NONLINEAR BSDES IN GENERAL FILTRATION WITH DRIVERS DEPENDING ON THE MARTINGALE PART OF A SOLUTION

TOMASZ KLIMSIAK AND MAURCY RZYMOWSKI

Abstract. In the present paper, we consider multidimensional nonlinear backward stochastic differential equations (BSDEs) with a driver depending on the martingale part $M$ of a solution. We assume that the nonlinear term is merely monotone continuous with respect to the state variable. As to the regularity of the driver with respect to the martingale variable, we consider a very general condition which permits path-dependence on ‘the future’ of the process $M$ as well as a dependence of its law (McKean-Vlasov-type equations). For such driver, we prove the existence and uniqueness of a global solution (i.e. for any maturity $T > 0$) to BSDE with data satisfying natural integrability conditions.

1. Introduction

Consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0})$, maturity $T > 0$, $\mathcal{F}_T$-measurable terminal condition $\xi$, and a driver

$$f : \Omega \times [0, T] \times \mathbb{R}^l \times \mathcal{M}_2^0(0, T; \mathbb{R}^l) \rightarrow \mathbb{R}^l$$

which is $\mathbb{F}$-progressively measurable with respect to the first two variables. Here $\mathcal{M}_2^0(0, T; \mathbb{R}^l)$ denotes the space of $l$-dimensional càdlàg $\mathbb{F}$-martingales starting at zero with square integrable supremum over $[0, T]$. In the present paper, we study backward stochastic differential equations (BSDEs) of the form

$$Y_t = \xi + \int_t^T f(r, Y_r, M) \, dr - \int_t^T dM_r, \quad t \in [0, T]. \quad (1.1)$$

A solution to (1.1) consists of an $l$-dimensional $\mathbb{F}$-adapted càdlàg process $Y$, and $M \in \mathcal{M}_2^0(0, T; \mathbb{R}^l)$ such that (1.1) holds $P$-a.s. We follow and generalize here the framework considered by Liang, Lyons and Qian in [14], and Bensoussan, Li and Yam in [4]. In these papers, the authors considered a special class of (1.1) with a driver of the form

$$f(r, Y_r, M) = g(r, Y_r, h(M)_r), \quad (1.2)$$

where $h$ is an operator, which maps $\mathcal{M}_2^0(0, T; \mathbb{R}^l)$ in the space $\mathcal{H}_2^2(0, T; \mathbb{R}^m)$ - consisting of $\mathbb{R}^m$-valued $\mathbb{F}$-progressively measurable processes that are square integrable with respect to $dt \otimes dP$ - and $g : [0, T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^l$.

Mathematics Subject Classification: Primary 60H20; Secondary 60G40
Keywords: backward stochastic differential equations, general filtration, global solutions
The purpose of the present paper is to address the conjecture made in [14] about the existence of a *global solution* to (1.1) with $f$ given by (1.2) under the following condition on the operator $h$: for some $\gamma_h > 0$, and any $t \in [0, T]$,  
\[ E \int_t^T |h(M)_r - h(N)_r|^2 \, dr \leq \gamma_h E \int_t^T d[M - N]_r, \quad M, N \in \mathcal{M}^2_0(0, T; \mathbb{R}^l). \tag{1.3} \]

Under this condition, and Lipschitz continuity of $g$ with respect to the second and the third variable, the authors proved in [14] that for small enough $T > 0$, there exists a solution to (1.1) - the so called *local solution*. For the existence of *global solutions* (i.e. for any $T > 0$) $h$ is additionally assumed in [14] to satisfy *local-in-time property*: for any $a < b$, $a, b \in [0, T]$, and any $M \in \mathcal{M}^2_0(0, T; \mathbb{R}^l)$,  
\[ h(M)_t = h(M^{a,b})_t, \quad t \in (a, b) \quad \text{with} \quad M^{a,b}_t := M_{(a \wedge t), a b}, \quad t \in [0, T]. \tag{1.4} \]

The above condition stands that the evaluation of $h(M)_t$ depends only on values of $M_s$, $s \in [t, t + \varepsilon)$ for however small $\varepsilon > 0$. In the paper, the authors conjectured (see the comments at the end of Section 3 in [14]) that regarding operator $h$, condition (1.3) alone is not sufficient to guarantee the existence of a *global solution* to (1.1), and suggested that the following operator may serve as an example supporting their conjecture:

\[ h_1(M)_t = E\left( \int_t^T d[M]_r |\mathcal{F}_t \right), \quad t \in [0, T]. \tag{1.5} \]

We see that the evaluation of $h_1(M)_t$ depends on the whole future, i.e. it depends on the values $M_s$, $s \in [t, T]$, so it does not share *local-in-time property*.

Recently, Cheridito and Nam in [8], considered even more general framework than presented above. They introduced the notion of backward stochastic equations (BSEs) of the form  
\[ Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T, \quad t \in [0, T] \tag{1.6} \]

for given generator $F$. If we take $F_t(Y, M) = \int_0^t f(r, Y_r, M) \, dr$, then the above BSE becomes BSDE (1.1). One of the results obtained in [8, Proposition 3.7] applies to BSDEs (1.1) with $f$ given by (1.2), and $h$ of a special form. However, a close inspection of the proof of [8, Proposition 3.7] reveals that it in fact applies not only to $h$ specified therein but to any $h$ satisfying (1.3) and the following additional condition:

if $M, N \in \mathcal{M}^2_0(0, T; \mathbb{R}^l)$ are strongly orthogonal, then $h(M + N) = h(M) + h(N). \tag{1.7}$

Therefore, from [8, Proposition 3.7] it follows that if $g$ is Lipschitz continuous with respect to the second and the third variable (uniformly in $t$), and $h$ satisfies (1.3),(1.7), then there exists a *global solution* to (1.1). Observe that operator $h_1$ satisfies the latter condition. Thereby, it does not support the conjecture stated in [14].

The main result of the present paper (Theorem 6.3) disproves the conjecture formulated in [14], and shows that in fact condition (1.3) and Lipschitz continuity of driver $g$ with respect to the second and the third variable (uniformly in $t$) guarantee the existence of a unique *global solution* to (1.1) with $f$ of the form (1.2). In fact, we shall prove our existence and uniqueness result of *global solutions* under considerable
weaker hypotheses, and in general framework (1.1). We assume that $f$ is merely continuous monotone in the second variable, i.e. for some $\mu \in \mathbb{R}$, and any $t \in [0, T)$,
\[
\langle f(t, y_1, M) - f(t, y_2, M), y_1 - y_2 \rangle \leq \mu |y_1 - y_2|^2, \quad y_1, y_2 \in \mathbb{R}^l, \quad M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l).
\]
Moreover, the growth of $f$ with respect to $y$ is subject to no restriction except the local behavior: for any $R > 0$,
\[
E\left( \int_0^T \sup_{|y| \leq R} |f(t, y, 0)| \, dt \right)^2 < \infty.
\]

Instead of (1.3), which only fits into the framework (1.2), we consider a more general condition of the following form: for any $t \in [0, T]$, $y \in \mathbb{R}^l$,
\[
E \int_t^T |f(r, y, M) - f(r, y, M')|^2 \, dr \leq \lambda E \int_t^T d[M - M']_r, \quad M, M' \in \mathcal{M}_0^2(0, T; \mathbb{R}^l),
\]
which holds in particular when $f$ is given by (1.2), $h$ satisfies (1.3), and $g$ is Lipschitz continuous with respect to $h(M)$. When $\mathbb{F}$ is generated by a Brownian motion $B$, $f$ is given by (1.2), and $h(M) = Z^M$, where $Z^M$ is a unique progressively measurable process such that $M_t = \int_0^t Z^M_r \, dB_r, \quad t \in [0, T]$, then our assumptions agree with the ones considered in [6] (with $p = 2$ in [6]; see also [12, Section 6], where reflected BSDEs on general filtered spaces are considered).

The organization of the paper is as follows. In Section 2 we introduce the basic notation and hypotheses. Section 3 is devoted to presenting several examples of driver $f$ satisfying (1.8). They show that the approach and framework considered in the present paper provide a unified way of treating a wide variety of seemingly disparate classes of BSDEs. The definition of a solution and a priori estimates are contained in Section 4. In Section 5 we give an existence and uniqueness result for BSDEs with a driver independent of $M$. Finally, Section 6 contains the proof of the paper’s main result. In Section 7, we demonstrate that the proof technique used in Section 6 may also be utilized to BSDEs of the form
\[
Y_t = \xi + \int_t^T f(r, Y, M) \, dr - \int_t^T dM_r, \quad t \in [0, T], \quad \tag{1.9}
\]
where now the driver may also depend on the future evaluation of $Y$. We prove that under the following Lipschitz type condition:

(A) There exists $L > 0$ such that for any $t \in [0, T]$, $M, M' \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$, $Y, Y' \in \mathbb{S}^2(0, T; \mathbb{R}^l)$
\[
E \int_t^T |f(r, Y, M) - f(r, Y', M')|^2 \, dr \leq L(E \int_t^T d[M - M']_r + E \int_t^T |Y_r - Y_r'|^2 \, dr),
\]
and standard integrability assumptions on the data, there exists a unique global solution to (1.9).

2. Preliminaries

2.1. Basic notation. Let $l \in \mathbb{N}$. For $x \in \mathbb{R}^l$, $|x|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ the natural inner product in $\mathbb{R}^l$. We set $\text{sgn}(x) = 1_{|x| \neq 0} x/|x|$. Let $(\mathcal{X}, \rho)$ be a
separable metric linear space. We denote by $L^2(\Omega, P; \mathcal{X})$ the set of $\mathcal{F}_T$-measurable $\mathcal{X}$-valued random variables $X$ such that

$$E\rho^2(X, 0) < \infty.$$ 

We let

$$\rho_{L^2}(X^1, X^2) := \left( E\rho^2(X^1, X^2) \right)^{1/2},$$

and if $\mathcal{X}$ is a normed space, then we let $\| \cdot \|_{L^2}$ denote the norm generated by $\rho_{L^2}$. By $\mathcal{S}_2^2(0, T; \mathcal{X})$, we denote the set of all $\mathcal{X}$-valued $\mathcal{F}$-progressively measurable processes $Y = (Y_t)_{t \in [0, T)}$, such that

$$E \sup_{0 \leq t \leq T} \rho^2(Y_t, 0) < \infty.$$ 

We set

$$\rho_{\mathcal{S}_2^2}(Y^1, Y^2) := \left( E \sup_{0 \leq t \leq T} \rho^2(Y^1_t, Y^2_t) \right)^{1/2}.$$

If $\mathcal{X}$ is a normed space, then by $\| \cdot \|_{\mathcal{S}_2^2}$ we denote the norm generated by $\rho_{\mathcal{S}_2^2}$. We let $\mathcal{M}_{loc}(0, T; \mathbb{R}^d)$ denote the set of all càdlàg processes $M = (M^i_0, \ldots, M^i_l)_{t \in [0, T]}$ such that for any $i \in \{1, \ldots, l\}$ process $M^i$ is a local $\mathcal{F}$-martingale. By $\mathcal{M}^2(0, T; \mathbb{R}^d)$ we denote the subset of $\mathcal{M}_{loc}(0, T; \mathbb{R}^d)$ consisting of processes $M$ such that

$$\| M \|_{M^2} := \left( E[|M|_T] \right)^{1/2} < \infty,$$

where $[M]_T = \sum_{i=1}^l [M^i]_T$ and $[M^i]$ is the square bracket of $M^i$, $i = 1, \ldots, l$. By $\mathcal{M}_{0,loc}^2(0, T; \mathbb{R}^d)$ (resp. $\mathcal{M}^2_0(0, T; \mathbb{R}^d)$) we denote the subspace of $\mathcal{M}_{loc}^2(0, T; \mathbb{R}^d)$ (resp. $\mathcal{M}^2(0, T; \mathbb{R}^d)$) consisting of those processes $M$ for which $M_0 = 0$. We let $\mathcal{H}_2^2(0, T; \mathcal{X})$ be the set of all $\mathcal{F}$-progressively measurable processes $X = (X_t)_{t \in [0, T]}$ taking values in $\mathcal{X}$, such that

$$E \int_0^T \rho^2(X_r, 0) \, dr < \infty.$$ 

We let

$$\rho_{\mathcal{H}_2^2}(X^1, X^2) := \left( E \int_0^T \rho^2(X^1_r, X^2_r) \, dr \right)^{1/2}.$$

When $\mathcal{X}$ is a normed space, then by $\| \cdot \|_{\mathcal{H}_2^2}$ we denote the norm generated by $\rho_{\mathcal{H}_2^2}$. By $B_2(0, T; L^2(\Omega, P; \mathbb{R}^d))$, we denote the set of all $\mathcal{F}$-adapted $\mathbb{R}^d$-valued processes $V = (V_t)_{t \in [0, T]}$, such that

$$\| V \|_{B_2} := \sup_{0 \leq t \leq T} \left( E|V_t|^2 \right)^{1/2} < \infty.$$

In the whole paper all relations between random variables hold $P$-a.s. For $\mathbb{R}^d$-valued processes $X$, $Y$ we write $X_t = Y_t$, $t \in [0, T]$ (or simply $X = Y$) iff

$$P(\exists t \in [0, T] \; X_t \neq Y_t) = 0.$$ 

For given $d \in \mathbb{N}$ we let $t^d$ denote $d$-dimensional Lebesgue measure on $\mathbb{R}^d$.
2.2. Assumptions on the data. The generator (driver) is the mapping
\[ f : \Omega \times [0, T] \times \mathbb{R}^l \times \mathcal{M}^2_0(0,T;\mathbb{R}^l) \rightarrow \mathbb{R}^l, \]
which is \( \mathbb{F} \)-adapted for fixed \( y \in \mathbb{R}^l, M \in \mathcal{M}^2_0(0,T;\mathbb{R}^l) \).

We shall need the following hypotheses:

\begin{itemize}
  \item [(H1)] \( E|\xi|^2 < \infty, E(\int_0^T |f(r,0,0)| dr)^2 < \infty, \)
  \item [(H2)] There exists \( \mu \in \mathbb{R} \) such that for any \( t \in [0, T], y, y' \in \mathbb{R}^l, M \in \mathcal{M}^2_0(0,T;\mathbb{R}^l), \)
    \( (y - y', f(t, y, M) - f(t, y', M)) \leq \mu|y - y'|^2, \)
  \item [(H3)] There exists \( \lambda > 0 \) such that for any \( t \in [0, T], y \in \mathbb{R}^l, M, M' \in \mathcal{M}^2_0(0,T;\mathbb{R}^l), \)
    \[ E \int_t^T |f(r, y, M) - f(r, y, M')|^2 dr \leq \lambda E \int_t^T d[M - M']_r, \]
  \item [(H4)] For every \( (t, M) \in [0, T] \times \mathcal{M}^2_0(0,T;\mathbb{R}^l) \) the mapping \( \mathbb{R}^l \ni y \rightarrow f(t, y, M) \)
    is continuous \( \mathbb{P} \)-a.s.,
  \item [(H5)] For each \( r > 0, \)
    \[ E\left( \int_0^T \psi_r(t) dt \right)^2 < \infty, \]
    where \( \psi_r(t) = \sup_{|y| < r} |f(t, y, 0) - f(t, 0, 0)|. \)
\end{itemize}

3. Examples of generators satisfying (H3)

In this section we shall give several examples of generators satisfying (H3). They show that the approach and framework considered in the present paper provides a unified way of treating a wide variety of seemingly disparate classes of BSDEs. We start with generators of the form (1.2) (Examples 3.1–3.7). Then BSDE (1.1) is of the following form
\[ Y_t = \xi + \int_t^T g(r, Y_r, h(M)_r) dr - \int_t^T dM_r, \quad t \in [0, T]. \] (3.1)

In fact we consider slightly more general case, when \( \mathbb{R}^m \) is replaced by the metric space \( \mathcal{X} \). In this case Lipschitz continuity of \( g \) with respect to the third variable and condition (1.3), which now admits the following form
\[ E \int_t^T \rho^2(h(M)_r, h(N)_r) dr \leq \gamma_h E \int_t^T d[M - N]_r, \quad M, N \in \mathcal{M}^2_0(0,T;\mathbb{R}^l), \] (3.2)

imply (H3). In Example 3.8 we show that more general than (3.1) framework (1.1) allows us in particular to cover a class of McKean-Vlasov-type equations.

**Example 3.1** (Classical BSDEs). In the pioneering paper [18] Pardoux and Peng introduced the notion of BSDEs, with a filtration \( \mathbb{F} \) that was assumed to be generated by a given \( d \)-dimensional Brownian motion \( B \), of the following form
\[ Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dB_r, \quad t \in [0, T]. \] (3.3)

A solution of equation (3.3) consists of a continuous process \( Y \in \mathcal{S}^2_0(0,T;\mathbb{R}^l), \) and

\[ Z \in \mathcal{H}^2_0(0,T;\mathbb{R}^l \otimes \mathbb{R}^d). \]

Observe that BSDE (3.1) becomes BSDE (3.3) if we consider \( \mathbb{F} \) as specified above, \( \mathcal{X} = \mathbb{R}^d \) and

\[ h(M) = Z^M, \]
where $Z^M$ is a unique process in $\mathcal{H}_x^2(0, T; \mathbb{R}^l \otimes \mathbb{R}^d)$ such that $M_t = \int_0^t Z^M_r dB_r$, $t \in [0, T]$. Clearly, $h$ satisfies (3.2).

**Example 3.2** (Lévy processes and Teugels martingales). In [15] Nualart and Schoutens considered filtration $\mathbb{F}$ generated by a $d$-dimensional Lévy process $(X_t)_{t \geq 0}$ with the Lévy triplet $(a, \sigma, \nu)$. They assumed additionally that for some $\alpha > 0$ and any $\varepsilon > 0$,

$$\int_{\mathbb{R} \setminus (\varepsilon, \varepsilon)} e^{\alpha |x|} \nu(dx) < \infty.$$  

Then there exists an orthonormal sequence $(H^i)_{i \geq 0} \subset \mathcal{M}^2(0, T; \mathbb{R}^l)$, built up from Teugels martingales, that furnishes the following representation property: for any $M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$ there exists a unique sequence $(Z^{M,i})_{i \geq 1} \subset \mathcal{H}_x^2(0, T; \mathbb{R}^l \otimes \mathbb{R}^d)$ such that

$$M_t = \sum_{i=1}^\infty \int_0^t Z^{M,i}_r dH^i_r, \quad t \in [0, T].$$

Based on this representation the authors in [15] introduced and studied BSDEs of the form

$$Y_t = \xi + \int_t^T f(r, Y_r, (Z^i_r)_{i \geq 1}) \, dr - \sum_{i=1}^\infty \int_t^T Z^i_r dH^i_r, \quad t \in [0, T]. \quad (3.4)$$

Observe that equation (3.1) becomes (3.4) if we take $\mathcal{X} = l^2 := \{(a_n)_{n \geq 1} : \| (a_n)_{n \geq 1} \|_2^2 := \sum_{n=1}^\infty a_n^2 < \infty\}$, and

$$h(M) = (Z^{M,i})_{i \geq 1}.$$

One easily checks that $h$ fulfills (3.2).

**Remark 3.3.** A more advanced model of the above type on general filtered spaces was considered in [12, Section 6] for reflected BSDEs (see also [9] for BSDEs).

**Example 3.4** (Poisson random measure). Assume that $B$ is a $d$-dimensional Brownian motion with respect to $\mathbb{F}$ and $N$ is a $d$-dimensional Poisson random measure on $E := \mathbb{R}^m \setminus \{0\}$ (independent of $B$), with an intensity measure $\ell \otimes \nu$ ($\nu = (\nu_1, \ldots, \nu_d)$) satisfying

$$\sum_{i=1}^d \int_E 1 \wedge |x|^2 \nu_i(dx) < \infty,$$

such that for any $B \in \mathcal{B}(E)$, $\tilde{N}([0, t], A) := N([0, t], A) - t\nu(A)$ is an $\mathbb{R}^d$-valued martingale with respect to $\mathbb{F}$. In this case any martingale $M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$ admits a unique representation (see e.g. [10, Section III.4])

$$M_t = \int_0^t Z^M_r dB_r + \int_0^t \int_E U_r(x) \tilde{N}(dr, dx) + L^M_t, \quad t \in [0, T],$$

where $Z^M \in \mathcal{H}_x^2(0, T; \mathbb{R}^l \otimes \mathbb{R}^d)$, $U \in \mathcal{H}_x^2(0, T; L^2(E, \nu; \mathbb{R}^1 \otimes \mathbb{R}^d))$, and $L^M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$ is strongly orthogonal to $B$ and $\tilde{N}$. There are numerous results in the literature (see e.g. [2, 13, 21, 23, 24, 25]) concerning BSDEs of the form

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r, U_r) \, dr - \int_t^T Z_r dB_r - \int_t^T \int_E U_r(x) \tilde{N}(dr, dx), \quad t \in [0, T]. \quad (3.5)$$
This equation fits into our framework again. To see this take $\mathcal{X} = \mathbb{R}^d \times L^2(E, \nu; \mathbb{R}^d \otimes \mathbb{R}^d)$ with natural metric $\rho$, and

$$h(M) = (Z^M, U^M).$$

(3.6)

One easily checks that (3.2) holds.

**Example 3.5** (Carré du champs operators). One of the major advantage of BSDE’s theory is that it provides an easy path for proving probabilistic interpretation of solutions to a class of semilinear PDEs. Let $\mathcal{F}$ be the filtration generated by a Markov process $(X_t)_{t \geq 0}$, say Feller process, associated to Feller generator $(A, D(A))$ on $\mathbb{R}^d$ with $C_c^\infty(\mathbb{R}^d) \subset D(A)$. It is well known that any such operator is of the form

$$Au(x) = \sum_{i,j=1}^d q_{ij}(x)u_{x_i,x_j}(x) + \sum_{i=1}^d b_i(x)u_{x_i}(x) + c(x)u(x)
+ \int_{\mathbb{R}^d} (u(x + y) - u(x) - 1_{\{|y| \leq 1\}}(\nabla u(x), y)) N(x, dy),$$

with $x$-dependent Lévy triplet $(b(x), Q(x), N(x, dy))$, and $c \leq 0$ (see e.g. [5]). When $(X_t)_{t \geq 0}$ is a diffusion process, i.e. $N \equiv 0$, then solutions to the Cauchy problem

$$- \partial_t u - Au = g(t, x, u, \sigma \nabla u), \quad u(T, \cdot) = \varphi,$$  

(3.7)

where $\sigma \cdot \sigma^T = Q$, are related to BSDEs of the form (3.3) with $\xi = \varphi(X_T)$ and $f(r, y, z) = \hat{f}(r, X_r, y, z)$ (see e.g. [1, 16, 11, 17, 22]). However, for general operator $A$ it is more natural to consider $\sqrt{\Gamma(u)}$ in place of $\sigma \nabla u$ in the nonlinear part of equation (3.7), where $\Gamma$ is the so called **carré du champs operator**:

$$\Gamma(u) := A(u^2) - 2u Au, \quad u \in D(A).$$

Solutions of such equations are related to BSDE (3.1), where $\xi, f$ are as in the foregoing, and

$$h(M)_r = \sqrt{\frac{d(M)_r}{dr}}$$

(see e.g. [3]). Observe that (3.2) is satisfied once again.

In all the above examples operator $h$ fulfills **local-in-time property** (1.4). Furthermore, it satisfies condition (1.7). Now, we shall give simple examples of operator $h$ which fulfills (1.3), and at the same time does not satisfy (1.4) nor (1.7). In the sequel we denote by $D(0, T; \mathbb{R}^l)$ the set of all càdlàg $\mathbb{R}^l$-valued functions on $[0, T]$.

**Example 3.6** (Beyond local-in-time property; path dependent BSDEs). Consider a functional $\phi: D(0, T; \mathbb{R}^l) \to \mathbb{R}$ such that for some $L > 0$ and any $a, b \in D(0, T; \mathbb{R}^l)$,

$$|\phi(a) - \phi(b)| \leq L \sup_{t \in [0, T]} |a(t) - b(t)|.$$ 

Then operator $h: \mathcal{M}^2_0(0, T; \mathbb{R}^l) \to \mathcal{H}^2_\mathcal{F}(0, T; \mathbb{R})$, defined by

$$h(M)_t = E(\phi(M_{t\wedge T} - M_t)|\mathcal{F}_t),$$

satisfies (1.3), however in general it does not satisfy (1.4) nor (1.7) (take e.g. $\phi(a) = \sup_{t \in [0, T]} |a(t)|$).
Example 3.7 (Beyond local-in-time property; anticipated BSDEs). Let $\zeta, \delta : [0, T] \to \mathbb{R}^+$ be continuous functions. With the notation of Example 3.4, we set

$$h(M)_t = (E(Z^M_{t+\zeta(t)}|\mathcal{F}_t), E(U^M_{t+\delta(t)}|\mathcal{F}_t)), \quad t \in [0, T],$$

where $(Z^M_t, U^M_t) = (\eta_t, \tilde{\eta}_t)$, $t \geq T$ for given stochastic processes $\eta, \tilde{\eta}$. BSDEs of the form (3.1) with $h$ as above were introduced (for Brownian filtration) by Peng and Yang in [20]. Observe that $h$ satisfies (3.2) and does not satisfy (1.4). If we consider in the definition of $h$ nonlinear expectation (see [19]), then condition (1.7) will be also violated.

The last example exhibits the advantage of the general framework (1.1) over the framework (3.1). By $\mathcal{P}_1(\mathcal{X})$ we denote the set of all probability measures $\mu$ on $\mathcal{X}$ for which

$$\int_{\mathcal{X}} \rho(x, 0) \mu(dx) < \infty.$$  

We let $W_\mathcal{X}$ denote the Wasserstein metric on $\mathcal{P}_1(\mathcal{X})$:

$$W_\mathcal{X}(\mu, \nu) = \inf \{ \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) P(dx, dy) : P \in \mathcal{P}_{\mu,\nu}(\mathcal{X} \times \mathcal{X}) \},$$

where $\mathcal{P}_{\mu,\nu}(\mathcal{X} \times \mathcal{X}) = \{ P \in \mathcal{P}_1(\mathcal{X}) \times \mathcal{P}_1(\mathcal{X}) : P(dx, \mathcal{X}) = \mu(dx), P(\mathcal{X}, dy) = \nu(dy) \}$.

Example 3.8 (McKean-Vlasov-type equations). Let $\mathcal{X} = D(0, T; \mathbb{R}^l)$, and $\rho$ be the Skorokhod metric on $D(0, T; \mathbb{R}^l)$. Consider a function

$$\hat{f} : \Omega \times [0, T] \times \mathbb{R}^l \times \mathcal{P}_1(\mathcal{X}) \to \mathbb{R}^l$$

such that for some $L > 0$ and any $t \in [0, T]$, $y \in \mathbb{R}^l$,

$$|\hat{f}(t, y, \mu) - \hat{f}(t, y, \nu)| \leq LW_\mathcal{X}(\mu, \nu), \quad \mu, \nu \in \mathcal{P}_1(\mathcal{X}).$$

For given martingale $M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$, and $t \in [0, T]$ we denote $M^t = M_{\cdot t} - M_t$. We set

$$f(t, y, M) := \hat{f}(t, y, \mathcal{L}(M^t)), \quad t \in [0, T], y \in \mathbb{R}^l, M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l).$$

Observe that

$$|f(r, y, M) - f(r, y, N)| \leq LW_\mathcal{X}(\mathcal{L}(M^r), \mathcal{L}(N^r)) \leq LE\rho(M^r, N^r) \leq LE \sup_{s \in [r, T]} |M^r_s - N^r_s|.$$  

Therefore, by the Burkholder-Davis-Gundy inequality, we easily obtain (H3). We may also get an interesting example of generator $f$ by taking $\mathcal{X} = (\mathbb{R}^l \otimes \mathbb{R}^l) \times L^2(E, \nu; \mathbb{R}^l)$ (cf. [7, 8]), and natural metric $\rho$ on $\mathcal{X}$ (under assumptions and notation of Example 3.4). Then, by using representation (3.6), we may define

$$f(t, y, M) := f(t, y, \mathcal{L}(Z^M_t, U^M_t)), \quad t \in [0, T], y \in \mathbb{R}^l, M \in \mathcal{M}_0^2(0, T; \mathbb{R}^l).$$

One easily checks that $f$ satisfies (H3).
4. Definition of a solution to BSDE and a priori estimates

**Definition 4.1.** We say that a pair \((Y, M)\) of \(\mathbb{F}\)-adapted processes is a solution of the backward stochastic differential equation with right-hand side \(f\) and terminal value \(\xi\) (BSDE(\(\xi, f\)) for short) if

(a) \(Y\) is càdlàg and \(M \in \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^l)\),
(b) \(\int_0^T |f(r, Y_r, M)_r| \, dr < +\infty\),
(c) \(Y_t = \xi + \int_t^T f(r, Y_r, M) \, dr - \int_t^T dM_r, \quad t \in [0, T]\).

**Proposition 4.2.** Assume that (H1)–(H3) are in force. Then there exists \(C > 0\), depending only on \(T, \mu^+, \lambda\), such that for any solution \((Y, M) \in \mathcal{S}^2(0, T; \mathbb{R}^l) \times \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^l)\) to BSDE(\(\xi, f\)),

\[
E \sup_{0 \leq t \leq T} |Y_t|^2 + E \int_0^T d[|M|_r] \leq CE \left( |\xi|^2 + \left( \int_0^T |f(r, 0, 0)| \, dr \right)^2 \right).
\]

**Proof.** All the constants in the proof labelled \(C_i\) for some \(i \in \mathbb{N}\), shall depend only on \(\lambda, \mu^+, T\). By Itô’s formula

\[
|Y_t|^2 + \int_t^T d[|M|_r] = |\xi|^2 + 2 \int_t^T \langle Y_r, f(r, Y_r, M) \rangle \, dr - 2 \int_t^T \langle Y_{t-}, dM_r \rangle.
\]

By (H2)

\[
\langle Y_r, f(r, Y_r, M) \rangle \leq |Y_r||f(r, Y_r, M) - f(r, Y_r, 0)| + \mu^+|Y_r|^2 + |\eta_r||f(r, 0, 0)|
\]

\[
\leq \frac{1}{2} (1 + \nu)|Y_r|^2 + \frac{1}{2(1 + \nu)} |f(r, Y_r, M) - f(r, Y_r, 0)|^2 + \mu^+|Y_r|^2 + |\eta_r||f(r, 0, 0)|,
\]

for every \(\nu \geq 0\). From (4.1), (4.2) we get

\[
|Y_t|^2 + \int_t^T d[|M|_r] \leq |\xi|^2 + (1 + \nu + 2\mu^+) \int_t^T |Y_r|^2 \, dr
\]

\[
+ \frac{1}{1 + \nu} \int_t^T |f(r, Y_r, M) - f(r, Y_r, 0)|^2 \, dr
\]

\[
+ 2 \int_t^T |Y_r||f(r, 0, 0)| \, dr - 2 \int_t^T \langle Y_{t-}, dM_r \rangle.
\]

Since \(Y \in \mathcal{S}^2(0, T; \mathbb{R}^l)\), and \(M \in \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^l)\), we have that \(\int_0^T \langle Y_{t-}, dM_r \rangle\) is a martingale. Thus, taking the expectation, and using (H3), we obtain

\[
E|Y_t|^2 + E \int_t^T d[|M|_r] \leq E|\xi|^2 + (1 + \nu + 2\mu^+) E \int_t^T |Y_r|^2 \, dr
\]

\[
+ \frac{\lambda}{1 + \nu} E \int_t^T d[|M|_r] + 2E \int_t^T |Y_r||f(r, 0, 0)| \, dr.
\]

Hence, for \(\nu := 2\lambda\), we get

\[
E|Y_t|^2 + \frac{1}{2} E \int_t^T d[|M|_r] \leq E|\xi|^2 + (1 + \nu + 2\mu^+) E \int_t^T |Y_r|^2 \, dr
\]

\[
+ 2E \int_t^T |Y_r||f(r, 0, 0)| \, dr.
\]
Set $\beta := (1 + \nu + 2\mu^+)$. By the Gronwall lemma
\[
E|Y_t|^2 \leq e^{\beta T}(E|\xi|^2 + 2E\int_0^T|Y_r||f(r,0,0)|dr).
\]
This combined with (4.5) yields
\[
E|Y_t|^2 + E\int_0^T d[M]_r \leq C_1EX,
\]
for some $C_1 > 0$, where $X := |\xi|^2 + 2\int_0^T|Y_r||f(r,0,0)|dr$. From (4.3), (H3), (4.6), and the Burkholder-Davis-Gundy inequality, we conclude that for any $\beta > 0$,
\[
E\sup_{0 \leq t \leq T} |Y_t|^2 \leq \frac{\lambda}{1 + \nu} E\int_0^T d[M]_r dr + C_2EX + 2E\sup_{0 \leq t \leq T} \int_t^T (Y_r, dM_r)
\]
\[
\leq \frac{\lambda}{1 + \nu} E\int_0^T d[M]_r dr + C_2EX + C_3E\sup_{0 \leq t \leq T} |Y_t| \bigg(\int_0^T d[M]_r \bigg)^{\frac{1}{2}}
\]
\[
\leq (\frac{1}{2} + C_3\beta^{-1}) E\int_0^T d[M]_r dr + C_2EX + C_3\beta E\sup_{0 \leq t \leq T} |Y_t|^2.
\]
Taking $\beta = \frac{1}{2C_3}$ we get that for some $C_4 > 0$,
\[
E\sup_{0 \leq t \leq T} |Y_t|^2 + E\int_0^T d[M]_r \leq C_4EX
\]
Observe that for any $\alpha > 0$,
\[
EX \leq \alpha E\sup_{0 \leq t \leq T} |Y_t|^2 + \alpha^{-1}E\left(|\xi|^2 + \left(\int_0^T |f(r,0,0)| dr\right)^2\right).
\]
Taking $\alpha = \frac{1}{2C_4}$ we conclude from (4.8) the desired inequality. $\Box$

When $l = 1$ we are able to say something more about integrability of the driver $f$.

**Proposition 4.3.** Suppose that $l = 1$. Assume that (H1)–(H3) are in force. Then there exists $C > 0$, depending only on $T, \mu^+, \lambda$, such that for any solution $(Y, M) \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{M}_0^+(0, T; \mathbb{R})$ to BSDE($\xi, f$),
\[
E\left(\int_0^T |f(r, Y_r, M)| dr\right)^2 \leq CE\left(|\xi|^2 + \left(\int_0^T |f(r, 0, 0)| dr\right)^2\right).
\]

**Proof.** Throughout the proof $C$ denotes a constant which can vary from line to line but depends only on $T, \mu^+, \lambda$. By the Itô-Meyer formula
\[
|Y_t| \leq |\xi| + \int_t^T \text{sgn}(Y_r)f(r, Y_r, M) dr - \int_t^T \text{sgn}(Y_{r-}) dM_r, \quad t \in [0, T].
\]
Hence
\[
|Y_t| \leq |\xi| + \int_t^T \text{sgn}(Y_r)f(r, Y_r, M) dr + \sup_{0 \leq t \leq T} \left|\int_t^T \text{sgn}(Y_{r-}) dM_r\right|, \quad t \in [0, T].
\]
By (H2)
\[-\text{sgn}(Y_r)(f(r, Y_r, M) - f(r, Y_r, 0)) + \mu^*|Y_r| \geq -\text{sgn}(Y_r)(f(r, Y_r, 0) - f(r, 0, 0)) + \mu^*|Y_r|\]
\[-|f(r, Y_r, M) - f(r, Y_r, 0)| - |f(r, 0, 0)| - \mu^*|Y_r|\]
\[= |\text{sgn}(Y_r)(f(r, Y_r, 0) - f(r, 0, 0)) + \mu^*|Y_r| - |f(r, Y_r, M) - f(r, Y_r, 0)| - |f(r, 0, 0)| - \mu^*|Y_r|\]
\[\geq |f(r, Y_r, 0) - f(r, 0, 0)| - \mu^*|Y_r|\]
\[-|f(r, Y_r, M) - f(r, Y_r, 0)| - |f(r, 0, 0)| - \mu^*|Y_r|\]
\[\geq |f(r, Y_r, 0)| - |f(r, Y_r, M) - f(r, Y_r, 0)| - 2|f(r, 0, 0)| - 2\mu^*|Y_r|.

From this inequality and (4.10) we infer that
\[
\left( \int_0^T |f(r, Y_r, 0)| \, dr \right)^2 \leq C \left( |\xi|^2 + \int_0^T |f(r, Y_r, M) - f(r, Y_r, 0)|^2 \, dr \right)
\[\quad + \left( \int_0^T |f(r, 0, 0)| \, dr \right)^2 + \int_0^T |Y_r| \, dr + \sup_{0 \leq t \leq T} \left| \int_t^T \text{sgn}(Y_r) \, dM_r \right|^2 \right)
\]
\[
\text{(4.11)}
\]
Thus, applying the expectation to (4.11), and using (H3) and Burkholder-Davis-Gundy inequality yields
\[
E\left( \int_0^T |f(r, Y_r, 0)| \, dr \right)^2 \leq CE\left( |\xi|^2 + \left( \int_0^T |f(r, 0, 0)| \, dr \right)^2 + \int_0^T |Y_r| \, dr + \int_0^T d[M_r] \right).
\]
\[
\text{(4.12)}
\]
Therefore, by (H3) and Proposition 4.2 we get the desired inequality.

5. BSDEs with Driver Independent of the Martingale Part

In the present section we shall give an existence result for BSDEs in case the driver \( f \) does not depend on the martingale part of a solution. The difficulty lies in the fact that \( f \) is assumed only monotone with respect to \( y \) with no assumptions on the growth. In the proof, among others, we adapt and combine the techniques applied in [17] and [6], where the authors considered BSDEs on Brownian filtration.

**Proposition 5.1.** Assume that \( f \) does not depend on \( M \) and (H1),(H2),(H4),(H5) are in force. Then there exists \((Y, M) - a solution to BSDE(\xi, f) - such that Y \in S^2(0, T; \mathbb{R}^l)\).

**Proof.** Since \( f \) does not depend on \( M \), (H5) is of the form
\[(H5) \text{ for any } r > 0, E\left( \int_0^T \psi_r(t) \, dt \right)^2 < \infty, \text{ where } \psi_r(t) = \sup_{|y| < r} |f(t, y) - f(t, 0)|.
\]
Without loss of generality, we may assume that \( \mu \leq 0 \). We divide the proof into three steps.

**Step 1.** In the first step, we need the following additional assumptions: there exist \( \alpha, \beta > 0 \) such that
\[|f(t, y)| \leq \alpha |f(t, 0)| + \beta, \quad t \in [0, T], y \in \mathbb{R}^l, \] \[
\text{(5.1)}
\]
Therefore, applying conditional expectation to both sides of (5.4) yields

\[ |\xi| + \sup_{0 \leq t \leq T} |f(t, 0)| \leq C \]  

(5.2)

In particular, the above conditions imply that \( |f| \) is bounded. Let \( \varrho \) be a non-negative smooth function on \( \mathbb{R}^l \) with support in \( B(0, 1) := \{ x \in \mathbb{R}^l : |x| \leq 1 \} \) such that \( \varrho(0) = 1 \), and \( \varrho(x) < 1, x \neq 0 \). We set

\[ f_n(t, y) := \int_{\mathbb{R}^l} f(t, x) \varrho_n(y - x) \, dx, \quad t \in [0, T], y \in \mathbb{R}^l, \]

where \( \varrho_n(x) = a_n \varrho(nx), \ a_n > 0, \) and

\[ \int_{\mathbb{R}^l} \varrho_n(x) \, dx = 1, \ n \geq 1. \]

It is well known that \( f_n \) is smooth and \( f_n \to f \) uniformly on compacts (both properties with respect to \( y \) under fixed \( \omega \in \Omega \) and \( t \in [0, T] \)). Observe that by assumptions made in the present step

\[ |f_n(t, y) - f_n(t, y')| \leq 2^l (B(0, 1))(\alpha C + \beta) \|
abla \varrho_n \|_\infty |y - y'|, \quad t \in [0, T], y, y' \in \mathbb{R}^l, \]

where \( l^l \) is \( l \)-dimensional Lebesgue measure. Furthermore, \( f_n \) satisfies (H2) with \( \mu \leq 0, \) and

\[ |f_n(t, y)| \leq (\alpha C + \beta). \]  

(5.3)

By [14, Theorem 4.1] the problem BSDE(\( \xi, f_n \)) admits a unique solution \((Y^n, M^n) \in \mathcal{S}^2(0, T; \mathbb{R}^l) \times \mathcal{M}_{\mathbb{F}}^2(0, T; \mathbb{R}^l) \). By Itô's formula

\[ e^r|Y^n|^2 + \int_0^T e^r|Y^n|^2 \, dr + \int_0^T e^r \, d[M^n]_r = e^T|\xi|^2 + 2 \int_0^T e^r(Y^n, f_n(r, Y^n)) \, dr \]

\[ - 2 \int_0^T e^r(Y^n, dM^n_r), \quad t \in [0, T]. \]  

(5.4)

Using (H2) (cf. the comment preceding (5.3)), we get

\[ 2(Y^n, f_n(r, Y^n)) \leq |Y^n|^2 + |f_n(t, 0)|^2. \]

Therefore, applying conditional expectation to both sides of (5.4) yields

\[ e^r|Y^n|^2 \leq E(e^T|\xi|^2 + \int_0^T e^r|f_n(r, 0)|^2 \, dr |\mathcal{F}_t). \]

From this, (5.2) and (5.3),

\[ |Y^n_t| \leq e^T[C + \sqrt{T}(\alpha C + \beta)], \quad t \in [0, T]. \]  

(5.5)

Let \( U^n_t = f_n(t, Y^n_t), \ t \in [0, T] \). By (5.3), Proposition 4.2, and the Burkholder-Davis-Gundy inequality,

\[ \sup_{n \in \mathbb{N}} \left( E \int_0^T |U^n|^2 \, dt + E \int_0^T d[M^n], + E \sup_{t \in [0, T]} |M^n_t|^2 \right) < \infty. \]

Since both \( \mathcal{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^l), L^2(\Omega, P; \mathbb{R}^l) \) are Hilbert spaces, they are reflexive. Therefore, by the Banach-Alaoglu theorem there exists a subsequence (not relabelled) such that \((U^n, M^n)\) converges weakly in the space \( \mathcal{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^l) \) to \((U, M)\) and \( M^n_T \) converges weakly in the space \( L^2(\Omega, P; \mathbb{R}^l) \) to \( N \). Let \( \tilde{M} \) be a càdlàg version of the martingale \( E(N|\mathcal{F}_t), \ t \in [0, T] \). We shall prove that for any \( \tau \in T \)

\[ M^n_T \to M_\tau, \ \text{weakly in} \ L^2(\Omega, P; \mathbb{R}^l). \]  

(5.6)
Let $X \in L^2(\Omega, P; \mathbb{R}^l)$, then

\[ E(M^n_T, X) = E(E(M^n_T | \mathcal{F}_r), X) = E(M^n_T, E(X | \mathcal{F}_r)) \rightarrow E(M_T, E(X | \mathcal{F}_r)) = E(M_T, X). \]

Observe that $\tilde{M} = M$ in $\mathcal{H}_T^2(0, T; \mathbb{R}^l)$. Indeed, let $Z \in \mathcal{H}_T^2(0, T; \mathbb{R}^l)$. Then $Z_r \in L^2(\Omega, P; \mathbb{R}^l)$ for a.e. $r \in [0, T]$. By (5.6),

\[ E(M^n_T, Z_r) \rightarrow E(M_T, Z_r), \quad \text{for a.e. } r \in [0, T]. \]

Therefore,

\[
E \int_0^T \langle \tilde{M}_r, Z_r \rangle \, dr = \lim_{n \to \infty} E \int_0^T \langle M^n_r, Z_r \rangle \, dr
\]

\[ = \lim_{n \to \infty} \int_0^T E(M^n_T, Z_r) \, dr = \int_0^T E(M_T, Z_r) \, dr = E \int_0^T \langle M_r, Z_r \rangle \, dr. \]

In the third equation we applied the Lebesgue dominated convergence theorem by using the following estimation

\[ E(M^n_T, Z_r) \leq \sup_{n \geq 1} E \sup_{t \leq T} |M^n_t|^2 + E|Z_r|^2. \]

The next convergence property we need is as follows: for any $t \in [0, T]$

\[
\int_t^T U^n_r \, dr \to \int_t^T U_r \, dr \quad \text{weakly in } L^2(\Omega, P; \mathbb{R}^l). \quad (5.7)
\]

This easily follows from the following calculation. Let $X \in L^2(\Omega, P; \mathbb{R}^l)$, then

\[
E(X, \int_t^T U^n_r \, dr) = E \int_t^T \langle X, E(U^n_r | \mathcal{F}_r) \rangle \, dr
\]

\[ = E \int_0^T (U^n_r, 1_{[t, T]}(r) E(X | \mathcal{F}_r)) \, dr \rightarrow E \int_0^T \langle U_r, 1_{[t, T]}(r) E(X | \mathcal{F}_r) \rangle \, dr
\]

\[ = E(X, \int_t^T U_r \, dr). \]

Set

\[ Y_t := \xi + \int_t^T U_r \, dr - \int_t^T dM_r, \quad t \in [0, T]. \]

By (5.6) and (5) for any $t \in [0, T]$

\[ Y^n_t \to Y_t \quad \text{weakly in } L^2(\Omega, P; \mathbb{R}^l). \quad (5.8) \]

We shall prove that $U = f(\cdot, Y)$ in $\mathcal{H}_T^2(0, T; \mathbb{R}^l)$. Let $X \in \mathcal{H}_T^2(0, T; \mathbb{R}^l)$. By (5.3), we have $f_n(\cdot, X) \to f(\cdot, X)$ in $\mathcal{H}_T^2(0, T; \mathbb{R}^l)$. From this and (H2) we conclude

\[ \limsup_{n \to \infty} E \int_0^T \{ Y^n_t - X_t, f_n(t, Y^n_t) - f(t, X_t) \} \, dt \leq 0. \quad (5.9) \]

By Ito’s formula

\[
2E \int_0^T \langle Y^n_t, f_n(r, Y^n_r) \rangle \, dr = E|Y^n_0|^2 - E|\xi|^2 + \int_0^T d[M^n]_r. \quad (5.10)
\]

Now, we apply the very well known fact saying that in Hilbert spaces the norm generated by the inner product is weakly lower semicontinuous. Thus, by (5.8)

\[ \liminf_{n \to \infty} E|Y^n_0|^2 \geq E|Y_0|^2, \]
and by (5.6)
\[
\lim_{n \to \infty} \inf E \int_0^T d[M^n]_r = \lim_{n \to \infty} \inf E|M^n|^2 dr \geq E|M_T|^2 dr = E \int_0^T d[M]_r.
\]
The above two inequalities when applied to (5.10) give
\[
\lim_{n \to \infty} \inf E \int_0^T \langle Y^n_t, f(t, Y^n_t) \rangle dt \geq E|Y_0|^2 - E|\xi|^2 + \int_0^T d[M]_r = E \int_0^T \langle Y_t, U_t \rangle dt.
\]
It follows from this and (5.9) that
\[
E \int_0^T \langle Y_t - X_t, U_t - f(t, X_t) \rangle dt \
\leq \lim_{n \to \infty} \inf E \int_0^T \langle Y^n_t - X_t, f_n(t, Y^n_t) - f(t, X_t) \rangle dt \leq 0.
\]
Let us choose \( X_t = Y_t - \varepsilon(U_t - f(t, Y_t)) \), \( t \in [0, T] \), \( \varepsilon > 0 \). Dividing (5.12) by \( \varepsilon \) and letting \( \varepsilon \to 0 \), we obtain
\[
E \int_0^T |U_t - f(t, Y_t)|^2 dt \leq 0.
\]
This concludes the proof of Step 1.

**Step 2.** In this step we dispense with (5.1) but we shall keep in force (5.2). Let \( a > 0 \) be such that
\[
c^{T/2}(|\xi| + \sqrt{T} \sup_{0 \leq t \leq T} |f(t, 0)|) \leq a.
\]
Let \( \theta_a \) be a smooth function on \( \mathbb{R}^l \) such that \( 0 \leq \theta_a \leq 1 \), \( \theta_a(y) = 1 \) for \( |y| \leq a \) and \( \theta_a(y) = 0 \) for \( |y| \geq a + 1 \). For \( n \in \mathbb{N} \) let us consider (cf. (H5))
\[
\hat{f}_n(t, y) = \theta_a(y) \cdot (f(t, y) - f(t, 0)) \cdot \frac{n}{\psi_{a+1}(t) \vee n} + f(t, 0).
\]
For each \( n \in \mathbb{N} \), \( \hat{f}_n \) satisfies (H1), (H4), (H5). Furthermore, \( \hat{f}_n \) also satisfies (H2) but with the positive constant \( \mu \). Indeed, let us take \( t \in [0, T] \), \( y, y' \in \mathbb{R}^l \) and assume that \( |y| \geq a + 1 \) and \( |y'| \geq a + 1 \). Then the inequality in (H2) is trivially satisfied with any \( \mu \geq 0 \) since \( \theta_a(y) = \theta_a(y') = 0 \). Suppose that \( |y| < a + 1 \). Since \( f \) satisfies (H2) with \( \mu \leq 0 \) we have
\[
\langle y - y', \hat{f}_n(t, y) - \hat{f}_n(t, y') \rangle \leq \frac{n(\theta_a(y) - \theta_a(y'))}{\psi_{a+1}(t) \vee n} \langle y - y', f(t, y) - f(t, 0) \rangle.
\]
By the very definition of \( \psi_{a+1} \), we have \( |f(t, y) - f(t, 0)| \leq \psi_{a+1}(t) \). Therefore
\[
\langle y - y', \hat{f}_n(t, y) - \hat{f}_n(t, y') \rangle \leq n\|\nabla \theta_a\|_\infty |y - y'|^2.
\]
Thus, \( \hat{f}_n \) satisfies (H2) with \( \mu = n\|\nabla \theta_a\|_\infty \). Furthermore, since \( \theta_a(y)|f(t, y) - f(t, 0)| \leq \psi_{a+1}(t) \), \( t \in [0, T] \), \( y \in \mathbb{R}^l \), we also have
\[
|\hat{f}_n(t, y)| \leq n + |f(t, 0)|, \quad t \in [0, T], \quad y \in \mathbb{R}^l.
\]

---

14 TOMASZ KLIMSIACK AND MAURYCY RZYMOWSKI
Therefore, by Step 1, the problem BSDE\((\xi, \hat{f}_n)\) admits a unique solution \((Y^n, M^n) \in S^2(0, T; \mathbb{R}^l) \times \mathcal{M}_0^2(0, T; \mathbb{R}^l)\). By Ito’s formula

\[
e^{\xi t} Y^n_t^2 + \int_t^T e^{r}d[M^n]_r + \int_t^T e^{r} |Y^n_r|^2 \, dr = e^{T} |\xi|^2 + 2 \int_{t}^{T} e^{r} \langle \hat{Y}_r^n, \hat{f}_n(r, Y^n_r) \rangle \, dr
- 2 \int_t^T e^{r}(Y^n_r, dM^n_r), \quad t \in [0, T].
\]

By (5.14), (5.15)

By (H2) (applied to \(f \circ Y^n\)) and the fact that \(\sup_{0 \leq t \leq T} |U_t| \leq 2a\), we have

\[
\langle U_r, f_{n+m}(r, Y^{n+m}_r) \rangle - f_n(r, Y^n_r) \leq 2a|f_{n+m}(r, Y^{n}_r) - f_n(r, Y^n_r)|.
\]

From this and (5.14), we conclude

\[
E \int_0^T d[V]_r \leq 4aE \int_0^T |f_{n+m}(r, Y^{n}_r) - f_n(r, Y^n_r)| \, dr.
\]

By (5.14), (5.15)

\[
E \sup_{0 \leq t \leq T} |U_t|^2 \leq 4aE \int_0^T |f_{n+m}(r, Y^{n}_r) - f_n(r, Y^n_r)| \, dr + E \sup_{0 \leq t \leq T} \left| \int_t^T \langle U_r, dV_r \rangle \right|.
\]

By the Burkholder-Davis-Gundy inequality and Young’s inequality there exists \(C_1 > 0\) such that

\[
E \sup_{0 \leq t \leq T} \left| \int_{t}^{T} \langle U_r, dV_r \rangle \right| \leq C_1 \left( \int_{0}^{T} |U_r|^2 \, d[V]_r \right)^{\frac{1}{2}} \leq C_1 E \sup_{0 \leq t \leq T} |U_t| \left( \int_0^T d[V]_r \right)^{\frac{1}{2}} \leq \frac{1}{2} E \sup_{0 \leq t \leq T} |U_t|^2 + C_2 E \int_0^T d[V]_r.
\]

By (5.16)–(5.18)

\[
E \sup_{0 \leq t \leq T} |U_t|^2 + E \int_0^T d[V]_r \leq 8a(C_1^2 + 1) E \int_0^T |f_{n+m}(r, Y^{n}_r) - f_n(r, Y^n_r)| \, dr.
\]
Observe that
\[|f_{n+m}(t, Y^n_t) - f_n(t, Y^n_t)| \leq 2\psi_n(t)1_{\{\psi_n(t) > n\}}.\]
Therefore, using (H5), we conclude that \((Y^n, M^n)\) is a Cauchy sequence in the space \(S^2(0, T; \mathbb{R}^d) \times \mathcal{M}_0^d(0, T; \mathbb{R}^d)\). Thus, there exists \((Y, M) \in S^2(0, T; \mathbb{R}^d) \times \mathcal{M}_0^d(0, T; \mathbb{R}^d)\) such that
\[
E\sup_{0 \leq t \leq T} |Y^n_t - Y_t|^2 + E \int_0^T d[M^n - M]_r \to 0, \quad n \to \infty. \tag{5.19}
\]
By the definition of a solution to BSDE \((\xi, f_n)\),
\[
Y^n_t = \xi + \int_t^T f_n(r, Y^n_r) \, dr - \int_t^T dM^n_r, \quad t \in [0, T]. \tag{5.20}
\]
By (5.19), up to subsequence, \(Y^n_t(\omega) \to Y_t(\omega)\) for \(\ell^1 \otimes P\)-a.e. \((t, \omega) \in [0, T] \times \Omega\). Thus, by (H4), \(f_n(\cdot, Y^n) \to f(\cdot, Y), \ell^1 \otimes P\)-a.e. Furthermore, since \(|Y^n_t| \leq a, t \in [0, T], n \geq 1\), we have that for any \(t \in [0, T]\),
\[
|f_n(t, Y^n_t)| \leq \psi_n(t) + |f(t, 0)|, \quad n \geq 1. \tag{5.21}
\]
Therefore, from (H5), and the Lebesgue dominated convergence theorem, we infer that
\[
\sup_{t \leq T} \left| \int_t^T f_n(r, Y^n_r) \, dr - \int_t^T f(r, Y_r) \, dr \right| \to 0, \quad n \to \infty. \tag{5.22}
\]
Thanks to this convergence and (5.19), we may pass to the limit in (5.20) getting that \((Y, M)\) is a solution to BSDE \((\xi, f)\).

**Step 3.** Finally, we shall dispense with (5.2). For each \(n \in \mathbb{N}\) set
\[
\xi_n = \Pi_n(\xi), \quad f_n(t, y) := f(t, y) - f(t, 0) + \Pi_n(f(t, 0)),
\]
where \(\Pi_n(y) = y \min(n, |y|)/|y|^{-1}\). Obviously,
\[
E|\xi - \xi_n|^2 \to 0, \quad E\left( \int_0^T |f(t, 0) - f_n(t, 0)| \, dt \right)^2 \to 0, \tag{5.23}
\]
as \(n \to \infty\). We see that for each \(n \in \mathbb{N}\), the pair \((\xi_n, f_n)\) satisfies the assumptions of Step 2. Therefore, for any \(n \in \mathbb{N}\), there exists a unique solution \((Y^n, M^n) \in S^2(0, T; \mathbb{R}^d) \times \mathcal{M}_0^d(0, T; \mathbb{R}^d)\) to BSDE \((\xi_n, f_n)\). Let
\[
F(r, y) := f_n(r, y + Y^n_r) - f_m(r, Y^m_r), \quad r \in [0, T], y \in \mathbb{R}^d.
\]
Observe that \((Y^n - Y^m, M^n - M^m)\) is a solution to BSDE \((\xi_n - \xi_m, F)\), and data \(\xi_n - \xi_m, F\) satisfy (H1)–(H2) (in place of \(\xi\) and \(f\), respectively). Thus, by Proposition 4.2
\[
E\sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^2 + E \int_0^T d[M^n - M^m]_r \leq C \left( E|\xi^n - \xi^m|^2 + E\left( \int_0^T |F(r, 0)| \, dr \right)^2 \right)
= C \left( E|\xi_n - \xi_m|^2 + \left( \int_0^T |f_n(r, 0) - f_m(r, 0)| \, dr \right)^2 \right). \tag{5.24}
\]
From this and (5.23), we conclude that

\[ E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^2 + E \int_0^T d[M^n - M^m]_r \to 0, \quad n, m \to \infty. \]

This means that \((Y^n, M^n)\) is a Cauchy sequence in \(S^2(0, T; \mathbb{R}^d) \times \mathcal{M}^2_\mu(0, T; \mathbb{R}^d)\). Thus, there exists \((Y, M) \in S^2(0, T; \mathbb{R}^d) \times \mathcal{M}^2_\mu(0, T; \mathbb{R}^d)\) such that

\[ E \sup_{0 \leq t \leq T} |Y^n_t - Y_t|^2 + E \int_0^T d[M^n - M]_r \to 0, \quad n \to \infty. \quad (5.25) \]

By the definition of a solution to BSDE\((\xi_n, f_n)\),

\[ Y^n_t = \xi_n + \int_t^T f_n(r, Y^n_r) \, dr - \int_t^T dM^n_r, \quad t \in [0, T]. \quad (5.26) \]

The proof shall be concluded by passing to the limit in (5.26). The only term that requires care is the integral of \(f_n\). By (5.25), there exists a subsequence (not relabelled) such that

\[ \sup_{t \leq T} |Y^n_t - Y_t| \to 0 \quad a.s. \]

Therefore, \(X := \sup_{n \geq 1} \sup_{t \leq T} |Y^n_t|\) is finite a.s. Observe that for any \(a > 0\), (5.21) holds on the set \(\{X \leq a\}\). Thus repeating the reasoning following (5.20) we get that (5.22) holds on the set \(\{X \leq a\}\). Since \(a > 0\) was arbitrary, and \(X\) is finite a.s., we easily deduce that the convergence (5.22) holds in probability \(P\). Using this and (5.25), and letting \(n \to \infty\) in (5.26), we get that \((Y, M)\) is a solution to BSDE\((\xi, f)\). \(\square\)

6. PROOF OF THE MAIN RESULT

**Lemma 6.1.** Assume that (H1)–(H5) are satisfied. Let \(H^1, H^2 \in \mathcal{M}^2_\mu(0, T; \mathbb{R}^d)\) and \((Y^i, M^i) \in S^2(0, T; \mathbb{R}^d) \times \mathcal{M}^2_\mu(0, T; \mathbb{R}^d)\) be a solution to BSDE\((\xi, f(\cdot, \cdot, H^1))\), \(i = 1, 2\). Let \(0 \leq a < b \leq T\) and \((b - a) \leq \frac{1}{2(1 + 16 M^* \mu)}\). Then there exists \(\beta \geq 1\) depending only on \(\lambda, \mu^*, (b - a)\) such that

\[ \sup_{a \leq t \leq b} E|Y^1_t - Y^2_t|^2 + E \int_a^b d[M^1 - M^2]_r \leq \beta E|Y^1_b - Y^2_b|^2 + \frac{1}{4} E \int_a^T d[H^1 - H^2]_r. \quad (6.1) \]

**Proof.** By Itô’s formula

\[ |Y^1_t - Y^2_t|^2 + \int_t^b d[M^1 - M^2]_r = |Y^1_b - Y^2_b|^2 \]

\[ + 2 \int_t^b \langle Y^1_r - Y^2_r, f(r, Y^1_r, H^1) - f(r, Y^2_r, H^2) \rangle \, dr \]

\[ - 2 \int_t^b \langle Y^1_r - Y^2_r, d(M^1_r - M^2_r) \rangle, \quad t \in [a, b]. \quad (6.2) \]
By (H2) for any \( \nu \geq 0 \),
\[
(Y_t^1 - Y_t^2, f(r, y_t^1, h^1) - f(r, y_t^2, h^2))
\leq |Y_t^1 - Y_t^2||f(r, y_t^1, h^1) - f(r, y_t^2, h^2)| + \mu^+|Y_t^1 - Y_t^2|^2
\leq \frac{1}{2}(1 + \nu)|Y_t^1 - Y_t^2|^2 + \frac{1}{2(1 + \nu)}|f(r, y_t^1, h^1) - f(r, y_t^2, h^2)|^2 + \mu^+|Y_t^1 - Y_t^2|^2.
\]
(6.3)

Due to the assumptions made on \((y^i, m^i)\) it is routine to verify that the process \( \int_0^b(Y_r^1 - Y_r^2)\,d(M_r^1 - M_r^2) \) is a martingale. Therefore, applying the expectation to (6.2) and (6.3), and combining these two inequalities, we get by (H3)
\[
E|Y_t^1 - Y_t^2|^2 + E \int_0^b d[M^1 - M^2] \leq E|Y_b^1 - Y_b^2|^2
+ \beta_\nu E \int_0^b |Y_r^1 - Y_r^2|^2 \,dr + \frac{\lambda}{1 + \nu}E \int_0^T d[H^1 - H^2],
\]
where \( \beta_\nu := (1 + \nu + 2\mu^+). \) By Gronwall’s lemma
\[
E|Y_t^1 - Y_t^2|^2 \leq e^{\beta_\nu(b-a)} \left( E|Y_b^1 - Y_b^2|^2 + \frac{\lambda}{1 + \nu}E \int_0^T d[H^1 - H^2] \right).
\]
(6.5)

This combined with (6.4) yields
\[
E|Y_t^1 - Y_t^2|^2 + E \int_0^b d[M^1 - M^2] \leq E|Y_b^1 - Y_b^2|^2
+ \beta_\nu(b-a)e^{\beta_\nu(b-a)} \left( E|Y_b^1 - Y_b^2|^2 + \frac{\lambda}{1 + \nu}E \int_0^T d[H^1 - H^2] \right)
+ \frac{\lambda}{1 + \nu}E \int_0^T d[H^1 - H^2] = (1 + \beta_\nu(b-a)e^{\beta_\nu(b-a)})E|Y_b^1 - Y_b^2|^2
+ \frac{\lambda}{1 + \nu}(1 + \beta_\nu(b-a)e^{\beta_\nu(b-a)})E \int_0^T d[H^1 - H^2].
\]

Set \( \nu := 16\lambda \) and \( \beta := 1 + \beta_\nu(b-a)e^{\beta_\nu(b-a)}. \) Then we get the desired inequality. \( \square \)

**Remark 6.2.** Observe that the last integral on the right-hand side of (6.1) is made over \([a, T]\) and not over \([a, b]\). This is the best we can get due to the assumption (H3).

**Theorem 6.3.** Assume that (H1)–(H5) are satisfied. Then there exists a solution \((Y, M)\) to BSDE\((\xi, f)\) such that \(Y \in \mathcal{S}^2(0, T; \mathbb{R}^d)\).

**Proof.** Set \((Y^0, M^0) = (0, 0). \) By Proposition 5.1 for each \(n \geq 1\) there exists a unique solution \((y^n, m^n) \in \mathcal{S}^2(0, T; \mathbb{R}^d) \times \mathcal{M}^0_0(0, T; \mathbb{R}^d)\) to BSDE\((\xi, f(\cdot, \cdot, M^{n-1}))\). Consider a partition of the interval \([0, T]\): \(0 = a_1 < a_2 < \cdots < a_{p-1} < a_p = T\), such that
\[
(a_{i+1} - a_i) \leq \frac{1}{2(1 + 16\lambda + \mu^+)} , \quad i = 1, \ldots, p - 1.
\]
Set
\[
b_i := E \int_{a_{i-1}}^{a_i} d[M^2 - M^1], \quad i = 1, \ldots, p, \quad b := \sum_{i=1}^p b_i.
\]
We let
\[
E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + E \int_{\alpha_i}^{a_{i+1}} d[M^{n+1} - M^n]_r \\
\leq \beta E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + \frac{1}{4} E \int_{\alpha_i}^{T} d[M^n - M^{n-1}]_r. \tag{6.6}
\]

In particular for any \( n \geq 2 \),
\[
\sup_{a_{p-1} \leq T} E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + E \int_{a_{p-1}}^{T} d[M^{n+1} - M^n]_r \leq \frac{1}{4} E \int_{a_{p-1}}^{T} d[M^n - M^{n-1}]_r. \tag{6.7}
\]

From this we conclude that
\[
\sup_{a_{p-1} \leq T} E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + E \int_{a_{p-1}}^{T} d[M^{n+1} - M^n]_r \leq \left( \frac{1}{4} \right)^{n-1} b_p \leq \left( \frac{1}{4} \right)^{n-1} b, \quad n \geq 2. \tag{6.8}
\]

We let
\[
I^M_n(k) = E \int_{a_{k-1}^{ak}} d[M^n - M^{n-1}]_r, \quad I^Y_n(k) = \sup_{a_{k-1} \leq T} E_{\alpha} Y_{n} - Y_{n-1},
\]

and
\[
I_n(k) = I^M_n(k) + I^Y_n(k).
\]

Under this notation (6.8) becomes
\[
I_{n+1}(p) \leq \left( \frac{1}{4} \right)^{n-1} b, \quad n \geq 2. \tag{6.9}
\]

Observe also that \( b_i = I^M_2(i), \ i = 1, \ldots, p \). By Lemma 6.1 and (6.9)
\[
I_{n+1}(p-1) \leq \beta E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + \frac{1}{4} E \int_{a_{p-2}}^{T} d[M^n - M^{n-1}]_r \\
\leq \beta \sup_{a_{p-1} \leq T} E_{\alpha} Y_{n+1} - Y_{n} \mid^2 + \frac{1}{4} E \int_{a_{p-1}}^{T} d[M^n - M^{n-1}]_r \\
+ \frac{1}{4} E \int_{a_{p-2}}^{a_{p-1}} d[M^n - M^{n-1}]_r \\
\leq \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4} E \int_{a_{p-2}}^{a_{p-1}} d[M^n - M^{n-1}]_r, \quad n \geq 2. \tag{6.10}
\]

Thus,
\[
I_{n+1}(p-1) \leq \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4} I^M_n(p-1), \quad n \geq 2.
\]

Consequently,
\[
I_{n+1}(p-1) \leq \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4} I^M_n(p-1) \leq \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4} I_n(p-1) \\
\leq \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4} \left( \left( \frac{1}{4} \right)^{n-2} b(1 + \beta) + \frac{1}{4} I^M_{n-1}(p-1) \right) \\
\leq 2 \left( \frac{1}{4} \right)^{n-1} b(1 + \beta) + \frac{1}{4^2} I^M_{n-1}(p-1).
\]

Continuing in this fashion, we get that for each \( n \geq 2 \),
\[
I_{n+1}(p-1) \leq (n - 1) \left( \frac{1}{4} \right)^{n-1} b(\beta + 1) + \left( \frac{1}{4} \right)^{n-1} b_{p-1} \leq n \left( \frac{1}{4} \right)^{n-1} b(1 + \beta). \tag{6.11}
\]
For the sake of clarity of the reasoning, we shall make one more step. By Lemma 6.1,
\[ I_{n+1}(p-2) \leq \beta E[Y_{a_{n-2}}^{n+1} - Y_{a_{n-2}}^n] + \frac{1}{4} E \int_a^T d[M^n - M^{n-1}] \]
\[ \leq \beta J_{n+1}^{Y}(p-1) + \frac{1}{4} \sum_{i=p-2}^n I_{n}^M(i), \quad n \geq 2. \quad (6.12) \]

By (6.9), (6.11) for each \( n \geq 2, \)
\[ I_{n+1}(p-2) \leq n(\frac{1}{4})^{n-1}b(1+\beta)^{(n) - 1}(\frac{1}{4})^{n-1}b(1+\beta) + (\frac{1}{4})^{n-1}b + \frac{1}{4} I_{n}^M(p-2). \]

Hence
\[ I_{n+1}(p-2) \leq 2n(\frac{1}{4})^{n-1}b(1+\beta)^2 + \frac{1}{4} I_{n}^M(p-2), \quad n \geq 2. \]

Therefore
\[ I_{n+1}(p-2) \leq 2n(\frac{1}{4})^{n-1}b(1+\beta)^2 + \frac{1}{4} I_{n}^M(p-2) \]
\[ \leq 2 \cdot (2n)(\frac{1}{4})^{n-1}b(1+\beta)^2 + \frac{1}{4} I_{n-1}^M(p-2). \]

Continuing in this fashion we get that for each \( n \geq 2, \)
\[ I_{n+1}(p-2) \leq 2n^2(\frac{1}{4})^{n-1}b(1+\beta)^2. \quad (6.13) \]

Proceeding as above for \( I_{n+1}(p-i) \) with \( i = 3, \ldots, p-1, \) we get
\[ I_{n+1}(k) \leq pn^p - k (\frac{1}{4})^{n-1}b(1+\beta)^p, \quad k = 1, \ldots, p, \quad n \geq 2. \quad (6.14) \]

Thus,
\[ \sup_{0\leq t \leq T} E[Y_{t}^{n+1} - Y_{t}^n]^2 + E \int_0^T d[M^n - M^n] \leq 2n^p(\frac{1}{4})^{n-1}b(1+\beta)^p =: q_n^2. \quad (6.15) \]

Clearly \( \sum_{n \geq 1} q_n \) is convergent. For \( Y \in B_{\mathbb{F}}(0,T;L^2(\Omega,\mathbb{P};\mathbb{R}^l)) \) and \( M \in \mathcal{M}^2_0(0,T;\mathbb{R}^l) \) we set
\[ \|(Y,M)\| := \sqrt{\sup_{0\leq t \leq T} E[Y_{t}]^2 + E \int_0^T d[M]}. \]

It is routine to verify that \( (B_{\mathbb{F}}(0,T;L^2(\Omega,\mathbb{P};\mathbb{R}^l)) \times \mathcal{M}^2_0(0,T;\mathbb{R}^l), \| \cdot \|) \) is a Banach space. Observe that for \( n < m, \)
\[ \|(Y^n,M^m) - (Y^n,M^n)\| \leq \sum_{i=n}^{m-1} q_i \leq \sum_{i=n}^{\infty} q_i. \quad (6.16) \]

Thus, \((Y^n,M^n)_{n \geq 1}\) is a Cauchy sequence in \( B_{\mathbb{F}}(0,T;L^2(\Omega,\mathbb{P};\mathbb{R}^l)) \times \mathcal{M}^2_0(0,T;\mathbb{R}^l). \)

By Itô’s formula
\[ |Y_{t}^m - Y_{t}^n|^2 + \int_0^T d[M^m - M^n] \]
\[ = 2 \int_0^T (f(r,Y_{r}^m,Y_{r}^{m-1}) - f(r,Y_{r}^n,Y_{r}^{n-1}), Y_{r}^m - Y_{r}^n) dr \]
\[ - 2 \int_0^T \langle Y_{r}^{m-n}, d(M^m - M^n) \rangle, \quad t \in [0,T]. \]
By (H2)

\[
(f(r, Y^m_r, M^{m-1}) - f(r, Y^n_r, M^{n-1}), Y^m_r - Y^n_r)
\leq (1 + \mu^*)|Y^m_r - Y^n_r|^2 + |f(r, Y^m_r, M^{m-1}) - f(r, Y^n_r, M^{n-1})|^2.
\]

(6.17)

This combined with the previous inequality yields

\[
E \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 + E \int_0^T d[M^m - M^n],
\]

\[
\leq 4(1 + \mu^*)E \int_0^T |Y^m_r - Y^n_r|^2 dr
\]

\[
+ 4E \int_0^T |f(r, Y^m_r, M^{m-1}) - f(r, Y^n_r, M^{n-1})|^2 dr
\]

\[
+ 4E \sup_{0 \leq t \leq T} E \left( \int_t^T (Y^m_r - Y^n_r) d(M^m_r - M^n_r) \right).
\]

By (H3) and the Burkholder-Davis-Gundy inequality there exists $C > 0$ such that

\[
E \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 + E \int_0^T d[M^m - M^n],
\]

\[
\leq 4(1 + \mu^*)E \int_0^T |Y^m_r - Y^n_r|^2 dr + 4\lambda E \int_0^T d[M^{m-1} - M^{n-1}],
\]

\[
+ 4CE \left( \int_0^T |Y^m_r - Y^n_r|^2 d[M^m_r - M^n_r] \right)^{1/2}
\]

\[
\leq 4(1 + \mu^*)E \int_0^T |Y^m_r - Y^n_r|^2 dr + 4\lambda E \int_0^T d[M^{m-1} - M^{n-1}],
\]

\[
+ \frac{1}{2} E \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 + 8C^2 E \int_0^T d[M^m - M^n].
\]

Hence,

\[
E \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 + E \int_0^T d[M^m - M^n],
\]

\[
\leq 8(1 + \mu^*)E \int_0^T |Y^m_r - Y^n_r|^2 dr + 8\lambda E \int_0^T d[M^{m-1} - M^{n-1}],
\]

\[
+ 16C^2 E \int_0^T d[M^m - M^n],
\]

\[
\leq (8\mu^* T + 8T + 16C^2) \|(Y^m, M^m) - (Y^n, M^n)\|^2
\]

\[
+ 8\lambda \|(Y^{m-1}, M^{m-1}) - (Y^{n-1}, M^{n-1})\|^2.
\]

From this and (6.16) we conclude that $(Y^n, M^n)_{n \geq 1}$ is a Cauchy sequence in the space $S_2(0, T; \mathbb{R}^d) \times \mathcal{M}_2^d(0, T; \mathbb{R}^d)$. Therefore, there exists a pair of càdlàg processes $(Y, M) \in S_2(0, T; \mathbb{R}^d) \times \mathcal{M}_2^d(0, T; \mathbb{R}^d)$ such that

\[
E \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 + E \int_0^T d[M^m - M^n] \rightarrow 0.
\]

(6.18)
By the definition of a solution to BSDE($\xi, f(\cdot, \cdot, M^{n-1})$),
\[ Y^n_t = \xi + \int_t^T f(r, Y^n_r, M^{n-1}) \, dr - \int_t^T dM^n_r, \quad t \in [0, T] \tag{6.19} \]
The final step in the proof thus shall be passing to the limit in (6.19). For this we have to cope with the nonlinear term of the equation. Clearly,
\[ |f(r, Y^n_r, M^{n-1}) - f(r, Y_r, M)| \leq |f(r, Y^n_r, M) - f(r, Y^n_r, M^{n-1})| + |f(r, Y^n_r, M) - f(r, Y_r, M)|. \tag{6.20} \]
By (H3) and (6.18)
\[ E \int_0^T |f(r, Y^n_r, M) - f(r, Y^n_r, M^{n-1})| \, dr \leq \sqrt{\lambda_T} \left( E \int_0^T d[m^{n-1} - M_r] \right)^{\frac{1}{2}} \to 0. \tag{6.21} \]
Now we shall focus on the second term on the right-hand side of (6.20). Let $(n_k)$ be a subsequence of $(n)$. Then by (6.18) there exists a further subsequence $(n_{k_l})$ such that
\[ \sum_{l=1}^\infty E \sup_{0 \leq t \leq T} |Y^{n_{k_l}}_t - Y_t| < \infty. \tag{6.22} \]
Set $l := n_{k_l}$. For $\nu > 0$ set
\[ A_\nu := \{ \omega \in \Omega : \exists t \in [0, T] \exists l \geq 1 |Y^l_t(\omega)| \geq \nu \}. \]
We have
\[ P\left( \int_0^T |f(r, Y^l_r, M) - f(r, Y_r, M)| \, dr > \varepsilon \right) \leq P(A_\nu) \]
\[ + P(1_{A_\nu} \int_0^T |f(r, Y^l_r, M) - f(r, Y_r, M)| \, dr > \varepsilon). \tag{6.23} \]
By (6.22)
\[ P(A_\nu) = P(\exists l \geq 1 \sup_{0 \leq t \leq T} |Y^l_t| \geq \nu) \]
\[ \leq P(\exists l \geq 1 \sup_{0 \leq t \leq T} |Y^l_t - Y_t| \geq \nu/2) + P(\sup_{0 \leq t \leq T} |Y_t| \geq \nu/2) \]
\[ \leq \frac{2}{\nu} E \sup_{0 \leq t \leq T} |Y_t| + \frac{2}{\nu} \sum_{l=1}^\infty E \sup_{0 \leq t \leq T} |Y^l_t - Y_t| \to 0, \quad \nu \to \infty. \tag{6.24} \]
By (6.22) and (H4)
\[ |f(r, Y^l_r, M) - f(r, Y_r, M)| \to 0, \quad l \to \infty, \quad \ell^1 \otimes P\text{-a.e.} \tag{6.25} \]
Observe that
\[ 1_{A_\nu}|f(r, Y^l_r, M) - f(r, Y_r, M)| \leq |f(r, Y_r, M) - f(r, Y_r, 0)| + |f(r, Y^l_r, M) - f(r, Y^l_r, 0)| \]
\[ + 1_{A_\nu}|f(r, Y^l_r, 0)| + 1_{A_\nu}|f(r, Y_r, 0)| \]
\[ \leq |f(r, Y_r, M) - f(r, Y_r, 0)| + |f(r, Y^l_r, M) - f(r, Y^l_r, 0)| \]
\[ + 2 \sup_{|y| \leq \nu} |f(r, y, 0)| ; \quad g^l(r) \]
By (H3) and (H5), the family $(g^l_r)_{l \geq 1}$ is uniformly integrable with respect to the measure $\ell^1 \otimes P$. From this, (6.25), and the Vitali convergence theorem, we conclude
that the second term on the right-hand side of (6.23) goes to zero as $l \to \infty$. Therefore, letting $l \to \infty$ in (6.23) we get
\[
\limsup_{l \to \infty} P\left( \int_0^T |f(r, Y_r^l, M) - f(r, Y_r, M)| dr > \varepsilon \right) \leq P(A_{\nu}).
\]
Then, letting $\nu \to \infty$ and using (6.24) we obtain
\[
\int_0^T |f(r, Y_r^l, M) - f(r, Y_r, M)| dr \to 0, \quad l \to \infty
\]
in probability $P$. Since $(n_k)$ was an arbitrary subsequence of $(n)$, we get that the above convergence holds with $l$ replaced by $n$. This combined with (6.20), (6.21) implies
\[
\sup_{0 \leq t \leq T} \left| \int_t^T f(r, Y_r^{n_k}, M^{n_k-1}) - \int_t^T f(r, Y_r, M) \right| dr \to 0, \quad l \to \infty
\]
in probability $P$. By using this convergence and (6.18), we let $n \to \infty$ in (6.19) to get
\[
Y_t = \xi + \int_t^T f(r, Y_r, M) dr - \int_t^T dM_r, \quad t \in [0, T].
\]
Thus, $(Y, M)$ is a solution to BSDE($\xi, f$).

\[
7. \text{ Lipschitz type condition on the driver}
\]

The purpose of this section is to demonstrate that the method presented in Section 6 also works for generators of the form
\[
f : [0, T] \times \Omega \times S^2(0, T; \mathbb{R}^l) \times \mathcal{M}_0^2(0, T; \mathbb{R}^l) \to \mathbb{R}^l.
\]
As usual we assume that for any $(Y, M) \in S^2(0, T; \mathbb{R}^l) \times \mathcal{M}_0^2(0, T; \mathbb{R}^l)$ process $f(\cdot, Y, M)$ is progressively measurable.

**Definition 7.1.** We say that a pair $(Y, M) \in S^2(0, T; \mathbb{R}^l) \times \mathcal{M}_0^2(0, T; \mathbb{R}^l)$ is a solution of BSDE($\xi, f$) if

(a) $Y$ is càdlàg,
(b) $\int_0^T |f(r, Y_r, M)| dr < +\infty$,
(c) $Y_t = \xi + \int_t^T f(r, Y_r, M) dr - \int_t^T dM_r, \quad t \in [0, T].$

As to the regularity of $f$ with respect to $Y$-variable, we consider an analogue of condition (H3). We formulate it as a one condition which expresses regularity of $f$ with respect to $Y$ and $M$-variable.

(A) There exists $L > 0$ such that for any $t \in [0, T]$, $M, M' \in \mathcal{M}_0^2(0, T; \mathbb{R}^l)$, $Y, Y' \in S^2(0, T; \mathbb{R}^l)$
\[
E \int_t^T |f(r, Y_r, M) - f(r, Y'_r, M')|^2 dr \leq L(E \int_t^T |M - M'|^2_r dr + E \int_t^T |Y_r - Y'_r|^2 dr).
\]

This is thus a type of Lipschitz continuity, so in some sense stronger than (H2), however it permits $f(t, Y, M)$ to depend not only on the future of the process $M$ but also on the future of the process $Y$. This generalization allows us in particular to consider in place of $Y_t$ in Examples 3.1–3.8, $Y_{t+\delta(t)}$ or $\mathcal{L}(Y_t)$. An existence result for (1.9) with $f$ satisfying condition of type (A) was provided in [8]. The authors,
however, assumed in [8] that \( f \) is of the special form (1.2) with \( h \) defined in Example 3.4.

**Theorem 7.2.** Assume that (H1) and (A) are in force. Then there exists a unique solution to BSDE\((\xi, f)\).

**Proof.** It is enough to repeat the arguments of Section 6 with small modifications. The proof of Lemma 6.1 may be repeated step by step with one exception that instead of (6.3) one should apply

\[
E \int_t^T (Y^1_r - Y^2_r, f(r, Y^1_r, H^1_r) - f(r, Y^2_r, H^2_r)) \, dr \\
\leq \frac{1}{2} (1 + \nu + L) E \int_t^T |Y^1_r - Y^2_r|^2 \, dr + \frac{L}{2(1 + \nu)} E \int_t^T d[H^1 - H^2]_r,
\]

which follows directly from (A). The whole proof of Theorem 6.3 up to (6.19) may be repeated under assumptions (H1), (A) with one exception that instead of (6.17) one should employ the following inequality

\[
E \int_0^T |(f(r, Y^m_r, M^{m-1}_r) - f(r, Y^n_r, M^{n-1}_r), Y^m_r - Y^n_r)| \, dr \\
\leq (1 + L) E \int_0^T |Y^m_r - Y^n_r|^2 \, dr + LE \int_0^T d[M^{m-1} - M^{n-1}]_r,
\]

which once again follows directly from (A). Here note that the existence of \((Y^n_r, M^n)\) follows from [8, Proposition 3.7]. Now we can easily pass to the limit in (6.19) by using condition (A) again. □

**Acknowledgements.** T. Klimsiak is supported by Polish National Science Centre: Grant No. 2016/23/B/ST1/01543. M. Rzymowski acknowledges the support of the Polish National Science Centre: Grant No. 2018/31/N/ST1/00417.

**References**

[1] Bally, V., Pardoux, É., Stoica, L.: Backward stochastic differential equations associated to a symmetric Markov process. *Potential Anal.* **22** (2005) 17–60.

[2] Barles, G., Buckdahn, R., Pardoux, É.: Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.* **60** (1997) 57–83.

[3] Barrasso, A., Russo, F.: Decoupled mild solutions of path-dependent PDEs and integro PDEs represented by BSDEs driven by cadlag martingales. *Potential Anal.* **53** (2020) 449–481.

[4] Bensoussan, A., Li, Y., Yam, S.: Backward stochastic dynamics with a subdifferential operator and non-local parabolic variational inequalities. *Stoch. Anal. Appl.* **128** (2018) 644–688.

[5] Böttcher, B., Schilling, R., Wang, J.: Lévy Matters III. Lévy-Type Processes: Construction, Approximation and Sample Path Properties. *Lecture Notes in Math.* **2099** Springer, Cham (2013).

[6] Briand, Ph., Delyon, B., Hu, Y., Pardoux, É., Stoica, L.: \( L^p \) solutions of Backward Stochastic Differential Equations. *Stochastic Process. Appl.* **108** (2003) 109–129.

[7] Buckdahn, R., Li, J., Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Process. Appl.* **119** (2009) 3153–3154.

[8] Cheridito, P., Nam, K.: BSE’s, BSDE’s and fixed point problems. *Ann. Probab.* **45** (2017) 3795–3828.

[9] Cohen, S.N., Elliott, R.J.: Existence, uniqueness and comparisons for BSDEs in general spaces. *Ann. Probab.* **40** (2012) 2264–2297.
[10] Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer, Berlin (2003).

[11] Klimsiak, T.: Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations. *J. Funct. Anal.* **268** (2015) 1205–1240.

[12] Klimsiak, T.: Reflected BSDEs on filtered probability spaces. *Stochastic Process. Appl.* **125** (2015) 4204–4241.

[13] Kruse, T., Popier, A.: BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics* **88** (4) (2016) 491–539.

[14] Liang, G., Lyons, T., Qian Z.: Backward stochastic dynamics on a filtered probability space. *Ann. Probab.* **39** (2011) 1422–1448.

[15] Nualart, D., Schoutens, W.: Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance. *Bernoulli* **7** (5) (2001) 761–776.

[16] Pardoux, É.: Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In Stochastic analysis and related topics, VI (Gel’o, 1996) *Progr. Probab.* **42** Birkhäuser Boston, Boston, MA (1998) 79–127.

[17] Pardoux, É.: BSDEs, weak convergence and homogenization of semi-linear PDEs. *Proc. Séminaires de Mathématiques Supérieures (Montréal, 1998)* F.H. Clarke and R.J. Stern (Eds.) Kluwer, Dordrecht (1999) 503–549.

[18] Pardoux, É., Peng, S.: Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14**(1) (1990) 55–61.

[19] Peng, S.: Nonlinear expectations, nonlinear evaluations and risk measures. *Stochastic Methods in Finance. Lecture Notes in Math.* **1856** Springer, Berlin (2004) 165–253.

[20] Peng, S., Yang, Z.: Anticipated backward stochastic differential equations. *Ann. Probab.* **37** (2009) 877–902.

[21] Quenez, M.-C., Sulem, A.: BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Process. Appl.* **123** (2013) 3328–3357.

[22] Rozkosz, A.: Backward SDEs and Cauchy problem for semilinear equations in divergence form. *Probab. Theory Related Fields* **125** (2003) 393–407.

[23] Royer, M.: Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Process. Appl.* **116** (2006) 1358–1376.

[24] Situ, R.: On solution of backward stochastic differential equations with jump and applications. *Stochastic Process. Appl.* **66** (1997) 209–236.

[25] Yin, J., Mao, X.: The adapted solution and comparison theorem for backward stochastic differential equations with Poisson jumps and applications. *J. Math. Anal. Appl.* **346** (2008) 345–358.

(Tomasz Klimsiak) INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSAW, POLAND, AND FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY, CHOPINA 12/18, 87-100 TORUŃ, POLAND, E-MAIL: tomas@mat.umk.pl

(Maurycy Rzymowski) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY, CHOPINA 12/18, 87-100 TORUŃ, POLAND, E-MAIL: maurycyrzymowski@mat.umk.pl