Renormalization Group for Large $N$ Strongly Commensurate Dirty Boson Model

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The large $N$ sigma model, in $D < 4$ space-time dimensions, with disorder a function of $d$ space dimensions, is analyzed via a renormalization group treatment. Critical exponents for average quantities are calculated, first to lowest order and then to all orders, in $\epsilon = D - 2 - \frac{d}{2}$. In particular, it is found that $\nu d = 2$. When $D = d + 1$, this model is equivalent to a large $N$ limit of the strongly commensurate dirty boson problem.

I. INTRODUCTION

The problem of the statistical mechanics of a system with quenched randomness is very interesting and difficult, combining the problems of disorder and interaction. We can consider two cases, the ordinary statistical mechanics of such a system, or the quantum statistical mechanics, in which a $d$ dimensional quantum system can be considered as a $d + 1$ dimensional classical system, with disorder correlated in the time direction. These correlations make the effects of disorder much stronger, as seen, for example, in the one dimensional disordered quantum Ising chain $[1]$, a problem which is probably the most well understood disordered quantum phase transition. There has also been much work on the quantum statistical mechanics of a system of repulsively interacting bosons in a disordered potential, known as the dirty boson problem $[2]$, but this problem is not as well understood.

An action that may be used to describe the dirty boson problem is

$$\int d^d x \, d t \left( - \partial_x \phi(x, t) \partial^x \phi(x, t) - \partial_t \phi(x, t) \partial^t \phi(x, t) + w(x) \phi(x, t) \partial_t \phi(x, t) - U(x) \phi \phi + \frac{g}{N} (\phi \phi)^2 \right)$$

(1)

Here, $U(x)$ and $w(x)$ are considered to be quenched random variables, with small fluctuations. Although the action above describes a $D = d + 1$ dimensional system, the randomness in $U(x)$ and $w(x)$ is a function of the $d$ dimensions $x$ only. This action is expected to describe a system with a number of different phases. If the average chemical potential $U$ is negative, the system is a Mott insulator, which has a gap. Increasing the chemical potential, the system then becomes a Bose glass, and finally a superfluid. The Bose glass is characterized by a gapless phase with infinite superfluid susceptibility $[10]$. In this phase, the bosons occupy localized states, with a localization length that diverges as the superfluid transition is approached.

If the boson density commensurates with a lattice, then $w(x)$ vanishes on average; this can be accomplished experimentally by tuning a chemical potential. In a typical experimental situation, however, there will still be fluctuations in $w(x)$. This case, with nonvanishing fluctuations in $w(x)$,
the weakly commensurate case. If \( w(x) = 0 \) identically, then we have the strongly commensurate case. In an experimental setup with either a dirty boson system or a Josephson junction array, this would require tuning the local chemical potential to be constant, so there is no local breaking of particle-hole symmetry due to disorder, while still having some fluctuations in the hopping term. This is obviously a more difficult task.

In this paper, we consider only the strongly commensurate case, where \( w(x) = 0 \) everywhere. Although less physically applicable to boson systems, we feel that the results on the strongly commensurate case are interesting in themselves, as well as being applicable to some other problems of quantum critical points. In particular, there exist quantum critical systems, with an \( O(3) \) order parameter, which in the disordered case may be well described by an \( N = 3 \) disordered rotor model like those consider in this paper. See work by Sachdev [4], and references therein to collinear quantum antiferromagnets.

In the strongly commensurate case, we will not see the Bose glass phase. As the chemical potential is increased, the system will go from a gapped Mott insulator phase to a gapless Griffiths phase, but the superfluid susceptibility will remain finite. One reason for considering the strongly commensurate case is related to the use of the large \( N \) limit and will be discussed later. Results will be described in a future work for the weakly commensurate case [3]. However, we feel that it is useful, in this paper, to mention the existence of the weakly commensurate problem, and the Bose glass found in that problem, to understand the different phase diagram with and without the \( w(x) \) term. The Bose glass phase, as we understand it, relies on the existence of the \( w(x) \) term, and so, contrary to what other authors suggest [5], we feel it is impossible to have such a phase in the strongly commensurate problem. Since the problem without the \( w(x) \) term can be understood simply as a rotor problem, it seems that the only possible phases are paramagnetic and ferromagnetic, with a Griffiths phase near the critical point.

In this paper, we consider a large \( N \) generalization of the strongly commensurate dirty boson problem [3]. We consider a system described by the following partition function

\[
\int \left( \prod_{x,t} \delta(\phi(x,t)\phi_i(x,t) - N\sigma^2(x)) \right) |d\phi_i| e^{-\int d^d x \int d^{(D-d)} t (\partial_\mu \phi_i(x,t) \partial^\mu \phi_i(x,t) + \partial_\mu \phi_i(x,t) \partial^\mu \phi_i(x,t))}
\]

(2)

where the field \( \phi_i \) has \( N \) components, and is a function of \( d \) space coordinates \( x \) and \( D-d \) time coordinates \( t \). The \( \delta \)-function constraint on the length of the spins is technically simpler and is not expected to alter the universality class from the quartic interaction considered above. The function \( \sigma^2(x) \) is a function of the spatial coordinate \( x \) only, and is related to the spatially varying \( U(x) \) considered in the previous equation.

The advantage of the large \( N \) formulation of the problem is that the system may be exactly solved for any fixed realization of disorder by solving a self-consistency equation. We may replace the \( \delta \)-function constraint by an integral over a Lagrange multiplier field \( \lambda(x) \) as

\[
\int |d\phi_i| |d\lambda| e^{-\int d^d x \int d^{(D-d)} t \partial_\mu \phi_i(x,t) \phi_i(x,t) + \int d^d x \int d^{(D-d)} t \lambda(x) (\phi \phi - \sigma^2(x))}
\]

(3)

2
where the integral for $\lambda(x)$ extends from $-i \infty$ to $+i \infty$. In the large $N$ limit, we can use a saddle point approximation for $\lambda$, the saddle point being found by the self-consistency equation

$$
\sigma^2(x) = \langle x, t = 0 | (-\partial^2_x + \lambda(x))^{-1} | x, t = 0 \rangle
$$

(4)

This equals

$$
\sigma^2(x) = \int d^{(D-d)}\omega \langle x, t = 0 | (-\partial^2_x + \omega^2 + \lambda(x))^{-1} | x, t = 0 \rangle
$$

(5)

We will write $\sigma^2 = \sigma_0^2 + \delta\sigma^2$, where $\sigma_0^2$ is a constant, which is tuned to drive the system through the phase transition, and $\delta\sigma^2$ is a random term. For small $\sigma_0^2$ the system is in the Mott insulator phase. For large $\sigma_0^2$ the system is in the superfluid phase.

In the above equations, we assume that the Green’s function on the right-hand side has been renormalized by subtracting a divergent quantity. That is, we will take a Pauli-Villars regularization for the Green’s function, and take the regulator mass to be very large, while adding an appropriate divergent constant to $\sigma^2$ on the left-hand side. The cutoff for the regulator is completely different from the cutoff for fluctuations in $\delta\sigma^2$ that will be introduced for the RG of the next section; the cutoff for the regulator will be much larger than the cutoff for fluctuations in $\delta\sigma^2$ and will be unimportant in the RG.

Note that in the large $N$ limit, it makes sense to define the theory for non-integer $D - d$, by doing the integral over $\omega$. For finite $N$, such a definition may have trouble, and indeed in the double dimensional expansion in $4 - d$ and $D - d$, there is some question about expanding around $D - d = 0$.

One disadvantage of the large $N$ expansion is that it is slightly difficult to include the terms linear in the time derivative, necessary to understand the weakly commensurate and incommensurate dirty boson problem. Consider the simple problem

$$
\int [d\phi] e^{-\int dt [\bar{\phi}(t) \partial_t \phi(t) - U(x) \bar{\phi}(t) \phi(t) + \bar{\phi}(t) (\bar{\phi}(t+\delta) - \bar{\phi}(t))]}
$$

(6)

where $\delta$ is a small number introduced for point-splitting. While for $U < 0$ everything is correct, for $U > 0$ the solution of the problem via the self-consistency equation becomes ill-defined for zero temperature, although not for arbitrarily small, non-zero, temperatures. The trouble is in the zero temperature limit of the problem in which we must consider frequencies $\omega$ arbitrarily close to 0. This will be further discussed in future work.

Having dropped terms linear in the time derivative, we do not expect to see a Bose glass phase. The original argument for the Bose glass phase was based on considering a system consisting of localized states for the bosons, with hopping between the localized states being neglected. It was then shown that for a random distribution of chemical potentials for each localized state, the spectrum of excitations would be gapless with constant density of excitations of low energy. These excitations would correspond to changing by one the number of particles in a given localized state. However, if one considers a system of localized states with action containing terms quadratic in the time derivative instead of linear, in the large $N$ limit, the density of low energy excitations goes to zero. In the Griffiths phase of the model we
are considering there are states at arbitrarily low energies, but the density of states vanishes as $e^{-cE^{-d}}$, where $c$ is some constant.

In the RG treatment, we will be considering the phase transition between the Griffiths phase and the superfluid phase, with weak randomness in $\sigma^2(x)$. By the Harris criterion $[12]$, weak randomness is irrelevant at the pure fixed point for $\nu d > 2$ and marginal for $\nu d = 2$. There is also a bound that for a stable disordered fixed point $\nu d \geq 2 [13]$. For $D < 4$, we find that $\nu = 1/(D - 2)$. Thus, there is a range of values of $D$ and $d$ satisfying $d/(D - 2) = 2$, at which the randomness is marginal. We are able to construct a renormalization group near any of these values. Note that for the quantum statistical mechanics of a $d$ dimensional system, where $D = d + 1$, we find that the disorder is marginal for $d = 2$ and relevant for $d > 2$. The renormalization group is not constructed by expanding down from an upper critical dimension. Instead, it is constructed by expanding upwards near a range of dimensions such that $d/(D - 2) = 2$. The RG is only valid for $D < 4$, as discussed in the next section.

Within the RG, we will proceed perturbatively, but will obtain exact results for all the various exponents for average quantities. One of the most striking results is that $\nu d = 2$. This implies that the system saturates the bound discussed above $[13]$. Similar results have been found for various other phase transitions. For example, exact results for the transverse field Ising model in $d=1$ lead to saturation of this bound $[1]$, with $\nu = 2$. Also, numerical simulations on the quantum Ising spin glass in 3+1 dimensions show this bound $[14]$. This may be a common feature of quantum phase transitions.

The next two sections will develop the renormalization group. The procedure will be to take a problem with fluctuations in $\sigma^2$ up to some cutoff $\Lambda$, and define another problem which includes fluctuations in $\sigma^2$ only up to a wavevector $\Lambda - \delta \Lambda$, such that we preserve the low-momentum correlation functions. In section II we will perform a one-loop Wilson-Fisher RG. To extend this to higher orders would require keeping track of many operators, while an alternative procedure discussed section III requires only keeping track of the renormalization of two quantities: $\sigma^2_0$ and $S$, where $S$ is a measure of the strength of fluctuations in the disorder $\delta \sigma^2$. In order to preserve the low-momentum correlation functions, we will require that the propagator used includes self-energy corrections due to the high-wavevector fluctuations, and also require a renormalization of the disorder strength to produce the same low-wavevector fluctuations in $\lambda$, up to a renormalization of the vertex. Requiring the same low-wavevector fluctuations in $\lambda$ makes the procedure consistent, and permits one to continue iterating the RG.

II. RENORMALIZATION GROUP

We proceed with a renormalization group acting directly on the self-consistency equation. The goal will be to start with a given problem which includes high wavevector fluctuations in $\sigma^2$, and find a related problem which includes fluctuations in $\sigma^2$ only up to a lower wavevector. We will require that the correlation functions in the related problem are equal, up to rescaling, to the correlation functions in the original problem averaged over the high wavevector fluctuations. This requirement is what will
define the RG.

At lowest order, considered in this section, this will require that \( \lambda \) is unchanged, up to rescaling, at low wavevectors. To this order, we can accomplish this goal by defining a new problem in the same form as the old problem, via a self-consistency equation with one parameter \( \lambda \); to higher orders this Wilson-Fisher procedure will be more complicated and we will instead use an alternative technique in the next section. To higher orders within the Wilson-Fisher approach one encounters operators such a spatially fluctuating gradient terms and \( \omega^2 \) terms, as well as non-Gaussian and momentum dependent distributions of the disorder.

In equation (4), we have broken \( \sigma^2(x) \) into two pieces: \( \sigma_0^2 + \delta\sigma^2(x) \), where \( \delta\sigma^2(x) \) has vanishing mean. In the perturbative \( \epsilon \) expansion being developed here, we can perturb in \( \delta\sigma^2(x) \) as it will have fluctuations at the critical point which are of order \( \epsilon \).

Consider the self-consistency equation (4), and define a cutoff \( \Lambda \) such \( \sigma^2 \) only has fluctuations for wavevectors less than \( \Lambda \). Formally, we can invert equation (4) to obtain \( \lambda \) as a function of \( \sigma^2 \) as follows. For small \( \delta\sigma^2 \), we can expand the right-hand side of (4) as a power series in \( \lambda \). At criticality, where \( \lambda \) vanishes for vanishing \( \delta\sigma^2 \), we find to lowest order

\[
\delta\sigma^2(p) = \int d^dk \frac{1}{(p-k)^2 + \omega^2} \left( \frac{1}{\Lambda^2} - \int dDk' \frac{1}{(p-k')^2 + \omega^2} \right) \lambda(p)
\]

This can be derived from a diagram similar to a polarization bubble as shown in figure 1.

This implies that as a formal inverse, to lowest order,

\[
\lambda(p) = c_1 \delta\sigma^2(p) p^{-d}
\]

where \( c_1 = \frac{1}{\pi^{D/2} (2-D/2)^{D/2}(D-2)/2} \). For small wavevector components of \( \delta\sigma^2 \), this formal inverse is ill-behaved, but for high wavevector components when \( D < 4 \), the procedure given will be correct to lowest order in \( \epsilon \). This is why we need \( D < 4 \) for the RG, as otherwise the polarization bubble is divergent for large wavevector instead of small wavevector.

Define a measure \( S \) of the strength of disorder, by assuming that \( \langle \delta\sigma^2(p)\delta\sigma^2(q) \rangle = (2\pi)^d \delta(p - q)S \)

where to lowest order we may take \( \delta\sigma^2 \) to be Gaussian distributed. Then, we find that at the cutoff \( \lambda \) has mean-square \( c_1^2 \Lambda^{8-2D}(2\pi)^d S \). We will define \( L = c_1^2 \Lambda^{8-2D} S \) to measure fluctuations in \( \lambda \). The effect of these components in \( \lambda \) will be to renormalize the propagator for the field \( \phi \), as well as to renormalize the vertex used to calculate the scattering of \( \phi \) off \( \lambda \) and to calculate \( \sigma^2 \). The self-energy is given by

\[
\Sigma(p, \omega) = \delta \Lambda \int_{k^2 = \Lambda^2} d^{d-1}k \frac{1}{(p+k)^2 + \omega^2} L
\]

See figure 2 for the appropriate diagram. This is equal to a constant, which may be absorbed into a renormalization of \( \lambda \), plus
where \( c_2 = (1 - 4/d)c_3 \) and \( c_3 = 2^{ \frac{d}{2} } \Lambda^{d-4} \). This implies that the propagator is renormalized to

\[
\frac{1}{(1 + \frac{\delta \Lambda}{\Lambda} c_2 L) \omega^2 + (1 + \frac{\delta \Lambda}{\Lambda} c_3 L) \omega^2}
\]

while the vertex renormalization, shown in figure 3, is given by

\[
\frac{\delta \Lambda}{\Lambda} c_3 L
\]

This implies that we can write a new self-consistency equation

\[
(1 - \frac{\delta \Lambda}{\Lambda} (c_3 - c_2)L) \sigma^2(x) = \int d^{D-d} \omega \langle x, t = 0 \rangle \left( -\partial_x^2 + (1 + \frac{\delta \Lambda}{\Lambda} c_3 L)(\omega^2 + \lambda(x)) \right)^{-1} |x, t = 0|
\]

where \( \sigma^2 \) now has only components at wavevectors less than \( \Lambda - \delta \Lambda \). Note that by removing the high wavevector components of \( \lambda \) in the vertex renormalization, we reduce the value of \( \sigma^2 \) on the left-hand side of equation (14). We rescale equation (14) by defining a new scaled \( \lambda \), scaling the integral over \( \omega \), and scaling the \( d \) space dimensions, to obtain

\[
(1 + \frac{\delta \Lambda}{\Lambda} (D-2))(1 - \frac{\delta \Lambda}{\Lambda} (c_3 - c_2)L) \sigma^2(x) = \int d^{D-d} \omega \langle x, t = 0 \rangle \left( -\partial_x^2 + \omega^2 + \lambda(x) \right)^{-1} |x, t = 0|
\]

Assuming that \( \sigma = \sigma_0^2 + \delta \sigma^2 \), with \( \delta \sigma^2 \) having a distribution given by equation (3), we can obtain RG equations for \( \sigma_0^2 \) and \( S \) by considering the scaling of \( \sigma \) under equation (13). We obtain

\[
\frac{\ln \sigma_0^2}{\ln \Lambda} = D - 2 - (c_3 - c_2)c_1^2 \Lambda^{8-2D}S(1 - \frac{D-d}{2})
\]

\[
\frac{1}{2} \frac{\ln S}{\ln \Lambda} = D - 2 - (c_3 - c_2)c_1^2 \Lambda^{8-2D}S(1 - \frac{D-d}{2}) - d/2
\]

Therefore, there is a fixed point of the RG flow, for \( D - 2 - d/2 = (c_3 - c_2)c_1^2 \Lambda^{8-2D}S(1 - \frac{D-d}{2}) \). The fixed point is stable in the \( S \) direction. It is unstable in the \( \sigma_0^2 \) direction. We find \( \frac{\ln \sigma_0^2}{\ln \Lambda} = d/2 \) at the fixed point, leading to the exponent \( \nu = 2/d \), which implies that the bound \( \nu d \geq 2 \) is saturated.

In fact, equation (14) is not quite correct. Due to scaling of the regulator mass, there is an additional constant term in the change of \( \sigma_0^2 \) with respect to \( \ln \Lambda \), or equivalently an extra term \( \frac{C}{\sigma_0^2} \) in the change of \( \ln \sigma_0^2 \) with respect to \( \ln \Lambda \). Here, \( C \) is some non-universal constant. This additional term only leads to a non-universal change in the critical value of \( \sigma_0^2 \), and no change in the critical exponents.

We can obtain the dynamic critical exponent by considering the different scaling of \( p^2 \) and \( \omega^2 \) in the propagator. This difference is \( (c_3 - c_2)c_1^2 \Lambda^{8-2D}S = \frac{D-2-d}{1-\frac{D-d}{2}} \). Therefore,
\[ z = 1 + \frac{D - 2 - d}{2 - D + d} = \frac{d/2}{2 + d - D} \quad (18) \]

The average Green’s function, \( \langle G(p, \omega) \rangle \), is

\[ \frac{1}{p^{\frac{d-2}{d-1}} + \omega^2} \quad (19) \]

This result for the dynamic critical exponent differs greatly from that expected in the weakly commensurate case, where scaling theory predicts \( z = d \). This is not due to \( 1/N \) corrections, but rather a big difference between weakly and strongly commensurate phase transitions.

III. RESULTS TO ALL ORDERS

Given the simplicity of the result \( \nu d = 2 \), one might suspect that the exponents for average quantities remain unchanged to all orders. A very simple argument shows that this is true. To obtain results to higher order, it is useful to use a formulation of the RG other than the Wilson-Fisher RG. We will proceed as follows. Consider a theory with quadratic fluctuations in \( \sigma^2 \) of some magnitude \( S \), with fluctuations ranging up to some wavevector \( \Lambda \), and with given \( \sigma_0^2 \). In this theory calculate the average Green’s function at low momentum, as well as the low momentum fluctuations in \( \lambda \), to some order in perturbation theory. The reason for considering fluctuations in \( \lambda \) is that this is what is needed to continue the computation of the Green’s function to higher orders. Then, for a theory with cutoff \( \Lambda - \delta \Lambda \), determine the appropriate value of \( S \) and appropriate coefficients of \( \partial^2_2 \) and \( \partial^2_t \) in the action, so as to reproduce the given low energy behavior to the same order in perturbation theory, with the same \( \lambda \) up to a renormalization of the scattering vertex.

This procedure is a well-established alternative to the Wilson-Fisher and Callan-Szymanzik techniques. It is discussed for example in the classic reference Domb and Green volume 6, as the second of two field theory techniques for performing an RG. Within this procedure, one needs only to keep two terms in the RG, the value of \( \sigma_0^2 \) and the value of \( S \), as these are the only two relevant terms. It is possible, of course, that higher loop corrections will make other operators relevant, and destabilize the fixed point; this is something that cannot be analyzed within this approach. However, this approach will yield results that agree with the Wilson-Fisher technique so long as the fixed point is stable. Let me emphasize this point: if no other operators become relevant, then this technique agrees with the Wilson-Fisher procedure to all orders. If other operators become relevant, then the procedure does not work and the fixed point has more than one unstable direction. What we show in this section is that if no other operators become relevant and the fixed point remains stable then the exponents are unchanged to all orders. What was shown in the previous section is that there is a stable fixed point to lowest order for \( D - d < 2 \), with exponents as given above. It is reasonable to assume that the fixed point remains stable to all orders, at least for some range of \( \epsilon \), so that the exponents remain unchanged to all orders at this fixed point.
In the case of the one loop calculation above, it was found that the vertex renormalization for $\lambda$, the vertex renormalization for $\sigma^2$, and the renormalization of $\omega^2$ all had the same coefficient. This will remain true to all orders. This is due to a Ward identity, explained at the end of this section. Therefore, the general RG equation to all orders will take the form

$$(1 - \frac{\delta \Lambda}{\Lambda} F_1(S, D, d)) \sigma^2(x) = \int d^{(D-d)} \omega \langle x, t = 0 \mid \left( (1 + \frac{\delta \Lambda}{\Lambda} F_2(S, D, d)) \partial^2_x + (1 + \frac{\delta \Lambda}{\Lambda} F_1(S, D, d)) (\omega^2 + \lambda(x)) \right)^{-1} \rangle |x, t = 0\rangle$$

(20)

where $F_1, F_2$ are some generic functions of disorder strength and dimensionality. By the same procedure of rescaling, and writing $\sigma^2 = \sigma_0^2 + \delta \sigma^2$, as before, we can obtain

$$\frac{d \ln \sigma_0^2}{d \ln \Lambda} = D - 2 - (F_1(S, D, d) - F_2(S, D, d))(1 - \frac{D - d}{2})$$

(21)

$$\frac{1}{2} \frac{d \ln S}{d \ln \Lambda} = D - 2 - (F_1(S, D, d) - F_2(S, D, d))(1 - \frac{D - d}{2}) - d/2$$

(22)

Then, although the fixed point of the RG flow will be at a different value of $S$ than was obtained to lowest order, the exponents will be unchanged from their lowest order values.

Let us consider why this Ward identity holds. First, consider the equivalence between the renormalization of $\sigma^2$ on the left-hand side of the self-consistency equation and the renormalization of $\lambda$ on the right-hand side. It is apparent that both renormalizations are vertex renormalizations and are the same by definition.

The more interesting aspect of the Ward identity is the equivalence between the renormalization of $\sigma^2$ and the renormalization of the $\omega^2$ term. This, however, is again almost true by definition. The term $\omega^2$ is constant in space, and a function only of frequency. The Green’s function for given frequency $\omega$ is constant under a shift $\omega^2 \rightarrow \omega^2 + \Delta$ and $\lambda \rightarrow \lambda - \Delta$, for some constant $\Delta$. For this property to hold under renormalization, we need the desired Ward identity. Alternatively we may say that for any given frequency $\omega$, the term $\omega^2$ plays exactly the same role in the Green’s function as a constant term in $\lambda$ would, and so the renormalizations must be equal.

For a more diagrammatic derivation of the identity, proceed as follows. The coefficient of the $\omega^2$ term can be obtained by differentiating the inverse of the average Green’s function with respect to $\omega^2$ at $\omega = 0$. In any given diagram for the average Green’s function, one can differentiate the inverse of any one of the propagators in the diagram with respect to $\omega^2$. This gives one a sum over different places to insert the derivative. However, each place one inserts gives a result for the diagram which is equivalent to placing a scattering vertex for $\lambda$ at that point, and thus the total renormalization of the $\omega^2$ vertex is the same as the renormalization of the $\lambda$ vertex. For example, in figure 2 for the one-loop self-energy, the renormalization of the $\omega^2$ term can be obtained by differentiating the inverse of the propagator in the loop with respect to $\omega^2$. However, this yields exactly the diagram in figure 3.

The existence of this Ward identity does not require the formulation of the RG used in this section. A similar identity would exist within a Wilson-Fisher renormalization group. This would then
connect the renormalization of the $\omega^2$ term to that of the $\sigma^2$ and $\lambda$ terms at lowest order as found in the last section. At higher orders, other terms would also be connected. For example, the existence of the symmetry would also require that a term such as $A(x)\omega^2$, that is, a spatially fluctuating $\omega^2$ term, would have a renormalization connected to that of a term $A(x)\lambda$, and a term $A(x)\sigma^2$. This last term would imply a spatially varying magnitude of fluctuations in the disorder $\delta\sigma^2$, which would mean a non-Gaussian distribution of the disorder. From the Ward identity, we would still have a connection between the renormalization of the $\omega^2$, $\sigma^2$ and $\lambda$ terms, which would be enough to obtain the desired results for critical exponents, to all orders.

The only thing the Ward identity does not guarantee is the stability and existence of the fixed point. It only guarantees the exponents if the fixed point exists and is stable. As $D - d \to 2$, the fixed point runs to stronger and stronger disorder. This will be discussed more in the conclusion.

IV. A HEURISTIC TREATMENT OF THE RG

We present a simple alternative treatment of the problem which supports the results of the RG and provides a physical motivation for it. First, let us present an alternative version of the Harris criterion. For the pure system, at criticality, the self-consistency equation is given by

$$\sigma^2 = \int \frac{1}{k^2 + \omega^2} \frac{1}{\Lambda^2} \frac{d^d k d^{(D-d)} \omega}{k^2 + \omega^2} \tag{23}$$

where we have now inserted a cutoff for high energy states of the field $\phi$. We find by doing the integral in equation (23) that $\sigma^2 \propto \Lambda^{(D-2)}$. However, if we consider a disordered system, we find that, after removing states with $k^2 + \omega^2 > \Lambda^2$, we are considering a case in which we have smeared out space over a length of order $1/\Lambda$, which corresponds to a $d$-dimensional volume $V$ of order $1/\Lambda^d$. The average value of the fluctuation in $\sigma^2$ over this volume is of order $V^{-1/2}$, or $\Lambda^{d/2}$. When the Harris criterion indicates that disorder is relevant, we find that $\Lambda^{d/2} > \Lambda^{(D-2)}$ for small $\Lambda$. This is to be interpreted as saying that there are not enough low energy states in the pure system to produce the required $\sigma^2$.

However, we expect that disorder will increase the low energy density of states, and we heuristically modify the Green’s function to $\frac{1}{k^a + \omega^2}$, with $a > 2$, and we change the cutoff to $k^a + \omega^2 < \Lambda^2$ so that we have

$$\sigma^2 = \int \frac{1}{k^a + \omega^2} \frac{1}{\Lambda^2} \frac{d^d k d^{(D-d)} \omega}{k^a + \omega^2} \tag{24}$$

In this case, we find by doing the integrals that $\sigma^2 \propto \Lambda^{2d/a - 2-d+D}$. However, we are now smearing the system out over a length of order $1/\Lambda^{(2/a)}$. For small $\Lambda$, this is less the length $1/\Lambda$ which we had in the pure system. This is to be expected, indicating the motion has become subdiffusive. This leads to an average of $\sigma^2$ over the $d$-dimensional volume of order $\Lambda^{(d/a)}$. Equating these two results for $\sigma^2$, one given by the integral in equation (24) and the other given by the volume average of $\sigma^2$, we find that
or \( a = d/(2 + d - D) \). This agrees with the value of \( a \) obtained by the RG treatment of the previous section. Further, if we imagine slightly leaving the critical point, we can imagine that the self-consistency equation gets replaced by \( \sigma^2 = \frac{1}{\sigma^2 + \omega^2 + \lambda} \), where \( \lambda \) and \( \sigma^2 \) are now taken spatially constant. If we slightly adjust \( \sigma^2 \) away from the critical value, we can calculate the change in \( \lambda \), and thus the correlation length of the system, and we find that we obtain \( \nu d = 2 \).

V. THE ZERO ENERGY WAVE FUNCTION

As we increase \( \sigma_0^2 \), with fixed randomness \( \delta \sigma^2 \), we find a sequence of different phases. Assume we are given \( \lambda(x) \) as a function of \( \sigma_0^2 + \delta \sigma^2 \). For small \( \sigma_0^2 \), the operator \((-\partial_x^2 + \lambda(x))\) has a gap, and there are no eigenstates of this operator with eigenvalues below the gap. For the case of the zero temperature quantum phase transition, where \( D = d + 1 \), this gap corresponds to an energy gap for the system. This is the Mott insulator phase discussed in the introduction. As \( \sigma_0^2 \) is increased the gap decreases and at a certain point the system enters a Griffiths phase. In this case, there are eigenstates at arbitrarily small energy, all of which are localized, and the average correlation function decays exponentially. At the critical \( \sigma_0^2 \), the average correlation function acquires power law behavior, and there appears a state at zero energy which is delocalized. Above the critical \( \sigma_0^2 \), particles begin to condense into this state. We would like to examine the wave function \( \alpha(x) \) of this state.

We will first give a physical argument for the form of the wave function. Then, we will derive this to first order in \( \epsilon \) through the RG. Unfortunately, this result cannot be derived to all orders in \( \epsilon \).

Arguing physically, the following should be a good description of this wave function. Consider a problem at short distances. From the RG, we know that, on a short scale, one can obtain \( \lambda \) directly from \( \sigma^2 \). So, the wave equation, \((-\partial_x^2 + \lambda)\alpha(x) = 0\), can be solved approximately on a short scale knowing only the local fluctuations in \( \sigma^2 \). However, the wave equation is homogeneous, and defines \( \alpha(x) \) only up to a multiplicative constant. So, the zero energy wavefunction in some region in space is defined by the local disorder, up to a multiplicative constant. This constant will be set by the behavior of the disorder on larger length scales. If we want to find the value of \( \alpha(x) \) at some point, we proceed as follows: consider short distance fluctuations to obtain \( \alpha(x) \) as a solution of a wave equation on some short scale. The overall multiplicative constant on this wave function is not known. So, \( \alpha(x) \) at a given point is known up to a multiplicative constant set by longer scales. Solving at a slightly longer length scale, we can find \( \alpha(x) \) up to a constant set at even longer length scales.

So, we would find that \( \alpha(x) \) is given by a product of these multiplicative constants on larger and larger length scales, where these constants will be drawn from some random distribution. This is simply a statement that the zero energy wavefunction, or the value of the condensate, is a multiplicatively renormalizable operator, which gets a random multiplicative renormalization at each step. Thus, \( \alpha(x) \) would have a log-normal distribution. Assuming scale invariance we would find that
\[ \alpha(x) = e^{\beta(x)} \]  

(26)

where, in order to produce the same fluctuations on all length scales, \( \beta \) has a Gaussian distribution with

\[ \langle \beta(p)\beta(q) \rangle = \frac{g}{p^d} \delta(p + q) \]  

(27)

where \( g \) is some constant measuring the strength of the disorder. In two space dimensions, we can give \( \beta \) the probability distribution

\[ e^{-\frac{1}{2g} \int d^2 x (\partial_x \beta(x))^2} \]  

(28)

Let us now derive this result from the RG. The zero energy wavefunction \( \alpha(x) \) can be obtained by considering the correlation function of \( \phi(x, t) \) and \( \overline{\phi}(y, t') \) where \( y \) is some point very far from \( x \). So, we must look at how the correlation function \( G(x, t; y, t') \) renormalizes under the RG, beyond the simple calculation of the average correlation function in the previous sections. The interesting part is the renormalization of \( \phi(x) \). Let us look at this problem using moments. That is, we will look at the average over disorder of the \( n \)-th moment of \( G(x, t; y, t') \). Let us start with the second moment for simplicity.

Consider what we may call a renormalization of a vertex. We must have both Green’s functions starting at the same point, \( x, t \). One may connect the two lines with a single scattering off of \( \lambda \), with momentum of order \( \Lambda \) running around the loop, with low momentum leaving the diagram. See figure 4 for the appropriate diagram. The result of this is that the vertex is renormalized under RG flow as

\[ \frac{d \ln V_2}{d \ln \Lambda} = c_3 L = c_3 c_1^2 \Lambda^{8-2D} S \]  

(29)

where \( V_2 \) is the vertex.

In general, for the \( n \)-th moment, there are \( \frac{n(n-1)}{2} \) ways to connect the lines at the vertex, and so we find

\[ \frac{d \ln V_n}{d \ln \Lambda} = \frac{n(n-1)}{2} c_3 L = \frac{n(n-1)}{2} c_3 c_1^2 \Lambda^{8-2D} S \]  

(30)

Interpreting this result, we find that the \( n \)-th moment of \( \alpha(x) \) at some point can be expressed, up to the multiplicative renormalization calculated here, as the \( n \)-th moment of \( \alpha(x) \) at that point with a smoothed disorder potential. Looking at the renormalization, though, we see that it is exactly the form expected for a log-normal distribution of \( \alpha(x) \). That is, if we took \( \alpha(x) \) at some point to be expressed as a random multiplicative renormalization of the value of \( \alpha(x) \) at that point calculated from a smoothed disorder potential, we would obtain a result of the form of equation (30).

Let us show that the moments are those expected from a log-normal distribution, as well as obtaining the value of \( g \), measuring fluctuations in \( \beta \), as defined above. Using \( \alpha(x) = e^{\beta(x)} \), computing the average of \( \alpha^n(x) \), over fluctuations in \( \beta \) down to some scale \( \Lambda \), we find

\[ \langle \alpha^n(x) \rangle = \int [d\beta(p)] e^{\int \xi^d p (n\beta(p) - \frac{\xi^d}{2}\beta(p)^2)} \]  

(31)
This is equal to
\[ e^{\frac{\pi}{2}} \int_{\Lambda} d^{d-2} \frac{d^2}{2^d} \int d\ln p \frac{d^2}{2^d} \]
\[ = e^{2 \pi d/2} \int_{\Lambda} d\ln p \frac{d^2}{2^d} \quad (32) \]

Finally, we see that
\[ \frac{d\ln(\langle \alpha^n \rangle)}{d\ln \Lambda} = \frac{\pi d/2}{\Gamma(d/2)} g n^2 \quad (33) \]

So we find, comparing equations (30) and (33), that
\[ g = \frac{c_3 L \Gamma(d/2)}{2\pi^{d/2}} \quad (34) \]

where we need to choose \( g \) to make the coefficients of the \( n^2 \) term in equations (30) and (33) the same. The term in \( n \) can be different, as this simply represent an overall scale for the wavefunction.

Given this form of the wave function, with log-normal fluctuations, the change in the low energy density of states should not be a surprise. Originally, the field \( \phi \) had the action
\[ \int d^d x d^d t \left( -\partial_\mu \overline{\phi}(x, t) \partial^\mu \phi(x, t) + \lambda(x) \overline{\phi} \phi \right) \quad (35) \]

Given the zero energy wave function, we can write this as
\[ -\int d^d x d^d t \alpha(x)^2 \partial_\mu (\alpha^{-1}(x) \overline{\rho}(x, t)) \partial^\mu (\alpha^{-1}(x) \rho(x, t)) \quad (36) \]

This leads to a random stiffness problem, which is related to the problem of disordered SUSY quantum mechanics [8] and the problem of Dirac fermions in a random vector potential [9]. Both of these problems are known to have an increase in the low energy density of states. In particular, for the Dirac problem in which there exists a dimensionless measure of disorder, the low energy density of states is a power law with continuously variable exponent.

VI. CONCLUSION

In conclusion, we have presented a renormalization group treatment of the disordered large \( N \) sigma model. We have calculated the exponents describing the average correlation functions. However, there is still much that we would like to know.

First, we have not considered the behavior of averages of higher moments of the correlation functions, or typical behavior of the correlation functions. These will be discussed in another publication [10]. The calculation of these higher moments is closely related to the calculation of the zero energy wave function. It is shown [11] that
\[ \langle G^n(p, \omega) \rangle \propto \langle G(p, \omega) \rangle^n p^{-n(n-1)c_3 L} \quad (37) \]
similar to the result for the wave function derived above. Here, $c_3$ and $L$ are the constants defined in section II, while the expectation value is an average of the $n$-th moment of the correlation function.

Also, there exists another RG treatment of the same system \cite{5}, based on a Callan-Symanzik type RG for the large $N$ system. In that work, no perturbatively accessible fixed point was found, and I do not fully understand why their technique fails. Of course, this work \cite{5} is not necessarily in contradiction with the results obtained here. Since no fixed point was found with their technique, this may simply be failure of technique, rather than indicating different results for the same system.

Let us consider the differences in techniques used in more detail. It is conceivable that the lack of a perturbatively accessible fixed point in that work \cite{5} is due to their use of a Callan-Symanzik technique, which is equivalent to the Wilson-Fisher RG only for a field theory which possesses a renormalizable continuum description. However, the $\delta$-function interaction is non-renormalizable as a continuum theory (in that work a quartic interaction, which is renormalizable, was used, but the technique of summing polarization bubbles used there made the results equivalent to the $\delta$-function theory) and the inverse polarization bubble used to calculate fluctuations in $\lambda$ from fluctuations in $\sigma^2$ grows rapidly in the ultraviolet which can lead to poor behavior of the diagrammatic expansion of the continuum theory. A perturbative RG implies an ordering by momenta, in which high momentum processes are calculated before low momentum processes. For the expansion to be valid, this ordering by momenta must be correct, in that corrections to high momentum processes due to low momentum effects must be small. I have checked this for my procedure; I am not aware if it is true for other techniques on the same problem.

Although results have been obtained for average quantities to all orders, one still may inquire about the radius of convergence of the expansion. It is apparent that if $(D - d) \geq 2$ then the dynamic critical exponent $z$ resulting from equation (18) becomes infinite. When $(D - d) > 2$, however, it is possible for the system to do a phase transition in a finite region in the $d$ dimensional space, since there is enough volume in the remaining $D - d$ dimensions. Thus, certainly when $D - d = 2$, and possibly before, the perturbation theory must break down. Further, results for the higher moments of correlation functions may have interesting behavior to higher order.

Also, we would like to understand $1/N$ corrections to this problem. To lowest order in $1/N$ and $\epsilon$ such corrections only modify the RG equations by changing the scaling dimension of $\sigma^2$ \cite{11} in the pure system, and thus changing the equation to

$$\frac{d \ln \sigma^2}{d \ln \Lambda} = D - 2 + \eta - (c_3 - c_2)c_1^2 A^{8 - 2D} S \left(1 - \frac{D - d}{2}\right)$$  \hspace{1cm} (38)

$$\frac{1}{2} \frac{d \ln S}{d \ln \Lambda} = D - 2 + \eta - (c_3 - c_2)c_1^2 A^{8 - 2D} S \left(1 - \frac{D - d}{2}\right) - d/2$$  \hspace{1cm} (39)

where in the physically interesting case of $d = 2, D = 3$ we find $\eta = \frac{32}{4\pi^2 2N}$. Then, we would expect to still find $\nu d = 2$ for the disordered critical point.
VII. ACKNOWLEDGEMENTS

I would like to thank David Huse for many useful discussions on disordered quantum critical points. I would also like to thank Shivaji Sondhi and Andy Green.

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FIG. 1. Polarization bubble. Thick lines represent either scattering vertex off $\lambda$ or scattering vertex used to define $\sigma^2$ in self-consistency equation.

FIG. 2. Self-energy correction due to fluctuations in $\lambda$. Joining the thick lines in a loop denotes averaging $\lambda$ over disorder in $\sigma^2$. Momentum of order $\Lambda$ flows around loop.

FIG. 3. Vertex correction due to fluctuations in $\lambda$. This represents both renormalization of vertex defining scattering off of $\lambda$ and renormalization of vertex defining $\sigma^2$ in self-consistency equation.
FIG. 4. Renormalization of vertex in computing higher moments of Green’s function. Two Green’s functions start at the same point. After Fourier transforming, this implies that they start with given total momentum. By including fluctuations in $\lambda$, with momentum of order $\Lambda$ running around the loop, one can define a renormalized vertex.