A MONOIDAL STRUCTURE ON THE CATEGORY OF RELATIVE HOPF MODULES

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Abstract. Let $B$ be a bialgebra, and $A$ a left $B$-comodule algebra in a braided monoidal category $C$, and assume that $A$ is also a coalgebra, with a not-necessarily associative or unital left $B$-action. Then we can define a right $A$-action on the tensor product of two relative Hopf modules, and this defines a monoidal structure on the category of relative Hopf modules if and only if $A$ is a bialgebra in the category of left Yetter-Drinfeld modules over $B$. Some examples are given.

Introduction

It is well-known that the category of corepresentations over a bialgebra $B$ in a braided monoidal category $C$ is monoidal. Now let $A$ be a left $B$-comodule algebra, and consider the category of relative Hopf modules $BC_A$. A relative Hopf module is always a left $B$-comodule, in fact we have a forgetful functor $BC_A \to BC$. The following natural question arises: is there a monoidal structure on $BC_A$ that is compatible with the one on $BC$, by which we mean that the forgetful functor is strongly monoidal.

Monoidal structures on a more general category, the category of Doi-Hopf modules, have been discussed in [5], in the particular case where $C$ is the category of vector spaces over a field $k$. A monoidal structure on $BC_A$ can be constructed if $A$ is a bialgebra and two additional compatibility conditions are satisfied. The aim of this paper is to present a more general result in the following direction: we will no longer assume that $A$ is a bialgebra: it will be sufficient that $A$ is at the same time an algebra and a coalgebra, and, moreover, we will assume that we have a left $B$-action $B \otimes A \to A$, which is not assumed to be associative or unital from the beginning. These additional structures (coalgebra and $B$-action) on $A$ allow us to define a right $A$-action on the tensor product of two relative Hopf modules, and on the unit object $1$ of $C$. Left $B$-coactions on these objects are supplied using the monoidal structure of the category of left $B$-comodules. Our main result, Theorem 2.1, states that the category of relative Hopf modules with this additional structure, is a monoidal category if and only if $A$ is a braided bialgebra, this is a bialgebra in the prebraided monoidal category $YD$ of left Yetter-Drinfeld modules. In the case where $C$ is the category of vector spaces, braided bialgebras and Hopf algebras appeared in the theory of Hopf algebras with a projection [13]. Observe also that braided Hopf algebras play an important role in the classification theory of pointed Hopf algebras, see for example [1] for a survey and [2] for the most recent developments.

In Section 3, we discuss some particular situations and examples. We first consider

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a quasitriangular Hopf algebra and its associated enveloping algebra braided group \( H \) as in [12]. It is well-known that \( H \) is a braided bialgebra, and therefore \( \mathcal{H}_H \) is a monoidal category. As a consequence, we find that the category of Long \( H \)-dimodules over a cocommutative Hopf algebra is monoidal. A second example is provided by a coquasitriangular Hopf algebra \((H, \sigma)\) and its associated (left) function algebra braided group \( H \), which is a braided bialgebra, so that the category of relative Hopf modules \( \mathcal{H}_H \) is monoidal. In the case where \( H \) is commutative and \( \sigma \) is trivial, this category is identified with the category of Yetter-Drinfeld modules over \( H \).

Finally, we look at the situation where the given left \( B \)-action on \( A \) is trivial, and we show that \((B, A, B)\) is a monoidal Doi-Hopf datum in the sense of [5] if and only if \( A \) is a braided bialgebra with trivial \( B \)-action.

1. Preliminary results

1.1. Braided monoidal categories. We assume that the reader is familiar with the basic theory of braided monoidal categories; for details, we refer to [9, 12]. In the sequel \( C \) is a (pre)braided monoidal category with tensor product \( \otimes : C \times C \to C \), unit object \( 1 \) and (pre)braiding \( c : \otimes \to \otimes \circ \tau \), where \( \tau : C \times C \to C \times C \) is the twist functor. For any two objects \( X, Y \) of \( C \) we denote \( c_{X,Y} \) by \( \begin{array}{c} X \ Y \ Z \\ Y \ Z \ X \end{array} \). Recall that a (pre)braiding \( c \) satisfies

\[
(1.1) \quad c_{X,Y \otimes Z} = \begin{array}{c} X \ Y \ Z \\ Y \ Z \ X \end{array} \quad \text{and} \quad c_{X \otimes Y, Z} = \begin{array}{c} X \ Y \ Z \\ Z \ X \ Y \end{array},
\]

for all objects \( X, Y, Z \in C \), and

\[
(1.2) \quad \begin{array}{ccc} X \ Y \ Z \\ Z \ Y \ X \end{array} = \begin{array}{ccc} X \ Y \ Z \\ Z \ Y \ X \end{array},
\]

the categorical version of the Yang-Baxter equation. Furthermore, \( c \) is natural in the sense that

\[
\begin{array}{c} M \ N \\ \uparrow \otimes \downarrow \\ V \ U \end{array} = \begin{array}{c} M \ N \\ \uparrow \otimes \downarrow \\ V \ U \end{array},
\]

for all morphisms \( f : M \to U \) and \( g : N \to V \) in \( C \). In particular, if we have a morphism \( X \otimes Y \to Z \) in \( C \), which is often denoted by \( \begin{array}{c} X \ Y \\ Z \end{array} \), then we have

\[
(1.3) \quad \begin{array}{c} T \ X \ Y \\ Z \ T \end{array} = \begin{array}{c} T \ X \ Y \\ Z \ T \end{array} \quad \text{and} \quad \begin{array}{c} X \ Y \ T \\ T \ Z \end{array} = \begin{array}{c} X \ Y \ T \\ T \ Z \end{array}.
\]
for all \( T \in C \). Similarly, if \( X \xrightarrow{T} Y \otimes Z \) is a morphism from \( X \) to \( Y \otimes Z \) in \( C \) then the naturality of \( c \) implies that

\[
X T \xrightarrow{T Y Z} Y T Z = X T Y Z \quad \text{and} \quad T X \xrightarrow{Y Z T} Y X Z T = T X Y Z T,
\]

for all \( T \in C \). For \( X \in C \), we identify \( 1 \otimes X \cong X \cong X \otimes 1 \) using the left and right unit constraints. By [9, Prop. XIII.1.2] we can also identify \( c_{1,X} \) and \( c_{X,1} \) with the identity morphism of \( X \) in \( C \), which will be denoted from now on by \( \text{Id}_X = X \).

In addition, all the results will be proved for strict monoidal categories, these are monoidal categories for which all the associativity, left and right unit constraints are identity morphisms. The results remain valid for an arbitrary monoidal category, since every monoidal category is equivalent to a strict one, see [9].

1.2. Braided bialgebras. An algebra in a monoidal category \( C \) is an object \( A \) of \( C \) endowed with a multiplication \( m_A : A \otimes A \to A \) and a unit morphism \( \eta_A : 1 \to A \) which are associative and unital up to the associativity and unit constraints. The multiplication and the unit of \( A \) will be denoted by \( A \) and \( 1 \). A coalgebra \( B \) in \( C \) is an algebra in the opposite category. The comultiplication \( \Delta_B : B \to B \otimes B \) and counit \( \varepsilon_B : B \to 1 \) will respectively be denoted by \( B \) and \( 1 \).

A bialgebra \( (B, m_B, \eta_B, \Delta_B, \varepsilon_B) \) in a prebraided monoidal category \( C \) is a 5-tuple \( (B, m_B, \eta_B, \Delta_B, \varepsilon_B) \), with \( (B, m_B, \eta_B) \) an algebra and \( (B, \Delta_B, \varepsilon_B) \) a coalgebra in \( C \) such that \( \Delta_B : B \to B \otimes B \) and \( \varepsilon_B : B \to 1 \) are algebra morphisms. Here \( B \otimes B \) has the tensor product algebra structure and \( 1 \) is viewed as an algebra in \( C \) through the left or right unit constraint. Explicitly, apart from \( \varepsilon_B \eta_B = \text{Id}_1 \), in diagrammatic notations the axioms for a bialgebra \( B \) in \( C \) read as follows,

\[
\begin{align*}
B B B & = B B B, \\
\frac{B}{B} & = \frac{B}{B}.
\end{align*}
\]

If \( B \) is a bialgebra in \( C \) then \( \mathcal{B}C \) (resp. \( \mathcal{B}C \)), the category of left \( B \)-modules (resp. left \( B \)-comodules) in \( C \) is a monoidal category. If \( X, Y \) are objects in \( \mathcal{B}C \) (resp. \( \mathcal{B}C \)) then \( X \otimes Y \) is a left \( B \)-module (resp. left \( B \)-comodule) via the action (resp.
coaction)

\[
\begin{pmatrix}
B & X & Y \\
X & Y
\end{pmatrix}
\quad \text{resp.} \quad
\begin{pmatrix}
X & Y \\
B & X & Y
\end{pmatrix},
\]

where \( \overline{B X} \) is our diagrammatic notation for the left action of \( B \) on \( X \), while \( \overline{X} \) is the notation used for the left \( B \)-coaction on \( X \), etc.

If \( B \) is a braided bialgebra, then we can consider algebras and coalgebras in \( B \mathcal{C} \) and \( B \mathcal{C} \). A (co)algebra in \( B \mathcal{C} \) is called a left \( B \)-comodule (co)algebra, and a (co)algebra in \( B \mathcal{C} \) is called a left \( B \)-module (co)algebra. More precisely, with notation as above, a left \( B \)-comodule (co)algebra in \( \mathcal{C} \) is a left \( B \)-comodule \( \mathcal{A} \) which is at the same time a (co)algebra in \( \mathcal{C} \) such that the (co)multiplication and the (co)unit are morphisms in \( B \mathcal{C} \), that is,

\[
(1.7)
\]

\[
\begin{pmatrix}
A \\
B
\end{pmatrix}
= \begin{pmatrix}
A \\
B
\end{pmatrix},
\]

in the comodule coalgebra case, and

\[
(1.8)
\]

\[
\begin{pmatrix}
1 \\
B & A
\end{pmatrix}
= \begin{pmatrix}
1 \\
B & A
\end{pmatrix},
\]

in the comodule algebra case.

In a similar way, a (co)algebra in \( B \mathcal{C} \) is called a left \( B \)-module (co)algebra, that is a left \( B \)-module with a (co)algebra structure in \( \mathcal{C} \) such that the (co)multiplication and (co)unit are morphisms in \( B \mathcal{C} \); this condition can be expressed in a diagrammatic way as follows:

\[
(1.9)
\]

\[
\begin{pmatrix}
B \\
A
\end{pmatrix}
= \begin{pmatrix}
B \\
A
\end{pmatrix},
\]

in the algebra case, and

\[
(1.10)
\]

\[
\begin{pmatrix}
B & A
\end{pmatrix}
= \begin{pmatrix}
B & A
\end{pmatrix},
\]

in the coalgebra case.
1.3. **Yetter-Drinfeld modules.** The category of left Yetter-Drinfeld modules $B^YD$ over a bialgebra $B$ in a braided monoidal category $C$ was introduced in [4]. It is a prebraided monoidal category, and it can be identified with a full subcategory of the left weak center of the monoidal category $B^C$ (see [4, Prop. 3.6.1]). We now give an explicit description.

A left Yetter-Drinfeld module is an object $X \in C$ with a left $B$-action and a left $B$-coaction satisfying the compatibility relation

\begin{equation}
\begin{array}{c}
\xymatrix@R=20pt@C=20pt{
B \\
X \\
B \\
X \\
B \\
X}
\end{array}
= \begin{array}{c}
\xymatrix@R=20pt@C=20pt{
B \\
X \\
B \\
X \\
B \\
X}
\end{array},
\end{equation}

where, from now on and in order to avoid confusion, we denote a $B$-action on a generic $X$, different from $A$ and $B$, by $B X$; similarly, for an object $X$ different from $A, B$ we denote a left coaction of $B$ on $X$ by $X B$.

Morphisms in $B^YD$ are morphisms in $C$ that are left $B$-linear and left $B$-colinear. The tensor and prebraiding on $B^YD$ are inherited from the tensor and prebraiding on the left weak center of $B^C$. Namely, the left $B$-action and left $B$-coaction on the tensor product $X \otimes Y$ of $X, Y \in B^YD$ is given by (1.6), and the prebraiding $\mathcal{C}$ is defined by

\begin{equation}
\mathcal{C}_{X,Y} = \begin{array}{c}
\xymatrix@R=20pt@C=20pt{
X \\
Y \\
X \\
Y}
\end{array}.
\end{equation}

Consequently, a bialgebra in $B^YD$ is an object $A$ in $B^YD$ that admits an algebra and a coalgebra structure in $C$ satisfying conditions (1.7-1.10) and

\begin{equation}
\mathcal{C}_{A,A} = \Id_A,
\end{equation}

where we used the notation from Section 1.2. Note that the above equations express the fact that the counit and the comultiplication of $A$ are algebra morphisms in $B^YD$. 


\begin{equation}
\begin{array}{c}
\xymatrix@R=20pt@C=20pt{
A \\
A \\
A}
\end{array} = \begin{array}{c}
\xymatrix@R=20pt@C=20pt{
A \\
A \\
A}
\end{array} = \begin{array}{c}
\xymatrix@R=20pt@C=20pt{
A \\
A \\
A}
\end{array} = \begin{array}{c}
\xymatrix@R=20pt@C=20pt{
A \\
A \\
A}
\end{array}.
\end{equation}
2. Main result

From now on, we assume that $B$ is a bialgebra in a braided monoidal category $\mathcal{C}$, and that $A$ is a left $B$-comodule algebra in $\mathcal{C}$ with left $B$-coaction $\Delta^B_A$. $B_C^A$ will be the notation for the category of right-left relative $(B, A)$-Hopf modules in $\mathcal{C}$. These are objects $X \in \mathcal{C}$ with a left $B$-coaction and a right $A$-action such that

$$(2.1) \quad X A \cdot B X = X A \cdot B X.$$

Morphisms in $B_C^A$ are morphisms in $\mathcal{C}$ that are left $B$-colinear and right $A$-linear. Observe that $A \in B_C^A$, with right $A$-action given by multiplication and left $B$-coaction via the $B$-comodule algebra structure.

We call $(B, A)$ an input monoidal Doi-Hopf datum if $B$ is a bialgebra, $A$ is a left $B$-comodule algebra and a coalgebra, and we have a morphism $B \otimes A \to A$. We do not assume that this makes $A$ into bialgebra or a left $B$-module coalgebra. Nevertheless, we still use the diagrammatic notation $\xymatrix{B \ar[r] & A}$ for the (not necessarily associative or unital) $B$-action $B \otimes A \to A$.

Now take two relative Hopf modules $X$ and $Y$. We know that $X \otimes Y$ is a left $B$-comodule using (1.6). Assume that $(B, A)$ is an input monoidal Doi-Hopf datum.

- We can define a right $A$-action on $X \otimes Y$ using the diagram

$$(2.2) \quad \xymatrix{X \ar[r] & Y \ar[r] & A}$$

- $\xymatrix{1}$ has a left $B$-coaction defined by the unit of $B$ and a right $A$-action defined by the counit of $A$.

- For $X, Y, Z \in B_C^A$, we can write down the associativity constraints $a_{X,Y,Z}$ and the unit constraints $l_X$ and $r_X$ in $\mathcal{C}$.

This provides part of the ingredients that are needed to define a monoidal structure on the category of relative Hopf Modules $B_C^A$. We will say that $(B_C^A, \otimes, \xymatrix{1 \ar[r] & a, l, r)$ is the input monoidal structure on $B_C^A$ defined by the input monoidal Hopf module datum $(B, A)$. The main result of this note is the following.

**Theorem 2.1.** The input monoidal structure on $B_C^A$ defined by the input monoidal Hopf module datum $(B, A)$ is a monoidal structure if and only if $A$ is a bialgebra in the prebraided monoidal category $B^D_Y$.

Before we present the proof of Theorem 2.1, we need some Lemmas.
Lemma 2.2. Let \((\mathcal{B} \otimes_{\mathcal{C}} A, 1, l, r)\) be the input monoidal structure on \(\mathcal{B} \otimes_{\mathcal{C}} A\) associated to an input monoidal Hopf module datum \((B, A)\). The following statements are equivalent.

1) \(1 \in \mathcal{B} \otimes_{\mathcal{C}} A\), \(l_X\) and \(r_X\) are morphisms in \(\mathcal{B} \otimes_{\mathcal{C}} A\), and the tensor product \(X \otimes Y\) satisfies the unit condition, for all \(X, Y \in \mathcal{B} \otimes_{\mathcal{C}} A\).

2) we have the following compatibility relations between the unit and counit morphisms of \(A\) and \(B\),

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
A \ A \\
\downarrow \quad \downarrow \\
1 \\
\end{array}
\end{array}
&= \begin{array}{c}
\begin{array}{c}
A \ A \\
\downarrow \quad \downarrow \\
1 \\
\end{array}
\end{array},
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
A \ A \\
\downarrow \\
B \\
\end{array}
\end{array}
&= \begin{array}{c}
\begin{array}{c}
A \ A \\
\downarrow \\
B \\
\end{array}
\end{array},
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
B \ A \\
\downarrow \\
A \\
\end{array}
\end{array}
&= \begin{array}{c}
\begin{array}{c}
B \ A \\
\downarrow \\
A \\
\end{array}
\end{array}.
\end{align*}
\]

(2.3)

In particular, if the input monoidal structure is monoidal, then the compatibility relations (2.3) hold.

Proof. We examine first when \(1\) is an object of \(\mathcal{B} \otimes_{\mathcal{C}} A\). Since \(\Delta_B\) and \(\varepsilon_B\) respect the unit of \(B\) it follows that \(1\) is always a left \(B\)-comodule in \(\mathcal{C}\) via the unit morphism of \(B\). Now, it can be easily checked that \(1\) is a right \(A\)-module in \(\mathcal{C}\) via the counit of \(A\) if and only if \(\varepsilon_A\) is an algebra morphism in \(\mathcal{C}\), and that the module-comodule compatibility relation holds in this case if and only if \(B \in \mathcal{B} \otimes_{\mathcal{C}} A\). We will call this relative Hopf module structure on \(B\) trivial, and denote \(B\) with this structure by \(B_{tr}\).

\(l_X\) is always left \(B\)-colinear; this follows from a simple inspection, and is due to the fact that the category \(\mathcal{B} \otimes_{\mathcal{C}} A\) is monoidal. \(l_X\) is right \(A\)-linear if and only if \(X \otimes A\) is in \(\mathcal{B} \otimes_{\mathcal{C}} A\) if and only if the fourth equality in (2.3) holds. To see the direct implication, take \(X = B_{tr}\) and then apply \(\varepsilon_B\) to the lower \(B\).
In a similar way, $r_X$ is always left $B$-colinear, and is right $A$-linear if and only if 

$$
\begin{array}{c}
\begin{array}{ccc}
X & A
\
\downarrow & & \downarrow \\
Y & = & Y
\
\downarrow & & \downarrow \\
X & & X
\end{array}
\end{array}
$$

Then it follows that the right unit constraint morphisms of all relative Hopf modules $X$ are in $^{B}C_A$ if and only if the fifth relation in (2.3) holds. For the direct implication, take $X = A$ in the above equality, and then compose it to the right by $\eta_A \otimes \text{id}_A$.

We are left to show the two final equalities in (2.3). We will see that they follow from the unit condition on the tensor product of $X, Y \in ^{B}C_A$. More precisely, the right $A$-module structure on $X \otimes Y$ respects the unit of $A$ if and only if

$$
\begin{array}{c}
\begin{array}{ccc}
X & Y
\
\downarrow & & \downarrow \\
X & = & X
\
\downarrow & & \downarrow \\
X & & X
\end{array}
\end{array}
$$

For the direct implication, take $X = A$, and compose the equality to the right by $\eta_A \otimes \text{id}_Y$. Consequently, the unit condition on $X \otimes Y$ is equivalent to (2.4). Now take $Y = A$ in the second equality of (2.4), and compose it to the right by $\eta_A$.

With the help of

\[
\begin{array}{c}
\begin{array}{ccc}
X & Y
\
\downarrow & & \downarrow \\
X & = & X
\
\downarrow & & \downarrow \\
X & & X
\end{array}
\end{array}
\]

and this is equivalent to the last equality in (2.3); for the direct implication, take $Y = B_{tr}$, and compose to the left $\text{id}_A \otimes \underline{\varepsilon}_B$. This finishes our proof. $\square$

Our next aim is to show that the left $B$-action on $A$ defines a left $B$-module algebra and a left $B$-module coalgebra structure on $A$. First we remark that the right $A$-module structure on $X \otimes Y$ $(X, Y \in ^{B}C_A)$ satisfies the associativity condition if
and only if

\[
\begin{array}{cc}
X & Y \\
\downarrow & \downarrow \\
X & Y
\end{array}
\]

Since \(X\) is a right \(A\)-module, this is equivalent to

\[
\begin{align*}
Y & A A \\
\downarrow & \downarrow \\
A & Y
\end{align*}
\]

(2.5)

\[
\begin{align*}
Y & A A \\
\downarrow & \downarrow \\
A & Y
\end{align*}
\]

For the direct implication, take \(X = A\), and compose to the right with \(\eta_A \otimes \text{id}_Y \otimes A \otimes A\).

In a similar way, it can be shown that the associativity constraints \(a_{X,Y,Z}\) are right \(A\)-linear if and only if

\[
\begin{align*}
Y & Z A \\
\downarrow & \downarrow \\
A & Y
\end{align*}
\]

(2.6)

\[
\begin{align*}
Y & Z A \\
\downarrow & \downarrow \\
A & Y
\end{align*}
\]

for all \(Y, Z \in \mathcal{B}_A\). The verification is left to the reader.

**Lemma 2.3.** Assume that the input monoidal structure \((\mathcal{B}_A, \otimes, \mathbb{1}, a, l, r)\) associated to an input monoidal Hopf module datum \((\mathcal{B}, A)\) is monoidal. Then the left \(B\)-action on \(A\) makes \(A\) into a left \(B\)-module algebra and a left \(B\)-module coalgebra.
Proof. We first show that $A$ is a left $B$-module. Taking $Y = Z = B_{tr}$ in (2.6), we obtain that

\[
\begin{array}{c}
\begin{array}{c}
B \\
A \ B \\
B
\end{array} & = & \begin{array}{c}
B \\
A \ B \\
B
\end{array}
\end{array}
\]

Composing this identity to the left with $\text{id}_A \otimes \varepsilon_B \otimes \varepsilon_B$, we find that

\[
\begin{array}{c}
\begin{array}{c}
B \\
A
\end{array} & = & \begin{array}{c}
B \\
A
\end{array}
\end{array}
\]

Together with the fifth equality in (2.3), this shows that $A$ is a left $B$-module.

Now take $Y = B_{tr}$ in (2.5), and compose at the left with $\varepsilon_B \otimes \text{id}_A$, to obtain the second equality in (1.9). Together with the last equality in (2.3), this tells us that $A$ is a left $B$-module algebra.

Now take $Y = A$ and $Z = B_{tr}$ in (2.6), and compose to the left with $\text{id}_A \otimes \text{id}_A \otimes \varepsilon_B$. Then we obtain

\[
\begin{array}{c}
\begin{array}{c}
A \\
A
\end{array} & = & \begin{array}{c}
A \\
A
\end{array}
\end{array}
\]

Now compose to the right with $\eta_A \otimes \text{id}_B \otimes \text{id}_A$. Taking into account that $\dfrac{1}{B \ A} = \dfrac{1}{B \ A}$, we find the second equality in (1.10). Combined with the fourth equality in (2.3), this tells us that $A$ is a left $B$-module coalgebra. \hfill \square

Our next step is to show that $A$ is a bialgebra in $\mathcal{B}YD$. To this end, we first point out that the tensor product $X \otimes Y$ of two objects $X, Y \in \mathcal{B}C_A$ satisfies the
compatibility relation for relative Hopf modules if and only if
\[
\begin{align*}
X & \quad Y & \quad A \\
\begin{array}{c}
B \\
X \\
Y \\
A
\end{array} & \quad = & \quad \begin{array}{c}
B \\
X \\
Y \\
A
\end{array}
\end{align*}
\] (2.7)

\[\text{Proposition 2.4. If the input monoidal structure } (B \mathcal{C}_A, \otimes, L, a, l, r) \text{ associated to an input monoidal Hopf module datum } (B, A) \text{ is monoidal, then } A \text{ is a bialgebra in } B \mathcal{Y}D.\]

Proof. It follows from our previous results that it suffices to show that \(A\) is a left \(B\)-comodule coalgebra and an object of \(B \mathcal{Y}D\), and that \(\Delta_A\) satisfies the last equality in (1.12).

We take \(X = Y = A\) in (2.7), and then compose to the right with \(\eta_A \otimes \eta_A \otimes \text{id}_A\). Using the first relation in (1.8) and the fact that \(A\) is a left \(B\)-module, we obtain the second equality in (1.7). The first equality in (1.7) was proved in Lemma 2.2 hence it follows that \(A\) is a left \(B\)-comodule coalgebra.

Now let \(X = A\) and \(Y = \text{tr}_B\) in (2.7), and compose to the right with \(\eta_A \otimes \text{id}_B \otimes \text{id}_A\), and to the left with \(\text{id}_B \otimes \text{id}_A \otimes \varepsilon_B\). Then we obtain that
\[
\begin{align*}
\begin{array}{c}
B \\
A \\
A
\end{array} & \quad = & \quad \begin{array}{c}
B \\
A \\
A
\end{array}
\end{align*}
\] (2.8)

which is precisely the required compatibility condition between the left \(B\)-action and left \(B\)-coaction on \(A\) that is needed to make \(A\) a left Yetter-Drinfeld module over \(B\) in \(\mathcal{C}\).

Finally, take \(Y = A\) in (2.7), and compose to the right with \(\eta_A \otimes \text{id}_A \otimes \text{id}_A\). Due to the first equality in (1.8) and the fact that \(A\) is a left \(B\)-module, it turns out that the last equality in (1.12) holds, and this finishes our proof. \(\Box\)

\[\text{Proof of Theorem 2.1. One implication has been proved in Proposition 2.4. Conversely, assume that } A \text{ is a bialgebra in } B \mathcal{Y}D. \text{ Since } B \mathcal{C} \text{ is a monoidal category } (B \text{ is a bialgebra in } \mathcal{C}), \text{ we have that } l_X, r_X \text{ and } a_{X,Y,Z} \text{ are left } B\text{-colinear, for all } X,Y,Z \in B \mathcal{C}_A. \text{ From Lemma 2.2 we know that } L \in B \mathcal{C}_A, \text{ that } l_X \text{ and } r_X \text{ are also right } A\text{-linear, and the right } A\text{-action on } X \otimes Y \text{ defined in (2.2) is unital. Thus it remains to be shown that } a_{X,Y,Z} \text{ is right } A\text{-linear, that } X \otimes Y \text{ is associative as a right } A\text{-module, and that it satisfies the compatibility condition for a relative Hopf module. Otherwise stated, we have to show that } 2.5 - 2.7 \text{ are satisfied.}\]
To prove (2.5) we compute, for all $Y \in B^C_A$,

\begin{align*}
Y \ A \ A \\
A \ Y
\end{align*}

as needed. The proof of (2.6) is similar, and is left to the reader. Observe that is essentially based on the fact that $A$ is a left $B$-module coalgebra.

Finally, for all $X, Y \in B^C_A$, we have that

\begin{align*}
X \ Y \ A \\
B \ X \ Y
\end{align*}
and this shows that (2.7) holds. In this computation, we freely used the fact that $Y$ is left $B$-comodule, that the multiplication on $B$ is associative, etc.
3. Examples

Let $\mathcal{C} = \mathfrak{g}\mathfrak{m}$, the category of vector spaces over a field $k$. A bialgebra in $\mathfrak{g}\mathfrak{m}$ is an ordinary $k$-bialgebra, and, for a left $B$-comodule algebra $A$, the category $B\mathcal{A} = B\mathcal{A}_A$ is the classical category of relative $(B, A)$ Hopf modules, see [10] [12].

It is well-known that particular examples of bialgebras (even Hopf algebras) in $\mathfrak{g}\mathfrak{m}$ can be obtained from quasitriangular or coquasitriangular Hopf algebras. We will study these two classes of examples in more detail.

For the definitions of a quasitriangular and a coquasitriangular Hopf algebra, we invite the reader to consult [9, 10, 12]. In the sequel, we denote the $R$-matrix of a quasitriangular Hopf algebra $H$ by $R = R_1 \otimes R_2 \in H \otimes H$, and the bilinear form that defines a coquasitriangular structure on a Hopf algebra $H$ by $\sigma : H \otimes H \rightarrow k$.

Let $(H, R)$ be a quasitriangular Hopf algebra with antipode $S$. The enveloping algebra braided group $H$ is equal to $H$ as an algebra, has the same unit and counit, but newly defined comultiplication $\Delta$:

$$\Delta(h) = h_1 S(R^2) \otimes R^1 \triangleright h_2,$$

where $\triangleright$ is the left adjoint action, that is, $h \triangleright h' = h_1 h' S(h_2)$, for all $h, h' \in H$. Now $H$ is a braided bialgebra in $\mathfrak{g}\mathfrak{m}$, the category of left representations of $H$, see [12] Ex. 9.4.9. For $(H, R)$ is quasitriangular, so we have a braided functor $F : H\mathcal{M} \rightarrow H\mathcal{YD}$, see [12] Lemma 7.4.4. For $M \in \mathcal{M}, F(M) = M$ with its original left $H$-action, and left $H$-coaction defined by

$$\lambda_M : M \rightarrow H \otimes M , \lambda_M(m) = R^2 \otimes R^1 \cdot m , \ m \in M .$$

A braided functor sends bialgebras to bialgebras, hence it follows that $H$ is a bialgebra in $H\mathcal{YD}$. This provides the following example.

Example 3.1. Let $(H, R)$ be a quasitriangular Hopf algebra with antipode $S$ and $H$ the enveloping algebra braided group of $H$. Then the category of relative Hopf modules $H\mathcal{M}_H$ is a monoidal category. The tensor product is the usual tensor product of vector spaces endowed with the left $H$-comodule structure given by the comultiplication $\Delta$ of $H$ and with the right $H$-module structure given by

$$(x \otimes y) \triangleright h = x \cdot (y_{-1} \triangleright h_1 S(R^2)) \otimes y_0 \cdot (R^1 \triangleright h_2),$$

for all $x \in X \in H\mathcal{M}_H$ and $y \in Y \in H\mathcal{M}_H$ (here $Y \triangleright y \mapsto y_{-1} \otimes y_0 \in H \otimes Y$ is the Sweedler notation for the left $H$-coaction on $Y$). The unit object is $k$ considered as a left $H$-comodule via the unit of $H$ and a right $H$-module via the counit of $H$, and the associativity, left and right unit constraints are those of $\mathfrak{g}\mathfrak{m}$.

Proof. We only point out that an object $X$ of $H\mathcal{M}_H$ is a left $H$-comodule and a right $H$-module ($H = H$ as algebras) for which the following compatibility relation holds:

$$(x \cdot h)_{-1} \otimes (x \cdot h)_0 = x_{-1} R^2 \otimes x_0 \cdot (R^1 \triangleright h),$$

for all $x \in X$ and $h \in H$. □

Corollary 3.2. If $H$ is a cocommutative Hopf algebra, then the category of right-left Long $H$-dimodules is a monoidal category.

Proof. If $H$ is cocommutative, then it is quasitriangular with $R = 1 \otimes 1$. In this particular situation, we have that $H = H$ as an ordinary bialgebra, and an object $X$ of $H\mathcal{M}_H$ is a left $H$-comodule and a right $H$-module such that

$$(x \cdot h)_{-1} \otimes (x \cdot h)_0 = x_{-1} \otimes x_0 \cdot h,$$

for all $x \in X$ and $h \in H$. Comparing to the right-left version of [6 Def. 16] we can conclude that, in this situation, the category of right-left $H$-dimodules is $H\mathcal{M}_H$. 


and so it is a monoidal category. Observe that the right $H$-module structure on the tensor product is given by the formula
\[(x \otimes y) \cdot h = x \cdot (y_1 \triangleright h_1) \otimes y_0 \cdot h_2 ,\]
for any two right-left $H$-dimodules $X,Y$, and any $x \in X$, $y \in Y$ and $h \in H$. □

We now move to the dual situation: let $(H,\sigma)$ be a coquasitriangular Hopf algebra with antipode $S$ ($S$ is then bijective, see [7]). The (left) function algebra braided group $H$ is equal to $H$ as a coalgebra, with the same unit and counit, and multiplication $\circ$ defined by
\[(3.3) \quad h \circ h' = \sigma(h'_2, S(h_1) h_3) h_4 h'_1,\]
for all $h, h' \in H$. Note that we mention explicitly that we consider the left handed version of the function algebra braided group; it is obtained from $H$ using the left version of the transmutation theory, see [3, Remark 4.3]. In [12, Ex. 9.4.10], the algebra with antipode $S$ is a bialgebra in $H_{\mathcal{M}}$ via the left adjoint coaction $\lambda$ given by the formula $\lambda(h) = S^{-1}(h_1) h_2 \otimes h_3$, for all $h \in H$. The multiplication is given by (3.3), and the other structure maps on $H$ coincide with the corresponding ones on $H$. $(H,\sigma)$ is a coquasitriangular Hopf algebra, hence there exists a braided monoidal functor $G : H_{\mathcal{M}} \to \mathcal{H}_{\mathcal{YD}}$. At the level of objects, $G(M) = M$, with the original left $H$-coaction, and left $H$-action given by the formula $h \cdot m = \sigma(m_{-1}, h) m_0$, for all $h \in H$ and $m \in M$. Consequently $H$ is a bialgebra in $\mathcal{H}_{\mathcal{YD}}$ via the left coadjoint coaction and $H$-action defined by $h \triangleright h' = \sigma(S^{-1}(h'_3) h_1', h) h'_2$, for all $h, h' \in H$. We then obtain the following result.

**Example 3.3.** Let $(H,\sigma)$ be a coquasitriangular Hopf algebra and $H$ the associated (left) function algebra braided group. Then $H_{\mathcal{M}}$ is a monoidal category with tensor product inherited from $H_{\mathcal{M}}$ and equipped with the additional right $H$-module structure given by
\[(x \otimes y) \cdot h = \sigma(S^{-1}(h_3) h_2, y_{-1}) x \cdot h_1 \otimes y_0 \cdot h_4 ,\]
for all $x \in X \in H_{\mathcal{M}}$, $y \in Y \in H_{\mathcal{M}}$ and $h \in H$. The unit object is $k$ considered as a left $H$-comodule via the unit of $H$ and a right $H$-module via the counit of $H$, and the associativity, left and right unit constraints are those of $H_{\mathcal{M}}$.

**Proof.** Everything follows from the above considerations and results. Observe that an object in $H_{\mathcal{M}}$ is an ordinary left $H$-comodule $M$ which is at the same time a right $H$-module, that is, $m \cdot 1 = m$ and
\[(m \cdot h) \cdot h' = \sigma(h'_{2}, S(h_1) h_3)m \cdot (h_2 h'_1),\]
for all $m \in M$, $h, h' \in H$, and such that the following compatibility relation holds,
\[(m \cdot h)_{-1} \otimes (m \cdot h)_0 = m_{-1} S^{-1}(h_3) h_1 \otimes m_0 \cdot h_2 ,\]
for all $m \in M$ and $h \in H$. □

**Remark 3.4.** A commutative Hopf algebra $H$ is coquasitriangular with trivial $\sigma$, $\sigma(h, h') = \varepsilon(h) \varepsilon(h')$. In this case, the (left) function algebra braided group associated to $H$ is $H$ itself with the left coadjoint coaction. Therefore, an object $M$ in $H_{\mathcal{M}}$ is a left $H$-comodule and a right $H$-module such that
\[(m \cdot h)_{-1} \otimes (m \cdot h)_0 = m_{-1} S^{-1}(h_3) h_1 \otimes m_0 \cdot h_2 ,\]
for all $m \in M$ and $h \in H$. Since $H$ is commutative it follows that the above condition is equivalent to the required compatibility relation for a right-left Yetter-Drinfeld module over $H$. Hence $\mathcal{H}_{\mathcal{YD}} = H_{\mathcal{M}}$. It is well-known, of course, that
the category of Yetter-Drinfeld modules is monoidal; but perhaps it is interesting to know that the category of Yetter-Drinfeld modules over a commutative Hopf algebra can be identified with a suitable category of relative Hopf modules.

Finally, we discuss the relationship with the monoidal structures that were discussed in [5]. Let $A$ be a left $B$-module algebra. Then $(B, A, B)$ is a Doi-Hopf datum in the sense of [3]. Monoidal Doi-Hopf data were introduced in [5], and it is easy to see that $(B, A, B)$ is monoidal if and only if $A$ is a bialgebra (in the category of vector spaces), and

\begin{equation}
ha[-1] \otimes \Delta(a[0]) = a(1)[-1]ha(2)[-1] \otimes a^{(1)}[0] \otimes a^{(2)}[0];
\end{equation}

\begin{equation}
\varepsilon_A(a)1_B = \varepsilon_A(a[0])a[-1],
\end{equation}

for all $h \in B$ and $a \in A$. Note that the left-right convention in [5] is different from ours. We have used the Sweedler notation, where indices between brackets refer to comultiplication and indices between square brackets refer to coaction.

**Proposition 3.5.** Let $A$ be a left $B$-comodule algebra, equipped with the trivial left $B$-action $h \cdot a = \varepsilon(h)a$. Then $(B, A, B)$ is a monoidal Doi-Hopf datum if and only if $A$ is bialgebra in $\mathcal{YD}$.

**Proof.** $A$ is a braided algebra if and only the following conditions hold,

1. $A$ is a left Yetter-Drinfeld module over $B$;
2. $A$ is a left $B$-(co)module (co)algebra;
3. $\varepsilon_A$ is an algebra map and $\Delta_A(1_A) = 1_A \otimes 1_A$;
4. $\Delta_A(ab) = a(1)(a(2)[-1] \cdot b(1)) \otimes a^{(2)}[0]b^{(2)}$.

First assume that $(B, A, B)$ is monoidal. (3) is satisfied since $A$ is a bialgebra; (4) simplifies to $\Delta(ab) = \Delta(a)\Delta(b)$, since the $B$-action on $A$ is trivial, and this is also satisfied. Three of the four conditions in (2) are satisfied, the only one that is left to prove is the fact that $A$ is a left $B$-comodule coalgebra. This follows from (3.5) and (3.4) (with $h = 1_B$). Applying $\varepsilon_A$ to (3.4), we find

\begin{equation}
ha[-1] \otimes a[0] = a(1)[-1]ha(2)[-1] \cdot \varepsilon_A(a(2)[0]) \otimes a^{(1)}[0] = a[-1]h \otimes a[0],
\end{equation}

which is precisely the compatibility relation for Yetter-Drinfeld modules, at least in the case where the $H$-action is trivial.

Conversely, assume that $A$ is a braided bialgebra. $A$ is a left $B$-comodule coalgebra, so (3.5) holds and

\begin{equation}
a[-1] \otimes \Delta(a[0]) = a(1)[-1]a^{(2)}[-1] \otimes a^{(1)}[0] \otimes a^{(2)}[0].
\end{equation}

Furthermore,

\begin{equation}
ha[-1] \otimes a[0] = a[-1]h \otimes a[0],
\end{equation}

since $A$ is a Yetter-Drinfeld module. Then

\begin{equation}
ha[-1] \otimes \Delta(a[0]) = a[-1]h \otimes \Delta(a[0]) = a(1)[-1]a^{(2)}[-1]h \otimes a^{(1)}[0] \otimes a^{(2)}[0].
\end{equation}

This proves that (3.4) holds, and the result follows. \qed

If $(B, A, B)$ is a monoidal Doi-Hopf data, then we have a monoidal structure on the category of relative Doi-Hopf modules, see [5, Prop. 2.1]. This monoidal structure coincides with the one that follows from Theorem 2.1.

**Example 3.6.** We end with a trivial example. Let $B = k$, and $A$ a $k$-algebra. The category of Yetter-Drinfeld modules $\mathcal{YD}$ is just the category of vector spaces $\mathcal{M}_k$, and a bialgebra in this category is an ordinary bialgebra. So we recover the...
classical result that monoidal structures on the category of representations of an algebra \( A \) are in one-to-one correspondence with bialgebra structures on \( A \).

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