How the Quantum Universe Became Classical

J.J.Halliwell

Blackett Laboratory
Imperial College
London SW7 2BZ
UK

Abstract

This is an informal introduction to the ideas of decoherence and emergent classicality, including a simple account of the decoherent histories approach to quantum theory. It is aimed at undergraduates with a basic appreciation of quantum theory. The emphasis is on simple physical ideas and pictures.
1. Introduction

Quantum theory has been around now for almost a century and it is probably fair to say that, in its most elaborate form, quantum field theory, it is the most spectacular physical theory every invented. It applies to and explains essentially all atomic phenomena and its predictions, some of which are remarkably precise, have been exceptionally well verified by experiment. Indeed, there is not one shred of experimental evidence to suggest that the basic structure of quantum theory is in any way incorrect. Suitably adapted, it is expected to apply to all types of matter and all types of forces holding matter together, from the very smallest scales of fundamental particles, right up to the cosmological scale, and possibly even to the entire universe. Any theory claiming to be fundamental must be expressed in the language of quantum theory. It is the fundamental theory.

Yet, despite being the fundamental theory of matter, quantum theory is not the theory of the ordinary everyday world of our immediate experience. Many of its features, such as superposition states and entanglement, defy intuition in a truly profound way, to the extent that that are even described as “paradoxical”. Furthermore, the mathematical objects of its vocabulary do not correspond directly to the things we see and measure in the world about us, and the question of the relationship between the formalism and physical reality – the interpretation of the theory – has been a hotly debated topic.

These tensions are due to the fact that the large scale world of immediate experience is best described by other, older areas of physics, namely the classical mechanics of Newton, devised in the 18th century, and the theory of thermodynamics, devised in the 19th century. These classical theories are theories of the ordinary everyday world, and the mathematical objects they deal with do correspond directly to things we see and measure. They are utterly different in their structure to quantum theory.

This then leads to a very interesting question. If large scale objects are made of atoms, and atoms are described by quantum theory, how do large collections of atoms come to be described by classical physics? In brief, how does classical physics emerge from quantum theory?

This question has been one of great interest to many researchers over the last ten or twenty years, and the purpose of this paper is to explain what this question means and how one can begin to solve it. The emphasis will be on simple physical ideas and pictures, not
on elaborate mathematics, in the hope that it will understandable to the broadest possible audience.

2. Classical Physics

A very wide variety of particle systems at ordinary scales are very accurately described by the classical mechanics of Newton. For, say, a single particle system, the description is very simple. The state of the system at each time is characterized by its position $x$ and momentum $p$ of each particle. If the forces are specified through a potential $V(x)$, then the evolution in time is described by Newton’s law

$$m\frac{d^2x}{dt^2} + V'(x) = 0 \quad (1)$$

We may solve this equations, on a computer if necessary, and the solution gives a curve from which we may see the position and momentum at any time, given the initial position and momentum. The laws describing this very simple system readily generalize to systems consisting of many particles with all sorts of complicated interactions between them. They describe a wide variety of phenomena at small scales almost as small as the atomic scale, right up to the scale of planets, stars and the whole universe.

The mathematical objects the Newtonian theory deals with correspond directly with our immediate experience of the world: position, momentum, acceleration, force and so on. One can think of a classical system as having “definite properties”, such as definite position and momentum, without contradiction. There is no sense in which the theory needs to be “interpreted”.

Classical mechanics also readily generalizes to stochastic theories. These are theories where one has incomplete information about some of the variables and they have to be modelled in terms of random processes. A simple example is Brownian motion, where a small particle, like a particle of pollen, is seen to jiggle around as a result of invisible collisions with molecules. The pollen particle approximately follows the equations of Newtonian mechanics, but with random fluctuations around that motion and also with dissipation. In one dimension, this is described by an equation of the form,

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + V'(x) = noise \quad (2)$$
Other important examples, where the fluctuations are more significant, are the stock market or the weather. In these stochastic theories we still think of the variables like position and momentum as taking definite values, even though those values are known imperfectly.

3. Quantum Theory

Classical physics was realized to be inadequate around the beginning of the 20th century when physicists started trying to understand the structure of atoms. In particular, it did not explain the striking fact that radiation emitted from atoms occurs only at discrete sets of frequencies. Niels Bohr in 1913 put forward a model of the atom in which the electrons orbited the nucleus at circular orbits of fixed size and the discrete nature of the radiated spectrum was explained by the transition between these orbits. Eventually, however, the founders of quantum theory made a much bolder and more far-reaching assumption, namely that elementary particles, like electrons, are best modelled by the mathematics of waves. This explained the discrete set of sizes for the orbits since a whole number of waves had to fit around the atom, and moreover, the possible frequencies of vibration of these waves explained the spectrum of radiation. Hence the discreteness comes from the natural relationship between waves and discrete sets of numbers.

Absolutely central to this description is an equation determining the precise form of these waves. This is Schrödinger’s famous equation,

\[ i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi \]  

perhaps the single most important equation in the whole of physics. It makes an utterly spectacular range of predictions and there is absolutely no doubt that it is “correct” in any sense of the word.

The solution to the Schrödinger equation is the wave function \( \psi(x, t) \), a complex-valued field which describes the possible frequencies of vibration of the atoms and the precise way in which they vibrate. However, although the vibrational modes appeared to be naturally related to the frequencies of atomic radiation, the introduction of the wave function as a description of atomic matter was completely new. This quantity does not relate directly to measurements, or to our intuitive understanding of the physical world. It requires an *interpretation*.
An interpretation of a theory becomes necessary when the mathematical objects in its vocabulary do not directly correspond to the things we measure or to our intuitive picture of the physical world. Classical mechanics does not need an interpretation, since the objects in its basic vocabulary – position, momentum, energy, force – correspond to our direct experience of the physical world and to the things we measure. In fact, the whole idea of interpretation is essentially meaningless in classical mechanics. However, quantum mechanics introduces a genuinely new concept, the wave function, which does not immediately correspond to anything we can measure or have a physical feeling for. In some ways, it is remarkable that quantum theory was discovered at all!

What do we mean by “interpretation”? This is a deep question which has some difficult philosophical aspects, but broadly, it comes down to two things. Firstly, it refers to the practical way in which physical predictions corresponding to measurements are extracted from the wave function. Secondly, and more philosophically, the phrase “interpretation” refers also to the picture of reality suggested by the theory. Quantum theory appears to involve a very strange picture of reality at the atomic level and this is one reason it is so difficult to really understand. Here, we will for the large part be concentrating on the first more pragmatic question, although it is probably reasonable to say that views on the second issue underlie all thinking in this area, if only unconsciously.

4. The Probability Formula

If the electrons around an atom are described in terms of a wave that is spread out all over the place, it became important to understand what this means, given that previously an electron was thought of as a point particle at a specific point in space. It was quickly realized in the development of quantum theory that the wave function $\psi(x)$ may be interpreted in terms of probability. Particles no longer could be thought of as being at definite points in space – there is only a probability of finding it at a certain point. Furthermore, the probability is given in terms of the wave function by the important formula

$$|\psi(x)|^2 = \text{probability of finding the particle at the point } x$$ (4)

However, the founders of quantum mechanics did not hit on this straight away. In the original paper of Max Born in 1926 [1], who first put forward this idea, we find the statement, that
the probability is proportional to a quantity he denotes $\Phi_{n,m}$ (essentially the same as $|\psi|$ in the notation used here). But then there is a footnote, clearly added at a later stage just before publication, stating, “more careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n,m}$”. Born still got the Nobel Prize anyway, despite this minor hitch!

Max Born may well have been guessing when he put forward this interpretation, but now, in the modern understanding of quantum theory, the theorem of Gleason confirms that the Born interpretation is essentially the only reasonable one [2]. The fact that the probabilities are given by the square of the wave function leads to much of the mystery and apparent paradox of quantum theory, that is, to the phenomena that are utterly different to classical phenomena. More about this shortly.

The wave description of matter and its probability interpretation automatically leads to the Heisenberg uncertainty principle. This says that the position $x$ and momentum $p$ of a particle cannot be precisely specified but suffer uncertainties $\Delta p$ and $\Delta x$ which obey the inequality

$$ (\Delta p)(\Delta x) \geq \frac{\hbar}{2} $$

where $\hbar$ is Planck’s constant and is of order $10^{-34} Js$. Planck’s constant is clearly a very small number compared to ordinary laboratory scales, which shows why this effect is not seen in classical physics. The uncertainty principle alone indicates that the description of physics systems suffers from statistical uncertainties, similar to those encountered in stochastic classical theories. However, what is important is that quantum mechanics is not just a classical stochastic theory, but exhibits new effects which are provably distinct from merely stochastic effects.

5. Superposition States and the Double-Slit Experiment

What makes quantum theory truly different from classical physics is the possibility of superposition states. The Schrödinger equation is a linear wave equation. This means that if $\psi_1$ and $\psi_2$ are solutions to it, describing two different physical situations, then the superposition state

$$ \psi = \psi_1 + \psi_2 $$

(6)
is also a solution to the equation – it is also a valid physical state. An example of such a superposition state is shown in Figure 1.

So far, there is nothing remarkable here, since most wave equations (like those obeyed by sound waves) have the same property. However, in quantum mechanics, the physics depends on the probabilities, which are given by the square of the wave function, $|\psi|^2$, and this does not behave a simple way for superposition states. For the superposition state Eq. (6), we have

$$|\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^* + \psi_1^*\psi_2$$

which means that the probability $|\psi|^2$ associated with the total wave function is not simply related to the probabilities $|\psi_1|^2$ and $|\psi_2|^2$ of the individual elements of the superposition. This leads to an important observation: that one of the essential differences between classical and quantum theory is that in the quantum theory, probabilities do not add up in the way we expect them to on the basis of classical intuition.

We can state this even more sharply. Mathematically, there is a close relationship between probability theory and logic. Loosely speaking, when probabilities can be assigned and add up properly, then we can make logical statements about the system that may be manipulated according to the rules of ordinary logic. This is sometimes called Boolean logic, or classical logic – basically, it is the logic of ordinary language. We probabilities cannot be assigned then the logical rules of ordinary language are not necessarily respected. So quantum theory has the possibility of violating the familiar logical structure of ordinary language. Hence, another way of expressing the essential difference between classical and quantum mechanics is to say that they involve different types of logic, and that quantum mechanics does not respect the logic of ordinary everyday language. The phrase “quantum logic” is sometimes used to describe the logical rules respected by quantum theory, but this does not help us to understand it.

An outstandingly important example of these ideas is the double slit experiment with electrons, first performed by Davisson and Germer in 1927 [3]. In this experiment, electrons are fired from a source towards a blind with two holes in it, and then detected at a screen beyond the holes, as shown in Figure 2. What the detector in fact measures is an interference pattern, and this experiment was a major piece of evidence in favour of the wavelike nature of matter.

The wave function $\psi(x)$ for particles that arrive at the screen at point $x$ is given by a
linear superposition of wave functions coming from hole 1 and hole 2. The probability of arriving at a certain point on the screen is given by the square of this,

\[ P_{12} = |\psi_1 + \psi_2|^2 \]  

and this indeed is what is measured by the detector.

If we were to apply our classical intuition to this experiment, we would be inclined to say that, since the particle hit the screen at \( x \) and since it was at an earlier time emitted from the source, then it must have passed through either slit number 1 or slit number 2. If this were the case, then we could assign probabilities \( P_1 = |\psi_1|^2 \) and \( P_2 = |\psi_2|^2 \) to the electrons coming from each slit, and the probability of hitting the screen would be simply

\[ P_1 + P_2 = |\psi_1|^2 + |\psi_2|^2 \]  

But this is not the same as Eq. (8), the result that is actually measured! So the probabilities don’t add up properly and we quite simply cannot make the logical deduction that the electron went through one hole or the other. The rules of ordinary logic do not apply! This simple experiment and its interpretation is perhaps the biggest challenge quantum theory makes on our intuition. (Interestingly, in a recent poll in Physics World, readers voted the double slit experiment with single electrons as “the most beautiful experiment in physics”\[4\].)

Other interesting challenges to our intuition become possible when we consider wave functions describing two particles. In particular, states of the form

\[ \Psi(x_1, x_2) = \phi(x_1)\chi(x_2) + \phi'(x_1)\chi'(x_2) \]  

are called entangled states and can often involve a degree of correlation between the particles which cannot be understood in classical terms. The EPR state is the most famous example of such a state\[5\].

6. The Copenhagen Interpretation

We now need to say a little bit more about the interpretation of quantum theory. The probability formula is not the end of the story, but it is part of a wider interpretational framework put forward by Bohr in the 1920s, which became known as the Copenhagen
interpretation. This set of ideas, which is widely accepted for most practical purposes, describes how quantum systems are affected by measurements [6].

In the strongest statement of the Copenhagen interpretation, it is asserted that the world is divided into two different regimes. There is the atomic scale, described by quantum mechanics, and the macroscopic, laboratory scale, which, to agree with our experience, must be described by the classical mechanics of Newton. In some sense we never in fact apprehend the quantum world directly, only its indirect effects on the macroscopic, classical world. The existence of a classical regime was thought to be necessary for the interpretation of the theory. This is already rather unsatisfactory since we would like quantum theory to be fundamental, yet we seem to have to postulate a division of the world into classical and quantum realms in order to understand quantum theory. It gets even more complicated.

The Copenhagen interpretation gives a specific mechanism for the way in which quantum systems are affected as a result of being measured. Suppose, for example, the quantum state of the system is a superposition state in which a single particle is localized about two different positions, as in Figure 1. If we perform a measurement which asks whether the particle is in a certain region, ∆, say, then, according to the Copenhagen interpretation, the effect of the measurement is that the wave function outside that region gets squashed to zero, leaving behind only the part within ∆. See Figure 3. This is the famous “collapse of the wave function” process. It is instantaneous and discontinuous. Furthermore, the precise way in which the wave function changes depends on what is measured – it changes differently, for example, if one measures momentum or energy instead of position.

Quantum systems therefore have two modes of evolution: smooth evolution according to the Schrödinger equation interspersed with abrupt jumps when measurements take place. Hence the evolution of quantum systems depends on whether they are being observed and what is being observed. David Mermin once wrote an article entitled, “Is the Moon Really There When No-One Looks?”, elaborating on this disturbing aspect of quantum systems [7]. The apparent observer-dependence of quantum theory is yet another fact that is utterly different to classical mechanics, since there is no inconsistency in maintaining that the classical systems of our immediate experience continue to exist whether or not we look at them, and that looking at them does not change them.
7. How is Classical Mechanics Different to Quantum Mechanics?

We now then come to the turning point of this article, which is to emphasize the difference between classical and quantum theory. These differences are best summarized in the following table.

| CLASSICAL MECHANICS                        | QUANTUM MECHANICS                     |
|--------------------------------------------|---------------------------------------|
| system state: $p, x$                       | system state: $\psi(x)$               |
| $p, x$ take definite values                | superpositions, entanglement          |
| classical (Boolean) logic                  | quantum logic                         |
| $m\ddot{x} + V'(x) = 0$                    | $i\hbar \frac{\partial \psi}{\partial t} = H\psi$ |
| observer-independent                       | depends on act of measurement         |
| no need for interpretation                 | Copenhagen interpretation             |

We have arrived at the following picture: Classical mechanics has a broad but finite degree of applicability. Quantum theory supersedes it as the description of nature in a much wider class of situations. Its description of the world is extremely different to that provided by classical mechanics. But they both describe the same phenomena. How can one reduce to the other? Furthermore, quantum mechanics needs a classical domain to assist its interpretation, yet this leads to a rather unsatisfactory dualistic view of the world.

It seems that the way out of this is to unashamedly assume that quantum theory is the fundamental description of everything, from the tiniest atom right up to the entire universe \[8\]. This explicitly drops the assumption of a separate external classical domain together with the rather cumbersome machinery describing quantum measurements in the Copenhagen interpretation. Our task, then, is to show that substantial parts of the universe appear to be classical even though they are described fundamentally by quantum mechanics.

This now brings us to the main point of this article, in which we consider how quantum theory becomes approximated by classical theory under certain conditions.
8. Emergent Classicality: Some Clues

In the standard quantum mechanics text books, it is often asserted that classical mechanics “emerges” from quantum mechanics when we take the quantum mechanical averages of position and momentum, \( \langle x \rangle, \langle p \rangle \). Indeed, it follows from the Schrödinger equation that \( \langle x \rangle \) obeys the equation of motion

\[
m \frac{d^2 \langle x \rangle}{dt^2} + \langle V'(x) \rangle = 0
\]

which is clearly very similar to the classical equation of motion Eq.(1) for \( \langle x \rangle \). This is the famous Ehrenfest theorem. But it is not exactly the same unless,

\[
\langle V'(x) \rangle = V'(\langle x \rangle)
\]

which is only true exactly for linear systems.

Another relevant fact is the evolution of wavepackets strongly concentrated about particular values of position and momentum (but consistent with the uncertainty principle). Such wavepackets satisfy Eq.(12) approximately, and under evolution according to the Schrödinger equation, follow approximately classical trajectories, at least for a period of time. However, these wavepackets typically spread out in time, which means that their positions and momentum become somewhat blurry unlike macroscopic classical objects. See Figure 4.

Despite their limitations, these results do however provide some useful clues. They show that there is an approximate classical determinism buried in the Schrödinger equation and that this might emerge if we look at the system in a sufficiently crude or averaged sort of way, that is, if we take a coarse-grained view of the dynamics. In simple terms, this is like looking at something from so far away that it is impossible to see the finer details of the structure. The necessity for coarse-graining in order to obtain classical physics is to be expected anyway because of the uncertainty principle, Eq.(5), since it is only by looking at position and momentum very coarsely that the effects of the uncertainty principle are not noticed.

However, the really important difference between classical and quantum systems is that quantum systems may exist in superposition states, exhibiting interference effects, and the Ehrenfest theorem says nothing about how these disappear.
9. Interferences and Classical States

The most important issue in understanding the appearance of classical mechanics from quantum mechanics is seeing how interferences are suppressed. This issue is most usefully rephrased in terms of the density matrix (and now we need to briefly get a bit more mathematical). For a system described in terms of a wave function $\psi(x)$, we define the density matrix by

$$
\rho(x, y) = \psi(x)\psi^*(y)
$$

(13)

So far there is nothing new in this. However, the density matrix is most useful when both quantum and statistical effects are separately present in the system. Suppose, for example, we don’t know exactly which quantum state the system is in, but we know only the probability $p_n$ that the system is in the quantum state $\psi_n(x)$. Then a very useful object is the density matrix

$$
\rho(x, y) = \sum_n p_n\psi_n(x)\psi^*_n(y)
$$

(14)

The quantum description of thermal equilibrium, for example, involves a state of this form. A state of this type, in which they are a combination of quantum and statistical effects is called a mixed state, whereas Eq.(13), which contains only quantum effects is called a pure state.

Now, the density matrix also gives a particularly useful picture of superposition states. Suppose the system is in a superposition state,

$$
\psi(x) = \frac{1}{\sqrt{2}} (\psi_1(x) + \psi_2(x))
$$

(15)

(where the $\frac{1}{\sqrt{2}}$ factor is for normalization). Then the associated density matrix is

$$
\rho(x, y) = \frac{1}{2} \left[ \psi_1(x)\psi^*_1(y) + \psi_2(x)\psi^*_2(y) + \psi_1(x)\psi^*_2(y) + \psi_2(x)\psi^*_1(y) \right]
$$

(16)

The last two terms represent the interferences, and because of them, we are not allowed to say that the system is in state $\psi_1$ or $\psi_2$ (because remember that the probabilities don’t add up properly). To get emergent classicality, what we want is for these terms to go away, and to get instead the mixed state,

$$
\rho(x, y) = \frac{1}{2} \left[ \psi_1(x)\psi^*_1(y) + \psi_2(x)\psi^*_2(y) \right]
$$

(17)
This density matrix does correspond to the statement that the system is in state $\psi_1$ or state $\psi_2$, with probability 1/2 for being in either state.

For the superposition state shown in Figure 1, a superposition of spatially localized states, the two density matrices Eqs.(16), (17) are shown in Figures 5 and 6.

10. Measurements, Decoherence and the Environment

The key question now is, how do the interference terms in the density matrix go away? The first question one might ask is whether the interference terms naturally go away under evolution according to the Schrödinger equation. There are a variety of ways of showing that this is in fact impossible. It is quite easy to show that the property of purity is preserved by the Schrödinger equation, so a pure state cannot go to a mixed state. Physically, this is because a mixed state involves statistical effects, whereas a pure state does not. To go from a pure state to a mixed state, statistical effects need to appear somehow, but the Schrödinger equation contains no such effects. (Although it is interesting to note that explicit modifications of the Schrödinger equation, which contain precisely such effects have been postulated [9].) Something more ingenious is therefore required.

At this stage, standard measurement theory in the Copenhagen approach gives a key insight. Recall that in the Copenhagen approach, it does not seem to be necessary to worry about the non-classical effects of superpositions. We just measure the system, whatever the state is, and get the result, which is then regarded as a “classical” thing. That is, when a quantum system is measured, in the Copenhagen sense, parts of superposition states go away. See Figure 3. But here, recall, we are trying to avoid talking about measurements because we are taking quantum theory to be universal. This brings us to an important idea. The standard machinery of quantum mechanics with the Schrödinger equation applies to isolated systems, but in practice, true isolation is extremely hard to achieve for macroscopic systems. The vast majority of macroscopic physical systems are not in fact isolated but are in continual interaction with their immediate surroundings – their environment, as we call it. The environment in effect continually “measures” macroscopic systems and it seems reasonable to suppose that it is this effect that habitually destroys interference, thereby rendering the systems classical. This effect is called decoherence and is absolutely central to our understanding of emergent classicality. See Figure 7.
11. System-Environment Models

These compelling physical ideas about decoherence are nicely illustrated in the classic calculation of Joos and Zeh \[10\] who considered the evolution of a massive particle, such as a dust grain, being continuously “measured” by its interactions with the surrounding gas molecules. They considered a particle with position \(x\) in interaction with an environment \(E\). The wave function for the combined system and environment is \(\psi(x, E)\) and this wave function describes the situation in which the dust grain is correlated with the surrounding molecules causing the measurements of it.

However, the key idea here is that the details of the environment state are not in fact of interest, so we average over it (an example of coarse-graining). The state of the system only is then described by an object called the reduced density matrix, obtained by averaging over environment states:

\[
\rho(x, y) = \sum_E \psi(x, E)\psi^*(y, E) \tag{18}
\]

This object is similar in interpretation to the usual density matrix in standard quantum theory. However, because we have averaged out part of the system it no longer evolves according to the Schrödinger equation. The whole system described by \(\psi(x, E)\) evolves according to the Schrödinger equation, but for the particle only, Joos and Zeh derived a so-called master equation, whose one-dimensional version is

\[
\frac{\partial \rho}{\partial t} = -\frac{i\hbar}{2m} \left( \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2} \right) - D(x - y)^2 \rho \tag{19}
\]

The first term on the right-hand side is easily recognized as the usual Schrödinger evolution. What is new is the second term, \(D(x - y)^2 \rho(x, y)\), describing the continuous measurement effect of the environment. Importantly, this term turns pure states into mixed states, exactly what we need.

How does this new term affect the dynamics? On short timescales we have

\[
\rho(x, y, t) \sim e^{-D(x-y)^2t} \rho(x, y, 0) \tag{20}
\]

which means that the interference terms with \(x \neq y\) decay exponentially fast, the effect we are looking for. In typical models,

\[
D = \frac{2m\gamma kT}{\hbar^2}
\]
where $T$ is the temperature of the environment and $\gamma$ the dissipation. The key thing is that $D$ goes like $1/\hbar^2$ so $D$ has the possibility of being extremely large for laboratory scales and hence the interference terms extremely small. For example, if we take the mass, temperature, lengthscale $|x - y|$, etc. to be of order 1 in cgs units, we find that the off-diagonal terms, representing interference, are very small indeed:

$$e^{-D(x-y)^2t} \sim \exp(-10^{40})$$

This shows rather decisively why interference effects are not observed on macroscopics scales! See Ref. [11] for a similar calculation. Note also that the decoherence effect may also be shown to kill the non-classical effects of entangled states, such as Eq. (10) [12, 13].

This example is by no means pathological. For most macroscopic systems decoherence is the most ruthless and efficient process in the whole of physics and is the main physical explanation as to how quantum systems become classical.

11. What does it mean for a quantum system to be approximately classical?

We now have a heuristic picture of how we can understand the emergence of classical mechanics from quantum theory. It is time to focus this into something more precise.

When we say that a system is classical, we mean that it can be described by variables $p, x$ which have definite values (or at least, probabilities) and that these variables evolve according to classical evolution equations. Classical evolution equations, such as Newton’s law, Eq. (1), involve two time derivatives. So to assert that the classical equation of motion is satisfied we need to specify position at three moments of time. This means that the whole notion of “classical” is tied up with the notion of a history – a sequence of events distributed in time.

This suggests that if we wish to decide whether a given quantum system is approximately classical, we do the following. We use quantum theory to attempt calculate the probability for a history of positions, $p(\alpha_1, t_1, \alpha_2, t_2 \cdots)$ and then see if it is strongly peaked about the classical evolution equations. See Figure 8. However, it is here that the interesting tension lies, because quantum theory resists the notion of history. This is for two reasons.

First, the uncertainty principle limits the precision to within which positions at different times may be specified (since this is equivalent to specifying position and momentum at
one time). This means that probabilities for histories cannot be perfectly peaked about a single classical path. However, if we look at positions that are sufficiently coarse-grained compared to the uncertainty principle limits, we might expect the probability to be quite strongly peaked about one path.

Secondly, and more fundamentally, as we have seen in the double slit experiment interference resists the assignment of probabilities to histories. However, we know we can get round this when there is a decoherence mechanism to kill the interferences.

It therefore appears to be a very worthwhile goal explore formulations of quantum mechanics which concentrates on looking at probabilities for histories, and which also incorporates a decoherence mechanism to make sure that those probabilities are well-defined.

12. The Decoherent Histories Approach to Quantum Theory

Everything said so far is very concisely and comprehensively synthesized in a new understanding of quantum theory which goes by the name of the decoherent histories approach (or consistent histories approach). This approach was invented by Robert Griffiths in 1984 and subsequently developed substantially by Roland Omnès. Murray Gell-Mann and James Hartle invented and developed a similar approach, in part independently. (See also Ref. for a recent review).

The decoherent histories approach is standard quantum mechanics but without the usual assumptions about classical domains or measurements. It assumes that everything is quantum, up to and including the entire universe, so there is no division of the world into classical and quantum regimes. Its just all quantum theory. Instead of measurements, the theory concentrates on finding those situations to which probabilities may be assigned. This is a much weaker and more objective notion and does not rely on a classical domain. We look at histories partly because they are the most general class of possible situations, but also because they are necessary to characterize classical behaviour we seek to explain the emergence of.

Here is a simple account of how it works. We need to extend the formalism of quantum theory so that it can describe a history – a series of events distributed in time. Suppose the system has an initial state $|\psi\rangle$ and we want to talk, in quantum mechanics, about a history in which the system evolves through a series of properties $\alpha_1, \alpha_2 \cdots \alpha_n$ at times $t_1, t_2 \cdots t_n$. 

Then this history is represented by the state

$$|\psi_\alpha\rangle = P_{\alpha_n}(t_n)P_{\alpha_{n-1}}(t_{n-1})\cdots P_{\alpha_1}(t_1)|\psi\rangle$$

(22)

Here the objects $P_\alpha(t)$ are projection operators at each moment of time characterizing the property we are interested in at that time. This is best thought of in terms of a diagram. Without going into detail, the state $|\psi_\alpha\rangle$ could represent, for example, the statement in quantum mechanics that the particle started out in the state $\psi(x)$ and then it passed through the gate $\alpha_1$ at time $t_1$, and then the gate $\alpha_2$ at $t_2$ and so on. See Figure 9. Following the normal rules of quantum theory, the probability for the entire history is

$$p(\alpha_1, \alpha_2, \cdots \alpha_n) = \langle \psi_\alpha | \psi_\alpha \rangle$$

(23)

which is nothing more than an extension to histories of the usual Born rule, Eq.(4).

However, as already stressed, probabilities cannot in general be assigned to histories because of interference. Interference means that probabilities will not add properly in the way we expect classical probabilities to do, so the probabilities are meaningless. Now here is the important point. What is new about the decoherent histories approach is that the theory also includes a test to see which sets of histories suffer quantum interference and which do not. The mathematical condition for no interference is that all distinct pairs of histories $\alpha, \alpha'$ in the set must have no overlap:

$$\langle \psi_{\alpha'} | \psi_{\alpha} \rangle = 0 \quad \text{for} \quad \alpha \neq \alpha'$$

(24)

The more physical significance of this condition is that it ensures that the probabilities add up properly, that is, that

$$p(A \text{ or } B) = p(A) + p(B)$$

(25)

for all exclusive pairs of distinct histories $A$ and $B$. When histories satisfy this condition, we call them a decoherent set of histories. They can be assigned probabilities and manipulated as if they were classical histories. There is no need to appeal to notions of measurement.

We know from the double slit experiment that that no-interference condition Eq.(24) or Eq.(25) will not be satisfied in general, so probabilities cannot always be assigned to histories. But it can be satisfied when there is an environment present to produce decoherence, since satisfaction of Eq.(24) is very similar to the diagonalization of the density matrix Eq.(20). When probabilities can be assigned we can talk about the properties of the system as if they
were real, classical things. Differently put, in the non-Boolean logic of quantum theory, we look for Boolean subsets.

So this is the histories approach in a nutshell: a formula for probabilities for histories and a simple condition to determine when those probabilities are meaningful.

This basic framework has been used extensively to analyze emergent classicality in a variety of different settings [19, 20]. For example, one can compute the probability that a particle passes through a series of spatial regions at a sequence of time, with an environment present to ensure decoherence. The probabilities for such histories are typically strongly peaked about classical equations of motion, not quite of the form Eq. (1), but with dissipation. Furthermore, as discussed above, the peaking cannot be perfect so there are fluctuations about the classical path (produced by the essentially random effects of the environment). Hence what emerges from quantum theory is not precisely Newtonian mechanics, Eq. (1), but a classical stochastic theory, like Eq. (2). For particles of sufficiently large mass, the stochastic effects are however negligible and a near-deterministic equation of motion is obtained.

The decoherent histories approach, however, offers much more than a coherent account of emergent classicality. Since the formalism does not rely in measurements it can be used to analyze conceptually tricky points in quantum theory where the debate centres around what is happening to the system when it is not being measured. Of course, the decoherent histories approach cannot provide answers where none exist any more than standard quantum theory can. However, the emphasis on finding situations where Boolean logic applies brings a certain clarity lacking in the Copenhagen interpretation. Often so-called “paradoxes” in quantum theory are seen to arise from a use of logic that, in the decoherent histories approach, would not be permitted [14, 15, 16, 17].

13. The Preferred Basis Problem

There is another important issue in the question of emergent classicality still to be addressed. This is the issue of why certain variables, such as position, momentum and energy appear to have a preferred role in describing the classical world, when quantum theory appears to indicate no preference. That is, in the decoherence process that produces classical behaviour, interferences between different values of position are killed, and so we get to assign probabilities to histories of position. Quantum theory doesn’t care if the state of the
system is localized in position or in a superposition of such localized states but somehow localized position states appear to be preferred.

When decoherence is produced by interactions with an environment, the provisional answer to this question is that the environment interacts with the system of interest through position. The interactions are local in position. So the environment effectively “measures” the position of the system and not some other funny variable.

But there is a lot more to it than that. The question relates to an even more general question, which is how do we decide where to draw the line between “system” and “environment”? In many situations there is no obvious environment to produce the decoherence process. This takes us right back to where we started, to the old Copenhagen interpretation, in which the world was arbitrarily divided into classical and quantum regimes.

Fortunately there is the beginnings of an answer to these issues, and it appears most clearly in the decoherent histories approach. The decoherent histories approach to quantum theory does not specifically require an environment to produce decoherence, just some sort of coarse-graining process. There is in fact a set of general physical principles that help us to decide which variables to concentrate on and which to ignore in the coarse graining process necessary to get decoherence, namely, conservation laws. For a system consisting of a collection of interacting particles, the total energy, momentum, number, charge etc. are conserved, that is, they are unchanged under time evolution. Importantly, conservation laws persist through to quantum theory (usually), and at least in simple situations in non-relativistic physics, energy, momentum, number etc. are conserved.

Now in quantum theory, conserved quantities have a very interesting property. This is that there can be no interference between different values of a conserved quantity. The reason for this is that to observe the interference effects between, say, states of different charge, one would need a measuring device which violates charge conservation, and this is impossible. Alternatively, if we have a superposition state of conserved quantities (as in Figure 10, for example), the different elements of the superposition can interfere only if they evolve into each other. But conserved quantities don’t evolve anywhere, even in the quantum theory so the different elements of the superposition never meet. So the superposition still “exists” in some sense, but its interference effects are never seen.

In the decoherent histories approach, this means that one can always assign probabilities to histories of conserved quantities. Or in other words, conserved quantities can always
be manipulated as if they were classical, even in the quantum theory. Now this is a very important point. It means that in the foggy landscape of quantum theory, the conserved quantities represent the mountain peaks that reach above the clouds, immune to weird quantum behaviour.

Conserved quantities are not the end of the story. They can be used to define quantities that are not exactly conserved, but approximately conserved, and so slowly varying. In particular, we can define the local densities associated with them: if energy, for example, is conserved, then the energy in a small region of the system can only change by flowing in or out of the region, so will tend to change slowly. See Figure 11. If exactly conserved quantities are decoherent then approximately conserved quantities will be approximately decoherent \[19, 21\]. This means that, to a very good approximation, we can assign probabilities to them, which will add up properly with only very small errors.

Hence the real question is not how to divide up the universe into classical and quantum, or system and environment, but into slowly and rapidly varying, and conservation laws are the fundamental physical principle that guides us in how to do this.

These ideas have been used to give what is perhaps the most general possible account of emergent classicality from quantum theory. That is, the approximate conservation of the local densities has been used to show that they are naturally decoherent, and to obtain classical equations of motion for them. These equations are hydrodynamics equations, of the Navier-Stokes form, for example \[21, 22\].

14. Summary

The essence of this article may be very concisely summarized as follows:
(a) Classical physics emerges from quantum theory at sufficiently coarse-grained scales when there is a natural division into fast and slow variables.
(b) Conservation laws are the guiding principle in making that division.

15. Other Literature

The point of this article has been to give a simple flavour of the physical ideas involved in understanding emergent classicality, mainly from the perspective of the decoherent histories...
approach. It is far from exhaustive and the selection of references is rather sparse. Much work has been done on many issues touched on here. See the review article Ref. 20 for a more extensive review of the decoherent histories approach, with references. Many of the key ideas about emergent classicality are nicely covered in the article by Hartle. See also the websites maintained by Brun 24 and by Joos 25, both very valuable and extensive sources of literature.

Acknowledgements

I have benefited from discussions with many people over the years on the topic of this paper, but special mention goes to Jim Hartle for many conversations over a very long period of time. I would also like to thank Neal Powell and Marko Ivin for their assistance in preparing the figures.

[1] Born, M., 1926, Z.Phys., 37, 863.
[2] Gleason, A.M., 1957, Jour.Math.Mech., 6, 885.
[3] Davisson, C., and Germer, L.H., 1927, Phys.Rev., 30, 705.
[4] http://physicsweb.org/article/world/15/9/2
[5] Einstein, A., Podolsky,B. and Rosen, N., 1935, Phys. Rev. 47, 777.
[6] Wheeler, J.A., and Zurek, W.H., 1983, Quantum Theory and Measurement (Princeton University Press, Princeton, NJ).
[7] Mermin, D., 1990, in Boojums All the Way Through: Communicating Science in a Prosaic Age (Cambridge University Press, Cambridge).
[8] Everett, H., 1957, Rev.Mod.Phys., 29, 454.
[9] Ghirardi, G.C., Rimini, A., and Weber,T., 1986, Phys.Rev. D34, 470.
[10] Joos, E. and Zeh, H.D., 1985, Z.Phys. B59, 223.
[11] Zurek, W.H., 1986, in Frontiers of Non-Equilibrium Statistical Mechanics, edited by G.T.Moore and M.O.Scully (Plenum, New York).
[12] Diosi, L., 2003, Lect. Notes Phys., 622 157.
[13] Dodd, P., and Halliwell, J.J., 2004, Phys.Rev., A69, 052105.
[14] Griffiths, R.B., 1984, *J.Stat.Phys.* **36**, 219.

[15] Griffiths, R.B., 2001, *Consistent Quantum Theory* (Cambridge University Press, Cambridge).

[16] Omnès, R., 1992, *Rev.Mod.Phys.*, **64**, 339.

[17] Omnès, R., 1999, *Understanding Quantum Mechanics* (Princeton University Press, Princeton).

[18] Gell-Mann, M., and Hartle, J.B., 1990, in *Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity*, Vol. VIII, edited by W. Zurek (Addison Wesley, Reading, MA).

[19] Gell-Mann, M., and Hartle, J.B, 1993, *Phys.Rev.* **D47**, 3345.

[20] Halliwell, J.J., 2003, e-print quant-ph/0301117

[21] Halliwell, J.J., 1999, *Phys.Rev.Lett.*, **83**, 2481.

[22] Halliwell, J.J., 2003, *Phys.Rev.*, **D68**, 025018.

[23] Hartle, J.B., 1994, e-print gr-qc/9404017.

[24] http://almaak.usc.edu/~tbrun/Data/decoherence_list.html

[25] http://www.decoherence.de

**Figure Captions**

**Figure 1.** A wave function consisting of a superposition of two wave functions, each localized around a different point in space.

**Figure 2.** The double slit experiment. Electrons are fired at a blind with two holes in it and detected at the screen. At the screen, the wave function of the system is of the form $\psi = \psi_1 + \psi_2$, where $\psi_1$ and $\psi_2$ represent the wave functions of electrons coming from slits 1 and 2 respectively. The detector measures an intensity proportional to $P_{12} = |\psi_1 + \psi_2|^2$. This is is not the same as $|\psi_1|^2 + |\psi_2|^2$, the result one might guess from classical expectations.

**Figure 3.** A schematic representation of the collapse of the wave function process which takes place as a result of a quantum measurement. A measurement localized around the right-hand peak in the superposition state causes the wave function to “collapse” to zero everywhere outside the measured region, leaving only the piece of the wave function within the measured region.

**Figure 4.** The spreading of the wave function. A wavepacket tightly peaked in position at $t = 1$ evolves along an approximately classical trajectory but tends to become more spread
out as time evolves.

**Figure 5.** The density matrix for a pure state, Eq.(16), for the superposition state of the form shown in Figure 1. The interference terms are represented by the two off-diagonal \((x \neq y)\) peaks.

**Figure 6.** The density matrix for the mixed state, Eq.(17). The off-diagonal terms representing interferences are absent compared to Figure 5.

**Figure 7.** An example of a system being continually monitored by its environment. The system is a collection of small particles, such as dust grains, and the environment consists of the molecules in the surrounding atmosphere which scatter off the dust grains, effectively “measuring” them.

**Figure 8.** The trajectory of a classical system is determined by seeing if it passes through a series of gates \(\alpha_1, \alpha_2, \alpha_3, \cdots\) at times \(t_1, t_2, t_3, \cdots\).

**Figure 9.** A schematic picture of the construction of the quantum state for a history \(|\psi_\alpha\rangle\). It represents the statement that the system starts in the initial wave function \(|\psi\rangle\) and then passes through the gates \(\alpha_1, \alpha_2, \alpha_3, \cdots\) at times \(t_1, t_2, t_3, \cdots\).

**Figure 10.** A wave function consisting of a superposition of two states, each strongly concentrated around different values of energy.

**Figure 11.** A volume \(V\) of a box of gas molecules. Since energy (for example) is conserved, the total energy within \(V\) can only change by particles flowing in or our of \(V\). If \(V\) is sufficiently large compared to the microscopic scale (and comoving with the average flow), then the total energy in \(V\) will be a slowly varying quantity, so will we approximately decoherent.

**About the Author**

J.J.Halliwell did his PhD research with Stephen Hawking at the University of Cambridge, graduating in 1986. After a Research Fellowship for a year at Christ’s College, Cambridge, he moved to the USA, spending two years at the Institute for Theoretical Physics, Santa Barbara, and three years at MIT, in Cambridge, Massachusetts. He came to Imperial College in 1992, initially as a Royal Society University Research Fellow, and shortly after was appointed lecturer. He has been a Professor of Theoretical Physics since 2002. His research interests are centred around quantum cosmology, quantum gravity and the related
problems these fields generate in the foundations of quantum theory.
This figure "fig1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig2.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig3.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig4.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig5.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig6.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig7.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig8.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig9.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig10.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1
This figure "fig11.jpg" is available in "jpg" format from:

http://arxiv.org/ps/quant-ph/0501119v1