Complexity of multivariate polynomial evaluation

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Abstract

We describe a method to evaluate multivariate polynomials over a finite field and discuss its multiplicative complexity.

Keywords: multiplicative complexity, complexity, multivariate polynomials, finite field, computational algebra

MSC: 12Y05, 94B27

1 Introduction

Many applications require the evaluation of multivariate polynomials over finite fields. For instance, the so called affine codes (also called evaluation or functional or algebraic geometry codes) are obtained evaluating a finite-dimensional linear subspace of $\mathbb{F}_q[x_1, \ldots, x_r]$ at a finite set $S \subseteq \mathbb{F}_q^r$ ([2], [4], [5] and several other papers). When the degree $n$ of the polynomials is small, and/or the number $r$ of variables is also small the direct computation is efficient, however as $n$, or $r$, or both become large, evaluation becomes an issue. The case of univariate polynomials was considered by several authors, see e.g [10], [9] and some recent papers ([14], [1]). In this paper we propose an evaluation method for multivariate polynomials which reduces significantly the multiplicative complexity and hence the computational burden.

Set $M_r(n) = \binom{n+r}{r}$, and let $p(x_1, \ldots, x_r)$ be a polynomial of degree $n$ in $r$ variables with coefficients in a finite field $\mathbb{F}_p$; the number of monomials occurring in $p(x_1, \ldots, x_r)$ is $M_r(n)$. We will consider the evaluation of $p(x_1, \ldots, x_r)$ at a point $a = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_p^m$, where $m$ is divisible by $s$. A direct evaluation of $p(\alpha_1, \ldots, \alpha_r)$ is obtained from the evaluation of the $M_r(n)$ distinct monomials, a task requiring $M_r(n) - r - 1$ multiplications. Therefore we perform $M_r(n) - 1$ multiplications, and a total number
\[ A_r(n) = M_r(n) - 1 \] of additions. The total number of required multiplications is
\[ P_r(n) = 2M_r(n) - r - 2 = 2 \frac{n^r}{r!} + \frac{r + 1}{(r - 1)!} n^{r-1} + \ldots - 1 - r, \]
however different computing strategies can require a significantly smaller number of multiplications. To the aim of developing some of these strategies, the polynomial \( p(x_1, \ldots, x_r) \) is written as a sum
\[ p(x_1, \ldots, x_r) = \sum_{i=0}^{s-1} \beta^i q_i(x_1, \ldots, x_r) \quad (1) \]
of \( s \) polynomials, where each \( q_i(x_1, \ldots, x_r) \) is a polynomial of degree \( n \) in \( r \) variables with coefficients in the prime field \( \mathbb{F}_p \), the value \( p(\alpha_1, \ldots, \alpha_r) \) can be obtained from the \( s \) values \( q_i(\alpha_1, \ldots, \alpha_r) \), \( 2s - 2 \) multiplications, and \( s \) additions in \( \mathbb{F}_{p^m} \). In these computations \( \beta \) and its powers constitute a basis of \( \mathbb{F}_{p^m} \). Therefore, we may restrict our attention to the evaluation at a point \( a \in \mathbb{F}_{p^m} \) of a polynomial \( q(x_1, \ldots, x_r) \), in \( r \) variables, of degree \( n \) over \( \mathbb{F}_p \).

As pointed out in [1], §2.1, the prime 2 is particularly interesting because of its occurrence in many practical applications, for example in error correction coding. Furthermore, in \( \mathbb{F}_2 \) multiplications are trivial. Therefore, we give first a description of our method in the easiest case, that is, over \( \mathbb{F}_2 \) and with two variables. Later, we generalize to any setting.

## 2 Our computational model

There are two kinds of multiplications that are involved in our computations: field multiplications in the coefficient field \( \mathbb{F}_p \) and in extension field \( \mathbb{F}_{p^m} \). We assign cost 1 to any of these, except for the multiplications by 0 or 1, that cost 0 in our model.

**Remark 1.** There can be multiplications that cost much less, such as squares in characteristic 2, but we still treat them as cost 1.

As customary, we assign cost 0 to any data reading.

We could count separately field sums, but our aim is to minimize the number of field multiplications, and so we use as implicit upper bound for the number of sums the value \( 2M_r(n) \), that is, twice the number of all monomials. We will not discuss of the number of sums any further.
We assume that an ordering on monomials is chosen once and for all, e.g. the degree lexicographical ordering (see [8]), so that our input data can be modeled as an \( \mathbb{F}_p \) string, any entry corresponding to a polynomial coefficient.

**Remark 2.** A well-established method to evaluate all monomials up to degree \( n \) at a given point is to start from degree-1 monomials and then iterate from degree-\( r \) monomials to degree-\( r+1 \) monomials, since the computations of any degree-\( r+1 \) monomial requires only one multiplication, once you have in memory all degree-\( r \) monomials.

We remark here that our algorithm accepts as input any polynomial of a given total degree and so our estimates are worst-case complexity, which translates in considering dense polynomials. Clearly, other faster methods could be derived for special classes of polynomials, such as sparse polynomials or polynomials with a predetermined algebraic structure.

We will not discuss the memory requirement of our methods, but one can see easily by inspecting the following algorithms that it is negligible compared to their computational effort.

### 3 The case \( r = 2, \ p = 2 \)

A polynomial \( P(x, y) \) of degree \( n \) in 2 variables over the binary field may be decomposed into a sum of 4 polynomials as

\[
P(x, y) = P_{0,0}(x^2, y^2) + xP_{1,0}(x^2, y^2) + yP_{0,1}(x^2, y^2) + xyP_{1,1}(x^2, y^2)
\]

\[
P(x, y) = P_{0,0}(x, y)^2 + xP_{1,0}(x, y)^2 + yP_{0,1}(x, y)^2 + xyP_{1,1}(x, y)^2.  \tag{2}
\]

where \( P_{i,j}(x, y) \) are polynomials of degree \( \left\lfloor \frac{n-i-j}{2} \right\rfloor \). Therefore the value of \( P(x, y) \) in the point \( a = (\alpha_1, \alpha_2) \in \mathbb{F}_2^2 \) can be obtained by computing the 4 numbers \( P_{i,j}(\alpha_1, \alpha_2) \), the monomial \( \alpha_1\alpha_2 \), performing 3 products \( \alpha_1 P_{1,0}(\alpha_1, \alpha_2), \alpha_2 P_{0,1}(\alpha_1, \alpha_2), \) and \( \alpha_1\alpha_2 P_{1,1}(\alpha_1, \alpha_2) \), and finally performing 3 additions. We observe that all \( P_{i,j} \)’s have the same possible monomials, i.e. all monomials of degree up to \( \left\lfloor \frac{n}{2} \right\rfloor \). There is no need to store separately \( P_{0,0}, P_{0,1}, P_{1,0}, P_{1,1} \), because the selection of any of these is obtained by a trivial indexing rule. The polynomials \( P_{i,j}(\alpha_1, \alpha_2) \) can be evaluated as sums of such monomials, which can be evaluated once for all. Therefore, \( P(\alpha_1, \alpha_2) \) is obtained performing (see Remark 2) a total number of

\[
P_r(n) = 4 + 3 + \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)}{2} - 3 \approx \frac{n^2}{8}
\]
multiplications, a figure considerably less than \( \frac{n^2}{2} \) as required by the direct
computation. However, the mechanism can be iterated, and the point is to
find the number of steps yielding the maximum gain, that is to find the most
convenient degree of the polynomials that should be directly evaluated. We
have the following:

**Theorem 1.** Let \( P(x, y) \) be a polynomial of degree \( n \) over \( \mathbb{F}_2 \), its evaluation
at a point \( (\alpha_1, \alpha_2) \in \mathbb{F}_2^2 \), performed by applying repeatedly the decomposition
(2), requires a number \( G_2(n, 2, L_{opt}) \) of products which asymptotically is
\[
G_2(n, 2, L_{opt}) \approx c\sqrt[7]{\frac{6}{7}n} \quad c < 5
\]
where \( L_{opt} \), the number of iterations yielding the minimum of \( G_2(n, 2, L) \), is
an integer included into the interval
\[
-\frac{1}{2} + \log_4(\sqrt[7]{6n}) + \epsilon < L_{opt} < \log_4(\sqrt[7]{6n}) + \epsilon',
\]
where \( \epsilon \) and \( \epsilon' \) are less than 1 and \( O(\frac{1}{\sqrt{n}}) \).

**Proof.** The polynomial \( P(x, y) \) is decomposed into the sum of 4 polynomials that are perfect squares over \( \mathbb{F}_2 \), each of which is the similarly
decomposed. Let \( P_{i,j}^{(L,h)}(x, y) \) denote the polynomials at the \( L \)-step of this
iterative process, with \( h \) varying from 1 to \( 4L-1 \). The number of polynomials
after \( L \) steps is \( 4L \), while their degrees are not greater than \( \lfloor \frac{n}{2L} \rfloor \).
The value \( P(\alpha_1, \alpha_2) \) is obtained performing backward the reconstruction
process obtaining at each step the values \( P_{i,j}^{(L,h)}(\alpha_1, \alpha_2) \) from the values
\( P_{i,j}^{(L-1,h)}(\alpha_1, \alpha_2) \), whereas the \( 4L \) numbers \( P_{i,j}^{(L,h)}(\alpha_1, \alpha_2), i, j \in \{0, 1\} \) and
\( h = 0, \ldots, 4L-1 \), are computed from the direct evaluation of \( M_2(\lfloor \frac{n}{2L} \rfloor) \) monomials using \( M_2(\lfloor \frac{n}{2L} \rfloor) - 3 \) multiplications.

Therefore the total number of multiplications necessary to obtain \( P(\alpha_1, \alpha_2) \)
is a sum of \( M_2(\lfloor \frac{n}{2L} \rfloor) - 3 \) with
- the number of squares
  \[
  \frac{4}{3}(4L - 1) = [4^{L-1} + 4^{L-2} + \ldots + 4^{L-L+1}]
  \]
- the number of multiplications of kind \( x^iy^j P_{i,j}(\alpha_1, \alpha_2) \)
  \[
  4L - 1 = 3[4^{L-1} + 4^{L-2} + \ldots + 4^{L-L}]
  \]
that is the total number is:

$$G_2(n, 2, L) = \frac{7}{3}(4^L - 1) + \frac{1}{2}\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right)\left(\left\lceil \frac{n}{2}\right\rceil + 2\right) - 3$$

The number of products required to evaluate $P(\alpha_1, \alpha_2)$ in this way is a function of $L$, and the values of $L$ that correspond to local minima are specified by the conditions

$$G_2(n, 2, L) \leq G_2(n, 2, L - 1) \quad \text{and} \quad G_2(n, 2, L) \leq G_2(n, 2, L + 1)$$

from which, it is straightforward to obtain the conditions

$$4^L - \frac{3}{56}n^2 > \frac{n}{2^L}\left(\left\lfloor \frac{3}{2}\right\rfloor - \left\{ \frac{n}{2^L}\right\}\right) + \left(\left\lfloor \frac{n}{2^L}\right\rfloor + \left\{ \frac{n}{2^L}\right\}\right)\left(\left\lfloor \frac{3}{2}\right\rfloor + \left\lceil \left\{ \frac{n}{2^L}\right\}-1\right\rfloor\right)$$

$$4^L - \frac{6}{7}n^2 < \frac{4n}{2^L}\left(\left\lfloor \frac{3}{2}\right\rfloor + \left\{ \frac{n}{2^L}\right\}-2\left(\left\lfloor \frac{2n}{2^L}\right\rfloor\right)\right) - \frac{2}{7}\left(\left\lceil \frac{n}{2^L}\right\rceil\right)^2 - \left(\left\lfloor \frac{2n}{2^L}\right\rfloor\right)^2$$

where $\{x\}$ denotes the fractional part of $x$. These inequalities show that there is only one minimum that corresponds to a value of $L$ such that

$$-\frac{1}{2} + \log_4\left(\sqrt{\frac{6}{7}n}\right) + \epsilon < L_{op} < \log_4\left(\sqrt{\frac{6}{7}n}\right) + \epsilon'$$

where $\epsilon$ and $\epsilon'$ are $O\left(\frac{1}{\sqrt{n}}\right)$. Therefore, the minimum value of $G_2(n, 2, L)$ is asymptotically

$$G_2(n, 2, L_{op}) \approx c\sqrt{\frac{7}{6}n}$$

where $c$ is a constant less than 5.

Remark 3. In the computations of our bounds we essentially compute separately each monomial. Hence our approach seems to be very efficient for the computation of several polynomials at the same point. This fact is exploited in the computation of the required number of multiplications when the polynomial coefficients are in $\mathbb{F}_2$. An application of equation (1) and Theorem 1 would give the asymptotic estimate

$$G_{2^s}(n, 2, L_{op}) \approx c\sqrt{\frac{7}{6}ns}$$

since the evaluation of any $q_i$ would cost $c\sqrt{\frac{7}{6}n}$. However, the polynomials $q_i(x, y)$ can be evaluated contemporarily. Therefore, computing the power
necessary to evaluate the polynomial at step \( L \) only once, this lead to a total number or required multiplications

\[
G_2(n, 2, L) = s \frac{7}{3}(4^L - 1) + \frac{(\lfloor \frac{3}{2} \rfloor + 1)(\lfloor \frac{3}{2} \rfloor + 2)}{2} - 3
\]

because only the reconstruction operations need to be repeated \( s \) times. By repeating the argument outlined in the proof of Theorem \( \text{7} \), the conclusion is that the optimal value of \( L \) depends also on \( s \) and asymptotically the required value of multiplications is

\[
G_2(n, 2, L_{\text{opt}}) \approx c' \sqrt[6]{\frac{n}{s}} .
\]

4 The case \( r > 2, p = 2 \)

The evaluation of a polynomial \( P(x_1, \ldots, x_r) \) in \( r \) variables can be done writing \( P \), similarly to equation (2), this polynomial as a sum of \( 2^r \) polynomials

\[
P(x_1, \ldots, x_r) = \sum_{i_1, \ldots, i_r \in \{0, 1\}} x_1^{i_1} \cdots x_r^{i_r} P_{i_1, \ldots, i_r}(x_1^2, \ldots, x_r^2)
\]

\[
= \sum_{i_1, \ldots, i_r \in \{0, 1\}} x_1^{i_1} \cdots x_r^{i_r} (P_{i_1, \ldots, i_r}(x_1, \ldots, x_r))^2 , \tag{3}
\]

where \( P_{i_1, \ldots, i_r}(x_1, \ldots, x_r) \) is a polynomial of degree \( \frac{n - \sum i_j}{2} \). The argument of Theorem \( \text{1} \) still applies, and the minimum number of steps is obtained in the following theorem.

**Theorem 2.** Let \( L_{\text{opt}} \) be the number of steps of this method yielding the minimum number of products, \( G_2(n, r, L_{\text{opt}}) \), required to evaluate a polynomial of degree \( n \) in \( r \) variables, with coefficients in \( \mathbb{F}_2 \). Then \( L_{\text{opt}} \) is an integer that asymptotically is included into the interval

\[
-\frac{1}{2} + \frac{1}{2^r} \log_2 \frac{2^r - 1}{r!(2^{r+1} - 1)} + \frac{\log_2 n}{2} \leq L_{\text{opt}} \leq \frac{1}{2} + \frac{1}{2^r} \log_2 \frac{2^r - 1}{r!(2^{r+1} - 1)} + \frac{\log_2 n}{2}
\]

that is \( L_{\text{opt}} \) is the integer closest to \( \frac{1}{2^r} \log_2 \frac{2^r - 1}{r!(2^{r+1} - 1)} + \frac{\log_2 n}{2} \). Asymptotically the minimum \( G_2(n, r, L_{\text{opt}}) \) is included into the interval:

\[
\frac{1}{\sqrt{2^r}} \sqrt{\frac{4^{2^r+1} - 1}{2^r - 1} n^{r/2}} < G_2(n, r, L_{\text{opt}}) < \sqrt{2^r} \sqrt{\frac{4^{2^r+1} - 1}{2^r - 1} n^{r/2}} .
\]
Proof. Using equation (3) the polynomial $P(x_1, \ldots, x_r)$ evaluated at the point $a = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_2^r$ can be obtained from the evaluation of all $P_{i_1,\ldots,i_r}(x_1, \ldots, x_r)$ at $a$, by evaluating $2^r$ monomials $\alpha_{i_1}^{i_1} \cdots \alpha_{i_r}^{i_r}$ (which require $2^r - r - 1$ multiplications), performing $2^r$ squaring, combining these factors with $2^r - 1$ multiplications, and finally adding the results.

We can iterate this procedure: at each step the number of polynomials $P_{i,j}$'s is multiplied by $2^r$ and their degrees are divided at least by 2. Therefore, after $L$ steps the number of polynomials is $2^{Lr}$ and their degrees are not greater than $\left\lfloor \frac{n}{2^L} \right\rfloor$. Once the $2^r$ numbers $P_{i_1,\ldots,i_r}(\alpha_1, \ldots, \alpha_r)$ are known, the total number of squaring is

$$2^{rL} + 2^{r(L-1)} + \cdots + 2^{r(L-L+1)} = \frac{2^r}{2^r - 1} (2^{rL} - 1)$$

and the number of products necessary to obtain $P(\alpha)$ is

$$(2^r - 1)[2^{(L-1)} + 2^{(L-2)} + \cdots + 2^{(L-L)}] = 2^{rL} - 1,$$

hence the total number of required multiplications is

$$\frac{2^{r+1} - 1}{2^r - 1}(2^{rL} - 1).$$

Since the total number of monomials in $r$ variables in a generic polynomial of degree $\left\lfloor \frac{n}{2^L} \right\rfloor$ is $M_r(\left\lfloor \frac{n}{2^L} \right\rfloor)$, then $M_r(\left\lfloor \frac{n}{2^L} \right\rfloor) - r - 1$ is the number of products necessary to evaluate all independent monomials. Therefore, the total number of multiplications for evaluating $P(a)$ is

$$G_2(n, r, L) = \frac{2^{r+1} - 1}{2^r - 1}(2^{rL} - 1) + M_r(n) - r - 1.$$

We look for the optimal value $L_{op}$ giving the minimum $G_2(n, r, L_{op})$. Since

$$M_r(\left\lfloor \frac{n}{2^L} \right\rfloor) = \frac{1}{r!} \left( \frac{n}{2^L} - \left\lfloor \frac{n}{2^L} \right\rfloor \right)^r \prod_{j=1}^r \left( 1 + \frac{j}{2^L - \left\lfloor \frac{n}{2^L} \right\rfloor} \right),$$

then $M_r(\left\lfloor \frac{n}{2^L} \right\rfloor)$ is an expression that is $\frac{1}{r!}(\frac{n}{2^L})^r + O(\frac{2^L}{n})$ asymptotically in $n$. The local optima are given by the values of $L$ such that

$$G_2(n, r, L) \leq G_2(n, r, L - 1) \quad \text{and} \quad G_2(n, r, L) \leq G_2(n, r, L + 1).$$

Then, considering the asymptotic expression

$$G_2(n, r, L) = \frac{2^{r+1} - 1}{2^r - 1}2^{rL} + \frac{1}{r!} \frac{n}{2^L},$$

we look for the optimal value $L_{op}$ giving the minimum $G_2(n, r, L_{op})$. Since
it is immediate to obtain the conditions

\[
2^{2rL} > \frac{1}{r!} \frac{2^r - 1}{2^r - 1} \left( \frac{n}{r^r} \right)
\]

\[
2^{2rL} < 2^r n^r \frac{1}{r!} \frac{2^r - 1}{2^r - 1},
\]

showing that asymptotically \( L_{op} \) must satisfy the inequalities

\[
-\frac{1}{2} \left( \log_2 \frac{2^r - 1}{r!(2^r - 1)} + \frac{\log_2 n}{2} \right) < L_{op} < \frac{1}{2} \left( \log_2 \frac{2^r - 1}{r!(2^r - 1)} + \frac{\log_2 n}{2} \right).
\]

Therefore \( L_{op} \) is the closest integer to

\[
\frac{1}{2} \left( \log_2 \frac{2^r - 1}{r!(2^r - 1)} + \frac{\log_2 n}{2} \right),
\]

and the total number of products asymptotically is included into the interval:

\[
\frac{1}{\sqrt{2^r}} \sqrt{4 \left( \frac{2^{r+1} - 1}{2^r - 1} \right) \frac{n^{r/2}}{r^!}} < G_2(n, r, L_{op}) < \sqrt{2^r} \sqrt{4 \left( \frac{2^{r+1} - 1}{2^r - 1} \right) \frac{n^{r/2}}{r!}}.
\]

\[
\square
\]

5 The case \( r \geq 2, p > 2 \)

A polynomial \( P(x_1, \ldots, x_r) \) of degree \( n \), in \( r \) variables over the field \( \mathbb{F}_p \), is simply decomposed into a sum of \( p^r \) polynomials as

\[
P(x_1, \ldots, x_r) = \sum_{i_1, \ldots, i_r \in \{0, 1, \ldots, p-1\}} x_1^{i_1} \ldots x_r^{i_r} P_{i_1, \ldots, i_r}(x_1^p, \ldots, x_r^p)
\]

\[
= \sum_{i_1, \ldots, i_r \in \{0, 1, \ldots, p-1\}} x_1^{i_1} \ldots x_r^{i_r} \left( P_{i_1, \ldots, i_r}(x_1, \ldots, x_r) \right)^p \quad (4)
\]

where \( P_{i_1, \ldots, i_r}(x_1, \ldots, x_r) \) is a polynomial of degree \( \left\lfloor \frac{n - \sum i_j}{p} \right\rfloor \). Therefore the polynomial \( P(x_1, \ldots, x_r) \) evaluated at the point \( a = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_p^{\alpha} \) can be obtained from the evaluation of all polynomials \( P_{i_1, \ldots, i_r}(x_1, \ldots, x_r) \) at \( a \), by evaluating the \( p^r \) monomials \( \alpha_1^{i_1} \ldots \alpha_r^{i_r} \) (which require \( p^r - r - 1 \) multiplications), performing \( p^r \) computations of \( p \)-powers, combining these factors with \( p^r \) multiplications, and finally adding all results.

The argument of Theorem 1 and 2 still applies, and the minimum number of steps is obtained in the following theorem.
Theorem 3. Let $L_{opt}$ be the number of steps of this method yielding the minimum number of products, $G_p(n, r, L_{opt})$, required to evaluate a polynomial of degree $n$ in $r$ variables, with coefficients in $\mathbb{F}_p$. Then $L_{opt}$ is an integer that asymptotically is included into the interval

$$-\frac{1}{2} + B + \frac{\log p}{2} n \leq L_{opt} \leq \frac{1}{2} + B + \frac{\log p}{2} n$$

where $B = \frac{1}{2} \log p \frac{(p-1)(p'-1)}{r!(2p'-1)}$, that is, $L_{opt}$ is the integer closest to $B + \frac{\log p}{2} n$. Asymptotically the minimum $G_p(n, r, L_{opt})$ is included into the interval:

$$2 \sqrt{\frac{p}{n}} \sqrt{\frac{p-1}{p'-1}} \frac{r/2}{n^{r/2}} < G_p(n, r, L_{opt}) < 2 \sqrt{p} \sqrt{\frac{p-1}{p'-1}} \frac{r/2}{n^{r/2}}.$$

Proof. Using equation (4) the polynomial $P(x_1, \ldots, x_r)$ evaluated at the point $a = (\alpha_1, \ldots, \alpha_r) \in \mathbb{F}_{p^m}$ can be obtained from the evaluation of all $P_{i_1, \ldots, i_r}(x_1, \ldots, x_r)$ at $a$, by evaluating $p^r$ monomials $\alpha_1^{i_1} \cdots \alpha_r^{i_r}$ (which require $p^r - r - 1$ multiplications), computing $p^r$ $p$-powers, combining these factors with $p^r - 1$ multiplications, and finally performing the required additions.

We can iterate this procedure: at each step the number of polynomials is multiplied by $p^r$ and their degrees are at least divided by $p$. Therefore, after $L$ steps the number of polynomials is $p^{rL}$ and their degrees are not greater than $\lfloor \frac{n}{2} \rfloor$. Once the $p^r$ numbers $P_{i_1, \ldots, i_r}(\alpha_1, \ldots, \alpha_r)$ are known, the total number of $p$-powers is

$$p^{rL} + p^{r(L-1)} + \cdots + p^{r(L-L+1)} = \frac{p^r}{p^r - 1} (p^{rL} - 1)$$

and the number of products necessary to obtain $P(\alpha)$ is

$$(p^r - 1) \left[ p^{r(L-1)} + p^{r(L-2)} + \cdots + p^{r(L-L)} \right] = p^{rL} - 1,$$

hence the total number of required multiplications is

$$\frac{2p^r - 1}{p^r - 1} (p^{rL} - 1).$$

The total number of multiplications for computing all the monomials of all the polynomials arising at step $L$ is $M_r(\lfloor \frac{n}{2r} \rfloor) - r - 1$, and further $(p - 2)(M_r(\lfloor \frac{n}{2r} \rfloor) - r - 1)$ products are necessary to provide every possible term
occurring in the polynomials at step $L$. As a consequence the total number of multiplications necessary to evaluate $P(a)$ is

$$\frac{2p^r - 1}{p^r - 1} (p^rL - 1) + (p - 1)(M_r\left(\left\lfloor \frac{n}{2L} \right\rfloor \right) - r - 1).$$

The same passages used in Theorem 2 allow us to conclude that $L_{op}$ is asymptotically identified by the chain of inequalities

$$\frac{\sqrt{1}}{p^r} \sqrt{\frac{(p - 1)(p^r - 1)n^r}{r!(2p^r - 1)}} \leq p^r L_{op} \leq \frac{\sqrt{1}}{p^r} \sqrt{\frac{(p - 1)(p^r - 1)n^r}{r!(2p^r - 1)}} \sqrt{p^r}$$

which written in the form

$$\frac{1}{2} + B + \frac{\log p n}{2} \leq L_{op} \leq \frac{1}{2} + B + \frac{\log p n}{2}$$

shows that the unique optimal value is the integer closest to $B + \frac{\log p n}{2}$, where $B = \frac{1}{2r} \log p \frac{(p - 1)(p^r - 1)}{r!(2p^r - 1)}$. The minimum number of multiplications is asymptotically included into the interval

$$\frac{2}{\sqrt{p^r}} \sqrt{(p - 1) \frac{2p^r - 1}{p^r - 1} \frac{1}{r!} n^{r/2}} < G_p(n, r, L_{op}) < \frac{2}{\sqrt{p^r}} \sqrt{(p - 1) \frac{2p^r - 1}{p^r - 1} \frac{1}{r!} n^{r/2}}.$$

**Remark 4.** Our proofs start with the evaluations of certain monomials. Hence they may be extended verbatim to other finite-dimensional linear subspaces of $\mathbb{F}_p[x_1, \ldots, x_r]$, just taking their dimension $\alpha$ as vector spaces instead of the integer $\binom{n+r}{r}$. For a suitable linear space $V$ in Theorem 3 we could get a bound of order $c_3 \sqrt{\alpha}$ with $c_3 \sim 2\sqrt{2p^r + 1}$. For instance, call $V(r, n)$ the linear subspace of $\mathbb{F}_p[x_1, \ldots, x_r]$ formed by all polynomials whose degree in each variable is at most $n$. We have $\dim(V(n, r)) = \binom{n+1}{r}$. In this case iterating this procedure we arrive at each step at a vector space $V(\lceil n/p^L \rceil, r)$. Taking $L$ such that $2p^L \sim p \dim(V(\lceil n/p^L \rceil, r))$, i.e. taking $L \sim \log_p n/2 + B$ with $B \sim \frac{1}{2r} \log_p (p)/2 \sim \frac{1}{2p} \sim -\frac{1}{2p} \log_p 2$, we get an upper bound of order $2\sqrt{2p^r + 1} n^{r/2}$.

### 6 Further remarks

The complexity of polynomial evaluation is crucial in the determination of the complexity of several computational algebra methods, such as the Buchberger-Moeller algorithm ([6][7]), other commutative algebra methods ([8]), the Berlekamp-Massey-Sakata algorithm ([11][12]).
In turn, these algorithms are the main tools used in algebraic coding theory (and in cryptography). This justifies our special interest in the finite field case. For example, the previous algorithms can be adapted naturally to achieve iterative decoding of algebraic codes and algebraic-geometry codes, see e.g. [13, 3]. Other versions can decode and construct more general geometric codes, see e.g. [2].

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