Gravitational effects on the neutrino oscillation in vacuum

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The propagation of neutrinos in a gravitational field is studied by developing a method of calculating a co-
variant quantum–mechanical phase in a curved space–time. The result is applied to neutrino propagation in the
Schwarzschild metric.

1. Introduction

The neutrino oscillation phenomenon has been discussed in the case of neutrinos propagating in
a flat space–time both in the plane wave approach for highly relativistic neutrinos [1] and in the wave
packet formalism [2]. In this note we present a
generalization of the plane wave approach to the
case of curved space–time [3–5].

In order to generalize the plane–wave approach
to the case of a curved space–time, it is useful to
recall the key points which are used to derive the
oscillation probability in the flat space–time:

• Neutrinos are produced at a space–time
point \( A(t_A, \vec{x}_A) \) as flavor eigenstates \( |\nu_\alpha\rangle = \sum_k U_{\alpha k} |\nu_k\rangle \), superpositions of the mass eigen-
states \( |\nu_k\rangle \). \( U \) is the unitary mixing matrix of
the neutrino fields.

• The propagation in the space–time of \( |\nu_k\rangle \) is
described by the quantum–mechanical phase of a
plane wave. For a neutrino propagating to
\( P(t, \vec{x}) \), the state evolves to
\( |\nu_k(t, \vec{x}; t_A, \vec{x}_A)\rangle = \exp(-i\Phi_k) |\nu_k\rangle \), where
\[
\Phi_k = E_k (t - t_A) - \vec{p}_k \cdot (\vec{x} - \vec{x}_A). \tag{1}
\]
The energy \( E_k \) and the momentum \( \vec{p}_k \) are related by the flat space–time mass–shell relation
\( E_k^2 - \vec{p}_k^2 = m_k^2 \).

• In order for the oscillation to occur and to be
observed, the interference among different mass
eigenstates has to be calculated in a specific
space–time point \( B(t_B, \vec{x}_B) \). This gives the space–
time flavor oscillation probability
\[
P_{\nu_\alpha \rightarrow \nu_\beta}(t_B, \vec{x}_B; t_A, \vec{x}_A) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2 \text{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta j} U_{\alpha j}^* U_{\beta k}^* \exp(-i\Delta\Phi_{kj}), \tag{2}
\]
where the phase shift \( \Delta\Phi_{kj} = \Phi_k - \Phi_j \) is due
to the interference between the \( k \)th and \( j \)th mass
eigenstates.

• In actual experiments, the time difference
\((t_B - t_A)\) is not measured, whereas the relative position \( |\vec{x}_B - \vec{x}_A| \) of the source and the detector
is known, and an oscillation probability in space
\( P_{\nu_\alpha \rightarrow \nu_\beta}(\vec{x}_B; \vec{x}_A) \) can be measured. In the plane
wave formalism, this can be taken care of consistently only for relativistic neutrinos by employing
the light–ray approximation, which consists in taking
\((t_B - t_A) = |\vec{x}_B - \vec{x}_A| \) in the calculation of the phase \( \Phi_k \) of each mass eigenstate. This corre-
ponds to calculate the quantum–mechanical phase of each \( |\nu_k\rangle \) along the light–ray path that
links \( A \) to \( B \). In this approximation, Eq.(1) becomes
\[
\Phi_k^L = (E_k - |\vec{p}_k|) |\vec{x}_B - \vec{x}_A|. \tag{3}
\]
Since we are dealing with relativistic neutrinos,
(m_k \ll E_k), we can approximate, to the first order, \(E_k \approx E_0 + O(m_k^2/(2 E_0))\), where \(E_0\) is the energy of a massless neutrino. This leads to the standard result for the phase shift

\[
\Delta \Phi_{kj}^L \approx \frac{\Delta m_{kj}^2}{2E_0} |\vec{x}_B - \vec{x}_A| = \frac{2\pi L_{\nu}(A,B)}{L_{\nu}^{osc}_{kj}},
\]

where \(\Delta m_{kj}^2 = m_k^2 - m_j^2\), \(L_{\nu}^{osc} = (4\pi E_0/\Delta m_{kj}^2)\) is the oscillation length and \(L_{\nu}(A,B) = |\vec{x}_B - \vec{x}_A|\) is the proper length in the flat space–time.

A generalization to a curved space–time is obtained by observing that the expression for the phase \(\Phi_k\) in Eq. (4) is a covariant quantity [3]:

\[
\Phi_k = \int_{A}^{B} g_{\mu\nu} p_{(k)^\mu}^\nu d\xi^\nu = \int_{A}^{B} p_{\mu}^{(k)} d\xi^\mu,
\]

where \(p_{(k)}^\mu = m_k d\xi^\mu/ds\) is the four–momentum of \(|\nu_k\), \(g_{\mu\nu}\) is the metric tensor, \(ds\) is the line element and \(p_{\mu}^{(k)} = g_{\nu\mu} p_{(k)^\nu}\) is the conjugate momentum to \(x^\mu\). The last expression in Eq. (5) is useful when the metric tensor does not depend on some of the coordinates \(x^\mu\); in that case, the corresponding components of \(p_{\mu}^{(k)}\) are constant of motion along the classical trajectory of the particle, while the components of \(p_{(k)}^{\nu}\) are not conserved quantities. The momentum \(p_{\mu}^{(k)}\) obeys the mass–shell condition in the curved metric

\[
m_k^2 = g_{\nu\mu} p_{(k)^\nu}^{\mu} = g_{\nu\mu} p_{\mu}^{(k)} p_{\nu}^{(k)}. \tag{6}
\]

Eq. (5) denotes the quantum–mechanical phase acquired by a particle traveling from \(A\) to \(B\) in a curved space–time. When calculated along the classical path, it corresponds to the classical action for a free particle.

The fact that the time difference between the production and detection points is not measured leads to the generalization to the curved space–time case of the light–ray approximation: the phase of each mass eigenstate must be calculated along the light–ray trajectory which links \(A\) to \(B\)

\[
\Phi_k^L = \left[ \int_{A}^{B} p_{\mu}^{(k)} d\xi^\mu \right]_{\text{light}}. \tag{7}
\]

This formulation is valid for highly relativistic neutrinos, since the light–ray approximation is viable only in this case.

2. Neutrino propagation in the Schwarzschild metric

Let us consider the propagation of neutrinos in the Schwarzschild metric. The line element in the coordinate frame \(x^\mu = (t, r, \theta, \phi)\) is

\[
ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \tag{8}
\]

where \(B(r) = [1 - (2GM)/r]\), \(G\) is the Newtonian constant and \(M\) is the mass of the source of the gravitational field. Since the gravitational field is isotropic, the classical orbits are confined to a plane, which we choose to be the equatorial plane \(\theta = \pi/2\). Therefore, we have \(d\theta = 0\). The relevant components of the canonical momentum are \(p_t^{(k)}\), \(p_r^{(k)}\) and \(p_\phi^{(k)}\) and they are related by the mass–shell relation as

\[
m_k^2 = \frac{1}{B(r)} (p_t^{(k)})^2 - B(r) (p_r^{(k)})^2 - \frac{(p_\phi^{(k)})^2}{r^2}. \tag{9}
\]

Since the metric tensor does not depend on the coordinates \(t\) and \(\phi\), the canonical momenta \(p_t^{(k)} \equiv E_k\) and \(p_r^{(k)} \equiv -J_k\) are constant of motion. With these definitions, the expression of the phase is

\[
\Phi_k^L = \int_{r_A}^{r_B} \left[ E_k \left( \frac{dt}{dr} \right)_0 - p_k(r) - J_k \left( \frac{d\phi}{dr} \right)_0 \right] dr. \tag{10}
\]

where \(p_k(r) \equiv -p_\phi^{(k)}\) and the differentials \((dt/dr)_0\) and \((d\phi/dr)_0\) are calculated along the light–ray path from \(A\) to \(B\). It is important to notice that \(E_k\) and \(J_k\), which are constants of motion for the geodesic trajectory of \(|\nu_k\), are not constant along the light–ray trajectory. Instead, the quantities \(E_0\) and \(J_0\) for a massless particle are constant along the light–ray path.

We briefly comment on the definition of the energies involved in the calculation of \(\Phi_k^L\). The constant of motion \(E_k\) is conveniently used in the calculation of the phase \(\Phi_k^L\), but it does not directly represent the energy of the neutrino in the gravitational field (except for \(r = \infty\), since \(p_k^L(r) = E_k/B(r)\)). Moreover, the correct definition of the production and detection energies depends on the reference frames where the physical processes occur. For definiteness, in our dis-
cussion we will choose the local reference frame, where \( E_k^{(loc)}(r) = |g_{tt}|^{-1/2}E_k = B(r)^{-1/2}E_k \).

2.1. Radial propagation

For neutrinos propagating in the radial direction, \( d\varphi = 0 \) and \( (dt/dr)_0 = \pm B(r)^{-1} \) where the + (−) sign refers to outward (inward) propagation. Deriving \( p_k(r) \) from the mass–shell relation of Eq. (2) with \( J_k = 0 \), the quantum mechanical phase of Eq. (10) becomes

\[
\Phi^L_k = \pm \int_{r_A}^{r_B} \left( E_k - \sqrt{E_k^2 - B(r)m^2_k} \right) \frac{dr}{B(r)}. \tag{11}
\]

We can now apply the relativistic expansion using the energy at infinity \( E_\infty \) as a reference value, i.e. \( m_k \ll E_k \), expanding \( E_k \approx E_\infty + O(m^2_k/(2E_\infty)) \), where \( E_\infty \) is the energy at infinity for a massless particle. Then, the phase of Eq. (11) is easily calculated and the phase shift which determines the oscillation is

\[
\Delta \Phi^L_{kj} \approx \frac{\Delta m^2_{kj}}{2E_\infty} |r_B - r_A|. \tag{12}
\]

This result does not depend on any assumption on the strength of the gravitational field.

The expression of the phase shift in Eq. (12) appears identical to that of the flat space–time case, Eq. (11). However, the gravitational effects are implicitly present in Eq. (12). When expressed in terms of the local energy and the proper distance, Eq. (12) becomes (for a weak field, i.e. \( GM \ll r \))

\[
\Delta \Phi^L_{kj} \approx \left( \frac{\Delta m^2_{kj} L_p(A,B)}{2E_\infty^{(loc)}(r_B)} \right) \times \left[ 1 - GM \left( \frac{1}{L_p(A,B)} \ln \frac{r_B}{r_A} - \frac{1}{r_B} \right) \right], \tag{13}
\]

where \( L_p(A,B) \approx r_B - r_A + GM \ln(r_A/r_B) \) is the proper distance (in the weak field limit).

The first parenthesis on the right–hand side in Eq. (13) is analogous to the flat space–time oscillation phase. The second parenthesis represents the correction due to the gravitational effects.

The proper oscillation length \( L^\text{osc}_{kj} \) which is obtained from Eq. (13) is increased because of the gravitational field, as expected.

2.2. Non–radial propagation

When the classical trajectory is not in the radial direction, the motion has an additional angular dependence. The light–ray differentials are \( (dt/dr)_0 = E_0B(r)^{-2}p_0(r)^{-1} \) and \( (d\varphi/dr)_0 = J_0r^{-2}B(r)^{-1}p_0(r)^{-1} \). The angular momentum \( J_k \) is related to \( E_k \), the impact parameter \( b \) and the velocity at infinity \( v_k^{(\infty)} \) as \( J_k = E_k b v_k^{(\infty)} \). Making use of the mass–shell condition Eq. (3) and the relativistic expansion \( m_k \ll E_k \), the phase becomes

\[
\Phi^L_k \approx \frac{m^2_k}{2E_0} \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - B(r)(b^2/r^2)}}. \tag{14}
\]

This expression is valid for any spherically symmetric (and time–independent) field.

As a specific example, we consider the propagation of the neutrino around the massive object. In the weak field limit, and for the typical situation where \( b \ll r_{A,B} \), the phase shift is

\[
\Delta \Phi^L_{kj} = \frac{\Delta m^2_{kj}}{2E_0} L_{AB} \left[ 1 - \frac{b^2}{2r_Ar_B} + \frac{2GM}{L_{AB}} \right], \tag{15}
\]

where \( L_{AB} = (r_A + r_B) \). As discussed for the radial case, a comparison of Eq. (15) with the flat space–time case, requires a correct identification of the proper distance and of the energies involved in the the physical problem. An interesting application of Eq. (15) to the possibility of gravitational lensing of neutrinos, has been discussed in Ref. [3].

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