TORSION OF SU(2)-STRUCTURES AND RICCI CURVATURE IN DIMENSION 5

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Abstract. Following the approach of Bryant [6], we study the intrinsic torsion of an SU(2)-structure on a 5-dimensional manifold deriving an explicit expression for the Ricci and the scalar curvature in terms of torsion forms and its derivative. As a consequence of this formula we prove that the $\alpha$-Einstein condition forces some special SU(2)-structures to be Sasaki-Einstein.

Introduction

In 1960 Sasaki introduced in [18] a new class of contact-metric structures which can be considered as an odd-dimensional counter-part of Kähler structures. This kind of geometry became known as Sasaki geometry and it is present today in many mathematical and physical contexts. In Sasakian geometry Einstein metrics play a central role and Sasaki-Einstein manifolds arise in many physical models. As general references for these topics see e.g. [1], [3], [4], [5], [13], [16], [15] and the references therein.

Since in dimension 5 Sasakian-Einstein metrics correspond to Killing spinors (see [12]), it is rather natural to study the larger class of SU(2)-structures induced by generalized Killing spinors. These structures were firstly investigated and called Hypo-structures by Conti and Salamon in [7], where they prove that any analytic Hypo-manifold can be realized as a hypersurface of a Calabi-Yau threefold.

In terms of differential forms a Hypo-structure is determined by a nowhere vanishing 1-form $\alpha$ and a triple of 2-forms $(\omega_1, \omega_2, \omega_3)$ satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad d\omega_1 = 0, \quad d(\omega_2 \wedge \alpha) = 0, \quad d(\omega_3 \wedge \alpha) = 0,$$

where $v$ is a 4-form such that $v \wedge \alpha \neq 0$ everywhere.

In [11] the authors introduce two new types of SU(2)-structures on 5-manifolds: nearly-Hypo structures are the natural structures inherited by an hypersurface of a nearly Kähler SU(3)-manifold, while double-Hypo structures are nearly-Hypo and Hypo simultaneously.

In this paper, following the same approach used by Bryant in [6] to compute the Ricci tensor of a $G_2$-structure, we write down an explicit formula for the scalar curvature and the Ricci tensor of the metric induced by an SU(2)-structure on a 5-manifold in terms of the intrinsic torsion (Theorems 3.4 and 3.8). As a direct
consequence of the formula of the scalar curvature, we have that if the Ricci tensor of a Hypo-structure \((\alpha, \omega_1, \omega_2, \omega_3)\) satisfies

\[
\text{Ric}(R_\alpha, R_\alpha) = 4,
\]

where \(R_\alpha\) is the Reeb vector field of \(\alpha\), then the Hypo-structure is Sasaki-\(\alpha\)-Einstein. This result slightly strengthens a previous result by Conti and Salamon (see [7]). The formula for the Ricci tensor has as a direct application the study of \(\alpha\)-Einstein metrics on contact-Hypo manifolds. The \(\alpha\)-Einstein metrics were introduced by Okumura in [17] in the context of contact-metric geometry and they are characterized by the equation

\[
\text{Ric} = \mu g + \lambda \alpha \otimes \alpha,
\]

where \(\lambda\) and \(\mu\) are constant. Sasaki-\(\alpha\)-Einstein metrics seem to be a natural generalization of Kähler-Einstein metrics to the odd dimension (see e.g [4]).

We prove that the \(\alpha\)-Einstein condition forces a double-Hypo structure to be Sasaki-Einstein (Proposition 4.3). Finally, as a corollary, we prove that if the almost Kähler cone of a 5-dimensional \(\alpha\)-Einstein SU(2)-manifold inherits a symplectic half-flat structure (see [10], [8] and [2]), then it is a Sasaki-\(\alpha\)-Einstein manifold (Corollary 4.4).

The present paper is organized as follows: In section 1 we recall some basic facts on SU(2)-structures and set up the algebraic preliminaries needed in the sequel. In section 2 we recall the properties of the intrinsic torsion of an SU(2)-structure proving some new formulae which will be useful in the next part of the paper. Section 3 is devoted to the main result. We describe the computational steps needed to reach it (and carried out with the aid of MAPLE) and we write down the formulae for the scalar curvature and the Ricci tensor. Then we prove the consequences obtained imposing the \(\alpha\)-Einstein condition.

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**Notation.** Given a manifold \(M\), we denote by \(\Lambda^r M\) the space of smooth \(r\)-forms on \(M\).

When a coframe \(\{e^1, \ldots, e^n\}\) is given, we will denote the \(r\)-form \(e^{i_1} \wedge \cdots \wedge e^{i_r}\) by \(e^{i_1 \ldots i_r}\).

Furthermore, when a contact form \(\alpha\) is fixed, we will denote by \(\beta^T\) the projection of an arbitrary differential form \(\beta\) onto the contact distribution \(\xi = \ker \alpha\).

The symbol \(\langle \cdot, \cdot \rangle\) will denote the scalar product induced on exterior forms by a Riemannian metric.

Finally in the indicial expression the symbol of sum over repeated indices is omitted.

**1. Five-dimensional SU(2)-structures**

Let \(M\) be a 5-dimensional smooth manifold and let \(\mathcal{L}(M) \to M\) be the GL(5)-bundle of linear frames on \(M\). An SU(2)-structure on \(M\) is by definition an SU(2)-reduction of \(\mathcal{L}(M)\). In terms of differential forms an SU(2)-structure may be characterized as follows
Proposition 1.1 \([17]\). SU(2)-structures on \(M\) are in one-to-one correspondence with quadruples \((\alpha, \omega_1, \omega_2, \omega_3)\), where \(\alpha\) is a nowhere vanishing 1-form, \(\omega_1, \omega_2, \omega_3 \in \Lambda^2 M\) satisfy
\[
\omega_i \wedge \omega_j = \delta_{ij} v \quad \text{for } i, j = 1, 2, 3
\]
for some 4-form \(v\) with \(v \wedge \alpha \neq 0\), and
\[
\iota_X \omega_1 = \iota_Y \omega_2 \implies \omega_3(X,Y) \geq 0.
\]

An SU(2)-structure \((\alpha, \omega_1, \omega_2, \omega_3)\) on \(M\) singles out a rank 4 distribution \(\xi = \ker \alpha \subset TM\). Note that for any \(r = 1, 2, 3\), the pair \((\xi, \omega_r)\) is a symplectic bundle over \(M\). Furthermore there exists a unique vector field \(R_\alpha\) on \(M\) satisfying
\[
\alpha(R_\alpha) = 1, \quad \iota_{R_\alpha} \omega_1 = 0.
\]

In analogy with the terminology used in contact geometry, we will refer to \(R_\alpha\) as the Reeb vector field associated to \((\alpha, \omega_1, \omega_2, \omega_3)\). Note that from the definition we also have
\[
\iota_{R_\alpha} \omega_2 = \iota_{R_\alpha} \omega_3 = 0.
\]

Definition 1.2. A differential form \(\gamma\) on \(M\) is said to be \(\alpha\)-transversal if it satisfies
\[
\iota_{R_\alpha} \gamma = 0.
\]
The set of \(\alpha\)-transversal \(p\)-forms on \(M\) is denoted by \(\Lambda^p_0 M\). Analogously \(S^p_0(M)\) will denote the set of \(\alpha\)-transversal symmetric \(p\)-tensors defined in the same way.

Remark 1.3. If we identify the vector bundle \(\xi^*\) dual to \(\xi\) with the subbundle of \(T^* M\) whose fibre over \(x\) is \(\{\phi \in T^*_x M \mid \phi(R_\alpha) = 0\}\), then \(\Lambda^p_0 M\) is identified with \(\Gamma(\Lambda^p_0 \xi^*)\).

We define the operators \(\star_r, r = 1, 2, 3\) on the transversal forms
\[
\star_r : \Lambda^1_0 M \to \Lambda^{4-r}_0 M
\]
by means of the relations
\[
\gamma \wedge \star_r \beta = \omega_r(\gamma, \beta) \frac{\omega_r^2}{2},
\]
for \(r = 1, 2, 3\), where the \(\omega_r\)'s are extended to exterior forms in the usual way.

Lemma 1.4. Let
\[
J_r : \Lambda^1 M \to \Lambda^1 M, \quad \text{for } r = 1, 2, 3,
\]
be the \(C^\infty(M)\)-linear endomorphisms defined by
\[
J_1(\phi) = \star_1(\omega_3 \wedge \star_1(\omega_2 \wedge \phi)),
J_2(\phi) = \star_2(\omega_1 \wedge \star_2(\omega_3 \wedge \phi)),
J_3(\phi) = \star_3(\omega_2 \wedge \star_3(\omega_1 \wedge \phi)),
\]
for any \(\phi \in \Lambda^1_0 M\) and by
\[
J_1(\alpha) = J_2(\alpha) = J_3(\alpha) = 0.
\]
Then for \(r = 1, 2, 3\) one has
\begin{itemize}
  \item \(J_r^2 = -I + R_\alpha \otimes \alpha\);
  \item \(\omega_r(J_r \beta, \gamma) = -\omega_r(\beta, J_r \gamma)\) for every 1-form \(\beta, \gamma\).
\end{itemize}

Proof. The statement is a consequence of the real version of Schur’s lemma (the proof is analogous to that of Proposition 2.1 of [2]).
Every $J_r$ induces an endomorphism of $TM$ (we denote it with the same symbol) in the following way

1. if $X$ is a smooth section of $\xi$, then we set $J_r(X) := -\tilde{\iota}_r^{-1}J_r(\tilde{\iota}_r X)$, where $\tilde{\iota}_r : \xi \to \xi^*$ is the duality on $\xi$ induced by $\omega_r$,
2. if $X = R_\alpha$ we set $J_r(X) = 0$.

In this way each $J_r$ is an $\omega$-compatible bundle complex structure on $\xi$.

Note that from the definition one easily obtains the quaternionic identities satisfied by $J_r \in \text{End}(TM)$:

\[ J_r J_s = -J_s J_r, \quad \text{for } r, s = 1, 2, 3, \; r \neq s \]

and

\[ J_1 J_2 = J_3. \]

At the dual level the $J_r$’s anticommute, but the composition satisfies $J_1 J_2 = -J_3$.

Furthermore we fix on $M$ the Riemannian metric $g$ defined by

\[ g = g^T + \alpha \otimes \alpha, \]

where

\[ g^T(X, Y) = \omega_1(X, J_1 Y) = \omega_2(X, J_2 Y) = \omega_3(X, J_3 Y). \]

Note that for any $X, Y \in \Gamma(\xi)$ we have

\[ g^T(J_1 X, J_1 Y) = g^T(J_2 X, J_2 Y) = g^T(J_3 X, J_3 Y) = g^T(X, Y). \]

Another direct consequence is that

\[ g(J_r X, J_r Y) = g(X, Y) - \alpha(X)\alpha(Y) \quad \text{for } r = 1, 2, 3. \]

The metric $g$ together with the orientation defined by $\alpha \wedge \omega^2_1$ induces the Hodge star operator in the usual way. Finally we denote by $*^T$ the transverse Hodge operator acting on the transverse forms so that

\[ \eta \wedge *^T \nu = g^T(\eta, \nu) \frac{\omega^2_1}{2}. \]

Note that

\[ *^T \omega_r = \omega_r \quad \text{for } r = 1, 2, 3 \]

and that for any transverse $p$-form $\gamma$ we have

\[ *^T \gamma = *(\alpha \wedge \gamma). \]

1.1. The standard model. Let $e^1, \ldots, e^5$ be the coframe dual to the canonical basis of $\mathbb{R}^5$. Then

\[ \begin{align*}
\alpha &= e^5, \\
\omega_1 &= e^{12} + e^{34}, \\
\omega_2 &= e^{13} - e^{24}, \\
\omega_3 &= e^{14} + e^{23}.
\end{align*} \]

(1.1)

define a linear SU(2)-structure on $\mathbb{R}^5$. In fact, given any linear SU(2)-structure on a vector space $V$, we can find a basis of $V^*$ with respect to which the structure forms take the standard form (1.1) (see [2]). Therefore it is useful to introduce the following notation:

\[ \omega_r = \frac{1}{2} e^i_{ij} e^i \wedge e^j. \]
The endomorphisms $J_1, J_2, J_3$ induced by the standard structure act on the canonical basis $e_1, \ldots, e_5$ as follows:

$$
\begin{align*}
J_1(e_1) &= e_2 & J_2(e_1) &= e_3 & J_3(e_1) &= e_4 \\
J_1(e_3) &= e_4 & J_2(e_4) &= e_2 & J_3(e_2) &= e_3.
\end{align*}
$$

Using this standard model one can easily check that, given an SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-dimensional manifold $M$,

$$(1.2) \quad \star_r \phi = \phi \wedge \omega_r,$$

for any $r = 1, 2, 3$ and transverse 1-form $\phi$ on $M$.

1.2. Decomposition of the Lie algebra $\mathfrak{so}(5)$. We use the $\epsilon$-notation introduced above to obtain the decomposition of the Lie algebra $\mathfrak{so}(5)$ of skew-symmetric $5 \times 5$ matrices in irreducible SU(2)-modules. Indeed

$$(1.3) \quad \mathfrak{so}(5) \simeq \mathfrak{su}(2) \oplus [R^4] \oplus [R]_1 \oplus [R]_2 \oplus [R]_3,$$

where a matrix $A = (a_{ij})$ lies in $\mathfrak{su}(2)$ if and only if

$$
\begin{align*}
\epsilon_{ij} a_{ij} &= 0 \\
a_{i5} &= a_{5i} = 0;
\end{align*}
$$

for every $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & v_1 \\
0 & 0 & 0 & 0 & v_2 \\
0 & 0 & 0 & 0 & v_3 \\
0 & 0 & 0 & 0 & v_4 \\
-v_1 & -v_2 & -v_3 & -v_4 & 0
\end{pmatrix}
$$

and for any $t \in \mathbb{R}$

$$(\{\tau\}_{r})_{ij} = t \epsilon_{ij}.$$

Note that we can alternatively write in compact form

$$
\{v\}_{ij} = \eta_{ijk} v_k,
$$

using the $\eta$-symbol

$$(1.4) \quad \eta_{ijk} = \delta_{ik} \delta_{j5} - \delta_{jk} \delta_{i5} \quad \text{for } i, j = 1 \ldots, 5, \ k = 1, \ldots, 4$$

we will need later.

2. Intrinsic torsion and special SU(2)-structures

Since the natural action of SU(2) on $\Lambda^p(\mathbb{R}^5)^*$ for every $p$, once an SU(2)-structure on a 5-manifold $M$ is fixed, we have a natural splitting of the space of forms of each degree. More precisely we have the following decomposition in irreducible SU(2)-modules:

$$
\begin{align*}
\Lambda^1 M &= (\alpha) \oplus \Lambda^1_0 M, \\
\Lambda^2 M &= \alpha \wedge \Lambda^1_0 M \oplus_{r=1}^3 (\omega_r) \oplus \Lambda^2_3 M, \\
\Lambda^3 M &= \Lambda^3_0 M \oplus_{r=1}^3 (\alpha \wedge \omega_r) \oplus \alpha \wedge \Lambda^2_3 M, \\
\Lambda^2_3 M &= \{\sigma \in \Lambda^2_0 M \mid \sigma \wedge \omega_r = 0 \text{ for } r = 1, 2, 3\}.
\end{align*}
$$
The previous decomposition allows us to define also a projection
\[ E : \Lambda^2 M \to \Lambda^2_0 M \]
explicitly defined by
\[ (2.1) \quad E(\phi) = \phi^T - \frac{1}{2} \sum_{r=1}^{3} (\phi^T \land \omega_r \land \alpha) \omega_r , \]
where \( \phi^T \) denotes the projection of \( \phi \) onto \( \Lambda^2_0 M \), i.e.
\[ \phi^T := \phi - \alpha \land \iota_{R\alpha} \phi . \]

\textbf{Remark 2.1.} Since \( E \) is the projection on the \(-1\) eigenspace of the diagonalizable operator \( *T \), the operator \( E \) restricted to \( \Lambda^2_0 M \) and \( *T \) commute, i.e.
\[ E(*T \beta) = *T E(\beta) \]
for every \( \beta \in \Lambda^2_0 M \). Moreover, if \( \psi \) is an arbitrary 3-form on \( M \), then we immediately have
\[ (2.2) \quad E(*\psi) = *T E(\iota_{R\alpha} \psi) . \]

\textbf{Remark 2.2.} Note that the elements of \( \Lambda^2_3 M \) are the sections of a subbundle of \( \Lambda^2(T^*M) \) isomorphic to the bundle associated to the SU(2)-reduction \( Q \) with respect to the adjoint representation of SU(2).

In the sequel we will use the following

\textbf{Proposition 2.3.} Let \( \sigma \in \Lambda^2_3 M \), then
1. \( *\sigma = -\sigma \land \alpha \),
2. \( J_r(\sigma) = \sigma \) for \( r = 1, 2, 3 \).

\textbf{Proof.} Since any element of the Lie algebra \( \mathfrak{su}(2) \) is SU(2)-conjugated to an element of a fixed Cartan subalgebra, remark 2.2 implies that for any \( x \in M \) there exists an SU(2)-local frame \( e_1, \ldots, e_5 \) near \( x \), such that
\[ \sigma = e^{12} - e^{34} \]
and the claim follows. \( \square \)

According to the decomposition of the exterior algebra the derivatives of the structure forms split as
\[ d\omega_r = \nu_r \land \omega_r + \sum_{j=1}^{3} f_{rj} \alpha \land \omega_j + \alpha \land \sigma_r , \]
\[ d\alpha = \alpha \land \nu_4 + \sum_{i=1}^{3} \phi_i \omega_i + \sigma_4 , \]
where \( \nu_i \in \Lambda^1_0 M, \sigma_i \in \Lambda^2_3 M \), for \( i = 1, \ldots, 4 \) and \( \phi_i, f_{ij} \) are smooth functions. Imposing \( d^2 = 0 \) one has
\[ f_{11} = f_{22} = f_{33} , \]
\[ f_{ij} = -f_{ji} \text{ for } i \neq j . \]
We will refer to \( \{\nu_i, \sigma_j, \phi_r, f_{uv}\} \) as the \textit{torsion forms} of the SU(2)-structure.
2.1. **Decomposition of symmetric 2-tensors.** In order to write the Ricci tensor of a 5-dimensional SU(2)-manifold in terms of its torsion forms, we must decompose the space of symmetric 2-tensors on $M$ in irreducible SU(2)-modules. We have

$$S^2(M) = \langle g^T \rangle \oplus \langle \alpha \otimes \alpha \rangle \oplus \bigoplus_{i=1}^3 \Sigma_i(M) \oplus (\alpha \otimes \Lambda^1_0 M).$$

where

$$\Sigma_1(M) = \{ h \in S^2_0(M) \mid J_1(h) = h, J_2(h) = J_3(h) = -h \},$$

$$\Sigma_2(M) = \{ h \in S^2_0(M) \mid J_2(h) = h, J_1(h) = J_3(h) = -h \},$$

$$\Sigma_3(M) = \{ h \in S^2_0(M) \mid J_3(h) = h, J_1(h) = J_2(h) = -h \}.$$

Let

$$\iota_r: \Sigma_r(M) \to \Lambda^3_3 M$$

be defined by

$$\iota_r(h_{lm} e^l \otimes e^m) = \frac{1}{2} \epsilon_{ik} h_{kj} e^i \wedge e^j.$$

It is immediate to verify that every $\iota_r$ is an isomorphism of SU(2)-representations.

2.2. **The almost Kähler cone and special SU(2)-structures.** In order to consider some interesting kind of SU(2)-structure on 5-manifolds, we first take the more general point of view of $U(n)$-structures on $(2n+1)$-manifolds. A $U(n)$-structure on a $(2n+1)$-dimensional manifold $M$ is determined by a triple $(\alpha, J, \omega)$, where $\alpha$ is a nowhere vanishing 1-form on $M$, $\omega$ is a 2-form such that $\alpha \wedge \omega \neq 0$, and $J \in \text{End}(TM)$ is such that

$$J^2 = -I + \alpha \otimes R_\alpha,$$

where $R_\alpha$ is the Reeb vector field (i.e. $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} \omega = 0$). Any $U(n)$-structure on $M$ induces a $U(n+1)$-structure on the cone $C(M) = M \times \mathbb{R}_t^+$ specified by

$$\kappa = t^2 \omega + t \alpha \wedge dt$$

and the $\kappa$-compatible almost complex structure $\tilde{J}$ defined by

$$\tilde{J}X = \begin{cases} JX & \text{if } X \in \Gamma(\ker \alpha) \\ -t \frac{\partial}{\partial t} & \text{if } X = R_\alpha. \end{cases}$$

Note that the 2-form $\kappa$ is closed (and hence symplectic) if and only if $\alpha$ and $\omega$ are related by

$$d\alpha = -2\omega.$$

In this case $\alpha$ is a contact form on $M$ and $(\kappa, \tilde{J})$ is an almost-Kähler structure on $C(M)$. A $U(n)$-structure is said to be Sasakian (Sasaki-Einstein) if $(C(M), \kappa, \tilde{J})$ is a Kähler (Calabi-Yau) manifold.

Now let us come back to the case of SU(2)-structures. First remark that an SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-dimensional manifold $M$ induces an SU(3)-structure on the cone $C(M)$. In fact, once a $U(3)$-structure $(\kappa, J)$ on a 6-dimensional manifold $N$ is given, in order to specify an SU(3)-structure it is sufficient to give a complex volume form $\varepsilon \in \Lambda^3_{3,0} N$ satisfying

$$\varepsilon \wedge \overline{\varepsilon} = -i \frac{4}{3} \kappa^3.$$
Hence on the cone $C(M)$ we may take
\[ \varepsilon = t^2(\omega_2 + i\omega_3) \wedge (t\alpha + i\alpha) . \]
This SU(3)-structure is integrable if and only if
\[
\begin{cases}
  d\alpha = -2\omega_1 \\
  d\omega_2 = 3\alpha \wedge \omega_3 \\
  d\omega_3 = -3\alpha \wedge \omega_2 ,
\end{cases}
\]
see e.g. [7].
Here we list some special SU(2)-structures which have been studied in the last years.

- **Hypo manifolds**: An SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ is said to be a Hypo-structure if the structure forms satisfy
  \[ d\omega_1 = 0 , \quad d(\alpha \wedge \omega_2) = 0 , \quad d(\alpha \wedge \omega_3) = 0 . \]
In terms of intrinsic torsion the Hypo condition reads as
\[
\begin{align*}
  \nu_1 &= 0 , \\
  f_{1i} &= 0 , \quad i = 1, 2, 3 , \\
  \sigma_1 &= 0 , \\
  \nu_2 &= \nu_3 = \nu_4 , \quad \phi_2 = \phi_3 = 0
\end{align*}
\]
and the other torsion forms are arbitrary. Hypo structures were first investigated by Conti and Salamon in [7]. The name is due to the fact that a real hypersurface of a Calabi-Yau 3-fold inherits a Hypo structure.

- **Contact-Hypo manifolds**: A Hypo structure is called contact Hypo if further the 1-form $\alpha$ is a contact form so that the SU(3)-structure on the cone $C(M)$ is actually almost Kähler, i.e.
  \[ d\alpha = -2\omega_1 , \quad d(\alpha \wedge \omega_2) = 0 , \quad d(\alpha \wedge \omega_3) = 0 . \]
This special SU(2)-structures are the subject of the recent paper [9]. In terms of torsion forms we have
\[
\begin{align*}
  \nu_1 &= 0 , \quad i = 1, 2, 3, 4 , \\
  f_{1i} &= 0 , \quad i = 1, 2, 3 , \\
  \sigma_1 &= \sigma_4 = 0 , \\
  \phi_1 &= -2 , \quad \phi_2 = \phi_3 = 0
\end{align*}
\]
(2.6)

- **Nearly Hypo manifolds**: These manifolds have been introduced in [11]. In this case the structure equations are
  \[ d\omega_2 = 3\alpha \wedge \omega_3 , \quad d(\alpha \wedge \omega_1) = -2\omega_1^2 \]
which in terms of torsion forms are
\[
\begin{align*}
  \phi_1 &= -2 , \quad \phi_3 = 0 , \quad \sigma_2 = 0 , \\
  \nu_4 &= \nu_3 = \nu_1 , \quad \nu_2 = 0 , \\
  f_{23} &= 3 , \quad f_{12} = f_{11} = 0 ,
\end{align*}
\]
the remaining torsion forms being arbitrary. Such a structure is inherited by any hypersurface of a nearly-Kähler SU(3)-manifold.

- **Double Hypo manifolds**: These manifolds have been introduced in [11], too. A double Hypo structure is an SU(2)-structure which is both Hypo and nearly Hypo. This kind of structures are characterized by the following equations
  \[ d\omega_1 = 0 , \quad d(\alpha \wedge \omega_2) = 0 , \quad d(\alpha \wedge \omega_1) = -2\omega_1^2 , \]
\[ d\omega_2 = 3\alpha \wedge \omega_3. \]

In this case the only non-vanishing torsion forms are \( \phi_1, f_{23}, \sigma_3, \sigma_4 \), where \( \phi_1 = -2, \quad f_{23} = 3, \quad \) and \( \sigma_3, \sigma_4 \) are arbitrary.

- **Sasaki-Einstein manifolds**: A Sasakian manifold is said to be Sasaki-Einstein if the induced Riemann metric is Einstein. In dimension 5 a Sasakian structure induces an SU(2)-structure \((\alpha, \omega_1, \omega_2, \omega_3)\) satisfying
  \[ d\alpha = -2\omega_1, \quad d\omega_2 = 3\alpha \wedge \omega_3, \quad d\omega_3 = -3\alpha \wedge \omega_2. \]

In terms of torsion forms these conditions read as
  \[ \phi_1 = -2, \quad f_{23} = 3 \]

and the other torsion forms vanish.

2.3. **Symplectic half-flat structures.** Let \( N \) be a 6-dimensional manifold. Any U(3)-structure \((\kappa, J)\) on \( N \) induces a natural connection \( \tilde{\nabla} \), called the Hermitian connection, uniquely determined by the following equations
  \[ \tilde{\nabla} J = 0, \quad \tilde{\nabla} \kappa = 0, \quad (T_{\tilde{\nabla}})^{1,1} = 0 \]

where \( (T_{\tilde{\nabla}})^{1,1} \) is the \((1,1)\)-part of the torsion of \( \tilde{\nabla} \). It turns out that the holonomy group of this connection is contained in SU(3) if and only if there exists \( \varepsilon \in \Lambda^3_{1,0} N \) satisfying
  \[
  \begin{cases}
  \varepsilon \wedge \overline{\varepsilon} = -i \frac{4}{3} \kappa^3 \\
  \mathcal{O}_J \varepsilon = 0
  \end{cases}
  \]

(see e.g. [10]). In this case we call \((N, \kappa, J, \varepsilon)\) a symplectic Calabi-Yau manifold ([10], [2]). Requiring further that the real part of \( \varepsilon \) is closed, we obtain an interesting subclass of manifolds lying in the intersection between symplectic and half-flat geometry, indeed they are called symplectic half-flat manifolds in [8].

Let us consider now a 5-dimensional SU(2)-manifold \((M, \alpha, \omega_1, \omega_2, \omega_3)\). We have the following

**Lemma 2.4.** Let \((\kappa, \tilde{J}, \varepsilon)\) be the SU(3)-structure on the cone \( C(M) \) associated to \((\alpha, \omega_1, \omega_2, \omega_3)\). Then \((\kappa, \tilde{J}, \varepsilon)\) is symplectic half-flat if and only if \((\alpha, \omega_1, \omega_2, \omega_3)\) is contact-Hypo with \( f_{23} = 3 \) and \( \sigma_3 = 0 \).

**Proof.** As already observed, the 2-form \( \kappa \) defined by (2.5) is closed if and only if \( d\alpha = -2\omega_1 \). This implies \( d\omega_1 = 0 \). In terms of torsion forms:

\[
\begin{array}{c}
\phi_1 = -2 \\
\phi_2 = \phi_3 = 0 \\
f_{1r} = 0 \\
\nu_1 = \nu_4 = 0 \\
\sigma_1 = \sigma_4 = 0. \\
\end{array}
\]

Now

\[
d\text{Re} \varepsilon = -3t^2 \omega_2 \wedge \alpha \wedge dt + t^3 \omega_2 \wedge \alpha - t^2 \omega_3 \wedge dt
\]

\[
= -3t^2 \omega_2 \wedge \alpha \wedge dt + t^3 \nu_2 \wedge \omega_2 \wedge \alpha - t^2 (\nu_3 \wedge \omega_3 + \sum_{r=1}^3 f_{3r} \omega_r \wedge \alpha + \sigma_3 \wedge \alpha) \wedge dt.
\]

\[\text{In [10] and [2] such structures are named Generalized Calabi-Yau, but this terminology is misleading because it is widely used with a different meaning, see [14].}\]
Therefore $d \Re \varepsilon = 0$ if and only if one has the extra-conditions
\[ f_{23} = 3, \quad \nu_2 = \nu_3 = 0, \quad \sigma_3 = 0. \]

\[ \square \]

**Remark 2.5.** If $(\alpha, \omega_1, \omega_2, \omega_3)$ is a SU(2)-structure on $M$ inducing a symplectic half-flat structure on $C(M)$, then defining
\[ \tilde{\alpha} = \alpha, \quad \tilde{\omega}_1 = \omega_1, \quad \tilde{\omega}_2 = -\omega_3, \quad \tilde{\omega}_3 = \omega_2, \]
we obtain a double-hypo structure on $M$. The remarkable fact is that the two structures induce the same metric.

3. **Ricci curvature of an SU(2)-structure**

Fix an SU(2)-reduction $Q$ of the linear frame bundle $\mathcal{L}(M)$, given by the quadruple $(\alpha, \omega_1, \omega_2, \omega_3)$. $Q$ can be viewed as a subbundle of the principal SO(5)-
bundle $p: \mathcal{F} \to M$ of the normal frames of the metric $g$ associated to the triple
$(\alpha, \omega_1, \omega_2, \omega_3)$. Consider on the bundle $\mathcal{F}$ the tautological $\mathbb{R}^5$-valued 1-form $w$ defined by $w[u](v) = u(p_u[v])$ for every $u \in \mathcal{F}$ and $v \in T_u\mathcal{F}$. On $\mathcal{F}$ we have also the\nLevi-Civita connection 1-form $\psi$ taking values in $\mathfrak{so}(5)$. Using the canonical basis\n$\{e_1, \ldots, e_5\}$ of $\mathbb{R}^5$ we will regard $w$ as a vector of $\mathbb{R}$-valued 1-forms on $\mathcal{F}$\n\[ w = w_1 e_1 + \cdots + w_5 e_5 \]
and $\psi$ as a skew-symmetric matrix of 1-forms, i.e. $\psi = (\psi_{ij})$. With this notation\nthe first structure equation relating $w$ and $\psi$\n\[ (3.1) \quad dw = -\psi \wedge w, \]
becomes $dw_i = -\psi_{ij} \wedge w_j$. Note that equation (3.1) simply means that $\psi$ is torsion-free.\nThe curvature of $\psi$ is by definition the $\mathfrak{so}(5)$-valued 2-form $\Psi = d\psi + \psi \wedge \psi$. In\nindex notation\n\[ \Psi_{ij} = d\psi_{ij} + \psi_{ik} \wedge \psi_{kj} = \frac{1}{2} R_{ijkl} w_k \wedge w_l. \]
We consider the pull-backs of $\psi$ and $w$ to $Q$ and denote them by the same symbols for\nthe sake of brevity. The intrinsic torsion of the SU(2)-structure measures the failure of $\psi$ to take values in $\mathfrak{su}(2)$. More precisely according to the splitting\n\[ \mathfrak{so}(5) = \mathfrak{su}(2) \oplus [\mathbb{R}^4] \oplus [\mathbb{R}]_1 \oplus [\mathbb{R}]_2 \oplus [\mathbb{R}]_3. \]
obtained above, $\psi$ decomposes as\n\[ \psi = \theta + [r] + [\mu_1]_1 + [\mu_2]_2 + [\mu_3]_3. \]
Thus $\theta$ is a connection 1-form on $Q$ which in general is not torsion-free. We shall\nregard $\tau$ as a 4-vector of 1-forms $\tau = \tau_i e_i$. Furthermore we can write\n\[ \tau_i = T_{ij} w_j, \quad \mu_r = M_r^i w_j \]
for $i = 1, 2, 3, 4$ and $r = 1, 2, 3$, where $T_{ij}$ and $M_r^i$ are smooth functions. Formula\n(3.1) now read as\n\[ dw_i = -\theta_{ij} \wedge w_j - \eta_{ijk} \tau_k \wedge w_j - \epsilon_{ij}^{1} \mu_1 \wedge w_j - \epsilon_{ij}^{2} \mu_2 \wedge w_j - \epsilon_{ij}^{3} \mu_3 \wedge w_j, \]
where the $\eta_{ijk}$’s are defined by (1.4).\nNow we have\n
**Lemma 3.1.** The following identities hold:
1. $[\mu_r]_r \wedge [\tau] + [\tau] \wedge [\mu_r]_r = [[\mu_r]_r \wedge \tau]$ for $r = 1, 2, 3$;
2. $[\tau] \wedge \theta + \theta \wedge [\tau] = [[\theta] \wedge \tau]$,
where in the expressions $[[\mu_r]_r \wedge \tau]$ and $\theta \wedge \tau$, $\tau$ is regarded as the $\mathbb{R}^5$-valued 1-form $\tau = (\tau_1, \ldots, \tau_4, 0)$.

We are ready to introduce the following quantities

$$D\theta = d\theta + \theta \wedge \theta + [\tau] \wedge [\tau] + \frac{1}{4} \sum_{r=1}^{3} [\epsilon^r_{ij} \tau_1 \wedge \tau_2]_r ,$$

$$D\tau = d\tau + \theta \wedge \tau + \sum_{r=1}^{3} [\mu_r]_r \wedge \tau ,$$

$$D\mu_1 = d\mu_1 - \frac{1}{4} \epsilon^1_{ij} \tau_1 \wedge \tau_2 - 2\mu_2 \wedge \mu_3 ,$$
$$D\mu_2 = d\mu_2 - \frac{1}{4} \epsilon^2_{ij} \tau_1 \wedge \tau_2 - 2\mu_3 \wedge \mu_1 ,$$
$$D\mu_3 = d\mu_3 - \frac{1}{4} \epsilon^3_{ij} \tau_1 \wedge \tau_2 - 2\mu_1 \wedge \mu_2 .$$

A direct computation gives that $D\theta$ takes values in $\text{su}(2)$; moreover lemma 3.1 implies

$$\Psi = d(\theta + [\tau] + [\mu_1]_1 + [\mu_2]_2 + [\mu_3]_3) + (\theta + [\tau] + [\mu_1]_1 + [\mu_2]_2 + [\mu_3]_3) \wedge (\theta + [\tau] + [\mu_1]_1 + [\mu_2]_2 + [\mu_3]_3)$$
$$= D\theta + [D\tau] + [D\mu_1]_1 + [D\mu_2]_2 + [D\mu_3]_3 .$$

In terms of the $w$-frame we shall write

$$D\theta_{ij} = \frac{1}{2} S_{ijkl} w_k \wedge w_l ,$$
$$D\tau_i = \frac{1}{2} T_{ijk} w_j \wedge w_k ,$$
$$D\mu_r = \frac{1}{2} N^r_{kl} w_k \wedge w_l ,$$

where the coefficients are smooth functions such that

$$S_{ijkl} = -S_{jikl} = -S_{ijtk} ,$$
$$T_{ijk} = -T_{ikj} ,$$
$$N^r_{kl} = -N^r_{lk} .$$

In terms of the functions just introduced, the components of the curvature tensor express as

$$R_{ijkl} = S_{ijkl} + \eta_{ijk} T_{hkl} + \epsilon^1_{ij} N^1_{kl} + \epsilon^2_{ij} N^2_{kl} + \epsilon^3_{ij} N^3_{kl} ,$$

where the $\eta_{ijk}$’s are the symbols defined in (1.4). Let $\text{Ric}_{ij} = R_{ikkj}$ be the components of the Ricci tensor of $g$. As an application of the Bianchi identities we have

the following theorem which gives a formula for the Ricci tensor and the scalar curvature $s = \text{Ric}_{ii}$ of $g$ in terms of intrinsic torsion.
Theorem 3.2. The Ricci tensor does not depend on the functions $S_{ijkl}$ and each component writes as

$$\text{Ric}_{ij} = \sum_{r=1}^{3} \left\{ \epsilon_{ik}^r N_{jk}^r + \epsilon_{jk}^r N_{ik}^r - \eta_{ijl} \epsilon_{lk}^r N_{rk}^r \right\} + \delta_{i5} \delta_{j5} T_{kk5} + T_{ij5}.$$ 

Consequently,

$$s = 2 \sum_{r=1}^{3} (\epsilon_{ik}^r N_{jk}^r) + 2T_{kk5}.$$ 

3.0.1. The scalar curvature in terms of torsion forms. Pulling back the structure forms to the SU(2)-bundle $\pi: Q \to M$, and using the frame $w_1, \ldots, w_5$, one gets the standard expression for $\alpha, \omega_1, \omega_2, \omega_3$:

$$\pi^*(\alpha) = w_5, \quad \pi^*(\omega_r) = \frac{1}{2} \epsilon_{ij}^r w_i \wedge w_j \text{ for } r = 1, 2, 3.$$ 

Applying the symmetries of the $\epsilon$-symbol, we have

**Proposition 3.3.** The derivatives of the structure forms are

$$d\pi^*(\alpha) = \tau_k \wedge w_k,$$

$$d\pi^*(\omega_1) = \epsilon_{ij}^1 \tau_i \wedge w_j \wedge w_5 - \epsilon_{ij}^2 \mu_3 \wedge w_i \wedge w_j + \epsilon_{ij}^3 \mu_2 \wedge w_i \wedge w_j,$$

$$d\pi^*(\omega_2) = \epsilon_{ij}^2 \tau_i \wedge w_j \wedge w_5 - \epsilon_{ij}^3 \mu_1 \wedge w_i \wedge w_j + \epsilon_{ij}^1 \mu_3 \wedge w_i \wedge w_j,$$

$$d\pi^*(\omega_3) = \epsilon_{ij}^3 \tau_i \wedge w_j \wedge w_5 - \epsilon_{ij}^1 \mu_2 \wedge w_i \wedge w_j + \epsilon_{ij}^2 \mu_1 \wedge w_i \wedge w_j.$$

Proposition 3.3 allows to write down the pull-backs of the torsion forms in terms of $T_{ij}, M_r^i$. A direct computation gives the following formulae

$$\pi^*(f_{11}) = \frac{1}{2} T_{ii},$$

$$\pi^*(f_{12}) = \frac{1}{2} \epsilon_{ij}^3 T_{ij} - 2M_5^3,$$

$$\pi^*(f_{13}) = -\frac{1}{2} \epsilon_{ij}^2 T_{ij} + 2M_5^2,$$

$$\pi^*(f_{23}) = \frac{1}{2} \epsilon_{ij}^1 T_{ij} - 2M_5^1,$$

$$\pi^*(\phi_1) = -\frac{1}{2} \epsilon_{ij}^1 T_{ij},$$

$$\pi^*(\phi_2) = -\frac{1}{2} \epsilon_{ij}^2 T_{ij},$$

$$\pi^*(\phi_3) = -\frac{1}{2} \epsilon_{ij}^3 T_{ij},$$

$$\pi^*(\nu_1) = (2\epsilon_{ij}^1 M_1^i + 2\epsilon_{ij}^3 M_3^i) w_j,$$

$$\pi^*(\nu_2) = (2\epsilon_{ij}^1 M_1^i + 2\epsilon_{ij}^3 M_3^i) w_j,$$

$$\pi^*(\nu_3) = (2\epsilon_{ij}^1 M_1^i + 2\epsilon_{ij}^3 M_3^i) w_j,$$

$$\pi^*(\nu_4) = T_{i5} w_i.$$
\[ \pi^*(\sigma_1) = \frac{1}{4}(\epsilon_{ip}(T_{pj} + T_{jp}) + \epsilon_{pq}^3(T_{pq} + T_{qp})) w_i \wedge w_j, \]
\[ \pi^*(\sigma_2) = \frac{1}{4}(\epsilon_{ip}(T_{pj} + T_{jp}) - \epsilon_{ip}^1\epsilon_{pq}^3(T_{pq} + T_{qp})) w_i \wedge w_j, \]
\[ \pi^*(\sigma_3) = \frac{1}{4}(\epsilon_{ip}(T_{pj} + T_{jp}) + \epsilon_{ip}^2\epsilon_{pq}^3(T_{pq} + T_{qp})) w_i \wedge w_j, \]
\[ \pi^*(\sigma_4) = (T_{ji} + \frac{1}{2}\epsilon_{pq}^r\epsilon_i^jT_{pq}) w_i \wedge w_j + T_{15} w_i \wedge w_5. \]

**Warning:** From now on we identify the structure and torsion forms with their pull-backs to the principal SU(2)-bundle \( Q \).

Combining these formulae with (3.2) we get the following

**Theorem 3.4.** The scalar curvature of the Riemannian metric induced by an SU(2)-structure with torsion \((f_{ij}, \phi_i, \nu_i, \sigma_i)\) on a 5-manifold is

\[
s = -5f_{11}^2 - \sum_{i=1}^{3} \phi_i^2 - 4\phi_1f_{23} + 4\phi_2f_{13} - 4\phi_3f_{12} + \sum_{i=1}^{3} d^*\nu_i - 2d^*\nu_4 - \sum_{i=1}^{3} \left( \frac{1}{2} |\nu_i|^2 - \frac{1}{2} |\sigma_i|^2 \right) + \langle \nu_1, \nu_2 \rangle + \langle \nu_1, \nu_3 \rangle - \langle \nu_1, \nu_4 \rangle + \langle \nu_2, \nu_3 \rangle - \langle \nu_2, \nu_4 \rangle - \langle \nu_3, \nu_4 \rangle - 2 \star (df_{11} \wedge \omega_i^2) - \sum_{i=1}^{4} \frac{1}{2} |\sigma_i|^2.
\]

As a direct consequence of the previous theorem we have the following characterization of the scalar curvature of some special structures:

- **Hypo manifolds:** \( s = -\phi_1^2 - 4\phi_1f_{23} - 2|\nu_1|^2 - \frac{1}{2} \sum_{i=2}^{4} |\sigma_i|^2 \);
- **Contact-Hypo manifolds:** \( s = -4 + 8f_{23} - \frac{1}{4} |\sigma_2|^2 - \frac{1}{2} |\sigma_3|^2 \);
- **Double Hypo manifolds:** \( s = 20 - \frac{1}{2} |\sigma_2|^2 - \frac{1}{2} |\sigma_4|^2 \);
- **Sasaki-Einstein manifolds:** \( s = 20 \).

Hence we have

**Corollary 3.5.** The scalar curvature of the metric induced by a double-Hypo structure is always less or equal to 20. Furthermore it is equal to 20 if and only if the double-Hypo structure is Sasaki-Einstein.

### 3.0.2. The Ricci curvature in terms of torsion forms.

According to the splitting (1.3) of symmetric 2-tensors, the Ricci curvature of a metric \( g \) associated to a SU(2)-structure on a 5-manifold decompose as follows

\[
\text{Ric} = \frac{\lambda}{4} g^T + \mu \alpha \otimes \alpha + R_{0}.
\]

We recall that the metric \( g \) is said to be \( \alpha \)-Einstein if

\[ R_{0} = 0. \]

(see e.g [4]).

From the decomposition of the Ricci tensor (3.2), the scalar curvature splits as

\[ s = \lambda + \mu. \]
A straightforward computation gives the following formulae which express $\lambda$ and $\mu$ in terms of torsion forms:

$$
\lambda = -4f^2_{11} - 2 \sum_{i=1}^{3} \phi_i^2 - 4\phi_1 f_{23} + 4\phi_2 f_{13} - 4\phi_3 f_{12} + \sum_{i=1}^{3} d^* \nu_i - d^* \nu_4 - \frac{1}{2} \sum_{i=1}^{3} |\nu_i|^2 \\
+ \langle \nu_1, \nu_2 \rangle - \langle \nu_1, \nu_3 \rangle - \langle \nu_1, \nu_4 \rangle + \langle \nu_2, \nu_3 \rangle - \langle \nu_2, \nu_4 \rangle - \langle \nu_3, \nu_4 \rangle - |\sigma_4|^2 \\
- \ast (df_{11} \wedge \omega_1^2)
$$

and

$$
\mu = -f^2_{11} + \sum_{i=1}^{3} \phi_i^2 - d^* \nu_4 - \frac{1}{2} \sum_{i=1}^{3} |\sigma_1|^2 + \frac{1}{2} |\sigma_4|^2 - \ast (df_{11} \wedge \omega_1^2) .
$$

As a consequence of these formulae we get the following

**Proposition 3.6.** Let $(M, \alpha, \omega_1, \omega_2, \omega_3)$ be a contact-Hypo manifold. Assume that the Ricci tensor of the metric induced by the Hypo-structure satisfies

$$
\text{Ric}(R_{\alpha}, R_{\alpha}) = 4 ;
$$

then $(M, \alpha, \omega_1, \omega_2, \omega_3)$ is Sasaki $\alpha$-Einstein.

**Proof.** For a Sasaki SU(2)-structure to be $\alpha$-Einstein is equivalent to be Hypo (see Theorem 14 in [7]), so we only need to prove that $(M, \alpha, \omega_1, \omega_2, \omega_3)$ is Sasaki. By equations (2.6), in the contact-Hypo case formula (3.3) reduces to

$$
\mu = 4 - \frac{1}{2} |\sigma_2|^2 - \frac{1}{2} |\sigma_3|^2.
$$

Then condition $\text{Ric}(R_{\alpha}, R_{\alpha}) = 4$ readily implies $\sigma_2 = \sigma_3 = 0$. Furthermore we have

$$
0 = d^2 \omega_2 = df_{23} \wedge \omega_3 \wedge \alpha
$$

which implies that $df_{23} = h \alpha$ for some $h \in C^\infty(M, \mathbb{R})$. Moreover

$$
0 = d^2 f_{23} = dh \wedge \alpha + h d\alpha = dh \wedge \alpha - 2h \omega_1
$$

implies $h = 0$. Hence $f_{23}$ is a constant function on $M$.

Let

$$
\tilde{\varepsilon} = e^{(f_{23}-1) \log t} (\omega_2 + i \omega_3) \wedge (t \alpha + i dt),
$$

then $\tilde{\varepsilon}$ is a closed $(3,0)$-form on the almost Kähler cone $C(M) = M \times \mathbb{R}^+$. Thus $C(M)$ is Kähler (see e.g. [2], remark 1.1) and consequently $(M, \alpha, \omega_1, \omega_2, \omega_3)$ is Sasaki.

**Remark 3.7.** Note that an SU(2)-structure satisfying the hypotheses of the proposition above gives rise to an Einstein metric $g$ if and only if the scalar curvature $s = \lambda + \mu$ is exactly 20. Indeed, $g$ is Einstein if and only if $\mu = \frac{4}{3}$ and the hypothesis $\text{Ric}(R_{\alpha}, R_{\alpha}) = 4$ means $\mu = 4$.

The main theorem is obtained using the following algorithm, analogous to the one used by Bryant in [6]:

- introduce the symbols $S_{ijk}$, $V^r_{ij}$ in the expressions of the derivatives of the $T_{ij}$ and $M^r_i$:

  $$
  dT_{ij} = T_{ik}\theta_{kj} + T_{kj}\theta_{ki} + S_{ijk}w_k, \\
  dM^r_i = M^r_k\theta_{ki} + V^r_{ik}w_k.
  $$


The “traceless part” of the Ricci curvature of the Riemannian metric induced by an SU(2)-structure with torsion \((f_{ij}, \phi_i, \nu_i, \alpha_i)\) on a 5-manifold is

\[
\text{Ric}_0 = \iota_1^{-1}(E(\Phi_1)) + \iota_2^{-1}(E(\Phi_2)) + \iota_3^{-1}(E(\Phi_3)) + \Phi_4 \odot \alpha ,
\]
where

\[ \Phi_1 = -\frac{1}{4} f_{11}\sigma_1 + \frac{1}{4} f_{12}\sigma_2 + \frac{1}{4} f_{13}\sigma_3 - f_{23}\sigma_4 + \phi_3\sigma_2 - \phi_2\sigma_3 - \phi_1\sigma_4 - \frac{1}{2}\nu_1 \wedge J_1 \nu_1 \\
+ \frac{1}{2}\nu_1 \wedge J_1 \nu_2 + \frac{1}{2}\nu_2 \wedge J_1 \nu_2 - \frac{1}{2}\nu_2 \wedge J_1 \nu_3 + \frac{1}{2}\nu_3 \wedge J_1 \nu_3 - \frac{1}{2}\nu_2 \wedge J_1 \nu_3 \\
+ \frac{1}{2}\nu_4 \wedge J_1 \nu_4 + \frac{1}{2} \iota_{R_4} d\sigma_1 - \frac{1}{2} dJ_1 \nu_1 + \frac{1}{2} dJ_1 \nu_2 + \frac{1}{2} dJ_1 \nu_3; \]

\[ \Phi_2 = -\frac{1}{2} f_{12}\sigma_2 - \frac{1}{2} f_{11}\sigma_2 + \frac{1}{2} f_{23}\sigma_3 + f_{13}\sigma_4 - \phi_3\sigma_1 + \phi_1\sigma_3 - \phi_2\sigma_4 + \frac{1}{2} \ast d\sigma_2 \\
+ \frac{1}{2} dJ_2 \nu_1 - \frac{1}{2} dJ_2 \nu_2 - \frac{1}{2} dJ_2 \nu_4 + \frac{1}{2} dJ_2 \nu_3 + \frac{1}{2} \nu_1 \wedge J_2 \nu_1 - \frac{1}{2} \nu_1 \wedge J_2 \nu_3 \\
- \frac{1}{2} \nu_2 \wedge J_2 \nu_2 + \frac{1}{2} \nu_2 \wedge J_2 \nu_4 + \frac{1}{2} \nu_4 \wedge J_2 \nu_4 + \frac{1}{2} \nu_3 \wedge J_2 \nu_3; \]

\[ \Phi_3 = -\frac{1}{2} (f_{13}\sigma_1 + f_{23}\sigma_2 + f_{11}\sigma_3) + \phi_2\sigma_1 - \phi_1\sigma_2 - \phi_3\sigma_4 + \frac{1}{2} \ast d\sigma_3 - f_{12}\sigma_4 \\
+ \frac{1}{2} (\nu_1 \wedge J_3 \nu_1 + \nu_2 \wedge J_3 \nu_2 - \nu_3 \wedge J_3 \nu_3) - \frac{1}{2} \nu_1 \wedge J_3 \nu_2 + \frac{1}{2} \nu_3 \wedge J_3 \nu_4 \\
+ \frac{1}{2} \nu_4 \wedge J_3 \nu_4 + \frac{1}{2} (dJ_3 \nu_1 + dJ_3 \nu_2 - dJ_3 \nu_3 - dJ_3 \nu_4); \]

\[ \Phi_4 = 3 (df_{11})^T - \frac{3}{2} f_{11}\nu_4 - \frac{1}{2} (d^* \sigma_4)^T - \frac{1}{2} f_{23}J_1 \nu_4 - \frac{1}{2} f_{12}J_2 \nu_4 \\
+ \frac{3}{2} (\phi_1 J_1 \nu_4 + \phi_2 J_2 \nu_4 + \phi_3 J_3 \nu_4 - \frac{1}{2} \iota_{R_4} (d\nu_1 + d\nu_2 + d\nu_3) \\
+ \iota_{R_4} (d\nu_4 + \ast (df_{12} \wedge \omega_3) - \ast (df_{13} \wedge \omega_2) + \ast (df_{23} \wedge \omega_1) + \frac{1}{2} \ast d\sigma_4) \\
- J_1 \iota_{R_4} (\ast d\sigma_1) - \frac{3}{2} J_2 \iota_{R_4} (\ast d\sigma_2) - J_3 \iota_{R_4} (\ast d\sigma_3) + \frac{1}{2} J_1 \iota_{R_4} (dJ_1 \nu_4) \\
+ \frac{1}{2} J_3 \iota_{R_4} (dJ_3 \nu_4); \]

and the operators \( \iota_r : \Sigma_r(M) \rightarrow \Lambda^2 \mathcal{M} \) and \( \mathbf{E} : \Lambda^2 \mathcal{M} \rightarrow \Lambda^2 \mathcal{M} \) are defined respectively in (2.4) and (2.1).

4. The Ricci Tensor of a Contact-Hypo Manifold

Let \((M, \alpha, \omega_1, \omega_2, \omega_3)\) be a contact-Hypo manifold. In view of the observations of subsection 2.2, \(\Phi_1, \Phi_2, \Phi_3, \Phi_4\) reduce to

\[ \Phi_1 = 0, \]

\[ \Phi_2 = (\frac{1}{2} f_{23} - 2) \sigma_3 + \frac{1}{2} \ast d\sigma_2, \]

\[ \Phi_3 = (-\frac{1}{2} f_{23} + 2) \sigma_2 + \frac{1}{2} \ast d\sigma_3, \]

\[ \Phi_4 = -\frac{1}{2} f_{23}(d^* \sigma_4)^T + \iota_{R_4} (d(\sigma_2 \wedge \omega_1)) - \frac{3}{2} J_2 \iota_{R_4} (\ast d\sigma_2) - J_3 \iota_{R_4} (\ast d\sigma_3). \]

Now we observe that

\[ E((\frac{1}{2} f_{23} - 2) \sigma_3 + \frac{1}{2} \ast d\sigma_2) = (\frac{1}{2} f_{23} - 2) \sigma_3 + \frac{1}{2} E(\ast d\sigma_2), \]

\[ E((-\frac{1}{2} f_{23} + 2) \sigma_2 + \frac{1}{2} \ast d\sigma_3) = (-\frac{1}{2} f_{23} + 2) \sigma_2 + \frac{1}{2} E(\ast d\sigma_3). \]

Moreover, using [2.2], for \( i = 2, 3 \), we get

\[ E(\ast d\sigma_i) = \ast^T E(\iota_{R_4} d\sigma_i) = \ast^T (\iota_{R_4} d\sigma_i) - \frac{1}{2} \sum_{r=1}^{3} \ast (\iota_{R_4} d\sigma_i \wedge \omega_r \wedge \alpha) \omega_r. \]
Consequently

$$E(\Phi_2) = \left(\frac{1}{2}f_{23} - 2\right)\sigma_3 + \frac{1}{2} * T (\iota_{R_0} d\sigma_2) - \frac{1}{4} \sum_{r=1}^{3} (\iota_{R_r} d\sigma_2 \wedge \omega_r \wedge \alpha) \omega_r,$$

(4.3)

$$E(\Phi_3) = \left(-\frac{1}{2}f_{23} + 2\right)\sigma_2 + \frac{1}{2} * T (\iota_{R_0} d\sigma_3) - \frac{1}{4} \sum_{r=1}^{3} (\iota_{R_r} d\sigma_3 \wedge \omega_r \wedge \alpha) \omega_r.$$

(4.4)

In order to write down the Ricci tensor of a contact-Hypo structure, we consider the following

**Lemma 4.1.** Let \((M, \alpha, \omega_1, \omega_2, \omega_3)\) be a contact-Hypo manifold, then

$$\Phi_4 = 3J_1 (df_{23})^T.$$

**Proof.** The lemma is essentially a consequence of the vanishing of \(d^2\). First of all note that, for any 3-form \(\gamma\), one can write \(-\gamma^T \gamma^T\) instead of \(\iota_{R_0} \gamma\). Hence in the contact-Hypo case one has

$$\Phi_4 = -\frac{1}{2} J_2 (d^* \sigma_2) - \gamma (df_{23} \wedge \omega_1)^T + \frac{3}{2} J_2 \gamma (df_2 \wedge \omega_2 + \sigma_3)^T.$$

(4.5)

Now

$$0 = d^2 \omega_2 = -\alpha \wedge (df_{23} + \omega_3 + d\sigma_2),$$

$$0 = d^2 \omega_3 = -\alpha \wedge (df_{23} + \omega_2 + d\sigma_3).$$

Hence

$$(d\sigma_2)^T = -(df_{23} \wedge \omega_3)^T = -(df_{23} \wedge \omega_3)^T = -J_3 \gamma (df_{23})^T,$$

$$(d\sigma_3)^T = (df_{23} \wedge \omega_2)^T = (df_{23} \wedge \omega_2)^T = J_2 \gamma (df_{23})^T.$$

For the first term of \((4.3)\) consider

$$d * \sigma_2 = -d \sigma_2 \wedge \alpha = -d \sigma_2 \wedge \alpha + 2 \sigma_2 \wedge \omega_1 = -(d\sigma_2)^T \wedge \alpha.$$

Thus

$$J_2 (d^* \sigma_2)^T = J_2 (\alpha \wedge (d\sigma_2)^T) = J_2 \gamma (df_{23})^T = -J_2 \gamma (df_{23})^T = J_1 (df_{23})^T.$$

Finally for the second term

$$* \gamma (df_{23} \wedge \omega_1)^T = * \gamma (df_{23} \wedge \omega_1)^T = * \gamma J_1 \gamma (df_{23})^T = -J_1 (df_{23})^T.$$

Therefore, keeping in mind the quaternionic relations of \(J_{r}'s\), one has

$$\Phi_4 = \left(-\frac{1}{2} + 1 + \frac{3}{2} + 1\right) J_1 (df_{23})^T = 3J_1 (df_{23})^T.$$

\(\square\)

Summarizing we have the following

**Proposition 4.2.** The “traceless part” of the Ricci tensor of a contact-Hypo manifold is given by the following formula

$$\text{Ric}_0 = \iota_2^{-1} \left(\frac{1}{2}f_{23} - 2\right)\sigma_3 + \frac{1}{2} \gamma (\iota_{R_0} d\sigma_2) - \frac{1}{4} \sum_{r=1}^{3} (\iota_{R_r} d\sigma_2 \wedge \omega_r \wedge \alpha) \omega_r + \iota_3^{-1} \left(-\frac{1}{2}f_{23} + 2\right)\sigma_2 + \frac{1}{2} \gamma (\iota_{R_0} d\sigma_3) - \frac{1}{4} \sum_{r=1}^{3} (\iota_{R_r} d\sigma_3 \wedge \omega_r \wedge \alpha) \omega_r + 3J_1 (df_{23})^T \circ \alpha.$$
Now we collect some consequences of this result.

**Proposition 4.3.** Let $(M, \alpha, \omega_1, \omega_2, \omega_3)$ be a double-Hypo 5-manifold. The metric induced by the SU(2)-structure is $\alpha$-Einstein if and only if $(M, \alpha, \omega_1, \omega_2, \omega_3)$ is Sasaki-Einstein.

**Proof.** The $\alpha$-Einstein condition means that the projection onto $\Lambda^2_3 M$ of $\Phi_1, \Phi_2$ and $\Phi_3$ vanishes. But in the double-Hypo case one has

$$\Phi_1 = -\sigma_4, \quad \Phi_2 = -\frac{1}{2}\sigma_3,$$

which lie in $\Lambda^2_3 M$, and the conclusion follows. □

**Corollary 4.4.** Let $(M, \alpha, \omega_1, \omega_2, \omega_3)$ be a 5-dimensional SU(2)-manifold. Assume that:

1. the SU(3)-structure induced on the cone $C(M) = M \times \mathbb{R}^+$ is symplectic half-flat,
2. the metric $g$ induced by $(\alpha, \omega_1, \omega_2, \omega_3)$ is $\alpha$-Einstein,

then $(M, \alpha, \omega_1, \omega_2, \omega_3)$ is Sasaki-Einstein.

**Proof.** Simply recall remark 2.5 and apply the previous proposition. □

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