Can the entanglement entropy be the origin of black-hole entropy?

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(December 25, 1996)

Abstract

Entanglement entropy is often speculated as a strong candidate for the origin of the black-hole entropy. To judge whether this speculation is true or not, it is effective to investigate the whole structure of thermodynamics obtained from the entanglement entropy, rather than just to examine the apparent structure of the entropy alone or to compare it with that of the black hole entropy. It is because entropy acquires a physical significance only when it is related to the energy and the temperature of a system. From this point of view, we construct a ‘thermodynamics of entanglement’ by introducing an entanglement energy and compare it with the black-hole thermodynamics. We consider two possible definitions of entanglement energy. Then we construct two different kinds of thermodynamics by combining each of these different
definitions of entanglement energy with the entanglement entropy. We find that both of these two kinds of thermodynamics show significant differences from the black-hole thermodynamics if no gravitational effects are taken into account. These differences are in particular highlighted in the context of the third law of thermodynamics. Finally we see how inclusion of gravity alter the thermodynamics of the entanglement. We give a suggestive argument that the thermodynamics of the entanglement behaves like the black-hole thermodynamics if the gravitational effects are included properly. Thus the entanglement entropy passes a non-trivial check to be the origin of the black-hole entropy.
I. INTRODUCTION

Understanding the origin of the black-hole entropy is one of the most fascinating problems in black-hole physics \[1,2\]. The concept of the black-hole entropy traces back to the work by Bekenstein, who pointed out that the behavior of the basic physical quantities describing stationary black holes has an analogous structure to that of ordinary thermodynamical systems \[3\]. This thermodynamical structure inherent in the black-hole theory is usually called the ‘black-hole thermodynamics’. In analogy with ordinary material systems, then, it is natural to expect that the black-hole entropy comes from microscopic degrees of freedom of a system including a black hole. This suggests that quantum theory of gravity should inevitably take part in the black-hole thermodynamics. In this sense, understanding the origin of the black-hole entropy shall provide us with important information on quantum gravity. This is one among the several reasons why the black-hole entropy needs to be understood at the fundamental level.

Let us recall basic properties of the black-hole thermodynamics by taking a simple example. We consider the one-parameter family of Schwarzschild black holes parameterized by the mass \( M_{BH} \). Here and throughout this paper, we assume that the relation analogous to the first law of thermodynamics holds for a black-hole system. In the present example, there is only one parameter \( M_{BH} \) characterizing a black hole. Therefore, this relation should be of the simplest form

\[
dE_{BH} = T_{BH} dS_{BH} \quad ,
\]

where \( E_{BH} \), \( S_{BH} \) and \( T_{BH} \) are quantities that are identified with the energy, the entropy and the temperature of a black hole, respectively. The relation Eq.(1.1) is called the first law of the black-hole thermodynamics \[3,4\]. Thus, if two of the quantities \( E_{BH} \), \( S_{BH} \) and \( T_{BH} \) are given, Eq.(1.1) determines the remaining quantity. For simplicity, let us call this procedure of defining energy, entropy and temperature which satisfy the first law (Eq.(1.1)) the ‘construction of thermodynamics’.
In the present example, $M_{BH}$ is the only parameter characterizing the family of black holes. Therefore the simplest combination which yields the dimension of energy is

$$E_{BH} \equiv M_{BH}c^2 .$$

(1.2)

This is the energy of the black hole.

There is also a natural choice for $T_{BH}$ [5]. Hawking showed that a black hole with surface gravity $\kappa$ emits thermal radiation of a matter field (which plays the role of a thermometer) at temperature $k_B T_{BH} = \hbar \kappa / 2\pi c$. Moreover one can show that if any matter field in a thermal-equilibrium state at any temperature is scattered by a black hole, it goes to another thermal-equilibrium state at a temperature closer to $T_{BH}$ [6]. Thus it is natural to define the temperature of a Schwarzschild black hole with mass $M_{BH}$ by

$$k_B T_{BH} = \frac{\hbar c^3}{8\pi GM_{BH}} ,$$

(1.3)

since $\kappa = c^4 / 4GM_{BH}$ [1].

From Eqs. (1.1)-(1.3), we can construct the thermodynamics for the Schwarzschild black holes. Thus we get an expression for $S_{BH}$ given by

$$S_{BH} = \frac{k_B c^3}{4\hbar G} A + C ,$$

(1.4)

where $A \equiv 16\pi G^2 M_{BH}^2 / c^4$ is the area of the event horizon and $C$ is some constant. Since a value of $C$ is not essential in our discussions, we shall set hereafter

$$C = 0 .$$

(1.5)

It is well-known that classically the area of the event horizon does not decrease in time just as the ordinary thermodynamical entropy. The result Eq. (1.4) looks reasonable in this sense. Indeed this observation was the original motivation for the introduction of the black-hole thermodynamics [3]. However, it is not clear to what extent $S_{BH}$ is related with the information as the ordinary thermodynamical entropy is. At this stage we would like to point out that the third law of thermodynamics does not hold for a black hole irrespective
of the choice of the value for $C$ as is seen from Eqs. (1.3) and (1.4). We shall come back to this point in §V B. In any case, understanding the origin of $S_{BH}$ is an important problem in black-hole physics.

There have been many attempts to understand the origin of the black-hole entropy [2]. Among them we shall concentrate only on the so-called entanglement entropy [8–12]. The aim of this paper is to judge whether entanglement entropy can be regarded as the origin of black-hole entropy. For this purpose it is effective to investigate the whole structure of the thermodynamics obtained from the entanglement entropy, rather than just to examine the apparent structure of the entropy alone. Thus we shall construct the ‘thermodynamics of entanglement’ and compare it with the black-hole thermodynamics. As is expected by the above example of the black-hole thermodynamics, we have to define either energy or temperature to construct the thermodynamics of entanglement. Combining it with the entanglement entropy, which is already at hand [8–11], the other is automatically defined by means of Eq. (1.1). In this paper we shall choose the option to define the entanglement energy firstly, deriving the entanglement temperature afterwards. We shall consider two possible definitions of entanglement energy. Therefore we can construct two different kinds of thermodynamics by combining each of these definitions of entanglement energy with the entanglement entropy. We show that neither of these thermodynamics is compatible with the black-hole thermodynamics if no gravitational effects are taken into account. After that, we see how inclusion of gravity alter the thermodynamics of the entanglement. We give a suggestive argument that they have a common behavior if gravitational effects are taken into account properly. Thus the entanglement entropy passes a non-trivial check to be the origin of the black-hole entropy.

The rest of the paper is organized as follows. In section II we review the concept of the entanglement entropy. In section III we propose two natural definitions of entanglement energy and present general formulas for calculating the energy. In section IV explicit expression for the entanglement energy are derived for some tractable models with the help of the formulas prepared in section III. In section V we construct the thermodynamics
of entanglement and compare it with the black hole thermodynamics from various angles. Section VI is devoted to the summary of our results.

II. ENTANGLEMENT ENTROPY

In this section we review the definition and basic properties of the entanglement entropy.

A. Definition of the entanglement entropy

Here we give a general definition of entanglement entropy, since it is not usually made clear in the literature.

Let $\mathcal{U}$ be a Hilbert space constructed from two Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ as

$$\mathcal{U} = \mathcal{V} \bar{\otimes} \mathcal{W}, \quad (2.1)$$

where $\bar{\otimes}$ denotes a tensor product followed by a suitable completion. We call an element $u \in \mathcal{U}$ prime if $u$ can be written as $u = v \otimes w$ with $v \in \mathcal{V}$ and $w \in \mathcal{W}$. For example, $u = v_1 \otimes w_1 + 2v_1 \otimes w_2 + v_2 \otimes w_1 + 2v_2 \otimes w_2$ is prime since $u$ can be represented as $u = (v_1 + v_2) \otimes (w_1 + 2w_2)$. On the other hand $u = v_1 \otimes w_1 + v_2 \otimes w_2$ is not prime if neither $v_1$ and $v_2$ nor $w_1$ and $w_2$ are linearly dependent. The entanglement entropy $S_{\text{ent}}: \mathcal{U} \to \mathbb{R}_+ = \{\text{non-negative real numbers}\}$ defined below can be regarded as a measure of the non-prime nature of an element of $\mathcal{U} = \mathcal{V} \bar{\otimes} \mathcal{W}$.

First of all, from an element $u$ of $\mathcal{U}$ with unit norm we construct an operator $\rho$ (‘density operator’) by

$$\rho v = (u, v)u \quad \forall v \in \mathcal{U}, \quad (2.2)$$

where $(u, v)$ is the inner product which is antilinear with respect to $u$. In this context $\rho$ represents a ‘pure state’.

From $\rho$ we define another operator (‘reduced density operator’) $\rho_W$ by
\[ \rho_{Wy} = \sum_{i,j} f_j(e_i \otimes f_j, \rho e_i \otimes y) \quad \forall y \in \mathcal{W}, \quad (2.3) \]

where \( \{e_i\} \) and \( \{f_j\} \) are orthonormal bases of \( \mathcal{V} \) and \( \mathcal{W} \) respectively. Note that

\[ \text{Tr}_\mathcal{W}(\rho_{Wy} A) = \text{Tr}_\mathcal{U}(\rho_{1} \otimes A) \quad (2.4) \]

for an arbitrary bounded operator \( A \) on \( \mathcal{W} \).

Finally we define the entanglement entropy with respect to \( \rho \) as

\[ S_{\text{ent}}[\rho] \equiv -k_B \text{Tr}_\mathcal{W}[\rho_{Wy} \ln \rho_{Wy}] \quad . \quad (2.5) \]

We can totally exchange the roles played by \( \mathcal{V} \) and \( \mathcal{W} \) in Eq.(2.3) and Eq.(2.5). The entanglement entropy is so defined as to be invariant under the exchange of \( \mathcal{V} \) and \( \mathcal{W} \) when \( \rho \) corresponds to a pure state, i.e., when \( \rho \) is given by Eq.(2.2). (See Appendix for the proof of this property.)

As a simple example, let us consider spin states for a system consisting of an electron and a proton. We take \( \mathcal{V} = \{|u\rangle, |d\rangle\} \) for an electron and \( \mathcal{W} = \{|U\rangle, |D\rangle\} \) for a proton, where ‘u’ and ‘U’ are for ‘up’, while ‘d’ and ‘D’ are for ‘down’. Then \( \mathcal{U} = \mathcal{V} \otimes \mathcal{W} \) is spanned by

\[ \{|u\rangle \otimes |U\rangle, |u\rangle \otimes |D\rangle, |d\rangle \otimes |U\rangle, |d\rangle \otimes |D\rangle\} \quad . \]

Now let us consider a state

\[ |\phi\rangle = (\alpha|u\rangle + \beta|d\rangle) \otimes (A|U\rangle + B|D\rangle) \quad , \]

\[ |\alpha|^2 + |\beta|^2 = |A|^2 + |B|^2 = 1 \quad , \]

which is clearly a prime state. According to Eq.(2.3), we then get

\[ \rho_e = \begin{pmatrix} |\alpha|^2 & \alpha \beta^* \\ \alpha^* \beta & |\beta|^2 \end{pmatrix} \quad . \]

Here ‘e’ is for ‘electron’. By a suitable diagonalization of this matrix, it is easy to see that \( S_{\text{ent}} = 0 \). We can exchange the roles between ‘electron’ and ‘proton’: we then get
\[ \rho_p = \begin{pmatrix} |A|^2 & AB^* \\ A^*B & |B|^2 \end{pmatrix}, \]

\(='p'\) is for ‘proton’) which again leads to \(S_{\text{ent}} = 0\).

On the contrary, an \(s\)-state

\[ |\phi'\rangle = \gamma |u\rangle \otimes |D\rangle + \delta |d\rangle \otimes |U\rangle, \]

\[ |\gamma|^2 + |\delta|^2 = 1, \gamma \delta \neq 0 \]

is not a prime state. For this state the reduced density operators are given by

\[ \rho_e = \begin{pmatrix} |\gamma|^2 & 0 \\ 0 & |\delta|^2 \end{pmatrix}, \rho_p = \begin{pmatrix} |\delta|^2 & 0 \\ 0 & |\gamma|^2 \end{pmatrix}. \]

Therefore we get \(S_{\text{ent}} = -k_B (|\gamma|^2 \ln |\gamma|^2 + |\delta|^2 \ln |\delta|^2) > 0\).

**B. Relevance to black hole entropy**

In the case of black-hole physics, the presence of the event horizon causes a natural decomposition of a Hilbert space \(\mathcal{F}\) of all states of matter fields to a tensor product of the state spaces inside and outside a black hole as

\[ \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2. \tag{2.6} \]

For example, let us take a scalar field. We can suppose that its one-particle Hilbert space \(\mathcal{H}\) is decomposed as

\[ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{2.7} \]

where \(\mathcal{H}_1\) is a space of mode functions with supports inside the horizon and \(\mathcal{H}_2\) is a space of mode functions with supports outside the horizon. Then we can construct new Hilbert spaces (‘Fock spaces’) \(\mathcal{F}, \mathcal{F}_1\) and \(\mathcal{F}_2\) from \(\mathcal{H}, \mathcal{H}_1\) and \(\mathcal{H}_2\), respectively, as
\[ \mathcal{F} \equiv C \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})_{\text{sym}} \oplus \cdots , \]
\[ \mathcal{F}_1 \equiv C \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1)_{\text{sym}} \oplus \cdots , \]
\[ \mathcal{F}_2 \equiv C \oplus \mathcal{H}_2 \oplus (\mathcal{H}_2 \otimes \mathcal{H}_2)_{\text{sym}} \oplus \cdots , \]

(2.8)

where \((\cdots)_{\text{sym}}\) denotes the symmetrization. Now these three Hilbert spaces satisfy the relation (2.6). Hence the entanglement entropy \(S_{\text{ent}}\) is defined by the procedure given at the beginning of this section (Eqs.(2.2)-(2.5)) for each state in \(\mathcal{F}\).

The entanglement entropy \(S_{\text{ent}}\) originates from a tensor product structure of the Hilbert space as Eq.(2.6), which is caused by the existence of the boundary between two regions (the event horizon) through Eq.(2.7). Furthermore the symmetric property of \(S_{\text{ent}}\) between \(\mathcal{V}\) and \(\mathcal{W}\) mentioned before also suggests that \(S_{\text{ent}}\) is related with a boundary between two regions. In fact \(S_{\text{ent}}\) turns out to be proportional to the area of such a boundary (a model for the event horizon) in simple models (see the next subsection). In view of Eq.(1.4) with Eq.(1.3), thus, the entanglement entropy has a nature similar to the black hole entropy.

The relevance of the entanglement entropy to the black hole entropy is also suggested by the following observation. Let us consider a free scalar field on a background geometry describing a gravitational collapse to a black hole. We compare the black hole entropy and the entanglement entropy for this system. We begin with the black hole entropy. In the initial region of the spacetime, there is no horizon and the entropy around this region can be regarded as zero. In the final region, on the other hand, there is an event horizon so that the black-hole possesses non-zero entropy. As for the entanglement entropy, the existence of the event horizon naturally divides the Hilbert space \(\mathcal{F}\) of all states of the scalar field into \(\mathcal{F}_1 \otimes \mathcal{F}_2\). Thus according to the argument in the previous subsection, the scalar field in some pure state possesses non-zero entanglement entropy. In this manner, we observe that the black-hole entropy and the entanglement entropy come from the same origin, i.e. the existence of the event horizon. This is the reason why the entanglement entropy is regarded as one of the potential candidates for the origin of the black-hole entropy.
C. Simple models

The relation between the entanglement entropy and the black hole entropy was analyzed in terms of simple tractable models by [8] and [9]. They considered a free scalar field on a flat spacelike hypersurface \( \Sigma = \mathbb{R}^3 \) embedded in a 4-dimensional Minkowski spacetime, and calculated the entanglement entropy for a division of \( \Sigma \) into two regions \( \Sigma_1 \) and \( \Sigma_2 \) with a common boundary \( B \). Here \( \Sigma_1, \Sigma_2 \) and \( B \) are, respectively, the models of the interior, the exterior of the black holes and the horizon. Ref. [8] chooses \( B \) to be a 2-dimensional flat surface, and the matter state to be the ground state, showing that the resulting entanglement entropy becomes proportional to the area of \( B \). Ref. [9] chooses \( B \) to be a 2-sphere in \( \mathbb{R}^3 \), and chooses \( \Sigma_1 \) and \( \Sigma_2 \) to be the interior and the exterior of the sphere. The matter state is chosen to be the ground state. Then it is shown that the resulting entanglement entropy is again proportional to the area of \( B \).

Both of the results can be expressed as

\[
S_{\text{ent}}[\rho_0] = \frac{k_B N_S}{a^2} A ,
\]

(2.9)

where \( \rho_0 \) is the ground-state density matrix, \( A \) is the area of the boundary, \( a \) is a cutoff length, and \( N_S \) is a dimensionless numerical constant of order unity. This coincides with (1.4) and (1.5) if the cut-off length \( a \) is chosen as

\[
a = \sqrt{\frac{4N_S \hbar G}{c^3}} = 2\sqrt{N_S} \ell_p ,
\]

(2.10)

where \( \ell_p \) is the Planck length. Here note that \( a \) depends only on the Planck length.

In this paper we adopt the same simple models to construct thermodynamics of entanglement, and to discuss its relevance to the black hole thermodynamics. There the relation (2.10) and the subsequent comment will play an important role.

III. ENTANGLEMENT ENERGY

In this section we define the entanglement energy to construct the thermodynamics of entanglement. We give two possible definitions of entanglement energy. The difference
between them comes from the difference in the way to formulate the reduction of a system (caused by, for instance, the formation of an event horizon). In the first definition (§III A and §III B), we assume that the state of a total system undergoes a change in the course of the reduction of the system (so that the density matrix of the system changes actually), while operators are regarded as unchanged. In the second definition (§III C and §III D), on the contrary, we assume that some operators drop out from the set of all observables, while the state is regarded as unchanged. Since at present we cannot judge whether and which one of these treatments reflects the true process of reduction, the best way is to investigate both options. As we shall see in §V, the universal behavior of the thermodynamics of entanglement does not depend on the choice of the entanglement energy.

To make our analysis a concrete one, we apply these definitions to the tractable models given in §II C. Let us consider a system described by a Fock space $\mathcal{F}$ constructed from a one-particle Hilbert space $\mathcal{H}$ in the previous section. Let $H_{tot}$ be a total Hamiltonian acting on $\mathcal{F}$. We assume that the Hamiltonian $H_{tot}$ is naturally decomposed as

$$H_{tot} = H_1 + H_2 + H_{int} \ ,$$

where $H_1$ and $H_2$ are parts acting on $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, and $H_{int}$ is a part representing the interaction of two regions.

**A. The first definition of the entanglement energy**

First let us consider the case in which the total density operator $\rho$ actually changes to the product of reduced density operators of each subsystems, $\rho_1$ and $\rho_2$, (when, for instance, an event horizon is formed), while the observables remain unchanged.

$\rho$ reduces to $\rho'$ given by

$$\rho' = \rho_1 \otimes \rho_2 \ .$$

It is easy to see that the entropy associated with this density matrix becomes
\[-k_B \text{Tr} \left[ \rho^I \ln \rho^I \right] = S_{\text{ent}}[\rho] + S'_{\text{ent}}[\rho], \quad (3.3)\]

where $S_{\text{ent}}[\rho]$ and $S'_{\text{ent}}[\rho]$ are entanglement entropy obtained through $\rho_1$ and $\rho_2$, respectively. $S_{\text{ent}}[\rho]$ and $S'_{\text{ent}}[\rho]$ are identical if $\rho$ is a pure state (see the argument below Eq. (2.5)).

It is clear that the partial systems labeled by ‘1’ and ‘2’ can be treated symmetrically: the symmetric property of the entanglement entropy for a pure state shows that it measures the entanglement between $\mathcal{F}_1$ and $\mathcal{F}_2$, so that it is symmetrical in nature. Accordingly, one can exchange the suffices ‘1’ and ‘2’ in the above formulas.

Since we are assuming that the observables do not change, we are led to the following first definition of entanglement energy:

\[E^I_{\text{ent}} \equiv \text{Tr} \left[ : H_{\text{tot}} : \rho^I \right], \quad (3.4)\]

where $:\ - :$ denotes the usual normal ordering (a subtraction of the ground state energy).

**B. Formula of $E^I_{\text{ent}}$ for the ground state**

What we should do next is to calculate $E^I_{\text{ent}}$ explicitly by choosing $\rho$ as the ground state of $H_{\text{tot}}$. We consider a free scalar field and discretize it with some spatial separation for regularization. Since the system thus obtained is equivalent to a set of harmonic oscillators, in this section, we give a formula of $E^I_{\text{ent}}$ for the ground state of coupled harmonic oscillators. In the next section we calculate $E^I_{\text{ent}}$ explicitly by using the formula.

Let us consider a system of coupled harmonic oscillators $\{q^A\}$ ($A = 1, \ldots, N$) described by the Lagrangian,

\[L = \frac{a}{2} \delta_{AB} \dot{q}^A \dot{q}^B - \frac{1}{2} V_{AB} q^A q^B. \quad (3.5)\]

Here $\delta_{AB}$ is Kronecker’s delta symbol[4]; $V$ is a real-symmetric, positive-definite matrix which

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\[1\] From now on, we choose the units $\hbar = c = 1$ and apply Einstein’s summation convention unless otherwise stated.
does not depend on \( \{ q^A \} \). We have introduced \( a(>0) \) as a fundamental length characterizing the system. The corresponding Hamiltonian becomes

\[
H_{\text{tot}} = \frac{1}{2a} \delta^{AB} p_A p_B + \frac{1}{2} V_{AB} q^A q^B ,
\]

(3.6)

where \( p_A = a \delta_{AB} \dot{q}^B \) is the canonical momentum conjugate to \( q^A \).

Firstly we calculate the wave function \( \langle \{ q^A \} | 0 \rangle \) of the ground state \( | 0 \rangle \). Note that Eq.(3.6) can be written as

\[
H_{\text{tot}} = \frac{1}{2a} \delta^{AB} (p_A + i W_{AC} q^C) (p_B - i W_{BD} q^D) + \frac{1}{2a} \text{Tr} \ W
\]

(3.7)

by using the commutation relation \([q^A, p_B] = i \delta^A_B\). Here \( W \) is a symmetric matrix satisfying \((W^2)_{AB} = aV_{AB}\). The ambiguity in sign is fixed by requiring \( W \) to be positive definite. Now \( \langle \{ q^A \} | 0 \rangle \) is given as the solution to

\[
\left( \frac{\partial}{\partial q^A} + W_{AB} q^B \right) \langle \{ q^A \} | 0 \rangle = 0 ,
\]

(3.8)

since \( p_A \) is expressed as \(-i \frac{\partial}{\partial q^A}\). The solution is

\[
\langle \{ q^A \} | 0 \rangle = \left( \det \frac{W}{\pi} \right)^{1/4} \exp \left( -\frac{1}{2} W_{AB} q^A q^B \right) ,
\]

(3.9)

which is normalized with respect to the standard Lebesgue measure \( dq^1 \cdots dq^N \). The corresponding density matrix \( \rho_0 \) corresponding to this ground state is represented as

\[
\langle \{ q^A \} | \rho_0 | \{ q'^B \} \rangle = \langle \{ q^A \} | 0 \rangle \langle 0 | \{ q'^B \} \rangle = \left( \det \frac{W}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} W_{AB} (q^A q^B + q'^A q'^B) \right] .
\]

(3.10)

Now we split \( \{ q^A \} \) into two subsystems, \( \{ q^a \} (a = 1, \cdots, n) \) and \( \{ q'^\alpha \} (\alpha = n+1, \cdots, N) \). (We assign the labels ‘1’ and ‘2’ to the former and the latter subsystems, respectively.) Then we obtain the reduced density matrix associated with the subsystem 2 (the subsystem 1), by taking the partial trace of \( \rho_0 \) w.r.t. the subsystem 1 (the subsystem 2):

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2 Thus \( \{ q^A \} \) are treated as dimension-free quantities in the present units.
\[
\langle \{ q^a \} | \rho_2 | \{ q^{b'} \} \rangle = \int \prod_{c=1}^{n} dq^c \langle \{ q^a, q^\alpha \} | \rho_0 | \{ q^b, q^{b'} \} \rangle \\
= \left( \det \frac{D'}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} D'_\alpha \beta (q^a q^\beta + q'^a q'^\beta) \right] \times \exp \left[ -\frac{1}{4} (B^T A^{-1} B)_\alpha \beta (q - q')^\alpha (q - q')^\beta \right] \\
= \left( \det \frac{A'}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} A'_a b (q^a q^b + q'^a q'^b) \right] \times \exp \left[ -\frac{1}{4} (B D^{-1} B^T)_{ab} (q - q')^a (q - q')^b \right],
\]
(3.11)

and

\[
\langle \{ q^a \} | \rho_1 | \{ q^b \} \rangle = \int \prod_{\gamma=n+1}^{N} dq^\gamma \langle \{ q^a, q^\alpha \} | \rho_0 | \{ q^b, q^\beta \} \rangle \\
= \left( \det \frac{A'}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} A'_a b (q^a q^b + q'^a q'^b) \right] \times \exp \left[ -\frac{1}{4} (B D^{-1} B^T)_{ab} (q - q')^a (q - q')^b \right],
\]
(3.12)

where \( A, B, D, A' \) and \( D' \) are defined by

\[
(W_{AB}) = \left( \begin{array}{cc} A_{ab} & B_{a\beta} \\ (B^T)_{ab} & D_{\alpha \beta} \end{array} \right), \quad A' = A - B D^{-1} B^T, \quad D' = D - B^T A^{-1} B.
\]
(3.13)

(The superscript \( T \) denotes transposition.) Here note that \( A^T = A \) and \( D^T = D \). Thus \( \rho' \) defined by Eq. (3.2) is represented as

\[
\langle \{ q^A \} | \rho' | \{ q'^B \} \rangle = \left( \det \frac{M}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} M_{AB} (q^A q^B + q'^A q'^B) \right] \times \exp \left[ -\frac{1}{4} N_{AB} (q - q')^A (q - q')^B \right],
\]
(3.14)

where

\[
(M_{AB}) = \left( \begin{array}{cc} A'_a b & 0 \\ 0 & D'_{\alpha \beta} \end{array} \right),
\]

\[
(N_{AB}) = \left( \begin{array}{cc} (B D^{-1} B^T)_{ab} & 0 \\ 0 & (B^T A^{-1} B)_{\alpha \beta} \end{array} \right).
\]
(3.15)

We can diagonalize \( M \) and \( N \) simultaneously by the following non-orthogonal transformation:
\[ q^A \rightarrow \tilde{q}^A \equiv (UM^{1/2})_B^A q^B, \quad (3.16) \]

where \( U \) is a real orthogonal matrix satisfying
\[
M^{-1/2}NM^{-1/2} = U^T \lambda U \ ,
\]
\[
\lambda = \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots
\end{pmatrix} . \quad (3.17)
\]

Now in terms of \( \{\tilde{q}^A\} \), \( H_{\text{tot}} \) is represented as
\[
H_{\text{tot}} = -\frac{1}{2a} (UMU^T)^{AB} \left( \frac{\partial}{\partial \tilde{q}^A} - \tilde{W}_{AC} \tilde{q}^C \right) \left( \frac{\partial}{\partial \tilde{q}^B} + \tilde{W}_{BD} \tilde{q}^D \right) + \frac{1}{2a} \text{Tr} W \ , \quad (3.18)
\]
thus,
\[
: H_{\text{tot}} : = -\frac{1}{2a} (UMU^T)^{AB} \left( \frac{\partial}{\partial \tilde{q}^A} - \tilde{W}_{AC} \tilde{q}^C \right) \left( \frac{\partial}{\partial \tilde{q}^B} + \tilde{W}_{BD} \tilde{q}^D \right) \ , \quad (3.19)
\]
where
\[
\tilde{W} \equiv UM^{-1/2}WM^{-1/2}U^T . \quad (3.20)
\]

Hence the density matrix \( \rho^I \) is expressed in terms of \( | \{ \tilde{q}^A \} \rangle \) as
\[
\langle \{ \tilde{q}^A \} | \rho^I | \{ \tilde{q}^B \} \rangle = \prod_{C=1}^N \pi^{-1/2} \exp \left[ -\frac{1}{2} \left( (\tilde{q}^C)^2 + (\tilde{q}^C')^2 \right) - \frac{1}{4} \lambda_C (\tilde{q}^C - \tilde{q}^C')^2 \right] . \quad (3.21)
\]

This density matrix is normalized with respect to the measure \( dq^1 \cdots dq^N \).

Now it is easy to calculate the entanglement energy. First the matrix components of
\( : H_{\text{tot}} : \rho^I \) with respect to \( \{\tilde{q}^A\} \) are given by
\[
\langle \{ \tilde{q}^A \} | : H_{\text{tot}} : \rho^I | \{ \tilde{q}^B \} \rangle = -\frac{1}{2a} \left\{ [UMU^T - iUM^{-1/2}VM^{-1/2}U^T]_{AB} \tilde{q}^A \tilde{q}^B \right.
\]
\[
+ \text{Tr} [W - N/2 - M] \left\} \prod_{C=1}^N \pi^{-1/2} \exp \left[ -\frac{1}{2} \left( (\tilde{q}^C)^2 \right) \right] . \quad (3.22)
\]

\[ ^3 \text{Einstein's summation convention is not applied to Eq.}(3.21) . \]
Hence from the definition (3.4) the entanglement energy $E_{ent}^I$ is expressed as

$$E_{ent}^I = \int \left( \prod_{C=1}^{N} d\tilde{q}^C \right) \langle \{ \tilde{q}^C \} | : H_{tot} : \rho^I | \{ \tilde{q}^B \} \rangle$$

$$= \frac{1}{4a} \text{Tr} \left[ VM^{-1} + M + N - 2W \right]. \quad (3.23)$$

Here we have used the formula $\int d\vec{x} \cdot \vec{A} \cdot \exp[-\vec{x} \cdot \vec{x}] = \frac{1}{2\pi N/2} \text{Tr} \vec{A}$, where $N$ is the dimension of $\vec{x}$. With the help of the identity $\text{Tr} [M + N] = \text{Tr} A + \text{Tr} D = \text{Tr} W$, we finally arrive at the following formula for $E_{ent}^I$

$$E_{ent}^I = \frac{1}{4a} \text{Tr} \left[ aVM^{-1} - W \right]$$

$$= \frac{1}{4} \text{Tr} \left[ VM^{-1} - W^{-1} \right]$$

$$= -\frac{1}{2} \text{Tr} \left[ V_{int}^T \tilde{B} \right]. \quad (3.24)$$

Here $V_{int}$ is a block in the matrix $V$ given by

$$(V_{AB}) = \begin{pmatrix} V^{(1)}_{ab} & (V_{int})_{a\beta} \\ (V_{int}^T)_{ab} & V^{(2)}_{a\beta} \end{pmatrix}, \quad (3.25)$$

and $\tilde{B}$ is a block in the matrix $W^{-1}$ given by

$$W^{-1} = \begin{pmatrix} (A^r)^{-1} & \tilde{B}^{a\beta} \\ (\tilde{B}^T) & (D^r)^{-1} \end{pmatrix}. \quad (3.26)$$

It is easy to see that

$$\tilde{B}^{a\beta} = -\left( (A^r)^{-1} BD^{-1} \right)^{a\beta} = -\left( A^{-1} BD^r \right)^{a\beta}. \quad (3.27)$$

C. The second definition of the entanglement energy

The second definition of entanglement energy follows when we regard that the operators connecting the two subsystems drop out from the set of observables (when, for instance, an event horizon is formed), while the state of the system is regarded as unchanged. To be
more precise, we assume that $H_1$ and $H_2$ remain to be observables but that $H_{int}$ is no longer an observable (see Eq.(3.1)). In this case it is natural to define the entanglement energy by

$$E^{II}_{ent} \equiv \text{Tr}[(: H_1 : + : H_2 :)\rho], \quad (3.28)$$

where the two normal orderings mean to subtract the minimum eigenvalues of $H_1$ and $H_2$ respectively.

**D. Formula of $E^{II}_{ent}$ for the ground state**

For the ground state of the system analyzed in §III B, we now evaluate the entanglement energy in the sense of Eq.(3.28).

Firstly we divide the Hamiltonian (3.6) into three terms as Eq.(3.1) (see Eq.(3.25)):

$$H_1 \equiv \frac{1}{2a} \delta^{ab} p_a p_b + \frac{1}{2} V^{(1)}_{ab} q^a q^b$$

$$= \frac{1}{2a} \delta^{ab} \left( p_a + iw^{(1)}_{ac} q^c \right) \left( p_b - iw^{(1)}_{bd} q^d \right) + \frac{1}{2a} \text{Tr} w^{(1)},$$

$$H_2 \equiv \frac{1}{2a} \delta^{\alpha\beta} p_\alpha p_\beta + \frac{1}{2} V^{(2)}_{\alpha\beta} q^\alpha q^\beta$$

$$= \frac{1}{2a} \delta^{\alpha\beta} \left( p_\alpha + iw^{(2)}_{\alpha\gamma} q^\gamma \right) \left( p_\beta - iw^{(2)}_{\beta\delta} q^\delta \right) + \frac{1}{2a} \text{Tr} w^{(2)},$$

$$H_{int} \equiv H_{tot} - H_1 - H_2$$

$$= V_{int} a^\beta q^\alpha q^\beta, \quad (3.29)$$

where $w^{(1)}$ and $w^{(2)}$ are, respectively, the positive square-roots of $aV^{(1)}$ and $aV^{(2)}$. Although there exists freedom in the way of the division, the above division seems to be the most natural one. Here and throughout this paper we adopt it.

Now it is convenient to diagonalize $W$ in (3.10) by a non-orthogonal transformation,

$$q^A \rightarrow \bar{q}^A \equiv \delta^{AB} (W^{1/2}) \text{ }_{BC} q^C. \quad (3.30)$$

In terms of $\{\bar{q}^A\}$, the density matrix $\rho_0$ corresponding to the ground state of $H_{tot}$ is written as
\begin{equation}
\langle \{ \bar{q}^A \} | \rho_0 | \{ \bar{q}^B \} \rangle = \prod_{C=1}^{N} \pi^{-1/2} \exp \left[ -\frac{1}{2} \left( (\bar{q}^C)^2 + (\bar{q}^C')^2 \right) \right].
\end{equation}

On the other hand the operator \( :H_1 : + :H_2 : \) is written as
\begin{equation}
: H_1 : + : H_2 : = -\frac{1}{2a} \delta^{AC} \delta^{BD} W_{CD} \left( \frac{\partial}{\partial \bar{q}^A} - \bar{w}_{AE} \bar{q}^E \right) \left( \frac{\partial}{\partial \bar{q}^B} + \bar{w}_{BF} \bar{q}^F \right),
\end{equation}
where \( \bar{w} \) is defined by
\begin{equation}
\bar{w} \equiv \delta_{AC} \left( W^{-1/2} w W^{-1/2} \right)^{CD} \delta_{DB},
\end{equation}
and
\begin{equation}
(w_{AB}) \equiv \begin{pmatrix}
w^{(1)}_{ab} & 0 \\
0 & w^{(2)}_{\alpha\beta}
\end{pmatrix}.
\end{equation}

From these expressions we obtain
\begin{align}
\langle \{ \bar{q}^A \} | ( : H_1 : + : H_2 : ) \rho_0 | \{ \bar{q}^B \} \rangle &= \frac{1}{2a} \left\{ [(\bar{w} + 1) W (\bar{w} - 1)]_{AB} \bar{q}^A \bar{q}^B \right.
\notag \noindent \left. - \text{Tr} [W (\bar{w} - 1)] \right\} \prod_{C=1}^{N} \pi^{-1/2} \exp \left[ -(\bar{q}^C)^2 \right].
\end{align}

Hence we arrive at the following expression \( E_{\text{ent}}^{II} \) for \( \rho_0 \)
\begin{align}
E_{\text{ent}}^{II} &= \int \left( \prod_{C=1}^{N} dq^C \right) \langle \{ \bar{q}^A \} | ( : H_1 : + : H_2 : ) \rho_0 | \{ \bar{q}^B \} \rangle 
\notag \noindent  
= \frac{1}{4a} \text{Tr} \left[ w^2 W^{-1} - W \right] - \frac{1}{2a} \text{Tr} \left[ w - W \right].
\end{align}

With the help of the relation \( \text{Tr}[w^2 W^{-1}] = \text{Tr}[a V M^{-1}] \) which follows from the definitions of \( w \) and \( M \), this formula is simplified as
\begin{align}
E_{\text{ent}}^{II} &= \frac{1}{4a} \text{Tr} \left[ a V M^{-1} - W \right] - \frac{1}{2a} \text{Tr} \left[ w - W \right] 
\notag \noindent  
= \frac{1}{2a} \text{Tr} \left[ w - W \right],
\end{align}
where Eq.(3.24) has been used to obtain the last line.

\textbf{IV. EXPLICIT EVALUATION OF THE ENTANGLEMENT ENERGY FOR SOME TRACTABLE MODELS IN FLAT SPACETIME}

With the help of the formulas derived in the previous section, we now calculate \( E_{\text{ent}}^{I} \) and \( E_{\text{ent}}^{II} \) explicitly to construct two kinds of the thermodynamics of entanglement for the simple models discussed in \( \S \text{II} \text{C} \).
We consider a free scalar field $\phi$ on 4-dimensional Minkowski space $\mathcal{M}_4$, described by the action

$$S = \int_{\mathcal{M}_4} \left\{ -\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}. \quad (4.1)$$

Now we divide the spatial section $\mathbb{R}^3$ of $\mathcal{M}_4$ into two disconnected regions $\Sigma_1$ and $\Sigma_2$ by a suitable 2-dimensional surface $B$. We consider two cases: One is the choice $B = \mathbb{R}^2$ and the other is $B = S^2$. We calculate $E_{\text{ent}}^I$ and $E_{\text{ent}}^{II}$ for each case, getting four results in total.

Stating the results first, they are summarized in the universal form

$$E_{\text{ent}} = \mathcal{N}_E \frac{\hbar c}{a^3} A, \quad (4.2)$$

where $A$ is the area of $B$, $a(>0)$ is a cutoff length and $\mathcal{N}_E$ is a dimensionless numerical constant.

A. Entanglement energy for the case of $B = \mathbb{R}^2$

First we take $B = \mathbb{R}^2$. Without loss of generality the resulting two half-spaces are represented as $\Sigma_1 = \{(x_1, x_2, x_3) : x_1 > 0\}$ and $\Sigma_2 = \{(x_1, x_2, x_3) : x_1 < 0\}$.\[8\]

Here some comments are in order. Since all the degrees of freedom on and across $B$, which is infinite, contribute to the entanglement energy, a suitable cut-off length $a(>0)$ should be introduced to avoid the ultra-violet divergence. For the same reason, the infra-red divergence is also anticipated in advance, since $B$ is non-compact in this model. The latter is taken care of by considering the massive case since the inverse of the mass characterizes a typical size of the spreading of the field. Clearly $a$ should be taken short enough in the unit of the Compton length of the field, $m^{-1}$, to obtain meaningful results. Therefore we shall only pay attention to the leading order in the limit $ma \to 0$ in the course of calculation as

\[4\] Here we follow the sign-convention of $\text{diag}(-,+,+,+)$.\[5\] We have recovered $\hbar$ and $c$ in Eq.(4.2).
well as in the final results. These remarks are valid in any model of this type, and the same remarks apply to the case of the entanglement entropy, too [8,9].

In order to calculate $E_{\text{ent}}^I$ for the present case, we first note that the term $V_{AB}q^Aq^B$ in Eq.(3.3) corresponds to the expression $\int \left[ (\nabla \phi)^2 + m^2 \phi^2 \right] d^3x$ read off from Eq.(4.1), which defines an operator $V(x,y)$ acting on a space $W = (\{\phi(\cdot)\}, d^3x)$. In order to use the formula (3.24), thus, we need the positive square-root and the inverse of $aV$. For this purpose, it is convenient to work in the momentum representation of $W$ given by

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k \exp[i\vec{k} \cdot \vec{x}],$$

$$\phi_{\vec{k}} = \int d^3x \phi(\vec{x}) \exp[-i\vec{k} \cdot \vec{x}].$$

(4.3)

The results are

$$V(\vec{x}, \vec{y}) = \int \frac{d^3k}{(2\pi)^3} (\vec{k}^2 + m^2) \exp[i\vec{k} \cdot (\vec{x} - \vec{y})],$$

$$W^{-1}(\vec{x}, \vec{y}) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} (\vec{k}^2 + m^2)^{-1/2} \exp[i\vec{k} \cdot (\vec{x} - \vec{y})].$$

(4.4)

Note that both $V(\vec{x}, \vec{y})$ and $W^{-1}(\vec{x}, \vec{y})$ are symmetric under the exchange of $\vec{x}$ and $\vec{y}$. (The cut-off must preserve this property.) Now the formula (3.24) gives

$$E_{\text{ent}}^I = -\frac{1}{2} \int_{y_1 < -a} d^2y \int_{x_1 > a} d^3x \int_{|k_1| < a^{-1}} \frac{d^3k}{(2\pi)^3} (\vec{k}^2 + m^2) \exp[i\vec{k} \cdot (\vec{y} - \vec{x})]$$

$$\times \int_{|k'_1| < a^{-1}} \frac{d^3k'}{(2\pi)^3} (\vec{k'}^2 + m^2)^{-1/2} \exp[i\vec{k'} \cdot (\vec{x} - \vec{y})].$$

(4.5)

where, as discussed above, a cut-off length $a$ was introduced in the integral.

Since the integrand is invariant under the translation along $B$, the integral with respect to $x_2$ and $x_3$ yields a divergent factor $A = \int_{\mathbb{R}^2} dx_2 dx_3$. Clearly this factor should be interpreted as the area of $B$. If this divergent integral $A$ is factored out, we obtain the following convergent expression for $E_{\text{ent}}^I$:

$$E_{\text{ent}}^I = -\frac{A}{2} \int_{-\infty}^{-a} dy_1 \int_{a}^{\infty} dx_1 \int_{|k_1| < a^{-1}} \frac{d^2k_\parallel}{(2\pi)^2} \int_{-a^{-1}}^{a^{-1}} \frac{dk_1}{2\pi} \int_{-a^{-1}}^{a^{-1}} \frac{dk'_1}{2\pi}$$

$$\times (k_\parallel^2 + k_1^2 + m^2)(k_\parallel^2 + k'_1^2 + m^2)^{-1/2} \exp[i(k_1 - k'_1)(y_1 - x_1)].$$

(4.6)
Here $\vec{k}$ is a 2-vector lying along $B$ and $k_1$, $k'_1$ are components normal to $B$ (if we make an obvious identification of $\mathbb{R}^3$ with its Fourier space). Let us change the variables from $x_1$ and $y_1$ to $z \equiv x_1 - y_1$ and $u \equiv (x_1 + y_1)/2$. Then $z$ and $u$ take values in the range $z \leq 2a$ and $-(\frac{a}{2} - a) \geq u \geq (\frac{a}{2} - a)$, respectively. Hence the integration with respect to $u$ yields

$$E_{\text{ent}}^I = -\frac{A}{2} \int_{2a}^{\infty} dz (z - 2a) \int_{-a}^{a-1} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} dk'_1 \times (k^2 + k_1^2 + m^2)(k^2 + k'_1^2 + m^2)^{-1/2} \exp[-i(k_1 - k'_1)z]$$

$$= -\frac{A}{2} \int_{2a}^{\infty} dz (z - 2a) \int_{-a}^{a-1} \frac{dk_1}{2\pi} \cos(k_1 z) \int_{m}^{\infty} \frac{d\kappa}{2\pi} \kappa \cos(k_1 z)$$

$$\times \int_{-\infty}^{\infty} \frac{dk'_1}{2\pi} \frac{1}{(k^2 + k'_1^2)^{-1/2}} \cos(k'_1 z) \tag{4.7}$$

in the leading order, where $\kappa$ is defined by $\kappa^2 = \vec{k}^2 + m^2$. Here note that in this expression, the integration with respect to $k'_1$ followed by that with respect to $\kappa$ leads to an infra-red divergence if we set $m = 0$, in accordance with our discussion at the beginning of this subsection.

Now let us recollect some formulas with the modified Bessel functions \[\text{[13]}\]:

$$K_0(x) = \int_{0}^{\infty} dt \frac{\cos t}{\sqrt{t^2 + x^2}},$$

$$\int_{x_0}^{\infty} dx xK_0(x) = x_0 K_1(x_0),$$

$$\int_{x_0}^{\infty} dx x^3 K_0(x) = x_0^3 K_1(x_0) + 2x_0^2 K_2(x_0). \tag{4.8}$$

With the help of these formulas $E_{\text{ent}}^I$ is written as

$$E_{\text{ent}}^I = -\frac{A}{2} \int_{2a}^{\infty} dz (z - 2a) \int_{-a}^{a-1} \frac{dk_1}{2\pi} \cos(k_1 z) \times \frac{1}{2\pi^2} \left[ \frac{m}{z} (k_1^2 + m^2) K_1(mz) + \frac{m^2}{z^2} K_2(mz) \right]$$

$$= -\frac{A}{4\pi^3a^3} [\alpha_1(ma) + \alpha_2(ma) + \alpha_3(ma)] \tag{4.9}$$

in the leading order. Here we have introduced

$$\alpha_1(x) \equiv -x \int_{2}^{\infty} d\xi \frac{\xi - 2}{\xi^4} K_1(x\xi) \left[ 2\xi \cos \xi + (\xi^2 - 2) \sin \xi \right],$$

$$\alpha_2(x) \equiv -2x^2 \int_{2}^{\infty} d\xi \frac{\xi - 2}{\xi^3} K_2(x\xi) \sin \xi,$$

$$\alpha_3(x) \equiv -x^3 \int_{2}^{\infty} d\xi \frac{\xi - 2}{\xi^2} K_1(x\xi) \sin \xi \tag{4.10}$$
A numerical evaluation shows

\[ [\alpha_1(x) + \alpha_2(x) + \alpha_3(x)] \sim 0.05 \quad \text{as} \quad x \to 0 \ . \]

This result is of the form of Eq. (4.2) with \( N_E \sim 0.05/4\pi^3 \approx 4.0 \times 10^{-4} \) in the limit \( ma \to 0 \).

In order to calculate \( E_{ent}^{II} \) by the formula (3.36) we use the expression for \( w \),

\[
w(\vec{x}, \vec{y}) = \int \frac{d^3k}{(2\pi)^3} \left( \vec{k}^2 + m^2 \right)^{1/2} \left\{ \theta(x_1)\theta(y_1) + \theta(-x_1)\theta(-y_1) \right\} \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \ . \quad (4.11)
\]

From this it follows that

\[
W(x, y) - w(x, y) = \int \frac{d^3k}{(2\pi)^3} \left( \vec{k}^2 + m^2 \right)^{1/2} \left\{ \theta(x_1)\theta(-y_1) + \theta(-x_1)\theta(y_1) \right\} \\
\times \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \ . \quad (4.12)
\]

Taking the trace of this expression, we get

\[
\text{Tr}(W - w) = \int d^3x \int d^3y \delta^3(\vec{x} - \vec{y}) [W(\vec{x}, \vec{y}) - w(\vec{x}, \vec{y})] \\
= A \int \frac{d^3k}{(2\pi)^3} \left( \vec{k}^2 + m^2 \right)^{1/2} \\
\times \left[ \int dx_1 dy_1 \delta(x_1 - y_1) \left\{ \theta(x_1)\theta(-y_1) + \theta(-x_1)\theta(y_1) \right\} \right] \\
= 0 \ . \quad (4.13)
\]

Therefore we get\[\]
\[
E_{ent}^{II} = E_{ent}^I \approx \frac{0.05A}{4\pi^3a^3} \ . \quad (4.14)
\]

\[
B. \text{ Entanglement energy for the case of } B = S^2
\]

Next we consider the case \( B = S^2 \), a sphere with radius \( R \). This is the same model as in the calculation of the entanglement entropy in Ref. \[\].

\[\]

\[
6 \text{ If we adopt another regularization scheme with the same cut-off length } a, \text{ the result may change. However, the change is in sub-leading order.}
\]
It should be noted that in this model, we can put \( m = 0 \) from the beginning without being bothered by the infra-red divergence. This is because \( B \) is compact as is discussed at the beginning of the previous subsection. Hence we simply put \( m = 0 \) in this subsection.

By introducing the polar coordinates, \( \phi(\vec{x}) \) is expanded by the spherical harmonics as

\[
\phi(r, \theta, \psi) = \sum_{l,m} \frac{\phi_{lm}(r)}{r} Z_{lm}(\theta, \psi),
\]

where \( Z_{lm} \) and \( Z_{l,-m} \) are the real parts of the spherical harmonics \( Y_{lm} \) and \( Y_{l,-m} \), respectively.

In terms of \( \phi_{lm} \) the potential term in Eq.(4.1) is written as

\[
\int (\nabla \phi)^2 d^3x = \sum_{l,m} \int_0^\infty \left[ r^2 \left( \frac{\partial}{\partial r} \left( \frac{\phi_{lm}(r)}{r} \right) \right)^2 + \frac{l(l+1)}{r^2} \phi_{lm}^2(r) \right] dr. \tag{4.17}
\]

Now we introduce a cut-off scale \( a \) as in the previous model to take care of the ultra-violet divergence. For that purpose we divide the radial coordinate \( r \) into a lattice as \( r_A = an \) \( (n = 1, 2, \cdots, N) \), with the identification \( R \equiv (n_B + 1/2)a \) for some \( n_B \). It is understood that the limit \( N \to \infty \) is taken in the final results. As is discussed at the beginning of the previous subsection, the entanglement energy is carried by the modes around \( B \), as the term ‘entanglement’ implies. We thus need to introduce the cut-off scale \( a \) only in the \( r \)-direction.

Under this regularization, the right-hand side of Eq.(4.17) (corresponding to \( V_{AB} q^A q^B \) in Eq.(3.5)) turns to

\[
\frac{1}{a} \sum_{l,m} \sum_{n=1}^N \left[ \left( n + \frac{1}{2} \right)^2 \left( \frac{\phi_{lm,n}}{n} - \frac{\phi_{lm,n+1}}{n+1} \right)^2 + \frac{l(l+1)}{n^2} \phi_{lm,n}^2 \right],
\]

where we have imposed the boundary condition \( \phi_{lm,N+1} \equiv 0 \). Thus \( V \) is written as the direct sum

\[
V = \bigoplus_{l,m} V^{(l,m)},
\]

where \( V^{(l,m)} \) is the \( N \times N \) matrix given by

\[
\begin{pmatrix}
\Sigma_1^{(l)} & \Delta_1^{(l)} \\
\Delta_1^{(l)} & \Sigma_2^{(l)} & \Delta_2^{(l)} \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \ddots \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
\Sigma_1^{(l)} & \Delta_1^{(l)} & \Delta_2^{(l)} & \cdots & \cdots \\
\Delta_1^{(l)} & \Sigma_2^{(l)} & \Delta_2^{(l)} & \cdots & \cdots \\
\Delta_2^{(l)} & \Delta_2^{(l)} & \Sigma_3^{(l)} & \cdots & \cdots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]
\[
\sum_n^{(l)} = 1 + \frac{1}{4n^2} + \frac{l(l+1)}{2n^2}, \\
\Delta_n^{(l)} = \frac{(n+1/2)^2}{2n(n+1)}.
\]  

(4.18)

Hence \( E_{\text{ent}} \) is written as

\[
E_{\text{ent}} = \sum_{l=0}^{\infty} (2l+1)E_{\text{ent}}^{(l)},
\]

where \( E_{\text{ent}}^{(l)} \) is defined by (3.24) or (3.36) with \( V \) replaced by \( V^{(l,m)} \).

Unfortunately, it is difficult to calculate \( E_{\text{ent}} \) analytically. So we evaluated it by numerical calculation. First we evaluated \( E_{\text{ent}}^{(l)} \) for fixed values of \( N \) and \( n_B (N > 30, 1 \leq n_B \leq 30) \). Next the summation with respect to \( l \) was performed up to \( l = l_{\text{max}} \), which is determined so that

\[
(2l_{\text{max}}+1)E_{\text{ent}}^{(l_{\text{max}})}/\sum_{l=0}^{l_{\text{max}}} (2l+1)E_{\text{ent}}^{(l)} \leq 10^{-3}.
\]

Then we repeated the above procedure for all values of \( n_B \) in the range \( 1 \leq n_B \leq 30 \). Finally we confirmed that \( N = 60 \) for \( E_{\text{ent}}^{I} \) and \( N = 200 \) for \( E_{\text{ent}}^{I} - E_{\text{ent}}^{II} \) are so large that the boundary condition \( \phi_{lm,N+1} \equiv 0 \) does not affect the results. The results are shown in Figure 1, where \( aE_{\text{ent}}^{I} \) and \( a(E_{\text{ent}}^{I} - E_{\text{ent}}^{II}) \) are written as functions of \( (R/a)^2 = (n+1/2)^2 \).

From this figure we see that \( aE_{\text{ent}}^{I} \) is almost proportional to \( (R/a)^2 \). Hence both \( E_{\text{ent}}^{I} \) and \( E_{\text{ent}}^{II} \) are proportional to \( R^2/a^3 \):

\[
E_{\text{ent}}^{I} \sim 0.35 \frac{R^2}{a^3}, \quad (4.19)
\]

\[
E_{\text{ent}}^{II} \sim 0.20 \frac{R^2}{a^3}. \quad (4.20)
\]

These results again confirm the relation Eq.(4.12).

V. COMPARISON: THE THERMODYNAMICS OF THE ENTANGLEMENT AND THE BLACK-HOLE THERMODYNAMICS
A. The thermodynamics of the entanglement in flat spacetime

We have introduced two possible definitions of entanglement energy, $E_{\text{ent}}^I$ and $E_{\text{ent}}^{II}$, in the previous section. By combining each of them with the entanglement entropy $S_{\text{ent}}$, we obtain two kinds of the thermodynamics of entanglement.

For this purpose, we consider an infinitesimal process in which the way of the division of the Hilbert space $\mathcal{H}$ into $\mathcal{H}_1$ and $\mathcal{H}_2$ is changed smoothly, with the ‘initial’ state $\rho_0$ being fixed. (See [IV A].) Let $dS_{\text{ent}}$ and $dE_{\text{ent}}$ be the resultant infinitesimal changes in the entanglement entropy and in the entanglement energy, respectively. We are dealing with a 1-parameter family of the infinitesimal changes for the entanglement. The parameter is chosen to be the area $A$ of $B$ for both of the models in [V A] and in [V B]. Thus, the construction of the thermodynamics means to use (see Eq.(I))

$$dE_{\text{ent}} = T_{\text{ent}}dS_{\text{ent}}$$

(5.1)

to determine $T_{\text{ent}}$, which is interpreted as the temperature of the entanglement. Combining (2.9) and (4.2) with Eq.(5.1), we thus get\footnote{In this section, we recover $\hbar$ and $c$.}

$$k_B T_{\text{ent}} = \frac{\mathcal{N}_E}{\mathcal{N}_S} \frac{a}{\hbar c} .$$

(5.2)

Note that the temperatures $T_{\text{ent}}$ obtained from the two definition of the entanglement energy (Eq.(3.4) and Eq.(3.28)) coincides up to numerical factors of order unity, because of the universal behavior of Eq.(4.2).

Let us interpret the thermodynamics of the entanglement given by (2.9), (4.2) and (5.2). It is helpful to introduce the quantities

$$n_{\text{ent}} \equiv \frac{A}{a^2} ,$$

$$e_{\text{ent}} \equiv \frac{\hbar c}{a} .$$

(5.3)
Here $n_{ent}$ is regarded as an effective number of degrees of freedom of matter on the boundary $B$, and $e_{ent}$ is a typical energy scale of each degree of freedom on $B$.

From Eqs. (2.9), (4.2) and (5.2), we find that

$$S_{ent} \sim k_B n_{ent},$$
$$E_{ent} \sim e_{ent} n_{ent},$$
$$k_B T_{ent} \sim e_{ent}.$$  \hspace{1cm} (5.4)$$

Therefore our results can be interpreted as follows:\textsuperscript{8}: The entanglement entropy is a measure for the number of microscopic degrees of freedom on the boundary $B$; the entanglement energy is a measure for the total energy carried by all of the degrees of freedom on $B$; the temperature of the entanglement is measure for the energy carried by each degree of freedom on $B$.

**B. Discrepancy between the thermodynamics of the entanglement in flat spacetime and the black-hole thermodynamics**

Now we compare these results with the case of black holes. For that purpose we express the black-hole thermodynamics in the same form as in the previous subsection.

Let us introduce the quantities

$$n_{BH} \equiv \frac{A}{l_p^2},$$
$$e_{BH} \equiv \frac{hc}{l_{pl}}.$$  \hspace{1cm} (5.5)$$

We can interpret that $n_{BH}$ corresponds to the effective number of degrees of freedom on the event horizon and $e_{BH}$ is a typical energy scale for each degree of freedom of matter on the horizon.

\textsuperscript{8} It is safer, however, to regard such an interpretation just as a convenient way of representing our results. This note in particular applies to the case of the black-hole thermodynamics (see Eq.(5.6)).
The black-hole thermodynamics can be recast in terms of these quantities as

\[ S_{BH} \sim k_B n_{BH} , \]

\[ E_{BH} \sim \gamma_{BH} e_{BH} n_{BH} , \]

\[ k_B T_{BH} \sim \gamma_{BH} e_{BH} , \]

(5.6)

where \( \gamma_{BH} \equiv l_{pl}/R \). The factor \( \gamma_{BH} \) can be understood as a magnification of energy due to an addition of gravitational energy or a red-shift factor of temperature since \( \sqrt{\gamma_{BH}} \sim l_{pl}/R \) at \( r \sim R + l_{pl}^2/R \), which corresponds to a stationary observer at the proper distance \( l_{pl} \) away from the horizon. Here \( R \) is the area radius of the horizon. Thus the following interpretation is possible\(^9\): The black-hole entropy is a measure for the number of the microscopic degrees of freedom on the event horizon; the black-hole energy is a measure at infinity for the total energy carried by all of the degrees of freedom on the event horizon; the black-hole temperature is a measure at infinity for the energy carried by each degree of freedom.

Now we compare the two types of thermodynamics characterized by Eq.(5.4) and Eq.(5.6), respectively. Both of them allow the interpretation that they describe the behavior of the effective microscopic degrees of freedom on the boundary \( B \), or on the horizon. Because of the factor \( \gamma_{BH} \), however, they are hardly understood in a unified picture. This strongly suggests that an inclusion of gravitational effects is necessary for agreement between them. In the remaining of this subsection we shall see that the discrepancy cannot be avoided by any means unless gravitational effects are taken into account for the thermodynamics of the entanglement. After that, a restoration of the agreement by gravity is discussed in the next subsection.

The discrepancy is highlighted in the context of the third law of thermodynamics. Both types of thermodynamics fail to follow the third law (when \( A \) is chosen as a control-parameter), but in quite different manners.

\(^9\) See the footnote after Eq.(5.4).
In Eq. (5.2), we see that $T_{\text{ent}}$ remains constant if $\frac{N_{e}}{N_{S}}$ is assumed to be constant.\textsuperscript{10} On the other hand, Eq. (2.9) shows that $S_{\text{ent}}$ tends to zero as $A \to 0$. Therefore we obtain the following $A$-dependence:

$$
S_{\text{ent}} \propto A ,
E_{\text{ent}} \propto A ,
k_{B}T_{\text{ent}} \propto A^{0} .
$$

(5.7)

The system behaves as though it is kept in touch with a thermal bath with temperature $T_{\text{ent}}$.

In contrast, for the black-hole thermodynamics, Eq. (1.3) and Eq. (1.4) along with Eq. (1.5) give the behavior (note that $A \propto M_{\text{BH}}^{2}$)

$$
S_{\text{BH}} \propto A ,
E_{\text{BH}} \propto \sqrt{A} ,
k_{B}T_{\text{BH}} \propto \frac{1}{\sqrt{A}} .
$$

(5.8)

Thus we see that $S_{\text{BH}} \to \infty$ as $T_{\text{BH}} \to 0$.

The discrepancy between Eq. (5.7) and Eq. (5.8) is quite impressive. On one hand, a well-known behavior (5.8) comes from the fundamental properties of the black-hole physics. On the other hand, the behavior characterized by Eq. (5.7) is also an universal one in any model of the entanglement: The zero-point energy of the system has been subtracted as Eq. (3.4) or Eq. (3.28), thus only the degrees of freedom on the boundary $B$ contributes to $E_{\text{ent}}$, yielding the behavior $E_{\text{ent}} \propto A$. The two definitions of $E_{\text{ent}}$ proposed here (Eq. (3.4) and Eq. (3.28)) look quite reasonable though other definitions may be possible. The result $E_{\text{ent}} \propto A$ also looks natural, being compatible with the concept of ‘entanglement’. At the

\textsuperscript{10} Here we are regarding the cut-off scale $a$ as the fundamental constant of the theory, not to be varied. However, see the discussions below.
same time, $S_{\text{ent}}$ also behaves universally as $S_{\text{ent}} \propto A$, which has been the original motivation for investigating the relation between $S_{\text{BH}}$ and $S_{\text{ent}}$ 10–12.

It is also interesting to investigate the cut-off dependence of both types of thermodynamics. From the viewpoint of the theory of the renormalization group 15, this dependence also deserves to be investigated.

First, for the case of the thermodynamics of entanglement, $a$ should be varied with $A$ being fixed. Hence, from Eqs.(2.9), (1.2) and (5.2), we see that

$$S_{\text{ent}} \propto a^{-2},$$
$$E_{\text{ent}} \propto a^{-3},$$
$$k_B T_{\text{ent}} \propto a^{-1}.$$  \hspace{1cm} (5.9)

When we regard $a$ instead of $A$ as the external control-parameter of the system, thus, the third law of thermodynamics follows: $S_{\text{ent}} \rightarrow 0$ as $T_{\text{ent}} \rightarrow 0$.

For the case of the black-hole thermodynamics, on the other hand, $l_p$ should be varied with $A = \frac{16\pi^3 G^2}{c} M_{\text{BH}}^2 = 16\pi l_p^4 \left( \frac{\hbar}{M_{\text{BH}} c} \right)^2$ being fixed13. Then we see from Eqs.(1.4) (with (1.5)), (1.2) and (1.3) that

$$S_{\text{BH}} \propto l_p^{-2},$$
$$E_{\text{BH}} \propto l_p^{-2},$$
$$k_B T_{\text{BH}} \propto l_p^0.$$  \hspace{1cm} (5.10)

Thus the third law of thermodynamics does not hold in this case, too. It is interesting to note that this behavior looks similar to that in the case of the entanglement with $A$ being varied (Eq.(5.7)).

Finally, for completeness let us look at the behavior of thermodynamics of entanglement and the black-hole thermodynamics when $M_{\text{BH}}$ and $E_{\text{ent}}$ are fixed, respectively. It becomes

11 There is no direct connection between $a$ and $l_p$. However they have a common property that both of them introduces a cut-off scale into a quantum matter field.
\[ S_{BH} \propto l_p^2, \]
\[ E_{BH} \propto l_p^0, \]
\[ k_B T_{BH} \propto l_p^{-2}. \]  \hspace{1cm} (5.11)

and

\[ S_{ent} \propto a, \]
\[ E_{ent} \propto a^0, \]
\[ k_B T_{ent} \propto a^{-1}. \]  \hspace{1cm} (5.12)

The third law fails to hold in this case.

The results Eq.(5.7) in comparison with Eq.(5.8), and Eq.(5.9) in comparison with Eq.(5.10) are summarized in Table I. Anyway, the thermodynamics of the entanglement in flat spacetime shows significant differences from the black-hole thermodynamics.

C. Restoration of the agreement by gravity

In this subsection let us discuss a possible restoration of the agreement between the thermodynamics of entanglement and the black-hole thermodynamics by considering gravitational effects.

Although the behavior (5.4) of the thermodynamics of the entanglement was derived by considering models in flat spacetime, it seems very reasonable that we regard the quantities \( S_{ent} \), \( E_{ent} \) and \( T_{ent} \) as those in a black-hole background measured by a stationary observer located at the proper distance \( a \) away from the horizon \(^{12}\). Since \( S_{BH} \), \( E_{BH} \) and \( T_{BH} \) in (5.6) are quantities measured at infinity, it is behavior of \( S_{ent} \), \( E_{ent} \) and \( T_{ent} \) at infinity that we have to compare with (5.3). \( S_{ent} \) at infinity probably has the same behavior as that measured by the observer near the horizon since a number of degrees of freedom seems independent of

\(^{12}\) The authors thank T. Jacobson for helpful comments on this point.
an observer’s view-point. That is consistent with the fact that the entanglement entropy on Schwarzschild background has the same behavior $S_{\text{ent}} \sim k_B n_{\text{ent}}$ [11]. On the other hand it seems natural to add the gravitational energy to the entanglement energy by replacing $E_{\text{ent}}$ with $\sqrt{-g_{tt}} E_{\text{ent}}$. Then the entanglement temperature is determined by use of the first law (5.1). Thus the inclusion of gravity may alter the behavior (5.4) to

$$S_{\text{ent}} \sim k_B n_{\text{ent}} \ ,$$

$$E_{\text{ent}} \sim \gamma_{\text{ent}} c_{\text{ent}} n_{\text{ent}} \ ,$$

$$k_B T_{\text{ent}} \sim \gamma_{\text{ent}} c_{\text{ent}} \ ,$$

(5.13)

where $\gamma_{\text{ent}} \equiv a/R$. The factor $\gamma_{\text{ent}}$ represents the gravitational magnification of the entanglement energy due to the addition of gravitational energy since on the corresponding Schwarzschild background $\sqrt{-g_{tt}} \sim a/R$ at $r \sim R + a^2/R$, which corresponds to a stationary observer at the proper distance $a$ away from the horizon (see the argument below (5.6) and Ref. [11,14]). Here $R$ is the area radius of the horizon. (5.13) shows a complete agreement with (5.6). Note that the last equality in (5.13) is consistent with an interpretation that the entanglement temperature is red-shifted by the factor $\gamma_{\text{ent}}$. Thus the inclusion of gravitational effects restores the agreement between the thermodynamics of the entanglement and the black-hole thermodynamics at least qualitatively.

VI. SUMMARY AND DISCUSSIONS

In this paper we have tried to judge whether the black-hole entropy can be understood as the entanglement entropy associated with the division of spacetime by the event horizon. Our strategy has been to look at the whole thermodynamical structures inherent in a black-hole system and models introduced in [8,9] to relate their entanglement entropy to the black-hole entropy. Following this strategy we have undertaken the construction of the thermodynamics of entanglement. For this purpose, after reviewing the basics of the entanglement entropy $S_{\text{ent}}$ [8–12], we have proposed two reasonable definitions of the entanglement energy $E_{\text{ent}}$. To
obtain explicit values of $E_{\text{ent}}$, we have prepared basic formulas for the entanglement energy. We have then estimated the entanglement energy by choosing two tractable models of a scalar field in the Minkowski space. The boundary $B$ has been chosen as, respectively, $B = \mathbb{R}^2$ and $B = \mathbb{S}^2$ in each model. We have thus found a common behavior independent of the definition of $E_{\text{ent}}$, $E_{\text{ent}} \propto A/a^3$, where $A$ is the area of the boundary and $a$ is a fundamental cut-off scale of the system. Getting $E_{\text{ent}}$ along with $S_{\text{ent}}$ in hand, we have constructed the thermodynamics of entanglement by postulating the first law of thermodynamics. In particular we have found that the temperature of the entanglement $T_{\text{ent}}$ is proportional to $1/a$. Finally we have compared the thermodynamics of entanglement with the black-hole thermodynamics from various angles.

Though both of them allow the interpretation that the degrees of freedom around the boundary $B$ (or the event horizon) carry the thermodynamical properties\textsuperscript{13}, it seems quite difficult to find further parallelism between them. The difficulty becomes clear in the context of the third law of thermodynamics. Namely both the response to the variation of $A$ (with $a$ or $l_p$ being fixed) and the response to the variation of $a$ or $l_p$ (with $A$ being fixed) are very different in these two types of thermodynamics (Table I). As is discussed in §VII, this discrepancy is expected to be a universal one, independent of the models to be considered. As for a system of the entanglement, both $S_{\text{ent}}$ and $E_{\text{ent}}$ become proportional to the area of the boundary, $A$, by the very nature of the entanglement. Then we get the universal behaviors of the thermodynamics of entanglement, Eq.(5.7). On the other hand, the behaviors Eq.(5.8) of the black-hole thermodynamics are well-established results. Though $S_{BH}$ is also proportional to the area of the event horizon, $A$, like $S_{\text{ent}}$, $E_{BH}$ is proportional to $\sqrt{A}$ rather than $A$.

What is the reason of the discrepancy? One simple answer is that the thermodynamics of the entanglement we obtained is in flat spacetime and does not include any effects of

\textsuperscript{13} Needless to say, this interpretation is nothing more than one concise way of grasping the situations among many possibilities.
gravity. We have discussed a possible restoration of the agreement between them by introducing gravitational effects. It is due to a magnification of the entanglement energy by an addition of gravitational energy and looks very reasonable for us. Thus we can expect that the thermodynamics of the entanglement behaves like the black-hole thermodynamics if gravitational effects are taken into account properly. Any way, the entanglement entropy passes a non-trivial check to be the black-hole entropy. Finally we mention that our expectation is based on a qualitative argument and that more quantitative and detailed analysis along this line is needed.

It will be also valuable to analyze the cut-off dependence of the thermodynamics of entanglement more systematically from the viewpoint of the renormalization group.

Acknowledgments

M.S. was supported by the Yukawa Memorial Foundation, the Japan Association for Mathematical Sciences and the Japan Society for Promotion of Science. H.K. was supported by the Grant-in-Aid for Scientific Research (C) from the Ministry of Education, Science, Sports and Culture of Japan (40161947).

APPENDIX A: SYMMETRIC PROPERTY OF THE ENTANGLEMENT ENTROPY FOR A PURE STATE

In this appendix we first give an abstract expression for the reduced density operators $\rho_V$ and $\rho_W$ corresponding to a pure state $u$ in $U = V \overline{\otimes} W$, which do not use the subtrace. Then with the help of them we prove that $S_{\text{ent}}$ obtained from $\rho_V$ and $\rho_W$ coincide with each other. We follow the notations in §II A.

Proposition 1 For an arbitrary element $u$ of $U = V \overline{\otimes} W$, there are antilinear bounded operators $A_u \in \overline{B}(V, W)$ and $A_u^* \in \overline{B}(W, V)$ such that

$$(A_u x, y) = (A_u^* y, x) = (u, x \otimes y)$$

(A1)
for $\forall x \in V$ and $\forall y \in W$.

Proof. Fix an arbitrary element $x$ of $V$. Then $(u, x \otimes y)$ gives a linear bounded functional of $y(\in W)$ since

$$|(u, x \otimes y)| \leq ||u|||x|||y||.$$

Hence by Riesz's theorem there is a unique element $z_{u,x}$ of $W$ such that

$$(z_{u,x}, y) = (u, x \otimes y) \quad (A2)$$

for $\forall y \in W$. Let us define $A_u$ by $A_u : x \to z_{u,x}$. It is evident that $A_u$ is an antilinear bounded operator from $V$ to $W$ since

$$||A_u x|| = ||z_{u,x}|| = ||u|||x||.$$

Exchanging the roles played by $V$ and $W$ in the above argument, it is shown that there is an antilinear bounded operator $A_u^*$ from $W$ to $V$ such that $(A_u^*, y) = (u, x \otimes y)$. \qed

Note that $A_u$ and $A_u^*$ defined above are written as

$$A_u x = \sum_j f_j (x \otimes f_j, u) \quad ,$$
$$A_u^* y = \sum_i e_i (e_i \otimes y, u) \quad (A3)$$

Using this expression, it is easily shown that

$$A_u^* A_u x = \sum_{ij} e_i (e_i \otimes f_j, (u, x \otimes f_j)u) \quad ,$$
$$A_u A_u^* y = \sum_{ij} f_j (e_i \otimes f_j, (u, e_i \otimes y)u) \quad . \quad (A4)$$

These coincide with $\rho_V$ and $\rho_W$, respectively, if $u$ has unit norm (see Eq. (2.3) and (2.3)). Therefore the following proposition says that $\rho_V$ and $\rho_W$ have the same spectrum and the same multiplicity and that entropy of them are identical.

Proposition 2 $\rho_u, V (\in B(V))$ and $\rho_u, W (\in B(W))$ defined by
\[ \rho_{u,\mathcal{V}} = A_u^* A_u \quad , \]
\[ \rho_{u,\mathcal{W}} = A_u A_u^* \quad \text{(A5)} \]

are non-negative, trace-class self-adjoint operators, where \( A_u \) and \( A_u^* \) are defined in Proposition \=[[\text{Ref}]](\text{Ref})\ for an arbitrary element \( u \) of \( \mathcal{U} \). The spectrum and the multiplicity of \( \rho_{u,\mathcal{V}} \) and \( \rho_{u,\mathcal{W}} \) are identical for all non-zero eigenvalues.

**Proof.** In general
\[ (x', \rho_{u,\mathcal{V}} x) = (A_u x, A_u x') \]
for \( \forall x, x' \in \mathcal{V} \) by definition. Therefore
\[ (x, \rho_{u,\mathcal{V}} x) = ||A_u x||^2 \geq 0 \quad \text{(A6)} \]
and
\[ \text{Tr}_{\mathcal{V}}(\rho_{u,\mathcal{V}}) = \sum_i (A_u e_i, A_u e_i) \\
= \sum_{i,j} (A_u e_i, f_j)(f_j, A_u e_i) \\
= \sum_{i,j} |(u, e_i \otimes f_j)|^2 \\
= ||u||^2, \quad \text{(A7)} \]
i.e. \( \rho_{u,\mathcal{V}} \) is non-negative and trace-class. In general a non-negative operator is self-adjoint and a trace-class operator is compact. Thus the eigenvalue expansion theorem for a self-adjoint compact operator says that all eigenvalues of \( \rho_{u,\mathcal{V}} \) are discrete except zero and have finite multiplicity. For a later convenience let us denote the non-zero eigenvalues and the corresponding eigenspaces as \( \lambda_i \) and \( \mathcal{V}_i \) \( (i = 1, 2, \cdots) \).

Similarly, it is shown that \( \rho_{u,\mathcal{W}} \) is non-negative and trace-class and that all eigenvalues of it are discrete except zero and have finite multiplicity.

Now \( \ker \rho_{u,\mathcal{W}} = \ker A_u^* \) since \( \rho_{u,\mathcal{W}} y = A_u A_u^* y \) and \( (y, \rho_{u,\mathcal{W}} y) = ||A_u^* y||^2 \) for an arbitrary element \( y \) of \( \mathcal{W} \) by definitions. Moreover, from (A1) it is evident that
\[ y \perp \text{Ran} A_u \Leftrightarrow y \in \ker A_u^*. \]

With the help of these two facts \( \mathcal{W} \) is decomposed as

\[ \mathcal{W} = \ker \rho_{u,\mathcal{W}} \oplus \overline{\text{Ran} A_u}. \quad (A8) \]

where the overline means to take a closure.

Moreover, it is easily shown by definitions that

\[ \rho_{u,\mathcal{W}} A_u x = \lambda_i A_u x, \]
\[ (A_u x, A_u x') = \lambda_i (x', x) \quad (A9) \]

for \( \forall \, x \in \mathcal{V}_i \) and \( \forall \, x' \in \mathcal{V}_{i'} \). Hence \( A_u \) maps the eigenspace \( \mathcal{V}_i \) to a eigenspace of \( \rho_{u,\mathcal{V}} \) with the same eigenvalue, preserving its dimension. Taking account of Eq. (A8), this implies that the spectrum and the multiplicity of \( \rho_{u,\mathcal{V}} \) and \( \rho_{u,\mathcal{W}} \) are identical for all non-zero eigenvalues. \( \square \)
REFERENCES

[1] See e.g. R.M. Wald, General Relativity (University of Chicago, Chicago, 1984), Chapter 12 and Chapter 14.

[2] E.g., V. Iyer and R. M. Wald, Phys. Rev. D52, 4430 (1995); V. P. Frolov, D. V. Fursaev and A. I. Zelnikov, Phys. Rev. D54, 2711 (1996); M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 72, 957 (1994); J. D. Brown and J. W. York, Jr., Phys. Rev. D47, 1420 (1993); S. W. Hawking and G. T. Horowitz, Phys. Rev. D51, 4302 (1995); G. ’t Hooft, Int. J. Mod. Phys. A11, 4623 (1996); S. Mukohyama, Mod. Phys. Lett. A11, 3035 (1996).

[3] J. D. Bekenstein, Phys. Rev. D7, 2333 (1973).

[4] R. M. Wald, Phys. Rev. D48, R3427 (1993); V. Iyer and R. M. Wald, Phys. Rev. D50, 846 (1994).

[5] S. W. Hawking, Comm. math. Phys. 43, 199 (1975).

[6] P. Panangaden and R. M. Wald, Phys. Rev. 16, 929 (1977); S. Mukohyama, gr-qc/9611017

[7] G. W. Gibbons and S. W. Hawking, Phys. Rev. D15, 2752 (1977).

[8] L. Bombelli, R. K. Loul, J. Lee and R. D. Sorkin, Phys. Rev. D34, 373 (1986).

[9] M. Srednicki, Phys. Rev. Lett. 71, 666 (1993).

[10] C. Callan and F. Wilczek, Phys. Lett. B333, 55 (1994).

[11] V. Frolov and I. Novikov, Phys. Rev. D48, 4545 (1993).

[12] For other recent discussions on the entanglement entropy, see e.g., J. S. Dowker, Class. Quantum. Grav. 11, L55 (1994); D. Kabat, Nucl. Phys. B453, 281 (1995); J. D. Bekenstein, gr-qc/9409013; S. Liberati, gr-qc/9601032; A. P. Balachandran, L. Chandra and A. Momen, hep-th/9512047; E. Benedict and S. Pi, Ann. Phys. 245, 209 (1996); R.
Muller and C. Lousto, D52, 4512; C. Hozhey, F. Larsen and F. Wilczek, Nucl. Phys. B424, 443 (1994); F. Larsen and F. Wilczek, Ann. Phys. 243, 280 (1995); S. R. Das, Phys. Rev. D51, 6901 (1995).

[13] E.g., M. Abramowitz and I. A. Stegun (ed.), Handbook of Mathematical Functions (Dover, New York, 1972), Chapter 9.

[14] J. W. York, Jr., Phys. Rev. D28, 2929 (1983); G. ’t Hooft, Nucl. Phys. B256, 727 (1985).

[15] S-K. Ma, Modern Theory of Critical Phenomena (Reading, W.A.Benjamin, 1976).
FIG. 1. The numerical evaluations for $aE_{ent}^I$ and $a(E_{ent}^I - E_{ent}^{II})$ for the case of $B = S^2$. They are shown as functions of $(R/a)^2$, where $R \equiv (n_B + 1/2)a$. We have taken $N = 60$ for $aE_{ent}^I$ and $N = 200$ for $a(E_{ent}^I - E_{ent}^{II})$. 
TABLE I. Comparison of two kinds of thermodynamics

| Entanglement in flat spacetime | Black-Hole |
|-------------------------------|------------|
| **Varied**                    | **Fixed**  |
| A                             | $l_{pl}$   |
| $S \propto A$                 | $\propto A$|
| $E \propto A$                 | $\propto A^{1/2}$|
| $T \propto A^{0}$             | $\propto A^{-1/2}$|

| **Varied**                    | **Fixed**  |
| $a$                           | A          |
| $S \propto a^{-2}$            | $\propto l_{pl}^{-2}$|
| $E \propto a^{-3}$            | $\propto l_{pl}^{-2}$|
| $T \propto a^{-1}$            | $\propto l_{pl}^{0}$|

| **Varied**                    | **Fixed**  |
| $a$                           | $l_{pl}$   |
| $E_{ent}$                     | $M$        |
| $S \propto a$                 | $\propto l_{p}^{2}$|
| $E \propto a^{0}$             | $\propto l_{p}^{0}$|
| $T \propto a^{-1}$            | $\propto l_{p}^{-2}$|