On quotients of affine superschemes over finite supergroups

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Abstract

In this article we consider sheaf quotients of affine superschemes by finite supergroups that act on them freely. More precisely, if a finite supergroup \( G \) acts on an affine superscheme \( X \) freely, then the quotient \( K\)-sheaf \( \tilde{X}/G \) is again an affine superscheme \( Y \), where \( K[Y] \simeq K[X]^G \). Besides, \( K[X] \) is a finitely presented projective \( K[X]^G \)-module.

Introduction

In the present article we prove that if a finite supergroup \( G \) acts on an affine superscheme \( X \) freely, then the sheaf quotient \( \tilde{X}/G \) is again affine and isomorphic to \( SSp R \), where \( R = K[X]^G \). Moreover, we also prove that \( K[X] \) is finitely presented projective \( R \)-module. This theorem generalizes the classical, purely even case (cf. [6, 5]). On the whole, we follow the ideas from [6] but there is a principal difference between purely even and super cases. In the classical case \( K[X] \) is always integral over \( R \). In the supercase it is not still true (see Example 3.1 below)! It happens as soon as the \( G \)-action is not free. In an equivalent formulation, for some finite supergroups 14-th Hilbert problem has the negative solution. To overcome this obstacle we exploit the freeness of our action and reduce the general case to the case, when \( G \) has not any proper normal supersubgroups.

1 Superalgebras and supermodules

In what follows all superalgebras are commutative. The category of commutative superalgebras with even morphisms is denoted by \( SAlg_K \). If \( A \in SAlg_K \), then the category of left (right) \( A \)-supermodules with even morphisms is denoted by \( A^-smod \) (respectively, \( smod^-A \)). Remind that \( A^-smod \simeq smod^-A \) [4]. More precisely, any \( M \in A^-smod \) has the structure of a right \( A \)-supermodule via \( ma = (-1)^{|a||m|}am \). Let \( M \) be a free \( A \)-supermodule of (finite) superrank \((m,n)\). Take elements \( m_1, \ldots , m_{m+n} \in M \) such that \( |m_i| = 0 \) iff \( 1 \leq i \leq m \), otherwise \( |m_i| = 1 \).

Lemma 1.1 The elements \( m_1, \ldots , m_{m+n} \) form a free basis of \( M \) iff their canonical images form a free basis of \( A/radA \)-supermodule \( M/(radA)M \).

Proof. By Nakayama’s lemma the supersubmodule \( N = \sum_{1 \leq i \leq m+n} Am_i \) coincides with \( M \) (cf. [1], Theorem 9.2.1(d)). Lemma 5.5 from [4] concludes the proof.
We say that an $A$-supermodule $M$ is finitely generated, if $M$ is an epimorphic image of a free $A$-supermodule of finite superrank. Besides, if the kernel of the above epimorphism is also finitely generated, then $M$ is called \textit{finitely presented}. It is obvious that $M$ is finitely generated as a supermodule iff it is finitely generated as a module.

\textbf{Lemma 1.2} A supermodule $M$ is finitely presented iff it is finitely presented as an $A$-module.

\textbf{Proof.} Let $\phi : A^n = \bigoplus_{1 \leq i \leq n} Ae_i \rightarrow M$ be an epimorphism of $A$-modules such that $\ker \phi$ is a finitely generated $A$-submodule of $A^n$. Consider a free $A$-supermodule $A^n|n$ with a basis $e_{i,\epsilon}$, $1 \leq i \leq n, \epsilon = 0, 1$. Denote $\phi(e_i)$ by $m_i$. Define the supermodule epimorphism $\psi : A^n|n \rightarrow M$ by $\psi(e_{i,\epsilon}) = m_i, \epsilon$, where $|m_i,\epsilon| = \epsilon$ and $m_i,0 + m_i,1 = m$. Since the elements $m_i$ generate $M$, we have $m_{i,0} = \sum_{1 \leq j \leq n} a_{ij}m_j, 1 \leq i \leq n, a_{ij} \in A$. A diagram

\[
\begin{array}{cc}
A^n & M \\
p \downarrow \phi & \psi \\
A^n|n & \\
\end{array}
\]

where $p(e_{i,0}) = \sum_{1 \leq j \leq n} a_{ij}e_j, p(e_{i,1}) = e_i - p(e_{i,0})$, is obviously commutative. Moreover, $p$ is an epimorphism and $\ker p$ contains a submodule $T$, generated by the elements

\[e_{i,0} - \sum_{1 \leq j \leq n} a_{ij}(e_{j,0} + e_{j,1}), 1 \leq i \leq n.\]

As $A^n|n/T$ is generated by the residue classes of $n$ elements $(e_{i,0} + e_{i,1})$, it follows that $p$ induces an isomorphism $A^n|n/T \simeq A^n$. In particular, the supersubmodule $\ker \psi = p^{-1}(\ker \phi)$ is finitely generated.

\textbf{Remark 1.1} If a superalgebra $A$ is finitely presented as a module over its supersubalgebra $B$, then $A$ is finitely presented as a $B$-superalgebra.

A superalgebra $A$ is called semi-local iff $A$ contains only finitely many maximal ideals. By the above, $A$ is semi-local iff $A_0$ is semi-local. Let $\mathcal{N}_1, \ldots, \mathcal{N}_t$ are all maximal ideals of $A$. It can be easily checked that \textit{Chinese reminder Theorem} holds for two-sided ideals of any (not necessary commutative) algebra or ring (see for example \cite{2}, II, §1, Proposition 5). Thus

\[A/\text{rad}A = A/ \bigcap_{1 \leq i \leq t} \mathcal{N}_i \simeq \prod_{1 \leq i \leq t} A/\mathcal{N}_i\]

is a direct product of fields. Conversely, if $A/\text{rad}A$ is a direct product of finitely many fields, then $A$ is semi-local. Besides, if $L$ is an $A$-module, then

\[L/(\text{rad}A)L \simeq \prod_{1 \leq i \leq t} L/\mathcal{N}_i L.\]

Let $A$ be a semi-local superalgebra and $B$ be its local supersubalgebra whose maximal ideal $\mathcal{M}$ is contained in $\text{rad}A$. Let $M$ be a free $A$-supermodule of finite superrank.

\textbf{Lemma 1.3} If $N$ is a $B$-supersubmodule of $M$ such that $AN = M$ and $B/M$ is an infinite field, then $N$ contains a free basis of $M$. 

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Proof. Using Lemma 1.1 one can replace \( A, M, N \) by \( A/radA, M/(radA)M \) and \((N + (radA)M)/(radA)M\) respectively. The final arguing can be copied from [3], III, §2, Lemma 4.7.

Let \( C \in SAlg_K \) and \( S \) is a multiplicative subset of \( C_0 \). One can define its left (right) \( S^{-1}C \)-supermodule of fractions \( S^{-1}M = S^{-1}C \otimes_C M \) (respectively, \( MS^{-1} = M \otimes_C S^{-1}C \)). It is clear that \( S^{-1}M \), considered as a right \( S^{-1}C \)-supermodule, is isomorphic to \( MS^{-1} \). The isomorphism is given by \( x \otimes m \mapsto (-1)^{|x||m|} m \otimes x, x \in S^{-1}C, m \in M \).

**Lemma 1.4** If \( M, N \in C - \text{smod} \), then \( S^{-1}C \)-supermodules

\[ S^{-1}M \otimes_C N, M \otimes_C S^{-1}N, S^{-1}M \otimes_{S^{-1}C} S^{-1}N \text{ and } S^{-1}(M \otimes_C N) \]

are canonically isomorphic each to other.

Proof. Routine checking (see also [2], II, §2, Proposition 18).

Let \( A \) be a superalgebra and \( B \) be its supersubalgebra. We say that \( A \) is an integral extension of \( B \) (or \( A \) is integral over \( B \)) iff \( A_0 \) is an integral extension of \( B_0 \). The following lemma is an obvious consequence of Lemma 1.2, [1].

**Lemma 1.5** If \( B \subseteq A \) is integral and \( P \) is a prime ideal of \( B \), then there is a prime ideal \( Q \) of \( A \) such that \( Q \cap B = P \) (in particular, \( PA \neq A \)). Moreover, \( P \) is maximal iff \( Q \) is maximal.

We say that \( Q \) lies over \( P \). Notice that a maximal ideal of the superalgebra of fractions \( A_P = (B_0 \setminus P_0)^{-1}A \) has a form \( Q_P \), where \( Q \) lies over \( P \). It infers that \( A_P \) is semi-local iff there are finitely many prime ideals of \( A \) those lie over \( P \). The last property is guaranteed for any \( P \), whenever \( A_0 \) is a finitely generated \( B_0 \)-module (cf. [2], V, §2, Proposition 3).

The proof of the following lemma can be copied from Proposition 8 and Proposition 9, [2], II, §3.

**Lemma 1.6** Let \( M \) be a maximal ideal of a superalgebra \( A \) and \( M \) be an \( A \)-supermodule. The canonical morphism \( M/M^0M \to M_M/M^0M_M \) is a supermodule isomorphism.

**Lemma 1.7** Let \( u : M \to N \) be a morphism of \( A \)-supermodules. If \( N \) is finitely generated, then \( u \) is surjective iff for any maximal ideal \( M \) of \( A \) the induced morphism \( M/M^0M \to N/MN \) is surjective.

Proof. Combine Lemma 1.6 with Lemma 1.5 from [4] and argue as in Proposition 11, [2], II, §3.

**Remark 1.2** The statements of the above lemmas are still true, even if \( M \) and \( N \) are \( A \)-modules.

**Proposition 1.1** Let \( \phi : B \to A \) be a superalgebra morphism such that \( A \) is finitely generated \( B \)-module. If the induced morphism \( SS\rho A \to SS\rho B \) is an inclusion of \( K \)-functors, then \( \phi \) is an epimorphism.

Proof. We use the following nice trick from [5], I, §5, 1.5. The diagonal morphism \( SS\rho A \to SS\rho A \times SS\rho B \ SS\rho A \) is an isomorphism. By Yoneda’s lemma the canonical superalgebra morphism \( A \otimes_B A \to A \) by \( a_1 \otimes a_2 \to a_1a_2, a_1, a_2 \in A \), is an isomorphism. It implies that for any maximal ideal \( N \) of \( B \) we have the isomorphism \( A/NA \otimes_B/N A/NA \to A/NA \). Comparing dimensions (over the field \( B/N \)) we see that \( B/N \to A/N \) is surjective. Lemma 1.7 concludes the proof.
2 Unipotent supergroups

In what follows a supersubgroup of an affine or algebraic supergroup is closed. We use notations and definitions from [4].

Let $G$ be an algebraic supergroup. It is called unipotent iff any simple $G$-supermodule is one dimensional and trivial. It is easy to see that $G$ is unipotent iff for any non-zero $G$-supermodule $V$ its invariant subspace $V^G$ is not zero also. By Proposition 6.2 from [4] there is a finite-dimensional $G$-supermodule $V$ such that $G$ is isomorphic to an supersubgroup of $GL(V)$. Since $G$ is unipotent, there is a flag of $G$-supersubmodules

$$0 \subseteq V_1 \subseteq \ldots \subseteq V_r = V$$

such that for all $i \geq 1$ $V_i/V_{i-1}$ is a trivial $G$-supermodule. Denote this flag by $V$. Consider the subfunctor $U(V) \subseteq GL(V)$ defined by

$$U(V)(A) = \{ g \in Stab_V(A)|g|_{V_i \otimes A} \text{ acts identically modulo } V_{i-1} \otimes A, i \geq 1 \}, A \in SAlg_K.$$

It is clear that $U(V)$ is a supersubgroup of $GL(V)$. In fact, there is a basis $v_1, \ldots, v_r$ of the superspace $V$ such that $|v_i| = 0$ if $1 \leq i \leq m$, otherwise $|v_i| = 1$, and a unique substitution $\sigma \in S_r$ with $\sigma(1) < \ldots < \sigma(m), \sigma(m+1) < \ldots < \sigma(r)$. Besides, $v_i$ generates $V_{\sigma(i)}/V_{\sigma(i)-1}$. The supersubgroup $U(V)$ is defined by $x_{ii} = 1$ and $x_{ji} = 0, 1 \leq i, j \leq r, \sigma(j) > \sigma(i)$. We also use for $U(V)$ the other notations, say $U_\sigma$ or $U_\sigma(m|n)$, where $n = \dim V_1 = r - m$. By the above, $G \leq U_\sigma$.

**Remark 2.1** A supergroup $U_\sigma$ is contained in $SL(V)$, that is $Ber(U_\sigma) = 1$. In fact, the berezinian $Ber$ induces a supergroup epimorphism $GL(V) \to GL(1|0) = G_m$. In particular,

$$K[U_\sigma] = K[x_{ij}|\sigma(i) < \sigma(j)] \simeq K[x_{ij}|1 \leq i < j \leq r] \simeq K[A^{m(m-1)/2} + n^{(n-1)/2}|mn].$$

**Lemma 2.1** Let $G$ be an algebraic supergroup. Assume that the superalgebra $K[G]$ has a $G$-supermodule (or equivalently, a right $K[G]$-supercomodule) filtration

$$0 \subseteq W_1 \subseteq W_2 \subseteq \ldots$$

such that $\bigcup_{i \geq 1} W_i = K[G]$ and each factor $W_i/W_{i-1}$ is a trivial $G$-supermodule. Then $G$ is unipotent.

Proof. Let $W$ be a simple $G$-supermodule and $f \in M^*, f \neq 0$. We have a supermodule morphism $g = (f \otimes id_{K[G]})\tau_W : W \to K[G]$ of the same parity as $f$. Since the preimages $g^{-1}(W_i)$ form a $G$-supermodule filtration of $W$ and $W$ is simple, we see that $W$ is isomorphic to a factor $W_i/W_{i-1}$.

**Proposition 2.1** If $G$ is isomorphic to a supersubgroup of $U_\sigma(m|n)$, then $G$ is unipotent.

Proof. One has to build an $U_\sigma$-supermodule filtration of $K[U_\sigma]$ as in Lemma 2.1. The superalgebra $R = K[U_\sigma]$ has a natural $\mathbb{N}$-grading $R = \bigoplus_{k \geq 0} R_k$, where each $R_k$ is a
the statement that the superalgebra morphism $\sigma: \text{epimorphism } U \to V$ is a simple supersubgroup of $G$ respectively. Besides, all sequential factors of this filtration are sums of trivial $U_\sigma$-supermodules. Proposition is proved.

**Corollary 2.1** If $G$ is unipotent, then any its supersubgroup and superfactor group is also unipotent.

The supergroups $U_1(2|0)$ and $U_1(1|1)$ are usually denoted by $G_a$ and $G_a^-$ respectively. Besides, $G_a$ is called even and $G_a^-$ is called odd one-dimensional additive supergroup.

**Lemma 2.2** Let $G$ be an algebraic supergroup and $N$ be its normal supersubgroup such that $N$ and $G/N$ are unipotent. Then $G$ is unipotent.

Proof. Let $V$ be a simple $G$-supermodule. We know that $V^N \neq 0$ and $V^N$ is the largest supersubspace of $V$ whose coefficient (super)space belongs to $K[G]^N = K[G/N]$. Thus $V = V^N$ is a simple $G/N$-supermodule. In particular, $V$ is one-dimensional and trivial.

**Lemma 2.3** Let $\pi: G \to H$ be an algebraic group epimorphism with the kernel $N$. If $L$ is a supesubgroup of $G$, then $L/L \cap N$ is canonically isomorphic to $\text{Im} \pi|_L$.

Proof. Notice that ker $\pi|_L = L \cap N$ and use Theorem 6.1 from [4].

**Proposition 2.2** If $G$ is unipotent, then $G$ has a series of normal supersubgroups $1 \leq N_1 \leq \ldots \leq N_t = G$ such that any factor $N_i/N_{i-1}$ belongs to $Z(G/N_{i-1})$ and isomorphic either to a supersubgroup of $G_a$ or to $G_a^-$. 

Proof. By Lemma 2.3 all we have to prove is that such series exists in $U_\sigma$. For any $k \geq 1$ define a superideal $I_k$ of $K[U_\sigma]$, generated by the elements $x_{ij}$ with $\sigma(j) - \sigma(i) \leq k$. It can easily be checked that $U_{\sigma,k} = V(I_k) \subset U_\sigma$. Indeed, the superalgebra $B_k = K[x_{ij}] / \sigma(j) - \sigma(i) \leq k$ is a Hopf supersubalgebra of $K[U_\sigma]$ and $U_{\sigma,k}$ coincides with the kernel of the epimorphism $U_\sigma \to SSp B_k$. In the same way,

$$U_{\sigma,k}/U_{\sigma,k+1} \simeq SSp B_{k+1}/B_{k+1}B_k^+ \simeq SSp K[x_{ij}] / \sigma(j) - \sigma(i) = k+1] \simeq (G_a)^s \times (G_a^-)^l,$$

where $s$ (respectively, $l$) is the number of even (respectively, odd) elements among $\{x_{ij} | \sigma(j) - \sigma(i) = k+1\}$. It remains to check that $U_{\sigma,k}/U_{\sigma,k+1} \leq Z(U_\sigma/U_{\sigma,k+1})$. It is equivalent to the statement that the superalgebra morphism

$$K[U_{\sigma,k}/U_{\sigma,k+1}] \otimes K[U_\sigma/U_{\sigma,k+1}] \to K[U_{\sigma,k}/U_{\sigma,k+1}],$$

superspace. Ascribe to any monomial $m = x_{i_1j_1} \cdots x_{i_kj_k} \in R_k \setminus 0$ the weight $\nu(m) = \sum_{1 \leq t \leq k}(\sigma(j_t) - \sigma(i_t))$. It is easy to see that

$$\tau_R(m) - m \otimes 1 \in \sum_{m' \in R_k, \nu(m') < \nu(m)} m' \otimes R + \sum_{0 \leq s \leq k-1} R_s \otimes R.$$

In particular, we have a $U_\sigma$-supermodule filtration

$$0 \subseteq K = R_{1,0} \subseteq R_{1,2} \subseteq \ldots \subseteq R_{k,t} \subseteq \ldots,$$

where

$$R_{k,t} = \bigoplus_{0 \leq s \leq k-1} R_s \bigoplus \bigoplus_{m \in R_k, \nu(m) \leq t} (Km), \quad k \geq 1, \quad 0 \leq t \leq (r-1)k.$$

Besides, all sequential factors of this filtration are sums of trivial $U_\sigma$-supermodules. Proposition is proved.
induced by \( \nu_1 \), coincides with the morphism \( f \to f \otimes 1, f \in K[U_{\sigma,k}/\tilde{U}_{\sigma,k+1}] \). The last one is dual to the projection

\[
U_{\sigma,k}/\tilde{U}_{\sigma,k+1} \times U_{\sigma}/\tilde{U}_{\sigma,k+1} \to U_{\sigma,k}/\tilde{U}_{\sigma,k+1}.
\]

Since

\[
\nu_1(x_{ij}) = \sum_{t_1,t_2,\sigma(i)\leq \sigma(t_1) < \sigma(t_2) \leq \sigma(j)} (-1)^{|x_{t_1,t_2}||x_{t_1,t_2}|} x_{t_1,t_2} \otimes x_{t_1,t_2} \otimes \delta U_{\sigma}(x_{t_2,j}),
\]

we obtain that \( \nu_1(x_{ij}) = x_{ij} \otimes 1 \) modulo \( B_{k+1}B_1^+ \). Proposition is proved.

\section{Proof of the main theorem}

Let \( G \) be an algebraic supergroup. Assume that \( G \) acts on an affine superscheme \( X \). Denote the corresponding morphism of (affine) superschemes \( X \times G \to X \) by \( \phi_0 \). For the reader’s convenience we remind some basic notations and facts from \cite{6}, III, §2-4. The squares

\[
\begin{array}{cccc}
X \times G \times G & \xrightarrow{\phi_0} & X \times G & \xrightarrow{\phi_1} & X \times G \\
\phi_2 & \downarrow & \downarrow \phi_1 & & \downarrow \phi_1 \\
X \times G & \xrightarrow{\phi_0} & X \times G & \xrightarrow{\phi_1} & X
\end{array}
\]

are cartesian, where \( \phi_1 = pr_X, \phi_2 = pr_{X \times G} \) and \( \phi_0(x, g, h) = (x, gh), x \in X(A), g, h \in G(A), A \in SAlg_k \). The morphism of superalgebras \( K[X] \to K[X] \otimes K[G] \), dual to \( \phi_0 \) (respectively, dual to \( \phi_1 \)), is denoted by \( \tau_X \) (respectively, by \( i_X \)). The supersubalgebra of (co)invariants \( K[X]^G = \ker(\tau_X - i_X) \) is denoted by \( R \). Since \( \phi_0 \) has a left inverse \( \sigma(x) = (x, 1), x \in X(A), A \in SAlg_k \), the couple \( (X, \phi_0) \) is a cokernel of the pair morphisms \( (\phi_0', \phi_1') \) (in the category of \( K \)-functors!). Dualizing we obtain a commutative diagram

\[
\begin{array}{cccc}
K[X] \otimes K[G]^\otimes & \xrightarrow{i_X} & K[X] \otimes K[G] & \xrightarrow{\tau_X} & K[X] \\
\delta_2 & \downarrow & \downarrow i_X & & \downarrow \tau_X \\
K[X] \otimes K[G] & \xrightarrow{i_X} & K[X] & \xrightarrow{\tau_X} & K[X] \otimes K[G]
\end{array}
\]

where \( \delta_0 = \tau_X \otimes id_{K[G]}, \delta_1 = id_{K[X]} \otimes \delta_G, \delta_2 = id_{K[X] \otimes K[G]} \otimes 1 \). Its horizontal lines are exact and the left square is composed from cocartesian squares those are dual to the above first and third cartesian ones. We call this diagram \textit{basic}.

From now on we assume that all supergroups are finite unless otherwise stated. Without loss of generality one can assume that \( K \) is algebraically closed. The \( K \)-functor morphism \( (\phi_1, \phi_0) : X \times G \to X \times SSp_R X \) is dual to the morphism of superalgebras

\[
\psi : K[X] \otimes_R K[X] \to K[X] \otimes K[G]
\]

defined as

\[
f \otimes h \to \sum f h_1 \otimes h_2, \tau_X(h) = \sum h_1 \otimes h_2, f, h, h_1 \in K[X], h_2 \in K[G]
\]
Lemma 3.1 If $G$ acts on $X$ freely, then $\psi$ is surjective.

Proof. Notice that $(\phi_1, \phi_0)$ is an injective $K$-functor morphism and $\text{Im} \psi$ contains $K[X] \otimes 1$. It remains to refer to Proposition 1.1.

Lemma 3.2 Let $B$ be a supersubalgebra of a superalgebra $A$. Then:
1) If $A$ is a finitely generated $B$-module, then $A$ is integral over $B$;
2) If $A$ is a finitely generated superalgebra and integral over $B$, then $A$ is a finitely generated $B$-module and $B$ is a finitely generated superalgebra.

Proof. To prove the first statement we fix a finite set of generators of $B_0$-module $A_0/B_1A_1$. Using Cayley-Hamilton’s theorem we see that for any $a \in A_0$ there is a unitary polynomial $f(t) \in B_0[t]$ such that $f(a)A_0 \subseteq B_1A_1$. In particular, $f(a) \in B_1A_1$ and since $AA_1$ is nil, it is done. For the second statement notice that $B_0$ is finitely generated. Since $A$ is a finitely generated $A_0$-module, it implies that $A$ is a finitely generated $B_0$-module. In particular, $B_1$ is a finitely generated $B_0$-module.

Proposition 3.1 Assume that $K[X]$ is a finitely generated $R$-module and $K[X]_0$ is a finitely generated algebra. Then $X/G \cong \text{SSp} \ R$, provided $G$ acts freely on $X$. Besides, $K[X]$ is a projective $R$-module.

Proof. One has to superize [9], III, 4.6. More precisely, we prove that $\psi$ is a superalgebra isomorphism and $K[X]$ is a projective $R$-module. By Lemma 1.4 one can replace $R$ and $K[X]$ by $R_P$ and $K[X]_P$, where $P$ is a prime ideal of $R$. In other words, one can assume that $R$ is local and $K[X]$ is semi-local. Since $\psi$ is a $K[X]$-supermodule morphism, Lemma 1.3 and Lemma 3.1 infer that there are elements $f_1, \ldots, f_{m+n} \in K[X]$, where $|f_i| = 0, 1 \leq i \leq m = \dim K[G]_0, |f_i| = 1, m+1 \leq i \leq m+n, n = \dim K[G]_1$, such that $\tau_X(f_1), \ldots, \tau_X(f_{m+n})$ form a basis of the free $K[X]$-supermodule $K[X] \otimes K[G]$. Let $V$ be a superspace of superdimension $(m, n)$ with a basis $v_1, \ldots, v_{m+n}$ such that $|v_i| = |f_i|, 1 \leq i \leq m+n$. Tensoring by $V$ the bottom line of the basic diagram we obtain a diagram

$$
\begin{array}{ccc}
K[X] \otimes K[G] \otimes^2 & \xrightarrow{\delta_0} & K[X] \otimes K[G] \\
\uparrow u_2 & & \uparrow u_1 \uparrow u_0 \\
V \otimes K[X] \otimes K[G] & \xrightarrow{V \otimes \tau_X} & V \otimes K[X] \\
& \xleftarrow{V \otimes \iota_X} & \downarrow \ i \\
& & V \otimes R
\end{array}
$$

where $u_0(v_i \otimes r) = f_i r, u_1(v_i \otimes f) = \tau_X(f_i)i_X(f), u_2(v_i \otimes t) = \delta_0(\tau_X(f_i))\delta_2(t)$. By definition, $u_1$ is an isomorphism of $K[X]$-supermodules. As in [9] we conclude that $u_2$ is an isomorphism (of superspaces) and therefore, $u_0$ is. In particular, $K[X]$ is a free $R$-supermodule and the elements $f_i$ form its basis. Returning to the general case, by Lemma 1.5 from [4] we obtain that $\psi$ is an isomorphism and $K[X]$ is a projective $R$-module by [9], Theorem A.2.4. By Lemma 1.5 (see also [2], I, §2, Proposition 1) $K[X]$ is a faithfully flat (left and right) $R$-module. Proposition 4.2, [4], concludes the proof.
Let a group $K$-sheaf $G$ acts on a $K$-sheaf $X$ freely. If $A, B \in SAlg_K$ and $B$ is a fpfp covering of $A$, then we denote $B \supseteq A$. Notice that $\supseteq$ is a (partial) direct order. If $X$ is a $K$-functor, the kernel of maps $X(A) \xrightarrow{X(\iota_1)} X(B \otimes_A B)$, where $i_1(a) = a \otimes 1, i_2(a) = 1 \otimes a, a \in A$, is denoted by $X(B, A)$ (see [1] [2] for more definitions and notations).

**Proposition 3.2** Let $N$ be a normal group $K$-subfunctor of $G$. Then the group $K$-sheaf $G/N$ acts freely on $Y = X/N$ and $Y/H \simeq X/G$.

Proof. Denote the "naive" factors

$$A \to G(A)/N(A), A \to X(A)/N(A), A \in SAlg_K,$$

by $H(n)$ and $Y(n)$ correspondingly. Consider $h \in H(A), y \in Y(A)$. There is a fpfp-covering $B \supseteq A$ such that $h' = H(y_B^n)(g) \in H(n)(B, A), y' = Y(y_B^n)(y) \in Y(n)(B, A)$. By the normality of $N$, the group functor $H(n)$ acts canonically on $Y(n)$. In particular, $y'h' \in Y(n)(B, A) \subseteq Y(B, A)$. Since $Y$ is a sheaf, one can define $yh = Y(y_B^n)^{-1}(y'h') \in Y(A)$. This definition does not depend on the choice of $B$. In fact, let $C$ be another fpfp-covering of $A$. Then $D = B \otimes_A C \supseteq B, C$ (see [1] [2]). Thus $D \supseteq A$. Set

$$H(y_B^n)(h) = h'', Y(y_B^n)(y) = y'', H(y_B^n)(h) = h'''', Y(y_B^n)(y) = y''''.''

We have

$$H(y_B^n)(h') = H(y_B^n)(h'') = h'''', Y(y_B^n)(y') = Y(y_B^n)(y'') = y''''.

It follows that $Y(y_B^n)(y'h') = Y(y_B^n)(y''''') = y''''. On the other hand, all morphisms $Y(y_B^n)$ are mono and therefore,

$$Y(y_B^n)^{-1}(y''''') = Y(y_B^n)^{-1}(y'h') = Y(y_B^n)^{-1}(y''''').$$

Similarly, one can prove that $H$ acts on $Y$ freely. To prove that the above action is functorial on the argument $A \in SAlg_K$ one can mimic the proof of Lemma 2.3 from [4].

Finally, let $\rho : X \to Z$ be a $K$-sheaf morphism such that $\rho(A)(xg) = \rho(A)(x)$ for all $x \in X(A), g \in G(A), A \in SAlg_K$. There is a unique morphism $\alpha : Y \to Z$ satisfying $\rho = \pi \alpha$, where $\pi : X \to Y$ is the canonical factor-morphism. More precisely, for any $y \in Y(A)$ and for a fpfp covering $B \supseteq A$ such that $Y(y_B^n)(y) = xN(B), x \in X(B)$, we set $\alpha(A)(y) = Z(y_B^n)^{-1}(\rho(B)(x))$. Comparing with the definition of the $H$-action on $Y$ we see that $\alpha$ is constant on $H$-orbits. In particular, there is a unique morphism $\beta : Y/H \to Z$ such that $\beta \pi' = \alpha$, where $\pi' : Y \to Y/H$ is the corresponding factor-morphism. In other words, morphism $\pi' \pi : X \to Y/H$ is the required factor-morphism. Theorem is proved.

**Remark 3.1** The same statement can be proved for dur $K$-sheafs.

**Lemma 3.3** An (not necessary finite) algebraic supergroup $G$ acts freely on an affine superscheme $X$ iff the ideal $J$ of $K[X] \otimes K[G]$, generated by the elements $\tau_X(f) - f \otimes 1, f \in K[X],$ contains $1 \otimes M$. 

Proof. If $1 \otimes M$ is not contained in $J$, then set $A = K[X] \otimes K[G]/J$ and define

$$\alpha(f) = f \otimes 1 + J, g(h) = 1 \otimes h + J, f \in K[X], h \in K[G].$$

It is obvious that $g \in \text{Stab}_{G(A)}(\alpha) \setminus 1$. Conversely, if $g \in \text{Stab}_{G(A)}(\alpha), \alpha \in X(A)$, then

$$\alpha \otimes g(\tau_X(f) - f \otimes 1) = 0$$

for any $f \in K[X]$, where $\alpha \otimes g(f \otimes h) = \alpha(f)g(h)$. The inclusion $1 \otimes M \subseteq J$ implies $g(M) = 0$.

Now, everything is prepared to prove the main theorem. At first, assume that $K[X]$ is finitely generated. Using induction on $[G]$ we prove that $K[X]$ is a finitely generated $K[X]^G$-module and then apply Proposition 3.1. If $G$ has a proper normal supersubgroup $N$, then by the inductive hypothesis $K[X]$ is a finitely generated $K[X]^N$-module. By Proposition 3.1 $X/N \simeq SSp \ K[X]^N$ and by Proposition 3.2 $G/N$ acts on $X/N$ freely. Since by Lemma 3.2 $K[X]^N$ is finitely generated, again the inductive hypothesis infers that $K[X]^N$ is a finitely generated $K[X]^G = (K[X]^N)^{G/N}$-module. Thus $K[X]$ is a finitely generated $K[X]^G$-module. So, it remains to prove that $K[X]$ is a finitely generated $K[X]^G$-module, whenever $G$ has no proper normal supersubgroups. In particular, $G$ is either connected or purely even and etale (cf. [4]). Assume that $G$ is connected and $G \neq 1$.

Proposition 3.3 If $\text{char}K = 0$, then $G \simeq G_a^\ast$.

Proof. As it was noticed in [4], $M = r + K[G]e$, where $r$ is the radical of $K[G]$ and $e$ is the sum of primitive idempotents belonging to $M$. Besides, $I_{G(0)} = \bigcap_{l \geq 0} M^l = K[G]e$. Since $G = G(0)$, it follows that $e = 0$ and $M = r = K[G]K[G]_1$. In particular, $M/M^2$ is purely odd that implies $\text{Lie}(G)_0 = 0$ and $\text{Lie}(G)_1^2 = 0$. In other words, $\text{Lie}(G)$ is abelian and as in [4] we conclude that $G$ is abelian. By Lemma 9.5, [4], for any finite-dimensional $G$-supermodule $V$ the equality $V^G = V^{\text{Lie}(G)}$ holds. In fact, $V^G$ is naturally identified with $\text{Hom}_G(K,V)$, where $K$ is regarded as one-dimensional trivial $G$-supermodule. Identify $\text{Lie}(G)$ with an odd abelian supersubalgebra $L$ of $gl(V)$. Then for all $x, y \in L$ we have $xy = -yx$. Let $A$ be an associative subalgebra of $\text{End}_K(V)$ without unit, generated by $L$. It is clear that $A^{\dim L + 1} = 0$ and by Engel’s theorem there is a vector $v \in V$ such that $Av = 0$. In particular, $G$ is unipotent. Proposition 2.2 concludes the proof.

Remark 3.2 Proposition 3.3 infers that over an algebraically closed field of characteristic zero, any finite supergroup is an extension of abelian unipotent supersubgroup by an even etale group. It seems to be very likely that such extension have to be split (for the classical case see Theorem 3.3 from [7]). We hope to check all details in a next article and to get rid of the assumption about the ground field to be algebraically closed.

Let $G = G_a^\ast$. Remind that

$$K[G] = K[t], |t| = 1, \delta_G(t) = t \otimes 1 + 1 \otimes t, \epsilon_G(t) = 0, s_G(t) = -t.$$

Lemma 3.4 A $G$-supermodule structure on a superspace $V$ is uniquely defined by an odd (locally finite) endomorphism $\phi : V \to V, \phi^2 = 0$. Precisely, $\tau_V(v) = v \otimes 1 + \phi(v) \otimes t$ and therefore, $V^G = \ker \phi$. 

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Proof. Straightforward calculations.

By Lemma 3.4 $\tau_X(f) = f \otimes 1 + \phi(f) \otimes t, f \in K[X]$, where $\phi \in \text{End}_K(K[X])_1$ and $\phi^2 = 0$. Since $\tau_X$ is a superalgebra morphism, we obtain that $\phi(f_1 f_2) = f_1 \phi(f_2) + (-1)^{|f_2|} \phi(f_1) f_2, f_1, f_2 \in K[X]$. In other words, $\phi$ is a right (odd) superderivation.

**Lemma 3.5** The supergroup $G$ acts on $X$ freely iff there is $f \in K[X]_1$ such that $\phi(f) \in K[X]^*$. 

Proof. By Lemma 3.3 $G$ acts on $X$ freely iff there are $h_1, \ldots, h_n \in K[X]_1$ and $f_1, \ldots, f_n \in K[X]_0$ such that $\sum_{1 \leq i \leq n} f_i \phi(h_i) = 1$. Thus $\phi(\sum_{1 \leq i \leq n} f_i h_i) = 1 - \sum_{1 \leq i \leq n} \phi(f_i) h_i \in K[X]^*$.

**Lemma 3.6** The superalgebra $K[X]$ is a free $R$-supermodule of rank 2.

Proof. Consider $g \in K[X]_1$ such that $g = \phi(f) \in K[X]^*$. Set $z = fg^{-1}$. Since $\phi(z) = 1$, for any $h \in K[X]$ we have $\phi(hz) = h - \phi(h)z$. Thus $K[X] = R + Rz$. If $h \in R \cap Rz$, then $h = rz, r \in R$, and therefore, $0 = \phi(h) = r$.

**Example 3.1** (see [4], section 10) Consider a $G = G_{\infty}$-supermodule $V$ with a basis $v_1, v_2, |v_1| = 0, |v_2| = 1$, such that $\phi(v_1) = v_2, \phi(v_2) = 0$ in the above notations. The symmetric superalgebra $S(V)$ has the induced $G$-supermodule structure by 

$$
\tau_S(V)(v_1^r) = v_1^r \otimes 1 + rv_1^{r-1}v_2 \otimes t, \quad \tau_S(V)(v_1^{r-1}v_2) = v_1 ^{r-1}v_2 \otimes 1, r \geq 0.
$$

By Lemma 3.3 the induced $G$-action on $X = SSp S(V)$ is not free. Moreover, $K[X] = S(V)$ is nor finitely generated $R$-module neither integral over $B$, provided $\text{char} K = 0$. In fact, the superalgebra $R = K \bigoplus (\bigoplus_{r \geq 1} K v_1^{r-1}v_2)$ is not finitely generated. In [4] it was also proved that $K[X]$ is not any flat $R$-module. Notice that the final conclusion in [4] is not completely correct. Indeed, Proposition 4.1 holds for free actions which is not the case.

Let $\text{char} K = p > 0$. An algebraic supergroup $H$ is called *infinitesimal supergroup* of hight 1 if $h^p = 0$ for any $h \in M$.

**Lemma 3.7** If $\text{char} K = p > 0$ and $G$ is connected, then $G$ is infinitesimal supergroup of hight 1. In particular, $K[X]$ is a finitely generated $K[X]^G$-module.

Proof. As above, $M = \ker \epsilon_G = r$. We have a series $1 \leq G_1 \leq G_2 \leq \ldots \leq G$, where each $G_n$ is a $n$-th infinitesimal supersubgroup (cf. [4, 5]). Since any $G_n$ is a normal supersubgroup of $G$, we have either $G_1 = 1$ and $K[G] = F(K[G]) = \{f^p | f \in K[G]\}$ (see the notice before Lemma 8.2, [4]), or $G_1 = G$. The equality $K[G] = F(K[G])$ implies $r = F(r)$. The nilpotence of $r$ infers $r = 0$ and $G = 1$. The last case $G = G_1$ is equivalent to $f^p = 0$ for any $f \in M$. Finally, for any $f \in K[X]$ we have $\tau_X(f) = f \otimes 1 + \sum f_i \otimes h_2$, where each $h_2$ belongs to $M$. Thus $\tau_X(f^p) = f^p \otimes 1$, that is $f^p \in K[X]^G$. Lemma 3.2 concludes the proof.

Now, let $G$ be even and etale. It is well known that $K[G] \simeq (K\Gamma)^*$, where $\Gamma$ is a finite group and $K\Gamma$ is its group algebra, endowed with Hopf algebra structure by $\delta_{K\Gamma}(\gamma) = \gamma \otimes \gamma, s_{K\Gamma}(\gamma) = \gamma^{-1}, \gamma \in \Gamma$ (see [5], part I 8.5, 8.21) and [8, 2.3, 6.4, or see [6].
Therefore, \( K[G] \) is generated by the idempotents \( e_\gamma \), such that \( e_\gamma(\gamma') = \delta_{\gamma,\gamma'} \) and

\[
e(e_\gamma) = \delta_{\gamma,1}, \delta_G(e_\gamma) = \sum_{\gamma' \in \Gamma} e_{\gamma'} \otimes e_{\gamma'-1,\gamma}, s_G(e_\gamma) = e_{-1,\gamma,\gamma} \in \Gamma.
\]

A vector superspace \( V \) is called \( \Gamma \)-supermodule iff it is a \( \Gamma \)-module and any \( \gamma \in \Gamma \) acts on \( V \) as an even operator. The category of \( \Gamma \)-supermodules with even morphisms is denoted by \( \Gamma - \text{smod} \). If \( V \in \Gamma - \text{smod} \), then it has a \( G \)-supermodule structure by

\[
\tau_V(v) = \sum_{\gamma \in \Gamma} \gamma v \otimes e_\gamma, v \in V.
\]

This correspondence defines an equivalence of categories. In particular, \( G \) acts on an affine superscheme \( X \) iff \( K[X] \) is a \( \Gamma \)-supermodule and any \( \gamma \in \Gamma \) acts as a superalgebra automorphism. Since

\[
K[X]^G = K[X]^\Gamma = K[X]_0^\Gamma \bigoplus K[X]_1^\Gamma
\]

this case is also done.

It remains to consider the case when \( K[X] \) is not finitely generated. Since any \( K[G] \)-supercomodule is locally finite, the superalgebra \( K[X] \) is a direct union of its finitely generated subalgebras \( B_i, i \in I \), such that each \( B_i \) is a \( G \)-supersubmodule of \( K[X] \). In other words, \( G \) acts on any \( \text{SSp} \, B_i \) and the canonical morphism \( \text{SSp} \, B_i \to X \) commutes with this action. Since \( \mathcal{M} \) is finite-dimensional, by Lemma 3.3 one can assume that \( G \) acts freely on each \( \text{SSp} \, B_i \). By the above, for any \( i \in I \) the superalgebra \( B_i \) is a faithfully flat (left and right) \( R_i \)-module and the canonical morphism \( B_i \otimes_{R_i} B_i \to B_i \otimes K[G] \) is an isomorphism. Thus \( K[X] = \lim_i B_i \) is a faithfully flat (left and right) \( R = \lim_i R_i \)-module (cf. Lemma 7.1, III, §3, [6]) and

\[
K[X] \otimes_R K[X] = \lim_i B_i \otimes_{R_i} B_i \simeq \lim_i B_i \otimes K[G] = K[X] \otimes K[G].
\]

Use Proposition 11, [2], I, §3, and the above isomorphism (of \( K[X] \)-modules) one can conclude that \( K[X] \) is finitely presented. Exercise 15 from [2], I, §2, implies that \( K[X] \) is also a projective \( R \)-module. Remark 1.1 and Proposition 4.2, [4], infer \( X/G \simeq \text{SSp} \, R \).

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