EXPONENTIAL STABILITY OF SDES DRIVEN BY FBM WITH MARKOVIAN SWITCHING

LITAN YAN$^{1,2}$, WENYI PEI$^1$ AND ZHENZHONG ZHANG$^{2,3,*}$

$^1$College of Information Science and Technology, Donghua University
Shanghai 201620, China
$^2$Department of Statistics, Donghua University
Shanghai 201620, China
$^3$Institute for Nonlinear Science, Donghua University
Shanghai 201620, China

(Communicated by Jaime San Martin)

Abstract. In this paper, we focus on the exponential stability of stochastic differential equations driven by fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$. Based on the generalized Itô formula and representation of the fBm, some sufficient conditions for exponential stability of a class of SDEs with additive fractional noise are given. Besides, we present a criterion on the exponential stability for the fractional Ornstein-Uhlenbeck process with Markov switching. A numerical example is provided to illustrate our results.

1. Introduction. To characterize the continuous dynamical system changes with the discrete state, the following stochastic differential equations (SDEs) have been developed

$$dX_t = f(X_t, r_t)dt + \sigma(X_t, r_t)dB_t,$$

where $\{r_t\}_{t \geq 0}$ is a Markov chain taking values in $S = \{1, 2, \ldots, N\}$ and $\{B_t\}_{t \geq 0}$ is a standard Brownian motion. The process $\{X_t, r_t\}$ is called a switching diffusion or a diffusion with switching. In the past thirty years, stability of stochastic hybrid systems has been considered extensively. For example, Yuan and Mao [30] consider the moment exponential stability of stochastic hybrid delayed systems with Lévy noise in mean square. Mao [18] discusses the exponential stability of general nonlinear stochastic hybrid systems. Some sufficient conditions for asymptotic stability in distribution of SDEs with Markovian switching are given by Yuan and Mao [29]. Most recently, Tan [27] focuses on the exponential stability of fractional stochastic systems with distributed delay driven by fractional Brownian motion. There are lots of work having been dedicated to Markovian switching. See [24, 3, 7, 17, 32, 23] and so forth. It is well known that if $H > 1/2$, $\{B^H_t\}_{t \geq 0}$ exhibits long range dependence and self-similarity. Because of these properties, $\{B^H_t\}_{t \geq 0}$ has been suggested as a useful tool in many fields, especially mathematical finance, network traffic analysis and pricing of weather derivatives. For example, fBm is used to model the dynamics of temperature in [6], and in [25], it is used to model the electricity prices in
the liberated Nordic electricity market. However, some statisticians find that it is better to model the pricing with hybrid system [see, e.g., \cite{12, 28}]. Hence, it is a natural question that under what conditions, SDEs driven by fBm with Markov switching have some exponential stability.

The main purpose of this paper is to consider the $p$th moment exponential stability of stochastic hybrid systems driven by fractional Brownian motion of the form:

$$\begin{aligned}
&dX_t = f(X_t, t, r_t)dt + \sigma(t, r_t)dB_t^H,
&X_0 = x_0,
\end{aligned}$$

(1)

where $\{r_t\}_{t \geq 0}$ is a Markov chain taking values in $\mathbb{S} = \{1, 2, ..., N\}$, $\{B_t^H\}_{t \geq 0}$ is a standard fractional Brownian motion. Moreover $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$.

This equation can be regarded as the result of the following $N$ equations:

$$\begin{aligned}
&dX_t = f(X_t, t, i)dt + \sigma(t, i)dB_t^H, \quad 1 \leq i \leq N, \\
&X_0 = x_0,
\end{aligned}$$

(2)

switching from one to another according to the movement of $\{r_t\}_{t \geq 0}$. Note that for each fixed $i$, $\sigma(t, i)$ is nonrandom.

From \cite{10, 8, 21, 14}, we know that there exists a $\mathbb{R}$-valued global solution satisfying Eq.(2), rather than Eq.(1), under suitable conditions, for each fixed $i \in \mathbb{S}$. Therefore, our first goal is to obtain the existence and uniqueness of the solution for Eq.(1). Then we discuss the $p$th moment exponential stability for Eq.(1).

Throughout this paper, unless otherwise specified, the fBm $\{B_t^H\}_{t \geq 0}$ generates a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ with $\mathcal{F}_t = \sigma\{B_s^H, 0 \leq s \leq t\}$. We let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be the complete probability space, with the filtration described above. Also let $C$ denote a general constant. Let $C^{2,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$ denote the family of all real value functions $f(x, t, i)$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$ which are continuously twice differentiable in $x$ and once differentiable in $t$. The Markov chain $\{r_t\}_{t \geq 0}$ is assumed to be independent of $\{B_t^H\}_{t \geq 0}$. Almost every sample path of the Markov chain $\{r_t\}_{t \geq 0}$ is assumed to be a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$.

The rest of the paper is organized as follows. In Section 2, we shall briefly revisit some basic facts regarding Markovian switching, stochastic integration and the Itô formula with respect to fBm, and some preliminaries for our main results. In Section 3, we shall show the existence and uniqueness of the solution for Eq.(1) firstly. Then we will discuss the sufficient conditions to guarantee the $p$th moment exponential stability. Next, in Section 4, we shall use the theory of Poisson equation and M-matrix to establish some criteria for the exponential stability. Then in Section 5, we will discuss the stability of switching fractional Ornstein-Uhlenbeck process. Finally, a numerical example will be given in Section 6.

2. Preliminaries.

2.1. Markov chain. Let $\{r_t\}_{t \geq 0}$ be a right-continuous Markov chain which takes values in a finite state space $\mathbb{S} = \{1, 2, ..., N\}$. The generator $\Gamma = (\gamma_{ij})_{N \times N}$ is given by

$$
P\{r_{t+\Delta} = j \mid r_t = i\} = \begin{cases} 
\gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\
1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j,
\end{cases}
$$

where $\Delta > 0$.

Here $\gamma_{ij}$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$.
Theorem 2.1. \cite{2} Let $P(t) = (P_{ij}(t))_{N \times N}$ be the transition probability matrix and $\Gamma = (\gamma_{ij})_{N \times N}$ be the generator of a finite Markov chain. Then
\[ P(t) = e^{-t\Gamma}. \]

Let $\Delta_{ij}$ be consecutive, left closed, right open intervals each having the length $\gamma_{ij}$ such that
\[ \Delta_{12} = [0, \gamma_{12}), \Delta_{13} = [\gamma_{12}, \gamma_{12} + \gamma_{13}), \Delta_{1n} = \left[ \sum_{j=1}^{n-1} \gamma_{1j}, \sum_{j=2}^{n} \gamma_{1j} \right), \ldots, \]
\[ \Delta_{21} = \left[ \sum_{j=2}^{n} \gamma_{1j}, \sum_{j=2}^{n} \gamma_{1j} + \gamma_{21} \right), \ldots, \]
\[ \Delta_{2n} = \left[ \sum_{j=2}^{n} \gamma_{1j} + \sum_{j=1,j\neq 2}^{n-1} \gamma_{2j}, \sum_{j=2}^{n} \gamma_{1j} + \sum_{j=1,j\neq 2}^{n} \gamma_{2j} \right), \ldots. \]

Then define $h : S \times \mathbb{R} \to \mathbb{R}$ by
\[ h(i, x) = \begin{cases} j - i, & \text{if } x \in \Delta_{ij}, \\ 0, & \text{otherwise}. \end{cases} \quad (3) \]

According to \cite{11, 26}, a continuous-time Markov chain $\{r_t\}_{t \geq 0}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ can be represented as a stochastic integral with respect to a Poisson random measure. Then
\[ dr_t = \int_{\mathbb{R}} h(r_{t-}, y) \nu(dt \times dy), \]
with initial condition $r_0 = i_0$, where $\nu(dt \times dy)$ is a Poisson random measure with intensity $dt \times m(dy)$. Here $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.

2.2. Fractional Brownian motion and Wick product. Given a finite time interval $[0, T]$ with arbitrary fixed horizon $T > 0$, and let $\{B_t^H\}_{t \geq 0}$ be a one-dimension standard fBm with Hurst parameter $H \in (1/2, 1)$, i.e. a centered Gaussian process with covariance function:
\[ \mathbf{E}(B_s^H B_t^H) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}), \quad s, t \geq 0. \]

Note that if $H = \frac{1}{2}$, then $\{B_t^H\}_{t \geq 0}$ is a standard Brownian motion. Moreover $\{B_t^H\}_{t \geq 0}$ has the following Wiener integral representation:
\[ B_t^H = \int_0^t K^H(t, s)dW_s, \]
where $\{W_t\}_{t \geq 0}$ is a Wiener process and $K^H(t, s)$ is the kernel function defined by
\[ K_H(t, s) = c_H s^{1-H} \int_0^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du, \]
where $c_H = \left( \frac{H(2H-1)}{B(2H,H-\frac{1}{2})} \right)^{\frac{1}{2}}$, in which $B(\cdot, \cdot)$ is the Beta function, and $t > s$. For more details about fBm, we refer the reader to \cite{21, 22, 1}.
Let $I$ be the set of all finite multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some $n \geq 1$ of non-negative integers. Denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\alpha! = \alpha_1! \cdots \alpha_n!$. For $n \geq 0$, define the Hermite polynomials by

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}),$$

and Hermite functions

$$\tilde{h}_n(x) = \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} h_n(x) e^{-\frac{x^2}{2}}.$$

Let $S(\mathbb{R})$ denote the Schwartz space of rapidly decreasing infinitely differentiable real-valued functions, and denote the dual space of $S(\mathbb{R})$ by $S'(\mathbb{R})$. Define

$$H_\alpha(\omega) = \prod_{i=1}^n h_{\alpha_i}(\tilde{h}_i(x), \omega),$$

the product of Hermite polynomials. Consider a square integrable random variable $F = F(\omega) \in L^2(S'(\mathbb{R}), \mathcal{F}, P)$.

Thus, according to [21, 13], every $F(\omega)$ admits a unique representation

$$F(\omega) = \sum_{\alpha \in I} c_\alpha H_\alpha(\omega),$$

and

$$\|F\|_{L^2(\omega)}^2 = \sum_{\alpha \in I} \alpha! h_\alpha^2 < \infty.$$

**Definition 2.2.** (Wick Product) For $F(\omega) = \sum_{\alpha \in I} c_\alpha H_\alpha(\omega)$ and $G(\omega) = \sum_{\beta \in I} d_\beta H_\beta(\omega)$, their Wick product is defined by

$$F \circ G(\omega) = \sum_{\alpha, \beta \in I} a_\alpha b_\beta H_{\alpha + \beta}(\omega) = \sum_{\gamma \in I} \left( \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) H_\gamma(\omega).$$

2.3. Malliavin derivative. Let $p \geq 1$, $L^p := L^p(\Omega, \mathcal{F}, P)$ be the space of all random variables $\Omega \to \mathbb{R}$, such that

$$\|F\|_p = \mathbb{E}(|F|^p)^{1/p} < \infty,$$

and let

$$L^2_0(\mathbb{R}^+) = \{ f|f : \mathbb{R}^+ \to \mathbb{R}, \|f\|^2_0 := \int_0^\infty \int_0^\infty f(s)f(t)\phi(s,t)dsdt < \infty \},$$

where $\phi(s,t) = H(2H - 1)|s - t|^{2H-2}$.

**Definition 2.3.** (Malliavin Derivative) Let $g \in L^2_0(\mathbb{R})$. The $\phi$-derivative of a stochastic variable $F \in L^p$ in the direction of $\Phi_\phi$ is defined by

$$D_{\Phi_\phi} F(\omega) = \lim_{\delta \to 0} \frac{1}{\delta} \left\{ F(\omega + \delta) - F(\omega) \right\},$$

if the limit exists in $L^p$. Moreover if there exists a process $(D^\phi_s F_s, s \geq 0)$ such that

$$D_{\Phi_\phi} F = \int_0^\infty D^\phi_s F_s^s ds \ a.s.,$$

for all $g \in L^2_0(\mathbb{R})$, then $F$ is said to be $\phi$-differentiable.
Theorem 2.5. (The Itô Formula) Let \( \mathcal{A}[0,T] \) be the family of stochastic process on \([0,T]\) such that \( F \in \mathcal{A}(0,T) \) if \( E|F|^2 < \infty \) and \( F \) is \( \phi \)-differentiable, the trace of \((D^2_{\phi}F_t, 0 \leq s \leq T, 0 \leq t \leq T)\) exists and \( E \int_0^T (D^2_{\phi}F_t)^2 ds < \infty \), and for each sequence of partitions \( \pi_n, n \in \mathbb{N}_+ \) such that \( \pi_n \to 0 \), as \( n \to \infty \). Moreover

\[
\sum_{i=0}^{n-1} E \left\{ \int_{t_i}^{t_{i+1}} (D^2_{\phi}F^\pi_t - D^2_{\phi}F_s) ds \right\}^2 \to 0,
\]

and

\[
E|F^\pi_t - F^\pi_s|^2 \to 0,
\]

as \( n \to \infty \). Here \( \pi_n : 0 = t_0^{(n)} < t_1^{(n)} < ... < t_n^{(n)} = T \).

Next, we define a stochastic integral with respect to fBm considered in [5].

**Definition 2.4.** Let \( \{F_t\}_{t \geq 0} \) be a stochastic process such that \( F \in \mathcal{A}(0,T) \), and define \( \int_0^T F_s dB_s^H \) by

\[
\int_0^T F_s dB_s^H = \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} F^\pi_{t_i} \circ (B_{t_{i+1}} - B_{t_i}),
\]

where \( |\pi| = \max\{t_{i+1} - t_i, i = 0, 1, \ldots, n - 1\} \).

**Remark 1.** (I) According to Theorem 3.6.1 in [5], if \( F \in \mathcal{A}(0,T) \), then the stochastic integral satisfies \( E \int_0^T F_s dB_s^H = 0 \), and

\[
E \left[ \int_0^T F_s dB_s^H \right]^2 = E \left[ \left( \int_0^T D^\phi_s F_s ds \right)^2 + |1_{[0,T]} F|_{\phi}^2 \right].
\]

What’s more, by Definition 3.4.1 in [5], the stochastic integral can be extended as follows

\[
\int_0^T F_s dB_s^H := \int_0^T F_t \circ W^H(t) dt,
\]

where \( F : \mathbb{R} \to (S)_{H}^T \) is a given function such that \( F_t \circ W^H(t) \) is \( dt \)-integrable in \((S)_{H}^T\). Here \((S)_{H}^T\) is the fractional Hida distribution space defined by Definition 3.1.11 in [5]. And in this extension, the integral on an interval \([0,T]\) can be defined by

\[
\int_0^T F_t dB_t^H := \int_0^T F_t I_{[0,T]}(t) dB_t^H.
\]

(II) For \( H = \frac{1}{2} \), the definition of stochastic integral \( \int_0^T F_s dB_s^H \) and \( \int_0^\infty F_s dB_s^H \) can find from textbooks (cf., Chapter 3 of Karatzas and Shreve [16]).

2.4. The Itô formula. At first, we shall review the results in [9] on the Itô formula for fBm. Then we will extend them to SDEs driven by fBm with Markovian switching.

**Theorem 2.5.** (The Itô Formula) Let \( \{F_u, 0 \leq u \leq T\} \) be a stochastic process in \( \mathcal{A}(0,T) \). Assume that there exists an \( \gamma > 1 - H \) and \( C > 0 \) such that

\[
E|F_u - F_v|^2 \leq C|u - v|^{2\gamma},
\]

where \( |u - v| \leq \delta \) for some \( \delta > 0 \) and

\[
\lim_{0 \leq u,v \leq t, |u - v| \to 0} E[D^\phi_{u}(F_u - F_v)]^2 = 0.
\]
Set $\sup_{0\leq s\leq T}|G_s|<\infty$ and $g=g(x,t)\in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$ with bounded derivatives. Moreover, for $\eta_t = \int_0^t F_u dB_u^H$, it is assumed that $\mathbb{E}\int_0^T |F_t D^\phi \eta_t| ds < \infty$ and $(\frac{\partial g}{\partial x}(s,\eta_t)F_s, s\in[0,T])$ is in $\mathcal{A}(0,T)$. Denote $x_t = \xi + \int_0^t G_u du + \int_0^t F_u dB_u^H$, $\xi \in \mathbb{R}$ for $t \in [0,T]$. Let $(\frac{\partial g}{\partial x}(s,x_t)F_s, s\in[0,T]) \in \mathcal{A}(0,T)$, $\mathbb{E}[\sup_{0\leq s\leq t}|G_s|] < \infty$. Then for $t \in [0,T],$

\[
g(x_t, t) = g(\xi, 0) + \int_0^t \frac{\partial g}{\partial s}(x_s, s)ds + \int_0^t \frac{\partial g}{\partial x}(x_s, s)G_s ds + \int_0^t \frac{\partial^2 g}{\partial x^2}(x_s, s)F_s D^\phi x_s ds.
\]

Here $D^\phi x_s$ is the Malliavin derivative defined in Definition 2.2.

In particular, for the process $X^{(i)}_t = X^{(i)}_0 + \int_0^t f(X^{(i)}_s, s, i)ds + \int_0^t \sigma(s, i)dB^H_s$, where $\sigma(s, i) \in L^2_\phi$ is a deterministic function, then for each fixed $i \in \mathbb{S}$, we have

\[
F(X^{(i)}_t, t, i) = F(X^{(i)}_0, 0, i) + \int_0^t \frac{\partial F}{\partial s}(X^{(i)}_s, s, i)ds + \int_0^t \frac{\partial F}{\partial x}(X^{(i)}_s, s, i)\sigma(s, i)dB^H_s + \int_0^t \frac{\partial^2 F}{\partial x^2}(X^{(i)}_s, s, i)\sigma(s, i)\left[\int_0^s \sigma(u, i)\phi(s, u)du\right] ds,
\]

where $\phi(s, u) = H(2H-1)|s - u|^{2H-2}$. Formally,

\[
dF(X^{(i)}_t, t, i) = F_t(X^{(i)}_t, t, i)dt + F_{xx}(X^{(i)}_t, t, i)\sigma(t, i)\left[\int_0^t \sigma(u, i)\phi(s, u)du\right] dt + F_x(X^{(i)}_t, t, i)f(X^{(i)}_t, t, i)ds + F_x(X^{(i)}_t, t, i)\sigma(t, i)dB^H_t.
\]

Let

\[
\mathcal{L}^{(i)}F(x, t, i) = F_t(x, t, i) + F_x(x, t, i)f(x, t, i) + F_{xx}(x, t, i)\sigma(t, i)\left[\int_0^s \sigma(u, i)\phi(s, u)du\right].
\]

Substituting (5) into (4), we get

\[
F(X^{(i)}_t, t, i) = F(X^{(i)}_0, 0, i) + \int_0^t \mathcal{L}^{(i)}F(X^{(i)}_s, s, i)ds + \int_0^t F_x(X^{(i)}_s, s, i)\sigma(s, i)dB^H_s.
\]

In the sequel of our paper, unless otherwise specified, we let the coefficients of Eq. (1) satisfy the conditions in Theorem 2.5, for each fixed $i \in \mathbb{S}$. Let $V(X_t, t, r_t) \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{S}\times\mathbb{R}_+)$. Then we will discuss an Itô's formula which reveals how $V$ maps $(X_t, t, r_t)$ into a new process $V(X_t, t, r_t)$. Here $\{X_t\}_{t\geq0}$ is a stochastic process with the stochastic differential (1).
Theorem 2.6. If \( V(X_t,t,r_t) \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+ \times S; \mathbb{R}_+) \), then
\[
V(X_t,t,r_t) = V(X_0,0,r_0) + \int_0^t \mathcal{L}V(X_s,s,r_s)ds
+ \int_0^t V_x(X_s,s,r_s)\sigma(s,r_s)dB^H_s
+ \int_0^t \int_\mathbb{R} (V(X_s,s,r_0 + h(r,s,l)) - V(X_s,s,r_s))\mu(ds,dl),
\]
where \( h \) is defined by (3), and \( \mu(ds,dl) = \nu(ds,dl) - \gamma(ds,dl)ds \) is a martingale measure, and \( \mathcal{L}V(x,t,i) \) is defined by
\[
\mathcal{L}V(x,t,i) = \mathcal{L}^{(i)}V(x,t,i) + \sum_{j=1}^N \gamma_{ij}V(x,t,j).
\]
Besides, for any time \( 0 < s < t < \infty \),
\[
\mathbf{E}V(X_t,t,r_t) = \mathbf{E}V(X_s,s,r_s) + \mathbf{E} \int_s^t \mathcal{L}V(X_u,u,r_u)du
+ \mathbf{E} \int_s^t V_x(X_u,u,r_u)\sigma(u,r_u)dB^H_u.
\]
Proof. For \( 0 < s < t < \infty \), let \( s < \tau_1 < \tau_2 < \ldots < \tau_n < t \) be all the times when the Markov chain has a jump. Applying Itô’s formula to \( V(X_t,t,r_t) \) on the intervals \( [s,\tau_1],[\tau_1,\tau_2],\ldots,[\tau_n,t] \), we get
\[
V(X_{\tau_{k-1}},\tau_{k-1},r_0) - V(X_s,s,r_0)
= \int_s^{\tau_{k-1}} \mathcal{L}^{(r_u)}V(X_u,u,r_0)du + \int_s^{\tau_{k-1}} V_x(X_u,u,r_0)\sigma(u,r_0)dB^H_u,
\]
\[
V(X_{\tau_{k+1}-1},\tau_{k+1}-1,r_{\tau_k}) - V(X_{\tau_k},\tau_k,r_{\tau_k})
= \int_{\tau_k}^{\tau_{k+1}-1} \mathcal{L}^{(r_u)}V(X_u,u,r_{\tau_k})du + \int_{\tau_k}^{\tau_{k+1}-1} V_x(X_u,u,r_{\tau_k})\sigma(u,r_{\tau_k})dB^H_u,
\]
\[
V(X_{t},t,r_{\tau_n}) - V(X_{\tau_n},\tau_n,r_{\tau_n})
= \int_{\tau_n}^t \mathcal{L}^{(r_u)}V(X_u,u,r_{\tau_n})du + \int_{\tau_n}^t V_x(X_u,u,r_{\tau_n})\sigma(u,r_{\tau_n})dB^H_u.
\]
Adding (8) to (10) over \( k \) from 1 to \( n-1 \), we get
\[
V(X_t,t,r_t) - V(X_s,s,r_s)
= \int_s^t \mathcal{L}^{(r_u)}V(X_u,u,r_u)du + \int_s^t V_x(X_u,u,r_u)\sigma(u,r_u)dB^H_u
+ \sum_{k=1}^n [V(X_{\tau_k},\tau_k,r_{\tau_k}) - V(X_{\tau_k},\tau_k,r_{\tau_k})]
= \int_s^t \mathcal{L}^{(r_u)}V(X_u,u,r_u)du + \int_s^t V_x(X_u,u,r_u)\sigma(u,r_u)dB^H_u
+ \int_s^t \int_\mathbb{R} (V(X_u,u,r_u + h(r_u,l)) - V(X_u,u,r_u))\gamma(ds,dl)du
+ \int_s^t \int_\mathbb{R} (V(X_u,u,r_0 + h(r_u,l)) - V(X_u,u,r_0))\mu(ds \times dl).
According to [26, 19], one has
\[
\int_{\mathbb{R}} \left[ V(X_u, u, i + h(i, l)) - V(X_u, u, i) \right] m(dl) = \sum_{j=1}^{N} \gamma_{ij} V(X_u, u, j).
\]
Thus
\[
V(X_t, t, r_t) - V(X_s, s, r_s) = \int_s^t \left( \mathcal{L}^{(r_u)} V(X_u, u, r_u) + \sum_{j=1}^{N} \gamma_{ij} V(X_u, u, j) \right) du
+ \int_s^t V_x(X_u, u, r_u) \sigma(u, r_u) dB^H_u
+ \int_s^t \int_{\mathbb{R}} \left[ V(X_u, u, r_u + h(r_u, l)) - V(X_u, u, r_u) \right] \mu(du \times dl).
\]
Consequently,
\[
V(X_t, t, r_t) = V(X_0, 0, r_0) + \int_0^t \left( \mathcal{L}^{(r_s)} V(X_s, s, r_s) \right) ds
+ \int_0^t V_x(X_s, s, r_s) \sigma(s, r_s) dB^H_s
+ \int_0^t \int_{\mathbb{R}} \left[ V(X_s, s, r_0 + h(r_s, l)) - V(X_s, s, r_s) \right] \mu(ds \times dl).
\]
Taking the expectation and by [18, 19], the desired result (7) follows. The proof is complete.

3. Some properties of solutions of Eq.(1). In this section, we will consider the existence and uniqueness of the solution of Eq.(1). Besides, the $p$th moment exponential stability conditions will be presented.

3.1. Existence and uniqueness. To ensure the existence and uniqueness of the solution, we shall impose the following basic assumptions.

**Assumption 1.** Let $f = f(x, t, i) : \mathbb{R} \times \mathbb{R}_+ \times S \to \mathbb{R}$ satisfy the following hypotheses.

(i) $f$ is measurable, and there exists $K > 0$ such that
\[
|f(x, t, i) - f(y, t, i)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R}, \quad \forall t \in \mathbb{R}_+, \quad \forall i \in S.
\]

(ii) There exists $\bar{K} > 0$ such that
\[
|f(x, t, i)| \leq \bar{K} (1 + |x|), \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+, \quad \forall i \in S.
\]

**Assumption 2.** Let $\sigma = \sigma(t, i) : \mathbb{R}_+ \times S \to \mathbb{R}$ satisfy the following hypotheses.

(iii) There exists $M > 0$, $\gamma > 1 - H$ such that $\sigma$ is bounded and
\[
|\sigma(t, i) - \sigma(s, i)| \leq M |t - s|^\gamma, \quad \forall t, s \in \mathbb{R}_+, \quad \forall i \in S.
\]

**Theorem 3.1.** Let Assumptions 1-2 hold. Then Eq.(1) has a unique solution.

**Proof.** Recall that the Markov chain $(r_t)_{t \geq 0}$ can be rewritten as
\[
r_t = r_0 + \sum_{n=1}^{\infty} \eta_n 1_{(r_n \leq t)},
\]
where \( r_{\tau_k} = i \), and \( \tau_{k+1} - \tau_k \) is exponentially distributed. The jump \( \eta_{k+1} = r_{\tau_{k+1}} - r_{\tau_k} \) is independent of the past. According to [21, 14], there exists a unique global solution to Eq.(2), for each \( i \in S \),

\[
X_t^{(i)} = X_0 + \int_0^t f(X_s^{(i)}, s, i)ds + \int_0^t \sigma(s, i)dB_s^H, \quad t > 0.
\]

Note the unique solution \( \{X_t^{(i)}\}_{t \geq 0} \) is a stochastic process without Markov switching. We denote the unique solution by \( \{X_t \}_{t \geq 0} \). For each \( k \in \mathbb{N} \), \( t \in [\tau_k, \tau_{k+1}) \), we have \( r_t = j \in S \). Thus, we obtain a sequence of solutions \( \{X_t^{r_{\tau_k} \rightarrow r_{k}}\}_{t \geq 0, k \in \mathbb{N}} \). According to Lemma 3.1 of [31], we construct the solution to Eq.(1) as follows.

For \( t \in [0, \tau_1) \), we define

\[
X_t = X_t^{X_0 \rightarrow r_0},
\]

where \( x_0 \) is the initial value.

Then for \( t \in [\tau_1, \tau_2) \), we define

\[
X_t = X_t^{X_{\tau_1} \rightarrow r_{\tau_2}}.
\]

Iteration for \( n \) times, for \( t \in [\tau_{n-1}, \tau_n) \), we then define

\[
X_t = X_t^{X_{\tau_{n-1}} \rightarrow r_{\tau_n}}.
\]

Thus, for any \( t \in [0, \infty) \), we have

\[
X_t = x_0 + \sum_{n=1}^{\infty} X_t^{X_{\tau_{n-1}} \rightarrow r_{\tau_n}} \mathbf{1}_{[\tau_{n-1}, \tau_n)}(t)
= X_0 + \int_0^t f(X_s, s, r_s)ds + \int_0^t \sigma(s, r_s)dB_s^H. \tag{11}
\]

Therefore \( \{X_t\}_{t \geq 0} \) is the solution of Eq.(1). Suppose \( \{X_t\}_{t \geq 0} \) and \( \{\hat{X}_t\}_{t \geq 0} \) are two global solutions of Eq.(1), which have the same initial conditions. By (11), it's easy to show that for any \( T \geq 0 \), \( t \in [0, T] \),

\[
X_t - \hat{X}_t = \int_0^t \left[ f(X_s, s, r_s) - f(\hat{X}_s, s, r_s) \right] ds.
\]

Using condition (i) one can show that

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_s - \hat{X}_s|^2 \right) \leq KT \int_0^t \mathbb{E} \left( \sup_{0 \leq u \leq s} |X_u - \hat{X}_u|^2 \right) ds, \quad t \in [0, T]. \tag{12}
\]

Then, according to the Gronwall inequality, (12) implies that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t - \hat{X}_t|^2 \right) = 0.
\]

Letting \( T \rightarrow \infty \), together with the continuity of sample path, one has \( P(X_t = \hat{X}_t \text{ for all } t \geq 0) = 1 \).

The proof is complete. \( \square \)
3.2. Exponential stability. In the sequel of this section, we will state one of the main criteria of this paper.

**Theorem 3.2.** Let Assumption 1-2 hold. If there exists a function $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+)$ and positive constants $a_1$, $a_2$, $b$ and $p \geq 1$, such that

$$a_1|x|^p \leq |V(x, t, i)| \leq a_2|x|^p,$$

(13)

and

$$L V(x, t, i) \leq -b|x|^p,$$

(14)

for all $x \in \mathbb{R}$, $t \geq 0$, $i \in S$. Then the solution of Eq.(1) is $p$th moment exponential stable, i.e.

$$\lim_{t \to \infty} \sup t \log(\mathbb{E}|X_t|^p) < 0.$$

**Proof.** By Theorem 3.1, we know that Eq.(1) has a unique solution, denoted by $\{X_t\}_{t \geq 0}$. Set

$$U(X_t, t, i) = e^{\lambda t}V(X_t, t, i),$$

where $\lambda \in (\eta, \frac{b}{a_2})$, $\eta > 0$. Then we get

$$LU = e^{\lambda t}(\lambda V + LV).$$

Applying the conditions (13) and (14), together with the generalized Itô's formula (7) and Theorem 6.1.10 of [5], we find that for any $t \in [0, T]$,

$$a_1 e^{\eta t} \mathbb{E}|X_t|^p \leq \mathbb{E}U(X_t, t, i)$$

$$= \mathbb{E}V(X_0, 0) + \mathbb{E} \int_0^t LU ds + \mathbb{E} \int_0^t e^{\lambda s}V_x \sigma dB^H_s$$

$$= \mathbb{E}V(X_0, 0) + \mathbb{E} \int_0^t e^{\lambda s}(\lambda V + LV) ds$$

$$\leq \mathbb{E}V(X_0, 0) + \mathbb{E} \int_0^t e^{\lambda s}(\lambda a_2 - b)|X_s|^p ds.$$

Then we obtain that

$$a_1 e^{\eta t} \mathbb{E}|X_t|^p \leq \mathbb{E}V(X_0, 0) + \mathbb{E} \int_0^t e^{\lambda s}(\lambda a_2 - b)|X_s|^p ds.$$ (15)

Dividing both sides of (15) by $a_1 e^{\eta t}$, noting that $\lambda a_2 - b < 0$, we obtain

$$\mathbb{E}|X_t|^p \leq \frac{e^{-\eta t}}{a_1} \mathbb{E}V(X_0, 0) + \frac{e^{-\eta t}}{a_1} \mathbb{E} \int_0^t e^{\lambda s}(\lambda a_2 - b)|X_s|^p ds$$

$$\leq \frac{e^{-\eta t}}{a_1} \mathbb{E}V(X_0, 0).$$

Consequently,

$$\sup_{t \in [0, T]} a_1 e^{\eta t} \mathbb{E}|X_t|^p \leq \mathbb{E}V(X_0, 0).$$

Letting $T \to \infty$ gives

$$\sup_{t \geq 0} \mathbb{E}|X_t|^p \leq \frac{e^{-\eta t}}{a_1} \mathbb{E}V(X_0, 0),$$

and the required assertion follows. The proof is complete \(\square\)
4. **Main results.** In this section we shall use the theory of Poisson equation and M-matrix to establish some criteria for the exponential stability. These criteria can be more useful and can be verified much more easily than the general one in the previous section. For the convenience of the reader, we will introduce some useful notation and basic properties on M-matrix firstly.

Let $B$ be a vector or matrix. By $B \geq 0$ we mean that all elements of $B$ are nonnegative. By $B \gg 0$, we mean that all elements of $B$ are positive. By $B > 0$ we mean $B \geq 0$ and at least one element of $B$ is positive. Moreover, we also write $B_1 \geq B_2, B_1 \gg B_2, B_1 > B_2$ if and only if $B_1 - B_2 \geq 0, B_1 - B_2 \gg 0, B_1 - B_2 > 0$, respectively.

**Definition 4.1.** (M-matrix). A square matrix $A = (a_{ij})_{N \times N}$ is called an M-matrix if $A$ can be expressed in the form $A = sI - B$ with some $B \geq 0$ and $s \geq \rho(B)$, where $I$ is the $N \times N$ identity matrix and $\rho(B)$ is the spectral radius of $B$. And further more, $A$ is called a nonsingular M-matrix if $s > \rho(B)$.

It is easy to see that if $A$ is a non-singular M-matrix then it has nonpositive off-diagonal and positive diagonal entries, that is

$$a_{ii} > 0, \quad \text{and} \quad a_{ij} \leq 0, \quad i \neq j.$$

There are many conditions equivalent to the statement that $A$ is a nonsingular M-matrix. Now we cite some of them for the use of this paper, and refer to [4] for more details.

**Lemma 4.2.** The following statements are equivalent.

1. $A$ is a nonsingular $N \times N$ M-matrix.
2. All of the leading principal minors of $A$ are positive; that is
   $$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{1k} & \cdots & a_{kk} \end{vmatrix} > 0 \text{ for every } k = 1, 2, \ldots, N.$$
3. $A$ is semipositive; that is, there exists $x \gg 0$ in $\mathbb{R}^N$ such that $Ax \gg 0$.
4. Every real eigenvalue of $A$ is positive.
5. $A$ is inverse-positive; that is $A^{-1}$ exists and $A^{-1} \geq 0$.

To discuss the stability, we impose the following assumption.

**Assumption 3.** Let $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^+)$ satisfy the following condition:

(iv) There exists positive constants $a_1, a_2, p \geq 1$ and $\beta_i \in \mathbb{R}$ such that

$$a_1|x|^p \leq |V(x, t, i)| \leq a_2|x|^p,$$

$$\mathcal{L}^{(i)}V(x, t, i) \leq \beta_i V(x, t, i),$$

for all $x \in \mathbb{R}, \ t \geq 0, \ i \in S$.

Here the constants $\beta_i$ could be positive or negative, which is different from Theorem 3.2. For the vector $\beta = (\beta_1, \ldots, \beta_N)^T$, we use $\text{diag}(\beta) = \text{diag}(\beta_1, \ldots, \beta_N)$ to denote the diagonal matrix generated by $\beta$ as usual. Next, we will give a very simple criterion.
Theorem 4.3. Assume that Assumptions 1-2 hold and there exists a function $V \in C^2(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+)$ such that Assumption 3 holds and

$$\sum_{i \in S} \mu_i \beta_i < 0,$$

where $(\mu_i)_{i \in S}$ is the invariant probability measure of $\{r_t\}_{t \geq 0}$. Then the solution of Eq. (1) is $p$th moment exponential stable.

Proof. Set $\Gamma = (\gamma_{ij})_{N \times N}$ be the generator of $\{r_t\}_{t \geq 0}$. Define $\bar{\beta}_i = \frac{1}{\theta} \beta_i$, where $\theta \in (0, 1)$. Let $\delta = -\sum_{i \in S} \mu_i \bar{\beta}_i = -\mu \bar{\beta}$. Let $\mathbb{1}$ denote the vector which all elements are 1. Thus,

$$\mu(\bar{\beta} + \delta \mathbb{1}) = \mu \bar{\beta} + \delta = -\delta + \delta = 0. \quad (16)$$

According to [28], (16) implies the Poisson equation:

$$\Gamma_c = \bar{\beta} + \delta \mathbb{1}. \quad (17)$$

Note that (17) has the solution $c = (c_1, \ldots, c_N)^T$. Then one has for $i \in S$,

$$-\delta = \bar{\beta}_i - \sum_{j=1}^N \gamma_{ij} c_j. \quad (18)$$

For each fixed $i \in S$, set $U(x, t, i) = (1 - \theta c_i)V(x, t, i)$ where $\theta \in (0, 1)$ is sufficiently small such that $1 - \theta c_i > 0$, for each $i \in S$.

Then we get

$$\mathcal{L}U(x, t, i) = (1 - \theta c_i)\mathcal{L}^iV(x, t, i) + \sum_{i \neq j} \gamma_{ij}(U(x, t, j) - U(x, t, i))$$

$$= (1 - \theta c_i)\mathcal{L}^iV(x, t, i) + \theta V(x, t, i) \sum_{i \neq j} \gamma_{ij}(c_j - c_i)$$

$$\leq (1 - \theta c_i)\theta V(x, t, i) \left[ \frac{1}{\theta} \beta_i - \sum_{i \neq j} \gamma_{ij} \frac{c_j - c_i}{1 - \theta c_i} \right] \quad (19)$$

$$= (1 - \theta c_i)\theta V(x, t, i) \left[ \bar{\beta}_i - \sum_{i \neq j} \gamma_{ij} \frac{c_j - c_i}{1 - \theta c_i} \right].$$

By [28], one has

$$\sum_{i \neq j} \gamma_{ij} \frac{c_j - c_i}{1 - \theta c_i} = \sum_{i \neq j} \gamma_{ij} c_j + \sum_{i \neq j} \gamma_{ij} \frac{\theta c_i c_j - c_i}{1 - \theta c_i}$$

$$= \sum_{j=1}^N \gamma_{ij} c_j - c_i + \sum_{i \neq j} \gamma_{ij} c_i (c_j - c_i) + \frac{\theta c_i^2 c_j - c_i}{1 - \theta c_i}$$

$$= \sum_{j=1}^N \gamma_{ij} c_j + \sum_{i \neq j} \gamma_{ij} \frac{c_i (c_j - c_i)}{1 - \theta c_i} + \theta c_i \sum_{j=1}^N \gamma_{ij} c_j + o(\theta). \quad (20)$$

Making use of (19) and (20), we obtain that

$$\mathcal{L}U(x, t, i) \leq (1 - \theta c_i)\theta V(x, t, i) \left[ \bar{\beta}_i - \sum_{j=1}^N \gamma_{ij} c_j + o(\theta) \right]. \quad (21)$$
Substituting (18) into (21), we get
\[ \mathcal{L}U(x, t, i) \leq (1 - \theta c_i)\mathcal{V}(x, t, i)[o(\theta) - \delta] = \kappa U(x, t, i), \]
where \( \kappa < 0. \)
Making use of Theorem 3.2, we can show that the solution of Eq.(1) is \( p \)th moment exponential stable. The proof is complete.

In the sequel of this section, we shall use the theory of M-matrix to establish a criterion.

**Proposition 1.** Assume that Assumptions 1-2 hold and there exists a function \( V \in C^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+) \) such that Assumption 3 is satisfied and the matrix \( -\Gamma + \text{diag}(\beta) \) is a nonsingular M-matrix. Then the solution of Eq.(1) is \( p \)th moment exponential stable.

**Proof.** By (6), we have
\[ \mathcal{L}f(x, t, i) = \mathcal{L}^{(i)} f(x, t, i) + \Gamma f(x, t, \cdot)(i), \]
where \( \Gamma f(x, t, \cdot)(i) = \sum_{j \neq i} \beta_{ij}(f(x, t, j) - f(x, t, i)) \). As the matrix \( -\Gamma + \text{diag}(\beta) \) is a nonsingular M-matrix, by Lemma 4.2, there exists a vector \( \xi = (\xi_1, \ldots, \xi_N)^T \gg 0 \) such that \( \lambda = (\lambda_1, \ldots, \lambda_N)^T = -\Gamma + \text{diag}(\beta) \xi \gg 0 \).
Take \( G(x, t, i) = V(x, t, i) \xi_i \), thus
\[ \mathcal{L}G(x, t, i) = \mathcal{L}^{(i)} V(x, t, i) + \Gamma \xi_i V(x, t, i) \]
\[ \leq (\beta_i \xi_i + \Gamma \xi_i) V(x, t, i) = -\lambda_i V(x, t, i) \]
\[ = -\left( \frac{\lambda_i}{\xi_i} \right) G(x, t, i) \leq - \min_{1 \leq i \leq N} \left( \frac{\lambda_i}{\xi_i} \right) G(x, t, i). \]
Note that \( \min_{1 \leq i \leq N} \left( \frac{\lambda_i}{\xi_i} \right) > 0. \) Then making use of Theorem 3.2, we can show that the solution of Eq.(1) is \( p \)th moment exponential stable. The proof is complete.

5. **Fractional Ornstein-Uhlenbeck processes.** In the previous sections, we provide some criteria for general SDEs driven by fractional Brownian motion with Markov switching. In this section, we will present a criterion for switching fractional Ornstein-Uhlenbeck process. Without loss generality, we consider the following Ornstein-Uhlenbeck process with Markov switching
\[
\begin{cases}
\text{d}X_t = -\alpha(r_t)X_t \text{d}t + \sigma(r_t) \text{d}B^H_t, \\
X_0 = x_0,
\end{cases}
\tag{22}
\]
where \( \{r_t\}_{t \geq 0} \) is a Markov chain taking values in \( \mathbb{S} = \{1, 2, \ldots, N\} \), \( \alpha(i) \) and \( \sigma(i) \) are constants for each fixed \( i \in \mathbb{S} \). \( \{B^H_t\}_{t \geq 0} \) is a standard fractional Brownian motion, independent of \( \{r_t\}_{t \geq 0} \). In order to simplify the proof, we assume that \( x_0 = 0. \) We first provide a useful lemma.

**Lemma 5.1.** Let \( \{r_t\}_{t \geq 0} \) be a right-continuous Markov chain which takes values in a finite state space \( \mathbb{S} = \{1, 2, \ldots, N\} \). Assume that it is irreducible and positive recurrent with invariant measure \( \mu. \) If \( \alpha(\cdot) : \mathbb{S} \to \mathbb{R} \) is a function verifying
\[ \sum_{i \in \mathbb{S}} \mu(i) \alpha(i) > 0. \]
Then there exists constants $C, c, \alpha > 0$ such that:

$$ce^{-\alpha t} \leq \mathbb{E}[e^{-\int_0^t \alpha(r_s) ds}] \leq Ce^{-\alpha t},$$

for any initial condition $r_0$ and every $t \geq 0$.

**Proof.** It is a consequence of Perron-Frobenius theorem and the study of eigenvalues. For further details, see also the proofs of [3] and Lemma 2.7 of [7].

Now we are able to give the desired criterion.

**Theorem 5.2.** The switching fractional Ornstein-Uhlenbeck process (22) is $p$th moment exponential stable, if $\sum_{i \in S} \mu_i \alpha(i) > 0$, where $\mu = (\mu_i)_{i \in S}$ is the invariant probability measure of $\{r_t\}_{t \geq 0}$.

**Proof.** It is well known that for each fixed $i \in S$,

$$X_t^{(i)} = e^{-\alpha(i)t} \left( x_0 + \sigma(i) \int_0^t e^{\alpha(i)u} dB_u^H \right)$$

$$= e^{-\alpha(i)t} \left( \sigma(i) \int_0^t e^{\alpha(i)u} dB_u^H \right), \quad t \geq 0.$$

Let $\Upsilon(t) = \int_0^t \alpha(r_s) ds$, $\sigma = \max\{|\sigma(i)|, i \in S\}$, and $X_t$ be the global solution of Eq. (22). By Theorem 3.1 and [15], one has

$$X_t = \int_0^t e^{-\Upsilon(t) - \Upsilon(u)} \sigma(r_u) dB_u^H$$

$$= e^{-\Upsilon(t)} \int_0^t e^{\Upsilon(u)} \sigma(r_u) dB_u^H, \quad t \geq 0. \quad (23)$$

We then see from (23) that

$$|X_t|^p = e^{-p\Upsilon(t)} \left| \int_0^t e^{\Upsilon(u)} \sigma(r_u) dB_u^H \right|^p, \quad t \geq 0.$$

Thus, by Hölder’s inequality, we can derive that

$$\mathbb{E}|X_t|^p \leq \left( \mathbb{E} \left| e^{-p\Upsilon(t)} \right|^m \right)^\frac{1}{m} \left[ \mathbb{E} \left| \int_0^t e^{\Upsilon(u)} \sigma(r_u) dB_u^H \right|^{np} \right]^\frac{1}{np}, \quad t \geq 0, \quad (24)$$

where $m > 1, 1/m + 1/n = 1$.

According to Lemma 5.1, (24) implies

$$\mathbb{E}|X_t|^p \leq C \left( e^{-\alpha pt} \right) \left| \int_0^t e^{\Upsilon(u)} \sigma(r_u) dB_u^H \right|^{np}, \quad t \geq 0, \quad (25)$$

where $\alpha > 0$, $C$ is a general positive constant. Making use of Theorem 1.1 of [20] and (25), in a similar way, we can derive that for any $T > 0$ and $t \in [0, T]$,

$$\mathbb{E}|X_t|^p \leq C \left( e^{-\alpha pt} \right) \left| \int_0^t e^{\Upsilon(u)} \sigma(r_u) dB_u^H \right|^{np} \leq C \left( e^{-\alpha pt} \sigma^p \right) \left( \int_0^t |e^{\alpha u}|^{\frac{1}{p}} du \right)^{Hp}.$$
≤ C \left( e^{-\alpha pt} \right) \left( e^{\alpha H t} - 1 \right)^{H_p}

≤ C \left( e^{-\alpha pt} \right) \left( e^{\alpha H t} \right)^{H_p}

≤ C \left( e^{-\alpha pt} \right) \left( e^{\hat{\alpha} H t} \right)^{H_p},

where \( \hat{\alpha} = \alpha - o(1) \) such that \( \exp\{\frac{\alpha}{H} t\} - 1 < \exp\{\frac{\hat{\alpha}}{H} t\} < \exp\{\frac{\alpha}{H} t\} \).

This implies that

\[
\sup_{0 \leq t \leq T} E|X_t| e^{(\alpha - \hat{\alpha}) pt} \leq C.
\]

Letting \( T \to \infty \), we have that

\[
\sup_{t \in [0, \infty)} E|X_t| e^{(\alpha - \hat{\alpha}) pt} \leq C.
\]

Consequently, the required assertion follows. The proof is complete. \( \square \)

6. An example. In this section we shall give a numerical example to illustrate our results.

Example 1. Let \( \{r_t\}_{t \geq 0} \) be a right-continuous Markov chain taking values in \( S = \{1, 2\} \) with generator \( \Gamma = (\gamma_{ij})_{2 \times 2} : \)

\[
-\gamma_{11} = \gamma_{12} = 1,
-\gamma_{22} = \gamma_{21} = 1.
\]

Consider a one-dimension stochastic process with Markovian switching of the form:

\[
\begin{cases}
  dX_t = -\alpha(r_t) X_t dt + \sigma(t, r_t) dB_t^H, \\
  X_0 = 1,
\end{cases}
\] (26)

on \( t \geq 0 \). Here we take \( H = 0.7 \) and

\[
\begin{cases}
  \alpha(i) = -0.1, & \sigma(t, i) = e^{-0.1t}, \quad i = 1, \\
  \alpha(i) = 0.3, & \sigma(t, i) = e^{-0.3t}, \quad i = 2.
\end{cases}
\]

Obviously, the coefficients of Eq.(26) satisfy Assumption 1-2. By Theorem 3.1, Eq.(26) has a unique solution \( \{X_t\}_{t \geq 0} \).

Set \( \varphi(x) = |x|^2 \). Thus

\[
\mathcal{L}(i)\varphi(x) = -2\alpha(i)\varphi(x) + 2\sigma(t, i) \int_0^t \sigma(u, i) \phi(t, u) du, \quad i \in S.
\]

Noting that there exists \( \eta > 0 \) such that

\[
\lim_{t \to \infty} \left[ \sigma(t, i) \int_0^t \sigma(u, i) \phi(t, u) du \right] < \lim_{t \to \infty} e^{-\eta t}, \quad i \in S.
\]

Note also that there exists \( \varepsilon \) sufficiently small and \( t_0 \) such that for \( t > t_0 \),

\[
2\varepsilon |X_t|^2 > e^{-\eta t}.
\]

Hence

\[
\mathcal{L}(i)\varphi(x) \leq -2(\alpha(i) - \varepsilon)\varphi(x), \quad i \in S.
\]

Compute that

\[
\sum_{i \in S} \mu_i(-2(\alpha(i) - \varepsilon)) = -2[(\alpha(1) - \varepsilon) + (\alpha(2) - \varepsilon)] = -2(0.2 - 2\varepsilon) < 0.
\]
It then follows from Theorem 4.3 that Eq. (26) is second moment exponential stable. Fig. (1)-(2) show a single path of the solution and the corresponding solution’s norm square, respectively.

**Figure 1.** A single path of solution.  
**Figure 2.** Norm square trajectory.

**Acknowledgments.** The authors are grateful to thank the anonymous reviewers for careful reading of the paper and for helpful comments that led to improvement of the first version of this paper. The research of L. Yan was partially supported by the National Natural Science Foundation of China (No. 11571071). The research of Z. Zhang was partially supported by the Humanities and Social Sciences Fund of Ministry of Education of China (No. 17YJA910004), the Natural Science Foundation of Shanghai (No. 19ZR1400600) and the Fundamental Research Funds for the Central Universities.

**REFERENCES**

[1] E. Alòs, O. Mazet and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.*, 29 (2001), 766–801.

[2] W. J. Anderson, *Continuous-Time Markov Chains: An Applications-Oriented Approach*, Springer Series in Statistics: Probability and its Applications, Springer-Verlag, New York, 1991.

[3] J.-B. Bardet, H. Guérin and F. Malrieu, Long time behavior of diffusions with Markov switching, *ALEA Lat. Am. J. Probab. Math. Stat.*, 7 (2010), 151–170.

[4] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[5] F. Biagini, Y. Z. Hu, B. Øksendal and T. S. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Probability and its Applications (New York), Springer-Verlag London, 2008.

[6] D. C. Brody, J. Syroka and M. Zervos, Dynamical pricing of weather derivatives, *Quant. Finance*, 2 (2002), 189–198.

[7] B. Cloez and M. Hairer, Exponential ergodicity for Markov processes with random switching, *Bernoulli*, 21 (2015), 505–536.

[8] T. E. Duncan, B. Maslowski and B. Pasik-Duncan, Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise, *Stochastic Process. Appl.*, 115 (2005), 1357–1383.

[9] T. E. Duncan, Y. Z. Hu and B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion. I. Theory, *SIAM J. Control Optim.*, 38 (2000), 582–612.

[10] T. E. Duncan, B. Pasik-Duncan and B. Maslowski, Fractional Brownian motion and stochastic equations in Hilbert spaces, *Stoch. Dyn.*, 2 (2002), 225–250.

[11] M. K. Ghosh, A. Arapostathis and S. I. Marcus, Ergodic control of switching diffusions, *SIAM J. Control Optim.*, 35 (1997), 1952–1988.
[12] J. D. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, *Econometrica*, 57 (1989), 357–384.
[13] H. Holden, B. Øksendal, J. Ubøe and T. S. Zhang, *Stochastic Partial Differential Equations*, 2nd edition, Universitext, Springer, New York, 2010.
[14] Y. Z. Hu, Itô Stochastic differential equations driven by fractional Brownian motion of Hurst parameter $H > 1/2$, *Stochastics*, 90 (2018), 720–761.
[15] G. Huang, H. M. Jansen, M. Mandjes, P. Spreeij and K. De Turck, Markov-modulated Ornstein-Uhlenbeck processes, *Adv. in Appl. Probab.*, 48 (2016), 235–254.
[16] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd edition, Graduate Texts in Mathematics, 113. Springer, New York, 1991.
[17] M. L. Li and F. Q. Deng, Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise, *Nonlinear Anal. Hybrid Syst.*, 24 (2017), 171–185.
[18] X. R. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.*, 79 (1999), 45–67.
[19] X. R. Mao and C. G. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.
[20] J. Mémé, Y. Mishura and E. Valkeila, Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion, *Statist. Probab. Lett.*, 51 (2001), 197–206.
[21] Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Lecture Notes in Mathematics, 1929. Springer-Verlag, Berlin, 2008.
[22] D. Nualart and A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.*, 53 (2002), 55–81.
[23] A. Rathinasamy and M. Balachandran, Mean-square stability of Milstein method for linear hybrid stochastic delay integro-differential equations, *Nonlinear Anal. Hybrid Syst.*, 2 (2008), 1256–1263.
[24] S. Cong, On almost sure stability conditions of linear switching stochastic differential systems, *Nonlinear Anal. Hybrid Syst.*, 22 (2016), 108–115.
[25] I. Simonsen, Measuring anti-correlations in the Nordic electricity spot market by wavelets, *Physica A: Statistical Mechanics and its Applications*, 322 (2003), 597–606.
[26] A. V. Skorohod, *Asymptotic Methods in the Theory of Stochastic Differential Equations*, Translations of Mathematical Monographs, 78. American Mathematical Society, Providence, RI, 1989.
[27] L. Tan, Exponential stability of fractional stochastic differential equations with distributed delay, *Adv. Difference Equ.*, 2014 (2014), 8 pp.
[28] G. G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Stochastic Modelling and Applied Probability, 63. Springer, New York, 2010.
[29] C. G. Yuan and X. R. Mao, Asymptotic stability in distribution of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.*, 103 (2003), 277–291.
[30] C. G. Yuan and X. R. Mao, Stability of stochastic delay hybrid systems with jumps, *Eur. J. Control*, 16 (2010), 595–608.
[31] Z. Z. Zhang, J. Y. Tong and L. J. Hu, Long-term behavior of stochastic interest rate models with Markov switching, *Insurance Math. Econom.*, 70 (2016), 320–326.
[32] W. N. Zhou, J. Yang, X. Q. Yang, A. D. Dai, H. S. Liu and J. Fang, pth Moment exponential stability of stochastic delayed hybrid systems with Lévy noise, *Appl. Math. Model.*, 39 (2015), 5650–5658.

Received December 2018; revised May 2019.

E-mail address: litanyan@dhu.edu.cn
E-mail address: peiwenyi@163.com
E-mail address: zzzhang@dhu.edu.cn