The Pion-Nucleon Sigma Term and the Goldberger-Treiman Discrepancy

James V. Steele\textsuperscript{1}, Hidenaga Yamagishi\textsuperscript{2} and Ismail Zahed\textsuperscript{1}

\begin{footnotesize}1 Department of Physics, SUNY, Stony Brook, New York 11794, USA; 2 Chome 11-16-502, Shimomeguro, Meguro, Tokyo, Japan. 153

(August 7, 2018)
\end{footnotesize}

The pion-nucleon sigma term is shown to be equal to the Goldberger-Treiman discrepancy at tree level. Its value estimated this way is very sensitive to the pion-nucleon coupling constant $g_{\pi NN}$. This relation, when combined with the pion-nucleon S-wave scattering lengths, yields a new determination of $g_{\pi NN}$ at tree level. The results of a one-loop analysis are also summarized determining an allowed range for the induced pseudoscalar coupling constant $g_P$.

Pion-nucleon interactions have been extensively investigated using dispersion relations and chiral symmetry. Most of the studies using chiral symmetry have relied on unphysical limits such as the soft pion limit \footnote{\label{fn1}See References.} or the chiral limit \footnote{\label{fn2}See References.}. A typical example is the pion-nucleon sigma term \footnote{\label{fn3}See References.}, the fraction of the nucleon mass due to the explicit breaking of chiral $SU(2) \times SU(2)$. The scattering amplitude is analytically continued to the unphysical Cheng-Dashen point \footnote{\label{fn4}See References.}, and chiral perturbation theory is applied.

An important exception to the above is Weinberg’s formula for pion-nucleon scattering \footnote{\label{fn5}See References.}, which yields the Tomozawa-Weinberg relations for the S-wave scattering lengths on shell \footnote{\label{fn6}See References.}. Recently, we have been able to extend this result to processes involving an arbitrary number of on-shell pions and nucleons \footnote{\label{fn7}See References.}. In this way, the pion-nucleon sigma term can be directly assessed. In particular, we find that at tree level the pion-nucleon sigma term is simply given by the Goldberger-Treiman discrepancy. The purpose of this letter is to give a derivation of this result, and discuss some of its quantitative aspects. We also review Weinberg’s formula in light of our result, and briefly discuss the effects of one-loop corrections.

The approach discussed in \footnote{\label{fn8}See References.} requires an extended $S$ matrix analysis for a concise quantum formulation that enforces both chiral symmetry and unitarity. However, since we are primarily interested here in a tree level result, we will use an equivalent but shorter route in terms of effective Lagrangians with some supplemental rules following from the complete analysis \footnote{\label{fn9}See References.}.

For the $SU(2) \times SU(2)$ symmetric part, we take the standard effective Lagrangian

\[ \mathcal{L}_1 = \frac{f_\pi^2}{4} \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \bar{\Psi} i \gamma_5 \frac{(g_A - 1)}{2} \Psi + m_\pi \left( \bar{\Psi} R U \Psi_L + \bar{\Psi}^\dagger L U^\dagger \Psi_R \right) + \frac{i}{2} (g_A - 1) \bar{\Psi}_R (\bar{\phi} U) U^\dagger \Psi_R \]

where $U$ is a chiral field, $\Psi = (\Psi_R, \Psi_L)$ is the nucleon field, and $\bar{\phi} = \gamma^\mu \partial_\mu$. In the low-energy limit, the scattering amplitude given by (1) is essentially unique, given that the isospin of the nucleon is $1/2$.

Ignoring isospin breaking and strong CP violation, the term which explicitly breaks chiral symmetry must be a scalar-isoscalar. The simplest non-trivial representation of $SU(2) \times SU(2)$ which contains such a term is (2,2), since (2,1) $\oplus$ (1,2) contains only isospinors, and (1,3) $\oplus$ (3,1) contains only isovectors. We therefore have,

\[ \mathcal{L}_2 = \frac{1}{4} f_\pi^2 m_\pi^2 \text{Tr} (U + U^\dagger) - \frac{m_\pi^2}{\Lambda} \bar{\Psi} \Psi \]

We assume that $\Lambda$ is non-vanishing as $m_\pi \to 0$, so that $\mathcal{L}_2$ vanishes in the chiral limit. The nucleon mass is $m_N = m_{\pi\pi} + m_\pi^2 / \Lambda$. The second term in $\mathcal{L}_2$ is usually dropped (e.g. in chiral perturbation theory), but it is essential to keeping the nucleons on shell and so we will retain it here.

From (2,2) it follows that the vector current is

\[ V_\mu^a(x) = i \frac{f_\pi^2}{8} \text{Tr} \left( \{ \tau^a, U^\dagger \} \partial_\mu U \right) + \text{h.c.} \]

the axial current is

\[ A_\mu^a(x) = i \frac{f_\pi^2}{8} \text{Tr} \left( \{ \tau^a, U^\dagger \} \partial_\mu U \right) + \text{h.c.} \]

and the scalar density is

\[ \sigma(x) = \frac{1}{m_\pi f_\pi} \mathcal{L}_2 = \frac{f_\pi^2}{4} \text{Tr} (U + U^\dagger) - \frac{1}{f_\pi \Lambda} \bar{\Psi} \Psi \]

We also introduce the PCAC pion field

\[ -\frac{i}{2} (g_A - 1) \bar{\Psi}_R (\bar{\phi} U) U^\dagger \Psi_L \]
\[ \pi^a(x) = \frac{1}{m_\pi^2 f_\pi} \partial^\mu A_\mu^a, \]
\[ = -i \frac{f_\pi}{4} \text{Tr} (\tau^a (U - U^\dagger)) + \frac{1}{f_\pi} \bar{\Psi} \gamma_5 \tau^a \Psi \quad (6) \]

and the one-pion reduced axial current
\[ J^a_{\mu}(x) = A_\mu^a + f_\pi \partial_\mu \pi^a, \]
\[ = g_A \bar{\Psi} \gamma_5 \tau^a \frac{1}{2} \Psi + \frac{1}{\Lambda} \partial_\mu \left( \bar{\Psi} \gamma_5 \tau^a \Psi \right) + O(\pi^3) \quad (7) \]

Between nucleon states of momentum \( p_i \) and an implicit spin dependence \( s_i \),
\[ \langle N(p_2) | A_\mu^a(x) | N(p_1) \rangle = \epsilon^{(p_2 - p_1) x} \]
\[ \times \bar{\psi}(p_2) (\gamma_\mu \gamma_5 G_1(t) + (p_2 - p_1)_{\mu} \gamma_5 G_2(t)) \frac{\tau^a}{2} u(p_1) \quad (8) \]

and
\[ \langle N(p_2) | J^a_{\mu}(x) | N(p_1) \rangle = \epsilon^{(p_2 - p_1) x} \]
\[ \times \bar{\psi}(p_2) \left( \gamma_\mu \gamma_5 G_1(t) + (p_2 - p_1)_{\mu} \gamma_5 G_2(t) \right) \frac{\tau^a}{2} u(p_1) \quad (9) \]

with \( t = (p_2 - p_1)^2 \) and \( G_2 \) is free of pion poles.

From (8), we also have \( \partial_\mu J^a_{\mu} = f_\pi (\gamma + m_\pi^2) \pi \). Hence,
\[ \langle N(p_2) | \pi^a(x) | N(p_1) \rangle = \langle N(p_2) | \pi^a_\text{in}(x) | N(p_1) \rangle \]
\[ - \frac{1}{f_\pi} \int d^4 y \Delta_N(x - y) \langle N(p_2) | \partial_\mu J^a_{\mu}(y) | N(p_1) \rangle \]
\[ = - \frac{1}{f_\pi} t - m_\pi^2 \bar{\psi}(p_2) \left( 2m_N G_1(t) + t G_2(t) \right) \]
\[ \times i \gamma_5 \frac{\tau^a}{2} u(p_1) \epsilon^{(p_2 - p_1) x} \quad (10) \]

where \( \pi^a_\text{in} \) is the incoming pion field, and we have used \( \langle N(p_2) | \pi^a_\text{in}(x) | N(p_1) \rangle = 0 \). This is a non-trivial requirement if the nucleon is a chiral soliton. This point will not be pursued further here.

It follows from (8),(9) that
\[ G_2(t) = \frac{1}{m_\pi^2 - t} \left( 2m_N G_1(t) + m_\pi^2 G_2(t) \right) . \quad (11) \]

Also by definition, eq. (10) is equal to
\[ -g_{\pi NN}(t) \frac{1}{t - m_\pi^2} \bar{\psi}(p_2) i \gamma_5 \tau^a u(p_1) \epsilon^{(p_2 - p_1) x} \quad (12) \]

and hence
\[ f_\pi g_{\pi NN}(t) = m_N G_1(t) + \frac{t}{2} G_2(t) \quad (13) \]

where \( g_{\pi NN} = g_{\pi NN}(m_\pi^2) \) is the pion-nucleon coupling constant. Extrapolating from \( t = m_\pi^2 \) to \( t = 0 \) gives the standard Goldberger-Treiman relation \( g_A m_N \sim f_\pi g_{\pi NN} \), where \( g_A = G_1(0) \). However one can do better. Substituting (10) at tree level into (8) gives
\[ G_1(t) = g_A \]
\[ G_2(t) = -\frac{2}{\Lambda} \quad (14) \]

Inserting (14) into (13) at \( t = m_\pi^2 \) gives the desired relation
\[ \sigma_{\pi N} \equiv \frac{m_\pi^2}{\Lambda} = g_A m_N - f_\pi g_{\pi NN} \quad (15) \]

between the pion-nucleon sigma term and the Goldberger-Treiman discrepancy. The one-loop corrections to (10) are of order \( m_N m_\pi^2 / (4\pi f_\pi)^2 \). They will be discussed below.

Numerically, there is a huge cancellation in the right hand side, and the value of \( \sigma_{\pi N} \) is very sensitive to \( g_{\pi NN} \). Using the central values for all experimentally measured quantities with \( (m_N, f_\pi) = (940, 92.4) \) MeV [10] and \( g_A = 1.2650(16)\) [11], we have \( \sigma_{\pi N} = -62 \) MeV for the value \( g_{\pi NN}^2 / 4\pi = 14.6(3) \) [12], whereas we have \( \sigma_{\pi N} = 17 \) MeV for the value \( g_{\pi NN}^2 / 4\pi = 12.80(36) \) [13]. Unfortunately, this sensitivity means that we cannot directly extract a reliable value of the sigma term from the existing data, although the tree level result [13] suggests a low value for the pion-nucleon coupling constant, in view of the current value \( \sigma_{\pi N} = 45 \pm 8 \) MeV [14]. We therefore turn to the relation with Weinberg’s formula for the pion-nucleon scattering amplitude \( i T \).

Taking \((k_1, a)\) as the incoming pion, and \((k_2, b)\) as the outgoing pion, with \( p_1 + k_1 = p_2 + k_2 \), the formula reads
\[ i T = i T_V + i T_S + i T_{AA} \quad (16) \]

where
\[ i T_V = -\frac{1}{f_\pi^2} k_1^\mu \epsilon^{bac} \langle N(p_2) | V_\mu^c(0) | N(p_1) \rangle \quad (17) \]
\[ i T_S = \frac{i}{f_\pi} m_N^2 \delta^{ab} \langle N(p_2) | \sigma(0) | N(p_1) \rangle \text{conn.} \quad (18) \]
\[ i T_{AA} = -\frac{1}{f_\pi^2} k_1^\mu k_2^\nu \int d^4 x e^{-k_1^{-} x} \]
\[ \times \langle N(p_2) | T^a_{\rho \nu} J^{a\rho}_{\mu}(x) J^{a\nu}_{\mu}(0) | N(p_1) \rangle \text{conn.} . \quad (19) \]

Substitution of (10) at tree level yields
\[ T_V = \frac{1}{f_\pi^2} i \epsilon^{abc} \bar{\psi}(p_2) k_1^\mu \frac{\tau^c}{2} u(p_1) \quad (20) \]
\[ T_S = \frac{m^2}{f_\pi^2} \delta^{ab} \bar{\psi}(p_2) u(p_1) \quad (21) \]
\[ T_{AA} = -\frac{1}{f_\pi^2} \bar{\psi}(p_2) \left( g_A k_2 + \frac{2m^2}{\Lambda} \right) \frac{\tau^b \tau^a}{4} \]
\[ \times \left( \frac{1}{p_1 + k_1 + m_N} \right) \left( g_A k_1 + \frac{2m^2}{\Lambda} \right) u(p_1) + (k_1, a) \leftrightarrow (k_2, b) . \quad (22) \]

The isospin structure is decomposed as \( T^{ba} = \delta^{ba} T^+ + i \epsilon^{bac} c T^- \) to give
\[ T^+ = T_S^+ + T_{AA}^+ \quad T^- = T_V^- + T_{AA}^- \]  

(23)

At threshold in the center of mass frame, the amplitudes \( T^\pm \) can be extrapolated from data and written as scattering lengths \( a^\pm \). Taking (23) at threshold,

\[
4\pi \left( 1 + \frac{m_\pi}{m_N} \right) a^+ = \frac{\sigma_{\pi N}}{f_\pi^2} \left( 1 - \frac{\sigma_{\pi N}}{m_N} \right) - \frac{1}{f_\pi^2 m_N} \frac{m_\pi^2}{4m_N^2 - m_\pi^2} (g_A m_N - \sigma_{\pi N})^2
\]

(24)

\[
4\pi \left( 1 + \frac{m_\pi}{m_N} \right) a^- = \frac{m_\pi}{2f_\pi^2} (1 - g_A^2) + \frac{2m_\pi}{4m_N^2 - m_\pi^2} (g_A m_N - \sigma_{\pi N})^2
\]

(25)

showing the corrections to the Tomozawa-Weinberg formula are small. Eqs. (15) and (25) give a direct relation between \( a^- \) and \( g_{\pi NN} \), which is

\[
4\pi \left( 1 + \frac{m_\pi}{m_N} \right) a^- = \frac{m_\pi}{2f_\pi^2} (1 - g_A^2)
\]

(26)

\[
= \frac{\sigma_{\pi N}}{f_\pi^2} \left( 1 - \frac{\sigma_{\pi N}}{m_N} \right) + \frac{m_\pi^2}{4f_\pi^2 m_N}(1 - g_A^2).
\]

(27)

Using \( a^+ = -(8 \pm 4) \times 10^{-3}/m_\pi \) [15] and the above value for \( a^- \) gives \( \sigma_{\pi N} = 2 \text{ MeV} \). (The other root \( \sigma_{\pi N} \sim m_N \) has been discarded). In [15], this corresponds to the value \( g_{\pi NN}^2/4\pi = 13.1 \), to be compared with 14.4.

The present analysis can be extended to one-loop by using power counting in \( 1/f_\pi \) [16]. In this context we have analyzed one-loop corrections to the above and they require a new subtraction constant in \( C_2 \). The extra piece of data necessary to fix this constant is \( g_\rho = m_\rho G_2(-0.88m_N^2) = 8.2 \pm 2.4 \) available from muon capture in hydrogen [10].

The loop corrections are in general small as can be seen in Fig. 1 by the shift from the tree level (dotted line) to the one-loop result at \( \sigma_{\pi N} = 0 \).

The exception is \( G_2 \) due to the large cancellation at tree level, since it is proportional to \( \sigma_{\pi N} \) in this case (eq. (14)). If we require that \( \sigma_{\pi N} \) is positive, the one-loop correction does not exceed 50%, and \( g_{\pi NN} \) is larger than the lower bound from [15]. We then obtain an inequality between \( \sigma_{\pi N}, g_{\pi NN} \) and \( g_\rho \), as indicated by the shaded area of Fig. 1. We therefore have,

\[ 12.4 \leq g_{\pi NN}^2/4\pi \leq 13.15 \quad \text{and} \quad 8.30 \leq g_\rho \leq 8.55 \]  

(28)

with \( 0 \leq \sigma_{\pi N} \leq 70 \text{ MeV} \), to one-loop. Our allowed range for \( g_\rho \) is to be compared with \( 8.44 \pm 0.16 \) from [17]. The justification of the supplementary rules, and details of the one-loop calculation will be given elsewhere [8].

FIG. 1. The dependence of the pseudoscalar coupling constant \( g_\rho \) on the \( \pi N \)-sigma term. The horizontal lines have \( g_{\pi NN}^2/4\pi = 12.4, 12.8, \text{ and } 13.2 \) respectively. The dotted line is the tree result for \( g_\rho \). Constraining the loop corrections of \( G_2 \) to be 50% or less gives the shaded region. See the text for further discussion.

Acknowledgements

This work was supported in part by the US DOE grant DE-FG-88ER40388.

[1] S.L. Adler, Phys. Rev. B 139 (1965) 1638; S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.
[2] R.F. Dashen, Phys. Rev. 133 (1966) 1245; R. Dashen and M. Weinstein, Phys. Rev. 133 (1969) 1291; H. Pagels, Phys. Rep. 16 (1975) 219; J. Gasser and H. Leutwyler,
Ann. Phys. 158 (1984) 142.

[3] J. Gasser, M.E. Sainio, and A. Švarc, Nucl. Phys. B 307 (1988) 779, and references therein.

[4] T.P. Cheng and R.F. Dashen, Phys. Rev. Lett. 26 (1971) 594.

[5] S. Weinberg, “Lectures on Elementary Particles and Quantum Field Theory,” Brandeis Summer Institute 1970, S. Deser, M. Grisaru, and H. Pendleton, MIT Press, 1970.

[6] Y. Tomozawa, Nuovo Cim. (Ser. X) 46A (1966) 707; S. Weinberg, Phys. Rev. Lett. 17 (1966) 616; 18 (1967) 188, 507.

[7] H. Yamagishi and I. Zahed, Ann. Phys. to be published.

[8] J.V. Steele, H. Yamagishi, and I. Zahed, in preparation.

[9] C.G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177 (1969) 2247.

[10] Particle Data Group, Phys. Rev. D 50 (1994) 1173.

[11] D. Dubbers, W. Mampe, and J. Döhner, Europhys. Lett. 11 (1990) 195.

[12] T.E.O. Ericson, et al., Phys. Rev. Lett. 75 (1995) 1046.

[13] F. Bradamante, et al., Phys. Lett. B 343 (1995) 431.

[14] J. Gasser, H. Leutwyler, and M.E. Sainio, Phys. Lett. B 253 (1991) 260.

[15] R. Koch, Nucl. Phys. A 448 (1986) 707.

[16] J. Bernabéu, Nucl. Phys. A 374 (1982) 593c.

[17] V. Bernard, N. Kaiser, and U.G. Meissner, Phys. Rev. D 50 (1994) 6899.