Progress on Feynman Integrals for $2 \rightarrow 3$ scattering at NNLO

Nikolaos Syrrakos$^{1,2}$

based on work in collaboration with D. Canko, C. Papadopoulos, C. Wever, A. Kardos and A. Smirnov

arXiv:2009.13917 [hep-ph] (JHEP)
arXiv:2012.10635 [hep-ph] (JHEP)

$^1$Institute of Nuclear and Particle Physics, NCSR Demokritos

$^2$School of Applied Mathematics and Physical Sciences, NTUA

Recent Developments in High Energy Physics and Cosmology
16-19 June 2021 (Virtual)
Table of Contents

1. Motivation
2. Method
3. Results
4. Summary and future work
5. Backup slides
Table of Contents

1 Motivation

2 Method

3 Results

4 Summary and future work

5 Backup slides
Motivation

- Indications for the need of physics Beyond the Standard Model mostly from Cosmology (e.g. is dark matter a new particle/ particles?).
- Absence of a clear signal for BSM physics from the LHC → Collider Physics at the Precision Frontier\(^1\).
- Measured cross sections are being compared to improved theoretical predictions looking for deviations from the SM → constraints on BSM physics.
- LHC Run 3 and HL-LHC Run will require at least NNLO corrections.
- NNLO → two-loop Feynman Diagrams → two-loop Feynman Integrals (FI).
- Numerical approaches not efficient (physical kinematics) → need for analytical results.

\(^1\)G. Heinrich, arXiv:2009.00516 [hep-ph]
Table of Contents

1 Motivation

2 Method

3 Results

4 Summary and future work

5 Backup slides
Computing Feynman Integrals

Introduce a family of two-loop FI for a specific scattering process (e.g. $2 \rightarrow 2, 2 \rightarrow 3$):

$$F_{a_1, \ldots, a_N}(\{p_j\}, \epsilon) = \int \left( \prod_{r=1}^{2} \frac{d^d k_r}{i \pi^{d/2}} \right) \frac{e^{2\epsilon \gamma_E}}{D_1^{a_1} \ldots D_N^{a_N}},$$

$$D_i = (c_{ij} k_j + f_{ij} p_j)^2, \quad d = 4 - 2\epsilon$$

Two pillars

1. Integrals of total derivatives wrt loop momenta vanish within DR. (IBP identities)
2. FI satisfy differential equations (DE) derived wrt kinematic invariants.
Computing Master Integrals

- IBP reduction identifies a minimal set of Master Integrals, called basis of MI $G$.
- Use DE to compute them.

$$\frac{\partial}{\partial s_{ij}} G = A(\{s_{ij}\}, \epsilon) G$$  \hspace{1cm} (2)

- Simplified DE (SDE)$^2$: introduce an external parameter $x$ and differentiate wrt it.

$$\partial_x G = A(\{s_{ij}\}, x, \epsilon) G$$  \hspace{1cm} (3)

- In general the matrix $A$ can be very complicated.
- Instead of $G$ use a special basis $^3 g = TG$

---

^3 G. Papadopoulos, JHEP 07 (2014), 088
J. M. Henn, Phys. Rev. Lett. 110 (2013), 251601
Computing Master Integrals

- Canonical SDE
  \[ \partial_x g = \epsilon M(\{s_{ij}\}, x)g \]  
  (4)

- \( \epsilon \) is fully factorised

- Differential matrix \( M(\{s_{ij}\}, x) \) is Fuchsian, i.e. it has simple poles in \( x \) (for all cases considered in this talk).

- Solved with recursive iterations in terms of Goncharov PLs (GPLs)
  \[ G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \]  
  (5)

  \[ G(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \]  
  (6)
Table of Contents

1 Motivation

2 Method

3 Results

4 Summary and future work

5 Backup slides
2 → 3 with one massive leg (e.g. W+2 jets)

Figure: The two-loop diagrams representing the top-sector of the planar pentabox family $P_1(74 \text{ MI}), P_2(75 \text{ MI})$ and $P_3(86 \text{ MI})$. All external momenta are incoming.

- IBP tools: FIRE6, KIRA2
Traditional approach

Semi-numerical approach

S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP 11 (2020), 117

Kinematics

- External momenta $q_i$, $i = 1 \ldots 5$
- $\sum_{i=1}^{5} q_i = 0$, $q_1^2 \equiv p_{1s}$, $q_i^2 = 0$, $i = 2 \ldots 5$
- $\{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := (q_i + q_j)^2$

$$d\mathbf{g} = \epsilon \sum_{a} d \log (W_a) \tilde{\mathbf{M}}_{a \mathbf{g}}$$ (7)

Very difficult to solve analytically!
SDE approach

Re-parametrize external momenta in terms of a dimensionless parameter $x$.

$$q_1 \to p_{123} - xp_{12}, \ q_2 \to p_4, \ q_3 \to -p_{1234}, \ q_4 \to xp_1$$  \hspace{1cm} (8)

Kinematics

- Underline momenta $p_i, \ i = 1 \ldots 5$
- $$\sum_{1}^{5} p_i = 0, \ p_i^2 = 0, \ i = 1 \ldots 5, \text{ with } p_{i \ldots j} := p_i + \ldots + p_j$$
- $$\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}, \text{ with } S_{ij} := (p_i + p_j)^2$$
SDE approach

- Much simpler canonical DE

### Equation 9

\[
\partial_x g = \epsilon \sum_b \frac{1}{x - l_b} M_b g
\]

- Naturally expressed in terms of GPLs.
- Boundary terms: need \( x \to 0 \) limit, Expansion-By-Regions\(^4\) (Chris Wever, Adam Kardos (Gsuite)).
- Re-write \( g = T G \)
- Asymptotic expansion around \( x \to 0 \) for \( G \).

---

\(^4\)B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C 72 (2012), 2139
Analytic solution up to $\mathcal{O}(\epsilon^4)$

\[
g = \epsilon^0 b_0^{(0)} + \epsilon \left( \sum G_a M_a b_0^{(0)} + b_0^{(1)} \right) \\
+ \epsilon^2 \left( \sum G_{ab} M_a M_b b_0^{(0)} + \sum G_a M_a b_0^{(1)} + b_0^{(2)} \right) \\
+ \epsilon^3 \left( \sum G_{abc} M_a M_b M_c b_0^{(0)} + \sum G_{ab} M_a M_b b_0^{(1)} + \sum G_a M_a b_0^{(2)} + b_0^{(3)} \right) \tag{10} \\
+ \epsilon^4 \left( \sum G_{abcd} M_a M_b M_c M_d b_0^{(0)} + \sum G_{abc} M_a M_b M_c b_0^{(1)} + \sum G_{ab} M_a M_b b_0^{(2)} + \sum G_a M_a b_0^{(3)} + b_0^{(4)} \right) \\
G_{ab\ldots} := G(l_a, l_b, \ldots ; x)
\]

- Pure solution: residue matrices $M_i$ are independent of the kinematics.
- Universally transcendental.
- Transcendental weight $\sim$ number of integrations.
Explicit expressions up to $\mathcal{O}(\epsilon^1)$

\[
gb_{72}^{P1} = \frac{3}{2} + \epsilon \left[ G \left( \frac{S_{45}}{S_{12}}, x \right) - G \left( \frac{S_{12} - S_{34}}{S_{12}}, x \right) \right.
\]
\[
- 3G(0, x) + G(1, x) - \log (S_{12} - S_{34}) - 2 \log (-S_{51}) \left] \right.
\] (11)

\[
gb_{73}^{P2} = \epsilon \left[ 3G \left( \frac{S_{12} + S_{23}}{S_{12}}, x \right) - 3G \left( \frac{S_{12} - S_{34}}{S_{12}}, x \right) \right.
\]
\[
- 3G \left( \frac{S_{45}}{S_{12}}, x \right) + 3G(1, x) - 3 \log (S_{12} - S_{34}) + 3 \log (-S_{51}) \left] \right.
\] (12)

\[
gb_{84}^{P3} = \frac{1}{2} + \epsilon \left[ \frac{5}{2} G \left( \frac{S_{45}}{S_{12}}, x \right) - \frac{3}{2} G \left( \frac{S_{12} - S_{34}}{S_{12}}, x \right) \right.
\]
\[
- \frac{5}{2} G \left( -\frac{S_{45}}{S_{23} - S_{45}}, x \right) - 2G(0, x) + \frac{5}{2} G(1, x) - \log (-S_{12}) - \frac{3}{2} \log (S_{12} - S_{34}) + \frac{3}{2} \log (-S_{51}) \left] \right.
\] (13)
### Numerics (N=32 digits, 1.9, 3.3, 2 sec) in GiNaC

| $P_1$ | $g_{72}$ | $\epsilon^0$: 3/2 |
|-------|----------|--------------------|
|       |          | $\epsilon^1$: -2.2514604753379400332169314784961 |
|       |          | $\epsilon^2$: -17.910593443812320786572184851867 |
|       |          | $\epsilon^3$: -26.429770706459534336624681550003 |
|       |          | $\epsilon^4$: 21.437938934510558345847354772412 |

| $P_2$ | $g_{73}$ | $\epsilon^0$: 1/2 |
|-------|----------|--------------------|
|       |          | $\epsilon^1$: 2.8124788185742741402751457351382 |
|       |          | $\epsilon^2$: 5.4813042746593704203645729908938 |
|       |          | $\epsilon^3$: 11.590234540689191439870956817546 |
|       |          | $\epsilon^4$: -5.9962816226829136730734255754596 |

| $P_3$ | $g_{84}$ | $\epsilon^0$: 1/2 |
|-------|----------|--------------------|
|       |          | $\epsilon^1$: 3.2780415861887284967738281876762 |
|       |          | $\epsilon^2$: 0.11455863130537720411162743574627 |
|       |          | $\epsilon^3$: -16.979642659429606120982671925458 |
|       |          | $\epsilon^4$: -48.101985355625914648042310964575 |
Table of Contents

1 Motivation
2 Method
3 Results
4 Summary and future work
5 Backup slides
Summary

- Fully analytic results for all 2-loop planar families.
- Pentagon (1-loop 5-point) with one massive leg to all orders.
- Results used for 2-loop QCD corrections to $Wb\bar{b}$ production\(^5\).
- Push to compute all MI for $2 \rightarrow 3$ with up to one massive leg at two loops. (Planar: known, Non-Planar: work in progress).
- Fully analytic results for first non-planar family (numerical\(^6\)).
- Automated framework for NNLO predictions following the NLO paradigm. (talk by D. Canko)

---

5. S. Badger, H. B. Hartanto and S. Zoia, [arXiv:2102.02516 [hep-ph]]
6. C. G. Papadopoulos and C. Wever, JHEP 02 (2020), 112
Future work

Compute remaining non-planar MI for two-loop $2 \rightarrow 3$ with one massive leg\(^7\),

\[^7\] Pure bases from S. Abreu, H. Ita, B. Page, W. Tschernow
This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Program Human Resources Development, Education and Lifelong Learning 2014 - 2020 in the context of the project “Higher order corrections in QCD with applications to High Energy experiments at LHC” -MIS 5047812.

Thank you for your attention!
Table of Contents

1 Motivation

2 Method

3 Results

4 Summary and future work

5 Backup slides
Automated tools

- Ginac\(^8\) for the numerical calculation of GPLs.
- PolyLogTools\(^9\) for the algebraic manipulation of GPLs.
- FIRE6\(^10\) and KIRA2\(^11\) for the IBP reduction.
- FIESTA4\(^12\) for Expansion-By-Regions.
- pySecDec\(^13\) and FIESTA4 for numerical computation of FI, used for cross-checking our results.

---

\(^8\) J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005), 177
\(^9\) C. Duhr and F. Dulat, JHEP **08** (2019), 135
\(^10\) A. V. Smirnov and F. S. Chuharev, arXiv:1901.07808 [hep-ph]
\(^11\) J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, arXiv:2008.06494 [hep-ph]
\(^12\) A. V. Smirnov, Comput. Phys. Commun. **204** (2016), 189-199
\(^13\) S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **222** (2018), 313-326
Scattering kinematics

- Results in Euclidean region (all initial kinematic invariants are negative).
- GPLs and solutions are real there.
- Analytic continuation to get results in physical regions for phenomenology.
- Fibration basis techniques (exploit symbol algebra and coproduct to analytically continue GPLs).
- Numerically: \( \{ S_{ij}, x \} \rightarrow \{ S_{ij} + i\delta_{ij}\eta, x + i\delta\eta \} \) for \( \eta \rightarrow 0 \).
- Constraints on \( \delta_{ij}, \delta_x \) from one-scale integrals and second graph polynomial of top sector FI.