Electromagnetic Duality, Quaternion and Supersymmetric Gauge Theories of Dyons

H. Dehnen and O. P. S. Negi

27th March 2022

Universität Konstanz
Fachbereich Physik
Postfach M 677
D-78457 Konstanz, Germany
Email: Heinz.Dehnen@uni-konstanz.de
ops_negi@yahoo.co.in

Abstract

Starting with the generalized potentials, currents, field tensors and electromagnetic vector fields of dyons as the complex complex quantities with real and imaginary counter parts as electric and magnetic constituents, we have established the electromagnetic duality for various fields and equations of motion associated with dyons in consistent way. It has been shown that the manifestly covariant forms of generalized field equations and equation of motion of dyons are invariant under duality transformations. Quaternionic formulation for generalized fields of dyons are developed and corresponding field equations are derived in compact and simpler manner. Supersymmetric gauge theories are accordingly reviewed to discuss the behaviour of dualities associated with BPS mass formula of dyons in terms of supersymmetric charges. Consequently, the higher dimensional supersymmetric gauge theories for N=2 and N=4 supersymmetries are analysed over the fields of complex and quaternions respectively.

1 Introduction

The asymmetry between electricity and magnetism became very clear at the end of 19th century with the formulation of Maxwell’s equations. Magnetic monopoles were advocated to symmetrize these equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge and accordingly the considerable literature has come in force. The fresh interests in this subject have been enhanced by ’t Hooft and Polyakov with the idea that the classical solutions having the properties of magnetic monopoles may be found in Yang - Mills gauge theories. Now it has become clear that monopoles are better understood in grand unified theories and supersymmetric gauge theories. Julia and Zee extended the ’t Hooft-Polyakov theory of...
monopoles and constructed the theory of non-Abelian dyons (particles carrying simultaneously electric and magnetic charges). The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen from the semi-classical quantization of global charge rotation degree of freedom of monopoles. In view of the explanation of CP-violation in terms of non-zero vacuum angle of world, the monopoles are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. Renewed interests in the subject of monopole has gathered enormous potential importance in connection of quark confinement problem in quantum chromodynamics, possible magnetic condensation leading to absolute color confinement in QCD, its role as catalyst in proton-decay, CP-violation, current grand unified theories and supersymmetric gauge theories.

There has been a revival in the formulation of natural laws within the framework of general quaternion algebra and basic physical equations. Quaternions were very first example of hyper complex numbers having the significant impacts on mathematics and physics. Moreover, quaternions are already used in the context of special relativity, electrodynamics, Maxwell’s equation, quantum mechanics, quaternion oscillator, gauge theories, supersymmetry and other branches of Physics and Mathematics. On the other hand supersymmetry is described as the symmetry of bosons and fermions. The quaternionic formulation of generalized electromagnetic fields of dyons has been developed in unique, simple, compact and consistent manner. Postulation of Heavisidian monopole immediately follows the structural symmetry between generalized gravito-Heavisidean and generalized electromagnetic fields of dyons.

In order to understand the theoretical existence of monopoles (dyons) and keeping in view their recent potential importance and the fact that the formalism necessary to describe them has been clumsy and is not manifestly covariant, in the present paper, we have revisited the generalized fields of dyons, electromagnetic duality, quaternion formulation, non-Abelian and supersymmetric gauge theories in a consistent manner. Starting with the idea of two four potentials and taking the generalized charge, current, field, potential and electromagnetic field tensor as complex quantities with real and imaginary parts of them as electric and magnetic constituents, we have discussed the manifestly covariant and dual invariant field theory of generalized electromagnetic fields of dyons. The electromagnetic duality between electric and magnetic constituents of dyons have been also established in terms of a duality matrix and Generalized Dirac Maxwell’s (GDM) equations, equation of motion, energy and momentum densities associated with dyons are shown to be invariant under duality transformations. It has been analysed that the dual invariance is maintained only with the introduction of magnetic charge and the present model has been shown to be consistent for point like monopoles too and the Chirality quantization parameter plays an important role in electromagnetic duality to obtain the Dirac quantization condition. Quaternion formulation of various field equations has also been discussed in terms its simple, compact and consistent analyticity and it has been shown that the quaternion field equations are not only covariant, it is invariant under quaternion transformations of different kinds and represents the self dual structure of dyonic fields. It has also been described the model of quaternion fields of dyons is valid in classical, quantum,
The algebra $\mathbb{H}$ of quaternion is a four-dimensional algebra over the field of real numbers $\mathbb{R}$ and a quaternion $\phi$ is expressed in terms of its four base elements as

$$\phi = \phi_\mu e_\mu = \phi_0 + e_1 \phi_1 + e_2 \phi_2 + e_3 \phi_3 (\mu = 0, 1, 2, 3),$$  \hspace{1cm} (1)

where $\phi_0, \phi_1, \phi_2, \phi_3$ are the real quartets of a quaternion and $e_0, e_1, e_2, e_3$ are called quaternion units and satisfies the following relations,

$$
e_0^2 = e_0 = 1, \quad e_j^2 = -e_0, \quad e_0 e_i = e_i e_0 = e_i (i = 1, 2, 3), \quad e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k (\forall i, j, k = 1, 2, 3),$$  \hspace{1cm} (2)

where $\delta_{ij}$ is the delta symbol and $\varepsilon_{ijk}$ is the Levi Civita three index symbol having value ($\varepsilon_{ijk} = +1$) for cyclic permutation, ($\varepsilon_{ijk} = -1$) for anti cyclic permutation and ($\varepsilon_{ijk} = 0$) for any two repeated indices. Addition and multiplication are defined by the usual distribution law $(e_j e_k)e_l = e_j (e_k e_l)$ along with the multiplication rules given by equation (2). $\mathbb{H}$ is an associative but non commutative algebra. If $\phi_0, \phi_1, \phi_2, \phi_3$ are taken as complex quantities, the quaternion is said to be a bi- quaternion. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$. We thus have $\phi = v + e_2 \omega (v, \omega \in \mathbb{C})$ and $v = \phi_0 + e_1 \phi_1$, $\omega = \phi_2 - e_1 \phi_3$ with the basic multiplication law changes to $ve_2 = -e_2 \bar{v}$. The quaternion conjugate $\bar{\phi}$ is defined as

$$\bar{\phi} = \phi_\mu e_\mu = \phi_0 - e_1 \phi_1 - e_2 \phi_2 - e_3 \phi_3 .$$  \hspace{1cm} (3)
In practice, $\phi$ is often represented as a $2 \times 2$ matrix $\phi = \phi_0 - i \vec{\sigma} \cdot \vec{\phi}$ where $e_0 = I$, $e_j = -i \sigma_j (j = 1, 2, 3)$ and $\sigma_j$ are the usual Pauli spin matrices. Then $\overline{\phi} = \sigma_2 \phi^T \sigma_2$ with $\phi^T$ the transpose of $\phi$. The real part of the quaternion $\phi_0$ is also defined as

$$Re \phi = \frac{1}{2} (\phi + \overline{\phi}) ,$$

(4)

where $Re$ denotes the real part and if $Re \phi = 0$ then we have $\phi = -\overline{\phi}$ and imaginary $\phi$ is known as pure quaternion written as

$$\phi = e_1 \phi_1 + e_2 \phi_2 + e_3 \phi_3 .$$

(5)

The norm of a quaternion is expressed as $N(\phi) = \phi \phi = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = \text{Det} (\phi) \geq 0$.

(6)

Since there exists the norm of a quaternion, we have division i.e. every $\phi$ has an inverse of a quaternion and is described as

$$\phi^{-1} = \frac{\overline{\phi}}{|\phi|} .$$

(7)

While the quaternion conjugation satisfies the following property

$$\overline{\phi_1 \phi_2} = \overline{\phi_2} \overline{\phi_1} .$$

(8)

The norm of the quaternion (6) is positive definite and enjoys the composition law

$$N(\phi_1 \phi_2) = N(\phi_1) N(\phi_2) .$$

(9)

Quaternion (11) is also written as $\phi = (\phi_0, \vec{\phi})$ where $\vec{\phi} = e_1 \phi_1 + e_2 \phi_2 + e_3 \phi_3$ is its vector part and $\phi_0$ is its scalar part. So, the sum and product of two quaternions are described as

$$(\alpha_0, \vec{\alpha}) + (\beta_0, \vec{\beta}) = (\alpha_0 + \beta_0, \vec{\alpha} + \vec{\beta}) ,$$

$$(\alpha_0, \vec{\alpha}) \cdot (\beta_0, \vec{\beta}) = (\alpha_0 \beta_0 - \vec{\alpha} \cdot \vec{\beta} , \alpha_0 \vec{\beta} + \beta_0 \vec{\alpha} + \vec{\alpha} \times \vec{\beta} ) .$$

(10)

Quaternion elements are non-Abelian in nature and thus represent a non-commutative division ring.

### 3 Fields Associated with Dyons

Starting with the idea of Cabbbing and Ferrari [45] of two four-potentials, a self-consistent, gauge covariant and Lorentz invariant quantum field theory of dyons have been developed [39, 40, 41, 42] on assuming the generalized charge, generalized current and generalized four-potential of dyons as a complex quantity with their real and imaginary parts as a electric and magnetic constituents i.e.

$$q = e - i g \quad (\text{generalized charge}) \quad (i = \sqrt{-1}),$$
$$J_\mu = J_\mu - i k_\mu \quad (\text{generalized four – current}) ,$$
$$V = A - i B \quad (\text{generalized four – potential}) ,$$

(11)
where $e$ is the electric charge, $g$ is the magnetic charge, $j$ is the electric four-current, $k$ is the magnetic four-current, $A$ is the electric four-potential and $B$ is the magnetic four-potential, introduction of second four potential gives rise to the removal of arbitrary string variables [1]. Singleton [46] described the necessity of second potential and called it as pseudo vector (hidden) potential to formulate the symmetric theory of classical electrodynamics in presence of electric and magnetic charges while the Hamiltonian formulation of the Theory containing electric and magnetic charges is also described by Barker-Granziani [47].

We recall now the symmetric Maxwell’s equations derived earlier [1] as the Generalized Dirac Maxwell’s (or GDM) equations for dyons given as

$$\vec{\nabla} \cdot \vec{E} = \rho_e, \quad \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{E}}{\partial t},$$

$$\vec{\nabla} \cdot \vec{H} = \rho_g, \quad \vec{\nabla} \times \vec{E} = -\vec{k} - \frac{\partial \vec{H}}{\partial t},$$

(12)

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field, $\rho_e$ is the charge source density due to electric charge, $\rho_m$ is the charge source density due to magnetic charge (monopole), $\vec{j}$ is the current source density due to electric charge and $\vec{k}$ is the current source density due to magnetic charge. We have used the notations $\{j\} = \{\rho_e, \vec{j}\}$, $\{k\} = \{\rho_m, \vec{k}\}$, $\{A\} = \{\phi_e, \vec{A}\}$ and $\{B\} = \{\phi_g, \vec{B}\}$ along with the metric as (+,−,−,−). We have also used the unit value of coefficients along with natural units $c = h = 1$ throughout the text. The electric and magnetic fields of dyons satisfying the GDM equations (12) are now expressed in terms of components of two four potentials in a symmetrical manner i.e.

$$\vec{E} = -\vec{\nabla} \phi_e - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{B},$$

$$\vec{H} = -\vec{\nabla} \phi_g - \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{A}.$$  

(13)

The vector field $\vec{\psi}$ associated with generalized electromagnetic fields of dyons is defined as

$$\vec{\psi} = \vec{E} - i \vec{H}$$

(14)

which reduces the four GDM equations (12) to following two differential equations as

$$\vec{\nabla} \cdot \vec{\psi} = \rho, \quad \vec{\nabla} \times \vec{\psi} = -i \vec{j} - i \frac{\partial \vec{\psi}}{\partial t}$$

(15)

where $\rho = \rho_e - i \rho_g$ is the generalized charge and $\vec{j} = \vec{j} - i \vec{k}$ is the generalized current source densities of dyons. These are regarded as the temporal and spatial components of generalized four current $\{J\} = \{\rho, \vec{j}\}$ of dyons. As such, we may write the generalized electromagnetic field vector $\vec{\psi}$ in the following manner
in terms of components of generalized four potential \( \{ V \} = \{ \varphi, V \} \) with \( \varphi = \phi_e - i \phi_g \) and \( \vec{V} = \vec{A} - i \vec{B} \) as

\[
\vec{\psi} = -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla} \varphi - i \vec{\nabla} \times \vec{\psi}.
\]

(16)

We may now write the following covariant forms of generalized Maxwell’s-Dirac equations of dyons i.e.

\[
\partial^\nu F_{\mu \nu} = F_{\mu \nu, \nu} = j, \quad \partial^\nu \tilde{F}_{\mu \nu} = \tilde{F}_{\mu \nu, \nu} = k,
\]

(17)

with

\[
F_{\mu \nu} = E_{\mu \nu} - \tilde{H}_{\mu \nu}, \quad \tilde{F}_{\mu \nu} = H_{\mu \nu} + \tilde{E}_{\mu \nu},
\]

\[
E_{\mu \nu} = A_{\mu, \nu} - A_{\nu, \mu} = \partial_{\nu} A - \partial A_{\nu}, \quad H_{\mu \nu} = B_{\mu, \nu} - B_{\nu, \mu} = \partial_{\nu} B - \partial B_{\nu},
\]

\[
\tilde{E}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \sigma \lambda} E^{\sigma \lambda}, \quad \tilde{H}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \sigma \lambda} H^{\sigma \lambda},
\]

(18)

where the tilde denotes the dual, \( E_{\mu \nu} \) and \( H_{\mu \nu} \) are the electromagnetic field tensors respectively due to the presence of electric and magnetic charges and \( \varepsilon_{\mu \nu \sigma \lambda} \) is the four index Levi-Civita symbol. Generalized electric and magnetic fields of dyons given by equations (13) may now directly be obtained from the components of field tensors \( F_{\mu \nu} \) and \( \tilde{F}_{\mu \nu} \) described in terms of two potential as,

\[
F_{0j} = E_j, \quad F_{jk} = \varepsilon_{jkl} H^l,
\]

\[
\tilde{F}_{0j} = -H_j, \quad \tilde{F}_{jk} = -\varepsilon_{jkl} E^l.
\]

(19)

Taking the curl of second part of equation (15) and using first part of equation (15), we obtain a new vector parameter \( \vec{S} \) (say) as

\[
\vec{S} = \Box \vec{\psi} = -\frac{\partial \vec{J}}{\partial t} - \vec{\nabla} \rho - i \vec{\nabla} \times \vec{J}.
\]

(20)

where \( \Box \) represents the D’Alembertian operator i.e.

\[
\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.
\]

(21)

Let us define the complex form of generalized field tensor of dyons as

\[
G_{\mu \nu} = F_{\mu \nu} - i \tilde{F}_{\mu \nu},
\]

(22)

which directly combines the two differential equations given by equation (17) in to the following covariant field equation of dyons i.e.

\[
\partial^\nu G_{\mu \nu} = G_{\mu \nu, \nu} = J,
\]

(23)

with \( G_{\mu \nu} = V_{\mu, \nu} - V_{\nu, \mu} = \partial_{\nu} V - \partial V_{\nu} \) is named as the generalized electromagnetic field tensors of dyons. The present model for generalized fields of dyons may also be explained [39, 40] in terms of the suitable Lagrangian density to obtain the field
equation described above along with the following form of Lorentz force equation of motion for dyons i.e.

$$f = m \frac{d^2 x_\mu}{d\tau^2} = \frac{1}{2} (q G^*_{\mu\nu} + q^* G_{\mu\nu}) u^\nu = \left( e F_{\mu\nu} + g \tilde{F}_{\mu\nu} \right) u^\nu$$  \(24\)

where \(m\) is the mass of particle, \(x_\mu\) is the position, \(\tau\) is the proper time, \((\ast)\) denotes the complex conjugate and \(\{u^\nu\}\) is the four-velocity of the particle. Equation \(24\) reduces to the following usual form of equation of motion for a particle carrying simultaneously electric and magnetic charges i.e.

$$m \ddot{x} = \vec{f} = e (\vec{E} + \vec{u} \times \vec{H}) + g (\vec{H} - \vec{u} \times \vec{E})$$  \(25\)

where \(\vec{E}\) and \(\vec{H}\) are electric and magnetic fields given by equation \(13\). Equation \(25\) may also be written accordingly \[40\] in terms of generalized charge and generalized vector field \(\psi\).

4 Electromagnetic Duality

The concept of duality has been receiving much attention \[17\] \[18\] \[19\] \[20\] \[49\] \[50\] \[51\] \[52\] \[53\] \[54\] in gauge theories, field theories, supersymmetry and super strings. Duality invariance is an old idea introduced a century ago in classical electromagnetism for Maxwell’s equations in vacuum as these were invariant not only under Lorentz and conformal transformations but also invariant under the following duality transformations,

$$\vec{E} = \vec{E} \cos \theta + \vec{H} \sin \theta, \quad \vec{H} = -\vec{E} \sin \theta + \vec{H} \cos \theta.$$  \(26\)

Dirac \[1\] put forward this idea and introduced the concept of magnetic monopole not only to symmetrize Maxwell’s equations but also to make them dual invariant. Consequently, the the GDM equations \(12\) are invariant under duality transformations \(26\). For a particular value of \(\theta = \frac{\pi}{2}\) we get the following discrete duality transformations,

$$\vec{E} \mapsto -\vec{H}; \quad \vec{H} \mapsto -\vec{E}.$$  \(27\)

and correspondingly we get the duality transformations for electric and magnetic charges i.e.

$$e \mapsto g; \quad g \mapsto -e.$$  \(28\)

Under these duality transformations the GDM equation \(12\) and equation \(25\) for the Lorentz force equation of motion are invariant. As such, the duality is preserved in the theory of simultaneous existence of electric and magnetic charges if we include transformations \(27\) to the electric and magnetic fields of dyons given by equation \(13\) along with the following transformations to the potential and current components i.e.,

$$\{A\} \mapsto \{B\}; \quad \{B\} \mapsto -\{A\}$$

$$\{j\} \mapsto \{k\}; \quad \{k\} \mapsto -\{j\}.$$  \(29\)

In general we can write the duality transformations in terms of duality matrix representation as
\[
\begin{pmatrix}
  e \\
  g
\end{pmatrix} = \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  e \\
  g
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \vec{E} \\
  \vec{H}
\end{pmatrix} \mapsto \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  \vec{E} \\
  \vec{H}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  A \\
  B
\end{pmatrix} \mapsto \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  A \\
  B
\end{pmatrix},
\]

\[
\begin{pmatrix}
  j \\
  k
\end{pmatrix} \mapsto \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  j \\
  k
\end{pmatrix}.
\]

Consequently, the covariant forms of GDM given by equations (17) are invariant under the transformations of duality along with the following transformations for the components of electromagnetic field tensors i.e.

\[
\begin{pmatrix}
  F_{\mu\nu} \\
  F_{\mu\nu}^\ast
\end{pmatrix} \mapsto \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  F_{\mu\nu} \\
  F_{\mu\nu}^\ast
\end{pmatrix}
\]

(31)

If we consider the generalized fields of dyons as complex quantities the duality transformations now take the following forms respectively for the electromagnetic complex field vector, generalized charge, potential, current and generalized electromagnetic field tensors of dyons as

\[
\vec{\psi} \mapsto \exp i\theta (\vec{\psi}) \\
q \mapsto \exp i\theta (q) \\
V \mapsto \exp i\theta (V) \\
J \mapsto \exp i\theta (J) \\
G_{\mu\nu} \mapsto \exp i\theta (G_{\mu\nu})
\]

(32)

Hence with these transformations equations the GDM equations (15), covariant field equation (23) and equation of motion (25) associated with dyons in complex representation are invariant. The energy and momentum densities of generalized electromagnetic field of dyons is described respectively as

\[
\frac{1}{2} |\vec{\psi}|^2 = \frac{1}{2}(|\vec{E}|^2 + |\vec{H}|^2); \quad \frac{1}{2i}(\vec{\psi}^\ast \times \vec{\psi}) = \vec{E} \times \vec{H}
\]

(33)

are invariant under duality transformations and the real and imaginary parts of

\[
\frac{1}{2} (\vec{\psi}^\ast)^2 = \frac{1}{2}(|\vec{E}|^2 - |\vec{H}|^2) - i \vec{E} \cdot \vec{H}
\]

(34)

are described respectively as the Lagrangian and the charge density density transform as the doublet under the duality group. The duality is thus maintained only with the introduction of magnetic charge which has been always a challenging new frontier and its existence is still questionable. The present model is still consistent for point like monopoles in view of covariant formulation and duality invariance. The coupling between two generalized charges \( q_i \) and \( q_k \) is described as

\[
q_i^* q_k = (\epsilon_j \epsilon_k + g_j g_k) - i(\epsilon_j g_k - \epsilon_k g_j) = \alpha_{jk} - i jk
\]

(35)
where the real part $\alpha_{jk}$ is called the electric coupling parameter (the Coulomb like term) responsible for the existence of either electric charge or magnetic monopole while the imaginary part $j_k$ is the magnetic coupling parameter and plays an important role for the existence of magnetic charge. Both of these parameters are invariant under the duality transformations. The parameter $j_k$ has also been named as Chirality quantization parameter \[2, 3, 39, 40\] for dyons and leads the following charge quantization condition i.e.

$$ j_k = \pm n \quad (n = 0, 1, 2, 3, .......) \quad (36) $$

where the half integral quantization is forbidden by chiral invariance and locality in commutator of the electric and magnetic vector potentials \[39\]. If we consider two dyons with $q_j = (e, 0)$ and $q_k = (0, g)$ the quantization condition \(36\) reduces to well known Dirac quantization condition $eg = \pm n$. If we do not consider Dirac particle as dyon the Dirac quantization condition loses its dual invariance. Thus dyon plays an important role in electromagnetic duality with the association of Chirality quantization parameter and it is important to consider the consistent quantum field theory for the simultaneous existence of electric and magnetic charges (dyons). Now it is also expected \[18\] that just as the energy density given by equation \(33\) respects the symmetry given by equation \(32\) for generalized vector field, on the similar grounds the mass formula for dyon also respects this symmetry \(32\) for the generalized charge so that

$$ M(e, g) = M(|e - ig|) = M(\sqrt{e^2 + g^2}) \quad (37) $$

which plays an important role in supersymmetric gauge theories.

## 5 Quaternion Formulation for Dyons

In section 3 we have reduced four sets of GDM \(12\) to two sets of field equation \(13\) and we have established a relation between potential and field by equation \(16\) and correspondingly derived a relation between current and the new field vector given by equation \(20\). These two equations are a kind of potential and current equation and have a direct analogue to quaternion formulation of dyons. To do this let us define the space time four differential operator as the quaternion in the following manner

$$ \Box = -i \frac{\partial}{\partial t} + e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} \quad (38) $$

which has been the quaternionic form of four differential operator $\{\partial_\mu\} = (-i \frac{\partial}{\partial t}, \vec{\nabla})$. Similarly we can define quaternionic form of generalized four potential of dyons as

$$ V = -i \varphi + e_1 V_x + e_2 V_y + e_3 V_z. \quad (39) $$

Now operating equation \(38\) to equation \(39\) and using the relations \(2\) for the quaternion units, we get
\[ \Box V = \psi \] (40)

where \( \varphi \) is again a quaternion defined as

\[ \psi = -\psi_t + ie_1\psi_x + ie_2\psi_y + ie_3\psi_z \] (41)

with

\[ \psi_t = \partial_t \varphi + \partial_x V_x + \partial_y V_y + \partial_z V_z. \] (42)

Here we have \( \psi = (\psi_x, \psi_y, \psi_z) \) and \( \psi_t = 0 \) because of Lorentz gauge conditions applied separately on electric and magnetic four potential. As such, the relation \( \Box \) represents the quaternion differential equation form of generalized potential of dyons. Let us take the any inhomogeneous quaternion differential equation as

\[ \Box f = b \] (43)

where \( \Box = -i \frac{\partial}{\partial t} - e_1 \frac{\partial}{\partial x} - e_2 \frac{\partial}{\partial y} - e_3 \frac{\partial}{\partial z} \) is the quaternion conjugate differential operator. This equation reduces to the GDM equations \( \Box \) by identifying quaternion function variable \( f \) as the \( \psi \) and the right hand side quaternion variable as the quaternion valued four current of dyons given by

\[ J = -i \rho + e_1 J_x + e_2 J_y + e_3 J_z. \] (44)

Similarly on operating equation \( \Box f = b \) to equation \( \Box f = b \) we get

\[ \Box J = S \] (45)

where

\[ S = -S_t + ie_1 S_x + ie_2 S_y + ie_3 S_z, \]

\[ S_t = \partial_t \rho + \partial_x J_x + \partial_y J_y + \partial_z J_z. \] (46)

The components of \( \Box S = (S_x, S_y, S_z) \) are given by equation \( \Box \) and \( S_t = \partial_t \rho + \partial_x J_x + \partial_y J_y + \partial_z J_z = 0 \) due to continuity equation applied individually on electric and magnetic four currents. Thus we have seen that there are four sets of GDM differential equation on real representation, two sets in complex representation and one set in quaternion representation. Thus the quaternion formulation of dyon is compact, simpler and manifestly covariant under quaternion Lorentz transformation. The theory corresponds to two dimensional representation in complex case and four dimensional representation to real case. In other words \( N = 1 \) quaternion representation maps to \( N = 2 \) dimensional complex and \( N = 4 \) dimensional real representation. Similarly, we may develop the quaternionic forms of other differential equations in simple, compact and consistent manner. The theory of quaternion variables presented here to the case of dyon is dual invariant as the quaternion quantities are self dual. Here we have used the bi quaternions instead of real quaternions to establish the relations among the complex parameters of dyons. This model of quaternions is valid for the classical and quantum dynamics of individual electric and magnetic charges in the absence of each other. Thus two dual theories are coupled together in this formalism and there is no difficulty to represent them in terms of complex components of a bi quaternion variables.
The free particle quaternion Dirac equation is described \cite{55, 56} as,

\[
(i \gamma^\mu \partial_\mu - m)\Psi(x,t) = 0
\]

(47)

where \( \Psi(x,t) = \begin{pmatrix} \Psi_a(x,t) \\ \Psi_b(x,t) \end{pmatrix} \) is the two component spinor and

\[
\Psi_a(x,t) = \Psi_0 + i \Psi_1; \quad \Psi_b(x,t) = \Psi_2 - i \Psi_3
\]

(48)

are the components of spinor quaternion \( \Psi \) and \( \gamma \) matrices are also defined in terms of quaternion units i.e.

\[
\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \gamma_j = \epsilon_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (j = 1, 2, 3)
\]

(49)

The set of pure quaternion field defined by equation (1) is invariant under the transformations

\[
\phi \rightarrow \phi' = U \phi U^\dagger, \quad U \in Q, \quad U U^\dagger = 1
\]

(50)

and accordingly the quaternion conjugate transforms like

\[
\overline{\phi'} = \overline{U \phi U} = U \overline{\phi} U = -U \overline{\phi} U = -\phi'.
\]

(51)

Any \( U \in Q \) has a decomposition like equation (50) and the quaternion differential equations of dyons discussed above are invariant under these transformations of a quaternion. Transformation equation (50) gives rise to a set \( \{U \in Q; \quad U U^\dagger = 1\} \sim SP(1) \sim SU(2) \). Though it has been emphasized earlier \cite{26, 27, 29} that the automorphic transformation of \( Q \)-fields are local but one can select it according to the representation. On the other hand a \( Q \)-field is subjected to more general \( SO(4) \) transformations

\[
\phi \rightarrow \phi' = U_1 \phi U_2, \quad U_1, U_2 \in Q, \quad U_1 U_1^\dagger = U_2 U_2^\dagger = 1.
\]

(52)

and the covariant derivative is described in terms of two \( Q \)- gauge fields i.e

\[
D_\mu \phi = \partial_\mu \phi + A \phi - \phi B
\]

(53)

with

\[
A' = U_1 A U_1^\dagger + (\partial_\mu U_1) U_1^\dagger; \\
B' = U_2 B U_2^\dagger + (\partial_\mu U_2) U_2^\dagger
\]

(54)

where \( A \) and \( B \) are used in our model as the four potential associates with electric and magnetic charge of dyons in Abelian gauge theory where the gauge transformations are Abelian and global and the quaternion covariant derivative given by equation (53) supports the idea of two four potential. As such it is possible to extend our theory to the case of non-Abelian gauge theory and we may write the
Dirac equation for dyons as follows by replacing the partial derivative to covariant
derivative i.e.

\[(i \gamma^\mu D_\mu - m) \Psi(x, t) = 0\]  \hspace{1cm} (55)

with the following definition

\[\left[D_\mu, D_\nu\right] \Psi = D_\mu(D_\nu \Psi) - D_\nu(D_\mu \Psi) = F_{\mu\nu} \Psi - \Psi \tilde{F}_{\mu\nu}\]  \hspace{1cm} (56)

where the gauge field strengths \(F_{\mu\nu}\) and \(\tilde{F}_{\mu\nu}\) are defined in section-3 for Abelian
gauge theory of generalized electromagnetic fields associated with dyons. Here we
can express now the four potentials (gauge potentials) in terms of quaternion as

\[A = A^0 e_0 + A^1 e_1 + A^2 e_2 + A^3 e_3,\]
\[B = B^0 e_0 + B^1 e_1 + B^2 e_2 + B^3 e_3.\] (57)

As such, the Abelian theory of dyons discussed in section-3 and section-4 can now
be restored by putting the conditions \(U_1 U_1 = U_2 U_2 = 1\) implying that \((A^0)' = (A^0)\)
and \((B^0)' = (B^0)\). It is also possible when we have \(A = \overline{A}\) and \(B = \overline{B}\). However
if we consider the imaginary quaternion i.e. \(A = -\overline{A}\) and \(B = -\overline{B}\) we have
the \(SU(2) \times SU(2)\) gauge structure and \(A = A^a e_a = A^1 e_1 + A^2 e_2 + A^3 e_3\) and
\(B = B^a e_a = B^1 e_1 + B^2 e_2 + B^3 e_3\). Thus with the implementation of condition
\(U_1 U_1 = U_2 U_2 = 1\) there are only the six gauge fields \(A^a\) and \(B^a\) associated with the
covariant derivative of Dirac equation (55). The transformation equation (52) is
continuous and isomorphic to \(SO(4)\) i.e.

\[\overline{\phi'} \phi' = (U_1 \phi U_2)(U_1 \phi U_2) = U_2 \overline{\phi} U_1 U_1 \phi U_2 = U_2 \overline{\phi} U_2 = \phi \phi.\] (58)

The resulting \(Q-\) gauge theory has the correspondence \(SO(4) \sim SO(3) \times SO(3)\)
and the spinor transforms as left and right component (electric or magnetic) spinors as

\[\Psi_e \mapsto (\Psi_e)' = U_1 \Psi_e,\]
\[\Psi_g \mapsto (\Psi_g)' = U_2 \Psi_g.\] (59)

The following split basis of quaternion units my also be considered as

\[u_0 = \frac{1}{2}(1 - i e_3), \hspace{1cm} u_0^* = \frac{1}{2}(1 + i e_3),\]
\[u_1 = \frac{1}{2}(e_1 + i e_2), \hspace{1cm} u_1^* = \frac{1}{2}(e_1 - i e_2).\] (60)

to constitute the \(SU(2)\) doublets. As such, we may classify the \(Q-\) classes into five
groups and can describe the theory accordingly. These five irreducible representations
of \(SO(4)\) are realized as

1. \((U_1, U_2) \mapsto SO(4) \mapsto (2, 2)\)
2. \((U_1, U_1) \mapsto SU(2) \mapsto (3, 1)\)
3. \((U_2, U_2) \mapsto SU(2) \mapsto (1, 3)\)
4. \((U_1, 1) \mapsto Spinor \mapsto (2, 1)\)
5. \((U_2, 1) \mapsto Spinor \mapsto (1, 2)\) (61)
and accordingly it is easier to develop a non-Abelian gauge theory of dyons. Here it is important to emphasize that the $\gamma-$ matrices are defined above in terms of quaternions and when we talk about quaternion local gauge transformations the $\gamma-$matrices are to be local and space time dependent. Such possibility of $\gamma-$matrices is already explored by Dehnen-Hitzer [57]. Here we have just described the existence of quaternion Dirac equation without deriving its solution and other consequences which will be further programme.

7 Non-Abelian Dyons

Now it is widely recognized that magnetic monopoles are better understood in non-Abelian gauge theories but if we apply the two potential approach to them, it leads to the existence of two different photons. This possibility may be seen only in Abelian gauge theories not in non Abelian gauge theories where it is not important to consider the complex gauge potential. Here the gauge potentials are subjected by one of the above mentioned five irreducible representations of quaternion transformations. In Abelian gauge theory the existence of second photon is minimized [39, 40] with the help of constancy condition which relates the electric and magnetic parameters and is given by

$$\frac{q}{e} = \frac{B}{A} = \frac{k}{j} = \frac{\vec{F}_{\mu\nu}}{\vec{F}_{\mu\nu}} = \tan \theta.$$  

(62)

Similar relation for $U_1$ and $U_2$ associated with quaternion transformations may be developed. Equation (62) shows that if dyons exists $\tan \theta$ should have finite value because for $\theta = 0$, the theory of electric charge particle exit while for $\theta = \frac{\pi}{2}$ the theory of magnetic charge particle exists. This may be one of the region that monopoles are not found as point like objects but are considered as extended objects with finite shape and size. First step in this direction for the theory of monopoles was put forward by 't Hooft-Polyakov [8, 9] who wrote their famous paper on the existence of a regular magnetic monopole solution in Georgi-Glashow model [6] with the gauge group $G = SO(3)$ is broken down to $U(1)$ by a Higgs field in the triplet representation. Their argument implies the existence of such regular monopoles in any unified gauge theory where the $U(1)$ of electromagnetism would be a sub group of some simple non Abelian group. We can analyzed this in terms of quaternion gauge groups too. The crucial difference between non Abelian monopole and Dirac monopole was that these monopoles appear as regular Solitons like solutions to the classical field equations and can not be avoidable. This type of monopole satisfies the Dirac quantization condition with the finite energy solutions. This may be one of the important region with the definition of duality at $\theta = \frac{\pi}{2}$ where only monopole can exist with the different origin from the electric charge. In nut shell we say that 't Hooft-Polyakov monopole solutions describe an object of finite size which from far away can not be distinguished from a Dirac monopole of charge $g \mapsto -\frac{1}{e}$. If we take the case $(U_2, U_2) \Rightarrow SU(2) \Rightarrow (1, 3)$ ( isomorphic to $SO(3)$ ) we can interpret our result analogous to the Georgi-Glashow model consisting of $SO(3)$ gauge, $B^a$ and Higgs triplet $\Phi^a$ in the theory of 't Hooft-Polyakov monopole and obtain following equations of motion

$$DF^{a\mu\nu} = \alpha \varepsilon^{abc} \Phi^b D^\nu \Phi^c, \quad (D_\mu D_\mu \Phi)^a = -\lambda \Phi^a (\Phi^b \Phi^b - v^2),$$

$$D_\mu \vec{F}^{a\mu\nu} = 0$$

(63)
where we introduce the Yang-Mills field strength

\[ F_{\mu\nu}^a = \partial_\mu B_{\nu}^a - \partial_\nu B_{\mu}^a + \alpha \varepsilon^{abc} B_{\mu}^b B_{\nu}^c, \]  

(64)

the covariant derivative

\[ D\Phi^a = \partial_\mu \Phi^a + \varepsilon^{abc} B_{\mu}^b \Phi^c, \]  

(65)

and the Higgs potential

\[ V(\Phi) = \frac{\lambda}{4} (\Phi^a \Phi^a - v^2)^2. \]  

(66)

There are a simple formulas \[48\] can be obtained accordingly for the resultant masses of gauge particles

\[ M(e, 0) = v |e| \]  

(67)

where \( e \) is the eigen value of electric charge of a massive eigenstate and \( v \) specifies the the magnitude of the vacuum expectation value of scalar Higgs field when we take \((U_1, U_1) \Rightarrow SU(2) \mapsto (3, 1)\) structure of a quaternion. Similar formula can be calculated for ’t Hooft-Polyakov monopole named as Bogomolny bound \[58\], is given by

\[ M(0, g) \geq v |g| \]  

(68)

which is possible in Prasad-Sommerfield limit \[59\], where the the ’t Hooft-Polyakov monopole and Julia-Jee dyon \[10\] solutions are generalized to vanish the self interaction of Higgs field and in this case we obtain the Bogomolny equation \( D\Phi = H \) which is the reduction form of quaternion equation \([10]\). Olive \[48\] showed that the Bogomolny bound \[58\] which is the self dual can be generalized to contain the higher value of magnetic (topological) charge and thus it is possible for each magnetic monopole Solitons to carry an electric charge \( e \). The electromagnetic duality hence gives rise the following bound on the mass of dyon for all angles \( \theta \) i.e.

\[ M \geq v (e \cos \theta + g \sin \theta). \]  

(69)

The sharpest bound is obtained when the right hand side has maximum value and it happens for \( e \sin \theta = g \cos \theta \). In other words \( \tan \theta = \frac{g}{e} \) which is the constancy condition \([62]\). Thus the electromagnetic duality given by equation \([62]\) implies a generalization of Bogomolny bound \([58]\) to give rise the mass of a dyon as

\[ M(e, g) = v |e - ig| = v \sqrt{(e^2 + g^2)}, \]  

(70)

which is known as BPS mass formula. This mass formula does not distinguish between the fundamental quantum particles and the magnetic monopoles, being applicable to all of them, like meson-Solitons democracy in Sine Gorden Model. The BPS mass formula is universal and is also invariant under electromagnetic duality transformations. The natural frame work for the realization of this symmetry in quantum field theory is the \( N = 4 \) supersymmetry \([60]\).
Finite quantum theories seem to exhibit a duality symmetry between electricity and magnetism as stated earlier. The \( \mathbb{Z}_2 \) duality transforms the coupling constant of the theory and its inverse and exchanges (non-Abelian) electric and magnetic charges. The spectrum due to BPS mass formula given by equation (70) is invariant under electromagnetic \( \mathbb{Z}_2 \) duality (\( e, g \) \( \mapsto \) \( g, -e \)) is the consequence of the fact that the formula for Bogomolny bound is invariant under duality and the spectrum saturates the bound. This observation prompted Montonen and Olive [49] to conjecture that there should be a dual (magnetic') description of the gauge theory where the elementary particles are the BPS monopole and the massive vector bosons appear as to be 'electric monopole'. It is speculated that the mass less Higgs could play the role of a Goldstone boson associated to the breaking of \( SO(2) \) symmetry down to \( \mathbb{Z}_2 \). Montonen and Olive duality is expressed as

\[
e \mapsto g = \pm \frac{1}{e} \quad \text{(in the units of } 4\pi \hbar)\]

A dyon is not invariant under \( CP- \) transformation as we get

\[
(e, g) \mapsto (-e, g)
\]

and gauge theories contain a parameter that breaks \( CP- \). This difficulty is removed by Witten [12] by adding a so called \( \vartheta \)-term (or vacuum angle) to the Yang Mills Lagrangian with out spoiling its renormalizability. Being a total derivative, it does not affect the classical equations of motion. It violates \( P- \) and \( CP-\) but not \( C- \), which makes it as a good candidate for generalizing the long range behaviour of the theory while maintaining the duality. By adding the \( CP- \)-violating term \( \frac{e\vartheta}{8\pi^2}g \) to the electric charge of dyon, the duality conjecture implies a richer dyonic spectrum and has been tested accordingly. With this conjecture and applying the Dirac quantization condition \( eg = 4\pi n_m \) where \( n_m \in \mathbb{Z} \), we get a general dyon as

\[
(e, g) \Rightarrow (n_e e + \frac{e\vartheta}{2\pi} n_m, \frac{4\pi}{e} n_m) \quad (\forall n_e, n_m \in \mathbb{Z})
\]

where \( n_e \), & \( n_m \) are the numbers of electric and magnetic charges presented in the system of a dyon respectively. Witten- effect [12] provides a physical meaning of electric charge \( q_e = (n_e e + \frac{e\vartheta}{2\pi} n_m) \) of a BPS monopole (dyon) to shift it by \( \vartheta \mapsto \vartheta + 2\pi \). As such, the net charge on a dyon becomes

\[
Q = (e, -ig) \mapsto n_e e + \frac{e\vartheta}{2\pi} n_m - i \frac{4\pi}{e} n_m \Rightarrow \\
Q = e \left[ n_e + n_m \left( \frac{\vartheta}{2\pi} - i \frac{4\pi}{e^2} \right) \right] \Rightarrow e(n_e + n_m \tau)
\]

where we have defined the complex parameter

\[
\tau = \left( \frac{\vartheta}{2\pi} - i \frac{4\pi}{e^2} \right)
\]

and accordingly the Bogomolny bound takes the following form
$M \geq ve |n_e + n_m \tau|$. \hspace{1cm} (76)

To write the Lagrangian density in terms of complex parameter $\tau$, it is now convenient to introduce the complex linear combination of gauge field strengths in the similar manner as we have introduced in the theory of dyons give by equation \[(22)\] i.e.

$$\vec{G}_{\mu\nu} = \vec{F}_{\mu\nu} - i\vec{\tilde{F}}_{\mu\nu}$$ \hspace{1cm} (77)

and we need accordingly the complex structure of gauge potential to obtain this gauge field strength. This gauge field strength may be obtained directly from quaternion gauge analyticity. When the ‘theta angle’ is included in the complex coupling constant $\tau$, the group of duality transformation is $SL(2, \mathbb{Z})$ acting on the vectors of electric and magnetic charges and projectively on the coupling constant.

It is to be noticed that the physics is periodic in $\vartheta$ with a period $2\pi$, we have the duality transformation ($\vartheta = 2\pi$),

$$T : \quad \tau \mapsto \tau + 1$$ \hspace{1cm} (78)

whereas the Montonen-Olive duality transformation \[(71)\] takes the following form (at $\vartheta = 0$) in terms of $\tau$,

$$S : \quad \tau \mapsto -\frac{1}{\tau}$$ \hspace{1cm} (79)

The operator $S$ and $T$ are the invertible operators and generate a discrete group. Equations \[(78)\] and \[(79)\] generate the group $SL(2, \mathbb{Z})$ of projective transformations for full duality transformations i.e.

$$\tau = \frac{a\tau + b}{c\tau + d} \quad \text{where} \ a, b, c, d \in \mathbb{Z}, \ \text{and} \ ad - bc = 1$$ \hspace{1cm} (80)

BPS mass formula is thus invariant under $S$ and $T$ dualities and plays an important role in the supersymmetric gauge theories as these dualities are investigated for the invariance of supersymmetry and super strings. We may now write

$$\left( \begin{array}{c} n_e \\ n_m \end{array} \right) \mapsto \left( \begin{array}{cc} a & -b \\ c & -d \end{array} \right) \left( \begin{array}{c} n_e \\ n_m \end{array} \right),$$ \hspace{1cm} (81)

and the BPS bound now takes the form

$$M^2 \geq 4v^2 \cdot (n_e, n_m) \frac{1}{Im.\tau} \left[ \begin{array}{cc} 1 & -Re.\tau \\ -Re.\tau & |\tau|^2 \end{array} \right] \left( \begin{array}{c} n_e \\ n_m \end{array} \right)$$ \hspace{1cm} (82)

which is invariant under $SL(2, \mathbb{Z})$ transformations. The candidates for the invariance of $S$ and $T$ duality are the $N = 2$ and $N = 4$ extended supersymmetric gauge theories which could be established respectively in terms of complex and quaternion representations. The $N = 2$ supersymmetry is crucial in that the electric and magnetic charges enter the central charges in the supersymmetry algebra. It is the external supersymmetry that explains the Bogomolny’s bound and provides
an exact quantum status to BPS states. As we have stated earlier that to write the GDM equations in compact simpler and self dual representation we need to develop the theory of quaternion variables. We can establish the generators of higher dimensional supersymmetry \[61\] by writing supercharge operators represented for \(N = 1\), \(N = 2\), \(N = 4\) supersymmetry respectively in terms of real number, complex number and quaternion hyper complex number systems. Only these numbers form a division ring and it is therefore impossible to extend the supersymmetric gauge theories consistently beyond \(N = 4\) as the the next number system octonions do not obey the law of associativity. The manifold associated to \(N = 1\), \(N = 2\), \(N = 4\) gauge theories are also related with the hyper complex number system namely Riemannian manifold for \(N = 1\) (i.e. the algebra of real number), Kähler manifold for \(N = 2\) (i.e. algebra of complex numbers) and the hyperKähler manifold for \(N = 4\) supersymmetry describes from the algebra of quaternions. This is the case when one describes one Chiral spinor representation in \(N = 1\), two Majorana spinors for \(N = 2\) and four spinors (two Majorana and two Weyl) in \(N = 4\) supersymmetric Yang Mills gauge theories. Correspondingly we have one supersymmetric charge spinor operator (only electric) in \(N = 1\), two charge spinors in \(N = 2\), electric and magnetic (dyon), but still incomplete to represent two component spinors for both charges and a dyon with two spinors for both electric and magnetic charges in \(N = 4\) supersymmetric gauge theories to represent the electromagnetic as well as \(S\) and \(T\) duality of \(SL(2,\mathbb{Z})\) gauge group. It is also believed that \(N = 4\) supersymmetric gauge theory is the complete structural theory to explain the dyon mass formula and supersymmetric generators represented therein have the direct one to one relation with this mass formula. It has already been shown \[61\] that the \(\alpha\) and \(\beta\) matrices introduced by Osborn \[62\] for the description of \(N = 4\) supersymmetric gauge theory have the same commutation and anti-commutation relations as that of quaternion elements. A quaternion may also be represented in terms of two complex structures of duality together in quantum field and supersymmetric gauge theories. So, we have tried to reformulate the duality and supersymmetric gauge theories in terms of hyper complex numbers over the fields of real, complex and quaternion number system. As such, it may be concluded that the quaternion formulation be adopted in a better way to understand the explanation of the duality conjecture and supersymmetric gauge theories as the candidate for the existence of monopoles and dyons where the complex parameters be described as the constituents of quaternion.

Acknowledgment: - We are thankful to German Academic Exchange Service (DAAD), Bonn, Germany for providing financial support to O. P. S. Negi to carry out this work at the Universität Konstanz under re-invitation programme.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. London, A133, 60 (1931) ; Phys. Rev., 74, 817 (1948).

[2] J. Schwinger, Science, 165, 757 (1978); Phys. Rev., 144, 1087 (1966); 151, 1048 (1966); 151, 1055 (1966); 173, 1536 (1968).

[3] D. Zwanzinger, Phys. Rev., 176, 1489 (1968) ; Phys. Rev., D3, 880 (1971).

[4] C. N. Yang and T. T. Wu, Nucl.Phys., B107, 365(1976); Phys. Rev., D14, 437 (1976).

[5] F. Rhrlich, Phys. Rev., 150, 1104 (1966).
[6] P. Goddard and D. Olive, Rep. on Progress in Phys., 41, 1357 (1978).

[7] N. Craigie, “Theory and Detection of magnetic Monopoles in Gauge Theories”, World Scientific, Singapur, 1986.

[8] G.’ t Hooft, Nucl.Phys., B79, 276 (1974).

[9] A. M. Polyakov, JETP lett., 20, 194 (1974).

[10] B. Julia and A. Zee, Phys. Rev., D11, 2227 (1975).

[11] J. Preskill, Ann. Rev. Nucl. Sci., 34, 461 (1984).

[12] E. Witten, Phys. Lett., B86, 283 (1979).

[13] Y. M. Cho, Phys. Rev., D 21, 1080 (1980).

[14] G. ’ t Hooft, Nucl. Phys., B190, 455 (1981); B153, 141 (1979).

[15] V. A. Rubakov, JETP Lett., B86, 645 (1981); Nucl. Phys., B203, 311 (1982).

[16] C. Dokos and T. Tomaras, Phys. Rev., D21, 2940 (1980).

[17] N. Seiberg and E. Witten, Nucl. Phys., B426, 19 (1994); hep-th/9407087.

[18] K. Intriligator and N. Seiberg, Nucl. Phys.(Proc. Suppl.), BC 45, 1-28 (1996); hep-th/9509066.

[19] M. E. Peskin, “Dualities in Supersymmetric Yang-Mills Theories”, hep-th/9703136.

[20] A. Kapustin and E. Witten, “Electric-Magnetic Duality And The Geometric Lang lands Program”, hep-th/0604151 and references there in.

[21] W. R. Hamilton, “Elements of quaternions”, Chelsea Publications Co., NY, (1969).

[22] L. Silberstein, Phil. Mag., 63, 790 (1912).

[23] P.G.Tait, “An elementary Treatise on Quaternions”, Oxford Univ. Press (1875).

[24] B.S.Rajput, S.R.Kumar and O.P.S.Negi, Lett. Nuovo Cimento, 34, 180 (1982); 36, 75 (1983).

[25] V. Majernik, PHYSICA, 39, 9 (2000) and references therein.

[26] S. L. Adler, “Quaternion Quantum Mechanics and Quantum Fields”, Oxford Univ. Press, NY, (1995).

[27] D. Finklestein, J. M. Jauch, S. Schiminovich and D. Speiser, J. Math. Phys., 4, 788 (1963).

[28] D. V. Duc and V. T. Cuong, Comm. in Physics, 8, 197 (1998).

[29] K. Morita, Prog. Theor. Phys., 67, 1860 (1982); 68, 2159 (1982).

[30] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Int. J. Theor. Phys., 32, 2099 (1993).
[31] J. Lukiersky and A. Nowicki, Fortsch. Physik, 30, 75 (1982); A.J.Davies, Phys. Rev., A49, 714 (1994).

[32] V. V. Kravchenkov, “Applied Quaternion Analysis”, Helderman Verlag, Germany (2003).

[33] A. Waser, AW-Verlag www.aw-verlag.ch (2000).

[34] M. F. Sohnius, Phys. Rep., 128, 53 (1985); P. Fayet and S. Ferrara, Phys. Rep., 32, 247 (1977).

[35] V. A. Kostelecky and D. K. Campbell, “Supersymmetry in Physics”, North Holland, (1985).

[36] A. Bilal, “Introduction to Supersymmetry”, hep-th/0101055.

[37] E. Witten, Nucl. Phys. B202, 513 (1982).

[38] C. M. Hull, ”The geometry of Supersymmetric Quantum Mechanics”, hep-th/9910028.

[39] B. S. Rajput and D. C. Joshi, Pramana(India), 13, 637 (1995); Hadronic J., 4, 1805 (1981).

[40] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Prog. Theo. Phys., 85, 157 (1991); Ind. J. of Pure and Appl. Phys., 28, 157 (1990); Int. J. Theor. Phys., 32, 2099 (1998).

[41] Shalini Bisht, P. S. Bisht and O. P. S. Negi, Nuovo Cimento, B113, 1449 (1998).

[42] O. P. S. Negi and B. S. Rajput, Lett. Nuovo Cimento, 37, 325 (1983); 34, 180 (1982); 36, 75 (1983).

[43] J. S. Dowker and R. A. Roche, Proc. of Phys. Soc., 92, 1 (1997); D. D. Cantani, Nuovo Cimento, 60, 67 (1980).

[44] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Ind. J. of Pure and Appl. Phys., 28, 157 (1990); 24, 543 (1993).

[45] N. Cabibbo and E. Ferrari, Nuovo Cimento, 23, 157 (1962).

[46] D. Singleton, Am. J. Phys., 64, 452 (1996); Int. J. Theor. Phys., 34, 37 (1995); 35, 2419 (1996).

[47] W. A. Barker and F. Granziani, Phys. Rev., D18, 3849 (1978); Am. J. Phys., 46, 1111 (1978).

[48] D. I. Olive, “Exact Electromagnetic Duality”, hep-th/9508089.

[49] C. Montonen and D. Olive, Phys.Lett., B72, 117 (1977).

[50] J. A. Mignaco, Brazilian J. Phys., 31, 235 (2001).

[51] P. D. Vecchia, “Duality in Supersymmetric Gauge theories”, hep-th/9608090; J. A. Harvey, ”Magnetic Monopoles, Duality and Supersymmetry”, hep-th/9603086.
[52] C. Gomez and R. Hernandez, “Electromagnetic Duality and Effective Field Theory”, hep-th/9510023; F. A. Bais, “To be or not to be? Magnetic Monopoles in Non-Abelian Gauge theories”, hep-th/0407197.

[53] L. Alvarez-Gaume and S. F. Hasan, Fortsch. Phys., 45, 159 (1997); hep-th/9701069; E. Kiritsis, “Supersymmetry and Duality in Field Theory and String Theory”, hep-th/9911525.

[54] S. V. Ketov, “Solitons, Monopoles & Dualities”, hep-th/9611209; Fifueroa-O Farril, “Electromagnetic Duality for Children”, hep-th/9710082.

[55] A. J. Davies, Phys. Rev., D41, 2628 (1990); A. Govorkov, Theor. Math. Phys., 68, 893 (1987).

[56] P. Roteli, Mod. Phys. Lett., A4, 933 (1989); A4, 1763 (1989).

[57] H. Dehnen and E. Hitzer, Int. J. Theor. Phys., 34, 1981 (1995); 33, 575 (1994).

[58] E. B. Bogomolny, Sov. J. Nucl. Phys., 24, 449 (1976).

[59] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett., 35, 760 (1975).

[60] D. Bak, K. Lee and P. Yi, Phys. Rev., D62, 0205009 (2000); J. P. Gauntlett, “Duality and Supersymmetric Monopoles”, hep-th/9705025.

[61] A. Das, S. Okubo and S. A. Pernice, “Higher Dimensional SUSY Quantum Mechanics”, hep-th=4/9612125; Mod. Phys. Lett., 12, 581 (1997); O.P.S. Negi, “Higher Dimensional Supersymmetry”, hep-th/0608019.

[62] H. Osborn, Phys. Lett., B83, 321 (1971); J. D. Blum, ”Supersymmetric Quantum Mechanics of N = 4 Yang Mills Theory”, hep-th/9401133, Phys. Lett., B333, 92 (1994).