THE QUIVERS OF THE HEREDITARY ALGEBRAS OF TYPE $\tilde{A}_n$

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Abstract. This is a translation of the diploma thesis of Thomas Brüstle,
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A quiver of type $\tilde{A_n}$ has the following form

\[
\begin{array}{c}
0 \\
1 \\
\vdots \\
n - 1 \\
n \\
\end{array}
\quad \text{with arbitrary orientation of the arrows.}
\]

The classification of (finite-dimensional) representations of quivers of this type is known. The goal of this thesis is, to describe the indecomposable representations and their morphisms by giving a quiver with relations. For this, we will restrict to the case of the quiver $K$ of the following form.

\[
K:
\]

for some $1 \leq g \leq n$.

$g$ is the number of arrow in clockwise orientation,

$h := n + 1 - g$ is the number of anti clockwise arrows.

The running example throughout the thesis will be the case $g = 3, n = 4$:

\[
K:
\]

1. Classification of the representations

We fix an algebraically closed field $k$ (commutative).

A representation $V$ of $K$ is a collection of $k$-vector spaces $V(x)$ for $x$ a vertex of $K$ and of $k$-linear maps $V(\gamma): V(x) \to V(y)$ for every arrow $\gamma: x \to y$ in $K$.

A morphism $\mu: V \to W$ between two representations $V$ and $W$ of $K$ is a collection of $k$-linear maps $\mu(x): V(x) \to W(x)$ for $0 \leq x \leq n$ such that we have $W(\gamma)\mu(x) = \mu(y)V(\gamma)$.
for every arrow $\gamma : x \to y$ in $K$. Composition of morphisms is defined as composition of the linear maps in every vertex.

The finite dimensional representations of $K$ form a $k$-category, we write $\text{rep} \, K$ for it. For $V, W \in \text{rep} \, K$ let $\text{Hom}(V, W)$ be the space of morphisms between $V$ and $W$.

To classify the representations we need another notion:

A *(clockwise) walk* in $K$ is a pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ with $p \leq q$. To any walk $w$ we associate a representation $V_w$ of $K$ as follows: Assume first $0 \leq p \leq n$ and set $i = i + (n + 1) \mathbb{Z}$ in $\mathbb{Z}/(n + 1) \mathbb{Z}$, let $E_w$ be a vector space with basis $e_{p}, e_{p+1}, \ldots, e_{q}$. Then we define

$$V_w(x) := \bigoplus_{i=x} e_i \subset E_w \quad \text{for } 0 \leq x \leq n$$

and

$$V_w(\beta_x) := \bigoplus_{i-x+1} e_i \to \bigoplus_{j-x} e_j \quad \text{for } 0 \leq x \leq g - 1$$

by setting

$$V_w(\beta_x)(e_i) = \begin{cases} e_{i+1} & \text{for } i < q \\ 0 & \text{if } i = q \end{cases}$$

and

$$V_w(\alpha_x) := \bigoplus_{i=x+1} e_i \to \bigoplus_{j=x} e_j \quad \text{for } g \leq x \leq n$$

through

$$V_w(\alpha_x)(e_i) = \begin{cases} e_{i-1} & \text{for } i > p \\ 0 & \text{if } i = p \end{cases}$$

For arbitrary $p \in \mathbb{Z}$ we set $V_{(p,q)} := V_{(p',q')}$ for $p' \in \{0, \ldots, n\}$ with $p' \equiv p \mod n + 1$ and $q' = q + p' - p$.

**Example.** (Recall that $g = 3, n = 4$). For $w = (4,10)$, $V_w$ looks as follows:

In addition to these, we consider representations $V_{d}^\lambda$ for any $\lambda \in k \cup \{\infty\}$, for any $d \in \mathbb{N} \setminus \{0\}$, given as follows:

$$V_{d}^\lambda(x) = k^d, \quad 0 \leq x \leq n, \quad V_{d}^\lambda(\gamma) = \begin{cases} \lambda \text{id}_d + J_d & \text{if } \gamma = \beta_{g-1} \\ \text{id}_d & \text{if } \gamma \neq \beta_{g-1} \end{cases}$$

\(^1\text{Wanderweg in the original.}\)
where $J_d$ is the following $d \times d$-matrix: $J_d = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \in k^{d \times d}$.

Then we have

**Proposition.** The representations $V_{(p,q)}$ with $0 \leq p \leq n$, $q \geq p$ and the $V^\lambda_d$ for $\lambda \in k \cup \{0\}$, $d \in \mathbb{N} \setminus \{0\}$ form a complete and irredundant list of indecomposables.

For a proof we refer to [1].

We introduce some notation. If $p \equiv 1, \ldots, g \mod n+1$ and $q \equiv g, g+1, \ldots, n \mod n+1$ then we call $w = (p, q)$ and $V^w$ post-projective. $P$ denotes the category whose objects are the post-projectives $V_{(p,q)}$.

If $p \equiv 0, g+1, g+2, \ldots, n \mod n+1$ and $q \equiv 0, 1, \ldots, g-1 \mod n+1$ we call $w = (p, q)$ pre-injective. The category whose objects are the pre-injectives $V_w$ is denoted by $I$.

The remaining representations from the list above are called regular. We write

- $R^0$ for the category with the objects $V_{(p,q)}$ with $p \equiv 1, \ldots, g \mod n+1$
- $R^\infty$ for the category with the objects $V_{(p,q)}$ with $q \equiv 0, g+1, \ldots, n \mod n+1$
- $R^\lambda$ for the category with the objects $V^\lambda_d$ for $\lambda \in k \cup \{0\}$, $d \in \mathbb{N} \setminus \{0\}$

## 2. The post-projectives

We start the study of the post-projective objects by “rolling up” the quiver $K$ (reversing arrows) to a quiver $\tilde{K}$:

Let $\tilde{K}$ be the quiver whose vertices are $\mathbb{Z}$, with arrows $i \xrightarrow{\alpha_i} i+1$ for all $i \in \mathbb{Z}$ with $i \equiv g, \ldots, n \mod n+1$ and arrows $i+1 \xrightarrow{\beta_i} i$ for all $i \in \mathbb{Z}$ with $i \equiv 0, 1, \ldots, g-1 \mod n+1$. $\tilde{K}$ determines a quiver $\mathbb{N} \tilde{K}$ whose vertices are the pairs $(r, s) \in \mathbb{N} \times \mathbb{Z}$ and whose arrows are given by $(r, \gamma) : (r, i) \to (r, j)$ as well as $(r, \gamma') : (r, j) \to (r+1, i)$ for any arrow $\gamma : i \to j$ of $\tilde{K}$ and any $r \in \mathbb{N}$. A picture for the quiver $\mathbb{N} \tilde{K}$ for the running example is in Figure [1]

We now define a functor $F : k \mathbb{N} \tilde{K} \to P$ (for any quiver $Q$ we use $kQ$ to denote the $k$-category of the paths of $Q$). We first give a map $F$ on the objects:

To any vertex $(r, s) \in \mathbb{N} \tilde{K}$ there is exactly one path of maximal length in $\mathbb{N} \tilde{K}$ ending at $(r, s)$ that is a composition of arrows $\gamma : (l, i) \to (l', i+1)$. The starting point $(0, p)$ of this path determines $p \in \mathbb{Z}$ with $p \equiv 1, \ldots, g \mod n+1$.

Analoguously, the path of maximal length ending at $(r, s)$ that is a composition of arrows $\gamma : (l, i) \to (l', i-1)$ starts at a point $(0, q)$ for some $q \in \mathbb{Z}$ with $q \equiv q, \ldots, n \mod n+1$. Using this, we set

$$F(r, s) := V_{(p,q)} \in P.$$

Facts: This map is surjective and the fibre of $(r, s)$ is the set of vertices $(r, s')$ with $s' \equiv s \mod n+1$
The functor $F$ is determined uniquely, if we associate to any arrow $\gamma : X \to Y$ in $\mathbb{N} \tilde{K}$ a morphism $F\gamma : FX \to FY$. For this, we consider for any vertex $(r, s)$ the arrows $\gamma_1 : (r, s) \to (r_1, s + 1)$ and $\gamma_2 : (r, s) \to (r_2, s - 1)$. If $F(r, s) = V(p,q)$ we have

$$F(r_1, s + 1) = \begin{cases} V(p,q+1) & \text{for } q \equiv g + 1, \ldots, n - 1 \mod n + 1 \\ V(p,q+1+g) & \text{for } q \equiv n \mod n + 1 \end{cases}$$

$$F(r_2, s - 1) = \begin{cases} V(p-1,q) & \text{for } p \equiv 2, \ldots, g \mod n + 1 \\ V(p-1-h,q) & \text{for } p \equiv 1 \mod n + 1 \end{cases}$$

Then we can define the morphisms $F\gamma_1$ and $F\gamma_2$:

$F\gamma_1 : F(r, s) \to F(r_1, s + 1)$ is the morphism sending $e_t$ to $e_t$ for $q \leq t \leq q$;

$F\gamma_2 : F(r, s) \to F(r_2, s - 1)$ is the morphism with $e_t \mapsto e_t$ for $p \leq t \leq q$ with $p \equiv 2, \ldots, g \mod n + 1$, if $p \equiv 1 \mod n + 1$, it sends $e_t$ to $e_{t+n+1}$.

Example. $F(1, 1) \xrightarrow{F(1, 1)} F(1, 2) \xleftarrow{F(0, 1)} F(0, 1)$.
For all arrows $\gamma, \gamma', \delta, \delta'$ of $\mathbb{N}\tilde{K}$ of the form

$\begin{align*}
&\gamma : Y_1 
\begin{array}{c}
\downarrow \\
\ast
\end{array}
\gamma' : Y_2, \\
X 
\begin{array}{c}
\downarrow \\
\ast
\end{array}
Z
\end{align*}$

we have $F\gamma' \circ F\gamma = F\delta' \circ F\delta$.

We will sometimes call this a \textbf{diamond} starting at $X$ and ending at $Z$, with arrows $X \xrightarrow{\gamma} Y_1 \xrightarrow{\gamma'} Y_2, X \xrightarrow{\delta} Z$.

Let $J$ be the ideal of $k\mathbb{N}\tilde{K}$ generated by the elements $\gamma'\gamma - \delta'\delta$ (whenever the four arrows $\gamma, \gamma', \delta, \delta'$ are in a diamond as above). Then $J$ is annihilated by $F$, hence $F$ induces a functor $\tilde{F} : k(\mathbb{N}\tilde{K}) \to P$ where $k(\mathbb{N}\tilde{K}) := k\mathbb{N}\tilde{K}/J$.

\textbf{Proposition.} For any post-projective $V_{(p,q)} \in P$ and any vertex $Y$ of $\mathbb{N}\tilde{K}$ we have:

$\begin{align*}
\Theta_{F_X = V_{(p,q)}} \text{Hom}(X,Y) &\xrightarrow{\sim} \text{Hom}(V_{(p,q)} , \overline{FY}) \\
(\mu_X)_{F_X = V_{(p,q)}} &\xrightarrow{\sim} \sum_{F_X = V_{(p,q)}} \overline{F}\mu_X
\end{align*}$

The fibres of $F$ define an equivalence relation $\sim$ on $\mathbb{N}\tilde{K}_v$ and on $\mathbb{N}\tilde{K}_a$ (where we use $Q_v$ to denote the vertices of a quiver $Q$ and $Q_a$ to denote the arrows of $Q$).

We then let $Q^P$ be the quiver with $Q^P_v := \mathbb{N}\tilde{K}_v/\sim$ and $Q^P_a := \mathbb{N}\tilde{K}_a/\sim$, denoting the equivalence class of $(r,s) \in \mathbb{N}\tilde{K}_v$ by $(r,s)_P$ and the equivalence class of $\gamma \in \mathbb{N}\tilde{K}_a$ by $\gamma_P$.

\footnote{This definition is not in the diploma thesis. It is introduced to shorten the wording at times. A diamond will consist of four objects and four arrows, with an object $X$ at the start and an object $Z$ at the end.}
We can visualize $Q^P$ as a tube. In our running example: $Q^P$ (page 8 of the thesis: note that in the thesis, the subscripts $P$ are not all there).

It is more convenient to cut open $Q^P$ along the line through $(0,3)_P$ and $(1,3)_P$:

\[ (0,0)_P, (0,1)_P, (0,2)_P, (0,3)_P \]
\[ (1,0)_P, (1,1)_P, (1,2)_P, (1,3)_P \]
\[ (2,3)_P, \ldots \]

\[ (0,3)_P, (0,4)_P, (1,3)_P, (1,4)_P, (2,3)_P, \ldots \]

\[ (0,0)_P, (0,1)_P, (0,2)_P, (0,3)_P \]
\[ (1,0)_P, (1,1)_P, (1,2)_P, (1,3)_P \]
\[ (2,3)_P, \ldots \]

**Figure 2.** $\tilde{K}$ for the running example, $g = 3, n = 4$

The functor $F : k \mathbb{N} \tilde{K} \to P$ determines, in a natural way, a functor $F_P : kQ^P \to P$ such that the map on the objectives is bijective.

In the calculations later we will restrict to certain subcategories $P_m$ of $P$. For this, we define for any $m \in \mathbb{N}$ the full subquiver $Q^P_m$ of $Q^P$ on the vertices $(r,s)_P$ with $0 \leq s \leq n$ and $0 \leq r \leq ghm$.

**Remark.** If we study $F_P : Q^P \to P$ more carefully, we can see that

\[ F_P(r,0)_P = V_{(g, \text{MOD}(r,g) + (n+1)[\text{DIV}(r,g) + \text{DIV}(r,h)+1] + \text{MOD}(r,h))} \]

where MOD and DIV are the functions associating to any pair $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ the uniquely defined numbers $\text{DIV}(p,q)$ and $\text{MOD}(p,q)$ with

\[ p = \text{DIV}(p,q) \cdot q + \text{MOD}(p,q) \quad \text{and} \quad 0 \leq \text{MOD}(p,q) < q \]

This expression becomes simpler if we restrict to numbers $r = ghm$ for $m \in \mathbb{N}$:

\[ F_P(ghm,0)_P = V_{(g, (n+1)(hm+gm+1))} = V_{(g, (n+1)(m(n+1)+1)+g)} \]

We then define $P_m$ as the category whose objects are the $F_pX$ for all vertices $X$ of $Q^P_m$. To any $V_{(p,q)} \in P$ there is $m \in \mathbb{N}$ such that $V_{(p,q)} \in P_m$.

To make the description more transparent, we now assume $n \geq 2$. The Proposition above leads to the following:
Korollar 2.1. Let $k(Q^P_m)$ be the $k$-category defined by the quiver $Q^P_m$ with the relations $\gamma'\gamma = \delta'\delta$ for all arrows $\gamma, \gamma', \delta, \delta'$ of $Q^P_m$ forming a diamond as above. Then $F_P$ induces an isomorphism

$$F_P : k(Q^P_m) \to P_m$$

3. The pre-injectives

The following construction allows us to reduce the description of the pre-injective objects to the post-projective ones: We denote by $Q_I^m$ the quiver with a vertex $(r, s)_I$ for any vertex $(r, s)_P$ of $Q^P_m$ and an arrow $\gamma_I : (r_2, s_2)_I \to (r_1, s_1)_I$ for any arrow $\gamma_P : (r_1, s_1)_P \to (r_2, s_2)_P$ of $Q^P_m$.

We view the transition from $Q^P_m$ to $Q_I^m$ as a reflection $S$: we set $S(r, s)_P = (r, s)_I$ and $S(r, s)_I = (r, s)_P$ (for all vertices of $Q^P_m$ resp. of $Q_I^m$).

To get a functor $F_I : kQ^I_m \to I$, we use an equivalent construction on $\text{rep} K$: Let $K^T$ be the quiver with vertices $K^T_v = K_v$ and arrows $\gamma^T : y \to x$ for every arrow $\gamma : x \to y$ in $K$. For every $V \in \text{rep} K$ we then define $V^T \in \text{rep} K^T$ using the dual space $V(x)^T$ in every vertex $x$ and the dual maps $V(\gamma)^T : V(y)^T \to V(x)^T$ for every arrow $\gamma^T : y \to x$ of $K^T$.

By setting $\mu^T = (\mu(0)^T, \ldots, \mu(n)^T) \in \text{Hom}(W^T, V^T)$ for any $\mu \in \text{Hom}(V, W)$ we get a functor $?^T : \text{rep} K \to \text{rep} K^T$ with $\text{Hom}(V, W) \to \text{Hom}(W^T, V^T)$ for all $V, W \in \text{rep} K$.

By tilting ("kippen") $K^T$ we can associate to any $V^T$ a representation $\overline{V} \in \text{rep} K$; in the example:

$$V^T \mapsto \overline{V}$$
In general, we denote by \( G \) the permutation of \( \{0, \ldots, n\} \) with \( Gx = gx \) for \( x = 0, \ldots, g \) and \( Gx = n + g + 1 - x \) for \( x = g + 1, \ldots, n \). \( G \) induces a bijection between the arrows of \( K^T \) and of \( K \), sending \( \gamma^T : y \to x \) to the arrow \( G\gamma : Gy \to Gx \) of \( K \).

We can then describe a functor \( G : \text{rep} K^T \to \text{rep} K \) as follows: For \( U \in \text{rep} K^T \) let \( GU \in \text{rep} K \) be defined by

\[
GU(x) = U(Gx) \quad x = 0, \ldots, n \quad GU(G\gamma) = U(\gamma^T) \quad \text{for any arrow } \gamma^T \text{ of } K^T;
\]

\( \nu \) to a morphism

\[
\nu = (\nu(0), \ldots, \nu(0)) \in \text{Hom}(U, U') \quad \text{we assign}
\]

\( GV = (\nu(G0), \ldots, \nu(Gn)) \in \text{Hom}(GU, GU') \).

The composition \( G : \text{rep} K^T \to \text{rep} K \) is then a functor \( \text{rep} K \to \text{rep} K \) with the following property:

For any \( V, W \in \text{rep} K \), the map \( \text{Hom}(V, W) \to \text{Hom}(W, V), \mu \mapsto \mu^T \), is an isomorphism.

We extend the permutation \( G \) to a function \( G : \mathbb{Z} \to \{0, \ldots, n\} \) by setting, for all \( z \in \mathbb{Z} \), \( Gz := G(\text{MOD}(z, n + 1)) \).

**Proposition.** Let \( p, q \in \mathbb{Z} \) with \( p \leq q \), let \( p', q' \in \mathbb{Z} \) with \( p' \equiv Gp \mod n + 1, q' \equiv Gq \mod n + 1 \) and \( q' - p' = q - p \). Then there is an isomorphism

\[
\varphi : \overline{V}_{(p,q)} \to \overline{V}_{(p',q')}
\]

**Proof.** The definition of \( V_{(p,q)} \) is given with the help of a vector space \( E_{(p,q)} \) with basis \( e_p, \ldots, e_q \). Let \( \hat{e}_p, \ldots, \hat{e}_q \) be the dual basis in \( E_{(p,q)}^T \). Then we have

\[
\overline{V}_{(p,q)}(x) = \bigoplus_{t=0}^{n} \overline{k}\hat{e}_t, \quad 0 \leq x \leq n.
\]

One checks that the map \( \varphi : \overline{V}_{(p,q)} \to \overline{V}_{(p',q')} \), given by \( \varphi(\hat{e}_{p+t} = e_{q'-t}, t = 0, \ldots, p - q \), is an isomorphism. \( \square \)

**Example.** We consider our running example, looking at

\[
\begin{array}{c}
V_{(3,9)} \xrightarrow{?} V_{(3,9)}^T \xrightarrow{G} V_{(3,9)} \xrightarrow{\varphi} V_{(4,10)} : \\
\end{array}
\]
For any $X \in Q_m^P$ with $F_P X = V_{(p,q)}$ we define the isomorphism $\varphi : V_{(p,q)} \to V_{(p',q')}$ by $\varphi_X$.

We are now ready to define a functor $F_I : kQ_m^I \to I$ by setting $F_I X = \varphi_{SX}(F_P SX)$ for any vertex $X$ of $Q_m^I$ and, for any arrow $\gamma : Y \to X$ of $Q_m^I$ we get a morphism $F_I \gamma : F_I Y \to F_I X$ by setting $F_I \gamma = \varphi_{SX} \circ F \gamma \circ \varphi_{SY}^{-1}$ with the arrow $S \gamma : SX \to SY$ of $Q_m^I$.

(Let $n \geq 2$.)

**Proposition.** Let $kQ_m^I$ be the $k$-category defined by the quiver $Q_m^I$ and the relations $\gamma' \gamma = s' s$ for any diamond from $X$ to $Z$ in $Q_m^I$, let $I_m$ be the $k$-category with objects $F_I X$ for all $X \in Q_m^I$. Then $F_I$ induces an isomorphism $F_I : kQ_m^I \to I_m$.

4. The regular ones

We now describe $R^0$. Let $Z$ be the quiver with vertices $(r,s) \in \mathbb{N} \times \mathbb{Z}$ and arrows $\pi(r,s) : (r+1,s+1) \to (r,s)$ and $\rho(r,s) : (r,s) \to (r+1,s)$ for every vertex $(r,s)$.

![Diagram of the quiver Z]

We define a map $F$ which associates to any vertex $(r,s)$ an element $F(r,s)$ of $R^0$: let $(r,s)$ be a vertex of $Z$, define $p \in \{1, \ldots, g\}$ by asking $p \equiv s - r \mod g$ and $q' \in \{0, \ldots, g-1\}$ by asking $q' \equiv s \mod g$; let $t$ be the number of $s' \in \mathbb{Z}$ with $s' \equiv 0 \mod g$ and $s-r \leq s' \leq s$.

![Diagram of the map F]

Then we define $F_{(r,s)} := V_{(p,q' + t(n+1))}$.

Next we associate morphisms to the arrows $\pi(r,s)$ and $\rho(r,s)$ (for any vertex $(r,s)$). To describe this, assume $F(r,s) = V_{(p,q)}$.

We let $F\pi(r,s)$ and $F\rho(r,s)$ be the following morphism:

- $F\pi(r,s) : F(r+1,s+1) \to F(r,s)$
  
  $$e_t \mapsto \begin{cases} e_t & \text{if } t = p, \ldots, q \\ 0 & \text{if } t > q \end{cases}$$

- $F\rho(r,s) : F(r,s) \to F(r+1,s)$
  
  $$e_t \mapsto \begin{cases} e_t & \text{if } t = p, \ldots, q \text{ and } p = 2, \ldots, g \\ e_{t+n+1} & \text{if } t = p, \ldots, q \text{ and } q > q \text{ with } p = 1 \end{cases}$$
**Example.** $F(2, 3) \xrightarrow{F\pi(1,2)} F(1, 2) \xrightarrow{F\rho(1,2)} F(2, 2)$:

We have thus determined uniquely a functor $F : kZ \to R^0$. It is surjective on objectives and we have

$$
F(r, s) = F(r, s + g) \\
F\pi(r, s) = F\pi(r, s + g) \\
F\rho(r, 1) = F\rho(r, s + g)
$$

for any vertex $(r, s)$. Analogously to the construction for the post-projective objects, we define a quiver $Q^0 : Z/\sim$, where the equivalence relation $\sim$ is given by the fibres of $F$; we denote the equivalence class of $(r, s)$ by $(r, s)_0$, we write $\pi_0(r, s)$ and $\rho_0(r, s)$ for the equivalence classes of $\pi(r, s)$ and of $\rho(r, s)$ respectively.

Let $Q_m^0$ be the full subquiver of $Q^0$ containing the vertices $(r, s)_0$ with $1 \leq s \leq g$ and $0 \leq r \leq gm + s$ (for $m \in \mathbb{N}$). In the example, the shape of $Q_m^0$ for even $m$ is the following:

As in Section 2, $F$ defines a functor $F_0 : kQ_m^0 \to R^0$; we thus define $R_m^0$ to be the category whose objects are $F_0X$, for all $X \in Q_m^0$.

**Proposition.** Let $kQ_m^0$ be the $k$-category defined by $Q_m^0$, with the relations

- $\pi'\rho = \rho'\pi$ for any four arrows $\pi$, $\pi'$, $\pi$, $\pi'$ of $Q_m^0$ in a diamond

\[
\begin{array}{ccc}
\pi & \xrightarrow{X} & Z \\
\rho & \xleftarrow{Y_1} & \pi' \\
\end{array}
\]

\[
\begin{array}{ccc}
\rho' & \xrightarrow{Y_2} & \pi' \\
\end{array}
\]
Then $F_0$ induces an isomorphism $F_0 : kQ^0_m \to R^0_m$.

For $R^\infty$, the construction is similar to the one of $R^0$, we leave out the steps and give the result immediately:

Let $Q^\infty$ be the quiver with the vertices $(r,s)_{\infty}$, $r \in \mathbb{N}$, $s = 0, \ldots, h-1$, and arrows $\pi^\infty(r,s):(r+1,s-1)_{\infty} \to (r,s)_{\infty}$ for $s = 1, \ldots, h-1$ and $\pi^\infty(r,s):(r+1,h-1)_{\infty} \to (r,s)_{\infty}$ for $s = 0$, as well as $\rho^\infty(r,s):(r,s)_{\infty} \to (r+1,s)_{\infty}$, for $r \in \mathbb{N}$. Let $Q^\infty_m$ be the full subquiver of $Q^\infty$ on the vertices $(r,s)_{\infty}$, for $s = 0, \ldots, h-1$ and $r = 0, \ldots, h(m+1)-s$.

In the running example, $Q^\infty_2$ looks as follows:

```
We define a functor $F^\infty : kQ^\infty_m \to R^\infty$ as follows: Let $(r,s)_{\infty} \in Q^\infty_m$ and

$$p_s = \begin{cases} 0 & \text{if } s = 0 \\ g + s & \text{if } s = 1, \ldots, h-1 \end{cases}$$

$q_{r,s} = g + \text{MOD}(r+s,h) + (n+1) \cdot \text{DIV}(r+s,h)$

With this, we set

$$F^\infty((r,s)_{\infty}) = V_{(p_s,q_{r,s})} \in R^\infty$$

For any arrow $\rho^\infty((r,s)_{\infty}) : (r,s)_{\infty} \to (r+1,s)_{\infty}$ of $Q^\infty_m$ we define a morphism $F^\infty \rho^\infty((r,s)_{\infty}) : V_{(p_s,q_{r,s})} \to V_{(p_{s+1},q_{r,s})}$ as follows:

$$F^\infty \rho^\infty((r,s)_{\infty})(e_t) = e_t \quad \text{for } t = p_s, \ldots, q_{r,s}$$

For any arrow $\pi^\infty((r,s)_{\infty}) : (r+1,s-1)_{\infty} \to (r,s)_{\infty}$ of $Q^\infty_m$ we define a morphism $F^\infty \pi^\infty((r,s)_{\infty}) : V_{(p_{s+1},q_{r,s})} \to V_{(p_{s+1},q_{r,s})}$ as follows:

If $s \in \{1, \ldots, h-1\}$:

$$F^\infty \pi^\infty((r,s)_{\infty})(e_t) = \begin{cases} 0 & \text{for } q_{s-1} \leq t \leq p_s \\ e_t & \text{for } t = p_{s+1}, \ldots, q_{r+s,s-1} \end{cases}$$

And if $s = 0$ we define $F^\infty \pi^\infty((r,0)_{\infty}) : V_{(n,q_{r+1,s-1})} \to V_{(n,q_{r,s})}$ by setting

$$F^\infty \pi^\infty((r,0)_{\infty})(e_t) = \begin{cases} 0 & \text{if } t = n \\ e_{t-(n+1)} & \text{for } t = n+1, \ldots, q_{r+1,s-1} \end{cases}$$
Example.

\[ F_\infty(1,1) \xrightarrow{F_\infty(1,1)} F_\infty(2,1) \xrightarrow{F_\infty(1,0)} F_\infty(1,0) \]

**Proposition.** Let \( R_m^\infty \) be the category with the objects \( F_\infty X \), for any \( X \in Q_m^\infty \) and let \( kQ_m^\infty \) be the \( k \)-category defined by the quiver \( Q_m^\infty \), subject to the relations \( \pi' \rho = \rho' \pi \), for all arrows \( \pi, \pi' \), \( \rho, \rho' \) of \( Q_m^\infty \) in a diamond \( Y \), as well as subject to the relations \( \rho(0,s') = 0 \) for all arrows \( (0,s') \) of \( Q_m^\infty \). Then \( F_\infty \) induces an isomorphism \( F_\infty : kQ_m^\infty \to R_m^\infty \).

Finally, we consider the \( k \)-category \( R_m^\lambda \) for every \( \lambda \in k \setminus \{0\} \) whose objects are the \( V_d^\lambda \), \( 1 \leq d \leq m + 2 \) (with \( m \in \mathbb{N} \)).

For this, we let \( Q_m^\lambda \) be the quiver with vertices \( (\lambda,1), \ldots, (\lambda,m+2) \) and with arrows \( \pi_\lambda(r) : (\lambda,r+1) \to (\lambda,r) \), \( \rho_\lambda : (\lambda,r) \to (\lambda,r+1) \) for \( r = 1, \ldots, m+1 \):

\[ Q_m^\lambda : \begin{array}{c}
(\lambda,1) \bullet \\
\pi_\lambda(1) \\
\rho_\lambda(1)
\end{array} \quad \cdots \quad \begin{array}{c}
(\lambda,m+1) \\
\pi_\lambda(m+1) \\
\rho_\lambda(m+1)
\end{array} \]

We define a functor \( F_\lambda : kQ_m^\lambda \to R_m^\lambda \) by setting

\[ F_\lambda(\lambda,r) = V_r^\lambda \quad r = 1, \ldots, m+2 \]

with morphisms (for \( r = 1, \ldots, m+1 \)):

\[ F_\lambda \pi_\lambda(r) \quad \text{given by } (1_r \ 0) \in k^{r+r+1} \text{ in every vertex of } K \]

\[ F_\lambda \rho_\lambda(r) \quad \text{given by } \begin{pmatrix} 0 \\ 1_r \end{pmatrix} \in k^{r+1 \times r} \text{ in every vertex of } K \]

**Proposition.** \( F_\lambda \) induces an isomorphism between the \( k \)-category defined by \( Q_m^\lambda \), subject to the relations

1. \( \pi_\lambda(r) \rho_\lambda(r) = \rho_\lambda(r+1) \pi_\lambda(r+1), \ r = 1, \ldots, m \)
2. \( \pi_\lambda(1) \rho_\lambda(1) = 0 \)

and the \( k \)-category \( R_m^\lambda \).
Remark. Condition (1) in the proposition above can also be written in the form

\[ \pi' \rho = \rho' \pi, \]  
for all arrows of \( Q_m^\lambda \) of the form

\[
\begin{array}{c}
X \\
\pi \\
Y_2 \\
\downarrow \pi' \\
\downarrow Z \\
\rho \\
\downarrow \rho'
\end{array}
\]

since \( X = Z \) is allowed.

5. Main Theorem

For \( m \in \mathbb{N} \) let \( Q_m \) be the quiver whose vertices are the vertices of the quivers \( Q_m^P, Q_m^I \) and \( Q_m^{\alpha}_{2m(n+1)} \), for \( \sigma \in k \cup \{ \infty \} \), the arrows are all the arrows of these three quivers, with in additional the “connecting” arrows \( t_0(x) \), \( \kappa_0(x) \) for \( x = 0, \ldots, g \), \( t_\infty(y) \), \( \kappa_\infty(y) \) for \( y = 0, g, g + 1, \ldots, n \) and \( t_\lambda, \kappa_\lambda \) for \( \lambda \in k \setminus \{ 0 \} \). Their definition is apparent from the figure below of \( Q_m \) below. (Remark by KB: The example \( g = h = 1 \) appears in Examples 5 and 6 in Chapter 8 of [1]).

By \( J_m K \) we denote the full subcategory of \( \text{rep} K \) whose objects are the union of the objects of \( P_m, I_m, R_m^\alpha_{2m(n+1)} \), \( \sigma \in k \cup \{ \infty \} \) (for \( m \in \mathbb{N} \)).

Remark. To any \( V \in \text{rep} K \) there exists \( m \in \mathbb{N} \) such that \( V \) is isomorphic to a direct sum of representations from \( J_m K \).

Our goal is to find an isomorphism \( \Phi_m \) (for any \( m \in \mathbb{N} \)) between the \( k \)-category defined by \( Q_m \) and certain relations and the \( k \)-category \( J_m K \). For this, we first define a functor \( \Phi_m : kQ_m \rightarrow J_m K \):

Let \( X \) be a vertex of \( Q_m \). We set

\[
\Phi_m X := \begin{cases} 
F_P X & \text{if } X \in Q_m^P \\
F_I X & \text{if } X \in Q_m^I \\
F_{\sigma} X & \text{if } X \in Q_m^{\alpha}_{2m(n+1)}, \quad \sigma \in k \cup \{ \infty \}
\end{cases}
\]

Analogously, if \( \gamma \) is an arrow of \( Q_m \), we set

\[
\Phi_m \gamma := \begin{cases} 
F_P \gamma & \text{if } \gamma \text{ is an arrow of } Q_m^P \\
F_I \gamma & \text{if } \gamma \text{ is an arrow of } Q_m^I \\
F_{\sigma} \gamma & \text{if } \gamma \text{ is in } Q_m^{\alpha}_{2m(n+1)}, \quad \sigma \in k \cup \{ \infty \}
\end{cases}
\]

We also need to define \( \Phi_m \) on the connecting arrows.

- For any arrow \( t_0(x) : X \rightarrow Y \) in \( Q_m, x = 0, \ldots, g \), define the morphism \( \Phi_m t_0(x) : V(p,q) \rightarrow V(p',q') \) (where \( V(p,q) = \Phi_m X \) and \( V(p',q') = \Phi_m Y \)) by setting

\[
\Phi_m t_0(x)(e_t) = e_t, \quad t = p, \ldots, q
\]

- For any arrow \( t_\infty(x) : X \rightarrow Y \) in \( Q_m, x = 0, g, \ldots, n \) we define a morphism \( \Phi_m t_\infty(x) : \Phi_m X = V(p,q) \rightarrow V(p',q') = \Phi_m Y \) by

\[
\Phi_m t_\infty(x)(e_{q-t}) = e_{q'-t} \quad \text{for } t = 0, \ldots, q-p.
\]
For $\lambda \in k \setminus \{0\}$ we define a morphism $\Phi_m|_\lambda : \Phi_m(ghm,0)_p = V_w \to \Phi_m(\lambda,2m(n+1) + 2)$ by giving the matrices w.r.t. the basis $e_g, \ldots, e_{(n+1)(m(n+1)+1)+g}$ of $E_w$ and the canonical basis of the spaces $k^{2m(n+1)+2}$. For this, we let (for $d \in \mathbb{N}$) $D_d\lambda \in k^{2m(n+1)+2 \times d}$ be the matrix

$$D_d(\lambda)_{ij} := \begin{cases} 0 & \text{for } i > j, \\ (j-1)_{i-1}^{\lambda} \lambda^{j-i} & \text{else} \end{cases} \quad D_d(\lambda) := \begin{pmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^{d-1} \\ 1 & 2\lambda & \ldots & \ldots & \ldots \\ 1 & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 1 & \ldots & \ldots & \ldots & 1 \end{pmatrix}$$
Taking into account the dimensions of the vector spaces $\Phi_m(ghm,0)_P(x)$ we define $\Phi_m\lambda(x)$ as

$$
\begin{cases}
D_{m(n+1)+2}(\lambda) & \text{if } x = g \\
D_{m(n+1)+1}(\lambda) & \text{if } x = 0, \ldots, g-1, g+1, \ldots, n
\end{cases}
$$

Then $\Phi_m\lambda$ is a morphism.

To define $\Phi_m$ on the remaining arrows of $Q_m$, we use the construction from Section 3. Let $S$ be the permutation of the vertices of $Q_m$ that corresponds to a reflection along the vertical (dashed) line in Figure 4. For every vertex $X$ of $Q_m^P$ or of $Q_m^\infty$ we use the definition of Section 3 for $(r,s)_0 \in Q_m^0$ and let $s_0 \in \{1, \ldots, g\}$ be such that $s_0 \equiv r - s \mod g$. Then we set $S(r,s)_0 = (r,s_0)_0$. For $(r,s)_\infty \in Q_m^\infty$ let $s_\infty \in \{0, \ldots, h-1\}$ be such that $s_\infty = -r - s \mod h$; then set $S(r,1)_\infty = (r,s_\infty)_\infty$; finally, we define $S(\lambda,r) = (\lambda,r)$ for all $(\lambda,r) \in Q_m^\lambda$, $\lambda \in k \setminus \{0\}$.

For every arrow $\gamma : X \to Y$ of $Q_m$ let $S\gamma$ be the arrow $S\gamma : SY \to SY$ of $Q_m$; then we have

$$
\begin{align*}
S\gamma_0(x) &= \kappa_0(x) & \text{for } x = 0, \ldots, g, \\
S\gamma_\infty(y) &= \kappa_\infty(y) & \text{for } y = 0, g, \ldots, n, \\
S\gamma_\lambda &= \kappa_\lambda & \text{for } \lambda \in k \setminus \{0\}.
\end{align*}
$$

**Proposition.** For every vertex $X$ of $Q_m$ there is an isomorphism

$$
\varphi_X : \overline{\Phi_mX} \overset{\sim}{\longrightarrow} \Phi_mSX
$$

**Proof.** First let $X$ be a vertex of one of the quivers $Q_m^P$, $Q_m^0$, $Q_m^\infty$ and let $\Phi_mX = V_{(p,q)}$.

By Section 3 we know that there is an isomorphism $\varphi = \varphi_X : \overline{V_{(p,q)}} \overset{\sim}{\longrightarrow} V_{(p',q')}$ given by $\varphi_X(e_{t+1}) = e_{q' - t}$, for $t = 0, \ldots, q - p$, for integers $p',q'$ with $p' \equiv Gq \mod n + 1$, $q' \equiv Gp \mod n + 1$ and $q' - p' = q - p$.

We have $\Phi_mSX = V_{(p',q')}$. For $X \in Q_m^P$, the claim follows from the fact that for any $V \in \text{rep } K$ there is an isomorphism between $\overline{V}$ and $V$.

Now let $X = (\lambda,r) \in Q_m^\lambda$, $\lambda \in k \setminus \{0\}$. We give an isomorphism $\varphi^{-1}_X : V_{r,\lambda} \overset{\sim}{\longrightarrow} \overline{V_{r,\lambda}}$, we then let $\varphi_X$ be the inverse morphism of this.

We describe $\varphi^{-1}_X$ using the matrices representing it with respect to the canonical bases $e_1, \ldots, e_r$ of $k^r$ and $\tilde{e}_1, \ldots, \tilde{e}_r$ of $(k^r)^\perp$: Using $\theta_r := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in k^{r \times r}$ let $\varphi^{-1}_X$ be given by

$$
\begin{cases}
\theta_r & \text{if } x = 0, g, g + 1, \ldots, n \\
\lambda \theta_r + \theta_r J_r & \text{if } x = 1, \ldots, g - 1
\end{cases}
$$

where $J_r = \begin{pmatrix} 0 & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in k^{r \times r}$

This proves the claim.
For any connecting arrow \( \gamma : X \to Y \) of \( Q_m \) with \( X \in Q_m^P \) and \( Y \in Q_{2m(n+1)}^\sigma \), we define
\[
\Phi_m S \gamma : \Phi_m SY \to \Phi_m SX \quad \text{as follows} \quad \Phi_m S \gamma = \varphi_X \circ \Phi_m \gamma \circ \varphi_Y^{-1}
\]
Hence we have associated a morphism to any arrow of \( Q_m \) and thus defined the functor \( \Phi_m : kQ_m \to J_m K \) uniquely. \( \square \)

To keep the description of the relations in the following statement simple, we assume \( n \geq 2 \) and introduce some notation to allow us to abbreviate the relations:

\[ Q^\infty_{2m(n+1)} \]

\[ (ghm, g)_P \]

\[ \pi_\infty^h \]

\[ \beta^h \]

\[ \alpha^h_P \]

\[ (g2m(n+1), g)_P \]

\[ Q^P_m \]

\[ (ghm, g)_P \]

\[ \alpha^g_P \]

\[ (ghm, g)_P \]

\[ \pi^0 \]

\[ \pi_0^0 \]

\[ \pi_\infty^0 \]

\[ Q^0_{2m(n+1)} \]

\[ (ghm, g)_P \]

\[ \beta^g_P \]

- \( \beta^g_P \) denotes the path of length \( g \) from \((ghm, g)_P\) to \((ghm, 0)_P\) composed by the arrows \((ghm, \beta_X)_P\).
- \( \alpha^h_P \) denotes the path of length \( h \) from \((ghm, g)_P\) to \((ghm, 0)_P\) composed by the arrows \((ghm, \alpha_X)_P\).

The definitions of \( \alpha^g_P, \beta^g_P, \pi^0_\infty, \rho^0_\infty, \pi^h_\infty \) and \( \rho^h_\infty \) can be understood from the picture above.

The “lowest” vertices of the tubes \( Q^\sigma_{2m(n+1)} \), \( \sigma \in k \cup \{ \infty \} \), i.e. the vertices
\[ (0, s)_0 \in Q^0_{2m(n+1)}, \ s = 1, \ldots, g \]
\[ (0, s')_0, \ s' = 0, \ldots, h - 1 \]
\[ (\lambda, 1) \in Q^{\lambda}_{2m(n+1)} \] are called the vertices at the mouth of the tubes. Finally, we write
\[ \varepsilon_\lambda = \rho_\lambda(2m(n+1) + 1)\pi_\lambda(2m(n+1) + 1) \] for \( \lambda \in k \setminus \{ 0 \} \).

**Theorem 1** (Hauptsatz). Let \( kQ_m \) be the \( k \)-category defined by \( Q_m \) and the following relations

\[ 3 \]The \( s' \) were equal to 1, \ldots, \( h \) on page 12.
(a) $\gamma' = \delta' \delta$ for all arrows $\gamma$, $\gamma'$, $\delta$, $\delta'$ of $Q_m$ in a diamond from $X$ to $Z$ with $X \neq (ghm, 0)_P$.

(b) $\pi_\sigma \rho_\sigma = 0$ for all $\sigma \in k \cup \{\infty\}$ and all arrows $\pi_\sigma$, $\rho_\sigma$ of $Q_{2m(n+1)}^I$ of the form $X^* \xrightarrow{\rho_0} Y \xrightarrow{\pi_0} Z^*$ with base points $X^*$ and $Z^*$.

(c1) $\iota_0(g) = \pi_0^g \iota_0(0) \beta^j_h \pi_0^g$ \quad $\iota_\infty(g) = \pi_\infty^h \iota_\infty(0) \alpha^j_p \pi_\infty^h$.

(c2) $\kappa_0(g) = \rho_0^g \kappa_0(0) \beta^j_I \rho_0^g$ \quad $\kappa_\infty(g) = \alpha^j_I \kappa_\infty(0) \rho_\infty^h$.

(d) $\iota_\lambda \alpha^j_p = \lambda \alpha^j_h \beta^j_h + \varepsilon \lambda \alpha^j_h \beta^j_h$ \quad $\alpha^j_h \kappa_\lambda = \lambda \beta^j_h \kappa_\lambda + \beta^j_h \varepsilon \lambda$.

(e) $\kappa_0(0) \iota_0(0) (ghm, \alpha_n)_P = 0 \quad (ghm, \alpha_n)_I \kappa_0(0) \iota_0(0) = 0$.

(f) $\kappa_\infty(0) (\rho_\infty^h \iota_\infty(0)) = \kappa_0(0) (\rho_0^g \iota_0(0))^{2m(n+1)+1-j \iota_0(0)}$.

(g) $\kappa_\lambda \varepsilon \lambda \iota_\lambda = \sum_{i=0}^j (\frac{2m(n+1)+1-j+i}{2m(n+1)+1-j}) \lambda \iota_0(0) (\rho_0^g \iota_0(0))^{-i \iota_0(0)}$

with $\lambda \in k \setminus \{0\}$ and $j = 0, \ldots, 2m(n+1)+1$.

Then $\Phi_m$ induces an isomorphism

$$\Phi_m : kQ_m \to J_m K$$

**REMARKS**

The diploma thesis ends here. This section contains a few remarks concerning the relations appearing in Theorem II.

**Remark.** In (a), we have all the mesh relations with two middle vertices. This includes meshes between different components. The latter are indicated by dashed red lines Figure 5.

(b) shows the relations at the mouths of the tubes. So these are the mesh relations involving only one middle vertex.

In (c1) and (c2), we have relations for arrows between $Q^I_m$ and $Q^\infty_{2m(n+1)}$ or $Q^\infty_{2m(n+1)}$ (respectively), as well as between any of these two tubes and $Q^I_m$. The paths/arrows start at the vertex $(ghm, g)_P$ and end at $(g2m(n+1), g)_0$ resp. at $(g2m(n+1), g)_\infty$ (the two relations at the left) or they start at $(g2m(n+1), g)_0$ resp. at $(g2m(n+1), g)_\infty$ and end at $(ghm, g)_I$ (the two relations at the right).

The relations in (d) link the homogeneous tubes with the post-projective and with the pre-projective component.

The relation to the left says that when you first compose all the $(ghm, \alpha_i)_P$ (starting at $(ghm, g)_P$ (of them) and then compose this with the arrow $\iota_\lambda$ to get to $Q^I_{2m(n+1)}$, it is the same as the sum of two paths, both starting all the $(ghm, \beta^j)_P$ first (of them) and the arrow $\iota_\lambda$, one is the multiple of this by $\lambda$, the other the uses $\varepsilon \lambda$, a path that just goes one arrow in this homogeneous tube (from vertex $(\lambda, 2m(n+1)+2)$ and back).

The relation to the right is dual to it, using $\kappa_\lambda$ to go from the homogeneous tube to $R^I_m$ and then the composition of all the $(ghm, \alpha_i)_I$ (of them) is the same as the sum of two paths, one starting with $\varepsilon \lambda$, then out of the tube and the composition of all the $(ghm, \beta^j)_I$ (of them), the other the multiple of $\lambda$ of the path that just goes out of the tube and then does the composition of all the $(ghm, \beta^j)_I$ (of them).
(e) In the diploma thesis, relation (e) uses $\beta_n$ twice, they are replaced by $\alpha_n$. These two relations link paths from the post-projective component $Q^P_{m}$ through $Q^0_{2m(n+1)}$ to the pre-injective component. In the left equation, the last of the $\alpha$’s is used, then the connecting arrows to $Q^0_{2m(n+1)}$ and from there to $Q^n_I$. This composition is zero. Dually, the composition of these two connecting arrows with the first of the $\alpha$’s, i.e. with $(ghm, \alpha_n)_I$ is zero.

Note that the two paths

$$\kappa_\infty(0)\iota_\infty(0)(ghm, \beta_0)_P, \ (ghm, \beta_0)_I\kappa_\infty(0)\iota_\infty(0)$$

are also zero. This follows from (e) and (f) (with $j = 1$), using the diamond relations of (a) iteratedly to push the path all the way down to include a triangle at the mouth of the tube (and then use (b)).

We also note that the four “other” compositions

$$\kappa_0(0)\iota_0(0)(ghm, \beta_0)_P \quad (ghm, \beta_0)_I\kappa_0(0)\iota_0(0),$$

$$\kappa_\infty(0)\iota_\infty(0)(ghm, \alpha_n)_P \quad (ghm, \alpha_n)_I\kappa_\infty(0)\iota_\infty(0)$$

are not zero!

(f) Relates paths from $(ghm, 0)_P$ to $(ghm, 0)_I$, passing around the ‘top layer’ of $Q^\infty_{2m(n+1)}$ resp. around $Q^0_{2m(n+1)}$ several times (adding up to $2m(n + 1) + 1$).

(g) Relates paths from $(ghm, 0)_P$ passing through a homogeneous tube, going into this homogeneous tube (with $\varepsilon$ used $j$ times) and on to $(ghm, 0)_I$ with a sum of paths passing through $Q^0_{m}$, going around this tube (at the ‘top layer’) several times, and multiplying by a scalar.
Figure 5. $Q_m$ for $g = 3$, $n = 4$
References

[1] P. Gabriel, A.V. Roiter, *Representations of finite-dimensional algebras*. Translated from the Russian. With a chapter by B. Keller. Reprint of the 1992 English translation. Springer-Verlag, Berlin, 1997. iv+177 pp.

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\[\text{in Brüstle's thesis: “Enzyklopädie der Sowjetischen Akademie der Wissenschaften, Springer-Verlag (in preparation)”}\]