Abstract. For a finite quiver without sinks, we establish an isomorphism in the homotopy category of \(B_\infty\)-algebras between the Hochschild cochain complex of the Leavitt path algebra and the singular Hochschild cochain complex of the corresponding finite dimensional algebra \(\Lambda\) with radical square zero. Combining this isomorphism with a description of the dg singularity category of \(\Lambda\) in terms of the dg perfect derived category of the Leavitt path algebra, we verify Keller’s conjecture for the singular Hochschild cohomology of \(\Lambda\). More precisely, we prove that there is an isomorphism in the homotopy category of \(B_\infty\)-algebras between the singular Hochschild cochain complex of \(\Lambda\) and the Hochschild cochain complex of the dg singularity category of \(\Lambda\).

We prove that Keller’s conjecture is invariant under one-point (co)extensions and singular equivalences with levels. Consequently, Keller’s conjecture holds for those algebras obtained inductively from \(\Lambda\) by one-point (co)extensions and singular equivalences with levels.

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1. Introduction

1.1. The background. Let $k$ be a field, and $\Lambda$ be a finite dimensional $k$-algebra. Denote by $\Lambda$-mod the abelian category of finite dimensional $\Lambda$-modules and by $D^b(\Lambda$-mod) its bounded derived category. Following [16, 53], the singularity category $D_{sg}(\Lambda)$ of $\Lambda$ is by definition the Verdier quotient category of $D^b(\Lambda$-mod) by the full subcategory of perfect complexes. It measures the homological singularity of the algebra $\Lambda$, and reflects the asymptotic behaviour of syzygies of $\Lambda$-modules.

It is well known that triangulated categories are less rudimentary than dg categories as the former are inadequate to handle many basic algebraic and geometric operations. The bounded dg derived category $D^b_{dg}(\Lambda$-mod) is a dg category whose zeroth cohomology
coincides with $D^b(\Lambda\text{-mod})$. Similarly, the \textit{dg singularity category} $S_{\text{dg}}(\Lambda)$ of $\Lambda$ \cite{42,12,15} is defined to be the dg quotient category of $D^b_{\text{dg}}(\Lambda\text{-mod})$ by the full dg subcategory of perfect complexes. Then the zeroth cohomology of $S_{\text{dg}}(\Lambda)$ coincides with $D_{\text{sg}}(\Lambda)$. In other words, the dg singularity category provides a canonical dg enhancement for the singularity category.

As one of the advantages of working with dg categories, their Hochschild theory behaves well with respect to various operations \cite{40,50,60}. We consider the Hochschild cochain complex $C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$ of the dg singularity category $S_{\text{dg}}(\Lambda)$, which has a natural structure of a $B_\infty$-algebra \cite{30}. Moreover, it induces a Gerstenhaber algebra structure \cite{28} on the Hochschild cohomology $\text{HH}^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$. The $B_\infty$-algebra structures on the Hochschild cochain complexes play an essential role in the deformation theory \cite{50} of categories. We mention that $B_\infty$-algebras are the key ingredients in the proof \cite{59} of Kontsevich’s formality theorem. We refer to \cite[Subsection 1.19]{52} for the relationship between $B_\infty$-algebras and Deligne’s conjecture.

The \textit{singular Hochschild cohomology} $\text{HH}^*_\text{sg}(\Lambda, \Lambda)$ of $\Lambda$ is defined as $\text{HH}^*_\text{sg}(\Lambda, \Lambda) := \text{Hom}_{D_{\text{sg}}(\Lambda^e)}(\Lambda^e, \Sigma^n(\Lambda))$, for any $n \in \mathbb{Z}$, where $\Sigma$ is the suspension functor of the singularity category $D_{\text{sg}}(\Lambda^e)$ of the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{\text{op}}$; see \cite{11,64,42}. By \cite{66}, there are two complexes $\overline{C}_{\text{sg},L}(\Lambda, \Lambda)$ and $\overline{C}_{\text{sg},R}(\Lambda, \Lambda)$ computing $\text{HH}^*_\text{sg}(\Lambda, \Lambda)$, called the \textit{left singular Hochschild cochain complex} and the \textit{right singular Hochschild cochain complex} of $\Lambda$, respectively. Moreover, both $\overline{C}_{\text{sg},L}(\Lambda, \Lambda)$ and $\overline{C}_{\text{sg},R}(\Lambda, \Lambda)$ have natural $B_\infty$-algebra structures, which induce the same Gerstenhaber algebra structure on $\text{HH}^*_\text{sg}(\Lambda, \Lambda)$.

There is a canonical isomorphism
\begin{equation}
\overline{C}_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \simeq \overline{C}_{\text{sg},R}(\Lambda, \Lambda)^{\text{op}}
\end{equation}
of $B_\infty$-algebras; see Appendix A. Here, for a $B_\infty$-algebra $A$ we denote by $A^{\text{op}}$ its opposite $B_\infty$-algebra; see Definition 5.5. We mention that the $B_\infty$-algebra structures on the singular Hochschild cochain complexes come from a natural action of the cellular chains of the spineless cacti operad introduced in \cite{36}.

The singular Hochschild cohomology is also called Tate-Hochschild cohomology in \cite{65,66,67}. The result in \cite{55} shows that the singular Hochschild cohomology can be viewed as an algebraic formalism of Rabinowitz-Floer homology \cite{21} in symplectic geometry.

1.2. \textbf{The main results.} Denote by $\Lambda_0$ the semisimple quotient algebra of $\Lambda$ modulo its Jacobson radical. Recently, Keller proves in \cite{42} that if $\Lambda_0$ is separable over $k$, then there is a natural isomorphism of graded algebras
\begin{equation}
\text{HH}^*_\text{sg}(\Lambda, \Lambda) \cong \text{HH}^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).
\end{equation}
This isomorphism plays a central role in \cite{33}, which proves a weakened version of Donovan-Wemyss’s conjecture \cite{24}.

Denote by $\text{Ho}(B_\infty)$ the homotopy category of $B_\infty$-algebras \cite{40}. In \cite[Conjecture 1.2]{42}, Keller conjectures that there is an isomorphism in $\text{Ho}(B_\infty)$
\begin{equation}
\overline{C}_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \cong C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).
\end{equation}
In particular, we have an induced isomorphism
\[ \text{HH}_*^{\text{sg}}(\Lambda, \Lambda) \xrightarrow{\sim} \text{HH}_*^{\text{sg}}(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)) \]
respecting the Gerstenhaber structures. A slightly stronger version of the conjecture claims that the induced isomorphism above coincides with the natural isomorphism achieved in [42].

Keller’s conjecture indicates that the deformation theory of the dg singularity category is controlled by the singular Hochschild cohomology, where the latter is usually much easier to compute than the Hochschild cohomology of the dg singularity category. For example, in view of the work [12, 26, 42], it would be of interest to study the relationship between the singular Hochschild cohomology and the deformation theory of Landau-Ginzburg models. We mention that Keller’s conjecture is analogous to the isomorphism
\[ C^*(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \cong C^*(D_{\text{dg}}(\Lambda-\text{mod}), D_{\text{dg}}(\Lambda-\text{mod})) \]
for the classical Hochschild cochain complexes; see [40, 50].

We say that an algebra \( \Lambda \) satisfies Keller’s conjecture, provided that there is an isomorphism (1.2) for \( \Lambda \). The following invariance theorem justifies Keller’s conjecture to some extent, as a reasonable conjecture is invariant under reasonable equivalence relations.

**Main Theorem I.** Let \( \Pi \) be another algebra. Assume that \( \Pi \) and \( \Lambda \) are connected by a finite zigzag of one-point (co)extensions and singular equivalences with levels. Then \( \Lambda \) satisfies Keller’s conjecture if and only if so does \( \Pi \).

Recall that a derived equivalence [54] between two algebras naturally induces a singular equivalence with levels. It follows that Keller’s conjecture is invariant under derived equivalences.

We leave some comments on the proof of Main Theorem I (= Theorem 9.4). It is known that both one-point (co)extensions of algebras [18] and singular equivalences with levels [63] induce triangle equivalences between the singularity categories. We observe that these triangle equivalences can be enhanced to quasi-equivalences between the dg singularity categories.

On the other hand, we prove that the singular Hochschild cochain complexes, as \( B_\infty \)-algebras, are invariant under one-point (co)extensions and singular equivalences with levels. For the invariance under singular equivalences with levels, the idea using a triangular matrix algebra is adapted from [40], while our argument is much more involved due to the colimits occurring in the consideration. For example, analogous to the colimit construction [66] of the right singular Hochschild cochain complex, we construct an explicit colimit complex for any \( \Lambda\Pi \)-bimodule \( M \). When \( M \) is projective on both sides, the constructed colimit complex computes the Hom space from \( M \) to \( \Sigma^i(M) \) in the singularity category of \( \Lambda\Pi \)-bimodules.

Let \( Q \) be a finite quiver without sinks. Denote by \( kQ/J^2 \) the corresponding finite dimensional algebra with radical square zero. The second main goal is to verify Keller’s conjecture for \( kQ/J^2 \). However, our approach is indirect, using the Leavitt path algebra \( L(Q) \) over \( k \) in the sense of [1, 6, 7]. We mention close connections of Leavitt path algebras with symbolic dynamic systems [2, 31, 19] and noncommutative geometry [57].

By the work [57, 20, 47], the singularity category of \( kQ/J^2 \) is closely related to the Leavitt path algebra \( L(Q) \). The Leavitt path algebra \( L(Q) \) is infinite dimensional as \( Q \) has
that cochain complex. Using the explicit description [65] of theorem for dg algebras yields an A
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trivial differential throughout this paper.

In particular, there are isomorphisms of Gerstenhaber algebras

$$\hat{\text{C}}_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$

In particular, there are isomorphisms of Gerstenhaber algebras

$$\text{HH}^*(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \rightarrow \text{HH}^*(L(Q), L(Q)) \rightarrow \text{HH}^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$

In Main Theorem II, the isomorphism $\Delta$ between the Hochschild cochain complex of the
Leavitt path algebra $L(Q)$ and the one of the dg singularity category $S_{\text{dg}}(\mathbb{k}Q/J^2)$ enhances
the link [57, 20, 47] between $L(Q)$ and $\mathbb{k}Q/J^2$ to the $B_\infty$ level. The approach to obtain
$\Delta$ is categorical, i.e., it relies on a description of $S_{\text{dg}}(\mathbb{k}Q/J^2)$ via the dg perfect derived
category of $L(Q)$. The isomorphism $\Upsilon$, which is inspired by [65] and is of combinatoric
flavour, establishes a brand new link between $L(Q)$ and $\mathbb{k}Q/J^2$. The primary tool to obtain
$\Upsilon$ is the homotopy transfer theorem [35] for dg algebras.

The composite isomorphism $\Delta \circ \Upsilon$ verifies Keller’s conjecture for the algebra $\mathbb{k}Q/J^2$, which seems to be the first confirmed case. Indeed, combining Main Theorem I and II, we
verify Keller’s conjecture for $\mathbb{k}Q/J^2$ for any finite quiver $Q$ (possibly with sinks).

Let us describe the key steps in the proof of Main Theorem II (= Theorem 9.5).

Using the standard argument for dg quotient categories [38, 25], we prove first that the
dg enhancement of the singularity category via acyclic complexes of injective modules [46]. Then using the explicit compact
generator [47] of the homotopy category of acyclic complexes of injective modules and the
general results in [40] on Hochschild cochain complexes, we infer the isomorphism $\Delta$.

The isomorphism $\Upsilon$ is constructed in a very explicit but indirect manner. The main
ingredients are the (non-strict) $B_\infty$-isomorphism (1.1), two strict $B_\infty$-isomorphisms and an explicit
$B_\infty$-quasi-isomorphism $(\Phi_1, \Phi_2, \ldots)$.

We introduce two new explicit $B_\infty$-algebras, namely the combinatorial $B_\infty$-algebra
$\overline{C}_{\text{sg},R}(Q, Q)$ of $Q$ constructed by parallel paths in $Q$, and the Leavitt $B_\infty$-algebra $\overline{C}^*(L, L)$
whose construction is inspired by an explicit projective bimodule resolution of $L = L(Q)$.

Let $E = kQ_0$ to be the semisimple subalgebra of $\Lambda$. We first observe that
$\overline{C}_{\text{sg},R}^*(\Lambda, \Lambda)$ is strictly $B_\infty$-quasi-isomorphic to $\overline{C}_{\text{sg},R,E}^*(\Lambda, \Lambda)$, the $E$-relative right singular Hochschild
cochain complex. Using the explicit description [65] of $\overline{C}_{\text{sg},R,E}^*(\Lambda, \Lambda)$ via parallel paths in
$Q$, we obtain a strict $B_\infty$-isomorphism between $\overline{C}_{\text{sg},R,E}^*(\Lambda, \Lambda)$ and $\overline{C}_{\text{sg},R}(Q, Q)$. We prove that
$\overline{C}_{\text{sg},R}(Q, Q)$ and $\hat{C}^*(L, L)$ are strictly $B_\infty$-isomorphic.

We construct an explicit homotopy deformation retract between $\hat{C}^*(L, L)$ and $\overline{C}_{E}^*(L, L)$, the normalized $E$-relative Hochschild cochain complex of $L$. Then the homotopy transfer
theorem for dg algebras yields an $A_\infty$-quasi-isomorphism

$$(\Phi_1, \Phi_2, \ldots): \hat{C}^*(L, L) \rightarrow \overline{C}_{E}^*(L, L).$$
This $A_\infty$-morphism is explicitly given by the brace operation of $\hat{C}^*(L, L)$. Using the higher pre-Jacobi identity, we prove that
\[(\Phi_1, \Phi_2, \cdots) : \hat{C}^*(L, L) \longrightarrow C^*_E(L, L)^{opp}\]
is indeed a $B_\infty$-morphism. Since the natural embedding of $C^*_E(L, L)$ into $C^*(L, L)$ is a strict $B_\infty$-quasi-isomorphism, we obtain the required isomorphism $\Upsilon$. The above steps are illustrated in the diagram (9.2) in the proof of Theorem 9.5.

1.3. The structure of the paper. The paper is structured as follows. In Section 2, we review basic facts and results on dg quotient categories. We prove in Subsection 2.2 that both one-point (co)extensions and singular equivalences with levels induce quasi-equivalences between the dg singularity categories of the relevant algebras.

We enhance a result in [46] to the dg level in Section 3. More precisely, we prove that the dg singularity category is essentially the same as the dg category of certain acyclic complexes of injective modules; see Proposition 3.1. The notion of Leavitt path algebras is recalled in Section 4. We prove that there is a zigzag of quasi-equivalences connecting the dg singularity category of $\Lambda = kQ/J^2$ to the dg perfect derived category of the opposite dg algebra $L^{op} = L(Q)^{op}$; see Proposition 4.2. Here, $Q$ is a finite quiver without sinks.

In Section 5, we give a brief introduction to $B_\infty$-algebras. We describe the axioms of $B_\infty$-algebras explicitly. We mainly focus on a special kind of $B_\infty$-algebras, the so-called brace $B_\infty$-algebras, whose underlying $A_\infty$-algebras are dg algebras as well as some of whose $B_\infty$-products vanish. We review some facts on Hochschild cochain complexes of dg categories and (normalized) relative bar resolutions of dg algebras in Section 6.

We recall from [66] the singular Hochschild cochain complexes and their $B_\infty$-structures in Section 7. We describe explicitly the brace operation on the singular Hochschild cochain complex and illustrate it with an example in Subsection 7.3. In Section 8, we prove that the (relative) singular Hochschild cochain complexes, as $B_\infty$-algebras, are invariant under one-point (co)extensions of algebras and singular equivalences with levels.

In Section 9, we prove that Keller’s conjecture is invariant under one-point (co)extensions of algebras and singular equivalences with levels; see Theorem 9.4. We formulate Theorem 9.5 and give a sketch of the proof.

In Section 10, we give a combinatorial description for the singular Hochschild cochain complex of $\Lambda = kQ/J^2$. We introduce the combinatorial $B_\infty$-algebra $\overline{C}^*_{sg,R}(Q, Q)$ of $Q$, which is strictly $B_\infty$-isomorphic to the (relative) singular Hochschild cochain complex of $\Lambda$; see Theorem 10.3. We introduce the Leavitt $B_\infty$-algebra $\hat{C}^*(L, L)$ in Section 11, and show that it is strictly $B_\infty$-isomorphic to $\overline{C}^*_{sg,R}(Q, Q)$, and thus to the (relative) singular Hochschild cochain complex of $\Lambda$; see Proposition 11.4.

Slightly generalizing a result in [32], we provide a general construction of homotopy deformation retracts for dg algebras in Section 12. Using this, we construct an explicit homotopy deformation retract for the bimodule projective resolutions of Leavitt path algebras; see Proposition 12.5. In Section 13, we apply the homotopy transfer theorem [35] for dg algebras to obtain an explicit $A_\infty$-quasi-isomorphism $(\Phi_1, \Phi_2, \cdots)$ from $\hat{C}^*(L, L)$ to $\overline{C}^*_E(L, L)$; see Proposition 13.7. In Section 14, we verify that $(\Phi_1, \Phi_2, \cdots)$ is indeed a $B_\infty$-morphism; see Theorem 14.1.
Appendix A gives a proof of the isomorphism (1.1); see Corollary A.9. This actually follows from a more general result on comparing the opposite $B_\infty$-algebra and the transpose $B_\infty$-algebra of a certain $B_\infty$-algebra; see Theorem A.6. More precisely, motivated by the Kontsevich-Soibelman minimal operad [45], we construct an explicit (non-strict) $B_\infty$-isomorphism between the opposite and the transpose $B_\infty$-algebras; see (A.9).

Throughout this paper, we work over a fixed field $k$. In other words, we require that all the algebras, categories and functors in the sequel are $k$-linear; moreover, the unadorned Hom and tensor are over $k$. We use $1_V$ to denote the identity endomorphism of the graded $k$-vector space $V$. When no confusion arises, we simply write it as $1$.

2. DG CATEGORIES AND DG QUOTIENTS

In this section, we recall basic facts and results on dg categories. The standard references are [37, 25]. We prove that both one-point (co)extensions of algebras and singular equivalences with levels induce quasi-equivalences between dg singularity categories.

For the fixed field $k$, we denote by $k$-$\text{Mod}$ the abelian category of $k$-vector spaces.

2.1. DG CATEGORIES AND DG FUNCTORS. Let $A$ be a dg category over $k$. For two objects $x$ and $y$, the Hom-complex is usually denoted by $A(x,y)$ and its differential is denoted by $d_A$. For a homogeneous morphism $a$, its degree is denoted by $|a|$. Denote by $Z^0(A)$ the ordinary category of $A$, which has the same objects as $A$ and its Hom-space is given by $Z^0(A(x,y))$, the zeroth cocycle of $A(x,y)$. Similarly, the homotopy category $H^0(A)$ has the same objects, but its Hom-space is given by the zeroth cohomology $H^0(A(x,y))$.

Recall that a dg functor $F: A \to B$ is quasi-fully faithful, if the cochain map

$$F_{x,y}: A(x,y) \longrightarrow B(Fx,Fy)$$

is a quasi-isomorphism for any objects $x, y$ in $A$. Then $H^0(F): H^0(A) \to H^0(B)$ is fully faithful. A quasi-fully faithful dg functor $F$ is called a quasi-equivalence if $H^0(F)$ is dense.

Example 2.1. Let $a$ be an additive category. Denote by $C_{dg}(a)$ the dg category of cochain complexes in $a$. A cochain complex in $a$ is usually denoted by $X = (\bigoplus_{p \in \mathbb{Z}} X^p, d_X)$ or $(X, d_X)$. The $p$-th component of the Hom-complex $C_{dg}(a)(X,Y)$ is given by the following infinite product

$$C_{dg}(a)(X,Y)^p = \prod_{n \in \mathbb{Z}} \text{Hom}_a(X^n, Y^{n+p}),$$

whose elements will be denoted by $f = \{f^n\}_{n \in \mathbb{Z}}$ with $f^n \in \text{Hom}_a(X^n, Y^{n+p})$. The differential $d$ acts on $f$ such that $d(f)^n = d_Y^{n+p} \circ f^n - (-1)^{|f|} f^{n+1} \circ d_X^n$ for each $n \in \mathbb{Z}$.

We observe that the homotopy category $H^0(C_{dg}(a))$ coincides with the classical homotopy category $K(a)$ of cochain complexes in $a$.

Example 2.2. The dg category $C_{dg}(k$-$\text{Mod})$ is usually denoted by $C_{dg}(k)$. Let $A$ be a small dg category. By a left dg $A$-module, we mean a dg functor $M: A \to C_{dg}(k)$. The following notation will be convenient: for a morphism $a: x \to y$ in $A$ and $m \in M(x)$, the resulting element $M(a)(m) \in M(y)$ is written as $a.m$. Here, the dot indicates the left $A$-action on $M$. Indeed, we usually identify $M$ with the formal sum $\bigoplus_{x \in \text{obj}(A)} M(x)$ with the above left $A$-action. The differential $d_M$ means $\bigoplus_{x \in \text{obj}(A)} d_M(x)$. 
We denote by $\mathcal{A}$-DGMod the dg category formed by left dg $\mathcal{A}$-modules. For two dg $\mathcal{A}$-modules $M$ and $N$, a morphism $\eta = (\eta_x)_{x \in \text{obj}(\mathcal{A})}: M \to N$ of degree $p$ consists of maps $\eta_x: M(x) \to N(x)$ of degree $p$ satisfying

$$N(a) \circ \eta_x = (-1)^{|a||p|} \eta_y \circ M(a)$$

for each morphism $a: x \to y$ in $\mathcal{A}$. These morphisms form the $p$-th component of $\mathcal{A}$-DGMod$(M,N)$. The differential is defined such that $d(\eta)_x = d(\eta_x)$. Here, $d(\eta_x)$ means the differential in $C_{\text{dg}}(k)$. In other words, $d(\eta_x) = d_{N(x)} \circ \eta_x - (-1)^{|p|} \eta_x \circ d_{M(x)}$.

For a left dg $\mathcal{A}$-module $M$, the suspended dg module $\Sigma(M)$ is defined such that $\Sigma(M)(x) = \Sigma(M(x))$, the suspension of the complex $M(x)$. The left $\mathcal{A}$-action on $\Sigma(M)$ is given such that $a.\Sigma(m) = (-1)^{|a||m|} \Sigma(a.m)$, where $\Sigma(m)$ means the element in $\Sigma(M(x))$ corresponding to $m \in M(x)$. This gives rise to a dg endofunctor $\Sigma$ on $\mathcal{A}$-DGMod, whose action on morphisms $\eta$ is given such that $\Sigma(\eta)_x = (-1)^{|\eta||\eta_x|}$.

**Example 2.3.** Denote by $\mathcal{A}^{\text{op}}$ the opposite dg category of $\mathcal{A}$, whose composition is given by $a \circ^{\text{op}} b = (-1)^{|a||b|} b \circ a$. We identify a left $\mathcal{A}^{\text{op}}$-module with a right dg $\mathcal{A}$-module. Then we obtain the dg category $\text{DGMod-}\mathcal{A}$ of right dg $\mathcal{A}$-modules.

For a right dg $\mathcal{A}$-module $M$, a morphism $a: x \to y$ in $\mathcal{A}$ and $m \in M(y)$, the right $\mathcal{A}$-action on $M$ is given such that $m.a = (-1)^{|a||m|} M(a)(m) \in M(x)$. The suspended dg module $\Sigma(M)$ is defined similarly. We emphasize that the right $\mathcal{A}$-action on $\Sigma(M)$ is identical to the one on $M$.

Let $\mathcal{A}$ be a small dg category. Recall that $H^0(\mathcal{A}\text{-DGMod})$ has a canonical triangulated structure with the suspension functor induced by $\Sigma$. The derived category $\text{D}(\mathcal{A})$ is the Verdier quotient category of $H^0(\mathcal{A}\text{-DGMod})$ by the triangulated subcategory of acyclic dg modules.

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. A triangulated subcategory $\mathcal{N} \subseteq \mathcal{T}$ is localizing if it is closed under arbitrary coproducts. For a set $\mathcal{S}$ of objects, we denote by $\text{Loc}(\mathcal{S})$ the localizing subcategory generated by $\mathcal{S}$, that is, the smallest localizing subcategory containing $\mathcal{S}$.

An object $X$ in $\mathcal{T}$ is compact if $\text{Hom}_\mathcal{T}(X, -): \mathcal{T} \to \text{k-Mod}$ preserves coproducts. Denote by $\mathcal{T}^c$ the full triangulated subcategory formed by compact objects. The category $\mathcal{T}$ is compactly generated, provided that there is a set $\mathcal{S}$ of compact objects such that $\mathcal{T} = \text{Loc}(\mathcal{S})$.

For example, the free dg $\mathcal{A}$-module $\mathcal{A}(x, -)$ is compact in $\text{D}(\mathcal{A})$. Indeed, $\text{D}(\mathcal{A})$ is compactly generated by these modules. The perfect derived category $\text{per}(\mathcal{A}) = \text{D}(\mathcal{A})^c$ is the full subcategory formed by compact objects.

The Yoneda dg functor

$$\text{Y}_\mathcal{A}: \mathcal{A} \to \text{DGMod-}\mathcal{A}, \quad x \mapsto \mathcal{A}(-, x)$$

is fully faithful. In particular, it induces a full embedding

$$H^0(\text{Y}_\mathcal{A}): H^0(\mathcal{A}) \to H^0(\text{DGMod-}\mathcal{A}).$$

The dg category $\mathcal{A}$ is said to be pretriangulated, provided that the essential image of $H^0(\text{Y}_\mathcal{A})$ is a triangulated subcategory of $H^0(\text{DGMod-}\mathcal{A})$. The terminology is justified by the evident fact: the homotopy category $H^0(\mathcal{A})$ of a pretriangulated dg category $\mathcal{A}$ has a canonical triangulated structure.

The following fact is well known; see [17, Lemma 3.1].
Lemma 2.4. Let $F: \mathcal{A} \to \mathcal{B}$ be a dg functor between two pretriangulated dg categories. Then $H^0(F): H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is naturally a triangle functor. Moreover, $F$ is a quasi-equivalence if and only if $H^0(F)$ is a triangle equivalence. □

In this sequel, we will identify quasi-equivalent dg categories. To be more precise, we work in the homotopy category $\text{Hodgcat}$ [58] of small dg categories, which is by definition the localization of $\text{dgcat}$, the category of small dg categories, with respect to quasi-equivalences. The morphisms in $\text{Hodgcat}$ are usually called dg quasi-functors. Any dg quasi-functor from $\mathcal{A}$ to $\mathcal{B}$ can be realized as a roof

$$\mathcal{A} \xleftarrow{F_1} \mathcal{C} \xrightarrow{F_2} \mathcal{B}$$

of dg functors, where $F_1$ is a cofibrant replacement, in particular, it is a quasi-equivalence. Recall that up to quasi-equivalences, every dg category might be identified with its cofibrant replacement; compare [25, Appendix B.5].

Assume that $\mathcal{B} \subseteq \mathcal{A}$ is a full dg subcategory. We denote by $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the dg quotient of $\mathcal{A}$ by $\mathcal{B}$ [38, 25]. Since we work over the field $k$, the simple construction of $\mathcal{A}/\mathcal{B}$ is as follows: the objects of $\mathcal{A}/\mathcal{B}$ are the same as $\mathcal{A}$; we freely add new endomorphisms $\varepsilon_U$ of degree $-1$ for each object $U$ in $\mathcal{B}$, and set $d(\varepsilon_U) = 1_U$. In other words, the added morphism $\varepsilon_U$ is a contracting homotopy for $U$; see [25, Section 3].

The following fact follows immediately from the above simple construction.

Lemma 2.5. Assume that $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ are full dg subcategories. Then there is a canonical quasi-equivalence

$$(\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C}) \sim \to \mathcal{A}/\mathcal{B}. \quad \Box$$

The following fundamental result follows immediately from [25, Theorem 3.4]; compare [51, Theorem 1.3(i) and Lemma 1.5].

Lemma 2.6. Assume that both $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. Then $\mathcal{A}/\mathcal{B}$ is also pretriangulated. Moreover, $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces a triangle equivalence

$$H^0(\mathcal{A})/H^0(\mathcal{B}) \sim \to H^0(\mathcal{A}/\mathcal{B}).$$

Here, $H^0(\mathcal{A})/H^0(\mathcal{B})$ denotes the Verdier quotient category of $H^0(\mathcal{A})$ by $H^0(\mathcal{B})$. □

We will be interested in the following dg quotient categories.

Example 2.7. For a small dg category $\mathcal{A}$, denote by $\mathcal{A}\text{-DGMod}^{\text{ac}}$ the full dg subcategory of $\mathcal{A}\text{-DGMod}$ formed by acyclic modules. We have the dg derived category

$$D_{\text{dg}}(\mathcal{A}) = \mathcal{A}\text{-DGMod}/\mathcal{A}\text{-DGMod}^{\text{ac}}.$$

The terminology is justified by the following fact: there is a canonical identification of $H^0(D_{\text{dg}}(\mathcal{A}))$ with $D(\mathcal{A})$; see Lemma 2.6. Then we have the dg perfect derived category

$$\text{per}_{\text{dg}}(\mathcal{A}) = D_{\text{dg}}(\mathcal{A})^c,$$

which is formed by modules becoming compact in $D(\mathcal{A})$.

Here, we are sloppy about the precise definition of $D_{\text{dg}}(\mathcal{A})$, since neither of the dg categories $\mathcal{A}\text{-DGMod}$ and $\mathcal{A}\text{-DGMod}^{\text{ac}}$ is small. However, by choosing a suitable universe $U$ and restricting to $U$-small dg modules, we can define the corresponding dg derived category $D_{\text{dg},U}(\mathcal{A})$; compare [51, Remark 1.22 and Appendix A]. We then confuse $D_{\text{dg}}(\mathcal{A})$ with the well-defined category $D_{\text{dg},U}(\mathcal{A})$. 

Example 2.8. Let $\Lambda$ be a $k$-algebra, which is a left noetherian ring. Denote by $\Lambda$-mod the abelian category of finitely generated left $\Lambda$-modules. Denote by $C^b_{dg}(\Lambda$-mod) the dg category of bounded complexes, and by $C^{b,ac}_{dg}(\Lambda$-mod) the full dg subcategory formed by acyclic complexes. The bounded dg derived category is defined to be

$$D^b_{dg}(\Lambda$-mod) = C^b_{dg}(\Lambda$-mod)/$C^{b,ac}_{dg}(\Lambda$-mod).$$

Similar as in Example 2.7, we identify $H^0(D^b_{dg}(\Lambda$-mod)) with the usual bounded derived category $D^b(\Lambda$-mod).

Denote by $\text{per}(\Lambda)$ the full subcategory of $D^b(\Lambda$-mod) consisting of perfect complexes. The singularity category [16, 53] of $\Lambda$ is defined to be the following Verdier quotient

$$D_{sg}(\Lambda) = D^b(\Lambda$-mod)/$\text{per}(\Lambda).$$

As its dg analogue, the dg singularity category [42, 12] of $\Lambda$ is given by the following dg quotient category

$$S_{dg}(\Lambda) = D^b_{dg}(\Lambda$-mod)/$\text{per}_{dg}(\Lambda).$$

Here, $\text{per}_{dg}(\Lambda)$ denotes the full dg subcategory of $D^b_{dg}(\Lambda$-mod) formed by perfect complexes. This notation is consistent with the one in Example 2.7, if $\Lambda$ is viewed as a dg category with a single object. By Lemma 2.6, we identify $D_{sg}(\Lambda)$ with $H^0(S_{dg}(\Lambda))$.

2.2. One-point (co)extensions and singular equivalences with levels. In this sub-section, we prove that both one-point (co)extensions [8, III.2] and singular equivalences with levels [63] induce quasi-equivalences between dg singularity categories of the relevant algebras. For simplicity, we only consider finite dimensional algebras and finite dimensional modules.

We first consider a one-point coextension of an algebra. Let $\Lambda$ be a finite dimensional $k$-algebra, and $M$ be a finite dimensional right $\Lambda$-module. We view $M$ as a $k$-$\Lambda$-bimodule on which $k$ acts centrally. The corresponding one-point coextension is an upper triangular matrix algebra

$$\Lambda' = \begin{pmatrix} k & M \\ 0 & \Lambda \end{pmatrix}.$$

As usual, a left $\Lambda'$-module is viewed as a column vector $\begin{pmatrix} V \\ X \end{pmatrix}$, where $V$ is a $k$-vector space and $X$ is a left $\Lambda$-module together with a $k$-linear map $\psi : M \otimes_\Lambda X \to V$; see [8, III.2]. We usually suppress this $\psi$.

The obvious exact functor $j : \Lambda'$-mod $\to \Lambda$-mod sends $\begin{pmatrix} V \\ X \end{pmatrix}$ to $X$. It induces a dg functor $j : D^b_{dg}(\Lambda'$-mod) $\to D^b_{dg}(\Lambda$-mod).

Lemma 2.9. The above dg functor $j$ induces a quasi-equivalence $\overline{j} : S_{dg}(\Lambda') \to S_{dg}(\Lambda)$.

Proof. We observe that the functor $j : \Lambda'$-mod $\to \Lambda$-mod sends projective $\Lambda'$-modules to projective $\Lambda$-modules. It follows that the above dg functor $j$ respects perfect complexes. Therefore, we have the induced dg functor $\overline{j}$ between the dg singularity categories. As in Example 2.8, we identify $H^0(S_{dg}(\Lambda'))$ and $H^0(S_{dg}(\Lambda))$ with $D_{sg}(\Lambda')$ and $S_{sg}(\Lambda)$, respectively.
Then we observe that \( H^0(\tilde{j}) : D_{\text{sg}}(\Lambda) \to D_{\text{sg}}(\Lambda) \) coincides with the triangle equivalence in [18, Proposition 4.2 and its proof]. By Lemma 2.4, we are done. □

Let \( N \) be a finite dimensional left \( \Lambda \)-module. The one-point extension is an upper triangular matrix algebra

\[
\Lambda'' = \begin{pmatrix} \Lambda & N \\ 0 & \mathbb{k} \end{pmatrix}.
\]

Similarly, a left \( \Lambda'' \)-module is denoted by a column vector \( \begin{pmatrix} Y \\ U \end{pmatrix} \), where \( U \) is a \( \mathbb{k} \)-vector space and \( Y \) is a left \( \Lambda \)-module endowed with a left \( \Lambda \)-module morphism \( \phi : N \otimes U \to Y \).

The exact functor \( i : \text{mod-} \Lambda \to \text{mod-} \Lambda'' \) sends a left \( \Lambda \)-module \( Y \) to an evidently-defined \( \Lambda'' \)-module \( \begin{pmatrix} Y \\ 0 \end{pmatrix} \). It induces a dg functor

\[
i : D^b_{\text{dg}}(\text{mod-} \Lambda) \longrightarrow D^b_{\text{dg}}(\text{mod-} \Lambda'').
\]

**Lemma 2.10.** The above dg functor \( i \) induces a quasi-equivalence \( \tilde{i} : S_{\text{dg}}(\Lambda) \xrightarrow{\sim} S_{\text{dg}}(\Lambda'') \).

**Proof.** The argument here is similar to the one in the proof of Lemma 2.9. As the functor \( i : \text{mod-} \Lambda \to \text{mod-} \Lambda'' \) sends projective \( \Lambda \)-modules to projective \( \Lambda'' \)-modules, the above dg functor \( i \) respects perfect complexes. Therefore, we have the induced dg functor \( \tilde{i} \) between the dg singularity categories. We observe that \( H^0(\tilde{i}) : D_{\text{sg}}(\Lambda) \to D_{\text{sg}}(\Lambda'') \) coincides with the triangle equivalence in [18, Proposition 4.1 and its proof]. Then we are done by applying Lemma 2.4. □

Let \( \Lambda \) and \( \Pi \) be two finite dimensional \( \mathbb{k} \)-algebras. For a \( \Lambda \)-\( \Pi \)-bimodule, we always require that \( \mathbb{k} \) acts centrally. Therefore, a \( \Lambda \)-\( \Pi \)-bimodule might be identified with a left module over \( \Lambda \otimes \Pi^{\text{op}} \).

Denote by \( \Lambda^e = \Lambda \otimes \Lambda^{\text{op}} \) the enveloping algebra of \( \Lambda \). Therefore, \( \Lambda \)-\( \Pi \)-bimodules are viewed as left \( \Lambda^e \)-modules. Denote by \( \Lambda^e\text{-mod} \) the stable \( \Lambda^e \)-module category modulo projective \( \Lambda^e \)-modules [8, IV.1], and by \( \Omega^n_{\Lambda^e}(\Lambda) \) the \( n \)-th syzygy of \( \Lambda \) for \( n \geq 1 \). By convention, we have \( \Omega^0_{\Lambda^e}(\Lambda) = \Lambda \).

The following terminology is modified from [63, Definition 2.1].

**Definition 2.11.** Let \( M \) and \( N \) be a \( \Lambda \)-\( \Pi \)-bimodule and a \( \Pi \)-\( \Lambda \)-bimodule, respectively, and let \( n \geq 0 \). We say that the pair \((M, N)\) defines a singular equivalence with level \( n \), provided that the following conditions are fulfilled.

1. The four one-sided modules \( \Lambda M, M \Pi, \Pi N \) and \( N \Lambda \) are all projective.
2. There are isomorphisms \( M \otimes_{\Pi} N \cong \Omega^n_{\Lambda^e}(\Lambda) \) and \( N \otimes_{\Lambda} M \cong \Omega^n_{\Pi^{\text{op}}}(\Pi) \) in \( \Lambda^e\text{-mod} \) and \( \Pi^{\text{op}}\text{-mod} \), respectively. □

**Remark 2.12.**

1. A stable equivalence of Morita type in the sense of [14, Definition 5.5A] is naturally a singular equivalence with level zero.
2. By [63, Theorem 2.3], a derived equivalence induces a singular equivalence with a certain level.
3. By [56, Proposition 2.6], a singular equivalence of Morita type, studied in [68], induces a singular equivalence with a certain level.
Assume that $M$ is a $\Lambda$-$\Pi$-bimodule such that both $\Lambda M$ and $M\Pi$ are projective. The obvious dg functor $M \otimes_{\Pi} - : D^b_{dg}(\Pi\text{-mod}) \to D^b_{dg}(\Lambda\text{-mod})$ between the bounded dg derived categories preserves perfect complexes. Hence it induces a dg functor

$$M \otimes_{\Pi} - : S_{dg}(\Pi) \to S_{dg}(\Lambda)$$

between the dg singularity categories.

Definition 2.11 is justified by the following observation.

Lemma 2.13. Assume that $(M,N)$ defines a singular equivalence with level $n$. Then the above dg functor $M \otimes_{\Pi} - : S_{dg}(\Pi) \to S_{dg}(\Lambda)$ is a quasi-equivalence.

Proof. We identify $H^0(S_{dg}(\Pi))$ with $D_{sg}(\Pi)$, and $H^0(S_{dg}(\Lambda))$ with $D_{sg}(\Lambda)$; see Example 2.8. Then $H^0(M \otimes_{\Pi} -)$ is identified with the obvious tensor functor $M \otimes_{\Pi} - : D_{sg}(\Pi) \to D_{sg}(\Lambda)$.

As noted in [63, Remark 2.2], the latter functor is a triangle equivalence, whose quasi-inverse is given by $\Sigma^n \circ (N \otimes_{\Lambda} -)$. Then we are done by Lemma 2.4. □

3. The dg singularity category and acyclic complexes

In this section, we enhance a result in [46] to show that the dg singularity category can be described as the dg category of certain acyclic complexes of injective modules.

We fix a $k$-algebra $\Lambda$, which is a left noetherian ring. We denote by $\Lambda\text{-Mod}$ the abelian category of left $\Lambda$-modules. For two complexes $X$ and $Y$ of $\Lambda$-modules, the Hom complex $C_{dg}(\Lambda\text{-Mod})(X,Y)$ is usually denoted by $\text{Hom}_{\Lambda}(X,Y)$. Recall that the classical homotopy category $K(\Lambda\text{-Mod})$ coincides with $H^0(C_{dg}(\Lambda\text{-Mod}))$.

Denote by $\Lambda\text{-Inj}$ the category of injective $\Lambda$-modules, and by $K(\Lambda\text{-Inj})$ the homotopy category of complexes of injective modules. The full subcategory $K_{ac}(\Lambda\text{-Inj})$ is formed by acyclic complexes of injective modules.

For a bounded complex $X$ of $\Lambda$-modules, we denote by $\phi_X : X \to iX$ its injective resolution. Then we have the following isomorphism

$$\text{Hom}_{K(\Lambda\text{-Inj})}(iX,I) \simeq \text{Hom}_{K(\Lambda\text{-Mod})}(X,I), \quad f \mapsto f \circ \phi_X, \quad (3.1)$$

for each complex $f \in K(\Lambda\text{-Inj})$. It follows that $iX$ is compact in $K(\Lambda\text{-Inj})$, if $X$ lies in $K^b(\Lambda\text{-mod})$; see [46, Lemma 2.1]. In particular, we have

$$\text{Hom}_{K(\Lambda\text{-Inj})}(i\Lambda,I) \simeq \text{Hom}_{K(\Lambda\text{-Mod})}(\Lambda,I) \simeq H^0(I). \quad (3.2)$$

Here, we view the regular module $\Lambda$ as a stalk complex concentrated in degree zero. We denote by $\text{Loc}(i\Lambda)$ the localizing subcategory of $K(\Lambda\text{-Inj})$ generated by $i\Lambda$.

Denote by $C_{ac}^{dg}(\Lambda\text{-Inj})$ the full dg subcategory of $C_{dg}(\Lambda\text{-Mod})$ formed by acyclic complexes of injective $\Lambda$-modules. We identify $H^0(C_{ac}^{dg}(\Lambda\text{-Inj}))$ with $K_{ac}(\Lambda\text{-Inj})$. Then $C_{dg}^{ac}(\Lambda\text{-Inj})^c$ means the full dg subcategory formed by complexes which become compact in $K_{ac}(\Lambda\text{-Inj})$.

The following result enhances [46, Corollary 5.4] to the dg level.

Proposition 3.1. There is a dg quasi-functor

$$\Phi : S_{dg}(\Lambda) \to C_{dg}^{ac}(\Lambda\text{-Inj})^c,$$

such that

$$H^0(\Phi) : D_{sg}(\Lambda) \to K_{ac}(\Lambda\text{-Inj})^c.$$

is a triangle equivalence up to direct summands.

The following immediate consequence will be useful.

**Corollary 3.2.** Assume that the $k$-algebra $\Lambda$ is finite dimensional. Then there is a zigzag of quasi-equivalences connecting $S_{dg}(\Lambda)$ to $C_{dg}^{ac}(\Lambda-\text{Inj})^c$.

**Proof.** By [18, Corollary 2.4], the singularity category $D_{sg}(\Lambda)$ has split idempotents. It follows that $H^0(\Phi)$ is actually a triangle equivalence. In view of Lemma 2.4, the required result follows immediately. □

Let $\mathcal{T}$ be a triangulated category. For a triangulated subcategory $\mathcal{N}$, we have the right orthogonal subcategory $\mathcal{N}^\perp = \{ X \in \mathcal{T} | \text{Hom}_{\mathcal{T}}(N, X) = 0 \text{ for all } N \in \mathcal{N} \}$ and the left orthogonal subcategory $\perp \mathcal{N} = \{ Y \in \mathcal{T} | \text{Hom}_{\mathcal{T}}(Y, N) = 0 \text{ for all } N \in \mathcal{N} \}$. The subcategory $\mathcal{N}$ is right admissible (resp. left admissible) provided that the inclusion $\mathcal{N} \hookrightarrow \mathcal{T}$ has a right adjoint (resp. left adjoint); see [13].

The following lemma is well known; see [13, Lemma 3.1].

**Lemma 3.3.** Let $\mathcal{N} \subseteq \mathcal{T}$ be left admissible. Then the natural functor $\mathcal{N} \to \mathcal{T} / \perp \mathcal{N}$ is an equivalence. Moreover, the left orthogonal subcategory $\perp \mathcal{N}$ is right admissible satisfying $\mathcal{N} = (\perp \mathcal{N})^\perp$. □

Denote by $L$ the full dg subcategory of $C_{dg}(\Lambda-\text{Mod})$ consisting of those complexes $X$ such that $\text{Hom}_\Lambda(X, I)$ is acyclic for each $I \in C_{dg}(\Lambda-\text{Inj})$. Similarly, denote by $M$ the full dg subcategory formed by $Y$ satisfying that $\text{Hom}_\Lambda(Y, J)$ is acyclic for each $J \in C_{dg}^{ac}(\Lambda-\text{Inj})$.

**Lemma 3.4.** The following canonical functors are all equivalences

1. $K(\Lambda-\text{Inj}) \sim \to K(\Lambda-\text{Mod})/H^0(L)$;
2. $K^{ac}(\Lambda-\text{Inj}) \sim \to K(\Lambda-\text{Mod})/H^0(M)$;
3. $K^{ac}(\Lambda-\text{Inj}) \sim \to K(\Lambda-\text{Inj})/\text{Loc}(i\Lambda)$;
4. $K(\Lambda-\text{Inj})/\text{Loc}(i\Lambda) \sim \to K(\Lambda-\text{Mod})/H^0(M)$,

which send any complex $I$ to itself, viewed as an object in the target categories.

**Proof.** The Brown representability theorem and its dual version yield the following useful fact: for a triangulated category $\mathcal{T}$ with arbitrary coproducts and a localizing subcategory $\mathcal{N}$ which is compactly generated, then the subcategory $\mathcal{N}$ is right admissible; if furthermore $\mathcal{N}$ is closed under products, then $\mathcal{N}$ is also left admissible; see [46, Proposition 3.3].

Recall from [46, Proposition 2.3 and Corollary 5.4] that both $K(\Lambda-\text{Inj})$ and $K^{ac}(\Lambda-\text{Inj})$ are compactly generated, which are both closed under coproducts and products in $K(\Lambda-\text{Mod})$. Moreover, we observe that $\perp K(\Lambda-\text{Inj}) = H^0(L)$ and $\perp K^{ac}(\Lambda-\text{Inj}) = H^0(M)$, where the orthogonal is taken in $K(\Lambda-\text{Mod})$. Then the above fact and Lemma 3.3 yield (1) and (2).

By the isomorphism (3.2), we infer that $K^{ac}(\Lambda-\text{Inj}) = \text{Loc}(i\Lambda)^\perp$, where the orthogonal is taken in $K(\Lambda-\text{Inj})$. Since $i\Lambda$ is compact in $K(\Lambda-\text{Inj})$, the subcategory $\text{Loc}(i\Lambda)$ is right admissible. It follows from the dual version of Lemma 3.3 that $K^{ac}(\Lambda-\text{Inj}) \subseteq K(\Lambda-\text{Inj})$ is left admissible satisfying $\perp K^{ac}(\Lambda-\text{Inj}) = \text{Loc}(i\Lambda)$. Then (3) follows from Lemma 3.3.

The functor in (4) is well defined, since $\text{Loc}(i\Lambda) \subseteq H^0(M)$. Then (4) follows by combining (2) and (3). □
Denote by $\mathcal{P}$ the full dg subcategory of $C^b_{\text{dg}}(\Lambda\text{-mod})$ formed by those complexes which are isomorphic to bounded complexes of projective $\Lambda$-modules in $D^b(\Lambda\text{-mod})$. Therefore, we might identify the singularity category $D^b_{\text{sg}}(\Lambda)$ with $K^b(\Lambda\text{-mod})/\text{H}^0(\mathcal{P})$.

**Lemma 3.5.** The canonical functor $K^b(\Lambda\text{-mod})/\text{H}^0(\mathcal{P}) \rightarrow K(\Lambda\text{-Mod})/\text{H}^0(\mathcal{M})$ is fully faithful, which induces a triangle equivalence up to direct summands

$$K^b(\Lambda\text{-mod})/\text{H}^0(\mathcal{P}) \xrightarrow{\sim} (K(\Lambda\text{-Mod})/\text{H}^0(\mathcal{M}))^c.$$  

**Proof.** The functor is well defined since we have $\mathcal{P} \subseteq \mathcal{M}$. The assignment $X \mapsto iX$ of injective resolutions yields a triangle functor $i: K(\Lambda\text{-mod}) \rightarrow K(\Lambda\text{-Inj})$. It induces the following horizontal functor.

$$K^b(\Lambda\text{-mod})/\text{H}^0(\mathcal{P}) \xrightarrow{i} K(\Lambda\text{-Inj})/\text{Loc}(i\Lambda) \xrightarrow{\sim} (K(\Lambda\text{-Inj})/\text{Loc}(i\Lambda))^c.$$  

The unnamed arrows are canonical functors. By [46, Corollary 5.4] the horizontal functor $i$ induces a triangle equivalence up to direct summands

$$K^b(\Lambda\text{-mod})/\text{H}^0(\mathcal{P}) \xrightarrow{\sim} (K(\Lambda\text{-Inj})/\text{Loc}(i\Lambda))^c.$$  

We claim that the diagram is commutative up to a natural isomorphism. Then we are done by Lemma 3.4(4).

For the claim, we take $X \in K^b(\Lambda\text{-mod})$ and consider its injective resolution $\phi_X: X \rightarrow iX$. We have the exact triangle

$$X \xrightarrow{\phi_X} iX \rightarrow \text{Cone}(\phi_X) \rightarrow \Sigma(X).$$  

The isomorphism (3.1) implies that $\text{Cone}(\phi_X)$ lies in $H^0(\mathcal{L}) \subseteq H^0(\mathcal{M})$. Therefore, $\phi_X$ becomes an isomorphism in $K(\Lambda\text{-Mod})/\text{H}^0(\mathcal{M})$, proving the claim. $\square$

We are now in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** By the equivalence in Lemma 3.4(2), the canonical dg functor

$$C^\text{ac}_{\text{dg}}(\Lambda\text{-Inj}) \rightarrow C(\Lambda\text{-Mod})/\mathcal{M}$$  

is a quasi-equivalence, which restricts to a quasi-equivalence on compact objects

$$C^\text{ac}_{\text{dg}}(\Lambda\text{-Inj})^c \rightarrow (C(\Lambda\text{-Mod})/\mathcal{M})^c.$$  

Here, for the precise definition of the dg quotient category $C(\Lambda\text{-Mod})/\mathcal{M}$, we have to consult [51, Remark 1.22]; compare Example 2.7.

By Lemma 2.5, we may identify $S_{\text{dg}}(\Lambda)$ with $C^b_{\text{dg}}(\Lambda\text{-mod})/\mathcal{P}$. By Lemma 3.5, the following canonical dg functor

$$C^b_{\text{dg}}(\Lambda\text{-mod})/\mathcal{P} \rightarrow (C_{\text{dg}}(\Lambda\text{-Mod})/\mathcal{M})^c$$  

is quasi-fully faithful, which induces a triangle equivalence up to direct summands between the homotopy categories. Combining them, we obtain the required dg quasi-functor. $\square$
4. Quivers and Leavitt path algebras

In this section, we recall basic facts on quivers and Leavitt path algebras. Using the main result in [47], we relate the dg singularity category of the finite dimensional algebra with radical square zero to the dg perfect derived category of the Leavitt path algebra. We obtain an explicit graded derivation over the Leavitt path algebra, which will be used in Subsection 12.2.

Recall that a quiver $Q = (Q_0, Q_1; s, t)$ consists of a set $Q_0$ of vertices, a set $Q_1$ of arrows and two maps $s, t: Q_1 \to Q_0$, which associate to each arrow $\alpha$ its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$, respectively. A vertex $i$ of $Q$ is a sink provided that the set $s^{-1}(i)$ is empty.

A path of length $n$ is a sequence $p = \alpha_n \cdots \alpha_2 \alpha_1$ of arrows with $t(\alpha_j) = s(\alpha_{j+1})$ for $1 \leq j \leq n - 1$. Denote by $l(p) = n$. The starting vertex of $p$, denoted by $s(p)$, is $s(\alpha_1)$ and the terminating vertex of $p$, denoted by $t(p)$, is $t(\alpha_n)$. We identify an arrow with a path of length one. We associate to each vertex $i \in Q_0$ a trivial path $e_i$ of length zero. Set $s(e_i) = i = t(e_i)$. Denote by $Q_i$ the set of paths of length $n$.

The path algebra $kQ = \bigoplus_{n \geq 0} kQ_n$ has a basis given by all paths in $Q$, whose multiplication is given as follows: for two paths $p$ and $q$ satisfying $s(p) = t(q)$, the product $pq$ is their concatenation; otherwise, we set the product $pq$ to be zero. Here, we write the concatenation of paths from right to left. For example, $e_t(p)p = p = pe_{s(p)}$ for each path $p$. Denote by $J = \bigoplus_{n \geq 1} kQ_n$ the two-sided ideal generated by arrows.

We denote by $\overline{Q}$ the double quiver of $Q$, which is obtained by adding for each arrow $\alpha \in Q_1$ a new arrow $\alpha^*$ in the opposite direction. Clearly, we have $s(\alpha^*) = t(\alpha)$ and $t(\alpha^*) = s(\alpha)$. The added arrows $\alpha^*$ are called the ghost arrows.

In what follows, we assume that $Q$ is a finite quiver without sinks. We set $\Lambda = kQ/J^2$ to be the corresponding finite dimensional algebra with radical square zero. Observe that $J^2$ is the two-sided ideal of $kQ$ generated by the set of all paths of length two.

The Leavitt path algebra $L = L(Q)$ [1, 6, 7] is by definition the quotient algebra of $k\overline{Q}$ modulo the two-sided ideal generated by the following set

\[ \{ \alpha\beta^* - \delta_{\alpha,\beta} e_{t(\alpha)} | \alpha, \beta \in Q_1 \text{ with } s(\alpha) = s(\beta) \} \cup \{ \sum_{\{\alpha \in Q_1 | s(\alpha) = i\}} \alpha^* \alpha - e_i | i \in Q_0 \}. \]

These elements are known as the first Cuntz-Krieger relations and the second Cuntz-Krieger relations, respectively.

If $p = \alpha_n \cdots \alpha_2 \alpha_1$ is a path in $Q$ of length $n \geq 1$, we define $p^* = \alpha^*_n \alpha^*_n \cdots \alpha^*_1$. We have $s(p^*) = t(p)$ and $t(p^*) = s(p)$. For convention, we set $e_i^* = e_i$. We observe that for paths $p, q$ in $Q$ satisfying $t(p) \neq t(q)$, $p^*q = 0$ in $L$. Recall that the Leavitt path algebra $L$ is spanned by the following set

\[ \{ e_i, p, p^*, \gamma^* \eta | i \in Q_0, p, \gamma, \text{ and } \eta \text{ are nontrivial paths in } Q \text{ with } t(\gamma) = t(\eta) \}; \]

see [61, Corollary 3.2]. In general, this set is not $k$-linearly independent. For an explicit basis, we refer to [3, Theorem 1].

The Leavitt path algebra $L$ is naturally $\mathbb{Z}$-graded by $|e_i| = 0$, $|\alpha| = 1$ and $|\alpha^*| = -1$ for $i \in Q_0$ and $\alpha \in Q_1$. We write $L = \bigoplus_{n \in \mathbb{Z}} L^n$, where $L^n$ consists of homogeneous elements of degree $n$. 
For each $i \in Q_0$ and $m \geq 0$, we consider the following subspace of $e_i Le_i$

$$X_{i,m} = \text{Span}_k \{ \gamma^* \eta \mid t(\gamma) = t(\eta), s(\gamma) = i = s(\eta), l(\eta) = m \}.$$ 

We observe that $X_{i,m} \subseteq X_{i,m+1}$, since we have

$$\gamma^* \eta = \sum_{\{ \alpha \in Q_1 \mid s(\alpha) = t(\eta) \}} (\alpha \gamma)^* \alpha \eta. \quad (4.1)$$

**Lemma 4.1.** The following facts hold.

1. The set $\{ \gamma^* \eta \mid t(\gamma) = t(\eta), s(\gamma) = i = s(\eta), l(\eta) = m \}$ is $k$-linearly independent.
2. We have $e_i Le_i = \bigcup_{m \geq 0} X_{i,m}$.

**Proof.** Using the grading of $L$, the first statement follows from [19, Proposition 4.1]. The second one is trivial. \qed

The following result is based on the main result of [47]. We will always view the $Z$-graded algebra $L = L(Q)$ as a dg algebra with trivial differential. Then $L^{\text{op}}$ denotes the opposite dg algebra. We view $\Lambda = kQ/J^2$ as a dg algebra concentrated in degree zero.

**Proposition 4.2.** Keep the notation as above. Then there is a zigzag of quasi-equivalences connecting $S_{dg}(\Lambda)$ to $\text{per}_{dg}(L^{\text{op}})$.

**Proof.** Recall that the injective Leavitt complex $I$ is constructed in [47], which is a dg $\Lambda$-$L^{\text{op}}$-bimodule. Moreover, it induces a triangle equivalence

$$\text{Hom}_\Lambda(I, -): K^{ac}(\Lambda-\text{Inj}) \xrightarrow{\sim} D(L^{\text{op}}),$$

which restricts to an equivalence

$$K^{ac}(\Lambda-\text{Inj})^c \xrightarrow{\sim} \text{per}(L^{\text{op}}).$$

Recall the identifications $H^0(C^{ac}_{dg}(\Lambda-\text{Inj})^c) = K^{ac}(\Lambda-\text{Inj})^c$ and $H^0(\text{per}_{dg}(L^{\text{op}})) = \text{per}(L^{\text{op}})$. Then combining the above restricted equivalence and Lemma 2.4, we infer that the dg functor

$$\text{Hom}_\Lambda(I, -): C^{ac}_{dg}(\Lambda-\text{Inj})^c \rightarrow \text{per}_{dg}(L^{\text{op}})$$

is a quasi-equivalence. Then we are done by Corollary 3.2. \qed

Set $E = kQ_0 = \bigoplus_{i \in Q_0} k e_i$, which is viewed as a semisimple subalgebra of $L^0$. Let $M$ be a graded $L$-$L$-bimodule. A graded map $D: L \rightarrow M$ of degree $-1$ is called a graded derivation provided that it satisfies the graded Leibniz rule

$$D(xy) = D(x)y + (-1)^{|x|}x D(y)$$

for $x, y \in L$; if furthermore it satisfies $D(e_i) = 0$ for each $i \in Q_0$, it is called a graded $E$-derivation.

Let $sk$ be the 1-shifted space of $k$, that is, $sk$ is concentrated in degree $-1$. The element $s1_k$ of degree $-1$ will be simply denoted by $s$. Then we have the graded $L$-$L$-bimodule $\bigoplus_{i \in Q_0} Le_i \otimes sk \otimes e_i L$, which is clearly isomorphic to $L \otimes_E sE \otimes_E L$.

**Lemma 4.3.** Keep the notation as above. Then there is a unique graded $E$-derivation

$$D: L \rightarrow \bigoplus_{i \in Q_0} Le_i \otimes sk \otimes e_i L$$

satisfying $D(\alpha) = -\alpha \otimes s \otimes e_{s(\alpha)}$ and $D(\alpha^*) = -e_{s(\alpha)} \otimes s \otimes \alpha^*$ for each $\alpha \in Q_1$. 


Definition 5.1. An endomorphism where the sign \( f \) between graded spaces, the tensor product \( f \) morphisms. For details, we refer to [39]. For two graded maps

\[ D: kQ \rightarrow \bigoplus_{i \in Q_0} L e_i \otimes s k \otimes e_i L \]

satisfying \( D(\alpha) = -\alpha \otimes s \otimes e_{s(\alpha)} \) and \( D(\alpha^*) = -e_{s(\alpha)} \otimes s \otimes \alpha^* \); consult the explicit bimodule projective resolution in [22, Chapter 2, Proposition 2.6]. It is routine to verify that \( D \) vanishes on the Cuntz-Krieger relations. Therefore, by the graded Leibniz rule, it vanishes on the whole defining ideal. Then \( D \) induces uniquely the required derivation \( D \). \( \square \)

The following observation will be useful in the proof of Proposition 13.7.

Remark 4.4. By the graded Leibniz rule, the graded \( E \)-derivation \( D \) has the following explicit description: for nontrivial paths \( \eta = \alpha_m \cdots \alpha_2 \alpha_1 \) and \( \gamma = \beta_p \cdots \beta_2 \beta_1 \) satisfying \( t(\eta) = t(\gamma) \), we have

\[
D(\gamma^* \eta) = -e_{s(\gamma)} \otimes s \otimes \gamma^* \eta - \sum_{p=1}^{m-1} (-1)^{l} \beta_1^* \cdots \beta_l^* \otimes s \otimes \beta_{l+1}^* \cdots \beta_p^* \alpha_m \cdots \alpha_1 + \sum_{l=1}^{m-1} (-1)^{m+p-l} \beta_1^* \cdots \beta_l^* \alpha_m \cdots \alpha_{l+1} \otimes s \otimes \alpha_l \cdots \alpha_1 + (-1)^{m+p} \gamma^* \eta \otimes s \otimes e_{s(\eta)}.
\]

Similarly, we have

\[
D(\gamma^*) = -e_{s(\gamma)} \otimes s \otimes \gamma^* - \sum_{l=1}^{m-1} (-1)^{l} \beta_1^* \cdots \beta_l^* \otimes s \otimes \beta_{l+1}^* \cdots \beta_p^*, \text{ and}
\]

\[
D(\eta) = \sum_{l=1}^{m-1} (-1)^{m-l} \alpha_m \cdots \alpha_{l+1} \otimes s \otimes \alpha_l \cdots \alpha_1 + (-1)^{m} \gamma \otimes s \otimes e_{s(\eta)}.
\]

5. A brief introduction to \( B_\infty \)-algebras

In this section, we give a brief self-contained introduction to \( B_\infty \)-algebras and \( B_\infty \)-morphisms. We are mainly interested in a class of \( B_\infty \)-algebras, called brace \( B_\infty \)-algebras, whose underlying \( A_\infty \)-algebras are dg algebras and some of whose \( B_\infty \)-products vanish.

5.1. \( A_\infty \)-algebras and morphisms. Let us start by recalling \( A_\infty \)-algebras and \( A_\infty \)-morphisms. For details, we refer to [39]. For two graded maps \( f: U \rightarrow V \) and \( f': U' \rightarrow V' \) between graded spaces, the tensor product \( f \otimes f': U \otimes U' \rightarrow V \otimes V' \) is defined such that

\[
(f \otimes f')(u \otimes u') = (-1)^{|f'|} f(u) \otimes f'(u'),
\]

where the sign \( (-1)^{|f'|} \) is given by the Koszul sign rule. We use \( 1 \) to denote the identity endomorphism.

Definition 5.1. An \( A_\infty \)-algebra is a graded \( \mathbb{k} \)-vector space \( A = \bigoplus_{p \in \mathbb{Z}} A^p \) endowed with graded \( \mathbb{k} \)-linear maps

\[
m_n: A^{\otimes n} \rightarrow A, \quad n \geq 1,
\]
of degree $2 - n$ satisfying the following relations

$$
\sum_{j=0}^{n-1} \sum_{s=1}^{n-j} (-1)^{j+s(n-j-s)} m_{n-s+1}(1 \otimes m_s \otimes 1 \otimes (n-j-s)) = 0, \quad \text{for } n \geq 1. \quad (5.1)
$$

In particular, $(A, m_1)$ is a cochain complex of $k$-vector spaces.

For two $A_\infty$-algebras $A$ and $A'$, an $A_\infty$-morphism $f = (f_n)_{n \geq 1} : A \to A'$ is given by a collection of graded maps $f_n : A^{\otimes n} \to A'$ of degree $1 - n$ such that, for all $n \geq 1$, we have

$$
\sum_{a+s+t=n \atop a,t \geq 0, s \geq 1} (-1)^{a+s} f_{a+1+t} (1 \otimes a \otimes m_s \otimes 1 \otimes t) = \sum_{r \geq 1 \atop i_1 + \cdots + i_r = n} (-1)^{r} m'_r (f_{i_1} \otimes \cdots \otimes f_{i_r}), \quad (5.2)
$$

where $\epsilon = (r - 1)(i_1 - 1) + (r - 2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1)$; if $r = 1$, we set $\epsilon = 0$. In particular, $f_1 : (A, m_1) \to (A', m'_1)$ is a cochain map.

The composition $g \circ_{\infty} f$ of two $A_\infty$-morphisms $f : A \to A'$ and $g : A' \to A''$ is given by

$$(g \circ_{\infty} f)_n = \sum_{r \geq 1 \atop i_1 + \cdots + i_r = n} (-1)^{r} g_{i_1} (f_{i_1} \otimes \cdots \otimes f_{i_r}), \quad n \geq 1,$$

where $\epsilon$ is defined as above.

An $A_\infty$-morphism $f : A \to A'$ is strict provided that $f_i = 0$ for all $i \neq 1$. The identity morphism is the strict morphism $f$ given by $f_1 = 1_A$. An $A_\infty$-morphism $f : A \to A'$ is an $A_\infty$-isomorphism if there exists an $A_\infty$-morphism $g : A' \to A$ such that the composition $f \circ_{\infty} g$ coincides with the identity morphism of $A'$ and $g \circ_{\infty} f$ coincides with the identity morphism of $A$. In general, an $A_\infty$-isomorphism is not necessarily strict; see Theorem A.6 for an example.

An $A_\infty$-morphism $f : A \to A'$ is called an $A_\infty$-quasi-isomorphism provided that $f_1 : (A, m_1) \to (A', m'_1)$ is a quasi-isomorphism between the underlying complexes. An $A_\infty$-isomorphism is necessarily an $A_\infty$-quasi-isomorphism.

**Remark 5.2.** Let $A$ be a graded $k$-space and let $sA$ be the 1-shifted graded space: $(sA)^i = A^{i+1}$. Denote by $(T^c(sA), \Delta)$ the tensor coalgebra over $sA$. It is well known that an $A_\infty$-algebra structure on $A$ is equivalent to a dg coalgebra structure $(T^c(sA), \Delta, D)$ on $T^c(sA)$, where $D$ is a coderivation of degree one satisfying $D^2 = 0$ and $D(1) = 0$. Accordingly, $A_\infty$-morphisms $f : A \to A'$ correspond bijectively to dg coalgebra homomorphisms $T^c(sA) \to T^c(sA')$. Under this bijection, the above composition $f \circ_{\infty} g$ of the $A_\infty$-morphisms $f$ and $g$ corresponds to the usual composition of the induced dg coalgebra homomorphisms; see [39, Lemma 3.6].

We mention that any dg algebra $A$ is viewed as an $A_\infty$-algebra with $m_n = 0$ for $n \geq 3$. In Subsection 13.2, we will construct an explicit $A_\infty$-quasi-isomorphism between two concrete dg algebras, which is a non-strict $A_\infty$-morphism, that is, not a dg algebra homomorphism between the dg algebras.

### 5.2. $B_\infty$-algebras and morphisms

The notion of $B_\infty$-algebras\(^1\) is due to [30, Subsection 5.2]. We unpack the definition therein and write the axioms explicitly. We are mainly

\(^1\)We remark that the letter ‘B’ stands for Baues, who showed in [10] that the normalized cochain complex $C^*(X)$ of any simplicial set $X$ carries a natural $B_\infty$-algebra.
concerned with a certain kind of $B_\infty$-algebras, called brace $B_\infty$-algebras; see Definition 5.6. We mention other references [62, 40] for $B_\infty$-algebras.

Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a graded space, and let $r \geq 1$ and $l, n \geq 0$. For any two sequences of nonnegative integers $(l_1, l_2, \ldots, l_r)$ and $(n_1, n_2, \ldots, n_r)$ satisfying $l = l_1 + \cdots + l_r$ and $n = n_1 + \cdots + n_r$, we define a $k$-linear map

$$\tau_{(l_1, \ldots, l_r; n_1, \ldots, n_r)} : A^{\otimes l_1} \otimes A^{\otimes n_1} \to (A^{\otimes l_1} \otimes \cdots \otimes A^{\otimes l_r}) \otimes (A^{\otimes n_1} \otimes \cdots \otimes A^{\otimes n_r})$$

by sending $(a_1 \otimes \cdots \otimes a_l) \otimes (b_1 \otimes \cdots \otimes b_n) \in A^{\otimes l} \otimes A^{\otimes n}$ to

$$(-1)^{\epsilon}(a_1 \otimes \cdots \otimes a_l \otimes b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes (a_{l_1 + \cdots + l_{r-1} + 1} \otimes \cdots \otimes a_l \otimes b_{l_1 + \cdots + n_{r-1} + 1} \otimes \cdots \otimes b_n),$$

where $\epsilon = \sum_{i=0}^{r-2} (|b_{n_1+\cdots+n_{i+1}}| + \cdots + |b_{n_1+\cdots+n_{r-1}+1}|)(|n_1+\cdots+n_{i+1}| + \cdots + |a_i|)$ with $n_0 = 0$.

If $l_i = 0$ for some $1 \leq i \leq r$ we set $A^{\otimes k_1} = k$ and $a_{l_1+\cdots+l_{i-1}+1} \otimes \cdots \otimes a_{l_1+\cdots+l_i} = 1 \in k$; similarly, if $n_i = 0$ we set $A^{\otimes n_i} = k$ and $b_{n_1+\cdots+n_{i-1}+1} \otimes \cdots \otimes b_{n_1+\cdots+n_i} = 1 \in k$. Here and later, we use the big tensor product $\otimes$ to distinguish from the usual $\otimes$ and to specify the space where the tensors belong to.

**Definition 5.3.** A $B_\infty$-algebra is an $A_\infty$-algebra $(A, m_1, m_2, \cdots)$ together with a collection of graded maps (called $B_\infty$-products)

$$\mu_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A, \quad p, q \geq 0$$

of degree $1-p-q$ satisfying the following relations.

1. The unital condition:

$$\mu_{1,0} = 1_A = \mu_{0,1}, \quad \mu_{k,0} = 0 = \mu_{0,k} \quad \text{for} \ k \neq 1.$$  \hspace{1cm} (5.3)

2. The associativity of $\mu_{p,q}$: for any fixed $k, l, n \geq 0$, we have

$$\sum_{r=1}^{l+n} \sum_{l_1+\cdots+l_r = l \atop n_1+\cdots+n_r = n} (-1)^{\epsilon_k l} \mu_{k,r}(1^{\otimes k} \otimes ((\mu_{l_1,n_1} \otimes \cdots \otimes \mu_{l_r,n_r}) \circ \tau_{(l_1,\ldots,l_r;n_1,\ldots,n_r)}))$$

$$= \sum_{s=1}^{k+l} \sum_{l_1+\cdots+l_s = k \atop l_1+\cdots+l_s = l} (-1)^{\epsilon_s l} \mu_{s,n}((\mu_{k_1,l_1} \otimes \cdots \otimes \mu_{k_s,l_s}) \circ \tau_{(k_1,\ldots,k_s;l_1,\ldots,l_s)} \otimes 1^{\otimes n}),$$  \hspace{1cm} (5.4)

where

$$\epsilon_1 = \sum_{i=1}^{r-1} (l_i + n_i - 1)(r - i) + \sum_{i=1}^{r-1} n_i(l_{i+1} + \cdots + l_r),$$

and

$$\epsilon_1 = \sum_{i=1}^{s} (k_i + l_i - 1)(n + s - i) + \sum_{i=1}^{s-1} l_i(k_{i+1} + \cdots + k_s).$$

The Leibniz rule for $m_n$ with respect to $\mu_{p,q}$: for any fixed $k, l \geq 0$, we have
\[
\sum_{r=1}^{k+l} \sum_{k_1 + \cdots + k_r = k \atop l_1 + \cdots + l_r = l} (-1)^{r} m_r(\mu_{k_1,l_1} \otimes \cdots \otimes \mu_{k_r,l_r}) \circ \tau(k_1, \ldots, k_r, l_1, \ldots, l_r) = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^{i} \eta^\prime_{k-r+i,l}(1^{\otimes i} \otimes m_r \otimes 1^{\otimes k-r-i} \otimes 1^{\otimes l}) \tag{5.5}
\]
\[+ \sum_{s=1}^{l} \sum_{i=0}^{l-s} (-1)^{i} \eta^\prime_{k,l-s+i}(1^{\otimes k} \otimes 1^{\otimes i} \otimes m_s \otimes 1^{\otimes l-i-s}),
\]
where
\[
\epsilon = \sum_{i=1}^{r} (k_i + l_i - 1)(r - i) + \sum_{i=1}^{r} l_i(k_i - 1 - \cdots - k_i),
\]
\[\eta^\prime_{i} = r(k - r - i + l) + i, \quad \text{and} \quad \eta^\prime_{i} = s(l - i - s) + k + i.
\]
We usually denote a $B_\infty$-algebra by $(A, m_n; \mu_{p,q})$.

A $B_\infty$-morphism from $(A, m_n; \mu_{p,q})$ to $(A', m'_n; \mu'_{p,q})$ is an $A_\infty$-morphism
\[f = (f_n)_{n \geq 1} : A \to A'
\]
satisfying the following identity for any $p, q \geq 0$:
\[
\sum_{r,s \geq 0} \sum_{i_1 + i_2 + \cdots + i_r = p \atop j_1 + j_2 + \cdots + j_s = q} (-1)^{r} \mu^\prime_{r,s}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes f_{j_1} \otimes \cdots \otimes f_{j_s}) = \sum_{l \geq 1} \sum_{m_1 + m_2 + \cdots + m_q = q} (-1)^{l} f_{l} \circ (\mu_{1,m_1} \otimes \cdots \otimes \mu_{t,m_t}) \circ \tau(l_1, \ldots, l_t, m_1, \ldots, m_t),
\tag{5.6}
\]
where
\[
\epsilon = \sum_{k=1}^{r} (i_k - 1)(r + s - k) + \sum_{k=1}^{s} (j_k - 1)(s - k), \quad \text{and}
\]
\[\eta = \sum_{k=1}^{r} m_k(p - l_1 - \cdots - l_k) + \sum_{k=1}^{t} (l_k + m_k - 1)(t - k).
\]
The composition of $B_\infty$-morphisms is the same as the one of $A_\infty$-morphisms. \hfill \Box

A $B_\infty$-morphism $f : A \to A'$ is strict if $f_i = 0$ for each $i \neq 1$. A $B_\infty$-morphism $f : A \to A'$ is a $B_\infty$-isomorphism if there exists an $B_\infty$-morphism $g : A' \to A$ such that the compositions $f \circ_\infty g = 1_{A'}$ and $g \circ_\infty f = 1_A$. A $B_\infty$-morphism $f : A \to A'$ is a $B_\infty$-quasi-isomorphism if $f_1 : (A, m_1) \to (A', m'_1)$ is a quasi-isomorphism.

Consider the category of $B_\infty$-algebras, whose objects are $B_\infty$-algebras and whose morphisms are $B_\infty$-morphisms. It follows from \cite{40} that the category of $B_\infty$-algebras admits a model structure, whose weak equivalences are precisely $B_\infty$-quasi-isomorphisms. We denote by $Ho(B_\infty)$ the homotopy category associated with this model structure. In particular, each isomorphism in $Ho(B_\infty)$ comes from a zigzag of $B_\infty$-quasi-isomorphisms.
Remark 5.4. Similar to Remark 5.2, a $B_\infty$-algebra structure on $A$ is equivalent to a dg bialgebra structure $(\Delta^c(sA), \Delta, D, \mu)$ on the tensor coalgebra $T^c(sA)$ such that $1 \in \mathbb{k} = (sA)^{\otimes 0}$ is the unit of the algebra $(T^c(sA), \mu)$; compare [10]. Precisely, for a $B_\infty$-algebra $(A, m_n; \mu_{p,q})$ we may define two family of graded maps $M_n$ and $M_{p,q}$ on $sA$ via the following two commutative diagrams:

\[
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{m_n} & A \\
\downarrow{} & & \downarrow{a} \\
(sA)^{\otimes n} & \xrightarrow{M_n} & sA
\end{array}
\]

and

\[
\begin{array}{ccc}
A^{\otimes p} \otimes A^{\otimes q} & \xrightarrow{\mu_{p,q}} & A \\
\downarrow{} & & \downarrow{a} \\
(sA)^{\otimes p} \otimes (sA)^{\otimes q} & \xrightarrow{M_{p,q}} & sA,
\end{array}
\]

where $s : A \to sA$ is the canonical map $a \mapsto a$ of degree $-1$. The maps $M_n$ and $M_{p,q}$ induce, respectively, the differential $D$ and the multiplication $\mu$ on $T^c(sA)$. For more details, we refer to Subsection A.1 of Appendix A.

Accordingly, an $A_\infty$-morphism between two $B_\infty$-algebras is a $B_\infty$-morphism if and only if its induced dg coalgebra homomorphism is a dg bialgebra homomorphism.

Definition 5.5. The \textit{opposite $B_\infty$-algebra} of a $B_\infty$-algebra $(A, m_n; \mu_{p,q})$ is defined to be the $B_\infty$-algebra $(A, m_n; \mu_{p,q}^{\text{opp}})$, where $\mu_{p,q}^{\text{opp}} = (-1)^{pq} \mu_{q,p} \circ \tau_{p,q}$ and $\tau_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes q} \otimes A^{\otimes p}$ is the isomorphism sending an element $a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q$ to

\[
(-1)^{|b_1| + \cdots + |b_q|(|a_1| + \cdots + |a_p|)} b_1 \otimes \cdots \otimes b_q \otimes a_1 \otimes \cdots \otimes a_p.
\]

Observe that $\tau_{p,q} = \tau_{(0,p,q),0}$, defined at the beginning of this subsection. We will simply denote $(A, m_n; \mu_{p,q}^{\text{opp}})$ by $A^{\text{opp}}$ when no confusion can arise. By definition, $A^{\text{opp}}$ and $A$ have the same $A_\infty$-algebra structure. Note that $(A^{\text{opp}})^{\text{opp}} = A$ as $B_\infty$-algebras.

The following new terminology will be convenient for us.

Definition 5.6. A $B_\infty$-algebra $(A, m_n; \mu_{p,q})$ is called a \textit{brace $B_\infty$-algebra}, provided that $m_n = 0$ for $n > 2$ and that $\mu_{p,q} = 0$ for $p > 1$. □

We mention that a brace $B_\infty$-algebra is called a \textit{homotopy G-algebra} in [29] or a \textit{Gerstenhaber-Voronov algebra} in [18, 27, 9]. The notion is introduced mainly as an algebraic model to unify the rich algebraic structure on the Hochschild cochain complex of an algebra.

The underlying $A_\infty$-algebra structure of a brace $B_\infty$-algebra is just a dg algebra. For a brace $B_\infty$-algebra, we usually use the following notation, called the \textit{brace operation} [29, 62]:

\[
a\{b_1, \ldots, b_p\} := (-1)^{p|a|+(p-1)|b_1|+(p-2)|b_2|+\cdots+b_{p-1}} \mu_{1,p}(a \bigotimes b_1 \otimes \cdots \otimes b_p) \tag{5.7}
\]

for any $a, b_1, \ldots, b_p \in A$. In particular, $a\{\emptyset\} = \mu_{1,0}(a \bigotimes 1) = a$ by (5.3). We will abbreviate $a\{b_1, \ldots, b_p\}$ and $a'\{c_1, \ldots, c_q\}$ as $a\{b_1, p\}$ and $a'\{c_1, q\}$, respectively.

The $B_\infty$-algebras occurring in this paper, except Appendix A, are all brace $B_\infty$-algebras; see Subsections 6.1 and 7.1. In the following remark, we describe the axioms for brace $B_\infty$-algebras explicitly, which will be useful later.
Remark 5.7. Let \((A, m_n; \mu_{p,q})\) be a brace \(B_\infty\)-algebra. Then the above \(B_\infty\)-relation (5.4) is simplified as (1) below, and the \(B_\infty\)-relation (5.5) splits into (2) and (3) below (corresponding to the cases \(k = 2\) and \(k = 1\), respectively).

(1) The higher pre-Jacobi identity:

\[
(a\{b_1,p\}\{c_{1,q}\}) = \sum (-1)^{q}a\{c_{1,i_1}, b_1\{c_{1,i_1+i_1+l_1}\}, c_{i_1+i_1+i_2}, b_2\{c_{i_2+i_1+i_2+l_2}\}, \ldots, c_{i_p}, b_p\{c_{i_p+i_1+i_p+l_p}\}, c_{i_p+i_p+1,q}\},
\]

where the sum is taken over all sequences of nonnegative integers \((i_1, \ldots, i_p; l_1, \ldots, l_p)\) such that

\[
0 \leq i_1 \leq i_1 + l_1 \leq i_2 \leq i_2 + l_2 \leq i_3 \leq \cdots \leq i_p + l_p \leq q
\]

and

\[
\epsilon = \sum_{l=1}^{p} \left( (|b_l| - 1) \sum_{j=1}^{i_l} (|c_j| - 1) \right).
\]

(2) The distributivity:

\[
m_2(a_1 \otimes a_2\{b_{1,q}\}) = \sum_{j=0}^{q} (-1)^{|a_2|} \sum_{i=1}^{(|b_l| - 1)} m_2((a_1\{b_{1,j}\}) \otimes (a_2\{b_{j+1,q}\})).
\]

(3) The higher homotopy:

\[
m_1(a\{b_{1,p}\}) = (-1)^{|a||b_l|-1} m_2(b_1 \otimes (a\{b_{2,p}\})) + (-1)^{\epsilon_{p-1}} m_2((a\{b_1,p-1\}) \otimes b_p)
\]

\[
= m_1(a\{b_{1,p}\}) - \sum_{i=0}^{p-1} (-1)^{\epsilon_{i}} a\{b_{1,i}, m_1(b_{i+1}), b_{i+2,p}\} + \sum_{i=0}^{p-2} (-1)^{\epsilon_{i+1}} a\{b_{1,i}, m_2(b_{i+1,i+2}), b_{i+3,p}\},
\]

where \(\epsilon_0 = |a|\) and \(\epsilon_i = |a| + \sum_{j=1}^{i} (|b_j| - 1)\) for \(i \geq 1\).

Remark 5.8. The opposite \(B_\infty\)-algebra \((A, m_n; \mu_{p,q}^{opp})\) of a brace \(B_\infty\)-algebra \(A\) is given by

\[
\mu_{0,1}^{opp} = \mu_{1,0}^{opp} = 1_A, \quad \mu_{p,1}^{opp}(b_1 \otimes \cdots \otimes b_p \otimes a) = (-1)^{\epsilon} \mu_{1,p}(a \otimes b_1 \otimes \cdots \otimes b_p),
\]

and \(\mu_{p,q}^{opp} = 0\) for other cases, where \(\epsilon = |a||b_l| + \cdots + |b_p| + p\). In general, the opposite \(B_\infty\)-algebra \(A^{opp}\) is not a brace \(B_\infty\)-algebra.

The following observation follows directly from Definition 5.3.

Lemma 5.9. Let \(A\) and \(A'\) be two brace \(B_\infty\)-algebras. A homomorphism of dg algebras \(f: (A, m_1, m_2) \to (A', m'_1, m'_2)\) becomes a strict \(B_\infty\)-morphism if and only if \(f\) is compatible with \(-\{\cdots\}A\) and \(-\{\cdots\}A'\), namely

\[
f(a\{b_1, \ldots, b_p\}) = f(a)\{f(b_1), \ldots, f(b_p)\}
\]

for any \(p \geq 1\) and \(a, b_1, \ldots, b_p \in A\). \(\square\)
Let \( f = (f_n)_{n \geq 1} : A \to A' \) be an \( A_\infty \)-morphism. We define \( \tilde{f}_n : (sA)^{\otimes n} \to A' \) by the following commutative diagram.

\[
A^{\otimes n} \xrightarrow{s \otimes} (sA)^{\otimes n} \xrightarrow{f_n} A'
\]

Namely, we have
\[
\tilde{f}_n(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n) = (-1)^{\sum_{i=1}^{n} (n-i)|a_i|} f_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n).
\] (5.8)

The advantage of using \( (\tilde{f}_n)_{n \geq 1} \) in Lemma 5.10 below, instead of using \( (f_n)_{n \geq 1} \), is that the signs become much simpler.

The following lemma will be used in the proofs of Theorem 14.1 and Proposition A.18. We will abbreviate \( sa_1 \otimes \cdots \otimes sa_n \) as \( sa_{1,n} \), and \( a\{b_1, \ldots, b_m\} \) as \( a\{b_1, m\} \).

**Lemma 5.10.** Let \( A \) and \( A' \) be two brace \( B_\infty \)-algebras. Assume that \( (f_n)_{n \geq 1} : A \to A' \) is an \( A_\infty \)-morphism. Then \( (f_n)_{n \geq 1} : A \to A^{\text{opp}} \) is a \( B_\infty \)-morphism if and only if the following identities hold for any \( p, q \geq 0 \) and \( a_1, \ldots, a_p, b_1, \ldots, b_q \in A \):

\[
\sum_{r \geq 1} \sum_{i_1 + \cdots + i_r = p} (-1)^r \tilde{f}_q(sb_{1,q}) \{ \tilde{f}_{i_1}(sa_{1,i_1}) \tilde{f}_{i_2}(sa_{i_1 + 1,i_1 + i_2}) \cdots, \tilde{f}_{i_r}(sa_{i_1 + \cdots + i_{r-1} + 1,p}) \} A'
\]

\[
= \sum (-1)^n \tilde{f}_t(sb_{1,i_1} \otimes sa_{1 \{ b_1,1,j_1+1 \} A}) \otimes sb_{j_1+1,i_2} \otimes s(a_2\{b_2,j_2+1 \} A) \otimes \cdots \otimes sb_{j_p} \otimes s(a_p \{ b_p,j_p+1 \} A) \otimes sb_{j_p+l_p+1,q}).
\] (5.9)

Here, the maps \( \tilde{f}_q \) and \( \tilde{f}_t \) are defined in (5.8); the sum on the right hand side is taken over all the sequences of nonnegative integers \( (j_1, \ldots, j_p, l_1, \ldots, l_p) \) such that

\[
0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q,
\]

and \( t = p + q - l_1 - \cdots - l_p \); the signs are determined by the identities

\[
\epsilon = (|a_1| + \cdots + |a_p| - p)(|b_1| + \cdots + |b_q| - q),
\]

\[
\eta = \sum_{i=1}^{p} (|a_i| - 1)(|b_1| - 1 + |b_2| - 1 + \cdots + (|b_j| - 1)).
\]

**Proof.** Since \( \mu_{s,r}' = 0 \) for \( s > 1 \) and \( (\mu_{1,r}' \text{opp}) = (-1)^r \mu_{1,r} \circ \tau_{r,1} \), the identity (5.6) becomes

\[
\sum_{r \geq 1} \sum_{i_1 + \cdots + i_r = p} (-1)^r \mu_{s,r}' \circ \tau_{r,1}(f_1 \otimes \cdots \otimes f_r \otimes f_q)
\]

\[
= \sum_{r \geq 1} (-1)^n f_{t+1} \circ (\mu_{m_1,n_1} \otimes \cdots \otimes \mu_{m_{r,n_{r}}} \circ \tau_{m_1,\ldots,m_{r,n_1},\ldots,n_{r}})
\]

\[
= \sum (-1)^n f_{t} \circ (\mu_{0,1}^{j_1} \otimes \mu_{1,l_1}^{j_2-j_1-l_1} \otimes \mu_{1,l_2}^{j_3-j_2-l_2} \cdots \otimes \mu_{1,l_p}^{j_{p+1}-l_{p+1}} \otimes \mu_{0,1}^{q-l_p-j_p}) \circ \tau_{m_1,\ldots,m_{r,n_1,\ldots,n_{r}}},
\]

where the sum on the right hand side of the last identity is taken over all the sequences of nonnegative integers \( (j_1, \ldots, j_p, l_1, \ldots, l_p) \) such that

\[
0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q.
\]
and $t = p + q - l_1 - \cdots - l_p$. The signs are determined by

$$
\epsilon_1 = \sum_{k=1}^{r} (i_k - 1)(r + 1 - k), \quad \text{and}
$$

$$
\eta_1 = \sum_{k=1}^{t} n_k(p - m_1 - \cdots - m_k) + \sum_{k=1}^{t} (m_k + n_k - 1)(t - k)
$$

$$
= \sum_{i=1}^{p} j_i + \sum_{i=1}^{p} l_i(t - j_i - l_1 - \cdots - l_{i-1} + i).
$$

We apply (5.10) to the element $(-1)^{\sum_{i=1}^{p} |a_i|(p+q-i)+\sum_{j=1}^{q} |b_j|(q-j)}(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q)$, where the sign $(-1)^{\sum_{i=1}^{p} |a_i|(p+q-i)+\sum_{j=1}^{q} |b_j|(q-j)}$ is added just in order to simplify the sign computation. Using (5.8), we obtain the required identity (5.9).

5.3. Gerstenhaber algebras. In this subsection, we recall the well-known relationship between $B_\infty$-algebras and Gerstenhaber algebras.

**Definition 5.11.** A Gerstenhaber algebra is the triple $(G, - \cup - , [- , -])$, where $G = \bigoplus_{n \in \mathbb{Z}} G^n$ is a graded $k$-space equipped with two graded maps: a cup product

$$
- \cup : G \otimes G \rightarrow G
$$

of degree zero, and a Lie bracket of degree $-1$

$$
[- , -] : G \otimes G \rightarrow G
$$

satisfying the following conditions:

1. $(G, - \cup - )$ is a graded commutative associative algebra;
2. $(G^{*+1}, [- , -])$ is a graded Lie algebra, that is

$$
[\alpha, \beta] = (-1)^{|\alpha|-1}\beta \alpha - (-1)^{|\beta|-1}\alpha \beta
$$

and

$$
(-1)^{|\alpha|-1}|\gamma|\gamma|\alpha \beta \gamma] + (-1)^{|\beta|-1}|\alpha|\alpha \beta \gamma] + (-1)^{|\gamma|-1}|\beta|\alpha \beta \gamma] = 0;
$$

3. the operations $- \cup -$ and $[- , -]$ are compatible through the graded Leibniz rule

$$
[\alpha, \beta \cup \gamma] = [\alpha, \beta] \cup \gamma + (-1)^{|\alpha|-1}\gamma \cup [\alpha, \gamma].
$$

The following well-known result is contained in [30, Subsection 5.2].

**Lemma 5.12.** Let $(A, m_n; \mu_{p,q})$ be a $B_\infty$-algebra. Then there is a natural Gerstenhaber algebra structure $(H^*(A, m_1), - \cup - , [- , -])$ on its cohomology, where the cup product $- \cup -$ and the Lie bracket $[- , -]$ of degree $-1$ are given by

$$
\alpha \cup \beta = m_2(\alpha, \beta);
$$

$$
[\alpha, \beta] = (-1)^{|\alpha|}\mu_{1,1}(\alpha, \beta) - (-1)^{|\alpha|-1}|\beta|\mu_{1,1}(\beta, \alpha).
$$

Moreover, a $B_\infty$-quasi-isomorphism between two $B_\infty$-algebras $A$ and $A'$ induces an isomorphism of Gerstenhaber algebras between $H^*(A)$ and $H^*(A')$. 

\[\square\]
**Remark 5.13.** A prior, the Lie bracket \([-,-]\) in Lemma 5.12 is defined on \(A\) at the cochain complex level. By definition, we have \([\alpha,\beta] = (-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,\alpha]\). It follows from (5.4) that \([-,-]\) satisfies the graded Jacobi identity (5.11). By (5.5) we have
\[
m_1([\alpha,\beta]) = [m_1(\alpha),\beta] + (-1)^{|\alpha|-1}[\alpha,m_1(\beta)],
\]
which ensures that \([-,-]\) descends to \(H^*(A)\). That is, \((A,m_1,[-,-])\) is a dg Lie algebra of degree \(-1\); see [30, Subsection 5.2]. By (5.6) we see that a \(B_\infty\)-morphism induces a morphism of dg Lie algebras between the associated dg Lie algebras.

We mention that the associated dg Lie algebras to \(B_\infty\)-algebras play a crucial role in deformation theory; see e.g. [50].

### 6. The Hochschild cochain complexes

In this section, we recall basic results on Hochschild cochain complexes of dg categories and (normalized) relative bar resolutions of dg algebras.

#### 6.1. The Hochschild cochain complex of a dg category

Recall that for a cochain complex \((V,d_V)\), we denote by \(sV\) the 1-shifted complex. For a homogeneous element \(v \in V\), the degree of the corresponding element \(sv \in sV\) is given by \(|sv| = |v| - 1\) and \(d_{sV}(sv) = -sd_V(v)\). Indeed, we have \(sV = \Sigma(V)\), where \(\Sigma\) is the suspension functor.

Let \(\mathcal{A}\) be a small dg category over \(k\). The Hochschild cochain complex of \(\mathcal{A}\) is the complex
\[
C^*(\mathcal{A},A) = \prod_{n \geq 0} \prod_{A_0,\ldots,A_n \in \text{obj}(\mathcal{A})} \text{Hom}(s\mathcal{A}(A_{n+1},A_n) \otimes \cdots \otimes s\mathcal{A}(A_1,A_0), A(A_0,A_n))
\]
with differential \(\delta = \delta_n + \delta_{ex}\) defined as follows. For any \(\phi \in \text{Hom}(s\mathcal{A}(A_{n+1},A_n) \otimes \cdots \otimes s\mathcal{A}(A_0,A_1), A(A_0,A_n))\) the **internal differential** \(\delta_n\) is
\[
\delta_n(\phi)(sa_{1,n}) = d_A\phi(sa_{1,n}) + \sum_{i=1}^n (-1)^{\epsilon_i} \phi(sa_{1,i-1} \otimes sd_A(a_i) \otimes sa_{i+1,n})
\]
and the **external differential** is
\[
\delta_{ex}(\phi)(sa_{1,n+1}) = (-1)^{(|a_{i-1}|-1)|\phi|} a_{i-1} \circ \phi(sa_{2,n+1}) + (-1)^{n+1} \phi(sa_{1,n}) \circ a_{n+1} - \sum_{i=2}^{n+1} (-1)^{\epsilon_i} \phi(sa_{1,i-2} \otimes s(a_{i-1} \circ a_i) \otimes sa_{i+1,n+1}).
\]
Here, \(\epsilon_i = |\phi| + \sum_{j=1}^{i-1} (|a_j| - 1)\) and \(sa_{i,j} := sa_i \otimes \cdots \otimes sa_j \in s\mathcal{A}(A_{n-i},A_{n-i+1}) \otimes \cdots \otimes s\mathcal{A}(A_{n-j},A_{n-j+1})\) for \(i \leq j\).

For any \(n \geq 0\), we define the following subspace of \(C^*(\mathcal{A},A)\)
\[
C^{\ast,n}(\mathcal{A}, A) := \prod_{A_0,\ldots,A_n \in \text{obj}(\mathcal{A})} \text{Hom}(s\mathcal{A}(A_{n+1},A_n) \otimes \cdots \otimes s\mathcal{A}(A_1,A_0), A(A_0,A_n)).
\]
We observe \(C^{\ast,0}(\mathcal{A}, A) = \prod_{A_0 \in \text{obj}(\mathcal{A})} \text{Hom}(k, A(A_0,A_0)) \simeq \prod_{A_0 \in \text{obj}(\mathcal{A})} A(A_0, A_0)\).

There are two basic operations on \(C^*(\mathcal{A},A)\). The first one is the **cup product**
\[
- \cup - : C^*(\mathcal{A},A) \otimes C^*(\mathcal{A},A) \longrightarrow C^*(\mathcal{A},A).
\]
For \(\phi \in C^{\ast,p}(\mathcal{A},A)\) and \(\varphi \in C^{\ast,q}(\mathcal{A},A)\), we define
\[
\phi \cup \varphi(sa_{1,p+q}) = (-1)^p \phi(sa_{1,p}) \circ \varphi(sa_{p+1,p+q}),
\]
where $\epsilon = (|a_1| + \cdots + |a_p| - p)\varphi$.

The second one is the brace operation

$$\{-\ldots, -\}: C^*(A, A) \otimes C^*(A, A)^{\otimes k} \to C^*(A, A)$$

defined as follows. Let $k \geq 1$. For $\varphi \in C^{*,m}(A, A)$ and $\phi_i \in C^{*,n_i}(A, A)$ ($1 \leq i \leq k$),

$$\varphi\{\phi_1, \ldots, \phi_k\} = \sum \varphi(1^{\otimes i_1} \otimes (s \circ \phi_1) \otimes 1^{\otimes i_2} \otimes (s \circ \phi_2) \otimes \cdots \otimes 1^{\otimes i_k} \otimes (s \circ \phi_k) \otimes 1^{\otimes i_{k+1}}),$$

(6.1)

where the summation is taken over the set

$$\{(i_1, i_2, \ldots, i_{k+1}) \in \mathbb{Z}_{\geq 0}^{(k+1)} \mid i_1 + i_2 + \cdots + i_{k+1} = m - k\}.$$ 

If the set is empty, we define $\varphi\{\phi_1, \ldots, \phi_k\} = 0$. Here, $s \circ \phi_j$ means the composition of $\phi_j$ with the natural isomorphism $s: A(A, A') \to sA(A, A')$ of degree $-1$ for suitable $A, A' \in \text{obj}(A)$. For $k = 0$, we set $-\{0\} = 1$. Observe that the cup product and the brace operation extend naturally to the whole space $C^*(A, A) = \prod_{n \geq 0} C^{*,n}(A, A)$.

It is well known that $C^*(A, A)$ is a brace $B_\infty$-algebra with

$$m_1 = \delta, \quad m_2 = -\mu - \delta, \quad \text{and} \quad m_i = 0 \quad \text{for} \quad i > 2;$$

$$\mu_{0,1} = \mu_{1,0} = 1, \quad \mu_{1,k}(\varphi, \phi_1, \ldots, \phi_k) = \varphi\{\phi_1, \ldots, \phi_k\}, \quad \text{and} \quad \mu_{p,q} = 0 \quad \text{otherwise.}$$

We refer to [30, Subsections 5.1 and 5.2] for details.

The following useful lemma is contained in [40, Theorem 4.6 b)].

**Lemma 6.1.** Let $F: A \to B$ be a quasi-equivalence between two small dg categories. Then there is an isomorphism

$$C^*(A, A) \to C^*(B, B)$$

in the homotopy category $\text{Ho}(B_\infty)$ of $B_\infty$-algebras. \hfill \Box

Let $A$ be a dg algebra. We view $A$ as a dg category with a single object, still denoted by $A$. In particular, the Hochschild cochain complex $C^*(A, A)$ is defined as above. The dg category $A$ might be identified as a full dg subcategory of $\text{per}_{dg}(A^{op})$ by taking the right regular dg $A$-module $A_A$. Then the following result follows from [40, Theorem 4.6 c)]; compare [50, Theorem 4.4.1].

**Lemma 6.2.** Let $A$ be a dg algebra. Then the restriction map

$$C^*(\text{per}_{dg}(A^{op}), \text{per}_{dg}(A^{op})) \to C^*(A, A)$$

is an isomorphism in $\text{Ho}(B_\infty)$. \hfill \Box

### 6.2. The relative bar resolutions

Let $A$ be a dg algebra with its differential $d_A$. Let $E = \bigoplus_{i \in \mathbb{Z}} \mathbb{k} e_i \subseteq A^0 \subseteq A$ be a semisimple subalgebra satisfying $d_A(e_i) = 0$ and $e_ie_j = \delta_{i,j}e_i$ for any $i, j \in \mathbb{Z}$. Set $(sA)^{\otimes e_0} = E$ and $T_E(sA) := \bigoplus_{n \geq 0} (sA)^{\otimes e^n}$.

Recall from [4] that the $E$-relative bar resolution of $A$ is the dg $A-A$-bimodule

$$\text{Bar}_E(A) := A \otimes_E T_E(sA) \otimes_E A$$

with the differential $d = d_{in} + d_{ex}$, where $d_{in}$ is the internal differential given by

$$d_{in}(a \otimes_E sa_{i,n} \otimes_E b) = d_A(a) \otimes_E sa_{i,n} \otimes_E b + (-1)^{e_i+1} a \otimes_E sa_{i,n} \otimes_E d_A(b)$$

$$- \sum_{i = 1}^n (-1)^{e_i} a \otimes_E sa_{i-1,n} \otimes_E s d_A(a_i) \otimes_E sa_{i+1,n} \otimes_E b$$
and $d_{ex}$ is the external differential given by
\[ d_{ex}(a \otimes_E sa_{i,n} \otimes_E b) = (-1)^i a \otimes_E sa_{i+1,n} \otimes_E b − (-1)^n a \otimes_E sa_{1,n} \otimes_E a_n b \]
\[ + \sum_{i=2}^n (-1)^i a \otimes_E sa_{1,i−2} \otimes_E sa_{i−1} \otimes_E sa_{i+1,n} \otimes_E b. \]

Here, $\epsilon_i = |a| + \sum_{j=1}^{i-1} (|a_j| − 1)$, and for simplicity, we denote $sa_i \otimes_E sa_{i+1} \otimes_E \cdots \otimes_E sa_j$ by $sa_{i,j}$ for $i < j$. The degree of $a \otimes_E sa_{1,n} \otimes_E b \in A \otimes_E (sA)^{\otimes E_n} \otimes_E A$ is
\[ |a| + \sum_{j=1}^n (|a_j| − 1) + |b|. \]

The graded $A$-$A$-bimodule structure on $A \otimes_E (sA)^{\otimes E_n} \otimes_E A$ is given by the outer action
\[ a(a_0 \otimes_E sa_{1,n} \otimes_E a_{n+1})b := aa_0 \otimes_E sa_{1,n} \otimes_E a_{n+1}b. \]

There is a natural morphism of dg $A$-$A$-bimodules $\varepsilon : \text{Bar}_E(A) \to A$ given by the composition
\[ \text{Bar}_E(A) \to A \otimes_E A \xrightarrow{\mu} A, \tag{6.2} \]
where the first map is the canonical projection and $\mu$ is the multiplication of $A$. It is well known that $\varepsilon$ is a quasi-isomorphism.

Set $\overline{A}$ to be the quotient dg $E$-$E$-bimodule $A/(E \cdot 1_A)$. We have the notion of normalized $E$-relative bar resolution $\overline{\text{Bar}}_E(A)$ of $A$. By definition, it is the dg $A$-$A$-bimodule
\[ \overline{\text{Bar}}_E(A) = A \otimes_E T_E(sA) \otimes_E A \]
with the induced differential from $\text{Bar}(A)$. It is also well known that the natural projection $\text{Bar}_E(A) \to \overline{\text{Bar}}_E(A)$ is a quasi-isomorphism.

Let $D(A^e)$ be the derived category of dg $A$-$A$-bimodules. Let $M$ be a dg $A$-$A$-bimodule. The Hochschild cohomology group with coefficients in $M$ of degree $p$, denoted by $\text{HH}^p(A, M)$, is defined as $\text{Hom}_{D(A^e)}(A, \Sigma^p(M))$, where $\Sigma$ is the suspension functor in $D(A^e)$. Since $\text{Bar}_E(A)$ is a dg-projective bimodule resolution of $A$, we obtain that
\[ \text{HH}^p(A, M) \cong \text{H}^p(\text{Hom}_{A \cdot A}(\text{Bar}_E(A), M), \delta), \quad \text{for } p \in \mathbb{Z} \]
where $\delta(f) := d_M \circ f − (-1)^{|f|} f \circ d$. We observe that there is a natural isomorphism, for each $i \geq 0$,
\[ \text{Hom}_{E \cdot E}((sA)^{\otimes E_i}, M) \xrightarrow{\sim} \text{Hom}_{A \cdot A}(A \otimes_E (sA)^{\otimes E_i} \otimes_E A, M) \tag{6.3} \]
which sends $f$ to the map $a_0 \otimes_E sa_{1,i} \otimes_E a_{i+1} \mapsto (-1)^{|a_0||f|} a_0 f(sa_{1,i})a_{i+1}$. It follows that
\[ \text{HH}^p(A, M) \cong \text{H}^p(\text{Hom}_{E \cdot E}(T_E(sA), M), \delta = \delta_m + \delta_{ex}), \]
where the differentials $\delta_m$ and $\delta_{ex}$ are defined as in Subsection 6.1.

We call $C_E^*(A, M) := (\text{Hom}_{E \cdot E}(T_E(sA), M), \delta)$ the $E$-relative Hochschild cochain complex of $A$ with coefficients in $M$. In particular, $C_E^*(A, A)$ is called the $E$-relative Hochschild cochain complex of $A$. Similarly, the normalized $E$-relative Hochschild cochain complex $\overline{C}_E^*(A, M)$ is defined as $\text{Hom}_{E \cdot E}(T_E(sA), M)$ with the induced differential. When $E = k$, we simply write $C_k^*(A, M)$ as $C^*(A, M)$ and write $\overline{C}_k^*(A, M)$ as $C^*(A, M)$.
When the dg algebra $A$ is viewed as a dg category $A$ with a single object, $C^*(A, A)$ coincides with $C^*(A, A)$. Thus, from Subsection 6.1, $C^*(A, A)$ has a $B_\infty$-algebra structure induced by the cup product $- \cup -$ and the brace operation $-\{-, \ldots, -\}$.

We have the following commutative diagram of injections.

$\begin{array}{c}
\mathcal{C}^*_E(A, A) \\ \nearrow \\
\mathcal{C}^*(A, A)
\end{array}$

$\begin{array}{c}
\mathcal{C}^*_E(A, A) \\ \nearrow \\
\mathcal{C}^*(A, A)
\end{array}$

Lemma 6.3. The $B_\infty$-algebra structure on $C^*(A, A)$ restricts to the other three smaller complexes $C^*_E(A, A), \mathcal{C}^*_E(A, A)$ and $\mathcal{C}^*(A, A)$. In particular, the above injections are strict $B_\infty$-quasi-isomorphisms.

Proof. It is straightforward to check that the cup product and brace operation on $C^*(A, A)$ restrict to the subcomplexes $C^*_E(A, A), \mathcal{C}^*_E(A, A)$ and $\mathcal{C}^*(A, A)$. Moreover, the injections preserve the two operations. Thus by Lemma 5.9, the injections are strict $B_\infty$-morphisms. Clearly, the injections are quasi-isomorphisms since all the complexes compute $\text{HH}^*(A, A)$.

This proves the lemma. \qed

7. The singular Hochschild cochain complexes

In this section, we recall the singular Hochschild cochain complexes and their $B_\infty$-structures. We describe explicitly the brace operation on the singular Hochschild cochain complex and illustrate it with an example.

7.1. The left and right singular Hochschild cochain complexes. Let $\Lambda$ be a finite dimensional $k$-algebra. Denote by $\Lambda^e = \Lambda \otimes \Lambda^{\text{op}}$ its enveloping algebra. Let $D_{\text{sg}}(\Lambda^e)$ be the singularity category of $\Lambda^e$. Following [11, 64, 42], the singular Hochschild cohomology of $\Lambda$ is defined as

$HH^n_{\text{sg}}(\Lambda, \Lambda) := \text{Hom}_{D_{\text{sg}}(\Lambda^e)}(\Lambda, \Sigma^n(\Lambda))$, for $n \in \mathbb{Z}$.

Recall from [66, Section 3] that the singular Hochschild cohomology $HH^*_\text{sg}(\Lambda, \Lambda)$ can be computed by the so-called singular Hochschild cochain complex.

There are two kinds of singular Hochschild cochain complexes: the left singular Hochschild cochain complex and the right singular Hochschild cochain complex, which are constructed by using the left noncommutative differential forms and the right noncommutative differential forms, respectively. We mention that only the left one is considered in [66] with slightly different notation; see [66, Definition 3.2]. We will first recall the right singular Hochschild cochain complex $\mathcal{C}^*_{\text{sg}, R}(\Lambda, \Lambda)$.

Throughout this subsection, we denote $\overline{\Lambda} = \Lambda / (k \cdot 1_{\Lambda})$. Recall that the graded $\Lambda$-$\Lambda$-bimodule of right noncommutative differential $p$-forms is defined as

$\Omega^p_{\text{nc}, R}(\Lambda) = (s\overline{\Lambda})^{\otimes p} \otimes \Lambda$. 

Observe that $\Omega^p_{nc,R}(\Lambda)$ is concentrated in degree $-p$ and that its bimodule structure is given by

$$a_0 \mapsto (s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_p \otimes a_{p+1})b = \sum_{i=0}^{p-1} (-1)^i s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_i a_{i+1} \otimes \cdots \otimes s\bar{a}_p \otimes a_{p+1}b$$

(7.1)

for $b, a_0 \in \Lambda$ and $s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_p \otimes a_{p+1} \in \Omega^p_{nc,R}(\Lambda)$.

Note that there is a $k$-linear isomorphism between $\Omega^p_{nc,R}(\Lambda)$ and the cokernel of the $(p+1)$-th differential

$$\Lambda \otimes (s\bar{X})\otimes^{p+1} \Lambda \xrightarrow{dx} \Lambda \otimes (s\bar{X})\otimes^p \Lambda$$

in $\text{Bar}(\Lambda)$ defined in Subsection 6.2. Then the above bimodule structure on $\Omega^p_{nc,R}(\Lambda)$ is inherited from this cokernel; compare [66, Lemma 2.5]. We have a short exact sequence of graded bimodules

$$0 \to \Sigma^{-1} \Omega^{p+1}_{nc,R}(\Lambda) \xrightarrow{d'} \Lambda \otimes (s\bar{X})\otimes^p \Lambda \xrightarrow{d''} \Omega^p_{nc,R}(\Lambda) \to 0$$

(7.2)

where $d'$ and $d''$ are given as follows

$$d'(s^{-1}x) = d_{ex}(1 \otimes x)$$

for any $x \in \Omega^{p+1}_{nc,R}(\Lambda)$

$$d'' = (\varpi \otimes 1_{s\bar{X}}^\otimes \otimes 1_{\Lambda}) \circ d_{ex}$$

where $\varpi : \Lambda \to s\bar{X}$ is the natural projection of degree $-1$.

Let $\mathcal{C}^s(\Lambda, \Omega^p_{nc,R}(\Lambda))$ be the normalized Hochschild cochain complex of $\Lambda$ with coefficients in the graded bimodule $\Omega^p_{nc,R}(\Lambda)$. Here, $\Lambda$ is viewed as a dg algebra concentrated in degree zero.

For each $p \geq 0$, we define a morphism (of degree zero) of complexes

$$\theta_{p,R} : \mathcal{C}^s(\Lambda, \Omega^p_{nc,R}(\Lambda)) \longrightarrow \mathcal{C}^s(\Lambda, \Omega^{p+1}_{nc,R}(\Lambda)), \quad f \mapsto 1_{s\bar{X}} \otimes f.$$ 

Here, we recall that $\mathcal{C}^m(\Lambda, \Omega^p_{nc,R}(\Lambda)) = \text{Hom}((s\bar{X})\otimes^m \Lambda, \Omega^p_{nc,R}(\Lambda))$, the Hom-space between non-graded spaces. Then for $f \in \mathcal{C}^m(\Lambda, \Omega^p_{nc,R}(\Lambda))$, the map $1_{s\bar{X}} \otimes f$ naturally lies in $\mathcal{C}^m(\Lambda, \Omega^{p+1}_{nc,R}(\Lambda))$, using the following identification

$$\Omega^{p+1}_{nc,R}(\Lambda) = s\bar{X} \otimes \Omega^p_{nc,R}(\Lambda).$$

We mention that when $1_{s\bar{X}} \otimes f$ is applied to elements in $(s\bar{X})\otimes^m \Lambda$, an additional sign $(-1)^{|f|}$ appears due to the Koszul sign rule.

The right singular Hochschild cochain complex $\mathcal{C}^s_{sg,R}(\Lambda, \Lambda)$ is defined to be the colimit of the inductive system

$$\mathcal{C}^s(\Lambda, \Lambda) \xrightarrow{\theta_{0,R}} \mathcal{C}^s(\Lambda, \Omega^1_{nc,R}(\Lambda)) \xrightarrow{\theta_{1,R}} \cdots \xrightarrow{\theta_{p,R}} \cdots.$$ 

(7.3)

We mention that all the maps $\theta_{p,R}$ are injective.

The above terminology is justified by the following observation.

**Lemma 7.1.** For each $n \in \mathbb{Z}$, we have an isomorphism

$$\text{HH}^n_{sg}(\Lambda, \Lambda) \simeq H^n(\mathcal{C}^s_{sg,R}(\Lambda, \Lambda)).$$
Proof. The proof is analogous to that of [66, Theorem 3.6] for the left singular Hochschild cochain complex. For the convenience of the reader, we give a complete proof.

Since the direct colimit commutes with the cohomology functor, we obtain that

$$H^n(\mathcal{C}_{sg,R}^*(\Lambda, \Lambda)) \simeq \lim_{\rightarrow \theta_{p,R}} HH^p(\Lambda, \Omega_{nc,R}^p(\Lambda)),$$

where the maps $\tilde{\theta}_{p,R}$ are induced by the above cochain maps $\theta_{p,R}$.

Applying the functor $HH^*(\Lambda, -)$ to the short exact sequence (7.2), we obtain a long exact sequence

$$\cdots \to HH^n(\Lambda, \Lambda \otimes (s\Lambda)^{\otimes p} \otimes \Lambda) \to HH^p(\Lambda, \Omega_{nc,R}^p(\Lambda)) \to HH^{n+1}(\Lambda, \Sigma^{-1}\Omega_{nc,R}^{p+1}(\Lambda)) \to \cdots .$$

Since $HH^{n+1}(\Lambda, \Sigma^{-1}\Omega_{nc,R}^{p+1}(\Lambda)$ is naturally isomorphic to $HH^n(\Lambda, \Omega_{nc,R}^p(\Lambda)$, the connecting morphism $c$ in the long exact sequence induces a map

$$\tilde{\theta}_{p,R}: HH^n(\Lambda, \Omega_{nc,R}^p(\Lambda) \to HH^n(\Lambda, \Omega_{nc,R}^{p+1}(\Lambda)).$$

We claim that $\tilde{\theta}_{p,R} = \tilde{\theta}_{p,R}$. Indeed, let $f \in HH^n(\Lambda, \Omega_{nc,R}^p(\Lambda)$). It may be represented by an element $\tilde{f} \in \text{Hom}_\Lambda(\Lambda \otimes (s\Lambda)^{\otimes p} \otimes \Lambda, \Omega_{nc,R}^p(\Lambda)$) such that $f \circ d_{ex} = 0$. We have the following diagram

$$\begin{array}{ccc}
\Lambda \otimes (s\Lambda)^{\otimes n+p+1} \otimes \Lambda & \xrightarrow{d_{ex}} & \Lambda \otimes (s\Lambda)^{\otimes n+p} \otimes \Lambda \\
\downarrow \tilde{f} & & \downarrow \tilde{f} \\
\Omega_{nc,R}^{p+1}(\Lambda) & \xrightarrow{d'} & \Lambda \otimes (s\Lambda)^{\otimes p} \otimes \Lambda \xrightarrow{d''} \Omega_{nc,R}^p(\Lambda),
\end{array}$$

where $\tilde{f}$ is given by the following formula

$$\tilde{f}(a \otimes s\pi_{1,n+p} \otimes b) = a \otimes f(1 \otimes s\pi_{1,n+p} \otimes b),$$

and $\tilde{f}$ is the morphism of $\Lambda$-$\Lambda$-bimodules uniquely determined by

$$\tilde{f}(1 \otimes s\pi_{1,n+p+1} \otimes 1) = \theta_{p,R}(f)(s\pi_{1,n+p+1});$$

compare (6.3). One can check that $f = d'' \circ \tilde{f}$ and $d' \circ \tilde{f} = (-1)^{|f|} \tilde{f} \circ d_{ex}$. This shows that $\tilde{f}$ is a lifting of $f$ along the normalized bar resolution $\overline{\text{Bar}}(\Lambda)$, that is, $\tilde{\theta}_{p,R}(f) = \tilde{f}$. Obviously, we have $\tilde{\theta}_{p,R}(f) = \tilde{f}$. This proves the claim.

By [44, Subsection 2.3], the above claim yields the desired isomorphism; also compare [42, Lemma 2.4].

There are two basic operations on $\mathcal{C}_{sg,R}^*(\Lambda, \Lambda)$. The first one is the cup product

$$- \cup_R -: \mathcal{C}_{sg,R}^*(\Lambda, \Lambda) \otimes \mathcal{C}_{sg,R}^*(\Lambda, \Lambda) \to \mathcal{C}_{sg,R}^*(\Lambda, \Lambda)$$

which is defined as follows: for $\varphi \in \mathcal{C}^{m-p}(\Lambda, \Omega_{nc,R}^p(\Lambda)$ and $\phi \in \mathcal{C}^{n-q}(\Lambda, \Omega_{nc,R}^q(\Lambda)$, we define

$$\varphi \cup_R \phi := (1_{s\Lambda}^{\otimes p+q} \otimes \mu) \circ (1_{s\Lambda}^{\otimes q} \otimes \varphi \otimes 1_{\Lambda}) \circ (1_{s\Lambda}^{\otimes m} \otimes \phi) \in \mathcal{C}^{m+n-p-q}(\Lambda, \Omega_{nc,R}^{p+q}(\Lambda),$$

(7.4)

where $\mu$ denotes the multiplication of $\Lambda$. When $\varphi \cup_R \phi$ is applied to elements in $(s\Lambda)^{\otimes m+n}$, an additional sign $(-1)^{mn+pq}$ appears due to the Koszul sign rule. In particular, if $p = q = 0$
we get the classical cup product on \( \overline{C}^*(\Lambda, \Lambda) \). Note that \( - \cup_R - \) is compatible with the colimit, hence it is well defined on \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \).

The second one is the brace operation

\[
-\{-, \ldots, -\}_R : \overline{C}^*_{sg,R}(\Lambda, \Lambda) \otimes \overline{C}^*_{sg,R}(\Lambda, \Lambda)^{\otimes k} \to \overline{C}^*_{sg,R}(\Lambda, \Lambda), \quad \text{for } k \geq 1,
\]

which is defined in Subsection 7.3 below; see Definition 7.8. It restricts to the classical brace operation on \( \overline{C}^*(\Lambda, \Lambda) \).

The following result is a right-sided version of [66, Theorem 5.1].

**Theorem 7.2.** The right singular Hochschild cochain complex \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \), equipped with \( \cup_R \) and \( -\{-, \ldots, -\}_R \), is a brace \( B_\infty \)-algebra. Consequently, \( (\text{HH}^p_{sg}(\Lambda, \Lambda), - \cup_R - \{\ldots, -\}_R) \) is a Gerstenhaber algebra.

We now recall the left singular Hochschild cochain complex \( \overline{C}^*_{sg,L}(\Lambda, \Lambda) \). The graded \( \Lambda-\Lambda \)-bimodule of left noncommutative differential \( p \)-forms is

\[
\Omega^p_{nc,L}(\Lambda) = \Lambda \otimes (s\Lambda)^{\otimes p},
\]

whose bimodule structure is given by

\[
b(a_0 \otimes s\alpha_1 \cdots \otimes s\alpha_p) \triangleright a_{p+1} = (-1)^p b a_0 a_1 \otimes s\alpha_2 \otimes \cdots \otimes s\alpha_p \otimes s\alpha_{p+1} + \\
\sum_{i=1}^p (-1)^{p-i} b a_0 \otimes s\alpha_1 \otimes \cdots \otimes s\alpha_i a_{i+1} \otimes \cdots \otimes s\alpha_{p+1}
\]

for \( b, a_{p+1} \in \Lambda \) and \( a_0 \otimes s\alpha_1 \cdots \otimes s\alpha_p \in \Omega^p_{nc,L}(\Lambda) \). It follows from [66, Lemma 2.5] that \( \Omega^p_{nc,L}(\Lambda) \) is also isomorphic, as graded \( \Lambda-\Lambda \)-bimodules, to the cokernel of the \((p+1)\)-th differential

\[
\Lambda \otimes (s\Lambda)^{\otimes p+1} \otimes \Lambda \xrightarrow{d_{ex}} \Lambda \otimes (s\Lambda)^{\otimes p} \otimes \Lambda
\]

in \( \overline{\text{Bar}}(\Lambda) \). In particular, we infer that \( \Omega^p_{nc,L}(\Lambda) \) and \( \Omega^p_{nc,R}(\Lambda) \) are isomorphic as graded \( \Lambda-\Lambda \)-bimodules.

The left singular Hochschild cochain complex \( \overline{C}^*_{sg,L}(\Lambda, \Lambda) \) is defined as the colimit of the inductive system

\[
\overline{C}^*(\Lambda, \Lambda) \xrightarrow{\theta_{0,L}} \overline{C}^*(\Lambda, \Omega_{nc,L}^1(\Lambda)) \xrightarrow{\theta_{1,L}} \cdots \xrightarrow{\theta_{p-1,L}} \overline{C}^*(\Lambda, \Omega_{nc,L}^p(\Lambda)) \xrightarrow{\theta_{p,L}} \cdots,
\]

where

\[
\theta_{p,L} : \overline{C}^*(\Lambda, \Omega_{nc,L}^p(\Lambda)) \to \overline{C}^*(\Lambda, \Omega_{nc,L}^{p+1}(\Lambda)), \quad f \mapsto f \otimes 1_{s\Lambda}.
\]

The cup product and brace operation on \( \overline{C}^*_{sg,L}(\Lambda, \Lambda) \) are defined in [66, Subsections 4.1 and 5.2]. Let us denote them by \( - \cup_L - \) and \( -\{-, \ldots, -\}_L \), respectively.

**Theorem 7.3.** ([66, Theorem 5.1]) The left singular Hochschild cochain complex \( \overline{C}^*_{sg,L}(\Lambda, \Lambda) \), equipped with the mentioned cup product and brace operation, is a brace \( B_\infty \)-algebra. Consequently, \( (\text{HH}^*_{sg}(\Lambda, \Lambda), - \cup_L - \{\ldots, -\}_L) \) is a Gerstenhaber algebra.

The above two Gerstenhaber algebra structures on \( \text{HH}^*_{sg}(\Lambda, \Lambda) \) are actually the same.

**Proposition 7.4.** The above two Gerstenhaber algebras \( (\text{HH}^*_{sg}(\Lambda, \Lambda), - \cup_L - \{\ldots, -\}_L) \) and \( (\text{HH}^*_{sg}(\Lambda, \Lambda), - \cup_R - \{\ldots, -\}_R) \) coincide.
Proof. By [66, Proposition 4.7], both $-\cup_L -$ and $-\cup_R -$ coincide with the Yoneda product on $\text{HH}^*_{\text{sg}}(\Lambda, \Lambda)$. Then we have $-\cup_L - = -\cup_R -$. By [67, Corollary 5.10], we infer that $[-,-]_R$ is isomorphic to a subgroup $G_A$ of the singular derived Picard group of $\Lambda$. Similarly, one proves that $[-,-]_L$ is also isomorphic to $G_A$. For more details, we refer to [67]. □

Remark 7.5. In Appendix A, we will prove that there is a (non-strict) $B_\infty$-isomorphism

$$
\mathcal{C}^*_{\text{sg},L}(\Lambda, \Lambda) \simeq \mathcal{C}^*_{\text{sg},R}(\Lambda^{\text{op}}, \Lambda^{\text{op}})^{\text{opp}},
$$

whose first component is the swap isomorphism

$$
T: \mathcal{C}^*_{\text{sg},L}(\Lambda, \Lambda) \longrightarrow \mathcal{C}^*_{\text{sg},R}(\Lambda^{\text{op}}, \Lambda^{\text{op}}),
$$

defined in (A.6). In particular, this $B_\infty$-isomorphism induces an isomorphism of Gerstenhaber algebras

$$
(\text{HH}^*_{\text{sg}}(\Lambda, \Lambda), -\cup_L - , [-,-]_L) \simeq (\text{HH}^*_{\text{sg}}(\Lambda^{\text{op}}, \Lambda^{\text{op}}), -\cup_R - , [-,-]^{\text{opp}}_R),
$$

where $[f,g]^{\text{opp}}_R = -[f,g]_R$.

In contrast to Proposition 7.4, we do not know whether the $B_\infty$-algebras $\mathcal{C}^*_{\text{sg},L}(\Lambda, \Lambda)$ and $\mathcal{C}^*_{\text{sg},R}(\Lambda, \Lambda)$ are isomorphic in $\text{Ho}(B_\infty)$. Actually, it seems that there is even no obvious natural quasi-isomorphism of complexes between them, although both of them compute the same $\text{HH}^*_{\text{sg}}(\Lambda, \Lambda)$.

7.2. The relative singular Hochschild cochain complexes. We will need the relative version of the singular Hochschild cochain complexes.

Let $E = \bigoplus_{i=1}^n k e_i \subseteq \Lambda$ be a semisimple subalgebra of $\Lambda$ with a decomposition $e_1 + \cdots + e_n = 1_\Lambda$ of the unity into orthogonal idempotents. Assume that $\varepsilon: \Lambda \rightarrow E$ is a split surjective algebra homomorphism such that the inclusion map $E \hookrightarrow \Lambda$ is a section of $\varepsilon$.

The following notion is slightly different from the one in Subsection 7.1. We will denote the quotient $E$-$E$-bimodule $\Lambda/(E \cdot 1_\Lambda)$ by $\overline{\Lambda}$. The quotient $k$-module $\Lambda/(k \cdot 1_\Lambda)$ will be temporarily denoted by $\overline{\Lambda}$ in this subsection. Identifying $\overline{\Lambda}$ with $\text{Ker}(\varepsilon)$, we obtain a natural injection

$$
\xi: \overline{\Lambda} \longrightarrow \overline{\Lambda}, \quad x + (E \cdot 1_\Lambda) \longmapsto x + (k \cdot 1_\Lambda)
$$

for each $x \in \text{Ker}(\varepsilon)$.

Consider the graded $\Lambda$-$\Lambda$-bimodule of $E$-relative right noncommutative differential $p$-forms

$$
\Omega^p_{\text{nc},R,E}(\Lambda) = (s\overline{\Lambda})^\otimes E \otimes E \Lambda.
$$

Similarly, $\Omega^p_{\text{nc},R,E}(\Lambda)$ is isomorphic to the cokernel of the differential in $\text{Bar}_E(\Lambda)$

$$
\Lambda \otimes_E (s\overline{\Lambda})^\otimes E \otimes E \Lambda \xrightarrow{d_{\text{ex}}} \Lambda \otimes_E (s\overline{\Lambda})^\otimes E \otimes E \Lambda.
$$

The $E$-relative right singular Hochschild cochain complex $\mathcal{C}^*_{\text{sg},R,E}(\Lambda, \Lambda)$ is defined to be the colimit of the inductive system

$$
\mathcal{C}^*_E(\Lambda, \Lambda) \xrightarrow{\theta_{0,R,E}} \mathcal{C}^*_E(\Lambda, \Omega^1_{\text{nc},R,E}(\Lambda)) \xrightarrow{\theta_{1,R,E}} \cdots \xrightarrow{\theta_{p,R,E}} \cdots,
$$

where

$$
\theta_{p,R,E}: \mathcal{C}^*_E(\Lambda, \Omega^p_{\text{nc},R,E}(\Lambda)) \longrightarrow \mathcal{C}^*_E(\Lambda, \Omega^{p+1}_{\text{nc},R,E}(\Lambda)), \quad f \longmapsto 1_{s\overline{\Lambda}} \otimes_E f. \tag{7.5}
$$
We have the natural (k-linear) projections
\[ \varpi^m: (s\Lambda)^{\otimes m} \rightarrow (s\Lambda)^{\otimes m}, \quad \text{for all } m \geq 0. \]
Denote by \( t_p \) the natural injection
\[ \Omega^p_{nc,R,E}(\Lambda) \hookrightarrow \Omega^p_{nc,R}(\Lambda), \]
induced by \( \xi \). We have inclusions
\[ \text{Hom}_{E}((s\Lambda)^{\otimes m+p}, \Omega^p_{nc,R,E}(\Lambda)) \hookrightarrow \text{Hom}((s\Lambda)^{\otimes m+p}, \Omega^p_{nc,R}(\Lambda)), \]
where the first inclusion is induced by the projection \( \varpi^{m+p} \), and the second one is given by \( \text{Hom}((s\Lambda)^{\otimes m+p}, t_p). \) Therefore, we have the injection
\[ \overline{C}^m_E(\Lambda, \Omega^p_{nc,R,E}(\Lambda)) \hookrightarrow \overline{C}^m(\Lambda, \Omega^p_{nc,R}(\Lambda)). \]
For any \( m \in \mathbb{Z} \), we have the following commutative diagram.
\[
\begin{array}{ccc}
C^m_E(\Lambda, \Lambda) & \xrightarrow{\theta_0,R,E} & \overline{C}^m_E(\Lambda, \Omega^1_{nc,R,E}(\Lambda)) \\
\downarrow & & \downarrow \\
C^m(\Lambda, \Lambda) & \xrightarrow{\theta_0,R} & \overline{C}^m(\Lambda, \Omega^1_{nc,R}(\Lambda)) \\
\end{array}
\]
It gives rise to an injection of complexes
\[ \iota: \overline{C}^*_{sg,R,E}(\Lambda, \Lambda) \hookrightarrow \overline{C}^*_{sg,R}(\Lambda, \Lambda). \]
We observe that the cup product and the brace operation on \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \) restrict to \( \overline{C}^*_{sg,R,E}(\Lambda, \Lambda) \). Thus \( \overline{C}^*_{sg,R,E}(\Lambda, \Lambda) \) inherits a brace \( B_{\infty}\)-algebra structure.

**Lemma 7.6.** The injection \( \iota: \overline{C}^*_{sg,R,E}(\Lambda, \Lambda) \hookrightarrow \overline{C}^*_{sg,R}(\Lambda, \Lambda) \) is a strict \( B_{\infty}\)-quasi-isomorphism.

**Proof.** Since \( \iota \) preserves the cup products and brace operations, it follows from Lemma 5.9 that \( \iota \) is a strict \( B_{\infty}\)-morphism.

It remains to prove that \( \iota \) is a quasi-isomorphism of complexes. The injection \( \xi: \Lambda \rightarrow \overline{\Lambda} \) induces an injection of complexes of \( \Lambda\)-\( \Lambda \)-bimodules
\[ \text{Bar}_E(\Lambda) \hookrightarrow \text{Bar}(\Lambda) = \bigoplus_{n \geq 0} \Lambda \otimes (s\Lambda)^{\otimes n} \otimes \Lambda. \]
Recall that \( \Omega^p_{nc,R}(\Lambda) \) is isomorphic to the cokernel of the external differential \( d_{ex} \) in \( \text{Bar}(\Lambda) \) and that \( \Omega^p_{nc,R,E}(\Lambda) \) is isomorphic to the cokernel of \( d_{ex} \) in \( \text{Bar}_E(\Lambda) \). We infer that both \( \overline{C}^*_{sg,R,E}(\Lambda, \Lambda) \) and \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \) compute \( \text{HH}^*_{sg}(\Lambda, \Lambda); \) compare [66, Theorem 3.6]. Therefore, the injection \( \iota \) is a quasi-isomorphism. \( \square \)

Similar, we define the \( E\)-relative left singular Hochschild cochain complex \( \overline{C}^*_{sg,E,L}(\Lambda, \Lambda) \) as the colimit of the inductive system
\[ \overline{C}^*(\Lambda, \Lambda) \xrightarrow{\theta_{0,L,E}} \overline{C}^*(\Lambda, \Omega^1_{nc,L,E}(\Lambda)) \xrightarrow{\theta_{1,L,E}} \cdots \xrightarrow{\theta_{p-1,L,E}} \overline{C}^*(\Lambda, \Omega^p_{nc,L,E}(\Lambda)) \xrightarrow{\theta_{p,L}} \cdots, \]
where $\Omega_{nc,L,E}^p(\Lambda) = \Lambda \otimes_E (s\Lambda)^{\otimes p}$ is the graded $\Lambda$-$\Lambda$-bimodule of $E$-relative left noncommutative differential $p$-forms and the maps
\[
\theta_{p,L,E} : C^*(E(\Lambda, \Omega_{nc,L,E}^p(\Lambda))) \to C^*(E(\Lambda, \Omega_{nc,L,E}^{p+1}(\Lambda))), \quad f \mapsto f \otimes_E 1_{s\Lambda}.
\]
(7.6)

We have an analogous result of Lemma 7.6.

Lemma 7.7. There is a natural injection $C^*_{sg,L,E}(\Lambda, \Lambda) \hookrightarrow C^*_{sg,L}(\Lambda, \Lambda)$, which is a strict $B_\infty$-quasi-isomorphism. □

7.3. The brace operation on the right singular Hochschild cochain complex. We will recall the brace operation $\{-\cdots,-\}^R$ on $C^*_{sg,R}(\Lambda, \Lambda)$. It might be carried over word by word from the left case, studied in [66, Section 5], but with different graph presentations.

We mention that, similar to the left case, the brace operation $\{-\cdots,-\}^R$ is induced from a natural action of the cellular chain dg operad of the spineless cacti operad [36].

Similar to [66, Figure 1], any element
\[
f \in C^{m-p}_{sg-R}(\Lambda, \Omega_{nc,R}^p(\Lambda)) = \text{Hom}((s\Lambda)^{\otimes m}, (s\Lambda)^{\otimes p} \otimes \Lambda)
\]
can be depicted by a tree-like graph and a cactus-like graph (cf. Figure 1):

- The tree-like presentation is the usual graphic presentation of morphisms in tensor categories (cf. e.g. [34]). We read the graph from top to bottom and left to right. We use the color blue to distinguish the special output $\Lambda$ and the other black outputs represent $s\Lambda$. The inputs $(s\Lambda)^{\otimes m}$ are ordered from left to right at the top but are labelled by $1, 2, \ldots, m$ from right to left. Similarly, the outputs $(s\Lambda)^{\otimes p} \otimes \Lambda$ are ordered from left to right at the bottom but are labelled by $0, 1, 2, \ldots, p$ from right to left. The above labelling is convenient when taking the colimit (7.7); see Figure 2.
- The cactus-like presentation is read as follows. The image of $0 \in \mathbb{R}$ in the red circle $S^1 = \mathbb{R}/\mathbb{Z}$ is decorated by a blue dot, called the zero point of $S^1$. The center of $S^1$ is decorated by $f$. The blue radius represents the special output $\Lambda$. The inputs $(s\Lambda)^{\otimes m}$ are represented by $m$ black radii (called inward radii) on the right semicircle pointing towards the center in clockwise. Similarly, the outputs $(s\Lambda)^{\otimes p} \otimes \Lambda$ are represented by $p$ black radii (called outward radii) on the left semicircle pointing outwards the center in counterclockwise. The cactus-like presentation is inspired by the spineless cacti operad.

![Figure 1](image-url)

Figure 1. The tree-like and cactus-like presentations of $f \in C^{m-p}_{sg-R}(\Lambda, \Omega_{nc,R}^p(\Lambda))$. 
Recall that the maps in the inductive system (7.3) of \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \) are given by

\[
\theta_{p,R}: \overline{C}^*_{sg,R}(\Lambda, \Omega^{p}_{nc,R}(\Lambda)) \rightarrow \overline{C}^*_{sg,R}(\Lambda, \Omega^{p+1}_{nc,R}(\Lambda)), \quad f \mapsto 1 \otimes f.
\]

That is, for any \( f \in \overline{C}^*_{sg,R}(\Lambda, \Omega^{p}_{nc,R}(\Lambda)) \) we have

\[
f = 1 \otimes f = 1 \otimes 2 \otimes f = \cdots = 1 \otimes m \otimes f = \cdots
\]

in \( \overline{C}^*_{sg,R}(\Lambda, \Lambda) \). Thus, any element \( f \in \overline{C}^{m-p}_{sg,R}(\Lambda, \Lambda) \) is depicted by Figure 2, where the straight line represents the identity map of \( s \Lambda \). Thanks to (7.7), we can freely add or remove the straight lines from the left side and from the top, respectively.

The tree-like and cactus-like presentations have their own advantages: it is much easier to read off the corresponding morphisms from the tree-like presentation (as we have seen from tensor categories), while it is more convenient to construct the brace operation using the cactus-like presentation as you will see in the sequel.

\[
\text{Figure 2. The colimit maps } \theta^*_{s,R}, \text{ where the straight line represents the identity map of } s \Lambda.
\]

For any \( k \geq 0 \), let us define the brace operation of degree \(-k\)

\[
{-\{ -, \ldots, - \}}_{R}: \overline{C}^*_{sg,R}(\Lambda, \Lambda) \otimes \overline{C}^*_{sg,R}(\Lambda, \Lambda)^{\otimes k} \rightarrow \overline{C}^*_{sg,R}(\Lambda, \Lambda).
\]

**Definition 7.8.** Let \( x \in \overline{C}^{m-p}_{sg,R}(\Lambda, \Omega^{p}_{nc,R}(\Lambda)) \) and \( y_i \in \overline{C}^{m_i-q_i}_{sg,R}(\Lambda, \Omega^{q_i}_{nc,R}(\Lambda)) \) for \( 1 \leq i \leq k \). Set \( m' = m - p \) and \( n'_r = n_r - q_r - 1 \) for \( 1 \leq r \leq k \). Then we define

\[
x\{y_1, \ldots, y_k\}_R \in \text{Hom}((s \Lambda)^{\otimes m + n_1 + n_2 + \cdots + n_k - k}, \Omega^{p+q_1+\cdots+q_k}_{nc,R}(\Lambda))
\]

as follows: for \( k = 0 \), we set \( x\{\emptyset\} = x \); for \( k \geq 1 \), we set

\[
x\{y_1, \ldots, y_k\}_R = \sum_{0 \leq j \leq k} (-1)^{k-j} B^{(i_1, \ldots, i_j)}_{(l_1, \ldots, l_{k-j})}(x; y_1, \ldots, y_k),
\]

where the summand \( B^{(i_1, \ldots, i_j)}_{(l_1, \ldots, l_{k-j})}(x; y_1, \ldots, y_k) \) is illustrated in Figure 4; where the extra sign \((-1)^{k-j}\) is added in order to make sure that the brace operation is compatible with the
colimit maps $\theta_{s,R}$. When the operation $B_{(l_1, \ldots, l_{k-j})}^{(i_1, \ldots, i_j)}(x; y_1, \ldots, y_k)$ applies to elements, an additional sign $(-1)^\epsilon$ appears due to Koszul sign rule, where

$$
\epsilon := \left( m' + \sum_{i=1}^{k} n'_i \right) \left( p + \sum_{i=1}^{k} q_i \right) + m' p + \sum_{i=1}^{k} n'_i q_i \\
+ \sum_{r=1}^{k-j} (n'_1 + \cdots + n'_r + l_r - 1) n'_r + \sum_{s=1}^{j} (n'_1 + \cdots + n'_{k-s+1} + m' - i_s - 1) n'_{k-s+1}.
$$

Figure 3. A cell in the spineless cacti operad.

Let us now describe Figure 4 in detail and how to read off $B_{(l_1, \ldots, l_{k-j})}^{(i_1, \ldots, i_j)}(x; y_1, \ldots, y_k)$.

(i) We start with the cell depicted in Figure 3 of the spineless cacti operad. As in Figure 2, we use the element $x$ to decorate the circle 1 of Figure 3 and similarly use the element $y_i$ to decorate the circle $i+1$ for $1 \leq i \leq k$.

(ii) The left semicircle of the circle 1 is divided into $p + 1$ arcs by the outward radii of $x$. For each $1 \leq r \leq k-j$, the red curve of the circle $r$ (decorated by $y_r$) intersects with the circle 1 at the open arc between the $(l_r - 1)$-th and $l_r$-th outward radii of $x$. The red curves are not allowed to intersect with each other.

(iii) On the right semicircle of the circle 1, we have $m$ intersection points of the $m$ inward radii of $x$ with the circle 1. Unlike (ii), for each $1 \leq r \leq j$ the red curve of the circle $k-r+1$ (decorated by $y_{k-r+1}$) intersects with the circle 1 exactly at the $i_r$-th intersection point.

(iv) We connect some inputs with outputs using the following rule.

- For each $1 \leq r \leq j$, connect the blue output of $y_{k-r+1}$ with the $i_r$-th inward radius of the circle 1 on the right semi-circle of the circle 1. Then starting from the blue dot (i.e. the zero point) of circle 1, walk counterclockwise along the red path (i.e. the outside of the red circles and the red curves) and record the inward and outward radii (including the blue radii) in order as a sequence $S$. When an outward radius is found closely behind an inward radius in $S$, we call this pair in-out.

- Let us define the following operation.

**Deletion Process:** Once the pair in-out appears in the sequence $S$, we connect the outward radius with the inward radius via a dashed arrow in Figure 4. Delete
this pair and renew the sequence $\mathcal{S}$. Then repeat the above operations iteratively until no pair in-out left in $\mathcal{S}$.

(v) After applying the above Deletion Process, we obtain a final sequence $\mathcal{S}$ with all outward radii preceding all inward radii. Finally, we translate the updated cactus-like graph into a tree-like graph by putting the inputs (in the final sequence) on the top and outputs on the bottom. We therefore get the $k$-linear map

$$B_{(i_1, \ldots, j_{k-j})}^{(i_1, \ldots, i_j)}(x; y_1, \ldots, y_k): (s\Lambda)^{\otimes u} \longrightarrow (s\Lambda)^{\otimes v} \otimes \Lambda,$$

where $u$ and $v$ are respectively the numbers of the inward radii and outward radii in the final sequence $\mathcal{S}$. See Example 7.9 below.

Figure 4. The summand $B_{(i_1, \ldots, j_{k-j})}^{(i_1, \ldots, i_j)}(x; y_1, \ldots, y_k)$ of $x\{y_1, \ldots, y_k\}_R$.

Note that $x\{y_1, \ldots, y_k\}_R$ is compatible with the colimit maps $\theta_{s,R}$ and thus it induces a well-defined operation (still denoted by $-\{\ldots,-\}_R$) on $\overline{C}_{s_{\text{nc},R}}(\Lambda, \Lambda)$. When $p = q_1 = \cdots = q_k = 0$, the above $x\{y_1, \ldots, y_k\}_R$ coincides with the usual brace operation on $\overline{C}^0(\Lambda, \Lambda)$; compare (6.1).

Example 7.9. Let

$$f \in \overline{C}^0(\Lambda, \Omega_{\text{nc},R}^3(\Lambda)) = \text{Hom}(s\Lambda)^{\otimes 5}, (s\Lambda)^{\otimes 3} \otimes \Lambda)$$
$$g_1 \in \overline{C}^0(\Lambda, \Omega_{\text{nc},R}^1(\Lambda)) = \text{Hom}(s\Lambda)^{\otimes 3}, s\Lambda \otimes \Lambda)$$
$$g_2 \in \overline{C}^0(\Lambda, \Omega_{\text{nc},R}^1(\Lambda)) = \text{Hom}(s\Lambda)^{\otimes 3}, (s\Lambda)^{\otimes 3} \otimes \Lambda)$$
$$g_3 \in \overline{C}^0(\Lambda, \Omega_{\text{nc},R}^3(\Lambda)) = \text{Hom}(s\Lambda)^{\otimes 2}, (s\Lambda)^{\otimes 3} \otimes \Lambda).$$
Then the operation $B_{(2,4)}^{(2,4)}(f; g_1, g_2, g_3)$ is depicted in Figure 5. It is represented by the following composition of maps (Here, we ignore the identity map $1_{sX} \otimes 1_{sX}$ on the left)

$$(1_{sX} \otimes g_1 \otimes 1_{sX} \otimes 1_{sX})(1_{sX} \otimes 2^{\otimes 2}_{sX} \otimes f)(2_{sX} \otimes 1_{sX} \otimes g_3 \otimes 1_{sX}): (sX)^{\otimes 4} \rightarrow (sX)^{\otimes 4} \otimes \Lambda$$

where $g: (sX)^{\otimes m} \rightarrow (sX)^{\otimes p} \otimes \Lambda \xrightarrow{1_{sX} \otimes \pi} (sX)^{\otimes p+1}$ and $\pi: \Lambda \rightarrow sX$ is the natural projection $a \mapsto sa$ of degree $-1$.

![Figure 5. An example of $B_{(2,4)}^{(2,4)}(f; g_1, g_2, g_3)$.](image)

8. $B_{\infty}$-Quasi-Isomorphisms Induced by One-Point (Co)Extensions and Bimodules

In this section, we prove that the (relative) singular Hochschild cochain complexes, as $B_{\infty}$-algebras, are invariant under one-point (co)extensions of algebras and singular equivalences with levels.

These invariance results are analogous to the ones in Subsection 2.2. However, the proofs here are much harder, since the construction of the singular Hochschild cochain complex is involved.

Throughout this section, $\Lambda$ and $\Pi$ will be finite dimensional $k$-algebras.

8.1. Invariance under one-point (co)extensions. Let $E = \bigoplus_{i=1}^{n} k e_i \subseteq \Lambda$ be a semisimple subalgebra of $\Lambda$. Recall that $\Lambda = \Lambda / (E \cdot 1_{\Lambda})$. We have the $B_{\infty}$-algebra $C_{\text{sg},R,E}(\Lambda, \Lambda)$ of the $E$-relative right singular Hochschild cochain complex of $\Lambda$.

Consider the one-point coextension $\Lambda' = \begin{pmatrix} k & M \\ 0 & \Lambda \end{pmatrix}$ in Subsection 2.2. Set $e' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and identify $\Lambda$ with $(1_{\Lambda'} - e')\Lambda'(1_{\Lambda'} - e')$. We take $E' = k e' \oplus E$, which is a semisimple subalgebra of $\Lambda'$. Set $\Lambda' = \Lambda' / (E' \cdot 1_{\Lambda'})$.

To consider the $E'$-relative right singular Hochschild cochain complex $C_{\text{sg},R,E'}(\Lambda', \Lambda')$, we identify $\Lambda'$ with $\Lambda' \oplus M$. Then we have a natural isomorphism for each $m \geq 1$

$$(sX)^{\otimes E m} \simeq (sX)^{\otimes E m} \oplus sM \otimes_E (sX)^{\otimes E m-1}.$$  (8.1)
The following decomposition follows immediately from (8.1).

\[ \text{Hom}_{E', E'}((sN)_{\otimes E'}^m, (sN')_{\otimes E'} \otimes E' \Lambda) \]
\[ \simeq \text{Hom}_{E, E}((sN)_{\otimes E}^m, (sN')_{\otimes E} \otimes E \Lambda) \oplus \text{Hom}_{p, E}(sM \otimes E (sN)_{\otimes E}^{m-1}, sM \otimes E (sN')_{\otimes E}^{m-1} \otimes E \Lambda) \]

We take the colimits along \( \theta_{p, R, E} \) for \( \Lambda' \), and along \( \theta_{p, E} \) for \( \Lambda \) in (7.5). Then the above decomposition yields a restriction of complexes

\[ \overline{C}^*_{sg, R, E'}(\Lambda', \Lambda') \to \overline{C}^*_{sg, R, E}(\Lambda, \Lambda). \]

It is routine to check that the above restriction preserves the cup products and brace operations, i.e. it is a strict \( B_\infty \)-morphism.

**Lemma 8.1.** Let \( \Lambda' \) be the one-point coextension as above. Then the restriction map \( \overline{C}^*_{sg, R, E'}(\Lambda', \Lambda') \to \overline{C}^*_{sg, R, E}(\Lambda, \Lambda) \) is a strict \( B_\infty \)-isomorphism.

**Proof.** The crucial fact is that \( sN' \otimes E' sM = 0 \). Then by the very definition, \( \theta_{p, R, E'} \) vanishes on the following component

\[ \text{Hom}_{p, E}(sM \otimes E (sN)_{\otimes E}^{m-1}, sM \otimes E (sN')_{\otimes E}^{m-1} \otimes E \Lambda). \]

It follows that taking the colimits, the restriction becomes an actual isomorphism. \( \square \)

We now consider the \( E \)-relative left singular Hochschild cochain complex \( \overline{C}^*_{sg, L, E}(\Lambda, \Lambda) \), and the \( E' \)-relative left singular Hochschild cochain complex \( \overline{C}^*_{sg, L, E'}(\Lambda', \Lambda') \). Using the natural isomorphism (8.1), we have a decomposition

\[ \text{Hom}_{E', E'}((sN)_{\otimes E'}^m, \Lambda' \otimes E' (sN')_{\otimes E'} \Lambda) \]
\[ \simeq \text{Hom}_{E, E}((sN)_{\otimes E}^m, \Lambda \otimes E (sN')_{\otimes E} \Lambda) \oplus \text{Hom}_{p, E}(sM \otimes E (sN)_{\otimes E}^{m-1}, sM \otimes E (sN')_{\otimes E}^{m-1} \otimes E \Lambda) \]
\[ \oplus \text{Hom}_{p, E}(sM \otimes E (sN)_{\otimes E}^{m-1}, M \otimes E (sN')_{\otimes E}). \]

Similar as above, the decomposition will give rise to a restriction of complexes

\[ \overline{C}^*_{sg, L, E'}(\Lambda', \Lambda') \to \overline{C}^*_{sg, L, E}(\Lambda, \Lambda), \]

which is a strict \( B_\infty \)-morphism.

Unlike the isomorphism in Lemma 8.1, this restriction is only a quasi-isomorphism.

**Lemma 8.2.** Let \( \Lambda' \) be the one-point coextension. Then the above restriction map \( \overline{C}^*_{sg, L, E'}(\Lambda', \Lambda') \to \overline{C}^*_{sg, L, E}(\Lambda, \Lambda) \) is a strict \( B_\infty \)-quasi-isomorphism.

**Proof.** It suffices to show that the kernel of the restriction map is acyclic. For this, we observe that the decomposition (8.2) induces a decomposition of graded vector spaces

\[ \overline{C}^*_{sg, L, E'}(\Lambda', \Lambda') \simeq \overline{C}^*_{sg, L, E}(\Lambda, \Lambda) \oplus X^* \otimes Y^*. \]

Here, \( X^* \) is the colimit of graded vector spaces along the maps
\[ \text{Hom}_{E}(sM \otimes E (sN)_{\otimes E}^{m-1}, k' \otimes sM \otimes E (sN')_{\otimes E}^{m-1} \rightarrow \text{Hom}_{E}(sM \otimes E (sN)_{\otimes E}^{m-1}, k' \otimes sM \otimes E (sN')_{\otimes E}^{m-1} \otimes E \Lambda) \]
which sends \( f \) to \( f \otimes E 1_{sN} \). Similarly, \( Y^* \) is the colimit along the maps
\[ \text{Hom}_{E}(sM \otimes E (sN)_{\otimes E}^{m-1}, M \otimes E (sN')_{\otimes E} \rightarrow \text{Hom}_{E}(sM \otimes E (sN)_{\otimes E}^{m-1}, M \otimes E (sN')_{\otimes E} \otimes E \Lambda) \]
sending \( f \) to \( f \otimes E 1_{sN} \).
We observe that $X^*$ is, as a graded vector space, isomorphic to the 1-shift of $Y^*$ by identifying $k e' \otimes sM$ with $sM$. Then we have

$$X^* \simeq \Sigma(Y^*). \quad (8.4)$$

The differential of $\overline{C}^*_E(\Lambda', \Lambda')$ induces a differential on the decomposition (8.3). Namely we have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Hom}_{E', E'}(s\Lambda'^{\otimes_{E'}m}, \Lambda' \otimes_{E'} s\Lambda'^{\otimes_{E'}p}) & \xrightarrow{\delta_{\Lambda'}} & \text{Hom}_{E, E}(\overline{s\Lambda}^{\otimes_{E}m}, \Lambda \otimes_{E} \overline{s\Lambda}^{\otimes_{E}p}) \\
\downarrow & & \downarrow \\
\text{Hom}_{E', E'}(s\Lambda'^{\otimes_{E'}m+1}, \Lambda' \otimes_{E'} s\Lambda'^{\otimes_{E'}p}) & \xrightarrow{\delta_{\Lambda'}} & \text{Hom}_{E, E}(\overline{s\Lambda}^{\otimes_{E}m+1}, \Lambda \otimes_{E} \overline{s\Lambda}^{\otimes_{E}p})
\end{array}
\]

where we write elements in the decomposition (8.3) as column vectors.

Let us explain the entries of the $3 \times 3$-matrix in (8.5).

(i) We observe that $\delta_{\Lambda'}$ restricts to a differential of the third component, denoted by $\delta_{Y'}$. Using the natural isomorphism $k e' \otimes sM \simeq sM$, the differential on the second component is given by $\Sigma(\delta_{Y'})$.

(ii) The differential $\delta_{\Lambda}$ is the external differential of $\overline{C}^*_E(\Lambda, \Lambda \otimes_{E} \overline{s\Lambda}^{\otimes_{E}p})$.

(iii) The differential $\tilde{\delta}$ is given by

$$\tilde{\delta}(f)(sx \otimes_E s\pi_{1,m}) = (-1)^{m-p}x \otimes_{\Lambda} f(s\pi_{1,m})$$

for any $f \in \text{Hom}_{E, E}(\overline{s\Lambda}^{\otimes_{E}m}, \Lambda \otimes_{E} \overline{s\Lambda}^{\otimes_{E}p})$.

(iv) The differential $\theta$ is given as follows: for any $f \in \text{Hom}_{E, E}(sM \otimes_{E} (s\Lambda)^{\otimes_{E}m-1}, ke' \otimes sM \otimes_{E} (s\Lambda)^{\otimes_{E}p-1})$, the corresponding element $\theta(f) \in \text{Hom}_{E, E}(sM \otimes_{E} (s\Lambda)^{\otimes_{E}m}, M \otimes_{E} (s\Lambda)^{\otimes_{E}p})$ is defined by

$$\theta(f)(sx \otimes E s\pi_{1,m}) = f(sx \otimes E s\pi_{1,m-1}) \otimes_{E} s\pi_{m}.$$

Here, we use the natural isomorphism $k e' \otimes sM \to M$ of degree one, and thus $\theta$ is a map of degree one. We observe that after taking the colimits, $\theta$ becomes the identity map

$$1 : X^* \to Y^*, \quad \Sigma(y) \mapsto y$$

using the identification (8.4).

Thus, the kernel of the restriction map is identified with the subcomplex

$$\left( X^* \oplus Y^*, \left( \begin{array}{cc} \Sigma(\delta_{Y'}) & 0 \\ 0 & \Sigma(\delta_{Y'}) \end{array} \right) \right),$$

which is exactly the mapping cone of the identity of $Y^*$. It follows that this kernel is acyclic, as required. \qed
Remark 8.3. The decomposition (8.3) induces an embedding of graded vector spaces

\[ C_{sg,L,E}(\Lambda, \Lambda) \rightarrow C_{sg,L,E}'(\Lambda', \Lambda'). \]

However, it is in general not a cochain map, since the differential \( \tilde{\delta} \) in the matrix of (8.5) is nonzero.

Let us consider the one-point extension \( \Lambda'' = \begin{pmatrix} \Lambda & N \\ 0 & k \end{pmatrix} \) in Subsection 2.2. We set \( e'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( E'' = E \oplus ke'' \subseteq \Lambda'' \). Set \( \Lambda'' = \Lambda''/(E'' \cdot 1_{\Lambda''}) \), which is identified with \( \Lambda \oplus N \).

We first consider the \( E \)-relative left singular Hochschild cochain complexes \( C_{sg,L,E}(\Lambda, \Lambda) \) and \( E'' \)-relative left singular Hochschild cochain complexes \( C_{sg,L,E'}(\Lambda'', \Lambda'') \).

The following result is analogous to Lemmas 8.1.

Lemma 8.4. Let \( \Lambda'' \) be the one-point extension as above. Then we have a strict \( B_\infty \) isomorphism

\[ C_{sg,L,E''}(\Lambda'', \Lambda'') \rightarrow C_{sg,L,E}(\Lambda, \Lambda). \]

Proof. The argument is similar as above. For example, we have a similar decomposition

\[ \text{Hom}_{E''}((s\Lambda')\otimes_E m, \Lambda'' \otimes_E (s\Lambda') \otimes_E p) \cong \text{Hom}_{E,E'}((s\Lambda)\otimes_E m \otimes_E (s\Lambda) \otimes_E p) \oplus \text{Hom}_{E,E'}((s\Lambda)\otimes_E (s\Lambda') \otimes_E p) \oplus \text{Hom}_{E,E'}((s\Lambda)\otimes_E m \otimes_E (s\Lambda) \otimes_E p). \]

We observe the crucial fact \( sN \otimes_E s\Lambda'' = 0 \). Then taking the colimit along \( \theta_{p,L,E''} \) in (7.6), the above rightmost component will vanish. This gives rise to the desired \( B_\infty \) isomorphism. \( \square \)

The following result is analogous to Lemma 8.2. We omit the same argument.

Lemma 8.5. Let \( \Lambda'' \) be the one-point extension as above. Then the obvious restriction

\[ C_{sg,R,E''}(\Lambda'', \Lambda'') \rightarrow C_{sg,R,E}(\Lambda, \Lambda) \]

is a strict \( B_\infty \)-quasi-isomorphism. \( \square \)

8.2. \( B_\infty \)-quasi-isomorphisms induced by a bimodule. We will prove that the \( B_\infty \) algebra structures on singular Hochschild cochain complexes are invariant under singular equivalences with levels. Indeed, a slightly stronger statement will be established in Theorem 8.6.

We fix a \( \Lambda \)-\( \Pi \)-bimodule \( M \), over which \( k \) acts centrally. Therefore, \( M \) is also viewed a left \( \Lambda \otimes \Pi^{\text{op}} \)-module. We require further that the underlying left \( \Lambda \)-module \( \Lambda M \) and the right \( \Pi \)-module \( M \Pi \) are both projective.

Denote by \( D_{sg}(\Lambda^e) \), \( D_{sg}(\Pi^e) \) and \( D_{sg}(\Lambda \otimes \Pi^{\text{op}}) \) the singularity categories of the algebras \( \Lambda^e \), \( \Pi^e \) and \( \Lambda \otimes \Pi^{\text{op}} \), respectively. The projectivity assumption on \( M \) guarantees that the following two triangle functors are well defined.

\[ - \otimes \Lambda : D_{sg}(\Lambda^e) \rightarrow D_{sg}(\Lambda \otimes \Pi^{\text{op}}) \]

\[ M \otimes \Pi - : D_{sg}(\Pi^e) \rightarrow D_{sg}(\Lambda \otimes \Pi^{\text{op}}) \] (8.6)
The functor $- \otimes \Lambda M$ sends $\Lambda$ to $M$, and $M \otimes \Pi -$ sends $\Pi$ to $M$. Consequently, they induce the following maps

$$HH^i_{sg}(\Lambda, \Lambda) \xrightarrow{\alpha^i} \text{Hom}_{D_{sg}(\Lambda \otimes \Pi^{op})}(M, \Sigma^i(M)) \xleftarrow{\beta^i} HH^i_{sg}(\Pi, \Pi)$$

for all $i \in \mathbb{Z}$. Here, we recall that the singular Hochschild cohomology groups are defined as

$$HH^i_{sg}(\Lambda, \Lambda) = \text{Hom}_{D_{sg}(\Lambda)}(\Lambda, \Sigma^i(\Lambda)) \quad \text{and} \quad HH^i_{sg}(\Pi, \Pi) = \text{Hom}_{D_{sg}(\Pi^{op})}(\Pi, \Sigma^i(\Pi)).$$

Moreover, these groups are computed by the right singular Hochschild cochain complexes $C^*_{sg,R}(\Lambda, \Lambda)$ and $C^*_{sg,R}(\Pi, \Pi)$, respectively; see Subsection 7.1 for details.

Under reasonable conditions, the bimodule $M$ induces an isomorphism between the above two right singular Hochschild cochain complexes.

**Theorem 8.6.** Let $M$ be a $\Lambda$-$\Pi$-bimodule such that it is projective both as a left $\Lambda$-module and as a right $\Pi$-module. Suppose that the two maps in (8.7) are isomorphisms for each $i \in \mathbb{Z}$. Then we have an isomorphism

$$C^*_{sg,R}(\Lambda, \Lambda) \cong C^*_{sg,R}(\Pi, \Pi)$$

in the homotopy category $\text{Ho}(B_{\infty})$ of $B_{\infty}$-algebras.

We postpone the proof until the end of this section, whose argument is adapted from the one developed in [40]. We will consider a triangular matrix algebra $\Gamma$, using which we construct two strict $B_{\infty}$-quasi-isomorphisms connecting $C^*_{sg,R}(\Lambda, \Lambda)$ to $C^*_{sg,R}(\Pi, \Pi)$.

We now apply Theorem 8.6 to singular equivalences with levels, in which case the two maps in (8.7) are indeed isomorphisms for each $i \in \mathbb{Z}$.

**Proposition 8.7.** Assume that $(M, N)$ defines a singular equivalence with level $n$ between $\Lambda$ and $\Pi$. Then the maps $\alpha^i$ and $\beta^i$ in (8.7) are isomorphisms for all $i \in \mathbb{Z}$. Consequently, there is an isomorphism $C^*_{sg,R}(\Lambda, \Lambda) \cong C^*_{sg,R}(\Pi, \Pi)$ in $\text{Ho}(B_{\infty})$.

It follows that a singular equivalence with a level gives rise to an isomorphism of Gerstenhaber algebras

$$HH^*_{sg}(\Lambda, \Lambda) \cong HH^*_{sg}(\Pi, \Pi).$$

We refer to [67] for an alternative proof of this isomorphism.

**Proof of Proposition 8.7.** By Theorem 8.6, it suffices to prove that both $\alpha^i$ and $\beta^i$ are isomorphisms. We only prove that the maps $\beta^i$ are isomorphisms, since a similar argument works for $\alpha^i$.

Indeed, we will prove a slightly stronger result. Let $\mathcal{X}$ (resp. $\mathcal{Y}$) be the full subcategory of $D_{sg}(\Pi^{op})$ (resp. $D_{sg}(\Lambda \otimes \Pi^{op})$) consisting of those complexes $X$, whose underlying complexes $X_{\Pi}$ of right $\Pi$-modules are perfect. The triangle functors

$$M \otimes \Pi - : \mathcal{X} \to \mathcal{Y} \text{ and } N \otimes \Lambda - : \mathcal{Y} \to \mathcal{X}$$

are well defined. We claim that they are equivalences. This claim clearly implies that $\beta^i$ are isomorphisms.
For the proof of the claim, we observe that for a bounded complex $P$ of projective $\Pi^e$-modules and an object $X$ in $\mathcal{X}$, the complex $P \otimes_{\Pi} X$ is perfect, that is, isomorphic to zero in $\mathcal{X}$. There is a canonical exact triangle in $\text{D}^b(\Pi^e\text{-mod})$

$$\Sigma^{n-1} \Omega^0_{\Pi^e}(\Pi) \to P \to \Pi \to \Sigma^n \Omega^0_{\Pi^e}(\Pi),$$

where $P$ is a bounded complex of projective $\Pi^e$-modules with length precisely $n$. Applying $- \otimes_{\Pi} X$ to this triangle, we infer a natural isomorphism

$$X \simeq \Sigma^n \Omega^0_{\Pi^e}(\Pi) \otimes_{\Pi} X$$
in $\mathcal{X}$. By the second condition in Definition 2.11, we have

$$N \otimes_{\Lambda} (M \otimes_{\Pi} X) \simeq \Omega^0_{\Pi^e}(\Pi) \otimes_{\Pi} X \simeq \Sigma^{-n}(X).$$

Similarly, we infer that $M \otimes_{\Pi} (N \otimes_{\Lambda} Y) \simeq \Sigma^{-n}(Y)$ for any object $Y \in \mathcal{Y}$. This proves the claim. \hfill $\square$

8.3. **A non-standard resolution and liftings.** In this subsection, we make preparation for the proof of Theorem 8.6. We study a non-standard resolution of $M$, and lift certain maps between cohomological groups to cochain complexes.

Recall from Subsection 6.2 the normalized bar resolution $\text{Bar}(\Lambda)$. It is well known that $\text{Bar}(\Lambda) \otimes_{\Lambda} M \otimes_{\Pi} \text{Bar}(\Pi)$ is a projective $\Lambda$-$\Pi$-bimodule resolution of $M$, even without the projectivity assumption on $M$. However, we will need another *non-standard* resolution of $M$; this resolution requires the projective assumption on the $\Lambda$-$\Pi$-bimodule $M$.

We denote by $\tilde{\text{Bar}}(\Lambda)$ the undeleted bar resolution

$$\cdots \to \Lambda \otimes (s\Lambda)^{\otimes m} \otimes \Lambda \xrightarrow{d_{ex}} \cdots \xrightarrow{d_{ex}} \Lambda \otimes (s\Lambda) \otimes \Lambda \xrightarrow{d_{ex}} \Lambda \otimes \Lambda \xrightarrow{\mu} s^{-1} \Lambda \to 0,$$ \hspace{1cm} (8.8)

where $\mu$ is the multiplication and $d_{ex}$ is the external differential; see Subsection 6.2. Here, we use $s^{-1} \Lambda$ to emphasize that it is of cohomological degree one. Similarly, we have the undeleted bar resolution $\tilde{\text{Bar}}(\Pi)$ for $\Pi$.

Consider the following complex of $\Lambda$-$\Pi$-bimodules

$$\mathcal{E} = \mathcal{E}(\Lambda, M, \Pi) := \text{Bar}(\Lambda) \otimes_{\Lambda} sM \otimes_{\Pi} \text{Bar}(\Pi).$$

We observe that $\mathcal{E}$ is acyclic. By using the natural isomorphisms

$$s^{-1}\Lambda \otimes_{\Lambda} sM \simeq M, \quad \text{and} \quad sM \otimes_{\Pi} s^{-1}\Pi \simeq M,$$

we obtain that the $(−p)$-th component of $\mathcal{E}$ is given by

$$\mathcal{E}^{-p} = \bigoplus_{i+j=p-1 \atop i,j \geq 0} \Lambda \otimes (s\Lambda)^{\otimes i} \otimes sM \otimes (s\Pi)^{\otimes j} \otimes \Pi \bigoplus \Lambda \otimes (s\Lambda)^{\otimes p} \otimes M \bigoplus M \otimes (s\Pi)^{\otimes p} \otimes \Pi$$

for any $p \geq 0$, and that $\mathcal{E}^1 = s^{-1}\Lambda \otimes_{\Lambda} sM \otimes_{\Pi} s^{-1}\Pi \simeq s^{-1}M$. In particular, we have

$$\mathcal{E}^0 \simeq (\Lambda \otimes M) \bigoplus (M \otimes \Pi),$$

$$\mathcal{E}^{-1} = (\Lambda \otimes sM \otimes \Pi) \bigoplus (\Lambda \otimes s\Lambda \otimes M) \bigoplus (M \otimes s\Pi \otimes \Pi).$$

The differential $\partial^{-p} : \mathcal{E}^{-p} \to \mathcal{E}^{-(p-1)}$ is induced by the differentials of $\text{Bar}(\Lambda)$ and $\text{Bar}(\Pi)$ in (8.8) via tensoring with $sM$. For instance, the differential $\partial^0 : \mathcal{E}^0 \to \mathcal{E}^1$ is given by

$$\Lambda \otimes M \bigoplus M \otimes \Pi \to M, \quad (a \otimes m \mapsto am, \quad m' \otimes b \mapsto m'b);$$
the differential $\partial^{-1} : \mathcal{B}^{-1} \to \mathcal{B}^{0}$ is given by the maps

\[
\begin{align*}
\Lambda \otimes sM \otimes \Pi & \longrightarrow (\Lambda \otimes M) \bigoplus (M \otimes \Pi), & (a \otimes sm \otimes b) & \longmapsto -a \otimes mb + am \otimes b, \\
\Lambda \otimes s\Lambda \otimes M & \longrightarrow \Lambda \otimes M, & (a \otimes s\alpha) & \otimes m & \longmapsto aa_1 \otimes m - a \otimes a_1 m, \\
M \otimes s\Pi \otimes \Pi & \longrightarrow M \otimes \Pi, & (m \otimes s\beta) & \otimes b & \longmapsto mb_1 \otimes b - m \otimes b_1 b.
\end{align*}
\]

Since $M$ is projective as a left $\Lambda$-module and as a right $\Pi$-module, it follows that all the direct summands of $\mathcal{B}^{-p}$ are projective as $\Lambda$-$\Pi$-bimodules for $p \geq 0$. We infer that $\mathcal{B}$ is an undeleted $\Lambda$-$\Pi$-bimodule projective resolution of $M$.

**Lemma 8.8.** For each $p \geq 1$, the cokernel $\text{Cok}(\partial^{-p-1})$ is isomorphic to

\[
\Omega^p_{\Lambda, \Pi}(M) := \bigoplus_{i+j=p-1} (s\Lambda)^{\otimes i} \otimes sM \otimes (s\Pi)^{\otimes j} \otimes \Pi \bigoplus (s\Lambda)^{\otimes p} \otimes M.
\]

In particular, $\Omega^p_{\Lambda, \Pi}(M)$ inherits a $\Lambda$-$\Pi$-bimodule structure from $\text{Cok}(\partial^{-p-1})$.

**Proof.** We have a $k$-linear map

\[
\gamma^{-p} : \Omega^p_{\Lambda, \Pi}(M) \xrightarrow{\text{1\times1}} \mathcal{B}^{-p} \longrightarrow \text{Cok}(\partial^{-p-1}),
\]

where the unnamed arrow is the natural projection and the first map $1 \otimes 1$ is given by

\[
\begin{align*}
& s\alpha_{1,i} \otimes sm \otimes s\beta_{1,j} \otimes b_{j+1} \longrightarrow 1 \otimes s\alpha_{1,i} \otimes sm \otimes s\beta_{1,j} \otimes b_{j+1} \\
& s\alpha_{1,p} \otimes m \longrightarrow 1 \otimes s\alpha_{1,p} \otimes m.
\end{align*}
\]

We observe that $\gamma^{-p}$ is surjective. Indeed, a typical element $a_0 \otimes s\alpha_{1,i} \otimes x$ represents the same image in $\text{Cok}(\partial^{-p-1})$ with the following element

\[
\sum_{k=0}^{i-1} (-1)^k \alpha_{1,k-1} \otimes s\alpha_{1,k+1} \otimes s\alpha_{k+2,i} \otimes x + (-1)^i \alpha_{1,i-1} \otimes a_i x.
\]

Here, $x$ lies in $sM \otimes (s\Pi)^{\otimes j} \otimes \Pi$ or $M$. Similarly, a typical element $m \otimes s\beta_{1,p} \otimes b_{p+1} \in M \otimes (s\Pi)^{\otimes p} \otimes \Pi$ represents the same image in $\text{Cok}(\partial^{-p-1})$ with

\[
\begin{align*}
1 \otimes (smb_1) & \otimes s\beta_{2,p} \otimes b_{p+1} + \sum_{k=1}^{p-1} (-1)^k \alpha \otimes sm \otimes s\beta_{1,k-1} \otimes s\beta_{k+1} \otimes b_{k+1} \otimes s\beta_{k+2,p} \otimes b_{p+1} \\
& + (-1)^p \alpha \otimes sm \otimes s\beta_{1,p-1} \otimes b_p b_{p+1}.
\end{align*}
\]

In both cases, the latter elements belong to the image of $\gamma^{-p}$.

On the other hand, we have a projection of degree $-1$

\[
\varpi^{-p+1} : \mathcal{B}^{-p+1} \to \Omega^p_{\Lambda, \Pi}(M)
\]

given by

\[
\begin{align*}
a_0 \otimes s\alpha_{1,i} \otimes sm \otimes s\beta_{1,j} \otimes b_{j+1} & \mapsto s\alpha_0 \otimes s\alpha_{1,i} \otimes sm \otimes s\beta_{1,j} \otimes b_{j+1} \\
a_0 \otimes s\alpha_{1,p-1} \otimes m & \mapsto s\alpha_0 \otimes s\alpha_{1,p-1} \otimes m \\
m \otimes s\beta_{1,p-1} \otimes b_p & \mapsto sm \otimes s\beta_{1,p-1} \otimes b_p.
\end{align*}
\]
We define a $\mathbb{k}$-linear map

$$\tilde{\eta}^{-p} = \varpi^{-p+1} \circ \partial^{-p} : \mathbb{E}^{-p} \to \Omega^p_{\Lambda-\Pi}(M).$$

In view of $\tilde{\eta}^{-p} \circ \partial^{-p-1} = 0$, we have a unique induced map

$$\eta^{-p} : \text{Cok}(\partial^{-p-1}) \to \Omega^p_{\Lambda-\Pi}(M).$$

One checks easily that $\eta^{-p} \circ \tilde{\gamma}^{-p}$ equals the identity. By the surjectivity of $\gamma^{-p}$, we infer that $\gamma^{-p}$ is an isomorphism. \hfill \square

**Remark 8.9.** The right $\Pi$-module structure on $\Omega^p_{\Lambda-\Pi}(M)$ is induced by the right action of $\Pi$ on $M$ and $\Pi$. The left $\Lambda$-module structure is given by

$$a_0 \triangleright (s\pi_{1,j} \otimes sm \otimes s\tilde{b}_{1,j} \otimes b_{j+1}) := (\pi \otimes 1^{\otimes p}) \circ \partial^{-p}(a_0 \otimes s\pi_{1,j} \otimes sm \otimes s\tilde{b}_{1,j} \otimes b_{j+1}),$$

$$a_0 \triangleright (s\tilde{a}_{1,p} \otimes m) := (\pi \otimes 1^{\otimes p}) \circ \partial^{-p}(a_0 \otimes s\tilde{a}_{1,p} \otimes m),$$

where $\pi : \Lambda \to \tilde{\Lambda}$ is the natural projection $a \mapsto s\tilde{a}$ of degree $-1$.

We have a short exact sequence of $\Lambda$-$\Pi$-modules; compare (8.20)

$$0 \to \Sigma^{-1}\Omega^p_{\Lambda-\Pi}(M) \to \mathbb{E}^{-p} \to \Omega^p_{\Lambda-\Pi}(M) \to 0,$$

where the map $1 \otimes 1$ is given in (8.9). Here, we always view $\Omega^p_{\Lambda-\Pi}(M)$ as a graded $\Lambda$-$\Pi$-bimodule concentrated in degree $-p$. By convention, we have $\Omega^0_{\Lambda-\Pi}(M) = M$.

Fix $p \geq 0$. Applying the functor $\text{Hom}_{\Lambda-\Pi}(-, \Omega^p_{\Lambda-\Pi}(M))$ to the resolution $\overline{\text{Bar}}(\Lambda) \otimes_{\Pi} \text{Bar}(\Pi)$, we obtain a cochain complex

$$\overline{C}^*(M, \Omega^p_{\Lambda-\Pi}(M))$$

computing $\text{Ext}^*(M, \Omega^p_{\Lambda-\Pi}(M))$. The space $\overline{C}^m(M, \Omega^p_{\Lambda-\Pi}(M))$ in degree $m$ is as follows:

$$\bigoplus_{i+j=m+p} \text{Hom} \left( (s\tilde{\Lambda})^\otimes i \otimes M \otimes (s\Pi)^\otimes j, \bigoplus_{k+l=p-1} (s\tilde{\Lambda})^\otimes k \otimes sM \otimes (s\Pi)^\otimes l \bigoplus_{k,j \geq 0} (s\tilde{\Lambda})^\otimes p \otimes M \right).$$

Recall that $\Omega^p_{\text{nc}, R}(\Lambda) = (s\tilde{\Lambda})^\otimes p \otimes \Lambda$ is the graded $\Lambda$-$\Lambda$-bimodule of right noncommutative differential $p$-forms. We have a natural identification

$$\text{HH}^*(\Lambda, \Omega^p_{\text{nc}, R}(\Lambda)) \simeq \text{Ext}^*(\Lambda, \Omega^p_{\text{nc}, R}(\Lambda)).$$

Consider the following triangle functor

$$- \otimes_{\Lambda} M : \mathbf{D}(\Lambda^e) \to \mathbf{D}(\Lambda \otimes \Pi^p).$$

Then we have a map

$$\alpha^*_p : \text{HH}^*(\Lambda, \Omega^p_{\text{nc}, R}(\Lambda)) \to \text{Ext}^*(\Lambda-\Pi(M, \Omega^p_{\text{nc}, R}(\Lambda) \otimes_{\Lambda} M) \to \text{Ext}^*(\Lambda-\Pi(M, \Omega^p_{\Lambda-\Pi}(M)),

where the second map is induced by the natural inclusion

$$\Omega^p_{\text{nc}, R}(\Lambda) \otimes_{\Lambda} M \to (s\tilde{\Lambda})^\otimes p \otimes M \to \Omega^p_{\Lambda-\Pi}(M).$$

We define a cochain map

$$\tilde{\alpha}_p : \overline{C}^*(\Lambda, \Omega^p_{\text{nc}, R}(\Lambda)) \to \overline{C}^*(M, \Omega^p_{\Lambda-\Pi}(M))$$

(8.11)
as follows: for any \( f \in \text{Hom}((s\Lambda)^{\otimes m},(s\Lambda)^{\otimes p} \otimes \Lambda) \) with \( m \geq 0 \), the corresponding map \( \bar{\alpha}_p(f) \in C^{m-p}(M,\Omega_{\Lambda-\Pi}^p(M)) \) is given by
\[
\bar{\alpha}_p(f)_{(s\Lambda)^{\otimes \cdot-1} \otimes M \otimes (s\Pi) \otimes x} = 0 \quad \text{if} \quad i \neq 0
\]
\[
\bar{\alpha}_p(f)(s\bar{\alpha}_{1,m} \otimes x) = f(s\bar{\alpha}_{1,m} \otimes \Lambda x)
\]
for any \( s\bar{\alpha}_{1,m} \otimes x \in (s\Lambda)^{\otimes m} \otimes M \).

Recall that the cochain complexes \( \overline{C}^*(\Lambda,\Omega^p_{nc,R}(\Lambda)) \) and \( \overline{C}^*(M,\Omega^p_{\Lambda-\Pi}(M)) \) compute \( \text{HH}^*(\Lambda,\Omega^p_{nc,R}(\Lambda)) \) and \( \text{Ext}^*_\Lambda(M,\Omega^p_{\Lambda-\Pi}(M)) \), respectively.

**Lemma 8.10.** The cochain map \( \bar{\alpha}_p \) is a lifting of \( \alpha^\ast_p \).

**Proof.** Since \( M \) is projective as a right \( \Pi \)-module, it follows that the tensor functor \( - \otimes \Lambda M \) sends the projective resolution \( \overline{\text{Bar}}(\Lambda) \) of \( \Lambda \) to a projective resolution \( \overline{\text{Bar}}(\Lambda) \otimes \Lambda M \) of \( M \).

Denote \( \Omega_{nc,R}(M) = \Omega^p_{nc,R}(\Lambda) \otimes \Lambda M \). Consider the complex
\[
\overline{C}_{k-\Pi}^*(M,\Omega^p_{nc,R}(M)) = \prod_{m \geq 0} \text{Hom}_{\text{nc}}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M))
\]
whose differential is induced by the differential of \( \text{Hom}_{\Lambda-\Pi}(\overline{\text{Bar}}(\Lambda) \otimes \Lambda M,\Omega^p_{nc,R}(M)) \) under the natural isomorphism
\[
\text{Hom}_{\Lambda-\Pi}(\Lambda \otimes (s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M)) \cong \text{Hom}_{\text{nc}}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M))
\]
\[
\xrightarrow{\cong} \text{Hom}_{\text{nc}}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M))
\]
\[
\xmapsto{f} (s\bar{\alpha}_{1,m} \otimes x \mapsto f(1_\Lambda \otimes s\bar{\alpha}_{1,m} \otimes \Lambda x)).
\]

The map \( \text{HH}^*(\Lambda,\Omega^p_{nc,R}(\Lambda)) \xrightarrow{-\otimes \Lambda M} \text{Ext}^*_\Lambda(M,\Omega^p_{nc,R}(M)) \) has the following lifting
\[
\alpha'_p: \overline{C}^*(\Lambda,\Omega^p_{\Lambda-\Pi}(\Lambda)) \rightarrow \overline{C}^*_{k-\Pi}(M,\Omega^p_{nc,R}(M)),
\]
which sends \( f \in \text{Hom}((s\Lambda)^{\otimes m},\Omega^p_{nc,R}(\Lambda)) \) to \( \alpha'(f) \in \text{Hom}_{k-\Pi}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M)) \) given by
\[
\alpha'_p(f)(s\bar{\alpha}_{1,m} \otimes x) = f(s\bar{\alpha}_{1,m} \otimes \Lambda x).
\]
We have an inclusion of complexes
\[
\iota: \overline{C}^*_{k-\Pi}(M,\Omega^p_{nc,R}(M)) \hookrightarrow \overline{C}^*(M,\Omega^p_{\Lambda-\Pi}(M))
\]
which is induced by the natural inclusion
\[
\text{Hom}_{k-\Pi}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M)) \rightarrow \text{Hom}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{nc,R}(M)) \rightarrow \text{Hom}((s\Lambda)^{\otimes m} \otimes M,\Omega^p_{\Lambda-\Pi}(M)).
\]
Observe that \( \bar{\alpha}_p = \iota \circ \alpha'_p \). It follows that \( \bar{\alpha}_p \) is a lifting of \( \alpha^\ast_p \). \( \square \)

Similarly, we have the following triangle functor
\[
M \otimes \Pi - : \text{D}(\Pi^c) \rightarrow \text{D}(\Lambda \otimes \Pi^op),
\]
and the corresponding map
\[
\beta^\ast_p: \text{HH}^*(\Pi,\Omega^p_{nc,R}(\Pi)) \xrightarrow{M \otimes \Pi -} \text{Ext}^*_\Lambda(M,\Omega^p_{nc,R}(\Pi)) \rightarrow \text{Ext}^*_\Lambda(M,\Omega^p_{\Lambda-\Pi}(M)),
\]
where the second map is induced by the following bimodule homomorphism
\[
M \otimes \Omega^p_{nc,R}(\Pi) \rightarrow \Omega^p_{\Lambda-\Pi}(M), \quad x \otimes \Omega_{s\bar{b}_{1,p} \otimes b_{p+1}} \rightarrow x \circ (s\bar{b}_{1,p} \otimes b_{p+1}). \quad (8.12)
\]
Here, the action $\triangleright$ is given by
\begin{align*}
x \triangleright (s_1 b_{1,p} \otimes b_{p+1}) &= s(xb_1) \otimes s_1 b_{1,p} \otimes b_{p+1} + \sum_{i=1}^{p-1} (-1)^i sx \otimes s_1 b_{1,i-1} \otimes s_1 b_{i+1} \otimes s_1 b_{i+2,p} \otimes b_{p+1}) \\
&\quad + (-1)^p sx \otimes s_1 b_{1,p-1} \otimes b_p b_{p+1},
\end{align*}
which is similar to (7.1).

We define a cochain map $	ilde{\beta}_p : C^\ast (\Pi, \Omega_{nc,R}(\Pi)) \rightarrow C^\ast (M, \Omega_{\Lambda-\Pi}(\Omega_{nc,R}(\Pi)))$

\begin{equation}
\text{(8.14)}
\end{equation}

as follows: for any map $g \in \text{Hom}((s\Pi)^{\otimes m}, (s\Pi)^{\otimes p} \otimes \Pi)$, the corresponding map $	ilde{\beta}_p(g) \in C^{m-p}(M, \Omega_{\Lambda-\Pi}(\Omega_{nc,R}(\Pi)))$ is given by
\begin{equation}
\text{(8.15)}
\end{equation}

for any $x \otimes s_1 b_{1,m} \in M \otimes (s\Pi)^{\otimes m}$, where the action $\triangleright$ is defined in (8.13).

We have the following analogous result of Lemma 8.10.

Lemma 8.11. The map $\tilde{\beta}_p$ is a lifting of $\beta_p^\ast$.

Proof. The tensor functor $M \otimes \Pi$ sends the projection resolution $\text{Bar}(\Pi)$ of $\Pi$ to the projective resolution $M \otimes \Pi \text{Bar}(\Pi)$ of $M$.

Consider the complex
\[
\mathcal{C}^\ast_{\Lambda-k}(M, M \otimes \Omega_{nc,R}(\Pi)) = \prod_{m \geq 0} \text{Hom}_{\Lambda-k}(M \otimes (s\Pi)^{\otimes m}, M \otimes \Omega_{nc,R}(\Pi)),
\]
which is naturally isomorphic to $\text{Hom}_{\Lambda-k}(M \otimes \Omega_{nc,R}(\Pi))$. Then the map
\[
\text{HH}^\ast(\Pi, \Omega_{nc,R}(\Pi)) \rightarrow \text{Ext}^\ast_{\Lambda \otimes \Pi}, \Omega_{nc,R}(\Pi)) (M, \otimes \Omega_{nc,R}(\Pi))
\]
has a lifting
\[
\beta_p^\ast : \mathcal{C}^\ast(\Pi, \Omega_{nc,R}(\Pi)) \rightarrow \mathcal{C}^\ast_{\Lambda-k}(M, M \otimes \Omega_{nc,R}(\Pi)),
\]
which sends $g \in \text{Hom}((s\Pi)^{\otimes m}, \Omega_{nc,R}(\Pi))$ to $\beta_p^\ast(g) \in \text{Hom}_{\Lambda-k}(M \otimes (s\Pi)^{\otimes m}, M \otimes \Omega_{nc,R}(\Pi))$
given by
\[
\beta_p^\ast(g)(x \otimes s_1 b_{1,m}) = x \otimes g(s_1 b_{1,m}).
\]
We have an inclusion of complexes
\[
\iota : \mathcal{C}^\ast_{\Lambda-k}(M, M \otimes \Omega_{nc,R}(\Pi)) \rightarrow \mathcal{C}^\ast(M, \Omega_{\Lambda-\Pi}(\Omega_{nc,R}(\Pi)))
\]
induced by the inclusion (8.12). By $\tilde{\beta}_p = \iota \circ \beta_p^\ast$, we conclude that $\tilde{\beta}_p$ is a lifting of $\beta_p^\ast$. □
8.4. A triangular matrix algebra and colimits. Denote by \( \Gamma = \begin{pmatrix} \Lambda & M \\ 0 & \Pi \end{pmatrix} \) the upper triangular matrix algebra. Set \( e_1 = \begin{pmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{pmatrix} \) and \( e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_\Pi \end{pmatrix} \). Then we have the following natural identifications:

\[
e_1 \Gamma e_1 \simeq \Lambda, \quad e_2 \Gamma e_2 \simeq \Pi, \quad e_1 \Gamma e_2 \simeq M, \quad \text{and} \quad e_2 \Gamma e_1 = 0. \tag{8.16}
\]

Denote by \( E = k e_1 \oplus k e_2 \) the semisimple subalgebra of \( \Gamma \). Set \( \overline{\Gamma} = \Gamma/(E \cdot 1_\Gamma) \). Consider the \( E \)-relative right singular Hochschild cochain complex \( \overline{C}_E^*(\Gamma, \Gamma) \).

Using (8.16), we identify \( \overline{\Gamma} \) with \( \overline{\Lambda} \oplus \overline{\Pi} \oplus M \). Here, we agree that \( \overline{\Lambda} = \Lambda/(k \cdot 1_\Lambda) \) and \( \overline{\Pi} = \Pi/(k \cdot 1_\Pi) \). Then we have

\[
s \overline{\Gamma} \otimes m \cong s \overline{\Lambda} \otimes m \bigoplus s \overline{\Pi} \otimes m \bigoplus \left( \bigoplus_{i, j \geq 0 \atop i + j = m - 1} s \overline{\Lambda} \otimes i \otimes sM \otimes s \overline{\Pi} \otimes j \right).
\]

For each \( m, p \geq 0 \), we have the following natural decomposition of vector spaces

\[
\text{Hom}_{E \cdot E}((s \overline{\Gamma}) \otimes E^m, (s \overline{\Gamma}) \otimes E^p \otimes E \Gamma) \\
\cong \text{Hom}((s \overline{\Lambda}) \otimes m, (s \overline{\Lambda}) \otimes p \otimes \Lambda) \bigoplus \text{Hom}((s \overline{\Pi}) \otimes m, (s \overline{\Pi}) \otimes p \otimes \Pi) \bigoplus \\
\bigoplus_{i, j \geq 0 \atop i + j = m - 1} \text{Hom}((s \overline{\Lambda}) \otimes i \otimes sM \otimes (s \overline{\Pi}) \otimes j, \bigoplus_{i', j' \geq 0 \atop i' + j' = p - 1} (s \overline{\Lambda}) \otimes i' \otimes sM \otimes (s \overline{\Pi}) \otimes j' \otimes \Pi \bigoplus (s \overline{\Lambda}) \otimes p \otimes M),
\]

which induces the following decomposition of graded vector spaces

\[
\overline{C}_E^*(\Gamma, \Omega_{nc,R,E}^p(\Gamma)) \cong \overline{C}^*(\Lambda, \Omega_{nc,R}^p(\Lambda)) \oplus \overline{C}^*(\Pi, \Omega_{nc,R}^p(\Pi)) \oplus \Sigma^{-1} \overline{C}^*(M, \Omega_{\Lambda,\Pi}^p(M)). \tag{8.18}
\]

We write elements on the right hand side of (8.18) as column vectors. The differential \( \overline{\delta}_\Gamma \) of \( \overline{C}_E^*(\Gamma, \Omega_{nc,R,E}^p(\Gamma)) \) induces a differential \( \overline{\delta} \) on the right hand side of (8.18). By a straightforward computation, we note that \( \overline{\delta} \) has the following form

\[
\overline{\delta} = \begin{pmatrix} \delta_\Lambda & 0 & 0 \\ 0 & \delta_\Pi & 0 \\ -s^{-1} \circ \overline{\alpha}_p & s^{-1} \circ \overline{\beta}_p & \Sigma^{-1}(\delta_M) \end{pmatrix}, \tag{8.19}
\]

where \( \delta_\Lambda, \delta_\Pi \) and \( \delta_M \) are the differentials of \( \overline{C}^*(\Lambda, \Omega_{nc,R}^p(\Lambda)) \), \( \overline{C}^*(\Pi, \Omega_{nc,R}^p(\Pi)) \) and \( \overline{C}^*(M, \Omega_{\Lambda,\Pi}^p(M)) \), respectively. The entry

\[
s^{-1} \circ \overline{\alpha}_p : \overline{C}^*(\Lambda, \Omega_{nc,R}^p(\Lambda)) \longrightarrow \Sigma^{-1} \overline{C}^*(M, \Omega_{\Lambda,\Pi}^p(M))
\]

is of degree one, which is the composition of \( \overline{\alpha}_p \) with the natural identification \( s^{-1} : \overline{C}^*(M, \Omega_{\Lambda,\Pi}^p(M)) \rightarrow \Sigma^{-1} \overline{C}^*(M, \Omega_{\Lambda,\Pi}^p(M)) \) of degree one. A similar remark holds for \( s^{-1} \circ \overline{\beta}_p \).
The decomposition (8.18) induces a short exact sequence of complexes

$$
0 \longrightarrow \Sigma^{-1}C^p(M, \Omega^p_{\Lambda, \Pi}(M)) \overset{\text{inc}}{\longrightarrow} C_E^p(\Gamma, \Omega_{\Lambda, \Pi}^p(M)) \overset{\text{res}_1}{\longrightarrow} \Sigma^{p+1}C^p(M, \Omega^p_{\Lambda, \Pi}(M)) \longrightarrow 0.
$$

(8.20)

Here, “res” denotes the corresponding projection.

In what follows, letting $p$ vary, we will take colimits of (8.20). For this end, we define

$$
\theta^M_p : C^p(M, \Omega^p_{\Lambda, \Pi}(M)) \longrightarrow C^p(M, \Omega^{p+1}_{\Lambda, \Pi}(M))
$$

as follows: for any $f \in C^p(M, \Omega^p_{\Lambda, \Pi}(M))$, we set

$$
\theta^M_p(f)(s_{\mathfrak{p}_1} \otimes m \otimes s_{\mathfrak{p}_1} b_{1,i}) = (-1)^{i+j}s_{\mathfrak{p}_1} \otimes f(s_{\mathfrak{p}_2} \otimes m \otimes s_{\mathfrak{p}_2} b_{1,i}),
$$

if $i \geq 1$; otherwise, we set

$$
\theta^M_p(f)(m \otimes s_{\mathfrak{p}_1} b_{1,i}) = 0.
$$

We observe that $\theta^M_p$ is indeed a morphism of cochain complexes for each $p \geq 0$.

We have the following commutative diagram of cochain complexes with row being short exact.

$$
\begin{array}{cccc}
\Sigma^{-1}C^p(M, \Omega^p_{\Lambda, \Pi}(M)) & \overset{\text{inc}}{\longrightarrow} & C_E^p(\Gamma, \Omega_{\Lambda, \Pi}^p(M)) & \overset{\text{res}_1}{\longrightarrow} \\
\downarrow{\theta^M_p} & & \downarrow{\theta^M_p} & \\
\Sigma^{-1}C^{p+1}(M, \Omega^{p+1}_{\Lambda, \Pi}(M)) & \overset{\text{inc}}{\longrightarrow} & C_E^{p+1}(\Gamma, \Omega^{p+1}_{\Lambda, \Pi}(M)) & \overset{\text{res}_1}{\longrightarrow} \\
\end{array}
$$

Similar to the definition of right singular Hochschild cochain complex in Subsection 7.1, we have an induction system of cochain complexes

$$
\Sigma C^p(M, M) \overset{\theta^M_p}{\longrightarrow} \cdots \overset{\theta^M_p}{\longrightarrow} C^p(M, \Omega^p_{\Lambda, \Pi}(M)) \overset{\theta^M_p}{\longrightarrow} C^p(M, \Omega^{p+1}_{\Lambda, \Pi}(M)) \overset{\theta^M_{p+1}}{\longrightarrow} \cdots,
$$

and denote its colimit by $C^p_{\text{sg}}(M, M)$.

**Lemma 8.12.** The cochain map $\theta^M_p$ is a lifting of the following connecting map

$$
\tilde{\theta}^M_p : \text{Ext}^n(\Lambda, \Pi)_{\Lambda, \Pi}(M)) \longrightarrow \text{Ext}^n(\Lambda, \Pi)_{\Lambda, \Pi}(M))
$$

in the long exact sequence obtained by applying the functor $\text{Ext}^n(\Lambda, \Pi)_{\Lambda, \Pi}(M)$ to (8.10). Consequently, for any $n \in \mathbb{Z}$ we have an isomorphism

$$
H^n(C^p_{\text{sg}}(M, M)) \cong \text{Hom}_{\mathbf{D}_{\text{sg}}(\Lambda \otimes \Pi^{op})}(M, \Sigma^n M).
$$

**Proof.** Since the direct colimit commutes with the cohomology functor, we have

$$
H^n(C^p_{\text{sg}}(M, M)) \cong \varinjlim_{\theta^M_p} \text{Ext}^n(\Lambda, \Pi)_{\Lambda, \Pi}(M),
$$

where the colimit map $\tilde{\theta}^M_p$ is induced by $\theta^M_p$. Apply the functor $\text{Ext}^n(\Lambda, \Pi)(M, -)$ to (8.10)

$$
\cdots \longrightarrow \text{Ext}^n(\Lambda, \Pi)(M, \mathcal{B}^{-p}) \longrightarrow \text{Ext}^n(\Lambda, \Pi)(M, \Omega^p_{\Lambda, \Pi}(M)) \longrightarrow \text{Ext}^{n+1}(\Lambda, \Pi)(M, \Sigma^{-1}\Omega^{p+1}_{\Lambda, \Pi}(M)) \longrightarrow \cdots
$$

Since $\text{Ext}^{n+1}(\Lambda, \Pi)(M, \Sigma^{-1}\Omega^{p+1}_{\Lambda, \Pi}(M))$ is naturally isomorphic to $\text{Ext}^n(\Lambda, \Pi)(M, \Omega^{p+1}_{\Lambda, \Pi}(M))$, the connecting morphism in the long exact sequence induces a map

$$
\tilde{\theta}^M_p : \text{Ext}^n(\Lambda, \Pi)(M, \Omega^p_{\Lambda, \Pi}(M)) \longrightarrow \text{Ext}^n(\Lambda, \Pi)(M, \Omega^{p+1}_{\Lambda, \Pi}(M)).
$$
We now show that $\tilde{\theta}_p^M = \tilde{\theta}_p^M$ using the similar argument as the proof of Lemma 7.1. We write down the definition of the connecting morphism $\tilde{\theta}_p^M$. Apply the functor $\text{Hom}_{A-P}(\text{Bar}(A) \otimes_A M \otimes_{\Pi} \text{Bar}(\Pi), -)$ to the short exact sequence (8.10). Then we have the following short exact sequence of complexes with induced maps

$$0 \rightarrow \Sigma^{-1}\mathcal{C}^i(M, \Omega^p_{A-P}(M)) \rightarrow \text{Hom}_{A-P}(\text{Bar}(A) \otimes_A M \otimes_{\Pi} \text{Bar}(\Pi), \mathbb{B}^{-p}) \rightarrow \mathcal{C}^i(M, \Omega^p_{A-P}(M)) \rightarrow 0.$$ (8.21)

Take $f \in \text{Ext}^i_{A-P}(M, \Omega^p_{A-P}(M))$. It may be represented by an element $f \in \mathcal{C}^i(M, \Omega^p_{A-P}(M))$ such that $\delta(f) = 0$ with $\delta'$ the differential of $\mathcal{C}^i(M, \Omega^p_{A-P}(M))$. Define

$$\tilde{f} \in \bigoplus_{i,j \geq 0} \text{Hom}(s\Lambda^\otimes i \otimes M \otimes s\Pi^\otimes j, \mathbb{B}^{-p})$$

such that

$$\tilde{f}(s\alpha_{i,i} \otimes m \otimes s\beta_{1,j}) = 1 \otimes f(s\alpha_{i,i} \otimes m \otimes s\beta_{1,j}).$$

We have that $f = \tilde{\eta}^{-p} \circ \tilde{f}$. We define $\tilde{f} \in \mathcal{C}^i(M, \Omega^p_{A-P}(M))$ such that

$$\tilde{f}(s\alpha_{i,i} \otimes m \otimes s\beta_{1,j}) = (-1)^n s\alpha_{1,1} \otimes f(s\alpha_{2,i} \otimes m \otimes s\beta_{1,j})$$

for $i \geq 1, j \geq 0$ and $i + j = n + p + 1$; otherwise for $i = 0$, we set $\tilde{f}(m \otimes s\beta_{1,n+p+1}) = 0$. We observe that

$$\partial^{-p-1} \circ (1 \otimes 1) \circ \tilde{f} = \delta''(\tilde{f}),$$

where $(1 \otimes 1)$ is defined in (8.9) and $\delta''$ is the differential of the middle complex in (8.21). Actually for $i = 0$ we have $\tilde{f}(m \otimes s\beta_{1,n+p+1}) = 0$ and

$$(\delta''(\tilde{f}))(1 \otimes m \otimes s\beta_{1,n+p+1} \otimes 1) = (-1)^n 1 \otimes (f(1 \otimes m \otimes_{\Pi} dx(1 \otimes s\beta_{1,n+p+1} \otimes 1)))$$

$$= 1 \otimes (\delta'(f))(1 \otimes m \otimes s\beta_{1,n+p+1} \otimes 1)$$

$$= 0.$$ (8.23)

Here $f$, $\tilde{f}$, $\delta''(\tilde{f})$ and $\delta'(f)$ are identified as $\Lambda$-$\Pi$-bimodule morphisms; compare (6.3). For $i \neq 0$, one can check directly that (8.22) holds. By the general construction of the connecting morphism, we have $\tilde{\theta}_p^M(f) = \tilde{f}$. Note that we also have $\tilde{\theta}_p^M(f) = \tilde{f}$. This shows that $\tilde{\theta}_p^M = \tilde{\theta}_p^M$.

Since $\text{Bar}(A) \otimes_A M \otimes_{\Pi} \text{Bar}(\Pi)$ is a projective resolution of $M$, by Lemma 8.8 and [42, Lemma 2.4], we have the following isomorphism

$$\lim_{\tilde{\theta}_p^M} \text{Ext}^i_{A-P}(M, \Omega^p_{A-P}(M)) \simeq \text{Hom}_{D_{\Lambda-P}(\Lambda \otimes_{\Pi} \Lambda)}(M, \Sigma^i M).$$

Combining the above two isomorphisms we obtain the desired isomorphism. \hfill \square

Recall from (8.7) the maps $\alpha^i$ and $\beta^i$. Analogous to [40, Lemma 4.5], we have the following result.
Proposition 8.13. Assume that the $\Lambda$-$\Pi$-bimodule $M$ is projective on each side. Then there is an exact sequence of cochain complexes

$$0 \longrightarrow \Sigma^{-1}C^{*}_{\text{sg}}(M, M) \overset{\text{inc}}{\longrightarrow} C^{*}_{\text{sg}, R; E}(\Gamma, \Gamma) \overset{\text{res}_{1} \text{res}_{2}}{\longrightarrow} C^{*}_{\text{sg}, R}(\Lambda, \Lambda) \oplus C^{*}_{\text{sg}, R}(\Pi, \Pi) \longrightarrow 0,$$

which yields a long exact sequence

$$\cdots \longrightarrow \text{HH}^{i}_{\text{sg}}(\Gamma, \Gamma) \overset{\text{res}_{1} \text{res}_{2}}{\longrightarrow} \text{HH}^{i}_{\text{sg}}(\Lambda, \Lambda) \oplus \text{HH}^{i}_{\text{sg}}(\Pi, \Pi) \overset{(-\alpha^{i}, \beta^{i})}{\longrightarrow} \text{Hom}_{D^{b}_{\text{sg}}(\Lambda \otimes \Pi^{\text{op}})}(M, \Sigma^{i} M) \longrightarrow \cdots.$$

Proof. The exact sequence of cochain complexes follows immediately from (8.20), since the three maps inc and res$_{i}$ ($i = 1, 2$) are compatible with the colimits. Then taking cohomology, we have an induced long exact sequence. However, it is tricky to prove that the maps $\alpha^{i}$ and $\beta^{i}$ do appear in the induced sequence. For this, we have to analyze the following induced long exact sequence of (8.20).

$$\cdots \longrightarrow \text{HH}^{i}(\Pi, \Omega^{p}_{\text{nc}, R; E}(\Gamma)) \overset{\text{res}_{1} \text{res}_{2}}{\longrightarrow} \text{HH}^{i}(\Lambda, \Omega^{p}_{\text{nc}, R}(\Lambda)) \oplus \text{HH}^{i}(\Pi, \Omega^{p}_{\text{nc}, R}(\Pi)) \overset{(-\alpha^{i}, \beta^{i})}{\longrightarrow} \text{Ext}^{i}_{\Lambda-\Pi}(M, \Omega^{p}_{\Lambda-\Pi}(M)) \longrightarrow \cdots.$$

(8.25)

Here, to see that the connecting morphism is indeed $(-\alpha^{i}, \beta^{i})$, we use the explicit description (8.19) of the differential, and apply Lemmas 8.10 and 8.11.

Note that we have the following commutative diagram

$$\begin{align*}
D^{b}(\Lambda^{e}) & \longrightarrow D^{b}(\Lambda \otimes \Pi^{\text{op}}) \longrightarrow D^{b}(\Pi^{e}) \\
D^{b}_{\text{sg}}(\Lambda^{e}) & \longrightarrow D^{b}_{\text{sg}}(\Lambda \otimes \Pi^{\text{op}}) \longrightarrow D^{b}_{\text{sg}}(\Pi^{e}),
\end{align*}$$

where the vertical functors are the natural quotients. This induces the following commutative diagram for each $p \geq 0$.

$$\begin{align*}
\text{HH}^{i}(\Pi, \Omega^{p}_{\text{nc}, R}(\Pi)) & \longrightarrow \text{Ext}^{i}_{\Lambda-\Pi^{\text{op}}}(M, \Omega^{p}_{\Lambda-\Pi}(M)) \overset{\alpha^{i}_{p}}{\longrightarrow} \text{HH}^{i}(\Lambda, \Omega^{p}_{\text{nc}, R}(\Lambda)) \\
\text{HH}^{i}_{\text{sg}}(\Pi, \Pi) & \longrightarrow \text{Hom}_{D^{b}_{\text{sg}}(\Lambda \otimes \Pi^{\text{op}})}(M, \Sigma^{i} M) \overset{\alpha^{i}}{\longrightarrow} \text{HH}^{i}_{\text{sg}}(\Lambda, \Lambda)
\end{align*}$$

Thus, by Lemmas 7.1 and 8.12 we have that

$$\alpha^{i} = \lim_{p} \alpha^{i}_{p} \quad \text{and} \quad \beta^{i} = \lim_{p} \beta^{i}_{p}$$

(8.26)

for any $i \in \mathbb{Z}$. Since the long exact sequence induced from (8.24) coincides with the colimit of (8.25), we are done.

Remark 8.14. We would like to stress that unlike [40, Lemma 4.5], the short exact sequence (8.24) does not have a canonical splitting. In other words, there is no canonical homotopy cartesian square as in [40, Lemma 4.5].
The reason is as follows. Note that for each \( p \geq 0 \), \((8.20)\) splits canonically as an exact sequence of graded modules, where the sections are given by the inclusions
\[
\text{inc}_1: C^r_* (\Lambda, \Omega^p_{nc,R}(\Lambda)) \longrightarrow C^r_* (\Gamma, \Omega^p_{nc,R,E}(\Gamma)) \\
\text{inc}_2: C^r_* (\Pi, \Omega^p_{nc,R}(\Pi)) \longrightarrow C^r_* (\Gamma, \Omega^p_{nc,R,E}(\Gamma)).
\]
We observe that \( \theta^\Gamma_p \circ \text{inc}_1 = \text{inc}_1 \circ \theta_p^\Lambda \). Taking the colimit, we obtain an inclusion of graded modules
\[
C^r_{\text{sg},R}(\Lambda, \Lambda) \longrightarrow C^r_{\text{sg},R,E}(\Gamma, \Gamma),
\]
which is generally not compatible with the differentials. We also have \( \theta^M_p \circ \tilde{\alpha}_p = \tilde{\alpha}_{p+1} \circ \theta^\Pi_p \).
Taking the colimit, we obtain a lifting at the cochain complex level
\[
\tilde{\alpha}: C^r_{\text{sg},R}(\Lambda, \Lambda) \longrightarrow C^r_{\text{sg}}(M, M)
\]
of the maps \( \alpha^i \).

However, the situation for \( \text{inc}_2 \) and \( \tilde{\beta}_p \) is different from \( \text{inc}_1 \). In general, we have
\[
\theta^\Gamma_p \circ \text{inc}_2 \neq \text{inc}_2 \circ \theta^\Pi_p \quad \text{and} \quad \theta^M_p \circ \tilde{\beta}_p \neq \tilde{\beta}_{p+1} \circ \theta^\Pi_p
\]
since for any \( f \in C^r_* (\Pi, \Omega^p_{nc,R}(\Pi)) \) we have
\[
(\theta^\Gamma_p \circ \text{inc}_2 - \text{inc}_2 \circ \theta^\Pi_p)(f) = 1_{sM} \otimes f
\]
and for \( f \in C^{m-p}_*(\Pi, \Omega^p_{nc,R}(\Pi)) \) we have
\[
((\theta^M_p \circ \tilde{\beta}_p)(f))(x \otimes s_{b_{1,m+1}}) = 0 \\
((\tilde{\beta}_{p+1} \circ \theta^\Pi_p)(f))(x \otimes s_{b_{1,m+1}}) = (-1)^{m-p} x \triangleright (b_1 \otimes f(s_{b_{2,m+1}})) \neq 0,
\]
where \( x \otimes s_{b_{m+1}} \) belongs to \( M \otimes \Pi^{\otimes m+1} \) and \( \triangleright \) is given in \((8.15)\). This means that the section \( \text{inc}_2 \) of \((8.20)\) is not compatible with \( \theta^\Gamma_p \) and \( \theta^\Pi_p \), we cannot take the colimit.

The above analysis also shows that we cannot lift the maps \( \beta^i \) at the cochain complex level canonically. This forces us to use the tricky argument in the proof of Proposition 8.13.

We are now in a position to prove Theorem 8.6.

**Proof of Theorem 8.6.** Since both the maps \( \alpha^i \) and \( \beta^i \) are isomorphisms, the long exact sequence in Proposition 8.13 yields a family of short exact sequences
\[
0 \longrightarrow \text{HH}^i_{\text{sg}}(\Gamma, \Gamma) \xrightarrow{(\text{res}_1, \text{res}_2)} \text{HH}^i_{\text{sg}}(\Lambda, \Lambda) \oplus \text{HH}^i_{\text{sg}}(\Pi, \Pi) \xrightarrow{(-\alpha^i, \beta^i)} \text{Hom}_{D_{\text{sg}}(\Lambda \otimes \Pi^{op})}(M, \Sigma^i M) \longrightarrow 0.
\]
In other words, we have the following commutative diagram
\[
\begin{array}{ccc}
\text{HH}^i_{\text{sg}}(\Gamma, \Gamma) & \xrightarrow{\text{res}_1} & \text{HH}^i_{\text{sg}}(\Lambda, \Lambda) \\
\text{res}_2 \downarrow & & \downarrow \alpha^i \\
\text{HH}^i_{\text{sg}}(\Pi, \Pi) & \xrightarrow{\beta^i} & \text{Hom}_{D_{\text{sg}}(\Lambda \otimes \Pi^{op})}(M, \Sigma^i M),
\end{array}
\]
which is a pullback diagram and pushout diagram, simultaneously. We infer that both \( \text{res}_i \) are isomorphisms. Then both projections
\[
\text{res}_1: C^r_{\text{sg},R,E}(\Gamma, \Gamma) \longrightarrow C^r_{\text{sg},R}(\Lambda, \Lambda) \quad \text{and} \quad \text{res}_2: C^r_{\text{sg},R,E}(\Gamma, \Gamma) \longrightarrow C^r_{\text{sg},R}(\Pi, \Pi)
\]
are quasi-isomorphisms. It is clear that they are both strict $B_\infty$-morphisms, and thus $B_\infty$-quasi-isomorphisms. This yields the required isomorphism in $\text{Ho}(B_\infty)$. \qed

9. Keller’s conjecture and the main results

Let $k$ be a field, and $\Lambda$ be a finite dimensional $k$-algebra. Denote by $\Lambda_0 = \Lambda/\text{rad}(\Lambda)$ the semisimple quotient algebra of $\Lambda$ by its Jacobson radical. Recall from Example 2.8 that $S_{\text{dg}}(\Lambda)$ denotes the dg singularity category of $\Lambda$.

Recently, Keller proves the following remarkable result.

**Theorem 9.1 ([42])**. Assume that $\Lambda_0$ is separable over $k$. Then there is a natural isomorphism of graded algebras between $\text{HH}^*_{\text{sg}}(\Lambda^{\text{op}}, \Lambda^{\text{op}})$ and $\text{HH}^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$.

The following natural conjecture is proposed by Keller.

**Conjecture 9.2 ([42])**. Assume that $\Lambda_0$ is separable over $k$. There is an isomorphism in the homotopy category $\text{Ho}(B_\infty)$ of $B_\infty$-algebras

$$C^*_\text{sg,L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \rightarrow C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$

(9.1)

Consequently, there is an induced isomorphism of Gerstenhaber algebras between $\text{HH}^*_{\text{sg}}(\Lambda^{\text{op}}, \Lambda^{\text{op}})$ and $\text{HH}^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$.

**Remark 9.3.** Indeed, there is a stronger version of Keller’s conjecture: the natural isomorphism in Theorem 9.1 lifts to an isomorphism between $C^*_\text{sg,L}(\Lambda^{\text{op}}, \Lambda^{\text{op}})$ and $C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$ in $\text{Ho}(B_\infty)$. Here, we treat only the above weaker version.

We say that an algebra $\Lambda$ satisfies Keller’s conjecture, provided that there is such an isomorphism (9.1) for $\Lambda$. It is not clear whether Keller’s conjecture is left-right symmetric. More precisely, we do not know whether $\Lambda$ satisfies Keller’s conjecture even assuming that $\Lambda^{\text{op}}$ does so; compare Remark 7.5.

The following invariance theorem provides useful reduction techniques for Keller’s conjecture. We recall from Subsection 2.2 the one-point coextension $\Lambda' = \begin{pmatrix} k & M \\ 0 & \Lambda \end{pmatrix}$ and the one-point extension $\Lambda'' = \begin{pmatrix} \Lambda \\ 0 \\ \Lambda \end{pmatrix}$ of $\Lambda$.

**Theorem 9.4.** The following statements hold.

1. The algebra $\Lambda$ satisfies Keller’s conjecture if and only if so does $\Lambda'$.
2. The algebra $\Lambda$ satisfies Keller’s conjecture if and only if so does $\Lambda''$.
3. Assume that the algebras $\Lambda$ and $\Pi$ are linked by a singular equivalence with a level. Then $\Lambda$ satisfies Keller’s conjecture if and only if so does $\Pi$.

**Proof.** For (1), we combine Lemmas 2.9 and 6.1 to obtain an isomorphism

$$C^*(S_{\text{dg}}(\Lambda'), S_{\text{dg}}(\Lambda')) \simeq C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda))$$

in the homotopy category $\text{Ho}(B_\infty)$. Note that $\Lambda^{\text{op}}$ is the one-point extension of $\Lambda^{\text{op}}$. Recall from Lemma 8.4 the strict $B_\infty$-quasi-isomorphism

$$C^*_\text{sg,L,E'}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \rightarrow C^*_\text{sg,L,E}(\Lambda^{\text{op}}, \Lambda^{\text{op}}).$$
Now applying Lemma 7.7 to both $\Lambda^{\text{op}}$ and $\Lambda'^{\text{op}}$, we obtain an isomorphism
$$C^*_{\text{sg},L}(\Lambda'^{\text{op}}, \Lambda'^{\text{op}}) \cong C^*_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}).$$
Then (1) follows immediately.

The argument for (2) is very similar. We apply Lemmas 2.10 and 6.1 to $\Lambda''$. Then we apply Lemma 8.2 to the opposite algebras of $\Lambda$ and $\Lambda''$.

For (3), we observe that by the isomorphism (1.1), Keller’s conjecture is equivalent to the existence of an isomorphism
$$C^*_{\text{sg},R}(\Lambda, \Lambda) \cong C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$
By Lemmas 2.13 and 6.1, we have an isomorphism
$$C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)) \cong C^*(S_{\text{dg}}(\Pi), S_{\text{dg}}(\Pi)).$$
Then we are done by Proposition 8.7.

□

The following result confirms Keller’s conjecture for an algebra $\Lambda$ with radical square zero. Moreover, it relates the singular Hochschild cochain complex of $\Lambda$ to the Hochschild cochain complex of the Leavitt path algebra.

**Theorem 9.5.** Let $Q$ be a finite quiver without sinks. Denote by $\Lambda = kQ/J^2$ the algebra with radical square zero, and by $L = L(Q)$ the Leavitt path algebra. Then we have the following isomorphisms in $\text{Ho}(B_\infty)$
$$C^*_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \xrightarrow{\Upsilon} C^*(L, L) \xrightarrow{\Delta} C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$
In particular, there are isomorphisms of Gerstenhaber algebras
$$HH^*_{\text{sg}}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \xrightarrow{\Upsilon} HH^*(L, L) \xrightarrow{\Delta} HH^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$

**Proof.** The isomorphism $\Delta$ is obtained as the following composite
$$C^*(L, L) \xrightarrow{\text{Lem. 6.2}} C^*(\text{per}_{\text{dg}}(L^{\text{op}}), \text{per}_{\text{dg}}(L^{\text{op}})) \xrightarrow{\text{Lem. 6.1 + Prop. 4.2}} C^*(S_{\text{dg}}(\Lambda), S_{\text{dg}}(\Lambda)).$$
Similarly, the isomorphism $\Upsilon$ is obtained by the following diagram
$$C^*(L, L) \xrightarrow{\text{Lem. 6.3}} C^*_{\text{sg},E}(L, L) \xrightarrow{\text{Thm. 10.3}} C^*_{\text{sg},R}(Q, Q)^{\text{op}} \xrightarrow{\text{Prop. 11.4}} \hat{C}^*_{\text{sg},R}(Q, Q)^{\text{op}} \xrightarrow{\text{Thm. 14.1}} \hat{C}^*_{\text{sg},R}(L, L)^{\text{op}} \xrightarrow{\text{App. A}} C^*_{\text{sg},L}(\Lambda^{\text{op}}, \Lambda^{\text{op}}) \xrightarrow{\text{Thm. 7.6}} C^*_{\text{sg},R,E}(\Lambda, \Lambda)^{\text{op}} \xrightarrow{\text{Lem. 7.6}} C^*_{\text{sg},E}(L, L) \xrightarrow{\text{Prop. 13.7}} \hat{C}^*(L, L)^{\text{op}}.$$
being a $B_{\infty}$-morphism is essentially using the higher pre-Jacobi identity of $\hat{C}^*(L,L)$. The isomorphisms of Gerstenhaber algebras follow from Lemma 5.12.

Denote by $\mathcal{X}$ the class of finite dimensional algebras $\Lambda$ with the following property: there exists some finite quiver $Q$ without sinks, such that $\Lambda$ is connected to $\mathbb{k}Q/J^2$ by a finite zigzag of one-point (co)extensions and singular equivalences with levels. For example, if $Q'$ is any finite quiver possibly with sinks, then $\mathbb{k}Q'/J^2$ clearly lies in $\mathcal{X}$.

We have the following immediate consequence of Theorems 9.4 and 9.5.

**Corollary 9.6.** Any algebra belonging to the class $\mathcal{X}$ satisfies Keller’s conjecture.

## 10. Algebras with radical square zero and the combinatorial $B_{\infty}$-algebra

Let $Q$ be a finite quiver without sinks. Let $\Lambda = \mathbb{k}Q/J^2$ be the corresponding algebra with radical square zero. We will give a combinatorial description of the singular Hochschild cochain complex of $\Lambda$; see Subsection 10.1. For its $B_{\infty}$-algebra structure, we describe it as the combinatorial $B_{\infty}$-algebra $\overline{C}'_{sg,R}(Q,Q)$ of $Q$; see Subsection 10.2.

### 10.1. A combinatorial description of the singular Hochschild cochain complex.

Set $E = \mathbb{k}Q_0$, viewed as a semisimple subalgebra of $\Lambda$. Then $\overline{X} = \Lambda/(E \cdot 1_\Lambda)$ is identified with $\mathbb{k}Q_1$. We will give a description of the $E$-relative right singular Hochschild cochain complex $\overline{C}'_{sg,R}(\Lambda,\Lambda)$ by parallel paths in the quiver $Q$.

For two subsets $X$ and $Y$ of paths in $Q$, we denote

$$X//Y := \{ (\gamma, \gamma') \in X \times Y \mid s(\gamma) = s(\gamma') \text{ and } t(\gamma) = t(\gamma') \}.$$  

An element in $Q_{m//}Q_p$ is called a *parallel path* in $Q$. We will abbreviate a path $\beta_m \cdots \beta_2 \beta_1 \in Q_m$ as $\beta_{m,1}$. Similarly, a path $\alpha_p \cdots \alpha_2 \alpha_1 \in Q_p$ is denoted by $\alpha_{p,1}$.

For a set $X$, we denote by $\mathbb{k}(X)$ the $\mathbb{k}$-vector space spanned by elements in $X$. We will view $\mathbb{k}(Q_{m//}Q_p)$ as a graded $\mathbb{k}$-space concentrated on degree $m-p$. For a graded $\mathbb{k}$-space $A$, let $s^{-1}A$ be the $(-1)$-shifted graded space such that $(s^{-1}A)^i = A^{i+1}$ for $i \in \mathbb{Z}$. The element in $s^{-1}A$ is denoted by $s^{-1}a$ with $|s^{-1}a| = |a| + 1$. Roughly speaking, we have $|s^{-1}| = 1$. Therefore, $s^{-1}\mathbb{k}(Q_{m//}Q_p)$ is concentrated on degree $m-p+1$.

We will define a $\mathbb{k}$-linear map (of degree zero) between graded spaces

$$\kappa_{m,p} : \mathbb{k}(Q_{m//}Q_p) \otimes s^{-1}\mathbb{k}(Q_{m//}Q_{p+1}) \rightarrow \text{Hom}_{E-E}((s\overline{X})^{\otimes E_m}, (s\overline{X})^{\otimes E_{p+1}} \otimes E \Lambda).$$

For $y = (\alpha_{m,1}, \beta_{p,1}) \in Q_{m//}Q_p$ and any monomial $x = s\alpha'_m \otimes_E \cdots \otimes_E s\alpha'_1 \in (s\overline{X})^{\otimes E_m}$ with $\alpha'_j \in Q_1$ for any $1 \leq j \leq m$, we set

$$\kappa_{m,p}(y)(x) = \begin{cases} (-1)^e s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E 1 & \text{if } \alpha_j = \alpha'_j \text{ for all } 1 \leq j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

For $s^{-1}y' = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\mathbb{k}(Q_{m//}Q_{p+1})$, we set

$$\kappa_{m,p}(s^{-1}y')(x) = \begin{cases} (-1)^e s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E \beta_0 & \text{if } \alpha_j = \alpha'_j \text{ for all } 1 \leq j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we denote $e = (m-p)p + \frac{(m-p)(m-p+1)}{2}$. 
Lemma 10.1. ([65, Lemma 3.3]) For any $m, p \geq 0$, the above map $\kappa_{m,p}$ is an isomorphism of graded vector spaces.

We define a graded vector space for each $p \geq 0$,

$$\kappa(Q//Q_p) := \prod_{m \geq 0} \kappa(Q_m//Q_p),$$

where the degree of $(\gamma, \gamma')$ in $Q_m//Q_p$ is $m - p$. We define a $k$-linear map of degree zero

$$\theta_{p,R}: \kappa(Q//Q_p) \rightarrow \kappa(Q//Q_{p+1}), \quad (\gamma, \gamma') \mapsto \sum_{\{\alpha \in Q_k | s(\alpha) = t(\gamma)\}} (\alpha\gamma, \alpha\gamma').$$

Denote by $\overline{C}_{s_p,R,0}^*(Q, Q)$ the colimit of the inductive system of graded vector spaces

$$\kappa(Q//Q_0) \xrightarrow{\theta_{0,R}} \kappa(Q//Q_1) \xrightarrow{\theta_{1,R}} \kappa(Q//Q_2) \xrightarrow{\theta_{2,R}} \cdots \xrightarrow{\theta_{p-1,R}} \kappa(Q//Q_p) \xrightarrow{\theta_{p,R}} \cdots .$$

Therefore, for any $m \in \mathbb{Z}$, we have

$$\overline{C}_{s_p,R,0}^m(Q, Q) = \lim_{\theta_{p,R}} \kappa(Q_{m+p}//Q_p).$$

We define a complex

$$\overline{C}_{s_p,R}^*(Q, Q) = \overline{C}_{s_p,R,0}^*(Q, Q) \oplus s^{-1}\overline{C}_{s_p,R,0}^*(Q, Q),$$

whose differential $\delta$ is induced by

$$\begin{pmatrix} 0 & D_{m,p}^p \end{pmatrix}: \kappa(Q_m//Q_p) \oplus s^{-1}\kappa(Q_{m+1}//Q_p) \rightarrow \kappa(Q_{m+1}//Q_p) \oplus s^{-1}\kappa(Q_{m+1}//Q_{p+1}).$$

For $(\gamma, \gamma') \in Q_m//Q_p$, we have

$$D_{m,p}((\gamma, \gamma')) = \sum_{\{\alpha \in Q_k | s(\alpha) = t(\gamma)\}} s^{-1}(\alpha\gamma, \alpha\gamma') - (-1)^{m-p} \sum_{\{\beta \in Q_k | t(\beta) = s(\gamma')\}} s^{-1}(\gamma\beta, \gamma\beta').$$

(10.3)

We implicitly use the identity $s^{-1}\theta_{p+1,R} \circ D_{m,p} = D_{m+1,p+1} \circ \theta_{p,R}$. Here if the set $\{\beta \in Q_k | t(\beta) = s(\gamma)\}$ is empty then we define $\sum_{\{\beta \in Q_k | t(\beta) = s(\gamma)\}} s^{-1}(\gamma\beta, \gamma\beta) = 0$.

Recall from Subsection 7.2 that $\Omega_{nc,R,E}^p(\Lambda) = (s\Lambda)^{\otimes p} \otimes E \Lambda$. Recall from (7.1) the left $\Lambda$-action $\bullet$. Note that we have

$$\beta_{p+1} \mapsto (s\beta_p \otimes_E \cdots \otimes_E s\beta_1 \otimes_E \beta_0) = \begin{cases} 0 & \text{if } \beta_0 \in Q_1 \\ (-1)^p s\beta_{p+1} \otimes_E \cdots \otimes_E s\beta_2 \otimes_E \beta_1 \beta_0 & \text{if } \beta_0 \in Q_0 \end{cases}$$

where $\beta_i \in Q_k = \overline{\Gamma}$ for $1 \leq i \leq p+1$. Then it is not difficult to show that the map (10.2) is compatible with the differential $\delta_{ex}$ of $\overline{C}_{s_p,R}^*(\Lambda, \Omega_{nc,R,E}^p(\Lambda))$. More precisely, the following diagram is commutative

$$
\begin{array}{ccc}
\text{Hom}_{E,E}((s\Lambda)^{\otimes m}, (s\Lambda)^{\otimes p} \otimes E \Lambda) & \xrightarrow{\delta_{ex}} & \text{Hom}_{E,E}((s\Lambda)^{\otimes m+1}, (s\Lambda)^{\otimes p} \otimes E \Lambda) \\
\kappa_{m,p} \otimes & & \kappa_{m+1,p} \otimes \\
\kappa(Q_m//Q_p) \oplus s^{-1}\kappa(Q_m//Q_{p+1}) & \xrightarrow{\begin{pmatrix} 0 & D_{m,p}^p \end{pmatrix}} & \kappa(Q_{m+1}//Q_p) \oplus s^{-1}\kappa(Q_{m+1}//Q_{p+1})
\end{array}
$$

where recall that the formula for $\delta_{ex}$ is given in Subsection 6.1.
The above commutative diagram allows us to take the colimit along the isomorphisms \( \kappa_{m,p} \) in Lemma 10.1. Therefore, we have the following result.

**Lemma 10.2.** The isomorphisms \( \kappa_{m,p} \) induce an isomorphism of complexes

\[ \kappa: \mathcal{T}_{sg,R}^*(Q, Q) \cong \mathcal{T}_{sg,R,E}^*(\Lambda, \Lambda). \]

### 10.2. The combinatorial \( B_{\infty} \)-algebra

In this subsection, we will transfer, via the isomorphism \( \kappa \), the cup product \( - \cup - \) and brace operation \( - \{ - \} \) of \( \mathcal{T}_{sg,R,E}^*(\Lambda, \Lambda) \) to \( \mathcal{T}_{sg,R}^*(Q, Q) \). We will provide an example for illustration.

By abuse of notation, we still denote the cup product and brace operation on \( \mathcal{T}_{sg,R}^*(Q, Q) \) by \( - \cup - \) and \( - \{ - \} \).

We will use the following non-standard sequences to depict parallel paths.

(i) We write \( s^{-1}x = s^{-1}(\alpha_m, \beta_p, 0) \in s^{-1}\mathcal{T}_{sg,R,0}^*(Q, Q) \) as

\[ \beta_0 \twoheadrightarrow \beta_1 \twoheadrightarrow \ldots \twoheadrightarrow \beta_p \twoheadrightarrow \alpha_m \twoheadrightarrow \ldots \twoheadrightarrow \alpha_2 \twoheadrightarrow \alpha_1. \]  

(ii) We write \( x = (\alpha_m, 1, \beta_p) \in \mathcal{T}_{sg,R,0}^*(Q, Q) \) as

\[ \beta_1 \twoheadrightarrow \ldots \twoheadrightarrow \beta_p \twoheadrightarrow \alpha_m \twoheadrightarrow \ldots \twoheadrightarrow \alpha_2 \twoheadrightarrow \alpha_1 \]

Here, all \( \alpha_1, \ldots, \alpha_m, \beta_0, \beta_1, \ldots, \beta_p \) are arrows in \( Q \).

The above sequences have the following feature: the left part consists of rightward arrows, and the right part consists of leftright arrows. Recall that \( \Omega_{nc,R,E}^*(\Lambda) = (s\Lambda)^{\otimes E} \otimes E \Lambda = (s\Lambda)^{\otimes E} \otimes E \Lambda + (s\Lambda)^{\otimes E} \otimes E E \), and that the leftmost arrow \( \beta_0 \) in (i) is an element in the tensor factor \( \Lambda \). To emphasize this fact, we color the arrow blue. These sequences will be quite convenient to express the cup product and brace operation on \( \mathcal{T}_{sg,R}^*(Q, Q) \), as we will see below.

Let us first describe \( - \cup - \) on \( \mathcal{T}_{sg,R}^*(Q, Q) \). Let

\[ s^{-1}x = s^{-1}(\alpha_m, 1, \beta_p, 0) = (\beta_0 \to \beta_1 \to \ldots \to \beta_p \to \alpha_m \to \ldots \to \alpha_1) \]

\[ s^{-1}y = s^{-1}(\alpha'_m, 1, \beta'_p, 0) = (\beta'_0 \to \beta'_1 \to \ldots \to \beta'_p \to \alpha'_m \to \ldots \to \alpha'_1) \]

be two elements in \( s^{-1}\mathcal{T}_{sg,R,0}^*(Q, Q) \). Let

\[ z = (\alpha_m, 1, \beta_p, 1) = (\beta_1 \to \ldots \to \beta_p \to \alpha_m \to \ldots \to \alpha_1) \]

\[ w = (\alpha'_m, 1, \beta'_p, 1) = (\beta'_1 \to \ldots \to \beta'_p \to \alpha'_m \to \ldots \to \alpha'_1) \]

be two elements in \( \mathcal{T}_{sg,R,0}^*(Q, Q) \). The cup product \( - \cup - \) is given by (C1)-(C4).

(C1) \( (s^{-1}x) \cup_R (s^{-1}y) = 0 \);

(C2) The cup product \( z \cup_R w \) is given by the following parallel path

\[ \delta_{s(\alpha_1), s(\beta'_1)}(\beta_1 \to \ldots \to \beta_p \to \alpha_m \to \ldots \to \alpha_1 \to \beta'_1 \to \ldots \to \beta'_p \to \alpha'_m \to \ldots \to \alpha'_1). \]
Here, we replace the subsequence $\alpha_1 \to \beta_1$ by $\delta_{\alpha,\beta}$ iteratively, till obtaining a parallel path, that is, the left part consists of rightward arrows and the right part consists of leftward arrows. More precisely, we have

$$z \cup_R w = \begin{cases} 
\prod_{i=1}^q \delta_{\beta_i,\alpha_i} (\alpha_{m,q+1}, \beta_{p,1}) & \text{if } q < m, \\
\prod_{i=1}^m \delta_{\beta_i,\alpha_i} (\alpha_{n,1}, \beta_{q,m+1}) & \text{if } q \geq m,
\end{cases}$$

(C3) $(s^{-1}x) \cup_R w$ is obtained by replacing $\alpha_1 \to \beta_1$ with $\delta_{\alpha,\beta}$, iteratively

$$\begin{array}{c}
\alpha \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_m \\
\beta \to \beta_1 \to \beta_2 \to \cdots \to \beta_p
\end{array}$$

Therefore, we have

$$(s^{-1}x) \cup_R w = \begin{cases} 
\prod_{i=1}^q \delta_{\beta_i,\alpha_i} s^{-1}(\alpha_{m,q+1}, \beta_{p,0}) & \text{if } q < m, \\
\prod_{i=1}^m \delta_{\beta_i,\alpha_i} s^{-1}(\alpha_{n,1}, \beta_{q,m+1}) & \text{if } q \geq m;
\end{cases}$$

(C4) $z \cup_R (s^{-1}y)$ is obtained by replacing $\alpha_1 \to \beta_1$ with $\delta_{\alpha,\beta}$, iteratively

$$\begin{array}{c}
\alpha \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_m \\
\beta \to \beta_1 \to \beta_2 \to \cdots \to \beta_p
\end{array}$$

Therefore, we have

$$(s^{-1}y) \cup_R z = \begin{cases} 
\prod_{i=1}^q \delta_{\beta_i,\alpha_i} s^{-1}(\alpha_{m,q+1}, \beta_{p,1}) & \text{if } q < m, \\
\prod_{i=1}^m \delta_{\beta_i,\alpha_i} s^{-1}(\alpha_{n,1}, \beta_{q,m+1}) & \text{if } q \geq m.
\end{cases}$$

Let us describe the brace operation $\{-,-,\}^*_R$ on $\overline{C}_{sg,R}(Q,Q)$ in the following cases (B1)-(B3).

(B1) For any $x \in \overline{C}_{sg,R}^*(Q,Q)$, we have

$$x\{y_1, \ldots, y_k\}_R = 0$$

if there exists some $1 \leq j \leq k$ with $y_j \in \overline{C}_{sg,R,0}^*(Q,Q) \subset \overline{C}_{sg,R}(Q,Q)$.

(B2) If $s^{-1}y_j \in s^{-1}\overline{C}_{sg,R,0}^*(Q,Q)$ is such that $y_j$ is a parallel path for each $1 \leq j \leq k$, and $s^{-1}x = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\overline{C}_{sg,R,0}(Q,Q)$, then

$$(s^{-1}x)\{s^{-1}y_1, \ldots, s^{-1}y_k\}_R = \sum_{1 \leq a+b=k, a,b \geq 0} \sum_{1 \leq i_1 < i_2 < \cdots < i_a \leq m} (-1)^{a+b} b_{(i_1, \ldots, i_a)}^{(i_1, \ldots, i_a)} (s^{-1}x; s^{-1}y_1, \ldots, s^{-1}y_k),$$

where $b_{(i_1, \ldots, i_a)}^{(i_1, \ldots, i_a)} (s^{-1}x; s^{-1}y_1, \ldots, s^{-1}y_k)$ is illustrated as follows

$$\begin{array}{c}
\beta_{i_1} \to \beta_{i_1} \to \cdots \to \beta_{i_1} \to y_1 \to \beta_{i_2} \to \cdots \to \beta_{i_2} \to y_2 \to \cdots \to \beta_{i_a} \to y_{i_a} \to \cdots \to \beta_{i_k} \to y_{i_k} \to \cdots \to \alpha_1
\end{array}$$

To save the space, we just use the symbol $y_j$ to indicate the sequence of the parallel path $y_j$ as in (10.5) for $1 \leq j \leq k$. We replace any subsequence $\alpha_1 \to \beta_1$ by $\delta_{\alpha,\beta}$ iteratively, and then arrive at a well-defined parallel path.
Let us explain the sign $(-1)^{a+\epsilon}$ appeared above. The sign

$$\epsilon = \sum_{r=1}^{b} (|s^{-1} y_r| - 1)(m + p - l_r + 1) + \sum_{r=1}^{a} (|s^{-1} y_{k-r+1}| - 1)(i_r - 1)$$

is obtained via the Koszul sign rule by re-ordering the positions ($\beta^*_l$ and $\alpha_j$ are of degree one) of the elements

$$\beta^*_0, \beta^*_1, \ldots, \beta^*_p, \alpha_m, \ldots, \alpha_1, y_1, y_2, \ldots, y_k;$$

and the extra sign $(-1)^a$ is to make sure that the brace operation is compatible with the colimit maps $\theta_{*,R}$.

(B3) If $s^{-1}y_j \in s^{-1}\mathcal{C}^+_R(Q,Q)$ is such that $y_j$ is a parallel path for each $1 \leq j \leq k$, and $x = (\alpha_p,1,\beta_{m,1}) \in \mathcal{C}^+_R(Q,Q)$, then

$$x\{s^{-1}y_1, \ldots, s^{-1}y_k\} = \sum_{a+b=k, a,b \geq 0} \sum_{1 \leq l_1 < l_2 < \cdots < l_a \leq m} \sum_{1 \leq l_1 < l_2 < \cdots < l_b \leq p} (-1)^{a+b} \delta_{(l_1,\ldots,l_a)}(x; s^{-1}y_1, \ldots, s^{-1}y_k),$$

where $\delta_{(l_1,\ldots,l_a)}(x; s^{-1}y_1, \ldots, s^{-1}y_k)$ is obtained from the following sequence by replacing $\frac{\alpha}{\beta} \rightarrow \frac{\delta_{\alpha,\beta}}{ \delta_{\delta_{\alpha,\beta}}} \text{iteratively}$

$$\beta_{l_1} \ldots \beta_{l_{a-1}} y_1 \beta_{l_1} \ldots \beta_{l_{b-1}} y_b \beta_{l_1} \ldots \beta_{l_{1}} y_k \alpha_{l_1} \ldots \alpha_{l_{a-1}} \alpha_{l_{1}} \ldots \alpha_{l_{b-1}} \alpha_{l_{1}},$$

and $\epsilon$ is the same as in (B2).

**Theorem 10.3.** The complex $\mathcal{C}^+_R(Q,Q)$, equipped with the cup product $- \cup_R -$ and brace operation $-\{-\} \cup_R -$, is a brace $B_{\infty}$-algebra. Moreover, the isomorphism $\kappa: \mathcal{C}^+_R(Q,Q) \rightarrow \mathcal{C}^+_R(E\Lambda,\Lambda)$ is a strict $B_{\infty}$-isomorphism.

The resulted $B_{\infty}$-algebra $\mathcal{C}^+_R(Q,Q)$ is called the **combinatorial $B_{\infty}$-algebra** of $Q$.

**Proof.** The above cup product $- \cup_R -$ and brace operation $-\{-\} \cup_R -$ on $\mathcal{C}^+_R(Q,Q)$ are transferred from $\mathcal{C}^+_R(E\Lambda,\Lambda)$ via the isomorphism $\kappa$; compare Theorem 7.2 and Lemma 10.2. More precisely, for any $x, y, y_1, \ldots, y_k \in \mathcal{C}^+_R(Q,Q)$ we may check

$$\kappa(x \cup_R y) + \kappa(y) = \kappa(x) \cup_R \kappa(y),$$

$$(-1)^{a+\epsilon} \kappa(\delta_{(l_1,\ldots,l_a)}(x; y_1, \ldots, y_k)) = (-1)^b \delta_{(l_1,\ldots,l_a)}(\kappa(x); \kappa(y_1), \ldots, \kappa(y_k)), \quad (10.6)$$

where $\epsilon$ is defined as in (B2) above.

We may check the first identity case by case. Let $x = (\alpha_{p,1}, \beta_{m,1})$ and $y = (\alpha_{n,1}, \beta_{q,1})$. Suppose first that $q < m$. Then for $z \in s\Lambda^{\otimes \ell m+n-q}$ we have

$$\kappa(x \cup_R y)(z) = \prod_{i=1}^{q} \kappa(\delta_{(l_1,\ldots,l_a)}(\alpha_{m,q+1}, \beta_{m,1}))(z)$$

$$= \left\{ \begin{array}{ll}
(-1)^{\ell q} \prod_{i=1}^{q} \delta_{\beta_{l_1}, \alpha} \cdot s\beta_p \otimes \cdots \otimes s\beta_1 \otimes 1, & \text{if } z = s\alpha_m \otimes \cdots \otimes s\alpha_{q+1} \otimes s\alpha_n \otimes \cdots \otimes s\alpha_1, \\
0, & \text{otherwise},
\end{array} \right.$$
where
\[
\epsilon_1 = (m + n - p - q)p + \frac{(m + n - p - q)(m + n - p + q + 1)}{2}.
\]
Here the first equality follows from (C2) and the second identity follows from the definition of \( \kappa \). Note that we have \( \kappa(x) \in \text{Hom}_{E,E}(s\Lambda^{\otimes E_m}, \Omega^p_{nc,R,E}(\Lambda)) \) and \( \kappa(y) \in \text{Hom}_{E,E}(s\Lambda^{\otimes E_n}, \Omega^q_{nc,R,E}(\Lambda)). \) By the definition of the cup product of \( C^\ast_{sg,R,E}(\Lambda, \Lambda) \) in (7.4), we have \( \kappa(x) \cup_R \kappa(y) \in \text{Hom}_{E,E}(s\Lambda^{\otimes E_m+n}, \Omega^{p+q}_{nc,R,E}(\Lambda)). \) One may check directly that
\[
\kappa(x) \cup_R \kappa(y) = (\theta_p+q-1,R,E \circ \cdots \circ \theta_{p+1},R,E \circ \theta_{p+1},R,E)(\kappa(x \cup_R y)).
\]
Thus we have \( \kappa(x \cup_R y) = \kappa(x) \cup_R \kappa(y) \) in \( C^\ast_{sg,R,E}(\Lambda, \Lambda) \). Similarly, we may check for \( q \geq m \).

We omit the routine verification for the other three cases, according to (C1), (C3) and (C4).

The second identity in (10.6) follows from the observation that the Deletion Process in Definition 7.8 exactly corresponds to the iterative replacement in (B2) and (B3). See Example 10.4 below for a detailed illustration.

\[\text{Figure 6. If } \beta_0 \in Q_1, \text{ then the left graph represents the element } g = s^{-1}(\alpha_m \cdots \alpha_1, \beta_p \cdots \beta_0) \in \overline{C}^\ast_{sg,R}(Q,Q). \text{ If } \beta_0 = s(\beta_1) \in Q_0, \text{ then it represents } g = (\alpha_m \cdots \alpha_1, \beta_p \cdots \beta_1) \in \overline{C}^\ast_{sg,R}(Q,Q). \text{ The map represented by the right graph is nonzero only if the elements in each internal edge coincide (i.e. } \alpha''_2 = \beta''_3, \alpha''_1 = \beta''_0, \alpha_2 = \beta'_0 \text{ and so on).}\]

**Example 10.4.** Consider the following four monomial elements in \( \overline{C}^\ast_{sg,R}(Q,Q) \)
\[
\begin{align*}
& s^{-1}x = s^{-1}(\alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1, \beta_3 \beta_2 \beta_1 \beta_0) \\
& s^{-1}y_1 = s^{-1}(\alpha'_3 \alpha'_2 \alpha'_1, \beta'_0) \\
& s^{-1}y_2 = s^{-1}(\alpha''_3 \alpha''_2 \alpha''_1, \beta''_3 \beta''_2 \beta''_1 \beta_0) \\
& s^{-1}y_3 = s^{-1}(\alpha'''_2 \alpha'''_1, \beta'''_3 \beta'''_2 \beta'''_1 \beta''_0).
\end{align*}
\]
According to (10.4), they may be depicted in the following way
\[
\begin{align*}
  s^{-1}x &= (\beta_0, \beta_1, \beta_3, \alpha_5, \alpha_4, \alpha_1, \rho_2, \rho_1) \\
  s^{-1}y_1 &= (\beta_0, \beta_3, \alpha_3, \alpha_1) \\
  s^{-1}y_2 &= (\beta_0, \beta_2, \alpha_3, \alpha_1) \\
  s^{-1}y_3 &= (\beta_0, \beta_2, \alpha_3, \alpha_1).
\end{align*}
\]

By Formula (B2), the operation \( b_{(2)}^{(2,4)}(s^{-1}x, s^{-1}y_1, s^{-1}y_2, s^{-1}y_3) \) is depicted by
\[
(\beta_0, \beta_1, \beta_3, \alpha_5, \alpha_4, \alpha_1, \rho_2, \rho_1, \alpha_3, \alpha_1, \rho_2, \rho_1, \alpha_3, \alpha_1).
\]

After replacing \( \xi \rightarrow \beta \) iteratively, we get
\[
\lambda(\beta_0, \beta_1, \beta_3, \alpha_5, \alpha_4, \alpha_1, \rho_2, \rho_1, \alpha_3, \alpha_1, \rho_2, \rho_1, \alpha_3, \alpha_1), \tag{10.7}
\]
where \( \lambda = \delta_{\alpha_1, \beta_2} \delta_{\alpha_2, \beta_3} \delta_{\alpha_4, \beta_5} \delta_{\alpha_5, \beta_4} \delta_{\alpha_1, \rho_2} \delta_{\alpha_2, \rho_1} \delta_{\alpha_3, \beta_1} \delta_{\alpha_4, \beta_2} \delta_{\alpha_5, \beta_3} \). Hence,
\[
b_{(2)}^{(2,4)}(s^{-1}x, s^{-1}y_1, s^{-1}y_2, s^{-1}y_3) = \lambda s^{-1}(\alpha_3 \alpha_2 \beta_1 \alpha_1, \beta_3 \beta_2 \beta_1 \beta_0, \beta_0). \tag{10.8}
\]

Let us check that \( \kappa \) preserves the brace operations. Note that
\begin{itemize}
  \item \( f := \kappa(s^{-1}x) \in C_{\mathcal{E}}(\Lambda, \Omega_{\text{nc,R}}(\Lambda)) \) is uniquely determined by
    \[
    so_5 \otimes so_4 \otimes so_3 \otimes so_2 \otimes so_1 \mapsto -s \beta_3 \otimes s \beta_2 \otimes s \beta_1 \otimes \beta_0,
    \]
    i.e. sending any other monomial to zero.
  \item \( g_1 := \kappa(s^{-1}y_1) \in C_{\mathcal{F}}(\Lambda, \Omega_{\text{nc,R}}(\Lambda)) \) is uniquely determined by
    \[
    so_3 \otimes so_2 \otimes so_1 \mapsto -s \beta_1 \otimes \beta_0;
    \]
  \item \( g_2 := \kappa(s^{-1}y_2) \in C_{\mathcal{F}}(\Lambda, \Omega_{\text{nc,R}}(\Lambda)) \) is uniquely determined by
    \[
    so_3 \otimes so_2 \otimes so_1 \mapsto s \beta_2 \otimes s \beta_3 \otimes s \beta_1 \otimes \beta_0;
    \]
  \item \( g_3 := \kappa(s^{-1}y_3) \in C_{\mathcal{F}}^{-1}(\Lambda, \Omega_{\text{nc,R}}(\Lambda)) \) is uniquely determined by
    \[
    so_3 \otimes so_2 \otimes so_1 \mapsto -s \beta_3 \otimes s \beta_2 \otimes s \beta_1 \otimes \beta_0.
    \]
\end{itemize}

By Figure 5 we have that the element
\[
B_{(2)}^{(2,4)}(\kappa(s^{-1}x), \kappa(s^{-1}y_1), \kappa(s^{-1}y_2), \kappa(s^{-1}y_3)) = B_{(2)}^{(2,4)}(f; g_1, g_2, g_3)
\]
is depicted by the graph on the right of Figure 6, which is uniquely determined by
\[
so_3 \otimes so_2 \otimes so_1 \mapsto \lambda(s \beta_3 \otimes s \beta_2 \otimes s \beta_1 \otimes \beta_0).
\]

Here \( \lambda \) is defined in (10.7). On the other hand, by (10.8) we have that \( \kappa(b_{(2)}^{(2,4)}(s^{-1}x, s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)) \) is uniquely determined by
\[
so_3 \otimes so_2 \otimes so_1 \mapsto -\lambda(s \beta_3 \otimes s \beta_2 \otimes s \beta_1 \otimes \beta_0).
\]
where we recall that the degree given by \( x \) for any \( \rho \) This defines the complex (\( \hat{\kappa} \)).

\[ \kappa(b_{(2,4)}^{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)) = -B_{(2)}^{(2,4)}(\kappa(s^{-1}x); \kappa(s^{-1}y_1), \kappa(s^{-1}y_2), \kappa(s^{-1}y_3)). \]

This verifies that \( \kappa \) preserves the brace operations.

### 11. The Leavitt \( B_\infty \)-algebra as an intermediate object

Let \( Q \) be a finite quiver without sinks. Let \( L = L(Q) \) is the Leavitt path algebra of \( Q \). In this section, we introduce the Leavitt \( B_\infty \)-algebra \( \hat{C}^*(L, L) \), which is an intermediate object connecting the singular Hochschild cochain complex of \( \mathbb{K}Q/J^2 \) to the Hochschild cochain complex of \( L \).

More precisely, we will show that the Leavitt \( B_\infty \)-algebra \( \hat{C}^*(L, L) \) is strictly \( B_\infty \)-isomorphic to \( \mathcal{C}_{sg,R}(Q, Q) \); see Proposition 11.4 below. In Sections 13 and 14, we will show that there is an explicit non-strict \( B_\infty \)-quasi-isomorphism between the two \( B_\infty \)-algebras \( \hat{C}^*(L, L) \) and \( \mathcal{C}_E(L, L) \). Namely, we have

\[ \mathcal{C}_{sg,R,E}(\Lambda, \Lambda) \leftarrow_{\kappa} \mathcal{C}_{sg,R}(Q, Q) \xrightarrow{\rho} \hat{C}^*(L, L) \xrightarrow{(\Phi_1, \Phi_2, \cdots)} \mathcal{C}_E(L, L), \]

where the left two maps are strict \( B_\infty \)-isomorphisms and the rightmost one is a non-strict \( B_\infty \)-quasi-isomorphism. Recall that the leftmost map \( \kappa \) is already given in Theorem 10.3.

#### 11.1. An explicit complex.

We define the following graded vector space

\[ \hat{C}^*(L, L) = \bigoplus_{i \in Q_0} e_i Le_i \oplus \bigoplus_{i \in Q_0} s^{-1}e_i Le_i, \]

where we recall that the degree \(|s^{-1}| = 1\). The differential \( \hat{\delta} \) of \( \hat{C}^*(L, L) \) is given by \((0 \ 0 \ \delta')\), where

\[ \delta'(x) = s^{-1}x - (-1)^{|x|} \sum_{\{\alpha \in Q_1 | b(\alpha) = i\}} s^{-1}\alpha^*x\alpha \]

for any \( x = e_i x e_i \in e_i Le_i \) and \( i \in Q_0 \). Note that we have \( \hat{\delta}(s^{-1}y) = 0 \) for \( y \in \bigoplus_{i \in Q_0} e_i Le_i \). This defines the complex \( (\hat{C}^*(L, L), \hat{\delta}) \).

Recall the complex \( \mathcal{C}_{sg,R}(Q, Q) \) from (10.1). We claim that there is a morphism of complexes

\[ \rho: \mathcal{C}_{sg,R}(Q, Q) \rightarrow \hat{C}^*(L, L) \]

given by

\[ \rho((\gamma, \gamma')) = \gamma'^*\gamma \quad \text{for} \ (\gamma, \gamma') \in Q_m//Q_p; \]

\[ \rho(s^{-1}(\gamma, \gamma')) = s^{-1}\gamma'^*\gamma \quad \text{for} \ s^{-1}(\gamma, \gamma') \in s^{-1}k(Q_m//Q_{p+1}). \]

Indeed, we observe that for \((\gamma, \gamma') \in Q_m//Q_p\)

\[ \rho(\theta_{p,R}(\gamma, \gamma')) = \sum_{\alpha \in Q_1} (\alpha \gamma')^*\alpha \gamma = \gamma'^*\gamma = \rho((\gamma, \gamma')), \]

where the second equality follows from \( \sum_{\{\alpha \in Q_1 | s(\alpha) = i\}} \alpha^*\alpha = e_i \). Similarly, we have

\[ \rho(\theta_{p,R}(s^{-1}(\gamma, \gamma'))) = \rho(s^{-1}(\gamma, \gamma')). \]
This shows that $\rho$ is well defined. Comparing $D_{m,p}$ in (10.3) and $\delta'$, it is easy to check that $\rho$ commutes with the differentials. This proves the claim. Moreover, we have the following result.

**Lemma 11.1.** The above morphism $\rho$ is an isomorphism of complexes.

*Proof.* This follows immediately from the definition of $\tilde{C}_{sg,R,0}^\epsilon(Q,Q)$ and Lemma 4.1. \hfill \Box

### 11.2. The Leavitt $B_\infty$-algebra.

We will define the cup product $-\cup'$ and brace operation $-\{\ldots,-\}'$ on $\hat{C}^*(L,L)$.

Recall from (4.1) that each element in $e_iLe_i \subset \hat{C}^*(L,L)$ can be written as a linear combination of the following monomials

$$\beta_1^* \beta_2^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1,$$

where $\beta_p \cdots \beta_2 \beta_1$ and $\alpha_m \alpha_{m-1} \cdots \alpha_1$ are paths in $Q$ with lengths $p$ and $m$, respectively. In particular, all $\beta_j$ and $\alpha_k$ belong to $Q_1$. Moreover, we require that $p \geq 1$ and $m \geq 0$, and that $t(\alpha_m) = s(\beta_p^*) = t(\beta_p)$. In case where $m = 0$, these $\alpha_i$’s do not appear. The monomial (11.1) has degree $m - p$.

Similarly, we write any element in $s^{-1}e_iLe_i \subset \hat{C}^*(L,L)$ as a linear combination of the following monomials

$$s^{-1} \beta_0^* \beta_1^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1,$$

where $\alpha_k, \beta_j \in Q_1$ for $1 \leq k \leq m$ and $0 \leq j \leq p$. The monomial (11.2) also has degree $m - p$. The difference here is that we require $p \geq 0$ and $m \geq 0$, since the $\beta_j$’s are indexed from zero.

The cup product $-\cup'$ on $\hat{C}^*(L,L)$ is defined by the following (C1’)-(C4’).

(C1’) For any $s^{-1}u \in s^{-1}e_iLe_i$ and $s^{-1}v \in s^{-1}e_jLe_j$ with $i,j \in Q_0$, we have

$$s^{-1}u \cup' s^{-1}v = 0;$$

(C2’) For any $u \in e_iLe_i$ and $v \in e_jLe_j$ with $i,j \in Q_0$, we have

$$u \cup' v = uv;$$

(C3’) For any $s^{-1}u \in s^{-1}e_iLe_i$ and $v \in e_jLe_j$ with $i,j \in Q_0$, we have

$$(s^{-1}u) \cup' v = s^{-1}uv;$$

(C4’) For any $u \in e_iLe_i$ and $s^{-1}v = s^{-1} \beta_0^* \beta_1^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1 \in s^{-1}e_jLe_j$ with $i,j \in Q_0$, we have

$$u \cup' s^{-1}v = \sum_{\alpha \in Q_1} s^{-1} \alpha^* u \alpha v = s^{-1} \beta_0^* u \beta_1^* \beta_2^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1.$$  

Here, we use the relations $\alpha \beta^* = \delta_{\alpha \beta} e_{t(\alpha)}$. Note that there is no Koszul sign caused by swapping $s^{-1} \beta_0^*$ with $u$, as the degree of $s^{-1} \beta_0^*$ is zero.

Then $\hat{C}^*(L,L)$ becomes a dg algebra with this cup product.

**Remark 11.2.** (1) It seems that we can not define the cup product naturally to $L \oplus s^{-1}L$. For instance, take $u \in e_iLe_j$ and $v \in e_jLe_i$ with $i,j \in Q_0$, $i \neq j$. When we
define \( u \cup' v = uv \) and extend the differential \( \delta' \): \( L \rightarrow s^{-1}L \) by \( \delta'(u) = s^{-1}u \) and \( \delta'(v) = s^{-1}v \), then we have

\[
\delta'(u \cup' v) = s^{-1}uv - (-1)^{|uv|} \sum_{\{\alpha \in Q_1, \ell(\alpha) = i\}} s^{-1}\alpha^* uv\alpha.
\]

But on the other hand, we have

\[
\delta'(u \cup' v + (-1)^{|u\cup' v|} u \cup' \delta'(v)) = s^{-1}uv + (-1)^{|u\cup' v|} s^{-1}v = s^{-1}uv + (-1)^{|u\cup' v|} \sum_{\alpha \in Q_1} s^{-1}\alpha^* uv\alpha = s^{-1}uv.
\]

So it is possible that \( \delta'(u \cup' v) \neq \delta'(u \cup' v + (-1)^{|u\cup' v|} u \cup' \delta'(v)) \). In other words, we could not obtain a dg algebra with the cup product and the differential.

(2) By \((C3')\) and \((C4')\), we may view \( \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \) as a bimodule over \( \bigoplus_{i \in Q_0} e_i L e_i \).

According to \((C1')\), \( \hat{C}^*(L, L) \) is the trivial extension ring; see [8, pp. 78].

Let \( v, u_1, \ldots, u_k \) be monomials in \( \hat{C}^*(L, L) \). Then the brace operation \( v\{u_1, \ldots, u_k\}' \) is defined by the following \((B1')-(B3')\).

\((B1')\) If \( u_j \in \prod_{i \in Q_0} e_i L e_i \subset \hat{C}^*(L, L) \) for some \( 1 \leq j \leq k \), then

\[
v\{u_1, \ldots, u_k\}' = 0.
\]

\((B2')\) If \( s^{-1}u_j \in \prod_{i \in Q_0} s^{-1}e_i L e_i \subset \hat{C}^*(L, L) \) for each \( 1 \leq j \leq k \), and

\[
s^{-1}v = s^{-1}\beta_0^{*}\beta_1^{*} \cdots \beta_p^{*}\alpha_m^{*} \cdots \alpha_1 \in \prod_{i \in Q_0} s^{-1}e_i L e_i \subset \hat{C}^*(L, L)
\]

then we define

\[
s^{-1}v\{s^{-1}u_1, \ldots, s^{-1}u_k\}' = \sum_{a+b=k, a,b \geq 0, 1 \leq l_1 < l_2 < \cdots < l_a \leq m, 1 \leq l_1 \leq l_2 \leq \cdots \leq l_b \leq p} (-1)^{a+b} h_{i_1, \ldots, i_a}^{(l_1, \ldots, l_b)}(s^{-1}v, s^{-1}u_1, \ldots, s^{-1}u_k),
\]

where \( h_{i_1, \ldots, i_a}^{(l_1, \ldots, l_b)}(s^{-1}v, s^{-1}u_1, \ldots, s^{-1}u_k) \in \prod_{i \in Q_0} s^{-1}e_i L e_i \) is defined as

\[
s^{-1}\beta_0^{*}\beta_1^{*} \cdots \beta_{l_1-1}^{*}u_1\beta_1^{*} \cdots \beta_{l_2-1}^{*}u_2\beta_2^{*} \cdots \beta_{l_3-1}^{*}u_3\beta_3^{*} \cdots \beta_{l_h-1}^{*}u_h\beta_h^{*} \cdots \beta_{l_p-1}^{*}\beta_p^{*}\alpha_m^{*} \cdots \alpha_1u_{b+1}\alpha_{i_{b+1}} \cdots \alpha_{i_1}u_{k-1}\alpha_{i_{k-1}} \cdots \alpha_2\alpha_1,
\]

and the sign

\[
\epsilon = \sum_{r=1}^{b} (|s^{-1}u_r| - 1)(m + p - l_r + 1) + \sum_{r=1}^{a} (|s^{-1}u_{k-r+1}| - 1)(i_r - 1)
\]

is obtained via the Koszul sign rule by reordering the elements \( \beta_i^* \) and \( \alpha_i \) are of degree one
(B3') If \( s^{-1} u_j \in \prod_{i \in Q_0} s^{-1} e_i L e_i \subset \hat{C}^s(L, L) \) for each \( 1 \leq j \leq k \), and \( v = \beta^*_1 \cdots \beta^*_p \alpha_m \cdots \alpha_1 \in \prod_{i \in Q_0} e_i L e_i \subset \hat{C}^s(L, L) \), then

\[
v\{s^{-1} u_1, \ldots, s^{-1} u_k\}' = \sum_{a+b=k, a,b \geq 0} (-1)^{a+\epsilon} \mathbb{L}_{(i_1, \ldots, i_b)}(v; s^{-1} u_1, \ldots, s^{-1} u_k),
\]

(11.4)

where \( \mathbb{L}_{(i_1, \ldots, i_b)}(v; s^{-1} u_1, \ldots, s^{-1} u_k) \) is defined as

\[
\beta^*_1 \beta^*_2 \cdots \beta^*_{l_1-1} u_1 \beta^*_{l_1} \cdots \beta^*_{l_2-1} u_2 \beta^*_{l_2} \cdots \beta^*_b \beta^*_{b-1} u_b \beta^*_{b} \cdots \beta^*_p \alpha_m \cdots \alpha_i u_{b+1} \cdots \alpha_i u_k \alpha_{i+1} \cdots \alpha_1.
\]

We are not allowed to insert any \( u_i \) between \( \beta^*_p \) and \( \alpha_m \); in case where \( m = 0 \), the insertion on the right of \( \beta^*_p \) is not allowed. If \( a = 0 \), there is no insertions into \( \alpha_k \)'s. Similarly, if \( b = 0 \), there is no insertions into \( \beta^*_p \)'s.

Since \( 1 \leq l_1 \leq l_2 \leq \cdots \leq l_b \leq p \), we are allowed to insert more than one \( u_i \)'s into \( s^{-1} v \) at the same position between \( \beta^*_p \) and \( \beta^*_j \) for some \( 1 \leq j \leq p \). For example, we might have the following insertion with \( l_2 = l_3 \)

\[
s^{-1} \beta^*_0 \beta^*_1 \cdots \beta^*_{l_1-1} u_1 \beta^*_{l_1} \cdots \beta^*_{l_3-1} u_2 u_3 \beta^*_{l_3} \cdots \beta^*_b \beta^*_{b-1} u_b \beta^*_{b} \cdots \beta^*_p \alpha_m \cdots \alpha_i u_{b+1} \cdots \alpha_i u_k \alpha_{i+1} \cdots \alpha_1.
\]

As \( 1 \leq i_1 < i_2 < \cdots < i_j \leq m \), we are not allowed to insert more than one \( u_i \)'s into \( s^{-1} v \) at the same position between \( \alpha_{j-1} \) and \( \alpha_j \) for some \( 1 \leq j \leq m \). For example, the following insertion is not allowed

\[
s^{-1} \beta^*_0 \beta^*_1 \cdots \beta^*_{i_1} \cdots \beta^*_{i_j} \cdots \beta^*_{i_j} \cdots \beta^*_p \alpha_m \cdots \alpha_i u_{b+1} \cdots \alpha_i u_k \alpha_{i+1} u_{k-2} \cdots \alpha_1.
\]

(2) The brace operation is well defined, that is, it is compatible with the second Cuntz-Krieger relations or (4.1). For the proof, one might use the following relation to swap the insertion of \( u_b \) into \( s^{-1} v \)

\[
\sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} \alpha^* u_b = \sum_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} u_b \alpha^* \alpha,
\]

where both sides are equal to \( \delta_{i,j} u_b \) for \( u_b \in e_j L e_j \). Proposition 11.4 will provide an alternative proof for the well-definedness.

(3) We observe that \( v\{s^{-1} u_1, \ldots, s^{-1} u_k\} \) in (11.4) is also defined for any \( v \in L \), not necessarily \( v \in \bigoplus_{i \in Q_0} e_i L e_i \). However, due to (2), it seems to be essential to require that all the \( u_j \)'s belong to \( \bigoplus_{i \in Q_0} e_i L e_i \).
It seems to be very nontrivial to verify directly that the above data define a $B_\infty$-structure on $\hat{C}^*(L, L)$. Instead, we use the isomorphism $\rho$ in Lemma 11.1 to show that the above data are transferred from those in $\hat{C}_{sg,R}^*(Q, Q)$.

**Proposition 11.4.** The isomorphism $\rho: \hat{C}_{sg,R}^*(Q, Q) \rightarrow \hat{C}^*(L, L)$ preserves the cup products and the brace operations. In particular, the complex $\hat{C}^*(L, L)$, equipped with the cup product $-\cup^\prime- \text{ and the brace operation }-\{ -, -, -\}^\prime$ defined as above, is a $B_\infty$-algebra.

The obtained $B_\infty$-algebra $\hat{C}^*(L, L)$ is called the Leavitt $B_\infty$-algebra, due to its closed relation to the Leavitt path algebra. Combining this result with Theorem 10.3, we infer that $\hat{C}^*(L, L)$ and $\hat{C}_{sg,R,E}^*(\Lambda, \Lambda)$ are strictly $B_\infty$-isomorphic.

**Proof.** By a routine computation, we verify that $\rho$ sends the formulae (C1)-(C4) to (C1')-(C4'), respectively. For example, replacing $\alpha \beta$ by $\delta_{\alpha, \beta}$ in (C2)-(C4) corresponds to the first Cuntz-Krieger relations $\alpha \beta^* = \delta_{\alpha, \beta} e_{(\alpha)}$ implicitly used in the multiplication of $L$ in (C2')-(C4').

It remains to check that $\rho$ is compatible with the brace operations. That is, $\rho$ sends the formulae (B1)-(B3) to (B1')-(B3'), respectively.

Let $x, y_1, \ldots, y_k$ be parallel paths either in $\hat{C}_{sg,R,0}^*(Q, Q)$ or in $s^{-1}\hat{C}_{sg,R,0}^*(Q, Q)$. If there exists some $y_j$ belonging to $\hat{C}_{sg,R,0}^*(Q, Q)$, then $x\{y_1, \ldots, y_k\}_R = 0$. Thus, we have

$$\rho(x\{y_1, \ldots, y_k\}_R) = 0 = \rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'_R.$$  

This shows that $\rho$ sends the formula (B1) to the formula (B1').

Let $x = s^{-1}(\alpha_{m,1}, \beta_{p,0}) \in s^{-1}\hat{C}_{sg,R,0}^*(Q, Q)$ and $y_1, \ldots, y_k \in s^{-1}\hat{C}_{sg,R,0}^*(Q, Q)$. Using the first Cuntz-Krieger relations $\alpha \beta^* = \delta_{\alpha, \beta} e_{(\alpha)}$, we infer that $\rho$ sends the summand $b_{(i_1, \ldots, i_j)}^{(i_1, \ldots, i_j)}(x; y_1, \ldots, y_k)$ of $x\{y_1, \ldots, y_k\}_R$ in (10.2) to the one $\rho(b_{(i_1, \ldots, i_k)}^{(i_1, \ldots, i_k)}(\rho(x); \rho(y_1), \ldots, \rho(y_k)))$ of $\rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'_R$ in (11.3). See Example 11.5 below for a detailed illustration. Thus we have

$$\rho(x\{y_1, \ldots, y_k\}_R) = \rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'_R.$$  

This shows that the formula (B2) corresponds to (B2') under $\rho$.

Similarly, if $x = (\alpha_{m,1}, \beta_{p,1}) \in \hat{C}_{sg,R,0}^*(Q, Q)$ and $y_1, \ldots, y_k \in s^{-1}\hat{C}_{sg,R,0}^*(Q, Q)$, we have

$$\rho(b_{(i_1, \ldots, i_k)}^{(i_1, \ldots, i_k)}(x; y_1, \ldots, y_k)) = \rho(b_{(i_1, \ldots, i_k)}^{(i_1, \ldots, i_k)}(\rho(x); \rho(y_1), \ldots, \rho(y_k)))$$

and thus $\rho(x\{y_1, \ldots, y_k\}_R) = \rho(x)\{\rho(y_1), \ldots, \rho(y_k)\}'_R$. This shows that $\rho$ sends (B3) to (B3').

**Example 11.5.** Consider the following monomial elements in $\hat{C}_{sg,R}^*(Q, Q)$ as in Example 10.4

$$s^{-1}x = s^{-1}(\alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1, \beta_3 \beta_2 \beta_1 \beta_0)$$

$$s^{-1}y_1 = s^{-1}(\alpha_3^2 \alpha_2 \alpha_1, \beta_2 \beta_1 \beta_0)$$

$$s^{-1}y_2 = s^{-1}(\alpha_3 \alpha_2^2 \alpha_1, \beta_3 \beta_2 \beta_1 \beta_0)$$

$$s^{-1}y_3 = s^{-1}(\alpha_2 \alpha_1^3, \beta_3 \beta_2 \beta_1 \beta_0).$$
Let us check that \( \rho \) preserves the brace operations. Note that
\[
\rho(s^{-1}x) = s^{-1} \beta_0^s \beta_1^s \beta_2^s \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1
\]
\[
\rho(s^{-1}y_1) = s^{-1} \beta_0^s \beta_1^s \alpha_3 \alpha_2 \alpha_1'^{t}
\]
\[
\rho(s^{-1}y_2) = s^{-1} \beta_0^s \beta_1^s \beta_2^s \beta_3^s \alpha_3 \alpha_2 \alpha_1
\]
\[
\rho(s^{-1}y_3) = s^{-1} \beta_0^s \beta_1^s \beta_2^s \beta_3^s \alpha_2 \alpha_1'^{m}.\]

Then by Formula (B2') we have that
\[
\mathcal{E}^{(2,4)}_{(2)}(\rho(s^{-1}x); \rho(s^{-1}y_1), \rho(s^{-1}y_2), \rho(s^{-1}y_3)) = s^{-1} \beta_0^s \beta_1^s \beta_2^s \beta_3^s \alpha_5 \alpha_4 \alpha_1\]
\[
\mathcal{E}^{(2,4)}_{(2)}(\rho(s^{-1}y_1); \rho(s^{-1}y_2), \rho(s^{-1}y_3)) = \lambda s^{-1} \beta_0^s \beta_1^s \beta_2^s \beta_3^s \alpha_3 \alpha_2 \alpha_1\]
\[
\rho(\mathcal{E}^{(2,4)}_{(2)}(s^{-1}x; s^{-1}y_1, s^{-1}y_2, s^{-1}y_3)) = \rho(\mathcal{E}^{(2,4)}_{(2)}(s^{-1}x); \rho(s^{-1}y_1), \rho(s^{-1}y_2), \rho(s^{-1}y_3)).\]

11.3. A recursive formula for the brace operation. We will give a recursive formula for the brace operation \(-\{-,\ldots,-\}'\) on \(C^*(L, L)\), which will be used in the proof of Proposition 13.7.

**Proposition 11.6.** Let \( v = \beta_1^s \cdots \beta_p^s \alpha_m \cdots \alpha_1 \in L \) be a monomial with \( \beta_i, \alpha_j \in Q_1 \) for \( 1 \leq i \leq p \) and \( 1 \leq j \leq m \), and let \( s^{-1}u_1, \ldots, s^{-1}u_k \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \) for \( k \geq 1 \). Suppose that \( s^{-1}u_k = s^{-1} \gamma_0 \tilde{u}_k \) with \( \gamma_0 \in Q_1 \) and \( \tilde{u}_k \in e_{(\gamma_0)} L e_{\delta(\gamma_0)} \). Then we have
\[
v\{s^{-1}u_1, \ldots, s^{-1}u_k\}'\]
\[
= \sum_{j=0}^{m-1} (-1)^{(j+|v|+1)+k+|u_k|} \left( (\beta_1^s \gamma_0^s)\{s^{-1}u_1, \ldots, s^{-1}u_k\}' \right) \cdot (\tilde{u}_k \beta_{j+1}^s \alpha_{m+1})
\]
\[
- \sum_{j=0}^{m-1} (-1)^{(j+|v|+k+|u_k|)} \delta_{\alpha_{j+1}, \gamma_0} \left( (\beta_1^s \alpha_{m,j+2})\{s^{-1}u_1, \ldots, s^{-1}u_k\}' \right) \cdot (\tilde{u}_k \alpha_{j+1}),\]
where \( \epsilon_k = |u_1| + \cdots + |u_k| \), and the dot indicates the multiplication of \( L \).

For the brace operation \( v\{s^{-1}u_1, \ldots, s^{-1}u_k\}' \) with \( v \in L \), we refer to Remark 11.3(3). Here, we write \( \alpha_{j,i} = \alpha_i \alpha_{j-1} \cdots \alpha_i \), \( \beta_{j,i} = \beta_j^s \beta_{j+1}^s \cdots \beta_i^s \) for any \( i \leq j \). Moreover, \( \beta_{1,0}^s \), \( \tilde{u}_k \alpha_{m,m+1} \) and \( \beta_{1,p}^s \alpha_{m+1} \) are understood as \( \gamma_0^s \), \( \tilde{u}_k \) and \( \beta_{1,p}^s \), respectively. In particular, the above proposition also works for \( v = \beta_1^s \cdots \beta_p^s \) and \( v = \alpha_m \cdots \alpha_1 \).

**Proof.** We only prove for \( m, p > 0 \). The cases where \( m = 0 \) or \( p = 0 \) can be proved in a similar way. We will compare the summands on the right hand side of (11.5) with the summands\( \mathcal{E}^{(1,\ldots,1)}_{(i_1,\ldots,i_n)}(v; s^{-1}u_1, \ldots, s^{-1}u_k) \) in (11.4). We analyze the position in \( v = \beta_1^s \beta_2^s \cdots \beta_p^s \alpha_m \alpha_{m-1} \cdots \alpha_1 \) where \( u_k \) is inserted according to Remark 11.3(1).
For any fixed $0 \leq j \leq p - 1$, the first term on the right hand side of (11.5)
\[-1\]^{(j+|\epsilon|+|\mu|)k+|\mu|}
\left( (\beta^*_v \gamma_0^*) \{ s^{-1} u_1, \ldots, s^{-1} u_k \} \right) \cdot (\tilde{u}_k \beta^*_j \alpha_{m,1})
eq \text{the following summands}
\sum_{1 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq k+1}
\left(-1\right)^{|\epsilon|+|\mu|} \beta_{l_1, \ldots, l_k} (v; s^{-1} u_1, \ldots, s^{-1} u_k),
\]since both of them are the sums of all insertions such that $u_k$ is inserted into $v$ at the position between $\beta^*_j$ and $\beta^*_j + 1$.

To complete the proof, we assume that the insertion of $u_k$ into $v$ is at the position between $\alpha_{j+1}$ and $\alpha_j$ for any fixed $0 \leq j \leq m - 1$. That is, we are concerned with the following summand
\[
\sum_{a+b=m, a,b \geq 0}
\left(-1\right)^{a+\epsilon} \beta_{l_1, \ldots, l_k} (v; s^{-1} u_1, \ldots, s^{-1} u_k).
\]
Here, $\epsilon$ is the same as in (11.4). We observe that
\[
\beta_{l_1, \ldots, l_k} (\beta^*_1 \alpha_{m,1}; s^{-1} u_1, \ldots, s^{-1} u_k) = \delta_{\alpha_{j+1}, \gamma_0} \beta_{l_1, \ldots, l_k} (\beta^*_1 \alpha_{m,j+2}; s^{-1} u_1, \ldots, s^{-1} u_k) \cdot (\tilde{u}_k \alpha_{j+1}),
\]
where the insertion of $u_1, \ldots, u_k$ into $\beta^*_1 \cdots \beta^*_j \alpha_m \cdots \alpha_{j+2}$ is involved in the latter term.

It follows that for each $0 \leq j \leq m - 1$, (11.6) equals
\[
-\left(-1\right)^{(j+1)k+|\mu|} \delta_{\alpha_{j+1}, \gamma_0} \left( (\beta^*_1 \alpha_{m,j+2}) \{ s^{-1} u_1, \ldots, s^{-1} u_k \} \right) \cdot (\tilde{u}_k \alpha_{j+1}).
\]
This is the second term on the right hand side of (11.5). Then the required identity follows immediately.

12. A HOMOTOPY DEFORMATION RETRACT AND THE HOMOTOPY TRANSFER THEOREM

In this section, we provide an explicit homotopy deformation retract for the Leavitt path algebra. We begin by recalling a construction of homotopy deformation retracts between resolutions.

12.1. A construction for homotopy deformation retracts. We will generalize a result in [32], which provides a general construction of homotopy deformation retracts between the bar resolution and a smaller projective resolution for a dg algebra.

The following notion is standard; see [49, Subsection 1.5.5].

**Definition 12.1.** Let $(V, d_V)$ and $(W, d_W)$ be two cochain complexes. A homotopy deformation retract from $V$ to $W$ is a triple $(\iota, \pi, h)$, where $\iota: V \rightarrow W$ and $\pi: W \rightarrow V$ are cochain maps satisfying $\pi \circ \iota = 1_V$, and $h: W \rightarrow W$ is a homotopy of degree $-1$ between $1_W$ and $\iota \circ \pi$, that is, $1_W = \iota \circ \pi + d_W \circ h + h \circ d_W$.

The homotopy deformation retract $(\iota, \pi, h)$ is usually depicted by the following diagram
\[
\begin{array}{c}
(V, d_V) \\
\iota
\end{array}
\xrightarrow{\pi}
\begin{array}{c}
(W, d_W) \\
\circ
\end{array}
\xrightarrow{h}
\]
Let $A$ be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i \in I} k e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$ for any $i, j \in I$. We consider the (normalized) $E$-relative bar resolution $\overline{\text{Bar}}_E(A)$, whose differential is denoted by $d$. The tensor-length of a typical element $y = a_0 \otimes_E s\overline{a}_{1,n} \otimes_E b \in A \otimes_E (s\overline{A})^{\otimes_{\mathbb{E}}} \otimes_E A$ is defined to be $n + 2$, where $s\overline{a}_{1,n}$ means $s\overline{a}_{1,n} \otimes_E s\overline{a}_{2,n} \otimes_E \cdots \otimes_E s\overline{a}_{n,n}$. The following natural map

$$s: A \otimes_E (s\overline{A})^{\otimes_{\mathbb{E}}} \otimes_E A \longrightarrow (s\overline{A})^{\otimes_{\mathbb{E}}+1} \otimes_E A$$

is of degree $-1$.

The following result is inspired by [32, Proposition 3.3].

**Proposition 12.2.** Let $A$ be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i \in I} k e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$. Assume that $\omega: \overline{\text{Bar}}_E(A) \rightarrow \overline{\text{Bar}}_E(A)$ is a morphism of dg $A$-bimodules satisfying $\omega(a \otimes_E b) = a \otimes_E b$ for all $a, b \in A$. Define a $\mathbb{k}$-linear map $h: \overline{\text{Bar}}_E(A) \rightarrow \overline{\text{Bar}}_E(A)$ of degree $-1$ as follows

$$h(a_0 \otimes_E s\overline{a}_{1,n} \otimes_E b) = \begin{cases} 0 & \text{if } n = 0; \\ \sum_{i=1}^{n} (-1)^{i+1} a_0 \otimes_E s\overline{a}_{1,i-1} \otimes_E \overline{\omega}(1 \otimes_E s\overline{a}_{1,n} \otimes_E b) & \text{if } n > 0. \end{cases}$$

Here, $e_i = |a_0| + |a_1| + \cdots + |a_{i-1}| + i - 1$, and $\overline{\omega}$ denotes the composition of $\omega$ with the natural map $s$ in (12.1). Then we have $d \circ h + h \circ d = 1_{\overline{\text{Bar}}_E(A)} - \omega$.

**Proof.** We use induction on the tensor-length. Let $a \in A$ and $y \in A \otimes_E (s\overline{A})^{\otimes_{\mathbb{E}}} \otimes_E A$. Then $a \otimes_E s(y)$ lies in $A \otimes_E (s\overline{A})^{\otimes_{\mathbb{E}}+1} \otimes_E A$. To save the space, we write $a \otimes_E s(y)$ as $a \otimes_E \overline{y}$.

Recall from Subsection 6.2 that $d = d_{in} + d_{ex}$, where $d_{in}$ is the internal differential and $d_{ex}$ is the external differential. We observe that $d_{in}(a \otimes_E \overline{y}) = d_A(a) \otimes_E \overline{y} + (-1)^{|a|+1} a \otimes_E d_{in}(y)$ and that $d_{ex}(a \otimes_E \overline{y}) = (-1)^{|a|}(ay - a \otimes_E \overline{d}_{ex}(y))$. Here, $ay$ denotes the left action of $a$ on $y$, and $d_{ex}$ (resp. $d_{in}$) is the composition of $d_{ex}$ (resp. $d_{in}$) with the map $s$ in (12.1). Then we have

$$d(a \otimes_E \overline{y}) = d_A(a) \otimes_E \overline{y} + (-1)^{|a|+1} a \otimes_E \overline{\omega}(y) + (-1)^{|a|} ay.$$  

(12.2)

From the very definition, we observe

$$h(a \otimes_E \overline{y}) = (-1)^{|a|+1}(a \otimes_E \overline{h}(y) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{y})).$$

Using the above two identities, we obtain

$$d \circ h(a \otimes_E \overline{y}) = (-1)^{|a|+1} d_A(a) \otimes_E \overline{h}(y) + a \otimes_E \overline{d} \circ h(y) - ah(y) + (-1)^{|a|+1} d_A(a) \otimes_E \overline{\omega}(1 \otimes_E \overline{y}) + a \otimes_E \overline{d} \circ \omega(1 \otimes_E \overline{y}) - a \omega(1 \otimes_E \overline{y}),$$

and

$$h \circ d(a \otimes_E \overline{y}) = (-1)^{|a|} d_A(a) \otimes_E \overline{h}(y) + (-1)^{|a|} d_A(a) \otimes_E \overline{\omega}(1 \otimes_E \overline{y}) + a \otimes_E \overline{h} \circ d(y) + a \otimes_E \overline{\omega}(1 \otimes_E \overline{d}(y)) + (-1)^{|a|} h(ay).$$
Using the fact $ah(y) = (-1)^{|a|}h(ay)$, we infer the first equality of the following identities

$$
(d \circ h + h \circ d)(a \otimes_E y) = a \otimes_E (d \circ h + h \circ d)(y) + a \otimes_E d \circ \omega(1 \otimes_E y) + a \otimes_E \bar{\omega}(1 \otimes_E \bar{d}(y)) - a \omega(1 \otimes_E y)
$$

Here, the second equality uses the induction hypothesis, and the third one uses the fact that $\omega$ respects the differentials and the left $A$-module structure. The last equality uses the following special case of (12.2)

$$
-y + d(1 \otimes_E y) + 1 \otimes_E \bar{d}(y) = 0.
$$

This completes the proof. \hfill \Box

**Remark 12.3.** We observe that the obtained homotopy $h$ respects the $A$-$A$-bimodule structures. More precisely, $h : \overline{\text{Bar}}_E(A) \to \Sigma^{-1}\overline{\text{Bar}}_E(A)$ is a morphism of graded $A$-$A$-bimodules.

The following immediate consequence of Proposition 12.2 is a slight generalization of [32, Proposition 3.3], which might be a useful tool in many fields to construct explicit homotopy deformation retracts. We recall from (6.2) the quasi-isomorphism $\varepsilon : \overline{\text{Bar}}_E(A) \to A$.

**Corollary 12.4.** Let $A$ be a dg algebra with a semisimple subalgebra $E = \bigoplus_{i \in I} k e_i \subseteq A^0 \subseteq A$ satisfying $d_A(e_i) = 0$ and $e_i e_j = \delta_{i,j} e_i$. Assume that $P$ is a dg $A$-$A$-bimodule and that there are two morphisms of dg $A$-$A$-bimodules

$$
\iota : P \to \overline{\text{Bar}}_E(A), \quad \pi : \overline{\text{Bar}}_E(A) \to P
$$

satisfying $\pi \circ \iota = 1_P$ and $\iota \circ \pi|_{A \otimes E A} = 1_{A \otimes E A}$. Then the pair $(\iota, \pi)$ can be extended to a homotopy deformation retract $(\iota, \pi, h)$, where $h : \overline{\text{Bar}}_E(A) \to \overline{\text{Bar}}_E(A)$ is given as in Proposition 12.2 with $\omega = \iota \circ \pi$.

In particular, the composition

$$
P \xrightarrow{\iota} \overline{\text{Bar}}_E(A) \xrightarrow{\pi} A
$$

is a quasi-isomorphism of dg $A$-$A$-bimodules. \hfill \Box

12.2. **A homotopy deformation retract for the Leavitt path algebra.** In this subsection, we apply the above construction to Leavitt path algebras. We obtain a homotopy deformation retract between the normalized $E$-relative bar resolution and an explicit bimodule projective resolution.

Let $Q$ be a finite quiver without sinks. Let $L = L(Q)$ be the Leavitt path algebra viewed as a dg algebra with trivial differential; see Section 4. Set $E = \bigoplus_{i \in Q_0} k e_i \subseteq L^0 \subseteq L$. We write $\overline{L} = L/(E \cdot 1_L)$. In what follows, we will construct an explicit homotopy deformation retract

$$
(P, \partial) \xrightarrow{\pi} (\overline{\text{Bar}}_E(L), d) \overset{h}{\circlearrowleft}.
$$

(12.3)
Let us first describe the dg $L$-$L$-bimodule $(P, \partial)$. As a graded $L$-$L$-bimodule,
\[ P = \bigoplus_{i \in Q_0} (Le_i \otimes \mathbb{k} \otimes e_i L) \oplus \bigoplus_{i \in Q_0} Le_i \otimes e_i L. \]
The differential $\partial$ of $P$ is given by
\[
\partial(x \otimes s \otimes y) = (-1)^{|x|} x \otimes y - (-1)^{|x|} \sum_{\{\alpha \in Q_1|s(\alpha) = i\}} x \alpha^x \otimes \alpha y,
\]
\[
\partial(x \otimes y) = 0,
\]
for $x \in Le_i$, $y \in e_i L$ and $i \in Q_0$. Here, $\mathbb{k}$ is the 1-dimensional graded $\mathbb{k}$-vector space concentrated in degree $-1$, and the element $s_1 \mathbb{k}$ is abbreviated as $s$.

The homotopy deformation retract (12.3) is defined as follows.

1. The injection $\iota: P \to \text{Bar}_E(L)$ is given by
\[
\iota(x \otimes y) = x \otimes_E y,
\]
\[
\iota(x \otimes s \otimes y) = - \sum_{\{\alpha \in Q_1|s(\alpha) = i\}} x \alpha^x \otimes_E s \alpha \otimes_E y,
\]
for $x \in Le_i$, $y \in e_i L$ and $i \in Q_0$.

2. The surjection $\pi: \text{Bar}_E(L) \to P$ is given by
\[
\pi(a' \otimes_E b') = a' \otimes b',
\]
\[
\pi(a \otimes_E s \otimes_E b) = aD(z)b,
\]
\[
\pi|_{L \otimes_E(sL) \otimes_E E>1} = 0,
\]
for $a' = a'e_i$ and $b' = e_ib'$ for some $i \in Q_0$, and any $a, b, z \in L$, where $D: L \to \bigoplus_{i \in Q_0} (Le_i \otimes \mathbb{k} \otimes e_i L)$ is the graded $E$-derivation of degree $-1$ in Lemma 4.3.

Here and also in the proof of Proposition 12.5, we use the canonical identification
\[
\bigoplus_{i \in Q_0} Le_i \otimes e_i L = L \otimes_E L, \quad x \otimes y \mapsto x \otimes_E y.
\]

3. The homotopy $h: \text{Bar}_E(L) \to \text{Bar}_E(L)$ is given by
\[
h(a_0 \otimes_E s \otimes_E \cdots \otimes_E s \otimes_E b) = \begin{cases} 0 & \text{if } n = 0; \\
(-1)^{\epsilon_n+1}a_0 \otimes_E s \otimes_E \cdots \otimes_E s \otimes_E b & \text{if } n > 0,
\end{cases}
\]
where $\epsilon_n = |a_0| + |a_1| + \cdots + |a_{n-1}| + n - 1$, and $\tau$ is the composition of $\iota \circ \pi$ with the natural isomorphism $s: L \otimes_E s \otimes_E \cdots \otimes_E s \otimes_E b \mapsto s \otimes_E \cdots \otimes_E s \otimes_E L$ of degree $-1$.

**Proposition 12.5.** The above triple $(\iota, \pi, h)$ defines a homotopy deformation retract in the abelian category of dg $L$-$L$-bimodules. In particular, the dg $L$-$L$-bimodule $P$ is a dg-projective bimodule resolution of $L$.

**Proof.** We first observe that $\iota$ and $\pi$ are morphisms of $L$-$L$-bimodules. Recall that the differential of $\text{Bar}_E(L)$ is given by the external differential $d_{ex}$ since the internal differential
\( d_{in} \) is zero; see Subsection 6.2. We claim that both \( \iota \) and \( \pi \) respect the differential. It suffices to prove the commutativity of the following diagram.

\[ \cdots \rightarrow 0 \rightarrow \bigoplus_{t \in Q_0} L e_t \otimes s \kappa \otimes e_i L \rightarrow L \otimes_E L \]
\[ \downarrow \iota \]
\[ \cdots \rightarrow L \otimes_E (s \ell) \otimes_E \mathbb{Z}_2 \otimes_E L \rightarrow L \otimes_E sL \otimes_E L \rightarrow L \otimes_E L \]
\[ \downarrow \pi \]
\[ \cdots \rightarrow 0 \rightarrow \bigoplus_{t \in Q_0} L e_t \otimes s \kappa \otimes e_i L \rightarrow L \otimes_E L \]

For the northeast square, we have
\[ d_{ex} \circ \iota (x \otimes s \otimes y) = - \sum_{\{\alpha \in Q_1 | s(\alpha) = i\}} d_{ex}(x \alpha^* \otimes_E s \alpha \otimes_E y) \]
\[ = \sum_{\{\alpha \in Q_1 | s(\alpha) = i\}} (-1)^{|x|+1} x \alpha^* \alpha \otimes_E y - (-1)^{|x|} x \alpha^* \otimes_E \alpha y \]
\[ = \partial(x \otimes s \otimes y) \]

where the third equality follows from the second Cuntz-Krieger relations.

For the southwest square, we have
\[ \pi \circ d_{ex}(a \otimes_E s \overline{y} \otimes_E s \overline{z} \otimes_E b) \]
\[ = (-1)^{|a|} \pi(ay \otimes_E s \overline{z} \otimes_E b) + (-1)^{|a|+|y|-1}(\pi(a \otimes_E s \overline{z} \otimes_E z b) - \pi(a \otimes_E s \overline{y} \otimes_E z b)) \]
\[ = (-1)^{|a|} ayD(z)b + (-1)^{|a|+|y|-1}aD(yz)w - (-1)^{|a|+|y|-1}aD(y)zb \]
\[ = 0, \]

where the last equality follows from the graded Leibniz rule of \( D \).

It remains to verify that the southeast square commutes, namely \( \partial \circ \pi = d_{ex} \). For this, we first note that
\[ \partial \circ \pi(a \otimes_E s \overline{\alpha} \otimes_E b) = - (-1)^{|a|+1} a \alpha \otimes b + (-1)^{|a|+1} \sum_{\{\beta \in Q_1 | s(\beta) = s(\alpha)\}} a \alpha \beta^* \otimes \beta b \]
\[ = (-1)^{|a|} a \alpha \otimes b - (-1)^{|a|} a \otimes ab \]
\[ = d_{ex}(a \otimes_E s \alpha \otimes_E b), \]

where \( \alpha \in Q_1 \) is an arrow, \( a \in L e_{t(\alpha)} \) and \( b \in e_{s(\alpha)} L \). For the second equality, we use the first Cuntz-Krieger relations \( \alpha \beta^* = \delta_{\alpha,\beta} e_{t(\alpha)} \). Similarly, we have \( \partial \circ \pi(a \otimes_E s \alpha^* \otimes_E b) = d_{ex}(a \otimes_E s \alpha^* \otimes_E b) \).

For the general case, we use induction on the length of the path \( w \) in \( a \otimes_E s w \otimes_E b \).

By the length of a path \( w \) in \( L \), we mean the number of arrows in \( w \), including the ghost arrows. We write \( w = \gamma \eta \) such that the lengths of \( \gamma \) and \( \eta \) are both strictly smaller than
that of \( w \). We have
\[
\partial \circ \pi (a \otimes_E s_\gamma \otimes_E b) = \partial (a D(\gamma) \eta b + (-1)^{|\gamma|} a \gamma D(\eta) b) \\
= \partial \circ \pi (a \otimes_E s_\gamma \otimes_E \eta b + (-1)^{|\gamma|} a \gamma \otimes_E s_\eta \otimes_E b) \\
= d_{ex} (a \otimes_E s_\gamma \otimes_E \eta b + (-1)^{|\gamma|} a \gamma \otimes_E s_\eta \otimes_E b) \\
= d_{ex} (a \otimes_E s_\gamma \otimes_E \eta b + (-1)^{|\gamma|} a \gamma \otimes_E s_\eta \otimes_E b),
\]
where the third equality uses the induction hypothesis, and the fourth one follows from \( d_{ex}^2 (a \otimes_E s_\gamma \otimes_E s_\eta \otimes_E b) = 0 \). This proves the required commutativity and the claim.

The fact \( \pi \circ \iota = 1 \) follows from the second Cuntz-Krieger relations. By Corollary 12.4, it follows that \( (\iota, \pi) \) extends to a homotopy deformation retract \( (\iota, \pi, h) \); moreover, the obtained \( h \) coincides with the given one. □

Remark 12.6. The following comment is due to Bernhard Keller: the above explicit projective bimodule resolution \( P \) might be used to give a shorter proof of the computation of the Hochschild homology of \( L \) in [5, Theorem 4.4].

12.3. The homotopy transfer theorem for dg algebras. We recall the homotopy transfer theorem for dg algebras, which will be used in the next section.

Theorem 12.7 ([35]). Let \((A, d_A, \mu_A)\) be a dg algebra. Let
\[
(V, d_V) \xleftarrow{\iota} (A, d_A) \xrightarrow{\pi} h
\]
be a homotopy deformation retract between cochain complexes (cf. Definition 12.1). Then there is an \( A_\infty \)-algebra structure \( (m_1 = d_V, m_2, m_3, \cdots) \) on \( V \), where \( m_k \) is depicted in Figure 7. Moreover, the map \( \iota : V \to A \) extends to an \( A_\infty \)-quasi-isomorphism \( (\iota_1 = \iota, \iota_2, \cdots) \) from the resulting \( A_\infty \)-algebra \( V \) to the dg algebra \( A \), where \( \iota_k \) is depicted in Figure 7.

In this paper, we only need the following special case of Theorem 12.7.

Corollary 12.8. Let \((A, d_A, \mu_A)\) be a dg algebra. Let
\[
(V, d_V) \xleftarrow{\iota} (A, d_A) \xrightarrow{\pi} h
\]
be a homotopy deformation retract between cochain complexes. We further assume that
\[ h_{\mu A}(a \otimes h(b)) = 0 = \pi_{\mu A}(a \otimes h(b)) \quad \text{for any } a, b \in A. \tag{12.5} \]
Then the resulting \(A_\infty\)-algebra \((V, m_1 = d_V, m_2, m_3, \cdots)\) is simply given by (cf. Figure 8)
\[
m_2(a_1 \otimes a_2) = \pi(\iota(a_1)\iota(a_2)),
\]
\[
m_k(a_1 \otimes \cdots \otimes a_k) = \pi(h(\cdots(h(\iota(a_1)\iota(a_2))\iota(a_3))\cdots)\iota(a_k)), \quad k > 3,
\]
where we simply write \(\iota(a) = \mu_A(\iota(a) \otimes \iota(b)).\)
Moreover, the \(A_\infty\)-quasi-isomorphism \((\iota_1 = \iota, \iota_2, \cdots)\) is given by (cf. Figure 8)
\[
\iota_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{\frac{k(k-1)}{2}} h(\cdots(h(\iota(a_1)\iota(a_2))\iota(a_3))\cdots)\iota(a_k)), \quad k \geq 2.
\]

**Remark 12.9.** Note that under the assumption (12.5), the formulae for the resulting \(A_\infty\)-algebra and \(A_\infty\)-morphism are highly simplified.

For \(k \geq 2\), we have the following recursive formula
\[
\iota_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{k-1} h(\iota_{k-1}(a_1 \otimes \cdots \otimes a_{k-1})\iota(a_k))
\]
and the following identity
\[
m_k(a_1 \otimes \cdots \otimes a_k) = (-1)^{\frac{(k-1)(k-2)}{2}} \pi(\iota_{k-1}(a_1 \otimes \cdots \otimes a_{k-1})\iota(a_k)).
\]

**Figure 8.** The \(A_\infty\)-product \(m_k\) and \(A_\infty\)-quasi-isomorphism \(\iota_k\).

### 13. An \(A_\infty\)-Quasi-Isomorphism for the Leavitt Path Algebra

In this section, we use the homotopy transfer theorem for dg algebras to obtain an explicit \(A_\infty\)-quasi-isomorphism between the two dg algebras \(\hat{C}^*(L, L)\) and \(\hat{C}^*_{\mathbb{L}}(L, L).\)

#### 13.1. An explicit \(A_\infty\)-quasi-isomorphism between dg algebras

In what follows, we apply the functor \(\text{Hom}_{L-L}(\cdot, L)\) to the homotopy deformation retract (12.3).

Recall from Section 11 the Leavitt \(B_\infty\)-algebra \(\hat{C}^*(L, L)\). We will use the identification
\[
\text{Hom}_{L-L}(P, L) = (\hat{C}^*(L, L), \hat{\delta})
\]
by the following natural isomorphisms
\[
\text{Hom}_{L-L}(Le_i \otimes e_i L, L) \cong e_i Le_i, \quad \phi \mapsto \phi(e_i \otimes e_i);
\]
\[
\text{Hom}_{L-L}(Le_i \otimes se_i L, L) \cong s^{-1}e_i Le_i, \quad \phi \mapsto (-1)^{|\phi|s^{-1}}\phi(e_i \otimes s \otimes e_i). \tag{13.1}
\]
It is straightforward to verify that the above isomorphisms are compatible with the differentials.
Recall that $E = \bigoplus_{i \in Q_0} k e_i$ and that the $E$-relative Hochschild cochain complex $\overline{C}_E^r(L, L)$ is naturally identified with $\text{Hom}_{L-L}(\overline{\text{Bar}}_E(L), L)$; compare (6.3). Under the above identifications, (12.3) yields the following homotopy deformation retract

\[
\begin{array}{c}
(\hat{C}^*(L, L), \hat{\delta}) \xrightarrow{\Phi} (\overline{C}_E^r(L, L), \delta) \xrightarrow{H} \bigcirc
\end{array}
\]

with $\Phi = \text{Hom}_{L-L}(\pi, L), \Psi = \text{Hom}_{L-L}(\iota, L)$ and $H = \text{Hom}_{L-L}(h, L)$ satisfying $\Psi \circ \Phi = 1_{\hat{C}^*(L, L)}$ and $\overline{1_{\text{Bar}}_E(L, L)} = \Phi \circ \Psi + \delta \circ H + H \circ \delta$.

As in Subsection 6.1, we denote the following subspaces of $\overline{C}_E^r(L, L)$ for any $k \geq 0$

\[
\overline{C}_E^{r, k}(L, L) = \text{Hom}_{E-E}((sL)^{\otimes_E^k}, L)
\]

\[
\overline{C}_E^{s, \geq k}(L, L) = \prod_{i \geq k} \text{Hom}_{E-E}((sL)^{\otimes_E^i}, L)
\]

\[
\overline{C}_E^{s, \leq k}(L, L) = \prod_{0 \leq i \leq k} \text{Hom}_{E-E}((sL)^{\otimes_E^i}, L).
\]

In particular, we have $\overline{C}_E^{r, 0}(L, L) = \text{Hom}_{E-E}(E, L) = \bigoplus_{i \in Q_0} e_i L e_i$.

Let us describe the above homotopy deformation retract (13.2) in more detail.

1. The surjection $\Psi$ is given by

\[
\Psi(x) = x \quad \text{for} \quad x \in \overline{C}_E^{r, 0}(L, L) = \bigoplus_{i \in Q_0} e_i L e_i;
\]

\[
\Psi(f) = -\sum_{\alpha \in Q_1} s^{-1} \alpha^* f(s\alpha) \quad \text{for} \quad f \in \overline{C}_E^{r, 1}(L, L);
\]

\[
\Psi(g) = 0 \quad \text{for} \quad g \in \overline{C}_E^{r, \geq 2}(L, L).
\]

2. The injection $\Phi$ is given by

\[
\Phi(u) = u \quad \text{for} \quad u \in \prod_{i \in Q_0} e_i L e_i \subset \hat{C}^*(L, L);
\]

\[
\Phi(s^{-1}u) \in \overline{C}_E^{r, 1}(L, L) \quad \text{for} \quad s^{-1}u \in \prod_{i \in Q_0} s^{-1} e_i L e_i \subset \hat{C}^*(L, L).
\]

(13.3)

where in the first identity we use the identification $\overline{C}_E^{r, 0}(L, L) = \bigoplus_{i \in Q_0} e_i L e_i$. The explicit formula of $\Phi(s^{-1}u)$ will be given in Lemma 13.1 below.

3. The homotopy $H$ is given by

\[
H[\overline{C}_E^{r, \leq 1}(L, L)] = 0
\]

\[
H(f)(s\overline{a}_{1,n}) = (-1)^\epsilon f(s\overline{a}_{1,n-1} \otimes_E i\pi(1 \otimes_E s\overline{a}_n \otimes_E 1))
\]

for any $f \in \overline{C}_E^{r, n+1}(L, L)$ with $n \geq 1$, where $\epsilon = 1 + |f| + \sum_{i=1}^{n-1}(|a_i| - 1)$ and for convenience we use the notation

\[
f(s\overline{a}_{1,n+1} \otimes_E x) := f(s\overline{a}_{1,n+1})x, \quad \text{for} \quad x \in L \text{ and } s\overline{a}_{1,n+1} \in (sL)^{\otimes_E^{n+1}},
\]

and we simply write $s\overline{a}_{1,n+1} := s\overline{a}_1 \otimes_E s\overline{a}_2 \otimes_E \cdots \otimes_E s\overline{a}_{n+1}$. 


The following lemma provides the formula of $\Phi(s^{-1}u)$.

**Lemma 13.1.** For any $s^{-1}u \in \bigoplus_{i \in Q_0} s^{-1}e_iLe_i \subset \hat{C}^*(L, L)$, we have

$$\Phi(s^{-1}u)(s\overline{v}) = (-1)^{|v||u|}uv\{s^{-1}u\}' ,$$

where $v \in L$ and $v\{s^{-1}u\}'$ is given by (11.4).

**Proof.** Let $v = \beta^*_i \cdots \beta^*_p\alpha_m \cdots \alpha_1 \in e_iLe_j$ be a monomial, where $i, j \in Q_0$. We assume that $m, p > 0$. Under the identification (13.1), the element $s^{-1}u$ corresponds to a morphism of $L$-$L$-bimodules of degree $|u| - 1$

$$\phi_{s^{-1}u} : Le_i \otimes sk \otimes e_iL \rightarrow L, \quad a \otimes s \otimes b \mapsto (-1)^{|a|+1}(-1)^{|u|+1}uv.$$ 

Then we have $\Phi(s^{-1}u)(s\overline{v}) = (\phi_{s^{-1}u} \circ \pi)(1 \otimes s\overline{v} \otimes 1)$. By Remark 4.4, we have

$$\Phi(s^{-1}u)(s\overline{v}) = (\phi_{s^{-1}u} \circ \pi)(1 \otimes s\overline{v} \otimes 1) = \phi_{s^{-1}u}(D(v))$$

$$= (-1)^{|u|}uv + \sum_{l=1}^{p-1} (-1)^{|l(l+1)|} \beta^*_1 \cdots \beta^*_l u\beta^*_{l+1} \cdots \beta^*_m \alpha_m \cdots \alpha_1$$

$$+ \sum_{l=1}^{m-1} (-1)^{|u|(m+p-l-1)+1} \beta^*_1 \cdots \beta^*_p \alpha_m \cdots \alpha_{l+1} u\alpha_l \cdots \alpha_1 + (-1)^{|v+1|}uv.$$ 

It follows from (B3') that

$$v\{s^{-1}u\}' = (-1)^{|v||u|}uv + \sum_{l=1}^{p-1} (-1)^{|v||u|+|u||l|} \beta^*_1 \cdots \beta^*_l u\beta^*_{l+1} \cdots \beta^*_m \alpha_m \cdots \alpha_1$$

$$+ \sum_{l=1}^{m-1} (-1)^{|u|+1+|u||v|-l} \beta^*_1 \cdots \beta^*_p \alpha_m \cdots \alpha_{l+1} u\alpha_l \cdots \alpha_1 - vu.$$ 

By comparing the signs of the above two formulae, we infer

$$\Phi(s^{-1}u)(s\overline{v}) = (-1)^{|v||u|}uv\{s^{-1}u\}'. $$

Similarly, one can prove this for either $p = 0$ or $m = 0$.

**Remark 13.2.** Note that for $\alpha \in Q_1$ we have

$$\Phi(s^{-1}u)(s\overline{v}) = \alpha \{s^{-1}u\}' = -\alpha u,$$

where the second identity is due to Remark 11.3 (3). The formula of $\Phi = \Phi_1$ will be generalized to $\Phi_k$ for $k > 1$ by using $\{\underbrace{-, \ldots, -}\}'$; see Proposition 13.7 below.

The following simple lemma on the homotopy $H$ will be used in Lemma 13.4 below.

**Lemma 13.3.** For any $\alpha \in Q_1$ and $f \in \overline{C}^{*,n+1}_E(L, L)$ with $n \geq 1$, we have

$$H(f)(s\overline{u} \otimes E \cdots \otimes E s\overline{u}_{n-1} \otimes E s\overline{v}) = 0.$$
Proof. By (13.4) we have
\[
H(f)(s\bar{\alpha}_{1,n-1} \otimes_E s\bar{\tau}) = (-1)^{c'}f(s\bar{\alpha}_{1,n-1} \otimes_E \bar{\tau}(1 \otimes_E s\bar{\tau} \otimes E 1)) = (-1)^{c'}f(s\bar{\alpha}_{1,n-1} \otimes_E s\bar{\tau}_{t(\alpha)} \otimes E s\bar{\tau}) = 0,
\]
where the last identity comes from the fact that $s\bar{\tau}_{t(\alpha)} = 0$ in $L = L/(E \cdot 1)$.

The following lemma shows that the homotopy deformation retract (13.2) satisfies the assumption (12.5) of Corollary 12.8.

**Lemma 13.4.** For any $g_1, g_2 \in \tilde{C}_E^*(L, L)$, we have
\[
H(g_1 \cup H(g_2)) = 0 = \Psi(g_1 \cup H(g_2)).
\]

**Proof.** Throughout the proof, we assume without loss of generality that $g_1 \in \tilde{C}_E^{*m}(L, L)$ and $g_2 \in \tilde{C}_E^{*n}(L, L)$ for some $m, n \geq 0$.

Note that if $n \leq 1$ then $H(g_2) = 0$ by (13.4) and the desired identities hold. So in the following we may further assume that $n \geq 2$.

Let us first verify $\Psi(g_1 \cup H(g_2)) = 0$. Since $\Psi(g) = 0$ for any $g \in \mathcal{C}_E^{* \geq 1}(L, L)$, we only need to verify $\Psi(g_1 \cup H(g_2)) = 0$ when $m = 0$ and $n = 2$. In this case, $g_1 \in \mathcal{C}_E^{*0}(L, L)$ is viewed as an element in $\bigoplus_{i \in Q_0} e_i \mathcal{L}$. Then we have
\[
\Psi(g_1 \cup H(g_2)) = - \sum_{\alpha \in Q_1} s^{-1}(\alpha^* g_1) \cdot \left( H(g_2)(s\alpha) \right) = 0,
\]
where the second identity follows from Lemma 13.3 since $\alpha \in Q_1$. In order to avoid confusion, we sometimes use the dot · to emphasize the multiplication of $L$.

It remains to verify $H(g_1 \cup H(g_2)) = 0$. For this, we have
\[
H(g_1 \cup H(g_2)) = (-1)^{c+c'+1} \sum_{\alpha \in Q_1,i} g_1(s\bar{\alpha}_{1,m}) \cdot H(g_2)(s\bar{\alpha}_{m+1,m+n-3} \otimes_E \bar{\tau}(s\bar{\alpha}_{m+n-2} \otimes_E 1)) = 0,
\]
where the last identity follows from Lemma 13.3 since $\alpha \in Q_1$; and
\[
\epsilon = |g_1| + |g_2| + \sum_{i=1}^{m+n-3} (|a_i| - 1) \quad \text{and} \quad \epsilon' = (|g_2| - 1) \left( \sum_{i=1}^{m} (|a_i| - 1) \right).
\]
Here to save space, we simply write $\pi(1 \otimes_E s\bar{\alpha}_{m+n-2} \otimes_E 1) = \sum_i b_i \otimes_E s \otimes_E c_i$ as we do not use the explicit formula.

Thanks to Lemma 13.4, we can apply Corollary 12.8 to the homotopy deformation retract (13.2). As a result, we obtain an $A_\infty$-algebra structure $(m_1 = \hat{\delta}, m_2, \cdots)$ on $\tilde{C}^*(L, L)$ and an
$A_{\infty}$-quasi-isomorphism ($\Phi_1 = \Phi, \Phi_2, \cdots$) from $(\widetilde{C}^*(L, L), m_1, m_2, \cdots)$ to $(\overline{C}^*_E(L, L), \delta, - \cup -)$. More precisely, we have the following recursive formulae for $k \geq 2$; see Remark 12.9

$$\Phi_k(u_1 \otimes \cdots \otimes u_k) = (-1)^{k-1} H(\Phi_{k-1}(u_1 \otimes \cdots \otimes u_{k-1}) \cup \Phi(u_k));$$  \hspace{1cm} (13.5)

$$m_k(u_1 \otimes \cdots \otimes u_k) = (-1)^{\frac{(k-1)(k-2)}{2}} \Psi(\Phi_{k-1}(u_1 \otimes \cdots \otimes u_{k-1}) \cup \Phi(u_k)).$$  \hspace{1cm} (13.6)

The following lemma provides some basic properties of $\Phi_k$.

**Lemma 13.5.** (1) For $k \geq 1$ we have

$$\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) \in \overline{C}^{s-1}_E(L, L)$$  \hspace{1cm} (13.7)

if $s^{-1}u_j \in \bigoplus_{i \in Q_0} s^{-1}e_iLe_i \subset \widetilde{C}^*(L, L)$ for all $1 \leq j \leq k$;

(2) For $k \geq 2$ we have

$$\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0$$  \hspace{1cm} (13.8)

if there exists some $1 \leq j \leq k$ such that $a_j \in \bigoplus_{i \in Q_0} e_iLe_i \subset \widetilde{C}^*(L, L)$.

**Proof.** Let us prove the first assertion by induction on $k$. For $k = 1$ it follows from (13.3). For $k > 1$, by (13.5) we have the following recursive formula

$$\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) = (-1)^{k-1} H(\Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1}) \cup \Phi(s^{-1}u_k)).$$

By the induction hypothesis, we have $\Phi(s^{-1}u_k), \Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1}) \in \overline{C}^{s-1}_E(L, L)$.

Then we obtain $\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) \in \overline{C}^{s-1}_E(L, L)$. It follows from (13.4) that $\Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k) \in \overline{C}^{s-1}_E(L, L)$.

Similarly, we may prove the second assertion by induction on $k$. For $k = 2$ we have

$$\Phi_2(a_1 \otimes a_2) = H(\Phi(a_1) \cup \Phi(a_2)).$$

By (13.4) we have $H|_{\overline{C}^{\leq 1}_E(L, L)} = 0$. It follows from (13.3) that $\Phi_2(a_1 \otimes a_2) = 0$ if $a_1$ or $a_2$ lies in $\bigoplus_{i \in Q_0} e_iLe_i \subset \widetilde{C}^*(L, L)$.

Now we consider the case for $k > 2$. By the induction hypothesis, we have $\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) = 0$ if there exists $1 \leq j \leq k - 1$ such that $a_j \in \bigoplus_{i \in Q_0} e_iLe_i$. Then by (13.5) we have $\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0$. Otherwise, by assumption $a_k$ must be in $\bigoplus_{i \in Q_0} e_iLe_i$.

Since the elements $a_1, \ldots, a_{k-1}$ are in $\bigoplus_{i \in Q_0} s^{-1}e_iLe_i$, by the first assertion we obtain $\Phi_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \in \overline{C}^{s-1}_E(L, L)$. By (13.5) again, we infer $\Phi_k(a_1 \otimes \cdots \otimes a_k) = 0$. \hfill $\square$

A prior, the higher $A_{\infty}$-products $m_k$ for $k \geq 3$ might be nonzero; see (13.6). We see from Lemma 13.5 that the maps $\Phi_k$ satisfy some nice degree conditions. This actually will lead to the fact that $m_k = 0$ for $k \geq 3$. Moreover, we will show that $m_2 = - \cup -$. Recall from Subsection 11.2 the cup product $- \cup -$ on $\widetilde{C}^*(L, L)$.

**Proposition 13.6.** The product $m_2$ on $\widetilde{C}^*(L, L)$ coincides with the cup product $- \cup -$.

Consequently, the collection of maps $(\Phi_1 = \Phi, \Phi_2, \cdots)$ is an $A_{\infty}$-quasi-isomorphism from the dg algebra $(\widetilde{C}^*(L, L), \delta', - \cup -)$ to the dg algebra $(\overline{C}^*_E(L, L), \delta, - \cup -)$.

**Proof.** Let us first prove that $m_2$ coincides with $- \cup -$. Let $u, v \in \prod_{i \in Q_0} e_iLe_i$. Then we view $s^{-1}u, s^{-1}v$ as elements in $\prod_{i \in Q_0} s^{-1}e_iLe_i$. We need to consider the following four cases corresponding to (C1’)-(C4’); see Subsection 11.2.
This shows that

It follows from Lemma 13.5 that \( \Phi_{u k} \) between the two dg algebras. In this subsection, we will give an explicit formula for \( \Phi_{u k} \) that

A tion 13.6 that we have an

13.2. The \( A_{\infty} \)-quasi-isomorphism via the brace operation. It follows from Proposition 13.6 that we have an \( A_{\infty} \)-quasi-isomorphism

(1) For (C1'), since \( \Phi(s^{-1}u), \Phi(s^{-1}v) \in \overline{C}_{E}^{s,1}(L,L) \) and \( \Psi_{|\overline{C}_{E}^{s,2}(L,L)} = 0 \), we have

\[
m_{2}(s^{-1}u \otimes s^{-1}v) = \Psi(\Phi(s^{-1}u) \cup \Phi(s^{-1}v)) = 0 = s^{-1}u \cup' s^{-1}v.
\]

(2) For (C2'), since \( \Psi(u) = u \) and \( \Psi(v) = v \), we have

\[
m_{2}(u \otimes v) = \Psi(\Phi(u) \cup \Phi(v)) = \Psi(uv) = uv = u \cup' v.
\]

(3) For (C3'), we have

\[
m_{2}(s^{-1}v \otimes u) = \Psi(\Phi(s^{-1}v) \cup \Phi(u)) = -\sum_{\alpha \in Q_{1}} s^{-1}\alpha^{*}\Phi(s^{-1}v)(s\alpha) \cdot u
\]

\[
= \sum_{\alpha \in Q_{1}} s^{-1}\alpha^{*}\alpha vu = s^{-1}v \cup' u,
\]

where the third identity follows from Remark 13.2; and the last identity is due to the second Cuntz-Krieger relations.

(4) Similarly, for (C4') we have

\[
m_{2}(u \otimes s^{-1}v) = \Psi(\Phi(u) \cup \Phi(s^{-1}v)) = -\sum_{\alpha \in Q_{1}} s^{-1}\alpha^{*}(u \cup \Phi(s^{-1}v))(s\alpha)
\]

\[
= \sum_{\alpha \in Q_{1}} s^{-1}\alpha^{*}u \alpha v = u \cup' (s^{-1}v),
\]

where the third identity follows from Remark 13.2.

This shows that \( m_{2} \) coincides with \(-\cup' -\).

Now let us prove \( m_{k} = 0 \) for \( k > 2 \). Assume by way of contradiction that \( m_{k}(u_{1} \otimes \cdots \otimes u_{k}) \neq 0 \) for some \( u_{1}, \ldots, u_{k} \). By (13.6), we have

\[
m_{k}(u_{1} \otimes \cdots \otimes u_{k}) = (-1)^{(k-1)(k-2)/2} \Psi(\Phi_{k-1}(u_{1} \otimes \cdots \otimes u_{k-1}) \cup \Phi(u_{k})), \tag{13.9}
\]

It follows from Lemma 13.5 that \( \Phi_{k-1}(u_{1} \otimes \cdots \otimes u_{k-1}) \in \overline{C}_{E}^{s,\leq 1}(L,L) \). Since \( \Psi_{|\overline{C}_{E}^{s,\geq 2}(L,L)} = 0 \), we infer that \( \Phi(u_{k}) \) must be in \( \overline{C}_{E}^{s,0}(L,L) = \bigoplus_{i \in Q_{0}} e_{i}Le_{i} \). Thus, by (13.9) again we get

\[
m_{k}(u_{1} \otimes \cdots \otimes u_{k})
\]

\[
= - (-1)^{(k-1)(k-2)/2} \sum_{\alpha \in Q_{1}} \alpha^{*}\Phi_{k-1}(u_{1} \otimes \cdots \otimes u_{k-1})(s\alpha) \cdot \Phi(u_{k})
\]

\[
= - (-1)^{(k+1)(k-2)/2} \sum_{\alpha \in Q_{1}} \alpha^{*}H(\Phi_{k-2}(u_{1} \otimes \cdots \otimes u_{k-2}) \cup \Phi(u_{k-1}))(s\alpha) \cdot \Phi(u_{k})
\]

\[
= 0
\]

where the third identity follows from Lemma 13.3. We have a contradiction. This shows that \( m_{k}(u_{1} \otimes \cdots \otimes u_{k}) = 0 \) for \( k > 2 \). \( \square \)

13.2. The \( A_{\infty} \)-quasi-isomorphism via the brace operation. It follows from Proposition 13.6 that we have an \( A_{\infty} \)-quasi-isomorphism

\[
(\Phi_{1} = \Phi, \Phi_{2}, \cdots) : (\overline{C_{E}}^{s}(L,L), \delta, -\cup' -) \longrightarrow (\overline{C}_{E}^{s}(L,L), \delta, -\cup -)
\]

between the two dg algebras. In this subsection, we will give an explicit formula for \( \Phi_{k} \).
Proposition 13.7. Let $k \geq 1$. For any $s^{-1}u_1, \ldots, s^{-1}u_k \in \bigoplus_{i \in Q_0} s^{-1}e_i L e_i \subset \hat{C}^*(L, L)$, we have
\[ \Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\tau) = (-1)^{\sum_{i=1}^{k} (|u_i|-1)(k-1)} v\{s^{-1}u_1, \ldots, s^{-1}u_k\}, \]
where $v \in L$ and $v\{s^{-1}u_1, \ldots, s^{-1}u_k\}$ is given by (11.4). Here, we denote $e_k = \sum_{i=1}^k |u_i|$. \[ \text{Proof.} \]
We prove this identity by induction on $k$. By Lemma 13.1 this holds for $k = 1$.

For $k > 1$ and $v = \beta_1^* \beta_2^* \cdots \beta_p^* \alpha_m \alpha_{m-1} \cdots \alpha_1 \in L$, we have
\[ \Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\tau) \]
\[ = (-1)^{k-1} H(\Phi_{k-1}(s\tau_{1,k-1}) \cup \Phi(s^{-1}u_k))(s\tau) \]
\[ = (-1)^{1+|u_k|+(k-1)}(\Phi_{k-1}(s\tau_{1,k-1}) \cup \Phi(s^{-1}u_k))(s\tau_{1,k-1}) \]
\[ = - \sum_{\alpha \in Q_1} \sum_{j=0}^{\text{p}+1} (-1)^{|u_k|+j} \Phi_{k-1}(s\tau_{1,k-1}) \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\Phi(s^{-1}u_k)(s\tau_{1,k-1}^0)) \cdot (\beta_{j+1}^* \alpha_{m,1}) \]
\[ + \sum_{\alpha \in Q_1} \sum_{j=0}^{m-1} (-1)^{|u_k|+(m+p-j)+(k-1)} \Phi_{k-1}(s\tau_{1,k-1}) \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\Phi(s^{-1}u_k)(s\tau_{1,k-1}^0)) \cdot (\alpha_j), \]
where the first identity follows from (13.5), the second one from (13.4), and the third one from Remark 4.4. Here, we simply write $\Phi_{k-1}(s\tau_{1,k-1}) = \Phi_{k-1}(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_{k-1})$, and write $\alpha_{j,i} = \alpha_j \alpha_{j-1} \cdots \alpha_i$, $\beta_{i,j}^* = \beta_{i-1,j}^* \cdots \beta_j^*$ for any $i < j$.

Write $u_k = \gamma_{00} \tilde{u}_k$ with $\gamma_0 \in Q_1$ and $\tilde{u}_k \in e_{t(\gamma_0)} L e_{s(\gamma_0)}$. Then by (11.4) and the case where $k = 1$, we have
\[ \Phi(s^{-1}u_k)(s\tau) = \alpha_k \phi(s^{-1}u_k) = -\alpha_\gamma \tilde{u}_k, \quad \text{for } \alpha \in Q_1. \]
Substituting this into (13.10), we get
\[ \Phi_k(s^{-1}u_1 \otimes \cdots \otimes s^{-1}u_k)(s\tau) \]
\[ = \sum_{j=0}^{\text{p}+1} (-1)^{|u_k|+(k-1)} \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\tilde{u}_k \beta_{j+1}^* \alpha_{m,1}) \]
\[ + \sum_{j=0}^{m-1} (-1)^{|u_k|+(m+p-j)+(k-1)} \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\tilde{u}_k \alpha_j) \]
\[ = (-1)^{k-1} \sum_{j=0}^{\text{p}+1} (-1)^{|u_k|+(k-1)} \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\tilde{u}_k \beta_{j+1}^* \alpha_{m,1}) \]
\[ + \sum_{j=0}^{m-1} (-1)^{|u_k|+(m+p-j)+(k-1)} \phi(s^{-1}u_k)(s\tau_{1,k-1}^0) \cdot (\tilde{u}_k \alpha_j) \]
\[ = (-1)^{k-1} \sum_{j=0}^{\text{p}+1} (-1)^{|u_k|+(k-1)} v\{s^{-1}u_1, \ldots, s^{-1}u_{k-1}\}. \]
Here, to save the space, we simply write $\{s^{-1}u_1, s^{-1}u_2, \ldots, s^{-1}u_{k-1}\}$ as $\{s\tau_{1,k-1}^0\}$. The first identity follows from the induction hypothesis, and the second identity exactly follows from Proposition 11.6. \[ \square \]

14. Verifying the $B_\infty$-morphism

The final goal is to prove that the $A_\infty$-quasi-isomorphism obtained in the previous section is indeed a $B_\infty$-morphism. The proof relies on the higher pre-Jacobi identity of the Leavitt
$B_\infty$-algebra $\tilde{C}^*(L, L)$; see Remark 5.7. For the opposite $B_\infty$-algebra $A^{\text{opp}}$ of a $B_\infty$-algebra $A$, we refer to Definition 5.5.

**Theorem 14.1.** The $A_\infty$-morphism $(\Phi_1, \Phi_2, \cdots)$ is a $B_\infty$-quasi-isomorphism from the $B_\infty$-algebra $\tilde{C}^*(L, L)$ to the opposite $B_\infty$-algebra $C^*_E(L, L)^{\text{opp}}$.

**Proof.** By Lemma 5.10 it suffices to verify the identity (5.9). That is, for any $x = u_1 \otimes u_2 \otimes \cdots \otimes u_p \in \tilde{C}^*(L, L)^{\otimes p}$ and $y = v_1 \otimes v_2 \otimes \cdots \otimes v_q \in \tilde{C}^*(L, L)^{\otimes q}$, we need to verify

$$
\sum_{r \geq 1} \sum_{r_1 + \cdots + r_p = r} (-1)^r \tilde{\Phi}_q(su_1, q) \{ \tilde{\Phi}_{r_1}(su_{1, i_1}), \tilde{\Phi}_{r_2}(su_{i_1 + 1, i_1 + 1}), \cdots, \tilde{\Phi}_{r_r}(su_{i_1 + \cdots + \epsilon - 1 + 1, p}) \} = \sum_{r \geq 1} \sum_{r_1 + \cdots + r_p = r} (-1)^r \tilde{\Phi}_q(su_1, q) \{ u_1 \{ v_{j_1 + 1, j_1 + 1}, \cdots, v_{j_r + 1, j_r + 1} \} \} \otimes \cdots \otimes v_{j_p + 1, j_p + 1}.
$$

(14.1)

where the sum on the right hand side is over all nonnegative integers $(j_1, \ldots, j_p, l_1, \ldots, l_p)$ such that

$$0 \leq j_1 \leq j_1 + 1 \leq j_2 \leq j_2 + 1 \leq \cdots \leq j_p \leq j_p + 1 \leq q,$$

and $t = p + q - l_1 - \cdots - l_p$; the signs are determined by the identities

$$\epsilon = (|u_1| + \cdots + |u_p| - 1)(|v_1| + \cdots + |v_q| - 1),$$

$$\eta = \sum_{i=1}^p (|u_i| - 1)(|v_1| - 1) + (|v_2| - 1) + \cdots + (|v_q| - 1).$$

Let us verify (14.1). Notice that if there exists $1 \leq j \leq p$ (or $1 \leq l \leq q$) such that $u_j$ (or $v_l$) lies in $\bigoplus_{i \in Q_0} e_i L e_i \subset \tilde{C}^*(L, L)$, then by (13.8) both the left and right hand sides of (14.1) vanish. So we may and will assume that all $u_j$’s and $v_l$’s are in $\bigoplus_{i \in Q_0} s^{-1} e_i L e_i \subset \tilde{C}^*(L, L)$.

It follows from (5.8) and Proposition 13.7 that for any $v_1, \ldots, v_q \in \bigoplus_{i \in Q_0} s^{-1} e_i L e_i$,

$$\tilde{\Phi}_q(su_1, q)(s\overline{a}) := (-1)^{|v_1|+|q-1|+|v_2|+|q-2|+\cdots+|v_q|+q-1} \chi \Phi_q(v_1 \otimes \cdots \otimes v_q)(s\overline{a})$$

$$= (-1)^{(|a|-1)(|v_1|+\cdots+|v_q|-q)} a\{v_1, v_2, \ldots, v_q\}'.$$

(14.2)

Here, we stress that the elements $su_1, \ldots, su_q$ are in the component $s(\bigoplus_{i \in Q_0} s^{-1} e_i L e_i)$ of $s\tilde{C}^*(L, L)$, rather than in $\bigoplus_{i \in Q_0} e_i L e_i \subset \tilde{C}^*(L, L)$.

It follows from (13.7) that $\Phi_q(su_1, q) \in C^*_E(L, L) = \text{Hom}_{E,E}(sL, L)$. Thus, by (6.1) we note that

$$\tilde{\Phi}_q(su_1, q) \{ \tilde{\Phi}_{r_1}(su_{1, i_1}), \tilde{\Phi}_{r_2}(su_{i_1 + 1, i_1 + 1}), \cdots, \tilde{\Phi}_{r_r}(su_{i_1 + \cdots + \epsilon - 1 + 1, p}) \} = 0$$

if $r \neq 1$. Therefore, the left hand side (denoted by LHS) of (14.1) equals

$$\text{LHS} = (-1)^{\epsilon} \tilde{\Phi}_q(su_1, q) \{ \tilde{\Phi}_p(su_{1, p}) \}.$$

Applying the above to elements $s\overline{a} \in sL$, we have

$$\text{LHS}(s\overline{a}) = (-1)^{\epsilon} \tilde{\Phi}_q(su_1, q) \{ \tilde{\Phi}_p(su_{1, p}) \}$$

$$= (-1)^{\epsilon+|a|-1(|v_1|+\cdots+|v_q|-q)} \tilde{\Phi}_q(su_1, q) \{ a\{u_1, \ldots, u_p\}' \}$$

(14.3)
where \( \epsilon_1 = (|a| - 1)(|u_1| + \cdots + |u_p| - p + |v_1| + \cdots + |v_q| - q) \), and the second and third identities follow from (14.2).

For the right hand side (denoted by RHS) of (14.1), using (14.2) again we have

\[
\text{RHS}(s\overline{a}) = \sum (-1)^{n+m} a\{v_{1,j_1}, u_1\{v_{j_1+1,j_1+l_1}\}', v_{j_1+l_1+1,j_2}, u_2\{v_{j_2+1,j_2+l_2}\}', v_{j_2+l_2+1}, \ldots, v_{j_p}, u_p\{v_{j_p+1,j_p+l_p}\}', v_{j_p+l_p+1,1}\}', (14.4)
\]

where \( \eta_1 = (|a| - 1)(|u_1| + \cdots + |u_p| - p + |v_1| + \cdots + |v_q| - q) \).

Comparing (14.3) and (14.4) with the higher pre-Jacobi identity in Remark 5.7 for the Leavitt \( B_\infty \)-algebra \( \hat{C}^*_{\infty}(L, L) \), we obtain

\[
\text{LHS}(s\overline{a}) = \text{RHS}(s\overline{a}).
\]

This verifies the identity (14.1), completing the proof. \( \square \)

**Appendix A. The Opposite \( B_\infty \)-algebra and the Transpose \( B_\infty \)-algebra**

In this appendix, we will prove that for any \( B_\infty \)-algebra \( (A, \mu; \mu_{p,q}) \) with \( \mu_{p,q} = 0 \) whenever \( p > 1 \), there is a (non-strict) \( B_\infty \)-isomorphism from the transpose \( B_\infty \)-algebra \( A^\text{tr} \) (see Definition A.2) to the opposite \( B_\infty \)-algebra \( A^\text{opp} \); see Theorem A.6. Consequently, we obtain the required isomorphism (1.1) between the singular Hochschild cochain complexes.

We leave a comment on the signs. During the preparation of this appendix, we made a strenuous effort to fix the signs in our computations by making use of the Koszul sign rule. Nevertheless, for the readers to understand the proofs, the signs may safely be skipped on a first reading.

**A.1. Some preparation.** In this subsection, we first fix the notation, and then recall the formulae which will be used later.

Let \( (A, m_n) \) be an \( A_\infty \)-algebra; see Subsection 5.1. For each \( n \geq 1 \), we define a linear map \( M_n: (sA)^\otimes n \to sA \) of degree 1 using the following commutative square

\[
\begin{array}{ccc}
A^\otimes n & \xrightarrow{m_n} & A \\
\downarrow s & & \downarrow s \\
(sA)^\otimes n & \xrightarrow{M_n} & sA,
\end{array}
\] (A.1)

where \( s: A \to sA \) is the natural map \( a \mapsto sa \) of degree \(-1\). The identity (5.1) is equivalent to

\[
\sum_{j=0}^{n-1} \sum_{t=1}^{n-j} M_{n-t+1} \circ (1_{sA}^\otimes j \otimes M_{t} \otimes 1_{sA}^\otimes n-j-t) = 0
\]

for \( n \geq 1 \); see [39, Subsection 3.6].

Similarly, an \( A_\infty \)-morphism \( (f_n)_{n \geq 1}: (A, m_n) \to (A', m_n') \) is equivalent to a collection of graded maps \( F_n: (sA)^\otimes n \to sA' \) of degree zero such that for all \( n \geq 1 \), we have (cf. (5.2))

\[
\sum_{j=0}^{n-t} \sum_{t=1}^{n-j} F_{n-t+1} \circ (1_{sA}^\otimes i_1 \otimes M_{t} \otimes 1_{sA}^\otimes n-j-t) = \sum_{r \geq 1, i_1 + \cdots + i_r = n} M_r' \circ (F_{i_1} \otimes \cdots \otimes F_{i_r}).
\] (A.2)
For any $B_\infty$-algebra $(A, m_n; \mu_{p,q})$ we define maps $M_{p,q}$ of degree 0 for $p, q \geq 0$ by the following commutative square

$$
\begin{array}{c}
A^\otimes p \otimes A^\otimes q \\
\downarrow{s_{p+q}} \\
(sA)^\otimes p \otimes (sA)^\otimes q \\
\downarrow{s} \\
A
\end{array}
$$

(A.3)

In particular, we have $M_{1,0} = 1_{sA} = M_{0,1}$ and $M_{k,0} = 0 = M_{0,k}$ for $k \neq 1$.

The axioms in Definition 5.3 may be rewritten with respect to $M_n$ and $M_{p,q}$. In the following remark, we write down the axioms explicitly for a $B_\infty$-algebra $(A, m_n; \mu_{p,q})$ with $\mu_{p,q} = 0$ for $p > 1$, which will be used later. The advantage of using $M_n$ and $M_{p,q}$ is that the sign computations are much simplified. For instance, compare (5.2) and (A.2).

Recall that for any $1 \leq i \leq j$, we use the following notation

$s_{a_{i,j}} := sa_i \otimes sa_{i+1} \otimes \cdots \otimes sa_j \in (sA)^{\otimes j-i}$.

Remark A.1. Let $(A, m_n; \mu_{p,q})$ be a $B_\infty$-algebra with $\mu_{p,q} = 0$ whenever $p > 1$. For any elements $a, b_1, \ldots, b_p, c_1, \ldots, c_q \in A$, we have the following identities.

1. The higher pre-Jacobi identity: for $p \geq 1, q \geq 1$, we have

$$M_{1,q}(M_{1,p}(sa \otimes sb_1 \otimes \cdots \otimes sb_p) \otimes sc_1 \otimes \cdots \otimes sc_q)$$

$$= \sum (-1)^t M_{1,t}(sa \otimes sc_{1,j_1} \otimes M_{1,l_1}(sb_1 \otimes sc_{j_1+1,j_1+l_1}) \otimes sc_{j_1+l_1+1,j_2} \otimes M_{1,l_2}(sb_2 \otimes sc_{j_2+1,j_2+l_2}) \otimes \cdots \otimes sc_{j_p} \otimes M_{1,l_p}(sb_p \otimes sc_{j_p+1,j_p+l_p}) \otimes sc_{j_p+l_p+1,q}).$$

2. The distributivity: for $p \geq 2$ and $q \geq 1$, we have

$$M_{1,q}(M_p(sb_1,p) \otimes sc_{1,q})$$

$$= \sum (-1)^t M_{t}(sc_{1,j_1} \otimes M_{1,l_1}(sb_1 \otimes sc_{j_1+1,j_1+l_1}) \otimes sc_{j_1+l_1+1,j_2} \otimes M_{1,l_2}(sb_2 \otimes sc_{j_2+1,j_2+l_2}) \otimes \cdots \otimes sc_{j_p} \otimes M_{1,l_p}(sb_p \otimes sc_{j_p+1,j_p+l_p}) \otimes sc_{j_p+l_p+1,q}).$$

In the above two identities, the sum on the right hand side of the equality is taken over all sequences of nonnegative integers $(j_1, \ldots, j_p; l_1, \ldots, l_p)$ such that

$$0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq j_3 \leq \cdots \leq j_p \leq j_p + l_p \leq q.$$ 

and we denote $t = p + q - l_1 - l_2 - \cdots - l_p$. The sign

$$\epsilon = \sum_{i=1}^{p}((|b_i| - 1)((|c_1| - 1) + (|c_2| - 1) + \cdots + (|c_{j_i}| - 1))$$

is obtained via the Koszul sign rule by reordering $sb_1 \otimes \cdots \otimes sb_p \otimes sc_1 \otimes \cdots \otimes sc_q$ into $sc_{1,j_1} \otimes sb_1 \otimes sc_{j_1+1,j_2} \otimes sb_2 \otimes \cdots \otimes sc_{j_{p-1}+1,j_p} \otimes sb_p \otimes sc_{j_p+1,q}$. 
(3) The higher homotopy: for \( p \geq 1 \), we have
\[
M_{1,p}(M_1(sa) \otimes sb_{1,p}) + \sum_{i=0}^{p-1} \sum_{t=1}^{p-i} (-1)^{\eta_1} M_{p-t+1}(sa \otimes sb_{1,i} \otimes M_t(sb_{i+1,i+t}) \otimes sb_{i+t+1,p})
\]
\[=
\sum_{i=0}^{p} \sum_{t=0}^{p-i} (-1)^{\eta_2} M_{p-t+1}(sb_{1,i} \otimes M_t(sa \otimes sb_{1,i+t+1}) \otimes sb_{i+t+1,p}),
\]
where \( \eta_1 = (|a| - 1) + (|b_1| - 1) + \cdots + (|b_i| - 1) \) and \( \eta_2 = (|a| - 1)(|b_1| - 1) + \cdots + (|b_i| - 1) \) are obtained via the Koszul sign rule. More precisely, \( \eta_1 \) is obtained since the degree one map \( M_t \) passes through \( sa \otimes sb_{1,i} \) from left to right and \( \eta_2 \) is by swapping \( sa \) with \( sb_{1,i} \).

We mention that for a brace \( B_\infty \)-algebra (i.e. \( m_i = 0 \) for \( i \geq 3 \) and \( \mu_{p,q} = 0 \) for \( p > 1 \)) the above three identities are equivalent to those in Remark 5.7.

A.2. The transpose \( B_\infty \)-algebra. We have defined the opposite \( B_\infty \)-algebra \( A^{\text{opp}} \) of a \( B_\infty \)-algebra \( A \) in Definition 5.5. In this appendix, we also need the following notion of the transpose \( B_\infty \)-algebra \( A^\text{tr} \) of \( A \).

**Definition A.2.** Let \( (A,m_n;\mu_{p,q}) \) be a \( B_\infty \)-algebra. We define the transpose \( B_\infty \)-algebra \( A^\text{tr} \) of \( A \) to be the \( B_\infty \)-algebra \( (A,m_n^\text{tr};\mu_{p,q}^\text{tr}) \), where
\[
m_n^\text{tr}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) := (-1)^{p_n} m_n(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1),
\]
\[
\mu_{p,q}^\text{tr}(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) := (-1)^q \mu_{p,q}(a_p \otimes \cdots \otimes a_1 \otimes b_q \otimes \cdots \otimes b_1),
\]
for any \( a_1, \ldots, a_p, b_1, \ldots, b_q \in A \). Here
\[
\epsilon_n = \frac{(n-1)(n-2)}{2} + \sum_{j=1}^{n-1} a_j(|a_{j+1}| + \cdots + |a_n|)
\]
\[
\epsilon = 1 + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \sum_{j=1}^{p-1} a_j(|a_{j+1}| + \cdots + |a_p|) + \sum_{j=1}^{q-1} b_j(|b_{j+1}| + \cdots + |b_q|).
\]

**Remark A.3.**

1. We explain the maps \( m_n^\text{tr} \) and \( \mu_{p,q}^\text{tr} \). Denote by \( O_n : A^\otimes n \to A^\otimes n \) the map sending \( a_1 \otimes \cdots \otimes a_n \in A^\otimes n \) to \( -1 \sum_{j=1}^{n-1} |a_j|(|a_{j+1}| + \cdots + |a_n|) a_n \otimes a_{n-1} \otimes \cdots \otimes a_1 \). Denote by \( \tilde{O}_n : (sA)^\otimes n \to (sA)^\otimes n \) the map sending \( sa_1 \otimes \cdots \otimes sa_n \in (sA)^\otimes n \) to \( -1 \sum_{j=1}^{n-1} |a_j|(|a_{j+1}| + \cdots + |a_n|) sa_n \otimes sa_{n-1} \otimes \cdots \otimes sa_1 \). We have the following diagram in which the right square is commutative and the left square commutes up to the sign \( (-1)^{\frac{n(n-1)}{2}} \).

\[
\begin{array}{c}
A^\otimes n \\
\downarrow s^\otimes n
\end{array}
\begin{array}{c}
O_n \\
\downarrow s
\end{array}
\begin{array}{c}
A^\otimes n \\
\downarrow s^\otimes n
\end{array}
\begin{array}{c}
m_n \\
\downarrow m
\end{array}
\begin{array}{c}
A
\end{array}
\]

Actually we have \( m_n^\text{tr} = (-1)^{\frac{(n-1)(n-2)}{2}} m_n \circ O_n \). Similarly, the map \( \mu_{p,q}^\text{tr} \) is determined by the following diagram in which the right square is commutative and the left square
where \( \epsilon \) is a strict \( k \)-module. Consider the \( (A, m_1, m_2; \{-\ldots, -\}) \) to be a brace \( B_{\infty} \)-algebra. Then the transpose \( B_{\infty} \)-algebra \( (A^{tr}, m_1^{tr}, m_2^{tr}; \{-\ldots, -\}^{tr}) \) is also a brace \( B_{\infty} \)-algebra given by

\[
m_1^{tr} = m_1, \quad m_2^{tr}(a \otimes b) = (-1)^{|a||b|} m_2(b \otimes a),
\]

where \( \epsilon' = k + \sum_{j=1}^{k-1} (|b_j| - 1)(|b_{j+1}| - 1) + \cdots + (|b_k| - 1) \). As dg algebras, \( (A^{tr}, m_1^{tr}, m_2^{tr}) \) coincides with the (usual) opposite dg algebra \( A^{op} \) of \( A \).

Let \( \Lambda \) be an algebra over a commutative ring \( k \) and \( \Lambda^{op} \) be the opposite algebra of \( \Lambda \). Consider the following two \( B_{\infty} \)-algebras

\[
(\overline{C}_{sg,R}(\Lambda, \Lambda), \delta, \cup_L; \{-\ldots, -\} )
\]

and

\[
(\overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op}), \delta, \cup_R; \{-\ldots, -\} )
\]

Consider the swap isomorphism (note that \( \Lambda = \Lambda^{op} \) as \( k \)-modules)

\[
T: \overline{C}_{sg,L}(\Lambda, \Lambda) \longrightarrow \overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op})
\]

which sends \( f \in \text{Hom}( (s\overline{\Lambda})^{\otimes m}, \Lambda \otimes (s\overline{\Lambda})^{\otimes p} ) \) to \( T(f) \in \text{Hom}( (s\overline{\Lambda})^{\otimes m}, (s\overline{\Lambda})^{\otimes p} \otimes \Lambda ) \) with

\[
T(f)(s\overline{m}_1 \otimes s\overline{m}_2 \otimes \cdots \otimes s\overline{m}_p) = (-1)^{m-p} (-1)^{\frac{m(m-1)}{2}} f(s\overline{m}_1 \otimes \cdots \otimes s\overline{m}_p) \otimes \Lambda
\]

Here, the \( k \)-linear map \( \tau_p: \Lambda \otimes (s\overline{\Lambda})^{\otimes p} \rightarrow (s\overline{\Lambda})^{\otimes p} \otimes \Lambda \) is defined as

\[
\tau_p(b_0 \otimes s\overline{b}_1 \otimes s\overline{b}_2 \otimes \cdots \otimes s\overline{b}_p) = (-1)^{\frac{p(p-1)}{2}} s\overline{b}_1 \otimes \cdots \otimes s\overline{b}_p \otimes b_0.
\]

**Lemma A.4.** Let \( \Lambda \) be a \( k \)-algebra, and \( \Lambda^{op} \) be the opposite algebra of \( \Lambda \). Then \( T \) becomes a strict \( B_{\infty} \)-isomorphism from the transpose \( B_{\infty} \)-algebra \( \overline{C}_{sg,L}(\Lambda, \Lambda)^{tr} \) to the \( B_{\infty} \)-algebra \( \overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op}) \).
A comparison theorem of $B_\infty$-algebras. The goal of this appendix is to prove the following result, which compares the transpose and the opposite $B_\infty$-algebras.

**Theorem A.6.** Let $(A,m_n;\mu_{p,q})$ be a $B_\infty$-algebra with $\mu_{p,q} = 0$ for $p > 1$. Then the identity morphism $1_A: A \to A$ extends to a (non-strict) $B_\infty$-isomorphism from the transpose $B_\infty$-algebra $A^{tr}$ to the opposite $B_\infty$-algebra $A^{opp}$ of $A$. 

**Proof.** It is straightforward to verify the following two identities

\[
T(g_1 \cup_R T(g_2) = (-1)^{|g_1||g_2|}T(g_2 \cup_L g_1),
\]

\[
T(f\{T(g_1),\ldots,T(g_k)\}) = (-1)^r T(f\{g_k,\ldots,g_1\}).
\]

where $\epsilon = k + \sum_{i=1}^{k+1}(|g_i| - 1)((|g_{i+1}| - 1) + \cdots + (|g_k| - 1))$. By (A.5) we have

\[
T(g_1 \cup^L_L g_2) = (-1)^{|g_1||g_2|}T(g_2 \cup_L g_1),
\]

\[
T(f\{g_1,\ldots,g_k\})^{tr} = (-1)^r T(f\{g_k,\ldots,g_1\} L).
\]

Combining the above identities, from Lemma 5.9 we obtain that $T$ is a strict $B_\infty$-isomorphism. □

Let $L$ be a dg $k$-algebra. Consider the brace $B_\infty$-algebra $(C^*(L,L),\delta,-\cup-;\{-,\ldots,-\})$ of Hochschild cochain complex; compare Subsection 6.1. Let $L^{op}$ be the opposite dg algebra of $L$. Similar to (A.6), let

\[
T: C^*(L,L) \to C^*(L^{op},L^{op})
\]

be the swap map sending $f \in C^*(L,L)$ to

\[
T(f)(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_m) = (-1)^{\epsilon} f(sa_m \otimes \cdots \otimes sa_2 \otimes sa_1),
\]

for any $a_1,a_2,\ldots,a_m \in L$, where $\epsilon = |f| + \sum_{i=1}^{m-1}(|a_i| - 1)(|a_{i+1}| - 1 + \cdots + |a_m| - 1)$. Here, we use the identification $L^{op} = L$ as dg $k$-modules.

**Lemma A.5.** The above isomorphism $T$ becomes a strict $B_\infty$-isomorphism from the transpose $B_\infty$-algebra $C^*(L,L)^{tr}$ to the $B_\infty$-algebra $C^*(L^{op},L^{op})$.

**Proof.** By Lemma 5.9 it suffices to verify the following two identities

\[
T(g_1 \cup^{tr} g_2) = T(g_1) \cup T(g_2)
\]

\[
T(f\{g_1,\ldots,g_k\})^{tr} = T(f\{T(g_1),\ldots,T(g_k)\}). \tag{A.7}
\]

By definition, $g_1 \cup^{tr} g_2 = (-1)^{|g_1||g_2|} g_2 \cup g_1$ and $f\{g_1,\ldots,g_k\}^{tr} = (-1)^{\epsilon} f\{g_k,\ldots,g_1\}$, where $\epsilon = k + \sum_{i=1}^{k+1}(|g_i| - 1)((|g_{i+1}| - 1) + \cdots + (|g_k| - 1))$. By a straightforward computation, we have

\[
T(g_1) \cup T(g_2) = (-1)^{|g_1||g_2|} T(g_2 \cup g_1)
\]

\[
T(f\{T(g_1),\ldots,T(g_k)\}) = (-1)^r T(f\{g_k,\ldots,g_1\}).
\]

This verifies (A.7). □
Remark A.7. Let us explain the motivation of Theorem A.6 and the rough idea of its proof. Recall from Lemma 5.12 that $m_2$ induces a graded commutative product (i.e. $m_2 = m_2^g$) on cohomology $H^*(A, m_1)$. This directly follows from the higher homotopy in Remark A.1 (take $p = 1$):

\begin{align}
    m_2(a \otimes b) - (-1)^{|a||b|}m_2(b \otimes a) \\
    = -\mu_{1,1}(m_1(a) \otimes b) - (-1)^{|a|}\mu_{1,1}(a \otimes m_1(b)) - m_1(\mu_{1,1}(a \otimes b)) 
\end{align}

(A.8)

where the right hand side of the equality provides an explicit homotopy of the graded commutativity of $m_2$.

The above equation (A.8) shows that the identity morphism of $H^*(A, m_1)$ is an algebra isomorphism from $(H^*(A, m_1), m_2^g)$ to $(H^*(A, m_1), m_2)$. We will lift this algebra isomorphism to an explicit $A_\infty$-isomorphism $(\Psi_k)_{k \geq 1}$ from the $A_\infty$-algebra $(A, m_1)$ to $(A, m_2)$ such that $\Psi_1$ is the identity morphism of $A$; see (A.9) and Remark A.12 below. It turns out that $(\Psi_k)_{k \geq 1}$ is exactly a $B_\infty$-isomorphism from $A^{tr}$ to $A^{opp}$.

Remark A.8. We do not know whether Theorem A.6 holds for arbitrary $B_\infty$-algebras.

As corollaries of Theorem A.6, we have the following two results.

Corollary A.9. Let $\Lambda$ be an algebra over a commutative ring $\mathbb{k}$. Let $\Lambda^{op}$ be the opposite algebra of $\Lambda$. Then there is a (non-strict) $B_\infty$-isomorphism between the $B_\infty$-algebra $\overline{C}_{sg,L}(\Lambda, \Lambda)$ and the opposite $B_\infty$-algebra $\overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op})^{opp}$.

Proof. By Lemma A.4 we get a strict $B_\infty$-isomorphism

$$T: \overline{C}_{sg,L}(\Lambda, \Lambda)^{tr} \rightarrow \overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op}).$$

By Remark A.3 (3), $T$ is also a strict $B_\infty$-isomorphism from $\overline{C}_{sg,L}(\Lambda, \Lambda) = (\overline{C}_{sg,L}(\Lambda, \Lambda)^{tr})^{tr}$ to $\overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op})^{tr}$. Applying Theorem A.6 to the $B_\infty$-algebra $\overline{C}_{sg,L}(\Lambda, \Lambda)$, we get a non-strict $B_\infty$-isomorphism from $\overline{C}_{sg,L}(\Lambda, \Lambda)^{opp}$ to $\overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op})^{opp}$.

Composing the above two $B_\infty$-isomorphisms, we obtain a (non-strict) $B_\infty$-isomorphism from $\overline{C}_{sg,L}(\Lambda, \Lambda)$ to $\overline{C}_{sg,R}(\Lambda^{op}, \Lambda^{op})^{opp}$.\hfill $\square$

Corollary A.10 ([43]). Let $L$ be a dg $\mathbb{k}$-algebra and $L^{op}$ be its opposite dg algebra. Then there is a (non-strict) $B_\infty$-isomorphism between the $B_\infty$-algebra $C^*(L, L)$ and the opposite $B_\infty$-algebra $C^*(L^{op}, L^{op})^{opp}$.

Proof. The proof is completely analogous to that of Corollary A.9, replacing Lemma A.4 by Lemma A.5.\hfill $\square$

Remark A.11. Keller [43] provides another proof of Corollary A.10 by using the intrinsic description of the $B_\infty$-algebra structures on Hochschild cochain complexes (cf. [41, Subsection 5.7]). We are very grateful to him for sharing his intuition on $B_\infty$-algebras, which essentially leads to Theorem A.6.

The remainder of this appendix will be devoted to the proof of Theorem A.6.

We first construct $\mathbb{k}$-linear maps $\Psi_k: (sA)^{\otimes k} \rightarrow sA$ of degree 0 such that $\Psi_1 = 1_{sA}$ and $\Psi_k$ involves the maps $M_{i,1}$ for $1 \leq i \leq k - 1$. Then we give two basic properties (see Lemma A.13 and Lemma A.17) of the maps $\Psi_k$, which play essential roles in our proof.
From now on, \((A, m_n; \mu_{p,q})\) is a \(B_\infty\)-algebra with \(\mu_{p,q} = 0\) whenever \(p > 1\) and the symbol 1 without any subscript stands for \(1_A\). Let us introduce a \(k\)-linear map of degree 0 for each \(k \geq 1\)

\[\Psi_k : (sA)^{\otimes k} \longrightarrow sA.\]

For \(k = 1\), we define \(\Psi_1 = 1\). For \(k > 1\), \(\Psi_k\) is defined by the recursive formula

\[\Psi_k = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{k-1}} M_{1,r} \circ (1 \otimes \Psi_{i_1} \otimes \Psi_{i_2} \otimes \cdots \otimes \Psi_{i_r}) \quad (A.9)\]

where the sum on the right hand side is taken over the set

\[\mathcal{I}_{k-1} = \{(i_1, i_2, \ldots, i_r) \mid r \geq 1 \text{ and } i_1, i_2, \ldots, i_r \geq 1 \text{ such that } i_1 + i_2 + \cdots + i_r = k - 1\}.\]

For instance, we have

\[\Psi_2 = M_{1,1}\]
\[\Psi_3 = M_{1,2} + M_{1,1} \circ (1 \otimes M_{1,1})\]
\[\Psi_4 = M_{1,3} + M_{1,2} \circ (1 \otimes M_{1,1} \otimes 1) + M_{1,2} \circ (1 \otimes 1 \otimes M_{1,1}) + M_{1,1} \circ (1 \otimes M_{1,2}) + M_{1,1} \circ (1 \otimes 1 \circ (1 \otimes M_{1,1})).\]

When \(\Psi_k\) is applied to elements in \((sA)^{\otimes k}\), additional signs appear due to the Koszul sign rule.

**Remark A.12.** The construction of the above maps \((\Psi_k)_{k \geq 1}\) is motivated from the Kontsevich-Soibelman minimal operad \(M\) introduced in [45, Section 5]. Roughly speaking, the \(n\)-th space \(M(n)\) for \(n \geq 1\) is a \(k\)-linear space spanned by planar rooted trees with \(n\)-vertices labelled by 1, 2, \ldots, \(n\) and some (possibly zero) number of unlabelled vertices (called neutral vertices). The neutral vertices are depicted by black circles in Figures.

Note that an algebra \(A\) over \(M\) has a natural \(B_\infty\)-algebra structure \((A, m_n; \mu_{p,q})\) such that \(\mu_{p,q} = 0\) for \(p \neq 1\), and \(\mu_{1,q}\) and \(m_n\) are given by the first and the second trees in Figure 9, respectively; compare (A.1) and (A.3).

For such a \(B_\infty\)-algebra \(A\), the summands of \(\Psi_k\) correspond bijectively to those trees \(T\) without neutral vertices in \(M(k)\) whose vertices are labelled in clockwise order (such labelling is unique). Note that the number of summands in \(\Psi_k\) is the Catalan number \(C_{k-1} = \frac{1}{k+1} \binom{2k-2}{k-1}\). For instance, the third tree in Figure 9 corresponds to the following summand in \(\Psi_6\)

\[M_{1,2} \circ (1 \otimes 1 \otimes M_{1,2} \circ (1 \otimes M_{1,1} \otimes 1)).\]

We point out that for the reader familiar with the theory of operads, all the proofs in the following may be done by graph computations; compare [45, Subsection 6.2] and [23]. In the present paper, we only provide purely algebraic proofs.

**A.4. The collection \((\Psi_1, \Psi_2, \cdots)\) as an \(A_\infty\)-morphism.** In this subsection, we prove that \((\Psi_1, \Psi_2, \cdots)\) is an \(A_\infty\)-morphism from \((A^n, m^n)\) to \((A, m_n)\); see Proposition A.14.

Recall from (A.4) that for \(n \geq 1\) the map \(M^{tr}_n : (sA)^{\otimes n} \rightarrow sA\) sends \(s_{a_1} \otimes \cdots \otimes s_{a_n}\) to \((-1)^{n-1+\epsilon_n} M_n(s_{a_n} \otimes s_{a_n-1} \otimes \cdots \otimes s_{a_1})\) with \(\epsilon_n = \sum_{j=1}^{n-1} |a_j| - 1) + \cdots + |a_n| - 1)\).

Based on the distributivity in Remark A.1 and the recursive formula (A.9) of \(\Psi_k\), we have the following result.
Lemma A.13. Let \((A, m; \mu_{p,q})\) be a \(B_\infty\)-algebra with \(\mu_{p,q} = 0\) for \(p > 1\). Then for any \(k \geq 2\) and \(a_1, \ldots, a_k \in A\) the following identity holds

\[ -\sum_{j=2}^{k} \Psi_{k-j+1} (M_j^{tr} (sa_{1,j}) \otimes sa_{j+1,k}) \]  

\[ = \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{j=1}^{r} \sum_{t=0}^{r-j} (-1)^{n} M_{r-t+1} (\Psi_{i_1} (sa_{2,i_1+1}) \otimes \cdots \otimes \Psi_{i_j} (sa_{1+i_j+1}+2, i_1+i_{j+1})) \]  

\[ \otimes M_{1,l} (sa_1 \otimes \Psi_{i_{j+1}} (sa_{1+i_j+2, i_1+i_{j+1}}) \otimes \cdots \otimes \Psi_{i_k} (sa_{1+i_k+i_j+2,k})), \]  

where we recall that \(M_{1,0} (sa_1) = sa_1\), and the sign \(\eta\) is obtained via the Koszul sign rule by reordering \(sa_{1,k}\) to \(sa_{2,i_1+i_j+1} \otimes sa_1 \otimes sa_{1+i_j+2,k}\), i.e.

\[ \eta = (|a_1| - 1)(|a_2| - 1) + |a_3| - 1 + \cdots + (|a_{i_1+i_j+i_k+1}| - 1). \]

We point out that the extra sign \((-1)^{n-1}\) of \(M_j^{tr}\) plays an important role in the proof of Lemma A.13. More precisely, it will be used to cancel the items \(T_j\) in (A.11). Note that for \(k = 2\) the identity (A.10) becomes \(-M_2^{tr} (sa_1 \otimes sa_2) = M_2 (sa_2 \otimes sa_1)\), which holds by the definition of \(M_2^{tr}\).

We make some preparation for the proof of Lemma A.13. For any fixed \(2 \leq j \leq k\), we denote

\[ T_j = \Psi_{k-j+1} (M_j^{tr} (sa_{1,j}) \otimes sa_{j+1,k}). \]

Then the left hand side of (A.10) is equal to \(-\sum_{j=2}^{k} T_j\).

Note that \(T_k = (-1)^{k-1+c} M_k (sa_k \otimes \cdots \otimes sa_1)\). For \(2 \leq j < k\), we have that

\[ T_j = \sum_{(i_1, \ldots, i_r) \in I_{k-j}} (-1)^{j-1+c} M_{1, r} \left( M_j (sa_{1,j}) \otimes \Psi_{i_1} (sa_{j+1,i_1}) \otimes \cdots \otimes \Psi_{i_r} (sa_{j+1+i_r+i_j+1}) \right) \]  

\[ = \sum_{(i_1, \ldots, i_r) \in I_{k-j}} \sum_{(p_1, \ldots, p_l) \in I_{k-j}} (-1)^{j-1+c} M_l \left( \Psi_{i_1}^{p_1} \otimes \cdots \otimes \Psi_{i_r}^{p_r} \otimes M_{1,l} (sa_{1-l} \otimes \Psi_{p_1+i_1}^j \otimes \cdots \otimes \Psi_{p_r+i_r}^j) \right) \]  

\[ \otimes \cdots \otimes \Psi_{l_{p_2}}^j \otimes M_{1,l_2} (sa_{j-1} \otimes \Psi_{l_{p_2+i_2}}^j \otimes \cdots \otimes \Psi_{l_{p_r+i_r}}^j) \otimes M_{1,l_1} (sa_1 \otimes \Psi_{l_{p_1+i_1}}^j \otimes \cdots \otimes \Psi_{l_{p_r+i_r}}^j). \]  

(A.11)
Here, to reduce the notational burden, we denote \(M_j(sa_{j,1}) = M_j(sa_j \otimes sa_{j-1} \otimes \cdots \otimes sa_1)\) and for any \(1 \leq l \leq r\) we denote

\[ \Psi^j_{i_l} = \Psi_{i_l}(sa_{j+i_l+\cdots+i_{l-1}+1,j+i_l+\cdots+i_l}). \]

The second identity follows from the distributivity in Remark A.1 for any fixed \((i_1, \ldots, i_r)\), where we denote \(t = j + r - l_1 - \cdots - l_j\) and the sequences \((p_1, \ldots, p_j; l_1, \ldots, l_j)\) are such that

\[ 0 \leq p_1 \leq p_1 + l_1 \leq p_2 \leq p_2 + l_2 \leq \cdots \leq p_j \leq p_j + l_j \leq r. \]

The sign \(\epsilon\) is obtained via the Koszul sign rule by reordering \(sa_{1,k}\) (note that \(\Psi_k\) and \(M_{1,l}\) are both of degree zero).

We denote the summands corresponding to \(p_1 = 0\) in (A.11) by

\[ T_j^0 = \sum_{(i_1, \ldots, i_r) \in I_{k-j+1}} \sum_{(p_1, \ldots, p_j)} (-1)^{j-1-\epsilon} M_t \left( M_{1,l_1}(sa_j \otimes \Psi^1_{i_1} \otimes \cdots \otimes \Psi^1_{i_{l_1}}) \otimes \Psi^j_{i_{l_1+1}} \otimes \cdots \otimes \Psi^j_{i_{p_2}} \right) \]

\[ M_{1,l_2}(sa_{j-1} \otimes \Psi^j_{i_{p_2+1}} \otimes \cdots) \cdots \otimes \Psi^j_{i_{p_j}} \otimes M_{1,l_j}(sa_1 \otimes \Psi^j_{i_{p_j+1}} \otimes \cdots) \cdots \otimes \Psi^j_{i_r}). \]  

(A.12)

The remaining summands (i.e. corresponding to \(p_1 \neq 0\) in (A.11) are denoted by \(T_j^{\neq 0}\). Thus, for \(2 \leq j < k\) we have

\[ T_j = T_j^0 + T_j^{\neq 0}. \]

For convenience, we write \(T_k^0 := T_k\) and \(T_k^{\neq 0} = 0.\)

We are now in a position to prove Lemma A.13.

**Proof of Lemma A.13.** We claim that \(T_j^0 = -T_{j-1}^{\neq 0}\) for \(3 \leq j \leq k\). Indeed, by definition \(T_{j-1}^{\neq 0}\) is equal to

\[ \sum_{(i_1, \ldots, i_r) \in I_{k-j+1}} \sum_{(p_1, \ldots, p_j)} (-1)^{j-2+\epsilon} M_t \left( \Psi^1_{i_1} \otimes \cdots \otimes \Psi^j_{i_{p_2+1}} \otimes M_{1,l_1}(sa_j \otimes \Psi^j_{i_{p_2+1}} \otimes \cdots \otimes \Psi^j_{i_{p_1+1}} \otimes M_{1,l_2}(sa_{j-1} \otimes \Psi^j_{i_{p_2+1}} \otimes \cdots) \cdots \otimes \Psi^j_{i_{p_j}} \otimes M_{1,l_j}(sa_1 \otimes \Psi^j_{i_{p_j+1}} \otimes \cdots) \cdots \otimes \Psi^j_{i_r} \right). \]

Recall that \(\Psi^j_{i_1} = \Psi_{i_1}(sa_{j+1} \otimes sa_{j+2} \cdots \otimes sa_{j+1-1})\). Replacing the term \(\Psi^j_{i_1}\) by (A.9) and then comparing with (A.12), we obtain that \(T_{j-1}^{\neq 0}\) is exactly equal to \(-T_j^0\). This proves the claim. We mention that the extra signs \((-1)^{n-1}\) in the definition \(M_{1,k}\) (see (A.4)) are implicitly used in the proof of the claim.

It follows from the above claim that the left hand side (denoted by LHS) of (A.10) equals

\[ \text{LHS} = - \sum_{j=2}^{k} (T_j^0 + T_j^{\neq 0}) = -T_2^0. \]

Consider \(j = 2\) in (A.12) and apply (A.9) to the terms \(M_{1,l_1}(sa_2 \otimes \Psi^2_{i_1} \otimes \cdots \otimes \Psi^2_{i_{l_1}})\). We obtain that

\[ T_2^0 = \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{p_2=1}^{r-p_2} \sum_{l_1=0}^{r-p_2+1} (-1)^{1+\epsilon} M_{l_1} \left( \Psi^2_{i_1} \otimes \cdots \otimes \Psi^j_{i_{p_2+1}} \otimes M_{1,l_1}(sa_1 \otimes \Psi^j_{i_{p_2+1}} \otimes \cdots \otimes \Psi^j_{i_{p_2+l_1}}) \right). \]
where recall that $\Psi_{i_l}^2 = \Psi_{i_l}(s_{a_{i_1} + \ldots + i_{l-1} + 2, i_1 + \ldots + i_l + 1})$ for $1 \leq l \leq r$. By comparing the signs

$$(-1)^{1+\epsilon} = (-1)^{1+|a_1|-1+|a_2|-1+\ldots+|a_k|-1+\ldots+|a_1+\ldots+a_{n+2}|-1}) = -(-1)^\eta,$$

we obtain that the right hand side of (A.10) is also equal to $-T_2^0$. This verifies (A.10). □

The following proposition essentially follows from Lemma A.13 and the higher homotopy in Remark A.1.

**Proposition A.14.** Let $(A, m_n; \mu_{p,q})$ be a $B_{\infty}$-algebra with $\mu_{p,q} = 0$ whenever $p > 1$. Then the above collection of maps $(\Psi_1, \Psi_2, \ldots)$ defines an $A_{\infty}$-morphism from $(A, m_1^r)$ to $(A, m_2).$

**Proof.** By (A.2), it suffices to verify the following identity for each $k \geq 1$

$$\sum_{j=0}^{k-1} \sum_{t=1}^{k-j} \Psi_{k-t+1}(1^{\otimes j} \otimes M_t^{lr} \otimes 1^{\otimes k-j-t}) = \sum_{(i_1, \ldots, i_r) \in I_k} M_r(\Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r}). \tag{A.13}$$

For $k = 1$, the above identity holds since $\Psi_1 = 1$. To make it easier for readers to follow the proof, we further verify the above identity (A.13) for $k = 2$. By the definition of $\Psi_k$ in (A.9) and $M_{j}^{lr}$ in (A.4) we observe that the identity (A.13) becomes

$$M_{1,1} \circ (M_1 \otimes 1) + M_{1,1} \circ (1 \otimes M_1) + M_2^{lr} = M_1 \circ M_{1,1} + M_2. \tag{A.14}$$

Recall from Remark A.1 the higher homotopy for $p = 1$

$$M_{1,1}(M_1(sa) \otimes sb) + (-1)^{|a|-1}M_{1,1}(sa \otimes M_1(sb)) = M_1(M_{1,1}(sa \otimes sb)) + M_2(sa \otimes sb) + (-1)^{|a|-1}M_2(sb \otimes sa). \tag{A.15}$$

Since $M_2^{lr}(sa \otimes sb) = (-1)^{|a|-1}M_2(sb \otimes sa)$, the identity (A.14) follows from (A.15). This verifies (A.13) for $k = 2$.

For $k > 2$, let us prove (A.13) by induction. The proof relies on Lemma A.13 and the higher homotopy as in the case where $k = 2$. By substituting (A.9) into the left hand side (denoted by LHS) of (A.13) we have that

$$\text{LHS} = \sum_{t=1}^{k} \sum_{(i_1, \ldots, i_r) \in I_{k-t}} M_{1,r}(M_t^{lr} \otimes \Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r}) + \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{j=1}^{r} \sum_{l=0}^{i_j-1} M_{1,r}(1^{\otimes l} \otimes M_t^{lr} \otimes 1^{\otimes (i_j-l-1)} \otimes \Psi_{i_j+1} \otimes \cdots \otimes \Psi_{i_r}).$$
Since \(i_j < k\), by the induction hypothesis we have

\[
\text{LHS} = \sum_{t=1}^{k} \sum_{(i_1, \ldots, i_r) \in I_{k-t}} M_{1,r}(M_t^{1r} \otimes \Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r}) + \sum_{(t_1, \ldots, t_r) \in I_{k-1}} \sum_{l=1}^{r} \sum_{(i_1, \ldots, i_r) \in I_{i_j}} M_{1,r-1}(1 \otimes \Psi_{i_1} \otimes \cdots \otimes \Psi_{i_{j-1}} \otimes M_l(\Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r})) = \sum_{t=1}^{k} \sum_{(i_1, \ldots, i_r) \in I_{k-t}} M_{1,r}(M_t^{1r} \otimes \Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r}) + \sum_{(t_1, \ldots, t_r) \in I_{k-1}} \sum_{l=1}^{r} \sum_{(i_1, \ldots, i_r) \in I_{i_j}} M_{1,r-1}(1 \otimes \Psi_{i_1} \otimes \cdots \otimes \Psi_{i_{j-1}} \otimes M_l(\Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r})),
\]

where the second identity just follows from rewriting the second sums.

We now apply the formula (A.16) to any element \(sa_1 \otimes sa_2 \cdots \otimes sa_k \in (sA)^{\otimes k}\). First, by the higher homotopy in Remark A.1 we have the following identity

\[
\sum_{(i_1, \ldots, i_r) \in I_{k-1}} M_{1,r}(M_t^{1r}(sa_1) \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_r}(-)) + \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{j=1}^{r} \sum_{t=1}^{r-j+1} (-1)^{\eta_1} M_{1,r-t+1}(sa_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{j-1}}(-) \otimes M_l(\Psi_{i_j}(-) \otimes \cdots \otimes \Psi_{i_{j+1}}(-))) \otimes \Psi_{i_{j+2}} \otimes \cdots \otimes \Psi_{i_r}(-) = \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{j=0}^{r} \sum_{t=0}^{r-j} (-1)^{\eta_2} M_{r-t+1}(\Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_j}(-) \otimes M_{1,t}(sa_1 \otimes \Psi_{i_{j+1}}(-) \otimes \cdots \otimes \Psi_{i_{j+1}}(-)) \otimes \cdots \otimes \Psi_{i_r}(-)).
\]

Here we simply write \(\Psi_{i_j}(sa_1 + \cdots + sa_{i-1}) + sa_{i+1} + \cdots + sa_k\) as \(\Psi_{i_j}(-)\) for \(1 \leq j \leq r\). The signs \(\eta_1\) and \(\eta_2\) are obtained via the Koszul sign rule, namely

\[
\eta_1 = (|a_1| - 1) + (|a_2| - 1) + \cdots + (|a_{i_1+\cdots+i_{j-1}+1}| - 1), \quad \eta_2 = (|a_1| - 1) + (|a_2| - 1) + \cdots + (|a_{i_1+\cdots+i_{j+1}}| - 1).
\]

Applying (A.9) to \(\Psi_{i_1}\) on the right hand side (denoted by RHS) of (A.13), we have

\[
\text{RHS} = \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{l=0}^{r} M_{r-t+1}(M_{1,t}(sa_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_r}(-))) \otimes \Psi_{i_{t+1}}(-) \otimes \cdots \otimes \Psi_{i_r}(-).
\]

Note that the above sums also appear on the right side of (A.17) corresponding to \(j = 0\). Substituting (A.17) and (A.18) into (A.16), we have

\[
\text{LHS} = \text{RHS} + \sum_{t=2}^{k} \sum_{(i_1, \ldots, i_r) \in I_{k-t}} M_{1,r}(M_t^{1r}(sa_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_r}(-))) + \sum_{(i_1, \ldots, i_r) \in I_{k-1}} \sum_{j=1}^{r} \sum_{t=0}^{r-j} (-1)^{\eta_2} M_{r-t+1}(\Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{j+1}}(-) \otimes M_{1,t}(sa_1 \otimes \Psi_{i_{j+1}}(-) \otimes \cdots \otimes \Psi_{i_{j+1}}(-)) \otimes \cdots \otimes \Psi_{i_r}(-)).
\]
By Lemma A.13 the last two terms of (A.19) equal zero. This verifies (A.13). □

A.5. **The collection** \((\Psi_1, \Psi_2, \cdots)\) **as a** \(B_\infty\)-**morphism.** In this subsection, we prove that \((\Psi_1, \Psi_2, \cdots)\) is a \(B_\infty\)-morphism from \(A^{tr}\) to \(A^{op}\); see Proposition A.18.

Recall from Remark A.3 (1) that the map \(M_{1,k}^{tr} : (sA) \otimes (sA)^{\otimes k} \to sA\) is defined as

\[
M_{1,k}^{tr}(sa \otimes sb_1 \otimes \cdots \otimes sb_k) = (-1)^{k+\epsilon} M_{1,k}(sa \otimes sb_k \otimes sb_{k-1} \otimes \cdots \otimes sb_1),
\]

where \(\epsilon\) is obtained via the Koszul sign rule by reversing the order \(sb_1 \otimes sb_2 \otimes \cdots \otimes sb_k\), i.e.

\[
\epsilon = \sum_{j=1}^{k-1} (|b_j| - 1)(|b_{j+1}| - 1) + \cdots + (|b_k| - 1).
\]

In particular, we have

\[
\Psi_1 \in k, \quad \Psi_2 \in k, \quad \text{and } \Psi \in k,
\]

which will be used to cancel the items \(\Psi_1^\infty, k\) and \(\Psi_2^\infty, k\), i.e. Lemmas A.15–A.17. Note that Lemmas A.15–A.16 are special cases of Lemma A.17.

**Lemma A.15.** Let \((A, m_n; \mu_{p,q})\) be a \(B_\infty\)-algebra with \(\mu_{p,q} = 0\) for \(p > 1\) and let \(a_1, b_1, b_2, \ldots, b_q \in A\).

(1) For \(q \geq 1\) the following identity holds

\[
\sum_{l=0}^{q} \Psi_{q-l+1} \left( M_{1,q-l}^{tr}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_l) \otimes sb_{l+1} \otimes \cdots \otimes sb_q \right) = 0. \tag{A.20}
\]

(2) For \(q \geq 2\) we have

\[
\sum_{l=1}^{q} M_{1,q-l}^{tr}(sa_1 \otimes \cdots \otimes sa_l) \otimes sa_{l+1} \otimes \cdots \otimes sa_q = 0. \tag{A.21}
\]

Before the proof of Lemma A.15, we would like to stress the importance of the extra sign \((-1)^k\) in \(M_{1,k}^{tr}\), which will be used to cancel the items \(S_l\) in (A.22) below. For instance, for \(q = 1\) the identity (A.20) becomes \(\Psi_2(sa_1 \otimes sb_1) + \Psi_1(M_{1,1}^{tr}(sa_1 \otimes sb_1)) = 0\). This holds since \(\Psi_2 = M_{1,1}\) and \(M_{1,1}^{tr} = -M_{1,1}\). For \(q = 2\) the identity (A.21) becomes

\[
M_{1,1}^{tr}(sa_1 \otimes sa_2) + M_{1,0}^{tr}(\Psi_2(sa_1 \otimes sa_2)) = 0,
\]

which follows since \(M_{1,1}^{tr} = -M_{1,1}\), \(M_{1,0}^{tr} = M_{1,0}\), and \(\Psi_2 = M_{1,1}\).

Let us first make some preparation for the proof of Lemma A.15. For \(q > 1\) and any fixed \(0 \leq l \leq q\), we denote

\[
S_l = \Psi_{q-l+1}(M_{1,l}^{tr}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_l) \otimes sb_{l+1} \otimes \cdots \otimes sb_q).
\]

In particular, we have \(S_0 = \Psi_{q+1}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_q)\) and \(S_q = M_{1,q}^{tr}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_q)\).

By (A.9) and the higher pre-Jacobi identity in Remark A.1, we have

\[
S_l = \sum_{(i_1, \ldots, i_{r}) \in \mathcal{I}_{q-l}} M_{1,r}(sa_1 \otimes sb_1, i_1) \otimes \Psi_{i_1}(sb_{l+1, l+i_1}) \otimes \cdots \otimes \Psi_{i_r}(sb_{l+1+i_1+\cdots+i_{r-1}, q})
\]

\[
= \sum_{(i_1, \ldots, i_{r}) \in \mathcal{I}_{q-l}} (-1)^{l+\epsilon} M_{1,n}(sa_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{p_1}}(-) \otimes M_{1,j_1}(sb_1 \otimes \Psi_{i_{p_1+j_1}}(-) \otimes \cdots \otimes \Psi_{i_r}(-))
\]

\[
\otimes \cdots \otimes \Psi_{i_{p_r}}(-) \otimes M_{1,j_1}(sb_1 \otimes \Psi_{i_{p_1+1}}(-) \otimes \cdots \otimes \Psi_{i_{p_1+j_1}}(-)) \otimes \cdots \otimes \Psi_{i_r}(-)) \tag{A.22}
\]

where the sequences \((p_1, \ldots, p_1; j_1, \ldots, j_1)\) are such that

\[
0 \leq p_1 \leq p_1 + j_1 \leq p_2 \leq p_2 + j_2 \leq \cdots \leq p_l \leq p_l + j_l \leq r,
\]
\( n = l + r - j_1 - \cdots - j_t \) and the sign \( \epsilon' \) is obtained via the Koszul sign rule by reordering \( sb_{1,t} \). Here to save the space, we simply write \( \Psi_{ij}(b_{t+i_1+\cdots+i_{j-1}+1,l+i_1+\cdots+i_j}) \) as \( \Psi_{ij}(-) \) for any \( 1 \leq j \leq r \).

We observe that \( S_l \) for \( 1 \leq l \leq q - 1 \) may split into two sums (depending on whether \( p_1 = 0 \) in (A.22)). We denote by \( S^0_l \) the sum corresponding to \( p_1 = 0 \), namely

\[
S^0_l = \sum_{(i_1, \ldots, i_r) \in I_q} \sum_{(j_1, \ldots, j_l)} (-1)^{l+\epsilon'} M_{1,n} \left( sa_1 \otimes M_{1,j_1} \left( sb_l \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{j_1}}(-) \right) \right)
\]

\[
\otimes \cdots \otimes \Psi_{i_{j_l}}(-) \otimes M_{1,j_1} \left( sb_1 \otimes \Psi_{i_{j_1+1}}(-) \otimes \cdots \otimes \Psi_{i_{j_{l+1}}(-)} \right) \otimes \cdots \otimes \Psi_{i_r}(-) \right).
\]

The remaining summands (i.e. \( p_1 \neq 0 \)) in (A.22) are denoted by \( S^\neq_l \). Thus, we have \( S_l = S^0_l + S^\neq_l \). Note that (using the definition of \( \Psi_k \) in (A.9))

\[
S^0_1 = -\Psi_{q+1}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_q) = -S_0
\]

\[
S^\neq_{q-1} = -M_{1,q}^{tr}(sa_1 \otimes sb_1 \otimes \cdots \otimes sb_q) = -S_q.
\]

We are now in a position to prove Lemma A.15.

**Proof of Lemma A.15.** We only provide the proof for (1). The proof for (2) is similar to that for (1). We have verified (A.20) for \( q = 1 \) above. Now we consider the case \( q > 1 \). We claim that \( S^\neq_l = -S^0_{l+1} \) for any \( 1 \leq l < q - 1 \). Indeed, we have

\[
S^\neq_l = \sum_{(i_1, \ldots, i_r) \in I_q} \sum_{(j_1, \ldots, j_l) \neq (j_1, \ldots, j_l)} (-1)^{l+\epsilon'} M_{1,n} \left( sa_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{j_1}}(-) \otimes M_{1,j_1} \left( sb_l \otimes \Psi_{i_{j_1+1}}(-) \otimes \cdots \otimes \Psi_{i_{j_{l+1}}(-)} \right) \otimes \cdots \otimes \Psi_{i_r}(-) \right).
\]

We apply (A.9) to \( \Psi_{i_1}(-) \) in the above sum. It is not difficult to see that \( S^\neq_l = -S^0_{l+1} \). Here the sign \( -1 \) is due to the difference of the extra signs \( (-1)^l \) in \( S^\neq_l \) and \( (-1)^{l+1} \) in \( S^0_{l+1} \). This proves the claim. We mention that the extra signs \( (-1)^q \) in the definition \( M_{1,q}^{tr} \) (see (A.4)) are implicitly used in the proof of the above claim.

Thus, we have that the left hand side of (A.20) is

\[
LHS = \sum_{l=0}^{q-1} S_l = S_0 + \sum_{l=1}^{q-1} \left( S^0_l + S^\neq_l \right) + S_q = (S_0 + S^0_1) + \sum_{l=1}^{q-2} \left( S^\neq_l + S^0_{l+1} \right) + (S^\neq_{q-1} + S_0) = 0.
\]

This verifies (A.20). \( \square \)

**Lemma A.16.** Let \((A, m_n; \mu_{p,q})\) be a \( B_\infty \)-algebra with \( \mu_{p,q} = 0 \) for \( p > 1 \). Then for \( q \geq 1 \) and any \( a_1, b_1, b_2, \ldots, b_q \in A \), we have the following identity

\[
(-1)^{\eta_q} M_{1,1} \left( \Psi_q(\mathcal{B}_1) \otimes \cdots \otimes \mathcal{B}_q \right) \otimes \mathcal{B}_1
\]

\[
= \sum_{j=0}^{q-1} \sum_{l=0}^{q-j} (-1)^{\eta_j} \Psi_{q-l+1} \left( \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_j \otimes \mathcal{B}_{j+1} \otimes \cdots \otimes \mathcal{B}_{l+1} \right),
\]

where \( \eta_j = (|a_1| - 1)(|b_1| - 1) + (|b_2| - 1) + \cdots + (|b_j| - 1)) \) for \( 1 \leq j \leq q \).
Proof. We prove this by induction on \( q \geq 1 \). For \( q = 1 \), the identity (A.23) holds since both sides are equal to \((-1)^{n_1}M_{1,1}(sb_1 \otimes sa_1)\).

For \( q > 1 \), by Lemma A.15 (1) we only need to consider \( 1 \leq j \leq q \) (removing \( j = 0 \)) and \( 0 \leq l \leq q - j \) on the right hand side of (A.23). It follows from (A.9) that the right hand side of (A.23) (denoted by RHS) equals

\[
\text{RHS} = \sum_{(i_1, \ldots, i_r) \in \mathbb{N}^r} \sum_{l=1}^{r+1} (-1)^{i_1} M_{1,r+1} \left( sb_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{r-1}}(-) \otimes \Psi_1(sa_1) \otimes \Psi_{i_r}(-) \otimes \cdots \otimes \Psi_{i_r}(-) \right)
+ \sum_{(i_1, \ldots, i_r) \in \mathbb{N}^r} \sum_{l=1}^{r} (-1)^{i_1} M_{1,r} \left( sb_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{r-1}}(-) \otimes \Psi_{i_l-1}(-) \otimes \cdots \otimes \Psi_{i_r}(-) \right)
\]

where \( k^l := i_1 + \cdots + i_{l-1} + 1 \) and for any \( 1 \leq k \leq r \) we denote

\[
\Psi_{ik}(-) = \Psi_{ik}(sb_{2+i_1+\cdots+i_{k-1}+1+i_{k+1}+\cdots+i_r});
\]

in the second identity we use the induction hypothesis since \( i_l < q \); and the signs \( \epsilon_1 \) and \( \epsilon_2 \) are obtained via the Koszul sign rule, namely \( \epsilon_1 = \eta_{1+i_1+\cdots+i_{l-1}} \) and \( \epsilon_2 = \eta_{1+i_1+\cdots+i_r} \).

Note that the left hand side (denoted by LHS) of (A.23) equals

\[
\text{LHS} = (-1)^{n_1} M_{1,1}(\Psi_q(s_{1, \ldots, q} \otimes sa_1))
= \sum_{(i_1, \ldots, i_r) \in \mathbb{N}^r} (-1)^{n_1} M_{1,1} \left( M_{1,r} \left( sb_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_r}(-) \right) \otimes sa_1 \right)
+ \sum_{(i_1, \ldots, i_r) \in \mathbb{N}^r} \sum_{l=1}^{r} (-1)^{i_1} M_{1,r+1} \left( sb_1 \otimes \Psi_{i_1}(-) \otimes \cdots \otimes \Psi_{i_{r-1}}(-) \otimes \Psi_{i_l-1}(-) \otimes sa_1 \otimes \Psi_{i_r}(-) \otimes \cdots \otimes \Psi_{i_r}(-) \right)
\]

where the last identity follows from the higher pre-Jacobi identity in Remark A.1 and the signs \( \epsilon_1 \) and \( \epsilon_2 \) are defined as in (A.24). By comparing the last identities of (A.24) and (A.25), we get \( \text{LHS} = \text{RHS} \).

More generally, we have the following property on the maps \( \Psi_k \), which makes \( (\Psi_1, \Psi_2, \cdots) \) a \( B_{\infty} \)-morphism from \( A^{tr} \) to \( A^{opp} \). We prove the following two identities (A.26) and (A.27) by simultaneous induction.

...
Lemma A.17. Let \((A, m_n; \mu_{p,q})\) be a \(B_\infty\)-algebra with \(\mu_{p,q} = 0\) for \(p > 1\). Then for \(p, q \geq 1\) and any elements \(a_1, \ldots, a_p, b_1, \ldots, b_q \in A\), we have the identity

\[
\sum_{(j_1=0, j_2, \ldots, j_p; l_1, \ldots, l_p)} (-1)^{\eta} \Psi_1 \left(M_{1,l_1}^{tr} (sa_1 \otimes sb_{1,l_1}) \otimes sb_{1+l_1, j_2} \otimes M_{1,l_2}^{tr} (sa_2 \otimes sb_{j_2+1, j_2+l_2}) \otimes \cdots \otimes sb_{j_p} \otimes M_{1,l_p}^{tr} (sa_p \otimes sb_{j_p+l_p+1, l_p}) \otimes sb_{l_p+1, q} \right) = 0,
\]

(A.26)

and the identity

\[
\sum_{(i_1, \ldots, i_r) \in I_{p,q}} (-1)^{\epsilon} M_{1,r} \left(\Psi_q (sb_{1,q}) \otimes \Psi_i (sa_1,i_i) \otimes \Psi_{i_2} (sa_{i_1+1,i_1+i_2}) \otimes \cdots \otimes \Psi_i (sa_{i,\ldots,i_r+1,p}) \right)
\]

\[
= \sum_{(j_1, \ldots, j_p; l_1, \ldots, l_p)} (-1)^{\eta} \Psi_1 \left(sb_{1,j_1} \otimes M_{1,l_1}^{tr} (sa_1 \otimes sb_{j_1+1,j_1+i_1}) \otimes sb_{j_1+l_1+1, j_2} \otimes M_{1,l_2}^{tr} (sa_2 \otimes sb_{j_2+1,j_2+l_2}) \otimes \cdots \otimes sb_{j_p} \otimes M_{1,l_p}^{tr} (sa_p \otimes sb_{j_p+l_p+1,l_p}) \otimes sb_{l_p+1,q} \right).
\]

(A.27)

In both identities, \((j_1, \ldots, j_p; l_1, \ldots, l_p)\) are sequences of integers such that \(j_1 = 0\) in (A.26)

\[
0 \leq j_1 \leq j_1 + l_1 \leq j_2 \leq j_2 + l_2 \leq \cdots \leq j_p \leq j_p + l_p \leq q;
\]

and \(t = p + q - l_1 - \cdots - l_p\). Here the signs are given by

\[
\epsilon = ((|a_1| - 1) + \cdots + (|a_p| - 1))( (|b_1| - 1) + \cdots + (|b_q| - 1)),
\]

\[
\eta = \sum_{i=1}^{p} (|a_i| - 1)((|b_1| - 1) + (|b_2| - 1) + \cdots + (|b_j| - 1)).
\]

Note that by (A.9) the left hand side of (A.27) just equals \((-1)^{t} \Psi_{p+1}(\Psi_q (sb_{1,q}) \otimes sa_{1,p})\).

Proof. For \(p = 1\), these follow from Lemmas A.15 (1) and A.16, respectively.

To make it easier for readers to follow the proof, we further verify the two identities for the special case \(p = 2, q = 1\). In this case, the left hand side of (A.26), denoted by LHS, becomes

\[
\text{LHS} = \Psi_3 (sa_1 \otimes sa_2 \otimes sb_1) + \Psi_2 (sa_1 \otimes M_{1,1}^{tr} (sa_2 \otimes sb_1))
\]

\[
+ (-1)^{|a_2|-1} (|b_1|-1) (\Psi_3 (sa_1 \otimes sb_1 \otimes sa_2) + \Psi_2 (M_{1,1}^{tr} (sa_1 \otimes sb_1) \otimes sa_2)) = M_{1,2} (sa_1 \otimes sa_2 \otimes sb_1) + M_{1,1} (sa_1 \otimes M_{1,1} (sa_2 \otimes sb_1))
\]

\[
+ (-1)^{|a_2|-1} (|b_1|-1) (M_{1,2} (sa_1 \otimes sb_1 \otimes sa_2) + M_{1,1} (sa_1 \otimes M_{1,1} (sb_1 \otimes sa_2)))
\]

\[
+ (-1)^{|a_2|-1} (|b_1|-1) M_{1,1} (M_{1,1}^{tr} (sa_1 \otimes sb_1) \otimes sa_2)
\]

By the higher pre-Jacobi identity in Remark A.1

\[
M_{1,1} (M_{1,1} (sa_1 \otimes sb_1) \otimes sa_2) = M_{1,2} (sa_1 \otimes sa_2 \otimes sb_1) + M_{1,1} (sa_1 \otimes M_{1,1} (sb_1 \otimes sa_2))
\]

\[
+ (-1)^{|a_2|-1} (|b_1|-1) M_{1,2} (sa_1 \otimes sa_2 \otimes sb_1).
\]

and \(M_{1,1}^{tr} = -M_{1,1}\), we have LHS = 0. This yields (A.26). Similarly, the identity (A.27) becomes

\[
M_{1,1} (sb_1 \otimes \Psi_2 (sa_1 \otimes sa_2)) + M_{1,2} (sb_1 \otimes sa_1 \otimes sa_2) = \Psi_3 (sb_1 \otimes sa_1 \otimes sa_2)
\]
for this case, since by \((A.26)\) we may assume that \(j_1 \neq 0\). The above identity directly follows from \((A.9)\).

More generally, let us prove these two identities by simultaneous induction on \(p \geq 1\). Assume that the two identities hold for any \(i < p\). We need to prove for \(i = p\). We first prove \((A.26)\). For this, denote the left hand side of \((A.26)\) by LHS. We claim that the following identity holds

\[
\text{LHS} = \sum_{l=0}^{q} (-1)^{l} \Psi_p \left( \Psi_{q-l+1}(M^{ur}_{1,l}(sa_1 \otimes sb_{1,q}) \otimes sb_{l+1,q}) \otimes sa_{2,p} \right) =: \text{RHS},
\]

where \(\epsilon' = (|a_2| - 1) + \cdots + (|a_p| - 1))(|b_1| - 1) + \cdots + (|b_q| - 1))\). Clearly, this claim implies that LHS = 0, since by Lemma A.15 (1) we have RHS = 0.

Let us prove the claim. Indeed, we have

\[
\text{RHS} = \sum_{l=0}^{q} \sum_{(j_1,...,j_l) \in I_{q-l}} \sum_{(1,...,i_r) \in I_{p-1}} (-1)^{l} M_{1,r} \left( M^{ur}_{1,t}(M^{ur}_{1,t}(-) \otimes \Psi^{b}_{j_1} \cdots \otimes \Psi^{b}_{j_t}) \otimes \Psi^{a}_{i_1} \otimes \Psi^{a}_{i_2} \otimes \cdots \otimes \Psi^{a}_{i_r} \right)
\]

\[
= \sum_{l=0}^{q} \sum_{(j_1,...,j_l) \in I_{q-l}} \sum_{(1,...,i_r) \in I_{p-1}} (-1)^{l} M_{1,n} \left( M^{ur}_{1,t}(-) \otimes \Psi^{a}_{i_1} \cdots \otimes \Psi^{a}_{i_{p-1}} \otimes M_{1,k_1}(\Psi^{a}_{i_1} \otimes \Psi^{a}_{i_{p-1}} \otimes \cdots \otimes \Psi^{a}_{i_{p-1+k_1}}) \right)
\]

\[
\otimes \cdots \otimes \Psi^{a}_{i_{p-1+k_1}} \otimes M_{1,k_1}(\Psi^{a}_{i_{p-1+k_1}} \otimes \Psi^{a}_{i_{p-1+k_2}} \otimes \cdots \otimes \Psi^{a}_{i_{p-1+k_2+k_1}}) \otimes \cdots \otimes \Psi^{a}_{i_r})
\]

(A.28)

where the second equality is due to the higher pre-Jacobi identity in Remark A.1, the sum without any subscript is taken over all \((k_1,...,k_t;p_1,...,p_t)\) such that

\[
0 \leq p_1 \leq p_1 + k_1 \leq p_2 \leq p_2 + k_2 \leq \cdots \leq p_t \leq p_t + k_t \leq r;
\]

and \(n = r + t - k_1 - \cdots - k_t\); where \(M^{ur}_{1,t}(-)\) is short for \(M^{ur}_{1,t}(sa_1 \otimes sb_{1} \otimes \cdots \otimes sb_{t})\), for any \(1 \leq m \leq t\) we simply write (setting \(i_0 = 0\))

\[
\Psi^{b}_{j_m} = \Psi_{j_m}(sb_{j_1+\cdots+j_{m-1}+l+1} \otimes \cdots \otimes sb_{j_1+\cdots+j_{m+l}})
\]

and for \(1 \leq m \leq r\) we write (setting \(j_0 = 0\))

\[
\Psi^{a}_{i_m} = \Psi_{i_m}(sa_{i_1+\cdots+i_{m-1}+2} \otimes \cdots \otimes sa_{i_1+\cdots+i_{m+1}}).
\]

Here, \(\epsilon'\) is given as above and similarly \(\epsilon''\) is obtained via the Koszul sign rule by reordering \(sa_{1,p} \otimes sb_{1,q}\).

By the induction hypothesis, we may apply \((A.27)\) to each term

\[
M_{1,k_1}(\Psi^{b}_{j_1} \otimes \Psi^{a}_{i_{p_1+1}} \otimes \cdots \otimes \Psi^{a}_{i_{p_1+k_1}}), \cdots, M_{1,k_t}(\Psi^{b}_{j_t} \otimes \Psi^{a}_{i_{p_t+1}} \otimes \cdots \otimes \Psi^{a}_{i_{p_t+k_t}})
\]

in the identity \((A.28)\). Then by \((A.9)\) again it is not difficult to see that RHS is further equal to LHS. This proves the claim.

Let us prove \((A.27)\) under the induction hypothesis. First, by \((A.26)\) we may assume that \(j_1 > 0\) on the right side of \((A.27)\). Denote the left hand side of \((A.27)\) by LHS. Similarly,
we have
\[
\text{LHS} = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_p} (M_{1,r})_1 \left( M_{1,l} \left( s_1 \otimes \Psi^b_{j_1} \otimes \Psi^b_{j_2} \otimes \cdots \otimes \Psi^b_{j_l} \right) \otimes \Psi^a_{i_1} \otimes \cdots \otimes \Psi^a_{i_r} \right)
\]
\[
= \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_p} \left( \sum_{(j_1, \ldots, j_l) \in \mathcal{I}_q} (M_{1,l})_1 \left( s_1 \otimes \Psi^a_{i_1} \otimes \cdots \otimes \Psi^a_{i_r} \otimes M_{1,k_1} \left( \Psi^b_{j_1} \otimes \Psi^a_{i_{p_1+1}} \otimes \cdots \otimes \Psi^a_{i_{p_1+k_1}} \right) \right) \right)
\]
\[
\otimes \Psi^a_{i_{p_1+k_1+1}} \otimes \cdots \otimes \Psi^a_{i_{p_r}} \otimes M_{1,k_1} \left( \Psi^b_{j_2} \otimes \Psi^a_{i_{p_2+1}} \otimes \cdots \otimes \Psi^a_{i_{p_2+k_2}} \right) \otimes \cdots \otimes \Psi^a_{i_{p_r}},
\]
(\ref{eq:A.29})
where the sum without any subscript is taken over all \((k_1, \ldots, k_l; p_1, \ldots, p_l)\) such that
\[
0 \leq p_1 \leq p_1 + k_1 \leq p_2 \leq p_2 + k_2 \leq \cdots \leq p_l \leq m + k_l \leq r;
\]
and \(n = r + t - k_1 - \cdots - k_t\).

Applying the induction hypothesis to each term
\[
M_{1,k_1} \left( \Psi^b_{j_1} \otimes \Psi^a_{i_{p_1+1}} \otimes \cdots \otimes \Psi^a_{i_{p_1+k_1}} \right), \ldots, M_{1,k_1} \left( \Psi^b_{j_1} \otimes \Psi^a_{i_{p_1+1}} \otimes \cdots \otimes \Psi^a_{i_{p_1+k_1}} \right)
\]
in the identity (\ref{eq:A.29}). More precisely, for any fixed indexes (on the right hand side of (\ref{eq:A.29}))
\[
j_1, j_2, \ldots, j_l; i_1, i_2, \ldots, i_r; k_1, k_2, k_3, \ldots, k_l; k_1, k_1+1, \ldots, k_1+k_1; k_2, k_2+1, \ldots, k_2+k_2; \ldots; i_1, i_1+1, \ldots, i_1+i_1\]
the sum \(N_1 := i_1+i_1+1+i_1+2+\cdots+i_1+i_1+1\) is fixed (although \(k_1, i_1+1, i_1+2, \ldots, i_1+k_1\) are not fixed). Thus, by the induction hypothesis the term \(M_{1,k_1} \left( \Psi^b_{j_1} \otimes \Psi^a_{i_{p_1+1}} \otimes \cdots \otimes \Psi^a_{i_{p_1+k_1}} \right)\) on (\ref{eq:A.29}) may be replaced by the right side of (\ref{eq:A.27}) since \(N_1 < p\). Similarly, we may do this, in turn, for \(M_{1,k_2} \left( \Psi^b_{j_2} \otimes \Psi^a_{i_{p_2+1}} \otimes \cdots \otimes \Psi^a_{i_{p_2+k_2}} \right), \ldots, M_{1,k_r} \left( \Psi^b_{j_r} \otimes \Psi^a_{i_{p_r+1}} \otimes \cdots \otimes \Psi^a_{i_{p_r+k_r}} \right)\).

Then using (\ref{eq:A.9}) again we see that LHS equals the right hand side of (\ref{eq:A.27}).

Now we prove that \((\Psi_1, \Psi_2, \ldots)\) is a \(B_\infty\)-morphism from the transpose \(B_\infty\)-algebra \(A^{tr}\) to the opposite \(B_\infty\)-algebra \(A^{opp}\) of \(A\).

**Proposition A.18.** Let \((A, m_n; \mu_{p,q})\) be a \(B_\infty\)-algebra with \(\mu_{p,q} = 0\) for \(p > 1\). Then the above \(A_\infty\)-morphism \((\Psi_1, \Psi_2, \cdots)\) is a \(B_\infty\)-morphism from the transpose \(B_\infty\)-algebra \(A^{tr}\) to the opposite \(B_\infty\)-algebra \(A^{opp}\) of \(A\).

**Proof.** Note that we may translate the brace operations of \(A^{tr}\) and \(A^{opp}\) using \(M_{l}^{\text{tr}}\) and \(M_{l}^{\text{opp}}\) for \(l \geq 0\); see (\ref{eq:A.3}) and compare (\ref{eq:A.3}). By Proposition A.14, we obtain that \((\Psi_1, \Psi_2, \cdots)\) is an \(A_\infty\)-morphism.

It remains to verify (\ref{eq:A.5}) for \((\Psi_1, \Psi_2, \cdots)\). Clearly, this directly follows from the identity (\ref{eq:A.27}) in Lemma A.17.

**A.6. The proof of the comparison theorem.** By Proposition A.18, there is a \(B_\infty\)-morphism
\[
(\Psi_1, \Psi_2, \cdots): sA^{tr} \rightarrow sA^{opp}
\]
such that \(\Psi_1 = 1_sA\). Recall that the underlying graded space of \(sA^{tr}\) and \(sA^{opp}\) is the same \(sA\).

To prove Theorem A.6, we will show that \((\Psi_1, \Psi_2, \cdots)\) is a \(B_\infty\)-isomorphism. Indeed, we will construct an explicit inverse \((\Phi_1, \Phi_2, \cdots)\).
We define a $k$-linear map $\Phi_k : (sA)^{\otimes k} \to sA$ of degree 0 for each $k \geq 1$ such that $\Phi_1 = 1$ and $\Phi_k$ for $k > 1$ is determined by the following recursive formula
\[
\Phi_k = \sum_{(i_1, \ldots, i_r) \in \mathcal{I}_{k-1}} M_{i_r}^{tr} (1 \otimes \Phi_{i_1} \otimes \Phi_{i_2} \otimes \cdots \otimes \Phi_{i_r}).
\] (A.30)

For instance, we have
\[
\begin{align*}
\Phi_2 &= M_{1,1}^{tr} = -M_{1,1} = -\Psi_2 \\
\Phi_3 &= M_{1,2}^{tr} + M_{1,1}^{tr} \circ (1 \otimes M_{1,1}^{tr}) \\
\Phi_4 &= M_{1,3}^{tr} + M_{1,2}^{tr} \circ (1 \otimes 1 \otimes M_{1,1}^{tr}) + M_{1,1}^{tr} \circ (1 \otimes M_{1,1}^{tr} \otimes 1) \\
&\quad + M_{1,1}^{tr} \circ (1 \otimes M_{1,2}^{tr}) + M_{1,1}^{tr} (1 \otimes M_{1,1}^{tr} \circ (1 \otimes M_{1,1}^{tr})).
\end{align*}
\]

From the viewpoint of the Kontsevich-Soibelman minimal operad $\mathcal{M}$ in Remark A.12, $\Phi_k$ is the sum of all the trees $T$ in $\mathcal{M}(k)$ such that the vertices are labelled in counterclockwise order; see the fourth tree in Figure 9.

We claim that
\[
\Phi \circ_\infty \Psi = 1_{sA^{tr}} \quad \text{and} \quad \Psi \circ_\infty \Phi = 1_{sA^{opp}},
\]
where $\circ_\infty$ is the composition of $A_\infty$-morphisms; see Definition 5.1. Indeed, it suffices to prove $\Phi \circ_\infty \Psi = 1_{sA^{tr}}$, as the proof of $\Psi \circ_\infty \Phi = 1_{sA^{opp}}$ is completely similar.

By definition, the identity $\Phi \circ_\infty \Psi = 1_{sA^{tr}}$ is equivalent to
\[
\sum_{(i_1, \ldots, i_r) \in \mathcal{I}_k} \Phi_r (\Psi_{i_1} \otimes \cdots \otimes \Psi_{i_r}) = 0 \quad \text{for any } k \geq 2.
\] (A.31)

Clearly, we have $\Phi_1 \Psi_1 = 1_{sA^{tr}}$. Let us prove the second identity (A.31) by induction on $k \geq 2$. For $k = 2$, the identity is clear since $\Phi_1 = 1_{sA} = \Psi_1$ and $\Phi_2 = -\Psi_2$. For $k > 2$, the left hand side (denoted by LHS) of (A.31) equals
\[
\sum_{(j_1, \ldots, j_l) \in \mathcal{I}_{k-1}} M_{i_l}^{tr} \left( \Psi_{i_1} \circ \Phi_{j_1} (\Psi_{i_2} \otimes \cdots \otimes \Psi_{i_{j_1+1}}) \circ \Phi_{j_2} (\Psi_{i_{j_1+2}} \otimes \cdots \otimes \Psi_{i_{j_1+j_2+1}}) \cdots \circ \Phi_{j_l} (\cdots \otimes \Psi_{i_r}) \right).
\] (A.32)

We apply the induction hypothesis to the terms $\Phi_{j_1} (\Psi_{i_2} \otimes \cdots \otimes \Psi_{i_{j_1+1}})$. More precisely, fix the following integers
\[
j_2, j_3, \ldots, j_l; \ i_1, i_{j_1+2}, i_{j_1+3}, \ldots, i_r.
\]
Since $i_1 + i_2 + \cdots + i_r = k$, the sum $N := i_2 + \cdots + i_{j_1+1}$ is fixed although $j_1, i_2, i_3, \ldots, i_{j_1+1}$ are not fixed. Thus, by the induction hypothesis the following identities hold (since $N < k$)
\[
\sum_{(i_2, i_3, \ldots, i_{j_1+1}) \in \mathcal{I}_N} \Phi_{j_1} (\Psi_{i_2} \otimes \Psi_{i_3} \otimes \cdots \otimes \Psi_{i_{j_1+1}}) = \begin{cases} 
0 & \text{if } N > 1 \\
1 & \text{if } N = 1.
\end{cases}
\]
This implies that
\[
\text{LHS} = \sum_{(j_1, j_2, \ldots, j_t) \in I^{r-1}} M_{i_1, i_2}^{fr}(\Psi_{i_1} \otimes 1 \otimes \Phi_{j_2}(\Psi_{i_3} \otimes \cdots \otimes \Psi_{i_{j_2+2}}) \otimes \cdots \otimes \Phi_{j_t}(\cdots \otimes \Psi_{i_r})). \tag{A.33}
\]

Similarly, we may apply the induction hypothesis to the terms, in turn,
\[
\Phi_{j_2}(\Psi_{i_3} \otimes \cdots \otimes \Psi_{i_{j_2+2}}), \ldots, \Psi_{j_t}(\Psi_{i_{j_2+\cdots+j_t-1+3}} \otimes \cdots \otimes \Psi_{i_r}).
\]

Afterwards, we obtain
\[
\text{LHS} = \sum_{i_1=1}^{k} M_{i_1, k-i_1}^{fr}(\Psi_{i_1} \otimes 1 \otimes \cdots \otimes 1),
\]
which corresponds to the summands of (A.32) with \(j_1 = j_2 = \cdots = j_t = 1\) and \(i_2 = i_3 = \cdots = i_r = 1\). Thus, by Lemma A.15 (2) we have LHS = 0. Therefore, \(\Phi \circ_\infty \Psi = 1_{sA^{tr}}\), completing the proof of Theorem A.6.

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