Thermodynamics of Soft Wall Model in Einstein-Maxwell-Gauss-Bonnet Gravity

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Abstract

We study the thermodynamics and deconfinement transition in soft wall model of QCD with Gauss-Bonnet corrections. The Gauss-Bonnet coupling terms modify the transition temperature of the system but qualitative features remain unaltered. We plot the chemical potential versus transition temperature for different values of Gauss-Bonnet couplings.

Keywords: AdS/CFT correspondence; Holographic QCD; Soft wall model.
1 Introduction

It has been a challenge to study the strongly coupled system such as Quantum Chromodynamics (QCD) analytically. In particular, the low energy behaviour of QCD is nonperturbative in nature. There is no hope to solve QCD in this regime by traditional perturbative methods. The lattice methods gives many important results about meson and baryon spectra but require the large amount of computational power and skills. Recently string inspired models are being widely used to study the low energy QCD and may interesting features of QCD such as chiral symmetry breaking and quark confinement are captured by these models.

Advent of AdS/CFT correspondence [1–4] has motivated to look for supergravity solutions [5] which can mimic the behaviour of realistic theories such as QCD. Some phenomenological models [6,12] has been developed to understand the low energy dynamics of QCD and such an approach is dubbed as holographic QCD or AdS/QCD. The dynamics of operators in the boundary theory is captured by equations of motion of the fields in the bulk and phenomena like chiral symmetry breaking and confinement/deconfinement transition also find their description in terms of dynamics of bulk fields [9,14,16,21]. The spectrum of mesons and baryons can be predicted [6,12,15,23,24] and seem to agree with experimental results.

In the simplest model of AdS/QCD, one assumes that the theory can be described by the five dimensional model living on the slice of the $AdS_5$ or some deformations of it [6,12,17]. This model known as hard wall model captures almost all essential features of QCD however fails to provide linear Regge trajectory. An improvement of the model so called as soft wall [7] introduced non dynamical dilaton field in the bulk action in order to achieve linear Regge trajectory.

In this paper we study the phase structure of QCD in soft wall model with Gauss-Bonnet corrections in the bulk theory. This is the extension of our previous work [22]. We consider the charged black hole solution of five dimensional Einstein-Maxwell theory with Gauss-Bonnet term and calculate the grand potential in this geometry. Gauss-Bonnet AdS black hole solution describes the high temperature quark gluon plasma phase of the holographic dual QCD. The low temperature confined phase is described by thermally charged Gauss-Bonnet AdS geometry. The difference of grand potentials in these two geometries is used to obtain the chemical potential versus temperature plot for the soft wall model for different values of Gauss-Bonnet coupling.
2 Charged Black Hole in Gauss-Bonnet Gravity

In this section we review the charged black hole solution of Einstein-Maxwell-Gauss-Bonnet-Gravity in five dimensions with negative cosmological constant [19,20]. The Euclidean action for Einstein-Maxwell theory with Gauss-Bonnet term is given by,

\[
S = - \int d^5 x \sqrt{g} \left\{ \frac{1}{2\kappa^2} (R - 2\Lambda + \alpha R_{GB}) - \frac{1}{4g^2} F^2 \right\},
\]  

(2.1)

where \( R_{GB} = R^2 - 4R_{MN}R^{MN} + R^{MNPQ}R_{MNPQ} \) is Gauss-Bonnet term and \( F^2 = F_{MN}F^{MN} \) is the field strength of the Maxwell field. The constant \( \kappa^2 \) is related to five dimensional Newton’s constant \( G_5 \) through relation \( \kappa^2 = 8\pi G_5 \) and the cosmological constant is taken as \( \Lambda = -6 \).

The Euler-Lagrange equations of motion with second order derivatives are given by,

\[
\frac{\partial L}{\partial g_{MN}} - \partial_P \left[ \frac{\partial L}{\partial (\partial g_{MN})} \right] + \partial_P \partial_Q \left[ \frac{\partial L}{\partial (\partial g_{MN})} \right] = 0,
\]  

(2.2)

which leads to equation of motion for \( g_{MN} \) as,

\[
G_{MN} + \alpha H_{MN} = \frac{\kappa^2}{g^2} T_{MN}
\]  

(2.3)

where,

\[
H_{MN} = 2(RR_{MN} - 2R_{MP}R_{N}^P - 2R^{PQ}R_{MPNQ} + R_{M}^{PQR}R_{NPQR}) - \frac{1}{2}g_{MN}R_{GB}
\]  

(2.4a)

\[
G_{MN} = R_{MN} - \frac{1}{2}Rg_{MN} + \Lambda g_{MN}
\]  

(2.4b)

\[
T_{MN} = F_{PM}F_{N}^P - \frac{1}{4}g_{MN}F^{PQ}F_{PQ}
\]  

(2.4c)

and for the gauge field

\[
\frac{1}{\sqrt{g}} \partial_M (\sqrt{g} F^{MN}) = 0.
\]  

(2.5)

The solution of gauge equation is given as,

\[
A_t(z) = i(\mu - Qz^2),
\]  

(2.6)

where \( i \) in front of solution is due to consideration of Eucliden spacetime. The constant \( \mu \) will be identified as quark chemical potential in the dual QCD and \( Q \) is related with black hole charge and the metric ansatz is,

\[
ds^2 = \frac{L^2}{z^2} \left( A^2 f(z) dt^2 + \frac{dz^2}{f(z)} + \sum_{i=1}^{3} dx_i^2 \right)
\]  

(2.7)
where $A^2 = \frac{1}{2} \left(\sqrt{1 - 8\alpha} + 1\right)$. The value of $A$ is fixed in such a way that it can lead to conformally flat metric at spatial infinity. Using equation of motion, the metric function $f(z)$ in Einstein-Maxwell-Gauss-Bonnet (EMGB) theory is given by,

\[ f_1(z) = \frac{1}{4\alpha} \left(1 - \sqrt{1 - 8\alpha(1 - mz^4 + q^2 z^6)}\right) \tag{2.8} \]

which, in the limit $\alpha \to 0$ leads to the metric of AdS RN black hole. Contracting equation (2.3) with $g^{MN}$, we get

\[-\frac{3}{2} R + 5\Lambda - \frac{\alpha}{2} R_{GB} = - \frac{\kappa^2}{4g^2} F_{ab} F^{ab} \tag{2.9} \]

which on substituting in (2.1), we get the final form of action as,

\[ S = - \int d^5x \sqrt{g} \left(\frac{-R + 4\Lambda}{\kappa^2}\right). \tag{2.10} \]

This equation shows that the dependence of Gauss-Bonnet coupling, $\alpha$ on action in five dimensional spacetime is not explicit but through the $\alpha$ dependence in the metric function.

To satisfy equation of motion (2.3), $q$ and $Q$ must be related as,

\[ q^2 = \frac{2 \kappa^2 Q^2}{3 g^2 A^2}. \tag{2.11} \]

The AdS/QCD correspondence relates five dimensional gravitational constant $2\kappa^2$ and five dimensional coupling constants to the rank of colour gauge group ($N_c$) and number of flavours ($N_f$) as,

\[ \frac{1}{2\kappa^2} = \frac{N_c^2}{8\pi^2} \quad \text{and} \quad \frac{1}{2g^2} = \frac{N_c N_f}{4\pi^2} \tag{2.12} \]

Let $z_+$ is outer horizon, thus $f_1(z_+) = 0$. This relation shows that the $m$ and $q$ are related as,

\[ m = \frac{1}{z_+^4} + q z_+^2. \tag{2.13} \]

The Hawking temperature of black hole with Gauss-Bonnet term can be written as,

\[ T = \frac{A f_1'(z_+)}{4\pi} = \frac{A}{\pi z_+} \left(1 - \frac{1}{2} q^2 z_+^6 \right). \tag{2.14} \]

Since the solutions of $A_t(z)$ are regular at the horizon, we impose Dirichlet boundary condition at the horizon $z_+$ as $A_t(z_+) = 0$, which leads to the relation $Q = \mu/z_+^2$. This gives the relation between $q$ and $\mu$ as,

\[ q^2 = 2\kappa^2 \mu^2 / 3g^2 z_+^4 A^2. \tag{2.15} \]
Using above relation and equation (2.14), we get a quadratic equation for $z_+$ and the solution for positive value of $z_+$ is given by,

$$z_+ = \frac{3A g^2}{2\kappa^2 \mu^2} \left( \sqrt{\frac{4 \kappa^2 \mu^2}{3 g^2} + \pi^2 T^2 - \pi T} \right).$$

(2.16)

3 Soft Wall Model Thermodynamics

We study effects of Gauss-Bonnet corrections to the free energy and phase structure of dual QCD in soft wall approach \cite{7,8}. The action in soft wall with Gauss-Bonnet term can be written as,

$$S = - \int d^5x \sqrt{g} e^{\phi} \left( -\frac{R + 4\Lambda}{\kappa^2} \right).$$

(3.1)

where $\phi$ is dilaton field given as $\phi(z) = -c z^2$. The dilaton field is considered to be non dynamical, therefore the equation of motion for gauge field remains unchanged.

The action in the high temperature deconfined phase is given by black hole solution of the EMGB theory;

$$S = - \int d^5x \sqrt{g} e^{-c z^2} \left( \frac{z^2 f''(z) - 8zf'(z) + 20f(z) + 4\Lambda}{\kappa^2} \right).$$

(3.2)

Since the integrals in the above action can not be done analytically, we expand the action upto first order in Gauss-Bonnet coupling, $\alpha$ and hence our results will be valid for small values of $\alpha$. The action upto first order in $\alpha$ is written as,

$$S_1 = - \frac{A}{\kappa^2} \int d^5x \frac{e^{-c z^2}}{z^5} \left[ (2q^2 z^6 - 4) + \alpha \left( 24m^2 z^8 - 120mq^2 z^{10} + 112q^4 z^{12} + 8q^2 z^6 + 40 \right) \right]$$

$$= - \frac{A}{\kappa^2} V_3 \int_0^\beta dt \int_\epsilon^{z_+} dz \frac{e^{-c z^2}}{z^5} \left[ (2q^2 z^6 - 4) + \alpha \left( 24m^2 z^8 - 120mq^2 z^{10} + 112q^4 z^{12} + 8q^2 z^6 + 40 \right) \right]$$

(3.3)

where $\beta$ is periodicity of Euclidean spacetime and defined as inverse of Hawking temperature $T$ and $V_3$ is three volume. The lower limit is UV cutoff and we shall be taking the limit as $\epsilon \to 0$ at the end of our calculation. Evaluation of integrals gives us,

$$S_1 = \frac{A V_3}{\kappa^2} \beta \left[ (1 - 10\alpha)F_1(z) + (1 + 4\alpha)F_2(z) + \alpha F_3(z) \right]_{\epsilon}^{z_+}$$

(3.4)
Here $F_1$, $F_2$ and $F_3$ are defined as,

\[
F_1(z) = \frac{\lambda}{z^d}(c z^2 - 1) + c^2 \text{Ei}(-c z^2)
\]

\[
F_2(z) = \frac{e^{-c z^2}}{c} q^2
\]

\[
F_3(z) = e^{-c z^2} \left[ 12 m^2 \left( \frac{1}{c^2} + \frac{z^2}{c} \right) - 60 m q^2 \left( \frac{2}{c^2} + \frac{z^4}{c} \right) + 56 q^4 \left( \frac{6}{c^4} + \frac{6 z^2}{c^3} + \frac{3 z^4}{c^2} + \frac{z^6}{c} \right) \right]
\]

and the exponential integral, Ei is given by $Ei(x) = -\int_{-x}^{\infty} dt e^{-t}/t$.

The low temperature phase of dual theory is given by thermal $\text{AdS}$ solution and the metric function is given by the replacement of the function $f_1$ by,

\[
f_2 = \frac{1}{4\alpha} (1 - \sqrt{1 - 8\alpha}).
\]

Using this metric function, the action upto order $\alpha$ for thermal $\text{AdS}$ is given by,

\[
S_2 = -\frac{A}{\kappa^2} \int d^5 x \frac{(40 \alpha - 4)e^{-c z^2}}{z^5} = -\frac{A}{\kappa^2} V_3 \int_{\epsilon}^{\infty} (40 \alpha - 4) e^{-c z^2}
\]

\[
= \frac{A}{\kappa^2} V_3 (1 - 10 \alpha) F_1(z) \bigg|_{\epsilon}^{\infty}
\]

where $\beta_1$ is periodicity of thermal $\text{AdS}$.

To study the transition from $\text{AdS}$ black hole to thermal $\text{AdS}$, we subtract the two actions and take the limit $\epsilon \to 0$. Comparing the the two geometries at radius $z = \epsilon$, we get,

\[
\beta_1 = \beta \sqrt{\frac{f_1}{f_2}}
\]

The difference between actions is given by,

\[
\Delta S = \lim_{\epsilon \to 0} \left( S_1 - \beta \sqrt{\frac{f_1}{f_2}} S_2 \right)
\]

\[
= \frac{AV_3}{\kappa^2} \beta \left[ (1 - 10\alpha) F_1(z_+) + (1 + 4\alpha) F_2(z_+) + \alpha F_3(z_+) \right.
\]

\[
- (1 + 4\alpha) F_2(0) - \alpha F_3(0) + \frac{m}{2} - 4\alpha \right].
\]

The last two terms in the expression arises from subtraction of $F_1$ terms in both the geometries and considering terms only upto order $\alpha$. Taking limit $\alpha \to 0$, we recover the results of [18].

The sign of $\Delta S$ determines the dominance of a particular geometry. Positive sign shows that
the thermal AdS is stable while the negative sign is indication of stability of AdS black hole geometry.

The thermodynamical grand potential and on-shell action are related by the relation, $\Omega = T S_{\text{on-shell}}$. Thus the difference of grand potentials per unit volume in the two geometries is given by,

$$\frac{\Delta \Omega}{V_3} = \frac{A}{\kappa^2} \beta \left[ \left( 1 - 10\alpha \right) F_1(z_+) + \left( 1 + 4\alpha \right) F_2(z_+) + \alpha F_3(z_+) - \left( 1 + 4\alpha \right) F_2(0) - \alpha F_3(0) + \frac{m^2}{2} - 4m \alpha \right]$$

(3.10)

We plot graphs of $\Delta \Omega/V_3$ vs Hawking temperature $T$ for different values of $\alpha$ and $\mu$ in Figure 1 and Figure 2. In Figure 3, we have plotted transition temperature vs chemical potential $\mu$, which we get by equating the grand potential equal to zero.

The plots of grand potential vs Hawking temperature, shows that increasing value of Gauss-Bonnet coupling $\alpha$ decreases the value of grand potential which shows the dependence of grand potential on coupling $\alpha$ is small but not negligible.

## 4 Summary

We have investigated the effects of Gauss-Bonnet couplings on the deconfinement transition in soft wall model of QCD. The effects alter the transition temperature but general features of the transition such as order of transition remains first order as is the case of holographic models.
Figure 3: Deconfinement temperature vs chemical potential plot at different values of $\alpha$.

The free energy of the system is evaluated by the difference in grand potentials in two phases. The chemical potential and transition temperature plot is obtained for small values of Gauss-Bonnet coupling. It would be interesting to see the dependence of transition temperature on large values of Gauss-Bonnet coupling.

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References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].

[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[4] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [hep-th/9803131].

[5] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, “Wilson loops, confinement, and phase transitions in large N gauge theories from supergravity,” JHEP 9806, 001 (1998) [hep-th/9803263].

[6] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, “QCD and a holographic model of hadrons,” Phys. Rev. Lett. 95, 261602 (2005) [hep-ph/0501128].

[7] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, “Linear confinement and AdS/QCD,” Phys. Rev. D 74, 015005 (2006) [hep-ph/0602229].

[8] C. P. Herzog, “A Holographic Prediction of the Deconfinement Temperature,” Phys. Rev. Lett. 98, 091601 (2007) [hep-th/0608151].

[9] C. A. Ballon Bayona, H. Boschi-Filho, N. R. F. Braga and L. A. Pando Zayas, “On a Holographic Model for Confinement/Deconfinement,” Phys. Rev. D 77, 046002 (2008) [arXiv:0705.1529 [hep-th]].

[10] E. Megias, H. J. Pirner and K. Veschgini, “QCD thermodynamics using five-dimensional gravity,” Phys. Rev. D 83, 056003 (2011) [arXiv:1009.2953 [hep-ph]].

[11] K. Veschgini, E. Megias and H. J. Pirner, “Trouble Finding the Optimal AdS/QCD,” Phys. Lett. B 696, 495 (2011) [arXiv:1009.4639 [hep-th]].

[12] L. Da Rold and A. Pomarol, “Chiral symmetry breaking from five dimensional spaces,” Nucl. Phys. B 721, 79 (2005) [hep-ph/0501218].
[13] A. Parnachev and D. A. Sahakyan, “Chiral Phase Transition from String Theory,” Phys. Rev. Lett. 97, 111601 (2006) [hep-th/0604173].

[14] T. Gherghetta, J. I. Kapusta and T. M. Kelley, “Chiral symmetry breaking in the soft-wall AdS/QCD model,” Phys. Rev. D 79, 076003 (2009) [arXiv:0902.1998 [hep-ph]].

[15] J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall, “Mesons in Gauge/Gravity Duals - A Review,” Eur. Phys. J. A 35, 81 (2008) [arXiv:0711.4467 [hep-th]].

[16] R. -G. Cai and J. P. Shock, “Holographic confinement/deconfinement phase transitions of AdS/QCD in curved spaces,” JHEP 0708, 095 (2007) [arXiv:0705.3388 [hep-th]].

[17] O. Andreev, “Cold Quark Matter, Quadratic Corrections and Gauge/String Duality,” Phys. Rev. D 81, 087901 (2010) [arXiv:1001.4414 [hep-ph]].

[18] C. Park, D. -Y. Gwak, B. -H. Lee, Y. Ko and S. Shin, “The Soft Wall Model in the Hadronic Medium,” Phys. Rev. D 84, 046007 (2011) [arXiv:1104.4182 [hep-th]].

[19] M. Cvetic, S. 'i. Nojiri and S. D. Odintsov, “Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity,” Nucl. Phys. B 628, 295 (2002) [hep-th/0112045].

[20] R. -G. Cai, “Gauss-Bonnet black holes in AdS spaces,” Phys. Rev. D 65, 084014 (2002) [hep-th/0109133].

[21] P. Zhang, “Linear Confinement for Mesons and Nucleons in AdS/QCD,” JHEP 1005, 039 (2010) [arXiv:1003.0558 [hep-ph]].

[22] S. Sachan and S. Siwach, “Thermodynamics of soft wall AdS/QCD at finite chemical potential,” Mod. Phys. Lett. A 27, 1250163 (2012) [arXiv:1109.5523 [hep-th]].

[23] G. F. de Teramond and S. J. Brodsky, “Hadronic spectrum of a holographic dual of QCD,” Phys. Rev. Lett. 94, 201601 (2005) [hep-th/0501022].

[24] D. K. Hong, T. Inami and H. -U. Yee, “Baryons in AdS/QCD,” Phys. Lett. B 646, 165 (2007) [hep-ph/0609270].