PERFORMANCE OF THE OPTIMIZED POST-ZEL’DOVICH APPROXIMATION FOR COLD DARK MATTER MODELS IN ARBITRARY FLRW COSMOLOGIES

TAKASHI HAMANA
Astronomical Institute, Tohoku University, Sendai 980-8578, Japan
Received 1998 August 13; accepted 1998 August 27; published 1998 September 15

ABSTRACT

We investigate the performance of the optimized post-Zel’dovich approximation in three cold dark matter cosmologies. We consider two flat models with $\Omega_0 = 1$ (SCDM) and with $\Omega_0 = 0.3$ (OCDM) and an open model with $\Omega_0 = 0.3$ (LCDM). We find that the optimization scheme proposed by Weiss, Gottlöber, & Buchert, in which the performance of the Lagrangian perturbation theory was optimized only for the Einstein–de Sitter cosmology, shows excellent performance not only for SCDM model but also for both OCDM and LCDM models. This universality of the excellent performance of the optimized post-Zel’dovich approximation is explained by the fact that a relation between the post-Zel’dovich order’s growth factor $E(a)$ and Zel’dovich order’s one $D(a)$, $E(a)D^2(a)$, is insensitive to the background cosmologies.

Subject headings: dark matter — large-scale structure of universe — methods: numerical

1. INTRODUCTION

The Zel’dovich approximation (Zel’dovich 1970) is known to be accurate even in the weakly nonlinear regime of the structure formation. However, when the shell crossing occurs, the validity of the approximation breaks down. Coles, Melott, & Shandrin (1993) introduced a truncated Zel’dovich approximation by smoothing the small-scale power in the initial conditions to remove the unwanted nonlinearity. Optimization of the smoothing schemes, the filter shapes and filter scales, and the performance of the approximation have been investigated by authors (Melott, Pellman, & Shandarin 1994; Melott, Buchert, & Weiss 1995). They found that the truncated Zel’dovich approximation does not provide a correct description of the internal structure and mass distribution of nonlinear structures like galaxy clusters, but it is accurate in locating their positions and thus reliably describes their spatial distribution. Therefore, the truncated Zel’dovich approximation is a powerful tool for studying the large-scale distribution of galaxy clusters, and it provides important constraints on models of cosmic structure formation (see, e.g., Borgani et al. 1995).

From a theoretical point of view, very recently, Takada & Futamase (1998) developed a formalism that allows one to investigate a relation between the large-scale quasi-nonlinear dynamics and the small-scale nonlinear dynamics within using an averaging method in the Lagrangian perturbation theory. They found that the small-scale dynamics only weakly affects the large-scale structure formation, and thus the truncated Lagrangian perturbation theory is a good approximation to investigate the large-scale structure formation.

The second-order correction to the truncated Zel’dovich approximation was introduced by Melott (1994), and Melott et al. (1995). They found that the second-order correction improves the performance of the approximation. In their study, Weiss, Gottlöber, & Buchert (1996) optimized the performance of the truncated second-order Zel’dovich approximation in the Einstein–de Sitter cosmology. They performed N-body simulations of the cold dark matter (CDM) and broken scale invariance models, and they compared the results with those obtained from the optimized approximation scheme. They found excellent performance of the optimized approximation down to scales close to the correlation length. However, since the optimization was performed only in the Einstein–de Sitter cosmology, it was not clear whether the same optimized scheme would perform as excellent in arbitrary Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies. Since the optimized approximation is a powerful tool to investigate the formation of large-scale structures, it is worthwhile to generalize the optimized scheme of the approximation to arbitrary FLRW cosmologies.

The main purpose of this Letter is to test the performance of the optimized second-order Zel’dovich approximation obtained by Weiss et al. (1996) (hereafter the optimized post-Zel’dovich approximation) in arbitrary FLRW cosmologies. We examine three CDM models, two flat models with $\Omega_0 = 1$ (SCDM) and with $\Omega_0 = 0.3$ (OCDM) and an open model with $\Omega_0 = 0.3$ (LCDM). From the results of the optimized post-Zel’dovich approximation, we calculate the two-point correlation functions. We do not perform the $N$-body simulation, but we compare the correlation functions with those predicted by a parameterized fitting formula that Peacock & Dodds (1996) use to predict the power spectrum of the nonlinear mass density field. We refer the reader to the above reference for details of the formula and its implementation. In their recent paper, Jenkins et al. (1998) compared the two-point correlation functions obtained from their very large $N$-body simulations with that predicted by fitting formula by Peacock & Dodds (1996) and found a good agreement between them over scales between ~0.1 and ~10 $h^{-1}$ Mpc.

The plan of this Letter is as follows. Section 2 gives a brief summary of the optimized post-Zel’dovich approximation. The performance of the approximation in arbitrary FLRW cosmologies is tested in § 3. Our Letter concludes in § 4 with discussions.

Throughout this Letter, we use a unit for which $c = 1$, and the scale factor $a$ is normalized to unity at the present epoch, i.e., $a_0 = 1$. The Hubble parameter $H$, density parameter $\Omega$, and normalized cosmological constant $\lambda$ are defined in the usual manner. Quantities of the present epoch and initial epoch are indicated by indices 0 and i, respectively.

2. SUMMARY OF THE OPTIMIZED POST-ZEL’DOVICH APPROXIMATION

The Zel’dovich and post-Zel’dovich approximation are regarded as subclasses of the first-order and second-order solution
of the Lagrangian perturbation theory, respectively (Buchert 1989, 1992). Since the Lagrangian perturbation theory has been thoroughly investigated by various authors (see, e.g., Buchert & Ehlers 1993; Buchert 1994; Bouchet et al. 1995; Catelan 1995; Sasaki & Kasai 1998), here we describe only the aspects that are directly relevant to this Letter.

Denoting the comoving Eulerian coordinates by $x$, and Lagrangian coordinates by $q$, the field of trajectories $x = F(q,a)$ up to the post-Zel’dovich order is

$$x = q + D(a)\mathbf{\nabla}\Psi^{(1)} + E(a)\mathbf{\nabla}\Psi^{(2)}, \quad (1)$$

with the time-dependent coefficients expressed in terms of the linear growth rate $D_i(a)$ (Peebles 1980)

$$D(a) = \frac{D_i(a)}{D_i(a)} - 1, \quad (2)$$

$$E(a) = \frac{3\Omega_0 H_0^2}{4} \left[ \frac{\alpha'}{(H'(a')})^3 \int da' D_i(\alpha') a''^2 \right], \quad (3)$$

where $H(a)$ is the Hubble parameter, $H(a) \equiv H_0(\Omega_0/\alpha^3 + \lambda_0 - K/\alpha^2)^{1/2}$. In the above expressions, we only take the fastest growing mode for each order. It is important to note that the relation between $D(a)$ and $E(a)$, $E(a)/D^2(a)$, is remarkably insensitive to the background cosmologies (Bouchet et al. 1992, 1995). The displacement potentials are obtained by solving iteratively two Poisson equations;

$$\Delta \Psi^{(1)} = -\delta_i, \quad (4)$$

$$\Delta \Psi^{(2)} = \Psi_{ij}^{(1)}\delta_{ij} - \Psi_{ij}^{(1)}\Psi_{kj}^{(1)}, \quad (5)$$

where $\delta_i$ is an initial density contrast field.

Next, we review the optimization scheme proposed by Weiss et al. (1996), which we adopt in this Letter. The high-frequency part of the Fourier transform of an initial density field is smoothed out by a Gaussian $k$-space filter with a characteristic smoothing scale $k_s$,

$$W(k) = \exp \left(-\frac{k^2}{2k_s^2}\right), \quad (6)$$

i.e., $P(k) \rightarrow P(k)W^2(k)$, where $P(k)$ is the power spectrum of the initial density field. Weiss et al. (1996) found that an optimal value of $k_s$ does not significantly depend on the form of $P(k)$ and is related to a scale of nonlinearity $k_{nl}$ by $k_s \sim 1.2k_{nl}$ with a little scatter. The quantity $k_{nl}$ is defined by

$$D_i^2(\alpha) \int_0^{k_{nl}} d^3k P(k) = 1. \quad (7)$$

We adopt the recommended value of $k_{nl} = 1.2k_{nl}$.

### 3. MODELS AND RESULTS

We examine three CDM models. Table 1 lists the models and gives their parameters. We use the CDM transfer function in Bardeen et al. (1986), with the scale-invariant ($n = 1$) primordial power spectrum. The shape parameter, $\Gamma$, in the spectrum defined by Sugiyama (1995), which we adopt, is

$$\Gamma = \Omega_c h \log [-1 + (2n/\Omega_c)], \quad (8)$$

where $\Omega_c$ is the baryonic matter density parameter and $h$ is the normalized Hubble constant, i.e., $H_0 = 100 h$ km s$^{-1}$ Mpc$^{-1}$. In all cases, the amplitude of primordial fluctuation is set such that the models reproduce the observed abundance of rich galaxy clusters of the present day. We adopt the values of $\alpha_s$ recommended by Eke, Cole, & Frenk (1996).

The initial density field is set on $128^3$ grid points for a cubic box of $L = 128 \times 128 \times 128h^{-1}$ Mpc a side, with a periodic boundary condition, where $\Delta x$ is the grid spacing. Here we use Bertschinger’s software COSMICS (Bertschinger 1995), with some modifications according to the shape parameter (eq. [8]), and smoothing of the power spectrum of the initial density field with the Gaussian filter (eq. [6]). In order to ensure the condition $\delta_i \ll 1$, which is required for deriving equation (4), the initial condition is set at a redshift $z_i = 10^4$. We solve the Poisson equations (4) and (5) via fast Fourier transformation. Then we move 128$^3$ particles having the initial Lagrangian coordinate on the grid, according to the post-Zel’dovich approximation, equation (1). Here, for the linear growth factor $D_i(a)$, we use the fitting formula of Carrol, Press, & Turner (1992), and the post-Zel’dovich order’s growth rate $E(a)$ is evaluated by numerical integrations.

We consider three cases of the box size. The grid spacings and particle masses are summarized in Table 2. The redshifts of realizations are chosen to be $z = 0, 1, 2$, and the scales of nonlinearity, $k_{nl}$, for each redshift for each model are presented in Table 3.

### Table 1: Summary of Model Parameters

| Model   | $\Omega_0$ | $\lambda_0$ | $\Omega_0 h^2$ | $h$ | $\alpha_s$ |
|---------|------------|--------------|----------------|-----|------------|
| SCDM    | 1.0        | 0.0          | 0.015          | 0.5 | 0.51       |
| OCDM    | 0.3        | 0.0          | 0.015          | 0.7 | 0.85       |
| ACDM    | 0.3        | 0.7          | 0.015          | 0.7 | 0.90       |

### Table 2: The Grid Spacing $\Delta x$ and Particle Mass

| Model   | Label | $\Delta x$ | Particle Mass |
|---------|-------|------------|---------------|
| SCDM    | S     | 0.5        | $3.47 \times 10^{10}$ |
|         | M     | 1.0        | $2.78 \times 10^{11}$ |
|         | L     | 2.0        | $2.22 \times 10^{12}$ |
| OCDM and ACDM | S | 0.75       | $3.51 \times 10^{10}$ |
|         | M     | 1.5        | $2.81 \times 10^{11}$ |
|         | L     | 3.0        | $2.25 \times 10^{12}$ |

### Table 3: The Scale of Nonlinearity $k_{nl}$ ($h$ Mpc$^{-1}$)

| Redshift of Realization $z$ | SCDM | OCDM | ACDM |
|-----------------------------|------|------|------|
| 0                           | 0.548| 0.303| 0.279|
| 1                           | 1.59 | 0.559| 0.605|
| 2                           | 3.62 | 0.933| 1.26 |
From the results of the realizations, we evaluate the two-point correlation function of the particles by adopting the direct estimator (Hockney & Eastwood 1988)

$$\xi(r) = \frac{N_c \Delta x^3}{N \delta V} - 1,$$

where $N_c$ is the number of pairs of particles with separations between $r - \Delta/2$ and $r + \Delta/2$, $\delta V$ is the volume of this shell, and $N_c$ is the number of particles taken as centers. The results are plotted in Figure 1.

It can be shown in Figure 1a that our results for the SCDM model, of course, agree well with that obtained by Weiss et al. (1996, see their Fig. 7). It can be also shown in Figures 1b and 1c that the comparable performance of the optimized post-Zel’’dovich approximation is also achieved in both OCDM and LCDM models. In all cases, the correlation functions obtained from the approximation are accurate down to the scales where the nonlinear correlation functions represent their nonlinear behavior, and below those scales they are depressed. Meanwhile the correlation length itself is underestimated only slightly by less than 1 $h^{-1}$ Mpc (this point has been also pointed out by

---

Fig. 1.—Two-point correlation functions compared with those predicted by the linear theory and fitting formula by Peacock & Dodds (1996). The results of our S, M, and L box realizations are shown by filled triangles, pluses and filled circles, respectively. Solid lines show the correlation function derived from the nonlinear power spectrum, and dashed lines are those predicted by the linear theory. (a) SCDM model; (b) OCDM model; (c) LCDM model. In all panels, from the top to bottom, the redshifts of realizations are $z = 0$, 1, and 2, respectively.
Weiss et al. (1996). For the high-redshift realization cases, the correlation functions agree well with those predicted by the linear theory down to very close to the correlation length.

4. DISCUSSION

In the last section, we found that the optimized post-Zel’dovich approximation by Weiss et al. (1996) performs excellently not only in SCDM model but also in both OCDM and ΛCDM models. This is, as a priori, not clear because in Weiss et al. (1996) the optimization was performed only in the Einstein–de Sitter background cosmology.

The universality of the excellent performance of the optimized post-Zel’dovich approximation may be explained as follows. The smoothing scale $k_s$ is chosen to be related to the scale of nonlinearity $k_{nl}$, and $k_{nl}$ is determined by the integral of the linearly evolved power spectrum of the initial density field, equation (7), i.e., the power spectrum evolves as $\propto D^2(a) \approx D^2(a)$. On the other hand, the post-Zel’dovich order’s displacement is evolved according to $E(a)$, and the unwanted nonlinearity in the initial data is removed by the Gaussian filter with the smoothing scale $k_s$. Thus, $k_{nl}$ relates to $D^2(a)$, and $k_{nl}$ relates to $E(a)$. Therefore, an optimized relation between $k_{nl}$, and $k_{nl}$ is determined by the relation between $D^2(a)$ and $E(a)$. As was pointed out by Bouchet et al. (1992, 1995), $E(a)/D^2(a)$ is very insensitive to the cosmologies. Therefore, once an optimized relation between $k_{nl}$, and $k_{nl}$ is found in a background cosmology, it is also optimized in arbitrary background cosmologies. We should also note that although we dealt only with the CDM model, it has been found that the optimized relation between $k_{nl}$ and $k_{nl}$ is insensitive to shapes of the power spectrum (Melott et al. 1994, 1995; Weiss et al. 1996).

From a computational point of view, the optimized post-Zel’dovich approximation is very time efficient; this enables us to compute many independent models within a reasonable CPU time. Although, the approximation cannot provide a correct description of the internal structure of nonlinear structures, it describes their spatial distribution well (Weiss et al. 1996).

As was shown in the last section, two-point correlation functions are accurate down to the scale close to the correlation length. Therefore, the approximation is a powerful tool to study the formation of large-scale structure on scales above the correlation length, such as the analysis of the cluster distribution (Borgani et al. 1995). It is also appropriate for the study of the gravitational lensing by the large-scale structures (Bertelmann & Schneider 1991).

Before closing this Letter, we propose to use the post-Zel’dovich approximation for setting an initial condition of $N$-body simulations. The post-Zel’dovich order’s solutions can be easily obtained from the Zel’dovich solutions, equations (3) and (5). Thus, the post-Zel’dovich approximation is as easy to implement as the Zel’dovich approximation. Therefore, one who adopts the Zel’dovich approximation to set the initial condition can easily include the post-Zel’dovich correction, which will improve the accuracy of the simulations.

The author would like to thank M. Kasai for valuable comments on this Letter, M. Morita for useful discussion on the Zel’dovich approximation, and P. Premadi for carefully reading and commenting the manuscript. He also would like to thank T. Futamase and M. Takada for providing their manuscript prior to publication and for useful discussion.

REFERENCES

Bardeen, J., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15
Bertelmann, M., & Schneider, P. 1991, A&A, 248, 349
Bertschinger, E. 1995, preprint (astro-ph/9506070)
Borgani, S., Plionis, M., Coles, P., & Moscardini, L. 1995, MNRAS, 277, 1191
Bouchet, F. R., Colombi, S., Hivon, E., & Juszkiewicz, R. 1995, A&A, 296, 575
Bouchet, F. R., Juszkiewicz, R., Colombi, S., & Pellat, R. 1992, ApJ, 394, L5
Buchert, T. 1989, A&A, 223, 9
———. 1992, MNRAS, 254, 729
———. 1994, MNRAS, 267, 811
Buchert, T., & Ehlers, J. 1993, MNRAS, 264, 375
Carrol, S. M., Press, W. H., & Turner, E. L. 1992, ARA&A, 30, 499
Catelan, P. 1995, MNRAS, 276, 115
Coles, P., Melott, A. L., & Shandarin, S. F. 1993, MNRAS, 260, 765
Eke, V. R., Cole, S., & Frenk, C. S. 1996, MNRAS, 282, 263
Hockney, R. W., & Eastwood, J. W. 1988, Computer Simulation Using Particles (Bristol: Adam Hilger)
Jenkins, A., et al. (The VIRGO Consortium). 1998, ApJ, 499, 20
Melott, A. L. 1994, ApJ, 426, L19
Melott, A. L., Buchert, T., & Weiss, A. G. 1995, A&A, 294, 345
Melott, A. L., Pellman, T. F., & Shandarin, S. F. 1994, MNRAS, 269, 626
Peacock, J. A., & Dodds, S. J. 1996, MNRAS, 280, L19
Peebles, P. J. E. 1980, The Large Scale Structure of the Universe (Princeton: Princeton Univ. Press)
Sasaki, M., & Kasai, M. 1998, Prog. Theor. Phys., 99, 585
Sugiyama, N. 1995, ApJS, 100, 281
Takada, M., & Futamase, T. 1998, Gen. Rel. Grav., submitted
Weiss, A. G., Gottlöber, S., & Buchert, T. 1996, MNRAS, 278, 953
Zel’dovich, Ya. B. 1970, A&A, 5, 84