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Improving Strategies via SMT Solving

Thomas Martin Gawlitza    David Monniaux

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Abstract

We consider the problem of computing numerical invariants of programs by
abstract interpretation. Our method eschews two traditional sources of impreci-
sion: (i) the use of widening operators for enforcing convergence within a finite
number of iterations (ii) the use of merge operations (often, convex hulls) at the
merge points of the control flow graph. It instead computes the least inductive
invariant expressible in the domain at a restricted set of program points, and
analyzes the rest of the code en bloc. We emphasize that we compute this induc-
tive invariant precisely. For that we extend the strategy improvement algorithm
of Gawlitza and Seidl [17]. If we applied their method directly, we would have to
solve an exponentially sized system of abstract semantic equations, resulting in
memory exhaustion. Instead, we keep the system implicit and discover strategy
improvements using SAT modulo real linear arithmetic (SMT). For evaluating
strategies we use linear programming. Our algorithm has low polynomial space
complexity and performs for contrived examples in the worst case exponentially
many strategy improvement steps; this is unsurprising, since we show that the
associated abstract reachability problem is \(\Pi^P_2\)-complete.

1 Introduction

Motivation     Static program analysis attempts to derive properties about the run-
time behavior of a program without running the program. Among interesting prop-
erties are the numerical ones: for instance, that a given variable \(x\) always has a value
in the range \([12, 41]\) when reaching a given program point. An analysis solely based
on such interval relations at all program points is known as interval analysis [11].
More refined numerical analyses include, for instance, finding for each program point
an enclosing polyhedron for the vector of program variables [13]. In addition to ob-
taining facts about the values of numerical program variables, numerical analyses
are used as building blocks for e.g. pointer and shape analyses.

However, by Rice’s theorem, only trivial properties can be checked automatically
[26]. In order to check non-trivial properties we are usually forced to use abstractions.
A systematic way for inferring properties automatically w.r.t. a given abstraction

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is given through the *abstract interpretation* framework of Cousot and Cousot \[12\]. This framework *safely over-approximates* the run-time behavior of a program.

When using the abstract interpretation framework, we usually have two sources of imprecision. The first source of imprecision is the abstraction itself: for instance, if the property to be proved needs a non-convex invariant to be established, and our abstraction can only represent convex sets, then we cannot prove the property. Take for instance the C-code:

\[
y = 0; \quad \text{if} \ (x < -1 \lor x > 1) \quad \text{if} \ (x = 0) \quad y = 1;
\]

No matter what the values of the variables \(x\) and \(y\) are before the execution of the above C-code, after the execution the value of \(y\) is 0. The invariant \(|x| \geq 1\) in the “then” branch is not convex, and its convex hull includes \(x = 0\). Any static analysis method that computes a convex invariant in this branch will thus also include \(y = 1\). In contrast, our method avoids enforcing convexity, except at the heads of loops.

The second source of imprecision are the safe but imprecise methods that are used for solving the *abstract semantic equations* that describe the abstract semantics: such methods safely over-approximate exact solutions, but do not return exact solutions in all cases. The reason is that we are concerned with abstract domains that contain infinite ascending chains, in particular if we are interested in numerical properties: the complete lattice of all \(n\)-dimensional closed real intervals, used for interval analysis, is an example. The traditional methods are based on Kleene fixpoint iteration which (purely applied) is not guaranteed to terminate in interesting cases. In order to enforce termination (for the price of imprecision) traditional methods make use of the widening/narrowing approach of Cousot and Cousot \[12\]. Grossly, widening extrapolates the first iterations of a sequence to a possible limit, but can easily overshoot the desired result. In order to avoid this, various tricks are used, including “widening up to” \[27, \text{Sec. 3.2}\], “delayed” or with “thresholds” \[6\]. However, these tricks, although they may help in many practical cases, are easily thwarted. Gopan and Reps \[25\] proposed “lookahead widening”, which discovers new feasible paths and adapts widening accordingly; again this method is no panacea. Furthermore, analyses involving widening are *non-monotonic*: stronger preconditions can lead to weaker invariants being automatically inferred; a rather non-intuitive behaviour. Since our method does not use widening at all, it avoids these problems.

**Our Contribution** We fight both sources of imprecision noted above:

- In order to improve the precision of the abstraction, we abstract sequences of if-then-else statements without loops en bloc. In the above example, we are then able to conclude that \(y \neq 0\) holds. In other words: we abstract sets of states only at the heads of loops, or, more generally, at a cut-set of the control-flow graph (a cut-set is a set of program points such that removing them would cut all loops).

- Our main technical contribution consists of a practical method for precisely computing abstract semantics of affine programs w.r.t. the template linear constraint domains of Sankaranarayanan et al. \[42\], with sequences of if-then-else statements which do not contain loops abstracted en bloc. Our method is based on a strict generalization of the strategy improvement algorithm of Gawlitza and Seidl \[17, 18, 21\]. The latter algorithm could be directly applied to the problem we solve in this article, but the size of its input would be
exponential in the size of the program, because we then need to explicitly enumerate all program paths between cut-nodes which do not cross other cut-nodes. In this article, we give an algorithm with low polynomial memory consumption that uses exponential time in the worst case. The basic idea consists in avoiding an explicit enumeration of all paths through sequences of if-then-else-statements which do not contain loops. Instead we use a SAT modulo real linear arithmetic solver for improving the current strategy locally. For evaluating each strategy encountered during the strategy iteration, we use linear programming.

- As a byproduct of our considerations we show that the corresponding abstract reachability problem is \( \Pi^p_2 \)-complete. In fact, we show that it is \( \Pi^p_2 \)-hard even if the loop invariant being computed consists in a single \( x \leq C \) inequality where \( x \) is a program variable and \( C \) is the parameter of the invariant. Hence, exponential worst-case running-time seems to be unavoidable.

Related Work  Recently, several alternative approaches for computing numerical invariants (for instance w.r.t. to template linear constraints) were developed:

Strategy Iteration  Strategy iteration (also called policy iteration) was introduced by Howard for solving stochastic control problems [29, 40] and is also applied to two-players zero-sum games [28, 31, 45] or min-max-plus systems [7]. Adjé et al. [2], Costan et al. [9], Gaubert et al. [16] developed a strategy iteration approach for solving the abstract semantic equations that occur in static program analysis by abstract interpretation. Their approach can be seen as an alternative to the traditional widening/narrowing approach. The goal of their algorithm is to compute least fixpoints of monotone self-maps \( f \), where \( f(x) = \min \{ \pi(x) \mid \pi \in \Pi \} \) for all \( x \) and \( \Pi \) is a family of self-maps. The assumption is that one can efficiently compute the least fixpoint \( \mu \pi \) of \( \pi \) for every \( \pi \in \Pi \). The \( \pi \)'s are the (min-)strategies. Starting with an arbitrary min-strategy \( \pi^{(0)} \), the min-strategy is successively improved. The sequence \( (\pi^{(k)})_k \) of attained min-strategies results in a decreasing sequence \( \mu \pi^{(0)} > \mu \pi^{(1)} > \ldots > \mu \pi^{(k)} \) that stabilizes, whenever \( \mu \pi^{(k)} \) is a fixpoint of \( f \) — not necessarily the least one. However, there are indeed important cases, where minimality of the obtained fixpoint can be guaranteed [1]. Moreover, an important advantage of their algorithm is that it can be stopped at any time with a safe over-approximation. This is in particular interesting if there are infinitely many min-strategies [2]. Costan et al. [3] showed how to use their framework for performing interval analysis without widening. Gaubert et al. [16] extended this work to the following relational abstract domains: The zone domain [33], the octagon domain [34] and in particular the template linear constraint domains [42]. Gawlitza and Seidl [17] presented a practical (max-)strategy improvement algorithm for computing least solutions of systems of rational equations. Their algorithm enables them to perform a template linear constraint analysis precisely — even if the mappings are not non-expansive. This means: Their algorithm always computes least solutions of abstract semantic equations — not just some solutions.

Acceleration Techniques  Gonnord [23], Gonnord and Halbwachs [24] investigated an improvement of linear relation analysis that consists in computing, when possible, the exact (abstract) effect of a loop. The technique is fully compatible with the use of widening, and whenever it applies, it improves both the precision and the
2 Basics

Notations  \(\mathbb{B} = \{0, 1\}\) denotes the set of Boolean values. The set of real numbers is denoted by \(\mathbb{R}\). The complete linearly ordered set \(\mathbb{R} \cup \{-\infty, \infty\}\) is denoted by \(\overline{\mathbb{R}}\).

We call two vectors \(x, y \in \mathbb{R}^n\) comparable iff \(x \leq y\) or \(y \leq x\) holds. For \(f : X \to \mathbb{R}^n\) with \(X \subseteq \mathbb{R}^m\), we set \(\text{dom}(f) := \{x \in X \mid f(x) \in \mathbb{R}^n\}\) and \(\text{fdom}(f) := \text{dom}(f) \cap \mathbb{R}^n\).

We denote the \(i\)-th row (resp. the \(j\)-th column) of a matrix \(A\) by \(A_i\) (resp. \(A_{j\cdot}\)). Accordingly, \(A_{i, j}\) denotes the component in the \(i\)-th row and the \(j\)-th column. We also use this notation for vectors and mappings \(f : X \to Y^k\).

Assume that a fixed set \(X\) of variables and a domain \(\mathbb{D}\) is given. We consider equations of the form \(x = e\), where \(x \in X\) is a variable and \(e\) is an expression over \(\mathbb{D}\). A system \(\mathcal{E}\) of (fixpoint) equations is a finite set \(\{x_1 = e_1, \ldots, x_n = e_n\}\) of equations, where \(x_1, \ldots, x_n\) are pairwise distinct variables. We denote the set \(\{x_1, \ldots, x_n\}\) of variables occurring in \(\mathcal{E}\) by \(X_{\mathcal{E}}\). We drop the subscript whenever it is clear from the context.

For a variable assignment \(\rho : X \to \mathbb{D}\), an expression \(e\) is mapped to a value \([e]_{\rho}\) by setting \([x]_{\rho} := \rho(x)\) and \([f(e_1, \ldots, e_k)]_{\rho} := f([e_1]_{\rho}, \ldots, [e_k]_{\rho})\), where \(x \in X\), \(f\) is a \(k\)-ary operator, for instance +, and \(e_1, \ldots, e_k\) are expressions. Let \(\mathcal{E}\) be a system of equations. We define the unary operator \([\mathcal{E}]\) on \(X \to \mathbb{D}\) by setting \(([\mathcal{E}])(x) := [e]_{\rho}\) for all \(x = e \in \mathcal{E}\). A solution is a variable assignment \(\rho\) such that \(\rho = [\mathcal{E}]\) holds. The set of solutions is denoted by \(\text{Sol}(\mathcal{E})\).
Let \( \mathbb{D} \) be a complete lattice. We denote the least upper bound and the greatest lower bound of a set \( X \subseteq \mathbb{D} \) by \( \bigvee X \) and \( \bigwedge X \), respectively. The least element \( \bigvee \emptyset \) (resp. the greatest element \( \bigwedge \emptyset \)) is denoted by \( \bot \) (resp. \( \top \)). We define the binary operators \( \lor \) and \( \land \) by \( x \lor y := \bigvee \{x, y\} \) and \( x \land y := \bigwedge \{x, y\} \) for all \( x, y \in \mathbb{D} \), respectively. For \( \square \in \{\lor, \land\} \), we will also consider \( x_1 \square \cdots \square x_k \) as the application of a \( k \)-ary operator. This will cause no problems, since the binary operators \( \lor \) and \( \land \) are associative and commutative. An expression \( e \) (resp. an equation \( x = e \)) is called monotone iff all operators occurring in \( e \) are monotone.

The set \( X \rightarrow \mathbb{D} \) of all variable assignments is a complete lattice. For \( \rho, \rho' : X \rightarrow \mathbb{D} \), we write \( \rho \triangleleft \rho' \) (resp. \( \rho \triangleright \rho' \)) iff \( \rho(x) < \rho'(x) \) (resp. \( \rho(x) > \rho'(x) \)) holds for all \( x \in X \). For \( d \in \mathbb{D}, d \) denotes the variable assignment \( \{x \mapsto d \mid x \in X\} \). A variable assignment \( \rho \) with \( \bot \triangleleft \rho \triangleleft \top \) is called finite. A pre-solution (resp. post-solution) of a system \( \rho \) is a variable assignment \( \rho \) such that \( \rho \leq [\mathcal{E}] \rho \) (resp. \( \rho \geq [\mathcal{E}] \rho \)) holds. For all \( \rho \in \mathbb{D} \), \( \mathcal{E} \) is a set of equations. This will cause no problems, since the binary operators \( \lor \) and \( \land \) are associative and commutative. An expression \( e \) (resp. an equation \( x = e \)) is called monotone iff all operators occurring in \( e \) are monotone.

### Linear Programming

We consider linear programming problems (LP problems for short) of the form \( \sup \{c^\top x \mid x \in \mathbb{R}^n, Ax \leq b\} \), where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \) are the inputs. The convex closed polyhedron \( \{x \in \mathbb{R}^n \mid Ax \leq b\} \) is called the feasible space. The LP problem is called infeasible iff the feasible space is empty. An element of the feasible space, is called feasible solution. A feasible solution \( x \) that maximizes \( c^\top x \) is called optimal solution.

LP problems can be solved in polynomial time through interior point methods \([32, 43]\). Note, however, that the running-time then crucially depends on the sizes of occurring numbers. At the danger of an exponential running-time in contrived cases, we can also instead rely on the simplex algorithm: its running-time is \( \text{uniform} \), i.e., independent of the sizes of occurring numbers (given that arithmetic operations, comparison, storage and retrieval for numbers are counted for \( \mathcal{O}(1) \)).

### SAT modulo real linear arithmetic

The set of SAT modulo real linear arithmetic formulas \( \Phi \) is defined through the grammar \( e := c \mid x \mid e_1 + e_2 \mid c \cdot e' \), \( \Phi := a \mid e_1 \leq e_2 \mid \Phi_1 \lor \Phi_2 \mid \Phi_1 \land \Phi_2 \mid \overline{\Phi} \). Here, \( c \in \mathbb{R} \) is a constant, \( x \) is a real valued variable, \( e, e', e_1, e_2 \) are real-valued linear expressions, \( a \) is a Boolean variable and \( \Phi, \Phi', \Phi_1, \Phi_2 \) are formulas. An interpretation \( I \) for a formula \( \Phi \) is a mapping that assigns a real value to every real-valued variable and a Boolean value to every Boolean variable. We write \( I \models \Phi \) for "\( I \) is a model of \( \Phi \)" i.e., \([c]I = c, [x]I = I(x), [e_1 + e_2]I = [e_1]I + [e_2]I, [c \cdot e']I = c \cdot [e']I\), and:

\[
I \models a \iff I(a) = 1 \quad \quad I \models e_1 \leq e_2 \iff [e_1]I \leq [e_2]I
\]
A formula is called \textit{satisfiable} iff it has a model. The problem of deciding, whether or not a given SAT modulo real linear arithmetic formula is satisfiable, is NP-complete. There nevertheless exist efficient solver implementations for this decision problem \cite{15}.

In order to simplify notations we also allow matrices, vectors, the operations \(\geq, <, >, \neq, =\), and the Boolean constants 0 and 1 to occur.

\section*{Collecting and Abstract Semantics}

The programs that we consider in this article use real-valued variables \(x_1, \ldots, x_n\). Accordingly, we denote by \(x = (x_1, \ldots, x_n)\) the vector of all program variables. For simplicity, we only consider elementary statements of the form \(x := Ax + b\), and \(Ax \leq b\), where \(A \in \mathbb{R}^{n \times n}\) (resp. \(\mathbb{R}^{k \times n}\)), \(b \in \mathbb{R}^n\) (resp. \(\mathbb{R}^k\)), and \(x \in \mathbb{R}^n\) denotes the vector of all program variables. Statements of the form \(x := Ax + b\) are called (affine) \textit{assignments}. Statements of the form \(Ax \leq b\) are called (affine) \textit{guards}. Additionally, we allow statements of the form \(s_1; \cdots; s_k\) and \(s_1 \mid \cdots \mid s_k\), where \(s_1, \ldots, s_k\) are statements. The operator \(\mid\) binds tighter than the operator \(|\), and we consider ; and \(\mid\) to be right-associative, i.e., \(s_1 \mid s_2 \mid s_3\) stands for \(s_1 \mid (s_2 \mid s_3)\), and \(s_1; s_2; s_3\) stands for \(s_1; (s_2; s_3)\). The set of statements is denoted by \texttt{Stmt}. A statement of the form \(s_1 \mid \cdots \mid s_k\), where \(s_i\) does not contain the operator \(|\) for all \(i = 1, \ldots, k\), is called \textit{merge-simple}. A merge-simple statement \(s\) that does not use the \(\mid\) operator at all is called \textit{sequential}. A statement is called \textit{elementary} iff it neither contains the operator \(|\) nor the operator \(:\).

The \textit{collecting semantics} \([s]: \mathbb{R}^n \rightarrow \mathbb{R}^n\) of a statement \(s \in \texttt{Stmt}\) is defined by

\[
[x := Ax + b]X := \{Ax + b \mid x \in X\}, \quad [Ax \leq b]X := \{x \in X \mid Ax \leq b\},
\]

\[
[s_1; \cdots; s_k] := [s_k] \circ \cdots \circ [s_1], \quad [s_1 \mid \cdots \mid s_k]X := [s_1]X \cup \cdots \cup [s_k]X
\]

for \(X \subseteq \mathbb{R}^n\). Note that the operators \(|\) and \(\mid\) are associative, i.e., \([s_1; s_2; s_3] = [s_1; (s_2; s_3)]\) and \(\[(s_1 \mid s_2) \mid s_3\] = \([s_1 \mid (s_2 \mid s_3)\)\] hold for all statements \(s_1, s_2, s_3\).

An (affine) \textit{program} \(G\) is a triple \((N, E, \texttt{st})\), where \(N\) is a finite set of \textit{program points}, \(E \subseteq N \times \texttt{Stmt} \times N\) is a finite set of control-flow edges, and \(\texttt{st} \in N\) is the \textit{start program point}. As usual, the \textit{collecting semantics} \(V\) of a program \(G = (N, E, \texttt{st})\) is the least solution of the following constraint system:

\[
V[\texttt{st}] \supseteq \mathbb{R}^n \quad V[v] \supseteq [s](V[u]) \quad \text{for all } (u, s, v) \in E
\]

Here, the variables \(V[v], v \in N\) take values in \(\mathbb{R}^n\). The components of the collecting semantics \(V\) are denoted by \(V[v]\) for all \(v \in N\).

Let \(\mathbb{D}\) be a complete lattice (for instance the complete lattice of all \(n\)-dimensional closed real intervals). Let the partial order of \(\mathbb{D}\) be denoted by \(\leq\). Assume that \(\alpha: \mathbb{R}^n \rightarrow \mathbb{D}\) and \(\gamma: \mathbb{D} \rightarrow \mathbb{R}^n\) form a Galois connection, i.e., for all \(X \subseteq \mathbb{R}^n\) and all \(d \in \mathbb{D}\), \(\alpha(X) \leq d\) iff \(X \subseteq \gamma(d)\). The \textit{abstract semantics} \([s]^\alpha: \mathbb{D} \rightarrow \mathbb{D}\) of a statement \(s\) is defined by \([s]^\alpha := \alpha \circ [s] \circ \gamma\). The \textit{abstract semantics} \(V^\alpha\) of an affine program \(G = (N, E, \texttt{st})\) is the least solution of the following constraint system:

\[
V^\alpha[\texttt{st}] \supseteq \alpha(\mathbb{R}^n) \quad V^\alpha[v] \supseteq [s]^\alpha(V^\alpha[u]) \quad \text{for all } (u, s, v) \in E
\]
Here, the variables $V^\sharp[v]$, $v \in N$ take values in $\mathcal{D}$. The components of the abstract semantics $V^\sharp$ are denoted by $V^\sharp[v]$ for all $v \in N$. The abstract semantics $V^\sharp$ safely over-approximates the collecting semantics $V$, i.e., $\gamma(V^\sharp[v]) \supseteq V[v]$ for all $v \in N$.

Using Cut-Sets to improve Precision Usually, only sequential statements (these statements correspond to basic blocks) are allowed in control flow graphs. However, given a cut-set $C$, one can systematically transform any control flow graph $G$ into an equivalent control flow graph $G'$ of our form (up to the fact that $G'$ has fewer program points than $G$) with increased precision of the abstract semantics. However, for the sake of simplicity, we do not discuss these aspects in detail. Instead, we consider an example:

Example 1 (Using Cut-Sets to improve Precision). As a running example throughout the present article we use the following C-code:

```c
int x_1, x_2; x_1 = 0; while (x_1 < 1000) { x_2 = -x_1; if (x_2 < 0) x_1 = -2 * x_1; else x_1 = -x_1 + 1; }
```

This C-code is abstracted through the affine program $G_1 = (N_1, E_1, st)$ which is shown in Figure 1(a). However, it is unnecessary to apply abstraction at every program point; it suffices to apply abstraction at a cut-set of $G_1$. Since all loops contain program point 1, a cut-set of $G_1$ is $\{1\}$. Equivalent to applying abstraction only at program point 1 is to rewrite the control-flow graph w.r.t. the cut-set $\{1\}$ into a control-flow graph $G$ equivalent w.r.t. the collecting semantic. The result of this transformation is drawn in Figure 1(b). This means: the affine program for the above C-code is $G = (N, E, st)$, where $N = \{st, 1\}$, $E = \{(st, x_1 := 0, 1), (1, s, 1)\}$, and

\[
\begin{align*}
  s' &= x_1 \leq 1000; x_2 := -x_1 & \quad s_1 &= x_2 \leq -1; x_1 := -2x_1 \\
  s_2 &= -x_2 \leq 0; x_1 := -x_1 + 1 & \quad s &= s'; (s_1 \mid s_2)
\end{align*}
\]

Let $V_1$ denote the collecting semantics of $G_1$ and $V$ denote the collecting semantics of $G$. $G_1$ and $G$ are equivalent in the following sense: $V[v] = V_1[v]$ holds for all program points $v \in N$. W.r.t. the abstract semantics, $G$ is, is we will see, strictly more precise than $G_1$. In general we at least have $V^\sharp[v] \subseteq V_1^\sharp[v]$ for all program points $v \in N$. This is independent of the abstract domain.\[\]
Template Linear Constraints In the present article we restrict our considerations to template linear constraint domains \[12\]. Assume that we are given a fixed template constraint matrix \( T \in \mathbb{R}^{m \times n} \). The template linear constraint domain is \( \mathbb{R}^n \). As shown by Sankaranarayanan et al. \[42\], the concretization \( \gamma : \mathbb{R}^m \to 2^{\mathbb{R}^n} \) and the abstraction \( \alpha : 2^{\mathbb{R}^n} \to \mathbb{R}^m \), which are defined by

\[
\gamma(d) := \{ x \in \mathbb{R}^n \mid TX \leq d \}, \quad \forall d \in \mathbb{R}^m,
\]

\[
\alpha(X) := \bigwedge \{ d \in \mathbb{R}^m \mid \gamma(d) \supseteq X \}, \quad \forall X \subseteq \mathbb{R}^n,
\]

form a Galois connection. The template linear constraint domains contain intervals, zones, and octagons, with appropriate choices of the template constraint matrix \[42\].

In a first stage we restrict our considerations to sequential and merge-simple statements. Even for these statements we avoid unnecessary imprecision, if we abstract such statements en bloc instead of abstracting each elementary statement separately:

**Example 2.** In this example we use the interval domain as abstract domain, i.e., our complete lattice consists of all \( n \)-dimensional closed real intervals. Our affine program will use 2 variables, i.e., \( n = 2 \). The complete lattice of all 2-dimensional closed real intervals can be specified through the template constraint matrix \( T = (-I \quad I)^\top \in \mathbb{R}^{4 \times 2} \), where \( I \) denotes the identity matrix. Consider the statements \( s_1 = x_2 := x_1, \ s_2 = x_1 := x_1 - x_2, \) and \( s = s_1; s_2 \) and the abstract value \( I = [0,1] \times \mathbb{R} \) (a 2-dimensional closed real interval). The interval \( I \) can w.r.t. \( T \) be identified with the abstract value \( (0,\infty,1,\infty)^\top \). More generally, w.r.t. \( T \) every 2-dimensional closed real interval \( [l_1,u_1] \times [l_2,u_2] \) can be identified with the abstract value \( (-l_1,-l_2,u_1,u_2)^\top \). If we abstract each elementary statement separately, then we in fact use \( [s_2]^I \circ [s_1]^I \) instead of \( [s]^I \) to abstract the collecting semantics \( [s] \) of the statement \( s = s_1; s_2 \). The following calculation shows that this can be important: \( [s]^2I = [0,0] \times [0,1] \neq [-1,1] \times [0,1] = [s_2]^2(([0,1] \times [0,1]) = ([s_2]^I \circ [s_1]^I)I \). The imprecision is caused by the additional abstraction. We lose the information that the values of the program variables \( x_1 \) and \( x_2 \) are equal after executing the first statement.

Another possibility for avoiding unnecessary imprecision in the above example would consist in adding additional rows to the template constraint matrix. Although this works for the above example, it does not work in general, since still only convex sets can be described, but sometimes non-convex sets are required (cf. with the example in the introduction).

Provided that \( s \) is a merge-simple statement, \([s]^2d\) can be computed in polynomial time through linear programming:

**Lemma 3** (Merge-Simple Statements). Let \( s \) be a merge-simple statement and \( d \in \mathbb{R}^m \). Then \([s]^2d\) can be computed in polynomial time through linear programming.

However, the situation for arbitrary statements is significantly more difficult, since, by reducing SAT to the corresponding decision problem, we can show the following:

**Lemma 4.** The problem of deciding, whether or not, for a given template constraint matrix \( T \), and a given statement \( s \), \([s]^\infty > \infty\) holds, is NP-complete.
Before proving the above lemma, we introduce \(\lor\)-strategies for statements as follows:

**Definition 1 (\(\lor\)-Strategies for Statements).** A \(\lor\)-strategy \(\sigma\) for a statement \(s\) is a function that maps every position of a \([\cdot]\)-statement, (a statement of the form \(s_0 | s_1\)) within \(s\) to 0 or 1. The application \(s\sigma\) of a \(\lor\)-strategy \(\sigma\) to a statement \(s\) is inductively defined by \(s\sigma = s\), \((s_0 | s_1)\sigma = s_{\sigma(\text{pos}(s_0 \mid s_1))}\), and \((s_0; s_1)\sigma = (s_0\sigma; s_1\sigma)\), where \(s\) is an elementary statement, and \(s_0, s_1\) are arbitrary statements. For all occurrences \(s'\), \(\text{pos}(s')\) denotes the position of \(s'\), i.e., \(\text{pos}(s')\) identifies the occurrence.

**Proof.** Firstly, we show containment in NP. Assume \([s]^{\geq \infty} > -\infty\). There exists some \(k\) such that the \(k\)-th component of \([s]^{\geq \infty}\) is greater than \(-\infty\). We choose \(k\) non-deterministically. There exists a \(\lor\)-strategy \(\sigma\) for \(s\) such that the \(k\)-th component of \([s\sigma]^{\geq \infty}\) equals the \(k\)-th component of \([s]^{\geq \infty}\). We choose such a \(\lor\)-strategy non-deterministically. By Lemma [3] we can check in polynomial time, whether the \(k\)-th component of \([s\sigma]^{\geq \infty}\) is greater than \(-\infty\). If this is fulfilled, we accept.

In order to show NP-hardness, we reduce the NP-hard problem SAT to our problem. Let \(\Phi\) be a propositional formula with \(n\) variables. W.l.o.g. we assume that \(\Phi\) is in normal form, i.e., there are no negated sub-formulas that contain \(\land\) or \(\lor\). We define the statement \(s(\Phi)\) that uses the variables of \(\Phi\) as program variables inductively by \(s(z) := z = 1\), \(s(\overline{z}) := z = 0\), \(s(\Phi_1 \land \Phi_2) := s(\Phi_1); s(\Phi_2)\), and \(s(\Phi_1 \lor \Phi_2) := s(\Phi_1) | s(\Phi_2)\), where \(z\) is a variable of \(\Phi\), and \(\Phi_1, \Phi_2\) are formulas. Here, the statement \(Ax = b\) is an abbreviation for the statement \(Ax \leq b; -Ax \leq -b\). The formula \(\Phi\) is satisfiable iff \([s(\Phi)]\mathbb{R}^n \neq \emptyset\). Moreover, even if we just use the interval domain, \([s(\Phi)]\mathbb{R}^n \neq \emptyset\) holds iff \([s(\Phi)]^{\geq \infty} > -\infty\) holds. Thus, \(\Phi\) is satisfiable iff \([s(\Phi)]^{\geq \infty} > -\infty\) holds.\[\square\]

Obviously, \([(s_1 | s_2); s] = [s_1; s | s_2; s]\) and \([s; (s_1 | s_2)] = [s; s_1 | s; s_2]\) for all statements \(s, s_1, s_2\). We can transform any statement \(s\) into an equivalent merge-simple statement \(s'\) using these rules. We denote the merge-simple statement \(s'\) that is obtained from an arbitrary statement \(s\) by applying the above rules in some canonical way by \([s]\). Intuitively, \([s]\) is an explicit enumeration of all paths through the statement \(s\).

**Lemma 5.** For every statement \(s\), \([s]\) is merge-simple, and \([s] = [[s]]\). The size of \([s]\) is at most exponential in the size of \(s\).\[\square\]

However, in the worst case, the size of \([s]\) is exponential in the size of \(s\). For the statement \(s = (s_1^{(1)} | s_1^{(2)}); \cdots; (s_k^{(1)} | s_k^{(2)})\), for instance, we get \([s] = [(a_1, \ldots, a_k) \in \{1, 2\}^k | s_1^{(a_1)}; \cdots; s_k^{(a_k)}]\). After replacing all statements \(s\) with \([s]\) it is in principle possible to use the methods of Gawlitza and Seidl [17] in order to compute the abstract semantics \(V^2\) precisely. Because of the exponential blowup, however, this method would be impractical in most cases.\[^2\]

Our new method that we are going to present avoids this exponential blowup: instead of enumerating all program paths, we shall visit them only as needed. Guided by a SAT modulo real linear arithmetic solver, our method selects a path through

\[^2\] Note that we cannot expect a polynomial-time algorithm, because of Lemma [3] even without loops, abstract reachability is NP-hard. Even if all statements are merge-simple, we cannot expect a polynomial-time algorithm, since the problem of computing the winning regions of parity games is polynomial-time reducible to abstract reachability [10].
We define the system \( E \). Here, for a system \( C \) to be the smallest set of inequalities that fulfills the following constraints:

- \( C \) contains the inequality \( x_{st,i} \geq \alpha_i(R^n) \) for every \( i \in \{1, \ldots, m\} \).
- \( C \) contains the inequality \( x_{c,i} \geq [s]^2_i(x, u, 1, \ldots, x, u, m) \) for every control-flow edge \((u, s, v) \in E \) and every \( i \in \{1, \ldots, m\} \).

We define the system \( \mathcal{E}(G) \) of abstract semantic inequalities to be the smallest set of inequalities that fulfills the following constraints:

- \( C \) contains the inequality \( x_{c,i} \geq [s]^2_i(x, u, 1, \ldots, x, u, m) \) for every control-flow edge \((u, s, v) \in E \).
- \( C \) contains the inequality \( x_{st,i} \geq \alpha_i(R^n) \) for every \( i \in \{1, \ldots, m\} \).

We define the system \( \mathcal{E}(G) \) of abstract semantic equations by \( \mathcal{E}(G) := \mathcal{E}(C(G)) \). Here, for a system \( C' = \{x_1 \geq e_{1,1}, \ldots, x_1 \geq e_{1,k_1}, \ldots, x_n \geq e_{n,1}, \ldots, x_n \geq e_{n,k_n}\} \) of inequalities, \( \mathcal{E}(C') \) is the system \( \mathcal{E}(C') = \{x_1 = e_{1,1} \lor \cdots \lor e_{1,k_1}, \ldots, x_n = e_{n,1} \lor \cdots \lor e_{n,k_n}\} \) of equations. The system \( \mathcal{E}(G) \) of abstract semantic equations captures the abstract semantics \( V^\sharp \) of \( G \):

**Lemma 6.** \((V^\sharp[v])_v = \mu[\mathcal{E}(G)](x_{v,i}) \) for all program points \( v, i \in \{1, \ldots, m\} \). \(Q.E.D.\)

**Example 7 (Abstract Semantic Equations).** We again consider the program \( G \) of Example 1. Assume that the template constraint matrix \( T \in \mathbb{R}^{2 \times 2} \) is given by \( T_1 = (1,0) \) and \( T_2 = (-1,0) \). Let \( V^\sharp \) denote the abstract semantics of \( G \). Then \( V^\sharp[1] = (2001, 2000)^\top \). \( \mathcal{E}(G) \) consists of the following abstract semantic equations:

\[
\begin{align*}
x_{st,1} &= \infty \\
x_{st,2} &= \infty
\end{align*}
\]

As stated by Lemma 6 we have \((V^\sharp[1])_1 = \mu[\mathcal{E}(G)](x_{1,1}) = 2001\), and \((V^\sharp[1])_2 = \mu[\mathcal{E}(G)](x_{1,2}) = 2000\).

**3 A Lower Bound on the Complexity**

In this section we show that the problem of computing abstract semantics of affine programs w.r.t. the interval domain is \( \Pi^P_2 \)-hard. \( \Pi^P_2 \)-hard problems are conjectured to be harder than both NP-complete and co-NP-complete problems. For further information regarding the polynomial-time hierarchy see e.g. Stockmeyer [44].

**Theorem 8.** The problem of deciding, whether, for a given program \( G \), a given template constraint matrix \( T \), and a given program point \( v \), \( V^\sharp[v] > -\infty \) holds, is \( \Pi^P_2 \)-hard.

**Proof.** We reduce the \( \Pi^P_2 \)-complete problem of deciding the truth of a \( \forall \exists \) propositional formula [46] to our problem. Let \( \Phi = \forall x_1, \ldots, x_n. \exists y_1, \ldots, y_m. \Phi' \) be a formula without free variables, where \( \Phi' \) is a propositional formula. We consider the affine
program $G = (N, E, \text{st})$, with program variables $x, x', x_1, \ldots, x_n, y_1, \ldots, y_m$, where $N = \{\text{st}, 1, 2\}$, and $E = \{(\text{st}, x := 0, 1), (1, s, 1), (1, x \geq 2^n, 2)\}$ with
\[
s = x' := x; \quad (x' \geq 2^{n-1}; x' := x' - 2^{n-1}; x_n := 1 \mid x' \leq 2^{n-1} - 1; x_n := 0); \ldots
\]

\[
(x' \geq 2^{n-1}; x' := x' - 2^{n-1}; x_1 := 1 \mid x' \leq 2^{1-1} - 1; x_1 := 0);
\]

$s(\Phi')$: $x := x + 1$

The statement $s(\Phi')$ is defined as in the proof of Lemma 3.

In intuitive terms: this program initializes $x$ to 0. Then, it enters a loop: it computes into $x_1, \ldots, x_n$ the binary decomposition of $x$, then it attempts to non-deterministically choose $y_1, \ldots, y_m$ so that $\Phi'$ is true. If this is possible, it increments $x$ by one and loops. Otherwise, it just loops. Thus, there is a terminating computation iff $\Phi$ holds.

Then $\Phi$ holds iff $V[2] \neq \emptyset$. For the abstraction, we consider the interval domain. By considering the Kleene-Iteration, it is easy to see that $V[2] \neq \emptyset$ holds iff $V^2[2] > -\infty$ holds. Thus $\Phi$ holds iff $V^2[2] > -\infty$ holds. □ □

4 Determining Improved Strategies

In this section we develop a method for computing local improvements of strategies through solving SAT modulo real linear arithmetic formulas.

In order to decide, whether or not, for a given statement $s$, a given $j \in \{1, \ldots, m\}$, a given $c$, and a given $d \in \mathbb{R}^m$, $[s]_j d > c$ holds, we construct the following SAT modulo real linear arithmetic formula (we use existential quantifiers to improve readability):

\[
\Phi(s, d, j, c) \equiv \exists v \in \mathbb{R} \cdot \Phi(s, d, j) \land v > c
\]

\[
\Phi(s, d, j) \equiv \exists x \in \mathbb{R}^n, x' \in \mathbb{R}^n \cdot Tx \leq d \land \Phi(s) \land v = T_j x'
\]

Here, $\Phi(s)$ is a formula that relates every $x \in \mathbb{R}^n$ with all elements from the set $\{s\} \{x\}$. It is defined inductively over the structure of $s$ as follows:

\[
\Phi(x := Ax + b) \equiv x' = Ax + b
\]
\[
\Phi(Ax \leq b) \equiv Ax \leq b \land x' = x
\]
\[
\Phi(s_1; s_2) \equiv \exists x'' \in \mathbb{R}^n \cdot \Phi(s_1)[x''/x'] \land \Phi(s_2)[x''/x]
\]
\[
\Phi(s_1 | s_2) \equiv (\sigma_{\text{pos}(s_1,s_2)} \land \Phi(s_1)) \lor (\sigma_{\text{pos}(s_1,s_2)} \land \Phi(s_2))
\]

Here, for every position $p$ of a subexpression of $s$, $a_p$ is a Boolean variable. Let $\text{Pos}_i(s)$ denote the set of all positions of $i$-subexpressions of $s$. The set of free variables of the formula $\Phi(s)$ is $\{x, x'\} \cup \{a_p \mid p \in \text{Pos}_i(s)\}$. A valuation for the variables from the set $\{a_p \mid p \in \text{Pos}_i(s)\}$ describes a path through $s$. We have:

**Lemma 9.** $[s]_j^p d > c$ holds iff $\Phi(s, d, j, c)$ is satisfiable. □

Our next goal is to compute a $\forall$-strategy $\sigma$ for $s$ such that $[s\sigma]_j^p d > c$ holds, provided that $[s]_j^p d > c$ holds. Let $s$ be a statement, $d \in \mathbb{R}^m$, $j \in \{1, \ldots, m\}$, and $c \in \mathbb{R}$. Assume that $[s]_j^p d > c$ holds. By Lemma 9, there exists a model $M$ of $\Phi(s, d, j, c)$. We define the $\forall$-strategy $\sigma_M$ for $s$ by $\sigma_M(p) := M(a_p)$ for all $p \in \text{Pos}_i(s)$. By again applying Lemma 9, we get $[s\sigma]_j^p d > c$. Summarizing we have:
Lemma 10. By solving the SAT modulo real linear arithmetic formula $\Phi(s,d,j,c)$ that can be obtained from $s$ in linear time, we can decide, whether or not $[s]_d^2 d > c$ holds. From a model $M$ of this formula, we can obtain a $\lor$-strategy $\sigma_M$ for $s$ such that $[s_{\sigma_M}]_{d}^2 d > c$ holds in linear time.

Example 11. We again continue Example 11 and 1. We want to know, whether $[s]_d^2(0,0)^T > 0$ holds. For that we compute a model of the formula $\Phi(s,(0,0))^T,1,0)$ which is written down in Figure 2. $M = \{ a_1 \mapsto 1 \}$ is a model of the formula $\Phi(s,(0,0))^T,1,0)$. Thus, we have $0 < [s_{\sigma_M}]_{d}^2(0,0)^T = [s',s_2]_{d}^2(0,0)^T$ by Lemma 10.

It remains to compute a model of $\Phi(s,d,j,c)$. Most of the state-of-the-art SMT solvers, as for instance Yices [14, 15], support the computation of models directly; if unsupposed, one can compute the model using standard self-reduction techniques.

The semantic equations we are concerned with in the present article have the form $x = e_1 \lor \cdots \lor e_k$, where each expression $e_i$, $i = 1, \ldots, k$, is either a constant or an expression of the form $[s]_{j}^2(x_1, \ldots, x_m)$. We now extend our notion of $\lor$-strategies in order to deal with the occurring right-hand sides:

Definition 2 ($\lor$-Strategies). The $\lor$-strategy for all constants is the 0-tuple $(\cdot)$. The application $c()$ of $(\cdot)$ to a constant $c \in R$ is defined by $c() := c$ for all $c \in R$. A $\lor$-strategy $\sigma$ for an expression $[s]_{j}^2(x_1, \ldots, x_m)$ is a $\lor$-strategy for $s$. The application $([s]_{j}^2(x_1, \ldots, x_m))\sigma$ of $\sigma$ to $[s]_{j}^2(x_1, \ldots, x_m)$ is defined by $([s]_{j}^2(x_1, \ldots, x_m))\sigma := [s_{\sigma}]_{j}^2(x_1, \ldots, x_m)$. A $\lor$-strategy for an expression $e = e_0 \lor e_1$, where, for each $i \in \{0,1\}$, $e_i$ is either a constant or an expression of the form $[s]_{j}^2(x_1, \ldots, x_m)$, is a pair $(p,\sigma)$, where $p \in \{0,1\}$ and $\sigma$ is a $\lor$-strategy for $e_p$. The application $e(p,\sigma)$ of $(p,\sigma)$ to $e = e_0 \lor e_1$ is defined by $e(p,\sigma) = e_p\sigma$. A $\lor$-strategy $\sigma$ for a system $E = \{x_1 = e_1, \ldots, x_n = e_n\}$ of abstract semantic equations is a mapping $\{x_i \mapsto \sigma_i \mid i = 1, \ldots, n\}$, where $\sigma_i$ is a $\lor$-strategy for $e_i$ for all $i = 1, \ldots, n$. We set $E(\sigma) := \{x_1 = e_1(\sigma(x_1)), \ldots, x_n = e_n(\sigma(x_n))\}$.

Using the same ideas as above, we can prove the following lemma which finally enables us to use a SAT modulo real linear arithmetic solver for improving $\lor$-strategies for systems of abstract semantic equations locally.
Lemma 12. Let \( x = e \) be an abstract semantic equation, \( \rho \) a variable assignment, and \( c \in \mathbb{R} \). By solving a SAT modulo real linear arithmetic formula that can be obtained from \( e, \rho \) and \( c \) in linear time, we can decide, whether or not \( \llbracket e \rrbracket \rho > c \) holds. From a model \( M \) of this formula, we can in linear time obtain a \( \lor \)-strategy \( \sigma_M \) for \( e \) such that \( \llbracket e \sigma_M \rrbracket \rho > c \) holds.

5 Solving Systems of Concave Equations

In order to solve systems of abstract semantic equations (see the end of Section 2) we generalize the \( \lor \)-strategy improvement algorithm of Gawlitza and Seidl [21] as follows:

Concave Functions A set \( X \subseteq \mathbb{R}^n \) is called convex iff \( \lambda x + (1 - \lambda)y \in X \) holds for all \( x, y \in X \) and all \( \lambda \in [0, 1] \). A mapping \( f : X \rightarrow \mathbb{R}^m \) with \( X \subseteq \mathbb{R}^n \) convex is called convex (resp. concave) iff \( f(\lambda x + (1 - \lambda)y) \leq (\text{resp.} \geq) f(x) + (1 - \lambda)f(y) \) holds for all \( x, y \in X \) and all \( \lambda \in [0, 1] \). Note that \( f \) is concave iff \( -f \) is convex. Note also that \( f \) is convex (resp. concave) iff \( f_i \) is convex (resp. concave) for all \( i = 1, \ldots, m \).

We extend the notion of convexity/concavity from \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) to \( \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \) as follows: Let \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \), and \( I : \{1, \ldots, n\} \rightarrow \{-\infty, \text{id}, \infty\} \). Here, \( -\infty \) denotes the function that assigns \( -\infty \) to every argument, \( \text{id} \) denotes the identity function, and \( \infty \) denotes the function that assigns \( \infty \) to every argument. We define the mapping \( f(I) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \) by \( f(I)(x_1, \ldots, x_n) := f(I(1)(x_1), \ldots, I(n)(x_n)) \) for all \( x_1, \ldots, x_n \in \mathbb{R}_+ \). A mapping \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \) is called concave iff \( f_i \) is continuous on \( \{x \in \mathbb{R}_+^n \mid f_i(x) > -\infty\} \) for all \( i \in \{1, \ldots, m\} \), and the following conditions are fulfilled for all \( I : \{1, \ldots, n\} \rightarrow \{-\infty, \text{id}, \infty\} \):

1. \( \text{fdom}(f(I)) \) is convex.
2. \( f(I)_{\text{fdom}(f(I))} \) is concave.
3. For all \( i \in \{1, \ldots, m\} \) the following holds: If there exists some \( y \in \mathbb{R}_+^n \) such that \( f_i(I)(y) \in \mathbb{R} \), then \( f_i(I)(x) < \infty \) for all \( x \in \mathbb{R}_+^n \).

A mapping \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \) is called convex iff \( -f \) is concave. In the following we are only concerned with mappings \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^m \) that are monotone and concave.

We slightly extend the definition of concave equations of Gawlitza and Seidl [21]:

Definition 3 (Concave Equations). An expression \( e \) (resp. equation \( x = e \)) over \( \mathbb{R} \) is called basic concave expression (resp. basic concave equation) iff \( \llbracket e \rrbracket \) is monotone and concave. An expression \( e \) (resp. equation \( x = e \)) over \( \mathbb{R} \) is called concave iff \( e = \bigvee E \), where \( E \) is a set of basic concave expressions.

The class of systems of concave equations strictly subsumes the class of systems of rational equations and even the class of systems of rational LP-equations as defined by Gawlitza and Seidl [17, 22] (cf. [21]).

For this paper it is important to observe that every system of abstract semantic equations (cf. Section 2) is a system of concave equations: For every statement \( s \), the expression \( \llbracket s \rrbracket_j(x_1, \ldots, x_m) \) is a concave expression, since (1) the expression
following conditions are fulfilled: (1) If \( \rho / E \) solution of a function \( P \) is a \( e \)-operator \( \vee \) case that a switch from one real linear arithmetic solver (cf. Section 4). Whether or not a strategy improvement operator is realized by a SAT modulo real linear arithmetic solver (cf. Section 4). Whether or not a \( \vee \)-strategy represents an improvement may depend on the current approximate. It can indeed be the case that a switch from one \( \vee \)-strategy to another \( \vee \)-strategy is only then profitable, when it is known, that the least solution is of a certain size. Hence, we talk about an improvement of a \( \vee \)-strategy w.r.t. an approximate:

**Definition 4** (Improvements). Let \( E \) be a system of monotone equations over a complete linear ordered set. Let \( \sigma, \sigma' \in \Sigma \) be \( \vee \)-strategies for \( E \) and \( \rho \) be a presolution of \( E(\sigma) \). The \( \vee \)-strategy \( \sigma' \) is called improvement of \( \sigma \) w.r.t. \( \rho \) if the following conditions are fulfilled: (1) If \( \rho / E(\sigma') \), then \( \| E(\sigma') \| \rho > \rho \). (2) For all \( \vee \)-expressions \( e \) occurring in \( E \) the following holds: If \( \sigma'(e) \neq \sigma(e) \), then \( \| e\sigma' \| \rho > \| e\sigma \| \rho \).

A function \( P_\vee \) which assigns an improvement of \( \sigma \) w.r.t. \( \rho \) to every pair \((\sigma, \rho)\), where \( \sigma \) is a \( \vee \)-strategy and \( \rho \) is a pre-solution of \( E(\sigma) \), is called \( \vee \)-strategy improvement operator.

In many cases, there exist several, different improvements of a \( \vee \)-strategy \( \sigma \) w.r.t. a pre-solution \( \rho \) of \( E(\sigma) \). Accordingly, there exist several, different strategy improvement operators. One possibility for improving the current strategy is known as all profitable switches \([4, 5]\). Carried over to the case considered here, this means: For the improvement \( \sigma' \) of \( \sigma \) w.r.t. \( \rho \) we have: \( \| E(\sigma') \| \rho = \| E(\sigma) \| \rho \), i.e., \( \sigma' \) represents the best local improvement of \( \sigma \) at \( \rho \). We denote \( \sigma' \) by \( P_\vee^{\text{profitable}}(\sigma, \rho) \) \([17, 18, 19, 22]\).

Now we can formulate the strategy improvement algorithm for computing least solutions of systems of monotone equations over complete linear ordered sets. This algorithm is parameterized with a \( \vee \)-strategy improvement operator \( P_\vee \). The input is a system \( E \) of monotone equations over a complete linear ordered set, a \( \vee \)-strategy \( \sigma_{\text{init}} \) for \( E \), and a pre-solution \( \rho_{\text{init}} \) of \( E(\sigma_{\text{init}}) \). In order to compute the least and not some arbitrary solution, we additionally assume that \( \rho_{\text{init}} \leq \mu[E] \) holds:
Algorithm 1 The Strategy Improvement Algorithm

Input:
- A system $\mathcal{E}$ of monotone equations over a complete linear ordered set
- A $\lor$-strategy $\sigma_{\text{init}}$ for $\mathcal{E}$
- A pre-solution $\rho_{\text{init}}$ of $\mathcal{E}(\sigma_{\text{init}})$ with $\rho_{\text{init}} \leq \mu[\mathcal{E}]$

\[
\sigma \leftarrow \sigma_{\text{init}}; \quad \rho \leftarrow \rho_{\text{init}}; \quad \text{while } (\rho \notin \text{Sol}(\mathcal{E})) \{ \sigma \leftarrow P_\lor(\sigma, \rho); \quad \rho \leftarrow \mu_{\geq \rho}[\mathcal{E}(\sigma)]; \} \quad \text{return } \rho;
\]

Lemma 13. Let $\mathcal{E}$ be a system of monotone equations over a complete linear ordered set. For $i \in \mathbb{N}$, let $\rho_i$ be the value of the program variable $\rho$ and $\sigma_i$ be the value of the program variable $\sigma$ in the strategy improvement algorithm after the $i$-th evaluation of the loop-body. The following statements hold for all $i \in \mathbb{N}$:

1. $\rho_i \leq \mu[\mathcal{E}]$.
2. $\rho_i \in \text{PreSol}(\mathcal{E}(\sigma_{i+1}))$.
3. If $\rho_i < \mu[\mathcal{E}]$, then $\rho_{i+1} > \rho_i$.
4. If $\rho_i = \mu[\mathcal{E}]$, then $\rho_{i+1} = \rho_i$. □

An immediate consequence of Lemma 13 is the following: Whenever the strategy improvement algorithm terminates, it computes the least solution $\mu[\mathcal{E}]$ of $\mathcal{E}$.

At first we are interested in solving systems of concave equations with finitely many strategies and finite least solutions. We show that our strategy improvement algorithm terminates and thus returns the least solution in this case at the latest after considering all strategies. Further, we give an important characterization for $\mu_{\geq \rho}[\mathcal{E}(\sigma)]$.

Feasibility In order to prove termination we define the following notion of feasibility:

Definition 5 (Feasibility ([21])). Let $\mathcal{E}$ be a system of basic concave equations. A finite solution $\rho$ of $\mathcal{E}$ is called \emph{(\mathcal{E}-)feasible} iff there exists $X_1, X_2 \subseteq X$ and some $k \in \mathbb{N}$ such that the following statements hold:

1. $X_1 \cup X_2 = X$, and $X_1 \cap X_2 = \emptyset$.
2. There exists some $\rho' \prec \rho|X_1$ such that $\rho' \cup \rho|X_2$ is a pre-solution of $\mathcal{E}$, and $\rho = [\mathcal{E}]^k(\rho' \cup \rho|X_2)$.
3. There exists a $\rho' \prec \rho|X_2$ such that $\rho' \prec ([\mathcal{E}]^k(\rho|X_1 \cup \rho'))|X_2$.

A finite pre-solution $\rho$ of $\mathcal{E}$ is called \emph{(\mathcal{E}-)feasible} iff $\mu_{\geq \rho}[\mathcal{E}]$ is a feasible finite solution of $\mathcal{E}$. A pre-solution $\rho \prec \infty$ is called feasible iff $e = -\infty$ for all $x = e \in \mathcal{E}$ with $[e]\rho = -\infty$, and $\rho|X'$ is a feasible finite pre-solution of $\{x = e \in \mathcal{E} \mid x \in X'\}$, where $X' := \{x \mid x = e \in \mathcal{E}, [e]\rho > -\infty\}$.

A system $\mathcal{E}$ of basic concave equations is called \emph{feasible} iff there exists a feasible solution $\rho$ of $\mathcal{E}$. □

The following lemmas ensure that our strategy improvement algorithm stays in the feasible area, whenever it is started in the feasible area.

Lemma 14 ([21]). Let $\mathcal{E}$ be a system of basic concave equations and $\rho$ be a feasible pre-solution of $\mathcal{E}$. Every pre-solution $\rho'$ of $\mathcal{E}$ with $\rho \leq \rho' \leq \mu_{\geq \rho}[\mathcal{E}]$ is feasible. □

Lemma 15 ([21]). Let $\mathcal{E}$ be a system of concave equations, $\sigma$ be a $\lor$-strategy for $\mathcal{E}$, $\rho$ be a feasible solution of $\mathcal{E}(\sigma)$, and $\sigma'$ be an improvement of $\sigma$ w.r.t. $\rho$. Then $\rho$ is a feasible pre-solution of $\mathcal{E}(\sigma')$. □
In order to start in the feasible area, we simply start the strategy improvement algorithm with the system $\mathcal{E} \vee -\infty := \{ x = e \vee -\infty \mid x = e \in \mathcal{E} \}$, a $\vee$-strategy $\sigma_{\text{init}}$ for $\mathcal{E} \vee -\infty$ such that $(\mathcal{E} \vee -\infty)(\sigma_{\text{init}}) = \{ x = -\infty \mid x = e \in \mathcal{E} \}$, and the feasible pre-solution $-\infty$ of $(\mathcal{E} \vee -\infty)(\sigma_{\text{init}})$.

It remains to determine $\mu_{\geq \rho}[\mathcal{E}]$. Because of Lemma 14 and Lemma 15 we are allowed to assume that $\rho$ is a feasible pre-solution of the system $\mathcal{E}$ of basic concave equations. This is important in our strategy improvement algorithm. The following lemma in particular states that we have to compute the greatest finite pre-solution.

**Lemma 16 ([21]).** Let $\mathcal{E}$ be a feasible system of basic concave equations with $e \neq -\infty$ for all $x = e \in \mathcal{E}$. There exists a greatest finite pre-solution $\rho^*$ of $\mathcal{E}$ and $\rho^*$ is the only feasible solution of $\mathcal{E}$. If $\rho$ is a finite pre-solution of $\mathcal{E}$, then $\rho^* = \mu_{\geq \rho}[\mathcal{E}]$. □

**Termination.** Lemma 16 implies that our strategy improvement algorithm has to consider each $\vee$-strategy at most once. Thus, we have shown the following theorem:

**Theorem 17.** Let $\mathcal{E}$ be a system of concave equations with $\mu[\mathcal{E}] < \infty$. Assume that we can compute the greatest finite pre-solution $\rho^*$ of each $\mathcal{E}(\sigma)$, if $\mathcal{E}(\sigma)$ is feasible. Our strategy improvement algorithm computes $\mu[\mathcal{E}]$ and performs at most $|\Sigma| + |X|$ strategy improvement steps. The algorithm in particular terminates, whenever $\Sigma$ is finite. □

### 6 Computing Greatest Finite Pre-Solutions

For all systems $\mathcal{E}$ of abstract semantic equations (see Section 2) and all $\vee$-strategies $\sigma$, $\mathcal{E}(\sigma)$ is a system of abstract semantic equations, where each right-hand side is of the form $[s]_j^T(x_1, \ldots, x_m)$, where $s$ is a sequential statement and $x_1, \ldots, x_m$ are variables. We call such a system of abstract semantic equations a system of basic abstract semantic equations. It remains to explain how we can compute the greatest finite solution of such a system — provided that it exists.

Let $\mathcal{E}$ be a system of basic abstract semantic equations with a greatest finite pre-solution $\rho^*$. We can compute $\rho^*$ through linear programming as follows:

We assume w.l.o.g. that every sequential statement $s$ that occurs in the right-hand sides of $\mathcal{E}$ is of the form $Ax \leq b; x := A'x + b'$, where $A \in \mathbb{R}^{k \times n}, b \in \mathbb{R}^k, A' \in \mathbb{R}^{m \times n}, b' \in \mathbb{R}^n$. This can be done w.l.o.g., since every sequential statement can be rewritten into this form in polynomial time. We define the system $\mathcal{C}$ of linear inequalities to be the smallest set that fulfills the following properties: For each equation

$$x = [Ax \leq b; x := A'x + b']^T(x_1, \ldots, x_m),$$

the system $\mathcal{C}$ contains the following constraints:

$$x \leq T_j A'(y_1, \ldots, y_n)^T + T_j b', \quad A_i(y_1, \ldots, y_n)^T \leq b_i \text{ for all } i = 1, \ldots, k$$

$$T_i(y_1, \ldots, y_n)^T \leq x_i \text{ for all } i = 1, \ldots, m$$

Here, $y_1, \ldots, y_n$ are fresh variables. Then $\rho^*(x) = \sup \{ \rho(x) \mid \rho \in \text{Sol}(\mathcal{C}) \}$. Thus $\rho^*$ can be determined by solving $|X_\mathcal{C}|$ linear programming problems each of which can be constructed in linear time. We can do even better by determining an optimal
solution of the linear programming problem \( \sup \left\{ \sum_{x \in X} \rho(x) \mid \rho \in \text{Sol}(C) \right\} \). Then the optimal values for the variables \( x \in X \) determine \( \rho^* \) (cf. Gawlitza and Seidl [17, 22]). Summarizing we have:

**Lemma 18.** Let \( \mathcal{E} \) be a system of basic abstract semantic equations with a greatest finite pre-solution \( \rho^* \). Then \( \rho^* \) can be computed by solving a linear programming problem that can be constructed in linear time. \( \square \)

**Example 19.** We again use the definitions of Example 7. Consider the system \( \mathcal{E} \) of basic abstract semantic equations that consists of the equations

\[
x_{1,1} = [s'; s_2]_1(x_{1,1}, x_{1,2}) \quad \quad x_{1,2} = [s'; s_1]_2(x_{1,1}, x_{1,2}),
\]

where \( s' := x_1 \leq 1000; x_2 := -x_1, s_1 := x_2 \leq -1; x_1 := -2x_1, \) and \( s_2 := -x_2 \leq 0; x_1 := -x_1 + 1 \). Our goal is to compute the greatest finite pre-solution \( \rho^* \) of \( \mathcal{E} \).

Firstly, we note that \( [s'; s_2] = [x_1 \leq 0; (x_1, x_2) := (-x_1 + 1, -x_1)] \) and \( [s'; s_1] = [(x_1, -x_1) \leq (1000, -1); (x_1, x_2) := (-2x_1, -x_1)] \) hold. Accordingly, we have to find an optimal solution for the following linear programming problem:

\[
\begin{align*}
\text{maximize } & \quad x_{1,1} + x_{1,2} \\
\text{subject to } & \quad x_{1,1} \leq -y_1 + 1 \quad x_{1,2} \leq 2y_1' \quad y_1 \leq 0 \quad y_1' \leq 1000 \quad y_1 \leq x_{1,1} \\
& \quad -y_1' \leq -1 \quad -y_1 \leq x_{1,2} \quad y_1' \leq x_{1,1} \quad -y_1' \leq x_{1,2}
\end{align*}
\]

An optimal solution is \( x_{1,1} = 2001, x_{1,2} = 2000, y_1 = -2000, \) and \( y_1' = 1000 \). Thus \( \rho^* = \{x_{1,1} \mapsto 2001, x_{1,2} \mapsto 2000\} \) is the greatest finite pre-solution of \( \mathcal{E} \). \( \square \)

Summarizing, we have shown our main theorem:

**Theorem 20.** Let \( \mathcal{E} \) be a system of abstract semantic equations with \( \mu[\mathcal{E}] < \infty \). Our strategy improvement algorithm computes \( \mu[\mathcal{E}] \) and performs at most \( |\Sigma| + |X| \) strategy improvement steps. For each strategy improvement step, we have to do the following:

1. Find models for \( |X| \) SAT modulo real linear arithmetic formulas, each of which can be constructed in linear time.

2. Solve a linear programming problem which can be constructed in linear time.

**Proof.** The statement follows from Lemmas 14, 15, 16, 18 and Theorem 17. \( \square \)

Our techniques can be extended straightforwardly in order to get rid of the pre-condition \( \mu[\mathcal{E}] < \infty \). However, for simplicity we eschew these technicalities in the present article.

### 7 An Upper Bound on the Complexity

In Section 5 we have provided a lower bound on the complexity of computing abstract semantics of affine programs w.r.t. the template linear domains. In this section we show that the corresponding decision problem is not only \( \Pi_2^p \)-hard, but in fact \( \Pi_2^p \)-complete:

**Theorem 21.** The problem of deciding, whether or not, for a given affine program \( G \), a given template constraint matrix \( T \), and a given program point \( v \), \( V^G[v] > -\infty \) holds, is in \( \Pi_2^p \).
Proof. (Sketch) We have to show that the problem of deciding, whether or not, for a given affine program \(G\), a given program point \(v\), and a given \(i \in \{1, \ldots, m\}\), \((V^G[v])_i = -\infty\) holds, is in \(\co-P_2 = \Sigma_2^P = \text{NP}^\text{NP}\). In polynomial time we can guess a \(\lor\)-strategy \(\sigma\) for \(E' := E(G)\) and compute the least feasible solution \(\rho\) of \(E'(\sigma)\) (see Gawlitza and Seidl [17]). Because of Lemma 4, we can use a NP oracle to determine whether or not there exists an improvement of the strategy \(\sigma\) w.r.t. \(\rho\). If this is not the case, we know that \(\rho \geq \mu_{\{E'\}}\) holds. Therefore, by Lemma 6 we have \(\rho(x_{v,i}) \geq (V^G[v])_i\). Thus we can accept, whenever \(\rho(x_{v,i}) = -\infty\) holds.

Finally, we give an example where our strategy improvement algorithm performs exponentially many strategy improvement steps. It is similar to the program in the proof of Theorem 8. For all \(n \in \mathbb{N}\), we consider the program \(G_n = (N, E, \text{st})\), where \(N = \{\text{st}, 1\}\), \(E = \{(\text{st}, x_1 := 0; y_1 := 1; y_2 := 2y_1; \ldots; y_n := 2y_n−1,1), (1,s,1)\}\), and

\[
s = x_2 := x_1; (x_2 \geq y_n; x_2 := x_2 − y_n | x_2 \leq y_n−1); \ldots
\]

\[
(x_2 \geq y_1; x_2 := x_2 − y_1 | x_2 \leq y_1−1); x_1 := x_1 + 1.
\]

It is sufficient to use a template constraint matrix that corresponds to the interval domain. It is remarkable that the strategy iteration does not depend on the strategy improvement operator in use. At any time there is exactly one possible improvement until the least solution is reached. All strategies for the statement \(s\) will be encountered. Thus, the strategy improvement algorithm performs \(2^n\) strategy improvement steps. Since the size of \(G_n\) is \(\Theta(n)\), exponentially many strategy improvement steps are performed.

8 Conclusion

We presented an extension of the strategy improvement algorithm of Gawlitza and Seidl [17, 18, 21] which enables us to use a SAT modulo real linear arithmetic solver for determining improvements of strategies w.r.t. current approximates. Due to this extension, we are able to compute abstract semantics of affine programs w.r.t. the template linear constraint domains of Sankaranarayanan et al. [42], where we abstract sequences of if-then-else statements without loops en bloc. This gives us additional precision. Additionally, we provided one of the few “hard” complexity results regarding precise abstract interpretation.

It remains to practically evaluate the presented approach and to compare it systematically with other approaches. Besides this, starting from the present work, there are several directions to explore. One can for instance try to apply the same ideas for non-linear templates [21], or to use linearization techniques [35].

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