ABSTRACT

In string theory, nilpotence of the BRS operator $\delta$ for the string functional relates the Chern-Simons term in the gauge-invariant antisymmetric tensor field strength to the central term in the Kac-Moody algebra. We generalize these ideas to p-branes with odd p and find that the Kac-Moody algebra for the string becomes the Mickelsson-Faddeev algebra for the p-brane.
1. Introduction

In a recent paper [1], the coupling of Yang-Mills fields to the heterotic string in bosonic formulation was generalized to extended objects of higher dimension (p-branes). In particular, it was noted that for odd p the Bianchi identities obeyed by the field strengths of the (p+1)-forms receive Chern-Simons corrections. In the case of the string (p=1), there is an equality between the coefficient \( n \) of the Chern-Simons term \( I_3(A) \) in the antisymmetric tensor field strength \( H_3 = dB_2 + nI_3(A) \), and the central charge \( n \) of the Kac-Moody algebra obeyed by certain operators \( T^a(\sigma) \) that appear in the gauge BRS transformations of the string functional [2]. The purpose of the present paper is to show that for 3-branes the coefficient of the Chern-Simons term is equal to the coefficient of an Abelian extension of a \( T^a(\sigma^j) \) algebra involving new generators \( T_i^a(\sigma^j) \), \( i, j = 1, 2, 3 \). The corresponding algebras have already appeared before in the context of anomalies [3,4,5,6] and are known in the mathematical literature as loop algebras with a Mickelson-Faddeev extension [7]. There is a straightforward generalization to \( p > 3 \) branes.

In string theory, the integer \( n \) also appears as a coefficient of the Wess-Zumino-Witten term in the action, and the operators \( T^a \) can be constructed from the action [2], which is invariant under simultaneous gauge variations of the background fields and the group coordinates. While this action is known for the p-branes[1], the operators \( T^a \) have not yet been constructed and examined. A second way to get the relation is to insist on the nilpotence of the gauge BRS transformations of the string field \( \Phi \) and background fields \( A \) etc. It is this second method which will here be generalized to the 3-brane.
2. Loop Space Algebras

In manifestly supersymmetric and $\kappa$-symmetric form the heterotic string can be formulated as a mapping from two dimensions to a target space parametrized by variables $X^\mu, \theta^a$ and $y^m$. We ignore $\theta$ from now on. $y^m$ are bosons parametrizing the group space. We take the $\sigma$-model point of view that there are also background fields present representing the massless bosonic excitations of the string. Consider the following BRS transformation:

$$\delta = \delta_1 + \delta_B$$  \hspace{1cm} (2.1)

Here $\delta_1$ is defined by:

$$\delta_1 = \prod_{\mu,m,\sigma'} \int dy^m(\sigma')dX^\mu(\sigma') \left\{ \left( \int d\sigma [-\omega^aT^a(\sigma) + \Lambda_\mu \frac{dX^\mu}{d\sigma}] \Phi \right) \frac{\delta}{\delta \Phi} \right\}$$  \hspace{1cm} (2.2)

where the ‘doubly functional’ derivative is defined by:

$$\frac{\delta}{\delta \Phi(X)} \Phi(X') = \prod_{\sigma} \delta^D[X(\sigma) - X'(\sigma)]$$  \hspace{1cm} (2.3)

and hence:

$$\delta_1 \Phi = \int d\sigma [-\omega^aT^a(\sigma) + \Lambda_\mu \frac{dX^\mu}{d\sigma}] \Phi$$  \hspace{1cm} (2.4)

In the above, $\delta_1$ is a BRS transformation which acts on functionals of the string field $\Phi$, which is itself a functional of the string variables $X^\mu(\sigma)$ and $y^m(\sigma)$. $\Phi$ is a string field, but we will ignore the problems of closed string field theory here (for reviews see e.g. [8] [9]) – in particular we ignore the dependence of $\Phi$ on the reparametrization ghost fields. The exterior derivative $d$ and the BRS operator $\delta$ are taken to be anticommuting in this paper. Our aim is to consider just the Yang-Mills part of the BRS transformations of the background fields and the corresponding transformation of the string field.
The variable $\sigma$ is the spacelike variable on the string world sheet. The operator $T^a(\sigma)$ is assumed here to depend only on $y^m(\sigma)$ and functional derivatives with respect to $y^m(\sigma)$. An example of $T^a(\sigma)$, for the case of the string, can be found in [2]. We shall alternate between component and form notation, for example setting $dX^\mu \Lambda_\mu = \Lambda_1$ etc. The part $\delta_1$ is not separately nilpotent. The part $\delta_B$ is separately nilpotent ($\delta_B^2 = 0$) and it acts only on the background fields $A^a_\mu(x)$ etc. These BRS transformations of the background fields are:

$$
\delta_B = \int d^Dx \left\{ D^{ab}_\mu \omega^b \frac{\delta}{\delta A^a_\mu} - \frac{1}{2} f^{abc} \omega^b \omega^c \frac{\delta}{\delta \omega^a} + \left[ -n A^a_\mu \partial_\nu \omega^a + \partial_\nu \Lambda_\mu \right] \frac{\delta}{\delta B_{\mu\nu}} 
\right. 
+ \left[ n \omega^a \partial_\mu \omega^a - \partial_\mu B_0 \right] \frac{\delta}{\delta A^a_\mu} + \frac{1}{6} n f^{abc} \omega^a \omega^b \omega^c \frac{\delta}{\delta B_0} \right\} \quad (2.5)
$$

Here $\Lambda_\mu$ is a ghost for the antisymmetric tensor field $B_{\mu\nu}$ and $B_0$ is a ‘ghost for ghost’ for the ghost $\Lambda_\mu$. The field $\omega^a$ is the Yang-Mills Faddeev-Popov ghost. In terms of fields this becomes, for example:

$$
\delta A^a_\mu = D^{ab}_\mu \omega^b = \partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c \quad (2.6)
$$

Alternatively we can use the notation:

$$
\delta A^a = -d\omega^a - f^{abc} A^b \omega^c \quad (2.7)
$$

$$
\delta \Lambda = n I^2_1 + dB_0 \quad (2.8)
$$

$$
\delta B_2 = n I^1_2 + d\Lambda \quad (2.9)
$$

\ldots

In the above the terms $I^i_{p+2-i}$ are the terms of ghost number $i$ that appear in the descent equations for the Yang-Mills fields. In our conventions the curvature
two-form is:

\[ F^a = dA^a + \frac{1}{2} f^{abc} A^b A^c \]  

(2.10)

and it transforms as:

\[ \delta F^a = f^{abc} F^b \omega^c \]  

(2.11)

The descent equations take the form:

\[ \delta I^{i}_{p+2-i} = dI^{i+1}_{p+1-i} \]  

(2.12)

so that

\[ I^0_4 = F^a F^a = dI^0_3 \]  

(2.13)

\[ I^0_3 = A^a dA^a + \frac{1}{3} f^{abc} A^a A^b A^c \]  

(2.14)

\[ I^1_2 = -A^a d\omega^a \]  

(2.15)

\[ I^2_1 = \omega^a d\omega^a \]  

(2.16)

\[ I^3_0 = \frac{1}{6} f^{abc} \omega^a \omega^b \omega^c \]  

(2.17)

Nilpotency follows easily using these. For example:

\[ \delta^2 B_2 = n\delta I^1_2 - d\delta \Lambda = 0 \]  

(2.18)

We note that:

\[ H^0_3 = dB_2 + nI^0_3 \]  

(2.19)

is gauge invariant:

\[ \delta H^0_3 = \delta_B H^0_3 = 0 \]  

(2.20)

\[ dH^0_3 = I^0_4 \]  

(2.21)

We assume that the background fields and their ghosts depend on \( X \) but not on \( y \), so that the action of \( T \) on the background fields and ghosts is trivial here. We
also assume that the action of $\delta_B$ on the operators $T$ is trivial, since they do not depend on the background fields. We further assume that the string field $\Phi$ does not depend on the background fields. Note that these operators have been defined so that $\delta_1$ acts only on $\Phi$ and $\delta_B$ acts only on the background fields. For example:

$$\delta_B \Phi = \delta_1 A^a_\mu = \delta_1 \Lambda_\mu = 0 \quad (2.22)$$

Calculation shows that nilpotency ($\delta^2 = 0$) of $\delta$ implies that the Kac-Moody algebra of the generators $T$ has a central term with coefficient $n$:

$$[T^a(\sigma), T^b(\sigma')] = f^{abc} T^c(\sigma) \delta(\sigma - \sigma') + 2n \delta^{ab} \frac{d}{d\sigma} \delta(\sigma - \sigma') \quad (2.23)$$

Now we want to generalize this string case to p-branes for odd $p$. The way that $\delta^2 \Phi = 0$ works is that the variation $\delta \Lambda_\mu = n \omega^a \partial_\mu \omega^a$ is compensated by the central term in the commutator (2.23). For higher p-branes the variation $\delta \Lambda_p = I_p^2$ always involves the field $A^a_\mu$ in addition to the ghosts $\omega^a$. Hence the analogue of (2.2) for p-branes must have an explicit dependence on $A^a_\mu$ as well as $\omega^a$ and $\Lambda_{\mu_1 \cdots \mu_p}$.

For example, for the 3-brane, we can accomplish this by writing:

$$\delta = \delta_3 + \delta_B \quad (2.24)$$

where $\delta_3$ acts on the 3-brane wave function $\Phi$

$$\delta_3 = \prod_{\mu,m,\sigma} \int dy^m(\sigma') dX^\mu(\sigma') \left\{ \left( \int d^3 \sigma \{-\omega^a T^a(\sigma) \right.ight.ight.

$$

$$-n \varepsilon^{ijk} d^{abc} \partial_\mu \omega^a A^b_\mu \Pi^\nu_{ij} T^c_k(\sigma) + \Lambda_{\mu \nu \lambda} \Pi^{\mu \nu \lambda} \Phi \left) \frac{d}{d\Phi} \right\} \quad (2.25)$$

Here we use the notation:

$$\Pi^\mu_{ij} = \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \quad (2.26)$$

$$\Pi^{\mu \nu \lambda} = \varepsilon^{ijk} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \frac{\partial X^\lambda}{\partial \sigma^k} \quad (2.27)$$

In the foregoing, $\delta_3$ is a BRS transformation which acts on $\Phi$, which is a functional
of the 3-brane variables $X^\mu(\sigma)$ and $y^m(\sigma)$. All the $T$ operators are again assumed to involve only functions of $y^m(\sigma)$ and $\delta y^m(\sigma)$ and hence the operators $T$ commute with $\delta B$. The background transformations are now:

$$\delta B = \int d^Dx \{ D_\mu^a \omega^b \frac{\delta}{\delta A^a_\mu} 
- \frac{1}{2} f^{abc} \omega^b \frac{\delta}{\delta \omega^c} + [n I_4^1(A, \omega) + d\Lambda_{\mu\nu\rho} \frac{\delta}{\delta B_{\mu\nu\rho}}] 
+ [n I_3^2(A, \omega) + dB_2]_{\mu\nu\lambda} \frac{\delta}{\delta \Lambda_{\mu\nu\lambda}} + \cdots + n I_0^5(\omega) \frac{\delta}{\delta B_0} \} \tag{2.28}$$

where

$$I_0^5 = d^{abc} A^a dA^b dA^c + \cdots \tag{2.29}$$

$$I_4^1 = -d^{abc} d\omega^a A^b dA^c + \frac{1}{4} d^{abc} d\omega^a A^b f^{cde} A^d A^e \tag{2.30}$$

$$I_3^2 = d^{abc} d\omega^a A^b d\omega^c \tag{2.31}$$

$$I_2^3 = -d^{abc} d\omega^a d\omega^b d\omega^c \tag{2.32}$$

$$I_1^4 = -\frac{1}{4} d^{abc} f^{cde} d\omega^a \omega^b d\omega^c \tag{2.33}$$

$$I_0^5 = -\frac{1}{40} d^{abc} f^{bde} f^{c\epsilon g} d\omega^a \omega^b d\omega^c f\omega^\epsilon f\omega^g \tag{2.34}$$

In particular:

$$\delta \Lambda_{\mu\nu\lambda} = -d^{abc} \partial_\mu \omega^a A^b \partial_\lambda \omega^c + \cdots \tag{2.35}$$

By calculation, one can show that the above $\delta$ is nilpotent if $T^a_1$ and $T^a_1$ satisfy the
Mickelsson-Faddeev algebra:

\[
[T^a(\sigma), T^b(\sigma')] = f^{abc} T^c(\sigma) \delta^3(\sigma - \sigma') - 2 nd^{abc} \epsilon^{ijk} \partial_i \delta^3(\sigma - \sigma') \partial_j T^c(\sigma') \tag{2.36}
\]

\[
[T^a(\sigma), T^b_i(\sigma')] = f^{abc} T^c_i(\sigma) \delta^3(\sigma - \sigma') + \delta^{ab} \partial'_i \delta^3(\sigma - \sigma') \tag{2.37}
\]

\[
[T^a_i(\sigma), T^b_j(\sigma')] = 0 \tag{2.38}
\]

One may verify that the Jacobi identities are satisfied by this algebra. Note the new kind of generator \(T^a\), which forms a (non-invariant) Abelian subalgebra of the \(T^a\) algebra. \(T^a\) transforms under the action of \(T^a\) like a Yang-Mills field.

The gauge invariant field strength associated with this nilpotent \(\delta B\) is:

\[
H_5 = dB_4 + n I^0_5 \tag{2.39}
\]

and it satisfies:

\[
\delta H_5 = \delta_B H_5 = 0 \tag{2.40}
\]

\[
dH_5 = I^0_6 = d^{abc} F^a F^b F^c \tag{2.41}
\]

3. Spacetime Algebras

If we take the term of \(\delta\) that is linear in the field \(\omega^a(x)\), then its algebra is also the Kac-Moody (p=1) or Mickelsson-Faddeev (p=3) algebra (pulled back). This works as follows. Define

\[
\delta = \int d^4 x \omega^a(x) T^a_{\text{tot}}(x) + \text{other terms} \tag{3.1}
\]

where the other terms are those which do not have exactly one field \(\omega\) in the numerator of the transformation.
Then nilpotence of $\delta$ implies that

$$
\frac{1}{2} \int d^D x \int d^D x' \omega^a(x) \omega^b(x') \{ [T^a_{\text{tot}}(x), T^b_{\text{tot}}(x')] 
- \delta^D(x - x') f^{abc} T^c_{\text{tot}}(x)] \} \Phi = n \int d^p \sigma I^2_p(X(\sigma))_{\mu_1...\mu_p} \Pi^{\mu_1...\mu_p} \Phi
$$

Using functional derivatives to peel off the two powers of $\omega$ in the above yields a ‘pulled back’ version of the algebra, which, for $p \geq 3$, has an $A$-dependent central extension, determined by the form of $I^2_p(X(\sigma))_{\mu_1...\mu_p}$. For $p = 1$ the extension can be chosen to be $A$-independent because $I^2_p$ can be chosen to be $A$-independent. The $A$-dependent extension for the $p = 3$ case is somewhat reminiscent of the situation in four-dimensional Yang-Mills field theory with fermions [5]. Explicitly for the 3-brane case we have:

$$
[T^a_{\text{tot}}(x), T^b_{\text{tot}}(x')] \Phi = \left\{ f^{abc} T^c_{\text{tot}}(x') \delta^D(x - x') 
+ 2n \left[ \int d^3 \sigma \delta^D(x - X(\sigma)) d^{abc} \partial^\lambda A^e_{\mu \nu} \partial_{\lambda} \delta^D(x - x') \right] \right\} \Phi
$$

4. Conclusion

Our motivation for this work was to see how the loop space algebra of the heterotic string can be generalized to the p-branes. One constructs a BRS transformation that transforms the background fields and the p-brane functional, and then demands that it be nilpotent.

For the string, this nilpotence relates the coefficient $n$ of the central extension of the Kac-Moody algebra of the operators $T^a$ formed from the group coordinates to the coefficient $n$ in the gauge invariant field strength

$$
H_3 = dB_2 + n I_3
$$

of the background Yang-Mills fields.
We have shown that for the 3-brane, it is necessary to introduce operators $T^a_i(\sigma)$ and $T^a(\sigma)$ which are formed from the group coordinates. These operators obey the well-known Mickelsson-Faddeev algebra familiar from anomaly analysis in four-dimensional theories with chiral fermions. In particular the operators $T^a_i(\sigma)$ transform like Yang-Mills fields under the action of $T^a(\sigma)$. We believe that the operators $T$ obtained by an analysis along the lines of [2] of the action in [1] should provide a realization of the Mickelsson-Faddeev algebra discussed here. Nilpotence of the BRS transformation of the 3-brane functional $\Phi$ relates the coefficient $n$ of the (non-invariant) Abelian extension of the algebra (2.36) to the parameter $n$ in the gauge invariant field strength

$$H_5 = dB_4 + n I_5$$

(4.2)

of the background Yang-Mills fields.

We anticipate that this procedure should easily generalize to higher $p$, and in particular to the heterotic 5-brane [10,11,12,13,1] which in fact provided the original impetus for the present paper.

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