One parameterisation to fit them all

Gustavo Arciniega,1,2 † Mariana Jaber,3 †† Luisa G. Jaime,4 ‡ and Omar A. Rodríguez-López,5 §

1 Centro Tecnológico Aragón, Universidad Nacional Autónoma de México, Av. Rancho Seco SN, Bosques de Aragón, Nezahualcóyotl, Estado de México, 57130 México
2 Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, A. P. 50-542, México, CDMX 04510, México
3 Institute for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5, 87-100 Toruń, Poland
4 Departamento de Física, Instituto Nacional de Investigaciones Nucleares, Apartado Postal 18-1027, Col. Escandón, Ciudad de México, 11801, México.
5 Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000, Ciudad de México, México

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ABSTRACT

Perhaps the most explored hypothesis for the accelerated cosmic expansion rate arise in the context of extra fields or modifications to General Relativity. A prevalent approach is to parameterise the expansion history through the equation of state, \( \omega(z) \). We present a parametric form for \( \omega(z) \) that can reproduce the generic behaviour of the most widely used physical models for accelerated expansion with infrared corrections. The present proposal has at most 3 free parameters which can be mapped back to specific archetypal models for dark energy. We analyze in detail how different combinations of data can constrain the specific cases embedded in our form for \( \omega(z) \). We implement our parametric equation for \( \omega(z) \) to observations from CMB, luminous distance of SNeIa, cosmic chronometers, and baryon acoustic oscillations identified in galaxies and in the Lyman-\( \alpha \) forest. We find that the parameters can be well constrained by using different observational data sets. Our findings point to an oscillatory behaviour which is consistent with an \( f(R) \)-like model or a unknown combination of scalar fields. When we let the three parameters vary freely, we find an EOS which oscillates around the phantom-dividing line, and, with over 99\% of confidence, the cosmological constant solution is disfavored.

Key words: – dark energy – cosmology: theory – cosmology: observations – cosmological parameters

1 INTRODUCTION

Ever since the discovery of the acceleration of the Universe (Perlmutter et al. 1999; Riess et al. 1998) (hinted previously in Roukema & Yoshi (1993)), cosmology has tried to answer the question of what makes the Universe accelerate. Currently, the most accepted explanation by the scientific community is the cosmological constant, \( \Lambda \), in the frame of General Relativity with an FLRW metric, which has become the concordance model known as \( \Lambda \)CDM (Aghanim et al. 2018). Several phenomena can be explained by using such a simple model; nevertheless, the physical nature of \( \Lambda \) remains unaddressed.

In the past few years, observations of different astrophysical sources have been used to measure the acceleration of the Universe. The results have brought with them even more uncertainty about the nature of dark energy. They show a discrepancy on the present value of the Hubble parameter derived when local measurements are used (Riess et al. 2019, Wong et al. 2019, Shajib et al. 2020)) with the \( H_0 \) value when derived by fitting the cosmological parameters assuming the concordance model in the cosmic microwave background (CMB) (Aghanim et al. 2018). Different estimations of the discordance place the discrepancy as high as 4.4-\( \sigma \) Riess et al. (2019) or even at the level of 5.3-\( \sigma \), according to (Wong et al. 2019) (see Verde et al. (2019) for a summary plot).

It is possible that a systematic miscalculation is behind this conundrum; nevertheless, the possibility of having some new physics is provocative. Many alternatives to the standard concordance model have been proposed (for a review of several of these alternatives, see Zumalacarregui (2020)). One alternative to explore the evolution of the Universe in a model-independent fashion way is by setting a parametric form for the equation of state (EOS), \( \omega \equiv P/\rho \) of the dark energy component.

In this framework, several parameterisations of the EOS have been proposed with the idea of simplifying the analysis of observations; however, we find either a lack of physical motivation or a stringent dependence of a particular model.

Interestingly enough, in an observational effort carried out by Zhao et al. (2017), a reconstruction of the EOS was presented. The evolution found by the authors shows oscillating behaviour around the phantom line. It is well known that such evolution can not be provided by using a single scalar field (either phantom or quintessence like). Nevertheless, modified gravity or some unknown combination of multiple scalar fields could provide the reconstructed EOS.

Currently, different kinds of parameterisations are used to provide dynamical dark energy, some of them are motivated by scalar fields and, since the work on Jaime et al. (2018), there is a proposal inspired by modified gravity. We present a different parameterisation that has the advantage of reproducing the generic behaviour for both cases.

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depending on the choice of the parameters. Using this parameterisation with current and future data, we could test the generic evolution of the equation of state of the accelerating mechanism.

2 PARAMETERISING THE COSMIC EXPANSION

The parameterisations of $\omega(z)$ in the literature are either mere mathematical descriptions, polynomials or Taylor expansions around $a_0$ or $z$, or an attempt to capture distinctive features for particular models. In Chevallier & Polarski (2001) the authors perform a Hamiltonian analysis that provides physical tools to build their proposal for $\omega(z)$. Their aim is to explore small deviations of the cosmological constant. In the light of the recent Hubble tension, our motivation is reproducing predictions given by different alternative physical scenarios while avoiding inherent theoretical complications for the implementation of certain models. This way, the parameterisations that can mimic well supported physical evolution of the EOS will provide a simpler way to study complicated theories.

The standard approach to this aim, is to take model by model and constrain the introduced free parameters against data to obtain conclusions for the chosen physical scenario. Instead of choosing a parametric form for each different model and performing the statistical analysis case by case, we propose a single framework for analyzing the generic behaviour of the most widely used physical models for accelerated expansion with, at most, three parameters. Our proposal includes alternative models that make modifications for late time while maintaining the EOS includes alternative models that make modifications for late time and “freezing in”, depending on if the slope when is going to high redshift. With this on mind, we introduce our proposal for $\omega(z)$, explain its mathematical properties and describe its capabilities to mimic archetypal models for the accelerated expansion. A particular interesting question for us is: do observations point to a $\omega(z)$ which crosses the phantom-line?

Our proposal is engaging in mimicking and analysing two paradigmatic scenarios: $f(R)$ modified gravity and quintessence/phantom models, but we will see that the parameterisation is not restricted to only these two cases.

We focus on $f(R)$ theories of gravity because they are very straightforward modifications of General Relativity, and have been widely studied over the past twenty years (see for instance Jaime et al. (2012) and references therein). In this kind of modification, the dependence of the Ricci scalar $R$ in the Hilbert-Einstein action is not linear, it is replaced by an arbitrary function of $R$. Several $f(R)$ models have been proposed to provide an alternative explanation to the acceleration of the Universe. In Jaime et al. (2014) the definition of the EOS for the geometric dark energy is discussed and it was shown that, in general, the generic evolution of $\omega(z)$ for the $f(R)$ models that are considered candidates for dark energy is oscillatory. In Jaime et al. (2018) some of us presented a parameterisation for the EOS in $f(R)$ that can reproduce in a very high precision the numerical results for some $f(R)$ models, nevertheless if we try to go at high redshift the oscillatory behaviour given by the parameterisation will bring errors if it is implemented it into Boltzmann codes. By using the present proposal such problems can be avoided while the generic behaviour is maintained.

Regarding scalar fields, the most popular proposal to provide an alternative explanation to the acceleration of the Universe is quintessence models. Such models can be separated into two kinds: “thawing out” and “freezing in”, depending on if the slope when is going to $z = 0$ is positive or negative Caldwell & Linder (2005). In Roy et al. (2018) the authors presented a parameterisation for several models directly in the scalar field. By using our proposal, the generic evolution of the cases presented in Roy et al. (2018) can be reproduced. Even more, as we elaborate in the following section, this can be accomplished by fixing one or two out of three free parameters.

3 ONE PARAMETERISATION TO FIT THEM ALL

The parameterisation we are proposing for the EOS, $\omega(z)$, is the following:

$$\omega(z) = -1 - A \exp(-z) (e^{\rho} - z - C),$$

(1)

where $A$, $n$ and $C$ are real numbers that can take positive or negative values.

A quick inspection let us notice that the present-day value is given by $\omega(z = 0) = \omega_0 = -1 + AC$, while the high-redshift value rapidly converges to $\omega(z \gg 0) = -1$, corresponding to a cosmological constant $\Lambda$ scenario avoiding high redshift divergences that can be present in other parameterisations (Huterer & Turner (2001); Weller & Albrecht (2002), Jaber & de la Macorra (2018)).

Given the form of equation (1), it is possible to mimic different dynamical dark-energy EOS which can be characterized by, at most, a single oscillation of the $\omega(z)$ at low redshifts.

We identify four well-posed cases that we name: exponential, quintessence/phantom, $f(R)$, and general, besides the standard case of a cosmological constant. In the next subsections, we provide an analytical analysis in order to explain the flexibility of the model.

3.1 Exponential: $n = C = 1$ (one-free parameter).

The simplest case of (1) is when $n = 1$, so that the parameterisation reduce to the expression:

$$\omega(z) = -1 + AC e^{-\rho}.$$  

(2)

Without lost of generality, we can fix $C = 1$, that is equivalent to rename $\Lambda \equiv AC$. In this case, if we do not perform the redefinition of $\Lambda$, the parameterisation will present a degeneration effect given by the product $AC$.

Nevertheless, it is possible to avoid the degeneration if the parameterisation, for this particular case, takes just one parameter instead of two. In this case, the generic behaviour is exponential. Figure 1 shows the evolution of the EOS for different values of the amplitude: $\Lambda = \pm 0.2$, $\pm 0.5$, $\pm 1$, $\pm 5$, where the black lines represent the positive values of $\Lambda$, while the grey lines show the negative $\Lambda$ values.

From 1 we see that the value of $\omega$ at $z = 0$ is different from $-1$ if $\Lambda \neq 0$. Also, the evolution of $\omega(z)$ goes monotonically to the asymptotic value $\omega = -1$ for some $z > 0$. How fast it converges to $-1$ depends on the value of $\Lambda$. This parameter controls both, the present value of the equation of state, and the epoch $z$, for which $\omega(z)$ is practically $-1$. In order to formally express the value of $z$ where we can consider that the EOS has reached the value $-1$, let us consider $|\rho| \ll 1$ so that $\omega(z) = -1 + |\rho| = -1 + \Lambda e^{-\rho}$. From here, we define $\tilde{z} \equiv \ln(\Lambda/|\rho|)$ as the redshift such that $\omega(\tilde{z}) \approx -1$, up to an $|\rho|$ for large $z$.

This characteristic will play a role in distinguishing this from other cases of study. In this particular, we anticipate that although the evolution within this model (Exponential) can be similar to the one obtained in the particular cases II and IV (compare figure 1 to figures 3 and 7), it is the value of $\omega_0$ and the asymptotic relaxation to $\omega(z) = -1$ which can potentially distinguish between them observationally.
Theamplitudeforallthecurvesis
\[ A_e \]
when it can be seen how the parameterisation is able to transit evolution going up (down) from that minimum (maximum). In this case the behaviour will be similar to the one presented in the case \( n = 1 \). It is important to remark that although the profiles depicted in figure 1, and 3 look similar, the analytic form is completely different so it is guaranteed that there is no degeneration with the Exponential-like parameterisation case. Even more, in this case \( \omega_0 = -1 \) if and only if \( C = 1 \), so any deviation of \( C = 1 \) will modify the value of \( \omega_0 \) around \(-1\). In the case that \( C \leq 2 \), the minimum (maximum) will be located at \( z > 0 \) so the evolution of the EOS will cross the phantom line a single time and the behaviour will have the characteristic shape that is expected in the Quintom models (see dot-dashed line in figure 3).

3.3 \( f(R) \)-like: \( n \in (0, 1) \) and \( C = 0 \) (two-free parameters).

In this case the parameter \( n \) in (1) takes any value between \( 0 < n < 1 \), while the parameter \( C \) is fixed to \( C = 0 \). In this way we are imposing \( \omega_0 = -1 \). Equation (1) can be written as
\[
\omega(z) = -1 - A e^{-\frac{z}{\omega_0}}(1 - C - z),
\]
(3)
where \( \omega(z) \) has a single minimum/maximum value located at \( z = 2 - C \). The generic evolution of the EOS is depicted in figure (2), where it can be seen how the parameterisation is able to transit from a Quintessence-like profile (Roy et al. (2018)) to a mixture of Quintessence and phantom like fields, known as Quintom (for a review of this models, see Cai et al. (2010)). In order to visualize this case, in figure 2 we have fixed the amplitude for all the curves to \( A = 1.2 \). The solid black line is when \( C = 1.45 \), dashed black line is when \( C = 1 \), dotted black line is when \( C = 0.5 \), and dot-dashed black line corresponds to \( C = -1.3 \). Gray lines are the same as black lines with \( A \rightarrow -A \).

Figure 1. Exponential: \( n = 1 \), equation (2) taking \( A \Rightarrow AC \). Solid black line \( A = 5 \), dashed black line \( A = 1 \), dotted black line \( A = 0.5 \), and dot-dashed black line \( A = 0.2 \). Gray lines are the same as black lines but with \( A \rightarrow -A \).

Figure 2. Equation of state \( \omega(z) \) for \( n = 0 \), Quintessence/Phantom-like case: The amplitude for all the curves is \( A = 1.2 \). The solid, dashed, dotted, and dot-dashed black lines correspond to \( C = 1.45 \), \( C = 1 \), \( C = 0.5 \) and \( C = -1.3 \), respectively. Gray lines are the equivalent to black lines for \( A \rightarrow -A \).

Figure 3. Equation of state \( \omega(z) \) for the Quintessence/Phantom-like case, \( n = 0 \): The amplitude \( A \) is fixed for all the curves, \( A = -0.2 \). The solid black line is with \( C = -9 \), dashed black line is with \( C = -5 \), dotted black line is with \( C = -2 \), and dot-dashed black line corresponds with \( C = 0 \). Gray lines are the same as black lines but with \( A \rightarrow -A \).

3.4 General-model: \( n \in (0, 1) \) (three-free parameters)

We will now consider the EOS (1), for \( n \) taking values in between the previous cases, \( i.e. \)
\[
\omega(z) = -1 - A e^{-\frac{z}{\omega_0}}(z^n - z - C), \quad 0 < n < 1.
\]
(5)
This equation (5) is the one we will be referring to as ‘General-model’ from now on.

In general, the EOS will present two characteristic behaviours: (1) two critical points with a local maximum and minimum (figure 6), and (2) a monotone function (figure 8). For all cases, the critical values are given by the two roots of the following expression:

\[ z^n - nz^{n-1} - z + (1 - C) = 0. \]  

\[ (6) \]

It is not surprising that the critical \( z \) values obtained from equation (6) depend only on \( n \) and \( C \), because \( A \) acts only as a homothety factor for the \( \omega(z) + 1 \) function. However, it is worth to mention that \( \omega(z) \) has at most two real critical points, that we will name \( z_1 \) and \( z_2 \), no matter the value of \( n \in (0, 1) \).

It could happen that the roots, \( z_1 \) and \( z_2 \), are complex. This is the case when \( \omega(z) \) is a monotonic function (figure 8), that resembles the exponential case \( (n = 1) \) (see figure 1), and the Quintessence/Phantom case \( (n = 0) \) for \( C \geq 2 \) (see figure 2).

When the critical points, \( z_1 \) and \( z_2 \), are real numbers, we get the generic behaviour of the EOS depicted in figure 6 with a local maximum and minimum. It will be helpful to understand how \( \omega(z) \) is modified by the parameter \( C \). As is shown in figure 8, the parameter \( C \) can either increase or lower the general \( \omega(z) \) value alongside with a subtle displacement into the \( z \)-axis direction. This feature allows us to put an anchor to the first critical point. Let us define \( z_c = \min(z_1, z_2) \) and demand that \( \omega(z) \) must be -1 at \( z_c \). Under that condition, \( z_c \) and \( C \) are forced to be

\[ z_c = n^{1/(1-n)} \quad \text{and} \quad C = n^n/(1-n) - z_c, \]

where \( z_c \) acts as the anchor point (figure 9), and can be compared with case II (figure 2), when \( n = 0 \) and \( C \sim 1 \), for example.

It is worth to mention that \( \omega(z) = -1 - A e^{-z} (z^n - z - C) \), for \( n > 1 \), has equivalent qualitative attributes to \( n \in (0, 1) \): (1) at most two critical points, and (2) monotone behaviour when \( z_1, z_2 \notin \mathbb{C} \). In particular, the \( n > 1 \) case can mimic or behave too similar to \( n \in (0, 1) \) case. This is why we limit our analysis to \( n \in (0, 1) \).

This will be reflected in the choice of priors for \( n \) parameter.

By constraining these sub-cases separately, we can statistically
where $\Omega_m + \Omega_r + \Omega_{DE} = 1$, $H_0 = 100\text{km/s/Mpc}$ is the Hubble constant, and $F_{DE}(z)$ is a function of redshift involving the specific form of $\omega_x$. For a Cosmological Constant, $\omega_x = -1$, and $F_A(z) = 1$. In general, we have:

$$F_{DE}(z) = \exp\left(\int_0^z \frac{3(1 + \omega_x(z'))}{1 + z'} dz'\right),$$

(8)

where $\omega_x$ is the one given by (1).

From equations (7) and (8) it is clear how data coming from cosmological distances can be used to constrain the free parameters in (1).

5 METHODS

5.1 Data

In order to probe the parameters in equation (1) we use different cosmological distance measurements, covering a wide range of redshifts: $0.02 \lesssim z \lesssim 1090$.

5.1.1 Baryon Acoustic Oscillations

The Baryon Acoustic Oscillations feature is an imprint on the spatial distribution of galaxies and luminous tracers. It was detected for the first time by (Colless et al. 2003; Eisenstein et al. 2005) and has been explored with increasing detail becoming a powerful tool for cosmology. It has consolidated as one of the most robust ways to probe late time dynamics of the Universe, as shown in several observational efforts like those carried by experiments like 6dF (Beutler et al. 2011), WiggleZ (Kazin et al. 2014), Dark Energy Survey (DES) (Abbott et al. 2005) and the SDSS consortium (Anderson et al. 2014; Alam et al. 2016; Dawson et al. 2016), finalizing with their latest and final report on (Alam et al. 2020). BAO is also one of the main features to be probed by experiments like the Dark Energy Spectroscopic Instrument (DESI) (Levi et al. 2013; Aghamousa et al. 2016a,b) and in the near future, Euclid (Laureijs et al. 2011).

In this work we use the spherically averaged BAO signature, in terms of the size $r_{BAO}(z)$:

$$r_{BAO}(z) \equiv \frac{r_s(z_d)}{D_V(z)},$$

(9)

where the comoving sound horizon at the baryon drag epoch is represented by $r_s(z_d)$, and the dilation scale, $D_V(z)$, contains information about the cosmology used in $H(z)$:

$$r_s(z_d) = \int_{z_d}^{\infty} \frac{dz}{H(z)\sqrt{3(1 + \tilde{R}(z)) + 1}},$$

(10)

$$D_V(z) = \left[\frac{z(1 + z)^2}{H(z)} D_A(z)^2\right]^{1/3},$$

(11)

where $\tilde{R}(z)$ is the baryon to photon ratio, defined by $\tilde{R}(z) = \frac{3\Omega_b(z)}{4\Omega_c(z)}$ and the angular diameter distance, $D_A(z)$, given by:

$$D_A(z) = \frac{1}{1 + z} \int_0^z \frac{dz'}{H(z')},$$

(12)

where we can see clearly how to use the BAO standard ruler to constrain the parameters in equation (1). The sound horizon, $r_s(z_d)$, depends upon the physics prior to the recombination era, given by $z_d \approx 1059$ Ade et al. (2016a) and the baryon to photon ratio, $R(z)$. However, the dilation scale, $D_V(z)$, is sensitive to the physics of

4 COSMOLOGICAL BACKGROUND

We model the accelerated expansion of the Universe in terms of a barotropic fluid, $\rho_x$, described in terms of the equation of state $\omega_x \equiv p_x/\rho_x$.

Within the validity of General Relativity for a flat Universe and a FLRW metric, we can express the Friedmann equation as:

$$H^2(z)/H_0^2 = \Omega_m(1 + z)^3 + \Omega_r(1 + z)^4 + \Omega_{DE} F_{DE}(z),$$

(7)
much lower redshifts, particularly to those probed by large scale structure experiments.

In this work we make use of the observational points from the six-degree-field galaxy survey (6dFGS Beutler et al. (2011)), Sloan Digital Sky Survey Data Release 7 (SDSS DR7 Ross et al. (2015)) and the reconstructed value (SDSS(R) Padmanabhan et al. (2012)), as well as the uncorrelated values reported in the complete BOSS sample SDSS DR12 (Alam et al. (2016)). We included the measurement done in the auto and cross-correlation of the Lymann-$\alpha$ Forest (Ly$\alpha$-F) measurements from the quasars sample of the 11th Data Release of the Baryon Oscillation Spectroscopic (BOSS DR11) Delubac et al. (2015); Font-Ribera et al. (2014). In total we cover the redshift range $0.106 < z < 2.36$. Since the volume surveyed by BOSS and WiggleZ Kazin et al. (2014) partially overlap Beutler et al. (2016), we do not use data from the latter in this work. As in this case all the measurements we are using are independent, we can write the $\chi^2_{BAO}$ in terms of the observed values $\mu_{BAO}^{obs}$ with their corresponding errors $\sigma_i$, and the predicted values $\mu_{BAO}^{th}$ as:

$$\chi^2_{BAO} = \sum_i \frac{(\mu_{BAO}^{obs}(z_i) - \mu_{BAO}^{th}(z_i))^2}{\sigma_i^2}$$  \hspace{1cm} (13)

5.1.2 Cosmic Chronometers

In Jimenez & Loeb (2002), the use of the relative ages of galaxies was proposed to track the expansion of the universe. This method was coined “cosmic chronometers”. In Moresco et al. (2011) the authors presented a new methodology by using the spectral properties of early-type galaxies and showed that including the effect of metallicity impacts their results by less than $2 - 3\%$, even when different initial mass functions are considered. The Cosmic Chronometers (CC) data gives a measurement of the expansion rate, $H(z)$, that does not depend on the cosmology model, unlike the case of BAO or Supernovae measurements. Another advantage lies in the fact, that, unlike the distance measurements, we do not rely on the integral of $H(z)$ to constrain the parameters in the EOS (1). It is convenient to write the expansion rate as:

$$H(z) = \frac{\dot{a}}{a} = -\frac{1}{1+z} \frac{dz}{dt}.$$  \hspace{1cm} (14)

With the relation of the Hubble parameter written in this way, it is possible to use the redshift of the galaxies that are taken as chronometers because its redshift can be measured with high accuracy. The differential expression for $dz$ and $dt$ helps to cancel out systematic errors and the possible effects given by the bias (see Moresco et al. (2012) for a detailed revision of the method).

In this work we use the sample compiled in Farooq & Ratra (2013), which covers the redshift range $0.07 < z < 2.3$, with 28 independent measurements of the Hubble parameter. The value of $\chi^2$ will be estimated as:

$$\chi^2_{CC} = \sum_i \frac{(H(z)^{obs}_{CC} - H(z)^{th}_{CC})^2}{\sigma_i^2}.$$  \hspace{1cm} (15)

where $(H(z)^{obs}_{CC})$ and $(H(z)^{th}_{CC})$ stands for observational values and predicted values of the theory, respectively.

5.1.3 Supernovae Ia

Type-Ia supernovae were crucial for the discovery of the accelerated expansion of the Universe and are angular cosmological probes. Ever since the discovery made by Perlmutter et al. (1999) and Riess et al. (1998), they had played a crucial role in discovering the cosmic acceleration and have consolidated as one of most useful and powerful tools to investigate the nature behind the cosmic acceleration. Several high quality samples have been released over the past decade (Kowalski et al. 2008; Hicken et al. 2009; Kessler et al. 2009; Amanullah et al. 2010a; Conley et al. 2011; Suzuki et al. 2012; Betoule et al. 2014; Scollnic et al. 2018).

The $\chi^2$ function of Supernovae Ia can be expressed as:

$$\chi^2_{SN} = \Delta \mu^T \cdot C^{-1} \cdot \Delta \mu$$  \hspace{1cm} (16)

where $\Delta \mu \equiv \mu_{obs}^i - \mu_{th}^i$. We take $C = D_{stat}$ and $\mu_{obs}^i$ from the compilation presented in (Suzuki et al. 2012)$^1$, and estimate $\mu_{th}^i$, the distance modulus of the luminosity distance, as:

$$\mu_i(z) = 5 \log_{10} \left[ (1+z)H_0 \int_0^z dz' H^{-1}(z') \right] + 25,$$  \hspace{1cm} (17)

where $H(z)$ contains the free parameters of (1) through eq. (7). Even though SNe provide a measurement of the luminosity distance as a function of redshift, their absolute luminosity is uncertain and is marginalized out, which also removes any constraints on $H_0$. For that reason we do not include $h$ as part of the parameter vector to be constrained during the analysis when we use only this sample, and we consider a given value $h = 0.7$, as was done in (Amanullah et al. 2010b; Suzuki et al. 2012).

This sample covers the range $0.028 < z < 1.03$ with a total of 557 data points.

5.1.4 Cosmic Microwave Background

In order to add information from the CMB we follow the strategy used by (Ade et al. 2016b), suggested in (Mukherjee et al. 2008). In Mukherjee et al. (2008) it was shown how to compress the information of CMB power spectra within few observable quantities such as the angular scale of sound horizon at last scattering, $l_A \equiv \pi/\theta_s$, the scaled distance to last scattering surface, $R \equiv \Omega_M H_0^2 d_A(z_s)$, the baryon density, $\Omega_b h^2$, and the scalar spectral index, $n_s$.

For correlated data, the $\chi^2$ estimator reads as:

$$\chi^2 = \sum_{i,j} (D_i - y(x_i|\theta)) Q_{ij} (D_j - y(x_j|\theta))$$  \hspace{1cm} (18)

where $Q_{ij} = C^{-1}_{ij}$, is the inverse of the covariance matrix of the data.

In the particular case of $\chi^2_{CMB}$, we have:

$$\chi^2_{CMB} = \gamma_{CMB}^T \cdot C^{-1}_{CMB} \cdot \gamma_{CMB}$$  \hspace{1cm} (19)

where $C^{-1}_{CMB}$ is the inverse of the covariance matrix and $\gamma_{CMB} = D_i - y(x_i|\theta)$ given in terms of the data vector, $D_i = \{ R, I_A, \omega_b, n_s \}$, and $y(x_i|\theta) = \{ R(z, \theta), I_A(z, \theta), \omega_b, n_s \}$, the theoretical prediction that depends on the free parameters: $\theta = \{ A, n, C, h, \Omega_M h^2, \Omega_b h^2 \}$.

In this case, the inverse of the covariance matrix, $C^{-1}$, is

$$\begin{array}{cccc}
R & I_A & \omega_b & n_s \\
\hline
R & 78470.9 & -41.8857 & 1.39247 \times 10^7 & 77926.2 \\
I_A & -12169.8 & 76.7608 & 3.3485 \times 10^6 & -1046.39 \\
\omega_b & 15122.3 & 12.5159 & 2.82752 \times 10^7 & -9366.69 \\
n_s & 52165.5 & -2.41088 & -5.77371 \times 10^6 & 94698.4
\end{array}$$

Data can be found in http://supernova.lbl.gov/Union/.
where we have chosen the more conservative compressed likelihood values from Planck TT + lowP marginalizing over the amplitude of the lensing power, $A_L$ as presented in (Ade et al. 2016b).

The angle of horizon at last scattering is defined to be

$$\theta_\ast = \frac{r_s(z_\ast)}{d_A(z_\ast)},$$

(21)

where $r_s(z_\ast)$ is the horizon size at the decoupling epoch ($z_\ast \approx 1089.95$ according to Planck (Ade et al. 2016a)), defined by the integral in equation (10) evaluated from $z_\ast$ to $\infty$, and $d_A(z_\ast)$ is the comoving distance to last scattering surface:

$$d_A(z_\ast) = \int_0^{z_\ast} \frac{dz'}{H(z')}.$$

(22)

Introduced in this way, we are making use of the position that the comoving distance to last scattering surface:

$$d_A(z_\ast) = \int_0^{z_\ast} \frac{dz'}{H(z')}.$$

(22)

In our analysis we combine the different measurements: BAO, CC, SNe and CMB by adding their respective $\chi^2$ functions, as they are all independent from each other and are probing different cosmic epochs.

In this manner, we write down the combination of all the data as:

$$\chi^2_{Total} = \chi^2_{BAO} + \chi^2_{CMB} + \chi^2_{CC} + \chi^2_{SNe},$$

(23)

where each function is defined as explained in section 5.1.

Furthermore we are interested in the sample of standard rulers, fixed in the CMB and detected in the clustering of luminous tracers via the BAO. This will be defined as the combination:

$$\chi^2_{BAO−CMB} = \chi^2_{BAO} + \chi^2_{CMB},$$

(24)

to explore the constraining power of acoustic oscillations.

Additionally, we want to investigate the constraints coming from late time observations, and to that end we define the function:

$$\chi^2_{late} = \chi^2_{BAO} + \chi^2_{CC} + \chi^2_{SNe},$$

(25)

where we ignore the CMB data.

Even more, we investigate the constraints in our free parameters from the CC, $\chi^2_{CC}$ (15), and the SNe samples (16), $\chi^2_{SNe}$ independently.

For $\Lambda CDM$, the energy density fraction for DE is constant and, we know that for a flat Universe, we can simply express it by the flatness condition, $\Omega_A = 1 - \Omega_m - \Omega_r$. However, with a different dynamics for dark energy, this cannot be assumed to be equal to the fiducial value provided by the Planck collaboration (Aghanim et al. 2018), for instance, for $\Lambda CDM$. This means that, in addition to $A$, $n$, and $C$, the free parameters in (1), we let the physical densities, $\Omega_c$, $h^2$, $\Omega_b h^2$, and the reduced Hubble constant, $h$, free.

The free parameters were varied within uniform priors: $A \in [-50, 50]$, $n \in [0, 1]$, $C \in [-20, 20]$, $\Omega_c h^2 \in [0.001, 0.2]$, $\Omega_b h^2 \in [0.005, 0.045]$, and the Hubble parameter $h \in [0.5, 0.8]$.

However, not all data samples have the same constraining power over different cosmological parameters. In particular, if CMB data is not included in the fitting process, we fix $\Omega_b h^2 = 0.0222$ to the value set by the Planck TT + lowP likelihood Ade et al. (2016a).

We individually optimize the parameters in each case by minimizing the $\chi^2$ statistic

$$\chi^2 = \chi^2_{min}/(d.o.f.),$$

as our statistical measure of the goodness of the fit.

Given that we obtained a good fit for all our models and likelihoods (see last column of Table 1), of order unity (close to 1), we proceed discussing our results as follows.

We report the 1–3σ confidence intervals for different combinations in parameter space: the parameters of equation (1) $A−n, n−C$ (Figures 10–11), and the parameters, $\Omega_b h^2 − h$, and $\Omega_c h^2 − h$ (Figures 12 and 13, respectively). Also, we report the individual uncertainties after

2 Our implementation will be made publicly available and a version of the code can be shared upon reasonable request to the authors.
we report the resulting dynamical behaviour of marginalization over the other dimensions, and these can be found and (25). general model (3.4) using the likelihoods described in (15), (16), (23), (24), and (25). Figure 10 shows the 1-3σ confidence levels in the parameter space \( n - C \) for the general model (3.4) using the likelihoods described in (15), (16), (23), (24), and (25).

Figure 11. 1,2,3σ confidence levels in the parameter space \( n - C \) for the general model (3.4) using the likelihoods described in (15), (16), (23), (24), and (25).

Figure 10 shows the 1-3σ joint confidence levels (CL) for the parameters \( n \) and \( A \) in the general model, (5), fitted by each set of observations. It is noticeable how different observations constrain differently the behaviour of \( n \). In particular, from the Cosmic Chronometers (CC) sample, its value is tightly constrained around \( n \approx 0 \), whereas for the joint likelihood BAO-CMB-SNe-CC, it is consistent with \( n \geq 0.4 \). In the same figure, for the case of equation (5) constrained with the CC sample, we see that the resulting dynamics agrees with that of a cosmic fluid with a dust-like equation of state \( \omega \approx 0 \) (see the figure depicted in the last column, second row of Table 2), which in turn is consistent with a low value for \( \Omega_m \). To further test this hypothesis we reanalyzed the sample fixing the value of \( \Omega_m \) to the one reported by Aghanim et al. (2018) \( (\Omega_m h^2 = 0.1197) \) confirming its impact on the resulting dynamics for the EOS. This result is shown in the lighter green contours of figures 10-11.

Figure 10b shows a close up to the region \( A \in [-1, 2] \) in the parameter space \( A - n \). Here we can appreciate better the fact that the BAO-CMB joint likelihood constraints tightly the value of \( A \) around the value \( A = 0.053 \pm 0.01 \). It is important to recall that, at 1σ level, the value \( A = 0 \) is excluded by all data sets and data combinations, which corresponds to and EOS \( \omega = -1 \), which recovers a cosmological constant model.

\[ C - n \] contour plots

From the \( n - C \) CL, figure 11, we notice that the value \( n = 0 \) is excluded at the 3-σ level by the SNe sample and the joint likelihoods BAO-SNe-CC and BAO-CMB-SNe-CC, this is, all the data sets analyzed which included the SNe sample. On the other hand, the CC sample, imposes very tight constraints on the value of \( n \approx 0 \). Let us point out again that for this particular result we recover a dust-like EOS with almost no matter \( (\Omega_m = 0.04) \), and that, in order to understand the effect of the parameters of (5) on the value of \( \Omega_m \), we rerun the analysis \( \Omega_m h^2 = 0.1197 \) (Planck TT+lowP). In this case, we find that the value of \( n \) is not constrained, allowing a uniform variation along the \( n \) axis. Similarly, for the joint acoustic oscillations sample, BAO-CMB, we find that these are insensitive to the value of \( n \).

Looking at the \( C \) axis of figure 11 we see that the joint likelihood BAO-CMB, constrains \( C \) around \( C \approx 2 \), while the sample of CC with the prior on \( \Omega_m \) from Planck, imposes the weakest constraints on this parameter around \( C \approx 0 \). However, from the same sample, without fixing the value for \( \Omega_m \), we find very tight constraints for \( C \approx 0.8 \). The supernovae sample, by its own, constrains \( C \) around \( C = 0 \), as we can see from the figure 11b.

When used in combination with other data sets, as in BAO-SNe-CC, we find that the value \( C = 0 \) is not excluded at the 3-σ level. For the joint analysis of all the data sets, we find that, even when \( C = 0 \) is excluded with 99.7% of confidence, we obtain a value for \( C < 0.5 \), which in turn implies a present value for \( \omega \) close to \( \omega = -1 \). The case \( C = 0 \), as discussed in section 4, gives a dynamics that is consistent with an \( f(R) \)-like expansion for \( \omega_0 = -1 \).

As it was mentioned before, the parameter \( n \) controls whether the parameterisation depicts one, two or no oscillations at all. It is worth to notice that the value \( n = 0 \) was excluded with 99.7% of confidence, by the full joint likelihood (BAO-CMB-SNe-CC), the SNe sample and the late time observations (BAO-SNe-CC likelihood). The acoustic oscillations joint likelihood, BAO-CMB, and the Cosmic Clocks sample with \( \Omega_m \) fixed, do not constrain \( n \) within the explored range, \( n \in [0, 1] \). On the other hand, the Cosmic Clocks sample by itself, fixes \( n = 0.003^{+0.025}_{-0.003} \).

\[ \Omega_b h^2 \] and \( \Omega_c h^2 \) contours.

To explore more carefully these possibilities, we perform the same analysis for the other three cases of our proposal: the particular case \( n = 1 \) for a non-oscillatory EOS (I: Exponential case, figure 1), the case \( n = 0 \) which allows \( \omega \) to cross only once the phantom dividing line, \( \omega = -1 \), (II: Quintessence/Phantom or Quintom, figures 2 and 3), and the case \( C = 0 \), which presents an oscillatory behaviour around \( \omega = -1 \) (III: \( f(R) \), figures 4 and 5).

By performing this analysis we can investigate if some of the features marked by the value of the parameters have a statistical
preference. We compare the particular cases I, II, III against the general model (IV), and with the concordance $\Lambda CDM$ scenario. Figures 12 and 13 summarize our results for this part of the analysis, along with the figures in table 2. In both figures we present the constraints on the different models using the total likelihood, i.e., $\chi^2_{Total}$ (23), and the acoustic oscillations observations, i.e., BAO-CMB (24). As it was detailed in section 3, each particular case of (1) is referred to as a different model since each choice is motivated by a specific dynamical behaviour.

Figure 12 presents the parameter space for the physical density of cold dark matter, $\Omega_c h^2$, and the Hubble parameter, $h$. The fist thing we notice in this case, is that the constraints are more extended for the BAO-CMB likelihood (represented by dotted contour lines) than for the combination of all data sets (shown in solid contour lines). Moving away from that observation to more specific, we see that different models agree with different values of $\Omega_c h^2$ and $h$.

Focusing first on the constraints from BAO-CMB, we see that $\Lambda CDM$ gives a higher $h$ and large amount of matter while the model Quintessence/Phantom, on the contrary, is consistent with a lower $h$ and smaller amount of matter. Using BAO-CMB data sets, on the space of cosmological parameters $\Omega_c h^2 - h$, it is not possible to distinguish the Exponential model from the general form of the parameterisation, or from the $f(R)$-like background expansion. Which is to say that the Exponential $(n = C = 1)$, $f(R)$-like $(C = 0)$, and the general form of the EOS are fully compatible with each other in the parameter space $\Omega_c h^2 - h$. However, we must remember that each one is quite distinctive from the other in the space of their respective parameters, $n$ and $C$. Interestingly, the Quintessence/Phantom model (II) is consistent with the low $H_0$ value reported by Planck collaboration, while the rest of the models (I Exponential, III $f(R)$, and IV, the general model) have an $H_0$ value consistent with the determination for the Hubble parameter using the Tip of the Red Giants Branch (TRGB) done by Freedman et al. (2019), which sits midway in the range defined by the current Hubble tension (and indicated by the orange shaded area around $h = 0.698$).

Now, from the joint constraints of all data sets, BAO-CMB-SNe-CC, we see that the confidence regions are reduced in $\Omega_c h^2 - h$ space, as compared to those obtained only from the acoustic oscillations. In particular, we find that the general form of the EOS (model IV) is consistent with a lower $\Omega_c h^2$, while $\Lambda CDM$ prefers a slightly higher value. Exponential (I) and Quintessence/Phantom (II) models agree with each other at the 1-$\sigma$ level, as well as the Exponential (I) and $f(R)$-like models. The models Quintessence/Phantom (II) and $f(R)$-like (III) are consistent with each other at the 2-$\sigma$ level. However, all the resulting CL lie within the uncertainties of the TRGB determination of $H_0$ (Freedman et al. 2019).

For the sake of clarity, we present the CL in $\Omega_c h^2 - h$ space in two separate figures. Figure 13a shows the resulting contours from the acoustic oscillations, BAO-CMB, while the resulting constraints from the combination of all data sets, BAO-CMB-SNe-CC, can be seen in figure 13b. Similarly to the $\Omega_c h^2 - h$ contours, in the $\Omega_c h^2 - h$ space. From figure 13a (top panel of 13) we find that the Quintessence/Phantom model (case II, $n = 0$) is consistent with a low value of $h$ which lies within Planck’s determination of $H_0$. In contrast, $\Lambda CDM$ is consistent with a higher value of $h$, which coincides with the local determination of $H_0$ made by SH0ES (Riess et al. 2019). Models I (Exponential, $n = C = 1$), III $(f(R)$ with $C = 0)$, and IV (general form of (5) with $n \in [0, 1]$), are consistent within each other with 99.7% of confidence. They share a value of $h$ in agreement with the TRGB central value. Even when these three models cannot be discerned in the $\Omega_c h^2 - h$ space using BAO-CMB data, they are quite distinctive in the values of their respective parameters ($n$ and $C$). All the CL shown in 13a lie around a central value for $\Omega_c h^2 \approx 0.0226$, close to Planck’s value for the baryonic content.

The lower panel of figure 13 shows the constraints obtained by using the full combination of data, BAO-CMB-SNe-CC, $\chi^2_{Total}$. There is more dispersion around the $\Omega_c h^2$ value compared to the BAO-CMB constraints. However, we see that the five models agree within 1-$\sigma$ level with each other, both in the $h$, and in the $\Omega_c h^2$ dimensions. At 3-$\sigma$ level, the constraints for $h$ lie within the uncertainties from the TRGB determination of $H_0$. More in detail, we observe that, at the 1-$\sigma$ level, $\Lambda CDM$ (gray contour) and the general model (red contour) do not overlap in figure 13b.

**Goodness of the fit**

Table 1 shows the BFV for all models separated by observational data set. Columns 2, 3, 4 give the BFV for the parameters $A$, $n$ and $C$ respectively with uncertainties at 3-$\sigma$, while columns 5, 6, 7, show the values $\omega_0$, $\Omega_{b0}^1$, and $H_0$ within 1-$\sigma$. The last column of this table shows the $\chi^2_{red}$.

From the $\chi^2_{red}$ values, we point out that all fits were very close or approximately of order unity. Also, we find little discrepancies between each other and, moreover, when analyzed data sets individually, we find better fits for our model than for $\Lambda CDM$.

On the other hand, we notice that from all cases, the best fit was obtained for model IV $(f(R)$-like) fitting the full likelihood, $\chi^2_{Total}$, equation (23). From Table 1 we see that this has a value of $\chi^2_{red} \approx 0.991$.

From all the cases analyzed, the poorest fits were obtained for BAO-CMB likelihood, equation (24). In this case, the least favourable result corresponds to model IV (General) with $\chi^2_{BAO-CMB}$ for which we find $\chi^2_{red} \approx 2.645$. In this particular situation, we need to keep on mind that the number of data points in our acoustic oscillations data set is of the same order of magnitude as the number of free parameters.

Similarly, in the case of the CC sample, we notice an over-fitting of the data points, resulting in $\chi^2_{red} \approx 0.594$, for the model II (Quintessence/Phantom). Even in the simplest model, i.e. $\Lambda CDM$, you would have a $\chi^2_{red} \approx 0.991$. 

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**Figure 12.** Confidence intervals for the space $\Omega_c h^2 - h$ for the models $\Lambda CDM$ ($A = 0$), Exponential $(n = C = 1)$, Quintessence/Phantom $(n = 0)$, $f(R)$ ($C = 0$), and the general form of our EOS ($n \in [0, 1]$). Joint likelihoods: BAO – CMB are shown in dotted contour lines and Total with solid contour lines. Vertical shaded zones show the different $H_0$ values from the CMB reported by Planck (Aghanim et al. 2018), the TRGB determination (Freedman et al. 2019), and the SH0ES experiment (Riess et al. 2019).
the $\chi^2_{red}$ is of the same order. This is to be expected, given the size or the error bars for this sample (see section 6.1).

Taking a closer look at results from Table 1 each data set at a time, we find that:

- The model IV (General) was the best fit for data sets SNe and the CC sample.
- The III ($f (R)$)-like) model, was the best fit for local data, BAO-SNe-CC, and also for the combination of all data sets.
- $\Lambda$CDM was the best fit only in the case of the acoustic oscillations sample, BAO-CMB.

To conservatively report our uncertainties for $A$, $n$, and $C$, we quote them within 3-$\sigma$ level, whereas the cosmological parameters $\Omega_{\Lambda}^{(0)}$ and $h$ are reported at 1-$\sigma$ to facilitate the comparison with other works. The full 1,2,3 $\sigma$ contours have been already discussed.

The particularly compact constraints obtained from the CC sample are explained due to the fact that we obtained a profile for the EOS that behaves as dust ($\omega \sim 0$) during its evolution, making this fluid not negligible and hence, being able to constrain the values of the parameters in equation (5) very strictly. We can notice this in the figure portrayed in the last column and second row of Table 2, where we notice clearly $\omega(z) = 0$ during $z \in [0.5, 1.5]$. As the density for a dust-like component is non-negligible during this epoch, their dynamics can be better constrained. As a counter example, we point out the dynamics we obtained for the supernovae sample, under model II (Quintessence/Phantom). This can be seen in detail in the first row and second column of Table 2. In this case, the resulting dynamics for the EOS is that of a phantom component: for $z \geq 0$ we have $\omega(z) \sim -1$. Since this results in a highly sub-dominant component, $\rho = (\rho^{(0)})(a/\dot{a}_0)^{-3(1+\omega)}$, the involved parameters are much less tightly constrained.

To directly show the resulting dynamics of our EOS for a given model and how it is constrained by different data sets, in Table 2 we

### Table 1. Best fit values by data set. Column 1 refers the model: I-Exponential, II-Quintessence/Phantom, III- $f (R)$ with $\omega (0) = -1$, IV-General case and we compare with $\Lambda$CDM in the last row for each observational set. Columns 2, 3 and 4 show the best fit values for the $A$, $n$ and $C$ parameters. Columns 5, 6 and 7 correspond to the values $\omega_0$, $\Omega_{\Lambda}^{(0)}$ and $H_0$ respectively. Column 8 shows the value of the reduced $\chi^2$.
Figure 13. Confidence intervals for the space $\Omega_0 h^2 - h$ for the models ACDE (A = 0), Exponential ($n = C = 1$), Quintessence/Phantom ($n = 0, f(R)$) ($C = 0$), and the general form of our EOS ($n \in [0, 1]$). Vertical shaded zones show the different $H_0$ values from the CMB reported by Planck (Aghanim et al. 2018), the TRGB determination (Freedman et al. 2019), and the SH0ES experiment (Riess et al. 2019).

depict the evolution of the EOS for each model according to its best fit values.

Table 2 shows all the different profiles for $\omega(z)$ that we obtain within its 3-$\sigma$ uncertainties (see appendix A). It is organized as follows: Columns 1, 2, 3, and 4, show the evolution for cases Exponential, Quintessence/Phantom, $f(R)$ with $\omega_0 = -1$, and the General model, respectively. Each row shows the resulting constraints from the different data sets and their combinations.

We particularly stress the general case and how the evolution of the EOS is shaped by the observational data sets. (Column 4) In the first row we notice that, for the $z$-range of the SNe sample, the EOS prefers values $\omega(z) < -1$. In the second row, interestingly, the CC allow to the EOS to take values very close to $\omega = 0$, this way the parameterisation could mimic the EOS for dust, i.e. matter, and we obtain $\Omega_0^{\text{CC}} = 0.039$. For this reason, in the contour plots of $A$ vs $n$ (Fig 10) and $n$ vs $C$ (Fig. 11) we include one case where we fix the value of $\Omega_0^{\text{CC}} = 0.258$. When the parameterisation is fitted by using BAO and the reduced CMB, it is very clear that the best fit of the EOS goes very close to $\omega(z) = -1$ but still $\omega_0 \neq -1$. For the late-time collection (BAO-CC-SNe) the evolution of $\omega(z)$ crosses twice the phantom-line, and shows an oscillatory behaviour. When the CMB is included (last row) the behaviour is very similar and the uncertainty is dramatically reduced.

6.1 Direct comparison to observations

Figure 14 shows the EOS for all the models constrained by the different observations used at 99.7% CL of the BVF. The evolution in the exponential case (green dotted line) goes very close to $\omega(z) = -1$, nevertheless as it approaches $z = 0$, the EOS goes toward higher values going to $\omega_0 = -1.05 \pm 0.006$. In model II, Quintessence/Phantom (blue dotted-dashed line) the EOS reach a value of $\omega_0 = -0.99^{+0.008}_{-0.009}$. In this case, the EOS does not cross the phantom line, which is in consistence with a single scalar field as in standard quintessence models. In the case of Model III, $f(R), \omega_0 = -1$ (orange dashed line), the value of the EOS was fixed to $\omega_0 = -1$, the evolution has the characteristic behaviour of $f(R)$ and the $\chi^2_{red}$ value is closer to 1 than any other model of the parameterisation, including the case $A = 0$ which corresponds to the standard $\Lambda$CDM model. Finally, the general model (red solid line), with three free parameters, shows an oscillatory behaviour that goes to $\omega_0 = -0.39^{+0.030}_{-0.031}$ at $z = 0$, crossing the phantom line twice.

However, $\omega(z)$ is not directly observable. Hence, to explore how distinguishable the models are from each other, we compare their fits to each set of data individually. Figure 15 shows the prediction for the cosmic distances $\mu(z) = m - M$, for supernovae, and $r_{BAO}(z)$ according to the best fit values obtained constraining the respective data sample assuming each one of the models from table 1. In each case we show the direct prediction for the observable quantity (either $\mu(z), r_{BAO}(z)$, or $H(z)/(1+z)$), and the ratio between the best fit for models I-IV to $\Lambda$CDM, $\Delta \mu = (D - D_{\Lambda\text{CDM}}) / D_{\Lambda\text{CDM}}$. In figure 15a we show $\mu(z) = m - M$ for the best fit obtained of all cases. We focus only on the fits done to Union 2.1 supernovae sample. The upper panel of the figure shows the predictions for $\mu = m - M$ vs $z$ according to the best fit values obtained for the models, along with the observational points with error bars. The bottom panel shows the ratio between each model’s prediction for $\mu(z|A, n, C)$, and $\mu(z)_{\Lambda\text{CDM}}$, this is, $\Delta \mu(z) = (\mu(z) - \mu(z)_{\Lambda\text{CDM}}) / \mu(z)_{\Lambda\text{CDM}}$. In this case we find differences of the order $\Delta \mu(z) = 0 - 0.3\%$. Figure 15b shows the evolution of $r_{BAO}(z)$ vs $z$ for our models, along with the observations.

Figure 14. (Color on line) Equation of state for the four models constrained by the full observational data set at 99.7% CL of the BVF. The evolution in the exponential case (green dotted line) goes very close to $\omega(z) = -1$, nevertheless as it approaches $z = 0$, the EOS goes toward higher values going to $\omega_0 = -1.05 \pm 0.006$. In model II, Quintessence/Phantom (blue dotted-dashed line) the EOS reach a value of $\omega_0 = -0.99^{+0.008}_{-0.009}$. In this case, the EOS does not cross the phantom line, which is in consistence with a single scalar field as in standard quintessence models. In the case of Model III, $f(R), \omega_0 = -1$ (orange dashed line), the value of the EOS was fixed to $\omega_0 = -1$, the evolution has the characteristic behaviour of $f(R)$ and the $\chi^2_{red}$ value is closer to 1 than any other model of the parameterisation, including the case $A = 0$ which corresponds to the standard $\Lambda$CDM model. Finally, the general model (red solid line), with three free parameters, shows an oscillatory behaviour that goes to $\omega_0 = -0.39^{+0.030}_{-0.031}$ at $z = 0$, crossing the phantom line twice.

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(a) CL in $\Omega_0 h^2 - h$ for the models I-IV and $\Lambda$CDM from BAO-CMB joint likelihood.

(b) CL in $\Omega_0 h^2 - h$ for the models I-IV and $\Lambda$CDM from BAO-CMB-SNe-CC joint likelihood.

One parameterisation to fit them all 11

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we used in this work. The bottom panel contains the ratio $\Delta\rho_{BAO}(z)$, from where we see that the Quintessence/Phantom model differs the most from the $\Lambda$CDM prescription. Nevertheless the difference is below $\Delta\rho_{BAO} \approx 1\%$ in all cases, reaching the maximum discrepancy around $z \approx 0.1$. This range of redshift ($0.05 < z < 0.4$) will be accurately measured using BAO in the Bright Galaxy Survey (BGS) (Ruiz-Macias et al. 2020) by DESI, making it possible to accurately differentiate between these models at low redshifts.

Figure 16 shows the evolution of $H(z)/(1+z)$ vs $z$ for our models and their best fit to the CC sample. We superimpose the observations with error bars. In this case, we notice bigger discrepancies between our models’ predictions for the quantity $H(z)/(1+z)$, and $\Lambda$CDM. $\Delta H(z)$ reaches a value of $\approx 10\%$. The exponential (case I) and $f(R)$-like models (case III) differ from $\Lambda$CDM in the same proportion, with up to a $\Delta H(z) \approx 3\%$ at $z = 0.1$, $\Delta H(z) \approx -1.5\%$ around $z \approx 0.5$, and $\Delta H(z) \approx +1.5\%$ at $z \approx 3$. On the other hand, Quintessence/Phantom (case II) and the General model show the same pattern in their discrepancies from $\Lambda$CDM, but with $\Delta H(z) \approx H(z)5\%$ at $z = 0.1$, $\Delta H(z) \approx -3\%$ around $z \approx 0.3$, $\Delta H(z) \approx +5\%$ at $z \approx 1.5$, and peaking at $z \approx 3$, with $\Delta H(z) \approx -10\%$. Something worth to be mentioned is the change in sign of $\Delta H(z)$, so even when the errors for this particular observable are systematic dominated and hence, bigger than for the other type of observations, the oscillatory behaviour in $\Delta H(z)$ can potentially help distinguishing them.

|   | I (Exponential) | II (Quint./Phantom) | III ($f(R)$, $\omega_0 = -1$) | IV (General) |
|---|----------------|---------------------|-----------------------------|-------------|
| SNe | ![Graph] | ![Graph] | ![Graph] | ![Graph] |
| CC | ![Graph] | ![Graph] | ![Graph] | ![Graph] |
| BAO-CMB | ![Graph] | ![Graph] | ![Graph] | ![Graph] |
| BAO-CC-SNe | ![Graph] | ![Graph] | ![Graph] | ![Graph] |
| Total | ![Graph] | ![Graph] | ![Graph] | ![Graph] |

Table 2. Equation of state of each model by data set. Column 1 corresponds to model I-Exponential, column 2 shows the EOS for model II-Quintessence/Phantom, column 3 is model III-$f(R)$ with $\omega_0 = -1$ and column 4 shows the General case, model IV. Each row stands for each data set combination (SNe, CC, BAO-CMB, Sn-CC-BAO and Sn-CC-BAO-CMB). The solid lines depict $\omega(z)$ and the contours around them are the $3\sigma$ error propagation (see appendix A). The color vertical shadow show the range of $z$ for each data set. In the plots vertical axis is $\omega(z) \in [-2.5, 0]$ and the horizontal axis corresponds to the redshift $z \in [0, 3]$. 

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7 CONCLUSIONS AND DISCUSSION

We present a new parameterisation that can reproduce the generic behaviour of the most widely used physical models for accelerated expansion with infrared corrections. Our mathematical form for $\omega(z)$ has at most three free parameters which can be mapped back to specific archetypal models for dark energy. We analyze in detail how different combinations of data can constrain the specific cases embedded in our form for $\omega(z)$, and report: the confidence intervals, individual uncertainties, resulting dynamics and statistical indicator of the goodness of the fit. We show that the parameters can be well constrained by different observational data sets and that all cases were good fits to the data.

With only one free parameter ($n = C = 1$, $A$ free), we can parameterise the expansion rate of the variety of models described in Roy et al. (2018). We call this case, Exponential model. With this, we obtain not only a much richer dynamics for dark energy than the simplified $\omega = \text{const.} \neq -1$ model, but also, a good fit to the data sets we employ, with $\chi^2_{\text{red}} = 1.764$ in the worst case, and $\chi^2_{\text{red}} = 0.988$, in the best.

With two free parameters ($n = 1$, $A$ and $C$, free), we are able to reproduce the generic expansion rate of minimally coupled scalar fields, such as quintessence. Depending on the sign of $\omega(z)$, the generic behaviour of the so called Phantom models could be described with this subset of parameters. This comprises one of the most explored models for DE which we can model with the same number of free parameters as in the widely used CPL (Chevallier & Polarski 2001; Linder 2003) parameterisation. Our fits to the data are competitive, and both parameters can be simultaneously constrained.

Using a different subset of only two free parameters from our model ($C = 0$, $A$, and $n$ free), we are able to mimic the generic expansion rate provided by cosmologically viable $f(R)$ theories of gravity (as shown in Jaime et al. (2014), and previously attempted.
around $\omega$ in (Jaime et al. 2018)). For this case in particular, referred to as $f(R)$-like and characterized by an oscillatory behaviour for the EOS around $\omega = -1$, we find the best fit of the whole sample, BAO-SNe-CC-CMB, with a $\chi^2_{red} = 0.991$.

Equally important is the fact that we can constrain our free parameters in all the cases studied without degeneracy, divergent evolution at high redshift nor rapidly oscillatory behaviour or other mathematical misbehaves.

When we let the three parameters vary freely testing the general form of $\omega(z)$, we can answer which dynamical behaviour is favoured by observations. In this case, we find as result, an EOS which oscillates around the phantom-dividing line, and, with over 99% of confidence, the cosmological constant solution is disfavored.

The strength of our proposal lies in its independence of a specific theoretical model. Hence, even when we argued that the simplest, theoretically-sustained, explanation behind an oscillatory profile for the cosmic distances to be able to differentiate among particular cases. We find that all cases are good fits to the data.

We analyze in detail how different combinations of data can constrain the specific cases embedded in our form for $\omega(z)$, and report the confidence intervals, individual uncertainties, resulting dynamics and statistical indicator of the goodness of our fits, as well as a comparison against the required increase in precision for observations of the cosmic distances to be able to differentiate among particular cases. We find that all cases are good fits to the data.

It is interesting to note that our best fit values for $H_0$ lie in between the values to be known in tension.

To summarise, in this work we have presented a single equation which is able to reproduce a variety of well motivated physical scenarios for cosmic expansion at late times, we probed its adequacy to be implemented to data and aim to provide the community with a simple framework to incorporate physically-motivated models into surveys and clustering analyses and better link observational phenomena and theoretical hypotheses for testing the nature of cosmic acceleration.

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DATA AVAILABILITY

All the observational data used in this work is of public knowledge.

The cosmic chronometers sample we used can be found in Farooq & Ratra (2013), as a compiled table of $z$, $H(z)$, and the related errors, $\sigma H(z)$.

Our BAO data points can be found in the respective reference. For the six-degree-field galaxy survey (6dFGS) (Beutler et al., 2011), the Sloan Digital Sky Survey Data Release 7 (SDSS DR7) (Ross et al., 2015), the reconstructed value SDSS(R) presented in (Padmanabhan et al., 2012), and the uncorrelated values of the complete BOSS sample SDSS DR12 are reported in (Alam et al., 2016). The measurement of the auto and cross correlation of the Lymann-$\alpha$ Forest (Lyo-F) measurements from quasars of the 11th Data Release of the Baryon Oscillation Spectroscopic (BOSS DR11) can be found in Delubac et al. (2015); Font-Ribera et al. (2014).

The compressed CMB likelihood with Planck $TT+lowP$ values can be found in (Ade et al. 2016b), and we have given the full form of the reduced matrix in section 5.1.

Our chosen Supernovae compilation was Union 2.1, presented in Suzuki et al. (2012) and which can be downloaded from http://supernova.lbl.gov/Union/.

Our numerical implementation will soon be made publicly available in the repository https://github.com/oarodriguez/cosmostat, but a version of the code can be shared upon reasonable request to the authors.

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From equation (1) we note that $\omega(z)$ is a function of parameters $A$, $n$, and $C$, and of redshift, $z$. From here, it follows that the uncertainty $\delta\omega(z)$ depends, in the same way, on the parameters, and their uncertainties $\delta A$, $\delta n$, and $\delta C$.

When the uncertainties are independent of each other, $\delta\omega(z)$ has a broader dispersion around the central point at every $z$.

From the individual errors, $\delta A$, $\delta n$, and $\delta C$, we estimate the propagated uncertainty in the resulting $\omega(z)$, computed as:

$$\delta\omega(z) = (\omega + 1) \left[ \frac{\delta A}{A} - \frac{\delta C}{(z^n - z - C)} + \frac{z^n \ln(z) \delta n}{(z^n - z - C)} \right].$$  \hfill (A1)

We use equation (A1), $\delta\omega(z)$, considering the uncertainties parameters $\delta A$, $\delta n$, and $\delta C$ as independent of $z$.

In the case that the uncertainties depend explicitly on $z$, equation (A1) will be an overestimation of $\delta\omega(z)$, which guarantees that the dynamical range for each EOS lies inside our estimated errors.

Using the uncertainties for $A$, $n$, and $C$, reported in column 5 from table 1, into equation (A1) for $\delta\omega(z)$, we calculated the 99.7% CL in figures of table 2.

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