A Deformed Matrix Model and the Black Hole Background in Two-Dimensional String Theory

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Abstract

We investigate how the exact 2D black-hole solution for the critical string theory should be described, at least perturbatively with respect to the inverse mass of the black hole, within the framework of matrix model. In particular, we propose a working hypothesis on the basis of which we can present plausible candidates for the necessary non-local field redefinition of the tachyon field and the deformation of the usual $c = 1$ matrix model with $\mu = 0$. We exhibit some marked difference in the properties of tachyon scattering of the deformed model from those of the usual $c = 1$ model corresponding to tachyon condensation. These results lead to a concrete proposal for the S-matrix for the tachyon black-hole scattering.

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1. Introduction

We now have some sound evidences for the fact that the standard hermitian matrix model in one-dimensional target space can be regarded in the double scaling limit as a solution of critical string theory. Namely, it corresponds to a special solution of the $\beta$ function condition with the condensation of tachyon and dilaton in a flat two-dimensional target space time. One of the great merits of matrix-model formulation of the critical string theory will be that it is in principle defined non-perturbatively with respect to string-coupling constant. Furthermore, it is really remarkable that $W_\infty$ symmetry structure associated with the ground ring in the CFT approach emerges in a simple and natural way associated with an algebraic structure of the special inverted oscillator hamiltonian of the matrix model.

There has been, however, a basic question remaining to be understood in the matrix model approach to critical string theory. In the CFT approach, one has an exact classical solution describing a black-hole solution to the critical string theory which is different from a solution with tachyon condensation mentioned above. In the matrix model approach, however, we have not established yet whether and how, if any, the black hole background can be included in the framework. Witten has proposed that the $c = 1$ matrix model should be interpreted as an extreme limit of the black hole in the sense that the black hole horizon recedes to infinity. This interpretation is indeed natural at least in the limit of vanishing tachyon condensation, or in terms of the 2D gravity picture, zero cosmological constant. But it raises a question about what is then the non-extreme black hole solution with finite mass. If matrix models in general were to be trulul representations of the solutions of critical string theory as we hope, there should exit a matrix model corresponding to the non-extremal black hole, since it must reproduce the genus expansion in the weak coupling regime and the black hole background is a perfectly good exact solution at zero genus.

The purpose of this paper is to present some of our recent efforts towards the identification of a matrix model describing the black hole background. Although our

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* For a very preliminary report, see [5] which contains some of the initial materials for the present work.
results are not yet completely conclusive, we believe that our various observations already contain enough materials which help to clarify the issues and to establish the foundations for further investigations.

To make the paper reasonably self-contained and to explain our viewpoint, we begin the next section with a brief review on the spacetime interpretation of the usual $c = 1$ matrix model and the properties of the black hole solution. We emphasize a trivial but crucial relationship between the mass of a black hole and the string coupling constant. Based on this review, we propose a working hypothesis in section 3. The hypothesis consists of two key ingredients, namely a non-local field redefinition of the tachyon field and a deformation from the usual $c = 1$ matrix model at $\mu = 0$. In section 4, we then present our candidate for the non-local field redefinition. In section 5, we discuss a plausible candidate for the deformed matrix model. The model has some peculiar properties which are markedly different from the usual $c = 1$ model and deserves for further study independently of the problem of black hole. We derive on-shell scattering amplitudes for the deformed model in the tree approximation. In the final section, we discuss a candidate for $S$-matrix elements for black hole and tachyon scattering in weak coupling perturbation theory and conclude by indicating the limitation of the present work and future perspective.

2. Space-Time Interpretation of the 1D Matrix Model and the Black-Hole Solution

2.1 $c = 1$ matrix model and its spacetime interpretation

Why and how a one-dimensional matrix model embodies string theory with 2-dimensional target space is very mysterious and still remains to be clarified. But, the reason for a two dimensional target space to appear is not difficult to see. Let us start from the standard double scaled Lagrangian for an hermitian matrix $M(t)$ in one dimension,

$$L(M) = \text{Tr}(\frac{1}{2}\{\dot{M}^2 - M^2\}). \quad (2.1)$$

A convenient description of this system is in terms of a collective field which is essentially a density function, $\rho(x, t) = \text{Tr}\delta(x - M(t))$ of the eigenvalues of $M$. The hamiltonian for

\text{footnote} II There are also related works [6][7][8][9][10] from which we however differ at various points.
the collective field $P_{\rho}$ is given by

\[
H = - \int dx \left[ \frac{1}{2} \partial_x P_{\rho} \partial_x P_{\rho} + \frac{1}{6} \pi^2 \rho^3 + (-\frac{1}{2} x^2 + \mu) \rho \right] \quad (2.2)
\]

\[
\frac{1}{2\pi} \int dx \int_{\alpha_+}^{\alpha_-} d\phi \phi(p, x), \quad (2.3)
\]

where $h(p, x) = (p^2 - x^2)/2 + \mu$ is the one-body hamiltonian corresponding to the free fermion picture for the matrix model, and $P_{\rho}$ is the conjugate momentum of the collective field. The parameter $-\mu$ is the fermi energy. The new fields $\alpha_{\pm}$ are combinations $\partial_x P_{\rho} \mp \pi \rho$, satisfying Poisson bracket relations $\{\alpha_+(x), \alpha_-(x')\} = \mp 2\pi \delta'(x - x')$. Polchinski\[13\] showed that in the semi-classical approximation, the variables $\alpha_{\pm}$ can be interpreted as the branches of profile function for the fermi surface in the classical phase space $(p, x)$.

The hamiltonian (2.3) can be regarded as describing a field theory in a two-dimensional target spacetime $(t, x)$. The equation of motion is essentially a Liouville equation which in terms of $\alpha_{\pm}$ reads

\[
\dot{\alpha}_{\pm} = -\partial_x h(\alpha_{\pm}, x). \quad (2.4)
\]

The classical ground state corresponds to a static solution

\[
\alpha_{\pm} = \pm \sqrt{x^2 - 2\mu \theta(x^2 - 2\mu)} \equiv \pm \alpha_0. \quad (2.5)
\]

If we further redefine a new shifted field $\zeta$ and its conjugate $\Pi_\zeta$ by

\[
\alpha_{\pm} = \pm \alpha_0 + \sqrt{\pi} \left( \frac{dx}{d\sigma} \right)^{-1} (\Pi_\zeta \mp \partial_\sigma \zeta)
\]

and a parametrized coordinate $\sigma$ by $x = \sqrt{2\mu} \cosh \sigma$ satisfying $dx/d\sigma = \alpha_0$, the hamiltonian (2.3) takes a more familiar looking form,

\[
H = \int_0^\infty d\sigma \left[ \frac{1}{2} (\Pi_\zeta^2 + (\partial_\sigma \zeta)^2) + \frac{\sqrt{\pi}}{12} \left( \frac{dx}{d\sigma} \right)^{-2} \{ (\Pi_\zeta - \partial_\sigma \zeta)^3 - (\Pi_\zeta + \partial_\sigma \zeta)^3 \} \right] \quad (2.7)
\]

where we dropped a c-number contribution. The boundary condition for $\zeta$ at $\sigma = 0$ is known to be the Dirichlet condition $\zeta(t, 0) = 0$. Thus, the system is interpreted as a local field theory for a massless scalar field $\zeta$ with cubic interaction terms, whose strength is spatially varying, being proportional to

\[
\left( \frac{dx}{d\sigma} \right)^{-2} = \frac{1}{2\mu \sinh^2 \sigma}. \quad (2.8)
\]
Here we emphasize that the emergence of linearized massless field is actually not a consequence of a specific form of the potential $V(M)$ of the matrix model. The massless nature of $\zeta$ is universal for arbitrary potential provided ground state solution is static. Only the coupling function depends on the form of potential.

The spatial dependence of coupling strength and the massless nature of the scalar field suggest to identify $\zeta$ to the massless tachyon field in the so-called linear dilaton vacuum which is an exact classical solution of 2D critical string theory. The linear dilaton vacuum corresponds to the classical background fields

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad (2.9)$$

$$\Phi = 2\sqrt{2}\phi, \quad (2.10)$$

$$T(x) = 0, \quad (2.11)$$

for the world sheet action $\star^\star$

$$S = \frac{1}{8\pi} \int d^2\xi (G_{\mu\nu}\partial X^\mu \partial X^\nu - \sqrt{2}R^{(2)}\Phi(X) + 2T(X)) \quad (2.12)$$

where we used a two-dimensional notation for the coordinates $X^\mu = (t, \phi)$ of the target Minkowski spacetime. The physical spectrum around this background consists of massless tachyon which is the only propagating degree and additional infinite number of “discrete states” with discrete imaginary momenta and energies as given by $ip_\phi = -\sqrt{2}(1 - j), \ p_t = -i\sqrt{2}m$ with $j = 1/2, 1, 3/2, 2, \ldots$ and $m = -j, -j + 1, \ldots, j$. It is well known that the discrete states generate an infinite symmetry algebra $W_\infty$.

The linearized equation for the tachyon fluctuation is the zero mode part of the Virasoro condition

$$L_0 T (= \bar{L}_0 T) = T, \quad (2.13)$$

$$L_0 = \frac{1}{2}(\partial_t^2 - \partial_\phi^2) - \sqrt{2}\phi. \quad (2.14)$$

The redefined tachyon field $\tilde{T} \equiv e^{\sqrt{2}\phi}T$ satisfies the massless Klein-Gordon equation. On the other hand, the coupling strength of strings is

$$g_{st} = e^{-\phi/2} = e^{-\sqrt{2}\phi}. \quad (2.15)$$

$\star\star$ We use the unit $\alpha' = 2$ which seems to be more-or-less standard in the literature of conformal field theory.
Thus, at least asymptotically, we can make following identification of the coordinates and fields,

\begin{align}
\sigma & \leftrightarrow \phi / \sqrt{2}, \\
t_0 & \leftrightarrow t / \sqrt{2}, \\
\zeta(t_0, \sigma) & \leftrightarrow \tilde{T}(t, \phi).
\end{align}

Here and what follows we always denote \( t_0 \) as the time coordinate of the matrix model. Under this identification, the appearence of a scaling parameter \( \mu \) can be naturally explained in the framework of critical string theory by assuming the presence of a tachyon condensation of the form

\[ T(x) \sim \mu e^{-\sqrt{2} \phi}, \]

which corresponds to a static solution of the linearized tachyon equation. (More precisely, one has to take account of the degeneracy of the static solutions as discussed in ref. \[16\].) In particular, that the string coupling constant has a constant factor \( g_{st} \propto \mu^{-1} \) is a direct consequence of the scaling relation with respect to a constant shift of the coordinate \( \phi \).

Now under the dictionary \(2.16\) and \(2.17\) for the coordinates, the correspondence of the momentum and energy is given by

\begin{align}
-ip_\phi & = -\sqrt{2} + \frac{1}{\sqrt{2}} ip_\sigma, \\
-ip_t & = \frac{1}{\sqrt{2}} pt_0.
\end{align}

The values of momentum and energy of the discrete states are then \( ip_\sigma = -2j, ip_0 = -2m \), in terms of matrix model. There indeed exit special operators with precisely these values of momentum and energy,

\[ A_{jm} \equiv (\frac{p + x}{2})^{j + m}(\frac{p - x}{2})^{-j - m}. \]

These are eigenoperators for the single-body hamiltonian \( h(p, x) \) with purely imaginary eigenvalue \(-2im\) and generate a \( W_\infty \) algebra \([17, 18, 19, 20, 21, 22]\). This implies that the Schrödinger equations and hence the collective field theory is invariant under special time dependent canonical transformation generated by \( e^{-2mt_0} A_{jm} \).
The emergence of the $W_\infty$ structure associated with the discrete states strengthens the above correspondence of matrix model and the critical string theory, since the discrete states are really the remnants of excitation modes of critical strings in higher dimensions. There is, however, a small puzzle here. The linearized equation for the scalar quanta $\zeta$ is valid even for finite $\sigma$, whereas the free massless tachyon equation (2.13) with (2.14) of string theory is only valid in the asymptotic region where the tachyon condensation (2.19) is neglected. When the condensation of tachyon is taken into account, the Virasoro condition is expected to be modified into, at least for sufficiently large $\phi$, \[ L_0(\mu)T \equiv \left[ \frac{1}{2}(\partial_t^2 - \partial_\phi^2) - \sqrt{2}\partial_\phi + \mu e^{-\sqrt{2}\phi} \right]T = T. \] (2.23)

Given this form of the tachyon equation, a very plausible resolution of the puzzle is that the field $\zeta$ is related through a non-local field redefinition of the following form

\[ T(t, \phi) = e^{-\sqrt{2}\phi} \int_0^\infty d\sigma \exp(-2\sqrt{2}\mu e^{-\phi/\sqrt{2}} \cosh \sigma)\gamma(i\partial_{t_0})\partial_\sigma \zeta(t_0, \sigma) \] (2.24)

where $\gamma(i\partial_{t_0}) = \gamma(-i\partial_{t_0})^*$ is an arbitrary weight function to be determined by the requirement of normalization. It is easy to check that this is an intertwining operator for the correspondence

\[ L_0(\mu)T \leftrightarrow \frac{1}{4}(\partial_{t_0}^2 - \partial_\sigma^2)\zeta, \] (2.25)

provided $\partial_\sigma^2 \zeta|_{\sigma=0} = 0$ which is guaranteed for the on-shell linearized solution satisfying the Dirichlet condition. This transformation was first pointed out by Moore and Seiberg\[19\] in order to connect the macroscopic loop operator to the collective field. For our later purpose, it is more convenient to cast it into a Fourier transform form

\[ \tilde{T}(t, \phi) = \int_{-\infty}^\infty dp \tilde{\zeta}(p)\gamma(p)K_{ip}(2\sqrt{\mu}e^{-\phi/\sqrt{2}})e^{-ipt_0} \] (2.26)

where

\[ \zeta(t_0, \sigma) = \int_{-\infty}^\infty \frac{dp}{p} \tilde{\zeta}(p)e^{-ipt_0} \sin p\sigma. \] (2.27)

The effect of this transformation on the S-matrix can be read off from the asymptotic behavior for large $\phi$,

\[ \tilde{T}(t, \phi) \sim \int dp \tilde{\zeta}(p)\gamma(p)e^{-ipt_0}(\Gamma(ip)\mu^{-ip/2}e^{ip\phi/\sqrt{2}} + \Gamma(-ip)\mu^{ip/2}e^{-ip\phi/\sqrt{2}}). \] (2.28)

\[ \uparrow \uparrow \] The result of section 7 of ref. \[23\] in fact suggests that this is an exact linearized equation.
This implies that the on-shell wave functions are in general multiplicatively renormalized by momentum-dependent factor, \( \gamma(p) \Gamma(\pm ip) \mu^{\mp ip/2} \) when we move from the collective field \( \zeta \) to the tachyon field \( \tilde{T} \). Requiring the unitarity of the S-matrix, however, the renormalization factor must be at most a pure phase. Then a natural choice for \( \gamma(p) \) would be \( \gamma(p) = \Gamma(\pm ip)^{-1} \). The continuum calculation of the correlation functions indeed suggests that an S-matrix element continued to the Euclidean region \( p \to i|q| \) take the form

\[
\left( \prod_{i=1}^{N} -\mu^{|q_i|/2} \frac{\Gamma(-|q_i|)}{\Gamma(|q_i|)} \right) A_{\text{coll}}(q_1, q_2, \ldots, q_N) \tag{2.29}
\]

where \( A_{\text{coll}}(q_1, q_2, \ldots, q_N) \) is an S-matrix element of the collective field \( \zeta \) continued to purely imaginary momenta and energies. (See, e. g., refs. [24], [25]). Remarkably enough, the on-shell amplitude \( A_{\text{coll}} \) behaves as polynomials [20] with respect to the absolute values of certain combinations of energies and hence does not exhibit any pole singularity even in the tree approximation. On the other hand, the multiplicative factor can be understood as the Euclidean continuation \( p \to \pm i|q| \) of the renomalization factor appeared in (2.28), \( \mu^{\mp ip/2} \frac{\Gamma(\pm ip)}{\Gamma(\mp ip)} \). It is important to note that although being as pure phases these factors have no physical effect in the Minkowski metric, they carry information of the background tachyon condensate. For the poles at \( |q| = 1, 2, 3, \ldots \), coming from the numerator of the multiplicative factor can be, in terms of the energy of the tachyon \( p_t = in/\sqrt{2}, (n = 1, 2, 3, \ldots) \), interpreted as the values of momenta at which one-particle tachyon wave resonates with the background tachyon condensation. If an incoming one-particle tachyon wave produces \( N - 1 \) outgoing tachyons, resonances are expected to occur when both the momentum and energy conservation laws hold,

\[
i\sqrt{2}p_t = -(r + N - 2), \quad N \geq 2, \quad r = 1, 2, 3, \ldots \tag{2.30}
\]

where \( r \) is the number of insertions of the operator \( e^{-\sqrt{2}\phi} \) corresponding to tachyon condensation and \( -2 \) originates from the vacuum charge carried by the \( \phi \) coordinate. From (2.29) using the energy conservation \( |q_N| = \sum_{i=1}^{N-1} |q_i| \), we see that the residue of the resonace pole at \( |q_N| = r + N - 2 \) is proportional to \( (\prod_{i=1}^{N-1} R_L(q_i))A_{\text{coll}} \) where

\[
R_L(q) = -\mu^{|q|/2} \frac{\Gamma(-|q|)}{\Gamma(|q|)} \tag{2.31}
\]

is nothing but the Euclidean-continued reflection coefficient for the tachyon wave.
2.2 black-hole background

Let us next briefly summarize the relevant properties of the exact black-hole solution of Witten. The solution is described by an $SL(2, R)/O(1, 1)$ (or $SL(2, R)/U(1)$ in the case of Euclidean black hole) gauged WZW model with $k = 9/4$. In view of the role played by the linearized tachyon equation in the usual $c = 1$ case as discussed above, we first focus on the Virasoro condition for tachyon fluctuation which has been discussed in detail by Dijkgraaf, Verlinde and Verlinde [23).

Let us parametrize the $SL(2, R)$ group manifold as

$$g = \begin{pmatrix} a & u \\ -v & v \end{pmatrix}, \quad uv + ab = 1. \quad (2.32)$$

The parameters $a, b$ are redundant because of gauge symmetry $g \rightarrow hgh, h \in O(1, 1)$. In terms of $u, v$, the Virasoro condition takes the form

$$L_0(u, v)T \equiv \frac{1}{k - 2}[(1 - uv)\partial_u \partial_v - \frac{1}{2}(u\partial_u + v\partial_v) - \frac{1}{2k}(u\partial_u - v\partial_v)^2]T = T. \quad (2.33)$$

In fact, the Virasoro operator $L_0(u, v)$ consists of two parts, $L_0 = -\Delta_0 + (u\partial_u - v\partial_v)^2/4$, where $\Delta_0$ is the Casimir operator of $SL(2, R)$. The on-shell tachyon corresponds to the continuous series representation of $SL(2, R)$ which has eigenvalues $\Delta_0 = -\lambda^2 - \frac{1}{4} \quad (\lambda = \text{real})$ and $-i\partial_t = 2i\omega$ with the on-shell condition $\lambda^2 = 9\omega^2$ at $k = 9/4$. When this equation is rewritten in a covariant form with background spacetime metric $G_{\mu\nu}$ and dilaton $\Phi$,

$$L_0 = -\frac{1}{2e\sqrt{G}}\partial^\mu e^\Phi \sqrt{G}G_{\mu\nu}\partial_\nu, \quad \text{we have “exact” (in the sense of $\alpha'$ expansion) expressions for the background fields},$$

$$ds^2 = \frac{k - 2}{2}[dr^2 - \beta^2(r)d\bar{t}^2], \quad (2.34)$$

$$\Phi = \log(\sinh r/\beta(r)) + a, \quad (2.35)$$

$$\beta(r) = 2(\coth^2 \frac{r}{2} - \frac{2}{k})^{-1/2}, \quad (2.36)$$

where the new coordinate $r$ and time $\bar{t}$ are defined by

$$u = \sinh \frac{r}{2}e^{\bar{t}}, \quad (2.37)$$

$$v = -\sinh \frac{r}{2}e^{-\bar{t}}. \quad (2.38)$$
The variables \((r, \bar{t})\) are good coordinates for describing the static exterior region outside event horizon sitting at \(r = 0\). The arbitrary constant \(a\) is related to the ADM mass of the black hole by

\[
M_{\text{bh}} = \sqrt{\frac{2}{k-2}} e^a
\]

as shown in ref. [3]. The expressions (2.34)\(\sim\)(2.36) reduce in the limit \(k \to \infty\) to the solution of a low energy approximation of the \(\beta\)-function condition. It should be kept in mind, however, that in 2D critical string theory there is no nontrivial systematic \(\alpha'\)-expansion, since the string coupling always vanishes in that limit.

The global geometric structure described by the exact metric (2.34) has recently been shown to be free of curvature singularity and consists of an infinite copies of a black-hole type spacetime connected by wormholes at \(uv = 1\). However, it should be noted that there still is a “dilaton singularity” at \(uv = 1\) where the string coupling \(g_{\text{st}} \sim e^{-\Phi/2}\) diverges. In terms of \(u, v, \Phi\) reads

\[
\Phi = \log[4\{(1 - uv)(1 - \frac{1 - uv}{uv} - \frac{2}{k})\}] + a.
\]

The region \(uv > 1\) defines a disjoint region with a naked singularity.

The free parameter \(a\) can be eliminated by making a scale transformation

\[
u \to \sqrt{M}^{-1} u, \quad v \to \sqrt{M}^{-1} v, \quad M \equiv e^a.
\]

which in turn introduces the black-hole mass parameter in a more familiar looking manner. Namely, at a price of eliminating the parameter \(a\), we have to replace \(1 - uv\) by \(M - uv\) in (2.23) and (2.40).

Here we emphasize a fact that since the string coupling is determined by the background dilaton field, we have a simple relation

\[
g_{\text{st}}(r = 0) \propto e^{-a/2} = M^{-1/2}
\]

This is to be contrasted with the dependence of \(g_{\text{st}} \propto \mu^{-1}\) on the cosmological constant in flat spacetime. This observation will play a crucial role in our proposal. It also casts doubts on previous works [7, 8, 9, 10] on the relation of black hole solution and matrix model.
Let us now consider the asymptotic behavior of the Virasoro condition and the dilaton field for $r \to \infty$ using $u \sim e^{r+t}$, $v \sim e^{-t}$.

\[ L_0 \sim \frac{1}{4(k-2)}(\partial_r^2 + \partial_t) + \frac{1}{4k}\partial_t^2, \quad (2.43) \]

\[ \Phi \sim r + a - \log 4. \quad (2.44) \]

Thus, the parameters $r, \bar{t}$ are identified asymptotically with the $\phi$ and $t$ for the linear dilaton background as

\[ \bar{t} \leftrightarrow \sqrt{\frac{1}{2k}}t = \frac{\sqrt{2}}{3}t, \quad (2.45) \]

\[ r \leftrightarrow \sqrt{\frac{2}{k-2}}\phi = 2\sqrt{2}\phi. \quad (2.46) \]

In terms of momentum and energy, the dictionary reads

\[ ip_{\phi} = -\sqrt{2} + i2\sqrt{2}\lambda = -\sqrt{2} + \frac{i}{\sqrt{2}}p_\sigma, \quad (2.47) \]

\[ ip_t = i\frac{2\sqrt{2}}{3}\omega = i\frac{1}{\sqrt{2}}p_\phi. \quad (2.48) \]

We see that there is a one-to-one correspondence of tachyon states between the black-hole and linear dilaton backgrounds.

The discrete states around the black-hole background have been studied by several authors [27, 28, 29]. In the Minkowski metric, the spectrum of the discrete states is isomorphic to that of the linear dilaton background with just the same values of momenta and energies determined by the condition for the discrete states for the latter through the above dictionary. In particular, the first nontrivial discrete state with zero energy ($j = 1, m = 0$ or $ip_{\phi} = -2\sqrt{2}, p_t = 0$) is identified with the operator associated with the mass of black hole, as can be seen from the first correction to the asymptotic behavior of the exact spacetime metric,

\[ ds^2 \sim \frac{k-2}{2}[dr^2 - \frac{4k}{k-2}(1 - \frac{4k}{k-2}e^{-r} + O(e^{-2r}))dt^2]. \quad (2.49) \]

It is crucial to note that the $\phi$ momentum is twice that of the operator corresponding to tachyon condensation. This explains, for example, the difference in the relation between
string coupling and the black hole mass or the fermi energy of the matrix model, \( g_{st} \sim M^{-1/2} \) versus \( \mu^{-1} \).

Finally, before going into our main issue we briefly describe the properties of the solutions of the Virasoro condition. The integral representation of the solution with definite \( \omega \) and \( \lambda \) is

\[
\int_C \frac{dx}{x} x^{-2i\omega} \left( \sqrt{M - uv} + \frac{u}{x} \right)^{-\nu_+} \left( \sqrt{M - uv} - vx \right)^{-\nu_+} \tag{2.50}
\]

with \( \nu_\pm = \frac{1}{2} - i(\lambda \pm \omega) \). Following ref. [23], we denote two independent solutions corresponding to the contours \( C_2 \equiv [-u\sqrt{M - uv}, 0], C_4 \equiv -\infty, \nu^{-1} \sqrt{M - uv} \) as \( y \equiv uv = -\sinh^2 \frac{r}{2} \)

\[
T_{C_2} = U_\omega^\lambda = e^{-2i\omega \bar{t}} F_\omega^\lambda(y), \tag{2.51}
\]

\[
T_{C_4} = V_\omega^\lambda = e^{-2i\omega \bar{t}} F_{-\omega}^\lambda(y), \tag{2.52}
\]

\[
F_\omega^\lambda(y) = (-y)^{-i\omega} B(\nu_+, \nu_-) F(\nu_+, \nu_- 1 - 2i\omega, y). \tag{2.53}
\]

The asymptotic behaviors of the solutions are, apart from an irrelevant common phase factor, for \( r \to 0 \) (horizon):

\[
U_\omega^\lambda \sim \beta(\lambda, \omega) \left( \frac{u}{\sqrt{M}} \right)^{-2i\omega}, \tag{2.54}
\]

\[
V_\omega^\lambda \sim \beta(\lambda, -\omega) \left( -\frac{v}{\sqrt{M}} \right)^{-2i\omega} \tag{2.55}
\]

and for \( r \to \infty \) (null infinities):

\[
F_\omega^\lambda \sim \alpha(\lambda, \omega) (-y)^{-\frac{1}{2} + i\lambda} + \alpha(-\lambda, \omega) (-y)^{-\frac{1}{2} - i\lambda} \tag{2.56}
\]

where

\[
\alpha(\lambda, \omega) = \frac{\Gamma(\nu_+) \Gamma(\bar{\nu}_- - \nu_+)}{\Gamma(\bar{\nu}_-)}, \tag{2.57}
\]

\[
\beta(\lambda, \omega) = B(\nu_+, \bar{\nu}_-). \tag{2.58}
\]

From these behaviors, we see that \( U_\omega^\lambda \) describes a scattering of a wave coming from past null infinity with the black hole, while \( V_\omega^\lambda \) describes a wave emitted by the white hole crossing the past event horizon. In the followings, we will deal with the solution \( U_\omega^\lambda \) since
we are interested in the S-matrix elements of tachyons, incoming from the asymptotic flat region at past null infinity and scattered out to future null infinity, which are the only legitimate observables in string theory. For this solution with on-shell condition \( \omega = 3\lambda (> 0) \), ratios of the coefficients appearing in the above asymptotic behaviours,

\[
R_B(\lambda) = \frac{\alpha(\lambda, \omega)}{\alpha(-\lambda, \omega)},
\]

\[
T_B(\lambda) = \frac{\beta(\lambda, \omega)}{\alpha(-\lambda, \omega)},
\]

are reflection and absorption coefficients, respectively, for the scattering of tachyon wave with a static black hole, and satisfy the unitarity relation

\[
|R_B|^2 + \frac{\omega}{\lambda}|T_B|^2 = 1.
\]

A prefactor \( \frac{\omega}{\lambda} \) in the lhs is necessary to account for the change of velocities of tachyon at the horizon and at the null infinity. This is caused by the violation of equivalence principle due to the presence of dilaton background.

3. Working Hypothesis

What lessons do we have to learn from the review of the spacetime interpretation of the \( c = 1 \) matrix model in attempting to extend that to black-hole background? First it is clear that there is no way of detecting nontrivial background dependence by simply looking at the linearized field equation of the scalar collective field \( \zeta \). We have seen that the linearized equation around an arbitrary static solution of the matrix model is reduced to a free massless Klein-Gordon equation by choosing the time-of-flight coordinate, irrespective of the form of the Hamiltonian of the matrix model. The background dependence of critical string theory, which is nontrivial even in two dimensions because of the coupling of dilaton, comes into the scene only after making a non-local field redefinition. The comparison as we made among the S-matrix elements of collective field theory, the asymptotic behavior of the redefined field, and the results on non-bulk correlation functions in the euclidean continuum approach, strongly suggests that the momentum dependent external leg factor of the continuum correlation functions should be interpreted as an indication of the necessity of a non-local field redefinition for the collective field \( \zeta \). We have argued
that although the external leg factor is a pure phase in the Minkowski space and hence is not observable, it encodes an important physical information about the background as manifested as resonance behavior at purely imaginary discrete values of momenta or energies. We note that this proposal is similar to that of Polyakov [14], but is in fact slightly different in that the residue of the resonance pole need not necessarily equal to the free field correlator. A very remarkable point of this interpretation is that the coefficient appearing in the solution of the single-particle problem simultaneously accounts for the resonance poles for the multiparticle amplitudes. As argued by Polyakov, we expect that the codimension 2 pole singularities may only appear in external legs. Presumably, the consistency of this peculiar behavior of the on-shell amplitudes is deeply related with the $W_\infty$ symmetry of the 2D string theory and the matrix model.

Now in the case of the black hole background, no reliable continuum calculation of the scattering amplitudes has been done yet. Naively, one may hope to apply the method of free-field representation of the WZW model in calculating the Euclidean correlation functions as in the same way as being applied to the Liouville model coupled with $c = 1$ matter, making an analytic continuation with respect to the number of insertions of the black-hole mass operator. A free-field representation of the $SL(2,R)/U(1)$ model has indeed been known from the work of Bershadski and Kutasov [28]. However, as far as we know, it is very difficult to reproduce even the two-point function such as the reflection coefficient $R_B$ in this approach. It seems that the perturbation with respect to the black-hole mass is much more dangerous than that of the tachyon condensation. See, however, ref. [30] for a different possibility.

In view of this situation, we would like to propose an ansatz as a working hypothesis that the above mentioned properties of the scattering amplitudes also extend to the case of a black-hole background. Namely, *the hypothesis is that the S-matrix elements are in general factorized products of external leg factors and the amplitudes of a collective field theory.* The external leg factors must be related to the asymptotic behavior of solutions of the linearized tachyon equation as above. On the other hand, the collective field theory should represent an appropriate deformation of the usual $c = 1$ matrix model at the extreme limit $\mu \to 0$ (zero fermi energy) since we are considering the background without tachyon condensation. We emphasize that we consider the nonzero cosmological constant
(fermi energy $\mu$) and the black hole mass $M$ to be two entirely different
 deformations of the same critical theory described by linear dilaton
 background. In this main point, we differ from some of related works\textsuperscript{[7, 8, 9]}. For other related issues, see also \textsuperscript{[31]}.

The deformation that we would like to consider must properly describe the
effect of finite black hole mass. Since the exterior region of black-hole
 where the scattering experiment is done can be regarded as static, we
 are entitled to assume that the deformed matrix model is time-translation
 invariant as in the usual case. It is also reasonable to expect that
 the collective amplitudes are essentially polynomials since all the
codimension 2 resonance poles should be only in the external leg factors.
 A crucial property that we have to maintain is that the string coupling
 is proportional to $M^{-1/2}$ with $M$ being the deformation parameter
 of the deformed matrix model corresponding to the black hole
 mass. We assume that the action of the deformed model is analytic
 with respect to $M$, since it appears analytically in the Virasoro
 condition.

Let us now study the possible resonance poles for the black hole
 background. From the asymptotic behavior \textsuperscript{(2.56)} and the
 associated reflection coefficient \textsuperscript{(2.59)}, we see that
 the positions of the resonance poles are

$$i 4 \lambda = i \frac{4}{3} \omega = i \sqrt{2} p_t = -2, -4, -6, \ldots. \quad (3.1)$$

This contrasts with the case of the usual $c = 1$ model where we have
 poles at all negative integers, \textsuperscript{(2.30)} of the corresponding
 energy. On the other hand, if we consider an amplitude for an
 incoming tachyon with producing $N - 1$ outgoing tachyons, the energy
 and momentum conservation laws are satisfied when the incoming
 tachyon energy obeys

$$i \sqrt{2} p_t = -(2r + N - 2) \quad (3.2)$$

where $r$ now counts the number of insertions of the black hole mass
 operator. The factor 2 multiplied to $r$ comes about because the
 momentum carried by the black hole mass is twice that of the
tachyon condensation. Comparing \textsuperscript{(3.1)} and \textsuperscript{(3.2)},
 we see that our hypothesis is consistent when only even $N$
 are allowed. This puts a stringent requirement
to our hypothesis: the on-shell scattering
 amplitudes of the deformed
 matrix model must
 vanish when $N = \text{odd}$. It is clear that the usual $c = 1$
 matrix model with the purely
 oscillatory potential can never
 conform to this property.
Our task now is firstly to establish the possibility of a non-local field redefinition connecting the Virasoro condition for the black hole background to the free massless Klein-Gordon equation, and secondly to find a deformed matrix model satisfying all of our criteria.

4. Non-Local Field Redefinition

We first deal with the non-local field redefinition. Consider the integral representation (2.50) with the contour $C_2$ which is appropriate for the scattering problem in the exterior region $(u > 0, v < 0)$.

\[
U_\omega^\lambda(u, v) = \int_{C_2} \frac{dx}{x} x^{-2\omega} (\sqrt{M - uv} + \frac{u}{x})^{-\nu} (\sqrt{M - uv - vx})^{-\nu_+}. \tag{4.1}
\]

Since the spectrum of the on-shell solution has a one-to-one correspondence through (2.47) and (2.48) with that of the free Klein-Gordon equation, it is natural to make the following change of integration variable,

\[
(\sqrt{M - uv} + \frac{u}{x})^{-1} (\sqrt{M - uv - vx}) = e^{-4t_0/3}, \tag{4.2}
\]
\[
(\sqrt{M - uv} + \frac{u}{x}) (\sqrt{M - uv - vx}) = e^{-4\sigma}. \tag{4.3}
\]

Then apart from an irrelevant numerical coefficient, the solution takes the form

\[
U_\omega^\lambda = \int_{-\infty}^{\infty} dt_0 \int_{0}^{\infty} d\sigma \delta \left( \frac{ue^{-2t_0/3} + ve^{2t_0/3}}{2} - \sqrt{M} \cosh 2\sigma \right) e^{-4\omega t_0/3} \cos 4\lambda \sigma. \tag{4.4}
\]

It is indeed easy to show that the $\delta$-function kernel here is an intertwining operator between the Virasoro operator (2.33) and the free Klein-Gordon operator for $k = 9/4$. We recognize the form of plane wave of the $\sigma$-derivative of $\zeta$ satisfying the Dirichlet boundary condition in the rhs. We note that, according to the dictionary (2.47), (2.48), (2.20) and (2.21), the correspondence of momentum and energy is

\[
p_\sigma = 4\lambda, \quad p_{t_0} = \frac{4}{3} \omega. \tag{4.5}
\]

Thus the non-local field redefinition now is given by

\[
T(u, v) = \int_{-\infty}^{\infty} dt_0 \int_{0}^{\infty} d\sigma \delta \left( \frac{ue^{-2t_0/3} + ve^{2t_0/3}}{2} - \sqrt{M} \cosh 2\sigma \right) \gamma(i\partial_{t_0}) \partial_\sigma \zeta(t_0, \sigma). \tag{4.6}
\]
where $\gamma(i\partial_{t_0})^* = \gamma(-i\partial_{t_0})$ is an arbitrary weight function to be fixed by normalization condition, as before.

In terms of the Fourier decomposition (2.27), it reads

$$T(u,v) = \int_{-\infty}^{\infty} dp \tilde{\zeta}(p)\gamma(p)U_{\omega(p)}^\lambda(u,v)$$

with $\omega(p) = 3p/2$, $\lambda(p) = p/2$. In particular, the asymptotic behavior for $y \to \infty$ is

$$T(u,v) = \int_{-\infty}^{\infty} dp \tilde{\zeta}(p)\gamma(p) \left[ (-y)^{-\frac{1}{2}+i\lambda(p)}\alpha(\lambda(p),\omega(p)) + (-y)^{-\frac{1}{2}-i\lambda(p)}\alpha(-\lambda(p),\omega(p)) \right]e^{-2i\omega(p)t}.$$  

(4.8)

This shows that an asymptotic wave packet of $\zeta$ field is transformed into a deformed wave packet of the tachyon field. Note that there is no singularity in the weight functions for real $p$.

A remarkable and somewhat puzzling fact in this transformation is that even if $\zeta$ is a plane wave satisfying a perfectly reflecting boundary condition, the transformed wave has both the reflected and absorbed parts. See (2.54) and (2.61). After all, naturalness of the transformation can only be judged when it is combined with the deformed matrix model.

We here remark that similar transformations as (4.4) have been known for sometime. However, we must emphasize that our interpretation is different from other works[6][7][8]. Most importantly, we do not identify the expression $\sqrt{M}\cosh 2\sigma$ with the eigenvalue coordinate of the usual $c=1$ matrix model. It would have implied $M \sim \mu$, which identifies the black hole mass with fermi energy, and contradicts with the relation between the string coupling and the black hole mass, $g_{st}(r=0) \sim M^{-1/2}$. As we will see in the next section, the correct interpretation is that $\sqrt{M}\cosh 2\sigma$ is nothing but the square of the eigenvalue coordinate of a deformed matrix model.

5. Deformed Matrix Model

5.1 Deformed Hamiltonian

In the limit of vanishing black hole mass, the black hole background reduces to the linear-dilaton vacuum. This is a singular limit in the sense that the string coupling diverges,
corresponding to the $c = 1$ matrix model with vanishing 2D cosmological constant $\mu = 0$, or zero fermi energy. Since, according to our hypothesis, the deformation corresponding to non-vanishing black hole mass cannot be described by the usual matrix model, we have to seek for other possible deformations than the one induced by a change of the fermi energy. We therefore assume that the fermi energy is kept exactly at zero, while the hamiltonian itself is modified,

$$h(p, x) \rightarrow h_M(p, x) = \frac{1}{2}(p^2 - x^2) + M\delta h(p, x).$$  \hfill (5.1)

We have here assumed that the deformation is described by a term linear in $M$. If that would not work, we would have to add higher order terms. Fortunately, however, we will have a natural candidate with only the linear term, satisfying our criteria.

The first requirement for $\delta h$ is a scaling property to ensure that the string coupling is proportional to $M^{-1/2}$. The general form of the collective hamiltonian \( (2.3) \) and the field shift around the classical ground state solution imply that the string coupling in general is proportional to \( (\frac{dx}{d\sigma})^{-2} \). Thus, our requirement is satisfied if the deformation operator $\delta h$ has a scaling property $\delta h(p, x) \rightarrow \rho^{-2}\delta h(p, x)$ under a global scale transformation $(p, x) \rightarrow (\rho p, \rho x)$. This leads to

$$\delta h(p, x) = \frac{1}{2x^2} f\left(\frac{p}{x}\right).$$  \hfill (5.2)

To infer the form of an undetermined function $f\left(\frac{p}{x}\right)$, we invoke the following observation on the algebraic property of the double scaled matrix model. As mentioned in section 2, the usual $c = 1$ hamiltonian $h = (p^2 - x^2)/2$ allows a set of the eigenoperators $A_{j,m}$ satisfying the Poisson bracket relation

$$\{h(p, x), A_{j,m}\} = 2m A_{j,m}$$  \hfill (5.3)

and the $W_\infty$ algebra

$$\{A_{j,m}, A_{j',m'}\} = (mj' - m'j)A_{j+j'-1,m+m'}.$$  \hfill (5.4)

The origin of this algebraic structure which is supposed to encode the extended nature of strings can be traced back to the existence of an $SL(2, R)$ algebra consisting of

$$L_1 = \frac{1}{4}(p^2 - x^2) = h(p, x),$$  \hfill (5.5)
The existence of a set of eigenoperators satisfying the $W_{\infty}$ algebra is related to this $SL(2, R)$ structure by

\[ A_{j,m} = L_{j+m} L_{-j-m}, \quad (L_\pm = L_3 \pm L_2) \]  

which close under the Poisson bracket since the Casimir invariant has a fixed value

\[ L_1^2 + L_2^2 - L_3^2 = \frac{3\hbar}{16}. \]  

(Here we include the Planck constant to indicate the effect of operator ordering.)

Since the spectrum of the discrete states of the black hole background is essentially the same as the usual $c = 1$ model in the Minkowski metric, it is natural to require that the deformed model should also share a similar algebraic structure. Fortunately, this is satisfied if the simplest choice $f = 1$ is made. Namely, the $SL(2, R)$ generators are now given by

\[ L_1(M) = \frac{1}{2} \hbar M(p, x) = \frac{1}{4} (p^2 - x^2 + \frac{M}{x^2}), \]  

\[ L_2(M) = -\frac{1}{4} (px + xp), \]  

\[ L_3(M) = \frac{1}{4} (p^2 + x^2 + \frac{M}{x^2}), \]

which satisfy

\[ L_1(M) + L_2(M)^2 - L_3(M)^2 = -\frac{M}{2} + \frac{3\hbar}{16}. \]  

We note that because of different constraint for the Casimir invariant the algebra of eigenoperators is now modified in an $M$-dependent way. Although further implications of this property, especially an understanding of its connection to the ground ring structure of the $SL(2, R)/O(1, 1)$ WZW model, must be left for future investigations, we believe that this already is a strong motivation for adopting this particular model as a serious candidate for our deformed matrix model.
5.2 Properties of the deformed matrix model

Let us proceed to study the properties of the deformed model:

\[ h_M(p,x) = \frac{1}{2}(p^2 - x^2) + \frac{M}{2x^2}. \]  

(5.14)

We assume here that \( M > 0 \) and will consider the case \( M < 0 \) in the final section, separately. The solution of the classical equation of motion with energy \( \epsilon \) is

\[ x^2(t_0) = -\epsilon + \sqrt{M + \epsilon^2 \cosh 2t_0}. \]  

(5.15)

The ground state corresponding to zero fermi energy is obtained by setting \( \epsilon = 0 \) and replacing the time variable \( t_0 \) by the time-flight coordinate \( \sigma \),

\[ x^2 = \sqrt{M \cosh 2\sigma}. \]

We recognize that this is precisely the quantity which appeared in the integral transformation (4.6) whose \( \delta \)-function gives us a relation between the black hole and the matrix model variables, \( x^2 = u e^{-2t_0/3} + v e^{2t_0/3} / 2 \). The coupling function is now determined to be

\[ g(\sigma) \equiv \sqrt{\frac{\pi}{12}} \left( \frac{dx}{d\sigma} \right)^{-2} = \frac{1}{48} \sqrt{\frac{\pi}{M}} \left( \frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 \sigma} \right). \]  

(5.16)

which explains the required relation with the black hole mass and the asymptotic property for large \( \sigma \).

We emphasize here that we are taking a double scaling limit which is different from the usual case. Namely, we fixed the fermi energy exactly at zero and tuned the form of the potential in a particular way. In terms of the original matrix notation, the potential is given by \( V(\Phi) = \text{Tr}(-\frac{1}{2} \Phi^2 + \frac{\tilde{M}}{2} \Phi^2) \). Then the genus zero free energy in the limit of vanishing scaling parameter, \( \tilde{M} \to 0 \), behaves like \( F \sim N^2 \tilde{M} \log \tilde{M}. \) The double scaling limit is thus the limit \( \tilde{M} \to 0, N \to \infty \) with \( M \equiv N^2 \tilde{M} \) being kept fixed. After the usual rescaling, \( x \equiv \sqrt{N} \times \) eigenvalue of \( \Phi \), the system is reduced to the free fermion system with the one-body potential \( -\frac{1}{2}x^2 + \frac{\tilde{M}}{2}x^2 \). Note that in the limit \( \tilde{M} \to 0 \) the potential approaches to the usual inverted harmonic oscillator potential with a repulsive \( \delta \)-function like singularity. Although the critical point itself is the same as in the usual model, apart from the \( \delta \)-function like infinite wall at the origin, the above limit defines a different massive continuum theory.

It seems worthwhile to point out that the \( \delta \)-function like potential at the origin in the limit \( \tilde{M} \to 0 \) conforms to Witten’s discussion [] of the ground ring structure of the
c = 1 Liouville theory with vanishing cosmological constant. According to his arguments on the connection of Liouville theory coupled with c = 1 matter to the matrix model, the negatively dressed (i.e., forbidden in Seiberg criterion) $W_\infty$ operators such as the one corresponding to the black hole mass are concentrated at the origin of the phase space $(p, x)$. This is indeed the case in our deformed model in the limit $M \to 0$, since the states are affected only near at the top of the fermi sea by the $\delta$ function potential in the semi-classical limit. It is an interesting open question whether this similarity can be made more precise. We should also note that this in turn indicates a danger in treating the black hole mass parameter as a perturbation to the model without the deformation term. It is impossible to reproduce the properties of our model with finite $M$ in any finite order of perturbation with respect to the mass parameter $M$.

Once the deformed model is identified as above, it is tempting to extend (4.6) to the unshifted collected field $\rho(t_0, x)$ as

$$T(u, v) = \int_{-\infty}^{\infty} dt_0 \int_{0}^{\infty} dx \delta\left(\frac{ue^{-2t_0/3} + ve^{2t_0/3}}{2} - x^2\right)\gamma(i\partial_{t_0})\rho(t_0, x)$$

A possible divergence at $x^2 \to \infty$ is eliminated by choosing $\gamma(0) = 0$ and only other possible source of singularity will be the other end point $x = 0$. However, due to the infinite repulsive wall at $x = 0$, we are guaranteed that $\rho(t_0, 0) = 0$. Hence, we do not expect any singularity in exact theory for (5.17). The physical meaning of this is not, however, clear to us at present.

Let us finally mention a few related previous works. The potential with a $1/x^2$ term has been discussed by Avan and one of the authors (A. J.)\cite{Avan} as an example of an integrable collective field theory other than the standard inverted harmonic oscillator model. There, the $SL(2, R)$ structure has also been noticed. We should also mention that Z. Yang\cite{Yang} has proposed that the black hole background should be described by the potential $-\frac{1}{2}(x - \frac{M}{2})^2$ at vanishing fermi energy. Superficially this might look similar to the present proposal, but is actually quite different. Firstly, Yang identified the tachyon field directly with a shifted collective field without any field redefinition and used it to exhibit a black hole type metric. This cannot be justified since the linearized equation is then always recasted in the free Klein-Gordon form by a change of coordinate. The metric is only an artifact of the choice of coordinates and does not take the crucial dilaton coupling into account.
In fact, because of two-dimensionality, the metric is quite arbitrary for massless Klein-Gordon equation. Secondly, the identification of the black hole mass in his proposal is in contradiction with the relation $g_{st} \sim M^{-1/2}$. Thirdly, the system is singular from the beginning because of the infinite attractive potential at the origin and the fermi energy being assumed exactly on top of the potential. In our case, the system is completely well defined as long as $M > 0$. In the final section, we will discuss the singular case $M < 0$ as identified with the background with naked singularity.

5.3 Tree level scattering of $\zeta$ quanta

We now proceed to study the scattering of $\zeta$ quanta and to see whether the amplitudes vanish for odd number of particles. We will stay within the tree approximation in the present paper. The most convenient way of deriving the scattering amplitude in the tree approximation is to use the classical fermi liquid picture \cite{13, 34}. In this picture, the $\zeta$ field is connected the profile functions $\alpha_{\pm}(t_0, x)$ satisfying the equation of motion (2.4). In the case of deformed model, the general solution to this equation is given in the following parametrized form containing an arbitrary function $\epsilon(s)$ describing the deviation of the fermi surface from its ground state form,

$$x(t_0, s) = \left[-\epsilon(s) + \sqrt{M + \epsilon^2(s)} \cosh 2(s - t_0)\right]^{1/2},$$

$$\alpha(t_0, s) = \frac{1}{x(t_0, s)} \sqrt{M + \epsilon^2(s)} \sinh 2(s - t_0).$$

On the other hand, the asymptotic form, for large $x$, of the profile function satisfies the following relation

$$\alpha_{\pm}(t_0, \sigma) = \pm x(\sigma) \left(1 - \frac{\psi_{\pm}(t_0 \pm \sigma)}{x^2(\sigma)}\right) + O\left(\frac{1}{x^2}\right).$$

The functions $\psi_{\pm}(t_0 \pm \sigma)$ represent incoming and outgoing waves, respectively. By comparing this definition with that of the $\zeta$ field, we have the relation,

$$(\partial_{t_0} \pm \partial_{\sigma})\zeta = \pm \frac{1}{\sqrt{\pi}} \psi_{\pm}(t_0 \pm \sigma)$$

for $t_0 \to \mp \infty$.

The relation between $\psi_+$ and $\psi_-$ can be established by studying the time delay. Let the times at which a parametrized point $s$ is passed by the incoming and outgoing waves
at a fixed value of large $\sigma$ be $t_1(\to -\infty)$ and $t_2(\to \infty)$, respectively. From (5.18) we have then

\begin{align*}
(M + \epsilon^2(s))^{1/4}e^{s-t_1} &= M^{1/4}e^\sigma, \\
(M + \epsilon^2(s))^{1/4}e^{t_2-s} &= M^{1/4}e^\sigma.
\end{align*}

(5.22)

(5.23)

This implies

\begin{equation}
t_1 + \sigma = t_2 - \sigma + \frac{1}{2} \log(1 + \frac{\epsilon^2(s)}{M}),
\end{equation}

(5.24)

and hence

\begin{equation}
\epsilon(s) = \psi_+(t_1 + \sigma) = \psi_-(t_2 - \sigma).
\end{equation}

(5.25)

Thus we get functional scattering equations connecting the incoming and outgoing waves

\begin{equation}
\psi_\pm(z) = \psi_\mp(z \mp \frac{1}{2} \log(1 + \frac{1}{M}\psi^2_\pm(z))).
\end{equation}

(5.26)

The result is similar to that of the usual $c = 1$ model, but manifests a crucial difference in that it is invariant under the change of sign of $\psi_\pm \to -\psi_\pm$. This clearly ensures that the particles participating in the scattering is even.

The explicit power series solution of (5.26) is

\begin{equation}
\psi_\pm(z) = \sum_{p=0}^{\infty} \frac{M^{-p}}{p!(2p+1)} \frac{\Gamma(1 \pm \frac{1}{2} \partial_z)}{\Gamma(1 - p \pm \frac{1}{2} \partial_z)} \psi^{2p+1}_\pm(z)
\end{equation}

(5.27)

which shows that the amplitudes are essentially polynomial with respect to the momenta without any singularity.

The scattering equation can also be rewritten in terms of energy momentum tensor

\begin{equation}
T_{\pm\pm}(z) = \frac{1}{2\pi} \psi^2_\pm(z),
\end{equation}

(5.28)

as

\begin{equation}
\int dz e^{i\omega z} T_{\pm\pm}(z) = \frac{M}{2\pi} \int dz e^{i\omega z} \frac{1}{1 \pm \frac{i\omega z}{2}} \left[(1 + \frac{2\pi}{M} T_{\pm\pm}(z))^{1 \pm \frac{i\omega z}{2}} - 1\right].
\end{equation}

(5.29)

One can easily check that this defines a canonical transformation by confirming that the Virasoro algebra (in the level of Poisson bracket) is preserved by this transformation. This relation for the energy momentum tensor is very similar to the one obtained recently by Verlinde and Verlinde [35] for the S-matrix of the $N = 24$ dilaton gravity. The
main difference is that in ref. [35] the relation holds for *integrals* of the energy momentum tensor while here it is a relation for the energy-momentum tensor itself. Of course, dilaton gravity and our deformed matrix model describe inequivalent physical systems. But, the similarity is striking and suggestive and might turn out to be of further significance.

5.4 Scattering and representations of collective field theory

The reader may wonder how the above result that the S-matrix elements are nonvanishing only for even number of particles could be compatible with the collective hamiltonian which has only a cubic interaction. To check the consistency, let us compute the S-matrix elements directly from the collective hamiltonian. This will be also useful for getting more insights on the nature of collective field theory in general.

The 3-point vertex in the hamiltonian in the interaction representation takes the form

\[
H_3(t_0) = \frac{\sqrt{\pi}}{12} \int_0^\sigma d\sigma (\frac{dx}{d\sigma})^{-2} \{(\Pi_\zeta - \partial^\sigma \zeta)^3 - (\Pi_\zeta + \partial^\sigma \zeta)^3\}
= -\frac{1}{12\pi} \int d^3k f(k_1 + k_2 + k_3)e^{-i(k_1 + k_2 + k_3)t_0} \alpha(k_1)\alpha(k_2)\alpha(k_3) \tag{5.30}
\]

where

\[
f(k) = \int_{-\infty}^{\infty} d\sigma (\frac{dx}{d\sigma})^{-2}e^{ik\sigma}, \tag{5.31}
\]

and we have defined the creation and annihilation operators, \(\alpha(k)\) with \(k < 0\) and \(k > 0\), respectively, by

\[
\Pi_\zeta \pm \partial_\sigma \zeta = \pm \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk e^{-i(k_0 \pm \sigma)} \alpha(k), \tag{5.32}
\]

\[
[\alpha(k_1), \alpha(k_2)] = k_1 \delta(k_1 + k_2). \tag{5.33}
\]

Thus the on-shell 3-point amplitude \((k_i > 0)\)

\[
- < 0|\alpha(k_1)\alpha(k_2) i \int dt_0 H_3(t_0) \alpha(-k_3)|0 > = ik_1k_2k_3f(0) \delta(k_1 + k_2 + k_3) \tag{5.34}
\]

vanishes if \(f(0) = 0\). Using the classical solution \(x(\sigma) = M^{1/4} \cosh^{1/2} \sigma\), we find

\[
f(k) = \frac{1}{4\sqrt{M}} \mathcal{P} \int_{-\infty}^{\infty} d\sigma (\frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 \sigma})e^{ik\sigma} \tag{5.35}
= \frac{1}{4\sqrt{M}} \pi k (\cosh \frac{nk}{2} - 1) \sinh \frac{nk}{2}
\]
which indeed leads to \( f(0) = 0 \). Here we adopted the principal-value prescription, 
\[ \mathcal{P}F(\sigma) \equiv (1/2)[F(\sigma + i\epsilon) + F(\sigma - i\epsilon)] \]
to deal with the singularity associated with the turning point in the semi-classical approximation to the matrix model. This has been argued to be the correct procedure in ref. [26].

Next let us consider the case of the 4-point amplitude,
\[
A(k_1, k_2, k_3; k_4) \equiv <0| \prod_{i=1}^{3} \alpha(k_i)(-\frac{1}{2}) \int dt_1 \int dt_2 T(H_3(t_1)H_3(t_2))\alpha(-k_4)|0> = \frac{1}{2\pi i} \delta(k_1 + k_2 + k_3 - k_4)k_1k_2k_3k_4
\times (F(k_2 + k_3) + F(k_3 + k_1) + F(k_1 + k_2))
\]
(5.36)

where
\[
F(k) = \mathcal{P} \int_{-\infty}^{\infty} d\ell \frac{k - \ell}{\ell} f(\ell)^2
\]
(5.37)
\[
= -\frac{\pi^2}{16M} \int_{-\infty}^{\infty} d\ell \ell^2 \frac{(\cosh \frac{\pi \ell}{2} - 1)}{\sinh^2 \frac{\pi \ell}{2}}.
\]
(5.38)

Here we used \( \mathcal{P} \frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x) \) and \( f(0) = 0 \). This is again a singular integral. Following [26], we use the \( \zeta \)-function regularization which for (5.38) amounts to subtract a quadratically divergent piece \( f d\ell \ell^2 \). We then find
\[
F(k) = \frac{2\pi}{3M}
\]
(5.39)

Hence,
\[
A(k_1, k_2, k_3; k_4) = \frac{-i}{M} \delta(k_1 + k_2 + k_3 - k_4)k_1k_2k_3k_4.
\]
(5.40)

This coincides with the result obtained from the scattering equation (5.27).

Let us next move to higher-point amplitudes. Using the formula in [26] and \( f(0) = 0 \), a 5-point amplitude corresponding to a Feynman diagram (out of 15 different diagrams) generally takes the following form
\[
A(k_1, k_2, k_3, k_4, k_5) \propto \int_{-\infty}^{\infty} d\ell \int_{-\infty}^{\infty} d\ell' f(\ell)f(\ell')f(\ell + \ell')
+ (k_1 + k_2)(k_4 + k_5) \int_{-\infty}^{\infty} d\ell \int_{-\infty}^{\infty} d\ell' \frac{1}{\ell \ell'} f(\ell)f(\ell')f(\ell + \ell')
\]
(5.41)

where \( k_1 + k_2 \) and \( k_4 + k_5 \) are the momenta of associated with two internal lines of a Feynman diagram (note this uniquely fixes the Feynman diagram). These are again
singular integrals. They can be most conveniently treated by returning to the coordinate space and using the principal-value prescription. For example, the first term is then given as

$$(2\pi)^2 \mathcal{P} \int_{-\infty}^{\infty} d\sigma \left( \frac{1}{\alpha_0^2(\sigma)} \right)^3 = \frac{(2\pi)^2}{\sqrt{M}} \mathcal{P} \int_{-\infty}^{\infty} d\sigma \frac{(\cosh 2\sigma)^3}{(\sinh 2\sigma)^6}$$

(5.42)

By making a change of integration variable $y = \sinh 2\sigma$, we find the above integral is proportional to $\mathcal{P} \int_{-\infty}^{\infty} dy (y^{-6} + y^{-4}) = 0$. The second term is also rewritten in the coordinate space

$$(2\pi)^2 \mathcal{P} \int_{-\infty}^{\infty} d\sigma \frac{1}{\alpha_0^2(\sigma)} (\partial^{-1}_\sigma \frac{1}{\alpha_0(\sigma)^2})^2 \propto \mathcal{P} \int_{-\infty}^{\infty} d\sigma (\frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 \sigma})(-\coth \sigma + \tanh \sigma)^2$$

(5.43)

which again allows a change of variables so that it vanishes. In general, one can see that this happens for all odd-point amplitudes. Typically, one will have the integral ($n = \text{positive integers}$)

$$\mathcal{P} \int_{-\infty}^{\infty} d\sigma \left( \frac{1}{\alpha_0(\sigma)} \right)^{2n+1} \propto \mathcal{P} \int_{-\infty}^{\infty} dy \frac{(1 + y)^n}{y^{4n+2}}$$

(5.44)

which all vanish by the principal value prescription.

However, we cannot deny the impression that the above calculations are rather awkward in using *ad hoc* prescriptions for dealing with singular integrals. Actually, the identical results can be obtained without singular integrals, if we use the dual representation in constructing the collective hamiltonian. Namely, we can introduce the collective field in a representation in which the momentum $p$ is diagonalized instead of the coordinate $x$. We introduce the profile function $\beta(p)$ of the fermi sea as a function of $p$. Denoting the two branches (corresponding to $x < 0$ and $x > 0$) of the fermi sea as $\beta_\pm$, the hamiltonian is now

$$H = \frac{1}{2\pi} \int dp \int_{\beta_+(p)}^{\beta_-(p)} dx h(p, x) \equiv H_+ - H_-,$$

(5.45)

$$H_\pm = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp [\frac{1}{2} p^2 \beta_\pm - \frac{1}{6} \beta_\pm^3 - \frac{M}{2\beta_\pm}].$$

(5.46)

Here we dropped the boundaries at $x \to \pm \infty$ because we are only interested in the dynamics of the fluctuation of the fermi sea and the boundaries do not participate in it. The Poisson bracket for $\beta_\pm$ and the equation of motion are, respectively,

$$\{\beta_\pm(p_1), \beta_\pm(p_2)\} = \pm 2\pi \partial_p \delta(p_1 - p_2),$$

(5.47)

26
\[ \dot{\beta}_\pm(t_0, p) = \partial_p h(p, \beta_\pm(t_0, p)) \]  

The classical ground state solution is given as

\[ \beta_\pm = \mp \sqrt{[p^2 + \sqrt{p^4 + 4M^2}]^{1/2}} \equiv \beta_0. \]  

To study the fluctuation, an appropriate shift of the field is

\[ \beta = \beta_0 - \sqrt{\pi}(\frac{dp}{d\sigma})^{-1}(\Pi_\eta \pm \partial_\eta) \]

where the relation with the time-of-flight coordinate and the p-coordinate is

\[ \frac{dp}{d\sigma} = \beta_0(p) + \frac{M}{\beta_0^3(p)} \equiv \tilde{\beta}_0(p) \]  

which is solved as

\[ p = \frac{M^{1/4}}{\cosh^{1/2}2\sigma} \] coinciding with \( \frac{dp}{d\sigma} \). Then the Hamiltonian becomes

\[ H = \int_{-\infty}^{\infty} d\sigma \left[ \frac{1}{2}(\Pi_\eta^2 + (\partial_\sigma \eta)^2) - \frac{\sqrt{\pi}}{2}(\frac{1}{6} - \frac{1}{2\beta_0^3}) \frac{1}{\beta_0^3}(p) \{ (\Pi_\eta \pm \partial_\sigma \eta)^3 - (\Pi_\eta - \partial_\sigma \eta)^3 \} \right] \]

\[ - \frac{M}{4\pi} \sum_{n=4}^{\infty} \left[ (-)^n \frac{\beta_0^3(p)\beta_0^3(p)}{\beta_0^3(p)\beta_0^3(p)} \right] \{ (\Pi_\eta \pm \partial_\sigma \eta)^n - (\Pi_\eta - \partial_\sigma \eta)^n \}. \]  

Note that we now have non-polynomial interactions, in contrast to the purely cubic interactions of the collective field theory in the \( x \)-representation. Since in the \( p \)-representation there is no mixing of left-right moving modes, we only retain the right-moving terms in the interaction terms, containing

\[ \Pi_\eta - \partial_\sigma \eta = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk \alpha(k) e^{-ik(t_0 - \sigma)}. \]  

After substituting the expressions for \( \beta_0, \tilde{\beta}_0 \), we get the following explicit forms for the 3- and 4-point functions,

\[ H_3(t_0) = \frac{M^{-1/2}}{12\pi} \int d^3ke^{-i(k_1+k_2+k_3)t_0}g_3(k_1+k_2+k_3)\alpha(k_1)\alpha(k_2)\alpha(k_3), \]  

\[ g_3(k) = \int_{-\infty}^{\infty} d\sigma e^{ik\sigma} \frac{\cosh 2\sigma(\cosh^2 2\sigma - 3)}{(\cosh^2 2\sigma + 1)} = -\frac{\pi k}{4} \frac{\sin k s_0}{\cosh \frac{k}{4}}, \]  

\[ H_4(t_0) = \frac{1}{4\pi M} \int d^4ke^{-i(k_1+k_2+k_3+k_4)t_0}g_4(k_1+k_2+k_3+k_4), \]  

\[ g_4(k) = \int_{-\infty}^{\infty} d\sigma e^{ik\sigma} \frac{\cosh^2 2\sigma}{(\cosh^2 2\sigma + 1)^3} \]

\[ = \frac{(k^2 + 16)\sqrt{2\pi} \sin k s_0}{256 \sinh \frac{\pi k}{4}} + \frac{\pi \sqrt{2}}{512 \sinh \frac{\pi k}{4}}(2\sqrt{2}k \cos k s_0 - 12 \sin k s_0), \]  

27
where $s_0 (> 0)$ is a solution of $\cosh 4s_0 = 3$. We see that all the integrals for the form factors are perfectly well defined. Using these results, we find again that the on-shell 3-point amplitude vanishes $g_3(0) = 0$. The 4-point amplitude now receives the contribution from the 4-point vertex, in addition to the one coming from a product of the 3-point functions. They are, respectively,

$$\frac{-i}{2M} \delta(k_1 + k_2 + k_3 - k_4) k_1 k_2 k_3 k_4 \times 4! \times g_4(0) \times 4!$$  \hspace{1cm} (5.57)

and

$$\frac{-i}{2M} \delta(k_1 + k_2 + k_3 - k_4) k_1 k_2 k_3 k_4 \times \frac{3}{16} \mathcal{P} \int_{-\infty}^{\infty} d\ell \frac{k_1 + k_2 + k_3 - \ell}{\ell} g_3(\ell)^2.$$  \hspace{1cm} (5.58)

Evaluating the integrals, we find that the coefficients after the products of momenta $k_i$’s are $(3 + 15\sqrt{2} s_0)/4$ and $(5 - 15\sqrt{2} s_0)/4$, respectively, which just reproduce the result (5.40).

We expect that similar cancellations occurring here must happen in higher order terms as well, though checking this explicitly in the Feynman graph expansion becomes increasingly tedious in the $p$-representation. It is not difficult, however, in general to convince ourselves of the equivalence with the method of Polchinski, using the well known general theorem that the computing $S$-matrix elements in the tree approximation is equivalent to obtaining a general solution to the classical equation of motion.

It is remarkable that two entirely different field theories give equivalent $S$-matrix elements. The fields $\zeta$ and $\eta$ can never be connected by a simple field redefinition. Remember that we have effectively interchanged the role of field and coordinate, although in the free fermion picture, this is simply a standard dual transformation interchanging $x$ and $p$. This explains why we have the same $S$-matrix elements in the usual $c = 1$ model for both cases of positive and negative cosmological constant (see the first reference of [1]). It might be useful to study other possible choice of the representation such that it manifests the vanishing of the $S$-matrix elements for odd number of particles. Also, it is a very challenging question whether a new symmetry of this kind can be generalized to higher dimensional solutions of critical string theory.

6. Discussions
6.1 S-matrix elements for tachyon black-hole scattering

Now we can proceed to discuss a plausible candidate for the S-matrix elements for tachyon black-hole scattering. Our basic working hypothesis requires us to apply the non-local integral transformation to each external line of a Green’s function of the collective field quanta $\zeta$ (or $\eta$) of the deformed matrix model. Let the Green’s function of the $\zeta$ in the momentum representation ($k = p_\sigma = 4\lambda, E = p_{\tau_0} = \frac{4}{3}\omega$) be

$$G(k_1, E_1, k_2, E_2, \ldots, k_n, E_n) \sim (\prod_{i=1}^n \frac{1}{k_i^2 - E_i^2 + i\epsilon})A(k_1, k_2, \ldots, k_n)$$ (6.1)

Then, the asymptotic behavior of the transformed Green’s function for the tachyon field $\tilde{T}$ is given by

$$\left\{ \prod_{i=1}^n \int_0^\infty dk_i \int_{-\infty}^\infty dE_i \gamma(E_i)\lambda_i e^{-2i\omega_i t_i}(\alpha(-\lambda_i, \omega_i)e^{-2i\lambda_i r_i} + \alpha(\lambda_i, \omega_i)e^{2i\lambda_i r_i}) \times \frac{1}{k_i^2 - E_i^2 + i\epsilon} \right\}A(k_1, k_2, \ldots, k_n)$$ (6.2)

The integrals over $E_i$’s pick up the pole at $E_i = k_i$ or $-k_i$, depending on $t_i > 0$ or $t_i < 0$, respectively. This implies that in the asymptotic limit $r_i \to \infty$, the wave packets of the tachyon field $\tilde{T}$ receive contribution from the first term. In other words, the incoming or outgoing wave packets is right moving or left moving respectively as it should be. If we choose a natural normalization such that the incoming waves are normalized to be unity, $|\gamma(E_i)\lambda_i\alpha(-\lambda_i, 3\lambda_i)| = 1$, the outgoing waves have the normalization

$$|\gamma(-E_i)\lambda_i\alpha(-\lambda_i, -3\lambda_i)| = \frac{\alpha(-\lambda_i, -3\lambda_i)}{\alpha(-\lambda_i, 3\lambda_i)} = |R_B(\lambda_i)|.$$ (6.3)

Remember the definition (2.52) of the reflection coefficient for the black hole. Thus, apart from pure phase factors, the outgoing external lines get multiplied the reflection coefficient.

In the static black-hole background geometry, part of the incoming wave from the past null infinity is absorbed into the future horizon. To have the complete set of the scattered waves, we therefore need to take the absorbed waves into account. The asymptotic behavior (2.54) shows that we have to multiply the absorption coefficient corresponding to that part.
The rules for these external leg factors can be most conveniently stated by using the set of creation and annihilation operators for the asymptotic states \( t \to \pm \infty \). Let the asymptotic in and out oscillator of the \( \zeta \) field be \( \alpha_{\pm}(\lambda) \) as before and those of the \( \tilde{T} \) fields on past infinity (\( I^+ \)), future infinity (\( I^- \)), and on future event horizon (\( H_+ \)), be \( \alpha_{\tilde{T}}^+, \alpha_{\tilde{T}}^- \) and \( \alpha_{H}^- \), respectively. By convention, the operators with \( \lambda > 0 \) (\( < 0 \)) are annihilation (creation) operators such that 
\[
\alpha_{\pm}(\lambda_1), \alpha_{\pm}(\lambda_2) = \lambda_1 \delta(\lambda_1 + \lambda_2)
\]
for each asymptotic region. Suppose the scattering operator of the \( \zeta \) fields is given by
\[
S_{\zeta} = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \prod_{j=1}^{\infty} \int_0^\infty d\lambda_i A(\lambda_1, \ldots, \lambda_m; \lambda_{m+1}, \ldots, \lambda_{m+n}) \prod_{i=1}^m \alpha_-(\lambda_i) \prod_{j=m+1}^{n+m} \alpha_+(-\lambda_j).
\]
(6.4)
Then, the scattering operator of the tachyon field \( \tilde{T} \) is
\[
S_{\tilde{T}} = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} (\prod_{j=1}^{\infty} \int_0^\infty d\lambda_i) A(\lambda_1, \ldots, \lambda_m; \lambda_{m+1}, \ldots, \lambda_{m+n}) \times 
\prod_{i=1}^m \left( |R_B(\lambda_i)|\alpha_{\tilde{T}}^-(\lambda_i) + \sqrt{\omega_i} |T_B(\lambda_i)|\alpha_{H}^- \right) \prod_{j=m+1}^{n+m} \alpha_{H}^-(-\lambda_j)
\]
(6.5)
The unitarity of \( S_{\tilde{T}} \), providing the “observations” for \( t \to +\infty \) are made on both \( I^- \) and \( H^- \), is ensured by the relation (2.61) and the unitarity of \( S_{\zeta} \).

It is natural to define a density operator for the out states on \( I^- \) given an initial state \( |i >_+ \) by
\[
\rho_S = \text{Tr}_{H^-}[S_{\tilde{T}} |i >_+ < i | S_{\tilde{T}}^\dagger]
\]
where the trace is taken only over the set of states on \( H^- \).

We would like, however, warn the reader that although the above S-matrix seems natural at least in the perturbative regime where we just consider scatterings in a static black-hole background by assuming the mass of black hole is sufficiently large, it is by no means clear how to interpret it in the nonperturbative regime, where we have to properly take into account the back reaction of the geometry itself. Our tentative identification of the black hole mass operator in the deformed matrix model seems to indicate the impossibility of changing the black hole mass by a dynamical process. This seems consistent with an analysis \cite{36} of this question using a low-energy effective field theory. Still, we have in general non zero absorption coefficient. Note, however, that the absorption coefficient vanishes in the low energy limit. Here it might be worthwhile to remind ourselves that the relation of the string coupling with the inverse black hole mass is analogous to the
situation in the soliton solutions of local field theories. In the case of soliton solutions, the perturbation theory in fact reproduces the effect of back reaction order by order. It is a very important question whether or not the perturbation expansion in the present case can also be interpreted as containing the effect of back reaction. Saying anything about possible nonperturbative meaning of our results would be dangerous without a more clear understanding on this question.

6.2 Naked singularity

So far, we have been treating only the case $M > 0$. When $M < 0$, the deformed matrix model has a singularity at the origin corresponding to an infinite attractive force. Correspondingly, the Virasoro condition contain a naked singularity at $uv = M$. Equivalently, by making a transformation $u \rightarrow \sqrt{|M|}u$, $v \rightarrow -\sqrt{|M|}v$, we can consider the same Virasoro condition \[2.33\] in the region $uv > 1$, $u > 0$, $v > 0$. In terms of the exact global geometry described by \[2.34\], this region is disconnected from the exterior region $uv < 0$. In this subsection, let us work in the latter choice of the coordinates.

Because of the singularity, a generic solution of the Virasoro condition exhibits a logarithmic singularity $\log(1 - uv)$ near the naked singularity. However, it is known that there is a particular solution which is regular at the naked singularity, given by, $(y = uv)$

$$W^\lambda_\omega = e^{-2i\omega t} \frac{\Gamma(\bar{\nu}_-)\Gamma(\nu_-) y^{-i\omega} F(\nu_+, \bar{\nu}_-; 1; 1 - y)}{\Gamma(\nu_+)}$$

(6.6)

$$W^\lambda_\omega = e^{-2i\omega t} \left[ \frac{\Gamma(2i\omega)\Gamma(\bar{\nu}_-) y^{-i\omega} F(\nu_+, \bar{\nu}_-; 1 - 2i\omega; y)}{\Gamma(\nu_+)} \right. + \frac{\Gamma(-2i\omega)\Gamma(\nu_-) y^{i\omega} F(\bar{\nu}_+, \nu_-; 1 + 2i\omega; y)}{\Gamma(\nu_+)} \right]$$

(6.7)

Using an integral representation for $W^\lambda_\omega$,

$$W^\lambda_\omega = e^{-2i\omega t} \int_0^1 dz \frac{z^{-\frac{1}{2} + i(\lambda - \omega)}(1 - z)^{-\frac{1}{2} - i(\lambda - \omega)}(1 - (1 - y)z)^{-\frac{1}{2} + i(\lambda + \omega)}}{\Gamma(\nu_+) \Gamma(\nu_-)}$$

(6.8)

and making a change of variable

$$e^{2\nu} = \left( \frac{1 - z}{z} \right)^{1/2} (1 - (1 - y)z)^{1/2},$$

(6.9)

$$e^{2\sigma} = \left( \frac{1 - z}{z} \right)^{-1/2} (1 - (1 - y)z)^{1/2},$$

(6.10)
we can find an intertwining representation which is similar to the black hole case. (A
numerical proportionality constant is again neglected.)

\[ W_{\omega}^{\lambda} = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} dt_0 \delta\left( \frac{ue^{-2t_0/3} - ve^{2t_0/3}}{2} \right) - \sinh 2\sigma)e^{-4i\omega t_0/3}e^{-4i\lambda \sigma} \]  

(6.11)

The \( \delta \)-function kernel in this result is a natural generalization of that appeared in (4.4). First, the minus sign in front of \( v \) is coming from the above transformation \( v \rightarrow -v \). Second, in terms of the time-of-flight coordinate, the classical ground solution for \( M < 0 \) case is

\[
\begin{align*}
x^2 &= \sqrt{|M|| \sinh 2\sigma|}, \\
p^2 &= \sqrt{|M|| \sinh 2\sigma + \frac{1}{\sinh 2\sigma}}. 
\end{align*}
\]  

(6.12) (6.13)

Thus, after returning to our scaled coordinates, the kernel takes the form \( \delta(\frac{ue^{-2t_0/3} + ve^{2t_0/3}}{2} - \epsilon(x)x^2) \) which is equivalent with the previous case \( M > 0 \), since in that case we can restrict the range \( x \) to the positive real axis because of the infinite repulsive potential at the origin. Thus the difference is that the integration range of \( x \) or \( \sigma \) is now \((-\infty, \infty)\) instead of the half infinite. Classically, an incoming particle with total energy larger than a finite critical value \( \epsilon_0 = -\sqrt{2|M|(< 0)} \) reaches to the singularity in a finite time. The extension to negative \( \sigma \) may be interpreted as a particle continues to move beyond the singularity. It then seems natural to suppose that the tachyon wave which is chosen to be regular at the naked singularity corresponds classical fermi liquid passing through the singularity of the potential in the above sense. It is then easy to apply again the fermi-liquid picture to obtain the scattering equation for the naked singularity.

Before doing that, let us briefly mention the asymptotic property of the solution. By examining the asymptotic form for large \( y \), we easily see that the waves are perfectly reflected with the pure phase reflection coefficient

\[ R_N(\lambda) = -2^{-8i\lambda} \frac{\Gamma(1 + 4i\lambda)\Gamma(\frac{1}{2} - 4i\lambda)}{\Gamma(1 - 4i\lambda)\Gamma(\frac{1}{2} + 4i\lambda)} \]  

(6.14)

This exhibits the resonance poles at all integer values of purely imaginary momenta \(-4i\lambda = 1, 2, \ldots, \) in sharp contrast with the black-hole case with event horizon. This difference is very mysterious to us. It seems as if the special choice of perfectly reflecting
boundary condition effectively shifts the momentum of the mass operator such that the resonance poles appear at the same values as in the case of the usual tachyon condensation, although the reflection coefficient is not the same. Whatever the interpretation of this peculiar behavior might be, now there seems to be no reason for expecting vanishing of the odd-point on-shell amplitudes.

Let us derive the scattering equation for this case based on the picture above. It is convenient to work with $x^2$ and $p^2$. For finite energy $\epsilon > \epsilon_0$, the solution of the classical equation of motion is given in the parametrized form by

$$x^2(t, s) = -\epsilon(s) + \sqrt{2|M| - \epsilon^2(s)} \sinh 2(s - t), \quad (t < \tau(s)) \quad (6.15)$$

$$= \epsilon(s) - \sqrt{2|M| - \epsilon^2(s)} \sinh 2(t + s - \tau), \quad (t > \tau(s)) \quad (6.16)$$

$$\alpha^2(t, s) = x^2(t, s) - \frac{1}{x^2(t, s)} + 2\epsilon(s) \quad (6.17)$$

where $\tau$ is determined by $\sinh 2(s - \tau) = \frac{\epsilon}{\sqrt{2|M| - \epsilon^2}}$. By comparing the asymptotic relation among the coordinate $\sigma$ and the times $t_1, t_2$ as before at which a particular parametrized point $s$ of the incoming and outgoing waves reach, we arrive at the relation

$$t_1 + \sigma = t_2 - \sigma + \frac{1}{2} \log(1 - \frac{\epsilon^2(s)}{2|M|}) - \sinh^{-1} \frac{\epsilon(s)}{\sqrt{2|M| - \epsilon^2(s)}}. \quad (6.18)$$

This shows that the scattering equation is

$$\psi_\pm(z) = \psi_\pm(z \mp \frac{1}{2} \log(1 - \frac{\psi^2_\pm(z)}{2|M|}) \pm \sinh^{-1} \frac{\psi_\pm(z)}{\sqrt{2|M| - \psi^2_\pm(z)}}. \quad (6.19)$$

The on-shell amplitudes are indeed non-vanishing for both odd and even numbers of particles. It is easy to check that the collective field theory in the $x$-representation has a well defined 3-point vertex which does not vanish on shell.

Although we are not so sure whether the difference in the properties of tachyon scattering for $M > 0$ and $M < 0$ is an evidence for our hypothesis, it is interesting that the deformed model discriminates the two cases very clearly, as should be expected from the exact linearized analysis of ref. [23].
6.3 Conclusion

We believe that we have clarified some of the most crucial steps in the problem of a 2D black hole background in the matrix-model approach to string theory. We have proposed a working hypothesis, based on which we could obtain several concrete results. However, it is also clear that we have left some of the crucial questions unanswered. We have already emphasized a few of them. To conclude the present paper, we will give a list of most important remaining problems.

1. **Physical picture**: One might have expected that if the black hole geometry is describable at all in the matrix model, it had to be related to the existence of the other side of the inverted harmonic oscillator potential. In fact, however, after all our experimentation, we have been forced to reach at a totally different picture that the black hole should rather correspond to a deformed matrix model with infinite repulsive wall at the origin, such that the other side of the potential becomes irrelevant. The effect of a horizon is mostly taken into account by the non-local field redefinition, and not by the form of the potential. It is desirable to have some intuitive explanation why this must be so if our conclusion is ever correct.

2. **Unified understanding of the field redefinition and the matrix model Hamiltonian**: The question is whether and how the matrix models themselves encode the information of non-local field redefinition which is already background dependent. We have seen that the characteristic properties of the deformed matrix model, in particular, its scattering amplitudes, are surprisingly consistent with the properties of the one-particle wave functions coming from the field redefinition. However, the coincidence looks rather accidental in view of lack of some more logical derivation. We have no rigorous uniqueness proof for the field redefinition nor for the deformed model and a unified derivation would illuminate both. In this unified theory, the factorized ansatz for the S-matrix might be only a first approximation.

3. **Tachyon condensation**: To gain some insight into the above question, it might be useful to study the effect of tachyon condensation in the black hole background. In the present work, we have assumed that the fermi energy is precisely zero. We
conjecture that the deformed matrix model with non zero fermi energy corresponds
to a black hole background with tachyon condensation. In our approach, we should
be able to generalize the non-local field redefinition so that it incorporates both (2.24)
and (4.6) depending upon the magnitude of $M$ and $\mu^2$. As in the low energy effective
field theory\cite{3,37}, the tachyon condensation provides an independent deformation
parameter of the theory.

4. \textit{Role of $SL(2,R)$ algebra and the discrete states}: We have mentioned the presence
of an $SL(2,R)$ algebraic structure in the deformed model as one of the motivations
to our proposal. Undoubtedly, that must be crucial for understanding the full
symmetry of the theory. We have also to mention the relevance of it for the question
of possible $W_\infty$ hair of the black hole \cite{31}. Concerning to the discrete states, we
do not have an adequate understanding on the connection between the spectrum
of the discrete states and the poles in the external leg factors in the case of the
Minkowskian black hole. In the usual $c=1$ case, they are intimately related.

5. \textit{Connection with the Euclidean black hole}: We have only discussed the Minkowskian
black hole. The significance of our results for the Euclidean black hole is not clear
to us. If the integral transformation is formally Wick-rotated, the periodicity in
imaginary $t_0$ is $3\pi$. The latter is to be identified by a standard argument with
the inverse Hawking temperature. To be consistent, the deformed model should
be the Euclidean version with that periodicity. The question then arises whether
our S-matrix is related to Hawking radiation or not. It might be useful to remind
that when the information on the future event horizon is summed up, our S-matrix
operator turns into an density operator as discussed in a previous subsection. Is it
possible that the density operator obtained in this way is related with the thermal
density operator when the contribution of the one-loop effect is taken into account?

We hope to return to some of these questions in the future.
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