SOME REMARKS ON NICHOLS ALGEBRAS

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Abstract. Two algebras can be attached to a braided vector space \((V, c)\) in an intrinsic way; the FRT-bialgebra and the Nichols algebra \(B(V, c)\). The FRT-bialgebra plays the rô le of the algebra of quantum matrices, whereas the rô le of the Nichols algebra is less understood. Some authors call \(B(V, c)\) a quantum symmetric algebra. The purpose of this paper is to discuss some properties of certain Nichols algebras, in an attempt to establish classes of Nichols algebras which are worth of further study.

1. Definitions and examples of Nichols algebras. Let \((V, c)\) be a braided vector space, that is, \(V\) is a finite-dimensional complex vector space and \(c: V \otimes V \to V \otimes V\) is an invertible solution of the braid equation: \((c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)\). There is a remarkable braided graded Hopf algebra \(B(V, c) = \bigoplus_{n \geq 0} B_n(V, c)\), which is connected, generated in degree one, with \(B_1(V) \simeq V\) as braided vector spaces, such that all its primitive elements have degree one; and which is unique with respect to these properties. Algebras of this kind appeared naturally in our approach to classification of pointed Hopf algebras \([AS1, AS3]\) but we quickly realized they were already known to several authors under various presentations. We first briefly recall different definitions of Nichols algebras and survey examples of classes of Nichols algebras that are known. A detailed exposition on Nichols algebras can be found in \([AS3]\).

We shall simply write \(B(V) = B(V, c)\) omitting the reference to \(c\) unless it is needed. We shall always assume that \(c\) is rigid, i. e. the associated map \(c^\#: V^* \otimes V \to V \otimes V^*\) is also bijective (this is the case in all the examples below). Here \(c^\# = (\text{ev}_V \otimes \text{id}_V^*) (\text{id}_V^* \otimes c \otimes \text{id}_V^*) (\text{id}_V^* \otimes V \otimes \text{ev}_V^*)\).

This remarkable braided Hopf algebra was first described by W. Nichols in his thesis \([N]\), as the invariant part of his "bialgebras of type one". In his honor, \(B(V)\) is called the Nichols algebra of the braided vector space \((V, c)\). There are several ways to present \(B(V)\).

Consider \(T(V) \otimes T(V)\) as an algebra with the product 'twisted' by \(c\). Then \(T(V)\) is a braided Hopf algebra, with the comultiplication uniquely defined by \(\Delta(v) = v \otimes 1 + 1 \otimes v, v \in V\). Let \(I(V)\) be the largest Hopf ideal generated by homogeneous elements of degree greater than 1; then \(B(V) := T(V)/I(V)\) satisfies all the properties listed above \([AS3]\ Prop. 2.2]\).

The vector space \(T(V)\) has another structure of coalgebra, the free coalgebra over \(V\); let us denote it by \(t(V)\). M. Rosso observed that it admits a 'quantum shuffle product', so that \(t(V)\) is also a braided Hopf algebra, called the quantum shuffle algebra. The canonical map \(\Omega: T(V) \to t(V)\) turns out to be a map of braided Hopf algebras; the image of \(\Omega\), that is the subalgebra of \(t(V)\) generated by \(V\), is the Nichols algebra of \(V\). The nilpotent part \(U_q^+(\mathfrak{g})\) of a quantized enveloping algebra was characterized in this way by Rosso as the Nichols algebra of a suitable braided vector space \(\mathfrak{g}\) \([Ro1, Ro2]\). Results in the same spirit were also obtained by J. A. Green \([Gr]\).

\textit{Date}: October 29, 2018.

1991 \textit{Mathematics Subject Classification}. Primary: 17B37. Secondary: 16W30.

This work was partially supported by CONICET, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC).
Now, the components of the graded map \( \Omega \), that is \( \Omega^{(m)} : T^m(V) \to T^m(V) \) are the so-called “quantum symmetrizers” defined through the action of the braid group on \( T^m(V) \). Therefore, the Nichols algebra of \((V, c)\) coincides with the quantum exterior algebra of \((V, -c)\), defined by S. L. Woronowicz [Wo]. Indeed, the quantum symmetrizers of \(-c\) are the quantum antisymmetrizers of \(c\).

G. Lusztig characterized \( U_q^+(\mathfrak{g}) \) as the quotient of a \( T(\mathfrak{h}) \) by the radical of an invariant bilinear form [L]. This is indeed a general fact; the ideal \( I(V) \) is always the radical of an invariant bilinear form [AG1].

M. Rosso found that the Nichols algebra \( \mathfrak{B}(V) \) of a braided vector space of diagonal type has always a "PBW-basis" in terms of the so-called Lyndon words [Ro3]. Related work was done by V. K. Kharchenko, who also studied abstractly Nichols algebras from various points of view [Kh1, Kh2, Kh3].

The following two questions arise from classification problems of Hopf algebras [AS1, AS3]. Answers to both questions are needed to classify Hopf algebras of certain types.

- Under which conditions on \((V, c)\) is \( \mathfrak{B}(V) \) finite-dimensional, respectively of finite Gelfand-Kirillov dimension?

- For those pairs with a positive answer to the preceding question, give an explicit presentation of \( \mathfrak{B}(V) \); that is, find a minimal set of generators of the ideal \( I(V) \).

The study of \( \mathfrak{B}(V) \) is very difficult; neither the subalgebra of the quantum shuffle algebra generated by \( V \), nor the Lyndon words, nor the ideal \( I(V) \) have an explicit description.

- In particular, we do not know if the ideal \( I(V) \) is finitely generated.

There is little hope to perform explicit computations with a computer program without a positive answer to this question. However, let \( \mathfrak{B}_r(V) = T(V)/J_r \), where \( J_r \) is the two-sided ideal generated by the kernels of \( \Omega^{(n)} \), \( n \leq r \); these are braided Hopf algebras and we have epimorphisms \( \mathfrak{B}_r(V) \to \mathfrak{B}(V) \) for all \( r \geq 2 \). Hence, if one of the algebras \( \mathfrak{B}_r(V) \) is finite-dimensional, or has finite Gelfand-Kirillov dimension, so does \( \mathfrak{B}(V) \). In the first case, under favorable hypothesis we may conclude that \( \mathfrak{B}_r(V) \simeq \mathfrak{B}(V) \), see [AG2, Th. 6.4].

There are several classes of braided vector spaces which seem to be of special interest.

- We say that \((V, c)\) is of diagonal type if there exists a basis \( x_1, \ldots, x_\theta \) of \( V \), and non-zero scalars \( q_{ij} \) such that \( c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \), \( 1 \leq i, j \leq \theta \).

Nichols algebras of these braided vector spaces appear naturally in the classification of pointed Hopf algebras with abelian coradical, and also in the theory of quantum groups.

Namely, let \((a_{ij})_{1 \leq i, j \leq \theta}\) be a generalized Cartan matrix; let \( \mathfrak{h} \) be a vector space with a a basis \( x_1, \ldots, x_\theta \), let \( q \) be a non-zero scalar and let \( c \) be given by

\[
(1.1) \quad c(x_i \otimes x_j) = q^{a_{ij}} x_j \otimes x_i, \quad 1 \leq i, j \leq \theta.
\]

Then \( \mathfrak{B}(\mathfrak{h}) = U_q^+(\mathfrak{g}) \) if \( q \) is not a root of 1 [L, Ro1, Ro2, and \( \mathfrak{B}(\mathfrak{h}) = u_q^+(\mathfrak{g}) \) if \( q \neq 1 \) is a root of 1 (under some hypothesis on the order of \( q \)) [Ro1, Ro2, M1].
We say that \((V, c)\) is of rack type if there exists a basis \(X\) of \(V\), a function \(\triangleright : X \times X \rightarrow X\) and non-zero scalars \(q_{ij}\) such that
\[
c(i \otimes j) = q_{ij} i \triangleright j \otimes i,
\]
i, j \in X. Then \((X, \triangleright)\) is a rack and \(q_{ij}\) is a rack 2-cocycle with coefficients in \(k^\times\), see for example [Gn1, AG2].

These braided vector spaces appear naturally in the classification of pointed Hopf algebras.

We say that \((V, c)\) is of Jordanian type if there exists a basis \(x_1, \ldots, x_\theta\) of \(V\), and a non-zero scalar \(q\) such that \(c(x_i \otimes x_1) = qx_1 \otimes x_i, c(x_i \otimes x_j) = (qx_j + x_{j-1}) \otimes x_i, 1 \leq i \leq \theta, 2 \leq j \leq \theta\).

These braided vector spaces appear in the classification of pointed Hopf algebras with coradical \(\mathbb{Z}\).

We say that \((V, c)\) is of Hecke type if \((c - q)(c + 1) = 0\), for some non-zero scalar \(q\).

We say that \((V, c)\) is of quantum group type if \(V\) is a module over some quantized enveloping algebra \(U_q(\mathfrak{g})\) and \(c\) arises from the action of the corresponding universal \(R\)-matrix.

Here is what is known about the problems stated above.

- \(\mathcal{B}(V) = T(V)\) generically. That is, consider the locally closed space of all \(c \in \text{End}(V \otimes V)\) which are invertible solutions of the braid equation; then the subset of those \(c\) such that \(\mathcal{B}(V, c) = T(V)\) contains a non-empty open subset.

- Assume that \((V, c)\) is of diagonal type, where the \(q_{ii}\)'s are positive and different from one, \(1 \leq i \leq \theta\). Then \(\mathcal{B}(V)\) has finite Gelfand-Kirillov dimension if and only if \(q_{ij}q_{ji} = a_{ij}^2\) for some Cartan matrix of finite type [Re2].

- Assume that \((V, c)\) is of diagonal type, that the \(q_{ii}\)'s are roots of 1 but not 1, and that \(q_{ij}q_{ji} = q_{ii}^{a_{ij}}\), \(1 \leq i, j \leq \theta\), where \(a_{ij} \in \mathbb{Z}, a_{ii} = 2, \text{ord} q_{ii} < a_{ij} \leq 0\) if \(1 \leq i \neq j \leq \theta\). Then \((a_{ij})_{1 \leq i,j \leq \theta}\) is a generalized Cartan matrix and, under suitable conditions, \(\mathcal{B}(V)\) has finite dimension if and only if \((a_{ij})_{1 \leq i,j \leq \theta}\) is of finite type [AS2].

In these two cases, the calculation of \(\mathcal{B}(V)\) is reduced to the calculation of \(\mathcal{B}(h)\) as in (1.1); but the last requires deep facts on representation theory, and the action of the quantum Weyl group defined by Lusztig.

- There are a few examples of \((V, c)\) of diagonal type with finite-dimensional \(\mathcal{B}(V)\), due to Nichols [N] and Graña [Gn1], besides those in the last item (they are listed in [AS3, Section 3.3]).

- If \(c\) is of Hecke type and \(q\) is not a root of 1, or if \(q = 1\), then the Nichols algebra is quadratic: \(\mathcal{B}(V) = \mathcal{B}_2(V)\). Furthermore, the quadratic dual is also a Nichols algebra: \(\mathcal{B}(V)^! = \mathcal{B}(V^*, -q^{-1}c^d)\); see Proposition 2.3 below.

- If \(c\) is of diagonal type, information about \(\det \Omega^{(m)}\) is given in [FG].

- There are a few examples of \((V, c)\) of rack type with finite-dimensional \(\mathcal{B}(V)\) [MS, Gn1, AG2, Gn2]. See the table in the Appendix for a flavor of the kind of algebras obtained.

- Almost nothing is known about Nichols algebras of quantum group type, except when they are of Hecke type. To begin with, it would be interesting to know what happens when \(V = L(n)\) is a highest weight module over \(U_q(\mathfrak{sl}_2)\), \(n \geq 3\) (if \(n = 1\) it is of Hecke type, if \(n = 2\) it seems to be known but I do not have a reference).
Nichols algebras of Jordanian type were not considered in the literature, to my knowledge. It is likely that the quantum Jordanian plane is a Nichols algebra of Jordanian type.

In conclusion, a Nichols algebra may be finite-dimensional or not, have finite Gelfand-Kirillov dimension or not, and there is no general technique, up to now, to explicitly decide for a given braided vector space, what is the case for its Nichols algebra\(^1\).

This indicates that Nichols algebras do not have to be studied through a general approach, but splitting the category of braided vector spaces in classes.

**Remark 1.1.** Another important question in the classification of Hopf algebras is the following. Let \( B = \bigoplus_{n \geq 0} B^n \) be a braided graded Hopf algebra, connected, generated in degree one, and denote by \( V \) the braided vector subspace \( B^1 \) of \( B \). Is it possible to conclude that \( B \) is the Nichols algebra of \( V \), i.e. that all its primitive elements have degree one, under some abstract conditions? Partial positive answers to this question are given in [AS4, Th. 7.6] (finite-dimensional case) and [AS5, Lemma 5.1] (finite Gelfand-Kirillov dimension case).

2. **Some properties of some Nichols algebras.** It is clear that no fine ring theoretical properties can be established for Nichols algebras in general. But there might be a suitable class of braided vector spaces whose Nichols algebras deserve attention from this point of view.

2.1. **Graded algebras.** We shall only consider graded algebras \( R = \bigoplus_{i \geq 0} R_i \), which are finitely generated and \( R_0 = \mathbb{C} \).

**Definition 2.1.** A graded algebra \( R \) is \( AS\)-regular if it has finite global dimension \( d \), finite Gelfand-Kirillov dimension and is \( AS\)-Gorenstein. Thus, \( \text{Ext}^i_R(\mathbb{C}, R) = 0 \), if \( i \neq d \), and \( = \mathbb{C} \) if \( i = d \).

This class of graded algebras has been intensively investigated in the last years; \( AS \) is in honor of Artin and Schelter. The study of the category of graded modules of such an algebra has a strong geometrical flavor; this is usually called a noncommutative projective space. In particular, the space of all "point modules" is a genuine projective space which provides important information on the full category. See [SH, SV]. The homological conditions are designed to insure good regularity properties; we refer again to [SH, SV] and references therein.

- When is a Nichols algebra \( AS\)-regular?

- Let \( \mathcal{B}(V) \) be a Nichols algebra which is a domain with finite Gelfand-Kirillov dimension. Is it \( AS\)-regular?

Some insight about these questions is explained in the next subsections.

2.2. **Koszul algebras.** We first recall some well-known facts about Koszul algebras.

A quadratic algebra is a graded algebra \( A = \bigoplus_{n \geq 0} A_n \) generated in degree one with relations in degree 2; that is \( A \simeq T(V)/(R) \), where \( V = A_1 \) and \( R \subset V \otimes V \) is the kernel of the multiplication. We shall denote \( A = (V, R) \). The quadratic dual of a quadratic algebra \( A = (V, R) \) is \( A^! = (V^*, R^\perp) \). By [LG],

\[
A^! \simeq E(A) := \text{the subalgebra of } \text{Ext}^*_A(\mathbb{C}, \mathbb{C}) \text{ generated by } \text{Ext}^1_A(\mathbb{C}, \mathbb{C}).
\]

\(^1\)Some techniques are available for specific classes of Nichols algebras, e.g. for Nichols algebras of diagonal type, as already said.
A graded Koszul algebra is a quadratic algebra $A$ such that $A^! \simeq \text{Ext}^\bullet(C, C)$ as graded algebras.

**Lemma 2.2.** Let $A$ be any graded connected algebra.

(a). [Sm 1.4 and 5.9]. If $\text{gldim} A < \infty$ then $\dim A^! < \infty$. The converse holds if $A$ is Koszul.

(b). [Sm 5.10]. If $A$ is Koszul and has finite global dimension, then $A$ is AS-Gorenstein if and only if $A^!$ is Frobenius.

We can now decide when a Nichols algebra of Hecke type is AS-regular.

**Proposition 2.3.** Let $(V, c)$ be a braided vector space and assume that $c$ satisfies a Hecke-type condition with label $q$, $q \neq 1$ or not a root of 1. Then $B(V)$ is AS-regular if and only if it has finite Gelfand-Kirillov dimension and the dimension of $B(V^*)$ is finite.

**Proof.** By [AA, Prop. 3.3.1], see also [AS3, Prop. 3.4], the Nichols algebra $B(V)$ is quadratic, and its quadratic dual is $B(V)^! = B(V^*)$, the Nichols algebra with respect to $-q^{-1}c^!$. By [Gu, Wa], $B(V)$ is Koszul. By Lemma 2.2 (a), if $B(V)$ is AS-regular then $\dim B(V^*) < \infty$. Conversely, assume that $\dim B(V^*) < \infty$. By Lemma 2.2 part (a), $\text{gldim} A < \infty$; and by part (b), $B(V)$ is AS-Gorenstein. Indeed, $B(V^*)$ is a braided Hopf algebra; hence it is Frobenius whenever finite-dimensional. □

**Remark 2.4.** I do not know if the hypothesis on the Gelfand-Kirillov dimension can be removed. There are examples of Koszul algebras where $A^!$ is finite-dimensional and Frobenius but $A$ has infinite GK-dimension. I am indebted to James Zhang for pointing this out to me. We do know the Hilbert series of $B(V^*)$:

$$H(B(V))(t) = \frac{1}{H(B(V^*), -q^{-1}c^!)(-t)},$$

by [BGS, Th. 2.11.1], and $H(B(V^*))$ is a polynomial.

**Remark 2.5.** D. Gurevich studied intensively Nichols algebras of Hecke type, provided ways to construct explicit examples, and classified those such that $H(B(V^*), -q^{-1}c^!)$ is a polynomial of degree two [Gu].

**Remark 2.6.** A quadratic Nichols algebra is not necessarily Koszul; see [R].

2.3. Nichols algebras related to quantum groups.

**Proposition 2.7.** Let $(a_{ij})_{1 \leq i,j \leq \theta}$ be a Cartan matrix of finite type, let $q$ be a non-zero scalar and let $(\mathfrak{h}, c)$ be the braided vector space as in [GL]. Then

(a). [GL Th. 4.7] $B(\mathfrak{h}) = U^+_q(\mathfrak{g})$ is AS-regular when $q$ is not a root of 1.

(b). [GK Th. 2.5] $B(\mathfrak{h}) = u^+_q(\mathfrak{g})$ is not AS-regular if $1 \neq q$ is a root of 1. □

It would be interesting to have another proof of (a) in the spirit of [DCK, GK]. Namely, to consider the algebra filtration given by the PBW-basis; the associated graded algebra is a quantum linear space [DCK], hence a Nichols algebra of Hecke type; therefore it has finite global dimension. Then lift this information by a spectral sequence argument. Note also that Proposition 2.7 extends to the multiparametric case without difficulties. In view of Propositions 2.3 and 2.7 it is tempting to suggest the following questions.

○ If $A = B(V)$, when is $E(A)$ also a Nichols algebra?

○ Is the graded algebra associated to the filtration given by the PBW-basis on the Lyndon words of $A = B(V)$, also a Nichols algebra?
Incidentally, it seems that the determination of the space of point modules for $U_q^+(g)$ has not been explicited in the literature.

2.4. Invariants. A natural question in noncommutative geometry is the study of spaces of invariants under group (or Hopf algebra) actions. We believe that a suitable setting to discuss it is when the noncommutative space corresponds to a Nichols algebra $B(V)$ with suitable properties. Specifically, we propose to study the subalgebra of invariants of $B(V)$ under a coaction of a Hopf algebra $H$.

The first step is to find a good amount of Hopf algebras $H$ such that $B(V)$ is an $H$-comodule algebra. It is well-known that $B(V)$ is a comodule braided Hopf algebra over the FRT-Hopf algebra $H(c)$ associated to the braided vector space $(V,c)$, see for example [T]; thus, $B(V)$ is a comodule algebra over any Hopf algebra quotient of $H(c)$. See [Mü2] for the classification of finite-dimensional quotients of $\mathbb{C}_q[G]$, $G$ a simple algebraic group.

We stress that these comodule algebra structures are "linear", that is, they preserve also the comultiplication of $B(V)$; many other structures may arise. Assume for example that $(V,c)=(V,\tau)$, where $\tau$ is the usual transposition. Then the Nichols algebra $B(V)$ is the symmetric algebra $S(V)$. The automorphism group of a polynomial algebra is much larger than the group of linear automorphisms, and the determination of the former is a classical open problem.

In the quantum case the situation is much more rigid. Indeed, assume that $q$ is not a root of 1. Then $\text{Aut}_{\text{alg}} U_q^+(g)$ coincides with $\text{Aut}_{\text{Hopf alg}} U_q^+(g) \simeq (T \times \text{Aut } \Delta)$, where $T$ is a maximal torus and $\Delta$ is the Dynkin diagram, if $g$ is of type $A_2$ [AD] or of type $B_2$ [AD], and conjecturally for all the types. Again, one is tempted to ask for the class of braided vector spaces $(V,c)$ such that $\text{Aut}_{\text{alg}} B(V) = \text{Aut}_{\text{Hopf alg}} B(V)$.

Appendix. For illustration, we collect some information about finite dimensional Nichols algebras of rack type. Below we consider braided vector spaces of rack type, with $q_{ij} = -1$, see [L2]. The rank of a Nichols algebra $B(V)$ is the dimension of $V$.

An affine rack is a rack $(A,g)$ where $A$ is a finite abelian group and $g \in \text{Aut } A$; then $ab = g(b)+(\text{id } -g)(a)$. The first four racks listed below are affine. A subset of a group stable under conjugation is a rack; so is the set of transpositions in $S_n$. The action for the rack of faces of the cube can be described either geometrically or as an extension.

There is no problem to find the space of relations in degree 2; it is the kernel of $c+\text{id}$. Relations in higher degree (not coming from those in degree 2) are more difficult to find, as said. For affine racks, a first step is given in [AG2] 6.13. Typical relations in degree 4 and 6 are respectively

$$ x_0 x_1 x_0 x_1 + x_1 x_0 x_1 x_0 = 0, \quad x_0 x_1 x_2 x_0 x_1 x_2 + x_2 x_0 x_1 x_2 x_0 x_1 + x_1 x_2 x_0 x_1 x_2 x_0 = 0. $$

These relations depend upon the order of $-g$. Most of the computations were done with help of a computer program. See [Gp2] for details. No explanation of the numbers appearing in the table is available until now, but there are some evident patterns.

Except for the racks of transpositions in $S_4$ and faces of the cube, all the other racks are simple (they do not project properly onto a non-trivial rack). Those two racks are extensions with the same base and fiber but they are not isomorphic. The similarities between the corresponding Nichols algebras are explained by a kind of Fourier transform, see [AG2] Ch. 5.

More examples of finite-dimensional Nichols algebras of rack type are given in [AG2] Prop. 6.8; they are not of diagonal type but they arise from Nichols algebras of diagonal type by the same kind of Fourier transform.
Rack | rk | Relations | dim $\mathcal{B}(V)$ | top
--- | --- | --- | --- | ---
$(\mathbb{Z}/3, v^g), g = 2$ (Transpositions in $S_3$) | 3 | 5 relations in degree 2 | $12 = 3 \cdot 2^2$ | 4 = $2^2$
$(\mathbb{Z}/5, v^g), g = 2$ | 5 | 10 relations in degree 2 1 relation in degree 4 | $1280 = 5 \cdot 4^4$ | 16 = $4^2$
$(\mathbb{Z}/7, v^g), g = 3$ | 7 | 21 relations in degree 2 1 relation in degree 6 | $326592 = 7 \cdot 6^9$ | 36 = $6^2$
$(\mathbb{Z}/2 \times \mathbb{Z}/2, v^g), g = (0 1)$ | 4 | 8 relations in degree 2 1 relation in degree 6 | 72 | 9 = $3^2$
Transpositions in $S_4$ | 6 | 16 relations in degree 2 | 576 | 12
Faces of the cube | 6 | 16 relations in degree 2 | 576 | 12
Transpositions in $S_5$ | 10 | 45 relations in degree 2 | 8294400 | 40

Acknowledgements. The author is grateful to J. Alev, F. Dumas and S. Natale for many conversations about various aspects of Nichols algebras; several of the questions in the text arise from discussions with them; and also to J. T. Stafford and J. Zhang for answers to some consultations. The author also thanks F. Dumas for his warm hospitality during a visit to the University of Clermont-Ferrand in March 2002 (when this work was began); to S. Catoiu for the kind invitation to the International Conference in Chicago; and to the IHES, where this paper was written.

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