ON QUADRATIC LIE ALGEBRAS WITH NON-TRIVIAL CENTER

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Abstract. Lie algebras with non-trivial center are studied and shown how can they be built from characteristic ideals. The nature of such characteristic ideals is elucidated and their construction is provided in detail. It is shown how to approach the study and classification of these Lie algebras through the theory of extensions via appropriate cocycles and representations. Also, necessary and sufficient conditions for the existence of invariant metrics are given. It is shown that any non-Abelian quadratic Lie algebra \( g \) with non-trivial center is of the form \( g = h \oplus a \oplus h^* \), where \( i(g) \simeq h^* \) and \( j(g) \simeq a \oplus h^* \) are two canonically defined Abelian ideals for \( g \) satisfying \( i(g)^\perp = j(g) \). The main examples are such that \( [g, g]^\perp = C(g) \neq \{0\} \).

Introduction

Let \( g \) be a finite-dimensional Lie algebra over a field \( F \) of characteristic zero. The Lie algebra \( g \) is said to be quadratic if it comes equipped with a non-degenerate, symmetric, bilinear form, \( B : g \times g \to F \), satisfying, \( B([x, y], z) = B(x, [y, z]) \), for any \( x, y \) and \( z \) in \( g \); \([x, y]\) being the Lie bracket of \( x \) and \( y \) in \( g \). The bilinear form \( B \) is said to be invariant if this property is satisfied.

A theorem by A. Medina and P. Revoy [6] states that an indecomposable, non-semisimple, quadratic Lie algebra \( g \) can be build up from a minimal isotropic ideal \( a^* \). This ideal, in turn, defines a subspace \( h \subset g \), via \( h = (a^*)^\perp/a^* \), in such a way that \( g \) gets decomposed in the form \( g = a \oplus h \oplus a^* \), with \( a \) and \( a^* \) being totally isotropic, dual to each other, and \( h^\perp = a \oplus a^* \), as in Witt decomposition.

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Even though the Medina-Revoy Theorem deals in a clever and general way with indecomposable, non-semisimple, quadratic Lie algebras, its proof depends on the choice of the minimal ideal $a^*$. This choice, however, is not unique. Besides, even though the subspace $h$ has the structure of a quadratic Lie algebra, it is not in general a Lie subalgebra of $g$. Moreover, $h$ might be further decomposed into mutually orthogonal subspaces, even if the algebra $g$ one started with was indecomposable.

Later on, I. Kath and M. Olbrich proposed an alternative approach in [5], by looking for a canonical isotropic ideal that might play the role of $a^*$; they succeeded in discovering a way to get a canonical ideal, though it might not be minimal. On the other hand, an important asset of [5] is that it brings to the foreground interesting ideas pointing toward the use of cochain complexes and cohomology techniques based on elementary group actions build up from morphisms and representations.

Indeed, one may pose the question of reconstructing the quadratic Lie algebra $g$ out of its characteristic ideals $i(g)$ and $j(g)$ with $h = g/j(g)$ and $a = i(g)/i(g)$, where $a$ inherits an inner product from $g$. Being $g$ quadratic one would also have an identification of $i(g)$ with $h^*$. One may approach the reconstruction problem as that of an extension of $h$ in two steps so as to guarantee that $h = g/j(g)$, and $a = i(g)/i(g)$; namely, start up with $h$ and end up with $g = h \oplus a \oplus h^*$ having $j(g) \simeq a \oplus h^*$ and $i(g) \simeq h^*$. To reconstruct the quadratic Lie algebra $g$ in this way, $a$ would have to carry an inner product that is to be extended to a non-degenerate, symmetric, invariant, bilinear form on $g$, with $a^\perp = h \oplus h^*$. On the other hand, an alternative approach has been followed in [7] via the so called double extension procedure, ending up
with a decomposition of $g$ similar to the obtained in the classical double extension.

For a general Lie algebra $g$, our main result applies well when the center $C(g)$ is non-trivial and $[g, g] \neq g$; these conditions are needed to produce the two characteristic Abelian ideals $i(g)$ and $j(g)$ with $[g, j(g)] \subset i(g)$. Furthermore, when $g$ carries an invariant metric, the first of these two conditions implies the second, since in that case, $C(g) \perp = [g, g]$. In particular, this explains the title of this work. As a matter of fact, it is proved in Thm. 3.1 that any non-Abelian quadratic Lie algebra $g$ with non-trivial center is of the form $g = h \oplus a \oplus h^*$, where $i(g) \simeq h^*$ and $j(g) \simeq a \oplus h^*$ are precisely the two canonically defined ideals for $g$ satisfying $i(g) \perp = j(g)$.

As an example, we have worked out in full detail in §4 the classification, up to isomorphism, of the Lie algebras $g = h \oplus a \oplus h^*$, for which $h$ is the 3-dimensional Heisenberg Lie algebra, $i(g)$ is isomorphic to $h^*$ acted on by the coadjoint representation, and $j(g)/i(g) \simeq \mathbb{F}^3$. It turns out that there are nine different families of such isomorphism classes (see Prop. 4.4), and only a finite set of specific representatives inside four of them admit an invariant metric (see Prop. 4.5).

1. Background on Abelian Extensions

1.1. Abelian Extensions of Lie algebras. Let $g$ be a Lie algebra with Lie bracket $[\cdot, \cdot] : g \times g \to g$ and let $j$ be an Abelian ideal of $g$. Let $h$ be a vector subspace complementary to $j$, so that $g = h \oplus j$. For each pair $x, y \in h$, let $[x, y]_h$ and $\Lambda(x, y)$ be the components of $[x, y]$ along $h$ and $j$, so that,

$$[x, y] = [x, y]_h + \Lambda(x, y).$$

One also obtains a representation $R$ of $h$ in $j$ via,

$$h \ni x \mapsto R(x) = (\text{ad}|_h(x))|_j = [x, \cdot]|_j \in \mathfrak{gl}(j).$$

Indeed, being given by the adjoint representation, $R$ satisfies,

$$R([x, y]) = \text{ad}([x, y])|_j = R(x) \circ R(y) - R(y) \circ R(x)$$

$$= \text{ad}([x, y]_h + \Lambda(x, y))|_j.$$

By restricting the adjoint action to the ideal $j$, we have $[\Lambda(x, y), \cdot]|_j \equiv 0$, since $\Lambda(x, y) \in j$ and $j$ is Abelian. Thus, for any pair $x, y \in h$,

$$\text{ad}([x, y])|_j = \text{ad}([x, y]_h)|_j = R([x, y]_h)$$

and therefore, for any $x$ and $y$ in $h$, we have,

$$R([x, y]_h) = R(x) \circ R(y) - R(y) \circ R(x).$$
Now, for any three elements in $\mathfrak{h}$, say $x$, $y$ and $z$, we have,

$$
[x, [y, z]] = [x, [y, z]_\mathfrak{h}] + \Lambda(x, [y, z]) + R(x)(\Lambda(x, y)) .
$$

Since the Lie bracket $[\cdot, \cdot]$ satisfies the Jacobi identity, it follows that,

\begin{align*}
\sum_{\{x, y, z\}} [x, [y, z]_\mathfrak{h}] &= 0, \\
\sum_{\{x, y, z\}} (R(x)(\Lambda(y, z)) - \Lambda([y, z]_\mathfrak{h}, x)) &= 0.
\end{align*}

The first equation states that $\mathfrak{h} = \mathfrak{g}/\mathfrak{j}$ is a Lie algebra under $[\cdot, \cdot]_\mathfrak{h}$, whereas the second equation states that $\Lambda$ is a 2-cocycle in the cochain complex $C(\mathfrak{h}; \mathfrak{j})$ of alternating multilinear maps $\mathfrak{h} \times \cdots \times \mathfrak{h} \to \mathfrak{j}$ into the $\mathfrak{h}$-module $\mathfrak{j}$ defined by the representation $R$. That is,

$$(d \Lambda)(x, y, z) = \sum_{\{x, y, z\}} \left\{ R(x)(\Lambda(y, z)) - \Lambda([y, z]_\mathfrak{h}, x) \right\} = 0.$$

Conversely, let $\mathfrak{h}$ be a Lie algebra with Lie bracket $[\cdot, \cdot]_\mathfrak{h}$ and let $\Lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{j}$ be a 2-cocycle with values in the $\mathfrak{h}$-module $\mathfrak{j}$ defined by a given representation $R : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{j})$. It is well known that the skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, defined on the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$ by means of,

\begin{align*}
[x, y] &= [x, y]_\mathfrak{h} + \Lambda(x, y), \\
[x, v] &= R(x)(v), \\
[v, w] &= 0,
\end{align*}

for any $x, y \in \mathfrak{h}$ and any $v, w \in \mathfrak{j}$, is a Lie bracket in $\mathfrak{g}$. One says that the Lie algebra $\mathfrak{g}$ so defined is an Abelian extension of $\mathfrak{h}$ associated to the representation $R : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{j})$ and the 2-cocycle $\Lambda$ (see [4]). We shall denote by $\mathfrak{h}(\Lambda, R)$ the Lie algebra defined on the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$ with Lie bracket as in (4) in terms of the given 2-cocycle $\Lambda$ with coefficients in the representation $R : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{j})$.

1.2. On the Isomorphism Class of an Abelian extension. It is well known that for a fixed representation $R : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{j})$, and hence, within a fixed cochain complex $C(\mathfrak{h}; \mathfrak{j})$, the isomorphism class of the Abelian extension defined by a 2-cocycle $\Lambda$ is completely determined by its cohomology class $[\Lambda]$ (see [4]). We are interested, however, in understanding how, the isomorphism class of the Abelian extension defined by $[\Lambda]$, might be preserved, even if $[\Lambda]$ is moved into a different cohomology class $[\Lambda'] = \gamma[\Lambda]$ under a group action that might move the cochain complex $C(\mathfrak{h}; \mathfrak{j})$ into $C(\mathfrak{h}; \mathfrak{j})'$ by moving the representation $R$ into $R' = \gamma R$. 
Thus, we shall address the question of finding the most general conditions on a linear map $\Psi : \mathfrak{h}(\Lambda, R) \to \mathfrak{h}(\Lambda', R')$ to be a Lie algebra isomorphism. We shall assume, however, that the Abelian ideal $\mathfrak{j}$ in $\mathfrak{h}(\Lambda, R)$ is the same as in $\mathfrak{h}(\Lambda', R')$; that is, we shall assume that $\mathfrak{j}$ is somehow canonically defined; e.g., as in §2.2 below. Under this assumption, $\Psi$ has the form,

\[
\begin{align*}
\Psi(x) &= g(x) + \Theta(x), \quad x \in \mathfrak{h}, \\
\Psi(v) &= \sigma(v), \quad v \in \mathfrak{j},
\end{align*}
\]

where $g : \mathfrak{h} \to \mathfrak{h}$, $\sigma : \mathfrak{j} \to \mathfrak{j}$ and $\Theta : \mathfrak{h} \to \mathfrak{j}$ are linear maps, with $g$ and $\sigma$ invertible. The isomorphism condition on $\Psi$ is that,

\[
\begin{align*}
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)) + \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)) + \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
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\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)), \\
\Theta(\mathfrak{d}(x, y)) &= \mathfrak{d}(\Theta(x), \Theta(y)),
\end{align*}
\]

where the Lie bracket $[\cdot, \cdot]$ is the one defined in $\mathfrak{g} = \mathfrak{h}(\Lambda', R')$. The answer to the question of when is $\mathfrak{h}(\Lambda, R)$ isomorphic to $\mathfrak{h}(\Lambda', R')$ is given in the following:

1.1. **Proposition.** Let $\Lambda$ and $\Lambda'$ be 2-cocyles within the chain complexes $C(\mathfrak{h}; \mathfrak{j})$ and $C(\mathfrak{h}; \mathfrak{j})'$, associated to the representations $R$ and $R'$ of $\mathfrak{h}$ in $\mathfrak{j}$, respectively. Two Abelian extensions $\mathfrak{g} = \mathfrak{h}(\Lambda, R)$ and $\mathfrak{g}' = \mathfrak{h}(\Lambda', R')$ defined in the underlying vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$ as in (4) are isomorphic, if and only if, there are linear maps $g \in \text{GL}(\mathfrak{h})$, $\sigma \in \text{GL}(\mathfrak{j})$ and $\Theta \in \text{Hom}_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{j})$, such that for all $x, y \in \mathfrak{h}$ and $v \in \mathfrak{j}$:

\[
\begin{align*}
g(\mathfrak{d}(x, y)) &= [g(x), g(y)]_\mathfrak{h} \\
\Theta(\mathfrak{d}(x, y)) + \sigma(\Lambda(x, y)) &= \Lambda'(g(x), g(y)) + R'(g(x))(\Theta(y)) - R'(g(y))(\Theta(x))
\end{align*}
\]

In other words, if and only if $g \in \text{Aut}(\mathfrak{h})$, $\sigma \in \text{GL}(\mathfrak{j})$ and $\Theta \in \text{Hom}_{\mathfrak{g}}(\mathfrak{h}, \mathfrak{j})$, satisfy,

\[
(\mathfrak{d}, \mathfrak{d}')(g, \sigma).\Lambda = \Lambda' + \mathfrak{d}'(g, \Theta), \quad (\mathfrak{d}, \mathfrak{d}')(g, \sigma).R = R',
\]

where, $\mathfrak{d}'$ is the differential map of the cochain complex $C(\mathfrak{h}; \mathfrak{j})'$ associated to the representation $R'$, and

\[
\begin{align*}
g, \Theta(\cdot) &= \Theta \circ g^{-1}(\cdot), \\
(g, \sigma).\Lambda(\cdot, \cdot) &= \sigma \circ R(g^{-1}(\cdot)) \circ \sigma^{-1}, \\
(g, \sigma).\Lambda(\cdot, \cdot) &= \sigma \circ R(g^{-1}(\cdot)) \circ \sigma^{-1}.
\end{align*}
\]

Proof. The proof that (7) follows from (6), is immediate using (5), whereas (8) simply rewrites (7) using the obvious group action defined in (9) and the well known definitions of the differential maps on $C(\mathfrak{h}; \mathfrak{j})$ and $C(\mathfrak{h}; \mathfrak{j})'$ in terms of $R$ and $R'$, respectively. \(\square\)
1.2. Remark. Let \( G = \text{Aut} \mathfrak{h} \times \text{GL}(j) \). This result states what is needed for a map \( \Psi \) to be an isomorphism from \( \mathfrak{h} \) into \( \text{Hom}_{\text{g}} \) Observe how \( \Psi \) depends on the data \( \text{Hom} \) into account that the group elements \((g, \sigma) \in \text{Aut} \mathfrak{h} \times \text{GL}(j)\) and \( \Theta \in \text{Hom}_{\text{F}}(\mathfrak{h}, j) \). We have proved that \( \Psi \) is a Lie algebra isomorphism if and only if the equations (7) (equivalently (8)) are satisfied, taking into account that the group elements \((g, \sigma) \in \text{Aut} \mathfrak{h} \times \text{GL}(j)\) act on the data \( \{\Theta, \Lambda, R\} \) according to (9), even though \( \Theta \) is a component of \( \Psi \) itself. The point is that \( \Theta \) combines with \((g, \sigma)\) so as to produce the coboundary term shown in (8). In particular, this is consistent with the following factorization of the isomorphism \( \Psi : \mathfrak{h}(\Lambda, R) \to \mathfrak{h}(\Lambda', R') \):

\[
(10) \quad \Psi = \begin{pmatrix} g & 0 \\ \Theta & \sigma \end{pmatrix} = \left( \begin{pmatrix} \text{Id}_h & 0 \\ \Theta \circ g^{-1} & \text{Id}_j \end{pmatrix} \circ \begin{pmatrix} g & 0 \\ 0 & \sigma \end{pmatrix}, \quad (g, \sigma) \in G. \right.
\]

Now, the next result is also a simple and straightforward computation from the definitions involved.

1.3. Proposition. Let \( d \) and \( d' \) be the differential maps in the cochain complexes \( C(\mathfrak{h}; j) \) and \( C(\mathfrak{h}; j)' \) defined by the representations \( R : \mathfrak{h} \to \mathfrak{gl}(j) \), and \( R' : \mathfrak{h} \to \mathfrak{gl}(j) \), respectively. Let \( \gamma = (g, \sigma) \in G = \text{Aut} \mathfrak{h} \times \text{GL}(j) \), and let \( \Phi(\gamma) : C(\mathfrak{h}; j) \to C(\mathfrak{h}; j) \) be the \( G \)-action given by \( \lambda \mapsto \Phi(\gamma)(\lambda) = \gamma \lambda \), where

\[
(\gamma \lambda)(\cdot, \ldots, \cdot) = \sigma \left( \lambda(g^{-1}(\cdot), \ldots, g^{-1}(\cdot)) \right).
\]

Then,

\[
(11) \quad d' \circ \Phi(\gamma) = \Phi(\gamma) \circ d \iff R' = \gamma \cdot R,
\]

where,

\[
\gamma \cdot R(x) = \sigma \circ R(g^{-1}(x)) \circ \sigma^{-1}, \quad \forall x \in \mathfrak{h}.
\]

1.4. Remark. The property \( d' \circ \Phi(\gamma) = \Phi(\gamma) \circ d \) states that \( d \lambda = 0 \) if and only if \( d'(\Phi(\gamma) \lambda) = 0 \). From the statement of Prop. 1.1, we conclude that \( \mathfrak{h}(\Lambda, R) \simeq \mathfrak{h}(\Lambda', R') \) if and only if \( R' = \gamma \cdot R \) and \( \Phi(\gamma) \Lambda = \Lambda' + d'(g, \Theta) \), which certainly makes \( d'(\Phi(\gamma) \Lambda) = 0 \) when \( d' \Lambda' = 0 \). That is, the subgroups defined by the cocycles and the coboundaries, respectively, are stable under the group action \( \Phi(\gamma) \). It is also clear that if \( R' \) is not in the \( G \)-orbit of \( R \), then \( \mathfrak{h}(\Lambda, R) \) cannot be isomorphic to \( \mathfrak{h}(\Lambda', R') \).
The next result deals with a special case of Prop. 1.1; namely, the case when the differential map in the cochain complex $C(h; j)$ is fixed because the representation $R$ is fixed. Most of the classical results for Abelian extensions are obtained within a single cohomology theory through a fixed representation. By comparing Prop. 1.1 with Prop. 1.5 below, it is clear that by restricting the framework to a single cohomology theory, there will be several Lie algebras in the isomorphism class of $\mathfrak{h}(\Lambda, R)$ that can never be reached by changing the cocycle $\Lambda$ in the form, $\Lambda \mapsto \Lambda' = \Phi(\gamma)\Lambda$ modulo a coboundary. This is the difference between (13) below and the most general relationship found in (8).

1.5. Proposition. Fix the representation $R : h \to \mathfrak{gl}(j)$ and restrict the group action to pairs $(g, \sigma)$ in the isotropy subgroup $G_R \subset G$ of $R$. Let $d$ be the differential map of the cochain complex $C(h; j)$. Then

$$d \circ \Phi(\gamma) = \Phi(\gamma) \circ d \iff \gamma = (g, \sigma) \in G_R;$$

that is, $d$ is $G_R$-equivariant. In particular, if $\Lambda$ and $\Lambda'$ are 2-cocycles, then $h(\Lambda, R) \simeq h(\Lambda', R)$ if and only if there are maps $g \in \text{Aut}(h)$, $\sigma \in \text{GL}(j)$ and $\Theta \in \text{Hom}(h, j)$, such that,

$$\Phi(\gamma)(\Lambda) = \Lambda' + d(g, \Theta), \quad \text{with} \quad \gamma = (g, \sigma) \in G_R.$$

2. Extensions Defined by Two Canonical Abelian Ideals

2.1. Isomorphisms of Abelian Extensions Defined by Two Canonical Ideals. Let $g$ be a Lie algebra with Lie bracket $[\cdot, \cdot]$. We shall show in §2.2 below how to define two characteristic ideals, $i = i(g)$ and $j = j(g)$ of $g$, satisfying the following properties:

$$[a, b] = [a, b] \quad \text{is abelian;} \quad (b) \quad i \subset j; \quad (c) \quad [g, j] \subset i.$$

We shall also see in Lemma 2.6 that if $g$ admits an invariant metric, then $i^1 = j$. For the time being, however, we shall first restrict ourselves to the properties (a), (b), and (c) in (14).

Let $h = g/j$, and decompose $g$ in the form $g = h \oplus j$. We shall also assume that $i \neq j$, and therefore, $a = j/i \neq \{0\}$. Thus, we may further decompose $g$ in the form $g = h \oplus a \oplus i$, and write its elements as,

$$g \ni x + v + \theta; \quad x \in h, \quad v \in a, \quad \theta \in i, \quad v + \theta \in j.$$

Using the fact that $j$, and hence $i$ by (b) in (14), are Abelian, we have,

$$[x + u + \theta, y + v + \eta] = [x, y] + [x, v + \eta] - [y, u + \theta],$$

where $[x, y] \in g = h \oplus j$, and $[x, v + \eta]$ and $[y, u + \theta]$ belong to $i$ because of (c) in (14). Now, decompose $\Lambda(x, y)$ in the form, $\Lambda(x, y) = \lambda(x, y) + \mu(x, y)$, with $\lambda(x, y) \in a$ and $\mu(x, y) \in i$, respectively. Moreover, the
representation \( R : \mathfrak{h} \to \mathfrak{gl}(j) = \mathfrak{gl}(a \oplus i) \) also decomposes by means of (c), into \([x, v] = \varphi(x)(v) \in i\) and \([x, \theta] = \rho(x)(\theta) \in i\), respectively, for any \( x \in \mathfrak{h} \). That is, (15) has now the following finer structure produced by the ideals \( i \) and \( j \):

\[
\begin{align*}
[x, y] &= [x, y]_{\mathfrak{h}} + \lambda(x, y) + \mu(x, y), \\
[x, v] &= \varphi(x)(v), \\
[x, \theta] &= \rho(x)(\theta), \\
[v, w] &= [v, \eta] = [\theta, \eta] = 0.
\end{align*}
\]

(16)

for any \( x, y \in \mathfrak{h} \), \( v, w \in a \) and \( \theta, \eta \in i \). In particular, it follows that, for each \( x \in \mathfrak{h} \), \( R(x) : a \oplus i \to a \oplus i \), is given by

\[
R(x) = \begin{pmatrix} 0 & 0 \\ \varphi(x) & \rho(x) \end{pmatrix} : \begin{pmatrix} v \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \varphi(x)(v) + \rho(x)(\theta) \end{pmatrix}.
\]

Moreover, the property (2) yields the following identities:

\[
\begin{align*}
\varphi([x, y]) &= \varphi([x, y]_{\mathfrak{h}}) = \rho(x) \circ \varphi(y) - \rho(y) \circ \varphi(x), \\
\rho([x, y]) &= \rho([x, y]_{\mathfrak{h}}) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x),
\end{align*}
\]

(18)

for all \( x, y \in \mathfrak{h} \). The equalities from the middle terms to the left hand sides follow from \( R([x, y]) = R([x, y]_{\mathfrak{h}}) \) which is a consequence of the fact that the ideal \( j \) is Abelian. We shall use, however, the equalities from the middle terms to the right hand sides. They state that \( \rho : \mathfrak{h} \to \mathfrak{gl}(i) \) is a representation of \( \mathfrak{h} \) and that \( \varphi \) is a 1-cocycle in \( C(\mathfrak{h}; \text{Hom}_{\mathbb{F}}(a, i)) \) for the representation \( \tilde{\rho} : \mathfrak{h} \to \mathfrak{gl}(\text{Hom}_{\mathbb{F}}(a, i)) \) defined by \( \tilde{\rho}(x)(\tau) = \rho(x) \circ \tau \) on cochains \( \tau : \mathfrak{h} \times \cdots \times \mathfrak{h} \to \text{Hom}_{\mathbb{F}}(a, i) \). Observe that the representation \( \tilde{\rho} : \mathfrak{h} \to \mathfrak{gl}(\text{Hom}_{\mathbb{F}}(a, i)) \), is no other than the natural tensor product representation in \( i \otimes a^* \simeq \text{Hom}_{\mathbb{F}}(a, i) \), when \( a \) is the trivial \( \mathfrak{h} \)-module.

These properties on \( \varphi \) and \( \rho \) are needed for making \( \mathfrak{g} \) into a Lie algebra. Indeed, in order to look at the information contained in Jacobi identity, one may compute Lie brackets of the form,

\[
[[x + u + \theta, y + v + \eta], z + w + \xi],
\]

and then take the corresponding cyclic sum. It is a straightforward matter to show that the Lie bracket above is equal to,

\[
[[x, y], z] + \varphi([x, y])(w) + \rho([x, y])(\xi) + \rho(z)(\varphi(y)(u)) + \rho(z)(\rho(y)(\theta)) - \rho(z)(\rho(x)(\eta)) - \rho(z)(\rho(x)(\eta))
\]

The corresponding cyclic sum of three terms like this involves Jacobi identity for the Lie bracket in \( \mathfrak{g} \) of three elements in \( \mathfrak{h} \) and cyclic sums over the triples \([x, u, \theta], (y, v, \eta), (z, w, \xi)\) of terms similar to the last six in this expression. It is easy to verify that one is left with a sum of
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terms belonging to $i$ which vanish identically because of the properties of the representation $R$ of $g$ just observed in terms of $\varphi$ and $\rho$.

On the other hand, by writing down $[[x, y], z]$ for the triple $\{x, y, z\}$ of elements from $h$, but this time in terms of the $h$-component $[\cdot, \cdot]_h$ of the Lie bracket $[\cdot, \cdot]$ in $g$, the Jacobi identity will produce cyclic sums of three different expressions corresponding to the components along the direct summands $h$, $a$ and $i$, since,

$$[[x, y], z] = [[x, y]_h + \lambda(x, y) + \mu(x, y), z]$$

$$= [[x, y]_h, z]_h + \lambda([x, y]_h, z) + \mu([x, y]_h, z)$$

$$- [z, \lambda(x, y) + \mu(x, y)]$$

$$= [[x, y]_h, z]_h + \lambda([x, y]_h, z)$$

$$+ \mu([x, y]_h, z) - \varphi(z)(\lambda(x, y)) - \rho(z)(\mu(x, y)).$$

We therefore end up with,

$$\sum_{\odot\{x,y,z\}} [[x, y]_h, z]_h = 0, \quad \sum_{\odot\{x,y,z\}} \lambda([x, y]_h, z) = 0, \quad \sum_{\odot\{x,y,z\}} \{ \mu([x, y]_h, z) - \varphi(z)(\lambda(x, y)) - \rho(z)(\mu(x, y)) \} = 0.$$

The first is just Jacobi identity for the Lie algebra $h = g/j$. The second one is easy to understand for the skew-symmetric bilinear map $\lambda: h \times h \to a$ taking values in the trivial $h$-module $a$. In fact, ordinary Lie algebra cohomology lets us write

$$(d \lambda)(x, y, z) = -\sum_{\odot\{x,y,z\}} \lambda([x, y]_h, z) = 0.$$

On the other hand, for the skew-symmetric bilinear map $\mu: h \times h \to i$ into the $h$-module defined by the representation $\rho: h \to \mathrm{gl}(i)$, we have,

$$(d \mu)(x, y, z) = \sum_{\odot\{x,y,z\}} (\rho(z)(\mu(x, y)) - \mu([x, y]_h, z)).$$

In particular, the third cyclic sum in (19) states that,

$$(d \mu)(x, y, z) + \sum_{\odot\{x,y,z\}} \varphi(z)(\lambda(x, y)) = 0.$$

We may interprete the term $\sum_{\odot\{x,y,z\}} \varphi(z)(\lambda(x, y))$ as the result of applying a map $e_\varphi: C^2(h; a) \to C^3(h; i)$, induced by $\varphi: h \to \text{Hom}_F(a, i)$ on the chain complexes involved, as follows:

$$C^2(h; a) \ni \lambda(\cdot, \cdot) \mapsto e_\varphi(\lambda)(x, y, z) = \sum_{\odot\{x,y,z\}} \varphi(x)(\lambda(y, z)) \in C^3(h; i),$$
where the cyclic sum in the right hand side is taken over the arguments, thus producing an alternating trilinear map $e_\varphi(\lambda) : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h} \to \mathfrak{i}$ from the initial alternating map $\lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$. In general, one may define a degree-one map $e_\varphi$ of cochain complexes by means of,

$$C^n(\mathfrak{h}; \mathfrak{a}) \ni \lambda \mapsto e_\varphi(\lambda) \in C^{n+1}(\mathfrak{h}; \mathfrak{i}),$$

where, for any $x_1, \ldots, x_{n+1}$ in $\mathfrak{h}$,

$$e_\varphi(\lambda)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \varphi(x_i)(\lambda(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})). \tag{21}$$

The following result elucidates the behavior of $e_\varphi$ with respect to the corresponding differential maps on $C(\mathfrak{h}; \mathfrak{a})$ and $C(\mathfrak{h}; \mathfrak{i})$ which, for the statement and proof, we shall denote by $d_\mathfrak{a}$ and $d_\mathfrak{i}$, respectively.

**2.1. Proposition.** Let $(\mathfrak{h}, [\cdot, \cdot; \mathfrak{h}])$ be a Lie algebra. Let $\mathfrak{a}$ be a finite-dimensional trivial $\mathfrak{h}$-module and let $\mathfrak{i}$ be the $\mathfrak{h}$-module given by the representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{i})$. Let $\tilde{\rho} : \mathfrak{h} \to \mathfrak{gl}(\text{Hom}_\mathfrak{g}(\mathfrak{a}, \mathfrak{i}))$ be the tensor product representation, so that $\tilde{\rho}(x)(T) = \rho(x) \circ T$, for any $x \in \mathfrak{h}$ and any $T \in \text{Hom}_\mathfrak{g}(\mathfrak{a}, \mathfrak{i})$. Let $\varphi \in C(\mathfrak{h}; \text{Hom}_\mathfrak{g}(\mathfrak{a}, \mathfrak{i}))$ be a 1-cocycle with coefficients in the representation $\tilde{\rho}$. The degree-one map of cochain complexes $e_\varphi : C(\mathfrak{h}; \mathfrak{a}) \to C(\mathfrak{h}; \mathfrak{i})$ defined by (21), satisfies,

$$( e_\varphi \circ d_\mathfrak{a} )|_{C^{n-1}(\mathfrak{h}; \mathfrak{a})} = - (d_\mathfrak{i} \circ e_\varphi)|_{C^{n-1}(\mathfrak{h}; \mathfrak{a})},$$

for each $n \in \mathbb{N}$; that is, the following diagram anticommutes:

$$\begin{array}{c}
C^{n-1}(\mathfrak{h}; \mathfrak{a}) \xrightarrow{d_\mathfrak{a}} C^n(\mathfrak{h}; \mathfrak{a}) \\
\downarrow e_\varphi \quad \downarrow e_\varphi \\
C^{n}(\mathfrak{h}; \mathfrak{i}) \xrightarrow{d_\mathfrak{i}} C^{n+1}(\mathfrak{h}; \mathfrak{i}).
\end{array}$$

**Proof.** Consider $j = \mathfrak{a} \oplus \mathfrak{i}$, and the representation $R : \mathfrak{h} \to \mathfrak{gl}(j)$, given by, $R(x)(v + \theta) = \varphi(x)(v) + \rho(x)(\theta), \forall v \in \mathfrak{a}$ and $\forall \theta \in \mathfrak{i}$ (see (17)). The differential map $d$ in the cochain complex $C(\mathfrak{h}; j)$ defined by the representation $R : \mathfrak{h} \to \mathfrak{gl}(j)$, satisfies

$$d = \iota_\mathfrak{a} \circ d_\mathfrak{a} + \iota_\mathfrak{i} \circ (e_\varphi \oplus d_\mathfrak{i}) \leftrightarrow (d_\mathfrak{a}, e_\varphi \oplus d_\mathfrak{i}),$$

where $\iota_\mathfrak{a} : \mathfrak{a} \to j$ and $\iota_\mathfrak{i} : \mathfrak{i} \to j$ are the inclusion maps. The decomposition $j = \mathfrak{a} \oplus \mathfrak{i}$ makes $C(\mathfrak{h}; j)$ to decompose as $C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i})$ and therefore, the cochain complex $(C(\mathfrak{h}; j), d)$ becomes isomorphic to $(C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i}), (d_\mathfrak{a}, e_\varphi \oplus d_\mathfrak{i}))$. Then,

$$( e_\varphi \circ d_\mathfrak{a} )|_{C^{n-1}(\mathfrak{h}; \mathfrak{a})} = - (d_\mathfrak{i} \circ e_\varphi)|_{C^{n-1}(\mathfrak{h}; \mathfrak{a})}, \quad \forall n \in \mathbb{N},$$

follows from the fact that $(d \circ d)|_{C^n(\mathfrak{h}; j)} = 0$. \qed
2.2. **Remark.** Write \( \Lambda = \lambda \oplus \mu \), for any \( \Lambda \in C(\mathfrak{h} ; j) \), with \( \lambda \in C(\mathfrak{h} ; \mathfrak{a}) \) and \( \mu \in C(\mathfrak{h} ; i) \). In view of Prop. 2.1 and the specific form of the representation \( R \) given in (17) in terms of \( \varphi \) and \( \rho \), we shall write

\[
(d \Lambda)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} R(x_i)(\Lambda(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}))
\]

\[
+ \sum_{i < j} (-1)^{i+j} \Lambda([x_i, x_j]_\mathfrak{h}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1})
\]

\[
= (d_\lambda \lambda)(x_1, \ldots, x_{n+1})
\]

\[
\oplus (e_\varphi(\lambda)(x_1, \ldots, x_{n+1}) + (d_\mu \mu)(x_1, \ldots, x_{n+1}))
\]

\[
\leftrightarrow \left( \begin{array}{c} (d_\lambda \lambda)(x_1, \ldots, x_{n+1}) \\ e_\varphi(\lambda)(x_1, \ldots, x_{n+1}) + (d_\mu \mu)(x_1, \ldots, x_{n+1}) \end{array} \right)
\]

\[
= \left( \begin{array}{c} d_\lambda \\ e_\varphi \end{array} \right) \left( \begin{array}{c} \lambda \\ \mu \end{array} \right)(x_1, \ldots, x_{n+1})
\]

In other words, the differential map \( d \) of \( C(\mathfrak{h} ; j) \) gets identified with the operator \( D_\varphi = \left( \begin{array}{cc} d_\lambda & 0 \\ e_\varphi & d_\mu \end{array} \right) \) that acts on \( C(\mathfrak{h} ; \mathfrak{a}) \oplus C(\mathfrak{h} ; i) \). From now on we shall omit the explicit reference to \( \mathfrak{a} \) in \( d_\lambda \) and to \( i \) in \( d_\mu \) and simply write \( d \), as their meaning is clear from the context of the operator \( D_\varphi \).

We may now summarize what we have done so far in this section in the following statement:

**2.3. Corollary.** Define a skew-symmetric bilinear map \([\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) on the underlying vector space \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i} \), by means of (16), where \( \lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a} \) and \( \mu : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{i} \) are 2-cochains in the complexes \( C(\mathfrak{h}; \mathfrak{a}) \) and \( C(\mathfrak{h}; \mathfrak{i}) \) associated to the trivial representation of \( \mathfrak{h} \) in \( \mathfrak{a} \) and to the representation \( \rho \) of \( \mathfrak{h} \) in \( \mathfrak{i} \), respectively. Let \( \varphi : \mathfrak{h} \to \text{Hom}_F(\mathfrak{a}, \mathfrak{i}) \) be a 1-cochain in the complex \( C(\mathfrak{h}; \text{Hom}_F(\mathfrak{a}, \mathfrak{i})) \) associated to the representation \( \tilde{\rho} : \mathfrak{h} \to \text{gl}(\text{Hom}_F(\mathfrak{a}, \mathfrak{i})) \) defined by \( \tilde{\rho}(x)(\tau) = \rho(x) \circ \tau \). Then, \([\cdot, \cdot]\) is a Lie algebra bracket on \( \mathfrak{g} \) if and only if

\[
(22) \quad d \lambda = 0, \quad d \mu + e_\varphi(\lambda) = 0, \quad d \varphi = 0,
\]

where \( e_\varphi : C(\mathfrak{h}; \mathfrak{a}) \to C(\mathfrak{h}; \mathfrak{i}) \) is the degree-one map of cochain complexes defined in (21). Moreover, (22) can be rewritten in terms of the differential map \( D_\varphi = \left( \begin{array}{cc} d_\lambda & 0 \\ e_\varphi & d_\mu \end{array} \right) \) acting on \( C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i}) \to C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i}) \), so that, \([\cdot, \cdot]\) is a Lie algebra bracket on \( \mathfrak{g} \) if and only if

\[
(23) \quad D_\varphi \left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \quad \text{and} \quad d \varphi = 0.
\]
Corollary to Prop. 1.1, we must have,

\[
(24) \quad \sigma = \begin{pmatrix} h & 0 \\ T & k \end{pmatrix}, \quad \text{with,} \quad h \in \text{GL}(a), \; k \in \text{GL}(i), \; T \in \text{Hom}_F(a, i).
\]

and we shall also write, \(\Theta(x) = \tau(x) \oplus \nu(x) \leftrightarrow \left(\begin{array}{c} \tau(x) \\ \nu(x) \end{array}\right)\) for the map \(\Theta : h \to a \oplus i\), in Prop. 1.1. Taking into account the finer structure brought by the decomposition \(j = a \oplus i\), the specific form of \(\sigma\) in (24) and the dependence of \(\Lambda\) and \(R\) on the data \((\lambda, \mu)\) and \((\phi, \rho)\), respectively, we shall write \(h(\lambda, \mu, \phi, \rho)\) instead of \(h(\Lambda, R)\) and state the following Corollary to Prop. 1.1 and Prop. 1.3:

2.4. Corollary. Suppose \(g = h(\lambda, \mu, \phi, \rho)\) and \(g' = h(\lambda', \mu', \phi', \rho')\) are two Lie algebras whose Lie brackets \([\cdot, \cdot]\) and \([\cdot, \cdot]\)' are defined on the underlying vector space \(h \oplus a \oplus i\), in terms of the data satisfying the conditions of Prop. 1.1 and Prop. 1.3. Furthermore, assume that \(i = a \oplus i\) are two canonically defined Abelian ideals satisfying \([g, j] \subset i\) and \([g', j'] \subset i\). Then, \(g\) and \(g'\) are isomorphic if and only if there exist \(g \in \text{Aut}(h), \; h \in \text{GL}(a), \; k \in \text{GL}(i)\) and linear maps \(\tau : h \to a, \; \nu : h \to i\) and \(T : a \to i\) such that

\[
\Phi(\gamma) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} + \begin{pmatrix} d & 0 \\ e_{\phi'} & d' \end{pmatrix} \begin{pmatrix} \tau \circ g^{-1} \\ \nu \circ g^{-1} \end{pmatrix} =: \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} + \text{D}'_{\phi'} \begin{pmatrix} \tau \circ g^{-1} \\ \nu \circ g^{-1} \end{pmatrix},
\]

(25) \(\Phi(\gamma) \phi = \phi' + d(T \circ h^{-1})\) and \(\Phi(\gamma) \rho = \rho'\),

where \(\gamma = (g, \sigma)\) and \(\sigma = \left(\begin{array}{c} h \\ T \\ k \end{array}\right)\), just as in (24), and

\[
\begin{pmatrix} \Phi(\gamma) \left(\begin{array}{c} \lambda \\ \mu \end{array}\right) \end{pmatrix} (x, y) = \begin{pmatrix} h \\ T \\ k \end{pmatrix} \begin{pmatrix} \lambda(g^{-1}(x), g^{-1}(y)) \\ \mu(g^{-1}(x), g^{-1}(y)) \end{pmatrix},
\]

\[
(\Phi(\gamma) \phi)(x) = k \circ \phi(g^{-1}(x)) \circ h^{-1},
\]

\[
(\Phi(\gamma) \rho)(x) = k \circ \rho(g^{-1}(x)) \circ k^{-1}.
\]

Moreover \(\text{D}'_{\phi'} \circ \Phi(\gamma) = \Phi(\gamma) \circ \text{D}_{\phi}\), if and only if the relations (25) hold true.

We may now close this section by showing how to produce in a canonical way the abelian ideals \(i\) and \(j\) of \(g\) that satisfy (14).
2.2. Definition of the Canonical Ideals \(i(\mathfrak{g})\) and \(j(\mathfrak{g})\) for a Lie Algebra \(\mathfrak{g}\). We shall adhere ourselves to the standard convention of writing,

\[ C(\mathfrak{g}) = \{ z \in \mathfrak{g} \mid [z, x] = 0, \text{ for all } x \in \mathfrak{g} \}, \]

for the center of the Lie algebra \(\mathfrak{g}\) with Lie bracket \([\cdot, \cdot]\). The descending central series \(\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^{\ell} \supset \cdots\) of \(\mathfrak{g}\) is defined by,

\[ \mathfrak{g}^0 = \mathfrak{g}; \quad \mathfrak{g}^\ell = [[\mathfrak{g}, \mathfrak{g}^{\ell-1}], \ell \geq 1. \]

The derived central series \(C_1(\mathfrak{g}) \supset C_2(\mathfrak{g}) \supset \cdots \supset C_{\ell-1}(\mathfrak{g}) \supset C_{\ell}(\mathfrak{g}) \supset \cdots\) of \(\mathfrak{g}\) is defined by,

\[ C_1(\mathfrak{g}) = C(\mathfrak{g}); \quad C_{\ell}(\mathfrak{g}) = \pi_{\ell-1}^{-1}C(\mathfrak{g}/C_{\ell-1}(\mathfrak{g})), \ell \geq 1, \]

where \(\pi_{\ell-1} : \mathfrak{g} \rightarrow \mathfrak{g}/C_{\ell-1}(\mathfrak{g})\) is the corresponding canonical projection.

The first result in this section might be well known for experts. Actually, we may refer the reader to [2], [6] or [8] for at least the fact that for any \(\ell \in \mathbb{N}\), \(C(\mathfrak{g}^{\ell-1}) = C_{\ell}(\mathfrak{g}) = (\mathfrak{g}^{\ell})^{\perp}\) when \(\mathfrak{g}\) admits an invariant metric. The main statement \(C_{\ell}(\mathfrak{g}) \subseteq C(\mathfrak{g}^{\ell-1})\) holds true in general with no need of any invariant metric at all and it can be proved in a straightforward manner by induction on \(\ell\). We may safely omit the details.

2.5. Proposition. Let \(\mathfrak{g}\) be a Lie algebra with Lie bracket \([\cdot, \cdot]\) and let \(C_{\ell}(\mathfrak{g})\) be the \(\ell\)-th ideal in its derived central series. Then,

\[ C_{\ell}(\mathfrak{g}) \subseteq C(\mathfrak{g}^{\ell-1}) := \{ x \in \mathfrak{g} \mid [x, \mathfrak{g}^{\ell-1}] = \{0\} \}, \forall \ell \in \mathbb{N}. \]

Furthermore, if \(\mathfrak{g}\) admits an invariant metric, then \(C_{\ell}(\mathfrak{g}) = C(\mathfrak{g}^{\ell-1}) = (\mathfrak{g}^{\ell})^{\perp}\), for all \(\ell \in \mathbb{N}\).

Now, the following Lemma defines, in a canonical way for any Lie algebra \(\mathfrak{g}\), the two ideals, \(i(\mathfrak{g})\) and \(j(\mathfrak{g})\), whose properties form the basis of this work.

2.6. Lemma. Let \(\mathfrak{g}\) be an Lie algebra over a field of characteristic zero and let \([\cdot, \cdot]\) be its Lie bracket. Define,

\[ i(\mathfrak{g}) = \bigoplus_{k \in \mathbb{N}} C_k(\mathfrak{g}) \cap \mathfrak{g}^k \quad \text{and} \quad j(\mathfrak{g}) = \bigcap_{k \in \mathbb{N}} (C_k(\mathfrak{g}) + \mathfrak{g}^k). \]

Then,

(i) \(i(\mathfrak{g}) \subset j(\mathfrak{g})\).
(ii) \(j(\mathfrak{g}) = C(\mathfrak{g}) + \bigoplus_{k \in \mathbb{N}} C_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k\).
(iii) \([\mathfrak{g}, j(\mathfrak{g})] \subset i(\mathfrak{g})\).
(iv) \(j(\mathfrak{g})\) is Abelian.
(v) If $\mathfrak{g}$ admits an invariant metric, then $i(\mathfrak{g}) = j(\mathfrak{g})$.

Proof. (i) Let $j, k, \ell \in \mathbb{N}$ be such that $j \leq k \leq \ell$. Then $C_k(\mathfrak{g}) \cap \mathfrak{g}^k \subset \mathfrak{g}^k \subset \mathfrak{g}^j \subset \mathfrak{g}^j + C_j(\mathfrak{g})$. Similarly, $C_k(\mathfrak{g}) \cap \mathfrak{g}^k \subset C_k(\mathfrak{g}) \subset C_\ell(\mathfrak{g}) \subset C_\ell(\mathfrak{g}) + \mathfrak{g}^\ell$. Thus, $C_k(\mathfrak{g}) \cap \mathfrak{g}^k \subset C_\ell(\mathfrak{g}) + \mathfrak{g}^\ell$, for all $k, \ell \in \mathbb{N}$. Whence, $C_k(\mathfrak{g}) \cap \mathfrak{g}^k \subset \bigcap_{\ell \in \mathbb{N}} (C_\ell(\mathfrak{g}) + \mathfrak{g}^\ell)$, and therefore, $\sum_{k \in \mathbb{N}} (C_k(\mathfrak{g}) \cap \mathfrak{g}^k) \subset \bigcap_{\ell \in \mathbb{N}} (C_\ell(\mathfrak{g}) + \mathfrak{g}^\ell)$.

(ii) By definition, $j(\mathfrak{g}) = (C_1(\mathfrak{g}) + \mathfrak{g}^1) \bigcap \left( \bigcap_{k \geq 2} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right)$. Observe that $C_1(\mathfrak{g}) \subset C_k(\mathfrak{g}) + \mathfrak{g}^k$, for all $k \geq 2$. Thus, $C_1(\mathfrak{g}) \subset \bigcap_{k \geq 2} (C_k(\mathfrak{g}) + \mathfrak{g}^k)$.

It is a well known fact that for any triple of vector subspaces $U, V$ and $W$ of $\mathfrak{g}$, $U \subset W \Rightarrow (U + W) \cap V = U + W \cap V$. We shall refer to this result as the $+ \cap$ distribution property. In particular,

$$j(\mathfrak{g}) = C_1(\mathfrak{g}) + \mathfrak{g}^1 \bigcap \left( \bigcap_{k \geq 2} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right) = C_1(\mathfrak{g}) + \mathfrak{g}^1 \bigcap (C_2(\mathfrak{g}) + \mathfrak{g}^2) \bigcap \left( \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right).$$

Now, $C_2(\mathfrak{g}) \subset C_k(\mathfrak{g}) + \mathfrak{g}^k$ (for all $k \geq 3$) $\Rightarrow C_2(\mathfrak{g}) \subset \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k)$. Thus, we apply the $+ \cap$ distribution property to $(C_2(\mathfrak{g}) + \mathfrak{g}^2) \bigcap \left( \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right)$ and get,

$$j(\mathfrak{g}) = C_1(\mathfrak{g}) + \mathfrak{g}^1 \bigcap \left( C_2(\mathfrak{g}) + \mathfrak{g}^2 \bigcap \left( \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right) \right).$$

Since $\mathfrak{g}^2$ is contained in $\mathfrak{g}^1$, we can apply again the $+ \cap$ distribution property to $\mathfrak{g}^1 \bigcap \left( C_2(\mathfrak{g}) + \mathfrak{g}^2 \bigcap \left( \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right) \right)$, and get,

$$j(\mathfrak{g}) = C_1(\mathfrak{g}) + \mathfrak{g}^1 \cap C_2(\mathfrak{g}) + \mathfrak{g}^2 \bigcap \left( \bigcap_{k \geq 3} (C_k(\mathfrak{g}) + \mathfrak{g}^k) \right).$$

It is clear how to make this argument inductive to conclude that,

$$j(\mathfrak{g}) = C_1(\mathfrak{g}) + \sum_{k \in \mathbb{N}} C_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k.$$
(iii) By (ii), we have:

\[ [\mathfrak{g}, \mathfrak{j}(\mathfrak{g})] = \sum_{k \in \mathbb{N}} [\mathfrak{g}, C_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k] \subset \sum_{k \in \mathbb{N}} C_k(\mathfrak{g}) \cap \mathfrak{g}^k = \mathfrak{i}(\mathfrak{g}). \]

(iv) Using (ii) and the + ∩ distribution property, we have,

\[ \mathfrak{j}(\mathfrak{g}) = C(\mathfrak{g}) + \sum_{k} C_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k \subset C(\mathfrak{g}) + \sum_{k \in \mathbb{N}} C(\mathfrak{g}^k) \cap \mathfrak{g}^k. \]

Let \( k, \ell \in \mathbb{N} \), with \( k \leq \ell \). Then \( \mathfrak{g}^k \supseteq \mathfrak{g}^\ell \) and \( C(\mathfrak{g}^k) \subset C(\mathfrak{g}^\ell) \). Let \( x \in C_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k \) and \( y \in C_{\ell+1}(\mathfrak{g}) \cap \mathfrak{g}^\ell \). Then \( [x, y] \in [C_{k+1}(\mathfrak{g}), \mathfrak{g}^k] \subset [C(\mathfrak{g}^k), \mathfrak{g}^k] \subset [C(\mathfrak{g}^\ell), \mathfrak{g}^\ell] = \{0\} \). Therefore, \( [\mathfrak{j}(\mathfrak{g}), \mathfrak{j}(\mathfrak{g})] = \{0\} \).

(v) Let \( B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \) be an invariant metric. This time we shall use the + ∩ distribution properties satisfied for any pair of vector subspaces \( V \) and \( W \) of a quadratic \( \mathfrak{g} \); namely, \( (V+W)^\perp = V^\perp \cap W^\perp \) and \( (V \cap W)^\perp = V^\perp + W^\perp \), respectively. Therefore,

\[ i(\mathfrak{g})^\perp = \left( \sum_{k \in \mathbb{N}} C_k(\mathfrak{g}) \cap \mathfrak{g}^k \right)^\perp = \bigcap_{k \in \mathbb{N}} \left( C_k(\mathfrak{g}) \cap \mathfrak{g}^k \right)^\perp = \bigcap_{k \in \mathbb{N}} \left( \mathfrak{g}^k + C_k(\mathfrak{g}) \right) = \mathfrak{i}(\mathfrak{g}). \]

\[ \square \]

**Note.** It is worth noting that the definition of \( \mathfrak{i}(\mathfrak{g}) \) given in Lemma 2.6 coincides with the characterization given in [5] when the Lie algebra \( \mathfrak{g} \) is nilpotent.

### 3. Quadratic Lie Algebras with Non-trivial Center

Let \( \mathfrak{g} \) be a quadratic Lie algebra with Lie bracket \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) and invariant metric \( B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \). Let \( \mathfrak{i} := \mathfrak{i}(\mathfrak{g}) \) and \( \mathfrak{j} := \mathfrak{j}(\mathfrak{g}) \) be the characteristic ideals defined in Lemma 2.6. We shall assume that \( C(\mathfrak{g}) \neq \{0\} \). It follows from the invariance and non-degeneracy of \( B \) that \( \mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}] \), and in fact, \( C(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp \). We also know from Lemma 2.6 that \( \mathfrak{i}^\perp = \mathfrak{i} \circ \mathfrak{i} \). Thus, \( \mathfrak{i} \) is a canonically defined isotropic ideal of \( \mathfrak{g} \). By Witt decomposition, \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i} \), with \( \mathfrak{h} \) isotropic and \( \mathfrak{i}^\perp = \mathfrak{a} \oplus \mathfrak{i} \). In particular, \( \mathfrak{a} \) is non-degenerate and \( \mathfrak{a}^\perp = \mathfrak{h} \oplus \mathfrak{i} \).

Observe that \( B|_{\mathfrak{h} \times \mathfrak{i}} : \mathfrak{h} \times \mathfrak{i} \to \mathbb{F} \) cannot degenerate and one may identify \( \mathfrak{i} \) with \( \mathfrak{h}^* \) (or else, identify \( \mathfrak{h} \) with \( \mathfrak{i}^* \), which at the end is a matter of convenience depending on the data one wants to start with). The identification of \( \mathfrak{i} \) with \( \mathfrak{h}^* \) goes as follows: Start with \( \theta \in \mathfrak{i} \) and consider the map \( \theta \to \theta^\theta \in \mathfrak{h}^* \) defined by \( \theta^\theta(x) = B(\theta, x) \), for any \( x \in \mathfrak{h} \). This
map is equivariant and intertwines the representation $\rho$ of $\mathfrak{h}$ in $\mathfrak{i}$ with the coadjoint representation $\text{ad}^*_\mathfrak{h}$ of $\mathfrak{h}$ in $\mathfrak{h}^*$. Indeed, since, $i^{\perp} = a \oplus i$,

$$(\rho(x)(\theta))^\flat(y) = B(\rho(x)(\theta), y) = B([x, \theta], y)$$

$$= -B(\theta, [x, y]) = -B(\theta, [x, y]_\mathfrak{h})$$

$$= -(\theta^\flat)([x, y]_\mathfrak{h}) = -(\theta^\flat) \circ \text{ad}_\mathfrak{h}(x)(y)$$

$$= (\text{ad}^*_\mathfrak{h}(x) \theta^\flat)(y).$$

**Convention.** Having assumed that $\mathfrak{g} = \mathfrak{h} \oplus a \oplus i$ is quadratic with invariant symmetric bilinear form $B$, we shall, from now on, identify $i$ with $\mathfrak{h}^*$ and will use the coadjoint representation in $\mathfrak{h}^*$ instead of $\rho$.

We now want to look at the decomposition of the representation $R: \mathfrak{h} \to \mathfrak{gl}(a \oplus \mathfrak{h}^*)$ into the map $\varphi : \mathfrak{h} \to \text{Hom}_F(a, \mathfrak{h}^*)$, and the coadjoint representation $\text{ad}^*_\mathfrak{h}: \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)$. Observe that for any $y \in \mathfrak{h}$,

$$B(\varphi(x)(v), y) = B([x, v], y) = -B(v, [x, y])$$

$$= -B(v, [x, y]_\mathfrak{h} + \lambda(x, y) + \mu(x, y))$$

$$= -B(v, \lambda(x, y)).$$

This says that $a^* \ni -\lambda(x, y)^\flat = B(\varphi(x)(\cdot), y)$ is related to the map, $\varphi(x)^* : \mathfrak{h} \to a^*$, through the following:

$$(26) \quad -\lambda(x, y)^\flat(v) = (\varphi(x)^*)(v) = (\varphi(x)(v))(y).$$

In other words, when the Lie algebra is quadratic, $\lambda$ can be built up from $\varphi$ or viceversa, and only one of the two is needed in specifying the Abelian extension of **Cor. 2.3**.

Finally, observe that for any triple $x$, $y$ and $z$ in $\mathfrak{h}$, we get:

$$\mu(x, y)^\flat(z) = B(\mu(x, y), z) = B([x, y], z)$$

$$= B(x, [y, z]) = B(x, \mu(y, z)) = \mu^\flat(y, z)(x).$$

Therefore, under the identification of $i$ with $\mathfrak{h}^*$, the 2-cochain $\mu$ with values in $\mathfrak{h}^*$, has the cyclic property,

$$\mu(x, y)(z) = \mu(y, z)(x).$$

It is worth noting that cyclic cochains are found in several related contexts; see for example [1] or [3]. The cyclic property appears here due to the existence of an invariant metric. We may now rephrase the main result on Abelian extensions for the case in which $\mathfrak{g} = \mathfrak{h}(\lambda, \mu, \varphi, \rho)$ admits an invariant metric $B$, by stating the following theorem:
3.1. Theorem. (1) Let \((\mathfrak{h}, [\cdot, \cdot]_\mathfrak{h})\) be a finite dimensional Lie algebra. Let \(\mathfrak{h}\) act on the dual vector space \(\mathfrak{h}^*\) through the coadjoint representation \(\text{ad}^*_\mathfrak{h} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)\). Let \(\mathfrak{a}\) be a finite dimensional trivial \(\mathfrak{h}\)-module. Let \(\text{ad}^*_\mathfrak{a} : \mathfrak{a} \to \mathfrak{gl}(\text{Hom}_\mathfrak{a}(\mathfrak{h}, \mathfrak{h}^*))\) be the tensor product representation on \(\mathfrak{h}^* \otimes \mathfrak{a}^* \simeq \text{Hom}_\mathfrak{a}(\mathfrak{h}, \mathfrak{h}^*)\) so that \(\text{ad}^*_\mathfrak{h}(x)(T) = \text{ad}^*_\mathfrak{a}(x) \circ T\), for all \(x \in \mathfrak{h}\) and all \(T \in \text{Hom}_\mathfrak{a}(\mathfrak{h}, \mathfrak{h}^*)\). Let \(\varphi : \mathfrak{h} \to \text{Hom}_\mathfrak{a}(\mathfrak{h}, \mathfrak{h}^*)\) be a 1-cocycle with coefficients in the representation \(\text{ad}^*_\mathfrak{h}\) and let \(e_\varphi : C(\mathfrak{h}, \mathfrak{a}) \to C(\mathfrak{h}, \mathfrak{h}^*)\), be the degree-one map of cochain complexes defined in (21). Finally, let \(D_\varphi : C(\mathfrak{h}, \mathfrak{a}) \oplus C(\mathfrak{h}, \mathfrak{h}^*) \to C(\mathfrak{h}, \mathfrak{a}) \oplus C(\mathfrak{h}, \mathfrak{h}^*)\) be the differential map,

\[
D_\varphi \left( \frac{\lambda}{\mu} \right) = \left( \frac{d\lambda}{d\mu + e_\varphi(\lambda)} \right).
\]

Given a 2-cochain \((\frac{\lambda}{\mu})\) define the following skew-symmetric bilinear map \([\cdot, \cdot]_{\{\lambda, \mu\}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} :\n
\forall x, y \in \mathfrak{h}, \quad [x, y]_{\{\lambda, \mu\}} = [x, y]_\mathfrak{h} + \lambda(x, y) + \mu(x, y),

\forall x \in \mathfrak{h}, \forall v \in \mathfrak{a}, \quad [x, v]_{\{\lambda, \mu\}} = \varphi(x)(v),

\forall x \in \mathfrak{h}, \forall \alpha \in \mathfrak{h}^*, \quad [x, \alpha]_{\{\lambda, \mu\}} = \text{ad}^*_\mathfrak{a}(x)(\alpha),

\forall u, v \in \mathfrak{a}, \forall \alpha, \beta \in \mathfrak{h}^*, \quad [u + \alpha, v + \beta]_{\{\lambda, \mu\}} = 0.

Then \([\cdot, \cdot]_{\{\lambda, \mu\}}\) defines a Lie bracket in \(\mathfrak{g}\) if and only if \((\frac{\lambda}{\mu}) \in \text{Ker} D_\varphi\).

(2) Let \(B_\alpha : \mathfrak{a} \times \mathfrak{a} \to \mathbb{F}\) be a symmetric and non-degenerate bilinear form on \(\mathfrak{a}\). Let \(B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}\) be the bilinear form defined by

\[
B(x + u + \alpha, y + v + \beta) = \alpha(y) + \beta(x) + B_\alpha(u, v),
\]

for all \(x, y \in \mathfrak{h}\), \(u, v \in \mathfrak{a}\) and \(\alpha, \beta \in \mathfrak{h}^*\). Let \(\varphi^* : \mathfrak{h} \to \text{Hom}_\mathfrak{a}(\mathfrak{h}, \mathfrak{a}^*)\) be the linear map defined by \((\varphi^*(x)(y))(v) = (\varphi(x)(v))(y)\), for all \(x, y \in \mathfrak{h}\) and \(v \in \mathfrak{a}\). Then \((\mathfrak{g}, [\cdot, \cdot]_{\{\lambda, \mu\}}, B)\) is a quadratic Lie algebra if and only if:

\[
\lambda(x, y)^\beta = -\varphi^*(x)(y), \quad \forall x, y \in \mathfrak{h},
\]

\[
\mu(x, y)(z) = \mu(y, z)(x), \quad \forall x, y, z \in \mathfrak{h}.
\]

Thus, \(B\) also depends on the data \(\{\lambda, \mu\}\). Furthermore, any non-Abelian quadratic Lie algebra with non-trivial center has this structure, where \(\mathfrak{h}^* \simeq i(\mathfrak{g})\) and \(j(\mathfrak{g}) \simeq \mathfrak{a} \oplus \mathfrak{h}^*\).

Proof. (1). The statement gives necessary and sufficient conditions for \([\cdot, \cdot]_{\{\lambda, \mu\}}\) to be a Lie bracket on \(\mathfrak{g}\). The general conditions have already been given in Cor. 2.3. We are applying it here to the special case in which \(i(\mathfrak{g}) \simeq \mathfrak{h}^*\) and \(\rho = \text{ad}^*_\mathfrak{a}\) are fixed, and \(\varphi\) is obtained from \(\lambda\) (or viceversa) due to the existence of an invariant metric in \(\mathfrak{g}\) as shown in (26).
(2). We want to show that any non-Abelian quadratic Lie algebra \( g \) with non-trivial center \( C(g) \) has its Lie bracket and its invariant metric given as in the statement. Let \( g \) be such an algebra. By Lemma 2.6.(ii), we know that \( \{0\} \neq C(g) \subset j(g) \), which implies that \( j(g) \neq \{0\} \). Also, Lemma 2.6.(v) says that \( j(g)^\perp = i(g) \). If \( i(g) = \{0\} \), the non-degeneracy of the invariant metric implies that \( g = j(g) \). But, Lemma 2.6.(iv), says that \( j(g) \) is Abelian, contrary to the hypothesis on \( g \). Therefore, \( i(g) \neq \{0\} \). Let \( a \) and \( h \) be two subspaces of \( g \) such that \( j(g) = a \oplus i(g) \), and \( g = h \oplus j(g) \). Then \( g = h \oplus a \oplus i(g) \). Now apply Cor. 2.3 to conclude that the Lie bracket on \( g \) is precisely the one given in the statement. Furthermore, the invariance and the non-degeneracy of the metric also imply that the 2-cocycle \( \Lambda \leftrightarrow (\lambda, \mu, \varphi) \) and the 1-cocycle \( \varphi \), satisfy the conditions given in (27).

\[ \square \]

4. Classification of 9-dimensional Lie Algebras

\( g = h \oplus a \oplus h^* \) where \( h \) is the 3-dimensional Heisenberg Algebra and \( a = F^3 \)

In this section we shall classify up to isomorphism all the 9-dimensional Lie algebras \( g \) defined for the case in which \( h = g/j(g) \) is the 3-dimensional Heisenberg Lie algebra, and \( j(g)/i(g) \simeq F^3 \). The example will illustrate how Cor. 2.3 and Thm. 3.1 work.

Let \( h = \text{Span}_F\{x_1, x_2, x_3\} \) be the 3-dimensional Heisenberg Lie algebra, with \( C(h) = \text{Span}_F\{x_3\} \), and \([x_1, x_2] = x_3 \). Let \( a = F^3 \) be the trivial \( h \)-module. Let \( \rho = \text{ad}^* : h \to gl(h^*) \) be the coadjoint representation of \( h \). Let \( \varphi \) be a 1 cocycle in \( C(h; \text{Hom}(F^3, h^*)) \) associated to the representation \( \text{ad}^T : h \to gl(\text{Hom}(F^3, h^*)) \) given by \( T \mapsto \text{ad}^T(x)(T) = \text{ad}^*(x) \circ T \). Let \( h(\lambda, \mu, \varphi) \) be the Lie algebra defined on \( g = h \oplus F^3 \oplus h^* \) by the data \((\lambda, \mu, \varphi)\) as in Cor. 2.3 with \( \rho = \text{ad}^* \) fixed, \( i(g) = h^* \) and \( j(g) = F^3 \oplus h^* \).

Let \( \{v_1, v_2, v_3\} \) be a basis for \( F^3 \) and let \( \{\omega^1, \omega^2, \omega^3\} \) be its dual basis. Let \( \{\theta^1, \theta^2, \theta^3\} \) be the basis of \( h^* \) dual to \( \{x_1, x_2, x_3\} \). Since \( \varphi(x) \in \text{Hom}_F(F^3, h^*) \simeq h^* \otimes (F^3)^* \), for all \( x \in h \), we may write,

\[ \varphi(x_j) = \sum_{i, t=1}^{3} \varphi^j_{it} \theta^j \otimes \omega^t, \quad 1 \leq j \leq 3. \]

Each linear map \( \varphi(x_j) : F^3 \to h^* \), has the associated matrix,

\[ \varphi(x_j) \leftrightarrow \varphi^j = \begin{pmatrix} \varphi^j_{11} & \varphi^j_{12} & \varphi^j_{13} \\ \varphi^j_{21} & \varphi^j_{22} & \varphi^j_{23} \\ \varphi^j_{31} & \varphi^j_{32} & \varphi^j_{33} \end{pmatrix}, \quad 1 \leq j \leq 3. \]
The Lie bracket of the Heisenberg Lie algebra \( h \), gives the following relations:
\[
\text{ad}^*(x_1)(\theta^1) = \text{ad}^*(x_1)(\theta^2) = 0,
\text{ad}^*(x_2)(\theta^1) = \text{ad}^*(x_2)(\theta^2) = 0,
\text{ad}^*(x_1)(\theta^3) = -\theta^2, \quad \text{ad}^*(x_2)(\theta^3) = \theta^1, \quad \text{ad}^*(x_3) = 0.
\]

Since, \( \varphi \in \text{Hom}_F(h, \text{Hom}_F(F^3, h^*)) \) is a 1-cocycle, it satisfies,
\[
(28) \quad \varphi([x_i, x_j]) = \text{ad}^*(x_i) \circ \varphi(x_j) - \text{ad}^*(x_j) \circ \varphi(x_i), \quad \forall \ 1 \leq i, j \leq 3.
\]

It is straightforward to prove that (28) is equivalent to
\[
\varphi^3 = -\begin{pmatrix}
\varphi_{13} & \varphi_{12} & \varphi_{11} \\
\varphi_{23} & \varphi_{22} & \varphi_{21} \\
0 & 0 & 0
\end{pmatrix};
\]
that is, for \( 1 \leq j \leq 3 \),
\[
\varphi^3_{1j} = -\varphi^1_{3j}, \quad \varphi^3_{2j} = -\varphi^2_{3j}, \quad \varphi^3_{3j} = 0,
\]
and, the entries of \( \varphi^3 \) are thus completely determined by the entries of \( \varphi^1 \) and \( \varphi^2 \). We now write the 2-cocycle \( \lambda : h \times h \rightarrow F^3 \), in the form,
\[
\lambda(x_1, x_2) = \lambda_{13} v_1 + \lambda_{23} v_2 + \lambda_{33} v_3,
\lambda(x_2, x_3) = \lambda_{11} v_1 + \lambda_{21} v_2 + \lambda_{31} v_3,
\lambda(x_3, x_1) = \lambda_{12} v_1 + \lambda_{22} v_2 + \lambda_{32} v_3,
\]
with \( \lambda_{ij} \in F \). Thus, \( \lambda \) gets identified with the \( 3 \times 3 \) matrix,
\[
(29) \quad \lambda \leftrightarrow \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{pmatrix}.
\]
Similarly, \( \mu : h \times h \rightarrow h^* \) can be written in the form,
\[
\mu(x_1, x_2) = \mu_{13} \theta^1 + \mu_{23} \theta^2 + \mu_{33} \theta^3,
\mu(x_2, x_3) = \mu_{11} \theta^1 + \mu_{21} \theta^2 + \mu_{31} \theta^3,
\mu(x_3, x_1) = \mu_{12} \theta^1 + \mu_{22} \theta^2 + \mu_{32} \theta^3,
\]
so that,
\[
(30) \quad \mu \leftrightarrow \begin{pmatrix}
\mu_{11} & \mu_{12} & \mu_{13} \\
\mu_{21} & \mu_{22} & \mu_{23} \\
\mu_{31} & \mu_{32} & \mu_{33}
\end{pmatrix}.
\]
From now on, we shall identify the bilinear maps \( \lambda \) and \( \mu \) with their corresponding matrices as in (29) and (30), respectively. Observe that
dλ = 0 is trivially satisfied for the 3-dimensional Heisenberg Lie algebra ℱ. Indeed, since [x₁, x₃] = [x₂, x₃] = 0 and [x₁, x₂] = x₃, then

\[ \lambda([x₁, x₂], x₃) + \lambda([x₂, x₃], x₁) + \lambda([x₃, x₁], x₂) = \lambda(x₃, x₃) = 0. \]

On the other hand, a straightforward computation shows that, dμ + eϕ(λ) = 0, if and only if,

\[
\begin{align*}
(\varphi^1\lambda)_{11} + (\varphi^2\lambda)_{12} - (\varphi^1\lambda)_{33} + \mu_{32} &= 0, \\
(\varphi^1\lambda)_{21} + (\varphi^2\lambda)_{22} - (\varphi^2\lambda)_{33} - \mu_{31} &= 0, \\
(\varphi^1\lambda)_{31} + (\varphi^2\lambda)_{32} &= 0.
\end{align*}
\]

These equations must retain their form after acting on them with elements of the group of transformations \( G \subset \text{Aut}(\mathfrak{q}) \times \text{GL}(\mathfrak{a} \oplus \mathfrak{b}^*) \) consisting of those \( \Psi \)'s from Cor. 2.4, that preserve the isomorphism class of \( \mathfrak{h}(\lambda, \mu, \varphi) \). In particular, we look for group elements \( (g, \sigma) \) of the form, \( g \in \text{Aut}(\mathfrak{h}) \) and,

\[
\sigma = \begin{pmatrix} h & 0 \\ T & k \end{pmatrix} \in \text{GL}(\mathbb{F}^3 \oplus \mathfrak{b}^*)
\]

where, \( h \in \text{GL}(\mathbb{F}^3) \), \( k \in \text{GL}(\mathfrak{b}^*) \) and \( T \in \text{Hom}_\mathbb{F}(\mathbb{F}^3, \mathfrak{b}^*) \). For any \( g \in \text{GL}(\mathfrak{h}) \), write, \( g(x_j) = \sum_{i=1}^{3} g_{ij} x_i \). It is easy to see that \( g \in \text{Aut}(\mathfrak{h}) \), if and only if its matrix has the form,

\[
g = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad \text{with}, \quad g_{33} = g_{11}g_{22} - g_{12}g_{21} \neq 0.
\]

Now recall that a necessary condition to preserve the isomorphism class of \( \mathfrak{h}(\lambda, \mu, \varphi) \) is that the representations \( R \) and \( R' \) be in the same \( \text{G} \)-orbit. Since,

\[ R(x) = \begin{pmatrix} 0 & 0 \\ \varphi(x) & \text{ad}^*(x) \end{pmatrix}, \quad \forall x \in \mathfrak{h}, \]

we have,

\[ \sigma \circ R(g^{-1}(x)) \circ \sigma^{-1} = R'(x), \quad \forall x \in \mathfrak{h}. \]

Thus, \( \varphi \) can be changed into \( \varphi' \) inside \( R' \), as follows:

\[
\varphi'(x) = k (\varphi(g^{-1}(x)) - \text{ad}^*(g^{-1}(x)) k^{-1} T) k^{-1},
\]

\[
\text{ad}^*(x) = k^* \text{ad}^*(g^{-1}(x)) k^{-1}. \]

It is straightforward to see that a linear map \( k \in \text{GL}_\mathbb{F}(\mathfrak{b}^*) \) satisfies the second condition in (36) if and only if:

\[
k = \begin{pmatrix} k_{33} g_{33} & g_{11} g_{12} & -k_{13} \\ g_{21} g_{22} & k_{23} & \end{pmatrix}, \quad g_{33} = g_{11}g_{22} - g_{12}g_{21}.
\]
Also recall from Cor. 2.4, how \( \lambda \) and \( \mu \) are transformed via, \( \Lambda' (\cdot) \mapsto \Lambda (\cdot) = \Phi (\gamma) \Lambda - d(\ast) \), where, \( \gamma = (g, \sigma) \), with \( g \in \text{Aut} \, \mathfrak{h} \) and \( \sigma \) as in (34):

\[
(\Phi (\gamma) (\Lambda) ) (\cdot, \cdot ) = \Phi (\gamma) \left( \begin{array}{c}
\lambda (\cdot, \cdot ) \\
\mu (\cdot, \cdot )
\end{array} \right) = \sigma \left( \begin{array}{c}
\lambda (g^{-1} (\cdot), g^{-1} (\cdot)) \\
\mu (g^{-1} (\cdot), g^{-1} (\cdot))
\end{array} \right)
= \left( \begin{array}{c}
h (\lambda (g^{-1} (\cdot), g^{-1} (\cdot))) \\
k (\mu (g^{-1} (\cdot), g^{-1} (\cdot))) + T \circ \lambda (g^{-1} (\cdot), g^{-1} (\cdot))
\end{array} \right)
\]

We shall now determine a representative set of canonical forms for \( \Lambda \), under the left action, \( \lambda \mapsto (g, h). \lambda \), with \( (g, h) \in \text{Aut}(\mathfrak{h}) \times \text{GL}(F^3) \),

\[
((g, h). \lambda ) (\cdot, \cdot ) = h (\lambda (g^{-1} (\cdot), g^{-1} (\cdot)) )
\]

In terms of the corresponding matrices, it is easy to see that,

\[
(g, h) . \lambda \leftrightarrow \frac{1}{\det g} \ h \ \lambda \ g^t, \ \text{and} \ \ (g, k) . \mu \leftrightarrow \frac{1}{\det g} \ k \ \mu \ g^t.
\]

4.1. Claim. There exists an isomorphism \( \Psi : \mathfrak{h} (\lambda, \mu, \varphi) \rightarrow \mathfrak{h} (\lambda', \mu', \varphi') \), where the matrix of \( \lambda' \) is upper triangular.

Proof. Let \( g \in \text{Aut}(\mathfrak{h}) \) as in (35). Write \( h (v_j) = \sum_{i=1}^3 h_{ij} v_i \), for \( h \in \text{GL}(F^3) \) and let \( w_j = \sum_{i=1}^3 \lambda_{ij} v_i \) (\( 1 \leq j \leq 3 \)). The \( j \)-th column vector of the matrix \( h \lambda \) is,

\[
\sum_{i=1}^3 (h \lambda)_{ij} v_i = \sum_{i=1}^3 h_{ij} \lambda_{ij} v_i = h \left( \sum_{\ell=1}^3 \lambda_{ij} v_i \right) = h (w_j).
\]

So, we may symbolically write the matrix \( h \lambda \) in the form

\[
h \lambda = ( h (w_1) \ | \ h (w_2) \ | \ h (w_3) ), \ \text{where}, \ w_j = \sum_{\ell=1}^3 \lambda_{ij} v_\ell.
\]

If \( \{w_1, w_2\} \) is a linearly independent set, we may choose \( h \in \text{GL}(F^3) \) in such a way that \( h (w_1) = v_1 \) and \( h (w_2) = v_2 \). That is,

\[
h \lambda = \begin{pmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & *
\end{pmatrix}.
\]

On the other hand, if \( \{w_1, w_2\} \) is a linearly dependent set; say \( w_2 = \alpha w_1 \), with \( w_1 \neq 0 \), we may choose \( h \in \text{GL}(F^3) \) in such a way that,

\[
h \lambda = \begin{pmatrix}
1 & \alpha & * \\
0 & 0 & * \\
0 & 0 & *
\end{pmatrix}.
\]
If \( w_1 = 0 \), but \( w_2 \neq 0 \), we may choose \( h \in \text{GL}(F^3) \) in such a way that,
\[
h \lambda = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}.
\]
At any rate, this analysis implies that one may choose \( h \in \text{GL}(F^3) \) in such a way that the matrix of \( h \lambda \) is upper triangular. Therefore, we might as well assume, right from the start, that the matrix of \( \lambda \) is upper triangular. □

4.2. Claim. There exists an isomorphism \( \Psi : h(\lambda, \mu, \varphi) \to h(\lambda', \mu', \varphi') \), where the matrix of \( \lambda' \) has the form:
\[
\begin{pmatrix} \lambda'_{11} & 0 & 0 \\ 0 & \lambda'_{22} & 0 \\ 0 & 0 & 0 \\
\end{pmatrix}, \quad \lambda'_{11}, \lambda'_{22} \in F.
\]

Proof. Choose \( g \in \text{Aut}(h) \) to be upper triangular, so as to have a lower triangular \( g^t \). Then, \( \lambda' = (\det g)^{-1} \lambda g^t \), and,
\[
\begin{pmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ 0 & \lambda'_{22} & \lambda'_{23} \\ 0 & 0 & \lambda'_{33} \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix} \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{pmatrix}
\]
\[
= \frac{1}{\det g} \begin{pmatrix} \lambda_{11}g_{11} & \lambda_{11}g_{21} + \lambda_{12}g_{22} & * \\ 0 & \lambda_{22}g_{22} & * \\ 0 & 0 & \lambda_{33}g_{33} \end{pmatrix}.
\]
If \( \lambda_{12} \neq 0 \), we may choose \( g_{22} \) so as to make \( \lambda'_{12} = 0 \). Thus, we may also assume from the start that,
\[
\lambda = \begin{pmatrix} \lambda_{11} & 0 & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}
\]
Now use the additional freedom for transforming the 2-cocycle \( \lambda \) by means of a coboundary as in Cor. 2.4; that is, \( \lambda \mapsto \lambda + d\tau \) for a 1-cochain so as to obtain \( d\tau : h \to F^3 \). In particular, \( \tau \) can be chosen in such a way that,
\[
d\tau = \begin{pmatrix} 0 & 0 & -\lambda_{13} \\ 0 & 0 & -\lambda_{23} \\ 0 & 0 & -\lambda_{33} \end{pmatrix}.
\]
Therefore, \( \lambda + d\tau \) takes the form
\[
\begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
In other words, it can always be assumed that in the isomorphism class of the Lie algebra associated to the data \((\lambda, \mu, \varphi)\), the matrix of \(\lambda\) has the form,

\[
\begin{pmatrix}
\lambda_{11} & 0 & 0 \\
0 & \lambda_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
\lambda_{11}, \lambda_{22} \in \mathbb{F}.
\]

(38)

\[\square\]

4.3. Lemma. The Lie algebra \(h(\lambda, \mu, \varphi)\) is isomorphic to \(h(\lambda', \mu', \varphi')\), where the matrix of \(\lambda'\) can be one, and only one, of the following:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Proof. Once \(\lambda'\) is brought into the form (38) it can be further multiplied on the left by a diagonal invertible matrix \(h\) and the ordered pair \((\lambda_{11}, \lambda_{22})\) formed by the diagonal entries, can be assumed to be either \((1, 1)\), \((1, 0)\), \((0, 1)\) or \((0, 0)\). As a matter of fact, the cases \((1, 0)\) and \((0, 1)\) define the same isomorphism class for a given \(h(\lambda, \mu, \varphi)\) because the transformation \(\lambda \mapsto h\lambda g^t = \lambda'\) (with \(\det g = 1\)) yields,

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Now, the matrix (38) with diagonal entries \((1, 1)\) lies in the same isomorphism class of \(h(\lambda', \mu', \varphi')\) where \(\lambda'\) is the identity matrix as claimed in the statement. In fact, we simply change \(\lambda\) by a coboundary coming from a linear map \(\tau' : h \to \mathbb{F}^3\), such that \(\tau'(x_3) = v_3\), so that \(d\tau'\) has the matrix \(\begin{pmatrix}0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1\end{pmatrix}\).

Finally, we claim that the Lie algebras \(h(\lambda, \mu, \varphi)\) and \(h(\lambda', \mu', \varphi')\) arising from the 2-cocycles \(\lambda\) and \(\lambda'\) whose matrices are \(\begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1\end{pmatrix}\) and \(\begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0\end{pmatrix}\), respectively, cannot be isomorphic. If they were isomorphic, there should be invertible maps (matrices) \(h\) and \(g\) such that \((\det g)^{-1} h g^t + d\tau' = \lambda'\). This however implies that \((\det g)^{-1} h g^t = \lambda' - d\tau'\), but the matrix of the latter is \(\begin{pmatrix}1 & 0 & 0 \\
0 & 0 & -\tau_{31} \\
0 & 0 & 0\end{pmatrix}\), which is not invertible. \[\square\]

To proceed with the classification of the isomorphism classes of Lie algebras \(h(\lambda, \mu, \varphi)\) we shall now find a set of representative canonical forms for the skew-symmetric bilinear maps \(\mu : h \times h \to h^*\) for each
of the canonical forms of $\lambda$ already found. Recall that $h(\lambda, \mu, \varphi)$ and $h(\lambda, \mu', \varphi')$ are in the same isomorphism class whenever,

$$\mu'(\cdot, \cdot) = k (\mu(g^{-1}(\cdot), g^{-1}(\cdot))) + T(\lambda(g^{-1}(\cdot), g^{-1}(\cdot)))),$$

where $T : \mathbb{F}^3 \rightarrow \mathfrak{h}^*$ is a linear map and the pair $(g, k) \in \text{Aut}(\mathfrak{h}) \times \text{GL}(\mathfrak{g}^*)$, satisfies the second condition in (36). In this case, we shall impose first the restriction that the pairs $(g, h) \in \text{Aut}(\mathfrak{h}) \times \text{GL}(\mathbb{F}^3)$, lie in the isotropy group of the found canonical forms for $\lambda$. For those $g$’s we then determine $k$ as in (37) and choose an appropriate $T$ so as to obtain the desired canonical forms for the matrix,

$$\mu' = \frac{1}{\det g} (k \mu g^t + T \lambda g^t).$$

Thus, we now proceed in a case by case fashion.

**Case 1:** $\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Choose $g$, $h$ and $k$ to be the identity maps. If the entries of $\mu$ are $\mu_{ij}$, choose $T : \mathbb{F}^3 \rightarrow \mathfrak{h}^*$ so that $T(v_j) = -\sum_{i=1}^{3} \mu_{ij} \theta^j$ ($1 \leq j \leq 3$). These choices make $\mathfrak{h}(\lambda, \mu, \varphi)$ isomorphic to $\mathfrak{h}(\lambda, 0, \varphi')$. The canonical form for $\mu$ is in this case $\mu = 0$.

**Case 2:** $\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This time choose a linear map $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ and change $\mu$ into $\mu + d \nu$, whose matrix takes the form,

$$\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \nu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \nu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \nu_{33} \end{pmatrix},$$

and $\nu_{13}, \nu_{23}, \nu_{33} \in \mathbb{F}$ are arbitrary. Thus, we may choose $\nu_{33} = -\mu_{33}$, $1 \leq j \leq 3$, to let $\mu + d \nu$ achieve the matrix form $\begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ \mu_{31} & \mu_{32} & 0 \end{pmatrix}$. Thus, we might as well assume that $\mu$ is already of this form. Finally, we may further transform this $\mu$ into $\mu + T \circ \lambda$ by choosing the linear map $T : \mathbb{F}^3 \rightarrow \mathfrak{h}^*$ in such a way that $T(v_1) = -\mu_{11} \theta^1 - \mu_{21} \theta^2 - \mu_{31} \theta^3$, $T(v_2) = T(v_3) = 0$. By taking $g$, $h$ and $k$, to be the identity maps in their corresponding domains, we conclude that $\mathfrak{h}(\lambda, \mu, \varphi)$ is in the same isomorphism class as $\mathfrak{h}(\lambda, \mu + T \circ \lambda, \varphi')$, where the matrix of $\mu + T \circ \lambda$ is,

$$\begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ \mu_{31} & \mu_{32} & 0 \end{pmatrix} + \begin{pmatrix} -\mu_{11} & 0 & 0 \\ -\mu_{21} & 0 & 0 \\ -\mu_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu_{12} & 0 \\ 0 & \mu_{22} & 0 \\ 0 & \mu_{32} & 0 \end{pmatrix}.$$

Therefore, we may always assume that, $\mu \leftrightarrow \begin{pmatrix} 0 & \mu_{12} & 0 \\ 0 & \mu_{22} & 0 \\ 0 & \mu_{32} & 0 \end{pmatrix}$. Now, let $(g, h) \in \text{Aut} \mathfrak{h} \times \text{GL}(\mathbb{F}^3)$ be such that $\lambda = (\det g)^{-1} h \lambda g^t$, where,
\[ \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] It is easy to see that \( g \) and \( h \) must be given by,

\[ h = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & h_{32} & h_{33} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} g_{11} & g_{21} & 0 \\ 0 & g_{22} & 0 \\ 0 & g_{23} & g_{33} \end{pmatrix}, \]

respectively. Moreover, since \((g, k)\) must fix the coadjoint representation of \( \mathfrak{h} \), it follows that \( k \) has the form (see (37)),

\[ k = \begin{pmatrix} g_{22}k_{33} & -g_{21}k_{33} & k_{13} \\ 0 & g_{11}k_{33} & k_{23} \\ 0 & 0 & k_{33} \end{pmatrix}. \]

Thus, \( \mu \leftrightarrow \begin{pmatrix} 0 & \mu_{12} & 0 \\ 0 & \mu_{22} & 0 \\ 0 & \mu_{32} & 0 \end{pmatrix} \) may be transformed into,

\[ \frac{1}{\det g} \mu \, g' = \frac{1}{\det g} \begin{pmatrix} 0 & \mu_{12}' & 0 \\ 0 & \mu_{22}' & 0 \\ 0 & \mu_{32}' & 0 \end{pmatrix}, \]

and \( \mathfrak{h}(\lambda, \mu, \varphi) \simeq \mathfrak{h}(\lambda', \mu', \varphi') \). Observe that,

\begin{align*}
\mu_{12}' &= g_{22}(g_{22}k_{33}\mu_{12} - 2g_{21}k_{33}\mu_{22} + k_{13}\mu_{32}), \\
\mu_{22}' &= g_{22}(g_{11}k_{33}\mu_{22} + k_{23}\mu_{32}), \\
\mu_{32}' &= g_{22}k_{33}\mu_{32}.
\end{align*}

If \((\mu_{22}, \mu_{32}) \neq (0, 0)\), we may choose \( g \) and \( k \), so as to make \( \mu_{12}' = 0 \). Therefore, we obtain two possible types of canonical forms for \( \mu \) when \( \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \); namely,

\[ \mu_1 = \begin{pmatrix} 0 & \mu_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{or else,} \]

\[ \mu_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & \mu_{32} & 0 \end{pmatrix}, \quad \text{with} \quad (\mu_{22}, \mu_{32}) \neq (0, 0). \]

We claim that for the canonical form of \( \lambda \) under consideration, the Lie algebra extensions \( \mathfrak{h}(\lambda, \mu_1, \varphi_1) \) and \( \mathfrak{h}(\lambda, \mu_2, \varphi_2) \), can never be isomorphic, regardless of the choices of \( \varphi_1 \) and \( \varphi_2 \). Were they isomorphic, there would be \( g, h, k, T \) and \( \nu \), with \((g, h).\lambda = \lambda, (g, k).\text{ad}^* = \text{ad}^*, \) an arbitrary map \( T \in \text{Hom}_F(\mathbb{R}^3, \mathfrak{h}^*) \), and an appropriate map \( \nu \in \text{Hom}_F(\mathfrak{h}, \mathfrak{h}^*) \), satisfying, \((g, k).\mu_1 + (g, T).\lambda + d\nu = \mu_2. \) A straightforward computation, however, shows that the matrix form of this
equation is,
\[
\frac{1}{\det g} \begin{pmatrix}
g_{11}T_{11} + g_{21}g_{22}k_{33}\mu_{12} & g_{22}^2k_{33}\mu_{12} \\
g_{11}T_{21} & 0 \\
g_{11}T_{31} & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \mu_{22} & 0 \\
0 & \mu_{32} & 0
\end{pmatrix},
\]
but this implies that \((\mu_{22}, \mu_{32}) = (0, 0)\), contrary to our assumption. Now, from (40) it is clear that \(\mu_{32} = 0\) if and only if \(\mu'_{32} = 0\). If \(\mu_{32} \neq 0\) we may choose \(k_{23} = -\mu_{32}^{-1}g_{11}k_{33}\mu_{22}\) to make \(\mu'_{22} = 0\). On the other hand, if \(\mu_{32} = 0\), we have \(\mu'_{22} = g_{11}g_{22}k_{33}\mu_{22}\). In this case, \(\mu_{22} \neq 0\), and one may choose \(g_{11}g_{22}k_{33}\) so as to have \(\mu'_{22} = 1\).

In summary, we have found four different canonical forms for \(\mu\); namely,
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

**Case 3:** \(\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\). In this case, the action given in (39), transforms \(\mu\) into \((\det g)^{-1}k\mu g^t\), where the pair \((g, k) \in \text{Aut}(h) \times \text{GL}(h^*)\) satisfies the second condition of (36). Since \(\lambda = 0\), the condition \(d\mu + e_\varphi(\lambda) = 0\) reduces to \(d\mu = 0\) and from (31), (32) and (33), we conclude that \(\mu_{31} = \mu_{32} = 0\).

Let \(g = \begin{pmatrix} A & 0 \\ 0 & g_{33} \end{pmatrix} \in \text{Aut}(h)\). From (37), we get \(k = k_{33}g_{33}\left((A^{-1})^t \begin{pmatrix} 0 & 0 \\ 0 & (g_{33})^{-1} \end{pmatrix}\right)\).

By considering \(\mu + d\nu\), for \(\nu : h \to h^*\), we might as well assume that right from the start \(\mu\) has the form \(\begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}\) with the obvious definition of the \(2 \times 2\) matrix \(M\). Therefore,
\[
\frac{1}{\det g} k\mu g^t = \frac{k_{33}}{g_{33}} \left((A^t)^{-1}MA^t \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).
\]

We may now choose \(A\) in such a way as to bring \((A^t)^{-1}MA^t\) into its Jordan canonical form. This proves that if \(\lambda = 0\), there will be four possible canonical forms for \(\mu\), each of which defines an isomorphism class of its own; namely,
\[
\begin{pmatrix}
\mu_{11} & 0 & 0 \\
0 & \mu_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \begin{pmatrix}
\mu_{11} & 0 & 0 \\
0 & \mu_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \begin{pmatrix}
\mu_{11} & 0 & 0 \\
0 & \mu_{11} & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix};
\]
\(\mu_{11} \neq \mu_{22}; \quad \mu_{21} \neq 0; \quad \mu_{11} \neq 0\).

Finally, we can modify the first three of these canonical forms by adding up a coboundary \(d\nu\), changing the given \(\mu\)'s into \(\mu + d\nu\)'s so as to put in an additional \(\mu_{11}\) diagonal entry in their lower right corners. Then, by an additional rescaling, we may further assume that \(\mu_{11} = 1\). We summarize our findings in the following:
4.4. Proposition. Under the hypotheses of this section, \( h(\lambda', \mu', \varphi') \) is isomorphic to \( h(\lambda, \mu, \varphi) \) under one and only one of the following set of canonical forms for the initial data \((\lambda', \mu', \varphi')\)

1. \[ \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \]
   \[ \varphi^{11} + \varphi^{22} - \varphi^{33} = 0, \]
   \[ \varphi^{11} + \varphi^{22} - \varphi^{33} = 0, \]
   \[ \varphi^{11} + \varphi^{22} = 0. \]

2. \[ \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \]
   \[ \varphi^{11} = 0, \quad \varphi^{11} = 0, \quad \varphi^{11} = 0. \]

3. \[ \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \]
   \[ \varphi^{11} = 0, \quad \varphi^{11} = 0, \quad \varphi^{11} = 0. \]

4. \[ \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \]
   \[ \varphi^{11} + 1 = 0, \quad \varphi^{11} = 0, \quad \varphi^{11} = 0. \]

Proof. What remains to be proved are the form of the matrix entries of the 1-cocycle \( \varphi \), but these are given by (31), (32) and (33). In each case, one only has to use the found canonical forms for \( \lambda \) and \( \mu \). \[□\]

Now, according to Thm. 3.1, a necessary condition for \( h(\lambda, \mu, \varphi) \) to admit an invariant metric, is that the 2-cochain \( \mu : h \times h \to h^* \) satisfies the cyclic condition, \( \mu(x, y)(z) = \mu(y, z)(x) \), for all \( x, y, z \in h \). This ammounts to have \( \mu_{11} = \mu_{22} = \mu_{33} \). In particular, the Lie algebras given in 2.3 and 3.1, of Prop. 4.4, cannot admit invariant metrics. In addition, taking into account (27) of Thm. 3.1 for the relationship between \( \lambda \) and \( \varphi \) that has to be satisfied when an invariant metric exists, we may now conclude the following:
4.5. **Proposition.** Under the hypotheses of this section, $\mathfrak{h}(\lambda, \mu, \varphi)$ admits an invariant metric if and only if,

1. $\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varphi^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $\varphi^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\varphi^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

2. $\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varphi^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\varphi^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\varphi^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3. $\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varphi^i = 0$ ($1 \leq j \leq 3$).

4. $\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varphi^i = 0$ ($1 \leq j \leq 3$).

**Proof.** We only have to use the conditions given in Thm. 3.1. In particular $\lambda$ determines $\varphi$. On the other hand, $\mu$ has to satisfy (27). Since $\mu$ is skew-symmetric, it follows that the matrix associated to $\mu$ has to be diagonal with its diagonal entries equal among themselves. This restriction rules out the possibilities given in 2.2, 2.3, 2.4, 3.1 and 3.2 of Prop. 4.4. \qed

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