On the fields due to line segments

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Abstract

The remarkable geometries of ellipsoidal equipotentials and their associated gradient fields, as produced by uniformly charged or current carrying straight-line segments, are discussed at an elementary level, motivated by recent treatments intended for introductory physics classes. Some effort is made to put the results into a broader conceptual and historical context. The equipotentials and vector fields were first obtained for the electrostatic problem by George Green in his famous 1828 essay. Related problems often appeared on the Mathematical Tripos examinations given at the University of Cambridge, and their solutions were widely disseminated by William Thomson (Lord Kelvin), Peter Guthrie Tait, and Edward Routh during the last half of the 19th century.

I. INTRODUCTION

There are a number of problems in electromagnetism where coordinates centered on the observation point simplify the integrations required to obtain either the potentials or the fields. The standard example is to show there is no electric field at any point inside of a uniformly charged spherical shell by choosing spherical polar coordinates centered on the point in question [3, 21]. For another illustration, the electric potential on the rim of a uniformly charged disk is most readily computed using such coordinates, as shown in Purcell and Morin [4], page 70, Eqn (2.30).

The magnetic field along the axis of a finite solenoid is also very easily obtained using polar coordinates about the observation point (again see [4], page 300). But Purcell and Morin do not compute \( \mathbf{E} \) for a uniformly charged, finite length, straight line segment using this method. Nor does the author of any other textbook in common use today, as far as I can tell, including Griffiths [5] and Jackson [6]. Even in his treatise [3] Maxwell does not solve the finite line segment problem using coordinates centered on the observation point.

II. A CHARGE SEGMENT

Recently, however, Zuo [7] has presented a derivation of the electric field produced by the straight-line segment, through the use of coordinates centered on the observation point. (The reader is encouraged to read [7] before proceeding.) In terms of the coordinates used by Zuo, the electric field integral reduces to

\[
\frac{1}{y} \int \hat{n}(\theta) \, d\theta
\]

where \( \hat{n}(\theta) \) is the local normal to a circle of fixed radius \( y \) whose center is the observation point. While Zuo might have found a novel way to do the calculation, it seems highly unlikely that there is no precedent given that this particular problem must have been studied by many people [12] during the 140+ years since Maxwell’s treatise first appeared [22].

In fact, this calculational method was already known to work very well for the line segment. It was discussed and widely disseminated by Edward Routh in the 19th century, and it was probably familiar to almost every student at that time who took the famous Mathematical Tripos examinations at the University of Cambridge during Routh’s unsurpassed coaching of students [13] for those examinations [23].
More specifically, an exact solution of the line segment problem, making use of calculus and coordinates centered on the observation point, was published more than 120 years ago as the very first example on pages 4-6, volume II, of Routh’s once widely-read books on analytical statics [8]. The solution includes two clearly drawn diagrams. Although Routh discussed the problem in the context of Newtonian gravity, the mathematics is exactly the same in that context as it is for the electrostatic problem. It is immediately evident that Routh’s and Zuo’s methods are identical.

Even earlier, William Thomson (Lord Kelvin) and Peter Guthrie Tait had published a solution to the same gravitational problem from the same point of view using only geometrical reasoning without calculus (as if following Newton’s lead [21]) but arriving at the same results [3] (Volume II, Section 481, pages 26-28) while making use of similar diagrams. More recently (i.e. only 60 or 50 years ago) the electric field was discussed from the same perspective in [10], pp 50-51, as well as in [11], volume I, pp 155-156, again with similar diagrams [24].

Still, even though the line segment problem has been solved several times before by almost exactly the same method, I would agree with Zuo that many people today do not know either the method or that it works so well for this problem. In any case, this is a remarkably simple continuous charge distribution where Gauss’ law does not trivially give the answer, but nevertheless the integral to obtain the electric field from Coulomb’s law is trivial to evaluate, in special coordinates, and therefore more easily computed than even the potential.

A. Green’s potential

Of course the electric potential can also be computed using the same coordinates. But the integral for the potential does not reduce simply to $\int d\theta$ as one might naively expect given the form of the electric field integral in (1). In contrast, the integral required for the potential in those same coordinates is

$$\int \frac{d\theta}{\cos \theta} = \ln \left(\frac{1 + \sin \theta}{\cos \theta}\right)$$

(2)

Referring to Zuo’s first figure, this immediately gives the result ($1/k = 4\pi \varepsilon_0$)

$$V = k\lambda \ln \left(\frac{b + r_b}{a + r_a}\right)$$

(3)

where $r_a$ and $r_b$ are the distances from the observation point to the left and right ends of the line segment, located on the $x$-axis at $a$ and $b$, respectively. With a little algebra [25] this potential can be rewritten as

$$V(s) = k\lambda \ln \left(\frac{s + L}{s - L}\right), \quad s = r_a + r_b$$

(4)

where $L$ is the length of the line segment ($L = b - a$ in Zuo’s coordinates). In this form, the equipotentials, which are given by constant $r_a + r_b$, are clearly just prolate ellipsoids of revolution about the axis of the line segment.

So far as I have been able to determine, the result (4) first appears under Article 12 in the brilliant 1828 essay by George Green [1] (see pp 68-69 in [2]). Commenting on Green’s essay several decades later, in 1870, N M Ferrers aptly summarized the situation in an Appendix to Green’s collected papers (p 329 in 2):

In the case of a straight line uniformly covered with electricity ... Denoting the extremities of the straight line by $S, H$, we know that the attraction of the line on $p$ may be replaced by that of a circular arc of which $p$ is the centre ... Hence the direction of the resultant attraction bisects the angle $SpH$, and the equipotential surface is a prolate spheroid of which $S, H$ are the foci.
Thus it would seem the essential features of both $\vec{E}$ and $V$ for the uniformly charged line segment were understood and fully appreciated as a consequence of Green’s work [26].

Today the role played by ellipsoidal equipotentials for the charged line segment is well-known [10–12, 14–16]. In my opinion, most physicists would agree that the geometry of these ellipsoids is the “hidden symmetry” that underlies the line segment problem [27].

It is also well-known that the normals to an ellipse will always bisect the angle formed by the $r_a$ and $r_b$ lines [28]. Thus the direction of the electric field for the uniformly charged line segment will also bisect this angle, since $\vec{E}$ is always normal to equipotential surfaces [29]. This agrees with Routh’s and Zuo’s conclusion based on the explicit integral (1). But in consideration of the well-known geometry of an ellipse, and the early work of Green, it is definitely not appropriate to say that a calculation using coordinates centered on the observation point (such as that by Routh or Zuo) is either the first or the only way the direction of the total electric field for this charge configuration can be graphically defined.

On the other hand a calculation based on a perspective from the observation point is technically sweet, and the resulting form for the potential, (4), the electric field may be computed in the usual way.

$$\vec{E} (\vec{r}) = - \vec{\nabla} V (s) = - \frac{dV}{ds} \vec{\nabla} s$$

$$= - \frac{dV}{ds} (\vec{\nabla} r_a + \vec{\nabla} r_b)$$

$$= - \frac{dV}{ds} (\vec{r}_a + \vec{r}_b)$$

To achieve this let $\vec{r}_a$ and $\vec{r}_b$ be vectors from the $a$ and $b$ ends of the line segment to the observation point; let $\vec{r}$ be the vector from the center of the segment to the observation point, and let $\vec{L}$ be the vector from point $a$ to point $b$. Then

$$\vec{r}_a = \vec{r} + \frac{1}{2} \vec{L}, \quad \vec{r}_b = \vec{r} - \frac{1}{2} \vec{L}$$

and

$$\vec{\nabla} r_{a,b} = \vec{\nabla} \sqrt{\left(\vec{r} + \frac{1}{2} \vec{L}\right) \cdot \left(\vec{r} + \frac{1}{2} \vec{L}\right)} = \left(\frac{\vec{r} \pm \frac{1}{2} \vec{L}}{\sqrt{\cdots}}\right)$$

so the gradients of $r_a$ and $r_b$ are simply unit vectors.

$$\vec{\nabla} r_a = \frac{\vec{r}_a}{r_a} \equiv \hat{r}_a, \quad \vec{\nabla} r_b = \frac{\vec{r}_b}{r_b} \equiv \hat{r}_b$$

Moreover, the magnitude of the electric field is now explicitly given in terms of $s$ and $\theta_{ab} = \arccos (\vec{r}_a \cdot \vec{r}_b)$, upon using

$$- \frac{dV}{ds} = \frac{2k\lambda L}{s^2 - L^2}$$

That is to say, the direction of $\vec{E} (\vec{r})$ is given just by the arithmetic average of the unit vectors $\hat{r}_a$ and $\hat{r}_b$. But these unit vectors form the equal-length sides of an isosceles triangle, and their vector sum therefore bisects the angle between them [11, 12]. This establishes yet again that $\vec{E}$ bisects the angle $\theta_{ab}$ between $\vec{r}_a$ and $\vec{r}_b$.

Moreover, the magnitude of the electric field is now explicitly given in terms of $s$ and $\theta_{ab} = \arccos (\vec{r}_a \cdot \vec{r}_b)$, upon using

$$\left| \vec{E} (\vec{r}) \right| = \left| \frac{dV}{ds} \right| \sqrt{(\vec{r}_a + \vec{r}_b) \cdot (\vec{r}_a + \vec{r}_b)}$$

$$= \frac{2kL |\lambda|}{s^2 - L^2} \sqrt{2 + 2\vec{r}_a \cdot \vec{r}_b}$$

$$= \frac{4kL |\lambda| \cos (\theta_{ab}/2)}{s^2 - L^2} \left(9\right)$$

Consequently I obtain the magnitude of the electric field in a different form than that exhibited by Routh and Zuo.
Now, this too is a well-known result (e.g. see [8, 10, 12, 14, 15]). The \( \frac{dV(s)}{ds} \) factor in \( |\vec{E}(\tau)| \) is constant on any of the equipotential ellipsoids, but the angle-dependent factor \( \cos(\theta_{ab}/2) \) varies, in general. Note that \( s > L \) for all those observation points that do not lie on the line segment itself.

Also note the transparent behavior of \( \vec{E} \) as given by (9) in some situations. For example, far away from the the line segment, \( r \gg L \), so \( s^2 - L^2 \approx s^2 \approx 4r^2 \) and \( \cos(\theta_{ab}/2) \approx \cos(0) = 1 \). Thus the field looks like a point charge, \( |\vec{E}(\tau)| \approx kL|\lambda|/r^2 \), as expected.

Also, for points \( \tau = \pm s \hat{L} \) with \( s > L \), i.e. collinear with the segment but outside of it, the field reduces to a well-known form. For such points, \( \cos(\theta_{ab}/2) = \cos(0) = 1 \) and \( s = 2r \).

While (9) is a simple result for \( |\vec{E}(\tau)| \), its behavior is not always completely transparent, and it is not obviously equivalent to the form given by Routh and Zuo. For instance, in the limit where the observation point transversely approaches some interior point on the straight line joining \( a \) and \( b \), the charged segment should be indistinguishable from an infinitely long straight line charge. That is to say, it should be true that \( y|\vec{E}(\tau)| \big|_{y \to 0} \to 2k\lambda \) where \( y \) is the “\( \perp \) distance” from the observation point to the line of charge. On the other hand, as interior points are approached, \( \lim_{y \to 0} \cos(\theta_{ab}/2) = \cos(\pi/2) = 0 \), so the \( s^2 - L^2 \) denominator in (9) better have a double zero and vanish like \( y \cos(\theta_{ab}/2) \) to obtain the correct limit. It does.

Although a coordinate-free proof from first principles might be challenging for an inexperienced student, it is nevertheless true that [30]

\[
(s^2 - L^2) \tan(\theta_{ab}/2) = 2kL \tag{10}
\]

where \( h \geq 0 \) is the \( \perp \) distance from the infinite straight line containing the segment to the point in question on the ellipse. Thus the result (9) may also be written as

\[
|\vec{E}(\tau)| = \frac{2k|\lambda \sin(\theta_{ab}/2)|}{h} \tag{11}
\]

This is the form obtained by Routh and Zuo directly from integration performed from the perspective of the observation point. The results (9) and (11) are therefore completely equivalent expressions for the same electric field. Still, because it can be somewhat painful to establish (10), and because the standard treatment of this problem involves first finding the potential and then finding \( \vec{E} \), this latter form for \( |\vec{E}| \) is not the one most likely to be found in intermediate or more advanced texts as routinely used today.

The result (11) has some features that nicely complement those of (9), and vice versa. As one rather obvious feature, (11) consists of a simple geometrical factor multiplying the field that would be produced by an infinitely long uniformly charged straight line (from \(-\infty \) to \(+\infty \)). That is,

\[
|\vec{E}(\tau)| = \left| \vec{E}_\infty \sin \left( \frac{\theta_{ab}}{2} \right) \right|, \quad |\vec{E}_\infty| = \frac{2k\lambda}{h} \tag{12}
\]

where again \( h \) is the \( \perp \) distance from the observation point to the infinite line containing the charged segment. The \( \sin(\theta_{ab}/2) \) geometrical factor brings to mind some other well-known examples of static fields [31]. The general form (but not the specific dependence on the angles) follows just from elementary dimensional analysis.

As a consequence of (12), the approach to any point in the interior of the line segment is now easy to understand, since \( \sin \left( \frac{\theta}{2} \right) \to 1 \) as \( h \to 0 \) for any point between \( a \) and \( b \). For this situation, (12) is more useful than (9).

However, for points \( \tau = \pm s \hat{L} \) with \( s > L \), it is necessary to take a careful limit of (12) to obtain the usual collinear result
since both \(\sin (\theta_{ab}/2) = 0\) and \(h = 0\) for such points. For this situation, (9) is easier to understand. Also, to see the point-like \(1/r^2\) behavior of the field for any distant point it is necessary to take a careful limit of (12) since
\[
\sin \left(\frac{\theta_{ab}}{2}\right) \to \sin (0) = 0 \text{ as } r \to \infty.
\]
Again, for this situation, (9) is more transparent.

A few more remarks are in order before closing this discussion of the electric field due to a uniformly charged line segment. For this problem, as in many others, knowing the direction of \(\vec{E}\) at any point permits the complete determination of \(\vec{E}\) just from one (non-zero!) component. In this case it is easy to find the component parallel to the direction of the segment. This component can be found without having to do any integrations.

For instance, if the segment is along the \(z\)-axis, in cylindrical coordinates, by azimuthal symmetry \(E_\phi = 0\), and
\[
(E_\rho, E_z) = \left(\left|\vec{E}\right| \sin \theta_E, \left|\vec{E}\right| \cos \theta_E\right) = (E_z \tan \theta_E, E_z),
\]
where \(\theta_E\) is the polar angle of the vector \(\vec{E}\) at the point in question. Now, \(E_z\) can be determined without actually having to do any integrations — the integrations are all eliminated by Dirac deltas. To see this note that \(V\) and \(\vec{E} = -\vec{\nabla} V\) both obey Poisson equations, namely,
\[
\nabla^2 V = -\frac{1}{\varepsilon_0} \rho, \quad \nabla^2 \vec{E} = \frac{1}{\varepsilon_0} \vec{\nabla} \rho \quad (13)
\]
where \(\rho\) is the local charge density. In free space then, without any boundaries,
\[
\vec{E} (\vec{r}) = \frac{-1}{4\pi \varepsilon_0} \int \frac{\vec{\nabla}_s \rho (\vec{s})}{|\vec{r} - \vec{s}|} d^3 s \quad (14)
\]
For a uniformly charged segment along the \(z\)-axis, between \(-L/2\) and \(L/2\), say, the charge density is given in terms of Heaviside step functions and Dirac deltas.
\[
\rho (x, y, z) = \lambda \theta \left(\frac{L}{2} - z\right) \theta \left(z + \frac{L}{2}\right) \delta (x) \delta (y) \quad (15)
\]
Thus the \(z\) component of \(\vec{\nabla} \rho\) consists of three-dimensional Dirac deltas.
\[
\partial_z \rho (x, y, z) = \lambda \delta \left(z + \frac{L}{2}\right) \delta (x) \delta (y) - \lambda \delta \left(z - \frac{L}{2}\right) \delta (x) \delta (y) \quad (16)
\]
So all three integrations in (14) are automatically eliminated for \(E_z\). The result for any observation point \(\vec{r}\) is
\[
E_z (\vec{r}) = \frac{\lambda}{4\pi \varepsilon_0} \left(\frac{1}{|\vec{r} - \frac{L}{2}\hat{z}|} - \frac{1}{|\vec{r} + \frac{L}{2}\hat{z}|}\right) \quad (17)
\]
This result along with the direction of \(\vec{E}\) at any point (as given by \(\hat{r}_a + \hat{r}_b\), say) may be used as an equivalent alternative to either (9) or (11). It is not surprising that (17) can also be found in [8] (see Volume II, page 5, Eqn(3)) and in [11] (see Volume I, page 155, Eqn(83)).

III. A CURRENT SEGMENT

Straight line segments carrying constant currents also lead to ellipsoidal equipotentials and associated magnetic vector fields. The Biot-Savart law applied to current \(I\) flowing along a directed line segment represented by the vector \(\vec{L}\) gives a magnetic field due to only the segment as follows:
\[
\vec{B} (\vec{r}) = \hat{L} \times \vec{C} (\vec{r}) \quad (18)
\]
\[
\vec{C} (\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{\vec{r} - \ell \hat{L}}{|\vec{r} - \ell \hat{L}|^3} d\ell \quad (19)
\]
where \(\vec{r}\) is a vector from the center of the segment to the observation point. That is to say, under the replacement \(\mu_0 I \to \lambda/\varepsilon_0\) this integral expression for the auxiliary vector field \(\vec{C} (\vec{r})\) is exactly the same as the Coulomb integral for the electric field \(\vec{E} (\vec{r})\) of the previous uniformly charged segment. Consequently \(\vec{C} (\vec{r})\) has the same geometric features as that previous \(\vec{E} (\vec{r})\), e.g. the direction \(\vec{C} (\vec{r})\) bisects the angle between \(\vec{r}_a\) and \(\vec{r}_b\), where these vectors are defined as in
from the ends of the line segment to the observation point.

The correspondence between $\mathbf{C}$ and the previous charged segment $\mathbf{E}$ also allows us to write

$$\mathbf{C} (\vec{r}) = \frac{\mu_0 I}{4\pi} \left( \frac{2L}{s^2 - L^2} \right) (\hat{r}_a + \hat{r}_b) \quad (20)$$

$$\mathbf{B} (\vec{r}) = \frac{\mu_0 I}{4\pi} \left( \frac{2}{s^2 - L^2} \right) \mathbf{L} \times (\hat{r}_a + \hat{r}_b) \quad (21)$$

Moreover,

$$\mathbf{C} (\vec{r}) = -\nabla U (s) = -\frac{dU (s)}{ds} \nabla s \quad (22)$$

$$U (s) = \frac{\mu_0 I}{4\pi} \ln \left( \frac{s + L}{s - L} \right), \quad s = r_a + r_b \quad (23)$$

where $U (s)$ becomes exactly the same as $V (s)$ in (11) upon replacing $\mu_0 I \to \lambda / \varepsilon_0$.

From these results it follows that $\mathbf{B}$ is in the usual form of a curl,

$$\mathbf{B} (\vec{r}) = \nabla \times \mathbf{A} (\vec{r}) \quad (24)$$

where the easily visualized vector potential due to the segment is

$$\mathbf{A} (\vec{r}) = \mathbf{L} U (s) \quad (25)$$

This $\mathbf{A}$ is constant on each ellipsoid of revolution confocal with $\mathbf{L}$.

After evaluating the cross products in (21) and using the identity (10), the result for $\mathbf{B}$ is a simple geometrical factor multiplying the field $\mathbf{B}_\infty$ that would be produced by an infinitely long straight-line current. That is,

$$\mathbf{B} (\vec{r}) = \sin \left( \frac{\vartheta_a - \vartheta_b}{2} \right) \sin \left( \frac{\vartheta_a + \vartheta_b}{2} \right) \mathbf{B}_\infty (\vec{r})$$

$$= \frac{1}{2} \left( \cos \theta_a - \cos \theta_b \right) \mathbf{B}_\infty (\vec{r}) \quad (26)$$

$$\mathbf{B}_\infty (\vec{r}) \equiv \frac{\mu_0 I}{2\pi h} \hat{\varphi} \quad (27)$$

where $\vartheta_a$ and $\vartheta_b$ are the polar angles for $\vec{r}_a$ and $\vec{r}_b$ as measured from an axis along $\mathbf{L}$, where $\hat{\varphi}$ is the azimuthal unit vector about that axis, and where $h$ is the $\bot$ distance from the observation point to that same axis. The general form of $\mathbf{B}$ (but not the specific dependence on the angles) again follows just from elementary dimensional analysis.

Of course, the magnetic field due to a straight-line segment of current is treated in many texts (e.g. [5], page 225, Example 5.5, and [16], pp 306-307, Example 10.1), although few if any of these treatments emphasize parallels between the calculation of $\mathbf{B}$ for this situation and the calculation of $\mathbf{E}$ for the charged line segment, as I have done here. However, the perspicacious reader of [7] and of the solution for the current segment exhibited in [5] will have noticed that both authors use exactly the same change of variable to evaluate the necessary integral, as well as an identical diagram.

IV. GENERALIZATIONS

A large class of other problems are solved by these same methods. In particular, since the equipotentials are ellipsoids, the solution for the uniformly charged line segment implicitly provides the solution for any charged conducting prolate ellipsoid of revolution. This too is a well-known fact [1, 3, 8–12, 14, 16]. Thus the above results can be used to describe exactly the potentials and electric fields for such ideal conductors.

Alternatively, the electrostatic results presented here may be used to describe analogous Newtonian gravitational fields around massive focaloids.

Finally, since complicated circuits are often well-approximated by a sequence of straight-line segments of various lengths, and since the magnetic field in such situations is just the sum of the $\mathbf{B}$‘s for the individual segments, my description for the magnetic field of a single segment may help to understand $\mathbf{B}$ for many circuits, even those for which the field lines are very complex [20].
Acknowledgments

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[20] M Lieberherr, “The magnetic field lines of a helical coil are not simple loops” Am. J. Phys. 78 (2010) 1117-1119.

[21] The “shell theorem” for electrostatics was anticipated in gravitational problems, as is common knowledge, and so was its proof from the perspective of the observation point, as demonstrated by Newton in the *Principia*, Section XII, Proposition LXX, Theorem XXX.

[22] It should be stressed that electrostatics had matured considerably for almost a century before Maxwell wrote his treatise. Coulomb presented his eponymous force law in 1785 and George Green wrote his remarkable, self-published essay on the subject in 1828 [1]. Moreover, for nearly a century prior to Coulomb’s work, before electrostatic problems were even expressed mathematically, Newton’s *Principia* (1687) led to investigations of equivalent problems for gravitating mass distributions. For an account of the early history, see [17, 18]. So it is quite possible that the straight line segment exercise has been around for over 300 years!
[23] Perhaps it is worth noting that Routh was the Senior Wrangler (i.e. he had the highest score) for the 1854 Mathematical Tripos examinations. Who had the second highest score that year? James Clerk Maxwell!

[24] The line segment problem and its solution, from the perspective of the observation point, do appear in at least one contemporary text, namely, P Gnädig, G Honyek, and K F Riley, 200 Puzzling Physics Problems: With Hints and Solutions, Cambridge University Press, 2001. See problem 117 (p 28) and solution 117 (pp 182-183). The solution given there is geometrical (essentially the same as that in [9]) and makes no explicit use of calculus.

[25] In addition to \( s = r_a + r_b \) and \( L = b - a \), let \( t = r_a - r_b \). Then
\[
\frac{b + r_b}{a + r_a} = \frac{2b + s + t}{2a + s - t} = \frac{a + b + L + s + t}{a + b - L + s - t}
\]
Now comparing the two right triangles, with horizontal sides \( a \) & \( b \), hypotenuses \( r_a \) & \( r_b \), and a common vertical side, gives the relations
\[
st = r_b^2 - r_a^2 = b^2 - a^2 = (a + b)L
\]
\[
a + b + L + s + t = \frac{1}{L} (s + L) (t + L)
\]
\[
a + b - L + s - t = \frac{1}{L} (s - L) (t + L)
\]
Therefore
\[
\frac{b + r_b}{a + r_a} = \frac{s + L}{s - L}
\]

[26] Also note that Green’s essay was published before either Tait or Routh were born, in the year when Thomson was four years old. Later, in his early 20s, Thomson would be instrumental in bringing attention to Green’s essay when he obtained and read a copy in 1845, four years after Green’s death.

[27] The importance of ellipsoidal geometry for various gravitational problems, including spheroidal mass distributions, was initiated by Newton (see Section XIII in the Principia especially Proposition XCI.

[28] That’s why \( a \) and \( b \) are called “focal points” — consider the law of reflection for elliptical mirrors [19]. Or for the math, see any decent text on Euclidean geometry, or even Wikipedia. Better yet, work it out for yourself! But if you do, note that it is best to use calculus instead of purely Euclidean geometric reasoning.

[29] The semi-infinite line case can be understood as the parabolic limit of an ellipsoid where one of the foci is taken to infinity. Indeed, the electric field geometry discussed by Zuo in one special semi-infinite case (also see problem 118, p 29, and solution 118, pp 183-184, in [24]) is immediately seen to amount to nothing more than a particular case of ray tracing for a parabolic mirror.

[30] This is just the tangent half-angle formula
\[
tan (\theta/2) = \frac{\sin \theta}{1 + \cos \theta}, \text{ where numerator and denominator have been expressed in terms of the area and perimeter of the relevant triangle.}
\]

[31] For example, using polar coordinates centered on the observation point, the magnetic field on the axis of a finite length solenoid, carrying a uniform azimuthal current/meter \( K \), is easily seen to differ from the infinite solenoid result by a simple geometrical factor ([4], page 300; [10], pp 502-503):
\[
\vec{B} (z) = \frac{1}{\pi} (\cos \theta_R - \cos \theta_L) \vec{B}_\infty = \sin \frac{1}{2} (\theta_L - \theta_R) \sin \frac{1}{2} (\theta_L + \theta_R) \vec{B}_\infty, \text{ where } \theta_{L,R} \text{ are polar angles for the left and right circular rims of the finite solenoid, as measured from the observation point on the axis of the solenoid, and where } \vec{B}_\infty = \mu_0 K \hat{z} \text{ is the constant field on the axis of an infinitely long solenoid, extending from } -\infty \text{ to } +\infty. \text{ Note that the same geometrical factor appears for the magnetic field of a straight-line current segment (see Eqn [26] in the text) with appropriate identification of the angles.