Lipatov’s QCD high energy effective action: past and future

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Abstract

In this contribution we briefly review some aspects of Lipatov’s gauge invariant QCD high energy effective action. The high energy effective provides a field theory framework to systematically factorize QCD scattering amplitudes and related theories in the limit of high center of mass energies. After a short review of the underlying concepts, we address the question how the high energy effective action can be used for actual calculations. As an explicit example we review the derivation of the gluon Regge trajectory up to 2 loop. We then review two topics of current interest: automatic amplitude generation of high energy effective action amplitudes and a discussion of the relation between the so-called Color-Glass-Condensate framework and the high energy effective action.

1 Introduction

Effective field theories are highly successful tools to study dedicated kinematic limits of quantum field theories. This is in particular true for the study of Quantum Chromodynamics (QCD), the fundamental theory of strong interactions. In 1995, Lev Lipatov formulated a gauge invariant high energy effective action [1, 2], which provides the effective field theory description of QCD in the high energy or Regge limit. This effective action is an effective field theory which describes the interaction between conventional QCD degrees of freedom, i.e. quarks, gluons and ghost, and a newly introduced degree of freedom, the reggeized gluon. The formulation of the high energy effective action by Lev Lipatov is not the only effective field theory description of the QCD high energy limit: most notably one should mention first attempts by Verlinde and Verlinde [3], see also [4] as well as an earlier version of an high energy effective action by Lev Lipatov [5] and collaborators [6, 7]. Another important framework, which has been developed around the same time as Lipatov’s high energy effective action, is provided by the Color Glass Condensate effective theory and its ensuing calculational framework. We will comment on recent results concerning the relation of both frameworks further down in the text.

The benefits of such a effective field theory framework are manifold: first and most important they provided a field theory definition of the underlying degrees of the freedom in the $t$-channel of high energy factorized QCD amplitudes: the reggeized gluon. While the
The Multi-Regge-Kinematics

bare reggeized gluon fields is introduced as a new auxiliary degree in the formulation of the high energy effective action, reggeization of the gluon, i.e. non-zero Regge trajectory arises within the high energy effective action as a direct consequence of the calculation of perturbative higher order corrections to the propagator of this new “auxiliary” degree of freedom. Introducing this degree of freedom is therefore crucial for the formulation of the high energy effective action. By definition this new degree of freedom is invariant under local gauge transformation which in turn essential to achieve gauge invariant factorization of scattering amplitudes in the high energy limit.

The outline of this short review is as follows: in section Sec. 2 we briefly review the formulation of the high energy effective action. In Sec. 3 we review essential ingredients for the use of the high energy effective action in actual calculations. In Sec. 4 we give a short overview of the determination of the 2-loop gluon trajectory. Sec. 5 provides a short introduction on amplitude generation for the high energy effective action, while in Sec. 6 we address the high energy effective action in the presence of a dense reggeized gluon field and its relation to the framework of the Color Glass Condensate effective theory. In Sec. 7 we summarize our results and most importantly highlight those results which we could not cover in this short review. In particular we would like to note that the topics covered in this review are highly biased by the own work of the author. This is mainly due to the limitations of the author and implies by no means that work not mentioned would be of less relevance.

2 The high energy effective action: derivation

The high energy effective action aims at a description of scattering amplitudes which exhibit regions where plus and minus momenta are strongly ordered. It is instructive to consider the example of an explicit scattering process $p_A + p_B \rightarrow X$ in the high energy limit (‘any mass scale’) $2/s \rightarrow 0$ with $s = (p_A + p_B)^2$ the center of mass energy squared. In this limit, the
momenta of scattering particles $p_{A,B}$ are close to the opposite sides of the light cone. It is therefore natural to define dimensionless light-cone vectors associated with these momenta. If $p_{A,B}$ are already light-like, these are simply obtained through the following re-scaling

$$n_{\pm} = \frac{2}{\sqrt{s}} p_{B,A}, \quad n_+ \cdot n_- = 2. \quad (1)$$

Plus- and minus components of momenta (and general four vectors) are defined through

$$k_{\pm} = n_{\pm} \cdot k = n_+ \cdot k = k_\pm. \quad (2)$$

This implies the following Sudakov decomposition of a four vector,

$$k = k_+ \frac{n_-}{2} + k_- \frac{n_+}{2} + k_0 = k_+ \frac{n_-}{2} + k_- \frac{n_+}{2} + k, \quad (3)$$

where $k$ denotes the momentum perpendicular to the 2 light-cone directions with $k^2 > 0$. For derivatives the following relations hold:

$$\partial_{\pm} x^\pm = 2 \quad \partial_{\mp} x^\mp = 0. \quad (4)$$

The kinematic limit of interest is then defined as follows: a genuine scattering process is organized into several sub-sectors of particles which are close, i.e. local, in rapidity $\eta = \frac{1}{2} \ln \frac{k^+}{k^-}$. These sectors are strongly ordered with respect to each other in both plus and minus momenta (and therefore in rapidity). The situation is most easily illustrated using the tree diagram depicted in Fig. 1: the momenta of the each of the sectors $k_i$ with $i = 0, \ldots, n$ obey the conditions $k_0^+ \gg k_1^+ \gg \ldots k_n^+$ and $k_0^- \ll k_1^- \ll \ldots k_n^-$. As a side remark we note that for real particle production, this condition corresponds to the well known Multi-Regge-Kinematics i.e. strong ordering in rapidity, while all transverse momenta are of the same order of magnitude.

### 2.1 The high energy effective action for gluons

Following closely [1, 2], the essential idea is to decompose the complete gluonic field $V_\mu$ field into a sum of fields $v_\mu^{(s)}$ and $A_\mu^{(s)}$, where the upper index ‘$s$’ denotes a certain sector $s$ which is local in rapidity, i.e. a sector which has no regions with strong ordering in plus and minus momenta. The index ‘$s$’ indicates that the fields $v_\mu^{(s)}$ is restricted to such a sector and for each such sector, there exists an individual field $v_\mu^{(s)}$. The fields $A_\mu^{(s)}$ differ from these fields: while the index ‘$s$’ indicates that these fields also couple to a certain rapidity sector, these fields never occur inside a certain sector. They rather transmit the interaction between different sectors. They are “non-local in rapidity”. They generally appear in the $t$-channels of scattering processes and do not appear as asymptotic states. Since the fields $A_\mu^{(s)}$ connect to sectors which are strongly ordered in plus and minus momenta with respect to typical momenta in the sector $s$, their coordinate dependence is strongly boosted with respect to the regarding light-cone direction. As a consequence, only certain components of the the fields $A_\mu^{(s)}$ are of relevance. In particular,

$$A^{(s)}(x) = A_+^{(s)}(x) \frac{n_+^\mu}{2} + A_-^{(s)}(x) \frac{n_-^\mu}{2}, \quad (5)$$
while the dependence of the fields on light-cone coordinates is frozen:

$$\partial_+ A^{(s)}_\mu (x) = 0 = \partial_- A^{(s)}_\mu .$$  \hspace{1cm} (6)

In the momentum representations, to which we will turn soon, this encodes strong ordering of different sectors (which the $A_\pm$ fields connect) with respect to their plus and minus momenta. As we will see below, the field $A_\mu$ directly yields the so-called reggeized gluon, which has been identified by Lipatov and collaborators as the fundamental degree of freedom in the $t$-channel of high energy factorized scattering amplitudes. Although the fields $A_\mu$ are initial not reggeized, i.e. they have no non-zero Regge trajectory, reggeization of this fields will result from the resummation of radiative corrections. Following [1, 2], we therefore will generally refer to the fields $A_\mu$ as “reggeized gluon” fields.

The essential goal is now to construct the effective action which describes the local interaction, i.e. inside the regarding sectors, between $v^{(s)}$ and $A^{(s)}_\pm$ fields and $v^{(s)}$ fields themselves. The interactions among the $v^{(s)}_\mu$ fields is described by the conventional QCD Lagrangian, with the implicit constraint that the resulting QCD dynamics must be local in rapidity. In addition to the QCD Lagrangian, one further requires a term which describes the interaction among $v^{(s)}_\mu$ and $A^{(s)}_\mu$ fields. The resulting Lagrangian of the effective theory is therefore obtained as the sum of the conventional QCD Lagrangian and a so-called induced Lagrangian $\mathcal{L}_{\text{ind.}}$, which couples the gluon field to the $A_\mu$ field:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}}[v^{(s)}_\mu, \bar{\psi}^{(s)}, \psi^{(s)}] + \mathcal{L}_{\text{ind.}}[v^{(s)}_\mu, A^{(s)}_\pm, A^{(s)}_\mp].$$  \hspace{1cm} (7)

Here the QCD Lagrangian is as usually given by

$$\mathcal{L}_{\text{QCD}}[v^{(s)}_\mu, \bar{\psi}^{(s)}, \psi^{(s)}] = \frac{1}{2} \text{tr} [G_{\mu\nu} G^{\mu\nu}] + \sum_{f=1}^{n_f} \bar{\psi}_f (i\tilde{D} - m_f) \psi_f .$$  \hspace{1cm} (8)

In the effective Lagrangian, vectors fields are generally understood as anti-hermitian matrices in the fundamental representation of SU($N_c$), i.e. $A^{\pm}(x) = -it^a A^a_\pm(x)$ and $v_\mu = -it^a v_\mu(x)$ with $t^a$ the SU($N_c$) generators in the fundamental representation $\text{tr}(t^a t^b) = \delta^{ab}/2$; the trace is therefore in the fundamental representation. Furthermore

$$D_\mu = \partial_\mu + g v_\mu, \quad G_{\mu\nu} = \frac{1}{g} [D_\mu, D_\nu] .$$  \hspace{1cm} (9)

From now on we will drop the index ‘$s$’ and implicitly assume that the fields of the effective action are limited to a certain sector. As a first naïve attempt to construct the induced Lagrangian, one might start with the term

$$\text{tr} [2v^\mu \partial^2 A_\mu] = \text{tr} [v_+ \partial^2 A_- + v_- \partial^2 A_+] ,$$  \hspace{1cm} (10)

which provides the simplest coupling between the 2 fields. The differential operator ‘$\partial^2$’ has been introduced to ensure that the field $A_\mu$ cannot appear as an asymptotic state: whenever the momentum of the $A_\mu$ is the on the mass-shell, $\partial^2 = 0$, the field will decouple from QCD dynamics. To obtain the correct normalization for the propagator of the $A_\mu$-field, one further introduces an corrective term such that the coupling between both fields reads

$$\text{tr} [(v_+ - A_+) \partial^2 A_- + (v_- - A_-) \partial^2 A_+] .$$  \hspace{1cm} (11)
The above term is sufficient to construct the Lagrangian of the effective theory, as long as we ignore questions related to gauge invariance: the term induces a transition between the conventional gluon field \( v_\mu \) inside a certain sector and the field \( A_\mu \), which connects sectors strongly ordered in rapidity with respect to each other. The strong ordering condition is imposed on the field \( A_\mu \) by both polarization and the kinematic constraint Eq. (6). For a gauge theory, the above proposal comes with an essential deficit: factorization in the high energy limit of a gauge independent quantity, i.e. a scattering amplitude, must happen in an gauge independent way. To achieve this, Lipatov proposed to define the reggeized gluon fields \( A_\mu \) to be invariant under local gauge transformations

\[
\delta_L A_\pm = [A_\pm, \chi_L] = 0, \tag{12}
\]

with \( \chi_L \) the parameter of local gauge transformation which decreases for \( x \to \infty \). QCD fields are on the other hand as usual subject to such local gauge transformations,

\[
\delta_L v_\pm = [D_\pm, \chi], \quad \delta_L \psi = -\chi, \quad D_\mu = \partial_\mu + g v_\mu. \tag{13}
\]

As a consequence, the proposal Eq. (11) for the coupling between the \( v^\mu \) and the \( A^\mu \) fields violates local gauge invariance. The gauge invariant high energy effective action proposed by Lipatov solves this problems by adding the minimal necessary terms to re-establish gauge invariance, leading to the following induced Lagrangian:

\[
\mathcal{L}_{\text{ind}}[v^\mu, A_+, A_-] = \left\{ \text{tr} \left[ (T_- [v(x)] - A_- (x)) \partial_+^2 A_+(x) \right] \right. \\
+ \left. \text{tr} \left[ (T_+ [v(x)] - A_+ (x)) \partial_-^2 A_-(x) \right] \right\}. \tag{14}
\]

The functionals \( T_\pm [v] \) can be obtained from the following operator definition

\[
T_\pm [v] = -\frac{1}{g} \frac{\partial_\pm}{1 + \frac{g}{\partial_\pm} v_\pm} = v_\pm - g v_\pm \frac{1}{\partial_\pm} v_\pm + g^2 v_\pm \frac{1}{\partial_\pm} v_\pm + \ldots, \tag{15}
\]

where the integral operator \( 1/\partial_\pm \) is implied to act on a unit constant matrix from the left. Boundary conditions of the \( 1/\partial_\pm \) are fixed through

\[
\frac{1}{1 + \frac{g}{\partial_\pm} v_\pm} = U[v_\pm (x)] = \mathcal{P} \exp \left( -\frac{g}{2} \int_{-\infty}^{x_\pm} dx' v_\pm (x') \right) \\
= 1 - \frac{g}{2} \int_{-\infty}^{x_\pm} dx' v_\pm (x') + \frac{g^2}{4} \int_{-\infty}^{x_\pm} \int_{-\infty}^{x_\pm} dx'' v_\pm (x') v_\pm (x'') + \ldots. \tag{16}
\]

The functional \( T[v_\pm] \) therefore generalizes the naïve coupling between \( A_\mu \) and \( v_\mu \) fields. In particular

\[
\delta_L \left( \text{tr} \left[ T[v_\pm] \partial^2 A_\pm \right] \right) = 0, \tag{17}
\]

and the induced Lagrangian Eq. (14) is therefore invariant under local gauge transformations.
2.2 The high energy effective action for quarks

Following the same ideas, one can write down the high energy effective action for quarks, meant to describe the high energy limit of processes with flavor exchange \[^8\]. The degree of freedom, \(Q_{\pm}\), which is used to factorize amplitudes in the high energy limit, is in this case not bosonic, but fermionic. In correspondence to Eq. (5) this field satisfies:

\[
\begin{align*}
\bar{\psi}_{\pm} Q_{\mp} &= 0, \\
\bar{Q}_{\mp} \psi_{\pm} &= 0,
\end{align*}
\]

while the dependence on light-cone coordinates is frozen,

\[
\partial_{\pm} Q_{\mp}(x) = 0 = \partial_{\pm} \bar{Q}_{\mp}(x) = 0,
\]

in accordance with Eq. (6). The complete effective Lagrangian is given by

\[
L_{\text{eff}} = L_{\text{QCD}}[v_{\mu}, \psi, \bar{\psi}] + \sum_{f=1}^{n_f} L_{\text{ind.}}^{(Q_f)}[\psi, \bar{\psi}, v_{\mu}, Q_{\pm}, \bar{Q}_{\pm}],
\]

where the induced Lagrangian \(L_{\text{ind.}}^{(Q)}\) describes again the coupling of the \(Q_{\pm}\) and \(\bar{Q}_{\pm}\) fields to QCD fields:

\[
\begin{align*}
L_{\text{ind.}}^{(Q_f)}[\psi, \bar{\psi}, v_{\mu}, Q_{\pm}, \bar{Q}_{\pm}] &= Q_{\pm}^f(x) \left( i\partial - m_f \right) \left[ U[v_+(x)] \bar{Q}_{\mp}^f(x) \psi_f(x) \right] \\
&+ \left[ \bar{Q}_{\pm}^f(x) - \bar{\psi}(x) U[v_+(x)] \right] \left( i\partial - m_f \right) Q_{\mp}^f(x) + \{" + " \leftrightarrow " - " \};
\end{align*}
\]

for the definition of the Wilson line operator \(U[v_{\pm}(x)]\), see Eq. (16). At leading order in perturbation theory, the Wilson line operator \(U[v_{\pm}(x)]\) is equal to unity and one finds a direct coupling between conventional quark fields \(\psi\) and reggeized quark fields \(Q_{\pm}\). The operator \(i\partial - m_f\) ensures again vanishing of the coupling for asymptotic states. Considering the all order Wilson line operator, the induced Lagrangian turns invariant under local gauge transformations, Eq. (13).

3 Working with the high energy effective action

The determination of scattering amplitudes requires at first the determination of Feynman rules, which has been pioneered in \[^9\]. The Feynman rules of the effective action Eq. (7) comprise in addition to the usual QCD Feynman rules, the propagator of the reggeized gluon and an infinite number of so-called induced vertices. The leading order induced vertices and propagators are summarized in Fig. 2. In general it is possible to avoid the presence of a direct transition vertex between gluon and reggeized gluon field. This is most easily achieved through a shift \(v_{\pm} \rightarrow v_{\pm} + A_{\pm}\) in the effective action which removes this transition vertex and gives a direct coupling of the reggeized gluon field as one of the external fields of conventional QCD vertices. We will comment on this option further below. Working with the direct transition vertex, the reggeized gluon propagator receives also a correction from such a direct transition, which is already included in the above reggeized gluon propagator. A similar set of Feynman rules can be derived for the quark high energy effective action. In the following we will focus on the high energy effective action for reggeized gluon exchange, while the bulk...
\[ q, a, \pm \]

\[ k, c, \nu \]

\[ q, a, \pm \]

\[ k, c, \nu \]

\[ \delta(\pm) = -\frac{i}{2} q^2 \delta_{ac} (n^\pm)^\nu, \quad \delta_{ab} = \frac{2 f_{c1 c2 a} q^2}{k_1^2} (n^\pm)^\nu_1 (n^\pm)^\nu_2, \]

\[ k_1^\pm + k_2^\pm = 0 \]

(a) (b) (c)

(d)

Figure 2: Feynman rules for the lowest-order effective vertices of the effective action. Wavy lines denote reggeized fields and curly lines gluons. Note that in comparison with the Feynman rules used in [10–13] we absorbed a factor \( \frac{1}{2} \) into the vertices which is compensated by changing the residue of the reggeized gluon propagator from \( \frac{1}{2} \) to 2.

of observations equally applies to the high energy effective action for reggeized quarks. For a very nice recent treatment of the high energy effective action for quarks see [14,15].

To start we note that the set of Feynman rules arising from the high energy effective actions Eqs. (7) and (20) cannot be used in a straightforward manner. This is immediately clear from the particular form of the effective action: They contain both the complete QCD Lagrangian and on top of it the coupling the auxiliary fields \( A_{\pm} \). The solution to this problem is naturally resolved through the crucial observation that all the fields inside the high energy effective action are to be understood as to be restricted to a certain region local in rapidity: No strong ordering in light-cone momenta occurs within such a sector; only the entire set of light-cone momenta of a certain sector is strongly ordered with respect to other sectors. Interaction between those sectors takes place through reggeized gluon exchange. A closely related problem arises due to the manifestation of so-called rapidity divergences in certain correlators, which are not regularized by standard methods such as e.g. dimensional regularization. Last but not least it is necessary to provide a suitable pole-description to the light-cone denominators, which appear in the induced vertices Fig. 2 to define properly integrations over loop momenta. We will shortly review these problems in the following and discuss commonly solutions to such problems employed in the literature.

3.1 Locality in rapidity and subtraction

The problem is most easily illustrated using a particular example: We consider the additional emission of a gluon in the the interaction of a quark with a reggeized gluon. For full details and explicit expressions we refer to [10,16]. The scattering amplitude can be obtained from
Figure 3: effective action diagrams for the process quark + reggeized gluon → quark + gluon

Figure 4: The central gluon production vertex, i.e. the Lipatov vertex, in terms of effective action diagrams

the high energy effective action through evaluating the sets of diagrams, shown in Fig. 3. The resulting scattering amplitude does not vanish in the strongly ordered region $p^+ \gg q^+$; it is therefore non-local in rapidity. Apart from the above matrix element Fig. 3, the effective action allows also for the construction of a scattering amplitude for the same process, which combines two different sectors, connected through an internal reggeized gluon line: the 2 sectors are in this case given by the coupling of a quark to a reggeized gluon and the emissions of a gluon from a reggeized gluon, the famous gluon production or Lipatov vertex, see [9] for a derivation of this vertex from the effective action. Analyzing the explicit expressions corresponding to Fig. 3, it is evident that Fig. 3 turns into Fig. 5 in the strongly ordered region $p^+ \gg q^+$, where gluon and quark are widely separated in rapidity. In this particular example, it would be straightforward to include an explicit cut-off on the final state that separates Fig. 3 from Fig. 4. With increasing complexity of the final states and/or the case of higher-loop corrections, this becomes rather cumbersome. It is therefore more suitable, to simple subtract diagrams with an internal reggeized gluon line from the regarding correlator, removing in this way the contribution which causes the non-locality in rapidity. For the above example one finds that the combination

$$\begin{align*}
\text{\includegraphics[width=0.2\textwidth]{diagram.png}} &= \text{\includegraphics[width=0.2\textwidth]{diagram.png}} - \text{\includegraphics[width=0.2\textwidth]{diagram.png}}
\end{align*}$$

is local in rapidity, in the sense that the resulting expression is not growing in the strongly ordered region $p^+ \gg q^+$ but instead assumes a constant value.

In general, the following set of rules can be formulated for the determination of a (sub-) amplitude, local in rapidity within the effective action, using perturbation theory:

a) determine the correlator of interest from the high energy effective action, to the desired
order in perturbation theory, with the reggeized gluon field treated as a background field

b) subtract disconnected contributions which contain internal reggeized gluon lines (which are already themselves subtracted) to the desired order in perturbation theory.

A similar situation arises for diagrams with multiple reggeized gluon exchange, such as Fig. 5, first line. In the limit where the squared center-of-mass energy $s = (p_a + k_1)^2$ tends to infinity,

\[
c_1, k_1 \quad c_2, k_2 = 1 + \quad +
\]

\[
c_1, k_1 \quad c_2, k_2 = -
\]

**Figure 5:** First line: Amplitude for 2 reggeized gluons coupling to a quark. Second line: Amplitude with the factorized contribution subtracted.

the contribution diverges logarithmically $1/k_1^\pm$. Similar to the previous example, the high energy limit of this correlator is captured by the corresponding diagram with an internal reggeized gluon exchange, such that the subtracted contribution, Fig. 5, second line, is local in rapidity in the sense that the expression vanishes up to terms $O(1/(k_1^\pm)^2)$ in the high energy limit [17], see also the discussion of a related observation in [18].

### 3.2 Pole prescription for induced vertices

At first sight there is no need to choose a particular pole prescription: the poles in $1/\partial_\pm$ arise from propagators of particles with significantly different rapidity, which emit gluons into the rapidity cluster under consideration. The corresponding particle virtualities $k_\pm$ are therefore large. Based on this argument, the pole prescription of these denominators can be ignored and one arrives directly at the induced vertices Fig. 2. Most notably, the color structure of these vertices is given by structure constants $f^{abc}$ only, which arise from commutators of SU($N_c$) generators of the fields $v(x) = -it^av^a(x)$ in the functional $T_{\pm}(v_{\pm})$. Since the color structure of induced vertices arises from commutators of SU($N_c$) generators, the color structure is independent of the representation of the fields $v(x)$. The resulting induced vertices are therefore universal. They do not depend on the color representation of the specific particle from which the gluons have been emitted.

Even though the overall result must be independent of the pole prescription due to the above arguments, it is nevertheless necessary to choose a certain prescription in actual calculations. To have consistency for the evaluation of different elements (which eventually one needs to combine) it is highly recommendable to do so at the level of the effective Lagrangian or equivalently the resulting Feynman rules. At first sight the problem appears to be well defined: Requiring convergence of the path ordered exponential Eq. (13) at $-\infty$ naturally defines a pole prescription of the light-cone denominator: it induces the replacement $\partial_\pm \to \partial_\pm + \epsilon$. There are various reasons why this is not the generally accepted treatment.
At the very first, using such a pole prescription, the effective Lagrangian is no longer hermitian and unitary of theory is formally lost (which itself is an essential requirement for any effective field theory description to study QCD high energy limit). In practice this problem can be overcome by replacing the path ordered exponential Eq. (13) by the combination

$$U[v_\pm(x)] \to \frac{1}{2} \left( U[v_\pm(x)] + U^\dagger[v_\pm(x)] \right).$$

(23)

A similar prescription has been adapted in the original formulation of the quark effective action [8]; for a related discussion in the case of the effective action for gluons see [14]. For the induced vertex Fig. 2.c, such a prescription leads to a principal value prescription, i.e.

$$\frac{1}{k^\pm_1} \to \frac{1}{2} \left( \frac{1}{k^+_1 + i\epsilon} + \frac{1}{k^-_1 - i\epsilon} \right) = \frac{k^\pm_1}{(k^\pm_1)^2 + \epsilon^2}. \quad (24)$$

Such a prescription is not only attractive from the point of view of hermiticity of the high energy effective action, it further provides vertices which are odd under $k^\pm_1 \to -k^\pm_1$ – a property in accordance with negative signature of reggeized gluon (which eventually arises from the reggeized gluon fields). While this pole prescription provides a simple and straightforward definition of the pole prescription and appears suitable for the use in calculations, it gives rise to certain symmetric color structures which cannot be expressed as commutators (and therefore enhances the color structure of the induced vertices in Fig. 2). Since symmetric color tensors depend in general on the representation of the SU(Nc) generators, universality of these vertices is therefore lost. To avoid this problem, in [20] a perturbative prescription has been developed which projects out order by order in perturbation theory symmetric color structures. The resulting pole prescription respects then Bose symmetry and universality of the induced vertices. For the $O(g)$ vertex, see Fig. 2.c, this corresponds to replacing the denominator $1/k^\pm_1$ by a Cauchy principal value. For the $O(g^2)$ vertex, see Fig. 2.c, the combinations of denominators $1/(k^+_3 k^+_1)$ and $1/(k^+_3 k^-_2)$ are replaced by functions $g^\pm_2(3, 2, 1)$ and $g^\pm_2(3, 1, 2)$ respectively with

$$g^\pm_2(i, j, m) = \left[ \frac{1}{[k^+_i][k^-_m]} + \frac{\pi^2}{3} \delta(k^+_i) \delta(k^-_m) \right], \quad \frac{1}{|k|} = \frac{1}{2} \left( \frac{1}{k + i0} + \frac{1}{k - i0} \right). \quad (25)$$

Similar functions can be constructed for the higher order induced vertices. One of the main advantages over simply defining all denominators as principal values is that the function $g_2$ (and corresponding higher order versions) satisfy eikonal identities in the sense of the theory of distributions i.e.

$$g^\pm_2(1, 2, 3) + g^\pm_2(2, 3, 1) + g^\pm_2(3, 1, 2) = 0. \quad (26)$$

Finally we would like to highlight that Lipatov himself proposed a certain pole-prescription in [19]

$$T_\pm[v_\pm] = \frac{1}{2} \mathcal{P} e^{-\frac{g}{4} \int_{-\infty}^{+\infty} dx^+ x^+ v_\pm} \left[ \mathcal{P} e^{\frac{g}{4} \int_{-\infty}^{+\infty} dx^+ x^+ v_\pm} + \mathcal{P} e^{-\frac{g}{4} \int_{-\infty}^{+\infty} dx^+ x^+ v_\pm} \right], \quad (27)$$

1Applying a consistent subtraction mechanism, the dependence on such color tensors must drop in the final result, i.e. on the level of the observable. It should be therefore possible to carry out actual higher order calculations within this prescription.
where $\mathcal{P}$ ($\bar{\mathcal{P}}$) denotes (anti-)path ordering. To the best of our knowledge the vertices resulting from this particular pole prescription have not been explored so far.

### 3.3 Regularization of high energy singularities

Last but not least it is necessary to address the problem that higher order corrections lead to so-called rapidity divergences which are not regularized by conventional methods, such as dimensional regularization. A very nice review of different regulators which can be used can be found in [15]. A very convenient choice is given by tilting the light-cone direction of the effective action slightly against the light cone through introducing a regulator $\rho$, evaluated in the limit $\rho \to \infty$

$$
\begin{align*}
  n^- &\to n_a = e^{-\rho} n^+ + n^- , \\
  n^+ &\to n_b = n^+ + e^{-\rho} n^- .
\end{align*}
$$

(28)

This regulator has the important advantage over other regulators (such as regulation by a hard cut-off or analytic regulations) since it respects gauge invariance properties of the high energy effective action.

### 4 The gluon Regge trajectory and the effective action

As an example of an explicit higher order calculations which combines all of the above aspects, we review in the following the determination of the gluon Regge trajectory from the high energy effective action, see [12,13] for full details. This calculation requires

- determination of the propagator of the reggeized gluon to the desired order in perturbation theory;
- a transition function for its rapidity divergences.

As a start it is necessary to determine the self-energies of the reggeized gluon. Within the subtraction procedure Sec. 3.1, this requires

(a) determination of the self-energy of the reggeized gluon from the effective action, with the reggeized gluon treated as a background field;

(b) subtraction of all disconnected contributions which contain internal reggeized gluon lines.

Using the pole prescription of [20], at 1-loop, all diagrams with internal reggeized gluon lines vanish, and no subtraction occurs; the contributing diagrams are

$$
\Sigma^{(1)} \left( \rho; \epsilon, \frac{q^2}{\mu^2} \right) = \text{1 loop} = \text{1 loop} + \text{1 loop} + \text{1 loop} + \text{1 loop} + \text{1 loop} .
$$

(29)

The first of the above diagrams is divergent and requires regularization. To achieve a consistent treatment, we tilt the light-cone direction for all diagrams and take the limit $\rho \to \infty$ after
the evaluation of integrals. Keeping only finite and divergent terms, one finds in \( d = 4 + 2\epsilon \) dimensions\(^2\):

\[
\frac{\Sigma^{(1)}(\rho; \epsilon, \frac{q^2}{\mu^2})}{(-2i\mathbf{q}^2)} = g^2 c_\Gamma \left( \frac{q^2}{\mu^2} \right)^\epsilon \left\{ \frac{i\pi - 2\rho}{\epsilon} - \frac{5 + 3\epsilon - \frac{n_f}{N_c}(2 + 2\epsilon)}{(1 + 2\epsilon)(3 + 2\epsilon)\epsilon} \right\},
\]

(30)

and

\[
g^2 = g^2 N_c \frac{\Gamma(1 - \epsilon)}{(4\pi)^{2+\epsilon}}, \quad c_\Gamma = \frac{\Gamma^2(1 + \epsilon)}{\Gamma(1 + 2\epsilon)}.
\]

(31)

To determine the 2-loop self energy it is now needed to subtract disconnected diagrams, whereas diagrams with multiple internal reggeized gluons can be shown to yield zero result. Schematically one has

\[
\Sigma^{(2)}(\rho; \epsilon, \frac{q^2}{\mu^2}) = \text{2 loop} - \text{1 loop}.
\]

(32)

The black blob denotes the unsubtracted 2-loop reggeized gluon self-energy. It is obtained through the direct application of the Feynman rules of the effective action (with the reggeized gluon itself treated as a background field). The resulting 2-loop integrals have been determined in [12,13]. The result for \( n_f \) flavor reads

\[
\Sigma^{(2)}(\rho; \epsilon, \frac{q^2}{\mu^2}) = (-2i\mathbf{q}^2) g^4 N_c^2 \left\{ - \left[ \frac{2}{\epsilon^2} + \frac{4(1 - \tau)}{\epsilon} + 4(1 - \tau)^2 - \frac{\pi^2}{3} \right] \rho^2 \\
+ \left[ \frac{1}{3\epsilon^2} + \frac{1}{9\epsilon} + \frac{\pi^2}{3\epsilon} - \frac{2\tau}{3\epsilon} + \frac{\pi^2(11 - 12\tau)}{18} + \frac{16}{27} - \frac{2}{9} \tau^2 + \frac{2}{3} \tau^2 - 2\zeta(3) \right] \rho \right\} \\
+ \frac{n_f}{N_c} \left( \frac{2}{3\epsilon} + \frac{n_f(6 - 36\tau)}{27\epsilon} + \frac{32 - 3\pi^2 - 12\tau + 36\tau^2}{27} \right) + \mathcal{O}(\epsilon) + \mathcal{O}(\rho^0).
\]

(33)

with \( \tau = 1 - \ln \frac{q^2 e^{\gamma_E}}{4\pi\mu^2} \). To obtain the gluon trajectory, we need to construct next the (bare) two-loop reggeized gluon propagators

\[
G(\rho; \epsilon, \frac{q^2}{\mu^2}) = \frac{i/2}{q^2} \left\{ 1 + \frac{i/2}{q^2} \Sigma(\rho; \epsilon, \frac{q^2}{\mu^2}) + \left[ \frac{i/2}{q^2} \Sigma(\rho; \epsilon, \frac{q^2}{\mu^2}) \right]^2 + \ldots \right\},
\]

(34)

with the reggeized gluon self energy

\[
\Sigma(\rho; \epsilon, \frac{q^2}{\mu^2}) = \Sigma^{(1)}(\rho; \epsilon, \frac{q^2}{\mu^2}) + \Sigma^{(2)}(\rho; \epsilon, \frac{q^2}{\mu^2}) + \ldots
\]

(35)

Apparently Eq. (29) is divergent in the limit \( \rho \to \infty \). In [10,11] it has been demonstrated by explicit calculations that these divergences cancel at one-loop level against divergences in the

\(^2\)In the original result present in [10] and reproduced in [16], a finite result for the second and third diagram has been erroneously included, see also [11–13]
couplings of the reggeized gluon to external particles. The entire one-loop amplitude is then found to be free of any high energy singularity in \( \rho \). Consistency of high energy factorization as formulated within the high energy effective action requires that such a cancellation holds also beyond one loop. To make this cancellation explicit, we introduce transition functions \( Z^{\pm} \).

For explicit examples we refer the reader to [10,11]. In particular we define the renormalized reggeized gluon propagator as

\[
G^R(\eta; \epsilon, q^2, \mu^2) = \frac{G(\rho; \epsilon, q^2, \mu^2)}{Z^+ \left( \eta, \rho; \epsilon, \frac{q^2}{\mu^2} \right) Z^- \left( \eta, \rho; \epsilon, \frac{q^2}{\mu^2} \right)},
\]

(36)

In their most general form these transition functions are parametrized as

\[
Z^{\pm} \left( \eta, \rho; \epsilon, \frac{q^2}{\mu^2} \right) = \exp \left[ \frac{\rho - \eta}{2} \omega \left( \epsilon, \frac{q^2}{\mu^2} \right) + f^{\pm} \left( \epsilon, \frac{q^2}{\mu^2} \right) \right].
\]

(37)

The coefficient of the \( \rho \)-divergent term defines the gluon Regge trajectory \( \omega(\epsilon, q^2) \),

\[
\omega \left( \epsilon, \frac{q^2}{\mu^2} \right) = \omega^{(1)} \left( \epsilon, \frac{q^2}{\mu^2} \right) + \omega^{(2)} \left( \epsilon, \frac{q^2}{\mu^2} \right) + \ldots,
\]

(38)

which is determined by the requirement that the renormalized reggeized gluon propagator must be free of high energy divergences, \( i.e. \) \( \rho \) independent. At one loop one obtains

\[
\omega^{(1)} \left( \epsilon, \frac{q^2}{\mu^2} \right) = - \frac{2g^2 \Gamma^2(1 + \epsilon)}{\Gamma(1 + 2\epsilon)\epsilon} \left( \frac{q^2}{\mu^2} \right)^\epsilon.
\]

(39)

The function \( f^{\pm}(\epsilon, q^2) \) parametrizes finite contributions and is, in principle, arbitrary. Symmetry of the scattering amplitude requires \( f^+ = f^- = f \), Regge theory suggests fixing it in such that terms which are not enhanced in \( \rho \) are entirely transferred from the reggeized gluon propagators to the vertices, to which the reggeized gluon couples. With

\[
f \left( \epsilon, \frac{q^2}{\mu^2} \right) = f^{(1)} \left( \epsilon, \frac{q^2}{\mu^2} \right) + f^{(2)} \left( \epsilon, \frac{q^2}{\mu^2} \right) + \ldots
\]

(40)

we obtain from Eq. (29)

\[
f^{(1)} \left( \epsilon, \frac{q^2}{\mu^2} \right) = \frac{g^2 \Gamma^2(1 + \epsilon)}{\Gamma(1 + 2\epsilon)} \left( \frac{q^2}{\mu^2} \right)^\epsilon \left[ 5 + 3\epsilon \frac{n_f}{3} \left( \frac{2 + 2\epsilon}{3 + 2\epsilon} \right) \right]
\]

(41)

and consequently

\[
\omega^{(2)} \left( \epsilon, \frac{q^2}{\mu^2} \right) = \frac{1}{\rho} \lim_{\rho \to \infty} \frac{\Sigma^{(2)}}{(-2iq^2)} + \frac{\rho^2}{2} \omega^{(1)} \left( \omega^{(1)} \right)^2 + 2\rho f^{(1)} \omega^{(1)},
\]

(42)

where we omitted at the right hand side the dependencies on \( \epsilon \) and \( q^2/\mu^2 \) and expanded \( \Sigma^{(1)} \) in terms of the functions \( \omega^{(1)} \) and \( f^{(1)} \). We obtain

\[
\omega^{(2)} (q^2) = \frac{(\omega^{(1)} (q^2))^2}{4} \left[ 11 \left( \frac{2n_f}{3N_c} + \frac{\pi^2}{3} \frac{67}{9} \right) + \left( \frac{404}{27} - 2\zeta(3) \right) \epsilon \right],
\]

(43)
in agreement with the results in the literature [21]. The parameters $\eta$ in the transition functions are arbitrary; their role is analogous to the renormalization scale in UV renormalization or the factorization scale in collinear factorization. It gives rise to a dependence of the reggeized gluon propagator on the the factorization parameter $\eta$ from which a renormalization group equation (RGE) results:

$$\frac{d}{d\eta} G_R(\eta; \epsilon, \frac{q^2}{\mu^2}) = \omega(\epsilon, \frac{q^2}{\mu^2}) G_R(\eta; \epsilon, \frac{q^2}{\mu^2}).$$

(44)

With the above choice for the function $f^{\pm}$, one has

$$G_R(0; \epsilon, \frac{q^2}{\mu^2}) = \frac{i/2}{q^2},$$

(45)

and

$$G_R(\eta; \epsilon, \frac{q^2}{\mu^2}) = \frac{i/2}{q^2} \exp \left[ \eta \cdot \omega(\epsilon, \frac{q^2}{\mu^2}) \right].$$

(46)

The value of the parameter of $\eta$ is to be fixed by the impact factors which describe coupling of the reggeized gluon to external particles. In the case of partonic scattering, the natural choice is given by $\eta = \ln \frac{s}{|t|}$ with $t = -q^2$ and $s$ the center-of-mass energy squared. In general the value of $\eta$ is however arbitrary and must be constrained through the evaluation of higher order corrections to impact factors. It goes without saying that Eq. (46) gives nothing but reggeization of the gluon.

5 Amplitude generation for the high energy effective action

For successful phenomenology of the QCD high energy limit, an efficient determination of high energy factorized QCD scattering amplitudes is of high importance. An important step in this direction has been achieved through the parton-level event generator KATIE [23]. In a nut-shell one uses the fact that conventional collinear tree-level scattering amplitudes can be very effectively determined via numerical Dyson-Schwinger recursion. The state of a reggeized gluon is then generated through an auxiliary quark-antiquark pair, which allows

![Figure 6: Left: A QCD scattering amplitude of arbitrary many external QCD states. A certain quark-antiquark pair is identified. Following the prescription given in the text, it is reduced to a reggeized gluon which then yields a corresponding scattering amplitude of a reggeized gluon with arbitrary many external partons [figure taken from [22]]](image)
to obtain the reggeized gluon state through a conventional collinear scattering amplitude. In more detail: Due to the kinematic constraint Eq. (6), the momenta of reggeized gluons satisfy $k \cdot n^\pm = 0$. Relating light-cone directions of reggeized gluons to actual momenta in a scattering process we have $n^\pm = \sqrt{2} \frac{p_{1,2}}{\sqrt{p_1 \cdot p_2}}$ and a reggeized gluon momentum can be parametrized as

$$k^\mu = x p_1^\mu + k^\mu, \quad k \cdot n^\pm = 0. \quad (47)$$

To generate the scattering amplitude of an off-shell reggeized gluon with quarks and gluons, the reggeized gluon is then embedded into a purely partonic scattering amplitude with the help of an auxiliary quark-antiquark pair with momenta $p_A^\mu$ and $p_{A'}^\mu$. Both auxiliary momenta are light-like $p_A^2 = 0 = p_{A'}^2$. In more detail one defines

$$p_A^\mu = \Lambda p_1^\mu + \alpha p_2^\mu + \beta \mathbf{k}. \quad (48)$$

The requirement $p_A^2 = 0$ fixes then $\alpha = \frac{\beta^2 k^2}{x(p_1 + q)^2}$ while $p_{A'}^2 = 0 = (p_A - k)^2$ yields $\beta = \frac{1}{1 + \sqrt{1 - x/\Lambda}}$. In this way the momenta $p_A$ and $p_{A'}$ fulfill the important requirement of light-likeness for all values of $\Lambda$ as well as $p_A + p_{A'} = k$. The desired scattering amplitude of a single reggeized gluons with arbitrary many partons is then obtained from the limit

$$\mathcal{M}(0 \rightarrow r^*(xp_1 + \mathbf{k}) + X) = \lim_{\Lambda \rightarrow \infty} \frac{|\mathbf{k}|}{\Lambda} \cdot \mathcal{M}(0 \rightarrow q(p_A(\Lambda)), \bar{q}(p_{A'}(\Lambda)) + X). \quad (49)$$

Within the resulting scattering amplitude, it is then straightforward to distinguish gluons which are emitted from the auxiliary quark line from those which are not. The latter play the rôle of the gluons which are generated by the QCD Lagrangian within the Lagrangian of the high energy effective action Eq. (7). The former yield the contributions stemming from the induced Lagrangian. To make this relation more explicit, it is useful to consider a typical propagator of the auxiliary quark line. With a generic momentum in the auxiliary quark line given by

$$p^\mu = (\Lambda + x \mathbf{p}) p_1^\mu + y p_2^\mu + \mathbf{p}, \quad (50)$$

one then defines

$$p' = p - p_{A'} = p - k + p_A$$

$$= (2\Lambda - x) p_1 + (y q + \alpha) p_2 + \mathbf{p} - \mathbf{k}(1 - \beta), \quad (51)$$

as the momentum flowing in the (auxiliary) quark propagator, if the external quark $p_A$ would given the momentum $k$ and the anti-quark momentum 0, i.e. the total momentum of the gluons emitted by the quark sum up to the reggeized gluon momentum. $p'$ is therefore the total momentum of gluons “so far” emitted from auxiliary quark line. In particular

$$\lim_{\Lambda \rightarrow \infty} p_1 \cdot p' = y p_2 p_1 \cdot q. \quad (52)$$

One then finds for the quark propagator with momentum $p$ in the limit $\Lambda \rightarrow \infty$:

$$\lim_{\Lambda \rightarrow \infty} \frac{\hat{f}}{p^2} = \lim_{\Lambda \rightarrow \infty} \frac{(\Lambda + x p) \hat{f}_1 + y p \hat{f}_2 + \mathbf{p}}{2(\Lambda + x p) y p p_1 \cdot p_2 + p^2} = \frac{\hat{f}_1}{2 p_1 \cdot p'} = \frac{\hat{f}_1}{2 p'^+}. \quad (53)$$

[^1]: We use the symbol $r^*$ to denote the external state of an (off-shell) reggeized gluon.
Making now use of well-known identities such as
\[ n^\mu \gamma_\mu n^\nu = 2 n^\mu \cdot n^\nu, \]
one re-obtains from these contributions the induced vertices\(^4\) of the high energy effective action, Fig. 2. The whole procedure can be generalized in a straightforward manner to the case of two reggeized gluons with opposite i.e. plus- and minus-polarization. In this way it is then possible to generate all tree-level amplitudes of the high-energy effective action. This is of particular use, since there exists very powerful algorithms which allow to generate the underlying scattering amplitudes numerically. It is therefore correct to claim that the problem of efficiently computing gauge invariant tree level scattering amplitudes within high energy factorization in the region of phase space where one reggeized gluon exchange provides a suitable approximation, has been completely solved in recent years, both analytically and numerically. For details we refer the interested reader to the original papers \([22,24–30]\). The method has been used in the recent past for a large number of phenomenological studies, but also for theoretical exploration, see for instance the calculation of transverse momentum dependent splitting kernels in \([31]\).

6 High energy effective action and high parton densities

An area of high phenomenological interest is given by the physics of high parton densities: occurrence of high parton densities is closely related to the physics of gluon saturation which is assumed to be relevant mechanism to ensure unitarity of QCD scattering amplitudes in the high energy limit. The bulk of theoretical efforts in recent years to describe this region of phase space within high energy factorization, can be summarized under the term “Color Glass Condensate effective field theory” (see \([32]\) for a review). In the following we summarize the essential results of \([33]\) which demonstrate of these efforts with the high energy effective action.

6.1 A special parametrization of the gluonic field

The starting point for this exploration has already been laid in the original publication \([1,2]\) by Lipatov himself: there it has been noted that while a shift in the gauge field \( v^\mu \rightarrow v^\mu + n^\mu_+ A_+ / 2 + n^\mu_- A_- / 2 \) provides a possibility to avoid the appearance of a direct transition vertex between reggeized gluon field and gluon field, Fig. 2.a, it mixes two fields with different gauge transformation properties: \( v_\mu \) transforms as a gauge field, while \( A_\pm \) is by definition invariant under local gauge transformations. The solution put forward in \([1,2]\) suggests the following parametrization of the gluonic field:

\[ V^\mu(x) = v^\mu(x) + \frac{n^\mu_+}{2} B_-(x) + \frac{n^\mu_-}{2} B_+(x), \]

where

\[ B_\pm[v_\mp] = U[v_\mp] A_\mp U^{-1}[v_\mp]. \]

\(^4\)In more detail: one first obtains the term corresponding to one particular ordering of the operator Eq. (15); summing over all possible orderings, the full color structure of Fig. 2 is obtained
and (inverse) Wilson line operators are defined as

\[ U[v_\pm] = \frac{1}{1 + \frac{g}{\partial_\pm} v_\pm} , \quad U^{-1}[v_\pm] = 1 + \frac{g}{\partial_\pm} v_\pm . \quad (57) \]

For the above composite field \( B_\pm[v_\mp] \), one finds the following gauge transformation properties:

\[ \delta_L B_\pm = \delta_L U[v_\mp] A_\pm U^{-1}[v_\mp] + U[v_\mp] A_\pm \delta_L U^{-1}[v_\mp] = [g B_\pm, \chi_L] , \quad (58) \]

and as a consequence the shifted gluonic field Eq. (55) transforms like a gluonic field:

\[ \delta V_\pm = [D_\pm, \chi] + [g B_\pm, \chi] = [D_\pm + g B_\pm, \chi] . \quad (59) \]

In the following we will use the above parametrization of the gluonic field to expand the high energy effective action for the quasi-elastic case around the reggeized gluon field \( A_+ \) which we treat as a strong classical background field \( g A_+ \sim 1 \). We therefore consider the high energy effective action for the quasi-elastic case

\[ S_{\text{eff}}^{\text{q.e.}} = S_{\text{QCD}} + S_{\text{ind.}}^{\text{q.e.}} \quad (60) \]

with

\[ S_{\text{ind.}}^{\text{q.e.}} = \int d^4 x \, \text{tr} \left( \{ T_-[v] - A_-(x) \} \partial^2 A_+(x) \right) . \quad (61) \]

Keeping fields \( A_+ \) to all orders and expanding in quantum fluctuations \( v_\mu \) and \( \psi, \bar{\psi} \) to quadratic order, we obtain

\[ S_{\text{eff}}^{\text{q.e.}} = \int d^4 x \, [ \mathcal{L}_0 + \mathcal{L}_1 - \text{tr} \left( A_- \partial^2 A_+ \right) ] + \mathcal{O}(v_\mu^3) , \quad (62) \]

with the kinetic term of the gluonic and quark field

\[ \mathcal{L}_0 = \text{tr} \left( -v^\mu [g_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}] v^\nu \right) + \bar{\psi} i \gamma^\nu \psi , \quad (63) \]

and the quadratic terms which describe interaction with the reggeized gluon field,

\[ \mathcal{L}_1 = g \left\{ \frac{i}{2} \bar{\psi} \gamma_\mu A_+ \psi + \text{tr} \left[ \partial_+ v_\mu [A_+, v^\mu] + 2 \partial_\mu v_-[v^\mu, A_+] + \partial^2 v_- \left[ \left( \frac{1}{\partial_-} v_- \right), A_+ \right] - v_- \left( \frac{1}{\partial_-} v_- \right) \partial^2 A_+ \right] \right\} . \quad (64) \]
6.2 Resummation of a dense reggeized gluon field

The quasi-elastic effective action allows then for the extraction of quark-quark-reggeized gluon (QQR) and gluon-gluon-reggeized gluon vertices,

\[ \Gamma_{\beta\alpha}(r, p) = -\frac{1}{2} f^{\beta\alpha\gamma} \]

(65)

with \( T_{ab}^c = -i f^{abc} \). The QQR and GGR vertices obey various properties:

\[ \Gamma_{\beta\gamma'}(r, p) n_{\gamma'} = 0 = \Gamma_{\beta\gamma'}(r, p), \]

\[ \Gamma_{\beta\gamma}(r, k) k_{\gamma'} \Gamma_{\gamma'\alpha}(k, p) = -p^+ \Gamma_{\beta\alpha}(r, p), \]

(67)

and

\[ n_{\nu} \cdot \Gamma_{\gamma\mu}^\nu(r, p) = 0 = \Gamma_{\gamma\mu}^\nu(r, p) \cdot n_{\mu}, \]

\[ \Gamma_{\gamma\mu}^\nu(r, k) \cdot (-g_{\alpha\gamma'}) \cdot \Gamma_{\alpha\mu}(k, p) = -p^+ \Gamma_{\gamma\mu}^\nu(r, p), \]

\[ r_{\nu} \cdot \Gamma_{\gamma\mu}^\nu(r, p) = 0 = \Gamma_{\gamma\mu}^\nu(r, p) \cdot p_{\mu}. \]

(68)

Note that the above GGR-vertex was already obtained in [1]. A potential disadvantage of this vertex, pointed out in [1], is that it can yield individual Feynman diagrams which contain singularities in overlapping channels and therefore violate the Steinmann-relations [34]. Since the vertex is obtained from an effective action which explicitly respects the Steinmann relations, such effects must cancel at the level of the observable. The properties Eq. (67) are of great use for the resummation of a strong reggeized gluon field \( gA_+ \sim 1 \) to all orders. Such an exercise has been carried out in [33]. As a result one obtains the following effective vertices which sum up the interaction of a quark and a gluon with an arbitrary
number of reggeized gluon fields:

\[
p \to r = \tau_{G,\nu\mu}^{ab}(p,-r) = -4\pi\delta(p^+ - r^+)\Gamma_{\nu\mu}(r,p)e^{-ix_0^+(p^- - r^-)}
\]

\[
\cdot \int d^2z e^{iz \cdot (p-r)} \left[ \theta(p^+) U^0(a)(z) - \delta^{ab} - \theta(-p^+) \left[ (U^{0b}(z))^\dagger - \delta^{ab} \right] \right], \quad (69)
\]

\[
p \to r = \tau_F(q,-r) = 2\pi\delta(p^+ - r^+)\hat{p}^+ e^{-ix_0^+(p^- - r^-)}
\]

\[
\cdot \int d^2z e^{iz \cdot (p-r)} \left[ \theta(p^+) [W(z) - 1] - \theta(-p^+) \left[ (W(z))^\dagger - 1 \right] \right]. \quad (70)
\]

To write down the above expressions, we introduced Wilson lines in the adjoint

\[
U^{ab}(z) = \text{P} \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dz^+ A_+^+ \right), \quad \tilde{A}_+ = -iT^{c}_{ab}A_+^c,
\]

and the fundamental representation

\[
W(z) = \text{P} \exp \left( -\frac{g}{2} \int_{-\infty}^{\infty} dz^+ A_+ \right), \quad A_+ = -it^{c}_{ij}A_+^c. \quad (72)
\]

The above vertices Eqs. (69) allow to construct partonic propagators in the presence of a (reggeized gluon) background field and can be compared with corresponding propagators derived for the resummation of a background field in the light-cone gauge [35–40]. It turns that the regarding propagators agree with each other, if the gluonic field is taken in the same gauge, i.e. light-cone gauge. The interpretation is however slightly different for the Wilson line which resum gluonic background field in light-cone gauge and reggeized gluon field respectively. For a more detailed discussion we refer the interested reader to [33]. As a first application it was possible to use the vertices Eq. (69) to re-derive the leading order Balitsky-JIMWLK equation from the high energy effective action: Calculating quantum fluctuations of an ensemble of Wilson lines of the reggeized gluon fields

\[
U^{ab}(z) = 2\text{tr} \left[ t^a W(z) t^b W^\dagger(z) \right], \quad (73)
\]

one finds for the evolution with respect to the regulator used for the high energy singularities, the Balitsky-JIMWLK equation

\[
-\Lambda_a \frac{d}{d\Lambda_a} \left[ W(x_1) \otimes \ldots \otimes W(x_n) \right] = \sum_{i,j=1} H_{ij} \left[ W(x_1) \otimes \ldots \otimes W(x_n) \right], \quad (74)
\]

with the Balitsky-JIMWLK Hamiltonian

\[
H_{ij} = \frac{\alpha_s G^2 (1 + \epsilon)}{2\pi^2 \Gamma(1 - \epsilon)} \left( \frac{4}{\pi \mu^2} \right)^\epsilon \int d^2+2\epsilon z e^{i(x_i - z) \cdot (x_j - z)} \left[ T_{i,L}^a T_{j,L}^a + T_{i,R}^a T_{j,R}^a - U^{ab}(z) \left( T_{i,L}^a T_{j,R}^b + T_{i,R}^a T_{j,L}^b \right) \right], \quad (75)
\]

where \( T_{i,L}^a \) and \( T_{i,R}^a \) are the group generators acting to the left (L) or to the right (R) on the Wilson line \( W(x_i) \),

\[
T_{i,L}^a[W(z_i)] \equiv t^a W(z_i), \quad T_{i,R}^a[W(z_i)] \equiv W(z_i)t^a. \quad (76)
\]
7 Conclusions

The topics covered in this short review provide apparently only a small subset of results achieved within the high energy effective action (with the choice heavily biased by the author’s own work). Among the results which we did not review here, are the effective action for Regge processes in gravity [41], earlier versions of an high energy effective action [5–7] as well as the determination of the $3 \to 3$ reggeized gluon transition kernel, needed for the NLO Bartels-Kwiecinski-Praszalowicz equation in [42] and the NLO corrections to the Mueller-Tang jet impact factor [43–45]. Another line of research has been developed in a series of papers [18, 46–53] dedicated to the study of production processes at central rapidities in the presence of multiple reggeized gluon exchange. The relation of the high energy effective action to the Color Glass Condensate approach as well as the further development of the underlying formulation has been explored in [54, 55] and continued in [56–60]. In parallel to the high energy effective action for gluons, the corresponding effective action for quarks has been explored in series of papers, at first mainly focusing on phenomenological applications in combination with transverse momentum dependent parton distribution functions [61–78], while recent efforts address the determination of higher order perturbative corrections [14,15].

When I met Lev Lipatov in 2006, I had just started working on my doctoral thesis at the II. Institute for Theoretical Physics at Hamburg University. While working with him was at first a challenge, I soon realized what a great teacher he was and how many insights he was generously sharing with me. I must admit that I could not always follow all of his explanations and that sometimes it took me years until I was able to finally understand the full impact of one or the other comment he made to me during these days. While the high energy effective action might be not among the achievements Lipatov is most famous for, I am deeply convinced that it constitutes an important part of his legacy. It allows us to benefit from his deep insights into the Regge limit of QCD even after he went away from us.

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