PROPERTY VALUES

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Abstract
By ascribing a complex anticommuting variable $\zeta$ to each basic property of a field it is possible to describe all the fundamental particles as combinations of only five $\zeta$ and understand the occurrence of particle generations. An extension of space-time $x$ to include property then specifies the ‘where, when and what’ of an event and allows for a generalized relativity where the gauge fields lie in the $x - \zeta$ sector and the Higgs fields in the $\zeta - \zeta$ sector.

Preamble
First of all let me say why I am so pleased to be present at this FestSchrift in honour of Girish’ and Bruce’s retirements. When I arrived in Tasmania one of my first tasks was to make contact with physicists in other Australian institutions. Amongst these Melbourne University had high priority and I was very glad to welcome Bruce as one of the earliest visitors to Hobart. Since then we have interacted many times and I have relished my own visits to Melbourne to give occasional seminars and find out what Joshi and McKellar were up to. Let me wish them both a long and happy retirement and say how much I have appreciated their support and friendship throughout the last 30 years! Retirement has much to recommend it and they might care to view their future condition as changing from ‘battery hen’ to ‘free-range chicken’, since they are no longer obliged to feed on grants in order to lay eggs.

1 Introduction
As most of you will have already surmised, the title of my talk has nothing to do with real estate and perhaps sounds all the more mysterious for it. In fact property values have everything to do with quantum fields and accurately reflect the contents of what follows. The motivation for this work is to be able to describe the ‘what’ as well as the ‘where-when’ of an event. To make an analogy with personality traits in humans, psychologists may characterise a person as optimistic/pessimistic, happy/sad, aggressive/submissive, etc. (A person will possess some combination or superposition over these trait states.) The same
sort of characterisation applies to a quantum field and is usually underlined by attaching a label to each distinct field. Thus we speak of an electron/positron field, a neutron/antineutron field, a quark/antiquark field, etc. and draw them in Feynman diagrams say with solid or dashed lines and arrows, plus legends if necessary, to distinguish them from one another. An event is some confluence between these labels with a possible interchange of traits, as specified by an interaction Lagrangian. Often we assume a symmetry group is operational which 'rotates' labels; this constrains the interchange of property and, if the symmetry is local, it can be gauged.

The basic idea I wish to put to you is that traits/labels are normally discrete: thus a field is either an electron or it is not; similarly for a proton, and so on. It makes sense to attach a separate coordinate to each such property and choose the coordinate to be anticommuting since its occupation number is either one or zero. (So a person can be pessimistic & sad & aggressive with a product of the three traits.) Furthermore if we make the coordinate $\zeta$ complex we can describe the converse property by simple conjugation. When applying this idea to quantum fields the question arises: how many coordinates are needed?

The fewer the better of course. From my investigations thus far [1][2] I have concluded that one can get a reasonable description of the known fundamental particle spectrum by using just five complex coordinates $\zeta^\mu; \mu = 0, 1, 2, 3, 4$. I have found [1] that four $\zeta$ are insufficient to account for the three known light generations and their features. The way generations arise in this context comes by multiplying traits by neutral products of other traits. For instance a person can be pessimistic or (pessimistic $\times$ sad-happy) or (pessimistic $\times$ aggressive-submissive) and so on. With $N$ $\zeta$ one potentially encounters $2^{N-1}$ generations of a particular property by forming such neutral products; this is interesting because it suggests that any anticommuting property scheme produces an even number of generations. This is not a cause for panic; all we know at present is that there exist 3 light neutrinos, so there may be other (possibly sterile) heavy neutrinos accompanied by other quarks and charged leptons.

The proposal therefore is to append a set of five complex anticommuting coordinates $\zeta^\mu$ to space-time $x^m$ which are to be associated with property. This is in contrast to traditional brane-string schemes which append yet unseen bosonic extra degrees of freedom to space-time. Now one of the most interesting aspects of fermionic degrees of freedom is that they act oppositely to bosonic ones in several respects: we are familiar with sign changes in commutation relations, statistical formulae and, most importantly, quantum loop contributions; but less well-known is that in certain group theoretical representations [3] the $SO(2N)$ Casimirs are continuations to negative $N$ of $Sp(2N)$ Casimirs so that anticommuting coordinates effectively subtract dimensions. The opposite (to bosons) loop sign is extra confirmation of this statement and this is put to great use in standard supersymmetry (SUSY) in order to resolve the fine-tuning problem. So it is not inconceivable that with a correct set of anticommuting property coordinates $\zeta$ appended to $x$ we might end up with zero net dimensions as it was presumably before the BIG BANG; at the very least that we may arrive at a scheme where fermions cancel quantum effects from bosons as in SUSY. For this
it is not necessary to embrace all the tenets of standard SUSY; indeed we shall contemplate an edifice where property coordinates are Lorentz scalar. In that respect they are similar to the variables occurring in BRST transformations for quantized vector gauge-fixed models — without implying any violation of the spin-statistics theorem for normal physical fields.

2 Property superfields

Now let us get down to the nuts and bolts of the construction and find out what the edifice will look like. We might be tempted to match the four space-time $x^\mu$ by four $\zeta^\mu$, but as we have shown elsewhere that is not enough — lepton generations do not ensue. However, all is not lost: we can add 5 $\zeta^\mu$ (which are sufficient), but only take half the states to get correct statistics. Associate each Lorentz scalar anticommuting numbers with a ‘property’ or ‘trait’; this invites us to consider symmetry groups like $SU(5)$ or $Sp(10)$ or $SO(10)$ to reshuffle the properties. (That these popular groups pop up is probably no accident.)

The only way I have found to obtain the well-established quantum numbers of the fundamental particle spectrum as superpositions over traits is to postulate the following charge $Q$ and fermion number $F$ assignments,

$$Q(\zeta^{0,1,2,3,4}) = (0, 1/3, 1/3, 1/3, -1); \quad F(\zeta^{0,1,2,3,4}) = (1, -1/3, -1/3, -1/3, 1).$$

Other properties are to built up as composites of these. For example $\zeta^4$ may be identified as a negatively charged lepton, but then so can its product with neutral combinations like $\bar{\zeta}_0\zeta_0$, $\bar{\zeta}_i\zeta_i$, ... strongly suggesting how generations can arise in this framework. We’ll return to this shortly.

Since the product of an even number of $\zeta$ is a (nilpotent) commuting property a Bose superfield $\Phi$ should be a Taylor series in even powers of $\zeta$, $\bar{\zeta}$. Similarly a superfield which encompasses fermions $\Psi_\alpha$ will be a series in odd powers of $\zeta$, $\bar{\zeta}$ — up to the 5th:

$$\Phi(x, \zeta, \bar{\zeta}) = \sum_{r, \bar{r}} (\zeta)^r \phi_{(r)}(\bar{\zeta})^r;$$
$$\Psi_\alpha(x, \zeta, \bar{\zeta}) = \sum_{r, \bar{r}} (\zeta)^r \psi_{(r)}(\bar{\zeta})^r.$$

So far as labels $\mu$ on $\zeta^\mu$ go, we can characterise

- label 0 as neutriniicity
- labels 1 - 3 as (down) chromicity
- label 4 as charged leptonicity
You will notice that these expansions produce too many states for \( \psi_\alpha \) and \( \phi \), viz. 512 properties in all, so they demand cutting down. An obvious tactic is to associate conjugation (\( \bar{\psi} \)) with the operation \( \zeta \leftrightarrow \bar{\zeta} \)

\[
\psi_{(r), (\bar{r})} = \psi_{(\bar{r}), (r)},
\]

corresponding to reflection along the main diagonal when we expand \( \Psi \) as per the table below. Indeed if we suppose that all fermion field components are left-handed the conjugation/reflection operation then automatically includes right-handed particle states. Even so there remain too many components and we may wish to prune more. One strategy is to notice that under reflection about the cross-diagonal the \( F \) and \( Q \) quantum numbers are not altered. We shall call this cross-diagonal reflection a duality (\( \times \)) transformation. For example,

\[
(\bar{\zeta}_\alpha \zeta^\mu \zeta^\nu)^\times = \frac{1}{3!} \epsilon^{\rho \sigma \tau \mu \nu} \zeta_\rho \zeta_\sigma \bar{\zeta}_\tau \cdot \frac{1}{4!} \epsilon_{\alpha \beta \gamma \delta} \zeta^\beta \zeta^\gamma \zeta^\delta \zeta^\epsilon.
\]

By imposing the antidual reflection symmetry: \( \psi_{(r), (\bar{r})} = -\psi_{(\bar{5} - r), (5 - r)} \) we roughly halve the remaining number of components. So whereas previously we had the separate set of neutrino states, for instance,

\[
\zeta^0, \zeta^0(\bar{\zeta}_4 \zeta^4), \zeta^0(\bar{\zeta}_4 \zeta^4)(\bar{\zeta}_4 \zeta^4), \zeta^0(\bar{\zeta}_4 \zeta^4)^2, (\bar{\zeta}_4 \zeta^4)^3, \zeta^0(\bar{\zeta}_4 \zeta^4)(\bar{\zeta}_4 \zeta^4)^2, \zeta^0(\bar{\zeta}_4 \zeta^4)(\bar{\zeta}_4 \zeta^4)^3
\]

antiduality sifts out half the combinations, namely:

\[
\zeta^0[1 - (\bar{\zeta}_4 \zeta^4)(\bar{\zeta}_4 \zeta^4)^3/6], \quad \zeta^0[\bar{\zeta}_4 \zeta^4] - (\bar{\zeta}_4 \zeta^4)^2/2],
\]

\[
\zeta^0[(\bar{\zeta}_4 \zeta^4) - (\bar{\zeta}_4 \zeta^4)^2/2], \quad \zeta^0[(\bar{\zeta}_4 \zeta^4)(\bar{\zeta}_4 \zeta^4) - (\bar{\zeta}_4 \zeta^4)^2/2]
\]

In particular as \( \bar{\zeta}_0 \zeta^1 1 \zeta^2 \zeta^3 \) and \( \bar{\zeta}_4 \zeta^0 1 \zeta^2 \zeta^3 \) are self-dual, imposing antiduality eliminates these unwanted states, a good thing since they respectively have \( F = 3 \) and \( Q = -2 \). Applying antiduality and focussing on left-\( \Psi \), the resulting square contains the following varieties of up (\( U \)), down (\( D \)), charged lepton (\( L \)) and neutrinos (\( N \)), where the subscript distinguishes between repetitions. In the \( \zeta \bar{\zeta} \) expansion table, \( \times \) are duals, \( * \) are conjugates:

| \( r \) | \( \bar{r} \) | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|-------|---|---|---|---|---|---|
| \( 0 \) |       | \( L_1 \), \( N_1 \), \( D_3^c \) | \( L_{2,3,}, N_{2,3}, D_{3,6,7}^c, U_3^c \) | \( L_{5}, D_{1}, U_1 \) | \( L_{6}, D_{2}, U_2 \) | \( \times \) | \( \times \) |
| \( 1 \) |       | * | * | * | * | * | * |
| \( 2 \) |       | * | * | * | * | * | * |
| \( 3 \) |       | * | * | * | * | * | * |
| \( 4 \) |       | * | * | * | * | * | * |

Observe that colour singlet and triplet fermions come in 4s, 6s and 8s which comfortably contain the known three generations. However you will see that whereas \( U, D \) in the first and second generation are bona fide weak isospin doublets, the third and fourth family \( U, D \) are accompanied by another exotic
colour triplet quark, call it \( X \) say, having charge \( Q = -4/3 \), and make up a weak isospin triplet. So this is a departure from the standard model! Specifically the weak isospin generators are:

\[
T_+ = \zeta^0 \partial_4 - \bar{\zeta}_4 \bar{\partial}^0, \quad T_- = \zeta^4 \partial_0 - \bar{\zeta}_0 \bar{\partial}^4;
\]

\[
2T_3 = [T_+, T_-] = \zeta^0 \partial_0 - \zeta^4 \partial_4 + \bar{\zeta}_4 \bar{\partial}^4 - \bar{\zeta}_0 \bar{\partial}^0, \text{ so we meet doublets like } (N_1, L_1), (U_1, D_1) \sim (\zeta^0, \zeta^4), \quad (-\bar{\zeta}_4, \bar{\zeta}_0)
\]

singlets like \( L_5, D_5 \sim (\zeta^0 \zeta^0 + \bar{\zeta}_4 \bar{\zeta}_4), \quad (\bar{\zeta}_0 \zeta^0 \bar{\zeta}_4 \bar{\zeta}_4), \)

triplets like \( (U_3, D_3, X_3) \sim (-\bar{\zeta}_4 \zeta^0, [\bar{\zeta}_0 \zeta^0 - \zeta_4 \zeta^4]/\sqrt{2}, \bar{\zeta}_0 \zeta^4) \).

## 3 Exotic Particles, Generations & the Mass Matrix

The \( U \)-states arise from combinations like \( \zeta^i \zeta^j \zeta^k \) lying in the \( \mathbb{H} \)-fold SU(5) combination \( \zeta^i \zeta^j \zeta^k \zeta^l \zeta^m \). (Note that \( \zeta^0 \zeta^4 \zeta^k \) has the exotic value \( F = 5/3 \) and cannot be identified with \( U^c \).) The \( D \)-states occur similarly, as do the \( N \)'s and \( L \)'s. However, observe that \( L_{5,6} \sim \bar{\zeta}_3 \zeta_2 \zeta_1 \) and \( D_{5,6,7,8} \sim \bar{\zeta}_k \) are nominally weak isosinglets, which again differs from the standard model. These mysterious states are definitely charged but do not possess any weak interactions, so their behaviour is very curious. One might even regard \( L_{5,6} \) like colourless charged baryons (antiproton-like) but really until one sees how these states mix with the usual weak isodoublet leptons it is dangerous to label them one thing rather than another without further research.

'Pentaquarks’ such as \( \Theta^+ \sim uudds \) & \( \Xi^{--} \sim ddss \) were recently discovered(?) with quite narrow widths and many people have advanced models to describe these new resonances as well as tetraquark mesons. But who is to say unequivocally that they are not composites of ordinary quarks and another \( U \)-quark or other \( D \)-family quarks which my scheme indicates? Further, I get a fourth neutrino, which could be essentially sterile and might help explain the mystery of \( \nu \) masses. This is clearly fertile ground for investigation and I have only scratched the surface here. Possibly conflicts with experiment may arise that will ultimately invalidate the entire property values scheme.

One of the first matters to be cleared up is the mass matrix and flavour mixing which affects quarks as well as leptons. If we assume it is due to a Higgs \( \Phi \) field’s expectation values, there are nine colourless possibilities having \( F = Q = 0 \) lying in an antidual boson superfield:

- one \( \phi_{(0)(0)} = \langle \phi \rangle = M \)
- one \( \phi_{(0)(4)} = \langle \phi_{1234} \rangle = H \) complex
- three \( \phi_{(1)(1)} = \langle \phi_0, \phi_4, \phi_t \rangle = A, B, C \)
- four \( \phi_{(2)(2)} = \langle \phi_{04}, \phi_{0k}, \phi_{4k}, \phi_{ij} \rangle = D, E, F, G \),
others being related by duality. With nine $\langle \phi \rangle$ the mass matrix calculation already becomes a difficult task! To ascertain what happens, consider the subset of quarks involving $U_{1-4}$ and $D_{1-4,7}$ interacting with anti-selfdual superHiggs.

The Lagrangian $\int d^5 \zeta d^5 \bar{\zeta} \bar{\Psi}(\Phi)\Psi$ produces $U$ and simplistic $D$ mixing matrices:

$$2M(U) \rightarrow \begin{pmatrix} 2M + F/\sqrt{3} & B + C/\sqrt{3} & -H^* & 0 \\ B + C/\sqrt{3} & 2M & 0 & 0 \\ -H & 0 & 2M + G/\sqrt{3} & 2C/\sqrt{3} \\ 0 & 0 & 2C/\sqrt{3} & 2M \end{pmatrix}$$

$$2M(D) \rightarrow \begin{pmatrix} 2M + E/\sqrt{3} & A + C/\sqrt{3} & -H^* & 0 & 0 \\ A + C/\sqrt{3} & 2M & 0 & 0 & 0 \\ -H & 0 & 2M + G/\sqrt{3} & 2C/\sqrt{3} & (E - F)/\sqrt{3} \\ 0 & 0 & 2C/\sqrt{3} & 2M & A - B \\ 0 & 0 & -(E + F)/\sqrt{3} & A - B & 2M - D \end{pmatrix}.$$ 

Similar expressions can be found for the leptons and quarks. Such matrices have to be diagonalised via unitary transformations $V(U)$ & $V(D)$. However the coupling of the weak bosons is different for isodoublets $U_{1,2}$ $D_{1,2}$ and isotriplets $U_{3,4}$ $D_{3,4}$ (and zero for the weak isosinglet $D_7$). So we can’t just evaluate $V^{-1}(U)V(D)$ for the unitary CKM matrix now. Rather the weak interactions connecting $U$ & $D$ mass-diagonalised quarks will not be quite unitary (because of the different coupling factors):

$$L_{\text{weak}}/g_w = W^+(\bar{U}_1D_1 + \bar{U}_2D_2) + \sqrt{2}W^+(\bar{U}_3D_3 + \bar{U}_4D_4 + \bar{D}_3X_3 + \bar{D}_4X_4)$$

$$+ W^-(\bar{D}_1U_1 + \bar{D}_2U_2) + \sqrt{2}W^-(\bar{D}_3U_3 + \bar{D}_4U_4 + \bar{X}_3D_3 + \bar{X}_4D_4) + W^3 \text{ terms}$$

So this is another departure from the standard picture: nonunitarity of the $3 \times 3$ CKM matrix is a test of the scheme. Also CP violation is an intrinsic feature of the property formalism because $\tilde{H}$ is naturally complex, unlike the other expectation values $A, B, C...M$. A more realistic attempt for getting the quark and lepton masses would be to abandon antiduality in the Higgs sector and use all 18 expectation values; otherwise it may prove impossible to cover the 12 or more orders of magnitude all the way from the electron neutrino to the top quark (and higher).

4 Generalized relativity

We know that gauge fields can transport/communicate property from one place to another so where are they? Maybe one can mimic the SUSY procedure and supergauge the massless free action for $\Psi$, without added complication of spin. But there is a more compelling way, which has the benefit of incorporating gravity. Construct a fermionic version of Kaluza-Klein (KK) theory [5], this time without worrying about infinite modes which arise from squeezing normal bosonic coordinates. These are the significant points of such an approach:
• One must introduce a fundamental length $\Lambda$ in the extended $X$, as property $\zeta$ has no dimensions; maybe this is the gravity scale $\kappa = \sqrt{8\pi G_N}$?

• Gravity (plus gauge field products) fall within the $x-x$ sector, gauge fields in $x-\zeta$ and the Higgs scalars must form a matrix in $\zeta-\zeta$.

• Gauge invariance is connected with the number of $\zeta$ so $SU(5)$ or $Sp(10)$ or perhaps a subgroup are indicated.

• There is no place for a gravitino as spin is absent ($\zeta$ are Lorentz scalar).

• There are necessarily a small finite number of modes.

• Weak left-handed SU(2) is associated with rotations of $\zeta$ not $\bar{\zeta}$ so may have something to with $\zeta$-analyticity.

The real metric specifies the separation in location as well as property: it tells us how ‘far apart’ and ‘different in type’ two events are. Setting $\bar{\zeta}_\mu \equiv \bar{\zeta}_\mu$,
$$ds^2 = dx^m dx^n g_{mn} + dx^m d\zeta^\mu G_{\mu n} + dx^n d\zeta^{\bar{\mu}} G_{\bar{\mu} n} + d\zeta^\mu d\zeta^{\bar{\mu}} G_{\mu \bar{\mu}}$$

where the tangent space limit corresponds to Minkowskian
$$G_{\mu \bar{\mu}} \to I_{\mu \bar{\mu}} = \eta_{\mu \bar{\mu}}, \quad G_{\mu \nu} \to I_{\mu \nu} = \Lambda^2 \delta_{\mu \nu},$$
multiplied at least by $(\bar{\zeta}\zeta)^5$ — to arrange correct property integration. Proceeding to curved space the components should contain the force fields, leading one to a ‘superbein’
$$\bar{E}^A_M = \begin{pmatrix} e^a_m & i\Lambda (A_m)^{a \alpha} \zeta^\mu \\ 0 & \Lambda \delta^a_\mu \zeta^\mu \end{pmatrix},$$

and the metric “tensor” arising from
$$ds^2 = dx^m dx^n g_{mn} + 2\Lambda^2 [d\bar{\zeta}^\mu - idx^m \bar{\zeta}^\mu (A_m)^{\mu}_{\bar{\kappa}}] \delta_{\mu \nu} [d\zeta^\nu + idx^m (A_n)^{\nu}_{\lambda} \zeta^\lambda];$$

$$g_{mn} = e^a_m e^b_n \eta_{ab}.$$  

Gauge symmetry corresponds to the special change $\zeta^\mu \to \zeta'^\mu = [\exp(i\Theta(x))]^{\mu}_{\nu} \zeta^\nu$ with $x' = x$. Given the standard transformation law
$$G_{\zeta m}(X) = \frac{\partial X'^R}{\partial x^m} \frac{\partial X'^S}{\partial \zeta} G'_{S R}(X')(-1)^{|R|} = \frac{\partial \zeta'_\alpha}{\partial \zeta} G'_\alpha m - \frac{\partial \zeta'_m}{\partial \zeta} G'_\zeta m,$$

this translates into the usual gauge variation (a matrix in property space),
$$A_m(x) = \exp(-i\Theta(x))[A'_m(x) - i\partial_m \exp(i\Theta(x))].$$

The result is consistent with other components of the metric tensor but does not fix what (sub)group is to be gauged in property space although one most certainly expects to take in the nonabelian colour group and the abelian electromagnetic group, so as to agree with physics.
5 Some notational niceties

When dealing with commuting and anticommuting numbers within a single coordinate framework $X^M = (x^m, \zeta^\mu)$ one has to be exceedingly careful with the order of quantities and of labels. I cannot stress this enough. (It took me six months and much heartache to get the formulae below correct.) For derivatives the rule is $dF(X) = dX^M(\partial F/\partial X^M) = dX^M \partial_ MF$, not with the $dX$ on the right, and for products of functions: $d(FG) = dFG + F dG + ...$. Coordinate transformations read $dX'^M = dX^N (\partial X'^M/\partial X^N)$, and in that particular order. Since $ds^2 = dX^N dX^M G_{MN}$, the symmetry property of the metric is $G_{MN} = (-1)^{[M][N]} G_{NM}$. Let $G^{LM} G_{MN} = \delta^L_N$ for the inverse metric, so $G^{MN} = (-1)^{|M|+|N|} G_{NM}$. The notation here is $|M| = 0$ for bosons and 1 for fermions.

Changing coordinate system from $X$ to $X'$, we have to be punctilious with signs and orders of products, things we normally never care about; the correct transformation law is

$$G_{NM}(X) = \left( \frac{\partial X'^R}{\partial X^M} \right) \left( \frac{\partial X^S}{\partial X^N} \right) G^S_{SR}(X') (-1)^{|N||R|+|M|}.$$  

Transformation laws for contravariant and covariant vectors read:

$$V'^M(X') = V^R(X) \left( \frac{\partial X'^M}{\partial X^R} \right) \text{ and } A'_M(X') = \left( \frac{\partial X^R}{\partial X'^M} \right) A_R(X),$$

in the order stated. Thus the invariant contraction is

$$V'^M(X') A'_M(X') = V^R(X) A_R(X) = (-1)^{|R|} A_R(X) V^R(X).$$

The inverse metric $G^{MN}$ can be used to raise and lower indices as well as forming invariants, so for instance $V_R \equiv V^S G_{SR}$ and $V'^R V'^S G_{SR} = V'^M V^N G_{MN}$.

The next issue is covariant differentiation; we insist that $A_{M,N}$ should transform like $T_{MN}$, viz.

$$T'_{MN}(X') = (-1)^{|S|+|N||R|} \left( \frac{\partial X^R}{\partial X'^M} \right) \left( \frac{\partial X^S}{\partial X'^N} \right) T_{RS}(X).$$

After some work we find that

$$A_{M,N} = (-1)^{|M||N|} A_{M,N} - A^L \Gamma_{[M,N,L]},$$

where the connection is given by

$$\Gamma_{[M,N,L]} \equiv \frac{[(-1)^{|L|+|M||N|} G_{LM,N} + (-1)^{|M||L|} G_{LN,M} - G_{MN,L}]/2}{(-1)^{|M||N|} \Gamma_{[N,M,L]}.}$$

Another useful formula is the raised connection

$$\Gamma_{MN}^K \equiv (-1)^{|L|(|M|+|N|)} \Gamma_{[M,N,L]} G^{LK} = (-1)^{|M||N|} \Gamma_{NM}^K,$$
whereupon one may write

\[ A_{M;N} = (-1)^{[M][N]}A_{M,N} - \Gamma_{MN} L A_L. \]

Similarly one can show that for double index tensors the true differentiation rule is

\[ T_{LM;N} \equiv (-1)^{[N][L]+[M]} T_{LM,N} - (-1)^{[M][N]} \Gamma_{LN} K T_{KM} - (-1)^{[L][M]+[N][K]} \Gamma_{MN} K T_{LK}. \]

As a nice check, the covariant derivative of the metric properly vanishes:

\[ G_{LM;N} \equiv (-1)^{[N][L]+[M]} G_{LM,N} - (-1)^{[L][M]} \Gamma_{LN,M} - \Gamma_{\{MN,L\}} = 0. \]

Moving on to the Riemann curvature we form doubly covariant derivatives:

\[ A_K;L,M - (-1)^{[L][M]} A_{K;M,L} \equiv (-1)^{[J][L]+[M]} R^{J}_{KLM} A_J \]

where one discovers that

\[ R^{J}_{KLM} \equiv (-1)^{[K][M]} (\Gamma_{KM} J)_L - (-1)^{[L][K]+[M]} (\Gamma_{KL} J)_M \\
+ (-1)^{[M][K]+[L]} \Gamma_{KM} N \Gamma_{NL} J - (-1)^{[K][M]+[L]} \Gamma_{KM} N \Gamma_{NM} J. \]

Evidently, \( R^{J}_{KLM} = (-1)^{[L][M]} R^{J}_{KML} \) and, less obviously, the cyclical relation takes the form

\[ (-1)^{[K][L]} R^{J}_{KLM} + (-1)^{[L][M]} R^{J}_{LMK} + (-1)^{[M][K]} R^{J}_{MKL} = 0. \]

The fully covariant Riemann tensor is \( R^{J}_{KLM} \equiv (-1)^{[J][L]+[K][M]} R^{N}_{KLM} G_{NJ} \) with pleasing features:

\[ R^{J}_{KLM} = -(-1)^{[L][M]} R^{J}_{JKLM} = -(-1)^{[J][K]} R^{J}_{KLM}, \]
\[ 0 = (-1)^{[J][L]} R^{J}_{JKLM} + (-1)^{[J][M]} R^{J}_{JLMK} + (-1)^{[J][K]} R^{J}_{JMKL} \]
\[ R^{J}_{JKLM} = -(-1)^{[J][K][L]+[M]} R^{J}_{LMJK}. \]

Finally proceed to the Ricci tensor and scalar curvature:

\[ R^{J}_{KM} \equiv (-1)^{[J][K][L]+[J][K]+[M]} G^{LJ} R^{J}_{JKLM} \\
= (-1)^{[L][K]+[L]+[M]} R^{L}_{KLM} = (-1)^{[K][M]} R_{MK}, \]

\[ R \equiv G^{MK} R_{KM}. \]

## 6 Curvatures of space-time-property

When Einstein produced his general theory of relativity with Grossmann he wrote all his expressions in terms of real variables. In order to avoid any confusion with complex variables, we shall copy him by writing everything in terms of real coordinates \( \xi, \eta \) rather than complex \( \zeta = (\xi + i\eta)/\sqrt{2} \). (Note that the real invariant is \( \zeta^* \zeta = \zeta \xi \eta \) and that a phase transformation of \( \zeta \) corresponds to a real rotation in \((\xi, \eta)\) space.) For simplicity consider just one extra pair (rather than five pairs) and the following two examples, which are complicated enough as it is.
(1) Decoupled property and space-time, but both curved:

\[ ds^2 = dx^m dx^n G_{mn}(x, \xi, \eta) + 2 i d\xi d\eta G_{\xi\eta}(x, \xi, \eta) \]

\[ \equiv dx^m dx^n g_{nm}(x)(1 + i f \xi \eta) + 2 \Lambda^2 d\xi d\eta(1 + i g \xi \eta) \]

from which we can read off the metric components \((G^{LM} G_{MN} = \delta^L_N)\)

\[
\begin{pmatrix}
G_{\xi\xi} & G_{\xi\eta} & G_{\eta\eta} \\
G_{\eta\xi} & G_{\eta\eta} & 0 \\
G_{\eta\eta} & 0 & 0
\end{pmatrix} = \begin{pmatrix}
g_{\xi\xi}(1 + i f \xi \eta) & 0 & 0 \\
0 & 0 & -i \Lambda^2(1 + i g \xi \eta) \\
0 & i \Lambda^2(1 + i g \xi \eta) & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
G^{\xi\xi} & G^{\xi\eta} & G^{\eta\eta} \\
G^{\eta\xi} & G^{\eta\eta} & 0 \\
G^{\eta\eta} & 0 & 0
\end{pmatrix} = \begin{pmatrix}
g^{\xi\xi}(1 - i f \xi \eta) & 0 & 0 \\
0 & 0 & -i(1 - i g \xi \eta)/\Lambda^2 \\
0 & i(1 - i g \xi \eta)/\Lambda^2 & 0
\end{pmatrix}.
\]

The non-zero connections in the property sector are

\[ \Gamma^\xi_{\xi\eta} = -\Gamma^\eta \xi = ig \xi, \quad \Gamma^\eta_{\eta\xi} = -\Gamma^\xi_{\eta\xi} = ig \eta \]

so

\[ R^\eta \xi_{\eta\eta} = -2i g(1 + i g \xi \eta) = -R^\xi \eta \xi \]

\[ R^\xi \xi_{\eta\eta} = -ig (1 + i g \xi \eta) = -R^\eta \eta \xi \]

so

\[ R_{\xi\xi} = -R_{\eta\eta} = 3i g (1 + i g \xi \eta), \]

Consequently the total curvature is given by

\[ R = G^{mn} R_{nm} + 2 G^{\eta\xi} R_{\xi\eta} = R^{(g)}(1 - i f \xi \eta) - 6 g/\Lambda^2. \quad (1) \]

Since \(\sqrt{-G} = -i \Lambda^2 \sqrt{-g} \cdot (1 + 2 i f \xi \eta)(1 + i g \xi \eta)\), we obtain an action

\[ I \equiv \frac{1}{2\Lambda^4} \int R \sqrt{-G} \cdot d^4 x d\eta d\xi = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \cdot \left[ R^{(g)} + \lambda \right] \]

where \(\kappa^2 = 8\pi G_N = \Lambda^2/(f + g)\), \(R^{(g)}\) is the standard gravitational curvature and \(\lambda = -6g(2f + g)/\Lambda^2(f + g)\) corresponds to a cosmological term.

(2) Our second example leaves property space flat (in the \(\eta, \xi\) sector) but introduces a U(1) gauge field \(A\), governed by the metric,

\[
\begin{pmatrix}
G_{\xi\xi} & G_{\xi\eta} & G_{\eta\eta} \\
G_{\eta\xi} & G_{\eta\eta} & 0 \\
G_{\eta\eta} & 0 & 0
\end{pmatrix} = \begin{pmatrix}
g_{\xi\xi}(1 + i f \xi \eta) + 2 i \Lambda^2 A_m A_n \eta & i \Lambda^2 A_m \xi & i \Lambda^2 A_m \eta \\
i \Lambda^2 A_n \xi & 0 & -i \Lambda^2 \\
i \Lambda^2 A_n \eta & i \Lambda^2 & 0
\end{pmatrix}.
\]

Simplify the analysis somewhat by going to flat (Minkowski) space first as there are then fewer connections. After some work \((F_{mn} \equiv A_{m,n} - A_{n,m})\) one obtains,

\[
\begin{align*}
\Gamma^\xi_{\xi\eta} &= \Gamma^\eta \xi = \Gamma^\xi_{\eta\xi} = 0, \\
\Gamma^\xi_{\eta\xi} &= \Gamma^\eta \xi = i \Lambda^2 A^I F_{\xi\eta \xi} / 2, \\
\Gamma^\xi_{m\xi} &= \Gamma^\eta m = - \Gamma^\eta \xi = A_m, \\
\Gamma^\xi_{m\xi} &= i \Lambda^2 F_{m\xi \xi} / 2, \\
\Gamma^\xi_{m\xi} &= i \Lambda^2 F_{m\eta \eta} / 2, \\
\Gamma^\xi_{m\xi} &= - A_m A_n \xi - (A_{m,n} + A_{n,m}) \eta / 2, \\
\Gamma^\xi_{m\xi} &= - A_m A_n \eta + (A_{m,n} + A_{n,m}) \xi / 2, \\
\Gamma^\xi_{m\xi} &= i \Lambda^2 (A_m F_{m\xi} + A_n F_{m\eta}) \xi.
\end{align*}
\]
Other Christoffel symbols can be deduced through symmetry of indices. Hence
\[ R_{km} = R^l_{klm} - R^l_{k\xi m} - R^l_{\eta km} = -i\Lambda^2(A_{k,l} + A_{l,k})F^l_{m}\xi\eta/2 + \text{total der.} \]
\[ R_{k\xi} = R^l_{kl\xi} + R^l_{k\xi \xi} + R^\eta_{k\xi \xi} = i\Lambda^2[F^l_{k,l}\xi/2 + A^l_{k,l}\eta] + \text{total der.} \]
\[ R_{k\eta} = R^l_{kl\eta} + R^l_{k\xi \eta} + R^\eta_{k\eta \eta} = i\Lambda^2[F^l_{k,\eta}/2 - A^l_{k,\xi}] + \text{total der.} \]
\[ R_{\xi\eta} = -\Lambda^4g_{kl}F^l_{m\xi\eta}/2 + \text{total der.} \]

Then covariantize by including the gravitational component \( g_{mn}(1 + if\xi\eta) \) to end up with the total curvature:
\[ R = G^{mn}R_{nm} + 2G^{m\xi}R_{\xi m} + 2G^{m\eta}R_{\eta m} + 2G^{\eta \xi}R_{\xi \eta} \]
\[ \rightarrow R^{(g)} - 3i\Lambda^2g^{km}g^{ln}F_{kl}F_{nm}\xi\eta/2 \]

Finally rescale \( A \) to identify the answer as electromagnetism + gravitation:
\[ \int R\sqrt{-G}.d^4x.d\eta.d\xi/4\Lambda^4 = \int d^4x\sqrt{-g} \left[ R^{(g)}/2\kappa^2 - F^k_{kl}/4 \right], \]

where \( \kappa^2 = \Lambda^2/f = 8\pi G_N \). It is a nice feature of the formalism that the gauge field Lagrangian arises from space-property terms — like the standard K-K model (from the tie-up between ordinary space-time and the fifth dimension).

Where do we go from these two examples? Well some generalizations come to mind:

- replace property couplings \( f \) and \( g \) by two fields (dilaton and Higgs),
- combine the two models; this should lead to gravity + em + cosmic const.,
- extend fully to five \( \zeta \); it is easy enough to incorporate the standard gauge model and one can even entertain a GUT \( SU(5) \) of some ilk,
- work out the particle mass spectrum from all the \( \langle \phi \rangle \).

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