6-Vertex Model on an Open String Worldsheet

Charles B. Thorn*

Institute for Fundamental Theory,
Department of Physics, University of Florida, Gainesville, FL 32611

Abstract

We propose boundary conditions on a two dimensional 6-vertex model, which is defined on the lightcone lattice for an open string worldsheet. We show that, in the continuum limit, the degrees of freedom of this 6-vertex model describe a target space coordinate compactified on a circle of radius $R$, which is related to the vertex weights. This conclusion had already been established for the case of a 6-vertex model on the worldsheet lattice for the propagator of a closed string. This exercise illustrates how the Bethe ansatz works in the presence of boundaries, at least of this particular type.

*E-mail address: thorn@phys.ufl.edu
1 Introduction

The lightcone worldsheet [1] lattice [2] provides a useful tool for analyzing the sum of planar diagrams in field theory [3–5] as well as in open string theory [2, 6, 7]. Once one commits to a lattice definition of the worldsheet theory, it no longer is necessary to limit worldsheet degrees of freedom to discretized versions of the continuum worldsheet fields. For example, the worldsheet fermion fields in the Ramond-Neveu-Schwarz model [8, 9] can be represented on the lattice by Ising spin variables [10]. One benefit of doing this is to eliminate the lattice fermion doubling problem.

In this spirit, one can use a 6-vertex model [11, 12] defined on the worldsheet lattice to realize a bosonic target space coordinate compactified on a circle [13]. In this context it is convenient to replace the standard rectangular lattice discussed in [13] by the diamond lattice arising in certain fishnet models of the worldsheet [14, 15]. For the closed string worldsheet this adaptation of the six vertex model to a diamond lattice was carried out in [16], where the continuum limit was carefully analyzed via the Bethe ansatz [17], and its connection to a compactified target space coordinate established. In this short article we include boundaries in the 6-vertex model to establish its equivalence in the continuum limit to a compactified target space coordinate on the open string worldsheet.

It turns out that the mathematical analysis of the periodic case discussed in [16], which closely follows the work of Bethe [17] and Yang and Yang [18] on one dimensional Heisenberg spin chains and the work of [11, 12] on six vertex models, can be easily adapted to include the case with boundaries of interest here.

We present our definition of the six-vertex model on an open string diamond lattice in the next section 2. In Section 3 we obtain the transfer matrix and construct its eigenstates using the Bethe ansatz. In section 4 we analyze the eigenvalue spectrum of this transfer matrix in the continuum limit. We conclude with comments and discussion in Section 5. In an appendix, we directly analyze our model for the special value of the vertex weight for which the transfer matrix is diagonalized by the states of a free fermion system. The availability of explicit formulas in this case provides an insightful confirmation of the results obtained in the main text.

2 Six Vertex Model on a Worldsheet Lattice

In this article we discuss the 6-vertex model on a diamond lattice, illustrated with charge conserving boundary conditions on the vertical boundaries in Fig.1. In the worldsheet interpretation we think of time as running vertically, and the arrows at the bottom and top of the lattice specify possible initial and final configurations of a worldsheet spin variable. Each link on this lattice contains an arrow, which can be thought of as specifying the direction of charge flow, each link carrying ±1 unit of charge. There are precisely six (planar) charge conserving vertices (see Fig. 2): Two with weight 1 in which each adjacent pair carries charge 0 into the vertex, and four with weight \( v \) in which two adjacent lines carry charge 2 into the vertex. A typical fishnet diagram with these vertices is shown in Fig. 1. The sum of
all allowed arrow configurations is thus seen to be equivalent to calculating the partition function for a 6-vertex model on a diamond lattice.

Figure 1: Diamond worldsheet lattice propagating $M$ units of $P^+ N$ steps in time. The left (right) figure shows even (odd) $M$. Charge conserving boundary conditions have been imposed.

To set up a six vertex model that is suitable for an open string worldsheet, it is important that the charge $Q$ whose flow is given by the vertex arrows is conserved at the boundaries. This is necessary if $Q$ is to be identified as the zero mode momentum of a compactified bosonic coordinate. A very natural choice is shown on the left of Fig. 1 for even $M$ and on the right of Fig. 1 for odd $M$. Notice that in the even case the vertices at the two boundaries are in step with each other, whereas in the odd case they are offset by one lattice step in time. These figures show how to define the transfer matrix element between the arrow

Figure 2: The six charge conserving vertices.
configurations at the top and bottom of each figure. Just as in the case of periodic boundary conditions, it is natural with the diamond lattice, to define the fundamental transfer matrix to cover two steps in time.

3 The Transfer Matrix and its Eigenvalues

The worldsheet lattice can be thought of as a discrete (imaginary) time evolution of a state which is a tensor product of $M$ two state systems, ("spins"), labeled by up and down arrows. Because of the diamond lattice configuration, the basic discrete evolution is two time steps, and we define each element of the $2^M \times 2^M$ transfer matrix $T$ as the product of vertex factors associated with the subgraph that connects a given row of arrows with the row two time steps above it. It is easy to see that the state with all arrows up or all arrows down is an eigenstate of the transfer matrix with eigenvalue $v^{M-1}$ for an open string worldsheet and $v^M$ for a closed string worldsheet. Because the transfer matrix conserves $Q$, we can work out its eigenstates independently in each charge sector. The state with all arrows up is the unique state with $Q = M$, and so it is automatically an eigenstate with eigenvalue $v^{M-1}$, compared to $v^M$ for the analogous state with periodic boundary conditions. The difference is explained by the fact that with open string boundaries there is one less vertex in two time steps.

3.1 One overturned arrow ($Q = M - 2$)

With one overturned arrow we can label each state by the location of that arrow $|j\rangle$, $j = 1, \ldots, M$. If $j$ is sufficiently far from both boundaries, in the bulk bulk of the worldsheet, the action of the transfer matrix is identical to that for periodic boundary conditions [16], but with one less power of $v$ on the right:

$$
T|j\rangle = \left\{ \begin{array}{ll}
|j + 2\rangle v^{M-1} + |j + 1\rangle v^{M-2} + |j - 1\rangle v^{M-2} + |j\rangle v^{M-3} & \text{for } j \text{ odd} \\
|j - 2\rangle v^{M-1} + |j + 1\rangle v^{M-2} + |j - 1\rangle v^{M-2} + |j\rangle v^{M-3} & \text{for } j \text{ even.}
\end{array} \right.
$$

By direct inspection, we find that the action of $T$ on the states with the overturned arrow close to the left boundary, with $j = 1, 2$, is given by:

$$
T|1\rangle = |1\rangle v^{M-2} + |2\rangle v^{M-2} + |3\rangle v^{M-1} \quad (2)
$$
$$
T|2\rangle = |1\rangle v^{M-1} + |2\rangle v^{M-3} + |3\rangle v^{M-2} \quad (3)
$$

When the overturned arrow is close to the right boundary, we need to consider separately the cases of $M$ even and odd:

$$
T|M\rangle = \left\{ \begin{array}{ll}
|M\rangle v^{M-2} + |M - 1\rangle v^{M-2} + |M - 2\rangle v^{M-1} & M \text{ even} \\
|M\rangle v^{M-2} + |M - 1\rangle v^{M-1} & M \text{ odd}
\end{array} \right.
$$
$$
T|M - 1\rangle = \left\{ \begin{array}{ll}
|M\rangle v^{M-1} + |M - 1\rangle v^{M-3} + |M - 2\rangle v^{M-2} & M \text{ even} \\
|M\rangle v^{M-2} + |M - 1\rangle v^{M-3} + |M - 2\rangle v^{M-2} + |M - 3\rangle v^{M-1} & M \text{ odd}
\end{array} \right.
$$
Because the transfer matrix acts locally, we can diagonalize its action in the bulk by the same spin wave construction as in the periodic case

\[ |k\rangle_0 = \sum_{j \text{ odd}} |j\rangle e^{ikj} + \xi(k) \sum_{j \text{ even}} |j\rangle e^{ikj} \]  

(6)

\[ \xi(k) \equiv iv \sin k + \sqrt{1 - v^2 \sin^2 k} \]  

(7)

with the eigenvalue \( v^{M-1}t(k) \) with

\[ t(k) = \left( \cos k + \frac{1}{v} \sqrt{1 - v^2 \sin^2 k} \right)^2, \]  

(8)

and for \(-\pi < k < \pi\), these are all independent.

The states (7) do not diagonalize the action of \( T \) near the boundaries. But because \( t(-k) = t(k) \) We can construct the eigenstates in the presence of these boundaries by taking a linear combination of the states \(|k\rangle_0 \) for \( k > 0 \) and \(-k \).

\[ |k\rangle = |k\rangle_0 + \eta(k)|-k\rangle \]  

(9)

\[ = |1\rangle(e^{ik} + \eta(k)e^{-ik}) + |2\rangle(\xi(k)e^{2ik} + \eta(k)\xi(-k)e^{-2ik}) + \cdots \]  

(10)

Focusing first on the left boundary, we apply \( T \) to the first few terms and collect the coefficient of \(|1\rangle \) to give the relation

\[ v^{M-1}t(k) = v^{M-2} + v^{M-1}\frac{\xi(k)e^{2ik} + \eta(k)\xi(-k)e^{-2ik}}{(e^{ik} + \eta(k)e^{-ik})} \]  

(11)

\[ \eta(k) = -\frac{(vt(k) - 1)e^{ik} - \eta(k)e^{2ik}}{(vt(k) - 1)e^{-ik} - \eta(k)e^{-2ik}} = -e^{2ik} \frac{vt - 1 - vz}{vt - 1 - vz^*} \]  

(12)

where we have used the definition \( z = \xi e^{ik} \). Next we note that

\[ z + v = v + (\cos k + i \sin k)(\sqrt{1 - v^2 \sin^2 k} + iv \sin k) \]  

\[ = (v \cos k + \sqrt{1 - v^2 \sin^2 k})i \sin k + v \cos^2 k + \cos k \]  

\[ = (v \cos k + \sqrt{1 - v^2 \sin^2 k})e^{ik} \]  

(13)

\[ e^{2ik} = \frac{v + z}{v + z^*} \]  

(14)

and we rewrite

\[ t(k) = \left( \cos k + \frac{1}{v} (\xi - iv \sin k) \right)^2 = \left( e^{-ik} + \frac{\xi}{v} \right)^2 = \frac{1}{v^2} e^{-2ik}(v + z)^2 = \frac{(v + z)(v + z^*)}{v^2} \]  

(15)

From these relations we see that \( \eta \) simplifies:

\[ \eta(k) = -\left( \frac{v + z}{v + z^*} \right) \left( \frac{v + z}{v + z^*} \right)(v + z^*) - vz(v + z^*) = -\frac{v + (1-v)z}{v + (1-v)z^*} \]  

(16)
On the other hand we can also determine \( \eta \) by applying \( \mathcal{T} \) to the first few terms on the right of the row of arrows

\[
|k\rangle = |M\rangle (e^{iMk} + \eta e^{-iMk}) + |M - 2\rangle (e^{i(M-2)k} + \eta e^{-i(M-2)k}) + |M - 1\rangle \xi(k)e^{i(M-1)k} + \eta(k)\xi^*(k)e^{-i(M-1)k} + \ldots, \quad M \text{ odd}
\]

and

\[
|k\rangle = |M\rangle (\xi(k)e^{iMk} + \eta\xi^*(k)e^{-iMk}) + |M - 1\rangle (e^{i(M-1)k} + \eta(k)e^{-i(M-1)k}) + \ldots, \quad M \text{ even}
\]

After applying \( \mathcal{T} \) and collecting the coefficient of \( |M\rangle \), we obtain the relation for \( M \) odd:

\[
vt = 1 + ve^{i(M-2)k} + \eta e^{-i(M-2)k} + \xi(k)e^{i(M-1)k} + \eta(k)\xi^*(k)e^{-i(M-1)k} e^{iMk} + \eta e^{-iMk}
\]

\[
\eta(k) = \frac{(vt - 1)e^{iMk} - ve^{i(M-2)k} - \xi^* e^{-i(M-2)k} - \xi e^{i(M-1)k}}{(vt - 1)e^{-iMk} - ve^{-i(M-2)k} - \xi e^{-i(M-1)k}} = -e^{2iMk} vt - 1 - e^{-2i}k(v + z)/v - 1 - (v + z)
\]

\[
\eta(k) = -e^{2iMk} \frac{z - 1 + 1/v}{z^* - 1 + 1/v} = \frac{z^*}{z} \eta^* e^{2iMk}
\]

which is identical to the result obtained with \( M \) odd. Thus for all \( M \) even and odd the quantization of \( k \) is given by the condition

\[
e^{2iMk} = \frac{z^*}{z} \eta^2(k), \quad \eta(k) = \frac{v + (1 - v)z}{v + (1 - v)z^*}.
\]

For periodic boundary conditions the quantization condition was the much simpler \( e^{iMk} = 1 \).

### 3.2 \( q \) overturned arrows \((Q = M - 2q)\)

Eigenstates with several overturned arrows, in the presence of boundaries, can again be constructed by taking linear combinations of the Bethe ansatz in the bulk which for \( q = 2 \) is
given by

\[
|k_1, k_2\rangle = \sum_{l \leq m} |l, m\rangle (\xi_l(k_1)\xi_m(k_2)e^{ik_1+imk_2} + A(k_1, k_2)\xi_l(k_2)\xi_m(k_1)e^{ik_2+imk_1})
\] (23)

\[
A(k_1, k_2) = -\frac{(1 - 1/v^2)z_2 - z_1 - z_1z_2/v - 1/v}{(1 - 1/v^2)z_1 - z_2 - z_1z_2/v + 1/v},
\] (24)

To economize notation we have affixed a subscript to \(\xi(k)\) such that \(\xi_l(k) = 1\) if \(l\) is odd and \(\xi_l(k) = \xi(k)\) if \(l\) is even. Also we have defined \(z_j \equiv \xi(j)e^{ik_j}\). For \(q\) overturned arrows, the Bethe ansatz is a sum over all permutations of the down arrows, and \(A(1, 2)\) replaced by an \(A_P\) for each permutation. \(A_P\) factors into a product of \(A(k, l)\) for each pair interchange needed to accomplish the permutation. When all down arrows are away from the boundaries, the action of \(T\) diagonalizes on these bulk states determining the eigenvalue of the transfer matrix to be

\[
T(k_1, \ldots, k_q) = v^{M-1} \prod_{j=1}^{q} t(k_j).
\] (25)

Since \(T\) is invariant under the reversal of any of the \(k_j \rightarrow -k_j\), we can take linear combinations with each distinct term having one or more of the \(k\)’s reversed to diagonalize the action of \(T\) near the boundaries.

All of the essential features are already contained in the case \(q = 2\), which we next analyze in detail, quoting the general result at the end. Fixing \(k_2\) for the moment we see by inspection that the combination

\[
|\psi_1\rangle = |k_1, k_2\rangle + \eta(k_1)|-k_1, k_2\rangle
\] (26)

will properly realize the boundary conditions on the left for the arrow associated with \(k_1\), when it is to the left of that associated with \(k_2\). When the order of the down arrows is reversed, as in the second term, the \(k_1\) dependence is then

\[
A(k_1, k_2)\xi_m(k_1)e^{imk_1} + \eta(k)A(-k_1, k_2)\xi_m(-k_1)e^{-imk_1} =
\]

\[
A(k_1, k_2) \left[ \xi_m(k_1)e^{imk_1} + \eta(k)\frac{A(-k_1, k_2)}{A(k_1, k_2)}\xi_m(-k_1)e^{-imk_1} \right]
\] (27)

We see that the role of \(\eta\) when the spin \(k_1\) is on the left is played by \(\eta(k_1)A(-k_1, k_2)/A(k_1, k_2) = \eta(k_1)A(k_2, k_1)A(-k_2, k_1)\) when \(k_1\) is on the right. It follows then that the boundary condition on the right will be met by the \(k_1\) arrow provided

\[
e^{2iMk_1} = \frac{z^*(k_1)}{z(k_1)}\eta^2(k_1)A(k_2, k_1)A(-k_2, k_1)
\] (28)

The symmetry of the \(k_2\) dependence under \(k_1 \rightarrow -k_2\) is important because it means that the construction

\[
|\psi_2\rangle = |k_1, -k_2\rangle + \eta(k_1)|-k_1, -k_2\rangle
\] (29)
leads to the same eigenvalue condition on $k_1$. To complete the construction we need to form

$$|\psi\rangle = |\psi_1\rangle + \eta(k_2) \frac{A(k_1, k_2)}{A(k_1, -k_2)} |\psi_2\rangle$$

(30)

which then satisfies the boundary conditions of the down arrow $k_2$ provided

$$e^{2iMk_2} = \frac{z^*(k_2)\eta^2(k_2)A(k_1, k_2)A(-k_1, k_2)}{z(k_2)}$$

(31)

The generalization to any number $q$ of overturned spins is now straightforward. The Bethe ansatz constructed along parallel lines leads to the quantization conditions

$$e^{2iMk_r} = \frac{z^*(k_r)\eta^2(k_r)}{z(k_r)} \prod_{s \neq r} A(k_s, k_r)A(-k_s, k_r)$$

(32)

We discuss the solution of these equations in the next section.

### 4 Analysis of the Eigenvalue Equation

The eigenvalue equation for the six-vertex model with boundaries (32) can be cast in a form similar to the eigenvalue equation with periodic boundary conditions analyzed in [16]. For comparison, recall that the eigenvalue equation in the periodic case with $M'$ arrows at each time slice, took the form

$$e^{iM'k_r} = \prod_{s \neq r} A(k_s, k_r)$$

(33)

To mimic the equation with boundaries we take $M' = 2M$, and take $q' = 2q$ down arrows where half of them are associated with the $q k_r > 0$ in (32), and the other half are associated with the negatives $-k_r$ of these. Then the periodic equations take the form

$$e^{2iMk_r} = A(-k_r, k_r) \prod_{s \neq r} A(k_s, k_r)A(-k_s, k_r), \quad \text{for } k_r, k_s > 0$$

(34)

$$e^{-2iMk_r} = A(k_r, -k_r) \prod_{s \neq r} A(k_s, -k_r)A(-k_s, -k_r)$$

$$= A(-k_r, k_r)^* \prod_{s \neq r} A(-k_s, k_r)^*A(k_s, k_r)^*$$

(35)

The first equation is of the form (32) with $\eta^2(k_r)z^*(k_r)/z(k_r)$ replaced by $A(-k_r, k_r)$, and the second is the complex conjugate of the first equation. Thus we can use the results of [16] to infer the continuum properties of the system with boundaries. To do this we analyze the equation

$$e^{2iMk_r} = e^{i\Theta(k_r)} \prod_{s \neq r} A(k_s, k_r) = -e^{i\Theta(k_r) + i \sum_{s \neq r} \theta(k_s, k_r)}$$

(36)

$$e^{i\Theta(k_r)} = -\frac{z^*(k_r)\eta^2(k_r)}{z(k_r)A(-k_r, k_r)}$$

(37)
where \( r, s = 1, \ldots, 2q \) and we constrain the solution to satisfy \( k_{2q+1-r} = -k_r \). The positive \( k_r \)'s will satisfy the open eigenvalue equation. Taking the logarithm we can present the equation to solve in the form

\[
k_r = \frac{\pi I_r}{M} + \frac{\Theta(k_r)}{2M} + \frac{1}{2M} \sum_{s \neq r} \theta(k_s, k_r)
\]

(38)

where the \( I_r \) are half odd integers satisfying the restriction \( I_{2q+1-r} = -I_r \). The second term on the right is the new feature of the equations compared to those analyzed in [16, 18].

### 4.1 The continuum limit \( M \to \infty \)

For analyzing these equations we map the \( k_j \) onto new variables \( \alpha_j \) for which \( A(k_j, k_l) \) depends only on the difference \( \alpha_j - \alpha_l \). This is accomplished by the map [18]

\[
\begin{align*}
  z &= \xi e^{ik} = \frac{e^{i\nu} - e^\alpha}{e^{i\nu + \alpha} - 1} \\
  e^{i\nu} &= \frac{1}{2\nu} + i\sqrt{1 - \frac{1}{4\nu^2}}.
\end{align*}
\]

(39)

Note that our parameter \( \nu \) is related to a similar parameter \( \mu = 2\nu \) in [18]. Here we restrict \( \infty > \nu \geq 1/2 \), for which \( e^{i\nu} \) is a pure phase. We note some special values of the mapping: \( \alpha = 0 \) corresponds to \( e^{ik}\xi = 1 \) which implies \( k = 0 \), and \( \alpha = \pm \infty \) map to \( k = \pm(\pi - 2\nu) \). (We are choosing \( k \) to be in the range \( -\pi < k < \pi \).) Thus the whole range \( -\infty < \alpha < \infty \) corresponds to \( -(\pi - 2\nu) < k < \pi - 2\nu \). Note that \( \nu \to \infty \) shrinks the range of \( k \) to 0, whereas \( \nu \to 1/2 \) represents the maximum range. It is straightforward to work out the following quantities in terms of the new variables:

\[
\begin{align*}
  \tan k &= \frac{\sin 2\nu \sinh \alpha}{\cos \nu - \cos 2\nu \cosh \alpha} \\
  \frac{dk}{d\alpha} &= \frac{\sin 3\nu}{2[\cosh \alpha - \cos 3\nu]} + \frac{\sin \nu}{2[\cosh \alpha - \cos \nu]} \\
  t(k) &= \left[ \cos k + \frac{1}{\nu} \sqrt{1 - \nu^2 \sin^2 k} \right]^2 = \frac{\cosh \alpha - \cos 3\nu}{\cosh \alpha - \cos \nu} \\
  A(k(\alpha), k(\beta)) &= -\frac{1 - e^{2\beta - \alpha - 4i\nu}}{e^{2\alpha} - e^{-4i\nu}} \equiv -e^{i\Theta(\alpha, \beta)} \\
  \theta(\alpha, \beta) &= 2 \arctan(\cot 2\nu \tanh((\beta - \alpha)/2)) \\
  \eta(k) &= z \frac{e^{\alpha} + e^{2i\nu}}{e^{\alpha + 2i\nu} + 1} \\
  A(-k(\alpha), k(\alpha)) &= -\frac{e^{4i\nu} - e^{2\alpha}}{e^{4i\nu + 2\alpha} - 1} \\
  e^{i\Theta(k(\alpha))} &= \frac{\sinh \alpha + i \sin 2\nu}{\sinh \alpha - i \sin 2\nu} \quad \Theta(\alpha) = 2 \arctan \frac{\sin 2\nu}{\sinh \alpha}
\end{align*}
\]

(40-43)
Using these equations, we can express the boundary conditions in the alternative form

\[ k(\alpha_r) = \frac{\pi I_r}{M} + \frac{\Theta(\alpha_r)}{2M} + \frac{1}{2M} \sum_{s \neq r} \theta(\alpha_s, \alpha_r), \]  

(44)

where the \( I_i \) are half-odd integers since \( q' = 2q \) is even. Different choices for these integers lead to different solutions for the set of \( k \)'s, and hence they provide us with a labeling of the eigenstates of the transfer matrix. Yang and Yang [18] have analyzed similar equations in their solution of the \( x, y \) Heisenberg spin chain, and their techniques for solving them in the limit \( M \to \infty \) can be directly applied. For easy comparison, we attempt as far as possible to adopt their notation.

4.2 The ground state with \( Q' = 2Q > 0 \)

The ground state of the worldsheet system is the eigenstate of the transfer matrix with maximal eigenvalue. For the problem with periodic boundary conditions this state corresponds to the choice of \( I_i \)'s symmetrically disposed about 0, with no gaps [18]. For application to open boundary conditions, the symmetry about 0 is automatic due to the constraint on the \( k_r \)'s. Thus the ground state corresponds to the choice

\[ I_r = r - q - \frac{1}{2}, \quad r = 1, 2, \ldots, 2q \]  

(45)

We remind the reader that our reference periodic system has \( M' = 2M \) arrows at each time step, \( q' = 2q \) of which are down. Thus the total charge of the reference system is \( Q' = M' - 2q' = 2M - 4q = 2Q \) where \( Q \) is the total charge of the open system. In the reference periodic system the total momentum \( P' = \sum_r k_r = (2\pi/M') \sum_r I_r \) is a good quantum number which can be nonzero in general. But for application to the open system \( P' = 0 \) due to the symmetry of the \( k_r \) about 0. Of course the actual open system has no conserved momentum because of the presence of boundaries.

We are interested in obtaining excitation energies of order \( 1/M \) above the ground state. If we try to calculate the total energy of these states, we would have to not only calculate the \( M' \to \infty \) behavior of the energy, which is proportional to \( M' \), but also corrections up to order \( 1/M' \). However, excitation energies may be obtained more simply by calculating energy differences \( \Delta E = E(Q') - E(0) \), as described in [16, 18]. The trick is to calculate \( \Delta E \) in the thermodynamic limit \( M \to \infty \) with \( J = Q'/M' = Q/M \) fixed and \( P' = 0 \). Of course for finite \( Q' \), we must examine the small \( J \) limit at the end of the calculation. We expect \( \Delta E \propto M'J^2 = Q^2/M' \), which shows the desired \( 1/M' \) dependence of the excitation energy.

In the thermodynamic limit the eigenvalue equation reduces to an integral equation for the density of eigenvalues \( R(\alpha) \). We define a kernel \( K \) and density function \( R(\alpha) \) by

\[ K(\alpha, \beta) = \frac{1}{2\pi} \frac{\partial \theta}{\partial \beta} = \frac{1}{2\pi} \frac{\sin 4\nu}{\cosh(\alpha - \beta) - \cos 4\nu}, \]

\[ R(\alpha) = \frac{2\pi}{M'} \frac{dj}{d\alpha}, \]  

(46)
and then the equation for the $k$'s as $M' \to \infty$ becomes

$$\frac{dk}{d\alpha} = R(\alpha) + \frac{1}{M'} \frac{d\Theta}{d\alpha} + \int_{-\alpha_+}^{\alpha_+} d\beta K(\alpha - \beta) R(\beta). \quad (47)$$

This equation has the same kernel $K$ as the one analyzed in [16], but the second term on the right is new to the system with boundaries. However this new term vanishes in the thermodynamic limit, so in the end, we can simply copy the results of from this paper.

The value chosen for $\alpha_+$ determines the characteristics of the eigenstate. For example, the eigenstate with maximum eigenvalue $T'$ for the transfer matrix corresponds to $\alpha_+ = \infty$. The values of $k$ at the limits of this range are $k = \pm(\pi - 2\nu)$ and $t(k) = 1$ for these values.

As long as $0 < \nu < \pi/2$, $t(k) > 1$ for all finite $\alpha$, so taking the whole range of $\alpha$ corresponds to including in the expression for $T'$ all values for $t$ greater than unity. For the continuum limit we are only interested in very large $\alpha_+$ since then the eigenvalues will be close (within $1/M'$) of the maximum eigenvalue.

From [16] we quote $\Delta E'$ of the reference periodic model

$$\Delta E' \sim \frac{\pi - \mu}{4aM'}Q'^2 = \frac{\pi - \mu}{2aM}Q^2 \quad (48)$$

Because the reference periodic system has doubled the number of $k$'s, this energy is twice the energy of the system with boundaries:

$$\Delta E \sim \frac{\pi - \mu}{4aM}Q^2 = \frac{T_0}{2P^+} \left[ \frac{\pi - \mu}{2}Q^2 \right] \quad (49)$$

In brief, the charge dependence of the energy for the open system is identical to that of the periodic system.

4.3 Particle-hole excitations

The particle-hole excitations in the reference periodic system also correspond to excitations of the open system. In these excitations the distribution of $I_r$'s is allowed to have gaps. Of course for energies of order $1/M$, these gaps must be close to the ends of the gapless distributions. For the periodic system the particle's and holes near opposite ends of the gapless distribution can be independently chosen. For the open system the constraint on the $k_r$ requires that they always occur in equal and opposite pairs of particles and holes. From [16] we quote the change in energy due to a particle-hole pair in the periodic system

$$\Delta E' = \frac{2\pi n}{M'a} = \frac{\pi n}{Ma} \quad (50)$$

where $n = |I_r - I_r^0|$ the integer $I_r^0$ has been replaced by the integer $I_r$. For the open system this excitation is matched by one where $-I_r^0$ is replaced by $-I_r$. This doubles the energy,
but the energy of the open system is half the energy of the reference periodic system so the 
energy change in the open system is

$$\Delta E = \frac{\pi n}{Ma} = T_0 \frac{\pi n}{P^+}$$  \hspace{1cm} (51)$$

For several particle-hole excitations $n_1, \ldots, n_k$, we simply replace $n$ by $N = \sum n_i$. Putting 
together all types of excitations we have the general expression for low-lying energy eigenvalues

$$\Delta E = \frac{T_0}{2P^+} \left[ \frac{\pi}{2} - \frac{\mu}{2} Q^2 + 2\pi N \right]$$  \hspace{1cm} (52)$$

5 Discussion and Concluding Remarks

Ref [16] established that in the periodic case the low lying spectrum of the six vertex model 
matched that of a compactified coordinate on the continuum closed string world-sheet, 
described by the action

$$S = \frac{1}{2} \int d\tau \int_0^{P^+} d\sigma (\dot{\phi}^2 - T_0^2 \phi'^2)$$  \hspace{1cm} (53)$$

with the equivalence relation

$$\phi \equiv \phi + 2\pi R.$$

This implies that the zero mode momentum conjugate to $\phi$ is quantized; $p = k/R$ with $k$ an integer. The associated energy is $k^2/(2R^2P^+)$ There is an associated winding number $l$ for which $\phi(p^+) - \phi(0) = 2\pi lR$ which is associated with the energy $4\pi^2 l^2 T_0^2 R^2/(2P^+)$. Since $Q$ is even for the periodic case, it is identified with $2k$. It then followed by comparison that $R^2 = [2T_0(\pi - 2\nu)]^{-1}$.

Now $\cos 2\nu = \text{Re} e^{2i\nu} = -1 + 1/2\nu^2$, so the limit $R \to \infty$ implies $\nu \to \pi/2$ or $\nu \to \infty$. The self dual radius $R_*^2 = 1/(2\pi T_0)$ corresponds to $\nu = 0$ or $\nu = 1/2$. Thus the range of couplings considered here $1/2 \leq \nu < \infty$ (for which the 6-vertex model is critical) produces circle radii $R_* \leq R < \infty$. Interestingly, small radii, $R < R_*$ are not accessible in the vertex model. For $\nu < 1/2$ the model is not critical and the continuum limit accordingly sends all 
excitations to infinite energy, i.e. there is no interesting continuum limit.

In this article we have obtained the low lying spectrum for the open string worldsheet lattice. There is of course no winding number, but the $Q$ dependence of the energy is exactly 
as in the closed string case, with the exception that $M$ can be odd, in which case $Q$ is odd. In 
the compactified coordinate interpretation, this implies that, under the shift $\phi \to \phi + 2\pi R$, 
the wave function of the open string is periodic when $M$ is even and antiperiodic when $M$ is 
odd.

We have therefore confirmed the expectation that the six vertex model on the diamond lattice provides a satisfactory discretization of a compactified target space coordinate for
both open and closed strings. This discretization may be particularly effective in monte carlo simulations of the sum of all planar diagrams of open string theory as advocated in [7].

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A The free fermion case \( v = 1/\sqrt{2} \)

As a useful check on our conclusions, we study the case \( v = 1/\sqrt{2} \) (\( \nu = \pi/4 \)) for which \( A = -1 \). Then the quantization conditions on the \( k_r \) decouple and reduce to

\[
e^{2iMk_r} = \frac{z^* (k_r)}{z(k_r)} \left( \frac{1 + z(\sqrt{2} - 1)}{1 - z^* (\sqrt{2} - 1)} \right)^2, \quad r = 1, \ldots, q
\]

\[
z(k) = \frac{e^{ik}}{\sqrt{2}} (i \sin k + \sqrt{1 + \cos^2 k})
\]

Here there is no need to use a reference periodic system, and no need to double the \( k_r \)’s. An eigenstate involves any number of overturned spins which can be independently assigned a momentum solving this equation. Just as with a free Fermi gas, the ground state is obtained by populating all the \( k_r \) with \( t(k_r) > 1 \). Dropping the index, we have, for \( v = 1/\sqrt{2} \),

\[
t(k) = \left( \cos k + \sqrt{1 + \cos^2 k} \right)^2
\]

and we see that \( t(k) = 1 \) for \( k = \pi/2 \), and \( t(k) > 1 \) for \( k < \pi/2 \). The low lying excitations all arise from altering the population of overturned spins with \( k \approx \pi/2 \), leaving the overturned spins with \( k - \pi/2 \) of order unity in their ground state configuration. To study the spectrum of these low lying excitations, put \( k = \delta + \pi/2 \). Then

\[
t(k) \to \left( -\sin \delta + \sqrt{1 + \sin^2 \delta} \right)^2 \sim 1 - 2\delta + O(\delta^2)
\]

\[
\Delta E = -\frac{\ln t}{2a} \sim \frac{\delta}{a} + O(\delta^2)
\]

Next we examine the quantization condition for \( k \approx \pi/2 \). We find \( z \to e^{3i\pi/4 + i\delta} + O(\delta^2) \) and the right side of the quantization condition becomes

\[
\frac{z^* (k)}{z(k)} \left( \frac{1 + z(\sqrt{2} - 1)}{1 - z^* (\sqrt{2} - 1)} \right)^2 \to -1 + O(\delta)
\]

Then the quantization equation reads

\[
e^{2iM(\delta + \pi/2)} = (-)^M e^{2iM\delta} = -1 + O(\delta)
\]

\[
\delta = \frac{(2I + 1)\pi}{2M} + O(\delta/M), \quad M \text{ even}
\]

\[
\delta = \frac{I\pi}{M} + O(\delta/M), \quad M \text{ even}
\]
where $I$ is any integer. Since $\delta$ starts out at order $1/M$, it is safe to drop the correction terms $O(\delta/M)$ to this solution. These results show immediately that particle-hole excitations, which leave the number of overturned arrows constant, change the energy by an integer multiple of $\pi/(Ma)$.

To compare energies in different charge sectors, we change the number of overturned arrow. Start with the lowest energy state with charge zero. This state requires that $M$ is even and there are $q = M/2$ overturned arrows, all populating all the levels with negative energy. We can increase the charge by $2n$ units by flipping the $n \ll M$ arrows at the top of the sea. These previously down arrows were contributing a negative energy, so flipping them increases the energy by the amount

$$\Delta E(2n) = \frac{\pi}{2Ma} [1 + 3 + \cdots + (2n - 1)] = \frac{\pi}{2Ma} [n(n + 1) - n] = \frac{\pi n^2}{2Ma} = \frac{\pi Q^2}{8Ma}$$

which agrees with our result (52) for $\mu = 2\nu = \pi/2$. Of course, when $M$ is even only sectors with even charge can appear.

To reach odd values of the charge we need to take $M$ odd. In this case there is no state of zero charge: the lowest energy states has $Q = \pm 1$. These two degenerate states are connected by the overturned arrow with $k = 0$, which is allowed when $M$ is odd. So start with the $Q = 1$ state. Then flipping $n \ll M$ arrows at the top of the sea reaches the state with $Q = 1 + 2n$. This increases the energy by

$$\frac{\pi}{Ma} [1 + 2 + \cdots + n] = \frac{\pi n(n + 1)}{2Ma} = \frac{\pi(Q - 1)(Q - 1 + 2)}{2Ma} = \frac{\pi Q^2}{8Ma} - \frac{\pi}{8Ma}$$

This is consistent with the result (52) but leaves open the possibility that there is a $Q$ independent shift in the energies between the cases with even and odd $M$. However, we know the $1/M$ contribution to the large $M$ behavior of the ground state energy in any sector of a free fermion system is determined by the well-known Casimir zero-point energy calculation. When $M$ is even the low energy frequencies are $(n+1/2)\pi/(aM)$ which is known to give a contribution of $-d\pi/(48aM)$ where $d$ is the number of Fermi fields: $d = 2$ for the present case of free charged fermions. When $M$ is odd, the Casimir zero-point energy is $+d\pi/(24aM)$. Thus the energy difference between energies in the even and odd $M$ sectors has the $1/M$ dependence

$$\frac{d\pi}{24aM} - \frac{-d\pi}{48aM} = \frac{d\pi}{16aM} \rightarrow \frac{\pi}{8aM}$$

for $d = 2$. The boundary terms must also match by locality. The bulk terms $\alpha M$ will have the same $\alpha$ for even and odd $M$, but of course $M$ itself will be different in even and odd sectors. So, as a consequence of these general arguments we can conclude that the low-lying energies are

$$E = \alpha M + \beta - \frac{\pi}{24aM} + \frac{\pi Q^2}{8aM} + \frac{\pi N}{aM}$$

where $N = \sum_l n_l$ is the total mode number of the particle hole excitations, where the $n_l$ are nonnegative integers. We now understand that $Q$ is even when $M$ is even and $Q$ is odd when $M$ is odd.
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