Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole

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Abstract
Aiming to search for a signal of space-time noncommutativity, we study a quasinormal mode spectrum of the Reissner–Nordström black hole in the presence of a deformed space-time structure. In this context we study a noncommutative (NC) deformation of a scalar field, minimally coupled to a classical (commutative) Reissner–Nordström background. The deformation is performed via a particularly chosen Killing twist to ensure that the geometry remains undeformed (commutative). An action describing a noncommutative scalar field minimally coupled to the RN geometry is manifestly invariant under the deformed $U(1)_\star$ gauge symmetry group. We find the quasinormal mode solutions of the equations of motion governing the matter content of the model in some particular range of system parameters which corresponds to a near extremal limit. In addition, we obtain a well defined analytical condition which allows for a detailed numerical analysis. Moreover, there exists a parameter range, rather restrictive though, which allows for obtaining a QNMs spectrum in a closed analytic form. We also argue within a semiclassical approach that NC deformation does not affect the Hawking temperature of thermal radiation.

Keywords: noncommutative scalar field, noncommutative gauge theory, Reissner–Nordström black hole, black hole quasinormal modes

(Some figures may appear in colour only in the online journal)
1. Introduction

Excitations of black holes may be provoked in different ways. They can appear as a result of merging of two black holes, as was the case with the recent LIGO experiment [1] that brought about the first detection of gravitational waves, or simply by infall of matter into a supermassive black hole. Once being perturbed, a black hole responds to a perturbation by going through a ringdown phase during which it emits gravitational waves. The ringdown phase of a black hole may broadly be divided into three stages, a short period of strong initial outburst of radiation, which is then followed by a long period of damped oscillations, dominated by quasinormal modes (QNMs). After QNMs damp out at later times, they become superseded by a power law behaviour of the field, known as late-time tails. Among the three stages mentioned, the second stage, characterized by quasinormal modes, is far more important than the other two, since it significantly dominates the perturbation signal. This makes QNMs a very important object of study, not only for the reason of searching for gravitational waves, but also for other reasons, such as the study of (in)stability of black holes [2, 3], the problem of black hole area quantization [4–10], AdS/CFT correspondence [11, 12] and others.

Since the second stage of the black hole ringdown, which is dominated by QNMs, is essentially independent of the details of a perturbation, it appears to be particularly suited for exploring the properties of black holes. This feature of QNMs, namely that they essentially do not depend on the details of a perturbation, but only on the parameters of a black hole, makes them a fairly convenient carrier of information on the properties of black holes. Black hole QNMs [2, 13–18] also provide key signatures of the gravitational waves. Moreover, the recent experimental discovery of gravitational waves including the ringdown phase arising from black hole mergers [1] have opened up new possibilities for the observations of QNMs. This raises to a more solid ground the idea of the gravitational wave astronomy as being able to probe the primordial universe.

Besides carrying the intrinsic information about black holes, it is reasonable to assume that QNMs may also carry information about the properties of the underlying space-time structure. In particular, if the underlying space-time structure is deformed in such a way that it departs from the usual notion of smooth space-time manifold, then this deviation should also in some way be imprinted in the QNMs’ spectrum.

There are abundant theoretical arguments which point toward a necessity for deforming a space-time manifold at higher energies. In particular, it is well known that the quantum theory and general relativity together lead to a noncommutative (NC) description of space-time [19–21]. There are many different methods that are specifically devised for dealing with the deformed structures of space-time. In this paper we use the method of deformation by a Drinfeld twist. We derive the differential calculus along the lines of [22, 23] and we use the Seiberg–Witten map [24, 25] to derive the NC gauge theory. The choice of twist is non-unique and there are various examples in the literature. Inspired by the \( \kappa \)-Minkowski space-time [26] with commutation relations

\[
[x^0, x^j] = x^0 x^j - x^j x^0 = iax^j, \quad [x^j, x^k] = 0, \quad (1.1)
\]

with an arbitrary constant \( a \), in our previous work [27, 28] we constructed the NC \( U(1)_a \) gauge theory based on the twist

\[
\mathcal{F} = e^{-\frac{a}{2}(\partial \otimes x^j + x^j \otimes \partial)}, \quad (1.2)
\]

with two commuting vector fields \( X_1 = \partial_i \) and \( X_2 = x^i \partial_j \) and indices \( j = 1, 2, 3 \). However, the vector field \( X_2 \) does not belong to the Poincaré algebra and thus, the deformed (twisted)
space-time symmetry of the constructed model is the twisted $gl(1, 3)$. To keep the space-time symmetry at the level of a twisted Poincaré symmetry, in this work we choose a twist of the form

$$\mathcal{F} = e^{-\frac{\alpha}{2} \left( \partial_t \otimes (x \partial_y - y \partial_x) - (x \partial_y - y \partial_x) \otimes \partial_t \right)}.$$  (1.3)

This twist we call ‘angular twist’ since the vector field $x \partial_y - y \partial_x$ is the generator of rotations around the $z$-axis. The twist (1.3) has additional nice properties when applied to some special curved backgrounds as we will explain through the paper.

In order to find experimental signatures of noncommutativity one can construct different field theory models on NC spaces. NC correction to Standard Model of particle physics have been intensively discussed in literature [29]. Various NC extensions of General Relativity with different phenomenological consequences have also been proposed [30]. Our goal in this paper is to study quasinormal modes of a NC scalar field in a fixed background of the Reissner–Nordström (RN) black hole. More precisely, we fix a particular NC deformation of space-time by ‘turning on’ the twist (1.3). In string theory, this step corresponds to turning on background fluxes. Then strings propagate in a fixed background determined by these fluxes. After fixing the NC deformation in our model, we study the propagation of scalar and U(1) gauge fields in the non-propagating geometry of RN black hole. In particular, we study the QNMs spectrum of the charged, massive scalar field. Our model is semiclassical in the following sense: the backreaction of the NC scalar field on the background geometry is neglected and therefore the background remains isotropic. We plan to address the problem of backreaction of the NC scalar field on the background geometry in our future work. Let us point out that, based on our previous work [31], we expect the first non-zero corrections (anisotropy) to the RN background to be of the second order in the NC parameter $\alpha$. In this way, they can safely be neglected in the present paper where the expansions are done only up to first order in the NC parameter $\alpha$. Finally, note that throughout the paper, exception being the end of section 6, we use the term ‘semiclassical’ in a somehow different context. Namely, our scalar field is not quantized in the sense of quantum field theory, but it feels the effects of space-time deformation (noncommutativity). Therefore the scalar field is noncommutative. The gravitational field is also not quantized, but in addition, it does not feel the noncommutativity.

Getting back to quasinormal modes, they were first introduced by Vishveshwara [13] as solutions to the wave equation that are purely ingoing at the horizon and purely outgoing at far infinity. The boundary conditions of pure ingoingness at the horizon and pure outgoingness at infinity that are imposed on the solutions of the wave equation reflect the intrinsic inability of black holes to sustain stable matter-energy configurations in the region around them. Over the years an extensive machinery was developed to calculate QNMs for various geometries and types of perturbations. The methods range from purely numerical [37–39], all the way through a set of various semianalytic and semiclassical methods, [36, 40–44] down to the purely analytic ones. However, there are very few cases that are amenable to an exact analytical treatment. Among them are the QNMs calculated in an analytically exact way for the case of scalar perturbations in the nonrotating BTZ background [16], as well as in the rotating BTZ background perturbed by massive scalars [45]. Fermionic perturbations in the same background were treated in [12, 46, 47] and vector perturbations in [12]. The quasinormal spectrum in the pure anti-de Sitter space-time was first found in [48] for a massless scalar field and for gravitational perturbations in four-dimensional AdS space-time was found in [49] and then generalized to arbitrary space-time dimensions in [50]. Exact analytical treatments

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3 For asymptotically anti-de Sitter geometries this definition changes a little bit, which effects the corresponding QNMs spectrum [32–36]. Here, we shall be interested in geometries that are asymptotically flat.
in de Sitter space searching for QNMs spectrum were carried out in [50] and [51] in arbitrary space-time dimensions. The analytical expressions for the quasinormal spectra of topological AdS black holes were found for the massive scalar field [52] and for gravitational perturbations [53]. Moreover, the exact solutions for quasinormal modes of the Gauss–Bonnet black hole were found in [54, 55]. Likewise, the analytical approximations for quasinormal modes of dilatonic black holes were found in [56] and [57] and for \(d\)-dimensional Schwarzschild black hole with Gauss–Bonnet term in [58].

Analytic treatment may also be possible in other settings, though for a limited ranges of system parameters, such as the near extremal regime of black holes [59–61], or different asymptotic limits, including large multipole number expansion or high overtone numbers [58, 62–67].

QNMs spectra in certain settings have been analysed in the presence of the noncommutative (NC) structure of space-time [68–70]. As we have already mentioned, in this paper we continue along these lines and consider quasinormal modes of a NC scalar field perturbing a black hole background of the Reissner–Nordström type.

To start with, in section 2 we introduce the NC space-time and the NC differential calculus. More details on the twist formalism and the twisted differential geometry are presented in the appendix. Using the twisted differential calculus, in section 3 we construct the NC U(1)\(_*\) gauge theory coupled to a charged scalar field. The propagating fields are the charged NC scalar field and the U(1)\(_*\) gauge field, while the gravitational field, describing the background is fixed. The background is arbitrary with one constraint: the vector fields entering the definition of the angular twist have to be the Killing vector fields for the chosen background. Using the twisted differential geometry, we write the action for our model, such that it is invariant under the NC U(1)\(_*\) gauge transformations. Then we use the Seiberg–Witten map [25] to expand the NC fields in terms of the corresponding commutative fields and obtain the expanded action. The action and the corresponding equations of motion are expanded up to first order in the NC parameter \(a\). In that way we also fix the U(1) gauge field to be the electromagnetic field of the Reissner–Nordström black hole. The scalar field remains the only propagating field and we calculate its equation of motion in this particular background. This equation is our starting point for discussing QNMs solutions in section 5. We do the QNMs analysis in the near extremal region, \((r_+ - r_-)/r_+ \to 0\). This approximation enables us to find a condition for QNMs frequencies analytically. However, the condition itself has to be solved numerically. We present our solutions at the end of sections 5 and 6. In addition, in a very special region of parameters the analytic solution of the QNMs condition can be calculated and we present it in section 6.

Both numerical and analytic solutions show that there is a Zeeman-like splitting of the QNMs frequencies for a fixed angular momentum number \(l\). Solutions depend on the magnetic number \(m\) in such a way that \(m\) always couples to the NC parameter \(a\). This property we discuss in section 6. Finally, for completeness we also calculate the changes in the Hawking temperature of this system (RN black hole and the NC scalar field). We find no changes. We end section 6 with some plans for future research.

2. Noncommutative space-time form the angular twist

Reissner–Nordström (RN) black hole is a well known solution of Einstein equations. It represents a charged non-rotating black hole and it is given by
Here $M$ is the mass of the RN black hole, while $Q$ is the charge of the RN black hole. This space-time is static and spherically symmetric, therefore it has four Killing vectors. Two of them are $\xi_1 = \partial_t$, and $\xi_2 = \partial_x = x\partial_y - y\partial_x$.

We have already mentioned in Introduction that the semiclassical approach used in this paper is based on the assumption that the geometry (gravitational field) is classical (it is not deformed by noncommutativity), while the scalar field propagating in the RN background ‘feels’ the effects of space-time noncommutativity, but it is not quantized in a sense of quantum deformed by noncommutativity), while the scalar field propagating in the RN background.

In order to realize this approach, we choose a Killing twist. The twist is given by

$$\mathcal{F} = e^{-i\theta^{AB}X_A \otimes X_B},$$

(2.5)

Here $\theta^{AB}$ is a constant antisymmetric matrix

$$\theta^{AB} = \left( \begin{array}{cc} 0 & a \\ -a & 0 \end{array} \right),$$

with an arbitrary constant $a$. Indices $A, B = 1, 2$, while $X_1 = \partial_t$, $X_2 = x\partial_y - y\partial_x$ are commuting vector fields, $[X_1, X_2] = 0$. This twist fulfils the requirements (A.1)–(A.3). We call (2.5) ‘angular twist’ because the vector field $X_2 = x\partial_y - y\partial_x$ is nothing else but a generator of rotations around $z$-axis, that is $X_2 = \partial_\varphi$. The vector fields $X_1$ and $X_2$ are two Killing vectors for the metric (2.4) and that is why the twist is called a Killing twist. In particular, the twist (2.4) does not act on the RN metric and it does not act on the functions of the RN metric. In this way we ensure that the geometry remains undeformed. Let us mention that the deformation of this type is a special case of deformations introduced in [71, 72]. Similar type of twist operators that lead to a Lie algebra-type of deformation of Minkowski space-time were also considered in [73].

Since the twist (2.5) introduces space-time noncommutativity, see (2.11), one might be concerned with a formulation of unitary quantum field theory [74]. However, it was shown in [75] that a careful formulation of quantum field theory with space-time noncommutativity leads to no problems with unitarity and causality.

Before we proceed with the construction of differential calculus, let us comment on the relation between the $\kappa$-Minkowski deformation introduced in [27, 28] and the twist we use in this paper. In the case of $\kappa$-Minkowski twist in [27, 28], the vector fields defining the twist were chosen to be $X_1 = \partial_t$, and $X_2 = x\partial_y - y\partial_x$. The vector field $X_2 = x\partial_y - y\partial_x$ does not belong to the Poincaré algebra, but instead to the algebra of general linear transformations $gl(1, 3)$. Therefore, the twisted symmetry of the $\kappa$-Minkowski space-time is the twisted general linear symmetry. In the case of (2.5) the vector fields defining the twist belong to the Poincaré algebra and the twisted symmetry of a NC Minkowski space-time obtained for the twist (2.5) is the twisted Poincaré symmetry. We will not be concerned with the deformation of Minkowski space-time in this paper. Just for the completeness, we write the twisted Poincaré Hopf algebra following from (2.5). We use hermitian generators. In coordinate representation they are:

$$P_\mu = -i\partial_\mu, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad \eta_{\mu\nu} = \text{diag}(+,-,-,-).$$

The twisted Poincaré Hopf algebra is

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu),$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}).$$

(2.6)
\[ \Delta F P_0 = P_0 \otimes 1 + 1 \otimes P_0, \]
\[ \Delta F P_3 = P_3 \otimes 1 + 1 \otimes P_3, \]
\[ \Delta F P_1 = P_1 \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes P_1 + P_2 \otimes \sin \left( \frac{a}{2} P_0 \right) - \sin \left( \frac{a}{2} P_0 \right) \otimes P_2, \]
\[ \Delta F P_2 = P_2 \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes P_2 - P_1 \otimes \sin \left( \frac{a}{2} P_0 \right) + \sin \left( \frac{a}{2} P_0 \right) \otimes P_1, \]
\[(2.7)\]
\[ \Delta F M_{01} = M_{01} \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes M_{01} + M_{02} \otimes \sin \left( \frac{a}{2} P_0 \right) - \sin \left( \frac{a}{2} P_0 \right) \otimes M_{02} - P_1 \otimes \frac{a}{2} M_{12} \cos \left( \frac{a}{2} P_0 \right) + \frac{a}{2} M_{12} \cos \left( \frac{a}{2} P_0 \right) \otimes P_1 - P_2 \otimes \frac{a}{2} M_{12} \sin \left( \frac{a}{2} P_0 \right) - \frac{a}{2} M_{12} \sin \left( \frac{a}{2} P_0 \right) \otimes P_2, \]
\[ \Delta F M_{02} = M_{02} \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes M_{02} - M_{01} \otimes \sin \left( \frac{a}{2} P_0 \right) + \sin \left( \frac{a}{2} P_0 \right) \otimes M_{01} - P_2 \otimes \frac{a}{2} M_{12} \cos \left( \frac{a}{2} P_0 \right) + \frac{a}{2} M_{12} \cos \left( \frac{a}{2} P_0 \right) \otimes P_2 + P_1 \otimes \frac{a}{2} M_{12} \sin \left( \frac{a}{2} P_0 \right) + \frac{a}{2} M_{12} \sin \left( \frac{a}{2} P_0 \right) \otimes P_1, \]
\[ \Delta F M_{03} = M_{03} \otimes 1 + 1 \otimes M_{03} = \frac{a}{2} P_3 \otimes M_{12} + \frac{a}{2} M_{12} \otimes P_3, \]
\[ \Delta F M_{12} = M_{12} \otimes 1 + 1 \otimes M_{12}, \]
\[ \Delta F M_{13} = M_{13} \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes M_{13} + M_{23} \otimes \sin \left( \frac{a}{2} P_0 \right) - \sin \left( \frac{a}{2} P_0 \right) \otimes M_{23} \]
\[ \Delta F M_{23} = M_{23} \otimes \cos \left( \frac{a}{2} P_0 \right) + \cos \left( \frac{a}{2} P_0 \right) \otimes M_{23} - M_{13} \otimes \sin \left( \frac{a}{2} P_0 \right) - \sin \left( \frac{a}{2} P_0 \right) \otimes M_{13} \]
\[ \varepsilon F (P_\mu) = 0, \quad \varepsilon F (M_{\mu \nu}) = 0, \]
\[ S^F (P_\mu) = -P_\mu, \quad S^F (M_{\mu \nu}) = -M_{\mu \nu}. \]
\[(2.8) \quad (2.9)\]

2.1. Twisted differential calculus

Let us analyse consequences of the twist (2.5) on the differential calculus. In this paper we shall just use the well known results in twisted differential geometry. More details on the twisted differential calculus can be found in appendix.

The \(*\)-product of functions is given by
\[ f \star g = \mu \left( \varepsilon \left( \delta f \otimes g - g \otimes \delta f \right) \right) + \mathcal{O}(a^2). \]
\[(10)\]

This \(*\)-product is noncommutative, associative and in the limit \( a \to 0 \) it reduces to the usual point-wise multiplication; the last property is guaranteed by (A.3) and the associativity is guaranteed by (A.1). In this way we obtain the noncommutative algebra of functions, i.e. the noncommutative space-time.

In the special case of coordinates, the \(*\)-commutation relations are
\[ [r \star x] = -iay, \]
\[ [r \star y] = iax. \]

\[(2.11)\]
while all other coordinates commute. These commutation relations resemble the $\kappa$-Minkowski space-time commutation relations, that is they are linear in coordinates.

Using the $\star$-product between functions and one-forms defined in (A.8) we obtain

\begin{align*}
    dt \star f &= f \star dt, \\
    dz \star f &= f \star dz, \\
    dx \star f &= \cos(i\alpha \partial_t) f \star dx + \sin(i\alpha \partial_t) f \star dy, \\
    dy \star f &= \cos(i\alpha \partial_t) f \star dy - \sin(i\alpha \partial_t) f \star dx
\end{align*}

(2.12)

for the $\star$-product of functions with basis one-forms. We used that $L_{X_2}(dx) = -dy$ and $L_{X_2}(dy) = dx$.

In the similar way, the wedge product between forms is deformed to the $\star$-wedge product (A.10)

$$\omega \wedge \star \omega' = \bar{f}_\alpha(\omega) \wedge \bar{f}_\alpha(\omega').$$

In the special case of basis 1-forms and (2.5) the wedge product is undeformed, that is

$$dx^\mu \wedge \star dx^{\nu} = dx^\mu \wedge dx^{\nu}.$$  

(2.13)

Since basis 1-forms anticommute the volume form remains undeformed

$$dt \wedge \star dx \wedge \star dy \wedge \star dz = dt \wedge dx \wedge dy \wedge dz = d^4x.$$  

(2.14)

The exterior derivative is the undeformed exterior derivative, see (A.11). The $\star$-derivatives following from (2.5) are given by

\begin{align*}
    \partial_\star t f &= \partial_t f, \\
    \partial_\star z f &= \partial_z f, \\
    \partial_\star x f &= \cos(\frac{i\alpha}{2} \partial_t) \partial_x f - \sin(\frac{i\alpha}{2} \partial_t) \partial_y f, \\
    \partial_\star y f &= \cos(\frac{i\alpha}{2} \partial_t) \partial_y f + \sin(\frac{i\alpha}{2} \partial_t) \partial_x f.
\end{align*}

(2.15)

Since the twist (2.5) is an Abelian twist, the cyclicity of integral holds

$$\int \omega_1 \wedge \star \cdots \wedge \star \omega_p = (-1)^{d_1 + \cdots + d_p} \int \omega_p \wedge \star \omega_1 \wedge \star \cdots \wedge \star \omega_{p-1},$$

with $d_1 + d_2 + \cdots + d_p = 4$. It can be shown that the twist (2.5) fulfills an even stronger requirement. Namely, one can check that the cyclicity holds also on $\star$-products of functions

$$\int d^4x f \star g = \int d^4x g \star f = \int d^4x f \cdot g.$$  

(2.16)

### 2.2. Hodge dual

In general, it is difficult to generalize the Hodge dual operation to NC spaces and NC gauge theories. For detailed discussion see [28, 76]. However, for the twist (2.5) the Hodge dual problem has a simple solution.

Let us first analyze the problem in the Minkowski space-time twisted by (2.5). Then we can define the NC Hodge dual of a 2-form $F = \frac{1}{2} F_{\mu\nu} \star dx^\mu \wedge \star dx^{\nu}$ in a ‘natural way’
\[ *_HF = *_H \left( \frac{1}{2} F_{\mu \nu} \star dx^\mu \wedge_* dx^\nu \right) = \frac{1}{2} F_{\mu \nu} \star \left( \varepsilon^{\mu \nu \rho \sigma} \eta_{\rho \sigma} \eta_{\theta \beta} dx^\theta \wedge_* dx^\beta \right) = \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \eta_{\rho \sigma} \eta_{\theta \beta} \left( F_{\mu \nu} \star dx^\theta \wedge_* dx^\beta \right). \]  

(2.17)

Let us further assume that the 2-form \( F \) is a NC field-strength tensor of some NC gauge theory. It transforms covariantly under NC \( U(1) \) gauge transformations

\[ F' = U_* \star F \star U_*^{-1}. \]

Here the finite NC gauge transformation is done with the matrix \( U_* \) and \( U_* \) can be a function of the commutative gauge parameter \( \alpha \). This fact and the explicit form of this dependance is of no relevance for the discussion of Hodge dual. Using (2.12) and (2.17) one can show that the transformation law of the Hodge-dual is given by

\[ *_HF' = U_* \star \left( *_HF \right) \star U_*^{-1}. \]  

(2.18)

That is, in the case of twist (2.5) the natural definition of the NC Hodge dual form transforms covariantly under the NC gauge transformations. This is an important result if one aims to construct a gauge invariant action for the NC gauge field, see the discussion in [28].

The construction of Hodge dual in a curved space-time, in particular in the RN background (2.4), we discuss in next section.

### 2.3. Angular twist in curved coordinates

We have already mentioned that the vector field \( X_2 \) is nothing else but the generator of rotations around the \( z \)-axis. Let us rewrite the twist (2.5) in the spherical coordinate system

\[ F = e^{-i \frac{1}{2} \theta^\alpha \delta_{\alpha, \beta} \hat{\partial}_{\beta}} = e^{-i \frac{1}{2} (\hat{\partial}_\theta \hat{\partial}_\phi - \hat{\partial}_\phi \hat{\partial}_\theta)}, \]  

(2.19)

with \( \alpha, \beta = t, \varphi \). Note that the twist has the same (2.19) form in the cylindrical coordinate system.

Now we rewrite all the formulas for the \( \star \)-product of functions and the differential calculus in spherical coordinate system. Here we present the most important results, the rest can be calculated easily:

\[ f \star g = \mu \left\{ e^{i \frac{1}{2} (\hat{\partial}_\theta \hat{\partial}_\phi - \hat{\partial}_\phi \hat{\partial}_\theta)} f \otimes g \right\} = fg + \frac{i a}{2} \left( \hat{\partial}_\phi (\hat{\partial}_\phi f) - \hat{\partial}_\theta g(\hat{\partial}_\theta f) \right) + O(a^2), \]  

(2.20)

\[ dx^\mu \star f = f \star dx^\mu = f dx^\mu, \]  

(2.21)

\[ \hat{\partial}_\mu \star f = \hat{\partial}_\mu f. \]  

(2.22)

Note that now \( x^\mu = (t, r, \theta, \varphi) \).

In the case of the Hodge dual we have to use the definition of the Hodge dual in curved space-time. This leads to
that is we obtained the commutative (undeformed) Hodge dual. In this calculation we used (2.21). In addition, we used that the metric tensor $g_{\mu \nu}$ does not depend on $t, \varphi$ coordinates and therefore $g_{\mu \nu} \star f = g_{\mu \nu} \cdot f$ for an arbitrary function $f$. In a more general case, when the twist $F$ is not a Killing twist for the space-time metric $g_{\mu \nu}$, we cannot use this definition of the Hodge dual, since in general $g_{\mu \nu} \star f \neq f \star g_{\mu \nu}$ and $g_{\rho \alpha} \star g_{\sigma \beta} \neq g_{\sigma \beta} \star g_{\rho \alpha}$. These will spoil the covariance of the $*HF$ under the NC gauge transformations and make the construction of NC gauge invariant action complicated, see [28, 76] for detailed discussion.

In the following we will work with the twist (2.19) and we will develop the NC scalar $U(1)_s$ gauge theory on the RN background.

### 3. Scalar $U(1)_s$ gauge theory

The twist (2.19) enables us to study the behaviour of a NC scalar field in a gravitational field of the Reissner–Nordström black hole.

Let us start from a more general action, describing the NC $U(1)_s$ gauge theory of a complex charged scalar field on an arbitrary background. The only requirement on the background is that $\partial_t$ and $\partial_\varphi$ are Killing vectors.

If a one-form gauge field $\hat{A} = A_\mu \star dx^\mu$ is introduced to the model through a minimal coupling, the relevant action becomes

$$S[\hat{\phi}, \hat{A}] = \int \left( d\hat{\phi} - i A_\mu \star \hat{\phi} \right) \wedge_\ast \star_H \left( d\hat{\phi} - i A_\mu \star \hat{\phi} \right)$$

$$- \int \frac{\mu^2}{4!} (\hat{\phi}^+ \wedge_\ast \hat{\phi}^\mu \wedge_\ast \hat{\phi}^\nu \wedge_\ast \hat{\phi}^a \wedge_\ast \hat{\phi}^b) - \frac{1}{4q^2} \int (\ast_H \hat{F}) \wedge_\ast \hat{F}. \quad (3.24)$$

The mass of the scalar field $\hat{\phi}$ is $\mu$, while its charge is $q$. The two-form field-strength tensor is defined as

$$\hat{F} = d\hat{A} - \hat{A} \wedge_\ast \hat{A} = \frac{1}{2} F_{\mu \nu} \star d x^\mu \wedge_\ast d x^\nu. \quad (3.25)$$

In order to write the mass term for the scalar field $\hat{\phi}$ geometrically, we introduced vierbein one-forms $e^a = e^a_\mu \star dx^\mu$ and $g_{\mu \nu} = \eta_{abc} e^a_\mu \star e^b_\nu$. In index notation, the action is of the form

$$S[\hat{\phi}, \hat{A}] = S_{\hat{\phi}} + S_\Lambda,$$

$$S_{\hat{\phi}} = \int d^4 x \sqrt{-g} \ast \left( g^{\mu \nu} \ast D_\mu \hat{\phi}^+ \ast D_\nu \hat{\phi} - \mu^2 \hat{\phi}^+ \ast \hat{\phi} \right), \quad (3.26)$$

$$S_\Lambda = - \frac{1}{4 q^2} \int d^4 x \sqrt{-g} \ast g^{\alpha \beta} \ast g^{\mu \nu} \ast F_{\alpha \mu} \ast F_{\beta \nu}. \quad (3.27)$$

The scalar field $\hat{\phi}$ is a complex charged scalar field transforming in the fundamental representation of NC $U(1)_s$. Its covariant derivative is defined as...
The components of NC field-strength tensor follow from (3.25)

\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu ; \hat{A}_\nu]. \]  

(3.28)

The background gravitational field \( g_{\mu\nu} \) is not specified for the moment. However, it is important that its Killing vectors are \( \partial_t \) and \( \partial_\phi \) since only in that case the action (3.27) has this simple form. Note that \( \ast \)-products in \( \sqrt{-g} \ast g^{\alpha\beta} \ast g^{\mu\nu} \) can all be removed since the twist (2.19) does not act on the metric tensor.

One can check that the actions (3.26) and (3.27) are invariant under the infinitesimal \( U(1) * \) gauge transformations\(^4\) defined in the following way:

\[ \delta_* \hat{\phi} = i \hat{\Lambda} \ast \hat{\phi}, \]
\[ \delta_* \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda} ; \hat{A}_\mu], \]
\[ \delta_* \hat{F}_{\mu\nu} = i[\hat{\Lambda} ; \hat{F}_{\mu\nu}], \]
\[ \delta_* g_{\mu\nu} = 0 \]  

(3.29)

with the NC gauge parameter \( \hat{\Lambda} \).

### 3.1. Seiberg–Witten map

There are different approaches to construction of NC gauge theories. In this paper we use the enveloping algebra approach [24] and the Seiberg–Witten (SW) map [25]. SW map enables to express NC variables as functions of the corresponding commutative variables. In this way, the problem of charge quantization in \( U(1) * \) gauge theory does not exist. In the case of NC Yang–Mills gauge theories, SW map guarantees that the number of degrees of freedom in the NC theory is the same as in the corresponding commutative theory. That is, no new degrees of freedom are introduced.

Using the SW-map NC fields can be expressed as function of corresponding commutative fields and can be expanded in orders of the deformation parameter \( a \). Expansions for an arbitrary Abelian twist deformation are known to all orders [77]. Applying these results to the twist (2.19), expansions of fields up to first order in the deformation parameter \( a \) follow. They are given by:

\[ \hat{\phi} = \phi - \frac{1}{4} \theta^{\sigma\rho} A_\rho (\partial_\sigma \phi + D_\sigma \phi), \]  

(3.30)

\[ \hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\sigma\rho} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu}), \]  

(3.31)

\[ \hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{2} \theta^{\rho\sigma} A_\rho (\partial_\sigma F_{\mu\nu} + D_\sigma F_{\mu\nu}) + \theta^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu}. \]  

(3.32)

\(^4\)The actions (3.26) and (3.27) are also invariant under the finite \( U(1) * \) transformations defined as:

\[ \hat{\phi}' = U_* \ast \hat{\phi}, \]
\[ \hat{A}'_{\mu} = -U_* \ast \partial_\mu U^{-1}_* + U_* \ast \hat{\Lambda}_\mu \ast U^{-1}_*, \]

with \( U_* = \psi_{\hat{\Lambda}} = 1 + i \hat{\Lambda} + \frac{1}{2} i \hat{\Lambda} \ast i \hat{\Lambda} + \ldots \).
The $U(1)$ covariant derivative of $\phi$ is defined as $D_\mu \phi = (\partial_\mu - i A_\mu) \phi$, while $D_\sigma F_{\mu \nu} = \partial_\sigma F_{\mu \nu}$ in the case of $U(1)$ gauge theory. The commutative complex scalar field $\phi$ can be decomposed into two real scalar fields $\phi_1, \phi_2$ in the usual way $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i \phi_2)$. It is important to note that the coupling constant $g$ between fields $\phi$ and $A_\mu$, the charge of $\phi$, is included into $A_\mu$, namely $A_\mu = q A_\mu$, compare also (3.27).

### 3.2. Expanded actions and equations of motion

Using the SW-map solutions and expanding the $\star$-products in (3.26) and (3.27) we find the action up to first order in the deformation parameter $a$. It is given by

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{4g^2} g^{\mu \rho} g^{\nu \sigma} F_{\mu \nu} F_{\rho \sigma} + g^{\mu \nu} D_\mu \phi^+ D_\nu \phi - \mu^2 \phi^+ \phi + \frac{1}{8q^2} g^{\mu \rho} g^{\nu \sigma} \theta^{\alpha \beta} (F_{\alpha \beta} F_{\mu \nu} F_{\rho \sigma} - 4 F_{\mu \alpha} F_{\nu \beta} F_{\rho \sigma}) + \frac{\mu^2}{2} \theta^{\alpha \beta} F_{\alpha \beta} \phi^+ \phi + \frac{\theta^{\alpha \beta}}{2} g^{\mu \nu} (\frac{1}{2} D_\mu \phi^+ F_{\alpha \beta} D_\nu \phi + (D_\mu \phi^+ F_{\alpha \alpha} D_\nu \phi + (D_\beta \phi^+ F_{\alpha \alpha} D_\nu \phi)) \right).$$

(3.33)

Now we vary the action (3.33) to calculate the equations of motion. Varying the action with respect to $\phi^+$ we obtain

$$g^{\mu \nu} \left( (\partial_\mu - i A_\mu) D_\nu \phi - \Gamma^\lambda_{\mu \nu} D_\lambda \phi \right) - \mu^2 \phi + \frac{\mu^2}{2} \theta^{\alpha \beta} F_{\alpha \beta} \phi + \frac{1}{4} \theta^{\alpha \beta} g^{\mu \nu} \left( (\partial_\mu - i A_\mu) (F_{\alpha \beta} D_\nu \phi) - \Gamma^\lambda_{\mu \nu} F_{\alpha \beta} D_\lambda \phi \right) - 2(\partial_\mu - i A_\mu) (F_{\alpha \beta} D_\nu \phi) + 2 \Gamma^\lambda_{\mu \nu} F_{\alpha \lambda} D_\beta \phi - 2(\partial_\beta - i A_\beta) (F_{\alpha \lambda} D_\nu \phi) \right) = 0.$$  

(3.34)

The $U(1)$ covariant derivative $D_\mu \phi$ has been defined in the previous subsection. Since the background metric $g_{\mu \nu}$ is not flat, the corresponding Christoffel symbols $\Gamma^\lambda_{\mu \nu}$ appear in the equation of motion.

Varying the action (3.33) with respect to $A_\lambda$ we obtain

$$\partial_\mu F^{\mu \lambda} + \Gamma^\mu_{\nu \rho} F^{\rho \lambda} + \theta^{\alpha \beta} \left( - \frac{1}{2} (\partial_\mu F_{\alpha \beta} F^{\rho \lambda}) + \Gamma^\rho_{\nu \rho} F_{\alpha \beta} F^{\mu \lambda} \right) + \partial_\mu (F^{\mu \lambda} F_{\alpha \beta}^\lambda) + \Gamma^\rho_{\mu \rho} F^{\mu \lambda} F_{\alpha \beta}^\lambda - \partial_\alpha (F^{\nu \lambda} F_{\nu \mu}^\nu) - \frac{1}{4} \partial_\lambda (F_{\nu \rho} F^{\nu \rho}) \right) = q^2 j^\lambda.$$  

(3.35)

with

$$j^\lambda = (1 - \frac{1}{4} \theta^{\alpha \beta} F_{\alpha \beta}) j^{(0)} - \frac{1}{2} \theta^{\alpha \beta} g^{\lambda \mu} \left( - F_{\nu \alpha} j_{\beta}^{(0)} - \partial_\alpha (D_\mu \phi^+ D_\beta \phi + D_\beta \phi^+ D_\mu \phi) \right) \right) - \frac{1}{4} \theta^{\alpha \beta} \left( - F_{\nu \alpha} j_{\beta}^{(0)} - g^{\mu \nu} \partial_\beta (D_\mu \phi^+ D_\nu \phi) \right) - \partial_\nu \left( g^{\mu \nu} (D_\mu \phi^+ D_\alpha \phi + D_\alpha \phi^+ D_\mu \phi) \right) - \Gamma^\rho_{\nu \rho} \left( g^{\mu \nu} (D_\mu \phi^+ D_\alpha \phi + D_\alpha \phi^+ D_\mu \phi) \right).$$  

(3.36)
The zeroth order current is defined as
\[ j_\lambda^0 = ig^{\mu\lambda}(D_\mu \phi^+ \phi - \phi^+ D_\mu \phi), \]
and \( j_\beta^0 = g_{\beta\lambda} j_\lambda^0 \). Once again, the Christoffel symbols corresponding to the metric \( g_{\mu\nu} \) appear in the equation.

4. Scalar field in the Reissner–Nordström background

Finally, let us specify the gravitational background to be that of a charged non-rotating black hole, the Reissner–Nordström (RN) black hole. Since we are interested in the QNMs of the scalar field, we will only consider the equation (3.34) and assume that the gravitational field \( g_{\mu\nu} \) and the \( U(1) \) gauge field \( A_\mu \) are fixed to be the gravitational field and the electromagnetic field of the RN black hole.

We write the RN metric tensor once again
\[
g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}
\]
with \( f = 1 - \frac{2MG}{r} + \frac{Q^2}{r^2} \) and \( M \) is the mass of the RN black hole, while \( Q \) is the charge of the RN black hole. The corresponding non-zero Christoffel symbols are:
\[
\Gamma^r_{tt} = f \frac{MG}{r} - \frac{Q^2 G}{r^2}, \quad \Gamma^r_{rr} = -\frac{MG}{r} - \frac{Q^2 G}{r^2}, \quad \Gamma^r_{\theta\theta} = -rf,
\]
\[
\Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta, \quad \Gamma^\phi_{\theta\phi} = \cot \theta, \quad \Gamma^\phi_{r\phi} = \Gamma^r_{\phi\phi} = \frac{1}{r}.
\]
The RN black hole is non-rotating, therefore the only non-zero component of the gauge field is the scalar potential
\[
A_0 = -\frac{qQ}{r},
\]
The corresponding electric field is given by
\[
F_{r0} = \frac{qQ}{r^2}.
\]
The only non-zero components of the NC deformation parameter \( \theta^{\alpha\beta} \) are \( \theta^{\phi\phi} = -\theta^{r\phi} = a \). Inserting all these into (3.34) gives the following equation
\[
\left( \frac{1}{r^2} \frac{\partial^2}{\partial t^2} - \Delta + (1-f) \frac{\partial^2}{\partial r^2} + \frac{2MG}{r^2} \frac{\partial}{\partial r} + 2iqQ \frac{1}{rf} \frac{\partial}{\partial \theta} - \frac{q^2 Q^2}{rf^2} - \mu^2 \right) \phi
+ \frac{aqQ}{r^2} \left( \frac{MG}{r} - \frac{GG^2}{r^2} \right) \frac{\partial}{\partial r} \frac{\partial}{\partial \phi} \phi = 0.
\]
Note that \( \Delta \) is the usual Laplace operator.

In order to solve this equation we assume an ansatz
\[
\phi_{lm}(t, r, \theta, \varphi) = R_{lm}(r)e^{-i\omega t}Y^m_l(\theta, \varphi)
\]
with spherical harmonics \( Y^m_l(\theta, \varphi) \). Inserting (4.43) into (4.42) leads to an equation for the radial function \( R_{lm}(r) \)

\[
fr''_{lm} + \frac{2}{r} \left( 1 - \frac{MG}{r} \right) R'_{lm} - \left( \frac{l(l+1)}{r^2} - \frac{1}{f} (\omega - \frac{qQ}{r^2})^2 + \frac{\mu^2}{r^2} \right) R_{lm} - \frac{im a qQ}{r^3} \left( \frac{MG}{r^2} - \frac{GQ^2}{r^2} \right) R_{lm} + r f'_{lm} = 0. 
\]

(4.44)

The zeroth order of this equation corresponds to the equation for the radial function \( R_{lm} \) in [61, 78].

5. Solutions for QNMs

We are interested in a special solution of equation (4.44), namely the quasinormal mode solution. This solution describes damped oscillations of a perturbed black hole, as the black hole goes through the ringdown phase. A set of the boundary condition which leads to this solution is the following: at the horizon, the QNMs are purely incoming, while in the infinity the QNMs are purely outgoing. In order to find the spectrum of QNMs in our model, let us firstly rewrite equation (4.44) in a more convenient way.

Taking into account that the outer and inner horizon of RN black hole are given by

\[
r_{\pm} = GM \pm \sqrt{G^2 M^2 - GQ^2},
\]

we introduce the following variables and abbreviations

\[
x = r - r_+, \quad \tau = r_+ - r_-, \\
k = 2 \omega r_+ - qQ, \quad \kappa = \frac{\omega k - \mu^2 r_+}{\sqrt{\omega^2 - \mu^2}}, \\
\Omega = \frac{\omega r_+ - qQ}{\sqrt{G^2 M^2 - GQ^2} r_+} = 2 \frac{\omega r_+ - qQ}{\tau}.
\]

(5.45)

(5.46)

Then the radial equation of motion (4.44) reduces to

\[
x(x + \tau) \frac{d^2 R}{dx^2} + \left( 2x + \tau - \frac{im a qQ}{r_+} \frac{x(x + \tau)}{x(x + 1)^2} \right) \frac{dR}{dx} - \left[ \frac{l(l+1)}{x(x + 1)^2} + \frac{\mu^2 r_+^2}{x(x + 1)^2} \right] R = 0.
\]

(5.48)

In order to simplify writing, instead of \( R_{lm} \) we just write \( R \) for the radial part of the scalar field, keeping in mind that it depends on \( l \) and \( m \).

The equation (5.48) can be treated analytically in the near extremal case \( \tau \ll 1 \) or \( \frac{Q}{\sqrt{G^2 M^2}} \rightarrow 1 \).

The strategy that we adopt here is similar to that in [61]. It allows us to analyse the equation (5.48) in two different regions, one being relatively far from the horizon, \( x \gg \tau \), and the other being relatively close to the horizon, \( x \ll 1 \). It is important to emphasize that these two regions have to be chosen in such a way as to ensure that their region of a common overlap exists (or is likely being close to exist). The restriction to a near extremal limit that we make, as well as an appropriate choice of a range of the system parameters makes this possible.

With a presence of a common overlapping region being guaranteed, we extrapolate the solutions obtained in two separate regions all the way up (or down) to this region of a common
overlapping. The subsequent comparison of the extrapolated solutions is then used to fix the unknown constants of integration.

In the region \( x \gg \tau \) equation (5.48) can be approximated by

\[
x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} = \left[ l(l+1) + \mu^2 r_+^2 (x+1)^2 - \omega^2 r_+^2 x^2 - 2\omega r_+ kx - k^2 \right] R = 0.
\]

(5.49)

In our analysis we set the scale of the NC parameter \( a \) to be of the order of the ‘extremality’ \( \tau \). In that case, all NC corrections in equation (5.49) vanish. This is due to the approximation being used, \( a \sim \tau \), and due to the region being considered, \( x \gg \tau \). Namely, being proportional to \( \tau \) and having powers of \( x \) in the denominator, these terms give rise to the least dominant contributions in the region \( x \gg \tau \). That is, they are higher order corrections in \( \xi \).

Next we consider the region \( x \ll 1 \). Picking up all relevant terms and introducing a new conveniently chosen variable \( y = -\frac{2}{x} \), from (5.48) we obtain

\[
y(1-y) \frac{d^2 R}{dy^2} + (1-2y) \frac{dR}{dy} + \left[ l(l+1) + \mu^2 r_+^2 \right] R = 0.
\]

(5.50)

We can immediately see that NC corrections survive in this region. In this way, our approximation results in NC corrections near the horizon where the gravitational field is strong. Far from the horizon, where the gravitational field is weak, the NC corrections are not present. These results are consistent with the assumption that in the strong gravitational field the space–time structure becomes deformed.

We point out that the approximation \( a \sim \tau \) was essential in obtaining the equations (5.49) and (5.50). Besides this one, an additional approximation has been used to get (5.49) and (5.50). This latter approximation is dictated by the character of the region in which the analysis has been made (\( x \gg \tau \) or \( x \ll 1 \)). However, note that since \( \tau \) can be arbitrarily small, it does not severely constrain the value of the NC parameter \( a \).

In the first region, \( x \gg \tau \), the substitution \( R(x) = e^{-\alpha x} x^\beta g(x) \) transforms equation (5.49) into a confluent hypergeometric equation with the solution

\[
R(x) = C_1(2\alpha)^{\frac{1}{2}+i\sigma} x^{-\frac{1}{2}+i\sigma} e^{-\alpha x} M\left(\frac{1}{2} + i\sigma + i\kappa, 1 + 2i\sigma, 2\alpha x\right)
+ C_2(2\alpha)^{\frac{1}{2}-i\sigma} x^{-\frac{1}{2}-i\sigma} e^{-\alpha x} M\left(\frac{1}{2} - i\sigma + i\kappa, 1 - 2i\sigma, 2\alpha x\right),
\]

(5.51)

where \( \alpha = i \sqrt{\omega^2 - \mu^2} r_+ \) and \( \sigma = \sqrt{k^2 - \mu^2 r_+^2 - (l + \frac{1}{2})^2} \) and \( M \) is the Kummer’s hypergeometric function. As a matter of fact, in order to get a confluent hypergeometric equation, the parameters \( \alpha \) and \( \beta \) can actually take two values each, \( \alpha = \pm i \sqrt{\omega^2 - \mu^2} r_+ \) and \( \beta = -\frac{1}{2} \pm i \sigma \), respectively. However, as far as the physical arguments go, not each of these values is acceptable. The important point to realise here is that the solutions obtained in two regions, \( x \gg \tau \) and \( x \ll 1 \), not only need to satisfy the proper boundary conditions, but they also need to match mutually in the region of their common overlap. This means that, when extrapolated, they have to converge to expressions that have the same analytical form and the proper choice of the parameter \( \beta \), as well as the parameters \( \lambda_1 \) and \( \lambda_2 \) (see below) appears to be crucial to this purpose. For these reasons, all these parameters have to be chosen carefully. In particular, \( \beta \) is chosen as \( \beta = -\frac{1}{2} + i \sigma \). If instead the parameters are not selected carefully,
two solutions obtained by extrapolation will not converge to the expressions of the same kind and thus two solutions will not be able to match in their common region of overlap. As for $\alpha$, it is the QNMs boundary condition at far infinity which determines the choice of its sign.

In the second region, $x \ll 1$, the substitution $R(y) = y^{\lambda_1}(1 - y)^{\lambda_2}F(y)$ transforms equation (5.50) into a hypergeometric equation with the solution

$$R(y) = y^{-i\frac{\Omega}{2}}(1 - y)^{(\frac{\Omega}{2} - k)}F\left(\frac{1}{2} + i\sigma - ik + \varrho, \frac{1}{2} - i\sigma - ik - \varrho, 1 - i\Omega; y\right)$$  (5.52)

where $\varrho = \frac{8m}{2\sigma}\left(\frac{GQ^2}{r_+^2} - \frac{GMqQ}{r_+^2}\right) + i\sqrt{\frac{G^2M^2 - GQ^2}{2\sigma}}\left(k + \frac{\Omega}{2}\right)$ and $\sigma = \sqrt{k^2 - \mu^2r_+^2 - \left(1 + \frac{1}{2}\right)^2}$. Note here that $\tau = \sqrt{G^2M^2 - GQ^2}/r_+$.

As in the previous situation, the parameters $\lambda_1$ and $\lambda_2$ for which the equation (5.50) reduces to a hypergeometric equation, namely $\lambda_1 = \pm i\frac{\Omega}{2}$ and $\lambda_2 = i\left(\frac{\Omega}{2} \pm k\right)$, are not unique. Instead, they may acquire two possible values each and, as already mentioned before, need to be selected carefully in order to meet the physical demands. The value of $\lambda_1$ is chosen with the minus sign, $\lambda_1 = -i\frac{\Omega}{2}$, so to comply with the QNMs boundary condition at the horizon. Furthermore, the parameter $\lambda_2$ has been selected as $\lambda_2 = i\left(\frac{\Omega}{2} - k\right)$. With this choice, the extrapolated solutions obtained in two regions, $x \gg \tau$ and $x \ll 1$, can be matched together.

Once $\lambda_1$ has been selected, the general solution of equation (5.50) may be written as a linear combination of two independent solutions of the hypergeometric equation. The requirement of having a purely ingoing wave at the horizon forces us to drop out one of these two independent functions, resulting in a solution of the form (5.52).

Extrapolation of both solutions to the mutual overlap region $\tau \ll x \ll 1$ leads to

$$R(x)_{\text{reg1}} \simeq C_1(2i\sqrt{\omega^2 - \mu^2 r_+})^{\frac{1}{2} + i\sigma}x^{\frac{1}{2} + i\sigma} + C_2(2i\sqrt{\omega^2 - \mu^2 r_+})^{\frac{1}{2} - i\sigma}x^{\frac{1}{2} - i\sigma}$$  (5.53)

and

$$R(x)_{\text{reg2}} \simeq (-1)^{-i\frac{\Omega}{2}}\left[\frac{\Gamma(1 - i\Omega)\Gamma(-2i\sigma - 2\varrho)}{\Gamma\left(\frac{1}{2} - i\sigma - ik - \varrho\right)}\left(\frac{1}{2} + i\sigma + ik - i\Omega - \varrho\right) - \frac{\Gamma(1 - i\Omega)\Gamma(2i\sigma + 2\varrho)}{\Gamma\left(\frac{1}{2} + i\sigma - ik + \varrho\right)}\left(\frac{1}{2} + i\sigma + ik + i\Omega + \varrho\right)\right] x^{-\frac{1}{2} + i\sigma}$$  (5.54)

By matching these two expressions, it is possible to fix the so far unknown integration constants $C_1$ and $C_2$. We obtain

$$C_1 = (-1)^{-i\frac{\Omega}{2}}\frac{\Gamma(1 - i\Omega)\Gamma(2i\sigma + 2\varrho)}{\Gamma\left(\frac{1}{2} + i\sigma - ik + \varrho\right)}\left(2i\sqrt{\omega^2 - \mu^2 r_+}\right)^{-\frac{1}{2} + i\sigma}$$  (5.55)

and

$$C_2 = (-1)^{-i\frac{\Omega}{2}}\frac{\Gamma(1 - i\Omega)\Gamma(-2i\sigma - 2\varrho)}{\Gamma\left(\frac{1}{2} - i\sigma - ik - \varrho\right)}\left(2i\sqrt{\omega^2 - \mu^2 r_+}\right)^{-\frac{1}{2} + i\sigma}.$$  (5.56)
Note that in obtaining the expression (5.54) we have used the linear transformation

\[
F(a, b, c; y) = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} (-y)^{-a} F(a, 1 - c + a, 1 - b + a; \frac{1}{y}) + \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} (-y)^{-b} F(b, 1 - c + b, 1 - a + b; \frac{1}{y}).
\]  
(5.57)

Taking the limit \( x \gg \tau \) makes \( 1/y \) approach zero, and one can use the standard Taylor expansion of the function \( F(a, b, c; y) \) around zero to get the most dominant contributions coming from the second region.

Consider again the solution (5.51) in the region \( x \gg \tau \). From the asymptotic behaviour of the confluent hypergeometric functions

\[
M(a, b, z) \xrightarrow{|z| \to \infty} \Gamma(b) \left( \frac{e^{z} a^{-b}}{\Gamma(a)} + \frac{(-z)^{-a}}{\Gamma(b - a)} \right) + O(|z|^{a+b-1})
\]  
(5.58)

the asymptotic behavior of the solution (5.51) follows

\[
R(x) \sim \left[ C_{1}(2\alpha)^{i\sigma} \frac{\Gamma(1 + 2i\sigma)}{\Gamma(\frac{1}{2} + i\sigma + i\kappa)} x^{-1 + i\kappa} + C_{2}(2\alpha)^{i\sigma} \frac{\Gamma(1 - 2i\sigma)}{\Gamma(\frac{1}{2} - i\sigma + i\kappa)} x^{-1 - i\kappa} \right] e^{\alpha x}
\]

\[
+ \left[ (-1)^{-1-i\sigma} C_{1}(2\alpha)^{-i\kappa} \frac{\Gamma(1 + 2i\sigma)}{\Gamma(\frac{1}{2} + i\sigma - i\kappa)} x^{-1-i\kappa} + (-1)^{-1+i\sigma} C_{2}(2\alpha)^{-i\kappa} \frac{\Gamma(1 - 2i\sigma)}{\Gamma(\frac{1}{2} - i\sigma - i\kappa)} x^{-1+i\kappa} \right] e^{-\alpha x} + O(x^{-1}).
\]  
(5.59)

Recalling the QNMs boundary condition of having a purely outgoing wave at far infinity, it is clear that the second square bracket in the expression (5.59) has to vanish. The equation (5.59) thus provides another constraint on the constants \( C_{1} \) and \( C_{2} \), namely

\[
C_{1} \frac{\Gamma(1 + 2i\sigma)}{\Gamma(\frac{1}{2} + i\sigma - i\kappa)} (-1)^{-i\sigma} + C_{2} \frac{\Gamma(1 - 2i\sigma)}{\Gamma(\frac{1}{2} - i\sigma - i\kappa)} (-1)^{i\sigma} = 0.
\]  
(5.60)

Combining this constraint with (5.55) and (5.56), the quantization condition emerges that determines the black hole QNMs spectrum. It is given by

\[
\frac{\Gamma(1 - 2i\sigma) \Gamma(-2i\sigma - 2\tilde{\rho})}{\Gamma(\frac{1}{2} - i\sigma - i\tilde{\rho}) \Gamma(\frac{1}{2} - i\sigma + ik - i\Omega - \tilde{\rho}) \Gamma(\frac{1}{2} - i\sigma - i\kappa)}
\]

\[
= \frac{\Gamma(1 + 2i\sigma) \Gamma(2i\sigma + 2\tilde{\rho}) r^{-2\tilde{\rho}}}{\Gamma(\frac{1}{2} + i\sigma - ik + i\tilde{\rho}) \Gamma(\frac{1}{2} + i\sigma + ik - i\Omega + \tilde{\rho}) \Gamma(\frac{1}{2} + i\sigma - i\kappa)}
\]

\[
\times \left( -2i \sqrt{\omega^{2} - \mu^{2} r^{2} r_{+}^{2}} \right)^{-2i\sigma}.
\]  
(5.61)

In general, this condition cannot be solved analytically. In the next section we discuss a special choice of parameters where the analytic solution is possible. In the following, we present some numerical results for the QNMs frequencies, obtained by Wolfram Mathematica and resulting from the analytic condition (5.61). In particular, we shall be concerned with the fundamental quasinormal mode. However, before we proceed to discuss the properties of QNMs frequencies and describe the behaviour of the fundamental mode in terms of certain system parameters, like the mass and charge of the scalar probe, we explain why the fundamental tone is so important.
To begin with, the fundamental mode in the QNMs spectrum has the frequency with the lowest absolute value of the imaginary part (with a sign of the imaginary part of course being such that it guarantees stability of a black hole). Since the imaginary part of the frequency is proportional to a decay rate of a perturbation, it is clear that the modes with larger imaginary part of the frequency will more quickly die out, and the fundamental mode will be the one that will dominate the perturbation signal. Therefore, in the forthcoming experiments that are going to utilise the current gravitational antennas (such as LIGO, VIRGO, LISA) for observing gravitational signals from black holes, the most dominant contribution to these signals will come from the fundamental mode of the QNMs spectrum.

From the reasons just explained, we give separately the dependence of the fundamental QNMs frequency $\omega$ (i.e. its real and imaginary part) on the charge of the scalar field $q$ and on the mass of the scalar field $\mu$. In order to make comparison, we plot the results for two choices of ‘extremality’: $Q_M = 0.9999$, corresponding to the ‘extremality’ parameter $\tau = 0.027\,889\,1617$ and $Q_M = 0.999\,999$, corresponding to $\tau = 0.002\,824\,4321$. For simplicity we set $G = 1$. When deriving (5.61) we used the approximation that the NC deformation parameter $a$ is of the same order as the extremality $\tau$. Therefore, in the case $Q_M = 0.9999$ we set $a = 0.1$ while $a = 0.01$ corresponds to the case $Q_M = 0.999\,999$. The $q$ dependence of $\text{Re}\,\omega$ and $\text{Im}\,\omega$ is presented in figures 1 and 1. In this case we assumed that the mass of the scalar field is fixed at $\mu = 0.05$.

Likewise, the dependance of the fundamental QNMs frequency $\omega$ on the mass of the scalar field $\mu$, for the fixed charge $q = 0.075$, is shown in figures 3 and 4.

It needs to be said that the calculations leading to results depicted in figures 1–4 were carried out\(^5\) for $l = 1$, as well as for the three values of $m$, namely $m = 0, \pm 1$. However, as readily seen from figures, the curves corresponding to three different values of $m$ cannot be distinguished, actually. Nevertheless, this does not mean that these three curves coincide identically. Just contrary, they are not identical, as can be easily verified simply by improving the resolution and by letting the graphs show a higher level of details. To that purpose, in the next section we consider the differences of frequencies $\omega^\pm = \omega_{m=\pm 1} - \omega_{m=0}$, which indeed appear to be nonvanishing. In this regard we observe that the deformation $a$ and the azimuthal quantum number $m$ always come in pair, implying that the mode with $m = 0$ actually corresponds to the limit $a \to 0$, that is the absence of a deformation. It is therefore clear that the differences $\omega^\pm$ encode the effect of a space-time deformation. This is what makes them interesting and worthy of a separate analysis, which we leave for the final section.

\(^5\) Note that $l = 0$ corresponds to a trivial situation where NC effects disappear.
Some previous works \cite{79, 80} on QNMs have studied the decay of the massive scalar field and found that there exist perturbations with arbitrary long life when the field mass has special values. These modes are called quasiresonances and they are characterized by the zero value of the imaginary part of the frequency. One may ask if such quasiresonances appear in our results. If they do, they should certainly be visible from figure 4. Although the graphs on figure 4 do not intersect the horizontal axis, it is though clearly seen that they approach the zero value as the mass of the field grows. The intersection point itself cannot be seen on figure 4 due to the Mathematica’s performance. Namely, when the value of Im $\omega$ approaches zero, the increment with which Mathematica performs calculation, and which corresponds to

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Dependance of Im $\omega$ on the charge $q$ of the scalar field with the mass $\mu = 0.05$, $l = 1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Dependance of Re $\omega$ on the mass $\mu$ of the scalar field with the charge $q = 0.075$, $l = 1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Dependance of Im $\omega$ on the mass $\mu$ of the scalar field with the charge $q = 0.075$, $l = 1$.}
\end{figure}
a numerical error, becomes of the order of the result itself, thus rendering a numerical approximation inadequate. This is why in our analysis we have been proceeding with a calculation only up to the point where the Mathematica starts to yield the results that are of order of a numerical error (and are thus unreliable). We plan to investigate this issue in more details in future work with the WKB and the continued fraction approach.

The real part of $\omega$ grows linearly with both $q$ and $\mu$, as expected. The imaginary part reaches the saturation point for larger $q$, while for larger $\mu$ it tends to zero, signaling the appearance of quasiresonances. Moreover, the figures clearly show that the closer is the system to reach the extremal conditions, the smaller is the value of $q$ at which the imaginary part of the QNMs frequency saturates. The $q$ dependence is in agreement with the numerical results for QNMs of the RN black hole presented in [81].

6. Discussion and final remarks

We have seen from the solutions (5.49) and (5.50) that the NC correction only appears in the Region 2, that is when $x \ll 1$. However, due to the matching procedure, we find a non-zero NC effect on the QNMs spectrum. The effect can be described as a Zeeman-like splitting in the spectrum, manifested by the coupling between the deformation parameter $a$ and the azimuthal (magnetic) quantum number $m$. In figures 5 and 6 we plot the dependance of frequency splitting $\omega^{\pm} = \omega_{m=\pm 1} - \omega_{m=0}$, of real/imaginary part of $\omega$ on the scalar field charge $q$. The plots are done for the following set of parameters: $Q/M = 0.999999$, $\mu = 0.05$ and $l = 1$. The green line represents $\omega^{+} = \omega_{m=1} - \omega_{m=0}$, while the red line represents $\omega^{-} = \omega_{m=-1} - \omega_{m=0}$.

We also show (see figures 7 and 8) the dependance of frequency splitting, $\omega^{\pm}$, on the scalar field mass $\mu$, for the following set of parameters: $Q/M = 0.999999$, $q = 0.075$ and $l = 1$. Again, the green line represents $\omega^{+} = \omega_{m=1} - \omega_{m=0}$, while the red line represents $\omega^{-} = \omega_{m=-1} - \omega_{m=0}$.

The frequency splitting is small, as expected. To have an idea of how small, one can estimate $\Delta \omega$ for the imaginary part of $\omega$ in the case of $q$ dependence from figures 1 and 6 and obtain $\Delta \omega \approx 10^{-6}$. However, the effect is very important qualitatively, since it predicts a Zeeman-like splitting of the QNMs spectrum in the presence of noncommutativity.

The phenomenon of frequency splitting can be made even more evident if we turn our attention to the graphs with $l = 2$.

From figures 9 and 10 we clearly see the frequency splitting, indicating lines that correspond to $m = \pm 2$ and $m = \pm 1$. Moreover, from all above graphs it is manifest that $\omega^{+} = -\omega^{-}$ and $\omega^{++} = -\omega^{--}$, a feature that was expected due to a parity symmetry. Here we use the
The frequency splitting manifests itself as a coupling between the deformation parameter \( a \) and the azimuthal (magnetic) quantum number \( m \). At first glance, a similar behaviour can be found in the QNMs spectrum of the Kerr black hole evaluated in the limit of slow rotation \([18, 42]\), where the magnetic quantum number \( m \) couples with the black hole angular momentum \( J \). Described feature would suggest the existence of a specific kind of duality between noncommutative and non-rotating systems on the one side and standard commutative systems on the other side.
This duality has already been observed in some lower dimensional systems [69, 70, 82]. However, a closer inspection shows that these two spectra are not equivalent or dual to each other, since in the case of rotating black hole in linear approximation, in addition to the term proportional to \( mJ \), there is another contribution, proportional to \( J \) alone, which is nonzero for \( m = 0 \). This is different from the dependence on the noncommutative scale \( a \) encountered in our analysis. This shows that a true relationship between these two settings (commutative and noncommutative) still needs to be found and we plan to address this problem in future work. What we have shown definitely is that

\[
\text{Figure 9. Dependance of frequency splitting of } Re \omega \text{ on the charge } q \text{ of the scalar field with the mass } \mu = 0.05, l = 2. 
\]

\[
\text{Figure 10. Dependance of frequency splitting of } Im \omega \text{ on the charge } q \text{ of the scalar field with the mass } \mu = 0.05, l = 2. 
\]

\[
\text{Figure 11. Dependance of } Re \omega \text{ on the charge } q \text{ of the scalar field with the mass } \mu = 0.05, l = 2. 
\]

and rotating systems on the other side. This duality has already been observed in some lower dimensional systems [69, 70, 82]. However, a closer inspection shows that these two spectra are not equivalent or dual to each other, since in the case of rotating black hole in linear approximation, in addition to the term proportional to \( mJ \), there is another contribution, proportional to \( J \) alone, which is nonzero for \( m = 0 \). This is different from the dependence on the noncommutative scale \( a \) encountered in our analysis. This shows that a true relationship between these two settings (commutative and noncommutative) still needs to be found and we plan to address this problem in future work. What we have shown definitely is that the

\[6\] Also more generally, a duality between generic noncommutative and commutative settings was claimed, with the gravitational background in the latter case being modified [83].
Dependence of $\text{Im} \omega$ on the charge $q$ of the scalar field with the mass $\mu = 0.05$, $l = 2$.

QNMs spectrum of a charged scalar field in the noncommutative RN BH background exhibits a Zeeman-like splitting.

So far we discussed properties of the numerical solution for the QNMs spectrum. Let us briefly comment on an analytical solution. The analytic solution is possible in a special regime of the system parameters: $\tau \ll 1$, $q^2 > G\mu^2$ and $\sigma > 1$. In this case the quantization condition can be written as

$$\frac{1}{\Gamma\left(\frac{1}{2} - i\sigma + ik - i\Omega - \tilde{\rho}\right)} = D(\omega, \tau) \left( -2i\sqrt{\omega^2 - \mu^2} \tau + \tau \right)^{-2i\sigma} \ll 1, \quad (6.62)$$

where the quantity $D(\omega, \tau)$ is defined as

$$D(\omega, \tau) = -\frac{\Gamma(2i\sigma + 2\rho)\Gamma(1 + 2i\sigma)\Gamma(\frac{1}{2} - i\sigma - ik - \rho)\Gamma(\frac{1}{2} - i\sigma + \rho)\Gamma(\frac{1}{2} + i\sigma + ik - i\Omega + \rho)\Gamma(\frac{1}{2} + i\sigma - i\Omega)}{\Gamma(1 - 2i\sigma)\Gamma(-2i\sigma - 2\rho)\Gamma(\frac{1}{2} + i\sigma - ik - \rho)\Gamma(\frac{1}{2} + i\sigma + ik + i\Omega + \rho)\Gamma(\frac{1}{2} + i\sigma + i\Omega)}.$$

This means that the value of the argument of $\Gamma$-function needs to be very close to any of the poles, implying

$$\frac{1}{2} - i\sigma + ik - i\Omega - \tilde{\rho} = -n + \eta(n)\epsilon + O(\epsilon^2), \quad (6.64)$$

where $\epsilon$ is a small parameter [61] which scales with the power of $\tau$. The quantum number $n$ is a nonnegative integer. It takes the role of an overtone number which characterizes the QNMs spectrum. The mode with $n = 0$ is the fundamental mode. In addition, the quantity $\eta(n)$ can be calculated [61, 84] as $\eta(n) = \frac{D(\omega, 0)}{(-1)^{n}m}$, where $D(\omega, 0)$ is (6.63) evaluated in the extremal limit $Q \rightarrow \sqrt{G\mu}$. The condition (6.64) directly gives the QNMs spectrum of the RN black hole perturbed by a charged matter and in a presence of deformed (noncommutative) structure of space-time

$$\omega = \frac{q^2 Q^2 - Q^2}{r_+^2} \left( Q^2 - \sqrt{Q^2 - \mu^2 r_+^2} - (l + \frac{1}{2})^2 \right) + \frac{am}{2r_+} \frac{G\mu Q^2 - GMQr_+}{\sqrt{Q^2 - \mu^2 r_+^2} - (l + \frac{1}{2})^2},$$

$$+ \frac{Q^2 Q^2 - Q^2}{r_+^2} \left( \frac{a^2 m^2}{2r_+} Q^2 - \sqrt{Q^2 - \mu^2 r_+^2} - (l + \frac{1}{2})^2 \right) - 2(a + \frac{1}{2}) + \frac{a^2 m^2}{2r_+} \frac{1}{\sqrt{Q^2 - \mu^2 r_+^2} - (l + \frac{1}{2})^2} \left( 90(n + \frac{1}{2})^2 - 11a^2 \right).$$

where $n = 0, 1, 2, ...$

Note that to get this formula, we have utilised the fact that the rhs of (6.62) contains the factor of $(-i)^{-2i\sigma} = e^{-i\pi\sigma}$, which may severely suppress the whole expression as long as the
value of $\sigma$ satisfies $\sigma > 1$. We also use the additional assumption that the actual spectrum slightly deviates from its classical value of $\omega = qQ/r_+$. In this case $\sigma$ is almost real and has enough large real part. Consequently the right hand side of (6.62) will come close to zero and so will the left hand side. This means that the argument of any $\Gamma$-function in the denominator of the lhs of (5.61) might reach the pole and the corresponding $\Gamma$-function may consequently blow up, essentially leading to three different branches for the spectrum. However, not each of these branches of the solution for the spectrum would be physically acceptable, in a sense that not all of them would comply to the requirement that the actual spectrum needs to be centered around the classical value of $\omega = qQ/r_+$. This argument singles out the appropriate $\Gamma$-function, which leads to the condition (6.62) and the spectrum (6.65).

Although derived under rather stringent conditions, the expression (6.65) may nevertheless indicate, at least qualitatively, certain landmark properties of the QNMs spectrum for the problem considered. In the commutative limit, $a \to 0$, and expanding up to first order in $\tau r_+$ the result (6.65) reduces to the solution found in [61] and it can be checked that by its general appearance, the structure of this solution is the same as that of the corresponding result obtained by the WKB method [78]. Moreover, a comparison of our results to those obtained in [81] by the continued fraction method and for the near extremal case ($Q/M$ close to 1) shows very similar patterns for the frequency curves $\omega$ versus $q$, obtained in these two approaches. We postpone the comparison of our results with the results of [85] for the subsequent paper where we plan to analyze the most general equation of motion (4.44) by using the method of continued fractions and WKB. Finally, note that, even in the commutative limit, our analytical result holds up to second order in $\tau r_+$, while the result of [61] is only valid up to first order in $\tau r_+$. Of course, we again note the appearance of the Zeeman-like splitting in the spectrum.

Finally, let us briefly discuss the impact of space-time deformation on Hawking temperature of the RN black hole. In particular, it is of interest to analyse whether it changes or not. To get the answer to this question, one may use the approach based on a semiclassical method of modeling Hawking radiation as a tunneling effect from the inside to the outside of the horizon [86, 87]. Up to now we have treated the scalar field $\phi$ as a classical, noncommutative field. In order to study the Hawking radiation, the scalar field $\phi$ has to be quantized. Following [75], we know that a careful formulation of NC quantum field theory leads to no problems with unitarity and causality.

The method for calculating changes in Hawking temperature consists of calculating the imaginary part of the classical Hamilton–Jacobi action $S$ which fixes [88–90] the tunneling probability amplitude $\Gamma$ through the relation $\Gamma = \exp(-\frac{1}{\hbar}\text{Im}S)$. In our analysis the required quantity is obtained by the Hamilton–Jacobi method [91, 92], which assumes that the field $\phi$ may be written as

$$\phi = \exp\left(-\frac{i}{\hbar}S + O\left(\frac{i}{\hbar}\right)^2\right),$$

(6.66)

where in our case $\phi$ satisfies equation (4.42). $S$ may be simplified by separating out the variables in terms of which the system is described. This is possible due to the symmetry properties of the system, which is made obvious by the existence of the two Killing vectors. Therefore, one may write $S = -Et + j\varphi + \mathcal{R}(r, \theta)$. Inserting (6.66) into (4.42) and using the above separation of $S$, gives rise to the equation for $\mathcal{R}$,

$$\frac{E^2}{f} + \frac{aqQ}{r^2}f(\xi)S(\partial_r\mathcal{R}) - \frac{1}{r^2}f(\xi) = f(\partial_r\mathcal{R})^2.$$

(6.67)
Here \( \Xi(\theta, \varphi) \) is the abbreviation for
\[
\Xi(\theta, \varphi) \equiv -\frac{1}{\hbar^2} \left( (\partial_\theta S)^2 + i\hbar \partial_\theta S + i\hbar \partial_\varphi S + \frac{1}{\sin^2 \theta} (\partial_\varphi S)^2 + i\hbar \frac{1}{\sin^2 \theta} \partial^2_\theta S \right).
\]

Solving (6.67) with respect to \( \partial_\varphi R \) gives
\[
\partial_\varphi R = \frac{aqQ}{2r^2} \partial_\varphi S \pm \sqrt{\left( \frac{E^2}{f^2} - \frac{1}{r^2} \Xi(\theta, \varphi) \right) \left( 1 + \frac{a^2 q^2 Q^2}{4r^4} \frac{(\partial_\varphi S)^2}{E^2} - \frac{1}{r^2} \Xi(\theta, \varphi) \right)}.
\]

Expanding with respect to the deformation parameter \( a \) and keeping terms up to order \( O(a^2) \) yields
\[
R(r) = \int_0^\pi dr \left( \frac{aqQ}{2r^2} \partial_\varphi S \pm \frac{1}{f} \sqrt{\frac{E^2}{r^2} - \frac{1}{r^2} \Xi(\theta, \varphi)} \left( 1 + \frac{f^2}{8r^4} \frac{a^2 q^2 Q^2 (\partial_\varphi S)^2}{E^2} - \frac{1}{r^2} \Xi(\theta, \varphi) \right) \right),
\]
with \( r_- < r_{in} < r_+ \) and \( r > r_+ \). What is needed is the imaginary part of the classical action \( S \) (and therefore of the function \( R \)). This can only come from the integration over the contour encircling the pole of the integrand. Since the pole is located at the zero of \( f \) (outer horizon), the calculation of the corresponding residue then clearly shows that the terms scaling with \( f \) and \( \sqrt{f} \) simply disappear. Therefore, the noncommutativity in space-time does not affect the Hawking temperature, it is shifted neither above nor below the standard Hawking temperature for the RN black hole.

This result is expected, since in our model the gravitational field (geometry) is not affected by the noncommutativity. It would be interesting to analyse effects of the NC scalar fields to the background geometry. Some results in this direction are presented in [93, 94], but they only concern Schwarzschild black hole background. The backreaction of the NC scalar field on the geometry could introduce (among other effects) a shift of the horizon, leading to a change of the Hawking temperature. This analysis we postpone for future work.

In addition to the analysis we performed here, one can straightforwardly include NC spinor and NC vector fields and calculate the QNMs spectrum of these fields. A more difficult task is to analyse the gravitational QNMs. For that one needs an action describing a NC gravitation field. There are various suggestions in the literature [31]. The main difficulty is certainly the fact that the first order correction in the deformation parameter vanishes. Therefore, the first non-vanishing correction to the NC gravitational action is second order in the deformation parameter, leading to more cumbersome calculations. All these effects we plan to investigate in our future work.

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Appendix. Twisted differential calculus

Space-time symmetries such as diffeomorphism symmetry of GR, or its subgroups Poincaré symmetry or conformal symmetry, are generated by vector fields. The Lie algebra of vector fields we label by $\Xi$ and its universal enveloping algebra by $U\Xi$. Note that $U\Xi$ is a Hopf algebra.

A well defined way to deform the symmetry Hopf algebra is via a Drinfeld twist [95]. The twist $F$ is an invertible element of $U\Xi \otimes U\Xi$ satisfying the following properties:

1. the cocycle condition
   \[(F \otimes 1)(\Delta \otimes id)F = (1 \otimes F)(id \otimes \Delta)F,\] (A.1)

2. normalization
   \[(id \otimes \epsilon)F = (\epsilon \otimes id)F = 1 \otimes 1,\] (A.2)

3. perturbative expansion
   \[F = 1 \otimes 1 + O(\lambda),\] (A.3)

where $\lambda$ is a deformation parameter. The last property insures that in the limit $\lambda \to 0$ the undeformed algebra $U\Xi$ is restored. We shall frequently use the notation (sum over $\alpha = 1, 2, \ldots \infty$ is understood)

\[F = f_\alpha \otimes f_\alpha, \quad F^{-1} = \bar{f}_\alpha \otimes \bar{f}_\alpha,\] (A.4)

where, for each value of $\alpha$, $f_\alpha$ and $\bar{f}_\alpha$ are two distinct elements of $U\Xi$ (and similarly $F^\alpha$ and $\bar{F}_\alpha$ are in $U\Xi$). The twist acts on the symmetry Hopf algebra and gives the twisted symmetry (as deformed Hopf algebra)

\[[\xi, \eta] = (\xi^\mu \partial_\mu \eta^\nu - \eta^\mu \partial_\mu \xi^\nu)\partial_\nu, \quad \Delta^F(\xi) = F\Delta(\xi)F^{-1} \quad \epsilon(\xi) = 0, \quad \bar{S} F(\xi) = F^\alpha S(f_\alpha) S(\bar{F}_\beta) \bar{f}_\beta.\] (A.5)

Here $\xi, \eta$ are vector fields belonging to $\Xi$. We see that after the twist deformation, the algebra remains the same, while in general the comultiplication (coproduct) and antipode change.

The whole deformation depends on formal parameters which control classical limit. Twisted (deformed) comultiplication leads to the deformed Leibnitz rule for the symmetry transformations when acting on product of fields.

The twist can be used to deform the commutative geometry of space-time (vector fields, 1-forms, exterior algebra of forms, tensor algebra) in a well defined way. More details on the twisted differential geometry can be found in [22].

We work with an Abelian twist in $D = 4$

\[F = e^{-\frac{i}{2} \theta^{AB} X_A \otimes X_B}.\] (A.6)

Here $\theta^{AB}$ is a constant antisymmetric matrix and indices $A, B = 1, \ldots, p$ with $p \leq 4$. $X_A$ are commuting vector fields $[X_A, X_B] = 0$. This requirement ensures that the cocycle condition is fulfilled.

Applying the inverse of the twist (A.6) to the usual point-wise multiplication of functions, $\mu(f \otimes g) = f \cdot g; f, g \in A$, we obtain the $*$-product of functions
\[ f \star g = \mu F^{-1}(f \otimes g) = \bar{P}^\alpha(f)\bar{I}_\alpha(g). \] (A.7)

The action of the twist (\( \bar{P}^\alpha \) and \( \bar{I}_\alpha \)) on the functions \( f \) and \( g \) is via the Lie derivative.

The product between functions and one-forms is defined as
\[ h \star \omega = \bar{P}^\alpha(h)\bar{I}_\alpha(\omega) \] (A.8)
with a function \( h \) and an arbitrary 1-form \( \omega \). The action of \( \bar{f}_\alpha \) on forms is given via the Lie derivative.

We often use the Cartan’s formula for the Lie derivative along the vector field \( \xi \) of an arbitrary form \( \omega \)
\[ L_\xi \omega = d_i \xi \omega + i_\xi d\omega. \] (A.9)
Here \( d \) is the exterior derivative and \( i_\xi \) is the contraction along the vector field \( \xi \).

Arbitrary forms form an exterior algebra with the wedge product. The \( \star \)-wedge product on two arbitrary forms \( \omega \) and \( \omega' \)
\[ \omega \wedge \star \omega' = \bar{P}^\alpha(\omega) \wedge \bar{I}_\alpha(\omega'). \] (A.10)

The usual (commutative) exterior derivative satisfies:
\[ d(f \star g) = df \star g + f \star dg, \]
\[ d^2 = 0, \]
\[ df = (\partial_\mu f) dx^\mu = (\partial_\mu^\star f) \star dx^\mu. \] (A.11)

The first property if fulfilled because the usual exterior derivative commutes with the Lie derivative which enters in the definition of the \( \star \)-product. Therefore, we will use the usual exterior derivative as the noncommutative exterior derivative. Note that the last line of (A.11) gives a definition of the \( \partial_\mu^\star \) derivatives.

The usual integral is cyclic under the \( \star \)-exterior products of forms
\[ \int \omega_1 \wedge_{\star} \omega_2 = (-1)^{d_1 d_2} \int \omega_2 \wedge_{\star} \omega_1, \] (A.12)
where \( d = \text{deg}(\omega) \), \( d_1 + d_2 = 4 \) provided that \( S^F(\bar{P}^\alpha)\bar{I}_\alpha = 1 \) holds. One can check that this indeed holds for any Abelian twist. The property (A.12) can be generalized to
\[ \int \omega_1 \wedge_{\star} \cdots \wedge_{\star} \omega_p = (-1)^{d_1 d_2 \cdots d_p} \int \omega_p \wedge_{\star} \omega_1 \wedge_{\star} \cdots \wedge_{\star} \omega_{p-1}, \] (A.13)
with \( d_1 + d_2 + \cdots + d_p = 4 \). We say that the integral is cyclic. This property is very important for construction of NC gauge theories.

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