Electromagnetic formulations for GTC simulation

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Abstract

A set of electromagnetic formulations with gyrokinetic ions and drift kinetic electrons for global gyrokinetic simulations is presented. Low frequency electromagnetic modes like drift-waves, AEs and MHD modes can be simulated using this framework. In the long wavelength and low frequency limit, ideal MHD equations and dispersion relation are found. Especially, the magnetic compressional term and the equilibrium current are carefully treated in this model. This model is successfully used to simulate the internal kink modes in DIII-D tokamaks.

1 Introduction

The electromagnetic gyrokinetic simulation model with drift kinetic electrons was derived. [1] Later, the equilibrium flow [2] and compressional magnetic perturbation [3] was included in the model. The model was successfully applied to various GTC simulations of micro-turbulence, Alfvén modes, and current or pressure driven MHD modes. [3–8] Recently, we used this model to carry out a benchmark of internal kink mode [9]. During this benchmark, we examined and improved this model by include more accurate compressional magnetic perturbation term and the equilibrium current, which are critical to obtain the correct kink instability. Especially, the $\delta B_\parallel$ effect was thought unimportant, because the low plasma $\beta$. However, the simulation found that $\delta B_\parallel$ can be comparable to $\delta B_{\perp}$ and the kink instability is very sensitive to the cancellation of stabilizing "drift-reversal" effect by the perpendicular force balance between perturbed pressure and $\delta B_{\parallel}$. The $\delta$ term representing non-orthogonality in Boozer coordinate was commonly neglected in simulations for the tokamak core. In the kink mode simulation we found that the $\delta$ term makes significant contribution to equilibrium current can be very important for current driven instabilities. An simple method is found to calculated the correct equilibrium current which is consistent with the magnetic field in Boozer coordinate system. Besides the kink mode benchmark, we used the improved model to simulate more than 5000 DIII-D cases which can exhibit kink instability to build a database. We used this database to train a surrogate model to predict the linear kink drive using experimental parameters, the results is presented in a separate paper [10].

This paper is organized as follows. In section 2, the model equations are given, in which the gyrokinetic ion and drift kinetic electron model are used. In the long wavelength limit, this model reduces to the ideal MHD model and the ideal MHD dispersion relation (the vorticity equation) can be found. In section 3 we show the simulation results from the kink mode benchmark and the results from over 5000 simulations of kink instability. The conclusion is given in Section 4.

2 Model equations

2.1 Gyrokinetic equations for $\delta f$ simulation

The gyrokinetic Vlasov equation is used to describe the evolution of distribution function in five-dimensional gyrocenter phase space, $f_s = f_s (R, v_\parallel, \mu, t)$

$$L f_s \equiv \left( \frac{\partial}{\partial t} + \dot{R} \cdot \nabla + \dot{v}_\parallel \frac{\partial}{\partial v_\parallel} \right) f_s (R, \mu, v_\parallel) = 0,$$

where $R$ is the gyrocenter position, $v_\parallel$ the gyrocenter parallel velocity along field line, $\mu$ the magnetic moment. The time evolution of $R$ and $v_\parallel$ are given by

$$\dot{R} = v_\parallel \frac{B^*}{B_{\parallel}} + v_E + v_g + v_b$$

(2)
\[
\dot{v}_\parallel = - \frac{1}{m_s} \frac{B_r^*}{B_\parallel^*} \cdot (\mu \nabla B_0 + Z_s \nabla \langle \phi \rangle - Z_s \nabla \langle \delta A_\perp \cdot \mathbf{v}_\perp \rangle),
\]
\[
\frac{Z_s}{m_s} \frac{\partial \langle \delta A_\parallel \rangle}{\partial t}.
\]

Here we have utilized the parallel-symplectic representation of the modern gyrokinetic formula [11, 12], where \( Z_s \) and \( m_s \) are the particle charge and mass, respectively. \( B_0 \) is the amplitude of equilibrium magnetic field, \( B_r^* = B_0^* + \delta B_\parallel = B_0 + \frac{m_s v_\parallel}{e} \nabla \cdot \mathbf{b}_0 + \delta B_\perp \), \( \mathbf{b}_0 = \mathbf{B}_0^*/B_0 \) is the unit vector along the field line, and \( B_\parallel^* = B_0^* \cdot \mathbf{b}_0 \). \( \phi \) is the electrostatic potential, and \( \delta A_\parallel \) and \( \delta A_\perp \) stand for the parallel and perpendicular components of the vector potential. \( \delta B_\parallel = \nabla \times (\delta A_\parallel \cdot \mathbf{b}_0) \), \( \delta B_\parallel = \mathbf{b}_0 \cdot (\nabla \times \delta A_\perp) \). The \( \mathbf{E} \times \mathbf{B} \) velocity, the gradient drift velocity and the drift velocity from \( \delta B_\parallel \) are

\[
\mathbf{v}_E = \frac{\mathbf{b}_0 \times \nabla \langle \phi \rangle}{B_\parallel^*},
\]
\[
\mathbf{v}_g = \frac{\mu}{Z_s B_\parallel^*} \mathbf{b}_0 \times \nabla B_0
\]
\[
\mathbf{v}_{b0} = - \frac{\mathbf{b}_0 \times \nabla \langle \delta A_\perp \cdot \mathbf{v}_\perp \rangle}{B_\parallel^*}.
\]

In the above equations, the operator \( \langle \cdots \rangle \) denotes gyro-averaging, \( \langle a \rangle (\mathbf{R}) = \frac{1}{\omega_0} \oint \eta \partial \zeta \int d\mathbf{x} a(\mathbf{x}) \delta (\mathbf{x} - \mathbf{R} - \mathbf{r}) \), with \( \zeta \) the phase angle. Note that \( \phi \) consists of perturbed electrostatic potential \( \phi_{pt} \) and the equilibrium potential \( \phi_{eq} \), and \( \delta A_\parallel \) consists of both perturbed part \( \delta A_{\parallel pt} \) and external equilibrium part \( \delta A_{\parallel eq} \) (e.g. RMP field).

The \( \delta f \) scheme is used for GTC simulation, in which we split the distribution function to an equilibrium part and a perturbed part, \( f_s = f_{0s} + \delta f_s \). The equilibrium part \( f_{0s} \) is the solution of equilibrium Vlasov equation,

\[
L_0 f_{0s} = - \frac{\partial f_{0s}}{\partial t} + \left( v_\parallel \frac{\delta B_\parallel}{B_\parallel} + \mathbf{v}_{E,eq} + \mathbf{v}_{g} \right) \cdot \nabla f_{0s} - \frac{\mu}{m_s} \frac{B_\parallel^*}{B_\parallel^*} \cdot \nabla B_0 \frac{\partial f_{0s}}{\partial v_\parallel} - \frac{\mu}{m_s} \frac{B_\parallel^*}{B_\parallel^*} \cdot \nabla \langle \phi_{eq} \rangle \frac{Z_s}{m_s} \frac{\partial f_{0s}}{\partial v_\parallel} - C_s f_{0s} = 0,
\]

where \( L_0 \) is the equilibrium propagator, and \( C_s \) is the collision operator for species ‘s’. And the governing equation of \( \delta f_s \) is given by

\[
L \delta f_s = -(L - L_0) f_{0s}
\]
\[
= - \left( v_\parallel \frac{\delta B_\parallel}{B_\parallel} + \mathbf{v}_{E,pt} + \mathbf{v}_{b0} \right) \cdot \nabla f_{0s}
\]
\[
+ \frac{1}{m_s} \left[ \frac{\mu B_\perp^*}{B_\parallel^*} \cdot \nabla B_0 + \frac{B_\perp^*}{B_\parallel^*} \cdot (Z_s \nabla \langle \phi_{pt} \rangle - Z_s \nabla \langle \delta A_\perp \cdot \mathbf{v}_\perp \rangle) + \frac{\delta B_\perp}{B_\parallel^*} \cdot (Z_s \nabla \langle \phi \rangle - Z_s \nabla \langle \delta A_\perp \cdot \mathbf{v}_\perp \rangle) \right]
\]
\[
+ Z_s \frac{\partial \langle \delta A_\parallel \rangle}{\partial t} \frac{\partial}{\partial v_\parallel} f_{0s}.
\]

The propagator in the left hand side of equation involves time derivatives of the phase space coordinates (\( \dot{\mathbf{R}}, \dot{v}_\parallel \)) given in (2) and (3) solved from Euler-Lagrange equation in gyrocenter space [12]. The difference between (2) and (3) and the Hamiltonian equations of motion[White&Chance 1984] in Boozer coordinate system is explained in Appendix B. From the perpendicular Ampère’s Law, we have [3, 13]

\[
\langle \delta A_\perp \cdot \mathbf{v}_\perp \rangle = - \frac{\mu}{Z_s} \langle \langle \delta B_\parallel \rangle \rangle = - \frac{\mu}{Z_s} \frac{1}{\pi \rho^2} \int_0^r r dr \int_0^{2\pi} \delta B_\parallel (\mathbf{R} - \mathbf{r}) d\zeta.
\]

Define the ion particle weight as \( w_s = \delta f_s/f_s \), then the evolution of \( w_s \) is given by \( \frac{dw_s}{dt} = -(1 - w_s) \frac{1}{2\tau_0} (L - L_0) f_{0s} \). The detailed formula of \( w_s \) equation can be found in Appendix B. The equilibrium distribution \( f_{0s} \) is solved from (5) and should be the function of constants of motion. In (6) we usually use approximation form of \( f_{0s} \) for simplicity. For thermal ions and electrons, the shifted Maxwellian distribution is appropriate,

\[
f_{0s} = \frac{n_s}{(2\pi T_s/m_s)^{3/2}} \exp \left( -\frac{m_s (v_\parallel - u_{i0s})^2}{2T_s} - \frac{\mu B_0}{T_s} \right).
\]
\( u_{\parallel 0s} \) is the equilibrium ion parallel flow velocity for species 's', \( n_{0s} = n_{eq0,s} \ast \exp (-q\phi_{eq}/T_s) \) is the equilibrium density of species 's', and \( n_{eq0,s} \) is the equilibrium density when \( \phi_{eq} = 0 \).

For energetic particle, we need to consider the particle source term in equilibrium Vlasov equation, and the steady-state slowing-down distribution can be chosen to model \( f_{0E P} \) [14,15].

\[
f_{0E P} = c \frac{n_{0E P} \delta F(v_0 - v)}{v^3 + v_c^3} \exp \left[ - \left( \frac{\Lambda - \Lambda_0}{\Delta \Lambda} \right)^2 \right].
\]

Here \( v_c \) is the critical velocity, \( v_0 \) is the birth velocity, \( H \) is the Heaviside step function, and \( c \) is the normalization factor. \( \Lambda = \mu B_0/E \) is the pitch angle, \( E = \mu B_0 + mv_c^2/2 \) is the kinetic energy, \( B_0 \) is the on-axis magnetic field strength. \( \Lambda_0 \) is the peak of the pitch angle, and \( \Delta \Lambda \) is the width of the pitch angle distribution.

### 2.2 Gyrokinetic field equations

The transformation from gyrocenter space distribution to particle space distribution up to the first order is given by [11,12]

\[
\delta F_s(x, v_\parallel, \mu, t) = \int dR \left[ \delta f_s(R, v_\parallel, \mu, t) + \left( G^R_1 \cdot \frac{\partial}{\partial R} + G^{v_\parallel}_1 \frac{\partial}{\partial v_\parallel} + G^\mu_1 \frac{\partial}{\partial \mu} \right) f_s(R, v_\parallel, \mu, t) \right] \delta (x - R - \rho).
\]

(7)

where \( \delta F_s(x, v_\parallel, \mu, t) \) is the perturbed distribution function in particle space, and \( G^R_1 \) is the first order generating vector along the \( \alpha \) direction in the Lie transformation method. The quasi-neutrality condition reads,

\[
0 = \sum_{s \neq e} Z_s \delta n_{pol,s} + \sum_{s \neq e} Z_s \delta \bar{n}_s - c \delta n_e,
\]

(8)

\( \delta \bar{n}_s \) is the gyro-averaged gyrocenter density,

\[
\delta \bar{n}_s(x) = \int dv \int dR \delta f_s(R) \delta (x - R - \rho),
\]

(9)

and \( \delta n_{pol} \) is the polarization density,

\[
\delta n_{pol,s} = \int dv \int dR \left( G^R_1 \cdot \frac{\partial}{\partial R} + G^{v_\parallel}_1 \frac{\partial}{\partial v_\parallel} + G^\mu_1 \frac{\partial}{\partial \mu} \right) f_s(R, v_\parallel, \mu, t) \delta (x - R - \rho).
\]

(10)

In (8), the electron gyro-radius and electron polarization density are neglected. The perpendicular Ampere’s law can be used to solve \( \delta B_\parallel \),

\[
\nabla \delta B_\parallel \times b_0 = \mu_0 \left( Z_s \sum_{s \neq e} \delta \bar{u}_\perp,s + Z_s \sum_{s \neq e} \delta u_{\perp pol,s} - c \delta \bar{u}_\perp,e \right).
\]

(11)

Similar to (8), \( \delta \bar{u}_\perp,s \) is the gyro-averaged perpendicular gyrocenter flow velocity,

\[
\delta \bar{u}_\perp,s(x) = \int dv v \parallel \int dR \delta f_s(R) \delta (x - R - \rho),
\]

(12)

and \( \delta u_{\perp pol,s} \) is the perpendicular polarization velocity,

\[
\delta u_{\perp pol,s} = \int dv v \parallel \int dR \left( G^R_1 \cdot \frac{\partial}{\partial R} + G^{v_\parallel}_1 \frac{\partial}{\partial v_\parallel} + G^\mu_1 \frac{\partial}{\partial \mu} \right) f_s(R, v_\parallel, \mu, t) \delta (x - R - \rho).
\]

(13)

The detailed expression of (7)(10) and (13) can be found in [12]. In general, the quasi-neutrality equation and perpendicular Ampere’s law can be solved by integral method [16]. In the limit of \( 1/(k_\perp L_p) \ll 1 \), the polarization terms reduce to simpler forms and the two equations are given by [3]

\[
\sum_{s \neq e} Z_s^2 n_s \frac{\delta \bar{\phi} - \delta \bar{\phi}_s}{T_s} = -\frac{1}{B_0} \left( \sum_{s \neq e} Z_s n_{s0} \left\{ \delta B_\parallel \right\}_s - c n_0 \left\{ \delta B_\parallel \right\}_e \right) = \sum_{s \neq e} Z_s \bar{n}_s - c n_e,
\]

(14)
where $\phi$ is the inductive potential, the method to solve $\phi_{ind}$ will be given in next section. The non-zonal part of parallel gyrokinetic Ampere's law is used to solve $\delta B_{||}$ and $(14)$ and $(15)$ in terms of $k_{||} \rho_{s}$. The above two equations can be decoupled, and the equation for $\delta B_{||}$ is simplified to

$$\frac{\delta B_{||}}{B_{0}} = - \frac{\beta_{e}}{2 + \beta_{e} + 2 \sum_{s \neq e} \beta_{s}} \left[ \frac{3}{2} \nabla_{\perp}^{2} \delta \phi \sum_{s \neq e} \frac{\beta_{s} Z_{s}}{\beta_{e} T_{s}} \rho_{s}^{2} + \frac{5}{4} \nabla_{\parallel}^{2} \delta \phi \sum_{s \neq e} \frac{\beta_{s} Z_{s}}{\beta_{e} T_{s}} \rho_{s}^{2} \right] + \frac{1}{P_{\perp 0e}} \left( \delta n_{e}^{(0)} T_{e} + n_{0e} \frac{\partial T_{e}}{\partial \psi} \delta \psi + \delta P_{\parallel e}^{na} + \sum_{s \neq e} \delta P_{\perp s} \right)$$

where $\delta P_{\parallel e}^{na} = \int d\nu \mu B_{0} \delta h_{e}$ is the non-adiabatic perpendicular electron pressure.

The non-zonal part of $\delta A_{||}$ is given by

$$\frac{\partial A_{||,nz}}{\partial t} = B_{0} \cdot \nabla \phi_{ind},$$

where $\phi_{ind}$ is the inductive potential, the method to solve $\phi_{ind}$ will be given in next section. The non-zonal part of parallel gyrokinetic Ampere's law is used to solve $\delta u_{||,nz}$

$$en_{e} \delta u_{||,nz} = \frac{1}{\mu_{0}} \nabla_{\perp}^{2} \delta A_{||,nz} + \sum_{s \neq e} Z_{s} n_{s} \delta \tilde{u}_{s,nz},$$

where $\delta \tilde{u}_{s} \parallel$ is the gyro-averaged parallel flow,

$$n_{s} \delta \tilde{u}_{s} \parallel (x) = \int d\nu \int dR \delta f_{s} (R) \delta (x - R - \rho)$$

The zonal part of $\delta A_{||}$ can be found simply from flux surface averaged Ampere's law,

$$- (\nabla_{\parallel}^{2} \delta A_{||})_{00} = \sum_{s \neq e} Z_{s} n_{s} \delta \tilde{u}_{00||,s} - cn_{e} \delta u_{00||,e},$$

the zonal parallel current can be directly obtained by integrating $\delta f_{s}$ and $\delta f_{e}$ following by the flux surface averaging.
2.3 Drift kinetic electron model

The drift kinetic electron model is used to describe electron dynamics. [1, 17, 18] The continuity equation of electrons can be obtained by integrating the Vlasov equation [3],

$$
\frac{\partial \delta n_e}{\partial t} + \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{le}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) - n_0 (\mathbf{v}_* + \mathbf{v}_E) \cdot \nabla \frac{B_0}{B_0} + \mathbf{B}_\perp \cdot \nabla \left( \frac{n_{0e} u_{le}}{B_0} \right) 
$$

$$
- \nabla \times \mathbf{B}_0 \cdot \left( \nabla \delta P_{le} + \frac{(\delta B_{le} - \delta B_{le}) \nabla B_0}{B_0} - n_{0e} e \nabla \delta \phi \right) + \nabla \cdot \left( \frac{\delta P_{le} \mathbf{b}_0 \times \mathbf{b}_0 \cdot \mathbf{b}_0}{e B_0} \right) 
$$

$$
+ \mathbf{B}_0 \cdot \nabla \left( \frac{n_{0e} \delta u_{ad}}{B_0} \right) + B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e}}{B_0} \right) + \frac{\delta n_e}{B_0} + \frac{\delta n_e}{B_0} \mathbf{b}_0 \times \nabla B_0 \cdot \nabla \phi + \frac{\delta n_e}{B_0} \nabla \times B_0 \cdot \nabla \phi 
$$

$$
- \frac{\mathbf{b}_0 \times \nabla \delta B_\parallel}{e} \cdot \nabla \left( \frac{\delta P_{le} + P_{\perp 0e}}{B_0^2} \right) \frac{\nabla \times \mathbf{b}_0 \cdot \nabla \delta B_\parallel}{e B_0^2} \left( \delta P_{le} + P_{\perp 0e} \right) = 0, 
$$

(23)

with \( \mathbf{v}_* = \mathbf{b}_0 \times \nabla (\delta P_{le} + \delta P_{\perp 0e}) \), and \( n_{0e} u_{le} = -\nabla \times \mathbf{B}_0 / (e \mu_0) + \sum_{\alpha \neq e} Z_{\alpha} n_{0e} u_{\alpha || e} / e \) denotes the electron equilibrium parallel flow. The continuity equation can also be written in conservative form,

$$
\frac{\partial \delta n_e}{\partial t} + \nabla \cdot \Gamma = 0, 
$$

(24)

where \( \Gamma \) is the particle flux, \( \Gamma = 2 \pi \int d r \psi \int d \mathbf{R} f_e B_0 / m_e \). It should be pointed out that the continuity equation directly integrated from the Vlasov equation is in gyrocenter space. Here we simply neglected electron FLR effect, therefore \( \delta n_e \) represents the electron density in real space. The difference of \( \delta B_\perp = B_0 + B_0 \mathbf{v}_E / (e \mu_0) + \sum_{\alpha \neq e} Z_{\alpha} n_{0e} u_{\alpha || e} / e \) denotes the electron equilibrium parallel flow. The continuity equation can also be written in conservative form,

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The equation for $\delta h_e$ is

$$L \delta h_e = - \delta L f_{0e} - L \delta f_{e}^{ad}$$

$$= - \left( v_\parallel \frac{\delta B_\perp}{B_\perp} + v_{E,pt} + v_\parallel \right) \cdot \nabla f_{0e}$$

$$+ \frac{1}{m_e} \left[ \mu B_\perp \cdot \nabla \left( B_\parallel - \frac{e \phi}{\mu} \right) - \frac{B^*}{B_\parallel} \cdot \nabla \left( \phi_{pt} - e \delta B_\parallel - e \frac{\partial A_\parallel}{\partial t} \right) \right] \frac{\partial}{\partial v_\parallel} f_{0e}$$

$$- \frac{\partial \delta f_{e}^{ad}}{\partial t} \left( v_\parallel \frac{\delta B_\perp}{B_\perp} + v_E + v_\parallel \right) \cdot \nabla \delta f_{e}^{ad}$$

$$+ \frac{1}{m_e} \left[ \frac{B^*}{B_\parallel} \cdot \left( \mu \nabla B_0 - e \nabla \phi + \mu \nabla B_\parallel \right) \right] \frac{\partial}{\partial v_\parallel} \delta f_{e}^{ad}$$

Note that the time derivative in the equation requires us to store the $\delta f_{e}^{ad}$ value at previous time step in the simulation. (27) (30) and (31) are coupled equations for which we use an iterative method to find out $\phi_{eff}$, $\delta f_{e}^{ad}$ and $\delta h_e$. First of all, $\phi_{eff}$ is obtained from (29), with $\delta n_e$ obtained from electron continuity equation. Then $\phi_{eff}$ is used in (27) for the initial iteration to obtain the leading order adiabatic distribution function $\delta f_{e}^{ad,(0)}$. Then $\delta f_{e}^{ad,(0)}$ and $\phi_{eff}$ are substituted to (31) to solve $\delta h_e^{(1)}$, the first order non-adiabatic part of electron density can be calculated by $\delta n_e^{na,(1)} = \int d\psi \delta h_e^{(1)}$. The first order of non-adiabatic $\phi_{eff}^{na,(1)}$ will be obtained with $\delta n_e^{na,(1)}$.

$$\frac{e \phi_{eff}^{na,(1)}}{T_e} = - \frac{e \delta n_e^{na,(1)}}{n_0}.$$  

In the next iteration, $\phi_{eff}^{ad} + \phi_{eff}^{na,(1)}$ is used in (27). The iterations can continue, with $\phi_{eff}$, $\delta f_{e}^{ad}$, and $\delta h$ obtained order by order. We can define the electron particle weight as $w_e = \delta h_e/f_e$, the detailed equation for $w_e$ is given in B. The adiabatic part of pressure can be found by integrating (27),

$$\delta P_{e}^{ad} = \int d\nu \mu B_0 \delta f_{e}^{ad}$$

$$= e n_0 \phi_{eff} + \frac{\partial P_{e0}^{ad}}{\partial \psi} \delta \psi + \frac{\partial P_{e0}^{ad}}{\partial \alpha} \delta \alpha - 2 \frac{\delta B_\parallel}{B_\parallel} P_{e0} - n_0 e \frac{\partial \phi_{eq}}{\partial \psi} \delta \psi,$$  

$$\delta P_{||e}^{ad} = \int d\nu m v_\parallel^2 \delta f_{e}^{ad}$$

$$= e n_0 \phi_{eff} + \frac{\partial P_{e0}^{ad}}{\partial \psi} \delta \psi + \frac{\partial P_{e0}^{ad}}{\partial \alpha} \delta \alpha - \frac{\delta B_\parallel}{B_\parallel} P_{e0} - n_0 e \frac{\partial \phi_{ad}}{\partial \psi} \delta \psi.$$  

The equilibrium pressure $P_{0e}$ in the above equations is defined as $P_{e0} = \int d\nu \mu B_0 f_{0e}$, $P_{||0e} = \int d\nu m v_\parallel^2 f_{0e}$. The non-adiabatic part of $\delta P$ can be obtained by substituting $\delta h_e$ in the above integrations. Another method to define the adiabatic and non-adiabatic perturbation is given in [19]. That method allows the adiabatic and non-adiabatic equations decoupled explicitly, and no iteration is needed to solve the equations.

The inductive electric potential is given by $\phi_{ind} = \phi_{eff} - \phi$. $\delta \psi$ and $\delta \alpha$ are the components of $\delta B_\perp$ along the $\psi$ direction and the unperturbed field line direction. From (19) and (26),

$$\frac{\partial \delta \psi}{\partial t} = - \frac{\partial \phi_{ind}}{\partial \alpha_0}$$

$$\frac{\partial \delta \alpha}{\partial t} = \frac{\partial \phi_{ind}}{\partial \psi_0}.$$  

In deriving (34b), $\nabla \times (\delta A_\parallel B_0) \approx \nabla (\delta A_\parallel B_0) \times B_0$ is assumed, i.e., the equilibrium flow is neglected. However this inaccuracy does not affect the system since $f_0$ does not depend on $\alpha_0$ in most scenarios, and $\delta \alpha$ will not appear in the simulation model.
2.4 Equations for linear ideal MHD simulation

For ideal MHD modes, the simulation model can be greatly simplified by several approximations. In the long wavelength limit, FLR effects can be neglected. $E_{||}$ and accordingly $\phi_{eff}$ are forced to be 0 everywhere.

In the long wavelength limit, we can keep the terms up to the order of $O(\epsilon)$ equation (32) and (33) stay unchanged except that $\phi_{bal}$.

And we can get the continuity equation for the gyrocenter charge density $\delta n = \delta n_e - Z_i \delta n_i / e$,

$$\frac{\partial \delta n_i}{\partial t} + B_0 \cdot \nabla \left( \frac{n_0 u_i}{B_0} \right) - n_0 v_i \cdot \nabla B_0 - \delta B_\perp \cdot \nabla \left( \frac{n_0 u_{i0}}{B_0} \right) = 0.$$ (35)

And we can get the continuity equation for the gyrocenter charge density $\delta n = \delta n_e - Z_i \delta n_i / e$,

$$\frac{\partial \delta n}{\partial t} + B_0 \cdot \nabla \left( \frac{n_0 u_\parallel}{B_0} \right) - n_0 v_\parallel \cdot \nabla B_0 - \delta B_\perp \cdot \nabla \left( \frac{n_0 u_{\parallel0}}{B_0} \right)$$

$$- \frac{\nabla \times B_0}{e B_0^2} \cdot \nabla \left( \frac{\delta P_\parallel}{B_0^2} \right) + \frac{\nabla \times B_0}{e B_0^2} \cdot \nabla \left( \frac{\delta P_\perp}{B_0^2} \right) + \nabla \cdot \left( \frac{\delta P_\parallel}{e B_0^2} \nabla \times b_0 \cdot b_0 \right) = 0.$$ (36)

where $\delta P = \delta P_e, P_0 = P_{0e}, n_0 = n_{0e},$ and $\delta u_\parallel = \delta u_{\parallel0}, u_{\parallel0} = u_{\parallel0} - Z_i u_{\parallel0}$. $\delta u_\parallel$ is still solved from parallel Ampere’s law,

$$\delta u_\parallel = \frac{1}{\mu_0 e n_0} \nabla^2 \delta A_\parallel.$$ (37)

$\delta A_\parallel$ and $\delta \psi$ is solved from electrostatic potential directly, $\phi_{ind} = -\phi$ in (19). The quasi-neutrality condition effectively reduces to

$$\frac{\omega^2}{v_A^2} \nabla^2 \phi + i B_0 \cdot \nabla \left( \frac{\nabla^2 \left( k_\parallel \phi \right)}{B_0} \right) + i \omega \mu_0 \left( i \omega \epsilon n + \nabla \cdot \delta J_\parallel \right) = 0.$$ (38)

In the low ion temperature limit, where $c$ is the speed of light, $v_A$ the Alfvén velocity, and $\epsilon_0$ the dielectric constant of vacuum. In the incompressible ideal MHD limit, (18) is given passively by the perpendicular force balance.

$$\frac{\delta B_\perp}{B_0} = - \frac{\beta_\parallel \delta P_\parallel}{2 P_{1,0}} = - \frac{\beta_\perp \delta P_{1,0}}{2 P_{1,0}}$$ (39)

(32) and (33) stay unchanged except that $\phi_{eff}$ is explicitly set to 0. Combining the quasi-neutrality condition in long wavelength limit (38) and the Ampere’s law (37), and using the fact $\phi_{ind} = -\phi$, we can get the following equation [2]

$$\omega^2 v_A^2 \nabla^2 \delta \phi + i B_0 \cdot \nabla \left( \frac{\nabla^2 \left( k_\parallel \phi \right)}{B_0} \right) + i \omega \mu_0 \left( i \epsilon \omega n + \nabla \cdot \delta J_\parallel \right) = 0.$$ (38)

Substituting (36) in the above equation, we obtain the dispersion relation for the system

$$0 = \frac{\omega^2}{v_A^2} \nabla^2 \delta \phi + i B_0 \cdot \nabla \left( \frac{\nabla^2 \left( k_\parallel \phi \right)}{B_0} \right) + i \omega \mu_0 \left[ B_0 \times \nabla \left( \frac{k_\parallel \phi}{B_0} \right) \right]$$

$$- \nabla \times B_0 \cdot \nabla \delta P_\parallel + \left( \frac{\delta P_\perp - \delta P_\parallel}{B_0} \nabla B_0 \right) + \nabla \cdot \left( \frac{\delta P_\parallel}{B_0} \nabla \times b_0 \cdot b_0 \right) - B_0 \times \nabla \delta B_\perp \cdot \nabla \left( \frac{P_{1,0}}{B_0^2} \right) + B_0 \times \nabla \delta B_\perp \cdot \nabla \delta B_\perp + \left( \frac{\nabla \times b_0}{B_0} \nabla \delta P_\parallel \right).$$

In the long wavelength limit, we can keep the terms up to the order of $O(\epsilon^2)$, and get the simplified dispersion relation,

$$0 = \frac{\omega^2}{v_A^2} \nabla^2 \delta \phi + i B_0 \cdot \nabla \left( \frac{\nabla^2 \left( k_\parallel \phi \right)}{B_0} \right) + i B_0 \times \nabla \left( \frac{\mu_0 J_{||0}}{B_0} \right)$$

$$- \nabla \times B_0 \cdot \nabla \delta P_\parallel + \left( \frac{\nabla \times B_0 \cdot \nabla \delta P_\parallel}{B_0^2} \right) + \nabla \delta P_\perp = \frac{\nabla \times B_0}{B_0} \cdot \nabla \delta B_\parallel + \left( \frac{\nabla \times b_0}{B_0} \nabla \delta P_\parallel \right).$$
(39) shows that the compressional magnetic perturbation $\delta B_\parallel /B_0$ is much smaller than the pressure perturbation $\delta P/P_0$ because $\beta_e \ll 1$. However, the $\delta B_\parallel$ drive can correct the $\delta P_\perp$ drive by cancelling out the "drift-reversal" effects from the grad-B drift associated with perpendicular diamagnetic current. We can use the perpendicular force balance relation given in (39) to rewrite the $\delta B_\parallel$ drive in the above equation,

$$\frac{b_0 \times \nabla P_{\perp 0}}{e B_0^2} \cdot \nabla \delta B_\parallel = -\frac{\mu_0 J_{\perp 0}}{e B_0^2} \cdot \nabla \delta P_\perp,$$

where $J_{\perp 0} = \frac{b_0 \times \nabla P_{\perp 0}}{e B_0^2}$ is the perpendicular diamagnetic current. After applying this relation, the stabilization effect associated with fast magnetosonic wave is explicitly removed. In the low beta limit, the perturbed pressure is isotropic $\delta P_\perp = \delta P_\parallel$. And by further assuming $k_\parallel \ll k_\perp$, the equation reduces to the commonly used ideal MHD dispersion relation [20, 21],

$$0 = \frac{\omega^2}{v_A^2} \nabla_\perp^2 \delta \phi + i B_0 \cdot \nabla \left( \frac{\nabla_\perp^2 (k_\parallel \phi)}{B_0} \right) + i b_0 \times \nabla \left( k_\parallel \phi \right) \cdot \nabla \left( \frac{\mu_0 J_{\parallel 0}}{B_0} \right)$$

$$- i \omega \mu_0 \frac{2b_0 \times \kappa}{B_0} \cdot \nabla \delta P.$$

(40)

There are some high order (up to $1/(k_\perp L_B)$) difference between the above dispersion relation and the one derived from MHD momentum equation, since we have dropped $O(1/(k_\perp L_B))$ term in Ampere’s law (37).

In the ideal MHD simulation, we have shown that incorporation of $\delta B_\parallel$ component in continuity equation is critical to the kink instability calculation. This important effect of $\delta B_\parallel$ is also found for other instabilities [3, 22, 23]. Another important factor which is important for kink modes is the equilibrium parallel current. In particular, we show that the poloidal variation of $J_{\parallel 0}$ which is normally ignored in other gyrokinetic simulations needs to be calculated accurately. In Appendix A we present the numerical method used in GTC for $J_{\parallel 0}$ calculation.

3 Kink mode simulation by GTC

![Figure 1: δ current](image)

We have used the ideal MHD model described in Section 2.4 to successfully carry out the verification and validation of the internal kink mode for DIII-D geometry. One DIII-D experimental shot is selected which
exhibits the characteristics of kink instability. Using the equilibrium constructed from the experiments, GTC and several other codes has been benchmarked on the kink instability with and without kinetic effects [9]. Then GTC is used to conduct more than 5000 simulations in which the equilibria are constructed from DIII-D shots. The simulation data are used to build a database and train a surrogate model based on deep learning methods. [10] In this section we show the important factors for simulating kink mode in Boozer coordinates and the physical conjecture we have learned from the simulations.

Since kink instability is driven by both parallel current and pressure gradient, it is necessary to evaluate \( J_\parallel \) accurately. In Boozer coordinate system, the equilibrium magnetic field and the equilibrium parallel current can be expressed as

\[
B_0 = \delta \nabla \psi + I \nabla \theta + g \nabla \zeta \\
J_\parallel = \frac{1}{\mu_0 J} \left(\frac{1}{2} (I' - \partial_\psi \delta) g + g' I\right),
\]

where \( J = (gg + I)/B_0^2 \) is the Jacobian of Boozer coordinates, \( \delta \) shows the non-orthogonality of the basis vectors of Boozer coordinates,

\[
\delta = -\frac{I \nabla \psi \cdot \nabla \theta + g \nabla \psi \cdot \nabla \zeta}{\nabla \psi \cdot \nabla \psi}
\]

Figure 2: Parallel current term \( \mu_0 J_\parallel/B_0 \)

The \( \delta \)-current in the benchmark case is shown in Figure 1. It can be seen \( \delta \) is small near the magnetic axis, and has a large \( m=1 \) component when \( \varepsilon \) is large. The current components from \( g' \) and \( I' \) do not vary along the flux surface, since \( g, I, \) and \( q \) are functions of \( \psi \) only. In previous simulations using Boozer coordinates, \( \delta \) is often neglected for simplicity. However, our simulation on the kink mode benchmark for DIII-D equilibrium shows that the neglect of \( \delta \) will lead the growth rate of kink instabilities to increase from \( 4.2 \times 10^4 \text{s}^{-1} \) to \( 22 \times 10^4 \text{s}^{-1} \). [9] The result shows the current contribution from \( \delta \) must be considered. Figure 2 shows that the equilibrium parallel current on the cross section. The poloidally varying part in Figure 2b comes from \( \partial_\psi \delta \). The poloidally varying parallel current can be at the same order with or even larger than the \( m=0 \) component, especially when \( \varepsilon \) is large. We have implemented the numerical method to evaluate the parallel current accurately for Boozer coordinates. The detailed method is described in Appendix A. In Figure 3 we show that given the benchmark equilibrium, the parallel current term calculated from different numerical methods matches very well. Meanwhile the kink instability is extremely sensitive to the parallel current. Even
using the 3 current profiles shown in Figure 3 would cause about 20% difference in the linear growth rate of kink mode.

Figure 3: Parallel current term from different methods implemented for Boozer coordinate (See Appendix A). The reference value shown is calculated from $\nabla \times B_0$ in cylindrical coordinate. The poloidal angle $\theta = 0$.

In section 2.4, we have discussed the physical effect of compressional magnetic perturbation $\delta B_{||}$ in the ideal MHD dispersion relation. In the simulation, significant effects from $\delta B_{||}$ is also observed. $\delta B_{||}$ appears in the equations of motion, the expressions of adiabatic pressures, and the explicitly in the continuity equation. The simulations show that the explicit $\delta B_{||}$ terms in continuity equation change the growth rate from $4.2 \times 10^4$ s$^{-1}$ to 0 (full stabilization). The $\delta B_{||}$ effect in other terms is almost negligible. On the other hand, The ratio of $\delta B_{||}$ to $\delta B_{\perp}$ is significant, as shown in Figure 4. For the kink eigenmode structure, $\delta B_{||}/\delta B_{\perp}$ can be as large as 0.35, which also indicates that $\delta B_{||}$ can not be simply neglected.

In order to build a surrogate GTC simulation model for future real-time plasma control system, we have used GTC to simulate 5758 equilibria chosen from DIII-D experiments to generate a database. GTC has the
capability to carry out large number of MHD simulations very efficiently. These simulations have been performed in 12 GTC runs, each of which simulates 500 experiments in 30 minutes using 2000 nodes of the Summit supercomputer (which has about 4700 nodes in total). Thanks to the database, we are able to investigate the kink stability dependence on different physical parameters for realistic experimental equilibriums. In Figure 5 we show the spider plots of kink instability with 6 important physical parameters. Figure 5a and Figure 5b correspond to different sets of equilibrium, constructed by EFIT01 and EFIT02, respectively. The $q$ profile of EFIT02 data is generally more reasonable since the motional Stark effect (MSE) diagnostics information is considered. In both figures, the $q = 1$ surface location and $\delta \beta_p$ are shown to be two relevant parameters. The data dependence on these two parameters shows that when $|\delta \beta_p|$ increase, the kink mode tends to be more unstable. When the $q = 1$ surface is close to the magnetic axis, the boundary between unstable and stable data in the parameter space becomes unclear. These observations are qualitatively consistent with the ideal MHD theory.

Figure 5: Spider plots of kink mode instability from GTC simulations with (a) EFIT01 equilibriums and (b) EFIT02 equilibriums

4 Conclusion

In this paper we formulate the nonlinear electromagnetic gyrokinetic model incorporating accurate perturbed magnetic compression and equilibrium flow. In the long wavelength limit, the ideal MHD dispersion relation is recovered. The model is used to study the current and pressure driven internal kink mode. The kink instability is sensitive to the correct implementation of $\delta B_\parallel$ and $J_{\parallel0}$. A large number of kink cases chosen from DIII-D experiments is successfully simulated to build a database for machine learning. The implementation of this model is important for later electromagnetic simulations and coupling of different modes with different scales.

Appendix A Calculation of $J_{\parallel0}$

In the GTC ideal MHD simulations in tokamaks, kink mode can be driven by parallel current and pressure gradient, the parallel current can be calculated numerically in several ways.

A.1 $J_{\parallel0}$ from $\delta$ current

We directly calculate $J_{\parallel0}$ from (41). The $\delta$-current is given by

$$\delta = -\frac{I_E^\psi + g^\psi_\psi}{g^\psi^\psi},$$

(42)
where $g^{\alpha \beta} = \nabla \alpha \cdot \nabla \beta$ is the element of contra-variant metric tensor. One can note that $g^{\phi \zeta} / g^{\psi \theta} \sim O (\epsilon^2)$. But in the calculation of $\delta$, $g^{\phi \zeta}$ cannot be simply neglected because $1/g \sim O (\epsilon^2)$, and the two terms are comparable in (42). Axisymmetry exists for 2D equilibriums, and the toroidal direction $\nabla \phi$ is perpendicular to $\nabla \psi$ and $\nabla \theta$. However a transformation has been applied to $\phi$ to construct the Boozer toroidal angle $\zeta$. For 2d equilibrium, the difference between $\zeta$ and $\phi$ is given by [24]

$$\nu = \phi - \zeta = \int \left( \frac{g J}{R^2} - q \right) d\theta.$$  

(43)

There is one degree of freedom to fully determine $\nu$ function, and we can simply choose $\nu (\theta = 0) = 0$. Now from $\phi (\psi, \theta) = \zeta + \nu$ we can obtain $\partial_{\psi} \phi = \partial_{\psi} \nu$, $\partial_{\theta} \phi = \partial_{\theta} \nu$, and $\partial_{\phi} \phi = 1$. Together with $R (\psi, \theta)$ and $Z (\psi, \theta)$, the covariant metric tensor is constructed,

$$g_{\alpha \beta} = \frac{\partial R}{\partial \alpha} \frac{\partial R}{\partial \beta} + R^2 \frac{\partial \phi}{\partial \alpha} \frac{\partial \phi}{\partial \beta} + \frac{\partial Z}{\partial \alpha} \frac{\partial Z}{\partial \beta},$$  

(44)

and accordingly the contra-variant metric tensor is constructed from the covariant one,

$$\left( \begin{array}{ccc} g^{\psi \psi} & g^{\psi \theta} & g^{\psi \zeta} \\ g^{\theta \psi} & g^{\theta \theta} & g^{\theta \zeta} \\ g^{\zeta \psi} & g^{\zeta \theta} & g^{\zeta \zeta} \end{array} \right) = \frac{1}{\Delta} \left( \begin{array}{ccc} g_{\psi \zeta} g_{\theta \zeta} - g_{\psi \theta} g_{\zeta \zeta} & g_{\psi \theta} g_{\zeta \zeta} - g_{\psi \zeta} g_{\zeta \theta} & g_{\psi \zeta} g_{\zeta \theta} - g_{\psi \theta} g_{\zeta \zeta} \\ g_{\psi \zeta} g_{\zeta \zeta} - g_{\psi \theta} g_{\theta \zeta} & g_{\psi \theta} g_{\theta \zeta} - g_{\psi \zeta} g_{\theta \theta} & g_{\psi \zeta} g_{\theta \theta} - g_{\psi \theta} g_{\theta \zeta} \\ g_{\psi \zeta} g_{\theta \zeta} - g_{\psi \theta} g_{\zeta \zeta} & g_{\psi \theta} g_{\zeta \zeta} - g_{\psi \zeta} g_{\zeta \theta} & g_{\psi \zeta} g_{\zeta \theta} - g_{\psi \theta} g_{\zeta \zeta} \end{array} \right),$$  

(45)

where $\Delta$ is the determinant of the covariant metric tensor, $\Delta = g_{\psi \psi} g_{\theta \theta} g_{\zeta \zeta} - g_{\psi \theta} g_{\theta \psi} g_{\zeta \zeta} - g_{\psi \zeta} g_{\zeta \psi} g_{\theta \theta} + 2 g_{\psi \psi} g_{\theta \theta} g_{\zeta \zeta}$. Note both covariant and the contra-variant metric tensor are symmetric, $g_{\alpha \beta} = g_{\beta \alpha}$, and $g^{\alpha \beta}$ = $g^{\beta \alpha}$. The disadvantage of this method is that, $\nu$ itself is a higher order term comparing to $\phi$ or $\zeta$, and often has large numerical errors near the separatrix. The numerical errors in $\delta$ term and therefore the error in the parallel current is amplified through (44) (45), and (42).

### A.2 $J_{||0}$ from pressure profile and $g$ current

The total equilibrium current can be expressed as

$$\mu_0 j = \mu_0 j^\zeta \hat{e}_\zeta + \mu_0 j^\theta \hat{e}_\theta,$$  

(46)

where $\hat{e}_\alpha = \partial \mathbf{r} / \partial \alpha$ is the covariant basis vector. Comparing it with the expression of $\nabla \times \mathbf{B}_0$ in Boozer coordinate, we have $\mu_0 j^\zeta = - \partial_{\psi} g / J$, then along with the equilibrium force balance equation $\mathbf{J}_{||0} \times \mathbf{B}_0 = \nabla p$ we have

$$j^\theta - j^\zeta = p',$$  

(47)

that is,

$$\mu_0 \frac{dp}{d\psi} = - \mu_0 j^\zeta - \frac{dq}{d\psi} / J, \quad \mu_0 \frac{dp}{d\psi} = - \mu_0 j^\zeta - \frac{dq}{d\psi} / J,$$  

(48)

and the the parallel current can be written as

$$\frac{\mu_0 J_{||0}}{B_0} = \frac{\mu_0}{B_0} (g j^\zeta + I j^\theta) = \frac{\mu_0 g}{B_0^2} \frac{dq}{d\psi} - \frac{\mu_0 g}{B_0^2} \frac{dp}{d\psi}. \quad \left[ 49 \right]$$

This method is much more straightforward and numerical friendlier than the first one. It only requires the two 1-dimensional derivatives and does not need calculation of high order terms like $\delta$ or $\nu$. But it requires fully consistent pressure profile with other field quantities like $g$ and $B$ field.

### Appendix B Numerical implementations in Boozer coordinate system

The covariant and contra-variant form of magnetic field in Boozer coordinate system are given by

$$\mathbf{B}_0 = q (\psi) \nabla \psi \times \nabla \theta - \nabla \psi \times \nabla \zeta$$

$$= \delta (\theta, \zeta) \nabla \psi + I (\psi) \nabla \theta + g (\psi) \nabla \zeta, \quad \left[ 50 \right]$$
which are frequently used in deriving the following equations. In (50), \( q \) is the safety factor, \( I \) the toroidal current, \( g \) the poloidal current. \( \delta \) current comes from the non-orthogonality of the coordinate system. In GTC we use \( \psi, \theta, \zeta \) and \( \rho_\parallel = m v_\parallel / (Z B_0) \) to update the gyrocenter location in phase space,

\[
\frac{\dot{\psi}}{Z B_0} = \frac{\rho_\parallel B_0^2 Z}{m} - \frac{\rho_\parallel B_0^2 Z}{m} (\rho_\parallel + \lambda) (I' - \partial_\delta \delta) + \frac{\rho_\parallel B_0^2 Z}{m} \left( -\delta \frac{\partial \lambda}{\partial \psi} + I \frac{\partial \lambda}{\partial \psi} \right) + \frac{1}{Z B_0} \left( \delta \frac{\partial B_0}{\partial \theta} - \delta \frac{\partial B_0}{\partial \psi} \right) + \frac{1}{Z B_0} \left( \delta \frac{\partial \psi}{\partial \psi} - \delta \frac{\partial \psi}{\partial \theta} \right)
\]

(51)

\[
\frac{\dot{\psi}}{Z B_0} = \frac{\rho_\parallel B_0^2 Z}{m} - \frac{\rho_\parallel B_0^2 Z}{m} (g' - \partial_\psi \delta) (\rho_\parallel + \lambda) + \frac{1}{Z B_0} \frac{1}{Z B_0} \left( g \frac{\partial B_0}{\partial \psi} - g \frac{\partial B_0}{\partial \theta} \right) + \frac{1}{Z B_0} \left[ I \frac{\partial \psi}{\partial \psi} - I \frac{\partial \psi}{\partial \theta} \right]
\]

(52)

\[
\frac{\dot{\theta}}{Z B_0} = \frac{\rho_\parallel B_0^2 Z}{m} - \frac{\rho_\parallel B_0^2 Z}{m} (g' - \partial_\psi \delta) (\rho_\parallel + \lambda) + \frac{1}{Z B_0} \frac{1}{Z B_0} \left( g \frac{\partial B_0}{\partial \psi} - g \frac{\partial B_0}{\partial \theta} \right) + \frac{1}{Z B_0} \left[ I \frac{\partial \psi}{\partial \psi} - I \frac{\partial \psi}{\partial \theta} \right]
\]

(53)

\[
\frac{\dot{\rho}_\parallel}{Z B_0} = \frac{1}{Z B_0} \left( -1 + (\rho_\parallel + \lambda) (g' - \partial_\psi \delta) - \left( \frac{\partial \lambda}{\partial \psi} - \frac{\partial \lambda}{\partial \theta} \right) \left( \frac{1}{Z B_0} \frac{\partial B_0}{\partial \theta} + \partial_\theta \phi_\parallel \right) \right) - \frac{1}{Z B_0} \left( q + (\rho_\parallel + \lambda) (I' - \partial_\theta \delta) + \left( \frac{1}{Z B_0} \frac{\partial B_0}{\partial \psi} - \frac{\partial \lambda}{\partial \theta} \right) \left( \frac{1}{Z B_0} \frac{\partial B_0}{\partial \psi} + \partial_\theta \phi_\parallel \right) \right)
\]

(54)

where \( I' = \partial_\delta I, g' = \partial_\psi g, \lambda = \langle \delta A_\parallel \rangle / B_0, D = g q + I + \rho_\parallel [(I' - \partial_\theta \delta) g - I (g' - \partial_\psi \delta)], \partial \psi / \partial B = \left( \mu + \rho_\parallel^2 \Omega \right), \phi_\parallel = \langle \phi \rangle + \frac{\mu}{Z} \langle \delta B_\parallel \rangle, \) and \( \partial_\lambda = B_0 \cdot \nabla \phi_{rad} / B_0. \) Sometimes, the parallel acceleration from electric field in \( \dot{\rho}_\parallel \) can be neglected for better numerical properties, and (54) becomes

\[
\frac{\dot{\rho}_\parallel}{Z B_0} = \frac{1}{Z B_0} \left( -1 + \rho_\parallel (g' - \partial_\psi \delta) \right) \left( \frac{1}{Z B_0} \frac{\partial B_0}{\partial \theta} \right) + \frac{1}{Z B_0} \left( -q - \rho_\parallel (I' - \partial_\theta \delta) \right) \left( \frac{1}{Z B_0} \frac{\partial B_0}{\partial \psi} \right)
\]

\[
+ \frac{\rho_\parallel^2 B_0 Z}{m} \left[ g' \lambda + g \partial_\psi \lambda - \lambda \partial_\psi \delta - \delta \partial_\psi \lambda \right] \frac{\partial_\theta B_0}{g_0} - \frac{\rho_\parallel^2 B_0 Z}{m} \left[ I' \lambda + I \partial_\psi \lambda - \lambda \partial_\psi \delta - \delta \partial_\psi \lambda \right] \frac{\partial_\theta B_0}{g_0}
\]

(55)

The equations (51)-(54) are derived from Euler-Lagrangean equation in gyrokinetic phase space. White et al [White1984] made an additional transformation to the parallel gyrocenter velocity to get rid of \( \delta \) term, and derived the equation of motion from Hamiltonian method. In fact, by dropping all \( \delta \) terms in equations (51)-(54), one obtains the identical equations of motion derived in [White 1984]. We choose not to do the transformation in order to keep the Maxwell equations as the standard gyrokinetic form. In practice, \( \delta \) is often not important, especially for simulations in large aspect ratio tokamaks.
If we use the shifted Maxwellian equilibrium distribution, the ion weight equation can be derived from (6) and is given by

$$\frac{d}{dt}w_i = (1 - w_i) \frac{1}{D} \left[ \partial_t \phi + \mu Z_i \delta B_i - g \partial_t \phi + \mu Z_i \delta B_i \right] (\kappa_i + \kappa_{v,i})$$

$$+ \frac{Z_i B_0}{D} \left[ \partial_v \lambda_{t} + \lambda_{e} \right] \left( I \partial_t \phi + \mu Z_i \delta B_i - g \partial_t \phi + \mu Z_i \delta B_i \right)$$

$$+ \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left( 1 \partial_t \phi + \mu Z_i \delta B_i - g \partial_t \phi + \mu Z_i \delta B_i \right)$$

$$+ \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left( 1 \partial_t \phi + \mu Z_i \delta B_i - g \partial_t \phi + \mu Z_i \delta B_i \right)$$

$$- \frac{Z_i B_0}{D} \partial_v \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left[ \frac{m_i v_i}{Z_i} \left( v_i - u_{i0} \right) \right]$$

$$+ \frac{1}{D} \left[ I \partial_t B \partial_v \phi + \mu Z_i \delta B_i \right]$$

$$+ \frac{1}{D} \left[ I \partial_t B \partial_v \phi + \mu Z_i \delta B_i \right]$$

$$- \frac{v_i B}{D} \left[ g \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) - I \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \right]$$

$$- \frac{Z_i B_0}{D} \partial_v \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left[ \frac{m_i v_i}{Z_i} \left( v_i - u_{i0} \right) \right]$$

$$+ \frac{\mu u_{i0} B_0}{D} \left[ \partial_v \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \right]$$

$$+ \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left( g \partial_t \phi + \mu Z_i \delta B_i \right)$$

$$- \partial_t \phi \left( \partial_v \lambda_{t} + \lambda_{e} \right) \left( g \partial_t \phi + \mu Z_i \delta B_i \right)$$

where $\kappa_x = -\partial_t \ln n_{x0} - \partial_v \ln T_x \left[ \frac{m_i (v_i - u_{i0})^2}{2 T_x} + \frac{N_x}{T_x} - 1.5 \right]$, $\kappa_{v,x} = -\frac{m_i (v_i - u_{i0})}{T_x} \partial_v u_{i0}$, and $\kappa_x + \kappa_{v,x} = -\partial_v \ln f_{x0}$ for shifted Maxwellian distribution. The ion equilibrium parallel flow is assumed as a function of $\psi$. $\phi_{nz}$ and $\phi_{00}$ are the non-zonal and zonal part of the electrostatic potential, respectively. The gyroaveraging operator $\langle \cdot \rangle$ and $\langle \langle \cdot \rangle \rangle$ are dropped for simplicity.
If we take the local Maxwellian distribution as the equilibrium electron distribution, the electron weight equation is given by

$$
\frac{dw_e}{dt} = \left(1 - \frac{\delta f_e^{ad}}{f_{0e}} - w_e\right) \left[ \frac{\partial}{\partial t} \frac{\delta f_e^{ad}}{f_{0e}} - v_{E,eq} \cdot \nabla \frac{\delta f_e^{ad}}{f_{0e}} - \frac{\mu B^*_0}{m_e B^*_||} \nabla B^*_0 \frac{\partial}{\partial \psi_0} \frac{\delta f_e^{ad}}{f_{0e}} + e \frac{B^*_0}{m_e B^*_||} \nabla \phi_{eq} \frac{\partial}{\partial \psi} \frac{\delta f_e^{ad}}{f_{0e}} \right]
$$

\begin{align}
+ \frac{b_0}{B^*_||} \times \nabla \left( \phi_{pt} - \frac{\mu \delta B||}{e} \right) \cdot \nabla \psi \frac{\partial \ln f_{0e}}{\partial \psi_0} \bigg|_{v_\perp} & \nonumber \\
+ \mathbf{v}_d \cdot \nabla \left( \frac{\epsilon \phi_{ind}^{ad}}{T_e} + \frac{\partial \ln f_{0e}}{\partial \psi_0} \bigg|_{v_\perp} \psi \frac{\partial \phi_{eq}}{\partial \psi_0} \delta \psi \right) & \\
+ \epsilon \frac{\delta B^*_\perp}{B^*_||} \left( \nabla \phi_{pt} - \frac{\mu}{e} \nabla \delta B|| \right) \frac{v_\parallel}{T_e} & \nonumber \\
= (1 - \frac{\delta f_e^{ad}}{f_{0e}} - w_e) \times (w_{drive} + w_{para} + w_{drift, ind} + w_{drift} + w_{de}) & \nonumber 
\end{align}

(58)

where the nonlinear part $\delta L \delta f_e^{ad}$ has been neglected, $\delta \phi_{ind}^{ad} = \phi_{eff}^{ad} - \phi$ is the adiabatic part of inductive potential. In Boozer coordinate system, the detailed implementation can be separated to serveral parts. The terms in the last line of (58) are defined as

$$
w_{drive} = \frac{b_0}{B^*_||} \times \nabla \left( \phi_{pt} - \frac{\mu}{e} \delta B|| \right) \cdot \nabla \psi \frac{\partial \ln f_{0e}}{\partial \psi_0} \bigg|_{v_\perp}
$$

$$
w_{para} = - \frac{\partial \delta f_e^{ad}}{\partial t} + \epsilon \frac{\delta B^*_\perp}{B^*_||} \nabla \left( \phi_{pt} - \frac{\mu}{e} \delta B|| \right) \frac{v_\parallel}{T_e}
$$

$$
= - \frac{1}{m_{0e}} \frac{\partial \delta n_e}{\partial t} - \left(1 - \frac{\mu B^*_0}{T_e} \right) \frac{\partial \delta B||}{B^*_0} \frac{1}{q} \frac{\partial \phi_{ind}^{ad}}{\partial \theta} \left( \kappa_e - \kappa_{n,e} \right) & \\
+ \epsilon \frac{B^*_0}{T_e} \left[ \frac{\partial v_\perp}{T_e} \delta \psi \left( \phi_{pt} + \phi_{eq} \right) \right] & \\
+ \phi \left( \phi_{pt} + \phi_{eq} \right) \left( \phi_{pt} - \frac{\mu}{e} \delta B|| \right) & \\
+ \phi \left( \phi_{pt} + \phi_{eq} \right) \left( \phi_{pt} - \frac{\mu}{e} \delta B|| \right) & \\
= (1 - \frac{\delta f_e^{ad}}{f_{0e}} - w_e) \times (w_{drive} + w_{para} + w_{drift, ind} + w_{drift} + w_{de}) & \nonumber 
$$

where we have used the relation (34a) and (34b).
where we have kept the terms of the order \(O(1/k_L L_p)\), and dropped the terms of the order \(O(1/k_L L_B)\).

\[
\begin{align*}
w_{\text{drift}} &= - \mathbf{v}_{E,eq} \cdot \nabla P_{ad} - \frac{\mathbf{J} \cdot \nabla f_{ad}}{\mathcal{A}_{\text{eff}}} \\
&= \frac{\partial \phi_{ad}}{\partial T_e} \left[ I \left( \frac{e}{T_e} \partial_{\psi} \phi_{ad} + \frac{\mu}{T_e} \partial_{\psi} B_\parallel - \kappa_e \partial_{\psi} \delta \psi - \frac{e}{T_e} \partial_{\psi} \phi_{ad} \partial_{\psi} \delta \psi \right) \right] \\
&- \frac{g}{\partial \psi} \left( \frac{e}{T_e} \partial_{\psi} \phi_{ad} + \frac{\mu}{T_e} \partial_{\psi} B_\parallel - \kappa_e \partial_{\psi} \delta \psi - \frac{e}{T_e} \partial_{\psi} \phi_{ad} \partial_{\psi} \delta \psi \right) \right] \\
w_{dv} &= - \frac{\mu B_0^2}{\mu_e B_0^2} \cdot \nabla P_{ad} \left( \frac{m_e B_0^2}{\mu_e} \left( \partial_{\psi} \phi_{ad} \partial_{\psi} \delta \psi \right) \right) \\
&= \left[ - \frac{\mu B_0}{\mu_e} \left( q \partial_{\psi} B_0 + \partial_{\psi} B_0 \right) + \frac{\mu v_n}{e} \right] \left( \frac{1}{D} \left( \partial_{\psi} \phi_{ad} \partial_{\psi} B_0 - (g' - \partial_{\psi}) \partial_{\psi} B_0 \right) \right) \\
&- \frac{v_n}{B_0} \delta \phi_{ad} \left( I \partial_{\psi} B_0 - g \partial_{\psi} B_0 \right) \left( \delta \psi \partial_{\psi} \ln \frac{m_{\text{eff}}}{\mu_B} \right) \\
&= \frac{\partial A_{NL}^0}{\partial t} = \delta B_\parallel \cdot \nabla \phi_{ad} - m_{\text{eff}} v_n \nabla \left( \delta u_{\parallel e} - \frac{P_{\parallel e} B_0^2}{B_0^3} \right) + \frac{P_{\parallel e} \delta B_\parallel}{\mu_{\parallel e}} \nabla \delta B_\parallel \tag{59}
\end{align*}
\]

This nonlinear \(A_{NL}^0\) is a nonlinear correction to the \(A_{\parallel n2}\) from (19).

The electron continuity equation is given by

\[
\frac{\partial n_e}{\partial t} = - (u_\parallel + w_{\text{drift}} + w_{\text{drive}} + w_{\text{eqc}} + w_\ast + w_{\parallel n} + w_{\perp n}), \tag{60}
\]

where the \(w\) terms represent the contribution from various flows. \(w_\parallel\) denotes the parallel perturbed flow,

\[
w_\parallel = B_0 \cdot \nabla \left( \frac{n_{\text{eff}} \delta u_{\parallel e}}{B_0} \right) = \frac{n_{\text{eff}}}{J_e B_0} \left( q \partial_{\psi} \delta u_{\parallel e} + \partial_{\psi} \delta u_{\parallel e} \right), \tag{61}
\]

where \(J_e\) is the Jacobian for Boozer coordinates \(J_e = (q + I)/B^2\). \(w_{\text{drive}}\) denotes the diamagnetic flow,

\[
w_{\text{drive}} = - n_0 \mathbf{v}_\ast \cdot \nabla B_0 + \frac{1}{J_e B_0^2} \left( \frac{\partial B_e}{\partial \psi} (I \partial_{\psi} \delta P_e + g \partial_{\psi} \delta P_e) + \frac{\partial B_\theta}{\partial \psi} (\partial_{\psi} \delta P_e - \delta \partial_{\psi} \delta P_e) + \frac{\partial B_\phi}{\partial \psi} (\partial_{\psi} \delta P_e - I \partial_{\psi} \delta P_e) \right), \tag{62}
\]

where \(\delta P_e\) is the total perturbed electron pressure.

\[
\delta P_e = \delta P_{ad} + \delta P_{\parallel e} + \delta P_{\perp e} + \delta P_{\parallel e} + \delta P_{\parallel e} + \delta P_{\parallel e}, \tag{63}
\]

\[\mu_{\parallel e} = 2m_{\text{eff}} \frac{\partial \phi_{ad}}{\partial \psi} + 2 \frac{\partial P_{\parallel e}}{\partial \psi} - \frac{3}{2} \frac{\delta B_0}{B_0} P_{\parallel e} - 2m_{\text{eff}} \frac{\partial \phi_{ad}}{\partial \psi} \delta \psi + \delta P_{\parallel e} + \delta P_{\parallel e} + \delta P_{\parallel e}.\]
$w_{\text{drift}}, w_s, w_{\text{snl}}, w_{\text{drift0}},$ and $w_{\text{eb0}}$ denotes the flow due to $E \times B$ drift velocity,

$$w_s + w_{\text{drift}} + w_{\text{snl}} + w_{\text{drift0}} + w_{\text{eb0}}$$

$$= B_0 \mathbf{v}_E \cdot \nabla \left( \frac{n_{0e} + \delta n_e}{B_0} \right) - (n_{0e} + \delta n_e) \mathbf{v}_E \cdot \nabla B_0$$

$$= \frac{1}{J B_0^2} \frac{\partial n_{0e}}{\partial \psi} \left( I \partial_\phi \phi_{pt} - g \partial_\theta \phi_{pt} \right)$$

$$+ \frac{2(n_{0e} + \delta n_e)}{J B_0^2} \left[ \frac{\partial B}{\partial \psi} (g \partial_\theta \phi_{pt} - I \partial_\phi \phi_{pt}) + \frac{\partial B}{\partial \theta} (\delta \partial_\phi \phi_{pt} - g \partial_\theta \phi_{pt}) + \frac{\partial B}{\partial \psi} (I \partial_\psi \phi_{pt} - \delta \partial_\theta \phi_{pt}) \right]$$

$$+ \frac{1}{J B_0^2} \left[ \frac{\partial n_{0e}}{\partial \psi} (I \partial_\phi \phi_{pt} - g \partial_\theta \phi_{pt}) + \frac{\partial \delta n_e}{\partial \theta} (g \partial_\psi \phi_{pt} - \delta \partial_\phi \phi_{pt}) \right]$$

$$+ \frac{2(n_{0e} + \delta n_e)}{J B_0^2} \left[ \frac{1}{I \partial_\phi S - \delta \partial_\theta (S - 1) \partial_\phi B_0 - \delta \partial_\theta (S - 1) \partial_\psi B_0) \right]$$

$$+ \frac{1}{J B_0^2} \left[ \frac{g \partial \delta n_e}{\partial \theta} \right]$$

$$+ \frac{1}{J B_0^2} \left[ \frac{\partial n_{0e}}{\partial \psi} (I \partial_\phi \phi_{pt} - g \partial_\theta \phi_{pt}) \right]$$

$$w_{\text{eqc}}$$

$$= \delta B_{\perp} \cdot \nabla \left( \frac{n_{0e} w_{\parallel 0}}{B_0} \right) - \nabla \times \mathbf{B}_0 \cdot \left( \nabla \delta P_{\|e} + \left( \frac{\delta P_{\|e} - \delta P_{\perp e}}{B_0} \right) \nabla B_0 \right) - n_{0e} e \nabla \phi + \frac{\delta n_e}{B_0^2} \nabla \times \mathbf{B}_0 \cdot \nabla \phi$$

where

$$\partial_\psi \delta P_{\|e} = e n_{0e} \partial_\psi \varphi_{c0} + e n_{0e} \partial_\psi \varphi_{c0} \partial_\psi \delta \psi - P_{\parallel 0e} \partial_\psi \delta B_0 \| B_0 - e n_{0e} \partial_\psi \varphi_{c0} \partial_\psi \delta \psi + \partial_\psi P_{\|e}^{\perp},$$

$$\partial_\theta \delta P_{\|e} = e n_{0e} \partial_\theta \varphi_{c0} + \partial_\theta \varphi_{c0} \partial_\psi \delta \psi - P_{\parallel 0e} \partial_\theta \delta B_0 \| B_0 - e n_{0e} \partial_\theta \varphi_{c0} \partial_\psi \delta \psi + \partial_\theta P_{\|e}^{\perp},$$

$$\delta P_{\perp e} - \delta P_{\|e} = \left( \frac{\delta P_{\perp e}^{\perp} - \delta P_{\|e}^{\perp}}{B_0} \right) - B_\parallel \delta P_{\perp e} \| B_0.$$
In general curliner coordinate system, the Laplacian operator is expressed as

$$S = \frac{\partial^2}{\partial \varphi^2}$$ is proportional to the parallel current. \( w_{\parallel} \) denotes the flow from magnetic compressional perturbation

$$w_{\parallel} = - \frac{\mathbf{b}_0 \times \nabla \delta B_\parallel}{e} \cdot \nabla \left( \frac{\delta P_{\perp e} + P_{\perp e}}{B_0^2} \right) - \frac{\nabla \times \mathbf{b}_0 \cdot \nabla \delta B_\parallel}{eB_0^2} (\delta P_{\perp e} + P_{\perp e})$$

$$= \frac{3P_{\perp e}^T}{eJ B_0^2} \left[ \frac{\partial \delta B_\parallel}{\partial \psi} \left( g \frac{\partial B_0}{\partial \theta} - I \frac{\partial B_0}{\partial \zeta} \right) + \frac{\partial \delta B_\parallel}{\partial \zeta} \left( \delta \frac{\partial B_0}{\partial \theta} - g \frac{\partial B_0}{\partial \psi} \right) + \frac{1}{eJ B_0^2} \left[ (q' - \partial_\psi) \frac{\partial \delta B_\parallel}{\partial \theta} - (I' - \partial_\psi) \frac{\partial \delta B_\parallel}{\partial \zeta} \right) \right]$$

(66)

Here \( P_{\perp e}^T = P_{\perp e} + \delta P_{\perp e} \) is the total perpendicular electron pressure, \( n_{\parallel e} \) is the perturbed electron parallel flow, \( P_{\perp e} = n_{\perp e} T_e + \delta P_{\perp e}^T + n_{\parallel e} \partial_\psi T_e \delta \psi - n_{\perp e} T_e \delta B_\parallel / B_0 \).

\( w_{\perp e} = \delta B_\perp \cdot \nabla \left( \frac{n_{\perp e} \delta v_{\parallel e}}{B_0} \right) \)

(67)

$$w_{\perp e} = - \frac{1}{J} \left[ \delta \left( \partial_\psi (\lambda_{\parallel e} + \lambda_{\perp e}) - \partial_\zeta (\lambda_{\parallel e} + \lambda_{\perp e}) \right) \right]$$

$$+ \left( \partial_\psi (\lambda_{\parallel e} + \lambda_{\perp e}) - \partial_\zeta (\lambda_{\parallel e} + \lambda_{\perp e}) \right) \delta S + \left( \partial_\psi (\lambda_{\parallel e} + \lambda_{\perp e}) - \partial_\zeta (\lambda_{\parallel e} + \lambda_{\perp e}) \right) \delta S$$

where \( \delta S \) is the perturbed electron parallel flow, \( \delta S = n_{\perp e} \delta v_{\parallel e} / B_0 \). In the continuity equation, we have dropped the terms at the order higher than \( O(k_{\perp} L_B) \), but we keep the second order derivative of equilibrium magnetic field \( \nabla S \), since it is the driving term for kink instability.

Appendix C  The implementation of Laplacian operator

In general curliner coordinate system, the Laplacian operator is expressed as

$$\nabla^2 f = \sum_{\alpha=1,2,3} \sum_{\beta=1,2,3} \frac{1}{J} \frac{\partial}{\partial \xi^\alpha} \left( J \nabla \xi^\beta \cdot \nabla \xi^\beta \frac{\partial}{\partial \xi^\beta} f \right),$$

(68)

where \( (\xi^1, \xi^2, \xi^3) = (\psi, \theta, \zeta) \), and \( J \) is the Jacobian. Then for Boozer coordinate system, we have

$$\nabla^2 f = g^{\psi \psi} \frac{\partial^2 f}{\partial \psi^2} + 1 \frac{\partial f}{\partial \psi} \left( \frac{\partial J g^{\psi \psi}}{\partial \psi} + \frac{\partial J g^{\psi \theta}}{\partial \theta} + \frac{\partial J g^{\psi \zeta}}{\partial \zeta} \right) + g^{\psi \theta} \frac{\partial^2 f}{\partial \psi \partial \theta} + 1 \frac{\partial f}{\partial \theta} \left( \frac{\partial J g^{\psi \theta}}{\partial \psi} + \frac{\partial J g^{\theta \theta}}{\partial \theta} + \frac{\partial J g^{\theta \zeta}}{\partial \zeta} \right) + g^{\psi \zeta} \frac{\partial^2 f}{\partial \psi \partial \zeta} + 1 \frac{\partial f}{\partial \zeta} \left( \frac{\partial J g^{\psi \zeta}}{\partial \psi} + \frac{\partial J g^{\theta \zeta}}{\partial \theta} + \frac{\partial J g^{\zeta \zeta}}{\partial \zeta} \right)$$

$$+ 2g^{\psi \theta} \frac{\partial^2 f}{\partial \psi \partial \theta} + 2g^{\psi \zeta} \frac{\partial^2 f}{\partial \psi \partial \zeta} + 2g^{\theta \zeta} \frac{\partial^2 f}{\partial \theta \partial \zeta}$$

(69)
Now we define the flux aligned coordinate \((\psi, \theta_0, \zeta_0)\), with \(\theta_0 = \theta - \zeta/q\), \(\zeta_0 = \zeta\). The expression of the Laplacian operator in the flux aligned coordinates is given by

\[
\nabla^2 f = g^{\psi \psi} \frac{\partial^2 f}{\partial \psi^2} + 1 \frac{\partial f}{\partial \psi} \left[ \frac{\partial J g^{\psi \psi}}{\partial \psi} + \frac{\partial J g^{\psi \theta}}{\partial \theta_0} \right] + \left( \frac{\partial^2 f}{\partial \theta_0^2} + 1 \frac{\partial f}{\partial \theta_0} \left[ \frac{\partial J g^{\theta \psi}}{\partial \psi} + \frac{\partial J g^{\theta \theta}}{\partial \theta_0} \right] + \frac{\partial J g^{\psi \psi}}{\partial \psi} + \frac{\partial J g^{\psi \theta}}{\partial \theta_0} \right) \left( J g^{\psi \psi} \right)
\]

If we assume \(k L_\phi \gg 1\), the parallel Laplacian operator can be obtained from full Laplacian operator by enforcing \(\partial_{\zeta_0} f = 0\),

\[
\nabla_1^2 f = g^{\psi \psi} \frac{\partial^2 f}{\partial \psi^2} + 1 \frac{\partial f}{\partial \psi} \left[ \frac{\partial J g^{\psi \psi}}{\partial \psi} + \frac{\partial J g^{\psi \theta}}{\partial \theta_0} \right] + \left[ \frac{\partial^2 f}{\partial \theta_0^2} + 1 \frac{\partial f}{\partial \theta_0} \left( J g^{\psi \psi} \right) \right].
\]

**References**

[1] I. Holod, W. L. Zhang, Y. Xiao, and Z. Lin, “Electromagnetic formulation of global gyrokinetic particle simulation in toroidal geometry,” *Physics of Plasmas*, vol. 16, no. 12, p. 122307, 2009. [Online]. Available: https://doi.org/10.1063/1.3273070

[2] W. Deng, Z. Lin, and I. Holod, “Gyrokinetic simulation model for kinetic magnetohydrodynamic processes in magnetized plasmas,” *Nuclear Fusion*, vol. 52, no. 2, p. 023005, Jan 2012. [Online]. Available: https://doi.org/10.1088/0029-5515/52/2/023005

[3] G. Dong, J. Bao, A. Bhattacharjee, A. Brizard, Z. Lin, and P. Porazik, “Gyrokinetic particle simulations of the effects of compressional magnetic perturbations on drift-alfvenic instabilities in tokamaks,” *Physics of Plasmas*, vol. 24, no. 8, p. 081205, 2017. [Online]. Available: https://doi.org/10.1063/1.4997788

[4] W. Deng, Z. Lin, I. Holod, X. Wang, Y. Xiao, and W. Zhang, “Gyrokinetic particle simulations of reversed shear alfvén eigenmode excited by antenna and fast ions,” *Physics of Plasmas*, vol. 17, no. 11, p. 112505, 2010. [Online]. Available: https://doi.org/10.1063/1.3496057

[5] H. S. Zhang, Z. Lin, I. Holod, X. Wang, Y. Xiao, and W. L. Zhang, “Gyrokinetic particle simulation of beta-induced alfvén eigenmode,” *Physics of Plasmas*, vol. 17, no. 11, p. 112505, 2010. [Online]. Available: https://doi.org/10.1063/1.3498761

[6] J. McClenaghan, Z. Lin, I. Holod, W. Deng, and Z. Wang, “Verification of gyrokinetic particle simulation of current-driven instability in fusion plasmas. i. internal kink mode,” *Physics of Plasmas*, vol. 21, no. 12, p. 122519, 2014. [Online]. Available: https://doi.org/10.1063/1.4905073

[7] D. Liu, W. Zhang, J. McClenaghan, J. Wang, and Z. Lin, “Verification of gyrokinetic particle simulation of current-driven instability in fusion plasmas. ii. resistive tearing mode,” *Physics of Plasmas*, vol. 21, no. 12, p. 122520, 2014. [Online]. Available: https://doi.org/10.1063/1.4905074

[8] I. Holod, D. Fulton, and Z. Lin, “Microturbulence in DIII-d tokamak pedestal. II. electromagnetic instabilities,” *Nuclear Fusion*, vol. 55, no. 9, p. 093020, Aug 2015. [Online]. Available: https://doi.org/10.1088/0029-5515/55/9/093020
[9] G. e. a. Brochard, “Verification and validation of gyrokinetic and kinetic-mhd simulations for internal kink instability in diii-d tokamak,” In preparation.

[10] G. Dong, X. Wei, J. Bao, G. Brochard, Z. Lin, and W. Tang, “Deep learning based surrogate model for first-principles global simulations of fusion plasmas,” 2021.

[11] A. Brizard, “Nonlinear gyrofluid description of turbulent magnetized plasmas,” Physics of Fluids B: Plasma Physics, vol. 4, no. 5, pp. 1213–1228, 1992. [Online]. Available: https://doi.org/10.1063/1.860129

[12] A. J. Brizard and T. S. Hahm, “Foundations of nonlinear gyrokinetic theory,” Rev. Mod. Phys., vol. 79, pp. 421–468, Apr 2007. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.79.421

[13] P. Porazik and Z. Lin, “Gyrokinetic simulation of magnetic compressional modes in general geometry,” Communications in Computational Physics, vol. 10, no. 4, p. 899–911, 2011.

[14] Y. Chen, W. Zhang, J. Bao, Z. Lin, C. Dong, J. Cao, and D. Li, “Verification of energetic-particle-induced geodesic acoustic mode in gyrokinetic particle simulations,” Chinese Physics Letters, vol. 37, no. 9, p. 095201, 2020. [Online]. Available: http://cpl.iphy.ac.cn/EN/abstract/article_105730.shtml

[15] J. D. Gaffey, “Energetic ion distribution resulting from neutral beam injection in tokamaks,” Journal of Plasma Physics, vol. 16, no. 2, p. 149–169, 1976.

[16] Z. Lin and W. W. Lee, “Method for solving the gyrokinetic poisson equation in general geometry,” Phys. Rev. E, vol. 52, pp. 5646–5652, Nov 1995. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevE.52.5646

[17] Z. Lin and L. Chen, “A fluid–kinetic hybrid electron model for electromagnetic simulations,” Physics of Plasmas, vol. 8, no. 5, pp. 1447–1450, 2001. [Online]. Available: https://link.aps.org/doi/10.1063/1.1356438

[18] Z. Lin, Y. Nishimura, Y. Xiao, J. Holod, W. L. Zhang, and L. Chen, “Global gyrokinetic particle simulations with kinetic electrons,” Plasma Physics and Controlled Fusion, vol. 49, no. 12B, pp. B163–B172, nov 2007. [Online]. Available: https://doi.org/10.1088/0741-3335/49/12b/s15

[19] J. Bao, D. Liu, and Z. Lin, “A conservative scheme of drift kinetic electrons for gyrokinetic simulation of kinetic-mhd processes in toroidal plasmas,” Physics of Plasmas, vol. 24, no. 10, p. 102516, 2017. [Online]. Available: https://doi.org/10.1063/1.4995455

[20] G. Y. Fu and H. L. Berk, “Effects of pressure gradient on existence of alfven cascade modes in reversed shear tokamak plasmas,” Physics of Plasmas, vol. 13, no. 5, p. 052502, 2006. [Online]. Available: https://doi.org/10.1063/1.2196246

[21] J. Bao, W. L. Zhang, D. Li, and Z. Lin, “Effects of plasma diamagnetic drift on alfven continua and discrete eigenmodes in tokamaks,” Journal of Fusion Energy, vol. 39, pp. 382–389, 2020. [Online]. Available: https://doi.org/10.1007/s10894-020-00275-0

[22] H. L. Berk and R. R. Dominguez, “Variational method for electromagnetic waves in a magneto-plasma,” Journal of Plasma Physics, vol. 18, no. 1, p. 31–48, 1977.

[23] W. Tang, J. Connor, and R. Hastie, “Kinetic-ballooning-mode theory in general geometry,” Nuclear Fusion, vol. 20, no. 11, pp. 1439–1453, nov 1980. [Online]. Available: https://doi.org/10.1088/0029-5515/20/11/011

[24] R. B. White, Theory of toroidally confined plasmas, 3rd ed. IMPERIAL COLLEGE PRESS, 2013.