A LITTLEWOOD-RICHARDSON FILTRATION AT ROOTS OF 1
FOR MULTIPARAMETER DEFORMATIONS OF SKEW SCHUR MODULES

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Let \( R \) be a commutative ring, \( q \) a unit of \( R \) and \( P \) a multiplicatively antisymmetric matrix with coefficients which are integer powers of \( q \). Denote by \( SE(q, P) \) the multiparameter quantum matrix bialgebra associated to \( q \) and \( P \). Slightly generalizing [H-H], we define a multiparameter deformation \( L_{\lambda/\mu} V_P \) of the classical skew Schur module. In case \( R \) is a field and \( q \) is not a root of 1, arguments like those given in [H-H] show that \( L_{\lambda/\mu} V_P \) is completely reducible, and its decomposition into irreducibles is \( \sum_{\nu} \gamma(\lambda/\mu; \nu)L_{\nu} V_P \), where the coefficients \( \gamma(\lambda/\mu; \nu) \) are the usual Littlewood-Richardson coefficients. When \( R \) is any ring and \( q \) is allowed to be a root of 1, we construct a filtration of \( L_{\lambda/\mu} V_P \) as an \( SE(q, P) \)-comodule, such that its associated graded object is precisely \( \sum_{\nu} \gamma(\lambda/\mu; \nu)L_{\nu} V_P \).

1. The Ingredients

1.1 Let \( N > 1 \) be a positive integer. Choose a unit \( q \) in a commutative ring \( R \); fix a matrix \( P = (p_{ij})_{i,j=1}^{N} \) where the \( p_{ij} \)'s are non-zero elements of \( R \) with the property
\[ p_{ij}p_{ji} = p_{ii} = 1 \quad \forall i, j = 1, \ldots, N. \]
Consider the free \( R \)-module \( V_P \) with basis \( \{u_1, \ldots, u_N\} \) and define an automorphism \( \beta_{q,P} \) on \( V_P \otimes V_P \) by the following rule:
\[
\beta_{q,P}(u_i \otimes u_j) = \begin{cases} 
  u_i \otimes u_i & \text{if } i = j \\
  q p_{ji} u_j \otimes u_i & \text{if } i < j \\
  q p_{ij} u_j \otimes u_i + (1 - q^2)u_i \otimes u_j & \text{if } i > j.
\end{cases}
\]
Then \( (V_P, \beta_{q,P}) \) is a YB pair in the sense of [H-H]. Moreover it satisfies the Iwahori's quadratic equation
\[
(id_{V_P \otimes V_P} - \beta_{q,P}) \circ (id_{V_P \otimes V_P} + q^{-2} \beta_{q,P}) = 0,
\]
as we can easily verify.

1.2 The multiparameter quantum matrix bialgebra \( SE(q, P) \) [S] is the algebra generated by the \( N^2 \) elements \( x_{ij} \) \( (i, j = 1, \ldots, N) \) with relations (for \( i < j \) and \( k < m \)):
\[
x_{ik}x_{im} = q p_{mk} x_{im} x_{ik} \quad x_{ik} x_{jk} = q p_{ij} x_{jk} x_{ik} \quad p_{mk} x_{im} x_{jk} = p_{ij} x_{jk} x_{im}
\]

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\[ p_{km}x_{ik}x_{jm} - p_{ij}x_{jm}x_{ik} = (q - q^{-1})x_{im}x_{jk}. \]

The coalgebra structure is given by the following comultiplication and counity:

\[ \Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}. \]

1.3 There is a natural \( SE(q, P) \)-comodule structure on \( V_P \) given by

\[ u_j \mapsto \sum_{i} u_i \otimes x_{ij}. \]

Consider the ideal \( \mathcal{B}_P^+ \) of \( SE(q, P) \) generated by all \( x_{ij} \) with \( i > j \) and put

\[ SB^+(q, P) = SE(q, P)/\mathcal{B}_P^+. \]

The relations between the generators in \( SB^+(q, P) \) are those given in (1.2) when we put \( x_{ij} = 0 \) for \( i > j \). In particular \( x_{ii} \) commutes with \( x_{jj} \) for all \( i, j \).

1.4 Henceforth the \( p_{ij} \)'s will be integer powers of \( q \). More precisely (cf. [R]) we shall take

\[ p_{ij} = q^{2(u_{ij} - u_{j-1} - u_{j-1} + u_{j-1} - 1)}, \]

where \( U = (u_{ij})_{i,j=1}^{N-1} \) is an appropriate alternating integer matrix. In this way we shall be in the situation of [C-V 1-2], where in fact an integer form of the multiparameter quantum function algebra is constructed. From now on, we shall skip all indices \( q, P \) in our notations as long as no ambiguity is likely.

1.5 We now begin reviewing some results of [H-H], freely adopting the notations in there. Starting from the YB pair \((V, \beta_V)\), we can construct some graded YB bialgebras. First of all the tensor algebra \( TV = \bigoplus_{i \geq 0} V^{\otimes i} = \bigoplus_{i \geq 0} T_i V \) with YB operator \( T(\beta_V) = \bigoplus_{i,j \geq 0} \beta_V(\chi_{ij}) \), where \( \chi_{ij} \) is the following element of \( S_{i+j} \):

\[ \chi_{ij} = \left( \begin{array}{cccccc} 1 & 2 & \cdots & i & i+1 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & \cdots & j \end{array} \right). \]

We recall that, if \( \sigma = \sigma_{i_1} \cdots \sigma_{i_r} \) is a reduced expression for an element \( \sigma \in S_k \), then it is well defined on \( T_k V \) the operator \( \beta_V(\sigma) = \beta_V(\sigma_{i_1}) \cdots \beta_V(\sigma_{i_r}) \), \( \beta_V(\sigma_j) \) being the map \( id_V^{\otimes j-1} \otimes \beta_V \otimes id_V^{\otimes k-j-1} \).

In order to describe the coproduct of \( TV \), for every sequence \( \alpha = (\alpha_1, \ldots, \alpha_s) \) of nonnegative integers with \( \sum_i \alpha_i = k \), define \( \Delta^s_{TV} \) to be the composite map \( TV \rightarrow T_{\alpha} V \rightarrow T_{\alpha} V \) of the \( s \)-th iteration of \( \Delta_T V \) and the projection onto \( T_{\alpha} V = V^{\otimes \alpha_1} \otimes \cdots \otimes V^{\otimes \alpha_s} \). Put

\[ S^\alpha = \{ \sigma \in S_k | \sigma(1) \prec \cdots \prec \sigma(\alpha_1), \sigma(\alpha_1 + 1) \prec \cdots \prec \sigma(\alpha_1 + \alpha_2), \ldots, \sigma(\sum_{i=1}^{s-1} \alpha_i + 1) \prec \cdots \prec \sigma(\sum_{i=1}^{s} \alpha_i) \}. \]
Finally note that $T$ will play a key role in what follows. The algebra that is, all the structure morphisms (including YB operators) are homomorphisms of $T$. We consider the symmetric and the exterior algebras $\Lambda V\Lambda$.

1.6 We consider the symmetric and the exterior algebras $SV$ and $\Lambda V$ of the YB pair $(V, \beta_V)$, which will play a key role in what follows. The algebra $SV$ is generated by $u_1, \ldots, u_N$ with relations

$$u_iu_j = p_{ij}qu_ju_i,$$

while $\Lambda V$ is the algebra on the same generators with relations

$$u_i\wedge u_i = 0, \quad p_{ij}qu_i\wedge u_j + u_j\wedge u_i = 0 \ (i < j).$$

So for every sequence $i = (i_1, \ldots, i_k)$ of elements in $[1, N]$ we have

$$u_{i_1}\wedge \cdots \wedge u_{i_k} = \begin{cases} 0 & \text{if there are repetitions in } i \\ \left(\prod_{r < t, \sigma(r) > \sigma(t)} -q^{-1}p_{i_{\sigma(r)}i_{\sigma(t)}}\right)u_{i_{\sigma(1)}}\wedge \cdots \wedge u_{i_{\sigma(r)}} & \text{if } i_1 < \cdots < i_k \text{ and } \sigma \in \mathcal{S}_k. \end{cases}$$

The $R$-modules $S_r V$ and $\Lambda_r V$ are free with bases, respectively,

$$\{u_{j_1} \cdots u_{j_r} \mid 1 \leq j_1 \leq \cdots \leq j_r \leq N\}, \quad \{u_{i_1}\wedge \cdots \wedge u_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq N\}.$$

1.7 Put $\gamma_V = -q^{-2}\beta_V$. Then the two YB operators $\beta_V, \gamma_V$ satisfy conditions (4.9) and (4.10) in [H-H], that is, $(V, \beta_V, \gamma_V)$ is a YB triple. From this follows (Theorem 4.10 in [H-H]) that $SV$ and $\Lambda V$ are graded YB bialgebras. Moreover there exist YB operators $\varphi_{SV}, \psi_{SV}$ on $SV$, and $\varphi_{\Lambda V}, \psi_{\Lambda V}$ on $\Lambda V$, for which $(SV, \varphi_{SV}, \psi_{SV})$ and $(\Lambda V, \varphi_{\Lambda V}, \psi_{\Lambda V})$ are YB algebra triples. In particular, the operator $\varphi_{\Lambda V}$ is defined by the relation $\varphi_{\Lambda V} \circ (p \otimes p) = (p \otimes p) \circ T(-\beta_V)$ where $p$ denotes the projection from $TV$ onto $\Lambda V$. The multiplicative structure on $\Lambda V$ is given by the fusion procedure, namely, by

$$m_{T,(\Lambda V)} = m_{\Lambda V} \circ \varphi_{\Lambda V}(\omega_i), \quad \omega_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+3 & \cdots & 2i \\ 1 & 3 & \cdots & 2i-1 & 2 & 4 & \cdots & 2i \end{pmatrix}.$$ Finally note that $TV$, $SV$ and $\Lambda V$ are $SE$-equivariant as YB bialgebras with YB algebra triples, that is, all the structure morphisms (including YB operators) are homomorphisms of $SE$-comodules.

1.8 A translation into our setting of Lemma 5.3 in [H-H] gives the following very useful equality.

**Lemma** For any $k \geq 0$ and any sequence $(i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq N$ we have:

$$\Delta_{\Lambda V}^{(1, \ldots, 1)}(u_{i_1}\wedge \cdots \wedge u_{i_k}) = \sum_{\sigma \in \mathcal{S}_k} \left( \prod_{r < t, \sigma(r) > \sigma(t)} -q^{p_{i_{\sigma(r)}i_{\sigma(t)}}}\right) u_{i_{\sigma(1)}} \otimes \cdots \otimes u_{i_{\sigma(k)}}.$$ In particular, for any $k$, $\Delta_{\Lambda V} : \Lambda V \rightarrow T_k V$ is a split injection.

1.9 We are now ready to introduce our multiparameter deformations of Schur modules. In fact all definitions and results in Section 6 of [H-H], stated for the "Jimbo case", still hold in our situation.
For all but Lemma 6.12 can be deduced from formal properties of graded YB bialgebras which are also equipped with a structure of YB algebra triple. The proof of Lemma 6.12, which depends directly on the definition of $\beta_V$, can be easily modified for our purposes.

Given a skew partition $\lambda/\mu$ with $l(\lambda/\mu) = s$ and $\lambda_1 = t$, denote by $d_{\lambda/\mu}(V)$ the Schur map, that is, the composite map

$$
\Lambda_{\lambda/\mu}V = \Lambda_{\lambda_1-\mu_1}V \otimes \cdots \otimes \Lambda_{\lambda_t-\mu_t}V \xrightarrow{\Delta^{(\lambda_1-\mu_1)} \otimes \cdots \otimes \Delta^{(\lambda_t-\mu_t)}} T_{\lambda/\mu}V = T_{\lambda_1-\mu_1}V \otimes \cdots \otimes T_{\lambda_t-\mu_t}V \xrightarrow{(-q^{-2}\beta_V)(\chi_{\lambda/\mu})} T_{\tilde{\lambda}/\tilde{\mu}}V = T_{\tilde{\lambda}_1-\tilde{\mu}_1}V \otimes \cdots \otimes T_{\tilde{\lambda}_s-\tilde{\mu}_s}V,
$$

where, as usual, $\tilde{\lambda}$ denotes the dual partition of $\lambda$, and $\chi_{\lambda/\mu}$ is the permutation defined in Section 6 of [H-H]. We illustrate such a permutation by the following example:

$$
\lambda = (5, 4, 2) \quad \mu = (2, 1) \quad \chi_{\lambda/\mu} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 6 & 8 & 2 & 5 & 7 & 1 & 3
\end{pmatrix}
$$

The image of the Schur map, denoted by $L_{\lambda/\mu}V$, is the Schur module of $V$ with respect to the skew partition $\lambda/\mu$. It is an $SE$-comodule, with coaction induced by the following coaction on $T_kV$:

$$
u_{j_1} \otimes \cdots \otimes \nu_{j_k} \mapsto \sum_{i_1, \ldots, i_k} (\nu_{i_1} \otimes \cdots \otimes \nu_{i_k}) \otimes x_{i_1j_1} \otimes \cdots \otimes x_{i_kj_k}.$$

1.10 The principal properties of $L_{\lambda/\mu}V$ are summarized in the following theorem, which one proves along the lines of Theorem 6.19 and Corollary 6.20 in [H-H].

**Theorem** Let $\lambda/\mu$ a skew partition with $l(\lambda/\mu) = s$. Then:

(i) $L_{\lambda/\mu}V$ is an $R$-free module, and for any $\sigma \in \mathcal{S}_N$, a free basis is the set

$$
L_{\lambda/\mu}Y(\sigma) = \{d_{\lambda/\mu}(V)(\xi_S) | S \in St_{\lambda/\mu}Y(\sigma)\}.
$$

Here $St_{\lambda/\mu}Y$ denotes the set of all standard tableaux in the alphabet $Y(\sigma) = \{\nu_{\sigma(1)} < \cdots < \nu_{\sigma(N)}\}$, and

$$
\xi_S = S(1, \mu_1 + 1) \wedge \cdots \wedge S(1, \lambda_1) \otimes \cdots \otimes S(s, \mu_s + 1) \wedge \cdots \wedge S(s, \lambda_s) \in \Lambda_{\lambda/\mu}V.
$$

(ii) Let $R'$ be a commutative ring and let $f : R \rightarrow R'$ be a homomorphism of commutative rings. Then we have an isomorphism of $SE'$-comodules

$$
L_{\lambda/\mu}(R' \otimes_R V) \simeq R' \otimes_R L_{\lambda/\mu}V, \quad E' = R' \otimes_R E.
$$
As a consequence of (ii), it will not be restrictive for us to take $R = \mathbb{Z}[Q, Q^{-1}]$, where $Q$ stands for an indeterminate.

1.11 We recall that an element of $\text{Tab}_{\lambda/\mu}Y(\sigma)$, the set of all tableaux of shape $\lambda/\mu$ with elements in $Y(\sigma)$, is said to be row-standard if its rows are strictly increasing, and column-standard if its columns are non-decreasing. A tableau is said to be standard if it is both row- and column-standard. Let $\text{Row}_{\lambda/\mu}Y(\sigma)$ denote the set of row-standard tableaux of shape $\lambda/\mu$ and with elements in $Y(\sigma)$. For every $S \in \text{Row}_{\lambda/\mu}Y(\sigma)$, the element $d_{\lambda/\mu}(V)(\xi_S)$ can be expressed as a linear combination of basis elements. The algorithm, called $\mathcal{R}_\sigma$, which does this is based on a descending induction with respect to a pseudo order defined in $\text{Tab}_{\lambda/\mu}Y(\sigma)$. Let $S$ and $S'$ be elements in $\text{Tab}_{\lambda/\mu}Y(\sigma)$. We say that $S \leq_\sigma S'$ if $\forall p, q$

$$\#\{(i, j) \in \Delta_{\lambda/\mu} | i \leq p, S(i, j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\} \geq \#\{(i, j) \in \Delta_{\lambda/\mu} | i \leq p, S'(i, j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\}.$$ 

The key steps of $\mathcal{R}_\sigma$ are the following:

1. Choose two adjacent lines in $S$ where there is a violation of column-standardness; we are in the situation of Proposition (1.12) below, and we can use Corollary (1.13). We get certain $S_i$'s such that $S_i < S$ for every $i$.
2. Reorder in increasing order $S_i(1, \mu_1 + 1) \land \cdots \land S_i(1, \lambda_1), \ldots, S_i(s, \mu_s + 1) \land \cdots \land S_i(s, \lambda_s)$ for each $i$; this operation produces a power of $q$ for every $S_i$ (cf. (1.6)).
3. Apply induction to each $S_i$.

$\mathcal{R}_\sigma$ is also called the "straightening law with respect to the ordering $u_{\sigma(1)} < \cdots < u_{\sigma(N)}".

1.12 Proposition Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be partitions with $\lambda \supset \mu$. Define $\gamma = \lambda - \mu$ and take $a, b$ nonnegative integers with $a + b < \lambda_2 - \mu_1$. Then the image of the composite map

$$\square_{(a, b)} : A_a V \otimes A_{\lambda_1 - a + \gamma_2 - b} V \otimes A_b V \xrightarrow{1 \otimes \Delta_{\lambda/\mu}} A_a V \otimes A_{\lambda_1 - a} V \otimes A_{\lambda_2 - b} V \otimes A_b V \xrightarrow{m \otimes m} A_{\gamma_1} V \otimes A_{\gamma_2} V = A_{\lambda/\mu} V$$

is contained in $\text{Im}(\square_{\lambda/\mu})$, where $\square_{\lambda/\mu}$ is given by

$$\sum_{\nu=0}^{\lambda_2 - \mu_1} A_{\gamma_1 + \gamma_2 - \nu} V \otimes A_\nu V \xrightarrow{\Delta \otimes 1} \sum_{\nu=0}^{\lambda_2 - \mu_1} A_{\gamma_1} V \otimes A_{\gamma_2 - \nu} V \otimes A_\nu V \xrightarrow{1 \otimes m} \sum_{\nu=0}^{\lambda_2 - \mu_1} A_{\gamma_1} V \otimes A_{\gamma_2} V.$$

Proof. Mimic the proof of Lemma 6.15 in [H-H].

1.13 Corollary Let $\lambda/\mu$ be a skew partition with $l(\lambda) = s$, $\sigma$ be an element of $S_N$ and $S$ be an element of $\text{Row}_{\lambda/\mu}Y(\sigma) \setminus \text{St}_{\lambda/\mu}Y(\sigma)$. Then there exist $S_1, \ldots, S_r \in \text{Row}_{\lambda/\mu}Y(\sigma)$ ($r \in \mathbb{N}$) with $S_i \leq_\sigma S$, $\forall i = 1, \ldots, r$ such that

$$\xi_S - \sum_{i} c_i \xi_{S_i} \in \text{Im}(\square_{\lambda/\mu}) = \text{Ker}(d_{\lambda/\mu}(V)).$$
for some $c_i \in \mathbb{Z}[q, q^{-1}]$. Here :

$$\boxtimes_{\lambda/\mu} = \sum_{i=1}^{s-1} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \boxtimes_{\lambda'/\mu'} \otimes 1_{i+2} \otimes \cdots \otimes 1_s, \; \lambda' = (\lambda_i, \lambda_{i+1}), \; \mu' = (\mu_i, \mu_{i+1}), \; 1_j = id_{\lambda_{j-\mu} \otimes v}.$$

**Proof.** Mimic the proof of Lemma 6.18 in [H-H]. \qed

**1.14** We want to stress a consequence of Theorem (1.10) and of all the machinery which allows to prove it. First of all note that the subcategory of $\mathcal{Y}B_R$ (cf. [H-H]) given by the YB pairs as in 1.1 is a preadditive one. Namely, let $P^1 = (p_{ij})_{i,j=1}^n$ and $P^2 = (p_{ij})_{i,j=1}^m$ be two multiplicatively antisymmetric matrices, and put $V_{P^1} = \langle u_1^1, \ldots, u_n^1 \rangle$, $V_{P^2} = \langle u_1^2, \ldots, u_m^2 \rangle$. We define a YB operator on $V_{P^1} \oplus V_{P^2}$ by means of the matrix $P = (p_{ij})_{i,j=1}^N$, $N = n + m$, defined as follows:

$$p_{ij} =
\begin{cases}
p_{ij}^1 & \text{for } i, j \in [1, N] \\
p_{ij}^2 & \text{for } i, j \in [n + 1, N] \\
1 & \text{for } i \in [1, n], j \in [n + 1, N] \text{ or } i \in [n + 1, N], j \in [1, n]
\end{cases}.$$

Then $\beta_P$ is a YB operator on $V_P = \langle u_1^1, \ldots, u_n^1, u_1^2, \ldots, u_m^2 \rangle$. Note that $V_P$ becomes in a natural way an $SE(q, P^1) \otimes SE(q, P^2)$-comodule.

Write for short $V_i = V_{P^i}$, $\beta_i = \beta_{P^i}$, for $i = 1, 2$, and let $\mu \subseteq \gamma \subseteq \lambda$ be partitions. Following [A-B-W], define two $R$-modules

$$M_\gamma(\Lambda_{\lambda/\mu}(V_1 \oplus V_2)) = \text{Im}(\sum_{\mu \subseteq \sigma \subseteq \lambda, \sigma \geq \gamma} \Lambda_{\sigma/\mu} V_1 \otimes \Lambda_{\lambda/\sigma} V_2 \rightarrow \Lambda_{\lambda/\mu}(V_1 \oplus V_2)),$$

$$M_\gamma(\Lambda_{\lambda/\mu}(V_1 \oplus V_2)) = \text{Im}(\sum_{\mu \subseteq \sigma \subseteq \lambda, \sigma \geq \gamma} \Lambda_{\sigma/\mu} V_1 \otimes \Lambda_{\lambda/\sigma} V_2 \rightarrow \Lambda_{\lambda/\mu}(V_1 \oplus V_2)),$$

where the indicated maps are obtained by tensoring the obvious maps

$$\Lambda_{\sigma, -\mu} V_1 \otimes \Lambda_{\lambda, -\sigma} V_2 \rightarrow \Lambda_{\lambda, -\mu}(V_1 \oplus V_2).$$

Let $M_\gamma(\Lambda_{\lambda/\mu}(V_1 \oplus V_2))$ and $M_\gamma(\Lambda_{\lambda/\mu}(V_1 \oplus V_2))$ be the images of the previous modules under the Schur map $d_{\lambda/\mu}(V_1 \oplus V_2)$. The following result holds as in the classical case:

**Theorem** The $R$-modules

$$L_{\lambda/\mu} V_1 \otimes L_{\lambda/\gamma} V_2, \; M_\gamma(L_{\lambda/\mu}(V_1 \oplus V_2))/M_\gamma(L_{\lambda/\mu}(V_1 \oplus V_2))$$

are isomorphic. Hence the $R$-modules $M_\gamma(L_{\lambda/\mu}(V_1 \oplus V_2))$, $\mu \subseteq \gamma \subseteq \lambda$, give a filtration of $L_{\lambda/\mu}(V_1 \oplus V_2)$, whose associated graded module is isomorphic to

$$\sum_{\mu \subseteq \gamma \subseteq \lambda} L_{\gamma/\mu} V_1 \otimes L_{\lambda/\gamma} V_2.$$

**Proof.** Follow verbatim the proof of Theorem II. 4.11 in [A-B-W]. \qed

Note that the isomorphism of the theorem is in fact an isomorphism of $SE(q, P^1) \otimes SE(q, P^2)$-comodules.
2. The Recipe

2.1 In this Section we let \( R = \mathbb{Z}[Q, Q^{-1}] \), \( Q \) an indeterminate, and take a multiplicatively antisymmetric matrix \( P = (p_{ij})_{i,j=1}^{N} \), and the YB pair \( (V_P, \beta_{Q,P}) \), where \( V_P = \langle u_1, \ldots, u_N \rangle \)

\[
\beta_{Q,P}(u_i \otimes u_j) = \begin{cases} 
    u_i \otimes u_i & \text{if } i = j \\
    Q_{p_{ij}} u_j \otimes u_i & \text{if } i < j \\
    Q_{p_{ij}} u_i \otimes u_j + (1 - Q^2) u_i \otimes u_j & \text{if } i > j
\end{cases}
\]

We are going to construct a filtration of \( L_{\lambda/\mu} V_P \) as an \( SE(Q,P) \)-comodule, such that the associated graded object is isomorphic to \( \sum_{\nu} \gamma(\lambda/\mu; \nu) L_{\nu} V_P \). As in the classical Littlewood-Richardson rule, here \( \gamma(\lambda/\mu; \nu) \) stands for the number of standard tableaux of shape \( \lambda/\mu \) filled with \( \tilde{\mu}_1 \) copies of 1, \( \tilde{\mu}_2 \) copies of 2, \( \tilde{\mu}_3 \) copies of 3, etc., such that the associated word (formed by listing all entries from bottom to top in each column, starting from the leftmost column) is a lattice permutation.

The construction is a suitable "deformation" of the one used in the first author’s doctoral thesis, Brandeis University 1984, as illustrated for instance in [B]. We again remark that owing to Theorem (1.10) (ii), the construction holds in fact for every commutative ring \( R \) and every choice of a unit \( q \in R \).

2.2 In order to embed \( L_{\lambda/\mu} V_P \) into a (non-skew) Schur module, let \( M = \mu_1 \) and consider another multiplicatively antisymmetric matrix \( P' = (p'_{ij})_{i,j=1}^{M} \), together with the YB pair \( (V_{P'}, \beta_{Q,P'}) \), where \( V_{P'} = \langle u'_{1}, \ldots, u'_{M} \rangle \) and \( \beta_{Q,P'} \) is defined similarly to (1) above. For convenience of notations, we shall denote \( V_P, u_i, V_{P'}, u'_i \) by \( V, i, V', \) and \( i' \), respectively.

It follows from Theorem (1.14) that the \( SE(Q,P') \otimes SE(Q,P) \)-comodule \( L_{\lambda}(V' \oplus V) \) is isomorphic to \( \sum_{\alpha \leq \lambda} L_{\alpha} V' \otimes L_{\lambda/\alpha} V \), up to a filtration.

Let \( (L_{\lambda}(V' \oplus V))_h \) denote the sub-\( R \)-module of \( L_{\lambda}(V' \oplus V) \) spanned by the tableaux in which \( h V' \)-indices occur. (In this section we identify tableaux and corresponding elements of Schur modules.) Then up to a filtration,

\[
(L_{\lambda}(V' \oplus V))_h \cong \sum_{\alpha \leq \lambda \mid |\alpha| = h} L_{\alpha} V' \otimes L_{\lambda/\alpha} V,
\]

as \( SE(Q,P') \otimes SE(Q,P) \)-comodules.

If \( (L_{\lambda}(V' \oplus V))_{\tilde{\mu}} \) denotes the sub-\( R \)-module of \( L_{\lambda}(V' \oplus V) \) spanned by the tableaux in which every \( i' \) occurs exactly \( \tilde{\mu}_i \) times, also:

\[
(\L_{\lambda}(V' \oplus V))_{\tilde{\mu}} \cong \sum_{\alpha \leq \lambda} (L_{\alpha} V')_{\tilde{\mu}} \otimes L_{\lambda/\alpha} V,
\]

as \( SE(Q,P) \)-comodules, up to a filtration.

Since the bottom piece of the filtration relative to (2) corresponds to the (lexicographically) largest partition \( \alpha \), namely \( \mu \), it follows:

\[
(L_{\lambda} V')_{\tilde{\mu}} \otimes L_{\lambda/\mu} V \xrightarrow{SE(Q,P)} (L_{\lambda}(V' \oplus V))_{\tilde{\mu}}.
\]
And \( rk(L_{\mu} V')_{\bar{\mu}} = 1 \) implies that
\[
L_{\lambda/\mu} V \xrightarrow{SE(Q, P')} (L_{\lambda}(V' \oplus V))_{\bar{\mu}},
\]
as wished.

Explicitly, the embedding sends the tableau \( d_{\lambda/\mu}(V)(a_1 \otimes \cdots \otimes a_s) \) to
\[
d_{\lambda}(V' \oplus V)((b^{(\mu_1)} \wedge a_1) \otimes \cdots \otimes (b^{(\mu_r)} \wedge a_r) \otimes a_{r+1} \otimes \cdots \otimes a_s), \quad r = l(\mu),
\]
where we write \( b^{(k)} \) for \( 1' \wedge 2' \wedge \cdots \wedge k' \in \Lambda_k V' \). Notice that \( b^{(k)} \) is a relative \( SB^+(Q, P') \)-invariant.

2.3 Let \( t = (t_{r1}, \ldots, t_{r1} \cdot t_{r2}, \ldots, t_{rs}; \ldots, t_{rs}) \) be a family of nonnegative integers such that
\[
\sum_{j=1}^{s} t_{ji} = \mu_j \quad \forall j = 1, \ldots, r.
\]

Let \( f \) denote the \( SE(Q, P') \)-equivariant composite map :
\[
\Lambda_{\mu_r} V' \otimes \cdots \otimes \Lambda_{\mu_1} V' \\
\downarrow \otimes \downarrow_{j=r} (\Lambda_{\lambda/\mu} V')
\]
\[
(\Lambda_{t_{r1}} V' \otimes \cdots \otimes \Lambda_{t_{r1}} V') \otimes \cdots \otimes (\Lambda_{t_{rs}} V' \otimes \cdots \otimes \Lambda_{t_{rs}} V')
\]
\[
\downarrow \otimes \otimes (\Lambda_{\lambda/\mu} V')
\]
\[
(\Lambda_{t_{r1}} V' \otimes \Lambda_{t_{r-1,1}} V' \otimes \cdots \otimes \Lambda_{t_{rs}} V') \otimes \cdots \otimes (\Lambda_{t_{t_{rs}} V' \otimes \cdots \otimes \Lambda_{t_{rs} \cdot t_{r-1, s} + \cdots + t_{rs}} V'
\]
\[
\downarrow \otimes \otimes \otimes (m^{(r)}_{\lambda/\mu} V')
\]
\[
\Lambda_{t_{r1} + t_{r-1,1} + \cdots + t_{rs}} V' \otimes \cdots \otimes \Lambda_{t_{rs} + t_{r-1, s} + \cdots + t_{rs}} V'
\]
where \( t_j = (t_{j1}, \ldots, t_{js}) \), \( m^{(r)}_{\lambda/\mu} : \Lambda V' \otimes \cdots \otimes \Lambda V' \to \Lambda V' \) is obtained by iterating the multiplication, and
\[
\omega_{rs} = \begin{pmatrix}
1 & 2 & 3 & \cdots & s & s+1 & s+2 & \cdots & 2s+1 & \cdots & rs \\
1 & r+1 & 2r+1 & \cdots & (s-1)r+1 & 2 & r+2 & \cdots & 3 & \cdots & rs
\end{pmatrix}
\]
(cf. items (1.5) and (1.7)).

As \( b^{(\mu_r)} \otimes \cdots \otimes b^{(\mu_1)} \) is a relative \( SB^+(Q, P') \)-invariant, also \( f(b^{(\mu_r)} \otimes \cdots \otimes b^{(\mu_1)}) \) is so. We denote the latter by \( b(t) \).

2.4 For every \( \nu \subseteq \lambda \) such that \( |\nu| = |\lambda| - |\mu| \), let \( B(\lambda/\nu) \) denote the set of all possible \( b(t) \) which satisfy the further equalities :
\[
\sum_{j=1}^{r} t_{ji} = \lambda_i - \nu_i \quad \forall i = 1, \ldots, s.
\]
For every $b \in B(\lambda/\nu)$, we call $\varphi(\nu, b)$ the restriction to $\Lambda_\nu V \otimes \{b\}$ of the following composite map

$$
\Lambda_\nu V \otimes \Lambda_{\lambda/\nu} V' \xrightarrow{\varphi(\lambda)} \Lambda(\nu) \otimes V' \xrightarrow{d_\lambda(V' \otimes V)} \Lambda_\lambda (V' \otimes V),
$$

where $\varphi(\lambda)$ is obtained by tensoring the morphisms

$$
\Lambda_\nu V \otimes \Lambda_{\lambda/\nu} V' \longrightarrow \Lambda_\lambda (V' \otimes V), \quad x \otimes y \mapsto x \wedge y, \quad i = 1, \ldots, s.
$$

**Proposition** The image of $\varphi(\nu, b)$ lies in $L_{\lambda/\mu} V \hookrightarrow L_\lambda (V' \otimes V)$.

**Proof.** As $\varphi(\nu, b)$ is $SE(Q, P') \otimes SE(Q, P)$-equivariant, and $b$ is a relative $SB^+(Q, P')$-invariant of $V'$-content $\tilde{\mu}$ (i.e., it contains $\tilde{\mu}$ copies of $i'$), each element of $Im(\varphi(\nu, b))$ is a relative $SB^+(Q, P')$-invariant of $V'$-content $\tilde{\mu}$. But then we are through, thanks to Lemma (2.5) below and to the fact that $d_\mu(V')<(\nu, \otimes \otimes b^{(\mu)})$ is the only canonical tableau of content $\tilde{\mu}$. \hfill \Box

**2.5 Lemma** For every partition $\alpha$, take in $L_\alpha V' \otimes_R Q(Q)$ the element

$$
C_\alpha = d_\alpha(V')((1' \wedge \cdots \wedge a_{1}' \wedge \cdots \wedge a_{l}'), \quad l = l(\alpha)
$$

($C_\alpha$ is sometimes called the ”canonical tableau of $L_\alpha V'$”). Then the relative $SB^+(Q, P')$-invariant elements of $L_\alpha V' \otimes_R Q(Q)$ are spanned (over $Q(Q)$) by $C_\alpha$.

**Proof.** Combine $(L_\alpha V')_\tilde{\alpha} = R \cdot C_\alpha$ with a multiparameter version of a suitable analogue of Theorem 6.5.2 in [P-W]. \hfill \Box

**2.6** For each $\nu \subseteq \lambda$ such that $\gamma(\lambda/\mu; \nu) \neq 0$, we wish to describe a subset of $B(\lambda/\nu)$, say $B'(\lambda/\nu)$, such that $\#B'(\lambda/\nu) = \gamma(\lambda/\mu; \nu)$. Let $T \in L_{\lambda/\nu} V'$ be a standard tableau, of content $\tilde{\mu}$, and such that its associated word, $as(T) = (a_1, \ldots, a_{|\mu|})$, is a lattice permutation. Then $\mu$ is the content of the transpose lattice permutation $(as(T))^t$. (Explicitely, $(as(T))^t = (\tilde{a}_1, \ldots, \tilde{a}_{|\mu|})$, where $\tilde{a}_i$ is the number of times $a_i$ occurs in $as(T)$ in the range $(a_1, \ldots, a_i)$.) Let $\bar{T}$ be the tableau obtained from $T$ by replacing every entry $a_i$ of $T$ by $\tilde{a}_i$. For each $i \in \{1, \ldots, s\}$ and each $j \in \{1, \ldots, r\}$, we set :

$$
t_{ji} = \# \text{ of } j \text{'s occuring in the } i \text{-th row of } \bar{T}.
$$

We denote by $b(T)$ the element $b(t) \in B(\lambda/\nu)$, corresponding to this choice of $t_{ji}$’s.

**2.7** Given any row-standard tableau $T$, we can consider the word $w(T)$ formed by writing one after the other all the rows of $T$, starting from the top. As all such words can be ordered lexicographically, we can say that $T <_{le} T'$ if and only if $w(T) <_{le} w(T')$. It is then easy to see that the following holds.

**Proposition** If we write $b(T) \in \Lambda_{\lambda/\nu} V'$ as a linear combination of row-standard tableaux, then

$$
b(T) = \pm Q^* T + \sum_k c_k T_k, \quad c_k \in \mathbb{Z}[Q, Q^{-1}],
$$

where $Q^*$ is the transpose of $Q$. 

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where \( \mathcal{Q}^* \) stands for a power of \( \mathcal{Q} \), and each \( T_k \) is a row-standard tableau \( <_{lex} T \).

Since there are exactly \( \gamma(\lambda/\mu; \nu) \) tableaux \( T \in L_{\lambda/\mu} V' \) which are standard, of content \( \hat{\mu} \), and such that \( as(T) \) is a lattice permutation, the above Proposition implies that the elements \( b(T) \) form a subset of \( B(\lambda/\nu) \) of cardinality \( \gamma(\lambda/\mu; \nu) \). It is precisely this subset which we call \( B'(\lambda/\nu) \).

### 2.8 Consider the family of elements of \( L_{\lambda/\mu} V \mapsto L_{\lambda}(V' \oplus V) : \)

\[ F = \{ \varphi(\nu, b)(x) \mid \gamma(\lambda/\mu; \nu) \neq 0, \ b \in B'(\lambda/\nu), \ and \ \varphi(\nu, b)(x) \text{ is a standard tableau} \} . \]

We claim that \( F \) is an \( R \)-basis of \( L_{\lambda/\mu} V \).

**Proposition** The elements of \( F \) are linearly independent over \( R \).

**Proof.** Suppose that there exist nonzero coefficients \( r_{\nu, b, x} \in R \) such that \( \sum_{x} r_{\nu, b, x} \varphi(\nu, b)(x) = 0 \), i.e., such that \( \sum_{\nu, b, x} r_{\nu, b, x} d_\lambda(V' \oplus V)(\varphi(\nu, b)(x \otimes b)) = 0 \) in \( L_{\lambda}(V' \oplus V) \). This is the same as

\[ \sum_{\nu, b} d_\lambda(V' \oplus V)(\varphi(\nu, b)(y_{\nu, b} \otimes b)) = 0, \]

where \( y_{\nu, b} = \sum_x r_{\nu, b, x} \). Let \( \nu_0 \) be the (lexicographically) smallest \( \nu \) occurring in (3). Order the set \( B'(\lambda/\nu_0) = \{ b(T_1), \ldots, b(T_p) \} \) as follows:

\[ b(T_i) < b(T_j) \text{ if and only if } w(T_i) <_{lex} w(T_j). \]

Let \( b_0 = b(T_0) \) be the highest \( b(T_i) \in B'(\lambda/\nu_0) \) occurring in \( \sum_{\nu, b} d_\lambda(V' \oplus V)(\varphi(\nu, b)(y_{\nu, b} \otimes b)) \). Clearly, \( d_\lambda(V' \oplus V)(\varphi(\nu_0, \lambda)(y_{\nu_0, b_0} \otimes b_0)) \) is not in general a linear combination of standard tableaux of \( L_{\lambda}(V' \oplus V) \), with respect to the order \( 1 < \cdots < N < 1' < \cdots < M' \), since violations of column-standardness may occur in \( b_0 \). Apply therefore to \( d_\lambda(V' \oplus V)(\varphi(\nu_0, \lambda)(y_{\nu_0, b_0} \otimes b_0)) \) the straightening law of \( L_{\lambda}(V' \oplus V) \) with respect to \( 1 < \cdots < N < 1' < \cdots < M' \). One gets (recall Proposition (2.7)):

\[ \pm \mathcal{Q}^* d_\lambda(V' \oplus V)(\varphi(\nu_0, \lambda)(y_{\nu_0, b_0} \otimes T_0)) + (\text{a linear combination of standard tableaux with V-shape } \nu_0 > 1) + (\text{a linear combination of standard tableaux with V-shape } = \nu_0 \text{ and V'-part } <_{lex} T_0 ). \]

Because of our choice of \( \nu_0 \) and \( b_0, \) (3) then implies that \( d_\lambda(V' \oplus V)(\varphi(\nu_0, \lambda)(y_{\nu_0, b_0} \otimes T_0)) = 0 \), i.e.,

\[ \sum_x r_{\nu_0, b_0, x} d_\lambda(V' \oplus V)(\varphi(\nu_0, \lambda)(x \otimes T_0)) = 0. \]

But this is a linear combination of standard tableaux in \( L_{\lambda}(V' \oplus V) \), with respect to the order \( 1 < \cdots < N < 1' < \cdots < M' \), so that \( r_{\nu_0, b_0, x} = 0 \) for each \( x \), which contradicts our assumption on the coefficients \( r_{\nu, b, x} \).

### 2.9 Corollary \( F \) is a basis for \( L_{\lambda/\mu} V \otimes_R \mathcal{Q}(\mathcal{Q}). \)
Proof. By definition of $F$, $\#F = rk(L_{\lambda/\mu} V)$. By Theorem (1.10)(ii), the latter rank is constant on all rings. So proposition (2.8) says that $F$ is a basis for the vector space $L_{\lambda/\mu} V \otimes R \mathbb{Q}(Q)$. □

2.10 Corollary $F$ is a basis for $L_{\lambda/\mu} V$.

Proof. It suffices to show that $F$ is a system of generators for $L_{\lambda/\mu} V$. Let $y \in L_{\lambda}(V' \oplus V)$ be any tableau of type

\[
\begin{array}{ccc}
1' & \cdots & \mu'_1 \\
1' & \cdots & \mu'_2 \\
1' & \cdots & \mu'_3 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\end{array}
\]

where the little circles stand for basis elements of $V$.

Since $y \in L_{\lambda/\mu} V$, Corollary (2.9) says that in the quotient field of $R$, there exist (unique) coefficients $q_{\nu,b,x}$, such that

\[
y = \sum_{\nu} q_{\nu,b,x} \varphi(\nu,b)(x).
\]

To both sides of (4), apply the straightening law with respect to $1 < \cdots < N < 1' < \cdots < M'$. In the left-hand side, only coefficients in $R$ occur. In the right-hand side, if $v_0$ denotes the smallest $V$-shape coupled with a nonzero $\sum_x q_{v_0,b,x}x$, and $b_0 = b(T_0)$ denotes the highest element of $B'(\lambda/v_0)$ (cf. ordering in the proof of Proposition (2.8)) occurring with a non zero $\sum_{x} q_{v_0,b,x}x$, we find that the term $\pm \mathbb{Q}^*d_{\lambda}(V' \oplus V)(\varphi_{v_0}(\lambda)(\sum_{x} q_{v_0,b_0,x}x \otimes T_0))$ must cancel with something in the left-hand side; since each $d_{v_0}(V)(x) \in L_{\nu} V$ is standard, it follows that $q_{v_0,b_0,x} \in R$ for every $x$.

Write next (4) as :

\[
y - \sum_{x} q_{v_0,b_0,x} \varphi(v_0,b_0)(x) = \sum_{(\nu,b) \neq (v_0,b_0)} \varphi(\nu,b)(\sum_{x} q_{\nu,b,x}).
\]

Reasoning for (4') as done for (4), it follows that $q_{v_0,b_1,x} \in R$, where $(v_1,b_1)$ is the pair $(\nu,b)$ coming immediately before $(v_0,b_0)$ in the total ordering :

$(\nu,b) < (\nu',b')$ if and only if either $\nu > \nu'$, or $\nu = \nu'$ and $b < b'$ in the ordering of $B'(\lambda/\nu)$ given in the proof of Proposition (2.8).

Repeating the argument as many times as necessary, the proof is completed. □

2.11 Theorem Up to a filtration, $L_{\lambda/\mu} V \simeq \sum_{\nu} \gamma(\lambda/\mu;\nu)L_{\nu} V$ as $SE(Q,P)$-comodules.

Proof. For every $\nu$ such that $\gamma(\lambda/\mu;\nu) \neq 0$, let $M_{\nu}$ denote the $R$-span (in $L_{\lambda}(V' \oplus V)$) of all elements $\varphi(\tau,b)(x)$ of $F$ such that $\tau \geq \nu$. Also let $\hat{M}_{\nu}$ denote the $R$-span of all $\varphi(\tau,b)(x)$ such that $\tau > \nu$. Clearly, we have the isomorphism of free $R$-modules :

\[
M_{\nu}/\hat{M}_{\nu} \xrightarrow{\psi_{\nu}} L_{\nu} V \oplus \cdots \oplus L_{\nu} V \quad (\gamma(\lambda/\mu;\nu) \text{ summands}).
\]
\{M_\nu\} will be the required filtration, if we show that each $\psi_\nu$ is an $SE(Q, P)$-isomorphism. In order to do so, it suffices to prove that for every fixed $b_0 \in B'(\lambda/\nu)$, and for every basis element $y \in \Lambda_\nu V$, $\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) \in \dot{M}_\nu$, where $\sum r_i d_\nu(V)(x_i)$ is obtained by application to the tableau $d_\nu(V)(y)$ of the straightening law of $L_\nu V$. Notice however that $\varphi(\nu, b_0)(y) \in L_{\lambda/\mu} V \subseteq L_\lambda(V' \oplus V)$ can be written in two ways:

(5) $\varphi(\nu, b_0)(y) = \sum_F r_{\tau, b, x} \varphi(\tau, b)(x)$,

by Corollary (2.10), and

(6) $\varphi(\nu, b_0)(y) = \sum r_i \varphi(\nu, b_0)(x_i) + L.C.$,

where $L.C.$ denotes a linear combination of tableaux, standard with respect to $1 < \cdots < N < 1' < \cdots < M'$, and with $V$-part $> \nu$. This last equality is obtained by eliminating in the $V$-part of $\varphi(\nu, b_0)(y)$ all violations of standardness, with respect to $1 < \cdots < N < 1' < \cdots < M'$. Comparing (5) and (6), it follows that

$$\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) \in \dot{M}_\nu$$

with $r_{\tau, b, x} = 0$ whenever $\tau \leq \nu$. Hence $\varphi(\nu, b_0)(y) - \varphi(\nu, b_0)(\sum r_i x_i) \in \dot{M}_\nu$ as wished. \hfill \Box

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