On the fixed point equation of a solvable 4D QFT model

Harald Grosse · Raimar Wulkenhaar

Dedicated to Prof. Eberhard Zeidler on the occasion of his 75th birthday

Abstract The regularisation of the $\lambda \phi^4$-model on noncommutative Moyal space gives rise to a solvable QFT model in which all correlation functions are expressed in terms of the solution of a fixed point problem. We prove that the non-linear operator for the logarithm of the original problem satisfies the assumptions of the Schauder fixed point theorem, thereby completing the solution of the QFT model.

Keywords quantum field theory · solvable model · Schauder fixed point theorem

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1 Introduction

This paper provides another key result in our long-term project on quantum field theory on noncommutative geometries. This project was strongly supported and influenced by Prof. Eberhard Zeidler. One of us (H.G.) spent a semester as Leibniz professor at the University of Leipzig and enjoyed very much the hospitality at the Max-Planck-Institute for Mathematics in the Sciences at Inselstraße, directed under Prof. Eberhard Zeidler. Shortly later the other one of us (RW) was Schloßmann fellow in the group of Prof. Eberhard Zeidler. Our project started in this time.

The first milestone was the proof of perturbative renormalisability [1], [2] of the $\lambda \phi^4$-model on Moyal space with harmonic propagation. Eberhard Zeidler was constantly interested in our work and played a decisive rôle in further development: He understood that our computation of the $\beta$-function [3] with the remarkable absence of the Landau ghost problem [4] could be of interest for Vincent Rivasseau who visited the MPI Leipzig in summer
2004. Eberhard Zeidler initiated a meeting of one of us (RW) with Vincent Rivasseau. This contact led to a first joint publication [3] which brought the perturbative renormalisation proof of [2] closer to the constructive renormalisation programme [4]. The growing group around Vincent Rivasseau progressed much faster: they reproved the renormalisation theorem in position space [7], derived the Symanzik polynomials [8], extended the method to the Gross-Neveu model [9] and so on [10].

The most important achievement started with a remarkable three-loop computation of the \( \beta \)-function by Margherita Disertori and Vincent Rivasseau [11] in which they confirmed that at a special self-duality point [12], the \( \beta \)-function vanishes to three-loop order. Eventually, Margherita Disertori, Razvan Gurau, Jacques Magnen and Vincent Rivasseau proved in [13] that the \( \beta \)-function vanishes to all orders in perturbation theory. The central idea consists in combining the Ward identity for an \( U(\infty) \) group action with Schwinger-Dyson equations.

We felt that the result of [13] goes much deeper: Using these tools it must be possible to solve the model! Indeed we succeeded in deriving a closed equation for the two-point function \( G(\alpha,\beta) =: G_{\alpha\beta} \) on the unit square \( 0 \leq \alpha, \beta < 1 \):

\[
G_{\alpha\beta} = 1 - \lambda \left( \frac{1 - \alpha}{1 - \alpha \beta} (L_\beta - \beta \mathcal{Y}) + \frac{1 - \beta}{1 - \alpha \beta} (L_\alpha - \alpha \mathcal{Y}) \right) + \frac{1 - \beta}{1 - \alpha \beta} (G_{\alpha\beta}/G_0 - 1) (L_\alpha - \alpha N_0) - \frac{\alpha(1 - \beta)}{1 - \alpha \beta} (L_\beta + N_\beta - N_0), \tag{1}
\]

where

\[
L_\alpha := \int_0^1 dp \frac{G_{\alpha\rho} - G_{0\rho}}{1 - \rho}, \quad \mathcal{M}_\alpha := \int_0^1 dp \frac{\alpha G_{\alpha\rho}}{1 - \alpha \rho}, \quad N_{\alpha\beta} := \int_0^1 dp \frac{G_{\beta\rho} - G_{\alpha\beta}}{1 - \rho}, \tag{2}
\]

and \( \mathcal{Y} = \lim_{\alpha \to 0} \frac{\mathcal{M}_\alpha - \mathcal{Y}}{\alpha} \). A solution would be the key step to compute all higher correlation functions. Unfortunately, all our attempts to solve this equation failed, forcing us to put the problem aside for two years.

During the QFT workshop in November 2011 in Leipzig, one of us (RW) had the chance to meet Eberhard Zeidler and to report about the programme: that we succeeded to reduce all difficulties of a QFT model to a single equation, but failed to solve it. Eberhard Zeidler immediately offered help. He studied the problem (1)±(2) during the following three weeks, unfortunately without success.

This exchange led to a renewed interest and a subsequent major breakthrough in spring 2012: We noticed that after suitable rescaling of \( G_{\alpha\beta} \) to \( G_{ab} \), now with \( a, b \in [0, \Lambda^2] \), the difference function \( D_{ab} = \frac{1}{2} (G_{ab} - G_{ab}) \) satisfies a linear singular integral equation of Carleman type [15] (the singular kernel is the \( N_{\alpha\beta} \)-integral in (2)). We proved in [16], and with corrections in [17] concerning a possible non-trivial solution of the homogeneous Carleman equation [18], [19], that given the boundary function \( G_{ab} \) with \( G_{ab} \equiv 1 \), the full two-point function \( G_{ab} \) reads

\[
G_{ab} = \frac{\exp(\lambda/2) |G_{\alpha\rho}|^2 e^{-\pi a/2} e^{-\pi b/2} \sin(\tau_\rho(a))}{|\lambda| \pi a}, \quad \tau_\rho(a) := \operatorname{arctan}_{[0, \pi]} \left( \frac{|\lambda| \pi a}{b + \frac{1 + \lambda \pi a}{\pi a}} \right). \tag{3}
\]
By $\mathcal{H}_a^\Lambda[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon \to 0} \left( \int_0^{a-\epsilon} + \int_{a+\epsilon}^\infty \right) \frac{f(x)}{x-a} dx$, we denote the finite (or truncated) Hilbert transform. We are mainly interested in the one-sided Hilbert transform $\mathcal{H}_a^\Lambda[f(\bullet)] := \lim_{\lambda \to \infty} \mathcal{H}_a^\Lambda[f(\bullet)]$. As shown in [17], this result is correct for $\lambda < 0$, which is the interesting case for reflection positivity [24]. For $\lambda > 0$ one has to multiply by a factor $(1 + \frac{\lambda^2}{\lambda^2 - \epsilon^2} (aC + F(b)))$, where $C$ is a constant and $F(b)$ an arbitrary function with $F(0) = 0$.

The symmetry condition $G_{ab} = G_{ba}$ of a two-point function leads for $a = 0$ and $\lambda < 0$ to the consistency condition (in the limit $\Lambda \to \infty$)

$$G_{ab} = G_{ba} = \frac{1}{1+b} \exp \left( -\lambda \int_0^b dt \int_0^\infty \frac{dp}{p} \left( \frac{1}{p} \right)^2 + \left( t + \frac{1 + \lambda \pi p}{G_{ab}(\lambda \pi p)} \right)^2 \right).$$

Equation (4) is a much simpler problem than (1)+(2). In [16] we already proved existence of a solution for $\lambda > 0$ via the Schauder fixed point theorem. This case turned out to be much less interesting than $\lambda < 0$: Reflection positivity is excluded for $\lambda > 0$ [27], and the formulae (1),(2) need to be corrected by a winding number [17].

The proof for $\lambda > 0$ given in [16] does not generalise to the opposite sign. In this paper we fill the gap and prove that (4) has a solution for $-\frac{1}{6} \leq \lambda < 0$. The key is to focus on the logarithm of $G_{ab}$, which is an unbounded function. We are able to control the divergence at $\infty$ and prove uniform continuity of the Hilbert transform on such spaces. For $-\frac{1}{6} \leq \lambda \leq 0$ we are able to verify the assumptions of the Schauder fixed point theorem so that (4) has a solution with good additional properties. We would like to warn the reader that the estimates are cumbersome.

The Schauder fixed point theorem is a central topic in Eberhard Zeidler’s book [24, Chap. 2], It follows from Brouwer’s fixed point theorem for which an elementary proof is given in [24, §77].

It is a pleasure to dedicate this paper to Prof. Eberhard Zeidler who showed constant interest in our programme and provided strategic help. From our early common interaction we were strongly supported by the MPI (and ESI in Vienna), which allowed our long-standing fruitful interaction. We congratulate Prof. Zeidler to his birthday and wish him many happy recurrences. We hope he enjoys the connection between quantum field theory [21,22,23] and non-linear functional analysis [24,25,26,27].

2 Logarithmically bounded functions

Consider the following vector space of real-valued functions

$$LB := \left\{ f \in C^1(\mathbb{R}_+) : f(0) = 0, |f'(x)| \leq \frac{C}{1+x} \text{ for some } C \geq 0 \right\}.$$

These functions vanish at zero and grow/decrease at most logarithmically at $\infty$. We equip $LB$ with the norm

$$\|f\|_{LB} := |f(0)| + \sup_{x \geq 0} |(1+x)f'(x)| \text{ for } f \in LB.$$

Indeed, $\|f\|_{LB} = 0$ means $f(0) = 0$ and $f'(x) = 0$, hence $f' = 0$ and thus $f(x) = 0$ everywhere. The additional $|f(0)|$ is redundant but makes it easier to formulate the proofs.

Proposition 1. $(LB, \| \cdot \|_{LB})$ is a Banach space.
Proof. Given a Cauchy sequence \((f_n)_{n \in \mathbb{N}}\) in \(LB\). This means \(f_n(0) = 0\) for every \(n\), and for every \(\varepsilon > 0\) there is \(N_\varepsilon \in \mathbb{N}\) with \(\|f_n - f_m\|_{LB} = \sup_{x \geq 0} |(1 + x)f'_n(x) - (1 + x)f'_m(x)| < \varepsilon\) for all \(m, n \geq N_\varepsilon\). This implies \(|(1 + x)f'_n(x) - (1 + x)f'_m(x)| < \varepsilon\) for every \(x \geq 0\). By the completeness of \(\mathbb{R}\), the sequence \(((1 + x)f'_n(x))_{n \in \mathbb{N}}\) converges at every \(x \geq 0\) and defines a limit function \((1 + x)g(x) := \lim_{n \to \infty} (1 + x)f'_n(x)\). Taking the limit \(m \to \infty\) above shows that

\[
| (1 + x)f'_n(x) - (1 + x)g(x) | < \varepsilon \quad \text{for every } x \text{ and } n \geq N_\varepsilon. \tag{*}
\]

Fix such \(n \geq N_\varepsilon\). By definition of \(f_n\) in \(LB\), the derivative \(x \mapsto (1 + x)f'_n(x)\) is continuous at every \(x\). This means that there is \(\delta_\varepsilon > 0\) such that \(|(1 + x)f'_n(x) - (1 + y)f'_n(y)| < \varepsilon\) for all \(y \geq 0\) with \(|x - y| < \delta_\varepsilon\). For such \(y\) it follows

\[
| (1 + x)g(x) - (1 + y)g(y) | \leq | (1 + x)g(x) - (1 + x)f'_n(x) | + | (1 + x)f'_n(x) - (1 + y)f'_n(y) | + | (1 + y)f'_n(y) - (1 + y)g(y) | < 3\varepsilon.
\]

Therefore, the limit function \(t \mapsto (1 + t)g(t)\) and hence \(t \mapsto g(t)\) is continuous. As such it can be integrated over any compact interval. We define a function \(f(x)\) by

\[
f(x) = \int_0^x dt \, g(t).
\]

This means \(f(0) = 0\), and by the fundamental theorem of calculus the function \(f\) is differentiable at every \(x \geq 0\), and \(f'(x) = g(x)\) is continuous. Expressing this as \((1 + x)g(x) = (1 + x)f'(x)\) we have proved with (*)

\[
|f_n(0) - f(0)| = 0, \quad |(1 + x)f'_n(x) - (1 + x)f'(x)| < \varepsilon \quad \text{for every } x \text{ and } n \geq N_\varepsilon.
\]

Hence, \((f_n)_{n \in \mathbb{N}}\) converges to a function \(f \in \mathcal{C}^1(\mathbb{R}_+)\) in the \(LB\)-norm. By construction we have \(f \in LB\), hence \((LB, \|\cdot\|_{LB})\) is complete. \(\Box\)

Consider for \(-\frac{1}{2} < \lambda < 0\) the following subset

\[
\mathcal{K}_\lambda = \left\{ f \in LB : f(0) = 0, \quad -\frac{1 - |\lambda|}{1 + x} \leq f'(x) \leq -\frac{1}{1 + x} \right\} \subseteq LB. \tag{7}
\]

Lemma 1. \(\mathcal{K}_\lambda\) is a norm-closed subset of the Banach space \(LB\).

Proof. The evaluation maps \(\tilde{e}_\nu, e_\nu : LB \to \mathbb{R}\), with \(\tilde{e}(f) = f(0)\) and \(e_\nu(f) = (1 + x)f'(x)\) are continuous maps from \(LB\) to \(\mathbb{R}\). Hence, the following subset is closed in \(LB\):

\[
\mathcal{K}_\lambda = \tilde{e}^{-1}(\{0\}) \cap \bigcap_{x \geq 0} ev^{-1}_x \left( \left[ -\left(1 - |\lambda|\right), -(1 - \frac{|\lambda|}{1 - 2|\lambda|}) \right] \right). \tag{8}\]

In the sequel we use implicitly the fact that the Hilbert transform of a function that simultaneously belongs for some \(p > 1\) to \(L^p([\Lambda^2, \infty])\) and to the \(\alpha\)-Hölder space on \([0, \Lambda^2]\) for some \(0 < \alpha < 1\) is again a Hölder-continuous function with the same Hölder exponent \(\alpha\). For functions on \([-\pi, \pi]\) this was proved by Priwaloff [28] for a variant of the Hilbert transform. This proof is easily generalised to \([0, \Lambda^2]\). The \(L^p\) condition is necessary for Hilbert transforms over \(\mathbb{R}\) and clearly extends to the one-sided Hilbert transform over \(\mathbb{R}_+\).
This means that for \( f \in \mathcal{K}_\lambda \) the following maps are well defined (possibly with integrals restricted to \([\varepsilon, \Lambda^2]\); the convergence on \( \mathbb{R}_+ \) will be verified in the following section):

\[
R f(a) := \frac{1 - |\lambda| \pi a e^{\varepsilon} e^{f(a)}}{e^{f(a)}}, \quad (8a)
\]

\[
T f(b) := - \log(1+b) + \int_{0}^{\pi} \frac{dt}{\lambda t} \left( \arctan \frac{b + R f(t)}{|\lambda| t} - \arctan \frac{R f(t)}{|\lambda| t} \right). \quad (8b)
\]

Formula \((8b)\) involves the standard branch of the arctan-function with range \([-\frac{\pi}{2}, \frac{\pi}{2}]\), related to the branch used in \((8a)\) by \(\arctan(x) = \frac{\pi}{2} - \arctan \frac{1}{x}\). Comparing with \((8a)\) at \(a = 0\), equivalent to \((8b)\), shows \(\log G_{0b} = (T \log G_{0e})(b)\).

In the following three sections we prove three main results (for a restricted set of \(\lambda\)):

1. That \(T\) maps \(\mathcal{K}_\lambda\) into itself, that \(T\) is norm-continuous on \(\mathcal{K}_\lambda\) and that the image \(T \mathcal{K}_\lambda \subseteq \mathcal{K}_\lambda\) is relatively compact.

**Theorem 1** For \(- \frac{1}{2} < \lambda < 0\), consider the map \(T\) defined by \((8b)\) on the subset \(\mathcal{K}_\lambda \subseteq \mathcal{K}_\lambda\) of the Banach space of logarithmically bounded function, see \((8a)\), \((8b)\) and \((9)\). Then for any \(f \in \mathcal{K}_\lambda\) one has

i) \(T f \in \mathcal{K}_\lambda\).

ii) \(T : \mathcal{K}_\lambda \rightarrow \mathcal{K}_\lambda\) is norm-continuous.

iii) The restriction of \(T \mathcal{K}_\lambda\) to any interval \([0, \Lambda^2]\) is relatively compact in norm-topology.

In particular, \(T\) has a fixed point \(f_\ast = T f_\ast \in \mathcal{K}_\lambda\) which we denote \(\log G_{0b} := f_\ast(b)\).

**Proof** The domain \(\mathcal{K}\) is also convex. Then i),ii),iii) are the requirements of the Schauder fixed point theorem \([24, \text{Chapter 2}]\) to guarantee existence of fixed point \(T f_\ast = f_\ast\). The proof of i),ii),iii) is given in the following subsections. \(\square\)

In this way we prove existence of function \(G_{0b} = G_{0b}\) which satisfies \((8a)\) for all \(0 \leq b \leq \Lambda\). For \(b > \Lambda^2\) there is possibly a discrepancy. Since both sides of \((8a)\) belong to \(\mathcal{K}_\lambda\) the error is \(\leq (1 + \Lambda^2)^{-\frac{1}{2\lambda}} - (1 + \Lambda^2)|\lambda|^{-1}\). To put it differently, for every \(\varepsilon > 0\) there is \(G_{0b} \in \exp \mathcal{K}_\lambda\) such that the difference between lhs and rhs of \((8a)\), and consequently also the difference between their derivatives, is \(< \varepsilon\). This statement means that \((8a)\) has a solution in \(\mathcal{C}_0^1(\mathbb{R}_+)\).

3 \(T\) preserves \(\mathcal{K}_\lambda\)

Integrating the definition \((8a)\) of \(\mathcal{K}_\lambda\) from \(a\) to \(x > a\) yields

\[\log \left( \frac{1 + a}{1 + x} \right)^{1 - |\lambda|} \leq f(x) - f(a) \leq \log \left( \frac{1 + a}{1 + x} \right)^{1 - \frac{|\lambda|}{1 - 2|\lambda|}},\]

and consequently (for \(x > a\))

\[\left( \frac{1 + a}{1 + x} \right)^{1 - \lambda} \leq e^{f(x)} e^{f(a)} \leq \left( \frac{1 + a}{1 + x} \right)^{1 - \frac{|\lambda|}{1 - 2|\lambda|}}, \quad \left( \frac{1 + x}{1 + a} \right)^{1 - \frac{|\lambda|}{1 - 2|\lambda|}} \leq e^{f(a)} e^{f(x)} \leq \left( \frac{1 + x}{1 + a} \right)^{1 - \lambda}, \quad (9)\]
which we reinterpt as
\[
\begin{aligned}
\frac{(1+a)^{1-x} \Gamma(2a)}{1+x} & \leq \frac{e^{f(x)}}{e^{f(a)}} \leq \frac{(1+a)^{1-x} \Gamma(2a)}{1+x} \quad \text{for } x > a ,
\end{aligned}
\]
\[
\leq \frac{(1+a)^{1-x} \Gamma(2a)}{1+x} \quad \text{for } x < a .
\]
(10)

We take the one-sided Hilbert transform:
\[
\mathcal{H}_a e^{f(x)} = \lim_{\varepsilon \to 0} \left\{ - \int_0^{a-\varepsilon} \frac{dx}{x-a} e^{f(x)} + \int_{a+\varepsilon}^\infty \frac{dx}{x-a} e^{f(x)} \right\}.
\]
(11)

The Hilbert transform (11) becomes maximal if for \( x > a \) we use the maximal \( \frac{e^{f(x)}}{e^{f(a)}} \) but for \( x < a \) the minimal \( \frac{e^{f(x)}}{e^{f(a)}} \). Conversely, the Hilbert transform becomes minimal if for \( x > a \) we use the minimal \( \frac{e^{f(x)}}{e^{f(a)}} \) but for \( x < a \) the maximal \( \frac{e^{f(x)}}{e^{f(a)}} \):
\[
\frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ - \int_0^{a-\varepsilon} \frac{dx}{x-a} (1+a)^{1-x} + \int_{a+\varepsilon}^\infty \frac{dx}{x-a} (1+a)^{1-x} \right\}
\leq \frac{\mathcal{H}_a e^{f(x)}}{e^{f(a)}} \leq \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ - \int_0^{a-\varepsilon} \frac{dx}{x-a} (1+a)^{1-x} + \int_{a+\varepsilon}^\infty \frac{dx}{x-a} (1+a)^{1-x} \right\}.
\]
(12)

Note that the analogue only for \( \mathcal{H}_a e^{f(x)} \) would not hold; in that case the opposite boundaries of \( \mathcal{K}_a \) would contribute to \( x < a \) versus \( x > a \), and there is no chance of a reasonable estimate! We can reformulate (13) as
\[
\frac{\mathcal{H}_a e^{f(x)}}{e^{f(a)}} \leq \frac{(1+a)^{1-x} \Gamma(2a)}{1+x} \leq \frac{(1+a)^{1-x} \Gamma(2a)}{1+x}.
\]
(13)

We prove the following result which covers a slightly more general case:

**Proposition 2** For any \( \mu < 1 \), with \( \mu \neq 0 \), and \( \beta > 0 \) one has
\[
\frac{\mathcal{H}_a e^{f(x)}}{e^{f(a)}} = -\cot(\pi \mu) + \frac{1}{\mu} e^{\mu-1} F_{2,1} \left( \mu, 1+\mu \mid \frac{\beta}{a+\beta} \right).
\]
(14)

**Proof** We use the following indefinite integrals:
\[
\int dx \frac{(\beta+x)\mu^{-1}}{x+c} = \frac{(\beta+x)\mu^{-1}}{1-\mu} 2F_{1} \left( 1, 1-\mu \mid -c+\beta x \right), \quad x > -c ,
\]
(15a)
\[
\int dx \frac{(\beta+x)\mu^{-1}}{\mu-a} = \frac{(\beta+x)\mu^{-1}}{\beta+a} 2F_{1} \left( \mu, 1+\mu \mid \frac{x+\beta}{a+\beta} \right), \quad x < a .
\]
(15b)

This is proved via \( x \)-differentiation using \( \frac{d}{dx} 2F_{1} \left( \frac{a, \beta}{x+\gamma} \mid x \right) = a \beta \frac{d}{dx} 2F_{1} \left( \frac{a+1, \beta+1}{x+\gamma} \mid x \right) \) and use of the recursion relations [23, §9.137] for the hypergeometric function. With a large cut-off \( \Lambda \) we have for \( \mu < 1 \)
\[
\pi \mathcal{H}_a e^{f(x)} = \lim_{\varepsilon \to 0, \Lambda \to \infty} \left\{ - \int_0^{a-\varepsilon} dx \frac{(\beta+x)\mu^{-1}}{x+c} + \int_{a+\varepsilon}^\Lambda dx \frac{(\beta+x)\mu^{-1}}{x+c} \right\}
\]
\[
\pi \mathcal{H}_a e^{f(x)} = \lim_{\varepsilon \to 0, \Lambda \to \infty} \left\{ - \int_0^{a-\varepsilon} dx \frac{(\beta+x)\mu^{-1}}{x+c} + \int_{a+\varepsilon}^\Lambda dx \frac{(\beta+x)\mu^{-1}}{x+c} \right\}
\]
\[
\pi \mathcal{H}_a e^{f(x)} = \lim_{\varepsilon \to 0, \Lambda \to \infty} \left\{ - \int_0^{a-\varepsilon} dx \frac{(\beta+x)\mu^{-1}}{x+c} + \int_{a+\varepsilon}^\Lambda dx \frac{(\beta+x)\mu^{-1}}{x+c} \right\}
\]
\[ = \lim_{\varepsilon \to 0, \lambda^2 \to 0} \left\{ -\frac{(\beta + x)^{\mu-1}}{\mu} \frac{\beta + x}{(\beta + a)^2} _2F_1 \left( 1, \mu \left| \frac{\beta + x}{\beta + a} \right| \right)^{a-\varepsilon} \right. \\
\left. - \frac{(\beta + a + x)^{\mu-1}}{1 - \mu} \frac{1 - \mu}{\beta + a} x \right\} \left| \lambda^2 - \varepsilon \right|_{\beta + a + x} \right\} \left( 1, 1 - \mu \left| \frac{\beta + a}{\beta + x} \right| \right)^{\alpha-\varepsilon} \right\}
\]
\[= \frac{\beta^\mu}{(\beta + a)^2} _2F_1 \left( 1, \mu \left| \frac{\beta}{\beta + a} \right| \right)
+ \lim_{\varepsilon \to 0} \left\{ -\frac{(\beta + a - \varepsilon)^\mu}{\beta + a} B(1, \mu) \frac{1}{1 + \mu} \frac{\beta + a - \varepsilon}{\beta + a} \\
+ (\beta + a + \varepsilon)^\mu \frac{B(1, \mu)}{2} \frac{1}{1 + \mu} \frac{\beta + a + \varepsilon}{\beta + a} \right\}, \text{ (16b)} \]

where the special values \( B(1, \mu) = \frac{\Gamma(1)}{\Gamma(1)} = \frac{1}{\mu} \) for the Beta function have been used. The limit \( \varepsilon \to 0 \) is controlled by the following result in [30] (already claimed, but not proved, in Ramanujan’s notebooks) for zero-balanced hypergeometric functions: If \( 0 < \alpha, \beta, \varepsilon, x < 1 \), then
\[ -\psi(\alpha) - \psi(\beta) - 2\gamma < B(\alpha, \beta) _2F_1 \left( \alpha, \beta \left| \frac{\alpha + \beta}{\alpha - x} \right| 0 \right) + \log(x) \]
\[< -\psi(\alpha) - \psi(\beta) - 2\gamma + \frac{x}{1 - x} \log \frac{1}{x}. \text{ (17)} \]

Here \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), and \( \gamma = -\psi(1) \) is the Euler-Mascheroni constant. Since
\[ \lim_{\varepsilon \to 0} \left\{ -\frac{(\beta + a - \varepsilon)^\mu}{\beta + a} \log \left( \frac{\varepsilon}{\beta + a} \right) + \frac{(\beta + a + \varepsilon)^\mu}{(\beta + a + \varepsilon)} \log \left( \frac{\varepsilon}{\beta + a + \varepsilon} \right) \right\} = 0, \]

we can add the corresponding log-terms to (16b) and use (17) to conclude that the two lines (16b) converge in the limit \( \varepsilon \to 0 \) to
\[ \lim_{\varepsilon \to 0} (16b) = (\beta + a)^{\mu-1} \left( \psi(\mu) - \psi(1 - \mu) \right) = -.(\beta + a)^{\mu-1} \pi \cot(\pi \mu), \text{ (18)} \]

where [29, §8.365.8] has been used. This finishes the proof.}

Inserting (14) for \( \beta = 1 \) and \( \mu = |\lambda|, \frac{|\lambda|}{1 - 2|\lambda|} \), respectively, into (13) gives the following bounds valid for any \( f \in \mathcal{X}_\lambda \):
\[ -\cot(|\lambda| \pi) + \frac{1}{|\lambda| \pi (1 + a)^{|\lambda|}} _2F_1 \left( 1, |\lambda| \left| \frac{1}{1 + |\lambda| (1 + a)} \right| \right) \leq \frac{\mathcal{X}_\lambda^{\text{eff}}}{\mathcal{X}_\lambda^{\text{eff}(*)}} \leq -\cot \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right) + \frac{1 - 2|\lambda|}{|\lambda| \pi (1 + a)^{|\lambda|}} _2F_1 \left( 1, \frac{|\lambda|}{1 - 2|\lambda|} \left| \frac{1}{1 + a} \right| \right). \text{ (18)} \]

Together with (10) taken at \( x = 0 \) we obtain for the function \( Rf \) defined in (8a) the following bounds:
\[ |\lambda| \pi a \cot \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right) + \frac{1 + a}{(1 + a)^{|\lambda|}} - \frac{(1 - 2|\lambda|) a}{(1 + a)^{|\lambda|}} _2F_1 \left( 1, \frac{|\lambda|}{1 - 2|\lambda|} \left| \frac{1}{1 + a} \right| \right) \leq (Rf)(a) \leq |\lambda| \pi a \cot(|\lambda| \pi) + \frac{1 + a}{(1 + a)^{|\lambda|}} - \frac{a}{(1 + a)^{|\lambda|}} _2F_1 \left( 1, |\lambda| \left| \frac{1}{1 + a} \right| \right). \text{ (19)} \]
Since \( {}_2F_1 \left( \frac{1}{1+|\lambda|}, \frac{1}{1+a} \right) \geq 1 \) we have \( \frac{a}{(1+a)^2} - \frac{a}{(1+a)^{3+2a}} {}_2F_1 \left( \frac{1}{1+|\lambda|}, \frac{1}{1+a} \right) \leq 0 \). This means that the upper bound is smaller than \( |\lambda| \pi \cot(|\lambda| \pi) + \frac{1}{(1+a)^{2+2a}} \leq |\lambda| \pi a \cot(|\lambda| \pi) + 1 \).

In the lower bound we use [9.137.12] to write the hypergeometric function as \( {}_2F_1 \left( \frac{1}{1+|\lambda|}, \frac{1}{1+|\lambda|+1+a} \right) = 1 + \frac{|\lambda|}{(1+|\lambda|)(1+a)} {}_2F_1 \left( \frac{1}{2+|\lambda|}, \frac{1}{1+a} \right) \). This gives, partly expressed in terms of \( |\lambda_r| := \frac{|\lambda|}{2|\lambda|} \),

\[
(\frac{|\lambda| \pi a}{1-2|\lambda|}) + 1 + |\lambda| F_{\lambda_r} (a) \leq (Rf) (a) \leq |\lambda| \pi a \cot(|\lambda| \pi) + 1,
\]

where

\[
F_{\lambda_r} (a) := \frac{1+2|\lambda_r|}{|\lambda_r|} \left( \frac{1+|\lambda_r|a}{(1+a)|\lambda_r|} - 1 \right) + \hat{F}_{\lambda_r} (a), \quad (20)
\]

\[
\hat{F}_{\lambda_r} (a) := (1-2|\lambda_r|) a - \frac{a}{(1+|\lambda_r|)(1+a)} {}_2F_1 \left( \frac{1}{2+|\lambda_r|}, \frac{1}{1+a} \right).
\]

We have to show that \( F_{\lambda_r} (a) \) is of positive mean for a certain integral. This is easy to check for a computer, but we want to make it rigorous. For a lower bound we can remove the numerator \( (1+2|\lambda_r|) \) in the middle line of (20). The remaining piece \( \hat{F}_{\lambda_r} (a) \) is positive for \( 0 < |\lambda_r| < 1 \) by a particular case of Bernoulli’s inequality. Then its \( |\lambda_r| \)-derivative reads

\[
\frac{d}{d|\lambda_r|} \left( \frac{1+|\lambda_r|a}{(1+a)|\lambda_r|} - 1 \right) = \frac{-1 + (1+a)|\lambda_r| - (1+|\lambda_r|a) \log((1+a)^{1+|\lambda|})}{|\lambda_r|^2 (1+a)|\lambda_r|}.
\]

Using again Bernoulli’s inequality, the numerator is \( \leq x - (1+x) \log(1+x) \) with \( x := (1+a)^{1+|\lambda|} - 1 \). The function \( x - (1+x) \log(1+x) \) vanishes at \( x = 0 \) and has negative derivative for any \( x > 0 \). Consequently, \( \frac{d}{d|\lambda_r|} \left( \hat{F}_{\lambda_r} (a) \right) \) is monotonously decreasing in \( |\lambda_r| \) (hence in \( |\lambda| \)) for any fixed \( a \).

We expand \( \hat{F}_{\lambda_r} (a) \) in the last line of (20) into a power series and take the \( |\lambda_r| \)-derivative:

\[
\frac{d}{d|\lambda_r|} \hat{F}_{\lambda_r} (a) = -2a + a \sum_{k=0}^{\infty} \frac{1}{(1+|\lambda_r|)^2(1+a)^{k+1}} < a \left( -2 + \frac{\pi^2}{6} \right),
\]

\[
\frac{d}{d|\lambda_r|} \left( |\lambda_r| \right) \leq \frac{a}{(1+a)|\lambda_r|} \left( -2 + \frac{\pi^2}{6} - \frac{\log(1+a)}{a} \hat{F}_{\lambda_r} (a) \right).
\]

Hence also \( F_{\lambda_r} (a) \) is decreasing in \( |\lambda_r| \), and sufficient for extending this decrease to \( F_{\lambda_r} (a) \) is \( \hat{F}_{\lambda_r} (a) \geq -(2 - \frac{\pi^2}{6}) \). Using identities and recursion formulae such as [29, §9.137.14+17 §9.131.1] for the hypergeometric function it is straightforward to compute and rearrange the derivatives of \( \hat{F} \):

\[
\hat{F}_{\lambda_r} (a) = 1 - 2|\lambda_r| - \frac{1}{(1+a)^2(1+|\lambda_r|)(2+|\lambda_r|)} {}_2F_1 \left( \frac{2}{3}, \frac{1}{1+|\lambda_r|}, \frac{1}{1+a} \right), \quad (21a)
\]

\[
\hat{F}_{\lambda_r} (a) = 2 \frac{|\lambda_r|}{a(1+a)^2(1+|\lambda_r|)(2+|\lambda_r|)} {}_2F_1 \left( \frac{2}{3}, \frac{1}{1+|\lambda_r|}, \frac{1}{1+a} \right). \quad (21b)
\]

From (21a) we conclude that \( \hat{F} \) is convex in \( a \) for any fixed \( |\lambda_r| \), and (21b) shows that \( \hat{F} \) starts negative near \( a = 0 \) and diverges (in case of \( |\lambda_r| < \frac{3}{2} \)) to \( +\infty \) for \( a \rightarrow \infty \). Together with convexity, there is a unique zero \( \hat{F}_{\lambda_r} (t_\lambda) = 0 \) at \( t_\lambda > 0 \) and a single and unique
Lemma 3 We prove:

Recall that

Proof

ries that \( F_{-\frac{1}{2}}(\frac{1}{2}) > 0 \) and \( F'_{-\frac{1}{2}}(\frac{1}{2}) < 0 \). By convexity, \( F_{\frac{1}{2}} \) lies above any tangent, and the intersection of the tangent \( F_{-\frac{1}{2}}(\frac{1}{2}) + (t - \frac{1}{2})F'_{-\frac{1}{2}}(\frac{1}{2}) \) with the tangent \( F_{\frac{1}{2}}(\frac{1}{2}) + (t - \frac{1}{2})F'_{\frac{1}{2}}(\frac{1}{2}) \) located at \( (0.50048, -0.296723) \) gives a lower bound for the global minimum. This value confirms \( F_{\lambda}(a) \geq -(2 - \frac{a^2}{\lambda}) \) first for \( \lambda_r = \frac{1}{a} \) and then, since \( F_{\lambda}(a) \) decreases in \( |\lambda_r| \), for all \( 0 \leq |\lambda_r| \leq \frac{1}{a} \). We have thus established:

Lemma 2 Let \(-\frac{1}{a} \leq \lambda \leq 0 \) and \( f \in \mathcal{F}_\lambda \). Then

\[
|\lambda|\pi\cot\left(\frac{|\lambda|\pi}{1 - 2|\lambda|}\right) + 1 + |\lambda|F(a) \leq (RF)(a) \leq |\lambda|\pi\cot(|\lambda|\pi) + 1, \quad \text{where}
\]

\[
F(a) := \frac{4 + a}{(1 + a)^\frac{2}{3}} - 4 + \frac{1}{(1 + a)^\frac{2}{3}} \left( \frac{a}{2} - \frac{4a}{5(1 + a)} \right) F_1\left(1, \frac{5}{4}; \frac{1}{1 + a}\right).
\]

We prove:

Lemma 3 The function \( F(a) \) defined in (23) has the following properties:

1. \( F(a) \) is monotonously increasing for \( a \geq \frac{1}{\lambda_r} \).
2. \( F(a) \) is convex for \( 0 \leq a \leq \frac{9}{4} \).
3. \( F(a) \) is concave for \( a \geq \frac{5}{2} \).
4. \( |F''(a)| < \frac{1}{16} \) for \( \frac{9}{4} \leq a \leq \frac{5}{2} \).
5. \( F(a) \geq 0 \) for \( a \geq \frac{4}{5} \).
6. \( F(a) \geq -\frac{1}{a} \) for all \( a \geq 0 \).

Proof Recall that \( F(a) = -\frac{1}{|\lambda_r|} + \frac{1}{(1 + a)^\frac{2}{3}} (\frac{1}{|\lambda_r|} + a + F_{\lambda}(a)) \big|_{|\lambda_r|=\frac{1}{a}} \). Differentiation gives with (21a)

\[
F'(a) = \frac{1}{(1 + a)^\frac{2}{3}} \left( \frac{1}{2} + \frac{9}{8} a - \frac{16}{45(1 + a)} F_1\left(2, \frac{5}{4}; \frac{1}{1 + a}\right) + \frac{a}{5(1 + a)} F_1\left(1, \frac{5}{4}; \frac{1}{1 + a}\right) \right).
\]

This implies the following estimate valid for \( a \geq \frac{1}{\lambda_r} \).

\[
F'(a) \geq \frac{1}{(1 + a)^\frac{2}{3}} \left( \frac{1}{2} + \frac{9}{8} a - \frac{32}{135} F_1\left(2, \frac{5}{4}; \frac{1}{1 + a}\right) + \frac{a}{5(1 + a)} F_1\left(1, \frac{5}{4}; \frac{1}{1 + a}\right) \right),
\]

which shows that \( F \) is monotonously increasing for all \( a \geq \frac{1}{\lambda_r} \). The second derivative reads with (21a) and (21b)

\[
F''(a) = \frac{1}{(1 + a)^\frac{2}{3}} \left( \frac{16 - 9a}{32} + \frac{8 - a}{9(1 + a)} F_1\left(2, \frac{5}{4}; \frac{1}{1 + a}\right) \right)
\]

\[
+ \frac{32}{9 \cdot 13a(1 + a)} F_1\left(2, \frac{5}{4}; \frac{1}{1 + a}\right) - \frac{5a}{36(1 + a)} F_1\left(1, \frac{5}{4}; \frac{1}{1 + a}\right).
\]
Using \( _2\text{F}_1\left(\frac{1}{4}, \frac{1}{4} \mid \frac{1}{1+a} \right) \leq _2\text{F}_1\left(\frac{3}{4}, \frac{3}{4} \mid \frac{1}{1+a} \right) \) and the lower bound 1 for the hypergeometric functions we have the following lower bound for \( F'' \):

\[
F''(a) \geq \frac{1}{(1+a)^2} \left( \frac{16}{9} - 9a \right) + \frac{2}{3} \frac{32}{3(1+a)} \left( \frac{32}{3(1+a)} + \frac{32}{117a(1+a)} \right).
\]

This proves that \( F(a) \) is convex for all \( 0 \leq a \leq 2.26204 \), and we have \( F''(a) \geq -\frac{1}{10} \) for all \( \frac{9}{4} \leq a \leq \frac{5}{2} \).

We derive the converse inequality for \( a \geq \frac{9}{4} \) by splitting the prefactor \( \frac{8-a}{a(1+a)} \) at \( \frac{9}{4} \). We estimate the positive hypergeometric functions by its value at \( \frac{9}{4} \) and the negative hypergeometric functions by 1:

\[
\begin{align*}
a > \frac{9}{4} : \quad F''(a) &\leq \frac{1}{(1+a)^2} \left( \frac{16}{9} - 9a \right) + \frac{2}{3} \frac{32}{3(1+a)} \left( \frac{32}{3(1+a)} + \frac{32}{117a(1+a)} \right) \\
&\quad + \frac{512}{(117)^2} _2\text{F}_1\left(\frac{2}{3}, \frac{5}{3} \mid \frac{4}{15} \right) + \frac{23}{117} _2\text{F}_1\left(\frac{2}{3}, \frac{4}{3} \mid \frac{13}{15} \right)
\end{align*}
\]

This proves that \( F \) is concave for \( a \geq 2.48142 \) and the upper bound \( F''(a) \leq 0.08 \) for all \( \frac{9}{4} \leq a \leq \frac{5}{2} \).

One has \( F(1) = 0.141693 \) and then a good upper bound for \( F(t_0) = 0 \) by the tangent to \( F \) at 1, \( F(1) + (t_0 - 1)F'(1) = 0 \). This shows \( t_0 < \frac{9}{4} \). The tangent to \( F \) at \( \frac{4}{3} \) has positive slope, the tangent at \( \frac{5}{2} \) has negative slope. This means that the value \( F(t_m) \) at the intersection of these tangents \( F\left(\frac{4}{3}\right) + (t_m - \frac{4}{3})F'(\frac{4}{3}) = F\left(\frac{5}{2}\right) + (t_m - \frac{5}{2})F'(\frac{5}{2}) \) gives a lower bound for \( F \). One finds \( t_m = 0.223714 \) and \( F(t_m) = -0.190334 \).

We have now collected all information to prove:

**Lemma 4**

\[
F(a) \geq S(a) := \begin{cases} 
F\left(\frac{4}{3}\right) + (a - \frac{4}{3})F'\left(\frac{4}{3}\right) & \text{for } 0 \leq a \leq \frac{4}{3} \\
F\left(\frac{5}{2}\right) + (a - \frac{5}{2})F'\left(\frac{5}{2}\right) & \text{for } \frac{4}{3} < a < 6 \\
F(6) & \text{for } a \geq 6
\end{cases}
\]

**Proof** The region \( 0 \leq a \leq \frac{4}{3} \) follows from convexity of \( F \), the region \( a \geq 6 \) because \( F \) is monotonously increasing for \( a \geq \frac{4}{3} \). In the intermediate region we have \( F(a) \geq S(a) \) at least for \( \frac{4}{3} \leq a \leq \frac{5}{2} \) because of convexity of \( F \). For \( \frac{5}{2} \leq a \leq 6 \) we know by concavity that

\[
F(a) \geq \frac{(a - \frac{5}{2})F(6) + (6 - a)F\left(\frac{4}{3}\right)}{6 - \frac{5}{2}} \quad \text{for all } \frac{5}{2} \leq a \leq 6.
\]

Inserting the numerical values one checks that the secant \( \frac{(a - \frac{5}{2})F(6) + (6 - a)F\left(\frac{4}{3}\right)}{6 - \frac{5}{2}} \) lies above the tangent \( F\left(\frac{4}{3}\right) + (a - \frac{4}{3})F'\left(\frac{4}{3}\right) \) for \( \frac{5}{2} \leq a \leq 6 \). There remains the gap \( \frac{5}{2} \leq a \leq \frac{5}{2} \) where \( F \) changes from convex to concave. Using the bound \( |F''(a)| < \frac{1}{10} \) in that region we have

\[
F(a) \geq F\left(\frac{5}{2}\right) + (t - \frac{5}{2})F'\left(\frac{5}{2}\right) - \frac{(t - \frac{5}{2})^2}{2} \cdot \frac{1}{10}
\]

for all \( \frac{5}{2} \leq a \leq \frac{5}{2} \). The parabola on the rhs lies above the tangent \( F\left(\frac{5}{2}\right) + (a - \frac{5}{2})F'\left(\frac{5}{2}\right) \). \( \square \)
Observe that (8b) implies \( T f(0) = 0 \) and

\[
T f'(b) = -\frac{1}{1+b} - |\lambda| \int_0^\infty \frac{dt}{(|\lambda|\pi t)^2 + (b + R f(t))^2}.
\]

The inequality of Lemma \( \text{[3]} \) together with the lower bound \( (23) \) are now used to derive bounds for \( T f'(b) \). The inequality \( R f(t) \leq 1 + |\lambda| \pi \cot(|\lambda| \pi) \) leads to

\[
f \in \mathcal{K}_\lambda \Rightarrow T f'(b) \geq -\frac{1}{1+b} + |\lambda| \int_0^\infty \frac{dt}{(|\lambda|\pi t)^2 + (1 + b + |\lambda| \pi t \cot(|\lambda| \pi))^2}
\]

\[
= -\frac{1 - |\lambda|}{1+b}.
\]

We thus confirm that \( T \) preserves the lower bound of \( \mathcal{K}_\lambda \). Proving that \( T \) preserves the other bound, i.e. \( T f'(b) + \frac{1 - |\lambda|}{1+b} \leq 0 \), is more difficult. We insert the inequality \( R f(t) \geq 1 + |\lambda| \pi \cot(|\lambda| \pi) + |\lambda| S(a) \) into (8b) and evaluate the pieces via

\[
\int \frac{dt}{(\alpha t)^2 + (\beta + \gamma)^2} = \frac{\arctan\left(\frac{\alpha t - \gamma}{\beta}\right)}{\alpha \beta}.\]

This gives for any \( f \in \mathcal{K}_\lambda \) and with partial use of \( |\lambda_r| := \frac{|\lambda|}{1 - 2|\lambda|} \):

\[
T f'(b) + \frac{1 - |\lambda|}{1+b} \leq \int_0^\infty \frac{dt}{(|\lambda| \pi t)^2 + (1 + b + |\lambda| \pi t \cot(|\lambda| \pi))^2} - \frac{|\lambda_r|}{\pi(1+b)}
\]

\[
= \int_0^\infty \frac{dt}{(|\lambda| \pi t)^2 + (1 + b + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2})) + (|\lambda| \pi t \cot(|\lambda| \pi))^2}
\]

\[
+ \int_0^\frac{\pi}{2} dt \left( \arctan \left( \frac{-1 + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2})}{\pi (b + 1 + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2}))} \right) \right)
\]

\[
+ \int_0^\frac{\pi}{2} dt \left( \arctan \left( \frac{-1 + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2})}{\pi (b + 1 + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2}))} \right) \right)
\]

\[
+ \frac{|\lambda_r|}{\pi(1+b)}
\]

\[
= \frac{6|\lambda|}{\pi(b + 1 + |\lambda| F(\frac{3}{2}) - \frac{3|\lambda|}{2} F'(\frac{3}{2}))}
\]

\[
= \frac{|\lambda_r|}{\pi(1+b)}.
\]

For \( 0 \leq |\lambda| \leq \frac{1}{4} \) we have \( \cot(|\lambda_r| \pi) \geq 1 \). We are therefore within the convergence domain of the arctan series, and Leibniz’ criterion gives upper and lower bounds:

\[
0 \leq x \leq 1 \Rightarrow x - \frac{x^3}{3} \leq \arctan x \leq x - \frac{x^3}{3} + \frac{x^5}{5}.
\]
For the sake of transparency we abbreviate
\[
\beta := \frac{b + 1}{|\lambda|\pi}, \quad \gamma := \cot(|\lambda|\pi),
\]
\[
\delta_1 := \frac{1}{2} F\left(\frac{1}{2}\right) + \frac{1}{2} \cot\left(\frac{1}{2}\right), \quad \delta_2 := \frac{1}{2} F\left(\frac{3}{2}\right) - \frac{1}{2} F\left(\frac{1}{2}\right), \quad \delta_3 := \frac{1}{2} F\left(\frac{1}{2}\right) - \frac{1}{2} F\left(\frac{3}{2}\right), \quad \delta_4 := \frac{1}{2} F\left(\frac{3}{2}\right) - \frac{1}{2} F\left(\frac{1}{2}\right), \quad \delta_5 := \frac{1}{2} F\left(\frac{1}{2}\right) + \frac{1}{2} F\left(\frac{3}{2}\right), \quad \delta_6 := \frac{1}{2} F\left(\frac{3}{2}\right).
\]

Then
\[
|\lambda|\pi^2\left( T f'(b) + \frac{1 - |\lambda|}{1 + b} \right)
\]
\[
\leq \left( \frac{1}{(2\beta + 2\delta_1 + \gamma)(\beta + \delta_2)} - \frac{1}{(2\beta + 2\delta_3 + \gamma)(\beta + \delta_4)} + \frac{1}{(2\beta + 2\delta_5 + \gamma)(\beta + \delta_6)} \right) + (\beta + \delta_1 + \frac{1}{2}\gamma)^3 (\beta + \delta_2 + 6\gamma)^3 (\beta + \delta_3 + 6\gamma)^3 (\beta + \delta_4 + 6\gamma)^3 (\beta + \delta_5 + 6\gamma)^3 (\beta + \delta_6 + 6\gamma)^3 (\beta + \delta_1)(\beta + \delta_2)(\beta + \delta_3)(\beta + \delta_4)(\beta + \delta_5)(\beta + \delta_6) \beta.
\]

(27)

In the last line, the coefficients \( c_k \) are polynomials in \( \gamma, |\lambda|, \pi \) and \( \frac{1}{|\lambda|^2} \). One finds with \( |\lambda|\cot(|\lambda|\pi) \geq \frac{1}{2} \) and \( |\lambda| - |\lambda| \geq 0 \) for all \( 0 \leq |\lambda| \leq \frac{1}{2} \):
\[
c_{18} = -|\lambda|\pi\delta_6 = -3.53|\lambda|, 
\]
\[
c_{17} = -29.01|\lambda| - 183.74|\lambda|\cot(|\lambda|\pi) - 20.25|\lambda| - 7.75|\lambda|, 
\]
\[
c_{16} = -101.11|\lambda| - 1318.89|\lambda|\cot(|\lambda|\pi) - 4264.94|\lambda|\cot^2(|\lambda|\pi) 
\quad - 54.78|\lambda| - 41.92|\lambda| - 156.99|\lambda| - 56.11|\lambda| 
\quad - 994.28|\lambda| - 355.99|\lambda|\cot(|\lambda|\pi), 
\]
\[
c_{15} = -426.99 \frac{|\lambda|\cot(|\lambda|\pi) - \frac{1}{2} - 191.62|\lambda| - 3914.61|\lambda|\cot(|\lambda|\pi)}{|\lambda|^2} 
\quad - 26296.4|\lambda|\cot^2(|\lambda|\pi) - 29.992|\lambda| - 285.75|\lambda| 
\quad - 2104.91|\lambda| - 1813.03|\lambda|\cot(|\lambda|\pi) - 214.93|\lambda| - 74.85|\lambda|\cot(|\lambda|\pi) 
\quad - 6717.04|\lambda| - 2214.36|\lambda|\cot(|\lambda|\pi) - 21721.1|\lambda| - 7195.88|\lambda|\cot^2(|\lambda|\pi), 
\]
\[
c_{14} = -5405 \frac{|\lambda|\cot(|\lambda|\pi) - \frac{1}{2} - 2729|\lambda|\cot(|\lambda|\pi) - 2729|\lambda|\cot^2(|\lambda|\pi) - 679.6|\lambda|\cot(|\lambda|\pi) - \frac{1}{2}}{|\lambda|^2} 
\quad - 17313 \frac{|\lambda|\cot(|\lambda|\pi) - 207.1|\lambda| - 651.1|\lambda|\cot(|\lambda|\pi)}{|\lambda|^2} 
\quad - 64754|\lambda|\cot^2(|\lambda|\pi) - 301494|\lambda|\cot^3(|\lambda|\pi) - 517659|\lambda|\cot^4(|\lambda|\pi) 
\quad + 1111|\lambda| - 636.23|\lambda| - 3350|\lambda|\cot(|\lambda|\pi) - 1229|\lambda| - 357.4|\lambda|\cot(|\lambda|\pi) - 914.9|\lambda|\cot(|\lambda|\pi) 
\quad - 13307|\lambda| - 10573|\lambda|\cot(|\lambda|\pi) - 45442|\lambda| - 34358|\lambda|\cot(|\lambda|\pi) - 18691|\lambda| - 2338|\lambda|\cot(|\lambda|\pi) 
\quad - 125556|\lambda| - 37587|\lambda|\cot(|\lambda|\pi) - 277886|\lambda| - 84001|\lambda|\cot^3(|\lambda|\pi). 
\]

(28)
All contributions are manifestly negative. That negativity continues to all $c_k$, but the expressions become of exceeding length. It does not make much sense to display these formulae. Instead we give in Figure 1 a graphical description of the coefficients $c_k$. We confirm that all

\begin{align*}
T f'(b) &\leq -\frac{1 - |\lambda|}{1 + b} & \text{for all } b \geq 0 \text{ and any } -\frac{1}{6} \leq \lambda \leq 0.
\end{align*}

This finishes the proof that $T$ maps $\mathcal{X}_\lambda$ into itself.

4 $T$ is uniformly continuous on $\mathcal{X}_\lambda$, but not contractive

Take $f, g \in \mathcal{L}B$ with $\|f - g\|_{\mathcal{L}B} := \delta$. This means $-\frac{\delta}{1+x} \leq f'(x) - g'(x) \leq \frac{\delta}{1+x}$ for all $x \in \mathbb{R}_+$. Integration from $a$ to $x > a$ yields

\begin{align*}
-\delta \log \frac{1+x}{1+a} &\leq f(x) - g(x) - f(a) + g(a) \leq \delta \log \frac{1+x}{1+a}.
\end{align*}
or \((1 + a)^{-\delta} \leq e^{f(x)} e^{g(a)} \leq \left(1 + \frac{1}{a}\right)^{-\delta}\). Together with (10) we deduce the following inequalities valid for \(x > a\):

\[
\max \left\{ \left(1 + \frac{x}{1 + a}\right)^{|\lambda| - 1}, \left(1 + \frac{x}{1 + a}\right)^{-\delta} e^{g(x)} e^{g(a)} \right\} \leq \frac{e^{f(x)}}{e^{f(a)}}
\]

\[
\leq \min \left\{ \left(1 + \frac{x}{1 + a}\right)^{-|\lambda| - 1}, \left(1 + \frac{x}{1 + a}\right)^{\delta} e^{g(x)} e^{g(a)} \right\}.
\]

We subtract \(\frac{e^{g(x)}}{e^{g(a)}} =: \left(1 + \frac{x}{1 + a}\right)^{\mu - 1}\) with \(|\lambda| \leq \mu \leq \frac{|\lambda|}{1 - 2|\lambda|}\). A careful discussion of \(\mu\) versus \(|\lambda| + \delta\) shows that for \(x > a\) one has

\[
- \left(1 + \frac{1}{1 + a}\right)^{-\frac{|\lambda|}{1 - 2|\lambda|}} \leq \frac{e^{f(x)}}{e^{f(a)}} - \frac{e^{g(x)}}{e^{g(a)}} \leq \left(1 + \frac{1}{1 + a}\right)^{\frac{|\lambda|}{1 - 2|\lambda|}} - \left(1 + \frac{1}{1 + a}\right)^{-\frac{|\lambda|}{1 - 2|\lambda|}}.
\]

(29)

Conversely, for \(x < a\) we start from \(\left(1 + \frac{a}{1 + x}\right)^{-\delta} \leq \frac{e^{f(x)}}{e^{f(a)}} \leq \left(1 + \frac{a}{1 + x}\right)^{\delta}\), which together with (10) leads to

\[
\max \left\{ \left(1 + \frac{a}{1 + x}\right)^{1 - \frac{|\lambda|}{1 - 2|\lambda|}}, \left(1 + \frac{a}{1 + x}\right)^{\delta} e^{g(x)} e^{g(a)} \right\} \leq \frac{e^{f(x)}}{e^{f(a)}}
\]

\[
\leq \min \left\{ \left(1 + \frac{a}{1 + x}\right)^{1 - |\lambda|}, \left(1 + \frac{a}{1 + x}\right)^{\delta} e^{g(x)} e^{g(a)} \right\}.
\]

We subtract \(\frac{e^{g(x)}}{e^{g(a)}} =: \left(1 + \frac{a}{1 + x}\right)^{-\mu}\) with \(|\lambda| \leq \mu \leq \frac{|\lambda|}{1 - 2|\lambda|}\). A careful discussion of \(\mu\) versus \(1 - |\lambda| - \delta\) shows that for \(x < a\) one has

\[
- \left(1 + \frac{a}{1 + x}\right)^{1 - |\lambda|} \leq \frac{e^{f(x)}}{e^{f(a)}} - \frac{e^{g(x)}}{e^{g(a)}} \leq \left(1 + \frac{a}{1 + x}\right)^{1 - |\lambda| - \delta}.
\]

(30)

With these preparations we can prove that \(\frac{\mathcal{H}_\lambda[e^{f(x)}]}{e^{f(a)}}\) varies slowly with \(f\):

**Lemma 5** For any \(f, g \in \mathcal{K}_\lambda\) with \(\|f - g\|_{LB} = \delta\), hence \(\delta \leq \frac{2|\lambda|^2}{1 - 2|\lambda|}\), one has for \(0 \leq |\lambda| < \frac{1}{\pi}\) the bound

\[
\left| \frac{\mathcal{H}_\lambda[e^{f(x)}]}{e^{f(a)}} - \frac{\mathcal{H}_\lambda[e^{g(x)}]}{e^{g(a)}} \right| \leq \delta \cdot \left( \zeta_{\lambda} + \frac{1}{|\lambda|} \frac{1 + a}{\pi} \frac{(1 + a)^{|\lambda| - 1} - |\lambda| \log(1 + a)}{|\lambda|(1 + a)^{|\lambda|}} \right),
\]

\[
\zeta_{\lambda} := \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{(k + |\lambda|)^2 + \frac{1}{(k - \frac{1}{2|\lambda|})^2}}.
\]

(31)
Proof We take the Hilbert transform of (32) and (33). The principal value limit can be weakened to improper Riemann integrals:

\[
\left| \mathcal{H}_a^{\text{pr}}[f(x)] - \mathcal{H}_a^{\text{pr}}[\varepsilon(x)] \right| \leq \int_0^a \frac{dx}{\pi} \frac{(1+k)}{1+a} \left| \frac{|\lambda|}{1+a} - \frac{1}{1+a} \right|
\]

\[
+ \int_a^\infty \frac{dx}{\pi} \frac{(1+k)}{1+a} \left| \frac{|\lambda|}{1+a} - \frac{1}{1+a} \right|
\]

\[
= \lim_{\varepsilon \to 0} \left( \int_0^{1-\varepsilon} \frac{dt}{\pi} \frac{1}{1-t} - \int_0^{1-\varepsilon} \frac{dt}{\pi} \frac{1}{1-t} \right)
\]

\[
+ \int_1^{\infty} \frac{dt}{\pi} \frac{1}{t-1} - \int_1^{\infty} \frac{dt}{\pi} \frac{1}{t-1} .
\]

(32)

The second integral \( I_2 \) known from [29, §3.231.6+§3.231.5]:

\[
I_2 = \cot \left( \frac{|\lambda|}{1-2|\lambda|} + \pi \right) - \cot \left( \frac{|\lambda|}{1-2|\lambda|} + \pi \right) - \frac{1}{\pi} \left( \psi \left( \frac{|\lambda|}{1-2|\lambda|} \right) - \psi \left( \frac{|\lambda|}{1-2|\lambda|} - \delta \right) \right)
\]

\[
= \frac{1}{\pi} \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \delta \right)
\]

\[
\leq \frac{\delta}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \delta \right) .
\]

(33a)

We have used the power series expansion [29, §8.363.3] for the difference of digamma functions. The result is uniformly bounded for \(|\lambda| < \frac{1}{2} |\lambda| \).

The first integral \( I_1 \) is evaluated with (51b) to

\[
I_1 = \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ \frac{(1+\varepsilon)|\lambda|}{|\lambda|} \right\} \left[ \psi \left( \frac{1}{1+|\lambda|} \right) - \psi \left( \frac{1+\varepsilon}{1+|\lambda|} \right) - \frac{(1+\varepsilon)|\lambda|+\delta}{|\lambda|+\delta} 2F_1 \left( \frac{1}{1+|\lambda|} \right) \frac{1}{1+|\lambda|+\delta} \right]
\]

\[
- \frac{1}{|\lambda|} \psi \left( \frac{1}{1+|\lambda|} \right) - \frac{1}{|\lambda|} \psi \left( \frac{1}{1+|\lambda|+\delta} \right) - \frac{1}{|\lambda|+\delta} 2F_1 \left( \frac{1}{1+|\lambda|} \right) \frac{1}{1+|\lambda|+\delta} \right]
\]

\[
= \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{ \frac{1}{1+|\lambda|+\delta} \right\} \left[ \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \frac{1}{k+1} - \delta \right]
\]

(33b)

Here we have expanded the hypergeometric functions into a power series and rearranged them to differences which admit the limit \( \varepsilon \to 0 \). The line (33b) is monotonic in \( a \) and thus can be estimated by its limit \( a \to \infty \). The same argument gives a possible uniform estimate of (33b),

\[
\frac{33a}{\pi} \leq \frac{\delta}{\pi} \sum_{k=1}^{\infty} \frac{1}{(k+|\lambda|)^2} , \quad \frac{33b}{\pi} \leq \frac{\delta}{\pi} \frac{1}{|\lambda|^2} , \quad \frac{33c}{\pi} + \frac{33b}{\pi} .
\]
The last estimate is enough for continuity, but not for contractivity. We write (33b) as a double integral:

\[
\frac{1 - (\frac{1}{\pi})^{\delta|\lambda|}}{|\lambda| \pi} - \frac{1 - (\frac{1}{\pi})^{\delta|\lambda|+\delta}}{(\delta|\lambda|+\delta) \pi} = \frac{1}{\pi} \int_0^1 dt \left( \frac{1}{|\lambda|-1 - t|\lambda|+\delta-1} \right) \int_0^1 dt \left( \frac{1}{|\lambda|-1 - t|\lambda|+\delta-1} \right)
\]

\[
= -\frac{1}{\pi} \int_0^1 dt \int_0^\delta d\xi \frac{d}{d\xi} |\lambda|+\xi-1 = \frac{1}{\pi} \int_0^1 dt \int_0^\delta d\xi (-\log t)|\lambda|+\xi-1
\]

\[
\leq \frac{\delta}{\pi} \int_0^1 dt (-\log t)|\lambda|+\xi-1 = \frac{\delta}{|\lambda|^2 \pi} \left( \frac{(1+a)|\lambda| - 1 - |\lambda| \log(1+a)}{(1+a)^2} \right). \tag{34}
\]

This gives together with (33a) and the estimate (33c) the claimed result. \(\square\)

Putting \(x = 0\) in (30) leads to

\[
\left| \frac{1}{e^{t(\alpha)}} - \frac{1}{e^{t(a)}} \right| \leq (1+a)^{1-|\lambda|} - (1+a)^{1-|\lambda|} |\lambda|^{1-\xi} = -\frac{\delta}{\pi} \frac{d}{d\xi} (1+t)^{1-|\lambda|-\xi}
\]

\[
= \int_0^\delta d\xi (1+t)^{1-|\lambda|-\xi} \log(1+t) \leq \delta (1+t)^{1-|\lambda|} \log(1+t). \tag{35}
\]

Together with Lemma 5 we have thus proved for the map \(R\) defined in (8):

**Proposition 3** Let \(0 \leq |\lambda| < \frac{1}{\pi}\). For any \(f, g \in \mathcal{K}_\lambda\) with \(\|f - g\|_{LB} = \delta\) one has the pointwise bound

\[
\left| (Rf)(t) - (Rg)(t) \right| \leq (\Delta R)^{(1)}(t) + (\Delta R)^{(2)}(t) + (\Delta R)^{(3)}(t),
\]

\[
(\Delta R)^{(1)}(t) := \delta \cdot (1+t)^{1-|\lambda|} \log(1+t), \tag{36a}
\]

\[
(\Delta R)^{(2)}(t) := \delta \cdot |\lambda| \pi \xi_\lambda, \tag{36b}
\]

\[
(\Delta R)^{(3)}(t) := \delta \cdot t \cdot \frac{(1+t)^{|\lambda|} - 1 - |\lambda| \log(1+t)}{|\lambda|(1+t)^{|\lambda|}}. \tag{36c}
\]

**Proposition 4** The map \(T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda\) is norm-continuous. More precisely, for \(-\frac{1}{\pi} \leq \lambda \leq 0\) one has

\[
\|Tf - Tg\|_{LB} \leq \|f - g\|_{LB} \cdot \sin^2 \left( \frac{|\lambda| \pi}{2|\lambda|} \right) \frac{(1 - \frac{1}{\pi})^{-1}}{|\lambda| \pi} \left( 1 + |\lambda| \pi + |\lambda|^2 \pi \xi_\lambda \right). \tag{37}
\]

The rhs ranges from 1.36788 \(\|f - g\|_{LB}\) for \(|\lambda| = 0\) to 4.09942 \(\|f - g\|_{LB}\) for \(|\lambda| = \frac{1}{\pi}\).

**Proof** The definition (24) gives for \(f, g \in \mathcal{K}_\lambda\)

\[
\|Tf - Tg\|_{LB} = \sup_{a \geq 0} \int_0^\infty dt \left( \frac{(1+a)Rg(t) - RF(t)}{((|\lambda| \pi t)^2 + (a + RF(t))^2)^{1/2}} \right) \left( \frac{(2a + RG(t) + RF(t))}{((|\lambda| \pi t)^2 + (a + RG(t))^2)^{1/2}} \right)
\]

\[
\leq \sum_{t=1}^\infty \sup_{a \geq 0} \int_0^\infty dt \left( \frac{(1+a)(\Delta R)^{(1)}(t)}{((|\lambda| \pi t)^2 + (a + 1 + |\lambda| \pi t \cos(\frac{|\lambda| \pi t}{1-|\lambda|})) (|\lambda| \pi t)^2 + (a + |\lambda| F(t))^2)^{1/2}} \right)^2. \tag{38}
\]
where we have inserted the lower bound $Rf(t) \geq 1 + |\lambda| \pi t \cot\left(\frac{\lambda|\pi|}{1-2|\lambda|}\right) + |\lambda| F(t)$ derived in Lemma 3. We write this as corresponding decomposition $\|Tf - Tg\|_{LB} \leq \sum_{i=1}^{n} \|Tf - Tg\|_{LB}^{(i)}$.

We start with the easiest contribution $\tau = 2$ where we substitute $u = |\lambda| \pi t$:

$$\|Tf - Tg\|_{UB}^{(2)} := \sup_{u \geq 0} \frac{2\delta}{\pi} \int_0^\infty du \left(\frac{(1+1)^2}{a^2 + (a+1 + |\lambda| F(u|\lambda|) + u \cot\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right))^2}\right).$$

There is no doubt that $F(t)$ is of positive mean also for this integral (the small-$u$-region is suppressed) so that it is safe to put $F(\cdot) \to 0$. We postpone this proof and temporarily work with the conservative estimate $1 + |\lambda| F(u|\lambda|) \geq h_k := 1 - |\lambda|$. This reduces the problem to a standard integral [3] §3.252.7:

$$\|Tf - Tg\|_{UB}^{(2)} = \sup_{a \geq 0} \int_0^\infty du \frac{2\delta \zeta_k (1+1) \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)}{(u+2a \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) + (\sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)(a+h_k))}| \delta \zeta_k \sin^2\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) h_k \pi 1 + \cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)$$

which becomes arbitrarily small for $\lambda \to 0$.

The contribution $\tau = 1$ is more difficult, but can be controlled. Again we expect $F(t)$ to be of positive mean. We postpone the proof and temporarily work with a conservative estimate $1 + |\lambda| F(u|\lambda|) \geq h_k := 1 - |\lambda|$. Then $(a+1) \leq \frac{a+b}{h_k}$ and consequently

$$\|Tf - Tg\|_{UB}^{(1)} \leq \sup_{a \geq 0} \frac{\delta}{h_k} \int_0^\infty dt \frac{2|\lambda| \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)(a+h_k) \cdot (1+t)^{1-|\lambda|} \log(1+t)}{(t+(a+h_k)) \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) \cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)}$$

$$= \frac{2\delta |\lambda|}{h_k \cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)} \sin^2\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) \sup_{A_k} \int_0^\infty dt \frac{A_k(a) \cdot (1+t)^{1-|\lambda|} \log(1+t)}{(t+A_k(a))^3},$$

where $A_k(a) := (a+h_k) \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) \cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)$. We use Young’s inequality

$$(A_k(a))^{\lambda} (1+t)^{1-\lambda} \leq \lambda A_k(a) + (1-\lambda) (1+t) = (1-\lambda) + (2\lambda - 1) A_k(a) + (1-\lambda) (t+A_k(a))$$

(41)

to write

$$\|Tf - Tg\|_{UB}^{(2)} \leq \frac{2\delta |\lambda|}{A_k h_k \cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)} \sin^2\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right) \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)(A_k(a))^{1-|\lambda|}$

$$\times \int_0^\infty dt \left( (1-\lambda) \log(1+t) + (1-\lambda) + (2\lambda - 1) A_k(a) + (1-\lambda) (t+A_k(a)) \right)$$

$$= \delta \cdot \frac{(1-\lambda)^{-1} \sin^2\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)}{\cos\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)} \sin\left(\frac{|\lambda| |\pi|}{1-2|\lambda|}\right)$$

(42)
where (after integration by parts)

\[ C_\lambda(x) := |\lambda| x^{-1-|\lambda|} \int_0^\infty dt \left( \frac{2(1-|\lambda|)}{(t+1)(t+x)} + \frac{(1-|\lambda|) + (2|\lambda| - 1)x}{(1+t)(t+x)} \right) \]

\[ = -|\lambda|^2 + |\lambda|(1 - 2|\lambda|) \frac{x^2}{x^4(1-|\lambda|)} + \frac{x^2 - (1-|\lambda|)x \log x}{x^4|\lambda|}. \]  \hspace{1cm} (43)

The maximum of \( C_\lambda \) is governed by the function \( \frac{\log x}{x^{1-|\lambda|}} \) which reaches \( \frac{1}{e} \) at \( x = e^{\pi/2} \). For the range of \( |\lambda| \) under consideration, this becomes huge so that all other terms except for \( x^{1-|\lambda|} \approx e \) become negligible. Therefore we expect

\[ \sup_x C_\lambda(x) \leq \frac{1 + |\lambda|}{e}. \]

A numerical investigation confirms this.

It remains the contribution from \( \tau = 3 \). There is a short cut resulting from the crude bound \( (\Delta R)^{(3)}(t) \leq \frac{\delta}{|\lambda|} = \frac{\Delta R^{(3)}(t)}{|\lambda|}. \) Inserting this relation into (39) gives

\[ ||T f - T g||^{(3)}_{LB} \leq \frac{\delta}{1 - \frac{\lambda}{\pi}} \frac{\sin^2 \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right)}{(|\lambda| \pi)^2} \frac{2}{1 + \cos \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right)}. \]  \hspace{1cm} (44)

We show that this naive bound is optimal. For that we start from Taylor’s formula

\[ (\Delta R)^{(3)}(t) = \delta |\lambda| \int_0^1 d\xi \frac{(1 - \xi)(\log(1+t))^2}{(1+t)^{1-2|\lambda|}}. \]

Up to an order \( |\lambda| \) error we may replace \( F(t) \to 0 \). Then

\[ ||T f - T g||^{(3)}_{LB} \leq \frac{\delta}{\cos \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right)} \frac{\sin \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right)}{|\lambda| \pi} \sup_{\lambda > 0} \hat{C}_\lambda(A_\lambda), \]

\[ \hat{C}_\lambda(A_\lambda) := 2|\lambda|^2 \int_0^1 d\xi \int_0^\infty dt \frac{(1 - \xi)A_\lambda(\log(1+t))^2(1+t)^{1-2|\lambda|}}{(1 + A_\lambda)^3}, \]

where \( A_\lambda := \frac{(1+\alpha)}{|\lambda| \pi} \sin \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right) \cos \left( \frac{|\lambda| \pi}{1 - 2|\lambda|} \right). \) Inserting Young’s inequality (41) we get:

\[ \hat{C}_\lambda(\alpha) = 2|\lambda|^2 \int_0^1 d\xi \int_0^\infty dt \left\{ (1 - \xi) (1 - 2|\lambda|)(1 - |\lambda|) \frac{(\log(1+t))^2}{(t+\alpha)^3} + \frac{\log(1+t)^2}{(t+\alpha)^3} + (1 - \xi) (2|\lambda| - 1) \alpha^{-2(1-\xi)|\lambda|} \frac{(\log(1+t))^2}{(t+\alpha)^3} \right\} \]
We need the following integrals

\[
2 \int_0^\infty dt \left\{ \alpha^{1-|\lambda|} \left( \frac{2\alpha^{|\lambda|} - 2}{(\log \alpha)^2} + \frac{\alpha^{|\lambda|} + 2|\lambda| - 1}{(\log \alpha)^2} - \frac{|\lambda|(1-|\lambda|)}{\log \alpha} \right) \right.
\]
\[
\times \left( \frac{(\log(1+t))^2}{(t+\alpha)^2} + \frac{(\log(1+t))}{(t+\alpha)^3} \right)
\]
\[
+ \alpha^2 - |\lambda| \left( \frac{4\alpha^{|\lambda|}}{(\log \alpha)^2} - \frac{\alpha^{|\lambda|} + 4|\lambda| - 1}{(\log \alpha)^2} + \frac{|\lambda|(1-2|\lambda|)}{\log \alpha} \right) \frac{(\log(1+t))}{(t+\alpha)^3} \right\},
\]

We need the following integrals

\[
\int_0^\infty dt \frac{(\log(1+t))^2}{(t+\alpha)^2} = \begin{cases} 
  \frac{2\zeta(1-\alpha)}{(1-\alpha)} & \text{for } 0 < \alpha < 1 \\
  \frac{2\zeta(1-\alpha)}{(1-\alpha)} + \frac{2\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} & \text{for } \alpha = 1 \\
  \frac{2\zeta(1-\alpha)}{(1-\alpha)} + \frac{2\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} & \text{for } \alpha > 1
\end{cases}
\]

\[
\int_0^\infty dt \frac{(\log(1+t))^2}{(t+\alpha)^3} = \begin{cases} 
  \frac{(1-\alpha)^2}{\log(\alpha)} & \text{for } 0 < \alpha < 1 \\
  \frac{(1-\alpha)^2}{\log(\alpha)} + \frac{2\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} & \text{for } \alpha = 1 \\
  \frac{(1-\alpha)^2}{\log(\alpha)} + \frac{2\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} & \text{for } \alpha > 1
\end{cases}
\]

We specify to \( \alpha > 1 \) (the other cases are analytic continuations):

\[
\tilde{C}_\lambda(\alpha) = \frac{\alpha^2}{(\alpha - 1)^2} \left\{ \left( 1 - \frac{1}{\alpha^{|\lambda|}} \right) \left( 1 + 2|\lambda|^2 \frac{\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} \right) \right.
\]
\[
\times \left( -\frac{8|\lambda|^2(1-\frac{1}{\alpha})}{(\log \alpha)^2} + \frac{2|\lambda|(1-\frac{1-2|\lambda|}{\alpha^{|\lambda|}})}{(\log \alpha)^2} - \frac{2|\lambda|(1-2|\lambda|)}{\alpha^{|\lambda|}} \right) \}
\]
\[
+ \frac{\alpha}{(\alpha - 1)^2} \left\{ \left( \frac{4|\lambda|^2(1-\frac{1}{\alpha})}{(\log \alpha)^2} - \frac{2|\lambda|(1-\frac{1}{\alpha})}{(\log \alpha)^2} + \frac{2|\lambda|(1-|\lambda|)}{\alpha^{|\lambda|}} \right) \}
\]
\[
+ \frac{2|\lambda|(1-\frac{1}{\alpha})}{(\log \alpha)^2} \right\} \times \left( 1 + 2|\lambda|^2 \frac{\zeta(1-\frac{1}{\alpha})}{(\log \alpha)^2} \right) \right\},
\]

This shows \( \lim_{\alpha \to \infty} \tilde{C}_\lambda(\alpha) = 1 \). The next-to-leading terms turn out to be \( 1 - \frac{\log x}{x} + \frac{2|\lambda|}{\log x} \), where \( x := \alpha^{|\lambda|} \). This function gets bigger 1 with a local maximum \( 1 + \frac{|\lambda|}{2} \) for \( |\lambda| \leq \frac{1}{6} \). A closer numerical simulation confirms this bound \( \sup_{x} \tilde{C}_\lambda(\alpha) \leq 1 + \frac{|\lambda|}{2} \) for all \( 0 \leq |\lambda| \leq \frac{1}{6} \). Inserted into (45) gives no improvement compared with the crude bound (44).

\[\square\]

5 Equicontinuity and Arzelà-Ascoli theorem

The remaining task is to prove a variant of the Arzelà-Ascoli theorem which establishes that if a subset \( \mathcal{F} \subseteq LB \) is equicontinuous and pointwise bounded, then \( \mathcal{F} \) is compact. We start with the equicontinuity:
Lemma 6 The subset $T \mathcal{X}_2 \subseteq LB$ is equicontinuous in the norm topology of LB. More precisely, given $\varepsilon > 0$ one has $|(1 + a)(T f)'(a) - (1 + b)(T f)'(b)| < \varepsilon$ for all $f \in \mathcal{X}_2$, and all $a, b \in \mathbb{R}^+$ with $|a - b| < \varepsilon$.

Proof We estimate via (24)

$$[(1 + a)(T f)'(a) - (1 + b)(T f)'(b)] = \int_b^a dx \frac{d}{dx}((1 + x)(T f)'(x)) \left| \int_b^a dx \int_0^\infty dt \frac{|\lambda|(1 + x)}{(|\lambda|\pi)^2 + (x + R f(t))^2} \right| \leq \int_b^a dx \int_0^\infty dt \frac{2|\lambda|(1 + x)\cot \frac{\pi}{2(|\lambda|\pi)}\pi}{(x + 1 - \frac{|\lambda|\pi}{\pi})^2(1 + \cos \frac{|\lambda|\pi}{1 - 2|\lambda|})}.$$ 

We have the following upper bound:

$$\int_0^\infty dt \frac{2|\lambda|(1 + x)(x + R f(t))}{((|\lambda|\pi)^2 + (x + R f(t))^2)^2} \leq \int_0^\infty dt \frac{2|\lambda|(1 + x)}{(x + 1 - \frac{|\lambda|\pi}{\pi})^2(1 + \cos \frac{|\lambda|\pi}{1 - 2|\lambda|})^2}$$

We ignore possible cancellations and add the upper bound $\int_0^\infty dt \frac{|\lambda|}{(|\lambda|\pi)^2 + (x + R f(t))^2}$ to establish the proof of $T \mathcal{X}_2 \subseteq \mathcal{X}_2$. Taking also the supremum in $x$ we conclude

$$\left|(1 + a)(T f)'(a) - (1 + b)(T f)'(b)\right| \leq |a - b| \frac{|\lambda|}{1 - 2|\lambda|} \left(1 + \frac{\sin \frac{|\lambda|\pi}{1 - 2|\lambda|}}{\frac{|\lambda|\pi}{1 - 2|\lambda|} - 2(1 - \frac{|\lambda|\pi}{\pi})^{-2}}\right).$$

The rhs is $\leq |a - b|$ for any $0 \leq |\lambda| \leq \frac{1}{5}$.

The standard Arzelá-Ascoli theorem concerns continuous functions on compact spaces. This can largely be generalised to $\mathcal{F}(X, Y)$ equipped with the compact-open topology relative to general Hausdorff spaces $X, Y$, see [31]. The idea is to prove that for an equicontinuous family $\mathcal{F}$, the compact-open topology and the pointwise topology coincide. Pointwise compactness of $\mathcal{F}(x)$ for every $x \in X$ implies compactness of $\prod_{x \in X} \mathcal{F}(x)$ by Tychonoff’s theorem, thus compactness of the equicontinuous family $\mathcal{F}$ in the compact-open topology.

We cannot make use of this setting because to prove continuity of $T$ we had to control the Hilbert transform via the global behaviour of functions in $\mathcal{X}_2$. It seems unlikely that this can be replaced by a local control in the compact-open topology.

Being forced to work in norm topology, the only chance to rescue Arzelá-Ascoli for equicontinuous families in $LB$ is to restrict to compact subsets of $\mathbb{R}^+$. This is not unreasonable because we worked originally over the cut-off space $[0, \Lambda^2]$. We find it necessary to reprove the Arzelá-Ascoli theorem for equicontinuous subsets of $LB$.

Lemma 7 The subset $T \mathcal{X}_2 \subseteq LB$ is relatively compact in the $\| \cdot \|_{LB}$ topology if restricted to any compact interval $[0, \Lambda^2]$. 

Proof Choose any $\Lambda^2 > 0$. The family $T\mathcal{X}_{\lambda} \subseteq LB$ is bounded and equicontinuous on $[0, \Lambda^2]$ with respect to $f \mapsto (1 + x)f'(x)$. On metric spaces such as $LB$, compactness is equivalent to sequentially compactness. We thus have to prove that any sequence $(f_k) \in T\mathcal{X}_{\lambda}$ has a $|| \cdot ||_{LB}$-convergent subsequence when restricted to $[0, \Lambda^2]$.

Given $\varepsilon > 0$, there is for every $0 < x < \Lambda^2$ an open $\varepsilon$-neighbourhood $U_\varepsilon(x) := \{ y \in \mathbb{R}_+ : |y - x| < \frac{\varepsilon}{3} \}$ which by the equicontinuity of $T\mathcal{X}_{\lambda}$ has the property that

$$|(1 + x)f'(s) - (1 + x)f'(x)| < \frac{\varepsilon}{3} \quad \text{for all } s \in U_\varepsilon(x) \text{ and all } f \in T\mathcal{X}_{\lambda}.$$ 

These $\{ U_\varepsilon(x) \}_{0 < x < \Lambda^2}$ form an open cover of $[0, \Lambda^2]$ which by the compactness of $[0, \Lambda^2]$ can be reduced to a finite subcover $\{ U_\varepsilon(x_i) \}_{i = 1, \ldots, N}$ (it is this step which does not work for $\mathbb{R}_+$).

Start at $x_1$ and note that $((1 + x_1)f''(x_1))_{k \in \mathbb{N}}$ is bounded for every member of the sequence $(f_k)$. By the Bolzano-Weierstraß theorem there is a subsequence $(f_{k_1}(x_1))_{k_1 \in \mathbb{N}}$ such that $((1 + x_1)^{f''}(x_1))_{k_1 \in \mathbb{N}}$ converges at $x_1$. Repeat this to construct a subsequence $(f_{k_2})_{k_2 \in \mathbb{N}}$ of $(f_{k_1})_{k_1 \in \mathbb{N}}$ such that both $((1 + x_1)^{f''}(x_1))_{k_2 \in \mathbb{N}}$ and $((1 + x_2)^{f''}(x_2))_{k_2 \in \mathbb{N}}$ converge. And so on. This eventually produces a subsequence $(f_{k_3})_{k_3 \in \mathbb{N}}$ of $(f_k)$ which has the property that $((1 + x)^{f''}(x_1))_{k_3 \in \mathbb{N}}$ converges for every $i = 1, \ldots, N$. We rename $(f_{k_3})_{k_3 \in \mathbb{N}} = (f_i)_{i \in \mathbb{N}}$ for simplicity.

Convergence implies that for every $i = 1, \ldots, N$ there is a $K_i(\varepsilon) \in \mathbb{N}$ such that

$$|(1 + x_1)^{f_i}(x) - (1 + x_1)^{f_i}(x_i)| < \frac{\varepsilon}{3} \quad \text{for all } \ell, m \geq K_i(\varepsilon).$$

Given any $x \in [0, \Lambda]$, choose one index $j \in \{1, \ldots, N\}$ such that $x \in U_\varepsilon(x_j)$. Then for any $\ell, m \geq K(\varepsilon) := \max_{i = 1, \ldots, N} K_i(\varepsilon)$ one has

$$|(1 + x)^{f_i}(x) - (1 + x)^{f_i}(x_i)| < \frac{\varepsilon}{3} \quad \text{for all } i = 1, \ldots, N.$$ 

In other words, any sequence $(f_k)_{k \in \mathbb{N}}$ in $T\mathcal{X}_{\lambda}$ has a subsequence $f_i \in \mathcal{X}$ such that $((1 + x)^{f''}(x))_{i \in \mathbb{N}}$ converges uniformly on any compact interval $[0, \Lambda^2]$ to a differentiable limit function which belongs to the closure $\overline{T\mathcal{X}_{\lambda}} \subseteq \mathcal{X}$.

6 Conclusions

In proving existence of a solution of (3) we closed a major gap in our programme to construct a solvable quantum field theory model in four dimensions. In (17) we have studied the numerical iteration of (4) in the spirit of the Banach fixed point theorem and convinced ourselves that the iteration converges numerically. As shown in Figure 2 there is perfect agreement between the numerical solution (at $\lambda = -\frac{1}{4\pi}$) and the analytically established fixed point domain $\exp(\mathcal{X}_{\lambda})$.

The numerical treatment (17) leaves no doubt that the solution $G_{\text{num}}$ inside $\exp(\mathcal{X}_{\lambda})$ is unique. It would be very desirable to prove this also analytically. As shown in the appendix where we prove that also $G_{\text{num}} = 1$ solves (4) for $\lambda < 0$, the restriction to $\exp(\mathcal{X}_{\lambda})$ is essential.
We slightly missed in Prop. 4 the contractivity criterion of the Banach fixed point theorem. If we knew the asymptotic exponent \( \lim_{b \to \infty} \frac{-\log G_0(b)}{\log(1 + b)} \) then we could considerably improve the bound (33b) by an integration from the other end. Another strategy would be to prove that, starting with the very good estimate

\[
T f^{(n)}(b) = f^{(n+1)}(b) \geq f^{(n)}.
\]

Together with the boundedness proved here, such a monotonicity would also imply uniqueness.

As discussed in [20] and [17] it is very important to know that \( G_0(b) \) is a Stieltjes function (see e.g. [32]). We have no doubt that this is true, but the proof is missing. The boundaries of \( \exp(K_{\lambda}) \) are Stieltjes and the numerical solution is parallel to these boundaries (Figure 2). We made recently some progress in this direction using results of this paper in an essential way: We can prove that any fixed point solution \( G_0(b) \) of (4) inside \( \exp(K_{\lambda}) \) has a holomorphic continuation \( z \mapsto G_0(z) \) to complex \( z \) with \( \Re(z) > -1 + |\lambda|/5 \) (in fact a bit more) and satisfies the anti-Herglotz property \( \Im(G_0(z)) \leq 0 \) for \( \Im(z) > 0 \) in that half space. To prove the Stieltjes property we have to extend these results to the cut plane \( \mathbb{C} \setminus [-\infty, 0] \), see [32]. The estimates proved in this paper will definitely be relevant for this step.

A The fixed point operator applied to the constant function

We have proved in sec. 3 that the operator \( T \) defined in (8b) maps \( \mathcal{K}_{\lambda} \) defined in (7) into itself. We add a small note showing the existence of fixed points outside \( \mathcal{K}_{\lambda} \). Concretely we show that \( T \) converges pointwise to 0 for \( \Lambda^2 \to \infty \). We have to reintroduce a finite cut-off \( \Lambda^2 \) to make sense of the Hilbert transform of \( \exp(0) = 1 \), namely \( \mathcal{H}_p^\Lambda(1) = \int \frac{1}{1 + b + |\lambda| \log(\Lambda^2/p)} \). We then have for (24)

\[
(T \mathcal{H}_p^\Lambda(1))'(b) := -\frac{1}{1 + b + |\lambda| \log(\Lambda^2/p)} \int_0^\Lambda \left( \frac{dp}{(\lambda |p|^2 + (b + 1 - |\lambda| \log(\Lambda^2/p))^2} \right)
\]
where we have substituted $\frac{\Lambda^2 - p^2}{p^2} = q$. We prove:

**Lemma 8** For $u > 0$ one has
\[
\int_0^\infty \frac{dq}{\pi^2 + (u(1 + q) - \log q)^2} = \frac{1}{u(u + 1)}.
\]

**Proof** We have
\[
\int_0^\infty \frac{dq}{\pi^2 + (u(1 + q) - \log q)^2} = \int_0^\infty \frac{dq}{2\pi i} \left( \frac{1}{-u(1 + q) + \log q + i\pi} + \frac{1}{u(1 + q)} \right)
\]
\[
- \left\{ \frac{1}{-u(1 + q) + \log q + i\pi} + \frac{1}{u(1 + q)} \right\}.
\]

The terms $\frac{1}{\pi i}$ are added to improve the decay at infinity. We put $z = qe^{i\phi}$ in the first $\{\ldots\}$ and $z = qe^{(2\pi - \epsilon)i}$ in the second $\{\ldots\}$. Then for $\epsilon \to 0$ we have
\[
\int_0^\infty \frac{dq}{\pi^2 + (u(1 + q) - \log q)^2} = \sum_{\epsilon < a < \pi} \left( \frac{1}{-u(1 + z) + \log (ize^{-i\pi})} + \frac{1}{u(1 + z)} \right).
\]

with $\Re z$ chosen as the cut of $\log(ze^{-i\pi})$. The decay at $\infty$ guarantees that the integral over the arc $c_\infty$ does not contribute. Therefore the residue theorem gives
\[
\int_0^\infty \frac{dq}{\pi^2 + (u(1 + q) - \log q)^2} = \sum_{\epsilon < a < \pi} \text{Res} \left( \frac{1}{-u(1 + z) + \log (ize^{-i\pi})} + \frac{1}{u(1 + z)} \right).
\]

For $z = |z|e^{i\phi}$ with $0 < \phi < \pi$ one has $\text{Im}(-u(1 + z) + \log(ze^{-i\pi})) = -u|z| \sin \phi - (\pi - \phi) < 0$. Therefore, the residue equation $0 = u(1 + z) + \log(ze^{-i\pi})$ has solutions only on the negative real axis: $z = -x$ and $u(1 + x) = \log x$ with unique solution $x = 1$. This gives
\[
\int_0^\infty \frac{dq}{\pi^2 + (u(1 + q) - \log q)^2} = \left( \frac{1}{-u + 1} \right) \left( \log \left( \frac{1}{u + 1} \right) \right) = \frac{1}{u(u + 1)}.
\]

Insertion into (A.1) gives
\[
(T0)(b) = -\frac{1}{|\Lambda|^2 + 1 + b} \quad \Rightarrow \quad (T0)(b) = \log \left( \frac{1}{1 + \frac{b}{1 + |\Lambda|^2}} \right),
\]

which is pointwise convergent to 0 for $\Lambda^2 \to \infty$. This means that $G_{0b} = \exp(0) = 1$ for all $b$ is a solution of $\{\text{solution}\}$ for $\lambda < 0$.

This solution is interesting in so far as the numerical investigation in [7] shows a phase transition at critical coupling constant $\lambda_c \approx -0.39$. For $\lambda < \lambda_c \leq 0$ we find qualitative agreement with $\exp(\mathcal{A}_\lambda^2)$, see Figure 3 whereas for $\lambda_c < \lambda$ we have $G_{0b} = 1$ in a whole neighbourhood of $b = 0$. This suggests that $\lambda_c$ locates the transition between solutions $G_{0b} \in \exp(\mathcal{A}_\lambda^2)$ and $G_{0b} = \exp(0) = 1$.

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