Asymptotic normality of randomized periodogram for estimating quadratic variation in mixed Brownian–fractional Brownian model

Ehsan Azmoodeh∗ Tommi Sottinen† and Lauri Viitasaari‡§

November 18, 2014

Abstract

We study asymptotic normality of the randomized periodogram estimator of quadratic variation in the mixed Brownian–fractional Brownian model. In the semimartingale case, i.e., when the Hurst parameter $H$ of the fractional part satisfies $H \in (3/4, 1)$ the central limit theorem holds. In the non-semimartingale case, i.e., when $H \in (1/2, 3/4]$, the convergence towards the normal distribution with a non-zero mean still holds if $H = \frac{3}{4}$, whereas for the other values, i.e. $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, the central convergence does not take place. We also provide Berry–Esseen estimates for the estimator.

Keywords: Central limit theorem; Multiple Wiener integrals; Malliavin calculus; Fractional Brownian motion; Quadratic variation; Randomized periodogram.

2010 AMS subject classification: 60G15, 60H07, 62F12.
JEL Classification: C13, G13.

1 Introduction and motivation

The quadratic variation, or the pathwise volatility, of stochastic processes is of paramount importance in mathematical finance. Indeed, it was the major discovery of the celebrated article by Black and Scholes [8] that the prices of

∗Mathematics Research Unit, Luxembourg University, P.O. Box L-1359, Luxembourg, ehsan.azmoodeh@uni.lu.
†Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, Finland, tommi.sottinen@iki.fi.
‡Department of Mathematics and System Analysis, Aalto University School of Science, Helsinki P.O. Box 11100, FIN-00076 Aalto, Finland, lauri.viitasaari@aalto.fi.
§Department of Mathematics, Saarland University, Post-fach 151150, D-66041 Saarbrücken, Germany.
financial derivatives depend only on the volatility of the underlying asset. In
the Black–Scholes model of geometric Brownian motion the volatility simply
means the variance. Later the Brownian model was extended to more general
semimartingale models. In [10, 11] Delbaen and Schachermayer gave the
final word on the pricing of financial derivatives with semimartingales. In all
these models, the volatility simply meant the variance or the semimartingale
quadratic variance. Now, due to the important article by Föllmer [13] it is
clear that the variance is not the volatility. Instead, one should consider
the pathwise quadratic variation. This revelation and its implications to
mathematical finance has been studied in, e.g., [6, 22].

An important class of pricing models is the mixed Brownian–fractional Brown-
ian model. This is a model where the quadratic variation is determined
by the Brownian part and the correlation structure is determined by the
fractional Brownian part. Thus this is a pricing model that captures the
long-range dependence while leaving the Black–Scholes pricing formulas in-
tact. The mixed Brownian–fractional Brownian model has been studied in
the pricing context e.g. in [1, 6, 7].

By the hedging paradigm, the prices and hedges of financial derivative de-
pend only on the pathwise quadratic variation of the underlying process.
Consequently, the statistical estimation of the quadratic variation is an im-
portant problem. One way to estimate the quadratic variation is to use
directly its definition by the so called realized quadratic variation. Although
the consistency result (see Subsection 2.1) does not depend on a specific
choice of the sampling scheme, the asymptotic distribution does. There are
numerous articles that study the asymptotic behavior of realized quadratic
variation, cf. [1, 3, 16, 14, 15] and references therein. Another approach,
suggested by Dzhaparidze and Spreij [12], is to use the randomized peri-
odogram estimator. In [12] the case of semimartingales was studied. In [2]
the randomized periodogram estimator was studied for the mixed Brownian–
fractional Brownian model and the weak consistency of the estimator was
proved. This article investigates the asymptotic normality of the randomized
periodogram estimator for the mixed Brownian–fractional Brownian model.

The rest of the paper is organized as follows. In Section 2 we briefly in-
troduce the two estimators for the quadratic variation already mentioned
above. In Section 3 we introduce the stochastic analysis for Gaussian pro-
cesses needed for our results. In particular we introduce Föllmer path-wise
calculus and Malliavin calculus. Section 4 contains our main results: the
central limit theorem for the randomized periodogram estimator and an
associated Berry–Esseen bound. Finally, some technical calculations are
defered into Appendix A and Appendix B.
2 Two methods for estimating quadratic variation

2.1 Using discrete observations: realized quadratic variation

It is well-known that (see \cite[Chapter 6]{21}) for a semimartingale \(X\), the bracket \([X,X]\) can be identified with

\[
[X,X]_t = \mathbb{P} \lim_{|\pi| \to 0} \sum_{t_k \in \pi} (X_{t_k} - X_{t_{k-1}})^2,
\]

where \(\pi = \{t_k : 0 = t_0 < t_1 < \cdots < t_n = t\}\) is a partition of the interval \([0,t]\), \(|\pi| = \max \{t_k - t_{k-1} : t_k \in \pi\}\), and \(\mathbb{P}\)-lim means convergence in probability. Statistically speaking, the sums of squared increments (realized quadratic variation) is a consistent estimator for the bracket as the volume of observations tends to infinity. Barndorff-Nielsen and Shephard \cite{3} studied precision of the realized quadratic variation estimator for a special class of continuous semimartingales. They showed that sometimes the realized quadratic variation estimator can be rather noisy estimator. So one should seek for new estimators of the quadratic variation.

2.2 Using continuous observations: randomized periodogram

Dzhaparidze and Spreij \cite{12} suggested another characterization of the bracket \([X,X]\). Let \(\mathbb{F}^X\) be the filtration of \(X\) and \(\tau\) be a finite stopping time. For \(\lambda \in \mathbb{R}\), define the periodogram \(I_\tau(X;\lambda)\) of \(X\) at \(\tau\) by

\[
I_\tau(X;\lambda) = \left| \int_0^\tau e^{i\lambda s} dX_s \right|^2
\]

\[
= 2 \text{Re} \int_0^\tau \int_0^t e^{i\lambda(t-s)} dX_s dX_t + [X,X]_\tau \quad \text{(by Ito formula).}
\]

(2.1)

Given \(L > 0\) and \(\xi\) be a symmetric random variable with a density \(g_\xi\), real characteristic function \(\varphi_\xi\), and independent of the filtration \(\mathbb{F}^X\). Define the randomized periodogram by

\[
\mathbb{E}_\xi I_\tau(X;L\xi) = \int_{\mathbb{R}} I_\tau(X;Lx) g_\xi(x) dx.
\]

(2.2)

If the characteristic function \(\varphi_\xi\) is of bounded variation, then Dzhaparidze and Spreij have shown that we have the following characterization of the bracket as \(L \to \infty\)

\[
\mathbb{E}_\xi I_\tau(X;L\xi) \xrightarrow{\mathbb{P}} [X,X]_\tau.
\]

(2.3)

Recently, the convergence \(2.3\) is extended in \cite{2} to some class of stochastic processes which contains non-semimartingales in general. Let \(W = \)
\{W_t\}_{t \in [0,T]} is a standard Brownian motion and \( B^H = \{B^H_t\}_{t \in [0,T]} \) is a fractional Brownian motion with Hurst parameter \( H \in \left( \frac{1}{2}, 1 \right) \), independent of the Brownian motion \( W \). Define the mixed Brownian-fractional Brownian motion \( X_t \) by

\[
X_t = W_t + B^H_t, \quad t \in [0,T].
\]

**Remark 2.1.** It is known that (see [9]) the process \( X \) is a \((IF^X, IP)\) semi-martingale, if \( H \in \left( \frac{3}{4}, 1 \right) \), and for \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \), \( X \) is not a semimartingale with respect to its own filtration \( IF^X \). Moreover in both cases we have

\[
[X, X]_t = [X]_t = IP- \lim_{|\pi| \to 0} \sum_{t_i \leq t} (X_{t_i} - X_{t_{i-1}})^2 = t. \tag{2.4}
\]

If the partitions in (2.4) are nested, then the convergence can be strengthened to almost sure convergence. Hereafter, we always assume that the sequence of partitions are nested.

Given \( \lambda \in \IR \), define the periodogram of \( X \) at \( T \) as (2.1), i.e.

\[
I_T(X; \lambda) = \left| \int_0^T e^{i\lambda t} dX_t \right|^2
= \left| e^{i\lambda T} X_T - i\lambda \int_0^T X_t e^{i\lambda t} dt \right|^2
= X_T^2 + X_T \int_0^T i\lambda (e^{i\lambda (T-t)} - e^{-i\lambda (T-t)}) X_t dt + \lambda^2 \int_0^T e^{i\lambda t} X_t dt \right|^2.
\]

Assume \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) be an another probability space and identify the \( \sigma \)-algebra \( \mathcal{F} \) by \( \mathcal{F} \otimes \{\phi, \tilde{\Omega}\} \) on the product space \((\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, P \otimes \tilde{P})\). Let \( \xi : \tilde{\Omega} \to \IR \) be a real symmetric random variable with density \( g_\xi \), and independent of the filtration \( IF^X \). Define for any positive real number \( L \) the randomized periodogram by

\[
E_\xi I_T(X; L\xi) := \int_{\IR} I_T(X; Lx) g_\xi(x) dx. \tag{2.5}
\]

Azmoodeh and Valkeila in [2] proved the following.

**Theorem 2.1.** Assume that \( X \) is a mixed Brownian-fractional Brownian motion, \( E_\xi I_T(X; L\xi) \) is the randomized periodogram given by (2.5) and

\[ E_\xi \xi^2 < \infty. \]

Then, as \( L \to \infty \) we have

\[ E_\xi I_T(X; L\xi) \xrightarrow{P} [X, X]_T. \]
3 Stochastic analysis for Gaussian processes

3.1 Path-wise Itô formula

Föllmer \cite{13} obtained a path-wise calculus for continuous functions with finite quadratic variation. The next theorem is essentially due to Föllmer. For a nice exposition, and its use in finance, see Sondermann \cite{23}.

Theorem 3.1. \cite{23} Let \( X : [0, T] \to \mathbb{R} \) be a continuous process with continuous quadratic variation \([X, X]_t\) and \( F \in C^2(\mathbb{R}) \). Then for any \( t \in [0, T] \), the limit of the Riemann-Stieltjes sums

\[
\lim_{|\pi| \to 0} \sum_{t_i \leq t} F_{X}(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) := \int_0^t F_{X}(X_s)dX_s,
\]

exists almost surely. Moreover, we have

\[
F(X_t) = F(X_0) + \int_0^t F_{X}(X_s)dX_s + \frac{1}{2} \int_0^t F_{XX}(X_s)d[X, X]_s. \quad (3.1)
\]

The rest of the section contains the essential elements of Gaussian analysis and Malliavin calculus that are used in this paper. See for instance the references \cite{17, 18} for further details. In what follows, we assume that all the random objects are defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

3.2 Isonormal Gaussian processes derived from covariance functions

Let \( X = \{X_t\}_{t \in [0, T]} \) be a centered, continuous Gaussian process on interval \([0, T]\) with \( X_0 = 0 \) and a continuous covariance function \( R_X(s, t) \). We assume that \( F \) is generated by \( X \). Denote by \( \mathcal{E} \) the set of real valued step functions on \([0, T]\) and let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = R_X(t, s), \quad s, t \in [0, T].
\]

For example, when \( X \) is a Brownian motion, then \( \mathcal{H} \) reduces to Hilbert space \( L^2([0, T], dt) \). However, in general \( \mathcal{H} \) is not a space of functions. For example, when \( X \) is a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \) (see \cite{20}). The mapping \( 1_{[0, t]} \to X_t \) can be extended to a linear isometry between \( \mathcal{H} \) and the Gaussian space \( \mathcal{H}_1 \) spanned by Gaussian process \( X \). We denote this isometry by \( \varphi \mapsto X(\varphi) \), and \( \{X(\varphi); \varphi \in \mathcal{H}\} \) is an isonormal Gaussian process in the sense of \cite{15} Definition 1.1.1, i.e. it is a Gaussian family with covariance function

\[
\mathbb{E}(X(\varphi_1)X(\varphi_2)) = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} = \int_{[0, T]^2} \varphi_1(s)\varphi_2(t)dR_X(s, t), \quad \forall \varphi_1, \varphi_2 \in \mathcal{E}.
\]
Let \( S \) be the space of smooth and cylindrical random variables of the form
\[
F = f(X(\varphi_1), \ldots, X(\varphi_n)),
\]
where \( f \in C_0^\infty(\mathbb{R}^n) \) (\( f \) and all its partial derivatives are bounded). For a random variable \( F \) of the form (3.2), we define its Malliavin derivative as the \( H \)-valued random variable
\[
DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(\varphi_1), \ldots, X(\varphi_n))\varphi_i.
\]
By iteration, one can define the \( m \)th derivative \( D^mF \), which is an element of \( L^2(\Omega; H^\otimes m) \), for every \( m \geq 2 \). For \( m \geq 1 \), \( D^m \) denotes the closure of \( S \) with respect to the norm \( \| \cdot \|_{m, 2} \), defined by the relation
\[
\|F\|_{m, 2}^2 = \mathbb{E}[|F|^2] + \sum_{i=1}^{m} \mathbb{E}(\|D_iF\|_{H^\otimes i}^2).
\]
Let \( \delta \) be the adjoint of the operator \( D \), also called the divergence operator. A random element \( u \in L^2(\Omega, H) \) belongs to the domain of \( \delta \), denoted \( \text{Dom}(\delta) \), if and only if it verifies
\[
|\mathbb{E}(DF, u)| \leq c_u \|F\|_{L^2},
\]
for any \( F \in \mathbb{D}^{1, 2} \), where \( c_u \) is a constant depending only on \( u \). If \( u \in \text{Dom}(\delta) \), then the random variable \( \delta(u) \) is defined by the duality relationship
\[
\mathbb{E}(F\delta(u)) = \mathbb{E}(DF, u),
\]
which holds for every \( F \in \mathbb{D}^{1, 2} \). The divergence operator \( \delta \) is also called the Skorohod integral because when the Gaussian process \( X \) is a Brownian motion, it coincides with the anticipating stochastic integral introduced by Skorohod in [18]. We will make use of the notation \( \delta(u) = \int_0^T u_t \delta X_t \).

For every \( q \geq 1 \), the symbol \( H_q \) stands for the \( q \)th Wiener chaos of \( X \), defined as the closed linear subspace of \( L^2(\Omega) \) generated by the family \( \{H_q(X(h)) : h \in \mathcal{H}, ||h||_{\mathcal{H}} = 1\} \), where \( H_q \) is the \( q \)th Hermite polynomial, defined as follows:
\[
H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}}).
\]
We write by convention \( H_0 = \mathbb{R} \). For any \( q \geq 1 \), the mapping \( I^X_q(h^{\otimes q}) = H_q(X(h)) \) can be extended to a linear isometry between the symmetric tensor product \( \mathcal{H}^{\otimes q} \) (equipped with the modified norm \( \sqrt{q!} \| \cdot \|_{\mathcal{H}^{\otimes q}} \)) and the \( q \)th Wiener chaos \( H_q \). For \( q = 0 \), we write by convention \( I^X_0(c) = c, c \in \mathbb{R} \).

For any \( h \in \mathcal{H}^{\otimes q} \), the random variable \( I^X_q(h) \) is customarily called a multiple Wiener integral of order \( q \). A crucial fact is that, when \( \mathcal{H} = L^2(A, \mathcal{A}, \nu) \),
where $\nu$ is a $\sigma$-finite and non-atomic measure on the measurable space $(A, \mathcal{A})$, then $\mathcal{F}_g^q = L^2_q(\nu^q)$, where $L^2_q(\nu^q)$ stands for the subspace of $L^2(\nu^q)$ composed of the symmetric functions. Moreover, for every $h \in \mathcal{F}_g^q = L^2_q(\nu^q)$, the random variable $I^q_h(t)$ coincides with the $q$-fold multiple Wiener-Itô integral of $h$ with respect to the centered Gaussian measure (with control $\nu$) generated by $X$ (see [18]). We will also make use of the following central limit theorem for sequences living in a fixed Wiener chaos (see [19]).

**Theorem 3.2.** Let $\{F_n\}_{n \geq 1}$ be a sequence of random variables in the $q$th Wiener chaos, $q \geq 2$, such that $\lim_{n \to \infty} E(F^2_n) = \sigma^2$. Then, as $n \to \infty$, the following asymptotic statements are equivalent:

(i) $F_n$ converges in law to $\mathcal{N}(0, \sigma^2)$.

(ii) $\|DF_n\|_H^2$ converges in $L^2$ to $q\sigma^2$.

To obtain Berry-Esseen type estimate, we shall use the following result from [17] Corollary 5.2.10.

**Theorem 3.3.** Let $\{F_n\}_{n \geq 1}$ be a sequence of elements in the second Wiener chaos such that $E(F^2_n) \to \sigma^2$ and $\text{Var}(DF_n)_H^2 \to 0$, as $n \to \infty$. Then, $F_n \xrightarrow{law} Z \sim \mathcal{N}(0, \sigma^2)$ and

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(F_n < x) - \mathbb{P}(Z < x) \right| \leq \frac{2}{E(F^2_n)} \sqrt{\text{Var}(DF_n)_H^2} + \frac{2|E(F^2_n) - \sigma^2|}{\max\{E(F^2_n), \sigma^2\}}.$$

### 3.3 Isonormal Gaussian process associated with two Gaussian processes

In this subsection, we briefly describe how two Gaussian processes can be embedded into an isonormal Gaussian process. Let $X_1$ and $X_2$ be two independent centered, continuous Gaussian processes, with $X_1(0) = X_2(0) = 0$ and continuous covariance functions $R_{X_1}$ and $R_{X_2}$ respectively. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the associated Hilbert spaces as explained in Subsection 3.2. The appropriate set $\tilde{\mathcal{E}}$ of elementary functions is the set of the functions that can be written as $\varphi(t, i) = \delta_{1i}\varphi_1(t) + \delta_{2i}\varphi_2(t)$, for $(t, i) \in [0, T] \times \{1, 2\}$, where $\varphi_1, \varphi_2 \in \mathcal{E}$, and $\delta_{ij}$ is the Kronecker’s delta. On the set $\tilde{\mathcal{E}}$, we define the following inner product

$$\langle \varphi, \psi \rangle_{\tilde{\mathcal{H}}} : = \langle \varphi(\cdot, 1), \psi(\cdot, 1) \rangle_{\mathcal{H}_1} + \langle \varphi(\cdot, 2), \psi(\cdot, 2) \rangle_{\mathcal{H}_2}$$

$$= \int_{[0,T]^2} \varphi(s, 1)\psi(t, 1)dR_{X_1}(s, t) + \int_{[0,T]^2} \varphi(s, 2)\psi(t, 2)dR_{X_2}(s, t). \quad (3.5)$$

Let $\tilde{\mathcal{H}}$ denote the Hilbert space which is the completion of $\tilde{\mathcal{E}}$ with respect to the above inner product. Notice that $\tilde{\mathcal{H}} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 \oplus \mathcal{H}_2$ is the
direct sum of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e. it is a Hilbert space consists of elements of the form of ordered pairs $(h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ equipped with the inner product $\langle (h_1, h_2), (g_1, g_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} := \langle h_1, g_1 \rangle_{\mathcal{H}_1} + \langle h_2, g_2 \rangle_{\mathcal{H}_2}$.

Now, for any $\varphi \in \tilde{\mathcal{E}}$, we define $X(\varphi) := X_1(\varphi(\cdot, 1)) + X_2(\varphi(\cdot, 2))$. Using the independence between $X_1$ and $X_2$, one infers that $\mathbb{E} \langle X(\varphi)X(\psi) \rangle = \langle \varphi, \psi \rangle_{\mathcal{H}}$ for all $\varphi, \psi \in \tilde{\mathcal{E}}$. Hence, the mapping $X$ can be extended to an isometry on $\tilde{\mathcal{E}}$, and therefore $\{X(h), h \in \tilde{\mathcal{E}}\}$ defines an isonormal Gaussian process associated to the Gaussian processes $X_1$ and $X_2$.

### 3.4 Malliavin calculus with respect to (mixed Brownian) fractional Brownian motion

The fractional Brownian motion $B^H = \{B_t^H\}_{t \in \mathbb{R}}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process with covariance function

$$
\mathbb{E}(B_t^H B_s^H) = R_H(s, t) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right). \tag{3.6}
$$

Let $\mathcal{H}$ denote the Hilbert space associated to the covariance function $R_H$, see Subsection 3.2. It is well known that for $H = \frac{1}{2}$ we have $\mathcal{H} = L^2([0, T])$, whereas for $H > \frac{1}{2}$ we have $L^2([0, T]) \subset L^2_H([0, T]) \subset \mathcal{H}$, where $\mathcal{H}$ is defined as the linear space of measurable functions $\varphi$ on $[0, T]$ such that

$$
||\varphi||^2_{\mathcal{H}} := \alpha_H \int_0^T \int_0^T |\varphi(s)||\varphi(t)||t - s|^{2H - 2} ds dt < \infty,
$$

where $\alpha_H = H(2H - 1)$.

**Proposition 3.1.** [18, Chapter 5] Let $\mathcal{H}$ denote the Hilbert space associated to the covariance function $R_H$ for $H \in (0, 1)$. If $H = \frac{1}{2}$, i.e $B^H$ is a Brownian motion, then for any $\varphi, \psi \in \mathcal{H} = L^2([0, T], dt)$ the inner product of $\mathcal{H}$ is given by the well known Itô isometry

$$
\mathbb{E} \left( B_t^\varphi B_t^\psi \right) = \langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^T \varphi(t)\psi(t) dt.
$$

If $H > \frac{1}{2}$, then for any $\varphi, \psi \in \mathcal{H}$, we have

$$
\mathbb{E} \left( B^H(\varphi)B^H(\psi) \right) = \langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \varphi(s)\psi(t)||t - s|^{2H - 2} ds dt, \tag{3.7}
$$

The following proposition establishes the link between pathwise integral and Skorokhod integral in Malliavin calculus associated to fractional Brownian motion, and will play an important role in our analysis.
Proposition 3.2. Let \( u = \{u_t\}_{t \in [0,T]} \) be a stochastic process in the space \( D^{1,2}([\delta]) \) such that almost surely
\[
\int_0^T \int_0^T |D_s u_t| |t - s|^{2H-2} ds dt < \infty.
\]
Then \( u \) is pathwise integrable, and we have
\[
\int_0^T u_t dB^H_t = \int_0^T u_t \delta B^H_t + \alpha H \int_0^T \int_0^T |D_s u_t| |t - s|^{2H-2} ds dt.
\]

For further uses, we also need the following ancillary facts related to the isonormal Gaussian process derived from covariance function of the mixed Brownian fractional Brownian motion. Assume that \( X = W + B^H \) stands for a mixed Brownian fractional Brownian motion with \( H > \frac{1}{2} \). We denote by \( \mathcal{H} \) the Hilbert space associated to the covariance function of the process \( X \) with the inner product \( \langle \cdot, \cdot \rangle_\mathcal{H} \). Then a direct application of relation (3.3) and Proposition 3.1 yields the following facts. We recall that the notations \( I^X_1 \) and \( I^X_2 \) in what follows stands for multiple Wiener integrals of order 1 and 2 with respect to isonormal Gaussian process \( X \), see Subsection 3.2.

Lemma 3.1. For any \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in L^2([0, T]) \), we have
\[
\mathbb{E} \left( I^X_1(\varphi)I^X_1(\psi) \right) = \langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^T \varphi(t)\psi(t) dt + \alpha H \int_0^T \int_0^T \varphi(s)\psi(t)|t - s|^{2H-2} ds dt.
\]
Moreover,
\[
\mathbb{E} \left( I^X_2(\varphi_1 \otimes \varphi_2)I^X_2(\psi_1 \otimes \psi_2) \right) = 2 \langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle_{\mathcal{H}} \otimes 2
\]
\[
= \int_{[0,T]^2} \varphi_1(s_1)\psi_1(s_1)\varphi_2(s_2)\psi_2(s_2) ds_1 ds_2 + \alpha H \int_{[0,T]^3} \varphi_1(s_1)\psi_1(s_1)\varphi_2(s_2)\psi_2(t_2)|t_2 - s_2|^{2H-2} ds_1 ds_2 dt_2
\]
\[
+ \alpha H \int_{[0,T]^3} \varphi_1(s_1)\psi_1(t_1)\varphi_2(s_2)\psi_2(s_1)|t_1 - s_1|^{2H-2} ds_1 dt_1 ds_1
\]
\[
+ \alpha^2 H \int_{[0,T]^4} \varphi_1(s_1)\psi_1(t_1)\varphi_2(s_2)\psi_2(t_2) \times |t_1 - s_1|^{2H-2}|t_2 - s_2|^{2H-2} ds_1 dt_1 ds_2 dt_2.
\]

4 Main results

Throughout this section, we assume that \( X = W + B^H \) stands for a mixed Brownian fractional Brownian motion with \( H > \frac{1}{2} \), unless otherwise mentioned. We denote by \( \mathcal{H} \) the Hilbert space associated to process \( X \) with the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \).
4.1 Central limit theorem

We start with the following fact which is one of our key ingredients.

**Lemma 4.1.** Let $\mathbb{E} \xi^2 < \infty$. Then the randomized periodogram of the mixed Brownian-fractional Brownian motion $X$ given by (2.5) satisfies in $\mathbb{E} \xi I_t(X; L \xi) = [X, X]_T + 2 \int_0^T \int_0^t \varphi(\xi(L(t-s)))dX_s dX_t \quad (4.1)$

where the iterated stochastic integral in the right hand side is understood in path-wise way, i.e. as the limit of the Riemann-Stieltjes sums, see Subsection 3.1.

Our first aim is to transform the pathwise integral in (4.1) into the Skorokhod integral. This is the topic of the next Lemma.

**Lemma 4.2.** Let $u_t = \int_0^t \varphi(L(t-s))dX_s$. Then $u \in \mathbb{D}^{1,2}$, and moreover

$$\int_0^T u_t dX_t = \int_0^T u_t \delta X_t + \alpha_H \int_0^T \int_0^T D_s(B^H) u_t |t-s|^{2H-2} ds dt,$$

where the stochastic integral in the right hand side is the Skorokhod integral with respect to mixed Brownian-fractional Brownian motion $X$, and $D_s(B^H)$ denote the Malliavin derivative operator with respect to the fractional Brownian motion $B^H$.

**Proof.** First, note that $u_t = u_t^W + u_t^B = \int_0^t \varphi(L(t-s))dW_s + \int_0^t \varphi(L(t-s))dB_t^H$.

Moreover, $\mathbb{E}(\int_0^T u_t^2 dt) < \infty$, such that $u_t \in \mathbb{D}^{1,2}$ for almost all $t \in [0, T]$ and $\mathbb{E}(\int_0^T (D_s u_t)^2 ds dt) < \infty$. Hence, $u \in \mathbb{D}^{1,2}$ by [18 Proposition 1.3.1]. On the other hand,

\[
\begin{align*}
\int_0^T u_t dX_t &= \int_0^T u_t^W dW_t + \int_0^T u_t^B dB_t^H \\
&= \int_0^T u_t^W dW_t + \int_0^T u_t^B dW_t + \int_0^T u_t^W dB_t^H + \int_0^T u_t^B dB_t^H \\
&= \int_0^T u_t^W \delta W_t + \int_0^T u_t^B \delta W_t + \int_0^T u_t^W \delta B_t^H + \int_0^T u_t^B \delta B_t^H \\
& \quad + \alpha_H \int_0^T \int_0^T D_s(B^H) u_t^B |t-s|^{2H-2} ds dt \\
&= \int_0^T u_t \delta W_t + \int_0^T u_t \delta B_t^H + \alpha_H \int_0^T \int_0^T D_s(B^H) u_t |t-s|^{2H-2} ds dt,
\end{align*}
\]
where we have used the independence between $W$ and $B^H$, Proposition 3.2 and the fact that for adapted integrands, the Skorohod integral coincides with the Itô integral. To finish the proof, we use the very definition of Skorohod integral, relation (3.3) to obtain that $\int_0^T u_t \delta W_t + \int_0^T u_t \delta B^H_t = \int_0^T u_t \delta X_t$.

We will also pose the following assumption for characteristic function $\varphi_\xi$ of the symmetric random variable $\xi$.

**Assumption 4.1.** The characteristic function $\varphi_\xi$ satisfies

$$\int_0^\infty |\varphi_\xi(x)|dx < \infty.$$  

**Remark 4.1.** Note that the Assumption 4.1 is satisfied for many distributions. Especially, if the characteristic function $\varphi_\xi$ is positive and the density function $g_\xi(x)$ is differentiable, we get, by applying Fubini’s Theorem and integration by parts, that

$$\int_0^\infty \varphi_\xi(x)dx = 2 \int_0^\infty \int_0^\infty \cos(yx)g_\xi(y)dydx = \pi g_\xi(0) < \infty.$$  

We continue with the following technical lemma which in fact provide the correct normalization for our central limit theorems.

**Lemma 4.3.** Consider the symmetric two variables function $\psi_L(s,t) := \varphi_\xi(L|t-s|)$ on $[0,T] \times [0,T]$. Then $\psi_L \in \mathcal{S}^{\otimes 2}$, and moreover, as $L \to \infty$ we have

$$\lim_{L \to \infty} L\|\psi_L\|_{\mathcal{S}^{\otimes 2}}^2 = \sigma^2_T < \infty,$$

(4.2)

where $\sigma^2_T := 2T \int_0^\infty \varphi_\xi^2(x)dx$ and is independent of the Hurst parameter $H$.

**Remark 4.2.** We point it out that the variance $\sigma^2_T$ in Lemma 4.3 is finite. This is a simple consequence of the Assumption 4.1 and the fact that the characteristic function $\varphi_\xi$ is bounded by one over the real line.

**Proof.** Throughout the proof $C$ denotes unimportant constant depending on $T$ and $H$ which may vary from line to line. First, note that clearly $\psi_L \in \mathcal{S}^{\otimes 2}$, since $\psi_L$ is a bounded function. In order to prove (4.2), we show that as $L \to \infty$, we have

$$\|\psi_L\|_{\mathcal{S}^{\otimes 2}}^2 \sim \frac{1}{L}.$$
Next, by applying Lemma 3.1, we obtain $\|\psi_L\|_{H^2}^2 = A_1 + A_2 + A_3$, where

$$A_1 := \int_{[0,T]^2} \varphi_L^2(L|t-s|)dt ds,$$

$$A_2 := \alpha_H \int_{[0,T]^4} \varphi(L|t-u|)\varphi(L|s-v|)|t-s|^{2H-2}du dv dt ds,$$

$$A_3 := \alpha_H^2 \int_{[0,T]^4} \varphi(L|t-u|)\varphi(L|s-v|)|t-s|^{2H-2}|v-u|^{2H-2}dv dt ds.$$

First, we show that $A_1 \sim \frac{1}{L}$. By change of variables $y = \frac{t}{L}$ and $x = \frac{s}{L}$, we obtain

$$A_1 = \frac{T^2}{L^2} \int_0^L \int_0^L \varphi_L^2(T|x-y|)dx dy.$$

Now, by applying L’Hôpital’s rule and some elementary computations, we obtain that

$$\lim_{L \to \infty} L^{-1} \int_0^L \int_0^L \varphi_L^2(T|x-y|)dx dy = \lim_{L \to \infty} 2 \int_0^L \varphi_L^2(T(L-x))dx = \frac{2}{T} \int_0^\infty \varphi_L^2(y)dy
$$

which is finite by Assumption 4.1. Consequently, we get

$$\lim_{L \to \infty} LA_1 = 2T \int_0^\infty \varphi_L^2(y)dy,$$

or in other words, $A_1 \sim \frac{1}{L}$. To complete the proof, it is shown in Appendix B that $\lim_{L \to \infty} L(A_2 + A_3) = 0.$

We also apply the following proposition. The proof is rather technical and it is postponed to Appendix A.

**Proposition 4.1.** Consider the symmetric two variables function $\psi_L(s,t) := \varphi_L(L|t-s|)$ on $[0,T] \times [0,T]$. Denote

$$\tilde{\psi}_L(t,s) = \frac{\psi_L(s,t)}{\sqrt{2} \parallel \psi_L \parallel_{H^2}^2}.$$

Then, for any value $H \in (\frac{1}{2},1)$, as $L \to \infty$, we have

$$I_X^X(\tilde{\psi}_L) \xrightarrow{law} \mathcal{N}(0,1).$$

Our main theorem is the following.

**Theorem 4.1.** Assume that the characteristic function $\varphi_\xi$ of the symmetric random variable $\xi$ satisfies in assumption 4.1 and let $\sigma_T^2 \xi$ be given by 4.2. Then, as $L \to \infty$, we have the following asymptotic statements:
1. if $H \in \left(\frac{3}{4}, 1\right)$, then

$$\sqrt{L}\left(\mathbb{E}I_T(X; L\xi) - [X, X]_T\right) \xrightarrow{\text{law}} N(0, \sigma^2_T).$$

2. if $H = \frac{3}{4}$, then

$$\sqrt{L}\left(\mathbb{E}I_T(X; L\xi) - [X, X]_T\right) \xrightarrow{\text{law}} N(\mu, \sigma^2_T),$$

where $\mu = 2\alpha_H T \int_0^\infty \varphi_\xi(x)x^{2H-2}dx$.

3. if $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, then for every $\epsilon > 0$

$$L^{2H-1-\epsilon}\left(\mathbb{E}I_T(X; L\xi) - [X, X]_T\right) \xrightarrow{P} 0.$$

**Proof.** First that by applying Lemmas 4.1 and 4.2 we can write

$$\mathbb{E}I_T(X; L\xi) - [X, X]_T = I_2^X(\psi_L) + \alpha_H \int_0^T \int_0^T \varphi_\xi(L|t-s)|t-s|^{2H-2}dsdt.$$

Consequently, we obtain

$$\sqrt{L}\left(\mathbb{E}I_T(X; L\xi) - [X, X]_T\right) = \sqrt{L} I_2^X(\psi_L)$$

$$+ \sqrt{L} \alpha_H \int_0^T \int_0^T \varphi_\xi(L|t-s)|t-s|^{2H-2}dsdt$$

$$:= A_1 + A_2.$$

Now, thanks to Proposition 4.1 for any value of $H \in \left(\frac{1}{2}, 1\right)$, we have

$$A_1 = \sqrt{L} \|\psi_L\|_{S^{\odot 2}} I_2^X(\psi^-_L) \xrightarrow{\text{law}} N(0, \sigma^2_T),$$

where $\sigma^2_T$ is given by (4.2). Hence, it remains to study the term $A_2$. Using change of variables $y = \frac{L}{T}s$ and $x = \frac{L}{T}t$, we obtain

$$\int_0^T \int_0^T \varphi_\xi(L|t-s)|t-s|^{2H-2}dsdt = T^{2H} L^{-2H}$$

$$\times \int_0^L \int_0^L \varphi_\xi(T|x-y|)|x-y|^{2H-2}dxdy,$$

where by the L’Hospital’s rule, we obtain

$$\lim_{L \to \infty} L^{-1} \int_0^L \int_0^L \varphi_\xi(T|x-y|)|x-y|^{2H-2}dxdy = 2T^{1-2H} \int_0^\infty \varphi_\xi(x)x^{2H-2}dx.$$
Note also that the integral in the right hand side of the above identity is finite by Assumption 4.1. Consequently, we obtain

\[
\lim_{L \to \infty} L^{2H-1-\alpha_H} \int_0^T \int_0^T \varphi_\xi(L|t-s)||t-s|^{2H-2}dsdt = 2\alpha_H T \int_0^\infty \varphi_\xi(x)x^{2H-2}dx = \mu. \tag{4.6}
\]

Therefore,

\[
\lim_{L \to \infty} A_2 = \lim_{L \to \infty} L^{\frac{3}{2}-2H} \mu,
\]

which converges to zero for \( H \in \left(\frac{3}{4}, 1\right) \), and proving the item 1 of the claim. Similarly, for \( H = \frac{3}{4} \), we obtain

\[
\lim_{L \to \infty} A_2 = \mu,
\]

which proves the item 2 of the claim. Finally, for the item 3, from (4.6), we infer that for every \( \gamma < 2H - 1 \), as \( L \to \infty \),

\[
L^\gamma \int_0^T \int_0^T \varphi_\xi(L|t-s)||t-s|^{2H-2}dsdt \to 0.
\]

Furthermore, for term \( I_2^X(\psi_L) \), we obtain

\[
L^\gamma \int_0^T \int_0^T \varphi_\xi(L|t-s)||t-s|^{2H-2}dsdt \to 0,
\]

as \( L \to \infty \). This is because, when \( H < \frac{3}{4} \) implies \( \gamma - \frac{1}{2} < 2H - \frac{3}{2} < 0 \) and moreover \( \sqrt{L} \lim_{L \to \infty} I_2^X(\psi_L) \xrightarrow{\text{law}} N(0, 1) \) and \( L^{\gamma - \frac{1}{2}} \to 0 \).

\[\Box\]

**Corollary 4.1.** When \( X = W \) is a standard Brownian motion, i.e. when the fractional Brownian motion part drops, then with similar arguments as in Theorem 4.1, we obtain

\[
\sqrt{L} \left( \mathbb{E}I_T(X; L \xi) - [X, X]_T \right) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_T^2),
\]

where \( \sigma_T^2 = 2T \int_0^\infty \varphi_\xi^2(x)dx \).

**Remark 4.3.**

1. Note that in the case \( H \in \left(\frac{1}{2}, \frac{3}{4}\right) \) and putting \( \epsilon = 0 \) in item 3 of Theorem 4.1, we obtain convergence towards a constant.

2. Note that the proof of Theorem 4.1 reveals that in the case \( H \in \left(\frac{1}{2}, \frac{3}{4}\right) \), for any \( \epsilon > \frac{3}{2} - 2H \), we have that, as \( L \to \infty \),

\[
\sqrt{L} \left( \mathbb{E}I_T(X; L \xi) - [X, X]_T \right) \xrightarrow{P} \infty,
\]

and, moreover

\[
L^{\frac{1}{2}-\epsilon} \left( \mathbb{E}I_T(X; L \xi) - [X, X]_T \right) \xrightarrow{P} 0.
\]
4.2 The Berry–Esseen estimates

As a consequence of the proof of Theorem 4.1, we obtain also the following Berry–Esseen bound for the semimartingale case.

**Proposition 4.2.** Let all the assumptions of Theorem 4.1 hold and $H \in (\frac{3}{4}, 1)$. Furthermore, let $Z \sim \mathcal{N}(0, \sigma_T^2)$, where the variance $\sigma_T^2$ is given by (4.2). Then there exists a constant $C$ (independent of $L$) such that for sufficiently large $L$, we have

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \sqrt{L}(\mathbb{E}_\xi(I_T(X; L \xi) - [X, X]^T) < x) - \mathbb{P}(Z < x) \right) \right| \leq C \rho(L)$$

where

$$\rho(L) = \max \left\{ L^{\frac{3}{2} - 2H}, \int_0^\infty \phi^2_\xi(Tz)dz \right\}.$$

**Proof.** By proof of Theorem 4.1 we have

$$\sqrt{L}(\mathbb{E}_0(I_T(X; L \xi) - [X, X]^T)) = \sqrt{L} I_2^X(\psi_L) + \sqrt{L} \alpha_H \int_0^T \int_0^T \phi_\xi(L|t - s|)|t - s|^{2H - 2} ds dt$$

$$=: A_1 + A_2$$

where

$$A_1 = \sqrt{2L} \| \psi_L \|_{H^{\otimes 2}} I_2^X(\tilde{\psi}_L).$$

Now, we know that the deterministic term $A_2$ converges to zero with the rate $L^{\frac{3}{2} - 2H}$ and the term $A_1 \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_T^2)$. Hence, in order to complete the proof, it is sufficient to show that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(A_1 < x) - \mathbb{P}(Z < x) \right| \leq C \rho(L).$$

Now, by using the proof of Proposition 4.1 in the Appendix A, we have

$$\sqrt{\text{Var}\|DF_L\|_{\mathcal{H}}^2} \leq L^{-\frac{3}{2}} \leq L^{\frac{3}{2} - 2H}.$$

Finally, using the notations of the proof of Lemma 4.3 we have

$$E(F_n^2) = L \| \psi_L \|_{H^{\otimes 2}}^2 = L \times (A_2 + A_3),$$

where $A_2 + A_3 \leq CL^{-2H}$. Consequently,

$$L \times (A_2 + A_3) \leq CL^{1 - 2H} \leq CL^{\frac{3}{2} - 2H}.$$
To complete the proof, we have
\[
LA_1 = \frac{T^2}{L} \int_0^L \int_0^L \varphi_\xi(T|x-y|) dy dx = \frac{T^2}{L} \int_0^L \int_{-x}^{L-x} \varphi_\xi^2(Tz) dz dx \\
= \frac{T^2}{L} \int_{-L}^L \int_{-z}^{L-z} \varphi_\xi^2(Tz) dz dx = T^2 \int_{-L}^L \varphi_\xi^2(Tz) dz
\]

This gives us
\[
LA_1 - \sigma_T^2 = 2T^2 \int_L^\infty \varphi_\xi^2(Tz) dz.
\]

Now, the claim follows by an application of Theorem 3.3.

**Remark 4.4.** Consider the case \( X = W \), i.e. \( X \) is a standard Brownian motion. In this case, the correction term \( A_2 \) in the proof of Theorem 4.1 disappears, and we have
\[
\mathbb{E}(F_L^2) = 2T^2 \int_L^\infty \varphi_\xi^2(Tx) dx.
\]

Furthermore, by applying L’Hopital’s rule twice and some elementary computations, it can be shown that
\[
\mathbb{E} \left[ \left\| DF_L \right\|_0^2 - \mathbb{E} \left\| DF_L \right\|_0^2 \right] \leq |\varphi_\xi(TL)| L^{-1}.
\]

Consequently, in this case, we obtain the Berry-Esseen bound
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{L} \left( \mathbb{E}_\xi (I_T(X; L \xi) - [X, X]_T) < x \right) - \mathbb{P} (Z < x) \right) \right| \leq C \rho(L),
\]

where
\[
\rho(L) = \max \left\{ \sqrt{\varphi_\xi(TL)} L^{-1}, \int_L^\infty \varphi_\xi^2(Tz) dz \right\}
\]

which is in fact better in many cases of interest. For example, if \( \varphi_\xi \) admits an exponential decay, then we obtain \( \rho(L) \leq e^{-cL} \) for some constant \( c \).

**Acknowledgments**

Azmoodeh is supported by research project F1R-MTH-PUL-12PAMP from University of Luxembourg and Lauri Viitasaari was partially funded by Emil Aaltonen Foundation. The authors are grateful to Christian Bender for useful discussions.
A Proof of Proposition 4.1

Denote \( F_L = I^X_2(\tilde{\psi}_L) \), and note that by definition of \( \tilde{\psi}_L \), we have \( \mathbb{E}(F^2_L) = 1 \). Hence, it is sufficient to prove that, as \( L \to \infty \), we have

\[
\mathbb{E} \left[ \|DF_L\|_H^2 - \mathbb{E}\|DF_L\|_H^{\otimes 2} \right]^2 \to 0.
\]

Now, using definition of Malliavin derivative, we get

\[
D_s F_L = 2 \mathbb{I}_1(\tilde{\psi}_L(s, \cdot)) = \sqrt{2} \|\psi_L\|_{\bar{\mathcal{H}}^2} \mathbb{I}_1(\varphi_\xi(L|s - \cdot|))
\]

For the rest of the proof \( C \) denotes unimportant constants and may vary from line to line. Furthermore, we also use the short notation

\[
K(ds, dt) = \delta_0(t - s)dsdt + \alpha_H|t - s|^{2H-2}dsdt,
\]

where \( \delta_0 \) denotes the Kronecker delta function, to denote the measure associated to the Hilbert space \( \bar{\mathcal{H}} \) generated by the mixed Brownian fractional Brownian motion \( X \). Furthermore, without loss of generality, we assume that \( \varphi_\xi \geq 0 \). Indeed, otherwise we simply approximate the integral by taking absolute values inside the integral which is consistent with Assumption 4.1. Now we have

\[
\|D_s F_L\|_{\bar{\mathcal{H}}}^2 = \frac{C}{\|\psi_L\|_{\bar{\mathcal{H}}}^{\otimes 2}} \int_0^T \int_0^T I^X_1(\varphi_\xi(L|u - \cdot|))I^X_1(\varphi_\xi(L|v - \cdot|))K(du, dv).
\]

Next using the multiplication formula for multiple Wiener integrals, we see that

\[
I^X_1(\varphi_\xi(L|u - \cdot|))I^X_1(\varphi_\xi(L|v - \cdot|)) = \langle \varphi_\xi(L|u - \cdot|), \varphi_\xi(L|v - \cdot|) \rangle_{\bar{\mathcal{H}}} + I^X_2(\varphi_\xi(L|u - \cdot|) \bar{\otimes} \varphi_\xi(L|v - \cdot|)) =: J_1(u, v) + J_2(u, v),
\]

where the term \( J_1 \) is deterministic and \( J_2 \) has expectation zero. Hence, we need to show that

\[
\mathbb{E} \left[ \frac{1}{\|\psi_L\|_{\bar{\mathcal{H}}}^{\otimes 2}} \int_0^T \int_0^T J_2(u, v)K(du, dv) \right]^2 \to 0.
\] (A.1)

Therefore, by applying Fubini’s Theorem, it suffices to show that, as \( L \to \infty \), we have

\[
\mathbb{E} \left[ J_2(u_1, v_1)J_2(u_2, v_2) \right] \times K(du_1, dv_1)K(du_2, dv_2) \to 0.
\] (A.2)
First, using isometry (iii) [13 page 9], we get that

\[
\mathbb{E}\left[J_2(u_1, v_1)J_2(u_2, v_2)\right] = 2 \int_{[0,T]^4} \left( \varphi_\xi(L|u_1 - \cdot|) \tilde{\varphi}_\xi(L|v_1 - \cdot|) \right) (x_1, y_1) \times \left( \varphi_\xi(L|u_2 - \cdot|) \tilde{\varphi}_\xi(L|v_2 - \cdot|) \right) (x_2, y_2) K(dx_1, dx_2)K(dy_1, dy_2).
\]

By plugging into (A.2), we obtain that it suffices to have

\[
\frac{1}{\|\psi_L\|^4_{\Delta^2}} \int_{[0,T]^8} \left( \varphi_\xi(L|u_1 - \cdot|) \tilde{\varphi}_\xi(L|v_1 - \cdot|) \right) (x_1, y_1) \times \left( \varphi_\xi(L|u_2 - \cdot|) \tilde{\varphi}_\xi(L|v_2 - \cdot|) \right) (x_2, y_2) \times K(dx_1, dx_2)K(dy_1, dy_2)K(du_1, dv_1)K(du_2, dv_2) \to 0.
\]

The rest of the proof is based on similar arguments as the proof of Lemma 4.3. Indeed, again by symmetric property of measures \(K(dx, dy)\) and functions \(\varphi_\xi(L|v_1 - \cdot|) \tilde{\varphi}_\xi(L|v_1 - \cdot|)\), we obtain five different terms, denoted by \(A_k, k = 1, 2, 3, 4, 5\), of forms

\[
A_1 = \int_{[0,T]^4} \varphi_\xi(L|u - x|)\varphi_\xi(L|u - y|) \varphi_\xi(L|v - x|)\varphi_\xi(L|y - v|) dx dy dv du
\]

\[
A_2 = \alpha_H \int_{[0,T]^5} \varphi_\xi(L|u - x_1|)\varphi_\xi(L|u - y_1|) \varphi_\xi(L|v - x_2|)\varphi_\xi(L|y - v_2|) \times |x_1 - x_2|^{2H-2} dx_1 dx_2 dy_1 dy_2 dv du
\]

\[
A_3 = \alpha_H^2 \int_{[0,T]^6} \varphi_\xi(L|u - x_1|)\varphi_\xi(L|u - y_1|) \varphi_\xi(L|v - x_2|)\varphi_\xi(L|y - v_2|) \times |x_1 - x_2|^{2H-2} dy_1 |y_1 - y_2|^{2H-2} dx_1 dx_2 dy_2 dv_1 dv du
\]

\[
A_4 = \alpha_H^3 \int_{[0,T]^7} \varphi_\xi(L|u_1 - x_1|)\varphi_\xi(L|v_1 - y_1|) \varphi_\xi(L|v_2 - x_2|)\varphi_\xi(L|y_2 - v_2|) \times |x_1 - x_2|^{2H-2} |y_1 - y_2|^{2H-2} |u_1 - v_1|^{2H-2} dx_1 dx_2 dy_1 dy_2 dv_1 du_1 dv
\]

\[
A_5 = \alpha_H^4 \int_{[0,T]^8} \varphi_\xi(L|u_1 - x_1|)\varphi_\xi(L|v_1 - y_1|) \varphi_\xi(L|u_2 - x_2|) \times \varphi_\xi(L|y_2 - v_2|) |x_1 - x_2|^{2H-2} |y_1 - y_2|^{2H-2} |u_1 - v_1|^{2H-2} |u_2 - v_2|^{2H-2} dx_1 dx_2 dy_1 dy_2 dv_1 du_1 dv_2 dv du.
\]

Next, we prove that \(A_3 \leq CL^{-3}\). First by change of variables, we obtain

\[
A_3 = CL^{-4H-2} \int_{[0,L]^6} \varphi_\xi(T|u - x_1|)\varphi_\xi(T|u - y_1|) \varphi_\xi(T|v - x_2|)\varphi_\xi(T|y_2 - v_2|) \times |x_1 - x_2|^{2H-2} |y_1 - y_2|^{2H-2} dx_1 dx_2 dy_1 dy_2 dv_1 dv du.
\]

Note that Assumption 4.1 implies that \(\int_0^L \varphi_\xi(T|x - y|) dx \leq C\) where the constant \(C\) does not depend on \(L\) and \(y\). Similarly, we have

\[
\int_0^L |x - y|^{2H-2} dx \leq CL^{2H-1}
\]
where again the constant $C$ is independent of $L$ and $y$. Moreover, we have $\varphi_\xi(T|u - v|) \leq 1$ for any $u, v \in \mathbb{R}$. Hence, we can estimate

$$A_3 \leq CL^{-4H-2} \int_{[0,L]^6} \varphi_\xi(T|u - x_1|) \varphi_\xi(T|u - y_1|) \varphi_\xi(T|v - x_2|) \varphi_\xi(T|y_2 - v|) \times \varphi_\xi(T|y_2 - v|) |x_1 - x_2|^{2H-2} |y_1 - y_2|^{2H-2} dx_1 dy_1 dy_2 du dv \leq CL^{-4H-2} \int_{[0,L]^6} 1 \times \varphi_\xi(T|u - y_1|) \varphi_\xi(T|v - x_2|) \varphi_\xi(T|y_2 - v|) \times |x_1 - x_2|^{2H-2} |y_1 - y_2|^{2H-2} dx_1 dy_1 dy_2 dv \leq CL^{-4H-2} \int_{[0,L]^4} \varphi_\xi(T|v - x_2|) \varphi_\xi(T|y_2 - v|) \times \left( \int_{[0,L]^2} \varphi_\xi(T|u - y_1|) |x_1 - x_2|^{2H-2} dx_1 \right) dx_2 dy_1 dy_2 \leq CL^{-4H-2} \times L^{2H-1} \int_{[0,L]^4} \varphi_\xi(T|v - x_2|) \varphi_\xi(T|y_2 - v|) \times |y_1 - y_2|^{2H-2} dx_2 dy_1 dy_2 dv = CL^{-2H-3} \int_{[0,L]^6} \varphi_\xi(T|y_2 - v|) |y_1 - y_2|^{2H-2} \times \left( \int_{0}^{L} \varphi_\xi(T|v - x_2|) dx_2 \right) dy_1 dy_2 \leq CL^{-2H-3} \int_{[0,L]^2} |y_1 - y_2|^{2H-2} \left( \int_{0}^{L} \varphi_\xi(T|y_2 - v|) dv \right) dy_1 dy_2 \leq CL^{-2H-3} \int_{[0,L]^2} |y_1 - y_2|^{2H-2} dy_1 dy_2 = CL^{-3}.$$

To conclude, treating $A_1, A_2, A_4$ and $A_5$ similarly, we deduce that

$$\sum_{k=1}^{5} |A_k| \leq CL^{-3}.$$

Hence, by applying $\|\psi_L\|_{\mathcal{S}^0}^2 \sim L^{-1}$, we obtain (A.3) which completes the proof.

**B  Analysis of the variance**

We have

$$A_2 = \alpha_H \int_{[0,T]^3} \varphi_\xi(L|t - u|) \varphi_\xi(L|s - u|) |t - s|^{2H-2} dt du,$$

$$A_3 = \alpha_H^2 \int_{[0,T]^4} \varphi_\xi(L|t - u|) \varphi_\xi(L|s - v|) |t - s|^{2H-2} |v - u|^{2H-2} dv du.$$
which, by change of variable, leads to

\[ A_2 = \alpha_H T^{2H+1} L^{-2H-1} \int_{[0,L]^3} \varphi_\xi(T|t-u|) \varphi_\xi(T|s-u|) |t-s|^{2H-2} dt ds du, \]

\[ A_3 = \alpha_H^2 T^{4H} L^{-4H} \int_{[0,L]^4} \varphi_\xi(T|t-u|) \varphi_\xi(T|s-v|) |t-s|^{2H-2} |v-u|^{2H-2} dv du ds. \]

We begin with the term \( A_2 \). Denote

\[ \tilde{A}_2(L) = \int_{[0,L]^3} \varphi_\xi(T|t-u|) \varphi_\xi(T|s-u|) |t-s|^{2H-2} dt ds du. \]

By differentiating, we get

\[
\frac{d\tilde{A}_2}{dL}(L) = 2 \int_{[0,L]^2} \varphi_\xi(T|L-u|) \varphi_\xi(T|u-v|) |L-v|^{2H-2} dv du \\
+ \int_{[0,L]^2} \varphi_\xi(T|L-u|) \varphi_\xi(T|L-v|) |u-v|^{2H-2} dv du \\
=: J_1 + J_2.
\]

First, we analysis the term \( J_1 \). Similarly to Appendix A, we assume that \( \varphi_\xi \geq 0 \). Hence, we have

\[
\frac{1}{2} J_1 = \int_{[0,L]^2} \varphi_\xi(T|L-u|) \varphi_\xi(T|u-v|) |L-v|^{2H-2} dv du \\
= \int_{[0,L]^2} \varphi_\xi(Tu) \varphi_\xi(Tu-vv) v^{2H-2} dv du \\
= \int_0^L \int_1^L \varphi_\xi(Tu) \varphi_\xi(Tu-vv) v^{2H-2} dv du \\
+ \int_0^L \int_0^1 \varphi_\xi(Tu) \varphi_\xi(Tu-vv) v^{2H-2} dv du \\
\leq \int_0^L \int_1^L \varphi_\xi(Tu) \varphi_\xi(Tu-vv) dv du \\
+ \int_0^L \int_0^1 \varphi_\xi(Tu) v^{2H-2} dv du \\
\leq C.
\]
For the term $J_2$, we write

$$J_2 = \int_{[0,L]^2} \varphi_\xi(T|L-u|)\varphi_\xi(T|L-v|)|u-v|^{2H-2}dudv$$

$$= \int_{[0,L]^2} \varphi_\xi(Tu)\varphi_\xi(Tv)|u-v|^{2H-2}dudv$$

$$= 2\int_0^L \int_0^L \varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}dudv$$

$$= 2\left(\int_0^t \int_0^t + \int_0^L \int_0^{t-1} + \int_1^L \int_1^t\right) \varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}dudv$$

$$=: J_{2,1} + J_{2,2} + J_{2,3}$$

Now, it is straightforward to show that $J_{2,1} + J_{2,2} \leq C$. Consequently, as $L \to \infty$, we obtain

$$A_2 \sim L^{-2H-1}\tilde{A}_2 \sim L^{-2H}(J_1 + J_{2,1} + J_{2,2} + J_{2,3})$$

where

$$L^{-2H}(J_1 + J_{2,1} + J_{2,2}) \sim L^{-2H}.$$ 

For the term $J_{2,3}$, we write

$$J_{2,3}(L) = \int_1^L \int_{t-1}^t \varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}dudv$$

so that

$$\frac{dJ_{2,3}}{dL}(L) = \int_{L-1}^L \varphi_\xi(TL)\varphi_\xi(Tv)(L-v)^{2H-2}dv$$

$$\leq C\varphi_\xi(TL).$$

Hence, by L’Hopital’s rule, we have $L^{-2H}J_{2,3} \sim L^{1-2H}\varphi_\xi(TL)$. On the other hand, we have $\varphi_\xi(TL) = o\left(L^{2H-2}\right)$, since $\varphi$ is integrable by Assumption 4.1. Hence $L^{-2H}J_{2,3} = o\left(L^{-1}\right)$, which shows that $\lim_{L\to\infty} LA_2 = 0$. Consequently, $A_2$ does not affect to the variance. The term $A_3$ is easier and it can be treated with similar elementary computations together with L’Hopital’s rule. As a consequence, we obtain $A_3 \sim L^{-2H}$, so that $\lim_{L\to\infty} LA_3 = 0$. Hence $A_3$ does not affect to the variance either which justifies (4.2).

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