Relativistic magnetohydrodynamics in one dimension

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Abstract

We derive a number of solution for one-dimensional dynamics of relativistic magnetized plasma that can be used as benchmark estimates in relativistic hydrodynamic and magnetohydrodynamic numerical codes.

First, we analyze the properties of simple waves of fast modes propagating orthogonally to the magnetic field in relativistically hot plasma. The magnetic and kinetic pressures obey different equations of state, so that the system behaves as a mixture of gases with different polytropic indices. We find the self-similar solutions for the expansion of hot strongly magnetized plasma into vacuum.

Second, we derive linear hodograph and Darboux equations for the relativistic Khalatnikov potential, which describe arbitrary one-dimensional isentropic relativistic motion of cold magnetized plasma and find their general and particular solutions. The obtained hodograph and Darboux equations are very powerful: system of highly non-linear, relativistic, time dependent equations describing arbitrary (not necessarily self-similar) dynamics of highly magnetized plasma reduces to a single linear differential equation.
I. INTRODUCTION

Expansion of plasma into vacuum is a basic problem in fluid mechanics that has a wide range of applications from heavy nuclei collisions to astrophysics. In nuclear physics, Belenkij and Landau [1] used the hydrodynamical approach to study multiparticle production in heavy ion collisions. A head-on collision of two highly relativistic nuclei creates a relativistically-compressed hot layer of quarkgluon plasma that expands quasi-one-dimensionally [2]. On a very different scale, a wide variety of astrophysical objects like jets from Active Galactic Nuclei (AGN) [3], Gamma Ray Burst (GRB) [4] and pulsar winds and magnetospheres of a special type of neutron stars - magnetars [5] - may contain relativistic strongly magnetized plasma, in which the energy density of the magnetic field dominates over the matter energy density (including kinetic and thermal energies), $B^2 \geq \rho c^2$, $P$. During explosions the strongly magnetized plasma created by the central source suddenly expands into a surrounding low density medium. In case of magnetar flares, the initial dissipation event creates relativistically hot, strongly magnetized fireball, somewhat analogous to Solar coronal mass ejections [6]; the fireball accelerates to relativistic velocities [7]. In long GRBs, when a hot magnetically dominated jet reaches the surface of the star it breaks into low density medium [4, 8]. Similar dynamics may occur in non-stationary outflows in AGNs as well [9].

In all the above mentioned cases, it is expected that at some distances from the source, the magnetic field is dominated by the toroidal component, while motion is preferentially radial. Thus, velocity is nearly perpendicular to magnetic field - this type of motion is sometimes called transverse magnetohydrodynamics. Equations of transverse MHD reduce to fluid equations, but with a complicated equation of state [10]. Qualitatively, such plasma behaves like a mixture of fluids with different adiabatic indices, $\Gamma = 2$ for the magnetic field, and some $\Gamma$ for the kinetic pressure, usually taken to be $\Gamma = 4/3$. Thus, many results of fluid dynamics, which assume a single adiabatic index, become invalid.

Numerical investigation of these phenomena requires the use of relativistic MHD codes that can handle high magnetization and high kinetic pressures exceeding the rest mass density. Exact, explicit non-linear solutions of fluid equations, and especially relativistic MHD equations, are rare. Yet they are important for benchmark estimates of the overall dynamical behavior in numerical simulations of relativistic flows and strongly magnetized
outflows in particular. In §II we find analytical expressions for self-similar expansion into vacuum of a hot magnetized plasma, considering both the Newtonian and relativistic cases with arbitrary ratios of kinetic and magnetic pressures to rest mass density.

In addition, at later times, when the whole initial state of plasma is affected by the expansion, the expansion dynamics becomes non-self-similar. A classical related problem is then the expansion of a slab of finite length.

Arbitrary isentropic one-dimensional motion of a fluid is fully integrable. Mathematically this is achieved by exchanging the independent variables \( \{t, x\} \) and dependent variables \( \{\rho, v\} \). As a result a system of two non-linear equations (of mass conservation and Euler’s law plus assumed isentropic equation of state) is reduced to a single linear equation for the (Legendre-transformed) Bernoulli (Khalatnikov) potential \( e.g. \) [11]. This is always possible if the coefficients in these equations do not depend explicitly on time and coordinate. In the case of isentropic fluid, the equation of state can be correspondingly inverted. This transformation is called the hodograph transformation [11]. In §IV we perform a hodograph transformation for relativistic cold magnetized plasma and derive the corresponding hodograph and Darboux equations.

II. RELATIVISTIC EXPANSION OF HOT MAGNETIZED PLASMA INTO VACUUM

Consider a relativistic one-dimensional flow of hot magnetized fluid along \( x \) direction, carrying magnetic field in perpendicular direction (so-called transverse MHD) The governing of the equations are [11]

\[
\begin{align*}
\partial_t (\gamma \rho) + \partial_x (\gamma \beta \rho) &= 0 \\
\partial_t T_{00} + \partial_x T_{0x} &= 0 \\
\partial_t T_{0x} + \partial_x T_{xx} &= 0 \\
T_{00} &= \gamma^2 w - P - B^2/2 \\
T_{0x} &= \gamma^2 \beta w \\
T_{xx} &= \gamma^2 \beta^2 w + P + B^2/2 \\
E &= \rho + 3P + B^2/2, \quad w = \rho + 4P + B^2
\end{align*}
\]
where $B$ is the proper (plasma frame) magnetic field divided by $\sqrt{4\pi}$, $\beta$ and $\gamma$ are fluid’s velocities in terms of the speed of light and Lorentz factors, $E$ is the proper energy density, $\rho$ is proper mass density and $w$ is proper enthalpy. The adiabatic index for the kinetic pressure is assumed to be $\Gamma = 4/3$.

Let us consider a semi-space $x < 0$ occupied by homogeneous hot plasma with density $\rho_0$, magnetic field $B_0$ and pressure $P_0$. The initial state is assumed to be homogeneous. (At time $t = 0$ a barrier located at $x = 0$ is removed; plasma starts expanding in the positive $x$ direction, while a rarefaction wave is launched in the negative $x$ direction. We introduce two dimensionless parameters describing the initial plasma magnetization and the ratio of kinetic to magnetic pressures:

$$\sigma_0 = \frac{B_0^2}{\rho_0}$$

$$\mathcal{B} = \frac{2P_0}{B_0^2}$$  \hspace{1cm} (2)

Parameter $\sigma_0$ \[12\] measures the importance of relativistic effects of magnetic fields: for $\sigma \geq 1$ the Alfvén velocity approaches the speed of light. Parameter $\mathcal{B}$ is the conventional plasma beta parameter: the ratio of kinetic and magnetic pressures.

The relevant speed of propagation of disturbances is the fast magnetosonic speed $c_f$ \[13\]

$$c_f^2 = \frac{3B^2 + 4P}{3(B^2 + 4P + \rho)} = v_A^2 + c_s^2 = \frac{(3 + 2\mathcal{B})\sigma}{3(1 + \sigma(1 + 2\mathcal{B}))}$$

$$v_A^2 = \frac{\sigma}{1 + \sigma(1 + 2\mathcal{B})}$$

$$c_s^2 = \frac{2\mathcal{B}\sigma}{3(1 + \sigma + 2\mathcal{B}\sigma)}$$  \hspace{1cm} (3)

where $v_A$ is Alfvén velocity and $c_s$ is sound speed. Note that for $\sigma = 1/2$, the fast speed is independent of $\mathcal{B}$ and equals $c_f = 1/\sqrt{3}$. In this case the contribution of kinetic pressure to total pressure is compensated by its contribution to effective mass density.

For transverse MHD (when the motion is perpendicular to the magnetic field), the induction equation and matter conservation require $B/B_0 = \rho/\rho_0$. Assuming that all the quantities depend on the self-similar variable $\eta = x/t$, Eqns. \[1\] give

$$\rho' = \frac{1 - \eta\beta}{\eta - \beta} \gamma^2 \rho_1 \beta'$$

$$\gamma^2(\eta - 2\beta + \eta\beta^2) \left(4P + \rho_1(\rho_0 + B_0^2\rho_1)\right) \beta' +$$

$$\left(\eta\beta(\rho_0 + 2B_0^2\rho_1) - \beta^2(\rho_0 + B_0^2\rho_1) - B_0^2\rho_1\right) \rho'_1 - (1 - 4\eta\beta - 3\beta^2)P' = 0$$  \hspace{1cm} (4)
where \( \rho_1 = \rho/\rho_0 \) and primes denote derivative with respect to \( \eta \) (note, that if the initial state is not homogeneous, the self-similar solutions may depend on combination \( x^\alpha/t \)).

Eliminating pressure and magnetic field in the initial state in favor of \( \sigma \) and \( B \), \( P = P_0(\rho/\rho_0)^{4/3} \), \( P_0 = (B\sigma/2)\rho_0 \), \( B_0 = \sqrt{\sigma\rho_0} \), we find

\[
(\eta - \beta)^2 - \frac{1 - \eta^2}{\gamma^2} \sigma \rho_1 - \frac{2}{3} \left(1 + 4\beta\eta - 3\eta^2 + \beta^2(\eta^2 - 3)\right) B\sigma \rho_1^{1/3} = 0
\]

(6)

Changing to Doppler factors

\[
\delta_\beta = \sqrt{\frac{1 + \beta}{1 - \beta}}
\]

\[
\delta_\eta = \sqrt{\frac{1 + \eta}{1 - \eta}}
\]

(7)

Eqns. (4-6) become

\[
(\delta_\beta^2 - \delta_\eta^2)^2 - 4\delta_\beta^2\delta_\eta^2 \sigma \rho_1 + \frac{4}{3} \left(\delta_\beta^4 - 4\delta_\beta^2\delta_\eta^2 + \delta_\eta^4\right) B\sigma \rho_1^{1/3} = 0
\]

(8)

\[
(\delta_\beta^2 + \delta_\eta^2) \rho_1 \frac{\partial \delta_\beta}{\partial \delta_\eta} + \delta_\beta (\delta_\beta^2 - \delta_\eta^2) \frac{\partial \rho_1}{\partial \delta_\eta} = 0
\]

(9)

It is more convenient to change to a new variable \( U = \sqrt{\rho_1 \sigma} \) and redefine parameter \( B_1 = (2/3)B\sigma^{2/3} \) (in this case the resulting equations (10-11) depend only on one parameter \( B_1 \); also, in the cold limit \( U \) becomes a four-velocity of Alfvén waves \( U = \beta A/\sqrt{1 - \beta^2} \) [14]).

In terms of variables \( \beta - U \), Eqns (8-9) become

\[
(\delta_\beta^2 - \delta_\eta^2)^2 - 4\delta_\beta^2\delta_\eta^2 U^2 + 2B_1 U^{2/3} \left(\delta_\beta^4 - 4\delta_\beta^2\delta_\eta^2 + \delta_\eta^4\right) = 0
\]

(10)

\[
(\delta_\beta^2 + \delta_\eta^2) U \frac{\partial \delta_\beta}{\partial \delta_\eta} - 2\delta_\beta (\delta_\beta^2 - \delta_\eta^2) \frac{\partial U}{\partial \delta_\eta} = 0
\]

(11)

Previously, Lyutikov [14] found simple analytical solution of Eqns. (10-11) for simple waves in cold magnetized plasma expanding into vacuum. If initially the plasma is at rest, and the Alfvén velocity in the unperturbed medium \( v_{A,0} \) is given by corresponding Doppler factor \( \delta_{A,0} = \sqrt{(1 + v_{A,0})/(1 - v_{A,0})} \), in the limit \( B_1 = 0 \), Eqns. (10-12) have solutions

\[
\delta_\beta = \delta_{A,0}^{2/3}\delta\eta^{2/3}, \quad \delta_A = \delta_{A,0}^{2/3}\delta\eta^{1/3}
\]

(12)

In case of hot plasma, \( B_1 \neq 0 \), Eq. (10) can be resolved for \( \delta_\beta \)

\[
\frac{\delta_\beta^2}{\delta_\eta^2} = \frac{1 + 4B_1 U^{2/3} + 2U^2 \pm 2U^{1/3} \sqrt{3B_1^2 U^{2/3} + U^{4/3} + U^{10/3} + B_1 (1 + 4U^2)}}{1 + 2B_1 U^{2/3}} \equiv f(U)
\]

(13)
This gives a general solution for the Doppler factor in terms of initial parameters \( B_1 = (2/3)B\sigma^{2/3} \) and local density \( U = \sqrt{\rho_1\sigma} \).

Eq. (11) then becomes

\[
\frac{\partial U}{\partial \ln \delta} = \frac{f(1 + f^2)U}{2f(1 - f^2) - (1 + f^2)U\partial_U f}
\]

or, using the explicit form of \( f \), Eq. (13),

\[
\frac{\partial \ln \delta}{\partial U} = -\frac{12B_1^2U^{2/3} + 9U^{4/3} + B_1(7 + 16U^2)}{3(1 + 2B_1U^{2/3}U^{2/3}/(B_1 + U^{4/3})(1 + 3B_1U^{2/3} + U^2)}
\]

Changing independent variables \( \delta_\eta \to U \), we find:

\[
\ln \delta = -\int U \frac{12B_1^2y^{2/3} + 9y^{4/3} + B_1(7 + 16y^2)}{3(1 + 2B_1y^{2/3}y^{2/3}/(B_1 + y^{4/3}))(1 + 3B_1y^{2/3} + y^2)} dy
\]

Eq. (16) gives a general solution for simple fast waves in relativistic magnetized fluid with arbitrary ratios of magnetic and kinetic pressure to rest mass, with kinetic part of the pressure obeying adiabatic law with \( \Gamma = 4/3 \). Eq. (16) expresses implicitly the density \( U = \sqrt{\rho_1\sigma} \) in terms of the self-similar variable \( \eta = x/t \).

The lower limit of integration in Eq. (16) can be found from the conditions on the front characteristics propagating into undisturbed plasma. The rarefaction wave propagates into the undisturbed medium with the local fast velocity (see Eq. (3)):

\[
c^2_{f,0} = \frac{B_1\sigma^{1/3} + \sigma}{1 + 3B_1\sigma^{1/3} + \sigma}
\]

This corresponds to

\[
\delta_{\eta,0} = \sqrt{\frac{1 - c^2_{f,0}}{1 + c^2_{f,0}}} \quad \quad U_0 = \frac{1}{2} \left( \frac{1}{\delta_{\eta,0}} - \delta_{\eta,0} \right)
\]

(note the signs of velocities in the definition of \( \delta_{\eta,0} \): the rarefaction wave propagates in the direction opposite to the flow). At the front of the rarefaction wave \( \delta_\eta = \delta_{\eta,0} \). Thus, a particular solution corresponding to expansion into vacuum of a medium initially at rest is

\[
\ln \frac{\delta_\eta}{\delta_{\eta,0}} = -\int_{U_0} U \frac{3(1 + 2B_1y^{2/3}y^{2/3}/(B_1 + y^{4/3})(1 + 3B_1y^{2/3} + y^2)}{12B_1^2y^{2/3} + 9y^{4/3} + B_1(7 + 16y^2)} dy
\]

Eq. (19) gives a solution for relativistic expansion of hot magnetized plasma into vacuum, see Fig. 1. Corresponding velocities are plotted in Fig. 2.
One can verify that Eq. 19 reproduces the known result for cold magnetized plasma. For zero kinetic pressure, \( B_1 = 0 \), we find
\[
\begin{align*}
    f(U) &= \delta_{\pm 1} \\
    \delta_U &= \delta_{U_0} \sigma^{1/3}
\end{align*}
\]
(20)
in agreement with 14.

For zero kinetic pressure, \( B_1 \to 0 \), relations 19-20 imply that the front of the rarefaction wave is located at \( \eta_{RW} = -\sqrt{\sigma/(1 + \sigma)} \), while for large kinetic pressure, \( B_1 \to \infty \), the front of the rarefaction wave is located at \( \eta_{RW} = -1/\sqrt{3} \). The vacuum interface in the limit \( B_1 \to \infty \) approaches the speed of light.

**FIG. 1.** Four-velocity of the relativistic self-similar expansion for different values of \( B_1 = (2/3)\sigma^{2/3}, B = 2P_0/B_0^2 \). Plotted is the value of \( U = \sqrt{\rho_1 \sigma} \) as a function of the self-similar parameter for various values of the kinetic pressure parameter \( B_1 \) and magnetization. **Left Panel:** \( \sigma = 1 \), **Right Panel** \( \sigma = 1/2 \). The upper curve, corresponding to \( B_1 = 0 \), is, in fact two coincident curves, plotted using the known explicit solution for relativistic expansion of magnetized cold gas into vacuum, Eq. (12), see also [14], and the implicit solutions (16) for zero kinetic pressure. In this case the front of the rarefaction wave is located at \( \eta = -\sqrt{\sigma/(1 + \sigma)} \). For \( \sigma = 1/2 \), the front of the rarefaction wave is located at \( \eta = -\sqrt{\sigma/(1 + \sigma)} = -1/\sqrt{3} \).

For a fixed values of \( \sigma > 1/2 \), the increase of kinetic pressure leads to larger values of negative \( \eta_{RW} \) (smaller absolute values of \( \eta_{RW} \)) due to effective increase in plasma inertia. For \( \sigma = 1/2 \), the front of the rarefaction wave is located at fixed \( \eta = -1/\sqrt{3} \) for any \( B_1 \). Finally, we note that the Riemann invariants in a transverse relativistic MHD, does not have a representation in simple functions.
Fig. 2. Velocities for relativistic expansion of hot plasma into vacuum $\sigma = 1$ (Left Panel), $\sigma = 1/2$ (Right Panel).

III. SELF-SIMILAR EXPANSION OF UNMAGNETIZED RELATIVISTIC FLUID

In this section we simplify the above relations for the case of simple waves in relativistic unmagnetized fluid. Adopting a polytropic EoS, we will calculate simple waves in plasma with adiabatic index of 4/3, yet for a finite ratio $P/\rho$, without assuming that the speed of sound equals $1/\sqrt{3}$. This is an important step since most codes use a single value of adiabatic index, but for finite ratios $P/\rho$.

For a finite ratio of $P/\rho$, the speed of sound equals

$$ c_s^2 = \frac{4P_0}{3(4P_0 + \rho_0)}, $$

while Eqns. (6) become

$$ (\eta - \beta)^2 - \frac{4P_0}{3\rho_0} \left(1 - 3\eta^2 + 4\eta\beta - (3 - \eta^2)\beta^2 \right) \rho_1^{1/3} = 0 $$

$$ \rho'_1 = \frac{1 - \eta\beta}{\eta - \beta} \gamma^2 \rho_1^{1/3}. $$

In case of negligible density $\rho_0 \to 0$, this immediately gives the ultra-relativistic limit $c_s = 1/\sqrt{3}$,

$$ \beta = \frac{2\eta \pm \sqrt{3(1 - \eta^2)}}{3 - \eta^2} $$

$$ \delta_\beta = \frac{\delta_\eta}{\sqrt{2 - \sqrt{3}}} $$

(24)
The front of the rarefaction wave is located at $\eta_{RW} = -1/\sqrt{3}$, while on the vacuum side expansion proceeds with the speed of light.

In the ultra-relativistic limit the flow lines $dx/dt = \beta$ are given by

$$\frac{t}{t_0} = \frac{1}{\sqrt{1-\eta^2}} \left( \frac{1+\eta}{1-\eta} \right)^{\sqrt{3}/2}$$

(25)

The characteristics satisfy $dx/dt = (\beta + c_s)/(1 + \beta c_s)$, which gives

$$\frac{t}{t_0} = \frac{1}{\sqrt{1-\eta^2}} \left( \frac{1+\eta}{1-\eta} \right)^{1/\sqrt{3}}$$

(26)

(the other characteristics are straight lines).

In case of finite density, introducing Doppler factors, Eqns. (22-23) give

$$\rho_1 = \frac{3^{1/3}}{2} \left( \frac{\delta_\eta^2 - \delta_\beta^2}{4\delta_\beta^2 \delta_\eta^2 - \delta_\beta^4 - \delta_\eta^4 \rho_0} \right)^{1/3}$$

$$\left( 3\delta_\eta^4 \delta_\beta - 16\delta_\eta^2 \delta_\beta^3 + 3\delta_\beta^4 \right) \delta_\beta + 4\delta_\eta \delta_\beta^4 = 0$$

(27)

(28)

Which can be integrated

$$\delta_\eta^2 = \frac{\sqrt{3} \delta_\beta^2 (C + \delta_\beta^{3\sqrt{3}})}{(3 + 2\sqrt{3})C - (3 - 2\sqrt{3})\delta_\beta^{3\sqrt{3}}}$$

(29)

The constant of integration $C$ in Eq. (29) can be found from the condition that at the front of the rarefaction wave, propagating with velocity $-c_{s,0}$ and located at $\eta_0 = -c_{s,0}, \delta_{\eta,0} = \sqrt{(1+\eta_0)/(1-\eta_0)}$ the fluid is at rest, $\beta = 0, \delta_\beta = 1$. This gives

$$C = -\sqrt{3} + (3 - 2\sqrt{3})\delta_{\eta,0}^2$$

$$\sqrt{3} - (3 + 2\sqrt{3})\delta_{\eta,0}^2$$

(30)

Equations (27, 29, 30) give an analytical solution to the problem of self-similar expansion of fluid with $\Gamma = 4/3$ into vacuum, valid for arbitrary ratios of kinetic pressure to mass density, see Fig. 3

IV. NON-SIMILAR EXPANSION OF COLD MAGNETIZED PLASMA: RELATIVISTIC HOODOGRAPH AND DARBOUX EQUATIONS

Relativistic hodograph transformation was derived by Belenkij & Landau for polytropic fluid. Here we first re-derive the corresponding equation for cold magnetized plasma
and then transform it to a normal form, where Riemann invariants are taken as independent variables. In case of regular fluids the internal energy and the corresponding hodograph equations are defined in term of temperature, which is zero in the case of cold magnetized fluid. As we will see below, the role of temperature is taken by the proper enthalpy.

The relativistic hodograph transformation is achieved by introducing the Khalatnikov potential $\phi$ [11]

$$\gamma \beta \tilde{w} = \partial_x \phi$$
$$\gamma \tilde{w} = -\partial_t \phi$$
$$\tilde{w} = \frac{w}{\rho}$$

(31)

In the non-relativistic limit the corresponding equations are the condition on potential flow

FIG. 3. Self-similar expansion of relativistic unmagnetized fluid with adiabatic index $\Gamma = 4/3$. Top to bottom $P_0/\rho_0 = 0.1, 1, 10, 100$. The curve $P_0/\rho_0 = \infty$ is given by Eq. (24). The vacuum interface propagates with the speed of light. In the ultra-relativistic limit, $P_0/\rho_0 \to \infty$ the front of the rarefaction wave is located at $\eta_{RW} = -1/\sqrt{3}$. 

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$$\tilde{w} = \frac{w}{\rho}$$

(31)

In the non-relativistic limit the corresponding equations are the condition on potential flow
and the Bernoulli equation. Below we drop the tilde sign over the enthalpy: \( w \) is then the enthalpy per unit mass:

\[
w = \frac{\rho + B^2}{\rho} = \frac{1}{1 - \beta_A^2}
\]

(32)

The differential of the Khalatnikov potential is

\[
d\phi = \partial_x \phi dx + \partial_t \phi dt = \gamma \beta w dx - \gamma w dt
\]

(33)

Next we employ Legendre transform of \( \phi \) with respect to variables \( \{x, t\} \). The transformed potential \( \chi \) becomes

\[
\chi = \phi - \gamma \beta wx + \gamma wt
\]

\[
d\chi = (t - \beta x)\gamma dw - (x - \beta t)w\gamma^2 d\beta
\]

(34)

Variables \( t \) and \( x \) are then expressed from the potential \( \chi \) as

\[
t = \gamma \frac{\partial \chi}{\partial w} - \beta \frac{\partial \chi}{\gamma w \partial \beta}
\]

\[
x = \beta t - \frac{1}{\gamma^2 w} \frac{\partial \chi}{\partial \beta}
\]

(35)

These equations can be modified if we introduce rapidity \( \beta = \tanh r \):

\[
t = \cosh r \frac{\partial \chi}{\partial w} - \frac{\sinh r}{w} \frac{\partial \chi}{\partial r}
\]

\[
x = \sinh r \frac{\partial \chi}{\partial w} - \frac{\cosh r}{w} \frac{\partial \chi}{\partial r}
\]

(36)

Comparing with Ref. [1] Eq. (4.12), we see that in cold magnetized plasma proper enthalpy \( w \) plays a role of a temperature, while the speed of sound \( c_s \) – Alfvén velocity - is given by

\[
c_s^2 = v_A^2 = \frac{\partial P}{\partial E} = \frac{B^2}{B^2 + \rho} = 1 - 1/w
\]

(37)

The relativistic hodograph equation is then obtained from the continuity equation by transforming to independent variables \( r \) and \( w \).

\[
\partial_r^2 \chi - w \partial_w \chi + (1 - w)w\partial_w^2 \chi = 0
\]

(38)

This is relativistic hodograph equation for one-dimensional motion of cold magnetized plasma. Eq. (38) reduces to the one obtained by Belenkij and Landau for hot fluid with a substitution \( w = (\rho + B^2)/\rho \to T \). A general separable solution of Eq. (38) is

\[
\chi = e^{Cr} w_2 F_1 (1 - C, 1 + C, 2, w)
\]

(39)
where, we remind, $r$ is the rapidity, $\beta = \tanh r$, and $\, _2F_1$ is the hypergeometric function.

Let us transform the hodograph equation (38) taking the Riemann invariants as independent variables. The Riemann invariants are

\[ J_1 = \log \delta_A^2 \delta_\beta \]
\[ J_2 = \log \delta_A^2 \delta_\beta \] (40)

where

\[ \delta_A = \sqrt{\frac{1 + \beta_A}{1 - \beta_A}} = \sqrt{\frac{1 + \sqrt{1 - 1/w}}{1 - \sqrt{1 - 1/w}}} \]
\[ \delta_\beta = \sqrt{\frac{1 + \beta}{1 - \beta}} = \sqrt{\frac{1 + \tanh r}{1 - \tanh r}} \] (41)

are corresponding Doppler factors.

Using Eqns (41), the Riemann invariants become

\[ J_1 = 2 \arctanh \sqrt{1 - 1/w} + r \]
\[ J_2 = 2 \arctanh \sqrt{1 - 1/w} - r \] (42)

Thus, the rapidity $r$ and the proper enthalpy are

\[ r = \frac{J_1 - J_2}{2} \]
\[ w = \cosh^2 \frac{J_1 + J_2}{4} \] (43)

Changing the independent variables in the hodograph equation (38), we derive the relativistic Darboux equation for the Khalatnikov potential as a function of the Riemann invariants:

\[ \partial_{J_1} \partial_{J_2} \chi + \frac{1}{4} \frac{\partial J_1 \chi + \partial J_2 \chi}{\sinh \frac{J_1 + J_2}{4}} = 0 \] (44)

In the non-relativistic limit $J_1 \to \beta + 2 \beta_A$, $J_2 \to 2 \beta_A - \beta$, $\sinh (J_1 + J_2)/2 \approx (J_1 + J_2)/2$ and Eq. (44) reduces to the non-relativistic Darboux equation for a fluid with adiabatic index of $\Gamma = 2$, Eq. (B3)

Using constancy of the Riemann invariants on the characteristics, one finds (cf., Eq. (45))

\[ \partial_{J_2} x = \tanh \left( \frac{J_1 - 3 J_2}{4} \right) \partial_{J_2} t \]
\[ \partial_{J_2} x = \tanh \left( \frac{3 J_1 - J_2}{4} \right) \partial_{J_2} t \] (45)
System (45) can be written as a single equation for time
\[
\partial_{J_1} \partial_{J_2} t + \frac{3}{4} \frac{\text{sech}^2 \left( \frac{J_1 - 3J_2}{4} \right) \partial_{J_1} t + \text{sech}^2 \left( \frac{3J_1 - J_2}{4} \right) \partial_{J_2} t}{\tanh \left( \frac{3J_1 - J_2}{4} \right) - \tanh \left( \frac{J_1 - 3J_2}{4} \right)} = 0 \quad (46)
\]
This is the relativistic analogue of the Darboux equation for the time variable. Solutions of Eqs. (46,45) give time and spatial coordinate as functions of two Riemann invariants. These solutions can then be inverted for \( J_{1,2}(x,t) \).

V. 1D EXPANSION OF A SLAB OF GAS: SOLUTIONS OF THE HODOGRAPH EQUATION

Consider a slab of gas initially at rest, occupying region \( 0 < x < L \) and expanding into vacuum \( x > L \). At \( x = 0 \) there is an impenetrable wall, Fig. 4. In the initial state the Riemann invariants are \( J_{1,0} = J_{2,0} = 2c_{s,0} \) where \( c_{s,0} = v_{A,0} \) is the sound (Alfvén) velocity in the initial state. The boundary conditions for this problem are [11, problem after parag. 105]: zero velocity at the wall and constancy of the first Riemann invariant on the characteristics that leaves the wall at the moment of reflection:

\[
\left. \frac{\partial \chi}{\partial \beta} \right|_{\beta=0} = 0 \\
\chi(J_1 = J_{1,0}) = 0 \quad (47)
\]

Numerical solutions for Alfvén and sound speeds are found and plotted in both the non-relativistic (as discussed in Appendix B) and relativistic case. In both cases finding a numerical solution involves inverting a pair of functions of the Riemann invariants, \( \{t(J_1, J_2), x(J_1, J_2)\} \), to obtain functions of position and time giving the Riemann invariants, \( \{J_1(x, t), J_2(x, t)\} \). In the non-relativistic case the initial functions \( \{t(J_1, J_2), x(J_1, J_2)\} \) are functions represented by Eq.’s (B8) and (B9) whereas in the relativistic case these functions are determined using numerical methods. The relevant quantities, Alfvén and sound speed, can then be expressed as functions of position and time. The numerical solutions in both relativistic and non-relativistic cases involve separate calculations for the regions of expansion affected by the reflection of the initial rarefaction wave at the position of the wall, \( x = 0 \), and the regions unaffected by this reflection (as well as the boundary between these regions). The former region will be referred to as the “complex” region and latter as the “simple” region.
B. Numerical Solutions of Relativistic Expansion

Initial conditions in the relativistic case are given, by initial Alfvén speed \( \beta_A(x,0) = \tanh(1) \), initial velocity \( \beta(x,0) = 0 \), the plasma occupies the region \( 0 < x < \tanh(1) \) with an impenetrable wall at \( x = 0 \) and the plasma expands into vacuum at \( t = 0 \). The initial conditions are such that reflection of the initial rarefaction wave occurs at \( t = 1 \). The Initial Alfvén speed, \( \tanh(1) \approx 0.762 \), was selected as a speed somewhat near the speed of light that gives convenient initial values for the Riemann invariants: \( 2 = J_{1,0} = J_{2,0} \).

In the case of relativistic expansion, no simple analogue to Eq. (B8) that satisfies Eq. (46) and gives time as a function of the Riemann invariants in the complex wave region is readily available. Therefore, in the complex region, time must be numerically calculated as a function of the two Riemann invariants, \( J_{1,2} \). The first step in this calculation is accomplished by numerically solving for a function satisfying the adjoint differential operator corresponding to Eq. (73) subject to boundary conditions as outlined in \[15\] Ch. 5, Sec. 2, (Eq.’s 3 and 3′), obtaining a solution \( R[J_1, J_2; \alpha, \beta] \). In the present calculations this function is solved numerically after specifying boundary values based on \( \alpha \) and \( \beta \). Exploiting the symmetry of the Riemann’s function, \( R[J_1, J_2; \alpha, \beta] \) \[15\] Ch. 5, Sec. 2, (Eq.’s 3 and 3′), one obtains an expression for time given as a function of \( J_1 \) and \( J_2 \) satisfying (46):

\[
t[J_1, J_2] = R\left[2 \tanh^{-1}(\beta_{A,0}) , 2 \tanh^{-1}(\beta_{A,0}) ; J_1, J_2 \right] \quad (48)
\]

The initial value of both Riemann invariants, \( 2 \tanh^{-1}(\beta_{A,0}) \), corresponds to the initial conditions of \( \beta_{t=t_0} = 0 \) and \( \beta_{A,t=t_0} = \beta_{A,0} \). Position can then be calculated as a function of the Riemann invariants by integrating by parts Eq. (45).

As a next step towards a numerical solution, \( J_2 \) is calculated as a function of \( J_1 \) and time by identifying the values of \( J_2 \) corresponding to a specified time through an interpolation method similar to the one used in the non-relativistic case, resulting in the calculation of a function \( J_2(J_1,t) \). In order to determine the numerical solutions in the complex region for a given time, a collection of points of the form \( (J_1, J_2(J_1;t), x[J_1, J_2(J_1;t)]) \), where we have emphasized only a parametric dependence on time, are calculated based on a number of \( J_1 \) values in the range \( (J_{1,x=0}(t), 2) \). This range represents the \( J_1 \) values in the complex region and \( J_{1,x=0}(t) \) represents the minimum value encountered at the point \( x = 0 \) subject to the condition \( t = R\left[2 \tanh^{-1}(\beta_{A,0}), 2 \tanh^{-1}(\beta_{A,0}) ; J_{1,x=0}(t), J_{1,x=0}(t) \right] \). This condition
is based on the requirement that \( \beta(0,t) = 0 = \tanh \left( \frac{J_{1,x=0} - J_{2,x=0}}{2} \right) \) and hence \( J_{1,x=0} = J_{2,x=0} \).

Having obtained a sufficient collection of points, the solutions

\[
\beta(x,t) = \tanh \left[ \frac{J_1(x,t) - J_2(x,t)}{2} \right]
\]

\[
\beta_A(x,t) = \tanh \left[ \frac{J_1(x,t) + J_2(x,t)}{4} \right]
\]

(49)

can be numerically calculated by interpolation. The functions \( \beta(x,t) \) and \( \beta_A(x,t) \) in the simple region of the expansion are known analytical functions of \( x \) and \( t \), more specifically of a single variable \( \eta = \frac{x-x_0}{t} \), given in Ref. [14] as

\[
\beta(\eta) = \frac{\delta_{A,0}^{4/3} \delta_{\eta}^{4/3} - 1}{1 + \delta_{A,0}^{4/3} \delta_{\eta}^{4/3}}
\]

\[
\beta_A(\eta) = 1 - 2 \frac{\delta_{\eta}^{2/3}}{\delta_{\eta}^{2/3} + \delta_{A,0}^{4/3}}
\]

(50)

The boundary, \( x_b(t) \), between the simple and complex regions of expansion is determined as a function of time in terms of \( \eta_b = \frac{x_b-x_0}{t} \) by the equation [14]

\[
t = \left( \delta_{A,0}^2 - 1 \right) \sqrt{\delta_{A,0}^4 - \frac{1}{(\delta_{A,0}^{8/3} - \delta_{\eta_b}^{4/3})^{3/2}}}
\]

(51)

Combining the interpolated complex region solutions and the simple region solutions given in Eq.’s (50), complete solutions are plotted in Figure (4).

Previously, the problem of non-self-similar expansion of magnetized plasma was considered numerically in Ref. [16]. Contrary to the initial claim in Ref. [16], the presence of the wall is detrimental to acceleration, as can be seen from Fig. 4. Magnetic pressure-driven acceleration proceeds most efficiently while a given fluid element is causally disconnected from the wall, during the self-similar stage discussed in Ref. [14].

VI. DISCUSSION

In this paper we derived a number of analytical results for one-dimensional expansion of magnetized gas into plasma. First, we found the self-similar expansion into vacuum of a hot magnetized plasma. In this case the total pressure has contribution both from magnetic field and from kinetic motion of particles. These two contribution obey different equations of state, so effectively, we considered relativistic self-similar expansion of a mixture of gasses with different adiabatic indices.
FIG. 4. Non-self-similar one-dimensional relativistic expansion of a slab of magnetized plasma, initially occupying $0 < x < 1$ and limited by a wall at $x = 0$. Expansion proceeds in the positive direction. Plots of (a) velocity and (b) Alfvén speed, along with related $\gamma$-factors (c-d) with respect to position measured from the wall at $x = 0$ for times $t = 1, 2, 5,$ and $10$. Velocities are measured as fractional multiples of $c$ and position, $x$, is measured in the same units as time multiplied by the speed of light. Initially the plasma occupies the region $0 < x < \beta_{A,0}$ (Note: $\beta_{A,0} = \tanh(1)$ in the present calculations) and expands into the vacuum occupying $x > \beta_{A,0}$ while an impenetrable wall is present at $x = 0$ (resulting in the reflection of the initial rarefaction wave at $t = 1$). The effect of finite distance from the wall occurs for times after $t = 1$ and can be observed as the discontinuities of the first derivatives in the velocities and Alfvén speeds, marking the separation between complex and simple regions of expansion.

Second, we derived relativistic hodograph and Darboux equations that describe arbitrary one-dimensional motion of magnetized plasma perpendicular to magnetic field. The obtained resulting hodograph and Darboux equations are very powerful: we reduced a system of highly...
non-linear, relativistic, time dependent equations describing arbitrary (not necessarily self-similar) dynamics of highly magnetized plasma to a single linear differential equation. Using semi-analytical methods we calculated evolution of the flow parameters.

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Appendix A: Non-Relativistic expansion of hot magnetized plasma into vacuum

In the non-relativistic limit, the equations of one-dimensional transverse MHD read

\[
\begin{align*}
\partial_t \rho + \partial_x (v \rho) &= 0 \\
\rho (\partial_t v + v \partial_x v) &= -\partial_x (B^2 / 2 + P) \\
\partial_t B + \partial_x (B \rho) &= 0 \\
P &= P_0 (\rho/\rho_0)^\Gamma
\end{align*}
\]  

(A1)

where \( P \) is kinetic pressure, \( B \) is magnetic field divided by \( \sqrt{4\pi} \), \( \rho \) is density is \( v \) is plasma velocity. The kinetic pressure obeys a polytropic equations of state with index \( \Gamma \). As the initial condition, we assume that at time \( t = 0 \) plasma occupies region \( x < 0 \), while at \( x > 0 \) the medium is a vacuum. Initial homogeneous density, magnetic fields and kinetic pressures are \( \rho_0, B_0 \) and \( P_0 \) correspondingly. At time \( t = 0 \) a barrier at \( x = 0 \) is removed and the plasma starts expanding into vacuum while a rarefaction wave propagates into the bulk plasma. In the initial state the Alfvén and sound velocities are

\[
\begin{align*}
v_{A,0}^2 &= B_0^2 / \rho_0 \\
c_{s,0}^2 &= \Gamma P_0 / \rho_0
\end{align*}
\]  

(A2)

Let us assume that all the quantities depend on self-similar combination \( \eta = z/t \). The system (A1) then reduces to

\[
\begin{align*}
(\eta - v) \rho_1' + v' \rho_1 &= 0 \\
(v - \eta)^2 &= v_{A,0}^2 \rho_1 + c_{s,0}^2 \rho_1^{\Gamma^{-1}}
\end{align*}
\]  

(A3)

where \( \rho_1 = \rho/\rho_0 \). Eq. (A3) can be resolved for velocity \( v(\rho_1) \),

\[
v = \eta \pm \sqrt{v_{A,0}^2 \rho_1 + c_{s,0}^2 \rho_1^{\Gamma^{-1}}},
\]  

(A4)

and an equation for \( \rho_1 \):

\[
\partial_\eta \rho_1 = \pm \frac{2\sqrt{v_{A,0}^2 + c_{s,0}^2 \rho_1^{\Gamma^{-2}}}^{3/2}}{3v_{A,0}^2 + c_{s,0}^2 \rho_1^{\Gamma^{-2}}(1 + \Gamma)}
\]  

(A5)
Eq. (A5) can be integrated
\[ \eta = C_1 \pm \left( 1 + \Gamma \frac{\rho_1}{\Gamma - 1} \sqrt{v_{A,0}^2 + c_{s,0}^2} \right) - 2 \left( \Gamma - 2 \right) \frac{v_{A,0}^2 \rho_1^{2(\Gamma - 1)/2}}{(3 - \Gamma)(\Gamma - 1) c_{s,0}^2} 2F_1 \left( \frac{1}{2}, \frac{3 - \Gamma}{2(2 - \Gamma)}; \frac{7 - 3\Gamma}{2(2 - \Gamma)}; -\frac{v_{A,0}^2 \rho_1^{2(\Gamma - 1)}}{c_{s,0}^2} \right) \]

where \( 2F_1 \) is a hypergeometric function. Signs in Eqns (A5) and (A6) correspond to the choice in (A4).

The constant of integration \( C_1 \) can be found from the condition that at the front of the rarefaction wave, which propagates with fast velocity in the undisturbed medium \( v_{f,0} \) and is located at \( \eta_{RW} = -v_{f,0} = -\sqrt{v_{A,0}^2 + c_{s,0}^2} \), the plasma density is undisturbed \( \rho_1 = 1 \):

\[ C_1 = \frac{2}{\Gamma - 1} \sqrt{v_{A,0}^2 + c_{s,0}^2} + 2 \left( \Gamma - 2 \right) \frac{v_{A,0}^2 \rho_1^{2(\Gamma - 1)/2}}{(3 - \Gamma)(\Gamma - 1) c_{s,0}^2} 2F_1 \left( \frac{1}{2}, \frac{3 - \Gamma}{2(2 - \Gamma)}; \frac{7 - 3\Gamma}{2(2 - \Gamma)}; -\frac{v_{A,0}^2 \rho_1^{2(\Gamma - 1)}}{c_{s,0}^2} \right) \]

In particular, for adiabatic index \( \Gamma = 5/3 \) the previous relations simplify

\[ \frac{\eta}{c_{s,0}} = \left( 2(1 + M_{A,0}^2)^{3/2} - (2 + 3M_{A,0}^2 \rho_1^{1/3}) \sqrt{1 + M_{A,0}^2 \rho_1^{1/3}} \right) \frac{1}{M_{A,0}^2} \]
\[ v = 2 \left( (1 + M_{A,0}^2)^{3/2} - (1 + M_{A,0}^2 \rho_1^{1/3})^{3/2} \right) c_{s,0} \]
\[ M_{A,0} = \frac{v_{A,0}}{c_{s,0}}, \]

see Fig. 5. The front of the rarefaction wave propagates with the fast speed and is located at \( \eta_{RW} = -c_{s,0} \sqrt{1 + M_{A,0}^2} \), while the vacuum interface, corresponding to \( \rho_1 = 0 \), propagates with the velocity

\[ v_{\text{vac}} = 2 \left( (1 + M_{A,0}^2)^{3/2} - 1 \right) c_{s,0} \]

For unmagnetized plasma Eq. (A8) gives

\[ \eta = (3 - 4\rho_1^{1/3})c_{s,0} \]
\[ v_{\text{vac}} = 3c_{s,0}, \]

while for cold magnetized plasma

\[ \eta = (2 - 2\sqrt{\rho_1})v_{A,0} \]
\[ v_{\text{vac}} = 2v_{A,0}, \]

in correspondence with the general solutions in media with adiabatic indices \( \Gamma = 5/3 \) and \( \Gamma = 2 \) [11, 17].
FIG. 5. Self-similar non-relativistic expansion of magnetized fluid into vacuum, Eq. [A8], $\Gamma = 5/3$. 

**Left Panel:** the value of the self-similar parameter in terms of fast velocity, $\eta/v_f,0 = \eta/\sqrt{c_{s,0}^2 + v_{A,0}^2}$ as a function of density $\rho_1 = \rho/\rho_0$. **Right Panel:** velocity in terms of sound velocity, $v/c_{s,0}$ as function of density. The front of the rarefaction wave is located at $\rho_1 = 1$, the vacuum interface is located at $\rho_1 = 0$. Different curves correspond to different values of $M_{A,0} = v_{A,0}/c_{s,0} = 0, 1, 10$ (top to bottom).

More, generally, in case of unmagnetized plasma, $v_{A,0} = 0$, and arbitrary adiabatic index $\Gamma$,

$$\eta = \left( \frac{2}{\Gamma - 1} - \frac{1 + \Gamma}{\Gamma - 1} \rho_1^{(\Gamma - 1)/2} \right) c_{s,0}. \tag{A12}$$

(So that the vacuum interface expands with $2c_{s,0}/(\Gamma - 1)$.)

The Riemann invariants [11] are

$$J_{\pm} = v \pm \int \frac{dp}{\rho v_f} = v \pm 2 \frac{c_{s,0} v_f^3}{c_{s,0}^3 v_{A,0}^2} \tag{A13}$$

where $p$ is total pressure, kinetic plus magnetic. Relations [A13] assume that both sound and Alfvén velocities are non-zero.

The Riemann invariants $J_{\pm}$ are constant along the corresponding characteristics $C_{\pm}$: $dz/dt = v \pm v_f$. We find then that the $C_-$ characteristics are straight lines $x = \eta t$, while $C_+$ characteristics are determined by

$$\frac{dz}{dt} = \left( 2 \frac{(1 + M^2)^{3/2}}{M^2} - \sqrt{1 + M^2 \rho_1^{1/3}(2 + M^2 \rho_1^{1/3})} \right) c_{s,0} \tag{A14}$$

where $\rho_1(z/t)$ should be found from Eq. [A8]. This gives a transcendental equation for the
characteristics:
\[
\left( \frac{dz}{dt} \right)^3 - \left( \eta + 4 \frac{(1 + M^2)^{3/2}}{M^2} \right) \left( \frac{dz}{dt} \right)^2 + \frac{1}{3} \left( \eta^2 + 8 \frac{(1 + M^2)^{3/2}}{M^2} c_{s,0} \eta + 16 \left( (1 + M^2)^3 - 1/9 \right) c_{s,0}^2 \right) \frac{dz}{dt} - \frac{1}{27} \left( \eta^3 + 12 \frac{(1 + M^2)^{3/2}}{M^2} c_{s,0} \eta^2 + 16(1 + M^2)^3 + 1 \frac{c_{s,0} \eta + 64 \sqrt{1 + M^2}(3 + 6M^2 + 4M^4 + M^2)}{M^4} c_{s,0}^3 \right)
\]
\[
(A15)
\]

Appendix B: Non-relativistic hodograph and Darboux equations for cold magnetized plasma

Let us next discuss non-relativistic one-dimensional motion of cold magnetized plasma. In this case the induction and continuity equations imply \( B/\rho = \text{constant} \), so that the pressure, \( P \propto B^2 \) is related to density via adiabatic law \( P \propto \rho^\Gamma \) with \( \Gamma = 2 \).

For polytropic index of \( \Gamma = 2 \), the corresponding hodograph equations becomes \([11, \S 105]\)
\[
w \partial_{\alpha} \chi - \partial_{\beta} \chi + \partial_\chi = 0
\]
\[
t = \frac{\partial \chi}{w}
\]
\[
x = vt - \frac{\partial \chi}{v}
\]
(B1)

where \( w = c_s^2 = v_A^2 \) is proper fluid enthalpy and \( c_s = v_A \) is the sound (Alfvén) speed.

A general solutions of the equation (B1) can be easily obtained. For example, a solution separable in \( w, \beta \) is \( \chi = e^{\alpha \beta} I_0(2\alpha \sqrt{w}) \). The main problem with solving the hodograph equation (B1) (or the corresponding Darboux equation (B3)) in a simple application of one-dimensional expansion of gas into vacuum, is in finding solutions that satisfy boundary conditions, one given on the characteristics and another at a fixed velocity.

Equation (B1) is often transformed taking Riemann invariants as independent variables \([17]\)
\[
J_1 = 2c_s + v
\]
\[
J_2 = 2c_s - v
\]
(B2)

\( J_1 \) is constant on the characteristics \( dx/dt = v + c \), while \( J_2 \) is constant on the characteristics \( dx/dt = v - c \). In the initial state \( J_1 = J_2 = 2c_{s,0} = J_{1,0} = J_{2,0} \).

In terms of \( \{J_1, J_2\} \), the hodograph equation becomes
\[
\partial_{J_1} \partial_{J_2} \chi + \mathcal{N} \partial_{J_1} \chi + \partial_{J_2} \chi = 0, \quad \mathcal{N} = 1/2
\]
(B3)
Eq. (B3) is referred to as Darboux equation. Darboux equation played an important role in
the development of fluid mechanics: analysis of the analogue of the Darboux equation lead
Riemann to the formulation of the theory of hyperbolic equations.

For integer values of the coefficient \( N = (1/2)(3 - \Gamma)/(\Gamma - 1) \), Darboux equation can be
reduced to a one-dimensional wave equation. For the case of interest, \( \Gamma = 2, N = 1/2 \), this
is not possible.

Equivalently, using constancy of the Riemann invariants on the characteristics, one find
\[
\partial_{J_2} x = \frac{3J_1 - J_2}{4} \partial_{J_2} t
\]
\[
\partial_{J_1} x = \frac{J_1 - 3J_2}{4} \partial_{J_2} t
\]
(B4)

On can then write Darboux equation for time variable:
\[
\partial_{J_1, J_2} t + \frac{3 \partial_{J_1} t \partial_{J_2} t}{2 J_1 + J_2} = 0.
\]
(B5)

The main mathematical difficulty in solving Darboux equation (B3) with boundary con-
ditions (47) is that one boundary condition is given on the characteristics, while the other
at a fixed values of velocity. For integer values of \( N \) give the solution; e.g. for \( N = 1 \), \( \chi = (J_2^2 - J_2^2)/(J_1 + J_2) \).

A general solutions of the Darboux equation (B3) can be expressed in terms of the
corresponding Riemann function (the analogue of the Green’s function) \[18, Eq. 3.71\]
\[
B(J_{1,0}, J_{2,0}, J_1, J_2) = \left( \frac{J_1 + J_2}{J_{1,0} + J_{2,0}} \right)^N 2F_1 \left( 1 - N, N, 1; -\frac{(J_{1,0} - J_1)(J_{2,0} - J_2)}{(J_1 + J_2)(J_{1,0} + J_{2,0})} \right)
\]
(B6)

It turns out that the corresponding Darboux equation for time, Eq. (B5) with boundary
condition \( t = t_0 \), the moment of reflection, when \( J_1 = J_2 = 2 \) can be solved explicitly [see
[19] Eq. 82.17]
\[
t = t_0 B(J_{1,0}, J_{2,0}, J_1, J_2)
\]
(B7)

(In passing we note that the equations of the one-dimensional fluid motion with adiabatic
index \( \Gamma = 2 \) are equivalent to shallow water equation. The corresponding problem of a dam
break with a finite lock length has been solve in Ref. [20].)

This gives
\[
t(J_1, J_2) = \frac{64}{\sqrt{J_1 + J_{1,0}} \sqrt{J_2 + J_{2,0}}} 2F_1 \left( \frac{3}{2}, \frac{3}{2}, 1, \frac{(J_{1,0} - J_1)(J_{2,0} - J_2)}{(J_1 + J_1)(J_{1,0} + J_{2,0})} \right)
\]
(B8)
where \( _2F_1 \) is the hypergeometric function.

Coordinate \( x \) then can then be derived from Eq. (B4):

\[
x = \frac{3J_1 + J_2}{4} t - \frac{1}{4} \int_{J_{2,0}}^{J_2} t dJ_2
\]

(B9)

In the numerical solution of the non-relativistic case, the initial conditions are such that the velocity, \( v(x,0) = 0 \), the Alfvén speed, \( c_A(x,0) = 1 \), the plasma occupies the region \( 0 < x < 1 \), and an impenetrable wall is present at \( x = 0 \). These conditions are exactly analogous to those given in Ref. [20] for the case of large Froud number. The front of the expansion of plasma propagates as \( x_{\text{Front}}(t) = 2t + 1 \), corresponding to the foremost \( J_2 \) characteristic, with Riemann invariant value \( J_2 = -2 \), emanating from \( x = 1 \) at \( t = 0 \). (We refer to characteristics for which \( \frac{dx}{dt} = v - c \) as \( J_2 \) characteristics and characteristics for which \( \frac{dx}{dt} = v + c \) as \( J_1 \) characteristics, the names thus referring to the Riemann invariant which is constant on the specified characteristic) The linear relationship governing the expansion front position as a function of time is obtained by integration of (Eq. 48). At time \( t = 1 \), the initial rarefaction wave reaches the wall at \( x = 0 \) and is reflected. Boundary conditions are such that plasma velocity is 0 at the wall location \( x = 0 \) (i.e., \( J_1 - J_2 = 0 \) when \( x = 0 \)). The constancy of the Riemann invariant \( J_2 = 2c - v \) on the characteristic \( \frac{dx}{dt} = v - c \), implies the leading rearward propagating characteristic emanating from \( x = 1 \) has value \( J_2 = 2 \) on the path \( x = 1 - t \). Correspondingly, the \( J_1 \) characteristic emanating from \( x = 1 \) at \( t = 0 \) has the value 2, satisfying the condition \( J_1 - J_2 = 0 \) when \( x = 0 \). All \( J_1 \) characteristics will have a constant value \( J_1 = 2 \) for \( t < 1 \) in the region \( \{ 1 - t < x < 2t + 1 \} \), satisfying the initial conditions of \( v(x,0) = 0 \) and \( c_A(x,0) = 1 \). For \( t > 1 \), the \( J_1 \) characteristics begin to influence the flow in the region \( 0 < x < x_b(t) \). The discontinuities in the first derivatives of the plots after \( t = 1 \) separate the simple and complex region of the flow. The conditions determined by eq.'s (B4) in the simple region where \( J_1 = 2 \) are

\[
\begin{align*}
\partial_{x} x = & \frac{1}{4} (6 - J_2) \partial_{t} t \\
\ & \xi = 1 + \frac{1}{4} (2 - 3J_2) t
\end{align*}
\]

(B10)

We can numerically solve for the position, \( x_b(t) \), of the boundary between the simple and complex regions. This point is given by the implicit equation

\[
x_b(t) = 1 + 2 \cdot \frac{2 - 3J_2(x_b,t)}{(2 + J_2(x_b,t))^{3/2}}
\]

(B11)
The solution for $J_2$ (and therefore also velocity and Alfvén speed) in the simple region, where $J_1 = 2$, can also be obtained from Eq. (B10), which gives

$$J_2(x, t) = \frac{1}{3} \left( 2 - 4 \left( \frac{x - 1}{t} \right) \right)$$  \hspace{1cm}  \text{(B12)}$$

The functions \{t(J_1, J_2), x(J_1, J_2)\} are given, in the complex region, by Eq.’s (B8) and (B9). The first step in numerically constructing the functions \{J_1(x, t), J_2(x, t)\} in the complex region is accomplished by numerically constructing a function $J_2(J_1, t)$ by interpolating values of $t(J_1, J_2)$ at numerous values of $J_2$ over the range $[-2, J_1]$ (this range covers all possible values of $J_2$ in the complex region). Having achieved computationally constructing a function $J_2(J_1, t)$, it is then possible to calculate position as a function of time and $J_1$: $x(J_1, t) = x(J_1, J_2(J_1, t))$. A procedure similar to that used in calculating $J_2(J_1, t)$ from $t(J_1, J_2)$ is then be employed to calculate $J_1(x, t)$ and thereby $J_2(x, t) = J_2(J_1(x, t), t)$. The procedure described allows for the Riemann invariants $J_{1,2}$, and hence $v(x, t)$ and $c_A(x, t)$ via Eq. (40), to be calculated in the complex region.

FIG. 6. Non-relativistic expansion of magnetized plasma limited by the wall at $x = 0$. Plots of the plasma velocity, $v(x, t)$, and Alfvén speed, $c(x, t)$, as functions of distance from the wall at $x = 0$ for one-dimensional flow satisfying Eq.’s (B1), (47). Initial conditions are such that gas is initially at rest occupying the region $0 < x < 1$ with Alfvén speed $c_A = 1$. An impenetrable wall is present at $x = 0$ and the gas expands into vacuum at $t = 0$. Times shown in this plot are: $t = 1, 2, 5, $ and $10$. 

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