DIRAC OPERATOR COUPLED TO INSTANTONS ON POSITIVE
DEFINITE 4 MANIFOLDS

JOÃO PAULO SANTOS

Abstract. We study the moduli space of instantons on a simply connected
positive definite four manifold by analyzing the classifying map of the index
bundle of a family of Dirac operators parametrized by the moduli space. As
applications we compute the cohomology ring for the charge 2 moduli space
in the rank stable limit.

1. Introduction

Let $X$ be a simply connected positive definite four manifold with base point $x_\infty \in X$
and let $E \to X$ be an $SU(r)$ vector bundle with second Chern class $c_2(E) = -k < 0$.
We will denote by $\mathcal{M}_k^r(X)$ the moduli space of self-dual connections on $E$,
framed at $x_\infty$. By results in Don83 and Fre82, $X$ is homeomorphic to a connected sum
of a number $q$ of copies of $\mathbb{CP}^2$. Let $X_q$ denote this manifold.

A Dirac operator $\not\partial : \Gamma(S^+ \to S^-)$ defines a $K$-theory class $[\ker \not\partial] - [\coker \not\partial]$
over $\mathcal{M}_k^r(X_q)$ whose fiber at each point $A \in \mathcal{M}_k^r(X_q)$ is given by the kernel and cokernel
of the operator $\not\partial$ coupled to the connection $A$. For $A$ a self-dual connection,
the operator $\not\partial$ is surjective, hence its kernel defines a vector bundle over
$\mathcal{M}_k^r(X_q)$, with classifying map $f : \mathcal{M}_k^r(X_q) \to BU(N)$ ($N = \dim \ker \not\partial = \text{Ind} \not\partial$).
In this paper we will study this family of maps (one for each Dirac operator) and use it to
obtain information about the topology of the moduli space $\mathcal{M}_\infty^r(X_q)$.

For $r' > r$, direct sum with a trivial rank $r' - r$ bundle induces a map $\mathcal{M}_k^r(X_q) \to \mathcal{M}_k^{r'}(X_q)$
and we define $\mathcal{M}_\infty^r(X_q) = \lim_{r \to \infty} \mathcal{M}_k^r(X_q)$. In San93 it was shown that, for
$q = 0$ ($X_0 = S^4$), $\text{Ind} \not\partial = k$ and the map $f : \mathcal{M}_\infty^r(S^4) \to BU(k)$ is a homotopy
equivalence. Our first result is (see also BS97)

Theorem 1.1. Let $\not\partial_\pm$ be the Dirac operators on $\mathbb{CP}^2$ with associated Chern classes
the generators $\pm 1 \in H^2(\mathbb{CP}^2)$. Then $\text{Ind} \not\partial_\pm = k$ and the associated map $f_+ \times f_- : \mathcal{M}_\infty^r(\mathbb{CP}^2) \to BU(k) \times BU(k)$ is a homotopy
equivalence.

These results have a nice interpretation as follows: In Tau89 Taubes showed that,
for $k' > k$, there is a glueing map $\mathcal{M}_k^r(X) \to \mathcal{M}_{k'}^r(X)$ such that the space
$\mathcal{M}_\infty^r(X) = \lim_k \mathcal{M}_k^r(X)$ is homotopy equivalent to $\text{Map}(X, BSU(r))$. When $r \to \infty$
$\text{Map}(X_q, BSU)$ is homotopic to the product of $q + 1$ copies of $BU$ so we have a map
$\mathcal{M}_\infty^r(X_q) \to \lim_{k'} \mathcal{M}_k^r \cong \bigotimes_{l=0}^q BU$. For $X_0, X_1$ this map is just the map $BU(k) \to BU$
and $BU(k) \times BU(k) \to BU \times BU$ respectively. For each $q$ we will choose $q + 1$ Dirac operators with index $k$ inducing a map $f : \mathcal{M}_k^r(X_q) \to \prod_{l=0}^{q} BU(k)$ such that the diagram
\[
\begin{array}{ccc}
\mathcal{M}_k^r(X_q) & \xrightarrow{f} & \prod_{l=0}^{q} BU(k) \\
\downarrow & & \downarrow \\
\mathcal{M}_\infty^r(X_q) & \xrightarrow{f} & \prod_{l=0}^{q} BU
\end{array}
\]
commutes and, when $r \to \infty$, the bottom map is a homotopy equivalence. We will show that

**Theorem 1.2.** For $k = 1, 2$ and any $q$, the map $\mathcal{M}_k^\infty(X_q) \to \prod_{k} BU$ induces a surjective map in cohomology.

We conjecture this result is true for all $k$:

**Conjecture 1.3.** The map $\mathcal{M}_k^\infty(X_q) \to \prod_{k} BU$ induces a surjective map in cohomology.

Theorem 1.2 will allow us to compute the cohomology ring of $\mathcal{M}_2^\infty(X_q)$:

**Theorem 1.4.** $H^*(\mathcal{M}_2(X_q))$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[C_1, C_2, S_1^i, S_2^i; 1 \leq i \leq q]$ (with deg $C_n, S_n^i = 2n$) by the ideal generated by
\[
\begin{align*}
C_1 S_1^i S_1^j + S_1^i S_2^j + S_2^i S_1^j, & \quad i \neq j \\
C_2 S_1^i S_1^j - S_2^i S_2^j, & \quad i \neq j \\
S_1^i S_1^j S_1^k, & \quad i \neq j \neq k \\
S_1^i S_2^j S_2^k, & \quad i \neq j \neq k
\end{align*}
\]

The strategy of the proofs will be to define a subspace $\Sigma_{q,k}^r \subset \mathcal{M}_k^r(X_q)$ and to compute the map induced in cohomology by the composition
\[
\Sigma_{q,k}^r \to \mathcal{M}_k^r(X_q) \to \prod_{l=0}^{q} BU(k)
\]

In section 2 we will define the moduli spaces $\mathcal{M}_k^r(X_q)$ and the classifying map $f : \mathcal{M}_k^r(X_q) \to \prod_{l=0}^{q} BU(k)$. Then in section 3 we will introduce the spaces $\Sigma_{q,k}^r$ and use them to prove theorem 1.1. In section 4 we define, for $q' < q$, maps $\pi^* : \mathcal{M}_k^r(X_{q'}) \to \mathcal{M}_k^r(X_q)$ and study some of their properties. These results will be used in section 5 to prove theorems 1.4 and 1.2.

2. The moduli space and the classifying map

Let $X_q$ denote the connected sum of $q$ copies of $\mathbb{CP}^2$. Fix a $C^\infty SU(r)$ vector bundle $E \to X_q$ with $c_2(E) = -k < 0$ and let $\mathcal{A}(E)$ denote the space of connections on $E$. A connection $A \in \mathcal{A}(E)$ is called self-dual if $F_A^- = 0$. These connections minimize
the Yang-Mills functional $\int |F_A|^2$ (see [DK84]). Let $G$ be the gauge group of automorphisms of the bundle $E$. Fix a point $x_\infty \in X_q$ and let $\mathcal{G}_0 = \{g \in G | g(x_\infty) = 1\}$. The moduli space $\mathcal{M}^q_X(X_q)$ of instantons framed at $x_\infty$ is the quotient of the space of self-dual connections by $\mathcal{G}_0$.

Given a $\text{Spin}^c$ structure on $X_q$ with associated line bundle $L$ and cohomology class $c \in H^2(X_q)$ ($c = w_2(X_q) \mod 2$), a choice of a connection $a$ in $L$ induces a Dirac operator $\partial_c : \Gamma(S^+) \to \Gamma(S^-)$ (see for example [LM89]). We will take $a$ to be the (unique) self-dual connection on $L$. Then, to each connection $A \in \mathcal{A}(E)$ we can associate a Dirac operator $\partial_{c,A} : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ coupled with $A$.

**Proposition 2.1.** The index of $\partial_{c,A}$ is given by

$$\text{ind} \partial_{c,A} = k + \frac{c_1(E) \cdot (c_1(E) + c)}{2} + r \frac{c^2 - q}{8}$$

**Proof.** It is a direct consequence of the formula

$$\text{ind} \partial_{c,A} = e^{\hat{\tau}} \cdot c\text{h}(E) \cdot \hat{A}(M)$$

for the index of a Dirac operator. \hfill \square

Notice that if $c^2 = \text{sig}(M) = q$ then $\text{ind} \partial_{c,A}$ is independent of the rank $r$ of $E$.

**Proposition 2.2.** There are exactly $2^q$ $\text{Spin}^c$ structures on $M$ such that $c^2 = q$, namely $c = (\pm 1, \ldots, \pm 1)$.

**Proof.** From Wu’s formulas,

$$w_2 = Sq^0 v_2 + Sq^1 v_1 + Sq^2 v_0 = v_2$$

where $v_2 \in H^2(M, \mathbb{Z}/2)$ satisfies $v_2 \cup x = Sq^2 x = x \cup x$ for any $x \in H^2(M, \mathbb{Z}/2)$ (see [MS74], theorem 11.14). From here it follows that

$$w_2(M) = (1, \ldots, 1) \in H^2(M, \mathbb{Z}/2)$$

Now the conditions $c^2 = \text{sig}(M) = q$ and $w_2 = c \mod 2$ imply $c = (\pm 1, \ldots, \pm 1)$. \hfill \square

**Proposition 2.3.** Let $X_q, \partial_{c,A}$ be as above and assume that $X_q$ has a metric with scalar curvature $s > 0$. Then if the connections $A,a$ are self-dual, the operator $\partial_{c,A}$ is surjective.

**Proof.** We want to show the cokernel of $\partial_{c,A}$ is zero. Hence we look at the Bochner-Weizenboch formula for the dual operator $\partial_{c,A}^*$:

$$\partial_{c,A} \partial_{c,A}^* = \nabla^* \nabla + \frac{1}{4} s + \mathfrak{R}$$

where $s$ is the scalar curvature of $X_q$ and $\mathfrak{R}$ is defined as follows: locally we can write $S^- = S_0 \otimes L^2 \otimes E$. Let $\sigma \in \Gamma(S_0)$, $v \in \Gamma(L^2 \otimes E)$. Then

$$\mathfrak{R}(\sigma \otimes v) = \sum_{j,k=1}^4 (e_j e_k \sigma) \otimes (F^- (e_j, e_k)v)$$

Here $F^-$ is the anti-self-dual part of the curvature on $L^2 \otimes E$ induced by $a,A$, which vanishes if $A,a$ are self-dual. Since $s > 0$ it follows that $\partial_{c,A}^*$ is injective which completes the proof. \hfill \square
Let $I \subset \mathcal{A}$ denote the space of self-dual connections, We have the diagram

$$
\begin{array}{c}
\mathcal{I} \times \mathcal{G} \Gamma(S^+ \otimes E) \\
\downarrow \phi_c \\
\mathcal{I}/\mathcal{G}_0 \\
\end{array}
$$

\begin{align*}
\phi_c([A, s]) &= [A, \phi_c(A, s)].
\end{align*}

Then the kernel of $\phi_c$ defines a rank $k$ vector bundle $[\text{Ker } \phi_c]$ over $\mathcal{I}/\mathcal{G}_0 = \mathcal{M}_k^r$ (see [Seg70]), with classifying map $f_c : \mathcal{M}_k^r \to BU(k)$.

Now we choose $q + 1$ specific $Spin^c$ structures: fix a basis $e_1, \ldots, e_q$ of $H^2(X_q)$. Then define $c_0 = (1, 1, \ldots, 1)$ and, for $l = 1, \ldots, q$,

$$
c_l = -e_l + \sum_{i \neq l} e_i
$$

The associated classifying maps define a map $f : \mathcal{M}_k^r(X_q) \to \prod_{l=0}^q BU(k)$. This choice of Chern classes $c_0, \ldots, c_q$ is justified by the following theorem (which will be proven at the end of next section):

**Theorem 2.4.** When $k, r \to \infty$, the map $f : \mathcal{M}_k^r(X_q) \to \prod BU$ is a homotopy equivalence.

3. Orbits

In this section we define the subspace $\Sigma^r_{q,k} \subset \mathcal{M}_k^r(X_q)$ and compute the composition $\Sigma^r_{q,k} \to \mathcal{M}_k^r \to \prod BU(k)$. As a corollary we prove theorems [11] and [24].

The group $\mathcal{G}$ of automorphisms of $E$ acts on $\mathcal{M}_k^r(X_q)$. For a connection $A \in \mathcal{A}(E)$ let $Or(A) \subset \mathcal{M}_k$ denote the orbit of $[A]$ under the action of $\mathcal{G}$. We will choose $(k-q-1)$ connections $A_j$ and define $\Sigma^r_{q,k}$ as the union of the orbits $Or(A_j)$:

To each $q$-tuple $J = (j_1, j_2, \ldots, j_q)$ with $\sum_j j_i = k$ we define a connection $A_J$ on $E$ as follows: fix an isomorphism $\phi : E \cong \bigoplus_a L_a$ where the line bundles $L_a$ satisfy

$$
\# \{ a | c_1(L_a) = e_i \} = \# \{ a | c_1(L_a) = -e_i \} = j_i , \# \{ a | c_1(L_a) = 0 \} = r - 2k
$$

($e_1, \ldots, e_q$ is a basis of $H^2$). That is, $E \cong L_{+e_1} \oplus L_{-e_1} \oplus \ldots \oplus L_{+e_q} \oplus L_{-e_q} \oplus L_0^{r-2k}$. Let $\lambda_a$ be the self-dual connection on $L_a$ and define $A_J = \phi^{-1}(\bigoplus_a \lambda_a \phi)$. We can now define $\Sigma^r_{q,k}$:

**Definition 3.1.** The space $\Sigma^r_{q,k}$ is the disjoint union

$$
\Sigma^r_{q,k} = \bigcup_{|J|=k} Or(A_J).
$$

Observe that for a different choice of isomorphism $\tilde{\phi} : E \cong \bigoplus_a L_a$, $A_J$ would differ by $\phi^{-1}\phi \in \mathcal{G}$. Hence $Or(A_J)$ depends only on $J$. We will often simplify the notation and write $Or(J)$ instead of $Or(A_J)$.
**Proposition 3.2.** Let $I_A$ be the stabilizer of $A$ under the action of $G$, $I_{[A]}$ be the stabilizer of $[A]$ under $G/G_0$ and $[I_A]$ the image of $I_A$ on $\mathfrak{M}_k$. Then these 3 groups are isomorphic and we have
\[
\text{Or}(A) \cong \frac{SU(r)}{I_A}
\]

**Proof.** It follows directly from the fact that the action of $G_0$ on $A(E)$ is free. □

**Proposition 3.3.** The stabilizer $I_{[A_j]} \subset G/G_0$ of $[A_j]$ is the intersection of the subgroup
\[
P(J) = U(j_1) \times U(j_1) \times \ldots \times U(j_q) \times U(r-j) \times U(r-2k),
\]
sitting in $U(r)$ diagonally, with $SU(r)$.

**Proof.** It is clear that $P(J) \subset I_{[A_j]}$. Let $g \in I_{A_j} \subset G$ (we are identifying $I_{A_j}$ and $I_{[A_j]}$: see proposition 3.2). Locally $(A_j)_{ab} = \delta_{ab}\lambda_a$ and we get the equation $d_g\delta_{ab} = g_{ab}(\lambda_a - \lambda_b)$ which implies $g_{ab}(d\lambda_a - d\lambda_b) = 0$. So, since $d\lambda_a$ is the harmonic 2-form representing $c_1(L_a)$, $g_{ab} = 0$ if $c_1(L_a) \neq c_1(L_b)$. This shows $g \in P(J)$ completing the proof. □

Hence, when $r \to \infty$, the orbit of $A_j$ is given by
\[
\text{Or}(J) \cong BU(j_1)^2 \times \ldots \times BU(j_q)^2
\]

The main result of this section is

**Theorem 3.4.** Let
\[
B\pi_c : BU(j_1)^2 \times \ldots \times BU(j_q)^2 \to BU(j_1) \times \ldots \times BU(j_q)
\]
be the projection map defined as follows: in each factor $BU(j_i)^2$, $B\pi_c$ is the projection onto the first component if $c \cup e_i = 1$ and onto the second component if $c \cup e_i = -1$. Then the restriction of $f$ to $\text{Or}(J)$, $f : \text{Or}(J) \to BU(k)$, is the composition of $B\pi_c$ with the Whitney sum map
\[
BU(j_1) \times \ldots \times BU(j_q) \to BU(k)
\]

Before we prove this theorem we need the following lemma analyzing the restriction of the vector space $[\text{Ker } \partial_i] \rightarrow \mathfrak{M}_k(X_q)$ to an orbit $\text{Or}(A_0)$:

**Proposition 3.5.** Let $G = \mathcal{G}/G_0$ and fix $A_0 \in \mathcal{I}$. Let $\rho : I_{[A_0]} \to I_{A_0}$ denote the isomorphism of proposition 3.2. Then $\rho$ induces a representation of $I_{[A_0]}$ on the vector space
\[
V = \{\psi \in \Gamma(S^+ \otimes E) | \partial_{A_0} \psi = 0\}
\]
The bundle $[\text{Ker } \partial_i]$ restricted to the orbit $\text{Or}(A_0)$ is isomorphic to the vector bundle associated to the principal bundle $G \rightarrow G/[I_{A_0}]$ and the representation $\rho$.

**Proof.** $\rho$ is well defined since, for $h \in I_{A_0}$, $\partial_{A_0} \psi = h^{-1} \partial_{A_0} h \psi$. We begin by defining a map
\[
\mathcal{G} \times V \xrightarrow{\partial} \mathcal{I} \times_{\partial_0} \Gamma
\]

\[
(g, \psi) \mapsto [g^{-1}A_0 g, g^{-1} \psi]
\]
(where \( g^{-1}A_0g \) is a shorthand for \( g^{-1} \circ \nabla_{A_0} \circ g \)). Observe that this map descends to give a map \( (G/G_0) \times I_{A_0} V \to \mathcal{I} \times G_0 \Gamma \) where \( G_0 \) acts on \( G \) on the right and \( I_{A_0} \) acts on \( G \) on the left and on \( V \) by \( \rho \). The statement of the proposition is that this bundle map, when restricted to \( Or(A) \), is an isomorphism onto the kernel of \( \partial_c \).

We first show surjectivity. Let \( A \in Or(A), \hat{\psi} \in \Gamma(S^+ \otimes E) \) be such that \( \partial_A \hat{\psi} = 0 \). Then \( A = g^{-1} \circ A_0 \circ g \) for some \( g \in G \) and \( g\hat{\psi} \in V \) because \( \partial_{A_0} g \hat{\psi} = g \partial_A \hat{\psi} = 0 \). Then \( (g, g\hat{\psi}) \mapsto [A, \hat{\psi}] \) which shows surjectivity.

Now suppose \( (g_1, \psi_1) \) and \( (g_2, \psi_2) \) have the same image, that is, \([g_1^{-1}A_0g_1, g_1^{-1}\psi_1] = [g_2^{-1}A_0g_2, g_2^{-1}\psi_2]\). Then, for some \( h \in G_0, (g_1h)^{-1}\psi_1 = g_2^{-1}\psi_2 \) and \( (g_1h)^{-1}A_0(g_1h) = g_2^{-1}A_0g_2 \), so \( g_2(g_1h)^{-1} \in I_{A_0} \). It follows that

\[
(g_1, \psi_1) \sim (g_1h, \psi_1) \sim (g_2(g_1h)^{-1}g_1h, g_2(g_1h)^{-1}\psi_1) = (g_2, \psi_2)
\]

This concludes the proof of the proposition. \( \square \)

We can now prove theorem 5.3

**Proof.** \( f \) is the composition of the classifying map \( G/I_{A_J} \to BI_{A_J} \) with the map \( B\rho : BI_{A_J} \to BG_0(V) \) induced by the representation \( \rho : I_{A_J} \to Gl(V) \). So we have to analyze this representation. Let \( \psi \in V \) and let \( g \in P(J) \). Recall

\[
E \cong L_{+e_i}^{\oplus j_1} \oplus L_{-e_i}^{\oplus j_1} \oplus \ldots \oplus L_{+e_q}^{\oplus j_q} \oplus L_{-e_q}^{\oplus j_q} \oplus L_{0}^{\oplus r - 2k}\]

Then we can write

\[
\psi = \psi_1^+ \oplus \psi_1^- \oplus \ldots \psi_q^+ \oplus \psi_q^- \oplus \psi_0
\]

\[
g = g_1^+ \oplus g_1^- \oplus \ldots \oplus g_q^+ \oplus g_q^- \oplus g_0
\]

with \( \psi_i^\pm \in \Gamma \left( L_{\pm e_i}^{\oplus j_i} \right) \) and equivalently for \( g_i^\pm \). Now \( \partial_A \psi = 0 \) hence \( \partial_{A_i^\pm} \psi_i^\pm = 0 \) where \( A_i^\pm = A_i^{\pm e_i} \) is the connection on \( L_{\pm e_i}^{\oplus j_i} \). Also \( \partial \psi_0 = 0 \) hence \( \psi_0 = 0 \) and \( g_0 \) acts trivially. From the index theorem it follows

\[
\text{ind} \partial_{A_{i^\pm}} = \frac{\pm c_i \cdot (c \pm e_i)}{2} = \frac{1 \pm c \cup e_i}{2}
\]

Now, \( c \cup e_i = \pm 1 \). Hence

- If \( c \cup e_i = +1 \) then \( \psi_1^- = 0 \) and \( \psi_1^+ = v \psi_1 \) with \( \psi_1 \in \Gamma(L_{+e_i}) \) and \( v \in C^{j_i} \);
  It follows that \( g_1^- \) acts trivially and \( g_1^+ \) acts by matrix multiplication on \( v \);
- If \( c \cup e_i = -1 \) then \( \psi_1^+ = 0 \) and \( \psi_1^- = v \psi_1 \) with \( \psi_1 \in \Gamma(L_{-e_i}) \) and \( v \in C^{j_i} \);
  It follows that \( g_1^+ \) acts trivially and \( g_1^- \) acts by matrix multiplication on \( v \).

Fixing \( \psi_1, \ldots, \psi_q \) as above gives an isomorphism \( V \cong C^{j_1} \oplus \ldots \oplus C^{j_q} \). Let

\[
P_c(J) = \left\{ U(j_1) \times \ldots \times U(j_q) \mid \det = 1 \right\}
\]

Then there is an obvious representation \( P_c(n) \) on \( V \cong C^{j_1} \oplus \ldots \oplus C^{j_q} \) which is trivial on the \( U(r - k) \) factor. It follows that \( \rho \) is the composition of \( \pi_c \) with direct sum \( P_c \to Gl(V) \).

Now, when \( r \to \infty \) we have

\[
G/I_{A_J} \cong BU(j_1)^{x_2} \times \ldots \times BU(j_q)^{x_2}
\]
and $BI_{A_j} \cong G/I_{A_j} \times BU \times 2$. The classifying map of $G \to G/I_{A_j}$ is then the identity on the $G/I_{A_j}$ component and the result follows. \qed

As a corollary we have

**Theorem 3.6.** For $X_1 = \mathbb{CP}^2$, and when $r \to \infty$, the map $f : \mathcal{M}_k(\mathbb{P}^2) \to BU(k) \times BU(k)$ is a homotopy equivalence.

**Proof.** Clearly the restriction of $f$ to $\Sigma_{1,k}^\infty$ is a homotopy equivalence. From theorem A.2 in the appendix, the inclusion $\Sigma_{1,k}^\infty \to \mathcal{M}_k(\mathbb{P}^2)$ is a homotopy equivalency. The result follows. \qed

As another corollary we can now prove theorem 2.4:

**Theorem 3.7.** When $k,r \to \infty$, the map $f : \mathcal{M}_k(\mathbb{P}^2) \to \prod BU$ is a homotopy equivalence.

**Proof.** We begin by writing $\mathcal{M}_\infty = \lim_n \mathcal{M}_n$. Let $[A_n] \in \mathcal{M}_n$ be the connection associated with $J = (n,n,\ldots,n)$. We claim that, when $n \to \infty$, the map

$$f_* : H_*(\text{Or}(A_n)) \to H_*(BU(n) \times \ldots \times BU(n))$$

induced by $f$ in homology is surjective. It follows that, when $n \to \infty$, the map

$$f_* : H_*(\mathcal{M}_\infty) \to H_*(BU(n) \times \ldots \times BU(n))$$

is surjective hence an isomorphism. Since $\mathcal{M}_\infty$ is simply connected, $f$ is a homotopy equivalence.

To prove the claim we recall that

$$\text{Or}(A_n) \cong \prod_{i=1}^q (BU(n) \times BU(n))$$

When $n \to \infty$, $\text{Or}(A_\infty)$ is a loop space and the homology ring is

$$H_*(\text{Or}(A_\infty)) \cong \mathbb{Z}[x^i_1, x^i_2 : 1 \leq i \leq q, m \geq 1]$$

$$H_*\left(\prod_{i=0}^q BU\right) \cong \mathbb{Z}[X^i_l : 0 \leq l \leq q, m \geq 1]$$

and, by theorem 3.4, $f_*$ is given by

$$f_*(x^i_1) = \left(\sum_{l=0}^q X^i_l\right) - X^i_m$$

$$f_*(x^i_2) = X^i_m$$

This map is clearly surjective. This completes the proof. \qed
4. Instantons and holomorphic bundles

The objective of this section is to define, for $q' < q$, embeddings $\pi^*: \mathcal{M}^r_q(X_{q'}) \to \mathcal{M}^r_q(X_q)$ and to investigate their properties. In particular we will show that

- $\pi^*(\text{Or}(A)) = \text{Or}(\pi^*A)$
- For $X_{q'} = S^4$ and $r \to \infty$, $f_c \circ \pi^*$ is a homotopy equivalence.

We begin by relating instantons with holomorphic bundles. For details see [Buc93].

The space $X_q \setminus \{x_{\infty}\}$ is isomorphic to the blow up $\mathbb{C}q$ of $\mathbb{C}^2$ at $q$ points $x_1, \ldots, x_q$, and given a self-dual connection $A$ on $E \to X_q$, its $(0, 1)$ part $A_{0,1}$ is a holomorphic structure on the bundle $E \to \mathbb{C}q$. Denote by $\mathcal{E}_A$ the resulting holomorphic bundle over $\mathbb{C}q$. $\mathbb{C}q$ can be compactified by adding a line $L_{\infty}$ at infinity. The resulting compact surface is the blow-up $\mathbb{C}P^2_q$ of $\mathbb{C}P^2$ at $q$ points. Then $\mathcal{E}_A$ extends to a holomorphic bundle over $\mathbb{C}P^2_q$, and the framing of the connection $A$ at $x_{\infty} \in X_q$ induces a framing of $\mathcal{E}$ at $L_{\infty}$. In [Buc93], [Mat00], it was shown that this correspondence is a real analytic isomorphism between moduli spaces:

**Definition 4.1.** Let $p: \mathbb{C}P^2_q \to X_q$ be the blow-up map sending $L_{\infty}$ to $x_{\infty}$. Let $\mathcal{M}^r_q(\mathbb{C}P^2_q)$ be the quotient by $\text{Aut}(p^*E)$ of the space of pairs $(\alpha^{0,1}, \phi)$, where $\alpha^{0,1}$ is a holomorphic structure on $p^*E \to \mathbb{C}P^2_q$, inducing a holomorphic bundle $\mathcal{E}$ trivial at $L_{\infty}$, and $\phi: \mathcal{E}|_{L_{\infty}} \to \mathcal{O}^{\mathbb{C}P^2}_{L_{\infty}}$ is a trivialization.

**Theorem 4.2.** Fix an isomorphism $h: E_{\infty} \to \mathbb{C}r$ and define a map $\Psi: \mathbb{M}^r_q(X_q) \to \mathcal{M}^r_q(\mathbb{C}P^2_q)$ by $\Psi([A]) = [p^*A^{0,1}, p^*h]$. Then $\Psi$ is an isomorphism of real analytic moduli spaces.

The isomorphism $\Psi$ has the following properties:

**Proposition 4.3.** Let $A_j = \bigoplus_a \lambda_a$ be the connection associated with the $q$-tuple $J$. For each $a$, $\lambda_a$ induces a holomorphic structure on $L_a$, defining a holomorphic line bundle $\mathcal{L}_a$. Then the holomorphic bundle $\mathcal{E}_{A_j}$ associated with the connection $A_j$ is isomorphic to the direct sum $\bigoplus_a \mathcal{L}_a$.

**Proof.** It follows from $p^*A^{0,1} = \bigoplus_a p^*\lambda_a^{0,1}$.

**Proposition 4.4.** Let $g \in \mathcal{G}/\mathcal{G}_0 \cong \text{Aut} E_{\infty} \cong SU(r)$. Then $\Psi([g^{-1}Ag]) = [p^*A^{0,1}, (p^*g) \circ (p^*h)] (p^*g \in \text{Aut}(\mathcal{O}^{\mathbb{C}P^2}_{L_{\infty}}))$.

**Proof.** It follows from $p^*(g^{-1}A^{0,1}g) = (p^*g)^{-1}(p^*A^{0,1})p^*g$.

Let $L_i \subset \mathbb{C}P^2_q$, $i = 1, \ldots, q$, denote the exceptional divisors. Then (see [Sau03])

**Theorem 4.5.** Let $I \subset \{1, \ldots, q\}$ and write $|I| = \#I$. Let $\pi_I: \mathbb{C}P^2_q \to \mathbb{C}P^2_{|I|}$ be the blow up at points $x_j, j \notin I$. $\pi_I$ induces embeddings

\[ \pi_I^*: \mathcal{M}^r_q(\mathbb{C}P^2_q) \to \mathcal{M}^r_q(\mathbb{C}P^2_{|I|}) \]

defined by $\pi_I^*([\alpha^{0,1}, \phi]) = [\pi_I^*\alpha^{0,1}, \phi]$, whose images are the bundles

\[ \pi_I^*(\mathcal{M}^r_q(\mathbb{C}P^2_q)) = \{ [\mathcal{E}, \phi] | \mathcal{E}|_{L_j} \text{ is trivial for } j \notin I \} \]
The inverse map $(\pi^*_q)^{-1} : \pi^*_q \mathcal{M}_k^*(\mathbb{CP}^2_{|I|}) \to \mathcal{M}_k^*(\mathbb{CP}^2_{|I|})$ can be described as follows: \( \mathbb{CP}^2_{|I|} \setminus \bigcup_{j \notin I} L_j \) is biholomorphic to \( \mathbb{CP}^2_{|I|} \) minus \( q - |I| \) points. Given a bundle over \( \mathbb{CP}^2_{|I|} \) restrict it to \( \mathbb{CP}^2_{q \setminus \bigcup L_j} \). Then this bundle extends in a unique way to give a bundle over \( \mathbb{CP}^2_{|I|} \). A corollary of this description is

**Proposition 4.6.** To a \( q \)-tuple \( J = (j_1, \ldots, j_q) \) such that \( j_i = 0 \) whenever \( i \notin I \) we can associate a \( (q - |I|) \)-tuple \( \tilde{J} \) formed by the entries \( j_i \) of \( J \) with \( i \in I \). Then \( \pi^*_q \text{Or}(\tilde{J}) = \text{Or}(J) \). In particular \( \text{Or}(J) \subset \pi^*_q \mathbb{M}_k^*(\mathbb{CP}^2_{|I|}) \).

**Proof.** From proposition 4.3 it follows that \( \pi^*_q \text{Or}(\tilde{J}) \subset \text{Or}(J) \). Now, from equation 2 \( \text{Or}(J) \subset \pi^*_q \mathbb{M}_k^*(\mathbb{CP}^2_{|I|}) \). Then \( (\pi^*_q)^{-1} \text{Or}(J) \subset \text{Or}(\tilde{J}) \) which completes the proof. \( \square \)

A direct consequence of proposition 4.4 is

**Proposition 4.7.** Let \( A \in \mathbb{M}_k(X_{|I|}) \). Then \( \pi^*_q(\text{Or}(A)) = \text{Or}(\pi^*_q A) \)

The last result of this section is

**Theorem 4.8.** Let \( f_c : \mathbb{M}_k(X_q) \to BU(k) \) be the classifying map of \([\text{Ker} \phi_c] \). Then the composition

\[
BU(k) \cong \mathbb{M}_k(S^4) \xrightarrow{\pi^*_b} \mathbb{M}_k(X_q) \xrightarrow{f_c} BU(k)
\]

is a homotopy equivalence.

**Proof.** We first prove the result for \( k = 1 \). Let \( \lambda \) be a self-dual (irreducible) connection on a rank 2 charge 1 bundle \( E_2 \) over \( S^4 \). Then we can define a self-dual connection \( A = (\lambda 0) \) on \( E = E_2 \oplus \mathcal{O}^{r-2} \). The stabilizer of \( A \) is the subgroup \( U(1) \times U(r - 2) \subset U(r) \) of matrices

\[
g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \\ g_{22} \end{pmatrix}
\]

where \( g_{22} \in U(r - 2) \). From proposition 4.7 \( \text{Or}(\pi^*_q A) = \pi^*_b(\text{Or}(A)) \). This shows that the maps \( f_c : \text{Or}(\pi^*_q A) \to BU(1) \) are homotopy equivalences. The result of the theorem for \( k = 1 \) then follows since the inclusion \( \text{Or}(A) \to \mathbb{M}_1(S^4) \) is a homotopy equivalence.
Now we prove the general case. Direct sum induces a map $\mathfrak{M}_{1}^{r_{1}}(X_{q}) \times \mathfrak{M}_{2}^{r_{2}}(X_{q}) \to \mathfrak{M}_{k_{1}+k_{2}}^{r_{1}+r_{2}}(X_{q})$ which is well defined when $r_{1}, r_{2} \to \infty$ (see theorem B.2 in the appendices). We have the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{M}_{1}(X_{0}) \times \ldots \times \mathfrak{M}_{1}(X_{0}) & \overset{h}{\longrightarrow} & \mathfrak{M}_{k}(X_{0}) \\
\pi_{0} \times \ldots \times \pi_{0} & & \pi_{0} \\
\mathfrak{M}_{1}(X_{q}) \times \ldots \times \mathfrak{M}_{1}(X_{q}) & \overset{h}{\longrightarrow} & \mathfrak{M}_{k}(X_{q}) \\
f_{c} \times \ldots \times f_{c} & & f_{c} \\
BU(1) \times \ldots \times BU(1) & \overset{\oplus}{\longrightarrow} & BU(k)
\end{array}
$$

We showed already that the left vertical maps give a homotopy equivalence. The bottom map (Whitney sum) is surjective in homology. It follows that the composition $f_{c} \circ \pi_{0}^{*}$ of the right vertical maps is surjective in homology. Hence it must be an isomorphism in homology. Since $BU(k)$ is simply connected, $f_{c} \circ \pi_{0}^{*}$ is a homotopy equivalence.

5. Applications

The objective of this section is to prove theorems 1.2 and 1.4. They will be consequences of the theorem

**Theorem 5.1.** Let $i: \Sigma_{q,k} \to \mathfrak{M}_{k}(X_{q})$ be the inclusion and let $K = \text{Ker}(i^{*} \circ f^{*} : H^{*}(\prod BU(k)) \to H^{*}(\Sigma_{q,k}))$.

Then, for $k = 1, 2$ the map $i^{*} : H^{*}(\mathfrak{M}_{k}(X_{q})) \to H^{*}(\Sigma_{q,k})$ is injective and the map $f^{*} : H^{*}(\prod BU(k)) / K \to H^{*}(\mathfrak{M}_{k}(X_{q}))$ is an isomorphism.

We begin by computing $K$. Recall that $\Sigma_{q,k} = \prod_{j=1}^{q} \text{Or}(J)$ and we have $\text{Or}(J) = \prod_{i=1}^{q} (BU(j_{i}) \times BU(j_{i}))$. We introduce some notation:

$$
H^{*}(\prod_{|J|=k} \text{Or}(J)) \cong \bigoplus_{|J|=k} \left[ t_{n}^{+}, t_{n}^{-} ; 1 \leq i \leq q, 1 \leq n \leq j_{i} \right]
$$

$$
H^{*}(\prod_{l=0}^{q} BU(k)) \cong \mathbb{Z} \left[ C_{n}^{l} ; 0 \leq l \leq q, n \geq 1 \right]
$$

It will be convenient to make a change of variable. With power series notation $(C^{l} = 1 + C_{1}^{l} + \ldots , C_{i}^{l} = 1 + C_{1}^{l} + \ldots )$ let

$$
C = C^{0} \quad j_{c}^{i} = c_{c}^{i+} \quad j_{s}^{i} = (c_{c}^{i+})^{-1} c_{c}^{i-}
$$

Then

$$
f^{*}(C) = \sum_{|J|=k} t_{c}^{i+} \ldots t_{c}^{i-} \quad f^{*}(S^{i}) = \sum_{|J|=k} j_{s}^{i+} \ldots j_{s}^{i-}
$$

$$
f^{*}(C) = \sum_{|J|=k} t_{c}^{i+} \ldots t_{c}^{i-} \quad f^{*}(S^{i}) = \sum_{|J|=k} j_{s}^{i+} \ldots j_{s}^{i-}
$$

(3)
We will also use in this section the following notation: Let $J_i$ be the $q$-tuple such that $j_i = 0$ for $l \neq i$ and $j_i = 1$. Then let $J_{ij} = J_i + J_j$.

**Proposition 5.2.** For $k = 1$, $K$ is generated by $S_1^1 S_1^1$, $i \neq j$. For $k = 2$, $K$ is generated by

$$
C_1 S_1^1 S_1^1 + S_1^1 S_1^2 + S_2^2 S_1^1, \ i \neq j \\
C_2 S_1^1 S_1^1 - S_2^2 S_1^1, \ i \neq j \\
S_1^1 S_1^1 S_1^1, \ i \neq j \neq k \\
S_1^1 S_1^1 S_1^1, \ i \neq j \neq k
$$

**Proof.** These elements are clearly in the kernel so we only have to show the map is injective if we mod out by this ideal. For $k = 1$ this is clearly the case so let $k = 2$. Then, for each $i$, the composition

$$
\mathbb{Z}[C_1, C_2, S_1^1, S_1^2] \rightarrow H^* \left( \prod BU(2) \right) \rightarrow H^* (\Sigma) \rightarrow Or(J_{ii})
$$

is an isomorphism. Hence the kernel is contained in the ideal $M_2$ generated by products $S_{i_1}^1 S_{i_2}^1, n_j = 1, 2, i_1 \neq i_2$. Now observe that

$$S_2^2 (C_1 S_1^1 S_1^1 + S_1^1 S_2^2 + S_2^2 S_1^1) + S_1^1 (C_2 S_1^1 S_1^1 - S_2^2 S_1^1) = S_1^1 ((S_2^2)^2 + C_1 S_1^1 S_1^1 + C_2 (S_1^1)^2)
$$

It follows that any element $x \in M_2$ can be written as

$$x = \sum_{i \neq j} (P_{ij} S_1^1 S_1^1 + Q_{ij} S_1^1 S_2^2) + k$$

where $P_{ij}, Q_{ij} \in \mathbb{Z}[C_1, C_2, S_1^1, S_1^2]$ and $k \in K$. We want to show that if $f^*(x) = 0$ then $P_{ij} = Q_{ij} = 0$. Fixing $i, j$, $i \neq j$, and letting $f_{ij}^*$ be the composition $H^* \left( \prod BU(2) \right) \rightarrow H^* (\Sigma) \rightarrow Or(J_{ij})$ we have

$$f_{ij}^* (x) = (f_{ij}^* (P_{ij}) + f_{ij}^* (Q_{ij}) c_1^1) s_1^1 s_1^1 = 0$$

But both $f_{ij}^* (P_{ij})$ and $f_{ij}^* (Q_{ij})$ are symmetric in $c_1^1, c_1^1$ hence we must have $f_{ij}^* (P_{ij}) = f_{ij}^* (Q_{ij}) = 0$. This clearly implies $P_{ij} = Q_{ij} = 0$. \hfill $\square$

Before we prove theorem 5.1 we need the following results proven in [San03]:

**Theorem 5.3.** Consider the ideals $K_i, K_{ij}$ generated by

$$K_i = \langle k_1, k_2 \rangle \subset \mathbb{Z}[a_1, a_2, k_1, k_2] \cong H^* (BU(2) \times 2)$$

$$K_{ij} = \langle x_1 x_2 \rangle \subset \mathbb{Z}[x_1, x_2, x_3, x_4] \cong H^* (BU(1) \times 4)
$$

Then, we have the isomorphism of abelian groups

$$H^* (M_2 (X_q)) \cong \mathbb{Z}[a_1, a_2] \oplus \bigoplus_i K_i \oplus \bigoplus_{i < j} K_{ij}
$$

The next theorem uses the notation in theorem 5.3 with $I = \emptyset, \{i\}, \{i, j\}$.

**Theorem 5.4.** Let $C \subset M_2 (X_2) \cong M_2 (\mathbb{C}P^2)$ be given by

$$C = \{ | \mathcal{E}, \phi \rangle \mathcal{E} \text{ is trivial in } \mathbb{C}P^2 \setminus (L_1 \cup L_2) \}
$$

There is a homotopy equivalence $g : C \rightarrow M_1 (X_1) \times M_1 (X_1)$ and
(1) Let \( i_0 : \pi_0^* \mathcal{M}_2(X_0) \rightarrow \mathcal{M}_2(X_0) \) be the inclusion. Then \( i_0^* : \mathbb{Z}[a_1, a_2] \rightarrow H^*(\pi_0^* \mathcal{M}_2) \) is an isomorphism.

(2) Let \( i_1 : \pi_1^* \mathcal{M}_2(X_1) \rightarrow \mathcal{M}_2(X_1) \) be the inclusion. Then \( i_1^* : \mathbb{Z}[a_1, a_2] \oplus K_i \rightarrow H^*(\pi_1^* \mathcal{M}_2) \) is an isomorphism and \( i_1^*(K_{kl}) = 0 \) for any \( k, l \) and any \( j \neq i \).

(3) Let \( i_{ij} : \pi_{ij}^* C \rightarrow \mathcal{M}_2(X_0) \) be the inclusion. Then the map \( i_{ij}^* : K_{ij} \rightarrow H^*(\pi_{ij}^* C) \) is injective and \( i_{ij}^*(K_{kl}) = 0 \) for different sets \( \{k, l\} \neq \{i, j\} \).

**Proposition 5.5.** \( \text{Or}(J_{ij}) \subset \pi_{ij}^* \mathcal{M}_2, \text{Or}(J_{ij}) \subset \pi_{ij}^* C \ (i \neq j) \), and these inclusions are homotopy equivalences.

**Proof.** From proposition 4.6 we have \( \pi_1^*(\Sigma_1) = \text{Or}(J_{ii}) \) and from theorem 1.2 it follows that the inclusion \( \text{Or}(J_{ij}) \rightarrow \pi_{ij}^* \mathcal{M}_2 \) is a homotopy equivalence. Since \( \text{Or}(1, 1) \subset C \ (\text{Or}(1, 1) \) is the orbit associated with \( J = (1, 1) \)\), proposition 4.6 implies \( \text{Or}(J_{ij}) = \pi_{ij}^* \text{Or}(1, 1) \subset \pi_{ij}^* C \). To show that this inclusion is a homotopy equivalence we need to look at the map \( g : C \rightarrow \mathcal{M}_1(X_1) \times \mathcal{M}_1(X_1) \). This map is defined as follows (compare with the discussion after theorem 4.6): the spaces \( \mathbb{C}P_2 \setminus L_i \ (i = 1, 2) \) are biholomorphic to \( \mathbb{C}P_1 \) minus one point. The bundles \( E_{\mathbb{C}P_2 \setminus L_i} \) extend uniquely to give bundles \( E_i \) over \( \mathbb{C}P_1 \ (i = 1, 2) \). Then \( g \) is defined by \( g(E) = (E_1, E_2) \). We have a commutative diagram

\[
\begin{array}{ccc}
\pi_{ij}^* \mathcal{M}_2(X_2) & \xrightarrow{\pi_{ij}^*} & \pi_{ij}^* C \\
\text{Or}(J_{ij}) & \xrightarrow{\pi_{ij}^*} & \text{Or}(1, 1) \\
| & \xrightarrow{g} & | \\
\Sigma_{1,1} & \xrightarrow{\cong} & \Sigma_{1,1} \times \Sigma_{1,1}
\end{array}
\]

We claim that \( g : \text{Or}(1, 1) \rightarrow \Sigma_{1,1} \times \Sigma_{1,1} \) is a homotopy equivalence. It will follow that the inclusion \( \text{Or}(J_{ij}) \rightarrow \pi_{ij}^* C \) is a homotopy equivalence. To prove the claim we recall that

\[
\text{Or}(1, 1) \cong \frac{SU(r)}{U(1) \times U(1) \times (r - 4)} \quad \Sigma_{1,1} \cong \frac{SU(r)}{U(1) \times U(1) \times (r - 2)}
\]

Then it follows from proposition 4.3 that the map \( \text{Or}(1, 1) \rightarrow \Sigma_{1,1} \) is the natural projection map. When \( r \rightarrow \infty \) we see that \( g : \text{Or}(1, 1) \rightarrow \Sigma_{1,1} \times \Sigma_{1,1} \) is a homotopy equivalence. This concludes the proof. \( \square \)

Now we prove theorem 5.3

**Proof.** First we observe that the map \( \iota^* : H^*(\mathcal{M}_2(X_0)) \rightarrow H^*(\Sigma) \) is injective. This follows easily from theorems 5.3 and 5.4. Hence we only need to show that \( f^* : H^*(\prod_{\Sigma} \mathbb{B}U(2)) \rightarrow H^*(\mathcal{M}_2) \) is surjective. We will divide the proof into several steps.

1. Let \( I_{ij} \subset \mathbb{Z}[C_1, C_2, S_1^1, S_2^1] \) be the ideal generated by \( S_1^1 S_2^1 \) and \( S_1^1 S_2^1 \). We claim that \( f^*(I_{ij}) \subset K_{ij} \). Let \( y \in I_{ij} \). Then write \( f^*(y) = x_0 + \sum x_i + \sum x_{ij} \) with \( x_0 \in \mathbb{Z}[a_1, a_2], x_i \in K_i \) and \( x_{ij} \in K_{ij} \). Then, for any \( i \), \( f^*(x_i) = 0 = f^*(x_0 + x_i) \) so \( x_0 = x_i = 0 \). Then, for any pair \( \{k, l\} \neq \{i, j\} \) we have \( f_{kl}^*(y) = 0 = f_{kl}^*(x + x_k + x_l + x_{kl}) = f_{kl}^*(x_{kl}) \) hence \( x_{kl} = 0 \). This completes the proof.
Proposition A.1. Let $(0 \to \mathcal{O})$ will use the monad description of self-dual connections on $X$ group $\text{Aut}(W_n)$ be $(A \to 1)$ a self-dual connection

Proof. $a \in r(5)$. Let

$$(3) \quad \text{Now let } I_1 \subseteq \mathbb{Z}[C_1, C_2, S_1, S_2] \text{ be the ideal generated by } S'_1, S'_2. \text{ We claim that } i^*_1 f^*(I_1) = i^*_1(K_1). \text{ Let } y \in I_1. \text{ Then write } f^*(y) = x_0 + \sum x_i + \sum x_{ij}.$$

Then, for any $l \neq i$, $i^*_l f^*(y) = 0 = i^*_l(x_0 + x_l)$ so $x_0 = x_l = 0$. Hence $i^*_l f^*(I_1) \subseteq i^*_l K_1$. Now, since $i^*_l f^*(I_1) \cong i^*_l(K_1)$ and both $S'_1, S'_2$ are sent to primitive elements, it follows that $i^*_l f^*(I_1) = i^*_l(K_1)$.

(4) Let $x \in K_1$. We claim that there is $y \in H^*(\prod BU(2))$ such that $f^*(y) = x$. Proof: choose $y_i \in I_1$ such that $i^*_i f^*(y_i) = i^*_i(x)$. Then $f^*(y_i) = x_i + \sum x_{kl}$. We only have to choose $y_{kl}$ such that $f^*(y_{kl}) = x_{kl}$ and define $y = y_i - \sum x_{kl}$.

(5) Let $x \in \mathbb{Z}[a_1, a_2]$. We claim that there is $y \in H^*(\prod BU(2))$ such that $f^*(y) = x$. From theorem [LS] the map $i^*_0 : \mathbb{Z}[C_1, C_2] \to H^*(S^2)$ is an isomorphism hence we can find $y_0 \in \mathbb{Z}[C_1, C_2]$ such that $i^*_0 f^*(y_0) = i^*_0(x)$. Then $f^*(y_0) = x + \sum x_i + \sum x_{ij}$ so we just choose $y_i \mapsto x_i, y_{ij} \mapsto x_{ij}$ and let $y = y_0 - \sum y_i - \sum y_{ij}$.

\[ \square \]

**Appendix A. Monads**

Here we show that the inclusion $\Sigma_{1,k} \to \mathfrak{M}^\infty_k(X_1)$ is a homotopy equivalence. We will use the monad description of self-dual connections on $X_1$ (see [Kin89]): let $U, W$ be $k$ dimensional vector spaces and consider the space $\hat{R}$ of configurations $(a_1, a_2, d, b, c)$ where $a_1, a_2 \in \text{Hom}(U, W), d \in \text{Hom}(W, U), b \in \text{Hom}(\mathbb{C}^*, W)$ and $c \in \text{Hom}(U, \mathbb{C}^*)$, satisfying the integrability condition $a_1 da_2 - a_2 da_1 + bc = 0$. The group $\text{Aut}(W) \times \text{Aut}(U)$ acts in $\hat{R}$ by

\[ (g, h) \cdot (a_1, a_2, d, b, c) = (ga_1 h^{-1}, ga_2 h^{-1}, hdg^{-1}, gb, ch^{-1}) \]

Then there is an open subset of $\hat{R}$, $R$ (the non-degenerate configurations) such that $\mathfrak{M}^\infty_k(X_1)$ is isomorphic to the quotient of $R$ by the action of $\text{Aut}(W) \times \text{Aut}(U)$.

**Proposition A.1.** Let $M_0$ be the set of equivalence classes of configurations of the form $(0, 0, 0, b, c)$. Then, as a subspace of $\mathfrak{M}^\infty_k(X_1) \cong \mathcal{M}^\infty_1(\mathbb{C}P^2)$, $M_0 = \Sigma_{1,k}$.

**Proof.** A self dual connection $A \in M_0 \subseteq \mathfrak{M}^\infty_k(X_1)$ induces a holomorphic bundle $\mathcal{E}$ on $\mathbb{C}P^2$ given by the homology of the sequence ($\mathcal{E} = \text{Ker } B/\text{Im } A$)

\[ (5) \quad U \otimes \mathcal{O}(-L_\infty) \oplus W \otimes \mathcal{O}(-L_\infty + L) \xrightarrow{A} V \otimes \mathcal{O} \xrightarrow{B} W \otimes \mathcal{O}(L_\infty) \oplus U \otimes \mathcal{O}(L_\infty - L) \]
the pullback of this bundle to
argument shows that
Theorem A.2. When \( r \to \infty \), the inclusion \( \iota : \Sigma_{1,k}^\infty \to \mathcal{M}_k^\infty(X_1) \) is a homotopy equivalence.

Proof. Consider the principal \( \text{Aut}(W) \times \text{Aut}(V) \) bundle \( R \to \mathcal{M}_k^\infty(X_1) \). Consider the pullback of this bundle to \( M_0 \). We have the diagram

\[
\begin{array}{ccc}
\iota^*R & \rightarrow & R \\
\downarrow & & \uparrow \\
M_0 & \rightarrow & \mathcal{M}_k^\infty
\end{array}
\]

In [BS97] it was shown that \( R \) becomes contractible when \( r \to \infty \) and the same argument shows that \( \iota^*R \) (the set of configurations of the form \( (0,0,0,b,c) \)) is also contractible. Applying the five lemma to the long exact sequences of homotopy groups associated to these principal bundles it follows that the map \( \iota^* \) is an isomorphism in all homotopy groups, hence \( \iota \) is a homotopy equivalence.

Appendix B. Direct sum

Here we will define a map \( \mathcal{M}_{k_1}^{r_1}(X_q) \times \mathcal{M}_{k_2}^{r_2}(X_q) \to \mathcal{M}_{k_1+k_2}^{r_1+r_2}(X_q) \) induced by direct sum and show that this map is well defined when \( r_1, r_2 \to \infty \).
For each pair \((r, k)\) fix an \(SU(r)\) bundle \(E_{r,k}\) over \(X_q\) with \(c_2(E_{r,k}) = k\). Then, to each isomorphism \(\phi : E_{k_1, r_1} \oplus E_{k_2, r_2} \to E_{k_1+k_2, r_1+r_2}\) we can associate a map

\[
h_\phi : \mathcal{M}_{k_1}(X_q) \times \mathcal{M}_{k_2}(X_q) \to \mathcal{M}_{k_1+k_2}(X_q)
\]

that sends a pair of connections \((A_1, A_2)\) to \(\phi^*(A_1 \oplus A_2)\). Since we act on the connections with the gauge group \(G_0\), \(h_\phi\) only depends on the value of \(\phi\) at the base point \(x_\infty \in X_q\). The following result is then an easy consequence of the connectedness of \(SU(r_1 + r_2)\):

**Proposition B.1.** Any two maps \(h_\phi_1, h_\phi_2\) are homotopic.

Sum with a trivial rank \(R\) bundle induces a map \(\mathcal{M}_k^r \to \mathcal{M}_k^{r+R}\). Then

**Theorem B.2.** The following diagram is homotopy commutative

\[
\begin{array}{ccc}
\mathcal{M}^{r_1}_{k_1} \times \mathcal{M}^{r_2}_{k_2} & \xrightarrow{h} & \mathcal{M}^{r_1+r_2}_{k_1+k_2} \\
(\mathcal{M}^{r_1+R}_{k_1} \times \mathcal{M}^{r_2+R}_{k_2}) & \xrightarrow{h} & \mathcal{M}^{r_1+r_2+2R}_{k_1+k_2} \\
\end{array}
\]

Hence the map \(h : \mathcal{M}^{\infty}_{k_1} \times \mathcal{M}^{\infty}_{k_2} \to \mathcal{M}^{\infty}_{k_1+k_2}\) is well defined.

**Proof.** It follows directly from proposition B.1.

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