Nondeterministic testing of Sequential Quantum Logic propositions on a quantum computer

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In the past few years it has been shown that universal quantum computation can be obtained by projective measurements alone, with no need for unitary gates. This suggests that the underlying logic of quantum computing may be an algebra of sequences of quantum measurements rather than an algebra of products of unitary operators. Such a Sequential Quantum Logic (SQL) was developed in the late 70's and has more recently been applied to the consistent histories framework of quantum mechanics as a possible route to the theory of quantum gravity. In this letter, I give a method for deciding the truth of a proposition in SQL with nonzero probability of success on a quantum computer.

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Formal logic plays a central role in the theory of classical computation and leads to many powerful results in computability [1], computational complexity [2] and the theory of programming languages [3]. On the other hand, quantum computing is not normally presented in formal terms, and one might wonder whether developing a logic of quantum computing would lead to similar insights. As a simple example of the sort of result that might be raised to the quantum domain, consider any computation in the classical circuit model that produces a one-bit output, i.e. a circuit for solving a decision problem. It is clear that such a computation simply evaluates the truth-value of a Boolean formula, given the truth-values of its inputs. This simple observation is directly related to the Cook-Levin theorem, which states that SATISFIABILITY - the problem of determining whether a Boolean formula has any satisfying assignments - is NP complete, and indeed that similar logical problems are complete for all complexity classes in the polynomial hierarchy. In turn, this means that some of the most important problems in computer science - whether P = NP and whether the polynomial hierarchy collapses - can be determined by the equivalence or non-equivalence in complexity of these logical decision problems.

The Extended Church-Turing thesis, that any physically reasonable model of computing can be simulated by a probabilistic Turing machine with only polynomial overhead, implies that such results are robust against the particular choice of physical realization used to build a computer. Hence, classical computer scientists do not have to learn the full details of Newtonian physics to study their subject. Instead, they can abstract away from the physics by realizing that "information is logical".

The discovery of quantum computing has compelled us to believe that in fact "information is physical". Physically realizable models of computing may be more powerful than those based on the abstraction of Boolean logic. This point is forcefully demonstrated by the existence of quantum algorithms that run exponentially faster than any known classical algorithm for solving the same problem [4]. It seems that the ability to evaluate polynomial length Boolean logic formulas does not capture the essence of what is efficiently computable in our universe, so perhaps information is not merely logical after all.

However, given the unifying power of logic in classical computer science, we should be reluctant to give up a logical notion of information processing altogether. Perhaps the shift from classical to quantum computing is simply a shift in the logic that underlies our models of computing. This has been suggested by Deutsch et. al. [5], who speculated that reversible classical logic should be expanded to include other unitary operations that occur in the circuit model of quantum computing, such as √NOT. These ideas have been formalized by dalla Chiara et. al. [6], but their relevance to the general theory of quantum computing, and particularly to quantum complexity theory, remains unclear at present [7].

The idea that the shift from classical to quantum requires a shift in logic is not new. It was proposed in a different context by Birkhoff and von Neumann in 1936 [8]. They were concerned with the logic of properties of quantum systems, so theirs is a logic of possible alternative measurements rather than of unitary operators. On the other hand, quantum computing requires a logic of processes, i.e. a description of how quantum data can be manipulated by sequences of operations. Classically, Boolean algebra can be viewed as both a logic of property and of process, the latter via the classical circuit model, but a priori the analogous quantum logics might be much less closely related. Nevertheless, the power of quantum computers may reside in an ability to efficiently decide propositions in some sort of quantum logic, more general than the Boolean logic underlying classical computing [18].

The advent of universal models of quantum computation based on measurements [8] suggests that the most natural logic of quantum computation might be an algebra of sequences of measurements rather than a logic of unitary operators. Such a Sequential Quantum Logic (SQL) was developed by Mittlestaedt and Stachow [4, 10].
and has more recently been extended by Isham et. al. [11] as a route to a quantum theory of gravity within the consistent histories approach to quantum mechanics. SQL is therefore a natural starting point for developing a logic of quantum computation based on measurements.

In this letter, we give a method for evaluating SQL propositions on a quantum computer with non-zero probability of success. It is inspired by recent applications of quantum information ideas to the Density Matrix Renormalization Group (DMRG) that is used extensively for numerical simulations of many-body systems [12]. We begin with a review of SQL, followed by a simple example of the algorithm to illustrate the general procedure and conclude with open problems suggested by this work.

The syntax of SQL is given by \( \langle L, S, \sqcap, \neg, (,) \rangle \), where \( L \) is a set of elementary propositions and \( S \) is the set of sequential propositions. Propositions in \( L \) are denoted \( a, b, c, \ldots \) to distinguish them from generic propositions in \( S \), denoted \( s, t, u, \ldots \). \( S \) is defined recursively by:

- If \( a \in L \) then \( a \in S \).
- If \( s, t \in S \) then \( (s \sqcap t) \in S \).
- If \( s \in S \) then \( \neg s \in S \).

The aim of SQL is to model sequences of two-outcome measurements, or tests, made on a quantum system. The operators \( \neg, \sqcap \) have the interpretation of negation and sequential conjunction, or “not” and “and then” respectively. An elementary test is associated with a pair \( \{a, \neg a\} \), representing the two possible outcomes. If the system is then subjected to another test represented by \( \{b, \neg b\} \) then the four outcomes are represented via the \( \sqcap \) operator as \( \{a \sqcap b, a \sqcap \neg b, \neg a \sqcap b, \neg a \sqcap \neg b\} \). Sequential conjunction differs from a regular conjunction in that \( a \sqcap b \) does not necessarily mean that \( a \) and \( b \) are ever both true at the same time. \( \neg \) is not assumed to be commutative, \( s \sqcap t \neq t \sqcap s \), but is associative, \( (s \sqcap t) \sqcap u = s \sqcap (t \sqcap u) = s \sqcap t \sqcup u \), which facilitates the removal of brackets from long expressions. Full details of the syntax of SQL can be found in [10], but the above is sufficient for our present purpose.

SQL is modeled by an algebra of operators acting on a Hilbert space \( \mathcal{H} \). Given a proposition \( s \), denote by \( [s] \) the operator on \( \mathcal{H} \) assigned to \( s \). Throughout, we assume that states are updated according to the Läders-von Neumann projection postulate after a measurement and states are left unnormalized. For the elementary propositions, recall that in quantum mechanics, elementary measurements are represented by sets of projection operators \( \{P_j\} \), where \( \sum_j P_j = I \) and \( I \) is the identity operator. If the state of the system prior to the measurement is \( |\Psi\rangle \) then, upon obtaining outcome \( P_n \), the post-measurement state is \( P_n |\Psi\rangle \). Therefore, it is natural to represent an elementary proposition \( a \) by a projection operator \( [a] = P \). The other outcome of the test, \( \neg a \), is represented by \( [-a] = I - [a] = I - P \). Given a state \( |\Psi\rangle \in \mathcal{H} \), the elementary proposition \( a \) can be tested by performing the measurement corresponding to the projectors \( [a] \) and \( [-a] \), obtaining either the state \( [a] |\Psi\rangle \) or the state \( [-a] |\Psi\rangle \) with probabilities given by the norm squared of the states.

Having determined the Hilbert space representation of elementary propositions, the operator \( [a \sqcap b] \) can be defined. Consider making a sequence of two measurements on \( |\Psi\rangle \) corresponding to sets of projectors \( \{P_j\} \) and \( \{Q_k\} \) respectively. Upon obtaining the outcome \( P_n \) of the first measurement followed by the outcome \( Q_m \) of the second, the state is updated to \( Q_m P_n |\Psi\rangle \). Thus, it is natural to set \( [a \sqcap b] = [b] [a] \). To complete the algebra, the definitions are extended to all sequential propositions, i.e. \( [\neg s] = I - [s] \) and \( [s \sqcap t] = [t] [s] \). For later convenience, the notation \( [s]^\dagger = [-s] \), \( [s]^* = [s] \) is also used.

Sequences of two-outcome measurements are described correctly by SQL, provided the measurement results of the entire sequence are retained. For example if two elementary propositions, \( a \) and \( b \), are tested sequentially on a state \( |\Psi\rangle \), then the four states \( [a \sqcap b] |\Psi\rangle \), \( [a \sqcap \neg b] |\Psi\rangle \), \( [-a \sqcap b] |\Psi\rangle \) and \( [-a \sqcap \neg b] |\Psi\rangle \) are obtained after the measurement with probabilities given by the norm squared of the states. However, coarse grainings of such sequences are not related to physical tests in such a simple manner. For example, the pair \( [a \sqcap b, \neg(a \sqcap b)] \) is generally not a possible two-outcome test in quantum mechanics. To see this, note that according to the Hilbert space model of SQL, the second outcome is associated with the operator \( I - [b] [a] \), which is the sum of the three operators \( [a \sqcap b], [a \sqcap \neg b] \) and \( [-a \sqcap \neg b] \) that appear in the fine grained version of the test described above. This means that the negative outcome of the coarse-grained test should result in a coherent superposition of the states \( [a \sqcap \neg b] |\Psi\rangle \), \( [-a \sqcap b] |\Psi\rangle \) and \( [-a \sqcap \neg b] |\Psi\rangle \) rather than the incoherent mixture that would be obtained by performing the fine grained test and discarding information about some of the outcomes. In fact, the operators \( \{[b] [a], (I - [b] [a])\} \) are not a pair of generalized measurement operators, since \( (b) [a]^\dagger (b) [a] + (I - [b] [a])^\dagger (I - [b] [a]) \neq I \) unless \([a] \) and \([b] \) commute, so there is no direct quantum mechanical implementation of the test \( [a \sqcap b, \neg(a \sqcap b)] \).

Another way of seeing this is to note that although \( [s\psi] = [s] [\psi] + [-s] [\psi] \) for any \( s \) and \( [\psi] \), the states \( [s] [\psi] \) and \( [-s] [\psi] \) are generally not orthogonal so they cannot be distinguished with certainty. However, it is possible to perform the test nondeterministically by introducing a third “failure” outcome, as in the unambiguous discrimination of nonorthogonal states [13]. Conditional on not obtaining this outcome, the state \( [s] [\psi] \) is obtained with probability \( N [\psi] [s] [\psi] \), and the state \( [-s] [\psi] \) is obtained with probability \( N [\psi] [-s] [\psi] \), where \( N \) is a positive constant. This is what is meant by the nondeterministic testing of a proposition \( s \) and we now describe an algorithm that achieves this by means of a simple ex-
Thus, for each $x$, stood that elementary proposition, i.e. it should always be understood that $x = a, b, c$. Suppose that each elementary proposition $x$, is assigned to a projector $|x⟩ = |ψ_x⟩⟨ψ_x|$ onto a state $|ψ_x⟩$ on $C^2$; the Hilbert space of a single qubit. Let $U_x$ be the unitary operator that acts as $U_x |1⟩ = |ψ_x⟩$, $U_x |0⟩ = |ψ_{¬x}⟩$ where $|ψ_{¬x}⟩$ is any representative state in the subspace orthogonal to $|ψ_x⟩$. Thus, for each $x$, we have $|x⟩ = |ψ_x⟩$, $U_x |0⟩ = |ψ_{¬x}⟩$ and $|ψ_{¬x}⟩ = U_x |0⟩$. Throughout the algorithm, ancillary qubits are used to coherently store the results of testing sub-propositions of $s$, i.e. $\{a, b, c\}$ at the beginning, followed by $\{a \cap b, c\}$, followed by $\{(a \cap b), c\}$, and finally $s$. It is convenient to use these sub-propositions as labels for the ancillas and to label the qubit that the output state ends up in by $f$. The procedure to test $s$ on a state $|\Psi⟩$ consists of two stages. The first stage is to prepare the state

$$
\sum_{j,k,m=0}^{1} |j⟩_a |k⟩_b |m⟩_c [e^{im}][b^k][a^j]|\Psi⟩_f , \tag{1}
$$

which we call a history state, since, in the computational basis, the qubits $a, b$ and $c$ encode which of the $2^3$ possible sequences of measurement outcomes have occurred. In other words, the $a, b$ and $c$ qubits contain full information about the outcomes of a sequence of tests: $\{a, ¬a\}$ followed by $\{b, ¬b\}$ followed by $\{c, ¬c\}$. The information we require about $s$ is encoded in the correlations between these qubits, and is obtained in the second stage by two rounds of DMRG-like operations, which compute history states for the sub-propositions of $s$, reducing the dimensionality of the Hilbert space until there is only a single qubit left which stores the result of the test $\{s, ¬s\}$.

The preparation stage begins with the state

$$
|\Psi⟩_a |\Phi^+⟩_{a,b} |\Phi^+⟩_{b,c} |\Phi^+⟩_{c,f} , \tag{2}
$$

where $|\Phi^+⟩ = |00⟩ + |11⟩$ and $a', b', c'$ are additional ancillas. By performing measurements followed by a unitary correction, this can be transformed into $|\Psi⟩_f$.

The first step is to perform a local unitary operation $U_unitary \otimes T$ on qubits $a$ and $a'$, where $T$ is the transpose in the computational basis. Then a parity measurement is performed on $a$ and $a'$, with outcomes corresponding to the projectors $P_1 = |00⟩⟨00|_{aa'} + |11⟩⟨11|_{aa'}$ and $P_2 = |01⟩⟨01|_{aa'} + |10⟩⟨10|_{aa'}$. If outcome $P_2$ occurs, then a unitary correction $U_{correct} X U_{unitary} †$ is applied to the qubit $b$, where $X = |0⟩⟨1| + |1⟩⟨0|$. It is the bit-flip operator. In both cases a CNOT operation is performed with $a$ as the control and $a'$ as the target. This disentangles $a'$ from the other qubits and it can then be discarded. The same procedure is then applied to the $b', b$ and $c, c'$ qubits, where, in the case of $c, c'$, the correction is applied to the qubit $f$ if $P_2$ occurs. This results in the state $|\Psi⟩_f$ and $\rightarrow$.

In the second stage, the first step is to compute $a \cap b$. This could be done if it were possible to apply a coherent AND gate, $A_{a,b} = |0⟩_{a,b}⟨00⟩ + |01⟩ + (10)_{ab} + |1⟩_{a,b}$ to $|\Psi⟩_f$, resulting in the state

$$
\sum_{j,k=0}^{1} |j⟩_a |k⟩_b |c⟩|a \cap b⟩ |\Psi⟩_f . \tag{3}
$$

This is a history state encoding the results of the sub-formulae $a \cap b$ and $c$. Unfortunately, it is not possible to implement $A_{a,b}$ directly, since the largest eigenvalue of $A_{a,b} A_{a,b}$ is greater than 1. Instead, a measurement can be performed that has two generalized measurement operators given by

$$
M_{a,b}^{(s)} = \frac{1}{\sqrt{3}} |00⟩_{ab} (|00⟩ + |01⟩ + |10⟩_{ab}) + |11⟩_{ab} (|11⟩_{ab} ) , \tag{3}
$$

$$
M_{a,b}^{(f)} = (I - M_{a,b}^{(s)})^{1/2} ,
$$

wherein obtaining the $M^{(s)}$ outcome, discarding the qubit $b$ and relabeling $a$ as $a \cap b$ gives the desired result. Since we know that SQL propositions cannot be tested deterministically, and the coherent AND is the only probabilistic stage of this algorithm, it cannot be possible to perform a correction if the $M^{(f)}$ outcome is obtained in general. Therefore, on obtaining $M^{(f)}$, the whole procedure has to be repeated from the beginning until a successful outcome is obtained.

The next step is to perform a bit-flip operation $|0⟩_{a∩b} (|1⟩_{a∩b} + |1⟩_{a∩b}) |0⟩_{a∩b}$ in order to negate the outcome of the test of $a \cap b$. This results in the state

$$
\sum_{j,k=0}^{1} |j⟩_a |k⟩_b |c⟩|¬(a \cap b)⟩ |\Psi⟩_f .
$$

Finally, we need to compute the sequential AND of $¬(a \cap b)$ and $c$. This is done by applying the operator $A_{¬(a \cap b),c}$, using the probabilistic procedure described above. This results in the state $\sum_{j,k=0}^{1} |j⟩_a |s⟩ |\Psi⟩_f$, where $s = ¬(a \cap b) \cap c$. Measuring the qubit $s$ in the computational basis performs the test $\{s, ¬s\}$ on the state $|\Psi⟩_f$, which is the desired result.

The generalization to more complicated propositions is straightforward, and this gives the full algorithm for deciding SQL propositions. This is analogous to solving a Boolean decision problem via a classical circuit with a single output. To obtain the full classical circuit model, one has to consider circuits with multiple outputs, which correspond to testing multiple Boolean propositions on the same inputs simultaneously. Likewise, to build a full model of computing based on SQL, one would need to consider how multiple propositions might be tested in the same computation. One way of doing this would be to use a quantum fan-out gate, $F(α|0⟩ + β|1⟩) = α|00⟩ + β|11⟩$, whenever the same data is needed by more than one proposition. This is similar to what is done in the classical circuit model, where classical fan-out gates are used to share information between the different propositions being tested. However, the disadvantage of this is that quantum fan-out does not have an interpretation within the standard formalism of SQL, so SQL would have to be extended to take this into account.
In this letter it was shown that an SQL proposition can be tested on a quantum computer with nonzero probability of success. More than this cannot be expected, since a pair \( \{ s, \neg s \} \) does not generally correspond to a physically implementable test in the Hilbert space model. This means that SQL cannot really be regarded as “the logic of quantum computing” in the same sense as Boolean logic is “the logic of classical computing”. For that, one needs to find a logic for which every quantum computation can be viewed as testing propositions in that logic and where the elementary operations of the computation correspond to the elementary connectives of the logic.

One might hope to achieve this by modifying the definition of sequential conjunction in SQL so that all pairs \( \{ s, \neg s \} \) do correspond to physically implementable tests. It is unclear how this can be achieved, but there is a natural definition of sequential exclusive OR (\( \oplus_{\text{seq}} \)) that always generates physical tests when combined with the \( \neg \) operator, namely \( [s \oplus_{\text{seq}} t] = [\neg t][s] + [t][\neg s] \). Unfortunately, exclusive OR and NOT do not constitute a universal set of gates in the classical circuit model, so they are probably not sufficient in the quantum case either, but one might hope to define a sequential conjunction satisfying a formula such as \( s \oplus_{\text{seq}} t = (\neg s \sqcap \neg t) \sqcup (s \sqcap t) \) in analogy with the defining formula of exclusive OR in Boolean logic.

Given that SQL is not the logic of quantum computing, it is interesting to ask what model of computing it is. For that, one needs to find a logic for which every quantum computation can be viewed as testing propositions in that logic and where the elementary operations of the computation correspond to the elementary connectives of the logic.

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[17] Very different approaches, directed towards the formal semantics side of computer science, can be found in [11].
[18] Interestingly, the enhanced power is not captured by the ability to decide propositions in Birkhoff-von Neumann quantum logic. I have constructed a model of computing capable of doing this in close analogy with the classical circuit model, but it can be simulated efficiently on a classical computer [11].
[19] The preparation of the history state is the only part of the algorithm that changes if the restriction to rank 1 projectors onto qubit states is removed. If, for each pair \( [x], [\bar{x}] \), either they are of equal rank or one of them is the identity and the other is the null projector, then there is a similar teleportation-like procedure to prepare the history state deterministically starting with \( |\Phi^+\rangle \), a maximally entangled state of appropriate dimension. If this is not the case then, since the history state just coherently records the outcomes of three successive measurements, it can be prepared deterministically without \( \mathcal{E} \) by starting from the state \( |\Psi\rangle \), introducing 3 ancilla qubits in the state \( |0\rangle \) and performing the unitary operations \( \sum_{k=0}^3 |j\rangle_d \otimes [x^{j\otimes 3}]. \)
[20] In particular, the analog of BQP in this model, PostBQP, is equal to the classical complexity class PP.