EFFECTIVE CONE OF THE BLOW UP OF THE SYMMETRIC PRODUCT OF A CURVE

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ABSTRACT. Let $C$ be a smooth curve of genus $g \geq 1$ and let $C^{(2)}$ be its second symmetric product. In this note we prove that if $C$ is very general, then the blow-up of $C^{(2)}$ at a very general point has non-polyhedral pseudo-effective cone. The strategy is to consider first the case of hyperelliptic curves and then to show that having polyhedral pseudo-effective cone is a closed property for families of surfaces.

INTRODUCTION

The study of the effective cone of the blow up $\tilde{S}$ of a projective surface $S$ at a smooth point $x \in S$ is connected with the calculation of Seshadri constants. Deciding when the (pseudo)effective cone of $\tilde{S}$ is polyhedral is an open problem even when $S$ is a toric surface. For instance, if the effective cone of the blow up of the weighted projective plane $\mathbb{P}(a,b,c)$ at a general point is not closed, then Nagata’s Conjecture holds for $abc$ points in $\mathbb{P}^2$, see [4] and [7–10] for recent results on blow ups of weighted projective planes. In [2] it has been shown that there exist toric surfaces whose blow up at a general point has non-polyhedral pseudo-effective cone. This result allows one to deduce that the pseudo-effective cone of the Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$ is not polyhedral for $n \geq 10$.

In this paper we focus on the second symmetric product $C^{(2)}$ of a positive genus curve $C$. In general it is not known if the effective cone of these surfaces is open. This would be true if the Nagata Conjecture holds, as shown in [3]. Our interest is in the blow up $\tilde{C^{(2)}}$ at a very general point $p \oplus q \in C^{(2)}$.

**Theorem 1.** Let $C$ be a very general curve of genus $g \geq 1$. Then the blow-up of the symmetric product $C^{(2)}$ at a very general point has non-polyhedral pseudo-effective cone.

In order to prove the theorem we first show, in Proposition 1.3, that having polyhedral pseudo-effective cone is a closed property for families of surfaces and then we prove the following.

**Theorem 2.** Let $C$ be a genus $g \geq 1$ hyperelliptic curve with hyperelliptic involution $\sigma$, let $p \in C$ and let $\tilde{C^{(2)}}$ be the blow-up of $C^{(2)}$ at $\sigma(p)$, $p \oplus \sigma(p)$. If the class of $\sigma(p) - p$ is non-torsion in $\text{Pic}^0(C)$ then $\text{Eff}(\tilde{C^{(2)}})$ is non-polyhedral.

When $C$ is an elliptic curve, its symmetric product is the Atiyah surface. In this case in [6] it has been proved that if $q - p$ is non-torsion, then $\tilde{C^{(2)}}$ contains

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infinite many negative curves. Therefore the pseudo-effective cone of $\tilde{C}^{(2)}$ is not polyhedral, and in [13] it is proved that the classes of the above mentioned curves (together with other two classes) indeed generate the pseudo-effective cone.

Our proof of Theorem 2 focuses on the quotient surface $\tilde{X}$ by the action of the hyperelliptic involution on both factors. We show that there is an irreducible curve $B$ on $\tilde{X}$ having self intersection $B^2 = 0$, whose class spans an extremal ray of the pseudo-effective cone of $\tilde{X}$, so that the latter cannot be polyhedral by [2, Proposition 2.3]. We then apply Proposition 1.1 to the double cover $\tilde{C}^{(2)} \to \tilde{X}$ to conclude that the pseudo-effective cone of $\tilde{C}^{(2)}$ is not polyhedral.

The paper is structured as follows. In Section 1 we recall some definitions and we prove some preliminary results about the effective cone of projective surfaces. In Section 2 we study the symmetric product $C^{(2)}$ of a curve, with particular emphasis on the case $C$ hyperelliptic. Section 3 is devoted to the proof of Theorem 1 and 2, while in Section 4 we prove some results in case $g(C) = 1$.

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1. Preliminaries

Let $k$ be an algebraically closed field of arbitrary characteristic. We recall some definitions (see for example [11, 12]). If $X$ is a normal projective irreducible variety over $k$, let $\text{Cl}(X)$ be the divisor class group and let $\text{Pic}(X)$ be the Picard group of $X$. As usual, we denote by $\sim$ the linear equivalence of divisors and by $\equiv$ the numerical equivalence. Recall that for Cartier divisors $D_1$, $D_2$, we have $D_1 \equiv D_2$ if and only if $D_1 \cdot C = D_2 \cdot C$, for any curve $C \subseteq X$. We let

$$N^1(X) := \text{Pic}(X)/\equiv$$

be the Néron-Severi group, i.e. the group of numerical equivalence classes of Cartier divisors on $X$. We denote by $\rho(X)$ the rank of $N^1(X)$ and by $N^1(X)_\mathbb{R} = N^1(X) \otimes \mathbb{R}$, $N^1(X)_\mathbb{Q} = N^1(X) \otimes \mathbb{Q}$. We define the pseudo-effective cone

$$\text{Eff}(X) \subseteq N^1(X)_\mathbb{R},$$

as the closure of the effective cone $\text{Eff}(X)$, i.e., the convex cone generated by numerical classes of effective Cartier divisors ([12, Def. 2.2.25]). We let $\text{Nef}(X) \subseteq N^1(X)_\mathbb{R}$ be the cone generated by the classes of nef divisors.

Proposition 1.1. Let $f: X \to Y$ be a finite surjective morphism of normal $\mathbb{Q}$-factorial projective varieties. If $\rho(X) = \rho(Y)$, then $f_*: N^1(X)_\mathbb{R} \to N^1(Y)_\mathbb{R}$ is an isomorphism such that $f_*(\text{Eff}(X)) = \text{Eff}(Y)$.

Proof. Since $Y$ is $\mathbb{Q}$-factorial, the image of $\text{Pic}(Y)$ in the Néron-Severi group $N^1(Y)$ has finite index. Over this subgroup the pullback is defined and the projection formula gives $f_* \circ f^* = n \cdot \text{id}$, where $n = \text{deg}(f)$. This, together with the hypothesis $\rho(X) = \rho(Y)$, imply that $f_*: N^1(X)_\mathbb{R} \to N^1(Y)_\mathbb{R}$ is an isomorphism whose inverse is $\frac{1}{n} f^*$. Then one concludes by the inclusions

$$f_*(\text{Eff}(X)) \subseteq \text{Eff}(Y) \text{ and } f^*(\text{Eff}(Y)) \subseteq \text{Eff}(X).$$

$\square$
Proposition 1.2. Let $X$ be a normal $\mathbb{Q}$-factorial algebraic surface with $g(X) \geq 3$ and positive light cone $Q \subseteq N^1(X)_{\mathbb{R}}$. Let $C_1,\ldots,C_n$ be irreducible curves of $X$. Then the following are equivalent:

(1) $Q \subseteq \text{Cone}([C_1],\ldots,[C_n])$;
(2) $\text{Eff}(X) = \text{Cone}([C_1],\ldots,[C_n])$.

Moreover if $\text{Eff}(X)$ is polyhedral then $\text{Eff}(X) = \text{Eff}(X)$ holds and both cones are generated by classes of negative curves.

Proof. We prove $(1) \Rightarrow (2)$. Let $[D]$ be a divisor class which generates an extremal ray of the effective cone. Then $D^2 < 0$ so that the hyperplane $D^1$ intersects $Q$ along its interior. As a consequence at least one of the $C_i$ satisfies $D \cdot C_i < 0$. Thus any effective multiple of $D$ contains $C_i$ into its support, so that $[D] = [C_i]$ up to multiples.

The implication $(2) \Rightarrow (1)$ is obvious. \hfill \Box

Proposition 1.3. Let $X \rightarrow B$ be a flat projective morphism of Noetherian schemes, whose general fiber is a normal $\mathbb{Q}$-factorial surface with Picard lattice isometric to the one of the special fiber $X_0$ over $0 \in B$. If the general fiber has polyhedral pseudoeffective cone, then the same holds for the special fiber.

Proof. If the Picard rank is $\leq 2$, then the pseudoeffective cone is polyhedral and there is nothing to prove. We then assume that the Picard rank is at least 3. By Proposition 1.2 the pseudo-effective cone of the general fiber is generated by finitely many classes of negative curves $C_1,\ldots,C_n$. By semicontinuity of cohomology dimension, each such curve $C_i$ degenerates to a, possibly reducible, curve of $X_0$. Let $C_{i_1},\ldots,C_{i_r}$ be the irreducible components of the degenerate curve. We claim that in the Néron-Severi space of the special fiber $X_0$ the following inclusions of cones hold

$$Q \subseteq \text{Cone}([C_i] : 1 \leq i \leq n) \subseteq \text{Cone}([C_{ij}] : 1 \leq i \leq n, 1 \leq j \leq r_i).$$

Indeed, by Proposition 1.2, the first inclusion holds true in the Néron-Severi space of the general fiber and, by the assumption on the Picard lattice of the special fiber, it holds as well on the Néron-Severi space of the special fiber. The second inclusion follows by the definition of the curves $C_{ij}$. Then, again by Proposition 1.2, one concludes that $\text{Eff}(X_0) = \text{Cone}([C_{ij}] : 1 \leq i \leq n, 1 \leq j \leq r_i)$. \hfill \Box

2. Symmetric product of a curve

Given a genus $g \geq 1$ curve $C$, we denote by $C^{(2)}$ its second symmetric product, that is the quotient of $C \times C$ by the involution $\tau$, defined by $(p,q) \mapsto (q,p)$, and we denote by $p \oplus q \in C^{(2)}$ the class of $(p,q) \in C \times C$.

From now on we assume that $C$ is hyperelliptic, we fix a hyperelliptic involution $\sigma$ and we denote by $p_1,\ldots,p_{2g+2} \in C$ its fixed points. Observe that $\sigma$ induces two commuting involutions $\sigma_1, \sigma_2$ on $C \times C$, each of which acts only on one coordinate. The group $G := (\sigma_1, \sigma_2, \tau)$ is isomorphic to $D_4$, with center generated by the composition $\sigma_1 \cdot \sigma_2$, that we still denote by $\sigma$ with abuse of notation. We have the
following diagram of degree two quotient morphisms

\[
\begin{array}{ccc}
C \times C & \longrightarrow & S \\
\downarrow & & \downarrow \\
C^{(2)} & \phi \longrightarrow & X \\
\downarrow & & \downarrow \\
\quad & \psi \longrightarrow & \mathbb{P}^2,
\end{array}
\]

where each vertical map is the quotient by \( \tau \), the first horizontal map on each line is the quotient by \( \sigma \), and the second one is the quotient by \( \sigma_1 \).

**Remark 2.1.** Let us consider the diagonal \( \Delta_+ := \{ p \oplus p \mid p \in C \} \) and the antidiagonal \( \Delta_- := \{ p \oplus \sigma(p) \mid p \in C \} \) in \( C^{(2)} \). We set \( C_+ := \phi(\Delta_+) \subseteq X \) and \( \Gamma := \psi(C_+) = \psi(C_-) \subseteq \mathbb{P}^2 \). From the above diagram we see that \( \Gamma \) is the image of the diagonal of \( \mathbb{P}^1 \times \mathbb{P}^1 \) via the double cover defined by \( ([s_0 : s_1], [t_0 : t_1]) \to [s_0t_0 : s_1t_1 : s_0t_1 + s_1t_0] \), so that it is the conic \( \Gamma = V(x_2^2 - 4x_1x_3) \subseteq \mathbb{P}^2 \).

Given a point \( p \in C \), consider the two curves \( \{ p \} \times C \) and \( C \times \{ p \} \) in \( C \times C \). On \( \mathbb{P}^1 \times \mathbb{P}^1 \) they are mapped to two lines on the two different rulings, while on \( C^{(2)} \) they are mapped to the curve \( C_p := \{ p \oplus q \mid q \in C \} \). We set \( B_p := \phi(C_p) \subseteq X \) and \( L_p := \psi(B_p) \subseteq \mathbb{P}^2 \). We observe that \( B_p \) is isomorphic to \( C \), while \( L_p \) is a line which is tangent to \( \Gamma \) at the image of \( p \oplus p \) (equivalently, at the image of \( \sigma(p) \oplus p \)) in \( \mathbb{P}^2 \). We finally remark that given the curve \( C_{\sigma(p)} := \{ \sigma(p) \oplus q \mid q \in C \} \), we have \( \phi(C_{\sigma(p)}) = \phi(C_p) = B_p \subseteq X \).

**Proposition 2.2.** The surface \( X \) is a double cover of the plane, branched along the union of \( 2g + 2 \) lines, tangent to the conic \( \Gamma \). It has \( (\frac{2g+2}{2}) \) singular points, namely the ordinary double points points \( \phi(p_i \oplus p_j) \), for \( 1 \leq i < j \leq 2g + 2 \). The equation of \( X \) in the weighted projective space \( \mathbb{P}(1,1,1,g+1) \) is

\[ x_2^2 + \prod_{i=1}^{2g+2} \ell_i = 0, \]

where \( \ell_1, \ldots, \ell_{2g+2} \in \mathbb{C}[x_1, x_2, x_3] \) are defining polynomials for the \( 2g + 2 \) lines.

**Proof.** The ramification divisor of \( \psi : X \to \mathbb{P}^2 \) consists of the images of points \( (p, q) \in C \times C \) such that the orbit of \( (p, q) \) with respect to \( (\sigma, \tau) \) equals the orbit with respect to the whole group \( G \), that is \( (\sigma(p), q) \in \{(p, q), (q, p), (\sigma(p), \sigma(q)), (\sigma(q), \sigma(p))\} \). The latter condition holds if and only if either \( p \) or \( q \) is a fixed point for \( \sigma \). Thus the ramification is the union of the curves \( B_{p_1} \cdots B_{p_{2g+2}} \). We conclude that the branch divisor of \( \psi \) is the union of the \( 2g + 2 \) lines \( L_{p_1}, \ldots, L_{p_{2g+2}} \) which, by Remark 2.1, are tangent to \( \Gamma \) at the images of \( p_j \oplus p_j \).

\[ \square \]

**Remark 2.3.** Since \( X \) is a hypersurface of a weighted projective space, by [5, Thm. 4.2.2] we have that \( q(X) = 0 \). In particular \( X \) is a weak del Pezzo of degree 2 if \( g = 1 \), it is a singular \( K3 \) when \( g = 2 \) and it is of general type when \( g \geq 3 \).

**Proposition 2.4.** Assume that \( C \) is a very general hyperelliptic curve of genus \( g \geq 1 \). Then both \( C^{(2)} \) and \( X \) have Picard rank 2 and their effective cones are generated by the classes of the images of the diagonal and the antidiagonal. The intersection matrices of these curves in \( C^{(2)} \) and of their images in \( X \) are

\[
\begin{pmatrix}
4 - 4g & 2g + 2 \\
2g + 2 & 1 - g
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 - 2g & 2g + 2 \\
2g + 2 & 2 - 2g
\end{pmatrix}
\]
respectively.

Proof. By [1, Chapter VIII, §5], $N^1(C) \cong \mathbb{Z} \oplus N^1(JC)$, so that [15, Prop. 3.4] implies that the Picard rank of $C$ is 2. As a consequence the Picard rank of $X$ is at most 2 and it is 2 because $N^1(X)$ contains two numerically independent classes.

The diagonal $\Delta$ and the antidiagonal $\Delta^-$ are both mapped to the conic $\Gamma$ of $\mathbb{P}^2$, tangent to the 2g+2 lines. Thus $C_+ + C_- = \phi(\Delta) + \phi(\Delta^-)$ is the pullback of $\Gamma$, so that $(C_+ + C_-)^2 = 8$. Since these two curves are numerically equivalent and intersect in 2g+2 points, we obtain the second matrix. To get the first matrix it is enough to observe that the double cover $C_2 \to X$ branches at $C_-$, which is the image of the antidiagonal. □

3. Proof of Theorem 2 and 1

Proof of Theorem 2. Let us fix a point $p \in C$, such that the class of $\sigma(p) - p$ is non-torsion, and let $\tilde{C} \to C$ be the blow-up at the point $p \oplus \sigma(p) \in \Delta_+ \cap C_\sigma(p)$, with exceptional divisor $E$. First of all observe that the point $p \oplus \sigma(p)$ is invariant for $\sigma$, so that the latter lifts to an involution on the blow-up $C$ that, by abuse of notation, we denote by the same symbol $\sigma$. Let $\tilde{\phi} : \tilde{C} \to \tilde{X} := \tilde{C}/(\sigma)$ be the quotient morphism. The involution $\sigma$ has two fixed points on the exceptional divisor $E$: the intersection point with the strict transform of $\Delta_-$, and one isolated point $x$, so that $\tilde{\phi}(x)$ is a singular point of $\tilde{X}$. We have a birational morphism $\eta : \tilde{X} \to X$ which is the contraction of $\tilde{\phi}(E)$, having self-intersection $-1/2$ in $\tilde{X}$. The map $\eta$ is a weighted blow-up at the point $\phi(\sigma(p) \oplus p)$ and can also be described as follows. Consider the blow-up $X_1 \to X$ at the point $\phi(p \oplus \sigma(p))$, with exceptional divisor $E_1$, and then the blow-up $X_2 \to X_1$, at the intersection point of $E_1$ with the strict transform of $B_p = \phi(C_p) = \phi(C_{\sigma(p)})$ (see Remark 2.1). Finally contract the strict transform of $E_1$, which is now a $(-2)$-curve (its image gives the singular point $\tilde{\phi}(x) \in \tilde{X}$).

We can resume the above discussion in the following commutative diagrams:

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\tilde{\phi}} & \tilde{X} \\
\downarrow & & \downarrow \\
C & \xrightarrow{\phi} & X \\
\end{array}
\quad
\begin{array}{ccc}
X_2 & \xrightarrow{\eta} & \tilde{X} \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\eta} & X \\
\end{array}
\]

We are going to show that the pseudo-effective cone of $\tilde{X}$ is not polyhedral. Observe that the strict transform of $B_p$ in $X$ is isomorphic to $B_p$ and hence to $C_{\sigma(p)}$ and to $C$. Therefore, by abuse of notation, we denote this strict transform by $C$. Since $\psi(B_p) \subseteq \mathbb{P}^2$ is the line $L_p$, tangent to the conic $\Gamma$, we have that $B_p^2 = 2$. By the description of $\eta : \tilde{X} \to X$, we are blowing up a point on $B_p$ and then the same point on its strict transform. Therefore $C^2 = 0$, and we can write

\[C \sim (\psi \circ \eta)^*L_p - 2\tilde{\phi}(E).\]

Let us compute now the restriction $\mathcal{O}_C(C)$. Since $L_p$ is a line in $\mathbb{P}^2$, the restriction of $(\psi \circ \eta)^*L_p$ to $C$ is the $\mathcal{O}_{\mathbb{P}^2}(1)$, so that it is equivalent to $\sigma(p) + p$. On the other hand, the restriction of $\tilde{\phi}(E)$ to $C$ corresponds to the point we are blowing-up in $\eta : \tilde{X} \to X$, i.e. to the image of $\sigma(p) \oplus p$ in $X$. Via the isomorphism $C_{\sigma(p)} \to C$,
the point \( \sigma(p) \oplus p \) corresponds to \( p \in C \), so that we conclude that the restriction of \( \tilde{\phi}(E) \) to \( C \) is \( p \). Summing up we obtain

\[
\mathcal{O}_C(C) = \mathcal{O}_C(\sigma(p) + p - 2p) = \mathcal{O}_C(\sigma(p) - p).
\]

Since we are assuming that \( \sigma(p) - p \) is non-torsion, we deduce that \( \mathcal{O}_C(C) \) is non-torsion, i.e. the class of \( D \) spans an extremal ray of \( \text{Eff}(\tilde{X}) \), so that this cone is not polyhedral. By Proposition 1.1 we conclude that the cone \( \text{Eff}(\tilde{C}^{(2)}) \) is not polyhedral. \( \square \)

**Remark 3.1.** In genus 2 the Abel-Jacobi map presents \( C^{(2)} \) as the blow-up of \( \text{Pic}^2C \) in the point \( \Omega \) that corresponds to the canonical class \( K_C \) of \( C \), with exceptional divisor \( \Delta_\pm \subseteq C^{(2)} \). So in this case we blow-up \( \text{Pic}^2C \) twice infinitely near at \( \Omega \). The map \( C \to \text{Pic}^2C \) given by \( x \mapsto [x + \sigma(p)] \) embeds \( C \) as a theta-divisor passing through \( \Omega \) and with tangent direction \( p + \sigma(p) \). So after the blow-up the proper transform of \( C \) is a curve of self-intersection 0. The restriction of \( C \) to \( C \) will be \( K_C - 2p \) (because we blow-up twice the same point \( p \) of \( C \)), so it is \( \sigma(p) - p \) as we claim in the proof of the general statement.

**Remark 3.2.** Assume that \( \sigma(p) - p \) is not torsion, so that the pseudo-effective cone \( \text{Eff}(\tilde{X}) \) is not polyhedral. If \( g = 1 \) we are going to show that on \( \tilde{X} \) there are infinitely many negative rays accumulating on \( C \) (see Proposition 4.9 and Figure 3). If \( g > 1 \), consider the intersection matrix of the classes \( \Delta_+, \Delta_-, E_1, C, E \) on \( \tilde{X} \):

\[
\begin{pmatrix}
2 - 2g & 2 + 2g & 1 & 2 & 0 \\
2 + 2g & -2g & 1 & 0 & 1 \\
1 & 1 & \frac{1}{2} & 1 & 0 \\
2 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & -\frac{1}{2}
\end{pmatrix}.
\]

We already know that \( C \) generates an extremal ray of \( \text{Eff}(\tilde{X}) \), and the same holds for the classes \( \Delta_+, \Delta_- \) and \( E \), since they have negative self-intersection. In particular \( \text{Eff}(\tilde{X}) \) (and hence also \( \text{Eff}(\tilde{C}^{(2)}) \)) has a polyhedral part (see Figure 1).

**Figure 1.** \( \text{Eff}(\tilde{X}) \), when \( g > 1 \)

**Question 3.3.** When \( C \) is hyperelliptic of genus \( g > 1 \), does \( \tilde{C}^{(2)} \) have infinitely many negative curves?
Proof of Theorem 1. Let $\pi : \mathcal{X} \to B$ be a flat family whose general fiber is a general genus $g$ curve $C$ and whose special fiber over $0 \in B$ is a general hyperelliptic curve $C_0$. Passing to the symmetric product one gets a new flat family with basis $B$. Blowing-up a section of the new family which cuts out $p \oplus \sigma(p)$ on $C_0^{(2)}$, with $\sigma(p) - p$ non-torsion, one concludes, by Theorem 2 and Proposition 1.3, that the pseudo-effective cone of the blow-up $\tilde{C}^{(2)}$ is non-polyhedral.

We remark that when $C$ and the point that we are blowing up are general, even if we know that $\overline{\text{Eff}}(\tilde{C}^{(2)})$ is not polyhedral, we do not know any negative class. Therefore it is natural to ask the following.

Question 3.4. When $C$ is general, does $\tilde{C}^{(2)}$ have infinitely many negative curves?

4. The genus one case

In this section we make the assumption that $C$ has genus 1. In particular we first show that in Theorem 1 also the opposite implication holds (see Theorem 4.8). Then we describe the rays of the pseudo-effective cone of $X$, both when it is polyhedral and when it is not (Proposition 4.9), and finally we give a planar model for the resolution $Z$ of $\tilde{X}$.

Remark 4.1. When $g(C) = 1$, the symmetric product $C^{(2)}$ is a ruled surface whose fibers correspond to the $g_1^2$'s of $C$. Observe that if we fix two points $p \neq q \in C$, they define a unique $g_1^2$, and hence a hyperelliptic involution $\sigma$. This implies that the antidiagonal $\Delta_- = \{ r \oplus \sigma(r) \mid r \in C \}$ is indeed a fiber.

Let us recall the following definition from [2, § 3].

Definition 4.2. An elliptic pair $(C, X)$ consists of a projective rational surface $X$ with log terminal singularities and an irreducible curve $C \subseteq X$, with arithmetic genus one, disjoint from the singular locus of $X$ and such that $C^2 = 0$.

The elliptic pair $(C, X)$ is a minimal elliptic pair if it does not contain irreducible curves $E$ such that $K \cdot E < 0$ and $C \cdot E = 0$.

Let us consider as before the blowing-up $\tilde{C}^{(2)} \to C^{(2)}$ at the point $p \oplus q \in \Delta_-$, with $p \neq q$, or equivalently at $p \oplus \sigma(p)$, where $\sigma$ is the involution exchanging $p$ and $q$. We denote by $E$ the exceptional divisor and by $\tilde{C}_p \subseteq \tilde{C}^{(2)}$ the strict transform of the curve $C_p := \{ p \oplus r \mid r \in C \} \subseteq C^{(2)}$. The involution $\sigma$ induces an involution on $\tilde{C}^{(2)}$, that we still denote by $\sigma$, whose ramification is the strict transform $\tilde{\Delta}_- \subseteq \tilde{C}^{(2)}$ of $\Delta_-$. We denote by $\tilde{\phi} : \tilde{C}^{(2)} \to \tilde{X} := \tilde{C}^{(2)}/(\sigma)$ the quotient morphism. Since the curve $\tilde{\phi}(\tilde{C}_p)$ is isomorphic to $C_p$ and hence to the curve $C$, by abuse of notation in what follows we will simply set $C := \tilde{\phi}(\tilde{C}_p)$.

Lemma 4.3. The pair $(C, \tilde{X})$ is a minimal elliptic pair.

Proof. The rationality of $\tilde{X}$ follows from Remark 2.3. From Proposition 2.2 we have that $X$ has 6 ordinary double points, and none of them lies on $B_p := \phi(C_p) = \phi(C_q)$. Therefore they give rise to 6 ordinary double points of $\tilde{X}$, disjoint from the curve $C$. Moreover the involution on $\tilde{C}^{(2)}$ has 2 fixed points on $E$, but only one of them is isolated. Its image is the seventh ordinary double point of $\tilde{X}$ (which does not lie on $C$). This proves that $(C, \tilde{X})$ is an elliptic pair. By Proposition 4.7 we can compute $K^2_{\tilde{X}} = 0$, so that by [2, Lemma 3.7] we conclude that $(C, \tilde{X})$ is minimal.
Remark 4.4. Let us consider a minimal resolution \( \pi: Z \to \tilde{X} \). Since \( C \subseteq \tilde{X} \) does not pass through the singular points, we have an isomorphic copy of \( C \) in \( Z \), that we still denote by \( C \). Therefore \( (C, Z) \) is a smooth minimal elliptic pair and in particular, by [2, Theorem 3.8], the Picard rank of \( Z \) is 10.

Notation 4.5. Before stating our next results about \( \tilde{X} \) and \( Z \) we need to fix some notation. First of all we are going to denote by \( L_i \subseteq \mathbb{P}^2 \), \( 1 \leq i \leq 4 \) the lines whose union is the branch locus of \( \pi: X \to \mathbb{P}^2 \), and by \( E_i \subseteq \tilde{X} \) and \( \tilde{E}_i \subseteq Z \) the strict transforms of \( L_i \) on \( \tilde{X} \) and \( Z \) respectively. By abuse of notation we denote by \( E \) the image \( \phi(E) \subseteq \tilde{X} \) and by \( E \subseteq Z \) its strict transform. Analogously we denote simply \( \Delta \), the curve \( \psi(\Delta) \subseteq \tilde{X} \) and by \( \tilde{\Delta} \) its strict transform in \( Z \). For any \( 1 \leq i < j \leq 4 \), we denote by \( E_{ij} \subseteq Z \) the \((-2)\)-curve over the singular point \( p_{ij} := L_i \cap L_j \subseteq \mathbb{P}^2 \), while the \((-2)\)-curve over the isolated singular point \( \phi(x) \in \tilde{X} \) is denoted by \( E' \).

Finally, for any \( (i, j, k) \in \{(1, 2, 3), (1, 3, 2), (2, 3, 1)\} \), we denote by \( L_{(ij)(k4)} \subseteq \mathbb{P}^2 \) the line through \( p_{ij} \) and \( p_{k4} \). Observe that over any of these \( L_{(ij)(k4)} \) we have two irreducible curves in \( X \), say \( E_{(ij)(k4)} \) and \( E'_{(ij)(k4)} \). We use the same notation for their strict transforms on \( \tilde{X} \), while we denote by \( E_{(ij)(k4)} \) and \( E'_{(ij)(k4)} \) their strict transforms on \( Z \).

We now recall that \( \Gamma = V(x_3^2 - 4x_1x_2) \subseteq \mathbb{P}^2 \), so that \( \Delta = V(x_1 + 2x_1x_3 - 2x_2x_3) \) (it corresponds to one of the two irreducible components over \( \Gamma \)). Moreover we can fix the tangent lines \( L_1, \ldots, L_4 \) to be \( V(x_1), V(x_2), V(x_1 + x_2 - x_3) \) and \( V(x_1 + x_2 + x_3) \) respectively. Then, by Proposition 2.2,

\[
X = V(x_1^2 - x_1x_2(x_1 + x_2 - x_3)(x_1 + x_2 + x_3)) \subseteq \mathbb{P}(1, 1, 1, 2),
\]

and \( L_{(12)(34)} = V(x_1 + x_2), L_{(13)(24)} = V(x_1 + x_2 - x_3), L_{(23)(14)} = V(x_1 - x_2 - x_3) \).

From these equations one can see that the 6 curves \( E_{(ij)(kl)} \) and \( E'_{(ij)(kl)} \) form a hexagon, and we can choose the labels in order to have

\[
E_{(12)(34)} = V(x_1 + x_2, x_1 + x_2 + x_3),
E_{(13)(24)} = V(x_1 - x_2 + x_3, x_1 + 2x_2^2 - 2x_2x_3),
E_{(23)(14)} = V(x_1 - x_2 - x_3, x_1 - 2x_2^2 - 2x_2x_3).
\]

It is now straightforward to check that these 3 curves are disjoint and do not meet \( \Delta \), so that the same holds for the strict transforms on \( \tilde{X} \) and on \( Z \) (analogously, \( E'_{(12)(34)}, E'_{(13)(24)} \) and \( E'_{(23)(14)} \) are disjoint and do not meet \( \Delta \)).

In Picture 2 we represent the intersection products of the negative curves described before. The red dots are the \((-2)\)-curves while the blue dots are the \((-1)\)-curves. When two dots are connected, the two corresponding curves have intersection product 1, otherwise their product is 0.

Remark 4.6. The lattice \( C^\perp \) in \( \text{Pic}(Z) \) is isomorphic to \( \tilde{E}_8 \). Since the eight \((-2)\)-curves described above are all disjoint, their classes span the sublattice \( A^\perp_8 \subseteq \tilde{E}_8 \).

Proposition 4.7. On \( \tilde{X} \) the following hold.

1. \( \text{Cl}(\tilde{X}) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})^2 \).

2. \( \text{Cl}(\tilde{X})_{\text{free}} \) is generated by \( E_1, E_{(12)(34)} \) and \( E \), which have the following intersection matrix

\[
\begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 0 & 0 \\
0 & 0 & -1/2
\end{pmatrix}
\]
We prove that $E$ is a line. The self-intersection of $E$ is of the form \((L - E)^2\), and that $\bar{Z}$ consists of the classes of the following curves: \(\bar{Z}_s\) are contracting seven $s$-curves of \(\bar{Z}\), which has Picard rank 10 (see Remark 4.4). Moreover the torsion part is of the form \((\mathbb{Z}/2\mathbb{Z})^s\), for some $0 \leq s \leq 7$, because the singularities of \(\bar{Z}\) are ordinary double points. A basis for the Picard group of \(\bar{Z}\) consists of the classes of the following curves: $\bar{E}_{12}, \bar{E}_{13}, \bar{E}_{14}, \bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{E}_{(12)(34)}, \bar{E}', \bar{E}$ because the corresponding intersection matrix is unimodular of rank 10. This implies that $\text{Cl}(\bar{X})$ is generated by their images, i.e. $E_1, E_2, E_3, E_4, E_{(12)(34)}, E$ (recall Notation 4.5). Observe that for any $i \neq j$, the class of $E_i - E_j$ is $2$-torsion because $2E_i$ is pullback of a line of $\mathbb{P}^2$. Since the class of $E_1 + E_2 + E_3 + E_4$ is linearly equivalent to the pullback of a conic, it is divisible by 2, so that also the class of $$(E_2 - E_1) + (E_2 - E_3) + (E_2 - E_4) = 4E_2 - (E_1 + E_2 + E_3 + E_4)$$ is divisible by 2, and in particular it is trivial. Therefore the class of $E_4$ is not needed to generate $\text{Cl}(\bar{X})$ and thus $s \leq 2$. On the other hand $E_1 - E_2 \neq E_1 - E_3$ so that $s = 2$.

We prove (2). Since $E$ is disjoint from $E_1$ and $E_{(12)(34)}$, we have $E_1 \cdot E = E_{(12)(34)} \cdot E = 0$. The self-intersection of $E_1$ is $1/2$ because $2E_1$ is the pullback of a line. The self-intersection of $E_{(12)(34)}$ is $0$ because its pullback in $Z$ is $1/2E_{12} + E_{(12)(34)} + 1/2E_{34}$, which has self-intersection $0$. Similarly one shows that $E^2 = -1/2$ and that $E_{(12)(34)} \cdot E_3 = 1/2$.

We prove (3). The equivalence $C \sim 2E_1 - 2E$ follows from equation (1), since $L_p \subseteq \mathbb{P}^2$ is a line tangent to $\Gamma$. By the ramification formula,

$$K_X = \psi^*K_{P^2} + R = \psi^*(-3L) + \psi^*(2L) = -\psi^*(L) = -2\eta(E_1).$$
Recall that the map $\eta: \tilde{X} \to X$ is obtained by blowing up twice (one time on the exceptional divisor) and then contracting the $(-2)$-curve. The contraction is crepant so that it does not affect the canonical class. From this we conclude that

$$K_{\tilde{X}} = \eta^* K_X + 2E = -2E_1 + 2E.$$ 

In order to prove (4) observe that on $Z$ the divisors $2\tilde{E}_{12}(34) + \tilde{E}_{34} + \tilde{E}_{12}$ and $\Delta - 2\tilde{E} + \tilde{E}'$ have both self-intersection 0 and their intersection product is 0. By the Hodge index theorem it follows that the classes of these divisors must be proportional. Since both classes have intersection product 1 with some curve, they are primitive in $\text{Pic}(Z)$. It follows that the two classes are equal, and one concludes by taking pushforward of these classes via $\pi: Z \to \tilde{X}$.

\[\Box\]

**Theorem 4.8.** With the notation above, the following are equivalent:

1. $\text{Eff}(\tilde{C}^{(2)})$ is rational polyhedral;
2. $\text{Eff}(\tilde{X})$ is rational polyhedral;
3. $\text{Eff}(Z)$ is rational polyhedral;
4. the class of $q - p$ has order $m < \infty$ in $\text{Pic}^0(C)$;
5. $\dim | -mK_Z| = 1$ and $\dim | -rK_Z| = 0$, for $0 \leq r < m$.

**Proof.** By Proposition 2.4, $\rho(\tilde{X}) = \rho(\tilde{C}^{(2)})$, so that the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 1.1. Since $\tilde{X}$ has Du Val singularities, the equivalence (2) $\Leftrightarrow$ (3) was proved in [2, Lemma 3.14]. We now prove the equivalence of (3) and (4). Since $(C, Z)$ is an elliptic pair, by [2, §3], the effective cone of $Z$ is rational polyhedral if and only if $C^+\cap Z$ is generated by the kernel of $\text{res}: C^+ \to \text{Pic}^0(C)$. In Remark 4.4 we have already seen that there are eight disjoint $(-2)$-curves in $\ker(\text{res})$. Thus $C^+\cap Z$ is spanned by elements of $\ker(\text{res})$ if and only if there exists an integer $m > 1$ such that the multiple $mC$ is in $\ker(\text{res})$, that is if $\text{res}(C)$ is of $m$-torsion. We conclude by observing that $\text{res}(C) = q - p$ (see the proof of Theorem 2).

Finally, from Proposition 4.7 $C \sim -K_{\tilde{X}}$, and since the curve $C$ is disjoint from the singular points, also on $Z$ we have $C \sim -K_Z$. The equivalence (4) $\Leftrightarrow$ (5) follows. \[\Box\]

We are now going to describe the extremal rays of $\text{Eff}(\tilde{X})$, both when it is polyhedral and when it is not. We remark that by Proposition 1.1 we can identify $\text{Eff}(\tilde{C}^{(2)})$ with $\text{Eff}(\tilde{X})$.

**Proposition 4.9.** Let $m > 1$ be the order of $q - p$ and let us consider the following classes:

$$D_n := 2n(n + 1)E_1 - 2nE_{(12)(34)} + (1 - 2n^2)E, \quad n \in \mathbb{Z}_{\geq 0}.$$ 

1. If $m = \infty$, then $\text{Eff}(\tilde{X}) = (-K_{\tilde{X}}, \Delta, D_0, D_1, \ldots, D_n, \ldots)$.
2. If $m < \infty$, then $\text{Eff}(\tilde{X}) = (-K_{\tilde{X}}, \Delta, D_0, D_1, \ldots, D_{\left\lfloor \frac{m}{m+1}\right\rfloor - 1}, \Gamma_m)$, where

$$\Gamma_m := mE_1 - E_{(12)(34)} + (1 - m)E.$$ 

**Proof.** (1) If $m = \infty$, we already know that the effective cone of $\tilde{X}$ is not polyhedral. A direct calculation shows that $D_n^2 = -1/2$ and $D_n \cdot K_{\tilde{X}} = -1$, for any $n \geq 0$. The divisors $2E_1$ and $2E_{(12)(34)}$ are Cartier, while $E$ is not Cartier. Since $E'$ is the only $(-2)$-curve intersecting $E$ and contracted by $\pi: Z \to \tilde{X}$, it follows that

$$R_n := [\pi^* D_n] = \pi^* D_n - \frac{1}{2} E'.$$
is a divisor with integer coefficients. Since \( R_n^2 = R_n \cdot K_Z = -1 \), by Riemann-Roch we conclude that \( R_n \) is linearly equivalent to an effective divisor. Moreover each \( R_n \) has non-negative intersection product with all the \((-2)\)-curves since \( R_n \cdot E_{ij} = 0 \), \( R_n \cdot E' = 1 \) and \( R_n \cdot \Delta_0 = 2n + 1 \). We claim that \( R_n \) is irreducible. Suppose that we can write \( R_n = C_1 + N \), where \( C_1 \) is an irreducible \((-1)\)-curve and \( N \) is a sum of \((-2)\)-curves. The condition \( R_n^2 = -1 \) implies that either \( R_n \cdot C_1 < 0 \) or \( R_n \cdot N < 0 \), but the latter would imply that \( R_n \) has negative intersection with at least one \((-2)\)-curve, a contradiction. Therefore \((C_1 + N) \cdot C_1 < 0 \), so that \( N \cdot C_1 = 0 \), which implies that also \( N^2 = 0 \). Since the intersection form is negative semidefinite on the components of \( N \), we deduce that \( N \) is indeed a multiple of \(-K \). Therefore \( R_n = C_1 - tK \), which gives \( R_n^2 > -1 \), again a contradiction. This proves the claim, and since \( D_n = \pi_\ast (R_n) \), it is irreducible too.

Consider now the cone \( C \), generated by \(-K_X, \Delta_\ast \) and \( D_n \), for \( n \geq 0 \). The following matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & -2
\end{pmatrix}, \quad \begin{pmatrix}
-2 & 1 \\
1 & -\frac{1}{2}
\end{pmatrix}, \quad \begin{pmatrix}
-\frac{1}{2} & 1 \\
\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]

give the intersection form on the edges \((-K_X, \Delta_\ast \), \((\Delta_\ast, D_0)\) and \((D_n, D_{n+1})\) (for any \( n \geq 0 \)) respectively. Since they are all negative semidefinite and the rays \( D_n \) accumulate on \(-K_X \), we conclude that \( C = \text{Eff}(X) \), which proves (1).

Let us prove (2). Observe that if \( m < \infty \), then \( -mK_Z \) defines an elliptic fibration which is extremal in the sense of Miranda-Persson [14]. According to [14, Thm. 4.1] the only extremal rational elliptic surface which contains eight disjoint \((-2)\)-curves is \( X_{11}(j) \), which has exactly two singular fibers of type \( I_0^* \). Thus, as soon as \( -mK_Z \) moves, two new \((-2)\)-curves appear, each of which is the unique curve of multiplicity two in the fiber \( I_0^* \). On \( \tilde{X} \) one of these curves is disjoint from \( \Delta_\ast \), so that its self-intersection is 0, while the other one intersects \( \Delta_\ast \) and has self-intersection \(-1/2\). The class of the latter is

\[
\Gamma_m := \frac{1}{2}(-mK_X - \Delta_\ast) = mE_1 - E_{(12)(34)} + (1 - m)E,
\]

and by the intersection matrix given in Proposition 4.7 we have that \( \Gamma_m \cdot D_n = 1/2(m - 1) - n \), which is non-negative if and only if \( n < \left\lfloor \frac{m}{2} \right\rfloor \).

\[\square\]

Figure 3. Eff \( \tilde{X} \) for \( m = \infty \) and \( m = 6 \)
Remark 4.10. We remark that the irreducible negative curves on $\tilde{C}^{(2)}$, first found in [6], are unexpected, meaning with this that the expected dimension of the linear system is negative. The images of these curves in $\tilde{X}$ are the curves $D_n$ of Proposition 4.9. They are still negative, of self-intersection $-\frac{1}{2}$, but we have seen that the round-down of the pull-back $R_n = [\pi^* D_n]$ on $Z$ is a $(-1)$-curve, and in particular it is expected.

Remark 4.11. Looking at Figure 2 we see that on $Z$ there are 8 disjoint $(-1)$-curves, namely $\bar{E}_1, \ldots, \bar{E}_4, \bar{E}_{(12)(34)}, \bar{E}_{(13)(24)}, \bar{E}_{(23)(14)}, \bar{E}$. If we contract all of them, $\bar{E}'$ becomes a $(-1)$-curve as well and if we contract also this one, we obtain a birational map $Z \to \mathbb{P}^2$. We denote by $q_i$, $q_{(ij)(kl)}$ and $q$ the images of $\bar{E}_i$, $\bar{E}_{(ij)(kl)}$ and $\bar{E}'$ respectively. The image of $\bar{E}_{12}$ is a line $l_{12}$, passing through $q_1, q_2$ and $q_{(12)(34)}$, and analogously $l_{13}$ and $l_{23}$. The image of $\bar{\Delta}$ is a conic passing through $q_1, \ldots, q_4$ and $q$. In Figure 4 we represent the configuration of the points that we are blowing up on $\mathbb{P}^2$ (together with the tangent direction to the conic at $q$) in order to obtain $Z$.

Therefore, for any $n > 0$, the curve $D_n$ appearing in Proposition 4.9 corresponds to a plane curve. We can compute that its degree is $3n^2 + n$, intersecting $R_n = \pi^* D_n - \frac{1}{2} \bar{E}'$ (see the proof of Proposition 4.9) with the pull-back of $E_{12}$, i.e. the class $\bar{E}_{12} + \bar{E}_1 + \bar{E}_2 + \bar{E}_{(12)(34)}$ in $Z$. In the same way we see that the multiplicity in $q_1$ is $n^2$, by taking the intersection with the pull-back of $E_1$, namely $\bar{E}_1 + \frac{1}{2}(\bar{E}_{12} + \bar{E}_{13} + \bar{E}_{14})$, and the same holds for $q_2, q_3, q_4$. Computing the intersection with $\bar{E}_{(12)(34)} + \frac{1}{2}(\bar{E}_{12} + \bar{E}_{34})$ we have that the multiplicity in $q_{(12)(34)}$ is $n^2 + n$, and the same holds for $q_{(13)(24)}$ and $q_{(23)(14)}$. Finally, the curve has multiplicity $n^2$ at $q$, and multiplicity $n^2 - 1$ at the point infinitely near to $q$, in the direction of the conic.
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