A Discrete Helgason-Fourier transform for Sobolev and Besov functions on noncompact symmetric spaces

Isaac Pesenson

Abstract. Let $f$ be a Paley-Wiener function in the space $L^2(X)$, where $X$ is a symmetric space of noncompact type. It is shown that by using the values of $f$ on a sufficiently dense and separated set of points of $X$ one can give an exact formula for the Helgason-Fourier transform of $f$. In order to find a discrete approximation to the Helgason-Fourier transform of a function from a Besov space on $X$ we develop an approximation theory by Paley-Wiener functions in $L^2(X)$.

1. Introduction and main results

Let $f$ be a smooth function in the space $L^2(X, dx)$, where $X$ is a symmetric space of noncompact type and $dx$ is an invariant measure. The notation $\hat{f}$ will be used for the Helgason-Fourier transform of $f$. The Helgason-Fourier transform $\hat{f}$ can be treated as a function on $\mathbb{R}^n \times B$ where $B$ is a certain compact homogeneous manifold and $n$ is the rank of $X$. Moreover, $\hat{f}$ belongs to the space $L^2(\mathbb{R}^n \times B; |c(\lambda)|^{-2} d\lambda db)$, where $c(\lambda)$ is the Harish-Chandra’s function, $d\lambda$ is the Euclidean measure and $db$ is the normalized invariant measure on $B$, $R = \mathbb{R}^n \times B$, $dy = |c(\lambda)|^{-2} d\lambda db$. The notation $\Pi_\omega \subset \mathbb{R}^n \times B$ will be used for the set of all points $(\lambda, b) \in \mathbb{R}^n \times B$, $\lambda \in \mathbb{R}^n$, $b \in B$, for which $\sqrt{\langle \lambda, \lambda \rangle} < \omega$, where $\langle \cdot, \cdot \rangle$ is the Killing form.

The Paley-Wiener space $PW_\omega(X), \omega > 0$, is defined as the set of all functions in $L^2(X)$ whose Helgason-Fourier transform has support in $\Pi_\omega$ and belongs to the space $L^2(\Pi_\omega; |c(\lambda)|^{-2} d\lambda db)$.

It is shown that if $f \in PW_\omega(X)$ is known only on a sufficiently dense and separated set of points of $X$ then there exists an exact formula for reconstruction the Helgason-Fourier transform $\hat{f}$. In order to extend this result and to find a discrete approximation to the Helgason-Fourier transform of a function from a
Besov space on $X$ we develop an approximation theory by Paley-Wiener functions in $L_2(X)$.

In the Section 2 we list some basic facts about harmonic Analysis on symmetric spaces of noncompact type (see [H1]-[H3]). One of the main results of the Section 3 is Corollary 3.1 which says that for a fixed $\omega > 0$ and a sufficiently dense and separated set of points $Z_\omega = \{x_j\}, x_j \in X$, there exists a set of functions \( \{\Theta_{x_j}\}, \Theta_{x_j} \in PW_\omega(X) \), such that for any $f \in PW_\omega(X)$ the following exact formula holds

\[ \hat{f} = \sum_{x_j \in Z_\omega} f(x_j) \Theta_{x_j}. \]  

This formula implies a quadrature rule which gives that for any compact measurable set $U \subset X$

\[ \int_U f \, dx = \sum_{x_j \in Z_\omega} f(x_j) w_j, \]

for all $f \in PW_\omega(X)$. Here the weights $w_j$ are given by the formulas

\[ w_j = \int_U \Theta_{x_j} \, dx. \]

In order to extend these results to non-Paley-Wiener functions we consider the following scheme. For a function $f \in L_2(X)$ we consider its orthogonal projection on the space $PW_\omega(X)$ which is the function

\[ f_\omega = \mathcal{H}^{-1} \left( \chi_{\omega} \hat{f} \right), \]

where $\chi_{\omega}$ is the characteristic function of the set $\Pi_\omega$ and $\mathcal{H}^{-1}$ is the inverse Helgason-Fourier transform. It is clear that for a general function $f \in L_2(X)$ the sum

\[ \hat{f}_\omega = \sum_{x_j \in Z_\omega} f_\omega(x_j) \Theta_{x_j} \]

gives just an approximation to the Helgason-Fourier $\hat{f}$ of $f$ and a natural problem is to measure a degree of such approximation. If $Z_\omega = \{x_j\}, x_j \in X$, is a set of points for which the formula (1.1) holds then by using the Plancherel Theorem and the formula $\hat{f}_\omega = \chi_{\omega} \hat{f}$ we obtain the following inequality

\[ \Phi(f, Z_\omega) = \left\| \hat{f} - \sum_{x_j \in Z_\omega} f_\omega(x_j) \Theta_{x_j} \right\|_{L_2(R, d\mu)} \leq \]

\[ \|\hat{f} - \chi_{\omega} \hat{f}\|_{L_2(R, d\mu)} + \left\| \hat{f}_\omega - \sum_{x_j \in Z_\omega} f_\omega(x_j) \Theta_{x_j} \right\|_{L_2(R, d\mu)} = \|f - f_\omega\|_{L_2(X)}. \]

This inequality shows that the error of approximation of $\hat{f}$ for a general function $f \in L_2(X)$ by a sum $\sum_{x_j \in Z_\omega} f_\omega(x_j) \Theta_{x_j}$ is controlled by the best approximation $E(f, \omega)$ of $f \in L_2(X)$ by Paley-Wiener functions

\[ E(f, \omega) = \inf_{g \in PW_\omega(X)} \| f - g \|_{L_2(X)} = \| f - f_\omega \|_{L_2(X)}, f \in L_2(X). \]
The corresponding approximation theory is developed in Sections 4 and 5. The main result of the Section 5 is Theorem 5.1 which describes a rate of approximation of \( f \in L^2(X) \) by Paley-Wiener functions in terms of Besov spaces \( B^\alpha_{\infty q}(X), 1 \leq q \leq \infty, \alpha > 0 \). The Besov spaces are described in terms of the one-parameter group generated by the positive square root \( \sqrt{-\Delta} \), where \( \Delta \) is the Laplace-Beltrami operator of an invariant metric on \( X \). We formulate here two particular cases of our main Theorem 5.2.

**Theorem 1.1.** There exists a constant \( C_0(X) \) and for every \( \omega > 0 \) there exist a separated set of points \( Z_\omega = \{ x_j \} \) and a set of functions \( \{ \Theta_{x_j} \}, \Theta_{x_j} \in PW_\omega(X) \), as in (1.1), such that for any \( f \) in the Sobolev space \( H^\alpha(X), \alpha > 0 \), the following holds

\[
\left( \int_0^\infty (\omega^\alpha \Phi(f,Z_\omega))^2 \frac{d\omega}{\omega} \right)^{1/2} \leq C_0(X) \| f \|_{H^\alpha(X)},
\]

where \( \Phi(f,Z_\omega) \) is defined in (1). Moreover, if the following relation holds for an \( 0 \leq \alpha \leq r, r \in \mathbb{N} \),

\[
\left\| \left( I - e^{is\sqrt{-\Delta}} \right) f \right\|_{L^2(X)} = O(s^\alpha), s \to 0,
\]

where \( e^{is\sqrt{-\Delta}} \) is the group generated by a positive square root from the operator \( -\Delta \), then

\[
\Phi(f;Z_\omega) = O(\omega^{-\alpha}), \omega \to \infty.
\]

The results of the Section 5 are obtained as consequences of an abstract Direct Approximation Theorem 4.4 which is proved in the Section 4. The Theorem 4.4 is an extension of the classical results by Peetre and Sparr [PS] about interpolation and approximation spaces in abelian quasi-normed groups. The reason we use the language of quasi-normed linear spaces is not because we want to achieve a bigger generality but because this language allows to treat simultaneously interpolation and approximation spaces [BL, PS]. To be more specific: the two main Theorems of this theory one of which gives a connection between interpolation and approximation spaces and another one which is known as the Power Theorem can be formulated only on the language of quasi-normed linear spaces and not on the language of normed linear spaces.

### 2. Harmonic Analysis on symmetric spaces

A Riemannian symmetric space of the noncompact type is a Riemannian manifold \( X \) of the form \( X = G/K \) where \( G \) is a connected semisimple Lie group with finite center and \( K \) is a maximal compact subgroup of \( G \). The Lie algebras of the groups \( G \) and \( K \) will be denoted respectively as \( \mathfrak{g} \) and \( \mathfrak{k} \). The group \( G \) acts on \( X \) by left translations. If \( e \) is the identity in \( G \) then the base point \( eK \) is denoted by \( 0 \). Every such \( G \) admits Iwasawa decomposition \( G = NAK \), where the nilpotent Lie group \( N \) and the abelian group \( A \) have Lie algebras \( \mathfrak{n} \) and \( \mathfrak{a} \) respectively. The dimension of \( \mathfrak{a} \) is known as the rank of \( X \). The letter \( M \) is usually used to denote the centralizer of \( A \) in \( K \) and the letter \( \mathcal{B} \) is commonly used for the homogeneous space \( K/M \).

Let \( \mathfrak{a}^* \) be the real dual of \( \mathfrak{a} \) and \( W \) be the Weyl’s group. We denote by \( \Sigma \) will be the set of restricted roots, and \( \Sigma^+ \) will be the set of all positive roots. The
notation $\mathfrak{a}^+$ has the following meaning

$$
\mathfrak{a}^+ = \{ h \in \mathfrak{a} | \alpha(h) > 0, \alpha \in \Sigma^+ \}
$$

and is known as positive Weyl’s chamber. Let $\rho \in \mathfrak{a}^*$ is defined in a way that $2\rho$ is the sum of all positive restricted roots. The Killing form $\langle , \rangle$ on $\mathfrak{a}$ defines a metric on $\mathfrak{a}$. By duality it defines a scalar product on $\mathfrak{a}^*$. We denote by $\mathfrak{a}^*_+$ the set of $\lambda \in \mathfrak{a}^*$, whose dual belongs to $\mathfrak{a}^+$. According to Iwasawa decomposition for every $g \in G$ there exists a unique $A(g) \in \mathfrak{a}$ such that

$$
g = n \exp A(g) k, \ k \in K, \ n \in N,
$$

where $\exp : \mathfrak{a} \to A$ is the exponential map of the Lie algebra $\mathfrak{a}$ to Lie group $A$. On the direct product $X \times B$ we introduce function with values in $\mathfrak{a}$ using the formula

$$
A(x, b) = A(u^{-1}g)
$$

where $x = gK, g \in G, b = uM, u \in K$.

For every $f \in C^\infty_0(X)$ the Helgason-Fourier transform is defined by the formula

$$
\hat{f}(\lambda, b) = \int_X f(x)e^{-(i\lambda + \rho)(A(x, b))} dx,
$$

where $\lambda \in \mathfrak{a}^*, b \in B = K/M$, and $dx$ is a $G$-invariant measure on $X$. This integral can also be expressed as an integral over the group $G$. Namely, if $b = uM, u \in K$, then

$$
\hat{f}(\lambda, b) = \int_G f(gK)e^{-i\lambda + \rho)(A(u^{-1}g))} dg.
$$

The invariant measure on $X$ can be normalized so that the following inversion formula holds for $f \in C^\infty_0(X)$

$$
f(x) = w^{-1} \int_{\mathfrak{a}^* \times B} \hat{f}(\lambda, b)e^{i(\lambda + \rho)(A(x, b))}|c(\lambda)|^{-2} d\lambda db,
$$

where $w$ is the order of the Weyl’s group and $c(\lambda)$ is the Harish-Chandra’s function, $d\lambda$ is the Euclidean measure on $\mathfrak{a}^*$ and $db$ is the normalized $K$-invariant measure on $B$. This transform can be extended to an isomorphism between the spaces $L_2(X, dx)$ and $L_2(\mathfrak{a}^*_+ \times B, |c(\lambda)|^{-2}d\lambda db)$ and the Plancherel formula holds true

$$
\|f\| = \left( \int_{\mathfrak{a}^*_+ \times B} |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right)^{1/2}.
$$

An analog of the Paley-Wiener Theorem is known which says in particular that a Helgason-Fourier transform of a compactly supported distribution is a function which is analytic in $\lambda$.

Denote by $T_x(X)$ the tangent space of $X$ at a point $x \in X$ and let $\exp_x : T_x(X) \to X$ be the exponential geodesic map i. e. $\exp_x(u) = \gamma(1), u \in T_x(X)$ where $\gamma(t)$ is the geodesic starting at $x$ with the initial vector $u : \gamma(0) = x, \frac{d \gamma(0)}{dt} = u$. In what follows we assume that local coordinates are defined by $\exp$.

By using a uniformly bounded partition of unity $\{ \varphi_{\nu} \}$ subordinate to a cover of $X$ of finite multiplicity

$$
X = \bigcup_{\nu} B(x_{\nu}, r),
$$
where $B(x_\nu, r)$ is a metric ball at $x_\nu \in X$ of radius $r$ we introduce Sobolev space $H^\sigma(X), \sigma > 0$, as the completion of $C_0^\infty(X)$ with respect to the norm

$$
\|f\|_{H^\sigma(X)} = \left( \sum_\nu \|\varphi_\nu f\|_{H^\sigma(B(y_\nu, r))}^2 \right)^{1/2}.
$$

The usual embedding Theorems for the spaces $H^\sigma(X)$ hold true.

The Killing form on $G$ induces an inner product on tangent spaces of $X$. Using this inner product it is possible to construct $G$-invariant Riemannian structure on $X$. The Laplace-Beltrami operator of this Riemannian structure is denoted as $\Delta$.

It is known that the following formula holds

$$
\hat{\Delta} f(\lambda, b) = - (\|\lambda\|^2 + \|\rho\|^2) \hat{f}(\lambda, b), f \in C_0^\infty(X),
$$

where $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$, $\|\rho\|^2 = \langle \rho, \rho \rangle$, $\langle \cdot, \cdot \rangle$ is the Killing form.

It is also known that the operator $(-\Delta)$ is a self-adjoint positive definite operator in the corresponding space $L^2(X, dx)$, where $dx$ is the $G$-invariant measure.

The regularity Theorem for the Laplace-Beltrami operator $\Delta$ states that domains of the powers $(-\Delta)^{\sigma/2}$ coincide with the Sobolev spaces $H^\sigma(X)$ and the norm (2.3) is equivalent to the graph norm $\|f\| + \|(-\Delta)^{\sigma/2}f\|$ (see [T3], Sec. 7.4.5.) Moreover, since the operator $\Delta$ is invertible in $L^2(X)$ the Sobolev norm is also equivalent to the norm $\|(-\Delta)^{\sigma/2}f\|$.

3. Paley-Wiener functions and their Discrete Helgason-Fourier transform

**Definition 3.1.** We will say that $f \in L^2(X, dx)$ belongs to the class $PW_\omega(X)$ if its Helgason-Fourier transform has compact support in the sense that $\hat{f}(\lambda, b) = 0$ a.e. for $\|\lambda\| > \omega$. Such functions will be also called $\omega$-band limited.

Using the spectral resolution of identity $P_\lambda$ we define the unitary group of operators by the formula

$$
e^{it\Delta} f = \int_0^\infty e^{it\tau} dP_\tau f, f \in L^2(X), t \in \mathbb{R}.
$$

Let us introduce the operator

$$
R_\sigma^\Delta f = \frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{i \pi(k-1/2)} \Delta f, f \in L^2(X), \sigma > 0.
$$

Since $\|e^{it\Delta} f\| = \|f\|$ and

$$
\frac{\sigma}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(k-1/2)^2} = \sigma,
$$

the series in (3.1) is convergent and it shows that $R_\sigma^\Delta$ is a bounded operator in $L^2(X)$ with the norm $\sigma$:

$$
\|R_\sigma^\Delta f\| \leq \sigma \|f\|, f \in L^2(X).
$$

The next theorem contains generalizations of several results from the classical harmonic analysis (in particular the Paley-Wiener theorem) and it follows essentially from our more general results in [P1], [P2], [P3] (see also [A], [Pa]).
Theorem 3.2. Let $f \in L_2(X)$. Then the following statements are equivalent:

1. $f \in PW_\omega(X)$;
2. $f \in H^\infty(X) = \bigcap_{k=1}^\infty H^k(X)$, and for all $s \in \mathbb{R}_+$ the following Bernstein inequality holds:

$$\|\Delta^s f\| \leq (\omega^2 + \|\rho\|^2)^s\|f\|;$$

3. $f \in H^\infty(X)$ and the following Riesz interpolation formula holds

$$\Delta^n f = \left( R^{\omega^2+\|\rho\|^2}_\Delta \right)^n f, n \in \mathbb{N};$$

4. For every $g \in L_2(X)$ the function $t \mapsto \langle e^{it\Delta}f, g \rangle, t \in \mathbb{R}$, is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type $\omega^2 + \|\rho\|^2$;

5. The abstract-valued function $t \mapsto e^{it\Delta}f$ is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type $\omega^2 + \|\rho\|^2$;

6. The solution $u(t), t \in \mathbb{R}$, of the Cauchy problem

$$i \frac{\partial u(t)}{\partial t} = \Delta u(t), u(0) = f, i = \sqrt{-1},$$

has a holomorphic extension $u(z)$ to the complex plane $\mathbb{C}$ satisfying

$$\|u(z)\|_{L_2(X)} \leq e^{(\omega^2+\|\rho\|^2)|\Im z|}\|f\|_{L_2(X)}.$$

Now we give new characterizations of the space $PW_\omega(X)$. We will need the following Lemma which was proved in [PI].

Lemma 3.3. If for some $f \in H^\infty(X)$ and a certain $\sigma > 0$ the upper bound

$$\sup_{k \in \mathbb{N}} (\sigma^{-k}\|\Delta^k f\|) = C(f, \sigma) < \infty,$$

is finite, then $C(f, \sigma) \leq \|f\|$ and the following inequality holds

$$\|\Delta^k f\| \leq \sigma^k\|f\|, k \in \mathbb{N}.$$

For a vector $f \in PW_\omega(X)$ the notation $\omega_f$ will be used for a smallest positive number such that $\Pi_{\omega_f}$ contains the support of the Helgason-Fourier transform $\hat{f}$. The following Theorem gives a new characterization of the Paley-Wiener spaces.

Theorem 3.4. A vector $f \in L_2(X)$ belongs to the space $PW_{\omega_f}(X), 0 < \omega_f < \infty$, if and only if $f$ belongs to the set $H^\infty(X)$, the limit

$$\lim_{k \to \infty} \|\Delta^k f\|^{1/k}$$

exists and

$$\lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \omega_f^2 + \|\rho\|^2.$$

Proof. If $f \in PW_{\omega_f}(X)$ then $f$ is obviously in $H^\infty(X)$ and for $\hat{f}$ we have

$$\|\Delta^k f\|^{1/k} = \left( \int_{\|\lambda\| < \omega_f} \int_{B} (\|\lambda\|^2 + \|\rho\|^2)^k |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right)^{1/2k} \leq \left( \omega_f^2 + \|\rho\|^2 \right)^{1/k},$$

and for $f \in H^\infty(X)$ it is obviously bounded.
which shows that
\[(3.8) \quad \lim_{k \to \infty} \|\Delta^k f\|^{1/k} \leq \omega_f^2 + \|\rho\|^2,\]

Now, assume that
\[(3.9) \quad \lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \sigma^2 + \|\rho\|^2 < \omega_f^2 + \|\rho\|^2,\]
which means that there exists a sequence \(k_j \to \infty\) for which
\[(3.10) \quad \lim_{k \to \infty} \|\Delta^{k_j} f\|^{1/k} = \sigma^2 + \|\rho\|^2 < \omega_f^2 + \|\rho\|^2.\]
Note that the following inequality holds
\[(3.11) \quad \|\Delta^m f\| \leq \pi\|\Delta^k f\|^{m/k}\|f\|^{1-m/k}, 0 \leq m \leq k.\]
Indeed, for any \(h \in L_2(X)\) the Kolmogorov-Stein inequality gives
\[
\left\|\left(\frac{d}{dt}\right)^m \langle e^{t\Delta} f, h \rangle\right\|_{C(R^1)} \leq \pi \left\|\left(\frac{d}{dt}\right)^k \langle e^{t\Delta} f, h \rangle\right\|_{C(R^1)}^{m/k}\|\langle e^{t\Delta} f, h \rangle\|_{C(R^1)}^{1-m/k},
\]
or
\[
\left\|\langle e^{i\Delta} \Delta^m f, h \rangle\right\|_{C(R^1)} \leq \pi \left\|\langle e^{i\Delta} \Delta^k f, h \rangle\right\|_{C(R^1)}^{m/k}\|\langle e^{i\Delta} f, h \rangle\|_{C(R^1)}^{1-m/k}.
\]
Applying the Schwartz inequality we obtain
\[
\|\langle e^{i\Delta} \Delta^m f, h \rangle\|_{C(R^1)} \leq \pi \|h\|^{m/k}\|\Delta^k f\|^{m/k}\|f\|^{1-m/k}\|h\|^{1-m/k} = \pi\|\Delta^k f\|^{m/k}\|f\|^{1-m/k}\|h\|.
\]
When \(t = 0\) it gives
\[
|\langle \Delta^m f, h \rangle| \leq \pi\|\Delta^k f\|^{m/k}\|f\|^{1-m/k}\|h\|.
\]
By choosing \(h\) such that \(\|\Delta^m f, h\| = \|\Delta^m f\|\) and \(\|h\| = 1\) we obtain (3.11). The assumption (3.10) and the inequality (3.11) imply that the quantity
\[(3.12) \quad \sup_{k \in \mathbb{N}} \left\{\left(\sigma^2 + \|\rho\|^2\right)^{-k}\|\Delta^k f\|\right\} = C(f, \sigma),
\]
is finite and the previous Lemma 8.3 gives the inequality
\[
\|\Delta^k f\| \leq \left(\sigma^2 + \|\rho\|^2\right)^k\|f\|, k \in \mathbb{N}.
\]
According to the Theorem 3.2 the last inequality shows that \(f \in PW_\sigma(X)\). Since \(\sigma < \omega_f\) this contradicts to the definition of \(\omega_f\).

Conversely, assume that the following holds
\[(3.13) \quad \lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \sigma^2 + \|\rho\|^2\]
for a certain \(\sigma > 0\). It would imply
\[(3.14) \quad \sup_{k \in \mathbb{N}} \left\{\left(\sigma^2 + \|\rho\|^2\right)^{-k}\|\Delta^k f\|\right\} < C(f, \sigma)
\]
for some \(C(f, \sigma) > 0\) and by Lemma 8.3 one would have
\[
\|\Delta^k f\| \leq \left(\sigma^2 + \|\rho\|^2\right)^k\|f\|, k \in \mathbb{N}.
\]
It shows that \(f \in PW_\sigma(X)\) and there exists an \(\omega_f \leq \sigma\). But as it was just shown, this fact implies (3.7) which together with (3.11) gives \(\omega_f = \sigma\). The Theorem is proved. \(\square\)

The above Lemma and the proof of the Theorem imply two other characterizations of the Paley-Wiener spaces.
Corollary 3.5. The following holds true:

1. a function $f \in L^2(X)$ belongs to $PW_\omega(X)$ if and only if $f \in H^\infty(X)$ and the upper bound

$$
\sup_{k \in \mathbb{N}} (\omega^2 + \|\rho\|^2)^{-k} \|\Delta^k f\| < \infty
$$

is finite,

2. a function $f \in L^2(X)$ belongs to $PW_\omega(X)$ if and only if $f \in H^\infty(X)$ and

$$
\lim_{k \to \infty} \|\Delta^k f\|^{1/k} = \omega^2 + \|\rho\|^2 < \infty.
$$

In this case $\omega = \omega_f$.

In [PT] the following Lemma was proved.

Lemma 3.6. There exists a natural number $N = N(X) \in \mathbb{N}$ such that for any sufficiently small $r > 0$ there exists a set of points $\{x_j\}$ from $X$ with the following properties:

1. the balls $B(x_j, r/4)$ are disjoint,
2. the balls $B(x_j, r/2)$ form a cover of $X$,
3. the multiplicity of the cover by balls $B(x_j, r)$ is not greater $N$.

We will use notation $Z = Z(r, N)$ for any set of points $\{x_j\} \in X$ which satisfies the properties (1)- (3) from the last Lemma and we will call such set a metric $(r, N)$-lattice of $X$.

If $\delta_{x_j}$ is a Dirac distribution at a point $x_j \in X$ then according to the inversion formula for the Helgason-Fourier transform we have

$$
\langle \delta_{x_j}, f \rangle = w^{-1} \int_{\mathbb{R}^* \times B} \hat{f}(\lambda, b)e^{i(\lambda + \rho)(A(x_j, b))}|c(\lambda)|^{-2}d\lambda db.
$$

It implies that if $f \in L^2(X)$ then the action on $\hat{f}(\lambda, b)$ of the Helgason-Fourier transform $\delta_{x_j}$ of $\delta_{x_j}$ is given by the formula

$$
\hat{f}(\lambda, b) \to \langle \delta_{x_j}, \hat{f} \rangle = w^{-1} \int_{\mathbb{R}^* \times B} e^{i(\lambda + \rho)(A(x_j, b))}\hat{f}(\lambda, b)|c(\lambda)|^{-2}d\lambda db.
$$

We introduce the notation $k^\omega_{x_j}$ for a function which is a restriction of the smooth function $\hat{\delta}_{x_j}$ to the set $\Pi_\omega$:

$$
k^\omega_{x_j} = \hat{\delta}_{x_j}|_{\Pi_\omega},
$$

and

$$
\langle k^\omega_{x_j}, \hat{f} \rangle = w^{-1} \int_{\mathbb{R}^* \times B} \chi_\omega(\lambda)e^{i(\lambda + \rho)(A(x_j, b))}\hat{f}(\lambda, b)|c(\lambda)|^{-2}d\lambda db, f \in L^2(X),
$$

where $\chi_\omega$ is the characteristic function of the set $\Pi_\omega$.

Theorem 3.7. There exists a constant $c(X)$ such that for any given $\omega > 0$, for every $(r, N)$-lattice $Z_\omega = Z(r, N)$ with

$$
r = c(X)(\omega^2 + \|\rho\|^2)^{-1/2},
$$

the set of functions $\{k^\omega_{x_j}\}, x_j \in Z_\omega$, is a frame in the space

$$
\Lambda_\omega = L^2(\Pi_\omega; |c(\lambda)|^{-2}d\lambda db)
$$
and there exists a frame \( \{ \Theta_{x_j} \} \) in the space \( PW_\omega(X) \) such that every \( \omega \)-band limited function \( f \in PW_\omega(X) \) can be reconstructed from a set of samples \( f(x_j) = \langle \delta_{x_j}, f \rangle \) by using the formula

\[
(3.19) \quad f = \sum_{x_j \in Z_\omega} f(x_j) \Theta_{x_j}.
\]

**Proof.** It was shown in [PI] that for any \( k > d/2 \) there exist constants \( C_1(X) > 0, C_2(X) > 0, r_0(X) > 0 \), such that for any for any \( k > d/2 \), any \( 0 < r < r_0(X) \) and any \((r,N)\)-lattice \( Z = Z(r,N) \) the following inequality holds true

\[
(3.20) \quad \|f\| \leq C_1(X) \left\{ r^{d/2} \left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2} + r^k \|\Delta f\| \right\},
\]

where \( f \in H^k(X), k > d/2, \) and

\[
\left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2} \leq C_2(X) \|f\|_{H^k(X)}, f \in H^k(X), k > d/2.
\]

Along with the Bernstein inequality \([6,14]\) it implies that there exist positive constants \( c(X), c_1(X), c_2(X) \), such that for every \( \omega > 0 \), every lattice \( Z_\omega = Z(r,N) \) with \( r = c(X)(\omega^2 + \|\rho\|^2)^{-1/2} \) and every \( f \in PW_\omega(X) \) the following inequalities hold true

\[
(3.21) \quad c_1(X) \left( \sum_{x_j \in Z_\omega} |f(x_j)|^2 \right)^{1/2} \leq r^{-d/2} \|f\|_2 \leq c_2(X) \left( \sum_{x_j \in Z_\omega} |f(x_j)|^2 \right)^{1/2}.
\]

An application of the Plancherel formula gives that there are constants \( A_1(X) > 0, A_2(X) > 0 \), such that for any \( f \in PW_\omega(X) \)

\[
(3.22) \quad A_1(X) \|f\|_{L_\omega} \leq \left( \sum_{x_j \in Z_\omega} \left| \langle \delta_{x_j}, \hat{f} \rangle_{L_\omega} \right|^2 \right)^{1/2} \leq A_2(X) \|\hat{f}\|_{L_\omega},
\]

where \( \langle \cdot, \cdot \rangle \) is the scalar product in the space \( L_\omega = L_2(\Pi_\omega; |c(\lambda)|^{-2}d\lambda db) \).

Since \( k_{x_j}^\omega = \delta_{x_j}|_{\Pi_\omega} \in L_\omega \) and since for \( f \in PW_\omega(X) \)

\[
\langle \delta_{x_j}, \hat{f} \rangle_{L_\omega} = \langle k_{x_j}^\omega, \hat{f} \rangle_{L_\omega},
\]

the inequalities \(3.22\) show that the set of functions \( \{k_{x_j}^\omega, x_j \in Z_\omega \} \), is a frame in the space \( L_\omega \).

It is known \([DS]\) that to construct a dual frame one has to consider the so called frame operator

\[
F(\hat{f}) = \sum_{x_j \in Z_\omega} \langle k_{x_j}^\omega, \hat{f} \rangle_{L_\omega} k_{x_j}^\omega.
\]
One can show that the frame operator $F$ is invertible and the formula
\[ (3.23) \hat{\Theta}_{x_j} = F^{-1} k_{x_j}^\omega \]
gives a dual frame $\Lambda_\omega$. A reconstruction formula of a function $f$ can be written in terms of the dual frame as
\[ (3.24) \hat{f} = \sum_{x_j \in Z_\omega} \langle \hat{\Theta}_{x_j}, \hat{f} \rangle_{\Lambda_\omega} k_{x_j}^\omega = \sum_{x_j \in Z_\omega} \langle k_{x_j}^\omega, \hat{f} \rangle_{\Lambda_\omega} \hat{\Theta}_{x_j}, \]
where inner product is taken in the space $\Lambda_\omega$.

Functions $\hat{\Theta}_{x_j}$ belong to the space $\Lambda_\omega = L_2([\omega, \omega])$. By extending them by zero outside of the set $\Pi_\omega$ we can treat them as functions in $L_2(\mathbb{R}, \|c(\lambda)\|^{-2}d\lambda db)$ and then by taking the inverse Helgason-Fourier transform $H^{-1}$ of both sides of the formula (3.24) we obtain the formula (3.19) of our Theorem because for $f \in PW_\omega(X)$ the Plancherel Theorem gives
\[ \langle k_{x_j}^\omega, \hat{f} \rangle_{\Lambda_\omega} = \langle \hat{\delta}_{x_j}, \hat{f} \rangle_{L_2(\mathbb{R}, d\mu)} = \langle \delta_{x_j}, f \rangle = f(x_j). \]
\[ \square \]

In the classical case when $X = \mathbb{R}$ is the one-dimensional Euclidean space we have
\[ (3.25) \hat{\delta}_{x_j}(\lambda) = e^{ix_j \lambda}. \]
In this situation the Theorem means that the complex exponentials $e^{ix_j \lambda}, x_j \in Z(r, N)$ form a frame in the space $L_2([-\omega, \omega])$. Note that in the case of a uniform point-wise sampling in the space $L_2(\mathbb{R})$ this result gives the classical sampling formula
\[ f(t) = \sum f(\gamma n \Omega) \frac{\sin(\omega(t - \gamma n \Omega))}{\omega(t - \gamma n \Omega)}, \Omega = \pi/\omega, \gamma < 1, \]
with a certain oversampling.

The formula (3.24) can be treated as a Discrete Helgason-Fourier transform in the following sense.

**Corollary 3.8.** There exists a constant $c(X)$ such that for any given $\omega > 0$, for every $(r, N)$-lattice $Z_\omega = Z(r, N)$ with
\[ r = c(X)(\omega^2 + \|\rho\|^2)^{-1/2}, \]
there exist functions $\{\hat{\Theta}_{x_j}\}, x_j \in Z_\omega$, in the space $PW_\omega(X)$ such that
\[ (3.26) \hat{f} = \sum_{x_j \in Z_\omega} f(x_j) \hat{\Theta}_{x_j}, \]
for all $f \in PW_\omega(X)$.

The formula (3.19) can be used to obtain the following quadrature rule for functions from $PW_\omega(X), \omega > 0$.

**Corollary 3.9.** There exists a constant $c(X)$ such that for any given $\omega > 0$, for every $(r, N)$-lattice $Z_\omega = Z(r, N)$ with
\[ r = c(X)(\omega^2 + \|\rho\|^2)^{-1/2}, \]
and for every compact $U \subset X$ there exist a set of numbers $\{w_j\}$ such that
(3.27) \[ \int_U f \, dx = \sum_{x_j \in \mathbb{Z}_\omega} f(x_j) w_{x_j}, \]

for all \( f \in PW_\omega(X) \). The weights \( w_{x_j} \) are given by the formulas

\[ w_{x_j} = \int_U \Theta_{x_j} \, dx, \]

where \( \Theta_{x_j} \) are the functions from (3.19).

As another consequence we obtain the following quadrature formula on the Fourier transform-side.

**Corollary 3.10.** For any measurable set \( V \subset \Pi_\omega \) there exist weights \( v_{x_j} = \int_V \hat{\Theta}_{x_j}(\lambda, b)|c(\lambda)|^{-2} \, d\lambda db \) such that

\[ \int_V \hat{f}(\lambda, b)|c(\lambda)|^{-2} \, d\lambda db = \sum_{x_j \in \mathbb{Z}_\omega} f(x_j) v_{x_j} \]

for all \( f \in PW_\omega(X) \).

4. A Direct Approximation Theorem for quasi-normed linear spaces

The goal of the section is to establish certain connections between interpolation spaces and approximation spaces which will be used later to develop an approximation theory on a symmetric space \( X \). The general theory of interpolation spaces can be found in [BL], [PS], [KPS], [T3]. The notion of approximation spaces and their relations to interpolation spaces can be found in [BL], Ch. 3 and 7, and in [PS]. As it was explained in the Introduction we use the language of quasi-normed linear spaces not because we want to achieve a bigger generality but because this language allows to treat simultaneously interpolation and approximation spaces.

Let \( E \) be a linear space. A quasi-norm \( \| \cdot \|_E \) on \( E \) is a real-valued function on \( E \) such that for any \( f, f_1, f_2 \in E \) the following holds true

1. \( \| f \|_E \geq 0; \)
2. \( \| f \|_E = 0 \iff f = 0; \)
3. \( \| f \|_E = \| -f \|_E; \)
4. \( \| f_1 + f_2 \|_E \leq C_E(\| f_1 \|_E + \| f_2 \|_E), C_E > 1. \)

We say that two quasi-normed linear spaces \( E \) and \( F \) form a pair, if they are linear subspaces of a linear space \( A \) and the conditions \( \| f_k - g \|_E \to 0 \), and \( \| f_k - h \|_F \to 0 \), \( f_k, g, h \in A \), imply equality \( g = h \). For a such pair \( E, F \) one can construct a new quasi-normed linear space \( E \cap F \) with quasi-norm

\[ \| f \|_{E \cap F} = \max(\| f \|_E, \| f \|_F) \]

and another one \( E + F \) with the quasi-norm

\[ \| f \|_{E + F} = \inf_{f = f_0 + f_1, f_0 \in E, f_1 \in F} (\| f_0 \|_E + \| f_1 \|_F). \]
All quasi-normed spaces \( H \) for which \( E \cap F \subset H \subset E + F \) are called intermediate between \( E \) and \( F \). A group homomorphism \( T : E \to F \) is called bounded if
\[
\|T\| = \sup_{f \in E, f \neq 0} \|Tf\|_F/\|f\|_E < \infty.
\]
One says that an intermediate quasi-normed linear space \( H \) interpolates between \( E \) and \( F \) if every bounded homomorphism \( T : E + F \to E + F \) which is a bounded homomorphism of \( E \) into \( E \) and a bounded homomorphism of \( F \) into \( F \) is also a bounded homomorphism of \( H \) into \( H \).

On \( E + F \) one considers the so-called Peetre’s \( K \)-functional
\[
(4.1) \quad K(f, t) = K(f, t, E, F) = \inf_{\beta_0 + f_1 \in E, f_1 \in F} (\|f_0\|_E + t\|f_1\|_F).
\]
The quasi-normed linear space \( (E, F)_{\theta,q}^K, 0 < \theta < 1, 0 < q \leq \infty, \) or \( 0 \leq \theta \leq 1, q = \infty \), is introduced as a set of elements \( f \) in \( E + F \) for which
\[
(4.2) \quad \|f\|_{\theta,q} = \left( \int_0^\infty (t^{-\theta} K(f, t))^{q} \frac{dt}{t} \right)^{1/q}.
\]
It turns out that \( (E, F)_{\theta,q}^K, 0 < \theta < 1, 0 < q \leq \infty, \) or \( 0 \leq \theta \leq 1, q = \infty \), with the quasi-norm \((4.2)\) interpolates between \( E \) and \( F \). The following Reiteration Theorem is one of the main results of the theory (see BL, PS, KPS, T3).

**Theorem 4.1.** Suppose that \( E_0, E_1 \) are complete intermediate quasi-normed linear spaces for the pair \( E, F \). If \( E_i \in K(\theta_i, E, F) \) which means
\[
K(f, t, E, F) \leq C\theta_i \|f\|_{E_i}, i = 0, 1,
\]
where \( 0 \leq \theta_i \leq 1, \theta_0 \neq \theta_1, \)
then
\[
(E_0, E_1)_{\eta,q}^K \subset (E, F)_{\theta,q}^K,
\]
where \( 0 < q < \infty, 0 < \eta < 1, \theta = (1 - \eta)\theta_0 + \eta\theta_1. \)

If for the same pair \( E, F \) and the same \( E_0, E_1 \) one has \( E_i \in J(\theta_i, E, F) \) that means
\[
\|f\|_{E_i} \leq C\|f\|_{E}^{1-\theta_i} \|f\|_{F}^{\theta_i}, i = 0, 1,
\]
where \( 0 \leq \theta_i \leq 1, \theta_0 \neq \theta_1, \)
then
\[
(E, F)_{\eta,q}^K \subset (E_0, E_1)_{\eta,q}^K,
\]
where \( 0 < q < \infty, 0 < \eta < 1, \theta = (1 - \eta)\theta_0 + \eta\theta_1. \)

It is important to note that in all cases which will be considered in the present article the space \( F \) will be continuously embedded as a subspace into \( E \). In this case \((4.1)\) can be introduced by the formula
\[
K(f, t) = \inf_{f_1 \in F} (\|f - f_1\|_E + t\|f_1\|_F),
\]
which implies the inequality
\[
(4.3) \quad K(f, t) \leq \|f\|_E.
\]
This inequality can be used to show that the norm \((4.2)\) is equivalent to the norm
\[
(4.4) \quad \|f\|_{\theta,q} = \|f\|_E + \left( \int_0^\infty (t^{-\theta} K(f, t))^{q} \frac{dt}{t} \right)^{1/q}, \varepsilon > 0,
\]
for any positive \( \varepsilon \).
Let us introduce another functional on \( E + F \), where \( E \) and \( F \) form a pair of quasi-normed linear spaces
\[
E(f, t) = \mathcal{E}(f, t, E, F) = \inf_{g \in F, \|g\|_F \leq t} \|f - g\|_E.
\]

**Definition 4.2.** The approximation space \( E_{\alpha,q}(E, F) \), \( 0 < \alpha < \infty, 0 < q \leq \infty \) is a quasi-normed linear spaces of all \( f \in E + F \) with the following quasi-norm
\[
(\int_0^\infty (t^\alpha E(f, t))^q \frac{dt}{t})^{1/q}.
\]

For a general quasi-normed linear spaces \( E \) the notation \((E)^\rho\) is used for a quasi-normed linear spaces whose quasi-norm is \( \| \cdot \|_\rho \).

The following Theorem describes relations between interpolation and approximation spaces (see [BL], Ch. 7).

**Theorem 4.3.** If \( \theta = 1/(\alpha + 1) \) and \( r = \theta q \), then
\[
(\mathcal{E}_{\alpha,q}(E, F))^0 = (E, F)^K_{\theta, q}.
\]

The following important result is known as the Power Theorem (see [BL], Ch. 7).

**Theorem 4.4.** Suppose that the following relations satisfied: \( \nu = \eta \rho_1/\rho \), \( \rho = (1 - \eta)\rho_0 + \eta \rho_1 \), and \( q = \rho r \) for \( \rho_0 > 0, \rho_1 > 0 \). Then, if \( 0 < \eta < 1, 0 < r \leq \infty \), the following equality holds true
\[
((E)^\rho_0, (F)^\rho_1)^K_{\eta, r} = ((E, F)^K_{\nu, q})^\rho.
\]

Theorem we prove next represents a very general version of a Jackson-type Theorem (see [KP], [P4], [P5]). Such type results are known as Direct Approximation Theorems.

**Theorem 4.5.** Suppose that \( \mathcal{P} \subset F \subset E \) are quasi-normed linear spaces and \( E \) and \( F \) are complete. Suppose that there exist \( C > 0 \) and \( \beta > 0 \) such that for any \( f \in F \) the following Jackson-type inequality verified
\[
t^\beta \mathcal{E}(t, f, \mathcal{P}, E) \leq C\|f\|_F, t > 0,
\]
then the following embedding holds true
\[
(E, F)^K_{\theta, q} \subset \mathcal{E}_{\theta \beta, q}(E, \mathcal{P}), 0 < \theta < 1, 1 < q \leq \infty.
\]

**Proof.** It is known ([BL], Ch.7) that for any \( s > 0 \), for
\[
t = K_\infty(f, s) = K_\infty(f, s, \mathcal{P}, E) = \inf_{f=f_1+f_2, f_1 \in \mathcal{P}, f_2 \in E} \max(||f_1\|_{\mathcal{P}} ||f_2\|_E)
\]
the following inequality holds
\[
s^{-1}K_\infty(f, s) \leq \lim_{\tau \to t-0} \inf \mathcal{E}(f, \tau, E, \mathcal{P}).
\]

Since
\[
K_\infty(f, s) \leq K(f, s) \leq 2K_\infty(f, s),
\]
the Jackson-type inequality (4.6) and the inequality (4.9) imply
\[
s^{-1}K(f, s, \mathcal{P}, E) \leq Ct^{-\beta}||f||_F.
\]
The equality \( (4.8) \) and inequality \( (4.10) \) imply the estimate
\[
(4.12) \quad t^{-\beta} \leq 2^\beta (K(f, s, P, E))^{-\beta}
\]
which along with the previous inequality gives the estimate
\[
K^{1+\beta}(f, s, P, E) \leq C s \| f \|_F
\]
which in turn imply the inequality
\[
(4.13) \quad K(f, s, P, E) \leq C s^{1+\beta} \| f \|_F
\]
At the same time one has
\[
(4.14) \quad K(f, s, P, E) = \inf_{f = f_0 + f_1, f_0 \in P, f_1 \in E} (\| f_0 \|_P + s \| f_1 \|_E) \leq s \| f \|_E,
\]
for every \( f \) in \( E \). The inequality \( (4.13) \) means that the quasi-normed linear space \((F)^{1+\beta}\) belongs to the class \( K(1^{1+\beta}, P, E) \) and \( (4.14) \) means that the quasi-normed linear space \( E \) belongs to the class \( K(1, P, E) \). This fact allows to use the Reiteration Theorem to obtain the embedding
\[
(4.15) \quad ((F)^{1+\beta}, E)^{K_{\theta,q}(1+\theta \beta)} \subset (P, E)^{K_{\theta,q}(1+\theta \beta)}
\]
for every \( 0 < \theta < 1, 1 < q < \infty \). But the space on the left is the space
\[
((E, (F)^{1+\beta})^{K_{\theta,q}(1+\theta \beta)},
\]
which is according to the Power Theorem is the space
\[
((E, (F)^{1+\beta})^{K_{\theta,q}(1+\theta \beta)},
\]
All these results along with the equivalence of interpolation and approximation spaces give the embedding
\[
(E, F)^{K_{\theta,q}} \subset ((P, E)^{K_{\theta,q}(1+\theta \beta)})^{1+\theta \beta} = \mathcal{E}_{\theta \beta,q}(E, P),
\]
which proves the Theorem. \( \square \)

5. Approximation by Paley-Wiener functions on noncompact symmetric spaces

The Helgason-Fourier transform can be treated as a unitary operator from the space \( L^2(X) \) onto the space \( L^2(a^*_+, |c(\lambda)|^{-2} d\lambda) \) of abstract-valued functions
\[
\hat{f}(\lambda, \cdot) : a^*_+ \to L^2(B, db).
\]
Define the support of \( \hat{f}(\lambda, \cdot) \) as the complement of the maximal open set \( \mathcal{U} \subset a^* \) such that \( \hat{f}(\lambda, \cdot) = 0 \) for almost all \( \lambda \in \mathcal{U} \).

Consider the space
\[
PW(X) = \bigcup_{t>0} PW_t(X),
\]
which is a quasi-normed linear space with respect to the quasi-norm
\[
(5.1) \quad \| f \|_{PW(X)} = \sup \left\{ \| \lambda \|^2 = \langle \lambda, \lambda \rangle : \lambda \in supp \hat{f}(\lambda, \cdot) \right\},
\]
where \( \langle \cdot, \cdot \rangle \) is the Killing form on \( a^* \).
The corresponding approximation functional takes the form
\[ E(f,t) = E(f,t,PW(X),L_2(X)) = \inf_{g \in PW(X)} \| f - g \|, f \in L_2(X). \]

The next goal is to introduce Besov spaces on \( X \) in terms of the group \( e^{is\sqrt{-\Delta}} \) generated in \( L_2(X) \) by the operator \( \sqrt{-\Delta} \). Consider a difference operator of order \( r \in \mathbb{N} \) as
\[ \left( I - e^{is\sqrt{-\Delta}} \right)^r f = (-1)^{r+1} \sum_{k=0}^{r} (-1)^{k-1} C_r^k e^{i k s \sqrt{-\Delta}} f, f \in L_2(X). \]
and the modulus of continuity which is defined as
\[ \Omega_r(f,s) = \sup_{s \leq \tau} \left\| \left( I - e^{i \tau \sqrt{-\Delta}} \right)^r f \right\|. \]

The Besov space \( B^\alpha_{q,r}(X), 1 \leq q \leq \infty, \alpha > 0, \) as an interpolation space between \( L_2(X) \) and Sobolev space \( H^r(X), \)
\begin{equation}
(5.2) \quad B^\alpha_{q,r}(X) = (L_2(X), H^r(X))_{\alpha/r,q}^K,
\end{equation}
where \( r \in \mathbb{N}, 0 < \alpha < r, 1 \leq q < \infty \), or \( 0 \leq \alpha \leq r, q = \infty \).

The fact that the Sobolev space \( H^r(X) \) is the domain of a self-adjoint operator \((-\Delta)^{r/2}\) implies (see \[BL, BB, KPS, T3\]) that this definition \[5.2\] is independent on \( r \) and it is the reason that \( r \) does not appear on the left side of the last formula. Furthermore the Besov norms can be given by the following formulas (see \[BL, BB, KPS, T3\])
\begin{equation}
(5.3) \quad \| f \|_{B^\alpha_{q,r}(X)} = \| f \| + \left( \int_0^\infty \left( s^{-\alpha} \Omega_r(f,s) \right)^q \frac{ds}{s} \right)^{1/q},
\end{equation}
where \( 0 < \alpha < r, 1 \leq q < \infty \) or
\begin{equation}
(5.4) \quad \| f \|_{B^\alpha_{r,\infty}(X)} = \| f \| + \sup_{0 < s < \infty} \left( s^{-\alpha} \Omega_r(f,s) \right),
\end{equation}
where \( 0 \leq \alpha \leq r, q = \infty \). Note that many other descriptions of Besov spaces on complete manifolds and symmetric spaces were given in \[S1, S3 \] and \[T1, T3\].

Our goal is to prove the following Theorem which gives description of Besov spaces in terms of the best approximations by Paley-Wiener functions.

**Theorem 5.1.** The following embedding holds true
\begin{equation}
(5.5) \quad B^\alpha_{q,r}(X) \subset \mathcal{E}_{\alpha,q}(PW(X),L_2(X)),
\end{equation}
where \( 0 < \alpha < \infty, 1 \leq q \leq \infty \). In other words there exists a constant \( C(X) \) such that for all \( f \in B^\alpha_{q,r}(X) \) the following inequality holds
\begin{equation}
(5.6) \quad \left( \int_0^\infty \left( s^\alpha \mathcal{E}(f,s) \right)^q \frac{ds}{s} \right)^{1/q} \leq C(X) \left( \| f \| + \left( \int_0^\infty \left( s^{-\alpha} \Omega_r(f,s) \right)^q \frac{ds}{s} \right)^{1/q} \right),
\end{equation}
for any \( 0 < \alpha < r \in \mathbb{N} \) if \( 1 \leq q < \infty \) and any \( 0 < \alpha \leq r \in \mathbb{N} \) if \( 1 \leq q \leq \infty \).

**Proof.** We have to prove the following embedding
\begin{equation}
(5.7) \quad (L_2(X), H^r(X))_{\alpha/r,q}^K \subset \mathcal{E}_{\alpha,q}(PW(X),L_2(X)),
\end{equation}
where \( \alpha < r \in \mathbb{N}, 1 < q \leq \infty \). In order to be able to apply our Theorem 4.4 we are going to verify the Jackson-type inequality \[4.3\].
Let \( \chi_t \) be the characteristic function of the set \( \Pi_t \). According to the Plancherel Theorem
\[
\mathcal{E}(f, t, PW(X), L_2(X)) = \inf_{g \in PW(X)} \| f - g \|_{L_2(X)} =
\]
(5.8)
\[
\inf_{g \in PW(X)} \| \hat{f} - \hat{g} \|_{L_2(a^* \times B, d\mu)},
\]
where \( d\mu = |c(\lambda)|^{-2} d\lambda db \). But it is obvious that the inf in the last formula is achieved exactly when \( \hat{g} = \chi_t \hat{f} \). Since the Sobolev space \( H^r(X) \) is the domain of \((-\Delta)^{r/2}\) we obtain the following inequalities for every \( f \in H^r(X) \)
\[
\mathcal{E}(f, t, PW(X), L_2(X)) =
\]
(5.9)
\[
\left( t^2 + \| \rho \|^2 \right)^{-r/2} \| f \|_r \leq t^{-r} \| f \|_r, r \in \mathbb{N},
\]
that shows that the Jackson-type inequality (4.6) is satisfied and \( \beta = r \). Thus, by Theorem 4.4 and (5.2) we obtain (5.5) and the claim follows. \( \square \)

As a consequence of this result in the case \( q = \infty \) we obtain the following Corollary.

**Corollary 5.2.** If for a function \( f \in L_2(X) \) the following relation holds
(5.10)
\[
\Omega_r(f, s) = O(s^\alpha), s \to 0.
\]
for an \( 0 < \alpha \leq r \in \mathbb{N} \), then the following relations holds
(5.11)
\[
\mathcal{E}(f, s, PW(X), L_2(X)) = O(s^{-\alpha}), s \to \infty.
\]

Let \( c(X) \) will be the same constant as in Theorem 3.4. We introduce the functional
(5.12)
\[
\Phi(f; Z_\omega) = \left\| \hat{f} - \sum_{x_j \in Z_\omega} f_\omega(x_j) \Theta_{x_j} \right\|_{L_2(a^* \times B, d\mu)}, \omega > 0,
\]
where \( Z_\omega = Z(r, N) \) is any lattice with \( r = c(X)(\omega^2 + \| \rho \|^2)^{-1/2} \) and \( \{ \Theta_{x_j} \} \) is the corresponding frame in the space \( \Lambda_\omega = L_2(\Pi_\omega; |c(\lambda)|^{-2} d\lambda db) \).

The following Theorem is a consequence of (1), (1.5), and Theorem 5.1.

**Theorem 5.3.** There exists a constant \( C_0(X) \) and for any \( \omega > 0 \) there exists a sequence of \( (r, N) \)-lattices \( Z_\omega = Z(r, N) \) with \( r = c(X)(\omega^2 + \| \rho \|^2)^{-1/2} \) such that for any \( f \in B_2^\alpha(X), 0 < \alpha < \infty, 1 \leq q \leq \infty \) the following inequality holds
(5.13)
\[
\left( \int_0^\infty (\omega^q \Phi(f; Z_\omega))^q \frac{d\omega}{\omega} \right)^{1/q} \leq C_0(X) \| f \|_{B_2^\alpha(X)},
\]
where functional \( \Phi(f; Z_\omega) \) is defined in (5.12).

Since the Sobolev space \( H^r(X) \) is the domain of the self-adjoint operator \((-\Delta)^{r/2}\) in the Hilbert space \( L_2(X) \) the general theory of interpolation spaces [KPS], [T3], implies the isomorphism
\[
B_2^\alpha(X) = (L_2(X), H^r(X))_{\alpha/r, 2}^K = H^\alpha(X).
\]
Using this fact we obtain the Theorem 1.1 as a consequence of Theorem 5.2 and Corollary 5.1.

6. Acknowledgment

I would like to thank the anonymous referee for constructive suggestions.

References

[A] N. Andersen, Real Paley-Wiener theorem for the inverse Fourier transform on a Riemannian symmetric space, Pacific J. Math., 213 (2004), 1-13.

[BL] J. Bergh, J. Lofstrom, Interpolation spaces, Springer-Verlag, 1976.

[BB] P. Butzer, H. Berens, Semi-Groups of operators and approximation, Springer, Berlin, 1967.

[DS] R. Duffin, A. Schaeffer, A class of nonharmonic Fourier series, Trans. AMS, 72, (1952), 341-366.

[H1] S. Helgason, A duality for symmetric spaces with applications to group representations, Adv. Math. 5, (1970), 1-154.

[H2] S. Helgason, Differential Geometry and Symmetric Spaces, Academic, N.Y., 1962.

[H3] S. Helgason, The Abel, Fourier and Radon transforms on symmetric spaces, Indag. Mathem., N. S., 16(3-4), 2005, 531-551.

[KP] S. Krein, I. Pesenson, Interpolation Spaces and Approximation on Lie Groups, The Voronezh State University, Voronezh, 1990, (Russian).

[KPS] S. Krein, Y. Petunin, E. Semenov, Interpolation of linear operators, Translations of Mathematical Monographs, 54. AMS, Providence, R.I., 1982.

[N] S. M. Nikolskii, Approximation of functions of several variables and imbedding theorems, Springer, Berlin, 1975.

[Pa] A. Pasquale, A Paley-Wiener theorem for the inverse spherical transform, Pacific J. Math., 193 (2000), 143-176.

[PS] J. Peetre, G. Sparr, Interpolation on normed Abelian groups, Ann. Mat. Pura Appl. 92 (1972), 217-262.

[P1] I. Pesenson, A sampling theorem on homogeneous manifolds, Trans. of AMS, Vol. 352(9), (2000), 4257-4270.

[P2] I. Pesenson, Deconvolution of band limited functions on noncompact symmetric spaces, Houston J. of Math., 32, No. 1, (2006), 183-204.

[P3] I. Pesenson, Frames in Paley-Wiener spaces on Riemannian manifolds, in Integral Geometry and Tomography, Contemp. Math., 405, AMS, (2006), 137-153.

[P4] I. Pesenson, Best approximations in a space of the representation of a Lie group, (Russian) Dokl. Akad. Nauk SSSR 302 (1988), no. 5, 1055-1058; translation in Soviet Math. Dokl. 38 (1989), no. 2, 384-388.

[P5] I. Pesenson, Approximations in the representation space of a Lie group, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1990, , no. 7, 43-50; translation in Soviet Math. (Iz. VUZ) 34 (1990), no. 7, 49–57.

[P6] I. Pesenson, Paley-Wiener approximations and discrete approximations in Sobolev and Besov spaces on manifolds, preprint.

[S1] L. Skrzypczak, Function spaces of Sobolev type on Riemannian symmetric manifolds, Forum Math. 3 (1991), no. 4, 339–353.

[S2] L. Skrzypczak, Some equivalent norms in Sobolev-Besov spaces on symmetric Riemannian manifolds, J. London Math. Soc. (2) 53 (1996), no. 3, 569–581.

[S3] L. Skrzypczak, Heat and harmonic extensions for function spaces of Hardy-Sobolev-Besov type on symmetric spaces and Lie groups, J. Approx. Theory 96 (1999), no. 1, 149–170.

[T1] H. Triebel, Spaces of Hardy-Sobolev-Besov type on complete Riemannian manifolds, Ark. Mat. 24, (1986), 299-337.

[T2] H. Triebel, Function spaces on Lie groups, J. London Math. Soc. 35, (1987), 327-338.

[T3] H. Triebel, Theory of function spaces II, Monographs in Mathematics, 84. Birkhuser Verlag, Basel, 1992.