RODRIGUES FORMULAS FOR THE MACDONALD POLYNOMIALS

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Abstract. We present formulas of Rodrigues type giving the Macdonald polynomials for arbitrary partitions \( \lambda \) through the repeated application of creation operators \( B_k, k = 1, \ldots, \ell(\lambda) \) on the constant 1. Three expressions for the creation operators are derived one from the other. When the last of these expressions is used, the associated Rodrigues formula readily implies the integrality of the \((q, t)\)-Kostka coefficients. The proofs given in this paper rely on the connection between affine Hecke algebras and Macdonald polynomials.

1. Introduction

The Macdonald polynomials \( J_\lambda(x; q, t) \) form a two-parameter basis for symmetric polynomials \([1]\). They play an important role in algebraic combinatorics and, in mathematical physics, they occur in particular in the wave functions of integrable quantum many-body models \([2]\). We shall show that the polynomials \( J_\lambda(x) \) in \( N \) variables can be constructed by acting with a string of creation operators \( B_k, k = 1, \ldots, N \) on the constant 1, and shall thereby give Rodrigues formulas for these polynomials. Such results were first obtained in \([3]\) in the limit case \( q = t^\alpha, t \to 1 \) of the Jack polynomials and proved rather useful \([4]\).

Three expressions \( B_k^{(i)}, i = 1, 2, 3 \), will be obtained for the creation operators of the Macdonald polynomials. Expression \( B_k^{(1)} \) will first be derived using the Pieri formula. The operator \( B_k^{(2)} \) will then be shown to be equal to the operator \( B_k^{(1)} \) and expression \( B_k^{(3)} \) will finally be obtained from \( B_k^{(2)} \) by observing that many terms in \( B_k^{(2)} \) (and hence in \( B_k^{(1)} \)) act trivially on the Macdonald polynomials \( J_\lambda \) associated to partitions with no more than \( k \) parts. Expression \( B_k^{(1)} \) was first derived in \([5]\) where in addition, the \( q \)-difference operator version of \( B_k^{(3)} \) was given as a conjecture. This third expression was also found by Kirillov and Noumi who provided two proofs \([6,7]\) of the fact that the operators \( B_k^{(3)} \) are creation operators for the Macdonald polynomials. It should be pointed out that the integrality of the \((q, t)\)-Kostka coefficients \([8]\) is an immediate consequence of the Rodrigues formula for \( J_\lambda(x) \) associated to \( B_k^{(3)} \). We shall derive this formula from the one involving the operators \( B_k^{(1)} \) by obtaining, as an intermediate step, the Rodrigues formula with the \( B_k^{(2)} \) as creation operators. Our proofs will rely in an essential way on the connection between affine Hecke algebras and Macdonald polynomials \([9,10]\). They will use in particular the fact that the Macdonald operators can be realized in terms of Dunkl-Cherednik operators. This is the main difference between the

\[\text{1 Other proofs of the integrality of the \((q, t)\)-Kostka coefficients have been given recently using different approaches by Garsia and Remmel \([11]\), Garsia and Tesler \([12]\), Knop \([13,14]\) and Sahi \([15]\).}\]
2. THE AFFINE HECKE ALGEBRA \( H(\tilde{W}) \)

Let \( \Lambda_N = \mathbb{Q}(q,t)[x_1, \ldots, x_N] \) be the ring of polynomials in the \( N \) variables \( x_1, \ldots, x_N \) with coefficients in \( \mathbb{Q}(q,t) \), the field of rational functions in the two indeterminates \( q \) and \( t \). The Weyl group \( W \cong S_N \) is generated by the transpositions \( s_i, i = 1, \ldots, N \). On \( x^\lambda = x_1^{\lambda_1} \cdots x_N^{\lambda_N} \) their action is such that

\[
s_i x^\lambda = x^{s_i \lambda} s_i, \tag{1}\n\]

where \( s_i \lambda = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_N) \). We denote by \( \Lambda_N^W \), the subring of all symmetric polynomials. We can extend the action of the Weyl group \( W \) on \( \Lambda_N \) to one of the affine Weyl group \( \tilde{W} \) by introducing the elements \( s_0 \) and \( \omega \pm 1 \) realized by:

\[
s_0 = s_N-1 \cdots s_2 s_1 s_2 \cdots s_{N-1} \tau_1 \tau_N^{-1}, \omega = s_N-1 \cdots s_1 \tau_1 = \tau_N s_N-1 \cdots s_1. \tag{2}\n\]

where \( \tau_i \), the shift operator, is such that

\[
\tau_i f(x_1, \ldots, x_N) = f(x_1, \ldots, qx_i, \ldots, x_N) \tag{3}\n\]

for any polynomial \( f \in \Lambda_N \).

The generators of \( \tilde{W} \) obey the fundamental relations:

\[
\begin{align*}
(i) \quad & s_i^2 = 1, & i = 0, 1, \ldots, N - 1, \\
(ii) \quad & s_i s_j = s_j s_i, & |i - j| \geq 2, \\
(iii) \quad & s_i s_j s_i = s_j s_i s_j, & |i - j| = 1, \\
(iv) \quad & \omega s_i = s_{i-1} \omega, & i = 0, 1, \ldots, N - 1.
\end{align*} \tag{4}\n\]

where the indices \( 0, 1, \ldots, N - 1 \) are understood as elements of \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \). In the case of the Weyl group \( W \), for any \( w \in W \), there is a smallest positive integer \( p \) such that \( w = s_{i_1} \cdots s_{i_p} \) (reduced decomposition). We say that \( p \) is the length \( L(w) \) of \( w \). Let \( v, w \in \tilde{W} \), in the Bruhat order, \( v \leq w \) if \( v \) is of the form \( v = s_{j_1} \cdots s_{j_q} \) with \( (j_1, \ldots, j_q) \) a subsequence of \( (i_1, \ldots, i_p) \).

The operators

\[
T_i = 1 + \frac{1 - t^{-1} x_i q^{1-x_i}}{1 - x_i q^{-1}} (s_i - 1), \tag{5}\n\]

for \( i = 1, \ldots, N - 1 \) and

\[
T_0 = 1 + \frac{1 - t^{-1} q^{-1} x_N}{1 - q^{-1} x_N} (s_0 - 1), \tag{6}\n\]
and $\omega^\pm 1$ realize on $\Lambda_N$ the affine Hecke algebra $H(\tilde{W})$ of $\tilde{W}$, that is they verify the defining relations

\begin{align}
(i) \quad (T_i - 1)(T_i + t^{-1}) &= 0, & i = 0, 1, \ldots, N - 1, \\
(ii) \quad T_iT_j &= T_jT_i, & \vert i - j \vert \geq 2, \\
(iii) \quad T_iT_jT_i &= T_jT_iT_j, & \vert i - j \vert = 1, \\
(iv) \quad \omega T_i &= T_{i-1}\omega, & i = 0, 1, \ldots, N - 1, \\
\end{align}

(7)

where again the indices are understood as elements of $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. The Dunkl-Cherednik operators $Y_1, \ldots, Y_N$ are constructed as follows from the generators of $H(\tilde{W})$:

\begin{equation}
Y_i = T_i \ldots T_{N-1}\omega T_{1}^{-1} \ldots T_{i-1}^{-1}.
\end{equation}

(8)

They form an Abelian algebra: $[Y_i, Y_j] = 0, \forall i, j \in 1, \ldots, N$. They also satisfy the following commutation relations with the $T_i$'s:

\begin{align}
T_iY_{i+1}T_i &= Y_i, \\
T_iY_jT_i &= Y_j, & j \neq i, i + 1.
\end{align}

(9)

Let $J = \{j_1, j_2, \ldots, j_\ell\}$ denote sets of cardinality $|J| = \ell$ made out of integers $j_\kappa \in \{1, \ldots, N\}, 1 \leq \kappa \leq \ell$ such that $j_1 < j_2 < \cdots < j_\ell$. We introduce the operators

\begin{equation}
Y_{J,u} = (1 - ut^{j_1-1}Y_{j_1}) \cdots (1 - ut^{j_\ell-1}Y_{j_\ell})(1 - uY_{j_1}^{-1})
\end{equation}

(10)

associated to such sets and labelled by a real number $u$. If $|J| = 0$, we define $Y_{J,u} = 1$.

For each $w = s_{i_1} \ldots s_{i_p} \in W$, $T_w$ is defined by

\begin{equation}
T_w = T_{i_1} \ldots T_{i_p}.
\end{equation}

(11)

Note that $T_w$ does not depend on the choice of the reduced decomposition of $w$.

The following relations between the generators of $H(W)$ and the variables $x_i$ will prove useful

\begin{align}
T_i x_i &= x_{i+1}T_i - x_{i+1}(1 - t^{-1}), \\
T_i x_{i+1} &= x_iT_i + x_{i+1}(1 - t^{-1}), \\
T_i x_j &= x_jT_i, & j \neq i, i + 1, \\
T_{i-1}^{-1}x_i &= x_{i+1}T_{i-1}^{-1} + x_i(1 - t), \\
T_{i-1}^{-1}x_{i+1} &= x_iT_{i-1}^{-1} - x_i(1 - t), \\
\omega x_i &= x_{i-1}\omega, & i \neq 1, \\
\omega x_1 &= qx_N\omega.
\end{align}

(12)

From (7) and (12), we see that the $x_i$'s, the $T_j$'s and $\omega^\pm 1$ form an algebra over the field $\mathbb{Q}(q, t)$. An element $O$ of this algebra will be said to be normally ordered if all the variables $x_i$'s have been moved to the left, that is if $O$ has been put in the form

\begin{equation}
O = \sum_\lambda x^\lambda O_\lambda,
\end{equation}

(13)

where $O_\lambda$ is in $\mathbb{Q}(q, t)[T_i, \omega^\pm 1]$. 
3. The Macdonald polynomials

Let $\lambda \in P \equiv \mathbb{N}^N$. We denote by $|\lambda| = \sum_i \lambda_i$, the degree of $\lambda$, and by $\ell(\lambda)$ the number of non-zero entries in $\lambda$. The dominance order on the set $P^+ \subseteq P$ of all partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, is $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i$. This ordering is extended to $P$ as follows \cite{16}. The orbit $W\lambda$ of $\lambda \in P$ under the action of the symmetric group $W \cong S_N$ will contain a unique partition $\lambda^+ \in P^+$. We denote by $w_\lambda$, the unique element of minimal length such that $w_\lambda \lambda = \lambda^+$. We have $\lambda \geq \mu$ if either $\lambda^+ > \mu^+$ or $\lambda^+ = \mu^+$ and $w_\lambda \leq w_\mu$ in the Bruhat order of $W$. Note that $\lambda^+$ is the unique maximum of $W\lambda$.

Homogeneous symmetric polynomials are labelled by partition $\lambda$ of their degree. In the remainder of this section, $\lambda$ always stands for a partition, that is $\lambda \in P^+$. Three standard bases for $\Lambda_W^N$, the space of symmetric functions, are:

(i) the power sum symmetric functions $p_\lambda$ which in terms of the power sums

$$p_i = \sum_k x_k^i,$$

are given by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots,$$ (15)

(ii) the monomial symmetric functions $m_\lambda$ which are

$$m_\lambda = \sum_{\text{distinct permutations}} x_{\lambda_1} x_{\lambda_2} \cdots$$ (16)

(iii) the elementary symmetric functions $e_\lambda$ which in terms of the $i^{th}$ elementary function

$$e_i = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i} = m_{(1^i)},$$ (17)

are given by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots.$$ (18)

The Macdonald polynomials can now be presented as follows. To the partition $\lambda$ with $m_i(\lambda)$ parts equal to $i$, we associate the number

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$ (19)

We define a scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on $\Lambda_W^N$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$ (20)

where $\ell(\lambda)$ is the number of parts of $\lambda$. The Macdonald polynomials $J_\lambda(x; q, t) \in \Lambda_W^N$ are uniquely specified by

(i) $\langle J_\lambda, J_\mu \rangle_{q,t} = 0$, if $\lambda \neq \mu$, (21)

(ii) $J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q,t) m_\mu$, (22)

(iii) $v_{\lambda\lambda}(q,t) = c_\lambda(q,t)$, (23)
where
\[ c_\lambda(q,t) = \prod_{s \in \lambda} (1 - q^{\alpha(s)}t^{\ell(s)+1}). \tag{24} \]

As usual \( \alpha(s) \) and \( \ell(s) \) denote the number of squares in the diagram associated to the partition \( \lambda \) that are respectively to the south and east of the square \( s \).

For \( r = 1, \ldots, N \), let \( M_N^r \) denote the Macdonald operator
\[ M_N^r = \sum_I t^{(N-r)r+(r-1)/2} \tilde{A}_I(x;t) \prod_{i \in I} \tau_i, \tag{25} \]
where the sum is over all \( r \)-element subsets \( I \) of \( \{1, \ldots, N\} \),
\[ \tilde{A}_I(x;t) = \prod_{i \in I} \frac{x_i - t^{-1}x_j}{x_i - x_j}, \tag{26} \]
and \( M_N^0 \equiv 1 \). These operators commute with each other, \([M_N^r, M_N^l] = 0\) and are diagonal on the Macdonald polynomials basis. From the Macdonald operators, one constructs
\[ M_N(X; q, t) = \sum_{r=0}^{N} M_N^r X^r, \tag{27} \]
with \( X \) an arbitrary parameter. With \( J \), a set of cardinality \(|J| = j\) we shall also use \( M_J(X; q, t) \) to represent the operator \( M_J(X; q, t) \) in the variables \( x_i, i \in J \).

The generating function \( M_N \) will play a crucial role in the following. Its action on \( J_\lambda(x; q, t) \) with \( \ell(\lambda) \leq N \) is given, remarkably, by
\[ M_N(X; q, t)J_\lambda(x; q, t) = a_\lambda(X; q, t)J_\lambda(x; q, t), \tag{28} \]
where
\[ a_\lambda(X; q, t) = \prod_{i=1}^{N} (1 + Xq^{\lambda_i}t^{N-i}). \tag{29} \]

From (27) we see that the eigenvalue of \( M_N^r \) on \( J_\lambda(x; q, t) \) is the coefficient of \( X^r \) in the polynomial (29).

It is known (see for instance [6]) that the Macdonald operators can be rewritten in terms of Dunkl-Cherednik operators on \( \Lambda_N^W \). In particular, we have that
\[ \text{Res} \ Y_{\{1,\ldots,N\},u} = M_N(-u; q, t), \tag{30} \]
where \( \text{Res} \ O \) means that \( O \) is restricted to act on \( \Lambda_N^W \).

4. Creation operators

We now give the expressions \( B_k^{(i)}; i = 1, 2, 3; k = 1, \ldots, N \) of the creation operators that we will derive in the remainder of the paper.

- Expression 1
\[ B_k^{(1)} = \frac{1}{(q^{-1}; t^{-1})_{N-k}} Y_{\{1,\ldots,N\},t^{k+1-N}q^{-1}e_k}, \tag{31} \]
where for \( n \) positive integer, \((a; q)_n = (1 - a)(1 - qa) \ldots (1 - q^{n-1}a)\) and \((a; q)_0 \equiv 1\).

- Expression 2

\[
P_k^{(2)} = \sum_{|I|=k} x_I \sum_{m=0}^{N-k} \sum_{|I'|=m} \frac{q^{-m}}{(t^{k-N+1})_m} t^{-d(I', t)} Y_{I,j', t^{i'-m}}.
\]  

(32)

The quantity \(d(I, J)\) entering in the above expression depends on nested subsets \( J \subset I \) of \( \{1, \ldots, N\} \) and is defined as follows. Order the elements of \( I \) so that \( I = \{i_0, \ldots, i_{\ell-1}\} \) with \( i_0 < i_1 < \cdots < i_{\ell-1} \). Let \( J = \{i_{j_0}, \ldots, i_{j_m}\} \subset I \) with its elements ordered so that \((i_{j_0}, \ldots, i_{j_m})\) is a subsequence of \((i_0, \ldots, i_{\ell-1})\), \(0 \leq j_\ell \leq \ell - 1; k = 1, \ldots, m\). We then define

\[
d(J, I) = \sum_{k=1}^{m} j_k - m(m - 1)/2.
\]

(33)

Note that the sum is over the indices that identify the elements of \( J \) in the reference set \( I \). If \(|J| = 0\), then \(d(J, I) = 0\).

- Expression 3

\[
B_k^{(3)} = \sum_{|I|=k} x_I Y_{I,t}. 
\]

(34)

We need to prove that these three sets of operators are such that

\[
B_k^{(i)} J_\lambda(x) = J_{\lambda + (k^i)}(x),
\]

(35)

if \(\ell(\lambda) \leq k\). If this is so, the following Rodrigues formula for the Macdonald polynomials associated to any partition \(\lambda\) are easily seen to hold

\[
J_\lambda(x; q, t) = (B_N^{(i)})^{\lambda_N} (B_{N-1}^{(i)})^{\lambda_{N-1} - \lambda_N} \cdots (B_1^{(i)})^{\lambda_1 - \lambda_2} \cdot 1.
\]

(36)

That the first expression has property (35) will follow from the Pieri formula which gives the action of the elementary symmetric functions \(e_k\) on the monic Macdonald polynomials \(P_\lambda = 1/c_\lambda(q, t)J_\lambda\). This formula reads

\[
e_k P_\lambda = \sum_\mu \Psi_{\mu/\lambda} P_\mu,
\]

(37)

where the sum is over all partitions \(\mu\) containing \(\lambda\) such that the set-theoretic difference \(\mu - \lambda\) is \(k\)-dimensional with the property that \(\mu_i - \lambda_i \leq 1, \forall i \geq 1\). With \(C_{\mu/\lambda}\) and \(B_{\mu/\lambda}\) respectively denoting the union of the columns and of the rows that intersect \(\mu - \lambda\), the coefficients \(\Psi_{\mu/\lambda}\) are given by

\[
\Psi_{\mu/\lambda} = \prod_{s \in C_{\mu/\lambda}} \frac{b_\mu(s)}{b_\lambda(s)}
\]

(38)

where

\[
b_\lambda(s) = \begin{cases} 
1-\frac{q^{\ell(s)}}{1-\frac{q^{\ell(s)+1}}{1-\frac{q^{\ell(s)+1}t^{\ell(s)}}{1}}}, & \text{if } s \in \lambda \\
1, & \text{otherwise}
\end{cases}
\]

(39)

We shall then construct \(B_k^{(2)}\) from \(B_k^{(1)}\), using the realization of the affine Hecke algebra \(H(\tilde{W})\) given in Section 2 and properties of the Dunkl-Cherednik operators. Upon proving the operator equality \(B_k^{(2)} = B_k^{(1)}\) we shall infer that \(B_k^{(2)}\) are indeed
creation operators. Last, we shall prove to conclude the derivation that \( B_k^{(3)} J_\lambda = B_k^{(2)} J_\lambda = J_{\lambda + (1^k)} \) on Macdonald polynomials with \( \ell(\lambda) \leq k \).

We start by giving some results that we will need in the sequel. First, a lemma that has to do with the normal ordering of some expressions (see (13)):

**Lemma 1.**

\[
Y_k x_\ell = \sum_{j \geq \ell} x_j O_j,
\]

with \( O_j \in \mathbb{Q}(q, t)[T_i, \omega^{\pm 1}] \). And,

\[
Y_k x_1 \ldots x_\ell = \begin{cases} 
qx_1 \ldots x_\ell & \text{if } \ell \geq k \\
\sum_{\ell < k} x_\mu O_\mu & \text{if } \ell < k
\end{cases}
\]

where all \( \lambda \)'s in the sum contain at least one non-zero part \( \lambda_j \) with \( j > \ell \), that is \( \lambda \not\in P^+ \).

It is easily proved by induction from (12). A corollary of (40) is:

**Corollary 2.** For any \( \lambda \in P, |\lambda| = \ell \), containing at least one non-zero part \( \lambda_j \) with \( j > \ell \), we have, for any \( k \),

\[
Y_k x^\lambda = \sum_{|\mu| = \ell} x^\mu O_\mu,
\]

where all \( \mu \)'s in the sum contain at least one non-zero part \( \mu_j \) with \( j > \ell \), that is \( \mu \not\in P^+ \).

This is seen from (40) by commuting first \( Y_k \) with one of the \( x_j \) with \( j > \ell \) and \( \lambda_j \neq 0 \).

Next a lemma about the normal ordering of expressions involving \( T_i \) and \( x^\lambda \).

**Lemma 3.**

(i) if \( s_i \lambda > \lambda \)

\[
T_i x^\lambda = x^\lambda T_i + \sum_{\mu < s_i \lambda} x^\mu O_\mu,
\]

(ii) if \( s_i \lambda < \lambda \)

\[
T_i x^\lambda = \sum_{\mu < \lambda} x^\mu O_\mu,
\]

(iii) if \( s_i \lambda = \lambda \)

\[
T_i x^\lambda = x^\lambda T_i.
\]

Proof. Since \( T_i \) commutes with all the variables except \( x_i \) and \( x_{i+1} \), it suffices to look at the action of \( T_i \) on \( x_i^{\lambda_i} x_{i+1}^{\lambda_i+1} \). The third case occurs when \( \lambda_i = \lambda_{i+1} \) and it is trivially verified that \( T_i(x_i x_{i+1})^{\lambda_i} = (x_i x_{i+1})^{\lambda_i} T_i \). From this result, in case (i) where \( \lambda_{i+1} > \lambda_i \), we see upon factoring \( (x_i x_{i+1})^{\lambda_i} \) that it suffices to consider the action of \( T_i \) on \( x_i^{\lambda_i-\lambda_{i+1}} \). Similarly, in case (ii), we see that we only need to consider how \( T_i \) acts on \( x_i^\lambda \). The proof is then straightforwardly completed using (12).
Lemma 4. If \( \mu \) and \( \lambda \) with \( \mu \neq \lambda \) are in the same orbit \( W \lambda^+ \) and such that \( L(w_\mu) \geq L(w_\lambda) \neq 0 \), then
\[
T_{w_\lambda} x^\mu = \sum_{\rho < \lambda^+} x^\rho O_\rho. \tag{46}
\]

Proof. The only non-trivial case is when \( L(w_\mu) = L(w_\lambda) \). In this case, we have from Lemma 3
\[
T_{w_\lambda} x^\mu = T_{i_1} \ldots T_{i_p} x^\mu = \sum_{\rho \leq w_{\mu, \lambda} \rho} x^\rho O_\rho, \tag{47}
\]
with \( w_{\mu, \lambda} \) some Weyl group element such that \( w_{\mu, \lambda} < w_\lambda \) in the Bruhat order. In order to have \( w_{\mu, \lambda} = w_\lambda \), case (i) of Lemma 3 would have to apply for every permutation \( s_{i_k} \) in the reduction of \( w_\lambda \), but this is impossible since it would require that \( s_{i_{p-k}}(s_{i_{p-k+1}} \ldots s_{i_p} \mu) > s_{i_{p-k+1}} \ldots s_{i_p} \mu \) for \( k = 1, \ldots, p - 1 \), in other words, it would demand that \( w_k \mu = \lambda^+ \) which can not be the case because \( \mu \neq \lambda \) by hypothesis. We thus have that all the \( \rho \)'s entering in (47) are such that \( \rho \leq w_{\mu, \lambda} \mu < \lambda^+ \), which proves the lemma.

Proposition 5. If a non-zero operator is of the form \( O = \sum_{|\mu| = k} x^\mu O_\mu \) with \( O_\mu = 0 \) when \( \mu \in P^+ \), there is at least one \( T_i \), \( i = 1, \ldots, N - 1 \), for which \( T_i O \neq OT_i \) and hence \( O \) is not symmetric.

Proof. Suppose that \( O \) is symmetric and of the form \( O = \sum_{|\mu| = k} x^\mu O_\mu \) with \( O_\mu = 0 \) when \( \mu \in P^+ \). There exists one term \( x^\lambda O_\lambda \) of \( O \) with \( O_\lambda \neq 0 \) and such that, either \( \mu^+ \neq \lambda^+ \) or \( \mu^+ = \lambda^+ \) and \( L(w_\mu) \neq L(w_\lambda) \), for all \( \mu \) such that \( O_\mu \neq 0 \) in the decomposition of \( O \). From Lemma 3 (which imply that \( T_w x^\mu = \sum_{\rho \leq \mu^+} x^\rho O_\rho \) for any \( w \in W \) and \( \mu \in P \)) and Lemma 4, we then have that
\[
T_{w_\lambda} O = x^\lambda T_{w_\lambda} O_\lambda + \sum_{\rho, \rho \neq \lambda^+} x^\rho O_\rho'. \tag{48}
\]
Since \( O \) is assumed to be symmetric, we also have
\[
T_{w_\lambda} O = OT_{w_\lambda} = \sum_{\mu} x^\mu O_\mu'' \tag{49},
\]
with \( O_\mu'' = 0 \) if \( \mu \in P^+ \). From (48) and (49), since \( T_{w_\lambda} O_\lambda \neq 0 \), we have a contradiction and hence Proposition 5 must be true.

Corollary 6. Two normally ordered symmetric operators \( A \) and \( B \), whose factors of \( x^{\lambda^+} \) are the same for any \( \lambda^+ \in P^+ \), are equal.

Since \( A - B \) is symmetric and does not have any part in \( x^{\lambda^+}, \forall \lambda^+ \in P^+ \), it must be zero from Proposition 5.

Theorem 7. For any partition \( \lambda \) with \( \ell(\lambda) \leq k \), the operators \( B_k^{(1)} \) act as follows on the Macdonald polynomials \( J_\lambda(x; q, t) \):
\[
B_k^{(1)} J_\lambda(x) = J_{\lambda+((1)_k)}(x). \tag{50}
\]

Proof. The following lemma is an immediate consequence of the Pieri formula.
Lemma 8. For $\lambda$ a partition such that $\ell(\lambda) \leq k$, the action of $e_k$ on $P_\lambda$ is given by

$$e_k P_\lambda = P_{\lambda+(1^k)} + \sum_{\mu \neq \lambda+(1^k)} \Psi_{\mu/\lambda} P_\mu,$$  \hspace{1cm} (51)

where all the $\mu$’s in the sum are such that $\mu_{k+1} = 1$.

Indeed, the only way to construct a $\mu$ with $\mu_{k+1} \neq 1$ is to add a 1 in each of the first $k$ entries of $\lambda$. From Lemma 8 and (28) and (29), we have

$$Y_{(1,\ldots,N), t^{k+1-N} q^{-1}} e_k P_\lambda = \prod_{i=1}^{k} (1 - t^{k+1-i} q^{\lambda_i})(q^{-1}; t^{-1})_{N-k} P_{\lambda+(1^k)}$$ \hspace{1cm} (52)

since the eigenvalues

$$a_\mu(-t^{k+1-N} q^{-1}; q, t) = \prod_{i=1}^{N} (1 - t^{k+1-i} q^{\mu_i-1}),$$ \hspace{1cm} (53)

of $Y_{(1,\ldots,N), t^{k+1-N} q^{-1}}$ on the $P_\mu$’s in (51) vanish if $\mu_{k+1} = 1$.

From the definition given in (24), it is easy to check that

$$\frac{c_{\lambda+(1^k)}}{c_{\lambda}} = \prod_{i=1}^{k} (1 - t^{k+1-i} q^{\lambda_i}).$$ \hspace{1cm} (54)

Using this result and passing from $P_\lambda$ to $J_\lambda$ we see that

$$B_k^{(1)} J_\lambda = \frac{1}{(q^{-1}; t^{-1})_{N-k}} Y_{(1,\ldots,N), t^{k+1-N} q^{-1}} e_k J_\lambda = J_{\lambda+(1^k)}$$ \hspace{1cm} (55)

when $\ell(\lambda) \leq k$. This proves Theorem 7.

We are going to order $B_k^{(1)}$ normally and thus move all the variables contained in $e_k(x)$ to the left. In doing so, we shall only focus on terms having $x^\lambda$ with $\lambda \in P^+$ on the left, knowing from Corollary 6, that we only need to symmetrize these terms in order to find the full expression. From Corollary 2, we see that the operators $Y_{(1,\ldots,k)} \lambda I$ will not have terms of the form $x^\lambda$ to the left whenever $I \neq \{1, \ldots, k\}$. Therefore, it only remains to consider

$$\frac{1}{(q^{-1}; t^{-1})_{N-k}} (1 - t^k q^{-1} Y_1) \cdots (1 - t q^{-1} Y_k)$$

$$\times (1 - q^{-1} Y_{k+1}) \cdots (1 - t^{k+1-N} q^{-1} Y_N) x_1 \ldots x_k.$$ \hspace{1cm} (56)

From (41) and Corollary 2, we see that the expression below is the only term of type $x^\lambda$ in (56) once the variables $x_i$’s have been moved to the left:

$$\frac{1}{(q^{-1}; t^{-1})_{N-k}} x_1 \ldots x_k (1 - t^k Y_1) \cdots (1 - t Y_k)$$

$$\times (1 - q^{-1} Y_{k+1}) \cdots (1 - t^{k+1-N} q^{-1} Y_N).$$ \hspace{1cm} (57)

Thus,

$$\frac{1}{(q^{-1}; t^{-1})_{N-k}} x_1 \ldots x_k Y_{(1,\ldots,k), t} Y_{(k+1,\ldots,N), t^{k+1-N} q^{-1}}$$ \hspace{1cm} (58)

is the only term of $B_k^{(1)}$ that has to the left a factor of the form $x^\lambda$. Before symmetrizing, we shall expand this last expression using the following lemma:
Lemma 9.

\[ Y_{\{1,\ldots,\ell\},t^{-\ell+q-1}} = \sum_{m=0}^{\ell} q^{-m} (q^{-1}; t^{-1})_{\ell-m} \sum_{I \subseteq \{1,\ldots,\ell\} \atop |I| = m} t^{-d(I,\{1,\ldots,\ell\})} Y_{I,t^{1-m}}. \quad (59) \]

Proof. We proceed by induction. The formula is easily seen to hold in the case \( \ell = 1 \). Assuming that (59) is true, we thus have

\[ Y_{\{1,\ldots,\ell+1\},t^{-\ell+q-1}} = Y_{\{1,\ldots,\ell\},t^{-\ell+q-1}} (1 - Y_{\ell+1,t^{-\ell}}) q^{-1} \]

which concludes the proof. In the derivation, we have used the following two properties of the quantity \( d(I,J) \): \( d(I,\{1,\ldots,\ell\}) = d(I,\{1,\ldots,\ell+1\}) \) if \( \ell + 1 \not\in I \) and \( d(I \cup \{ \ell + 1 \},\{1,\ldots,\ell + 1\}) = d(I,\{1,\ldots,\ell\}) + \ell - m \).

With the help of Lemma 9 and of the identity \( Y_{\{1,\ldots,k\},t^m} = Y_{\{1,\ldots,k\} \cup I,t^{1-m}} \) if \( |I| = m \), expression (58) can be recast in the form

\[ x_1 \ldots x_k \sum_{m=0}^{N-k} \sum_{I \subseteq \{k+1,\ldots,N\} \atop |I| = m} q^{-m} \frac{1}{(q^{-1}; t^{1-m})_m} t^{-d(I,\{k+1,\ldots,N\})} Y_{\{1,\ldots,k\} \cup I,t^{1-m}}. \quad (61) \]

We shall now give an expression in normal order (see (62)) which has (61) as its only term of type \( x^\lambda t^\ell \). By Corollary 6, in order to show that this expression coincides with \( B_k^{(1)} \) we shall only need to prove that it is symmetric.

Theorem 10.

\[ B_k^{(1)} = B_k^{(2)} \equiv \sum_{|I|=k} x_I \sum_{m=0}^{N-k} \sum_{I' \subseteq I \atop |I'| = m} q^{-m} \frac{1}{(q^{-1}; t^{1-m})_m} t^{-d(I',\{1,\ldots,N\})} Y_{I \cup I',t^{1-m}}. \quad (62) \]

hence \( B_k^{(2)} \) is also such that \( B_k^{(2)} J_\lambda = J_{\lambda+(1)} \) for \( \ell(\lambda) \leq k \).
Proof. We first give some expressions that are easily checked to commute with $T_i$ from properties (7),(9) and (12). With $f \in \Lambda_N^W$, 

(I) $(T_i - 1)(x_i + x_{i+1})f = 0$

(II) $(T_i - 1)x_i x_{i+1}f = 0$

(III) $(T_i - 1)(1 - uY_i)(1 - uY_{i+1})f = 0$

(IV) $(T_i - 1)\left[x_i(1 - uY_i) + x_{i+1}(1 - uY_{i+1})\right]f = 0$

(V) $(T_i - 1)\left[(1 - uY_i) + t^{-1}(1 - uY_{i+1})\right]f = 0$

That $B_k^{(2)}$ has expression (61) has its only term of type $x^\lambda^+$ is obvious. In view of the remark made before Theorem 10, we now only need to verify that the operators $B_k^{(1)}$ as defined in (32) (and (62)) are symmetric, in other words that they satisfy $(T_i - 1)B_k^{(2)}f = 0$, $\forall i = 1, \ldots, N - 1$. Since two sets, $I$ and $I'$, enter in the definition of $B_k^{(2)}$ we proceed by looking at all the inclusion possibilities of the indices $i$ and $i + 1$ into these two sets and then examine the particular terms in $B_k^{(2)}$ that are affected by the action of $T_i$. What we find using (63) is that these terms are separately or pairwise symmetric. Indeed consider the cases:

(i) $i, i + 1 \notin I$ and $i, i + 1 \notin I'$: $T_i$ trivially commutes.

(ii) $i, i + 1 \notin I$ and $i \notin I', i + 1 \notin I'$: add the case $\bar{I} = I, \bar{I}' = (I' \cup \{i + 1\}) \setminus \{i\}$

which is such that $d(\bar{I}', \bar{I}') = d(I', I') + 1$. The symmetry follows from (V).

(iii) $i, i + 1 \notin I$ and $i, i + 1 \in I'$: $T_i$ commutes owing to (III).

(iv) $i \in I, i + 1 \notin I$ and $i + 1 \notin I'$: add the case $\bar{I} = (I \cup \{i + 1\}) \setminus \{i\}, \bar{I}' = I'$ which is such that $d(\bar{I}', \bar{I}') = d(I', I')$. The symmetry is verified with the help of (IV).

(v) $i \in I, i + 1 \notin I$ and $i + 1 \in I'$: add the case $\bar{I} = (I \cup \{i + 1\}) \setminus \{i\}, \bar{I}' = (I' \cup \{i\}) \setminus \{i + 1\}$ which is such that $d(\bar{I}', \bar{I}') = d(I', I')$. The symmetry is then confirmed using (I) and (III).

All the other cases are immediate consequences of these cases and of (II). Theorem 10 is thus seen to hold.

When going from $B_k^{(2)}$ to $B_k^{(3)}$, it is useful to obtain $\text{Res} B_k^{(2)}$, that is the $q$-difference operator version of $B_k^{(2)}$. The next lemma gives the essential step.

Lemma 11.

$$\text{Res} \sum_{|I| = k} \sum_{\substack{T' \subseteq T \quad |T'| = m \quad I' \subseteq I \quad |I'| = m}} t^{-d(I', I')}Y_{I \cup I', I - m} = \sum_{|I| = k} \sum_{\substack{T' \subseteq T \quad |T'| = m \quad I' \subseteq I \quad |I'| = m}} \tilde{A}_{I \cup I'} M_{I \cup I'} (-t^{1-m}; q, t).$$

(64)

Proof. We first give two formulas that we will need. The first one is a well known identity and the second is a special case of a formula given by Garsia and Tesler (12, Proposition 3.1).

Formula 12.

$$e_m(1, t^{-1}, \ldots, t^{-N+1}) = t^{m(m-1)/2} t^{-m(N-1)} \begin{bmatrix} N \\ m \end{bmatrix}_t,$$

(65)
where the $q$-binomial coefficient is
\[
\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.
\] (66)

**Formula 13.** With $N = |J \cup J^c|$, we have that, for any $k = 0, \ldots, N$ and $m = 0, \ldots, N - k$,
\[
\sum_{|J|=k} x_J \sum_{J' \subseteq J^c \atop |J'|=m} \tilde{A}_{J,J'} = t^{-m(N-k-m)} \left[ \begin{array}{c} N-k \\ m \end{array} \right] \sum_{|J|=k} x_J.
\] (67)

Given that
\[
\sum_{|J'|=m \atop J' \subseteq J^c} t^{-d(J',J')} = t^{m(m-1)/2} e_m(1, t^{-1}, \ldots, t^{-(N-k)+1}),
\] (68)
for any subset $J$ of $\{1, \ldots, N\}$ of cardinality $k$, the following lemma follows from Formulas 12 and 13.

**Lemma 14.** With $N = |J \cup J^c|$, we have that, for any $k = 0, \ldots, N$ and $m = 0, \ldots, N - k$,
\[
\sum_{|J|=k} x_J \sum_{J' \subseteq J^c \atop |J'|=m} t^{-d(J',J')} = \sum_{|J|=k} x_J \sum_{|J'|=m \atop J' \subseteq J^c} \tilde{A}_{J,J'}.
\] (69)

We now return to the proof of (64). Since both sides are symmetric, it will suffice to show that the coefficients of $\tau_1 \ldots \tau_\ell$ for $\ell \leq N$ are identical on both sides of the equation.

To that end, let
\[
I = L \cup J, \quad I' = \bar{L} \cup J',
\]
\[
L \subseteq \{1, \ldots, \ell\}, \quad \bar{L} = \{1, \ldots, \ell\} \setminus L,
\]
\[
J, J' \subseteq \{1, \ldots, N\} \setminus \{1, \ldots, \ell\} = J \cup \bar{J},
\]
\[
J \cap \bar{J} = \phi, J' \subseteq \bar{J},
\] (70)
and define
\[
[\ell, k] = \begin{cases} 
\ell & \text{if } \ell \leq k \\
\ell - k & \text{if } \ell > k
\end{cases}
\] (71)

The only place in the l.h.s. of (64) where the operator product $\tau_1 \ldots \tau_\ell$ will occur is in $\text{Res} Y_1 \ldots Y_\ell$ (see Proposition 6.1 and formula 7.20). The coefficient of $\tau_1 \ldots \tau_\ell$ in this expression is $\tilde{A}_{\{1, \ldots, \ell\}}$. With this knowledge and the help of (10),(25) and (27), we find that the coefficients of $\tau_1 \ldots \tau_\ell$ on both sides of (64) are respectively:
\[
\text{l.h.s.}|_{\tau_1 \ldots \tau_\ell} = t^{\ell(\ell-1)/2 \left( -t^{k-\ell+1} \right)} \tilde{A}_{\{1, \ldots, \ell\}}
\]
\[
\times \sum_{n=0}^{[\ell, k]} x_L \sum_{|J|=k-n} x_J \sum_{|L\cup J'|=m} t^{-d(J',J)}
\] (72)
and

\[ \text{r.h.s.}\big|_{\tau_1 \ldots \tau_\ell} = t^{\ell(\ell-1)/2} (-t^{1-m})^{\ell} t^{(m+k-\ell)} A_{\{1, \ldots, \ell\}} \]

\[ \times \sum_{n=0}^{[\ell,k]} \sum_{|L|=n} x_L \sum_{|J|=k-n} x_J \sum_{|I\cup I'|=m} \tilde{A}_{I\cup I'}^{J}, \quad (73) \]

with

\[ \tilde{A}_{J} = \prod_{j \in J} \frac{x_j - t^{-\frac{1}{2}} x_j}{x_i - x_j}. \quad (74) \]

We have used in (72) the fact that \( d(I', I^c) = d(J', \bar{J}) \) and in (73), the identity \( \tilde{A}_{I\cup I'}^{J} = \tilde{A}_{\{1, \ldots, \ell\}} A_{I\cup I' \setminus \{1, \ldots, \ell\}} \). It is then immediate to see that the equality

\[ \text{l.h.s. of (64)}|_{\tau_1 \ldots \tau_\ell} = \text{r.h.s. of (64)}|_{\tau_1 \ldots \tau_\ell} \quad (75) \]

holds, since after trivial simplifications it is seen to amount to

\[ \sum_{n=0}^{[\ell,k]} \sum_{|L|=n} x_L \left( \sum_{|J|=k-n} x_J \sum_{|I'|=m-\ell+n} t^{-d(I', \bar{J})} \right) = \]

\[ \sum_{n=0}^{[\ell,k]} \sum_{|L|=n} x_L \left( \sum_{|J|=k-n} x_J \sum_{|I'|=m-\ell+n} \tilde{A}_{I\cup I'}^{J} \right). \quad (76) \]

and hence to follow from Lemma 14.

Once Lemma 11 is proved, the connection between \( B_k^{(2)} \) and \( B_k^{(3)} \) is readily obtained.

**Theorem 15.** For any partition \( \lambda \), such that \( \ell(\lambda) \leq k \), the actions of \( B_k^{(2)} \) and \( B_k^{(3)} \) on the Macdonald polynomials \( J_\lambda(x) \) coincide:

\[ B_k^{(3)} J_\lambda(x) = B_k^{(2)} J_\lambda(x) = J_{\lambda+(1^k)}(x). \quad (77) \]

This is shown to be true with the help of the following lemma

**Lemma 16.** Let \( |I| = k \) and \( |I'| = m, I' \subseteq I^c \).

\[ M_{I'\cup I'}(-t^{1-m}; q, t) J_\lambda(x; q, t) = 0, \quad (78) \]

if \( \ell(\lambda) \leq k \) and \( m > 0 \).

Proof. Denote by \( x(I) \) the set of variables \( \{x_i, i \in J\} \). The Macdonald polynomials are known \([\square]\) to enjoy the property according to which

\[ J_\lambda(x(I), x(I^c)) = \sum_{\mu, \nu} \tilde{f}_\mu^\lambda J_\mu(x(I)) J_\nu(x(I^c)) \quad (79) \]

with \( \tilde{f}_\mu^\lambda = 0 \) unless \( \mu \subset \lambda \) and \( \nu \subset \lambda \) and in particular if \( \ell(\mu) \) or \( \ell(\nu) \) is greater than \( k \).
Since \( M_{I\cup I'}(-t^{1-m};q,t) \) is a \( q \)-difference operator depending only of the variables \( x_i, i \in I \cup I' \), we see from (79) that
\[
M_{I\cup I'} J_\lambda(x) = \sum_{\mu, \nu} \tilde{f}_{\mu \nu} J_\mu(x((I \cup I')^c)) M_{I\cup I'} J_\nu(x(I \cup I')).
\]
(80)

The proof of Lemma 16 is then completed by observing from (28)-(29) that
\[
M_{I\cup I'}(-t^{1-m};q,t) J_\nu(x(I \cup I')) = \prod_{i=1}^{k+m} (1 - q^{\nu_{k+1}} t^{k+1-i}) J_\nu(x(I \cup I')) = 0,
\]
whenever \( m > 0 \), since \( \nu_{k+1} = 0 \).

Theorem 15 is thus an immediate consequence of this lemma since, by (64), we have that:
\[
\left( \sum_{|I|=k} x_I \sum_{I' \subseteq I^c \cap I^{1-m}} t^{-d(I', I^{1-m})} Y_{I\cup I', I^c} \right) J_\lambda(x; q, t) = 0,
\]
if \( \ell(\lambda) \leq k \) and \( m > 0 \); this only leaves the \( m = 0 \) term of \( B^{(2)}_k \), which coincides with \( B^{(3)}_k \).

5. Conclusion

As already mentioned in the introduction, the integrality of the \((q,t)\)-Kostka coefficients follows quite straightforwardly from the Rodrigues formula (36) when the operators \( B^{(3)}_k \) are used as creation operators. This is explained in [6, 7, 10]. Other interesting properties of \( B^{(3)}_k \) have been conjectured in [5]. In particular, a formula that would give the action of these operators on arbitrary Macdonald polynomials has been proposed: it looks like a deformation of the Pieri formula and, if true, would imply that the \( N \) operators Res \( \sum_{|I|=m} x_I Y_{I, I^{1-m+1}} \), \( m = 1, \ldots, N \) form a commuting set for any \( \kappa \in \mathbb{R} \). We hope that the constructive approach presented in this paper will allow one to make progress towards proving these conjectures and unravelling the algebra of which the creation operators are part of.

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