Loop Groups and twisted $K$-theory III

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Abstract

This is the third paper of a series relating the equivariant twisted $K$-theory of a compact Lie group $G$ to the “Verlinde space” of isomorphism classes of projective lowest-weight representations of the loop groups. Here, we treat arbitrary compact Lie groups. In addition, we discuss the relation to semi-infinite cohomology, the fusion product of Conformal Field theory, the rôle of energy and the topological Peter-Weyl theorem.

Introduction

In [FHT1, FHT2] the twisted equivariant $K$-theory of a compact Lie group was described in terms of positive energy representations of its loop group. There, we assumed that the group was connected, with torsion-free fundamental group. Here, we remove those restrictions; we also relax the constraints on the twisting, assuming only its regularity. Additional constraints allow the introduction of an energy operator, matching the rotation of loops, and lead to the positive energy representations relevant to conformal field theory. Finer restrictions on the twisting lead to a structure of 2-dimensional topological field theory, the “Verlinde TFT” [FHT1]. This is not discussed here, but we do prove two of the key underlying results: we identify the fusion product with the topologically constructed TFT bilinear form with the duality pairing between irreducible representations at opposite levels.

Capturing the Verlinde ring topologically lets us revisit, via twisted $K$-theory, some constructions on representations that were hitherto assumed to rely on the algebraic geometry of loop groups. Thus, restriction to and induction from the maximal torus in twisted $K$-theory recover of semi-infinite restriction and induction of Feigin and Frenkel [FF] on representations. The energy operator comes from the natural circle action on the quotient stack of $G$, under its own conjugation action. The numerator in the character formula can be obtained by dualising the Gysin inclusion of the identity in $G$. Next, the cup-product action of $R(G)$ on $K^\tau(G)$ corresponds to the fusion of Conformal Field Theory, defined via holomorphic induction. Finally, we discuss the Borel-Weil theorem for the “annular” flag variety of a product of two copies of the loop group, interpreted now as a topological Peter-Weyl theorem. This last result can be interpreted as a computation of the TFT bilinear form mentioned earlier, but in addition, it can be further extended to an index theorem for generalised flag varieties of loop groups, in which twisted $K$-theory provides the topological side. We refer to [FHT3, §8] for a verification of this result in the special case of connected groups with free $\pi_1$, and to [12] for further developments concerning higher twistings of $K$-theory.

The paper is organised as follows. Chapter I states the main theorems and describes the requisite technical specifications. Two examples are discussed in Chapter II: the first relates our theorem in the case of a torus to the classical spectral flow of a family of Dirac operators, while the second
recalls the Dirac family associated to a compact group \([\text{FH12}]\), whose loop group analogue is the “non-abelian spectral flow” implementing our isomorphism. Chapter III computes the twisted \(K\)-theory \(K^\tau_G(G)\) topologically, by reduction to the maximal torus and its normaliser in \(G\). Chapter IV reviews the theory of loop groups and their lowest-weight representations; the classification of irreducibles in \([10]\) reproduces the basis for \(K^\tau_G(G)\) constructed in Chapter III. The Dirac family in Chapter V assigns a twisted \(K\)-class to any (admissible) representation of the loop group, and this is shown to recover the isomorphism already established by our classification. Chapter VI gives the topological interpretation of some known constructions on loop group representations as discussed above. Appendix A reviews the diagram automorphisms of simple Lie algebras and relates our definitions and notation with those in Kac \([K]\).

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Index of Notation

Groups

- \(G, G_1\): Compact Lie group and its identity component
- \(T, N\): Maximal torus and its normaliser in \(G\)
- \(W, W_1\): Weyl groups \(N/T, N \cap G_1/T\) of \(G\) and \(g\)
- \(G(f), N(f)\): Centralisers in \(G\) and \(N\) of the connected component of \(f\) \((\S 6)\)
- \(\mathfrak{g}, \mathfrak{t}, \mathfrak{g}_c, \mathfrak{t}_c\): Lie algebras and their complexifications
- \(n \subset \mathfrak{g}_c\): Sub-algebra spanned by the positive root vectors
- \(\langle \cdot \rangle, \{\xi_a\}\): Basic inner product on \(\mathfrak{g}\) (when semi-simple); orthonormal basis \((\S 4)\)
- \(\rho, \theta \in \mathfrak{t}^*\): Half-sum of positive roots, highest root \((\S 4)\)
- \(h^\vee\): (for simple \(g\)) Dual Coxeter number \(\rho\theta + 1\) \((\S 4)\)
- \(\varepsilon; \mathfrak{g}, \mathfrak{t}\): Diagram automorphism of \(g\); \(\varepsilon\)-invariant sub-algebras \((\S 7)\)
- \(W, T\): Weyl group of \(g\); torus \(\exp(\mathfrak{t})\) \((\S 7)\)
- \(\rho, \theta \in \mathfrak{t}^*\): Half-sum of positive roots in \(\mathfrak{g}\), highest \(g\)-weight of \(g//g\) \((\S 9)\)
- \(\mathfrak{R}, \mathfrak{R}^\vee, \Lambda, \Lambda^\vee, \Lambda^\tau\): Root and co-root lattices, weight lattices of \(T\) and \(T\); lattice of \(\tau\)-affine weights

Loop Groups

- \(LG, L_f G\): Smooth loop group and twisted loop group \((\S 11)\)
- \(LG^\tau\): Central extension by \(\mathbb{T}\) with cocycle \(\tau\)
- \(L_g, L_f g\): Smooth loop Lie algebras
- \(L'_g, L'_G\): Laurent polynomial Lie algebra \((\S 8)\), loop group \((\S 16)\)
- \(N_{\text{aff}} = \Gamma_f N\): Group of (possibly \(f\)-twisted) geodesic loops in \(N\) \((\S 6)\)
- \(W_{\text{aff}}(g, f), W_{\text{aff}}\): \(f\)-twisted affine Weyl group of \(g\), extended affine Weyl group \(N_{\text{aff}}//\mathbb{T}\)
- \(\mathfrak{a}, \mathfrak{a}_c\): Alcove of dominant \(\xi \in \mathfrak{t}\) with \(\theta(\xi) \leq 1\), resp. \(\theta(\xi) \leq 1/r\)
- \(\tau \cdot \mathfrak{a}^* \subset \mathfrak{t}\): Product of the centre of \(g\) and the \([\tau]\)-scaled alcoves on simple factors \((\S 10)\)
Twistings
\( \tau; [\tau] \) 2-cocycle on \( LG \); level in \( H^3_G(G; \mathbb{Z}) \) (§2.1)
\( \kappa' \) Linear map \( H_1(\mathbb{T}) \rightarrow H^2(B\mathbb{T}) \) defined from \([\tau]\) (§6)
\( \sigma, \sigma \) \( LG \)-cocycle of the Spin modules for \( LG \), \( LF \) (§1.6)
\( \sigma(t), \sigma(t) \) W-cocycle for the spinors on \( t \) and \( t \) (§1.6)
\( \tau', \tau'' \) Twisting for the \( W_{aff} \)-action on \( \Lambda^\tau \); shifted twisting \( \tau' - \sigma(t) \) (§6)

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I Statements

Throughout the paper, cohomology and \( K \)-theory have integer coefficients, if no others are specified. \( K \)-theory has \textit{compact supports}; however, for proper actions of non-compact groups, or for stacks in general, this refers to the quotient space. For a twisting \( \tau \) on \( X \), the twisted \( K \)-theory will be denoted \( K^\tau(X) \). This is a \( \mathbb{Z}/2 \)-graded group, whose two components are denoted \( K^{\tau_0}(X), K^{\tau_+}(X) \). For a central extension \( G^\tau \) of \( G \), the Grothendieck group of \( \tau \)-projective representations is denoted by \( R^\tau(G) \); it is a module over the representation ring \( R(G) \).\(^1\)

1. Main theorems

(1.1) \textit{Simply connected case}. The single most important special case of our result concerns a simple, simply connected compact Lie group \( G \). Central extensions of its smooth loop group \( LG \) by the circle group \( \mathbb{T} \) are classified by their \textit{level}, the Chern class \( c_1 \in H^2(LG) = \mathbb{Z} \) of the underlying circle bundle. These extensions are equivariant under loop rotation. Among the projective

\(^1\)Note that, when \( \tau \) is graded, this module can have an odd component, cf. §1.3
relations \( \psi \) equivariance in (ii) is the absence of symmetry in the level (see energy do not exist. While (iii) is merely a question of choosing the correct sign, an obstruction to The energy operator cannot be defined without (ii), and without (iii), representations of positive
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with no counterpart in \( K \)

Theorems 3 and 5 instead). However, it has the virtue of explaining the shift between level and

regular twisting, as the projective cocycle of the positive energy spinors on \( \mathcal{G} \)

Here, \( G \) acts on itself by conjugation, \( h^\vee \) is the dual Coxeter number of \( G \), \( [k + h^\vee] \) is interpreted as a twisting class in \( H^3_G(\mathbb{Z}) \cong \mathbb{Z} \) for equivariant K-theory, while \( d := \dim G \) is a degree-shift: the two sides are supported in degrees 0, resp. \( d \mod 2 \). The ring structure on K-theory is the convolution (Pontryagin) product. The isomorphism is established by realising both sides as quotient rings of \( R(G) \), via holomorphic induction on the loop group side, and via the Thom push-forward from the identity in \( G \), on the \( K \) side.

(1.2) General groups. The isomorphism between the two sides and the relation between level and twisting cannot be described so concisely for general compact Lie groups. This is due to the presence of torsion in the group \( H^3 \) of twistings, to an additional type of twistings classified by \( H^1_G(\mathbb{Z}/2) \), related to gradings of the loop group, and to the fact that the two sides need not be quotients of \( R(G) \). For a construction of the map via a correspondence induced by conjugacy classes, we refer to [12] (see also §6.10 here). Ignoring the difficulties for a moment, there still arises a natural isomorphism between the twisted equivariant K-groups of \( G \) and those of the category of positive energy representations at a shifted level, provided that:

(i) we use \( \mathbb{Z}/2 \)-graded representations;
(ii) we choose a central extension of \( LG \) which is equivariant under loop rotation;
(iii) the cocycle of the extension satisfies a positivity condition.

The energy operator cannot be defined without (ii), and without (iii), representations of positive energy do not exist. While (iii) is merely a question of choosing the correct sign, an obstruction to equivariance in (ii) is the absence of symmetry in the level (see [15]). This can only happen for tori — whose loop groups, ironically, have a simple representation theory.

This formulation is unsatisfactory in several respects. The loop group side involves the energy, with no counterpart in \( K_2(G) \); instead, a rotation-equivariant version of the latter will be more relevant. There is also the positivity restriction, whereas the topological side is well-behaved for regular twistings ([2]. There is, finally, the unexplained “dual Coxeter” shift.

We now formulate the most canonical statement. This need not be the most useful one (see Theorems 3 and 5 instead). However, it has the virtue of explaining the shift between level and twisting, as the projective cocycle of the positive energy spinors on \( LG \). Gradings in (i), if not originally present in the twisting \( \tau \), are also imposed upon us by the spinors whenever the Ad-representation of \( G \) does not spin.

(1.3) Untwisted loop groups. Let \( G \) be any compact Lie group and \( LG^\tau \) a smooth \( \mathbb{T} \)-central extension of its loop group. We allow \( LG \) to carry a grading, or homomorphism to \( \mathbb{Z}/2 \); this is classified by an element of \( H^1(G;\mathbb{Z}/2) \) and is notionally incorporated into \( \tau \). An Ad-invariant \( L^2 \) norm on \( LG \) defines the graded Clifford algebra\(^4\) \( \text{Cliff}(LG^\tau) \), generated by odd elements \( \psi(\mu), \mu \in LG^\tau \), with relations \( \psi(\mu)^2 = \|\mu\|^2 \).

\(^2\)A simple statement can be given when \( G \) is connected and \( \pi_1(G) \) is free [FHT3 §6], precisely because both sides are quotients of \( R(G) \).
\(^3\)Twisting for loop groups [15] and for K-theory mean different things, but both uses are well-entrenched.
\(^4\)This algebra should really be based on half-forms on the circle, which carry a natural bilinear form.
A $\tau$-representation of $LG$ is a graded representation of $LG^\tau$ on which the central circle acts by the natural character. We are interested in complex, graded $\tau$-representations of the crossed product $LG \ltimes \text{Cliff}(Lg^*)$, with respect to the co-adjoint action. Graded modules for $\text{Cliff}(Lg^*)$ can be viewed as $b$-projective representations of the odd vector space $\psi(Lg^*)$, where $b$ is the $L^2$ inner product, so we are considering $(\tau, b)$-representations of the graded super-group $LG_b := LG \ltimes \psi(Lg^*)$. Subject to a regularity restriction on $\tau$, an admissibility condition on representations will ensure their complete reducibility \((2)\).

A super-symmetry of a graded representation is an odd automorphism squaring to 1. Let $R^{\tau+0}(LG_b)$ be the free $\mathbb{Z}$-module of graded admissible representations, modulo super-symmetric ones, and $R^{\tau+1}(LG_b)$ that of representations with a super-symmetry, modulo those carrying a second super-symmetry anti-commuting with the first. These should be regarded as the $LG_b$-equivariant $K^\tau$-groups of a point. The reader should note that defining $K$-theory for graded algebras a delicate matter in general \([3]\); the shortcut above, also used in \([\text{FHT3}, \S 4]\), relies on the semi-simplicity of the relevant categories of modules.

Since $K^\tau_G(G)$ is a $K_G(G)$-module, it carries in particular an action of the representation ring $R(G)$. Fusion with $G$-representations defines an $R(G)$-module structure on $R^\tau(LG_b)$; the definition is somewhat involved, and we must postpone it until \([16]\). Here is our main result.

**Theorem 2.** For regular $\tau$, there is a natural isomorphism of (graded) $R(G)$-modules $R^\tau(LG_b) \cong K^\tau_G(G)$, where $K$-classes arise by coupling the Dirac operator family of Chapter \([\mathcal{V}]\) to admissible $LG_b$-modules.

**1.4 Remark.** For twistings that are suitably transgressed from $BG$, both sides carry isomorphic Frobenius ring structures. The portion of the product structure that exists for any regular twisting is discussed in \([16]\). A geometric construction of the duality pairing is described in \([17]\).

**1.5 Twisted loop groups.** When $G$ is disconnected, there are twisted counterparts of these notions. For any $f \in G$, the twisted loop group $L_f G$ of smooth maps $\gamma : \mathbb{R} \to G$ satisfying $\gamma(t + 2\pi) = f \gamma(t)f^{-1}$ depends, up to isomorphism, only on the conjugacy class in $\pi_0 G$ of the component $f G_1$ of $f$. Let $[f G_1] \subset G$ denote the union of conjugates of $f G_1$; the topological side of the theorem is $K^\tau_G([f G_1])$, while the representation side involves the admissible representations of $L_f G \ltimes \psi(L_g^*)$.

**1.6 Removing the spinors.** A lowest-weight spin module $S$ for $\text{Cliff}(Lg^*)$ (see \([2,9]\)) carries an intertwining projective action of the loop group $LG$. Denoting by $\sigma$ (or $\pi$, in the $f$-twisted case) the projective cocycle of this action and by $d$ the dimension of the centraliser $G^\times f$, a Morita isomorphism

$$R^\tau(L_f G_b) \cong R^{\tau-d}(L_f G)$$

results from the fact that an admissible, graded $\tau$-module of $L_f G \ltimes \text{Cliff}(Lg^*)$ has the form $H \otimes S$, for a suitable $(\tau - d)$-representation $H$ of $L_f G$, unique up to canonical isomorphism. Note in particular the dimension shift by $d$, from the parity of the Clifford algebra. We obtain the following reformulation of Theorem 2.

**Theorem 3.** For regular $\tau$, there is a natural isomorphism $K^\tau_G([f G_1]) \cong R^{\tau-d}(L_f G)$.

The loop group may well acquire a grading from the spinor twist $\sigma$ even if none was present in $\tau$; if so, $R^{\tau-d}(L_f G)$ is built from graded representations, as in \([13]\).

**1.8 Classifying representations.** In proving the theorems, we compute both sides of the isomorphism in Theorem 3. More precisely, we compute the twisted $K$-theory by reduction to the torus and the Weyl group, and produce an answer which agrees with the classification of admissible representations in terms of their lowest weights. In fact, twisted $K$-theory allows for an attractive formulation of the lowest-weight classification for disconnected (loop) groups, as follows.

Choose a maximal torus $T \subset G$ which, along with a dominant chamber, is stable under $f$-conjugation. (Such tori always exists, see Proposition \([7,2]\)). Recall that the extended affine Weyl
group $W_{\text{aff}}$ for $L_f G$, the group of $f$-twisted loops in the normaliser $N$ of $T$. Let $T \subset T$ denote the subtorus centralised by $f$, and $\Delta^\tau$ the set of its $\tau$-affine weights. The conjugation action of $L_f N$ on $T$ descends to an action of $W_{\text{aff}}$ on $\Delta^\tau$, which preserves the subset $\Delta^\tau_{\text{reg}}$ of regular weights. A tautological twisting $\tau'$ is defined for this action, because every weight defines a $\mathbb{T}$-central extension of its centraliser in $W_{\text{aff}}$ (see [10,4] for details). Finally, after projection to the finite Weyl group $W = N/T$, $W_{\text{aff}}$ also acts on the Lie algebra $\mathfrak{t}$ of $T$.

**Theorem 4.** The category of graded, admissible $\tau$-representations of $L_f G \ltimes \text{Cliff}(L_f g^*)$ is equivalent to that of $\tau'$-twisted $W_{\text{aff}} \ltimes \text{Cliff}(\mathfrak{t})$-modules on $\Delta^\tau_{\text{reg}}$.

It follows that the corresponding $K$-groups agree. We reduce Theorem 4 in [10] to the well-known cases of simply connected compact groups and tori.

Computing both sides is a poor explanation for a natural isomorphism, and indeed we improve upon this in Chapter V by producing a map from representations to $K$-classes using families of Dirac operators. The construction bypasses Theorem 4 and ties in beautifully with Kirillov’s orbit method, recovering the co-adjoint orbit and line bundle that correspond to an irreducible representation.

Another offshoot of this construction emerges in relation with the semi-infinite cohomology of Feigin and Frenkel [11], for which we give a topological model (Theorem 14.11). For integrable representations, the Euler characteristic of semi-infinite $L_n$-cohomology becomes the restriction from $G$ to $T$, on the $K$-theory side. While this can also be checked by computing both sides, our Dirac family gives a more natural proof, providing the same rigid model for both.

(I.9) *Loop rotation.* Assume now that the extension $LG^\tau$ carries a lifting of the loop rotation action on $LG$. It is useful to allow fractional lifts, that is, actions on $LG^\tau$ of a finite cover $\mathbb{T}$ of the loop rotation circle; such a lift always exists when $G$ is semi-simple (Remark 15.3). If so, admissible $\tau$-representations carry an intertwining, semi-simple action of this new $\mathbb{T}$. Schur’s lemma implies that the action is unique up to an overall shift on any irreducible representation.

In this favourable situation, we can incorporate the loop rotation into our results. The requisite object on the topological side is the quotient stack of the space $A$ of $g$-valued smooth connections on the circle by the semi-direct product $\mathbb{T} \ltimes LG$, the loop group acting by gauge transformations and $\mathbb{T}$ by loop rotation. We denote the twisted $K$-theory of this stack by $K^\tau_f(G_C)$. This notation, while abusive, emphasises that the $\mathbb{T}$-action makes it into an $R(\mathbb{T})$-module; its fibre over 1 is the quotient by the augmentation ideal of $R(\mathbb{T})$. The following formulation, while awkward, has the virtue of being canonical; there is no natural isomorphism of $K^\tau_f(G_C)$ with $K^\tau_f(G) \otimes R(\mathbb{T})$.

**Theorem 5.** If the regular twisting $\tau$ is rotation-equivariant, $K^\tau_f(G_C)$ is isomorphic to the $R^\tau$-group of graded, admissible, representations of $\mathbb{T} \ltimes LG$ (cf. [13]. It is a free module over $R(\mathbb{T})$, and its fibre over 1 is isomorphic to $K^\tau_f(G)$.

A noteworthy complement to Theorem 5 is that $K^\tau_f(G_C)$ contains the Kac numerator formula for $LG^\tau$-representations: see [15,6]. It would be helpful to understand this as a twisted Chern character, just as the the Kac numerator at $q = 1$ is the Chern character for $K^\tau_f(G)$ [14,11].

2. **Technical definitions**

In this section, we describe our regularity conditions on the central extension $LG^\tau$ and define the class of admissible representations. There is a topological and an analytical component to regularity.

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5A further positivity condition (15.5) on $\tau$ ensures that the spectrum of this action is bounded below, and the real infinitesimal generator of the intertwining action is then called the energy.
For a twisted loop group $L$ we call

$$H^2_{LG}(\mathbb{T}) \to H^3_{LG}(\mathbb{Z})$$

the last group is purely topological, and equals $H^3(BLG) \cong H^3_{LG}(G_1)$. When $g$ is semi-simple, the smooth-cochain group cohomology $H^2_{LG}(\mathbb{R})$ vanishes \cite[VIII]{FS}, and $[\tau]$ then determines the central extension $LG^\tau$, up to isomorphism. In any case, restricting to a maximal torus $T \subset G$ and writing $H^2_T$ for $H^2(BT)$, we obtain a class in

$$H^2_T(T) = H^1(T) \otimes H^2_T \oplus H^3(T).$$

For classes arising from central extensions, it turns out that the $H^3(T)$ component vanishes. In view of the isomorphism $H^1(T) \cong H^2_T$, we make the following

### 2.2 Definition.

We call $\tau$ topologically regular iff $[\tau]$ defines a non-singular bilinear form on $H^1(T)$.

For a twisted loop group $L_f/G$, topological regularity is detected instead by the $f$-invariant subtorus $T \subset T$ in an $f$-stable maximal torus $T$ as in \cite[1.8]{LR}. Restricting $[\tau]$ there leads to a bilinear form on $H^1(T)$, and regularity refers to the latter. In the next section, we will see how the bilinear form captures the commutation in $LT^\tau$ of the constant loops $T$ with the group of components $\pi_1 T$.

### 2.3 Analytic regularity.

This condition, which holds in the standard examples, concerns the centre $z \subset g$, and ensures that the topologically invisible summand $L_3/z$ does not affect the classification of representations of $LG^\tau$. Split $L_3$ into the constants $z$ and their $L^2$-complement $V$, and observe that $LG$ is the semi-direct product of the normal subgroup $\exp(V)$ by the subgroup $\Gamma$ of loops $\gamma$ whose velocity $d_\gamma \cdot \gamma^{-1}$ has constant $z$-projection. Because the action of $\Gamma$ on $V$ factors through the finite group $\pi_0 G$, invariant central extensions of $\exp(V)$ have a preferred continuation to $LG$.

### 2.4 Definition.

$\tau$ is analytically regular iff it is the sum of an extension of $\Gamma$ and a Heisenberg extension of $\exp(V)$, and, moreover, the Heisenberg cocycle $\omega : \Lambda^2 V \to i\mathbb{R}$ has the form $\omega(\xi, \eta) = b(S\xi, \eta)$, for some skew-adjoint Fredholm operator $iS$ on $V$.

The standard example\cite{S} has $S = id/dt$, an unbounded operator, so we really ask that $S/(1 + \sqrt{S^*S})$ should be Fredholm. We need to tame $\omega$ for the Dirac constructions in Chapter \cite[V]{V}. For twisted loop groups, the analytic constraints refer to $L_f \beta/\beta^f$.

### 2.5 Linear splittings. Restricted to any simple summand in $g$, every extension class is a multiple of the basic one in \cite[3.1]{3.1} and is detected by the level $[\tau]$. However, the extension cocycle $\omega : \Lambda^2 Lg \to i\mathbb{R}$ depends on a linear splitting of the extension

$$0 \to i\mathbb{R} \to Lg^\tau \to Lg \to 0. \tag{2.6}$$

For the unique $g$-invariant splitting, $S$ is a multiple of $id/dt$. Preferred splittings for the twisted loop groups also exist; they are discussed in \cite[9]{9}. Hence, subject to topological regularity, and using the preferred splittings, the second part of Condition (2.4) holds for the entire Lie algebra. Varying the splitting by a representable linear map $Lg \to i\mathbb{R}$, that is, of the form $\eta \mapsto \omega(\xi, \eta)$, changes $S$ by an inner derivation. We assume now that such a splitting has been chosen.

### 2.7 Remark.

(i) $S$ must vanish on $z$, because the latter exponentiates to a torus, over which any $\mathbb{T}$-extension is trivial. The Heisenberg condition allows $\ker S \cap L_3$ to be no larger. Combining this with the discussion of simple summands shows that, for regular $\tau$, $\ker S$ is the Lie algebra of a full-rank compact subgroup of $LG$. This is the constant copy of $G$, for the standard splitting.

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\footnote{This is the only possibility for $\text{Diff}(S^1)$-equivariant extensions \cite[VIII]{FS}.}
(2.8) Lowest-weight representations. The semi-positive spectral projection of $S$ is an $\omega$-isotropic subalgebra $\mathfrak{P} \subset L_{g;}$; we call it the positive polarisation. The strictly positive part $\mathfrak{U} \subset \mathfrak{P}$ is a Lie ideal, and $\ker S \otimes \mathbb{C}$ is isomorphic to $\mathfrak{P}/\mathfrak{U}$. A linear splitting in $\mathfrak{P}$ restricts to a Lie algebra splitting over $\mathfrak{P}$. A lowest weight $\tau$-representation of $L_{g}$ is one generated by an irreducible module of $\ker S$, which is killed by the lifted copy of $\mathfrak{U}$ in $\mathfrak{P}$.

The lowest-weight condition depends on $S$ and on the splitting of $\mathfrak{P}$ over the centre $\mathfrak{Z}$. However, if we insist on integrability of the representation to the identity component of the loop group (see §8.5), lowest-weight modules are irreducible, unitarisable, and their Hilbert space completions are unchanged under a representable variation of that splitting.

(2.9) Admissible representations. A projective representation of $L_{G}$ is called admissible if it decomposes as a finite-multiplicity sum of Hilbert space completed lowest-weight representations of the Lie algebra. Assuming topological regularity, any integrable lowest-weight representation of $L_{g}$ exponentiates to an action of the identity component of $L_{g}$ on the Hilbert space completion. This then induces an admissible representation of the full loop group. Moreover, at fixed level, there are finitely many irreducibles, up to isomorphism; see §10.

There is a similar notion of lowest-weight and admissibility for $\text{Cliff}(L_{g}^{\ast})$-modules, using the same polarisation. (Note that $\mathfrak{U}$ is $b$-isotropic). As in the finite-dimensional case, there are one or two isomorphism classes of lowest-weight representations, according to whether $\dim \mathfrak{g}$ is even or odd, and they are irreducible. The numbers are switched if we ask for graded representations; any of the graded irreducibles is called a spin module. The $K$-theory of graded, admissible $\text{Cliff}(L_{g}^{\ast})$-modules (as in §1.3) is $\mathbb{Z}$, in degrees $\dim G$ (mod 2). The two spin modules, in the even case, differ by parity-reversal, and represent opposite generators of $K^{0}$. (In the odd case, two opposite generators come from the two choices of a super-symmetry on the irreducible spin module.)

2.10 Remark. The algebraic approach to representations starts from the Laurent polynomial loop algebra $L'g$ and the finite-multiplicity sums of integrable lowest-weight modules of $L'g \times \text{Cliff}(L'g^{\ast})$. These are the Harish-Chandra modules underlying our admissible representations. However, as our Dirac construction of $K$-classes involves the smooth loop group and its unitary representations, we must work more analytically.

II Two examples

We recall from §1.9 two examples relevant to the construction of the Dirac operator families in Chapter IV, which relate representations to $K$-theory classes. The first concerns the group $LT$ of loops in a torus; the second is a finite-dimensional Dirac family, which leads to an interpretation of our theorem as an infinite-dimensional Thom isomorphism.

3. Spectral flow over a torus

(3.1) The circle [APS]. Let $D := d/d\theta$ be the one-dimensional Dirac operator on the complex Hilbert space $L := L^{2}(S^{1};\mathbb{C})$, acting as in on the Fourier mode $e^{int\theta}$. For any $\xi \in \mathbb{R}$, the modified operator
$\mathcal{D}_\xi := \mathcal{D} + i\xi$ has the same eigenvectors, but with shifted spectrum $i(n + \xi)$. Let $M : \mathcal{L} \to \mathcal{L}$ be the operator of multiplication by $e^{i\theta}$. The relation $M^{-1}\mathcal{D}_\xi M = \mathcal{D}_{\xi+1}$ shows that the family $\mathcal{D}_\xi$, parametrised by $\xi \in \mathbb{R}$, descends to a family of operators on the Hilbert bundle $\mathbb{R} \times \mathbb{Z} \mathcal{L}$ over $\mathbb{R}/\mathbb{Z}$ ($M$ generates the $\mathbb{Z}$-action on $\mathcal{L}$).

Following the spectral decomposition of $\mathcal{D}_\xi$, we find that one eigenvector crosses over from the negative to the positive imaginary spectrum as $\xi$ passes an integer value. Thus, the dimension of the positive spectral projection, although infinite, changes by 1 as we travel once around the circle $\mathbb{R}/\mathbb{Z}$. This property of the family $\mathcal{D}_\xi$ is invariant under continuous deformations and captures the following topological invariant. Recall that the interesting component of the space $\text{Fred}^a$ of skew-adjoint Fredholm operators on $\mathcal{L}$ has the homotopy type of the small unitary group $U(\infty)$; in particular, $\pi_1 \text{Fred}^a = \mathbb{Z}$. Weak contractibility of the big unitary group allows us to trivialise our Hilbert bundle on $\mathbb{R}/\mathbb{Z}$, uniquely up to homotopy; so our family defines a map from the circle to $\text{Fred}^a$, up to homotopy. This map detects a generator of $\pi_1 \text{Fred}^a$.

### (3.2) Generalisation to a torus

A metric on the Lie algebra $t$ of a torus $T$ defines the Clifford algebra $\text{Cliff}(t^*)$, generated by the dual $t^*$ of $t$. Denote by $\psi(\mu)$ the Clifford action of $\mu \in t^*$ on a complex, graded, irreducible spin module $S(t) = S^+(t) \oplus S^-(t)$ [R]. Let $L^\pm = L^2(T) \otimes S^\pm(t)$, denote by $\mathcal{D}$ the Dirac operator $\sum \partial/\partial \theta^a \otimes \psi^a$ on $\mathcal{L} := L^+ \oplus L^-$, and consider the family of operators parametrised by $\mu \in t^*$,

$$\mathcal{D}_\mu = \mathcal{D} + i\psi(\mu) : L^+ \to L^-.$$  

Let $\Pi = (2\pi)^{-1} \log 1$ be the integer lattice in $t$, isomorphic to $\pi_1 T$. For a weight $\lambda \in \Pi^* := \text{Hom}(\Pi; \mathbb{Z})$, let $M_\lambda : \mathcal{L} \to \mathcal{L}$ be the operator of multiplication by the associated character $\tau^\lambda : t \mapsto t^\lambda$. The relation $M_{-\lambda} \circ \mathcal{D}_\mu \circ M_{\lambda} = \mathcal{D}_{\mu+\lambda}$ shows that $\mathcal{D}_\mu$ descends to a family of fibre-wise operators on the Hilbert bundle $t^* \times_T \mathcal{L}$ over the dual torus $T^* := t^*/T$. Here, $\Pi^*$ acts on $t^*$ by translation and on $\mathcal{L}$ via the $M$. As before, contractibility of the unitary group leads to a continuous family of Fredholm operators over $T^*$. When $\ell := \dim t$ is odd, we choose a self-adjoint volume form $\omega \in \text{Cliff}(t^*)$. This commutes with all the $\psi^a$ and converts $\mathcal{D}_\mu$ to a skew-adjoint family $\omega \cdot \mathcal{D}_\mu$ of operators acting on $L^+$. Thus, in every case, we obtain a class in $\text{K}^i(T^*)$. This is the $\text{K}$-theoretic volume form; more precisely, it is a Fredholm model for the Thom push-forward of the identity in $T^*$.

### (3.3) Representations and twisted $K$-classes

Relating this construction to our concerns requires a bit more structure, in the form of a linear map $\tau : \Pi \to \Pi^*$ (not related to the metric). A central extension $\Gamma^\tau$ of the product $\Gamma := \Pi \times T$ by the circle group $\mathbb{T}$ is defined by the commutation rule

$$ptp^{-1} = t \cdot i^{\tau(p)} \quad p \in \Pi, t \in T \quad \text{and} \quad i^{\tau(p)} \in \mathbb{T}. \quad (3.4)$$

The group $\Gamma^\tau$ has a unitary representation on $L^2(T)$, with $T$ acting by translation and $\Pi$ by the $M_{\tau(p)}$’s. If $\tau$ has full rank, $L^2(T)$ splits into a finite sum of irreducible $\Gamma$-representations $\text{F}_{[\lambda]}$, each of them comprising the weight spaces of $T$ in a fixed residue class $[\lambda] \in \Pi^*/\tau(\Pi)$. Moreover, these are all the unitary $\tau$-irreducibles of $\Gamma$, up to isomorphism. (This will be shown in [10].)

Now, $\tau$ also induces a map $T \to T^*$, where-under the pull-back of $\mathcal{L}$ splits, according to the splitting of $L^2(T)$ into the $\text{F}_{[\lambda]}$. Each component carries the lifted Dirac family $\mathcal{D}_\xi := \mathcal{D} + i\psi(\tau(\xi))$, descending to a spectral flow family over $T$. Except at the single value $\exp(\tau^{-1}[-\lambda]) \in T$ of the parameter, $\mathcal{D}_\xi$ is invertible on the fibres $\text{F}_{[\lambda]} \otimes \mathbf{S}$.

All families $\text{F}_{[\lambda]} \otimes \mathbf{S}$ have the same image in $\text{K}^i(T)$, but this problem is cured by remembering the $T$-action, as follows. Instead of viewing the $\mathcal{D}_\xi$ as families over $T$, we interpret them as $\tau$-equivariant Fredholm families parametrised by $t$. Now, $t$ is a principal $\Pi$-bundle over the torus $T$, equivariant for the trivial action of $T$ on both, and the central extension $\tau T$ defines a twisting for the $T$-equivariant $K$-theory of $T$ [FHTI]. Classes in $\text{K}_{\tau}^T(T)$ are then described by $\Gamma$-equivariant

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7The components of essentially positive and essentially negative Fredholm operators are contractible.
families of Fredholm operators, parametrised by $t$, on $\tau$-projective unitary representations of $\Gamma$; twisted $K$-classes are represented by skew-adjoint families. Thus, our families $D_\xi : F[\alpha] \otimes S^+ \to F[\alpha] \otimes S^-$ (respectively $\varpi : D_\xi$ on $F[\alpha] \otimes S^+$ in odd dimensions) give classes in $K^{\tau-\ell}(T)$. A special case of our main theorem asserts that, when $\tau$ is regular, these classes form a $\mathbb{Z}$-basis of the twisted $K$-groups in dimension $\ell \mod 2$, while the other $K$-groups vanish.

3.5 Remark. The inverse map from $K^{\tau+f}(T)$ to representations of $\Gamma^\tau$ ought to be “integration over $t$” from $K^{\tau+f}(t)$ to $R^{\tau+0}(\Gamma)$. This is consistent with our interpretation of our main theorem as an infinite-dimensional Thom isomorphism, on the space of connections over the circle (Chapter V). However, we only know how to define the last group in terms of $C^*$-algebras.

(3.6) Direct image interpretation. Here, we give a topological meaning for the family $(D_\bullet, L)$; this will be used in (3.7). We claim it represents the image of the unit class $[1]$ under the Gysin map

$$p_* : K^0(T) \to K^{\tau-\ell}(T).$$

To define $p_*$, we must trivialise the lifted twisting $p^*\tau$. Recall that the twisting $\tau$ for the (trivial) $T$-action on $T$ is the groupoid defined from the action of $\Gamma^\tau$ on $t$. The matching model for $p^*\tau$ on $T = t/\Pi$ comes from the restricted extension $\Pi^\tau$, and this is trivialised by its construction (3.4).

The class $[1]$ then corresponds to the trivial line bundle on $t$ with trivial $T$-action.

We now give an equivalent, but more concrete model for $p_*$. Replace $K^\tau(T)$ by $K_\ell^\tau(T \times T)$, where $T$ translates the second factor; the projection $P$ to the first factor replaces $p$. If we represent $T$ by the $\Gamma$-action groupoid on $t \times T$, where $\Pi$ and $T$ act by translation on $t$, resp. $T$, then the twisting $P^*\tau$ is represented by the action of $\Gamma^\tau$ on $t \times T$.

Call $0(\tau)$ the trivial line bundle on $t \times T$, but with the translation action of $T$ and with $\Pi$-action via the operators $M_{\tau(t)}$. The two assemble to a $\tau$-action of $\Gamma$, so $0(\tau)$ gives a class in $K^{\tau+0}_\ell(T \times T)$. We claim that this is the image of $[1]$ under the trivialisation of $p^*\tau$. Indeed, our model for $p^*\tau$ as the action of $\Pi^\tau$ on $t$ maps to the model for $P^*\tau$ by inclusion at $t \times \{1\}$; thereunder, $0(\tau)$ restricts to the trivial bundle with trivial $T$-action.

The Gysin image $p_*[1]$ is now represented by any $\Gamma^\tau$-invariant family of Dirac operators on the fibres of $P$, and $(D_\bullet, L)$ is an example of this.

(3.7) Relation to the loop group $LT$. Decompose $LT$ as $\Gamma \times \exp(V)$, where $V = Lt \oplus t$. Central extensions of $\exp(V)$ by the circle group $\mathbb{T}$ are classified by skew 2-forms $\omega$ on $V$. We choose a regular such form, in the sense of [2,4] together with a positive isotropic subspace $\mathcal{U} \subset V_C$. There exists then, up to isomorphism, a unique irreducible, unitary projective Fock representation $F$ of $\exp(V)$ which contains a vector annihilated by $\mathcal{U}$. The sum of $\Gamma^\tau$ and our extension of $\exp(V)$ is a $\mathbb{T}$-central extension $LT^\tau$ of $LT$, whose irreducible admissible representations are isomorphic to the $F[\alpha] \otimes F$. Our construction assigns to each of these a class in $K^{\tau+f}_\ell(T)$.

We will extend this construction and resulting correspondence between $LT$-representations and twisted $K$ classes to arbitrary compact groups $G$. Observe, by factoring out the space of based loops, that $\Gamma^\tau$-equivariant objects over $t$ are in natural correspondence to $LT^\tau$-equivariant ones over the space $\mathcal{A}$ of $t$-valued connection forms on the circle, for the gauge action; and it is in this form that our construction of the Dirac spectral flow generalises. The explicit removal of the Fock factor $F$ has no counterpart for non-abelian groups, and the same effect is achieved instead by coupling the Dirac operator to the spinors on $Lt/t$.

4. A finite-dimensional Dirac family

We now recall from [FHT2] the finite-dimensional version of our construction of twisted $K$-classes from loop group representations (Chapter V). For simplicity, we take $G$ to be simple and simply connected. Choosing a dominant Weyl chamber in $t$ defines the nilpotent algebra $n$ spanned by
positive root vectors, the highest root \( \theta \) and the Weyl vector \( \rho \), the half-sum of the positive roots. Roots and weights live in \( t' \), a weight \( \lambda \) defines the character \( e^{i\lambda} : T \rightarrow \mathbb{T} \), sending \( e^\xi \in T \) to \( e^{i\lambda(\xi)} \).

The basic invariant bilinear form \( \langle \mid \rangle \) on \( g \) is normalised so that the roots have square-length 2. Define the structure constants \( f^a_{bc} \) by \( \{ \xi_a, \xi_b \} = f^a_{bc} \xi_c \), in an orthonormal basis \( \{ \xi_a \} \) of \( g \) with respect to this bilinear form.\(^8\) Note that \( f^b_{ac} f^c_{ad} = 2 h' \delta_{bd} \), where \( h' = \rho \theta + 1 \) is the dual Coxeter number. Let \( \text{Cliff}(g^*) \) be the Clifford algebra generated by elements \( \psi^a \) dual to the basis \( \xi_a \), satisfying \( \psi^a \psi^b + \psi^b \psi^a = 2 \delta^{ab} \), and let \( S = S^+ \oplus S^- \) be a graded, irreducible complex module for it. This is unique up to isomorphism and (if \( \dim g \) is even) up to parity switch. There is a unique action of \( g \) on \( S \) compatible with the adjoint action on \( \text{Cliff}(g^*) \); the action of \( \xi_a \) can be expressed in terms of Clifford generators as

\[
\sigma_a = -\frac{1}{4} f_{bc} \cdot \psi^b \psi^c.
\]

It follows from the Weyl character formula that \( S \) is a sum of \( 2^{[\dim t'/2]} \) copies of the irreducible representation \( V_{-\rho} \) of \( g \) of lowest weight \((-\rho)\). The lowest-weight space is a graded \( \text{Cliff}(t') \)-module; for dimensional reasons, it is irreducible.

**4.1 The Dirac operator.** Having trivialised the Clifford and Spinor bundles over \( G \) by left translation, consider the following operator on spinors, called by Kostant \([K1]\) the “cubic Dirac operator”:

\[
\mathcal{D} = R_a \otimes \psi^a + \frac{1}{3} \sigma_a \cdot \psi^a = R_a \otimes \psi^a - \frac{1}{12} f_{abc} \psi^a \psi^b \psi^c,
\]

where \( R_a \) denotes the right translation action of \( \xi_a \) on functions. Let also \( T_a = R_a + \sigma_a \) be the total right translation action of \( \xi_a \) on smooth spinors.

**4.3 Proposition.** \([\mathcal{D}, \psi^b] = 2T_b; \ [\mathcal{D}, T_b] = 0.\]

**Proof.** The second identity expresses the right-invariance of the operator, while the first one follows by direct computation:

\[
[\mathcal{D}, \psi^b] = R_a \otimes [\psi^a, \psi^b] + \frac{1}{3} \sigma_a \cdot [\psi^a, \psi^b] - \frac{1}{3} [\sigma_a, \psi^b] \cdot \psi^a
\]

\[
= 2R_b + \frac{2}{3} \sigma_b - \frac{1}{3} f^b_{ca} \cdot \psi^c \psi^a
\]

\[
= 2(R_b + \sigma_b).
\]

**4.4 The Laplacian.** The Peter-Weyl theorem decomposes \( L^2(G; S) \) as \( \bigoplus_{\lambda} V_{-\lambda}^\ast \otimes V_{-\lambda} \otimes S \), where the sum ranges over the dominant weights \( \lambda \) of \( g \). Left translation acts on the left, \( R_a \) on the middle and \( \sigma_a \) on the right factor. Hence, \( \mathcal{D} \) acts on the two right factors alone. As a consequence of (4.3), the Dirac Laplacian \( \mathcal{D}^2 \) commutes with the operators \( T_\bullet \) and \( \psi^\ast \). As these generate \( V_{-\lambda} \otimes S \) from its \((-(\lambda + \rho))\)-weight space, \( \mathcal{D}^2 \) is determined from its action there. To understand this action, rewrite \( \mathcal{D} \) in a root basis of \( g \),

\[
\mathcal{D} = R_j \otimes \psi^j + \frac{1}{3} \sigma_j \psi^j + R_\alpha \otimes \psi^{-\alpha} + R_{-\alpha} \otimes \psi^{\alpha} + \frac{1}{3} (\sigma_\alpha \psi^{-\alpha} + \sigma_{-\alpha} \psi^{\alpha}),
\]

where the \( j \)'s label a basis of \( t \) and \( \alpha \) ranges over the positive roots. The commutation relation

\[
[\sigma_{-\alpha}, \psi^{\alpha}] = \psi(-2i\rho),
\]

where summation over \( \alpha \) has been implied, converts (4.5) to

\[
\mathcal{D} = R_j \otimes \psi^j + \frac{1}{3} \sigma_j \psi^j - \frac{2i}{3} \psi(\rho) + R_\alpha \otimes \psi^{-\alpha} + R_{-\alpha} \otimes \psi^{\alpha} + \frac{1}{3} (\sigma_\alpha \psi^{-\alpha} + \psi^\alpha \sigma_{-\alpha}),
\]

and the vanishing of all \( \alpha \)-terms on the lowest weight space leads to the following

\(^8\)We use the Einstein summation convention, but will also use the metric to raise or lower indexes as necessary, when no conflict arises.
4.6 Proposition. (i) $\mathcal{D} = -i\psi(\lambda + \rho)$ on the $-(\lambda + \rho)$-weight space of $\mathcal{V}_- \otimes \mathcal{S}$.
(ii) $\mathcal{D}^2 = -(\lambda + \rho)^2$ on $\mathcal{V}_- \otimes \mathcal{S}$.

(4.7) The Dirac family. Consider now the family $\mathcal{D}_\mu := \mathcal{D} + i\psi(\mu)$, parametrised by $\mu \in \mathfrak{g}^*$. Conjugation by a suitable group element brings $\mu$ into the dominant chamber of $t^*$. From (4.6), we obtain the following relations, where $\langle T|\mu \rangle$ represents the contraction of $\mu$ with $T \in \mathfrak{g}^* \otimes \text{End}(\mathcal{V} \otimes \mathcal{S})$, in the basic bilinear form (the calculation is left to the reader).

4.8 Corollary. (i) $\mathcal{D}_\mu = i\psi(\mu - \lambda - \rho)$ on the lowest weight space of $\mathcal{V}_- \otimes \mathcal{S}$.
(ii) $\mathcal{D}_\mu^2 = -(\lambda + \rho - \mu)^2 + 2i(\mu|\mu) - 2(\lambda + \rho|\mu)$.

(4.9) The kernel. Because $i(T|\mu) \leq \langle \lambda + \rho|\mu \rangle$, with equality only on the $-(\lambda + \rho)$-weight space, $\mathcal{D}_\mu$ is invertible on $\mathcal{V}_- \otimes \mathcal{S}$, except when $\mu$ is in the co-adjoint orbit $\mathcal{O}$ of $(\lambda + \rho)$. In that case, the kernel at $\mu \in \mathfrak{g}^*$ is that very weight space, with respect to the Cartan sub-algebra $t_\mu$ and dominant chamber defined by the regular element $\mu$. This is the lowest-weight line of $\mathcal{V}_- \otimes \mathcal{S}$, with coefficients in the natural line bundle $\mathcal{O}(-\lambda - \rho)$. Sending $\mathcal{V}_- \otimes \mathcal{S}$ to this class defines a linear map

$$R(G) \to K^\dim_{\mathfrak{g}^*}(G).$$

(4.10) Topological interpretation. The family of operators $\mathcal{D}_\mu$ on $\mathcal{V}_- \otimes \mathcal{S}$ is a compactly supported $K$-cocycle on $\mathfrak{g}^*$, equivariant for the co-adjoint action of $G$. As before, when $\dim \mathfrak{g}$ is odd, we use the volume form $\varpi$ to produce the skew-adjoint family $\varpi \mathcal{D}_\mu$, which represents a class in $K^1_{G}$. Our computation of the kernel identifies these classes with the Thom classes of $\mathcal{O} \subset \mathfrak{g}^*$, with coefficients in the natural line bundle $\mathcal{O}(-\lambda - \rho)$. Sending $\mathcal{V}_- \otimes \mathcal{S}$ to this class defines a linear map

$$R(G) \to K^\dim_{\mathfrak{g}^*}(G).$$

There is another way to identify this map. Deform $\mathcal{D}_\mu$ to $i\psi(\mu)$ via the (compactly supported Fredholm) family $\varepsilon \cdot \mathcal{D} + i\psi(\mu)$. At $\varepsilon = 0$ we obtain the standard Thom class of the origin in $\mathfrak{g}^*$, coupled to $\mathcal{V}_- \otimes \mathcal{S}$. Therefore, our construction is an alternative rigid implementation of the Thom isomorphism $K^0_{G}(0) \cong K^\dim_{\mathfrak{g}^*}(G)$.

The inverse isomorphism is the push-forward from $\mathfrak{g}^*$ to a point. In view of our discussion, this expresses $\mathcal{V}_- \otimes \mathcal{S}$ as the Dirac index of $\mathcal{O}(-\lambda - \rho)$ over $\mathcal{O}$, leading to the Dirac index version of the Borel-Weil-Bott theorem. The affine analogue of the Thom isomorphism (4.11) is Theorem 3, equating the module of admissible projective representations with a twisted $K_{G}(G)$.

(4.12) Application to Dirac induction. For later use, we record here the following proposition; when combined with the Thom isomorphisms and the resulting twists, it gives the correct version of Dirac induction for any compact Lie group $G$ (not necessarily connected). Let $N \subset G$ be the normaliser of the maximal torus $T$. We have a restriction map $K_{G}(\mathfrak{g}^*) \to K_{N}(t^*)$ and an “induction” $K_{N}(t^*) \to K_{G}(\mathfrak{g}^*)$ (Thom push-forward from $t^*$ to $\mathfrak{g}^*$, followed by Dirac induction from $N$ to $G$).

4.13 Proposition. The composition $K_{G}(\mathfrak{g}^*) \to K_{N}(t^*) \to K_{G}(\mathfrak{g}^*)$ is the identity.

Proof. Express the middle term as $K_{G}(G \times_{N} t^*)$, with the left action of $G$ on the induced space. The map from $G \times_{N} t^*$ to $\mathfrak{g}^*$ sends $(g, \mu)$ to $g\mu g^{-1}$. Since $N$ meets every component of $G$ (Prop. 7.2), this map is a diffeomorphism over regular points. Every class in $K_{G}(\mathfrak{g}^*)$ is the Thom push-forward of a class $[V] \in K_{G}(0)$. Deforming this to $\mathcal{D} + i\psi(\mu)$ leads to a class supported on a regular orbit; a fortiori, our composition is the identity on such classes, hence on the entire $K$-group. □
III Computation of twisted $K_G(G)$

In this chapter, we compute the twisted $K$-theory $K^\tau_G(G)$ by topological methods, for arbitrary compact Lie groups $G$ and regular twistings $\tau$. A key step is the reduction to the maximal torus, Proposition 7.2. Our answer takes the form of a twisted $K$-theory of the set of regular affine weights at level $\tau$, equivariant under the extended affine Weyl group (6.3) [6.4]. This action has finite quotient and finite stabilisers, and the $K^\tau$-theory is a free abelian group of finite rank.

For a detailed discussion of foundational questions on twisted $K$-theory, we refer to [FHT1] and the references therein.

5. A “Mackey decomposition” lemma

We recall from [FHT1] the following construction and generalisation of Lemma 2.14 in [FHT3], which will be a key step in our computation of $K_G(G)$. This is a topological form of the Mackey decomposition of irreducible representations of a group, restricted to a normal subgroup; the analogy will be particularly relevant in 10.6.

(5.1) Construction. Let $H$ be a compact group, acting on a compact Hausdorff space $X$, $\tau$ a twisting for $H$-equivariant $K$-theory, $M \subseteq H$ a normal subgroup acting trivially on $X$. The following data can be extracted from this:

(i) an $H$-equivariant family, parametrised by $X$, of $T$-central extensions $M^\tau$ of $M$;
(ii) an $H/M$-equivariant covering space $p : Y \to X$, whose fibres label the isomorphism classes of irreducible, $\tau$-projective representations of $M$;
(iii) an $H$-equivariant, tautological projective bundle $\mathbb{P}R \to Y$, whose fibre $\mathbb{P}R_y$ at $y \in Y$ is the projectivised $\tau$-representation of $M$ labelled by $y$;
(iv) a class $[R] \in K^\mathbb{P}R_H(Y)$, represented by $R$;
(v) a twisting $\tau'$ for the $H/M$-equivariant $K$-theory of $Y$, and an isomorphism of $H$-equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.

Items (iii) and (v) are only defined up to canonical isomorphism. Note that, if $M^\tau$ is abelian, as will be the case in our application, then $\mathbb{P}R = Y$, which can be taken to represent the zero twisting. However, $[R]$ is not the identity class $[1]$, because of the non-trivial $M$-action on the fibres.

5.2 Lemma (Key Lemma). The twisted $K$-theories $K^\tau_{H/M}(Y)$ and $K^\tau_H(X)$ are naturally isomorphic. ⊓⊔

Recall that the isomorphism is induced by the composition below:

\[ K^\tau_{H/M}(Y) \to K^\tau_H(Y) \cong K^p\tau-\mathbb{P}R_H(Y) \otimes [R] \to K^p\tau_H(Y) \overset{p^*}{\to} K^\tau_H(X). \]  

The inverse map lifts a class from $K^\tau_H(X)$ to $K^p\tau_H(Y)$, tensors with the dual class $[R^\vee] \in K^{-\mathbb{P}R}_H(Y)$ and, finally, extracts the $M$-invariant part. The last step requires a bit of care, as it involves a Morita isomorphism associated to two different projective bundle models for the same twisting.

6. Computation when the identity component is a torus

To ensure consistency of notation when the identity component $G_1$ is a torus $T$, we write $N$ for $G$ and $W$ for $\pi_0N$. Denoting, for any $f \in N$, by $N(f)$ the stabiliser in $N$ of the component $fT$, we can decompose $K^\tau_N(N)$ as a sum over representatives $f \in N$ of the conjugacy classes in $W$:

\[ K^\tau_N(N) \cong \bigoplus_f K^\tau_{N(f)}(fT). \]  

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(6.2) The identity component. With $H = N$ and $M = X = T$ in construction 5.1, Lemma 5.2 gives $K_{N}^{∗}(T) = K_{W}^{∗}(Y)$. It is easy to describe the bundle $p : Y \to T$. A twisting class $[\tau] \in H_{N}^{3}(T)$ restricts to $H_{T}^{3}(T)$, hence to $H^{1}(T) \otimes H_{T}^{2}$; and contraction with the first factor gives a map $\kappa^{*} : H_{1}(T) \to H_{T}^{2}$. This gives a translation action of $\Pi := \pi_{1}T$ on the set $\Lambda^{*}$ of $\tau$-affine weights of $T$, and $Y$ is the associated bundle $t \times_{\Pi} \Lambda^{*}$. If $\kappa^{*}$ is injective, as per our regularity condition (2.2), $Y$ is a union of copies of $t$, labelled by $\Lambda^{*}/\kappa^{*}(\Pi)$, and integration along $t$ gives

$$K_{W}^{∗}(Y) = K_{W}^{∗-\sigma(t)-\dim T}(\Lambda^{*}/\kappa^{*}(\Pi)),$$

where the down-shift $\sigma(t)$ in the twisting is defined by a $W$-equivariant Thom class of $t$, represented by a choice of spinors $S(t)$ with projective $W$-action.

(6.3) Affine Weyl action. We restate this by observing that the class $[\tau] \in H_{N}^{3}(T)$ has a “leading term” in $H_{W}^{3}(T; H_{T}^{2})$, with respect to the Hochschild-Serre spectral sequence $E_{p,q}^{2} = H_{W}^{p}(T; H_{T}^{2}) \Rightarrow H_{N}^{p+q}(T)$. This term captures the $W$-action on the covering $Y$ of $T$, but, more importantly, defines an affine action on $\Lambda^{*}$ of the extended affine Weyl group $W \ltimes \Pi$, extending the action of $\Pi$. Comparing orbits and stabilisers gives an equivalence of categories of equivariant bundles, and hence an isomorphism of the desired form,

$$K_{W}^{∗-\sigma(t)}(\Lambda^{*}/\kappa^{*}(\Pi)) = K_{W \ltimes \Pi}^{∗}(\Lambda^{*}).$$

(6.4) A general component. Let now $T^{f}$ be the $T$-centraliser of $f \in N$ and $T$ its identity component. Then, $fT$ is a homogeneous space, with discrete isotropy, for the combined action of $N(f)/T$ by conjugation and of $T := t^{f}$ by translation. We thus have an $N(f) \times T$-isomorphism

$$fT \cong [(N(f)/T) \times T]/W_{aff};$$

(6.5)

the stabiliser $W_{aff}$ of $f$ is expressed, by projection to $N(f)/T$, as a group extension

$$1 \to \Pi \to W_{aff} \to W^{f} \to 1,$$

(6.6)

where $\Pi := \pi_{1}T$ and $W^{f} := [N(f)/T]^{f}$ is itself an extension of $W^{f}$ by the finite group $[T/T]^{f}$:

$$1 \to [T/T]^{f} \to W^{f} \to W^{f} \to 1.$$

Exactness on the right follows from the vanishing of $H_{1}^{f}(T/T)$; that, in turn, follows from the absence of $f$-invariants in $\pi_{1}(T/T)$.

With $X = fT$, $H = N(f)$ and $M = T$ in (5.1), an $N(f)$-equivariant twisting $\tau$ defines a covering space $Y \to fT$, with fibres the sets $\Lambda^{*}$ of $\tau$-affine weights of $T$. Via (6.5), this cover is associated to an affine action of $W_{aff}$ on $\Lambda^{*}$, which is classified by the leading component of $[\tau] \in H_{N(f)}^{3}(fT)$ in

$$H_{N(f)/T}^{1}(fT; H_{T}^{2}) \cong H_{W_{aff}}^{1}(H_{T}^{2}).$$

(6.7)

6.8 Theorem. (i) We have a natural isomorphism $K_{N(f)}^{∗}(fT) = K_{W_{aff}}^{∗}(\Lambda^{*} \times T)$.

(ii) If $\tau$ is regular, this is also $K_{W_{aff}}^{∗-\sigma(t)-\dim T}(\Lambda^{*})$, and is free, of finite rank over $Z$.

Proof. The first part is Lemma 5.2. Provided that all stabilisers of $W_{aff}$ on $\Lambda^{*}$ are finite, part (ii) follows from (i) by integration along $T$ and $\sigma(t)$ is the twisting of the equivariant Thom class.

Now, $\Pi \subset W_{aff}$ has finite index, and acts on $\Lambda^{*}$ by translation, via the linear map $\kappa^{*} : \Pi \to \Lambda^{*}$ defined by restricting $[\tau]$ to $H_{N(T)}^{3}(T)$. Topological regularity of $\tau$ implies finiteness of the quotient $\Lambda^{*}/W_{aff}$ and of all stabilisers. □
6.9 Remark. (i) Considering the action of $N(f) \times \mathfrak{t}$ on $fT$ leads to the presentation
\[ fT \cong N(f) \ltimes \mathfrak{t}/N_{\text{aff}} \]
where the stabiliser $N_{\text{aff}}$ of $f$ fits now in an extension $1 \to T \to N_{\text{aff}} \to W_{\text{aff}} \to 1$.
(ii) Without Lemma 5.2, the isomorphism (6.5) identifies $K_{N(f)}^\tau(fT)$ with $K_{N_{\text{aff}}}^\tau(\mathfrak{t})$; the right-hand side has a sensible topological interpretation, because the group action is proper, resulting in a stack of the type studied in [FHT]. It is tempting to integrate along $\mathfrak{t}$ to land in the $N_{\text{aff}}$-equivariant twisted $K$-theory of a point. However, no topological definition of $K$-theory that we know allows this operation (cf. Remark 5.5); this could perhaps be done by $C^*$-algebra methods.
(iii) For a loop group interpretation of $N_{\text{aff}}$, $W_{\text{aff}}$ and its action on $\Delta^\tau$, see Remark 6.13 below.

(6.10) Induction from conjugacy classes. The following result (with Theorem 7.9 in the next section) is the basis for our original construction of twisted $K$-classes. For each element of the natural basis of Theorem 6.8ii, it selects a distinguished $N(f)$-conjugacy class in $fT$.\(^9\) We shall revisit this when discussing the Dirac families in Chapter V.

6.11 Proposition. If $\tau$ is regular, $K_{N(f)}^\tau(fT)$ is spanned by classes supported on single $N(f)$-orbits.

Proof. An affine action of $W_{\text{aff}}$ on $\mathfrak{t}$ is inherited from the conjugation×translation action of the ambient group $(N(f)/T) \times \mathfrak{t}$. There is also a $W_{\text{aff}}$-action on the affine copy $\Delta^\tau \otimes \mathbb{R}$ of $\mathfrak{t}^*$, defined above by $[\tau]$. Such actions are classified by the groups
\[
H^1_{W_{\text{aff}}}([\mathfrak{t}]) \cong \text{Hom}_{\widetilde{W}_{\text{aff}}}((\Pi, \mathfrak{t})) \quad \text{and} \quad H^1_{W_{\text{aff}}}([\mathfrak{t}^*]) \cong \text{Hom}_{\tilde{W}_{\text{aff}}}((\Pi, \mathfrak{t}^*)),
\]
respectively; $\tilde{W}_{\text{aff}}$ acts by conjugation. The first class is the natural map $\Pi \to \mathfrak{t}$; the second, the map $\kappa^\tau \otimes \mathbb{R}$. Hence, the two actions of $W_{\text{aff}}$ are isomorphic by some translate $\kappa_\mu^\tau : \mathfrak{t} \to \mathfrak{t}^*$ of $\kappa^\tau \otimes \mathbb{R}$.

A class in $K_{W_{\text{aff}}}^\tau([\mathfrak{t}])$ can be pushed forward to $K_{W_{\text{aff}}}^\tau(\Delta^\tau \times \mathfrak{t})$ using the graph of the inverse map $(\kappa_\mu^\tau)^{-1}$. Under (6.8i), its image in $K_{N(f)}^\tau(fT)$ is supported on a single conjugacy class, if the original lived on a single $W_{\text{aff}}$-orbit. \(\square\)

6.12 Remark. (i) $\kappa_\mu^\tau$ descends to an affine isogeny from $fT$ to $\Delta^\tau \otimes T$, preserving the actions of $\tilde{W}_{\text{aff}}$. The quotient spaces $fT/\tilde{W}_{\text{aff}}$ and $fT/N(f)$ are isomorphic, and the conjugacy classes in Prop. 6.11 lie in the fibre of this isogeny over the base-point $\Delta^\tau$ of the second torus.\(^10\) Specifically, a class in the (twisted) $K_{W_{\text{aff}}}^*(\Delta^\tau)$ supported on a $W_{\text{aff}}$-orbit $\Omega$ corresponds to one in $K_{N(f)}^\tau(fT)$ with support at the single $\tilde{W}_{\text{aff}}$-orbit $fT \exp ((\kappa_\mu^\tau)^{-1}\Omega)$.
(ii) An ambiguity in the set of orbits results from our freedom in identifying the affine spaces $\mathfrak{t}$ and $\mathfrak{t}^*$: we are free to translate by the $\mathfrak{t}^*$-invariant part of $\mathfrak{t}$.

6.13 Relation to loop groups. The isomorphism (6.5) implies that $W_{\text{aff}}$ is $\pi_1$ of the homotopy quotient of $fT$ by $N(f)$. The latter is equivalent to the classifying space $BL_{\mathcal{I}}N$: this is best revealed by the gauge action of $L_{\mathcal{I}}N$ on the contractible space of connections on the principal $N$-bundle over the circle, with holonomies in $[fT]$; fixing the fibre over a base-point, the space of holonomies becomes $fT$, while the residual symmetry group is $N(f)$. All in all, $W_{\text{aff}} = \pi_0L_{\mathcal{I}}N$.

Every component of $L_{\mathcal{I}}N$ contains loops of minimal length, so the subgroup $\Gamma_{\mathcal{I}}N \subset L_{\mathcal{I}}N$ of $f$-twisted geodesic loops is an extension of $W_{\text{aff}}$ by $\mathcal{T}$. This $\Gamma_{\mathcal{I}}N$ is in fact isomorphic to $N_{\text{aff}}$: to equate them, interpret the presentation of $fT$ in Remark 6.9i as the quotient of $N(f) \times \mathfrak{t}$ the set of flat bundles over the interval based at the endpoints and with constant connection forms, under the gauge action of $\Gamma_{\mathcal{I}}N$.

\(^9\)Up to an overall ambiguity, see Remark 6.12.
\(^10\)Note that $fT$ covers $fT/\mathcal{T}$, which in turn surjects onto the conjugacy classes.
The action of \( \text{W}_{\text{aff}} \) on \( \Delta^T \) and its twisting \( \tau' \) also have a loop group description. The connection picture above gives an equivalence between the smooth groupoids associated to the actions of \( N(f) \) on \( fT \) and of \( \text{N}_{\text{aff}} \) on \( 1 \) (via \( \text{W}_{\text{aff}} \)). We are interested in twistings coming from \( \mathbb{T} \)-central extensions of \( \text{N}_{\text{aff}} \). In that case, the action of \( \text{W}_{\text{aff}} \) on \( \Delta^T \) arises from the conjugation action of \( \text{N}_{\text{aff}} \) on \( T \) in the central extension. The subgroup of \( (\text{N}_{\text{aff}})^T \) stabilising a weight \( \lambda \in \Delta^T \) is an extension of \( (\text{W}_{\text{aff}})^T \); pushing out via \( \lambda \) gives a \( \mathbb{T} \)-central extension of \( (\text{W}_{\text{aff}})^T \), and these extensions assemble to the twisting \( \tau' \).

7. General compact groups

For any compact \( G \), we will describe \( K_C^T(G) \) in terms of the maximal torus \( T \) of \( G \) and its normaliser \( N \). We must first recall some facts about disconnected groups; readers focusing on the connected case may skip ahead to (7.8) We keep the notations of (6)

(7.1) **Diagram automorphisms.** Choose a set of simple root vectors in \( \mathfrak{g} \), satisfying, along with their conjugates and the simple co-roots, the standard \( \mathfrak{sl}_2 \) relations.

7.2 **Proposition.** Every outer automorphism of \( \mathfrak{g} \) has a distinguished implementation, called diagram automorphism, which preserves \( \mathfrak{t} \) and its dominant chamber and permutes the simple root vectors.

**Proof.** The variety \( G/N \) of Cartan sub-algebras in \( \mathfrak{g} \) has the rational cohomology of a point, so any automorphism of \( \mathfrak{g} \) fixes a Cartan sub-algebra, by the Lefschetz theorem. Composing with a suitable inner automorphism ensures that we preserve \( \mathfrak{t} \) and the dominant chamber. Conjugation by \( T \) provides the freedom needed to permute the simple root vectors without scaling.

7.3 **Remark.** The use of Lefschetz’s theorem is justified, because any automorphism of \( \mathfrak{g} \) lifts to the orientable double cover of \( G/N \), which is the variety of oriented Cartan sub-algebras.

7.4 **Corollary.** \( G \) can be reduced to an extension of \( \pi_0 G \) by the centre of the identity component \( G_1 \).

**Proof.** The subgroup of \( G \)-elements whose Ad-action on \( \mathfrak{g} \) is a diagram automorphisms meets every component of \( G \), and meets \( G_1 \) in its centre. This is our reduction.

(7.5) **Conjugacy classes in \( G \).** The push-out of (7.4) to a maximal torus \( T \) is called a quasi-torus \( Q_T \subset G \); it meets every component of \( G \) in a translate of \( T \). \( Q_T \) depends on \( T \) and a choice of dominant chamber. Choose \( f \in Q_T \); its Ad-action on the dominant chamber must fix some interior points, so \( f = \tau f \) contains \( \mathfrak{g} \)-regular elements. The identity component \( \mathbb{T} \) of the invariant subgroup \( T^f \) is then a maximal torus of the centraliser \( G^f \) of \( f \).

Call \( W = N/T, W_1 = (N \cap G_1)/T \) the Weyl groups of \( G \) and \( G_1 \); we have \( W = \pi_0 G \ltimes W_1 \), by (7.4). Call \([f]\) the image of \( f \) in the quotient \( fT/T \) by \( T \)-conjugation. Conjugation by \( N(f) \), the subgroup of \( N \) preserving the component \( fT \), descends to an action of the group \( W^f = \pi_0 N(f) \) on \( fT/T \). Let \( \widetilde{W} := W^f \cap W_1 \), and \( \tilde{W} \) its extension by \([T/T]^f \) restricted from the \( \tilde{W} \) of (6).6.

7.6 **Lemma.** (i) The space of conjugacy classes \( fG_1/G_1 \) is \( (fT/T) / \).

(ii) The Weyl group of \( G^f_1 \) is an extension by \( \pi_0 T^f \) of the \( W \)-stabiliser of \( [f] \).

**Proof.** Part (i) reformulates Theorem 2.2 of [BD]; indeed, \( fT/T \) is the quotient of \( fT^f \) under conjugation by \([T/T]^f \), whence it follows that \( (fT/T)/W \cong fT/W \). That is the description in [BD].

The normaliser of \( \mathbb{T} \) in \( G_1^f \) is \( N \cap G_1^f \), by regularity, and exactness of \( 1 \rightarrow T^f \rightarrow N \cap G_1^f \rightarrow W \) implies (ii).

7.7 **Remark.** Translation by \( f \) identifies \( fT/T \) with the co-invariant torus \( T_f \) (quotient of \( T \) by the sub-torus \( \{ x f x^{-1} f^{-1} \} \)). The \( W \)-action on \( fT/T \) is affine under the quotient \( W \)-action on \( T_f \). However, the two \( W \)-actions agree when \( f \) is a diagram automorphism \( \varepsilon \): \( W \) indeed isomorphic to the Weyl group of \( \mathfrak{g} \) (Appendix A).
The Weyl map $\omega$. Decompose $K^\tau_G(G) = \bigoplus_f K^\tau_{G(f)}(fG_1)$ over representatives $f \in Q_T$ of conjugacy classes in $\pi_0$; $G(f)$ denotes the stabiliser of the component $fG_1$. The $G(f)$-equivariant map

$$\omega: G(f) \times_{N(f)} fT \to fG_1, \quad g \times ft \mapsto g \cdot ft \cdot g^{-1}$$

induces two morphisms in twisted $K$-theory, restriction $\omega^*$ and induction$^{11}$ $\omega_*$:

$$K^\tau_{N(f)}(fT) \cong K^\tau_{G(f)}(G(f) \times_{N(f)} fT) \xrightarrow{\omega^*} K^\tau_{G(f)}(fG_1).$$

7.9 Theorem. The composition $\omega_* \circ \omega^*$ is the identity.

Consequently, $K^\tau_{G(f)}(fG_1)$ is a summand in $K^\tau_{N(f)}(fT)$, split as an $R(G)$-module. To identify it, we will call a weight in $\Delta^\tau$ regular if it corresponds to a regular conjugacy class in $fG_1$, under the isomorphism $\kappa^\tau_f : \mathfrak{k} \to \mathfrak{k}$ in the proof of (6.11). Clearly, this condition is preserved by $W_{\text{aff}}$.

7.10 Theorem. $K^\tau_{G(f)}(fG_1)$ is the summand in $K^\tau_{N(f)}(fT)$ corresponding to the regular weights:

$$K^\tau_{G(f)}(fG_1) = K^\tau_{W_{\text{aff}}}(\pi^\tau_{\mathfrak{g}}).$$

7.11 Remark. $W_{\text{aff}}$ is called the $f$-twisted, extended affine Weyl group of $G$. Regular weights are those not fixed by any affine reflection of $L_f\mathfrak{g}$ (§10.4 A.9). Reaching ahead a bit, this can be seen as follows. The action of $N_{\text{aff}}$ on $\Delta^\tau$ (Remark 6.13) is part of the co-adjoint action of $L_fG$ on an affine slice (at level $[\tau]$) in $L_f(g)^\tau$ (§11.8). The latter is isomorphic the space of connections over the circle with holonomy in $fG_1$, with the gauge action. That, in turn, is equivalent to the conjugation $G(f)$-action on $fG_1$. Regular are those weights whose co-adjoint Lie algebra stabiliser is minimal, namely $\mathfrak{t}$. Singular weights will be fixed by some Weyl reflection in their $L_f\mathfrak{g}$-stabiliser, which is an affine Weyl reflection. Some facts about simple algebras are recalled in Appendix A.

Proof of (7.9). The quotient spaces $fT/N(f)$ and $fG_1/G(f)$ are isomorphic under $\omega$ (Lemma 7.6). We shall show that $\omega_* \circ \omega^*$ is the identity on small neighbourhoods of conjugacy classes: a Mayer-Vietoris argument then implies that the map is a global isomorphism. However, $K^\tau_{N(f)}(fT)$ is spanned by classes induced from single orbits (Prop. 6.11). Their $\omega_*$-images are fixed by $\omega_* \circ \omega^*$, so the theorem follows.

We need a local model for the Weyl map. We work near $f$, which was arbitrary in $Q_T$. Because $\mathfrak{T}$ contains regular elements, $\mathfrak{N}^f := N \cap \mathfrak{g}^f$ is the normaliser of $\mathfrak{T}$ in $\mathfrak{g}^f$. Now, the translate $f \cdot \exp(\mathfrak{g}^f)$ is a local slice for $G_1$-conjugation near $f$, while $f \cdot \exp(\mathfrak{t})$ is one for $T$-conjugation in $Q_T$. Therefore, a local, $\mathfrak{g}^f$-equivariant model for $\omega$ is the Dirac induction map of (4.12)

$$\mathfrak{g}^f \times_{\mathfrak{N}^f} \mathfrak{t} \to \mathfrak{g}^f,$$

(7.12)

and our claim reduces to Proposition 4.13.

Proof of (7.10). We use the construction 6.11 of $K$-classes from single conjugacy classes. Let $f \in Q_T$ and observe, from $fT \cong f \cdot \mathfrak{g}$ and Lemma 7.6, that the Weyl group of $\mathfrak{g}^f$ is identified with the stabiliser in $W_{\text{aff}}$ of the associated weight. Singular weights are then fixed by the Weyl reflection in some $\mathfrak{s}_2$ centralising $f$, and their $K$-classes are killed by the local induction (7.12). Near a regular $f$, on the other hand, the local model for $\omega$ is an isomorphism, so regular weights contribute non-zero generators in $K^\tau_G(G)$.

$^{11}$The names are justified in A.3.
IV Loop groups and admissible representations

In this chapter, we summarise some basic facts about loop groups, twisted loop groups and their Lie algebras, as well as the classification of admissible representations in terms of the action on affine regular weights of the extended affine Weyl group. The key result, Thm. [10.2] is certainly known, but does not seem to appear in the literature in this form (but see [TL] for simple groups). This combines the theorem of the lowest weight with Mackey’s irreducibility criterion.

We shall need to distinguish between representations of the polynomial loop algebras and their Hilbert space completions, and we convene to mark uncompleted spaces by a prime.

8. Refresher on affine algebras

(8.1) Affine algebras. We use the notations of [1] in particular, \( \mathfrak{g} \) is now simple. The Fourier polynomial loop algebra \( L'g_C \) has the Fourier basis \( \xi_a(m) = z^m \xi_a \). Its basic central extension \( \hat{L}'g := i\mathbb{R} K \oplus L'g \), with central generator \( K \), is defined by the 2-cocycle sending \( \xi \wedge \eta \in \Lambda^2 L'g \) to \( K \cdot \text{Res}_{z=0}(d\xi|\eta) \). The affine Lie algebra \( \hat{L}'g = L'g \oplus i\mathbb{R} E \) arises by adjoining a new element \( iE \), where the energy \( E \) satisfies \([E, K] = 0\) and \([E, \xi(n)] = n\xi(n)\), for any \( \xi \in \mathfrak{g} \). Unlike \( L'g, \hat{L}'g \) carries an ad-invariant bilinear form, extending the basic one on \( \mathfrak{g} \):

\[
\langle k_1 K + \xi_1 + e_1 E | k_2 K + \xi_2 + e_2 E \rangle \mapsto \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_1(t) | \xi_2(t) \rangle \, dt + k_1 e_2 + k_2 e_1.
\]

(8.3) Lowest-weight modules. A projective representation of \( L'g \) has level \( k \) if it extends to a strict representation of \( \hat{L}'g \) in which \( K \) acts as the scalar \( k \). This means that we can choose the action \( R_a(m) \) of \( \xi_a(m) \) so that

\[
[R_a(m), R_b(n)] = f_{ab} R_c(m + n) + km\delta_{ab}\delta_{m,-n}.
\]

Call \( \mathfrak{h} := i\mathbb{R} K \oplus t \oplus i\mathbb{R} E \) a Cartan sub-algebra of \( \hat{L}'g \), and let \( \mathfrak{n} := \bigoplus_{n \geq 0} z^n g_C \oplus n \subset L'g_C \). A lowest weight vector in an \( \hat{L}'g \)-module \( \mathcal{H}' \) is an \( \mathfrak{h} \)-eigenvector killed by \( \mathfrak{n} \). Call \( \mathcal{H}' \) a lowest weight module, with lowest weight \((k, -\lambda, m)\), if it is generated by a lowest weight vector \( v \) of that \((K, t, E)\)-weight. The factorisation \( U(\hat{L}'g) = U(\mathfrak{n}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{m}) \) shows that \( \mathcal{H}' \) is generated by \( \mathfrak{n} \) from \( v \). Defining the positive alcove \( a \subset t \) as the subset of dominant elements \( \xi \) satisfying \( \theta(\xi) \leq 1 \), we have the following:

8.4 Proposition. In a \((k, -\lambda, m)\)-lowest-weight module, the weight \((k, \omega, n)\) of any other \( \mathfrak{h} \)-eigenvector satisfies \( n - m \in \mathbb{Z} \) and \((\omega + \lambda)(\xi) + n > m\), for \( \xi \) inside \( a \).

Proof. All weights of \( \mathfrak{n} \) verify these conditions, with \( \lambda, m = 0 \). □

(8.5) Integrable modules. A lowest-weight module is integrable if the action exponentiates to the associated simply connected loop group.\(^{12}\) Integrable representations are unitarisable, completely reducible, and the irreducible ones are parametrised by their lowest weights \((k, -\lambda, m)\), in which \( k \) must be a non-negative integer and \( \lambda \) a dominant \( T \)-weight satisfying \( \lambda \cdot \theta \leq k \), in the basic inner product. These weights correspond to points of the scaled alcove \( k \cdot a \).

\(^{12}\) A precise definition is a bit delicate, but there are some simple equivalent Lie algebra conditions [K III]; for instance, it suffices that the action should exponentiate on all root \( \mathfrak{sl}_2 \)-subgroups [K VIII].
Spinors. The complex Clifford algebra \( \text{Cliff}(L^*_\mathfrak{g}) \) is generated by the odd elements \( \{ \psi^a(m) \} \) dual to \( \{ \xi_a(m) \} \), satisfying

\[
\psi^a(m) \psi^b(n) + \psi^b(n) \psi^a(m) = 2 \delta_{ab} \delta_{m-n}.
\]

Choose an irreducible, \( \mathbb{Z}/2 \)-graded, positive energy module \( S' \) of \( \text{Cliff}(L^*_\mathfrak{g}) \). As a vector space, this can be identified with the graded tensor product \( S(0) \otimes \Lambda^* (z \mathfrak{g}_C[z]) \), for an irreducible, graded spin module \( S(0) \) of \( \text{Cliff}(\mathfrak{g}) \). \( S' \) carries a hermitian metric, in which \( \psi^a(n)^* = \psi^a(-n) \); so \( \psi(\mu) \) is self-adjoint for \( \mu \in L^*_\mathfrak{g} \). There are obvious actions of \( \mathfrak{g} \) and \( E \) on \( S' \), intertwining with \( \text{Cliff}(L^*_\mathfrak{g}) \). The lowest \( E \)-eigenvalue is 0, achieved on \( S(0) \otimes 1 \). Setting

\[
K \mapsto h^\vee, \quad \xi_a(m) \mapsto \sigma_a(m) := -\frac{1}{4} \sum_{p+q=m} f^a_{bc} \psi^b(p) \psi^c(q)
\]

extends them to an action of \( \hat{L}^*_\mathfrak{g} \), with intertwining relation \( [\sigma_a(m), \psi^b(n)] = f^a_{bc} \psi^c(m+n) \). One derives \( S7 \) by considering the adjoint representation \( \hat{L}^*_\mathfrak{g} \) to the orthogonal Lie algebra \( so_{\text{res}}(L^*_\mathfrak{g}) \), “restricted” as in \( PS \) with respect to the splitting \( L^*_\mathfrak{g}_C = z \mathfrak{g}_C[z] \oplus \mathfrak{g}_C[z^{-1}] \). Formula \( S7 \) is then the quadratic expression of the spin representation of \( so_{\text{res}} \) in terms of Clifford generators \( KS \).

The following key result follows from the Kac character formula. It is part of affine algebra lore; but see \( [\text{FH12}] \) for a proof.

8.8 Proposition. As a representation of \( \hat{L}^*_\mathfrak{g} \), \( S' \) is a sum of copies of the integrable irreducible representation of level \( h^\vee \) and lowest weight \( (-\rho) \). The lowest weight space, which is isomorphic to the multiplicity space, is also the \( \mathfrak{g} \)-lowest-weight space in \( S(0) \), and is a graded, irreducible \( \text{Cliff}(t^\vee) \)-module.

9. Twisted affine algebras

The loop algebras \( L_\mathfrak{g} \) have twisted versions, arising from the automorphisms of non-trivial principal \( G \)-bundles over the circle. They are closely related to the outer automorphisms of \( \mathfrak{g} \) and to the twisted simple affine algebras in Tables (Aff 2,3) of \( \text{K} \): each of those is a central extension of a twisted loop algebra, plus an outer derivation \( E \).

9.1 The algebra \( L_{\varepsilon} \mathfrak{g} \) of loops in \( \mathfrak{g} \) twisted by an automorphism \( \varepsilon \) depends, up to isomorphism, only on the conjugacy class of \( \varepsilon \) in the outer automorphism group of \( \mathfrak{g} \). Thanks to Proposition \( J2 \), we may as well assume that \( \varepsilon \) is a diagram automorphism. When \( \mathfrak{g} \) is simple, this will have order 1, 2 or 3; in general, we insist that the order \( r \) should be finite. This leads to an attractive algebraic model for \( L_{\varepsilon} \mathfrak{g} \) as the invariant part of a copy of \( L_\mathfrak{g} \), based on the \( r \)-fold cover \( \sqrt[r]{S} \) of the unit circle \( S^1 \), under the Galois automorphism which rotates the cover by \( 2\pi/r \) and applies \( \varepsilon \) point-wise. To find the geometric meaning of this construction, let \( G_1 \) be the simply connected group with Lie algebra \( \mathfrak{g} \) and \( G = \mathbb{Z}/r \ltimes G_1 \). The quotient of the trivial bundle \( \sqrt[r]{S^1} \times G \) under the action of \( \mathbb{Z}/r \) which rotates the circle and left-translates the fibres \( G \) is a principal \( G \)-bundle \( P \) over \( S^1 \), and its Lie algebra of gauge transformations is \( L_{\varepsilon} \mathfrak{g} \).

9.2 Standard form. Let \( \mathfrak{g} \) be simple, for the rest of this section. The (smooth) twisted affine algebra \( \hat{L}_{\varepsilon} \mathfrak{g} \) is the invariant part of \( \hat{L}_\mathfrak{g} \), in our construction above. Its structure is described in \( \text{K} \), VI, VIII]. Inherited from the ambient \( \hat{L}_\mathfrak{g} \) is a linear decomposition \( \hat{L}_{\varepsilon} \mathfrak{g} = i \mathbb{R} K \oplus L_\mathfrak{g} \oplus i \mathbb{R} E \). We now rescale \( K \) and \( E \) to \( r \times r \), resp. \( 1/r \times 1/r \) the originals. Then, \( E \) is the natural generator for the rotation of the unit (downstairs) circle, while the bilinear form \( S2 \) is still ad-invariant. Using the standard connection \( \nabla_0 \) on \( P \), descended from the trivial connection on \( \sqrt[r]{S^1} \), the 2-cocycle of the central extension \( \hat{L}_{\varepsilon} \mathfrak{g} := i \mathbb{R} K \oplus L_{\varepsilon} \mathfrak{g} \) is again expressed as an integral over the unit circle, and the Lie bracket takes the following form:

\[
[\xi, \eta](t) = [\xi(t), \eta(t)] + \frac{K}{2\pi r} \int_0\langle \nabla_0 \xi | \eta \rangle.
\]

\(^{13}\)More precisely, the basic central extension of the sub-algebra of twisted, \( \mathfrak{g} \)-valued Fourier polynomials.
(9.4) **Lowest-weight modules.** The rôles of \( t, h \) and \( \mathfrak{N} \) are taken over by their Galois invariants within the ambient \( L'_G \); we denote them by underlines. The structure of weights and roots parallel the untwisted case,\(^\text{14}\) details are summarised in Appendix A.

In a level \( k \) representation of \( \hat{L}'_G \), \( K \) acts as the scalar \( k \). A lowest weight vector is an \( h \)-eigenvector killed by \( \overline{\mathfrak{N}} \), and a lowest-weight module is one generated by a lowest weight vector. We call such a module integrable if the action of all the root \( \mathfrak{sl}_2 \) sub-algebras of \( \hat{L}'_G \) is so; in that case, the module is unitarisable, and the Lie algebra action exponentiates to one of \( \hat{L}_G \) on the Hilbert space completion. Integrable representations are semi-simple, and the irreducible ones are parametrised by their level \( k \) and their lowest weight \((k, -\lambda)\), in which \( \lambda \) is dominant and satisfies \( \hat{\lambda} \cdot \lambda \leq k/r \).

The underlined Doppelgänger for \( \rho, \theta \) and \( a \) require a comment: \( \overline{\rho} \) has the obvious meaning, the half-sum of positive roots for \( \overline{\mathfrak{g}} := \mathfrak{g}'/\mathfrak{g} \), but \( \underline{\rho} \), which cuts out \( \mathfrak{g} \) from the dominant chamber of \( \mathfrak{t} \) by the relation \( \underline{\rho}(\xi) \leq 1/r \), is not the highest root of \( \underline{\mathfrak{g}} \), but rather the highest weight of \( \mathfrak{g}/\overline{\mathfrak{g}} \). Therewith, the analogue of Proposition 8.4 holds true.

A geometric sense in which \( \underline{a} \) plays the rôles of \( a \) is the following. Let \( A \) denote the space of smooth connections on the bundle \( P \); the quotients \( A / \hat{L}_G \) (by gauge transformations) and \( \varepsilon G_1 / G_1 \) (by conjugation) are isomorphic by the holonomy map. The classification \( [A^7] \) of twisted conjugacy classes gives the following.

9.5 **Proposition.** Every smooth connection on \( P \) is a smooth gauge transform of \( \nabla_0 + \xi dt \), for a unique \( \xi \in \underline{a} \). That is, \( \underline{a} \) is a global slice for \( \hat{L}_G \): \( \underline{a} \cong A / \hat{L}_G \).

\(^{14}\)Corollary \([A^9]\) below imposes a small distinction for the weight lattice for twisted SU(2\( \ell + 1 \)); see \([A^10] \).

(9.6) **The Clifford algebra.** A basis for \( \hat{L}'_G \) suited to calculations arises from a complex orthonormal \( \varepsilon \)-eigen-basis \( \{ \xi_a \} \) of \( \mathfrak{g}_C \), so chosen that the indexing set carries an involution \( a \leftrightarrow \bar{a} \) with \( \xi_{\bar{a}} = -\xi_a \). If \( \varepsilon(\bar{a}) \in \mathbb{Z}/r \) corresponds to the \( \varepsilon \)-eigenvalue of \( \xi_a \), then \( \{ \xi_a(m) \} \) forms a basis of \( \mathfrak{g}_a \), as \( m + \varepsilon(\bar{a})/r \) ranges over \( \mathbb{Z} \). Raising and lowering indexes involves a bar; for instance, the relations in the complex Clifford algebra of \( L'_G \) are \( [\psi^a(m), \psi^b(n)] = 2\delta_{ab} \cdot \delta_{m-n} \).

A positive energy, graded spin module \( S' \) can be identified, as a vector space, with \( S(0) \otimes \wedge^\bullet (\overline{\mathfrak{g}}) \), for a graded spin module \( S(0) \) of \( \text{Cliff}(\overline{\mathfrak{g}}) \). As in \([S^7] \), the obvious actions of \( \underline{\mathfrak{g}} \) and \( E \) extend to a lowest-weight representation of \( \hat{L}'_G \), with a bar in the raised index, but, remarkably, with the same \( h^{\nu'} \). As in Prop. 8.8, the representation can be identified using the Kac character formula; its lowest-weight space is the lowest \( \underline{\mathfrak{g}} \)-weight space in \( S(0) \), has pure weight \( -\underline{\rho} \) and is a graded irreducible \( \text{Cliff}(\mathfrak{t}') \)-module.

(9.7) **The loop group.** The extensions in \([S^1] \) and \([J^3] \) are so normalised as to generate all central extensions of \( L_G \) by the circle group \( T \). Call \( L_G \) the twisted loop group of \( G_1 \).

9.8 **Proposition.** \( \overline{L}_G \) is the Lie algebra of a basic central extension \( \overline{L}_G \) of \( L_G \), with central circle parametrised by \( \{ z^k | |z| = 1 \} \), and whose Chern class generates \( H^2(L_G; \mathbb{Z}) = \mathbb{Z} \).

Proof. The untwisted case is handled in \([PS] \), so we focus on \( r > 1 \). Being the space of sections of a \( G_1 \)-bundle over \( S^1 \), \( L_G \) is connected and simply connected. Further, \( \pi_2 L_G = H^1(S^1; \pi_3 G_1) = \mathbb{Z} \), and Hurewicz gives us \( H^2(L_G; \mathbb{Z}) = \mathbb{Z} \). Since \( \pi_2 L_G = H^1(\sqrt{S^1}; \pi_3 G_1) \), the restriction \( H^2(L_G) \to H^2(L_G) \) has index \( r \). Our extension of \( L_G \) will be the \( r \)-th root of the restriction of \( \overline{L}_G \), the basic extension of the ambient, untwisted loop group. Having fixed the cocycle \([J^3] \), the obstructions to existence and uniqueness of this root are topological, living in \( H^2 \) and \( H^1 \) of \( L_G \) with \( \mathbb{Z}/r \)-coefficients, respectively; and they vanish as seen. Finally, we have a semi-direct decomposition \( L_G \cong \mathbb{Z}/r \rtimes L_G \), and the \( \varepsilon \)-action on \( L_G \) preserves the cocycle \([J^3] \), so it lifts to an automorphism action on the central extension (again, by vanishing of the topological obstructions). We let \( \overline{L}_G = \mathbb{Z}/r \rtimes L_G \).
9.9 Corollary. The basic extension \( L_εG \) restricts trivially to the constant subgroup \( G_1^ε \), except when \( G_1 = \text{SU}(2ℓ + 1) \) and \( r = 2 \), in which case \( G_1^ε = \text{SO}(2ℓ + 1) \), and we obtain the Spin\(^r\)-extension.

Proof. The flag variety \( L_εG/G_1^ε \) is simply connected, with no \( H^3 \). (This follows, for instance, from its Bruhat stratification by even-dimensional cells.) The Leray sequence for the fibre bundle \( L_εG/G_1^ε \) shows that \( H^2(L_εG/G_1^ε) \) surjects onto \( H^2(G_1^ε) \). However, \( G_1^ε \) is simply connected, save in the cases listed, whence the result. □

10. Representations of \( L_fG \)

We now classify the admissible representations of the loop groups at levels \( τ - σ \) for which \( τ \) is regular, in terms of the affine Weyl action on regular weights.

(10.1) Notational refresher. Let \( f \) be an element of the quasi-torus \( Q_f \) and call \( L_fG \) is the \( f \)-twisted smooth loop group of \( G \) ([15]), \( τ \) a regular central extension and \( g \) the extension defined by the spin module \( S \) of \( L_fG \) ([16]). Gradings are incorporated into our twistings. The extended affine Weyl group \( W_{\text{aff}} = π_0L_fN \) acts on \( Δ^τ \) by conjugating the central extension of \( G \) and a tautological twisting \( τ^f \) is defined for this action, wherein each \( τ \)-affine weight defines a \( T \)-central extension of its stabiliser in \( W_{\text{aff}} \) ([6,13]). We now restate Theorem 4 without Clifford algebras; it is the lowest-weight classification of representations, enhanced to track the action of the components of \( L_fG \).

10.2 Theorem. (i) The category of admissible representations of \( L_fG \) of level \( τ - σ \) is equivalent to that of \( W_{\text{aff}} \)-equivariant, \( τ' - σ(1) \)-twisted vector bundles over \( Δ^reg \).

(ii) The \( K \)-groups of graded admissible representations are naturally isomorphic to the twisted equivariant \( K \)-theories \( K^τ' - σ(1) + ε(Δ^reg) \).

The reader may wish to consult the simple Example [10] where \( G = N \). In general, the equivalence in (i) arises as follows. A regular weight \( µ \) defines a polarisation of \( L_fg \), which selects, for each admissible representation \( H \), a lowest-weight space in \( H \otimes S \) with respect to \( L_fG \times \text{Cliff}(L_fg^*) \). The \((-µ)\)-eigen-component under \( T \) of this lowest weight-space is a \( \text{Cliff}(1) \)-module, and factoring out the spinors on \( t \) gives the fibre of our vector bundle at \( µ \in Δ^τ \).

Dirac induction provides the inverse equivalence. Each \( µ \in Δ^reg \) defines a regular co-adjoint orbit \( O_µ \subset L_f(g^*)^τ \) over which a twisted representation of the \( W_{\text{aff}} \)-stabiliser defines a \((τ - σ(1))\)-twisted, \( L_fG \)-equivariant vector bundle. The Dirac index of this bundle along \( O_µ \), coupled to the highest-weight spinors, is the desired representation of \( L_fG \). Its level \( (τ - σ) \) arises from the shift by the level \( σ(1) - σ \) of the highest-weight spinors on \( L_fg/\mathfrak{t} \).

Dirac induction in infinite dimensions is only a heuristic notion, but can be realised in this case by the Borel-Weil construction, as a space of holomorphic sections \([16,15]\). We will review that in \([16]\) where it is needed, but we will make no use of it this section.

Proving (10.2) requires some preparation. Split \( g \) into its centre \( 3 \) and derived sub-algebra \( g' \).

10.3 Proposition. \( L_fg' \) splits canonically into a sum of simple, possibly twisted loop algebras. Central extensions of \( L_fg \) are sums of extensions of \( L_f3 \) and of the simple summands.

Proof. In the decomposition of \( g' \) into simple ideals, \( f \)-conjugation permutes isomorphic factors. To a cycle \( C \) of length \( ℓ(C) \) in this permutation, we assign one copy of the underlying simple summand \( g(C) \) and the automorphism \( ε(C) := \text{Ad}(f)^ℓ \). This is a diagram automorphism of \( g(C) \), whose fixed-point sub-algebra is isomorphic to that of \( \text{Ad}(f) \) on the summand \( g(C)^{ε(C)} \) in \( g' \). Then, \( L_fg' \) is isomorphic to the sum of loop algebras \( L_{ε(C)}g(C) \), with the loops parametrised by the \( (C)\)-fold cover of the unit circle. The splitting arises from the eigenspace decomposition of \( \text{Ad}(f) \) on \( g(C)^{ε(C)} \). As the summands are simple ideals, uniqueness is clear. The splitting of the extension follows from the absence of one-dimensional characters of the simple summands. □
(10.4) More on \( W_{\text{aff}} \). The proposition splits \( \mathfrak{t} \) into \( \frac{1}{2} := g' \) and the sum of the Cartan sub-algebras \( \mathfrak{g}(C) \). Call \( \tau \cdot g \in \mathfrak{t} \) the product of \( g \) and the positive alcoves \( [\tau]^x \) in \( \mathfrak{g}(C) \), scaled by the simple components of the level \( [\tau]^x \), and let \( \tau \cdot g^* \) be its counterpart in \( t^* \) in the basic inner product on \( g' \). Reflection about the walls of \( \tau \cdot g^* \) generate a normal subgroup \( W_{\text{aff}}(g, f) \subset W_{\text{aff}} \), under whose action the transforms of the alcove are distinct and tessellate \( \mathfrak{t}^* \). The two groups agree when \( G \) is simply connected, but in general we have an exact sequence, split by the inclusion of \( \pi \) in \( W_{\text{aff}} \) as the stabiliser of \( \tau \cdot a \),

\[
1 \to W_{\text{aff}}(g, f) \to W_{\text{aff}} \to \pi := \pi_0 L_f G \to 1.
\]

(10.5) Regular are those weights not lying on any alcove wall (Remark 7.11). The alcoves correspond to positive root systems on \( L_f g \) which are conjugate to the standard one \( (\mathfrak{g}, \mathfrak{h}) \), the simple roots being the outward normals to the walls. The positive root spaces span a polarisation of \( L_f g' \); the various polarisations, plus the original one on \( L_f 3 \), are conjugate under \( \Gamma_f N \subset L_f G \), so they define the same class of admissible representations.

(10.6) Mackey decomposition in K-theory. Let \( H \) be a group, \( M \) a normal subgroup, \( \nu \) a central extension of \( H \). Conjugation leads to an action of \( H/M \) on isomorphism classes of \( \nu \)-representations of \( M \). Let \( Y \) be a family of isomorphism classes, satisfying the conditions

(i) \( Y \) is stable under \( H/M \);
(ii) Every point in \( Y \) has finite stabiliser in \( H/M \);
(iii) The \( M \)-automorphisms of any representation in \( Y \) are scalars.

There is a tautological projective vector bundle \( \mathbb{P}^R \) over \( Y \), whose fibre \( \mathbb{P}^R_y \) at \( y \in Y \) is the projective space on a representation of isomorphism type \( y \). Its uniqueness up to canonical isomorphism, and hence \( H \)-equivariance, follow from condition (iii). The bundle defines a \( \mathbb{T} \)-central extension of the action groupoid of \( H^\nu \) on \( Y \). This central extension is split over \( M^\nu \), so dividing out by the latter gives a central extension, or twisting, \( \nu' \) for the \( H/M \)-action on \( Y \).

Call an \( H \)-representation \( Y \)-admissible if its restriction to \( M \) is a finite-multiplicity sum of terms of type in \( Y \), with only finitely many \( H/M \)-orbit types. For instance, this includes all induced representations \( \text{Ind}_M^K(R_y) \). The same construction as in Lemma 5.2 establishes the following:

10.7 Proposition. The category of \( Y \)-admissible representations of \( H \) is equivalent to that of \( \nu' \)-twisted, \( H/M \)-equivariant vector bundles over \( Y \), supported on finitely many orbits.

In this equivalence, a \( M \)-representation \( H \) is sent to the bundle whose fibre at \( y \) is \( \text{Hom}^M(R_y, H) \). Conversely, to a bundle over \( Y \) we associate its space of sections. The relation to Construction 9.1 can be made explicit by choosing a representation \( H \) of \( H^\nu \) containing all elements of \( Y \). The projective bundle \( \mathbb{P} \text{Hom}^M(R, H) \) over \( Y \) gives a model for the twisting \( \nu' \) of the \( H/M \)-action.

Proof of 10.2. The unitary lowest-weight representations of the Lie algebra correspond to the admissible ones of the simply connected cover of the identity component \( (L_f G)_1 \). For the simple summands, integrable representations are classified by lowest-weights [K]. Analytic regularity of \( \tau \) on the centre \( L_f 3 \cong \frac{1}{2} \oplus L_f 3/\frac{1}{2} \) means that the second summand has a unique irreducible lowest-weight representation. Unitary irreducibles of \( 3 \) are labelled by the points of the \( \tau \)-affine dual space. Descent of representations to \( (L_f G)_1 \) is controlled by an integrality constraint imposed by \( \mathcal{T} \) parametrising the admissible irreducibles of \( (L_f G)_1 \) by their lowest weights \( (\lambda - \rho) \) over \( \Delta_{\text{reg}+} := \Delta_{\text{reg}} \cap \tau \cdot a^* \).

As \( W_{\text{aff}}(g, f) \) acts freely on \( \Delta_{\text{reg}+}^\tau \) and the orbits are in bijection with the points in \( \Delta_{\text{reg}+}^\tau \), we get an identification

\[
K_{W_{\text{aff}}}^{-\sigma}(\Delta_{\text{reg}}^\tau) = K_{\pi}^{-\sigma}(\Delta_{\text{reg}+}^\tau).
\]

(10.8)
We apply Proposition [10.7] to $H = L_f G$, $M = (L_f G)_{\tau}$, $v = \tau - \sigma$, $Y = \Delta_{\text{reg}+}$. The actions of $\pi$ described in [10.4] and [10.7] do match, because the (sign-reversed) lowest weight $(\sigma, \rho)$ of $S$ is $\pi$-invariant. To conclude the proof, it remains to identify the $\pi$-twistings $\nu'$ and $\tau' - \sigma(1)$.

The subgroup of $N_{\text{aff}}$ lying over $\pi$ preserves the lowest-weight space in any $\nu$-representation $H$ of $L_f G$, and so the projective action of $\pi$ on the resulting lowest-weight bundle over $Y$ represents $\nu'$. Similarly, a model for $\tau'$ arises from the action of $\pi$ on the lowest-weight space in $H \otimes S$, distributed over the (sign-reversed) eigenvalues in $\Delta_{\text{reg}+}$. The second bundles differs from the first by a factor of $S(1)$, and this represents the twisting $\sigma(1)$.

(10.9) Example: $G = N$. Let $V := L_f T \otimes \mathbf{1}$ and $L_f N \cong N_{\text{aff}} \rtimes \exp(V)$, as in [23]. Regularity of $\pi$ confines us to sums of Heisenberg extensions of $V$ and topologically regular extensions $\Gamma_f N^\tau$. The lowest-weight module $F$ of $\exp(V)$ carries a (projective) intertwining action of $\pi_0 N_{\text{aff}}$. An admissible representation $H$ of $L_f N$ factors then as $F \otimes \Hom_k(F; H)$, where the second factor is (the $\ell^2$ completion of) a weight module of $N_{\text{aff}}$, which means that it is $T$-semi-simple, of finite type. Our classification now becomes the following, more precise

10.10 Proposition. Global sections give an equivalence from the category of $W_{\text{aff}}$-equivariant, $\tau'$-twisted vector bundles on $\Delta^\tau$ with that of weight $\tau$-modules of $N_{\text{aff}}$.

It is understood here that $T^\tau$ acts with weight $\lambda$ on the fibre at $\lambda \in \Delta^\tau$. The proposition follows directly from Prop. [10.7]. Weight modules split into irreducibles, which are induced from stabilisers of single weights.

V From representations to K-theory

To an admissible representation $H$ of $L\Gamma$ at fixed level $\tau - \sigma$, we assign a family of Fredholm operators parametrised by an affine copy of $L\mathfrak{g}^*$, equivariant for the affine action of the loop group $L\Gamma$ at the shifted level $\tau$. The underlying space of the family is $H \otimes S$, and the operator family is the analogue of the one in [11] but is based on the Dirac-Ramond operator. We recall this operator in [11] and reproduce the calculation [11, 12] of its Laplacian, which we extend to twisted algebras. Our family defines an $L\Gamma$-equivariant twisted K-theory class over $L\mathfrak{g}^*$, which we identify, when $H$ is irreducible, with the Thom push-forward of the natural line bundle on a single, integral co-adjoint orbit. The passage from representation to orbit and line bundle is an inverse of Kirillov’s quantisation of co-adjoint orbits. The affine copy of $L\mathfrak{g}^*$ carrying our family can be identified with the space of $\mathfrak{g}$-connections over the circle with the gauge action, leading to an interpretation of our family as a cocycle for $K^2(T)$. (G).

11. The affine Dirac operator and its square

Let $\mathfrak{g}$ be simple and let $H'$ be a lowest weight module for $\widehat{L}\mathfrak{g}$, with lowest weight $(k, -\lambda, 0)$. Consider the following formally skew-adjoint operator on $H' \otimes S'$:

$$D = D_0 := R_\sigma(m) \otimes \psi^a(-m) + \frac{1}{3} \cdot \sigma_\sigma(m) \psi^a(-m).$$

(11.1)

This is known to physicists as the Dirac-Ramond operator [11]; in the mathematical literature, it may have been first considered by Taubes [11], and, more recently, studied in detail by Landweber [11], based on Kostant’s compact group analogue. Denote by $T_\sigma(m)$ the total action $R_\sigma(m) + \sigma_\sigma(m)$ of $\xi_\sigma(m)$ on $H' \otimes S'$, and let $k^\lambda := k + h^\lambda$.

11.2 Proposition. $[D, \psi^b(n)] = 2T_b(n)$, $[D, T_b(n)] = -nk^\lambda \cdot \psi^b(n)$. 

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We postpone the proof for a moment and explore the consequences. Clearly, the commutation action of the Dirac Laplacian $\mathbb{D}^2$ on the $T_\bullet$ and the $\psi$ agrees with that of $-2k^\vee E$. Normalize the total energy operator $E$ on $\mathcal{H}' \otimes S'$ to make it vanish on its lowest eigenspace $\mathcal{H}(0) \otimes S(0)$. This last space is $\mathbb{D}$-invariant, and the only terms in (11.1) to survive on it are those with $m = 0$. These sum to the Dirac operator for $\mathfrak{g}$, acting on its representation $\mathcal{H}(0)$. The latter squares to $-(\lambda + \rho)^2$. Since $\mathcal{H}' \otimes S'$ is generated by the actions of the $T_\bullet$ and the $\psi$ on $\mathcal{H}(0) \otimes S(0)$, the following formula for the Dirac Laplacian results:

$$\mathbb{D}^2 = -2k^\vee E - (\lambda + \rho)^2.$$  

(11.3)

In particular, $\mathbb{D}$ is invertible, with discrete, finite multiplicity spectrum.

11.4 Remark. Because the $\sigma$ are expressible in terms of the $\psi$, the Dirac operator (11.1) is expressible in terms of the operators $T_\bullet$ and $\psi$ alone. Define the level $k^\vee$ universal enveloping algebra of $\mathcal{L}' \mathfrak{g}$, $U_{k^\vee}(\mathcal{L}' \mathfrak{g}) := U(\mathcal{L}' \mathfrak{g})/(\mathcal{K} - k^\vee)$. Then, $\mathbb{D}$ is an odd element in a certain completion\textsuperscript{15} of the “semi-direct tensor product” of $\text{Cliff}(\mathcal{L}' \mathfrak{g}^*)$ by $U_{k^\vee}(\mathcal{L}' \mathfrak{g})$, acting via ad. The first equation in (11.2) determines $\mathbb{D}$ uniquely, because no odd elements of the completed algebra commute with all the $\psi$. However, a definite lifting $T_\bullet$ of $\mathcal{L} \mathfrak{g}$ into $U_{k^\vee}(\mathcal{L}' \mathfrak{g})$ has been chosen here. This shows up more clearly in the next section, where we consider the family of $\mathcal{D}'$s parametrised by all possible linear splittings of the central extension $\tilde{\mathcal{L}} \mathfrak{g}$ (cf. also (13.3)).

Proof of (11.2). The first identity follows by adding the two lines below, in which summation over $m \in \mathbb{Z}$ is implied, in addition to the Einstein convention:

$$\left[ R_a(m) \otimes \psi^a(-m), \psi^b(-n) \right] = 2R_b(-n),$$

$$\left[ \sigma_a(m)\psi^a(-m), \psi^b(-n) \right] = 2\sigma_b(-n) + f_{ac}^b \psi^c(m-n)\psi^a(-m) = 6\sigma_b(-n).$$

The second identity in (11.2) follows from the first. Indeed:

$$\left[ [\mathbb{D}, T_b(n)], \psi^c(p) \right] = [\mathbb{D}, [T_b(n), \psi^c(p)]] - [T_b(n), [\mathbb{D}, \psi^c(p)]]$$

$$= f_{da}^b \left[ \mathbb{D}, \psi^d(p+n) \right] - 2 \left[ T_b(n), T_c(p) \right]$$

$$= 2f_{da}^b T_a(p+n) - 2f_{da}^b T_d(p+n) - 2nk^\vee \cdot \delta_{bc} \delta_{n+p}$$

$$= -2nk^\vee \cdot \delta_{bc} \delta_{n+p}$$

$$= -nk^\vee \left[ \psi^b(n), \psi^c(p) \right],$$

whence we conclude that the odd operator $[\mathbb{D}, T_b(n)] + nk^\vee \psi^b(n)$ commutes with all the $\psi$; hence it is zero, as explained in Remark (11.4). \hfill \Box

(11.5) The twisted case. With the same notation and the same definition (11.1) of $\mathbb{D}$, we have

$$\left[ \mathbb{D}, \psi^b(n) \right] = 2T_b(n), \quad [\mathbb{D}, T_b(n)] = -nk^\vee \cdot \psi^b(n);$$  

(11.6)

and we obtain, as before, the formula for the Dirac Laplacian:

$$\mathbb{D}_0^2 = -2k^\vee E - (\lambda + \rho)^2.$$  

(11.7)

\textsuperscript{15}The most natural completion is that containing infinite sums of normal-ordered monomials, of bounded degree and energy; this acts on all lowest weight modules of $\mathcal{L}' \mathfrak{g} \ltimes \psi(\mathcal{L}' \mathfrak{g}^*)$
The level hyperplanes. The co-adjoint action \((11.9)\) preserves the fixed-level hyperplanes \(i\). Identifying this action with the gauge action on the space \(H\), space completion \(12.2\) Proposition. The assignment \(\lor\) affine action at level \(k\)

\[\hat{\mathcal{D}} := \mathcal{D} + E\psi^\delta + K\psi^\kappa,\]

commutes with the (new) total action \(T\) of \(\hat{L} g\) and satisfies a simpler formula \(\hat{\mathcal{D}}^2 = -(\lambda + \rho)^2\), whose verification we leave to the reader.

12. The Dirac family on a simple affine algebra

We now assume the representation \(H'\) of \(\hat{L}' g\) to be integrable; it is then unitarisable, and its Hilbert space completion \(H\) carries an action of the smooth loop group \(L G\). Furthermore, \(k^\vee > 0\).

(12.1) The level hyperplanes. The co-adjoint action \((11.9)\) preserves the fixed-level hyperplanes \(i k^\vee K^* + \hat{L} g^* \subset \hat{L} g^*.\) Ignoring \(\delta\) leads to the affine action at level \(k^\vee\) on \(L g^*\). The correspondence

\[i k^\vee K^* + \mu \leftrightarrow d/dt + \mu/k^\vee\]

identifies this action with the gauge action on the space \(A\) of \(g\)-valued connections on the circle.

12.2 Proposition. The assignment \(\mu \mapsto \mathcal{D}_\mu := \mathcal{D} + i\psi(\mu), from L g^* to End(\mathcal{H}' \otimes S'), intertwines the affine action at level \(k^\vee\) with the commutator action.

Proof. \([T(\xi), \mathcal{D}_\mu] = k^\vee \psi([E, \xi]) + i [\sigma(\xi), \psi(\mu)] = i \psi(-k^\vee d\xi/dt + ad^\vee(\mu)), as desired.\]

(12.3) The Laplacian. Formulae \((11.2)\) and \((11.3)\) give

\[
\begin{align*}
\hat{\mathcal{D}}^2_\mu &= \mathcal{D}^2 + i [\mathcal{D}, \psi(\mu)] - \psi(\mu)^2 \\
&= -2k^\vee E - (\lambda + \rho)^2 + 2i \langle T | \mu \rangle - \mu^2 \\
&= -2 (k^\vee E - i \langle T | \mu \rangle + \langle \lambda + \rho | \mu \rangle) - (\lambda + \rho - \mu)^2.
\end{align*}
\]

When \(\mu \in t^*\), we can view this formula as a generalisation of \((11.3)\), as follows. The first term in \((12.4)\) is \(-2k^\vee E_{\mu}\), with a modified energy operator

\[E_{\mu} = E - i \langle T | \mu/k^\vee \rangle + \langle \lambda + \rho | \mu/k^\vee \rangle.\]

This is associated to the connection \(d/dt + \mu/k^\vee\) in the same way that \(E\) is associated to the trivial connection: they intertwine correctly with the action of \(L g\). Furthermore, \(E_{\mu}\) is additively normalised so as to vanish on the \(- (\lambda + \rho)\)-weight space within \(\mathcal{H}(0) \otimes S(0)\). As we are about to see, when \(\mu/k^\vee \in a^*\), that weight space is the lowest eigenspace for the Dirac Laplacian on \(H \otimes S\).
(12.5) The Dirac kernels. To study a general \( D_\mu \), we conjugate by a suitable loop group element to bring \( \mu \) into \( k^V a^* \). As \( D_\mu \) now commutes with \( t \) and \( E \), we can evaluate (12.4) on a weight space of type \((\omega, n)\), where \( T(\mu) = i(\omega|\mu) \), and obtain

\[
D_\mu^2 = -2(\kappa^V n + (\omega + \lambda + \rho|\mu)) - (\lambda + \rho - \mu)^2
\]  

(12.6)

Now, a weight of \( H \otimes S \) splits as \((\omega, n) = (\omega_1, n_2) + (\omega_2, n_2)\), into weights of \( H \) and \( S \). Proposition (3.1) asserts that \((\omega_1 + \lambda) \cdot \mu + k^V n_1 \geq 0\), with equality only if \( \mu/k^V \) is on the boundary of \( a^* \), or else if \( \omega = -(\lambda + \rho) \) and \( n = 0 \). But then, (12.6) cannot only vanish if, additionally, \( \mu = \lambda + \rho \). Since that lies in the interior of \( k^V a^* \), we obtain the following.

12.7 Theorem. The kernel of \( D_\mu \) is nil, unless \( \mu \) is in the affine co-adjoint orbit of \((\lambda + \rho)\) at level \( k^V \). If so, \( \ker D_\mu \) is the image, under the same transformation, of the \(-(\lambda + \rho)\)-weight space in \( H(0) \otimes S(0) \).

The last space is the product of the lowest-weight space \( C_v \) of \( H(0) \) with that of \( S(0) \); the latter is a graded, irreducible Cliff(\( t \))-module. As in finite dimensions, the more canonical statement is that the kernels of the \( D_\mu \) on the “critical” co-adjoint orbit \( D \) of \( \lambda + \rho \) in \( ik^V K^* + Lg^* \) assemble to a vector bundle isomorphic to \( S(N)(-\lambda - \rho) \), the normal spinor bundle twisted by the natural line bundle on \( D \). This vector bundle has a natural continuation to a neighbourhood of \( D \) as the lowest eigen-bundle of \( D_\mu \). We can describe the action of \( D_\mu \) there, when \( \mu \) moves a bit off \( D \).

12.8 Theorem. Let \( \mu \in D \), \( \nu \in N_\mu \) a normal vector to \( D \) at \( \mu \) in \( A \). The Dirac operator \( D_{\mu + \nu} \) preserves \( \ker(D_{\mu}) \), on which it acts as Clifford multiplication \( i\psi(\nu) \).

(12.9) Twisted K-theory class. Proposition (12.2) shows that our constructions are preserved by the action of \( LG \), so the Fredholm bundle \((H \otimes S, D_\mu)\) over \( Lg^* \) defines a twisted, \( LG\)-equivariant K-theory class supported on \( D \). Formula (12.6) bounds the complementary spectrum of \( D_\mu \) away from zero, so the embedding of the lowest eigenbundle induces an equivalence of twisted, \( LG\)-equivariant K-theory classes in some neighbourhood of \( D \). Proposition (12.8) identifies the \( K \)-class with the Thom push-forward of the line bundle \( O(-\lambda - \rho) \), from \( D \) to \( Lg^* \). Finally, identifying the level \( k^V \) hyperplane in \( Lg \) with \( A \) as in (12.1) and using the holonomy map from \( A \) to \( G \) interprets our Dirac family as a class in \( K_C^\infty(G) \), in degree \( \dim g \) (mod \( 2 \)).

(12.10) Twisted affine algebras. The results extend verbatim to twisted affine algebras, if we use the presentation \( Lg \) of \( \mathfrak{g} \). Let \( A_\varepsilon \) be the space of smooth connections on the \( G \)-bundle of type \( \varepsilon \) and recall the distinguished connection \( \nabla_0 \) of (19.2).

12.11 Proposition. (i) The identification of the affine hyperplane \( ik^V K^* + Lg^* \subset A_\varepsilon \) with \( \nabla_0 + \mu/k^V \) is equivariant for the action of \( Lg \).

(ii) The assignment \( \mu \mapsto D + i\psi(\mu) \) intertwines the affine co-adjoint and commutator actions.

(iii) Formula (12.4) for \( D_\mu^2 \), and its consequences (12.7) and (12.8), carry over, with \( \rho \) replaced by \( \varepsilon \).

13. Arbitrary compact groups

We now extend the construction of the Dirac family, and the resulting map from representations to twisted \( K \)-classes, to the space \( A_P \) of connections on a principal bundle \( P \) over the circle, with arbitrary compact structure group \( G \). The Lie algebra \( L_P G \) of the loop group \( L_P G \) of gauge transformations splits into a sum of abelian and simple loop algebras, and the central extension preserves the splitting (Prop. 10.3). To assemble the families for the individual summands, we must still discuss the abelian case and settle their equivariance under the non-trivial components of \( L_P G \).
components which carry the holonomies of $P$. Thus, $D_{\mu}$ and $S_{\tau}$ variant under $L$. The kernel is identified as before: it is supported on the affine subspace $iK^* + \lambda + L_3^* \oplus \mathfrak{z}^*$ of $\tilde{L}_3^*$. This is a single co-adjoint orbit of the identity component of $LZ$, and the family represents the Thom push-forward of the $LZ$-equivariant line bundle $O(-\lambda)$, from that orbit to the ambient space.

(13.2) Spectral flow over $Z$. The positive polarisation $\mathfrak{h} \subset L_3^* \oplus \mathfrak{z}^*$ leads to vector space identifications $S' \cong S(0) \otimes \Lambda^*(\mathfrak{h})$ and $F' \cong \text{Sym}(\mathfrak{h})$. Decomposing $D_{\mu} = D_{\mu}^3 + D_{LZ/3}^3$ into zero-modes and $\mathfrak{h}$-modes, we recognise in the first term is the Dirac family of $\mathfrak{h}$ lifted to $\mathfrak{z}^*$ and restricted to the single summand $C_{-\lambda} \subset F_{[-\lambda]}$ whereas $D_{LZ/3} = \partial + \partial^*$, for the Koszul differential

$$\partial : \text{Sym}^p(\mathfrak{h}) \otimes \Lambda^q(\mathfrak{h}) \rightarrow \text{Sym}^{p+1}(\mathfrak{h}) \otimes \Lambda^{q-1}(\mathfrak{h}).$$

Thus, $D_{\mu}$ is quasi-isomorphic to the finite-dimensional family $(C_{-\lambda}, D_{\mu}^3)$ over $\mathfrak{z}^*$. The induced $LZ$-module will have the form $F' \otimes F_{[-\lambda]}$, and dropping the factor $\Lambda^*(\mathfrak{h}) \otimes F'$, which is equivalent to $C$, recovers our spectral flow family of (13.3).

(13.3) Characterisation of $D_{\mu}$. Proposition 12.2 ensures the equivariance of our Dirac family for the connected part of the loop group. When $G$ is not simply connected, we must extend this to the other components. This is accomplished by an intrinsic characterisation of $D_{\mu}$. We restate relations (11.2) and (11.6) in a “coordinate-free” way:

$$[D_{\mu}, \psi(\nu)] = 2\langle T | \nu \rangle + 2i\langle \mu | \nu \rangle \quad [D_{\mu}, T(\xi)] = \psi(\text{ad}_{\xi}(k^*K^* - i\mu)),$$

where the bracket in the first equation is contraction in the bilinear form $[\cdot, \cdot]$. Observe now that the first formula expresses the total action of $\nu$ on $H \otimes S$, in the lifting of $L_{PG}$ to $L_{PG}$ defined by the line $ik^*K^* + \mu \subset L_{PG}$. In the second formula, we have used the co-adjoint action of $L_3$. As explained in Remark 11.4, the first relation uniquely determines $D_{\mu}$, and we conclude

13.4 Proposition. The assignment $\mu \mapsto D_{\mu}$ is equivariant under all compatible automorphisms of $L_{PG}$, $H$ and $S$ which preserve the bilinear form on $L_{PG}$. □

(13.5) Coupling to representations. The Dirac family $D_{\mu}$ lives on an affine copy of $L_{PG}^*$, namely the hyperplane over $i \in i\mathbb{R}$ in the projection $(L_{PG}^*)^* \rightarrow i\mathbb{R}$, dual to the central extension (2.6). We transport it to $A_P$ by identifying the two as $L_{PG}$-affine spaces. For the simple factors, this is described in (12) but on the abelian part, there is an ambiguity: we can translate by the Lie algebra of the centre of $L_{PG}$. Under the holonomy map, this ambiguity matches the one encountered in (6.12(ii)), when we identified $\tau \cdot \mathfrak{a}^*$ with the space of holonomies. Note, however, that the regularity and singularity of the affine weights matches the one of the underlying (twisted) conjugacy classes in $G$, irrespective of the chosen identification.

Coupling $D_{\mu}$ to graded, admissible representations results in twisted $K$-classes on $A_P$, equivariant under $L_{PG}$. This is also an Ad-equivariant twisted $K$-classes over $G$, supported on the components which carry the holonomies of $P$. 

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13.6 Proposition. The isomorphism of Theorem 3 is induced by the Dirac family map, from admissible representations to $K$-classes.

Proof. This follows by comparing the Dirac kernels to the classification of irreducibles by their lowest-weight spaces in [10] and again with the basis of $K_G^*(G)$ described in Proposition 6.11. 

VI Variations and Complements

This chapter exploits the correspondence between representations and $K$-classes to produce analogues of known constructions in representation theory in purely topological terms.

14. Semi-infinite cohomology

In this section, we give alternative formulae (14.3), (14.10) for the Dirac operator $D$. With the Lie algebra cohomology results of Bott [B] and Kostant [K1] and with Garland’s loop group analogues [G], the new formulae explain the magical appearance of the kernel on the correct orbit. The relative Dirac operators of [K2] and [L] allow us to interpret the morphisms $\omega_*$ and $\omega^*$ of §7 in terms of well-known constructions for affine algebras, namely semi-infinite cohomology and semi-infinite induction [FF].

We work here with polynomial loop algebras and lowest-weight modules; for simplicity, we omit $f$-twist, underlines and the primes from the notation. We shall also use $\text{ad}^\vee$ to denote the co-adjoint action of a Lie algebra on its dual, reserving the “$*$” for hermitian adjoints.

(14.1) Lie algebra cohomology. The triangular decomposition $Lg_C = n \oplus t_c \oplus n^*$ factors the spin module as $S = S(t^*) \otimes N^*$. The action of $n^*$ on a lowest-weight module $H$ leads to a Chevalley differential on the Lie algebra cohomology complex, 

$$\partial : H \otimes \Lambda^k n^* \to H \otimes \Lambda^{k+1} n^*,$$

$$\partial = R_{-\alpha} \otimes \psi^\alpha + \frac{1}{2} \psi^\alpha \cdot \text{ad}^\vee_{-\alpha}$$ (14.2)

where we have used a root basis of $n^*$ and its dual basis $\psi^\alpha$ of Clifford generators. Let $\partial^*$ be the hermitian adjoint of $\partial$, and denote by $D^t$ the t-Dirac operator with coefficients in the representation $H \otimes N^* \otimes C_\rho$ of $T$.

14.3 Proposition. $D = \partial + \partial^* + D_{-\rho}^t$, and $D_{-\rho}^t$ commutes with $\partial + \partial^*$.

Proof. Commutation is obvious. It is also clear that the $R$-terms on the two sides agree; so, it remains to compare the Dirac $(\sigma \psi)/3$-term in [11.1] with $\psi \cdot \text{ad}^\vee/2 + (\psi \cdot \text{ad}^\vee)^*/2$, plus the ad-term in $D^t$. Now, all three terms have cubic expressions in the Clifford generators, and we will check their agreement. We have

$$\frac{1}{2} \psi^\alpha \cdot \text{ad}^\vee_{-\alpha} = \frac{1}{4} \sum_{\alpha,\beta,\gamma} f_{\alpha\beta\gamma} \psi^\alpha \psi^\beta \psi^\gamma,$$

$$\frac{1}{2} (\psi^\alpha \cdot \text{ad}^\vee_{-\alpha})^* = \frac{1}{4} \sum_{\alpha,\beta,\gamma} f_{\alpha\beta\gamma} \psi^{\gamma} \psi^{-\beta} \psi^{-\alpha}.$$

Disregarding the order of the generators, their difference contains precisely the terms in $\sigma \psi/3$ involving two positive roots and a negative one, respectively two negative roots and a positive one; whereas the ad-term in $D^t$ similarly collects the $\sigma \psi/3$-terms involving exactly one $t^*$-element. Clearly, this accounts for all terms in $\sigma \psi/3$. We have thus shown that the symbols of these operators agree in (a completion of) $\Lambda^3(Lg^*)$. 

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The difference between the two must then be a linear $\psi$-term. However, both operators commute with the maximal torus $T$ and with the energy $E$; so the difference is $\psi(\mu)$, for some $\mu \in \mathcal{U}_C$. A quick computation gives, for $\nu \in t^*$,
\[
[\bar{\partial}, \psi(\nu)] = [\bar{\partial}^\ast, \psi(\nu)] = 0,
\]
\[
[D^\dagger_{-\rho}, \psi(\nu)] = 2T(\nu) = [D, \psi(\nu)];
\]
so $[\psi(\mu), \psi(\nu)] = 0$ for all $\nu$, and it follows that $\mu = 0$. \hfill \Box

(14.4) The Dirac kernels. Proposition 14.3 gives a new explanation for the location of $\ker D_\mu$. If $H$ is irreducible with lowest weight $(-\lambda)$, we have, on $H \otimes \Lambda^q \mathfrak{m} \otimes S(t^*)$,
\[
\ker (\bar{\partial} + \bar{\partial}^\ast) \cong H^q (\overline{\mathfrak{m}}; H) \otimes S(t^*) = \bigoplus_{(\ell(\nu))=q} C_{w(-\lambda-\rho)+\rho} \otimes S(t^*),
\]
embedded in the Lie algebra complex as harmonic co-cycles; the sum ranges over the elements of length $q$ in the affine Weyl group of $\mathfrak{g}$. If $\mu \in t^*$, then $D^\dagger_{-\rho} = D^\dagger_{\mu-\rho} + i\psi(\mu)$ commutes with $(\bar{\partial} + \bar{\partial}^\ast)$, so $\ker D_\mu$ is also the kernel of $D^\dagger_{\mu-\rho}$ on $D^\dagger_{\mu-\rho}$ $\otimes$ $S(t^*)$. Clearly, the latter is non-zero precisely when $\mu$ is one of the $w(\lambda + \rho)$; otherwise, it follows that the highest eigenvalue of $D^2_\mu$ is the negative squared distance to the nearest such point, in agreement with 12.3.

(14.6) Semi-infinite cohomology. A similar construction applies to a decomposition of a rather different kind. Splitting $L^\bullet_{\mathfrak{g}} = L^n \oplus L_{t^*} \oplus L\delta$ gives a factorisation
\[
S(L^\star_{\mathfrak{g}}) = S(L^t_{\star}) \otimes \Lambda^{\infty / 2 + *}(L^t_\star),
\]
where the right-most factor is the exterior algebra on the non-negative Fourier modes in $n^*$ and the duals of the negative ones, the latter carrying degree $(-1)$ \cite{CGZ, LT}. A formula similar to 14.2 defines a differential $\bar{\partial}$ for semi-infinite Lie algebra cohomology, acting on $H \otimes \Lambda^{\infty / 2}(L^t_{\star})$.

With the same $H$, the semi-infinite cohomology can be expressed as a sum of positive energy Fock spaces $F \otimes C_\mu$ for $L/\mathfrak{t}$, on which $T$ acts with weight $\mu$:
\[
H^{\infty / 2 + q}(L; H) = \bigoplus_{(\ell(\nu))=q} F \otimes C_{w(-\lambda-\rho)+\rho}.
\]
Because the splitting of $L_{\mathfrak{g}}$ was $LT$-equivariant, $LT$ acts on $\Lambda^{\infty / 2}(L^\star_{\mathfrak{g}})$; it commutes with $\bar{\partial}$, so acts on the cohomology; but the non-trivial components shift the degree. Passing to Euler characteristics, we can collect terms into irreducible representations $F \otimes F_{[\mu]}$ of $LT$ \cite{B} and obtain a sum over the finite Weyl group\footnote{In the $f$-twisted case, this is the extension $\overline{W}$ of \cite{6.3}, and the $F_{[\mu]}$ are the irreducible $\tau$-modules of $\overline{\Pi} \times \mathcal{L}$.}
\[
\sum_q (-1)^q H^{\infty / 2 + q}(L; H) = \sum_{w \in W} \varepsilon(w) \cdot F \otimes F_{[w(-\lambda-\rho)+\rho]}.
\]
(14.9) Relative Dirac operator. Define $D^{\mathcal{L}^q / \mathcal{L}^t} := \bar{\partial} + \bar{\partial}^\ast$; its index is given by (14.8).

14.10 Proposition. $\bar{\partial} = D^{\mathcal{L}^t} + D^{\mathcal{L}^q / \mathcal{L}^t}$, and the three operators commute. \hfill \Box

The proof is very similar to the one of Prop. 14.3; see \cite{L} for more help. Similarly, we have $D^{\mathcal{L}^q} = D^{\mathcal{L}^q / \mathcal{L}^t} + D^{\mathcal{L}^q / \mathcal{L}^t}$, and the three operators commute when $\mu \in L^t$. As in 14.4 it follows that the restriction to $L^t$ of our Dirac family on $H \otimes S$ is stably equivalent to $D^{\mathcal{L}^t}$, acting on the alternating sum in (14.8). Comparing this with the construction (6.11) of $K$-classes from conjugacy classes and with the local model of the Weyl map (7.8), we obtain the following
14.11 Theorem. Under the Dirac family construction Chapter $[\text{A}]$ the semi-infinite $L_n$-Euler characteristic, from $R^{*}\sigma(LG)$ to $R^*(LT)$, corresponds to the Weyl restriction $\omega^* : K_G^\tau(G) \to K_T^\tau(T)$. $\square$

14.12 Remark. (i) In the twisted case, this applies to the restriction $\omega^* : K_{G(f)}^\tau(fG_1) \to K_T^\tau(fT)$.
(ii) We have used $LT$ for simplicity, but the result applies to $LN$, which preserves the relative Dirac $D^L_{\delta/L}$ (though not the semi-infinite differential $\partial$). We then detect the restriction to $K_{N(f)}^\tau(fT)$.

15. Loop rotation, energy and the Kac numerator

(15.1) Conditions for rotation-equivariance. The admissible loop group representations of greatest interest admit a circle action intertwining with the loop rotations (14.9). This will be the case iff the following two conditions are met:

(i) The loop rotation action lifts to the central extension $LG^\tau$, and the differential $\delta : H^2_G(G_1) \to H^2(T \times BT) \otimes H^1(G_1)$ in the Leray sequence for the projection to $BT$. This is a smooth stack, with compact quotient and proper diff-

terential $\delta$.

A lifting in (i) defines a semi-direct product $T \times LG^\tau$. Subject to condition (ii), the Borel-Weil con-

struction $\text{[PS]}$ shows that all of admissible representations carry actions of the identity component of this product, and the $T$-action is determined up to an overall shift on each irreducible. The action can be extended to the entire loop group as in $\text{[II]}$ and this leads to the same classification of irreducibles, but with the extra choice of normalisation for the circle action.

With respect to condition (i), it is convenient to allow fractional circle actions: that is, we allow the circle of loop rotations to be replaced by some finite cover. A lifting of the rotation action to $LG^\tau$ refines the level $[\tau]$ to a class in $H^3(B(T \times LG))$. The obstruction to such a refinement is the differential $\delta_2 : H^2_G(G_1) \to H^2(T \times BT) \otimes H^1(G_1)$ in the Leray sequence for the projection to $BT$. All torsion obstruction vanish when $T$ is replaced by a suitable finite cover. Rationally, $H^2_G(G_1)$ is the invariant part of $H^2(T)$ under the Weyl group $W$ of $G$, and for the torus we have the following.

15.2 Lemma. A class in $H^3(T \times BT)$ lifts to a rotation-equivariant one iff its component in $H^1(T) \otimes H^2_T$ is symmetric.

Proof. The differential $\delta_2$ vanishes on the $H^*(T)$ factor, and is determined its effect on $H^2_T$: this is mapped isomorphically onto $H^2(BT) \otimes H^1(T)$. On $H^3(T \times BT)$, this becomes the anti-symmetrisation map $H^1(T) \otimes H^2_T \to H^2(T) = \Lambda^2 H^1(T)$. $\square$

15.3 Remark. For semi-simple $G$, symmetry is ensured by Weyl invariance.

Adding loop rotations to the landscape leads to the quotient stack of the space $A$ of smooth connec-

tions by the action of $T \times LG$. This is a smooth stack, with compact quotient and proper stabiliser, but which, unlike the quotient stack $G_C$ of $G$ by its own Ad-action, cannot be presented as a quotient of a manifold by a compact group. The $K$-theory of such stacks is discussed in $\text{[FHT1]}$. Let $\Lambda^\tau = \Lambda^\tau \otimes \Bbb{Z} \delta$ be the level $\tau$ slice of the affine weight lattice $\text{[A,9]}$.

15.4 Proposition. We have isomorphisms $R^{\tau-\sigma}(T \times LG) \cong K_{W_{all}}^{\tau-\tau}(\Lambda^\tau) \cong K_{\tau}^{\tau+\text{dim} a}(G_1)$, obtained by tracking the loop rotation in Thm. $\text{[Z,10]}$ and in the Dirac family.

The middle group is a free $R_T$-module, with the generator acting on $\Lambda^\tau$ by $\delta$-translation. Killing the augmentation ideal forgets the circle action in the outer groups and $\delta$ in the middle group, and recovers the isomorphisms in Theorems 3 and 4.

Proof. The argument is a repetition of (5.2), (6.8) and (7.10), with the extra $T$-action. The main difference is that we are now dealing with the $K$-theories of some smooth, proper stacks, which are no longer global quotients, but only locally so. However, the proofs of (5.2) and (7.10) proceed via a local step, which continues to apply, globalised using the Mayer-Vietoris principle. $\square$
Proof. The natural choices for the Fredholm operator $S$ defining the Lie algebra cocycle in (24) are multiples of the derivative $-\text{id}/dt$; the polarisation $\mathfrak{P}$ is then the semi-positive Fourier part of $L_\mathfrak{g}$. With those choices, lowest-weight modules of $L_\mathfrak{g}$ carry a bounded-below energy operator $E$, unique up to additive normalisation, generating the intertwining loop rotation action. If the restriction to $H^1(T) \otimes H^2_T$ of $[\tau]$ is symmetric, loop rotations lift fractionally to $LG^\tau$; and, if that same bilinear form is positive, $E$ is bounded below on admissible $\tau$-representations of the group.

This generalises easily to the twisted loop groups $L_pG$ of gauge transformations of a principal bundle $P$ over $S^1$. The diffeomorphisms of the bundle $P$ which cover the loop rotation form an extension of the rotation group $T$ by $L_pG$; any connection on $P$ whose holonomy has finite order gives a fractional splitting of this extension. This group replaces the $T \times LG$ of the trivial bundle case. The topological constraint for rotation-equivariance of a extension $\tau$ is now the symmetry of the map $\kappa^\tau$ in 16.4.

15.6 The Kac numerator. For the remainder of this section, we make the simplifying assumption that $G$ is connected, with $\pi_1G$ free. Positive energy representations of $L_\mathfrak{g}$ are then determined by their restriction to the subgroup $T \times G$ of circle rotations and constant loops; moreover, loop rotations extend to a trace-class action of the semi-group $\{q \in \mathbb{C}^* | |q| < 1\}$. If $H$ is irreducible with lowest-weight $(-\lambda)$, the value of its character at $q \in \mathbb{C}$ and $g \in G$ is given by the Kac formula [K]

$$\text{Tr}(qg|H) = \frac{\sum \varepsilon(\mu) \cdot q^{\|\mu\|^2/2} \cdot \text{Tr}(g|V_{\mu-\mu})}{\Delta(g;q)},$$

where $\mu$ ranges over the dominant regular affine Weyl transforms of $(\lambda + \rho)$ at level $[\tau]$, $\varepsilon(\mu)$ is the signature of the transforming affine Weyl element, $V_\mu$, the $G$-representation with lowest weight $\mu$, $\|\mu\|^2 := (\kappa^\tau)^{-1}(\mu)\mu$ defined by the level $[\tau]$, and the Kac denominator for $(L_\mathfrak{g}, g)$

$$\Delta(g;q) = \prod_{n>0} \text{det}(1 - q^n \cdot \text{ad}(g))$$

independent of $\lambda$ and $\tau$, representing the (super)character of spinors on $L_\mathfrak{g}/g$. We shall now see how (15.7) is detected by our $K_T$-group.

Including the identity $e \in G$ defines a Gysin map

$$\text{Ind} : R^T-\sigma(g)(T \times G) \to K_T^{+\dim G}(G_C),$$

with $\tau$ on the left denoting the restricted twisting and $\sigma(g)$ the Thom twist of the adjoint representation. Dualising over $R_T$, while using the bases of irreducible representations to identify $K_T^{+\dim G}(G_C)$ with its $R_T$-dual, leads to an $R_T$-module map

$$\text{Ind}^* : K_T^{+\dim G}(G_C) \to \text{Hom}_\mathbb{Z}\left(R^T-\sigma(g)(G); R(T)\right);$$

the right-hand side is the $R(T)$-module of formal sums of (twisted) $G$-irreducibles with Laurent polynomial coefficients. The choice of basis gives an indeterminacy of an overall power of $q$ for each irreducible, which must be adjusted to give an exact match in the following theorem. Let $[H]$ be the $K_T^{+\dim G}(G_C)$-class corresponding to $H$.

15.8 Theorem. $\text{Ind}^* [H]$ is the Kac numerator in (15.7).

Proof. The theorem is a consequence of two facts. First is the relation

$$q^{\|\lambda + \rho\|^2/2} \cdot \text{Ind}(V_{-\lambda}) = \varepsilon(\mu) \cdot q^{\|\mu\|^2/2} \cdot \text{Ind}(V_{\mu-\mu}),$$

(15.9)
holding for any $\mu$ in the affine Weyl orbit of $(\lambda + \rho)$. Second is the fact that, with our simplifying assumption that $G$ is connected with free $\pi_1$, the twisted $K$-class $\text{Ind}(V_{-\lambda})$ corresponds to an irreducible representation of $LG^\varepsilon$. (There are no affine Weyl stabilisers of regular weights.)

We can check (15.9) by restriction to the maximal torus $T$. The Weyl denominator is the Euler class of the inclusion $T \subset G$; multiplying by it while using the Weyl character formula converts the Kac numerator for $(Lg, g)$ to that of $(Lt, t)$, and we are reduced to verifying the theorem for the torus (with $\varepsilon(\mu) = 1$ and without $\rho$-shifts, as the affine Weyl group is now the lattice $\pi_1 T$).

The twisting $\tau$ defines a line bundle $L$ over the representation ring of the stabiliser over $T$.

The stabiliser itself is a bundle of groups with fibre $\mathbb{T} \times T$ and holonomy around a loop $\gamma \in \Pi$ given by the automorphism

$$q^m \mu\lambda \mapsto q^{m+\langle \lambda | \gamma \rangle} \mu\lambda,$$

where $q \in \mathbb{T}$, $t \in T$ and $\lambda : \pi_1 T \to \mathbb{Z}$ is an integral weight. For $L$, this must vary by multiplication by a unit $q^{\phi(\lambda, \gamma) \cdot \mu \kappa^{\tau}(\gamma)}$. (The exponent $\kappa^{\tau}(\gamma)$ of $t$ is detected by restriction to $q = 1$.) We claim that the only option, up to automorphism, is $\phi(\lambda, \gamma) = \langle \kappa^{\tau}(\gamma) | \gamma \rangle / 2$, resulting in the holonomy

$$q^{m+\langle \lambda | \gamma \rangle^2 / 2} \mu\lambda \mapsto q^{m+\langle \lambda + \kappa^{\tau}(\gamma) | \gamma \rangle^2 / 2} \cdot \mu(\lambda + \kappa^{\tau}(\gamma)),$$

travelling around $\gamma$ shows that induction from the characters $q^{\langle \lambda | \gamma \rangle^2 / 2} \mu\lambda$ and $q^{\langle \lambda + \kappa^{\tau}(\gamma) | \gamma \rangle^2 / 2} \cdot \mu(\lambda + \kappa^{\tau}(\gamma))$ of the stabiliser must lead to the same twisted $K$-class, which proves (15.9) for the torus and hence our theorem. To check the claim, note the two relations

$$\phi(\lambda + \mu, \gamma) = \phi(\mu, \gamma),$$
$$\phi(\lambda, \gamma + \gamma^\prime) = \phi(\lambda, \gamma') + \phi(\lambda, \gamma) + \langle \kappa^{\tau}(\gamma) | \gamma' \rangle$$

the first, by computing the holonomy of $\mu^{\langle \lambda + \mu | \gamma \rangle} = \mu^\lambda \mu^\mu$ in two different ways (using the module structure of $L$) and the second, from the homomorphism condition. These imply that $\phi(\lambda, \gamma) = \langle \kappa^{\tau}(\gamma) | \gamma \rangle / 2$, modulo a linear $\gamma$-term; but the latter can be absorbed by a shift $t^\lambda \mapsto t^{\langle \lambda | \gamma \rangle}$ in $T$-characters, representing an automorphism of $L$.

15.10 Remark. This can be generalised to twisted loop groups and their disconnected versions, but to determine a representation uniquely, we must restrict it to a larger subgroup of the loop group, meeting every torsion component in a translate of the maximal torus. We expect in that case to recover the extension of the Kac character due to Wendt [W].

16. Fusion with $G$-representations

For positive energy representations, the fusion product of conformal field theory defines an operation $*: R(G) \otimes R^\tau(LG) \to R^\tau(LG)$. We will now recall its construction and prove its agreement with the $R(G)$-action on $K_G^\tau(G)$. For notational clarity, we treat the untwisted loop groups, the twisted result following by judicious insertion of underlines and $f$-subscripts.

(16.1) Example: $G_1$ is a torus. Recall from (2.6) that, when $G = N$, $LN \cong \Gamma N \times \exp(Lt \ominus t)$, where $\Gamma N = N_{\text{aff}}$ is the subgroup of geodesic loops. Evaluating geodesic loops at a point $x$ in the circle gives a homomorphism $E_x : LN \to N$. If $V$ is a finite-dimensional $N$-representation, the pull-back $E^*_x V$ is an admissible $LN$-representation, and fusing with $V$ is simply tensoring with $E^*_x V$.

Note that $h$ is not the “evaluation at $x$” homomorphism on the whole of $LN$; the latter would not lead to admissible representations. For non-abelian $G_1$, we need the more complicated definition that follows, essentially moving the base-point $x$ inside the disk.

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17This $L$ is the free $\mathbb{Z}$-module over the covering $Y$ in Construction 5.1.
Then, finishing of higher cohomologies of this bundle. Let $\gamma$ be a complex annulus, with an interior base-point $x$. The obvious group $\text{Hol}(A; G_C)$ of holomorphic maps with smooth boundary values acts on $H$, by restriction to the inner boundary, on $V$ by evaluation at $x$, and maps into a copy of $LG_C$ by restriction to the outer boundary. G. Segal defines the fusion of $H$ with $V$ along $A$ as the \textit{holomorphic induction}

$$H \ast V_x := \text{Ind}_{\text{Hol}(A; G_C)}^{LG_C}(H \otimes V_x),$$

by which we mean the space of right $\text{Hol}(A; G_C)$-invariant holomorphic maps from $LG_C$ to $H \otimes V$. Conjecturally, this is a completion of an admissible representation.

The rigorous implementations of this construction that we know are algebraic. The direct product $\hat{H}$ of energy eigenspaces in $H$ is a representation of the \textit{Laurent polynomial loop group} $L'G_C := G_C[z, z^{-1}]$. After evaluation at $z = x$, $L'G_C$ also acts on $V$. The completion of $L'G_C$ at $z = \infty$ is the group of formal Laurent loops $G_C((w)) (w = z^{-1})$. Its algebraic, positive energy $\tau - \sigma$-modules are completely reducible, and the irreducibles are precisely the direct sums $H'$ of energy eigenspaces in irreducible admissible representations $H$ of $LG$. Constructing the induced representation now from \textit{algebraic} functions, the following important lemma permits the subsequent definition.

16.4 \textbf{Lemma.} $\text{Ind}_{L'G_C}^{G_C((w))}(\hat{H} \otimes V)$ is a finitely reducible, positive-energy representation of $G_C((w))$.

16.5 \textbf{Definition.} The fusion product $H \ast V_x$ is $\text{Ind}_{L'G_C}^{G_C((w))}(\hat{H} \otimes V)$.

Using brackets to denote the associated $K$-classes, the fusion is identified by the following

16.6 \textbf{Theorem.} In $K_0^G(G)$ with its topological $R(G)$-action, $[H \ast V_x] = [H] \otimes [V]$.

The proof of this theorem requires some preliminary constructions.

(16.7) \textit{Borel-Weil construction.} We need to review the construction of $H$ by algebraic induction from a Borel-like subgroup (called the Iwahori subgroup), but minding the group $\pi_0LG$ of components. This is neatly accomplished by a construction due to Beilinson and Bernstein.

The \textit{quasi-iwahori subgroup} $Q_l \subset L'G_C$ is the normaliser of $\mathfrak{nil}$; it meets every component of $L'G_C$ in a translate of the standard Iwahori subgroup. Killing $\exp(\mathfrak{nil})$ converts $Q_l$ into a subgroup $Q_L \subset (N_{\text{aff}})_C$, which plays the rôles of a (complex) quasi-torus for the loop group. $Q_L$ is the normaliser of $\mathfrak{nil}$ in $(N_{\text{aff}})_C$. There is a Cartesian square

$$
\begin{array}{ccc}
Q_L & \rightarrow & N_{\text{aff}} \\
\downarrow & & \downarrow \\
\pi_0LG & \rightarrow & W_{\text{aff}},
\end{array}
$$

where the bottom horizontal arrow is the splitting of \cite{10.5} defined by the positive alcove.

Call $\mathcal{U}$ the algebraic vector bundle over the full flag variety $X' := L'G_C/Q_l$ whose fibre at a coset $\gamma Q_l$ is the space $H'/\mathfrak{nil}H'$ of co-invariants in $H'$, with respect to the conjugated nilpotent $\mathfrak{nil}^\gamma := \gamma \mathfrak{nil} \gamma^{-1}$. (This fibre is isomorphic to the lowest-weight space for the opposite polarisation.) Then, $\hat{H}$ is the space of algebraic sections of $\mathcal{U}$ over $X'$. A result of Kumar \cite{Ku} ensures the vanishing of higher cohomologies of this bundle.

---

\footnote{Experts will know that, when $G$ is not semi-simple, these algebraic loop groups are highly non-reduced group (ind)-schemes, and their formal part must be included in the discussion.}

\footnote{To see the problem, recall that every representation of a \textit{connected} compact Lie group is holomorphically induced from a Borel subgroup $B$. This fails in the disconnected case, where induction from the quasi-Borel $Q_T \cdot B$ is required instead.}
16.8 Remark. (i) $Q_1$ acts (projectively) on the space $U := \mathbf{H}/\mathfrak{m} \mathbf{H}$, which defines a projective $L'G_\mathfrak{c}$-vector bundle over $X'$; "unprojectivising" this bundle at level $\tau - \sigma$ results in $U$.

(ii) The same prescription defines $U$ over the "thicker" flag variety $X := G_\mathfrak{c}(\mathfrak{w})/Q_1$, and its sections there lead to the "thin" version $\mathbf{H}'$ of the same representation.

(16.9) Derived induction. The fibre of $U$ at 1 is a representation of $Q_1$ which factors through $Q_L$, and whose highest weights are in $\Lambda^+_{\text{aff}}$, as discussed in [10]. We now study the "derived induction" $R\text{Ind}$ from $Q_L$-modules to $L\mathbf{G}$-modules, by which we mean the Euler characteristic over $X$ of a vector bundle associated to a general $(\tau - \sigma)$-module of $Q_L$. By [10] again,

$$\tau - \sigma R(Q_L) \cong \tau - \sigma(0) K_\tau(\Lambda^\tau),$$

with the action and twistings defined there, and we claim that $R\text{Ind}$ is the result of the direct image map, followed by restriction to the regular part:

$$\tau - \sigma(0) K_\tau(\Lambda^\tau) \rightarrow \tau - \sigma(0) K_{W_{\text{aff}}}(\Lambda^\tau) \rightarrow \tau - \sigma(0) K_{W_{\text{aff}}}(\Lambda^\tau_{\text{reg}}).$$

From [10] and the vanishing of higher cohomology, this is known for weights in $\Lambda^+_{\text{aff}}$. Because $W_{\text{aff}} \cong \pi \ltimes W_{\text{aff}}(\mathfrak{g})$ and $\tau \cdot a^w$ is a fundamental domain for $W_{\text{aff}}(\mathfrak{g})$, it suffices to show that $R\text{Ind}$ is anti-symmetric under this last group and that weights on the walls of $\tau \cdot a^w$ induce 0. Both statements follow from Bott’s reflection argument [11] applied to the simple affine reflections.

Proof of (16.4). $Q_1$ acts on $V$ by evaluation at $z = x$; calling $V_x$ the associated vector bundle over $X$, transitivity of induction shows that

$$\text{Ind}^G_{G_C(\mathfrak{w})} (\widetilde{\mathbf{H}} \otimes V) \cong \Gamma(X; U \otimes V_x).$$

and the Lemma now follows from Theorem 4 of [11].

Proof of (16.6). Theorem 4 of [11] also ensures the vanishing of higher cohomologies when $V$ is small. We will identify $V \ast V_x$ by deforming $V_x$. Scaling $x \rightarrow 0$ deforms the action of $Q_1$ on $V_x$ into the representation $V_0$, pulled back from the quotient map $Q_1 \rightarrow Q_L$. More precisely, any pointwise evaluation $L \mathbf{G} \rightarrow G$ embeds $Q_L$ into $N$, and $V_0$ is obtained from $V$ under $Q_L \rightarrow N \subset G$. The Euler characteristic of the bundle $U \otimes V_x$ is unchanged under deformation, because of the rigidity of admissible representations of $G(\mathfrak{w})$, and we conclude that

$$\mathbf{H} \ast V_x \cong R\text{Ind}(U \otimes V_0).$$

To prove the theorem, we must show that $R\text{Ind} : R(Q_L) \rightarrow K_G^s$ is an $R(G)$-module map, under the inclusion $Q_1 \subset G$. Factoring $R\text{Ind}$ as in (16.10), this property is clear for the second step, restriction to $\Lambda^+_{\text{aff}}$, since that is nothing but the map $\omega^\ast$ of [7,10]. A different description makes the same obvious for the first step, the direct image. Indeed,

$$\tau - \sigma(0) K_{W_{\text{aff}}}(\Lambda^\tau) \cong K_{\tau^\ast}(t), \quad \tau - \sigma(0) K_{\tau}(\Lambda^\tau) \cong K_{\tau}(t),$$

as in Remark 6.9. The direct image map becomes now induction along the inclusion $Q_L \subset N_{\text{aff}}$, and this is clearly a module homomorphism under $R(N)$, to which $N_{\text{aff}}$ maps by evaluation.

17. Topological Peter-Weyl theorem

We now describe a topological version of the Peter-Weyl theorem for loop groups; beyond its entertainment value, the result can be used to confirm that the bilinear form in the Frobenius algebra $K_G^s$ of [FHT] agrees with the natural duality pairing in the Verlinde ring, as claimed in [FHT3, §8].

$^{20}$And the techniques of [11], which reduce this to a finite type problem.

$^{21}$This interpretation is only available for twistings that are transgressed from $BG$. 34
(17.1) Compact groups. One version of the Peter-Weyl theorem asserts that the two-sided regular representation of a compact group $G$ — the space of functions on $G$, under its translation actions on the left and on the right — is (a topological completion of) the direct sum $\bigoplus V \otimes V^*$, ranging over the irreducible finite-dimensional modules $V$. A variation of this, for a central extension $G^\tau$ by $\mathbb{T}$, describes the space of sections of the associated line bundle over $G$ as the corresponding sum over irreducible $\tau$-representations.

Qua $G \times G$-module, the regular representation of $G$ is induced from the trivial $G$-module, under the diagonal inclusion $G \subseteq G \times G$. For finite $G$, the result can be expressed in terms of equivariant $K$-theory: it asserts that the trivial representation $[1] \in R(G)$ maps, under diagonal inclusion, to the class $\sum[V \otimes V^*] \in R(G \times G)$. To see this topological direct image more clearly, identify $R(G)$ with $K_{G \times G}(G)$, with the left-right action, and push forward to a point with $G \times G$ action. In the presence of a twisting $\tau$ for $R(G)$, we map $[1] \in R(G)$ instead to $R^{\tau \times (-\tau)}(G \times G)$. In constructing this last push-forward, we have used the natural trivialisation of the sum of a central extension $\tau$ of $G$ with its opposite, so that the required twisting on $K_{G \times G}(G)$ is canonically zero.

17.2 Remark. When $\tau$ is graded, our formulation of Peter-Weyl conceals a finer point. The module $R^\tau(G)$ of graded representations has now an odd component $R^{\tau+1}(G)$, defined from the super-symmetric representations [FHT3 §4]. These are graded $G$-modules with a commuting action of the rank one Clifford algebra $\text{Cliff}(1)$. The contribution of such a super-symmetric representation $V$ to the Peter-Weyl sum is the (graded) tensor product $V \otimes_{\text{Cliff}(1)} V^*$ over $\text{Cliff}(1)$, and not over $\mathbb{C}$. However, this does match the cup-product

$$R^{\tau+1}(G) \otimes R^{\tau+1}(G) \to R^{\tau \times (-\tau) + 0}(G \times G);$$

indeed, the (graded) tensor product $V \otimes_{\mathbb{C}} V^*$ has a commuting $\text{Cliff}(2)$ action, and defines an element of $K^2$, which is indeed where the cup-product initially lands [LM]. Tensoring over $\text{Cliff}(1)$ instead of $\mathbb{C}$ is the Morita identification of complex $\text{Cliff}(2)$-modules with vector spaces, which implements the Bott isomorphism to $K^0$.

(17.3) Loop groups. Before discussing the loop group analogue of this, let us recall the algebraic Peter-Weyl theorem for loop groups, a special case of the Borel-Weil theorem of [T]. As in the preceding section, denote by $G_{\mathbb{C}}(\!(z\!))$ and $G_{\mathbb{C}}(\!(w\!))$ be the two Laurent completions of the loop group $LG$ at the points 0 and $\infty$ on the Riemann sphere. The Laurent polynomial loop group $L'G_{\mathbb{C}} = G[z, z^{-1}]$ embeds in both (with $w = z^{-1}$). The quotient variety $Y := G(\!(w\!)) \times L'/G_{\mathbb{C}} \times G(\!(z\!))$ for the diagonal action is a homogeneous space for the product of the two loop groups, which should be thought regarded as a generalised flag variety. For any twisting $\tau$, the product $0(\tau - \sigma) \boxtimes 0(\sigma - \tau)$ of the opposite line bundles on the two factors carries an action of $L'G_{\mathbb{C}}$, so it descends to an (algebraic) line bundle on $Y$. A special case of the Borel-Weil-Bott theorem of [T] asserts that, as a representation of $G(\!(w\!)) \times G(\!(z\!))$,

$$\Gamma(Y; 0(\tau - \sigma) \boxtimes 0(\sigma - \tau)) \cong \bigoplus_{H} H^\prime \otimes \overline{H},$$

with the sum ranging over the lowest-weight representations $H$ for $G_{\mathbb{C}}(\!(w\!))$ at level $\tau - \sigma$.

(17.4) Topological interpretation. The topological construction in [T] breaks down for infinite compact groups, but remarkably, it does carry over to loop groups. To start with, the diagonal self-embedding of $G$ leads to a Gysin map

$$\iota_* : K_G(G) \to K_{G \times G}(G \times G),$$

with the Ad-action in both cases. This models the diagonal $LG \to LG \times LG$ when $G$ is connected.

In general, the restriction of $\iota_*$ to $K_G(G)$ corresponds to $LG$, whereas its restriction to $K_{G(f)}(f G)$,

\[\text{for a Lie group, the direct sum describes the polynomial functions.}\]
as in §7.8, captures the diagonal embedding for the twisted loop group \( L_f G \). Similarly, for any \( \tau \), we get a map

\[
i_\tau^* : K_G(G) \to K_{G \times G}(G \times G),
\]

(17.5)
cancelling the pulled-back twisting by the same observation as before: the sum of extensions \( \tau + (-\tau) \) is trivial on the diagonal \( LG \).

17.6 Remark. To construct \( i_\tau^* \), we replace \( K_G(G) \) with the isomorphic group \( K_{G \times G}(G \times G) \), the action being now

\[
(g_1, g_2). (x, y) = (g_1 x g_1^{-1}, g_1 y g_2^{-1}).
\]

The isomorphism with \( K_G(G) \) arises by restriction to the diagonal \( G \)'s. The map \( G \times G \to G \times G \) inducing \( i_\tau^* \) sends \((x, y)\) to \((x, y^{-1} xy)\). Note that the relative tangent bundle of this map is (stably) equivariantly trivial, and there is a preferred relative orientation, if we use the same dual pair of Spin modules on each pair of \( g \)'s.

17.7 Theorem (Peter-Weyl for Loop Groups). When \( \tau \) is regular, we have

\[
i_\tau^*(1) = \sum_H [H \otimes H^*],
\]

summing over the irreducible admissible representations \( H \) of \( LG \), in the correspondence of Theorem 3. The analogue holds for each twisted loop group of \( G \).

Without using Theorem 3, we can assert that \( i_\tau^*(1) \) has a diagonal decomposition in the basis of \( K_G^\tau(G) \) produced from regular affine Weyl orbits and irreducible representations of the centralisers (Theorem 7.10), and the complex-conjugate basis for \( K_G^{-\tau}(G) \). The two formulations are of course related by Theorem 10.2. To state this more precisely, we use the “anti-diagonal” class \([\Delta^-]\) on \( \Delta^\tau \times \Delta^{-\tau} \), which is identically 1 on pairs \((\lambda, -\lambda)\) and null elsewhere. It is equivariant for the diagonal \( W_{aff} \)-action. Also let \( \tau'' = \tau' - \sigma(\tau) \).

17.8 Lemma. The sum in (17.7) corresponds to the direct image of \([\Delta^-]\) under the direct image map

\[
K_{W_{aff}}(\Delta_{reg}^\tau \times \Delta_{reg}^{-\tau}) \to K_{W_{aff} \times W_{aff}}(\Delta_{reg}^\tau \times \Delta_{reg}^{-\tau}).
\]

Proof. Replacing both sides with the sets of orbits, represented by weights \( \mu \in \Delta_{reg}^\tau \) and stabilisers \( \pi_\mu \subset W_{aff} \), we get the direct sum over \( \mu \) of the diagonal push-forwards

\[
R(\pi_\mu) \to R^{\tau''}(\pi_\mu) \otimes R^{-\tau''}(\pi_\mu),
\]

and apply the topological Peter-Weyl theorem to each \( \pi_\mu \).

For a torus \( T \), the representation categories of \( LT^\tau \) and \( \Gamma^\tau = (\Pi \times T)^\tau \) are equivalent, and \( i_\tau^* \) captures the Peter-Weyl theorem for \( \Gamma^\tau \): diagonal induction of the trivial representation to \( \Gamma^\tau \times \Gamma^{-\tau} \) leads to the sum in (17.7). This result generalises to every group \( N_{aff} \) of \((f\text{-twisted})\) geodesic loops in \( N \), and is the basis for the general proof. To convert it into a topological statement, we will factor both the algebraic and the topological induction (direct image) maps into two steps, with the second step being described by Lemma 17.8. Agreement of the other, first step is then verified by a Dirac family construction akin to §3. As the general case may be obscured by the requisite notational clutter, we handle the torus first.
Lemma 17.8. preserves the diagonal cp[y of $f T$ anti-diagonal $\Delta$]

Example: $G$ with the same pro-

\[ K^0_T(T) \xrightarrow{\text{diag}_s} K_{T \times T}^{\tau \times (-\tau - \ell)}(T) \xrightarrow{\text{diag}_s} K_{T \times T}^{\tau \times (-\tau + 0)}(T \times T), \tag{17.10} \]

along the obvious diagonal morphisms. Describing $\text{diag}_s$ is easy. Double use the Key Lemma \[5.2\] with the same $

\[ K_{T \times T}^{\tau \times (-\tau - \ell)}(T) \cong K^0(\Lambda^\tau \times \Pi \Lambda^{-\tau}), \]

\[ K_{T \times T}^{\tau \times (-\tau + 0)}(T \times T) \cong K^0(\Lambda^\tau / \Pi \times \Lambda^{-\tau} / \Pi). \tag{17.11} \]

Moreover, $\text{diag}_s$ becomes the direct image between the groups on the right, which is the map in Lemma \[17.8\]

In view of Lemma \[17.8\] we must check that $B\text{diag}_s[1]$ in the middle group of \[17.10\] is the anti-diagonal $[\Delta^-]$. We have a commutative square

\[
\begin{array}{c}
K^0_T(T) \xrightarrow{B\text{diag}_s} \tau \times (-\tau) K_{T \times T}^{\tau - \ell}(T) \\
\uparrow \quad \quad \quad \uparrow \\
K^0(T) \quad \quad \quad P_\ast \quad \quad \quad K_{T}^{\tau - \ell}(T)
\end{array}
\]

with the vertical arrows being the pull-backs, along the projection of $BT$ to a point and the multiplica-

\[ p \times \mu \]

Our anti-diagonal class, in the upper right, is the pull-back of the sum of the irreducible classes in $K_{T}^{\tau - \ell}(T)$. But we identified this in \[3.6\] with $p_\ast[1]$, as desired.

Proof of \[17.7\]. Fix a twisting element $f$ in the quasi-torus; we prove the theorem for $L_f G$. We use the notation of \[B\] and \[D\] except that we write $G$ for $G(f)$, $N$ for $N(f)$, $W$ for $W^f$ for simplicity.

Step 1. In view of the commutative square, in which $\omega_s(1) = 1$,

\[
\begin{array}{c}
K_N(fT) \xrightarrow{i_s} K_{N \times N}^{\tau \times (-\tau)}(fT \times fT) \\
\downarrow \omega_s \quad \quad \quad \downarrow \omega_s \\
K_G(fG_1) \xrightarrow{i_s} K_{G \times G}^{\tau \times (-\tau)}(fG_1 \times fG_1)
\end{array}
\]

it suffices to settle the upper $i_s$: that is, we may assume $G = N$.

Step 2. Let $\delta(N)$ be the left equaliser of the two projections $N^2 \Rightarrow N / T$. Its Ad-action on $fT^2$

the push-forward from upper to lower $K$-groups. We are reduced to showing that $B\text{diag}_s[1] \in K_{\delta(N)}^{\tau \times (-\tau - \ell)}(fT)$ is the anti-diagonal class in the upper right group.
Step 3. Call $\delta(N_{aff})$ the left equaliser of the projections $N_{aff} \times N_{aff} \rightrightarrows W_{aff}$. The presentation \((6.5)\) of $fT$ as a homogeneous space for $N \times \mathfrak{t}$ leads to the isomorphisms

$$K_N(fT) \cong K_{N_{aff}}^{\tau \times (-\tau)}(\mathfrak{t}),$$
$$K_{\delta(N)}^{\tau \times (-\tau)}(fT) \cong K_{\delta(N_{aff})}^{\tau \times (-\tau)}(\mathfrak{t}),$$

as flagged in Remark 6.9. The twisting $\tau \times (-\tau)$ is null on the diagonal copy of $N_{aff}$ in $\delta(N_{aff})$, but trivialising it in relation to the other twistings is the key step in finding $B_{\text{diag}}$.  

Step 4. Call $\mathcal{O}(\tau)$ the line bundle over $\mathcal{T} \cong \delta(N_{aff})/N_{aff}$ descended from the line bundle of the extension $\tau \times (-\tau)$ of $\delta(N_{aff})$. This $\mathcal{O}(\tau)$ carries a projective action of $\delta(N_{aff})$, by left translations, and its space of sections over $\mathcal{T}$ is, by definition, the representation $\text{Ind}[1]$ induced from $\mathcal{C}$ under the embedding $N_{aff} \subset \delta(N_{aff})^{\tau \times (-\tau)}$. This is the sought-after class $[\Delta^-]$ in \((17.13)\). 

Step 5. Finally, we show that, under the standard trivialisation of the extension $\tau \times (-\tau)$ over $N_{aff}$, the direct image of $[1]$ along the topological induction

$$K_{N_{aff}}^{\tau \times (-\tau) + 0}(\mathfrak{t}) \xrightarrow{B_{\text{diag}}} K_{\delta(N_{aff})}^{\tau \times (-\tau) - \ell}(\mathfrak{t})$$

is represented by the Dirac family on $\mathfrak{t}$ coupled to $\text{Ind}[1]$. This implies its agreement with $[\Delta^-]$. The argument merely repeats the discussion in \(3.6\) after observing that $[1]$ corresponds to the class of $\mathcal{O}(\tau)$ in the chain of isomorphisms

$$[1] \in K_{N_{aff}}(\mathfrak{t}) \cong K_{N_{aff}}^{\tau \times (-\tau)}(\mathfrak{t}) \cong K_{\delta(N_{aff})}^{\tau \times (-\tau)}(\mathcal{T} \times \mathfrak{t}).$$

\[\square\]

Appendix

A. Affine roots and weights in the twisted case

We recall here the properties of diagram automorphisms, which lead to a concrete description of the twisted affine algebras in terms of simple, finite-dimensional ones. The connection between the two questions is due to Kac, to which we refer for a complete discussion \(K\) \(\S 7.9\) and \(\S 7.10\); but we reformulate the basic facts more conveniently for us.

(A.1) When $\mathfrak{g}$ is simple, the order of a diagram automorphism $\varepsilon$ is $r = 1, 2$ or $3$. We assume that $\varepsilon \neq 1; \mathfrak{g}$ must then be simply laced. We summarise the relevant results from \(K\).

- The invariant sub-algebra $\mathfrak{g} := \mathfrak{g}^\varepsilon$ is simple, with Cartan sub-algebra $\mathfrak{t} := t^\varepsilon$ and Weyl group $W := W^\varepsilon$. 
- The simple roots are the restrictions to $\mathfrak{t}$ of those of $\mathfrak{g}$. 
- The ratio of long to short root square-lengths in $\mathfrak{g}$ is $r$, save for $\mathfrak{g} = su(3)$, when $\mathfrak{g} = su(2)$. 
- The $\varepsilon$-eigenspaces are irreducible $\mathfrak{g}$-modules. The two $\varepsilon \neq 1$-eigenspaces are isomorphic when $\mathfrak{g} = so(8)$ and $r = 3$. 

(A.2) The weight $\vartheta$. Denote by $\vartheta$ the highest weight of $\mathfrak{g}/\mathfrak{g}$, and let $a_0 = 2$ when $\mathfrak{g} = su(2\ell + 1)$ and $r = 2$; else, let $a_0 = 1$. Then, $\vartheta/a_0$ is the short dominant root of $\mathfrak{g}$. (When $\mathfrak{g} = su(2\ell + 1)$, $\mathfrak{g} = so(2\ell + 1)$ and $\mathfrak{g}/\mathfrak{g}$ is $\text{Sym}^2 \mathbb{R}^{2\ell+1}/\mathbb{R}$, whose highest-weight is twice the short root.) The basic inner product on $\mathfrak{g}$ restricts to $a_0$ times the one on $\mathfrak{g}$; so $\vartheta^2 = 2a_0/\ell$.  

A.3 Remark. With reference to \([K, VI]\), we have $\vartheta = \sum a_i \alpha_i - a_s \alpha_s$, where $s = 0$, except when $\mathfrak{g} = su(2\ell + 1)$, in which case $s = 2\ell$: if so, our $\vartheta$ differs from $\theta$ in loco citato.
(A.4) Twisted affine Weyl group. Denote by \( \mathfrak{g} \) the simplex of dominant elements \( \xi \in \mathfrak{t} \) satisfying \( \bar{\theta}(\xi) \leq 1/r \). The \( \varepsilon \)-twisted affine Weyl group \( W_{aff}(\mathfrak{g}, \varepsilon) \) is generated by the reflections about the walls of \( \mathfrak{g} \). Let \( \mathfrak{R}' \subset \mathfrak{t} \) correspond to the root lattice \( \mathfrak{R} \) in \( \mathfrak{t}^* \) under the \( \mathfrak{g} \)-basic inner product.

A.5 Proposition (K Props. 6.5 and 6.6). \( W_{aff}(\mathfrak{g}, \varepsilon) \) is the semi-direct product of the \( \mathfrak{R}' \)-translation group by \( W \). Its action on \( \mathfrak{t} \) has \( \mathfrak{g} \) as fundamental domain. The \( W_{aff}(\mathfrak{g}, \varepsilon) \)-stabiliser of any point in \( \mathfrak{g} \) is generated by the reflections about the walls containing it.

Proof. This follows from the analogous result for the untwisted affine algebra based on the Langlands dual to \( \mathfrak{g} \), in which \( \mathfrak{g} \) is the fundamental alcove and \( \mathfrak{R}' \) the co-root lattice.

(A.6) Twisted conjugacy classes. When \( G \) is simply connected, the points of \( \mathfrak{g} \) parametrise the conjugacy classes in \( \mathfrak{g} \). The alcove \( \mathfrak{g} \) fulfils the same role for the \( \varepsilon \)-twisted conjugation \( g : h \mapsto g \cdot h \cdot \varepsilon(g)^{-1} \).

A.7 Proposition. If \( G \) is simply connected, every \( \varepsilon \)-twisted conjugacy class in \( \mathfrak{g} \) has a representative \( \exp(2\pi \xi) \), for a unique \( \xi \in \mathfrak{g} \). The twisted centraliser of \( \exp(2\pi \xi) \) in \( G \) is connected, and its Weyl group is isomorphic to the stabiliser of \( \xi \) in \( W_{aff}(\mathfrak{g}, \varepsilon) \).

Proof. For the first part, we must show, given (A.5) and (A.6), that the integer lattice of \( T_\varepsilon \) in \( t_\varepsilon \cong \mathfrak{t} \) is \( \mathfrak{R}' \), and that the \( \varepsilon \)-twisted action of \( W \) on \( t_\varepsilon \) is the obvious one. Now, the first lattice is the image, in the quotient \( t_\varepsilon \) of \( t \), of the integer (co-root) lattice \( \mathfrak{R}' \subset t \) of \( T \). As \( T \) is simply laced, \( \mathfrak{R}' \) is identified with the root lattice \( \mathfrak{R} \) in \( t^* \) by the basic inner product, so the integer lattice of \( T_\varepsilon \) is also the image of \( R \) in \( t^* \). But, by (A.1), this agrees with the root lattice \( \mathfrak{R} \) of \( \mathfrak{g} \). Concerning \( W \), since that is the Weyl group of \( G' \), we can find \( \varepsilon \)-invariant representatives for its elements, and their \( \varepsilon \)-action coincides with the usual one.

Connectedness of twisted centralisers, for simply connected \( G \), is due to Borel [Bo]. Moreover, because maximal tori are maximal abelian subgroups, \( T^\varepsilon \) is connected as well; and (A.6) identifies the Weyl groups of centralisers as desired.

A.8 Remark. Connectedness of \( T^\varepsilon \) can also be seen directly, as follows. Clearly, the \( \varepsilon \)-fixed point set \( \exp(\mathfrak{a}') \) in the simplex \( \exp(\mathfrak{a}) \), is connected. By regularity of \( \mathfrak{t} \), every component of \( T^\varepsilon \) contains a regular element. This must be conjugate to some \( a \in \exp(\mathfrak{a}) \), hence of the form \( w(a) \), with \( w \in W \) and \( a \in \exp(\mathfrak{a}) \). Invariance under \( \varepsilon \) implies \( w(a) = \varepsilon(w(a)) = \varepsilon(w)\varepsilon(a) \). As \( a \) and \( \varepsilon(a) \) are both in \( \mathfrak{a} \) and regular, it follows that \( w = \varepsilon(w) \) and \( a = \varepsilon(a) \), so \( w(a) \) is in the \( W \)-image of \( \exp(\mathfrak{a}) \), hence in \( T \).

(A.9) Affine roots and weights. The sub-algebra \( \mathfrak{h} = i\mathbb{R}K \oplus \mathfrak{t} \oplus i\mathbb{R}E \) plays the role of a Cartan subalgebra of \( \tilde{L}'_{aff} \). The affine roots, living in \( \mathfrak{h}^* \), are the \( \mathfrak{h} \)-eigenvalues of the adjoint action on \( \tilde{L}'_{aff} \). Define the elements \( \delta \) and \( K^* \) of \( \mathfrak{h}^* \) by \( \delta(E) = 1/r \), \( K^*(K) = 1 \), \( \delta(K) = \delta(\mathfrak{t}) = K^*(\mathfrak{t}) = K^*(E) = 0 \). The simple affine roots are the simple roots of \( \mathfrak{g} \), plus \( \delta - \theta_\mathfrak{g} \); their \( \mathbb{Z} \)-span is the affine root lattice \( \mathfrak{R}_{aff} \). The positive roots are sums of simple roots. The standard nilpotent sub-algebra \( \mathfrak{m} \) is the sum of the positive root spaces, and a triangular decomposition \( \tilde{L}'_{aff} = \mathfrak{g} + \mathfrak{m} + \mathfrak{n} \) is inherited from \( \tilde{L}'_{aff} \).

The simple co-roots are those of \( \mathfrak{g} \), plus \( (K - \beta')/a_0 \), where \( \beta' \), the long dominant co-root of \( \mathfrak{g} \), satisfies \( \lambda(\beta') = (\lambda(\theta))/r \). The restriction \( \widetilde{T} \) of the basic central extension (9.8) to \( T \) is the quotient of \( \mathbb{R}K \oplus \mathfrak{t} \) by the affine co-root lattice \( \mathfrak{R}'_{aff} \).

The weight lattice \( \Lambda \) of \( \tilde{L}', G \), in \( \mathfrak{h}^* \), is the integral dual of \( \mathfrak{R}'_{aff} \), and comprises the characters of \( \widetilde{T} \).

Calling \( \Delta \) the (simply connected) weight lattice of \( \mathfrak{g} \), we have

\[
\tilde{\Lambda} = \begin{cases} 
\mathbb{Z}K^* \oplus \Delta & \text{if } \mathfrak{g} \neq \text{su}(2\ell + 1), \\
2\mathbb{Z}K^* \oplus \Delta^+ \cup (2\mathbb{Z} + 1)K^* \oplus \Delta^- & \text{if } \mathfrak{g} = \text{su}(2\ell + 1)
\end{cases}
\]
the superscript indicating the value of the character on the central element of Spin(2ℓ + 1). The affine weight lattice \(\hat{\Lambda}\) includes, in addition, the multiples of \(\delta\), giving the energy eigenvalue.

The dominant weights pair non-negatively with the simple co-roots; this means that \((k, \lambda, x)\) is dominant iff \(\lambda\) is \(g\)-dominant and \(\lambda \cdot \theta \leq k/r\). The affine Weyl group \(W_{\text{aff}}(g, \varepsilon)\) preserves the constant level hyperplanes, and its lattice part \(R_{\text{aff}}(A.5)\) acts by \(k\)-fold translation at level \(k\). Every positive-level weight has a unique dominant affine Weyl transform. Regular weights are those not fixed by any reflection in \(W_{\text{aff}}(g, \varepsilon)\). The important identity \(\langle \rho | \theta \rangle + \theta^2 / 2 = h^\vee / r\) [K VI] implies that an integral weight \((k, \lambda, x)\) is dominant iff the shifted weight \((k + h^\vee, \lambda + \rho, x)\) is dominant regular.

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