A_\alpha \rangle = \frac{(1 + |y|^2)^2}{1 - |y|^2} \langle y, \alpha \rangle. \quad (E_2.3)$$

Substitute (E$_2$.3) into (E$_2$.2). We now have a formula for $A_\alpha$ in terms of $y$:

$$A_\alpha = (1 + |y|^2) \left( \frac{2\langle y, \alpha \rangle y}{1 - |y|^2} + \alpha \right). \quad (E_2.4)$$

Differentiating equation (E$_2$.4) along $\alpha$ we can see that

$$B_\alpha = 2 \left( \frac{1 + |y|^2}{1 - |y|^2} \langle y, \alpha \rangle + \frac{yA_\alpha}{1 - |y|^2} \right).$$

Differentiate this equation once again along $\alpha$. Then we find that

$$\partial_\alpha B_\alpha - \frac{1}{2} B_\alpha^2 = \frac{6|A_\alpha|^2}{(1 - |y|^2)^2}.$$ From equation (6) it follows that

$$C = \frac{6}{(1 - |y|^2)^2}$$
in this example. In particular, $C$ may be nonzero. Note that in this example $C$ is real everywhere.

4 Second Integrability Condition

The case $C = 0$ has been already worked out in the previous section. Now assume that $C \neq 0$ at a given point. A priori, $C$ is a quaternion-valued function. But now we are going to prove the following

**Lemma 4** The function $C$ is real-valued.

This lemma does not follow form equation (1). We need to use directly the fact that $f$ takes lines to circles.

**Proof.** Take an arbitrary point $x_0 \in U$ and a vector $\alpha$ at $x_0$. The line $l$ passing through $x_0$ in the direction of $\alpha$, is mapped to a circle under $f$. Let $t \mapsto x = x(t)$ be the affine parameterization of $l$ such that $x(0) = x_0$ and $\dot{x}(0) = \alpha$.

First assume that $B_\alpha$ is real at $x_0$. Then $f(l)$ must be a line. Therefore, $\text{Im}(B_\alpha)$ must vanish everywhere on $l$. It follows that $C$ is also real in this case.
Suppose now that \( \text{Im}(B_\alpha) \neq 0 \). By Proposition 5.4 from [5] the center of the circle \( f(l) \) is
\[
f(x) - (\text{Im}(B_\alpha(x)))^{-1}A_\alpha(x).
\]
Hence this expression does not depend on the choice of \( x \in l \). Differentiate it by \( t \):
\[
A_\alpha + (\text{Im}(B_\alpha))^{-1} \left( \frac{1}{2} \text{Im}(B_\alpha^2) + \text{Im}(C)|A_\alpha|^2 \right) (\text{Im}(B_\alpha))^{-1}A_\alpha - (\text{Im}(B_\alpha))^{-1}B_\alpha A_\alpha = 0.
\]
It follows that \( \text{Im}(C) = 0 \). □

The first integrability condition is a partial differential equation on \( B \). Let us try to write down the integrability condition for this equation.

We will use the language of quaternion-valued differential forms. By definition, the algebra of quaternion-valued differential forms is obtained from the algebra of ordinary real-valued differential forms by taking the tensor product with the quaternions over reals. In particular, this definition says how to multiply quaternion-valued forms. As was already mentioned, \( A \) and \( B \) are thought of as quaternion-valued 1-forms.

From the first integrability condition it follows that
\[
dB = \frac{1}{2} B \wedge B + \frac{1}{3} C(A \wedge \bar{A}). \tag{6′′}
\]
Take the differential of the both parts of this equation, using relations \( d^2B = 0 \) and \( dA = 0 \):
\[
dC \wedge A \wedge \bar{A} = \frac{C}{2} (B \wedge A \wedge \bar{A} - A \wedge \bar{A} \wedge B). \tag{7}
\]

**Proposition 5** Suppose that the values of the 1-forms \( A \) and \( B \) and the function \( C \) are given at some point, and the map \( \alpha \mapsto A_\alpha \) is one-to-one at this point. If an analytic solution \( f \) of equations (2) with these initial values of \( A \), \( B \) and \( C \) exists, then it is unique.

**Proof.** Since \( A \) is non-degenerate, the operator of the right wedge multiplication by \( A \wedge \bar{A} \) is invertible. This can be verified by a simple direct computation. Therefore, equation (7) expresses all first derivatives of \( C \) through \( A \) and \( B \). Consider this equation together with equations (2) and (6′′). We obtain a system of partial differential equations which expresses all first derivatives of \( A \), \( B \) and \( C \) in terms of the values of \( A \), \( B \) and \( C \) only. Therefore, by successive applications of this system we can express all higher derivatives of \( A \), \( B \) and \( C \) at the given point through the initial values at this point. The proposition now follows. □

Note also that \( A \), \( B \) and \( C \) determine the 3-jet of \( f \). Namely, from equations (2) and (6′′) it follows that
\[
f = f(x_0) + A_{x-x_0} + \frac{B_{x-x_0}A_{x-x_0}}{2} + \left( \frac{1}{2} B_{x-x_0}^2 + C|A_{x-x_0}|^2 \right) A_{x-x_0} + \ldots
\]
In particular, Proposition 5 can be reformulated as follows:

**Proposition 6** If there exists a diffeomorphism \( f : U \subseteq \mathbb{R}^4 \to V \subseteq \mathbb{R}^4 \) with a given 3-jet at some point \( x_0 \in U \) such that the image of each line segment lying in \( U \) is an arc of a circle or a line segment in \( V \), then such diffeomorphism is unique.
5 Admissible 3-jets

From now on we identify the space of preimage with the space of image and assume that the distinguished point is the origin, \( f(0) = 0 \) and \( A_\alpha = \alpha \) for all vectors \( \alpha \) at 0. This may be achieved by a linear change of variables in the preimage composed with a translation in the image. Thus if \( x = x^0 + ix^1 + jx^2 + kx^3 \) denotes the natural quaternionic coordinate, then \( A = dx = dx^0 + idx^1 + jdx^2 + kdx^3 \).

Equation (7) imposes a restriction on \( B \) due to the fact that \( C \) is real. Let us find all admissible values of \( B \) at 0 by solving (7) as a linear system on \( B \) and \( dC/C \). We have \( A \wedge A = dx \wedge dx = i \omega_1 + j \omega_2 + k \omega_3 \) where \( \omega_1, \omega_2 \) and \( \omega_3 \) are real 2-forms given in coordinates by

\[
\begin{align*}
\omega_1 &= 2(dx^1 \wedge dx^0 + dx^3 \wedge dx^2), \\
\omega_2 &= 2(dx^2 \wedge dx^0 + dx^1 \wedge dx^3), \\
\omega_3 &= 2(dx^3 \wedge dx^0 + dx^2 \wedge dx^1).
\end{align*}
\]

Denote by \( \gamma \) the real 1-form \( 2dC/C \) and suppose that \( B = B^0 + iB^1 + jB^2 + kB^3 \) where \( B^0, B^1, B^2 \) and \( B^3 \) are real 1-forms. Equation (7) can be now rewritten as the following system:

\[
\begin{align*}
2(B^2 \wedge \omega_3 - B^3 \wedge \omega_2) &= \gamma \wedge \omega_1, \\
2(B^3 \wedge \omega_1 - B^1 \wedge \omega_3) &= \gamma \wedge \omega_2, \\
2(B^1 \wedge \omega_2 - B^2 \wedge \omega_1) &= \gamma \wedge \omega_3.
\end{align*}
\]

Denote by \( B^\nu_\nu \) and \( \gamma_\nu \) \((\nu = 0, 1, 2, 3)\) the components of the 1-forms \( B^\nu \) and \( \gamma \) so that

\[
B^\nu = \sum_{\nu=0}^{3} B^\nu_\nu dx^\nu, \quad \gamma = \sum_{\nu=0}^{3} \gamma_\nu dx^\nu.
\]

System (8) yields the following equations:

\[
\begin{align*}
2(-B^3_0 + B^3_2) &= -\gamma_1, \\
2(-B^2_0 - B^3_3) &= \gamma_0, \\
2(B^3_0 - B^0_0) &= -\gamma_3, \\
2(B^0_0 + B^1_3) &= \gamma_2, \\
2(-B^1_0 + B^1_3) &= \gamma_2, \\
2(B^0_0 + B^2_3) &= \gamma_0, \\
2(-B^2_0 - B^2_3) &= -\gamma_1, \\
2(-B^3_0 + B^2_3) &= -\gamma_3, \\
2(B^3_0 - B^0_0) &= -\gamma_2, \\
2(B^0_0 + B^1_3) &= \gamma_1, \\
2(-B^1_0 - B^2_3) &= \gamma_0.
\end{align*}
\]

Exclude \( \gamma_\nu \) from these equations:

\[
\begin{align*}
B^1_0 &= B^3_2 = B^2_3, \\
B^0_0 &= B^1_3 = -B^2_3, \\
B^2_0 &= B^3_1 = -B^3_1, \\
B^3_0 &= B^2_1 = -B^1_1.
\end{align*}
\]

The relations displayed above mean that \( B_\alpha = p(\alpha) + \alpha q \) where \( p \) is a real-valued 1-form and \( q \) is a quaternion. Hence the 3-jet at 0 of \( f \) should be as follows:

\[
x + \frac{(p(x) + xq)x}{2} + \frac{\left(\frac{3}{2}(p(x) + xq)^2 + C|x|^2\right)x}{6}.
\]

**Proposition 7** For an arbitrary real-valued linear function \( p \) on \( \mathbb{R}^4 \) and an arbitrary quaternion \( q \), a local diffeomorphism with 3-jet (9) exists. Moreover, it can be chosen to be one of the classical projections.
PROOF. First consider a map given by the implicit equation

\[ x = \frac{y}{1 - \frac{C}{6} |y|^2}. \]

This is clearly a map from example 2. It has the 3-jet \( x + \frac{C}{6} |x|^2 x \). This is a partial case of (9) where \( p = q = 0 \). To achieve other given values of \( p \) and \( q \), compose this map with the following M"obius transformation:

\[ y \mapsto 2q^{-1} \left( 1 - \frac{pq}{1 - \frac{1}{2p(y)}} \right)^{-1} - 2q^{-1}. \]

It is easy to see that the composition has 3-jet (9). □

Concluding remarks. We should now make things add up in the proof of theorem 1.

We have always assumed that \( \partial_\alpha A_\alpha = B_\alpha A_\alpha \) the multiplication by \( B_\alpha \) being from the left. The case of the right multiplication is completely analogous. The only difference is that for \( C = 0 \) we would have the right quaternionic Hopf fibration instead of the left one. In general there is an open subset \( U' \) of \( U \) such that the multiplication by \( B_\alpha \) is either form the left everywhere on \( U' \) or from the right everywhere on \( U' \). Therefore theorem 1 holds on \( U' \). By the uniqueness theorem for analytic functions it holds then on \( U \).

If \( C = 0 \) everywhere on \( U \), then by Proposition 3 we have a quaternionic Hopf fibration. Otherwise \( C \) nowhere vanishes on some open subset of \( U \). Then by Propositions 3 and 4 the map \( f \) is a classical projection on this subset. By the uniqueness theorem \( f \) is a classical projection on \( U \).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO 100 ST. GEORGE STREET, TORONTO ON M5S3G3 CANADA
E-mail address: vtimorin@math.toronto.edu

References

[1] K.Y. Lam: Some new results in composition of quadratic forms, Invent. Math. 79 (1985), 467-474
[2] G.S. Khovanskii: Foundations of Nomography, “Nauka”, Moscow, 1976 (Russian)
[3] A.G. Khovanskii: Rectification of circles, Sib. Mat. Zh., 21 (1980), 221–226
[4] F.A. Izadi: Rectification of circles, spheres, and classical geometries, PhD thesis, University of Toronto, (2001)
[5] V.A. Timorin Rectification of Circles and Quaternions, Michigan Math. J. 51 (2003)