UNIVERSALITY OF AFFINE SEMI-GROUPS ON SUPERCYCLICITY OF THE SEQUENCE OPERATORS

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Abstract: In this paper, show that for all supercyclic strongly continuous sequence of operators semi-group on a complex $\mathcal{F}^j$-space, every $T^j_{(1+\epsilon)}$ with $\epsilon > -1$ is supercyclic. Also, the sets of all supercyclic vectors $T^j_{(1+\epsilon)}$ with $\epsilon > -1$ are precisely the sets of supercyclic vectors of the entire semi-group.

Keywords: hypercyclic semi-groups; hypercyclic operators; supercyclic operators; supercyclic semi-groups.

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1. INTRODUCTION

Unless stated, the spaces in this paper are over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Topological spaces are assumed to be Hausdorff. As usual $\mathbb{Z}_+, \mathbb{N}$ and $\mathbb{R}_+$ are positive integers and real numbers respectively. The symbol $L(X)$ denoted the space of continuous linear sequence of operators on a topological vectors space $X$, while $X'$ is the space of continuous linear functionals on $X$. As usual, for $T^j \in L(X)$, the dual sequence of operators $\hat{T}^j: X' \to X'$ is defined by the formula $\sum f_j(\hat{T}^j)f_j(x) = \sum f_j(T^jx)$ for $x \in X$ and $f_j \in X'$.
suppose that an affine map $T^j$ on a vector space $X$ is a map of the forms $T^j x = u^j + S^j x$, where $u^j$ are fixed vector in $X$ and $S^j: X \to X$ is linear. Clearly, $T^j$ are continuous iff $S^j$ are continuous. The symbols $A_j(X)$ stands for the space of continuous affine maps on a topological vector space $\mathcal{F}^j$ –space are complete metrizable topological vector space. Recall that a families $\sum_j \mathcal{F}^j = \sum_j \{T^j_a\}_{a \in A_j}$ of continuous maps from a space $X$ to a space $Y$ is called universal if there is $x \in X$ for which $\{T^j_a x: a \in A_j\}$ is dense in $Y$ and such an $x$ is called a universal element for $\mathcal{F}^j$. Let the symbol $\mathcal{U}(\mathcal{F}^j)$ denote the set of universal elements for $\mathcal{F}^j$. If $X$ is a topological space $X$ to itself is called a semi-groups if $(T^j)_0 = I$ and $T^j_{2(\varepsilon + 1)} = T^j_{(1+\varepsilon)} T^j_{(1+\varepsilon)}$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Say that a semi-groups $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ are strongly continuous if $(1+\varepsilon) \mapsto T^j_{(1+\varepsilon)} x$ are continuous as a map from $\mathbb{R}_+$ to $X$ for every $x \in X$ and say that $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ is jointly continuous if $((1+\varepsilon), x) \mapsto T^j_{(1+\varepsilon)} x$ are continuous as map from $\mathbb{R}_+ \times X$ to $X$.

If $X$ is a topological vector space, semi-group $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ a linear semi-group if $\sum_j T^j_{(1+\varepsilon)} \in L(X)$ for every $(1+\varepsilon) \in \mathbb{R}_+$ and $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ is called an affine semi-group if $\sum_j T^j_{(1+\varepsilon)} \in A_j(X)$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Recall that $T^j \in L(X)$ are called hypercyclic if $\mathcal{U} \sum_j (T^j) \neq \emptyset$ and elements of $\mathcal{U} \sum_j (T^j)$ are called hypercyclic vectors. A universal linear semi-groups $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ are called hypercyclic and its universal elements are called hypercyclic vectors for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$. If $\sum_j T^j_{(1+\varepsilon)} \in L(X)$, then universal elements of the family $\{z \sum_j (T^j)^n x: z \in \mathbb{K}, n \in \mathbb{Z}_+\}$ are called supercyclic vectors for $T^j$ and $T^j$ are called supercyclic if it has a supercyclic vector. Similarly, if $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ are linear semi-group, then universal element of the families $\{z \sum_j T^j_{(1+\varepsilon)} x: z \in \mathbb{K}, (1+\varepsilon) \in \mathbb{Z}_+\}$ are called supercyclic vector for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$.
and the semi-group is called supercyclic if it has a supercyclic vector. [1] and references therein have been covered the concept of Hypercyclicity and super-cyclicity, also discuss the relation between the supercyclicity of a linear semi-group and supercyclicity of the individual members of the semi-group. The hypercyclicity version of the question was treated by Conejero, Müller and Peris [2], who proved that for every strongly continuous hypercyclic linear semi-groups \( \sum_j \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) on an \( \mathcal{F}^j \)-space, all \( T^j_{(1+\varepsilon)} \) with \( \varepsilon > -1 \) is hypercyclic and \( \mathcal{U} \sum_j \left( T^j_{(1+\varepsilon)} \right) = \mathcal{U} \sum_j \left( \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right) \).

Virtually the same proof works in the following much more general setting.

**Theorem A.** Let \( \sum_j \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a hypercyclic jointly continuous linear semi-group on all topological vector space \( X \). Then all \( T^j_{(1+\varepsilon)} \) with \( \varepsilon > -1 \) is hypercyclic and

\[
\mathcal{U} \sum_j \left( T^j_{(1+\varepsilon)} \right) = \mathcal{U} \sum_j \left( \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right).
\]

The stronger condition of joint continuity coincides with the strong continuity in the case when \( X \) is an \( \mathcal{F}^j \)-space due to a straightforward application of the Banach–Steinhaus theorem.

If \( X \) has no subsets different from \( \emptyset \) and \( X \) then it called connected, which are closed and open and it is called simply connected if for any continuous map \( f_j: \mathbb{T} \to X \), there is continuous maps \( F^j: \mathbb{T} \times [0,1] \to X \) and \( x_0 \in X \) such that \( F^j(z,0) = f_j(z) \) and \( F^j(z,1) = x_0 \) for any \( z \in \mathbb{T} \). Next, \( X \) is called locally path connected at \( x \in X \) if for any neighborhood \( U \) of \( x \), there is a neighborhood \( V \) of \( x \) such that for any \( y \in V \), there is a continuous map \( f_j: [0,1] \to X \) satisfying \( f_j(0) = x \), \( f_j(1) = y \) and \( f_j([0,1]) \subseteq U \).

**Proposition 1.1.** Let \( X \) be a topological space and \( \sum_j \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a jointly continuous semi-group on \( X \) such that:

1. \( \{ \sum_j T^j_{(1+\varepsilon)}(u^j) : (1 + \varepsilon) \in [0,(1 + \varepsilon)] \} \) are nowhere dense in \( X \) for every \( \varepsilon > -1 \) and \( u^j \in X \);
(2) for every $\varepsilon > -1$ and $x \in \mathcal{U} \sum_j \left( \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right)$, there is $Y_{(1+\varepsilon)x} \subseteq X$ such that $Y_{(1+\varepsilon)x}$ is connected, locally path connected, simply connected and

$$\left\{ \sum_j T^j_{(1+\varepsilon)}x : (1 + \varepsilon) \in [0, (1 + \varepsilon)] \right\} \subseteq Y_{(1+\varepsilon)x} \subseteq \mathcal{U} \sum_j \left( \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right).$$

Then $\mathcal{U} \sum_j \left( T^j_{(1+\varepsilon)} \right) = \mathcal{U} \sum_j \left( \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right)$ for every $\varepsilon > -1.$

In [3] the supercyclicity version of Theorem A holds. They have produced the following example to explain that it fails in the case $\mathbb{K} = \mathbb{R}$.

**Example B**. Let $X$ be a Banach space over $\mathbb{R}$, $\sum_j \left( T^j_{(1+\varepsilon)} \right)_{(1+\varepsilon) \in \mathbb{R}_+}$ be a hypercyclic linear semi-group on $X$ and $(A_j)_{(1+\varepsilon) \in \mathbb{R}_+}$ be the linear sequence of operators with the matrices $A_j = \left( \begin{array}{cc} \cos(1+\varepsilon) & \sin(1+\varepsilon) \\ -\sin(1+\varepsilon) & \cos(1+\varepsilon) \end{array} \right)$ Then $\sum_j \left( (A_j)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)} \right)_{(\varepsilon+1) \in \mathbb{R}_+}$ are supercyclic linear semi-group on $\mathbb{R}^2 \times X$, while $\sum_j \left( (A_j)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)} \right)$ are non-supercyclics whenever $\frac{(1+\varepsilon)}{\pi}$ is rational.

**Proposition C**. Let $X$ be a complex topological vector space and $\sum_j \left( T^j_{(1+\varepsilon)} \right)_{(1+\varepsilon) \in \mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on $X$ such that $T^j_{(1+\varepsilon)} - \lambda_j I$ has dense range for every $\varepsilon > -1$ and every $\lambda_j \in \mathbb{C}.$ Then each $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic.

Furthermore, the set of supercyclic vectors for $T^j_{(1+\varepsilon)}$ does not determine by the choice of $\varepsilon > -1$ and matches with the set of supercyclic vectors of the entire semi-group.

To show that in the case $\mathbb{K} = \mathbb{C}$, the supercyclicity version of Theorem A holds without any additional assumptions, we can applying Proposition 1.1. with the same results in [9] and considering the induced action on subsets of the projective space.

**Theorem 1.2**. Let $X$ be a complex topological vector space and $\sum_j \left( T^j_{(1+\varepsilon)} \right)_{(1+\varepsilon) \in \mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on $X$. Then all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is
supercyclic and the set of supercyclic vectors of $T^{j}_{(1+\varepsilon)}$ coincides with the set of supercyclic vectors of $\sum_j \{T^{j}_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$.

Clearly that any supercyclic jointly continuous linear semi-group on a complex topological vector $X$ satisfies conditions of Proposition C or has a closed invariant hyperplane $Y$ clearly that any supercyclic jointly continuous linear semi-group on a complex topological vector $X$ either satisfies conditions of Proposition C or has a closed invariant hyperplane $Y$. In the other case the topic reduces to the generalization of Theorem A to affine semi-groups see e.g.,[4].

**Theorem 1.3.** Let $X$ be a topological vector space and $\sum_j \{T^{j}_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ be a universal jointly continuous affine semi-group on $X$. Then all $T^{j}_{(1+\varepsilon)}$ with $\varepsilon > -1$ is universal and $\mathcal{U} \sum_j (T^{j}_{(1+\varepsilon)}) = \mathcal{U} \sum_j \{T^{j}_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$.

**2. A Dichotomy for Superacyclic Linear Semi-Groups**

An analogue of the following result for individual supercyclic sequence of operators.

**Proposition 2.1.** Let $X$ be a complex topological vector space and $\sum_j \{T^{j}_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ be a supercyclic strongly continuous linear semi-group on $X$. Then either $\sum_j (T^{j}_{(1+\varepsilon)} - \lambda_j I)(X)$ is dense in $X$ for every $\varepsilon > -1$ and $\lambda_j \in \mathbb{C}$ or there is a closed hyperplane $H$ in $X$ such that $\sum_j T^{j}_{(1+\varepsilon)}(H) \subseteq H$ for every $(1 + \varepsilon) \in \mathbb{R}_+$.

The most of the section is devoted to the proof of Proposition 2.1. Need several elementary lemmas. Recall that subsets $B_j$ of vector space $X$ are called balanced if $\lambda_j x \in B_j$ for every $x \in B_j$ and $\lambda_j \in \mathbb{K}$ such that $|\lambda_j| \leq 1$.

**Lemma 2.2.** Let $K$ be a compact subset of an infinite dimensional topological vector space and $X$ such that $0 \notin K$. Then $\Lambda = \{\lambda_j x : \lambda_j \in \mathbb{K}, x \in K\}$ is a closed nowhere dense subset of $X$.

**Proof.** Closeness of $\Lambda$ in $X$ is a straightforward exercise. Assume that $\Lambda$ is not
nowhere dense. Since $\Lambda$ is closed, its interior $L$ is non-empty. Since $K$ is closed and $0 \notin K$, we can find a non-empty balanced open set $U$ such that $U \cap K = \emptyset$. Clearly $\lambda_j x \in L$ whenever $x \in L$ and $\lambda_j \in \mathbb{K}, \lambda_j \neq 0$. Since $U$ is open and together with the latter property of $L$ implies that the open set $W^j = L \cap U$ is non-empty. Taking into account the definition of $\Lambda$, the inclusion $L \subseteq \Lambda$, the equality $U \cap K = \emptyset$ and the fact that $U$ is balanced, see that every $x \in W^j$ can be written as $x = \lambda_j y$, where $y \in K$ and $\lambda_j \in \mathbb{D} = \{z \in \mathbb{K}: |z| \leq 1\}$. Since both $K$ and $\mathbb{D}$ are compact, $Q = \{\lambda_j y: \lambda_j \in \mathbb{D}, y \in K\}$ is a compact subset of $X$. Since $X \subseteq Q$, $W^j$ is a non-empty open set with compact closure. Such a set exists [5] only if $X$ is finite dimensional. This contradiction completes the proof.

The following lemma is a particular case in [6].

**Lemma 2.3.** suppose that $X$ is complex topological vector space such that $2 \leq \dim X < \infty$. Therefore $X$ is not uphold supercyclic strongly continuous linear semi-groups.

**Lemma 2.4.** Let $X$ be an infinite dimensional topological vector space, $\lambda_j \in \mathbb{K}$, $(1 + \varepsilon)_0 > 0$ and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon) \in \mathbb{R}_+}$ be a strongly continuous linear semi-group such that $T^j_{(1+\varepsilon)} = \lambda_j I$. Then $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon) \in \mathbb{R}_+}$ are not supercyclics.

**Proof.** Let $x \in X \setminus \{0\}$. It suffices to show that $x$ is not a supercyclic vector for

$$\sum_j \left(\frac{T^j}{(1+\varepsilon)}\right)_{(1+\varepsilon) \in \mathbb{R}_+}.$$

First, consider the case $\lambda_j = 0$. By the strong continuity, there is $\varepsilon > -1$ such that $0 \notin K = \left\{\sum_j T^j_{(1+\varepsilon)} x: (1 + \varepsilon) \in [0, (\varepsilon + 1)]\right\}$ and $K$ is a compact subset of $X$. By Lemma 2.2, $\left\{z \sum_j T^j_{(1+\varepsilon)} x: z \in \mathbb{K}, (\varepsilon + 1) \in [0, (\varepsilon + 1)]\right\}$ is nowhere dense in $X$. Take $n \in \mathbb{N}$ such that $n(1 + \varepsilon) \geq (1 + \varepsilon)_0$. Since $T^j_{(1+\varepsilon)} = 0$ and $n(1 + \varepsilon) \geq (1 + \varepsilon)_0$, have $\sum_j \left(T^j\right)^n_{(1+\varepsilon)} = \sum_j T^j_{(1+\varepsilon)n} = 0$. Then $Y = \sum_j \left(T^j_{(1+\varepsilon)X}\right) \neq X$. In particular, $Y$ is nowhere dense in $X$. Clearly, $\sum_j T^j_{(1+\varepsilon)} x \in Y$ whenever $\varepsilon > -1$. Hence $\left\{z \left(T^j\right)^n: (1 + \varepsilon) \in \mathbb{R}_+, z \in \mathbb{K}\right\}$ is contained in $A_j \cup Y$ and therefore is nowhere dense in $X$. Thus $x$ is not a supercyclic
vector for $\sum_j \left\{ T_j^{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}$

Assume that $\lambda_j \neq 0$. Then $T_j^{(1+\varepsilon)} x = \lambda_j^n x \neq 0$ for every $n \in \mathbb{Z}_+$. Hence each of the compact sets $K_n = \{ z(T_j)^n x : z \in \mathbb{C}, (1 + \varepsilon)_0 n \leq (1 + \varepsilon) \leq (1 + \varepsilon)_0 (n + 1) \}$ with $n \in \mathbb{Z}_+$ does not contain 0. By Lemma $2.2,$ the sets $\sum_j (A_j)_n = \{ z \sum_j T_j^{(1+\varepsilon)} x : z \in \mathbb{C}, (1 + \varepsilon)_0 n \leq (1 + \varepsilon) \leq (1 + \varepsilon)_0 (n + 1) \}$ are nowhere dense in $X$. On the other hand, for every $(1 + \varepsilon) \in [(1 + \varepsilon)_0 n, (1 + \varepsilon)_0 (n + 1)]$, $\sum_j T_j^{(1+\varepsilon)} x = \sum_j T_j^{(1+\varepsilon)} T_j^{(1+\varepsilon)} = \sum_j \lambda_j T_j^{(1+\varepsilon)} x$ and therefore $\sum_j (A_j)_n = \sum_j (A_j)_{n+1}$ for each $n \in \mathbb{Z}_+$. Hence $\{ z \sum_j T_j^{(1+\varepsilon)} x : (1 + \varepsilon) \in \mathbb{R}_+, z \in \mathbb{K} \}$, which is clearly the union of $(A_j)_n$, coincides with $\sum_j ((A_j)_1)_n$ and therefore is nowhere dense. Thus $x$ is not a supercyclic vector for $\sum_j \left\{ T_j^{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}$.

**Lemma 2.5.** Let $X$ be a complex topological vector space and $\sum_j \left\{ T_j^{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}$ be a supercyclic strongly continuous linear semi-group on $X$. Let also $(\varepsilon + 1)_0 > 0$ and $\lambda_j \in \mathbb{C}$. Then the space $Y = \left( \sum_j (T_j^{(1+\varepsilon)} - \lambda_j I)(X) \right)$ either coincides with $X$ or is a closed hyperplane in $X$.

**Proof.** Using the semi-group property, it is easy to see that $Y$ is invariant for all $T_j^{(1+\varepsilon)}$. Factoring $Y$ out, arrive to a supercyclic strongly continuous linear semi-group $\sum_j \left\{ S_j^{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}$ acting on $X/Y$, where $\sum_j S_j^{(1+\varepsilon)} (x + Y) = \sum_j T_j^{(1+\varepsilon)} x + Y$. Obviously, $\sum_j S_j^{(1+\varepsilon)} = \sum_j \lambda_j I$. If $X/Y$ is infinite dimensional, arrive to a contradiction with Lemma 2.4. If $X/Y$ is finite dimensional and $\dim X/Y \geq 2$, we obtain a contradiction with Lemma 2.3. Thus $\dim X/Y \leq 1$, as required.

**Proof of Proposition 2.1.** Assume that there is $\varepsilon > -1$ and $\lambda_j \in \mathbb{K}$ such that $\sum_j (T_j^{(1+\varepsilon)} - \lambda_j I)(X)$ are not denses in $X$. By Lemma 2.5, $H = \left( \sum_j (T_j^{(1+\varepsilon)} - \lambda_j I)(X) \right)$ are closed hyperplanes in $X$. It is easy to see that $H$ is invariant for all $T_j^{(1+\varepsilon)}$. 
The following lemma provides some extra information on the second case in Proposition 2.1.

**Lemma 2.6.** Let $X$ be a complex topological vector space and \( \sum_j \{ T^j_{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a strongly continuous linear semi-group on $X$. Assume also that there is a closed hyperplane $H$ in $X$ such that \( T^j_{(1+\varepsilon)}(H) \subseteq H \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \) and let \( f_j \in X' \), be such that \( H = \ker f_j \). Then there exists $w \in \mathbb{C}$ such that \( e^{w(1+\varepsilon)} \sum_j (T^j_{(1+\varepsilon)}) f_j = \sum_j f_j \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \).

**Proof.** Since $H = \ker f_j$ is invariant for every $T^j_{(1+\varepsilon)}$, there is unique function $\varphi_j : \mathbb{R}_+ \to \mathbb{C}$ such that \( \sum_j \hat{T}^j_{(1+\varepsilon)} f_j = \sum_j \varphi_j (1+\varepsilon) f_j \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \). Pick $u^j \in X$ such that $f_j(u^j) = 1$. Then \( \sum_j \left( \hat{T}^j_{(1+\varepsilon)} f_j \right) (u^j) = \sum_j f_j \left( T^j_{(1+\varepsilon)} u^j \right) = \sum_j \varphi_j (1+\varepsilon) \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \). Since \( \sum_j \{ T^j_{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) is strongly continuous, $\varphi_j$ is continuous. The semi-group property for \( \sum_j \{ T^j_{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) implies the semi-group property for the dual sequence of operators: \( (T^j_0) = 1 \) and \( \sum_j \hat{T}^j_{2(1+\varepsilon)} = \sum_j \hat{T}^j_{(1+\varepsilon)} \hat{T}^j_{(1+\varepsilon)} \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \). Together with the equality \( \sum_j \hat{T}^j_{(1+\varepsilon)} f_j = \sum_j \varphi_j (1+\varepsilon) f_j \), it implies that $\varphi_j(0) = 1$ and $\varphi_j \sum_j (2(1+\varepsilon)) = \sum_j \varphi_j (1+\varepsilon) \varphi_j (1+\varepsilon)$ for every \( (1+\varepsilon) \in \mathbb{R}_+ \). The latter and the continuity of $\varphi_j$ means that there is $w \in \mathbb{C}$ such that $\varphi_j \sum_j (1+\varepsilon) = e^{-w(1+\varepsilon)}$ for each \( (1+\varepsilon) \in \mathbb{R}_+ \). Thus \( e^{w(1+\varepsilon)} \sum_j (\hat{t}^j_{(1+\varepsilon)}) f_j = \sum_j f_j \) for \( (1+\varepsilon) \in \mathbb{R}_+ \), as required.

**3. SUPERCYCLICITY VS. UNIVERSALITY OF AFFINE MAPS**

Relate the supercyclicity of an operators or a semi-groups in the case of the existence of an invariant hyperplane and the universality of an affine maps or an affine semi-groups. Begin with the following generic lemma.

**Lemma 3.1.** Let $X$ be a topological vector space, $u^j \in X, f_j \in X' \setminus \{0\}, f(u^j) = 1$ and $H = \ker f_j$. 

UNIVERSALITY OF AFFINE SEMI-GROUPS

Assume also that \( \sum_j \{ T^j_a \}_{a \in A_j} \) is a family of continuous linear sequence of operators on \( X \) such that \( \sum_j T^j_a f_j = \sum_j f_j \) for each \( a \in A_j \). Then the families \( \mathcal{F}^j = \{ z T^j_a : z \in \mathbb{K}, a \in A_j \} \) are universals if and only if the families \( \mathcal{G}^j = \{ R_a \}_{a \in A_j} \) of affine maps \( R_a : H \to H, \ R_a x = (T^j_a u^j - u^j) + T^j_a x \) are universals on \( H \). Moreover, \( x \in X \) is universal for \( \mathcal{F}^j \) if and only if \( x = \lambda_j(u^j + w) \), where \( \lambda_j \in \mathbb{K}\setminus\{0\} \) and \( w \) is universal for \( \mathcal{G}^j \). Next, if \( A_j = \mathbb{Z}_+ \) and \( T^j_a = (T^j_1)^a \) for every \( a \in \mathbb{Z}_+ \), then \( R_a = R^a_1 \) for every \( a \in \mathbb{Z}_+ \). Finally, if \( A_j = \mathbb{R}_+ \) and \( \sum_j \{ T^j_a \}_{a \in \mathbb{R}_+} \) is strongly continuous linear semi-group, then \( \sum_j \{ T^j_a \}_{a \in \mathbb{R}_+} \) is strongly continuous affine semi-group.

**Proof.** Since \( T^j_a(H) \subseteq H \) for every \( a \), vectors from \( H \) cannot be universal for \( \mathcal{F}^j \). Obviously, they also do not have the form \( \lambda_j(u^j + w) \) with \( \lambda_j \in \mathbb{K}\setminus\{0\} \) and \( w \in H \).

Let \( x_0 \in X \setminus H \). Then \( f_j(x_0) \neq 0 \) and therefore \( x = \frac{x_0}{f_j(x_0)} \in u^j + H \). Since \( T^j_a(u^j + H) \subseteq u^j + H \) for every \( a \in A_j \), \( O = \{ T^j_a x : a \in A_j \} \subseteq u^j + H \). It is straightforward to see that \( x_0 \) is universal for \( \mathcal{F}^j \) if and only if \( O \) is dense in \( u^j + H \). That is, \( x_0 \) is universal for \( \mathcal{F}^j \) if and only if \( x \) is universal for the families \( \{ Q_a \}_{a \in A_j} \), where each \( Q_a : u^j + H \to u^j + H \) is the restriction of \( T^j_a \) to the invariant subset \( u^j + H \). Obviously, the translation map \( \Phi : H \to u^j + H, \) \( \Phi(y) = u^j + y \) is a homeomorphism and \( R_a = \Phi^{-1} Q_a \Phi \) for every \( a \in A_j \). It follows that \( x_0 \) is universal for \( \mathcal{F}^j \) if and only if \( \Phi^{-1} x = x - u^j \) is universal for \( \mathcal{G}^j \). Denoting \( w = x - u^j \), see that the latter happens if and only if \( x_0 = f_j(x_0)(u^j + w) \) with \( w \in U(\mathcal{G}^j) \).

Since \( Q_a \) are the restrictions of \( T^j_a \) to the invariant subset \( u^j + H \) and \( R_a \) are similar to \( Q_a \) with the similarity independent on \( a \), \( \{ R_a \} \) inherits all the semi-group or continuity properties from \( \{ T^j_a \} \). The proof is complete.

**Lemma 3.2.** Let \( X \) be a topological vector space, \( u^j \in X, \ f_j \in X \setminus \{0\}, \) and \( H = \ker f_j \).
Then \( T^j \in L(X) \) satisfying \( \sum_j \tilde{T}^j f_j = \sum_j f_j \) is supercyclic if and only if the map \( R : H \to H \), \( Rx = (T^j u^j - u^j) + T^j x \) is universal. Moreover, \( x \in X \) is a supercyclic vector for \( T^j \) if and only if \( x = \lambda_j (u^j + w) \), where \( \lambda_j \in \mathbb{K}\backslash\{0\} \) and \( w \in U(R) \).

**Lemma 3.3.** Let \( X \) be a topological vector space, \( u^j \in X \), \( f_j \in X^\prime \backslash\{0\} \), \( f_j(u^j) = 1 \) and \( H = \ker f_j \). Then a strongly continuous linear semi-group \( \{R_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) on \( X \) satisfying \( \sum_j T^j_{(1+\varepsilon)} (\tilde{T}^j_{(1+\varepsilon)} f_j) = \sum_j f_j \) for \( (1+\varepsilon) \in \mathbb{R}_+ \) is supercyclic if and only if the strongly continuous affine semi-group \( \{R_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) on \( H \) defined by \( R_{(1+\varepsilon)} x = (T^j_{(1+\varepsilon)} u^j - u^j) + T^j_{(1+\varepsilon)} x \) are universals. Moreover, \( x \in X \) is a supercyclic vector for \( \sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) if and only if \( x = \lambda_j (u^j + w) \), where \( \lambda_j \in \mathbb{K}\backslash\{0\} \) and \( w \in U(\{R_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}) \).

### 4. Universal of Affine Semi-Groups

The following verification is the routine proof:

**Lemma 4.1.** Let \( X \) be a topological vector space, \( \sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a collection of continuous affine maps on \( X \), \( \sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a collection of continuous linear sequence of operators on \( X \) and \( (1+\varepsilon) \mapsto w_{(1+\varepsilon)} \) be a map from \( \mathbb{R}_+ \) to \( X \) such that \( \sum_j T^j_{(1+\varepsilon)} x = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)} x \) for every \( (1+\varepsilon) \in \mathbb{R}_+ \) and \( x \in X \). Then \( \sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) are affine semi-group if and only if \( \sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) are linear semi-groups, \( w_0 = 0 \) and

\[
  w_{2(1+\varepsilon)} = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)} w_{(1+\varepsilon)} \quad \text{for every} \quad (1+\varepsilon) \in \mathbb{R}_+. \tag{1}
\]

Moreover, the semi-group \( \sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) is strongly continuous if and only if \( \sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) are strongly continuous and the map \( (1+\varepsilon) \mapsto w_{(1+\varepsilon)} \) is continuous. Finally, the semi-group \( \sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+} \) are jointly continuous if and only if
\[ \sum_j \left\{ S_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \] are jointly continuous and the map \( (1 + \varepsilon) \mapsto w_{(1+\varepsilon)} \) is continuous.

**Lemma 4.2.** Let \( X \) be a topological vectors space and \( \sum_j \left\{ T_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) be a universal strongly continuous affine semi-group on \( X \). Then \( \sum_j \left( I - T_j^{\varepsilon} \right) (X) \) is dense in \( X \) for every \( \varepsilon > -1 \).

**Proof.** Assume the contrary. Then there is \( \varepsilon > -1 \) such that \( Y_0 \neq X \), where \( Y_0 = \sum_j \left( I - T_j^{\varepsilon} \right) (X) \). Let \( Y \) be a translation of \( Y_0 \), containing \( 0: Y = Y_0 - u_0^j \) with \( u_0^j \in Y_0 \). It is easy to see that, factoring out the closed linear subspace \( Y \), arrive to the universal strongly continuous affine semi-group \( \sum_j \left\{ F_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) on \( X/Y \), where \( F_j^{\varepsilon} (x + Y) = T_j^{\varepsilon} x + Y \) for every \((1 + \varepsilon) \in \mathbb{R}^+ \) and \( x \in X \). By definition of \( Y \), the linear part of \( F_j^{\varepsilon} \) is \( I \). Let \( \beta + \varepsilon \in X/Y \) be a universal vector for \( \sum_j \left\{ F_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \). By Lemma 4.1, there is a strongly continuous linear semi-group \( \left\{ G_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) on \( X/Y \) and a continuous map \((1 + \varepsilon) \mapsto \gamma_{(1+\varepsilon)} \) from \( \mathbb{R}^+ \) to \( X/Y \) such that \( \gamma_0 = 0 \), \( F_j^{\varepsilon} \beta = G_{(1+\varepsilon)} \beta + \gamma_{(\varepsilon+1)} \) and \( \gamma_{r+(1+\varepsilon)} = \gamma_r + G_r \gamma_{(1+\varepsilon)} = \gamma_{(1+\varepsilon)} + G_{(1+\varepsilon)} \gamma_r \) for every \( \beta \in X/Y \) and \( r, (1 + \varepsilon) \in \mathbb{R}^+ \).

Using these relations and the equality \( G_{(\varepsilon+1)} = I \), obtain that \( F_j^{\varepsilon+n(1+\varepsilon)} (\beta + \varepsilon) = F_j^{\varepsilon} (\beta + \varepsilon) + n \gamma_{(1+\varepsilon)} \) for every \( n \in \mathbb{Z}^+ \) and \((\varepsilon + 1) \in \mathbb{R}^+ \). It follows that \( \left\{ \sum_j F_j^{\varepsilon} (\beta + \varepsilon): (1 + \varepsilon) \in \mathbb{R}^+ \right\} = K + \mathbb{Z}^+ \gamma_{(1+\varepsilon)} \) where \( K = \left\{ \sum_j F_j^{\varepsilon} (\beta + \varepsilon): (1 + \varepsilon) \in [0, (1 + \varepsilon)] \right\} \).

Since \( (\beta + \varepsilon) \) is universal for \( \sum_j \left\{ F_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \), by the last display, \( O = K + \mathbb{Z}^+ \gamma_{(1+\varepsilon)} \) is dense in \( X/Y \). Since \( O \) is closed as a sum of a compact set and a closed set, \( O = X/Y \). On the other hand, \( O \) does not contain \( -(1 + \varepsilon) \gamma_{(1+\varepsilon)} \) for any sufficiently large \( \varepsilon > -1 \). This contradiction completes the proof.

**Lemma 4.3.** Let \( X \) be a topological vector space, \( x \in X \), \( \varepsilon > -1 \) and \( \sum_j \left\{ T_j^{\varepsilon} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) be a universal affine semi-group on \( X \). Assume also that \( \sum_j T_j^{\varepsilon} x = \sum_j S_j^{\varepsilon} x + w_{(1+\varepsilon)} \),
where \( \sum_j \left\{ s^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) strongly continuous linear semi-group on \( X \) and \((1+\varepsilon) \mapsto w_{(1+\varepsilon)} \) is a continuous map from \( \mathbb{R}_+ \) to \( X \). Then \( \sum_j \left\{ s^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) is hypercyclic.

Moreover, \( \mathcal{U}(\sum_j \left\{ s^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}) \cap (w_{(1+\varepsilon)} + \sum_j ( I - \sum_j s^j_{(1+\varepsilon)} ) ( X ) ) \neq \emptyset \) for every \( \varepsilon > -1 \).

**Proof.** Let \( x \in \mathcal{U}(\sum_j \left\{ t^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+}) \) and fix \( \varepsilon > -1 \). By Lemma 4.2, \( \sum_j (T^j_{(1+\varepsilon)} - I) ( X ) \) are dense in \( X \). Hence \( O = \{ \sum_j (T^j_{(1+\varepsilon)} - I) x : (1+\varepsilon) \in \mathbb{R}_+ \} \) are dense in \( X \). Using the semi-group property of \( \sum_j \left\{ t^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) and \( \sum_j \left\{ s^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) together with (1), get

\[
\sum_j (T^j_{(1+\varepsilon)} - I) x = \sum_j s^j_{(1+\varepsilon)} s^j_{(1+\varepsilon)} x + \sum_j s^j_{(1+\varepsilon)} w_{(1+\varepsilon)} + w_{(1+\varepsilon)} - \sum_j s^j_{(1+\varepsilon)} x - w_{(1+\varepsilon)}
\]

for every \((1+\varepsilon) \in \mathbb{R}_+ \). By the above display, \( O \) is exactly the St-orbit of \( w_{(1+\varepsilon)} - \sum_j ( I - s^j_{(1+\varepsilon)} ) x \). Since \( O \) is dense in \( X \), \( w_{(1+\varepsilon)} - \sum_j ( I - s^j_{(1+\varepsilon)} ) x \in w_{(1+\varepsilon)} + \sum_j ( I - s^j_{(1+\varepsilon)} ) ( X ) \) is hypercyclic vector for \( \left\{ s^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) and therefore \( \mathcal{U} \sum_j \left\{ \left\{ t^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \right\} \cap w_{(1+\varepsilon)} + \sum_j ( I - s^j_{(1+\varepsilon)} ) ( X ) \neq \emptyset \).

**Lemma 4.4.** Let \( X \) be a topological vector space and \( \sum_j \left\{ t^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}_+} \) be an affine semi-group on \( X \). Then for every \((1+\varepsilon)_1, \ldots, (1+\varepsilon)_n \in \mathbb{R}_+ \) and every \( z_1, \ldots, z_n \in \mathbb{K} \) satisfying \( z_1 + \cdots + z_n = 1 \), the map \( \sum_j s^j = z_1 \sum_j T^j_{(1+\varepsilon)_1} + \cdots + z_n \sum_j T^j_{(1+\varepsilon)_n} \) commutes with every \( \sum_j T^j_{(1+\varepsilon)} \).

**Proof.** It is easy to verify that for every affine maps \( A_j : X \to X \) and every \( x_1, \ldots, x_n \in X \),

\[
\sum_j A_j (z_1 x_1 + \cdots + z_n x_n) = z_1 \sum_j A_j x_1 + \cdots + z_n \sum_j A_j x_n
\]

provided \( z_j \in \mathbb{K} \) and \( z_1 + \cdots + z_n = 1 \).
Let \( (1 + \varepsilon) \in \mathbb{R}_+ \). By the above display,
\[
\sum_j T_j^{(1+\varepsilon)} S^j = z_1 \sum_j T_j^{(1+\varepsilon)} T_j^{(1+\varepsilon)} x + \cdots + z_n \sum_j T_j^{(1+\varepsilon)} T_j^{(1+\varepsilon)} n.
\]
Since \( \sum_j T_j^{(1+\varepsilon)} \) commute with each other, get
\[
\sum_j T_j^{(1+\varepsilon)} S^j = z_1 \sum_j T_j^{(1+\varepsilon)} T_j^{(1+\varepsilon)} x + \cdots + z_n \sum_j T_j^{(1+\varepsilon)} T_j^{(1+\varepsilon)} n = \sum_j S_j T_j^{(1+\varepsilon)}.
\]

**Lemma 4.5.** Let \( X \) be a topological vector space, \( \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) be universals strongly continuous affine semi-group on \( X \) and \( x \in \mathcal{U} \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \). Then \( \Lambda(x) \subseteq \mathcal{U} \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \), where
\[
\Lambda(x) = \{ z_1 \sum_j T_j^{(1+\varepsilon)} x + \cdots + z_n \sum_j T_j^{(1+\varepsilon)} n \mid x \in \mathbb{N}, (1 + \varepsilon)_j \in \mathbb{R}_+, z_j \in \mathbb{K}, z_1 + \cdots + z_n = 1 \}.
\]

**Proof.** Let \( n \in \mathbb{N}, (1 + \varepsilon)_1, \ldots, (1 + \varepsilon)_n \in \mathbb{R}_+, z_1, \ldots, z_n \in \mathbb{K} \) and \( z_1 + \cdots + z_n = 1 \). Have to show that \( x \in \mathcal{U} \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \), where \( A_j = z_1 \sum_j T_j^{(1+\varepsilon)} + \cdots + z_n \sum_j T_j^{(1+\varepsilon)} \). By Lemma 4.4, \( A \) commutes with all \( T_j^{(1+\varepsilon)} \). Since \( x \in \mathcal{U} \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \), it suffices to verify that \( A_j(X) \) are dense in \( X \). By Lemma 4.1, write \( \sum_j T_j^{(1+\varepsilon)} y = \sum_j S_j^{(1+\varepsilon)} y + w^{(1+\varepsilon)} \) for every \( y \in X \), where \( \sum_j \{ S_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) are strongly continuous linear semi-group on \( X \) and \( (1 + \varepsilon) \mapsto w^{(1+\varepsilon)} \) is a continuous map from \( \mathbb{R}_+ \) to \( X \). By Lemma 4.3, \( \sum_j \{ S_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) are hypercyclic, every non-trivial linear combination of members of a hypercyclic strongly continuous linear semi-group has dense range. Thus \( B_j = z_1 \sum_j S_j^{(1+\varepsilon)} + \cdots + z_n \sum_j S_j^{(1+\varepsilon)} n \) has dense range. Since \( A_j(X) \) are translation of \( B_j(X) \), \( A_j(X) \) is also dense in \( X \), which completes the proof.

**Proof of Theorem 1.3.** Let \( X \) be a topological vectors space and \( \sum_j \{ T_j^{(1+\varepsilon)} \}_{(1+\varepsilon) \in \mathbb{R}_+} \) be a universal jointly continuous affine semi-group on \( X \). Lemmas 4.1 and 4.3 provide a hypercyclic
jointly continuous linear semi-groups on $X$. By Theorem A, a hypercyclic continuous linear operator is exist on $X$. Since no such thing exists on a finite dimensional topological vectors space [7], $X$ is infinite dimensional.

Condition (1) of Proposition 1.1 is satisfied since any compact subspaces of an infinite dimensional topological vector spaces are nowhere dense [4]. Let $x \in \mathcal{U} \sum \{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$. By Lemma 4.5, the set $A(x)$ defined in (4.2) entirely of universal vectors for $\sum_j \{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$. Clearly, $\{\sum_j T_j^{(1+\varepsilon)} x : (1 + \varepsilon) \in \mathbb{R}_+ \} \subseteq A(x)$. By its definition, $A(x)$ is an affine subspace of $X$. $A(x)$ satisfies all requirements for the set $\mathcal{Y}^{(1+\varepsilon)}x$ (for every $\varepsilon > -1$) according to condition (2) in Proposition 1.1 and every affine subspace of a topological vectors space is connected and simply connected, By Proposition 1.1, $\mathcal{U} \sum_j (T_j^{(1+\varepsilon)}) = \mathcal{U} \sum_j (\{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+})$ for every $\varepsilon > -1$, as required.

5. PROOF OF THEOREM 1.2

Let $X$ be a complex topological vector space and $\sum_j \{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on $X$ in [8]. Have to prove that all $\sum_j T_j^{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic and the sets of supercyclic vectors of $T_j^{(1+\varepsilon)}$ coincides with the set of supercyclic vectors of $\sum_j \{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$. If $T_j^{(1+\varepsilon)} - \lambda_j I$ has dense range for every $\varepsilon > -1$ and every $\lambda_j \in \mathbb{C}$, then Proposition C provides the required result. Otherwise, by Proposition 2.1, there is a closed hyperplane $H$ in $X$ invariant for all $T_j^{(1+\varepsilon)}$. By Lemma 2.6, there are $f_j \in X$ and $(\beta + \varepsilon) \in \mathbb{C}$ such that $H = \ker f_j$ and $\sum_j e^{(1+\varepsilon)(\beta+\varepsilon)}(T_j^{(1+\varepsilon)})f_j = f_j$ for every $(1 + \varepsilon) \in \mathbb{R}_+$. Clearly $\{e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ is a jointly continuous supercyclic linear semi-group on $X$ with the same sets $S_j$ of supercyclic vectors as the original semi-group $\sum_j \{T_j^{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$. Fix $u^j \in X$ satisfying $f_j(u^j) = 1$. Now fix $\varepsilon > -1$ and $v^j \in S^j$. Have
to show that \( v^j \) is supercyclic for \( \sum_j T^j_{(1+\varepsilon)} \). By Lemma 3.3, applied to the semi-group \( \left\{ e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \), write \( v^j = \lambda_j (u^j + y) \), where \( \lambda_j \in \mathbb{K} \setminus \{0\} \) and \( y \) is a universal vector for the jointly continuous affine semi-group \( \left\{ R_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) on \( H \) defined by the formula \( R_{(1+\varepsilon)} x = w_{(1+\varepsilon)} + e^{(1+\varepsilon)(\beta+\varepsilon)} T^j_{(1+\varepsilon)} x \) with \( w_{(1+\varepsilon)} = (e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j T^j_{(1+\varepsilon)} - I) u^j \). By Theorem 1.3, \( y \) is universal for \( R_{(1+\varepsilon)} \). So, \( v^j = \lambda_j (u^j + y) \) is a supercyclic for \( e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j T^j_{(1+\varepsilon)} \) (by Lemma 3.2) and therefore \( v^j \) is a supercyclic vector for \( T^j_{(1+\varepsilon)} \). The proof is complete.

6. Remarks

The following example shows that the hypercyclicity of the underlying linear semi-group is not implies universality of a strongly continuous affine semi-group see e.g.[4].

Example 6.1. Consider the backward weighted shift \( T^j \in L(\ell_2) \) with the weight sequence \( \{e^{-2n}\}_{n \in \mathbb{N}} \). That is, \( T^j e_0 = 0 \) and \( T^j e_n = e^{-2n} e_{n-1} \) for \( n \in \mathbb{N} \), where \( \{e^n\}_{n \in \mathbb{Z}^+} \) is the standard basis of \( \ell_2 \). Then the jointly continuous linear semi-groups \( \sum_j \left\{ e^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) with \( \sum_j e^j_{(1+\varepsilon)} = e^{(1+\varepsilon) \ln \sum_j (1+T^j)} \) are hypercyclic. Moreover, there exists a continuous map \( (1+\varepsilon) \mapsto w_{(1+\varepsilon)} \) from \( \mathbb{R}_+ \) to \( \ell_2 \) such that \( \sum_j \left\{ T^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) are jointly continuous non-universal affine semi-group, where \( \sum_j T^j_{(1+\varepsilon)} x = w_{(1+\varepsilon)} + \sum_j e^j_{(1+\varepsilon)} x \) for \( x \in \ell_2 \).

Proof. Since \( T^j \), being compacts weighted backward shift, is quasinilpotent, the sequence of operators \( \ln(I + T^j) \) are well defined and bounded and \( \sum_j \left\{ e^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) are jointly continuous linear semi-group. Moreover, \( S^j_1 = I + T^j \) are hypercyclic [9] as a sum of the identity sequence of operators and a backward weighted shift. Hence \( \sum_j \left\{ e^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) are hypercyclic.
Let $u^j \in \ell_2$, $u^j_n = (n+1)^{-1}$ for $n \in \mathbb{Z}_+$. For each $(1+\varepsilon) \in \mathbb{R}_+$, let $w_{(1+\varepsilon)} = v^j_{(1+\varepsilon)}(T^j)u^j$, where $v^j_{(1+\varepsilon)}(z) = \sum_{n=1}^{\infty} \frac{(1+\varepsilon)^{n+2}}{n!} z^n$. Since $T^j$ are quasinilpotents, $v^j_{(1+\varepsilon)}(T^j)$ are well defined bounded linear sequence of operators and the map $(1+\varepsilon) \mapsto v^j_{(1+\varepsilon)}(T^j)$ are sequence of operators-norm continuous. Hence $(1+\varepsilon) \mapsto w_{(1+\varepsilon)}$ is continuous as a map from $\mathbb{R}_+$ to $\ell_2$. It is easy to verify that $w_0 = 0$, $w_1 = u^j$ and $w_{2(1+\varepsilon)} = S^j_{(1+\varepsilon)}w_{(1+\varepsilon)} + w_{(1+\varepsilon)}$ for every $\varepsilon \geq -1$. By Lemma 4.1, $\sum_1 \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ is a jointly continuous affine semi-group, where $T^j_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + S^j_{(1+\varepsilon)}x$. It remains to show that $\sum_1 \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon) \in \mathbb{R}_+}$ is non-universal. Assume the contrary. Since $w_1 = u^j$ and $S^j_1 = I + T^j$, Lemma 4.3 implies that the coset $\sum_1 (u^j + T^j(\ell_2))$ (must contain a hypercyclic vector for $I + T^j$. This however is not the case as shown in [9].

Recall that a topological space $X$ is called a Baire space if the intersection of any countable collection of dense open subsets of $X$ is dense in $X$.

**Remark 6.2.** Let $X$ be a topological vector space and $S^j \in L(X)$ be hypercyclic. If $u^j \in (I - S^j)(X)$, then the affine map $\sum_1 T^j x = \sum_1 (u^j + S^j)x$ is universal. Indeed, let $w \in X$ be such that $u^j = w - S^j w$. It is easy to show that $\sum_1 (T^j)^n x = w + \sum_1 (S^j)^n (x - w)$ for every $x \in X$ and $n \in \mathbb{N}$. Thus $x$ is universal for $T^j$ if and only if $x - w$ is universal for $S^j$.

Since $X$ is separable metrizable and Baire, so a standard Baire category type argument shows that the set of $u^j \in X$ is a dense $G_\delta$-subse for which the affine map $\sum_1 T^j x = \sum_1 (u^j + S^j x)$ is universal t. Example 6.1 shows that this set can differ from $X$.Recall that $X$ is locally convex topological vectors space and barrelled then every closed convex balanced subsets $B_j$ of $X$ satisfying $X = \bigcup_{n=1}^{\infty} n(B_j)$ contain a neighborhood of 0. As have mentioned in the introduction, the joint continuity of a linear semi-group follows from the strong continuity if the underlying space $X$ is an $\mathcal{T}^j$-space. The same is true for wider classes of topological
vector spaces. For instance, it is sufficient for \( X \) to be a Baire topological vector space or a barreled locally convex topological vector space. Thus the following observation holds true.

**Remark 6.3.** The joint continuity condition in Theorems A, 1.2 and 1.3 can be substituted by the strong continuity, on condition that \( X \) is Baire or barrelled and locally convex.

For general topological vectors space however strong continuity of a linear semi-group does not imply joint continuity. Example 6.4 explains that if the joint continuity condition is changed by the strong continuity then the theorem A fails. Recall that the Fréchet space \( L^2_{\text{loc}}(\mathbb{R}^+) \) consists of the scalar valued functions \( \mathbb{R}^+ \), square integrable on \([0, (1 + \varepsilon)]\) for each \( \varepsilon > -1 \). The dual space \( L^2_{\text{loc}}(\mathbb{R}^+) \) can be naturally interpreted as the space \( L^2_{\text{fin}}(\mathbb{R}^+) \) of square integrable scalar valued functions \( \mathbb{R}^+ \) with bounded support. The duality between \( L^2_{\text{loc}}(\mathbb{R}^+) \) and \( L^2_{\text{fin}}(\mathbb{R}^+) \) is provided by the natural dual pairing \( \sum_j \langle f_j, g_j \rangle = \int_0^\infty \sum_j f_j(t)g_j(t)dt \).

Obviously the linear semi-group \( \sum_j \left\{ S^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) of backward shifts \( \sum_j S^j_{(1+\varepsilon)}f_j(x) = \sum_j f_j(x + (1 + \varepsilon)) \) is strongly continuous and therefore jointly continuous on the Fréchet space \( L^2_{\text{loc}}(\mathbb{R}^+) \). It follows that the same semi-group is strongly continuous on \( L^2_{\sigma,\text{loc}}(\mathbb{R}^+) \) being \( L^2_{\text{loc}}(\mathbb{R}^+) \) endowed with the weak topology.

**Example 6.4.** Let \( X = L^2_{\sigma,\text{loc}}(\mathbb{R}^+) \) and \( \sum_j \left\{ S^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) be the above strongly continuous semi-group on \( X \). Then there are \( f_j \in X \) hypercyclics for \( \sum_j \left\{ S^j_{(1+\varepsilon)} \right\}_{(1+\varepsilon) \in \mathbb{R}^+} \) such that \( f_j \) are non-hypercyclic for \( S^j_1 \).

**Proof.** Let \( H \) be the hyperplane in \( L^2[0,1] \) consisting of the functions with zero Lebesgue integral. Fix a norm-dense countable subsets \( A_j \) of \( H \). One can easily construct \( f_j \in L^2_{\text{loc}}(\mathbb{R}^+) \) such that

(a) for every \( n \in \mathbb{N} \), the function \( (f_j)_n : [0,1] \to K, (f_j)_n(1 + \varepsilon) = f_j(n + (1 + \varepsilon)) \) belongs to \( A_j \);

(b) for every \( n \in \mathbb{N} \) and \( h_1, \ldots, h_n \in A_j \), there is \( m \in \mathbb{N} \) such that \( h_j = (f_j)_{m+j} \) for \( 1 \leq j \leq \)}
For \((1 + \varepsilon) \in \mathbb{R}^+\), let \(\chi_{(1+\varepsilon)} \in X' = L^2_{\text{fin}}(\mathbb{R}^+)\) be the indicator function of the interval 

\([ (1 + \varepsilon), (2 + \varepsilon) ]: \chi_{(1+\varepsilon)}(1 + \varepsilon) = 1\) if \((1 + \varepsilon) \leq (1 + \varepsilon)^2 + 1\) and \(\chi_{(1+\varepsilon)}((1 + \varepsilon)) = 0\) otherwise. By (a), \((S^j_1)^n f_j \in \ker \chi_0\) for every \(n \in \mathbb{N}\) and therefore \(f_j\) are not hypercyclic vector for \(S^j_1\). It remains to show that \(f_j\) are hypercyclic vector for \(\sum_j \{S^j_1\}^{(1+\varepsilon)}\) acting on \(X\). Using (a) and (b), we see that the Fréchet space topology closure of the orbits \(\{S^j_1 f_j: (1 + \varepsilon) \in \mathbb{R}^+\}\) is exactly the sets

\[ O = \bigcup_{(1+\varepsilon)\in[0,1]} \bigcap_{n \in \mathbb{Z}^+} \ker \chi_{(1+\varepsilon)+n}. \]

In order to show that \(f_j\) are hypercyclic for \(\sum_j \{S^j_1\}^{(1+\varepsilon)}\) acting on \(X\), it suffices to verify that \(O\) is dense in \(L^2_{\text{loc}}(\mathbb{R}^+)\). Assume the contrary. Then there exists a weakly open sets \(W^j\) in \(L^2_{\text{loc}}(\mathbb{R}^+)\), such that \(W^j \cap O = \emptyset\). That means, there are no linearly dependent \((\varphi_j)_1, \ldots, (\varphi_j)_m \in L^2_{\text{fin}}(\mathbb{R}^+)\) and \((1 + \varepsilon)_1, \ldots, (1 + \varepsilon)_m \in \mathbb{R}\) such that

\[ \max_{1 \leq j \leq m} \left| \sum_j (1 + \varepsilon)_j - \langle g, \varphi_j \rangle \right| \geq 1 \quad \text{for all } g^j \in O. \]

Let \(k \in \mathbb{N}\) be such that all \(\varphi_j\) vanishes on \([k, \infty)\). Pick any \(0 < (1 + \varepsilon)_0 < \cdots < (1 + \varepsilon)_m < 1\). Note that for every \(l \in \{0, \ldots, m\}\), the restrictions of the functionals \(\varphi_j\) to \(\cap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}\) are not linearly independent. Indeed, otherwise can find \(h_0 \in \cap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}\) such that \(\langle h_0, \varphi_j \rangle = (1 + \varepsilon)_j\) for \(1 \leq j \leq m\). It is easy to see that there is \(h \in L^2_{\text{loc}}(\mathbb{R}^+)\) such that \(h|_{[0,k]} = h_0|_{[0,k]}\), \(h|_{[k+1, \infty]}\) and \(\langle h, \chi_{(1+\varepsilon)_l+k-1} \rangle = \langle h, \chi_{(1+\varepsilon)_l+k} \rangle = 0\). Then \(\langle h, \varphi_j \rangle = (1 + \varepsilon)_j\) for \(1 \leq j \leq m\) and \(h \in \cap_{n=0}^\infty \ker \chi_{(1+\varepsilon)_l+n} \subseteq O\). We have arrived to a contradiction with the above display.

The fact that \(\varphi_j\) are not linearly independents on \(\cap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}\) implies that there is a
non-zero \((g_j)_l \in \text{span}\{(\varphi_j)_1, \ldots, (\varphi_j)_m\} \cap \text{span}\{\chi_{(1+\varepsilon)t}, \ldots, \chi_{(1+\varepsilon)t+k}\}\). Since \(\chi_{(1+\varepsilon)t+r}\) are all linearly independent, \((g_j)_0, \ldots, (g_j)_m\) are \(m + 1\) linearly independent vectors in the \(m\)-dimensional space \(\text{span}\{(\varphi_j)_1, \ldots, (\varphi_j)_m\}\). Completes the proof.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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