Generalized normal forms of infinitesimal symplectic and contact transformations in the neighbourhood of a singular point

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Abstract. Definition of generalized normal form for a system of ODEs corresponding to an infinitesimal symplectic or contact transformation near a singular point, with an arbitrary polynomial unperturbed part, and a method of its finding are introduced. Applicability of the introduced method to studying the critical phenomena in non-ideal media is shown. As examples, generalized normal forms for the equations of state of a mixture of non-ideal gases and non-ideal multicomponent plasma are considered within the framework of perturbation theory. In particular, it is shown that the lowest order perturbation effects in the Debye-Hückel hydrogen plasmas are classified by only three constant parameters.

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1. Introduction

In the introduction, some necessary facts from symplectic and contact geometry are given. For a complete overview see [1, Ch 1, §§ 6, 7] and references therein.

A symplectic form on a manifold $M$ of even dimension $s = 2n$ is a closed 2-form $\omega$ of maximal rank, i.e., $d\omega = 0$ and $\omega^n \neq 0$ everywhere on $M$. There exists a local coordinate system $z_1, \ldots, z_{2n}$ for which $\omega = \sum_{i=1}^{2n} dz_{n+i} \wedge dz_i$ (see [1, p 154]). Such coordinates are called canonical coordinates.

A transformation $f$ (or an infinitesimal transformation $X$, respectively) of $M$ is said to be symplectic if $f^* \omega = \omega$ ($L_X \omega = 0$), where $f^*$ denotes the linear mapping transposed to $df$, and $L_X$ is the Lie derivative along the vector field $X$.

In canonical coordinates, an infinitesimal symplectic transformation is given by

$$ X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial z_i} \frac{\partial}{\partial z_i} - \frac{\partial H}{\partial z_{n+i}} \frac{\partial}{\partial z_{n+i}}, \quad (1) $$

where $H$ is an arbitrary differentiable function. $H$ is called the Hamiltonian, and infinitesimal transformation (1) is called a Hamiltonian vector field. The corresponding system of ODEs is called a Hamiltonian system.

Defining the Poisson bracket of two functions $\Phi$ and $\Psi$ by $X_{\{\Phi, \Psi\}} = [X_\Phi, X_\Psi]$, in canonical coordinates we have the classical formula

$$ \{\Phi, \Psi\} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i} \frac{\partial \Psi}{\partial z_{n+i}} - \frac{\partial \Phi}{\partial z_{n+i}} \frac{\partial \Psi}{\partial z_i}. $$

A contact form on a manifold $M$ of odd dimension $s = 2n + 1$ is an open cover $\{U_\alpha\}$ of $M$ together with a system of 1-forms $\omega_\alpha$ of maximal rank, i.e., $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ everywhere on $U_\alpha$, such that $\omega_\alpha = \varphi_{\alpha\beta} \omega_\beta$ on $U_\alpha \cap U_\beta$ for some functions $\varphi_{\alpha\beta} \neq 0$.

For any point $p \in U_\alpha$, there exists a coordinate neighbourhood $U_p \subset U_\alpha$ with coordinates $z_1, \ldots, z_{2n+1}$ in which $\omega_\alpha = dz_{2n+1} - \sum_{i=1}^{n} z_{n+i} dz_i$ (see [1, p 150]). Just as in the symplectic case, we call such coordinates canonical.

A transformation $f$ of $M$ is said to be contact if for all $\alpha, \beta$, $f^* \omega_\alpha = \Lambda_{\alpha\beta} \omega_\beta$, where $\Lambda_{\alpha\beta} \neq 0$ everywhere on $f^{-1}(U_\alpha) \cap U_\beta$. Accordingly, an infinitesimal transformation $X$ is called contact if $L_X \omega_\alpha = \lambda_\alpha \omega_\alpha$ for some functions $\lambda_\alpha$.

In canonical coordinates, an infinitesimal contact transformation is given by

$$ X_H = \sum_{i=1}^{n} \left[ - \frac{\partial H}{\partial z_{n+i}} \frac{\partial}{\partial z_i} + \left( \frac{\partial H}{\partial z_i} + z_{n+i} \frac{\partial H}{\partial z_{2n+1}} \right) \frac{\partial}{\partial z_{n+i}} \right] + $$

$$ + \left( H - \sum_{j=1}^{n} z_{n+j} \frac{\partial H}{\partial z_{n+j}} \right) \frac{\partial}{\partial z_{2n+1}}, \quad (2) $$

where $H$ is an arbitrary differentiable function. The function $H$ is sometimes called a contact Hamiltonian, or simply the Hamiltonian.
In the contact case, the Poisson bracket in canonical coordinates has the view
\[
\{\Phi, \Psi\} = \Phi \frac{\partial \Psi}{\partial z_{2n+1}} - \Psi \frac{\partial \Phi}{\partial z_{2n+1}} + 
\sum_{i=1}^{n} \left( \frac{\partial \Phi}{\partial z_{i}} + z_{n+i} \frac{\partial \Phi}{\partial z_{2n+1}} \right) \frac{\partial \Psi}{\partial z_{n+i}} - \left( \frac{\partial \Psi}{\partial z_{i}} + z_{n+i} \frac{\partial \Psi}{\partial z_{2n+1}} \right) \frac{\partial \Phi}{\partial z_{n+i}}. \tag{3}
\]

From now on, only functions that are analytic in some neighbourhood \(U\) of the origin in \(\mathbb{R}^s\) are considered. We denote the algebra of analytic functions in \(U\) by \(\mathbb{R}_U(z)\), and the algebra of polynomial functions in \(z_1, \ldots, z_s\) by \(\mathbb{R}[z]\).

Diffeomorphisms of \(\mathbb{R}^s\) onto itself of the form \(\exp(X_P)\) with \(P \in \mathbb{R}[z]\) generate the subgroup \(\mathfrak{A}_s\) in the group of symplectic (if \(s = 2n\)), or contact (if \(s = 2n+1\)), transformations of \(\mathbb{R}^s\). The algebra of infinitesimal symplectic, or contact, transformations is closed under the action of this subgroup, and the Hamiltonian functions are transformed according to the law
\[
H \to \exp(\hat{P})(H),
\]
where \(\hat{P} (\cdot) = \{P, \cdot\}\).

Let the origin be a singular point of the vector field \(X_H\), i.e. \(X_H|_{z=0} = 0\), and let the corresponding Hamiltonian have the view
\[
H = H_0 + H_1 \quad (H_0 \in \mathbb{R}[z], \ H_1 \in \mathbb{R}_U(z)). \tag{4}
\]
We call \(H_0\) the unperturbed part, and \(H_1\) the perturbation of the Hamiltonian \(H\).

In the first four sections of the present article, the general problem of bringing the perturbation of \(H\) into the simplest form (normal form) by transformations from the group \(\mathfrak{A}_s\) that do not change the unperturbed part of \(H\) is being solved. Such problems often arise in studying Hamiltonian and nearly Hamiltonian systems. As a result, there are many different ad hoc methods and definitions of normal forms (see, for instance, [2, 3, 4]) that require the unperturbed Hamiltonian to satisfy certain conditions such as quadraticity, homogeneity, etc. But such requirements can be excessive in some physical problems. The generalized normal form introduced in this article is free of such restrictions. It is worth noting that the contact case was earlier considered only in the simplest, generic, situation. However, it turns out that real physical problems, for instance thermodynamics of plasmas, often refer to the non-generic cases. In section 5 we introduce some applications of the method of generalized normal forms to thermodynamics of non-ideal media.

2. Basic definitions

As usual, the sets of positive and nonnegative integers will be denoted by \(\mathbb{N}\) and \(\mathbb{Z}_+\) respectively.

Let \(n \in \mathbb{N}\) and \(s = 2n\), or \(2n + 1\), be fixed.
Definition 1. A vector $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{N}^s$ is called a weight of the variable $z = (z_1, \ldots, z_s)$ if $\gcd(\gamma_1, \ldots, \gamma_s) = 1$. If, moreover, for some natural $\sigma \geq 2$, $\gamma_1 + \gamma_{n+1} = \ldots = \gamma_n + \gamma_{2n} = \sigma$, and in the case of odd $s$, $\gamma_{2n+1} = \sigma$, we say that $\gamma$ is a canonical weight.

Definition 2. Given a power series (or a polynomial) $P = \sum_{\nu \in \mathbb{Z}^n} a_{\nu} z^\nu$, the generalized order (the generalized degree) of $P$ with weight $\gamma$ is the least (respectively, the greatest) of numbers $\nu \cdot \gamma = \nu_1 \gamma_1 + \ldots + \nu_s \gamma_s$ such that $a_{\nu} \neq 0$ if $P \neq 0$, and $\infty$ (or 0, respectively) otherwise.

Definition 3. A polynomial $P \in \mathbb{R}[z]$ is said to be quasi-homogeneous with weight $\gamma$ if either its generalized order and generalized degree with weight $\gamma$ are equal or $P$ is identically zero.

Definition 4. Given an arbitrary power series $P = \sum_{\nu \in \mathbb{Z}^n} a_{\nu} z^\nu$, define its $[k]$-jet with weight $\gamma$ by $J_{\gamma}^k(P) = \sum_{\nu, \gamma_0 = 0}^k a_{\nu} z^\nu$.

Denote the generalized order of $P$ with weight $\gamma$ by $\ord_\gamma P$, the generalized degree by $\deg_\gamma P$, and the vector space of all $[k]$-jets with weight $\gamma$ by $J_{\gamma}^k$.

Consider a power series $H(z) = H_0(z) + H_1(z)$ with the unperturbed part $H_0 \in \mathbb{R}[z]$ and the perturbation $H_1 \in \mathbb{R}_{U}(z)$ such that for some fixed canonical weight $\gamma$, $\ord_\gamma H_1 > \deg_\gamma H_0 \geq \ord_\gamma H_0$.

For brevity, denote $\delta = \ord_\gamma H_1 - \ord_\gamma H_0$ and $\Delta = \ord_\gamma H_1$.

Define an inner product on polynomials by

$$\langle P, Q \rangle = P(\partial) Q(z) \big|_{z=0} \quad (P, Q \in \mathbb{R}[z], \partial = (\partial/\partial z_1, \ldots, \partial/\partial z_s)), \quad \text{for all} \quad ij \in \mathbb{N},$$

and fix some linear order $\succ$ on the set of monomials in $z$.

For any integer $d \geq \Delta$, define a vector space $\mathfrak{N}_d \subset J_{\gamma}^{[d+\Delta]}$ as follows:

$$\mathfrak{N}_d = \left\{ J_{\gamma}^{[d+\Delta]}(\{F, H_0\}) : F \in \mathbb{R}[z], \ord_\gamma F \geq d + \delta - \Delta + \sigma \right\},$$

and denote the set of leading monomials of the elements of $\mathfrak{N}_d$ with respect to $\succ$ by $\mathrm{LM}(\mathfrak{N}_d)$.

Lemma 1. For any integer $d \geq \Delta$, there exists a basis $\mathfrak{B}_d = \{G_i\} \subset \mathfrak{N}_d$ such that $\langle G_i, \mathrm{LM}(G_j) \rangle = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta. Herewith, the sets $\mathrm{LM}(\mathfrak{B}_d)$ and $\mathrm{LM}(\mathfrak{N}_d)$ are the same.

Proof. The proof is by induction. Let $\{B_i\}$ be an arbitrary basis for $\mathfrak{N}_d$. Setting $G_1 = B_1/\langle B_1, \mathrm{LM}(B_1) \rangle$ gives $\mathrm{LM}(G_1) = \mathrm{LM}(B_1)$ and $\langle G_1, \mathrm{LM}(G_1) \rangle = 1$.

Suppose that for some $k \in \mathbb{N}$ and for all $i, j \leq k$, we have $\langle B_i, \mathrm{LM}(B_j) \rangle = \delta_{ij}$. Denote $G_{k+1}' = B_{k+1} - \sum_{i=1}^k \langle B_{k+1}, \mathrm{LM}(B_i) \rangle B_i$, so that $\langle G_{k+1}', \mathrm{LM}(B_i) \rangle = 0$. Then setting $G_{k+1} = G_{k+1}' / \langle G_{k+1}', \mathrm{LM}(G_{k+1}') \rangle$ and $G_i = B_i - \langle B_i, \mathrm{LM}(G_{k+1}) \rangle G_{k+1}$ for all $i \leq k$ gives $\mathrm{LM}(G_i) = \mathrm{LM}(B_i)$, and hence $\langle G_i, \mathrm{LM}(G_j) \rangle = \delta_{ij}$ for all $i, j \leq k + 1$, which completes the induction and proves the first statement of the lemma.

Since $\mathfrak{N}_d$ is finite-dimensional, the second statement follows at once from the fact that all of the elements of $\mathfrak{B}_d$ have different leading monomials. \qed
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Definition 5. Define a minimal resonant set of index $d$ as a complement of $\mathcal{R}_d$ in the set of monomials of generalized degree $k$, where $d \leq k < d + \delta$.

Denote the vector space spanned by a minimal resonant set of index $d$ by $\mathfrak{R}_d$. Then given a basis $G$ for some $\mathfrak{R}_d$ as in Lemma 1, the formula

$$\pi_d(P) = P - \sum_i \angle(P, \text{LM}(G_i)) G_i$$

(8)

defines a projection of the vector space of $[d + \delta - 1]$-jets of generalized order greater than or equal to $d$ onto $\mathfrak{R}_d$.

3. Normal form theorem

Theorem 1. For any integer $N \geq \Delta$, there exists a transformation from $A_s$ that brings $H$ into the form

$$\tilde{H} = H_0 + \sum_{k=0}^{\lfloor(N-\delta)/\delta\rfloor} R_k + S,$$

(9)

where $R_k \in \mathfrak{R}_{\Delta+k\delta}$ and ord, $S > N$.

Proof. Let $F \in \mathbb{R}[z]$ and ord, $F \geq \delta + \sigma$. Then the transformation $\exp(X_F) \in A_s$ brings $H$ into the form

$$\exp(\tilde{F})(H) = H_0 + H_1 + \{F, H_0\} + \ldots,$$

where ord, $\{F, H_0\} \geq \Delta$, and the dots denote terms of generalized order greater than or equal to $\Delta + \delta$.

More generally, if ord, $F \geq k \delta + \sigma$ for some $k \in \mathbb{N}$, then in the same formula ord, $\{F, H_0\} \geq \Delta + (k-1)\delta$, and the generalized order of terms denoted by dots is greater than or equal to $\Delta + k\delta$.

Suppose that $H = H_0 + \sum_{k=0}^{m-1} R_k + S_m$, where $R_k \in \mathfrak{R}_{\Delta+k\delta}$, ord, $S_m \geq \Delta + m\delta$, for some $m \in \mathbb{Z}_+$. The proof is by induction on $m$ with $S_0 = H_1$ for $m = 0$.

Write $S_m$ as $S_m = P_m + Q_m$, where $P_m = J_R^{\lfloor(A+(m+1)\delta-1)/(\delta)\rfloor}(S_m)$. Since $P_m - \pi_{\Delta+m\delta}(P_m) \in \mathfrak{R}_{\Delta+m\delta}$, where $\pi_{\Delta+m\delta}$ and $\mathfrak{R}_{\Delta+m\delta}$ are as defined in (7)–(8), there exists a polynomial $F_m$ of generalized order greater than or equal to $(m+1)\delta + \sigma$ such that $P_m + J_R^{\lfloor(A+(m+1)\delta-1)/(\delta)\rfloor}(\{F_m, H_0\}) = \pi_{\Delta+m\delta}(P_m)$. Denote $R_m = \pi_{\Delta+m\delta}(P_m)$. Then the transformation $\exp(X_{F_m}) \in A_s$ brings $H$ into the form $H_0 + \sum_{k=0}^{m+1} R_k + S_{m+1}$, where ord, $S_{m+1} \geq \Delta + (m+1)\delta$.

By induction, $m$ can be made arbitrarily large, and in particular, equal to $\lceil(N-\Delta)/\delta\rceil$, so that ord, $S_{m+1} \geq \Delta +[((N-\Delta)/\delta) + 1]\delta > N$. Since at each step the normalizing transformation has the form $\exp(X_{F_m}) \in A_s$, and the number of steps is finite, the composition of these transformations lies in $A_s$.

Definition 6. We say a Hamiltonian $H$ with the unperturbed part $H_0$, as well as the corresponding vector field, to be in generalized normal form up to generalized degree $N$ if it has the form (9).
4. The case of quasi-homogeneous unperturbed part

In case where the unperturbed Hamiltonian is quasi-homogeneous for some canonical weight, there is an alternative way of describing generalized normal forms in terms of what we call the ”resonant sets”. This method requires no linear order on monomials and is stated as follows.

Denote the vector space of quasi-homogeneous polynomials in \( z \) of generalized degree \( k \in \mathbb{Z}_+ \) with weight \( \gamma \) over \( \mathbb{R} \) by \( \mathbb{R}_\gamma^{|k|} \).

Let \( H_0 = H_0^{[x]} \in \mathbb{R}_\gamma^{|x|} \), where \( \gamma \) is a canonical weight, and \( \chi \geq \sigma \). Then for each \( k \in \mathbb{Z}_+ \), the linear operator \( \hat{H}_\gamma^{[x]} = \{ H_\gamma^{[x]}, \cdot \} \) maps \( \mathbb{R}_\gamma^{|k|} \) to \( \mathbb{R}_\gamma^{|k+\chi-\sigma|} \), and its conjugate with respect to the inner product \((\cdot, \cdot)\) has the form

\[
\hat{H}_\gamma^{[x]} = \sum_{i=1}^{n} z_{n+i} \left( \frac{\partial H_\gamma^{[x]}}{\partial z_i} \right)^* - z_i \left( \frac{\partial H_\gamma^{[x]}}{\partial z_{n+i}} \right)^*,
\]

in the symplectic case, and

\[
\hat{H}_\gamma^{[x]} = z_{2n+1} H_\gamma^{[x]} - \left( \frac{\partial H_\gamma^{[x]}}{\partial z_{2n+1}} \right)^* + \sum_{i=1}^{n} z_{n+i} \left( \frac{\partial H_\gamma^{[x]}}{\partial z_i} + z_{n+i} \frac{\partial H_\gamma^{[x]}}{\partial z_{2n+1}} \right)^* - \left( z_i + z_{2n+1} \frac{\partial}{\partial z_{n+i}} \right) \left( \frac{\partial H_\gamma^{[x]}}{\partial z_{n+i}} \right)^*,
\]

in the contact case. Here \( P^* = P(\partial) \) for each \( P \in \mathbb{R}[z] \).

**Definition 7.** We call the equation

\[
\hat{H}_\gamma^{[x]}(P) = 0
\]

for \( P \in \mathbb{R}[z] \) the resonance equation, and its solutions resonant polynomials.

Denote the vector space of resonant polynomials by \( \mathcal{R}_\gamma \).

By properties of the inner product \((\cdot, \cdot)\), it follows that \( \mathcal{R}_\gamma \) splits into direct sum of the orthogonal subspaces \( \mathcal{R}_\gamma^{|k|} \) spanned by quasi-homogeneous resonant polynomials of generalized degree \( k \in \mathbb{Z}_+ \).

**Definition 8.** We call a set of quasi-homogeneous polynomials \( \mathcal{S}_\gamma^{|m|} = \{ S_j \} \) a resonant set of generalized degree \( m \) with weight \( \gamma \) if for some basis \( \{ R_i \} \) of \( \mathcal{R}_\gamma^{|m|} \), \( \det(\{ \langle R_i, S_j \rangle \}) \neq 0 \). Given a resonant set \( \mathcal{S}_\gamma^{|m|} \) for each generalized degree \( m \in \mathbb{Z}_+ \), we call their union, \( \mathcal{S}_\gamma = \bigcup_{m=0}^{\infty} \mathcal{S}_\gamma^{|m|} \), a resonant set. If, moreover, \( \mathcal{S}_\gamma \) consists only of monomials, we say that it is a minimal resonant set.

Clearly, the definition of a resonant set does not depend on the choice of basis for \( \mathcal{R}_\gamma^{|k|} \). Note that in the case of a quasi-homogeneous unperturbed part, a minimal resonant set in the sense of Definition \( \mathcal{S}_\gamma \) is also a minimal resonant set in the sense of Definition \( \mathcal{S}_\gamma \) and vice versa.

**Theorem 2.** For each integer \( N \geq \Delta \) and arbitrarily chosen by \( H_\gamma^{[x]} \) resonant set \( \mathcal{S}_\gamma \), there exists a transformation from \( \mathcal{A}_\gamma \) that brings \( H \) to the form

\[
\tilde{H} = H_0 + \sum_{k=\Delta}^{N} R_\gamma^{|k|} + S,
\]
where $R_{\gamma}^{[k]} \in \text{span}(G_{\gamma}^{[k]})$ and $\text{ord}_{\gamma} S > N$.

Proof. The theorem follows from Definition 8 and the Fredholm alternative for finite-dimensional spaces. Detailed proof in the symplectic case (in the contact case, the proof is exactly the same) is given in [5].

For the symplectic case, a close result to Theorem 2 can be found in [4]. However, the normal form equations given there are very complicated, since the unperturbed part at each step of normalization involves normalized terms of the perturbation (for further information, see [6]). The quasi-homogeneous symplectic case was first analyzed in [5], where the canonicity condition on the weight was stated and definition of a resonant set was introduced.

5. Applications of generalized normal forms in thermodynamics

5.1. Thermal equations of state

In thermodynamics, relations between macroscopic parameters are described by the equations of state. Equations that provide a relationship between the temperature $T$, the volume concentrations $n_i$ ($i = 1, k$) of the system components, where $k$ denotes the number of components, the pressure $P$, and other generalized thermodynamic forces, are called thermal equations of state. For example,

$$P = T \sum_{i=1}^{k} n_i,$$

$$P = T \sum_{i=1}^{k} n_i - \frac{\sqrt{\pi}}{3 T^{1/2}} \left( \sum_{i=1}^{k} n_i q_i^2 \right)^{3/2}$$

are, respectively, thermal equations of state of the ideal gas and the classical Debye-Hückel plasma written in ESU CGS units, where $T$ is considered in erg (see, for instance, [7, pp 151, 282]). Here, $q_i$ denotes the electric charge of a particle of $i$-th type.

There are two approaches to describing thermodynamical properties of non-ideal media, the rigorous one treating media as a system of nuclei and electrons interacting by Coulomb’s law, and the semi-empirical one that consists in constructing qualitative models, in which the general form of functional relationships is established on theoretical arguments, and the numerical coefficients are determined experimentally [8, s 8]. Method of generalized normal forms allows to reduce significantly the number of unknown coefficients in such qualitative models of non-ideal media.

Consider a thermal equation of state of the form

$$H(P, T, n) = 0, \quad n = (n_1, \ldots, n_k).$$

Using the Gibbs-Duhem equation (see, for instance, [9 p 322])

$$\frac{dP}{s} - \frac{dT}{s} - \sum_{i=1}^{k} n_i d\mu_i = 0,$$
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where $s$ denotes the entropy density, and $\mu_i$ the chemical potential of the $i$-th type of particles, (13) is seen to be a first order partial differential equation for $P$. Its characteristic equations are determined by infinitesimal contact transformation (2):

$$
\begin{align*}
\dot{P} &= H - \sum_i n_i \frac{\partial H}{\partial n_i} \\
\dot{n}_i &= n_i \frac{\partial H}{\partial P} \\
\dot{T} &= 0
\end{align*}
$$

(14)

where subsystem (14) contains only variables that enter (13). Here, the dot denotes differentiating with respect to some parameter that does not have a physical sense itself. Note that the quantities $T, n_j/\sum_i n_i, H/\sum_i n_i$ are integrals of (14), and the surface given by (13) is integral for this system.

Suppose that $H$ can be represented in the form (4)–(5), where $z_i = \mu_i, z_{k+1} = T, z_{k+i} = n_i, z_{2(k+1)} = s, z_{2(k+1)+1} = P \ (i = 1, k)$. Then the above theory is applicable, so by Theorem 1, $H$ can be put into normal form $\tilde{H}$ up to a desired degree by a contact transformation, with the surface $H = 0$ transformed to $\tilde{H} = 0$.

By definition, variables $s$ and $\mu_i$ do not enter a thermal equation of state. So if we do not want them to appear in $\tilde{H}$, it is necessary to consider the transformations that depend only on $P, T,$ and $n_i$. This follows from the form of the Poisson bracket (3).

5.2. Perturbation theory for ideal gases

Equation of state of a mixture of non-ideal gases is usually written in the form of a virial expansion, also called the Mayer cluster expansion:

$$
P = T \sum_{i=1}^{k} n_i + \sum_{|m| \geq 2} B_m n^m, \quad m = (m_1, \ldots, m_k).
$$

(15)

$B_m$ are called virial coefficients and characterized by contributions of the interactions within $|m|$-particle groups, or clusters, of molecules of gas.

Suppose that $B_m$ are analytic functions of temperature near the point $T = 0$. Then the following theorem holds.

**Theorem 3.** For each integer $M \geq 2$, all the virial coefficients $B_m$ with $|m| \leq M$ in expansion (15) can be eliminated by a contact transformation.

**Proof.** In generalized degree 2 with weight $\gamma = (1, \ldots, 1, 2)$, where 2 is the weight of $P$, equation of state (15) can be written as

$$
P = T \sum_{i=1}^{k} n_i + \sum_{i,j=1}^{k} b_{ij} n_i n_j
$$

Here $b$ is a constant symmetric matrix. Clearly, the contact transformation $\tilde{P} = P - \sum_{i,j=1}^{k} b_{ij} n_i n_j/2, \tilde{\mu}_i = \mu_i - \sum_{j=1}^{k} b_{ij} n_j$ brings it into the form (11). Thus, the
part of the second virial coefficients that does not depend on temperature can always be eliminated by a contact transformation.

Let us prove the statement for terms of the perturbation of generalized degree greater than 2. For this purpose write out the resonance equation (10) for the unperturbed Hamiltonian $H_{\gamma}^2 = P - T \sum_{i=1}^{k} n_i$ corresponding to (11), taking into account that the perturbation depends only on $P$, $T$, and $n_i$:

$$P \frac{\partial R}{\partial P} + \sum_{i=1}^{k} n_i \frac{\partial R}{\partial n_i} - R = 0 \quad (R \in \mathbb{R}[P, T, n_i]).$$

Monomials of the form $PT^l$ and $n_i T^l$ make up a basis of solutions of this equation, and hence by Theorem 2, normal form of the perturbation. In particular, terms of degree higher than one in $n_i$ are absent in normal form.

Theorem 3 agrees with a well known fact that the media described by virial expansions (for example, the Van der Waals gas) permit continuous transitions between any two states without ever separating into two different phases [7, pp 298, 299]. Apparently, for the same reasons as claimed in Theorem 3, the virial expansion turned out to be useless for describing critical phenomena, such as the gas-liquid-solid triple point (for further discussion, see [10, pp 105–110]).

5.3. Perturbation theory for classical Debye-Hückel plasmas

Equation of state (12) of $k$-component plasma holds under assumptions of electrical neutrality and smallness of the Coulomb interaction in comparison to the kinetic energy [7, pp 279, 280]:

$$\sum_{i=1}^{k} n_i q_i = 0, \quad \sum_{i=1}^{k} n_i \ll \left(\frac{T}{\langle q^2 \rangle}\right)^3.$$  \hspace{1cm} (16)

The unperturbed Hamiltonian corresponding to (12) can be chosen as

$$H_0 = T \left( P - T \sum_{i=1}^{k} n_i \right)^2 - \frac{\pi}{9} \left( \sum_{i=1}^{k} n_i q_i^2 \right)^3.$$  \hspace{1cm} (17)

Without loss of generality, assume that $q_1 = 0$, $q_2 > 0$, and $q_3 < 0$. Then the change of variables $\tilde{n}_1 = \sum_{i=1}^{k} n_i$, $\tilde{n}_2 = \sum_{i=1}^{k} n_i q_i$, $\tilde{n}_3 = (\pi/9)^{1/3} \sum_{i=1}^{k} n_i q_i^2$, $\tilde{n}_j = n_j$, where $j = 4, k$; is nondegenerate and can be extended to a contact transformation. For this purpose, choose $\tilde{\mu}_i$ that satisfy $\tilde{n}_i d\tilde{\mu}_i = n_i d\mu_i$ ($i = 1, k$), and leave $P$, $T$, and $s$ unchanged. Notice that condition (16) takes the form $\tilde{n}_2 = 0$.

Let us denote

$$z_1 = \tilde{\mu}_1, \quad z_j = \tilde{\mu}_{j+1}, \quad z_k = T, \quad z_{k+1} = \tilde{n}_1, \quad z_{k+j} = \tilde{n}_{j+1}, \quad z_{2k} = s, \quad z_{2k+1} = P \quad (j = 2, k - 1).$$  \hspace{1cm} (18)

In new variables, the Hamiltonian (17) is written

$$H_0 = z_k \left( z_{2k+1} - z_k z_{k+1} \right)^2 - z_{k+2}^3.$$
Inequality (16.1), which takes the form \( z_{k+2} \ll z_k^3 \) according to (18), implies a natural condition on weight \( \gamma \) of the variable \( z : \gamma_{k+2} = 3 \gamma_k \). It follows in turn from (12) that \( \gamma_{2k+1} = 4 \gamma_k \). Furthermore, (18) implies \( \gamma_{k+j} = \gamma_{k+1} (j = 2, k-1) \). The remaining components of \( \gamma \) are determined by the canonicity condition. Thus, in the case under consideration, the natural canonical weight has the form \( \gamma = (1, \ldots, 1, 3, \ldots, 3, 4) \), and \( H_0 \) is a quasi-homogeneous polynomial in \( z \) of generalized degree 9 with weight \( \gamma \).

**Theorem 4.** The minimal resonant set \( \mathcal{S} \) corresponding to a lexicographical order determined by the order \( z_{2k+1} \gg z_k \gg z_{k+1} \gg \ldots \gg z_{2k-1} \) consists of the following monomials:

\[
(i) \quad z_k^{i_1} \cdots z_{2k-1}^{i_{2k-1}},
(ii) \quad z_{2k+1}^{i_{2k+1}} z_{k+j}, \text{ where } j = 1, k-1, \text{ and } z_{2k+1}^{3} z_k^{i_k+1};
(iii) \quad z_{2k+1}^{i_{2k+1}} z_{k+1}^{i_{k+1}} \cdots z_{2k-1}^{i_{2k-1}}, \text{ except } z_{2k+1}^{3} z_{k+2}^{2},
\]

where \( i_{2k+1}, i_k, i_{k+1}, \ldots, i_{2k-1} \in \mathbb{Z}_+ \).

**Proof.** For all \( i_{2k+1}, i_k, i_{k+1}, \ldots, i_{2k-1} \in \mathbb{Z}_+ \), we have

\[
\{H_0, z_{2k+1}^{i_{2k+1}} z_k^{i_k} \cdots z_{2k-1}^{i_{2k-1}} \} = z_{2k+1}^{i_{2k+1}-1} z_k^{i_k} \cdots z_{2k-1}^{i_{2k-1}} \times \left[ 2 \left( \frac{i_{2k+1}}{2} + i_{k+1} + \ldots + i_{2k-1} - 1 \right) z_{2k+1}^{2} z_k - 2 (i_{k+1} + \ldots + i_{2k-1} - 1) z_{2k+1}^{2} z_k z_{k+1} - i_{2k+1} z_k^{2} z_{k+1}^{3} + 2 i_{2k+1} z_k^{3} z_{k+2}^{3} \right].
\]

This implies that all monomials divisible by \( z_{2k+1}^{2} z_k \), except \( z_{2k+1}^{3} z_k^{i_k+1} z_{k+j} \) for \( j = 1, k-1 \) and \( z_{2k+1}^{3} z_k^{i_k+1} \), do not belong to \( \mathcal{S} \).

Furthermore, in case where \( i_{2k+1} + 2 i_{k+1} + \ldots + 2 i_{2k-1} = 2 \), the right-hand side of (19) is identically zero if \( i_{2k+1} = 0 \). Reducing one by one the leading monomials in the Poisson bracket (19) in the case where \( i_{2k+1} = 2 \) and \( i_{k+j} = 0 \) (\( j = 1, k-1 \)), we come to the following equality:

\[
\{H_0, z_k^{i_k} (z_{2k+1} - z_k z_{k+1})^{2} \} = 4 z_k^{i_k} (z_{2k+1} z_{k+2}^{3} - z_k z_{k+1} z_{k+2}^{3}).
\]

This implies that the monomial \( z_{2k+1} z_{k+2}^{3} \) does not belong to \( \mathcal{S} \). For \( i_k \geq 1 \), further reduction of the leading monomial gives identical zero.

Collecting all the results together, we come to the statement of the theorem. \( \square \)

From now on, we consider, for simplicity, the plasma consisting of the hydrogen atoms, protons and electrons. We denote the corresponding parameters by the subindices a, p, and e. In this case, \( q_a = 0, q_p = -q_e = e \), where \( e \) is the elementary charge, \( n_e = n_p \), and

\[
\begin{align*}
&z_1 = \mu_a, \quad z_2 = \frac{3^{2/3}}{2 \pi^{1/3} c^2} (\mu_p + \mu_e - 2 \mu_a), \quad z_3 = T, \\
&z_4 = n_a + 2 n_p, \quad z_5 = \frac{2 \pi^{1/3} c^2}{3^{2/3}} n_p, \quad z_6 = s, \quad z_7 = P.
\end{align*}
\]
Theorem 4 gives the following generalized normal form of the equation of state:

\[ z_3(z_7 - z_3 z_4)^2 - z_5^3 + \sum_{i+3j+3k \geq 10} h^{[i,j,k]}_{[i,j,k]} z_3^i z_4^j z_5^k + \sum_{4i+3j+3k \geq 10} h^{[i,j,k]}_{[i,j,k]} z_7 z_4 z_5^k + \]

\[ + \sum_{i=1}^{\infty} [h^{[3,i,0,0]}_{[3,i,0,0]} z_7^3 + (h^{[1,i+2,1,0]}_{[1,i+2,1,0]} z_4 + h^{[1,i+2,0,1]}_{[1,i+2,0,1]} z_5) z_7 z_3^2 z_4^i] = 0, \quad (21) \]

where \( h^{[1,0,0,3]}_{[1,0,0,3]} = 0 \).

Let us consider the case, where the perturbation consists of elements of the minimal resonant set \( \mathcal{G} \) of generalized degree 10. According to (21) and the physical restriction \( z_7/z_3 \to z_4 \) as either \( z_3 \to \infty \) or \( z_5 \to 0 \), the form of such perturbation is determined uniquely up to three constant coefficients

\[ H_{\gamma}^{[10]} = -z_5 (a z_4 + b z_5 + c z_3^3) (z_7 - z_3 z_4). \quad (22) \]

Note that if we considered an arbitrary perturbation (not in normal form), we should have additionally taken into account the terms \( z_7^2 z_3^2 \) and \( z_7 z_3^6 \).

In order to find out the physical meaning of \( a, b, \) and \( c \), first look at the behaviour of the Debye–Hückel plasma \( (a, b, c = 0) \) ignoring (16). By integrating (14), we come to the following expressions for \( z_1 \) and \( z_2 \):

\[ z_1 = T \ln z_4 + C_1, \quad z_2 = -3^{2/3} \pi^{1/6} e^{\sqrt{2 \alpha z_4 / (1 + \alpha) T}} + C_2, \]

where \( \alpha = n_p / (n_a + n_p) \) is called the ionization coefficient. Recall that \( \alpha \) is an integral of (14). Since at low values of the concentration \( z_4 \), plasma turns to a mixture of ideal gases, \( C_1 \) and \( C_2 \) can be determined by replacing the chemical potentials of components in (20) with the chemical potentials of ideal gases that have the form

\[ \mu_{\text{id}} = T \ln \left[ \frac{n}{Z} \left( \frac{2 \pi \hbar^2}{m T} \right)^{3/2} \right], \]

where \( n \) denotes the concentration, \( Z \) the partition function of a particle, and \( m \) the particle mass [7, p 163]. For electrons and protons \( Z = 2 \), and for the hydrogen atoms, for simplicity, we take \( Z = 4 \exp(Ry/T) \), where \( Ry \approx 2.18 \cdot 10^{-11} \) erg is the Rydberg constant. This partition function ignores all the energy levels of the hydrogen atom but the lowest one. For more precision the Planck-Larkin partition function should be used.

Solve the equation of chemical equilibrium that takes the form

\[ z_1 + 3^{-2/3} 2 \pi^{1/3} e^2 z_2 = 0 \quad (23) \]

to obtain

\[ \alpha = \left[ 1 + n_p \left( \frac{2 \pi \hbar^2 m_a}{m_e m_p T} \right)^{3/2} \exp \left( \frac{Ry}{T} - \frac{e^3 \sqrt{8 \pi n_p}}{T^{3/2}} \right) \right]^{-1}, \quad (24) \]

which implies the effect of the ionization potential decreasing as \( n_p \) increases. Note that this expression is equivalent to the classical Saha formula with the partition function of the hydrogen atom replaced by \( Z = 4 \exp(Ry/T - e^3 \sqrt{8 \pi n_p/T^{3/2}}) \). Such approach to perturbation theory of plasmas consisting in perturbing the hydrogen atom partition
Figure 1: $\alpha(n_0)$ and $P(n_0)$ in the Debye-Hückel model at $T = 20\,000$ K (——), $T = 23\,000$ K (– – –), and $T = 26\,000$ K (- - - -).

function is usually called a chemical model and used, for instance, in [11] where the Planck-Larkin and the nearest neighbour partition functions are used in this context (for further information on chemical model, see [12, ss 5.1, 5.2]).

Denote $n_0 = n_a + n_p$ and rewrite (12) as follows

$$P = T n_0 (1 + \alpha) - \frac{\sqrt{8 \pi e^3}}{3 T^{1/2}} (\alpha n_0)^{3/2}. \quad (25)$$

Then (24) and (25) together with $n_p = \alpha n_0$ give a complete thermodynamical description of Debye-Hückel plasma.

According to (24), in the low density region where (16) is satisfied, the Saha approximation is valid. As the density $n_0$ increases the ionization coefficient stabilizes at first, and then after $n_0$ exceeds the value $\approx 8 \cdot 10^{21}$ particles/cm$^3$, rapidly tends to 1 (see figure 1a). At temperatures less than $\approx 23\,000$ K the curve $P(n_0)$ has two branches where $(\partial P/\partial n_0)_T > 0$ (see figure 1b) which corresponds to splitting of the plasma into two stable phases. The thin one is weakly ionized and called the dielectric gas phase and the dense is strongly ionized and called the metal gas phase. This phenomenon is called the plasma phase transition. For temperatures greater than $\approx 23\,000$ K metal gas phase becomes unstable since $(\partial P/\partial n_0)_T$ is negative on the corresponding branch. An overview of relevant theoretical results can be found in [12, s 6.4] and references therein. It should be noted that plasma phase transition has not yet been detected experimentally. For this reason it is often called hypothetical in literature.

As can be seen from figure 1b, the density $n_0$ is bounded above. Beyond this bound the pressure becomes negative, which is meaningless. Apparently, at higher densities a thermodynamical description of plasmas is impossible due to absence of quasi-static states which are the only considered in thermodynamics.

System (14) for perturbed Debye-Hückel plasmas with the perturbation (22) can be integrated explicitly as well. The results for small perturbations are given in figure 2. The coefficients $a$ and $b$ affect the pressure and the concentrations of phases at the point...
Figure 2: $\alpha(n_0)$ and $P(n_0)$ in perturbed Debye-Hückel plasmas at $T = 23000$ K for (a)–(b): $a = 0$ (——), $a T/e^4 \approx -0.04$ (---), and $a T/e^4 \approx 0.04$ (-----), $b, c = 0$; (c)–(d): $b = 0$ (——), $b T/e^2 \approx -0.06$ (---), and $b T/e^2 \approx 0.06$ (-----), $a, c = 0$; (e)–(f): $c = 0$ (——), $c e^2 T \approx 0.035$ (---), and $c e^2 T \approx 0.07$ (-----), $a, b = 0$.

of phase transition without having a significant effect on stability of the dense phase (see figures 2a–2d). The coefficient $a$ has a greater impact on properties of the thin phase while $b$ on the properties of the dense phase. At the same time the critical temperature increases with $c$, which affects stability of the dense phase, as shown on figures 2e, 2f.
Our conclusions on influence of the coefficients \(a, b\) and \(c\) are based on the properties of graphs of \(\alpha(n_0)\) and \(P(n_0)\) at fixed temperatures. Let us show that they are invariant under contact transformations. There are four monomials of generalized degree 5, namely, \(z_7 z_3, z_5^2 z_3, z_3^2 z_4\) and \(z_2^3 z_5\), but only two of them give non-zero Poisson brackets with \(H_0\) of generalized degree 10,

\[\{z_3 z_7, H_0\} = -3 z_3 z_5^3 + 2 z_2^2 (z_7 - z_3 z_4)^2, \quad \{z_3^5, H_0\} = 2 z_3^6 (z_7 - z_3 z_4).\]

The corresponding 1-parameter groups of contact transformations have the form

\[
\begin{align*}
[z_4 &\to e^{\lambda z_4} z_4, \quad z_5 \to e^{\lambda z_3} z_5, \quad z_6 \to e^{\lambda z_3} (z_6 + \lambda z_7), \quad z_7 \to e^{\lambda z_3} z_7, \\
[z_6 &\to z_6 + 5 \lambda z_4^2, \quad z_7 \to z_7 + \lambda z_3^2],
\end{align*}
\]

where the omitted variables are left unchanged, and \(\lambda\) is a parameter. Both transformations preserve the equation of chemical equilibrium (23), since they do not change the chemical potentials. The first transformation represents the uniform scaling of pressure and concentrations by the scale factor \(e^{\lambda z_3}\), so it does not affect our considerations. The second transformation shifts pressure by the value \(\lambda z_3^2\), which leads to violation of equality of \(P\) to zero at zero density and hence has no physical sense, so the term \(z_6^3 (z_7 - z_3 z_4)\) is restricted in any perturbation of the Debye-Hückel model.

6. Conclusion

As can be seen from the above examples, the introduced method of generalized normal forms is a flexible instrument for studying the critical phenomena in various physical systems. It can be used whenever there is a natural symplectic or contact form. To our knowledge, thermodynamic equations of state have never been considered from this point of view. In the context of infinitesimal contact transformations and the corresponding systems of ODEs, normal forms were previously studied only in the case of quasi-homogeneous unperturbed part \(H_0\) of generalized degree 2 with weight \(\gamma = (1, \ldots, 1, 2)\) with generic coefficients (see [13, p 37] and references therein).

In both examples the unperturbed Hamiltonian \(H_0\) was a quasi-homogeneous polynomial with a canonical weight, and a generalized normal form was obtained rather easily. However, it is not so for all interesting systems. For example, the Hamiltonian

\[H_0 = \frac{m p_1^2 + \lambda_1 q_1^{2m}}{2m} - \frac{n p_2^2 + \lambda_2 q_2^{2n}}{2n},\]

where \(\lambda_1, \lambda_2 > 0, m, n \in \mathbb{N}, n > m > 1\), is not quasi-homogeneous for any canonical weight. Such a Hamiltonian arises in studying the interaction between two nonlinear oscillators at the level of perturbation. The problem of stability of the equilibrium for a system with such an unperturbed part was partially solved in [14] by methods of KAM theory: for sufficiently high orders of the perturbation, the origin is Lyapunov stable. At low orders it is natural to deal with a normal form instead of an arbitrary perturbation, since it contains a significantly smaller number of terms. In the general case, explicit formulas for minimal resonant sets turn out to be very complicated. Nevertheless,
Lemma 1 gives an algorithm, which can be used to obtain the leading terms of the Poisson brackets, and hence a minimal resonant set, for given \( m, n, N \), and linear order on monomials. For example, the minimal resonant set for \( m = 2, n = 3, N = 11 \), and a lexicographical order determined by the order \( p_1 > q_1 > p_2 > q_2 \) consists of monomials \( p_2^3 q_1^5 q_2^7 \), where \( 7 \leq j_1 + i_2 \leq 11 \) and \( j_1 \geq i_2 \), except the following: \( q_1^3 q_2^6, q_1^3 q_2^7, q_1^3 q_2^8, p_2 q_1 q_2^6, p_2 q_1 q_2^7, p_2 q_1 q_2^8, p_2 q_1 q_2^9, p_2 q_1 q_2^{10}, p_2 q_1 q_2^{11}, p_2 q_1 q_2^{12}, p_2 q_1 q_2^{13}, p_2 q_1 q_2^{14}, p_2 q_1 q_2^{15}, p_2 q_1 q_2^{16}, p_2 q_1 q_2^{17}, p_2 q_1 q_2^{18}, p_2 q_1 q_2^{19}, p_2 q_1 q_2^{20}, p_2 q_1 q_2^{21}, p_2 q_1 q_2^{22}, p_2 q_1 q_2^{23}, p_2 q_1 q_2^{24}, p_2 q_1 q_2^{25}, p_2 q_1 q_2^{26}, p_2 q_1 q_2^{27}, p_2 q_1 q_2^{28}, p_2 q_1 q_2^{29}, p_2 q_1 q_2^{30}, p_2 q_1 q_2^{31}, p_2 q_1 q_2^{32}, p_2 q_1 q_2^{33}, p_2 q_1 q_2^{34}, p_2 q_1 q_2^{35}, p_2 q_1 q_2^{36}, p_2 q_1 q_2^{37}, p_2 q_1 q_2^{38}, p_2 q_1 q_2^{39}, p_2 q_1 q_2^{40}, p_2 q_1 q_2^{41}, p_2 q_1 q_2^{42}, p_2 q_1 q_2^{43}, p_2 q_1 q_2^{44}, p_2 q_1 q_2^{45}, p_2 q_1 q_2^{46}, p_2 q_1 q_2^{47}, p_2 q_1 q_2^{48}, p_2 q_1 q_2^{49}, p_2 q_1 q_2^{50}, p_2 q_1 q_2^{51}, p_2 q_1 q_2^{52}, p_2 q_1 q_2^{53}, p_2 q_1 q_2^{54}, p_2 q_1 q_2^{55}, p_2 q_1 q_2^{56}, p_2 q_1 q_2^{57}, p_2 q_1 q_2^{58}, p_2 q_1 q_2^{59}, p_2 q_1 q_2^{60}, p_2 q_1 q_2^{61}, p_2 q_1 q_2^{62}, p_2 q_1 q_2^{63}, p_2 q_1 q_2^{64}, p_2 q_1 q_2^{65}, p_2 q_1 q_2^{66}, p_2 q_1 q_2^{67}, p_2 q_1 q_2^{68}, p_2 q_1 q_2^{69}, p_2 q_1 q_2^{70}, p_2 q_1 q_2^{71}, p_2 q_1 q_2^{72}, p_2 q_1 q_2^{73}, p_2 q_1 q_2^{74}, p_2 q_1 q_2^{75}, p_2 q_1 q_2^{76}, p_2 q_1 q_2^{77}, p_2 q_1 q_2^{78}, p_2 q_1 q_2^{79}, p_2 q_1 q_2^{80}, p_2 q_1 q_2^{81}, p_2 q_1 q_2^{82}, p_2 q_1 q_2^{83}, p_2 q_1 q_2^{84}, p_2 q_1 q_2^{85}, p_2 q_1 q_2^{86}, p_2 q_1 q_2^{87}, p_2 q_1 q_2^{88}, p_2 q_1 q_2^{89}, p_2 q_1 q_2^{90}, p_2 q_1 q_2^{91}, p_2 q_1 q_2^{92}, p_2 q_1 q_2^{93}, p_2 q_1 q_2^{94}, p_2 q_1 q_2^{95}, p_2 q_1 q_2^{96}, p_2 q_1 q_2^{97}, p_2 q_1 q_2^{98}, p_2 q_1 q_2^{99}, p_2 q_1 q_2^{100}, \). This set consists of 134 monomials, while in an arbitrary perturbation there are 1155 terms at these degrees.

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