RESTRICTIONS ON FORCINGS THAT CHANGE COFINALITIES

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Abstract. In this paper we investigate some properties of forcing which can be considered “nice” in the context of singularizing regular cardinals to have an uncountable cofinality. We show that such forcing which changes cofinality of a regular cardinal, cannot be too nice and must cause some “damage” to the structure of cardinals and stationary sets.

1. Introduction

In this paper we examine a few properties of forcing notions which limit their ability to change cofinalities of regular cardinals without collapsing them. This work is the result of trying to understand whether □(κ) can fail at the first inaccessible cardinal. A natural approach would be to start with a model where □(κ) holds for all non-weakly compact cardinals, and we have a partial □ sequence on the singular cardinals, and then to change the cofinalities of many cardinals to an uncountable cofinality without extending the □ sequence at those cardinals.

The standard approach would be to find a forcing that changes cofinalities, while being decomposable to ensure that any □(κ) added for μ < κ cannot be threaded in the second part of the forcing, then to iterate it up to some cardinal with sufficient reflection properties and use those in order to reflect any potential square sequence to a smaller cardinal and thread. The simplest way to guarantee that we do not add threads to a □ sequence would be to take a σ-closed forcing. This led to the question what sort of properties can forcing notions have while changing the cofinality of a cardinal to an uncountable cofinality.

In section 2 we prove that proper forcing cannot change cofinalities without collapsing, in particular σ-closed forcings cannot solve the motivating problem, and in section 3 we make some observations related to PCF theory that suggest that changing many cofinalities at once must come at the price of adding ω-sequences. The fourth section is dedicated to investigating properties related to homogeneity, when restricting our attention to forcings that change cofinalities to an uncountable one. We show that under some relatively weak additional hypotheses, the existence of such forcing affects the cardinal structure of V and implies that the cardinal that changes its cofinality to uncountable cofinality must be preceded by stationary many cardinals that change their cofinalities as well. We also show that under some further natural assumptions this cardinal must be at least Mahlo of high order, thus revealing some of its large cardinals properties already in V.

The conclusion we can make from the work presented here, is that a forcing that changes the cofinality of κ to be uncountable cannot be too nice in some aspect.

Throughout this paper we work in ZFC and any large cardinals assumption will be stated explicitly. Our notation is standard: for a forcing notion P and two conditions p, q ∈ P, p ≤ q means that p is stronger than q. All the definitions and properties of stationary sets, proper forcing and PCF, that are needed in this paper, are taken from Jech ([Jec03]) and the Handbook of Set Theory ([FK10]).

2. Proper Forcing

Theorem 1. Let κ is regular and P is a forcing preserving Sκκ, such that || cf(κ) = ˇµ > ω. Then || |κ| = |µ|.

Proof. Fix in V a partition of Sκκ into κ stationary sets, Sγ for γ < κ. Let G ⊆ P be a V-generic filter, and s ∈ V[G] a continuous and cofinal function from µ to κ. We define f: µ → κ by f(β) = α if and only if β ∈ Sα, and f(β) = 0 if there is no such α. By the assumption that stationary subsets of Sκκ are preserved, V[G] ⊨ Sα is stationary for every α. Since rng s is a club in κ, rng s ∩ Sα ̸= ∅ for all α < κ, and therefore f is onto κ.

Corollary 2. If P is a proper forcing which changes the cofinality of a regular cardinal κ, then P collapses κ.

Proof. If P changes the cofinality of κ to ω then by picking any cofinal s of order type ω we define in V a club in [κ]ω. {A ∈ [κ]ω | s ⊆ A}. But this club is disjoint of every subset of [κ]ω ∩ V. Therefore P is not proper.

If P is proper then it preserves the stationarity of subsets of Sκκ. If P changes the cofinality of κ to be uncountable, then κ is collapsed, as shown in the theorem above.
Corollary 3. If $\mathbb{P}$ is $\sigma$-closed and $\mathbb{P}$ changes the cofinality of a regular cardinal to $\omega_1$, then it collapses it.

One might look at the definition of the Prikry forcing, and attempt to generalize it to uncountable cofinalities by using countable (or longer) approximations of a generic club. The above shows that this cannot work, and explains the need in Magidor and Radin forcings to resort to finite conditions. We finish this section with a corollary showing the usefulness of this theorem in a context where cardinals are not necessarily preserved.

Corollary 4. If $\mathbb{P}$ is a $\sigma$-closed forcing, $\lambda$ singular and $\mathbb{P}$ collapses $\lambda^+$, then $\mathbb{P}$ collapses $\lambda$. In particular it is impossible to change the successor of a singular cardinal with a $\sigma$-closed forcing.

Proof. Since $\lambda^+$ is not a cardinal, its cofinality is some $\mu < \lambda$. By $\sigma$-closure it follows that $\mu > \omega$, and so $\lambda^+$ is collapsed to $\mu$, and so is $\lambda$.

3. PCF related restrictions

Gitik showed in his paper [Git86] that it is possible (using preparatory forcing) to have a model in which a regular cardinal $\kappa$ changes its cofinality to an uncountable cofinality without adding any bounded sets to $\kappa$. Since this forcing is $\kappa^+$-c.c. and changes the cofinality of $\kappa$ to be uncountable, it is also $\sigma$-distributive. By iterating Gitik’s forcing finitely many times, it is possible to get a model in which there is a forcing that changes the cofinality of finitely many regular cardinals to some unbounded cofinality without adding any $\omega$-sequences of ordinals: if $\kappa_1, \ldots, \kappa_n$ are suitable large cardinals (e.g. measurable cardinals of Mitchell order $\omega_1$), we iterate the preparatory forcing for each one, and the resulting model has a forcing changing the cofinalities of each $\kappa_n$ to be uncountable, without adding unnecessary bounded sets, and without adding any new $\omega$-sequences of ordinals.

The next theorem shows that we cannot immediately generalize this to infinitely many cardinals at the same time, with similar large cardinals consistency strength.

Theorem 5. Let $\kappa_n$ be an increasing sequence of regular cardinals, and let $\mathbb{P}$ be a forcing which changes the cofinality of every $\kappa_n$ to $\mu$. Then either $\mathbb{P}$ adds an $\omega$-sequence of ordinals, or for every $\lambda \in \text{pcf}(\{\kappa_n \mid n < \omega\})$, $\Vdash_\mathbb{P} \text{cf}(\lambda) = \mu$.

Remark. The assumption that $\mathbb{P}$ does not add an $\omega$-sequence can be replaced by the assumption that $\mu$ is uncountable and $(\prod_{n<\omega} \kappa_n)^\mathbb{P}$ is bounding in $(\prod_{n<\omega} \kappa_n)^{\mathbb{P}[\mathbb{G}]}$.

Proof. Let $G \subseteq \mathbb{P}$ be a $\mathbb{V}$-generic filter. Suppose that $\mathbb{P}$ does not add any $\omega$-sequences, note that in particular this implies that $\mu > \omega$. We will show that in $\mathbb{V}[\mathbb{G}]$ there is a scale of length $\mu$ in $\prod_{n<\omega} \kappa_n$, from this will follow that if $U$ is an ultrafilter on $\omega$, then the cofinality of $(\prod_{n<\omega} \kappa_n)/U$ is $\mu$ and therefore if $\lambda$ was the cofinality of the product in $\mathbb{V}$, it must be that $\mathbb{V}[\mathbb{G}] \models \text{cf}(\lambda) = \mu$.

First we observe that we did not add any $\omega$-sequences, and therefore $\prod_{n<\omega} \kappa_n$ is the same object in $\mathbb{V}$ and $\mathbb{V}[\mathbb{G}]$, as well if $U$ is an ultrafilter on $\omega$ in $\mathbb{V}$, then $U$ is still an ultrafilter in $\mathbb{V}[\mathbb{G}]$.

Let $\gamma_n: \mu \rightarrow \kappa_n$, be a continuous and cofinal function, and denote by $G_{\alpha}(n) = \gamma_n(\alpha)$. We have that for each $\alpha < \mu$, $G_{\alpha} \in \mathbb{V}$ as it is an ultrafilter on $\omega$, and for each $\alpha < \mu$ such that $f(n) < G_{\alpha}(n)$. Therefore there is some $\alpha$ such that for all $n$, $f(n) < G_{\alpha}(n)$, as wanted.

It seems evident from Shelah’s covering theorem [She94, Chapter 7, Lemma 4.9] that changing all the cofinalities in $\text{pcf}(\{\kappa_n \mid n \in \omega\})$ to be $\mu$, without collapsing cardinals and without adding $\omega$-sequences of ordinals will require very large cardinals, much larger than $\omega_1$.

Corollary 6. Let $\kappa_n$ be a sequence of regular cardinals, $\omega_\alpha = \sup\{\kappa_n \mid n < \omega\}$, and $\mathbb{P}$ a forcing changing the cofinality of each $\kappa_n$ to $\omega_1$ and preserves cofinalities of regular cardinals above $\kappa$. Then $\mathbb{P} \times \mathbb{P}$ collapses $\omega_1$.

Proof. Using the notation of the previous proof, we know that the $G_{\alpha}$’s added by $\mathbb{P}$ have the property that for some large enough $\alpha$, $G_{\alpha}$ is not bounded by any function in $(\prod_{n<\omega} \kappa_n/U)^\mathbb{P}$, where $U \in \mathbb{V}$ is some ultrafilter such that $\text{cf}(\prod_{n<\omega} \kappa_n/U) = \lambda > \kappa$ (note that $U$ might not be an ultrafilter in $\mathbb{V}[\mathbb{G}]$ and the reduced product, as calculated in $\mathbb{V}[\mathbb{G}]$ might not be a linear order anymore, which is exactly why we cannot say that $G_{\alpha}$ is bounding). If that did not happen, then $\lambda$ had changed its cofinality, contrary to the assumption.

Let $G_{\alpha}$ be a $\mathbb{P}$-name for $G_{\alpha}$. Let $H_0 \times H_1$ be a $\mathbb{V}$-generic filter for $\mathbb{P} \times \mathbb{P}$, if $\omega_1$ is not collapsed in $\mathbb{V}[H_0 \times H_1]$, then on a club $C \subseteq \omega_1$ we have that $G_{\alpha}^{H_0}$ and $G_{\alpha}^{H_1}$ are identical. Pick $\alpha$ large enough such that $G_{\alpha}$ is not bounded by the product as above, and $\alpha \in C$. But this is a contradiction since $G_{\alpha} \notin \mathbb{V}$, but $H_0$ and $H_1$ are mutually generic over $\mathbb{V}$.

Note that virtually the same proof shows that this result holds also for different forcing notions $\mathbb{P}$ and $\mathbb{Q}$. If both $\mathbb{P}$ and $\mathbb{Q}$ change the cofinality of each $\kappa_n$ to $\omega_1$ while preserving cofinalities above $\kappa_n$, then $\mathbb{P} \times \mathbb{Q}$ collapses $\omega_1$. 
Corollary 7. Let $\mathbb{P}$ be Magidor forcing for adding a cofinal sequence of order type $\omega_1 \cdot \omega$ to a measurable cardinal of Mitchell order $\omega_1 + 1$. Then $\mathbb{P} \times \mathbb{P}$ collapses $\omega_1$.

Proof. Let $\kappa$ be the measurable cardinal with $o(\kappa) = \omega_1 + 1$ for which we are going to add a cofinal sequence of order type $\omega_1 \cdot \omega$. Note that although $\mathbb{P}$ changes the cofinality of $\omega$ many cardinals to $\omega_1$ (the cardinals in the places $\omega_1, \omega_1 \cdot 2, \ldots$ in the Magidor sequence), since this sequence is not in the ground model we cannot just apply the previous corollary.

We overcome this difficulty by observing that $\mathbb{P}$ can be decomposed into an iteration of a Prikry forcing and a forcing that changes the cofinality of the cardinals in the Prikry sequence to $\omega_1$. The first part adds a Prikry sequence relative to the measure $U \subseteq \mathcal{P}(\kappa)$ such that $o(U) = \omega_1$, without adding any bounded sets. Let $\mathbb{P} = \mathbb{Q} \ast \mathbb{R}$ where $\mathbb{Q}$ is the Prikry forcing. First we observe that $\mathbb{P}$ is $\kappa$-centered, therefore $\mathbb{P} \times \mathbb{P}$ is $\kappa$-centered as well, and so $\mathbb{P} \times \mathbb{P}$ does not change any cofinalities above $\kappa$, and consequently neither does any of these forcings.

Let $H_1 \times H_2$ be a generic filter for $\mathbb{Q} \times \mathbb{Q}$. Then in $V[H_1 \times H_2]$ there are forcing notions $\mathbb{R}^{H_1}$, $\mathbb{R}^{H_2}$ and each of then changes the cofinality of a cofinal sequence of cardinals to $\omega_1$. By density arguments, there are unbounded many cardinals that appear in both the Prikry sequence of $H_1$ and $H_2$ so we can apply for them the corollary as above.

Remark. For every $\alpha \leq \omega_1$, if $\mathbb{P}$ is the Magidor forcing for a measurable of Mitchell order $\alpha$, then $\mathbb{P} \times \mathbb{P}$ preserves all cardinals.

4. Amalgamable and Reflective Forcing

Some of the nicest properties that a forcing that adds subsets to some cardinal $\kappa$ could have is being weakly homogeneous, and not adding bounded subsets to $\kappa$. The Prikry forcing is indeed very nice in this aspect, and at the same time it changes the cofinality of $\kappa$ to $\omega$. We want to know about the interaction between homogeneity, not adding bounded sets and changing cofinalities to an uncountable one. We do this in this section by defining some weaker properties and investigating them.

Recall that a forcing $\mathbb{B}$ is called weakly homogeneous if for every $p, q \in \mathbb{B}$ there is an automorphism $\pi$ of $\mathbb{B}$ such that $\pi q$ is compatible with $p$. We are only interested in the case where $\mathbb{B}$ is a complete Boolean algebra, since there are $\mathbb{P}$ and $\mathbb{P}'$ such that $\mathbb{P}$ is weakly homogeneous and $\mathbb{P}'$ is rigid (having no non-trivial automorphisms) and both have the same Boolean completion (which is weakly homogeneous).

Theorem 8. The following are equivalent for a complete Boolean algebra $\mathbb{B}$:

1. $\mathbb{B}$ is weakly homogeneous.
2. If $G$ is $V$-generic for $\mathbb{B}$ then for every $p \in \mathbb{B}$ there $V$-generic $H$ such that $p \in H$ and $V[G] = V[H]$.
3. For every $\varphi(x_1, \ldots, x_n)$ in the language of forcing, and $u_1, \ldots, u_n \in V$, $1_{\mathbb{B}}$ decides the truth value of $\varphi(u_1, \ldots, u_n)$.

Proof. The following is an outline of the proof. The implication from (1) to (3) is a well-known theorem. Assuming (3) holds, then taking $\varphi(\bar{p})$ to be “There is a generic filter $H$ such that $p \in H$, and $\forall x, x \in V[H]"$, is decided by $1_{\mathbb{B}}$ and of course it must be decided positively (where genericity can be easily formulated using the set of dense subsets of $\mathbb{B}$ in $V$ as a parameter of $\varphi$).

Finally, (2) to (1) follows from the theorem of Vopěnka and Hájek, that if $V[H] = V[G]$ where $G, H$ are $V$-generic filters for a complete Boolean algebra $\mathbb{B}$, then there is an automorphism $\pi$ of $\mathbb{B}$ such that $\pi^m G = H$. In particular if $q \in G$ and $p \in H$ then $\pi q$ is compatible with $p$ (see [Gri75, Section 3.5, Theorem 1] for a full discussion about the theorem).

Of course, the second and third conditions may apply to any notion of forcing, and they are preserved by passing on to the Boolean completion. So checking them would suffice in order to show that the Boolean completion of $\mathbb{P}$ is weakly homogeneous.

A natural weakening of the second condition is to say that for every $p \in \mathbb{P}$ there is some $H$ which is $V$-generic and $p \in H$, without requiring $V[G] = V[H]$. It is unclear whether or not this weakening is indeed equivalent to the above conditions, but it does entail that the maximum condition of $\mathbb{P}$ decides some statements about the ground model, more specifically if $\varphi(x, \bar{u}_1, \ldots, \bar{u}_n)$ is a statement which is upwards absolute then $1_{\mathbb{P}}$ decides the truth value of $\forall x \varphi(x, \bar{u}_1, \ldots, \bar{u}_n)$. In particular statements about singularity of a cardinal are examples of being decided by $1_{\mathbb{P}}$ under this condition.

Further weakening the second condition is the following property:

Definition 9. We say that a forcing $\mathbb{P}$ is amalgamated by a forcing $\mathbb{Q}$ if for every $p_1, p_2 \in \mathbb{P}$ there exists some $q \in \mathbb{Q}$ such that whenever $G$ is a $V$-generic filter for $\mathbb{Q}$, and $r \in G$ then there are $H_1, H_2 \in V[G]$ which are $V$-generic for $\mathbb{P}$ and $p_i \in H_i$. If $\mathbb{P}$ is amalgamated by itself then we say that it is amalgamated.
Some basic and immediate observations:

1. \( P \) is amalgamated by \( P \times P \).
2. If \( P \) is amalgamated by \( Q \) then for all \( R, P \) is amalgamated by \( Q \times R \).
3. If \( Q \) amalgamates \( P \), then there is a complete embedding of \( P \) into [the Boolean completion of] \( Q \). Therefore properties like distributivity and chain conditions are inherited from \( Q \) to \( P \).
4. Every weakly homogeneous forcing is amalgamable.
5. Radin forcing is amalgamable, although not weakly homogeneous.

We proceed to weaken the second “nice” property, not adding bounded subsets. Instead we are interested in predicting the changes \( P \) introduces to the stationary subsets of a small cardinal.

**Definition 10.** Let \( P \) be a forcing notion, \( \kappa > \mu > \omega \) two regular cardinals. We say that \( P \) is \((\kappa, \mu)\)-reflective if there exists a \( P \)-name \( \dot{c} \) and a Boolean homomorphism \( h: P(\kappa)/NS_\kappa \to P(\mu)/NS_\mu \) such that:

1. \( \Vdash_P \dot{c}: \dot{\mu} \to \dot{\kappa} \) is a continuous and cofinal function,
2. For every stationary \( S \subseteq \kappa \) in the ground model, \( \Vdash_P h([S]_{NS_\kappa}) = [\dot{c}^{-1}(S)]_{NS_\mu} \), and
3. \( \text{rng } h \) is \( \kappa \text{-c.c.} \) (as a subalgebra of \( P(\mu)/NS_\mu \)). We will call \( \text{rng } h \) the reflected forcing of \( P \).

**Remark.** If \( P \) is a \((\kappa, \mu)\)-reflective forcing then \( h \), as above, is unique. To see this, note that if \( \dot{c}_1, \dot{c}_2 \) are two names for continuous cofinal functions from \( \mu \) to \( \kappa \), then \( \Vdash_P \text{rng } \dot{c}_1 \triangle \text{rng } \dot{c}_2 \) is non-stationary. Therefore the symmetric difference between \( \dot{c}_1^{-1}(S) \) and \( \dot{c}_2^{-1}(S) \) is non-stationary as well.

We denote by \( R_\kappa \) the reflected forcing of \( P \). If we force with \( R_\kappa \), then we add a generic ultrafilter \( U_\mu \) on \( \mu \), and \( h^{-1}(U_\mu) = U \) is an ultrafilter on \( \kappa \). Moreover, since \( h \) is in the ground model, \( V[U] = V[U_\mu] \). Finally, we denote by \( (M_\kappa, E) \) the generic ultrapower of \( V \) by \( U \), \( V^* / U \) as defined in \( V[U] \). This model need not be well-founded, since \( U \) is often not \( \sigma \)-closed. If \( F: S \to V \) is a function in \( V \), where \( S \subseteq \kappa \) is stationary, we denote by \( [F]_E \) the equivalence class of \( F \) in the model \( M \). When the context is clear, we will omit all the subscripts from the notations and write \( R, M, \kappa \), etc.

We aim to show that \((\kappa, \mu)\)-reflective forcings must change cofinalities below \( \kappa \), and that cofinality-changing forcing which are amalgamable and distributive are reflective. One immediate consequence would be that Magidor or Radin forcings which change cofinalities to an uncountable, must “do some damage” to the structure of cardinals below \( \kappa \). This remark is also applicable to cofinality-changing forcings which collapse \( \kappa \).

**Theorem 11.** Suppose that \( \kappa > \mu > \omega \) are regular cardinals, and let \( P \) be a forcing changing the cofinality of \( \kappa \) to be \( \mu \). If \( P \) can be amalgamated by a forcing \( \mathcal{Q} \) which does not add subsets to \( \mu \), then \( \mathcal{P} \) is \((\kappa, \mu)\)-reflective.

**Proof.** \( P \) is amalgamated by a forcing which does not add new subsets to \( \mu \), therefore \( P \) has this property as well, so it neither adds, nor destroys stationary subsets of \( \mu \). Let \( \dot{c} \) be a name such that \( \Vdash_P \dot{c}: \dot{\mu} \to \dot{\kappa} \) continuous and cofinal. For \( S \subseteq \kappa \) stationary we define:

\[
h([S]_{NS_\kappa}) = \{ T \subseteq \mu \mid \exists p \in P : p \Vdash_P T = NS_\mu, \dot{c}^{-1}(S) \}.
\]

To see that \( h \) is well-defined we first note that since \( P \) does not add new subsets to \( \mu \), there is always such \( T \), so \( h([S]_{NS_\kappa}) \) is a non-empty set.

If \( T', T \subseteq \mu \) are stationary and \( T, T' \in h([S]) \) then there are \( p, p' \in P \) such that \( p \Vdash_P T = NS_\mu, \dot{c}^{-1}(S) \) and \( p' \) forces the same about \( T' \). Then there is some \( q \in Q \) and \( H \subseteq Q \) which is \( V \)-generic such that \( q \in H \) and there are \( G, G' \in V[H] \) which are \( V \)-generic for \( P \) and \( p \in G, p' \in G' \). In \( V[H] \) let \( c = c^G \) and \( c' = c'^{G'} \), consider now \( D = \{ \beta < \mu \mid c(\beta) = c'(\beta) \} \), then \( D \) is a club in \( \mu \), and \( D \cap T = D \cap T' \) therefore \( T \triangle T' \) is non-stationary, and since \( Q \) does not add new subsets to \( \mu \), \( T \triangle T' \) is non-stationary in \( V \), and \( h \) is indeed well-defined (of course that if \( T \in h([S]_{NS_\kappa}) \) and \( T \triangle T' \) is non-stationary then \( T' \in h([S]_{NS_\kappa}) \)).

So we have that \( h \) is a function with the wanted domain and codomain, and it is easy to see that it is a Boolean homomorphism and that the first two conditions hold. It remains to show why \( \text{rng } h \) is \( \kappa \text{-c.c.} \), this follows from the following observation that \( 2^\mu < \kappa \). Otherwise \( c^G \) can be encoded as a new sequence in \( 2^{\kappa \times \mu} \) which is a contradiction to the assumption on \( P \) and \( Q \). \( \Box \)

**Corollary 12.** If \( P \) is a weakly homogeneous forcing which changes the cofinality of \( \kappa \) to \( \mu > \omega \), and \( P \) does not add new subsets of \( \mu \) then \( P \) is \((\kappa, \mu)\)-reflective.

To improve readability we shall write \( h(S) \) instead of \( h([S]_{NS_\kappa}) \) as well treat this as a stationary subset of \( \mu \), rather than an equivalence class of stationary sets.

From the following theorems we shall learn that a weakly homogeneous forcing which changes the cofinality of \( \kappa \) to be uncountable, must add bounded subsets. So the two nice properties do not co-exist with changing cofinality to be uncountable.
Theorem 13. If $\mathbb{P}$ is $(\kappa, \mu)$-reflective, then $\mathbb{P}$ must change the cofinality of some $\lambda < \kappa$.

Proof. Let $h$ be the function witnessing the reflective property. Forcing with the reflected forcing of $\mathbb{R}$, we obtain a generic ultrafilter $U$ over $\kappa$, we define $M = M_p$, and $j: V \to M$ the canonical elementary embedding. We digress from the main proof, to prove two lemmas about $M$.

Lemma 14. Let $d$ be the diagonal function on $\kappa$, then $M \models \sup j''\kappa = [d]$.

Proof of Lemma. Suppose that $M \models [f] E [d]$, then there is some $S \subseteq U$ that for every $\alpha \in S$, $f(\alpha) < d(\alpha) = \alpha$. We will find $S_0 \subseteq S$ such that $h(S_0) \neq \emptyset$ and $j''S_0 \subseteq \beta < \kappa$, and therefore $h(S_0) \Vdash \mathbb{R}[f] E j(\beta)$.

In $V[G]$, where $G$ is a generic filter for $\mathbb{P}$, the function $f$ restricted to $S \cap \mathcal{E}$ is regressive on stationary set, so there is a stationary $T \subseteq S \cap \mathcal{E}$ in $V[G]$ and $\beta < \kappa$, such that $f(\alpha) < \beta$ for every $\alpha \in T$. So, back in $V$, the set $S_0 = f^{-1}(\beta)$ is stationary and $h(S_0) \neq \emptyset$. □

Lemma 15. Let $\alpha$ be an ordinal such that there is $p \in \mathbb{P}$ such that $p \Vdash \text{cf}(\dot{\alpha}, \mu)$. Then $\sup j''\alpha = j(\alpha)$. In particular the critical point of $j$ is at least $\mu$.

Proof of Lemma. Let $f: S \to \alpha$, for $S \in U$. We want to find $S_0 \subseteq S$, $h(S_0) \neq \emptyset$ such that $f''S_0 \subseteq \beta$ for some $\beta < \alpha$.

Let $\eta < \mu$, $p \Vdash \text{cf}(\check{\alpha}) = \check{\eta}$ and $G$ be a $V$-generic filter for $\mathbb{P}$ such that $p \in G$. For every $\beta < \alpha$, let $S_0 = f^{-1}(\beta)$. In $V[G]$ let $\{\beta_i \mid i < \eta\}$ be a cofinal sequence in $\alpha$. We know that $h(S) = \bigcup\{h(S_{\beta_i}) \mid \eta < \eta\}$, so it cannot be the case that for every $i$, $h(S_{\beta_i})$ is non-stationary. Therefore at least one $S_{\beta_i}$ is stationary with $h(S_{\beta_i}) \neq \emptyset$. From this it follows by induction that $j(\alpha) = \alpha$ for all $\alpha < \mu$. □

We return to the proof of the theorem: assume towards a contradiction that $\mathbb{P}$ does not change cofinalities below $\kappa$, therefore $h(S^n_\mu) = S^n_\mu$ for all $\eta < \mu$. We now assume that $S_n^\mu \in U$, we will show that

$$M \models \text{cf}(\check{\kappa}) = \text{cf}(\sup j''\kappa) = \omega,$$

and therefore $V[U] \models \text{cf}(\kappa) = \omega$ which is impossible since $R$ is $\kappa$-c.c., so $\sup \text{cf}(\check{\kappa}) = \check{\kappa}$ (because a $\kappa$-c.c. forcing cannot change the cofinality of a regular cardinal $\kappa$).

Now pick some $F: S^n_\mu \to \omega$ such that $F(\alpha)$ is a sequence cofinal in $\alpha$. Since $S^n_\mu \in U$, by Łoś theorem $[F]$ is a cofinal $j(\omega)$ sequence at $[d]$ in $M$. By Lemma 15, $j(\omega) = \omega$, and therefore $M \models [F] = \{\{F(\alpha) \mid n < \omega\}$ and $M \models [F](\alpha) = \check{\omega}$. This is a contradiction, as mentioned above, and therefore we changed some cofinalities below $\kappa$. □

The following is a relatively straightforward consequence of the above proof. We extend it in Theorem 18, but it is worth proving on its own.

Corollary 16. Suppose $\mathbb{P}$ is a $(\kappa, \mu)$-reflective forcing such that $\Vdash \kappa$ is a strong limit cardinal. Then $\kappa$ is a Mahlo cardinal in $V$.

Proof. We aim to show that $S = \kappa \cap \text{Reg}$ is stationary in $\kappa$, and moreover $h(S) = \mu$. This is equivalent to saying that for every $U$ induced by the generic filter for the reflected forcing, $\mathbb{R}$, we have that $V^\kappa/U = M \models \text{cf}(\check{\omega}) = \check{\mu}$. This is a consequence of Łoś theorem, similar to the previous proof. So it is enough to show that for each $\alpha < \kappa$, $M \models j(\alpha) < \text{cf}(\check{d})$.

From Lemma 17 it will follow that $V[U] \models |j(\alpha)| < \mu$. By the $\kappa$-c.c. of $\mathbb{R}$ we know that in $V[U]$ the cofinality of $\{x \mid M \models x < \check{d}\}$ (as a linear order, which is not necessarily well-founded) is $\kappa$. □

Lemma 17 (Magidor). If $\mathbb{P}$ is $(\kappa, \mu)$-reflective and $\mathbb{R}$ is the reflected forcing, then $V^\mathbb{R} \models |j(\alpha)| \leq (\alpha^\mu)^{V^\mathbb{R}}$.

Proof. We define an equivalence relation on functions from $\kappa$ to $\alpha$:

$$F \sim G \iff h\{\alpha < \kappa \mid F(\alpha) = G(\alpha)\} = \mu.$$

This is an equivalence relation because $h$ is required to be a Boolean homomorphism. We have that $F \sim G$ if and only if there exists $\check{p} \in \mathbb{P}$ such that $p \Vdash F|\check{c} = NS \check{G}|\check{c}$, equivalently $F \sim G$ if and only if $\check{1}_\mathbb{P} \Vdash F|\check{c} \neq NS \check{G}|\check{c}$. And we have that there are only $(\alpha^\mu)^{V^\mathbb{R}}$ possible values for this restriction. □

Theorem 18. Let $\mathbb{P}$ be a $(\kappa, \mu)$-reflective forcing, and assume that $\Vdash \kappa$ is strong limit. We define in $V$ the following stationary $T = \{\alpha < \kappa \mid \exists p \in \mathbb{P} : p \Vdash \text{cf}(\check{\alpha}) < \check{\mu}\}$. In $V$ the following statements hold:

- For every $\eta < \mu$, $\exists p \in \mathbb{P}$ such that $p \Vdash \text{cf}(\check{\alpha}) < \check{\mu}$.
- $\kappa$ is $\mu$-Mahlo.
- $\square(\kappa, < \mu)$ fails.
Proof. Let $\eta < \mu$, $\langle S_i \mid i < \eta \rangle$ be a sequence of stationary subsets of $T$. Since $\eta < \mu$, by Lemma 15, $j(\eta) = \eta$. Let us show that for every $i$, $M \models j(S_i) \cap [d] = \emptyset$. Assume otherwise, then there is some $i < \eta$ and some $C \in M$ a club in $M$ such that $M \models C \cap j(S_i) \cap [d] = \emptyset$. Let $E = \{ \alpha \mid j(\alpha) \in C \}$, we claim that $E$ is a $T$-club in $V[U]$. We start by showing that $E$ is unbounded. Let $\alpha_0 < \kappa$, and pick by induction $\xi_0 \in C$ and $\alpha_n < \kappa$ such that $\xi_0 < j(\alpha_n+1) < \xi_n+1$. Note that by the $\kappa_c.c.$ of $R$ we may pick those $\alpha_n$ in $V$. So for $\alpha_\omega = \sup \{ \alpha_n \mid n < \omega \}$, $j(\alpha_\omega) \in C$ since $\forall \xi < j(\alpha_n) \exists \xi > \xi_0, \xi \in C$, since $j(\alpha_\omega) = \sup \{ j(\alpha_n) \mid n \in \omega \}$.

Now let us show that $E$ is $T$-closed i.e. for every $\alpha \in T$ if $E(\alpha)$ is unbounded then $\alpha \in E$. Again, by Lemma 15, if $\alpha \in T$ then $\sup j^\alpha = j(\alpha)$ so if $E(\alpha)$ is unbounded then $M \models \forall \xi < j(\alpha) \exists \xi_0 \in C \setminus \xi$ and therefore $j(\alpha) \in C$, as wanted.

But $E \cap S_i = \emptyset$ by the definition of $E$, and this is impossible since $S_i$ in stationary in $V^R$, because $R$ is $\kappa_c.c.$ Therefore in $M$ there is some ordinal below $j(\kappa)$ which reflects all the stationary sets in $j(\{ S_\alpha \mid \alpha < \eta \})$, and by elementarity the sequence have a common reflection point below $\kappa$, as wanted.

Note that the common reflection point is $[d]$. By the proof of Corollary 16, the set $S = \kappa \cap \text{Reg} \in U$, so by L"{o}s theorem there is a regular common reflection point. Moreover, $h(T) = \mu$ since every element in the club $\text{acc}(\dot{C})$ is of cofinality less than $\mu$. So we have that $S \cap T \in U$ which again by L"{o}s theorem implies that every $< \mu$ stationary subsets of $T \cap \text{Reg}$ reflect in some $\nu \in T \cap \text{Reg}$. Now, by induction on $\alpha < \mu$ we get that $\kappa$ is $\mu$-Mahlo.

To see $\square(\kappa, < \mu)$ fails, assume towards contradiction that $C = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\kappa, < \mu)$-sequence in $V$. $M \models j(C)$ is a $\square(\langle j(\kappa), < j(\mu) \rangle)$-sequence, so it has a $[d]$-th element. Let $C \in j(C)[d]$ and as above, define $E$ to be $\{ \alpha < \kappa \mid j(\alpha) \in C \}$, then as above $E$ is a $T$-closed set. For every $\alpha$, if $\text{cf}^V(\alpha) = \omega$ then $\alpha \in T$. Since $R$ is $\kappa_c.c.$, there are unboundedly many $\alpha$ such that $\alpha \in \text{acc} E$ and $\text{cf}^V(\alpha) = \omega$, so $j(\alpha) = \sup j^\alpha$. For every such $\alpha$, $C \cap j(\alpha) = D_\alpha \in j(C)[j(\alpha)]$, since $C_\alpha$ has $< \mu$ members, $j(\alpha) = \text{acc} j(\alpha) = \text{acc} \langle j(D) \mid D \in C_\alpha \rangle$, so there is some $E_\alpha$ such that $j(E_\alpha) = D_\alpha$. For every $\alpha < \beta$ in $\text{acc} E \cap (S_\alpha^V)^V$, $E_\alpha$ is an initial segment of $E_\beta$ and therefore $F = \bigcup \{ E_\alpha \mid \alpha \in \text{acc} E \cap (S_\alpha^V)^V \}$ is a thread added by a $\kappa_c.c.$ forcing, which is impossible as the next lemma shows.

Lemma 19. Suppose that $C = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\kappa, < \eta)$ sequence and $\eta < \kappa$. If $\mathbb{P}$ is $\kappa_c.c.$, then $\mathbb{P}$ does not thread to $C$.

Proof. Assume otherwise, and let $T$ be the thread added by $\mathbb{P}$, namely $T$ is a club such that for every $\alpha \in \text{acc} T$, $T \cap \alpha \in C_\alpha$. Since $\mathbb{P}$ is $\kappa_c.c.$ there is some club $T' \subseteq T$ in the ground model, we define for each $\alpha \in \text{acc} T'$, $C'_\alpha = \{ C \in C_\alpha \mid T' \cap \alpha \subseteq C \}$. Then we can define a tree such that $C'_\alpha$ is the $\text{otp}(\text{acc} T' \cap \alpha)$-th level of the tree, with end extensions as the tree order.

But each level has size strictly less than $\eta$, and therefore by a theorem of Kurepa the tree has a branch (see [Kan03, Proposition 7.9]), but this branch is a thread of $C$ in the ground model, which is a contradiction to the definition of a $\square(\kappa, < \eta)$ sequence.

5. Some Questions

Question. In Theorem 5 we saw that we cannot change the cofinalities of infinitely many cardinals to a constant cofinality without adding $\omega$-sequences, or changing the cofinalities of all the cardinals in the pcf. Is it possible to do just without adding $\omega$-sequences? As remarked after the proof, this seems to require much larger cardinals than just measurables of high Mitchell order.

Question. Continuing from the previous question, we might ask what happens if we have $\mathbb{P}$ which changes the cofinality of a regular $\kappa_\mu$ to $\mu_\nu$, rather than a constant $\mu$. Will this forcing necessarily add an $\omega$-sequence, or change the cofinalities of the entire pcf?

We have seen that there is no $(\kappa, \mu)$-reflective forcing where $\kappa$ is the least inaccessible, and in particular there is no weakly homogeneous forcing that changes the cofinality of the first inaccessible to $\omega_1$ without adding subsets to $\omega_1$. The following conjectures set the conditions imposed in the main theorems of section 4 are somewhat optimal.

Conjecture. It is consistent, relative to the existence of large cardinals, that there is a weakly homogeneous forcing which changes the cofinality of the first inaccessible to $\omega$ without adding bounded subsets.

Conjecture. It is consistent, relative to the existence of large cardinals, that there is $(\kappa, \omega_1)$-reflective forcing where $\kappa$ is the first $\omega_1$-Mahlo cardinal.

It is clear that the Radin (or Magidor) forcing notions are not weakly homogeneous, since the cardinals that change cofinality are picked generically. It is also seems reasonable that Gitik’s iteration is also not weakly homogeneous. Which raises the following question:

Question. Can there exist a weakly homogeneous forcing that change a cofinality of a cardinal to uncountable cofinality without collapsing it?
6. Acknowledgments

The authors would like to thank Menachem Magidor for many helpful discussions and in particular for his proof of Lemma 17, which is a crucial ingredient in the proof of Corollary 16. The authors would also like to thank Joel D. Hamkins for his helpful observation regarding the proof of Theorem 8, that deciding statements about the ground model is equivalent to having many generic filters, and therefore the Vopěnka-Hajek theorem can be applied. Final thanks go to Monroe Eskew for asking the question leading to Corollary 4, providing the authors with a different context for applying the first theorem.

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