WEIGHTED ORTHOGONAL POLYNOMIALS-BASED GENERALIZATION OF WIRTINGER-TYPE INTEGRAL INEQUALITIES FOR DELAYED CONTINUOUS-TIME SYSTEMS

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Abstract. In the past three years, many researchers have proven and/or employed some Wirtinger-type integral inequalities to establish less conservative stability criteria for delayed continuous-time systems. In this present paper, we will investigate weighted orthogonal polynomials-based integral inequalities which is a generalization of the existing Jensen’s inequalities and Wirtinger-type integral inequalities.

Key words. Wirtinger-type integral inequalities (WTIIs); delayed continuous-time systems; weighted orthogonal polynomials (WOPs).

AMS subject classifications. 15A45, 34K38, 35A23

1. Introduction. Time delays are inherent in many nature’s processes and systems, for example, spread of infectious diseases and epidemics [24], population dynamics systems [13], neural networks [11, 10], vehicle active suspension [27], and biological and chemical systems [1, 32]. Since time delays are generally regarded as one of main sources of instability and poor performance [28, 9], the stability analysis issue of time-delay systems is important and has received considerable attention (see [31, 30, 15, 25, 26] and the references therein).

Most of the results on stability analysis of delayed continuous-time systems are obtained by the Lyapunov-Krasovskii functional (LKF) approach [4]. A key step of the LKF approach is how to construct LKF and to bound its derivative. It is well-known that an indispensable part of LKF is some integer items like

\[ \mathcal{I}_m(w_t) := \int_a^b (s-a)^mw_t^T(s)Rw_t(s)ds, \quad t \geq 0, \]

where \( w_t : [a, b] \to \mathbb{R}^n \) is defined by \( w_t(s) = w(t+s) \) for all \( s \in [a, b] \), \( w : [0, +\infty) \to \mathbb{R}^n \) is a continuous function, \( R \) is a real symmetric positive definite matrix, and \( m \) is a nonnegative integer. It is clear that \( \mathcal{I}_0(w_t) = \int_a^b w_t^T(s)Rw_t(s)ds \) and

\[ \mathcal{I}_m(w_t) = m! \int_a^b \int_{\theta_1}^{\theta_m} \int_{\theta_1}^{\theta_m} \cdots \int_{\theta_1}^{\theta_m} w_t^T(s)Rw_t(s)dsd\theta_m \cdots d\theta_1 \]

for \( m \geq 1 \). Since

\[ \frac{d}{dt} \mathcal{I}_0(w_t) = w_t^T(b)Rw_t(b) - w_t^T(a)Rw_t(a) \]

and

\[ \frac{d}{dt} \mathcal{I}_m(w_t) = (b-a)^m w_t^T(b)Rw_t(b) - m\mathcal{I}_{m-1}(w_t), \quad m \geq 1, \]

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the conservativeness of resulting stability criterion relies mainly on the lower bound to \( \mathcal{I}_{m-1}(w_t) \) for \( m \geq 1 \). Usually, the so-called Jensen’s inequalities (JIs) [3, 22, 2] are applied to bound \( \mathcal{I}_k(w_t) \) for any nonnegative integer \( k \).

Recently, some new integral inequalities, spectrally Wirtinger-type integral inequalities (WTIIs), have been proposed to improve Jensen’s inequalities (i.e., to give more accurate lower bounds of \( \mathcal{I}_m(w_t) \) or \( \mathcal{I}_m(w_t) \)) (see [3, 22, 2, 18, 19, 21, 20, 17, 14, 8, 29, 30, 11, 12, 15, 23, 16, 33, 34, 31, 7] and the references therein). It is shown by Gyurkovics [5] that the lower bound of \( \mathcal{I}_0(w_t) \) given in [18] is more accurate than ones in [11, 12], while the estimations to \( \mathcal{I}_0(w_t) \) obtained in [18, 29] are equivalent.

In this present paper, we aim in reducing the conservativeness of LKF approach by investigating new integral inequalities based on weighted orthogonal polynomials (WOPs) which is a generalization of those JIs and WTIIs mentioned above as special cases.

This paper is organized as follows: In Section 2, we will first introduce a class of WOPs, and thereby investigate WOPs-based inequality inequalities. Discussions of the relation between the WOPs-based inequality inequalities and the JIs and WTIIs in [3, 22, 2, 18, 19, 21, 20, 17, 14, 8, 29, 30, 11, 12, 15, 23, 16] will be presented in Section 3. We will conclude the results of this paper in Section 4.

**Notations:** The notations used throughout this paper are fairly standard. Let \( \mathbb{R}^{n \times m} \) be the set of all \( n \times m \) matrices over the real number field \( \mathbb{R} \). For a matrix \( X \in \mathbb{R}^{n \times n} \), the symbols \( X^{-1} \) and \( X^T \) denote the inverse and transpose of \( X \), respectively.

Set \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \) and \( X^{-T} = (X^{-1})^T \). The Kronecker product, \( A \otimes B \), of two matrices \( A = [a_{ij}] \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times q} \) is the \( mp \times nq \) block matrix:

\[
\begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

Denote by diag(\( \cdots \)) and col(\( \cdots \)) the (block) diagonal matrix and column matrix formed by the elements in brackets, respectively.

### 2. WOPs-based integral inequalities

In this section we will investigate novel WOPs-based integral inequalities, which is a generalization of many JIs and WTIIs in literature.

**2.1. WOPs.** If \( p(s) = \sum_{k=0}^{N} a_{k}s^k \) and \( a_N \neq 0 \), then we say \( p(s) \) is a polynomial of degree \( N \). Let \( \mathbb{R}[s]_N \) denote the linear space of polynomials with real coefficients of degree not exceeding \( N \). Set \( f_k(s) = (s-a)^k, k = 0, 1, 2, \ldots, N \). Then \( \{f_k(s)\}_{k=0}^{N} \) is a basis of \( \mathbb{R}[s]_N \). For an arbitrary but fixed nonnegative integer \( m \), define an inner product, \( \langle \cdot, \cdot \rangle_m \), on \( \mathbb{R}[s]_n \) by

\[
\langle p(s), q(s) \rangle_m = \int_a^b (s-a)^m p(s)q(s)ds
\]

for any \( p(s), q(s) \in \mathbb{R}[s]_N \). Let \( \{p_{km}(s)\}_{k=0}^{N} \) be the orthogonal basis of \( \mathbb{R}[s]_N \) which is obtained by applying the Gram-Schmidt orthogonalization process to the basis \( \{f_k(s)\}_{k=0}^{N} \), that is,

\[
p_{0m}(s) = f_0(s), \quad p_{1m}(s) = f_i(s) - \sum_{j=0}^{i-1} \frac{\langle f_i, f_{jm} \rangle}{\langle f_{jm}, f_{jm} \rangle} p_{jm}(s), \quad i = 1, 2, \ldots, N,
\]
where

\[(2.3) \quad g_{ijm} = (f_i(s), p_{jm}(s))_m, \quad \chi_{jm} = (p_{jm}(s), p_{jm}(s))_m.\]

Then \(\{p_{km}(s)\}_{k=0}^{N} \) are WOPs with the weight function \((s - a)^m\). Furthermore, (2.2) can be written as the following matrix form:

\[(2.4) \quad F_N(s) = G_Nm P_Nm(s)\]

with

\[P_Nm(s) = \text{col}(p_{0m}(s), p_{1m}(s), \ldots, p_{Nm}(s)), \]

\[F_N(s) = \text{col}(f_0(s), f_1(s), \ldots, f_N(s)),\]

where \(G_Nm\) be the \((N + 1) \times (N + 1)\) unit lower triangular matrix with the \((i, j)\)-th entry equal to \(\frac{g_{i-1,j-1,m}}{\chi_{j-1,m}}\) for any \(i > j\).

### 2.2. WOPs-based integral inequalities

To prove WOPs-based integral inequalities, the following property on Kronecker product of matrices is required.

**Lemma 2.1.** [6] If \(A, B, C\) and \(D\) are matrices of appropriate sizes, then \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\).

Based on the previous preparation, now we can investigate the following WOPs-based integral inequalities which give lower bounds of \(I_m(w_t)\).

**Theorem 2.2.** For given integers \(N \geq 0\) and \(m \geq 0\), a symmetric positive definite matrix \(R \in \mathbb{R}^{n \times n}\), and a continuous function \(\omega : [a, b] \rightarrow \mathbb{R}^n\), the following inequality holds:

\[(2.5) \quad I_m(w_t) \geq F_N^T(w_t)(\Xi_Nm \otimes R)F_Nm(w_t)\]

with

\[(2.6) \quad F_Nm(w_t) = \int_a^b (s - a)^m(F_N(s) \otimes w_t(s))ds,\]

\[(2.7) \quad \Xi_Nm = G^{-T}_{Nm} \Lambda^{-1}_{Nm} G^{-1}_{Nm},\]

\[(2.8) \quad \Lambda_Nm = \text{diag}(\chi_{0m}, \chi_{1m}, \chi_{2m}, \ldots, \chi_{Nm}),\]

and \(F_N(s), \chi_{km}\) and \(G_Nm\) are defined as previously.

**Proof.** Set

\[z(s) = w_t(s) - \sum_{k=0}^{N} \chi_{km} p_{km}(s) \pi_{km}(w_t)\]

with

\[(2.9) \quad \pi_{km}(w_t) = \int_a^b (s - a)^m p_{km}(s) w_t(s)ds.\]
Then it follows from (2.1), (2.3) and the orthogonality of \{p_{km}(s)\}_{k=0}^{\mathcal{N}} under the weight function \((s - a)^m\) that

\[ I_m(z) = I_m(w_t) - \sum_{k=0}^{\mathcal{N}} \lambda_{km}^{-1} \pi_{km}(w_t) R \pi_{km}(w_t). \]

This, together with \(I_m(z) \geq 0\), implies that

\[
I_m(w_t) \geq \sum_{k=0}^{\mathcal{N}} \lambda_{km}^{-1} \pi_{km}(w_t) R \pi_{km}(w_t),
\]

(2.10)

where

\[ \Pi_{\mathcal{N}m}(w_t) = \text{col}(\pi_{0m}(w_t), \pi_{1m}(w_t), \ldots, \pi_{\mathcal{N}m}(w_t)). \]

Since \(G_{\mathcal{N}m}\) is a unit lower triangular matrix, it follows from (2.4), (2.6) and (2.9) that

\[
\Pi_{\mathcal{N}m}(w_t) = \int_a^b (s - a)^m (P_{\mathcal{N}m}(s) \otimes w_t(s))ds
\]

This, together with (2.10) and Lemma 2.1, completes the proof. □

Since the inequality (2.5) is obtained by using the WOPs (2.2), we will refer to (2.5) as WOPs-based integral inequalities.

3. Discussions. In this section we will discuss the relation between the WOPs-based integral inequalities in Theorem 2.2 and the JIs and WTIIs in [3, 22, 2, 18, 19, 21, 20, 17, 14, 8, 29, 30, 11, 12, 15, 23, 16].

When \((\mathcal{N}, m) = (2, 0)\), by employing the symbolic operations of MATLAB, one can easily check that

\[ F_2(s) = \begin{bmatrix} 1 \\ s - a \\ (s - a)^2 \end{bmatrix}, \quad G_{20}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a - b}{(b - a)^2} & 1 & 0 \\ a - b & 1 & 1 \end{bmatrix}, \]

and hence

\[ A_{20} = \frac{1}{b - a} \text{diag}(1, \frac{12}{(b - a)^2}, \frac{180}{(b - a)^4}), \]

\[ F_{20}(w_t) = \begin{bmatrix} \int_a^b w_t(s)ds \\ \int_a^b \int_a^b w_t(s)dsd\alpha \\ \int_a^b \int_b^\beta \int_a^b w_t(s)dsd\alpha d\beta \end{bmatrix}, \]

\[ \Xi_{20}(R) = \frac{1}{b - a} (\delta_1 R \delta_1^T + 3\delta_2 R \delta_2^T + 5\delta_3 R \delta_3^T), \]
where
\[
\delta_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} 1 \\ \frac{2}{a-b} \\ 0 \end{bmatrix}, \quad \delta_3 = \begin{bmatrix} 1 \\ \frac{6}{(b-a)^2} \\ \frac{b}{(b-a)^2} \end{bmatrix}.
\]

This, together with Theorem 2.2, yields the following result.

**Corollary 3.1.** When \((N', m) = (2, 0)\), the inequality (2.5) turns into \([15, (13)]\), that is,
\[
\mathcal{I}_0(w_t) \geq \frac{1}{b-a} (\Omega_0^T \Omega_0 + 3 \Omega_1^T \Omega_1 + 5 \Omega_2^T \Omega_2),
\]
where
\[
\begin{align*}
\Omega_0 &= \int_a^b w_t(s) ds, \\
\Omega_1 &= \int_a^b w_t(s) ds - \frac{2}{b-a} \int_a^b \int_a^b w_t(s) ds d\alpha, \\
\Omega_2 &= \int_a^b w_t(s) ds - \frac{6}{b-a} \int_a^b \int_a^b w_t(s) ds d\alpha \\
&\quad + \frac{12}{(b-a)^2} \int_a^b \int_a^b \int_a^b w_t(s) ds d\sigma d\beta.
\end{align*}
\]

Similar to Corollary 3.1, the following several corollaries can be easily derived from Theorem 2.2.

**Corollary 3.2.** When \((N', m) = (1, 0)\), the inequality (2.5) turns into the so-called Wirtinger-based integral inequality \([15, (8)]\), that is,
\[
\mathcal{I}_0(w_t) \geq \frac{1}{b-a} (\Omega_0^T \Omega_0 + 3 \Omega_1^T \Omega_1),
\]
where \(\Omega_0\) and \(\Omega_1\) are defined as in Corollary 3.1.

**Corollary 3.3.** When \(N = 0\), the inequality (2.5) turns into the celebrated Jensen’s inequalities (see \([3]\) and \([22]\) for the cases \(m = 0\) and \(m = 1\), respectively; and \([2, Lemma 1]\) for the special case \((a, b) = (-d, 0))\), that is,
\[
\int_a^b \cdots \int_a^b w_t^T(s) R w_t(s) ds d\theta_m \cdots d\theta_1 \\
\geq \frac{(m+1)!}{(b-a)^{m+1}} \tilde{\Omega}_m^T \tilde{\Omega}_m,
\]
where
\[
\tilde{\Omega}_m = \int_a^b \cdots \int_a^b w_t(s) ds d\theta_m \cdots d\theta_1.
\]

**Corollary 3.4.** When \(N = 1\), the inequality (2.5) turns into
\[
\int_a^b \cdots \int_a^b w_t^T(s) R w_t(s) ds d\theta_m \cdots d\theta_1 \\
\geq \frac{(m+1)!}{(b-a)^{m+1}} \left( \tilde{\Omega}_m^T R \tilde{\Omega}_m + (m+3)(m+1) \Sigma_m^T \Sigma_m \right),
\]
where
\[ \Sigma_m = \tilde{\Omega}_m - \frac{m+2}{b-a} \tilde{\Omega}_{m+1}, \]

and \( \tilde{\Omega}_m \) and \( \tilde{\Omega}_{m+1} \) are defined as in Corollary 3.3.

**Corollary 3.5.** When \((N, m) = (1, 1)\), the inequality (2.5) turns into [15, (16)], that is,

\[ I_1(w_t) \geq \frac{2}{(b-a)^2} (\Omega_3^T R \Omega_3 + 8 \Omega_4^T R \Omega_4), \tag{3.5} \]

where
\[ \Omega_3 = \int_a^b \int_a^b w_t(s) d\alpha d\beta, \]
\[ \Omega_4 = \int_a^b \int_a^b w_t(s) d\alpha \left( \frac{3}{b-a} \int_a^b \int_a^b w_t(s) d\alpha d\beta \right). \]

**Corollary 3.6.** When \( m = 0 \) and \((a, b) = (-h, 0)\), the inequality (2.5) turns into the so-called Bessel–Legendre inequality [20, Lemma 3] (i.e., [19, Lemma 3]), that is,

\[ I_0(w_t) \geq \frac{1}{h} \sum_{k=0}^N (2k+1) \tilde{\Omega}_k^T R \tilde{\Omega}_k, \tag{3.6} \]

where \( \tilde{\Omega}_k = \int_{-h}^0 L_k(s) w_t(s) ds \), and \( \{L_k(s)\}_{k=0}^N \) is the Legendre orthogonal polynomials defined in [20, Definition 1].

If we replace \( w_t \) by \( \dot{w}_t \) in Corollaries 3.1–3.5, then the following several results can be obtained.

**Corollary 3.7.** When \((N, m) = (2, 0)\) and \( w_t \) is replaced by \( \dot{w}_t \), the inequality (2.5) turns into [15, (24)], that is,

\[ I_0(\dot{w}_t) \geq \frac{1}{b-a} (\Theta_0^T R \Theta_0 + 3 \Theta_1^T R \Theta_1 + 5 \Theta_2^T R \Theta_2), \tag{3.7} \]

where
\[ \Theta_0 = w_t(b) - w_t(a), \]
\[ \Theta_1 = w_t(b) + w_t(a) - \frac{2}{b-a} \int_a^b w_t(s) ds, \]
\[ \Theta_2 = w_t(b) - w_t(a) + \frac{6}{b-a} \int_a^b w_t(s) ds \]
\[ - \frac{12}{(b-a)^2} \int_a^b \int_a^b w_t(s) d\alpha d\beta. \]

**Corollary 3.8.** When \((N, m) = (1, 0)\) and \( w_t \) is replaced by \( \dot{w}_t \), the inequality (2.5) turns into [18, Corollary 5] (i.e., [17, Lemma 2.1] and [21, Lemma 2.1] or [15, (23)]), that is,

\[ I_0(\dot{w}_t) \geq \frac{1}{b-a} (\Theta_0^T R \Theta_0 + 3 \Theta_1^T R \Theta_1), \tag{3.8} \]
where $\Theta_0$ and $\Theta_1$ are defined as in Corollary 3.7.

**Corollary 3.9.** When $N = 0$ and $w_t$ is replaced by $\dot{w}_t$, the inequality (2.5) turns into the celebrated Jensen’s inequalities (see [3] and [22] for the cases $m = 0$ and $m = 1$, respectively), that is,

$$
\int_{a}^{b} \int_{\theta_1}^{b} \cdots \int_{\theta_m}^{b} \dot{w}_t^T(s) R \dot{w}_t(s) ds d\theta_m \cdots d\theta_1 \\
\geq \frac{(m+1)!}{(b-a)^{m+1}} \tilde{\Theta}_m^T R \tilde{\Theta}_m,
$$

(3.9)

where $\tilde{\Theta}_0 = w_t(b) - w_t(a)$ and

$$
\tilde{\Theta}_m = \frac{(b-a)^m}{m!} w_t(b) - \tilde{\Omega}_{m-1}, m \geq 1.
$$

**Corollary 3.10.** When $N = 1$ and $w_t$ is replaced by $\dot{w}_t$, the inequality (2.5) turns into

$$
\int_{a}^{b} \int_{\theta_1}^{b} \cdots \int_{\theta_m}^{b} \dot{w}_t^T(s) R \dot{w}_t(s) ds d\theta_m \cdots d\theta_1 \\
\geq \frac{(m+1)!}{(b-a)^{m+1}} \left( \tilde{\Theta}_m^T R \tilde{\Theta}_m + (m+1)(m+3) \Psi_m^T R \Psi_m \right),
$$

where

$$
\Psi_m = -\frac{(b-a)^m}{(m+1)!} w_t(b) - \tilde{\Theta}_{m-1} + \frac{m+2}{b-a} \tilde{\Theta}_m,
$$

and $\tilde{\Theta}_m$ and $\tilde{\Theta}_{m-1}$ are defined as in Corollary 3.9.

**Corollary 3.11.** When $(N, m) = (1, 1)$ and $w_t$ is replaced by $\dot{w}_t$, the inequality (2.5) turns into [15, (25)], that is,

$$
\mathcal{I}_1(\dot{w}) \geq 2\Theta_3^T R \Theta_3 + 4\Theta_4^T R \Theta_4,
$$

(3.10)

where

$$
\Theta_3 = w_t(b) - \frac{1}{b-a} \int_{a}^{b} w_t(s) ds,
$$

$$
\Theta_4 = w_t(b) + \frac{2}{b-a} \int_{a}^{b} w_t(s) ds - \frac{6}{(b-a)^2} \int_{a}^{b} w_t(s) ds d\alpha.
$$

**Remark 1.** Based on (3.2), Park et al. [14, Corollary 1] derived

$$
\mathcal{I}_1(w_t) \geq \frac{2}{(b-a)^2} (\Omega_3^T R \Omega_3 + 2\Omega_4^T R \Omega_4).
$$

(3.11)

Clearly, the inequality (3.5) is more accurate than (3.11).

**Remark 2.** It has been proven by Gyurkovics in [5, Corollary 9] that Corollary 3.8 is equivalent to [29, Lemma 4] in term of establishing stability criteria for delayed continuous-time systems. By a similar approach, we can show that Corollary 3.7 is
equivalent to [30, Lemma 1]. However, unlike [29, Lemma 4] and [30, Lemma 1], no free-weighting matrix is involved in Corollaries 3.7 and 3.8.

**Remark 3.** It has been proven by Gyurkovics in [5, Theorem 6] that the inequality in [11, Lemma 2.4] (i.e., [12, (12)]) is more conservative than (3.8). By a similar approach, one can prove that the inequality [12, (13)] is more conservative than (3.10).

**Remark 4.** The inequalities in Corollaries 3.4 and 3.10 are more accurate than ones in [8, Lemmas 5 and 6], respectively, since the coefficients of the second item on the right-hand side of the inequalities in [8, Lemmas 5 and 6] is

\[
\frac{m(m+1)}{(b-a)(m+3)}
\]

which is smaller than

\[
\frac{m(m+1)^2(m+3)}{(b-a)^2m+1}
\]

in Corollaries 3.4 and 3.10.

**Remark 5.** Note that Corollary 3.8 refines the inequality proposed in [16, Lemma 5], in which the second term of the righthand side is

\[
\frac{\pi^2}{4} \Theta T^1 R \Theta^1
\]

which is less than or equal to

\[
\frac{3\Theta T^1 R \Theta^1}{4}
\]

So, Corollary 3.8 is less conservative than [16, Lemma 5].

**Remark 6.** If \((s-a)^k\) is replaced by \((b-s)^k\) for all positive integer \(k\) throughout this paper, then we can obtain new WOPs-based integral inequalities like (2.5), which is a generalization of [18, Corollary 4], [23, (3.1) and (3.8)] and [15, (18) and (26)].

Corollaries 3.1–3.11 imply that Theorem 2.2 contains the corresponding results of [3, 22, 2, 18, 19, 21, 20, 17] as special cases, while Remarks 1–6 present that Theorem 2.2 improves the corresponding results of [14, 8, 29, 30, 11, 12, 15, 23, 16]. Therefore, Theorem 2.2 is a generalization of these literature.

**4. Conclusion.** In this paper, we have provided WOPs-based integral inequalities which encompass and/or improve the corresponding inequalities in [3, 22, 2, 18, 19, 21, 20, 17, 14, 8, 29, 30, 11, 12, 15, 23, 16]. From these literature, it is clear that the WOPs-based integral inequalities obtained in this paper have potential applications in establishing less conservative stability criteria for delayed continuous-time systems. This will be proceeded in our future work.

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