A curvature-driven effective attraction in multicomponent membranes

Matthew F. Demers,† Rastko Sknepnek,‡ and Monica Olvera de la Cruz

†Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, IL 60208
‡Department of Materials Science and Engineering, Northwestern University, Evanston, IL 60208

(Dated: May 1, 2014)

We study closed liquid membranes that segregate into three phases due to differences in the chemical and physical properties of its components. The shape and in-plane membrane arrangement of the phases are coupled through phase-specific bending energies and line tensions. We use simulated annealing Monte Carlo simulations to find low-energy structures, allowing both phase arrangement and membrane shape to relax. The three-phase system is the simplest one in which there are multiple interface pairs, allowing us to analyze interfacial preferences and pairwise distinct line tensions. We observe the system’s preference for interface pairs that maximize differences in spontaneous curvature. From a pattern selection perspective, this acts as an effective attraction between phases of most disparate spontaneous curvature. We show that this effective attraction is robust enough to persist even when the interface between these phases is the most penalized by line tension. This effect is driven by geometry and not by any explicit component-component interaction.

I. INTRODUCTION

Multicomponent liquid membranes are pervasive in nature, and their phase behavior is important in many biological processes. Such membranes are of interest not only as constituents of complex biological systems but also as rich pattern-forming systems in themselves. They provide a fertile ground for exploring the role of geometry in surface pattern formation at the nanoscale, which is an important aspect of functional materials. They have been studied as basic models for biological systems [1, 2] and as promising candidates in the rational design of biocompatible materials [3]. Their promise is due in large part to their versatile phase behavior. In a liquid membrane, the membrane’s constituents are free to rearrange. Although mixing is entropically favorable, this freedom has energetic consequences when there are multiple kinds of membrane components, since different components may have different chemical and physical properties [4]. When the system adapts its arrangement to its conditions, varying shapes and component patterns can result.

Patterns of segregated domains have been directly observed in experimentally created multicomponent giant unilamellar vesicles [5–10]. These experiments agree with theoretical results in which patterns arise in response to differing bending properties of the membrane’s segregated phases [11–16]. These bending properties can include bending rigidities, saddle-splay moduli, and spontaneous (preferred) curvatures. When these properties are phase-specific, the phase arrangement is coupled to the membrane shape.

A key aspect of surface pattern selection is the system’s response to geometric constraint. Constraints include fixed surface area, since the energy required to compress a liquid membrane is orders of magnitude larger than the energy to bend it, and closure, since tears or holes in the membrane would lead to energetically unfavorable exposure of hydrophobic tails to surrounding water. For a closed membrane that does not ex-

change material with its environment, there may also be a constraint on enclosed volume. This does not apply in all cases; for example, lipid bilayers have been observed to adjust their volume by forming transient pores [17]. We use a constant volume assumption here for the sake of simplicity and note that relaxing this constraint does not qualitatively affect reported results. Under these constraints, a multicomponent membrane may not be able to fully indulge the preferences of all of its constituents simultaneously.

Thus a strongly-segregated membrane may face this frustration: on the one hand, immiscibility (realized as a line tension between differing component domains) favors macroscopic segregation and interfaces with minimal length [18]; on the other hand, when subject to geometric constraints, component bending preferences may be more favorably accommodated by arrangements with extended interfaces, including arrangements with multiple domains. In this sense it is the matter of the length and location of interfaces that becomes the battlefront between these competing preferences.

Most studies so far have focused on two-phase systems. In two-phase systems there is only one phase pair, hence only one type of interface. Introducing a third phase is a nontrivial extension. The presence of a third phase provides three phase pairs and thus introduces the crucial feature of interfacial preferences: the three-phase system has freedom not only over length and location of interfaces but also over which phase pairs are brought into contact. This turns out to be an important avenue through which geometry can influence surface pattern. Furthermore, since the line tension between differing phase pairs need not be the same, a three-phase system allows for an additional type of control parameter, one which can produce qualitatively different pattern behavior.

In this work we analyze energy-minimizing structures of closed three-phase membranes. We observe that system favors interface pairs which maximize differences in spontaneous curvature. A pronounced effect of this preference is an effective attraction between phases of most disparate spontaneous curvature, an indirect interaction driven by mutual response to geometric constraints.

Fig. 1 illustrates the basic mechanism. Blocks of type I, II, III (red, green, blue) represent constituents with dif-
different preferred curvatures $C_I$, $C_{II}$, $C_{III}$. Possible four-
interface arrangements are shown. The arrangement which
maximizes the curvature difference at the boundaries is able
to most closely accomodate its constituents’ preferences when
closed.

Figure 1: (Color online) A cartoon representation of curvature-driven
interfacial preference. Blocks of type $I$, $II$, $III$ (red, green, blue)
represent constituents with different preferred curvatures $C_I$, $C_{II}$,
$C_{III}$. In this example $|C_I - C_{II}| > |C_I - C_{III}| > |C_{II} - C_{III}|$.
Possible 4-interface arrangements of these constituents are shown.
The arrangement which maximizes the curvature difference at the
boundaries (bottom middle) is able to most closely accomodate its
constituents’ preferences when closed.

We present cases of equal line tension and cases when line
tension is pairwise preferential, showing that the geometry-
induced attraction is robust enough to persist even in cases
when it is most penalized by line tension. This study is fo-
cused on characterizing the effective attraction. A full descrip-
tion of the wide paltet of observed patterns will be published
elsewhere

II. MODEL

A. Continuum model

We model the membrane as a two-dimensional surface.
This approximation is appropriate for structures whose lat-
eral dimensions are much larger than the membrane thickness.
The system is considered in the strong-segregation regime,
with each surface element identified with one of three phases.
Each phase has a specific, predetermined spontaneous curva-
ture, and interfaces between different phases are penalized
with a line tension. The membrane is assumed to retain a
closed spherical topology. We assume that the volume occu-
pied by the membrane is conserved, as is the area occupied
by each phase.

Our simulation searches for minimum energy configura-
tions – that is, shape and phase arrangements which satisfy
the area and volume constraints and minimize the total en-
ergy. This energy is written as the sum of a bending term $F_b$
and a line tension term $F_c$.

The bending energy takes the form of a Helfrich functional
\[ F_b = \sum_j \int 2\kappa (H - C_j)^2 \, dS_j \] (1)
where $j = 1, 2, 3$ counts phases, $H$ is the mean curvature, $C_j$
is the spontaneous curvature of phase $j$, and $\kappa$ is the bend-
ing rigidity. This term penalizes surface shapes whose local curvature deviates from its preferred value $C_j$. Since $C_j$
depends on the local composition $j$, the bending energy couples
shape and composition. Note that we do not give an index to
bending rigidity $\kappa$ since we are assuming that the phases are
equally bendable.

Note also that the Helfrich functional typically includes a
Gaussian curvature term weighted by saddle-splay modulus
$\tilde{\kappa}$. However, as with $\kappa$, we are assuming that all $\tilde{\kappa}$’s are equal.
Then, since the topology of the surface remains fixed, by the
Gauss-Bonnet theorem \[21\] the Gaussian curvature term in-
tegrates to a constant independent of the particular shape and
therefore does not affect the energy. If the $\tilde{\kappa}$’s were not equal,
the term would need to be retained. Phase-specific differences
in bending rigidity $\kappa$ or saddle-splay modulus $\tilde{\kappa}$ provide addi-
tional mechanisms for pattern formation \[13, 22\] and domain
budding \[23\].

In monolayers, the spontaneous curvature has an intuitive
physical origin related to the properties of the head and tail
groups \[4\]. Roughly speaking, molecules with larger effective
head sizes and smaller effective tail sizes tend to pack into sur-
faces of negative mean curvature, much as cones would pack.
While the actual curvature properties depend on more than
steric considerations, modeling lipid types with spontaneous
curvatures has been shown to agree well with experiments
\[24\]. In lipid bilayers, spontaneous curvature can arise from
the effects of asymmetry between the inner and outer layers
\[25, 27\]. Our model is therefore directly applicable to mono-
layers, such as emulsions in which oil droplets are surrounded
by multiple surfactant types, and to bilayers whose phases
with distinct spontaneous curvatures are conserved, such as
vesicles whose inner layer is uniform but whose outer layer
is multiphase \[28\]. To extend the model to general multicomp-
ponent bilayers would require treatment of separate bilayer
properties.

It is worth contrasting our deformation energy with that
of solid-like membranes. There energy penalties associated
with stretching and shear deformations can lead to buckling
\[29, 30\] and nonlinear conformation fluctuations \[31\] in the
homogeneous case, and in the two-component case elongated
interconnected domains can arise \[22, 32, 33\]. In lipid vesicles
with one fluid and one solid-like phase, complexes of stripes
and polygonal domains have been observed experimentally
\[34\]. These are outside the scope of our current study since
none of our three phases include resistance to shear; it would
be interesting to explore interfacial preferences in these systems.

Line tension energy is given by [35]

\[ \mathcal{F}_\ell = \sum_{i \neq j} \int_{\partial_{ij}} \lambda_{ij} d\ell \quad (2) \]

where \( i, j = 1, 2, 3 \), \( \lambda_{ij} \) is the line tension between phases \( i \) and \( j \), \( d\ell \) is a line element along \( i-j \) interfaces, and the line integral is calculated along the boundary \( \partial_{ij} \) between phases \( i \) and \( j \). This term favors segregation since it penalizes interfaces between domains.

Recently a novel class of molecules called linactants has been synthesized, which behave as 2-D analogues of surfactants: they migrate laterally in a fluid layer to boundaries between immiscible phases, mediating between the immiscible constituents. Small amounts of linactant added to fluid layers have been found to reduce the line tension between immiscible phases [36]. This suggests that the line tension between phases can be viewed as a tunable design parameter.

The surface area and volume constraints can be written as

\[ \int dS_j = A_j \quad (j = 1, 2, 3) \quad \text{and} \quad \int dV = V_0, \]

where \( A_j \) is the prescribed surface area of phase \( j \), and \( V_0 \) is the prescribed system volume. Note that there is an area constraint for each phase; the system’s total surface area and its composition fractions are both conserved.

It is convenient to work with dimensionless parameters. Therefore, we measure energy in units of bending rigidity \( \kappa \), lengths in units of \( R_0 \), the radius of a reference sphere whose volume is \( V_0 \), curvatures in units of \( 1/R_0 \), and line tension in units of \( \kappa/R_0 \). In these units, there remain nine independent input parameters: three spontaneous curvatures, three areas, and three line tensions. While this may seem like a daunting parameter space, we note that there is a great deal of symmetry and physical analog, and we believe our choice of parameter pairs for simulation is representative and can provide useful insight into the phenomena at hand.

### B. Discrete model

Our numerical method is based on a triangulation of the surface. Each vertex is identified with a phase type and surface area elements are represented by polygonal patches centered at vertices. The discrete analogue of the bending energy, Eq. (1), is computed as a sum over vertices, with the discrete mean curvature near each vertex \( v \) computed as [37].

\[ H_{\text{discrete}}(v) = \frac{1}{2} \sum_{i} \frac{|e_{ij}|}{A_j} \psi_i, \quad (3) \]

where \( i \) indexes edges for which \( v \) is an end point, \( |e_{ij}| \) is the length of the \( j^{\text{th}} \) edge, \( \psi_i \) is the dihedral angle between two triangles sharing the \( i^{\text{th}} \) edge, \( j \) indexes triangles containing \( v \), and \( A_j \) is the area of the \( j^{\text{th}} \) triangle.

To compute discrete line tension, Eq. (2), we sum over all adjacent vertex pairs of different phase types [35]:

\[ \sum_{(i,j)} \lambda_{ij} \left( |c_{ij}^{(1)} - m_{ij}| + |c_{ij}^{(2)} - m_{ij}| \right), \]

where \((i,j)\) denotes pairs of adjacent vertices \( i \) and \( j \) that have different phase types, \( c_{ij}^{(1)} \) and \( c_{ij}^{(2)} \) are the centers of the two triangles that have \( i \) and \( j \) as vertices, and \( m_{ij} \) is the midpoint of their shared edge. Note that this differs from the approach used by Hu, et al. [12], as we take into account local triangle distortions when computing interface length.

The surface area constraint is enforced by penalizing differences between \( A(i) \), the surface area associated with a vertex \( i \), and the prescribed per-vertex value, \( F_s = \sum_i \frac{\sigma}{2} (A(i) - A_0(i))^2 \). The volume constraint is also enforced by a harmonic potential penalty.

Throughout all simulations, we set \( \sigma = 2 \times 10^5 \) and \( \nu = 10^4 \). These values were tuned to ensure that, by the end of a run, the total volume and surface areas differed from their target values by less than 1%. Using \( \phi_i \) to denote the composition fraction of phase \( i \), the prescribed surface area of phase \( i \) is \( 1.05 \times 4\pi \phi_i \). This is 5% larger than the area of a sphere of identical volume, excess being necessary to avoid the trivial case of an undeformable sphere.

To find low-energy configurations, simulated annealing Monte Carlo simulations were performed following a linear cooling protocol. Three types of Monte Carlo moves were used: (i) a surface move, which attempts to perturb the position of a vertex by \( |\Delta \vec{r}_i| = 2.5 \times 10^{-3} \), (ii) a phase-swap move, which attempts to exchange the phase type for two randomly selected vertices, and (iii) an edge flip, which cuts an edge shared by two triangles and reattaches it so that it spans the opposite previously-unattached vertices [38 39]. All moves were accepted according to the Metropolis rules. \( 4 \times 10^6 \) sweeps, with a sweep defined as an attempted move of each vertex, were performed for each parameter set. Phase swap moves were performed every five and edge flip moves every ten sweeps. Initial configurations were random triangulations of a sphere, constructed from regular, Caspar-Klug triangulations [40] such that vertex moves were constraint to a sphere and edge flips were performed at a very high temperature to ensure high acceptance rates. Any moves or edge flips that would result in unphysical edge crossings were rejected. The resulting configuration had a large number of vertices with coordination different than six. Finally, vertex types were randomly permuted. Each vertex was assumed to have a hard core of diameter \( l_{\text{min}} = 0.093 \), and each edge was endowed with a tethering potential with maximum length \( l_{\text{max}} = 0.157 \) such that \( l_{\text{max}}/l_{\text{min}} = 1.688 \). These values were chosen to be tight enough to prevent membrane self-intersection but slack enough to allow edge-flips [35 41].

Since the system’s energy landscape is complicated, the computed configurations are not guaranteed to represent global minima. However, because independent runs starting from different random initial configurations reproduce the same qualitative features, we can regard the results as typical.

Patterns are classified using graphs: each colored domain is identified with a colored node, and two nodes are linked if they share an interface. Then two configurations are considered to represent the same pattern if their graphs are isomorphic.
III. RESULTS AND DISCUSSION

A sample of the varied patterns is provided in Fig. 2 where we present a phase diagram for a $\frac{1}{10} : \frac{1}{10} : \frac{8}{10}$ mixture of components $I : II : III$ (red:green:blue). In this diagram, we vary $C_{III}$ (spontaneous curvature of the blue phase) and $\lambda_{I,III}$ (line tension between red and blue phases). We fix spontaneous curvatures $C_I = 0$, $C_{II} = -1.0$ and line tensions $\lambda_{I,II} = \lambda_{I,III} = 1.0$. Results for each parameter set are classified according to pattern. We define pattern in terms of the number and arrangement of colored domains, irrespective of shape.

![Figure 2](image)

**Figure 2**: (Color online) Phase diagram for composition ratio $I : II : III = \frac{1}{10} : \frac{1}{10} : \frac{8}{10}$, varying spontaneous curvature $C_{III}$ and line tension $\lambda_{I,III}$. All other parameters are fixed, with $C_I = 0.0$, $C_{II} = -1.0$, and $\lambda_{I,II} = \lambda_{I,III} = 1.0$. Phases $I, II, III$ are shown in red (medium), green (lightest), and blue (darkest), respectively. Each example vesicle is shown twice: on the upper right, all phases are visible; on the lower left, one phase has been removed so that the pattern can be more clearly seen. Gray squares denote assorted structures with multiple green and red domains. All snapshots were generated with the Visual Molecular Dynamics (VMD) package [42] and rendered with the Tachyon ray-tracer [43].

For present purposes, we note two features. The first is that a preferential line tension opens the possibility of qualitatively different pattern behavior and thus can be an important control parameter in the system’s pattern selection. Furthermore, we observe a transition near $C_{III} = -1.0$, which is the dividing line where the spontaneous curvature of phase $III$ switches from middle to largest in magnitude.

In Fig. 3 we analyze in detail the $\lambda_{I,III} = 1.75$ row from the diagram in Fig. 2. Note that $I – III$ interfaces predominate in the region where $C_{III} < C_{II}$, even though such interfaces are nearly twice as costly as $I – II$ and $II – III$ interfaces.

![Figure 3](image)

**Figure 3**: (Color online) Analysis of a row of the diagram shown in Fig. 2 for $\lambda = 1.75$. Composition fraction is set to $\frac{1}{10} : \frac{1}{10} : \frac{8}{10}$ and $C_{III}$ is varied; $C_I = 0.0$, $C_{II} = -1.0$, $\lambda_{I,II} = \lambda_{I,III} = 1.0$ are kept fixed. *Upper*: Configurations with colored squares indicating pattern grouping. Phases $I, II, III$ are shown in red (medium), green (lightest), and blue (darkest), respectively. *Middle*: Length of each interface type to total interface length ratio $\eta$ as a function of $C_{III}$. Interface fraction is represented by height between curves. *Bottom*: Spontaneous curvatures, with red, green, and blue lines corresponding to spontaneous curvatures of phases of type $I$, $II$, and $III$, respectively.

In Figs. 4 and 5, we show that these interfacial preferences hold over wide slices of parameter space. Fig. 4 examines a $\frac{1}{4} : \frac{3}{4}$ composition ratio where all line tensions are set to $1.0$, with $C_I = -1.6$ and $C_{II} = -1.0$. In this row we observe that the spontaneous curvature of phase $III$ (blue) transitions from smallest in magnitude to middle to largest. Green-blue ($I – III$) interfaces are most favored on the left, where blue is largest in magnitude and green is smallest. In the middle, green ($II$) is smallest in magnitude and red ($I$) and blue ($III$) are close; here red-green ($I – II$) and blue-green ($I – III$) interfaces are about equally favorable, and red-blue ($I – III$) interfaces are disfavored. On the right, blue is largest in magnitude and red is smallest, and red-blue interface is most favored. Red and green are closest in magnitude, and red-green interfaces are disfavored. The interfaces between phases of most disparate spontaneous curvature are most favored.

In Figs. 5 and 6, we show that these interfacial preferences hold over wide slices of parameter space. Fig. 5 examines a $\frac{1}{4} : \frac{3}{4}$ composition ratio where all line tensions are set to $1.0$, with $C_I = -1.6$ and $C_{II} = -1.0$. In this row we observe that the spontaneous curvature of phase $III$ (blue) transitions from smallest in magnitude to middle to largest. Green-blue ($I – III$) interfaces are most favored on the left, where blue is largest in magnitude and green is smallest. In the middle, green ($II$) is smallest in magnitude and red ($I$) and blue ($III$) are close; here red-green ($I – II$) and blue-green ($I – III$) interfaces are about equally favorable, and red-blue ($I – III$) interfaces are disfavored. On the right, blue is largest in magnitude and red is smallest, and red-blue interface is most favored. Red and green are closest in magnitude, and red-green interfaces are disfavored. The interfaces between phases of most disparate spontaneous curvature are most favored.

In Figs. 5 and 6, we show that these interfacial preferences hold over wide slices of parameter space. Fig. 5 examines a $\frac{1}{4} : \frac{3}{4}$ composition ratio where all line tensions are set to $1.0$, with $C_I = -1.6$ and $C_{II} = -1.0$. In this row we observe that the spontaneous curvature of phase $III$ (blue) transitions from smallest in magnitude to middle to largest. Green-blue ($I – III$) interfaces are most favored on the left, where blue is largest in magnitude and green is smallest. In the middle, green ($II$) is smallest in magnitude and red ($I$) and blue ($III$) are close; here red-green ($I – II$) and blue-green ($I – III$) interfaces are about equally favorable, and red-blue ($I – III$) interfaces are disfavored. On the right, blue is largest in magnitude and red is smallest, and red-blue interface is most favored. Red and green are closest in magnitude, and red-green interfaces are disfavored. The interfaces between phases of most disparate spontaneous curvature are most favored.
Spontaneous curvatures, with red, green, and blue lines corresponding to spontaneous curvature of phases I, II, and III, respectively. Interface fraction is represented by height between curves. Lower: Spontaneous curvatures, with red, green, and blue lines corresponding to spontaneous curvature of phases I, II, and III, respectively. The composition fraction is set to \( \frac{1}{3} : \frac{1}{3} : \frac{1}{3} \) and \( C_I = -1.6 \), \( C_{II} = -1.0 \), \( \lambda_{I,II} = \lambda_{I,III} = \lambda_{II,III} = 1.0 \) are kept fixed.

Figure 4: Upper: Configurations with colored squares indicating pattern grouping. Phases I, II, III are shown in red (medium), green (lightest), and blue (darkest), respectively. Middle: Length of each interface type to total interface length ratio \( \eta \) as a function of \( C_I \). Interface fraction is represented by height between curves. Lower: Spontaneous curvatures, with red, green, and blue lines corresponding to spontaneous curvature of phases I, II, and III, respectively. The composition fraction is set to \( \frac{1}{3} : \frac{1}{3} : \frac{1}{3} \) and \( C_I = -1.6 \), \( C_{II} = -1.0 \), \( \lambda_{I,II} = \lambda_{I,III} = \lambda_{II,III} = 1.0 \) are kept fixed.

Figure 5: (Color online) Length of I – III (red-blue) interface as a fraction of total interface, for composition ratio I : II : III = \( \frac{1}{3} : \frac{1}{3} : \frac{1}{3} \), varying spontaneous curvatures \( C_{II} \) and \( C_{III} \). All other parameters are fixed. \( C_I = -1.0 \), and \( \lambda_{I,II} = \lambda_{I,III} = \lambda_{II,III} = 1.0 \).

Figure 6: (Color online) Length of I – III (red-blue) interface as a fraction of total interface, for composition ratio I : II : III = \( \frac{1}{3} : \frac{1}{3} : \frac{1}{3} \), varying spontaneous curvatures \( C_I \) and \( C_{III} \). All other parameters are fixed. \( C_I = -1.0 \), and \( \lambda_{I,II} = \lambda_{I,III} = \lambda_{II,III} = 1.0 \).

We observe the fraction of the I – III interface predominating in regions where \( C_I > C_{II} > C_{III} \) or \( C_I < C_{II} < C_{III} \). Conversely, I – III interfaces are strongly disfavored in regions where \( |C_I - C_{II}| < |C_I - C_{III}| \) and \( |C_I - C_{II}| < |C_{II} - C_{III}| \).

We point out that for large values of line tension or spontaneous curvatures, well beyond the range used in this study, one observes interesting budding effects. Budding plays an important role in biological systems and has been investigated in a number of studies. Effects of interfacial preference on budding in a three-phase liquid membrane will be addressed elsewhere.

In conclusion, we have seen that mutual response to geometry acts as an effective attraction between phases of most disparate spontaneous curvature in a three-phase liquid membrane. This effect arises indirectly through the coupling of deformation and compositional arrangement, rather than through a direct component-component interaction. Nonetheless, it is robust enough to compete with a countervailing line tension. In some cases it results in predominance of interfaces between phases least miscible by pure line tension considerations. Therefore, this system provides an example where geometric constraints, rather than direct interactions, can dominate its conformation. Our findings suggest that an intricate interplay between the geometry and composition can lead to a rich phase behavior of complex fluid membranes. We hope that our results will stimulate further experimental and theoretical work on these rich systems.

We would like to thank S. Patala, F. Solis, and C.K. Thomas for useful discussions. Numerical simulations were in part performed using the Northwestern University High Performance Computing Cluster Quest. MFD and MO thank the financial support of the Air Force Office of Scientific Research (AFOSR) under Award No. FA9550-10-1-0167. RS and MO
thank the financial support of the US Department of Energy Award DEFG02-08ER46539 and the Office of the Director of Defense Research and Engineering (DDR&E).

[1] L. Morelli, K. Uriu, S. Ares, and A. Oates, Science 336, 187 (2012).
[2] T. Chang, Artificial cells: biotechnology, nanomedicine, regenerative medicine, blood substitutes, bioencapsulation, and cell/stem cell therapy, vol. 1 (World Scientific Pub Co Inc, 2007).
[3] A. Sharma and U. Sharma, International Journal of Pharmaceutics 154, 123 (1997).
[4] J. N. Israelachvili, Intermolecular and Surface Forces (Academic Press, 1992), 2nd ed.
[5] S. Veatch and S. Keller, Biophysical Journal 85, 3074 (2003).
[6] T. Baumgart, S. Hess, W. Webb, et al., Nature 425, 821 (2003).
[7] S. L. Veatch and S. L. Keller, BBA-Mol. Cell Res. 1746, 172 (2005).
[8] K. Bacia, P. Schwille, and T. Kurzchalia, Proceedings of the National Academy of Sciences of the United States of America 102, 3272 (2005).
[9] M. Guhteti, M. Mldzianoski, and S. Hess, Biophysical Journal 93, 2011 (2007).
[10] J. de la Serna, J. Perez-Gil, A. Simonsen, and L. Bagatolli, Journal of Biological Chemistry 279, 40715 (2004).
[11] E. Gutlederer, T. Gruhn, and R. Lipowsky, Soft Matter 5, 3303 (2009).
[12] J. Hu, T. Weikl, and R. Lipowsky, Soft Matter 7, 6092 (2011).
[13] W. Gó´zd´z, G. Gompper, and F. Jülich, Colloids and Surfaces-A-Physicochemical and Engineering Aspects 208, 241 (2002).
[14] C. Funkhouser, F. Solis, and K. Thornton, Physical Review E 76, 011912 (2007).
[15] J. Harden, F. MacKintosh, and P. Olmsted, Physical Review E 72, 011903 (2005).
[16] F. Solis, C. Funkhouser, and K. Thornton, EPL (Europhysics Letters) 82, 38001 (2008).
[17] S. Gillmor and P. Weiss, The Journal of Physical Chemistry B 112, 13629 (2008).
[18] C. Huang and M. Olvera de la Cruz, Phys. Rev. E 53, 812 (1996).
[19] M. F. Demers, R. Sknepnek, and M. Olvera de la Cruz (2012), in preparation.
[20] W. Hellrich, Zeitschrift Fur Naturforschung C-a Journal Of Biosciences C 28, 693 (1973), ISSN 0939-5075.
[21] M. Do Carmo and M. Do Carmo, Differential geometry of curves and surfaces, vol. 2 (Prentice-Hall Englewood Cliffs, NJ, 1976).
[22] R. Sknepnek, G. Vernizzi, and M. Olvera de la Cruz, Soft Matter 8, 636 (2012).
[23] Z. Yao, R. Sknepnek, C. K. Thomas, and M. Olvera de la Cruz (2012), submitted.
[24] S. Gruner, The Journal of Physical Chemistry 93, 7562 (1989).
[25] S. Svetina and B. Zekš, European Biophysics Journal 17, 101 (1989).
[26] L. Miao, U. Seifert, M. Wortis, and H.-G. Döbereiner, Phys. Rev. E 49, 5389 (1994).
[27] H. Döbereiner, O. Selchow, and R. Lipowsky, European Biophysics Journal 28, 174 (1999).
[28] T. Taniguchi, Physical review letters 76, 4444 (1996).
[29] H. Seung and D. Nelson, Physical Review A 38, 1005 (1988).
[30] J. Lidmar, L. Mirny, and D. R. Nelson, Physical Review E 68, 051910 (2003).
[31] H. Yoon and J. M. Deutsch, Phys. Rev. E 56, 3412 (1997).
[32] G. Vernizzi, R. Sknepnek, and M. Olvera de la Cruz, Proceedings of the National Academy of Sciences 108, 4292 (2011).
[33] R. Sknepnek and M. Olvera de la Cruz, Phys. Rev. E 85, 050501 (2012).
[34] V. Gordon, P. Beales, Z. Zhao, C. Blake, F. MacKintosh, P. Olmsted, M. Cates, S. Egelhaaf, and W. Poon, Journal of Physics: Condensed Matter 18, L415 (2006).
[35] T. Kohyama, D. Kroll, and G. Gompper, Physical Review E 68, 061905 (2003).
[36] S. Trabelsi, S. Zhang, T. Lee, and D. Schwartz, Physical review letters 100, 37802 (2008).
[37] M. Meyer, M. Desbrun, P. Schröder, and A. Barr, Visualization and mathematics 3, 34 (2002).
[38] V. Kazakov, I. Kostov, and A. Migdal, Physics Letters B 157, 295 (1985).
[39] A. Billoire and F. David, Nuclear Physics B 275, 617 (1986).
[40] D. L. D. Caspar and A. Klug, Cold Spring Harbor Symposia on Quantitative Biology 27, 1 (1962).
[41] G. Gompper and D. Kroll, The European Physical Journal E: Soft Matter and Biological Physics 1, 153 (2000).
[42] W. Humphrey, A. Dalke, and K. Schulten, Journal of Molecular Graphics 14, 33 (1996).
[43] J. Stone, Master’s thesis, Computer Science Department, University of Missouri-Rolla (1998).
[44] O. Pornillos, J. Garrus, and W. Sundquist, Trends in cell biology 12, 569 (2002).
[45] J. Bonifacino and B. Glick, Cell 116, 153 (2004).
[46] S. Welsch, S. Miller, I. Romero-Brey, A. Merz, C. Bleck, P. Walther, S. Fuller, C. Antony, J. Krijnse-Locker, and R. Bartenschlager, Cell host & microbe 5, 365 (2009).
[47] F. Jülicher and R. Lipowsky, Phys. Rev. Lett. 70, 2964 (1993).
[48] F. Jülicher and R. Lipowsky, Physical Review E 53, 2670 (1996).
[49] P. B. Sunil Kumar, G. Gompper, and R. Lipowsky, Phys. Rev. Lett. 86, 3911 (2001).
[50] M. Laradji and P. Kumar, The Journal of chemical physics 123, 224902 (2005).