Holographic conductivity of holographic superconductors with higher order corrections

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Abstract

We analytically as well as numerically disclose the effects of the higher order correction terms in the gravity and in the gauge field on the properties of $s$-wave holographic superconductors. On the gravity side, we consider the higher curvature Gauss-Bonnet corrections and on the gauge field side, we add a quadratic correction term to the Maxwell Lagrangian. We show that for this system, one can still obtain an analytical relation between the critical temperature and the charge density. We also calculate the critical exponent and the condensation value both analytically and numerically. We use a variational method, based on the Sturm-Liouville eigenvalue problem for our analytical study, as well as a numerical shooting method in order to compare with our analytical results. For a fixed value of the Gauss-Bonnet parameter, we observe that the critical temperature decreases with increasing the nonlinearity of the gauge field. This implies that the nonlinear correction term to the Maxwell electrodynamics make the condensation harder. We also study the holographic conductivity of the system and disclose the effects of Gauss-Bonnet and nonlinear parameters $\alpha$ and $b$ on the superconducting gap. We observe that for various values of $\alpha$ and $b$, the real part of conductivity is proportional to the frequency per temperature, $\omega/T$, as frequency is enough large. Besides, the conductivity has a minimum in the imaginary part which is shifted toward greater frequency with decreasing the temperature.

1 Introduction

The correspondence between the gravity in a $d$-dimensional anti-de Sitter (AdS) spacetime and the conformal field theory (CFT) residing on the $(d - 1)$-dimensional boundary of this spacetime, well-known as AdS/CFT correspondence, provides an established method for calculating correlation functions in a strongly interacting field theory using a dual classical gravity description \cite{1}. It has been confirmed that this duality can be applied for solving the problem of high temperature superconductors in condensed matter physics \cite{2}. This
is due to the fact the high temperature superconductors are basically in a strong coupling regime, and thus one expects that the holographic method could give some insights into the pairing mechanism in these systems. Understanding the mechanism of high temperatures superconductors has long been a mystery problem in modern condensed matter physics. Recently, it was suggested that it is logical to understand the properties of high temperature superconductors on the boundary of spacetime by considering a classical general relativity in one higher dimensions. This idea is called the holographic superconductors (HSC) [3, 4, 5] and has got a lot of attentions in the past decade. According to the HSC proposal, in the gravity side, a Maxwell field and a charged scalar field are introduced to describe the \( U(1) \) symmetry and the scalar operator in the dual field theory, respectively. This holographic model undergoes a phase transition from black hole with no hair (normal phase/conductor phase) to the case with scalar hair at low temperatures (superconducting phase) [6].

Nowadays, the investigations on the HSC have attracted considerable attention and become an active field of research. Let us review some works in this direction. In the background of Schwarzschild AdS black holes in Einstein gravity, the properties of HSC have been explored in [7, 8, 9, 10, 11, 12, 13, 14]. The studies were also generalized to higher order gravity theories such as Gauss-Bonnet gravity [15, 16, 17, 18, 19]. It was argued that the critical temperature of the HSC decreases with increasing the backreaction, although the effect of the Gauss-Bonnet coupling is more subtle: the critical temperature first decreases then increases as the coupling tends towards the Chern-Simons value in a backreaction dependent fashion [18]. It was confirmed that the critical exponent of the condensation in Gauss-Bonnet HSC still obeys the mean field theory and has the value \( 1/2 \) [17]. Other studies on the holographic superconductor have been carried out in (see for example [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] and references therein).

It is also interesting to investigate the electrical conductivity of HSC in the dual CFT as a function of frequency. In the AdS/CFT correspondence, the electrical conductivity can be computed by looking at the linear response of the system to fluctuations of the fields \( A_x \) and \( g_{tx} \) in the bulk. These fluctuations are dual to the electric current \( J_x \) and energy current \( T_{tx} \) operators in the CFT. In the context of linear Maxwell field, the conductivity of HSC were computed in [2, 4, 5]. In the presence of nonlinear electrodynamics, the conductivity of HSC have been investigated in [46, 47]. Also, in the context of Born-Infeld nonlinear electrodynamics, the optical properties of Lifshitz HSC has been explored in [18]. It was demonstrated that this superconductor exhibits metamaterial property in low frequency of the external electric field for certain region of nonlinear parameter. The effects of the Weyle coupling parameter and Lifshitz dynamic exponent on the conductivity of HSC have been explored in [49]. In Ref. [50], a rotating BTZ black holes was considered as the gravity dual to \((1+1)\) dimensional superconductor. In this case, depending of the angular momentum on the conductivity has been investigated. Recently, the authors of [51] have analytically computed the holographic conductivity of HSC in the presence of Born-Infeld nonlinear electrodynamic by considering the backreaction of the matter field on the bulk metric. Further investigations on the holographic conductivity of HSC have been performed in [52, 53].

In this work, we will address the effects of the higher order corrections on the holographic conductivity of the s-wave HSC. On the gravity side, we will consider the Gauss-Bonnet curvature correction terms which is most general action in the 5D spacetime and
on the gauge field side we add the quadratic nonlinear gauge term. We shall investigate
the effects of these correction terms on the imaginary and real parts of the electrical
conductivity of the system. With these correction terms, especially including a Gauss-
Bonnet correction to the 5D action, we have the most general action with second-order
field equations in 5D [54], which provides the most general models for the s-wave HSC.
Furthermore, in an effective action approach to the string theory, the Gauss-Bonnet term
 corresponds to the leading order quantum corrections to gravity, and its presence guarantees
a ghost-free action[55]. The purpose of this work is to analytically as well as
numerically explore the effects of these correction terms on the properties of s-wave HSC.

The plan of the work is as follows. In section 2 we will set up our model of the HSC
in Gauss-Bonnet gravity with nonlinear electrodynamics in the probe limit and drive the
equations of motion. In section 3 we analytically as well as numerically compute the
relationship between the critical temperature and the charge density of Gauss-Bonnet
HSC. In section 4 we study condensation operator near the critical temperature using
analytical and numerical method. In section 5 we investigate the electrical conductivity
of the HSC in Gauss Bonnet gravity with nonlinear correction term to the Maxwell field.
In particular, we shall find the ratio of the gap frequency in conductivity to the critical
temperature. At last, we summarize and discuss our results in section 6.

2 HSC in Gauss-Bonnet gravity with nonlinear electrodynamics

We consider the 5D Einstein-Gauss-Bonnet gravity in the background of AdS spaces which
is described by the action [56],

\[
S = \int d^5x \sqrt{-g} \left[ R - 2\Lambda + \frac{\alpha}{2} \left( R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \right) + L_M \right],
\]

where \( \Lambda = -6/l^2 \) is the cosmological constant of 5-dimensional AdS spacetime with
radius \( l \), \( \alpha \) is the Gauss-Bonnet coefficient with dimension \( (\text{length})^2 \), \( R_{\mu\nu\rho\sigma} \), \( R_{\mu\nu} \) and \( R \) are the Riemann curvature tensor, Ricci tensor, and the Ricci scalar, respectively. For
congvenience, hereafter we set the AdS radius \( l = 1 \). We consider the Lagrangian density
of the matter field, \( L_M \), as

\[
L_M = L_{NL} - |\nabla \psi - iqA\psi|^2 - m^2|\psi|^2.
\]

where \( \psi \) is a scalar field, \( q \) and \( m \) are, respectively, the charge and the mass of the scalar
field, and the Lagrangian density of the nonlinear electrodynamics is given by [57] [58]

\[
\mathcal{L}_{NL} = \mathcal{F} + b\mathcal{F}^2 + O(\mathcal{F}^4),
\]

where \( \mathcal{F} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \) is the Maxwell Lagrangian and \( b \) is a parameter. The term \( b\mathcal{F}^2 \) is
the first order leading nonlinear correction term to the Maxwell field. There are several
motivation for choosing the nonlinear Lagrangian in the form of [59]. First, the series
expansion of the three well-known Lagrangian of nonlinear electrodynamics such as Born-
Infeld, Logarithmic and Exponential nonlinear electrodynamics have the form of [59].
Second, calculating one-loop approximation of QED, it was shown [60] that the effective
Lagrangian is given by [59]. Besides, if one neglect all other gauge fields, one may arrive
at the effective quadratic order of $U(1)$ as $F^2$ \cite{61,62}. Furthermore, considering the next order correction terms in the heterotic string effective action one can obtain the $F^2$ term as a corrections to the bosonic sector of supergravity, which has the same order as the Gauss-Bonnet term \cite{61,62,63,64}.

\[ L_{\text{cor}} = \beta \left[ \alpha \left( R^2 - 4R^{\mu \nu} R_{\mu \nu} + R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} \right) + b(F^{\mu \nu} F_{\mu \nu})^2 \right]. \] (4)

The field equations can be obtained by varying action (1) with respect to the metric $g_{\mu \nu}$, the scalar field $\psi$, and and the gauge field $A_\mu$. We find

\[
R_{\mu \nu} - \frac{(R - 2\Lambda)}{2} g_{\mu \nu} - \frac{\alpha}{2} \left\{ \frac{1}{2} g_{\mu \nu} \left( R^2 - 4R^{\rho \sigma} R_{\rho \sigma} + R^{\kappa \lambda \rho \sigma} R_{\kappa \lambda \rho \sigma} \right) 
- 2RR_{\mu \nu} + 4R_{\mu \lambda} R^{\lambda \nu} + 4R_{\mu \nu \rho \sigma} R^{\rho \sigma} - 2R_{\mu}^{\rho \sigma \lambda} R_{\nu \rho \sigma \lambda} \right\} = T_{\mu \nu},
\] (5)

\[
(\nabla_\mu - iqA_\mu)(\nabla_\nu - iqA^\nu)\psi - m^2 \psi = 0,
\] (6)

\[
\nabla_\mu [(1 + 2bF) F^{\mu \nu}] = iq \left[ \psi^* (\nabla_\nu - iqA^\nu) \psi - \psi (\nabla_\nu + iqA^\nu) \psi^* \right],
\] (7)

where $T_{\mu \nu}$ is the matter-stress tensor

\[
T_{\mu \nu} = \frac{1}{2} \left( F + bF^2 \right) g_{\mu \nu} - 2(1 + 2bF) F_\mu F_\nu - \frac{1}{2} m^2 |\psi|^2 g_{\mu \nu} - \frac{1}{2} g_{\mu \nu} |\nabla \psi - iqA\psi|^2
+ \frac{1}{2} |(\nabla_\nu - iqA^\nu)\psi (\nabla_\mu + iqA^\mu)\psi^* + \mu \leftrightarrow \nu].
\] (8)

The metric of a planar Schwarzschild-AdS black hole in 5D is \cite{65}

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2 + dz^2), \] (9)

with

\[ f(r) = \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 - 4\alpha \left( 1 - \frac{r^4}{r^4} \right)} \right). \] (10)

The Hawking temperature at the horizon can be written in the form

\[ T = \frac{f'(r)}{4\pi} = \frac{r_+}{\pi}. \] (11)

It is worthwhile to note that in the limit $r \to \infty$, we can obtain

\[ f(r) \sim \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 - 4\alpha} \right], \] (12)

so we can introduce the effective AdS radius as

\[ L_{\text{eff}}^2 = \frac{2\alpha}{1 - \sqrt{1 - 4\alpha}}. \] (13)

We choose the gauge and the scalar fields in the form [2]

\[ A_\mu = (\phi(r), 0, 0, 0, 0), \quad \psi = \psi(r), \] (14)
Inserting the metric (9) and the gauge and scalar fields (14) in the field equations (6) and (7), we arrive at

\[ \partial_r^2 \psi + \left( \frac{3}{r} + \frac{\partial_r f}{f} \right) \partial_r \psi + \left( \frac{\partial^2}{f^2} - \frac{m^2}{f} \right) \psi = 0, \tag{15} \]

\[ \partial_r^2 \phi + \frac{3}{r} \left( 1 - 2b(\partial_r \phi)^2 \right) \partial_r \phi - \frac{2\psi^2 \phi}{f} (1 - 3b(\partial_r \phi)^2) = 0. \tag{16} \]

The horizon radius is defined as the root of \( f(r_+) = 0 \). The regularity condition for the gauge field \( A_t \) on the horizon \( r_+ \), implies the boundary condition \( \phi(r_+) = 0 \), which substituting in Eq. (15) yields

\[ \psi(r_+) = \frac{\partial_r f(r_+)}{m^2} \partial_r \psi(r_+). \tag{17} \]

Near the AdS boundary \((r \to \infty)\) the asymptotic behaviors of the solutions are given by

\[ \phi(r) = \mu - \frac{\rho}{r^2}, \tag{18} \]

\[ \psi(r) = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}}. \tag{19} \]

where \( \Delta_{\pm} = 2 \pm \sqrt{4 + m^2 L_{\text{eff}}^2} \). It is clear that we should have \( m^2 L_{\text{eff}}^2 \geq -4 \). The value of \( \Delta_{\pm} \) depend on \( \tilde{m}^2 = m^2 L_{\text{eff}}^2 \). For example, setting \( \tilde{m}^2 = -3 \), we have \( \Delta_+ = 3 \) and \( \Delta_- = 1 \). The coefficients \( \psi_{\pm} \) correspond to the vacuum expectation values of the condensate operator, namely \( \psi_{\pm} = \langle O_{\pm} \rangle \), where \( O_{\pm} \) is the dual operator to the scalar field with the conformal dimension \( \Delta_{\pm} \). Following [2], we can impose the boundary condition in which either \( \psi_- \) or \( \psi_+ \) vanishes, so that the theory is stable in the asymptotic AdS region. In what follow, we set \( \psi_- = 0 \) and take \( \psi_+ = \langle O_+ \rangle \) non zero. The interpretation of the parameters \( \mu \) and \( \rho \), also comes from the gauge/gravity dictionary and are, respectively, interpreted as the chemical potential and charge density of the conformal field theory on the boundary.

### 3 Relation between critical temperature and charge density

In this section, we are going to study the critical temperature of HSC, when the higher order corrections to the gravity side as well as the gauge field is taken into account. We shall continue our study both analytically and numerically and compare the two method with each other.

#### 3.1 Analytical method

Here, we analytically obtain the relation between the critical temperature and the charge density of Gauss-Bonnet HSC. To do this, we first transform the coordinate \( r \) to \( z \), such that \( z = r_+/r \). Using this new coordinates, the equations of motion (15) and (16) can be rewritten as

\[ \partial_z^2 \phi - \frac{1}{z} \partial_z \phi + \frac{6b z^3}{r_+^2} (\partial_z \phi)^3 - \frac{2\psi^2 \phi r_+^2}{f z^4} + \frac{6b(\partial_z \phi)^2 \phi \psi^2}{f} = 0, \tag{20} \]
\[ \partial^2 \psi + \left( \frac{\partial f}{\partial z} - \frac{1}{z} \partial_z \psi + \frac{r_+^2}{z^4} \phi^2 - \frac{m^2}{f} \right) \psi = 0. \] (21)

Near the critical temperature \((T = T_c)\) we have \(\psi = 0\), and thus equation (20) reduces to

\[ \partial^2 \phi - \frac{1}{z} \partial_z \phi + \frac{6bz^3}{r_+^2} (\partial_z \phi)^3 = 0. \] (22)

Solving the above equation for the small value of nonlinear parameter \(b\), we find

\[ \phi(z) = \lambda r_+ c (1 - z^2) \left[ 1 - \frac{b \lambda^2}{2} \xi(z) \right] + O(b^2), \] (23)

where

\[ \xi(z) = (1 + z^2)(1 + z^4), \quad \lambda = \frac{\rho}{r_+^3 c}. \] (24)

Next, we consider the boundary conditions for \(\psi\) near the critical point \(T \to T_c\). We assume \(\psi\) has the following form

\[ \psi(z) \big|_{z \to 0} \sim \frac{\langle O \rangle}{r_+^3} z^3 F(z), \] (25)

where \(F(z)\) is the trial function near the boundary \(z = 0\), which satisfies the boundary conditions \(F(0) = 1\) and \(F'(0) = 0\). Substituting Eqs. (23) and (25) in Eq. (21) one arrives at

\[ F'' + p(z) F' + q(z) F + \lambda^2 w(z) F = 0, \] (26)

where the prime now indicates the derivative with respect to \(z\), and \(p(z), q(z)\) and \(w(z)\) read

\[ p(z) = \frac{3(1 - \sqrt{1 - 4\alpha + 4\alpha z^4}) - 12\alpha + 20\alpha z^4}{z[1 - 4\alpha + 4\alpha z^4 - \sqrt{1 - 4\alpha + 4\alpha z^4}]}, \] (27)

\[ q(z) = \frac{1}{z^2} \left[ \frac{3(1 - 4\alpha - 4\alpha z^4 - \sqrt{1 - 4\alpha + 4\alpha z^4})}{\sqrt{1 - 4\alpha + 4\alpha z^4} - 1 + 4\alpha - 4\alpha z^4} + \frac{2m^2 \alpha}{\sqrt{1 - 4\alpha + 4\alpha z^4} - 1} \right], \] (28)

\[ w(z) = \frac{4\alpha^2(1 - z^2)^2(1 - \frac{b}{2} \lambda^2 \xi(z))^2}{(1 - \sqrt{1 - 4\alpha + 4\alpha z^4})^2}. \] (29)

It is a matter of calculations to convert Eq. (26) to the standard form of the Sturm-Liouville equation

\[ (T(z) F'(z))' - Q(z) F(z) + \lambda^2 P(z) F(z) = 0, \] (30)

where,

\[ T(z) = \frac{z^3}{2\sqrt{\alpha}} \left( \sqrt{1 - 4\alpha + 4\alpha z^4} - 1 \right) \approx z^3 \sqrt{\alpha} (z^4 - 1)[1 - \alpha(z^4 - 1)], \] (31)

\[ Q(z) = -T(z) q(z) \approx -3z \sqrt{\alpha} (3z^4 + 6\alpha z^4 - 7\alpha z^8), \] (32)

\[ P(z) = T(z) w(z) \approx \frac{\sqrt{\alpha} z^3(z^2 - 1)(1 + \alpha(z^4 - 1))(1 - \frac{b}{2} \lambda^2 \xi(z))^2}{z^2 + 1}. \] (33)
In the above equations we have only kept the terms up to order $\alpha^{3/2}$. Next, we perform a perturbative expansion $b\lambda^2$ and retain only the terms that are linear in $b$ such that

$$b\lambda^2 = b(\lambda^2|_{b=0}) + O(b^2),$$

where $\lambda^2|_{b=0}$ is the value of $\lambda^2$ for $b = 0$. Thus we can rewrite Eq. (33) as

$$P(z) \approx \sqrt{\alpha}z^3(2z^2 - 1)(1 + \alpha(z^4 - 1))(1 - b(\lambda^2|_{b=0})\xi(z)).$$

Employing the Sturm-Liouville eigenvalues problem, the eigenvalues of Eq. (30) can be obtained by varying the following function

$$\lambda^2 = \int_0^1 dz(T(z)(F'(z))^2 + Q(z)F^2(z)) \int_0^1 dzP(z)F^2(z),$$

where we also choose $F(z) = 1 - az^2$ and $m^2 = -3/L_{eff}^2$ to appraise this expression. At last, using Eqs. (11) and (24), for $T \sim T_c$, one can obtain

$$T_c = \zeta \rho^{1/3},$$

where $\zeta = \frac{1}{\pi \lambda_{min}}$ and $\lambda_{min}$ is the minimum eigenvalue which can be obtained by variation of Eq. (36). Our strategy, in the analytical method, for calculating the critical temperature for condensation is to minimize the function (36) with respect to the coefficient $a$ by fixing other parameters of the model such as $b$ and $\alpha$. Then, we obtain $\lambda_{min}$ and hence the maximum value of $T_c/\rho^{1/3}$ can be deduced through relation (37). As an example, we bring the details of our calculation for $\alpha = 0.01$ and $b = 0.01$. In this case Eq. (36) reduces to

$$\lambda^2 = \frac{0.150900 - 0.226000a + 0.117119a^2}{0.003844 - 0.003350a + 0.000911a^2},$$

whose minimum is $\lambda_{min} = 25.6427$ at $a = 0.747087$. And thus according to Eq. (37), the critical temperature becomes $T_c = 0.185363\rho^{1/3}$. In tables 1, 2 and 3, we summarize our results for $\lambda_{min}$ and $\zeta$ for different values of the parameters $\alpha$, $a$ and $b$. This table shows that for a small and fixed value of $\alpha$, with increasing the nonlinear parameter $b$, the value of $\zeta = T_c/\rho^{1/3}$ decreases as well. As we shall see in the next section this results is in a very good agreement with the numerical results.

### 3.2 Numerical method

Now, we numerically investigate the critical behavior of the HSC in Gauss-Bonnet gravity with quadratic correction term to the gauge field. For the numerical study we employ the shooting method [67]. For simplicity we assume $r_+ = 1$, and thus Eqs. (20) and (21) for $\phi$ and $\psi$ reduces to

$$\partial_z^2 \phi \frac{1}{z} \partial_z \phi + 6bz^3(\partial_z^2 \phi)^3 \frac{2\psi^2 \phi}{fz^4} + 6b(\partial_z \phi)^2 \frac{\phi \psi^2}{f} = 0, \quad (39)$$

$$\partial_z^2 \psi + \left(\frac{\partial_z f}{f} - \frac{1}{z}\right) \partial_z \psi + \frac{1}{z^4} \left(\frac{\phi^2}{f^2} - \frac{m^2}{f}\right) \psi = 0. \quad (40)$$
Figure 1: The behavior of $\psi(z)$ versus $z$ for Gauss-Bonnet HSC and for $\tilde{m}^2 = -3$ and different $\alpha$ and $b$.

Near the horizon ($z = 1$), we can expand $\phi$ and $\psi$ as

$$\phi \approx \phi'(1)(1 - z) + \frac{\phi''(1)}{2}(1 - z)^2 + \ldots,$$

$$\psi \approx \psi(1) + \psi'(1)(1 - z) + \frac{\psi''(1)}{2}(1 - z)^2 + \ldots,$$

while near the AdS boundary ($z = 0$), they behave like

$$\phi \approx \mu - \rho z^2,$$

$$\psi \approx \psi_- z^\Delta - \psi_+ z^\Delta.$$

Table 1: Comparison of analytical and numerical values of $\zeta = T_c/\rho^{1/3}$ for $\alpha = 0.0001$.

| $b$  | $a$  | $\lambda_{\text{min}}^2$ | $\zeta_{SL} \left( = \frac{1}{\pi \lambda_{\text{min}}^{1/3}} \right)$ | $\zeta_{\text{Numerical}}$ |
|------|------|--------------------------|---------------------------------|-----------------------------|
| 0    | 0.721772 | 18.2331         | 0.196204                  | 0.197957                   |
| 0.01 | 0.747560 | 25.0682         | 0.186064                  | 0.181057                   |
| 0.02 | 0.777479 | 36.1392         | 0.175060                  | 0.165642                   |
| 0.03 | 0.809805 | 55.2025         | 0.163126                  | 0.151671                   |

We calculate $\phi''(1)$, $\psi'(1)$, and $\psi''(1)$ in the term of $\psi(1)$ and $\phi'(1)$ by using the equations of motion for $\phi$ and $\psi$, namely Eqs. (39) and (40), respectively. Since near the critical point $\psi$ is very small, thus we choose $\psi(1) = 0.0001$. Our strategy for using the shooting method is as follows. For specific value of the reduced scalar field mass $\tilde{m}^2$, we
can perform numerical calculation near the horizon boundary with one shooting parameter $\phi'(1)$ to get proper solutions at the infinite boundary. For specific values of $\phi'(1)$, we impose the boundary condition $\psi_- = 0$. We also calculate the analytical values and numerical values of $\zeta$ for different $b$. We compare our numerical results with analytical in tables 1, 2 and 3.

In Fig. 1, we plot $\psi$ versus $z$ for three first boundary condition $\phi'(1)$, $\tilde{m}^2 = -3$ and different values of Gauss-Bonnet coefficient $\alpha$ and nonlinear parameter $b$. In the absence of quadratic correction term ($b = 0$), our results exactly coincide with those presented in [17, 20]. The acceptable diagram for us is the red one in each plot since there is nothing in the bulk to effect on speed of the wave, so the diagram of $\psi$ will be stable. From tables 1 − 3, it is evident that when $b$ becomes larger the condensation gets harder. Similar behavior can be seen for the fixed value of $b$ and different values of $\alpha$, namely the critical temperature reduces and condensation becomes harder when the Gauss-Bonnet coupling parameter $\alpha$ gets larger.

### 4 Critical exponent and condensation values

In this section, our aim is to calculate the critical exponent of HSC with first order correction terms in gravity and gauge field. Again, we continue our studying both analytically and numerically.

#### 4.1 Analytical method

We would like to obtain the critical exponent and the condensation values of the condensation operator near the critical temperature using the analytical method. Inserting Eq. (25) into Eq. (20), we get

$$\partial_z^2 \phi - \frac{1}{z} \partial_z \phi + \frac{6b z^3}{r_+^2} (\partial_z \phi)^3 = \frac{\langle O_+ \rangle^2}{r_+^4} B \phi,$$

for $b = 0.01$. We compare our numerical results with analytical in tables 1, 2 and 3.

| $b$  | $a$    | $\lambda_{\min}^2$ | $\zeta_{\text{SL}}$ | $\zeta_{\text{Numerical}}$ |
|------|--------|---------------------|----------------------|-----------------------------|
| 0    | 0.720561 | 18.5392             | 0.195660             | 0.196843                    |
| 0.01 | 0.747087 | 25.6427             | 0.185363             | 0.179433                    |
| 0.02 | 0.777900 | 37.2577             | 0.174173             | 0.163592                    |
| 0.03 | 0.811046 | 57.4907             | 0.162026             | 0.149279                    |

Table 2: Comparison of analytical and numerical values of $\zeta = T_c / \rho^{1/3}$ for $\alpha = 0.01$.

| $b$  | $a$    | $\lambda_{\min}^2$ | $\zeta_{\text{SL}}$ | $\zeta_{\text{Numerical}}$ |
|------|--------|---------------------|----------------------|-----------------------------|
| 0    | 0.709061 | 21.5679             | 0.190787             | 0.186114                    |
| 0.01 | 0.743348 | 31.6982             | 0.178928             | 0.162657                    |
| 0.02 | 0.783517 | 49.9342             | 0.165876             | 0.141983                    |
| 0.03 | 0.823574 | 85.6808             | 0.151601             | 0.124005                    |

Table 3: Comparison of analytical and numerical values of $\zeta = T_c / \rho^{1/3}$ for $\alpha = 0.1$. 

In this section, our aim is to calculate the critical exponent of HSC with first order correction terms in gravity and gauge field. Again, we continue our studying both analytically and numerically.

#### 4.1 Analytical method

We would like to obtain the critical exponent and the condensation values of the condensation operator near the critical temperature using the analytical method. Inserting Eq. (25) into Eq. (20), we get

$$\partial_z^2 \phi - \frac{1}{z} \partial_z \phi + \frac{6b z^3}{r_+^2} (\partial_z \phi)^3 = \frac{\langle O_+ \rangle^2}{r_+^4} B \phi,$$

for $b = 0.01$. We compare our numerical results with analytical in tables 1, 2 and 3.

| $b$  | $a$    | $\lambda_{\min}^2$ | $\zeta_{\text{SL}}$ | $\zeta_{\text{Numerical}}$ |
|------|--------|---------------------|----------------------|-----------------------------|
| 0    | 0.720561 | 18.5392             | 0.195660             | 0.196843                    |
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| $b$  | $a$    | $\lambda_{\min}^2$ | $\zeta_{\text{SL}}$ | $\zeta_{\text{Numerical}}$ |
|------|--------|---------------------|----------------------|-----------------------------|
| 0    | 0.709061 | 21.5679             | 0.190787             | 0.186114                    |
| 0.01 | 0.743348 | 31.6982             | 0.178928             | 0.162657                    |
| 0.02 | 0.783517 | 49.9342             | 0.165876             | 0.141983                    |
| 0.03 | 0.823574 | 85.6808             | 0.151601             | 0.124005                    |

Table 3: Comparison of analytical and numerical values of $\zeta = T_c / \rho^{1/3}$ for $\alpha = 0.1$. 

In this section, our aim is to calculate the critical exponent of HSC with first order correction terms in gravity and gauge field. Again, we continue our studying both analytically and numerically.

#### 4.1 Analytical method

We would like to obtain the critical exponent and the condensation values of the condensation operator near the critical temperature using the analytical method. Inserting Eq. (25) into Eq. (20), we get

$$\partial_z^2 \phi - \frac{1}{z} \partial_z \phi + \frac{6b z^3}{r_+^2} (\partial_z \phi)^3 = \frac{\langle O_+ \rangle^2}{r_+^4} B \phi,$$
and,
\[ B = \frac{2z^2}{\int} \left( 1 - \frac{3bz^4(\partial_z \phi)^2}{r^2_+} \right) F^2(z). \tag{46} \]

Near the critical temperature, \( \langle O_+ \rangle^2 \) is a very small and thus we can expand \( \phi(z) \) as
\[ \frac{\phi(z)}{r_+} = \lambda (1 - z^2) \left[ 1 - \frac{b\lambda^2}{2} \xi(z) \right] + \frac{\langle O_+ \rangle^2}{r^2_+} \chi(z). \tag{47} \]

where \( \chi \) satisfies the following boundary condition
\[ \chi(1) = \chi'(1) = 0. \tag{48} \]

With the help of Eq. (47), and comparing the coefficient of \( \langle O_+ \rangle^2 \) on both sides, Eq. (45) leads to
\[ \chi''(z) - \frac{\chi'}{z} + 72b\lambda^2 z^5 \chi' = \lambda B (1 - z^2) \left( 1 - \frac{b}{2} \lambda^2 \xi(z) \right). \tag{49} \]

From Eq. (49) we figure out, in the limit \( z \to 0 \), the following equation
\[ \chi''(0) = \frac{\chi'(0)}{z} \bigg|_{z \to 0}. \tag{50} \]

It is a matter of calculations to show that Eq. (49) can be written
\[ \frac{d}{dz} \left( e^{12b\lambda^2 z^6} \frac{\chi'}{z} \right) = \lambda \frac{2z^3}{r^2_+} e^{12b\lambda^2 z^6 (1 - \frac{b}{2} \lambda^2 \Gamma(z))} \left( 1 + z^2 \right)^3 (1 + \alpha(1 - z^4)) F^2(z), \tag{51} \]

where \( \Gamma(z) = 1 + z^2 + z^4 + 25z^6 \). Integrating both sides of the above equation in the interval \([0, 1]\) and using the boundary condition (48), we arrive at
\[ \frac{\chi'}{z} \bigg|_{z \to 0} = -\frac{\lambda}{r^2_+} A, \tag{52} \]

where
\[ A \approx \int_0^1 \frac{2z^3 F^2(z) \left[ 1 - \frac{b}{2} \lambda^2 (1 + z^2 + z^4 + z^6) \right] [1 - \alpha(1 - z^4)]}{1 + z^2} dz. \]

Combining Eqs. (18) and (17), we achieve
\[ \frac{\mu}{r_+} - \frac{\rho}{r^2_+} z^2 = \lambda (1 - z^2) \left[ 1 - \frac{b\lambda^2}{2} \xi(z) \right] + \frac{\langle O_+ \rangle^2}{r^4_+} (\chi(0) + z\chi'(0) + \frac{z^2}{2} \chi''(0) + ...), \tag{53} \]

where in the last step we have expanded \( \chi(z) \) around \( z = 0 \). Equating the coefficients of \( z^2 \) on both sides of Eq. (53), we find
\[ \frac{\rho}{r^2_+} = \lambda \left( 1 + \frac{\langle O_+ \rangle^2}{r^4_+} A \right). \tag{54} \]

Using the fact that \( \lambda = \rho/r^2_+ \) as well as definition (11), we can obtain the order parameter \( \langle O_+ \rangle \) near the critical temperature \( T_c \) as
Figure 2: The condensate operator $<\mathcal{O}_+>$ as a function of temperature for different values of $b$ and various values of $\alpha$, where we have set $<\mathcal{O}_->=0$ and $\tilde{m}^2=-3$.

| $b$  | $a$    | $\lambda_{\text{min}}^2$ | $\gamma_{\text{SL}}$ | $\gamma_{\text{Numerical}}$ |
|------|--------|---------------------------|------------------------|-------------------------------|
| 0    | 0.721772 | 18.2331                   | 7.70525                | 12.4879                       |
| 0.01 | 0.747560 | 25.0682                   | 9.05565                | 17.4138                       |
| 0.02 | 0.777479 | 36.1392                   | 14.4103                | 24.5663                       |

Table 4: The analytical and numerical results for the condensation operator for $\alpha = 0.0001$.

$$
\langle \mathcal{O}_+ \rangle = \gamma \pi^3 T_c^3 \sqrt{1 - \frac{T}{T_c}},
$$

(55)

where

$$
\gamma = \sqrt{\frac{6}{A}}.
$$

(56)

From Eq. (55) we observe that the critical exponent has the mean field value $1/2$, which is independent of the nonlinear parameter $b$ and Gauss-Bonnet parameter $\alpha$. It is worth noting that $\langle \mathcal{O} \rangle$ is zero at $T = T_c$ and condensation occurs for $T < T_c$. We shall back to calculation the condensation value $\gamma$ in the next subsection.

### 4.2 Numerical method

We use the numerical method to explore the behaviour of the condensate operator $\langle \mathcal{O}_+ \rangle$ in terms of temperature for different values of $\alpha$ and $b$ (see Fig. 2). These curves are obtained by the shooting method which we described in the previous section. As one can see from this figure there is a critical temperature $T_c$ below which the condensate appears, then rises quickly as the system is cooled and finally goes to a constant for sufficiently low temperatures. This behaviour is qualitatively similar to that obtained in BCS theory and observed in many materials. Now we are going to study the condensation operator $\langle \mathcal{O}_+ \rangle$ in the close neighborhood of the superconductor critical temperature to compute the critical exponents and the condensation value $\gamma$ of the Gauss-Bonnet HSC with quadratic nonlinear electromagnetic. For this purpose, we first take the logarithmic of Eq. (55). We arrive at

$$
\log \left( \frac{\langle \mathcal{O}_+ \rangle}{T_c^3} \right) = \log (\pi^3 \gamma) + \frac{1}{2} \log \left( 1 - \frac{T}{T_c} \right).
$$

(57)
| \( b \) | \( a \) | \( \lambda^2_{\text{min}} \) | \( \gamma_{\text{SL}} \) | \( \gamma_{\text{Numerical}} \) |
|---|---|---|---|---|
| 0  | 0.720561 | 18.5392 | 7.72419 | 11.2411 |
| 0.01 | 0.747087 | 25.6427 | 9.44846 | 15.9088 |
| 0.02 | 0.777900 | 37.2577 | 15.6715 | 23.5892 |

Table 5: The analytical and numerical results for the condensation operator for \( \alpha = 0.01 \).

| \( b \) | \( a \) | \( \lambda^2_{\text{min}} \) | \( \gamma_{\text{SL}} \) | \( \gamma_{\text{Numerical}} \) |
|---|---|---|---|---|
| 0  | 0.709061 | 21.5679 | 7.90303 | 4.05071 |
| 0.01 | 0.743348 | 31.6982 | 9.81189 | 10.0376 |
| 0.02 | 0.783517 | 49.9342 | 37.6568 | 17.7555 |

Table 6: The analytical and numerical results for the condensation operator for \( \alpha = 0.1 \).

We have plotted the behaviour of the above function in Fig. (3). From this figure, we observe that the numerical results are fitted to the above analytic form in the vicinity of the critical temperature. We summarize our results in Fig. (3) and also tables 4 – 6 for different values of \( b \) and \( \alpha \). We see that for a fixed value of \( \alpha \), the condensation operator \( \gamma \) increases with increasing \( b \), while for a fixed value of \( b \), it decreases with increasing \( \alpha \).

5 Holographic Conductivity

In this section, we study the energy gap in the holographic superconductor phase which is constructed on the boundary of the background spacetime. In particular, we investigate the influence of the Gauss-Bonnet and nonlinear parameters on the superconducting gap. In order to do this, we must compute the electrical conductivity of holographic superconductor by turning on a small perturbation \( \delta A_x = A_x(r) \exp(-i\omega t) \) to the gauge field in the bulk where \( \omega \) is the frequency. At linearized order in perturbation (\( \delta A_x \)), the equation of motion for the gauge field \( A_x(r) \), which obeys Eq. (7), is

\[
\partial^2_r A_x + \left[ \frac{1}{r} \left( 1 - 6b\partial_r \phi^2 \right) + \frac{\partial_r f}{f} + \frac{4b\phi \partial_r \phi \psi^2}{f} \right] \partial_r A_x - \left[ \frac{2\psi^2}{f} \left( 1 - b\partial_r \phi^2 \right) - \frac{\omega^2}{f^2} \right] A_x = 0.
\]

(58)

In the absence of the nonlinear correction (\( b = 0 \)), this differential equation reduces to the Maxwell case as presented in \([2, 3]\). To determine the conductivity, we need the asymptotic (\( r \to \infty \)) form of the second order differential equation (58), which may be obtained as

\[
\partial^2_r A_x + \frac{3}{r} \partial_r A_x + \frac{\omega^2 L_{\text{eff}}^4}{r^4} A_x + \ldots = 0,
\]

(59)

which admits the following solution near the boundary

\[
A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r^2} + \frac{\omega^2 L_{\text{eff}}^4 \ln(Kr)}{2r^2} A_x^{(0)} + \ldots,
\]

(60)

where \( A_x^{(0)}, A_x^{(1)} \) are two constant and \( K \) is also a constant parameter with length dimension which is considered for a dimensionless logarithmic argument.
Figure 3: These figures show the behavior of $\log \left(\frac{<O_+>}{T_c^3}\right)$ versus $\log \left(1 - T/T_c\right)$ near the critical temperature $T_c$ for different values of $b$ and $\alpha$. The numerical results are highlighted in the filled squares.

According to AdS/CFT correspondence, the two point correlation function of the current operators in a system is given by its on shell action where the action is evaluated on the equations of motion. Here, the on shell action is

$$S_{\text{O.s.}} \equiv \int_{r_+}^{r_\infty} dr \int d^4 x \sqrt{-g} \mathcal{L},$$

which, in the quadratic approximation for the gauge field perturbation becomes

$$S_{\text{O.s.}} = \int d^4 x \int_{r_+}^{r_\infty} dr \left\{ -\frac{1}{2} r f(r) \left( 2\psi(r)^2 - \frac{\omega^2}{f(r)} - \frac{bw^2 \partial_r \phi^2}{f(r)} \right) A_x(r)^2 + f(r) \left( 1 + b \partial_r \phi \right) \partial_r A_x(r)^2 \right\}.$$

After performing an integration by parts and using Eq. (58), we get

$$S_{\text{O.s.}} = \int d^4 x \left[ -\frac{1}{2} r f(r) \left( 1 + b \partial_r \phi \right) \partial_r A_x(r) A_x(r) \right] \Bigg|_{r=r_\infty}.$$

Substituting Eqs. (12), (18) and (60) in the above expression, one arrives at

$$S_{\text{O.s.}} = \int d^4 x \left[ \frac{A_x^{(0)} A_x^{(1)}}{L^2_{\text{eff}}} - \frac{\omega^2 L^2_{\text{eff}} A_x^{(0)} A_x^{(1)}}{4} + \frac{1}{2} \frac{\omega^2 L^2_{\text{eff}} \ln(Kr) A_x^{(0)}}{L^2_{\text{eff}} r^2} + \frac{A_x^{(1)} A_x^{(0)}}{L^2_{\text{eff}} r^2} \right]$$

$$- \frac{\omega^2 L^2_{\text{eff}} A_x^{(0)} A_x^{(1)}}{4 r^2} - \frac{\omega^2 L^6_{\text{eff}} \ln(Kr) A_x^{(0)}}{8 r^2} + \frac{\omega^2 L^6_{\text{eff}} \ln(Kr) A_x^{(0)} A_x^{(1)}}{r^2}$$

$$+ \frac{\omega^4 L^6_{\text{eff}} \ln(Kr) A_x^{(0)}}{4 r^2} + O \left( \frac{1}{r^3} \right) \Bigg|_{r=r_\infty},$$

thus we can obtain $S_{\text{O.s.}}$ as follows

$$S_{\text{O.s.}} = \int d^4 x \left[ \frac{A_x^{(0)} A_x^{(1)}}{L^2_{\text{eff}}} - \frac{\omega^2 L^2_{\text{eff}} A_x^{(0)} A_x^{(1)}}{4} + \frac{1}{2} \frac{\omega^2 L^2_{\text{eff}} \ln(Kr) A_x^{(0)}}{L^2_{\text{eff}} r^2} \right],$$

in which logarithmic divergences appears. In order to cancel out this divergence, we obtain the boundary counterterm as described in Appendix by using
Figure 4: The real part of the conductivity as a function of frequency for different values of $b$ and $\alpha$. Each figure is plotted for different temperature $T/T_c$.

**Skenderis’s method of holographic renormalization** [68]. Therefore, the finite on shell action may be written as

$$S = S_{o.s.} + S_{c.t.},$$  \hspace{1cm} (66)

where the gauge invariant counterterm $S_{c.t.}$ is given by Eq. (79). Now, we can obtain the current operator in the boundary field theory [2, 3] as

$$\langle J_x \rangle = \frac{1}{A_x} \frac{\delta S}{\delta A_x} = \frac{2}{L^2} \frac{1}{L^2} A_x^{(1)} - \frac{\omega^2 L^2}{2} A_x^{(0)}.$$  \hspace{1cm} (67)

According to Ohm’s law, the electrical conductivity can be expressed as

$$\sigma (\omega) = \frac{\langle J_x \rangle}{E_x},$$  \hspace{1cm} (68)

where $E_x = -\partial_t \delta A_x$. Hence, using the current (67), the holographic conductivity is given by

$$\sigma = \frac{2i A_x^{(1)}}{\omega L^2 A_x^{(0)}} + \frac{i \omega L^2}{2}.$$  \hspace{1cm} (69)
Consequently, the holography conductivity is calculated by solving numerically a differential equation (58) such that the infalling boundary condition is imposed at the event horizon

$$A_x(r) = \exp\left(-\frac{i\omega}{4\pi T}\right) S(r),$$

in which $T$ is the Hawking temperature and

$$S(r) = 1 + a_1(r - r_+) + a_2(r - r_+)^2 + \ldots,$$

where $a_1, a_2, \ldots$ are calculated by Taylor series expansion of equation (58) around the horizon.

The numerical results for holographic conductivity associated with the condensation operator $\langle O_+ \rangle$ are plotted in Figs. 4-7. The real and imaginary parts of electrical conductivity versus frequency are illustrated at different temperature below $T_c$ in Figs. 4 and 5 respectively. As one can see from Fig. 4, the superconducting gap is opened below the critical temperature which became deeper with decreasing the temperature. Besides, for various values of $\alpha$ and $b$, the real part of conductivity is proportional to the frequency per temperature, $\omega/T$, as frequency is enough large. According to the Fig.
Figure 6: The real part of the conductivity as a function of $\omega/\langle O^+ \rangle^{1/3}$, at low temperatures, around $T \approx 0.1T_c$.

The divergence in imaginary part, at $\omega = 0$, points out a delta function in the real part, $\text{Re}[\sigma]$, at $\omega = 0$ which is not played in the Fig. 4. As one can see from Fig. 5, the holographic conductivity of HSC has a minimum in the imaginary part. Thus with decreasing the temperature the minimum in the imaginary part shifts toward greater frequency for various values of the Gauss-Bonnet and nonlinear parameters.

To study formation of the superconducting gap with changing $\alpha$ and $b$ at low temperature, e.g., $T \approx 0.1T_c$, the real and imaginary parts of holographic conductivity as a function of $\omega/\langle O^+ \rangle^{1/3}$ are plotted in Figs. 6 and 7, respectively. For a fixed value of Gauss-Bonnet coefficient $\alpha$, the energy gap ($\omega/\langle O^+ \rangle^{1/3}$) enlarges with increasing the nonlinear parameter $b$. It is evident from Figs. 6 and 7, that the energy gap of HSC for various $\alpha$ exhibits different behavior based on the nonlinear correction $b$. In case of the Maxwell field ($b = 0$), the superconducting energy gap decreases as $\alpha$ increases at low temperature (see Fig. 6(d)). When we take into account the nonlinear correction $b$, the energy gap of HSC increases with increasing $\alpha$ (see Figs. 6(d)-6(g)). From Fig. 7, we see that for a fixed value of $\alpha$, the minimum of imaginary part of conductivity goes to the larger value of $\omega/\langle O^+ \rangle^{1/3}$ when $b$ increases. In the absence of correction ($b = 0$), it decreases with enhancing the Gauss-Bonnet coefficient (Fig. 6(d)). Besides, for a fixed value of $b$, the minimum of $\text{Im}[\sigma]$ increases with increasing $\alpha$, while for a fixed value of $\alpha$, ...
Figure 7: The imaginary part of the conductivity as a function of $\omega/\langle O+ \rangle^{1/3}$, at low temperatures, around $T \approx 0.1Tc$.

it increase with increasing $b$.

6 Conclusions

In this paper, we continue the studies on the $s$-wave holographic superconductors (HSC) by taking into account the higher correction terms both to the gravity side as well as the gauge field side of the system. We considered the Gauss-Bonnet HSC when the Maxwell Lagrangian has a nonlinear correction term and is written in the form $\mathcal{L} = \mathcal{F} + b\mathcal{F}^2$, where $\mathcal{F}$ is the Maxwell lagrangian. We have provided several motivations for choosing this kind of Lagrangian for the gauge field. For example, all well-known nonlinear Lagrangian has a series expansion which their first two terms are exactly in the above form.

First, we have analytically as well as numerically investigated the relation between critical temperature of phase transition and charge density which depends on both the Gauss-Bonnet parameter $\alpha$ and the nonlinear parameter $b$. For this purpose, we employed the analytical Sturm-Liouville eigenvalue problem and the numerical shooting method. We find out that for a fixed value of $\alpha$, with increasing the nonlinear parameter $b$, the value of $T_c/\rho^{1/3}$ decreases as well. This implies that when $b$ becomes larger the condensation
gets harder. Similar behavior can be seen for the fixed value of $b$ and different values of $\alpha$, namely the critical temperature decreases and the condensation becomes harder when the Gauss-Bonnet coupling parameter $\alpha$ gets larger. We confirmed that this results are in a very good agreement with our numerical results. Then, we obtained the critical exponent of the Gauss-Bonnet HSC with nonlinear gauge field. We observed that the critical exponent has the mean field value $1/2$, which is independent of the nonlinear parameter $b$ and Gauss-Bonnet parameter $\alpha$.

Then, we explored, numerically, the holographic conductivity of the system. For this purpose, we plotted the real and imaginary parts of electrical conductivity versus $\omega/T$ and $\omega/\langle O_+ \rangle^{1/3}$ for $T < T_c$. We observed that the superconducting gap is opened below the critical temperature which became deeper with decreasing the temperature. Interestingly enough, we found that for different values of $\alpha$ and $b$, and for large frequency, the real part of conductivity is proportional to $\omega/T$. We observed that the holographic conductivity of HSC has a minimum in the imaginary part. Besides, with decreasing the temperature the minimum in the imaginary part shifts toward greater frequency for various values of the Gauss-Bonnet parameter $\alpha$ and nonlinear gauge field parameter $b$. Furthermore, for a fixed value of $b$, the minimum of $\text{Im}[\sigma]$ increases with increasing $\alpha$, while for a fixed value of $\alpha$, it increase with increasing $b$.

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Appendix: Holographic renormalization

To construct the boundary counterterm action, we utilize a holographic renormalization method of Skenderis which was presented in [68, 69, 70]. To apply this method, the spacetime metric takes the form

$$ds^2 = G_{ab}d\xi^a d\xi^b = -\frac{L_{\text{eff}}}{\varrho}X(\varrho)d\tau^2 + \frac{L_{\text{eff}}^2d\varrho^2}{4\varrho X(\varrho)} + \frac{L_{\text{eff}}^2}{\varrho}(dx^2 + dy^2 + dz^2),$$

which relates to the metric [9] via $\tau = t/L_{\text{eff}}$, $\varrho = L_{\text{eff}}^2/r$ and the asymptotic ($\varrho \to 0$) metric function is $X(\varrho)\varrho \to 0 = 1$. Hence, asymptotically metric becomes

$$ds^2 = \frac{L_{\text{eff}}^2d\varrho^2}{4\varrho X(\varrho)} + h_{\mu\nu}dx^\mu dx^\nu,$$

where

$$h_{\mu\nu} = \frac{L_{\text{eff}}^2}{\varrho}\gamma_{\mu\nu}^0.$$
According to the electromagnetic contribution, one can evaluate the on-shell action as

\[
S = \int_M d^5\xi \sqrt{-G} \left[ F + bF^2 \right] \\
= \int_M d^5\xi \sqrt{-G} \frac{1}{2} F_{ab} \left[ (1 + 2bF) F^{ab} \right] - \int_{\partial M} d^4x \sqrt{-h} A_\mu F^\mu (1 + 2bF) \\
= -L_{\text{eff}} \int_{\varrho = \epsilon} d^4x \sqrt{-\gamma^0} A_\mu \partial_\varrho A_\nu \gamma^{0\mu\nu} (1 + 2bF),
\]

(75)

where \( \epsilon \) is a small constant parameter. On the new coordinates [72], the gauge field equation of bulk motion near boundary is given by

\[
\varrho^2 \frac{d^2 A_i}{d\varrho^2} + \frac{1}{4} \partial_0^2 A_i + \mathcal{O}(\varrho^2) = 0,
\]

(76)

where \( \partial_0^2 \) points out to the wave operator of boundary metric \( \gamma^0_{\mu\nu} \) and the general solution of this equation is

\[
A_i = A^0_i + A^1_i \varrho + \psi \ln(\varrho),
\]

(77)

where \( \psi = -1/4\partial_0^2 A^0_i \). With \( A_i \) at hand, the on-shell electromagnetic action can be written as

\[
S = -L_{\text{eff}} \int_{\varrho = \epsilon} d^4x \sqrt{-\gamma^0} A^0_i \left( A^1_i - \frac{1}{4} \partial_0^2 A^0_i - \frac{1}{4} \ln(\epsilon) \partial_0^2 A^0_i \right),
\]

(78)

which is logarithmically divergent. Now, according to Ref. [68], in order to determine the counterterm action, we first invert the solution (77) to give \( A^0_i = A_i + \mathcal{O}(\epsilon) \). Thus, the counterterm action is obtained as

\[
S_{\text{c.t.}} = -\frac{L_{\text{eff}}}{4} \ln(\epsilon) \int_{\varrho = \epsilon} d^4x \sqrt{-\gamma^0} A_i \partial_0^2 A_i \\
= -\frac{L_{\text{eff}}}{4} \ln(\epsilon) \int_{\varrho = \epsilon} d^4x \frac{1}{2} F_{\mu\nu} F^{\mu\nu}.
\]

(79)

It is notable to mention that since we consider \( A_i = A_i(\varrho) \exp(-i\omega L_{\text{eff}} \tau) \), one can calculate

\[
\partial_0^2 A_i = \omega^2 L_{\text{eff}}^2 A_i.
\]

(80)

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