Some exact solutions of KdV-Burgers-Kuramoto equation

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Abstract

Some exact solutions of KdV-Burgers-Kuramoto (KBK) equation are derived by the ansaz and tanh methods. Also, the most general Lie point symmetry group of the KBK equation are presented using the basic Lie symmetry method. As well as, the non-classical and weak symmetries of this equation, as well as the corresponding similarity reductions, are investigated. Finally, the classical and non-classical symmetries of KBK and KdV-Burgers (KB) equations are compared.

1. Introduction

One of the ways to describe the important, physical phenomena is by using Traveling wave solutions of nonlinear partial differential equations (NLPDE). The extended tanh-function method [1], the special, truncated expansion method [2] has been introduced to find exact solutions of NLPDEs. One of the important ways to find algebraic solutions is to use the tanh method developed in recent years. Later, the other forms of the tanh method have been developed that the solution of the Riccati equation is used to the hyperbolic tan function in the tanh method [3].

Symmetry methods for differential equations were originally developed by SOPHUS LIE in [4]. These methods are undoubtedly so useful and algorithmic for analyzing and solving linear and non-linear differential equations [5]. Furthermore, many other types of analytical solutions of PDEs can be obtained via the Lie group method.

Classifying group invariant solutions, detecting the linearized transformations, reducing the order of ordinary differential equations and mapping the solutions to other solutions are other important applications of the Lie groups in the theory of differential equations. Moreover, a class of PDEs, which is invariant under a given group of transformations, can be obtained based on the Lie symmetry method (for more details consult [6–8]). Finding more solutions for a given equation has been a motivation to generalizing the classical symmetries to the non-classical ones. BLUMAN and COLE [9] developed the non-classical method to study the symmetry reductions of the heat equation. In the non-classical symmetry method, the invariance surface condition is added to the original equation. Since then, a large number of papers have been devoted to the study of non-classical symmetries of non-linear PDEs in both one and several dimensions (see [10, 11]).

In addition to classical and non-classical symmetries, OLVER and ROSENAU in [12] have introduced another type of symmetries, called weak symmetries, which are the generalization of non-classical symmetries. Indeed, two types of symmetries can be defined: strong and weak. A strong symmetry group is a group of transformations $G$ on the space of independent and dependent variables with the following two properties:

(a) The $G$ elements transform solutions of the system into other solutions, and

(b) The $G$- invariant solutions of the system are found from a reduced system of differential equations involving the fewer number of independent variables than the original system. The weak symmetry group of the system is a group of transformations which satisfy the reduction property (b), but no longer transform the solutions into solutions.
In this paper, we investigate the KdV-Burgers-Kuramoto equation

\[
\text{KBK: } u_t + uu_x + \alpha u_{xx} + \beta u_x^3 + \gamma u_{xxxx} = 0, \tag{1.1}
\]

where \(\alpha, \beta\) and \(\gamma\) are arbitrary constants. Physical applications of this equation especially in hydrodynamic and optic have motivated various papers. This equation was first introduced by KURAMUTO in 1976 for modeling several phenomena which are simultaneously involved in non-linearity, dissipation, dispersion, and instability. This equation has a key role in modeling physical processes in unstable systems \([13]\). Describing turbulence process is particularly used, in which long waves in a viscous fluid flowing down an inclined plane, as well as unstable drift waves in plasma and stress waves in fragmented porous media \(\text{Find more information in [14]}\).

Solving the KBK equation involves different analytical methods. Several methods such as Weiss-Tabor-Carnevale transformation, tanh-function method, trial-function method, and homogeneous balance method are applied for obtaining the exact solutions of the KBK equation \(\text{Find more information in [14]}\).

In order to examine the problems of the flow of liquids containing gas bubbles, the fluid flow in elastic tubes, the control equation can be reduced to the so-called KdV-Burgers equation in the following way and as defined by SU and GARDNER \([15]\):

\[
\text{KB: } u_t + uu_x + \alpha u_{xx} + \beta u_x^3 = 0. \tag{1.2}
\]

We can consider equation (1.2) as KBK equation with \(\gamma = 0\). Some well-known equations which are similar to equation (1.1) are listed in the following table:

| Change in constants of (1.1) | Resulted equation | Some applications |
|-----------------------------|-------------------|------------------|
| \(\alpha = \gamma = 0\) | KdV               | Shallow waves, Integrable systems |
| \(\beta = \gamma = 0\) | Burgers           | Fluid mechanics  |
| \(\gamma = 0\)            | KdV-Burgers       | Flow of liquids containing gas bubbles |
| \(\beta = 0\)            | Kuramoto- Sivashinsky | The large scale properties of spatiotemporal chaos weak turbulence, quasi-periodic regime |

It is outstanding that nonlinear phenomena are critical in a variety of scientific fields, particularly in fluid mechanics, solid state physics science, plasma physics, plasma waves, and compound material science. An assortment of strategies, for example, the inverse scattering method, Hirota s bilinear method, are used to obtain exact solutions. The tanh method is a ground-breaking arrangement technique for the calculation of correct voyaging wave arrangements \([1–3, 16]\). Huibin and Kelin \([17]\) presented a power arrangement in tanh as a conceivable arrangement and substituted this extension straightforwardly into a higher-arrange KdV condition.

The present paper is structured as follows: section 2 is devoted to the traveling wave solutions. In section 3, the complete investigation of the classical symmetries of the KBK equation has been done. In section 4, the non-classical symmetries of the KBK equation are determined. In section 5, the weak symmetries of the KBK equation are discussed. Finally, the classical and non-classical symmetries of the KB and KBK equations are compared in section 6.

2. Traveling wave Solutions of KdV—Burgers - Kuramoto (KBK) equation

In this section, we are using the most important ansatz method (tanh-function method) to obtain exact traveling wave solutions of this nonlinear equation of PDEs. So, we introduce a new variable \(\tau = \tanh(c_0 + c_1t + c_2x)\) and the ansatz \(u = A_{11} + A_{13}t + A_{12}t^2 + A_{13}t^3\), where \(A_{ij}\) and \(c_i\) are arbitrary constants. Substituting expansions into the KBK equation, we obtain the following algebraic equations:

\[
\begin{align*}
\gamma c_1^4 (c_2^2 - 1)u^3(\tau) + c_1^4 (c_2^2 - 1)^2(12 \gamma c_0 \tau - \beta u'(\tau) \\
+ c_1^4 (c_2^2 - 1)(6c_1^4(6c_1^4 - \beta)\tau^2 + c_1^4(\alpha - 6c_1^4 - \beta)\tau + c_1^4(\alpha - 8c_1^4 - \beta))u''(\tau) \\
+ (24c_1^4(\gamma^3 - 6c_1^4 - \beta + 2c_1^4(\alpha - 8c_1^4 - \beta) + \gamma(2c_1^4 - \beta - u) - c_1^4)u'(\tau) = 0.
\end{align*}
\]

With linear algebra and required simplifications, leads to

**Solution 1:** For \(\alpha = -76a_1^2\gamma, \beta = a_1x + a_2t + a_3\) and \(\beta = 0\), then

\[
u(x, t) = 120a_1^3\gamma \tanh \theta^3 + \frac{90a_1^2\gamma}{19} \tanh \theta - \frac{a_2}{a_1},
\]

**Solution 2:** For \(\alpha = 76a_1^2\gamma/11, \beta = a_1x + a_2t + a_3\) and \(\beta = 0\), then

\[
u(x, t) = 120a_1^3\gamma \tanh \theta^3 - \frac{270a_1^2\gamma}{209} \tanh \theta - \frac{a_2}{a_1},
\]
Solution 3: For \( \alpha = \beta^2/16\gamma, \theta = a_1 t + a_2 - \beta x/8\gamma, \eta = \beta^3/\gamma^2 \) and \( \beta\gamma \neq 0 \), then
\[
u(x, t) = \frac{15\varepsilon}{64}(\tanh \theta - \varepsilon \tanh \eta - \tanh \eta) + \frac{512a_1\gamma + 9\beta\eta}{64\beta},
\]
Solution 4: For \( \alpha = \beta^2/16\gamma, \theta = \beta x/8\gamma, \eta = a_1 t + a_2 \) and \( \gamma \neq 0 \), then
\[
u(x, t) = \frac{\alpha\beta}{4\gamma(\cosh \eta + \cos \theta^2 - 1)}(3 \cosh \eta^4 - 6 \cosh \eta^2 \cos \theta^2 - 4(\cosh \eta^6 + \cos \theta^6)
+ 3(\cosh \eta^4 + \cos \theta^4) + 36 \cosh \eta^2 \cos \theta^2 - \frac{15}{2}(\cosh \eta^2 + 1) \sin \theta
+ 15 \cos \theta^2 \sin \theta - 18(\cosh \eta^2 + \cos \theta^2^2) + 11),
\]
Solution 5: For \( \alpha = 47\beta^2/144\gamma, \theta = a_1 t + a_2 + 24\beta\gamma x \) and \( \beta\gamma \neq 0 \), then
\[
u(x, t) = \frac{5\varepsilon\alpha\beta}{188\gamma}(\tanh \theta^3 + 3\varepsilon \cosh^{-2} \theta^2 + 3 \tanh \theta) - \frac{24a_1\gamma}{\beta},
\]
Solution 6: For \( \alpha = 73\beta^2/256\gamma, \theta = a_1 t + a_2 - 32\beta\gamma x \) and \( \beta\gamma \neq 0 \), then
\[
u(x, t) = \frac{15\varepsilon\alpha\beta}{1168\gamma}(\tanh \theta^3 + 4\varepsilon \cosh^{-2} \theta^2 + 3 \tanh \theta) - \frac{3a_1\gamma}{\beta},
\]
where the \( a_i \) are constants and \( \varepsilon = \pm 1 \).
Also, SAYED et al [18] obtained some other traveling wave solutions as:
\[
u(x, t) = 15\lambda \sech^2 \rho(\beta - 8\gamma \lambda \tanh \rho), \quad \rho = \lambda(x - kt + c)
\]
and KUDRYASHOV in [19] presented some other solutions that interested readers could refer to them for seeing more details. In figure 1, solution 1 to 6 have been plotted for some constants coefficients.

### 3. Classical symmetries of KdV-Burgers-Kuramoto (KBK) equation

In this section, we explain the procedure of finding lie symmetry of a system partial differential equations general. then, we use this method for KBK equation [6, 8, 20]. Let a system of partial differential equation wit
dependent and $q$ independent variables has been given such that $\Delta_{\mu}(x, u^{(\mu)}) = 0$, $\mu = 1, \ldots, r$, where $u^{(\mu)}$ is defined as $u^{(\mu)} = \partial^{\mu} / \partial x^j$, $j = 0, \ldots, n$, $i = 1, \ldots, q$. The one-parameter Lie group of transformations $x^i = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2)$, $\partial^{\mu} = u^{(\mu)} + \varepsilon \phi^{(\mu)}(x, u) + O(\varepsilon^2)$, for $i = 1, \ldots, p$, $\alpha = 1, \ldots, q$, where $\xi^i = (\partial \xi^i / \partial \varepsilon)_{\varepsilon=0}$ and $\phi^{(\mu)} = (\partial \phi^{(\mu)} / \partial \varepsilon)_{\varepsilon=0}$ are given. The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the general vector field $X = \sum_{i=1}^{p} \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^{q} \phi^{(\mu)}(x, u) \partial_{u^{(\mu)}}$; characteristics of the vector field $X$ are given by the functions $Q^{(\mu)}(x, u^{(\mu)}) = \phi^{(\mu)}(x, u) - \sum_{\alpha=1}^{q} \xi^i(x, u) \partial u^{(\mu)} / \partial x^i$, $\alpha = 1, \ldots, q$. Assume that the symmetry generator associated to (1.1) is given by

$$ X = \xi(x, t, u) \partial_x + \eta(x, t, u) \partial_t + \phi(x, t, u) \partial_u, \quad (3.1) $$

The fourth order prolongation $X$ is obtained as following the vector field:

$$ X^{(4)} = X + \phi^x \partial_{u_x} + \phi^t \partial_{u_t} + \phi^u \partial_{u_u} + \phi^{xx} \partial_{u_{xx}} + \phi^{xy} \partial_{u_{yx}} + \phi^{xt} \partial_{u_{xt}} + \phi^{ux} \partial_{u_{ux}} + \phi^{yt} \partial_{u_{yt}} + \phi^{uy} \partial_{u_{uy}}, \quad (3.2) $$

with coefficients $\phi^t = D_t Q + \xi u_x + \eta u_t$ and $\phi^u = D_t (D_t Q) + \xi u_u + \eta u_u$, where $Q = \phi - \xi u_x - \eta u_t$, is the characteristic of the vector field $X$ given by (3.1) and $D_t$ represents total derivative. The subscripts of $u$ are the derivatives with respect to the respective coordinates. Also recall the above could be $x$ and $t$ coordinates. Considering theorem (6.5) in [6], $X^{(4)}[u_t + u_{ux} + \alpha u_{xx} + \beta u_{xt} + \gamma u_{tt}]|_{(1.1)} = 0$. Hence, we obtain

$$ \phi^t + \phi u_x + \phi^x u + \alpha \phi_{xx} + \beta \phi_{xt} + \gamma \phi_{tt} = 0 \mod (1.1). \quad (3.3) $$

Substituting the coefficient functions $\phi^t$, $\phi^x$, $\phi^u$, $\phi^xt$, $\phi^tt$ into invariance condition (3.1), we obtain a polynomial equation including of partial derivative $u(x, t)$, where the coefficient are derivatives of $\xi$ and $\eta$ due to the fact that $\xi$ and $\eta$ are functions of $x$, $t$ and $u$; we are able to set individual coefficients to zero, so the following system of equation is obtained as a result $\xi_t = \xi^t = \eta_t = \eta^t = 0$, $\eta_x = \eta^x = \xi_x = \xi^x$. By solving the above system we deduce that $\xi(x, t) = a t + c_3$, $\eta(x, t) = c_3$ and $\phi(x, t) = c_3$, where $c_1, c_2$ and $c_3$ are arbitrary constants.

As a result, we have:

**Theorem 3.1.** The KBK equation admits a three-dimensional Lie algebra $\mathfrak{g}$ of infinitesimal symmetries spanned by the vector fields $v_1 = \partial_x$, $v_2 = \partial_t$ and $v_3 = t \partial_x + \partial_u$. The commutator table of $\mathfrak{g}$ is given as table 1.

### 3.1. Invariant solutions and its classification

Equation (1.1) can be regarded as a submanifold of the jet space $J^4$. Thus, due to [6], we can find the most general group of invariant solutions of equation (1.1). By exponentiating the infinitesimal symmetries of equation (1.1), we obtain the one parameter groups $\text{Fl}^k$ generated by $\mathbf{V}$ for $k = 1, \ldots, 3$ as follows

$$ \text{Fl}_1^k(x, t, u) = (x + \varepsilon_1, t, u), \quad \text{Fl}_2^k(x, t, u) = (x + \varepsilon_2, u), \quad \text{Fl}_3^k(x, t, u) = (x + \varepsilon_3 t + \varepsilon_1 t, u + \varepsilon_2). $$

Recall that, in general terms, a family of so-called invariant solutions correspond to each parameter subgroup of the symmetry group of a system. So, we can state the following theorem:

**Theorem 3.2.** Let $u = f(x, t)$ be a solution of equation (1.1), then the following function are solution of (1.1) either

$$ \text{Fl}_1^k f(x, t) = f(x + \varepsilon_1, t), \quad \text{Fl}_2^k f(x, t) = f(x, t + \varepsilon_2), \quad \text{Fl}_3^k f(x, t) = f(x + \varepsilon_3 t, t + \varepsilon_1 t + \varepsilon_1) - \varepsilon_3. $$

Thus, for the arbitrary combination $\mathbf{v} = c_1 v_1 + c_2 v_2 + c_3 v_3 \in \mathfrak{g}$, the KBK equation has the solution $u = f(x + \varepsilon_1 t + \varepsilon_2 t + \varepsilon_3 t, t + \varepsilon_1 t + \varepsilon_2)$, where $\varepsilon_i$ are arbitrary real numbers.
Since any linear combination of infinitesimal generators is also an infinitesimal generator, the differential equation contains infinitely different subgroups. In order to complete understanding the invariant solutions, it is necessary and significant to determine which subgroups would give essentially different types of solutions. However, a well-known standard procedure [8, 20–24] allows the classification of all the 1D subalgebras into the subsets of conjugate subalgebras. This involves constructing the adjoint representation group, which introduces a conjugate relation in the set of all the 1D subalgebras. In fact, 1D subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. If we take only one representative of any class of equivalent subalgebras, an optimal set of subalgebras is created. The corresponding set of invariant solutions is then the minimal list, thus we can get all the other invariant solutions of 1D subalgebras simply via transformations.

Each \( v_i, i = 1, \ldots, 3 \), of the general member’s Lie algebra, symmetries generates an adjoint transformation \( \text{Ad}(\exp(\varepsilon v_i)) \) determined by the Lie series \( \text{Ad}(\exp(\varepsilon v_i)) = v_j - \varepsilon [v_i, v_j] + \varepsilon^2 [v_i, [v_i, v_j]]/2 \ldots \), where \([v_i, v_j]\) is the Lie algebra communicator, \( \varepsilon \) is a parameter and \( i, j = 1, \ldots, 3 \) (see [8]). In Table 2, all the adjoint representations of the Lie group of the KBK equation is presented, with the entry indicating \( \text{Ad}(\exp(\varepsilon v_i))v_j \). So, we can show that:

**Theorem 3.3.** An optimal system of 1D Lie subalgebras of KBK equation (1.1) is provided by those generated by

\[
(1) \ v_1, \quad (2) \ v_2, \quad (3) \ v_3, \quad (4) \ v_2 + \alpha v_3,
\]

where \( \alpha \) is an arbitrary nonzero constant.

**Proof.** Let \( \mathfrak{g} \) is the symmetry algebra of equation (1.1) with adjoint representation given in Table 2 and \( v = a_1v_1 + a_2v_2 + a_3v_3 \) is a nonzero vector field of \( \mathfrak{g} \). We will simplify as many of the coefficients \( a_i, i = 1, \ldots, 3 \) as possible through proper adjoint applications on \( v \). Consider the following cases:

**Case 1:** First, assume \( a_3 \neq 0 \). By scaling \( v \) if necessary, we can assume \( a_3 = 1 \). Using the adjoint table (Table 2), if we act on \( v \) with \( \text{Ad}(\exp(-a_2v_3)) \), the coefficient of \( v_3 \) can be vanished: \( v' = v_1 + a_2v_3 \). Then we apply \( \text{Ad}(\exp(a_2v_3)) \) on \( v' \) to cancel the coefficient of \( v_1 \). So, \( v \) reduces to (1).

**Case 2:** Assume \( a_1 = 0 \). If \( a_2 = 0 \), by scaling \( v \) if necessary, we can assume that \( a_2 = 1 \). If \( a_3 = 0 \), \( v \) reduces to (4); otherwise we deduce (2). Finally, if \( a_2 = 0 \), by scaling \( v \) if necessary, we can assume that \( a_3 = 1 \), which introduces (3).

### Table 2. Adjoint representation of infinitesimal symmetries of KBK equation.

| Ad     | \( v_1 \) | \( v_2 \) | \( v_3 \) |
|--------|-----------|-----------|-----------|
| \( v_1 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |
| \( v_2 \) | \( v_1 \) | \( v_2 \) | \( v_3 - \varepsilon v_4 \) |
| \( v_3 \) | \( v_1 \) | \( v_2 + \varepsilon v_3 \) | \( v_3 \) |

3.2. Similarity Reduction of KBK equation

The coordinates \((x, t, u)\) expressed The KBK equation (1.1); so, we will obtain a new coordinate independent invariant \((r, w)\) corresponding to the infinitesimal symmetry generator. By implementing the chain rule obtained the equation’ expression in the new coordinate.

**For \( v_1 \):** In this case, the characteristic equation has the form \( dx/1 = dt/0 = du/0 \), the resulted differential invariants are: \( r = t \) and \( w = u \). If we consider \( w \) as a function of \( r \), equation (1.1) reduces to \( w''(r) = 0 \).

**For \( v_2 \):** In this case, we have the characteristic equations \( dx/0 = dt/1 = du/0 \). So, we obtain the differential invariants \( w = u \) and \( r = x \). By considering \( w \) as a function of \( r \) the KBK equation reduces to:

\[ nw'' + \alpha w'' + \beta w'' + \gamma w(4) = 0. \]

**For \( v_3 \):** In this case we have the characteristic equations \( dx/t = dt/0 = du/1 \); we obtain the differential invariants \( r = t \) and \( w = u - x/t \). If we treat \( w \) as function of \( r \), the KBK equation reduces to:

\[ w'' - w/r = 0 \]. By solving this equation, we have \( u(x, t) = (x + c)/t \).
For \( v_2 + v_3 \): In this case we have the characteristic equations \( dx/t = dt/1 = du/1 \). So, we obtain the differential invariants \( r = t^2 - 2x \) and \( w = u - t \). By considering \( w \) as a function of \( t \) the KBK equation reduces to: \( 16\gamma w^{(4)} - 8\beta w^{(3)} + 4\alpha w'' - 2ww' + 1 = 0 \).

4. Non-classical symmetries of KBK equation

Beside the classical symmetries, we can use the non-classical symmetry method to find some other solutions for a system of PDEs and ODEs. Here, we follow the method used by CAI GUOLIANG et al, for obtaining the non-classical symmetries of the Burgers-Fisher equation based on compatibility of evolution equations [10].

For the non-classical method, we must add the invariance surface condition to the equation, and then apply the classical symmetry method. For this purpose, We consid the classical symmetry method. For this purpose, We consid the classical symmetry method.

As a result the governing equation has the following form:

\[
D_1(u_t) = D_1(-u u_x - \alpha u_{xx} - \beta u_x - \gamma u_x),
\]

\[
D_2(\varphi - \xi u_x) = -u u_x - \alpha u_{xx} - \beta u_x - \gamma u_x,
\]

\[
D_3(\varphi - \xi u_x) = \xi u_x - \varphi - u D_x(\varphi - \xi u_x) - \alpha D_x(\varphi - \xi u_x) - \beta D_x(\varphi - \xi u_x) - \gamma D_x(\varphi - \xi u_x).
\]

Substituting \( \xi u_x \) to both sides, we can get:

\[
\varphi' = \xi u_x - \varphi - u x^2 - \alpha \varphi x^2 - \beta \varphi x^3 - \gamma \varphi x^4 + \alpha \xi u_x + \beta \xi u_x + \gamma \xi u_x + \xi u_{xx} + \xi u_{xt}.
\]

In addition we have:

\[
D_3(u_t) = D_3(-u u_x - \alpha u_{xx} - \beta u_x - \gamma u_x),
\]

\[
u_{xt} = -u u_{xx} - u^2 - \alpha u_x - \beta u_x - \gamma u_x.
\]

As a result the governing equation has the following form:

\[
\varphi' + \varphi u_x + u x^2 + \alpha \varphi x^2 + \beta \varphi x^3 + \gamma \varphi x^4 = 0.
\] (4.1)

Where \( \varphi' \), \( \varphi'\), \( \varphi x' \), and \( \varphi x' \) are:

\[
\varphi' = D_1(\varphi - \xi u_x - \eta u_t), \quad \eta u_t = D_1(\varphi - \xi u_x) + \xi u_{xt}, ...
\]

\[
\varphi x' = D_x(\varphi - \xi u_x) + \xi u_x.
\]

By applying the Lie symmetry method and solving resulted in determining equations, we find the following results:

\[
\xi = \xi_1 + \xi_2, \quad \varphi = \varphi, \quad \varphi x = \varphi x,
\] (4.2)

when \( \xi_1 = 1 \) and \( \xi_2 \neq 1 \); Therefore, \( \sigma_1 = 1 - tu_x - u_t \) and \( \sigma_2 = u_t \), when \( \xi_1 = 1 \) and \( \xi_2 = 1 \). Thus when \( \xi_1 = 1 \) and \( \xi_2 = 1 \), the symmetries are \( \sigma_1 = 1 - tu_x - u_t \) and \( \sigma_2 = u_t \). When \( \xi_1 = 1 \) and \( \xi_2 = 1 \), the symmetries are \( \sigma_1 = 1 - tu_x - u_t \) and \( \sigma_0 = u_t \), when \( \xi_1 = 1 \) and \( \xi_2 = 1 \), the symmetries are \( \sigma_0 = 1 - u_t \) and \( \sigma_0 = tu_x + u_t \).

As a result we have obtained these new similarity reduced equations for the equation (1.1).

- \( \sigma_1 = \sigma_2 = 1 - tu_x - u_t \): Substituting it into \( \sigma(u) = 0 \) gives: \( u(x, t) = t + F(t^2 - 2x) \) where \( F \) is arbitrary function. So we have the similarity reduced equation as \( 16\gamma F^{(4)} - 8\beta F^{(3)} + 4\alpha F'' - 2FF' + 1 = 0 \).

- \( \sigma_2 = \sigma_3 = u_t \): Substituting it into \( \sigma(u) = 0 \) gives: \( u(x, t) = F(t) \) where \( F \) is arbitrary function, by using equation (1.1), we conclude that \( F(t) \) is an arbitrary constant.

- \( \sigma_3 = 1 - tu_x - u_t \): Substituting it into \( \sigma(u) = 0 \) gives: \( u(x, t) = t + 1 + F(2t + t^2 - 2x) \) where \( F \) is arbitrary function. So we have the similarity reduced equation as \( 16\gamma F^{(4)} - 8\beta F^{(3)} + 4\alpha F'' - 2FF' + 1 = 0 \).

- \( \sigma_4 = 1 - tu_x - u_t \): Substituting it into \( \sigma(u) = 0 \) gives: \( u(x, t) = F(x) \) where \( F \) is an arbitrary function. By substituting the given expression in equation (1.1), we obtain a trivial similarity reduced equation.

- \( \sigma_5 = \sigma_6 = u_t \): Substituting it in \( \sigma(u) = 0 \) gives: \( u(x, t) = F(x) \) where \( F \) is an arbitrary function, by using equation (1.1) is given \( F(\xi) \) which must be satisfy in \( FF' + \alpha F'' + \beta F^{(3)} + \gamma F^{(4)} = 0 \).
In this paper, we obtained some traveling wave solutions of the KBK equation derived by implementing the

As declared before, we can regard the KB equation as a particular case of the KBK equation

Substituting this expression in equation (1.1) leads to an incompatible similarity-reduced equation.

Case II: \( \eta = 0 \). In this case, without loss of generality we can set: \( \xi = 1 \). So, we have \( u_x = \varphi \) and

\[
A(x, t, u) = -u\varphi - \alpha\varphi_x - \beta\varphi_{xx} - \gamma\varphi_x^2.
\]

Subsisting this in the determining equation

\[
A_{\varphi_{xx}} + A_u \varphi_x - A_x = 0,
\]

gives:

\[
\varphi_t + 2\varphi^2 + \alpha\varphi_{xx} + \beta\varphi_{xxx} + \gamma\varphi_x^2 = 0.
\]

By assuming \( \varphi = \varphi(x, t) \) above equation changes into the following:

\[
\varphi_t + 2\varphi^2 + \alpha\varphi_{xx} + \beta\varphi_{xxx} + \gamma\varphi_x^2 = 0.
\]

So we have: \( \varphi = 1/(t + c) \), where \( c \) is an arbitrary constant. As a result we have the solution

\[
u(x, t) = x/(t + c) + F(t)
\]

for (1.1), where \( F \) is an arbitrary function.

5. On the weak symmetries of the KBK equation

Now, we perform the weak symmetry method in two examples to suggest a way to find more solutions, as follows

Example 5.1. Consider the one-parameter group \( (x, t, u) \rightarrow (\lambda x, \lambda t, \lambda u) \), with \( X = x\partial_x + u\partial_u \) as infinitesimal generator. For characterizing equation we have: \( dx/x = dt/0 = du/u \). So, related differential invariants are \( r = t, w = u/x, u_x = xw, u_t = w, u_{xx} = u_x = u_x^2 = 0 \).

Substituting above expressions in (1.1) gives: \( u_t + u_{xx} = 0 \). If we consider \( x \) as the parametric coordinate then we find \( w_t + w_{xx} = 0 \), as the reduced equation. As a result we find the solution \( u(x, t) = x/(t + c) \) for (1.1), where \( c \) is an arbitrary constant.

Example 5.2. Consider the one-parameter group \( G \) in the form \( (x, t, u) \rightarrow (\lambda x, \lambda t, \lambda u) \), as the infinitesimal generator. For characterizing equation we have: \( dx/x = dt/0 = du/u \). So, the related differential invariants are \( r = x/t \) and \( w = u \).

Substituting above expressions in (1.1) gives: \( (ww_t - rw_t)t^5 + \omega w_t^5 + \beta w_{tt}t = 0 \). If we consider \( t \) as the parametric coordinate then we find that \( w \) must be satisfy in all the following equations \( \beta w_{tt} = 0, \omega w_t = 0 \) and \( \omega w_{tt} - rw_t \). As a result, we obtain the constant solution for (1.1).

6. Applications

As declared before, we can regard the KB equation as a particular case of the KBK equation (when \( \gamma = 0 \)). By obtaining the classical and non-classical symmetries of the KB equation, We conclude they are the same as the KBK equation obtained in the previous sections. Indeed, the above investigation can also state for the KB equation. Notice that the obtained similarity solutions and related similarity-reduced equations in these equations are not necessarily the same. For instance, as shown above, the classical symmetry transformation \( (x, t, u) \rightarrow (\lambda x, \lambda t, \lambda u) \) reduces the KBK equation to \( \omega w_t^5 + \beta w_{tt} + \omega w_t + \gamma w_{tt} = 0 \), whereas its similarity-reduced equation for KB is \( \omega w_t^5 + \beta w_{tt} + \omega w_t = 0 \).

7. Conclusions

In this paper, we obtained some traveling wave solutions of the KBK equation derived by implementing the ansatz methods (the tanh-function method). Also, the largest possible set of symmetries is an important example of a non-linear dynamical system: the KdV-Burger-Kuramoto (KBK) equation presented. By applying the classical symmetry method, we found the classical symmetries. Further, we obtained many other symmetries of the KBK equation as well as the corresponding similarity-reduced equations via non-classical and weak symmetry methods. Finally, we deduced that the classical and non-classical symmetries of the KB equation (as a particular case of the KBK equation) are the same as those of the KBK equation.
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