NULL-FINITE SETS IN TOPOLOGICAL GROUPS AND THEIR APPLICATIONS

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ABSTRACT. In the paper we introduce and study a new family of “small” sets which is tightly connected with two well known σ-ideals: of Haar-null sets and of Haar-meager sets. We define a subset $A$ of a topological group $X$ to be null-finite if there exists a convergent sequence $(x_n)_{n \in \omega}$ in $X$ such that for every $x \in X$ the set \[ \{ n \in \omega : x + x_n \in A \} \] is finite. We prove that each null-finite Borel set in a complete metric Abelian group is Haar-null and Haar-meager. The Borel restriction in the above result is essential as each non-discrete metric Abelian group is the union of two null-finite sets. Applying null-finite sets to the theory of functional equations and inequalities, we prove that a mid-point convex function $f : G \to \mathbb{R}$ defined on an open convex subset $G$ of a metric linear space $X$ is continuous if it is upper bounded on a subset $B$ which is not null-finite and whose closure is contained in $G$. This gives an alternative short proof of a known generalization of Bernstein-Doetsch theorem (saying that a mid-point convex function $f : G \to \mathbb{R}$ defined on an open convex subset $G$ of a metric linear space $X$ is continuous if it is upper bounded on a non-empty open subset $B$ of $G$). Since Borel Haar-finite sets are Haar-meager and Haar-null, we conclude that a mid-point convex function $f : G \to \mathbb{R}$ defined on an open convex subset $G$ of a complete linear metric space $X$ is continuous if it is upper bounded on a Borel subset $B \subset G$ which is not Haar-null or not Haar-meager in $X$. The last result resolves an old problem in the theory of functional equations and inequalities posed by Baron and Ger in 1983.

INTRODUCTION

In 1920 Steinhaus [29] proved that for any measurable sets $A, B$ of positive Haar measure in a locally compact Polish group $X$ the sum $A + B := \{ a + b : a \in A, b \in B \}$ has non-empty interior in $X$ and the difference $B - B := \{ a - b : a, b \in B \}$ is a neighborhood of zero in $X$.

In [8] Christensen extended the “difference” part of the Steinhaus results to all Polish Abelian groups proving that for a Borel subset $B$ of a Polish Abelian group $X$ the difference $B - B$ is a neighborhood of zero if $B$ is not Haar-null. Christensen defined a Borel subset $B \subset X$ to be Haar-null if there exists a Borel $\sigma$-additive probability measure $\mu$ on $X$ such that $\mu(B + x) = 0$ for all $x \in X$.

A topological version of Steinhaus theorem was obtained by Pettis [25] and Picard [26] who proved that for any non-meager Borel sets $A, B$ in a Polish group $X$ the sum $A + B$ has non-empty interior and the difference $B - B$ is a neighborhood of zero.

In 2013 Darji [9] introduced a subideal of the $\sigma$-ideal of meager sets in a Polish group, which is a topological analog to the $\sigma$-ideal of Haar-null sets. Darji defined a Borel subset $B$ of a Polish group $X$ to be Haar-meager if there exists a continuous map $f : K \to X$ from a non-empty compact metric space $K$ such that for every $x \in X$ the preimage $f^{-1}(B + x)$ is meager in $K$. By [18], for every Borel subset $B \subset X$ which is not Haar-meager in a Polish Abelian group $X$ the difference $B - B$ is a neighborhood of zero.

It should be mentioned that in contrast to the “difference” part of the Steinhaus and Picard–Pettis theorems, the “additive” part can not be generalized to non-locally compact groups: by [23], [18] each non-locally compact Polish Abelian group $X$ contains two closed sets $A, B$ whose sum $A + B$ has empty interior but $A, B$ are neither Haar-null nor Haar-meager in $X$. In the Polish group $X = \mathbb{R}^\omega$, for such sets $A, B$ we can take the positive cone $\mathbb{R}_\omega^+$.

Steinhaus-type theorems has significant applications in the theory of functional equations and inequalities. For example, the “additive” part of Steinhaus Theorem can be applied to prove that a mid-point convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if it is upper bounded on some measurable set $B \subset \mathbb{R}^n$ of positive Lebesgue measure (see e.g. [20] p.210)). We recall that a function $f : G \to \mathbb{R}$ defined on a convex subset of a linear space is mid-point convex if $f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}$ for any $x, y \in G$. Unfortunately, due to the example of Matoušková and Zelený [23] we know that the “additive” part of the Steinhaus theorem does not extend to non-locally

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compact Polish Abelian groups. This leads to the following natural problem whose “Haar-null” version was posed by Baron and Ger in [3, P239].

**Problem** (Baron, Ger, 1983). *Is the continuity of an additive or mid-point convex function \( f : X \to \mathbb{R} \) on a Banach space \( X \) equivalent to the upper boundedness of the function on some Borel subset \( B \subset X \) which is not Haar-null or not Haar-meager?*

In this paper we give the affirmative answer to the Baron–Ger Problem applying a new concept of a null-finite set, which will be introduced in Section 2. In Section 3 we show that non-null-finite sets \( A \) in metric groups \( X \) possess a Steinhaus-like property: if a subset \( A \) of a metric group \( X \) is not null-finite, then \( A - A \) is a neighborhood of \( \theta \) and for some finite set \( F \subset X \) the set \( F + (A - A) \) is a neighborhood of zero in \( X \). In Sections 5 and 6 we shall prove that Borel null-finite sets in Polish Abelian groups are Haar-meager and Haar-null. On the other hand, in Example 7.1 we show that the product \( G \) of \( \omega \) additive functional \( f \) (in the sense of Lutsenko–Protasov [22]) are null-finite. In Section 9 we apply null-finite sets to prove that an each non-discrete metric Abelian group is the union of two null-finite sets and also observe that sparse sets (in the sense of Lutsenko–Protasov [22]) are null-finite. In Section 10 we apply null-finite sets to prove that an additive functional \( f : X \to \mathbb{R} \) on a Polish Abelian group is continuous if it is upper bounded on some subset \( B \subset X \) which is not null-finite in \( X \). In Section 11 we generalize this result to additive functions with values in Banach and locally convex spaces. Finally, in Section 12 we prove a generalization of Bernstein–Doetsch theorem and next apply it in the proof of a continuity criterion for mid-point convex functions on complete metric linear spaces, thus answering the Baron–Ger Problem. In Section 12 we pose some open problems related to null-finite sets.

1. **Preliminaries**

*All groups considered in this paper are Abelian.* The neutral element of a group will be denoted by \( \theta \). For a group \( G \) by \( G^* \) we denote the set of non-zero elements in \( G \). By \( C_n = \{ z \in \mathbb{C} : z^n = 1 \} \) we denote the cyclic group of order \( n \).

By a *complete metric group* we understand an Abelian group \( X \) endowed with a (complete) invariant metric \( \| - \cdot \| \). The invariant metric \( \| - \cdot \| \) determines (and can be recovered from) the *prenorm* \( \| - \cdot \| \) defined by \( \| x \| := \| x - \theta \| \). So, a metric group can be equivalently defined as a group endowed with a prenorm.

Formally, a *prenorm* on a group \( X \) is a function \( \| - \cdot \| : X \to \mathbb{R}^+_0 := [0, \infty) \) satisfying three axioms:

- \( \| x \| = 0 \) iff \( x = \theta \);
- \( \| x \| = \| -x \| \);
- \( \| x + y \| \leq \| x \| + \| y \| \)

for any \( x, y \in X \); see [1, §3.3].

Each metric group is a topological group with respect to the topology, generated by the metric.

All linear spaces considered in this paper are over the field \( \mathbb{R} \) of real numbers. By a *metric linear space* we understand a linear space endowed with an invariant metric.

A non-empty family \( \mathcal{I} \) of subsets of a set \( X \) is called *ideal of sets on \( X \) if \( \mathcal{I} \) satisfies the following conditions:

- \( X \notin \mathcal{I} \);
- for any subsets \( J \subset I \subset X \) the inclusion \( I \in \mathcal{I} \) implies \( J \in \mathcal{I} \);
- for any sets \( A, B \in \mathcal{I} \) we have \( A \cup B \in \mathcal{I} \).

An ideal \( \mathcal{I} \) on \( X \) is called a *\( \sigma \)-ideal* if for any countable subfamily \( \mathcal{C} \subset \mathcal{I} \) the union \( \bigcup \mathcal{C} \) belongs to \( \mathcal{I} \).

A topological space is *Polish* if it is homeomorphic to a separable complete metric space. A topological space is *analytic* if it is a continuous image of a Polish space.

2. **Introducing null-finite sets in topological groups**

In this section we introduce the principal new notion of this paper.

A sequence \( (x_n)_{n \in \omega} \) in a topological group \( X \) is called a *null-sequence* if it converges to the neutral element \( \theta \) of \( X \).
Definition 2.1. A set $A$ of a topological group $X$ is called \textit{null-finite} if there exists a null-sequence $(x_n)_{n \in \omega}$ in $X$ such that for every $x \in X$ the set $\{n \in \omega : x + x_n \in A\}$ is finite.

Null-finite sets in metrizable topological groups can be defined as follows.

Proposition 2.2. For a non-empty subset $A$ of a metric group $X$ the following conditions are equivalent:

1. $A$ is null-finite;
2. there exists an infinite compact set $K \subset X$ such that for every $x \in X$ the intersection $K \cap (x + A)$ is finite;
3. there exists a continuous map $f : K \to X$ from an infinite compact space $K$ such that for every $x \in X$ the preimage $f^{-1}(A + x)$ is finite.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, so it is enough to prove (3) $\Rightarrow$ (1). Assume that $f : K \to X$ is a continuous map from an infinite compact space $K$ such that for every $x \in X$ the preimage $f^{-1}(A + x)$ is finite. Fix any point $a \in A$. It follows that for every $x \in X$ the fiber $f^{-1}(a + x) \subset f^{-1}(A + x)$ is finite and hence the image $f(K)$ is infinite. So, we can choose a sequence $(y_n)_{n \in \omega}$ of pairwise distinct points in $f(K)$. Because of the compactness and metrizability of $f(K)$, we can additionally assume that the sequence $(y_n)_{n \in \omega}$ converges to some point $y_\infty \in f(K)$. Then the sequence $(x_n)_{n \in \omega}$ of points $x_n := y_n - y_\infty$, $n \in \omega$, is a null-sequence witnessing that the set $A$ is null-finite.

Null-finite sets are “small” in the following sense.

Proposition 2.3. For any null-finite set $A$ in a metric group $X$ and any finite subset $F \subset X$ the set $F + A$ has empty interior in $X$.

Proof. Since $A$ is null-finite, there exists a null-sequence $(x_n)_{n \in \omega}$ in $X$ such that for every $x \in X$ the set $\{n \in \omega : x + x_n \in A\}$ is finite.

To derive a contradiction, assume that for some finite set $F$ the set $F + A$ has non-empty interior $U$ in $X$. Then for any point $u \in U$ the set $\{n \in \omega : u + x_n \in F + A\} \subset \{n \in \omega : u + x_n \in F + A\}$ is infinite. By the Pigeonhole Principle, for some $y \in F$ the set $\{n \in \omega : u + x_n \in y + A\}$ is infinite and so is the set $\{n \in \omega : -y + u + x_n \in A\}$. But this contradicts the choice of the sequence $(x_n)_{n \in \omega}$.

Now we present some examples of closed sets which are (or are not) null-finite. We recall that for a group $G$ by $G^*$ we denote the set $G \setminus \{0\}$ of non-zero elements of $G$.

Example 2.4. Let $(G_n)_{n \in \omega}$ be a sequence of finite groups. The set $A := \prod_{n \in \omega} G_n^*$ is null-finite in the compact metrizable topological group $G = \prod_{n \in \omega} G_n$ if and only if $\lim_{n \to \infty} |G_n| \neq \infty$.

Proof. First, we assume that $\lim_{n \to \infty} |G_n| = \infty$ and show that the set $A$ is not null-finite in $G$. Given a null-sequence $(x_n)_{n \in \omega}$ in the compact topological group $G$, we should find an element $a \in G$ such that the set $\{n \in \omega : a + x_n \in A\}$ is infinite.

It will be convenient to think of elements of the group $G$ as functions $x : \omega \to \bigoplus_{n \in \omega} G_n$ such that $(x(k))_{k \in \omega} \in G_k$ for all $k \in \omega$.

Taking into account that $\lim_{n \to \infty} |G_k| = \infty$ and $\lim_{n \to \infty} x_n = \theta \in G$, we can inductively construct an increasing number sequence $(n_k)_{k \in \omega}$ such that $n_0 = 0$ and for every $k \in \mathbb{N}$ and $m \geq n_k$ the following two conditions are satisfied:

1. $x_m(i) = \theta$ for all $i < n_{k-1}$;
2. $|G_{n_k}| \geq k + 3$.

We claim that for every $i \geq n_1$ the set $\{x_{n_k}(i) : k \in \omega\}$ has cardinality $< |G_i|$. Indeed, given any $i \geq n_1$, we can find a unique $j \geq 1$ such that $n_j \leq i < n_{j+1}$ and conclude that $x_{n_k}(i) = \theta$ for all $k \geq j + 1$ (by the condition (1)). Then the set $\{x_{n_k}(i) : k \in \omega\} = \{\theta\} \cup \{x_{n_k}(i) : k \leq j\}$ has cardinality $\leq 1 + (j + 1) < j + 3 = |G_i|$. The last inequality follows from $i \geq n_k$ and condition (2). Therefore, the set $\{x_{n_k}(i) : k \in \omega\}$ has cardinality $< |G_i|$ and we can choose a point $a_i \in G_i \setminus \{-x_{n_k}(i) : k \in \omega\}$ and conclude that $a_i + x_{n_k}(i) \in G_i^*$ for all $k \in \omega$. Then the element $a = (a_i)_{i \in \omega} \in G$ has the required property as the set $\{n \in \omega : a + x_n \in A\} \supset \{n_k\}_{k \in \omega}$ is infinite.

Next, assuming that $\lim_{k \to \infty} |G_k| \neq \infty$, we shall prove that the set $A$ is Haar-finite in $G$. Since $\lim_{k \to \infty} |G_k| \neq \infty$, for some $l \in \omega$ the set $\Lambda = \{k \in \omega : |G_k| = l\}$ is infinite.

Since $\lim_{n \to \infty} (1 - \frac{1}{n})^n = 0$, there exists $n \in \mathbb{N}$ such that $(1 - \frac{1}{n})^n < \frac{1}{l}$ and hence $l(l-1)^n < l^n$. This implies that for every $k \in \Lambda$ the set $G_k(G_k^*)^n := \{(x + a_1, \ldots, x + a_n) : x \in G_k, a_1, \ldots, a_n \in G_k^*\} has cardinality $|G_k(G_k^*)^n| \leq l(l-1)^n < l^n = |G_k^*|$. Consequently, the (compact) set $G \cdot A^n = \{(x + a_1, \ldots, x + a_n) : x \in \text{null-sequence (x_n)_{n \in \omega}}$ in $X$ such that for every $x \in X$ the set $\{n \in \omega : x + x_n \in A\}$ is finite.
G, a₁, . . . , aₙ ∈ A} is nowhere dense in the compact topological group Gⁿ. Now Theorem 2.5 ensures that A is null-finite in G.

Theorem 2.5. A non-empty subset A of a Polish group X is null-finite in X if for some n ∈ ℤ the set X·Aⁿ := {(x + a₁, . . . , x + aₙ) : x ∈ X, a₁, . . . , aₙ ∈ A} is meager in Xⁿ.

Proof. Let K(X) be the space of non-empty compact subsets of X, endowed with the Vietoris topology. For a compact subset K ⊂ X let Kⁿ = {(x₁, . . . , xₙ) ∈ Kⁿ : |{x₁, . . . , xₙ}| = n} the set of n-tuples consisting of pairwise distinct points of K.

By Mycielski-Kuratowski Theorem [19, 19.1], the set Wₙ(K) := {K ∈ K(X) : Kⁿ ∩ (X·Aⁿ) = ∅} is comeager in K(X) and hence contains some infinite compact set K. We claim that for every x ∈ X the set K ∩ (x + A) has cardinality < n. Assuming the opposite, we could find n pairwise distinct points a₁, . . . , aₙ ∈ A such that x + aᵢ ∈ K for all i ≤ n. Then (x + a₁, . . . , x + aₙ) ∈ Kⁿ ∩ X·Aⁿ, which contradicts the inclusion K ∈ Wₙ(A). By Proposition 2.4 the set A is null-finite.

3. A Steinhaus-like properties of sets which are not null-finite

In [29] Steinhaus proved that for a subset A of positive Lebesgue measure in the real line, the set A − A is a neighborhood of zero. In this section we establish three Steinhaus-like properties of sets which are not null-finite.

Theorem 3.1. If a subset A of a metric group X is not null-finite, then

1. the set A − A is a neighborhood of θ in X;
2. each neighborhood U ⊂ X of θ contains a finite subset F ⊂ U such that F + (A − A) is a neighborhood of θ;
3. If X is Polish (and A is analytic), then A − A is not meager in X (and (A − A) − (A − A) is a neighborhood of θ in X).

Proof. 1. Assuming that A − A is not a neighborhood of θ, we could find a null-sequence (xₙ)ₙ∈ω contained in X \ (A − A). Since A is not null-finite, there exists a ∈ X such that the set Ω = {n ∈ ω : a + xₙ ∈ A} is infinite. Then a ∈ (a + xₙ)ₙ∈Ω ⊂ A and hence xₙ = (a + xₙ) − a ∈ A − A for all n ∈ Ω, which contradicts the choice of the sequence (xₙ)ₙ∈ω.

2. Fix a decreasing neighborhood base (Uₙ)ₙ∈ω at zero in X such that U₀ ⊂ U. For the proof by contradiction, suppose that for any finite set F ⊂ U the set F + (A − A) is not a neighborhood of zero. Then we can inductively construct a null-sequence (xₙ)ₙ∈ω such that xₙ ∈ Uₙ \ ⋃ₙ≤i<n (xᵢ + A − A) for all n ∈ ω. Observe that for each z ∈ X the set {n ∈ ω : z + xₙ ∈ A} contains at most one point. Indeed, in the opposite case we could find two numbers k < n with z + xₙ, z + xₚ ∈ A and conclude that z ∈ xₚ + A and hence xₚ ∈ z + A ⊂ xₚ − A + A, which contradicts the choice of the number xₚ. The sequence (xₙ)ₙ∈ω witnesses that the set A is null-finite in X.

3. Finally assume that the group X is Polish. To derive a contradiction, assume that A − A is meager. Since the map δ : X × X → X, δ : (x, y) → x − y, is open, the preimage δ⁻¹(A − A) is meager in X × X. Observe that δ⁻¹(A − A) = X·A² where X·A² = {(x + a, x + b) : x ∈ X, a, b ∈ A}. By Theorem 2.5 the set A is null-finite, which is a desired contradiction. This contradiction shows that the set A − A is not meager in X. If the space A is analytic, then so is the subset A − A of X. Being analytic, the set A − A has the Baire Property in X; see [19, 29.14]. So, we can apply the Pettis-Picard Theorem [19, 9.9] and conclude that (A − A) + (A − A) is a neighborhood of θ in X.

The last statement of Theorem 3.1 does not hold without any assumptions on A and X.

Example 3.2. Let f : C₂ → C₂ be a discontinuous homomorphism on the compact Polish group X = C₂ (thought as the vector space C₂ over the two-element field ℤ/2ℤ). Then the subgroup A = f⁻¹(θ) is not null-finite in X (by Proposition 2.3), but A − A = A = (A − A) − (A − A) is not a neighborhood of θ in X.

A combinatorial characterization of null-finite sets in compact metric groups

In compact metric groups null-finite sets admit a purely combinatorial description.

Proposition 4.1. A non-empty subset A of a compact metric group X is null-finite if and only if there exists an infinite set I ⊂ X such that for any infinite subset J ⊂ I the intersection ∩ₓ∈J (A − x) is empty.
Proof. To prove the “only if” part, assume that \( A \subset X \) is null-finite. So, there exists a null-sequence \((x_n)_{n \in \omega}\) such that for every \( x \in X \) the set \( \{ n \in \omega : x + x_n \in A \} \) is finite. It follows that for every \( x \in X \) the set \( \{ n \in \omega : x_n = x \} \) is finite and hence the set \( I := \{ x_n \}_{n \in \omega} \) is infinite. We claim that this set has the required property. Indeed, assuming that for some infinite subset \( J \subset I \) the intersection \( \bigcap_{x \in J} (A - x) \) contains some point \( a \in X \), we conclude that the set \( \{ n \in \omega : a + x_n \in A \} \) contains the set \( \{ n \in \omega : x_n \in J \} \) and hence is infinite, which contradicts the choice of the sequence \((x_n)_{n \in \omega}\).

To prove the “if” part, assume that there exists an infinite set \( I \subset X \) such that for every infinite set \( J \subset I \) the intersection \( \bigcap_{x \in J} (A - x) \) is empty. By the compactness of the metric group \( X \), some sequence \((x_n)_{n \in \omega}\) of pairwise distinct points of the infinite set \( I \) converges to some point \( x_\infty \in X \). Then the null-sequence \((x_n)_{n \in \omega}\) consisting of the points \( z_n = x_n - x_\infty \), \( n \in \omega \), witnesses that \( A \) is null-finite. Assuming the opposite, we would find a point \( a \in X \) such that the set \( \{ n \in \omega : a + z_n \in A \} \) is infinite. Then the set \( J := \{ x \in I : a - x_\infty + x \in A \} \subset \{ x_n : n \in \omega, a - x_\infty + x_n \in A \} \) is infinite, too, and the intersection \( \bigcap_{x \in J} (A - x) \) contains the point \( a - x_\infty \) and hence is not empty, which contradicts the choice of the set \( I \). \( \square \)

Following Lutsenko and Protasov \[22\], we define a subset \( A \) of an infinite group \( X \) to be sparse if for any infinite set \( I \subset X \) there exists a finite set \( F \subset I \) such that \( \bigcap_{x \in F} (x + A) \) is empty. By \[22\] Lemma 1.2 the family of sparse subsets of a group is an invariant ideal on \( X \) (in contrast to the family of null-finite sets).

**Proposition 4.2.** Each sparse subset \( A \) of a non-discrete metric group \( X \) is null-finite.

Proof. Being non-discrete, the metric group \( X \) contains a null-sequence \((x_n)_{n \in \omega}\) consisting of pairwise distinct points. Assuming that the sparse set \( A \) is not null-finite, we can find a point \( a \in X \) such that the set \( \Omega := \{ n \in \omega : a + x_n \in A \} \) is infinite. Since \( A \) is sparse, for the infinite set \( I := \{ -x_n : n \in \Omega \} \) there exists a finite set \( F \subset I \) such that the intersection \( \bigcap_{x \in F} (x + A) \supset \bigcap_{m \in \Omega} (-x_n + A) \ni a \) is empty, which is not possible as this intersection contains the point \( a \). \( \square \)

5. **Null-finite Borel sets are Haar-meager**

In this section we prove that each null-finite set with the universal Baire property in a complete metric group is Haar-meager.

A subset \( A \) of a topological group \( X \) is defined to have the **universal Baire property** (briefly, \( A \) is a uBP-set) if for any function \( f : K \to X \) from a compact metrizable space \( K \) the preimage \( f^{-1}(A) \) has the Baire property in \( K \), which means that for some open set \( U \subset K \) the symmetric difference \( U \triangle f^{-1}(A) \) is meager in \( K \). It is well-known that each Borel subset of a topological group has the universal Baire property.

A uBP-set \( A \) of a topological group \( X \) is called **Haar-meager** if there exists a continuous map \( f : K \to X \) from a compact metrizable space \( K \) such that \( f^{-1}(x + A) \) is meager in \( K \) for all \( x \in X \). By \[9\], for a complete metric group \( X \) the family \( \text{HM}_X \) of subsets of Haar-meager uBP-sets in \( X \) is an invariant \( \sigma \)-ideal on \( X \). For more information on Haar-meager sets, see \[9\], \[11\], \[10\], \[18\].

**Theorem 5.1.** Each null-finite uBP-set in a complete metric group is Haar-meager.

Proof. To derive a contradiction, suppose that a null-finite uBP-set \( A \) in a complete metric group \((X, \| \cdot \|)\) is not Haar-meager. Since \( A \) is null-finite, there exists a null-sequence \((a_n)_{n \in \omega}\) such that for every \( x \in X \) the set \( \{ n \in \omega : x + a_n \in A \} \) is finite.

Replacing \((a_n)_{n \in \omega}\) by a suitable subsequence, we can assume that \( \| a_n \| < \frac{1}{n} \) for all \( n \in \omega \). For every \( n \in \omega \) consider the compact set \( K_n := \{ \theta \} \cup \{ a_m \}_{m \geq n} \subset X \). The metric restriction on the sequence \((a_n)_{n \in \omega}\) implies that the function

\[
\Sigma : \prod_{n \in \omega} K_n \to X, \quad \Sigma : (x_n)_{n \in \omega} \mapsto \sum_{n \in \omega} x_n,
\]

is well-defined and continuous (the proof of this fact can be found in \[18\]; see the proof of Theorem 2).

Since the set \( A \) is not Haar-meager and has uBP, there exists a point \( z \in X \) such that \( B := \Sigma^{-1}(z + A) \) is not meager and has the Baire Property in the compact metrizable space \( K := \prod_{n \in \omega} K_n \). Consequently, there exists a non-empty open set \( U \subset K \) such that \( U \cap B \) is comeager in \( U \). Replacing \( U \) by a smaller subset, we can assume that \( U \) is of basic form \( \{ b \} \times \prod_{m \geq j} K_m \) for some \( j \in \omega \) and some element \( b \in \prod_{m < j} K_m \). It follows that for every \( x \in K_j \setminus \{ \theta \} \) the set \( \{ b \} \times \{ x \} \times \prod_{m \geq j} K_m \) is closed and open in \( K \) and hence the set \( C_x := \{ y \in \prod_{m \geq j} K_m : \{ b \} \times \{ x \} \times \{ y \} \subset U \cap B \} \) is comeager in \( \prod_{m \geq j} K_m \). Then the intersection \( \bigcap_{x \in K_j \setminus \{ \theta \}} C_x \) is comeager and hence contains some point \( c \). For this point we get the inclusion \( \{ b \} \times (K_j \setminus \{ \theta \}) \times \{ c \} \subset B \).

Let \( v \in K \) be a unique point such that \( \{ v \} = \{ b \} \times \{ \theta \} \times \{ c \} \). For any \( m \geq j \) the point \( a_m \) belongs to \( K_j \) and
the inclusion $\{b\} \times \{a_m\} \times \{c\} \subset B$ implies that $\Sigma(v) + a_m \in z + A$. Consequently, the element $u := -z + \Sigma(v)$ has the property $u + a_m \in A$ for all $m \geq j$, which implies that the set $\{n \in \omega : u + a_n \in A\}$ is infinite. But this contradicts the choice of the sequence $(a_n)_{n \in \omega}$.

6. Null-finite Borel sets are Haar-null

In this section we prove that each universally measurable null-finite set in a complete metric group is Haar-null.

A subset $A$ of a topological group $X$ is defined to be universally measurable if $A$ is measurable with respect to any $\sigma$-additive Radon Borel probability measure on $X$. It is clear that each Radon Borel subset of a topological group is universally measurable. We recall that a measure $\mu$ on a topological space $X$ is Radon if for any $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$.

A universally measurable subset $A$ of a topological group $X$ is called Haar-null if there exists a $\sigma$-additive Radon Borel probability measure $\mu$ on $X$ such that $\mu(x + A) = 0$ for all $x \in X$. By [8], for any complete metric group $X$ the family $\mathcal{HN}_X$ of subsets of universally measurable Haar-null subsets of $X$ is an invariant $\sigma$-ideal in $X$. For more information on Haar-null sets, see [8, 16, 17] and also e.g. [13, 15, 23, 28].

**Theorem 6.1.** Each universally measurable null-finite set $A$ in a complete metric group $X$ is Haar-null.

**Proof.** To derive a contradiction, suppose that the null-finite set $A$ is not Haar-null in $X$. Since $A$ is null-finite, there exists a null-sequence $(a_n)_{n \in \omega}$ such that for each $x \in X$ the set $\{n \in \omega : x + a_n \in A\}$ is finite. Replacing $(a_n)_{n \in \omega}$ by a suitable subsequence, we can assume that $\|a_n\| \leq \frac{1}{n}$ for all $n \in \omega$.

Consider the compact space $\Pi := \prod_{n \in \omega} (0, 1, \ldots, 2^n)$ endowed with the product measure $\lambda$ of uniformly distributed measures on the finite discrete spaces $\{0, 1, 2, 3, \ldots, 2^n\}$, $n \in \omega$. Consider the map

$$\Sigma : \Pi \to X, \quad \Sigma : (p_i)_{i \in \omega} \mapsto \sum_{i=0}^{\infty} p_i a_i.$$

Since $\|p_i a_i\| \leq 2^i \|a_i\| \leq 2^i \frac{1}{i} = \frac{1}{i}$, the series $\sum_{i=0}^{\infty} p_i a_i$ is convergent and the function $\Sigma$ is well-defined and continuous.

Since the set $A$ is not Haar-null, there exists an element $z \in X$ such that the preimage $\Sigma^{-1}(z + A)$ has positive $\lambda$-measure and hence contains a compact subset $K$ of positive measure. For every $n \in \omega$ consider the subcube $\Pi_n := \prod_{i=0}^{n-1} (0, 1, 2, \ldots, 2^i) \times \prod_{i=n}^{\infty} (1, 2, \ldots, 2^i)$ of $\Pi$ and observe that $\lambda(\Pi_n) \to 1$ as $n \to \infty$.

Replacing $K$ by $K \cap \Pi_l$ for a sufficiently large $l$, we can assume that $K \subset \Pi_l$. For every $m \geq l$ let $s_m : \Pi_l \to \Pi$ be the “back-shift” defined by the formula $s_m((x_i)_{i \in \omega}) := (y_i)_{i \in \omega}$ where $y_i = x_i$ for $i \neq m$ and $y_i = x_i - 1$ for $i = m$.

**Claim 6.2.** For any compact set $C \subset \Pi_l$ of positive measure $\lambda(C)$ and any $\varepsilon > 0$ there exists $k \geq l$ such that for any $m \geq k$ the intersection $C \cap s_m(C)$ has measure $\lambda(C \cap s_m(C)) > (1 - \varepsilon) \lambda(C)$.

**Proof.** By the regularity of the measure $\lambda$, the set $C$ has a neighborhood $O(C) \subset \Pi$ such that $\lambda(O(C) \setminus C) < \varepsilon \lambda(C)$. By the compactness of $C$, there exists $k \geq l$ such that for any $m \geq k$ the shift $s_m(C)$ is contained in $O(C)$. Hence $\lambda(s_m(C) \setminus C) \leq \lambda(O(C) \setminus C) < \varepsilon \lambda(C)$ and thus $\lambda(s_m(C) \cap C) = \lambda(s_m(C)) - \lambda(s_m(C) \setminus C) > \lambda(C) - \varepsilon \lambda(C) = (1 - \varepsilon) \lambda(C)$.

Using Claim 6.2 we can choose an increasing number sequence $(m_k)_{k \in \omega}$ such that $m_0 > l$ and the set $K_\infty := \bigcap_{k \in \omega} s_{m_k}(K)$ has positive measure and hence contains a point $\tilde{b} := (b_i)_{i \in \omega}$. It follows that for every $k \in \omega$ the point $\tilde{b}_k := s^{-1}_{m_k}(\tilde{b})$ belongs to $K \subset \Sigma^{-1}(z + A)$.

Observe that $\Sigma(\tilde{b}_k) = \Sigma(\tilde{b}) + a_{m_k} \in z + A$ and hence $-z + \Sigma(\tilde{b}) + a_{m_k} \in A$ for all $k \in \omega$, which contradicts the choice of the sequence $(a_n)_{n \in \omega}$.

**Remark 6.3.** After writing the initial version of this paper we discovered that a result similar to Theorem 6.1 was independently found by Bingham and Ostaszewski [6, Theorem 3].

7. The $\sigma$-ideal generated by (closed) Borel null-finite sets

In this section we introduce two new invariant $\sigma$-ideals generated by (closed) Borel null-finite sets and study the relation of these new ideals to the $\sigma$-ideals of Haar-null and Haar-meager sets.

Namely, for a complete metric group $X$ let

- $\sigma\mathcal{NF}_X$ be the smallest $\sigma$-ideal containing all Borel null-finite sets in $X$;
• \( \sigma_{\text{NF}} X \) be the smallest \( \sigma \)-ideal containing all closed null-finite sets in \( X \);
• \( \sigma_{\text{HN}} X \) be the smallest \( \sigma \)-ideal containing all closed Haar-null sets in \( X \).

Theorems 5.1 and 6.1 imply that \( \sigma_{\text{NF}} X \subset \sigma_{\text{HN}} X \) and \( \sigma_{\text{NF}} X \subset \text{HN}_X \cap \text{HM}_X \). So, we obtain the following diagram in which an arrow \( A \to B \) indicates that \( A \subset B \).

\[
\begin{array}{ccc}
\sigma_{\text{NF}} X & \xrightarrow{\subset} & \sigma_{\text{HN}} X \\
\downarrow & & \downarrow \\
\sigma_{\text{NF}} X & \xrightarrow{\subset} & \text{HN}_X \cap \text{HM}_X
\end{array}
\]

In Examples 7.1 and 7.2 we show that the \( \sigma \)-ideal \( \sigma_{\text{NF}} X \) is strictly smaller than \( \sigma_{\text{HN}} X \) and the ideal \( \sigma_{\text{NF}} X \) is not contained in \( \sigma_{\text{HN}} X \).

**Example 7.1.** The closed set \( A = \prod_{n \geq 2} C_n^* \) in the product \( X = \prod_{n \geq 2} C_n \) of cyclic groups is Haar-null but cannot be covered by countably many closed null-finite sets. Consequently, \( A \in \sigma_{\text{HN}} X \setminus \sigma_{\text{NF}} X \).

**Proof.** The set \( A = \prod_{n=2}^\infty C_n^* \) has Haar measure

\[
\prod_{n=2}^\infty \frac{|C_n^*|}{|C_n|} = \prod_{n=2}^\infty \frac{n-1}{n} = 0
\]

and hence is Haar-null in the compact Polish group \( X \).

Next, we show that \( A \) cannot be covered by countably many closed null-finite sets. To derive a contradiction, assume that \( A = \bigcup_{n \in \omega} A_n \) where each \( A_n \) is closed and null-finite in \( X \). By the Baire Theorem, for some \( n \in \omega \) the set \( A_n \) has non-empty interior in \( A \). Consequently, we can find \( m > 2 \) and \( a \in \prod_{n=2}^{m-1} C_n^* \) such that \( \{a\} \times \prod_{n=m}^\infty C_n^* \subset A_n \). By Example 7.2 the set \( \prod_{n=m}^\infty C_n^* \) is not null-finite in \( \prod_{n=m}^\infty C_n \), which implies that the set \( A_n \supset \{a\} \times \prod_{n=m}^\infty C_n^* \) is not null-finite in the group \( X \). But this contradicts the choice of \( A_n \). \( \square \)

Our next example shows that \( \sigma_{\text{NF}} X \not\subset \sigma_{\text{HN}} X \) for some compact Polish group \( X \).

**Example 7.2.** For any function \( f : \omega \to [2, \infty) \) with \( \prod_{n \in \omega} \frac{f(n)-1}{f(n)} > 0 \), the compact metrizable group \( X = \prod_{n \in \mathbb{N}} C_{f(n)} \) contains a null-finite \( G_\delta \)-set \( A \subset X \) which cannot be covered by countably many closed Haar-null sets in \( X \). Consequently, \( A \notin \text{NF}_X \setminus \sigma_{\text{HN}} X \).

**Proof.** For every \( n \in \omega \) let \( g_n \) be a generator of the cyclic group \( C_{f(n)} \). In the compact metrizable group \( X = \prod_{n \in \omega} C_{f(n)} \) consider the null-sequence \( (x_n)_{n \in \omega} \) defined by the formula

\[
x_n(i) = \begin{cases} 
\theta & \text{if } i \leq n \\
g_i & \text{if } i > n.
\end{cases}
\]

Consider the closed subset \( B = \prod_{n \in \omega} C_{f(n)}^* \) in \( X \) and observe that it has positive Haar measure, equal to the infinite product

\[
\prod_{n \in \omega} \frac{f(n)-1}{f(n)} > 0.
\]

It is easy to see that for every \( n \in \omega \) the set \( C_{f(n)}^* \cap \{\theta\} \) is not equal to \( C_{f(n)}^* \cap (g_n + C_{f(n)}^*) \), which implies that the intersection \( B \cap (x_n + B) \) is nowhere dense in \( B \). Consequently, the set \( A = B \setminus \bigcup_{n \in \omega} (x_n + B) \) is a dense \( G_\delta \)-set in \( B \).

We claim that the set \( A \) is null-finite. Given any \( a \in X \) we should prove that the set \( \{n \in \omega : a + x_n \in A\} \) is finite. If \( a \notin B \), then the open set \( X \setminus B \) is a neighborhood of \( a \) in \( X \). Since the sequence \( (a + x_n)_{n \in \omega} \) converges to \( a \in X \setminus B \), the set \( \{n \in \omega : a + x_n \in B\} \supset \{n \in \omega : a + x_n \in A\} \) is finite.

If \( a \in B \), then \( \{n \in \omega : a + x_n \in A\} \subset \{n \in \omega : (B + x_n) \cap A \neq \emptyset\} = \emptyset \) by the definition of the set \( A \).

Next, we prove that the \( G_\delta \)-set \( A \) cannot be covered by countably many closed Haar-null sets. To derive a contradiction, assume that \( A \subset \bigcup_{n \in \omega} F_n \) for some closed Haar-null sets \( F_n \subset X \). Since the space \( A \) is Polish, we can apply the Baire Theorem and find \( n \in \omega \) such that the set \( A \cap F_n \) has non-empty interior in \( A \) and hence its closure \( \overline{A \cap F_n} \) has non-empty interior in \( B = \prod_{n \in \omega} C_{f(n)}^* \). It is easy to see that each non-empty open subset of \( B \) has positive Haar measure in \( X \). Consequently, the set \( \overline{A \cap F_n} \) has positive Haar measure, which is not possible as this set is contained in the Haar-null set \( F_n \). \( \square \)
Remark 7.3. Answering a question posed in a preceding version of this paper, Adam Kwela [21] constructed two compact null-finite subsets $A, B$ on the real line, whose union $A \cup B$ is not null-finite. This means that the family of subsets of (closed) Borel null-finite subsets on the real line is not an ideal, and the ideal $\sigma_{\text{NF}} \mathbb{R}$ contains compact subsets of the real line, which fail to be null-finite.

8. DECOMPOSING NON-DISCRETE METRIC GROUPS INTO UNIONS OF TWO NULL-FINITE SETS

By Theorem 6.1 and the countable additivity of the family of Borel Haar-null sets [3], the countable union of Borel null-finite sets in a complete metric group $X$ is Haar-null in $X$ and hence is not equal to $X$. So, a complete metric group cannot be covered by countably many Borel null-finite sets. This result dramatically fails for non-Borel null-finite sets.

**Theorem 8.1.** Each non-discrete metric group $X$ can be written as the union $X = A \cup B$ of two null-finite subsets $A, B$ of $X$.

**Proof.** Being non-discrete, the metric group $X$ contains a non-trivial null-sequence, which generates a non-discrete countable subgroup $Z$ in $X$.

Let $Z = \{z_n\}_{n \in \omega}$ be an enumeration of the countable infinite group $Z$ such that $z_0 = \theta$ and $z_n \neq z_m$ for any distinct numbers $n, m \in \omega$. By induction we can construct sequences $(u_n)_{n \in \omega}$ and $(v_n)_{n \in \omega}$ in $Z$ such that $u_0 = v_0 = \theta$ and for every $n \in \mathbb{N}$ the following two conditions are satisfied:

1. $|u_n| \leq \frac{1}{n}$ and $u_n \notin \{z_i + z_j + v_k : i, j \leq n, k < n\} \cup \{z_i + z_j + u_k : i, j \leq n, k < n\}$;
2. $|v_n| \leq \frac{1}{n}$ and $v_n \notin \{z_i + z_j + v_k : i, j \leq n, k < n\} \cup \{z_i + z_j + v_k : i, j \leq n, k < n\}$.

At the $n$-th step of the inductive construction, the choice of the points $u_n, v_n$ is always possible as the ball $\{z \in Z : \|z\| \leq \frac{1}{n}\}$ is infinite and $u_n, v_n$ should avoid finite sets.

After completing the inductive construction, we obtain the null-sequences $(u_n)_{n \in \omega}$ and $(v_n)_{n \in \omega}$ of pairwise distinct points of $Z$ such that for any points $x, y \in Z$ the intersection $\{x + u_n\}_{n \in \omega} \cap \{y + u_n\}_{n \in \omega}$ is finite and for every distinct points $x, y \in Z$ the intersections $\{x + u_n\}_{n \in \omega} \cap \{y + v_m\}_{m \in \omega}$ and $\{x + v_n\}_{n \in \omega} \cap \{y + v_m\}_{m \in \omega}$ are finite.

Using these facts, for every $n \in \omega$ we can choose a number $i_n \in \omega$ such that the set $\{z_n + v_k : k \geq i_n\}$ is disjoint with the set $\{z_i + u_m : i < n, m \in \omega\} \cup \{z_i + u_m : i \leq n, m \in \omega\}$ and the set $\{z_n + u_k : k \geq i_n\}$ is disjoint with the set $\{z_i + u_m : i < n, m \in \omega\} \cup \{z_i + v_m : i \leq n, m \in \omega\}$.

We claim that the set $A := \bigcup_{n \in \omega} \{z_n + u_n : m > i_n\}$ is null-finite in $Z$. This will follow as soon as we verify the condition:

1. For any $z \in Z$ the set $\{n \in \omega : z + v_n \in A\}$ is finite.

Find $j \in \omega$ such that $z = z_j$ and observe that for every $n \geq j$ the choice of the number $i_n$ guarantees that $\{z_n + u_n\}_{m > i_n} \cap \{z_j + v_n\}_{n \in \omega} = \emptyset$, so $\{n \in \omega : z + v_n \in A\}$ is contained in the finite set $\bigcup_{n < j} \{z_n + v_m\}_{m \in \omega} \cap \{z_n + u_n\}_{n \in \omega}$ and hence is finite.

Next, we show that the set $B := Z \setminus A$ is null-finite in $Z$. This will follow as soon as we verify the condition

1. For any $n \in \omega$ the set $\{m \in \omega : z + u_m \in B\}$ is finite.

Find a number $n \in \omega$ such that $z = z_n$ and observe that $\{m \in \omega : z + u_m \in B\} = \{m \in \omega : z_n + u_m \notin A\} \subseteq \{m \in \omega : m \leq i_n\}$ is finite. Therefore, the sets $A$ and $B = Z \setminus A$ are null-finite in $Z$.

Using Axiom of Choice, choose a subset $S \subseteq X$ such that for every $x \in X$ the intersection $S \cap (x + Z)$ is a singleton. It follows that $X = S + Z = (S + A) \cup (S + B)$. We claim that the sets $S + A$ and $S + B$ are null-finite in $X$. This will follow as soon as for any $x \in X$ we check that the sets $\{n \in \omega : x + z_n \in S + A\}$ and $\{n \in \omega : x + u_n \in S + B\}$ are finite. Since $X = S + Z$, there exist elements $s \in S$ and $z \in Z$ such that $x = s + z$. Observe that if for $n \in \omega$ we get $x + v_n \in S + A$, then $s + z + v_n = t + a$ for some $t \in S$ and $a \in A \subseteq Z$. Subtracting $a$ from both sides, we get $t = s + z - a \in S + Z$ and hence $t = s$ as $S \cap (S + Z)$ is a singleton containing both points $t$ and $s$. So, the inclusion $x + v_n = s + z + v_n \in S + A$ is equivalent to $z + v_n \in A$. By analogy we can prove that $x + u_n \in S + B$ is equivalent to $z + u_n \in B$. Then the sets

\[ \{n \in \omega : x + v_n \in S + A\} = \{n \in \omega : z + v_n \in A\} \quad \text{and} \quad \{n \in \omega : x + u_n \in S + B\} = \{n \in \omega : z + u_n \in B\} \]

are finite by the properties (3) and (4) of the null-finite sets $A, B$.

Proposition 2.3 and Theorem 8.1 imply the following corollary.

**Corollary 8.2.** Let $X$ be a non-discrete metric group.

\square
(1) Each null-finite subset of $X$ is contained in some invariant ideal on $X$.
(2) For any ideal $I$ on $X$ there exists a null-finite set $A \subset X$ such that $A \notin I$.

Proof. 1. For any null-finite set $A$ in $X$ the family
\[ \mathcal{I}_A = \{ I \subset X : \exists F \in [X]^{<\omega} I \subset FA \} \]
is an invariant ideal on $X$ whose elements have empty interiors in $X$ by Proposition 2.3. We recall that by $[X]^{<\omega}$ we denote the ideal of finite subsets of $X$.

2. By Theorem 8.1, the group $X$ can be written as the union $X = A \cup B$ of two null-finite sets. Then for any ideal $I$ on $X$ one of the sets $A$ or $B$ does not belong to $I$. \qed

9. Applying null-finite sets to additive functionals

In this section we apply null-finite sets to prove a criterion of continuity of additive functionals on metric groups.

A function $f : X \to Y$ between groups is called additive if $f(x + y) = f(x) + f(y)$ for every $x, y \in X$. An additive functional into the real line is called an additive functional.

**Theorem 9.1.** An additive functional $f : X \to \mathbb{R}$ on a metric group $X$ is continuous if it is upper bounded on a set $B \subset X$ which is not null-finite.

Proof. Suppose that the functional $f$ is not continuous. Then there exists $\varepsilon > 0$ such that $f(U) \not\subseteq (-\varepsilon, \varepsilon)$ for each neighborhood $U \subset X$ of zero. It follows that for every $n \in \omega$ there is a point $x_n \in X$ such that $\|x_n\| \leq \frac{1}{2^n}$ and $|f(x_n)| > \varepsilon$. Observe that, $\|2^n x_n\| \leq 2^n \cdot \|x_n\| \leq \frac{1}{2}$ and $|f(2^n x_n)| = 2^n |f(x_n)| > 2^n \varepsilon$. Choose $\varepsilon_n \in \{1, -1\}$ such that $\varepsilon_n f(2^n x_n)$ is positive and put $z_n := \varepsilon_n 2^n x_n$. Then $f(z_n) > 2^n \varepsilon$ and $\|z_n\| \leq \frac{1}{2^n}$.

We claim that the null-sequence $(z_n)_{n \in \omega}$ witnesses that the set $B$ is null-finite. Given any point $x \in X$ we need to check that the set $\Omega := \{ n \in \omega : x + z_n \in B \}$ is finite.

Let $M := \sup f(B)$ and observe that for every $n \in \Omega$ we have
\[ M \geq f(x + z_n) = f(x) + f(z_n) \geq f(x) + 2^n \varepsilon, \]
which implies $2^n \leq \frac{1}{\varepsilon} (M - f(x))$. Hence the set $\Omega$ is finite. Therefore, the null-sequence $(z_n)_{n \in \omega}$ witnesses that the set $B$ is null-finite, which contradicts our assumption. \qed

Theorems 9.1 and 5.1 imply

**Corollary 9.2.** An additive functional $f : X \to \mathbb{R}$ on a complete metric group is continuous if and only if it is upper bounded on some uBP set $B \subset X$ which is not Haar-meager.

An analogous result for Haar-null sets follows from Theorems 9.1 and 6.1

**Corollary 9.3.** An additive functional $f : X \to \mathbb{R}$ on a complete metric group is continuous if and only if it is upper bounded on some universally measurable set $B \subset X$ which is not Haar-null.

10. Applying null-finite sets to additive functions

In this section we prove some continuity criteria for additive functions with values in Banach spaces or locally convex spaces.

**Corollary 10.1.** An additive function $f : X \to Y$ from a complete metric group $X$ to a Banach space $Y$ is continuous if for any linear continuous functional $y^* : Y \to \mathbb{R}$ the function $y^* \circ f : X \to \mathbb{R}$ is upper bounded on some set $B \subset X$ which is not null-finite in $X$.

Proof. Assuming that $f$ is discontinuous, we can find $\varepsilon > 0$ such that for each neighborhood $U \subset X$ of $0$ we get $f(U) \not\subseteq B_\varepsilon := \{ y \in Y : \|y\| < \varepsilon \}$. Then for every $n \in \omega$ we can find a point $x_n \in X$ such that $\|x_n\| \leq \frac{1}{2^n}$ and $|f(x_n)| \geq \varepsilon$. Since $\|2^n x_n\| \leq 2^n \|x_n\| \leq \frac{1}{2}$, the sequence $(2^n x_n)_{n \in \omega}$ is a null-sequence in $X$. On the other hand, $\|f(2^n x_n)\| = 2^n \|f(x_n)\| \geq \varepsilon$ for all $n \in \omega$, which implies that the set $\{ f(2^n x_n) \}_{n \in \omega}$ is unbounded in the Banach space $Y$. By the Banach-Steinhaus Uniform Boundedness Principle [14, 3.15], there exists a linear continuous functional $y^* : Y \to \mathbb{R}$ such that the set $\{ y^* \circ f(2^n x_n) : n \in \omega \}$ is unbounded in $\mathbb{R}$. By our assumption, the additive functional $y^* \circ f : X \to \mathbb{R}$ is upper bounded on some set $B \subset X$ which is not null-finite in $X$. By Theorem 9.1 the additive functional $y^* \circ f$ is continuous and thus the sequence $(y^* \circ f(2^n x_n))_{n \in \omega}$ converges to zero and hence cannot be unbounded in $\mathbb{R}$. This is a desired contradiction, completing the proof. \qed
Corollary \[10.1\] admits a self-generalization.

**Theorem 10.2.** An additive function \( f : X \to Y \) from a complete metric group \( X \) to a locally convex space \( Y \) is continuous if for any linear continuous functional \( y^* : Y \to \mathbb{R} \) the function \( y^* \circ f : X \to \mathbb{R} \) is upper bounded on some set \( B \subseteq X \) which is not null-finite in \( X \).

**Proof.** This theorem follows from Corollary \[10.1\] and the well-known fact [27, p.54] that each locally convex space is topologically isomorphic to a linear subspace of a Tychonoff product of Banach spaces. \( \square \)

Combining Theorem \[10.2\] with Theorems \[5.1\] and \[6.1\] we derive two corollaries.

**Corollary 10.3.** An additive function \( f : X \to Y \) from a complete metric group \( X \) to a locally convex space \( Y \) is continuous if for any linear continuous functional \( y^* : Y \to \mathbb{R} \) the function \( y^* \circ f : X \to \mathbb{R} \) is upper bounded on some \( uBp \)-set \( B \subseteq Y \) which is not Haar-meager in \( Y \).

**Corollary 10.4.** An additive function \( f : X \to Y \) from a complete metric group \( X \) to a locally convex space \( Y \) is continuous if for any linear continuous functional \( y^* : Y \to \mathbb{R} \) the function \( y^* \circ f : X \to \mathbb{R} \) is upper bounded on some universally measurable set \( B \subseteq Y \) which is not Haar-null in \( Y \).

11. Applying null-finite sets to mid-point convex functions

In this section we apply null-finite sets to establish a continuity criterion for mid-point convex functions on linear metric spaces.

A function \( f : C \to \mathbb{R} \) on a convex subset \( C \) of a linear space is called mid-point convex if

\[
    f\left(\frac{x + y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)
\]

for any \( x, y \in C \). Mid-point convex functions are alternatively called Jensen convex.

**Theorem 11.1.** A mid-point convex function \( f : G \to \mathbb{R} \) defined on an open convex subset \( G \subseteq X \) of a metric linear space \( X \) is continuous if and only if \( f \) is upper bounded on some set \( B \subseteq G \) which is not null-finite in \( X \) and whose closure \( \overline{B} \) is contained in \( G \).

**Proof.** The “only if” part is trivial. To prove the “if” part, assume that \( f \) is upper bounded on some set \( B \subseteq G \) such that \( \overline{B} \subseteq G \) and \( B \) is not null-finite in \( X \). We need to check the continuity of \( f \) at any point \( c \in G \). Shifting the set \( G \) and the function \( y \) by \( c \), we may assume that \( c = 0 \). Also we can replace the function \( f \) by \( f - f(\theta) \) and assume that \( f(\theta) = 0 \). In this case the mid-point convexity of \( f \) implies that \( f(\frac{x}{2} + \frac{y}{2}) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) \) for every \( x \in G \) and \( n \in \mathbb{N} \).

To derive a contradiction, suppose that the function \( f \) is not continuous at \( \theta \). Then there exists \( \varepsilon > 0 \) such that \( f(U) \not\subseteq (-\varepsilon, \varepsilon) \) for any open neighborhood \( U \subseteq G \). For every \( n \in \omega \) consider the neighborhood \( U_n := \{ x \in G : \|x\| \leq \frac{1}{n}, x \in (\frac{1}{n+1}G) \cap (-\frac{1}{n+1}G) \} \) and find a point \( x_n \in U_n \) such that \( |f(x_n)| \geq \varepsilon \). Replacing \( x_n \) by \(-x_n \), if necessary, we can assume that \( f(x_n) \geq \varepsilon \) (this follows from the inequality of Jensen convexity of \( f \)).

Since \( x_n \in \frac{1}{n}G \), for every \( k \leq n \) the point \( 2^k x_n \) belongs to \( G \) and has prenorm \( \|2^k x_n\| \leq 2^k \|x_n\| \leq \frac{2^k}{n} = \frac{1}{n} \) in the metric linear space \( (X, \| \cdot \|) \). This implies that \((2^k x_n)_{n \in \omega} \) is a null-sequence in \( X \). Since the set \( B \) is not null-finite, there exists \( a \in X \) such that the set \( \Omega := \{ n \in \mathbb{N} : a + 2^n x_n \in B \} \) is infinite. Then \( a \in \overline{B} \subseteq G \). Choose \( k \in \mathbb{N} \) so large that \( 2^{-k+1} a \in -G \) (such number \( k \) exists as \( -G \) is a neighborhood of \( \theta \)).

Observe that for every number \( n \in \Omega \) with \( n > k \) the mid-point convexity of \( f \) ensures that

\[
    2^{n-k} \varepsilon \leq 2^{n-k} f(x_n) \leq f(2^{n-k} x_n) = f\left(\frac{2^{-k+1} a + 2^{-k+1} a + 2^{-k+1} x_n}{2}\right) \leq \frac{1}{2} f(-2^{-k+1} a) + \frac{1}{2} f\left(2^{-k+1} a + 2^{-k+1} x_n\right) \leq \frac{1}{2} f(-2^{-k+1} a) + 2^{-k} f(a + 2^n x_n)
\]

and hence

\[
    \sup_{n \in \Omega} f(B) \geq \sup_{n \in \Omega} f(a + 2^n x_n) \geq \sup_{k < n \in \Omega} (2^n \varepsilon - 2^{k-1} f(-2^{-k+1} a)) = \infty,
\]

which contradicts the upper boundedness of \( f \) on \( B \). \( \square \)

Theorem \[11.1\] implies the following generalization of the classical Bernstein-Doetsch Theorem \[5\], due to Mehl [24]; see also a survey paper of Bingham and Ostaszewski [7].

**Corollary 11.2.** A mid-point-convex function \( f : G \to \mathbb{R} \) defined on an open convex subset \( G \) of a metric linear space \( X \) is continuous if and only if \( f \) is upper-bounded on some non-empty open subset of \( G \).
Combining Theorem [11.1] with Theorems [5.1] and [6.1] we obtain the following two continuity criteria, which answer the Problem of Baron and Ger [3, P239].

Corollary 11.3. A mid-point convex function \( f : G \to \mathbb{R} \) defined on a convex subset \( G \) of a complete metric linear space \( X \) is continuous if and only if it is upper bounded on some \( uBP \)-set \( B \subset G \) which is not Haar-meager in \( X \).

Proof. Assume that \( f \) is upper bounded on some \( uBP \)-set \( B \subset G \) which is not Haar-meager in \( X \).

Write the open set \( G \) of \( X \) as the union \( U = \bigcup_{n \in \omega} F_n \) of closed subsets of \( X \). By [9], the countable union of \( uBP \) Haar-meager sets is Haar-meager. Since the set \( B = \bigcup_{n \in \omega} B \cap F_n \) is not Haar-meager, for some \( n \in \mathbb{N} \) the subset \( B \cap F_n \) of \( B \) is not Haar-meager. By Theorem 5.1, \( B \cap F_n \) is not null-finite. Since \( B \cap F_n \subset F_n \subset G \) and \( f \) is upper bounded on \( B \cap F_n \), we can apply Theorem [11.1] and conclude that the mid-point convex function \( f \) is continuous.

Corollary 11.4. A mid-point convex function \( f : G \to \mathbb{R} \) defined on a convex subset \( G \) of a complete metric linear space \( X \) is continuous if and only if it is upper bounded on some universally measurable set \( B \subset G \) which is not Haar-null in \( X \).

Proof. The proof of this corollary runs in exactly the same way as the proof of Corollary 11.3 and uses the well-known fact [8] that the countable union of universally measurable Haar-null sets in a complete metric group is Haar-null.

12. Some Open Problems

In this section we collect some open problems related to null-finite sets.

It is well-known that for a locally compact Polish group \( X \), each Haar-meager set in \( X \) can be enlarged to a Haar-meager \( F_\omega \)-set and each Haar-null set in \( X \) can be enlarged to a Haar-null \( G_\delta \)-set in \( X \). Those “enlargement” results dramatically fail for non-locally compact Polish groups, see [13, 10]. We do not know what happens with null-finite sets in this respect.

Problem 12.1. Is each Borel null-finite subset \( A \) of a (compact) Polish group \( X \) contained in a null-finite set \( B \subset X \) of low Borel complexity?

By [28] (resp. [12, 4.1.6]), each analytic Haar-null (resp. Haar-meager) set in a Polish group is contained in a Borel Haar-null (resp. Haar-meager) set. On the other hand, each non-locally compact Polish group contains a coanalytic Haar-null (resp. Haar-meager) set which cannot be enlarged to a Borel Haar-null (resp. Haar-meager) set, see [13] (resp. [10]).

Problem 12.2. Is each (co)analytic null-finite set \( A \) in a Polish group \( X \) contained in a Borel null-finite set?

Our next problem ask about the relation of the \( \sigma \)-ideal \( \sigma N^X \) generated by null-finite sets to other known \( \sigma \)-ideals.

Problem 12.3. Let \( X \) be a Polish group.

1. Is \( \sigma H^X \subset \sigma N^X \)?
2. Is \( \sigma N^X = H^X \cap M^X \)?

The negative answer to both parts of Problem 12.3 would follow from the negative answer to the following concrete question.

Problem 12.4. Can the closed Haar-null set \( \prod_{n=2}^\infty C_n \) in the group \( X = \prod_{n=2}^\infty C_n \) be written as a countable union of Borel null-finite sets in \( X \)?

Problem 12.5. For an infinite Polish group \( X \) and an ideal \( I \in \{ \sigma N^X, \sigma M^X \} \) evaluate the standard cardinal characteristics of \( I \):

- \( \text{add}(I) = \min \{ |J| : J \subset I \text{ and } \bigcup J \notin I \} \);
- \( \text{non}(I) = \min \{ |A| : A \subset X \text{ and } A \notin I \} \);
- \( \text{cov}(I) = \min \{ |J| : J \subset I \text{ and } \bigcup J = X \} \);
- \( \text{cof}(I) = \min \{ |J| : \forall I \ni J \exists J \subset J \} \).

The cardinal characteristics of the \( \sigma \)-ideals \( H^X, M^X, \sigma H^X \) on Polish groups are well-studied [4, 2] and play an important role in Modern Set Theory.
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