Multiple Source Replacement Path Problem

MANOJ GUPTA, IIT Gandhinagar
RAHUL JAIN∗, Goldman Sachs, Bangalore
NITIKSHA MODI, IIT Gandhinagar

One of the classical line of work in graph algorithms has been the Replacement Path Problem: given a graph \( G \), \( s \) and \( t \), find shortest paths from \( s \) to \( t \) avoiding each edge \( e \) on the shortest path from \( s \) to \( t \). These paths are called replacement paths in literature. For an undirected and unweighted graph, (Malik, Mittal, and Gupta, Operation Research Letters, 1989) and (Hershberger and Suri, FOCS 2001) designed an algorithm that solves the replacement path problem in \( O(m + n) \) time\(^1\). It is natural to ask whether we can generalize the replacement path problem: can we find all replacement paths from a source \( s \) to all vertices in \( G \)? This problem is called the Single Source Replacement Path Problem.

Recently (Chechik and Cohen, SODA 2019) designed a randomized combinatorial algorithm that solves the Single Source Replacement Path Problem in \( O(m\sqrt{n} + n^2) \) time. One of the questions left unanswered by their work is the case when there are many sources, not one. When there are \( n \) sources, the combinatorial algorithm of (Bernstein and Karger, STOC 2009) can be used to find all pair replacement path in \( O(mn + n^2) \) time. However, there is no result known for any general \( \sigma \). Thus, the problem we study is defined as follows: given a set of \( \sigma \) sources, we want to find the replacement path from these sources to all vertices in \( G \). We give a randomized combinatorial algorithm for this problem that takes \( O(m\sqrt{n\sigma} + \sigma n^2) \) time. This result generalizes both results known for this problem. Our algorithm is much different and arguably simpler than (Chechik and Cohen, SODA 2019). Like them, we show a matching conditional lower bound using the Boolean Matrix Multiplication conjecture.

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1 INTRODUCTION

One of the classical line of work in graph algorithms is the replacement path problem. The general setting for this problem is as follows: we are given a graph \( G \) and two vertices \( s \) and \( t \). We want to output the length of all shortest paths avoiding edges on the \( st \) path. Note that we just want to output the length of these paths – not the path itself. These paths are called replacement paths in the literature.

Replacement paths were first investigated due to their relation with auction theory, where they were used to compute the Vickrey pricing of edges owned by selfish agents [21, 24]. One can also generalize the replacement path problem and ask to output \( k \) – not just one – replacement paths [3, 16, 30]. This is also called the \( k \)-replacement path problem.

∗The work was done when the author was a student at IIT Gandhinagar.
\(^1\)\( O \) notation hides \( \text{polylog} \) \( n \) factor.
The main open question left behind by these two works is the case when there are |S| = σ sources. In this paper, we solve this question by showing the following theorem:

**Theorem 1.** There is a randomized combinatorial algorithm that solves the Msrp problem in $O(m\sqrt{n} + \sigma n^2)$ time.

The reader can see that there are two combinatorial results for the above problem. If $\sigma = 1$, then we have the result of Chechik and Cohen [8]. And when $\sigma = n$, we have the result of Bernstein and Karger [4]. Our result generalizes both these results. Additionally, we extend the conditional lower first presented in [8] by giving a combinatorial reduction from Boolean Matrix Multiplication (BMM) to Msrp problem.

**Theorem 2.** For a combinatorial algorithm Msrp(n, m) with runtime of $T(n, m)$, there is a combinatorial algorithm for BMM(n, m) problem with runtime of $O(\sqrt{n}T(\sigma(n), o(m)))$.

### 1.1 Related Work

Bernstein and Karger [4] solved the Msrp problem when $\sigma = n$. As mentioned previously, their aim was to build a single edge fault tolerant distance oracle of size $O(n^2)$ with a query time of $O(1)$. Demestrescu et al. [11] were the first to design this distance oracle of size $O(n^2)$ and query time $O(1)$. However, they did not specify the running time of their algorithm. Bernstein and Karger [4] answered this question. Other related work in this area are [5, 9, 10, 12].

The fault-tolerant distance oracle itself can be generalized when there is a single source or $\sigma$ many sources. Bilo et al. [6] designed a distance oracle of size $O(\sigma^{1/2}n^{3/2})$ that can answer single fault queries in $O(\sqrt{n}n^2)$ time. Gupta and Singh [20] reduced the query time of this oracle to $O(1)$.

Other related problems include the fault tolerant subgraph problem. The aim of this problem is to find a subgraph of $G$ such that the shortest path from $s \in S$ is preserved in the subgraph after any edge deletion. Parter and Peleg [27] designed an algorithm to compute single fault tolerant subgraph with $O(\sqrt{n})$ edges. They also showed that their result can be easily extended to multiple sources with $O(\sigma^{1/2}n^{3/2})$ edges. This result was later extended to dual fault by Parter [16] with $O(\sqrt{n})$ edges. Gupta and Khan [19] extended the above result to multiple sources with $O(\sigma^{1/2}n^{3/2})$ edges. All the above results are optimal due to a result by Parter [26] which states that a multiple source $k$ fault tolerant subgraph requires $O(\sigma^{1/2}n^{3/2})$ edges. There is only one positive result known for a general $\sigma$, Bodwin et al. [7] showed the existence of a $k$ fault tolerant subgraph of size $O(k^{1/2}\sigma n^{3/2} - 1^{2k})$.

### 2 Previous Approach: Chechik and Cohen [8]

Before we dive into our approach, let us look at the previous approach to the problem. To this end, let us first define few terms.

1. Let $st$ denote the shortest path from $s$ to $t$ in $G$.
2. Let $P$ be the shortest replacement path from $s$ to $t$ avoiding an edge $e$ on the $st$ path. $SUFFIX(P)$ denotes the suffix of $P$ from the point it leaves the original $st$ path.

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Algorithm 1: Algorithm for finding a replacement path $P$ from $s$ to $t$ avoiding $e$ assuming that $SUFFIX(P)$ is long

1. $d(s, t, e) \leftarrow \infty$; // $d(s, t, e)$ will be equal to $|P|$ at the end of the algorithm.
2. for each pair $u, v \in L$ do
   3.    if $su, uv$ and $vt$ avoids $e$ then
      4.        $d(s, t, e) \leftarrow \min\{d(s, t, e), |su| + |uv| + |vt|\}$
   5. return $d(s, t, e)$

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The reader can check that the time taken by Algorithm 1 for a fixed $(t, e)$ pair is $O(n)$ (assuming that checking if $su, uv$ and $vt$ contain $e$ takes $O(1)$ time). Since there are $O(n^2)$ such pairs, the algorithm takes $O(n^3)$ time. However, remember that Chechik and Cohen [8] solve the Msrp problem in $O(m\sqrt{n} + n^2)$ time. Thus, we need to reduce this running time from $O(n^3)$ to $O(n^2)$. The above
simple algorithm is the heart of their paper ("the double pivot case") and reducing the time to \(\tilde{O}(n^2)\) require some more modification to the simple algorithm – we don’t describe this modification as it is technically heavy and orthogonal to the approach used by us. We refer the interested reader to [8] for details.

3 OUR APPROACH

We now describe our approach for the case when there is only one source, that is \(s = 1\). In between our explanation, we will also point out the differences between [8] and our result.

We will use the classical result of [21–23], that can find the replacement path from \(s\) to \(t\) in \(\tilde{O}(m+n)\) time. An efficient algorithm will then be to run this algorithm for each \((s,t)\) pair where \(t \in V\), giving a running time of \(\tilde{O}(mn)\). Since we want a better running time, we restrict the use of the result of [21–23] to only landmark vertices \(L\). Thus, we find the replacement path from \(s\) to vertices in \(L\). Since there are \(\tilde{O}(\sqrt{n})\) landmark vertices, this gives a running time of \(\tilde{O}(m\sqrt{n}+n\sqrt{n})\). We now use the set \(L\) to find replacement paths for all other vertices. We first note the first difference between [8] and our result, the use of landmark vertices is completely different from that in [8]. We use the classical result to find the replacement path between \(s\) to all landmark vertices, unlike [8] where the properties of landmark vertices are used (in Algorithm 1).

We now describe our strategy to find a replacement path \(P\) from \(s\) to \(t\) avoiding \(e\) if \(\text{Suffix}(P)\) is sufficiently long. We claim the following results (which we show using Lemma 9): If \(\text{Suffix}(P)\) is of length \(\tilde{O}(\sqrt{n})\), then with a high probability a landmark vertex, say \(v\), will lie on \(\text{Suffix}(P)\) such that \(st\) path does not contain \(e\). Note that this result is similar to the one used by Chechik and Cohen [8] – here we are arguing about \(st\) path only. Thus, our algorithm to find replacement path (whose suffix is sufficiently long) is as follows:

**Algorithm 2:** Algorithm for finding a replacement path \(P\) from \(s\) to \(t\) avoiding \(e\) assuming that \(\text{Suffix}(P)\) is long

1. \(d(s,t,e) \leftarrow \infty; / \text{If } d(s,t,e) \text{ will be equal to } |P| \text{ at the end of the algorithm.}\)
2. for each \(v \in L\) do
   3. if \(st\) avoids \(e\) then
      4. Let \(p\) be the length of the replacement path from \(s\) to \(v\) avoiding \(e\).
      5. \(d(s,t,e) \leftarrow \min\{d(s,t,e), p + |vt|\}\)
   6. return \(d(s,t,e)\)

In the above algorithm, we use the fact that we have already found replacement paths from \(s\) to all the landmark vertices. Even though we know that some landmark vertex lies on \(\text{Suffix}(P)\), we do not know which one. Thus, we have to scan all of \(L\) to find the vertex which lies in \(\text{Suffix}(P)\). This process itself takes \(\tilde{O}(\sqrt{n})\) time. This implies that it would take \(\tilde{O}(n\sqrt{n})\) time to find all replacement paths of \(t\), which in turns implies an \(\tilde{O}(n^2\sqrt{n})\) running time. Since we cannot afford such a running time, we use the following scaling trick. This scaling trick is the second difference between [8] and our paper. This trick greatly simplifies the algorithm as well as the analysis in our paper.

We look at an edge \(e\) at a distance of \([2^k, 2^{k+1}]\) from \(s\) on \(st\) path (assume that \(2^k \geq \sqrt{n}\log n\)). One can argue that the suffix of the replacement path, say \(P\), avoiding \(e\) will have length \(\geq 2^k\). This is because \(e\) itself is at a distance \(\geq 2^k\) from \(t\). Since \(|\text{Suffix}(P)| > \tilde{O}(\sqrt{n})\), we know that one of our landmark vertex, say \(e\), lies on \(\text{Suffix}(P)\). If we can find \(e\) then we can use Algorithm 2 to find the length of \(P\). At the same time, we don’t have enough time to look at all the landmark vertices to find \(e\). Here comes our main idea. Since \(|\text{Suffix}(P)| \geq 2^k\), we can choose a smaller set of landmark vertices \(L_k\) of size \(\tilde{O}(\frac{k}{2})\) (a set where each vertex is sampled with probability \(\frac{1}{2}\)). We show (using Lemma 9) that there exists a landmark vertex \(v \in L_k\) such that \(v\) lies on \(\text{Suffix}(P)\) and \(st\) path avoids \(e\). Since, we have already found all replacement paths for vertices in \(L_k\) (using [21–23]), we can use \(L_k\) (instead of \(L\)) to find replacement paths to \(t\) when the edges are in the range \([2^k, 2^{k+1}]\).

Thus, we use the same algorithm in Algorithm 2, but use the landmark set \(L_k\) for edges that are at a distance of \([2^k, 2^{k+1}]\) from \(t\). As the value of \(k\) increases the number of vertices in \(L_k\) decreases. The end effect is that we take the same amount of time to process edges in the range \([2^k, 2^{k+1}], [2^{k+1}, 2^{k+2}], [2^{k+2}, 2^{k+3}], \ldots, [n/2, n]\) from \(t\). In general, the reader will see that it will take \(\tilde{O}(n)\) time to process all the edges in any given range. This would imply a running time of \(\tilde{O}(n)\) for finding all replacement paths of \(t\) whose suffix is long. This approach reduces the running time from \(\tilde{O}(m\sqrt{n}+n^2\sqrt{n})\) to \(\tilde{O}(m\sqrt{n}+n^2)\).

Note that we cannot use the above approach for edges which are near to \(t\) in \(st\) path (that is, those edges whose distance from \(t\) is \(\leq \sqrt{n}\log n\)) or those replacement paths that have short suffixes. It turns out that dealing with these replacement paths is relatively easy. Even Chechik and Cohen [8] have a simple approach for these paths. In our paper, we design another algorithm to deal with these replacement paths. This completes the description of our algorithm for the single source case.

The above approach gives us a simple algorithm for finding replacement paths from a single source to all vertices. We then extend our result to multiple sources. Ideally, we would have liked to use the result of [21–23] to find all replacement paths from all sources to vertices in \(L\). However, this does not give us the required running time. To overcome this barrier, we show that we can adapt the result of Bernstein and Karger [4] to find all replacement paths between all sources and vertices in \(L\) (in the required running time). This completes the overview of our approach.

4 NOTATION

We use the following notation throughout the paper:

- Unless stated otherwise, \(uw\) will denote the shortest path between the vertex \(u\) and \(v\) in the graph \(G\). \(|uv|\) denotes the length of this shortest path.
- Let \(e = (u, v)\) be an edge on \(st\) path such that \(u\) is closer to \(s\) than \(v\). We will abuse notation and use \(se\) to denote \(su\) path and \(et\) to denote \(vt\) path.
- \(st \circ e\) is the shortest path from \(s\) to \(t\) avoiding \(e\). \(|st \circ e|\) is the length of this shortest path.
- \(uw + vy\) denotes the concatenation of two paths, one ending at \(v\) and other starting at \(u\).
Let $P$ be a path from $s$ to $t$, not necessarily the shortest path. A sub-path $w$ on $P$ will be denoted by $P[u, v]$.

- The shortest path tree of a vertex $v \in V$ is denoted by $T_v$. The shortest path tree can be built by performing Breadth First Search (Bfs) algorithm from $v$.
- $d(s, t, e)$: In our algorithm, we want to find the shortest replacement path from each source to each vertex. $d(s, t, e)$ is the length of the replacement path (from $s$ to $t$ avoiding $e$) calculated by our algorithm. We will normally initialize $d(s, t, e) = \infty$ and prove that at the end of the algorithm $d(s, t, e) = |st - e|$. Also, $d(s, t)$ will denote the length of the shortest $st$ path. Formally, $d(s, t) = d(s, t, \emptyset) = |st|$.
- The term with a high probability means with a probability $\geq 1 - \frac{1}{n^2}$ where $c \geq 1$.
- To save space, we have omitted proofs in this extended abstract. The concerned reader may read the proof in the full version of the paper[18].

5 PRELIMINARIES

We first sample a set of random vertices which we call as landmark vertices.

**Definition 3.** (Landmark vertices) Let $L_k$ be a set of vertices sampled randomly from $G$ with a probability of $\frac{1}{2k} \log \frac{n}{k}$ where $0 \leq k \leq \log \sqrt{n}$. Let $L = \bigcup_{k=0}^{\log \sqrt{n}} L_k$. Along with these vertices, $L$ also contains all source nodes.

The following lemma bounds the number of vertices in $L$.

**Lemma 4.** The size of $L_k$ is $\tilde{O}(\sqrt{n\log n})$ with a very high probability. Thus, the size of $L$ is $\tilde{O}(\sqrt{n\log n})$ with a very high probability.

**Proof.** Let $X_k$ be a random variable denoting the size of $L_k$. The expected size of $X_k$ is, $E[X_k] = \frac{\sqrt{n\log n}}{k}$. Using Chernoff’s bound, we know that $P[X_k \geq (1 + \delta)E[X_k]] \leq e^{-\frac{\delta^2}{3}E[X_k]}$ where $\delta \geq 1$. Putting $\delta = \log n$, we get $P[X_k \geq (1 + \log n)\frac{\sqrt{n\log n}}{k}] \leq e^{-\frac{\log n^2\log n}{2k}}$. Since $2^k$ can at most be $\sqrt{n\log n}$, we get $P[X_k \geq (1 + \log n)\frac{\sqrt{n\log n}}{k}] \leq e^{-\frac{\log n^2\log n}{2k}}$. Using union bound, the probability that the size of $L_k$ is $\tilde{O}(\sqrt{n\log n})$ is $\leq (\log \sqrt{n\log n} \times n^{-4/3}) \leq n^{-1}$ (where the last inequality is true for a high enough value of $n$).

Let us first build some elementary data-structure that will help us in our algorithm. Using Breadth First Search (Bfs) algorithm, we can find the shortest path from $s$ to all other vertices in $G$ in $O(m + n)$ time. We will assume that at the end of the Bfs algorithm, we will find the distance from $s$ to every other vertex in $G$, that is $d(s, v) = |sv|$. Also, assume that we obtain the shortest path tree of $s$, that is $T_s$, as the output of Bfs algorithm. We store $d(s, v)$ in a hash-table for efficient retrieval. To this end, we use the following data structure:

**Lemma 5.** (Pagh and Rodler [25]) There exists a randomized hashtable with constant look-up time in the worst case and constant insertion time in expectation.

For each landmark vertex $r$, we find the shortest path from $r$ to every other vertex in $G$. This can again be done using Bfs algorithm and the total running time is $\tilde{O}((m + n)\sqrt{n})$ as the number of landmark vertices is $\tilde{O}(\sqrt{n})$. We store the length of the shortest path from $r$ to every other vertex $v$, that is $d(r, v)$, in a hash-table.

Using the result of [21–23], we know that all replacement paths from $s$ to a landmark vertex $r$ can be found in $\tilde{O}(m + n)$ time. If there is only one source, that is $s = 1$, then we can use this result to find all replacement paths between the single source $s$ and all landmark vertices. For a single source case, since the number of landmark vertices is $\tilde{O}(\sqrt{n})$, the time taken to find all replacement paths is $\tilde{O}((m + n)\sqrt{n})$.

However, when there are many sources, the above strategy gives a running time of $\tilde{O}((m + n)\sqrt{n})$ where the second and third multiplicand represent the number of sources and the number landmark vertices. We cannot afford such a huge running time. So, we adapt the result of Bernstein and Karger [4] and show that it can be used to find all replacement paths from all sources to all landmark vertices in $O(m\sqrt{n} + sn^2)$ time. Given this result (which we will show in Section 8), for each $s \in S$ and $r \in L$, we store $d(s, r, e)$ in a hash-table for each $e \in sr$ path.

Lastly, we used the following classical result to compute least common ancestors quickly:

**Lemma 6.** (See [2] and its references) Given any tree $T_v$ (on $n$ vertices) rooted at $v$, we can build a data-structure of size $O(v)$ in $O(n)$ time which can find least common ancestor i.e. $LCA(x, y)$ where $x, y \in T_v$ in $O(1)$ time.

Remember that we want to find all replacement paths from $s$ to $t$ for each $s \in S$ and $t \in V$. Fix a $s \in S$ and $t \in V$. We partition the edges on the $st$ path into two sets, far and near.

- (k-Far Edges) Edges which are at a distance $|2^{k+1}\sqrt{\frac{n}{n}} \log n, 2^{k+2}\sqrt{\frac{n}{n}} \log n|$ away from $t$ on $st$ path (where $0 \leq k \leq \log \sqrt{n})$.
- (Near Edges) Edges which are at a distance $< 2\sqrt{\frac{n}{n}} \log n$ away from $t$ on $st$ path.

6 FAR EDGES

Fix a source $s \in S$ and $t \in V$. Assume that we are trying to find a replacement path for a $k$-far edge $e$ on $st$ path. Since the replacement path avoids $e$, it has to diverge from the st path before $e$. We now again look at the suffix of a replacement path and describe its properties.

**Definition 7.** Let $P$ be the shortest replacement path from $s$ to $t$ avoiding an edge $e$ on the st path. Then, $SUFX(P)$ denotes the suffix of $P$ from the point it leaves the original st path.

Since $P$ avoids $e$ on st path, it has to diverge from this path before edge $e$. $SUFX(P)$ is the sub path of $P$ that starts from this diverging vertex. We now make an important observation about the length of $SUFX(P)$.

**Observation 8.** If $P$ is the shortest replacement path from $s$ to $t$ avoiding a $k$-far edge $e$ on st path, then $|SUFX(P)| > 2^{k+1}\sqrt{\frac{n}{n}} \log n$.

The above observation holds because of the following simple argument: as $SUFX(P)$ start before $e$ on st path, its length should
be \( \geq \) length of \( et \) path. We now claim that there exists a vertex of \( \mathcal{L}_k \) on \( \text{Suffix}(P) \). This can be shown easily using elementary probability.

**Lemma 9.** Let \( \mathcal{P} \) be the set of all replacement paths from \( s \in S \) to \( t \in V \) that avoid a far edge. Given any path \( P \in \mathcal{P} \) such that \( P \) avoids a \( k \)-far edge \( e \) on \( st \) path, with a high probability there exists a vertex \( r \in \mathcal{L}_k \) on \( \text{Suffix}(P) \) such that the distance of \( r \) to \( t \) on \( \text{Suffix}(P) \) is \( \leq 2^k \sqrt{\frac{n}{\sigma}} \log n \).

**Proof.** Fix a \( P \in \mathcal{P} \). Since \( P \) avoids a \( k \)-far edge on \( st \) path, by Observation 8, \( |\text{Suffix}(P)| \geq 2^{k+1} \sqrt{\frac{n}{\sigma}} \log n \). Let \( X_P \) be the event that there does not exist a vertex of \( \mathcal{L}_k \) at a distance \( \leq 2^k \sqrt{\frac{n}{\sigma}} \log n \) from \( t \) on \( \text{Suffix}(P) \). Then the probability that \( X_P \) occurs is \( P[X_P] = (1 - \frac{4}{3^k})^{2^{k+1} \sqrt{\frac{n}{\sigma}} \log n} \leq \frac{1}{3^k} \). Note that the size of \( \mathcal{P} \) is \( \leq n^3 \sigma \leq n^3 \). Thus, the probability that \( X_P \) occurs for any \( P \in \mathcal{P} \) is \( P[\cup_{P \in \mathcal{P}} X_P] \leq \frac{1}{3^k} \). \( \Box \)

We now use a simple algorithm (See Algorithm 3) to find the shortest replacement path from \( s \) to \( t \) avoiding a \( k \)-far edge \( e \).

**Algorithm 3:** Algorithm for finding a replacement path for a \( k \)-far edge \( e \)

1. \( d(s, t, e) \leftarrow \infty; \)
2. for each \( r \in \mathcal{L}_k \) do
3. \( \text{if } d(r, t) \leq 2^k \sqrt{\frac{n}{\sigma}} \log n \text{ then} \)
4. \( \text{let } \mathcal{S} = \{ \text{vertices } v \text{ s.t. } d(s, v) + d(v, r) + d(r, t) \}; \)
5. return \( d(s, t, e) \)

Let \( P \) be the replacement path from \( s \) to \( t \) avoiding a \( k \)-far edge \( e \). Using Lemma 9, we know that there exists a landmark vertex \( r \in \text{Suffix}(P) \) such that the distance of \( r \) to \( t \) on \( \text{Suffix}(P) \) is \( \leq 2^k \sqrt{\frac{n}{\sigma}} \log n \). Thus, the shortest path from \( r \) to \( t \), that is \( rt \), has length \( \leq 2^k \sqrt{\frac{n}{\sigma}} \log n \). We first claim that this path cannot pass through \( e \). This is due to the fact that \( e \) is a \( k \)-far edge and the shortest path from \( e \) to \( t \) is \( \geq 2^{k+1} \sqrt{\frac{n}{\sigma}} \log n \). Given such a \( r \in \mathcal{L}_k \), finding the replacement path becomes easy, it is \( d(s, r, e) + d(r, t) \). We have already calculated both these terms in the preprocessing phase.

However, our algorithm does not know this particular \( r \) beforehand. So, it tries all the vertices in \( \mathcal{L}_k \) and finds the required \( r \). The running time of the above algorithm for a fixed \( t \) and a \( k \)-far edge is \( \tilde{O}(\frac{\sqrt{n} \sigma}{\sqrt{\sigma}}) \). Since there can be at most \( 2^k \sqrt{\frac{n}{\sigma}} \log n \) \( k \)-far edges on \( st \) path, the total time taken to find the replacement path for all \( k \)-far edges for a fixed \( t \) is \( \tilde{O}(n) \). Since \( k \leq \log \sqrt{\sigma \sigma} \), the total time taken to find replacement path for all far edges in \( st \) path is \( \tilde{O}(n) \). Thus, we can find replacement path for each far edge in \( st \) path for each \( s \in S \) and \( t \in V \) in \( \tilde{O}(\sigma n^2) \) time.

7 NEAR EDGES

There can be two types of replacement path that avoids a near edge \( e \) on a \( st \) path where \( s \in S \) and \( t \in V \).

1. Small replacement path \( |st \cup e| \leq |se| + 2\sqrt{\frac{n}{\sigma}} \log n \)
2. Large replacement path \( |st \cup e| > |se| + 2\sqrt{\frac{n}{\sigma}} \log n \)

We say that first set have small replacement paths avoiding a near edge, while the second set of paths have large replacement paths avoiding a near edge.

7.1 Small Replacement Paths avoiding a near edge

In this section, we will find all small replacement paths from \( s \) to \( t \) that avoid a near edge. To this end, we will make an auxiliary graph \( G_s \). This graph will encode the shortest path from \( s \) to other vertices \( v \in V \) avoiding near edges on \( st \) path. After making this graph, we will run Dijkstra’s algorithm on it. At the end of this section, we will show that the output of Dijkstra’s algorithm will give us all small replacement paths.

**Construction of the auxiliary graph:** The graph \( G_s \) contains a single source node \([s]\). For each \( t \in V \), there is a node \([t]\) in \( G_s \). For each near edge \( e \in st \) path, there is a node \([t, e]\) in \( G_s \). We will now add edges in this graph. There is an edge from \([s]\) to \([v]\) with weight \(|se|\) for each \( v \in V \). There is an edge from \([v]\) to \([t, e]\) of weight 1 if \( e \) does not lie in \( sv \) path and \( v \) is a neighbor of \( t \). For each \([v, e]\), there is an edge from \([v, e]\) to \([t, e]\) with weight 1 if \( t \) is a neighbor of \( v \).

**Size of the auxiliary graph:** Let us first find the number of vertices in \( G_s \). For each \( t \in V \), there is a node \([t]\) and \([t, e]\) where \( e \) is a near edge on \( st \) path. Thus, for each \( t \in V \), we add \( \tilde{O}(\sqrt{n} \sigma) \) vertices in \( G_s \). Thus, the total number of vertices in \( G_s \) is \( \tilde{O}(n \sqrt{n}) \). Let us now calculate the number of edges in \( G_s \). There may be an edge from \([s]\) to every other node in \( G_s \). For each \( v \in V \), there may be an edge from \([v]\) to \([t, e]\) where \( t \) is a neighbor of \( v \). But there are only \( \tilde{O}(\sqrt{n} \sigma) \) vertices of type \([t, .]\). This implies that the total number of edges of \([v]\) is \( \tilde{O}(\sigma \deg(v) \sqrt{n}) \). Similarly, there are at most \( \tilde{O}(\deg(v)) \) edges out of node \([v, .]\). This implies that the total number of edges in \( G_s \) is \( \tilde{O}(n \sqrt{n} \sigma + \sum_{v \in V} \deg(v) \sqrt{n} \sigma + \sum_{v \in V} \deg(v)) = \tilde{O}(m \sqrt{n} \sigma + n \sqrt{n} \sigma) \).

**Time taken to construct the auxiliary graph:** Let us now try to find the time taken to construct the graph \( G_s \). We can find all near edges on \( st \) path in \( \tilde{O}(\sqrt{n} \sigma) \) time using \( T \). Thus, creating the nodes in the graph takes \( \tilde{O}(n \sqrt{n}) \) time. Let us now find the time taken to add an edge in the graph. For each \([v]\), we need to add an edge from \([v]\) to \([t, e]\) if \( e \notin sv \) path and \( t \) is a neighbor of \( v \). We can check if \( e \) lies in \( sv \) path by using LCA query in \( T \). Thus, adding the edge takes \( O(1) \) time. Similarly adding an edge out of \([v, .]\) also takes \( O(1) \) time. Thus, the time taken to make \( G_s \) is proportional to the number of vertices and edges in \( G_s \).

**Time to run Dijkstra’s algorithm in the auxiliary graph:** We now run Dijkstra’s algorithm in \( G_s \). Let \( w[t, e] \) be the weight of the path from \([s]\) to \([t, e]\) returned by Dijkstra’s algorithm. We then set \( d(s, t, e) \leftarrow \min\{d(s, t, e), w[t, e]\} \). The time taken to run Dijkstra’s algorithm in \( G_s \) is \( \tilde{O}(m \sqrt{n} \sigma + n \sqrt{n} \sigma) \). Thus, the total time taken to construct all \( \sigma \) auxiliary graphs and run Dijkstra’s algorithm in them is \( \tilde{O}(m \sqrt{n} \sigma + n \sigma^2) \) time.
Proof of Correctness: We are now ready to prove the correctness of our algorithm. To this end, we show the following (for the proof, see [18]):

Lemma 10. Fix a \( t \in V \) and \( s \in S \). Let \( P \) be a replacement path avoiding a near edge \( e \) on \( st \) path. If \( |P| \leq |se| + 2.57 \log n \), then our algorithm sets \( d(s,t,e) \) to \(|P|\).

7.2 Large Replacement Paths avoiding a near edge
Let \( P \) be a replacement path from \( s \) to \( t \) avoiding \( e \) such that \(|st\circ e| > |se| + 2.57 \log n\). We will first prove a simple observation:

Lemma 11. Let \( P \) be a replacement path from \( s \) to \( t \) avoiding a near edge \( e \) such that \(|P| > |se| + 2.57 \log n\). Then \(|\text{SUFFIX}(P)| > 2\sqrt{n} \log n\).

Proof. We claim that the suffix of \( P \) will start from a vertex before \( e \) on \( st \) path. Let this vertex be \( z \). Then \(|sz| \leq |se|\). Also, \(|P| = |sz| + |\text{SUFFIX}(P)|\). But \(|P| > |se| + 2.57 \log n\). This implies that \(|sz| + |\text{SUFFIX}(P)| > |se| + 2.57 \log n\). Since \(|sz| \leq |se|\), it follows that \(|\text{SUFFIX}(P)| > 2\sqrt{n} \log n\).

Since \(|\text{SUFFIX}(P)| > 2\sqrt{n} \log n\), with a high probability, there exists a landmark vertex \( r \in L_0 \) such that the distance of \( r \) to \( t \) on \( \text{SUFFIX}(P) \) is \( \leq \sqrt{n} \log n \). The proof for this is similar to Lemma 9. We state this lemma without proof.

Lemma 12. Let \( P \) be the set of all large replacement paths from \( s \in S \) to \( t \in V \) that avoid a near edge. Given any path \( P \in P \) such that \( P \) avoids a near edge \( e \) on \( st \) path, with a high probability, there exists a vertex \( r \in L_0 \) such that the distance of \( r \) to \( t \) on \( \text{SUFFIX}(P) \) is \( \leq \sqrt{n} \log n \).

Now, we will find the \( r \) stated in the above lemma. Once we find this \( r \), we can calculate \( d(s,t,e) \) as follows: \( d(s,t,e) = d(s,r,e) + d(r,t,e) \). We would have liked to write \( d(r,t) \) instead of \( d(r,t,e) \) as we have not calculated \( d(r,t,e) \) beforehand. In the following lemma, we will show that \( e \notin rt \), implying that \( d(r,t,e) = d(r,t) \).

Lemma 13. Let \( P \) be the shortest replacement path from \( s \) to \( t \) avoiding a near edge \( e \) on \( st \) path such that \(|P| > |se| + 2.57 \log n\). Then there exists a landmark vertex \( r \in L_0 \) such that \( st \circ e = sr \circ e + rt \) and \( e \notin rt \).

Proof. By Lemma 11, \(|\text{SUFFIX}(P)| > 2\sqrt{n} \log n\). Using Lemma 12, there exists a landmark vertex \( r \in L_0 \) such that the distance of \( r \) to \( t \) on \( \text{SUFFIX}(P) \) is \( \leq \sqrt{n} \log n \). Thus, \( st \circ e = sr \circ e + rt \).

Also we claim that \(|rt \circ e| \leq \sqrt{n} \log n\), since the distance from \( r \) to \( t \) on \( \text{SUFFIX}(P) \) is \( \leq \sqrt{n} \log n \). We will now show that \( rt \circ e = rt \), that is \( e \) does not lie on \( rt \) path. Assume for contradiction that \( e \) lies on the \( rt \) path. Let \( e = (u,v) \). Since \( e \) lies on both \( st \) and \( rt \) path,
results in this section can be seen as the generalization of the result by Bernstein and Karger [4].

We sample another set of vertices which we call as centers (to differentiate them from landmark vertices). Let \( C_0 \) be the set of centers sampled with the probability \( \frac{1}{\sqrt{\frac{n}{\sigma}}} \) where \( 0 \leq k \leq \log \sqrt{\frac{n}{\sigma}} \).

Thus (similar to Lemma 4), with a high probability, \( |C_k| = O\left(\frac{\sqrt{n\sigma}}{\sigma}\right) \).

A center is said to have priority \( k \) if it lies in \( C_k \). Additionally, we add all vertices of \( S \) in \( C_0 \). Like landmark vertices (similar to Lemma 4), the total number of centers is \( O(\sqrt{n\sigma}) \). We run BFs algorithm from each center \( c \) and find the shortest path tree \( T_c \). This takes \( O(m\sqrt{n\sigma}) \) time.

Fix a source \( s \in S \) and a vertex \( r \in L \). We can go over the path from a source \( s \) to \( r \) to find a center with the highest priority. We then move from \( s \) to this highest priority center, finding a list of centers with priority in ascending order. Let \( c_1 \) be the first center in the \( sr \) path. Then \( c_2 \) be the next center with a higher priority than \( c_1 \). This continues till we reach the highest center on the path \( sr \). Then, we find the list of centers in descending order of priority. There are at most \( O(\log n) \) centers thus found. Since we are just walking on the path \( sr \) in this procedure, the time taken is the size of the path which is \( O(n) \). Since there are \( O(\sigma\sqrt{n\sigma}) \) pairs of possible \( s \) and \( r \), the total time taken to find the list of centers is \( O(n\sigma\sqrt{n\sigma}) = O(\sigma n^2) \) time. These centers naturally form an interval in the \( sr \) path, which we define next:

**Definition 15.** (Interval on a \( sr \) path) Let \( sr \) be a path such that \( s \in S \) and \( r \in L \). Assume that we find the centers \( c_1, c_2, \ldots, c_t \) on this path, then we say that the path can be divided into intervals \( sc_1, c_1c_2, \ldots, c_tc_r \).

Note that we have to find the replacement path from a source \( s \) to a landmark vertex \( r \) avoiding an edge \( e \). To this end, we first find the pair of centers \( c_1 \) and \( c_2 \) between which \( e \) lies in \( sr \) path. The replacement path can be of the following three types:

- It passes through \( c_1 \).
- It passes through \( c_2 \).
- It avoids the interval \( c_1c_2 \).

The above observation is named path cover lemma in [4].

**Lemma 16.** (Path Cover Lemma) Given a source \( s \) and a landmark vertex \( r \), for any edge \( e \) on the \( sr \) path, let \( c_1c_2 \) be the centers between which \( e \) lies in \( sr \) path. Then

\[
\begin{align*}
    sr \circ e &= \min \left\{ \begin{array}{ll}
        sc_1 + c_1r \circ e, \\
        sc_2 + e + c_2r, \\
        sr \circ [c_1c_2]
    \end{array} \right. \\
\end{align*}
\]

where the last distance represents the shortest path from \( s \) to \( r \) avoiding the interval \( c_1c_2 \).

The non-trivial part of the first and the second term in \( sr \circ e \) is \( c_1r \circ e \) and \( sc_2 \circ e \). In the first term, we want to find a replacement path from a center to a landmark vertex and in the second term we want to find a replacement path from a source to a center.

As in [4], we club together first two terms of the path cover lemma.

**Definition 17.** (Mtc, Minimum through centers) Given any source \( s \) and a vertex \( r \in L \). For any edge \( e \) on the \( sr \) path, let \( c_1c_2 \) be the centers between which \( e \) lies in \( sr \) path. Then

\[
Mtc(s, r, e) = \min \left\{ \begin{array}{ll}
    sc_1 + c_1r + e, \\
    sc_2 + e + c_2r
    \end{array} \right.
\]

Thus, Mtc, minimum through centers, defines first two terms in the path cover lemma. We first calculate the Mtc term. To calculate the Mtc term, we have to find a replacement path from a center to a landmark vertex and find a replacement path from a source to a center.

Let us do the second part first.

### 8.1 Finding the replacement path from a source to a center

The second term in the Mtc mandates us to find the replacement path from each source \( s \) to each center \( c \). However, we need not find this replacement path for each edge on \( sc \) path. We need find the replacement path only for those edges that lie in the interval ending at \( c \) on the \( sc \) path. To formalize this, let us show the following lemma (for the proof, see [18]):

**Lemma 18.** Let \( xy \) be an interval in \( sr \) path. Assume that the priority of \( x \) is \( k \) and the priority of \( y \) is greater than priority of \( x \). Then \( |xy| = \tilde{O}(\sqrt{\frac{n}{\sigma}}) \).

If \( c \) has priority \( k \), then by the above Lemma 18, we just need to find the replacement path for the first \( \tilde{O}(\sqrt{n}) \) edges on \( cs \) path. This is because we are sure that any interval ending at \( c \) will contain \( \tilde{O}(\sqrt{n}) \) edges if \( c \) has priority \( k \). In the ensuing discussion, we will assume that we are finding the replacement path from \( c \) to \( s \) for the first \( \tilde{O}(\sqrt{n}) \) edges on \( cs \) path.

If \( P \) is a small replacement path from \( s \) to \( c \) avoiding a near edge, then we have already found it in Section 7.1. Thus, our aim will be to find following replacement paths.

- (1) \( e \) is a far edge on \( sc \) path.
- (2) \( e \) is a near edge but \( |P| > |e| + 2\sqrt{\frac{n}{\sigma}} \log n \).

We now show an important result which binds these replacement paths ((for the proof, see [18]).

**Lemma 19.** Let \( P \) be a replacement path from \( s \) to \( c \in C \) avoiding \( e \).

- (a) If \( e \) is an i-far edge on \( sc \) path, then there exists a vertex \( c' \in C_i \) on \( Suffix(P) \) such that \( |c'c| \leq 2^i \sqrt{\frac{n}{\sigma}} \log n \) and \( e \notin c'c \).
- (b) If \( e \) is a near edge and \( |P| > |e| + 2\sqrt{\frac{n}{\sigma}} \log n \), then there exists a vertex \( c' \in C_0 \) on \( Suffix(P) \) such that \( |c'c| \leq \sqrt{\frac{n}{\sigma}} \log n \) and \( e \notin c'c \).

We will make an auxiliary graph \( G_3 \) that will find all the required replacement paths from \( s \) to each center. This graph will encode replacement path from \( s \) to every center. After making this graph, we will run Dijkstra’s algorithm on it. At the end of this section, we will show that the output of Dijkstra’s algorithm will give us all required replacement paths. For the construction and correctness of this auxiliary graph, we refer the reader to the [18].
Till now, we have found all replacement path that will be used in the second term of \( M_{ tc } \). We will now try to find replacement paths that will be used in the first term of \( M_{ tc } \).

8.2 Finding the replacement path from a center to a landmark vertex

Now, we calculate the first term of \( M_{ tc } \). Let \( P \) be a replacement path from a source \( s \) to a landmark vertex \( r \) that is passing through the center \( c \). To calculate the first term in \( M_{ tc } \), we should calculate \(| s r | + | r e |\) if \( e \) lies in the interval starting with \( c \) on \( s r \) path. This implies that the center \( c \) lies in \( s r \) path and \( e \) lies in the interval starting with \( c \) on \( c r \) path.

If \( e \) has priority \( k \), then by Lemma 18, on any \( s r \) path, we are sure to find a center of priority \( k + 1 \) at a distance of \( \tilde{O}(2^k \sqrt{\frac{n}{\sigma}}) \) from \( c \). Thus, we just need to find the replacement path for edges till a distance of \( \tilde{O}(2^k \sqrt{\frac{n}{\sigma}}) \) from \( c \).

Before moving ahead, let us first make an important observation. We need to find a replacement path from \( c \) to \( r \) avoiding \( e \) only if there exists a replacement path from some source \( s \) to \( r \) avoiding \( e \) that passes through \( c \). Otherwise, there is no need to even find a replacement path from \( c \) to \( r \) avoiding \( e \).

Remember that a replacement path from a source to \( r \) avoiding \( e \) and passing through \( c \) can be of three types:

1. Small replacement path that avoids a near edge.
2. Large replacement path that avoids a near edge.
3. Replacement path that avoids a far edge.

Let us look at the first set of replacement paths as we have already found these paths in Section 7.1.

8.2.1 Small replacement paths avoiding a near edge. For each \( r \in L \), we have already found the replacement path from each source to \( r \). We can enumerate all the edges on each of these replacement paths too. Remember that the algorithm in Section 7.1 only finds the length of the small replacement path, not the replacement path itself. However, we can use Dijkstra’s algorithm to find the actual path too. The time taken to enumerate a path is equal to the length of the path. Since, there are \( \sigma \) sources, \( \tilde{O}(\sqrt{\frac{n}{\sigma}}) \) vertices in \( L \), and only \( \tilde{O}(\sqrt{\frac{n}{\sigma}}) \) near edges, we need to enumerate \( \tilde{O}(\sigma \sqrt{\frac{n}{\sigma}} \sqrt{\frac{n}{\sigma}}) = \tilde{O}(\sigma n) \) paths. Since there can be \( n \) edges on each path, the total time to enumerate all the paths is \( \tilde{O}(\sigma n^2) \).

We can pre-process each enumerated replacement path to find whether a vertex lies on it – this can be done using LCA queries. For a landmark vertex \( r \) and a center \( c \), we can check if there exists a small replacement path from a source to \( r \) passing through \( c \) and avoiding a near edge \( e \). To this end, we will first check all enumeration that represents a replacement path from a source to \( r \) avoiding \( e \). There are \( \sigma \) such enumerations. If \( e \) lies in any of the enumerations, then there is a replacement path avoiding a near edge of small length passing through \( c \). Thus, we can find a small replacement path from \( c \) to \( r \) avoiding a near edge \( e \) in \( \tilde{O}(\sigma) \) time. Once again we reiterate, that we will find this replacement path only if there is a replacement path from a source to \( r \) avoiding \( e \) that passes through \( c \). If there is no such path, then there is no need to find a small replacement path from \( c \) to \( r \) avoiding a near edge \( e \).

Given a \((c, r, e)\), the time taken to find a small replacement path is \( \tilde{O}(\sigma) \). Since there are \( \tilde{O}(\sqrt{\frac{n}{\sigma}}) \) centers, \( \tilde{O}(\sqrt{\frac{n}{\sigma}}) \) landmark vertices and \( \tilde{O}(\sqrt{\frac{n}{\sigma}}) \) near edges, the total time taken to find all small replacement paths for all possible \((c, r, e)\) tuples is \( \tilde{O}(\sqrt{\frac{n^2}{\sigma^2}}) = \tilde{O}(\sigma n^2) \). We store the length of all small replacement paths in \( \mathcal{A}(c, r, e) \) (where \( c \in C \) and \( r \in L \) and \( e \) is a near edge on \( c r \) path) in a hash-table for efficient retrieval.

8.2.2 Other replacement paths. Once we have dealt with small replacement paths, two other types of replacement paths are left. Remember that this replacement path \( P \) is from a source \( s \) to a landmark vertex \( r \) that is passing through the center \( c \). Also, \( e \) lies in \( c r \) path. And our aim is to find the length of the sub-path \( P[c, r] \).

The replacement path \( P \) can be of two types:

1. \( P \) avoids a far edge \( e \) on \( s r \) path and passes through \( c \).
2. \( e \) is a near edge but \( P \) passes through \( c \).

Since \( e \) is a near edge on \( s r \) path, it is also a near edge in \( c r \) path. Thus, even the subpath \( P[c, r] \) satisfies, \(| P[c, r]| > |c e| + 2 \sqrt{\frac{n}{\sigma}} \log n \).

Thus, we need to find a replacement path \( P[c, r] \) such that:

1. \( P[c, r] \) avoids a far edge \( e \) on \( c r \) path.
2. \( e \) is a near edge but \( |P[c, r]| > |c e| + 2 \sqrt{\frac{n}{\sigma}} \log n \).

Lemma 20. Let \( c \) lie on \( s r \) path such that \( e \in c r \). Let \( P \) be a replacement path from \( s \) to \( r \) avoiding \( e \) and passing through \( c \). Let \( P[c, r] \) be the corresponding replacement path from \( c \) to \( r \) avoiding \( e \) such that \( (a) \) \( e \) is a far edge on \( c r \) path or \( (b) \) \( e \) is a near edge and \(|P[c, r]| > |c e| + 2 \sqrt{\frac{n}{\sigma}} \log n \). Then (1) \(|\text{Suffix}(P[c, r])| > 2 \sqrt{\frac{n}{\sigma}} \log n \) and (2) \( \text{there exists a vertex } r' \in L \text{ in } \text{Suffix}(P[c, r]) \text{ such that } r \notin r' \).

Proof. See [18] for proof.

With all armoury at hand, we are now ready to find the required replacement paths from a center to all landmark vertices. Fix a center \( c \) with priority \( k \). We will now find the replacement path from \( c \) to each landmark vertex. Also remember that we want to find replacement path avoiding all edges at a distance \( \tilde{O}(\sqrt{2^k \frac{n}{\sigma}}) \) from \( c \). We will now create an auxiliary graph that will help us in finding all the required replacement paths. We start the construction and correctness of this auxiliary graph, we refer the reader to [18].

Given the result in this section (Section 8.2) and the result in Section 8.1, we can now calculate the first two terms in the path cover lemma (See Lemma 16). In the ensuing discussion, we will be calculating a replacement path from a source to a landmark vertex that avoids an entire interval.

8.3 Replacement path avoiding an interval

In this section, we will find the replacement path from \( s \) to \( r \) avoiding the interval that contains edge \( e \). To this end, we use the concept of bottleneck vertex (adapted as bottleneck edge for our purpose)
We will now crucially use this property to make another auxiliary graph. We first observe that the bottleneck edge will have the highest value among all edges in the interval and then find the replacement path avoiding the bottleneck edge. This takes time. The hardest part is point (3) in above enumeration. Thus, two things are left now: find the bottleneck edge for each interval and then find the replacement path avoiding the bottleneck edge.

8.3.1 Finding Bottleneck edge in each interval in sr path. We now show how to find a bottleneck edge in the i-th interval of the path sr. We first observe that the bottleneck edge will have the highest Mtc value among all edges in the i-th interval. This is true as by Lemma 22, the last term for each edge in the interval is the same. So, to find a bottleneck edge, we should look at the edge in the interval which maximizes the first two terms, that is the Mtc value.

To find the bottleneck edge of the i-th interval, we just need to go over each edge in the i-th interval and check the Mtc value (whose constituents we have already calculated). This takes \( O(n) \) time for all intervals on the sr path. Since there are \( \sigma \) sources and \( O(\sqrt{n}\sigma) \) vertices in \( L \), the total time taken to find bottleneck edges is \( O(n\sqrt{n}\sigma) = O(\sigma n^2) \).

8.3.2 Finding the replacement path avoiding the bottleneck edge in sr path. Let us now find the replacement path \( P \) avoiding the bottleneck edge of the i-th interval on the sr path. Let \( e \leftarrow B[s, r, i] \), that is \( e \) is the bottleneck edge of the i-th interval in sr path. If \( P \) is a small replacement path avoiding a near edge on sr path, then we would have already found it in Section 7.1. Thus, our focus will be to find \( P \) when:

1. \( P \) avoids a far bottleneck edge \( e \).
2. \( P \) avoids a near bottleneck edge \( e \) but \( P > |se| + \sqrt{\frac{\sigma}{\delta}} \log n \).

By Lemma 20, there exists a \( r' \) in \( \text{Suffix}(P) \) such that \( e \notin r'r \). We will now crucially use this property to make another auxiliary graph \( G_s \).

Construction of the auxiliary graph: This graph \( G_s \) contains a source vertex \( s \). There is a vertex \( [r] \) for each \( r \in L \). For a bottleneck edge of interval \( i \) sr path, there is a vertex \( [s, r, i] \) where \( 1 \leq \log n \). We now find the edges in \( G_s \). There is an edge from \( [s] \) to \( [r] \) with weight \( |sr| \). If \( B[s, r, i] \) happen to be a near edge whose replacement path has small weight, then we add an edge from \( [s] \) to \( [s, r, i] \) with appropriate weight (see Section 7.1). Else, there can be three types of edges to \( [s, r, i] \).

(1) Edge from \( [s] \) to \( [s, r, i] \) of weight \( \text{Mtc}(s, r, B)[s, r, i] \). This edge represents the first two terms in the path cover lemma for the bottleneck edge \( B[s, r, i] \) (we have already calculated these in Section 8.1 and 8.2).

(2) For each \( r' \in L \), there is an edge from \( [s] \) to \( [s, r, i] \) with weight \( \text{Mtc}(s, r', B)[s, r, i] + |r'r| \) if \( B[s, r, i] \) does not lie in \( r'r \).

(3) If \( B[s, r, i] \) lies in the \( j^{th} \) interval on the sr' path, then there is an edge from \( [s, r', j] \) to \( [s, r, i] \) with weight \( |r'r| \) if \( B[s, r, i] \) does not lie in \( r'r \) path.

This completes the construction of \( G_s \).

Size of the auxiliary graph: The number of nodes of type \([r]\) in \( G_s \) is \( \tilde{O}(\sqrt{n}\sigma) \). The number of nodes in \( G_s \) of type \([s, r, i]\) is \( \tilde{O}(\sqrt{n}\sigma) \) since there are \( O(\sqrt{n}\sigma) \) landmark vertices and \( n \) interval in any sr path. We now find the number of edges in \( G_s \). For each vertex \([s, r, i]\), there are at most \( \tilde{O}(\sqrt{n}\sigma) \) from the source \([s]\) (due to point (2) in above enumeration). Also, there are at most \( \tilde{O}(\sqrt{n}\sigma) \) edges from other vertices in \( L \) (due to point (3) in above enumeration). Thus, the total number of edges in \( G_s \) is \( \tilde{O}(\sqrt{n}\sigma) = O(n^2) \).

Time taken to construct the auxiliary graph: For each source, we have already found the bottleneck edge of each interval in sr path in Section 8.3.1 (where \( r \) is a landmark vertex). Thus, adding vertices in \( G_s \) takes \( O(1) \) time. If there is a small replacement path from \([s]\) to \([s, r, i]\), then we have already found it in Section 7.1 and can be added in \( O(1) \) time. We add an edge from \([s]\) to \([s, r, i]\) with \( \text{Mtc}(s, r, B)[s, r, i] \). Again, we have calculated the Mtc term in Section 8.1 and 8.2. So, we can add this edge in \( O(1) \) time. For each \( r' \in L \), we add an edge from \([s]\) to \([s, r, i]\) if \( e \notin r'r \). Again, this edge can be added in \( O(1) \) time. The hardest part is point (3) in the above enumeration. For each interval \([s, r', j]\), we first need to check if \( B[s, r, i] \) lies in \( j^{th} \) interval in sr' path. This can be done by first finding if \( B[s, r, i] \) lies in sr' path – by doing LCA queries in \( T_r \). If \( B[s, r, i] \) lies in sr' path, then we can calculate the distance of \( B[s, r, i] \) relative to \( s \) and \( r' \). This can be done easily as we have already stored distances from \( s \) to all other vertices in the graph in \( d(s, \cdot) \) (in the pre-processing phase). Thus, all edges in \( G_s \) can be added in \( O(1) \) time. Thus, the time taken to construct \( G_s \) is equal to the worst case size of \( G_s \), that is \( O(n^2) \).

Time taken to run Dijkstra’s algorithm in the auxiliary graph: We run Dijkstra’s algorithm in \( G_s \) to find the shortest replacement path for each bottleneck edge. We set \( d(s, r, B)[s, r, i] \) to the weight of the shortest path from \([s]\) to \([s, r, i]\) as returned by Dijkstra’s algorithm in \( G_s \). The time taken by Dijkstra’s algorithm in \( G_s \) is \( O(n^2) \). Since there are \( \sigma \) such graphs, the total time taken is \( O(\sigma n^2) \).

Proof of Correctness: We now prove the correctness of the above algorithm (for the proof, see [18]).

Lemma 23. Let \( P \) be the shortest path from \( s \in S \) to \( r \in L \) avoiding the bottleneck edge in the i-th interval of sr path. Then Dijkstra’s algorithm in \( G_s \) correctly finds \( P \).

Thus, we claim the main theorem of the paper:
Theorem 24. There is a randomized combinatorial algorithm that solves the MSRP problem in $O(m\sqrt{n}\sigma + \sigma^2)$.

For lack of space, we present our lower bound in Theorem 2 in [18].

References

[1] Yehuda Afek, Anat Bremler-Barr, Haim Kaplan, Edith Cohen, and Michael Merritt. Restoration by path concatenation: fast recovery of MPLS paths. Distributed Computing, 13(4):273–283, 2002.

[2] Michael A. Bender and Martin Farach-Colton. The LCA problem revisited. In LATIN 2000: Theoretical Informatics, 4th Latin American Symposium, Punta del Este, Uruguay, April 10–14, 2000, Proceedings, pages 88–94, 2000.

[3] Aaron Bernstein. A nearly optimal oracle for approximating replacement paths and k shortest simple paths in general graphs. In Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms, pages 742–755. Society for Industrial and Applied Mathematics, 2010.

[4] Aaron Bernstein and David Karger. A nearly optimal oracle for avoiding failed vertices and edges. In Proceedings of the forty-first annual ACM symposium on Theory of computing, pages 101–110. ACM, 2009.

[5] Aaron Bernstein and David R. Karger. Improved distance sensitivity oracles via random sampling. In Shang-Hua Teng, editor, Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2008, San Francisco, California, USA, January 20-22, 2008, pages 34–43. SIAM, 2008.

[6] Davide Bilò, Keerti Choudhary, Luciano Gualà, Stefano Leucci, Merav Parter, and Guido Proietti. Efficient oracles and routing schemes for replacement paths. In 35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France, pages 13:1–13:15, 2018.

[7] Greg Bodwin, Fabrizio Grandoni, Merav Parter, and Virginia Vassilevska Williams. Preserving distances in very faulty graphs. In 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland, pages 73:1–73:14, 2017.

[8] Shiri Chechik and Sarel Cohen. Near optimal algorithms for the single source replacement paths problem. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, pages 2090–2109, 2019.

[9] Shiri Chechik, Sarel Cohen, Amos Fiat, and Haim Kaplan. $(1+\epsilon)$-approximate single-sink f-sensitivity distance oracles. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 1479–1496, 2017.

[10] Shiri Chechik, Michael Langberg, David Peleg, and Liam Roditty. Fault tolerant spanning trees for general graphs. SIAM J. Comput., 39(7):1429–1452, 2010.

[11] Camil Demetrescu, Mikkel Thorup, Rezaul Alam Chowdhury, and Vijaya Ramachandran. Oracles for distances avoiding a failed node or link. SIAM J. Comput., 37(5):1299–1318, 2008.

[12] Ran Duan and Seth Pettie. Dual-failure distance and connectivity oracles. In Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2009, New York, NY, USA, January 4-6, 2009, pages 506–515, 2009.

[13] Yuval Emek, David Peleg, and Liam Roditty. A near-linear-time algorithm for computing replacement paths in planar directed graphs. ACM Transactions on Algorithms (TALG), 6(4):64, 2010.

[14] David Eppstein. Finding the k shortest paths. In FOCS, pages 154–165. IEEE Computer Society, 1994.

[15] François Le Gall. Powers of tensors and fast matrix multiplication. In International Symposium on Symbolic and Algebraic Computation, ISSAC ’14, Kobe, Japan, July 23-25, 2014, pages 296–303, 2014.

[16] Zvi Galil and Moshe Lewenstein. Improved algorithms for the k simple shortest path problem and the replacement paths problem. Information Processing Letters, 109(7):352–355, 2009.

[17] Fabrizio Grandoni and Virginia Vassilevska Williams. Improved distance sensitivity oracles via fast single-source replacement paths. In Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on, pages 748–757. IEEE, 2012.

[18] Manoj Gupta, Rahul Jain, and Nitiksha Modi. Multiple source replacement path problem. arXiv preprint arXiv:2005.09262, 2020.

[19] Manoj Gupta and Aditi Singh. Generic single edge fault tolerant exact distance oracle. In 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic, pages 72:1–72:15, 2018.

[20] John Hershberger and Subhash Suri. Vickrey prices and shortest paths: What is an edge worth? In 42nd Annual Symposium on Foundations of Computer Science, FOCS 2001, 14-17 October 2001, Las Vegas, Nevada, USA, pages 252–259, 2001.

[21] Kavindra Malik, Ashok K Mittal, and Santosh K Gupta. The k most vital arcs in the shortest path problem. Operations Research Letters, 8(4):223–227, 1989.