Compliance in Real Time Multiset Rewriting Models

Max Kanovich1,6 Tajana Ban Kirigin2 Vivek Nigam3,4 Andre Scedrov5,6 and Carolyn Talcott7

1 University College, London, UK, m.kanovich@ucl.ac.uk
2 University of Rijeka, Department of Mathematics, HR, bank@math.uniri.hr
3 Federal University of Paraíba, João Pessoa, Brazil, vivek@ci.ufpb.br
4 fortiss, Germany, nigam@fortiss.org
5 University of Pennsylvania, Philadelphia, USA, scedrov@math.upenn.edu
6 National Research University Higher School of Economics, Moscow, Russia
7 SRI International, USA, clt@csl.sri.com

Abstract. The notion of compliance in Multiset Rewriting Models (MSR) has been introduced for untimed models and for models with discrete time. In this paper we revisit the notion of compliance and adapt it to fit with additional nondeterminism specific for dense time domains. Existing MSR with dense time are extended with critical configurations and non-critical traces, that is, traces involving no critical configurations. Complexity of related non-critical reachability problem is investigated. Although this problem is undecidable in general, we prove that for balanced MSR with dense time the non-critical reachability problem is PSPACE-complete.

1 Multiset Rewriting Systems with Real Time

We follow [18] in formalizing dense time in the multiset rewriting framework.

Assume a finite first-order typed alphabet, Σ, with variables, constants, function and predicate symbols. Terms and formulas are constructed as usual (see [11]) by applying symbols of correct type (or sort).

If $P$ is a predicate of type $τ_1 \times τ_2 \times \cdots \times τ_n \rightarrow o$, where $o$ is the type for propositions, and $u_1, \ldots, u_n$ are terms of types $τ_1, \ldots, τ_n$, respectively, then $P(u_1, \ldots, u_n)$ is a fact. A fact is grounded if it does not contain any variables. We assume that the alphabet contains the constant $z : \text{Nat}$ denoting zero and the function $s : \text{Nat} \rightarrow \text{Nat}$ denoting the successor function. Whenever it is clear from the context, we write $n$ for $s^n(z)$ and $(n + m)$ for $s^n(s^m(z))$.

Additionally, we allow an unbounded number of fresh values [6,10] to be involved.

In order to specify timed systems, to each fact we attach a timestamp denoting time. Timestamped facts are of the form $F@t$, where $F$ is a fact and $t \in \mathbb{R}$ is a non-negative real number called timestamp. Similarly, time variables denoting timestamps, such as variable $T$ in $F@T$, range over non-negative real numbers.

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For simplicity, instead of timestamped facts, we often simply say facts. Also, when we want to emphasize a difference between a fact \( F \), and a timestamped fact \( \text{timestamp} \), we say that \( F \) is an \textit{untimed fact}.

There is a special predicate symbol \( \text{timestamp} \) with arity zero, which will be used to represent global time. For example, the fact \( \text{timestamp} \) denotes that the current global time of the system is 10.4.

Given \( \text{timestamp} \), we say that a fact \( F \) is a \textit{future fact} when its timestamp is \( t_F > t \), and a fact \( F \) is a \textit{past fact} when \( t_F < t \), and a fact \( F \) is a \textit{present fact} when \( t_F = t \).

A \textit{configuration} is a multiset of ground timestamped facts,

\[
S = \{ \text{timestamp}, F_1 @ t_1, \ldots, F_n @ t_n \}
\]

with a single occurrence of a \( \text{timestamp} \) fact.

Configurations are to be interpreted as states of the system. Configurations are modified by multiset rewrite rules which can be interpreted as actions of the system. There is only one rule, \( \text{Tick} \), that modifies global time:

\[
\text{timestamp} \mathbin{\rightarrow} \text{timestamp}(T + \epsilon) \tag{1}
\]

where \( T \) is a time variable and \( \epsilon \) can be instantiated by any non-negative real number. We also write \( \text{Tick}_\epsilon \) when we refer to the \( \text{Tick} \) rule \( (1) \) for a specific \( \epsilon \). Applied to a configuration, \( \{ \text{timestamp}, F_1 @ t_1, \ldots, F_n @ t_n \} \), \( \text{Tick}_\epsilon \) advances global time by \( \epsilon \), resulting in configuration \( \{ \text{timestamp}(t + \epsilon), F_1 @ t_1, \ldots, F_n @ t_n \} \).

We point out that the \( \text{Tick} \) rule changes only the timestamp of the fact \( \text{timestamp} \), while the remaining facts in the configuration (those different from \( \text{timestamp} \)) are unchanged.

The remaining rules are \textit{instantaneous} as they do not modify global time, but may modify the remaining facts of configurations (those different from \( \text{timestamp} \)). Instantaneous rules have the form:

\[
\text{timestamp} @ T, W_1 @ T_1, \ldots, W_p @ T_p, F_1 @ T_1', \ldots, F_n @ T_n' | C \mathbin{\rightarrow} \exists X \cdot \{ \text{timestamp} @ T, W_1 @ T_1, \ldots, W_p @ T_p, Q_1 @ (T + D_1), \ldots, Q_m @ (T + D_m) \} \tag{2}
\]

where \( D_1, \ldots, D_m \) are natural numbers, \( W = \{ W_1 @ T_1, \ldots, W_p @ T_p \} \) is a multiset of timestamped facts, possibly containing variables, and \( C \) is the guard of the rule which is a set of constraints involving the time variables appearing in the rule’s pre-condition, \textit{i.e.} the variables \( T, T_1, \ldots, T_p, T_1', \ldots, T_n' \).

Constraints may be of the form:

\[
T > T' + N \quad \text{and} \quad T = T' + N \tag{3}
\]

where \( T \) and \( T' \) are time variables, and \( N \in \mathbb{N} \) is a natural number.

Here, and in the rest of the paper, the symbol \( \pm \) stands for either \( + \) or \( - \), that is, constraints may involve addition or subtraction.

We use \( T' \geq T' + N \) to denote the disjunction of \( T > T' + N \) and \( T = T' + N \). All time variables in the guard of a rule are assumed to appear in the rule’s pre-condition.
Finally, the variables $X$ that are existentially quantified in the rule (Equation 2) are to be replaced by fresh values, also called nonces in protocol security literature \[6,10\]. As in our previous work \[13\], we use nonces whenever a unique identification is required, for example for some protocol session or transaction identification.

A rule $W \mid C \rightarrow \exists X.W'$ can be applied to a configuration $S$ if there is a ground substitution $\sigma$, where the variables in $X$ are fresh, such that $W\sigma \subseteq S$ and $C\sigma$ is true. The resulting configuration is $\left((S \setminus W) \cup W'\right)\sigma$.

More precisely, given some rule $r$, an instance of a rule is obtained by substituting all variables appearing in the pre- and post-condition of the rule with constants. This substitution applies to variables appearing in terms inside facts, variables representing fresh values, as well as time variables used in specifying timestamps of facts. An instance of an instantaneous rule can only be applied if all the constraints in its guard are satisfied.

In order to express timed properties of the system, besides being attached to the rules, constraints may be attached to configurations. In particular, constraints may be used to express specific timed properties of configurations. For example,

$$\text{Time} @ T, \text{Deadline}(p) @ T', W \mid \{T + 7 = T'\}$$

represents a configuration where a deadline of process $p$ is in 7 time units.

Following \[10\] we say that a fact is consumed by some rule $r$ if that fact occurs more times in $r$ on the left side than on the right side. A fact is created by some rule $r$ if that fact occurs more times in $r$ on the right side than on the left side. Hence, $F_1 @ T'_1, \ldots, F_n @ T'_n$ are consumed by the rule (6) and $Q_1 @ (T + D_1), \ldots, Q_m @ (T + D_m)$ are created by that rule. In a rule, we usually color red the consumed facts and blue the created facts.

We write $S \rightarrow_r S'$ for the one-step relation where configuration $S$ is rewritten to $S'$ using an instance of rule $r$. For a set of rules $R$, we define $S \rightarrow^R S'$ as the transitive reflexive closure of the one-step relation on all rules in $R$. We elide the subscript $R$, when it is clear from the context, and simply write $S \rightarrow S'$.

**Definition 1.** A timed MSR system with dense time $T$ is a set of rules containing only instantaneous rules (Eq. 2) and the Tick rule (Eq. 1).

A trace of a timed MSR is constructed by a sequence of rules. A finite trace of a timed MSR $T$ starting from an initial configuration $S_0$ is a sequence

$$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n$$

where $S_i \rightarrow_r S_{i+1}$ for some $r_i \in T$, for all $i \in \{0, \ldots, n\}$. Infinite traces can also be considered, as in \[20\], but in this paper only finite traces will be used.

Notice that by the nature of multiset rewriting there are various aspects of non-determinism in the model. For example, different actions and even different instantiations of the same rule may be applicable to the same configuration $S$, which may lead to different resulting configurations $S'$.

There is the additional non-determinism in the dense time model with respect to the discrete time model used in \[20\], provided by the choice of $\varepsilon$, representing the non-negative real value of time increase. While in the discrete time model, time is advancing
using the rule
\[ Time@T \rightarrow Time@(T + 1), \]
where time always advances by one time unit, in the dense time model, using the rule (Eq. 1), time can advance by any non-negative real value \( \varepsilon \).

**Remark 1.** Notice that the consecutive time advancements \( \text{Tick}_{\varepsilon_1} \) and \( \text{Tick}_{\varepsilon_2} \) applied to some configuration have the same effect of the single tick \( \text{Tick}_{\varepsilon} \), for arbitrary \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon = \varepsilon_1 + \varepsilon_2 \).

Indeed, this is a property of the multiset rewriting formalism itself. In this context, above property reflects the continuity of time in the physical world.

With this property in mind, in any trace we can replace consecutive ticks
\[
S_0 \rightarrow_{\text{Tick}_{\varepsilon_1}} S_1 \rightarrow_{\text{Tick}_{\varepsilon_2}} \cdots \rightarrow_{\text{Tick}_{\varepsilon_n}} S_n
\]
with a single tick
\[
S_0 \rightarrow_{\text{Tick}_{(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)}} S_n,
\]
and vice versa, without compromising the semantics of the process that is being modelled.

### 1.1 Balanced Systems

The balanced condition \([23]\) is necessary for decidability of problems such as reachability studied in \([13,21,18]\) as well as the problem introduced in Section 2.

**Definition 2.** A timed MSR with dense time \( T \) is balanced if for all instantaneous rules \( r \in T \), \( r \) creates the same number of facts as it consumes, that is, instantaneous rules (Eq. 2) are of the form:

\[
Time@T, W_1, F_{1}@T_1', \ldots, F_n@T_n' \mid C \rightarrow \exists X_1, [ Time@T, W, Q_1@T + D_1, \ldots, Q_n@T + D_n ] ,
\]

where \( W \) is a multiset of timestamped facts.

By consuming and creating facts, rewrite rules can increase and decrease the number of facts in configurations throughout a trace. However, in balanced MSR systems, the number of facts in configurations in a trace is constant, as states the following proposition.

**Proposition 1.** Let \( T \) be a balanced timed MSR with dense time. Let \( S_0 \) be an initial configuration with exactly \( m \) facts. For all traces \( P \) of \( T \) starting with \( S_0 \), all configurations \( S_i \) in \( P \) have exactly \( m \) facts.

**Proof.** Since all the rules in \( T \) are balanced, rule application does not effect the number of facts in a configuration. That is, enabling configuration has the same number of facts as the resulting configuration. Hence, throughout the trace, all configurations have the same number of facts as the initial configuration \( S_0 \). \( \Box \)
2 Quantitative Temporal Properties

2.1 Goals, Critical Configurations and Non-critical Traces in MSR Systems with Dense Time

In order to define quantitative temporal properties, we review the notion of critical configurations and compliant traces from our previous work [22] and introduce reachability problem for MSR systems with dense time which considers critical configurations.

Definition 3. Critical configuration specification $CS$ (resp. a goal $GS$) is a set of pairs

$$\{ \langle S_1, C_1 \rangle, \ldots, \langle S_n, C_n \rangle \}.$$  

Each pair $\langle S_j, C_j \rangle$ is of the form:

$$\langle \{ F_1@T_1, \ldots, F_p@T_p \}, C_j \rangle$$

where $T_1, \ldots, T_p$ are time variables, $F_1, \ldots, F_p$ are facts (possibly containing variables) and $C_j$ is a set of time constraints involving only the variables $T_1, \ldots, T_p$.

Given a critical configuration specification $CS$ (resp. a goal $GS$), we classify a configuration $S$ as a critical configuration w.r.t $CS$ (resp. goal configuration w.r.t. $GS$) if for some $1 \leq i \leq n$, there is a grounding substitution, $\sigma$, such that:

- $S_\sigma \subseteq S$;
- All constraints in $C_\sigma$ are satisfied;

where substitution application ($S_\sigma$) is defined as usual [11], i.e., by mapping time variables in $S$ to natural numbers, nonce names to nonce names (renaming of nonces) and non time variables to terms.

For simplicity, when the corresponding critical configuration specification or goal is clear from the context, we will elide it and use terminology critical or goal configuration.

Notice that nonce renaming is assumed as the particular nonce name should not matter for classifying a configuration as a critical or a goal configuration. Nonce names cannot be specified in advance, since these are freshly generated in a trace, i.e. during the execution of the process being modelled.

Moving from discrete to dense time is not straightforward w.r.t. the notion of a compliant, i.e., non-critical trace. Consider, for example, a trace in a timed MSR with dense time, containing the following configurations and a Tick:

$$\begin{align*}
\text{Time}@1.5, F@3.5 & \rightarrow_{\text{Tick}_3} \text{Time}@4.5, F@3.5
\end{align*}$$

which could potentially be considered as non-critical w.r.t. with the critical configuration specification:

$$\begin{align*}
\text{Time}@T, F@T_1 \mid \{ T_1 = T \}
\end{align*}$$

as it doesn’t contain any critical configurations. However, a trace containing rules:

$$\begin{align*}
\text{Time}@1.5, F@3.5 & \rightarrow_{\text{Tick}_2} \text{Time}@3.5, F@3.5 & \text{Time}@4.5, F@3.5
\end{align*}$$
would not be non-critical w.r.t. the same critical configuration specification since it contains the critical configuration \{ Time@3.5, F@3.5 \}. Above traces differ only in the representation of time flow and they model the same real-time process. In reality, due to continuity of time, the process would reach such a critical state, *i.e.* it would not skip over this undesired state. Clearly, this inconsistency is not what we want in our model.

As the above example suggests, in the setting with dense time it is particularly important that the notion of a non-critical trace is properly defined. While in systems with discrete time, time can increase only by one time unit at a time, when time is dense, time can increase by any value, however small, and however large. That is how we model the natural continuous aspect of time we know in our everyday life. In particular, recall Remark [1] illustrating how the continuity of time flow is implicitly embedded in the MSR formalism. Namely, given arbitrary $\varepsilon > 0$ and any positive $\varepsilon_1 < \varepsilon$, there exists $\varepsilon_2 > 0$ such that the time *Tick* for $\varepsilon$ has the same effect as the *Tick* for $\varepsilon_1$ followed by the *Tick* for $\varepsilon_2$. That is, if

\[
S_0 \xrightarrow{T_{\text{Tick}}_{\varepsilon}} S_1
\]

then

\[
S_0 \xrightarrow{T_{\text{Tick}}_{\varepsilon_1}} S_2 \xrightarrow{T_{\text{Tick}}_{\varepsilon_2}} S_1 .
\]

Clearly, $\varepsilon = \varepsilon_1 + \varepsilon_2$ holds. Relying on above property, we now define which traces may be considered as compliant in the dense time setting.

**Definition 4.** Given a timed MSR with dense time $T$ and a critical configuration specification $CS$, a trace $P$ of $T$ is non-critical if no critical configuration is reached along any trace obtained by replacing any subtrace $S_i \xrightarrow{T_{\text{Tick}}_{\varepsilon}} S_{i+1}$ of $P$ with $S_i \xrightarrow{T_{\text{Tick}}_{\varepsilon_1}} S' \xrightarrow{T_{\text{Tick}}_{\varepsilon_2}} S_{i+1}$ for arbitrary $\varepsilon_1 < \varepsilon$, such that $\varepsilon = \varepsilon_1 + \varepsilon_2$ holds.

Above decomposition of the *Tick* rules, in all possible ways of consecutive *Ticks*, ensures that the continuity of time and the notion of non-critical traces are well combined.

On the other hand, however, checking whether a given trace in a system with dense time is non-critical is potentially more challenging than in the untimed setting [23] and models with discrete time [22,20]. Testing whether a trace is non-critical in models with dense time requires potentially checking through an infinite number of traces. This could possibly effect the complexity of the corresponding non-critical reachability problem. Fortunately, we can rely on our equivalence relation among configurations, *i.e.* on our technical machinery called circle-configurations, with respect to this issue as well. We show this result in Section 3.1.

### 2.2 Verification Problem

**Definition 5.** [Non-critical reachability problem]

Given a timed MSR $T$, a goal $GS$, a critical configuration specification $CS$ and an initial configuration $S_0$, is there a non-critical trace, $P$, that leads from $S_0$ to a goal configuration?
Our complexity results, for a given MSR $T$, an initial configuration $S_0$, a critical configuration specification $CS$ and a goal $GS$, mention the value $D_{\text{max}}$ which is an upper-bound on the natural numbers appearing in $S_0$, $T$, $CS$ and $GS$, which is syntactically inferred from timestamps and numbers appearing in facts, rules and constraints of $S_0$, $T$, $CS$ and $GS$.

For the complexity results for non-critical reachability problem (bisimulation of non-critical traces) with dense time we define immediate successors for configurations, motivated by the non-determinism in the model related to the choice of the positive real number $\varepsilon$ used in the $\text{Tick}_\varepsilon$ rule. Namely, unless some restrictions are imposed on a trace by some time sampling, $\text{Tick}_\varepsilon$ rule is applicable to every configuration, and for every $\varepsilon > 0$. However, the choice of $\varepsilon$ is important as it may have different effects on representation of time in a trace. Consider, for example, configuration

$$S = \{ \text{Time@2}, F@0.4, G@2.5, H@1 \}.$$  

Applying a $\text{Tick}$ rule to $S$ for any $\varepsilon < 0.4$ has the same effect w.r.t time constraints satisfied by the resulting configuration, regardless of a particular $\varepsilon < 0.4$ used. In fact, it has no effect in that sense, since the same set of constraints is satisfied by the resulting configuration as by configuration $S$. Advancing time in $S$ by $\varepsilon = 0.4$ is different. Resulting configuration

$$S' = \{ \text{Time@2.4}, F@0.4, G@2.5, H@1 \}.$$  

satisfies e.g. constraint $T' = T$, related to facts $\text{Time@T}$ and $G@T'$, which is not satisfied by $S$. Now, applying a $\text{Tick}_\varepsilon$ to $S'$ for any $\varepsilon > 0$ would change the set of constraints satisfied by the resulting configuration $S''$. The set of constraints satisfied by $S''$ will depend on the value of $\varepsilon$. For example, for $\varepsilon = 0.35$ constraint $T > T' + 2$, where $T, T'$ relate to facts $\text{Time@T}$ and $F@T'$, would be satisfied in $S'' (2.45 > 0.4+2)$ and $T = T' + 2$ would not, while for $\varepsilon = 0.3$, constraint $T = T' + 2$ would hold.

With the above consideration on the importance on how much the time advances by a single $\text{Tick}$ rule, we define the following, successor, relation among configurations.

**Definition 6.** Given a timed MSR $T$ with dense time, and a natural number $d$, let $C_d$ be a set of all constrains containing natural numbers up to $d$:

$$C_d = \{ T > T' \pm N, \ T \geq T' \pm N, \ T = T' \pm N \mid N \leq d \}.$$  

We say that configuration $S_2$ is an immediate successor of configuration $S_1$ w.r.t. $d$ if

i) There exists $\varepsilon > 0$ such that $S_1 \xrightarrow{\text{Tick}_\varepsilon} S_2$;

ii) $S_1$ and $S_2$ do not satisfy the same set of constraints from $C_d$, where variables $T$ and $T'$ refer to timestamps of same facts from $S_1$ and $S_2$;

iii) For all $\varepsilon' > 0$, $\varepsilon' < \varepsilon$ if $S_1 \xrightarrow{\text{Tick}_{\varepsilon'}} S'$ then $S'$ satisfies the same constraints from $C_d$ either as $S_1$ or as $S_2$.

When $S_2$ is an immediate successor of $S_1$ w.r.t. $d$ we write $S_1 \xrightarrow{\text{Tick}_d} S_2$.

When $d$ is clear from the context we simply say that $S_2$ is an immediate successor of $S_1$ and write $S_1 \xrightarrow{\text{Tick}_{d}} S_2$. 
Notice that in the above example, \{\text{Time}@2.05, F@0.4, G@2.1, H@1\} is an immediate successor of \(S\), while configuration \{\text{Time}@2.4, F@0.4, G@2.5, H@1\} is not because, e.g.,

\[
\begin{align*}
\{\text{Time}@2, F@0.4, G@2.5, H@1\} & \longrightarrow_{\text{Tick}@0.05} \{\text{Time}@2.05, F@0.4, G@2.5, H@1\} \longrightarrow_{\text{Tick}@0.35} \\
\{\text{Time}@2.4, F@0.4, G@2.5, H@1\}
\end{align*}
\]

where all of the above configurations satisfy different time constraints.

In general, the immediate successor of a configuration is not unique. For example, \{\text{Time}@2.15, F@0.4, G@2.5, H@1\} and \{\text{Time}@2.3, F@0.4, G@2.5, H@1\} are both immediate successors of \(S'\). On the other hand, the immediate successor of \{\text{Time}@2.15, F@0.4, G@2.5, H@1\} is unique, \{\text{Time}@2.4, F@0.4, G@2.5, H@1\}.

There is a clear connection between non-critical traces and immediate successor configurations. Notice that if neither \(S_i\) nor its immediate successor configuration \(S_{i+1}\) is critical, then the condition on non-critical traces given in Definition \[\text{III}\] is satisfied.

**Proposition 2.** Let \(T\) be a timed MSR with dense time, and \(d\) a natural number. Let \(S \longrightarrow_{\text{Tick}@\epsilon} S'\). If \(S\) and \(S'\) are not critical w.r.t. some critical configuration specification \(CS\) involving constraints form \(C_d\), then for any \(\epsilon' > 0\), \(\epsilon' < \epsilon\), the configuration \(S''\) such that \(S \longrightarrow_{\text{Tick}@\epsilon_1} S'' \longrightarrow_{\text{Tick}@\epsilon_2} S'\), is not critical.

**Proof.** Let \(S \longrightarrow_{\text{Tick}@\epsilon} S'\), and assume neither \(S\) nor \(S'\) is critical. Let

\[
S \longrightarrow_{\text{Tick}} S'' \longrightarrow_{\text{Tick}} S'.
\]

Since \(S'\) is an immediate successor of \(S\), as per Definition \[\text{VI}\], such configuration \(S''\) satisfies the same set of constraints form \(C_d\) as either \(S\) or \(S'\). This includes the constrains used in \(CS\). Since both \(S\) and \(S'\) are not critical, \(S''\) is not critical as well. \(\Box\)

### 3 Complexity Results for Balanced Timed MSR with Dense Time

Reachability and the related problems for MSR are undecidable in general \[\text{[14]}\]. However, by imposing some restrictions on the form of the rewrite rules, such as using only balanced rules and bounding the size of facts, these problems become decidable, even in timed models with fresh values.

A summary of related complexity results is shown in Table \[\text{I}\].

In this section we investigate the complexity of the non-critical reachability problem for balanced systems with facts of bounded size.

In this new setting with dense time, the non-critical reachability problem combines quantitative temporal properties defined for timed MSR with the refined notion of compliance. Our results rely heavily on the abstractions called circle-configurations. As we will show in Section \[\text{3.1}\] circle-configurations and the related time advancement rules, \(\text{Next}\), are defined in such a way to reflect similar characteristics related to advancement of time in dense time models.
Table 1: Summary of the complexity results for the reachability and non-critical reachability problems. These results also hold for MSR models with fresh values.

| MSR          | Reachability Problem | Non-critical Reachability |
|--------------|----------------------|---------------------------|
| Balanced     |                      |                           |
| untimed      | PSPACE-complete [23,13] | PSPACE-complete [23,13]   |
| discrete time| PSPACE-complete [15]  | PSPACE-complete [15]      |
| real time    | PSPACE-complete [19]  | PSPACE-complete new!       |
| Not necessarily balanced | Undecidable [14]       | Undecidable [14]          |

As discussed above, we assume a bound, $k$, on the size of facts. However, we do not impose an upper bound on the values of timestamps. Also, our timed MSRs with dense time are constructed over $\Sigma$, a finite alphabet with $J$ predicate symbols and $E$ constant and function symbols and can involve an unbounded number of fresh values.

3.1 Circle-configurations

In order to handle dense time, and in particular for our complexity results, in our previous work [18] we introduced an equivalence relation among configurations. We now review main ideas behind this machinery. For a more detailed exposition of this approach see [18].

The equivalence of configurations involves an upper bound $D_{max}$ on the numeric values mentioned in the specification of the considered system and problems: We set $D_{max}$ to be a natural number such that $D_{max} > n + 1$ for any number $n$ (both real or natural) appearing in the timestamps of the initial configuration, or the $N$s and $D_i$s in constraints (Eq.3) or rules (Eq.2) of the timed MSR, in goal and critical configuration specification.

Notice that immediate successor configurations also involve an upper bound, $d$, on natural numbers appearing in time constraints. For a given problem, we will extract the value $D_{max}$ as described above, and we will consider immediate successor configurations w.r.t. the same bound $D_{max}$.

Configurations are defined as equivalent if they contain the same (untimed) facts, up to nonce renaming, and if they satisfy the exact same set of constraints. When we say that some configurations satisfy the same constraint, we intend to say that time variables of that constraint refer to the same facts in both configurations.

**Definition 7.** Given a timed MSR $T$ with dense time, a goal $\mathcal{G}S$, a critical configuration specification $\mathcal{C}S$ and an initial configuration $S_0$, let $D_{max}$ be an upper bound on the numeric values appearing in $T$, $\mathcal{G}S$, $\mathcal{C}S$ and $S_0$. Let

\[ S = \{ Q_1@t_1, Q_2@t_2, \ldots, Q_n@t_n \} \quad \text{and} \quad \tilde{S} = \{ \tilde{Q}_1@\tilde{t}_1, \tilde{Q}_2@\tilde{t}_2, \ldots, \tilde{Q}_n@\tilde{t}_n \} \]

(6)
be two configurations written in canonical way where the two sequences of timestamps
\( t_1, \ldots, t_n \) and \( \tilde{t}_1, \ldots, \tilde{t}_n \) are non-decreasing. (For the case of equal timestamps, we
sort the facts in alphabetical order, if necessary.) We say that configurations \( S \) and \( \tilde{S} \)
are equivalent configurations if the following conditions hold:
(i) There is a bijection \( \sigma \) that maps the set of all nonce names appearing in configuration
\( S \) to the set of all nonce names appearing in configuration \( \tilde{S} \), such that \( Q_i \sigma = \tilde{Q}_i \),
for each \( i \in \{1, \ldots, n\} \); and
(ii) Configurations \( S \) and \( \tilde{S} \) satisfy the same constraints, that is:
\[
\begin{align*}
t_i &> t_j + D \quad \text{iff} \quad \tilde{t}_i > \tilde{t}_j + D, \\
t_i &= t_j + D \quad \text{iff} \quad \tilde{t}_i = \tilde{t}_j + D,
\end{align*}
\]
for all \( 1 \leq i \leq n, 1 \leq j \leq n \) and \( D \leq D_{\text{max}} \).
When \( S \) and \( \tilde{S} \) are equivalent we write \( S \sim_{D_{\text{max}}} \tilde{S} \), or simply \( S \sim \tilde{S} \).

As we already pointed out, when we say that \( S \) and \( \tilde{S} \) satisfy the same constraints,
we mean that the time variables in the constraint refer to the same facts \( Q_i \) and \( \tilde{Q}_i \), up to
nonce renaming.

Notice that no configuration is equivalent to its immediate successor configuration.

In [13] we also introduced an illustrative representation of the above equivalence
relation, called circle-configuration.

**Definition 8.** Let \( T \) be a timed MSR with dense time, \( G S \) a goal, \( C S \) a critical configuration
specification and \( S_0 \) an initial configuration. Let \( D_{\text{max}} \) be an upper bound on the numeric
values appearing in \( T \), \( G S, C S \) and \( S_0 \), and
\( S = \{ F_1 \oplus t_1, F_2 \oplus t_2, \ldots, F_n \oplus t_n, \text{Time} \oplus t\} \).
The pair \( A_S = (\Delta_S, U_S) \) is the circle-configuration of the configuration \( S \) defined as follows. The \( \delta \)-configuration of \( S \), \( \Delta_S \), is:
\[
\Delta_S = \left\{ \{ P^1_1, \ldots, P^1_{m_1} \}, \delta_{1,2}, \{ P^2_1, \ldots, P^2_{m_2} \}, \delta_{2,3}, \ldots, \delta_{j-1,j}, \{ P^j_1, \ldots, P^j_{m_j} \} \right\}
\]
where \( \{ P^1_1, \ldots, P^1_{m_1}, P^2_1, \ldots, P^2_{m_2} \} = \{ F_1, \ldots, F_n, \text{Time} \} \),
timestamps of facts \( P^i_1, \ldots, P^i_{m_i} \) have the same integer part, \( t^i, \forall i = 1, \ldots, j \), and
\[
\delta_{i,i+1} = \begin{cases} 
  t^{i+1} - t^i, & \text{if } t^{i+1} - t^i \leq D_{\text{max}}, \\
  \infty, & \text{otherwise}
\end{cases}, \quad i = 1, \ldots, j - 1.
\]
The unit circle of \( S \), \( U_S \), is:
\[
U_S = \left\{ \{ Q^0_1, \ldots, Q^0_{m_0} \}, \ldots, \{ Q^1_1, \ldots, Q^1_{m_1} \}, \ldots, \{ Q^k_1, \ldots, Q^k_{m_k} \} \right\}
\]
where \( \{ Q^0_1, \ldots, Q^0_{m_0}, Q^1_1, \ldots, Q^1_{m_1} \} = \{ F_1, \ldots, F_n, \text{Time} \} \),
timestamps of facts in the same class, \( Q^1_1, \ldots, Q^1_{m_1} \) have the same decimal part, \( \forall i = 0, \ldots, k \),
timestamps of facts \( Q^0_1, \ldots, Q^0_{m_0} \) are integers, and the classes are ordered in the increasing order,
i.e., \( \text{dec}(Q^i_j) < \text{dec}(Q^i_{j'}) \) for all \( i \neq j \), where \( 1 \leq i \leq m_i, 1 \leq j \leq m_j, 0 \leq l \leq k, 1 \leq l' \leq k \).
We write \( U_S(Q^i_j) = i \) to denote the class in which the fact \( Q^i_j \) appears in \( U_S \).
Fig. 1: Unit Circle

Fig. 2: Circle-Configuration

For simplicity, we sometimes write \( A \) and \( \langle \Delta, U \rangle \) instead of \( A_S \) and \( \langle \Delta_S, U_S \rangle \), when the corresponding configuration is clear from the context.

We graphically represent a unit circle as shown in Figure 1. The class marked with the subscript \( Z \), \( \{Q_i^0, \ldots, Q_m^0\}_Z \), is called the zero point and is marked as the (green) ellipse at the top of the circle. The remaining classes are placed on the circle as the (red) squares ordered clockwise starting from the zero point. From the above graphical representation, given in Figure 1, it can easily be seen that the decimal part of the timestamp of the fact \( Q_1^1 \) is smaller than the decimal of the timestamp of the fact \( Q_2^1 \), while the decimal part of the timestamps of the facts \( Q_i^1 \) and \( Q_i^2 \) are equal. The exact points where the classes are placed on the circle are not important, only their relative positions matter. As an example, the circle-configuration of configuration \( \{M \@_{3.01}, R \@_{3.11}, P \@_{4.12}, Time \@_{11.12}, Q \@_{12.58}, S \@_{14}\} \) for \( D_{\text{max}} = 3 \) consists of the \( \delta \)-configuration

\[
\Delta_{S_1} = (\{M, R\}, 1, \{P\}, \infty, \{Time\}, 1, \{Q\}, 2, \{S\} )
\]

and the unit circle

\[
[\{S\}_Z, \{M\}, \{R\}, \{P, Time\}, \{Q\}],
\]

as illustrated in Figure 2.

Notice that, although the graphical representation of the circle-configuration is very illustrative, a circle-configuration is given as a pair of sequences containing a finite number of symbols. Although these sequences do not contain any real numbers, they provide enough information related to satisfaction of time constraints, which is necessary e.g. for rule application. Circle-configurations are, hence, an elegant representation of configurations, considering that timestamps range over dense, real time domain and that there is no upper bound on the values of timestamps.

When compared to the equivalence relation between configurations (Definition 7), circle-configurations contain an additional bit of information. While for the equivalence relation only relative differences between concrete values of timestamps of facts are important, because of the zero point on the unit circle, circle-configurations may differentiate configurations based on the decimal part of their timestamps. For example, configurations \( \{Time \@_{1.12}, Q \@_{1.54}, S \@_{2.4}\} \) and \( \{Time \@_{1.12}, Q \@_{1.66}, S \@_{2.52}\} \) are equivalent, but have different unit circles, related only to the placement of facts at the zero point.

In [18] we have shown how the notion of circle-configurations corresponds to equivalence relation between configurations. In particular, configurations corresponding...
- Time in the zero point and not in the last class in the unit circle, where $n \geq 0$:

Rule 0:

- Time alone and not in the zero point nor in the last class in the unit circle:

Rule 1:

- Time not alone and not in the zero point nor in the last class in the unit circle:

Rule 2:

- Time not alone and in the last class in the unit circle which may be at the zero point:

Rule 3:

Fig. 3: Rewrite Rules for Time Advancement using Circle-Configurations.

to the same circle-configuration are equivalent. We are, therefore, able to say that a circle-configuration $\langle \Delta, U \rangle$ corresponding to a configuration $S$ satisfies a constraint $c$ if the configuration $S$ satisfies constraint $c$. We also say that a rule is applicable to a circle-configuration if that rule is applicable to the corresponding configuration. Furthermore, we say that a circle-configuration is critical iff it is the circle-configuration of a critical configuration. Analogously, we say that a circle-configuration is a goal circle-configuration iff it is the circle-configuration of a goal configuration.

In [18] we show in detail how both instantaneous rules and the time advancement over circle-configurations are compiled and applied (for more details see [18, Section 4.2]). For an instantaneous rule $r$, we write $[r]$ for the corresponding rewrite rule over circle-configurations.

Time advancement rule Tick is represented with a set of Next rules, shown in Figure 3 and Figure 4. For a given circle-configuration, exactly one of the 8 Next rules applies, depending on the position of the fact Time on the unit circle $U$ with respect to the remaining facts. For example, if the fact Time is alone on the unit circle (and not at the zero point, nor in the last class), time advancement is modelled by placing Time in the next class (clock-wise), see Rule 1. If we want to advance time from a circle-configuration where Time is in a class on a unit circle together with other facts (and not at the zero point, nor in the last class), we would place Time alone on the unit circle, at any point just before the next class (clock-wise) on the unit circle, see Rule 2. Cases when Time is in the last class, in addition to changes in the unit circle, require updating of the $\delta$-configuration of the resulting circle-configuration, see Figure 4.

Since, application of a Next rule changes the placement of the fact Time on the unit circle w.r.t. remaining facts, the enabling and the resulting circle-configurations are
- Time alone and in the last class in unit circle - Case 1: $m > 0, k \geq 0, n \geq 0$ and $\delta_1 > 1$:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, (\text{Time}) \cup P_1, \ldots, P_k)
\]

Rule 4:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, (\text{Time}) \cup P_1, \ldots, P_k)
\]

- Time alone and in the last class in unit circle - Case 2: $m > 0, k \geq 1$ and $n \geq 0$:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, (\text{Time}) \cup P_1, \ldots, P_k)
\]

Rule 5:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}, Q_1, \ldots, Q_n), 1, (\text{Time}) \cup P_1, \ldots, P_k)
\]

- Time alone and in the last class in unit circle - Case 3: $k > 0$ and $\gamma_{n+1}$ is the truncated time of $\delta_{n+1}$ + 1:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}), \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \gamma_{n+1}, (\text{Time}), \delta_1, P_1, \ldots, P_k)
\]

Rule 6:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}), \delta_1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \gamma_{n+1}, (\text{Time}), \delta_1, P_1, \ldots, P_k)
\]

- Time alone and in the last class in unit circle - Case 4: $k \geq 1$ and $\gamma_{n+1}$ is the truncated time of $\delta_{n+1}$ + 1:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}), 1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \gamma_{n+1}, (\text{Time}), 1, P_1, \ldots, P_k)
\]

Rule 7:

\[
\Delta = (\ldots, P_{-1}, \delta_{-1}, (\text{Time}), 1, P_1, \ldots, P_k) \quad \Delta' = (\ldots, P_{-1}, \gamma_{n+1}, (\text{Time}), 1, P_1, \ldots, P_k)
\]

Fig. 4: (Cont.) Rewrite Rules for Time Advancement using Circle-Configurations.

different. Moreover, they represent configurations that may not not be equivalent. In fact resulting configuration is either equivalent to the enabling configuration or is its immediate successor.

Correspondence to immediate successors refines our previous result [18] Lemma 1, stating that to a single \textit{Tick} rule corresponds a sequence of \textit{Next} rules, and, vice versa, a sequence of \textit{Next} rules represents a single \textit{Tick} rule for an adequately chosen value $\varepsilon$ of time advancement. Here, we show how \textit{Next} rule relates to \textit{Tick$\downarrow$} rule.

**Proposition 3.** Let $T$ be an MSR with dense time, $GS$ a goal, $CS$ a critical configuration specification and $S_0$ an initial configuration. Let $D_{\max}$ be an upper bound on the numeric values appearing in $T$, $GS$, $CS$ and $S_0$, and consider immediate successors of configurations w.r.t. the set of constraints from $C_{D_{\max}}$. If $A_1 \xrightarrow{\text{Next}} A_2$ then $S_1 \xrightarrow{\text{Tick$\downarrow$}} S_2$, or $S_1 \equiv S_2$ (in case Next is Rule 0, for $n = 0$, Figure 3 or Rule 4, for $n = 0$, Figure 4).

If $S_1 \xrightarrow{\text{Tick$\downarrow$}} S_2$ then $A_1 \xrightarrow{\text{Next$\uparrow$}} A_2$, $n \in \{1, 2, 3\}$.

**Proof.** Both circle-configurations (i.e., equivalence of configurations) and immediate successor configurations are defined w.r.t. an upper bound $D_{\max}$. We set the value of $D_{\max}$ to be an upper bound on numeric values in $T$, $CS$ and $S_0$ and refer to the same bound $D_{\max}$ in both cases. Let $A_1$ and $A_2$ be the circle-configurations of the configurations $S_1$ and $S_2$, respectively.
Notice that, as per Definition 8, facts in the same class on the unit circle satisfy some constraint of the form \( T_1 = T_2 \pm D \), while facts placed in different classes on the unit circle satisfy some constraint of the form \( T_1 < T_2 \pm D \).

Let \( A_1 \xrightarrow{\text{Next}} A_2 \). Then, as illustrated in Figure 5 and Figure 6, application of any of the 8 \textit{Next} rules, changes the placement of the fact \textit{Time} of the unit circle from one class to another.

There are two possibilities. In one case fact \textit{Time} is moved from a class containing some fact \( F \) to a new class (see Rules 0, 2, 3). In the other case there exists some fact \( F \) in \( S_1 \) such that \textit{Time} and \( F \) are in different classes in \( S_1 \), but in the same class in \( S_2 \) (see Rules 1, 4-7). Configurations \( S_1 \) and \( S_2 \) do not satisfy the same constraints referring to facts \textit{Time} and \( F \), except in the two cases shown below:

In the case shown to the left (Figure 5, Rule 0, for \( n = 0 \)) fact \textit{Time} is the only fact placed at zero point, while in the case shown to the right (Figure 6, Rule 4, for \( n = 0 \)) there are no facts at the zero point and \textit{Time} is alone in the last class of the unit circle. Only in this two cases configurations are equivalent, \( S_1 \equiv S_2 \).

Moreover, fact \textit{Time} is placed clock-wise, either to a position immediately following its previous position, but before any existing class, or it is places exactly to the first class clock-wise. This ensures that there are no ”intermediate” configurations, i.e. that \( S_2 \) is an immediate successor of \( S_1 \), i.e., \( S_1 \xrightarrow{\text{Tick}_{1S}} S_2 \), except in above two cases.

Conversely, if \( S_2 \) is an immediate successor of \( S_1 \), then \( S_1 \) is transformed into \( S_2 \) by means of a \textit{Tick}_{1S} rule. Then, when representing this time advancement with circle-configurations, the placement of the facts different from \textit{Time} on the unit circle of \( A_1 \) does not change. At the same time, the change in placement of the fact \textit{Time} on the unit circle should be such to satisfy the condition of immediate successor configuration w.r.t the corresponding configurations. Figure 5 and Figure 6 illustrate exactly such change in the placement of \textit{Time} on the unit circle, updating the \( \delta \)-configuration as well, when necessary. The change in placement of the fact \textit{Time} on the unit circle represents a minimal (or the exact) time advancement such that some constraint is no longer satisfied. Above two exceptions, related to the placement of the fact \textit{Time} at the zero point, require 2 or 3 \textit{Next} rules, as shown below:

Intermediate circle-configurations correspond to the configurations equivalent to the first one, but not to the final one. \( \square \)

The above result ensures that the representation of time advancement on circle-configurations using \textit{Next} rules is sound and complete. To \textit{Next} rules correspond \textit{Tick}_{1S} rules, and conversely, any \textit{Tick}_{2} rule can be decomposed into a finite number of \textit{Tick}_{2} rules (see Remark 1), each of which corresponds to one, two or three \textit{Next} rules.

We have considered traces over circle-configurations and showed that obtained traces over circle-configurations are a sound and complete representation of the set of
traces over concrete configurations with dense time. Notice that circle-configurations are symbolic form, containing only untimed facts, a few auxiliary symbols and a bounded number of natural numbers. The are no real numbers included, and yet there is enough information for the sound and faithful representation of timed systems with dense time. This means that we can search for solutions of some problems symbolically, that is, without writing down the explicit values of the timestamps, i.e., the real numbers, in a trace.

In [18] we investigated reachability problem which did not involve critical configurations. The notion of a non-critical trace in a timed MSR with dense time has not been investigated yet. Since we now address the non-critical reachability problem which involves non-critical traces, for our complexity results for timed MSR with dense time, we need to show that searching for traces in a symbolic form, using circle-configurations, is sound and complete also with respect to compliance, i.e., preserves non-critical traces.

The notion of non-critical traces over circle-configurations is not as complicated and delicate as the notion of a non-critical traces over configurations in systems with dense time, given in Definition 4. Recall that the Tick rule can be instantiated for any non-negative real value \( \varepsilon \), denoting an arbitrary advancement of time, which can cause “skipping” over critical configurations. Such a phenomena does not appear in traces over circle-configurations where Next rules are used for time advancement. Following Proposition 2 and Proposition 3, there is no issue of “skipping” over critical circle-configurations with the time advancement Next. When a Next rule is applied, the configuration corresponding to the resulting circle-configuration is an immediate successor of the configuration corresponding to the enabling configuration, or equivalent to it. That is, each of 8 Next rules corresponds to a time advancement that is just enough, or exactly enough, so that some time constraint involving the global time is no longer satisfied. In such a way, a single Next rule models either the minimal or the exact advancement of time for which the equivalence class changes. Since there is no “skipping” over circle-configurations, there is no need for decomposition of time advancements Next, as is the case with the Tick rule. Hence, the related notion of compliance, i.e., non-critical traces, is straightforward.

**Definition 9.** Let \( T \) be a timed MSR with dense time and \( CS \) a critical configuration specification. A trace over corresponding circle-configurations is non-critical if it does not contain any critical circle-configuration.

Recall that the notion of a non-critical trace in timed MSR with dense time potentially involves checking compliance through an infinite number of traces. Fortunately, this is not the case for non-critical traces over circle-configurations. Since there is no “skipping” over circle-configurations when using Next rules, there is no need for decomposition of time advancements Next, as is the case with the Tick rule. Smaller advancements of time would have either the exact same effect or no effect on a corresponding equivalence class. On the other hand, larger advancements of time are modelled by a sequence of several Next rules. This is essential for the complexity of the problems involving non-critical traces, and we, therefore, rely on non-critical traces over circle-configurations when searching for the solutions of our problems involving timed MSR with dense time.
The following proposition states that such a bisimulation is sound and complete w.r.t. application of rules and non-critical traces.

**Proposition 4.** Given any timed MSR $T$ with dense time, a goal $G$, a critical configuration specification $CS$ and an initial configuration $S_0$, any non-critical trace starting from the given initial configuration $S_0$ to a goal configuration can be conceived as a non-critical trace over circle-configurations, starting from initial circle-configuration $A_0$ and reaching a goal circle-configuration.

**Proof.** In our previous work [18, Theorem 2] we have shown a related bisimulation result for the reachability problem. Here we need to also address critical configurations. In particular, we must check time advancements more carefully in order to provide non-critical traces.

To the given set of instantaneous rules of timed MSR with dense time $T$, $r$, correspond the rules over circle-configurations, so that

$$\tilde{R} = \{ [r] : r \in R \} \cup \text{Next}$$

is the set of rules over circle-configurations. Let $A_0$ be the circle-configuration of $S_0$.

In [18, Theorem 2] we have shown that the equivalence among configurations is well defined with respect to application of rules. Namely, we have shown that for any instantaneous rule $r$, it is the case that $S_1 \rightarrow_r S_2$ if and only if $A_1 \rightarrow_r A_2$, that is:

$$S_1 \rightarrow_r S_2 \quad \iff \quad A_1 \rightarrow_r A_2$$

where $A_1$ and $A_2$ are circle-configurations of the configurations $S_1$ and $S_2$, respectively. Also, it is the case that $S_1 \rightarrow_{Tick} S_2$ if and only if $A_1 \rightarrow_{Next} A_2$, that is:

$$S_1 \rightarrow_{Tick} S_2 \quad \iff \quad A_1 \rightarrow_{Next} A_2$$

Again, $A_1$ and $A_2$ are circle-configurations of the configurations $S_1$ and $S_2$, respectively. Notice that to each $Tick$ rule in the trace over configurations corresponds a (possibly empty) sequence of $Next$ rules in the matching trace over circle-configurations.

Using induction on the length of a subtrace we can easily show that any trace of a timed MSR can be represented as a trace over corresponding circle-configurations, and vice versa, as shown below:

$$P : \quad S_j = S_0 \rightarrow_{r_1} \ldots \rightarrow_{r_{i-1}} S_{i-1} \rightarrow_{r_i} S_i \rightarrow_{r_{i+1}} \ldots \rightarrow_{r_{l-1}} S_l$$

$$P' : \quad A_j = A_0 \rightarrow_{r'_1} \ldots \rightarrow_{r'_{i-1}} A_{i-1} \rightarrow_{r'_i} A_i \rightarrow_{r'_{i+1}} \ldots \rightarrow_{r'_{l-1}} A_l$$

where $r'_i$ is either the instantaneous rule $[r_i]$ over circle-configurations, one or more $Next$ rules as given in Figures 3 and 4, or an empty rule.
We can easily conclude that bisimulation preserves goals. Since $A_i$ is the circle-configuration of $S_i$, it immediately follows that $S_i$ is a goal configuration iff $S_i$ is a goal circle-configuration.

It remains to show that bisimulation preserves non-critical traces. For that purpose we decompose multiple $\text{Next}$ rules in $\mathcal{P}'$. As per Proposition 3 the following correspondences for one or none applications of $\text{Next}$ rules holds:

\[
\begin{align*}
S\rightarrow_{\text{Tick}} S & \quad S \equiv S' \quad S \equiv S' \\
A \rightarrow_{\text{Next}} A' & \quad A \rightarrow_{\text{Next}} A' \quad A \rightarrow_{\text{Next}} A' = A
\end{align*}
\]

We can hence consider corresponding traces $\mathcal{P}$ and $\mathcal{P}'$ as:

\[
\begin{align*}
\mathcal{P} : & \quad S_0 \rightarrow_{r_1} \cdots \rightarrow_{r_{i-1}} S_i \rightarrow_{r_i} S_{i+1} \rightarrow_{r_{i+1}} \cdots \rightarrow_{r_{n-1}} S_n \\
\mathcal{P}' : & \quad A_0 \rightarrow_{r'_1} \cdots \rightarrow_{r'_{i-1}} A_i \rightarrow_{r'_i} A_{i+1} \rightarrow_{r'_{i+1}} \cdots \rightarrow_{r'_{n-1}} A_n
\end{align*}
\]

where $r'_i$ is either the instantaneous rule $\lbrack r_i \rbrack$ over circle-configurations, one $\text{Next}$ rule as given in Figures 3 and 4, or an empty rule, and all $r_i \text{Tick}$ rules are either $\text{Tick}_{IS}$ rules or $\text{Tick}$ rules for which enabling and resulting configurations are equivalent.

If the trace $\mathcal{P}$ is non-critical, all configurations $S_i$ are not critical. Recall that a circle-configuration is critical iff the corresponding configuration is critical. Hence the corresponding circle-configurations $A_i$ are not critical as well. Then the above trace $\mathcal{P}'$ contains no critical circle-configuration and is therefore non-critical (Definition 9).

For the other direction, assume the trace $\mathcal{P}'$ is non-critical. Then all circle-configurations $A_i$ are not critical, and hence configurations $S_i$ are not critical. As per Definition 4 we must consider decompositions of $\text{Tick}$ rules in $\mathcal{P}$.

In the case $S_i \rightarrow_{\text{Tick}} S_{i+1}$ and $S_i \equiv S_{i+1}$, following Remark 1 such decompositions do not contain critical configurations since $S_i$ and $S_{i+1}$ are not critical.

In the other case, $S_i \rightarrow_{\text{Tick}_{IS}} S_{i+1}$. Then, from the Proposition 2 we can conclude that there are no critical configurations $S'$ such that $S_i \rightarrow_{\text{Tick}} S' \rightarrow_{\text{Tick}} S_{i+1}$, $\forall i$.

\[\square\]

### 3.2 PSPACE-Completeness of Non-critical Reachability Problem

PSPACE-hardness of non-critical reachability problem can be inferred from our previous work [23] and [19] by considering non-critical reachability problem with no critical configurations.

**Proposition 5.** The non-critical reachability problem timed MSR $A$ with dense time is PSPACE-hard.

For non-critical reachability problem we need to construct a non-critical trace from the given initial configuration to a goal configuration. As per Proposition 5 instead of non-critical traces over configurations of a given timed MSR with dense time, we can consider non-critical traces over circle-configurations.
The following lemma establishes a criteria related to the length of traces, that is an upper bound on the number of different circle-configurations.

**Lemma 1.** Let $T$ be a timed MSR with dense time constructed over a finite alphabet $\Sigma$ with $J$ predicate symbols and $E$ constant and function symbols. Let $\mathcal{CS}$ a critical configuration specification, $\mathcal{GS}$ a goal, $S_0$ be an initial configuration with $m$ facts, $k$ an upper bound on the size of facts and $D_{\text{max}}$ an upper bound on the numeric values appearing in $T$, $\mathcal{CS}$, $\mathcal{GS}$ and $S_0$.

Then the number of different circle-configurations, denoted by $\text{L}(m,k,D_{\text{max}})$, is

$$\text{L}(m,k,D_{\text{max}}) \leq J^m(E + 2mk)^m m(D_{\text{max}} + 2)^{m-1}.$$ 

**Proof.** A circle-configuration consists of a $\delta$-configuration $\Delta$:

$$\Delta = \left\{ \{Q^1_1, \ldots, Q^1_m\}, \delta_1, 2, \{Q^2_1, \ldots, Q^2_m\}, \ldots, \delta_{j-1}, j, \{Q^j_1, \ldots, Q^j_m\} \right\}$$

and unit circle $U$:

$$U = \left\{ \{Q^0_{m_0}, Q^n_{m_0}\}, \{Q^1_1, \ldots, Q^1_m\}, \ldots, \{Q^j_1, \ldots, Q^j_m\} \right\}.$$ 

In each component, $\Delta$ and $U$, there are $m$ facts, therefore there are $m$ slots for predicate names and at most $mk$ slots for constants and function symbols. Constants can be either constants in the initial alphabet $\Sigma$ or names for fresh values (nonces). Following [13] and Definition [7] we need to consider only $2mk$ names for fresh values (nonces). Whenever an action creates some fresh values, instead of new constants that have not yet appeared in the trace, we use nonce names from this fixed set, different from any constants in the enabling configuration. In that way, we are able to simulate an unbounded number of nonces using a set of only $2mk$ nonce names.

For $\delta_{i,i+1}$, only the time differences up to $D_{\text{max}}$ have to be considered together with the symbol $\infty$, and there are at most $m - 1$ slots for time differences $\delta_{i,j}$ in $\Delta$.

Finally, for each $\delta$-configuration, there are at most $m^m$ unit circles as for each fact $F$ we can assign a class, $U(F)$, and there are at most $m$ classes.

Since, as per Lemma [1] there are only $L(m,k,D_{\text{max}})$ different circle-configurations, a non-critical trace $P$ of length greater than $L(m,k,D_{\text{max}})$ necessarily contains the same circle-configuration $C$ twice, that is, there is a loop in the trace. Hence, there is a shorter plan that is the solution to the same non-critical reachability problem. Therefore, we can nondeterministically search for plans of length bounded by $L(m,k,D_{\text{max}})$.

**Theorem 1.** Assume $\Sigma$ a finite alphabet with $J$ predicate symbols and $E$ constant and function symbols, $T$ a MSR with dense time constructed over $\Sigma$, an initial configuration $S_0$ with $m$ facts, $\mathcal{CS}$ a critical configuration specification, $\mathcal{GS}$ a goal, $k$ an upper-bound on the size of facts, and $D_{\text{max}}$ an upper-bound on the numeric values in $S_0$, $T$, $\mathcal{CS}$ and $\mathcal{GS}$. Let functions $N$, $X$ and $G$ run in Turing space bounded by a polynomial in $m$, $k$, $\log_2(D_{\text{max}})$ and return 1, respectively, when a rule in $T$ is applicable to a given circle-configuration, when a circle-configuration is critical with respect to $\mathcal{CS}$, and when a circle-configuration is a goal circle-configuration with respect to $\mathcal{GS}$. 


There is an algorithm that, given an initial configuration $S_0$, decides whether non-critical trace in $T$ from $S_0$ to some goal configuration and the algorithm runs in space bounded by a polynomial in $m, k$ and $\log_2(D_{\max})$.

The polynomial is in fact $\log_2(L(m, k, D_{\max}))$.

Proof. We adapt the non-deterministic algorithm used in [19, Theorem 7.3] in order to obtain non-critical traces. The algorithm accepts whenever there is a non-critical trace which starts from $S_0$ and reaches a goal configuration. We then apply Savitch’s Theorem to determinize this algorithm. That is, we rely on the fact that PSPACE and NPSPACE are the same complexity class [29].

Instead of searching for traces over concrete configurations, for the PSPACE result we rely on the equivalence among configurations and Proposition 4 which enable us to search for non-critical traces over circle-configurations, constructed using the rules $[r]$, for $r \in T$ and the Next rules.

Because of Lemma 1, it suffices to consider traces of size bounded by the number of different circle-configurations, $L(m, k, D_{\max})$ (stored in binary). Recall that

$$L(m, k, D_{\max}) \leq J^m(E + 2mk)^mk^m(D_{\max} + 2)^{(m-1)}.$$

Let $i$ be a natural number such that $0 \leq i \leq L(m, k, D_{\max}) + 1$. The algorithm starts with $i = 0$ and $A_0$ set as the circle-configuration of $S_0$, and iterates the following sequence of operations:

1. If $A_i$ is a critical circle-configuration, i.e., if $X(A_i) = 1$, then return FAIL, otherwise continue;
2. If $A_i$ is a goal circle-configuration, i.e., if $G(A_i) = 1$, then return ACCEPT, otherwise continue;
3. If $i \geq L(m, k, D_{\max})$, then ACCEPT; else continue;
4. Non-deterministically guess an action, $r$, from $T$ applicable to $A_i$, i.e., such an action $r$ that $N(r, A_i) = 1$. If so replace $A_i$ with the circle-configuration $A_{i+1}$ resulting from applying the action $[r]$ to the circle-configuration $A_i$. Otherwise FAIL;
5. Set $i = i + 1$.

We now show that this algorithm runs in polynomial space. The greatest number reached by the counter is $L(m, k, D_{\max})$, which stored in binary encoding takes space $\log(L(m, k, D_{\max}) + 1)$ bounded by:

$$m \log(J) + mk \log(E + 2mk) + m \log m + (m - 1) \log(D_{\max} + 2).$$

Therefore, to store the values of the step-counter, one only needs space that is polynomial in the given inputs.

Also, any circle-configuration, $A_i$, can be stored in space that is polynomial to the given inputs. Namely, $A_i$ is of the form $(\Delta, \mathcal{U})$, with

$$\Delta = \{Q^1_1, \ldots, Q^1_m, \delta_{1,2}, \{Q^2_1, \ldots, Q^2_m\}, \ldots, \delta_{j-1,j}, \{Q^j_1, \ldots, Q^j_m\}\}$$

$$\mathcal{U} = \{(Q^0_1, \ldots, Q^n_{m_0}) \in \mathcal{Z}, \{Q^1_1, \ldots, Q^m_m\}, \ldots, \{Q^j_1, \ldots, Q^j_m\}\}.$$

Values of the truncated time differences, $\delta_{i,j}$, are bounded, so each $\delta$-configuration $\Delta$ can be stored in space $mk + (m - 1)(D_{\max} + 2)$. Each unit circle $\mathcal{U}$ contains $m$ facts,
so it can be stored in space $mk + (m - 1)$, using a symbol for separating classes at most $(m - 1)$ times. Hence, each circle-configuration can be stored in space that is polynomially bounded with respect to the inputs.

Finally, in step 4, the algorithm needs to store the action $r$. This is done by remembering two circle-configurations. Moving from one circle-configuration to another is achieved by updating the facts, updating the positions of facts and the corresponding truncated time differences. Hence, step 3 can be performed in space polynomial to $m, k, \log_2(D_{max})$ and the sizes of $\mathcal{X}, \mathcal{N}$ and $\mathcal{G}$. Recall that functions $\mathcal{X}, \mathcal{N}$ and $\mathcal{G}$ run in space polynomial to the inputs.

\[ \square \]

**Corollary 1.** The non-critical reachability problem for balanced timed MSR with dense time is PSPACE-complete when assuming a bound on the size of facts.

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