On $\tau$-tilting finiteness of blocks of Schur algebras

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Abstract

In this paper, we determine the $\tau$-tilting finiteness for some blocks of (classical) Schur algebras. Combining with the results in [W3], we get a complete classification of $\tau$-tilting finite Schur algebras. As a refinement, we also give a complete classification of $\tau$-tilting finite blocks of the Schur algebra $S(2, r)$.

Keywords: $\tau$-tilting modules, $\tau$-tilting finiteness, blocks, Schur algebras.

1 Introduction

Let $n, r$ be positive integers and $\mathbb{F}$ an algebraically closed field of characteristic $p > 0$. We take an $n$-dimensional vector space $V$ over $\mathbb{F}$ with a basis $\{v_1, v_2, \ldots, v_n\}$. We denote by $V^{\otimes r}$ the $r$-fold tensor product $V \otimes_\mathbb{F} V \otimes_\mathbb{F} \ldots \otimes_\mathbb{F} V$. Then, $V^{\otimes r}$ has a $\mathbb{F}$-basis given by

$$\{v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r} \mid 1 \leq i_j \leq n \text{ for all } 1 \leq j \leq r\}.$$

Let $G_r$ be the symmetric group on $r$ symbols and $\mathbb{F}G_r$ its group algebra. Then, $G_r$, and hence also $\mathbb{F}G_r$, act on the right on $V^{\otimes r}$ by place permutations of the subscripts, that is, for any $\sigma \in G_r$,

$$(v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r}) \cdot \sigma = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \ldots \otimes v_{\sigma(i_r)}.$$

We call the endomorphism ring $\text{End}_{\mathbb{F}G_r} (V^{\otimes r})$ the Schur algebra (see [Ma, Section 2]) and denote it by $S_{\mathbb{F}}(n, r)$, or simply by $S(n, r)$.

One of the important roles of $S(n, r)$ is that the module category of $S(n, r)$ is equivalent to the category of $r$-homogeneous polynomial representations of the general linear group $\text{GL}_n(\mathbb{F})$. Based on this well-known fact, $q$-Schur algebras, infinitesimal Schur algebras, Borel-Schur algebras, etc., appear as derivatives. In addition, $S(n, r)$ is known to be a quasi-hereditary algebra that is closely related to the highest weight category and plays a significant role in Lie theory. Actually, the representation theory of $S(n, r)$ has been well-studied in the past few decades. We refer to [Dc], [DEMN], [DN], [EH1], [EH2], [EL], [Ma], etc., for more properties of Schur algebras.

In the representation theory of finite-dimensional algebras, the notion of progenerators is crucial because it characterizes the equivalence between the module categories of two algebras, in which case these two algebras are called Morita equivalent. In the 1970s, a generalization called tilting modules (together with tilting complexes) is discovered

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during the development of Morita theory for derived categories and has been extensively studied by many mathematicians, such as [BB], [HU], [Ric], [RS], etc. In recent years, the \(\tau\)-tilting theory introduced by Adachi-Iyama-Reiten [AIR] has drawn more and more attention, where \(\tau\) denotes the Auslander-Reiten translation. A central notion of \(\tau\)-tilting theory is the class of support \(\tau\)-tilting modules, which is a completion to the class of tilting modules from the viewpoint of mutation. Moreover, support \(\tau\)-tilting modules are in bijection with several important objects in representation theory, such as two-term silting complexes, functorially finite torsion classes, and left finite semibricks. One may look at [AI], [As], [AHMW], [BST], [DIRRT] and [EJR] for more materials.

Similar to the representation-finiteness of algebras, a modern analog called \(\tau\)-tilting finiteness is introduced by Demonet-Iyama-Jasso [DIJ]. Here, a finite-dimensional algebra \(A\) is called \(\tau\)-tilting finite if it has only finitely many isomorphism classes of basic support \(\tau\)-tilting modules. In the \(\tau\)-tilting finite case, the set of support \(\tau\)-tilting modules has a nice behavior, for example, the set admits bijections to the set of all torsion classes [DIJ] and the set of all semibricks [As]. So far, the \(\tau\)-tilting finiteness for several classes of algebras is known, including [Ad], [AAC], [MI], [MS], [Mo], [PI], [W1], [W2], [Zi], etc.

Since the representation type of \(S(n, r)\) over an algebraically closed field \(F\) of characteristic \(p > 0\) is completely determined, we may consider the \(\tau\)-tilting finiteness of \(S(n, r)\) as a new property. We mention that the \(\tau\)-tilting finiteness of most Schur algebras has been determined in [W3], and the only remaining cases are displayed in \((\star)\) below. So, our first aim in this paper is to solve these open cases.

\[
(\star) \begin{cases} 
  p = 2, n = 2, r = 8, 17, 19; \\
  p = 2, n = 3, r = 4; \\
  p = 2, n \geq 5, r = 5; \\
  p \geq 5, n = 2, p^2 \leq r \leq p^2 + p - 1.
\end{cases}
\]

**Theorem 1.1** (Theorem 3.6). Let \(S(n, r)\) be the Schur algebra over \(F\).

1. If \(p = 2\), then \(S(2, 8), S(2, 17)\) and \(S(2, 19)\) are \(\tau\)-tilting finite.
2. If \(p = 2\), then \(S(3, 4)\) is \(\tau\)-tilting finite.
3. If \(p = 2\), then \(S(n, 5)\) is \(\tau\)-tilting infinite for any \(n \geq 5\).
4. If \(p \geq 5\), then \(S(2, r)\) is \(\tau\)-tilting finite for any \(p^2 \leq r \leq p^2 + p - 1\).

According to the work in [W3] and the above results, we have obtained a complete classification of \(\tau\)-tilting finite Schur algebras. We may give a collection in Appendix A to provide a complete list of \(\tau\)-tilting finite Schur algebras. Besides, the number of isomorphism classes of basic support \(\tau\)-tilting modules for \(\tau\)-tilting finite \(S(n, r)\)’s over \(p = 2, 3\) is completely determined, see also Appendix A.

In order to understand a \(\tau\)-tilting finite Schur algebra \(S(n, r)\), it is enough to find the basic algebra \(\overline{S(n, r)}\) of \(S(n, r)\). We point out that finding the basic algebra \(\overline{S(n, r)}\)
has been completed by several previous works, including [El], [DEMN] and [W3]. In the proof of Theorem 1.1, one can see that (for example) if \( p = 2 \), then
\[
S(2, 8) \simeq L_5, \quad S(2, 17) \simeq L_5 \oplus A_2 \oplus F \oplus F, \quad S(2, 19) \simeq L_5 \oplus D_3 \oplus F \oplus F,
\]
where \( A_m, D_m \) and \( L_5 \) are defined in subsection 3.1. Then, we notice that the \( \tau \)-tilting finiteness of \( S(n, r) \) is always reduced to that of indecomposable blocks of \( S(n, r) \) (This is the main strategy to prove Theorem 1.1). However, the \( \tau \)-tilting infiniteness of \( S(n, r) \) does not imply the \( \tau \)-tilting infiniteness of the blocks of \( S(n, r) \). In other words, it is possible that a \( \tau \)-tilting infinite \( S(n, r) \) has a \( \tau \)-tilting finite block. This motivates us to try to give a classification of \( \tau \)-tilting finite blocks of Schur algebras, i.e.,

**Problem 1.2.** Give a complete classification of \( \tau \)-tilting finite blocks of Schur algebras.

The second aim of this paper is to give a partial answer to Problem 1.2. Namely, we determine all \( \tau \)-tilting finite blocks of \( S(2, r) \) as follows.

**Theorem 1.3** (Theorem 3.9 and Section 3.1). Let \( B \) be an indecomposable block of \( S(2, r) \).

1. If \( p = 2 \), then \( B \) is \( \tau \)-tilting finite if and only if \( B \) is Morita equivalent to one of \( F, \ A_2, \ D_3, \ K_4 \) and \( L_5 \).
2. If \( p \geq 3 \), then \( B \) is \( \tau \)-tilting finite if and only if \( B \) is Morita equivalent to one of \( F, \ A_m (2 \leq m \leq p) \) and \( D_{p+1} \).

In order to prove Theorem 1.3, a crucial statement we use (see [EH1]) is that if two indecomposable blocks \( B \) of \( S(2, r) \) and \( B' \) of \( S(2, r') \) have the same number of simple modules over the same field, then \( B \) and \( B' \) are Morita equivalent. However, such a phenomenon does not appear in the case of \( S(n, r) \) with \( n \geq 3 \). One may easily find a counterexample in Appendix A for example, comparing \( S(3, 4) \) and \( S(3, 5) \) over \( p = 2 \).

This paper is organized as follows. In Section 2, we first review some basic concepts of \( \tau \)-tilting theory and silting theory. Then, we introduce the sign decomposition which is the main tool we use in this paper to show the \( \tau \)-tilting finiteness of \( D_m \). Last, we recall the definitions and properties of Schur algebras. In Section 3, we present our main results and proofs.

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## 2 Preliminaries

We recall that any finite-dimensional basic algebra \( A \) over an algebraically closed field \( F \) is isomorphic to a bound quiver algebra \( FQ/I \), where \( FQ \) is the path algebra of a finite quiver \( Q \) and \( I \) is an admissible ideal of \( FQ \). We call \( Q \) the quiver of \( A \). We refer to [ASS] for more background on quiver representation theory and the representation theory of finite-dimensional algebras.
We first review some definitions and properties of \(\tau\)-tilting theory and silting theory, which are needed for this paper.

### 2.1 \(\tau\)-tilting theory

Let \(\mod A\) be the category of finitely generated right \(A\)-modules and \(\proj A\) the category of finitely generated projective right \(A\)-modules. For any \(M \in \mod A\), we denote by \(\add(M)\) (resp., \(\Fac(M)\)) the full subcategory of \(\mod A\) whose objects are direct summands (resp., factor modules) of finite direct sums of copies of \(M\). We denote by \(A^{\op}\) the opposite algebra of \(A\) and by \(|M|\) the number of isomorphism classes of indecomposable direct summands of \(M\). We recall that the Nakayama functor \(\nu = D(-)^*\) is induced by the dualities

\[ D = \Hom_{\mathcal{F}}(-, F) : \mod A \leftrightarrow \mod A^{\op} \quad \text{and} \quad (-)^* = \Hom_{A}(-, A) : \proj A \leftrightarrow \proj A^{\op}. \]

Then, for any \(M \in \mod A\) with a minimal projective presentation

\[ P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0, \]

the Auslander-Reiten translation \(\tau M\) is defined by the exact sequence:

\[ 0 \rightarrow \tau M \rightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0. \]

**Definition 2.1** \([\AIR, \text{Definition 0.1}]\). Let \(M \in \mod A\).

1. \(M\) is called \(\tau\)-rigid if \(\Hom_A(M, \tau M) = 0\).
2. \(M\) is called \(\tau\)-tilting if \(M\) is \(\tau\)-rigid and \(|M| = |A|\).
3. \(M\) is called support \(\tau\)-tilting if \(M\) is a \(\tau\)-tilting \((A/\langle e \rangle)\)-module with respect to an idempotent \(e\) of \(A\).

For a pair \((M, P)\) with \(M \in \mod A\) and \(P \in \proj A\), it is called a support \(\tau\)-tilting pair if \(M\) is \(\tau\)-rigid, \(\Hom_A(P, M) = 0\) and \(|M| + |P| = |A|\). Obviously, \((M, P)\) is a support \(\tau\)-tilting pair if and only if \(M\) is a \(\tau\)-tilting \((A/\langle e \rangle)\)-module and \(\add(P) = \add(eA)\).

We denote by \(\tau\text{-rigid } A\) the set of isomorphism classes of \(\tau\)-rigid \(A\)-modules, and by \(s\tau\text{-tilt } A\) (resp., \(\tau\text{-tilt } A\)) the set of isomorphism classes of basic support \(\tau\)-tilting (resp., \(\tau\)-tilting) modules. Then, it is known from \([\AIR]\) that \(\tau\text{-tilt } A \subseteq s\tau\text{-tilt } A \subseteq \tau\text{-rigid } A\).

**Definition 2.2.** An algebra \(A\) is called \(\tau\)-tilting finite if \(\tau\text{-tilt } A\) is a finite set. Otherwise, \(A\) is called \(\tau\)-tilting infinite.

**Proposition 2.3** \([\DL, \text{Corollary 2.9}]\). An algebra \(A\) is \(\tau\)-tilting finite if and only if one of (equivalently, any of) the sets \(\tau\text{-rigid } A\) and \(s\tau\text{-tilt } A\) is a finite set.

It is shown in \([\AIR, \text{Section 2.2}]\) that every \(\tau\)-rigid \(A\)-module \(M\) provides a torsion class \(\Fac M\) in \(\mod A\). If moreover, \(A\) is \(\tau\)-tilting finite, then all torsion classes in \(\mod A\) can be obtained in this way, see \([\DL, \text{Theorem 3.8}]\). According to the inclusions between torsion classes, we may define a partial order on \(s\tau\text{-tilt } A\), that is,
\[ M \leq N \iff \text{Fac } M \subseteq \text{Fac } N, \]

for \( M, N \in \mathfrak{s}\tau\text{-tilt } A \). We denote by \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) \) the Hasse quiver of \( \mathfrak{s}\tau\text{-tilt } A \) with respect to the partial order \( \leq \). From the viewpoint of mutation, it is known that arrows of \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) \) are explicitly described by left mutations of support \( \tau \)-tilting modules, see \([\text{AIR}], \text{Section 2.4}\) for details. Then, the following statement implies that an algebra \( A \) is \( \tau \)-tilting finite if we can find a finite connected component in \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) \).

**Proposition 2.4** ([AIR, Corollary 2.38]). If the Hasse quiver \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) \) contains a finite connected component \( \Delta \), then \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) = \Delta \).

We may give an example to illustrate the constructions above.

**Example 2.5.** Let \( A = \mathbb{F}(1 \xrightarrow{\alpha} 2)/\langle \alpha \beta, \beta \alpha \rangle \). We denote by \( S_1, S_2 \) the simple \( A \)-modules and by \( P_1, P_2 \) the indecomposable projective \( A \)-modules. We observe that

\[ S_1 \simeq 1, \quad S_2 \simeq 2, \quad P_1 \simeq \frac{1}{2}, \quad P_2 \simeq \frac{2}{1}. \]

Here, we describe \( A \)-modules via their composition factors. For instance, we denote the simple module \( S_i \) by \( i \) and then, \( \frac{1}{2} = \frac{S_1}{S_2} \) is an indecomposable module \( M \) with a unique simple submodule \( S_2 \) such that \( M/S_2 \simeq S_1 \). We deduce that

- \( \tau \)-tilting \( A = \{ P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2 \} \),
- \( \mathfrak{s}\tau \)-tilting \( A = \{ P_1 \oplus P_2, P_1 \oplus S_1, S_2 \oplus P_2, S_1, S_2, 0 \} \),

and therefore, \( A \) is \( \tau \)-tilting finite. Then, the Hasse quiver \( \mathcal{H}(\mathfrak{s}\tau\text{-tilt } A) \) is given as follows,

\[
\begin{array}{cccccccccc}
P_1 \oplus P_2 & \twoheadrightarrow & P_1 \oplus S_1 & \twoheadrightarrow & S_1 & \twoheadrightarrow & 0 \\
S_2 & \twoheadrightarrow & S_2
\end{array}
\]

We recall some reduction methods for the \( \tau \)-tilting finiteness of \( A \). First, we have

**Proposition 2.6** ([DIRRT, Theorem 5.12], [DIJ, Theorem 4.2]). If \( A \) is \( \tau \)-tilting finite,

1. the quotient algebra \( A/I \) is \( \tau \)-tilting finite for any two-sided ideal \( I \) of \( A \),
2. the idempotent truncation \( eAe \) is \( \tau \)-tilting finite for any idempotent \( e \) of \( A \).

**Proposition 2.7** (e.g., [Ad, Theorem 3.1]). Let \( A = \mathbb{F}Q/I \) be a bound quiver algebra presented by quiver \( Q \) and admissible ideal \( I \). If the quiver \( Q \) contains

\[
\begin{array}{ccc}
\circ & \rightarrow & \circ \\
\circ & \rightarrow & \circ
\end{array}
\]

as a subquiver, then \( A \) is \( \tau \)-tilting infinite.
Second, it is worth mentioning that Eisele, Janssens and Raedschelders \[EJR\] provided a powerful reduction theorem, as shown below.

**Proposition 2.8** (\[EJR, Theorem 1\]). Let \(I\) be a two-sided ideal generated by central elements which are contained in the Jacobson radical of \(A\). Then, there exists a poset isomorphism between \(s\tau\text{-tilt} A\) and \(s\tau\text{-tilt}(A/I)\).

Lastly, the following statement is obvious.

**Proposition 2.9.** Assume that \(A\) is decomposed into \(A_1, A_2, \ldots, A_\ell\) as (not necessary indecomposable) blocks. Then, \(A\) is \(\tau\)-tilting finite if and only if \(A_i\) is \(\tau\)-tilting finite for any \(1 \leq i \leq \ell\). If this is the case, then we have

\[
\#s\tau\text{-tilt} A = \prod_{i=1}^{\ell} \#s\tau\text{-tilt} A_i.
\]

### 2.2 Silting theory

We denote by \(C^b(\text{proj} A)\) the category of bounded complexes of projective \(A\)-modules and by \(K^b(\text{proj} A)\) the corresponding homotopy category which is triangulated. We denote by \(D^b(\text{mod} A)\) the bounded derived category of the abelian category \(\text{mod} A\).

**Definition 2.10** (\[AI, Definition 2.1\]). A complex \(T \in K^b(\text{proj} A)\) is called presilting if

\[
\text{Hom}_{K^b(\text{proj} A)}(T, T[i]) = 0 \text{ for any } i > 0.
\]

A presilting complex \(T\) is called silting if \(\text{thick } T = K^b(\text{proj} A)\), where \(\text{thick } T\) is the smallest full triangulated subcategory containing \(T\) which is closed under direct summands. Moreover, a silting complex \(T\) is called tilting if \(\text{Hom}_{K^b(\text{proj} A)}(T, T[i]) = 0 \text{ for any } i < 0\).

We introduce the silting mutation of silting complexes, see \[AI, Definition 2.30\]. Let \(T = X \oplus Y\) be a basic silting complex in \(K^b(\text{proj} A)\) with a direct summand \(X\). We take a minimal left \(\text{add}(Y)\)-approximation \(\pi\) and a triangle

\[
X \to Z \to X' \to X[1].
\]

Then, \(\mu_X(T) := X' \oplus Y\) is again a basic silting complex in \(K^b(\text{proj} A)\), see \[AI, Theorem 2.31\]. We call \(\mu_X(T)\) the left (silting) mutation of \(T\) with respect to \(X\). Dually, we can define the right mutation \(\mu_X(T)\) of \(T\) with respect to \(X\), which is also silting.

Recall that a complex in \(K^b(\text{proj} A)\) is called two-term if it is isomorphic to a complex \(T\) concentrated in degree 0 and \(-1\), i.e.,

\[
(T^{-1} \xrightarrow{d_{-1}} T^0) = \cdots \to 0 \to T^{-1} \xrightarrow{d_{-1}} T^0 \to 0 \to \cdots.
\]

We denote by \(2\text{-silt} A\) the set of isomorphism classes of basic two-term silting complexes in \(K^b(\text{proj} A)\). There is a partial order \(\leq\) on the set \(2\text{-silt} A\) which is introduced by \[AI, Theorem 2.11\]. For any \(T, U \in 2\text{-silt} A\), we have \(U \leq T\) if and only if \(\text{Hom}_{K^b(\text{proj} A)}(T, U[1]) = 0\). Then, we denote by \(\mathcal{H}(2\text{-silt} A)\) the Hasse quiver of \(2\text{-silt} A\) which is compatible with the irreducible left mutation of silting complexes.
Proposition 2.11 ([AI, Lemma 2.25, Theorem 2.27]). Let $T = (T^{-1} \to T^0) \in 2\text{-silt} \, A$. Then, we have $\text{add} \, A = \text{add} \, (T^0 \oplus T^{-1})$ and $\text{add} \, T^0 \cap \text{add} \, T^{-1} = 0$.

Next, we explain the connection between $\tau$-tilting theory and silting theory.

Theorem 2.12 ([AIR, Theorem 3.2]). There exists a poset isomorphism between $\mathcal{s}_\tau\text{-tilt} \, A$ and $2\text{-silt} \, A$. More precisely, the bijection is given by mapping a two-term silting complex $T$ to its $0$-th cohomology $H^0(T)$, and the inverse is given by $M \mapsto (P_1 \oplus P \xrightarrow{f,0} P_0)$, where $(M,P)$ is the corresponding support $\tau$-tilting pair and $P_1 \xrightarrow{f} P_0 \twoheadrightarrow M \twoheadrightarrow 0$ is the minimal projective presentation of $M$.

Suppose that $P_1, P_2, \ldots, P_n$ are pairwise non-isomorphic indecomposable projective $A$-modules. We denote by $[P_1], [P_2], \ldots, [P_n]$ the isomorphism classes of indecomposable complexes concentrated in degree $0$. Clearly, the classes $[P_1], [P_2], \ldots, [P_n]$ in the triangulated category $K^b(\text{proj} \, A)$ form a standard basis of the Grothendieck group $K_0(K^b(\text{proj} \, A))$. If a two-term complex $T$ in $K^b(\text{proj} \, A)$ is written as $\left( \bigoplus_{i=1}^n P_i^{\oplus b_i} \rightarrow \bigoplus_{i=1}^n P_i^{\oplus a_i} \right)$, then the class $[T]$ can be identified by an integer vector $g(T) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n) \in \mathbb{Z}^n$, which is called the $g$-vector of $T$. Then, we have the following statement.

Proposition 2.13 ([AIR, Theorem 5.5]). The above map $T \mapsto g(T)$ from $2\text{-silt} \, A$ to $\mathbb{Z}^n$ is an injection.

We give an example here.

Example 2.14. Let $A = \mathbb{F}(1 \xrightarrow{\alpha} 2)/\langle \alpha \beta, \beta \alpha \rangle$. In Example 2.5, the Hasse quiver $\mathcal{H}(\mathcal{s}_\tau\text{-tilt} \, A)$ is described. By applying Theorem 2.12, the Hasse quiver $\mathcal{H}(2\text{-silt} \, A)$ is

One can use this to verify Proposition 2.11. We conclude that the set of $g$-vectors is $\{g(T) \mid T \in 2\text{-silt} \, A\} = \{(1,1), (2,-1), (1,-2), (-1,2), (-2,1), (-1,-1)\}$. 

where the \(g\)-vectors are illustrated in \(\mathbb{Z}^2\) as follows,

\[
\begin{array}{c|c|c|c|c}
 & x_1 & & \\
\hline
-2 & 1 & -1 & 2 \\
\hline
 & \bullet & & \bullet \\
\hline
 & \bullet & & \bullet \\
\end{array}
\]

2.3 Sign decomposition

In this subsection, we introduce the notion of \textit{sign decomposition} of the set \(2\text{-silt} A\), which is originally considered in \[Ao\] and generalized in \[AHIKM\]. In fact, the main effect of sign decomposition is that the restriction of the set of \(g\)-vectors for the original algebra to each orthant can be described by the set of \(g\)-vectors for a simpler algebra.

Now, we define \([m, n] := \{m, m+1, \ldots, n\}\) for two integers \(1 \leq m \leq n\).

We recall that \(A = \mathbb{F}Q/I\) is a bound quiver algebra, where the vertex set of \(Q\) is given by \([1, n] = \{1, \ldots, n\}\). We denote by \(e_i\) the primitive idempotent of \(A\) corresponding to the vertex \(i\), and by \(P_i := e_iA\) the corresponding indecomposable projective \(A\)-module.

We denote by \(M(n) := \{\epsilon = (\epsilon(1), \ldots, \epsilon(n)) : [1, n] \rightarrow \{\pm 1\}\}\) the set of all maps from \([1, n]\) to \(\{\pm 1\}\), which endows an involution \(\epsilon \mapsto -\epsilon\) given by \((-\epsilon)(i) = -\epsilon(i)\) for all \(i \in [1, n]\). We sometimes simply use \(\epsilon(i) = +\) (resp., \(\epsilon(i) = -\)) to indicate \(\epsilon(i) = +1\) (resp., \(\epsilon(i) = -1\)) without causing confusion. For each \(\epsilon \in M(n)\), let \(Z^n_\epsilon\) be the area determined by the linear inequality \(\epsilon(i)x_i > 0\) for all \(i \in [1, n]\), that is, \(Z^n_\epsilon := \{x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \epsilon(i)x_i > 0, \ i \in [1, n]\}\).

For example, we have \(M(2) = \{(+, +), (+, -), (-, +), (-, -)\}\) and \(Z^2_\epsilon\) with \(\epsilon \in M(2)\) is illustrated as follows,

\[
\begin{array}{c|c|c|c|c}
 & x_1 & & \\
\hline
 & x_2 & & \\
\hline
 & \mathbb{Z}^2_{(+, +)} & & \\
\hline
 & \mathbb{Z}^2_{(+, -)} & & \\
\hline
 & \mathbb{Z}^2_{(-, +)} & & \\
\hline
 & \mathbb{Z}^2_{(-, -)} & & \\
\end{array}
\]

We consider a subset of \(2\text{-silt} A\) which plays a central role in the sign decomposition. For each \(\epsilon \in M(n)\), we define \(2\text{-silt}_\epsilon A := \{T \in 2\text{-silt} A \mid g(T) \in \mathbb{Z}^n_\epsilon\}\).

More generally, for a given subset \(M\) of \(M(n)\), we define \(2\text{-silt}_M A := \bigcup_{\epsilon \in M} 2\text{-silt}_\epsilon A\). Note that the union is disjoint by the definition of \(\mathbb{Z}^n_\epsilon\).
Proposition 2.15. We have $\text{2-silt } A = \text{2-silt}_{M(n)} A$.

Proof. It is obvious that the set on the right is contained in the set on the left. Then, we assume that $T = (T^{-1} \to T^0) \in \text{2-silt } A$. By Proposition 2.11, each indecomposable projective $A$-module $P_i$ lies in precisely one of $\text{add } T^0$ and $\text{add } T^{-1}$. In other words, the $i$-th entry of $g(T)$ is either positive or negative for all $1 \leq i \leq n$. Hence, there must exist a map $\epsilon \in M(n)$ such that $T \in \text{2-silt } A$.

Proposition 2.16 ([AIR, Theorem 2.14]). Let $M$ be a subset of $M(n)$ and $-M := \{ -\epsilon \mid \epsilon \in M \}$. Then, the duality $(-)^* = \text{Hom}_A(-, A)$ gives a bijection

$$2\text{-silt}_M A \xleftarrow{\sim} 2\text{-silt}_{-M} A^{\text{op}}.$$

Next, we associate an upper triangular matrix algebra $A_{\epsilon}$ to each $\epsilon \in M(n)$.

Definition 2.17. For an arbitrary map $\epsilon \in M(n)$, let $e_{\epsilon, +} := \sum_{\epsilon(i) = +} e_i$ and $e_{\epsilon, -} := \sum_{\epsilon(i) = -} e_i$. Then, we define an upper triangular matrix algebra

$$A_{\epsilon} := \begin{pmatrix} e_{\epsilon, +} & e_{\epsilon, +} \\ 0 & e_{\epsilon, -} \\ \frac{A_{\epsilon}}{J_+} & \frac{A_{\epsilon}}{J_-} \end{pmatrix},$$

where $J_+$ (resp., $J_-$) is a two-sided ideal of $e_{\epsilon, +} A e_{\epsilon, +}$ (resp., $e_{\epsilon, -} A e_{\epsilon, -}$) consisting of all $x$ such that $x$ is in the Jacobson radical and $xy = 0$ (resp., $yx = 0$) for all $y \in e_{\epsilon, +} A e_{\epsilon, -}$.

We denote by $e'_1, e'_2, \ldots, e'_n$ the primitive idempotents of $A_{\epsilon}$, which are naturally corresponding to $e_1, e_2, \ldots, e_n$ of $A$, respectively. Under this correspondence, we can regard that the quiver of $A_{\epsilon}$ has $[1, n]$ as the vertex set. If we set $P'_i := e'_i A_{\epsilon}$ for all $i \in [1, n]$, the correspondence $[P_i] \mapsto [P'_i]$ induces an isomorphism $K_0(K^b(\text{proj } A)) \to K_0(K^b(\text{proj } A_{\epsilon}))$ of Grothendieck groups. Based on the construction of the algebra $A_{\epsilon}$, we have

$$\text{Hom}_A(e_{\epsilon, -} A, e_{\epsilon, +} A) = e_{\epsilon, +} A e_{\epsilon, -} \simeq e'_{\epsilon, +} A e'_{\epsilon, -} = \text{Hom}_A(e'_\epsilon, - A_{\epsilon}, e'_\epsilon, + A_{\epsilon}),$$

where $e'_{\epsilon, +} := \sum_{\epsilon(i) = +} e'_i$ and $e'_{\epsilon, -} := \sum_{\epsilon(i) = -} e'_i$, and this gives rise to a bijection

$$\{ T = (T^{-1} \to T^0) \in K^b(\text{proj } A) \mid g(T) \in \mathbb{Z}_n \} \xymatrix{\sim\ar@{<->}[r] & } \{ U = (U^{-1} \to U^0) \in K^b(\text{proj } A_{\epsilon}) \mid g(U) \in \mathbb{Z}_n \}$$

over two-term complexes in the homotopy categories. Moreover, if we restrict our interest to two-term silting complexes, then all information about $2\text{-silt}_{\epsilon} A$ can be obtained from the algebra $A_{\epsilon}$, as shown below.

Proposition 2.18 ([AIHKM]). For each $\epsilon \in M(n)$, we have an isomorphism

$$2\text{-silt } A \xymatrix{\sim\ar@{<->}[r] & } 2\text{-silt } A_{\epsilon},$$

which preserves the $g$-vectors of two-term silting complexes. In particular, $A$ is $\tau$-tilting finite if $A_{\epsilon}$ is $\tau$-tilting finite for all $\epsilon \in M(n)$. 

9
One may use Example 2.14 to understand the above isomorphism.

**Example 2.19.** Recall that $A = \mathbb{F}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1) / \langle \alpha \beta, \beta \alpha \rangle$ and the set of $g$-vectors of two-term silting complexes is given in Example 2.14. We observe that the algebras $A_{\epsilon}$ for $\epsilon \in \mathcal{M}(2) = \{ (+, +), (+, -), (-, +), (-, -) \}$ are given by

$$A_{(+, +)} = \mathbb{F}(1 \rightarrow 2), \quad A_{(+, -)} = \mathbb{F}(1 \rightarrow 2), \quad A_{(-, +)} = \mathbb{F}(1 \leftarrow 2), \quad A_{(-, -)} = \mathbb{F}(1 \leftarrow 2).$$

Then, the corresponding subsets $\{ g(T) \mid T \in \mathbf{2\text{-}\text{silt}}_{A_{\epsilon}} \}$ are given by

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathbb{Z}^2_{(+, +)} \\
x_1 \\
\end{array}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathbb{Z}^2_{(+, -)} \\
x_1 \\
\end{array}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathbb{Z}^2_{(-, +)} \\
x_1 \\
\end{array}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\mathbb{Z}^2_{(-, -)} \\
x_1 \\
\end{array}
\end{array}
\end{array}$$

Next, we explain that the sign decomposition is compatible with tilting mutation of $A$ regarded as a tilting complex concentrated on degree 0. We fix an integer $j \in [1, n]$. We denote by $T := \mu^{-}_{j}(A)$ the left tilting mutation of $A$ with respect to $(0 \rightarrow P_{j})$ for the indecomposable projective $A$-module $P_{j}$. We observe that $T$ is of the form $\bigoplus_{i=1}^{n} T_{i}$ with $T_{i} := (0 \rightarrow P_{i})$ for $i \neq j$ and $T_{j} = (T_{j}^{-1} \rightarrow T_{j}^{0})$ is an indecomposable two-term complex in $K^{b}(\text{proj} A)$ such that $\text{add}(P_{j}) = \text{add}(T_{j}^{-1})$.

Assume that $T$ is a tilting complex. By Rickard’s theorem, its endomorphism algebra $\text{End}_{K^{b}(\text{proj} A)}(T)$, written as $\mu^{-}_{j}(A)$, is derived equivalent to $A$. More precisely, we have a triangle equivalence

$$F: D^{b}(\text{mod} A) \sim D^{b}(\text{mod} \mu^{-}_{j}(A)) \quad (2.1)$$

mapping $T \mapsto \mu^{-}_{j}(A)$ so that each indecomposable direct summand $T_{i}$ of $T$ is sent to the corresponding indecomposable projective $\mu^{-}_{j}(A)$-module $P'_{j}$. Besides, it naturally induces an isomorphism $K_{0}(K^{b}(\text{proj} A)) \cong K_{0}(K^{b}(\text{proj} \mu^{-}_{j}(A)))$ of Grothendieck groups by $[T_{i}] \mapsto [P'_{j}]$ for all $1 \leq i \leq n$. In this way, we can identify the vertex set of the quiver of $\mu^{-}_{j}(A)$ with $[1, n]$, so that $\mathcal{M}(n)$ is compatible between $A$ and $\mu^{-}_{j}(A)$.

Now, let $\mathcal{M}(n)_{j,-}$ (resp., $\mathcal{M}(n)_{j,+}$) be the subset of $\mathcal{M}(n)$ consisting of all maps $\epsilon$ satisfying $\epsilon(j) = -$ (resp., $\epsilon(j) = +$). We have the following statement.

**Proposition 2.20.** The triangle equivalence in (2.1) induces a bijection

$$2\text{-}\text{silt}_{\mathcal{M}(n)_{j,-}} A \xrightarrow{1 \leftarrow 1} 2\text{-}\text{silt}_{\mathcal{M}(n)_{j,+}} \mu^{-}_{j}(A). \quad (2.2)$$

We point out that this bijection has been shown in [AMN, Lemma 4.6] for a class of algebras called Brauer tree algebras, and the proof can be directly generalized to any finite-dimensional algebra $A$. For the convenience of readers, we give a proof here.
Proof of Proposition 2.20. Recall that $\mu_{P_j}(A)$ (resp., $\mu_{P_j}^+(A)$) denotes the left (resp., right) silting mutation of $A$ with respect to $(0 \to P_j)$. Similar to [AMN] Lemma 4.5, one can show that

$$2\text{-silt}_{M(n)_{j,-}} A = \{ U \in 2\text{-silt} A \mid A[1] \leq U \leq \mu_{P_j}(A) \},$$

(2.3)

and

$$2\text{-silt}_{M(n)_{j,+}} A = \{ V \in 2\text{-silt} A \mid \mu_{P_j}^+(A[1]) \leq V \leq A \},$$

(2.4)

with respect to the partial order $\leq$ on $2\text{-silt} A$.

Assume that $\mu_{P_j}(A)$ is tilting and $\mu_j^-(A) = \text{End}_{k^{\text{proj}}(A)}(\mu_{P_j}(A))$ as defined before. We denote by $P_i^j$ $(1 \leq i \leq n)$ the indecomposable projective modules of $\mu_j^-(A)$. Then, the triangle equivalence $F$ in (2.1) satisfies

$$F(\mu_{P_j}(A)) \simeq \mu_j^-(A) \text{ and } F(A[1]) \simeq \mu_{P_j}[1](\mu_j^-(A)[1]).$$

Since the triangle equivalence $F$ preserves $\leq$ and mutations, it gives an isomorphism

$$\sim \{ U \in 2\text{-silt} A \mid A[1] \leq U \leq \mu_{P_j}(A) \}$$

$$\sim \{ V \in 2\text{-silt} \mu_j^-(A) \mid \mu_{P_j}[1](\mu_j^-(A)[1]) \leq V \leq \mu_j^-(A) \}.$$  

Then, we get the desired bijection following (2.3) and (2.4). \qed

Furthermore, the above result can be generalized as follows. Let $J$ be a subset of $[1, n]$ and $e_J := \sum_{j \in J} e_i$. We denote by $T := \mu_{P_j}(A)$ the left silting mutation of $A$ with respect to $(0 \to P_J)$ for the projective $A$-module $P_j := e_J A$. By the definition of silting mutation, it is of the form $\bigoplus_{i=1}^n T_i$, where $T_i := (0 \to P_i)$ for all $i \notin J$ and $T_i = (T_i^{-1} \to T_i^0)$ is an indecomposable two-term complex such that $\text{add}(P_i) = \text{add}(T_i^{-1})$ for every $i \notin J$.

Now, we assume that $T$ is tilting. We denote by $\mu_j^+(A) := \text{End}_{k^{\text{proj}}(A)}(T)$ the endomorphism algebra of $T$, and by $P_i^j$ the indecomposable projective $\mu_j^+(A)$-module corresponding to $T_i$ for all $1 \leq i \leq n$. Then, we can naturally identify the vertex set of the quiver of $\mu_j^+(A)$ with $[1, n]$, so that $M(n)$ is compatible between $A$ and $\mu_j^+(A)$.

Let $M(n)_{\epsilon,-}$ (resp., $M(n)_{\epsilon,+}$) be the subset of $M(n)$ consisting of all maps $\epsilon$ satisfying $\epsilon(J) = \{-\}$ (resp., $\epsilon(J) = \{+\}$).

Corollary 2.21. Under the above setting, we have a bijection

$$2\text{-silt}_{M(n)_{\epsilon,-}} A \overset{\sim}{\leftrightarrow} 2\text{-silt}_{M(n)_{\epsilon,+}} \mu_j^+(A).$$

(2.5)

In particular, the above sets are finite if one of $A$ and $\mu_j^+(A)$ is $\tau$-tilting finite.

Proof. Note that Proposition 2.20 is precisely the case when $J = \{j\}$, and a proof of this statement can be done by a similar way. We omit the details. \qed
At the end of this subsection, it is better to give an example.

**Example 2.22.** Let $A = \mathbb{F}(1 \overset{\alpha}{\rightarrow} 2 \overset{\beta}{\rightarrow} 3)$ be a path algebra. We denote by $P_i$ the indecomposable projective $A$-modules corresponding to the vertices $i = 1, 2, 3$. If we choose $J = \{1, 3\}$, then $P_J = P_1 \oplus P_3$. Then, the left silting mutation $T := \mu_{\overline{P}_J}(A)$ of $A$ with respect to $(0 \rightarrow P_J)$ is given by $T = T_1 \oplus T_2 \oplus T_3$, where

$$T_1 := (P_1 \overset{\alpha}{\rightarrow} P_2), T_2 := (0 \rightarrow P_2) \text{ and } T_3 := (P_3 \overset{\beta}{\rightarrow} P_2).$$

It is not difficult to check that $T$ is a tilting complex. In addition, the endomorphism algebra of $T$, written as $\mu_{\overline{J}}(A)$, is

$$
\begin{pmatrix}
\mathbb{F} & \mathbb{F} & 0 \\
0 & \mathbb{F} & 0 \\
0 & \mathbb{F} & \mathbb{F}
\end{pmatrix},
$$

and it is isomorphic to the path algebra $\mathbb{F}(1 \overset{\alpha}{\rightarrow} 2 \overset{\beta}{\rightarrow} 3)$. We describe the bijection in Corollary 2.21 between the Hasse quivers as follows,

![Hasse quivers](image_url)

where the bullets indicate elements in $2\text{-silt}_{M(n),J}A$ and $2\text{-silt}_{M(n),J,\mu_{\overline{J}}}(A)$, respectively.

### 2.4 Schur algebras

In this subsection, we review some basics related to the representation theory of symmetric groups and Schur algebras. One may refer to some textbooks, such as [Ja], [Ma] and [Sa], for more details.

Let $r$ be a natural number and $\lambda = (\lambda_1, \lambda_2, \ldots)$ a sequence of non-negative integers. We call $\lambda$ a partition of $r$ if $\sum_{i \in \mathbb{N}} \lambda_i = r$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$, and the elements $\lambda_i$ are called parts of $\lambda$. If there exists an $n \in \mathbb{N}$ such that $\lambda_i = 0$ for all $i > n$, we denote $\lambda$ by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and call it a partition of $r$ with at most $n$ parts. We denote by $\Omega(n, r)$ the set of all partitions of $r$ with at most $n$ parts. It is well-known that $\Omega(n, r)$ admits the dominance order $\geq$ and the lexicographic order $>$, we omit the definitions.

We denote by $G_r$ the symmetric group on $r$ symbols and by $\mathbb{F}G_r$ the group algebra of $G_r$. Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $r$ gives a Young subgroup $G_\lambda$ of $G_r$ defined as

$$G_\lambda := G_{\lambda_1} \times G_{\lambda_2} \times \cdots \times G_{\lambda_n}.$$
Then, the permutation $\mathbb{F}G_r$-module $M^\lambda$ is $1_{G_\lambda} \uparrow^{G_r}$, where $1_{G_\lambda}$ denotes the trivial module for $G_\lambda$ and $\uparrow$ denotes induction. It is known that the permutation module $M^\lambda$ has a unique submodule which is isomorphic to the so-called Sepecht module $S^\lambda$. Furthermore, there is a unique indecomposable direct summand of $M^\lambda$ containing $S^\lambda$, which is called Young module and is denoted by $Y^\lambda$. In this way, each $M^\lambda$ with $\lambda \in \Omega(n, r)$ is a direct sum of Young modules $Y^\lambda$ with $\lambda \in \Omega(n, r)$.

Let $S(n, r)$ be the Schur algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$. Since each $M^\lambda$ with $\lambda \in \Omega(n, r)$ can be regarded as a direct summand of $V^\otimes r$ (e.g., [Ve, Section 1.6]), we may construct the basic algebra $S(n, r)$ of $S(n, r)$ as follows. Let $B$ be a block of the group algebra $\mathbb{F}G_r$ labeled by a $p$-core $\omega$. It is well-known that a partition $\lambda$ belongs to $B$ if and only if $\lambda$ has the same $p$-core $\omega$. We define

$$ S_B := \text{End}_{\mathbb{F}G_r} \left( \bigoplus_{\lambda \in B \setminus \Omega(n, r)} Y^\lambda \right). $$

Then, the basic algebra $S(n, r)$ is given by $\bigoplus_{B} S_B$ taken over all blocks of $\mathbb{F}G_r$. Moreover, $S_B$ is a direct sum of blocks of $S(n, r)$.

We recall some constructions on the blocks of $S(2, r)$. In order to avoid confusion of symbols, we use $\mathcal{B}$ to identify a block of $S(2, r)$ and we denote by $|\mathcal{B}|$ the number of simple $\mathcal{B}$-modules.

**Lemma 2.23** ([EH1, Theorem 13]). Let $\mathcal{B}$ and $\mathcal{B}'$ be two indecomposable blocks of $S(2, r)$ and $S(2, r')$ over the same field. If $|\mathcal{B}| = |\mathcal{B}'|$, then $\mathcal{B}$ and $\mathcal{B}'$ are Morita equivalent.

Based on the above lemma, it is useful to find the quiver of $S(2, r)$. Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be two partitions of $r$, we define two non-negative integers $s := \lambda_1 - \lambda_2$ and $t := \mu_1 - \mu_2$. We denote by $v^s$ the vertex in the quiver of $S(2, r)$ corresponding to the Young module $Y^{(\lambda_1, \lambda_2)}$ with $s = \lambda_1 - \lambda_2$. Let $n(v^s, v^t)$ be the number of arrows from $v^s$ to $v^t$. Then, it is shown in [EH2] that $n(v^s, v^t) = n(v^t, v^s)$ and $n(v^s, v^t)$ is either 0 or 1. We have the following recursive algorithm for computing $n(v^s, v^t)$.

**Lemma 2.24** ([EH2, Proposition 3.1]). Suppose that $p$ is a prime number and $s > t$. Let $s = s_0 + ps'$ and $t = t_0 + pt'$ with $0 \leq s_0, t_0 \leq p - 1$ and $s', t' \geq 0$.

1. If $p = 2$, then

$$ n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0 = 1 \text{ or } s_0 = t_0 = 0 \text{ and } s' \equiv t' \text{ mod } 2, \\ 1 & \text{if } s_0 = t_0 = 0, t' + 1 = s' \not\equiv 0 \text{ mod } 2, \\ 0 & \text{otherwise}. \end{cases} $$

2. If $p > 2$, then

$$ n(v^s, v^t) = \begin{cases} n(v^{s'}, v^{t'}) & \text{if } s_0 = t_0, \\ 1 & \text{if } s_0 + t_0 = p - 2, t' + 1 = s' \not\equiv 0 \text{ mod } p, \\ 0 & \text{otherwise}. \end{cases} $$

13
At the end of preparation, we mention that the classification for the representation type of Schur algebras is useful. In the following, some semi-simple cases are contained in the representation-finite cases. We may distinguish the semi-simple cases following [DN]. Namely, the Schur algebra $S(n, r)$ over a field $\mathbb{F}$ of characteristic $p > 0$ is semi-simple if and only if $p > r$ or $p = 2, n = 2, r = 3$.

**Proposition 2.25** ([Er, DEMN]). Let $p > 0$ be the characteristic of $\mathbb{F}$. Then, $S(n, r)$ is representation-finite if and only if $p = 2, n = 2, r = 5, 7$ or $p \geq 2, n = 2, r < p^2$ or $p \geq 2, n \geq 3, r < 2p$; (infinite-)tame if and only if $p = 2, n = 2, r = 4, 9, 11$ or $p = 3, n = 2, r = 9, 10, 11$ or $p = 3, n = 3, r = 7, 8$. Otherwise, $S(n, r)$ is wild.

3 Main results

3.1 Some blocks of Schur algebras

In order to give a complete classification of $\tau$-tilting finite Schur algebras, we need to recall some indecomposable blocks of Schur algebras which are constructed in [Er], [DEMN], [Xi] and [W3]. We present these blocks here as bound quiver algebras and we will give specific references when we use them. We shall determine their $\tau$-tilting finiteness and also find the number of support $\tau$-tilting modules for $\tau$-tilting finite cases appearing in the list in Appendix A, see Proposition 3.2, Lemma 3.4 and Proposition 3.5.

We first focus on the representation-finite blocks. Let $A_m := \mathbb{F}Q/I$ ($2 \leq m \in \mathbb{N}$) be the bound quiver algebra presented by the following quiver with relations,

$$Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-2}} m - 1 \xrightarrow{\alpha_{m-1}} m,$$

$$I : (\alpha_1\beta_1, \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \beta_i\alpha_i - \alpha_{i+1}\beta_{i+1} | 1 \leq i \leq m - 2).$$

Then, some related results are given as follows.

**Proposition 3.1** ([DR, Theorem 2.1]). Each representation-finite indecomposable block of $S(n, r)$ is Morita equivalent to $A_m$ for some $2 \leq m \in \mathbb{N}$.

**Proposition 3.2** ([W3, Theorem 3.2], see also [Ao, Theorem 5.6]). For any positive integer $m \geq 2$, we have $\#\text{sr-tilt } A_m = \binom{2m}{m}.$

We have known from [W3] that all tame blocks of tame Schur algebras are $\tau$-tilting finite. Thus, we consider some wild blocks, where the wildness is given in [DEMN].

**Definition 3.3.** We define bound quiver algebras $K_4$, $M_4$, $L_5$ and $N_5$ as follows.

1. $K_4 := \mathbb{F}Q/I$ is presented by

$$Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \text{ and } I : \left\langle \alpha_1\beta_1, \alpha_2\beta_2, \beta_3\alpha_3, \alpha_1\alpha_2\alpha_3, \beta_3\beta_2\beta_1, \beta_1\alpha_1\alpha_2 - \alpha_2\alpha_3\beta_3, \beta_2\beta_1\alpha_1 - \alpha_3\beta_3\beta_2 \right\rangle.$$
(2) $\mathcal{M}_4 := \mathbb{F}Q/I$ is presented by
\[
Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \text{ and } I : \left\langle \alpha_1 \beta_1, \beta_2 \alpha_3, \alpha_1 \alpha_2, \beta_2 \beta_1, \alpha_1 \alpha_3 \beta_3, \alpha_3 \beta_1 \alpha_1 - \alpha_2 \beta_2 \right\rangle. \quad (3.3)
\]

(3) $\mathcal{L}_5 := \mathbb{F}Q/I$ is presented by
\[
Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5 \text{ and } I : \left\langle \alpha_1 \beta_1, \alpha_1 \alpha_4, \beta_3 \alpha_3, \beta_2 \alpha_4, \beta_4 \alpha_4, \beta_4 \beta_1, \beta_4 \alpha_2 \beta_2, \alpha_1 \alpha_2 \alpha_3 \beta_3, \alpha_2 \beta_2 \alpha_1 \alpha_1 - \beta_1 \alpha_1 \alpha_2 \beta_2, \alpha_1 \alpha_2 \beta_2, \beta_2 \beta_1 \alpha_1 - \beta_1 \alpha_1 \alpha_2 \beta_2 \right\rangle. \quad (3.4)
\]

(4) $\mathcal{N}_5 := \mathbb{F}Q/I$ is presented by
\[
Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5 \text{ with } I : \left\langle \alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_3 \beta_3, \beta_4 \alpha_4, \alpha_1 \alpha_2 \alpha_3 \beta_3, \beta_4 \beta_3 \beta_2 \beta_1, \beta_2 \alpha_2 - \alpha_3 \alpha_4 \beta_3, \alpha_2 \alpha_3 \beta_4 - \beta_1 \alpha_1 \alpha_2 \alpha_3, \beta_3 \beta_2 \beta_1 \alpha_1 - \alpha_4 \beta_4 \beta_3 \beta_2 \right\rangle. \quad (3.5)
\]

The $\tau$-tilting finiteness of $\mathcal{K}_4$, $\mathcal{M}_4$, $\mathcal{L}_5$ and $\mathcal{N}_5$ is declared as follows.

Lemma 3.4. The algebras $\mathcal{K}_4$, $\mathcal{M}_4$, $\mathcal{L}_5$ are $\tau$-tilting finite, and $\mathcal{N}_5$ is $\tau$-tilting infinite. Furthermore, we have the following table.

|     | $\mathcal{K}_4$ | $\mathcal{M}_4$ | $\mathcal{L}_5$ |
|-----|----------------|----------------|----------------|
| $\#\text{st-tilt } A$ | 136 | 152 | 1656 |

Proof. In order to show the numbers for $\mathcal{K}_4$, $\mathcal{M}_4$ and $\mathcal{L}_5$, it is enough to find all two-term silting complexes of them according to Theorem 2.12. This is equivalent to finding all $g$-vectors for them, since a two-term silting complex $T$ is uniquely determined by its $g$-vector $g(T)$ as explained in Proposition 2.13. To do this, we use a computer to calculate the $g$-vectors directly. We refer to [https://infinite-wang.github.io/Notes/] for a complete list of $g$-vectors for $\mathcal{K}_4$, $\mathcal{M}_4$ and $\mathcal{L}_5$. Besides, one may verify the number for $\mathcal{K}_4$ by [W3, Proposition 4.1] in which the number is determined by a different way.

Next, let $e := e_2 + e_3 + e_4$ be an idempotent of $\mathcal{N}_5$. We look at the idempotent truncation $e \mathcal{N}_5 e$ and define $B$ to be the quotient algebra of $e \mathcal{N}_5 e$ modulo the two-sided ideal generated by $\alpha_3$ and $\beta_2$. Then, $B$ is presented by

\[
F \left( \alpha \bigcirc \longrightarrow \circ \longrightarrow \circ \bigcirc \beta \right) / \left\langle \alpha^2, \beta^2 \right\rangle.
\]

This is a gentle algebra and it is $\tau$-tilting infinite following [P1, Proposition 3.3]. Hence, $\mathcal{N}_5$ is $\tau$-tilting infinite by Proposition 2.6 (2). \qed
Lastly, we define $D_m := \mathbb{F}Q/I$ ($m \geq 3$) by the following quiver and relations,

$$\begin{align*}
Q : & \quad 1 \xleftarrow{\alpha_1} 2 \xrightarrow{\beta_1} 3 \xrightarrow{\mu_3/v_3} 4 \xrightarrow{\mu_4/v_4} \cdots \xrightarrow{\mu_{m-2}/v_{m-2}} m - 1 \xrightarrow{\mu_{m-1}/v_{m-1}} m \\
I : & \quad \langle \alpha_2\beta_2, \alpha_1\beta_1, \alpha_2\beta_1\alpha_1, \beta_1\alpha_2\beta_2, \alpha_2\mu_3, \alpha_1\mu_3, \nu_3\beta_2, \nu_3\beta_1, \mu_3\nu_3 - \beta_1\alpha_1, \\ & \quad \mu_i\mu_{i+1}, \nu_i+1\nu_i, \nu_i\mu_i - \mu_i+1\nu_{i+1}, 3 \leq i \leq m - 2 \rangle.
\end{align*}$$

(3.6)

It is shown in [DEMN, 3.4] that $D_3, D_4$ are tame and $D_m$ ($m \geq 5$) is wild. We have already known from [W3, Lemma 3.3] that $D_3, D_4$ are $\tau$-tilting finite and the numbers of support $\tau$-tilting modules are 28, 114, respectively. More generally, we show that $D_m$ is $\tau$-tilting finite for any $m \geq 3$.

**Proposition 3.5.** For an arbitrary integer $m \geq 3$, the algebra $D_m$ is $\tau$-tilting finite.

**Proof.** Let $D'_m$ be the quotient algebra of $D_m$ modulo the two-sided ideal generated by central elements $\beta_1\alpha_1, \alpha_1\beta_2\alpha_2\beta_2$ and $\nu_i\mu_i$ ($3 \leq i \leq m - 1$). Then, $D'_m$ is presented by the same quiver $Q$ in (3.6) with the following relations,

$$\langle \alpha_2\beta_2, \alpha_1\beta_1, \alpha_2\beta_1\alpha_1, \beta_1\alpha_2\beta_2, \alpha_2\mu_3, \alpha_1\mu_3, \nu_3\beta_2, \nu_3\beta_1, \alpha_1\beta_2\alpha_2\beta_1, \mu_3\nu_3 - \beta_1\alpha_1, \\ \mu_i\mu_{i+1}, \nu_i+1\nu_i, \nu_i\mu_i - \mu_i+1\nu_{i+1}, 3 \leq i \leq m - 2 \rangle.$$

We observe that the indecomposable projective $D'_m$-modules are

$$P'_1 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P'_2 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P'_3 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P'_i \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P'_m \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array},$$

where $4 \leq i \leq m - 1$. Using Proposition 2.8 with Theorem 2.12, we have a bijection

$$\text{2-silt} D_m \xrightarrow{\text{1-1}} \text{2-silt} D'_m.$$

In particular, $D_m$ is $\tau$-tilting finite if and only if $D'_m$ is $\tau$-tilting finite.

On the other hand, let $B_m$ be a symmetric algebra defined by the same quiver $Q$ in (3.6) with the following relations,

$$\langle \alpha_2\beta_2, \alpha_1\beta_1, \alpha_2\mu_3, \alpha_1\mu_3, \nu_3\beta_2, \nu_3\beta_1, \alpha_2\beta_1\alpha_1, \beta_1\alpha_2\beta_2, \alpha_1\mu_3 - \beta_1\alpha_1, \\ \mu_i\mu_{i+1}, \nu_i+1\nu_i, \nu_i\mu_i - \mu_i+1\nu_{i+1}, 3 \leq i \leq m - 2 \rangle.$$

Let $e_i$ be the primitive idempotent of $B_m$ corresponding to the vertex $i \in [1, n]$ and $P_i := e_iB_m$ the indecomposable projective $B_m$-modules. We observe that

$$P_1 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P_2 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P_3 \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P_i \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad P_m \simeq \begin{array}{c}
1 \\
2 \\
3
\end{array},$$

where $4 \leq i \leq m - 1$. Using Proposition 2.8 with Theorem 2.12, we have a bijection

$$\text{2-silt} B_m \xrightarrow{\text{1-1}} \text{2-silt} B'_m.$$
where \(4 \leq i \leq m - 1\). Since the algebra \(D_m'\) is precisely a quotient algebra of \(B_m\) modulo the two-sided ideal generated by \(\beta_1 \alpha_1, \alpha_1 \beta_2 \alpha_2 \beta_1\) and \(\nu_i \mu_i\) (\(3 \leq i \leq m - 1\)), it is enough to show that \(B_m\) is \(\tau\)-tilting finite by Proposition 2.18 (1).

For \(m = 3\), it is easy to check that \(B_3\) is \(\tau\)-tilting finite and \(#\text{silt} B_3\) = 32. Next, we show that \(B_m\) is \(\tau\)-tilting finite by induction on \(m\). We assume that \(m \geq 4\) and \(B_k\) is \(\tau\)-tilting finite for every \(k \in [3, m - 1]\). We feel free to use the sign decomposition introduced in Section 2.3. Let \(M := M(m)\) be the set of all maps from \([1, m]\) to \(\{\pm 1\}\). Suppose that \(M^\tau\) is a union of \(M^-^0\) and \(M^-^k\) with \(k \in [3, m - 1]\), where

- \(M^-^0 := \{\epsilon \in M \mid \epsilon(3) = -1\text{ and }\epsilon(j) \neq \epsilon(j + 1)\text{ for all }j \in [3, m - 1]\}\);
- \(M^-^k := \{\epsilon \in M \mid \epsilon(k) = \epsilon(k + 1) = -1\text{ and }\epsilon(j) \neq \epsilon(j + 1)\text{ for all }j \in [k + 1, m - 1]\}\).

We define \(M^\tau := \{-\epsilon \mid \epsilon \in M^-\}\). It is obvious that \(M = M^- \cup M^\tau\). (For example, if \(m = 5\), then \(M^- = \{(\epsilon(1), \epsilon(2), -1, +1, -1), (\epsilon(1), \epsilon(2), -1, +1, -1), (\epsilon(1), \epsilon(2), 3, -1, -1)\}\), and the number of maps in \(M^-\) is 16.) Since \(B_m \simeq (B_m')^\text{op}\), we have a bijection

\[
2\text{-silt}_{M^-} B_m \xrightarrow{\text{1-1}} 2\text{-silt}_{M^+} B_m
\]

by Proposition 2.16. Then, \(B_m\) is \(\tau\)-tilting finite if and only if \(2\text{-silt}_{M^-} B_m\) is finite, if and only if \(2\text{-silt}_{M^k} B_m\) are finite for all \(k \in \{0\} \cup [3, m - 1]\).

(1) We fix \(k \in [3, m - 1]\). Let \(\epsilon\) be a map in \(M^k\) and

\[
B := (B_m)_\epsilon = \left(\begin{array}{cc}
e_{+} \epsilon_{-} B_{m} \epsilon_{-} & \epsilon_{+} B_{m} \epsilon_{-} \\
J_{+} & 0
\end{array}\right)
\]

as defined in Definition 2.17. We denote by \(\epsilon'_1, \ldots, \epsilon'_n\) the idempotents of \(B\) corresponding to \(\epsilon_1, \ldots, \epsilon_n\) of \(B_m\), respectively. Then, we have a bijection between \(2\text{-silt}_{B_m}\) and \(2\text{-silt}_{B}\) by Proposition 2.18. Thus, we mainly observe the algebra \(B\) for our purpose.

Since \(\epsilon(k) = \epsilon(k + 1) = -1\), \(\epsilon_k B_m \epsilon_{k+1}\) and \(\epsilon_{k+1} B_m \epsilon_k\) are included in \(\epsilon_{-} B_m \epsilon_{-}\), while they are also contained in \(J_{-}\). By the definition of \(B\), we have \(\epsilon'_k B \epsilon'_{k+1} = 0 = \epsilon'_{k+1} B \epsilon'_k\).

In particular, there are no arrows between \(k\) and \(k + 1\) in the quiver of \(B\). We observe that both \(\epsilon' := \sum_{i=1}^{k} \epsilon_i\) and \(f := 1 - \epsilon'\) are central idempotents of \(B\), and hence \(B\) is decomposed into two blocks \(B' := \epsilon' B \epsilon'\) and \(B'' := f B f\). Thus, we have

\[
2\text{-silt}_{B} \xrightarrow{\text{1-1}} 2\text{-silt}_{\epsilon'_{|[1,k]} B'} \times 2\text{-silt}_{\epsilon'_{|[k+1,m]} B''}
\]

(3.7)

On the one hand, we find that the block \(B'\) is isomorphic to \((B_k)_{\epsilon'_{|[1,k]}}\) for the algebra \(B_k\) with respect to the restriction \(\epsilon'_{|[1,k]}\). Then, the set \(2\text{-silt}_{\epsilon'_{|[1,k]}} B'\) is bijective to \(2\text{-silt}_{\epsilon'_{|[1,k]}} B_k\) by Proposition 2.18. Thus, \(2\text{-silt}_{\epsilon'_{|[1,k]}} B'\) is finite by our induction hypothesis that \(B_k\) is \(\tau\)-tilting finite. On the other hand, since \(\epsilon\) satisfies \(\epsilon(k + 1) = -1\) and \(\epsilon(j) \neq \epsilon(j + 1)\) for all \(j \in [k + 1, m - 1]\), we have \(\epsilon'_{k+\ell} B \epsilon'_{k+r} \subseteq \epsilon'_{\ell} B \epsilon'_{\ell+r} = 0\) for any odd \(\ell\) and even \(r\) in \([1, m - k]\). Based on this fact, we deduce that the block \(B''\) is isomorphic to the path algebra \(\mathbb{F}Q_{m-k}\) of the following quiver \(Q_{m-k}\):

```
\[
\begin{array}{ccccccc}
& k + 1 & \rightarrow & k + 2 & \rightarrow & k + 3 & \cdots & m - 1 & \rightarrow & m \\
\end{array}
\]
```
which is \( \tau \)-tilting finite. (In fact, it is representation-finite.) Thus, we deduce that \( \text{2-silt}_k \mathcal{B} \) is a finite set by (3.7). Also, the set \( \text{2-silt}_k \mathcal{B}_m \) is finite following Proposition 2.18. Since \( \epsilon \in \mathcal{M}_k^- \) is arbitrary, the set \( \text{2-silt}_{\mathcal{M}_k^-} \mathcal{B}_m \) is finite.

(2) We show that the set \( \text{2-silt}_{\mathcal{M}_0} \mathcal{B}_m \) is finite. Let \( J \) be the set of odd numbers in \([3, m]\). We define \( \mathcal{M}_{J^-} := \{ \epsilon \in \mathcal{M} : \epsilon(J) = \{-1\} \} \) and \( \mathcal{M}_{J^+} := \{-\epsilon : \epsilon \in \mathcal{M}_{J^-}\} \). Since \( \mathcal{M}_0^- \subseteq \mathcal{M}_{J^-} \) holds, it is enough to see the finiteness of the set \( \text{2-silt}_{\mathcal{M}_{J^-}} \mathcal{B}_m \).

Let \( \mu_{P_j}^J(\mathcal{B}_m) \) be the left silting mutation of \( \mathcal{B}_m \) with respect to \((0 \to P_j)\) for the projective module \( P_j = \bigoplus_{j \in J} P_j \). Then, it is automatically tilting since silting complexes coincide with tilting complexes over symmetric algebras \[\text{[AA, Example 2.8]}\]. Following Corollary 2.21 we obtain a bijection

\[
\text{2-silt}_{\mathcal{M}_{J^-}} \mathcal{B}_m \xrightarrow{\mu_{P_j}^J} \text{2-silt}_{\mathcal{M}_{J^-}} \mathcal{B}_m \quad (3.8)
\]

where \( \mu_{P_j}^J(\mathcal{B}_m) := \text{End}_{\text{Ker}(\text{proj-}A)}(\mu_{P_j}^J(\mathcal{B}_m)) \). We may look at the right hand side in (3.8). By direct calculation, \( \mu_{P_j}^J(\mathcal{B}_m) \) is of the form \( \bigoplus_{i=1}^n T_i \), where \( T_i = (0 \to P_i) \) for \( i \notin J \) and

\[
T_i := \begin{cases} 
(P_3 \to P_1 \oplus P_2 \oplus P_3) & \text{for } i = 3, \\
(P_i \to P_{i-1} \oplus P_{i+1}) & \text{for } i = 5, 7, \ldots, 2 \left\lfloor \frac{m+1}{2} \right\rfloor - 1, \\
(P_m \to P_{m-1}) & \text{for odd } m,
\end{cases}
\]

where \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \). Then, \( \mu_{P_j}^J(\mathcal{B}_m) \) is presented by

\[
\begin{array}{cccccccc}
1 & \alpha_1 & \beta_3 & \alpha_3 & \beta_2 & \alpha_2 & \beta_1 & 3 \\
\beta_3 & \alpha_3 & \beta_2 & \alpha_2 & \beta_1 & \alpha_1 & \beta_3 & \alpha_3 & \beta_2 & \alpha_2 & \beta_1 & \alpha_1 \\
2 & \beta_3 & \alpha_3 & \beta_2 & \alpha_2 & \beta_1 & \alpha_1 & \beta_3 & \alpha_3 & \beta_2 & \alpha_2 & \beta_1 & \alpha_1 \\
\end{array}
\]

with

\[
\begin{aligned}
&\beta_1 \alpha_3, \alpha_3 \alpha_2, \alpha_2 \beta_1, \alpha_1 \beta_2, \beta_2 \beta_3, \beta_3 \alpha_1, \alpha_2 \mu_3, \alpha_1 \mu_3, \nu_3 \beta_1, \nu_3 \beta_2, \\
&\alpha_1 \beta_1 - \alpha_3 \beta_3, \alpha_2 \beta_2 - \beta_3 \alpha_3, \beta_1 \alpha_1 - \mu_3 \nu_3, \beta_2 \alpha_2 - \beta_1 \alpha_1, \\
&m \mu_{i+1} \nu_{i+1} \nu_i \mu_i - m \mu_{i+1} \nu_{i+1}, 3 \leq i \leq m - 2.
\end{aligned}
\]

We describe the indecomposable projective \( \mu_{P_j}^J(\mathcal{B}_m) \)-modules \( P''_i \) as follows,

\[
P''_1 \simeq 2, \quad P''_2 \simeq 1, \quad P''_3 \simeq 1, \quad P''_m \simeq 1
\]

where \( 4 \leq i \leq m - 1 \). Then, we find that \( \mu_{P_j}^J(\mathcal{B}_m) \) is a symmetric algebra with radical cube zero and it is \( \tau \)-tilting finite following from \[\text{[AA, Theorem 1.1]}\]. In particular, both sets in (3.8) are finite as desired. Thus, we have shown that \( \text{2-silt}_{\mathcal{M}_k} \mathcal{B}_m \) are finite for all \( k \in \{0\} \cup [3, m - 1] \). Therefore, \( \mathcal{B}_m \) is \( \tau \)-tilting finite, and so are \( \mathcal{D}_m' \) and \( \mathcal{D}_m \). \( \square \)

As we mentioned in Lemma 3.4, we can use a computer to find all \( g \)-vectors for \( \mathcal{D}_m \).

For the convenience of readers, we may give some information as follows,

| \( m \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| \#2-silt \( \mathcal{D}_m \) | 28 | 114 | 456 | 1816 | 4012 | 13238 | 45238 | 151568 |
3.2 A complete classification of $\tau$-tilting finite Schur algebras

We recall that most cases are determined in [W3] and the remaining cases are

\[
\begin{cases}
p = 2, n = 2, r = 8, 17, 19; \\
p = 2, n = 3, r = 4; \\
p = 2, n \geq 5, r = 5; \\
p \geq 5, n = 2, p^2 \leq r \leq p^2 + p - 1.
\end{cases}
\]

We have the following result. Recall that $\overline{S(n, r)}$ is the basic algebra of $S(n, r)$.

**Theorem 3.6.** Let $S(n, r)$ be the Schur algebra over $\mathbb{F}$.

1. If $p = 2$, then $S(2, 8)$, $S(2, 17)$ and $S(2, 19)$ are $\tau$-tilting finite.
2. If $p = 2$, then $S(3, 4)$ is $\tau$-tilting finite.
3. If $p = 2$, then $S(n, 5)$ is $\tau$-tilting infinite for any $n \geq 5$.
4. If $p \geq 5$, then $S(2, r)$ is $\tau$-tilting finite for any $p^2 \leq r \leq p^2 + p - 1$.

**Proof.** (1) We recall from [W3, Section 4.1] that

\[
S(2, 8) \simeq L_5, S(2, 17) \simeq L_5 \oplus A_2 \oplus F \oplus F \text{ and } S(2, 19) \simeq L_5 \oplus D_3 \oplus F \oplus F.
\]

By Proposition 3.2, Lemma 3.4 and Proposition 3.5 we have already known that $A_2$, $D_3$ and $L_5$ are $\tau$-tilting finite. Thus, we conclude that $S(2, 8)$, $S(2, 17)$ and $S(2, 19)$ are $\tau$-tilting finite. More precisely, the number of support $\tau$-tilting modules are 1656, 39744 and 185472, respectively.

(2) We recall from [DEMN, 3.6] that $S(3, 4)$ over $p = 2$ is Morita equivalent to $M_4$. Then, the assertion immediately follows from Lemma 3.4.

(3) By the definition of Schur algebras, $S(n, 5)$ with $n \geq 6$ is always Morita equivalent to $S(5, 5)$ over the same field. Then, it is shown in [Xi, Proposition 3.8] that $S(5, 5)$ over $p = 2$ is Morita equivalent to $N_5 \oplus A_2$. Since $N_5$ is $\tau$-tilting infinite by Lemma 3.4 so is $S(n, 5)$ with $n \geq 5$ over $p = 2$.

(4) We look at the case $S(2, r)$ with $p^2 \leq r \leq p^2 + p - 1$ over $p \geq 5$. We use Lemma 2.24 to understand the quiver of the basic algebra $S(2, r)$ of $S(2, r)$. We notice that the number of vertices in the quiver increases regularly when $r$ increases, see Table II.

Let $B$ be an indecomposable block of $S(2, r)$. By looking at the quiver of $S(2, r)$ as displayed in Table II, we find that if $p^2 \leq r \leq p^2 + p - 1$, then $1 \leq |B| \leq p + 1$. We have the following observations.

- For any $1 \leq m \leq p$, we can find an indecomposable block $B'$ of $S(2, r')$ with $1 \leq r' \leq p^2 - 1$, such that $|B'| = m$, see [Er, Proposition 5.1]. Then, $B'$ is Morita equivalent to $F$ if $m = 1$ and $B'$ is Morita equivalent to $A_m$ if $2 \leq m$ by Proposition 2.25 and Proposition 3.1.
- We notice from [DEMN, 3.4] that $S(2, p^2)$ has a block $B'$ satisfying $|B'| = p + 1$, which is Morita equivalent to the algebra $D_{p+1}$.
Table 1: the quiver of $S(2,r)$ over $p \geq 5$

|   |   |   |   |
|---|---|---|---|
| $2p - 1$ | $4p - 1$ | $p^2 - p - 1$ | $p^2 + p - 1$ |

```
0 {2p - 2} {2p} {4p - 2} \rightarrow p^2 - 3p \rightarrow p^2 - p - 2 \rightarrow p^2 - p

\quad p^2 + p \rightarrow p^2 + p - 2

1 \rightarrow 2p - 3 \rightarrow 2p + 1 \rightarrow 4p - 3 \rightarrow p^2 - 3p + 1 \rightarrow p^2 - p - 3 \rightarrow p^2 - p + 1

\quad p^2 + p + 1 \rightarrow p^2 + p - 3

2 \rightarrow 2p - 4 \rightarrow 2p + 2 \rightarrow 4p - 4 \rightarrow p^2 - 3p + 2 \rightarrow p^2 - p - 4 \rightarrow p^2 - p + 2

\quad p^2 + p + 2 \rightarrow p^2 + p - 4

\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots

p - 3 \rightarrow p + 1 \rightarrow 3p - 3 \rightarrow 3p + 1 \rightarrow p^2 - 2p - 3 \rightarrow p^2 - 2p + 1 \rightarrow p^2 - 3

\quad p^2 + 1

p - 2 \rightarrow p \rightarrow 3p - 2 \rightarrow 3p \rightarrow p^2 - 2p - 2 \rightarrow p^2 - 2p \rightarrow p^2 - 2

\quad p^2 \rightarrow p^2 - 1

p - 1 \quad 3p - 1 \quad p^2 - 2p - 1
```
Then, by Lemma 2.23, we have $B \simeq \mathbb{F}$ if $|B| = 1$, $B \simeq A|B|$ if $2 \leq |B| \leq p$ and $B \simeq D_{p+1}$ if $|B| = p + 1$. We conclude that $S(2, r)$ with $p^2 \leq r \leq p^2 + p - 1$ contains only $\mathbb{F}$, $A_m$ $(2 \leq m \leq p)$ and $D_{p+1}$ as indecomposable blocks. Since $A_m$ is obviously $\tau$-tilting finite, the problem in this case is reduced to the $\tau$-tilting finiteness of $D_{p+1}$. Then, we get the assertion since we have already shown in Proposition 5.3 that $D_{p+1}$ is $\tau$-tilting finite.

Now, the $\tau$-tilting finiteness of $S(n, r)$ is completely determined. As a summary, we may give a complete list of $\tau$-tilting finite Schur algebras, see Appendix A. Besides, we are able to determine the number of support $\tau$-tilting modules for any $\tau$-tilting finite $S(n, r)$ over $p = 2, 3$, see also Appendix A.

### 3.3 $\tau$-tilting finiteness of blocks of Schur algebras

As we have seen in the previous subsection, $\tau$-tilting infiniteness of a Schur algebra does not imply $\tau$-tilting infiniteness of its blocks. For instance, the Schur algebra $S(5, 5)$ over $p = 2$ is $\tau$-tilting infinite but it contains a $\tau$-tilting finite block which is Morita equivalent to $A_2$, see Theorem 3.6 (3). Thus, it is natural to ask the $\tau$-tilting finiteness for indecomposable blocks of Schur algebras. Namely,

**Problem 3.7.** Give a complete classification of $\tau$-tilting finite blocks of Schur algebras.

In this subsection, we provide some materials toward a solution of the above problem. We first give a general result.

**Proposition 3.8.** Let $B$ be an indecomposable block of $S(n, r)$.

1. If $|B| = 1$, then $B$ is Morita equivalent to $\mathbb{F}$.
2. If $|B| = 2$, then $B$ is Morita equivalent to $A_2$.

In particular, $B$ is $\tau$-tilting finite if $|B| \leq 2$.

**Proof.** We recall that the basic algebra $\overline{S(n, r)}$ of $S(n, r)$ is given by $\bigoplus S_B$ with

$$S_B = \text{End}_{\mathbb{F}G_r} \left( \bigoplus_{\lambda \in B \cap \Omega(n, r)} Y_{\lambda} \right),$$

where the sum is taken over all blocks of $\mathbb{F}G_r$.

1. If $|B| = 1$, then there is only one corresponding Young module. We may denote this unique Young module by $Y_{\lambda}$. Then, $Y_{\lambda} = S_{\lambda}$ is simple such that $\text{End}_{\mathbb{F}G_r} \left( Y_{\lambda} \right) \simeq \mathbb{F}$.

2. If $|B| = 2$, then there are only two corresponding Young modules, say, $Y_{\lambda}$ and $Y_{\mu}$. We may assume that $\lambda \triangleright \mu$. Then, the decomposition matrix $[S_{\lambda} : D_{\mu}]$ (11a) is of form

$$\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} \begin{pmatrix}
1 \\
\ast \\
1
\end{pmatrix},$$

such that the quiver of $B = \text{End}_{\mathbb{F}G_r} \left( Y_{\lambda} \oplus Y_{\mu} \right)$ must be of form $\circ \rightarrow \circ \circ$. We notice that any finite-dimensional algebra with quiver $\circ \rightarrow \circ \circ$ is representation-finite from Bongartz and Gabriel [BG]. Thus, $B$ is Morita equivalent $A_2$ by Proposition 3.4. \qed
Next, we may give a partial answer for Problem 3.7, that is, we completely determine the \( \tau \)-tilting finiteness for blocks of \( S(2, r) \). Many parts of the result have been given in the previous subsection.

**Theorem 3.9.** Let \( \mathcal{B} \) be an indecomposable block of \( S(2, r) \). Then, the followings hold.

1. If \( p = 2 \), then \( \mathcal{B} \) is \( \tau \)-tilting finite if and only if \( \mathcal{B} \) is Morita equivalent to one of \( F, \mathcal{A}_2, \mathcal{D}_3, \mathcal{K}_4 \) and \( \mathcal{L}_5 \).
2. If \( p \geq 3 \), then \( \mathcal{B} \) is \( \tau \)-tilting finite if and only if \( \mathcal{B} \) is Morita equivalent to one of \( F, \mathcal{A}_m (2 \leq m \leq p) \) and \( \mathcal{D}_{p+1} \).

**Proof.** We have already shown in the previous section that \( \mathcal{B} \) is \( \tau \)-tilting finite if \( \mathcal{B} \) is Morita equivalent to one of \( F, \mathcal{A}_m (2 \leq m \leq p) \), \( \mathcal{K}_4 \), \( \mathcal{L}_5 \) and \( \mathcal{D}_{p+1} \). Next, we show the necessity. We denote by \( S(2, r) \) the basic algebra of \( S(2, r) \) and we assume that \( \mathcal{B} \) is \( \tau \)-tilting finite. We may also assume \( |\mathcal{B}| \geq 3 \) by Proposition 3.8.

(1) Let \( p = 2 \). We recall from [Er] that \( S(2, 4) \) is isomorphic to \( \mathcal{D}_3 \) and \( S(2, 6) \) is isomorphic to \( \mathcal{K}_4 \). Also, it is given by [W3] that \( S(2, 8) \) is isomorphic to \( \mathcal{L}_5 \). By Lemma 2.23, we conclude that \( \mathcal{B} \) is Morita equivalent to \( \mathcal{D}_3, \mathcal{K}_4, \mathcal{L}_5 \) if \( |\mathcal{B}| = 3, 4, 5 \), respectively. By using Lemma 2.24, we find that the quiver of \( S(2, 10) \) is of form

```
  ◦ ↓  ◦  ←  ◦  ↓  ◦  ←  ◦  ◦  ↓  ◦  ←  ◦
```

with 6 vertices and it contains a \( \tau \)-tilting infinite subquiver, see Proposition 2.7. Also, according to Lemma 2.24 we find that if \( |\mathcal{B}| > 6 \), then the quiver of \( \mathcal{B} \) is obtained by adding extra vertices to the above quiver. Hence, \( \mathcal{B} \) is \( \tau \)-tilting infinite if \( |\mathcal{B}| \geq 6 \).

(2) Let \( p = 3 \). We recall from [Er] that \( S(2, 6) \) is isomorphic to \( \mathcal{A}_3 \oplus F \) and \( S(2, 9) \) is isomorphic to \( \mathcal{D}_4 \oplus F \). By Lemma 2.23 we conclude that \( \mathcal{B} \) is Morita equivalent to \( \mathcal{A}_3 \) if \( |\mathcal{B}| = 3 \) and \( \mathcal{B} \) is Morita equivalent to \( \mathcal{D}_4 \) if \( |\mathcal{B}| = 4 \). Similarly, by using Lemma 2.24 we find that the quiver of the principal block of \( S(2, 12) \) is of form

```
  ◦ ↓  ◦  ←  ◦  ↓  ◦  ←  ◦
```

with 5 vertices and it contains a \( \tau \)-tilting infinite subquiver. Therefore, \( \mathcal{B} \) is \( \tau \)-tilting infinite if \( |\mathcal{B}| \geq 5 \).

(3) Let \( p \geq 5 \). We have explained in the proof of Theorem 3.6 that if \( |\mathcal{B}| = m \) with \( 3 \leq m \leq p \), then \( \mathcal{B} \) is Morita equivalent to \( \mathcal{A}_m \); if \( |\mathcal{B}| = p+1 \), then \( \mathcal{B} \) is Morita equivalent to \( \mathcal{D}_{p+1} \). Now, it is not difficult to find in Table 1 that if \( |\mathcal{B}| \geq p+2 \), then \( \mathcal{B} \) must contain

```
  ◦ ↓  ◦  ←  ◦
```

as a subquiver. This implies that \( \mathcal{B} \) is \( \tau \)-tilting infinite if \( |\mathcal{B}| \geq p+2 \).
A A complete list of $\tau$-tilting finite Schur algebras

Since $S(n, r)$ is an idempotent truncation (resp., a quotient) of $S(N, r)$ for any $N > n$ (resp., $S(n, n+r)$), both $S(N, r)$ and $S(n, n+r)$ are $\tau$-tilting infinite if $S(n, r)$ is $\tau$-tilting infinite. Thus, we only need to consider the cases with small $n$ and $r$. Then, it needs two steps to give a complete list of $\tau$-tilting finite Schur algebras. The first step is to construct the basic algebra of $S(n, r)$ with small $n$ and $r$, while the second step is to check the $\tau$-tilting finiteness of these basic algebras. One may gradually enlarge $n$ and $r$, and repeat the second step until one can find a complete classification.

As we mentioned in the introduction, there is nothing new in this paper toward the first step since the work in [Er], [DEMN] and [W3] provide enough materials for this paper. We have already introduced most of the needed algebras in the previous section, but we still need the following three cases. We recall from [Er] that

- $U_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by
  \[
  Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \quad \text{and} \quad I : \langle \alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_1 \alpha_2 \alpha_3, \beta_3 \beta_2 \beta_1, \alpha_3 \beta_3 - \beta_2 \alpha_2 \rangle;
  \]

- $R_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by
  \[
  Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \quad \text{and} \quad I : \langle \alpha_1 \beta_1, \alpha_1 \alpha_2, \beta_2 \beta_1, \alpha_2 \beta_2 - \beta_1 \alpha_1, \alpha_3 \beta_3 - \beta_2 \alpha_2 \rangle;
  \]

- $H_4 := \mathbb{F}Q/I$ is the bound quiver algebra given by
  \[
  Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \quad \text{and} \quad I : \langle \alpha_1 \beta_1, \alpha_1 \alpha_2, \alpha_2 \beta_2, \alpha_1 \alpha_3, \beta_3 \beta_3 - \beta_1 \alpha_1 - \beta_2 \alpha_2 \rangle.
  \]

Toward the second step, we get some new results in this paper. For the convenience of readers, we recall some facts as follows. Here, the numbers for $R_4, H_4, U_4$ have been given in [W3, Lemma 3.3] and [W3, Proposition 4.4].

| $A$ | $\mathbb{F}$ | $A_2$ | $A_3$ | $D_3$ | $R_4$ | $H_4$ | $K_4$ | $U_4$ | $M_4$ | $L_5$ |
|-----|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\#\tau$-tilt $A$ | 2 | 6 | 20 | 28 | 88 | 96 | 114 | 136 | 136 | 152 | 1656 |

In fact, one may refer to

https://infinite-wang.github.io/Notes/ for a complete list of $g$-vectors of the above algebras.

Next, we use some visual tables to display the complete classification. In the following tables, we claim that the color purple means $\tau$-tilting finite, the color red means $\tau$-tilting infinite, the capital letter $S$ means semi-simple, the capital letter $F$ means representation-finite, the capital letter $T$ means tame and the capital letter $W$ means wild. In particular, we use Proposition [2.9] to calculate the number $\#\tau$-tilt $S(n, r)$ for a $\tau$-tilting finite $S(n, r)$.  

23
### A.1 The \( \tau \)-tilting finiteness of \( S(n, r) \) over \( p = 2 \)

We list all \( \tau \)-tilting finite \( S(n, r) \) over \( p = 2 \) as follows.

| \( n \) | \( r \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) | \( 13 \) | \( \ldots \) |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2     |       | S   | S   | T   | F   | F   | W   | T   | T   | T   | W   | W   | W   | W   | W   |

| \( n \) | \( r \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) | \( 13 \) | \( \ldots \) |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3     |       | S   | F   | F   | F   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   |
| 4     |       | S   | F   | F   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   |
| 5     |       | S   | F   | F   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   |
| \vdots |       | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |

| \( S(n, r) \) | The basic algebra of \( S(n, r) \) | Morita equivalence | \#\( \tau \)-tilt \( S(n, r) \) |
|--------------|----------------|-----------------|------------------|
| \( S(2, 1) \) | \( F \) | \( \simeq S(n, 1) \) for any \( n \geq 3 \) | 2 |
| \( S(2, 2) \) | \( A_2 \) | \( \simeq S(n, 2) \) for any \( n \geq 3 \) | 6 |
| \( S(2, 3) \) | \( F \oplus F \) | | 4 |
| \( S(2, 4) \) | \( D_3 \) | | 28 |
| \( S(2, 5) \) | \( A_2 \oplus F \) | | 12 |
| \( S(2, 6) \) | \( K_4 \) | | 136 |
| \( S(2, 7) \) | \( A_2 \oplus F \oplus F \) | | 24 |
| \( S(2, 8) \) | \( L_5 \) | | 1656 |
| \( S(2, 9) \) | \( D_3 \oplus F \oplus F \) | | 112 |
| \( S(2, 11) \) | \( D_3 \oplus A_2 \oplus F \) | | 336 |
| \( S(2, 13) \) | \( K_4 \oplus A_2 \oplus F \) | | 1632 |
| \( S(2, 15) \) | \( K_4 \oplus A_2 \oplus F \oplus F \) | | 3264 |
| \( S(2, 17) \) | \( L_5 \oplus A_2 \oplus F \oplus F \) | | 39744 |
| \( S(2, 19) \) | \( L_5 \oplus D_3 \oplus F \oplus F \) | | 185472 |
| \( S(3, 3) \) | \( A_2 \oplus F \) | \( \simeq S(n, 3) \) for any \( n \geq 4 \) | 12 |
| \( S(3, 4) \) | \( M_4 \) | | 152 |
| \( S(3, 5) \) | \( U_4 \) | | 136 |
| \( S(4, 5) \) | \( U_4 \oplus A_2 \) | | 816 |

### A.2 The \( \tau \)-tilting finiteness of \( S(n, r) \) over \( p = 3 \)

| \( n \) | \( r \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) | \( 13 \) | \( \ldots \) |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2     |       | S   | S   | F   | F   | F   | F   | F   | F   | T   | T   | T   | W   | W   | W   |
| 3     |       | S   | S   | F   | F   | W   | T   | T   | T   | W   | W   | W   | W   | W   | W   |
| 4     |       | S   | S   | F   | F   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   |
| 5     |       | S   | S   | F   | F   | W   | W   | W   | W   | W   | W   | W   | W   | W   | W   |
| \vdots |       | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
We list all \( \tau \)-tilting finite \( S(n, r) \) over \( p = 3 \) as follows.

| \( S(n, r) \) | The basic algebra of \( S(n, r) \) | Morita equivalence | \#\( \tau \)-tilt \( S(n, r) \) |
|----------------|-----------------------------------|--------------------|------------------------|
| \( S(2, 1) \)  | \( F \)                           | \( \simeq S(n, 1) \) for any \( n \geq 3 \) | 2                     |
| \( S(2, 2) \)  | \( F \oplus F \)                  | \( \simeq S(n, 2) \) for any \( n \geq 3 \) | 4                     |
| \( S(2, 3) \)  | \( \mathcal{A}_2 \)               |                     | 6                     |
| \( S(2, 4) \)  | \( \mathcal{A}_2 \oplus F \)     | \( \simeq S(2, 5) \) | 12                    |
| \( S(2, 6) \)  | \( \mathcal{A}_3 \oplus F \)     | \( \simeq S(2, 7) \) | 40                    |
| \( S(2, 8) \)  | \( \mathcal{A}_3 \oplus F \oplus F \) |                     | 80                    |
| \( S(2, 9) \)  | \( D_4 \oplus F \)               |                     | 228                   |
| \( S(2, 10) \) | \( D_4 \oplus F \oplus F \)      |                     | 456                   |
| \( S(2, 11) \) | \( D_4 \oplus \mathcal{A}_2 \)   | \( \simeq S(n, 3) \) for any \( n \geq 4 \) | 684                   |
| \( S(3, 3) \)  | \( \mathcal{A}_3 \)               |                     | 20                    |
| \( S(3, 4) \)  | \( \mathcal{A}_2 \oplus F \oplus F \) |                     | 24                    |
| \( S(3, 5) \)  | \( \mathcal{A}_2 \oplus \mathcal{A}_2 \oplus F \) |         | 72                    |
| \( S(3, 7) \)  | \( \mathcal{R}_4 \oplus \mathcal{A}_2 \oplus \mathcal{A}_2 \) |         | 3168                  |
| \( S(3, 8) \)  | \( \mathcal{R}_4 \oplus H_4 \oplus \mathcal{A}_2 \) |         | 50688                 |
| \( S(4, 4) \)  | \( \mathcal{A}_3 \oplus F \oplus F \) | \( \simeq S(n, 4) \) for any \( n \geq 5 \) | 80                    |
| \( S(4, 5) \)  | \( \mathcal{A}_3 \oplus \mathcal{A}_2 \oplus F \) |         | 240                   |
| \( S(5, 5) \)  | \( \mathcal{A}_3 \oplus \mathcal{A}_3 \oplus F \) | \( \simeq S(n, 5) \) for any \( n \geq 6 \) | 800                   |

### A.3 The \( \tau \)-tilting finiteness of \( S(n, r) \) over \( p \geq 5 \)

| \( n \) | \( p \) | \( 1 \sim p - 1 \) | \( p \sim 2p - 1 \) | \( 2p \sim p^2 - 1 \) | \( p^2 \sim p^2 + p - 1 \) | \( p^2 + p \sim \infty \) |
|---------|--------|-------------------|-------------------|-------------------|------------------|-------------------|
| 2       |        | \( S \)           | \( F \)           | \( F \)           | \( W \)           | \( W \)           |
| 3       |        | \( S \)           | \( F \)           | \( W \)           | \( W \)           | \( W \)           |
| 4       |        | \( S \)           | \( F \)           | \( W \)           | \( W \)           | \( W \)           |
| 5       |        | \( S \)           | \( F \)           | \( W \)           | \( W \)           | \( W \)           |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |

Depending on the prime number \( p \), we can determine the structure for the basic algebra \( S(n, r) \) of \( \tau \)-tilting finite \( S(n, r) \) as follows.

- In the case of \( n \geq 2 \) and \( 1 \leq r \leq p - 1 \), \( S(n, r) \) is isomorphic to a finite direct sum of copies of \( F \).
- In the case of \( n \geq 2 \) and \( p \leq r \leq 2p - 1 \), \( S(n, r) \) is isomorphic to a finite direct sum of copies of \( F \) and copies of \( \mathcal{A}_m \) with \( 2 \leq m \in \mathbb{N} \).
- In the case of \( n = 2 \) and \( 2p \leq r \leq p^2 + p - 1 \), \( S(2, r) \) is isomorphic to a finite direct sum of copies of \( F \), copies of \( \mathcal{A}_m(2 \leq m \leq p) \) and copies of \( D_{p+1} \).

In fact, the multiplicities of \( F \), \( \mathcal{A}_m \) and \( D_{p+1} \) in each case are determined by the decomposition matrix of the symmetric group \( G_r \) over \( p \), which is still open.
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