Spin Structure Function of the Virtual Photon Beyond the Leading Order in QCD

KEN SASAKI* and TSUNEUO UEMATSU†
Dept. of Physics, Faculty of Engineering, Yokohama National University
Yokohama 240-8501, JAPAN
Dept. of Fundamental Sciences, FIHS, Kyoto University
Kyoto 606-8501, JAPAN

Abstract

Polarized photon structure can be studied in the future polarized $e^+e^-$ colliding-beam experiments. We investigate the spin-dependent structure function of the virtual photon $g_1^\gamma(x, Q^2, P^2)$, in perturbative QCD for $\Lambda^2 \ll P^2 \ll Q^2$, where $-Q^2 (-P^2)$ is the mass squared of the probe (target) photon. The analysis is performed to next-to-leading order in QCD. We particularly emphasize the renormalization scheme independence of the result. The non-leading corrections significantly modify the leading log result, in particular, at large $x$ as well as at small $x$. We also discuss the non-vanishing first moment sum rule of $g_1^\gamma$, where $O(\alpha_s)$ corrections are computed.

*e-mail address: sasaki@ed.ynu.ac.jp
†e-mail address: uematsu@phys.h.kyoto-u.ac.jp
1 Introduction

In recent years there has been growing interest in the study of a polarized photon structure function. The information on the spin structure of the photon would be provided by the resolved photon process in polarized electron and proton collision in the polarized version of HERA [1, 2]. More directly, the spin-dependent structure function of photon $g_1^\gamma$ can be measured by the polarized $e^+ e^-$ collision in the future linear colliders (Fig.1).

From the theoretical viewpoint, the first moment of a photon structure function $g_1^\gamma$ has recently attracted attention in the literatures [3, 4, 5, 6, 7] in connection with its relevance for the axial anomaly, which has also played an important role in the QCD analysis of the spin structure of the nucleon. Our aim here is to carry out the QCD computation of photon’s polarized structure function at the same level of unpolarized case. Here we note that the two-loop splitting functions of DGLAP equation or equivalently the two-loop anomalous dimensions have recently been calculated [8, 9], and we can perform the next-to-leading order QCD analysis for the polarized photon structure function. Actually there has already been an analysis of spin-dependent structure function $g_1^\gamma$ for the real photon target by Stratmann and Vogelsang [10].

In this paper we shall investigate the polarized virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ to the next-leading order (NLO) in QCD, in the kinematical region:

$$\Lambda^2 \ll P^2 \ll Q^2$$  \hspace{1cm} (1.1)

where $-Q^2 (-P^2)$ is the mass squared of the probe (target) photon, and $\Lambda$ is the QCD scale parameter. We can base our arguments either on DGLAP-type $Q^2$ evolution equation for the parton distributions or on the framework of operator product expansion (OPE) and the renormalization group (RG) method. The unpolarized virtual photon structure functions $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ were studied in the leading order (LO) [11] and in the NLO [12, 13]. And the parton contents of virtual photon were studied in ref.[14, 15] and the target mass effect of unpolarized and polarized virtual photon structure in LO was discussed in ref.[16].

The advantage to study the virtual photon target is that we can calculate the
whole structure function entirely up to NLO by the perturbative method. On the other hand, for the real photon target \cite{17, 18} we can calculate the perturbative pieces, but not the non-perturbative contributions which may be estimated, for example, by vector-dominance model \cite{19}. The perturbative pieces for the real photon target can be reproduced from the result for the virtual photon case.

In the next section we discuss the polarized photon structure functions. Next we present the two theoretical frameworks based on OPE (section 3) and on DGLAP parton model approach (section 4). In section 5, the sum rule for the first moment of $g_1^\gamma$ will be evaluated up to the order of $\alpha_s$. The numerical analysis of $g_1^\gamma$ will be given in section 6. The final section is devoted to the conclusion and discussion.

\section{Polarized photon structure functions}

Let us consider the forward virtual photon scattering amplitude (Fig.2),

\[ T_{\mu\nu\rho\tau}(p, q) = i \int d^4x d^4yd^4ze^{i\vec{q} \cdot \vec{x}} e^{i\vec{p} \cdot (\vec{y} - \vec{z})} \langle 0 | T(J_{\mu}(x) J_{\nu}(0) J_{\rho}(y) J_{\tau}(z)) | 0 \rangle. \quad (2.1) \]

Its absorptive part is related to the structure tensor $W_{\mu\nu\rho\tau}(p, q)$ for the photon with mass squared $p^2 = -P^2$ probed by the photon with $q^2 = -Q^2$:

\[ W_{\mu\nu\rho\tau}(p, q) = \frac{1}{\pi} \text{Im} T_{\mu\nu\rho\tau}(p, q). \quad (2.2) \]

The anti-symmetric part, $W^A_{\mu\nu\rho\tau}$, which is antisymmetric under the interchange of $\mu$ and $\nu$, can be decomposed as

\[ W^A_{\mu\nu\rho\tau} = \epsilon_{\mu\nu\lambda\sigma} q^{\lambda} \epsilon_{\rho\sigma\beta\gamma} \frac{1}{p \cdot q} g_1^\gamma + \epsilon_{\mu\nu\lambda\sigma} q^{\lambda}(p \cdot q) \epsilon_{\rho\sigma\beta\gamma} \frac{1}{(p \cdot q)^2} g_2^\gamma, \quad (2.3) \]

which gives two spin-dependent structure functions, $g_1^\gamma(x, Q^2, P^2)$ and $g_2^\gamma(x, Q^2, P^2)$. For a real photon, $g_2^\gamma$ is identically zero, and there exists only one spin structure function, $g_1^\gamma(x, Q^2)$. On the other hand, for the off-shell or virtual photon ($P^2 \neq 0$) target, we have two spin-dependent structure functions $g_1^\gamma$ and $g_2^\gamma$. More detailed argument on the structure functions is given in the Appendix D. The $g_1^\gamma$ is related
to the structure function $W_4^\gamma$, which was discussed some years ago in [20, 21], such that $g_1^\gamma(x, Q^2) \equiv 2W_4^\gamma(x, Q^2)$. Here we note that the LO QCD correction was first studied by one of the authors in [22] and later in [23, 3].

First we note that the same framework used in the analysis of the nucleon spin structure functions can be applied in our case. We can either base our argument on the OPE supplemented by the RG method, or on the DGLAP type parton evolution equations. It should be noted the next-to-leading order analysis is now possible since the two-loop anomalous dimensions of twist-2 operators $R_n^i$ relevant for the spin-dependent structure function (or equivalently two-loop parton splitting functions) were calculated independently by two groups, by Mertig-van Neerven [8] and by Vogelsang [9].

### 3 Theoretical framework based on OPE

In our previous paper [24], we based our argument on the QCD improved parton model approach. Here we start with theoretical framework based on the OPE and RG method. Applying OPE for the product of two electromagnetic currents we get

$$i \int d^4x e^{iq \cdot x} T(J_\mu(x)J_\nu(0))^A = -i\varepsilon_{\mu\nu\lambda\sigma} q^\lambda \sum_{n=odd} \left( \frac{2}{Q^2} \right)^n q_{\mu_1} \cdots q_{\mu_{n-1}} \sum_i C_{1,n}^i R_{1,i}^{\sigma\mu_1 \cdots \mu_{n-1}}$$

$$-i(\epsilon_{\mu\rho\lambda\sigma} q_\rho q^\sigma - \epsilon_{\nu\rho\lambda\sigma} q_\mu q^\rho - q^2 \epsilon_{\mu\nu\lambda\sigma}) \sum_{n=odd} \left( \frac{2}{Q^2} \right)^n q_{\mu_1} \cdots q_{\mu_{n-2}} \sum_i C_{2,n}^i R_{2,i}^{\lambda\sigma\mu_1 \cdots \mu_{n-2}}.$$  (3.1)

For polarized deep inelastic scattering, the twist-2 and twist-3 operators: $R_1^i$, $R_2^i$ contribute to the structure functions in the scaling limit. For $g_1^\gamma$ only twist-2 operators are relevant. Now we can write down the moment sum rule for $g_1^\gamma$:

$$\int_0^1 dx x^{n-1} g_1^\gamma(x, Q^2, P^2) = \sum_{i=\psi, G, NS, \gamma} C_n^i (Q^2 / \mu^2, \bar{g}(\mu^2), \alpha) \langle \gamma(p)|R_n^i(\mu^2)|\gamma(p)\rangle.$$  (3.2)

where $R_n^i$ and $C_n^i$ are the twist-2 operators and their coefficient functions (Hereafter we suppress the index 1 for twist-2 operators), with $\mu$ being the renormalization point and $\alpha = e^2 / 4\pi$, the QED coupling constant. $\psi$, $G$, $NS$ and $\gamma$ stand for singlet quark, gluon, non-singlet quark and photon, respectively. The relevant twist-2 operators

---

3.
\( R^i_n \ (i = \psi(S), G, NS, \gamma) \) are given by [22]:

\[
R_{\psi}^{\sigma_{\mu_1 \cdots \mu_{n-1}}} = i^{n-1}\bar{\psi} \gamma^{[\sigma} D^{\mu_1} \cdots D^{\mu_{n-1}]} \gamma_5 \ 1 \psi - \text{trace terms},
\]

(3.3)

\[
R_{G}^{\sigma_{\mu_1 \cdots \mu_{n-1}}} = \frac{1}{4} i^{n-1} \epsilon^{[\sigma_{\alpha\beta\gamma]} G^{\alpha_{\mu_1}} D^{\mu_2} \cdots D^{\mu_{n-1}]} G^{\beta\gamma} - \text{trace terms}
\]

(3.4)

\[
R_{NS}^{\sigma_{\mu_1 \cdots \mu_{n-1}}} = i^{n-1}\bar{\psi} \gamma^{[\sigma} D^{\mu_1} \cdots D^{\mu_{n-1}]} \gamma_5 (Q_{ch}^2 - 1) \psi - \text{trace terms}
\]

(3.5)

\[
R_{\gamma}^{\sigma_{\mu_1 \cdots \mu_{n-1}}} = \frac{1}{4} i^{n-1} \epsilon^{[\sigma_{\alpha\beta\gamma]} F^{\alpha_{\mu_1}} \partial^{\mu_2} \cdots \partial^{\mu_{n-1}]} F^{\beta\gamma} - \text{trace terms}
\]

(3.6)

where \{ \} means complete symmetrization over the Lorentz indices \( \sigma_{\mu_1 \cdots \mu_{n-1}} \), \( D^\mu \) denotes covariant derivative, 1 is an \( n_f \times n_f \) unit matrix and \( Q_{ch}^2 \) is the square of the \( n_f \times n_f \) quark-charge matrix, with \( n_f \) being the number of flavors. Here we note that the essential feature in the above equation is the appearance of photon operators \( R_{\gamma} \) in addition to the hadronic operators.

For \(-p^2 = P^2 \gg \Lambda^2\), we can calculate the photon matrix elements of the hadronic operators perturbatively. Choosing \( \mu^2 \) to be close to \( P^2 \), we get, to the lowest order,

\[
\langle \gamma(p) | R^i_n(\mu) | \gamma(p) \rangle = \frac{\alpha}{4\pi} \left( -\frac{1}{2} K^0_{n,i} \ln \frac{P^2}{\mu^2} + A^i_n \right), \quad i = \psi(S), G, NS
\]

(3.7)

where \( K^0_{n,i} = (K^0_n)^i \) are one-loop anomalous dimension matrix elements between the photon and hadronic operators. On the other hand, in the leading order of the QED coupling constant, \( \alpha \), we have for the photon operator \( R^i_n \):

\[
\langle \gamma(p) | R^i_n(\mu) | \gamma(p) \rangle = 1
\]

(3.8)

It should be noted that the finite term \( A^i_n \) depends on the renormalization scheme for the operators \( R^i_n \). Putting \( \mu^2 = -p^2 = P^2 \), we have

\[
\langle \gamma(p) | R^i_n(\mu) | \gamma(p) \rangle |_{\mu^2 = P^2} = \frac{\alpha}{4\pi} A^i_n,
\]

(3.9)

and the \( n \)-th moment with this choice \( \mu^2 = P^2 \) in (3.2) becomes

\[
\int_0^1 dx x^{n-1} g_1^i(x, Q^2, P^2)
\]

(3.10)
The evolution factor in the last equation is found to be [17]:
\[
T \exp \left[ \int \frac{\gamma_n(g)}{\beta(g)} \, dg \right] = \left( \frac{M_n}{X_n} \right)^{0/1} (3.11)
\]
where
\[
M_n(Q^2/P^2, \bar{g}(P^2)) = T \exp \left[ \int \frac{\gamma_n(g)}{\beta(g)} \, dg \right]
\]
\[
X_n(Q^2/P^2, \bar{g}(P^2), \alpha) = \int \frac{\gamma_n(g')}{\beta(g')} \, d\bar{g} \left[ \frac{\gamma_n(g')}{\beta(g')} \right]
\]
with \( \gamma_n \) and \( K_n \) the hadronic anomalous dimension matrix and the off-diagonal element representing the mixing between the photon and hadron operators (see Appendix A). Thus we get
\[
\int_0^1 dx x^{n-1} \bar{g}_1(x, Q^2, P^2) = \frac{\alpha_s}{4\pi} A_n \cdot M_n(Q^2/P^2, \bar{g}(P^2)) C_n(1, \bar{g}(Q^2)) + X_n(Q^2/P^2, \bar{g}(P^2), \alpha) \cdot C_n(1, \bar{g}(Q^2)) + C^\gamma_n (3.13)
\]
with
\[
A_n = (A_n^V, A_n^G, A_n^{NS}). (3.14)
\]
The coefficient functions are given by (see Appendix C),
\[
C_n(1, \bar{g}) = \begin{pmatrix}
C_n^V(1, \bar{g}) \\
C_n^G(1, \bar{g}) \\
\bar{C}_n^{NS}(1, \bar{g})
\end{pmatrix} = \begin{pmatrix}
\delta_{\psi}(1 + \frac{\bar{g}^2}{16\pi^2} B^n_{\psi}) \\
\delta_{\psi}(1 + \frac{\bar{g}^2}{16\pi^2} B^n_{G}) \\
\delta_{NS}(1 + \frac{\bar{g}^2}{16\pi^2} B^n_{NS})
\end{pmatrix}
\]
\[
C^\gamma_n(1, \bar{g}, \alpha) = \frac{\alpha_s}{4\pi} \delta_{\gamma} B^n_{\gamma} \quad (3.15)
\]
with \( \delta_{\psi} = < e^2 > = \sum_{i=1}^{n_f} e_i^2 / n_f, \delta_{NS} = 1, \delta_{\gamma} = 3n_f < e^4 >= 3 \sum_{i=1}^{n_f} e_i^4 \).

We then derive the following formula for the moments:
\[
\int_0^1 dx x^{n-1} \bar{g}_1(x, Q^2, P^2) = \frac{\alpha_s}{4\pi} \frac{1}{2\beta_0} \left[ \sum_{i=+, -, NS} A_i^n + \sum_{i=+, -, NS} B_i^n \left\{ 1 - \frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right\} \right] (3.16)
\]
Here $\alpha_s(Q^2) = \bar{g}^2(Q^2)/4\pi$ is the QCD running coupling constant. In (3.16), we have defined

$$\tilde{P}_n^i = K^0_n P^n_i C_n(1, 0) \quad (i = +, -, NS).$$

where $P^n_i$'s are projection operators given in the Appendix A. The coefficients $A^n_i$, $B^n_i$, and $C^n$ are computed from the NLO perturbative calculation, and are given by

$$A^n_i = -K^0_n \sum_j \frac{P^n_j \tilde{\gamma}^{(1)}_n P^n_i}{2\beta_0 + \lambda^n_j - \lambda^n_i} C_n(1, 0) \frac{1}{\lambda^n_i / 2\beta_0} - K^n_0 \frac{\beta_1}{\beta_0} P^n_i C_n(1, 0) \frac{1 - \lambda^n_i / 2\beta_0}{\lambda^n_i / 2\beta_0}$$

$$+ K^n_1 P^n_i C_n(1, 0) \frac{1}{\lambda^n_i / 2\beta_0} - 2\beta_0 A^n_i P^n_i C_n(1, 0)$$

$$B^n_i = K^n_0 \sum_j \frac{P^n_j \tilde{\gamma}^{(1)}_n P^n_i}{2\beta_0 + \lambda^n_j - \lambda^n_i} C_n(1, 0) \frac{1}{1 + \lambda^n_i / 2\beta_0}$$

$$+ K^n_0 P^n_i \left( \frac{\delta^B_n G_n}{\delta^B_n B^B_{NS}} \right) \frac{1}{1 + \lambda^n_i / 2\beta_0} - K^n_0 \frac{\beta_1}{\beta_0} P^n_i C_n(1, 0) \frac{\lambda^n_i / 2\beta_0}{1 + \lambda^n_i / 2\beta_0}$$

$$C^n = 2\beta_0 (\delta^B_n + A^n_i \cdot C_n(1, 0))$$

where $\lambda^n_i (i = +, -, NS)$ are eigenvalues of the 1-loop anomalous dimension matrix $\tilde{\gamma}^{(0)}_n$ and are given in the Appendix A. $\beta_0$ and the $\beta_1$ are the one- and two-loop $\beta$ functions, and $\beta_0 = 11 - 2n_f/3$ and $\beta_1 = 102 - 38n_f/3$.

All the quantities necessary to evaluate $\tilde{P}_n^i$, $A^n_i$, $B^n_i$, and $C^n$ are now known and will be presented in Appendix. Two-loop results \cite{8, 9} have been calculated in the $\overline{MS}$ scheme \cite{25}. Actually the expressions of Eqs.(3.16) and (3.17)-(3.20) are the same in form as the ones obtained before by one of the authors and Walsh for the case of the virtual photon structure function $F_2^\gamma$ \cite{12}. The explicit expressions of the one-loop and two-loop anomalous dimensions \cite{8, 9} as well as one-loop coefficient functions \cite{20, 27, 28, 29, 30, 8, 9} are given in the Appendix B and C.

Eq.(3.16) is our main result of the present paper. The first term is the LO result, and the remaining terms are the NLO QCD corrections.

Now let us examine the renormalization scheme independence of the coefficients; $A^n_i$, $B^n_i$ and $C^n$. As in the unpolarized case, $B^n_i$ can be written in terms of a
scheme-independent combination of 2-loop anomalous dimensions and 1-loop coefficient functions in the hadronic sector. Using the scheme-independent coefficients $R_{2,n}^i$ (31, 32, 33), we can write

$$B_n^i = L_n^i R_{2,n}^i \quad (i = +, -, NS)$$  \hspace{1cm} (3.21)

where the explicit form for $R_{2,n}^i$ is given in Eqs.(9)-(12) of ref.[31] (see also [12]) and

$$L_n^i = \tilde{\beta}_{\psi}^n \frac{1}{1 + \lambda_i^n/2\beta_0}$$  \hspace{1cm} (3.22)

which is the coefficient of the leading-log term. The scheme independence of $B_n^i$ follows from these two equations.

Regarding $C_n$, we first consider the photon matrix elements of the renormalized quark and gluon operators. The finite matrix elements $A_n$ and the tree-level coefficient functions $C_n(1,0)$ are given by

$$A_n = 6(e^2, 0, e^4) - e^2 > 2 \tilde{A}_{\psi}^n$$  \hspace{1cm} (3.23)

$$C_n(1, 0) = \begin{pmatrix} <e^2> \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (3.24)

Hence we have

$$A_n \cdot C_n(1, 0) = 6(e^4 > \tilde{A}_{\psi}^n)$$  \hspace{1cm} (3.25)

Noting that

$$B_{\gamma}^n = \frac{2}{n_f} B_{G}^n, \quad \delta_{\gamma} = 3n_f < e^4 >$$  \hspace{1cm} (3.26)

we find $C_n$ equal to be

$$C_n = 12\beta_0 < e^4 > (B_{G}^n + \tilde{A}_{\psi}^n)$$  \hspace{1cm} (3.27)

Since the combination $B_{G}^n + \tilde{A}_{\psi}^n$ is scheme independent [23], so is $C_n$. In fact, in the $\overline{MS}$ scheme [8], the gluon matrix elements of quark operators ($i = \psi, NS$) read

$$\langle p, s | R_{\mu_1 \cdots \mu_{n-1}}^i (\mu^2) | p, s \rangle = A_i (p^2, \mu^2, g) \left[ \{ s^\sigma p_{\sigma}^\mu \cdots p_{\nu}^{\mu_{n-1}} \} - \text{traces} \right] \quad (i = \psi, NS)$$

$$A_{\psi} = \frac{g^2}{16\pi^2} \left( \frac{1}{2} \gamma_{\psi}^{(0)} \ln \frac{-p^2}{\mu^2} + \tilde{A}_{\psi}^n \right)$$  \hspace{1cm} (3.28)
where the finite matrix element $\tilde{A}_{nG}^{\psi}$ is given in the parton language as \[8\]:

$$\tilde{A}_{nG}^{\psi} = \int_{0}^{1} dx \, x^{n-1} a_{S,qq}^{(1)}(x)$$

(3.29)

where

$$a_{S,qq}^{(1)}(x) = T_f \left[ (4 - 8x) \left\{ \ln x + \ln(1 - x) \right\} \right]$$

(3.30)

with $T_f = n_f/2$. Thus we get

$$\tilde{A}_{nG}^{\psi} = 2n_f \left[ \frac{n-1}{n(n+1)} S_1(n) + \frac{4}{(n+1)^2} - \frac{1}{n^2} - \frac{1}{n} \right].$$

(3.31)

Therefore, from (3.27) we finally arrive at

$$C^n = 24\beta_0 f < e^4 > \left[ \frac{2}{n} - \frac{4}{n+1} - \frac{2}{n^2} + \frac{4}{(n+1)^2} \right]$$

(3.32)

which is consistent with the Box diagram calculation.

On the other hand, in the RG scheme adopted by Kodaira \[28\] which is the momentum subtraction scheme, we have for $n \geq 3$,

$$\tilde{A}_{nG}^{\psi} = 0$$

(3.33)

with the coefficient function given by

$$B_{G}^{n} = 2n_f \left[ \frac{2}{n} - \frac{4}{n+1} - \frac{2}{n^2} + \frac{4}{(n+1)^2} \right].$$

(3.34)

For $n = 1$, due to the Adler-Bell-Jackiw anomaly

$$\tilde{A}_{n=1G}^{\psi} = -2n_f$$

(3.35)

and

$$B_{G}^{n=1} = 0 ,$$

(3.36)

by definition, since we have no gauge-invariant gluon operator for $n = 1$. Combining this with the result for $\tilde{A}_{nG}^{\psi}$, we arrive at the same result for $C^n$.

The scheme-independence of the remaining coefficients $A_i^n$ follows from the above arguments on $B^n_i$ and $C^n$ and the physically measurable moments given in Eq.\((3.16)\).
4 QCD improved parton model approach

We now turn to the analysis based on the QCD improved parton model [34] using the DGLAP parton evolution equations.

Let $q^i_\pm(x, Q^2, P^2)$, $G^i_\pm(x, Q^2, P^2)$, $\Gamma^i_\pm(x, Q^2, P^2)$ be quark with $i$-flavor, gluon, and photon distribution functions with $\pm$ helicities of the longitudinally polarized virtual photon with mass $-P^2$ [24]. Then the spin-dependent parton distributions are defined as

$$\Delta q^i \equiv q^i_+ + q^i_- - q^i_- - q^i_-,$$

$$\Delta G^\gamma \equiv G^\gamma_+ - G^\gamma_-,$$

$$\Delta \Gamma^\gamma \equiv \Gamma^\gamma_+ - \Gamma^\gamma_-.$$ (4.1)

In the leading order of the electromagnetic coupling constant, $\alpha = e^2/4\pi$, $\Delta \Gamma^\gamma$ does not evolve with $Q^2$ and is set to be $\Delta \Gamma^\gamma(x, Q^2, P^2) = \delta(1 - x)$. The quark and gluon distributions $\Delta q^i$ and $\Delta G^\gamma$ satisfy the following evolution equations:

$$\frac{d\Delta q^i(x, Q^2, P^2)}{d\ln Q^2} = \int_x^1 \frac{dy}{y} \left\{ \sum_j \Delta \tilde{P}_{qq'}(\frac{x}{y}, Q^2) \Delta q^{j'}(y, Q^2, P^2) + \Delta \tilde{P}_{qG}(\frac{x}{y}, Q^2) \Delta G^\gamma(y, Q^2, P^2) \right\} + \Delta \tilde{P}_{q\gamma}(x, Q^2, P^2),$$ (4.3)

$$\frac{d\Delta G^\gamma(x, Q^2, P^2)}{d\ln Q^2} = \int_x^1 \frac{dy}{y} \left\{ \Delta \tilde{P}_{Gq}(\frac{x}{y}, Q^2) \Delta q^i(y, Q^2, P^2) + \Delta \tilde{P}_{G\gamma}(\frac{x}{y}, Q^2) \Delta G^\gamma(y, Q^2, P^2) \right\} + \Delta \tilde{P}_{G\gamma}(x, Q^2, P^2),$$ (4.4)

where $\Delta \tilde{P}_{AB}$ is a polarized splitting function of $B$-parton to $A$-parton, defined as $\Delta \tilde{P}_{AB} \equiv P_{A,B_+} - P_{A,B_-}$ ($= P_{A,B_+} - P_{A,B_-}$, due to parity conservation in QCD and QED).

For later convenience we use, instead of $\Delta q^i$, the flavor singlet and non-singlet combinations defined as follows:

$$\Delta q^i_S \equiv \sum_i \Delta q^i,$$ (4.5)

$$\Delta q^i_{NS} \equiv \Delta q^i - \frac{\Delta q^i_S}{n_f}$$ (4.6)
so that $\sum_i \Delta q_{iS} = 0$ and $n_f$ is the number of relevant active quark flavors. The quark-quark splitting function $\Delta \bar{P}_{q'q}$ is made up of two pieces, the one representing the case that $j$-quark splits into $i$-quark without through gluon, and the other one through gluon, and may be expressed as

$$\Delta \bar{P}_{q'q} = \delta_{ij} \Delta \bar{P}_{qq} + \frac{1}{n_f} \Delta \bar{P}_{qq}^S$$

(4.7)

where the second term is representing the splitting through gluon, and $\Delta \bar{P}_{qq}$ and $\Delta \bar{P}_{qq}^S$ are both independent of quark flavor, $i$ and $j$. It is noted that by construction $\Delta \bar{P}_{qq}^S$ is relevant for the evolution of flavor-siglet $\Delta q_S^\gamma$ and first appears in the order of $\alpha_s^2$.

In the QCD improved parton model, which is based on the factorization theorem, the polarized virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ is expressed as

$$g_1^\gamma(x, Q^2, P^2) = \int_x^1 \frac{dy}{y} \left\{ \sum_i \Delta q_i(y, Q^2, P^2) C_i^i \left( \frac{x}{y}, Q^2 \right) + \Delta G(y, Q^2, P^2) C_G^G \left( \frac{x}{y}, Q^2 \right) \right\} + C_\gamma^\gamma(x, Q^2), \quad (4.8)$$

where $C_i$, $C_G$, and $C_\gamma$ are the coefficient functions of $i$-quark, gluon, and photon, respectively, and are independent of target photon mass $P^2$. Up to one-loop level they are given by,

$$C_i(z, Q^2) = e_i^2 \left\{ \delta(1-z) + \frac{\alpha_s(Q^2)}{4\pi} B_q(z) \right\}$$

(4.9)

$$C_G(z, Q^2) = <e^2> \left\{ 0 + \frac{\alpha_s(Q^2)}{4\pi} B_G(z) \right\}$$

(4.10)

$$C_\gamma(z, Q^2) = \frac{\alpha}{4\pi} 3n_f <e^4> B_\gamma(z)$$

(4.11)

where $<e^2> = \sum_i e_i^2/n_f$ and $<e^4> = \sum_i e_i^4/n_f$. It is noted that $B_q(z)$ in Eq.(4.9) is independent of the quark flavor $i$. Since $\sum_i \Delta q^i C^i$ is rewritten as

$$\sum_i \Delta q^i C^i = \sum_i \left\{ \Delta q_{iS} + \frac{\Delta q^i}{n_f} \right\} C^i$$

(4.12)
we obtain
\[ g_1^\gamma(x, Q^2, P^2) = \int_x^1 \frac{dy}{y} \left\{ \Delta q_3^\gamma(y, Q^2, P^2) \, C^S(\frac{x}{y}, Q^2) + \Delta G^\gamma(y, Q^2, P^2) \, C^G(\frac{x}{y}, Q^2) \\
+ \Delta q_{NS}^\gamma(y, Q^2, P^2) \, C^{NS}(\frac{x}{y}, Q^2) \right\} + \Delta^r(x, Q^2) \] (4.13)

where we have defined
\[ C^S(z, Q^2) \equiv < e^2 > \left\{ \delta(1 - z) + \frac{\alpha_s(Q^2)}{4\pi} B_S(z) \right\} \] (4.14)
\[ C^{NS}(z, Q^2) \equiv \delta(1 - z) + \frac{\alpha_s(Q^2)}{4\pi} B_{NS}(z) \] (4.15)
\[ \Delta q_{NS}^\gamma \equiv \sum_i e_i^2 \Delta q_{iNS}^\gamma \] (4.16)

and \( B_S(z) = B_{NS}(z) = B_q(z) \). From Eqs.(4.3)–(4.7) and (4.16), the evolution equations for \( \Delta q_3^\gamma, \Delta G^\gamma, \) and \( \Delta q_{NS}^\gamma \) are now given by

\[ \frac{d}{d \ln Q^2} \Delta q_3^\gamma(x, Q^2, P^2) = \int_x^1 \frac{dy}{y} \left\{ \Delta \tilde{P}_{qg}(\frac{x}{y}, Q^2) + \Delta \tilde{P}_{gq}(\frac{x}{y}, Q^2) \right\} \Delta q_3^\gamma(y, Q^2, P^2) \\
+ n_f \Delta \tilde{P}_{qG}(\frac{x}{y}, Q^2) \Delta G^\gamma(y, Q^2, P^2) \right\} + \sum_i \Delta \tilde{P}_{q_i^\gamma}(x, Q^2), \] (4.17)

\[ \frac{d}{d \ln Q^2} \Delta G^\gamma(x, Q^2, P^2) = \int_x^1 \frac{dy}{y} \left\{ \Delta \tilde{P}_{Gq}(\frac{x}{y}, Q^2) \Delta q_3^\gamma(y, Q^2, P^2) \\
+ \Delta \tilde{P}_{GG}(\frac{x}{y}, Q^2) \right\} \Delta G^\gamma(y, Q^2, P^2) \right\} + \Delta \tilde{P}_{G^\gamma}(x, Q^2), \] (4.18)

\[ \frac{d}{d \ln Q^2} \Delta q_{NS}^\gamma(x, Q^2, P^2) = \int_x^1 \frac{dy}{y} \Delta \tilde{P}_{qg}(\frac{x}{y}, Q^2) \Delta q_{NS}^\gamma(y, Q^2, P^2) \\
+ \sum_i e_i^2 \{ \Delta \tilde{P}_{q_i^\gamma}(x, Q^2) - \frac{1}{n_f} \sum_j \Delta \tilde{P}_{q_j^\gamma}(x, Q^2) \} \]. (4.19)

Introducing a row vector \( \Delta q^\gamma = (\Delta q_3^\gamma, \Delta G^\gamma, \Delta q_{NS}^\gamma) \), the above evolution equations, Eqs.(4.17)–(4.19) are expressed in a compact matrix form.

\[ \frac{d}{d \ln Q^2} \Delta q^\gamma(x, Q^2) = \Delta k(x, Q^2) + \int_x^1 \frac{dy}{y} \Delta q^\gamma(y, Q^2, P^2) \Delta P(\frac{x}{y}, Q^2) \] (4.20)
where the elements of a row vector \( \Delta k = (\Delta K_S, \Delta K_G, \Delta K_{NS}) \) are

\[
\Delta K_S \equiv \sum_i \Delta \tilde{P}_{q'\gamma}, \quad \Delta K_G \equiv \Delta \tilde{P}_{G\gamma} \\
\Delta K_{NS} \equiv \sum_i e_i^2 \{ \Delta \tilde{P}_{q'\gamma} - \frac{1}{n_f} \sum_j \Delta \tilde{P}_{q''\gamma} \}
\]

(4.21)

Since \( \Delta \tilde{P}_{q'\gamma} \) is proportional to \( e_i^2 \), it is easily seen that \( \Delta K_S \) and \( \Delta K_{NS} \) have factors\( n_f \langle e^2 \rangle \) and\( n_f (\langle e^4 \rangle - \langle e^2 \rangle^2) \), respectively. The \( 3 \times 3 \) matrix \( \Delta P(z, Q^2) \) is written as

\[
\Delta P(z, Q^2) = \begin{pmatrix}
\Delta P_{qq}^S(z, Q^2) & \Delta P_{Gq}(z, Q^2) & 0 \\
\Delta P_{Gq}(z, Q^2) & \Delta P_{GG}(z, Q^2) & 0 \\
0 & 0 & \Delta P_{qq}^{NS}(z, Q^2)
\end{pmatrix}
\]

(4.22)

where

\[
\Delta P_{qq}^S \equiv \Delta \tilde{P}_{qq} + \Delta \tilde{P}_{qq}^S, \quad \Delta P_{Gq} \equiv f \Delta \tilde{P}_{Gq} \\
\Delta P_{Gq} \equiv \Delta \tilde{P}_{Gq}, \quad \Delta P_{GG} \equiv \Delta \tilde{P}_{GG}, \quad \Delta P_{qq}^{NS} \equiv \Delta \tilde{P}_{qq}
\]

(4.23)

Once we get the information on the coefficient functions in Eqs. (4.9)–(4.11) and parton splitting functions in Eqs. (4.21)–(4.23), we can predict the behavior of \( g_1^\gamma(x, Q^2, P^2) \) in QCD. The NLO analysis is now possible since the spin-dependent one-loop coefficient functions and two-loop parton splitting functions are available [8, 9]. There are two methods to obtain \( g_1^\gamma(x, Q^2, P^2) \) in NLO. In one method, we use the parton splitting functions up to two-loop level and we solve numerically \( \Delta q^\gamma(x, Q^2, P^2) \) in Eq. (4.20) by iteration, starting from the initial quark and gluon distributions of the virtual photon at \( Q^2 = P^2 \). The interesting point of studying the virtual photon with mass \( -P^2 \) is that when \( P^2 \gg \Lambda^2 \), the initial parton distributions of the photon are completely known up to the one-loop level in QCD. Then inserting the solved \( \Delta q^\gamma(x, Q^2, P^2) \) into Eq. (4.13), and together with the known one-loop coefficient functions we can predict \( g_1^\gamma(x, Q^2, P^2) \) in NLO.

The other method, which is more common than the former, is by making use of the inverse Mellin transformation. From now on we follow the latter method. First we take the Mellin moments of Eq. (4.13),

\[
\int_0^1 dx x^{n-1} g_1^\gamma(x, Q^2, P^2) = \Delta q^\gamma(n, Q^2, P^2) \cdot C(n, Q^2) + C^\gamma(n, Q^2),
\]

(4.24)
where we have defined the moments of an arbitrary function \( f(x) \) as
\[
f(n) \equiv \int_0^1 dx x^{n-1} f(x). \tag{4.25}
\]
Comparing Eq. (4.24) with Eqs. (3.10) and (3.15), we can easily see the correspondence between the quantities in the QCD improved parton model and those in the framework of OPE as follows:
\[
\left[ \Delta q^\gamma(n, Q^2, P^2) \right]_i = \sum_{j=S,G,NS,\gamma} \langle \gamma(p)| R^j_n(\mu^2 = P^2)|\gamma(p) \rangle \left\{ T \exp \left[ \int_{q(Q^2)}^{g(P^2)} dg \frac{\gamma_n(g)}{\beta(g)} \right] \right\}_{ji} \quad (i = S, G, NS) \tag{4.26}
\]
\[
C(n, Q^2) = C_n(1, \bar{g}) \tag{4.27}
\]
\[
C^\gamma(n, Q^2) = C^\gamma_n(1, \bar{g}, \alpha) \tag{4.28}
\]
Henceforse we omit the obvious \( n \)-dependence for simplicity. We expand the splitting functions \( \Delta k(Q^2) \) and \( \Delta P(Q^2) \) in powers of the QCD and QED coupling constants as
\[
\Delta k(Q^2) = \frac{\alpha}{2\pi} \Delta k^{(0)} + \frac{\alpha \alpha_s(Q^2)}{(2\pi)^2} \Delta k^{(1)} + \cdots \tag{4.29}
\]
\[
\Delta P(Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \Delta P^{(0)} + \left[ \frac{\alpha_s(Q^2)}{2\pi} \right]^2 \Delta P^{(1)} + \cdots \tag{4.30}
\]
and introduce \( t \) instead of \( Q^2 \) as the evolution variable [33],
\[
t \equiv \frac{2}{\beta_0} \ln \frac{\alpha_s(P^2)}{\alpha_s(Q^2)]. \tag{4.31}
\]
Then, taking the Mellin moments of the both sides in Eq. (4.20), we find that \( \Delta q^\gamma(t)(= \Delta q^\gamma(n, Q^2, P^2)) \) satisfies the following inhomogenous differential equation [33, 34]:
\[
\frac{d\Delta q^\gamma(t)}{dt} = \frac{\alpha}{2\pi} \left\{ \frac{2\pi}{\alpha_s} \Delta k^{(0)} + \left[ \Delta k^{(1)} - \frac{\beta_1}{2\beta_0} \Delta k^{(0)} \right] + \mathcal{O}(\alpha_s) \right\}
+ \Delta q^\gamma(t) \left\{ \Delta P^{(0)} + \frac{\alpha_s}{2\pi} \left[ \Delta P^{(1)} - \frac{\beta_1}{2\beta_0} \Delta P^{(0)} \right] + \mathcal{O}(\alpha_s^2) \right\} \tag{4.32}
\]
where we have used the fact the QCD effective coupling constant $\alpha_s(Q^2)$ satisfies
\begin{equation}
\frac{d\alpha_s(Q^2)}{d\ln Q^2} = -\beta_0 \frac{\alpha_s(Q^2)^2}{4\pi} - \beta_1 \frac{\alpha_s(Q^2)^3}{(4\pi)^2} + \cdots
\end{equation}
with $\beta_0 = 11 - 2n_f/3$ and $\beta_1 = 102 - 38n_f/3$. Note that the $P^2$ dependence of $\Delta q^\gamma$ solely comes from the initial condition (or boundary condition) as we will see below.

We look for the solution $\Delta q^\gamma(t)$ in the following form:
\begin{equation}
\Delta q^\gamma(t) = \Delta q^\gamma(0)(t) + \Delta q^\gamma(1)(t)
\end{equation}
where the first (second) term represents the solution in the LO (NLO). First we discuss about the initial conditions of $\Delta q^\gamma$.

In Section 3, we have observed that for $-p^2 = P^2 \gg \Lambda^2$ the photon matrix elements of the hadronic operators $R_n^i$ ($i = \psi(S), G, NS$) can be calculated perturbatively. Choosing the square of the renormalization point $\mu^2$ to be close to $P^2$, we obtained, to the lowest order
\begin{equation}
\langle \gamma(p) | R_n^i(\mu) | \gamma(p) \rangle = \frac{\alpha}{4\pi} \left( -\frac{1}{2} K_{n,i}^0 \ln \frac{P^2}{\mu^2} + A_n^i \right), \quad i = \psi(S), G, NS .
\end{equation}
The $K_{n,i}^0$-terms and $A_n^i$-terms represent the operator mixing between the hadronic operators and photon operators in the LO and NLO, respectively. The operator mixing implies that there exists quark distribution in the photon. When we renormalize the photon matrix elements of the hadronic operators at $\mu^2 = P^2$, we obtain
\begin{equation}
\langle \gamma(p) | R_n^i(\mu) | \gamma(p) \rangle|_{\mu^2 = P^2} = \frac{\alpha}{4\pi} A_n^i
\end{equation}
which shows that, at $\mu^2 = P^2$, quark distribution exists in the photon, not in the LO but in the NLO. Thus we have
\begin{equation}
\Delta q^\gamma(0)(0) = 0, \quad \Delta q^\gamma(1)(0) = \frac{\alpha}{4\pi} A_n .
\end{equation}
Explicit expressions of $A_n$ in the $\overline{\text{MS}}$ scheme are given in Section 3.

With these initial conditions, we obtain for the solution $\Delta q^\gamma(t)$ of Eq.(4.32),
\begin{equation}
\Delta q^\gamma(0)(t) = \frac{4\pi}{\alpha_s(t)} a \left\{ 1 - \left[ \frac{\alpha_s(t)}{\alpha_s(0)} \right]^{1 - \frac{2\Delta \rho^{(0)}(0)}{\beta_0}} \right\}
\end{equation}
\[ \Delta q^{\gamma(1)}(t) = -2a \left\{ \int_{0}^{t} d\tau e^{(\Delta P^{(0)} - \frac{i}{2}\beta_0)v\tau} \left[ \Delta P^{(1)} - \frac{\beta_1}{2\beta_0} \Delta P^{(0)} \right] e^{-\Delta P^{(0)}\tau} \right\} e^{\Delta P^{(0)}t} \\
+ b \left\{ 1 - \left[ \frac{\alpha_s(t)}{\alpha_s(0)} \right] \frac{2\Delta P^{(0)}}{\beta_0} \right\} + \Delta q^{\gamma(1)}(0) \left[ \frac{\alpha_s(t)}{\alpha_s(0)} \right] \frac{2\Delta P^{(0)}}{\beta_0} \] (4.39)\

where
\[ a = \frac{\alpha}{2\pi\beta_0} \Delta k^{(0)} \frac{1}{1 - 2\Delta P^{(0)}} \] (4.40)
\[ b = \left\{ \frac{\alpha}{2\pi} \left[ \Delta k^{(1)} - \frac{\beta_1}{2\beta_0} \Delta k^{(0)} \right] + 2a \left[ \Delta P^{(1)} - \frac{\beta_1}{2\beta_0} \Delta P^{(0)} \right] \right\} \frac{-1}{\Delta P^{(0)}} \] (4.41)

It is noted that the parton distributions \( \Delta q^{\gamma}(t) \) do depend on the initial conditions \( \Delta q^{\gamma}(0) = (\alpha/4\pi)A_n \), but we have seen in Section 3 that the structure function \( g^{\gamma}_1(x, Q^2, P^2) \) itself is independent of \( \Delta q^{\gamma}(0) \) in NLO in QCD.

The moments of the splitting functions are related to the anomalous dimensions of operators as follows:
\[ \Delta P^{(0)} = -\frac{1}{4} \gamma_n^0, \quad \Delta P^{(1)} = -\frac{1}{8} \gamma_n^{(1)} \] (4.42)
\[ \Delta k^{(0)} = \frac{1}{4} K_n^0, \quad \Delta k^{(1)} = \frac{1}{8} K_n^1 \] (4.43)

The evaluation of \( \Delta q^{\gamma(0)}(t) \) and \( \Delta q^{\gamma(1)}(t) \) in Eqs. (4.38) and (4.39) can be easily done by introducing the projection operators \( P_i^n \) such as
\[ \Delta P^{(0)} = -\frac{1}{4} \gamma_n^0 = -\frac{1}{4} \sum_{i=+, -, NS} \lambda_i^n P_i^n, \quad i = +, -, NS \] (4.44)
\[ P_i^n P_j^n = \begin{cases} 0 & i \neq j, \\ P_i^n & i = j, \\ \sum_i P_i^n = 1 \end{cases} \] (4.45)

where \( \lambda_i^n \) are the eigenvalues of the matrix \( \gamma_n^0 \). Then the solution \( \Delta q^{\gamma}(t) \) of Eq. (4.32) in the NLO is written as
\[ \Delta q^{\gamma}(t)/\left[ \frac{\alpha}{8\pi\beta_0} \right] = \frac{4\pi}{\alpha_s(t)} K_n^0 \sum_i P_i^n \frac{1}{1 + \frac{\lambda_i^n}{2\beta_0}} \left\{ 1 - \left[ \frac{\alpha_s(t)}{\alpha_s(0)} \right]^{1 + \frac{\lambda_i^n}{2\beta_0}} \right\} \]
\[ + \left\{ K_n^1 \sum_i P_i^n \frac{1}{\lambda_i^n/2\beta_0} + \frac{\beta_1}{\beta_0} K_n^0 \sum_i P_i^n \right\} \left( 1 - \frac{1}{\lambda_i^n/2\beta_0} \right) \]
\[- K_n^0 \sum_{j,i} \frac{P_n^m \gamma_n^{(1)} P_i^m}{2\beta_0 + \lambda_j^n - \lambda_i^n} \frac{1}{\lambda_i^n/2\beta_0} - 2\beta_0 A_n \sum_i P_i^n \right] \\
+ \left\{ K_n^0 \sum_{i,j} \frac{P_n^m \gamma_n^{(1)} P_j^m}{2\beta_0 + \lambda_j^n - \lambda_i^n} \frac{1}{1 + \lambda_i^n/2\beta_0} - \frac{\beta_1}{\beta_0} K_n^0 \sum_i P_i^n \lambda_i^n/2\beta_0 \right\} \\
\times \left\{ 1 - \left[ \frac{\alpha_s(t)}{\alpha_s(0)} \right]^{1+\lambda_i^n/2\beta_0} \right\}
+ 2\beta_0 A_n \right) \tag{4.46}
\]

Now inserting the above solution of $\Delta q(t)$ into Eq.(1.24) and together with the information on the coefficient functions in Eq. (3.15), we reproduce the same formula for the moments of $g_1^\gamma(x, Q^2, P^2)$ given in Eqs.(3.16)-(3.20), as for the case of OPE approach in the NLO.

## 5 Sum rule for the first moment of $g_1^\gamma(x, Q^2, P^2)$

The polarized structure function $g_1^\gamma$ of the real photon satisfies a remarkable sum rule \[ \int_0^1 g_1^\gamma(x, Q^2) dx = 0 \] \tag{5.1}

Now we can ask what happens to the first moment of the virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$. This can be studied by taking the $n \rightarrow 1$ limit of (3.16).

Note that we have the following eigenvalues of the one-loop anomalous dimension matrix $\gamma_{n=1}^0$:

\[ \lambda_{n=1}^1 = 0, \quad \lambda_{n=1}^n = -2\beta_0, \quad \lambda_{n=1}^{NS} = 0. \] \tag{5.2}

Physically speaking, the zero eigenvalues $\lambda_{n=1}^1 = \lambda_{n=1}^{NS} = 0$ correspond to the conservation of the axialvector current at one-loop order, which has the counterpart for the unpolarized structure function $F_2$; the conservation of energy momentum tensor, $\lambda_{n=2}^n = 0$. The other eigenvalue $\lambda_{n=1}^n = -2\beta_0$, which is negative, is rather an artifact of continuation of the anomalous dimension of the gluon operators to $n = 1$, since there is no twist-2 gluon operator exists for $n = 1$, in the RG scheme in
which only gauge-invariant operators are allowed. But $n = 1$ gluon operator exists in the so called Adler-Bardeen scheme \cite{38, 39}. In fact, in the QCD improved parton model approach, there is no reason why the $n = 1$ moment of the polarized gluon distribution should not be considered \cite{40}.

In the coefficients

$$
\tilde{P}_i^n \frac{1}{1 + \lambda_i^n/2\beta_0}, \quad A_i^n, \quad B_i^n \quad (i = +, -, NS) \quad (5.3)
$$

the special points \eqref{5.2} would develop the singularities at $n = 1$, since in those coefficients there exist the factors

$$
\frac{1}{\lambda_i^n}, \quad \frac{1}{\lambda_{NS}^n}, \quad \frac{1}{1 + \lambda_i^n/2\beta_0}. \quad (5.4)
$$

Now if we take the limit of $n$ going to 1, we have

$$
\tilde{P}_i^n \frac{1}{1 + \lambda_i^n/2\beta_0} \to 0 \quad (i = +, -, NS)
$$

$$
A_i^+ \to \text{finite}, \quad A_i^- \to 0, \quad A_i^{NS} \to \text{finite}
$$

$$
B_i^+ \to 0, \quad B_i^- \to \text{finite}, \quad B_i^{NS} \to 0. \quad (5.5)
$$

However, $A_i^+, A_i^{NS},$ and $B_i^-$ are multiplied by the following vanishing factors

$$
\left\{ 1 - \left( \frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{\lambda_i^n/2\beta_0} \right\}, \quad \left\{ 1 - \left( \frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{\lambda_{NS}^n/2\beta_0} \right\}, \quad \left\{ 1 - \left( \frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{\lambda_i^n/2\beta_0 + 1} \right\}, \quad (5.6)
$$

respectively, and thus the terms proportional to $\tilde{P}_i^n, A_i^n,$ and $B_i^n$ in Eq.\eqref{5.10} all vanish in the $n = 1$ limit. Note that these vanishing factors are specific to the case of the virtual photon target, and that such factors do not appear when the target is real photon.

Thus the first three terms in the 1st moment vanish irrespective of the RG scheme. So we get

$$
\int_0^1 dx g_1^\gamma(x, Q^2, P^2) = \frac{\alpha}{4\pi} \frac{1}{2\beta_0} C^{n = 1} + \mathcal{O}(\alpha_s) \quad (5.7)
$$

Now let us consider $C^{n = 1}$, which is given by

$$
C^{n = 1} = 12\beta_0 < e^4 > (B_G^n + \tilde{A}^\psi_{nG})|_{n = 1} \quad (5.8)
$$
As we have seen, the combination \((B_G^n + \tilde{A}_n G)\) is renormalization-scheme independent [25]. The results in the \(\overline{\text{MS}}\) scheme [8, 9, 29] are
\[
B_{G}^{n=1} = 0, \quad \tilde{A}_{n=1}^G = -2n_f \tag{5.9}
\]
The same results have been obtained by Kodaira [28] in the framework of OPE and RG method. He set \(B_{G}^{n=1} = 0\), observing that there is no gauge-invariant twist-2 gluon operator for \(n = 1\) and obtained \(\tilde{A}_{n=1}^G = -2n_f\) from the Adler-Bell-Jackiw anomaly. In the end, we have for the sum rule of the virtual photon structure function \(g_{\gamma 1}\),
\[
\int_{0}^{1} dx g_{\gamma 1}(x, Q^2, P^2) = -\frac{3\alpha}{\pi} \sum_{i=1}^{n_f} e_i^4 + \mathcal{O}(\alpha_s), \tag{5.10}
\]
where
\[
Q^2 \gg P^2 \gg m_i^2, \Lambda^2, \quad i = 1, \cdots, n_f \tag{5.11}
\]
with \(m_i\), the mass of \(i\)-th flavor quark, and \(n_f\), the number of active flavors.

Now it should be pointed out that we can further pursue the QCD corrections of order \(\alpha_s\) to the first moment of \(g_{\gamma 1}\). In the above equation for the first moment, the leading order is \(\mathcal{O}(1)\) not of order \(1/\alpha_s(Q^2)\), which is the case for the general moments. So we now go to the order \(\alpha_s\) QCD correction.

First we take the renormalization scheme of Kodaira [28]. We write down the first moment of \(g_{\gamma 1}(x, Q^2, P^2)\):
\[
\int_{0}^{1} dx g_{\gamma 1}(x, Q^2, P^2) = C_1^\psi(Q^2/P^2, \bar{g}(P^2), \alpha)\langle \gamma(p) | R_1^\psi(\mu^2 = P^2) | \gamma(p) \rangle \\
+ C_1^{NS}(Q^2/P^2, \bar{g}(P^2), \alpha)\langle \gamma(p) | R_1^{NS}(\mu^2 = P^2) | \gamma(p) \rangle \tag{5.12}
\]
Here it should be emphasized that because of the absence of the gauge-invariant \(n = 1\) gluon and photon operators, the mixing problem becomes much simpler. The coefficient functions can be given by
\[
\begin{pmatrix}
C_1^\psi(Q^2/P^2, \bar{g}(P^2), \alpha) \\
C_1^{NS}(Q^2/P^2, \bar{g}(P^2), \alpha)
\end{pmatrix}
= T \exp \int_{\bar{g}(Q^2)}^{(P^2)} dg' \tilde{\gamma}_1(g') \begin{pmatrix}
C_1^\psi(1, \bar{g}(Q^2), \alpha) \\
C_1^{NS}(1, \bar{g}(Q^2), \alpha)
\end{pmatrix} \tag{5.13}
\]
where \(\tilde{\gamma}_1(g)\) is a \(2 \times 2\) diagonal matrix:
\[
\tilde{\gamma}_1(g) = \begin{pmatrix}
\gamma_{\psi\psi}(g) & 0 \\
0 & \gamma_{NS}(g)
\end{pmatrix}. \tag{5.14}
\]
Here anomalous dimensions are expanded in powers of the coupling constant:

\[
\gamma(g) = \gamma^{(0)} \frac{g^2}{16\pi^2} + \gamma^{(1)} \left( \frac{g^2}{16\pi^2} \right)^2 + \mathcal{O}(g^6),
\]  

where

\[
\gamma^{(0)}_{\psi\psi} = \gamma^{(0)}_{NS} = 0, \quad \gamma^{(1)}_{\psi\psi} = 24 C_F T_f = 24 \cdot \frac{N_c^2 - 1}{2N_c} \cdot \frac{n_f}{2} = 16n_f, \quad \gamma^{(1)}_{NS} = 0,
\]

and the coefficient functions are

\[
C^\psi_1 (1, \tilde{g}(Q^2), \alpha) = (1 - \frac{3}{4} C_F \frac{\alpha_s(Q^2)}{\pi}) = 1 - \frac{\alpha_s(Q^2)}{\pi},
\]

\[
C^{NS}_1 (1, \tilde{g}(Q^2), \alpha) = 1 - \frac{3}{4} C_F \frac{\alpha_s(Q^2)}{\pi} = 1 - \frac{\alpha_s(Q^2)}{\pi}
\]

\[
T \exp \int_{\tilde{g}(Q^2)} \frac{\tilde{g}(P^2)}{\beta(g')} \frac{\tilde{g}_1(g')}{\beta(g')} = 1 - \frac{1}{16\pi^2 2\beta_0} \left[ \tilde{g}^2(P^2) - \tilde{g}^2(Q^2) \right].
\]

Here we have the finite matrix element of the quark operators between the virtual photon states:

\[
\langle \gamma(p) | R_{n=1} (\mu^2 = P^2) | \gamma(p) \rangle = \frac{\alpha}{4\pi} A^i_{n=1},
\]

where

\[
A_n = 6 (< e^2 >, 0, < e^4 > - < e^2 >) \tilde{A}_\psi_n G.
\]

Now we recall Kodaira’s statement that the bare Green’s function for the \( n = 1 \) case does not receive divergent corrections but the finite correction connected with Adler-Bell-Jackiw anomaly:

\[
\tilde{A}_\psi_{n=1} G = -4T(R) = -2n_f.
\]

Putting all these equations together, we finally obtain the \( \mathcal{O}(\alpha_s) \) QCD correction:

\[
\int_0^1 dx \tilde{g}(x, Q^2, P^2) = -\frac{3\alpha}{\pi} \left[ \sum_{i=1}^{n_f} e_i^4 \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) \right] - \frac{2}{\beta_0} \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \left( \frac{\alpha_s(P^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right) + \mathcal{O}(\alpha_s^2).
\]

This result is perfectly in agreement with the one obtained by Narison, Shore and Veneziano in ref.[5], apart from the overall sign for the definition of \( \tilde{g} \).
Now we show that we obtain the same result for the first moment of $g_1^n(x, Q^2, P^2)$ in the $\overline{\text{MS}}$ scheme. Although there exist no gauge-invariant twist-2 gluon- and photon-operators for $n = 1$, the $\overline{\text{MS}}$ calculation of the anomalous dimensions gives non-zero results for $\gamma_{GG}^{n=1}$ and $\gamma_{G\psi}^{n=1}$. Thus the $\overline{\text{MS}}$-scheme results rather correspond to the QCD parton model approach where the first moments of the gluon and photon distributions are as well defined as the other distributions.

Including the gluon and photon operators, let us start with Eq.(3.13) for the first moment of $g_1^n(x, Q^2, P^2)$ in $\overline{\text{MS}}$ scheme:

$$
\int_0^1 dx g_1^n(x, Q^2, P^2) = \frac{\alpha_s}{4\pi} A_1 \cdot M_1(Q^2/P^2, \tilde{g}(P^2)) \cdot C_1(1, \tilde{g}(Q^2)) + X_1(Q^2/P^2, \tilde{g}(P^2), \alpha) \cdot C_1(1, \tilde{g}(Q^2)) + C_\gamma (5.23)
$$

We expand $A_1$ in powers of $\tilde{g}^2(P^2)$ as

$$
A_1 = A_1^{(0)} + \frac{\tilde{g}^2(P^2)}{16\pi^2} A_1^{(1)} + \cdots \quad (5.24)
$$

where $A_1^{(0)}$ is given in Eq.(3.23). Using the $\overline{\text{MS}}$ results for the $n = 1$ anomalous dimensions, we obtain up to the order $O(\alpha_s^2)$

$$
M_1 = \left( \begin{array}{cccc} 1 - \frac{1}{2\pi^2} & \frac{1}{16\pi^2} & [\tilde{g}^2(P^2) - \tilde{g}^2(Q^2)] \times 24C_F T_f & \cdots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 1 \end{array} \right) \quad (5.25)
$$

where the second column is irrelevant since $B_1^G = 0$ in $\overline{\text{MS}}$ scheme and thus $C_G^G(1, \tilde{g})$ starts in $O(\alpha_s^2)$. Now it is easy to see that the first term of Eq.(5.23), to be more specific, $(\frac{\alpha_s}{4\pi} A_1^{(0)} M_1 C_1(1, \tilde{g}(Q^2)))$ gives the same result as in Eq.(5.22).

Let us now consider the contributions of other terms. If $A_1^{\psi(1)}$ and $A_1^{NS(1)}$ in the second term in Eq.(5.24) remain non-zero, then they give the $O(\tilde{g}^2)$ contribution. But $A_1^{\psi(1)} = A_1^{NS(1)} = 0$ due to the non-renormalization theorem [1] for the triangle anomaly, so its contribution is at most in $O(\alpha_s^2)$. The contribution of the second term in Eq.(5.23) is also in $O(\alpha_s^2)$, since $K_1^0 = K_1^{(1)} = 0$ and we expect

$$
K_\psi^{(2),n=1} = K_{NS}^{(2),n=1} = 0 \quad (5.26)
$$

for the three-loop mixing anomalous dimensions which are implied from the fact that the three-loop $\gamma_{\psi G}^{(2),n=1} = 0$ [1].
Finally we expand the third term of Eq.(5.23), $C^1$, as

$$C^1(1, \bar{g}, \alpha) = \frac{\alpha}{4\pi} \delta \left[ B^{(0),n=1}_\gamma + \frac{\bar{g}^2}{16\pi^2} B^{(1),n=1}_\gamma + \cdots \right] \quad (5.27)$$

where $B^{(0),n=1}_\gamma = B^{n=1}_\gamma$ in Eq.(3.15). We already know that $B^{(0),n=1}_\gamma = 0$ in the $\overline{\text{MS}}$ scheme. On the other hand, the two-loop ($O(\alpha_s^2)$) coefficient function for the polarized gluon has been calculated in the $\overline{\text{MS}}$ scheme by Zijlstra and van Neerven [30]. It is made up of two terms, one proportional to factor $C_F T_f n_f$ and the other proportional to factor $C_A T_f n_f$. The first moments of both terms turn out to vanish. It can be shown that the two-loop ($O(\alpha_s)$) coefficient function for the polarized photon, $B^{(1)}_\gamma$, is obtained from the two-loop gluon coefficient function, by picking up the term with the $C_F T_f n_f$ factor and by modifying the group factors. Thus we conclude that the first moment of $B^{(1)}_\gamma$ is zero. In the end, $C^1$ does not give $O(1)$ nor $O(\alpha_s)$ contributions to the first moment of $g_1^\gamma(x,Q^2,P^2)$. This means that we arrive at the same result for the 1st moment of $g_1^\gamma$ given in (5.22) in the $\overline{\text{MS}}$ scheme.

6 Numerical analysis

We now perform the inverse Mellin transform of the equation (3.16) to get $g_1^\gamma$ as a function of $x$. The $n$-th moment is denoted as

$$M(n,Q^2,P^2) = \int_0^1 x^{n-1} g_1^\gamma(x,Q^2,P^2) dx. \quad (6.1)$$

Then by inverting the moments (6.1) we get

$$g_1^\gamma(x,Q^2,P^2) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M(n,Q^2,P^2)x^{-n}dn \quad (6.2)$$

In order to have better convergence of the numerical integration, we change the contour in the complex $n$-plane from the vertical line connecting $C-i\infty$ with $C+i\infty$ (C is an appropriate positive constant), introducing a small positive constant $\varepsilon$, to

$$n = C - \varepsilon|y| + iy, \quad -\infty < y < \infty \quad (6.3)$$

Hence we have

$$g_1^\gamma(x,Q^2,P^2) = \frac{1}{2\pi i} \int_0^\infty M(C - \varepsilon y + iy,Q^2,P^2)e^{-\varepsilon y+i y \log(x)}(i - \varepsilon)dy$$
\[
+ \frac{1}{2\pi i} \int_{-\infty}^{0} M(C + \varepsilon y + iy, Q^2, P^2) e^{-(C + \varepsilon y + iy) \log(x)} (i + \varepsilon) dy
\]
\[
= \frac{1}{\pi} \int_{0}^{\infty} \left[ \text{Re} \{ M(z, Q^2, P^2) e^{-z \log(x)} \} - \varepsilon \text{Im} \{ M(z, Q^2, P^2) e^{-z \log(x)} \} \right] dy
\]
\[
z = C - \varepsilon y + iy.
\]

(6.4)

In Fig. 3 we have plotted, as an illustration, the result for \( n_f = 3, Q^2 = 30 \text{ GeV}^2 \) and \( P^2 = 1 \text{ GeV}^2 \) for the QCD scale parameter \( \Lambda = 0.2 \text{ GeV} \). The vertical axis corresponds to
\[
g_1^\gamma(x, Q^2, P^2, \frac{3\alpha}{\pi} n_f < e^4 > \ln \frac{Q^2}{P^2}.
\]

(6.5)

Here we have shown four cases; the Box (tree) diagram contribution;
\[
g_1^{\gamma(\text{Box})}(x, Q^2, P^2) = (2x - 1) \frac{3\alpha}{\pi} n_f < e^4 > \ln \frac{Q^2}{P^2},
\]

(6.6)

the Box diagram contribution including non-leading correction ignoring quark mass
\[
g_1^{\gamma(\text{Box, non-leading})}(x, Q^2, P^2) = \frac{3\alpha}{\pi} n_f < e^4 > \left[ (2x - 1) \ln \frac{Q^2}{P^2} - 2(2x - 1)(\ln x + 1) \right],
\]

(6.7)

the leading-order (LO) QCD correction and the next-to-leading order (NLO) QCD correction. We observe that the NLO QCD correction is significant at large \( x \) as well as at low \( x \). We have also studied other examples with different \( Q^2 \) and \( P^2 \).

In Fig. 4 we have plotted the case for \( Q^2 = 100 \text{ GeV}^2 \) with \( P^2 = 1 \text{ GeV}^2 \). Another case for \( Q^2 = 30 \text{ GeV}^2 \) with \( P^2 = 3 \text{ GeV}^2 \) is shown in Fig. 5. We have not seen any sizable change for the normalized structure function (6.5) for these different values of \( Q^2 \) and \( P^2 \). We examined the \( n_f = 4 \) case as well. It is observed that the normalized structure function is insensitive to the number of active flavors. Here we have not directly taken into account the heavy quark mass dependence, but rather confined ourselves to the above kinematical region. It turns out from the numerical analysis as well as from theoretical arguments that, as \( P^2 \) increases, the NLO QCD result approaches the Box contribution including the non-leading correction, as in the unpolarized structure function [11, 12].
Now let us consider the real photon case $P^2 = 0$. The structure function can be decomposed as
\begin{equation}
g_1^\gamma(x, Q^2) = g_1^\gamma(x, Q^2)|_{\text{pert.}} + g_1^\gamma(x, Q^2)|_{\text{non-pert.}}. \tag{6.8}
\end{equation}

The second term can only be computed by some non-perturbative method like lattice QCD, or estimated by vector meson dominance model (VMD). The first term, the point-like piece, can be calculated in a perturbative method. Actually, it can formally be recovered in our analysis by setting $P^2 = \Lambda^2$ in Eq.(3.16). In Fig.6, we have plotted the point-like piece of $g_1^\gamma$ of the real photon. The LO QCD result coincides with the previous result obtained by Sasaki in [22]. The NLO result is qualitatively consistent with the analysis by Stratmann and Vogelsang [10]. Finally, a comment on the $n = 1$ limit of the real photon structure function is in order. In the case of the unpolarized structure function $F_2^\gamma$ we have a singularity of $\mathcal{A}_n^\gamma$ at $n = 2$ due to the vanishing of $\lambda_n$ at $n = 2$ which leads to the negative structure function [18]. As discussed in refs.[42, 43, 44] we have to introduce some regularization prescription to recover positive structure function. For the polarized case we do not have such complication at $n = 1$ as we have seen in sect. 5.

## 7 Conclusion

Here in the present paper, we have investigated virtual photon’s spin structure function $g_1^\gamma(x, Q^2, P^2)$ for the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$, in the next-to-leading order in QCD. We presented our arguments both in the framework of OPE supplemented by RG method and in the DGLAP equation approach. The results are shown to be independent of the renormalization scheme.

The first moment of $g_1^\gamma$ for the virtual photon is non-vanishing in contrast to the vanishing first moment for the real photon case. We can go a step further to the order $\alpha_s$ which has turned out to reproduce the previous result of Narison, Shore and Veneziano [8], and the result is RG scheme-independent.

The numerical evaluation of $g_1^\gamma$ by the inverse Mellin transform was performed. The result shows that the NLO QCD corrections are significant at large $x$ and also at small $x$. The numerical analysis can also be applied to the point-like component.
of the real photon structure function. The result is qualitatively consistent with the previous analysis.

Although we have neglected in our kinematical region, we should also consider the power corrections of the form \((P^2/Q^2)^k(k = 1, 2, \cdots)\), which are arising from the target mass effects as well as from higher-twist effects.

In the present paper we only presented the result for the polarized photon structure function \(g_1^\gamma\) itself. In the course of the parton model analysis, we also obtain the polarized parton distributions \([3, 46]\) of the longitudinally polarized photon, for the case of virtual photon, which will be discussed elsewhere.

As a future subject, it would be intriguing to study another spin structure function \(g_2^\gamma\) which only exists for off-shell photon. In the OPE language, the twist-2 as well as twist-3 operators contribute to the QCD effects for \(g_2^\gamma\), which are now under investigation.

---

**Acknowledgement**

We thank G. Altarelli, J. Kodaira, G. Ridolfi, G. M. Shore, G. Veneziano and W. Vogelsang for valuable discussions. Part of this work was done while one of us (T.U.) was at CERN. He thanks the CERN Theory Division for the kind hospitality. This work is partially supported by the Monbusho Grant-in-Aid for Scientific Research No.(C)(2)-09640342 and No.(C)(2)-09640345.
Appendix

A  Notation for anomalous dimensions

To the lowest order in $\alpha$, the anomalous dimension matrix has the form

$$\gamma_n(g, \alpha) = \begin{pmatrix} \hat{\gamma}_n(g^2) & 0 \\ K_n(g^2, \alpha) & 0 \end{pmatrix}$$  \hspace{1cm} (A.1)

where $\hat{\gamma}_n(g^2)$ is the usual $3 \times 3$ anomalous dimension matrix in the hadronic sector

$$\hat{\gamma}_n(g) = \begin{pmatrix} \gamma_{\psi\psi}^n(g) & \gamma_{G\psi}^n(g) & 0 \\ \gamma_{G\psi}^n(g) & \gamma_{GG}^n(g) & 0 \\ 0 & 0 & \gamma_{NS}^n(g) \end{pmatrix},$$  \hspace{1cm} (A.2)

and $K_n(g, \alpha)$ is the three-component row vector

$$K_n(g, \alpha) = ( K_{\psi}^n(g, \alpha), \ K_{G}^n(g, \alpha), \ K_{NS}^n(g, \alpha) )$$ \hspace{1cm} (A.3)

representing the mixing between photon and three hadronic operators.

The anomalous dimensions are expanded as

$$\hat{\gamma}_n(g) = \frac{g^2}{16\pi^2} \hat{\gamma}_n^0 + \frac{g^4}{(16\pi^2)^2} \hat{\gamma}_n^{(1)} + O(g^6),$$ \hspace{1cm} (A.4)

$$K_n(g, \alpha) = -\frac{e^2}{16\pi^2} K_{\psi}^0 - \frac{e^2 g^2}{(16\pi^2)^2} K_n^{(1)} + O(e^2 g^4).$$ \hspace{1cm} (A.5)

The one-loop anomalous dimension matrix $\hat{\gamma}_n^0$ can be expressed in terms of its eigenvalues $\lambda_i^n$ ($i = +, -, NS$) as

$$\hat{\gamma}_n^0 = \sum_{i=+, -, NS} \lambda_i^n P_i^n$$ \hspace{1cm} (A.6)

where

$$\lambda_{\pm}^n = \frac{1}{2} \left\{ \gamma_{\psi\psi}^{0,n} + \gamma_{GG}^{0,n} \pm \left[ (\gamma_{\psi\psi}^{0,n} - \gamma_{GG}^{0,n})^2 + 4 \gamma_{\psi G}^{0,n} \gamma_{G\psi}^{0,n} \right]^{1/2} \right\}$$ \hspace{1cm} (A.7)

$$\lambda_{NS}^n = \gamma_{NS}^{0,n}$$ \hspace{1cm} (A.8)
and $P^n_i$ are the corresponding projection operators,

$$
P^\pm_n = \frac{1}{\lambda^+ - \lambda^-} \begin{pmatrix} \gamma^{0,n}_{\psi\psi} - \lambda^n & \gamma^{0,n}_{\psi G} & 0 \\ \gamma^{0,n}_{\psi G} & \gamma^{0,n}_{G G} - \lambda^n & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.9}
$$

$$
P^m_{NS} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.10}
$$

B  Explicit expressions for anomalous dimensions

B.1  One-loop order

\begin{align*}
\gamma^{0,n}_{\psi\psi} &= \gamma^{0,n}_{NS} = 2C_F \left[ -3 - \frac{2}{n(n+1)} + 4S_1(n) \right], \\
\gamma^{0,n}_{\psi G} &= -8T_f \frac{n-1}{n(n+1)}, \\
\gamma^{0,n}_{G G} &= -4C_F \frac{n+2}{n(n+1)}, \\
\gamma^{0,n}_{G G} &= 2C_A \left[ -\frac{11}{3} - \frac{8}{n(n+1)} + 4S_1(n) \right] + \frac{8}{3}T_f. \tag{B.4}
\end{align*}

where

$$
S_1(n) = \sum_{j=1}^{n} \frac{1}{j}. \tag{B.5}
$$

and

$$
C_A = 3, \quad C_F = \frac{4}{3}, \quad T_f = T_R n_f = n_f/2 \tag{B.6}
$$

with $n_f$ being the number of flavors.

\begin{align*}
K^0_n &= (K^0_{\psi}, 0, K^0_{NS}), \\
K^0_{\psi} &= 24 n_f < e^2 > \frac{n-1}{n(n+1)}, \tag{B.7}
\\
K^0_{NS} &= 24 n_f ( < e^4 > - < e^2 >^2) \frac{n-1}{n(n+1)}. \tag{B.9}
\end{align*}
B.2 Two-loop order \([8, 9]\)

B.2.1 Non-singlet sector

\[ \gamma^{(1), n}_{NS} = 8C_F^2 A^n_{NS} + 8C_A C_F B^n_{NS} + 8C_F T_f D^n_{NS}, \]  

with

\[
A^n_{NS} = -\frac{3}{8} + \frac{5}{n} - \frac{5}{n+1} - \frac{3}{n^2} - \frac{2}{(n+1)^2} + \frac{1}{n^3} - \frac{3}{(n+1)^3} \\
+ (-1)^n \left\{ -\frac{4}{n} + \frac{4}{n+1} + \frac{2}{n^2} + \frac{2}{(n+1)^2} - \frac{2}{n^3} + \frac{2}{(n+1)^3} \right\} \\
+ S_1(n) \left( \frac{2}{n^2} - \frac{2}{(n+1)^2} \right) + S_2(n) \left( 3 - \frac{2}{n} + \frac{2}{n+1} + 4S_1(n) \right) \\
+ S'_2 \left( \frac{n}{2} \right) \left( \frac{2}{n} - \frac{2}{n+1} - 4S_1(n) \right) - S'_3 \left( \frac{n}{2} \right) + 8\tilde{S}(n) \tag{B.10}
\]

\[
B^n_{NS} = -\frac{17}{24} - \frac{187}{18} \frac{1}{n} + \frac{187}{18} \frac{1}{n+1} + \frac{17}{6} \frac{1}{n^2} - \frac{5}{6} \left( \frac{1}{n+1} \right)^2 - \frac{1}{n^3} + \frac{1}{(n+1)^3} \\
+ (-1)^n \left\{ \frac{2}{n} - \frac{2}{(n+1)} - \frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{n^3} - \frac{1}{(n+1)^3} \right\} \\
+ \frac{67}{9} S_1(n) + S_2(n) \left( -\frac{11}{3} + \frac{2}{n} - \frac{2}{n+1} - 4S_1(n) \right) \\
+ S'_2 \left( \frac{n}{2} \right) \left( -\frac{1}{n} + \frac{1}{n+1} + 2S_1(n) \right) + \frac{1}{2} S'_3 \left( \frac{n}{2} \right) - 4\tilde{S}(n) \tag{B.11}
\]

\[
D^n_{NS} = \frac{1}{6} + \frac{22}{9} \frac{1}{n} - \frac{22}{9} \frac{1}{n+1} - \frac{2}{3} \frac{1}{n^2} + \frac{2}{3} \frac{1}{(n+1)^2} \\
- \frac{20}{9} S_1(n) + \frac{4}{3} S_2(n) \tag{B.12}
\]

where

\[
S_2(n) = \sum_{j=1}^{n} \frac{1}{j^2}, \quad S_3(n) = \sum_{j=1}^{n} \frac{1}{j^3}, \quad \tilde{S}(n) = \sum_{j=1}^{n} \frac{(-1)^j}{j^2} S_1(j), \tag{B.14}
\]

and

\[
S'_2 \left( \frac{n}{2} \right) = \frac{1 + (-1)^n}{2} S_2 \left( \frac{n}{2} \right) + \frac{1 - (-1)^n}{2} S_2 \left( \frac{n-1}{2} \right) \tag{B.15}
\]

\[
S'_3 \left( \frac{n}{2} \right) = \frac{1 + (-1)^n}{2} S_3 \left( \frac{n}{2} \right) + \frac{1 - (-1)^n}{2} S_3 \left( \frac{n-1}{2} \right). \tag{B.16}
\]
B.2.2 Singlet sector

[1] $\gamma_{\psi\psi}^{}$: 
\[ \gamma_{\psi\psi}^{(1),n} = \gamma_{\psi\psi}^{(1),n} + \gamma_{PS,\psi\psi}^{(1),n} \]  
with 
\[ \gamma_{PS,\psi\psi}^{(1),n} = 8C_F T_f \left\{ -\frac{2}{n} + \frac{2}{n+1} - \frac{2}{n^2} + \frac{6}{(n+1)^2} + \frac{4}{n^3} + \frac{4}{(n+1)^3} \right\} \]  
(B.17)

[2] $\gamma_{\psi G}^{}$: 
\[ \gamma_{\psi G}^{(1),n} = 8C_F T_f D_{\psi G} + 8C_A T_f E_{\psi G} \]  
with 
\[ D_{\psi G}^n = S_1^2(n)\left(\frac{8}{n} - \frac{8}{n+1} - \frac{4}{(n+1)^2}\right) + S_1^2(n)\left(\frac{8}{n} - \frac{8}{n+1} - \frac{4}{(n+1)^2}\right) + S_1^2(n)\left(\frac{8}{n} - \frac{8}{n+1} - \frac{4}{(n+1)^2}\right) + S_1^2(n)\left(\frac{8}{n} - \frac{8}{n+1} - \frac{4}{(n+1)^2}\right) \]  
(B.18)

E_{\psi G}^n = \frac{24}{n} + \frac{22}{n+1} - \frac{24}{n^2} + \frac{24}{(n+1)^2} + \frac{4}{n^3} + \frac{24}{(n+1)^3} \]  
(B.19)

\[ -S_1(n)\left(\frac{2}{n} - \frac{4}{n+1}\right) - S_1'(n)\left(\frac{2}{n} - \frac{4}{n+1}\right) \]  
(B.20)

\[ + S_2(n)\left(\frac{2}{n} - \frac{4}{n+1}\right) - S_2'(n)\left(\frac{2}{n} - \frac{4}{n+1}\right) \]  
(B.21)

[3] $\gamma_{G\psi}^{}$: 
\[ \gamma_{G\psi}^{(1),n} = 8C_F^2 A_{G\psi}^n + 8C_A C_F B_{G\psi}^n + 8C_F T_f D_{G\psi}^n \]  
with 
\[ A_{G\psi}^n = \frac{17}{2} - \frac{4}{n+1} - \frac{2}{n^2} - \frac{1}{2(n+1)^2} - \frac{1}{n^3} - \frac{1}{(n+1)^3} \]  
(B.22)
\[ B^n_{G\psi} = -\frac{41}{9} \frac{1}{n} - \frac{35}{9} \frac{1}{n+1} + \frac{4}{n^2} - \frac{38}{3} \frac{1}{(n+1)^2} - \frac{4}{n^3} - \frac{6}{(n+1)^3} \]
\[ + S_1(n) \left( \frac{10}{3} \frac{1}{n} + \frac{1}{3n+1} + \frac{4}{n^2} \right) - S_1^2(n) \left( \frac{2}{n} - \frac{1}{n+1} \right) \]
\[ - S_2(n) \left( \frac{2}{n} - \frac{1}{n+1} \right) + S_2' \left( \frac{n}{2} \right) \left( \frac{2}{n} - \frac{1}{n+1} \right) \]
\[ + \left[ 1 + (-1)^n \right] \left( \frac{2}{n} - \frac{1}{n+1} \right) \left( 2S_2(n) - S_2' \left( \frac{n}{2} \right) - \zeta(2) \right) \]  
(B.23)
B.2.3 $K^{(1)}_n$

$$K^{1}_{n} = (K^{1,n}_\psi, K^{1,n}_G, K^{1,n}_{NS}) ,$$
(B.30)

with

$$K^{1,n}_\psi = -3 n_f <e^2 > C_F 8 D^n_{\psi G} ,$$
(B.31)

$$K^{1,n}_G = -3 n_f <e^2 > C_F 8 (D^n_{GG} - 1) ,$$
(B.32)

$$K^{1,n}_{NS} = -3 n_f (<e^4 > - <e^2 >^2) C_F 8 D^n_{\psi G} .$$
(B.33)

B.2.4 Anomalous dimensions at $n = 1$ ($\overline{\text{MS}}$ scheme)

$$\gamma^{0,n=1}_{NS} = \gamma^{0,n=1}_{\psi\psi} = 0$$
(B.34)

$$\gamma^{0,n=1}_{\psi G} = 0 ,$$
(B.35)

$$\gamma^{0,n=1}_{G \psi} = -6 C_F$$
(B.36)

$$\gamma^{0,n=1}_{GG} = -\frac{22}{3} C_A + \frac{8}{3} T_f = -2 \beta_0$$
(B.37)

$$K^{0,n=1}_{NS} = K^{0,n=1}_{\psi} = 0$$
(B.38)

$$\gamma^{(1),n=1}_{NS} = 0$$
(B.39)

$$\gamma^{(1),n=1}_{\psi\psi} = 24 C_F T_f$$
(B.40)

$$\gamma^{(1),n=1}_{\psi G} = 0$$
(B.41)

$$\gamma^{(1),n=1}_{G \psi} = 18 C_F^2 - \frac{142}{3} C_A C_F + \frac{8}{3} C_F T_f$$
(B.42)

$$\gamma^{(1),n=1}_{GG} = 8 C_F T_f + \frac{40}{3} C_A T_f - \frac{68}{3} C_A^2 = -2 \beta_1$$
(B.43)

$$K^{(1),n=1}_{\psi} = K^{(1),n=1}_{G} = K^{(1),n=1}_{NS} = 0$$
(B.44)
C Coefficient Functions

$C_n(C_n^\gamma)$ is the coefficient function of the hadronic (photon) operators [17]:

$$C_n(1, g(Q^2)) = \left( \begin{array}{c} \delta_\psi \left( 1 + \frac{g^2(Q^2)}{16\pi^2} B_n^\psi \right) \\ \delta_\psi \frac{g^2(Q^2)}{16\pi^2} B_n^G \\ \delta_{NS} \left( 1 + \frac{g^2(Q^2)}{16\pi^2} B_{NS}^n \right) \end{array} \right)$$  \quad (C.1)

$$C_n(1, g(Q^2), \alpha) = \frac{e^2}{16\pi^2} \delta_\gamma B_n^\gamma$$  \quad (C.2)

and

$$B_n^\gamma = \left( \frac{2}{n_f} \right) B_n^G$$  \quad (C.3)

C.1 \hspace{1em} \textbf{MS scheme} [8, 9]

\begin{align*}
B_n^\psi = B_{NS}^n &= C_F \left[ \left( \frac{2}{n} + \frac{2}{n+1} + 3 \right) S_1(n-1) + 4 \sum_{j=1}^{n-1} \frac{1}{j} S_1(j) - 4 S_2(n-1) \\
& \quad \quad + \frac{6}{n} - 9 \right] \quad (C.4) \\
B_n^G &= 4 T_f \left[ - \frac{n-1}{n(n+1)} (S_1(n) + 1) - \frac{1}{n^2} + \frac{2}{n(n+1)} \right] \quad (C.5)
\end{align*}

C.2 \hspace{1em} \textbf{Momentum subtraction} [27, 28]

\begin{align*}
B_n^\psi = B_{NS}^n &= C_F \left[ -2 - \frac{3}{n} + \frac{8}{n+1} + \frac{4}{n^2} - \frac{4}{(n+1)^2} \\
& \quad \quad + 3 S_1(n) - 8 S_2(n) \right] \quad (C.6) \\
B_n^G &= 8 T_f \left[ \frac{1}{n} - \frac{2}{n+1} - \frac{1}{n^2} + \frac{2}{(n+1)^2} \right] \quad n \geq 3 \\
B_{G=1}^n &= 0 \quad (C.7)
\end{align*}
D Tensor decomposition of virtual photon-photon amplitude

After using parity invariance, time-reversal invariance, and gauge invariance, Brown and Muzinich \cite{20} have shown that there are eight independent tensors, in other words, eight-invariant amplitudes for virtual photon-photon scattering. Those eight independent tensors, which are free from kinematic singularities and kinematic zeros, are given in Eqs.(A3)- (A10) of Ref.\cite{20}.

Using these tensors \((I_i)_{\mu\nu\rho\tau}\), the absorptive part of the forward virtual photon-photon scattering amplitude \(W_{\mu\nu\rho\tau}\) is decomposed as

\[
W_{\mu\nu\rho\tau} = \sum_{i=1}^{8} (I_i)_{\mu\nu\rho\tau} A_i(w, t_1, t_2) \quad (D.1)
\]

where the \(A_i\) are the invariant amplitudes and

\[
w = p \cdot q, \quad t_1 = q^2 = -Q^2, \quad t_2 = p^2 = -P^2 . \quad (D.2)
\]

In order to implement crossing symmetry under \(q \rightarrow -q\) and \(\mu \leftrightarrow \nu\), we form the even combinations, \(I_1, I_2 + I_3, I_4, I_5, I_7 + I_8\), and the odd combinations, \(I_2 - I_3, I_7 - I_8, I_6' = 2I_6 - 3wI_7 - wI_8 + (t_1t_2/w)(I_2 - I_3)\). It is noted that the combinations \(I_2 - I_3\) and \(I_7 - I_8\) are anti-symmetric under the interchange of \(\mu \leftrightarrow \nu\) and \(\rho \leftrightarrow \tau\), while the rest of the combinations are symmetric. In terms of these crossing-even and -odd combinations, the amplitude \(W = \sum_{i=1}^{8} I_i A_i\) is rearranged as follows:

\[
\sum_{i=1}^{8} I_i A_i = I_1 A_1 + \frac{1}{2}(I_2 + I_3)(A_2 + A_3) + I_4 A_4 + I_5 A_5 \\
+ \frac{1}{2}(I_7 + I_8)(A_7 + A_8 + 2wA_6) + \frac{1}{2}I_6' A_6 \\
+ \frac{1}{2}(I_2 - I_3)(A_2 - A_3 - \frac{t_1t_2}{w}A_6) + \frac{1}{2}(I_7 - I_8)(A_7 - A_8 + wA_6) \quad (D.3)
\]

Now \(g_1^{\gamma} = 2W_4^{\gamma}\) is written in terms of invariant amplitudes as

\[
g_1^{\gamma} \propto a_{1111} - a_{1-11-1} \quad (D.4)
\]

\[
g_1^{\gamma} = w^2(A_2 - A_3 - \frac{t_1t_2}{w}A_6) - t_1t_2(A_7 - A_8 + wA_6) \quad (D.5)
\]
which is obtained from Eq.(A14) of Ref.\cite{20}. Here $a_{1111}$ ($a_{1-11-1}$) represents the $s$-channel helicity amplitude for $(+1)\gamma \pm (\pm 1)\gamma \longrightarrow (+1)\gamma \pm (\pm 1)\gamma$. It is noted that $(A_2 - A_3 - (t_1 t_2/w)A_6)$ is the invariant amplitude associated with $(I_2 - I_3)$, while $(A_7 - A_8 + wA_6)$ is associated with $(I_7 - I_8)$. In the limit $t_2 = p^2 = 0$ or in the case that $w = p \cdot q, t_1 = q^2 \gg t_2 = p^2$, the second term $t_1 t_2 (A_7 - A_8 + wA_6)$ does not contribute to $g_1^\gamma$.

In fact we observe that the tensor $(I_2 - I_3) \equiv I_-$ is associated to $g_1^\gamma$ while $(I_7 - I_8) \equiv J_-$ is associated to $g_2^\gamma$. It can be shown that

$$
\epsilon_{\mu\nu\lambda\sigma} q^\lambda \epsilon_{\rho\tau} \sigma^\beta p_\beta = \frac{1}{p \cdot q} I_-, \quad (D.6)
$$

and in the limit of $-q^2, p \cdot q \gg -p^2$

$$
[\epsilon_{\mu\alpha\lambda\sigma} q_\nu q^\alpha - \epsilon_{\nu\alpha\lambda\sigma} q_\mu q^\alpha - \epsilon_{\mu\nu\lambda\sigma} q^2] \epsilon_{\rho\tau} \sigma^\beta p_\beta p^\lambda = J_. \quad (D.7)
$$

Now using an identity

$$
g_{\mu\nu} \epsilon_{\alpha\beta\gamma\delta} = g_{\mu\alpha} \epsilon_{\nu\beta\gamma\delta} + g_{\mu\beta} \epsilon_{\alpha\nu\gamma\delta} + g_{\mu\gamma} \epsilon_{\alpha\beta\nu\delta} + g_{\mu\delta} \epsilon_{\alpha\beta\gamma\nu} \quad (D.8)
$$

we get

$$
\epsilon_{\mu\nu\lambda\sigma} q^\lambda (p \cdot q \epsilon_{\rho\tau} \sigma^\beta p_\beta - \epsilon_{\rho\alpha\beta\delta} p^\sigma q^\alpha) = - \left[\epsilon_{\mu\alpha\lambda\sigma} q_\nu q^\alpha - \epsilon_{\nu\alpha\lambda\sigma} q_\mu q^\alpha - \epsilon_{\mu\nu\lambda\sigma} q^2\right] \epsilon_{\rho\tau} \sigma^\beta p_\beta p^\lambda \quad (D.9)
$$

Hence we have from Eq.(2.3)

$$
W_{\mu\nu\rho\tau} = \frac{1}{(p \cdot q)^2} [(I_-)_{\mu\nu\rho\tau} g_1^\gamma - (J_-)_{\mu\nu\rho\tau} g_2^\gamma]. \quad (D.10)
$$

Finally it is interesting to see the relation between the polarized photon structure functions $g_1^\gamma$ and $g_2^\gamma$ and polarized nucleon structure functions $g_1$ and $g_2$. By introducing the polarization vectors, $\epsilon^{*\rho}$ and $\epsilon^\tau$ for the target photon just like those for the gluon target discussed by Gabrieli and Ridolfi\cite{46}, we have

$$
i W^A_{\mu\nu} = \epsilon^{*\rho} W_{\mu\nu\rho\tau} \epsilon^\tau = W_{\mu\nu\rho\tau} \frac{1}{2} (\epsilon^{*\rho} \epsilon^\tau - \epsilon^{*\tau} \epsilon^\rho) = W_{\mu\nu\rho\tau} \left(-\frac{i}{2\sqrt{|p^2|}} \epsilon^{*\rho\gamma\delta} p_\gamma s_\delta\right). \quad (D.11)
$$
where $s$ is the longitudinal spin vector for the target photon. After using the relation $p \cdot s = 0$, we get

$$ W^{A}_{\mu\nu} = \frac{\sqrt{|p^2|}}{p \cdot q} \left[ \epsilon_{\mu\nu\lambda\sigma} q^\lambda s^\sigma g_1^\gamma + \epsilon_{\mu\nu\lambda\sigma} q^\lambda (p \cdot q s^\sigma - q \cdot s p^\sigma) \frac{g_2^\gamma}{p \cdot q} \right] $$

(D.12)

which, apart from the factor $\sqrt{|p^2|}$, has exactly the same form as Eq.(2.4) of Kodaira et al. [27] which defines the polarized nucleon structure functions $g_1$ and $g_2$, and also as Eq.(9) in Ref. [46].
References

[1] D. Barber, in proceedings of the “Zeuthen Workshop on the Prospects of Spin Physics at HERA”, DESY 95-200, p.76, eds. J. Blömlein and W. D. Nowak.

[2] M. Stratmann and W. Vogelsang, Z. Phys. C74 (1997) 641.

[3] S. D. Bass, Int. J. Mod. Phys. A7 (1992) 6039.

[4] A. V. Efremov and O. V. Teryaev, Phys. Lett. B240 (1990) 200.

[5] S. Narison, G. M. Shore and G. Veneziano, Nucl. Phys. B391 (1993) 69; G. M. Shore and G. Veneziano, Mod. Phys. Lett. A8 (1993) 373; G. M. Shore and G. Veneziano, Nucl. Phys. B381 (1992) 23.

[6] A. Freund and L. M. Sehgal, Phys. Lett. B341 (1994) 90.

[7] S. D. Bass, S. J. Brodsky and I. Schmidt, Phys. Lett. B437 (1998) 424.

[8] R. Mertig and W. L. van Neerven, Z. Phys. C70 (1996) 637.

[9] W. Vogelsang, Phys. Rev. D54 (1996) 2023; Nucl. Phys. B475 (1996) 47.

[10] M. Stratmann and W. Vogelsang, Phys. Lett. B386 (1996) 370.

[11] T. Uematsu and T. F. Walsh, Phys. Lett. 101B (1981) 263.

[12] T. Uematsu and T. F. Walsh, Nucl. Phys. B199 (1982) 93.

[13] G. Rossi, Phys. Rev. D29 (1984) 852.

[14] M. Drees and R. M. Godbole, Phys. Rev. D50 (1994) 3124.

[15] M. Glück, E. Reya and M. Stratmann, Phys. Rev. D51 (1995) 3220; Phys. Rev. D54 (1996) 5515.

[16] P. Mathews and V. Ravindran, Int. J. Mod. Phys A11(1996) 2783.

[17] W. A. Bardeen and A. J. Buras, Phys. Rev. D20 (1979) 166.
[18] D. W. Duke and J. F. Owens, *Phys. Rev* **D22** (1980) 2280.

[19] E. Witten, *Nucl. Phys.* **B120** (1977) 189.

[20] R. W. Brown and I. J. Muzinich, *Phys. Rev.* **D4** (1971) 1496.

[21] M. A. Ahmed and G. G. Ross, *Phys. Lett.* **B59** (1975) 369.

[22] K. Sasaki, *Phys. Rev.* **D22** (1980) 2143; *Prog. Theor. Phys. Suppl.* **77** (1983) 197.

[23] A. V. Manohar, *Phys. Lett.* **B219** (1989) 357.

[24] K. Sasaki and T. Uematsu, “Polarized Virtual Photon Structure Function in the Next-to-leading Order in QCD”, preprint, [hep-ph/9812380](https://arxiv.org/abs/hep-ph/9812380).

[25] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, *Phys. Rev.* **D18** (1978) 3998.

[26] J. Kodaira, S. Matsuda, T. Muta, K. Sasaki and T. Uematsu, *Phys. Rev.* **D20** (1979) 627.

[27] J. Kodaira, S. Matsuda, K. Sasaki and T. Uematsu, *Nucl. Phys.* **B159** (1979) 99.

[28] J. Kodaira, *Nucl. Phys.* **B165** (1980) 129.

[29] G. T. Bodwin and J. Qiu, *Phys. Rev* **D41** (1990) 2755.

[30] E. B. Zijlstra and W. L. van Neerven, *Nucl. Phys.* **B417** (1994) 61.

[31] W. A. Bardeen and A. J. Buras, *Phys. Lett.* **86B** (1979) 61.

[32] A. J. Buras, *Rev. Mod. Phys.* **52** (1980) 199.

[33] M. Glück and E. Reya, *Phys. Rev.* **D25** (1982) 1211.

[34] G. Altarelli, *Phys. Rep.* **81** (1982) 1.
[35] W. Furmanski and R. Petronzio, Z. Phys. C11 (1982) 293.

[36] M. Glück and E. Reya, Phys. Rev. D28 (1983) 2749;
    M. Glück, E. Reya and A. Vogt, Phys. Rev. D45 (1992) 3986.

[37] M. Fontannaz and E. Pilon, Phys. Rev. D45 (1992) 382.

[38] G. Altarelli and G. G. Ross, Phys. Lett. B212 (1988) 39.

[39] R. D. Ball, S. Forte and G. Ridolfi, Phys. Lett. B378 (1996) 255.

[40] G. Altarelli and B. Lampe, Z. Phys. C47 (1990) 315.

[41] S. L. Adler and W. A. Bardeen, Phys. Rev. 182 (1969) 1517.

[42] W. A. Bardeen, Proc. 10-th Int. Symposium on Lepton and Photon Interaction at High Energies, (Bonn, 1981), ed. W. Pfeil (Bonn University, 1981) p.432.

[43] I. Antoniadis and G. Grunberg, Nucl. Phys. B213 (1983) 445.

[44] D. W. Duke, Proc. XIV Int. Symposium on Multiparticle Dynamics, (Lake Tahoe, 1983) eds. P. Yager and J. F. Gunion (World Scientific, 1984) p.124.

[45] M. Glück, M. Stratmann and W. Vogelsang, Phys. Lett. B337 (1994) 373.

[46] A. Gabrieli and G. Ridolfi, Phys. Lett. B417 (1998) 369.
Figure Captions

Fig. 1  Deep inelastic scattering on a polarized virtual photon in polarized $e^+e^-$ collision, $e^+e^- \rightarrow e^+e^- +$ hadrons (quarks and gluons). The arrows indicate the polarizations of the $e^+$, $e^-$ and virtual photons. The mass squared of the “probe” (“target”) photon is $-Q^2$ ($-P^2$) ($\Lambda^2 \ll P^2 \ll Q^2$).

Fig. 2  Forward scattering of a virtual photon with momentum $q$ and another virtual photon with momentum $p$. The Lorentz indices are denoted by $\mu, \nu, \rho, \tau$.

Fig. 3  Polarized virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ to the next-to-leading order (NLO) in units of $(3A_n f < e^4 > /\pi) \ln(Q^2/P^2)$ for $Q^2 = 30$ GeV$^2$, and $P^2 = 1$ GeV$^2$ and the QCD scale parameter $\Lambda = 0.2$ GeV with $n_f = 3$ (solid line). We also plot the leading order (LO) result (long-dashed line), the Box (tree) diagram (2dash-dotted line) and the Box including non-leading contribution, Box (NL) (short-dashed line).

Fig. 4  Virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ for $Q^2 = 100$ GeV$^2$, and $P^2 = 1$ GeV$^2$ with $\Lambda = 0.2$ GeV, $n_f = 3$.

Fig. 5  Virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ for $Q^2 = 30$ GeV$^2$, and $P^2 = 3$ GeV$^2$ with $\Lambda = 0.2$ GeV, $n_f = 3$.

Fig. 6  Point-like piece of the real photon structure function $g_1^\gamma(x, Q^2)$ in NLO for $Q^2 = 30$ GeV$^2$ with $\Lambda = 0.2$ GeV, $n_f = 3$ (solid line). Also plotted are the LO result (long-dashed line) and the Box (tree) diagram contribution (short-dashed line).
\[ e^-(e^+) \quad \Rightarrow \quad (\Leftarrow) \]

\[ q^2 = -Q^2 < 0 \]

\[ p^2 = -P^2 < 0 \quad \text{“target”} \]

\[ \Rightarrow \quad (\Leftarrow) \]

\[ \text{“probe”} \]

\[ \text{Fig. 1} \]
\[ g_1^\gamma(x, Q^2, P^2) <e^4_{\pi}> \ln \frac{Q^2}{P^2} \]

Fig. 3

\[ Q^2 = 30 \text{ GeV}^2 \]
\[ P^2 = 1 \text{ GeV}^2 \]
\[ n_f = 3 \]
\[ g_1^\gamma(x, Q^2, P^2)/3n_f <e^4>_{\pi} \ln Q^2/P^2 \]

\[ Q^2 = 100 \text{ GeV}^2 \]
\[ P^2 = 1 \text{ GeV}^2 \]
\[ n_f = 3 \]

Fig. 4
$Q^2 = 30 \text{ GeV}^2$

$P^2 = 3 \text{ GeV}^2$

$n_f = 3$

Fig. 5
\[ Q^2 = 30 \text{ GeV}^2 \quad n_f = 3 \]

\[ g_1^r(x, Q^2) / 3n_f \langle e^4 \rangle \ln Q^2 / \Lambda^2 \]

- Real Photon
- \(\text{Box}\)
- \(\text{LO}\)
- \(\text{NLO}\)

Fig. 6