Some results about regularity and monotonicity of the speed for excited random walks in small dimensions

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Abstract
Using renewal times and Girsanov’s transform, we prove that the speed of the excited random walk is differentiable of all orders with respect to the bias parameter on $(0, 1)$ in every dimension $d \geq 2$. For the case of the critical point $0$, using a special method, we also prove that the speed is differentiable and the derivative is positive at critical point $0$ for every dimension $2 \leq d \neq 3$. We also give some results of monotonicity for $m$-excited random walk when $m$ large enough and $m = +\infty$.

1 Introduction

1.1 Excited random walk
An $m$-excited random walk ($m$-ERW) with bias parameter $\beta \in [0, 1]$ is a discrete time nearest neighbor random walk $(Y_n)_{n \geq 0}$ on the lattice $\mathbb{Z}^d$ obeying the following rule: when at time $n$ the walk is at a site it has already visited more than $m - 1$ times before time $n$, it jumps uniformly at random to one of the $2d$ neighboring sites. On the other hand, when the walk is at a site it has not visited more than $m - 1$ before time $n$, it jumps with probability $(1+\beta)/2d$ to the right, probability $(1-\beta)/2d$ to the left, and probability $1/(2d)$ to the other nearest neighbor sites. From the description of $m$-ERW, the law $\mathbb{P}_{m,\beta}$ of $m$-ERW with is the probability on the path space $(\mathbb{Z}^d)^\mathbb{N}$, defined by:

- $\mathbb{P}_{m,\beta}(Y_0 = 0) = 1$,
- $\mathbb{P}_{m,\beta}[Y_{n+1} - Y_n = \pm e_i | Y_0, \ldots, Y_n] = \frac{1}{2d}$ for $2 \leq i \leq d$,
- if $Y_n$ has been exactly visited $k - 1$ times before time $n$ and we denote this event by $\{Y_n \notin k\}$ then
  \[ \mathbb{P}_{m,\beta}[Y_{n+1} - Y_n = \pm e_1 | Y_0, \ldots, Y_n] = \begin{cases} \frac{1+\beta}{2d} & \text{for } 1 \leq k \leq m, \\ \frac{1-\beta}{2d} & \text{for } k > m. \end{cases} \]
When \( m = 1 \) we recover excited random walk (ERW) that was introduced in 2003 by I. Benjamini and D.B. Wilson [4]. We also denote \( \mathbb{P}_\beta \) the law of ERW and \( N_n(\beta) \) is the numbers of visited points by ERW at time \( n \). When \( m = +\infty \) we obtain simple random walk with bias \( \beta \) (SRW), we denote \( R_n \) is the number of visited points by SRW at time \( n \). One proved that (see [18], [13]) a.s. \( R(\beta) := \lim_{n \to +\infty} \frac{R_n}{n} \).

In 2007, J. Bérand and A. Ramírez [5] and in 2012, with the different approach, M. Menshikov, S. Popov, A. Ramírez and M. Vachkovskaia [15] proved a law of large numbers and a central limit theorem for the excited random walk for \( d \geq 2 \), namely:

- (Law of large numbers). There exists \( v = v(\beta, d), 0 < v < +\infty \) such that a.s.
  \[
  \lim_{n \to \infty} n^{-1}Y_n \cdot e_1 = v.
  \]

- (Central limit theorem). There exists \( \sigma = \sigma(\beta, d), 0 < \sigma < +\infty \) such that \( \left( n^{-1/2}(Y_{\lfloor nt \rfloor} \cdot e_1 - v(\lfloor nt \rfloor)), t \geq 0 \right) \) converges in law as \( n \to +\infty \) to a Brownian motion with variance \( \sigma^2 \).

Our main result about regularity for the excited random walk is the following:

**Theorem 1.1.** For \( d \geq 2 \), let \( v_n(\beta) \) be the speed at time \( n \) and \( v(\beta) \) be the speed of ERW.

- The speed \( v(\beta) \) is infinitely differentiable on \((0, 1)\) i.e. \( v(\beta) \in C^\infty(0, 1) \) and
  \[
  \frac{\partial^k v}{\partial \beta^k}(\beta) = \lim_{n \to \infty} \frac{\partial^k v_n}{\partial \beta^k}(\beta) \text{ for every } k \in \mathbb{N}, \beta > 0.
  \]

  The derivative is expressed in function of renewal time:
  \[
  \frac{\partial v}{\partial \beta}(\beta) = \frac{1}{d} \frac{\hat{E}_\beta N_\tau}{\mathbb{E}_\beta \tau} + \beta \left( \frac{\hat{E}_\beta(N_\tau V_\tau)\mathbb{E}_\beta \tau - \hat{E}_\beta N_\tau \hat{E}_\beta(\tau V_\tau)}{\mathbb{E}_\beta \tau^2} \right) \text{ for } \beta > 0,
  \]
  where
  \[
  E_i = X_{i+1} - X_i, \quad V_n = \sum_{i=0}^{n-1} \frac{E_i Y_i}{1 + \beta E_i}.
  \]

- For \( d = 2 \) or \( d \geq 4 \), the speed is differentiable at \( \beta = 0 \) and the derivative at 0 is positif, and such that
  \[
  \lim_{\beta \to 0} \frac{v(\beta)}{\beta} = \lim_{n \to \infty} \frac{\partial v_n}{\partial \beta}(0),
  \]
  with
  \[
  \lim_{n \to \infty} \frac{\partial v_n}{\partial \beta}(0) = \frac{1}{d} \lim_{n \to \infty} \frac{R_n}{n} = \frac{1}{d} R(0) > 0 \text{ for } d \geq 4, \text{ and equals 0 for } d = 2.
  \]

- For \( d = 3 \) then
  \[
  \limsup_{\beta \to 0} \frac{v(\beta)}{\beta} \leq \lim_{n \to \infty} \frac{\partial v_n}{\partial \beta}(0) = \frac{1}{d} R(0).
  \]
R. van der Hofstad and M. Holmes proved in [9] that the speed of ERW $v$ is strictly increasing in $\beta$ for $d \geq 9$ and is increasing in a neighbourhood of 0 for $d = 8$ relying on the lace expansion technique. Using the same expansion technique, it is shown in [10], Th. 2.3 that the speed is in an appropriate sense continuous in the drift parameter $\beta$ if $d \geq 6$ and even differentiable if $d \geq 8$. In our paper, we prove that the speed is infinitely differentiable on $(0, 1)$ for all $d \geq 2$ using renewal times and Girsanov’s transform. We are also interested in the derivative at the critical point $\beta = 0$. When the derivative at 0 is positive and is continuous at 0 then the velocity is monotonic in a neighbourhood of 0. The existence of the derivative at 0 of the speed of a random process and it’s relation with the diffusion constant of the equilibrium state play an important role in mathematical physics. This problem is known as ”Einstein relation for random process”, (see for instance the work of N. Gantert, P. Mathieu and A. Piatnitski [8], see also [3], [11], [12], [14]). In [8], the authors use renewal times by $\tau_n \sim \frac{n}{\lambda}$ and they use Markov property of random process to prove the existence of the derivative of the speed at 0. However, we have not yet known how to use this technique for ERW when the Markov property disappears and the increments $\{Y_{[\tau_n,\tau_{n+1}]}\}_{n \geq 0}$ are not independent anymore where $\{Y_{[n,n+p]}\} = \{Y_{n+i} - Y_n, 1 \leq i < p\}$. In our present work, we use a special method to prove the existence of the derivative at 0 but it is still open for the case $d = 3$. It is also open for the existence of derivative in order 2 at 0, even the continuity of the derivative at 0.

In Theorem 1.1 we have the formula of the derivative of the speed that is expressed in function of renewal time:

$$\frac{\partial v}{\partial \beta}(\beta) = \frac{1}{d} E_\beta N_\tau + \frac{\beta}{d} \frac{E_\beta(N_\tau V_\tau)E_\beta\tau - E_\beta N_\tau E_\beta(\tau V_\tau)}{(E_\beta\tau)^2}$$

for $\beta > 0$.

We have

$$\frac{\partial v}{\partial \beta}(0) = \lim_{\beta \to 0} \frac{1}{d} E_\beta N_\tau.$$

Then, if we want to prove the continuity of the derivative at 0, it has to prove that:

$$\lim_{\beta \to 0} \left| \frac{\partial v}{\partial \beta}(\beta) - \frac{\partial v}{\partial \beta}(0) \right| = \lim_{\beta \to 0} \frac{\partial v}{\partial \beta}(\beta) = \lim_{\beta \to 0} \frac{\beta}{d} \frac{E_\beta(N_\tau V_\tau)E_\beta\tau - E_\beta N_\tau E_\beta(\tau V_\tau)}{(E_\beta\tau)^2} = 0.$$

For this reason, we want to estimate the renewal time $\tau$ in function of bias $\beta$. On the other hand, the speed at time $n$ is that: $v_n(\beta) = \frac{d}{d} E_\beta \left( \frac{N_n}{n} \right)$.

Take the derivative in $\beta$, we get:

$$\frac{\partial v_n}{\partial \beta}(\beta) = \frac{1}{d} E_\beta \left( \frac{N_n}{n} \right) + \frac{\beta}{d} E_\beta \left( \frac{N_n V_n}{n} \right).$$

Then,

$$\lim_{\beta \to 0} \left| \frac{\partial v}{\partial \beta}(\beta) - \frac{\partial v}{\partial \beta}(0) \right| = \lim_{n \to \infty} \lim_{\beta \to 0} \left| \frac{\partial v_n}{\partial \beta}(\beta) - \frac{1}{d} E_\beta \left( \frac{N_n}{n} \right) \right| = \lim_{n \to \infty} \frac{\beta}{d} \sqrt{\text{Var}_\beta V_n} \frac{\text{Var}_\beta N_n}{n} \sqrt{\text{Var}_\beta V_n}.$$
Note that

\[
\frac{\text{Var}_\beta V_n}{n} = \frac{E_\beta \left[ \left( \sum_{i=0}^{n-1} \frac{E_{i,Y_1}}{1+\beta e_1} \right)^2 \right]}{n} = \frac{\sum_{i=0}^{n-1} E_\beta \left[ \left( \frac{E_{i,Y_1}}{1+\beta e_1} \right)^2 \right]}{n} \leq \frac{1}{(1-\beta)^2}.
\]

We imply that if

\[
\lim_{\beta \to 0} \limsup_{n \to \infty} \left( \beta^2 \frac{\text{Var}_\beta N_n}{n} \right) = 0
\]

then the derivative \( \frac{2}{\beta^2} \) is continuous at 0.

For \( m-\text{ERW} \), we have some results about the monotonicity as follows:

**Theorem 1.2.** Let \( R(\beta) \) be the limit of \( \frac{R_n}{n} \), where \( R_n \) is the number of the visited point at time \( n \) by the simple random walk with bias \( \beta \) (resp. \( m = +\infty \)). Let \( v(m, \beta) \) be the speed of \( m-\text{ERW} \) with bias \( \beta \). Then,

- \( R(\beta) \) is monotonic in \( \beta \in [0, 1] \).
- For \( d \geq 4 \), for every \( 0 < \beta_0 < \beta_1 < 1 \), there exists an integer \( m_0 = m(\beta_0, \beta_1) \) large enough such that \( v(m, \beta) \) is monotonic on \( [\beta_0, \beta_1] \) for every \( m \geq m_0 \).

Let us explain the organization of this paper.

In Section 2 we present the renewal structure for random walks, we also recall some important properties of renewal times.

In Section 3 we give the proof of Theorem 1.1. We use Girsanov’s transform with renewal time to prove that the speed of ERW is infinitely differentiable for bias \( \beta \) positive. For the existence of the derivative at critical point 0, we use a special method for ERW.

In Section 4 we prove Theorem 1.2. Firstly, we introduce the coupling of \( m-\text{ERW} \) with stationary random walk with bias \( \beta \). For the monotonicity of the range of simple random walk (resp. \( m = +\infty \)) we see that a similar proof can apply for the case random walk in random environment on integer (i.e. in dimension \( d = 1 \)). For the dimension \( d \geq 2 \) we have known yet how to prove this problem. For monotonicity of the speed when \( m \) large enough, we use the coupling method and a key Lemma presented in this section.

## 2 The renewal structure

We define the renewal times for a random walk. Let \( \{Y_n\}_{n \geq 0} \) be a random walk on \( \mathbb{Z}^d \).

**Definition 2.1.** We present the definition given in [5] and [15]. For every \( u > 0 \), we set:

\[
T_u = \min\{k \geq 1 : Y_k \cdot e_1 \geq u\}.
\]

We define

\[
\overline{D} = \inf\{m \geq 0 : Y_m \cdot e_1 < Y_0 \cdot e_1\}.
\]
Moreover, we define two sequences of $\mathcal{F}_N$-stopping times $\{S_n : n \geq 0\}$ and $\{D_n : n \geq 0\}$ as follows: Let $S_0 = 0$, $R_0 = Y_0 \cdot e_1$ and $D_0 = 0$. We define by induction on $k \geq 0$

\[
\begin{align*}
S_{k+1} &= T_{R_{k+1}} \\
D_{k+1} &= \mathcal{D} \circ \theta_{S_{k+1}} + S_{k+1} \\
R_{k+1} &= \sup\{Y_i \cdot e_1 : 0 \leq i \leq D_{k+1}\},
\end{align*}
\]

where $\theta$ is the canonical shift on the space of trajectories. Let

$$
\kappa = \inf\{n \geq 0 : S_n < \infty, D_n = \infty\},
$$

with the convention that $\inf\{\emptyset\} = \infty$. We define the first renewal time as follows:

$$
\tau_1 = S_\kappa.
$$

Next, we define by induction on $n \geq 1$, the sequence of renewal times $\tau_1, \tau_2, \ldots$ as follows:

$$
\tau_{n+1} = \tau_n + \tau_1(Y_{\tau_n+}).
$$

Next, we define $D_i^{(0)} = D_i$ and $S_i^{(0)} = S_i$ and for every $k \geq 1$ two sequences $D_i^{(k)}$ and $S_i^{(k)}$ w.r.t. the trajectory $(Y_{\tau_n^+})$, of the same way that the sequences $D_i$ and $S_i$ are defined w.r.t. $(Y)$. For example, $S_0^{(1)}, R_0^{(1)} = Y_1 \cdot e_1, D_0^{(1)} = 0$ and we define by induction on $i \geq 0$,

\[
\begin{align*}
S_{i+1}^{(1)} &= T_{R_i^{(1)}+1} \\
D_{i+1}^{(1)} &= \mathcal{D} \circ \theta_{S_i^{(1)}+1} + S_{i+1}^{(1)} \\
R_{i+1}^{(1)} &= \sup\{Y_i \cdot e_1 : 0 \leq i \leq D_{i+1}^{(1)}\}.
\end{align*}
\]

For every $k \geq 1$ and $j \geq 0$ such that $S_j^{(k)} < \infty$, we need to introduce the $\sigma$-algebra $\mathcal{G}_j^{(k)}$ of the events up to $S_j^{(k)}$ as the smallest $\sigma$-algebra containing all of the sets of the form $\{\tau_1 \leq n_1\} \cap \{\tau_2 \leq n_2\} \cap \ldots \{\tau_k \leq n_k\} \cap A$, where $n_1 < n_2 < \ldots < n_k$ are integers and $A \in \mathcal{F}_{n_k+S_j^{(0)} \circ \theta_{n_k}}$.

The signification of renewal times is given by the following lemma:

**Lemma 2.2.** The first renewal time $\tau_1$ is the first time when the random walk attend the hyperplane $\{y \cdot e_1 = Y_{\tau_1} \cdot e_1\}$, and after that it do not come back anymore behind this hyperplane:

$$
\tau_1 = \inf\{n \geq 0 : \sup_{0 \leq i < n} Y_i \cdot e_1 < Y_n \cdot e_1 \leq \inf_{n \leq i} Y_i \cdot e_1\}.
$$

**Proof.** By the definition of two sequences $S_i$ and $D_i$ we have: $S_0 = D_0 = 0 < S_1 < D_1 < S_2 < D_2 < \ldots$. Because $D_i, S_i$ are integers, then $\lim_{i \to \infty} S_i = \lim_{i \to \infty} D_i = +\infty$. We set

$$
\tau_1' = \inf\{n \geq 0 : \sup_{0 \leq i < n} Y_i \cdot e_1 < Y_n \cdot e_1 \leq \inf_{n \leq i} Y_i \cdot e_1\},
$$

we prove that $\tau_1 = \tau_1'$. Firstly, it is clear that $\tau_1 \in \{n \geq 0 : \sup_{0 \leq i < n} Y_i \cdot e_1 < Y_n \cdot e_1 \leq \inf_{n \leq i} Y_i \cdot e_1\}$ so $\tau_1' \leq \tau_1$. On the other hand, there exists an integer $i_0$ such that $S_{i_0} \leq \tau_1' <
non-Markovian process by using renewal times. With this change, for Markov process, Lemma 2.4 is still true, but for ERW and non-Markovian process, it is not anymore true. This is a difficulty when we want to study

About the existence of renewal times and the existence of the moments in all orders for excited random walk, we have the following key lemma proved in [5], [15]:

**Lemma 2.3.** Let $(Y_n)_{n \geq 0}$ be a random walk with drift $\beta \in [0,1]$ fixed and let $(\tau_k, k \geq 1)$ be the matched renewal times. Then, there exists $C, \alpha > 0$ such that for every $n \geq 1$,

$$\sup_{k \geq 1} \mathbb{P}_\beta[\tau_{k+1} - \tau_k > n|G_0^{(k)}] \leq Ce^{-n^\alpha} \text{ a.s.}$$

In particular, for every $k \geq 1$ and $p \geq 1$, we have $\tau_k < \infty$ p.s and $\mathbb{E}_\beta[(\tau_{k+1} - \tau_k)^p] < \infty$.

The lemma above give an estimation of renewal times for every drift fixed. We know that, when $\beta = 0$, there does not exist the renewal times, we would like to estimate the renewal times when $\beta$ converges to 0. It is a interesting and difficult question. These renewal times are used in many models to prove the law of large numbers and to prove the Einstein’s relation. This problem in mathematical physic that is studied the first time by the greatest physician Albert Einstein in [7]. Recently this problem appears in the works of mathematicians, for example in [2], [8],... Einstein’s relation means to study the relation between the diffusion constant at the equilibrium state and the derivative of the speed of stochastic process at the critical point $\beta = 0$. A property very important of renewal times is that, they cut a trajectory of the random walk into the independent increments as the following lemma (see [5] and [15]):

**Lemma 2.4.** Under the probability $\mathbb{P}_\beta$, the random variables $(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)_{k \geq 1}$ and $(X_{\tau_1}, \tau_1)$ are independent and $(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)_{k \geq 1}$ have the same law of $(X_{\tau_1}, \tau_1)$ under the probability $\mathbb{P}_\beta$ conditionally on $\bar{D} = \infty$, write $\mathbb{P}_\beta(\cdot) = \mathbb{P}_\beta(|\bar{D} = \infty)$.

From Lemmas 2.3 et 2.4 and with the notation $\tau = \tau_1$, we have $\mathbb{P}_\beta[\tau \geq n] < Ce^{-n^\alpha}$. We note that Lemma 2.4 is not anymore true for the model of generalized excited random walk (see [15]), and also for the case the definition of renewal times is modified as in [17], [8]. We want to estimate the moments of $\tau$ in function of $\beta$ by the question that there exists an integer $k$ such that $\sup_{\beta \in (0,1]} \beta^k \mathbb{E}_\beta \tau < \infty$, where $\sup_{\beta \in (0,1]} \beta^2 \mathbb{E}_\beta \tau^2 < \infty$? We are interested in the case $k = 2$, and we would like to find a definition of renewal times to obtain the estimation of $k = 2$. With the definition 2.1, it is difficult to estimate $\tau$, and it is useful to change a little the definition of $\tau$, for example, in [8] the authors allow the random walk to come back behind the hyperplane (in Lemma 2.2) of a distant $\lambda = \frac{e}{e_1}$, this means that we redefine

$$\bar{D} = \inf\{m \geq 0 : Y_m \cdot e_1 < Y_0 \cdot e_1 - \lambda\}.$$
3 Proof of Theorem 1.1

We repeat some notations necessary.

• \((Y_n)_{n \in \mathbb{Z}}\) are the coordinate maps on \(\mathbb{Z}^d\) and \(\mathbb{P}_\beta\) is the law of the excited random walk. The speed is \(v = v(\beta)\),

• \((e_1, e_2, ..., e_d)\) is the canonical generators of the group \(\mathbb{Z}^d\),

• \(\{\tau_n\}\) is the sequence of renewal times,

• \(X_n = Y_n \cdot e_1, Z_n = (Y_n \cdot e_2, Y_n \cdot e_3, ..., Y_n \cdot e_d)\), \(E_n = X_{n+1} - X_n\),

• \(E_n = E_n - \mathbb{E}_\beta E_n, E_n' = E_n - v\), \(V_n = \sum_{j=0}^{n-1} \mathbb{E}_\beta 1_{Y_j \in \mathbb{E}}\),

• The speed at the time \(n\), the speed of the excited random walk and the derivative of the speed at the time \(n\) respectively are

\[
v_n(\beta) = \mathbb{E}_\beta \left( \frac{X_n}{n} \right), \quad v(\beta) = \frac{\hat{\mathbb{E}}_\beta X_\tau}{\hat{\mathbb{E}}_\beta \tau}, \quad \frac{\partial v_n}{\partial \beta} = \frac{\mathbb{E}_\beta (X_n V_n)}{n}.
\]

3.1 The existence of the derivative of the speed for \(\beta > 0\)

Remark 3.1. Using renewal times, we have that

\[
\lim_{n \to \infty} v_n(\beta) = v(\beta), \quad 0 = \lim_{n \to \infty} \mathbb{E}_\beta \left( \frac{X_n'}{n} \right) = \frac{\hat{\mathbb{E}}_\beta X_\tau}{\hat{\mathbb{E}}_\beta \tau} = \frac{\hat{\mathbb{E}}_\beta (X_\tau - \sum_{j=0}^{n-1} \mathbb{E}_\beta \mathcal{E}_j)}{\hat{\mathbb{E}}_\beta \tau}
\]

\[\mathbb{P} - a.s. \lim_{n \to \infty} \frac{X_n'}{n} = \frac{\hat{\mathbb{E}}_\beta X_\tau'}{\hat{\mathbb{E}}_\beta \tau} = \frac{\hat{\mathbb{E}}_\beta (X_\tau - v \tau)}{\hat{\mathbb{E}}_\beta \tau} = 0.
\]

We deduce from these equalities that \(\hat{\mathbb{E}}_\beta X_\tau = 0\) and \(\hat{\mathbb{E}}_\beta X_\tau' = \hat{\mathbb{E}}_\beta [\sum_{j=0}^{\tau-1} (\mathbb{E}_\beta \mathcal{E}_j)]\).

3.1.1 The existence of the limits of the derivatives at finite times

To prove the point 1 of Theorem 1.1 we need the following lemmas:

Lemma 3.2.

\[
\sup_{n \geq 1} \mathbb{E}_\beta \left( \frac{\max_{0 \leq i \leq n} |X_i'|^2}{n} \right) := C_1(\beta) < +\infty \tag{2}
\]

\[
\sup_{n \geq 1} \mathbb{E}_\beta \left( \frac{\max_{0 \leq i \leq n} |X_i|^2}{n} \right) := C_2(\beta) < +\infty \tag{3}
\]
Proof. Firstly, we prove that

$$\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq [na]} |X'_i|^2 \right)}{n} := C'_1(\beta) < +\infty$$

where $a = \hat{\tau}\beta$. We define

$$S'_i = X'_{na}$$

Set $S'_i = X'_{\tau}$, we have that

$$\max_{0 \leq i \leq [na]} X_i^2 \leq \max_{0 \leq i \leq [na]} S_i^2 + (\tau_n - [na])^2 + \sum_{j=0}^{n-1} (\tau_{j+1} - \tau_j)^2.$$ 

Because that $\max_{0 \leq i \leq [na]} X_i^2$ attains max at $i_0$ then either $i_0 \in [\tau_n, [na]]$ or there exists $j_0$ such that $i_0 \in [\tau_{j_0}, \tau_{j_0+1})$ By the inequality as follows

$$(\tau_n - [na])^2 = [(\tau_n - na + na - [na])]^2 \leq 2[(\tau_n - na)^2 + 1],$$

we get

$$\mathbb{E}_\beta \left( \max_{0 \leq i \leq [na]} |X'_i|^2 \right) \leq \max_{0 \leq i \leq n} S'_i^2 + 2\mathbb{E}_\beta (\tau^2) + 2(n - 1)\hat{\mathbb{E}}_\beta (\tau - a)^2 + (n - 1)\hat{\mathbb{E}}_\beta (\tau^2) + 2.$$

Note that $\{S'_i\}$ is the martingale then

$$\mathbb{E}_\beta (\max_{0 \leq i \leq n} S_i'^2) \leq 4\mathbb{E}_\beta \left( S_n'^2 \right) = 4\mathbb{E}_\beta (X'_\tau^2) + 4(n - 1)\hat{\mathbb{E}}_\beta (X'_\tau^2) \leq 4\mathbb{E}_\beta (\tau^2) + 4(n - 1)\hat{\mathbb{E}}_\beta (\tau^2).$$

Therefore,

$$\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq [na]} |X'_i|^2 \right)}{n} \leq \sup_{n \geq 1} \left( \frac{4n + 2}{n} \hat{\mathbb{E}}_\beta (\tau^2) + \frac{2n - 2}{n} \hat{\mathbb{E}}_\beta (\tau - a)^2 + \frac{2}{n} \right) < +\infty.$$

We now consider the sequence of integers $\{p_n\}$ such that $[p_n a] \leq n < [(p_n + 1)a]$ then $n/p_n \to a$. We deduce that

$$\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} |X'_i|^2 \right)}{n} \leq \sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq [(p_n+1)a]} |X'_i|^2 \right)}{(p_n + 1)} \times \frac{(p_n + 1)}{n} \leq \infty.$$

It is similar to prove that

$$\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} \bar{X}_i^2 \right)}{n} = C_2(\beta) < +\infty;$$

$$\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} V_i^2 \right)}{n} = C_3(\beta) < +\infty.$$
Lemma 3.3.

\[
\sup_{n,p \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq p} \left( X'_{\tau_n+i} - X'_{\tau_n} \right)^2 \right)}{p} = C_4(\beta) < +\infty
\]  

\[
\sup_{n \geq 1} \sup_{0 < p \leq \lfloor n/a \rfloor} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq p} \left( X'_{\tau_n+i} - X'_{\tau_n-i} \right)^2 \right)}{p} = C_5(\beta) < +\infty.
\]

We have similarly the result for the sequences \{\overline{X}_n\} and \{V_n\}.

Proof. From Lemma 3.2 we get

\[
\sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} X_i^2 \right)}{n} \leq \sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} X_i^2 1_{D=0} \right)}{n\mathbb{P}(D=0)}
\]

\[
\leq \frac{1}{\mathbb{P}(D=0)} \sup_{n \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq n} X_i^2 \right)}{p} < +\infty.
\]

Therefore

\[
\sup_{n,p \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq p} \left( X'_{\tau_n+i} - X'_{\tau_n} \right)^2 \right)}{n} = \sup_{n \geq 1} \sup_{p \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq p} (X'_i)^2 \right)}{n}
\]

To prove (5), we consider

\[
\sup_{0 < p \leq \lfloor n/a \rfloor} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq \lfloor pa \rfloor} \left( X'_{\tau_n} - X'_{\tau_n-i} \right)^2 \right)}{p}
\]

For \(0 < p \leq \lfloor n/a \rfloor\) then \(\lfloor pa \rfloor \leq pa \leq (n/a) \cdot a = n\). Set \(S'_i = X'_{\tau_n+i} - X'_{\tau_n-i}\) so that

\[
\max_{0 \leq i \leq \lfloor pa \rfloor} \left( X'_{\tau_n} - X'_{\tau_n-i} \right)^2 \leq \max_{0 \leq i \leq p} S'_i^2 + \sum_{j=0}^{p-1} (\tau_{n-j} - \tau_{n-j-1})^2 + (\tau_n - \tau_{n-p} - \lfloor pa \rfloor)^2.
\]

We deduce from the inequality above that

\[
\mathbb{E}_\beta \left[ \max_{0 \leq i \leq \lfloor pa \rfloor} \left( X'_{\tau_n} - X'_{\tau_n-i} \right)^2 \right] \leq \mathbb{E}_\beta \left( \max S'_i^2 \right) + \sum_{j=0}^{p-1} \mathbb{E}_\beta (\tau_{n-j} - \tau_{n-j-1})^2 + \mathbb{E}_\beta (\tau_n - \tau_{n-p} - \lfloor pa \rfloor)^2
\]

\[
\leq 4\mathbb{E}_\beta (S'_p^2) + p\mathbb{E}_\beta (\tau^2) + 2p\mathbb{E}_\beta [(\tau - a)^2] + 2
\]

\[
\leq 4p\mathbb{E}_\beta (X^2) + p\mathbb{E}_\beta (\tau^2) + 2p\mathbb{E}_\beta [(\tau - a)^2] + 2.
\]

Therefore,

\[
\sup_{p \geq 1} \frac{\mathbb{E}_\beta \left( \max_{0 \leq i \leq \lfloor pa \rfloor} \left( X'_{\tau_n} - X'_{\tau_n-i} \right)^2 \right)}{p} := C(\beta) < \infty.
\]
Let $p \leq [n/a]$ and $0 \leq i \leq p$, there exists a sequence $\{p_n\}$ such that $[p_n a] < p < [(p_n + 1)a]$.

Because $a \geq 1$ and $[p] \leq [n/a] \leq \frac{n}{a} = n$ then $[(p_n + 1)a] \leq [\frac{n}{a}]a \leq n$. So, we have

$$
\lim_{p \to 1} \frac{E_\beta \left( \max_{0 \leq i \leq [(p_n + 1)a]} (X_{r_n}^t - X_{r_n - i}^t)^2 \right)}{p}
\leq \sup_{p \geq 1} \frac{E_\beta \left( \max_{0 \leq i \leq [(p_n + 1)a]} (X_{r_n}^t - X_{r_n - i}^t)^2 \right) \cdot p + 1}{p}
\leq C(\beta) \cdot \frac{p + a}{ap} \leq C'(\beta) < +\infty.
$$

\[\blacksquare\]

**Lemma 3.4.** $\lim_{n \to +\infty} \left| \frac{1}{n} E_\beta(X_{r_n}^t V_{r_n}) - \frac{1}{n} E_\beta(X_{[n/a]}^t V_{[n/a]}) \right| = 0.$

**Proof.** Using the inequality $|(a + \delta)(b + \delta) - ab| \leq |a\delta| + |b\delta| + |\delta^2|$, we have that

$$
\left| \frac{1}{n} E_\beta(X_{r_n}^t V_{r_n}) - \frac{1}{n} E_\beta(X_{[n/a]}^t V_{[n/a]}) \right| \leq \frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \, |V_{r_n}| \right)
+ \frac{1}{n} E_\beta \left( |X_{r_n}^t| \cdot \sum_{j = [n/a]}^{\tau_n - 1} E_j \right)
+ \frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \right) \cdot \left( \sum_{k = [n/a]}^{\tau_n - 1} E_k \right)
\leq \sqrt{\frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j^2 \right)} \cdot \sqrt{\frac{1}{n} E_\beta \left( |V_{r_n}|^2 \right)}
+ \sqrt{\frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j^2 \right)} \cdot \sqrt{\frac{1}{n} E_\beta \left( \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \right)^2 \right)}
$$

There exist two finite constants $C(\beta), C'(\beta)$ depending only on $\beta$ such that

- For all $n \geq 1$ then $\frac{1}{n} E_\beta(V_{r_n}^2) = \frac{1}{n} E_\beta(V_{r_n}^2) + \frac{n}{n} E_\beta(V_{r_n}^2) \leq C(\beta)$;
- For all $n \geq 1$ then $\frac{1}{n} E_\beta(X_{r_n}^t V_{r_n}^2) = \frac{1}{n} E_\beta(X_{r_n}^t V_{r_n}^2) + \frac{n}{n} E_\beta(X_{r_n}^t V_{r_n}^2) \leq C'(\beta)$.

We need prove that

$$
\lim_{n \to +\infty} \frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \right)^2 = 0.
$$

In fact, we have that

$$
\frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \right)^2 \leq \frac{1}{n} E_\beta \left( (\tau_n - [n/a])^2 1_{|\tau_n - [n/a]| \geq \epsilon n} \right) + \frac{1}{n} E_\beta \left( \sum_{j = [n/a]}^{\tau_n - 1} E_j \right)^2 1_{|\tau_n - [n/a]| \leq \epsilon n}
$$

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\[ = L_1 + L_2. \]

Here, \( L_1, L_2 \) are respectively the first and the second terms of the site on the right hand. Estimate two terms to get

\[
L_1 \leq \frac{2}{n} \mathbb{E}_\beta \left[ \left( (\tau_n - na)^2 + 1 \right)_1 |\tau_n - [na]| \geq \varepsilon n \right]
\]

\[
\leq \sqrt{\frac{1}{n^2} \mathbb{E}_\beta \left[ (\tau_n - na)^4 \right]} \mathbb{P} \left( |\tau_n - [na]| \geq \varepsilon n \right) + \frac{2}{n} \mathbb{P} \left( |\tau_n - [na]| \geq \varepsilon n \right).
\]

Because \( \sup_{n \geq 1} \frac{1}{n^2} \mathbb{E}_\beta \left[ (\tau_n - na)^4 \right] < +\infty \) and \( \lim_{n \to +\infty} \mathbb{P} \left( |\tau_n - [na]| \geq \varepsilon n \right) = 0 \) then \( \lim_{n \to +\infty} L_1 = 0 \). On the other hand

\[
L_2 \leq \varepsilon \frac{\mathbb{E}_\beta \left[ \max_{0 \leq i \leq \varepsilon n} (X'_{\tau_n} - X'_{\tau_n - i})^2 + \max_{0 \leq i \leq \varepsilon n} (X'_{\tau_n + i} - X'_{\tau_n})^2 \right]}{\varepsilon n}
\]

\[
\leq \varepsilon C_4(\beta).
\]

For all \( \sigma > 0 \) choose \( \varepsilon = \frac{\sigma}{C_4(\beta)} \) so that \( L_2 \leq \sigma \).
Then we get

\[
\limsup_{n \to +\infty} \frac{1}{n} \mathbb{E}_\beta \left[ \left( \sum_{j=[na]}^{\tau_n-1} \mathcal{E}'_j \right)^2 \right] \leq \sigma \text{ for all } \sigma > 0.
\]

Therefore

\[
\lim_{n \to +\infty} \frac{1}{n} \mathbb{E}_\beta \left[ \left( \sum_{j=[na]}^{\tau_n-1} \mathcal{E}'_j \right)^2 \right] = 0.
\]

It is similar to prove that

\[
\lim_{n \to +\infty} \frac{1}{n} \mathbb{E}_\beta \left[ \left( \sum_{j=[na]}^{\tau_n-1} \frac{\mathcal{E}_j \mathcal{E}_j}{1 + \beta \mathcal{E}_j} \right)^2 \right] = 0.
\]

This finishes the proof of Lemma. \( \square \)

**Corollary 3.5.**

\[
\lim_{n \to +\infty} \mathbb{E}_\beta \left[ \frac{X'_{[na]} V_{[na]}}{n} \right] = \lim_{n \to +\infty} \mathbb{E}_\beta \left[ \frac{X'_{\tau_n} V_{\tau_n}}{n} \right] = \hat{\mathbb{E}}_\beta (X'_{\tau} V_{\tau}).
\]

We now prove the existence of the limit \( \frac{\partial p}{\partial \beta} (\lambda) \). Let \( \{p_n\} \) be the sequence such that \([p_n a] \leq n \leq (p_n + 1) a\) then \( \lim_{n \to +\infty} \frac{n}{p_n} = a \). So, we have

\[
\left| \mathbb{E}_\beta \left( \frac{X'_n V_n}{n} - \frac{X'_{[p_n a]} V_{[p_n a]}}{n} \right) \right| \leq \frac{(n - [p_n a])^2}{n} + \left| \frac{n - [p_n a]}{n} \right| \mathbb{E}_\beta \left| X'_n \right| + \left| \frac{n - [p_n a]}{n} \right| \mathbb{E}_\beta \left| V_n \right|.
\]

\[
\leq a^2 \frac{2}{n} + a \mathbb{E}_\beta \left| \frac{X'_n}{n} \right| + a \mathbb{E}_\beta \left| \frac{V_n}{n} \right|.
\]
When \( n \) goes to infinitely then \( \frac{X'_n}{n} \) and \( \frac{V_n}{n} \) go to 0. So that
\[
\lim_{n \to +\infty} \mathbb{E}_\beta \left( \frac{X'_n V_n}{n} \right) = \lim_{n \to +\infty} \mathbb{E}_\beta \left( \frac{X'_{[p_n a]} V_{[p_n a]}}{p_n} \right) = \lim_{n \to +\infty} \mathbb{E}_\beta \left( \frac{X'_{[p_n a]} V_{[p_n a]}}{p_n} \right) \cdot \frac{p_n}{n} = \hat{\mathbb{E}}_\beta(X'_\tau V_\tau). \frac{1}{a} = \frac{\hat{\mathbb{E}}_\beta(X'_\tau V_\tau)}{\mathbb{E}_\beta(\tau)} = \frac{\hat{\mathbb{E}}_\beta((X_\tau - \tau v)V_\tau)}{\mathbb{E}_\beta(\tau)}.
\]

Therefore,
\[
\lim_{n \to +\infty} \frac{\partial v_n(\beta)}{\partial \beta} = \lim_{n \to +\infty} \mathbb{E}_\beta \left( \frac{X_n V_n}{n} \right) = \lim_{n \to +\infty} \mathbb{E}_\beta \left( \frac{X'_n V_n}{n} \right) = \hat{\mathbb{E}}_\beta((X_\tau - \tau v)V_\tau). (6)
\]

### 3.1.2 Girsanov transform

In this section we prove the existence of the speed using the Girsanov transform. Firstly, we need a lemma as follows:

**Lemma 3.6.** For all \( \sigma \in (0, 1] \) then
\[
\sup_{t \in [\sigma, 1]} \mathbb{P}_t(\tau > n) \leq C e^{n^{-\alpha}}.
\]

Where \( C, \alpha \) are positive constants depending only on \( \sigma \).

**Proof.** To prove this lemma, repeating the proof of Lemma 2.3 (see in [15]) with note that the constants \( \lambda, h, r \) in [15] of general excited random walk depending only on \( \sigma \) for a excited random walk with the law \( \mathbb{P}_t \) where \( t \in [\sigma, 1] \).

Let \( \beta_0, \beta \in (0, 1) \) and we set
\[
M_n(\beta) := \frac{d\mathbb{P}_\beta}{d\mathbb{P}_0}|_{F_n} = \prod_{i=0}^{n-1} \left( 1 + \beta \mathcal{E}_i 1_{Y_n \notin \bar{\mathcal{A}}} \right)
\]
\[
M_n(\beta, \beta_0) := \frac{d\mathbb{P}_\beta}{d\mathbb{P}_{\beta_0}}|_{F_n} = \prod_{i=0}^{n-1} \left( \frac{1 + \beta \mathcal{E}_i 1_{Y_i \notin \bar{\mathcal{A}}}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \notin \bar{\mathcal{A}}}} \right)
\]

To prove the existence of the speed we need the following lemma

**Lemma 3.7.** Consider a \( \sigma \)-algebra \( F_\tau \) that is defined by
\[
F_\tau = \{ A \in F : \forall n, \exists B_n \in F_n \text{ such that } A \cap \{ \tau = n \} = B_n \cap \{ \tau = n \} \}.
\]
then \( \tau \) is \( F_\tau \)-mesurable, (\( D = \infty \)) \( \in F_\tau \) and
\[
\frac{d\mathbb{P}_\beta}{d\mathbb{P}_{\beta_0}}|_{F_\tau} = M_\tau(\beta, \beta_0). \frac{\mathbb{P}_\beta(D = \infty)}{\mathbb{P}_0(D = \infty)} (7)
\]
So, we get
\[ \frac{d\mathbb{P}_\beta}{d\mathbb{P}_{\beta_0}}|_{\mathcal{F}_\tau} = M_\tau(\beta, \beta_0). \frac{\mathbb{P}_\beta(D = \infty)}{\mathbb{P}_{\beta_0}(D = \infty)} \]
and
\[ \frac{d\hat{\mathbb{P}}_\beta}{d\mathbb{P}_{\beta_0}}|_{\mathcal{F}_\tau} = M_\tau(\beta, \beta_0). \]

A direct consequence is that
\[ \mathbb{E}_{\beta_0}[M_\tau(\beta, \beta_0)] = 1 \text{ and } \mathbb{E}_{\beta_0}[M_\tau(\beta, \beta_0)] = \frac{\mathbb{P}_\beta(D = \infty)}{\mathbb{P}_{\beta_0}(D = \infty)}. \]

Using the Girsanov transform, we get the formula of the speed:
\[ v(\beta) = \frac{\mathbb{E}_\beta X_\tau}{\mathbb{E}_\beta \tau} = \frac{\mathbb{E}_{\beta_0}[X_\tau M_\tau(\beta, \beta_0)]}{\mathbb{E}_{\beta_0}[\tau M_\tau(\beta, \beta_0)]]} \]

On the other hand,
\[ \frac{\partial}{\partial \beta} [M_\tau(\beta, \beta_0)] = \frac{\partial}{\partial \beta} \left[ \prod_{i=0}^{\tau-1} \left( \frac{1 + \beta \mathcal{E}_i 1_{Y_i \neq}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \neq}} \right) \right] = \left[ \sum_{i=0}^{\tau-1} \left( \frac{\mathcal{E}_i 1_{Y_i \neq}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \neq}} \right) \right] M_\tau(\beta, \beta_0). \]

Set \( V_\tau = \sum_{i=0}^{\tau-1} \left( \frac{\mathcal{E}_i 1_{Y_i \neq}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \neq}} \right) \) then
\[ \int_{\beta_0}^{\beta} M_\tau(t, \beta_0) V_\tau(t) dt = \int_{\beta_0}^{\beta} \frac{\partial}{\partial t} M_\tau(t, \beta_0) dt = M_\tau(\beta, \beta_0) - M_\tau(\beta_0, \beta_0) = M_\tau(\beta, \beta_0) - 1. \]
Therefore,
\[ v(\beta) = \frac{\mathbb{E}_{\beta_0} [X_\tau \left( 1 + \int_{\beta_0}^{\beta} M_\tau(t, \beta_0) V_\tau(t) dt \right)]}{\mathbb{E}_{\beta_0} [\tau \left( 1 + \int_{\beta_0}^{\beta} M_\tau(t, \beta_0) V_\tau(t) dt \right)]} \]

To prove the existence of the derivative, we apply the Fubini’s theorem as follows:
**Theorem 3.8** (Fubini’s theorem). Let $\mu, \nu$ be the $\sigma$–finite measures. If either

$$ \int_A \left( \int_B |f(x,y)| \nu(dy) \right) \mu(dx) < +\infty \text{ or } \int_B \left( \int_A |f(x,y)| \mu(dx) \right) \nu(dy) < +\infty $$

then $\int_{A \times B} f(x,y) (\mu \times \nu)(dxdy) < +\infty$ and

$$ \int_{A \times B} f(x,y) (\mu \times \nu)(dxdy) = \int_A (\int_B f(x,y) \nu(dy)) \mu(dx) = \int_B (\int_A f(x,y) \mu(dx)) \nu(dy). $$

To apply the Fubini’s theorem, let $\beta \in (\beta_0 - \delta, \beta_0 + \delta) \subset (0, 1)$ we observe that

$$ \beta_0^\beta (\mathbb{E}_{\beta_0}|X_\tau V_\tau M_\tau|) dt \leq \int_{\beta_0}^\beta \mathbb{E}_{\beta_0} \left( \tau^2 M_\tau \frac{1}{1-t} \right) dt $$

$$ \leq \frac{1}{1 - \beta_0 - \delta} \int_{\beta_0}^\beta |\mathbb{E}_t(\tau^2)| dt < +\infty. $$

The last inequality above is implied since $\sup_{t \in (\beta_0 - \delta, \beta_0 + \delta)} \mathbb{P}_t(\tau > n) \leq C e^{-n^\alpha}$ then

$$ \sup_{t \in (\beta_0 - \delta, \beta_0 + \delta)} \mathbb{E}_t(\tau^2) < +\infty. $$

It remains to prove that $\hat{\mathbb{E}}_{\beta_0}(X_\tau V_\tau M_\tau)$ is continuous in $\beta$, this is true if let an interval $J \subset (0, 1)$ we have that $\{(X_\tau V_\tau M_\tau)_{\beta \in J}\}$ is uniformly integrable. Let $\beta_1 \in J$, observe that

- $|X_\tau V_\tau M_\tau| \leq C \tau^2 M_\tau$ for some constant $C$;
- $\lim_{\beta \to \beta_1} (\tau^2 M_\tau)(\beta) = \tau^2 M_\tau(\beta_1)$;
- $\lim_{\beta \to \beta_1} \hat{\mathbb{E}}_{\beta_0}[(\tau^2 M_\tau)(\beta)] = \hat{\mathbb{E}}_{\beta_0}[(\tau^2 V_\tau M_\tau)(\beta_1)].$

Indeed, $\hat{\mathbb{E}}_{\beta_0}[(\tau^2 M_\tau)(\beta)] = \hat{\mathbb{E}}_{\beta_0}\left[ \int_{\beta_1}^\beta (\tau^2 V_\tau M_\tau)(t) dt \right] + \hat{\mathbb{E}}_{\beta_0}[(\tau^2 V_\tau M_\tau)(\beta_1)]$ and

$$ \hat{\mathbb{E}}_{\beta_0}\left[ \int_{\beta_1}^\beta (\tau^2 V_\tau M_\tau)(t) dt \right] = \int_{\beta_1}^\beta \hat{\mathbb{E}}_{\beta_0}\left[ (\tau^2 V_\tau M_\tau)(t) \right] dt $$

$$ \leq \int_{\beta_1}^\beta \hat{\mathbb{E}}_t(\tau^3) dt \leq C(\beta - \beta_1) \to 0 \text{ as } \beta \to \beta_1. $$

From the observation above we imply that $\{\tau^2 M_\tau\}_{\beta \in J}(\beta)$ and also $\{X_\tau V_\tau M_\tau\}_{\beta \in J}(\beta)$ is uniformly integrable then $\hat{\mathbb{E}}_{\beta_0}[(X_\tau V_\tau M_\tau)(\beta)]$ is continuous. \[\square\]

We rewrite the formular of the speed:

$$ v(\beta) = \frac{\hat{\mathbb{E}}_{\beta_0}\left[ X_\tau \left( 1 + \int_{\beta_0}^\beta M_\tau(t, \beta_0)V_\tau(t) dt \right) \right]}{\hat{\mathbb{E}}_{\beta_0}\left[ \tau \left( 1 + \int_{\beta_0}^\beta M_\tau(t, \beta_0)V_\tau(t) dt \right) \right]} = \frac{\hat{\mathbb{E}}_{\beta_0} X_\tau + \int_{\beta_0}^\beta \mathbb{E}_{\beta_0} (X_\tau M_\tau(t, \beta_0)V_\tau(t) dt)}{\hat{\mathbb{E}}_{\beta_0} \tau + \int_{\beta_0}^\beta \mathbb{E}_{\beta_0} (\tau M_\tau(t, \beta_0)V_\tau(t) dt)}, $$

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Set \( A := \hat{E}_{\beta_0} X_t + \int_{\beta_0}^\beta [E_{\beta_0} (X_t M_r (t, \beta_0) V_r (t)) dt] \) and \( B := \hat{E}_{\beta_0} \tau + \int_{\beta_0}^\beta [E_{\beta_0} (\tau M_r (t, \beta_0) V_r (t)) dt] \)

Taking the derivative we obtain:
\[
\frac{\partial v}{\partial \beta} (\beta) = \frac{E_{\beta_0} (X_t M_r (\beta, \beta_0) V_r (\beta)) B - E_{\beta_0} (\tau M_r (\beta, \beta_0) V_r (\beta)) A}{B^2}.
\]

As \( \beta = \beta_0 \),
\[
\frac{\partial v}{\partial \beta} (\beta_0) = \frac{E_{\beta_0} (X_t V_r (\beta_0)) \hat{E}\beta_0 \tau - E_{\beta_0} (\tau V_r (\beta_0)) \hat{E}\beta_0 X_t}{(E_{\beta_0} \tau)^2}.
\]

Therefore, for all \( \beta \in (0, 1) \) we have
\[
\frac{\partial v}{\partial \beta} (\beta) = \frac{E_{\beta} (X_t V_r) \hat{E}\beta \tau - E_{\beta} (\tau V_r) \hat{E}\beta X_t}{(E_{\beta} \tau)^2} = \frac{E_{\beta} [(X_t - v \tau)V_r] \hat{E}\beta \tau}{(E_{\beta} \tau)}. \tag{8}
\]

From 6 and 8 we get that
\[
\lim_{n \to +\infty} \frac{\partial v_n}{\partial \beta} (\beta) = \frac{\partial v}{\partial \beta} (\beta) = \frac{E_{\beta} [(X_t - v \tau)V_r] \hat{E}\beta \tau}{(E_{\beta} \tau)}.
\]

It is similar to prove that for \( k \geq 1 \) and \( \beta > 0 \), there exists a \( k \)-th derivative of the speed such that
\[
\lim_{n \to +\infty} \frac{\partial^k v_n}{\partial \beta^k} (\beta) = \frac{\partial^k v}{\partial \beta^k} (\beta).
\]

If we write the speed in the form \( v(\beta) = \frac{\beta}{d} \frac{\hat{E}\beta N_r}{E_{\beta} \tau} \), we can get the formula 1 of the derivative. We proved the first point of Theorem 1.1.

### 3.2 The existence of the derivative at the critical point 0

We denote the event \( \{ Y_n \notin \{ Y_{n-1}, Y_{n-2}, ..., Y_{n-k} \} \} \) by \( \{ Y_n \notin_k \} \) with the convention that if \( n \leq k \) then the event \( \{ Y_n \notin \{ Y_{n-1}, Y_{n-2}, ..., Y_{n-k} \} \} \) is the event \( \{ Y_n \notin \} \). Set \( N_n^{(k)} := 1_{Y_0 \notin_k} + 1_{Y_1 \notin_k} + ... + 1_{Y_{n-1} \notin_k} \). We need the following lemma:

**Lemma 3.9.** There exists a non negative constant \( N^{(k)}(\beta) \) such that \( \mathbb{P}_\beta - a.s. \)
\[
\lim_{n \to +\infty} \frac{N_n^{(k)}}{n} = N^{(k)}(\beta).
\]

**Proof.** The above result is easy to verify by considering two following cases:

If \( \beta > 0 \) then there exists a sequence of renewal times \( \{ \tau_n \} \) and the sequence \( \{ N^{(k)}_n - N^{(k)}_{n-1} \} \) is independent. It is similar as the law of large number for \( \frac{X_n}{n} \) we also have
\[
\lim_{n \to +\infty} \frac{N^{(k)}_n}{n} = N^{(k)}(\beta).
\]
If \( \beta = 0 \) then \((\mathbb{Z}^d)^n, \theta, \mathbb{P}_0)\) is a system ergodic where \( Y_n \circ \theta = Y_{n+1} - Y_1 \). For \( n \geq k \) then \( \{Y_n \not\in \mathcal{F}_k\} = \{Y_k \circ \theta^{n-k} \not\in \mathcal{F}_k\} \), therefore

\[
\lim_{n \to \infty} \frac{N_n^{(k)}}{n} = \lim_{n \to \infty} \frac{\sum_{j=0}^{n-k-1} 1_{Y_j \not\in \mathcal{F}_k} + \sum_{i=0}^{n-k-1} 1_{Y_i \circ \theta^i \not\in \mathcal{F}_k}}{n} \\
= \lim_{n \to \infty} \frac{\sum_{i=0}^{n-k-1} 1_{Y_i \circ \theta^i \not\in \mathcal{F}_k}}{n - k} \cdot \frac{n - k}{n} = \mathbb{P}_0(Y_k \not\in \mathcal{F}).
\]

\( \square \)

Observe that when \( k \) increases then \( 1_{Y_n \not\in \mathcal{F}_k} \) decreases and \( \lim_{k \to \infty} 1_{Y_n \not\in \mathcal{F}_k} = 1_{Y_n \not\in \mathcal{F}} \). Set \( N_n = 1_{Y_0 \not\in \mathcal{F}} + 1_{Y_1 \not\in \mathcal{F}} + \ldots + 1_{Y_{n-1} \not\in \mathcal{F}} \) then \( \mathbb{P}_\beta \)-a.s. we have \( N(\beta) := \lim_{n \to \infty} \frac{N_n}{n} \). We will prove a result as follows:

**Lemma 3.10.** When \( k \) tend to infinity, \( N^{(k)}(\beta) \) decreases to \( N(\beta) \) and uniformly for \( d \geq 4 \).

**Proof.** Indeed, we have

\[
|\mathbb{P}_\beta(Y_n \not\in \mathcal{F}_k) - \mathbb{P}_\beta(Y_n \not\in \mathcal{F})| = \mathbb{P}_\beta[Y_n \not\in \mathcal{F}_k] \cap [(Y_n = Y_{n-k-1}) \cup (Y_n = Y_{n-k-2}) \ldots \cup (Y_n = Y_0)]
\]

\[
\leq \sum_{j=0}^{n-k-1} \mathbb{P}_\beta(Y_n = Y_j) \leq \sum_{j=0}^{n-k-1} \mathbb{P}_\beta(Z_n = Z_j) = \mathbb{P}(Z_{k+1} = 0) + \mathbb{P}(Z_{k+2} = 0) + \ldots + \mathbb{P}(Z_n = 0)
\]

\[
\leq \sum_{j=k+1}^{n} \mathbb{P}(Z_j = 0). \tag{9}
\]

From the sequences \((Z_k)_{k \in \mathbb{Z}}, (\eta_k)_{k \in \mathbb{Z}}, (\xi_k)_{k \in \mathbb{Z}}, (\zeta_k)_{k \in \mathbb{Z}}\), we can construct the ERW \((Y_n)_{n \geq 0}\) just as in the first construction. We also define the sequence \((\tilde{Z}_k)_{k \in \mathbb{Z}}\) as the sequence of "moves" of \( Z \). More precisely, \((\tilde{Z}_k)_{k \in \mathbb{Z}}\) is the unique sequence such that:

\[
\forall n \geq 0, \quad Z_n = \begin{cases} 
\tilde{Z}_{\sum_{i=0}^{n-1} (1 - \eta_i)} & \text{if } n \geq 0; \\
\tilde{Z}_{\sum_{i=0}^{n-1} (1 - \eta_i)} & \text{if } n < 0. 
\end{cases} \tag{10}
\]

\( \eta_k := 1_{Z_k = \tilde{Z}_{k+1}} \) Set \( \eta_i = 1_{Z_i = \tilde{Z}_{i+1}} \) and \( U = \sum_{i=0}^{n-1} \eta_i \). We define \((\tilde{Z}_k)_{k \in \mathbb{Z}}\) as the sequence of "moves" of \( Z \), see (10). Using [18], page 75, we obtain \( \mathbb{P}(\tilde{Z}_i = 0) \sim \frac{d}{d + 1} \). We have

\[
\mathbb{P}(Z_n = 0) = \sum_{k=0}^{n} \mathbb{P}(\tilde{Z}_k = 0), C_{n-k}^m \left( \frac{1}{d} \right)^{n-k} \left( \frac{d - 1}{d} \right)^k
= \sum_{k=0}^{\lfloor n \rfloor} \mathbb{P}(\tilde{Z}_k = 0), C_{n-k}^m \left( \frac{1}{d} \right)^{n-k} \left( \frac{d - 1}{d} \right)^k \\
+ \sum_{k=\lceil \frac{n}{d+1} \rceil}^{n} \mathbb{P}(\tilde{Z}_k = 0), C_{n-k}^m \left( \frac{1}{d} \right)^{n-k} \left( \frac{d - 1}{d} \right)^k.
\]

We estimate the first term:

\[
\sum_{k=0}^{\lfloor n \rfloor} \mathbb{P}(\tilde{Z}_k = 0), C_{n-k}^m \left( \frac{1}{d} \right)^{n-k} \left( \frac{d - 1}{d} \right)^k \leq \sum_{k=0}^{\lfloor n \rfloor} C_{n-k}^m \left( \frac{1}{d} \right)^{n-k} \left( \frac{d - 1}{d} \right)^k
= \mathbb{P} \left( \frac{U - \frac{n}{d}}{\sqrt{n}} \leq -\frac{\sqrt{n}}{2d} \right) \sim \int_{-\infty}^{-\frac{\sqrt{n}}{2d}} e^{-\frac{x^2}{2}} dx \text{ converging to } 0 \text{ when } n \to \infty.
\]
On the other hand
\[ \sum_{k=\frac{n}{2d}+1}^{n} \mathbb{P}(\tilde{Z}_k = 0) C_n^{n-k} \left( \frac{1}{d} \right)^{n-k} \left( \frac{d-1}{d} \right)^k \sim C n^{-\frac{d-1}{2d}} \]

since
\[ \frac{1}{2} \leq \sum_{k=\frac{n}{2d}+1}^{n} C_n^{n-k} \left( \frac{1}{d} \right)^{n-k} \left( \frac{d-1}{d} \right)^k \leq 1 \]

and
\[ \mathbb{P}(\tilde{Z}_k = 0) \sim C' n^{-\frac{d-1}{2d}} \text{ for } \frac{n}{2d} \leq k \leq n. \]

From (9) we get,
\[ |\mathbb{P}_\beta(Y_n \notin \varnothing_k) - \mathbb{P}_\beta(Y_n \notin \varnothing)| \leq C \sum_{j=k+1}^{\infty} j^{-\frac{d-1}{2d}}. \]

It implies
\[ \left| \mathbb{E}_\beta \left( \frac{N_n^{(k)}}{n} \right) - \mathbb{E}_\beta \left( \frac{N_n}{n} \right) \right| \leq \frac{1}{n} \sum_{i=0}^{n-1} |\mathbb{P}_\beta(Y_i \notin \varnothing_k) - \mathbb{P}_\beta(Y_i \notin \varnothing)| \]
\[ \leq C \sum_{j=k+1}^{\infty} j^{-\frac{d-1}{2d}}. \]

Let n converge to infinity then \( |N^k(\beta) - N(\beta)| \leq C \sum_{j=k+1}^{\infty} j^{-\frac{d-1}{2d}}. \) If \( d \geq 4 \) then \( N^k \) converges uniformly to \( N \) in \( \beta \).

Now, we return to the proof of the point 2 of Theorem 1.1. By \( \frac{n(\beta)}{\beta} = N(\beta) \), to prove the existence of the derivative at 0 we need to prove that \( N(\beta) \) is continuous at 0. It is known that \( N^k \) converges uniformly to \( N \) in \( \beta \) for \( d \geq 4 \), then there is just one thing left is to show that \( N^k(\beta) \) is continuous at 0. Indeed,
\[ |\mathbb{P}_\beta(Y_n \notin \varnothing_k) - \mathbb{P}_0(Y_n \notin \varnothing_k)| \]
\[ = |\mathbb{E}_0 \left[ 1_{Y_n \notin \varnothing_k} \prod_{j=0}^{n-1} (1 + \mathcal{E}_j \beta 1_{Y_j \notin \varnothing}) \right] - \mathbb{E}_0 \left[ 1_{Y_n \notin \varnothing_k} \prod_{j=0}^{n-k} (1 + \mathcal{E}_j \beta 1_{Y_j \notin \varnothing}) \right]| \]
\[ \leq \mathbb{E}_0 \left[ 1_{Y_n \notin \varnothing_k} \prod_{j=0}^{n-k} (1 + \mathcal{E}_j \beta 1_{Y_j \notin \varnothing}) \right] \left| \prod_{j=n-k+1}^{n-1} (1 + \mathcal{E}_j \beta 1_{Y_j \notin \varnothing}) - 1 \right| \]
\[ \leq [(1 + \beta)^{k-1} - 1] \mathbb{E}_0 \left[ 1_{Y_n \notin \varnothing_k} \prod_{j=0}^{n-k} (1 + \mathcal{E}_j \beta 1_{Y_j \notin \varnothing}) \right] = [(1 + \beta)^{k-1} - 1] \mathbb{P}_0(Y_n \notin \varnothing_k) \]
\[ \leq [(1 + \beta)^{k-1} - 1]. \]

Hence, \[ \left| \mathbb{E}_\beta \left( \frac{N_n^{(k)}}{n} \right) - \mathbb{E}_0 \left( \frac{N_n^{(k)}}{n} \right) \right| \leq [(1 + \beta)^{k-1} - 1] \text{ and } |N^k(\beta) - N^k(0)| \leq [(1 + \beta)^{k-1} - 1]. \]
This implies that \( N^k(\beta) \) is continuous at 0. Therefore, for \( d \geq 4 \) then
\begin{itemize}
  \item $N^k(\beta)$ converges uniformly to $N(\beta)$ when $k \to \infty$,
  \item $N^k(\beta)$ is continuous for every $k > 1$.
\end{itemize}

We deduce that $N(\beta)$ is continuous at 0, it means that

$$\lim_{\beta \to 0} \frac{v(\beta)}{\beta} = \frac{1}{d} N(0) = \frac{1}{d} \lim_{n \to \infty} \frac{R_n}{n} = \frac{1}{d} R(0).$$

Notice that $R_n = \mathbb{E}_0(N_n)$

For $d = 2$, $R(0) = N(0) = 0$, see in [13]. Let $\sigma > 0$, on one hand, since $N^k(0)$ decreases to $N(0)$ when $k \to \infty$ then there exists $k_0$ such that $N^k(0) < \sigma$ for all $k \geq k_0$. On the other hand, $N^{k_0}(\beta)$ is continuous at 0 then there exists $\beta_0 > 0$ such that $|N^{k_0}(\beta) - N^{k_0}(0)| < \sigma$ for all $\beta < \beta_0$. Since $N(\beta) \leq N^{k_0}(\beta)$, $N(\beta) \leq N^{k_0}(\beta) \leq N^{k_0}(0) + \sigma < 2\sigma$ for all $\beta < \beta_0$.

This implies that $\lim_{\beta \to 0} N(\beta) = N(0) = 0$. Therefore $\lim_{\beta \to 0} \frac{v(\beta)}{\beta} = 0$.

For $d = 3$, because of $N(\beta) \leq N^k(\beta)$ for all $k > 1$ then

$$\limsup_{\beta \to 0} N(\beta) \leq \limsup_{k \to \infty} N^k(\beta) = \lim_{k \to \infty} N^k(0) = \frac{1}{d} R(0).$$

\section{The proof of Theorem 1.2}

\subsection{Coupling of random walks}

The method coupling is usually used for the proof about the recurrence or the monotonicity of random walks. In [4], the authors coupled an excited random walk with a simple symmetric random walk to prove the recurrence of ERW. In [4], this method is used to prove the monotonicity by coupling tree random walks, two biased random walks with bias respectively $\beta$ and $\beta + \varepsilon$ on the Galton-Watson tree, a simple symmetric random walk with bias $\beta$ on $\mathbb{Z}$. In this proof, we will couple a $m$-ERW with bias $\beta$ with a stationary random walk that we will construct as follows:

Let $(Z_n)_{n \in \mathbb{Z}}$ be a simple random walk on $\mathbb{Z}^{d-1}$, where $Z_0 = 0$, a.s. such that it does not move at every site with probability $\frac{1}{d}$, it jumps from a site to every next site with probability $\frac{1}{d}$. Consider a sequence $(\xi_n, \zeta_n)_{n \in \mathbb{Z}}$ of independent couple, independent of $Z$ and satisfies:

$$\xi_n \sim \text{Ber} \left( \frac{1}{2} \right), \quad \zeta_n \sim \text{Ber} \left( \frac{1 + \beta}{2} \right) \quad \text{and} \quad \xi_n \leq \zeta_n.$$

We denote $\{Z_n \notin \} = \text{the event} \{Z_n \notin Z(\infty,n)\}$.

Now, we will construct two random walks $(Y_n)_{n \geq 0}$ and $(\overline{Y}_n)$ as follows:

Let $(Y_n)_{n \geq 0}$ be a random walk with $Y_0 = 0$, a.s., the vertical component $(Y_n \cdot e_2, Y_n \cdot e_3, \ldots, Y_n \cdot e_d) = Z_n$ and the horizontal component $X_n = Y_n \cdot e_1$ such that:

\begin{itemize}
  \item If $Y_n$ has not been visited more than $m - 1$ times before time $n$ ($\{Y_n \notin \overline{m}\}$), then $X_{n+1} - X_n = (2\zeta_n - 1)1_{Z_n = Z_{n+1}}$.
  \item If not ($\{Y_n \in \overline{m}\}$), then $X_{n+1} - X_n = (2\zeta_n - 1)1_{Z_n = Z_{n+1}}$.
\end{itemize}
With the construction above, \((Y_n)\) is a \(m\)-ERW with bias \(\beta\). Now, we will construct a random walk \((\overline{Y}_n)\). We set \(\overline{Y}_0 = 0\) a.s. The vertical component is \((\overline{Y}_n \cdot e_2, \overline{Y}_n \cdot e_3, \ldots, \overline{Y}_n \cdot e_d) = Z_n, n \geq 0\). The horizontal component is defined as follows:

- If \(Z_n\) is new, i.e. on the event \(\{Z_n \notin \}\), on pose \(\overline{X}_{n+1} - \overline{X}_n = (2\zeta_n - 1)1_{Z_n=Z_{n+1}}\);
- If \(Z_n\) is old i.e. on the event \(\{Z_n \in \}\), we set \(\overline{X}_{n+1} - \overline{X}_n = (2\xi_n - 1)1_{Z_n=Z_{n+1}}\).

We obtain that \(\overline{Y}\) is a stationary random walk. With the coupling above, we have \(\overline{X}_{n+1} - \overline{X}_n \leq X_{n+1} - X_n\) and we deduce that if \((\tau_n)_{n \geq 1}\) are renewal times of \(\overline{Y}\) then they are also renewal times of \(m\)-ERW \(Y\). This property is used to prove the monotonicity of the speed of \(m\)-ERW when \(m\) is large enough.

### 4.2 The monotonicity of the range of the simple random walk

Let \(\mathbb{P}_\beta^s\) be the law of the simple random walk with the drift \(\beta\) on \(\mathbb{Z}^d\) starting from 0 \((Y_0 := 0\) under \(\mathbb{P}_\beta^s\)-a.s.). The range of the random walk at the time \(n\) is: \(R_n(\beta) = 1_{Y_0 \in Y_{[1,\infty)}} + Y_{2k} = 0\) and \(0 \notin Y_{[1,2k]}\)

Set \(R(\beta) = \lim_{n \to \infty} \frac{E_s R_n}{n}\). We will prove that \(R(\beta)\) is increasing in \(\beta \in [0, 1]\).

Firstly, for the range of the simple random walk, we have a known result as follows (see [18], [6]):

\[
R(\beta) = \mathbb{P}_\beta^s[Y_0 \notin Y_{[1,\infty)}]
\]

Then, we obtain:

\[
1 - R(\beta) = \mathbb{P}_\beta^s[\exists n > 0 \text{ such that } Y_n = Y_0 = 0] = \mathbb{P}_\beta^s \left\{ \bigcup_{k=1}^{\infty} [Y_{2k} = 0 \text{ and } 0 \notin Y_{[1,2k]}] \right\} = \sum_{k=1}^{\infty} \mathbb{P}_\beta^s \{ [Y_{2k} = 0 \text{ and } 0 \notin Y_{[1,2k]}] \}. \tag{11}
\]

On the other hand, we see that the trajectories with \(2k\) steps \(\{y_0 = 0, y_1, y_2, \ldots, y_{2k-1}, y_{2k} = 0\}\) start from the origin and return at the origin at the time \(2k\) whose number of jumps to the left equal to the number of jumps to right that we denote equal to \(a_1\). Therefore

\[
\mathbb{P}_\beta^s \{ [Y_{2k} = 0 \text{ et } 0 \notin Y_{[1,2k]}] \} = \sum_{\{y_0=0, y_1, \ldots, y_{2k}=0\}} \left( \frac{1 + \beta}{2d} \right)^{a_1} \left( \frac{1 - \beta}{2d} \right)^{a_1} \left( \frac{1}{2d} \right)^{a_2} \text{ where } 2a_1 + a_2 = 2k
\]

\[
= \sum \left( \frac{1 - \beta^2}{(2d)^2} \right)^{a_1} \left( \frac{1}{2d} \right)^{a_2}. \tag{12}
\]

From (11) and (12), we imply that \(1 - R(\beta)\) is decreasing then \(R(\beta)\) is increasing in \(\beta\).
4.3 The monotonicity of the speed of excited random walk with several identical cookies

**Lemma 4.1.** Let $J$ be an interval of $\mathbb{R}$ and \( \{X_n(\beta)\}_{\beta \in J, n \geq 1}, \{X(\beta)\}_{\beta \in J} \) the families of positive random variables. Under supposition that

1. for every $n$, \( \{X_n(\beta)\}_{\beta \in J} \) is uniformly integrable,
2. \( \{X(\beta)\}_{\beta \in J} \) is uniformly integrable,
3. \( X_n(\beta) \) converges in probability to \( X(\beta) \), uniformly in $\beta$: for every $\varepsilon > 0$,

\[
\lim_{n \to +\infty} \sup_{\beta \in J} P(|X_n(\beta) - X(\beta)| > \varepsilon) = 0.
\]

Then, \( \lim_{n \to +\infty} \sup_{\beta \in J} |\mathbb{E}(X_n(\beta)) - \mathbb{E}(X(\beta))| = 0 \) if and only if \( \{X_n(\beta)\}_{n \in \mathbb{N}, \beta \in J} \) is uniformly integrable.

We construct two random walks, the stationary random walk \( \{\overline{Y}_n\} \) and the $m$-excited random walk \( \{Y_n\} \) as the coupling in the section 4.1. We assume in this section that all of \( \{\eta_i\}_{i \geq 0}, \{\xi_i\}_{i \geq 0} \) and \( \{\zeta_i\}_{i \geq 0} \). Set \( D(\omega) = \{n \in \mathbb{Z}, X_{(-\infty, n-1]}(\omega) < X_n(\omega) \leq X_{[n, +\infty)}(\omega)\} \) and \( N(\omega, dk) = \sum_{n \in \mathbb{Z}} \sigma_n(\omega) \mathbb{1}_{n \in D(\omega)} \). We consider

\[
W = \{\omega \in \Omega, N(\omega, (-\infty, 0]) = N(\omega, [0, +\infty)) = \infty\}.
\]

Let \( \{\tau_n\} \) be the sequence of renewal times of the walk \( \{\overline{Y}_n\} \) such that \(-\infty < ... < \tau_2 < \tau_1 < \tau_0 \leq 0 < \tau_1 < \tau_2 < ... < +\infty\). By the construction of \( \{\overline{Y}_n\} \), the speed is following:

\[
\overline{v}(\beta) = \lim_{n \to +\infty} \frac{\overline{X}_n}{n} = \frac{\beta}{d} \mathbb{P}(Z_n \neq \emptyset).
\]

Using the idea of proof in [15] to have that, when $d \geq 4$ we have \( \overline{v}(\beta) > 0 \) and

**Lemma 4.2.** Let \( \{\overline{Y}_n\}_{n \in \mathbb{Z}} \) be a stationary random walk with drift $\beta \in [0, 1]$ fixed and \( \{\tau_k, k \in \mathbb{Z}\} \) is the sequence of the renewal times respectively. Then, there exists $C, \alpha > 0$ such that for every $n \in \mathbb{Z}$,

\[
\sup_{k \in \mathbb{Z}} \mathbb{P}_\beta[|\tau_{k+1} - \tau_k| \geq \delta n^\alpha] \leq C e^{-n^\alpha} \text{ a.s.}
\]

In particular, for every $k \in \mathbb{Z}$ and $p \geq 1$ we have that \( \tau_k < \infty \) p.s and \( \mathbb{E}_\beta[(\tau_{k+1} - \tau_k)^p] < \infty \).

Set \( \overline{D}_+ = \{\overline{X}_n \geq 0 \text{ for all } n \geq 0\}, \overline{D}_- = \{\overline{X}_n < 0 \text{ for all } n < 0\}, D = \overline{D}_+ \cap \overline{D}_- \) and \( \bar{\theta} = \bar{\theta}_+ \). It is similar as for simple random walk with drift $\beta$, we believe that: \( \mathbb{P}((\overline{D}_+) = c_1(\beta) \beta^2, \mathbb{P}(\overline{D}_-) = c_2(\beta) \beta^2 \) where for some positive constants $c_0, c_3$ then $c_0 \leq c_1(\beta), c_2(\beta) \leq c_3$. In fact, we don’t use these properties for the proof of Theorem 1.2 then we don’t prove them. To prove Theorem 1.2 we need one more lemma as follows:

**Lemma 4.3.** \( \mathbb{P}(D) > 0 \) and \( \mathbb{P}(W) = 1 \). Under \( \hat{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot | \overline{D}_+, \overline{D}_-) \) i.e \( \tau_0 = 0 \) the sequence \( \{\tau_{n+1} - \tau_n\}_{n \in \mathbb{Z}} \) is stationary. Moreover, the triples \( (\Omega, \mathbb{P}, \theta) \) and \( (\Omega, \hat{\mathbb{P}}, \hat{\theta}) \) are ergodic systems.
Proof. The random walk $\mathbf{Y}$ has these speeds:

$$\nabla(\beta) = \lim_{n \to +\infty} \frac{\nabla \cdot \mathbf{e}_1}{n} = \frac{\beta}{d} \mathbb{P}(Z_0 \notin \theta) > 0 \quad \text{and} \quad \nabla_-(\beta) = \lim_{n \to -\infty} \frac{\nabla \cdot \mathbf{e}_1}{n} = -\frac{\beta}{d} \mathbb{P}(Z_0 \notin \theta) < 0.$$  

Using the idea in [15] we get $\mathbb{P}(D) > 0$. Because $\mathbf{Y}$ is stationary, $\mathbb{P}(D) > 0$ implies that $\mathbb{P}(W) = 1$. Now, we prove the remain part of Lemma 4.3 $(\Omega, \mathbb{P}, \theta)$ is the standard ergodic system. We will prove it also is true for $(\Omega, \hat{\mathbb{P}}, \hat{\theta})$. First, we prove that $\hat{\mathbb{P}}$ is invariant under $\hat{\theta}$. Take any set $A \subset W$. Without loss of generality, suppose that $A \subset (0 \in D)$, then we have:

$$\hat{\theta} \circ \hat{\mathbb{P}}(A) = \hat{\mathbb{P}}(\hat{\theta}^{-1}A) = \frac{\mathbb{P}(\theta^{-1}A, D)}{\mathbb{P}(D)}$$

$$= \frac{\sum_{k \geq 1} \mathbb{P}(\theta^{-1}A, \tau_1 = k, D)}{\mathbb{P}(D)}$$

$$= \frac{\sum_{k \geq 1} \mathbb{P}(A, \tau_1 = -k, D)}{\mathbb{P}(D)}$$

$$= \frac{\mathbb{P}(A)}{\mathbb{P}(D)} = \hat{\mathbb{P}}(A).$$

Next, we prove that for any set $A \subset W$ such that $\hat{\theta}^{-1}A = A$ then $\hat{\mathbb{P}}(A) = 0$ or 1. Indeed, set $\hat{\Omega} := (0 \in D)$ and $B := A \cap \hat{\Omega} \subset W$. Note that $\hat{\theta}^{-1}(\hat{\Omega}) = W$, so that $\hat{\theta}^{-1}A = \hat{\theta}^{-1}B$. This in turn implies that $\hat{\theta}^{-1}B \cap \hat{\Omega} = \hat{\theta}^{-1}A \cap \hat{\Omega} = A \cap \hat{\Omega} = B$.

We will prove that $\hat{\theta} \left[ \hat{\theta}^{-1}B \right] = \hat{\theta}^{-1}B$. Using the ergodicity of $(\Omega, \mathbb{P}, \theta)$, it follows that $\mathbb{P}(\hat{\theta}^{-1}B) = 0$ or 1, and

$$\hat{\mathbb{P}}(A) = \hat{\mathbb{P}}(\hat{\theta}^{-1}A) = \hat{\mathbb{P}}(\hat{\theta}^{-1}B) = \frac{\mathbb{P}(\hat{\theta}^{-1}B \cap \hat{\Omega})}{\mathbb{P}(\hat{\Omega})} = 0 \text{ or } 1.$$  

So, to finish the proof we only need to prove that $\hat{\theta} \left[ \hat{\theta}^{-1}B \right] = \hat{\theta}^{-1}B$.

Firstly, we show that $\hat{\theta} \left[ \hat{\theta}^{-1}B \right] \subset \hat{\theta}^{-1}B$. Take $x \in \hat{\theta}^{-1}B$. Then $\hat{\theta}x \in B$. If $\tau_1(x) > 1$, then $\hat{\theta}(\tau_1x) = \hat{\theta}x \in B \Rightarrow \tau_1x \in \hat{\theta}^{-1}B$. If $\tau_1(x) = 1$, then $\tau_1x = \hat{\theta}x \in B = \hat{\theta}^{-1}B \cap \hat{\Omega} \Rightarrow \tau_1x \in \hat{\theta}^{-1}B$.

It remains to prove that $\hat{\theta}^{-1}B \subset \hat{\theta} \left[ \hat{\theta}^{-1}B \right]$. Take $x \in \hat{\theta}^{-1}B$ then $x = \hat{\theta} \left( \theta_1-x \right)$ and we will prove that $\theta_1-x \in \hat{\theta}^{-1}B \Leftrightarrow \hat{\theta} (\theta_1-x) \in B$ If $x \in \Omega$, then $\hat{\theta} (\theta_1-x) = x \in \hat{\theta}^{-1}B \cap \hat{\Omega} = B$. If $x \notin \Omega$, then $\hat{\theta} (\theta_1-x) = \hat{\theta}x \notin B$. Because the sequence of renewal times of the stationary random walk $\mathbf{Y}$ is also for $Y$ the $m-$ cookies excited random walk with bias parameter $\beta$. The increments $\{Y_{\tau_n, \tau_{n+1}}\}_{n \in \mathbb{Z}}$ are disjoint, by the construction of the coupling $\mathbf{Y}$ and $Y$, so we have

$$X_{\tau_{k+1}} - X_{\tau_k} = X_{\tau_k} \circ \hat{\theta}_k.$$

$$\nabla X_{\tau_{k+1}} - \nabla X_{\tau_k} = \nabla X_{\tau_k} \circ \hat{\theta}_k.$$
From the equations above and using the ergodicity of \((\Omega, \hat{P}, \hat{\theta})\), we apply that \(\hat{P}\)-a.s. there exist \(\nu(\beta), v(m, \beta) > 0\) such that

\[
v(m, \beta) = \lim_{n \to \infty} \frac{X_n}{n} = \frac{\hat{E}X}{\hat{P}}.
\]

Remark that if \(\frac{X_n}{n}\) converges \(\hat{P}\)-a.s. to \(v(m, \beta)\) then it is true for \(\mathbb{P}\)-a.s. Indeed, there exists a set \(\hat{A} \subset D\) and \(\mathbb{P}(\hat{A}) = 1\) such that for all \(\omega \in \hat{A}\), \(\frac{X_n}{n}(\omega)\) converges to \(v\). We suppose that there exists a subset \(B \subset \Omega\) such that \(\hat{\theta}B \subset \hat{A}\). If \(\mathbb{P}(B) > 0\), by \(B = \bigcup_{k=1}^{\infty}(B, T = k)\) then there exists \(k\) such that \(\mathbb{P}(B, T = k) > 0\). This implies that \(\mathbb{P}(\theta_k(B, T = k)) = \mathbb{P}(B, T = k) > 0\).

Moreover, \(\mathbb{P}(B) > 0\) and \(\hat{\mathbb{P}}(\hat{A}) = \frac{\hat{P}(\hat{A})}{\mathbb{P}(B)} > 0\). This is contradictory with the supposition that \(\hat{P}(\hat{A}) = 1\). Therefore, \(\mathbb{P}(B) = 0\), letting \(B = \hat{\theta}^{-1}(\hat{A}^c)\), this implies that \(\mathbb{P}(\hat{\theta}^{-1}(A)) = 1\). For all \(\omega \in \hat{\theta}^{-1}(A)\) then \(\frac{X_n}{n}(\omega)\) converges to \(v\) so \(\frac{X_n}{n}(\omega)\) converges to \(v\). It means that \(\frac{X_n}{n}\) converges to \(v\) almost surely under \(\mathbb{P}\).

Now, to prove the monotonicity of the speed on \([\beta_0, 1]\) we need to couple the stationary walk \(\overline{Y}\) with bias parameter \(\beta_0\) with the \(m\)-excited random walk \(Y\) with bias parameter \(\beta\) where \(\beta \geq \beta_0\). We consider the sequences as follows: \(\{\eta_i\}_{i \geq 0}, \{\xi_i\}_{i \geq 0}, \{\zeta_i\}_{i \geq 0}\) and \(\{\chi_i\}_{i \geq 0}\) such that the random vectors of the sequence \(\{(\eta_i, \xi_i, \zeta_i, \chi_i)\}_{i \in \mathbb{Z}}\) are independent. Two sequences \(\{\eta_i\}_{i \in \mathbb{Z}}\) and \(\{(\xi_i, \xi_i, \chi_i, \chi_i)\}_{i \in \mathbb{Z}}\) are independent. On the other hand, the vector \((\xi_i, \xi_i, \chi_i, \chi_i)\) satisfies:

- \(\eta_i \sim \text{Ber}(1/2), \xi_i = \zeta_i \sim \text{Ber}(1/2), \chi_i \sim \text{Ber}(1+\beta_0)/2, \zeta_i \sim \text{Ber}(1+\beta)/2\).
- Set \(\mathbb{P}(\overline{\xi}_i = x, \overline{\xi}_i = y, \zeta_i = z) = p_{xyz}\) where \(x, y, z \in \{0; 1\}\) then \(p_{111} = \frac{1}{2}, p_{011} = \frac{\beta_0}{2}, p_{000} = \frac{1-\beta}{2}\) and for other cases \(p_{xyz} = 0\).

### 4.3.1 Girsanov’s transform

The couple \((\overline{Y}, Y)\) takes its values in the space \(U = (\mathbb{Z}^d)^{\mathbb{Z}} \times (\mathbb{Z}^d)^{\mathbb{N}}\). Consider \(U^* = \{(\overline{y}_n)_{n \in \mathbb{Z}} \times (y_m)_{m \in \mathbb{N}}, \overline{y}_0 = y_0, z_n, \varepsilon_m \in \{0; 1\}\) for \(i \in \mathbb{N}, \overline{y}_i = z_i\) and \(\overline{z}_i = \varepsilon_i\) if \(y_i \notin \mathbb{N}\). We denote \(\mathbb{P}_{m, \beta}\) the law of the couple \((\overline{Y}, Y)\) then

\[
q_m(m, \beta) := \mathbb{P}_{\beta_0, \beta}[\overline{y}_{n+1} = \overline{y}_{n+1}, Y_{n+1} = y_{n+1}|(Z_i = z_i)_{i < 0}, \overline{Y}_0 = Y_0 = 0, ..., \overline{Y}_n = \overline{y}_n, Y_n = y_n] = \frac{1}{d} \left[1 - \frac{1 + \beta_0}{2} + \frac{1 + \beta}{2} \frac{m}{2} + \frac{1 - \beta}{2} \frac{m}{2} \right].
\]

Moreover,

\[
\mathbb{P}_{m, \beta}[\overline{Y}_0 = Y_0 = 0, ..., \overline{Y}_n = \overline{y}_n, Y_n = y_n, \overline{Y}_{n+1} = \overline{y}_{n+1}, ..., \overline{Y}_{n+k} = \overline{y}_{n+k}|(Z_i = z_i)_{i < 0}] = \mathbb{P}_{m, \beta}[\overline{Y}_0 = Y_0 = 0, ..., \overline{Y}_n = \overline{y}_n, Y_n = y_n|\overline{(Z_i = z_i)_{i < 0}}] \times \mathbb{P}_{m, \beta}[\overline{Y}_{n+1} = \overline{y}_{n+1}, ..., \overline{Y}_{n+k} = \overline{y}_{n+k}|(Z_i = z_i)_{i < 0}, \overline{Y}_0 = Y_0 = 0, ..., \overline{Y}_n = \overline{y}_n, Y_n = y_n].
\]
Set

\[ Q_n(m, \beta) = q_n(m, \beta) (\Upsilon, Y), \mathcal{F}_n = \sigma\{ (Y_i)_{i \in \mathbb{Z}}, (Y_m)_{0 \leq m \leq n} \}, M_n(\beta) = \prod_{i=0}^{n-1} \frac{Q_i(m, \beta)}{Q_i(1, \beta_0)}. \]

We deduce that

\[ \frac{dP_{m, \beta}}{dP_{1, \beta_0}} |_{\mathcal{F}_n} = M_n(m, \beta), \frac{dP_{m, \beta}}{dP_{1, \beta_0}} |_{\mathcal{F}_\tau} = M_\tau(m, \beta). \]

We get the formula of the speed for \( m \)-excited random walk \( Y \):

\[ v(m, \beta) = \frac{\beta}{d} \frac{\hat{E}_{m, \beta}(N^m_\tau)}{E_{1, \beta_0} \tau}, \]

\[ \frac{\partial v}{\partial \beta}(m, \beta) = \frac{1}{d} \frac{\hat{E}_{1, \beta_0}(N^m_\tau M_\tau(m, \beta))}{E_{1, \beta_0} \tau} + \frac{\beta}{d} \frac{\hat{E}_{1, \beta_0}(N^m_\tau M_\tau(m, \beta)V_\tau(m, \beta))}{E_{1, \beta_0} \tau}. \]

where

\[ V_\tau(m, \beta) = \frac{\partial}{\partial \beta} \frac{M_\tau(m, \beta)}{M_\tau(m, \beta)}. \]

Taking \( m \to \infty \), by Lemma 4.1 we get that \( \frac{\partial v}{\partial \beta}(m, \beta) \) converges to \( \frac{1}{d} \) uniformly in \( \beta \in [\beta_0, 1] \) when \( m \) tends to infinity. This finishes the proof of Theorem. \( \square \)

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