TRANSFORMATIONS OF HYPERGEOMETRIC MOTIVES

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ABSTRACT. We consider algebraic transformations of hypergeometric functions from a geometric point of view. Hypergeometric functions are shown to arise from the deRham realization of a hypergeometric motive. The \( \ell \)-adic realization of the motive gives rise to hypergeometric character sums over finite fields. This helps to unify and explain some recent results about transformations of hypergeometric character sums.

1. INTRODUCTION.

The transformation theory of hypergeometric functions goes back at least to Gauss and Kummer. A transformation of hypergeometric functions is an identity

\[
\binom{a}{b} R(z) = C(z) \binom{\alpha}{\beta} S(z)
\]

for rational functions \( R(z), S(z) \) and an algebraic function \( C(z) \). Kummer discovered many such identities by difficult calculations. Later Riemann deduced some of Kummer’s identities by proving that the hypergeometric differential equation is the unique second order analytic differential equation with three regular singularities at \( z = 0, 1, \infty \) and with prescribed local monodromy about these points. Following Katz, we say that the hypergeometric differential equation is rigid.

More recently, character sums over finite fields have been introduced which are analogs of hypergeometric functions, and many transformation identities have been discovered for them, analogous to the formulas known for the complex-analytic hypergeometric functions. The point of this paper is to explain that this is not an accident; both of these hypergeometric functions are manifestations (or realizations) of a hypergeometric motive. The complex-analytic function reflects the Hodge-deRham realization, whereas the finite-field hypergeometric function reflects the \( \ell \)-adic étale realization. There are also \( p \)-adic crystalline realizations.

The theme of this paper is that the transformations of hypergeometric functions are really geometric in origin—they are transformations of motives. Thus, one gets transformation formulas in all three realizations. However, the assertion that all transformation identities of hypergeometrics are geometric in origin is really an imprecise conjecture. In certain cases it follows readily from standard conjectures (Hodge, Tate) about algebraic cycles. For GKZ systems, see [60]. Nonetheless, we prove a weaker form of a consequence of this in theorem 7.1. This allows one to deduce from a complex-analytic transformation formula, a corresponding identity for finite-field hypergeometrics, up to an unspecified Galois twist.

In stating that all transformations arise from geometry, it should be understood that these geometric correspondences are very diverse. All \( \binom{a}{b} \) hypergeometric motives occur in the cohomology of the families of curves

\[
y^N = x^i(1-x)^j(1-\lambda x)^k
\]

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for nonnegative integers $N, i, j, k$. In principle, any transformation can be expressed as a correspondence among various of these curves, but this is often unnatural. Hypergeometric motives occur in the cohomology of other families of algebraic varieties. Example: recently, Yifan Yang and the second author of this paper have discovered some transformations of hypergeometric equations using the theory of Shimura curves, \cite{61}. The motives are then families of abelian varieties with quaternion multiplication. These will be explored in more detail in a subsequent publication.

The word \textit{hypergeometric} is used in a general sense in this paper. See section 3. We do not consider \textit{confluent} hypergeometrics. Finite field analogs of these are given e.g., by Kloosterman sums. Their $\ell$-adic sheaves have wild ramification. The Hodge-deRham story involves irregular Hodge theory, which has undergone a rapid development recently, see \cite{22}, \cite{25}, \cite{65}. In this paper, only tame ramification and regular singularities are permitted.

A word on our use of the term \textit{motivic}. Generally speaking, one expects to have a category of motivic sheaves with the formalism of the 6 functors and realizations into various cohomology theories. We will explain in section 16 the formalism we use. Also, the term \textit{hypergeometric motive} has already appeared and there is even a package in Magma for computing with these. Those hypergeometric motives are special cases of the ones considered here.

An outline of this paper: In section 2 we recall the relations between regular singular differential equations and monodromy. Section 3 is a general discussion of sheaves attached to hypergeometric functions. In section 4 this is specialized to the classical $\, _2F_1$ function. Section 5 analyzes the cohomology of a family of curves relevant to this paper. In section 6 we give the definition of hypergeometric motives used in this paper. In 7 we prove a theorem that allows one to deduce a transformation formula for $\ell$-adic sheaves, knowing one for the corresponding $\mathcal{D}$-module. Section 8 reviews the formalism on $\ell$-adic sheaves in application to character sums. Our main theorem 8.1 is proved there. Section 9 explains Katz’s theory of rigid local systems. This is specialized to Appell-Lauricella systems in 10. In sections 11, 12, 13 examples are given of transformation formulas related respectively to rigidity, arithmetic triangle groups, and elliptic curves. In section 14 we discuss a transformation formula for an Appell-Lauricella system arising from the Picard family of curves. Appendices 15, 16 explain the formalism of local systems: over $\mathbb{C}$, $\ell$-adic, and motivic.

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2. DIFFERENTIAL EQUATIONS AND MONODROMY

Riemann introduced the idea of monodromy into the study of analytic differential equations. Given a representation of the fundamental group

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, x) \to \text{GL}_2(\mathbb{C})$$

there is a unique second order rational differential equation with regular singular points $z = 0, 1, \infty$ with the property that if $f_1(z), f_2(z)$ is a basis of holomorphic solutions at $x$ then analytic continuation around a loop $\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, x)$ yields the linear transformation

$$
\begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix} \to \rho(\gamma) \begin{pmatrix}
  f_1 \\
  f_2
\end{pmatrix}.
$$

Since

$$\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, x) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1 \gamma_2 \gamma_3 = 1 \rangle,$$

to give the monodromy representation is equivalent to giving two-by-two matrices $\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3)$ such that $\rho(\gamma_1)\rho(\gamma_2), \rho(\gamma_3) = 1$. These are well-defined up to simultaneous conjugation by an element
of $GL_2(C)$. As Katz observes, Riemann proved a stronger result. Namely, it suffices to give the Jordan canonical forms of $\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3)$ to reconstruct the hypergeometric differential equation. Actually Riemann only considered the case when these were semisimple, so equivalent to diagonal matrices, with eigenvalues $\exp(2\pi i \alpha)$ and $\exp(2\pi i \alpha')$; he called the $\alpha, \alpha'$ the exponents at the singular point. They are well-defined up to permutation and adding $Z$. This stronger property, that a differential equation is determined by the Jordan forms of the monodromies at the singular points, is what Katz ([47]) calls rigidity.

If $X$ is a nonsingular algebraic variety over $C$ and

$$\rho : \pi_1(X^{an}, x) \to GL_n(C)$$

is a representation, we get a local system $V$ on the analytic space $X^{an}$. By the Riemann-Hilbert correspondence, this is the solution sheaf to a differential equation, unique up to isomorphism,

$$\nabla : V \to \Omega^1_{X/C} \otimes_{O_X} V$$

with regular singular points at infinity (see [19]). In [43] a proof is given of the regularity theorem: that the differential equations for the periods of algebraic varieties have regular singular points.

### 3. Hypergeometric Sheaves

The word hypergeometric will be understood in a generalized sense: they include the $pFq$, the Pochhammer equations, Appell-Lauricella equations. The most general form of these are the GKZ (Gelfand-Kapranov-Zelevinski) systems, [30].

Generally speaking, a hypergeometric function is one that

1. has power-series expansions in special form: $\Gamma$-series;
2. satisfies a (regular) holonomic system of differential equations;
3. has Euler integral expressions;
4. is attached to a motivic sheaf.

Because of 4 above, we expect realizations of hypergeometric systems. Let the motivic sheaf $\mathcal{H}$ be defined on $X/S$ where $X$ is a smooth $S$-scheme with $S = O_F[1/N]$, $F$ = an algebraic number field, $O_F$ its ring of integers, $N \geq 1$ an integer. A typical case is

$$X = \mathbb{P}^1 - \{\text{a finite number of points}\} \quad \text{or} \quad X = \mathbb{P}^N - \{\text{a finite number of hyperplanes}\}.$$ 

We expect

**Betti.** A Betti realization: $\mathcal{H}_{\sigma, \mathbb{Z}}$ on $X^{an}$, a local system of constructible $\mathbb{Z}$-modules on the analytic space $X^{an} = X^{an}_{\sigma}$, attached to each embedding $\sigma : R \to C$.

**HdR.** A Hodge-deRham realization: $\mathcal{H}_{dR}$ on $X$ which is a locally free sheaf in the Zariski topology with an integrable connection

$$\nabla : \mathcal{H}_{dR} \to \mathcal{H}_{dR} \otimes_{O_X} \Omega^1_{X/S}.$$ 

For each embedding $\sigma : R \to C$,

$$\mathcal{H}_{\sigma, C} := \mathcal{H}_{\sigma, \mathbb{Z}} \otimes_{\mathbb{Z}} C = \text{Ker}(\nabla^{an}_\sigma),$$

the sheaf of analytic solutions to the algebraic differential equation $\nabla$. There is a comparison isomorphism

$$\text{comp}_\sigma : \mathcal{H}_{\sigma, \mathbb{Z}} \otimes_{\mathbb{Z}} C \cong \mathcal{H}^{an}_{\sigma, dR} := \mathcal{H}_{dR} \otimes_{O_X} O_{X^{an}_{\sigma}}.$$ 

Written in a local flat frame for $\mathcal{H}_{\sigma, \mathbb{Z}}$, the above isomorphism is given by a matrix whose entries are analytic functions. These are the hypergeometric functions. They can be expressed as Euler
integrals, and are periods of these motives. Typically this structure extends to a variation of Hodge structures, or are projections of these onto character eigenspaces.

\(\ell\)-adic. For each good prime \(\ell\), an \(\ell\)-adic realization: this is a lisse \(\mathbb{Q}_\ell\)-sheaf \(\mathcal{H}_\ell\) on \(X_{\text{et}}[1/\ell]\) whose Frobenius traces give a function

\[X[1/\ell](F_{\mathbb{F}_q^n}) \ni x \mapsto \text{Tr}(\text{Frob}_x|\mathcal{H}_{\ell,x}) \in \mathbb{Q}_\ell.\]

These functions are finite-field analogs of hypergeometric functions. This theory was developed principally by Katz; see his works [45], [46], [47], [48]. For the \(\ell\)-adic version of GKZ see [27].

Crys. Finally there are \(p\)-adic crystalline realizations, and relations to \(p\)-adic Hodge theory. This is not as developed as the previous three. Recently, a Frobenius structure has been established for GKZ systems, see forthcoming works of Fu/Wan/Zhang, [28].

The above are related by a series of compatibilities, which will not be written. See [36].

Note that there are irregular differential equations of hypergeometric type, the confluent hypergeometrics. The Hodge-deRham realizations then belong to irregular Hodge theory. They are related to character sums involving additive as well as multiplicative characters of finite fields, and hence their \(\ell\)-adic sheaves have wild ramification at infinity. In this note we will simplify the discussion by only considering tamely ramified sheaves, and only from the \(D\)-module and \(\ell\)-adic point of view.

We will use the word transformation as follows. This is an identity

\[f^*\mathcal{M}_1 = g^*\mathcal{M}_2 \otimes \mathcal{K}\]

where \(\mathcal{M}_1\) is a hypergeometric sheaf on \(X_1\), \(\mathcal{M}_2\) is a hypergeometric sheaf on \(X_2\), and \(\mathcal{K}\) is a sheaf on \(X\), in a diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & X_1 \\
| & | & |
\downarrow & & \downarrow g
\end{array}
\]

4. THE CLASSICAL HYPERGEOMETRIC EQUATION

See [44, 6.0, 6.1, 6.8.0]. For any scheme \(T\), we denote by \(\lambda\) the coordinate on \(\mathbb{A}^1_T\) and by \(S_T\) the open set where \(\lambda(1-\lambda)\) is invertible. Given any sections \(a, b, c, \in \Gamma(T, \mathcal{O}_T)\) we denote by \(E(a, b, c)\) the free \(\mathcal{O}_{S_T}\)-module of rank 2 with basis \(e_0, e_1\), and integrable \(T\)-connection

\[
\begin{align*}
\nabla \left( \frac{d}{d\lambda} \right) (e_0) &= e_1 \\
\nabla \left( \frac{d}{d\lambda} \right) (e_1) &= -\frac{(c-(a+b+1)\lambda)}{\lambda(1-\lambda)} e_1 + \frac{ab}{\lambda(1-\lambda)} e_0.
\end{align*}
\]

Horizontal sections of the dual of \(E(a, b, c)\) over an open set \(U \subset S_T\) can be identified with sections \(f \in \Gamma(U, \mathcal{O}_U)\) which satisfy the differential equation

\[
\lambda(1-\lambda) \left( \frac{d}{d\lambda} \right)^2 + (c-(a+b+1)\lambda) \frac{df}{d\lambda} - abf = 0.
\]

These hypergeometric equations are two-dimensional factors of the the cohomology of the family of curves \(y^N = x^a(x-1)^b(x-\lambda)^c\). In effect, the Euler integral representation
\[ F(\alpha, \beta; \gamma; \lambda) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta - \gamma)} \int_1^\infty x^{\alpha - \gamma} (x - 1)^{\gamma - \beta - 1} (x - \lambda)^{-\alpha} dx \]

shows that the solutions to the differential equation are given by periods of those curves.

Given integers \( N, a, b, c \) greater than zero. Let \( Y(N; a, b, c) \) be the nonsingular projective model of the affine curve in \((x, y)\)-space defined by the equation \( y^N = x^a(x - 1)^b(x - \lambda)^c \). This is a family of curves depending on the parameter \( \lambda \neq 0, 1 \). We consider the family of curves

\[ f : Y(N; a, b, c) \to U := \mathbb{P}^1 - \{0, 1, \infty\}. \]

We have the Gauss-Manin connection

\[ \nabla : H^1_{DR}(Y(N; a, b, c)/U) \to \Omega^1_{U/C} \otimes_{O_U} H^1_{DR}(Y(N; a, b, c)/U). \]

The following theorem gives the structure of this, at least in the generic fiber \( \text{Spec}(\mathbb{C}(\lambda)) \to U \). Let

\[ X(N; a, b, c) = \text{Spec}(\mathbb{C}(\lambda)[x, y, 1/y]/(y^N - x^a(x - 1)^b(x - \lambda)^c)). \]

This is an open affine subset where \( y \) is invertible. It is affine and smooth of relative dimension one over \( \mathbb{C}(\lambda) \). The map \((x, y) \to y\) is a finite étale covering of

\[ A^1_{\mathbb{C}(\lambda)} - \{0, 1, \lambda\} := \text{Spec}(\mathbb{C}(\lambda)[x, (x(1 - (x - \lambda))^{-1}]). \]

For any root of unity \( \xi \in \mu_N \) there is an automorphism of \( X(N; a, b, c) \) given by \((x, y) \mapsto (x, \xi y)\). This gives the Galois group of the covering

\[ \pi : X(N; a, b, c) \to A^1_{\mathbb{C}(\lambda)} - \{0, 1, \lambda\}. \]

Note that the \( dx/y^m \) defines an element in the character eigenspace \( H^1_{DR}(X(N; a, b, c)/\mathbb{C}(\lambda))^{\chi(m)} \) where \( \chi(t)(\xi) = \xi^{-t} \) (the inverse of Katz’s convention).

**Proposition 4.1.** ([44, 6.8.6]) Suppose that \( N \) does not divide \( a, b, c, a + b + c \). Then for any integer \( k \geq 1 \) which is invertible modulo \( N \) the map

\[ e_0 \mapsto \text{the class of } \frac{dx}{y^k}, \]

\[ e_1 \mapsto \nabla \left( \frac{d}{d\lambda} \right) \left( \text{the class of } \frac{dx}{y^k} \right) \]

induces an isomorphism

\[ E \left( \frac{kc}{N}, \frac{k(a + b + c)}{N} - 1, \frac{k(a + c)}{N} \right) \cong H^1_{DR}(X(N; a, b, c)/\mathbb{C}(\lambda))^{\chi(k)}. \]

Note that this gives only the part of the cohomology belonging to primitive characters modulo \( N \).

The local system \( R^1 f_* \mathbb{C} \) on \( U^{an} \) underlies a polarized variation of Hodge structures of weight 1. The \( H^1_{DR}(X(N; a, b, c)/\mathbb{C}(\lambda))^{\chi(k)} \) are the modules with connection that correspond to the rank 2 local system \( (R^1 f_* \mathbb{C})^{\chi(k)} \). Note that \( (R^1 f_* \mathbb{C})^{\chi(k)} \) does not correspond to a variation of Hodge structures unless the character \( \chi(k) \) is real. Nonetheless there are period mapping attached to this situation (see [21]).

To consider the \( \ell \)-adic realization, let \( S = \text{Spec}(R_N) \), where \( R_N = \mathbb{Z}[\zeta_N, 1/N] \). The eigenspaces \((R^1 f_* \mathbb{Q}_\ell)^{\chi(k)}\) then give the \( \ell \)-adic realization, where \( f : Y(N; a, b, c) \to U \) is as before but now as schemes over \( S \). The étale topology is understood here.
5. Cohomology of Cycloelliptic Curves

5.1. A cycloelliptic curve is the projective nonsingular model of

\[ y^N = x^i (1 - x)^j (1 - \lambda_1 x)^{k_1} \cdots (1 - \lambda_r x)^{k_r}. \]

A more symmetric numbering is to take

\[ X^{[N,i]}_\lambda = X : y^N = \prod_{j=0}^{r+1} (x - \lambda_j)^{i_j}, \quad i = (i_0, \ldots, i_{r+1}). \]

At first, we examine this over \( \mathbb{C} \), with fixed values of the parameters \( \lambda_1, \ldots, \lambda_r \), and we use \( X \) to denote the corresponding Riemann surface. The natural projection \( p : X \to P \) sending \( (x, y) \mapsto x \) makes \( X \) into a Galois \( \mu_N \)-branched covering of \( P = \mathbb{P}^1_x \). We define the action as \( y \mapsto \zeta_N y, \zeta_N = \exp(2\pi i/N) \). The branching occurs over a subset of \( S = \{ \lambda_0, \ldots, \lambda_{r+1}, \infty \} \supset S_0 = \{ \lambda_0, \ldots, \lambda_{r+1} \} \).

In our set-up, the branching will be over all of \( S \). We let \( T = p^{-1}(S) \subset X \), which is the subset of \( X \) where \( y \neq 0, \infty \). We let \( X^o = X - T, P^o = P - S \). These are affine smooth curves and the projection \( p^o : X^o \to P^o \) is an étale \( \mu_N \)-covering. We have a Cartesian square

\[
\begin{array}{ccc}
X^o & \longrightarrow & X \\
p^o \downarrow & & \downarrow p \\
Y & \longrightarrow & P
\end{array}
\]

The cohomology decomposes

\[ H^1(X, \mathbb{C}) = \bigoplus_{\chi: \mu_N \to \mathbb{C}^*} H^1(X, \mathbb{C})^\chi \]

where the sum is over the characters \( \chi \) and the superscript refers to the \( \chi \)-eigenspace. One can replace the coefficients \( \mathbb{C} \) by a smaller field, e.g., \( K_N = \mathbb{Q}(\mu_N) \). The sheaf sequence

\[
0 \longrightarrow j'_! \mathbb{C}_{X^o} \longrightarrow \mathbb{C}_{X} \longrightarrow \mathbb{C}_{T} \longrightarrow 0
\]
gives

\[
\cdots \longrightarrow H^i_c(X^o, \mathbb{C}) \longrightarrow H^i(X, \mathbb{C}) \longrightarrow H^i(T, \mathbb{C}) \longrightarrow \cdots
\]

which shows that

\[ H^1_c(X^o, \mathbb{C}) = H^1(X, \mathbb{C}) \oplus \mathbb{C}^{#T-1} \]

where the first summand is pure of weight 1, and the second factor is pure of weight 0 (of Hodge type \( (0,0) \)). This decomposes into eigenpaces for \( \chi \in \tilde{\mu}_N \).

Projecting the above sheaf sequence by \( p \) we get

\[
\begin{array}{ccc}
0 & \longrightarrow & p_* j'_! \mathbb{C}_{X^o} \\
& = & \downarrow = \\
0 & \longrightarrow & \bigoplus_{\chi \in \tilde{\mu}_N} j'_! L_{\chi} \\
& = & \downarrow = \\
0 & \longrightarrow & \bigoplus_{\chi \in \tilde{\mu}_N} \tilde{L}_{\chi} \longrightarrow \bigoplus_{\chi \in \tilde{\mu}_N} (p_* \mathbb{C}_{T})^\chi \longrightarrow 0
\end{array}
\]

For each character \( \chi, L_{\chi} \) is a rank 1 \( \mathbb{C} \)-local system on \( P^o \), \( \tilde{L}_{\chi} \) is a constructible sheaf of \( \mathbb{C} \)-vector spaces on \( P \), and \( j' : P^o \to P \) is the inclusion. We have

\[ (p_* \mathbb{C}_{T})^\chi = \bigoplus_{\mathfrak{s} \in S} (p_* \mathbb{C}_{T})^\chi_{\mathfrak{s}}. \]
By Leray, we get
\[ H^1(X, \mathcal{C})^\chi = H^1(P, L_\chi), \quad H^1_c(X^\circ, \mathcal{C})^\chi = H^1_c(P^\circ, L_\chi). \]
By choosing a root of unity \( \zeta_N = \exp(2\pi i/N) \) we can identify \( \overline{\mu_N} = \mathbb{Z}/N \). Then the local system \( L_\chi \) belonging to the character \( \chi_k(\zeta_N) = \zeta_k^t \) is the subsheaf
\[ L_\chi = \mathcal{C}y^k \subset \mathcal{O}^\text{hol}_{\overline{P}^\circ}, \]
where \( y \) is any branch of \( \sqrt[n]{\prod_{j=0}^{t-1}(x - \lambda_j)^{i_j}} \).

**Theorem 5.1.** Assume that for each \( j \), \( i_j \not\equiv 0 \) mod \( N \) and that \( i_0 + \ldots + i_{r+1} \not\equiv 0 \) mod \( N \). Then for each primitive character \( \chi \in \overline{\mu_N}^{\text{prim}} \),
\[ H^1_c(P^\circ, L_\chi) = H^1_c(X^\circ, \mathcal{C})^\chi = H^1(X, \mathcal{C})^\chi = H^1(P, \tilde{L}_\chi). \]
The above space has dimension \( r + 1 \).

**Proof.** To show the first claim, it suffices to show that for all \( \chi \in \overline{\mu_N}^{\text{prim}} \), and for each \( s \in S \), we have
\[ (p_\ast \mathcal{C}_T)^{\chi s}_s = 0. \]
To prove the second claim, the Euler characteristic
\[ \chi_c(P^\circ, L_\chi) = 2 - \#S = 2 - (r + 3) = -(r + 1), \]
since \( L_\chi \) is a local system of rank 1. Under these hypotheses, we will see that each \( L_\chi \) is a nontrivial local system, and therefore \( H^1_c(P^\circ, L_\chi) = 0 \) for \( i = 0, 2 \). Note that \( H^0_c = 0 \) because \( P^\circ \) is not compact; by duality \( H^2_c(L) = H^0(L^*) \), and the latter is zero because \( L \) is nontrivial. Thus \( \dim H^1_c(P^\circ, L_\chi) = r + 1 \).

For each divisor \( d \) of \( N \), let \( \overline{\mu_d} \subset \overline{\mu_N} \) be the subset of those characters that factor \( \mu_N \to \mu_d \to \mathbb{C}^\times \), where the first map is \( \zeta \mapsto \zeta^{N/d} \). The primitive characters are those that do not factor for any divisor \( d < N \). For each \( s \in S \) we let \( d_s = \gcd(N, i_s) \) if \( s \) is a finite point, and for \( s = \infty \), \( d_\infty = \gcd(N, i_0 + \ldots + i_{r+1}) \). By our hypothesis, each \( d_s < N \). We will show, that as \( \mu_N \) representation
\[ (p_\ast \mathcal{C}_T)^{\chi s}_s = \text{Ind}^N_{N/d_s}(1) = \sum_{\chi \in \overline{\mu_d}} \chi. \]
That being so, no primitive character appears in any of these, so \( (p_\ast \mathcal{C}_T)^{\chi s}_s = 0 \) for primitive characters.

In more detail: the equation for the curve can be written \( y_N = \prod_{s \in S_0} t_s^{i_s} \), where \( t_s = x - \lambda_s \) is a local parameter at \( s \in S_0 \). In the local ring at \( s \) this is \( y_N = (\text{unit})t_s^{i_s} \), so to analyze the ramification above \( s \in S_0 \), we can consider the equation \( y_N = t_s^{i_s} \). For the ramification at \( \infty \), we use the parameter \( t_\infty = 1/x \), and the local equation is \( y_N = t_\infty^{i_0 + \ldots + i_{r+1}} \).

Writing, for each \( s \in S_0 \), \( N = Nsd_s, i_s = j_s d_s; N = N_\infty d_\infty; \sum_{s \in S_0} i_s = j_\infty d_\infty \), we see from the factorization
\[ y^N - t_s^{i_s} = (y^{N_s})^{d_s} - (t_s^{i_s})^{d_s} = \prod_{\omega \in \mu_{d_s}} (y^{N_s} - \omega t_s^{j_s}) \]
that the fiber of \( p \) above \( s \in S \) consists of \( d_s \) points, each totally ramified of degree \( N_s \). This is because \( \gcd(j_s, N_s) = 1 \), and each local curve \( y^{N_s} - \omega t_s^{j_s} = 0 \) is isomorphic to a disk, say by the map \( u \mapsto (y, t) = (u_j^{\omega^{i_j} N_s}, u^{N_s}) \). If \( t \in p^{-1}(s) \) the fiber \( \mathcal{C}_s \) is stabilized by the subgroup \( \mu_{N/d_s} \), and since the action of \( \mu_N \) is transitive on \( p^{-1}(s) \) we see that as a \( \mu_N \) representation, \( (p_\ast \mathcal{C}_T)^s \) is the induced module \( \text{Ind}^N_{N/d_s}(1) \), as claimed.

Each \( L_\chi \) is a nontrivial local system, if \( \chi = \chi_k \) is primitive. One can see this by considering the local monodromy around any point \( s \in S \). Analytic continuation of \( y^k \) around \( s \) is given by the character \( (\zeta_N)^{kis} \not\equiv 1 \) if \( \gcd(k, N) = 1 \), since \( i_s \not\equiv 0 \) mod \( N \). \( \square \)
The above theorem is valid for any algebraically closed base-field \(k\), where analytic cohomology is replaced by étale cohomology, that is, for \(H^1_c(X^\circ, \mathbb{Q}_\ell)^{\chi}\), provided that the characteristic of \(k\) is prime to \(N\ell\). The proof is exactly the same (replace disks by the Henselian local rings). The only nontrivial point to observe is that all the local systems are tame.

Here is a picture:

\[ X : y^6 = x^2(1 - x)^2(1 - \lambda x)^3 \]

5.2. Now we consider the dependence of the curves on the parameters \(\lambda = \{\lambda_0, \ldots, \lambda_{r+1}\}\). Let

\[ D(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j), \quad h(x) = \prod_{i=0}^{r+1}(x - \lambda_i). \]

Let

\[ R_N = \mathbb{Z}[\mu_N, 1/N], S_N = R_N[\lambda, D(\lambda)^{-1}], T_N = S_N[x, h^{-1}]. \]

Let

\[ U = \text{Spec}(S_N) = \mathbb{A}_{R_N}^{r+1} - \{D(\lambda) = 0\}, \text{coordinates } \lambda. \]

Let

\[ P_0^U = \text{Spec}(T_N) = \mathbb{A}_U^1 - \{h = 0\}, \text{coordinate } x. \]

There is an evident \(R_N\)-morphism \(u : P_0^U \to U\). Let

\[ X^\circ = \text{Spec}T_N[y, y^{-1}]/(y^N - \prod_{j=0}^{r+1}(x - \lambda_j)^{i_j}). \]

The natural map \(p^\circ : X^\circ \to P_0^U\) sending \((x, y) \to x\) is an étale \(\mu_N\)-covering. The affine curve \(X^\circ\) is the open subset of the projective, nonsingular model \(X = X^{[N : i]}\) where \(y \neq 0\). The composite map \(\alpha := u \circ p^\circ : X^\circ \to U\) sends the curve to its corresponding \(\lambda\) value. We omit reference to the ring of constants \(R_N\) when it is clear. Here is a picture (\(S = \{0, 1, 1/\lambda, \infty\}\)): 
Let \( f(x) = f_1(x) = \prod_{j=0}^{r+1} (x - \lambda_j)^{i_j} \in T_N \). This defines a morphism \( f : P^\circ_U \to G_m \), and we have a Cartesian diagram (schemes over \( R_N \))

\[
\begin{array}{ccc}
X^\circ & \xrightarrow{g} & G_m \\
p^\circ \downarrow & & \downarrow N \\
P^\circ_U & \xrightarrow{f} & G_m
\end{array}
\]

We get

\[ f^* N, \bar{Q}_\ell = \bigoplus_{\chi} f^* K(\chi)_\ell = p^*_u g^* \bar{Q}_\ell = p^*_u \bar{Q}_\ell, \]

where the sum is over all the characters \( \chi : \mu_N \to \bar{Q}_\ell^\times \), \( K(\chi) \) is the Kummer sheaf, see appendix 16. The lisse sheaf on \( U \) given by \( R^1 \alpha! \bar{Q}_\ell \) gives the cohomologies of the curves in each fiber, viz.,

\[ (R^1 \alpha! \bar{Q}_\ell)_{\tilde{\lambda}} = H^1_c(X^\circ_{\tilde{\lambda}}, Q_\ell) \]

for each geometric point \( \tilde{\lambda} \) on \( U \). Since \( p^\circ \) is finite, we have

\[ R^1 \alpha! \bar{Q}_\ell = R^1 u_1 p^*_u \bar{Q}_\ell = \bigoplus_{\chi} R^1 u_1 f^* K(\chi)_\ell. \]

This justifies our taking \( R^1 u_1 f^* K(\chi)_\ell \) as the \( \ell \)-adic realization of a hypergeometric sheaf.

**Definition 5.2.** Let \( K_N = \mathbb{Q}(\zeta_N) \) and \( \chi : \mu_N \to K_N^\times \) be a primitive character. Assume that \( N \) does not divide any \( i_j \) or \( i_0 + \ldots + i_{r+1} \). In the notations above, we define

\[ \mathcal{P}[i/N, \chi] := R u_1 f^* K(\chi) \]

in \( DA(U, K_N) \).

We ought to define this as \( R^1 u_1 f^*_i K(\chi) \), but this requires a \( t \)-structure on our motives, only conjecturally available. In our case, \( R^1 u_1 f^*_i K(\chi)_\ell \) for \( i \neq 1 \), so this is harmless. One can also make use of other theories of motives that do have \( t \)-structures, e.g., Nori motives.

Symbolically, we can write this as

\[ Jac(X/U)^\chi, \]

where \( Jac(X/U) \to U \) is the abelian scheme of the relative Jacobians of the curves in the fibers. Note that

\[ Jac(X/U)^{\text{prim}} = \bigoplus_{\chi \in \mu_N^{\text{prim}}} Jac(X/U)^\chi, \]
is meaningful as an abelian scheme up to isogeny, but the individual summands only make sense as motives.

6. Hypergeometric motives

For the main properties of the fundamental group, see [1]. Let $F$ be a finite extension field of $\mathbb{Q}$, $R = O_F[1/N]$ the localization of the ring of integers of $F$ for an integer $N \geq 1$. Let $S = \text{Spec}(R)$. We let $\eta = \text{Spec}(F)$, the generic point of $S$, and $\bar{\eta} = \text{Spec}(ar{F})$ for an algebraic closure of $F$. Let $U/S$ be an irreducible separated scheme, smooth and of finite type over $S$, with geometrically connected fibers.

As a first approximation, by a *motivic sheaf* on $U$ we mean the following:

$$\mathcal{H} = (\mathcal{H}_B, \mathcal{H}_{DR}, \mathcal{H}_\ell)$$

where

1. $\mathcal{H}_B$ is a local system of finite-dimensional $\mathbb{Q}$-vector spaces on $U^\text{an}$.
2. $\mathcal{H}_{DR}$ is a locally free $\mathcal{O}_U$-module with an integrable connection $\nabla : \mathcal{H}_{DR} \to \Omega^1_U \otimes_{\mathcal{O}_U} \mathcal{H}_{DR}$ with regular singular points at infinity (i.e., relative to a smooth compactification of $U$).
3. For each prime number $\ell$ prime to the residual characteristics of $U$, $\mathcal{H}_\ell$ is a lisse $\mathbb{Q}_\ell$-sheaf on $U_{\text{et}}$.

These are subject to a number of properties, of which we single out the comparison isomorphisms:

1. There is an isomorphism $\mathcal{H}_B \otimes_{\mathbb{Q}} \mathbb{C} \cong \text{Ker}(\nabla^\text{an})$ where $\text{Ker}(\nabla^\text{an})$ is the sheaf of solutions of the analytic differential equation attached to $\nabla$.
2. For each prime $\ell$ not dividing $N$, there is an isomorphism of fields $\iota : \mathbb{Q}_\ell \cong \mathbb{C}$ and an isomorphism

$$\mathcal{(H_\ell)^{\text{an}}} \cong \mathcal{H}_B \otimes_{\mathbb{Q}} \mathbb{C}$$

of $\mathbb{C}$-local systems on $U^\text{an}$.

Comparison 2 has the following meaning: The lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{H}_\ell$ on $U$ is equivalent to a representation

$$\rho : \pi_1(U, \bar{\eta}) \to \text{GL}(V)$$

for a finite-dimensional $\mathbb{Q}_\ell$-vector space $V$, where the left-hand side is the étale fundamental group. There is a canonical map $\pi(U^\text{an}, u) \to \pi_1(U, \bar{\eta})$ where the left-hand side is the usual fundamental group, and where $u \in U(\mathbb{C})$ is a base-point which lies over the base-point $\bar{\eta}$. That is, if $\bar{\eta} : \text{Spec}(K) \to U$ is the base-point attached to an algebraically closed field, then $u$ is the composite $\text{Spec}(\mathbb{C}) \to \text{Spec}(K) \to U$ for an embedding $K \subset \mathbb{C}$. By Riemann’s existence theorem, $\pi_1(U, \bar{\eta})$ is the profinite completion of $\pi(U^\text{an}, u)$, and in particular, the image is dense in the profinite topology of the target.

Via this isomorphism, we obtain

$$\rho^\text{an} : \pi(U^\text{an}, u) \to \text{GL}(V) \cong \text{GL}(V_\mathbb{C}), \quad V_\mathbb{C} : = V \otimes_{\mathbb{Q}_\ell} \mathbb{C},$$

where the last isomorphism comes from $\iota$. This defines the $\mathbb{C}$-local system $(\mathcal{H}_\ell)^{\text{an}}$. Statement 2 is that this is isomorphic to the $\mathbb{C}$-local system $\mathcal{H}_B \otimes_{\mathbb{Q}} \mathbb{C}$. This $\mathbb{C}$-local system is equivalent by statement 1 to the differential equation $\mathcal{H}_{DR}$.

In Section 16 we describe more precisely the triangulated categories of motivic sheaves.

7. Comparison Theorem

The main result of this section is to show that a transformation identity among hypergeometric differential equations implies a similar one among finite field hypergeometric functions, up to twisting by a Galois character. As before, let $F$ be a finite extension field of $\mathbb{Q}$, $R = O_F[1/N]$ the localization of the ring of integers of $F$ for an integer $N \geq 1$. Let $S = \text{Spec}(R)$. We let $\eta = \text{Spec}(F)$, the generic point of $S$, and $\bar{\eta} = \text{Spec}(ar{F})$ for an algebraic closure of $F$. Let $U/S$ be an irreducible separated scheme, smooth
and of finite type over $S$, with geometrically connected fibers. We can choose a geometric generic point $\xi : \text{Spec}(F(U)) \to U$ which lies over $\eta$, where $F(U)$ is the function field of $U$. We let $U_\eta$ and $U_\bar{\eta}$ be the schemes over $\text{Spec} F$ and $\text{Spec} \bar{F}$ obtained from $U$ by base-change.

We consider two geometrically irreducible lisse $\mathbb{Q}_\ell$-sheaves (for the étale topology) $\mathcal{F}$, $\mathcal{G}$ on $U$. These are equivalent to two $\ell$-adic representations

$$\rho_{\mathcal{F}} : \pi_1(U, \bar{\xi}) \to \text{GL}(V), \quad \rho_{\mathcal{G}} : \pi_1(U, \bar{\xi}) \to \text{GL}(W)$$

for finite-dimensional $\mathbb{Q}_\ell$-vector spaces $V$, $W$. Geometrically irreducible means: the restrictions

$$\rho_{\mathcal{F}}|_{U_\eta}, \rho_{\mathcal{G}}|_{U_\eta}$$

of $\pi_1(U_\eta, \bar{\xi})$ are irreducible. Note that we have a surjective homomorphism

$$\pi_1(U_\eta, \bar{\xi}) \to \pi_1(U, \bar{\xi})$$

and an exact sequence

$$0 \to \pi_1(U_\eta, \bar{\xi}) \to \pi_1(U_\eta, \bar{\xi}) \to \text{Gal}(F/F) \to 0.$$

Now let $\varphi : R \to \mathbb{C}$ be an embedding. We obtain a scheme $U_{\varphi, \mathbb{C}}$ over $\mathbb{C}$. This also defines an analytic space $U_{\varphi}^{an} := U_{\varphi}(\mathbb{C})$. Since $\varphi$ will be fixed, we will drop it from the notation. There is a canonical map

$$\pi_1(U^{an}, u) \to \pi_1(U_{\mathbb{C}}, u)$$

(left-hand side: topological fundamental group; right-hand side, the étale fundamental group) which identifies the right-hand side with the profinite completion of left-hand side (Riemann’s existence theorem). In particular, this map has dense image. Here we can take $u$ to be the geometric $u : \text{Spec}(\mathbb{C}) \to U$

$$\text{Spec}(\mathbb{C}) \to \text{Spec}(\bar{F}(U)) = \bar{\xi} \to U$$

where the first arrow is induced by some embedding $\bar{\varphi} : \bar{F}(U) \to \mathbb{C}$ which extends $\varphi$. It is known that there are isomorphisms $\pi_1(U_{\mathbb{C}}, u) = \pi_1(U_{\bar{\eta}}, u) = \pi_1(U_{\bar{\eta}}, \bar{\xi})$ induced by $\bar{\varphi}$. The first holds because $\bar{F}$ has characteristic 0; the second is a change in base-point. Choose an isomorphism $\iota : \mathbb{Q}_\ell \cong \mathbb{C}$. The theorem that follows will not depend on this artificial choice.

Composing all these, we get representations

$$\rho_{\mathcal{F}}^{an} : \pi_1(U^{an}, u) \to \pi_1(U_{\mathbb{C}}, u) = \pi_1(U_{\bar{\eta}}, \bar{\xi}) \xrightarrow{\rho_{\mathcal{F}}} \text{GL}(V) \cong \text{GL}(V_C)$$

where $V_C = V \otimes_{\mathbb{Q}_\ell, \iota} \mathbb{C}$. We get a similar story for $\rho_{\mathcal{G}}^{an}$. We let $\mathcal{F}$ and $\mathcal{G}$ be the $\mathbb{C}$-local systems on $U^{an}$ that arise from these representations of the fundamental group. Also $D(\mathcal{F})$ and $D(\mathcal{G})$ the regular holonomic $\mathcal{D}$-modules (=connections with regular singular points) corresponding to these by the Riemann-Hilbert correspondence.

**Theorem 7.1.** Under these assumptions (and $\mathcal{F}$, $\mathcal{G}$ geometrically irreducible), if the local systems $\mathcal{F}$ and $\mathcal{G}$ on $U^{an}$ are isomorphic (equivalently if the $\mathcal{D}$-modules $D(\mathcal{F})$ and $D(\mathcal{G})$ are isomorphic), then there is a continuous character $\chi : \text{Gal}(\bar{F}/F) \to \mathbb{Q}_\ell^\times$, such that $\mathcal{G}_\eta \cong \mathcal{F}_\eta \otimes \chi$.

**Proof.** There is a matrix $M : V_C \to W_C$ that intertwines the representation of $\pi_1(U^{an}, u)$ given by $\mathcal{F}$ and $\mathcal{G}$. Then $\iota^{-1}(M) : V \to W$ is a matrix that intertwines the representations of $\pi_1(U^{an}, u)$ in $\text{GL}(V)$ and $\text{GL}(W)$. But $\pi_1(U^{an}, u)$ has dense image in $\pi_1(U_{\bar{\eta}}, \bar{\xi})$ and the representations given by $\mathcal{G}$ and $\mathcal{F}$ on $V$, $W$ are continuous. Thus by continuity $\iota^{-1}(M)$ will intertwine those representations. Therefore the representations

$$\rho_{\mathcal{F}}|_{U_{\bar{\eta}}}, \quad \rho_{\mathcal{G}}|_{U_{\bar{\eta}}}$$
of $\pi_1(U_{\eta, \bar{\eta}})$ are isomorphic. From the exact sequence above, and the fact that these representations are isomorphic we get, by the lemma below, a character $\chi : \text{Gal}(F/F) \to \mathbb{Q}_\ell^*$ and an isomorphism
\[
(\rho_\varphi \mid U_{\eta}) = (\rho_\varphi \mid U_{\bar{\eta}}) \otimes \chi
\]
as representations of $\pi_1(U_{\eta, \bar{\xi}})$.

The following is well-known.

**Lemma 7.2.** Given an exact sequence of groups
\[
0 \longrightarrow H \xrightarrow{a} G \xrightarrow{b} G/H \longrightarrow 0
\]
and two finite-dimensional representations $\rho : G \to \text{GL}(V)$ and $\sigma : G \to \text{GL}(W)$ where $V, W$ are vector spaces over an algebraically closed field $k$. Suppose that $\rho \mid H$ and $\sigma \mid H$ are irreducible and isomorphic. Then there is a character $\chi : G/H \to \text{GL}_1(k) = k^*$, such that $\sigma = \rho \otimes \chi := \rho \otimes (\chi \circ b)$. If $k$ is a topological field and $\rho, \sigma$ are continuous representations, then $\chi$ is a continuous character.

We can give a stronger version of this theorem if we assume in addition that $U/S$ has a section, and that $U/S$ is the complement in $Z/S$ of a divisor with normal crossings $D/S$, where $Z/S$ is proper and smooth. We also assume that $\ell$ is invertible on $U$ and $S$. Under those assumptions, then we have an exact sequence ([1, Ch. XIII, Prop. 4.3, and Examples 4.4])
\[
0 \longrightarrow \pi_1^S(U_{\eta}, \bar{\xi}) \longrightarrow \pi_1^S(U, \bar{\xi}) \longrightarrow \pi_1(S, \bar{\eta}) \longrightarrow 0.
\]
Here $\mathbb{L}$ is a set of primes invertible on $S$. $\pi_1^S(U_{\eta})$ is the pro-$\mathbb{L}$- quotient of $\pi_1(U_{\eta})$. If $K$ is the kernel of the canonical homomorphism $\pi_1(U, \bar{\xi}) \to \pi_1(S, \bar{\eta})$ and $N \subset K$ is the smallest normal subgroup such that $K/N$ is a pro-$\mathbb{L}$-group, then $N \subset \pi_1(U, \bar{\xi})$ is a normal subgroup, and we denote $\pi_1^S(U, \bar{\xi}) = \pi_1(U, \bar{\xi})/N$. If we assume that the representations $\rho_\varphi$ and $\rho_\varphi$ factor through $\pi^S_1(U, \bar{\xi})$ and are geometrically irreducible, we can conclude that there exists a continuous character $\chi : \pi_1(S, \bar{\eta}) \to \mathbb{Q}_\ell^*$ such that $\mathcal{G} \cong \mathcal{F} \otimes \chi$ on $U$.

8. **Character sums and $\ell$-adic sheaves**

8.1. Now let $X_0$ be a scheme separated and of finite type over a finite field $\mathbb{F}_q$, with $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_q$. If $x \in |X|$ is a closed point, then the residue field $k(x)$ is a finite extension of $\mathbb{F}_q$ whose degree we denote by $\text{deg}(x)$, so $k(x)$ has $q^\text{deg}(x)$ elements. If $\varphi \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$, $\varphi(x) = x^q$ is the Frobenius substitution, the induced action of $\varphi$ on $X(\mathbb{F}_q)$ coincides with the action of the Frobenius morphism $F : X \to X$ on $|X|$; this is the morphism that sends the point with coordinates $x$ to the point with coordinates $x^q$. The fixed points of iterates of the Frobenius can be identified with $\mathbb{F}_q^*$-rational points, $X^{F^n} = X_0(\mathbb{F}_q^n)$.

Let $\ell$ be a prime number with $\ell \nmid q$. Given any constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on $X_0$, the fiber $\mathcal{F}_x$ in a geometric point $\bar{x}$ over any $x \in X$ is a finite dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of $\text{Gal}(k(\bar{x})/k(x))$. If $X_0$ is geometrically irreducible with generic point $\eta$, then to give a lisse $\mathcal{F}$ on $X_0$ is equivalent to give a continuous representation
\[
\rho : \pi_1(X_0, \bar{\eta}) \to \text{GL}(\mathcal{F}_{\bar{\eta}}) \sim \text{GL}_n(\mathbb{Q}_\ell).
\]
If $x$ is a closed point, there are automorphisms $\text{Frob}_x := \text{Frob}_{q^\text{deg}(x)}$ of the fibers $\mathcal{F}_x$. Let $K_0 \in D^b(\mathbb{Q}_\ell)$ and we define $K := K_0 \otimes_{\mathbb{F}_q} \mathbb{F}_q \in D^b(\mathbb{Q}_\ell)$. Then we have automorphisms $\text{Frob}_x$ on the fibers of the constructible $\mathbb{Q}_\ell$-sheaves $H^i(K_0)_x$. By definition,
\[
\text{Tr}(\text{Frob}_x \mid K_0) := \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Frob}_x \mid H^i(K_0)_x).
We have the Grothendieck-Lefschetz fixed-point formula:
\[ \sum_{x \in X^F} \Tr(Frob_x \mid K_0) = \sum_{n \in \mathbb{Z}} (-1)^n \Tr(F^*_q \mid H^i_c(X, K)), \]
where \( F = F_q \) is the Frobenius endomorphism of \( X \).

For all \( n \geq 1 \), we can consider the functions on \( X_0(F_q^n) \) given by
\[ X_0(F_q^n) \ni x \mapsto t_{K,n}(x) := \Tr(Frob_x \mid K \otimes_{F_q} F_q^n). \]

These functions are well-defined on the Grothendieck group \( K_0(X_0, \mathbb{Q}_\ell) \) of constructible \( \mathbb{Q}_\ell \)-sheaves on \( X_0 \). This latter group is the free abelian group on the simple perverse sheaves on \( X_0 \). It follows from the Čebotarev density theorem that the trace functions determine the image of \( K \) in that group in the sense that if \( K, L \in D_b^c(X_0, \mathbb{Q}_\ell) \) and for all \( n \geq 1 \) the trace functions are equal, i.e., \( \forall x \in X_0(F_q^n), t_{K,n}(x) = t_{L,n}(x) \), then \( K \) and \( L \) give the same class in \( K_0(X_0, \mathbb{Q}_\ell) \). In particular \( K = L \) if \( K \) and \( L \) are simple perverse. For all of this, see the first sections of [50].

8.2. As Deligne pointed out ([16, Ch. 6]), many character sums over finite fields can be interpreted as functions
\[ X_0(F_q) \ni x \mapsto t_K(x) = \Tr(Frob_x \mid K) \]
for suitable \( K \). This is so for the various hypergeometric character sums considered by many authors (see for instance [11], [23], [27], [29], [32], [46]).

Let \( \psi : F_q^* \to \mathbb{Q}_q^* \) be a nontrivial additive character. There is a lisse \( \mathbb{Q}_\ell \)-sheaf of rank one \( L \psi \) on \( \mathbb{G}_a = \mathbb{A}^1 \) (Artin-Schreier sheaf), with the property that \( \Tr(Frob_{x}) \mid L_{\psi,x} = \psi(\Tr_{F_q^*/F_q}(x)) \) if \( d = \deg(x) \).

Let \( \chi : F_q^* \to \mathbb{Q}_q^* \) be a multiplicative character. There is a lisse \( \mathbb{Q}_\ell \)-sheaf of rank one \( L \chi \) on \( \mathbb{G}_m = \mathbb{A}^1 - \{ 0 \} \) (Kummer sheaf), with the property that \( \Tr(Frob_{x}) \mid L_{\chi,x} = \chi(\Tr_{F_q^*/F_q}(x)) \) if \( d = \deg(x) \). If \( f : S \to \mathbb{G}_a \) (resp., \( f : S \to \mathbb{G}_m \)) is a morphism, then we define \( L_{\psi,f} = f^* L_{\psi} \) on \( S \) (resp., \( L_{\chi,f} = f^* L_{\chi} \)).

For the sheaf \( L_{\psi} \otimes L_{\chi} \) on \( \mathbb{G}_m \) we have that \( H^i_{\dR}(\mathbb{G}_m, L_{\psi} \otimes L_{\chi}) \) is zero unless \( i = 1 \); when \( i = 1 \) it is one-dimensional, and by the trace formula above, \( F^* \) on \( H^1 \) is multiplication by the Gauss sum
\[ -g(\psi; \chi) = - \sum_{x \in \mathbb{F}_q^*} \psi(x) \chi(x). \]

8.3. Now let \( \alpha_1, \ldots, \alpha_a \) and \( \beta_1, \ldots, \beta_b \) be two disjoint unordered lists of multiplicative characters of \( \mathbb{F}_q^* \), at least one of \( a, b \) nonzero. We allow some of the characters to be trivial, and we allow repetition in the lists. Then Katz [46, Ch. 8] defines a geometrically irreducible \( \mathbb{Q}_\ell \)-sheaf (a hypergeometric sheaf of type \((a, b)\))
\[ \mathcal{H}(\psi; \alpha_i ; \beta_j) \text{ on } \mathbb{G}_m / \mathbb{F}_q \]
with the property that if \( E / \mathbb{F}_q \) is a finite extension field, and \( t \in \mathbb{G}_m(E) = E^\times \),
\[ \Tr(Frob_{t}) \mid \mathcal{H}(\psi; \alpha_i ; \beta_j)_{t} = (-1)^{a+b-1} \sum_{V(n,m,t)(E)} \psi_E(\sum_i x_i - \sum_j y_j) \prod_i \alpha_i,E(x_i) \prod_j \beta_j,E(y_j) \]
\[ = (-1)^{a+b-1} \sum_{A \in E^\times} A(t) \prod_i g(\psi_E, \Lambda \alpha_i,E) \prod_j g(\bar{\psi}_E, \Lambda \bar{\beta}_i,E) \]
where in the last expression, the sum ranges over the multiplicative characters of \( E^\times \); this is the Fourier expansion of the first expression in terms of the characters of \( E^\times \). \( \psi_E = \psi \circ \Tr_{E/F_q} \), \( \chi_E = \chi \circ N_{E/F_q} \), and the first sum is over the \( E \)-rational points of the variety
\[ V(n,m,t) : \sum_{i=1}^a x_i = \sum_{j=1}^b y_j. \]
Without loss of generality, we can take $a \geq b$. Then this sheaf is of rank $a$ and is pure of weight $a + b - 1$. It is lisse on $\mathbb{G}_m$ if $a \neq b$; if $a = b$ it is lisse on $\mathbb{G}_m - \{1\}$, but tame at $\{0, 1, \infty\}$. The local monodromies of these sheaves are determined explicitly in terms of the characters $\alpha_i, \beta_j$ and their multiplicities. When $a > b$ the sheaves $\mathcal{H}(\psi; \alpha_i; \beta_j)$ have wild ramification at $\infty$. The cohomology of these sheaves is given by

$$H^i_c(\mathbb{G}_m \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{H}(\psi; \alpha_i; \beta_j)) = 0, \quad \text{unless } i = 1; \text{ when } i = 1, \text{the dimension is } 1,$$

and

$$\text{Tr}(\text{Frob}_\ell | H^1_c(\mathbb{G}_m \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{H}(\psi; \alpha_i; \beta_j))) = \prod_i (\deg(g(\psi, \alpha_i), E)) \prod_j (\deg(\tilde{\psi}_E, \tilde{\Lambda}_{\beta_i,E})).$$

Also useful are the determinant formulas; see [46, Theorem 8.12.2]. These hypergeometric sheaves are rigid in the sense to be recalled below. When the list of the $\beta_i$ is empty, we get the Kloosterman sums/sheaves explored in detail in [45].

In [11] a variant of the above hypergeometric sum is introduced, where the summation is taken over the subvariety $W(n, m, t) \subset V(n, m, t)$ where $\sum_i x_i = \sum y_j$. This has the advantage of removing the additive character $\psi$ and hence the wild ramification in the hypergeometric sheaf, hence giving objects that can be defined over the integers, not just over finite fields.

When $a = b$ it is shown in [48, Section 4] that there is a canonical twist $\mathcal{H}^{\text{can}}(\alpha_i; \beta_j) = \mathcal{H}(\psi; \alpha_i; \beta_j) \otimes \Phi$, by a $\bar{\mathbb{Q}}_\ell$-valued character $\Phi$ of $\text{Gal}(F_q/F_q)$, which is independent of $\psi$. This is given by $\Phi = (1/A)^{\text{deg}}$ where

$$A = \prod_i g(\psi; \alpha_i)g(\bar{\psi}, \bar{\beta}_i) = \prod_i g(\psi; \alpha_i)g(\psi, \bar{\beta}_i)(-1).$$

The character sum for $\mathcal{H}^{\text{can}}(\alpha_i; \beta_j)$ then involves a twist by a product of Jacobi sums:

$$J(F_q; \mu, \nu) = \sum_{x \in F_q} \mu(x)\nu(1-x)$$

($\mu(0) = 0, \mu(1) = 1, \mu(1) = 1$). When $a = b$, $\mathcal{H}^{\text{can}}(\alpha_i; \beta_j)$ globalizes to give an object over the integers in a suitable cyclotomic number field, with the Jacobi sum globalizing to a Hecke character.

For the canonical twist, we have

$$\text{Tr}(\text{Frob}_\ell | H^1_c(\mathbb{G}_m \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{H}^{\text{can}}(\alpha_i; \beta_j))) = -1.$$

8.4. When $a = b = 2$ these are the finite field analogs of the $2F_1$. This can be seen as follows: First, for $a = b = 1$ and $t \in \mathbb{F}_q^*$ an easy calculation shows that

$$\text{Tr}(\text{Frob}_t | \mathcal{H}(\psi; \alpha; \beta) \xi) = -g(\psi; \alpha \bar{\beta}) \alpha(t)(\beta/\alpha)(t-1)$$

Using $J(\mu, \nu)g(\psi; \mu) = g(\psi; \mu)g(\psi; \nu)$ if $\mu \neq 1$, we see that the sum associated to the canonical twist of this is

$$\text{Tr}(\text{Frob}_t | \mathcal{H}^{\text{can}}(\alpha; \beta) \xi) = \frac{1}{J(\alpha, \beta/\alpha)} \alpha(t)(\beta/\alpha)(1-t).$$

For $a = b = 2$,

$$\text{Tr}(\text{Frob}_t | \mathcal{H}(\psi; \alpha_1, \alpha_2; \beta_1, \beta_2) \xi) = -\sum_{x_1, x_2 = t} \psi(x_1)\psi(x_2)\bar{\psi}(y_1)\bar{\psi}(y_2)\alpha_1(x_1)\alpha_2(x_2)\bar{\beta}_1(y_1)\bar{\beta}_2(y_2)$$

which can be rewritten as

$$-\sum_{(t_1, t_2), t, t_2 = t} \left( \sum_{\frac{x_1}{y_1} = t_1} \psi(x_1 - y_1)\alpha_1(x_1)\bar{\beta}_1(y_1) \right) \left( \sum_{\frac{x_2}{y_2} = t_2} \psi(x_2 - y_2)\alpha_2(x_2)\bar{\beta}_2(y_2) \right).$$
The inner sums are just cases of \( a = b = 1 \). This reduces to
\[
C \sum_{t_1 t_2 = t} \alpha_1(t_1)(\beta_1/\alpha_1)(t_1 - 1)\alpha_2(t_2)(\beta_2/\alpha_2)(t_2 - 1), \quad C = -g(\psi; \alpha_1 \bar{\beta}_1)g(\psi; \alpha_2 \bar{\beta}_2).
\]
Replacing \( t_2 = t/t_1 \), this simplifies to
\[
C \alpha_2(t) \sum_{t_1} \lambda(t_1)\mu(t_1 - 1)\nu(t - t_1), \quad \lambda = \alpha_1\beta_2^{-1}, \mu = \beta_1\alpha_1^{-1}, \nu = \beta_2\alpha_2^{-1},
\]
which is indeed a finite field analog of \( {}_2F_1 \).

If \( \chi : \mu_N \to \mathbb{Q}_l^* \) is a primitive character, and \( \lambda = \chi^a, \mu = \chi^b, \nu = \chi^c \) the above sum becomes essentially
\[
\sum_\nu \chi(t^\nu(t_1 - 1)^b(t-t_1)^c).
\]
These sums occur when counting the \( \mathbb{F}_q \)-rational points on the smooth projective model of the curve
\[
y^N = x^a(x - 1)^b(t - x)^c.
\]
Because
\[
\mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2)[1] = \mathcal{H}^{\text{can}}(\alpha_1; \beta_1)[1] \ast \mathcal{H}^{\text{can}}(\alpha_2; \beta_2)[1],
\]
(convolution) a calculation similar to the above gives

**Lemma 8.1.**

\[
\text{Tr}(\text{Frob}_t|\mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2)) = A\alpha_2(t) \sum_{t_1} \lambda(t_1)\mu(1-t_1)\nu(t-t_1), \quad \lambda = \alpha_1\beta_2^{-1}, \mu = \beta_1\alpha_1^{-1}, \nu = \beta_2\alpha_2^{-1},
\]
where \( A = -\left[\beta_2\bar{\alpha}_2\right](-1)\left[J(\alpha_1, \beta_1 \bar{\alpha}_1)J(\alpha_2, \beta_2 \bar{\alpha}_2)\right]^{-1}\).

**Corollary 8.2.**

\[
\text{Tr}(\text{Frob}_t|\mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2)) = -\frac{\beta_2\bar{\alpha}_2(-1)J(\alpha_1 \bar{\beta}_2, \beta_1 \bar{\alpha}_1 \bar{\beta}_2)}{J(\alpha_1, \beta_1 \bar{\alpha}_1)J(\alpha_2, \beta_2 \bar{\alpha}_2)}.
\]

The sheaf \( \mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2) \) is a rank 2 local system on \( U = \mathbb{G}_m - \{1\} \) but the stalk at \( t = 1 \) has rank 1. This is because the dimension of the invariants under inertia \( I(1) \) is one-dimensional. In fact, the local monodromy at 1 is a pseudoreflection of determinant \( \beta_1 \beta_2/\alpha_1 \alpha_2 \), and every pseudoreflection has a codimension one space of invariants.

It is sometimes convenient to think of the sheaves \( \mathcal{H}^{\text{can}}(\alpha; \beta_j) \) as living on all of \( \mathbb{P}^1 \). They will always be understood as \( j_* (\mathcal{H}^{\text{can}}(\alpha; \beta_j) | U) \) where \( j : \mathbb{G}_m - \{1\} \to \mathbb{P}^1 \). It will be useful to calculate the Frobenius traces at the other singular points, viz., \( 0, \infty \), when these stalks are nonzero.

We remark:

**Proposition 8.3.** If \( \psi \) is the canonical additive character character \( x \mapsto e^{2\pi i/p}\text{tr}(x) \) of \( \mathbb{F}_q \), we have
\[
\text{Tr}(\text{Frob}_t|\mathcal{H}(\psi; \alpha_1, \alpha_2; \beta_1, \beta_2)) = \alpha_2\beta_2(-1)\alpha_2(t) \cdot \mathbb{P}^1 \left[ \frac{\alpha_2\bar{\beta}_2}{\alpha_2\bar{\alpha}_1}; t \right],
\]
\[
= \alpha_2\beta_2(-1)\alpha_2(t) \cdot J(\mathbb{F}_q; \alpha_2 \bar{\beta}_1, \beta_1 \alpha_1) \mathbb{P}^1 \left[ \frac{\alpha_2\bar{\beta}_2}{\alpha_2\bar{\alpha}_1}; t \right],
\]
the finite field \( n+1 \mathbb{F}_n \)- and \( n+1 \mathbb{F}_n \)-functions in the monograph [29, Chapter 4]. Since \( \alpha_1, \alpha_2 \) and \( \beta_1, \beta_2 \) are disjoint unordered lists of multiplicative characters, we can rewrite the character sum as
\[
\text{Tr}(\text{Frob}_t|\mathcal{H}(\psi; \alpha_1, \alpha_2; \beta_1, \beta_2)) = \alpha_2\beta_2(-1)\alpha_2(t) \cdot J(\mathbb{F}_q; \alpha_2 \bar{\beta}_1, \beta_1 \alpha_1) \mathbb{P}^1 \left[ \frac{\alpha_2\bar{\beta}_2}{\alpha_2\bar{\alpha}_1}; t \right].
\]
More generally, for \(a = b = n\), if \(\psi\) is the canonical additive character character \(x \mapsto e^{2\pi i/p \cdot \text{tr}(x)}\) of \(F_q\), we have

\[
\text{Tr}(\text{Frob}_p|\mathcal{H}(\psi; \alpha; \beta)_t) = \alpha_n \beta_n (-1)^n \alpha_n(t) \cdot C \cdot n! \prod_{n=1}^{n+1} \left[ \frac{\alpha_n \beta_n}{\alpha_n \alpha_1 \cdots \alpha_n \beta_{n-1}} ; t \right],
\]

the finite field \(n+1\)-function in [29], where \(C = -\prod_{i=1}^{n-1} g(\psi; \alpha_i \beta_i)\).

8.5. In chapter [29, Chapter 5] it is shown how, given rational numbers \(a_i, b_j, \lambda \in \mathbb{Q}\), one can attach a collection of hypergeometric functions over finite residue fields \(F_p\) (varying in \(p\))

\[
n+1 = \prod_{n=1}^{n+1} \left[ \frac{t_p(a_1)}{t_p(b_1)} \cdots \frac{t_p(a_{n+1})}{t_p(b_n)} ; \lambda; q(p) \right],
\]

where \(p\) runs through all unramified prime ideals of \(\mathbb{Q}(\zeta_N)\) with \(N\) being the least positive common denominator of all \(a_i\) and \(b_j\). The symbols \(t_p(a_{j+})\) etc. represent characters of the finite field \(F_p\) with \(q(p)\) elements. The Frobenius traces of the \(\ell\)-adic realization of the hypergeometric motives can be expressed in terms of these functions. This is done in [29, Theorem 1.1, Chapter 6.2].

We will generalize this result. Recall the notation from section 5.2. We let \(|U|\) be the set of closed points of \(U\) (= maximal ideals of \(S_N\)). Let \(x \in |U|\) be a closed point. This lies over a prime ideal \(p \subset R_N\). We let \(F_q = R_N/p\), where \(q = Np \equiv 1 \mod N\). Then \(\kappa(x) = \mathcal{O}_{U,x}/m_x = F_{q^\alpha}\) is a finite extension of \(F_q, d = \deg(x)\).

To calculate the Frobenius trace at \(x\) it is convenient to base-extend all our schemes, originally over \(R_N\) to the finite field \(F_q = R_N/p\). Then each element \(\lambda\) of the set \(U(F_{q^\alpha})\) is a vector \((\lambda_0, \ldots, \lambda_{r+1}) \in F_{q^\alpha}\), and we define \(h(x, \lambda) \in F_{q^\alpha}[x]\) by specializing \(h(x)\) in section 5.2 to these values of \(\lambda\). The fiber

\[
u^{-1}(\lambda) = A^1_x - \{h(x, \lambda) = 0\}.
\]

We define \(\bar{\lambda}\) as the geometric point \(\bar{\lambda} : \text{Spec}(F_q) \to \text{Spec}(F_{q^\alpha}) \xrightarrow{\lambda} U\). That is, we regard each of our finite fields as contained in a fixed algebraic closure \(F_q\).

Choose a primitive character \(\chi : \mu_N \to \mathbb{Q}_\ell^*\). For each finite field \(F_q\) such that \(q \equiv 1 \mod N\) we define the character

\[
\chi_q : F_q^\times \to \mathbb{Q}_\ell^*, \quad t \mapsto \chi(t^{(q-1)/N}),
\]

which is meaningful since \(t^{(q-1)/N} \in \mu_N\).

**Theorem 8.1.** Let \(K_N = \mathbb{Q}(\zeta_N)\). Assume that \(N\) does not divide any \(i_j\) or \(i_0 + \ldots + i_{r+1}\). Then there is a lisse \(\mathbb{Q}_\ell\)-adic sheaf \(\mathcal{P}[i/N, \chi]_\ell\) on \(U[1/\ell]\) corresponding to a representation

\[
\sigma_\ell : \pi_1(U, \bar{\xi}) \to GL_{r+1}(\mathbb{Q}_\ell)
\]

whose Frobenius traces for \(\lambda \in U(F_{q^\alpha})\) are given by

\[
\text{Tr}(\text{Frob}_\lambda|\mathcal{P}[i/N, \chi]_\ell, \bar{\xi}) = -\mathcal{P}[i/N, \chi; \lambda; q^{\deg(\lambda)}] = -\sum_{x \in F_{q^\alpha} : h(x, \lambda) \neq 0} \chi_{q^{\deg(\lambda)}}(f_i(x)).
\]

Here \(\bar{\xi} = \text{Spec}(\overline{K_N(\lambda)})\) is a geometric generic point. This sheaf is punctually pure of weight 1.

**Proof.** We calculate the Frobenius trace \(\text{Frob}_\lambda\) in the geometric fiber at \(\bar{\xi}\) in the \(\ell\)-adic realization of the motive

\[
\mathcal{P}[i/N, \chi] := R\mu_i f_i^* K(\chi).
\]
By the Grothendieck-Lefschetz formula, the trace $\text{Frob}_\lambda$ on $R \ell_! f^*_1 K(\chi)_\ell$, i.e., the alternating sum

$$\sum_{i=0}^{2} (-1)^i \text{Tr}(\text{Frob}_\lambda | R \ell_! f^*_1 K(\chi)_\ell) = \sum_{i=0}^{2} (-1)^i \text{Tr}(\text{Frob}_\lambda | H^i_c(u^{-1}(\lambda), f^*_1 K(\chi)_\ell)),$$

is the sum

$$\sum_{x \in \text{u}^{-1}(\lambda)(\mathbb{F}_q^d)} \text{Tr}(\text{Frob}_x | f^*_1 K(\chi)_\ell).$$

The local traces are $\chi_{q^d}(f_1(x))$, where $d = \deg(\lambda)$. On the other hand we have shown that have $H^i_c = 0$ for $i \neq 1$, and that the dimension of $H^1_c$ is $r + 1$. Therefore the local system $R \ell_! f^*_1 K(\chi)_\ell$ gives us a representation $\sigma_\ell : \pi_1(U, \xi) \to GL_r(\mathbb{Q}_\ell)$ and

$$-\text{Tr}(\text{Frob}_\lambda | R \ell_! f^*_1 K(\chi)_\ell) = \sum_{x \in \mathbb{F}_q^d \atop h(x, \lambda) \neq 0} \chi_{q^{\deg(\lambda)}}(f_1(x))$$

as claimed. That it is punctually pure of weight 1 follows from the fact that

$$H^1_c(u^{-1}(\lambda), f^*_1 K(\chi)_\ell) = H^1_c(X_{\lambda, x}^*, \mathbb{Q}_\ell) = H^1(X_{\lambda, x}, \mathbb{Q}_\ell)$$

for any primitive character. On the right-hand side is the cohomology of a projective, nonsingular curve, so this is pure of weight 1.

It is worth noting that the classes of Frobenius elements of the closed points in $U$ are dense in $\pi_1(U, \xi)$. Because the representation on $H^1$ of the smooth projective curve preserves the symplectic cup-product up to similitude, the representation preserves a Hermitian form up to similitude. For instance, when $N = 3$, $r = 2$, $i_0 = \ldots = i_3 = 1$, we get the Picard family of curves. The monodromy of this family is in the group $SU(2, 1)$. See section 14.

### 9. Rigid local systems

9.1. We give a brief survey of the main results of [47]. We consider local systems $\mathcal{F}$ on an open subset $U = \mathbb{A}^1 - S = \mathbb{P}^1 - (S \cup \infty)$ where $S$ is a finite set of points. This is studied in two contexts:

1. The $\mathcal{D}$-module setting. Then $\mathcal{F} = \mathcal{V}$ is a local system of $\mathbb{C}$-vector spaces on $U^{\text{an}}$ which is the solution sheaf to a (integrable) algebraic differential equation

$$\nabla : \mathcal{V} \to \Omega^1_{U/\mathbb{C}} \otimes_{\mathcal{O}_X} \mathcal{V}$$

with regular singular points at $S$. This is equivalent to the monodromy representation $(n = \# S)$

$$\rho : \pi_1(U^{\text{an}}, x) \cong \langle \gamma_1, \ldots, \gamma_n, \gamma_\infty | \gamma_1 \ldots, \gamma_n \gamma_\infty = 1 \rangle \to \text{GL}(\mathcal{V}_x) \sim \text{GL}_n(\mathbb{C}).$$

2. The $\ell$-adic setting. Then $\mathcal{F}$ is a lisse $\mathbb{Q}_\ell$-sheaf on $U$. Here the ground field $k$ is algebraically closed. The sheaf $\mathcal{F}$ is equivalent to a continuous representation

$$\rho : \pi_1(U, x) \to \text{GL}(\mathcal{F}_x) \sim \text{GL}_n(\mathbb{Q}_\ell)$$

where the $\pi_1$ refers to the profinite fundamental group in a geometric point $x$.

Let $X$ be a projective smooth connected curve over $k$ and $x \in X$ is a point. By a disk at $x$ we mean either a subset $D_<(x) \subset X$ containing $x$ and homeomorphic to a disk in the complex plane (in case 1 above), or the strict henselization $D_<(x) = \text{Spec}(\mathcal{O}_{X,x}^\text{h})$ of the local ring at $x$ (case 2). In both cases, we let $D_+(x)$ be the punctured disk, i.e., $D_<(x) - \{x\}$. Given a local system $\mathcal{F}$ on $X - S$ where $S = \{s_1, \ldots, s_m\}$ is a finite set of points, we get by restriction local systems $\mathcal{F}(s)$ on $D_+(s)$, $s \in S$. We regard these $\mathcal{F}(s)$ up
to isomorphism. In case 1, the datum \( \mathcal{F}(s) \) is equivalent to a representation of \( \pi_1(D(s)^+) \sim \mathbb{Z} \), hence to a matrix \( T_s \), well-defined up to conjugacy. In case 2 we obtain a representation of the inertia group
\[
\rho(s) : I(s) \to \text{GL}_n(\mathbb{Q}_\ell)
\]
well defined up to conjugation. Especially important are those representations that factor through the tame inertia \( I(s) \to I(s)^{\text{tame}} = \overline{\mathbb{Z}}(1)_{\text{not } p} = \lim_{\rightarrow N} \mu_N(\overline{k}(s)) \).

**Definition 9.1.** We say that a local system \( \mathcal{F} \) on \( U = X - S \) is rigid if given any other local system \( \mathcal{G} \) on \( U = X - S \) such that for all \( s \in S \) there are isomorphisms \( \mathcal{F}(s) \cong \mathcal{G}(s) \) on \( D(s)^+ \). \( \mathcal{F} \) is isomorphic to \( \mathcal{G} \) as local systems on \( U \).

We only consider this notion in the case of genus zero, i.e., \( X = \mathbb{P}^1 \). In the analytic case this has the concrete interpretation as follows. The local system \( \mathcal{F} \) is equivalent to giving complex matrices \( M_1, \ldots, M_m \) such that \( M_1 \cdots M_m = 1 \), and similarly \( \mathcal{G} \) is equivalent to giving complex matrices \( N_1, \ldots, N_m \) such that \( N_1 \cdots N_m = 1 \). That they are locally isomorphic at each \( s \) means that there are invertible matrices \( A_i \) such that \( N_i = A_i M_i A_i^{-1} \) for \( i = 1, \ldots, m \). Rigidity means that there is a single invertible matrix \( B \) such that \( N_i = B M_i B^{-1} \) for \( i = 1, \ldots, m \).

9.2. A related notion of cohomological rigidity is introduced. Let \( j : U \to \mathbb{A}^1 \) be the inclusion of a nonempty open subset (schemes over \( k = \overline{k} \)). Let \( h : \mathbb{A}^1 \to \mathbb{P}^1 \) be the inclusion. Let \( \mathcal{F} \) be an irreducible lisse \( \mathbb{Q}_\ell \)-sheaf on \( U \). Then \( K = j_* \mathcal{F}[1] \), as an element of \( D^c_\mathbb{Z}(\mathbb{A}^1, \mathbb{Q}_\ell) \), is an irreducible perverse and nonpunctual sheaf. For such a sheaf we have the index of rigidity
\[
\text{rig}(\mathcal{F}) := \chi(\mathbb{P}^1, h_* j_* \text{End}(\mathcal{F})).
\]
We say that such a sheaf is cohomologically rigid if \( \text{rig}(\mathcal{F}) = 2 \).

**Theorem 9.2.** [47, Thm. 5.0.2]. Let \( \mathcal{F} \) be a cohomologically rigid local system on an open set \( U \) as above. If \( \mathcal{G} \) is another lisse \( \mathbb{Q}_\ell \)-sheaf on \( U \) which is locally isomorphic to \( \mathcal{F} \) at all points \( s \) of \( \mathbb{P}^1 - U \) in the sense that the representations of inertia \( I(s) \) given by \( \mathcal{F}(s) \) and \( \mathcal{G}(s) \) are isomorphic, we have an isomorphism of lisse \( \mathbb{Q}_\ell \)-sheaves on \( U \), \( \mathcal{F} \cong \mathcal{G} \).

Note that this theorem requires an algebraically closed ground field \( k \). If one starts from \( \mathcal{F} = \mathcal{F}_0 \otimes_k \overline{k} \), and \( \mathcal{G} = \mathcal{G}_0 \otimes_k \overline{k} \) for lisse \( \mathcal{F}_0, \mathcal{G}_0 \) defined over a nonalgebraically closed field \( k \), then the rigidity conclusion is that \( \mathcal{G}_0 \cong \mathcal{F}_0 \otimes \Phi \) for a \( \mathbb{Q}_\ell \)-valued character \( \Phi \) of \( \text{Gal}(\overline{k}/k) \) (assuming \( \mathcal{F} \) and \( \mathcal{G} \) are irreducible).

Define the category \( \mathcal{T}_\ell \) as the full subcategory of constructible \( \mathbb{Q}_\ell \)-sheaves \( \mathcal{F} \) on \( \mathbb{A}^1 \), such that

1. \( \mathcal{F} \) is middle extension: there exists a dense open set \( j : U \to \mathbb{A}^1 \) such that \( j^* \mathcal{F} \) is lisse and irreducible and \( j_* j^* \mathcal{F} \cong \mathcal{F} \).
2. \( \mathcal{F} \) is tame: \( j^* \mathcal{F} \) is tamely ramified at each point of \( \mathbb{P}^1 - U \).
3. \( \mathcal{F} \) has at least two finite singularities: there are at least two distinct points of \( \mathbb{A}^1 \) where \( \mathcal{F} \) fails to be lisse.

One of the main results of [47] is a classification of objects of \( \mathcal{T}_\ell \) which are are

1. lisse on \( \mathbb{A}^1 - \{\alpha_1, \ldots, \alpha_n\} \) \((n \geq 2; \alpha_1, \ldots, \alpha_n \) fixed geometric points).
2. cohomologically rigid, and
3. all eigenvalues of all local monodromies are \( N \)th roots of unity. Here \( N \) is an integer invertible in \( k \).
9.3. The main results are Theorems 5.2.1, 5.5.4, and 8.4.1 of [47]. First it is shown that every such object is obtained, starting from objects of (generic) rank one in $T_\ell$, by repeated iteration of two constructions

1. $\mathcal{F} \mapsto \text{MT}_L(\mathcal{F})$ (middle tensor product), where $L$ is a rank one object in $T_\ell$.
2. $\mathcal{F} \mapsto \text{MC}_\chi(\mathcal{F})$ (middle convolution), where $\chi : \pi_1^{\text{tame}}(\mathbb{G}_m/k) \to \mu_N(Q_\ell)$ is a nontrivial character, with corresponding Kummer sheaf $\mathcal{L}_\chi$.

The effect of these operations on the local monodromies is determined in 3.3.6 and 3.3.7 of loc. cit. The rank one objects are tensor products of translated Kummer sheaves

$$\bigotimes_i \mathcal{L}_{\chi_i(x - \alpha_i)}.$$ 

In Theorem 8.4.1 these rigid local systems are given a motivic interpretation. The motives are eigenspaces of the cohomology of certain hypersurfaces. Define

$$R_{N,\ell} := \mathbb{Z}[\zeta_N, 1/N\ell]$$

and

$$S_{N,n,\ell} := R_{N,\ell}[T_1, \ldots, T_n][1/\Delta], \quad \Delta = \prod_{i \neq j}(T_i - T_j).$$

One fixes an embedding $R_{N,\ell} \to \bar{Q}_\ell$ (equivalently, a primitive $N$th root of unity in $\bar{Q}_\ell$). For each $r \geq 0$ define

$$A_r(n, r + 1)_{R_{N,\ell}} = \text{Spec}(R_{N,\ell}[T_1, \ldots, T_n, X_1, \ldots, X_{r+1}][1/\Delta_{n,r}])$$

where

$$\Delta_{n,r} = \prod_{i \neq j}(T_i - T_j) \prod_{a,j}(X_a - T_j) \prod_k(X_{k+1} - X_k)$$

$(i, j \in \{1, \ldots, n\}, a \in \{1, \ldots, r + 1\}, k \in \{1, \ldots, r\})$. When $r = 0$ the last factor is the empty product, interpreted as 1. In $\mathbb{G}_m \times A(n, r + 1)_{R_{N,\ell}}$ consider the hypersurface $\text{Hyp}(e, f)$ with equation

$$Y^N = \left(\prod_{a,i}(X_a - T_i)^{e(a,i)}\right)\left(\prod_{k=1}^r(X_{k+1} - X_k)^{f(k)}\right)$$

where the integers $e(a,i)$ are arbitrary, and none of the integers $f(k)$ is divisible by $N$. Let

$$\pi : \text{Hyp}(e, f) \to (\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$$

be the map

$$(Y, T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_{r+1}).$$

In other words, we regard $X_{r+1}$ as the coordinate on the target $\mathbb{A}^1$, and we think of $\text{Hyp}(e, f)$ as a family of hypersurfaces in $(Y, X_1, \ldots, X_r)$-space parametrized by $(T_1, \ldots, T_n, X_{r+1})$. In this setup, the parameter $X_{r+1} = \lambda$ is distinguished; we get local systems on the $\lambda$-line minus the points with coordinates $\lambda = T_1, \ldots, \lambda = T_n$.

Fix one faithful character $\chi : \mu_N(R_{N,\ell}) \to \bar{Q}_\ell^\times$. These roots of unity act on $\text{Hyp}(e, f)$ in the obvious way: $Y \mapsto \zeta Y$. Katz proves that the sheaves $R^i\pi_!\bar{Q}_\ell$ on $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ are lisse and tame. The eigenspace $(R^i\pi_!\bar{Q}_\ell)^\chi$ is nonvanishing only when $i = r$, and mixed in integral weights in the interval $[0, r]$. The weight $r$ quotient of that sheaf, denoted $\mathcal{H}_{\chi r}$, if nonzero, when restricted to every geometric fiber of $(\mathbb{A}^1 - \{T_1, \ldots, T_n\})_{S_{N,n,\ell}}$ is geometrically irreducible and cohomologically rigid, all of whose local monodromies have eigenvalues that are $N$th roots of unity.
He shows also that every such rigid local system arises this way. Also he notes that because these rigid local systems belong to universal families on the open subset \( V_{N,\ell} \) of affine space of dimension \( n + 1 \), \( \text{Spec}(R_{N,\ell}[T_1, \ldots, T_{n+1}]) \), where

\[
\Delta = \prod_{i<j}(T_i - T_j) \neq 0,
\]

the representation of \( \pi_1(\mathbb{A}^1 - \{\alpha_1, \ldots, \alpha_n\}, \bar{x}) \) afforded by any rigid local system in \( \mathcal{T}_\ell \) extends to a representation of \( \pi_1(V_{N,\ell}, \bar{x}) \). Geometrically, this fundamental group is Artin’s braid group on \( n + 1 \) letters.

10. APPELL-LAURICELLA SYSTEMS

10.1. This is the case \( r = 1 \) of the above constructions. We get a family of curves in the \((X_1, Y)\)-plane

\[
Y^N = \prod_i (X_1 - T_i)^{e(1,i)} \prod_i (X_2 - T_i)^{e(2,i)} (X_2 - X_1)^{f(1)}.
\]

The factor \( \prod_i (X_2 - T_i)^{e(2,i)} \) comes from the base, i.e., \( \text{Spec}(S_{N,n,\ell}) \) and its effect on \( \mathcal{H}_{r=1} \) is a twist by the Kummer sheaf

\[
\bigotimes_i \mathcal{L}_{X_2,e(X_2-T_i)}.
\]

This can be omitted, and so we are considering the family of curves in the \((X_1, Y)\)-plane \((\lambda = X_2)\)

\[
Y^N = \prod_{i=1}^n (X_1 - T_i)^{e_i} (\lambda - X_1)^f.
\]

We let \((T_1 = \alpha_1, \ldots, T_n = \alpha_n)\) take on fixed values in an algebraically closed field, and we get a one-parameter family of curves \( C_\lambda \). The stalk of \( \mathcal{H}_{r=1} \) at a geometric point \( \lambda \) is then

\[
(H^1_\ell(C_\lambda, \mathbb{Q}_\ell)^\chi)_{r=1} = H^1(\tilde{C}_\lambda, \mathbb{Q}_\ell)^\chi.
\]

where \( \tilde{C}_\lambda \) is the projective nonsingular model of \( C_\lambda \). We can calculate this as follows. Consider the projection \( \rho : \tilde{C}_\lambda \to \mathbb{P}^1 \) where the target has affine coordinate \( X_1 \). This is a Galois \( \mu_N \)-covering over \( U = \mathbb{P}^1 - \{\alpha_1, \ldots, \alpha_n, \lambda, \infty\} \). \( \rho_* \mathbb{Q}_\ell \) is a constructible sheaf on \( \mathbb{P}^1 \), lisse of rank \( N \) on \( U \). We have a decomposition into eigenspaces \((\chi \text{ is a primitive character of } \mu_N)\)

\[
\rho_* \mathbb{Q}_\ell = \bigoplus_{j=0}^N \rho_* \mathbb{Q}_\ell^j.
\]

Then \( \rho_* \mathbb{Q}_\ell^j = \mathbb{Q}_\ell \), and for any \( \phi \neq 1 \), over the open set \( U \), we get the Kummer sheaf

\[
\rho_* \mathbb{Q}_\ell^\phi \mid U = \mathcal{L}_{\phi(\prod_{i=1}^n (X_1-\alpha_i)^{e_i} (\lambda - X_1), f)} \mid U.
\]

The behavior at the ramification points \( \{\alpha_1, \ldots, \alpha_n, \lambda, \infty\} \) depends on the nature of the integers \( e_1, \ldots, e_n, f \) modulo \( N \). The simplest choice is to assume that each \( e_i, f \) and \( e_1 + \ldots + e_n + f \) are relatively prime to \( N \). Then

\[
\rho_* \mathbb{Q}_\ell^j = j_* \mathcal{L}_{\phi(\prod_{i=1}^n (X_1-\alpha_i)^{e_i} (\lambda - X_1), f)}
\]

where \( j : U \to \mathbb{P}^1 \) is the inclusion.
10.2. Curves with equations \( Y^N = \prod_{i=1}^{n} (X_1 - \alpha_i)^{\ell} (\lambda - X_1)^f \) are sometimes called cycloelliptic. It is more natural to think of them as depending simultaneously on the parameters \( \{\alpha_1, \ldots, \alpha_n, \lambda\} \). As such, they define local systems on the space of parameters \( \{\alpha_1, \ldots, \alpha_n, \lambda\} \) minus the hyperplanes where two of these coordinates agree. Deligne and Mostow [17] made an extensive study of the corresponding monodromy groups. Also, the structure of the \( D \)-modules for these families is worked out by Holzapfel in part 2 of the book [35] (see also [34]). To our knowledge, the \( \ell \)-adic (or \( p \)-adic) story of these is only partially available. Lei Fu studies \( \ell \)-adic analogs of GKZ hypergeometric systems in [27]. Katz’s theory is for local systems in one variable, i.e., on the line. This explains the singling out of the distinguished parameter \( \lambda \).

11. EXAMPLES: RIGIDITY

11.1. Recall that the group of automorphisms of \( \mathbb{P}^1 \) that permute the set 0, 1, \( \infty \) has order 6 and is generated by \( \text{inv}(x) = 1/x \) and \( g(x) = 1 - x \).

**Lemma 11.1.** If \( a = b \),

\[
\text{inv}^* \mathcal{H}^{\text{can}}(\alpha_i; \beta_j) = \mathcal{H}^{\text{can}}(\beta_j; \alpha_i).
\]

**Proof.** We have

\[
\mathcal{H}^{\text{can}}(\alpha_i; \beta_j)[1] = \mathcal{H}^{\text{can}}(\alpha_i; \beta_1)[1] * \cdots * \mathcal{H}^{\text{can}}(\alpha_1; \beta_0)[1].
\]

Since \( \text{inv} \) is an automorphism of the algebraic group \( G_m \), it commutes with this convolution, so it suffices to prove the lemma when \( a = 1 \). The equality clearly holds over \( U = G_m - \{1\} \) geometrically (i.e., over an algebraically closed field) since both sides have the same monodromy at 0, 1, \( \infty \). By rigidity, we get

\[
\text{inv}^* \mathcal{H}^{\text{can}}(\alpha_1 \beta_1) = \mathcal{H}^{\text{can}}(\beta_1; \alpha_1) \otimes C^{\text{deg}}
\]

for a constant \( C \in \mathbb{Q}_\ell \). We see that \( C = 1 \) by comparing the trace of Frobenius of both sides and using the elementary identity \( J(\mu, \nu) = \nu(-1)J(\bar{\mu} \bar{\nu}, \nu) \) (see equation (1) in section (8.4)). Since the equality holds over \( U \) it holds over all \( \mathbb{P}^1 \) by applying \( j_* \) for \( j : U \to \mathbb{P}^1 \).

The situation for \( g(x) = 1 - x \) is more complicated. This is not an automorphism of the group \( G_m \), and it does not commute with convolution.

**Lemma 11.2.** Let \( g(x) = 1 - x \). Then over the open set \( U = G_m - \{1\} \) there is an isomorphism

\[
g^* \mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2) \otimes \mathcal{L}_{\alpha_2}(x-1) \cong \mathcal{H}^{\text{can}}(\beta_1 \beta_2 \bar{\alpha}_1 \bar{\alpha}_2, 1; \beta_1 \bar{\alpha}_2, \beta_2 \bar{\alpha}_2) \otimes C^{\text{deg}}
\]

for an explicitly computable \( C \in \mathbb{Q}_\ell \).

**Proof.** This argument assumes semisimple monodromy. Both sides have monodromy

\[
\begin{pmatrix}
\beta_1 \beta_2 \bar{\alpha}_1 \bar{\alpha}_2 & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \alpha_1 \bar{\alpha}_2 \end{pmatrix}, \quad \begin{pmatrix} \beta_1 \alpha_2 & 0 \\ 0 & \beta_2 \alpha_2 \end{pmatrix}
\]

at 0, 1, \( \infty \) respectively. By rigidity, we have an isomorphism as above, with the constant \( C \) to be computed. This is done by evaluating the trace of Frobenius of both sides at 1/2. By lemma (8.1) the traces of Frobenius at 1/2 for \( g^* \mathcal{H}^{\text{can}}(\alpha_1, \alpha_2; \beta_1, \beta_2) \otimes \mathcal{L}_{\alpha_2}(x-1) \) and \( \mathcal{H}^{\text{can}}(\beta_1 \beta_2 \bar{\alpha}_1 \bar{\alpha}_2, 1; \beta_1 \bar{\alpha}_2, \beta_2 \bar{\alpha}_2) \) are respectively

\[
A \alpha_2 (1/2) \bar{\alpha}_2 (-1/2) \sum_{t_1} \lambda(t_1) \mu(1 - t_1) \nu(1/2 - t_1), \quad \lambda = \alpha_1 \beta_2^{-1}, \mu = \beta_1 \alpha_1^{-1}, \nu = \beta_2 \alpha_2^{-1}
\]

\[
B \sum_{u_1} \mu(u_1) \lambda(1 - u_1) \nu(1/2 - u_1)
\]
where $A, B$ are computed constants involving Jacobi sums (lemma (8.1)). Replacing $t_1 \mapsto -t_1$ in the first formula gives

$$A \alpha_1 (-1) \sum_{t_1} \lambda(t_1) \mu(t_1 + 1) \nu(-1/2 - t_1),$$

then replacing $t_1 \mapsto u_1 - 1$ gives $C = A \beta_2(-1)/B$ times the second sum.

If $\alpha_2 \neq 1$, we cannot assert an isomorphism over all of $G_m$. The right-hand side has a one-dimensional stalk at 1 whereas $L_{\alpha_2(x-1)}$ has a zero-dimensional stalk there. The map $g$ exchanges the singular points 0, 1, but we cannot use it to compute the Frobenius action on $H^{\can}(\alpha_1, \alpha_2; \beta_1, \beta_2) \circ 0$ from the action of Frobenius on $H^{\can}(\beta_1 \beta_2 \bar{a}_1 \bar{a}_2, 1; \beta_1 \bar{a}_1, \beta_2 \bar{a}_2)_1$. Recall that we view these sheaves as living on $\mathbb{P}^1$, extending via $j_*$ where $j : U \to \mathbb{P}^1$. However, it will work if $\alpha_2 = 1$.

**Corollary 11.3.** There is an isomorphism on all of $\mathbb{P}^1$:

$$g^* H^{\can}(\alpha_1, 1; \beta_1, \beta_2) \cong H^{\can}(\beta_1 \beta_2 \bar{a}_1 \bar{a}_2, 1; \alpha_1, \beta_2 \bar{a}_2)_0.$$

for an explicitly computable $C \in \bar{Q}_t$.

**Proof.** The map $g$ is an automorphism of $U$ and it extends to an automorphism of $\mathbb{P}^1$. Over $U$ we have proved the isomorphism, so we can apply $j_*$ to the equation. Note that this argument would fail if we had the additional factor $L_{\alpha_2(x-1)}$ because $j_*$ does not commute with tensor product in general ($\otimes C^{\deg}$ is not a problem). 

11.2. Here is an example of a quadratic transformation. Let $\varepsilon$ be the Legendre character of $F_q^*$ ($q$ is odd; $\varepsilon^2 = 1, \varepsilon \neq 1$). We let $\beta_1, \beta_2$ be characters of $F_q^*$ such that none of $\beta_1^2, \beta_2^2, \beta_1 \beta_2 \varepsilon$ is 1.

**Proposition 11.4.** Let $\mathcal{H} := H^{\can}(\varepsilon, 1; \beta_1, \beta_2)$, and $\mathcal{K} := H^{\can}(\beta_1 \beta_2 \varepsilon, 1; \beta_1^2, \beta_2^2)$. Then we have an equality of trace functions $t_{\mathcal{H}}(x^2) = C t_{\mathcal{K}}((x + 1)/2)$ for an explicitly computable $C \in \bar{Q}_t$.

**Proof.** The sheaf $\mathcal{H} := H^{\can}(\varepsilon, 1; \beta_1, \beta_2)$ has monodromy

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \beta_1 \beta_2 \varepsilon \end{pmatrix}, \begin{pmatrix} \beta_1^{-1} & 0 \\ 0 & \beta_2^{-1} \end{pmatrix}$$

respectively at $0, 1, \infty$. Let $[2] : G_m \to G_m$ be the map $t \mapsto t^2$. Then $[2]^* \mathcal{H}$ is lisse on $G_m$ except possibly at $0, -1, 1, \infty$, where the monodromies are respectively

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \beta_1 \beta_2 \varepsilon \end{pmatrix}, \begin{pmatrix} \beta_1^{-1} & 0 \\ 0 & \beta_2^{-1} \end{pmatrix}.$$

Since $\varepsilon^2 = 1$, the first one is the identity matrix, and thus $[2]^* \mathcal{H}$ is lisse at 0. Let $h : G_m \to G_m$ be the map $h(t) = (t + 1)/2$. Then $h^* \mathcal{K}$ has monodromy at $-1, 1, \infty$ given by the last three matrices above. By rigidity, we have a geometric isomorphism $h^* \mathcal{K} \cong [2]^* \mathcal{H}$, and since these are irreducible, they are isomorphic up to a twist $h^* \mathcal{K} \cong [2]^* \mathcal{H} \otimes \Phi$, for a character $\Phi$ of $\text{Gal}(F_{\bar{q}}/F_q)$ whose value on a Frobenius generator is a unit $C$ in $\bar{Q}_t$. We can compute this number in several ways. One way is to observe that $[2](1) = 1^2 = h(1)$. Since $[2]$ induces an isomorphism from the henselization of the local ring at 1 to the henselization of the local ring at 1, the Frobenius actions on the stalks $\mathcal{H}_1$ and $[2]^* \mathcal{H}_1$ coincide. On the other hand $h$ is an isomorphism, so the Frobenius actions on $\mathcal{K}_1$ and $(h^* \mathcal{K})_1$ coincide. Therefore, $C$ will be the ratio of the traces of Frobenius on $\mathcal{H}_1$ and $\mathcal{K}_1$. In general (see corollary 8.2), for $\mathcal{F} = H^{\can}(\alpha_1, \alpha_2; \beta_1, \beta_2)$

$$t_{\mathcal{F}}(1) = \text{Tr}(\text{Frob}_1|\mathcal{F}_1) = \frac{\beta_2 \bar{a}_2 (-1) J(\alpha_1 \beta_2, \beta_1 \bar{a}_1 \bar{a}_2)}{J(\alpha_1, \beta_1 \bar{a}_1) J(\alpha_2, \beta_2 \bar{a}_2)}.$$
We get an equality of hypergeometric character sums $t_{\mathcal{H}}(x^2) = Ct_{\mathcal{K}}((x + 1)/2)$, where $C = t_{\mathcal{H}}(1)/t_{\mathcal{K}}(1)$.

This identity is the analog of one of Kummer’s quadratic transformations. In the language of Riemann’s $P$-function, this is

$$P\left[\begin{array}{ccc}
0 & \infty & 1 \\
0 & a & 0 \\
\frac{1}{2} & b & \frac{1}{2} - a - b
\end{array}\right] = P\left[\begin{array}{ccc}
0 & \infty & 1 \\
0 & 2a & 0 \\
\frac{1}{2} - a - b & 2b & \frac{1}{2} - a - b
\end{array}\right].$$

Note that this is less precise than the corresponding formula given by Kummer. The above is an equality of hypergeometric character sums

$$B(4)_{2\mathbb{P}^1} \left[\begin{array}{cc}
A & B \\
\varepsilon & z^2
\end{array}\right] = \frac{g(\varepsilon B)g(\varepsilon A)}{g(\varepsilon)g(\varepsilon AB)} \ J_{2\mathbb{P}^1} \left[\begin{array}{cc}
A^2 & B^2 \\
\varepsilon AB & z^2 + 1/2
\end{array}\right].$$

Equivalently,

$$2\mathbb{P}^1 \left[\begin{array}{cc}
A & B \\
\varepsilon & z^2
\end{array}\right] = \frac{g(\varepsilon A\varepsilon B)}{g(\varepsilon, \varepsilon AB)} \ J_{2\mathbb{P}^1} \left[\begin{array}{cc}
A^2 & B^2 \\
\varepsilon AB & z^2 + 1/2
\end{array}\right].$$

To derive these identities, we also use the formula

$$g(\chi^2)g(\varepsilon) = \chi(4)g(\chi)g(\varepsilon\chi),$$

for any character $\chi$.

11.3. The general pattern of these identities is in the shape $t_{\mathcal{H}}(R(x)) = Ct_{\mathcal{K}}(R(x))$ for two (rigid) local systems $\mathcal{H}, \mathcal{K}$ and rational functions $R(x), S(x)$. Here $C \in \mathbb{Q}_\ell$, but in fact, the constant $C$ is an algebraic number. The dependence of $C$ on the various parameters appearing in $\mathcal{H}, \mathcal{K}$ is an interesting problem. In the previous example, the expression for $C = C(q; \varepsilon, \beta_1, \beta_2)$ in terms of Jacobi sum shows that, in an appropriate sense,

1. For a fixed prime $p$, $C(p^\varepsilon; \varepsilon, \beta_1, \beta_2)$ is a $p$-adic analytic function of the $\beta_1, \beta_2$. This follows from the Gross-Koblitz formula for Gauss sums, [33].

2. For fixed $\beta_1, \beta_2$, $C(q; \varepsilon, \beta_1, \beta_2)$ defines a Hecke character (an automorphic form for $\text{GL}_1$) of a cyclotomic field, [63], [64].

12. Examples: Arithmetic Triangle Groups

The basic idea here is based on the following observation: Let $X_1 := \Gamma_1\setminus \mathbb{H}^+$, $X_2 := \Gamma_2\setminus \mathbb{H}^+$ be the Riemann surfaces obtained as quotients of the suitably compactified complex upper half plane by triangle groups $\Gamma_1, \Gamma_2$, respectively. Further assuming $\Gamma_1 \subset \Gamma_2$, we obtain a covering of corresponding Riemann surfaces $X_1 \to X_2$. There is a hypergeometric DE attached to a triangle group: the DE belonging to the Schwarz uniformization. The Schwarzian differential equations pull back under coverings. This is slightly complicated by the fact that the Schwarzian DE is a third order equation for the ratio $y_1/y_2$ of a second order DE which is only well-defined up to a twist. This means that extra factors can occur in the formulae for the second order equations.

We say a triangle group $\Gamma \subset \text{SL}_2(\mathbb{R})$ is arithmetic if it arises from a quaternion algebra over a totally real number field. These have been classified by Takeuchi, see [59]. A vast generalization appears in the work of Deligne and Mostow in [17].
Here we give some examples arising from arithmetic triangle groups.

**Example 12.1** (A cubic formula from the groups \((2, 4, 8)\) and \((2, 3, 8)\).). We have the cubic transformation between the hypergeometric functions:

\[
2F1\left[\frac{\frac{17}{48}}{2}, \frac{x(x-9)2}{(x+3)^3}; x \right] = \left(1 + \frac{x}{3}\right)^{1/16} 2F1\left[\frac{\frac{3}{16}}{2}; x \right].
\]

In the language of Riemann’s \(P\)-function, this is

\[
\left(\frac{3}{x+3}\right)^{1/16} P\left[\begin{array}{ccc} 0 & \infty & 1 \\ 1/3 & 1/4 & x \\ \frac{1}{12} & 1/16 & \frac{1}{4} \end{array}\right] = P\left[\begin{array}{ccc} 0 & \infty & 1 \\ 1/2 & 1/16 & x \end{array}\right].
\]

Set \(t = \frac{x(x-9)2}{(x+3)^3}\). Then the values of \(x\) for \(t = 0, 1, \infty\), are as follows:

\[
\begin{array}{c|c|c|c}
  t & 0 & 1 & \infty \\
  x & 0, 9, 9 & 1, 1, \infty & -3, -3, -3 \\
\end{array}
\]

In terms of Katz’s hypergeometric sheaves and finite field hypergeometric functions, we have the following results. For a given prime \(p \equiv 1 \mod 48\), let \(\eta\) be any primitive character of \(\mathbb{F}_p^*\) of order 48. Then we have

\[
t_H\left(\frac{x(x-9)2}{(x+3)^3}\right) = t_K(x),
\]

where \(H := H^{can}(\varepsilon, 1; \eta, \eta^{17})\), and \(K := K^{can}(\varepsilon, 1; \eta, \eta^0) \otimes (K^{can}(1; \eta^3) \otimes L_{\eta^3, 1-x/3})\). Let \(f(z) = z(z-9)^2/(z+3)^3\). For any \(z \in \mathbb{F}_p\) with \(f(z) \neq 0, 1, \) and \(\infty\), we have

\[
2F1\left[\eta \eta^{17} \varepsilon; f(z)\right] = \eta^3 (1 + z/3) 2F1\left[\eta^3 \eta^0 \varepsilon; z\right].
\]

The sheaf \(H\) has monodromies

\[
\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \eta^6 \end{pmatrix}, \begin{pmatrix} \eta & 0 \\ 0 & \eta^{17} \end{pmatrix}
\]

at \(t = 0, 1\) and \(\infty\), respectively. Let \(g(x) = x(x-9)^2/(x+3)^3\). Then \(g^*H\) has monodromies as follows:

\[
\begin{array}{c|c|c|c|c}
  x & 0 & 9 & 1 & \infty \\
  \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \eta^3 \\ 0 & \eta^6 \end{pmatrix} & \begin{pmatrix} 1 & 0 \eta^{12} \\ 0 & \eta^9 \end{pmatrix} \\
\end{array}
\]

Therefore, \(g^*H \otimes (H^{can}(1; \eta^3) \otimes L_{\eta^3, x/(x+3)})\) and \(K^{can}(\varepsilon, 1; \eta, \eta^3)\) have the monodromies

\[
\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \eta^3 \\ 0 & \eta^0 \end{pmatrix}, \begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^9 \end{pmatrix}
\]

at \(0, 1, \) and \(\infty\), respectively. This give us the identity between the traces of Frobenius and thus the finite hypergeometric functions.
Example 12.2. [31, Entry (116)] Goursat showed the following cubic transformation of hypergeometric functions

\[ 2F_1 \left[ \begin{array}{c}
0 \\
2a + \frac{1}{3}
\end{array} \right| \frac{27x(1-x^2)}{(1+3x)^3} = (1+3x)^3 2F_1 \left[ \begin{array}{c}
3a \\
2a + \frac{1}{3}
\end{array} \right| x. \]

When \( a = \frac{2n-1}{2n} \) for a positive integer \( n \), the function \( f(x) \) gives the covering map from the curve associated to the arithmetic triangle group \((2,6n,12n)\) to the curve associated to the arithmetic triangle group \((2,3,12n)\).

In the language of Riemann’s \( P \)-function, this is

\[
\left( \frac{1}{1+3x} \right)^{3a} P \left[ \begin{array}{c}
0 \\
\frac{1}{6} - 2a
\end{array} \right| \frac{27x(1-x)^2}{(1+3x)^2} = P \left[ \begin{array}{c}
0 \\
\frac{1}{6} - 2a
\end{array} \right| \frac{1}{3} - 4a
\]

Set \( f(x) = 27\frac{x(1-x)^2}{(1+3x)^2} \). Then the values of \( x \) for \( f(x) = 0, 1, \infty \), are as follows:

| \( f(x) \) | \( x \) |
|---|---|
| 0 | 0, 1, 1 |
| 1 | 1/9, 1/9, \infty |
| \( \infty \) | \(-1/3, -1/3, -1/3\) |

In terms of hypergeometric sheaves and finite field hypergeometric functions, we have the following results. For a given prime \( p \equiv 1 \mod 6 \), let \( \eta \) be any primitive character of \( \mathbb{F}_p^* \) of order 6, and \( \alpha \) be any character with \( \alpha^6 \neq 1 \). Then we have

\[ t_H(f(x)) = t_K(x), \]

where \( H := H^{\text{can}}(\alpha^2; \eta, 1; \alpha, \alpha^2 \eta) \), and \( K := K^{\text{can}}(\alpha^2; \eta, 1; \alpha, \alpha^3 \zeta) \otimes (K^{\text{can}}(1; \alpha^3) \otimes L_{\alpha^3, 1 - 3x}) \). For any \( z \in \mathbb{F}_p^* \) with \( f(z) \neq 0, 1, \) and \( \infty \), we have

\[
\left. \begin{array}{c}
\eta^3 (f(z)) \end{array} \right| 2F_1 \left[ \begin{array}{c}
\alpha^3 \\
\alpha^2 \eta
\end{array} \right| z} = \left. \begin{array}{c}
\alpha \\
\alpha^2 \eta
\end{array} \right| 2F_1 \left[ \begin{array}{c}
\varepsilon \alpha^3 \\
\varepsilon \alpha \eta^2
\end{array} \right| f(z). \]

Example 12.3. [62, Equation (28)] Similarly, we have the finite field version of the following degree-10 algebraic transformation:

\[
\left( 1 - 57x - 1029x^2 + 50421x^3 \right)^{1/28} 2F_1 \left[ \begin{array}{c}
\frac{5}{84} \\
\frac{19}{7}
\end{array} \right| 27x \right]
\]

\[ = 2F_1 \left[ \begin{array}{c}
\frac{29}{84} \\
\frac{6}{7}
\end{array} \right| \frac{-27x^2(1 - 27x)(3 - 49x)^7}{4(1 - 57x - 1029x^2 + 50421x^3)^5} \right]
\]

For a given prime \( p \equiv 1 \mod 84 \), let \( \eta \) be any primitive character of \( \mathbb{F}_p^* \) of order 84. Let

\[ f(z) = 1 - 57z - 1029z^2 + 50421z^3, \quad g(z) = -z^2(1 - 27z)(3 - 49z)^7. \]

For any \( z \in \mathbb{F}_p^* \) with \( f(z)/g(z) \neq 0, 1, \) and \( \infty \), we have

\[
\left. \begin{array}{c}
\eta^{10} (f(z)) \end{array} \right| 2F_1 \left[ \begin{array}{c}
\eta \\
\eta^{29}
\end{array} \right| \frac{27 g(z)}{4 f(z)^3}. \]
13. EXAMPLES: ELLIPTIC CURVES

(See [57], [58]). We consider the differential equations satisfied by the periods of families of elliptic curves. Let

$$y^2 = 4x^3 - g_2x - g_3$$

be the Weierstrass family of elliptic curves. $\Delta = g_2^3 - 27g_3^2$ the discriminant. When $D \neq 0$ this is an elliptic curve. One has the differentials of the first and second kind

$$\omega = \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \quad \eta = \frac{xdx}{\sqrt{4x^3 - g_2x - g_3}} = \frac{x dx}{y}.$$ 

These generate the deRham cohomology of the curve. Recall

**Proposition 13.1.** ([15, Proposition 2.5]) Above $S = \text{Spec}(\mathbb{Z}[2^{-1}, 3^{-1}])$ there is a moduli scheme for pairs $(E, \omega)$ consisting of a curve of genus 1 together with an invariant invertible differential one form. This scheme is

$$\tilde{M} = \text{Spec}(\mathbb{Z}[2^{-1}, 3^{-1}])[g_2, g_3]$$

with universal curve (in nonhomogeneous coordinates) $y^2 = 4x^3 - g_2x - g_3$ with invariant differential $\omega = dx/y$.

In Deligne’s formulaire, singular curves are permitted. Precisely, a curve of genus 1 over a base $T$ is a proper and flat morphism of finite presentation $p : E \to T$ together with a section $e$ contained in the open subset of smoothness of $p$ whose geometric fibers are reduced irreducible curves of arithmetic genus 1. The fibers are of three types:

1. An elliptic curve, i.e. proper, smooth connected of genus 1;
2. a projective line in which two distinct points have been identified (cubic in $\mathbb{P}^2$ with an ordinary double point);
3. a projective line in which two infinitely near points have been identified (cubic in $\mathbb{P}^2$ with a cusp).

We let $M \subset \tilde{M}$ be the open set where $\Delta \neq 0$, and $f : E \to M$ be the universal Weierstrass elliptic curve. $f$ is a smooth morphism. We get the deRham cohomology sheaves on $M$,

$$H^i_{\text{DR}}(E/M) := \mathbb{R}^i f_* \Omega^\bullet_{E/M}.$$ 

There is a filtration

$$0 \longrightarrow f_* \Omega^1_{E/M} \longrightarrow H^1_{\text{DR}}(E/M) \longrightarrow R^1 f_* \mathcal{O}_E \longrightarrow 0.$$ 

Locally on $M$, $H^1_{\text{DR}}(E/M)$ is spanned as an $\mathcal{O}_M$-module by $\omega, \eta$, with the submodule $f_* \Omega^1_{E/M}$ spanned by $\omega$. There is an integrable Gauss-Manin connection

$$\nabla : H^1_{\text{DR}}(E/M) \to \Omega^1_{M/S} \otimes_{\mathcal{O}_M} H^1_{\text{DR}}(E/M).$$

which has regular singularities at infinity. Note that $\tilde{M} - M$ is not a divisor with normal crossings, but we can compactify $M$ in such a way that the divisor at infinity is a normal crossings divisor. The corresponding morphism of analytic spaces is denoted $f^\text{an} : E^\text{an} \to M^\text{an}$. We have

$$H^1_{\text{DR}}(E^\text{an}/M^\text{an}) = R^1 f^\text{an}_* \mathcal{C} \otimes_{\mathcal{C}} \mathcal{O}_{M^\text{an}}, \quad R^1 f^\text{an}_* \mathcal{C} = \text{Ker}(\nabla^\text{an}).$$

When clear in context, we omit the superscript $\text{an}$ for a morphism of analytic spaces.

We can describe this Gauss-Manin connection explicitly as follows. Let $U$ be an analytic set homeomorphic with a unit disk in the complex $u$-plane. We consider a family $f : E \to U$ of elliptic curves in Weierstrass form with holomorphic functions $g_2(u), g_3(u)$ with $\Delta(u) = g_2(u)^3 - 27g_3(u)^2 \neq 0$ at all points $u \in U$. This family can be regarded as the base-change the universal $E^\text{an} \to M^\text{an}$ by a morphism $U \to M^\text{an}$. 


At any given point \( u_0 \in U \), \( H^1(E_{u_0}, C) = (R^1 f_* C)_{u_0} \), and the elements can be represented by differentials of the first and second kind modulo exact differentials. Thus we can represent the generators of this two-dimensional vector space by differential forms \( \omega = dx/y, \eta = xdx/y \). The Gauss-Manin connection gives a lifting of the derivation \( \partial/\partial u \) to an endomorphisms of the sheaf \( H^1_{DR}(E/U) \). Concretely we extend the action of differentiation by \( u \) to a derivation \( D_u \) of the ring \( \mathbb{O}(U)[x,y]/(y^2 - 4x^3 + g_2(u)x + g_3(u)) \) by setting \( D_u(x) = 0 \). In this way we get a differential equation

\[
\frac{d}{du} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{24\Delta} \begin{bmatrix} -2\Delta' & 18\delta \\ -3g_2\delta & 2\Delta' \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad \Delta' = \frac{d\Delta}{du}, \quad \delta = 3g_3 \frac{dg_2}{du} - 2g_2 \frac{dg_3}{du}.
\]

Since \( E/U \) is topologically trivial, we can choose a 1-cycle \( 0 \neq \gamma \in H_1(E_{u_0}, \mathbb{Z}) \) which gives a section \( \gamma(u) \) of the sheaf \( H_1(E, \mathbb{Z}) \sim \mathbb{Z}^2 \) on \( U \). Then the periods \( h_1(u) = \int_{\gamma(u)} \omega, h_2(u) = \int_{\gamma(u)} \eta \) give a basis of local holomorphic solutions to the above differential equation.

**Example.** Elliptic curves with \( j \)-invariant = \( j \). Recall that if \( j \neq 0, 1 \), the most general solution to the equation \( g_2^3/\Delta = j \) in any field \( k \) of characteristic \( \neq 2, 3 \) is of the form \( g_2 = t\xi^2, g_3 = t\xi^3, t = 27j(j - 1)^{-1}, \xi \in k^* \). (see [38, Lemma 1]). We can apply this to \( k = \mathbb{Q}(j) \), with \( g_2 = g_3 = t \) we get a family of elliptic curves over \( \mathbb{P}_j^1 \setminus \{0, 1, \infty\} \) with \( j \)-invariant = \( j \). We can call this a universal elliptic curve, although strictly speaking it does not represent the obvious functor (for this we need the modular stack or orbifold quotient \( \text{SL}_2(\mathbb{Z})\backslash \Delta \)). Nonetheless we will consider this family \( E \to \mathbb{P}_j^1 \setminus \{0, 1, \infty\} \) as a scheme over \( A = \mathbb{Z}[2^{-1}, 3^{-1}][j, (j(j - 1))^{-1}] \). That is

\[
E = \text{Proj} \left( A[x, y, z]/(y^2z - 4x^3 + txz^2 + tz^3) \right), \quad t = 27j(j - 1)^{-1}.
\]

See [56]. Consider the hypergeometric differential equation

\[
\frac{d^2\omega}{dx^2} + \frac{1}{x} \frac{d\omega}{dx} + \frac{(31/144)x - (1/36)}{x^2(x - 1)^2} \omega = 0.
\]

In terms of Riemann’s P-function, this is

\[
P \begin{bmatrix} 0 & \infty & 1 \\ -1/6 & 0 & 1/4 \\ 1/6 & 0 & 3/4 \end{bmatrix} = x^{-1/6}(x - 1)^{1/4} P \begin{bmatrix} 0 & \infty & 1 \\ 0/12 & 0 & x \\ 1/3 & 1/12 & 1/2 \end{bmatrix}.
\]

As was known classically, this is the differential equation for the periods in this family of elliptic curves. Moreover, the monodromy matrices around the singular points \( x = \infty, 0, 1 \) are respectively

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

These generate \( \text{SL}_2(\mathbb{Z}) \). In fact, if \( y_1(x), y_2(x) \) are two linearly independent (multivalued) holomorphic solutions to the differential equation, the ratio \( y_1(x)/y_2(x) \) is the Schwarzian function for this situation, i.e., it is essentially the inverse of the \( j \)-function, \( j : \Delta \to \mathbb{C} \). A classical reference: [26]. See Stiller’s papers for a modern exposition.

**Example.** Modular families. For a congruence subgroup \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) we have a modular curve \( X_\Gamma \) whose analytic space is \( \Gamma \backslash \Delta \). When \( \Gamma \) has no nontrivial elements of finite order, there is a universal elliptic curve \( E_\Gamma \to X_\Gamma \). If \( \Gamma_1 \subset \Gamma_2 \) are two such subgroups, there is a morphism \( u : X_{\Gamma_1} \to X_{\Gamma_2} \), and a map

\[
\varphi : u^* E_{\Gamma_2} \to E_{\Gamma_1}
\]

of elliptic curves over \( X_{\Gamma_1} \). This is an isogeny, and therefore it induces an isomorphism of the \( \mathcal{D} \)-modules and the \( \ell \)-adic representations (since we ignore torsion). Thus we obtain transformations of the corresponding motivic sheaves. If \( \Gamma \) has elliptic points, the situation is more complicated. We do not have
universal families. For instance, consider \( \Gamma(2) \subset \Gamma(1) = \text{SL}_2(\mathbb{Z}) \). The covering of modular curves is 
\( X(2) = \mathbb{P}^1_\lambda \to X(1) = \mathbb{P}^1_j \) given by

\[
    j = \frac{27\lambda^2(\lambda - 1)^2}{4(\lambda^2 - \lambda + 1)^3}.
\]

Pulling back the elliptic curve with \( j \)-invariant \( j \) by this map does not give the universal Legendre curve \( y^2 = x(1-x)(1-\lambda x) \). The corresponding transformation of hypergeometric equations has a Kummer twist:

\[
    _2F_1 \left[ \frac{1}{12}, \frac{5}{12}; \frac{27\lambda^2(\lambda - 1)^2}{4(\lambda^2 - \lambda + 1)^3} \right] = (1 - \lambda + \lambda^2)^{1/4} _2F_1 \left[ \frac{1}{2}, \frac{1}{2}; \lambda \right].
\]

**Example.** The AGM transform. See [14]. Gauss discovered the following transformation of elliptic integrals during his investigations of the arithmetic geometric mean (AGM). Let

\[
    F(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2} _2F_1 \left[ \frac{1}{2}, \frac{1}{2}; k^2 \right]
\]

then

\[
    F \left( \frac{2\sqrt{k}}{1+k} \right) = (1+k)F(k).
\]

Let \( J_m \) be the family of curves \( y^2 = (1-x^2)(1-m^2x^2) \), and define

\[
    m = \frac{2\sqrt{k}}{1+k}, \quad x = \frac{(1+k)z}{1+kz^2}, \quad y = \frac{1-kz^2}{(1+kz^2)^2}w := Cw,
\]

then the above equation becomes

\[
    C^2(w^2 = (1-z^2)(1-kz^2)).
\]

Moreover,

\[
    \frac{dx}{y} = (1+k)\frac{dz}{w}.
\]

This can be understood as follows (see [18]). Let

\[
    M_4 = \text{Spec}\mathbb{Z}[i, 1/2, \sigma, (\sigma(\sigma^4 - 1))^{-1}].
\]

This is the moduli scheme for \( \Gamma(4) \subset \text{SL}_2(\mathbb{Z}) \).

The universal elliptic curve for this is

\[
    E_\sigma : y^2 = x(x-1)(x-\lambda), \quad \lambda = (\sigma + \sigma^{-1})^2/4.
\]

This curve is isomorphic with the Jacobi quartic

\[
    C_\sigma : y^2 = (1-\sigma^2x^2)(1-x^2/\sigma^2).
\]

via the change of variables

\[
    X = \frac{\sigma^2 + 1}{2\sigma^2} \cdot \frac{x - \sigma}{x - 1/\sigma}, \quad Y = \frac{\sigma^4 - 1}{4\sigma^3} \cdot \frac{y}{(x - 1/\sigma)^2}
\]

(see [55]).

A rescaling \( x \mapsto \sigma x \) gives the equivalent curve \( y^2 = (1-\sigma^4x^2)(1-x^2) \), which shows that we can view this curve as the pull-back via the projection

\[
    M_4 \to M_{2,4} := \text{Spec}\mathbb{Z}[1/2, k, (k(k^2 - 1))^{-1}], \quad \sigma \mapsto \sigma^2 = k
\]
of the quartic \( J_k : w^2 = (1 - z^2)(1 - k^2 z^2) \) on \( M_{2,4} \). \( M_{2,4} \) is the moduli scheme for the group

\[
\Gamma_{2,4} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod 4, c \equiv 0 \mod 4, b \equiv 0 \mod 2 \right\}
\]

Clearly \( \Gamma(4) \subset \Gamma_{2,4} \subset \Gamma(2) \). Here \( \Gamma(2) \) is the subgroup of \( \Gamma(2) \) defined by \( a \equiv d \equiv 1 \mod 4 \). Then \( \Gamma(2) \equiv \Gamma(2)/\pm 1 \). The quotient \( \Gamma(2) / \Gamma(4) \) is the \( \lambda \)-line. We have \( \Gamma(2) / \Gamma(4) \cong \mathbb{Z} / 2 \times \mathbb{Z} / 2 \), and \( \Gamma_{2,4} / \Gamma(4) \cong \mathbb{Z} / 2 \). The map \( \Gamma(4) / \Gamma \rightarrow \Gamma(2) / \Gamma \) is defined by

\[
\lambda = (\sigma + \sigma^{-1})^2 / 4.
\]

Then \( k(\tau) \) is a Hauptmodul for \( \Gamma_{2,4} \). The transformation for the AGM is defined by the map \( k(\tau) \mapsto k(2\tau) \). More precisely, it is the correspondance defined by the algebraic relation relating \( k(\tau), k(2\tau) \), viz.,

\[
k(\tau) = \frac{2 \sqrt{k(2\tau)}}{1 + k(2\tau)}.
\]

Define two function \( p(\sigma) = \sigma^2, q(\sigma) = 2\sigma / (1 + \sigma^2) \) both mapping \( M_4 \rightarrow M_{2,4} \). Then

\[
X = \frac{(1 + \sigma^2)x}{1 + \sigma^2x^2}, \quad Y = \frac{(1 - \sigma^2x^2)y}{(1 + \sigma^2x^2)^2}
\]

defines an isogeny \( p^* J_k \cong E_\sigma \rightarrow q^* J_k \).

**Example.** The Borwein’s cubic transform.

\[
\binom{1}{2} \binom{x}{1} : 1 - x^3 = \frac{3}{1 + 2x} \binom{1}{2} \binom{1 - x}{1 + 2x^{-1}}^3,
\]

proved by Borwein and Borwein \([12], [13]\) as a cubic analogue of Gauss’ quadratic AGM. Just as Gauss’s formula relates \( \tau \) with \( 2\tau \), Borweins’ formula relates \( \tau \) with \( 3\tau \).

A finite-field analog of this was proved in \([29]\):

**Theorem 13.2.** For \( p \equiv 1 \mod 3 \) prime, and let \( \omega \) be a primitive cube root of unity and let \( \eta_3 \) be a primitive cubic character in \( \mathbb{F}_p \). If \( \lambda \in \mathbb{F}_p \) satisfies \( 1 + 2\lambda \neq 0 \), then

\[
\binom{\eta_3}{\eta_3^2} \binom{x}{\varepsilon} : 1 - \lambda^3 = \frac{2}{1 + 2\lambda} \binom{\eta_3}{\eta_3^2} \binom{1 - \lambda}{1 + 2\lambda}^3.
\]

This has the geometric meaning as follows. Let \( M_3^0 \) be the coarse moduli space for \( \Gamma_0(3) \). This group is an arithmetic triangle group with two cusps and one elliptic point of order 3. The elliptic point means that there is no universal elliptic curve for \( \Gamma_0(3) \), but the family of curves

\[
C : y^2 + xy + (1/27)t^2 = x^3
\]
is a curve with a rational 3-torsion point, namely \((0, 0)\), which has the correct \(j\)-invariant. Here \(t\) is a Hauptmodul for \(\Gamma_0(3)\). Let \(M_3\) be the moduli space for \(\Gamma_3\), the principal congruence subgroup of level 3. The projection \(M_3 \to M_3^0\) is \(z \mapsto z^3\) in suitable coordinates. Let

\[p(z) = 1 - z^3, \quad q(z) = \left(\frac{1 - z}{1 + 2z}\right)^3\]

Then, there is an isogeny \(q^*C \equiv p^*C\). In fact, dividing the left-hand side by the subgroup generated by \((0, 0)\) gives the right-hand side. The map

\[z(\tau) \mapsto \frac{1 - z}{1 + 2z}(3\tau)\]

is induced by the Atkin-Lehner involution \(W_3\).

This transformation can be deduced from another transformation of a 2-variable hypergeometric, as will be explained in the next section.

14. The Picard Family of Curves

We finish with one example of a two-variable Appell-Lauricella equation. The treatment here is only a sketch, with full details to appear elsewhere. Unexplained notation is taken from the quoted papers.

The family of quartic curves

\[y^3 = x(1 - x)(1 - \lambda x)(1 - \mu x)\]

depending on parameters \(\lambda, \mu\) with \(\lambda \neq 0, 1, \mu \neq 0, 1, \lambda \neq \mu\) is a family of genus 3 curves whose Jacobian varieties have endomorphism rings containing \(\mathbb{Z}[\omega]\), where \(\omega = \exp(2\pi i/3)\). These were first studied by Picard, see [52], [53], [35], [34], [54]. The space

\[\mathbb{C}^2 - \{\lambda = 0, \lambda = 1, \mu = 0, \mu = 1, \lambda = \mu\} = \Gamma\backslash \mathbb{B}_2\]

is the set of \(\mathbb{C}\)-points of the Shimura variety of PEL type representing principally polarized abelian 3-folds with an embedding of \(\mathbb{Z}[\omega]\) into the endomorphism algebra. Here \(\mathbb{B}_2 \subset \mathbb{C}^2\) is the open unit ball, and \(\Gamma \subset SU(2, 1; \mathbb{Z}[\omega])\) is the congruence subgroup of the points of unitary group with signature \((2, 1)\) with coordinates in the Eisenstein integers which satisfy

\[\gamma \equiv 1 \mod (1 - \omega)\]

The Jacobians of the Picard family form the universal family of abelian varieties over this space.

The periods of integrals in this family satisfy an Appell-Lauricella differential equation ([3], [5], [4]). Remarkably, this was shown by Picard, who also computed the monodromy, in effect discovering the Picard-Lefschetz formula. He observed that the monodromy preserved a Hermitian form of signature \((2, 1)\).

In [49], Koike and Shiga studied Appell’s \(F_1\)-hypergeometric function in two variables to establish a new three-term arithmetic geometric mean result (AGM), related to Picard modular forms.

Let \(x, y \in \mathbb{C}\), and let \(\omega\) be a primitive cubic root of unity. Then
Theorem 14.1.  

\( F_1 \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 | 1 - x^3, 1 - y^3 \right] \)

\[ \frac{3}{1 + x + y} F_1 \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \left| \frac{1 + \omega x + \omega^2 y}{1 + x + y} \right. \right. \]  

The following is a finite-field analog of the formula of Koike and Shiga, proved in [24].

Theorem 14.2. Let \( p \equiv 1 \pmod{3} \) be prime, let \( \omega \) be a primitive cubic root of unity, and let \( \eta_3 \) be a primitive cubic character in \( \overline{\mathbb{F}}_p^\times \). If \( \lambda, \mu \in \mathbb{F}_p \) satisfy \( 1 + \lambda + \mu \neq 0 \), then

\[ \mathbb{P}^2 \left[ \eta_3; \eta_3 \eta_3 \epsilon ; 1 - \lambda^3, 1 - \mu^3 \right] = \mathbb{P}^2 \left[ \eta_3; \eta_3 \eta_3 \epsilon ; \left( \frac{1 + \omega \lambda + \omega^2 \mu}{1 + \lambda + \mu} \right)^3, \left( \frac{1 + \omega^2 \lambda + \omega \mu}{1 + \lambda + \mu} \right)^3 \right]. \]

The key observation in [24] is that a truncation of the Appell-Lauricella series gives the Hasse invariant of the Picard curve. The equality of the finite field analog reduces to comparing the number of rational points of the two transforms of the Picard curves.

The main result of [49] can be formulated as follows: There is a diagram

\[
\begin{array}{c}
M_p \\
\downarrow p \\
M_{\sqrt{-3}}
\end{array}
\quad \begin{array}{c}
M_\sqrt{3} \\
\downarrow q \\
M_{\sqrt{-3}}
\end{array}
\]

where \( M_{\sqrt{-3}} \) is a compactification of the PEL Shimura variety attached to the group

\( \Gamma(\sqrt{-3}) = \{ \gamma \in \text{SU}(2, 1; \mathbb{Z}[[\omega]]) \mid \gamma \equiv 1 \pmod{\pi} = (1 - \omega) \}. \)

An open subset \( M_{\sqrt{-3}}^\circ \) of the \( \mathbb{C} \)-points of this is the quotient \( \Gamma(\sqrt{-3})\backslash \mathbb{B}_2 \). In fact \( M_{\sqrt{-3}} = \mathbb{P}^2 \) with coordinates \( \xi_0, \xi_1, \xi_2 \). We have \( \xi_\mu = \theta_\mu(u, v)^3 \) for certain explicit theta functions depending on \( (u, v) \in \mathbb{B}^2 \). The rational map \( \mathbb{P}^2 \to \mathbb{P}^2 \) given by \( (\lambda_0, \lambda_1, \lambda_2) \to (\lambda_0^3 = \xi_0, \lambda_3^3 = \xi_1, \lambda_2^3 = \xi_2) \) corresponds to a congruence subgroup

\( \Gamma(\theta) \subset \Gamma(\sqrt{-3}) \)

of index 9. That is, there is a compactification of \( \Gamma(\theta)\backslash \mathbb{B}_2 \) which is \( \mathbb{P}^2 \) with coordinates \( (\lambda_0, \lambda_1, \lambda_2) \). This is denoted \( M_\theta \) in the diagram above.

The Jacobians of the Picard curve

\( C(\xi) : y^3 = x^3 - \xi_0 - \xi_1(x - \xi_0)(x - \xi_1) \)

form the universal abelian variety \( A(\xi) \) over an open set \( M_{\sqrt{-3}}^\circ \subset M_{\sqrt{-3}} \). We can write this family as \( A(u, v) \) to emphasize its dependence on \( (u, v) \in \mathbb{B}_2 \).

One of the main results of [49] is

Theorem 14.3. There is an isogeny of degree 27, \( a : q^* A \to p^* A \) covering the map \( (u, v) \mapsto (\sqrt{-3}u, 3v) \) of \( \mathbb{B}_2 \).
In affine coordinates $x = \xi_1/\xi_0, y = \xi_2/\xi_0$ and $w = \lambda_1/\lambda_0, z = \lambda_2/\lambda_0$ the maps are given by

$$(x, y) = p(w, z) = (1 - w^3, 1 - z^3)$$

$$(x, y) = q(w, z) = \left(\frac{1 + \omega w + \omega^2 z}{1 + w + z}\right)^3, \left(\frac{1 + \omega^2 w + \omega z}{1 + w + z}\right)^3$$

This is their isogeny formula, which is deduced from transformation properties of theta functions. Since the Appell-Lauricella differential equation is the DE for the periods of the Picard family, this isogeny formula essentially proves 14.1. One should observe that the entire picture above should be valid as schemes over $\mathbb{Z}[[\omega, 1/3]]$ and for that reason, taking $\ell$-adic coefficients, one obtains theorem 14.2. For this one needs the theory of compactified PEL Shimura varieties over integer rings. Also note that this gives a transformation formula for the corresponding $p$-adic Appell-Lauricella, via the machinery of crystalline cohomology. This will be discussed on another occasion.

15. Appendix: Local systems

15.1. C-local systems and differential equations. If $X$ is a nonsingular algebraic variety over $\mathbb{C}$ we let $X^{an} = X(\mathbb{C})$ be the set of complex points with the classical topology. For simplicity, assume that $X^{an}$ is connected.

Recall the following dictionary: The following categories are equivalent:

- **D1.** Local systems of finite-dimensional $\mathbb{C}$-vector spaces $V$ on $X^{an}$.
- **D2.** Representations $\rho : \pi_1(X^{an}, x) \to GL_n(V)$ on finite-dimensional $\mathbb{C}$-vector spaces $V$.
- **D3.** Holomorphic integrable connections $\nabla : V^{an} \to \Omega^1_{X^{an}/\mathbb{C}} \otimes _{O^{an}_X} V^{an}$

where $V^{an}$ is a locally free $O^{an}_X$ sheaf of finite rank.

- **D4.** Integrable algebraic connections $\nabla : V \to \Omega^1_{X/\mathbb{C}} \otimes _{O_X} V$

where $V$ is a locally free $O_X$ sheaf of finite rank, and which have regular singular points “at infinity”.

Some comments:

1. The *morphisms* in each of these categories are the obvious ones.
2. The integrability condition is that the composed map (curvature)

$$\nabla : V \to \Omega^1_{X/\mathbb{C}} \otimes _{O_X} V \to \Omega^1_{X/\mathbb{C}} \otimes _{O_X} \Omega^1_{X/\mathbb{C}} \otimes _{O_X} V \to \Omega^2_{X/\mathbb{C}} \otimes _{O_X} V$$

is 0. We then get both algebraic and holomorphic deRham complexes

$$\Omega^*_{X/\mathbb{C}} \otimes _{O_X} V.$$  

In the analytic case, this deRham complex is a resolution of the sheaf $V$, by the holomorphic Poincaré lemma.

3. We call these connections differential equations. D3 is essentially due to Frobenius. D4 is called the Riemann-Hilbert correspondence.

4. Regular singular points means this: Let $\bar{X} \subset X$ be a compactification such that $\bar{X} - X = D$ is a divisor with normal crossings (exists by Hironaka’s theorem). Then there is a locally free sheaf $\tilde{V}$ on $\bar{X}$ extending $V$ and a connection

$$\nabla : \tilde{V} \to \Omega^1_{X/\mathbb{C}}(\log D) \otimes _{O_X} \tilde{V}$$
extending $\nabla$. When $\dim X = 1$ this is equivalent to Fuchs’ growth conditions at singular points of the differential equations.

5. In the language of $\mathcal{D}_X$-modules, connection with regular singular points $=$ regular holonomic $\mathcal{D}_X$-module which is coherent (hence locally free) as an $\mathcal{O}_X$-module.

The functors go like this:
1. $1 \Rightarrow 2$: $V \mapsto V_x$ which is a $\pi_1(X^{\text{an}}, x)$-module.
2. $1 \Rightarrow 3$: $V \mapsto \mathcal{V} = \mathcal{V} \otimes \mathcal{O}_{X^{\text{an}}}$ with connection $\nabla = 1 \otimes d$.
3. $4 \Rightarrow 3$: $\mathcal{V} \mapsto \mathcal{V} \otimes \mathcal{O}_X \mathcal{O}_{X^{\text{an}}} = \mathcal{V}^{\text{an}}$, with the obvious connection. A proof of the regularity theorem can be found in [19] and [43].

15.2. $\ell$-adic local systems. A reference: [20]. Fix a prime number $\ell$. In this section: scheme $=$ a separated noetherian scheme on which $\ell$ is invertible. We are interested in constructible $\mathbb{Q}_\ell$-sheaves on $X$, in particular, those that are lisse. In this section: the étale topology is understood.

An $\ell$-adic representation of a profinite group $\pi$ on a $\mathbb{Q}_\ell$-vector space $V$ is a homomorphism

$$
\sigma : \pi \to \text{GL}(V)
$$

such that there is a finite subextension $E/\mathbb{Q}_\ell$ and an $E$-structure $V_E$ on $V$ such that $\sigma$ factorizes in a continuous homomorphism $\pi \to \text{GL}(V_E)$.

Recall that a geometric point $\bar{x}$ of a scheme $X$ is a morphism of the spectrum of an algebraically closed field denoted $k(\bar{x})$. It is localized in $x \in X$ if its image is $x$.

If $X$ is connected and pointed by a geometric point $\bar{x}$, the functor

$$
\mathcal{F} \mapsto \text{the } \pi_1(X, \bar{x}) \text{-module } \mathcal{F}_{\bar{x}}
$$

is an equivalence of categories between the categories of

1. lisse $\mathbb{Q}_\ell$-sheaves on $X$;
2. $\ell$-adic representations of $\pi_1(X, \bar{x})$.

Here $\pi_1(X, \bar{x})$ is Grothendieck’s fundamental group. Especially if $X = \text{Spec}(k)$ is a field, the category of lisse $\mathbb{Q}_\ell$-sheaves on $X$ is equivalent to the category of $\ell$-adic representations of $\text{Gal}(k/k)$.

16. Appendix: Motivic Sheaves

For an introduction to the theory of motives, see André’s book [2]. Fix a field $k$. The idea is to construct a rigid $\otimes$-category of mixed motives $\text{MM}(k)_F$ with coefficients in $F$, a field of characteristic 0, together with realization functors into various cohomology theories (Betti, deRham, $\ell$-adic étale, $p$-adic crystalline). The pure objects should constitute a semisimple subcategory $\mathcal{M}(k)_F$ which is Grothendieck’s category of motives for numerical equivalence. For a smooth projective variety $X$, an idempotent $e \in \text{Corr}(X)_F$, the correspondence ring of $X$, and an integer $i$, we have an element $eh(X)(i) \in M(k)_F$. We think $h(X)$ of this as representing the cohomology of $X$, and the integer $i$ represents a Tate twist. The idempotent could be for instance a Künneth projector onto a factor $h^i(X)$. In any realization, this becomes (e.g., for étale cohomology) $eH_{et}(X \otimes \bar{k}, \overline{\mathbb{Q}}_l)(i)$.

What is usually constructed is a triangulated $\otimes$-category $\text{DM}(k)$. The realization functors then map to various derived categories (e.g., of $\overline{\mathbb{Q}}_l - \text{Gal}(\bar{k})$-vector spaces). One hopes for a $t$-structure on $\text{DM}(k)$ whose heart is $\text{MM}(k)_F$, and such that $\text{DM}(k) = D^b(\text{MM}(k)_F)$. The situation today is that there are various constructions of $\text{DM}(k)$ (Voevodsky, Hanamura, Levine) but the existence of a $t$-structure is a conjecture.

One extends this construction to motivic sheaves. Given a scheme $S$, there is a $\otimes$-triangulated category $\text{DA}(S)$ of motivic sheaves on $S$. For instance, given a morphism $f : X \to S$, one wishes for objects
There are at least two formalisms of motivic sheaves available, one closely related to Voevodsky’s triangulated category of mixed motives, one related to Nori’s motives, see Arapura’s paper [6]. See also the papers of Huber, [36], [37]. We will follow the exposition in Ayoub’s papers, to which we refer the reader for further information. This whole theory is built upon Voevodsky’s theory of triangulated categories of mixed motives, see [51].

16.1. Let $X$ be a noetherian scheme. There is a tensor triangulated category $\DA(X)$ whose objects will be called relative motives over the scheme $X$. We briefly recall its construction. Let $\Sm / X$ be the category of smooth $X$-schemes of finite type, endowed with the étale topology. We let $\Shv(\Sm / X)$ be the category of étale sheaves of $\Q$-vector spaces on $\Sm / X$. Given a smooth $X$-scheme $Y \to X$, we let $\Q_{\text{et}}(Y) = \Q_{\text{et}}(Y \to X)$ be the étale sheaf associated to the presheaf defined by
\[
\Q(Y)(-) := \Q(\Hom_{\Sm / X}(-, Y)).
\]

The category $\DA(X)$ is defined in two steps:

1. The category of effective motives $\DA_{\text{eff}}(X)$ is defined as the Verdier quotient of the derived category $D(\Shv(\Sm / X))$ by the smallest triangulated category closed under infinite sums and containing all complexes $[\Q_{\text{et}}(\BA^1_X) \to \Q_{\text{et}}(Y)]$. The object $\Q_{\text{et}}(Y)$, viewed as an element of $\DA_{\text{eff}}(X)$, is denoted $\text{M}_{\text{eff}}(Y)$. It is called the effective homological motive associated to $Y \to X$. We denote $\text{M}_{\text{eff}}(\id_X : X \to X)$ by $\mathbb{1}_X$. It is the unit for tensor product in $\DA_{\text{eff}}(X)$.

2. $\DA(X)$ is obtained from $\DA_{\text{eff}}(X)$ by formally inverting the operation $T_X \otimes -$, where $T_X$ is the Tate object. This is defined as
\[
T_X = \ker \left( [\Q_{\text{et}}(\BA^1_X) \to \Q_{\text{et}}(X)] \to \Q_{\text{et}}(\id_X : X \to X) \right),
\]
where $o(X)$ is the zero section of $\BA^1_X$. Note that the Tate motive is defined as $\Q_X(1) := T_X[-1]$.

The tensor product on $\DA(X)$ makes it a closed monoidal symmetric category with unit object $\mathbb{1}_X$.

It can be shown that, for $X = \Spec(k)$ the spectrum of a perfect field, we have an equivalence of categories $\DA(k) \simeq \text{DM}(k)$, where $\text{DM}(k)$ is Voevodsky’s category of mixed motives with rational coefficients.

There is a variant of the above where the sheaves of $\Q$-vector spaces are replaced by $\Lambda$-modules, notation: $\DA(X, \Lambda)$. The important case for us is when $\Q \in \Lambda$. The unit object is also denoted $\Lambda_X(0)$, and the Tate objects $\Lambda_X(n)$.

In [7, 8], it is shown that one has the full machinery of Grothendieck’s six operations on the triangulated categories $\DA(X)$. Tensor product and Hom, $\otimes_X$ and $\text{Hom}_X$;

- inverse and direct image: $f^*, f_*$, for $f : X \to Y$ a morphism of noetherian schemes;

and

- compact supports: $f_!, f^!$, $f : X \to Y$ a quasi-projective morphism of noetherian schemes, as well as nearby and vanishing cycle functors. Moreover, it was shown in [9, 10] that there are realization functors, compatible with the above functors.

**Betti.** Let $k$ be a field and $X$ a scheme of finite type over $k$. Let $\sigma : k \hookrightarrow \C$ be an embedding. Then there is symmetric monoidal unitary functor
\[
\Betti_{X, \sigma} : \DA(X) \to \text{D}(\Xan)
\]
which commutes in the obvious sense with the above functors when restricted to compact objects, e.g., if \( f : Y \to X \) is a morphism of finite type of quasi-projective \( k \)-schemes of finite type, then there are natural isomorphisms

\[
(f^{an})^* \circ \text{Betti}_{X,\sigma} \cong \text{Betti}_{Y,\sigma} \circ f^*.
\]

The triangulated subcategory of compact objects \( \text{DA}_{\text{cp}}(X) \) is generated by the quasi-projective \( Y \to X \). On that subcategory, we have an isomorphism

\[
\text{Betti}_{X,\sigma}(\text{Hom}(A, B)) \cong \text{Hom}(\text{Betti}_{X,\sigma}(A), \text{Betti}_{X,\sigma}(B)).
\]

**Hodge-deRham.** For the precise statements, see [41].

**Étale.** See [10]. Let \( E/\mathbb{Q} \) be a finite extension field. For each prime number \( \ell \) there is a functor

\[
\Omega^e_{S,\ell} : \text{DA}^e(S, E) \to \text{D}^e(S, E \otimes \mathbb{Q}_\ell)
\]

from the category of constructible motives on \( S \) with \( E \)-coefficients, to the derived category of constructible \( E \otimes \mathbb{Q}_\ell \)-adic sheaves on \( S \). The validity of this theorem depends on certain broad technical hypotheses on \( S \), which are valid for all the schemes appearing in this paper. This functor is compatible with the 6 operations above, as well as nearby and vanishing cycle sheaves. See also [39], [40].

**Crystalline.** This is not yet available.

Currently under development, there is also a theory of perverse objects, see [42].

In this paper, \( \Lambda = \Lambda_N = K_N = \mathbb{Q}(\mu_N) \). If \( G \) is a finite group acting on a motivic sheaf \( M \) over any scheme in which \#\( G \) is invertible, then for any idempotent \( e \) in the group-ring \( \Lambda[G] \) there is an image \( eM \). If \( \chi \) the character of a irreducible representation and \( e = (1/\#G) \sum_{g \in G} \chi^{-1}(g) \cdot g \), then \( eM \) is denoted \( M^\chi \).

This paper makes use of the Kummer motives \( K(\chi) \) on \( \mathbb{G}_m \), attached to a character \( \chi : G \to \Lambda_N^* \). We consider the étale covering \([N] : \mathbb{G}_m \to \mathbb{G}_m \), where \( \mathbb{G}_m \) is viewed as a scheme over \( \text{Spec}(R_N) \), with \( R_N = \mathbb{Z}[\zeta_N, 1/N] \), i.e., \( \mathbb{G}_m = \text{Spec}R_N[t, t^{-1}] \). \( G \) is the Galois group of this covering, which may be canonically identified with \( \mu_N = \mu_N(R_N) \) via Kummer theory: if \( t \) is the coordinate on \( \mathbb{G}_m \), then for any \( \sigma \in G, \sigma t^{1/N} = \zeta_\sigma t^{1/N} \) for a root of unity \( \zeta_\sigma \), independent of the choice of \( t^{1/N} \).

The Kummer motive is defined by the formula

\[
K(\chi) = ([N]_* \Lambda_{\mathbb{G}_m}(0))^\chi
\]

in \( \text{DA}(\mathbb{G}_m, \Lambda_N) \). We have

\[
[N]_* \Lambda_{\mathbb{G}_m}(0) = \bigoplus_{\chi : G \to \Lambda_N^*} K(\chi).
\]

We have the realizations:

**Betti.** For each embedding \( \varphi : R_N \to \mathbb{C} \) we get an isomorphism \( \varphi : G = \mu_N(\Lambda_N) \cong \mu_N(\mathbb{C}) \). Then \( K(\chi)_{\varphi,B} \) is the \( \mathbb{C} \)-local system on the analytic space \( \mathbb{C}^x \) defined by the character

\[
\varphi \circ \chi : \pi_1(\mathbb{C}^x, 1) = \mathbb{Z} \to \mathbb{C}^x : k \mapsto \varphi(\chi(\varphi^{-1}(\exp(2\pi ik/N))))).
\]

**Hodge-deRham.** \( H^1_{dR}(\mathbb{G}_m/R_N) \) is the \( H^1 \) of the complex \([d : \mathcal{O}_X \to \Omega^1_{X/R_N}] \), \( X = \mathbb{G}_m \). This is a free \( R_N \)-module generated by \( dt \). Then we have the Gauss-Manin connection

\[
\nabla : N_* \mathcal{O}_X \to N_* \mathcal{O}_X \otimes \Omega^1_{Y/R_N},
\]

where \( N : X = \mathbb{G}_m \to Y = \mathbb{G}_m \) is the map \( s \mapsto t^N \). We have

\[
N_* \mathcal{O}_X = \bigoplus_{i \in \mathbb{Z}/N} \mathcal{O}_{Y} s^i = \bigoplus_{\chi \in \mathbb{Z}/N} (N_* \mathcal{O}_X)^\chi.
\]
The identification $i \leftrightarrow \chi$ is given by Kummer theory. For a fixed character $\chi$, there is a unique $i \in \mathbb{Z}/N$ such that $\sigma^i = \chi(\sigma) s^i$, for all $\sigma \in G$.

Now for any $i$ we define a connection on the free rank 1 module $\mathcal{O}_Y s^i$ by the formula

$$\nabla : \mathcal{O}_Y s^i \to \mathcal{O}_Y s^i \otimes \Omega^1_{Y/R_N}, \quad \nabla(f s^i) = \left( t \frac{df}{dt} + \frac{i}{N} \right) \frac{dt}{t} \otimes s^i$$

which follows from $ds^i = (i/N) s^i dt/t$.

**Étale.** We have a canonical epimorphism $\pi_1 \to \text{Gal}(\mathbb{Q}(\zeta_N, s)/\mathbb{Q}(\zeta_N, t)) = G$ where $\pi_1 = \pi_1(\mathbb{G}_m, \bar{\eta})$ is Grothendieck’s fundamental group, $\bar{\eta} = \text{Spec}(\mathbb{Q}(\zeta_N, t))$. Choose an embedding $\varphi : \mu_N \to \mathbb{Q}_l^*$ for a prime number $\ell$ prime to $N$. Composing the above epimorphism with $\varphi \circ \chi$ we get a character of $\pi_1$, which defines the $\ell$-adic local system $K(\chi)_{\varphi, \ell}$.

In our application, we will need the Frobenius traces of the Kummer sheaves. If $t \in \mathbb{G}_m(F_q)$ is a point $q \equiv 1 \mod N$, then

$$\text{Tr}(\text{Frob}_t | (K(\chi)_{\varphi, \ell})) = \varphi(\chi(t^{(q-1)/N})).$$

Note that $t \mapsto t^{(q-1)/N}$ which sends $F_q^\times \to \mu_N$ is the character giving the canonical action of Frobenius on the Kummer extension:

$$\text{Frob}_t(\sqrt[q]{t}) = t^{(q-1)/N} \sqrt[q]{t}.$$

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