The Elliptical Potential Lemma for General Distributions with an Application to Linear Thompson Sampling

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In this note, we introduce a general version of the well-known elliptical potential lemma that is a widely used technique in the analysis of algorithms in sequential learning and decision-making problems. We consider a stochastic linear bandit setting where a decision-maker sequentially chooses among a set of given actions, observes their noisy rewards, and aims to maximize her cumulative expected reward over a decision-making horizon. The elliptical potential lemma is a key tool for quantifying uncertainty in estimating parameters of the reward function, but it requires the noise and the prior distributions to be Gaussian. Our general elliptical potential lemma relaxes this Gaussian requirement which is a highly non-trivial extension for a number of reasons; unlike the Gaussian case, there is no closed-form solution for the covariance matrix of the posterior distribution, the covariance matrix is not a deterministic function of the actions, and the covariance matrix is not decreasing with respect to the semidefinite inequality. While this result is of broad interest, we showcase an application of it to prove an improved Bayesian regret bound for the well-known Thompson sampling algorithm in stochastic linear bandits with changing action sets where prior and noise distributions are general. This bound is minimax optimal up to constants.

Key words: Elliptical Potential Lemma, Stochastic Linear Bandit, Thompson Sampling

1. Introduction

In sequential linear prediction problems, the classical elliptical potential lemma is a key technique to quantify the decrease in the uncertainty of the model as more observations are obtained. This lemma was first introduced by Lai et al. (1982) to analyze stochastic regression and was later applied to study the sequential ridge forecaster (Cesa-Bianchi and Lugosi 2006) and in proving regret bounds for variants of the stochastic linear bandit problem (Auer 2002, Dani et al. 2008, Chu et al. 2011, Abbasi-Yadkori et al. 2011, Agrawal and Goyal 2013, Li et al. 2019). To state the elliptical potential lemma, let $A_1, A_2, \cdots$ be a sequence of vectors in $\mathbb{R}^d$ that satisfy $\|A_t\|_2 \leq 1$ for all $t \geq 1$. For a fixed constant $\lambda$ with $\lambda \geq 1$, define the sequence of matrices $\{\Sigma_t\}_{t \geq 0}$ as follows:

$$
\Sigma_1^{-1} := \lambda I_d, \quad \Sigma_t^{-1} := \lambda I_d + \sum_{\tau=1}^{t-1} A_\tau A_\tau^\top.
$$
The elliptical potential lemma then asserts that
\[ \sum_{t=1}^{T} A_t^\top \Sigma_t A_t \leq 2 \log \frac{\det \Sigma_1}{\det \Sigma_{T+1}} \leq 2d \log \left( \frac{T}{\lambda d} \right). \] (1.1)

Recently, Carpentier et al. (2020) presented a new proof for this inequality that additionally yields similar bounds for \( \sum_{t=1}^{T} A_t^\top \Sigma_t^p A_t \) for any \( p > 0 \).

In this paper, we generalize (1.1) from a different perspective. Notice that, intuitively, \( \Sigma_t \) captures how much information is available in each direction in a linear model. Specifically, let \( \Theta^* \) be sampled from \( \mathcal{N}(0, \lambda^{-1}I_d) \) and assume that, for each \( t \geq 1 \), an outcome \( Y_t = \langle A_t, \Theta^* \rangle + \varepsilon_t \) is observed where \( \varepsilon_t \) is a standard Gaussian noise, independent of the past. It is well-known that \( \Sigma_t \) is the covariance matrix of the posterior distribution of \( \Theta^* \) conditional on the data available up to time \( t - 1 \), namely, \( A_1, Y_1, \ldots, A_{t-1}, Y_{t-1} \).

The primary contribution of this note is to generalize the elliptical potential bound in (1.1) to any arbitrary prior and noise distributions. This generalization is non-trivial, compared to the Gaussian case, for a number of reasons. First, unlike the Gaussian case, there is no closed form solution for the covariance matrix of the posterior distribution. Second, this covariance matrix is a deterministic function of \( A_1, A_2, \ldots, A_{t-1} \) in the Gaussian case but in general it is a function of the whole history \( A_1, Y_1, A_2, Y_2, \ldots, A_{t-1}, Y_{t-1}, A_t \). Third, the covariance matrix of the posterior distribution for the Gaussian case is non-increasing with respect to semidefinite inequality (i.e., \( \Sigma_1 \succeq \Sigma_2 \succeq \cdots \)) but this property breaks down in the general case. Because of the first two reasons, the covariance update equation \( \Sigma_t^{-1} = \lambda I_d + \sum_{\tau=1}^{t-1} A_\tau A_\tau^\top \) is incorrect for the posterior covariance for general distributions. However, Eq. (1.1) still holds as an algebraic inequality, for example see (Dani et al. 2008, Abbasi-Yadkori et al. 2011). In contrast, our result is for an updated version of Eq. (1.1) that reflects the true covariance matrices.

The secondary contribution of this note is to showcase an application of the aforementioned generalization of the elliptical potential lemma in combination with the proof techniques in (Dong and Van Roy 2018, Kalkanli and Özgür 2020) to prove an \( \mathcal{O}(d\sqrt{T\log T}) \) bound for the Bayesian regret of the well-known linear Thompson sampling (LinTS) algorithm. This result is proved under mild distributional assumptions and allows the action sets to change at each round. This result extends the regret bound of Dong and Van Roy (2018) as they require action sets to be fixed (which excludes for example the \( k \)-armed contextual bandit problem). Our result also generalizes the bound of Kalkanli and Özgür (2020) by relaxing the Gaussian assumption. We note that the above comparison is only made for results that provide the tightest regret bound of \( \mathcal{O}(d\sqrt{T\log T}) \). In fact, Russo and Van Roy (2014) study LinTS with changing action sets, general bounded prior, and sub-Gaussian noise distributions. They prove a Bayesian regret bound of
\[ O(d \log T \sqrt{T}) \] which is worse than our regret bound and the bounds of Dong and Van Roy (2018), Kalkanlı and Özgür (2020) by a factor of \( \sqrt{\log T} \).

Our general elliptical potential lemma is presented in Section 2 and its application to the Bayesian regret of LinTS is provided in Section 3. Proofs are deferred to Sections A to B.

2. Elliptical Potential for General Distributions

In this section, we present our main result. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}\) be an increasing sequence of \(\sigma\)-algebras that are meant to encode the information available up to time \(t\). Let \(\Theta^* : \Omega \rightarrow \mathbb{R}^d\) be the true parameters vector and assume that \(\|\Theta^*\|_2 \leq 1\) almost surely. Furthermore, let \(A_1, A_2, \cdots : \Omega \rightarrow \mathbb{R}^d\) be a sequence of random vectors such that for all \(t \geq 1\), \(A_t\) is \(\mathcal{F}_t\)-measurable and \(\|A_t\|_2 \leq 1\) almost surely. More information about \(\Theta^*\) is then made available sequentially through a sequence of observations \(Y_1, Y_2, \cdots : \Omega \rightarrow \mathbb{R}\) where \(Y_t\) is \(\mathcal{F}_{t+1}\)-measurable and

\[ \mathbb{E}[Y_t \mid \mathcal{F}_t, \Theta^*] = \langle \Theta^*, A_t \rangle \quad \text{and} \quad \text{Var}(Y_t \mid \mathcal{F}_t, \Theta^*) \leq \sigma^2, \]

for all \(t \geq 1\) almost surely. We denote the posterior covariance matrix of \(\Theta^*\) at time \(t\) by \(\Gamma_t\), that is

\[ \Gamma_t := \text{Var}(\Theta^* \mid \mathcal{F}_t). \]

It follows from the definition that \(\Gamma_t\) is a stochastic positive semi-definite matrix in \(\mathbb{R}^{d \times d}\) that is \(\mathcal{F}_t\)-adapted. Notice that, nonetheless, it is \textit{not} true in general that \(\Gamma_{t+1} \preceq \Gamma_t\). To see this, let \(d = 1\) and \(\Theta^* \in \{0, 1/4, 3/4\}\) be such that the prior distribution of \(\Theta^*\) satisfies \(\mathbb{P}(\Theta^* = 1/4) = 3p\) and \(\mathbb{P}(\Theta^* = 3/4) = p\) for some small \(p > 0\). Also, define \(A_t := 1\) for all \(t \geq 1\) and assume \(Y_t\) is a Bernoulli random variable with mean \(\Theta^*\). We further let \(\mathcal{F}_t\) be the smallest \(\sigma\)-algebra generated by \(Y_1, \cdots, Y_{t-1}\). In this case, it is easy to see that \(\Gamma_1 = \text{Var}(\Theta^* \mid \mathcal{F}_1) = \text{Var}(\Theta^*)\) can be made arbitrarily small by choosing a sufficiently small \(p > 0\). In this case, notice that, whenever \(Y_1 = 1\), the distribution of \(\Theta^*\) conditional on \(\mathcal{F}_2\) is uniform over \(\{1/4, 3/4\}\) which gives us \(\Gamma_2 = 1/4 > \Gamma_1\).

This can be shown by noting that \(\mathbb{P}(\Theta^* = 1/4 \mid Y_1 = 1) = 0\) and

\[ \frac{\mathbb{P}(\Theta^* = 1/4 \mid Y_1 = 1)}{\mathbb{P}(\Theta^* = 3/4 \mid Y_1 = 1)} \cdot \frac{\mathbb{P}(Y_1 = 1 \mid \Theta^* = 1/4)}{\mathbb{P}(Y_1 = 1 \mid \Theta^* = 3/4)} = \frac{3p \cdot \frac{1}{4}}{p \cdot \frac{3}{4}} = 1. \]

We can, however, apply the law of total variance to get

\[ \mathbb{E}[\Gamma_{t+1} \mid \mathcal{F}_t] \preceq \mathbb{E}[\Gamma_{t+1} \mid \mathcal{F}_t] + \text{Var}(\mathbb{E}[\Theta^* \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t) = \text{Var}(\Theta^* \mid \mathcal{F}_t) = \Gamma_t. \]

This inequality only shows that \(\Gamma_t\) decreases in expectation but does not tell us \textit{how much} the expected variance decreases at each round. The next lemma provides a stronger bound. The proofs of this lemma and other results of this section are postponed to Section A.
Lemma 2.1 (Stochastic variance reduction) Whenever the above-mentioned assumptions hold, for all $t \geq 1$, we have

$$\mathbb{E} \left[ \Gamma_{t+1} \mid F_t \right] \preceq \Gamma_t - \frac{\Gamma_t^\top A_t A_t^\top \Gamma_t}{\sigma^2 + A_t^\top \Gamma_t A_t}$$

almost surely.

Lemma 2.1 demonstrates that the posterior covariance decays in expectation. As we discussed by the above example, this does not necessarily hold for $\Gamma_t$ almost surely. In fact, this is the most challenging roadblock in establishing a general version of the elliptical potential lemma as one can increase $\sum_{t=1}^T A_t^\top \Gamma_t A_t$ by defining $A_t$’s adaptively, to be aligned with high variance directions. The following lemma which is the main technical contribution of this note introduces a methodology to overcome this difficulty.

Lemma 2.2 For $x > 0$ and positive semi-definite matrix $\Sigma$, define $f(\Sigma, x) = \log \det(\mathbb{I} + x \Sigma)$. Then, $f(\cdot, \cdot)$ satisfies the following properties:

1. For any fixed $x > 0$, $f(\cdot, x)$ is a concave function on the positive semi-definite cone.
2. If $\Sigma$ is an invertible and positive semidefinite matrix then $f(\Sigma, x)$ satisfies the following variational representation

$$f(\Sigma, x) = \log \det \left( \Sigma^\frac{1}{2} \left( \Sigma^{-1} + x \mathbb{I} \right) \Sigma^\frac{1}{2} \right)$$

$$= \sup_{\Lambda \preceq x \mathbb{I}} \log \det \left( \Sigma^\frac{1}{2} \left( \Sigma^{-1} + \Lambda \right) \Sigma^\frac{1}{2} \right). \tag{2.1}$$

3. For any vector $V \in \mathbb{R}^d$, we have

$$\log(1 + V^\top \Sigma V) + f(\Sigma', x) \leq f(\Sigma, x + V^\top V) \tag{2.2}$$

where $\Sigma' := \Sigma - \Sigma V V^\top \Sigma = \Sigma^\frac{1}{2} \left( \mathbb{I} - \frac{V V^\top \Sigma^\frac{1}{2}}{1 + V^\top \Sigma V} \right) \Sigma^\frac{1}{2}$.

Using this result, we are now ready to conclude this section by stating our elliptical potential inequality for general distributions.

Theorem 2.1 (Elliptical Potential for General Distributions) Under the above assumptions, the following inequality holds,

$$\mathbb{E} \left[ \sum_{t=1}^T A_t^\top \Gamma_t A_t \right] \leq 2 \max(\sigma^2, 1) \log \det(\mathbb{I} + T \Gamma_1) .$$
Algorithm 1 Linear Thompson sampling (LinTS)
1: for $t = 1, 2, \cdots$ do
2: Observe the actions set $\mathcal{A}_t \subseteq \mathbb{R}^d$.
3: Sample $\Theta_t \sim \mathbb{P}(\Theta^* | \mathcal{F}_t)$. 
4: $\tilde{A}_t \leftarrow \arg \max_{A \in \mathcal{A}_t} \langle A, \Theta_t \rangle$
5: Observe reward $Y_t$.
6: end for

3. Linear Thompson Sampling

In this section, we apply Theorem 2.1 to show that, up to constants, Linear Thompson Sampling (LinTS) achieves an optimal prior-independent Bayesian regret. This statement is stronger than the bound in (Dong and Van Roy 2018) as it allows for changing action sets and it is more general than (Kalkanlı and Özgür 2020) since it does not require Gaussian assumption for the prior and noise distributions. As noted before, Russo and Van Roy (2014) also study LinTS with changing action sets and without Gaussian assumptions for the prior or noise distributions, but their Bayesian regret bound is not optimal for this class of problems. Specifically, there is an additional $\sqrt{\log T}$ factor in their bound compared to the one we provide here.

First, let $\bar{\Theta}_t$ be the posterior mean of $\Theta^*$ at time $t$. We also denote by $A^*_t$ and $\tilde{A}_t$ the optimal arm and the selected arm at time $t$ respectively. Now notice that the expected regret at time $t$ can be expressed as $\mathbb{E}[(\Theta^* - \bar{\Theta}_t)^\top A^*_t]$. In order to bound this, we utilize the idea in the proof of Proposition 5 in (Russo and Van Roy 2016) which was later generalized by Kalkanlı and Özgür (2020). This idea avoids constructing confidence sets around $\Theta^* - \bar{\Theta}_t$ that introduce an additional $\sqrt{\log T}$ term. We bring this idea and a slightly modified proof for that here. The proof of all results in this section is deferred to Section B.

Lemma 3.1 Let $X, Z$ be two random vectors in $\mathbb{R}^d$. Then, we have

$$\mathbb{E}[X^\top Z]^2 \leq d \text{Tr} (\mathbb{E}[XX^\top] \mathbb{E}[ZZ^\top]).$$

Notice that this lemma does not require independence between $X$ and $Z$. Therefore, one can set $X := \Theta^* - \bar{\Theta}_t$ and $Z := A^*_t$. Then, the main step in the proof is observing that $\mathbb{E}[XX^\top | \mathcal{F}_t] = \Gamma_t$, $\mathbb{E}[ZZ^\top | \mathcal{F}_t] = \mathbb{E}[\tilde{A}_t A^*_t^\top | \mathcal{F}_t]$, and $X$ and $\tilde{A}_t$ are independent conditional on $\mathcal{F}_t$.

Theorem 3.1 Let $\Theta^*$ be such that $\|\Theta^*\|_2 \leq 1$ almost surely and $\mathcal{F}_t$ be the $\sigma$-algebra generated by $(\mathcal{A}_t, \tilde{A}_t, Y_1, A_2, \cdots, A_t, \tilde{A}_t)$. Furthermore, assume that

$$\mathbb{E}[Y_t | \mathcal{F}_t, \Theta^*] = \langle \Theta^*, \tilde{A}_t \rangle \quad \text{and} \quad \text{Var}(Y_t | \mathcal{F}_t, \Theta^*) \leq \sigma^2,$$
almost surely. Then, the following regret bound holds for LinTS (Algorithm 1) when it has access to the true prior and noise distributions:

\[
\text{BayesRegret}(T, \pi^{\text{LinTS}}) \leq \sqrt{2 \max(\sigma^2, 1) d T \log \det(1 + T \Gamma_1)}.
\]  

(3.1)

**Remark 3.1** The assumption that \( \|\Theta^*\|_2 \leq 1 \) almost surely implies that \( \Gamma_1 \preceq I \). Hence, we have the trivial bound \( \log \det(1 + T \Gamma_1) \leq d \log(1 + T) \) which in turn leads to

\[
\text{BayesRegret}(T, \pi^{\text{LinTS}}) \leq d \sqrt{2 \max(\sigma^2, 1) T \log(1 + T)}.
\]

**Remark 3.2** As shown in (Hamidi and Bayati 2020), the assumption that LinTS uses the true prior distribution for \( \Theta^* \) is crucial, as the Bayesian regret of LinTS can grow linearly for \( \exp(C d) \) rounds for some constant \( C > 0 \) under a mild distributional mismatch.

**Remark 3.3** An interesting aspect of this result is that it does not require the noise to be bounded or sub-Gaussian. Having a bounded second moment suffices for Eq. (3.1) to hold. For the special case of \( k \)-armed (and non-contextual) bandits, Bubeck et al. (2013) show that when noise has a bounded second moment one can obtain matching regret bounds as when noise is sub-Gaussian. It is an open question whether their proof technique can be adapted to the setting we study here, without extending Eq. (1.1). Moreover, Bubeck et al. (2013) use a UCB type algorithm with a modified mean reward estimator based on robust statistics. It is intriguing that Theorem 3.1 does not require modifying LinTS.

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Appendix A: Proof of Section 2

Proof of Lemma 2.1. Let \( E_t[\cdot] \) be the shorthand for \( E[\cdot | \mathcal{F}_t] \). First, we prove the claim for \( \Theta^* \) with \( E_t[\Theta^*] = 0 \). It suffices to prove that

\[
V^T E_t[\Gamma_{t+1}] V = E_t \left[ \text{Var} \left( \langle \Theta^*, V \rangle \middle| \mathcal{F}_{t+1} \right) \right] \leq V^T \Gamma_t V - \frac{(A_t^T \Gamma_t V)^2}{\sigma^2 + A_t^T \Gamma_t A_t}
\]

for any fixed vector \( V \in \mathbb{R}^d \). Denoting by \( \mathcal{F}_t^A \) the set of \( \mathcal{F}_t \)-adaptable random variables, we have

\[
E_t \left[ \text{Var} \left( \langle \Theta^*, V \rangle \middle| \mathcal{F}_{t+1} \right) \right] = E_t \left[ \inf_{W \in \mathcal{F}_t} E_t \left[ \left( \langle \Theta^*, V \rangle - W \right)^2 \middle| \mathcal{F}_{t+1} \right] \right]
\]
we just proved (for the case almost surely, and \( \mathbb{E} \mu \)) by putting all the above together, we get the desired result. Finally, whenever whenever first term, using the assumption \( \mathbb{E} \).

Next, we can simplify each of the two expectations on the right hand side of Eq. (A.1). For the first term, using the assumption \( \mathbb{E}_t[\Theta^*] = 0 \), we have

\[
\mathbb{E}_t[\langle \Theta^*, V \rangle] = \mathbb{E}_t[V^\top \Theta^* \Theta^* V] = V^\top \Gamma_t V.
\]

For the second expectation, the numerator can also be computed in the following way

\[
\mathbb{E}_t[\langle \Theta^*, V \rangle Y_t] = \mathbb{E}_t\left[\mathbb{E}[\langle \Theta^*, V \rangle Y_t \mid \mathcal{F}_t, \Theta^*]\right] = \mathbb{E}_t\left[\langle \Theta^*, V \rangle \cdot \mathbb{E}[Y_t \mid \mathcal{F}_t, \Theta^*]\right] = \mathbb{E}_t\left[\langle \Theta^*, V \rangle \langle \Theta^*, A_t \rangle\right] = \mathbb{E}_t\left[A_t^\top \Theta^* \Theta^* V\right] = A_t^\top \Gamma_t V.
\]

Finally, for the denominator of the second expectation we have

\[
\mathbb{E}_t[Y_t^2] = \mathbb{E}_t\left[\text{Var}(Y_t \mid \mathcal{F}_t, \Theta^*) + \mathbb{E}[Y_t \mid \mathcal{F}_t, \Theta^*]^2\right] \leq \sigma^2 + \mathbb{E}_t[\langle \Theta^*, A_t \rangle^2] = \sigma^2 + A_t^\top \Gamma_t A_t.
\]

By putting all the above together, we get the desired result. Finally, whenever \( \mathbb{E}_t[\Theta^*] \neq 0 \), define \( \mu^* := \Theta^* - \mathbb{E}_t[\Theta^*] \) and \( Z_t := Y_t - \langle \mathbb{E}_t[\Theta^*], A_t \rangle \). Note that \( \text{Var}(\mu^* \mid \mathcal{F}_t) = \Gamma_t, \mathbb{E}[Z_t \mid \mathcal{F}_t, \Theta^*] = \langle \mu^*, A_t \rangle \) almost surely, and \( \mathbb{E}[\text{Var}(Z_t \mid \mathcal{F}_t, \Theta^*)] = \mathbb{E}[\text{Var}(Y_t \mid \mathcal{F}_t, \Theta^*)] \leq \sigma^2 \). Therefore, we can apply the result we just proved (for the case \( \mathbb{E}_t[\Theta^*] = 0 \)) to \( \mu^* \) and \( Z_t \) and get

\[
\mathbb{E}_t[\text{Var}(\mu^* \mid \mathcal{F}_{t+1})] \leq \Gamma_t - \frac{\Gamma_t \Gamma_t A_t^\top \Gamma_t}{\sigma^2 + A_t^\top \Gamma_t A_t}.
\]

Combining this by the fact that \( \mathbb{E}_t[\Gamma_{t+1}] = \mathbb{E}_t[\text{Var}(\mu^* \mid \mathcal{F}_{t+1})] \), we conclude the result for \( \Theta^* \) and \( Y_t \).

\( \square \)
Proof of Lemma 2.2. The concavity of \( f(\cdot, x) \) follows from the fact that \( \log \det(\cdot) \) is concave over the positive semi-definite cone, see (Boyd et al. 2004, page 74), and \( f(\cdot, x) \) is obtained by composing \( \log \det(\cdot) \) with a linear function of \( \Sigma \).

The variational representation can be obtained by noting that \( \log \det(\cdot) \) is increasing with respect to the positive semi-definite order \( \preceq \).

We now turn to proving Eq. (2.2). We first assume that \( \Sigma \) is invertible. In this case, we have \( \Sigma^{-1} = \Sigma^{-1} + VV^T \), using Sherman–Morrison formula. From Eq. (2.1) and using \( \det(AB) = \det(A) \det(B) \), we get that

\[
f(\Sigma, x + V^T V) = \sup_{\Lambda \prec (x + V^T V)I} \log \det(\Sigma^\frac{1}{2} (\Sigma^{-1} + \Lambda) \Sigma^\frac{1}{2}) \geq \sup_{\Lambda' \prec xI} \log \det(\Sigma^\frac{1}{2} (\Sigma^{-1} + VV^T + \Lambda') \Sigma^\frac{1}{2}) = \sup_{\Lambda' \prec xI} \log \det(\Sigma^\frac{1}{2} (\Sigma^{-1} + \Lambda') \Sigma^\frac{1}{2}) = \log \det(\Sigma) + \log \det(\Sigma^{-1} + xI)
\]

\[
= \log \det(\Sigma') - \log \det \left( \frac{\Sigma^\frac{1}{2} VV^T \Sigma^\frac{1}{2}}{1 + V^T V} \right) + \log \det(\Sigma^{-1} + xI)
\]

\[
\overset{(b)}{=} \log \det(\Sigma') - \log \left( 1 - \frac{V^T \Sigma V}{1 + V^T V} \right) + \log \det(\Sigma^{-1} + xI)
\]

\[
= \log \det(\Sigma') + \log \left( 1 + V^T \Sigma V \right) + \log \det(\Sigma^{-1} + xI)
\]

\[
= \log \left( 1 + V^T \Sigma V \right) + \log \det \left( (\Sigma^\frac{1}{2} (\Sigma^{-1} + xI) \Sigma^\frac{1}{2} \right)
\]

\[
= \log \left( 1 + V^T \Sigma V \right) + f(\Sigma', x).
\]

The inequality (a) uses the triangle inequality

\[
\|\Lambda' + V^T V\|_{op} \leq \|\Lambda'\|_{op} + \|V^T V\|_{op} = \|\Lambda'\|_{op} + VV^T.
\]

and the equality (b) is obtained by observing that \( \det(I + ZZ^T) = 1 + Z^T Z \) for any vector \( V \).

It only remains to prove Eq. (2.2) for a non-invertible matrix \( \Sigma \). In this case, for \( \epsilon > 0 \), we define \( \Sigma_{\epsilon} = \Sigma + \epsilon I \) and \( \Sigma'_{\epsilon} := \Sigma_{\epsilon} - \frac{\Sigma V V^T \Sigma_{\epsilon}}{1 + V^T \Sigma_{\epsilon} V} \). Clearly, \( \Sigma_{\epsilon} \) is invertible. Therefore, we can apply Eq. (2.2) to \( \Sigma_{\epsilon} \) to obtain

\[
\log(1 + V^T \Sigma_{\epsilon} V) + f(\Sigma'_{\epsilon}, x) \leq f(\Sigma_{\epsilon}, x + V^T V).
\]

The claim then follows the continuity of the above expressions with respect to \( \epsilon \) on \([0, \infty)\). \( \square \)
Proof of Theorem 2.1. Without loss of generality, we can assume that $\sigma \leq 1$ to simplify the analysis. Otherwise, we can re-scale each action $A_t$ and the noise by a factor $1/\max(\sigma, 1)$ and under this transformation the property $\|A_t\| \leq 1$ continues to hold.

Now, notice that since $\|\Theta^\star\|_2 \leq 1$ and $\|A_t\| \leq 1$ almost surely, we have $A_t^\top \Gamma_t A_t \leq 1$ for all $t \in [T]$ almost surely. Next, the fact that $x \leq 2 \log(1+x)$ for all $x \in [0,1]$ implies that

$$A_t^\top \Gamma_t A_t \leq 2 \log \left(1 + A_t^\top \Gamma_t A_t\right). \quad (A.2)$$

We now prove the main result inductively. For $T = 1$, it suffices to note that

$$\mathbb{E} \left[ \sum_{t=2}^{T} A_t^\top \Gamma_t A_t \middle| A_1, Y_1 \right] \leq 2 \log \det (I + (T-1)\Gamma_2)$$

almost surely. Using the concavity of $\log \det(\cdot)$, it follows from Jensen’s inequality and Lemma 2.1 that

$$\mathbb{E} \left[ \sum_{t=2}^{T} A_t^\top \Gamma_t A_t \middle| A_1, Y_1 \right] \leq 2 \log \det (I + (T-1)\Gamma_2)$$

For $T > 1$, we can use the induction hypothesis for $T - 1$ and get that

$$\mathbb{E} \left[ \sum_{t=2}^{T} A_t^\top \Gamma_t A_t \middle| A_1, Y_1 \right] \leq 2 \log \det (I + (T-1)\Gamma_2)$$

almost surely. Using the concavity of $\log \det(\cdot)$, it follows from Jensen’s inequality and Lemma 2.1 that

$$\mathbb{E} \left[ \sum_{t=2}^{T} A_t^\top \Gamma_t A_t \middle| A_1, Y_1 \right] \leq 2 \log \det (I + (T-1)\Gamma_2)$$

where $\Gamma'_1 := \Gamma_1 - \frac{\Gamma_1^\top A_1^\top \Gamma_1}{1 + A_1^\top \Gamma_1 A_1}$. Finally, we apply Eq. (2.2) in Lemma 2.2 and Eq. (A.2) to get that

$$\mathbb{E} \left[ \sum_{t=1}^{T} A_t^\top \Gamma_t A_t \right] \leq 2 \mathbb{E} \left[ \log \left(1 + A_1^\top \Gamma_1 A_1\right) + f(\Gamma'_1, T-1)\right]$$

$$\leq 2 \mathbb{E} \left[ f(\Gamma_1, T)\right]$$

$$= 2 \log \det (I + T\Gamma_1).$$

\[\square\]

Appendix B: Proofs of Section 3

Proof of Lemma 3.1. First, we observe that for any unitary matrix $U$, if one defines $X' := UX$ and $Z' := UZ$, we have that

$$\mathbb{E} [X^\top Z] = \mathbb{E} [X'^\top Z'].$$

and

\[
\text{Tr}(E[X'X'] E[Z'Z']) = \text{Tr}(E[UXX^TU^T] E[ZZ^TU^T])
\]

\[
= \text{Tr}(UE[X^T]U^TUE[ZZ^T]U^T)
\]

\[
= \text{Tr}(U^TUE[X^T]U^TUE[ZZ^T])
\]

\[
= \text{Tr}(E[X^T]E[ZZ^T]).
\]

These equalities imply that it suffices to prove the statement for \((X', Z')\) instead of \((X, Z)\). Now we choose \(U\) so that \(E[Z'Z'^T] = U E[ZZ^T] U^T\) is diagonal. This can be done through the singular value decomposition of \(E[ZZ^T]\). Then, notice that

\[
d \text{Tr}(E[X'X'] E[Z'Z']) = d \sum_{i=1}^{d} E[X'^2] E[Z'^2]
\]

\[
\geq d \sum_{i=1}^{d} E[X'_i Z'_i]^2
\]

\[
= \left( \sum_{i=1}^{d} 1 \right) \left( \sum_{i=1}^{d} E[X'_i Z'_i]^2 \right)
\]

\[
\geq E \left[ \sum_{i=1}^{d} X'_i Z'_i \right]^2
\]

\[
= E[X'^T Z']^2,
\]

where the inequalities are deduced from the Cauchy-Schwartz inequality. \(\square\)

**Proof of Theorem 3.1.** First, observe that \((\Theta^*, A^*_t)\) and \((\tilde{\Theta}_t, \tilde{A}_t)\) are exchangeable conditional on \((F_t, A_t)\). Then, defining \(\mu_t := E[\Theta^* | F_t, A_t]\), we have that

\[
\text{BayesRegret}(T, \pi_{\text{LinTS}}) = \sum_{t=1}^{T} E \left[ \langle \Theta^*, A^*_t \rangle - \langle \Theta^*, \tilde{A}_t \rangle \right]
\]

\[
= \sum_{t=1}^{T} E \left[ \langle \Theta^*, A^*_t \rangle \right] - E \left[ E \left[ \langle \Theta^*, \tilde{A}_t \rangle \mid F_t \right] \right]
\]

\[
= \sum_{t=1}^{T} E \left[ \langle \Theta^*, A^*_t \rangle \right] - E \left[ E [\Theta^* \mid F_t], E [\tilde{A}_t \mid F_t] \right]
\]

\[
= \sum_{t=1}^{T} E \left[ \langle \Theta^*, A^*_t \rangle \right] - E \left[ \langle \mu_t, E [A^*_t \mid F_t] \rangle \right]
\]

\[
= \sum_{t=1}^{T} E \left[ \langle \Theta^*, A^*_t \rangle - \langle \mu_t, A^*_t \rangle \mid F_t \right].
\]
Define $\Gamma_t := \mathbb{E}\left[ (\Theta^* - \mu_t)(\Theta^* - \mu_t)^\top \mid \mathcal{F}_t \right]$. Then, it follows from Lemma 3.1 and the independence of $\bar{A}_t$ and $\Theta^*$ conditional on $\mathcal{F}_t$ that

$$\text{BayesRegret}(T, \pi_{\text{LinTS}}) \leq \sqrt{d} \sum_{t=1}^T \mathbb{E}\left[ \text{Tr}(\Gamma_t \cdot \mathbb{E}[A_t^* A_t^{*\top} \mid \mathcal{F}_t]) \right]^{\frac{1}{2}}$$

$$= \sqrt{d} \sum_{t=1}^T \mathbb{E}\left[ \text{Tr}(\Gamma_t \cdot \mathbb{E}[\bar{A}_t \bar{A}_t^\top \mid \mathcal{F}_t]) \right]^{\frac{1}{2}}$$

$$= \sqrt{d} \sum_{t=1}^T \mathbb{E}\left[ \text{Tr}(\mathbb{E}[\Gamma_t \cdot \bar{A}_t \bar{A}_t^\top \mid \mathcal{F}_t]) \right]^{\frac{1}{2}}$$

$$= \sqrt{d} \sum_{t=1}^T \mathbb{E}\left[ \text{Tr}(\Gamma_t \cdot \bar{A}_t^\top \bar{A}_t \mid \mathcal{F}_t) \right]^{\frac{1}{2}}$$

$$\leq \sqrt{dT} \mathbb{E}\left[ \sum_{t=1}^T \bar{A}_t^\top \Gamma_t \bar{A}_t \right]^{\frac{1}{2}},$$

where the last two inequalities are obtained by applying the Cauchy-Schwartz inequality. The desired result follows from Theorem 2.1. \qed