Chapter 2
In Search of the Book of Nature

Abstract The Book of Nature is an ancient metaphor for knowledge, where the universe can be read like a book for understanding. It is written in the language of mathematics, giving birth to science. A modern interpretation is that human knowledge generation rests on an act of translation. Aspects of the physical world are translated into abstract, formal notions. These can be manipulated by the mind to gain insights into the workings of the world. This raises many philosophical questions, as it posits the existence and interaction of three worlds: the physical (space, time, and energy), the mental (consciousness), and the abstract (Platonism).

Level of mathematical formality: medium.

One of the main wellsprings of modern thought can be traced back to an obscure and secretive religious cult flourishing around 500 B.C.E. The Pythagoreans were a very unlikely origin of ideas that would influence the progress of human understanding of the world. They can be seen to have initiated a transformation in knowledge seeking, away from myth and superstition towards abstract truths which can be uncovered and grasped by the human mind.

Pythagoras, of whom very little is known, founded a religion of which the main principles were the transmigration of souls and the sinfulness of eating beans (see Russell 2004, for a list of other bizarre rules of the Pythagorean order). Nonetheless, a crucial element in their thinking was the realization that a mathematical reality underpins the physical. This is reflected in their motto “All is number” (Boyer 1968, p. 49). Thought to have coined the term “mathematics” (Heath 1981, p. 11), the Pythagoreans begin with the study of the subject for its own sake. Indeed, Aristotle would later credit the Pythagoreans for being the first to take up and advance mathematics, next to understanding the principles of mathematics “as being the principles of all things” (Kirk and Raven 1957, p. 236f.). Furthermore, they were associated with the analysis of the four sciences, which will later be known as the quadrivium: arithmetic, geometry, music, and astronomy. Although Pythagoras himself is often seen as a founder of mathematics and physics still today, it is unclear if such accomplishments should actually be credited only to him personally (Huffman 2011).
Moreover, the Pythagoreans also had a great influence on philosophy, as their ideas molded Plato’s thinking, and through him, reached out to all of Western philosophy: “[...] what appears as Platonism is, when analyzed, found to be in essence Pythagoreanism” (Russell 2004, p. 45). The conception of an abstract, eternal world, revealed to the intellect but not to the senses, finds a new expression in Plato’s notion of a perfect realm of ideas and forms. His vision of these abstract entities yields both an ontology and an epistemology: Platonic Ideas are not constructs of the human mind and the belief in their objective nature implies the existence of a domain of reality harboring them, a third realm next to the physical world perceived by the senses, and the inner thought world of consciousness. True knowledge is only attainable because of the mind’s ability to access this otherworldly sphere, and thus any empirical evidence must always be prone to fallibility.

The emergence of this worldview, where the regularities in the physical world are explicable through the structures in the abstract world, finds its metaphorical incarnation as the Book of Nature. The mind of God, the master-mathematician, is revealed to humans in this way. This conviction, that the coherence of the universe is explained by equations and can hence be comprehended by the human mind, echoes over the ages: “Mathematics is the door and key to the sciences” (Roger Bacon in 1267); “This book [the universe] is written in the mathematical language [...]” (Galileo Galilei in 1623); “Mathematical and mechanical principles are the alphabet in which God wrote the world” (Robert Boyle in 1744); “In every specific natural science there can be found only so much science proper as there is mathematics present in it” (Immanuel Kant in 1900); “Mathematics is the foundation of all exact knowledge of natural phenomena” (David Hilbert in 1900). (All quotes are taken from Hanson 2010, p. 193, except the first and last ones, which are found in Wolfram 2002, p. 859.)

Nonetheless, the rigorous and systematic description of physical processes aided by the use of analytical tools—the true mathematization of nature—can be seen to have started to emerge roughly four centuries ago. By introducing the idea of elliptical orbits into celestial motion, Johannes Kepler was able to solve the ancient mystery of planetary behavior. He thus demonstrated “mathematics’ genuine physical relevance to the heavens—its capacity to disclose the actual nature of the physical motions. Mathematics was now established not just as an instrument for astronomical prediction, but as an intrinsic element of astronomical reality” (Tarnas 1991, p. 257). Synthesizing Kepler’s laws of planetary motion, Galileo Galilei’s laws of dynamics, and René Descartes’s laws of motion and mechanistic philosophy, Isaac Newton was able to construct a single comprehensive mathematical framework, describing the general motion of matter under the action of forces. It is seamlessly able to describe terrestrial and celestial phenomena, explaining everything known about motion with a handful of mathematical equations. This body of work, which is seen by some as the beginning of modern physics (Russell 2004; Tarnas 1991), laid the foundations for what has come to be known in physics today as classical mechanics. Since this turning point in history, the understanding of the world has forever been transformed. Science is now seen as the effort to capture the processes of nature in formal mathematical representations.
But what is it exactly that bestows mathematics with such power? Why is it the blueprint for reality? And how topical are the musings about a Platonic world of mathematical forms?

2.1 A Modern Edition of the Book of Nature

The cornerstone of the scientific knowledge-generating process called science can be understood as an act of translation: quantifiable aspects of reality are transformed into formal, abstract representations which are hosted in the mind. Thus parts of reality become intelligible and the formal encodings foster novel insights. This enterprise can be understood as the quest of the natural sciences. In Fig. 2.1 a rough sketch of this idea is presented. Guided by observation, measurement, and reflection, a natural system of a given reality domain is encoded into a formal representation. Aided by the rules pertaining to the chosen abstract model, for instance, logical consistency or symmetrical regularities, novel insights about the behavior or characteristics of the natural system can be found, allowing predictions to be made. This newly decoded information can then be compared with experimental outcomes, lending validity to the formal representation as a model of the natural system. These ideas are reflected in the words of Paul A. M. Dirac (quoted in Goenner 2004, p. 6):

The successful development of science requires a proper balance between the method of building up from observations and the method of deducing by pure reasoning from speculative assumptions […].

Still today, this interplay of the physical with the abstract is emphasized by scientists, for instance, as observed in Davies (2014, p. 83):

The history of physics is one of successive abstractions from daily experience and common sense, into a counterintuitive realm of mathematical forms and relationships, with a link to the stark sense data of human observation that is long and often tortuous.

Nonetheless, for many scientists this cycle of translation is implicit and the focus is placed heavily on the details of the abstract realm. Albert Einstein remarked (as quoted in Schieber 2008, p. 97):

I am convinced that we can discover by means of pure mathematical constructions the concepts and the laws connecting them with each other, which furnish the key to the understanding of natural phenomena. Experience may suggest the appropriate mathematical concepts, but they most certainly cannot be deduced from it. Experience remains, of course, the sole criterion of the physical utility of a mathematical construction. But the creative principle resides in mathematics. In a certain sense, therefore I hold it true that pure thought can grasp reality, as the ancients dreamed.

To illustrate how effective this act of translation has become, in the following, some examples of physical theories are presented. The mathematical formalism is kept brief for the moment. Only later on will examples of full-blown analytical machineries be unveiled.
2.1.1 Classical Mechanics

A simple example of this recipe, described in Fig. 2.1, can be found when applied to Newtonian mechanics. The reality domain is restricted to be comprised of a system of \( n \) (unit) point masses in three-dimensional Euclidean space \( \mathbb{R}^3 \), described by their locations and velocities. These observables can exist in physically distinct states and are represented by two sets of \( N = 3n \) numbers. Conceptually, the encoding of the observables is accomplished by mapping the abstract states into the points of the space \( \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N \), also referred to as state- or phase-space. In detail, each particle’s location is formally captured by a differentiable mapping, called a motion, \( x_i: I \to \mathbb{R}^3, i = 1, \ldots, n \), where \( I \subset \mathbb{R} \) is a time interval. Thus any configuration of the positions of a mechanical system of \( n \) points is captured by the motion \( X: \mathbb{R} \to \mathbb{R}^N \), where \( X \) is the vector constructed from all \( x_i \). Taking the derivative of \( X \) with respect to time yields the velocity vector \( \dot{X} \). The derivative’s abstract capacity to measure how a function changes as a result of changes in its input, encodes the physical notion of displacement with respect to time. Newton’s equation is a function \( F: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \) such that

\[
\ddot{X} = F(X, \dot{X}, t),
\]

and it is the basis for his mechanics. Once the initial conditions are specified, i.e., \( X(t_0) \) and \( \dot{X}(t_0) \), a theorem relating to ordinary differential equations guarantees the existence and uniqueness of the solution of (2.1), see, for instance Blanchard et al. (2011). Decoding this equation reveals that the initial positions and velocities alone determine the acceleration forces emerging in the system. This is the predictive power of the formal representation, captured by a system of ordinary differential equations: the specification of the evolution of the physical system in time. The abstract rules relating to the mathematics of the infinitesimal are a concrete example of what the
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arrow on the far right-hand side in Fig. 2.1 is alluding to. What is today known as calculus, was first formally spelled out by Newton and, independently, Gottfried Wilhelm Leibniz.\(^1\) A general reference introducing classical mechanics is Arnold (1989).

### 2.1.2 Classical Electrodynamics

The formal rules relating to derivatives also have the capacity to breathe life into another fundamental set of equations. Taking the reality domain to encompass interactions between electric charges and currents, an extension of the classical Newtonian model results in a unified theory describing all electromagnetic phenomena with great precision. On the side of the abstract formulation, the notion of derivatives is extended to apply to vectors fields, which is the subject of vector calculus. Both the electric and magnetic fields, \(E\) and \(B\), respectively, find a formal encoding as functions \(f: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3\). The main mathematical actor is a vector differential operator, referred to as \(\nabla\).

\[
\nabla := \sum_{i=1}^{3} \hat{e}_i \frac{\partial}{\partial x_i}.
\] (2.2)

The \(\hat{e}_i\) are a standard basis in \(\mathbb{R}^3\) and the partial derivative, \(\partial / \partial x_i\), denotes differentiation with respect to the variable \(x_i\). The symbol “:=” identifies the expression on the left-hand side as a novel term defined by the quantities on the right-hand side.

What are today known as Maxwell’s equations, a particular set of partial differential equations, is perhaps one of the most important aggregations of empirical facts in the history of physics. James Clerk Maxwell built on the experimental observations and insights gained, among others, by André-Marie Ampère, Jean-Baptiste Biot, Charles-Augustin de Coulomb, Michael Faraday, Carl Friedrich Gauss, Hermann von Helmholtz, Hans Christian Ørsted, Siméon Denis Poisson, and Félix Savart, next to contributing his own (Panat 2003). An early form of Maxwell’s equations was published between 1861 and 1862, but only two decades later Oliver Heaviside provided the mathematical tools to elegantly group the four equations together into the distinct set still used today.\(^2\) The modern form of the four equations are based on the following expressions, building variations on the theme of the derivative

\[
\dot{E}, \dot{B}, \nabla \cdot E, \nabla \cdot B, \nabla \times E, \text{ and } \nabla \times B.
\] (2.3)

\(^{1}\)Historically, the question of who discovered calculus first caused a major intellectual controversy at the time.

\(^{2}\)Heaviside was not the only scientist grappling with these problems. Heinrich Hertz was doing similar work, and the reformulated Maxwell equations became known for some years as the “Hertz-Heaviside equations.” The young Einstein referred to them as “Maxwell–Hertz equations,” and, today, the legacy of Heaviside and Hertz has been lost to history (Nahin 2002, p. 111ff.). In addition, Heaviside and Josiah Willard Gibbs both developed vector calculus independently of each other during the same period.
The resulting equations are an explicit example of an encoding scheme illustrated in Fig. 2.1. All observable electromagnetic phenomena find their formal representation in four simple equations\(^3\)

\[
\begin{align*}
\nabla \cdot E &= \rho \quad \text{(2.4a)} \\
\nabla \cdot B &= 0 \quad \text{(2.4b)} \\
\nabla \times E &= -\dot{B} \quad \text{(2.4c)} \\
\nabla \times B &= J + \dot{E} \quad \text{(2.4d)}
\end{align*}
\]

Decoding these abstract expressions enables the physical manipulation of all electromagnetic manifestations and fosters technological innovation. Indeed, simply by the power of mathematical consistency, a new feature of reality is uncovered: decoding Maxwell’s equations also reveals that the electric and magnetic forces are in fact expressions of a single overarching electromagnetic force. Moreover, a further application of the operator \(\nabla \times\), called \textit{curl}, on (2.4c) and (2.4d), yields novel wave equations describing the propagation of electromagnetic waves traveling at the speed of light in a vacuum. Maxwell, understanding the connection between electromagnetic waves and light, thus unified the theories of electromagnetism and optics. For a general reference see Jackson (1998).

2.1.3 \textit{Mathematical Physics}

Today, it is hardly possible for a layperson to distinguish modern physics from pure mathematics. The merger of mathematics and physics has reached an unprecedented level. Even if by looking back at history this development seems natural, indeed inevitable, there were times when people disagreed. Johann Wolfgang von Goethe saw the necessity of keeping physics and mathematics independent. Physics should strive to understand the divine forces of nature, unaffected by the characteristics of mathematics. In reverse, mathematics should not be restricted or tainted by the outer world, as it is an immaculate tune of the spirit (Schottenloher 1995). An influential proponent of the idea that physics is not in need of mathematics to be a successful endeavor was Faraday. He developed a field theory describing electrical and magnetic forces without the aid of mathematics (Schottenloher 1995). Notwithstanding, the mathematization of physics marched on, culminating in today’s level of integration. This can be witnessed, for example, in mathematical textbooks aimed at introducing physicists to the mathematical methods. For instance, seen in the 1200 pages of Arfken et al. (2012). Or in a book written by the mathematical physicists Roger Penrose. It is an ambitious and comprehensive account of the physics describing the universe—a thousand page tour de force—focusing essentially on the underlying mathematical theories, giving a good impression of how far this enterprise has come,\(^3\)

\(^3\)The charge and current densities are captured by \(\rho\) and \(J\), respectively. Constants relating to the choice of electromagnetic units are ignored and set to one.
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and how complete the Book of Nature has become (Penrose 2004). The following is a selection from the table of contents, highlighting the mathematical context:

| Mathematical Context |
|----------------------|
| The Pythagorean theorem; Euclid’s postulates; Hyperbolic geometry: conformal picture; Solving equations with complex numbers; Convergence of power series; Caspar Wessel’s complex plane; How to construct the Mandelbrot set; Geometry of complex algebra; The idea of the complex logarithm; Multiple valuedness, natural logarithms; Complex powers; Higher derivatives, C1-smooth functions; The rules of differentiation; Integration; Complex smoothness, holomorphic functions; Contour integration; Power series from complex smoothness; Analytic continuation; Conformal mappings; The Riemann sphere; The genus of a compact Riemann surface; The Riemann mapping theorem; Fourier series; Functions on a circle; Frequency splitting on the Riemann sphere; The Fourier transform; Frequency splitting from the Fourier transform; Hyperfunctions; Complex dimensions and real dimensions; Smoothness, partial derivatives; Vector Fields and 1-forms; The Cauchy-Riemann equations; The algebra of quaternions; Geometry of quaternions; Clifford algebras; Grassmann algebras; Manifolds and coordinate patches; Scalars, vectors, and covectors; Grassmann products; Integrals of forms; Exterior derivative; Tensors; Complex manifolds; Groups of transformations; Subgroups and simple groups; Linear transformations and matrices; Determinants and traces; Eigenvalues and eigenvectors; Representation theory and Lie algebras; Tensor representation spaces, reducibility; Orthogonal groups; Unitary groups; Symplectic groups; Parallel transport; Covariant derivative; Curvature and torsion; Geodesics, parallelograms, and curvature; Lie derivative; Symplectic manifolds; Some physical motivations for fibre bundles; The mathematical idea of a bundle; Cross-sections of bundles; The Clifford bundle; Complex vector bundles, (co)tangent bundles; Projective spaces; Non-triviality in a bundle connection; Bundle curvature; Finite fields; Different sizes of infinity; Cantor’s diagonal slash; Puzzles in the foundations of mathematics; Turing machines and Gödel’s theorem; Euclidean and Minkowskian 4-space; The symmetry groups of Minkowski space; Hyperbolic geometry in Minkowski space; Non-commuting variables; Unitary structure, Hilbert space, Dirac notation; Spin and spinors; Higher spin: Majorana picture; Infinite-dimensional algebras; The Weyl curvature hypothesis; Killing vectors, energy flow—and time travel!; The algebra and geometry of supersymmetry; Higher-dimensional space-time; The magical Calabi-Yau spaces, M-theory; The chiral input to Ashtekar’s variables; Loop variables; The mathematics of knots and links; Spin networks; Theories where geometry has discrete elements; Conformal group, compactified Minkowski space; Twistors as higher-dimensional spinors; Twistor sheaf cohomology.

4Indeed, Penrose has contributed to many of these topics, as can be see, for instance, in the book summarizing his work (Huggett et al. 1998).
Penrose also acknowledges an intimate relationship of the two realms seen in Fig. 2.1 (Penrose 2004, p. 1014):

The interplay between mathematical ideas and physical behavior has been a constant theme in this book [“The Road to Reality”]. Throughout the history of physical science, progress has been made through finding the correct balance between, on the one hand, the strictures, temptations, and revelations of mathematical theory and, on the other, precise observation of the actions of the physical world, usually through carefully controlled experiments.

It is an interesting observation, that the most fruitful branch of mathematics appears to be geometry (Schottenloher 1995; Huggett et al. 1998; Barndorff-Nielsen and Jensen 1999; Frankel 1999; Gray 1999; Nakahara 2003; Atiyah et al. 2010). Indeed, the mathematician Marcel Grossmann introduced Einstein into the field of differential geometry, which would turn out to be the mathematical foundation of general relativity (Goenner 2005). Moreover, “[o]ne of the remarkable developments of the last decade is the penetration of topological concepts into theoretical physics” (Tom W.B. Kibble quoted in Nash and Sen 1983, back cover).

### 2.1.4 Mathematics from Physics

In the last decades, the pursuit of new physical theories has also spawned and nurtured new results in mathematics, namely topology, a field of study developed from geometry analyzing concepts of space and transformation. This is a remarkable cross-fertilization. Not only does the formal encoding of aspects of the natural world necessitate structures in the abstract realm which lead to the discovery of novel (decoded) features back in the physical world, now, crucially, these encoded remnants act as a guiding principle by which new structures in the abstract world are uncovered.

In 1994, the physicists Nathan Seiberg and Edward Witten introduced an equation within the context of quantum field theory⁵ (in detail, supersymmetric⁶ Yang-Mills theory,⁷ Seiberg and Witten 1994a, b), that had a great impact on the mathematical field of topology, namely the research of four-dimensional manifolds. This prompted the mathematician and Fields Medalist Simon Kirwan Donaldson to remark (Donaldson 1996):

In the last three months of 1994 a remarkable thing happened: this research [in 4-manifold topology] was turned on its head by the introduction of a new kind of differential-geometric equation by Seiberg and Witten: the space of a few weeks long-standing problems were solved, new and unexpected results were found, along with simpler new proofs of existing ones, and new vistas for research opened up.

A few years earlier, Witten’s work on topological quantum field theory provided new insights for the mathematical field of knot theory (Witten 1989). He showed

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⁵See Sects. 3.1.4, 3.2.2.1, 4.2, and 10.1.1.
⁶Discussed in Sect. 4.3.2.
⁷The topic of Sect. 4.2.
how the invariant of an oriented knot, the Jones polynomial, can be obtained by considering geometric insights of Chern–Simons theory.\textsuperscript{8} Indeed, Witten, who had graduated with a degree in history, was the first physicist to be awarded the Fields Medal in 1994, a prestigious award for outstanding discoveries in mathematics. The mathematician Michael Atiyah commented on Witten’s dual impact on physics and mathematics (Atiyah 1991, p. 31):

> Although he is definitely a physicist (as his list of publications clearly shows) his command of mathematics is rivaled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by his brilliant application of physical insight leading to new and deep mathematical theorems.

More on the history and details of the entwinement of physics and topology can be found in Nash (1999).

Another example of theoretical physics pollinating mathematics is monstrous moonshine. Next to being a rather peculiar name for a mathematical theory, it ties up some very exotic mathematical concepts with the help of string theory.\textsuperscript{9} The mathematical structure called monster group\textsuperscript{10} was independently postulated in 1973 by the two mathematicians Bernd Fischer and Robert L. Griess. It is a structure with

\[ M_s = 808017424794512875 \\
886459904961710757 \\
005754368000000000 \]

\[ (2.5) \]

elements in it. However, only in 1982 its existence was proved (Griess 1982). Barehanded, without the aid of a computer, Griess constructed the monster group by associating it with a 196,883-dimensional vector space. It was known, that if the group should exist, it would only become manifest at certain specific numbers of dimensions \( M_{d_i} \) (see, for instance Conway and Norton 1979): \( M_{d_1} = 196,883 \), \( M_{d_2} = 21,296,876 \), \( M_{d_3} = 8,426,093,265 \), \ldots A true oddity at the fringes of mathematics.

Unrelated, in another corner of mathematics, number theorists were analyzing a mysterious object, also inspired by string theory, called the modular function. Variants of this function would later turn out to be the unexpected key to solving Fermat’s Last

\textsuperscript{8}A quantum field theory theory built around the concept of the Chern–Simons form, an integrable geometric object on a manifold (Chern and Simons 1974). James Simons left academia in 1978 to set up Renaissance Technologies, a multi-billion quantitative hedge fund management company, heavily staffed with employees with non-financial backgrounds but with detailed scientific knowledge. They recruited researchers from the fields of cryptoanalysis and computerized speech recognition. For decades, Renaissance Technologies operated one of the most successful, albeit highly secretive, funds in the business, and the wealth that Simons has amassed also finances his many philanthropic pursuits. See Patterson (2010).

\textsuperscript{9}String theory will be introduced in Sect. 4.3.2.

\textsuperscript{10}Groups are discussed in detail at the beginning of Chap. 3 and in Sect. 3.1.2.
Theorem,\textsuperscript{11} a famous conjecture which had remained unproven since 1637 (Wiles 1995). Back in the late 1970s, a strange coincidence was noticed. The modular function introduced by Felix Klein, called the $j$-function, was known to be expressible as a Fourier series (Rankin 1977). John H. Conway and Simon Norton discovered an unexpected connection between the monster group and the $j$-function: the Fourier coefficients could be expressed as linear combinations of the dimension numbers $M_{di}$. This conjectured relationship was called monstrous moonshine (Conway and Norton 1979), at a time the existence of the monster group was still unproven.

As if this intimate connection between two very separate fields in mathematics was not puzzling enough, the techniques used to prove this kinship would come from an even more unexpected source: modern theoretical physics. Richard Borcherds proved the conjecture (Borcherds 1992), an achievement that would later also win him a Fields Medal. Using a theorem from the mathematical framework of string theory, he catapulted exotic topics from mathematics into the limelight, intriguing mathematicians and physicists alike. Indeed, before his discovery he would lament about the “new and esoteric algebraic structure” he introduced in 1986, called a vertex operator algebra (Du Sautoy 2008, p. 347):

I was pretty pleased with it at the time but after a few years I got a bit disillusioned, because it was obvious that nobody else was really interested in it. There is no point in having an idea that is so complicated that nobody can understand it. I remember I used to give talks on vertex algebras, and usually nobody turned up. Then there was this one time when I got a really big audience. But there had been a misprint, and the title read “vortex algebras,” not “vertex algebras.” The audience was made up of fluid physicists, and when they realized it was a misprint, they weren’t interested either in what I had to say.

This quote\textsuperscript{12} can be found along with a gripping account of the history and the events that conspired, leading to Borcherds’ proof in Du Sautoy (2008). Fascinatingly, the cross-pollination of mathematics and physics continued. Indeed, the initial notion of vertex operators originated in string theory, inspiring a proper mathematical formalization yielding the concept of Borcherds’ vertex algebra, which, in turn, could help to underpin some major ideas in modern physics and string theory (Gebert 1993).

But next to the success of mathematics in the sciences, and the mysterious connections between mathematic and physics, what does this all really mean? What is revealed about the nature of reality and the nature of mathematics?

\section*{2.2 Seeking Meaning}

Albeit simple, clear-cut, and seemingly straightforward, the conceptual categorization sketched in Fig. 2.1 already suffices to open Pandora’s Box of epistemic and ontic puzzles. The power of mathematics can be understood in its capacity to mirror the

\begin{itemize}
  \item No three positive integers $a$, $b$, and $c$ can satisfy the equation $a^n + b^n = c^n$ for any integer value of $n$ greater than two.
  \item Seen on p. 347.
\end{itemize}
structure of reality. As a consequence, two main themes emerge. First, questions about the reality status of the abstract Platonic world of ideas re-emerge. Then, crucially, an intermediary between the physical and the abstract worlds is required: a translating entity, responsible for the encoding and decoding. Such a vessel for abstract thought conjures up a third world, bringing consciousness to the center stage. Now, nested in the physical reality, a mental world, containing the mind’s reality content, appears. A schism, making it necessary to delimit between inner and outer worlds, an objective and subjective reality. In summary, the implicit assumptions underlying Fig. 2.1 are:

- The existence of a physical reality governed by regularities.
- The emergence of living structures inside this concrete world.
- The formation of a mind within these beings, i.e., a set of cognitive faculties, harboring an inner, mental world.
- The existence of a Platonic realm of abstractions.

This all culminates in knowledge about physical reality spontaneously becoming manifest in the mind: the workings of the natural world are wondrously uncovered when quantifiable subsets thereof are mapped into formal descriptions and are subjected to the constraints governing the abstract reality. The mind’s ability to access the Platonic world, meaning the emergence of abstract ideas within the mind’s reality, is in effect a conduit for the Platonic realm to enter the physical world.13

This discussion can be framed in the broader context given by the philosophy of mathematics, as one of the main tenets deals with mathematical realism. Regarding the ontological status of mathematics, mathematical anti-realists would deny that mathematical entities exist independently of the human mind. In other words, they posit that humans do not discover mathematical truths, but invent them. The three main schools of thought in the philosophy of mathematics, existing around the end of the nineteenth and the beginning of the 20th Century, were all anti-realist, and thus anti-Platonist, reflecting the general philosophical and scientific outlook of the time which tended toward the empirical. Logicism (Frege 1884) is the program aimed at reducing mathematics to logic, an idea dating back to Leibniz. Formalism understands mathematics as a formal game in which symbols are manipulated according to fixed rules and axioms (associated with David Hilbert). Finally, intuitionism assumes that mathematics is a creation of the human mind—it is essentially an activity of mental construction—with implications for logic, set theory and elementary arithmetic (the first piece of intuitionistic mathematics in a widely read international journal is Brouwer 1919, but the idea originates from his 1907 dissertation). Brouwer’s mathematical philosophy of intuitionism can be seen as a challenge to the then-prevailing formalism of Hilbert. Indeed, the intuitionistic critique of classical mathematics required a revisionist stance toward the existing body of mathematical knowledge (Horsten 2012). In contrast, Hilbert’s program was aimed at a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent (Hilbert 1922, although the ideas can be traced back at least to Hilbert 1899). A, at times bitter, foundational controversy ensued between

13 More on the interaction of these three worlds can be found in Chapter 1 of Penrose (2004).
Brouwer and Hilbert, and in 1921, Hilbert’s favorite student, the mathematician and physicist Hermann Weyl, would side with Brouwer (Weyl 1921).

However, in the years before the Second World War serious objections had been raised against each of the three anti-Platonist programs in the philosophy of mathematics. Regarding logicism, Bertrand Russell, a mathematician and philosopher like Frege, had discovered a contradiction in one of Frege’s basic laws, demonstrating that the axioms he was employing to formalize his logic were inconsistent. This challenged the foundations of set theory and is known as Russell’s paradox (Russell 1902). Let $R$ be the set of all sets which are not members of themselves. If $R$ is a member of itself, then by definition it must not be a member of itself. Similarly, if $R$ is not a member of itself, then by definition it must be a member of itself. Symbolically

$$R = \{ x; x \notin x \}, \text{ then } R \in R \iff R \notin R. \quad (2.6)$$

Frege abandoned his logicist program, but Russell continued with Alfred North Whitehead. Together they wrote the monumental three-volume *Principia Mathematica* (Whitehead and Russell 1910, 1912, 1913), hoping to achieve what Frege had been unable to do. By devising new abstractions (a hierarchy of “types” of sets), they tried to banish the paradoxes of naive set theory. Although the *Principia* was, and still is, a hugely influential book, the questions of whether mathematics can be reduced to logic, or whether it can only be reduced to set theory, remain open (Irvine 2010). In any case, Russell and Whitehead’s program would soon receive the final blow, shattering their dreams of a paradox-free foundation of mathematics. This same fate would also befall Hilbert’s program. With the failure of formalism and logicism, the face of mathematics would forever be changed. This upheaval was achieved single-handedly by Kurt Gödel, yet another mathematician and philosopher (a colorful account of his momentous work can be found in Hofstadter 1999, 2007).

Both the logicists and formalists, like most mathematicians, placed their faith in the precept that the edifice of mathematics is built on a rock-solid foundation. Mathematics must posses two qualities:

- Consistency: a statement is true because there is a proof of the statement.
- Completeness: if a statement is true there is a proof of the statement.

Gödel’s shocking revelations were centered around an act of translation. He devised a mechanism which assigns natural numbers to terms and formulas of a formal theory (Gödel 1931). Relying on the unique representation of natural numbers as products of powers of primes, Gödel was able to encode the whole *Principia Mathematica* into numbers. In essence, any pattern of symbols representing abstract formulas in a formal theory can be assigned a unique integer number. Vice versa, any number can be decoded to reveal the sequence of symbols it corresponds to. These Gödel numbers—“arithmetizations” of strings of symbols—translate elaborate manipulations of abstract symbols, as found in the *Principia*, into simple number-crunching. Now, starting from a set of axioms of a theory, the undertaking of finding a series of formulas leading to a proof has a number-theoretic counterpart. The trouble comes in the guise of self-referentiality. Consider a formula expressed within the *Prin-
cipia, stating: “The integer $g$ does not correspond to a formula provable within the Principia.” This statement, denoted as $S$, is innocuous and unsurprising, as there is no reason to believe that every integer can be decoded into a meaningful, let alone provable, formula. However, the consequences are disastrous when $g$ is taken to correspond to $S$. This self-references amounts to the statement $G$: “This statement is unprovable.” Gödel’s first incompleteness theorem (Gödel 1931) spelled doom for the logicist program, showing that the Principia could not be both consistent and complete. If $G$ is provable, then it is false, and if $G$ is not provable, then it is true. The act of translation has, again, the capacity to teleport a given domain into a new realm where powerful novel possibilities can be unlocked, both desirable and undesirable.

The first incompleteness theorem can be restated as follows: all consistent axiomatic formulations of number theory include undecidable propositions. This did not bode well for Hilbert, who set out to prove the consistency of, for instance, the set of axioms of mathematical analysis in classical arithmetic, going back to Giuseppe Peano. A possible loophole of regaining consistency by the use of higher mathematics was also closed by Gödel’s second incompleteness theorem (Gödel 1931): no formal system extending basic arithmetic can be used to prove its own consistency. In other words, if number theory is consistent, then a proof of this fact does not exist using the methods of first-order logic, as axiomatized by Peano arithmetic. Hilbert’s program fails: higher mathematics cannot be interpreted in a purely instrumental way.

Finally, the last anti-Platonist program, intuitionism, simply faded out of fashion. The initial enthusiasm for the intuitionistic critique of classical mathematics and the alternative that it propose was dampened, as it became clear what this approach entailed for higher mathematics. Namely, a drastically unfamiliar and complicated theory. Thus room was created for a renewed interest in the prospects of Platonistic views about the nature of mathematics. Notably, Frege, Gödel, and Russell were advocates of this idea. In the words of Gödel (quoted in Kennedy 2012):

I am under the impression that after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigor and that the result will then be […] that the Platonistic view is the only one tenable.

The influential mathematician Godfrey H. Hardy expressed a similar conviction (Hardy 1967, p. 123):

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our “creations,” are simply the notes of our observations.

Other notable and prolific mathematicians have espoused similar views. On the 16th of January 1913, Hardy received a letter (Selin 2008, p. 1868) from “an unknown Hindu clerk” (Hardy 1937, p. 144). Srinivasa Ramanujan, twenty-six at the time, had sent him a list of mathematical theorems out of the blue. One equation read (Hardy 1937, p. 143):
\[
\frac{1}{1 + \frac{\exp(-2\pi \sqrt{5})}{\exp(-4\pi \sqrt{5})}} = \left[ \frac{\sqrt{5}}{1 + \sqrt{\frac{5}{2} \left( \frac{\sqrt{5} - 1}{2} \right)^2} - 1} - \frac{\sqrt{5} + 1}{2} \right] \exp \frac{2\pi}{\sqrt{5}}. \tag{2.7}
\]

Hardy was taken by surprise (Hardy 1937, p. 144):

I had never seen anything in the least like them [the three formulas in the form of (2.7)] before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

The quote also echoes his Platonist conviction, namely that true mathematics is discovered by the human mind. Hardy became Ramanujan’s mentor and brought him to England the next year. After having achieved world-wide fame, Ramanujan would die six years later due to an array of ailments, like tuberculosis and vitamin deficiency (Selin 2008, p. 1868). Ramanujan was a unique mathematician. He was an autodidact with no formal tuition, unaware of most of existing Western mathematics. These circumstances resulted in him unwittingly discovering, i.e., rediscovering, a wealth of known mathematics. In Hardy’s words (Hardy 1937, p. 145):

It was inevitable that a very large part of Ramanujan’s work should prove on examination to have been anticipated. He had been carrying an impossible handicap, a poor and solitary Hindu pitting his brains against the accumulated wisdom of Europe. […] I should estimate that about two-thirds of Ramanujan’s best Indian work was rediscovery […]

Ramanujan was also an idiosyncratic mathematician. For one, he had an intimate relationship with numbers. A famous anecdote is given by Hardy (1937, p. 147):

I remember going to see him once when he was lying ill in Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped that it was not an unfavorable omen. “No,” he replied, “it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.”

Moreover, only in his later years was he introduced to the idea of proof in mathematics. He mostly mixed reasoning with intuition to reach his insights (Hardy 1937, p. 147). Ramanujan was also a Platonist and believed in the divine nature and reality of mathematics (Hardy 1937, p. 139):

Ramanujan used to say that the goddess of Namakkal inspired him with the formulae in dreams.

The goddess, also known as Namagiri of Namakkal, is just one divine manifestation in the vast and rich Hindu pantheon of deities. Another quote attributed to Ramanujan reads (Pickover 2005, p. 1):

An equation means nothing to me unless it expresses a thought of God.

\[1729 = 12^3 + 1^3 = 10^3 + 9^3.\]
Hardy would compare Ramanujan to mathematical geniuses like Leonhard Euler and Carl Gustav Jacob Jacobi (Hardy 1937, p. 149). Indeed, he saw his own role as mathematician only in the context given by Ramanujan (Kanigel 1992, p. 358):

Paul Erdős has recorded that when Hardy was asked about his greatest contribution to mathematics, he unhesitatingly replied, “The discovery of Ramanujan.”

Hardy was very fond of Ramanujan. He described his collaboration with him as “the one romantic incident in [his] life” (Hardy 1937, p. 138). It is hard to imagine what mathematical knowledge could have been discovered by Ramanujan, had he been formally educated in mathematics and spared from the redundant task of rediscovering known theorems.

Paul Erdős was another influential and idiosyncratic mathematician holding Platonist views (Aigner and Ziegler 2010, preface):

Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős said that you need not believe in God but, as a mathematician, you should believe in The Book.

He led a peculiar life (Dunham 1994, p. 9f.):

[Erdős was] the 20th Century’s most prolific, and perhaps most eccentric, mathematician. Even in a profession in which unusual behavior is accepted as something of the norm, Erdős is legendary. For instance, so sheltered was this young scholar that only at the age of 21 [...] did he first butter his own bread. [...] Equally unusual is that Erdős has no permanent residence. Instead, he travels around the globe from one mathematical research center to another, living out of a suitcase and trusting that at each stop someone will put him up for the night. As a result of his incessant wanderings, this vagabond mathematician has collaborated with more colleagues, and published more joint papers, than anyone in history.

Due to his prolific output, friends created the Erdős number as a humorous tribute to him (Goffman 1969). Defined as zero for Erdős himself, every collaborator with a joint paper gets assigned the Erdős number 1. Likewise, an Erdős number 2 denotes an author who published a mathematical paper with a person having Erdős number 1. Due to the occasional blurring of clear boundaries between scientific fields, some researchers in physics, chemistry, and medicine also have low Erdős numbers. For instance, Einstein has Erdős number 2. In general, this number is a reflection of the tight-kit nature of the collaboration network in academia, an example of the small world phenomenon, summed up as “six degrees of separation” (discussed in Sect. 5.2.3). Another small-world network is that of movie actors, where the Bacon number is an application of the Erdős number concept to actors, centered around Kevin Bacon. Finally, the Bacon–Erdős number is the sum of a person’s Erdős and Bacon numbers. As an example, the mathematician Steven Strogatz, co-author of a seminal paper on small-world networks, has a Bacon–Erdős number of 4, as he appeared as himself in a documentary about “six degrees of separation” featuring Bacon.

To conclude this section, all proposed mathematical anti-realist programs faced serious problems, leaving Platonism as a sound, albeit philosophically challeng-
ing, option. As discussed, many of the greatest mathematical minds subscribed to Platonism.

2.2.1 Shut Up and Calculate!

At this point the discussion threatens to be come intractable. For one, there are many conceptions of what Platonism is really supposed to mean. Perhaps the most extreme view comes in the guise of mathematical monism. A view espoused by Max Tegmark, a cosmologist, in his mathematical universe hypothesis. This radical Platonist view states (Tegmark 2008, 2014): Our external physical reality\(^{15}\) is in actual fact a mathematical structure. In effect, not only mathematical anti-realists would disagree with ideas espoused as Platonism, but also the various Platonists factions among themselves. Furthermore, one key criticism is the following. Platonic realism posits the existence of mathematical objects that are independent of the mind and language, which bear no spatiotemporal relations to anything. In contrast, flesh and blood mathematicians are physically localized in space and time. The so-called epistemological argument against Platonism is the question how human beings can attain knowledge of abstract objects. Is the human mind capable of penetrating the border between the physical and an eternal realm of existence? This puzzle is captured in the directed link labeled \(M_1\) in the schematic illustration seen in Fig. 2.2, where the interplay between these various modes of reality is summarized. There have been many responses by Platonists to this challenge, followed by more arguments against Platonism. The problem now is manifold. Why should anyone be convinced by either view, if not based on pure belief or intuition? Is an argument outlining a problem sufficient to prove an idea false? And, what then, is a tenable alternative? A general reference relating to Platonism and the philosophy of mathematics is Linnebo (2013).

Even if one chooses to ignore discussions relating to the ontological status of mathematics and epistemological inquiries about how the human mind can access mathematical structures, another problem emerges. The urgent question is why mathematics plays such a crucial and essential role for science? Perhaps since Galileo our best theories from the natural sciences are expressed with true mathematical rigor. This enigma obviously blurs the clear demarcation line between philosophy and science, making it difficult to retreat to the safety of objective inquiry. What is the ontic and epistemic status of the connection between mathematics and the workings of nature, captured by the relationship \(M_2\) in Fig. 2.2? One possible interpretation is in terms of the concept of entelechy,\(^{16}\) where the physical world is an actualization or

\(^{15}\)“I use the word [reality] to mean the ultimate nature of the outside physical world that we’re part of […]” (Tegmark 2014, p. 14).

\(^{16}\)The term was coined by Aristotle to describe the dichotomy between potentiality and actuality. Leibniz adapted the concept in a way that gave rise to the notion of energy used today in physics. See also Sect. 15.2.
manifestation of the potential abstractions residing in the Platonic realm. The mental agent acts as a bridge from the physical back into the abstract, denoted as $M_1$.

In a nutshell, the assumptions underlying Fig. 2.1, detailed in the list given on p. 53, can be restated as follows, given a structured external reality and ignoring the lack of knowledge related to the emergence and nature of the human mind:

1. There exists an abstract realm of objects transcending physical reality (ontology).
2. The human mind possesses a quality that allows it to access this world and acquire information (epistemology).
3. The structures in the abstract world map the structures in the physical (structural realism, see Sect. 6.2.2).

The idea of structural realism holds that the physical domain of a true theory corresponds to a mathematical structure. Or stated more cautiously, it is a “belief in the existence of structures in the world to which the laws of mathematical physics may approximately correspond” (Falkenburg 2007, p. 2). The term universal structural realism has been used for the hypothesis that the physical universe is isomorphic to a mathematical structure (Tegmark 2008), leading to Tegmark’s mathematical universe hypothesis. This would be one explanation for the puzzle in Fig. 2.2, referred to by $M_2$: the abstract and the physical worlds are the two sides of the same coin. However, such lofty radical ideas can be hard to stomach, making other, more modest and benign mysteries appear more tempting. For instance, why is there a correspondence or kinship between the realms to being with?

Many scientist abhor the idea of a reality existing beyond space and time. Being pragmatic, such ideas are viewed as ultimately futile and unnecessary baggage in any theory of the world. Specifically, the claim is “that purely philosophical considerations on ontology are fruitlessly speculative and ill-founded and have no value in the light of ‘real scientific findings’” (Kuhlmann 2010, p. 186). The power then, of mathematics in the natural sciences lies in the simple fact that it works. “Shut up and calculate!” is the rallying cry.\footnote{Originally this maxim goes back to the physicist Nathaniel David Mermin (Mermin 1990, p. 199), as a response to the persistent philosophical challenges posed by quantum theory.} Such an inclination reflects an instrumentalist
outlook. Theories are seen as mere conceptual tools for predicting, categorizing, and classifying observable phenomena. Assigning a reality to unobservables has no merit. Moreover, the genuine content of science is not to be found at the level of theories (Duhem 1991).

Essentially, these are debates relating to the old question of scientific realism, a belief in the content of theories and models, regarding both observable and unobservable aspects of the world as described by science. In detail, it is a commitment metaphysically “to the mind-independent existence of the world investigated by the sciences,” semantically to “a literal interpretation of scientific claims about the world,” and epistemologically “to the idea that theoretical claims […] constitute knowledge of the world” (Chakravartty 2013). In this sense, instrumentalist epistemologies of science can also be understood as being anti-realist. Historically, in the first half of the 20th Century, empiricism came predominantly in the form of variations of instrumentalism. Vocal advocates of this idea were the logical empiricists (or logical positivists), philosophers often associated with the notorious Vienna Circle. However, facing opposition from influential scholars, some even from within the Circle—the likes of Norwood Hanson, Thomas Kuhn, Karl Popper, Hilary Putnam, and Willard Van Orman Quine—the demise of logical empiricism was inevitable.18 In 1967 the philosopher John Passmore reported that: “Logical positivism, then, is dead, or as dead as a philosophical movement ever becomes” (as quoted in Creath 2013). This was followed by the resurrection of realism. However, as always, the demarcation lines are anything but clear. Instrumentalists can be non-realist, i.e., taking an agnostic stance as to whether parts of a physical theory have a correlate in reality. Moreover, structural realism is a very specific and restrictive type of realism: the real nature of things can never be known, only the way things are related to one another has true meaning. Indeed, in the ontic version of structural realism, relations are all that exist, without assuming the existence of individual things (French and Ladyman 2003). So the world is made up solely of structures, a network of relations without relata. See Sects. 6.2.2 and 10.4.1.

Be that as it may, “Shut up and calculate!” can help one avoid becoming stuck in philosophical mires. In the case at hand, it can be understood as encouraging the inquiry into the specifics of mathematics that makes this formalism such a powerful tool for science—abandoning musings about meaning and implications. And indeed, there is one primary mathematical ingredient that distinguishes itself among others:

The notion of symmetry, formally encoded as a principle of invariance, is singly one of the most powerful tools in unearthing novel and deep insights into the structure of the universe.

18See Sect. 9.1.1 for more details.
Conclusion

The Book of Nature has been found: The human mind can access the world of abstractions, which mirror the structures of the physical world. Consciousness is the translator. One archetypical example of these conceptualizations—representing a narrative arc in the Book of Nature—is found in the notion of symmetry, a fundamental cornerstone of physics. Symmetry will be the accompanying theme of the next two chapters. “Shut Up and Calculate!” allows the philosophical analysis to be postponed—for the moment.

The reader not wishing to dive into the particularities of the mathematics relating to symmetry and the corresponding physical concepts describing, for instance,

- conservation laws (Sect. 3.1),
- the speed of causality (Sect. 3.2.1),
- the classification of elementary particles (Sect. 3.2.2),
- unification schemes, ranging from the standard model of particle physics to string/M-theory (Chap. 4),

next to the historical embedding of some core ideas (Sects. 4.2 and 4.3), can jump to one of the following locations:

- Chapter 5: Unearthing the second volume in the Book of Nature Series, related to the algorithmic understanding of complexity, allowing for a further classification of human knowledge generation.
- Chapter 6: The new understanding of complexity, i.e., the science of simple rules.
- Chapter 7: Applying complexity thinking to finance and economics, concluding Part I and representing the highest point on the mountain of knowledge—reached before the downfall.
- Chapter 8: The beginning of Part II, glimpsing the first signs of uncertainty and confusion.
- Chapter 12: The start of Part III, transitioning to new horizons.
- Chapter 15: The final analysis.

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