The Lattice-Theoretic Essence of Property Directed Reachability Analysis *

Mayuko Kori, Natsuki Urabe, Shin-ya Katsumata, Kohei Suenaga, and Ichiro Hasuo

The Graduate University for Advanced Studies (SOKENDAI), Hayama, Japan
National Institute of Informatics, Tokyo, Japan
Kyoto University, Kyoto, Japan

Abstract. We present LT-PDR, a lattice-theoretic generalization of Bradley’s property directed reachability analysis (PDR) algorithm. LT-PDR identifies the essence of PDR to be an ingenious combination of verification and refutation attempts based on the Knaster–Tarski and Kleene theorems. We introduce four concrete instances of LT-PDR, derive their implementation from a generic Haskell implementation of LT-PDR, and experimentally evaluate them. We also present a categorical structural theory that derives these instances.

Keywords: property directed reachability analysis · model checking · lattice theory · fixed point theory · category theory

1 Introduction

Property directed reachability (PDR) (also called IC3) introduced in [9,13] is a model checking algorithm for proving/disproving safety problems. It has been successfully applied to software and hardware model checking, and later it has been extended in several directions, including fbPDR [25,26] that uses both forward and backward predicate transformers and PrIC3 [6] for the quantitative safety problem for probabilistic systems. See [14] for a concise overview.

The original PDR assumes that systems are given by binary predicates representing transition relations. The PDR algorithm maintains data structures called frames and proof obligations—these are collections of predicates over states—and updates them. While this logic-based description immediately yields automated tools using SAT/SMT solvers, it limits target systems to qualitative and non-deterministic ones. This limitation was first overcome by PrIC3 [6] whose target is probabilistic systems. This suggests room for further generalization of PDR.

* The authors are supported by ERATO HASUO Metamathematics for Systems Design Project (No. JPMJER1603). MK is a JSPS DC fellow and supported by JSPS KAKENHI Grant (No. 22J21742). KS is supported by JST CREST Grant (No. JP-MJCR2012) and JSPS KAKENHI Grant (No. 19H04084).
In this paper, we propose the first lattice theory-based generalization of the PDR algorithm; we call it \textit{LT-PDR}. This makes the PDR algorithm apply to a wider class of safety problems, including qualitative and quantitative. We also derive a new concrete extension of PDR, namely one for Markov reward models.

We implemented the general algorithm LT-PDR in Haskell, in a way that maintains the theoretical abstraction and clarity. Deriving concrete instances for various types of systems is easy (for Kripke structures, probabilistic systems, etc.). We conducted an experimental evaluation, which shows that these easily-obtained instances have at least reasonable performance.

**Preview of the Theoretical Contribution** We generalize the PDR algorithm so that it operates over an arbitrary complete lattice $L$. This generalization recasts the PDR algorithm to solve a general problem $\mu F \leq^? \alpha$ of over-approximating the least fixed point of an $\omega$-continuous function $F: L \to L$ by a safety property $\alpha$. This lattice-theoretic generalization signifies the relationship between the PDR algorithm and the theory of fixed points. This also allows us to incorporate quantitative predicates suited for probabilistic verification.

More specifically, we reconstruct the original PDR algorithm as a combination of two constituent parts. They are called \textit{positive LT-PDR} and \textit{negative LT-PDR}. Positive LT-PDR comes from a witness-based proof method by the \textit{Knaster–Tarski fixed point theorem}, and aims to verify $\mu F \leq^? \alpha$. In contrast, negative LT-PDR comes from the \textit{Kleene fixed point theorem} and aims to refute $\mu F \leq^? \alpha$. The two algorithms build up witnesses in an iterative and nondeterministic manner, where nondeterminism accommodates guesses and heuristics. We identify the essence of PDR to be an ingenious combination of these two algorithms, in which intermediate results on one side (positive or negative) give informed guesses on the other side. This is how we formulate LT-PDR in §3.3.

We discuss several instances of our general theory of PDR. We discuss three concrete settings: Kripke structures (where we obtain two instances of LT-PDR), Markov decision processes (MDPs), and Markov reward models. The two in the first setting essentially subsume many existing PDR algorithms, such as the original PDR \cite{9, 13} and Reverse PDR \cite{25, 26}, and the one for MDPs resembles PrIC3 \cite{6}. The last one (Markov reward models) is a new algorithm that fully exploits the generality of our framework.

In fact, there is another dimension of theoretical generalization: the derivation of the above concrete instances follows a \textit{structural theory of state-based dynamics and predicate transformers}. We formulate the structural theory in the language of \textit{category theory} \cite{3, 23}—using especially \textit{coalgebras} \cite{18} and \textit{fibrations} \cite{19}—following works such as \cite{8, 15, 22, 28}. The structural theory tells us which safety problems arise under what conditions; it can therefore suggest that certain safety problems are unlikely to be formulatable, too. The structural theory is important because it builds a mathematical order in the PDR literature, in which theoretical developments tend to be closely tied to implementation and thus theoretical essences are often not very explicit. For example, the theory is useful in classifying a plethora of PDR-like algorithms for Kripke structures (the original, Reverse PDR, fbPDR, etc.). See §5.1.
The Lattice-Theoretic Essence of Property Directed Reachability Analysis

We present the above structural theory in §4 and briefly discuss its use in the derivation of concrete instances in §5. We note, however, that this categorical theory is not needed for reading and using the other parts of the paper.

There are other works on generalization of PDR [17,24], but our identification of the interplay of Knaster–Tarski and Kleene is new. They do not accommodate probabilistic verification, either. See Appendix A for further discussions.

Preliminaries

Let \((L, \leq)\) be a poset. \((L, \leq)^{\text{op}}\) denotes the opposite poset \((L, \geq)\). Note that if \((L, \leq)\) is a complete lattice then so is \((L, \leq)^{\text{op}}\). An \(\omega\)-chain (resp. \(\omega^{\text{op}}\)-chain) in \(L\) is an \(\mathbb{N}\)-indexed family of increasing (resp. decreasing) elements in \(L\). A monotone function \(F: L \rightarrow L\) is \(\omega\)-continuous (resp. \(\omega^{\text{op}}\)-continuous) if \(F\) preserves existing suprema of \(\omega\)-chains (resp. infima of \(\omega^{\text{op}}\)-chains).

2 Fixed-points in Complete Lattices

Let \((L, \leq)\) be a complete lattice and \(F: L \rightarrow L\) be a monotone function. When we analyze fixed points of \(F\), pre/postfixed points play important roles.

Definition 2.1. A prefixed point of \(F\) is an element \(x \in L\) satisfying \(Fx \leq x\). A postfixed point of \(F\) is an element \(x \in L\) satisfying \(x \leq Fx\). We write \(\text{Pre}(F)\) and \(\text{Post}(F)\) for the set of prefixed points and postfixed points of \(F\), respectively.

The following results are central in fixed point theory. They allow us to under/over-approximate the least/greatest fixed points.

Theorem 2.2. A monotone endofunction \(F\) on a complete lattice \((L, \leq)\) has the least fixed point \(\mu_F\) and the greatest fixed point \(\nu_F\). Moreover,

1. (Knaster–Tarski [30]) The set of fixed points forms a complete lattice. Furthermore, \(\mu_F = \bigwedge \{x \in L \mid Fx \leq x\}\) and \(\nu_F = \bigvee \{x \in L \mid x \leq Fx\}\).
2. (Kleene, see e.g. [5]) If \(F\) is \(\omega\)-continuous, \(\mu_F = \bigvee_{n \in \mathbb{N}} F^n \bot\). Dually, if \(F\) is \(\omega^{\text{op}}\)-continuous, \(\nu_F = \bigwedge_{n \in \mathbb{N}} F^n \top\). \(\square\)

Thm. 2.2.2 is known to hold for arbitrary \(\omega\)-cpos (complete lattices are their special case). A generalization of Thm. 2.2.2 is the Cousot–Cousot characterization [11], where \(F\) is assumed to be monotone (but not necessarily \(\omega\)-continuous) and we have \(\mu_F = F^n \bot\) for a sufficiently large, possibly transfinite, ordinal \(\kappa\).

In this paper, for the algorithmic study of PDR, we assume the \(\omega\)-continuity of \(F\). Note that \(\omega\)-continuous \(F\) on a complete lattice is necessarily monotone.

We call the \(\omega\)-chain \(\bot \leq F \bot \leq \cdots\) the initial chain of \(F\) and the \(\omega^{\text{op}}\)-chain \(\top \geq F \top \geq \cdots\) the final chain of \(F\). These appear in Thm. 2.2.2

Thm. 2.2.2 and 2.2.2 yield the following witness notions for proving and disproving \(\mu_F \leq \alpha\), respectively.

Corollary 2.3. Let \((L, \leq)\) be a complete lattice and \(F: L \rightarrow L\) be \(\omega\)-continuous.

1. (KT) \(\mu_F \leq \alpha\) if and only if there is \(x \in L\) such that \(Fx \leq x \leq \alpha\).
2. (Kleene) \(\mu_F \not\leq \alpha\) if and only if there is \(n \in \mathbb{N}\) and \(x \in L\) such that \(x \leq F^n \bot\) and \(x \not\leq \alpha\). \(\square\)
By Cor. 2.3.1, proving \( \mu F \leq \alpha \) is reduced to searching for \( x \in L \) such that 
\[ Fx \leq x \leq \alpha. \]
We call such \( x \) a KT (positive) witness. In contrast, by Cor. 2.3.2, disproving \( \mu F \leq \alpha \) is reduced to searching for \( n \in \mathbb{N} \) and \( x \in L \) such that 
\[ x \leq F^n \perp \text{ and } x \not\leq \alpha. \]
We call such \( x \) a Kleene (negative) witness.

**Notation 2.4.** We shall use lowercase (Roman and Greek) letters for elements of \( L \) (such as \( \alpha, x \in L \)), and uppercase letters for (finite or infinite) sequences of \( L \) (such as \( X \in L^* \) or \( L^\omega \)). The \( i \)-th (or \((i - j)\)-th when subscripts are started from \( j \)) element of a sequence \( X \) is designated by a subscript: \( X_i \in L \).

### 3 Lattice-Theoretic Reconstruction of PDR

Towards the LT-PDR algorithm, we first introduce two simpler algorithms, called positive LT-PDR (§3.1) and negative LT-PDR (§3.2). The target problem of the LT-PDR algorithm is the following:

**Definition 3.1 (the LFP-OA problem \( \mu F \leq ? \alpha \)).** Let \( L \) be a complete lattice, \( F : L \rightarrow L \) be \( \omega \)-continuous, and \( \alpha \in L \). The lfp over-approximation (LFP-OA) problem asks if \( \mu F \leq \alpha \) holds; the problem is denoted by \( \mu F \leq ? \alpha \).

**Example 3.2.** Consider a transition system, where \( S \) be the set of states, \( \iota \subseteq S \) be the set of initial states, \( \delta : S \rightarrow \mathcal{P}S \) be the transition relation, and \( \alpha \subseteq S \) be the set of safe states. Then letting \( L := \mathcal{P}S \) and \( F := \iota \cup \bigcup_{s \in (-)} \delta(s) \), the lfp over-approximation problem \( \mu F \leq ? \alpha \) is the problem whether all reachable states are safe. It is equal to the problem studied by the conventional IC3/PDR [9, 13].

Positive LT-PDR iteratively builds a KT witness in a bottom-up manner that positively answers the LFP-OA problem, while negative LT-PDR iteratively builds a Kleene witness for the same LFP-OA problem. We shall present these two algorithms as clear reflections of two proof principles (Cor. 2.3), each of which comes from the fundamental Knaster–Tarski and Kleene theorems.

The two algorithms build up witnesses in an iterative and nondeterministic manner. The nondeterminism is there for accommodating guesses and heuristics. We identify the essence of PDR to be an ingenious combination of these two algorithms, in which intermediate results on one side (positive or negative) give informed guesses on the other side. This way, each of the positive and negative algorithms provides heuristics in resolving the nondeterminism in the execution of the other. This is how we formulate the LT-PDR algorithm in §3.3.

The dual of LFP-OA problem is called the gfp-under-approximation problem (GFP-UA): the GFP-UA problem for a complete lattice \( L \), an \( \omega^{op}\)-continuous function \( F : L \rightarrow L \) and \( \alpha \in L \) is whether the inequality \( \alpha \leq \nu F \) holds or not, and is denoted by \( \alpha \leq?\nu F \). It is evident that the GFP-UA problem for \((L, F, \alpha)\) is equivalent to the LFP-OA problem for \((L^{op}, F, \alpha)\). This suggests the dual algorithm called LT-OpPDR for GFP-UA problem. See Rem. 3.24 later.
3.1 Positive LT-PDR: Sequential Positive Witnesses

We introduce the notion of KT \( \omega \) witness—a KT witness (Cor. 2.3) constructed in a sequential manner. Positive LT-PDR searches for a KT \( \omega \) witness by growing its finitary approximations (called KT sequences).

Let \( L \) be a complete lattice. We regard each element \( x \in L \) as an abstract presentation of a predicate on states. The inequality \( x \leq y \) means that the predicate \( x \) is stronger than the predicate \( y \). We introduce the complete lattice \([n,L]\) of increasing chains of length \( n \in \mathbb{N} \), whose elements are \((X_0 \leq \cdots \leq X_{n-1})\) in \( L \) equipped with the element-wise order. We similarly introduce the complete lattice \([\omega,L]\) of \( \omega \)-chains in \( L \). We lift KT\( \omega \) predicate witnesses to \( \omega \)-chains in \( L \). This view is justified by the partial order relation \( \leq \) between KT sequences defined below.

The Lattice-Theoretic Essence of Property Directed Reachability Analysis 5

**Definition 3.5 (KT sequence).** Let \( L,F,\alpha \) be as in Def. 3.1. Define \( \Delta \alpha := (\alpha \leq \alpha \leq \cdots) \). A KT sequence is an extreme example is \( x \leq x \leq \cdots \) with a KT witness \( x \). KT\( \omega \) witnesses embrace the spectrum of such different sequential witnesses for \( \mu F \leq \alpha \), those which mix routine constructions (i.e. application of \( F \)) and heuristic guesses.

**Theorem 3.4.** Let \( L,F,\alpha \) be as in Def. 3.1. There exists a KT witness (Cor. 2.3) constructed by growing KT witnesses—a KT witness (Cor. 2.3) constructed by growing KT witnesses. A full proof (via Galois connections) is in the appendix.

Concretely, a KT witness \( x \) yields a KT\( \omega \) witness \( x \leq x \leq \cdots \); a KT\( \omega \) witness \( X \) yields a KT witness \( \bigvee_{n \in \omega} X_n \). A full proof (via Galois connections) is in the appendix.

The initial chain \( \bot \leq F \bot \leq \cdots \) is always a KT\( \omega \) witness for \( \mu F \leq \alpha \). There are other KT\( \omega \) witnesses whose growth is accelerated by some heuristic guesses—an extreme example is \( x \leq x \leq \cdots \) with a KT witness \( x \). KT\( \omega \) witnesses embrace the spectrum of such different sequential witnesses for \( \mu F \leq \alpha \), those which mix routine constructions (i.e. application of \( F \)) and heuristic guesses.

**Definition 3.5 (KT sequence).** Let \( L,F,\alpha \) be as in Def. 3.1. A KT sequence for \( \mu F \leq \alpha \) is a finite chain \( (X_0 \leq \cdots \leq X_{n-1}) \), for \( n \geq 2 \), satisfying

1. \( X_{n-2} \leq \alpha \); and
2. \( X \) is a prefixed point of \( F^\# \), that is, \( FX_i \leq X_{i+1} \) for each \( i \in [0,n-2] \).

A KT sequence \( (X_0 \leq \cdots \leq X_{n-1}) \) is conclusive if \( X_{j+1} \leq X_j \) for some \( j \).

KT sequences are finite by definition. Note that the upper bound \( \alpha \) is imposed on all \( X_i \) but \( X_{n-1} \). This freedom in the choice of \( X_{n-1} \) offers room for heuristics, one that is exploited in the combination with negative LT-PDR (3.3).

We take KT sequences as finite approximations of KT\( \omega \) witnesses. This view shall be justified by the partial order relation \( \leq \) between KT sequences defined below.

**Definition 3.6 (order \( \leq \) between KT sequences).** We define a partial order relation \( \leq \) on KT sequences as follows: \( (X_0,\ldots,X_{n-1}) \leq (X_0',\ldots,X_{m-1}') \) if \( n \leq m \) and \( X_j \geq X_j' \) for each \( 0 \leq j \leq n-1 \).
The order \( X_j \geq X'_j \) represents that \( X'_j \) is a stronger predicate (on states) than \( X_j \). Therefore \( X \preceq X' \) expresses that \( X' \) is a longer and stronger / more determined chain than \( X \). We obtain \( KT^\omega \) witnesses as their \( \omega \)-suprema.

**Theorem 3.7.** Let \( L, F, \alpha \) be as in Def. 3.1. The set of \( KT \) sequences, augmented with the set of \( KT^\omega \) witnesses \( \{ X \in [\omega, L] \mid F^\#X \leq X \leq \Delta \alpha \} \) and ordered by the natural extension of \( \leq \), is an \( \omega \)-cpo. In this \( \omega \)-cpo, each \( KT^\omega \) witness \( X \) is represented as the suprema of an \( \omega \)-chain of \( KT \) sequences, namely \( X = \bigvee_{n \geq 2} X|_n \) where \( X|_n \in [n, L] \) is the length \( n \) prefix of \( X \).

**Proposition 3.8.** Let \( L, F, \alpha \) be as in Def. 3.1. There exists a \( KT^\omega \) witness if and only if there exists a conclusive \( KT \) sequence.

**Proof.** (\( \Rightarrow \)): If there exists a \( KT^\omega \) witness, \( \mu F \leq \alpha \) holds by Cor. 2.3 and Thm. 3.4. Therefore, the “informed guess” (\( \mu F \leq \mu F \)) gives a conclusive \( KT \) sequence. (\( \Leftarrow \)): When \( X \) is a conclusive \( KT \) sequence with \( X_j = X_{j+1} \), \( X_0 \leq \cdots \leq X_j = X_{j+1} = \cdots \) is a \( KT^\omega \) witness.

The proposition above yields the following partial algorithm that aims to answer positively to the LFP-OA problem. It searches for a conclusive \( KT \) sequence.

**Definition 3.9 (positive LT-PDR).** Let \( L, F, \alpha \) be as in Def. 3.1. Positive LT-PDR is the algorithm shown in Alg. 1, which says ‘True’ to the LFP-OA problem \( \mu F \leq \alpha \) if successful.

The rules are designed by the following principles.

**Valid** is applied when the current \( X \) is conclusive.

**Unfold** extends \( X \) with \( \top \). In fact, we can use any element \( x \) satisfying \( X_{n-1} \leq x \) and \( FX_{n-1} \leq x \) in place of \( \top \) (by the application of **Induction** with \( x \)). The condition \( X_{n-1} \leq \alpha \) is checked to ensure that the extended \( X \) satisfies the condition in Def. 3.6.1.

**Induction** strengthens \( X \), replacing the \( j \)-th element with its meet with \( x \). The first condition \( X_k \not\leq x \) ensures that this rule indeed strengthens \( X \), and the second condition \( F(X_{k-1} \wedge x) \leq x \) ensures that the strengthened \( X \) satisfies the condition in Def. 3.6.2 that is, \( F^\#X \leq X \) (see the proof in Appendix J.11).

**Theorem 3.10.** Let \( L, F, \alpha \) be as in Def. 3.1. Then positive LT-PDR is sound, i.e. if it outputs ‘True’ then \( \mu F \leq \alpha \) holds.

Moreover, assume \( \mu F \leq \alpha \) is true. Then positive LT-PDR is weakly terminating (meaning that suitable choices of \( x \) when applying **Induction** make the algorithm terminate).

The last “optimistic termination” is realized by the informed guess \( \mu F \) as \( x \) in **Induction**. To guarantee the termination of LT-PDR, it suffices to assume that the complete lattice \( L \) is well-founded (no infinite decreasing chain exists in \( L \)) and there is no strictly increasing \( \omega \)-chain under \( \alpha \) in \( L \), although we cannot hope for this assumption in every instance (§ 5.2, 5.3).
Input: An instance \((\mu F \leq \alpha)\) of the LFP-OA problem in \(L\)
Output: ‘True’ with a conclusive KT sequence
Data: a KT sequence \(X = (X_0 \leq \cdots \leq X_{n-1})\)
Initially: \(X := (\bot \leq F \bot)\)
repeat (do one of the following)
  | Valid If \(X_{j+1} \leq X_j\) for some \(j < n - 1\), return ‘True’ with the conclusive KT sequence \(X\).
  | Unfold If \(X_{n-1} \leq \alpha\), let \(X := (X_0 \leq \cdots \leq X_{n-1} \leq \top)\), appending \(\top\)
  | Induction If some \(k \geq 2\) and \(x \in L\) satisfy \(X_k \not\leq x\) and \(F(X_{k-1} \wedge x) \leq x\), let \(X := X[X_j := X_j \wedge x]_{2 \leq j \leq k}\).
until any return value is obtained;

Algorithm 1: positive LT-PDR

Input: An instance \((\mu F \leq \alpha)\) of the LFP-OA problem in \(L\)
Output: ‘False’ with a conclusive Kleene sequence
Data: a Kleene sequence \(C = (C_0, \ldots, C_{n-1})\)
Initially: \(C := ()\)
repeat (do one of the following)
  | Candidate Choose \(x \in L\) such that \(x \not\leq \alpha\), and let \(C := (x)\).
  | Model If \(C_0 = \bot\), return ‘False’ with the conclusive Kleene sequence \(C\).
  | Decide If there exists \(x\) such that \(C_0 \leq Fx\), then let \(C := (x, C_0, \ldots, C_{n-1})\).
until any return value is obtained;

Algorithm 2: negative LT-PDR

Input: An instance \((\mu F \leq \alpha)\) of the LFP-OA problem in \(L\)
Output: ‘True’ with a conclusive KT sequence, or ‘False’ with a conclusive Kleene sequence
Data: \((X; C)\) where \(X\) is a KT sequence \((X_0 \leq \cdots \leq X_{n-1})\), and \(C\) is a Kleene sequence \((C_i, C_{i+1}, \ldots, C_{n-1})\) \((C\) is empty if \(n = i)\).
Initially: \((X; C) := (\bot \leq F \bot; ())\)
repeat (do one of the following)
  | Valid If \(X_{j+1} \leq X_j\) for some \(j < n - 1\), return ‘True’ with the conclusive KT sequence \(X\).
  | Unfold If \(X_{n-1} \leq \alpha\), let \((X; C) := (X_0 \leq \cdots \leq X_{n-1} \leq \top; ()))\).
  | Induction If some \(k \geq 2\) and \(x \in L\) satisfy \(X_k \not\leq x\) and \(F(X_{k-1} \wedge x) \leq x\), let \((X; C) := (X[X_j := X_j \wedge x]_{2 \leq j \leq k}; C)\).
  | Candidate If \(C = ()\) and \(X_{n-1} \not\leq \alpha\), choose \(x \in L\) such that \(x \leq X_{n-1}\) and \(x \not\leq \alpha\), and let \((X; C) := (X; (x))\).
  | Model If \(C_1\) is defined, return ‘False’ with the conclusive Kleene sequence \((\bot, C_{i+1}, \ldots, C_{n-1})\).
  | Decide If \(C_i \leq F X_{i-1}\), choose \(x \in L\) satisfying \(x \leq X_{i-1}\) and \(C_i \leq Fx\), and let \((X; C) := (X; (x, C_i, \ldots, C_{n-1}))\).
  | Conflict If \(C_i \not\leq F X_{i-1}\), choose \(x \in L\) satisfying \(C_i \not\leq x\) and \(F(X_{i-1} \wedge x) \leq x\), and let \((X; C) := (X[X_j := X_j \wedge x]_{2 \leq j \leq i}; (C_{i+1}, \ldots, C_{n-1}))\).
until any return value is obtained;

Algorithm 3: LT-PDR
Lemma 3.11. Let $L, F, \alpha$ be as in Def. 3.1. If $\mu F \leq \alpha$, then for any KT sequence $X$, at least one of the three rules in Algorithm 3 is enabled.

Moreover, for any KT sequence $X$, let $X'$ be obtained by applying either Unfold or Induction. Then $X \preceq X'$ and $X \neq X'$.

Theorem 3.12. Let $L, F, \alpha$ be as in Def. 3.1. Assume that $\leq$ in $L$ is well-founded and $\mu F \leq \alpha$. Then, any non-terminating run of positive LT-PDR converges to a $KT_\omega$ witness (meaning that it gives a $KT_\omega$ witness in $\omega$-steps). Moreover, if there is no strictly increasing $\omega$-chain bounded by $\alpha$ in $L$, then positive LT-PDR is strongly terminating.

3.2 Negative PDR: Sequential Negative Witnesses

We next introduce Kleene sequences as a lattice-theoretic counterpart of proof obligations in the standard PDR. Kleene sequences represent a chain of sufficient conditions to conclude that certain unsafe states are reachable.

Definition 3.13 (Kleene sequence). Let $L, F, \alpha$ be as in Def. 3.1. A Kleene sequence for the $LFP-OA$ problem $\mu F \leq \alpha$ is a finite sequence $(C_0, \ldots, C_{n-1})$, for $n \geq 0$ ($C$ is empty if $n = 0$), satisfying

1. $C_j \leq F C_{j-1}$ for each $1 \leq j \leq n - 1$;
2. $C_{n-1} \not\leq \alpha$.

A Kleene sequence $(C_0, \ldots, C_{n-1})$ is conclusive if $C_0 = \perp$. We may use $i$ ($0 \leq i \leq n$) instead of 0 as the starting index of the Kleene sequence $C$.

When we have a Kleene sequence $C = (C_0, \ldots, C_{n-1})$, the chain of implications $(C_j \leq F^j \perp) \implies (C_{j+1} \leq F^{j+1} \perp)$ hold for $0 \leq j < n - 1$. Therefore when $C$ is conclusive, $C_{n-1}$ is a Kleene witness (Cor. 2.3.2).

Proposition 3.14. Let $L, F, \alpha$ be as in Def. 3.1. There exists a Kleene (negative) witness if and only if there exists a conclusive Kleene sequence.

Proof. ($\Rightarrow$): If there exists a Kleene witness $x$ such that $x \leq F^n \perp$ and $x \not\leq \alpha$, $(\perp, F \perp, \ldots, F^n \perp)$ is a conclusive Kleene sequence. ($\Leftarrow$): Assume there exists a conclusive Kleene sequence $C$. Then $C_{n-1}$ satisfies $C_{n-1} \leq F^{n-1} \perp$ and $C_{n-1} \not\leq \alpha$ because of $C_{n-1} \leq FC_{n-2} \leq \cdots \leq F^{n-1} C_0 = F^{n-1} \perp$ and Def. 3.13.2.

This proposition suggests the following algorithm to negatively answer to the LFP-OA problem. It searches for a conclusive Kleene sequence. The algorithm updates a Kleene sequence until its first component becomes $\perp$.

Definition 3.15 (negative LT-PDR). Let $L, F, \alpha$ be as in Def. 3.1. Negative LT-PDR is the algorithm shown in Alg. 3 which says ‘False’ to the LFP-OA problem $\mu F \leq \alpha$ if successful.
The rules are designed by the following principles.

**Candidate** initializes \( C \) with only one element \( x \). The element \( x \) has to be chosen such that \( x \not\leq \alpha \) to ensure Def. 3.13.2.

**Model** is applied when the current Kleene sequence \( C \) is conclusive.

**Decide** prepends \( x \) to \( C \). The condition \( C_0 \leq Fx \) ensures Def. 3.13.1.

**Theorem 3.16.** Let \( L, F, \alpha \) be as in Def. 3.1.

1. Negative LT-PDR is sound, i.e. if it outputs ‘False’ then \( \mu F \not\leq \alpha \).
2. Assume \( \mu F \not\leq \alpha \) is true. Then negative LT-PDR is weakly terminating (meaning that suitable choices of \( x \) when applying rules **Candidate** and **Decide** make the algorithm terminate). \( \square \)

### 3.3 LT-PDR: Integrating Positive and Negative

We have introduced two simple PDR algorithms, called positive LT-PDR (§3.1) and negative LT-PDR (§3.2). They are so simple that they have potential inefficiencies. Specifically, in positive LT-PDR, it is unclear that how we choose \( x \in L \) in Induction, while in negative LT-PDR, it may easily diverge because the rules **Candidate** and **Decide** may choose \( x \in L \) that would not lead to a conclusive Kleene sequence. We resolve these inefficiencies by combining positive LT-PDR and negative LT-PDR. The combined PDR algorithm is called LT-PDR, and it is a lattice-theoretic generalization of conventional PDR.

Note that negative LT-PDR is only weakly terminating. Even worse, it is easy to make it diverge—after a choice of \( x \) in **Candidate** or **Decide** such that \( x \not\leq \mu F \), no continued execution of the algorithm can lead to a conclusive Kleene sequence. For deciding \( \mu F \leq \alpha \) efficiently, therefore, it is crucial to detect such useless Kleene sequences.

The core fact that underlies the efficiency of PDR is the following proposition, which says that a KT sequence (in positive LT-PDR) can quickly tell that a Kleene sequence (in negative LT-PDR) is useless. This fact is crucially used for many rules in LT-PDR (Def. 3.20).

**Proposition 3.17.** Let \( C = (C_i, \ldots, C_{n-1}) \) be a Kleene sequence \((2 \leq n, 0 < i \leq n-1)\) and \( X = (X_0 \leq \cdots \leq X_{n-1}) \) be a KT sequence. Then

1. \( C_i \not\leq X_i \) implies that \( C \) cannot be extended to a conclusive one, that is, there does not exist \( C_0, \ldots, C_{i-1} \) such that \( (C_0, \ldots, C_{n-1}) \) is conclusive.
2. \( C_i \not\leq FX_{i-1} \) implies that \( C \) cannot be extended to a conclusive one.
3. There is no conclusive Kleene sequence with length \( n - 1 \). \( \square \)

The proof relies on the following lemmas.

**Lemma 3.18.** Any KT sequence \((X_0 \leq \cdots \leq X_{n-1})\) over-approximates the initial sequence: \( F^i \perp \leq X_i \) holds for any \( i \) such that \( 0 \leq i \leq n-1 \). \( \square \)

**Lemma 3.19.** Let \( C = (C_i, \ldots, C_{n-1}) \) be a Kleene sequence \((0 < i \leq n-1)\) and \((X_0 \leq \cdots \leq X_{n-1})\) be a KT sequence. The following satisfy \( 1 \iff 2 \Rightarrow 3 \).
1. The Kleene sequence $C$ can be extended to a conclusive one.
2. $C_i \leq F^i \perp$.
3. $C_i \leq F^j X_{i-j}$ for each $j$ with $0 \leq j \leq i$. \hfill $\square$

Using the above lattice-theoretic properties, we combine positive and negative LT-PDRs into the following $LT$-$PDR$ algorithm. It is also a lattice-theoretic generalization of the original PDR algorithm. The combination exploits the mutual relationship between KT sequences and Kleene sequences, exhibited as Prop. 3.17 for narrowing down choices in positive and negative LT-PDRs.

**Definition 3.20 (LT-PDR).** Given a complete lattice $L$, an $\omega$-continuous function $F : L \rightarrow L$, and an element $\alpha \in L$, LT-PDR is the algorithm shown in Alg. 3 for the LFP-OA problem $\mu F \leq \alpha$.

The rules are designed by the following principles.

(Valid, Unfold, and Induction): These rules are almost the same as in positive LT-PDR. In Unfold, we reset the Kleene sequence because of Prop. 3.17.3. Occurrences of Unfold punctuate an execution of the algorithm: between two occurrences of Unfold, a main goal (towards a negative conclusion) is to construct a conclusive Kleene sequence with the same length as the $X$.

(Candidate, Model, and Decide): These rules have many similarities to those in negative LT-PDR. Differences are as follows: the Candidate and Decide rules impose $x \leq X_i$ on the new element $x$ in $(x, C_{i+1}, \ldots, C_{n-1})$ because Prop. 3.17.1 tells us that other choices are useless. In Model, we only need to check whether $C_1$ is defined instead of $C_0 = \perp$. Indeed, since $C_1$ is added in Candidate or Decide, $C_1 \leq X_1 = F \perp$ always holds. Therefore, $2 \Rightarrow 1$ in Lem. 3.19 shows that $(\perp, C_1, \ldots, C_{n-1})$ is conclusive.

(Conflict): This new rule emerges from the combination of positive and negative LT-PDRs. This rule is applied when $C_i \not\leq X_{i-1}$, which confirms that the current $C$ cannot be extended to a conclusive one (Prop. 3.17.2). Therefore, we eliminate $C_i$ from $C$ and strengthen $X$ so that we cannot choose $C_i$ again, that is, so that $C_i \not\leq (X_i \land x)$. Let us explain how $X$ is strengthened. The element $x$ has to be chosen so that $C_i \not\leq x$ and $F(X_{i-1} \land x) \leq x$. The former dis-inequality ensures the strengthened $X$ satisfies $C_i \not\leq (X_i \land x)$, and the latter inequality implies $F(X_{i-1} \land x) \leq x$. One can see that Conflict is Induction with additional condition $C_i \not\leq x$, which enhances so that the search space for $x$ is narrowed down using the Kleene sequence $C$.

Canonical choices of $x \in L$ in Candidate, Decide, and Conflict are $x := X_{n-1}, x := X_{i-1}$, and $x := FX_{i-1}$, respectively. However, there can be cleverer choices; e.g. $x := S \setminus (C_i \setminus FX_{i-1})$ in Conflict when $L = PS$.

**Lemma 3.21.** Each rule of LT-PDR, when applied to a pair of a KT and a Kleene sequence, yields a pair of a KT and a Kleene sequence. \hfill $\square$

**Theorem 3.22 (correctness).** LT-PDR is sound, i.e. if it outputs ‘True’ then $\mu F \leq \alpha$ holds, and if it outputs ‘False’ then $\mu F \not\leq \alpha$ holds. \hfill $\square$
Many existing PDR algorithms ensure termination if the state space is finite. A general principle behind is stated below. Note that it rarely applies to infinitary or quantitative settings, where we would need some abstraction for termination.

**Proposition 3.23 (termination).** LT-PDR terminates regardless of the order of the rule-applications if the following conditions are satisfied.

1. **Valid** and **Model** rules are immediately applied if applicable.
2. \((L, \leq)\) is well-founded.
3. Either of the following is satisfied: a) \(\mu F \leq \alpha\) and \((L, \leq)\) has no strictly increasing \(\omega\)-chain bounded by \(\alpha\), or b) \(\mu F \not\leq \alpha\).

Cond. 1 is natural: it just requires LT-PDR to immediately conclude ‘True’ or ‘False’ if it can. Cond. 2–3 are always satisfied when \(L\) is finite.

Thm. 3.22 and Prop. 3.23 still hold if **Induction** rule is dropped. However, the rule can accelerate the convergence of KT sequences and improve efficiency.

**Remark 3.24 (LT-OpPDR).** The GFP-UA problem \(\alpha \leq^1 \nu F\) is the dual of LFP-OA, obtained by opposing the order \(\leq\) in \(L\). We can also dualize the LT-PDR algorithm (Alg. 3), obtaining what we call the LT-OpPDR algorithm for GFP-UA. Moreover, we can express LT-OpPDR as LT-PDR if a suitable involution \(\neg: L \to L^{op}\) is present. See Appendix B for further details; see also Prop. 4.3.

### 4 Structural Theory of PDR by Category Theory

Before we discuss concrete instances of LT-PDR in §5, we develop a structural theory of transition systems and predicate transformers as a basis of LT-PDR. The theory is formulated in the language of category theory [3, 18, 19, 23]. We use category theory because 1) categorical modeling of relevant notions is well established in the community (see e.g. [2, 8, 18, 19, 27]), and 2) it gives us the right level of abstraction that accommodates a variety of instances. In particular, qualitative and quantitative settings are described in a uniform manner.

Our structural theory (§4) serves as a backend, not a frontend. That is,

– the theory in §4 is important in that it explains how the instances in §5 arise and why others do not, but
– the instances in §5 are described in non-categorical terms, so readers who skipped §4 will have no difficulties following §5 and using those instances.

#### 4.1 Categorical Modeling of Dynamics and Predicate Transformers

Our interests are in instances of the LFP-OA problem \(\mu F \leq^1 \alpha\) (Def. 3.1) that appear in model checking. In this context, 1) the underlying lattice \(L\) is that of predicates over a state space, and 2) the function \(F: L \to L\) arises from the dynamic/transition structure, specifically as a predicate transformer. The
Table 1: Categorical modeling of state-based dynamics and predicate transformers

| a transition system as a coalgebra [18] in the base category $\mathbb{B}$ of sets and functions |
|---|
| objects $X, Y, \ldots$ in $\mathbb{B}$ | sets (in our examples where $\mathbb{B} = \text{Set}$) |
| an arrow $f: X \rightarrow Y$ in $\mathbb{B}$ | a function (in our examples where $\mathbb{B} = \text{Set}$) |
| a functor $G: \mathbb{B} \rightarrow \mathbb{B}$ | a transition type $G = \mathcal{P}$ for Kripke structures (§5.1), $G = (D(\rightarrow) + 1)^{\text{set}}$ for MDPs (§5.2), etc. |
| a coalgebra $\delta: S \rightarrow GS$ in $\mathbb{B}$ [18] | a transition system (Kripke structure, MDP, etc.) |
| a fibration $p: E \rightarrow \mathbb{B}$ [19] that equips sets in $\mathbb{B}$ with predicates |
| the fiber category $E_S$ over $S$ in $\mathbb{B}$ | the lattice of predicates over a set $S$ |
| the pullback functor $l^*: E_Y \rightarrow E_X$ for $l: X \rightarrow Y$ in $\mathbb{B}$ | substitution $P(y) \mapsto P(l(x))$ in predicates $P \in E_Y$ over $Y$ |
| a lifting $\dot{G}: E \rightarrow E$ of $G$ along $p$ | logical interpretation of the transition type $G$ (specifies e.g. the may vs. must modalities) |
| the predicate transformer, whose fixed points are of our interest |
| the composite $\delta^*\dot{G}: E_S \rightarrow E_S$ | the predicate transformer associated with the transition system $\delta$ |

categorical notions in Table 1 model these ideas (state-based dynamics, predicate transformers). This modeling is well-established in the community.

Our introduction of Table 1 here is minimal, due to the limited space. See Appendix C and the references therein for more details.

**Transition Systems as Coalgebras** State-based transition systems are modeled as coalgebras in the base category $\mathbb{B}$ [18]. We use a functor $G: \mathbb{B} \rightarrow \mathbb{B}$ to represent a transition type. A $G$-coalgebra is an arrow $\delta: S \rightarrow GS$, where $S$ is a state space and $\delta$ describes the dynamics. For example, a Kripke structure can be identified with a pair $(S, \delta)$ of a set $S$ and a function $\delta: S \rightarrow \mathcal{P}S$, where $\mathcal{P}S$ denotes the powerset. The powerset construction $\mathcal{P}$ is known to be a functor $\mathcal{P}: \text{Set} \rightarrow \text{Set}$; therefore Kripke structures are $\mathcal{P}$-coalgebras. For other choices of $G$, $G$-coalgebras become different types of transition systems, such as MDPs (§5.2) and Markov Reward Models (§5.3).

**Predicates Form a Fibration** Fibrations are powerful categorical constructs that can model various indexed entities; see e.g. [19] for its general theory. Our use of them is for organizing the lattices $E_S$ of predicates over a set $S$, indexed by the choice of $S$. For example, $E_S = 2^S$—the lattice of subsets of $S$—for modeling qualitative predicates. For quantitative reasoning (e.g. for MDPs), we
use $\mathcal{E}_S = [0, 1]^S$, where $[0, 1]$ is the unit interval. This way, qualitative and quantitative reasonings are mathematically unified in the language of fibrations.

A *fibration* is a functor $p: \mathcal{E} \to \mathcal{B}$ with suitable properties; it can be thought of as a collection $(\mathcal{E}_S)_{S \in \mathcal{B}}$ of fiber categories $\mathcal{E}_S$—indexed by objects $S$ of $\mathcal{B}$—suitably organized as a single category $\mathcal{E}$. Notable in this organization is that we obtain the pullback functor $l^* : \mathcal{E}_Y \to \mathcal{E}_X$ for each arrow $l : X \to Y$ in $\mathcal{B}$. In our examples, $l^*$ is a *substitution* along $l$ in predicates—$l^*$ is the monotone map that carries a predicate $P(y)$ over $Y$ to the predicate $P(l(x))$ over $X$.

In this paper, we restrict to a subclass of fibrations (called $\text{CLat}_\wedge$-fibrations) in which every fiber category $\mathcal{E}_S$ is a complete lattice, and each pullback functor preserves all meets. We therefore write $P \leq Q$ for arrows in $\mathcal{E}_S$; this represents logical implication, as announced above. Notice that each $f^*$ has a left adjoint (lower adjoint in terms of Galois connection), which exists by Freyd’s adjoint functor theorem. The left adjoint is denoted by $f_*$.

We also consider a *lifting* $\hat{G} : \mathcal{E} \to \mathcal{E}$ of $G$ along $p$; it is a functor $\hat{G}$ such that $p\hat{G} = Gp$. See the diagram on the right. It specifies the *logical interpretation* of the transition type $G$. For example, for $G = \mathcal{P}$ (the powerset functor) from the above, two choices of $\hat{G}$ are for the *may* and *must* modalities. See e.g. [2][15][21][22].

**Categorical Predicate Transformer** The above constructs allow us to model predicate transformers—$F$ in our examples of the LFP-OA problem $\mu F \leq^? \alpha$—in categorical terms. A *predicate transformer* along a coalgebra $\delta : S \to GS$ with respect to the lifting $\hat{G}$ is simply the composite $\mathcal{E}_S \xrightarrow{\hat{G}} \mathcal{E}_GS \xrightarrow{\delta^*} \mathcal{E}_S$, where the first $\hat{G}$ is the restriction of $\hat{G} : \mathcal{E} \to \mathcal{E}$ to $\mathcal{E}_S$. Intuitively, 1) given a *postcondition* $P$ in $\mathcal{E}_S$, 2) it is first interpreted as the predicate $GP$ over $GS$, and then 3) it is pulled back along the dynamics $\delta$ to yield a *precondition* $\delta^*GP$. Such (backward) predicate transformers are fundamental in a variety of model checking problems.

### 4.2 Structural Theory of PDR from Transition Systems

We formulate a few general safety problems. We show how they are amenable to the LT-PDR (Def. 3.20) and LT-OpPDR (Rem. 3.21) algorithms.

**Definition 4.1 (backward safety problem, BSP).** Let $p$ be a $\text{CLat}_\wedge$-fibration, $\delta : S \to GS$ be a coalgebra in $\mathcal{B}$, and $G : \mathcal{E} \to \mathcal{E}$ be a lifting of $G$ along $p$ such that $\hat{G}_X : \mathcal{E}_X \to \mathcal{E}_{GX}$ is $\omega^p$-continuous for each $X \in \mathcal{B}$. The backward safety problem for $(\mathcal{E}_S, \delta, \alpha \in \mathcal{E}_S)$ in $(p, G, \hat{G})$ is the GFP-UA problem for $(\mathcal{E}_S, \alpha \land \delta^*\hat{G} \iota)$, that is,

$$
\iota \leq^? \nu x. \alpha \land \delta^*\hat{G} x.
$$

(2)

Here, $\iota$ represents the initial states and $\alpha$ represents the safe states. The predicate transformer $x \mapsto \alpha \land \delta^*\hat{G} x$ in (2) is the standard one for modeling safety currently safe ($\alpha$), and the next time $x$ ($\delta^*\hat{G} x$). Its $gfp$ is the safety property; (2) asks if all initial states ($\iota$) satisfy the safety property. Since the backward safety problem is a GFP-UA problem, we can solve it by LT-OpPDR (Rem. 3.21).
Additional assumptions allow us to reduce the backward safety problem to LFP-OA problems, which are solvable by LT-PDR, as shown on the right.

The first case requires the existence of the left adjoint to the predicate transformer $\delta^*G_S : \mathcal{E}_S \to \mathcal{E}_S$. Then we can translate BSP to the following LFP-OA problem. It directly asks whether all reachable states are safe.

**Proposition 4.2 (forward safety problem, FSP).** In the setting of Def. 4.1, assume that each $\delta^*G_X : \mathcal{E}_X \to \mathcal{E}_{G\delta^*}$ preserves all meets. Then by letting $H_S : \mathcal{E}_{GS} \to \mathcal{E}_S$ be the left adjoint of $G_S$, the BSP (2) is equivalent to the LFP-OA problem for $(\mathcal{E}_S, \iota \lor H_S\delta^*\alpha)$:

$$
\mu x. \iota \lor H_S\delta^*x \leq^? \alpha.
$$

This problem is called the forward safety problem for $(\iota, \delta, \alpha)$ in $(p, G, \hat{G})$. □

The second case assumes that the complete lattice $\mathcal{E}_S$ of predicates admits an involution operator $\neg : \mathcal{E}_S \to \mathcal{E}_{op}^S$ (cf. Appendix B).

**Proposition 4.3 (inverse backward safety problem, IBSP).** In the setting of Def. 4.1 assume further that there is a monotone function $\neg : \mathcal{E}_S \to \mathcal{E}_{op}^S$ satisfying $\neg \circ \neg = id$. Then the backward safety problem (2) is equivalent to the LFP-OA problem for $(\mathcal{E}_S, (\neg \alpha) \lor (\neg \circ \delta^*\hat{G} \circ \neg)(\neg))$, that is,

$$
\mu x. (\neg \alpha) \lor (\neg \circ \delta^*\hat{G} \circ \neg) \leq^? \neg \iota.
$$

We call (4) the inverse backward safety problem for $(\iota, \delta, \alpha)$ in $(p, G, \hat{G})$. Here $(\neg \alpha) \lor (\neg \circ \delta^*\hat{G} \circ \neg(\neg))$ is the inverse backward predicate transformer. □

When both additional assumptions are fulfilled (in Prop. 4.2 & 4.3), we obtain two LT-PDR algorithms to solve BSP. One can even simultaneously run these two algorithms—this is done in fbPDR [25, 26]. See also §5.1.

5 **Known and New PDR Algorithms as Instances**

We present several concrete instances of our LT-PDR algorithms. The one for Markov reward models is new (§5.3). We also sketch how those instances can be systematically derived by the theory in §4; details are in Appendix D.

5.1 **LT-PDRs for Kripke Structures: PDR$^F$-$Kr$ and PDR$^{IB}$-$Kr$**

In most of the PDR literature, the target system is a Kripke structure that arises from a program’s operational semantics. A Kripke structure consists of a set $S$ of states and a transition relation $\delta \subseteq S \times S$ (here we ignore initial states and atomic propositions). The basic problem formulation is as follows.
Definition 5.1 (backward safety problem (BSP) for Kripke structures). The BSP for a Kripke structure \((S, \delta)\), a set \(\iota \in 2^S\) of initial states, and a set \(\alpha \in 2^S\) of safe states, is the GFP-UAP problem \(\iota \subseteq^? \nu \cdot s . \alpha \wedge F' \cdot x\), where \(F': 2^S \to 2^S\) is defined by \(F'(A) := \{s \mid \forall s' . ((s, s') \in \delta \Rightarrow s' \in A)\}\).

It is clear that the GFP in Def. 5.1 represents the set of states from which all reachable states are in \(\alpha\). Therefore the BSP is the usual safety problem.

The above BSP is easily seen to be equivalent to the following problems.

Proposition 5.2 (forward safety problem (FSP) for Kripke structures). The BSP in Def. 5.1 is equivalent to the LFP-OA problem \(\mu . \iota \vee F'' . x \leq^? \alpha\), where \(F'': 2^S \to 2^S\) is defined by \(F''(A) := \bigcup_{s \in A} \{s' \mid (s, s') \in \delta\}\).

Proposition 5.3 (inverse backward safety problem (IBSP) for Kripke structures). The BSP in Def. 5.1 is equivalent to the LFP-OA problem \(\mu \neg \iota \vee \neg F' (\neg x) \leq^? \neg \alpha\), where \(\neg : 2^S \to 2^S\) is the complement function \(A \mapsto S \setminus A\).

Instances of LT-PDR The FSP and IBSP (Prop. 5.2–5.3), being LFP-OA, are amenable to the LT-PDR algorithm (Def. 3.20). Thus we obtain two instances of LT-PDR; we call them \(\text{PDR}^{F, Kr}\) and \(\text{PDR}^{IB, Kr}\). \(\text{PDR}^{IB, Kr}\) is a step-by-step dual to the application of LT-OpPDR to the BSP (Def. 5.1)—see Rem. 3.24.

We compare these two instances of LT-PDR with algorithms in the literature. If we impose \(|C_i| = 1\) on each element \(C_i\) of Kleene sequences, the \(\text{PDR}^{F, Kr}\) instance of LT-PDR coincides with the conventional IC3/PDR. In contrast, \(\text{PDR}^{IB, Kr}\) coincides with \textit{Reverse PDR} in [25, 26]. The parallel execution of \(\text{PDR}^{F, Kr}\) and \(\text{PDR}^{IB, Kr}\) roughly corresponds to fbPDR [25, 26].

Structural Derivation The equivalent problems (Prop. 5.2–5.3) are derived systematically from the categorical theory in [4, 2]. Indeed, using a lifting \(\mathcal{P} : 2^S \to 2^P^S\) such that \(A \mapsto \{A' \mid A' \subseteq A\}\) (the \textit{must modality} \(\square\)), \(F'\) in Def. 5.1 coincides with \(\delta^* \mathcal{P}\) in [2]. The above \(\mathcal{P}\) preserves meets (cf. the modal axiom \(\square (\varphi \wedge \psi) \equiv \square \varphi \wedge \square \psi\), see e.g. [7]; thus Prop. 5.1 derives the FSP. Finally, \(\neg\) in Prop. 5.3 allows the use of Prop. 4.3. More details are in Appendix D.

5.2 LT-PDR for MDPs: \(\text{PDR}^{IB, MDP}\)

The only known PDR-like algorithm for \textit{quantitative} verification is \textit{PrIC3} [6] for Markov decision processes (MDPs). Here we instantiate LT-PDR for MDPs and compare it with PrIC3.

An MDP consists of a set \(S\) of states, a set \(\text{Act}\) of actions and a transition function \(\delta\) mapping \(s \in S\) and \(a \in \text{Act}\) to either \(*\) (‘the action \(a\) is unavailable at \(s\)’) or a probability distribution \(\delta(s)(a)\) over \(S\).

Definition 5.4 (IBSP for MDPs). The inverse backward safety problem (IBSP) for an MDP \((S, \delta)\), an initial state \(s_i \in S\), a real number \(\lambda \in [0, 1]\), and a set \(\alpha \subseteq S\) of safe states, is the LFP-OA problem \(\mu x . \iota \vee F' x \leq^? d_{\alpha, \lambda}\). Here \(d_{\alpha, \lambda} : S \to [0, 1]\) is the predicate such that \(d_{\alpha, \lambda}(s_i) = \lambda\) and \(d_{\alpha, \lambda}(s) = 1\) otherwise. \(F' : [0, 1]^S \to [0, 1]^S\) is defined by \(F'(d)(s) = 1\) if \(s \notin \alpha\), and \(F'(d)(s) = \max\{\sum_{s' \in S} d(s') \cdot \delta(s)(a)(s') \mid a \in \text{Act}, \delta(s)(a) \neq *\}\) if \(s \in \alpha\).
The function $F'$ in Def. 5.4 is a Bellman operator for MDPs—it takes the average of $d$ over $\delta(s)(a)$ and takes the maximum over $a$. Therefore the lfp in Def. 5.4 is the maximum reachability probability to $S \setminus \alpha$; the problem asks if it is $\leq \lambda$. In other words, it asks whether the safety probability—of staying in $\alpha$ henceforth, under any choices of actions—is $\geq 1 - \lambda$. This problem is the same as in [6].

**Instance of PDR** The IBSP (Def. 5.4) is LFP-OA and thus amenable to LT-PDR. We call this instance $PDR_{IB-MDP}$; See Appendix E for details.

$PDR_{IB-MDP}$ shares many essences with PrIC3 [6]. It uses the operator $F'$ in Def. 5.4, which coincides with the one in [6, Def. 2]. PrIC3 maintains frames; they coincide with KT sequences in $PDR_{IB-MDP}$.

Our Kleene sequences correspond to obligations in PrIC3, modulo the following difference. Kleene sequences aim at a negative witness (§3.2), but they happen to help the positive proof efforts too (§3.3); obligations in PrIC3 are solely for accelerating the positive proof efforts. Thus, if PrIC3 cannot solve these efforts, we need to check whether obligations yield a negative witness.

**Structural Derivation** One can derive the IBSP (Def. 5.4) from the categorical theory in §4.2. Specifically, we first formulate the BSP $\neg d_\alpha \leq^2 \nu x. d_\alpha \wedge \delta^* G x$, where $\hat{G}$ is a suitable lifting (of $G$ for MDPs, Table 1) that combines average and minimum, $\neg$: $[0,1]^S \rightarrow [0,1]^S$ is defined by $(\neg d)(s) := 1 - d(s)$, and $d_\alpha$ is such that $d_\alpha(s) = 1$ if $s \in \alpha$ and $d_\alpha(s) = 0$ otherwise. Using $\neg$: $[0,1]^S \rightarrow [0,1]^S$ in the above as an involution, we apply Prop. 4.3 and obtain the IBSP (Def. 5.4).

Another benefit of the categorical theory is that it can tell us a forward instance of LT-PDR (much like $PDR_{F-Kr}$ in §5.1) is unlikely for MDPs. Indeed, we showed in Prop. 4.2 that $\hat{G}$’s preservation of meets is essential (existence of a left adjoint is equivalent to meet preservation). We can easily show that our $\hat{G}$ for MDPs does not preserve meets. See Appendix G.

### 5.3 LT-PDR for Markov Reward Models: $PDR_{MRM}$

We present a PDR-like algorithm for *Markov reward models (MRMs)*, which seems to be new, as an instance of LT-PDR. An MRM consists of a set $S$ of states and a transition function $\delta$ that maps $s \in S$ (the current state) and $c \in \mathbb{N}$ (the reward) to a function $\delta(s)(c): S \rightarrow [0,1]$; the last represents the probability distribution of next states.

We solve the following problem. We use $[0, \infty]$-valued predicates—representing accumulated rewards—where $[0, \infty]$ is the set of extended nonnegative reals.

**Definition 5.5 (SP for MRMm).** The safety problem (SP) for an MRM $(S, \delta)$, an initial state $s_\iota \in S$, $\lambda \in [0, \infty]$, and a set $\alpha \subseteq S$ of safe states is $\mu x. F'(x) \leq^2 d_{s_\iota, \lambda}$. Here $d_{s_\iota, \lambda}: S \rightarrow [0,\infty]$ maps $s_\iota$ to $\lambda$ and others to $\infty$, and $F': [0,\infty]^S \rightarrow [0,\infty]^S$ is defined by $F'(d)(s) = 0$ if $s \notin \alpha$, and $F'(d)(s) = \sum_{s' \in S, c \in \mathbb{N}} (c + d(s')) \cdot \delta(s)(c)(s')$ if $s \in \alpha$. 

The function $F'$ accumulates expected reward in $\alpha$. Thus the problem asks if the expected accumulated reward, starting from $s_\iota$ and until leaving $\alpha$, is $\leq \lambda$.

**Instance of PDR** The SP (Def. 5.5) is LFP-OA thus amenable to LT-PDR. We call this instance $\text{PDR}^{\text{MRM}}$. It seems new. See Appendix F for details.

**Structural Derivation** The function $F'$ in Def. 5.5 can be expressed categorically as $F'(x) = d_\alpha \land \delta^* \hat{G}(x)$, where $d_\alpha : S \to [0, \infty]$ carries $s \in \alpha$ to $\infty$ and $s \not\in \alpha$ to 0, and $\hat{G}$ is a suitable lifting that accumulates expected reward. However, the SP (Def. 5.5) is not an instance of the three general safety problems in §4.2. Consequently, we expect that other instances of LT-PDR than $\text{PDR}^{\text{MRM}}$ (such as $\text{PDR}^{\text{F-Kr}}$ and $\text{PDR}^{\text{IB-MDP}}$ in §5.1) are hard for MRMs.

6 Implementation and Evaluation

**Implementation LTPDR** We implemented LT-PDR in Haskell. Exploiting Haskell’s language features, it is succinct (~50 lines) and almost a literal translation of Alg. 3 to Haskell. Its main part is presented in Appendix K. In particular, using suitable type classes, the code is as abstract and generic as Alg. 3.

Specifically, our implementation is a Haskell module named LTPDR. It has two interfaces, namely the type class $\text{CLat} \tau$ (the lattice of predicates) and the type $\text{Heuristics} \tau$ (the definitions of $\text{Candidate}$, $\text{Decide}$, and $\text{Conflict}$). The main function for LT-PDR is $\text{ltPDR} : \text{CLat} \tau \to \text{Heuristics} \tau \to (\tau \to \tau) \to \tau \to \text{IO}(\text{PDRAnswer} \tau)$, where the second argument is for a monotone function $F$ of type $\tau \to \tau$ and the last is for the safety predicate $\alpha$.

Obtaining concrete instances is easy by fixing $\tau$ and $\text{Heuristics} \tau$. A simple implementation of $\text{PDR}^{\text{F-Kr}}$ takes 15 lines; a more serious SAT-based one for $\text{PDR}^{\text{F-Kr}}$ takes $\sim$130 lines; $\text{PDR}^{\text{IB-MDP}}$ and $\text{PDR}^{\text{MRM}}$ take $\sim$80 lines each.

**Heuristics** We briefly discuss the heuristics, i.e. how to choose $x \in L$ in $\text{Candidate}$, $\text{Decide}$, and $\text{Conflict}$, used in our experiments. The heuristics of $\text{PDR}^{\text{F-Kr}}$ is based on the conventional PDR [9]. The heuristics of $\text{PDR}^{\text{IB-MDP}}$ is based on the idea of representing the smallest possible $x$ greater than some real number $v \in [0, 1]$ (e.g. $x$ taken in $\text{Candidate}$) as $x = v + \epsilon$, where $\epsilon$ is a symbolic variable. This implies that $\text{Unfold}$ (or $\text{Valid}$, $\text{Model}$) is always applied in finite steps, which further guarantees finite-step termination for invalid cases and $\omega$-step termination for valid cases (see Appendix H for more detail). The heuristics of $\text{PDR}^{\text{MRM}}$ is similar to that of $\text{PDR}^{\text{IB-MDP}}$.

**Experiment Setting** We experimentally assessed the performance of instances of LTPDR. The settings are as follows: 1.2GHz Quad-Core Intel Core i7 with 10 GB memory using Docker, for $\text{PDR}^{\text{IB-MDP}}$; Apple M1 Chip with 16 GB memory for the other. The different setting is because we needed Docker to run PrIC3 [6].

**Experiments with $\text{PDR}^{\text{MRM}}$** Table 2a shows the results. We observe that $\text{PDR}^{\text{MRM}}$ answered correctly, and that the execution time is reasonable. Further performance analysis (e.g. comparison with [20]) and improvement is future...
work; the point here, nevertheless, is the fact that we obtained a reasonable MRM model checker by adding ~80 lines to the generic solver LTPDR.

**Experiments with PDR**

Table 2 shows the results. Both PrIC3 and our PDR solve a linear programming (LP) problem in **Decide**. PrIC3 uses Z3 for this; PDR uses GLPK. PrIC3 represents an MDP symbolically, while PDR do so concretely. Symbolic representation in PDR is possible—it is future work. PrIC3 can use four different interpolation generalization methods, leading to different performance (Table 2).

We observe that PDR outperforms PrIC3 for some benchmarks with smaller state spaces. We believe that the failure of PDR in many instances can be attributed to our current choice of a generalization method (it is the closest to the linear one for PrIC3). Table 2 suggests that use of polynomial or hybrid can enhance the performance.

**Experiments with PDR**

Table 2 shows the results. The benchmarks are mostly from the HWMCC'15 competition [1], except for latch0.smv and counter.smv (our own). IC3ref vastly outperforms PDR in many instances. This is hardly a surprise—IC3ref was developed towards superior performance, while PDR’s emphasis is on its theoretical simplicity and genericity. We nevertheless see that PDR solves some benchmarks of substantial size, such as power2bit8.smv. This demonstrates the practical potential of LT-PDR, especially in view of the following improvement opportunities (we will pursue them as future work): 1) use of well-developed SAT solvers (we currently use toysolver for its good interface but we could use Z3); 2) allowing $|C_i| > 1$, a technique discussed in §5.1 and implemented in IC3ref but not in PDR; and 3) other small improvements, e.g. in our CNF-based handling of propositional formulas.

**Ablation Study**

To assess the value of the key concept of PDR (namely the positive-negative interplay between the Knaster–Tarski and Kleene theorems (§3.3)), we compared PDR with the instances of positive and negative LT-PDR (§3.1, 3.2) for Kripke structures.

Table 2 shows the results. Note that the value of the positive-negative interplay is already theoretically established; see e.g. Prop. 3.17 (the interplay detects executions that lead to nowhere). This value was also experimentally witnessed: see power2bit8.smv and simpleTrans.smv, where the one-sided methods made wrong choices and timed out. One-sided methods can be efficient when they get lucky (e.g. in counter.smv). LT-PDR may be slower because of the overhead of running two sides, but that is a trade-off for the increased chance of termination.

**Discussion**

We observe that all of the studied instances exhibited at least reasonable performance. We note again that detailed performance analysis and improvement is out of our current scope. Being able to derive these model checkers, with such a small effort as ~100 lines of Haskell code each, demonstrates the value of our abstract theory and its generic Haskell implementation LTPDR.

---

4 https://github.com/arminbiere/aiger
5 https://github.com/msakai/toysolver
Table 2: experimental results for our **PDR**\textsuperscript{F-Kr}, **PDR**\textsuperscript{IB-MDP}, and **PDR**\textsuperscript{MRM}

(a) Results with **PDR**\textsuperscript{MRM}. The MRM is from [4, Example 10.72], whose ground truth expected reward is \(\frac{4}{3}\). The benchmarks ask if the expected reward (not known to the solver) is \(\leq 1.5\) or \(\leq 1.3\).

| Benchmark | Result | Time |
|-----------|--------|------|
| DieByCoin\(^{\dagger,1.5}\) | True | 6.01 ms |
| DieByCoin\(^{\dagger,1.3}\) | False | 43.1 µs |

(b) Results with **PDR**\textsuperscript{F-Kr} in comparison with IC3ref, a reference implementation of [9] ([https://github.com/arbrad/IC3ref](https://github.com/arbrad/IC3ref)). Both solvers answered correctly. Timeout (TO) is 600 sec.

| Benchmark | Result | PDR\textsuperscript{F-Kr} | IC3ref |
|-----------|--------|-----------------|--------|
| latch0.smv | 2\(^{\dagger}\) True | 317 µs | 270 µs |
| counter.smv | 3\(^{\dagger}\) | False | 1.620 s | 3.27 ns |
| power2bit8.smv | 2\(^{\dagger}\) True | 1.516 s | 4.13 ms |
| ndista128.smv | 2\(^{\dagger}\) True | TO | 73.1 ms |
| shift1add256.smv | 2\(^{\dagger}\) True | TO | 174 ms |

(c) Results with **PDR**\textsuperscript{IB-MDP} (an excerpt of Table 3). Comparison is against PrIC3 [6] with four different interpolation generalization methods (none, linear, polynomial, hybrid). The benchmarks are from [6]. \(|S|\) is the number of states of the benchmark MDP. “GT pr.” is for the ground truth probability, that is the reachability probability \(Pr_{max}(s, \not\in (S\setminus \alpha))\) computed outside the solvers under experiments. The solvers were asked whether the GT pr. (which they do not know) is \(\leq \lambda\) or not; they all answered correctly. The last five columns show the average execution time in seconds. – is for “did not finish,” for out of memory or timeout (600 sec.).

| Benchmark | \(|S|\) | GT pr. | \(\lambda\) | PDR\textsuperscript{IB-MDP} | PrIC3 |
|-----------|--------|--------|------------|-----------------|--------|
| Grid | \(10^2\) | 1.2 \(E^{-1}\) | 0.3 | 0.31 | 1.31 | 19.34 | – | – |
| Grid | \(10^3\) | 4.4 \(E^{-10}\) | 0.3 | 122.29 | – | – | – | – |
| BRP | | | | 0.1 | 18.52 | 56.55 | 594.89 | 722.38 |
| ZeroConf | \(10^3\) | 0.5 | 0.035 | 1.36 | 11.68 | 258.09 | – | – |
| Chain | \(10^2\) | 0.394 | 0.45 | <0.1 | 0.91 | 0.91 | 0.91 | – | – |
| DoubleChain | \(10^3\) | 0.215 | 0.4 | – | 80.83 | 0.93 | – | – |
| | | | 0.35 | 177.12 | 115.98 | – | – | – |
| | | | | 0.3 | 88.27 | 66.89 | 557.68 | – | – |

(d) Ablation experiments: LT-PDR (**PDR**\textsuperscript{F-Kr}) vs. positive and negative LT-PDRs, implemented for the FSP for Kripke structures. The benchmarks are as in Table 2b except for a new micro benchmark **simpleTrans.smv**. Timeout (TO) is 600 sec.

| Benchmark | Result | LT-PDR positive negative |
|-----------|--------|--------------------------|
| latch0.smv | True | 317 µs | 1.68 ms | TO |
| power2bit8.smv | True | 1.516 s | TO | TO |
| counter.smv | False | 1.620 s | TO | 2.88 µs |
| **simpleTrans.smv** | False | 295 µs | TO | TO |
7 Conclusions and Future Work

We have presented a lattice-theoretic generalization of the PDR algorithm called LT-PDR. This involves the decomposition of the PDR algorithm into positive and negative ones, which are tightly connected to the Knaster–Tarski and Kleene fixed point theorems, respectively. We then combined it with the coalgebraic and fibrational theory for modeling transition systems with predicates. We instantiated it with several transition systems, deriving existing PDR algorithms as well as a new one over Markov reward models. We leave instantiating our LT-PDR and categorical safety problems to derive other PDR-like algorithms, such as PDR for hybrid systems [29], for future work.

We will also work on the combination of our work and the theory of abstract interpretation [10,12]. Our current framework axiomatizes what is needed of heuristics, but it does not tell how to realize such heuristics (that differ a lot in different concrete settings). We expect abstract interpretation to provide some general recipes for realizing such heuristics.
References

1. The 8th competitive event for hardware model checkers (HW MCC'15) (2015), http://fmv.jku.at/hwmcc15/
2. Aguirre, A., Katsumata, S.: Weakest preconditions in fibrations. Electronic Notes in Theoretical Computer Science 352, 5 – 27 (2020), http://www.sciencedirect.com/science/article/pii/S1571066120300487, the 36th Mathematical Foundations of Programming Semantics Conference, 2020
3. Awodey, S.: Category Theory. Oxford Logic Guides, Oxford Univ. Press (2006)
4. Baier, C., Katoen, J.: Principles of model checking. MIT Press (2008)
5. Baranga, A.: The contraction principle as a particular case of kleene’s fixed point theorem. Discret. Math. 98(1), 75–79 (1991)
6. Batz, K., Junges, S., Kaminski, B.L., Katoen, J., Matheja, C., Schröer, P.: PrIC3: Property directed reachability for MDPs. In: Lahiri, S.K., Wang, C. (eds.) Computer Aided Verification - 32nd International Conference, CAV 2020, Proceedings, Part II. pp. 512–538 (2020), https://doi.org/10.1007/978-3-030-53291-8_27
7. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. No. 53 in Tracts in Theor. Comp. Sci. (2001)
8. Bonchi, F., König, B., Petrisan, D.: Up-to techniques for behavioural metrics via fibrations. In: Schewe, S., Zhang, L. (eds.) 29th International Conference on Concurrency Theory, CONCUR 2018. LIPIcs, vol. 118, pp. 17:1–17:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2018). https://doi.org/10.4230/LIPIcs.CONCUR.2018.17
9. Bradley, A.R.: SAT-based model checking without unrolling. In: Jhala, R., Schmidt, D.A. (eds.) Verification, Model Checking, and Abstract Interpretation - 12th International Conference. VMCAI 2011. Proceedings. pp. 70–87 (2011), https://doi.org/10.1007/978-3-642-18275-4_7
10. Cousot, P., Cousot, R.: Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In: Graham, R.M., Harrison, M.A., Sethi, R. (eds.) Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, 1977. pp. 238–252. ACM (1977). https://doi.org/10.1145/512950.512973
11. Cousot, P., Cousot, R.: Constructive versions of Tarski’s fixed point theorems. Pacific Journal of Mathematics 82(1), 43–57 (1979)
12. Cousot, P., Cousot, R.: Systematic design of program analysis frameworks. In: Aho, A.V., Zilles, S.N., Rosen, B.K. (eds.) Conference Record of the Sixth Annual ACM Symposium on Principles of Programming Languages, 1979. pp. 269–282. ACM Press (1979). https://doi.org/10.1145/567752.567778
13. Eén, N., Mishchenko, A., Brayton, R.K.: Efficient implementation of property directed reachability. In: Bjesse, P., Slobodová, A. (eds.) International Conference on Formal Methods in Computer-Aided Design, FMCAD 2011. pp. 125–134. FMCAD Inc. (2011), http://dl.acm.org/citation.cfm?id=2157675
14. Gurflinkel, A.: IC3, PDR, and friends (2015), https://arieg.bitbucket.io/pdf/gurfinkel_ssft15.pdf
15. Herrlida, C., Jacobs, B.: Structural induction and coinduction in a fibrational setting. Inf. Comput. 145(2), 107–152 (1998), https://doi.org/10.1006/inco.1998.2725
16. Herrlich, H.: Topological functors. General Topology and its Applications 4(2), 125–142 (1974)
17. Hoder, K., Bjørner, N.: Generalized property directed reachability. In: Cimatti, A., Sebastiani, R. (eds.) Theory and Applications of Satisfiability Testing - SAT 2012 - 15th International Conference, 2012. Proceedings. pp. 157–171 (2012), https://doi.org/10.1007/978-3-642-31612-8_13
18. Jacobs, B.: Introduction to Coalgebra: Towards Mathematics of States and Observation. Cambridge Tracts in Theoretical Computer Science, vol. 59. Cambridge University Press (2016), https://doi.org/10.1017/CBO9781139823187
19. Jacobs, B.P.: Categorical Logic and Type Theory. Studies in logic and the foundations of mathematics, vol. 141. North-Holland (2001), http://www.elsevierdirect.com/product.jsp?isbn=9780444508539
20. Katoen, J., Khattri, M., Zapreev, I.S.: A Markov reward model checker. In: Second International Conference on the Quantitative Evaluation of Systems (QEST 2005). pp. 243–244. IEEE Computer Society (2005), https://doi.org/10.1109/QEST.2005.2
21. Komorida, Y., Katsumata, S., Hu, N., Klin, B., Hasuo, I.: Codensity games for bisimilarity. In: 34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019. pp. 1–13. IEEE (2019), https://doi.org/10.1109/LICS.2019.8785691
22. Kori, M., Hasuo, I., Katsumata, S.: Fibrational initial algebra-final coalgebra coincidence over initial algebras: Turning verification witnesses upside down. In: Haddad, S., Varacca, D. (eds.) 32nd International Conference on Concurrency Theory. CONCUR 2021. LIPIcs, vol. 203, pp. 21:1–21:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2021), https://doi.org/10.4230/LIPIcs.CONCUR.2021.21
23. Mac Lane, S.: Categories for the Working Mathematician. Springer, Berlin, 2nd edn. (1998)
24. Rinetzky, N., Shoham, S.: Property directed abstract interpretation. In: Jobstmann, B., Leino, K.R.M. (eds.) Verification, Model Checking, and Abstract Interpretation - 17th International Conference, VMCAI 2016. Proceedings. pp. 104–123 (2016), https://doi.org/10.1007/978-3-662-49122-5_5
25. Seufert, T., Scholl, C.: Combining PDR and reverse PDR for hardware model checking. In: Madsen, J., Coskun, A.K. (eds.) 2018 Design, Automation & Test in Europe Conference & Exhibition, DATE 2018. pp. 49–54. IEEE (2018), https://doi.org/10.23919/DATE.2018.8341978
26. Seufert, T., Scholl, C.: fbPDR: In-depth combination of forward and backward analysis in property directed reachability. In: Teich, J., Fummi, F. (eds.) Design, Automation & Test in Europe Conference & Exhibition, DATE 2019. pp. 456–461. IEEE (2019), https://doi.org/10.23919/DATE.2019.8714819
27. Sokolova, A.: Probabilistic systems coalgebraically: A survey. Theor. Comput. Sci. 412(38), 5095–5110 (2011), https://doi.org/10.1016/j.tcs.2011.05.008
28. Sprunger, D., Katsumata, S., Dubut, J., Hasuo, I.: Fibrational bisimulations and quantitative reasoning. In: Cirstea, C. (ed.) Coalgebraic Methods in Computer Science - 14th IFIP WG 1.3 International Workshop, CMCS 2018, Colocated with ETAPS 2018. Revised Selected Papers. pp. 190–213 (2018), https://doi.org/10.1007/978-3-030-00389-0_11
29. Suenaga, K., Ishizawa, T.: Generalized property-directed reachability for hybrid systems. In: Beyer, D., Zufferey, D. (eds.) Verification, Model Checking, and Abstract Interpretation - 21st International Conference, VMCAI 2020. Proceedings. pp. 293–313 (2020), https://doi.org/10.1007/978-3-030-39322-9_14
30. Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics 5(2), 285–309 (1955)
A  Further Discussion of Related Work

We discuss some other works on generalization of PDR. Hoder and Bjørner [17] gave an abstract formulation of (the original) PDR, abstracting away implementation details (such as SAT-related ones) and presenting the algorithm itself as a transition system (an “abstract transition system” as they call it). Their notion of predicate transformer is an instance of our forward predicate transformer (Prop. 4.2). They also identified an invariant of frames, and our definition of KT sequence (Def. 3.5) is inspired by it. Another theoretical study of PDR is by Rinetzky and Shoham [24]. They studied PDR using abstract interpretation and showed a mapping between PDR configurations and elements of what they call cartesian trace semantics. In both of these works [17, 24], the formulated PDR algorithms target at Kripke structures, and do not accommodate quantitative verification. They are instances of our LT-PDR especially for categorical safety problems introduced in §5.1, similarly to the original PDR. Moreover, our view of PDR as collaborative searches for KT and Kleene witnesses is not explicit in [17, 24].

B  LT-OpPDR (Rem. 3.24)

Recall that the GFP-UA problem $\alpha \leq ? \nu F$ for $(L,F,\alpha)$ is defined to be the LFP-OA problem for $(L^{\text{op}},F,\alpha)$. Hence we can solve the GFP-UA problem by executing the LT-PDR algorithm over $L^{\text{op}}$. We call this algorithm LT-OpPDR; in other words, LT-OpPDR is obtained by opposing each inequality in LT-PDR.

Although LT-OpPDR is a formal dual of LT-PDR, applying the PDR-like algorithm for solving GFP-UA problems seems to be new.

When $L$ admits a duality by involution $\neg : L \to L^{\text{op}}$, the GFP-UA problem in $L$ can be formulated as the LFP-OA problem in $L$ (not in $L^{\text{op}}$ as in the above).

Proposition B.1. Let $L,F,\alpha$ be as in Def. 3.7. Assume that there is a monotone function $\neg : L \to L^{\text{op}}$ satisfying $\neg \circ \neg = \text{id}_L$. Then the GFP-UA problem $\alpha \leq ? \nu F$ in $L$ is equivalent to the LFP-OA problem $\mu(\neg F \circ \neg) \leq ? \neg \alpha$ in $L$.

Proof. This is a consequence of a more general statement about translating LFP-OA problem by isomorphisms. Let $L$ be a complete lattice, $\alpha$ be an element in $L$, and $F : L \to L$ be an $\omega$-continuous function. For any complete lattice $L'$ with an order-preserving isomorphism $f : L \cong L'$, LFP-OA problem $\mu F \leq ? \alpha$ in $L$ is equivalent to LFP-OA problem $\mu(f \circ F \circ f^{-1}) \leq ? f(\alpha)$ in $L'$.

In this case, we can invoke the LT-PDR algorithm over $(L,\neg \circ F \circ \neg,\neg \alpha)$ to solve the GFP-UA problem $\alpha \leq ? \nu F$. We however note that the execution steps of LT-OpPDR over $(L,F,\alpha)$, i.e. LT-PDR over $(L^{\text{op}},F,\alpha)$, and the execution steps of LT-PDR over $(L,\neg \circ F \circ \neg,\neg \alpha)$ are essentially the same; the configuration at each execution step is mutually convertible by the involution $\neg$. 

C Structural Theory of PDR by Category Theory, Further Categorical Preliminaries

Here we provide more details on the categorical modeling in §A.1.

A fibration $p : E \to B$ is a functor that models indexing and substitution. That is, a functor $p : E \to B$ can be seen as a family of categories $(E_X)_{X \in B}$ indexed by $B$-objects. Categories with different indices are connected by substitution functors. In our examples, the base category $B$ is that of sets and functions; and the total category $E$ models “predicates” over $B$ objects. We review a minimal set of definitions and results on fibrations. A good reference is [19] here we quote some definitions and examples given in §2.1 of [22]; see also [2] and [28].

**Definition C.1.** (fibre, fibration: [22] §2.1) Let $p : E \to B$ be a functor. For each $X \in B$, the fibre $E_X$ over $X$ is the category with objects $P \in E$ such that $pP = X$ and morphisms $f : P \to Q$ such that $pf = \text{id}_X$.

A morphism $f : P \to Q$ in $E$ is cartesian if it satisfies the following universality: for each $g : R \to Q$ in $E$ and $k : pR \to pP$ in $B$ with $pg = pf \circ k$, there exists a unique morphism $h : R \to P$ satisfying $g = f \circ h$ and $ph = k$ (see the diagram above).

The functor $p : E \to B$ is a fibration if, for each $Q \in E$ and each $l : X \to pQ$ in $B$, there exists $l^*Q \in E$ and a morphism $\overline{l} : l^*Q \to Q$ such that $p\overline{l} = l$ and $\overline{l}$ is cartesian. The functor $p : E \to B$ is an opfibration if $p^{op} : E^{op} \to B^{op}$ is a fibration. A functor that is both a fibration and an opfibration is called a bifibration.

When $p$ is a fibration, the correspondence from $Q$ to $l^*Q$ described above induces the substitution functor $l^* : E_Y \to E_X$ which replaces the index. The following characterization of bifibrations is useful for us: a fibration $p$ is a bifibration if and only if each substitution functor $l^* : E_Y \to E_X$ (often called a pullback) has a left adjoint $l_* : E_X \to E_Y$ (often called a pushforward).

**Definition C.2** (lifting [22] §2.1). Let $p : E \to B$ be a functor. We say that an endofunctor $\hat{G}$ on $E$ is a lifting of $G$ along $p$ if $p \circ \hat{G} = G \circ p$. For an object $S \in B$, we write $G_S : E_S \to E_{GS}$ for the restriction of $G$ to fibres.

To manipulate complete lattices along a transition function, we focus on a certain class of posetal fibrations called $\text{CLat}_\Lambda$-fibrations. They can be seen as topological functors [10] whose fibres are posets. Many categories arising from spacial and logical structures naturally determine $\text{CLat}_\Lambda$-fibrations.

**Definition C.3** ($\text{CLat}_\Lambda$-fibration [22], §2.1). A $\text{CLat}_\Lambda$-fibration is a fibration $p : E \to B$ such that each fibre $E_X$ is a complete lattice and each substitution $f^* : E_Y \to E_X$ preserves all meets $\land$. In each fibre $E_X$, the order is denoted by $\leq_X$ or $\leq$. Its least and greatest elements are denoted by $\bot_X$ and $\top_X$; its join and meet are denoted by $\lor$ and $\land$. 
The above simple axioms of \( \text{CLat}_\Lambda \)-fibrations induce many useful structures \([21, 28]\). One of them is that a \( \text{CLat}_\Lambda \)-fibration is always a bifibration whose push-forwards \( f_* \) arise essentially by Freyd’s adjoint functor theorem.

**Example C.4 (\( \text{CLat}_\Lambda \)-fibration \([23, \S 2.1]\)).** \( \text{Pred} \to \text{Set} \) The forgetful functor \( \text{Pred} \to \text{Set} \) is a \( \text{CLat}_\Lambda \)-fibration. Here \( \text{Pred} \) is the category of predicates; objects are sets with predicates \( (P \subseteq X) \), and morphisms \( f : (P \subseteq X) \to (Q \subseteq Y) \) are functions \( f : X \to Y \) satisfying \( f(P) \subseteq Q \).

(Domain fibration \( d^\Omega \)) For each complete lattice \( \Omega \), we introduce a \( \text{CLat}_\Lambda \)-fibration \( d^\Omega : \text{Set}/\Omega \to \text{Set} \) as follows. Here, we write \( \text{Set}/\Omega \) for the lax slice category with objects \((X, f : X \to \Omega)\) of a set and a function (an “\( \Omega \)-valued predicate on \( X \)”). We shall often write simply \( f : X \to \Omega \) for the pair \( (X, f) \). Its morphisms from \( f : X \to \Omega \) to \( g : Y \to \Omega \) are functions \( h : X \to Y \) such that \( f \leq_X g \circ h \), as shown above, where the order \( \leq_X \) is the pointwise order between functions of the type \( X \to \Omega \); the same order \( \leq_X \) defines the order in each fiber \((\text{Set}/\Omega)_X = \text{Set}(X, \Omega)\). Then \( d^\Omega \) is the evident forgetful functor, extracting the upper part of the above triangle. Following \([2\text{, Def. 4.1}]\), we call \( d^\Omega \) a domain fibration (from the lax slice category).

### D Structural Derivation of Instances of LT-PDR (\( \S 5 \))

In \( \S 5 \) for each instance of LT-PDR, we only sketched its structural derivation from the categorical theory in \( \S 1.2 \). Here we give a systematic exposition to the structural derivation.

We discuss concrete instances of our PDR framework. In its course, known PDR variations are organized in a unified categorical language; we also derive a new variation.

These concrete instances are formulated in a domain fibration \( d^\Omega \) for varying \( \Omega \) (Ex. C.4 see right). Given a complete lattice \( \Omega \), a set functor \( G \), and a monotone \( G \)-algebra \( \tau : G\Omega \to \Omega \) (see Def. D.1 below), we obtain a setting \((d^\Omega, G, \hat{G})\) of a safety problem (\( \S 4.2 \)). Specifically, \( \hat{G} \) is the lifting of \( G \) defined by the given monotone \( G \)-algebra \( \tau \), see Lem. D.2 below.

**Definition D.1 (monotone algebra \([2]\)).** Let \( G : \text{Set} \to \text{Set} \) be a functor and \( \Omega \) be a complete lattice. We call \( \tau : G\Omega \to \Omega \) a monotone \( G \)-algebra over \( \Omega \) if \( i \leq_X i' \Rightarrow \tau \circ Gi \leq_G Xi \) holds for all \( X \in \text{Set} \) and all \( i, i' \in \text{Set}(X, \Omega) \).

**Lemma D.2 (\([8]\)).** In the setting of Def. D.1, a monotone \( G \)-algebra \( \tau \) gives rise to the lifting \( \hat{G} : \text{Set}/\Omega \to \text{Set}/\Omega \) given by \( \hat{G}(x) = \tau \circ Gx \). \( \square \)

One benefit of this framework \((d^\Omega, G, \hat{G})\) is that we may easily get an involution appeared in Prop. 1.3. From a monotone function \( \neg \) : \( \Omega \to \Omega^{op} \) satisfying \( \neg \circ \neg = \text{id} \), we can define \( \neg : E_S \to E_S^{op} \) mapping \( f : S \to \Omega \) to \( \neg \circ f : S \to \Omega \). All involutions appeared in this section can be defined in this way.
D.1 LT-PDR for Kripke Structures: PDR\textsuperscript{F-Kr} and PDR\textsuperscript{IB-Kr}

We instantiate the structural theory in §4.2 to derive LT-PDR algorithms for Kripke structures. We then compare them with IC3/PDR \[9, 13\] and Reverse PDR \[25, 26\].

In most of the PDR literature, the target system is a Kripke structure that arises from a program’s operational semantics. A Kripke structure consists of a set $S$ of states and a transition relation $\delta \subseteq S \times S$ (we ignore initial states and atomic propositions). We regard $\delta$ as a function $S \rightarrow \mathcal{P}S$; it is thus a coalgebra of the powerset functor $\mathcal{P}$ on Set.

To employ the theory in §4.2, we next choose a complete lattice and a monotone $\mathcal{P}$-algebra. Consider the complete lattice $\mathbb{2} = \{\bot, \top\}$, and the monotone $\mathcal{P}$-algebra $\tau$ and $\longrightarrow$:

$$\mathcal{P}\mathbb{2} \rightarrow \mathbb{2}$$

mapping $A$ to $\bigwedge_{a \in A} a$. Then we obtain the triple $(\mathbb{2}, \mathcal{P}, \tau)$ as a setting of the safety problems in §4.2. Note that Set$/\mathbb{2}$ is isomorphic to the category of predicates Pred (see Ex. C.4).

We are ready to consider the backward safety problem for the transition system. Let $\alpha \subseteq S$ be a set of safe states. The backward safety problem for $(\iota, \delta, \alpha)$ in $(\mathbb{2}, \mathcal{P}, \tau)$ is the GFP-UA problem

$$\iota \leq\triangleleft \nu x. \alpha \land \delta^*\hat{\mathcal{P}}x \land (5)$$

This is the problem checking whether the initial states are always in the safe states after any steps. This setting satisfies both of the additional assumptions imposed in Prop. 4.2 and 4.3 (namely the existence of an adjoint and an involution). Therefore, we have two LT-PDR algorithms to solve (5) as below.

**FSP for Kripke Structures (Prop. 5.2)** Notice that the lifting $\hat{\mathcal{P}}S$, which maps $X \subseteq S$ to $\mathcal{P}X \subseteq \mathcal{P}S$, has a left adjoint $\bigcup : \text{Pred}_{\mathcal{P}S} \rightarrow \text{Pred}_S$. Then by Prop. 4.2 the backward safety problem (5) can be solved by the LT-PDR algorithm for the forward safety problem

$$\mu x. \iota \lor \bigcup \delta x \leq\triangleleft\alpha \land (6)$$

The forward predicate transformer $F'' = \iota \lor \bigcup \delta (-)$ on predicates over $S$ expands to $F''(x) = \iota \lor \bigcup_{s \in x} \delta(s)$, which has already appeared in Example 3.2.

**IBSP for Kripke Structures (Prop. 5.3)** Notice that the complete lattice of predicates over $S$ admits an involution defined by set complement $\neg : \mathcal{P}S \rightarrow \mathcal{P}S$. Then by Prop. 4.3 the backward safety problem (5) can be solved by the LT-PDR algorithm for the inverse backward safety problem

$$\mu x. \neg\alpha \lor \neg\delta^*\hat{\mathcal{P}}(\neg x) \leq\triangleleft\neg\iota \land (7)$$

The function $F'' = \neg\alpha \lor \neg\delta^*\hat{\mathcal{P}}(\neg x)$ that appears in the above expands to $F''(x) = (S \setminus \alpha) \cup \{s \mid \exists s' \in \delta s. s' \in x\}$. 
D.2 LT-PDR for MDPs: $\text{PDR}^{\text{IB-MDP}}$

We instantiate the theory in §4.2 to derive an LT-PDR algorithm for Markov decision processes (MDP for short). We then compare it with the probabilistic model checking algorithm $\text{PrIC3}$ [6].

An MDP consists of a set $S$ of states, a set $\text{Act}$ of actions and a transition function $\delta$ mapping $s \in S$ and $a \in \text{Act}$ to $\delta(s)(a)$ representing a probability distribution of next states. We model the transition function of the MDP as a coalgebra $\delta: S \to GS$ of $G := (D(\cdot) + 1)^{\text{Act}}$, where $D$ is the finite probability distribution endofunctor on $\text{Set}$ [27]. The case $\delta(s)(a) = * \in [0,1]$ means that the action $a$ is not available at $s$.

To employ the theory in §4.2, we next choose a complete lattice $\Omega$ and a monotone $G$-algebra over $\Omega$. Consider the complete lattice $[0,1]$ of the real numbers in the unit interval with the usual order, and the monotone algebra $\tau: G [0,1] \to [0,1]$ mapping $f$ to $\min\{\sum_{n \in [0,1]} n \cdot f(a)(n) \mid a \in \text{Act}, fa \neq *\}$ (note that $\min\{\} = 1$). Then we obtain the triple $(d[0,1],G,\hat{\delta})$ as a setting of the safety problems in §4.2. We note that $\hat{\delta}$ does not have a left adjoint (see Appendix G). We therefore cannot apply Prop. 4.2 to the current setting.

We are ready to consider the backward safety problem for MDPs. Let $s_\iota \in S$ be an initial state, and $\alpha \subseteq S$ be a set of safe states. We convert $s_\iota, \lambda$ and $\alpha$ to $[0,1]$-valued predicates $d_\iota,\lambda$ and $d_\alpha$: $d_\iota$ maps $s_\iota$ to $\lambda$ and others to 1, and $d_\alpha$ maps $s \in \alpha$ to 1 and $s \notin \alpha$ to 0. We use the involution $\neg: [0,1]^S \to [0,1]^S$ defined by $(\neg d)(s) := 1 - d(s)$, too. Then the backward safety problem for $(\neg d_\iota, \lambda, \delta, d_\alpha)$ in $(d[0,1],G,\hat{\delta})$ is the GFP-UA problem

$$\neg d_\iota,\lambda \leq^? \nu x. d_\alpha \wedge \delta^* \hat{\delta} x.$$ (8)

This is the problem whether the probability of being at $\alpha$ all the time is greater than or equal to $1 - \lambda$ under any choices of actions in the MDP.

**IBSP for MDPs (Def. 5.4)** Note that the complete lattice of $[0,1]$-valued predicates over $S$ admits the above involution $\neg$. Then by Prop. 4.3, the BSP in (8) can be solved by LT-PDR for the inverse backward safety problem

$$\mu x. \neg d_\alpha \vee \neg \delta^* \hat{\delta} (\neg x) \leq^? d_{e,\lambda}.$$ (9)

The precise algorithm is in Appendix E. The function $F' := \neg d_\alpha \vee \neg \delta^* \hat{\delta}(\neg -)$ used in (9) expands as follows (recall $\delta^* \hat{\delta}(d) = \tau \circ Dd \circ \delta$):

$$F'(d)(s) = \begin{cases} 1 & (s \notin \alpha) \\ \max\{\sum_{s' \in S} ds' \cdot \delta(s)(a)(s') \mid a \in \text{Act}, \delta(s)(a) \neq *\} & (s \in \alpha). \end{cases}$$ (10)

This is a standard Bellman Operator for MDPs.
D.3 LT-PDR for Markov Reward Models: PDR$^{\text{MRM}}$

We instantiate an LT-PDR algorithm for Markov Reward Models (MRM for short), which is seemingly new. As we said in §5.3, the safety problem we will define is not an instance of the theory in §4.2 (especially Prop. 4.1–4.3).

An MRM consists of a set $S$ of states and a transition function $\delta$ mapping $s \in S$ and $c \in \mathbb{N}$ to a function $\delta(s)(c) : S \to [0,1]$ that represents probability distribution of next states. We model the transition function of the MRM as a coalgebra $\delta : S \to G S$ of the endofunctor $G := \mathcal{D}((-) \times \mathbb{N})$, where $\mathcal{D}$ is introduced in §D.2.

Note that the above definition accommodates another definition of MRM, namely as an MC $(S, \delta_{\text{MC}} : S \to D S)$ with a reward function $\text{rew} : S \to \mathbb{N}$ [4]. Specifically, we can set $\delta(s)(c)(s') := \delta_{\text{MC}}(s)(s')$ if $c = \text{rew}(s)$ and $\delta(s)(c)(s') := 0$ otherwise.

We next define a complete lattice and a monotone $G$-algebra. Consider the complete lattice $[0, \infty]$ of the extended nonnegative real numbers with the usual order, and the monotone $G$-algebra $\tau : G[0, \infty] \to [0, \infty]$ defined by $\tau(\mu) = E_{(r,n) \sim \mu}[r + n]$. It takes the expectation of $r + n$ for the distribution $\mu \in G[0, \infty] = \mathcal{D}([0, \infty] \times \mathbb{N})$ (see [2] Ex. 6.6 for further details). Then we obtain the triple $(d^{[0,\infty]}, G, \tau)$, as a setting of the safety problem.

SP for MRMs (Def. 5.5) Using the above data, we obtain the function $\mathbf{F}'(x) = d_{\alpha} \land \delta^* G(x)$ in §5.3. This $\mathbf{F}'$ can be concretely described as in Def. 5.5.

E LT-PDR Algorithm PDR$^{\text{IB-MDP}}$ for MDPs in §5.2

See Algorithm 4. It solves the IBSP for MDPs (Def. 5.2).

F LT-PDR Algorithm PDR$^{\text{MRM}}$ for MRMs in §5.3

See Algorithm 5. It solves the SP for MRMs (Def. 5.5).

G No Adjunction in §5.2

In general, a component $\mathcal{G}_S : (\text{Set}/[0,1])_S \to (\text{Set}/[0,1])_{(DS+1)^\text{Act}}$ of the lifting $\mathcal{G}$ does not have a left adjoint since $\mathcal{G}_S$ may not preserve $\land$. It follows
The Lattice-Theoretic Essence of Property Directed Reachability Analysis 29

Input: \((d_ι, λ, δ, d_α)\)

Output: ‘True’ with a conclusive KT sequence, or ‘False’ with a conclusive Kleene sequence

Data: \((X; C)\) where \(X\) is a KT sequence \(X_0 \leq \cdots \leq X_{n-1}\) and \(C\) is a Kleene sequence \((C_i, C_{i+1}, \ldots, C_{n-1})\) in \((\text{Set}/[0,1])^s\) (\(C\) is empty if \(n = i\)).

Initially: \((X; C) = (0 \leq F'(0); ())\)

repeat
  \(\text{Valid}\) If \(X_{j+1} \leq X_j\) for some \(j < n - 1\), return ‘True’ with the conclusive KT sequence \(X\).

  \(\text{Unfold}\) If \(X_{n-1} \leq d_ι, λ\) (i.e. \(X_{n-1}(s_i) \leq λ\)), let \((X; C) := (X_0 \leq \cdots \leq X_{n-1} \leq 1; ())\).

  \(\text{Induction}\) If some \(k \geq 2\) and \(x \in L\) satisfy \(X_k \not\leq x\) and \(F'(X_{k-1} \land x) \leq x\), let \((X; C) := (X[X_j := X_j \land x]_{2 \leq j \leq k}; C)\).

  \(\text{Candidate}\) If \(C_i = ()\) and \(X_{n-1} \not\leq d_ι, λ\) (i.e. \(X_{n-1}(s_i) > λ\)), choose \(x: S \to [0, 1]\) satisfying \(x \leq X_{n-1}\) and \(x \not\leq d_ι, λ\), and let \((X; C) := (X; (x))\).

  \(\text{Model}\) If \(C_1\) is defined, return ‘False’ with the conclusive Kleene sequence \((0, C_1, \ldots, C_{n-1})\).

  \(\text{Decide}\) If \(C_i \leq F'X_{i-1}\) (i.e. for all \(s \in α\), there exists \(a \in \text{Act}\) such that \(C_i(s) \leq \sum_{s' \in S} X_{i-1} s' \cdot (s)(a)(s')\)), choose \(x: S \to [0, 1]\) satisfying \(x \leq X_{i-1}\) and \(C_i \leq Fx\), and let \((X; C) := (X; (x, C_i, \ldots, C_{n-1}))\).

  \(\text{Conflict}\) If \(C_i \not\leq F'X_{i-1}\) (i.e. there exists \(s \in α\) such that \(C_i(s) > \sum_{s' \in S} X_{i-1} s' \cdot (s)(a)(s')\) for all \(a \in \text{Act}\)), choose \(x: S \to [0, 1]\) satisfying \(C_i \not\leq x\) and \(F'(X_{i-1} \land x) \leq x\), and let \((X; C) := (X[X_j := X_j \land x]_{2 \leq j \leq i}; (C_{i+1}, \ldots, C_{n-1}))\).

until any return value is obtained;

Algorithm 4: LT-PDR Algorithm PDR\textsuperscript{IB-MDP} for MDPs
Heuristics for PDR

The above two do not coincide in general.

\[ H \]

\[ \text{Heuristics for PDR} \]

\[ \text{IB-MDP} \]

\[ \text{The above two do not coincide in general.} \]

\[ \text{⊓ ⊔} \]

\[ \text{\algorithmname} \]

\[ \text{\algorithmtitle: \text{LT-PDR algorithm} \text{PDR}^{\text{MRM}} \text{ for a Markov reward model}} \]

\[ \text{from the following calculation} \ (f,g : S \rightarrow [0,1] \text{ and } d : \text{Act} \rightarrow (DS + 1)) : \]

\[ (\hat{G}(f \land g))(d) = \langle \tau \circ (D(f \land g) + 1)^{\text{Act}} \rangle(d) \]

\[ = \min\{ \sum_{r \in [0,1]} r \cdot \sum_{da(s) | a \in \text{Act}, da \neq *}(da)(s) \mid \min(f_{s,gs}=r) \} \]

\[ (\hat{G}f \land \hat{G}g)(d) = \min\{ \sum_{r \in [0,1]} r \cdot \sum_{da(s), s=r} (da)(s) \sum_{r \in [0,1]} r \cdot \sum_{gs=r} (da)(s) \mid a \in \text{Act}, da \neq * \} \]

The above two do not coincide in general. \( \square \)

\section{H \ Heuristics for PDR\textsuperscript{IB-MDP} in \( \S 6 \)}

The algorithm \text{PDR}\textsuperscript{IB-MDP} in Alg.4 (except for \text{Induction}) is determined by heuristics, i.e. the way of choosing \( x : S \rightarrow [0,1] \) in \text{Candidate}, \text{Decide}, and
Conflict. The following is the heuristics of PDR\textsuperscript{IB-MDP} used in \cite{J.1}. We use a symbolic free variable $\epsilon$ for a positive margin, and define $a + \epsilon \leq b$ by $a < b$ and $a < b + \epsilon$ by $a \leq b$ for each $a, b \in [0, 1]$.

(Candidate): If $C = ()$ and $X_{n-1}(s_i) > \lambda$, let $(X; C) := (X; (x))$ where $x : S \rightarrow [0, 1]$ maps $s_i$ to $\lambda + \epsilon$ and others to 0.

(Decide): If $C \leq F'(X_{i-1})$ (i.e. for all $s \in \alpha$, there exists $a_s \in \text{Act}$ such that $C_i(s) \leq \sum_{s' \in S} X_{i-1} s' \cdot \delta(s)(a_s)(s')$), let $(X; C) := (X; (x, C_i, \ldots, C_{n-1}))$ where $x : S \rightarrow [0, 1]$ is defined as follows. Let $a_s \in \text{Act}$ be an action for $s \in \alpha$ satisfying $C_i(s) \leq \sum_{s' \in S} X_{i-1} s' \cdot \delta(s)(a_s)(s')$, and $V$ be the set $\{s' \in S \mid \delta(s)(a_s)(s') \neq 0 \text{ for some } s \in \text{supp}(C_i) \cap \alpha\}$. Then we define $x$ as

$$x(s) := \begin{cases} 0 & \text{if } s \notin V \\ x_s & \text{if } s \in V \text{ and } x_s = X_{i-1} s \\ x_s + \epsilon & \text{otherwise} \end{cases}$$

where $x_s$ is determined by solving the following linear program: find $(x_s)_{s \in V}$ that minimize $\sum_{s \in V} (2 - X_{i-1}s)x_s$ subject to $\{v_s \leq \sum_{s' \in S} x_{s'} \cdot \delta(s)(a_s)(s') \mid s \in \text{supp}(C_i) \cap \alpha, C_i(s) = v_s \text{ or } C_i(s) = v_s + \epsilon \text{ for some } v_s \in [0, 1]\} \cup \{0 \leq x_s \leq X_{i-1}s \mid s \in V\}$.

(Conflict): If $C_i > F'(X_{i-1})$ (i.e. there exists $s \in \alpha$ such that $C_i(s) > \sum_{s' \in S} X_{i-1}s' \cdot \delta(s)(a)(s')$ for all $a \in \text{Act}$), $A := \{s \in \alpha \mid C_i(s) > \sum_{s' \in S} X_{i-1}s', \delta(s)(a)(s') \text{ for all } a \in \text{Act}\}$ is not empty. Then let $(X; C) := (X[X_j := X_j \land x_{i \leq j \leq i}; C_{i+1}, \ldots, C_{n-1}) \mid X : S \rightarrow [0, 1] \text{ maps } s \notin A \text{ to } 1, s \in A \text{ with } C_i(s) = v + \epsilon \text{ to } v, \text{ and others to } F'X_{i-1}(s)$.

Note that $C_i(s)$ is always $v \in [0, 1]$ or $v + \epsilon$ for some $v \in [0, 1]$ by rules defined above. When applying Conflict, each values of $\epsilon$ in the Kleene sequence $C$ can be implicitly determined as small enough ones so that all conditions in rules (e.g. $C_i \leq X_i$ and $C_i \leq F'(X_{i-1})$) hold. By this fact the heuristics above is valid for Alg. \cite{J.1} The heuristics of PDR\textsuperscript{MRM} in \cite{J.1} is similarly designed.

I Full Experiment Results for PDR\textsuperscript{IB-MDP}

See Table \cite{I}

J Omitted Proofs

J.1 Proof of Cor. 2.3

\textit{Proof}. 1) easy. 2) By Thm. 2.2 we have the following.

$$\mu F \not\leq \alpha$$

$\Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $F^n \not\leq \alpha$

$\Leftrightarrow$ there exists $n \in \mathbb{N}$ and $x \in L$ such that $x \leq F^n \perp$ and $x \not\leq \alpha$.

\hfill $\Box$
Table 3: Results with $\text{PDR}^{\text{IB-MDP}}$. Comparison is against PrIC3 [6] with four different interpolation generalization methods (none, linear, polynomial, hybrid). The benchmarks are from [6]. $|S|$ is the number of states of the benchmark MDP.

“GT pr.” is for the *ground truth probability*, that is the reachability probability $Pr_{\text{max}}(s_i \models \phi(S \setminus \alpha))$ computed outside the solvers under experiments. The solvers were asked whether the GT pr. (which they do not know) is $\leq \lambda$ or not; they all answered correctly. The last five columns show the average execution time in seconds. – is for “did not finish,” for out of memory or timeout (600 sec.)

| Benchmark   | $|S|$ | $Pr_{\text{max}}(s_i \models \phi(S \setminus \alpha))$ | $\lambda$ | $\text{PDR}^{\text{IB-MDP}}$ w/o | lin | pol | hyb |
|-------------|------|---------------------------------------------------|---------|-------------------------------|-----|-----|-----|
| Grid 10²    | 0.3  | 0.31                                              | 1.31    | 19.34                         | –   | –   | –   |
|             | 0.2  | 0.48                                              | 1.75    | 24.62                         | –   | –   | –   |
| Grid 10³    | 0.3  | 112.29                                            | –       | –                             | –   | –   | –   |
|             | 0.2  | 136.46                                            | –       | –                             | –   | –   | –   |
| BRP 10³     | 0.01 | 18.52                                             | 56.55   | 594.89                        | 722.38 | –   | –   |
|             | 0.005| 1.36                                              | 11.68   | 238.09                        | –   | –   | –   |
| ZeroConf 10⁴| 0.9  | –                                                 | –       | 0.58                          | 0.51 | –   | –   |
|             | 0.75 | –                                                 | –       | 0.55                          | 0.46 | –   | –   |
|             | 0.52 | –                                                 | –       | 0.48                          | 0.46 | –   | –   |
|             | 0.45 | 0.014                                             | <0.1    | <0.1                          | <0.1 | <0.1| –   |
| Chain 10³   | 0.394| 0.9                                               | 72.37   | 0.91                          | 0.70 | –   | –   |
|             | 0.35 | 177.12                                            | 115.98  | –                             | –   | –   | –   |
|             | 0.3  | 88.27                                             | 66.89   | 557.68                        | –   | –   | –   |
| Chain 10⁴   | 0.394| 0.9                                               | –       | 0.86                          | 0.63 | –   | –   |
|             | 0.48 | –                                                 | –       | 0.84                          | –   | –   | –   |
|             | 0.4  | –                                                 | –       | 0.84                          | –   | –   | –   |
|             | 0.3  | –                                                 | –       | –                             | –   | –   | –   |
| Chain 10¹²  | 0.394| 0.9                                               | –       | 0.91                          | –   | –   | –   |
|             | 0.4  | –                                                 | –       | 0.89                          | –   | –   | –   |
| DoubleChain 10³ | 0.215| 0.9                                               | –       | 1.83                          | 1.99 | –   | –   |
|             | 0.3  | –                                                 | –       | 1.88                          | 1.96 | –   | –   |
|             | 0.216| –                                                 | –       | 139.76                        | –   | –   | –   |
|             | 0.15 | 7.46                                              | –       | –                             | –   | –   | –   |
| DoubleChain 10⁴ | 0.22 | 0.9                                               | –       | 1.83                          | 2.47 | –   | –   |
|             | 0.3  | –                                                 | –       | 2.11                          | 2.00 | –   | –   |
|             | 0.24 | –                                                 | –       | 2.01                          | –   | –   | –   |
J.2 Proof of Thm. 3.4

Proof. Since $L$ is a complete lattice, we consider a monotone function $\text{sup} : [\omega, L] \to L$ mapping $X$ to $\bigvee_{i \in \omega} X_i$, which has the upper (i.e. right) adjoint $\Delta : L \to [\omega, L]$.

Since $F$ is $\omega$-continuous, the monotone function $F^\#$ is a lifting of $F$ along sup, that is, $\text{sup} \circ F^\# = F \circ \text{sup}$ holds. Now one can easily check that we can restrict sup and $\Delta$ to functions between $\text{Pre}(F^\#)$ and $\text{Pre}(F)$, and a general result in category theory [15, Thm 2.14] tells us that the restrictions (denoted as $\text{Pre}(\text{sup})$ and $\text{Pre}(\Delta)$ in the above diagram) again form a Galois connection. Note that the initial chain of $F^\#$ is $\mu F^\#$, and is mapped to $\mu F$ by $\text{Pre}(\text{sup})(\mu F^\#) = \mu F$. We will use this fact in some later proofs.

Then $\text{Pre}(\Delta)$ maps a KT witness to a KT$^\omega$ witness, and $\text{Pre}(\text{sup})$ maps a KT$^\omega$ witness to a KT witness. \hfill \Box

J.3 Proof of Thm. 3.7

Proof. Assume $X_0 \preceq X_1 \preceq \cdots$ is an $\omega$-chain of KT sequences augmented with KT$^\omega$ witnesses. Then the suprema of this chain exist: its $j$-th element is the infimum of $\{X_i^j \mid X_i^j \text{ is defined}\}$ in $L$. The suprema compose $F^\#_n$ or $F^\#$-algebra and each element is less than or equal to $\alpha$. \hfill \Box

J.4 Proof of Thm. 3.10

Proof. (sound) easy by Cor. 2.3, Thm. 3.4 and Prop. 3.8 (weakly terminating). If $\mu F \preceq \alpha$ then the algorithm weakly terminates by the following procedure (skip Induction when we cannot apply the rule): $(\bot \preceq F \bot) \xrightarrow{\text{Unfold}} (\bot \preceq F \bot \preceq \mu F) \xrightarrow{\text{Unfold}} (\bot \preceq F \bot \preceq \mu F \preceq \top) \xrightarrow{\text{Induction}} (\bot \preceq F \bot \preceq \mu F)$. \hfill \Box

J.5 Proof of Lem. 3.11

Proof. When we cannot apply both Valid and Unfold, we can apply Induction by choosing $\mu F$ as $x$. \hfill \Box
J.6 Proof of Thm. 3.12

Proof. (non-termination) Since $L$ is well-founded, a non-terminating run $X^0 \preceq X^1 \preceq \cdots$ infinitely extends the length of KT sequences. Therefore, by Thm. 3.7, the supremum of the $\omega$-chain becomes a KT $\omega$ witness.

(strong termination) Assume there is a run of positive LT-PDR which does not terminate. Let $X^i$ be the $i$-step KT sequence and $X^i_j := \top$ when $|X^i| \leq j$. The $\omega$-chain $\bigwedge_{i \in \mathbb{N}} X^i$ is under $\alpha$ so it converges in some index $j$: $\bigwedge_{i \in \mathbb{N}} X^i_j = \bigwedge_{i \in \mathbb{N}} X^j_j + 1$.

We further assume that there is no $i \in \mathbb{N}$ such that $X^i_j = X^i_j + 1$. Then for each $i \in \mathbb{N}$, there exists $i'$ such that $X^i_j \geq X^i_j' + 1$ since $\bigwedge_{i \in \mathbb{N}} X^i_j = \bigwedge_{i \in \mathbb{N}} X^j_j + 1$.

Now $X^i_j' \neq X^i_j''$ so $X^i_j' < X^i_j''$ holds. Applying it repeatedly, we have $X^i_0 > X^i_0'' > \ldots$. This contradicts well-foundedness. $\square$

J.7 Proof of Thm. 3.16

Proof. 1) easy by Cor. 2.3 and Prop. 3.14. 2) By Thm. 2.2, there exists $n \in \mathbb{N}$ such that $F^j_n \perp \not\leq \alpha$. Then negative LT-PDR terminates when we choose $x$ in Candidate and Decide so as to get the conclusive Kleene sequence $(\perp, F_\perp, \ldots, F^n_\perp)$. $\square$

J.8 Proof of Prop. 3.17

Proof. 1) If $C_i \not\leq X_i$ then $C_i \not\leq F^i_\perp$ by Lem. 3.18 Lem. 3.19 (not 2) ⇒ not 1) concludes the proof.

2) Considering $j = 1$, Lem. 3.19 (not 3) ⇒ not 1) concludes the proof.

3) By Lem. 3.18 and the KT sequence $(X_0 \preceq \cdots \preceq X_{n-1})$, we have $F^{n-2}_\perp \preceq X_{n-2} \preceq \alpha$. Letting $i = n - 2$, Lem. 3.19 (not 2) ⇒ not 1) concludes the proof. $\square$

J.9 Proof of Lem. 3.18

Proof. In the proof of Thm. 3.4, we showed $\mu F^\#$ is the initial chain of $F$. Therefore, each prefixed point of $F^\#$ is greater than or equal to the initial chain of $F$.

This fact leads to the over-approximation of KT sequences. $\square$

J.10 Proof of Lem. 3.19

Proof. (1 ⇒ 2): $C_i \leq F C_{i-1} \leq \cdots \leq F^i C_0 = F^i \perp$. (2 ⇒ 1): It is true since $(\perp, F_\perp, \ldots, F^{i-1}_\perp, C_i, \ldots, C_{n-1})$ is a conclusive Kleene sequence. (2 ⇒ 3): $F^i \perp \leq F^j X_{i-j}$ by Lem. 3.18 $\square$
J.11 Proof of Lem. 3.21

Proof. Preservation of Kleene sequences is easily proved. We prove the preservation of KT sequences by checking each condition in Def. 3.5.

1. The initial $X$ satisfies $X_{n-2} \leq \alpha$ because $\bot \leq \alpha$. Rules except for Unfold cannot increase $X$, especially $X_{n-2}$, and Unfold also preserves $X_{n-2} \leq \alpha$.

2. The LT-PDR algorithm starts from $X = (\bot \leq F_{\bot}) \in [2, L]$ composing $F^\#_2$-algebra. All rules which update $X$ are the following:

   - (Unfold): For each $n, m$ with $n \leq m \leq \omega$, let $a$ denote the functor from $[m, L] \to [n, L]$ which shortens sequences by cutting large elements. Each $a$ has a right adjoint $r$ which appends a sequence by $\top \in L$.

     The rule sends $X \in [n, L]$ to $rX \in [n+1, L]$ by $r$ and we show $r$ sends $F^\#_n$-algebra to $F^\#_{n+1}$-algebra. As the same discussion in the proof of Thm. 3.4 Thm 2.14 in [15] yields the following since $F^\#_{n+1}$ is a lifting of $F^\#_n$ along $a$:

     $$F^\#_n \xrightarrow{\text{Unfold}} [n, L] \xrightarrow{a} [n+1, L] \xrightarrow{r} F^\#_{n+1} \xrightarrow{\text{Pre}} \xrightarrow{\text{Pre}} F^\#_{n+1}.$$ 

     Thus, $r$ preserves algebras.

   - (Induction, Conflict): These two rules preserve prefixed points of $F^\#$ because $F(X_{k-1} \wedge x) \leq x$ iff $F^\#_n (r \Delta x \wedge X) \leq r \Delta x$ ($\Delta : L \to [k+1, L]$ and $r : [k+1, L] \to [n, L]$) by the following:

     $$F(X_{k-1} \wedge x) = \sup aF^\#_n (X \wedge r \Delta x) \leq x$$

     $$aF^\#_n (X \wedge r \Delta x) \leq \Delta x \xrightarrow{\sup} \Delta : L \to [k+1, L]$$

     $$F^\#_n (X \wedge r \Delta x) \leq r \Delta x \xrightarrow{a + r : [k+1, L] \to [n, L]}$$

J.12 Proof of Prop. 3.23

Proof. Note that all rules in LT-PDR change the current data $(X; C)$. Since we have well-foundedness and the length of sequences in the data is always finite, Unfold or Model will be applied within finite steps.

When $\mu F \not\leq \alpha$ is true, there exists a conclusive Kleene sequence by Cor. 2.3 and Prop. 3.23. Letting $n$ be the length of the sequence, by Prop. 3.17.3 there is no KT sequence with length $n+1$. Thus the algorithm will terminate by Model in finite steps.

When $\mu F \leq \alpha$ is true and $(L, \leq)$ has no strictly increasing $\omega$-chain bounded by $\alpha$, we cannot apply Unfold infinitely. Thus the algorithm will terminate by Unfold within finite steps.
J.13 Proof of Prop. 4.2

Proof.

\[ \iota \leq \nu x. \alpha \land \delta^* \hat{G}x \]

iff there exists a coalgebra \( x \leq \alpha \land \delta^* \hat{G}x \) in \( E_S \) satisfying \( \iota \leq x \)

iff there exists \( x \) in \( E_S \) satisfying \( \iota \leq x \leq \alpha \) and \( x \leq \delta^* \hat{G}x \)

iff there exists \( x \) in \( E_S \) satisfying \( \iota \leq x \leq \alpha \) and \( \hat{F} \delta x \leq x \)

iff there exists an algebra \( \iota \lor \hat{F} \delta x \leq x \) in \( E_S \) satisfying \( x \leq \alpha \)

iff \( \mu x. \iota \lor \hat{F} \delta x \leq \alpha \).

\[ \Box \]

K Haskell Source Code for LT-PDR

The following is our Haskell implementation `ltPDR`.

class (Show a) => CLat a where
type Info a -- auxiliary information
    leq :: a -> a -> IO (Bool, Info a)
    bot :: a -> a -- include dummy argument
    top :: a -> a
    meet :: a -> a

    type KTSeq a = [a] -- [X_{n-1}, ..., X_1=f bot]
    type KleeneSeq a = Stack a -- Stack (n-i) [C_i, ..., C_{n-1}]

    newtype PDRConfig a = KTKl (KTSeq a, KleeneSeq a) deriving (Show)
    data PDRAnswer a = Valid (KTSeq a) | InValid (KleeneSeq a) deriving (Show)

    type (CLat a) => Heuristics a = Heuristics { f_candidate :: a -> a -> Info a -> IO a, f_decide :: a -> a -> (a -> a) -> Info a -> IO a, f_conflict :: a -> a -> (a -> a) -> Info a -> IO a }

    -- check whether \( \mu F \leq \alpha \)
    ltPDR :: forall a. CLat a => Heuristics a -> (a -> a) -> a -> IO (PDRAnswer a)

    let init = KTKl ([f $ bot alpha], stackNew) in
        loop init

    where
        loop :: PDRConfig a -> IO (PDRAnswer a)
        loop (KTKl (xs, cs)) = do
            rst <- sequence $ \[fst <$> leq (xs !! i) (xs !! (i+1)) | i <- [0..(length xs - 2)]\]
            if or rst
                then return $ Valid xs
            else if length xs == naturalToInt (stackSize cs)
                then return $ InValid cs
            else do
                (result1, solver1) <- leq (head xs) alpha
                if result1
                    then loop $ KTKl (top alpha:xs, stackNew)
                else case stackPop cs of
                        Nothing -> do
                            x <- f_candidate heuristics (head xs) alpha solver1
                            loop $ KTKl (xx, stackPush cs x)
                        Just (cs', ci) ->
                            let sizeOfcs = naturalToInt $ stackSize cs in
                            let xi1 = xs !! sizeOfcs in
do
                                (result2, solver2) <- leq ci (f xi1)
                                if result2
                                    then loop $ KTKl (top alpha:xs, stackNew)
                                else case stackPop cs of
                                        Nothing -> do
                                            x <- f_candidate heuristics (head xs) alpha solver1
                                            loop $ KTKl (xx, stackPush cs x)
                                        Just (cs', ci) ->
                                            let sizeOfcs = naturalToInt $ stackSize cs in
then do
  x <- f_decide heuristics x1 ci f solver2
  loop $ KT k1 (xs, stackPush cs x)
else do
  x <- f_conflict heuristics x1 ci f solver2
  let size0fxs = length xs
  let xs' = zipWith (h size0fc x size0fxs x) xs [0..]
  loop $ KT k1 (xs', cs')

h :: Int -> Int -> a -> a -> Int -> a
h size0fce size0fxs x x' i = if i == size0fxs - 1 || i < size0fce - 1 then x' else meet x x'