In this paper, we employ Meyer wavelets to characterize multiplier spaces between Sobolev spaces without using capacity. Further, we introduce logarithmic Morrey spaces $M^\tau_{t,r,p}(\mathbb{R}^n)$ to establish the inclusion relation between Morrey spaces and multiplier spaces. By wavelet characterization and fractal skills, we construct a counterexample to show that the scope of the index $\tau$ of $M^\tau_{t,r,p}(\mathbb{R}^n)$ is sharp. As an application, we consider a Schrödinger type operator with potentials in $M^\tau_{t,r,p}(\mathbb{R}^n)$.
For a compact set \( e \subset \mathbb{R}^n \), the capacity \( \text{cap}(e, H^{t,p}) \) on \( e \) is defined by
\[
\text{cap}(e, H^{t,p}) = \inf \left\{ \|u\|_{H^{t,p}([0,1]^n)} : u \in \mathcal{S}, u \geq 1 \text{ on } e \right\},
\]
where \( \mathcal{S} \) is the Schwartz class of rapidly decreasing smooth functions on \( \mathbb{R}^n \).

Lemma 1.2. \((\text{[10]})\) Given \( r > 0 \) and \( t \geq 0 \).
(i) For \( 1 < p < n/(r + t) \), \( f \in X_{t,p}^r(\mathbb{R}^n) \) if and only if
\[
\sup_{e \subset \mathbb{R}^n} \left( \frac{\|(-\Delta)^{t/2} f\|_{L^p(e)}}{(\text{cap}(e, H^{t+p,p}))^{1/p}} + \frac{\|f\|_{L^p(e)}}{(\text{cap}(e, H^{t,p}))^{1/p}} \right) < \infty.
\]
(ii) For \( 1 < p < n/r \) and any cube \( Q \) with length less than 1, the capacity \( \text{cap}(Q, H^{r,p}) \) is less than \( C|Q|^{1-pr/n} \).

Our motivation is based upon the following consideration. For complicated compact sets, it is very difficult to compute the capacity. The main aim of this paper is to give some wavelet characterizations and introduce some sufficient conditions which can be verified easily. Precisely, for \( r > 0, t \geq 0 \) and \( t + r < 1 < p < n/(r + t) \), we will give a characterization of the multipliers from \( H^{t+p,p}(\mathbb{R}^n) \) and \( H^{r,p}(\mathbb{R}^n) \) by Meyer wavelets without using capacity. See Theorems 3.3. Also our method can be applied to study the relation between multiplier spaces and Morrey spaces. To deal with the case \( t > 0 \), we have to introduce the almost local operator \( T' \). See §2.

Lemma 1.2 implies that the multiplier space \( X_{t,p}^r(\mathbb{R}^n) \) is a subspace of Morrey space \( M_{t,p}^r(\mathbb{R}^n) \). It is natural to ask if the reverse inclusion relation holds. Unfortunately, for \( t = 0 \), the imbedding \( X_{t,p}^r(\mathbb{R}^n) \subset M_{t,p}^r(\mathbb{R}^n) \) is not an isomorphism. In [3], P. G. Lemarié gave a counterexample to show that when \( n - 2r \) is an integer, \( X_{t,p}^r(\mathbb{R}^n) \neq M_{t,2}^r(\mathbb{R}^n) \). Recently, P. G. Lemarié [9] and Yang-Zhu [23] constructed some counterexamples for \( t = 0 \) and \( 1 < p < n/r \).

For \( t > 0 \), we have to consider the action of the differentiation, so we can not construct counterexample like the case \( t = 0 \) in [23]. Our counterexample in Theorem 5.3 depends on our wavelet characterization, Theorem 3.3 and fractal skills. From this counterexample, we can see that the product of \( f \in M_{t,p}^r(\mathbb{R}^n) \) and \( g \in H^{t+p,p}(\mathbb{R}^n) \) may produce a blow up phenomenon of logarithmic type on fractal sets with Hausdorff dimension \( n - p(r + t) \). To eliminate this defect, we introduce a logarithmic type Morrey space \( M_{t,p}^{\tau,0}(\mathbb{R}^n) \) and prove that for \( \tau > 1/p' \),
\[
M_{t,p}^{\tau,0}(\mathbb{R}^n) \subset X_{t,p}^r(\mathbb{R}^n) \subset M_{t,p}^r(\mathbb{R}^n) = M_{t,p}^{00}(\mathbb{R}^n),
\]
where \( r > 0, t \geq 0 \) and \( 1 < p < n/(r + t) \). See §4.

It should be pointed out that, in the above inclusion relation, the scope of \( \tau \) is \( (1/p', \infty) \), where \( p' \) is the conjugate number of \( p \). In §5, our counterexample implies that, for \( 0 < \tau \leq 1/p' \), there exists some function \( f \in M_{t,p}^{\tau,0}(\mathbb{R}^n) \), but \( f \notin X_{t,p}^r(\mathbb{R}^n) \). See §5 for
the details. Theorems 5.3 and 5.4 illustrate the difference between Morrey spaces and multiplier spaces.

Another motivation is that the results about multipliers on Sobolev spaces can be applied to the study on partial differential equations. For example, in [11], V. Maz’ya and I. E. Verbitsky considered the multipliers from $H^{1,2}(\mathbb{R}^n)$ to $H^{-1,2}(\mathbb{R}^n)$. For a Schrödinger operator $L = I - \Delta + V$, they got many sufficient and necessary conditions such that $V$ is a multiplier from $H^{1,2}(\mathbb{R}^n)$ to $H^{-1,2}(\mathbb{R}^n)$. For more information, we refer the readers to [8, 10, 11, 12] and the references therein.

As an application of our results, we consider the solution in Sobolev spaces $H^{t+r,p}(\mathbb{R}^n)$ to the equation:

$$
(I + (-\Delta)^{r/2} + V)f = g,
$$

where $g \in H^{t,p}(\mathbb{R}^n)$ and $V \in M^{T}_{r,p}(\mathbb{R}^n)$ with $r > 0, t \geq 0, 1 < p < n/(r + t), \tau > 1/p'$. If $V$ is a function of Hölder class, one usual method to deal with equation (1.1) is the boundedness of Calderón-Zygmund operators. As a function in the logarithmic Morrey spaces $M^{T}_{r,p}(\mathbb{R}^n)$, $V$ may be not a $L^\infty$ function. In §6, by Theorem 4.8 we prove that for $V(x) \in M^{T}_{r,p}(\mathbb{R}^n)$, the above equation (1.1) has an unique solution in the Sobolev space $H^{t+r,p}(\mathbb{R}^n)$.

In this paper, we use four tools in analysis. One is the multi-resolution analysis introduced by Y. Meyer and S. Mallat in 1990s. The other is the almost local operator $T'$. See §2. By the projection operators generated from multi-resolution analysis, S. Dobynski (cf. [4]) obtained a decomposition of the product of two functions. In order to adapt to our needs, we make some modification to Dobynski’s decomposition. The third main skills are to use combination atoms and to introduce some differential methods. The forth main skill is to choose special functions such that their wavelet coefficients restricted on some fractal sets. See §4 and §5.

Our paper is organized as follows. In §2, we state some notations and known results which will be used throughout this paper. In §3, we give a wavelet characterization of the multiplier spaces $X^{t}_{r,p}(\mathbb{R}^n)$. In §4, we introduce a class of logarithmic Morrey spaces $M^{T}_{r,p}(\mathbb{R}^n)$ and get a very simple sufficient condition of $X^{t}_{r,p}(\mathbb{R}^n)$. In §5, for $M^{T}_{r,p}(\mathbb{R}^n)$, we construct a counterexample to prove the sharpness of the scope of the index $\tau$ obtained in §4. In the last section, we consider an application to PDE problem.

2. Some preliminaries

In this section, we state some notations, knowledge and preliminary lemmas which will be used in the sequel. Firstly, we recall some background knowledge of wavelets and multi-resolution analysis.

We will adopt real-valued tensor product wavelets to study the multiplier spaces in this paper. Let $E_n = \{0, 1\}^n \setminus \{0\}$. For $\epsilon = 0$ (respectively, $\epsilon \in E_n$), we assume that $\Phi^\epsilon(x)$ is
a scaling function (respectively, wavelet). In the proof, we use only Meyer wavelets and regular Daubcheis wavelets. We say a Daubcheis wavelet is regular if it has sufficient vanishing moment until order $m$ and $\Phi^m(x) \in C_0^m([−2^M, 2^M])$, where the regularity exponent $m$ is large enough and $M$ is determined by $m$, see [13] for more details. For any $\varepsilon \in \{0, 1\}^n$, $j \in \mathbb{N}$ and $k \in \mathbb{Z}^n$, we denote $\Phi^m_{\varepsilon,j}(x) = 2^{m/2}\Phi^m(2^jx − k)$. In addition we define

$$\Lambda_n = \{(\varepsilon, j, k) : \varepsilon \in \{0, 1\}^n, j \in \mathbb{N}, k \in \mathbb{Z}^n \text{ and } \varepsilon \neq 0, \text{ if } j > 0\}.$$ 

For fixed tempered distribution $f$, if we use wavelets which is sufficient regular, then we can define $f_{\varepsilon,j} = \langle f, \Phi^m_{\varepsilon,j} \rangle$. And the wavelet representation $f = \sum_{(\varepsilon,j) \in \Lambda_n} f_{\varepsilon,j} \Phi^m_{\varepsilon,j}$ holds in the sense of distribution.

Let $\{V_j, j \in \mathbb{Z}\}$ be an orthogonal multi-resolution in $L^2(\mathbb{R})$ with the scaling function $\Phi^0(x)$. Denote by $W_j$ the orthogonal complement space of $V_j$ in $V_{j+1}$, that is, $W_j = V_{j+1} \ominus V_j$. Let $\{\Phi^1(x − k), k \in \mathbb{Z}\}$ be an orthogonal basis in $W_0$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$, denote $\Phi^m(\varepsilon) = \prod_{i=1}^{n} \Phi^m(\varepsilon_i)$. For $V_j = \{f(x) = \sum_{k \in \mathbb{Z}^n} f^0_{j,k} \Phi^m_{j,k}(x), \text{ where } \{f^0_{j,k}\}_{k \in \mathbb{Z}^n} \in L^2\}$ and $W_j = \{f(x) = \sum_{\varepsilon \in \Lambda_n, k \in \mathbb{Z}^n} f^\varepsilon_{j,k} \Phi^m_{j,k}(x), \text{ where } \{f^\varepsilon_{j,k}\}_{\varepsilon \in \mathbb{Z}^n} \in L^2\}$, we have

**Lemma 2.1.** $\{V_j, j \in \mathbb{Z}\}$ is an orthogonal multi-resolution with the scaling function $\Phi^0(x)$. $W_j$ is the orthogonal complement space of $V_j$ in $V_{j+1}$, that is, $W_j = V_{j+1} \ominus V_j$. Further $\{\Phi^m_{j,k}, (\varepsilon, j, k) \in \Lambda_n\}$ is an orthogonal basis in $V_0 \oplus_{j \geq 0} W_j = L^2(\mathbb{R}^n)$.

Denote by $P_j$ and $Q_j$ the projection operators from $L^2(\mathbb{R}^n)$ to $V_j$ and $W_j$, respectively. By Lemma 2.1 S. Dobynski got a decomposition of the product of two functions $f$ and $g$, which is similar to Bony’s paraproduct (see [4]). Denote

$$\bar{\Lambda}_n = \{(\varepsilon, \varepsilon', j, k, k'), \varepsilon, \varepsilon' \in \{0, 1\}^n \setminus \{0\}, j \geq 0, k, k' \in \mathbb{Z}^n, (\varepsilon, k) \neq (\varepsilon', k')\}.$$ 

By the projection operators $P_j$ and $Q_j$, we divide the product of $f(x)$ and $g(x)$ into the following terms.

$$f(x)g(x) = P_0(f)P_0(g) + \sum_{j \geq 0} P_j(f)Q_j(g) + \sum_{j \geq 0} Q_j(f)P_j(g)$$

$$+ \sum_{(\varepsilon,j) \in \Lambda_n} f^\varepsilon_{j,k}g_{\varepsilon',j,k'} \Phi^m_{j,k}(x)\Phi^m_{j,k'}(x) + \sum_{(\varepsilon,j) \neq (\varepsilon',j)} f^\varepsilon_{j,k}g_{\varepsilon',j,k'} \left(\Phi^m_{j,k}(x)\right)^2.$$ 

To facilitate our use, we make a modification to the above decomposition and use special wavelets for different cases. Let $N$ be a positive integer. We decompose the product
For the proof, we refer the readers to [20, 24].

\[ f(x)g(x) = \sum_{j=1}^{\infty} \left[ P_{j+1}(f)P_{j+1}(g) - P_j(f)P_j(g) \right] + P_0(f)P_0(g) \]

(2.1)

\[ = \sum_{j=N}^{\infty} \left[ Q_j(f)Q_j(g) + P_j(f)Q_j(g) + Q_j(f)P_j(g) \right] + P_N(f)P_N(g) \]

and the term \( \sum_{j=N}^{\infty} Q_j(f)P_j(g) \) can be decomposed as

(2.2)

\[ \sum_{j=N}^{\infty} Q_j(f)P_j(g) = \sum_{j=N}^{\infty} Q_j(f) \sum_{i=1}^{N} Q_{j-i}(g) + \sum_{j=N}^{\infty} Q_j(f)P_{j-N}(g) \]

For any \( j \in \mathbb{N} \) and \( k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n \), let \( Q_{j,k} = \prod_{s=1}^{n} [2^{-i}k_s, 2^{-i}(k_s + 1)] \) and denote by \( \Omega \) the set of all dyadic cubes \( Q_{j,k} \). For arbitrary set \( Q \), we denote by \( \tilde{Q} \) the \( 2^{M+1} \)-multiple of \( Q \). Finally, let \( \chi(x) \) be the characteristic function of the unit cube \( Q_0 \) and \( \tilde{\chi} \) be the characteristic function of \( \tilde{Q}_0 \).

In 1970s, H. Triebel introduced Triebel-Lizorkin spaces \( F_{r}^{p,q}(\mathbb{R}^n) \) (17). Many function spaces can be seen as the special cases for \( F_{r}^{p,q}(\mathbb{R}^n) \). For example, \( F_{r}^{p,2}(\mathbb{R}^n) \) is the fractional Hardy space. For \( 1 < p < \infty \), \( F_{r}^{p,2}(\mathbb{R}^n) \) are the Sobolev spaces \( H^{s,p}(\mathbb{R}^n) \). For \( p = \infty \), \( F_{\infty}^{r,2}(\mathbb{R}^n) \) is the fractional BMO space \( BMO^r(\mathbb{R}^n) \) which is defined by \( BMO^r(\mathbb{R}^n) := (I - \Delta)^{-r/2}BMO(\mathbb{R}^n) \), where \( I \) is the unit operator, \( \Delta \) is the Laplace operator. Here \( BMO(\mathbb{R}^n) \) denotes the non-homogeneous bounded mean oscillation space which is defined as the set of the functions such that

\[ \sup_{|Q|=1} |f_Q| = \sup_{|Q|=1} \frac{1}{|Q|} \left| \int_Q f(x)dx \right| \leq C \]

and

\[ \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2dx < \infty. \]

For \( 1 \leq p < \infty \) and \( r \in \mathbb{R} \), it is well known that \( \left( F_{r}^{p,2}(\mathbb{R}^n) \right)' = F_{r'}^{-2}(\mathbb{R}^n) \). The following lemma gives a characterization of \( F_{r}^{p,2}(\mathbb{R}^n) \) by Meyer wavelets and regular Daubechies wavelets. For the proof, we refer the readers to [20, 24].
Lemma 2.2. (i) For $1 \leq p < \infty$ and $|r| < m$, using Meyer wavelets or $m$-regular Daubechies wavelets, we have the following equivalent characterizations,
\[
g(x) = \sum_{(e, j, k) \in A_n} g_{j,k}^e \Phi_{j,k}^e(x) \in \mathcal{F}_{p}^{r,2}(\mathbb{R}^n)
\]
\[
\iff \left\| \left( \sum_{(e, j, k) \in A_n} 2^{j(n/2)}\|g_{j,k}^e\|^2 \chi(2^j - k) \right)^{1/2} \right\|_{L^p} < \infty
\]
\[
\iff \left\| \left( \sum_{(e, j, k) \in A_n} 2^{j(n/2)}\|g_{j,k}^e\|^2 \chi(2^j - k) \right)^{1/2} \right\|_{L^p} < \infty.
\]
(ii) Given $|r| < m.$ $g(x) = \sum_{(e, j, k) \in A_n} g_{j,k}^e \Phi_{j,k}^e(x) \in \mathcal{F}_{\infty}^{r,2}(\mathbb{R}^n)$ if and only if there exists $1 < p < \infty$ such that for any $Q \in \Omega,$
\[
\left\| \sum_{e \in \mathcal{E}_n, Q \subset Q} 2^{j(n/2)}\|g_{j,k}^e\|^2 \chi(2^j - k) \right\|_{L^p} \leq C|Q|^{1/p}.
\]

The wavelet characterizations of function spaces have been studied by many authors. In Chapters 5 and 6 of [13], Y. Meyer established wavelet characterizations for many function spaces. In [22], Q. Yang, Z. Cheng and L. Peng considered wavelet characterization of Lorentz type Triebel-Lizorkin spaces and Lorentz type Besov spaces. In [20], Q. Yang introduced the wavelet definition of Besov type Morrey spaces. W. Yuan, W. Sickel and D. Yang considered the atomic decomposition for Besov type Morrey spaces and Triebel-Lizorkin type Morrey spaces in [24].

Morrey spaces $M_{r,p}(\mathbb{R}^n)$ were introduced by Morrey in 1938 and play an important role in the research of partial differential equations. In 2003, Wu and Xie [19] proved that generalized Morrey spaces are also generalization of $Q$-type spaces. In recent 20 years, $Q$-type spaces are studied extensively (cf [6, 15, 20, 24]).

Let $f_{i,Q}$ be the mean value of $(I - \Delta)^{1/2}f$ on a cube $Q$,
\[
f_{i,Q} = \frac{1}{|Q|} \int_{Q} (I - \Delta)^{1/2}f(x)dx.
\]
The Morrey spaces $M_{r,p}(\mathbb{R}^n)$ are defined as follows.

Definition 2.3. For $1 \leq p < \infty$ and $r, t \geq 0$, the Morrey space $M_{r,p}(\mathbb{R}^n)$ is defined as the set of the functions $f$ such that $\sup_{Q \in \Omega} |f_{i,Q}| \leq C$ and
\[
\int_{Q} |(I - \Delta)^{1/2}f(x) - f_{i,Q}|^p dx \leq C|Q|^{1-p(r+n)/n},
\]
where $Q$ is any cube in $\mathbb{R}^n$ with $|Q| \leq 1.$

In [15, 24], the authors proved that Morrey spaces $M_{r,p}(\mathbb{R}^n)$ can be also characterized by wavelets. We state it as the following theorem and refer to [24] for the proof.
Theorem 2.4. Given \( t \in \mathbb{R}, 1 < p < \infty \) and \( 0 \leq p(r + t) < n \),

\[
f(x) = \sum_{(r,j,k) \in \Lambda_r} f_{r,j,k}^x \Phi_{r,j,k}(x) \in M_{r,p}^p(\mathbb{R}^n)
\]

if and only if for any \( Q \in \Omega \) with \(|Q| \leq 1\),

\[
\left( \sum_{r \in E_n, Q, \mu < Q} 2^{\ell(n+2m)j} |f_{r,j,k}^x|^r \chi(2^j x - k) \right)^{p/2} \leq C |Q|^{1-p(r+t)/n}.
\]

By Lemmas 1.2 and 2.2, the multiplier spaces \( X_{r,p}^r(\mathbb{R}^n) \) are also subspaces of Morrey spaces \( M_{r,p}^p(\mathbb{R}^n) \).

Lemma 2.5. Given \( r > 0, t \geq 0 \) and \( 1 < p < n/(r+t) \). If \( f \in X_{r,p}^r(\mathbb{R}^n) \), then \( f(x) \in M_{r,p}^p(\mathbb{R}^n) \).

Now we give two lemmas about the fractional BMO spaces \( BMO'(\mathbb{R}^n) \). In the first lemma, we prove that Morrey spaces \( M_{r,p}^p(\mathbb{R}^n) \) are subspaces of \( BMO'(\mathbb{R}^n) \).

Lemma 2.6. For \( r > 0, t \geq 0 \) and \( 1 < p < n/(r+t) \), \( M_{r,p}^p(\mathbb{R}^n) \subset BMO'(\mathbb{R}^n) \).

Proof. For any dyadic cube \( Q \), we have

\[
\left( \sum_{r \in E_n, Q, \mu < Q} 2^{\ell(n+2m)j} |f_{r,j,k}^x|^r \chi(2^j x - k) \right)^{p/2} \leq |Q|^{p(r+t)/n} \left( \sum_{r \in E_n, Q, \mu < Q} 2^{\ell(n+2m)j} |f_{r,j,k}^x|^r \chi(2^j x - k) \right)^{p/2} \leq C |Q|^{1-p(r+t)/n} \leq C |Q|.
\]

Lemma 2.7. Suppose \( r > 0 \) and \( f = \sum_{(r,j,k) \in \Lambda_r} f_{r,j,k}^x \Phi_{r,j,k}(x) \in BMO'(\mathbb{R}^n) \). The wavelet coefficients of \( f \) satisfy

\[
|f_{r,j,k}^x| \leq C 2^{(r-n)/2}, \forall \epsilon \in [0,1]^n, \ j \in \mathbb{N}, \ k \in \mathbb{Z}^n.
\]

Proof. Take \( j \in \mathbb{N} \) and \( k \in \mathbb{Z}^n \). We consider two cases \( \epsilon \in E_n \) and \( \epsilon = 0 \) separately.

(i) For any \( \epsilon \in E_n \), by the definition of \( BMO'(\mathbb{R}^n) \), we get

\[
\left( \sum_{r \in E_n, Q, \mu < Q} 2^{\ell(n+2m)j} |f_{r,j,k}^x|^r \chi(2^j x - k) \right)^{p/2} \leq C 2^{-jn}.
\]

It is easy to see that \( |f_{r,j,k}^x| \leq C 2^{(r-n)/2} \).

(ii) For \( \epsilon = 0 \),

\[
f_{r,j,k}^0 = \left\langle \sum_{(r',j',k') \in \Lambda_r} f_{r',j',k'}^\epsilon \Phi_{r',j',k'} \right\rangle = \left\langle \sum_{f < j} f_{r,j,k}^\epsilon \Phi_{r,j,k} \right\rangle = \left\langle \sum_{f < j} f_{r,j,k}^\epsilon \Phi_{r,j,k} \right\rangle.
\]
Because \( \left| \sum_{j<k} f_{j,k}^e \Phi_{j,k}^e(x) \right| \leq C 2^j \), we have \( \left| f_{j,k}^0 \right| \leq C \left( 2^j, \left| \Phi_{j,k}^0(x) \right| \right) \leq C 2^{\beta(n/2)} \). \( \square \)

Let \( \Psi^1 \) and \( \Psi^2 \) be two functions in \( C_0^\infty([-2^{M+1}, 2^{M+1})^n) \) with vanishing moments \( \int x^\alpha \Psi(x) dx = 0 \), where \(|x| \leq \mu \) and \( i = 1, 2 \). Denote

\[ a_{j,k,j',k'} = \langle \Psi_{j,k}^1, \Psi_{j',k'}^2 \rangle. \]

The following lemma can be found in Chapter 8 of [13] or Chapter 6 of [20].

**Lemma 2.8.** Given \(|\mu| \leq m \). For \(|s| < \mu \), the coefficients \( a_{j,k,j',k'} \) satisfy the following condition:

\[ |a_{j,k,j',k'}| \leq C 2^{-j-2|k|/2} \left( 2^{-j} + 2^{-j} + |k| 2^{-j} + k' 2^{-j} \right)^{n+s}. \]  

By wavelet characterization of \( H^{s,p}(\mathbb{R}^n) \), the continuity of Calderón-Zygmund operators on \( H^{s,p}(\mathbb{R}^n) \) is equivalent to the following lemma. We refer the readers to [13] [14] [20] for the proof.

**Lemma 2.9.** Suppose \( s > |r| \) and \( g(x) = \sum_{(\alpha, j,k) \in \Lambda_n} g_{j,k} \Phi_{j,k}^e(x) \in H^{s,p}(\mathbb{R}^n) \). Let \( \tilde{g}_{j,k} = \sum_{(\alpha, j,k) \in \Lambda_n} \tilde{a}_{j,k,j',k'} \Phi_{j,k}^e \). If the coefficients \( \tilde{a}_{j,k,j',k'} \) satisfy the condition (2.3), then we have

\[ \int \left( \sum_{(\alpha, j,k) \in \Lambda_n} 2^{j(n+2r)} \tilde{g}_{j,k}^2 \chi(2^j - k) \right)^{p/2} dx \leq C \int \left( \sum_{(\alpha, j,k) \in \Lambda_n} 2^{j(n+2r)} |g_{j,k}|^2 \chi(2^j - k) \right)^{p/2} dx. \]

We say that \( T \) is a local operator if there exists some constant \( C > 1 \) such that for all \( x \in \mathbb{R}^n \) and \( r > 0 \), \( T \) maps a distribution with the support \( B(x, r) \) to another distribution supported on the ball \( B(x, Cr) \). If \( t/2 \) is not a non-negative integer, the operator \( (I - \Delta)^{t/2} \) is not a local operator. Now we use wavelets to construct some special fractional differential operators \( T^t \), which are almost local operators and will be used in the proof of our main result.

**Definition 2.10.** For \( t \geq 0 \) and \( h(x) = \sum_{(\alpha, j,k) \in \Lambda_n} h_{j,k} \Phi_{j,k}^e(x) \), we define an operator \( T^t \) corresponding to the kernel \( K^t(x,y) = \sum_{(\alpha, j,k) \in \Lambda_n} 2^{-j t} \Phi_{j,k}^e(x) \Phi_{j,k}^e(y) \) as

\[ T^t h(x) = \sum_{(\alpha, j,k) \in \Lambda_n} 2^{-j t} h_{j,k} \Phi_{j,k}^e(x). \]

It is easy to prove that \( T^0 \) is the identity operator and \( \|T^t h\|_{L^p} = \|h\|_{H^{s,p}} \) for \( 1 < p < \infty \). Furthermore, we have

**Lemma 2.11.** Suppose \( t \geq 0 \). For any \( Q_{j,k} \in \Omega \) and \( x \in Q_{j,k} \), \( 2^{j(n/2-t)} |h_{j,k}^0| \leq CMT^t h(x) \), where \( M \) is the Hardy-Littlewood maximal operator.
Proof. If \( t = 0 \), the proof was given by Meyer [13]. Now we consider the case \( t > 0 \). Since 
\[
T^{-t}\Phi^0(x) = \int K^{-t}(x, y)\Phi^0(y)dy,
\]
it is easy to verify that 
\[
|T^{-t}\Phi^0(x)| \leq C(1 + |x|)^{-\nu t}.
\]

By the fact that \( t > 0 \), we have 
\[
2^{jn/2-t}h_{jk}^0 = 2^{jn/2-t} \left< T^t h(x), T^{-t}\Phi^0_jk(x) \right> = 2^{jn/2} \left< T^t h(x), (T^{-t}\Phi^0)_jk(x) \right>.
\]

Hence we can get 
\[
|2^{jn/2-t}h_{jk}^0| = 2^{jn/2} \left| \left< T^t h(x), (T^{-t}\Phi^0)_jk(x) \right> \right|
\leq C 2^{jn/2} \int |T^t h(x)| 2^{jn/2} \frac{dx}{(1 + |2^j x - k|)^{\nu t}}
\leq C 2^{jn} \left( \int |T^t h(x)| dx + \sum_{j=1}^{\infty} \int_{2^{-1} \leq |2^j x - k| \leq 2^j} |T^t h(x)| \frac{dx}{(1 + |2^j x - k|)^{\nu t}} \right)
\leq C 2^{jn} \left( 2^{-jn} M(T^t h)(x) + \sum_{j=1}^{\infty} 2^{-jn} M(T^t h)(x) 2^{-jn} \right)
\leq CM(T^t h)(x).
\]

This completes the proof of Lemma[2.11]. \( \square \)

In the rest of this section, we give a decomposition of Sobolev spaces associated with combination atoms. For \( |r| < m \) and \( g(x) = \sum_{(r, j, k) \in \Lambda} g_{jk}^x \Phi_{jk}^r(x) \), denote 
\[
S_r g(x) = \left( \sum_{(r, j, k) \in \Lambda} 2^{(2r + n)|g_{jk}^x|^2 (2^j x - k)} \right)^{1/2}
\]
and for \( t = 0 \), denote also \( \bar{S} g(x) = S_0 g(x) \).

Definition 2.12. Given \( r \in \mathbb{R}, \lambda > 0 \). For arbitrary measurable set \( E \), we say that \( g(x) \) is a \((r, \lambda, E)\)-combination atom, if \( \text{supp}(S_r g) \subset E \) and \( S_r g(x) \leq \lambda \). If \( E \) is a dyadic cube, then \( g(x) \) is called a \((r, \lambda, E)\)-atom.

In [21], Q. Yang introduced the combination atom decomposition of Lebesgue spaces. In this paper, we need a similar result for Sobolev spaces.

Theorem 2.13. If \( 1 < p < \infty, |r| < m \) and \( \|g\|_{H^p \ell^r} \leq 1 \), there exists a series of \((r, 2^v, E_r)\)-combination atoms \( g_r(x) \) such that \( \sum_{v \in \mathbb{N}} 2^v |E_r| \leq C \).

Proof. Denote 
\[
\bar{S}_r g(x) = \left( \sum_{(r, j, k) \in \Lambda} 2^{(2r + n)|g_{jk}^x|^2 (2^j x - k)} \right)^{1/2}.
\]
For $v \geq 1$, let $E_v = \{ x : S \cdot r \cdot g(x) > 2^v \}$. By wavelet characterization of Sobolev spaces, we have $\sum_{v \in \mathbb{N}} 2^{\nu v} |E_v| \leq C$. Let $E_v = \bigcup_{l} Q^v_{l}$, where $Q^v_{l}$ are disjoint maximal dyadic cubes with $|Q^v_{l}| \leq 1$. Let $\mathfrak{V}_{v,l}$ be the set of dyadic cubes contained in $Q^v_{l}$ but not in $E_{v+1}$. $\mathfrak{V}_v = \bigcup_{l} \mathfrak{V}_{v,l}$ and $\mathfrak{V}_0 = \Omega \setminus \bigcup_{v \geq 1} \mathfrak{V}_v$. Let $E_0 = \{ x \in Q : Q \not\subset \mathfrak{V}_0 \}$ and we can write also $E_0 = \bigcup_{l} Q^0_{l}$, where $Q^0_{l}$ are disjoint maximal dyadic cubes in $\Omega$. The related set $\mathfrak{V}_{0,l}$ is defined as $\mathfrak{V}_{0,l} = \{ Q \subset Q^0_{l} \}$.

For any $v \geq 0$, we write $g_v(x) = \sum_{Q \in \mathfrak{V}_{0,l}} g^v_{Q,l} \Phi^v_{Q,l}(x)$ and $g_v(x) = \sum_{Q \in \mathfrak{V}_{0,l}} g^v_{Q,l} \Phi^v_{Q,l}(x)$. Then $g_v(x)$ is a desired combination atom. This completes the proof. \(\square\)

3. Wavelet characterization of the multiplier spaces

In this section, we use Meyer wavelets to characterize the multiplier spaces $X^{s}_{r,p}(\mathbb{R}^n)$. For any $g \in H^{s}(\mathbb{R}^n)$, let $g^{\Phi}_{j,k} = \left\langle g(x), 2^j (\Phi^s(x))^{2}(2^j x - k) \right\rangle$. Let $\Phi(x)$ be a function satisfying $\Phi(x) \geq 0$, $\Phi(x) \in C_{0}^{\infty}(B(0, 1))$ and $\int \Phi(x) dx = 1$. For any $g \in H^{s}(\mathbb{R}^n)$, define $g_{j,k}^{\Phi} = \sum_{Q \in \mathfrak{V}_{0,l}} g_{Q,l}^v \Phi^v_{Q,l}(x)$. The function spaces $S^{s}_{r,p}(\mathbb{R}^n)$ and $S^{s}_{r,p}(\mathbb{R}^n)$ are defined as follows.

**Definition 3.1.** Given $r > 0$, $t \geq 0$ and $r + t < 1 < p < n/(r + t)$.

(i) We say $f(x) \in S^{s}_{r,p}(\mathbb{R}^n)$ if $f(x) = \sum_{(r,j,k) \in \Lambda_n} f^r_{j,k} \Phi^r_{j,k}(x)$ and

$$\left( \sum_{(r,j,k) \in \Lambda_n} 2^{\nu(r+2v)} ||g^{\Phi}_{j,k}(x)\Phi^r_{j,k}(x)||_p^2 \right)^{1/p} \leq C,$$

where $g \in H^{s+t,p}(\mathbb{R}^n)$ and $\|g\|_{H^{s+t,p}(\mathbb{R}^n)} \leq 1$.

(ii) We say $f(x) \in S^{\Phi}_{r,p}(\mathbb{R}^n)$ if $f(x) = \sum_{(r,j,k) \in \Lambda_n} f^{\Phi}_{j,k} \Phi^\Phi_{j,k}(x)$ and

$$\left( \sum_{(r,j,k) \in \Lambda_n} 2^{\nu(r+2v)} ||g^{\Phi}_{j,k}(x)\Phi^\Phi_{j,k}(x)||_p^2 \right)^{1/p} \leq C,$$

where $g \in H^{s+t,p}(\mathbb{R}^n)$ and $\|g\|_{H^{s+t,p}(\mathbb{R}^n)} \leq 1$.

Now we give a wavelet characterization of the multiplier space $X^{s}_{r,p}(\mathbb{R}^n)$. Let $\Phi^r(x)$ and $\Phi^\Phi(x)$, $r \in E_n$ be the scaling function and wavelet functions, respectively. For $(r, j, k)$, $(r', j', k')$, $(r'', j'', k'') \in \Lambda_n$ and $l \in \mathbb{Z}^n$, let

$$d_{r,j,k,l}^{r',r''} = \left\langle \Phi^r_{j,k+l}(x) \Phi^r_{j,k}(x), \Phi^r_{j',k'}(x) \right\rangle$$

and

$$d_{r,j,k,0,l}^{r',r''} = \left\langle \Phi^r_{j,k}(x)^2 - 2^{\nu(r)(2^j x - k)}, \Phi^r_{j',k'}(x) \right\rangle.$$

Furthermore, for $0 \leq s \leq N$, $r, r' \in E_n$, $l \in \mathbb{Z}^n$ and $s + |r - r'| + |l| \neq 0$, let

$$d_{r,j,k,l}^{s,r',r''} = \left\langle \Phi^s_{j,k}(x) \Phi^r_{j,k+l}(x), \Phi^r_{j',k'}(x) \right\rangle.$$

The following lemma is obtained in [13].
Lemma 3.2. There exist sufficient big integers $N, N_1$ and $N_2$ such that $\min\{N, N_1, N_2\} > 8n + 8m$ and the following estimates hold.

(i) If $(e, j, k), (e', j', k') \in \Lambda_n, l \in \mathbb{Z}^n$ and $j \geq j'$, then
\[ |a^{e',e}_j| \leq C (1 + |l|)^{-N_1} 2^{n(j/j'-1)} (1 + |k' - 2^{j'-j}k|)^{-N_2}. \]

(ii) If $(e, j, k), (e'', j', k') \in \Lambda_n, 0 \leq s \leq N, e' \in E_n, l \in \mathbb{Z}^n$ and $j \geq j'$, then
\[ |a^{e'',e}_j| \leq C (1 + |l|)^{-N_1} 2^{n(j/j'-1)} (1 + |k' - 2^{j'-j}k|)^{-N_2}. \]

(iii) If $(e, j, k), (e', j', k') \in \Lambda_n, l \in \mathbb{Z}^n$ and $j < j'$, then
\[ |a^{e',e}_j| \leq C (1 + |l|)^{-N_1} 2^{n(j/j'-1)} (1 + |k' - 2^{j'-j}k|)^{-N_2}. \]

(iv) If $(e, j, k), (e'', j', k') \in \Lambda_n, 0 \leq s \leq N, e' \in E_n, l \in \mathbb{Z}^n$ and $j < j'$, then
\[ |a^{e'',e}_j| \leq C (1 + |l|)^{-N_1} 2^{n(j/j'-1)} (1 + |k' - 2^{j'-j}k|)^{-N_2}. \]

Theorem 3.3. For $t \geq 0, r > 0$ and $t + r < 1 < p < n/(t + r)$, there exist two equivalent relations between $X_{r,p,p}^t(\mathbb{R}^n)$ and $M_{r,p}^t(\mathbb{R}^n)$.

(i) $f \in X_{r,p,p}^t(\mathbb{R}^n)$ if and only if $f \in M_{r,p}^t(\mathbb{R}^n)$ and $f \in S_{r,p}^t(\mathbb{R}^n)$.

(ii) $f \in X_{r,p,p}^t(\mathbb{R}^n)$ if and only if $f \in M_{r,p}^t(\mathbb{R}^n)$ and $f \in S_{r,p}^t(\mathbb{R}^n)$.

Proof. Let $\Phi^0$ and $\Phi^\varepsilon$ be the scaling function and wavelet functions of Meyer wavelets, respectively. There exists an integer $N \geq 3$ such that $\int x^\varepsilon \Phi^0(x) \Phi^\varepsilon (2^N x - k) dx = 0, \forall k \in \mathbb{Z}^n, \alpha \in \mathbb{N}^n, \forall \varepsilon \in E_n$. Denote by $\Lambda_{e,n}$ the set
\[ \{ (s, e', l), 0 \leq s \leq N, e' \in E_n, l \in \mathbb{Z}^n, |l| \leq 2^{(M+2+r)n} \text{ and if } s = 0, \text{ then } (0, e, 0) \neq (0, e', 0) \}. \]

Let $h(x)$ be any function in $H^{-t,p}(\mathbb{R}^n)$. We prove that for $f \in M_{r,p}^t(\mathbb{R}^n) \cap S_{r,p}^t(\mathbb{R}^n), f h \in H^{-t-r',p}(\mathbb{R}^n)$ and
\[ \|f h\|_{H^{-t-r,p}(\mathbb{R}^n)} \leq C \|f\|_{M_{r,p}^t(\mathbb{R}^n) \cap S_{r,p}^t(\mathbb{R}^n)} \|h\|_{H^{-r,p}(\mathbb{R}^n)}. \]

In fact, if the above inequality holds, we have
\[ \|f\|_{X_{r,p,p}^t(\mathbb{R}^n)} = \sup_{\|g\|_{H^{r,p}(\mathbb{R}^n)} \leq 1} \|fg\|_{H^{r,p}(\mathbb{R}^n)} = \sup_{\|g\|_{H^{r,p}(\mathbb{R}^n)} \leq 1} \sup_{\|\|g\|_{H^{r,p}(\mathbb{R}^n)} \leq 1} |\langle f g, h \rangle| \leq \sup_{\|\|g\|_{H^{r,p}(\mathbb{R}^n)} \leq 1} \sup_{\|\|g\|_{H^{r,p}(\mathbb{R}^n)} \leq 1} \|fh\|_{H^{r,p}(\mathbb{R}^n)} \|g\|_{H^{r,p}(\mathbb{R}^n)} \leq \|f\|_{M_{r,p}^t(\mathbb{R}^n) \cap S_{r,p}^t(\mathbb{R}^n)} \|h\|_{H^{r,p}(\mathbb{R}^n)}. \]

Hence we can get $X_{r,p,p}^t(\mathbb{R}^n) \subset M_{r,p}^t(\mathbb{R}^n) \cap S_{r,p}^t(\mathbb{R}^n)$. 
Now we begin to prove the above inequality. At first, we give a wavelet decomposition of the product of $fh$. Denote
\[
Z^n_N = \{ l = (l_1, \ldots, l_n) \in \mathbb{Z}^n, 0 \leq l_i \leq 2^N - 1, i = 1, \ldots, n \}.
\]
For $\varepsilon \in E_n$, $l \in \mathbb{Z}^n$, $|l| \leq 2^{(M+2)n}$ and $(\varepsilon', l) \in \Lambda_{\varepsilon,l_n}$, we denote
\[
T_{1,\varepsilon}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} f^0_{j,k+l} h^\varepsilon_{j,k} \Phi^0_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x);
\]
\[
T_{2,0,\varepsilon}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} f^0_{j,k+l} h^\varepsilon_{j,k} \Phi^0_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x);
\]
\[
T_{3,\varepsilon}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sum_{\varepsilon, \varepsilon' \in E_n} f^\varepsilon_{j,k+l} h^\varepsilon_{j,k} \Phi^\varepsilon_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x);
\]
\[
T_{4,\varepsilon}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sum_{\varepsilon \in E_n} \sum_{\varepsilon' \in E_n} f^\varepsilon_{j,k+l} h^\varepsilon_{j,k} \Phi^\varepsilon_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x);
\]
\[
T_{5}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sum_{\varepsilon \in E_n} \sum_{\varepsilon' \in E_n} f^\varepsilon_{j,k+l} h^\varepsilon_{j,k} \Phi^\varepsilon_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x).
\]
Further, for any $\varepsilon, \varepsilon' \in E_n$, $0 < s \leq N$ or $s = 0$ and $\varepsilon \neq \varepsilon'$, denote
\[
T_{2,\varepsilon,\varepsilon'}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} f^\varepsilon_{j,k+l} h^\varepsilon_{j,k} \Phi^\varepsilon_{j,k+l}(x) \Phi^\varepsilon_{j,k}(x).
\]
By the formulas (2.1) and (2.2), we can divide $f(x)h(x)$ into the sum of the above five terms, that is,
\[
f(x)h(x) = \sum_{\varepsilon \in E_n} T_{1,\varepsilon}(x) + \sum_{0 \leq s \leq N/L, \varepsilon' \in E_n} T_{2,\varepsilon,\varepsilon'}(x) + \sum_{\varepsilon \in E_n} T_{3,\varepsilon}(x) + \sum_{\varepsilon \in E_n} T_{4,\varepsilon}(x) + T_5(x)
\]
\[
= \sum_{i=1}^5 T_i(x).
\]
If $g(x) \in H^{s+r}(\mathbb{R}^n)$, $g(x) = \sum_{(\varepsilon, k) \in \Lambda_{\varepsilon,k}} g^\varepsilon_{j,k} \Phi^\varepsilon_{j,k}(x)$. For $\varepsilon, \varepsilon' \in E_n$ and $0 \leq s \leq N$, we define
\[
T_{1,\varepsilon} = \int T_{1,\varepsilon}(x) g(x) dx \quad \text{and} \quad T_{2,\varepsilon,\varepsilon'} = \int T_{2,\varepsilon,\varepsilon'}(x) g(x) dx.
\]
Let
\[
T_{1,1,\varepsilon,l} = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |f^0_{j,k+l}||h^\varepsilon_{j,k}||a^\varepsilon_{j,k,l}||g^\varepsilon_{j,k,l}|;
\]
\[
T_{1,2,\varepsilon,l} = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |f^0_{j,k+l}||h^\varepsilon_{j,k}||a^\varepsilon_{j,k,l}||g^\varepsilon_{j,k,l}|;
\]
\[
T_{2,1,\varepsilon,\varepsilon'}, l = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sum_{\varepsilon, \varepsilon' \in E_n} |f^\varepsilon_{j,k+l}||h^\varepsilon_{j,k}||a^\varepsilon_{j,k,l}||g^\varepsilon_{j,k,l}|;
\]
and
\[
T_{2,2,\varepsilon,\varepsilon'}, l = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sum_{\varepsilon, \varepsilon' \in E_n} |f^\varepsilon_{j,k+l}||h^\varepsilon_{j,k}||a^\varepsilon_{j,k,l}||g^\varepsilon_{j,k,l}|.
\]
It is easy to see that
\[
|T_{1,\varepsilon}| \leq \sum_{l \in \mathbb{Z}^n} (T_{1,1,\varepsilon,l} + T_{1,2,\varepsilon,l}) \quad \text{and} \quad |T_{2,\varepsilon,\varepsilon'}| \leq \sum_{l \in \mathbb{Z}^n} (T_{2,1,\varepsilon,\varepsilon', l} + T_{2,2,\varepsilon,\varepsilon', l}).
\]
Let \( S_j g_j (x) = \sum_{j' > k} 2^{j'(n/2 + r)} g_{j', k} \chi(2^{j'} x - k') \). For fixed \( x \), there is only one \( k' \) such that \( \chi(2^{j'} x - k') \neq 0 \) and the number of \( j' \) in the sum is finite. Then the operator \( S_j g_j (x) \) is equivalent to the following one:

\[
S_j g_j (x) = \left( \sum_{j' > k} 2^{j'(n/2 + r)} |g_{j', k}|^2 \chi(2^{j'} x - k') \right)^{1/2}.
\]

Let \( M \) be the Hardy-Littlewood maximal function. Then if \( j < j' \), \( \forall x \in Q_{j,k} \), we have

\[
\sum_{k'} (1 + |k - 2^{j'} k'|)^{-N_2} 2^{j'(-n/2 + r)} g_{j', k'} \leq C 2^{-j} MS_j g_j (x).
\]

Now we estimate the quantities \( T_{1,1,e,l}, T_{1,2,e,l}, T_{2,1,e,e',l} \) and \( T_{2,2,e,e',l} \) separately.

(1) For \( j' \leq j \), we have

\[
T_{1,1,e,l} = \sum_{j,k} \sum_{j' > k} |f_{j,k}^e| \|H_{j,k,l,f}^e \| \|g_{j', k'}^e\|.
\]

By Lemma 3.2 we know

\[
|d_{j,k,l,f}^e| \leq (1 + |l|)^{-N_1} (1 + |k' - 2^{j'} k|)^{-N_2} 2^{n j'/2 + j' - j}.
\]

By Lemma 2.1 \( f \in M_{r, t} (\mathbb{R}^n) \subset BMO^s (\mathbb{R}^n) \) implies \( |f_{j,k}^e| \leq C 2^{(t-r)/2} \). Now we can get

\[
\begin{align*}
\sum_{l \in \mathbb{Z}^n} T_{1,1,e,l} & \leq \sum_{l \in \mathbb{Z}^n} \sum_{j', k, j} \int 2^{3n j'/2 + (j' - j) + (r-n/2) j} \sum_k \left( \sum_e \left| \sum_k 2^{j' (n/2 + r)} g_{j', k'}^e \right|^2 \right)^{1/2} \frac{|H_{j,k,l,f}^e| (2^{j'} x - k')}{(1 + |l|)^{N_1} (1 + |k' - 2^{j'} k|)^{N_2}} dx \\
& \leq \sum_{j' \geq j} \int 2^{3n j'/2 + (j' - j) + (r-n/2) j} \sum_k \left( \sum_e \left| \sum_k 2^{j' (n/2 + r)} g_{j', k'}^e \right|^2 \right)^{1/2} 2^{-j' (t + n/2)} dx \\
& \quad \times \sum_k \frac{|H_{j,k,l,f}^e| (2^{j'} x - k')}{(1 + |k' - 2^{j'} k|)^{N_2}} 2^{j' (n/2 - r)/2} 2^{j' (n/2 + r)} dx \\
& \leq \int \sum_{j' \geq j} 2^{3n j'/2 - j' (t + n/2) + (j' - j) + (r-n/2) j + j' (n/2 + r)} 2^{-j' s} S_{t+ \epsilon} S_{t+ \epsilon} g_j (x) M (S_{- \epsilon} h_j) (x) dx.
\end{align*}
\]

Because \( 0 < t + r < 1 \),

\[
\begin{align*}
\sum_{l \in \mathbb{Z}^n} T_{1,1,e,l} & \leq \sum_{j' \geq j} 2^{(j' - j) + (r-n/2) j + j' + s} \int S_{t+ \epsilon} S_{t+ \epsilon} g_j (x) M (S_{- \epsilon} h_j) (x) dx \\
& \leq \sum_{j' \geq j} 2^{(j' - j)(1-r+s)} \|g\|_{H^{1-r+s}/(\mathbb{R}^n)} \|h\|_{H^{1-r+s}/(\mathbb{R}^n)} \\
& \leq C \|g\|_{H^{1-r+s}/(\mathbb{R}^n)} \|h\|_{H^{1-r+s}/(\mathbb{R}^n)}.
\end{align*}
\]

Now we estimate the term

\[
T_{2,1,e,e',e''} (x) = \sum_{j,k} \sum_{j' > k} |f_{j,k}^e| \|H_{j+k,j,k}^e \| \|g_{j', k'}^e\|.
\]
Because $f \in M^{r,p}_{c,p}(\mathbb{R}^n) \subset BMO^r(\mathbb{R}^n)$, by Lemma 2.7, we have $|f^0_{j,k+1}| \leq C 2^{r(n/2)j}$. By Lemma 5.2,
\[ |a^{r',s',x',x}_{j,k,l,f,k'}| \leq C(1 + |l|)^{-N}(1 + |k' - 2^{-f} j|)^{-N} 2^{n_j/2 + s + (f - j)}.
\]
We can get, similarly,
\[
\sum_{l \leq 2^j} T_{2,1,s,x',x'}(x) = \sum_{l \leq 2^j} \sum_{j_k,f} \sum_{j_k,f' \geq 0} f^0_{j,k} ||H^{r'}_{j_k,2^{j_k}+l}||a^{r',s',x',x'}_{j_k,j_{k,l},f,k'}||g^{r'}_{f,k'}|
\]
\[
\leq C \sum_{l \leq 2^j} \sum_{j_k,f} \sum_{j_k,f' \geq 0} 2^{(r-n/2)(2n_j/2 + j - j')} \frac{||H^{r'}_{j_k,2^{j_k}+l}||g^{r'}_{f,k'}|}{(1 + |l|)^{N}(1 + |k' - 2^{-f} j|)^{N2}}
\]
\[
\leq C \sum_{l \leq 2^j} \sum_{j_k,f} \sum_{j_k,f' \geq 0} 2^{(r-j)(1-t-r)} \int S_{t+r}(g_f)(x) MS_{j+k}(x) dx
\]
\[
\leq C ||S_{t+r}g_f||_{L^p(\mathbb{R}^n)} ||MS_{j+k}||_{L^{p'}(\mathbb{R}^n)}
\]
\[
\leq C ||g||_{H^{r,s,p}(\mathbb{R}^n)} ||h||_{H^{r',s'}(\mathbb{R}^n)}.
\]
(2) If $f' > j$, the estimates of the terms $T_{1,2,s,e,l}$ and $T_{2,2,s,e',f}$ are easier than those of $T_{1,1,e,l}$ and $T_{1,2,s,e',f}$. For example, we estimate the term
\[
T_{1,2,s,e,l} = \sum_{j_k} \sum_{0 \leq j_e < f', x' = k'} |f^0_{j,k+1}| ||H^{r'}_{j_k,2^{j_k}+l}||g^{r'}_{f,k'}|.
\]
Because $f \in M^{r,p}_{c,p}(\mathbb{R}^n) \subset BMO^r(\mathbb{R}^n)$, by Lemma 2.7, $|f^0_{j,k+1}| \leq C 2^{(r-n/2)j}$. By Lemma 5.2,
\[ |a^{r',s',x',x}_{j,k,l,f,k'}| \leq C(1 + |l|)^{-N}(1 + |k' - 2^{-f} j|)^{-N} 2^{n_j/2 + s + (f - j)}.
\]
Because $2^{n_j} \int 1(2^{j} x - k) dx = 1$, we can obtain
\[
\sum_{l \leq 2^j} T_{1,2,s,e,l} \leq C \sum_{l \leq 2^j} \sum_{0 \leq j_k < f', x' = k'} 2^{(r-n/2)(2n_j/2 + s + (f - j))} \frac{||H^{r'}_{j_k,2^{j_k}+l}||g^{r'}_{f,k'}|}{(1 + |l|)^{N}(1 + |k' - 2^{-f} j|)^{N2}}
\]
\[
\leq C \sum_{l \leq 2^j} \sum_{j_k} \sum_{0 \leq j_e < f', x' = k'} \int 2^{(r-n/2)(2n_j/2 + s + (f - j))} \frac{||H^{r'}_{j_k,2^{j_k}+l}||g^{r'}_{f,k'}|}{(1 + |l|)^{N}(1 + |k' - 2^{-f} j|)^{N2}} dx
\]
\[
\leq C \int 2^{(r-n/2)(2n_j/2 + s + (f - j))} \frac{||H^{r'}_{j_k,2^{j_k}+l}||g^{r'}_{f,k'}|}{(1 + |l|)^{N}(1 + |k' - 2^{-f} j|)^{N2}} dx
\]
\[
\leq C ||g||_{H^{r,s,p}(\mathbb{R}^n)} ||h||_{H^{r',s'}(\mathbb{R}^n)}.
\]
The estimate for $T_{2,2,s,e',f}(x)$ can be obtained similarly. By the same methods used in (1) and (2), we can get the estimate of the term $T_5$. We omit the details.

(3) Now we consider the term $T_{3,e}$. We have the following claim.
Claim 1: Given $r > 0$, $t \geq 0$ and $t + r < 1 < p < n/(t + r)$. If $f \in BMO'(\mathbb{R}^n)$, then
\[
|\langle T_{3,x}(x), g(x) \rangle| \leq C \|g\|_{H^{t,p}((\mathbb{R}^n))} \|h\|_{H^{t-r,p}((\mathbb{R}^n))}.
\]
In fact, for $l \in \mathbb{Z}^n_{\Lambda}$, $(e, j, k) \in \Lambda_{\alpha}$ and $l' \in \mathbb{Z}^n$, let
\[
g^{e,f}_{j+N,2^n} = 2^{-n/2} \left( \Phi^0_{j,k+t}(x) \Phi^e_{j+N,2^n}(x), g(x) \right).
\]
We have
\[
|\langle T_{3,x}(x), g(x) \rangle| = \left| \sum_{(e,j,k) \in \Lambda_{\alpha}} \sum_{l \in \mathbb{Z}^n} 2^{j(n+2r+1)} \sum_{l' \in \mathbb{Z}^n} \left( \sum_{p \in \mathbb{Z}} 2^{j(n/2-\eta)} |H_{j,k+p}^{e,f}_{j+N,2^n}(x)| \right)^2 \chi(2^j x - k) \right| dx.
\]
Because $(F_t^{e,f}(\mathbb{R}^n))^* = BMO'(\mathbb{R}^n)$, by Lemma 2.11 we have
\[
|\langle T_{3,x}(x), g(x) \rangle| \leq C \sum_{(e,j,k) \in \Lambda_{\alpha}} \|f\|_{BMO'} \int \left( \sum_{l \in \mathbb{Z}^n} 2^{j(n+2r+1)} \left( \sum_{l' \in \mathbb{Z}^n} \left( 1 + |l'| \right)^N |g^{e,f}_{j+N,2^n}(x)| \right)^2 \chi(2^j x - k) \right) dx.
\]
Because
\[
\|g^{e,f}_{j+N,2^n}(x)\| = 2^{-n/2} \left( \Phi^0_{j,k+t}(x) \Phi^e_{j+N,2^n}(x), \sum_{(e',j',k') \in \Lambda_{\alpha}} g^{e',f}_{j',k'} \Phi(x)^{e',f}_{j',k'} \right) = 2^{-n/2} \left( \Phi^0_{j,k+t}(x) \Phi^e_{j+N,2^n}(x), \sum_{(e',j',k') \in \Lambda_{\alpha}, |l'| - N \in \mathbb{Z}} g^{e',f}_{j',k'} \Phi(x)^{e',f}_{j',k'} \right),
\]
we can get
\[
\int \left( \sum_{(e,j,k) \in \Lambda_{\alpha}} 2^{j(n+2r+1)} \left( \sum_{l' \in \mathbb{Z}^n} \left( 1 + |l'| \right)^N |g^{e,f}_{j+N,2^n}(x)| \right)^2 \chi(2^j x - k) \right) dx \leq \sum_{|l'| - N \in \mathbb{Z}} MS_{t+r} g_f(x).
\]
Hence we have
\[
\int \left( \sum_{(e,j,k) \in \Lambda_{\alpha}} 2^{j(n+2r+1)} \left( \sum_{l' \in \mathbb{Z}^n} \left( \sum_{p \in \mathbb{Z}} 2^{j(n/2-\eta)} |H_{j,k+p}^{e,f}_{j+N,2^n}(x)| \right)^2 \chi(2^j x - k) \right)^{p/2} dx \leq C \|g\|_{H^{t,p}}.
\]
Finally, we obtain, by Hölder’s inequality,
\[
| \langle T_{3r}(x), g(x) \rangle | \leq C \| f \|_{L^{\infty}(\mathbb{R}^n)} \| g \|_{L^{p \rightarrow r}} \| MT^r h \|_{r'}
\]
\[
\leq C \| f \|_{L^{\infty}(\mathbb{R}^n)} \| g \|_{L^{p \rightarrow r}} \| h \|_{H^{r \rightarrow r'}}.
\]
This completes the proof of Claim 1.

In order to deal with the term \( T_{4r}(x) = \sum_{(r, jk) \in \Lambda_n} f_{jk}^r h_{jk}^r (\Phi_{jk}^r(x))^2 \), we need the following estimate.

**Claim 2:** For \( r > 0 \), \( t \geq 0 \) and \( t + r < 1 < p < n/(r + t) \), \( f(x) \in S_{t,p}^r(\mathbb{R}^n) \) if and only if
\[
| \langle T_{4r}(x), g(x) \rangle | \leq C \| g \|_{H^{r,p}(\mathbb{R}^n)} \| h \|_{H^{r \rightarrow r'}(\mathbb{R}^n)}.
\]

In fact, for \( g \in H^{r+p}(\mathbb{R}^n) \), \( \langle T_{4r}(x), g(x) \rangle = \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \). By the definitions of \( S_{t,p}^r(\mathbb{R}^n) \) and \( H^{r+p}(\mathbb{R}^n) \), using Hölder’s inequality, we have
\[
| \langle T_{4r}(x), g(x) \rangle | \leq \int \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) dx
\]
\[
\leq \int \left( \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) \right)^{1/2} dx
\]
\[
\leq \left( \int \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) \right)^{1/2} \left( \int \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) \right)^{1/2} \right)^{1/p}
\]
\[
\leq \left( \int \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) \right)^{1/p} \left( \int \sum_{(r, jk) \in \Lambda_n} 2^{jn/2} f_{jk}^r \phi_{jk}^r h_{jk}^r \chi(2^j x - k) \right)^{1/p} \right)^{1/p'}
\]
Because \( f \in S_{t,p}^r(\mathbb{R}^n) \), we can see that
\[
| \langle T_{4r}(x), g(x) \rangle | \leq C \| g \|_{H^{r,p}(\mathbb{R}^n)} \| h \|_{H^{r \rightarrow r'}(\mathbb{R}^n)}.
\]
Conversely, let \( \tau_{jk}^r = f_{jk}^r 2^{jn/2} \phi_{jk}^r h_{jk}^r \) and \( \tau(x) = \sum_{(r, jk) \in \Lambda_n} \tau_{jk}^r \Phi_{jk}^r(x) \). Denote \( h = |\tau|^{-2p} \). For \( h(x) = \sum_{(r, jk) \in \Lambda_n} h_{jk}^r \Phi_{jk}^r(x) \), we write \( h_t(x) \) as the function
\[
h_t(x) = \sum_{(r, jk) \in \Lambda_n} 2^r h_{jk}^r \Phi_{jk}^r(x) := \sum_{(r, jk) \in \Lambda_n} (h_t)_{jk}^r \Phi_{jk}^r(x).
\]
It is easy to see that \( h \in L^{p'}(\mathbb{R}^n) \) is equivalent to \( h_t \in H^{r+p}(\mathbb{R}^n) \).
By the wavelet characterization of $H^s_p(\mathbb{R}^n)$, we get

\[
\int \left( \sum_{(r,j,k) \in \Lambda_n} 2^{j(n+2)} |g_{r,j,k}^0| \frac{1}{2^j} |f_{r,j,k}^0|^2 \chi(2^j x - k) \right)^{p/2} \, dx
\]

\[
= \int \left( \sum_{(r,j,k) \in \Lambda_n} 2^{jm} |f_{r,j,k}^0|^2 \chi(2^j x - k) \right)^{p/2} \, dx
\]

\[
= \int |r|^p \, dx = \int \tau \, dx
\]

\[
= \sum_{(r,j,k) \in \Lambda_n} 2^{j(n+2)} f_{r,j,k}^0 \phi^r_{j,k} h_{j,k}^r
\]

\[
= \sum_{(r,j,k) \in \Lambda_n} 2^{j/2} f_{r,j,k}^0 \phi^r_{j,k} (h_0)^r_{j,k}
\]

Further, we can deduce that

\[
\int \left( \sum_{(r,j,k) \in \Lambda_n} 2^{j(n+2)} |g_{r,j,k}^0| \frac{1}{2^j} |f_{r,j,k}^0|^2 \chi(2^j x - k) \right)^{p/2} \, dx = \langle T_{4,\epsilon}(x), g(x) \rangle
\]

\[
\leq C \|g\|_{L^p(\mathbb{R}^n)} \|h\|_{H^{s+\epsilon'}(\mathbb{R}^n)}
\]

\[
\leq C \|g\|_{L^p(\mathbb{R}^n)} \|\tau\|_{L^{p'}(\mathbb{R}^n)}
\]

Hence $\|\tau\|_{L^p} < \infty$ and $f(x) \in S_{s,p}(\mathbb{R}^n)$. This completes the proof of (i) of this theorem.

For the proof of (ii), similarly, we divide the product $f(x) h(x)$ into the following terms

- $T_{1,\epsilon}(x) = \sum_{j \in \mathbb{R}, k \in \mathbb{Z}^n} \sum_{r \in \mathbb{Z}} f_{r,j,k}^0 \phi^r_{j,k} h_{j,k}^r \phi^r_{j,k}(x)$

- $T_{2,0,\epsilon}(x) = \sum_{j \in \mathbb{R}, k \in \mathbb{Z}^n} \sum_{r \in \mathbb{Z}} f_{r,j,k}^0 \phi^r_{j,k} \phi^r_{j,k+1}(x) \phi^r_{j,k}(x)$

- $T_{3,\epsilon}(x) = \sum_{j \in \mathbb{R}, k \in \mathbb{Z}^n} \sum_{r \in \mathbb{Z}} f_{r,j,k}^0 \phi^r_{j,k} \phi^r_{j,k+1}(x) \phi^r_{j,k+1}(x)$

- $T_4(x) = \sum_{j \in \mathbb{R}, k \in \mathbb{Z}^n} f_{r,j,k}^0 h_{j,k}^r 2^{-m} \phi(2^j x - k)$

- $T_5(x) = \sum_{k \in \mathbb{Z}^n} \sum_{r \in \mathbb{Z}} f_{r,j,k}^0 \phi^r_{j,k} \phi^r_{j,k+1}(x) \phi^r_{j,k+1}(x)$

For $\epsilon, \epsilon' \in E_n$, $0 < s \leq N$ or $s = 0$ and $\epsilon \neq \epsilon'$, denote

\[
T_{2,\epsilon,\epsilon'}(x) = \sum_{j \in \mathbb{R}, k \in \mathbb{Z}^n} \sum_{r \in \mathbb{Z}} f_{r,j,k}^\epsilon h_{j,k}^r \phi^\epsilon_{j,k} \phi^\epsilon_{j,k+1}(x) \phi^\epsilon_{j,k}(x)
\]

By the same method used in the proof of (i), we can get the conclusion.
4. A logarithmic condition for multipliers

By Lemma 2.3, we know that the multiplier space $X^r_{r,p}(\mathbb{R}^n) \subset M^r_{r,p}(\mathbb{R}^n)$. In this section, we consider the reverse inclusion relation. At first we introduce a logarithmic Morrey spaces.

**Definition 4.1.** Fix $1 < p < n/(r + t)$ and $\tau \geq 0$. We say $f(x) \in M^r_{r,p}(\mathbb{R}^n)$ if $\sup_{|Q|=1} |f|_Q \leq C$ and
\[
\int_Q |(I - \Delta)^{\frac{t}{2}}f(x) - f_x|_Q^p \, dx \leq C |1 - \log_2 |Q||^p |Q|^{1 - p(r + t)/n},
\]
for any cube $Q$ with $|Q| \leq 1$.

Similar to Theorem 2.4, we have the following wavelet characterization of $M^r_{r,p}(\mathbb{R}^n)$.

**Theorem 4.2.** Given $t, \tau \geq 0, r > 0, 1 < p < n/(r + t)$. $f(x) = \sum_{(r,j,k) \in \Lambda_n} f_{r,j,k} \Phi_{r,j,k}(x)$ belongs to the logarithmic Morrey spaces $M^r_{r,p}(\mathbb{R}^n)$ if and only if
\[
\int_Q \left( \sum_{r \in E, \Omega, j < Q} 2^{j(n+2)t} |f_{r,j,k}|^2 \chi(2^j x - k) \right)^{\frac{p}{2}} \, dx \leq C (1 - \log_2 |Q|)^{-p\tau} |Q|^{1 - p(r + t)/n},
\]
where $Q \in \Omega$ with $|Q| \leq 1$.

**Proof.** Similar to that of Theorem 2.4, the proof of this theorem can be obtained by the characterization of Triebel-Lizorkin spaces (see Lemma 2.2). We omit the detail. \(\square\)

In [7], C. Fefferman established the following relation:
\[
M^r_{r,p}(\mathbb{R}^n) \subset X^r_{r,p}(\mathbb{R}^n) \subset M^r_{r,p}(\mathbb{R}^n),
\]
where $q > p > 1$. In this section, we use wavelet characterization to give a logarithmic type inclusion. Let $r > 0, t \geq 0, 1 < p < n/(r + t)$ and $\tau > 1/p'$. We prove that $M^r_{r,p}(\mathbb{R}^n)$ is a subspace of $X^r_{r,p}(\mathbb{R}^n)$ in Theorem 4.3. Hence, for $q > p$,
\[
M^r_{r,p}(\mathbb{R}^n) \subset M^r_{r,p}(\mathbb{R}^n) \subset M^r_{r,p}(\mathbb{R}^n) = M^r_{r,p}(\mathbb{R}^n).
\]

**Lemma 4.3.** If $\tau > 0, r > 0, t \geq 0$ and $1 < p < n/(r + t)$, $f \in M^r_{r,p}(\mathbb{R}^n)$ implies that $|f_{r,j,k}| \leq C 2^j(\log 2)^{-t} (1 + j)^{-\tau}$.

**Proof.** Because $f \in M^r_{r,p}(\mathbb{R}^n)$, then for any dyadic cube $Q \in \Omega$,
\[
\int_Q \left( \sum_{r \in E, \Omega, j < Q} 2^{j(n+2)t} |f_{r,j,k}|^2 \chi(2^j x - k) \right)^{\frac{p}{2}} \, dx \leq C (1 - \log_2 |Q|)^{-p\tau} |Q|^{1 - p(r + t)/n}.
\]
Proof.

We have

\[
\begin{aligned}
\int_Q \left( \sum_{\beta \in E, Q \subset \beta} 2^{j(n-2r)}|f_{j,k}^\beta|^2 \chi(2^j x - k) \right)^{p/2} dx \\
\leq C |Q|^{(r+1)/n} \int_Q \left( \sum_{\beta \in E, Q \subset \beta} 2^{j(n-2r)}|f_{j,k}^\beta|^2 \chi(2^j x - k) \right)^{p/2} dx \\
\leq C |Q|^{(r+1)/n} |Q|^{-p(r+1)/n} (1 - \log_2 |Q|)^{-T_p} \\
\leq C |Q|(1 - \log_2 |Q|)^{-T_p}.
\end{aligned}
\]

For \( \varepsilon \in E_n, j \in \mathbb{N} \) and \( k \in \mathbb{Z}^n \), take \( Q = Q_{j,k} \). By the wavelet characterization of \( \text{BMO}^r(\mathbb{R}^n) \), we get

\[
\int_Q (2^{j(n-2r)}|f_{j,k}^\beta|^2 \chi(2^j x - k))^{p/2} dx \leq C 2^{-jn} (1 - \log_2 2^{-jn})^{-T_p} \leq C 2^{-jn}(1 + j)^{-T_p}.
\]

Then we have \( |Q_{j,k}|(2^{j(n/2-r)}|f_{j,k}^\beta|^2)^{p} \leq C 2^{-jn}(1 + j)^{-T_p} \) and \( |f_{j,k}^\beta| \leq C 2^{j(r-n/2)+1} \leq C 2^{j(r-n/2)}(1 + j)^{-r} \).

When \( \varepsilon = 0 \),

\[
f_{j,k}^0 = \sum_{\beta \in E, Q \subset \beta} f_{j,k}^\beta \Phi_{j,k}^\beta = \sum_{\beta \in E, Q \subset \beta} f_{j,k}^\beta \Phi_{j,k}^\beta
\]

Since \( \left| \sum_{\beta \in E, Q \subset \beta} f_{j,k}^\beta \Phi_{j,k}^\beta(x) \right| \leq C 2^{(r-n/2)+1} \leq C 2^{(r-n/2)+1} \leq C 2^{(r-n/2)+1} \),

\[
|f_{j,k}^0| \leq C \left( 2^{j+1} \right)^{-r}, \quad |\Phi_{j,k}^0| \leq C 2^{j(r-n/2)+1} \leq C 2^{j(r-n/2)+1} \leq C 2^{j(r-n/2)+1} \leq C 2^{j(r-n/2)+1} \leq C 2^{j(r-n/2)+1}.
\]

\[\square\]

For \( \beta = (\beta_1, \beta_2, \cdots, \beta_n), \beta_i \in \mathbb{N}_+ \), define \( f^\beta_0(x) = (\partial/\partial x)^\beta f \). We have the following two lemmas.

**Lemma 4.4.** For \( \tau > 0, r > 0, \beta \geq 0 \) and \( 1 < p < n/(r + t) \), the function \( f \in \text{M}_r^{\beta, \tau}(\mathbb{R}^n) \) implies its derivative \( f_0 \in \text{M}_r^{\beta_0, \tau}(\mathbb{R}^n) \), where \( |\beta| = \sum_{i=1}^n \beta_i \).

**Proof.** If \( f \in \text{M}_r^{\beta, \tau}(\mathbb{R}^n) \) and \( f(x) = \sum_{(\epsilon, j, k) \in \Lambda_E} f_{j,k}^\beta \Phi_{j,k}^\beta(x) \), by Theorem 4.2 we have

\[
\int \left( \sum_{(\epsilon, j, k) \in \Lambda_E} 2^{j(n+2r)}|f_{j,k}^\beta|^2 \chi(2^j x - k) \right)^{p/2} dx \leq C (1 - \log_2 |Q|)^{-pT} |Q|^{1-p(r+1)/n}.
\]
Denote by \( f_{x,k}^{\beta,\gamma} \) the wavelet coefficients of \( f_0(x) \). We can get \( f_{x,k}^{\beta,\gamma} = 2^{j\beta} f_{x,j}^\gamma \) and

\[
\int \left( \sum_{e \in E_u, Q_e \subset Q} 2^{j\beta+2r-2^\gamma} |f_{x,j}^\gamma|^2 \chi(2^j x - k) \right)^{p/2} dx \leq \int \left( \sum_{e \in E_u, Q_e \subset Q} 2^{j\beta+2\gamma} |f_{x,j}^\gamma|^2 \chi(2^j x - k) \right)^{p/2} dx \leq C (1 - \log_2 |Q|)^{-p/2} |Q|^{1-\frac{2(p+1)\beta}{n}}.
\]

This implies that \( f_0(x) \in M^p_{\epsilon,\beta,p} (\mathbb{R}^n) \). \( \square \)

To get the sufficient condition for multiplier spaces, we need to consider carefully the relationship of different dyadic cubes relative to combination atoms. Because of this reason, we use Daubechies wavelets in the rest of this section.

If \( g_\nu(x) \) is a \((m, 2^\nu, E_u)\)-combination atom, then we denote the number of biggest dyadic cubes in \( E_u \) by \( i_1 \). Denote by \( F_{u,0} \) the set \( \{ i \in \mathbb{N}, i = 1, \cdots, i_1 \} \). If \( i \in F_{u,0} \), we denote such cube by \( Q_{u,i} \). The volume of \( Q_{u,i} \) is denoted by \( 2^{-n_j} \) and that is, \( |Q_{u,i}| = 2^{-n_j} \). Denote \( E_{u,1} = E_u \setminus \bigcup_{i \in F_{u,0}} Q_{u,i} \).

We denote the number of biggest dyadic cubes in \( E_{u,1} \) by \( i_2 \). Denote by \( F_{u,2} \) the set \( \{ i \in \mathbb{N}, i = 1, \cdots, i_2 \} \). If \( i \in F_{u,2} \), we denote such cube by \( Q_{u,i} \). The volume of \( Q_{u,i} \) is denoted by \( 2^{-n_j} \), that is, \( |Q_{u,i}| = 2^{-n_j} \). Denote \( E_{u,2} = E_{u,1} \setminus \bigcup_{i \in F_{u,2}} Q_{u,i} \).

We continue this process until there exists some \( s \) such that \( E_{u,s+1} \) is empty. For \( s' \geq s+1 \), we denote \( i_{s'} = 0 \) and \( F_{u,s'} \) and \( E_{u,s'} \) are empty sets. Otherwise we continue until infinity. Then \( E_u = \bigcup_{s \geq 2, i \in F_{u,s}} Q_{u,s,i} \) and \( g_u(x) = \sum_{s \geq 2, i \in F_{u,s}} g_{u,s,i}(x) \), where \( g_{u,s,i}(x) = \sum_{Q_{u,s,i} \subset Q_{u,i}} g_{\nu}^{s,i} f_{x,k}^{s,i}(x) \).

To compute the norm of \( f(x)g_u(x) \), we need to find out a special set of dyadic cubes denoted by \( \{ Q_{u,s,i,k} \}_{i,j,k} \) such that \( \text{supp} g_u \subset \bigcup_{\{ s,j,k \} \in E_u} Q_{u,s,i,k} \) is nearly \( L^\infty \) function on \( Q_{u,s,i,k} \) and satisfies the estimate of Lemma 4.5. We divide such process into the following three steps.

Step 1. For \( \forall s \geq 1, \forall i \in F_{u,s} \) and \( k \in \mathbb{Z}^n \) with \( |k| < \sqrt{n^{s(M+3)n}} \), we denote \( (i,k) \in G_{u,s} \). Denote \( F_u = \{ (s,i,k), s \geq 1, (i,k) \in G_{u,s} \} \). For \( \forall (s,i,k) \in F_u \), we denote \( Q_{u,s,i,k} = 2^{-s-1} + k + Q_{u,s,i} \). For \( \forall s \geq 1 \), denote \( E^0_{u,s} = \bigcup_{(i,k) \in G_{u,s}} Q_{u,s,i,k} \). In the next step, we choose a special subcover to the support of \( g_u(x) \).

Step 2. We define now \( H_{u,s} (s \geq 1), E^1_{u,s} (s \geq 1) \) and \( G_{u,s} (s \geq 2) \).

For \( s = 1 \), denote \( H_{u,1} = G_{u,1} \) and \( E^1_{u,1} = E^0_{u,1} \). For \( s = 2 \), denote \( (i,k) \in G_{u,2} \), if there exists \( 0 \leq j \leq j_{u,1} \), \( l \in \mathbb{Z}^n \) such that \( Q_{j,l} \subset \bigcup_{|k| \leq \sqrt{n^{2(M+3)n}}} Q_{u,2,i,k} \) and \( < f_{x,k}^{s,i}(x) > 0 \). We know that \( \bigcup_{(i,k) \in G_{u,2}} Q_{u,2,i,k} \subset E^1_{u,1} \). Denote \( H_{u,2} = G_{u,2} \setminus G_{u,2} \) and \( E^1_{u,2} = \bigcup_{(i,k) \in H_{u,2}} Q_{u,2,i,k} \).
For \( s = 3 \), denote \((i, k) \in G_{n,1}\), if there exists \( 0 \leq j \leq j_{n,2}, l \in \mathbb{Z}^n \) such that \( Q_{j,l} \subset \bigcup_{|k - j| \leq \sqrt{2}^{n+2m}} Q_{n,2,i,k} \) and \(<g_u(x), \Phi_{j,k}^0(x)> \neq 0 \). We know that \( \bigcup_{(i,k) \in G_{n,3}} Q_{n,3,i,k} \subset \bigcup_{1 \leq s \leq 2} E_{n,s}^1 \)

Denote \( H_{n,3} = G_{n,3} \setminus G_{n,3}^1 \) and \( E_{n,3}^1 = \bigcup_{(i,k) \in H_{n,3}} Q_{n,3,i,k} \).

We continue this process until infinity. For \( s \geq 2 \), maybe, a party of \( G_{n,s}^1, H_{n,s} \) and \( E_{n,s}^1 \) are empty set.

Step 3. Let \( H_n = \{(s, i, k), s \geq 1, (i, k) \in H_{n,s}\} \). It is easy to see that the support of \( g_u(x) \) is contained in \( \bigcup_{s \geq 1} E_n^{1,s} \).

For a \((t+r, 2^n, E_n)\)-combination atom \( g_u(x) \) and \( g_{j,k}^0 = <g_u, \Phi_{j,k}^0> \), we have the following estimate.

**Lemma 4.5.** Given \( r > 0, t \geq 0, 1 < p < n/(t + r) \) and \( s \geq 1 \). For \((i, m') \in H_{n,s} \) and \( Q_{j,k} \subset Q_{n,s,i,m'} \), we have \( |g_{j,k}^0| \leq C 2^{n} 2^{-j(r + n)/2} \).

**Proof.** By the definition of \( H_{n,s} \), we have

\[
g_{j,k}^0 = \langle g_u(x), \Phi_{j,k}^0(x) \rangle
= \left\| \sum_{(i', j, k') \in H_n} \sum_{f \geq j_{n,s}} g_{j,k}^0(x), \Phi_{j,k}^0(x) \right\|
= \left\| \sum_{(i', j, k') \in H_n} \sum_{f \geq j_{n,s}} g_{j,k}^0(x), \Phi_{j,k}^0(x) \right\|.
\]

Because \( g_u \) is a \((m, 2^n, E_n)\)-combination atom, \( S_{t+r}(g_u)(x) \leq C 2^n \). Hence for every \((i', j', k')\),

\[
|g_{j,k}^0| \leq C 2^n 2^{-j(r + n)/2}.
\]

We can obtain

\[
|g_{j,k}^0| \leq \left\| \sum_{f \geq j_{n,s}} 2^{-j(r + n)/2} 2^{-j_{n,s}} 2^n \right\|
\]

\[
\leq C 2^n \int_{f \geq j_{n,s}} 2^{-j(r + n)/2} 2^{-j_{n,s}} dx
\]

\[
\leq C 2^n 2^{-j(r + n)/2}.
\]

\(\square\)

**Theorem 4.6.** Suppose that \( \tau > 1/p, t \geq 0, r > 0 \) and \( 1 < p < n/(r + t) \). If \( f \in M_{r,p}^{\tau}(\mathbb{R}^n) \) and \( g_u \) is a \((t + r, 2^n, E_n)\)-combination atom, then for \( s \geq 1 \), \((i, m') \in H_{n,s} \) and \( Q = Q_{n,s,i,m'} \), we have \( \|f g_u\|_{L^p(\mathbb{R}^n)} \leq C(1 + j_{n,s})^{-\tau} 2^n |Q|^{1/p} \).

**Proof.** First, for \( t \geq 0 \), we prove \( \|f g_u\|_{L^p(\mathbb{R}^n)} \leq C j_{n,s}^{-\tau} 2^n |Q|^{1/p} \). Let \( f(x) = \sum_{(i', j, k) \in H_n} f_{j,k}^i \Phi_{j,k}^i(x) \) and \( g_u(x) = \sum_{(i', j, k) \in H_n} g_{j,k}^i \Phi_{j,k}^i(x) \). Denote by \( N_n \) the set

\[
\{(i, j, k, l), \epsilon, \epsilon' \in E_n, (i, k) \neq (i', k + l), j \in \mathbb{Z}, k, l \in \mathbb{Z}^n, |l| \leq \sqrt{2}^{(|M|+2)|n|}\}.
\]
Because of the wavelet characterization of Sobolev spaces, we obtain

We divide the rest of the proof into three steps.

Step 1. For \( r \geq 0 \), we estimate the norm \( \| \cdot \|_{L^p(Q)} \) for the terms \( T_i(x) \), \( i = 1, 2, \cdots, 6 \). By the wavelet characterization of Sobolev spaces, we obtain

\[
f = \sum_{(r,j,k) \in \Lambda_n} f_{r,j,k}^e \Phi_{r,j,k}^e(x) \in L^p \quad \text{if and only if} \quad \left\| \sum_{(r,j,k) \in \Lambda_n} 2^{mj} |f_{r,j,k}^e|^2 \chi(2^j x - k) \right\|_{L^p}^{1/2}.
\]

Denote by \( S_0(f) \) the operator \( \left( \sum_{(r,j,k) \in \Lambda_n} 2^{mj} |f_{r,j,k}^e|^2 \chi(2^j x - k) \right)^{1/2} \). We have \( \|f\|_{L^p} \approx \|S_0(f)\|_{L^p} \).

1. Because \( f \in M_{1,0,0}^r \), by Lemma 4.3, \( |f_{r,j,k}^0| \leq (1 + j_{n,r})^{-r-2(r-n/2)j_{n,r}} \). By Lemma 4.5 we have \( |g_{r,j,k}^0| \leq C 2^{n-r(r+n/2)j_{n,r}} \). Hence we can get

\[
S_0(T_1)(x) = \left( \sum_{(Q, s) \in [Q_{n,r}], \sum_{Q_{n,r}} |Q_{n,r}|} 2^{2s} |f_{r,j,k}^0|^2 \chi(2^{2s} x - k) \right)^{1/2} \leq \left( \sum_{(Q, s) \in [Q_{n,r}], \sum_{Q_{n,r}} |Q_{n,r}|} 2^{2s} (1 + j_{n,r})^{-r-2(r-n/2)j_{n,r}} 2^{2s} 2^{2(n-r)j_{n,r}} \chi(2^{2s} x - k) \right)^{1/2}.
\]

Because \( j_{n,r} \geq 0 \), we have

\[
S_0(T_1)(x) \leq (1 + j_{n,r})^{-r-2n} \left( \sum_{(Q, s) \in [Q_{n,r}], \sum_{Q_{n,r}} |Q_{n,r}|} \chi(2^{2s} x - k) \right)^{1/2}.
\]

and

\[
\|T_1\|_{L^p(Q)} = \|S_0(T_1)\|_{L^p(Q)} \leq C (1 + j_{n,r})^{-r-2n} |Q|^{1/p}.
\]
(2) Now we estimate \( T_2(x) = \sum_{(e, j, k) \in \Lambda_{n,2}} f_{j,k}^0 \). Because \( j \geq j_{n,2} \geq 0 \), we have
\[
|f_{j,k}^0| \leq (1 + j_{n,2})^{-\tau} 2^{(r-n/2)j} \leq (1 + j_{n,2})^{-\tau} 2^{(r-n/2)j}.
\]
Let
\[
\Lambda_{n,2} = \{ (e, j, k) \in \Lambda_n \mid j \geq j_{n,2}, \forall l \leq \sqrt{2} (M+2)^n, |\text{supp}(\Phi_{j,k+l}^0(x)) \cap Q| \neq 0 \}.
\]
Then we can get
\[
S_0(T_2)(x) = \left( \sum_{(e, j, k) \in \Lambda_{n,2}} |f_{j,k}^0| \right)^{1/2} \leq C \left( \sum_{(e, j, k) \in \Lambda_{n,2}} 2^{2j} (1 + j)^{-\tau} 2^{2j(2^{r-n/2} - |e|)} |g_{j,k}^e|^2 \right)^{1/2} \leq C(1 + j_{n,2})^{-\tau} S_{1+r}(g_{n})(x).
\]
Because \( g_{n}(x) \) is a \((t + r, 2^n, E_n)\)-combination atom,
\[
\|S_0(T_2)\|_{L^p(Q)} \leq C(1 + j_{n,2})^{-\tau} \|S_{1+r}(g_{n})\|_{L^p(Q)} \leq C(1 + j_{n,2})^{-\tau} 2^n |Q|^{1/p}.
\]

(3) Since \( g_{n}(x) \) is a \((t + r, 2^n, E_n)\)-combination atom, for \( s \geq 1 \), \((i, m') \in H_{n,s}, j \geq 0 \) and \( Q = Q_{n,i,m'} \),
\[
|g_{j,k}^0| \leq C 2^{n-j/2} 2^{-(r+i)} = 2^{n-j/2} |Q|^{(r+i)/n}.
\]
Let
\[
\Lambda_{n,3} = \{ (e, j, k) \in \Lambda_n \mid j \geq j_{n,3}, \forall l \leq \sqrt{2} (M+2)^n, |\text{supp}(\Phi_{j,k+l}^0(x)) \cap Q| \neq 0 \}.
\]
We have, by \( j_{n,3} \geq 0 \),
\[
S_0(T_3)(x) = \left( \sum_{(e, j, k) \in \Lambda_{n,3}} 2^{j} |g_{j,k}^0| \right)^{1/2} \leq C 2^{n} |Q|^{(r+i)/n} \left( \sum_{(e, j, k) \in \Lambda_{n,3}} 2^{j} |f_{j,k}^e|^2 \right)^{1/2}.
\]
By the fact that \( j \geq j_{n,3} \) and \( f \in M_{r,p}^n(\mathbb{R}^n) \), we get
\[
\|S_0(T_3)\|_{L^p(Q)} \leq C 2^n |Q|^{(r+i)/n} (- \log_2 |Q|)^{-\tau} |Q|^{1/p-(r+i)/n} \leq C 2^n j_{n,3}^{-\tau} |Q|^{1/p}.
\]

(4) Now we estimate the term \( T_4(x) \). Let
\[
\Lambda_{n,4} = \{ (e, j, k) \in \Lambda_n \mid j \geq j_{n,4}, \forall (e, e', j, k, l) \in \Lambda_n, |\text{supp}(\Phi_{j,k+l}^{e'}(x)) \cap Q| \neq 0 \}.
\]
Because \( f \in M_{p,\rho}^\infty(\mathbb{R}^n) \), we have
\[
|f^e_{j,k}| \leq C(1 + j)^{-\tau}2^{j\rho(j-r-n/2)} \leq C(1 + j)^{-\tau}2^{j\rho(r+t-n/2)}
\]
and
\[
S_0(T_s)(x) = \left\{ \sum_{(e,j,k) \in \Lambda_n^0} 2^j(1 + j)^{-\tau}2^{j\rho(2r+2l-n)}2^{j\rho}g^e_{j,k} \chi(2^j x - k) \right\}^{1/2}
\leq C(1 + j_{u,v})^{-\tau}S_{r+t}(g_n)(x).
\]
Then we can get, by the fact that and \( g_n \) is a \((t + r, 2^n, E_n)\)-combination atom,
\[
\|S_0(T_s)\|_{L^r(Q)} \leq C(1 + j_{u,v})^{-\tau}\|S_{r+t}(g_n)\|_{L^r(Q)} \leq C^2(1 + j_{u,v})^{-\tau}|Q|^{1/p}.
\]
(5) Now we estimate the term \( T_5(x) = \sum_{(e,j,k) \in \Lambda_n^0} f^e_{j,k}g^e_{j,k} \left( (\Phi^e_{j,k}(x))^2 - 2^{\rho j/2}\Phi^0_{j,k}(x) \right) \).
Because the function \((\Phi^e_{j,k}(x))^2 - 2^{\rho j/2}\Phi^0_{j,k}(x)\) plays the role as that of \( \Phi^e_{j,k}(x) \), we have
\[
\|S_0(T_5)\|_{L^p(Q)} \leq C^2(1 + j_{u,v})^{-\tau}|Q|^{1/p}.
\]
(6) To estimate the term \( T_6(x) \), we take \( h \in L^{r'}(Q) \). Let
\[
\Lambda^Q_{n,0} = \left\{ (e,j,k) \in \Lambda_n \mid j \geq j_{u,v}, |\text{supp}(\Phi^0_{j,k}(x)\Phi^e_{j,k}(x)) \cap Q| \neq 0 \right\}.
\]
By the orthogonality of the wavelet function, we have
\[
< T_6, h > = \left\{ \sum_{(e,j,k) \in \Lambda_n^0} f^e_{j,k}g^e_{j,k} 2^{j\rho/2}\Phi^0_{j,k}, h \right\} = \sum_{(e,j,k) \in \Lambda_n^0} f^e_{j,k}g^e_{j,k} 2^{j\rho/2}\Phi^0_{j,k}.
\]
Then
\[
| < T_6, h > | \leq \int \sum_{(e,j,k) \in \Lambda_n^0} 2^n|f^e_{j,k}|g^e_{j,k}|(2^n h_{j,k})(2^j x - k)dx
\leq \int \sum_{(e,j,k) \in \Lambda_n^0} 2^n|f^e_{j,k}|g^e_{j,k}|(2^j x - k)M(h)(x)dx.
\]
By Hölder’s inequality and \( j \geq j_{u,v} \geq 0 \), it can be deduced that
\[
\sum_{(e,j,k) \in \Lambda_n^0} 2^n|f^e_{j,k}|g^e_{j,k}|(2^j x - k)
\leq \left( \sum_{(e,j,k) \in \Lambda_n^0} 2^{(n-2l+2r)}|f^e_{j,k}|g^e_{j,k}|(2^j x - k) \right)^{1/2} \left( \sum_{(e,j,k) \in \Lambda_n^0} 2^{(n+2l+2r)}|f^e_{j,k}|g^e_{j,k}|(2^j x - k) \right)^{1/2}
\leq C^2|Q|^{(r+t)/n} \left( \sum_{(e,j,k) \in \Lambda_n^0} 2^{(n-2l+2r)}|f^e_{j,k}|g^e_{j,k}|(2^j x - k) \right)^{1/2}.
Finally we can get

\[
| < T_b, h > | \leq C 2^n |Q|^{(r+1)/n} \left\| \sum_{(x, j, k) \in \Lambda^Q_n} 2^{(n-2r-2m)/2} f_{x, k}^0 \| \chi(2^{j} x - k) \right\|^{1/2}_{L^p(Q)} \leq 2^n |Q|^{(r+1)/n} \left| 1 - \log_2 |Q|^{-1} \right| |Q|^{1/p-(r+1)/n} \leq C 2^n |Q|^{1/p}(1 + j_{n,s})^{-\tau}.
\]

Because \( h \) takes over all functions in \( L^p(Q) \), it is obvious that \( \|S_0(T_b)\|_{L^p(Q)} \leq C(1 + j_{n,s})^{-\tau} 2^n |Q|^{1/p} \).

**Step 2.** Assume that \( 0 \leq t < 1 \). We need to prove for \( g(x) = \sum_{(x, j, k) \in \Lambda_n} g_{x, k} \Phi_{x, k} \) and \( S_t(g)(x) = \left( \sum_{(x, j, k) \in \Lambda_n} 2^{(2n+2m)/2} f_{x, k}^0 \| g_{x, k}^0 \| \chi(2^{j} x - k) \right)^{1/2} \), \( \|S_t(f g_a)\|_{L^p(Q)} \leq C 2^n (1 + j_{n,s})^{-\tau} |Q|^{1/p} \). The index sets \( \Lambda^Q_{n,s} \), \( i = 1, 2, \cdots, 6 \) are the same as Step 1.

1. For the term \( T_1 \), we have

\[
S_t(T_1)(x) = \left( \sum_{(x, j, k) \in \Lambda^Q_{n,s}} 2^{j+n+2m} 2^{j} x, k \right)^{1/2}
\leq \left( \sum_{(x, j, k) \in \Lambda^Q_{n,s}} 2^{j+n+2m} (1 + j_{n,s})^{-\tau} \right)^{1/2}
\leq (1 + j_{n,s})^{-\tau} 2^n \left( \sum_{(x, j, k) \in \Lambda^Q_{n,s}} \chi(2^{j} x - k) \right)^{1/2}.
\]

Then we have \( \|T_1\|_{L^p(Q)} = \|S_t(T_1)\|_{L^p(Q)} \leq C(1 + j_{n,s})^{-\tau} 2^n |Q|^{1/p} \).

2. For the term

\[
T_2(x) = \sum_{(x, j, k) \in \Lambda_n} f_{x, k}^0 g_{x, k} \Phi_{x, k}^0(x) \Phi_{x, k}^e(x),
\]

we have

\[
S_t(T_2)(x) = \left( \sum_{(x, j, k) \in \Lambda^Q_{n,s}} 2^{j+n+2m} 2^{j} x, k \right)^{1/2}
\leq C \left( \sum_{(x, j, k) \in \Lambda^Q_{n,s}} 2^{j+n+2m} (1 + j)^{-\tau} 2^{j-2(r+1)/2} \| g_{x, k}^0 \| \chi(2^{j} x - k) \right)^{1/2}
\leq C (1 + j_{n,s})^{-\tau} S_{t}(g_a)(x).
\]

Because \( g_a(x) \) is a \( (t + r, 2^u, E_u) \)-combination atom, we have

\[
\|S_t(T_2)\|_{L^p(Q)} \leq C(1 + j_{n,s})^{-\tau} \|S_t(g_a)\|_{L^p(Q)} \leq C(1 + j_{n,s})^{-\tau} 2^n |Q|^{1/p}.
\]
(3) Because \(g_s(x)\) is a \((t + r, 2^n, E_u)\)-combination atom, for \(s \geq 1\), \((i, m') \in H_{u, r}\) and \(Q = Q_{u, r, i, m'}\), we can get \(|g^{0}_{i,j,k}| \leq C^{2u} \left| 2^{-n/2} \right| 2^{-(r+i)} = 2^{-n/2} |Q|^{(r+i)/2}\) and

\[
S_i(T_3)(x) = \left( \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^{j(\nu+2)} |g^{0}_{j,k}|^2 |f^{e}_{j,k}|^2 \chi(2^j x - k) \right)^{1/2} \\
\leq C^{2u} \left( \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^{j(\nu+2)} |f^{e}_{j,k}|^2 \chi(2^j x - k) \right)^{1/2} \\
\leq C^{2u} |Q|^{(r+i)/2} \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^{j(\nu+2)} |f^{e}_{j,k}|^2 \chi(2^j x - k). \]

From the fact that \(j \geq j_{u, r}\) and \(f \in M^{r,\nu}_{t,p}(\mathbb{R}^n)\), we deduce that

\[
\|S_i(T_3)\|_{L^p(Q)} \leq C^{2u} |Q|^{(r+i)/2} (\log_2 |Q|)^{-\nu} |Q|^{1-p(r+i)/2} \leq C^{2u} j_{u, r}^{-\nu} |Q|^{1/p}. \]

(4) For the term \(T_4(x)\), because \(f \in M^{r,\nu}_{t,p}(\mathbb{R}^n)\) and \(g_u\) is a \((t + r, 2^n, E_u)\)-combination atom, we have \(|f^{e}_{j,k}| \leq C(1 + j)^{-\nu} 2^{-n/2}\) and

\[
S_i(T_4)(x) = \left( \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^{j(\nu+2)} (1 + j)^{-\nu} 2^n |g^{e}_{j,k}|^2 \chi(2^j x - k) \right)^{1/2} \leq C(1 + j_{u, r})^{-\nu} S_{i+r}(g_u)(x). \]

Then we can get

\[
\|S_i(T_4)\|_{L^p(Q)} \leq C(1 + j_{u, r})^{-\nu} \|S_{i+r}(g_u)\|_{L^p(Q)} \leq C^{2u} (1 + j_{u, r})^{-\nu} |Q|^{1/p}. \]

(5) Now we estimate the term

\[
T_5(x) = \sum_{(e,j,k) \in \Lambda^0_{a,i}} \int_{2^j \mathbb{R}^n} f^{e}_{j,k} g^{e}_{j,k} \left( (\Phi^{e}_{j,k}(x))^2 - 2^n \right) \chi(2^j x - k) dx. \]

Because the function \((\Phi^{e}_{j,k}(x))^2 - 2^n\Phi^{e}_{j,k}(x)\) plays the role as that of \(\Phi^{e}_{j,k}(x)\), we have \(\|S_i(T_5)\|_{L^p(Q)} \leq C^{2u} (1 + j_{u, r})^{-\nu} |Q|^{1/p}. \)

(6) For the term \(T_6(x)\), we take \(h \in H^{r,\nu}(Q)\). By the orthogonality of the wavelet functions, we have

\[
\langle T_6, h \rangle = \left( \sum_{(e,j,k) \in \Lambda^0_{a,i}} f^{e}_{j,k} g^{e}_{j,k} 2^{j/2} \Phi^{0}_{j,k} \right) h = \sum_{(e,j,k) \in \Lambda^0_{a,i}} f^{e}_{j,k} g^{e}_{j,k} 2^{j/2} h^{0}_{j,k}. \]

We can get

\[
|\langle T_6, h \rangle| \leq \int \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^{j/2} g^{e}_{j,k} f^{e}_{j,k} 2^j h^{0}_{j,k} \chi(2^j x - k) dx \leq \int \sum_{(e,j,k) \in \Lambda^0_{a,i}} 2^n |g^{e}_{j,k}| |f^{e}_{j,k}| |h^{0}_{j,k}| M(h)(x) \chi(2^j x - k) dx. \]
By Hölder’s inequality, we have

\[
\sum_{(\epsilon,j,k)\in\mathcal{A}_{\alpha}^0} 2^{n|\epsilon|} |f_{j,k}^\alpha| |g_{j,k}^\epsilon| \chi(2^j x - k) \\
\leq \left( \sum_{(\epsilon,j,k)\in\mathcal{A}_{\alpha}^0} 2^{(n-2r-2\tau)|\epsilon|} |f_{j,k}^\epsilon| \chi(2^j x - k) \right)^{1/2} \left( \sum_{(\epsilon,j,k)\in\mathcal{A}_{\alpha}^0} 2^{(n+2r+2\tau)|\epsilon|} |g_{j,k}^\epsilon| \chi(2^j x - k) \right)^{1/2} \\
\leq S_{n+r}(g_{\alpha})(x) \left( \sum_{(\epsilon,j,k)\in\mathcal{A}_{\alpha}^0} 2^{(n-2r)|\epsilon|} |f_{j,k}^\epsilon| \chi(2^j x - k) \right)^{1/2}.
\]

Because \( j \geq j_{u,s} \) implies that \( 2^{-\beta(t+r)} \leq 2^{-j_{u,s}(t+r)} \), we can get

\[
|T_h, h| \leq C 2^{n|\alpha|} \left( \sum_{(\epsilon,j,k)\in\mathcal{A}_{\alpha}^0} 2^{(n-2r-2\tau)|\epsilon|} |f_{j,k}^\epsilon| \chi(2^j x - k) \right)^{1/2} \|h\|_{L^p(Q)} \\
\leq C 2^{n|\alpha|} |Q|^{(r+r)/n} \log_2 |Q|^{-\tau}|Q|^{1/p-(r+\tau)/n} \\
\leq C 2^{n|\alpha|} |Q|^{1/p}(1 + j_{u,s})^{-\tau}.
\]

Because \( h \) takes over all functions in \( H^{-\sigma,p}(Q) \), we can get \( \|S_{\sigma}(T_h)\|_{L^p(Q)} \leq C(1 + j_{u,s})^{-\tau} 2^n|Q|^{1/p} \).

This completes the proof for the case \( 0 \leq t < 1 \).

**Step 3.** Now we consider the case \( t \geq 1 \). In this case, there exists an integer \( [t] \) such that \([t] \leq t < [t] + 1\). For any \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \) with \( |\alpha| = \sum_{i=1}^n \alpha_i = [t] \), the derivative \( \frac{\partial^\nu}{\partial x^\nu} \) of the product \( f \cdot g_\alpha \) can be represented as

\[
\frac{\partial^\nu}{\partial x^\nu}(f \cdot g_\alpha) = \sum_{\gamma=\beta} C_{\alpha,\beta,\gamma} \left( \frac{\partial^\beta}{\partial x^\beta} f(x) \right) \frac{\partial^\gamma}{\partial x^\gamma} g_\alpha(x),
\]

where \( |\alpha| = |\beta| + |\gamma| \). Denote \( \frac{\partial^\beta}{\partial x^\beta} f(x) \) by \( f_{\beta}(x) \) and denote \( \frac{\partial^\gamma}{\partial x^\gamma} g_\alpha(x) \) by \( g_{\alpha,\gamma}(x) \). Applying the conclusion in Step 1, we only need to prove

\[
\|f_{\beta} g_{\alpha,\gamma}\|_{H^{-\sigma,p}} \leq C r \|Q\|^{1/p},
\]

that is, \( \|S_{t-[t]}(f_{\beta} g_{\alpha,\gamma})\|_{L^p} \leq C 2^{n+\tau}|Q|^{1/p} \).

If \( g_\alpha \) is a \((t, r, 2^\alpha, E_\alpha)\)-combination atom, then

\[
g_{\alpha,\gamma}(x) = \sum_{Q_{\alpha}\in F_{\alpha,\gamma}} g_{j,k}^{\epsilon} \left( \frac{\partial}{\partial x^\nu} \right)^{|\nu|/2} \Phi_{j,k}^\epsilon(2^j x - k).
\]
Hence
\[
S_{t-\eta} (g_{u,\gamma})(x) = \left( \sum_{(r, j, k) \in \Lambda_u} 2^{jn(2r-2\eta)} |g_{f_{jk}}| \phi^2(2^j x - k) \right)^{1/2} \\
= \left( \sum_{(r, j, k) \in \Lambda_u} 2^{jn(2r)} |g_{f_{jk}}| \phi^2(2^j x - k) \right)^{1/2} \\
= S_t (g_u)(x).
\]

On the other hand, if \( f \in M_{\tau, \alpha}^p (\mathbb{R}^n) \), then
\[
\int_Q \left( \sum_{r \in \mathbb{Z}, Q \subset Q} 2^{jn(2r-2\eta)} |f_{jk}| \phi^2(2^j x - k) \right)^{p/2} \ dx \leq C (1 - \log Q)^{-p} |Q|^{1 - p(\eta + 1)/n}
\]
and \( (f_{jk})^p_{jk} = 2^{np} f_{jk}^p \). We have
\[
\int_Q \left( \sum_{r \in \mathbb{Z}, Q \subset Q} 2^{jn(2r-2\eta)} |f_{jk}| \phi^2(2^j x - k) \right)^{p/2} \ dx \leq \frac{C (1 - \log Q)^{-p} |Q|^{1 - p(\eta + 1)/n}}{2^{np} f_{jk}^p},
\]
that is, \( f_{jk} \in M_{\tau, \alpha}^p(\mathbb{R}^n) \).

For any cube \( Q \), the function \( f_{jk} \) and any dyadic cube \( Q \subset S_u \), we divide the product \( f_{jk} \cdot g_{u,\gamma} \) into the following parts.

\[
T_1^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} \sum_{l \leq \sqrt{2^{n + 2\eta}}} f_{j+l, k}^\eta \phi_{j+l, k}(x) \phi^{\eta}_{j_k}(x); \\
T_2^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} \sum_{l \leq \sqrt{2^{n + 2\eta}}} f_{j+l, k}^\eta \phi_{j+l, k}(x) \Phi^{\eta}_{j_k}(x); \\
T_3^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} \sum_{l \leq \sqrt{2^{n + 2\eta}}} f_{j+l, k}^\eta \Phi_{j+l, k}(x) \phi^{\eta}_{j_k}(x); \\
T_4^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} \sum_{l \leq \sqrt{2^{n + 2\eta}}} f_{j+l, k}^\eta \Phi_{j+l, k}(x) \Phi^{\eta}_{j_k}(x); \\
T_5^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} \sum_{l \leq \sqrt{2^{n + 2\eta}}} f_{j+l, k}^\eta \Phi_{j+l, k}(x) \left( \Phi^{\eta}_{j_k}(x) \right)^2 \phi_0^{\eta}(x); \\
T_6^{\eta, \gamma} (x) = \sum_{(r, j, k) \in \Lambda_u, \eta + 2j \geq j_k} f_{j+l, k}^\eta \phi_{j+l, k} \phi_0^{\eta}(x).
\]

Similar to the method used in the case \( 0 \leq t < 1 \), we can complete the proof of the case \( t \geq 1 \). This completes the proof of this theorem. \( \Box \)

By Theorem 4.6, we can get the following lemma.
Lemma 4.7. Given $r > 0, t \geq 0, 1 < p < n/(r + t)$ and $\tau > 1/p'$. If $f \in M_{r,p}^{t,\tau}(\mathbb{R}^n)$ and $g_a$ is a $(t + r, 2^n, E_a)$-combination atom, then

$$
\|fg_a\|_{H^{\tau,p}} \leq C(1 + u)^{-\tau}2^{|E_a|^{1/p}}.
$$

Theorem 4.8. Given $r > 0, t \geq 0, 1 < p < n/(r + t)$ and $\tau > 1/p'$. If $f \in M_{r,p}^{t,\tau}(\mathbb{R}^n)$, then $f \in X_{r,p}^t(\mathbb{R}^n)$.

Proof. By Lemma 4.7 we have

$$
\|fg\|_{H^{\tau,p}} \leq \sum_{a \geq 0} \|fg_a\|_{H^{\tau,p}} \\
\leq C \sum_{a \geq 0} (1 + u)^{-\tau}2^{|E_a|^{1/p}} \\
\leq C \left( \sum_{a \geq 0} (1 + u)^{-\tau} \right)^{1/p'} \left( \sum_{a \geq 0} 2^{|E_a|} \right)^{1/p} \\
\leq C \|g\|_{H^{\tau,p}}.
$$

□

5. The sharpness for the multiplier spaces $M_{r,p}^{t,\tau}(\mathbb{R}^n)$

In this section, applying our wavelet characterization of multiplier spaces and fractal theory, we prove that the scope of the index of $M_{r,p}^{t,\tau}(\mathbb{R}^n)$ obtained in Theorem 4.8 is sharp for $r + t < 1$. Precisely, by Meyer wavelets, we construct a counterexample to show that Theorem 4.8 is not true for the case $0 \leq \tau \leq 1/p'$.

Our key idea is to construct a group of sets $C_s$ composed by special dyadic cubes and fractal set $H$ with Hausdorff dimension $n - (t + r)p$. Denote by $S_s$ the union $\bigcup_{Q \in C_s} Q$ and $H = \bigcap_{s \geq 0} S_s$. By the above dyadic cubes $S_s, s \geq 0$, we construct a special $L^p$ function $g(x)$, which is bounded on $S_s \setminus S_{s+1}$ for all $s \geq 1$. The fractional integration $I_{r+\tau}g(x)$ bumps on the fractal set $H$. Then we construct a multiplier $f(x)$ such that its wavelet coefficients are based on these special dyadic cubes $C_s$ for all $s \geq 1$. Applying our wavelet characterization of multiplier spaces, we prove that the product of the above multiplier $f(x)$ and the function $I_{r+\tau}g(x)$ will go out the desired space $H^{r,p}(\mathbb{R}^n)$.

For the above purpose, we give first another characterization of $X_{r,p}^t(\mathbb{R}^n)$ associated with the fractional integration. Let $\Psi(x) \geq 0, \Psi(x) \in C_0^\infty(B(0, 2))$ with $\Psi(x) = 1$ on $B(0, 1)$. We know that $\tilde{g} \in H^{r+\tau,p}(\mathbb{R}^n)$ if and only if there exists $g \in L^p(\mathbb{R}^n)$ such that $\tilde{g}(x) = J_{1+r,\tau}g(x) = \int g(y)\Psi(|x - y|)|x - y|^{r+\tau}dy$. For $g \in L^p(\mathbb{R}^n)$, let $g_{j,k} = \int (2^{-j} + |y - 2^{-j}k|)^{r+\tau}g(y)dy$. We define the following function space.
Definition 5.1. Given $r > 0$, $t \geq 0$ and $1 < p < n/(r + t)$. For $f(x) = \sum_{(c,j,k) \in \Lambda_n} f_{\epsilon, j,k}^c(x)$, $f \in Q^r_{t,p}(\mathbb{R}^n)$ if and only if

\[ \int \left( \sum_{(c,j,k) \in \Lambda_n} 2^{j(n+2t)} |g_{j,k}|^2 |f_{\epsilon, j,k}^c|^2 \right)^{p/2} dx \leq C, \]

where $g \in L^p(\mathbb{R}^n)$ and $g(x) \geq 0$.

Let $f(x) = \sum_{(c,j,k) \in \Lambda_n} f_{\epsilon, j,k}^c(x)$ and $h(x) = \sum_{(c,j,k) \in \Lambda_n} h_{\epsilon, j,k}^c(x)$. Define

\[ T_\Phi(f,h)(x) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} 2^m f_{\epsilon, j,k}^c h_{\epsilon, j,k}^c \Phi(2^j x - k). \]

Similar to Claim 2 in Theorem 3.3 we can get

Lemma 5.2. Given $t \geq 0$, $r > 0$ and $t + r < 1 < p < n/(t + r)$. Let $g \in H^{t+r,p}(\mathbb{R}^n)$ and $h \in H^{t-r,p}(\mathbb{R}^n)$. The function $f(x) \in S_{t,r,p}(\mathbb{R}^n)$ if and only if

\[ \langle T_\Phi(f,h), g \rangle \leq C \|g\|_{H^{t+r,p}(\mathbb{R}^n)} \|h\|_{H^{t-r,p}(\mathbb{R}^n)}. \]

Now we give another characterization of $X^r_{t,p}(\mathbb{R}^n)$.

Theorem 5.3. Given $t \geq 0$, $r > 0$ and $t + r < 1 < p < n/(t + r)$. $f \in X^r_{t,p}(\mathbb{R}^n)$ if and only if $f \in M^r_{t,p}(\mathbb{R}^n)$ and $f \in Q^r_{t,p}(\mathbb{R}^n)$.

Proof. By modifying the coefficients such that $f_{\epsilon, j,k}^c h_{\epsilon, j,k} = |f_{\epsilon, j,k}^c| h_{\epsilon, j,k}^c$, we could suppose $g(y) \geq 0$. By Theorem 3.3(ii) and Lemma 5.2 we know that $f \in S_{r,p}(\mathbb{R}^n)$ is equivalent to

\begin{align*}
(5.1) & \quad \int \int \sum_{(c,j,k) \in \Lambda_n} |f_{\epsilon, j,k}^c| |h_{\epsilon, j,k}^c| 2^m \Phi(2^j x - k) \Psi(|x - y|) \frac{g(y)}{|x - y|^{p - (t+r)}} dy dx \\
& \quad \leq C \|g\|_{L^p} \|h\|_{H^{t-r,p}}, \forall g \geq 0.
\end{align*}

By a calculation of the integral associated to $dx$, we get

\[ \int 2^m \Phi(2^j x - k) \Psi(|x - y|) |x - y|^{t+r-n} dx = (2^j + |y - 2^{-j}k|)^{t+r-n}. \]

So (5.1) is equivalent to the following inequality:

\begin{align*}
(5.2) & \quad \int \sum_{(c,j,k) \in \Lambda_n} |f_{\epsilon, j,k}^c| |h_{\epsilon, j,k}^c|(2^j + |y - 2^{-j}k|)^{t+r-n} g(y) dy \\
& \quad \leq C \|g\|_{L^p} \|h\|_{H^{t-r,p}}, \forall g \geq 0.
\end{align*}

That is to say

\begin{align*}
(5.3) & \quad \int \int \sum_{(c,j,k) \in \Lambda_n} 2^m |f_{\epsilon, j,k}^c| |h_{\epsilon, j,k}^c| \Phi(2^j x - k) \frac{g(y)}{(2^j + |y - 2^{-j}k|)^{t+r}} dxdy \\
& \quad \leq C \|g\|_{L^p} \|h\|_{H^{t-r,p}}, \forall g \geq 0.
\end{align*}
Let $S_{\cdot}\cdot h(x) = \left( \sum_{(s,l,k) \in A_s} 2^{(n-2)l} |f_{s,l,k}|^2 \chi(2l x - k) \right)^{1/2}$. We have $\|S_{\cdot}\cdot h\|_{p'} \approx \|h\|_{H^{-\alpha}}$. Then the inequality (5.3) is equivalent to

$$
\int \left( \sum_{(s,l,k) \in A_s} 2^{(n+2)l} |f_{s,l,k}|^2 \chi(2l x - k) \right)^{p/2} \, dx \leq C \|g\|_p^p, \forall g \geq 0.
$$

The above inequality is equivalent to $f \in Q_{r,p}^{1,2}$. □

**Theorem 5.4.** If $0 \leq \tau \leq \frac{1}{p}$, there exists $f \in M_{r,p,\infty}^{1,2}(\mathbb{R}^n)$ such that $f$ does not belong to $X_{r,p}^\tau(\mathbb{R}^n)$.

**Proof.** We use Meyer wavelets and suppose that $\varepsilon = (1, 1, \cdots, 1)$ and $\Phi^{\varepsilon}(0) > 0$. First of all, we construct a group of self similar cubes $\{Q_i\}$ such that the limitation is a set with Hausdorff measure $n - (t + r)p$

We construct two series of integers $\{\tau_s\}_{s \geq 1}$ and $\{v_s\}_{s \geq 1}$ such that $v_1 = \max \left\{ \left[ \frac{2\varepsilon}{n-(t+r)p} \right], 2^{(M+2)n} \right\}$, $1 \leq \tau_s < \left[ \frac{2\varepsilon}{n-(t+r)p} \right] \leq v_s \leq \max \left\{ \left[ \frac{2\varepsilon}{n-(t+r)p} \right], 2^{(M+2)n} \right\}$. Denote $\sigma_s = \sum_{1 \leq l \leq s} \tau_l$ and $u_s = \sum_{1 \leq l \leq s} v_l$. We take $\tau_s$ such that there exists $C > 0$ satisfying that $2^{n\tau_s - (n-(t+r)p)u_s} \geq 2^{-nu_s}$ and $\lim_{s \to +\infty} 2^{n\tau_s - (n-(t+r)p)u_s} / C = n - (t + r)p = \lim_{s \to +\infty} \frac{\sigma_s}{u_s}$. For $\tau_s$ and $v_s$, denote $l_s = (l_{s,1}, l_{s,2}, \cdots, l_{s,n}) \in \mathbb{Z}_{+}^n$ if $l_s \in \mathbb{Z}^n$ and for $i = 1, 2, \cdots, n$, we have $0 \leq l_{s,i} < 2^{\tau_i - 1}$ or $0 \leq l_{s,i} + 2^{\tau_i - 1} - 2^{v_s} < 2^{\tau_i - 1}$. Denote $\Omega_1 = \mathbb{Z}_{+}^1$ and for $s \geq 2$, denote $k_s \in \mathbb{Z}_{+}^n$ such that $k_s = 2^{v_s} k_{s-1} + 1_{l_s}$.

We divide the unit dyadic cube $\Omega_0 = [0, 1]^n$ into $2^{v_1}$ dyadic cubes. Then we reserve the $2^{v_1}$ dyadic cubes which are near the $2^n$ vertex points, that is, we reserve $C_1 = \left\{ Q_{\cdot, j_1} : l_1 \in \mathbb{Z}_{+}^1 \right\}$ and denote $x \in S_1$, if there exists $l_1 \in \mathbb{Z}_{+}^1$ such that $x \in Q_{\cdot, j_1}$.

For the dyadic cube $Q_{\cdot, j_1} \in C_1$, we divide it into $2^{v_2}$ dyadic cubes, we reserve the $2^{v_2}$ dyadic cubes which are near the $2^n$ vertex points, that is, we reserve $C_{2, l_1} = \left\{ Q_{2 l_1, j_2, k_2} : k_2 = 2^{v_2} l_1 + 1_{l_1}, l_1 \in \mathbb{Z}_{+}^1 \right\}$ and denote $x \in S_2$, if there exists $l_2 \in \mathbb{Z}_{+}^2$ such that $x \in Q_{2 l_1, j_2, k_2}$.

We continue this process until infinity, we get a series of dyadic cubes $C_s$ and sets $S_s$.

We know that $|S_s| = 2^{n(\sigma_s - n)}$ and the limitation of $|S_s|$ is a fractal set $H$ with Hausdorff dimension $n - (t + r)p$.

For $s \geq 1$, let $f_s(x) = \sum_{Q_{n,k_s} \subset C_s} f_{n,k_s} \Phi_{n,k_s}(x)$, where $f_{n,k_s} = s^{-1/p'} 2^{-nu_s} 2^{(t+r)u_s}$. Let $f(x) = \sum_{s \geq 1} f_s(x)$. Applying the wavelet characterization of Morrey spaces, $f \in M_{r,p,\infty}^{1,2}(\mathbb{R}^n)$. In fact, for $s, l \geq 1$ and any cube $Q$ with $2^{-nu_{s+1}} \leq |Q| < 2^{-nu_s}$, we have $|Q \cap (S_{s+1}) \cap (S_{s+1})| \leq \varepsilon |Q|$. Q.E.D.
Hence we get
\[
\begin{align*}
\int_Q \left( \sum_{Q_{n,k} \subset Q} 2^{nu(n+2)l^\alpha} f_{Q_{n,k}} \chi(2^n x - k_s) \right)^{p/2} \, dx \\
\leq \int_{Q(S_{s+1})} \left( \sum_{Q_{n,k} \subset Q} 2^{nu(n+2)l^\alpha} 2^{-2mu} 2^{2(r+\nu_1)} \chi(2^n x - k_s) \right)^{p/2} \, dx \\
+ \sum_{j=1}^\infty \int_{Q(S_{s+1} \setminus S_{s+1})} \left( \sum_{Q_{n,k} \subset Q} 2^{nu(n+2)l^\alpha} 2^{-2mu} 2^{2(r+\nu_1)} \chi(2^n x - k_s) \right)^{p/2} \, dx \\
\leq s^{-p/(r+\nu_1)pu} 2^{-mu} + C \sum_{j=1}^\infty (s + 1)^{-p/(r+\nu_1)pu} 2^{-n(\sigma_j - \sigma_1) - mu} \\
\leq (1 - \log_2 |Q|)^{-p/(r+\nu_1)pu} |Q|^{-(r+\nu_1)pn}.
\end{align*}
\]

Let \( \delta \) be a sufficient small positive real number. For \( \forall x \in [0, 1]\setminus S_1 \), then \( g(x) = 1 \). For \( s \geq 1 \) and \( x \in S_{s+1} \setminus S_{s+1} \), we take \( g(x) = s^{-1/p} [\log_2(1 + s)]^{-(1+\delta)/p} 2^{(r+\nu_1)x} \). Then we have \( g \in L^p([0, 1]^n) \). In fact,
\[
\begin{align*}
\int_{[0,1]^n} |g(x)|^p dx &= \int_{[0,1]^n \setminus S_1} g^p(x) dx + \sum_{j=1}^\infty \int_{S_{j+1} \setminus S_{j+1}} g^p(x) dx \\
&\leq C + C \sum_{j=1}^\infty \int_{S_{j+1} \setminus S_{j+1}} s^{-(1+\delta)/2} 2^{(r+\nu_1)x} dx \\
&\leq C \sum_{j=1}^\infty s^{-1} [\log_2(1 + s)]^{-(1+\delta)/p} 2^{(r+\nu_1)x} \\
&\leq C.
\end{align*}
\]

Now we estimate the coefficients \(|g_{u,k}|\). We divide the estimate into two cases.

(1) For \( \text{dist} (Q_{u,k}, S_{s+1}) \leq 2^{-u_s} \),
\[
\begin{align*}
g_{j,k} &= \int_{S_{s+1}} g(y)(2^{-u_1} + |y - 2^{-u_k} k_s|)^{(r+n)} dy \\
&+ \sum_{1 \leq \alpha \leq n} \int_{S_{j+1} \setminus S_{j+1}} \Gamma^{1/p} [\log_2(1 + b)]^{-(1+\delta)/p} 2^{2^{u(n+1)/p}(2^{-u} + |y - 2^{-u_k} k_s|)^{(r+n)}} dy \\
&\geq s^{-1/p} [\log_2(1 + s)]^{-(1+\delta)/p} + \sum_{1 \leq \alpha \leq n} s^{-1/p} [\log_2(1 + s)]^{-(1+\delta)/p} \\
&\geq s^{1/p} [\log_2(1 + s)]^{-(1+\delta)/p}.
\end{align*}
\]

(2) For \( \text{dist} (Q_{u,k}, S_{s+1}) > 2^{-u_s} \), there exists \( l < s \) such that \( Q_{u_k} \subset S_{l+1} \setminus S_l \). It is easy to see that \( g_{j,k} \) is equivalent to \( l^{-1/p} [\log_2(1 + l)]^{-(1+\delta)/p} \).
Finally, we have

\[ \int \left( \sum_{(r, j, k) \in A_n} 2^{r(n+2)} |g_{r, j, k}|^2 |f_{r, j, k}|^2 \chi(2^r x - k) \right)^{p/2} dx \]

\[ \geq C \sum_{s \geq 2} \left( \sum_{\text{dist}(Q_{r, s, 0}) \leq 2^{-2u_s}} 2^{r(n+2)} |g_{r, s, 0}|^2 |f_{r, s, 0}|^2 \chi(2^r x - k) \right)^{p/2} dx \]

\[ \geq C \sum_{s \geq 2} \left( \sum_{\text{dist}(Q_{r, s, 0}) \leq 2^{-2u_s}} 2^{r(n+2)} 2^{2r/\tau} \left( \log_2 (1 + s) \right)^{-2(1+\delta)/p} 2^{-2u_s} 2^{2r(\tau - 1)} \chi(2^r x - k) \right)^{p/2} dx \]

\[ \geq \sum_{s \geq 2} \left( \log_2 (1+s) \right)^{-2(1+\delta)/p} 2^{2r(\tau - 1)} \]

\[ \geq \sum_{s \geq 2} \left( \log_2 (1+s) \right)^{-2(1+\delta)/p} = \infty. \]

This completes the proof. \( \square \)

6. An application to Schrödinger type operators with non-smooth potentials

In [11], the multipliers from \( H^{1,2}(\mathbb{R}^n) \) to \( H^{-1,2}(\mathbb{R}^n) \) were studied by V. Maz’ya and I. E. Verbitsky. For a Schrödinger operator \( L = \Delta + V \), they established many sufficient and necessary conditions such that \( V \) is a multiplier from \( H^{1,2}(\mathbb{R}^n) \) to \( H^{-1,2}(\mathbb{R}^n) \). In this section, we give an application of the wavelet characterization of \( X^r_{r,p}(\mathbb{R}^n) \) to the Schrödinger type operator \( I + (-\Delta)^{r/2} + V \).

For \( V \in M^{s,p}_{r,p}(\mathbb{R}^n) \), \( (\tau > 1/p') \) and \( g(x) \in H^{r,p}(\mathbb{R}^n) \), we want to find a solution \( f \in H^{r,r+p}(\mathbb{R}^n) \) to the equation

\[ (I + (-\Delta)^{r/2} + V) f(x) = g(x). \] (6.1)

**Remark 6.1.** Fixed \( r > 0, \tau \geq 0 \) and \( 1 < p < n/(r + \tau) \).

(i) If there exists a \( \delta > 0 \) such that \( \|V\|_{C^{\tau+\delta}} \) is sufficient small, according to the continuity of Calderón-Zygmund operator \( (I + (-\Delta)^{r/2})(I + (-\Delta)^{r/2} + V)^{-1} \), the equation (6.1) can be solved easily. But if we consider a non smooth potential \( V \in M^{s,p}_{r,p}(\mathbb{R}^n) \), applying the same proof in Lemmas 2.6 and 2.7, it is possible that \( V \) is not a \( L^\infty \) function.

(ii) The condition \( \tau > 1/p' \) can not be weaken to \( \tau \geq 1/p' \). In fact, according to our counterexample in §5, if \( r + t < 1 \), there exists some \( V \in M^{s,1/p'}_{r,p}(\mathbb{R}^n) \) such that the operator \( I + (-\Delta)^{r/2} + V \) is not continuous from \( H^{r+p}(\mathbb{R}^n) \) to \( H^{r,p}(\mathbb{R}^n) \).

Now, we use our sufficient condition of multiplier spaces \( X^r_{r,p}(\mathbb{R}^n) \) to get the solution of the equation (6.1). We need the following two operators. For \( r, r > 0 \), let \( T_{r,r} = [I + (-\Delta)^{r/2}]I + (-\Delta)^{r/2} \) and \( S_{r,r} = [I + (-\Delta)^{r/2}]V T_{r,r}^{-1} \). In the following lemma, we prove that the operator \( S_{r,r} = [I + (-\Delta)^{r/2}]V T_{r,r}^{-1} \) is bounded on \( L^p(\mathbb{R}^n) \).
Lemma 6.2. Given \( r > 0, t \geq 0, 1 < p < n/(r + t) \) and \( \tau > 1/p' \). If \( V(x) \in \mathcal{M}^{p'}_{r,t,p} (\mathbb{R}^n) \), the operator \( S_{r,t} \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) with the operator norm less than \( C_{r,t,\tau,p} \|V\|_{\mathcal{M}^{p'}_{r,t,p}} \), where \( C_{r,t,\tau,p} \) denotes a constant associated with \( t, r, \tau, p \).

Proof. By Theorem 4.8, for any \( f \in L^p(\mathbb{R}^n) \), we have
\[
\|S_{r,t}f\|_{L^p} \leq \|(I + (-\Delta)^{1/2}VT_{r,t}^{-1}f)\|_{L^p} \leq \|VT_{r,t}^{-1}f\|_{L^p} \leq C_{r,t,\tau,p}\|V\|_{\mathcal{M}^{p'}_{r,t,p}}\|f\|_{L^p}.
\]
Then \( S_{r,t} \) is a bounded operator on \( L^p(\mathbb{R}^n) \) with the norm less than \( C_{r,t,\tau,p} \|V\|_{\mathcal{M}^{p'}_{r,t,p}} \). \( \square \)

In the following lemma, we prove that \((I + S_{r,t})\) is invertible in \( L^p(\mathbb{R}^n) \) and the inverse \((I + S_{r,t})^{-1}\) can be written formally as \( \sum_{n=0}^{\infty} (-1)^n S_{r,t}^n \).

Lemma 6.3. Given \( r > 0, t \geq 0, 1 < p < n/(r + t) \) and \( \tau > 1/p' \). If \( \|V(x)\|_{\mathcal{M}^{p'}_{r,t,p}} < 1/C_{r,t,\tau,p} \), the operator \( I + S_{r,t} \) is invertible in \( L^p(\mathbb{R}^n) \).

Proof. By Lemma 6.2, the operator \( S_{r,t} \) is bounded on \( L^p(\mathbb{R}^n) \). Hence for any \( f \in L^p(\mathbb{R}^n) \),
\[
\left\| \sum_{n=0}^{\infty} (-1)^n S_{r,t}^n \right\|_{L^p} \leq \sum_{n=0}^{\infty} \|S_{r,t}^n\|_{L^p} \leq \sum_{n=0}^{\infty} (C_{r,t,\tau,p}\|V\|_{\mathcal{M}^{p'}_{r,t,p}})^n\|f\|_{L^p}.
\]
If \( \|V\|_{\mathcal{M}^{p'}_{r,t,p}} < 1/C_{r,t,\tau,p} \), the above series is convergent in \( L^p(\mathbb{R}^n) \). Further,
\[
(I + S_{r,t}) \left[ \sum_{n=0}^{\infty} (-1)^n S_{r,t}^n \right] = \sum_{n=0}^{\infty} (-1)^n S_{r,t}^n - \sum_{n=1}^{\infty} (-1)^n S_{r,t}^n = I.
\]
Similarly, we can also get
\[
\left[ \sum_{n=0}^{\infty} (-1)^n S_{r,t}^n \right] (I + S_{r,t}) = I, \text{ that is, the operator } I + S_{r,t} \text{ is invertible in } L^p(\mathbb{R}^n).
\]
\( \square \)

Theorem 6.4. Given \( r > 0, t \geq 0, 1 < p < n/(r + t) \) and \( \tau > 1/p' \). If \( \|V\|_{\mathcal{M}^{p'}_{r,t,p}} < 1/C_{r,t,\tau,p} \), then for \( g(x) \in H^{r+\tau,p}(\mathbb{R}^n) \), there exists a unique solution \( f(x) \in H^{r+\tau,p}(\mathbb{R}^n) \) for equation (6.1).

Proof. Because \( g \in H^{r+\tau,p}(\mathbb{R}^n) \), we have \( \tilde{g} = (I + (-\Delta)^{1/2})g(x) \in L^p(\mathbb{R}^n) \). By Lemma 6.3 the operator \( I + S_{r,t} \) is invertible in \( L^p(\mathbb{R}^n) \). Hence we can get there exists a unique solution to the following equation in \( L^p(\mathbb{R}^n) \),
\[
(6.2) \quad (I + S_{r,t})\tilde{f} = \tilde{g},
\]
where \( \tilde{g} \in L^p(\mathbb{R}^n) \). Hence for above \( \tilde{g} \in L^p(\mathbb{R}^n) \),
\[
\tilde{f} = (I + S_{r,t})^{-1}\tilde{g} = (I + S_{r,t})^{-1}(I + (-\Delta)^{1/2})g(x) \in L^p(\mathbb{R}^n)
\]
is a solution to the equation (6.2). Write \( f = (I + (-\Delta)^{1/2})^{-1}(I + (-\Delta)^{1/2})^{-1}\tilde{f} \). Then \( \tilde{f}(x) \in L^p(\mathbb{R}^n) \) is equivalent to \( f \in H^{r+\tau,p}(\mathbb{R}^n) \). It is easy to verify that \( f \) is a solution to the equation (6.1). This completes the proof. \( \square \)
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