The Invertibility of $U$-Fusion Cross Gram Matrices of Operators

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Abstract. Finding matrix representations is an important part of operator theory. Calculating such a discretization scheme is equally important for the numerical solution of operator equations. Traditionally in both fields, this was done using bases. Recently, frames have been used here. In this paper, we apply fusion frames for this task, a generalization motivated by a block representation, respectively, a domain decomposition. We interpret the operator representation using fusion frames as a generalization of fusion Gram matrices. We present the basic definition of $U$-fusion cross Gram matrices of operators for a bounded operator $U$. We give necessary and sufficient conditions for their (pseudo-)invertibility and present explicit formulas for the (pseudo-)inverse. More precisely, our attention is on how to represent the inverse and pseudo-inverse of such matrices as $U$-fusion cross Gram matrices. In particular, we characterize fusion Riesz bases and fusion orthonormal bases by such matrices. Finally, we look at which perturbations of fusion Bessel sequences preserve the invertibility of the fusion Gram matrix of operators.

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1. Introduction and Motivation

For the representation (and modification) of functions a standard approach is using orthonormal bases (ONBs). It can be hard to find a 'good' orthonormal basis, in the sense that it sometimes cannot fulfill given properties, as formally expressed, e.g. in the Balian–Low theorem [35]. For solving this problem, frames were introduced by Duffin and Schaeffer [28] and widely developed by many authors [17,22,26,30]. In recent years, frames have been the focus of active research, both in theory [2,18,31] and applications [11,19,24,29]. Also, several generalizations have been investigated, e.g. [1,3,4,46,49], among them fusion frames [20,21,28,33], which are the topic of this paper.
Recently, both in applications, e.g. numerical simulations in acoustics, and also in the mathematical theory it has been observed that frames are not only useful for analyzing signals or functions, but also for the matrix representation of operators. On an abstract level, it is well known that for orthonormal bases, operators can be uniquely described by a matrix representation [34]. We have shown that an analogous result holds for frames and their duals [8,10]. For a bounded, linear operator $O$ and frames $\Psi, \Phi$, the infinite matrix $[M_{\Phi,\Psi}(O)]_{k,l} = \langle O\psi_l, \phi_k \rangle$ acts as a bounded operator on $\ell^2$ since $M_{\Phi,\Psi}(O) = C_{\Phi}O D_{\Psi}$. One question that remains is can we find a stable way to find a block representation of the operator?

For a numerical treatment of operator equations, used, for example, for solving integral equations in acoustics [40] and stochastic [37], the involved operators have to be discretized to be handled numerically. The (Petrov–)Galerkin approach [32] is a particular and well-known way for this discretization, corresponding to the above definition. Frames were also used in numerics [36], in particular in an adaptive approach [25,47]. In [12,13], sufficient and necessary conditions of the invertibility of such matrices are investigated.

Note that the system matrix of the identity is the cross Gram matrix of the two sequences $\{\psi_k\}_{k \in I}$ and $\{\phi_k\}_{k \in I}$. Therefore, in [15], the concept of matrix representation of operators using frames is reinterpreted as a generalization of the Gram matrix to investigate the inverses. As the concept of domain decomposition is a particularly relevant topic in this field, the extension of the approach to operator representations to fusion frame is very useful [42].

In this paper, we, therefore, look at those $U$-fusion cross Gram matrices. In particular, we investigate the (pseudo-)invertibility of $U$-fusion Gram matrices of operators. In Sect. 2, we review basic notations and preliminaries, where we also state and prove, in passing, two generalized results for the characterization of fusion Riesz bases. In Sect. 3, we give necessary as well as sufficient conditions for the (pseudo-)invertibility of the $U$-cross Gram matrices, characterize fusion orthonormal bases and fusion Riesz bases by those properties and give formulas for the (pseudo-)inverses. Finally, in Sect. 4, some stability results are discussed. Finally, in Sect. 4, some stability results are discussed.

2. Preliminaries and Notations

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space, $I$ a countable index set and $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$ and $\{e_i\}_{i \in I}$ an orthonormal basis for $\mathcal{H}$. The orthogonal projection on a subspace $V \subseteq \mathcal{H}$ is denoted by $\pi_V$. We will denote the set of all linear and bounded operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ by $B(\mathcal{H}_1, \mathcal{H}_2)$ and for $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, it is represented by $B(\mathcal{H})$. We denote the range and the null space of an operator $U$ by $\text{ran}(U)$ and $\text{ker}(U)$, respectively. For a closed range operator $U \in B(\mathcal{H}_1, \mathcal{H}_2)$, the pseudo-inverse of $U$ is defined as the unique operator $U^\dagger \in B(\mathcal{H}_2, \mathcal{H}_1)$ satisfying

$$\text{ker}(U^\dagger) = \text{ran}(U)^\perp, \quad \text{ran}(U^\dagger) = \text{ker}(U)^\perp, \quad \text{and} \quad UU^\dagger U = U.$$
The operator $U$ has closed range if and only if $U^*$ has closed range and $(U^*)^\dagger = (U^\dagger)^*$, see, e.g. [22, Lemma 2.4.1, Lemma 2.5.2].

The space of all compact operators (which corresponds to the closure of finite-rank operators) on $\mathcal{H}$ is denoted by $K(\mathcal{H})$. The Schatten $p$-class of $\mathcal{H}$ for $0 < p < \infty$ consists of all operator $T$, such that its singular values $\{\lambda_n\}_{n \in I}$ belong to $\ell^p$. It is denoted by $S_p(\mathcal{H})$ which is a Banach space with the norm

$$
\|T\|_p = \left( \sum_n |\lambda_n|^p \right)^{\frac{1}{p}}.
$$

An operator $T \in B(\mathcal{H})$ is called trace class if $\text{trace}(T) := \sum_{i \in I} \langle Te_i, e_i \rangle < \infty$, for every orthonormal basis $\{e_i\}_{i \in I}$ for $\mathcal{H}$. It is shown that $T$ is trace class if and only if $T \in S_1(\mathcal{H})$. Also, the class of Hilbert–Schmidt operators of $\mathcal{H}$ is denoted by $S_2(\mathcal{H})$: the operator $T \in S_2(\mathcal{H})$ if and only if $\|T\|_2^2 = \sum_{i \in I} \|Te_i\|^2 < \infty$. It is well known that $K(\mathcal{H})$ and $S_p(\mathcal{H})$ are two-sided $*$-ideal of $B(\mathcal{H})$, that is, a Banach algebra under the norm (1) and the finite rank operators are dense in $(S_p(\mathcal{H}), \|\cdot\|_p)$. Moreover, for $T \in S_p(\mathcal{H})$, one has $\|T\|_p = \|T^*\|_p$ and $\|T\| \leq \|T\|_p$. If $S_1 \in B(\mathcal{H}, \mathcal{H}_1)$ and $S_2 \in B(\mathcal{H}_2, \mathcal{H})$, then $\|S_1T\|_p \leq \|S_1\|\|T\|_p$ and $\|TS_2\|_p \leq \|S_2\|\|T\|_p$. For more information about these operators, see [34,43,44,50].

2.1. Fusion Frames

We now review some definitions and primary results of fusion frames. For more information, see [20,21,33].

For each sequence $\{W_i\}_{i \in I}$ of closed subspaces in $\mathcal{H}$, the space

$$
\left( \bigoplus_{i \in I} W_i \right)_{\ell^2} = \left\{ \{f_i\}_{i \in I} : f_i \in W_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\},
$$

with the inner product

$$
\left\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \right\rangle = \sum_{i \in I} \langle f_i, g_i \rangle,
$$

is a Hilbert space.

We now give the central definition of fusion frames:

**Definition 2.1.** Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i \in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. The sequence $\{(W_i, \omega_i)\}_{i \in I}$ is called a fusion frame for $\mathcal{H}$ if there exist constants $0 < A_W \leq B_W < \infty$ such that

$$
A_W \|f\|^2 \leq \sum_{i \in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B_W \|f\|^2, \quad (f \in \mathcal{H}).
$$

The constants $A_W$ and $B_W$ are called fusion frame bounds. If we have the upper bound, we call $\{(W_i, \omega_i)\}_{i \in I}$ a Bessel fusion sequence. A fusion frame is called tight, if $A_W$ and $B_W$ can be chosen to be equal, and Parseval if $A_W = B_W = 1$. If $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i, \omega_i)\}_{i \in I}$ is called $\omega$-uniform. A fusion frame $\{(W_i, \omega_i)\}_{i \in I}$ is said to be a fusion orthonormal basis if $\mathcal{H} = \bigoplus_{i \in I} W_i$ and it is called a Riesz decomposition of $\mathcal{H}$ if for every
\( f \in \mathcal{H} \) there is a unique choice of \( f_i \in W_i \) such that \( f = \sum_{i \in I} f_i \). A family of subspaces is called complete if \( \text{span}(W_i) = \mathcal{H} \).

It is clear that every fusion orthonormal basis is a Riesz decomposition of \( \mathcal{H} \). Moreover, a family \( \{W_i\}_{i \in I} \) of closed subspaces of \( \mathcal{H} \) is a fusion orthonormal basis if and only if it is a 1-uniform Parseval fusion frame \([20]\).

Furthermore, the synthesis operator \( T_W : (\sum_{i \in I} \bigoplus W_i)_{\ell^2} \rightarrow \mathcal{H} \) for a Bessel fusion sequence \( W = \{(W_i, \omega_i)\}_{i \in I} \) is defined by
\[
T_W(\{f_i\}_{i \in I}) = \sum_{i \in I} \omega_i f_i.
\]
The adjoint operator \( T_W^* : \mathcal{H} \rightarrow (\sum_{i \in I} \bigoplus W_i)_{\ell^2} \) which is called the analysis operator is given by
\[
T_W^* f = \{\omega_i \pi_{W_i} f\}_{i \in I}, \quad (f \in \mathcal{H}).
\]
Both are bounded by \( \sqrt{B} \).

If \( W = \{(W_i, \omega_i)\}_{i \in I} \) is a fusion frame, the fusion frame operator \( S_W : \mathcal{H} \rightarrow \mathcal{H} \), which is defined by \( S_W f = T_W T_W^* f = \sum_{i \in I} \omega_i^2 \pi_{W_i} f \), is bounded (with bound \( B \)), invertible and positive \([20,33]\).

Every Bessel fusion sequence \( V = \{(V_i, v_i)\}_{i \in I} \) is called a Gavruta-dual of \( W = \{(W_i, \omega_i)\}_{i \in I} \), if
\[
f = \sum_{i \in I} \omega_i v_i \pi_{V_i} S_W^{-1} \pi_{W_i} f, \quad (f \in \mathcal{H}),
\]
for more details, see \([33]\). From here on, for simplicity we say dual instead of Gavruta-dual. The sequence of subspaces \( \widehat{W} := \{(S_W^{-1} W_i, \omega_i)\}_{i \in I} \) is also a fusion frame for \( \mathcal{H} \) and a dual of \( W \), called the canonical dual of \( W \) \([20,33]\). Rephrasing that, a Bessel fusion sequence \( V = \{(V_i, v_i)\}_{i \in I} \) is a dual of a fusion frame \( W = \{(W_i, \omega_i)\}_{i \in I} \) if and only if
\[
T_V \phi_{VW} T_W^* = I_{\mathcal{H}},
\]
where the bounded operator \( \phi_{VW} : (\sum_{i \in I} \bigoplus W_i)_{\ell^2} \rightarrow (\sum_{i \in I} \bigoplus V_i)_{\ell^2} \) is given by
\[
\phi_{VW}(\{f_i\}_{i \in I}) = \{\pi_{V_i} S_W^{-1} f_i\}_{i \in I}
\]
and \( \|\phi_{VW}\| \leq \|S_W^{-1}\| \). Also, \( V \) is called a pseudo-dual of \( W \) if \( T_V \phi_{VW} \) is an invertible operator, see \([23]\) for the discrete case. Another approach to duality \([38,39]\) uses any bounded operator \( O : (\sum_{i \in I} \bigoplus W_i)_{\ell^2} \rightarrow (\sum_{i \in I} \bigoplus V_i)_{\ell^2} \). Starting with two fusion frames the duality is defined analogously to (2), i.e. \( T_V OT_W^* = I_{\mathcal{H}} \). We stick to the Gavruta duals, but all results herein can be adapted to this other definition of duality.

Let \( \{W_i\}_{i \in I} \) be a family of closed subspaces of \( \mathcal{H} \) and \( \{\omega_i\}_{i \in I} \) a family of weights. We say that \( \{(W_i, \omega_i)\}_{i \in I} \) is a fusion Riesz basis for \( \mathcal{H} \) if \( \text{span}(W_i) = \mathcal{H} \) and there exist constants \( 0 < C \leq D < \infty \) such that for each finite subset \( J \subseteq I \) and all \( (f_j \in W_j, j \in J) \) we have
\[
C \sum_{j \in J} \|f_j\|^2 \leq \left\| \sum_{j \in J} \omega_j f_j \right\|^2 \leq D \sum_{j \in J} \|f_j\|^2.
\]
The next theorem explores fusion Riesz bases with respect to local frames and their operators.

**Theorem 2.2.** [20] Let \( \{W_i\}_{i \in I} \) be a fusion frame for \( \mathcal{H} \) and \( \{e_{ij}\}_{j \in J} \) be an orthonormal basis for \( W_i \), for each \( i \in I \). Then the following conditions are equivalent:

1. \( \{W_i\}_{i \in I} \) is a Riesz decomposition of \( \mathcal{H} \).
2. The synthesis operator \( T_W \) is one to one.
3. The analysis operator \( T_W^* \) is onto.
4. \( \{W_i\}_{i \in I} \) is a fusion Riesz basis for \( \mathcal{H} \).
5. \( \{e_{ij}\}_{i \in I, j \in J} \) is a Riesz basis for \( \mathcal{H} \).

For some results, we need a version of Theorem 2.2 formulated for general family of subspaces:

**Proposition 2.3.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a family of closed subspaces and \( \{f_{ij}\}_{j \in J} \) be a Riesz basis for \( W_i \), for each \( i \in I \) with bounds \( A_i \) and \( B_i \), respectively, such that
\[
0 < \inf_{i \in I} A_i \leq \sup_{i \in I} B_i < \infty.
\]

Then the following conditions are equivalent:

1. \( W \) is a Riesz decomposition of \( \mathcal{H} \).
2. The synthesis operator \( T_W \) is bounded and bijective.
3. The analysis operator \( T_W^* \) is bounded and bijective.
4. \( W \) is a fusion Riesz basis for \( \mathcal{H} \).
5. \( \{\omega_i f_{ij}\}_{i \in I, j \in J} \) is a Riesz basis for \( \mathcal{H} \).

**Proof.**

(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) by Theorem 2.2 for any family of closed subspaces (as those conditions imply a fusion frame property).

(4) \( \Leftrightarrow \) (5) by relating inequality (4) for fusion Riesz basis \( W \) to the one for Riesz bases \( \{w_{ij}\}_{i \in I, j \in J} \), see, e.g. [22, Theorem 3.3.7].

(4) \( \Leftrightarrow \) (2) Equation (4) is equivalent to \( T_W \) being injective, having closed range and being bounded. By the completeness we have \( (4) \Rightarrow (2) \). By the surjectivity of \( T_W \), we have completeness and so \( (2) \Rightarrow (4) \). \( \square \)

The following characterizations of fusion Riesz bases will be used frequently in this note, which is a generalization of [45, Theorem 2.2] to non-uniform fusion frames.

**Proposition 2.4.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a fusion frame in \( \mathcal{H} \). Then the following are equivalent:

1. \( W \) is a fusion Riesz basis.
2. \( S_W^{-1} W_i \perp W_j \) for all \( i, j \in I, i \neq j \).
3. \( \omega_i^2 \pi_{W_i} S_W^{-1} \pi_{W_j} = \delta_{ij} \pi_{W_j} \) for all \( i, j \in I \).

**Proof.**

(1) \( \Rightarrow \) (2) was proved in [20, Proposition 4.3].
(2) \( \Rightarrow \) (1) Suppose that \( \{e_{ij}\}_{j \in J} \) is an orthonormal basis for \( W_i \), for all \( i \in I \). Then for any \( f \in \mathcal{H} \), we have
\[
\sum_{i \in I} \sum_{j \in J_i} |\langle f, \omega_i e_{ij} \rangle|^2 = \sum_{i \in I} \omega_i^2 \|\pi_{W_i} f\|^2.
\]
It easily follows that $\{e_{ij}\}_{i \in I, j \in J_i}$ is a weighted frame [9] (with weights $\omega_i > 0$) for $\mathcal{H}$ with the frame operator $S_W$. Moreover, by Proposition 2.3, it is enough to show that $\{\omega_i e_{ij}\}_{i \in I, j \in J_i}$ is a Riesz basis for $\mathcal{H}$ or equivalently that the sequences $\{\omega_i e_{ij}\}_{i \in I, j \in J_i}$ and $\{S^{-1}_W \omega_i e_{ij}\}_{i \in I, j \in J_i}$ are biorthogonal. This immediately follows from the reconstruction formula

$$f = \sum_{i \in I} \sum_{j \in J_i} \langle f, S^{-1}_W \omega_i e_{ij} \rangle \omega_i e_{ij}, \quad (f \in \mathcal{H}).$$

(2) ⇒ (3) Suppose that $\{(W_i, \omega_i)\}_{i \in I}$ is a fusion frame in $\mathcal{H}$ and $f, g \in \mathcal{H}$. By (2), we obtain

$$\langle \pi_i, S^{-1}_W \pi_j f, g \rangle = \langle S^{-1}_W \pi_i f, \pi_j, g \rangle = 0 \quad (i \neq j).$$

In particular, suppose $\{e_{ij}\}_{j \in J_i}$ is an orthonormal basis for $W_i$, then we know by the argument above that the sequence $\{\omega_i e_{ij}\}_{i \in I, j \in J_i}$ is a Riesz basis for $\mathcal{H}$ with the frame operator $S_W$. Hence, $\{S^{-1}_W \omega_i e_{ij}\}_{i \in I, j \in J_i}$ is an orthonormal basis for $\mathcal{H}$. Let $f, g \in \mathcal{H}$ and take

$$\pi_i, f = \sum_{j \in J_i} c_{ij} e_{ij}, \quad \pi_i, g = \sum_{j \in J_i} d_{ij} e_{ij},$$

for all $i \in I$, for some $\{c_{ij}\}_{j \in J_i}$ and $\{d_{ij}\}_{j \in J_i}$ in $\ell^2$. Then for all $i \in I$ we have

$$\langle \omega^2_i \pi_i, S^{-1}_W \pi_i f, g \rangle = \left\langle S^{-1/2}_W \omega_i \pi_i f, S^{-1/2}_W \omega_i \pi_i g \right\rangle = \sum_{j \in J_i} c_{ij} S^{-1/2}_W \omega_i e_{ij}, \sum_{k \in J_i} d_{ik} S^{-1/2}_W \omega_i e_{ik} = \sum_{j, k \in J_i} c_{ij} d_{ik} \left\langle S^{-1/2}_W \omega_i e_{ij}, S^{-1/2}_W \omega_i e_{ik} \right\rangle = \sum_{j \in J_i} c_{ij} \sum_{k \in J_i} d_{ik} e_{ik} = \langle \pi_i f, g \rangle.$$

So, $\omega^2_i \pi_i, S^{-1}_W \pi_i = \pi_i$.

(2) ⇒ (3) By the assumption, for all $i \in I$ we have

$$\pi_i = \pi_i, S^{-1}_W S_W = \sum_{j \in I} w^2_i \pi_i, S^{-1}_W \pi_j = w^2_i \pi_i, S^{-1}_W \pi_i.$$

(3) ⇒ (2) Let $f \in W_i$ and $g \in W_j$, where $i \neq j$. Then

$$\langle S^{-1}_W f, g \rangle = \langle S^{-1}_W \pi_i f, \pi_j g \rangle = \langle \pi_i, S^{-1}_W \pi_i f, g \rangle = 0.$$

All items in Proposition 2.4 include a dependency on weights, for the second, the weight is included in the definition of $S_W$. So, the question arises how dependent the Riesz property is on the considered weights.
Remark 2.5. Note that weights are not included in the definition of the Riesz decomposition. Obviously, we have that \( f = \sum_{i \in I} f_i \) with unique \( f_i \)s, if and only if \( f = \sum_{i \in I} w_i f_i \) with the same uniqueness.

Using Proposition 2.3, the following lemma immediately follows.

Lemma 2.6. Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) and \( V = \{(W_i, v_i)\}_{i \in I} \) be a family of subspaces in \( \mathcal{H} \) with different weights. Then \( W \) is a fusion Riesz basis if and only if \( V \) is a fusion Riesz basis.

It is worthwhile to mention that, by the fusion frame definition, the family of weights belongs to \( \ell_+^\infty \) assuming that the subspaces are non-zero, see [39, Remark 2.4]. Moreover, if \( W = \{(W_i, w_i)\}_{i \in I} \) is a fusion Riesz basis, then (4) shows that

\[
\sqrt{C} \leq w_i \leq \sqrt{D}, \quad (i \in I).
\]

This could also be seen as direct consequence of Proposition 2.3, as the Riesz decomposition property, e.g. item (1) is independent of the weight.

In the sequel, for a given fusion Riesz basis \( W = \{(W_i, w_i)\}_{i \in I} \), we denote by \( W' \) the 1-uniform family of subspaces \( \{(W_i, 1)\}_{i \in I} \).

We will use the following criterion for the invertibility of operators.

Proposition 2.7. [34] Let \( F : \mathcal{H} \to \mathcal{H} \) be invertible on \( \mathcal{H} \). Suppose that \( G : \mathcal{H} \to \mathcal{H} \) is a bounded operator and \( \|Gh - Fh\| \leq v\|h\| \) for all \( h \in \mathcal{H} \), where \( v \in [0, \|F^{-1}\|) \). Then \( G \) is invertible on \( \mathcal{H} \) and \( G^{-1} = \sum_{k=0}^{\infty} [F^{-1} (F - G)]^k F^{-1} \).

3. U-Fusion Cross Gram Matrix of Operators

In this section, we extend the notion of cross Gram matrices [15] to fusion frames and discuss on their invertibility.

We interpret the representation of operators using fusion frames [14] as a generalization of the Gram matrix of operators.

Definition 3.1. Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a Bessel fusion sequence for \( \mathcal{H} \) and \( V = \{(V_i, v_i)\}_{i \in I} \) a fusion frame for \( \mathcal{H} \). For \( U \in B(\mathcal{H}) \), the matrix operator \( G_{U,W,V} : (\sum_{i \in I} \oplus W_i)_{\ell^2} \to (\sum_{i \in I} \oplus W_i)_{\ell^2} \) given by

\[
G_{U,W,V} = \phi_W V^* U T_W
\]

is called the U-fusion cross Gram matrix. If \( U = I_{\mathcal{H}} \), it is called fusion cross Gram matrix and denoted by \( G_{W,V} \). We use \( G_W \) for \( G_{W,W} \), the so-called fusion Gram matrix.

Note that

\[
[G_{U,W,V}(f_i)]_j = \phi_W \left\{ v_j \pi_{V_j} U \sum_i w_i f_i \right\}_{j \in I} = \sum_i w_i v_j \pi_{W_j} S^{-1}_V \pi_{V_j} U f_i \bigg|_{B_{j,i}} = \sum_i B_{j,i} f_i.
\]
where \( B_{j,i} : W_i \to W_j \). Therefore, \( G_{U,W,V} \) is a block-matrix of operators \([6]\), which motivates the name (cross)Gram matrix.

Clearly, using (3), \( U \)-fusion cross Gram matrices are well defined and
\[
\| G_{U,W,V} \| = \| \phi_{WV} T^*_V U T_W \|
\leq \| \phi_{WV} \| \| T^*_V \| \| U \| \| T_W \|
\leq \| S^{-1}_V \| \| U \| \sqrt{B_W B_V}
\leq \frac{\sqrt{B_W B_V}}{A_V} \| U \|.
\]

We have chosen to use \( G_{U,W,V} = \phi_{WV} T^*_V U T_W \) instead of the ‘naive’ \( G_{U,W,V} = T^*_V U T_W \). The reason for that is that, by this definition, \( G_{U,W,V} \) maps \( \sum_{i \in I} \bigoplus W_i \) into itself and is a projection as in the Hilbert space case (albeit an oblique one, see Remark 3.3).

We can represent an operator \( U \in B(H) \) from its \( U \)-fusion cross Gram matrix. Suppose \( W \) is a dual fusion frame of \( V \), then
\[
T_W G_{U,W,V} T^*_W S^{-1}_W = T_W \phi_{WV} T^*_V U T_W T^*_W S^{-1}_W = U. \tag{5}
\]

From the ideal property of \( S_p(H) \) in \( B(H) \), the following is apparent.

**Corollary 3.2.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) and \( V = \{(V_i, v_i)\}_{i \in I} \) be fusion frames in \( H \). The operator \( U \) is compact, respectively, Schatten-\( p \) (for \( 1 \leq p \leq \infty \)) if and only if \( G_{U,W,V} \) is.

**Remark 3.3.** Note that as in the discrete Hilbert space frame case \( G_{V,W} = G^2_{V,W} \) and so this is an oblique projection whenever \( V \) is a dual fusion frame of \( W \). Also, \( T_U G_{V,W} = T_V \), and \( G_{V,W} (\phi_{VW} T^*_W) = \phi_{VW} T^*_W \), but in general the operator is not self-adjoint. Using the above identities and the definition of \( G_{V,W} \), we achieve
\[
\ker (G_{V,W}) = \ker (T_V)
\]
and
\[
\text{ran} (G_{V,W}) = \left\{ \{ \pi_i S^{-1}_W f_i \}_{i \in I} : \{ f_i \}_{i \in I} \in \text{ran} (T^*_W) \right\}.
\]

Moreover,
\[
\phi_{VW} G_{U,W,V}^* = G_{U^*,V,W}^* \phi_{WV}.
\]

In the next result, we are going to characterize Gram matrices of fusion orthonormal bases.

**Proposition 3.4.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a fusion frame. The following are equivalent.

1. \( W \) is a fusion orthonormal basis.
2. \( G_W = I \sum_{i \in I} \bigoplus W_i \).
3. \( G_{W^*} = I \sum_{i \in I} \bigoplus W_i^* \).

---

1 This could be called a generalized subband matrix, motivated by system identification applications \([41]\).
Proof. (1) ⇒ (2) Assume that \( W \) is a fusion orthonormal basis. Applying (6) for all \( f = \{ f_i \}_{i \in I} \subseteq \bigoplus_{i \in I} W_i \), we have

\[
\mathcal{G}_W f = \phi_W W^* T_W f
\]

\[
= \phi_W \left\{ \sum_{j \in I} w_j f_j \right\}
\]

\[
= \left\{ \frac{1}{w_i^2} \sum_{j \in I} w_j f_j \right\}
\]

\[
= \{ \pi_W f_i \} \in I.
\]

(2) ⇒ (1) If \( \mathcal{G}_W = \phi_W W^* T_W T_W^{-1} \), then \( \phi_W W^* \) is invertible and so, \( W \) is a fusion Riesz basis. By Proposition 2.4, we obtain

\[
S_W = S_W T_W T_W^{-1}
\]

\[
= \sum_{i \in I} \pi_W, T_W T_W^{-1}
\]

\[
= \sum_{i \in I} w_i^2 \pi_W, S_W^{-1} \pi_W, T_W T_W^{-1}
\]

\[
= T_W \phi_W W^* T_W T_W^{-1}
\]

\[
= T_W \mathcal{G}_W T_W^{-1} = I_H.
\]

(1) ⇒ (3) It follows from (1) ⇒ (2) and the fact that \( W \) is a fusion orthonormal basis if and only if \( W' \) is a fusion orthonormal basis.

(3) ⇒ (1) If \( \mathcal{G}_{W'} = \phi_{W'} W' T_W T_W^* \), then \( \phi_{W'} W' = \phi_{W'} W' \) is invertible and so, \( W' \) is a fusion Riesz basis. It is enough to show that \( W' \) is a Parseval fusion frame by Proposition 3.23 of [20].

\[
S_{W'}^2 = T_W T_W T_W T_W^* = T_W \mathcal{G}_W T_W T_W^* = S_{W'}.
\]

Now, the invertibility of \( S_{W'} \) implies \( S_{W'} = I_H \).

3.1. Invertibility of \( U \)-Cross Gram Matrices

We now discuss the relationship between the invertibility of Gram matrices and their associated operators.

Proposition 3.5. Let \( W = \{ (W_i, \omega_i) \}_{i \in I} \) be a fusion frame in \( H \) and \( U \in B(H) \) an invertible operator. The following are equivalent:

1. \( W \) is a fusion Riesz basis.
2. \( \mathcal{G}_{U, W, W} \) is onto.
3. \( \mathcal{G}_{U, W, W} \) is one to one.
4. \( \mathcal{G}_{U, \widetilde{W}, W} \) is invertible.
5. \( \mathcal{G}_{U, W, W} \) is onto.
6. \( \mathcal{G}_{U, \widetilde{W}, W} \) is one to one.
Proof. (1) ⇒ (2) For every fusion Riesz basis $W$, $\phi_{WW}$ is invertible. Indeed, by Proposition 2.4, we get
\[ \phi_{WW}\{f_i\}_{i \in I} = \{\pi_W S_W^{-1} f_i\}_{i \in I} = \left\{ \frac{1}{w_i^2} f_i \right\}_{i \in I} \] (6)
for all $\{f_i\}_{i \in I} \subseteq \bigoplus_{i \in I} W_i$.

(2) ⇒ (1) If $G_{U,W,W} = \phi_{WW} T_W U T_W$ is onto, then $\phi_{WW} = \phi_{WW}$ is invertible. Hence, $T_{WW}^* U T_W = \phi_{WW}^{-1} G_{U,W,W}$ is onto. Thus, $W$ is a fusion Riesz basis.

(1) ⇒ (4) Using Theorem 2.9 of [5], $\tilde{W}$ is also a fusion Riesz basis and so by (6), $T_{\tilde{W}}$ is invertible by Theorem 2.2. Hence, $G_{U,\tilde{W},\tilde{W}} = \phi_{\tilde{W}} T_{\tilde{W}} U T_{\tilde{W}}$ is invertible by the invertibility of $U$ and $\phi_{\tilde{W}}\{f_i\}_{i \in I} = \{S_W^{-1} f_i\}_{i \in I}$, for all $\{f_i\}_{i \in I} \subseteq \bigoplus_{i \in I} W_i$.

(4) ⇒ (1) Note that $\phi_{\tilde{W}}$ is invertible by the definition of $\phi_{\tilde{W}}$. Also, $T_{\tilde{W}}$ is one to one since $G_{U,\tilde{W},\tilde{W}} = \phi_{\tilde{W}} T_{\tilde{W}} U T_{\tilde{W}}$ is invertible and, therefore, $\tilde{W}$ is a fusion Riesz basis by Theorem 2.2. By Theorem 2.9 of [5], $W$ is also a fusion Riesz basis. Applying Theorem 2.2, $T_{\tilde{W}}^*$ and $T_{\tilde{W}}$ are invertible and it immediately follows the desired result.

(1) ⇔ (3) ⇔ (5) ⇔ (6) can be proved in an analogue way. □

Similar to the question, when the inverse of a frame multiplier is a multiplier again [7,16], we can show that for fusion Riesz bases the operator keeps its structure. To state our results in a accessible way, we need a new definition:

Definition 3.6. Let $W = \{(W_i, w_i)\}_{i \in I}$ be a fusion frame and $V = \{(V_i, v_i)\}_{i \in I}$ a Bessel fusion sequence. The alternate cross-fusion frame operator $L_{VW} : \mathcal{H} \rightarrow \mathcal{H}$ is defined by
\[ L_{VW} = T_V \phi_{WW}^* T_W^*. \]
We denote $L_{WW}$ as $L_W$.

It follows that $\|L_{VW}\| \leq \sqrt{\frac{B_W B_V}{A_W}}$ and
\[ (L_{VW})^* = \sum_{i \in I} \pi_W S_W^{-1} \pi_W. \]
Obviously, $L_{VW}$ is an invertible operator if and only if $V$ is a pseudo-dual of $W$. In particular, $L_{VW} = I_{\mathcal{H}}$ if and only if $V$ is a dual of $W$.

In the following, we summarize the basic properties of $L_W$.

Proposition 3.7. Let $W = \{(W_i, w_i)\}_{i \in I}$ be a fusion frame. Then $L_W$ is positive, self-adjoint and invertible operator.

Proof. It is easy to see that $\phi_{WW}$ is self-adjoint and so, $L_W$ is self-adjoint. Moreover, for all $f \in \mathcal{H}$ we have
\[ \langle L_W f, f \rangle = \left\langle \sum_{i \in I} w_i^2 \pi_W S_W^{-1} \pi_W f, f \right\rangle \]
\[
\begin{align*}
&= \sum_{i \in I} \left\langle w_i \cdot S_W^{-1/2} \pi_w, f, w_i \cdot S_W^{-1/2} \pi_w, f \right\rangle \\
&= \left( \sum_{i \in I} w_i^2 \left\| S_W^{-1/2} \pi_w \right\|^2 \right) \\
&\geq \left\| S_W^{-1/2} \right\|^{-2} \left\| T_W \pi_w \right\|^2 \geq \frac{A_W}{\left\| S_W^{-1/2} \right\|^2} \left\| f \right\|^2,
\end{align*}
\]

where \( A_W \) is a lower bound of the fusion frame \( W \). This shows that \( L_W \) is positive and an invertible operator in \( B(\mathcal{H}) \).

As an easy consequence of Proposition 2.4, we state

**Corollary 3.8.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a fusion Riesz basis. Then \( L_W = S_W \).

**Theorem 3.9.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) and \( V = \{(V_i, \nu_i)\}_{i \in I} \) be fusion frames in \( \mathcal{H} \) and \( U \) an invertible operator in \( B(\mathcal{H}) \). Then the following assertions hold.

1. If \( \mathcal{G}_{U,W,W} \) is invertible, then \( W \) is a fusion Riesz basis and

\[ \mathcal{G}_{U,W,W}^{-1} = \mathcal{G}_{S_W^{-1},U^{-1},S_W^{-1},W,W}^{-1}. \]

2. If \( \mathcal{G}_{U,V,W} \) is invertible and \( V \) a dual of \( W \), then \( V \) is a fusion Riesz basis and \( \mathcal{G}_{U,V,W}^{-1} = \mathcal{G}_{U^{-1},V,W} \).

3. If \( \mathcal{G}_{U,W,V} \) is invertible and \( V \) a pseudo-dual of \( W \), then \( W \) is a fusion Riesz basis and

\[ \mathcal{G}_{U,W,V}^{-1} = \mathcal{G}_{L_W V S_W, W, W}^{-1}. \]

**Proof.**

1. By Proposition 3.5 \( W \) is a fusion Riesz basis. Using Corollary 3.8 follows that

\[ \mathcal{G}_{U,W,W} \mathcal{G}_{S_W^{-1},U^{-1},S_W^{-1},W,W} = \phi_W W T_W^* U T_W \phi_W W T_W^* S_W^{-1} U^{-1} S_W^{-1} T_W \]

\[ = \phi_W W T_W^* S_W^{-1} T_W = I(\Sigma_{i \in I} \Theta \bar{W}_i)_{\ell^2}. \]

With the same we have \( \mathcal{G}_{S_W^{-1},U^{-1},S_W^{-1},W,W} \mathcal{G}_{U,W,W} = I(\Sigma_{i \in I} \Theta \bar{W}_i)_{\ell^2}. \)

2. According to (2), we have \( T_V \phi_W W T_W^* = I_{\mathcal{H}}. \) In addition, the invertibility of \( \mathcal{G}_{U,V,W} \) implies that \( \phi_W W T_W^* \) has a right inverse. Using Proposition 2.2, \( V \) is a fusion Riesz basis and, therefore,

\[ \phi_V W T_W^* T_V = I(\Sigma_{i \in I} \Theta V_i)_{\ell^2}. \]

Hence,

\[ \mathcal{G}_{U,V,W} \mathcal{G}_{U^{-1},V,W} = \phi_V W T_W^* U T_V \phi_V W T_W^* U^{-1} T_V \]

\[ = \phi_V W T_W^* T_V = I(\Sigma_{i \in I} \Theta V_i)_{\ell^2}. \]

Similarly, \( \mathcal{G}_{U^{-1},V,W} \mathcal{G}_{U,V,W}^{-1} = I(\Sigma_{i \in I} \Theta W_i)_{\ell^2}. \)

3. It follows from Corollary 3.8.

\[ \square \]
Repeating the previous argument and using (2) leads to a characterization for fusion Riesz bases due to $U$-fusion cross Gram matrices.

**Theorem 3.10.** Let $V = \{(V_i,v_i)\}_{i \in I}$ be a dual fusion frame of $W = \{(W_i,\omega_i)\}_{i \in I}$ in $\mathcal{H}$. The following are equivalent:

1. $V$ is a fusion Riesz basis.
2. $G_{V,W} = I ([\sum_{i \in I} \Phi V_i])_{\ell^2}$. 
3. $G_{V,W}$ has a left inverse.

Consider $W = \{(W_i,\omega_i)\}_{i \in I}$, $V = \{(V_i,v_i)\}_{i \in I}$ and $Z = \{(Z_i,z_i)\}_{i \in I}$ as fusion frames in $\mathcal{H}$ and $U_1, U_2 \in B(\mathcal{H})$, it is obvious to see that

$$G_{U_1,W,V} G_{U_2,W,Z} = G_{U_1 U_2, W,V}.$$  

In particular, if $V = \{(V_i,v_i)\}_{i \in I}$ is a dual of $Z = \{(Z_i,z_i)\}_{i \in I}$, then

$$G_{U_1,V,W} G_{U_2,V,Z} = G_{U_1 U_2, V,W}.$$  

As a special case, if $U$ is invertible and $V$ a fusion Riesz basis such that $V$ is a dual of $W$, then $G_{U,V,W}$ has an inverse in the form of Gram matrices

$$(G_{U,V,W})^{-1} = G_{U^{-1},V,W}.$$  

The above identity is also proved in Theorem 3.9.

### 3.2. Pseudo-inverses

In the following, we discuss the pseudo-inverse of $U$-fusion cross Gram matrices with closed range, and under some conditions we represent their pseudo-inverse as a $U$-fusion cross Gram matrix again motivated by the discrete frame case [15]. In the following, we state a sufficient condition for a $U$-fusion cross Gram matrix having closed range.

**Lemma 3.11.** Let $V = \{(V_i,v_i)\}_{i \in I}$ be a dual fusion frame of $W = \{(W_i,\omega_i)\}_{i \in I}$ in $\mathcal{H}$ and $U$ an operator in $B(\mathcal{H})$ such that $UV = \{(UV_i,v_i)\}_{i \in I}$ is also a fusion frame in $\mathcal{H}$. Then $G_{U,V,W}$ has closed range and

$$\text{ran } (G_{U,V,W}) = \text{ran } (\phi_{V,W} T_W^*) .$$

**Proof.** Dual condition (2) implies that $\phi_{V,W} T_W^*$ has closed range. Thus,

$$\text{ran } (G_{U,V,W}) = \text{ran } (\phi_{V,W} T_W^* U \Phi V)$$

$$= \text{ran } (\phi_{V,W} T_W^* T_{VV})$$

$$= \text{ran } (\phi_{V,W} T_W^* ) .$$

\[ \square \]

The assumptions in the above lemma are fulfilled, in particular, for $U$ being invertible [33].

**Theorem 3.12.** Let $V = \{(V_i,v_i)\}_{i \in I}$ be a dual fusion frame of $W = \{(W_i,\omega_i)\}_{i \in I}$ in $\mathcal{H}$, also let both $U \in B(\mathcal{H})$ and $G_{U,V,W}$ have closed range. The following are equivalent:

1. $G_{U,V,W}^{\dagger} = G_{U^{-1},V,W}$.
2. $\text{ran } (\phi_{V,W} T_W^* U^*) = \text{ran } (T_{V,U}^*)$ and $\text{ran } (\phi_{V,W} T_W^* U) = \text{ran } (T_{V,U}^*)$.  


(3) \( \phi_{VW} T^*_W U^* = T^*_V S^{-1}_V U^* \) and \( \phi_{VW} T^*_W U = T^*_V S^{-1}_V U \).

**Proof.** (2) \( \Rightarrow \) (1) Applying (2) it follows that

\[
G_{U,V,W} G_{U^*,V,W} G_{U,V,W} = \phi_{VW} T^*_W U^* T_V \phi_{VW} T^*_W U^* T_V \phi_{VW} T^*_W U^* T_V
\]

\[
= \phi_{VW} T^*_W U^* U T_V
\]

\[
= G_{U,V,W}.
\]

In addition, by (2) \( T_W \phi_{VW}^* \) is surjective. Hence, we have

\[
\text{ran} \left( G_{U^*,V,W} \right) = \text{ran} \left( \phi_{VW} T^*_W U^* T_V \right)
\]

\[
= \text{ran} \left( \phi_{VW} T^*_W U^* \right)
\]

\[
= \text{ran} \left( T^*_V U^* \right)
\]

\[
= \text{ran} \left( T^*_V U^* T_W \phi_{VW}^* \right)
\]

\[
= \text{ran} \left( G^*_{U,V,W} \right).
\]

Moreover,

\[
\ker \left( G_{U,V,W} \right) = \ker \left( \phi_{VW} T^*_W U^* T_V \right)
\]

\[
= \ker \left( U^* T_V \right)
\]

\[
= \ker \left( \left( T^*_V (U^*)^\perp \right)^\perp \right)
\]

\[
= \ker \left( T^*_V U^* \right)^\perp
\]

\[
= \ker \left( \phi_{VW} T^*_W U \right)^\perp
\]

\[
= \ker \left( U^* T_W \phi_{VW}^* \right)
\]

\[
= \ker \left( G^*_{U,V,W} \right).
\]

Thus, (1) is obtained. The converse follows immediately from the above identities. To show (2) \( \Rightarrow \) (3) using Douglas’s theorem [27], there is an operator \( C \in B(H) \) such that

\[
\phi_{VW} T^*_W U = T^*_V U C.
\]

So,

\[
\phi_{VW} T^*_W U = T^*_V S^{-1}_V T_V T^*_V U C
\]

\[
= T^*_V S^{-1}_V T_V \phi_{VW} T^*_W U = T^*_V S^{-1}_V U.
\]

The other identity is obtained similarly. The converse is clear.

Note that fusion Riesz bases satisfy the assumptions of Theorem 3.12. These assumptions might seem obvious, but are not [33]. In particular, the failure to fulfill this equality makes the concept of duality of fusion frames interesting, so fusion frames are not ’just another’ generalization of frames.

**Remark 3.13.** In the proof of Theorem 3.12(1), we can replace the fusion frame appeared in the right side by any fusion frame \( Z \) such that \( V \) is its dual. More precisely, let \( W \) and \( Z \) be fusion frames and \( V \) is a dual of both \( W \) and \( Z \). Then

\[
G^*_{U,V,W} = G_{U^*,V,Z},
\]

if and only if \( \phi_{VZ} T^*_Z U^* = T^*_V S^{-1}_V U^* \) and \( \phi_{VZ} T^*_Z U = T^*_V S^{-1}_V U \).
Using a similar argument as Theorem 3.12 we obtain the following.

**Theorem 3.14.** Let \( W = \{(W_i, \omega_i)\}_{i \in I} \) be a fusion frame in \( \mathcal{H} \), also let \( U \in B(\mathcal{H}) \) and \( \mathcal{G}_{U,W,W} \) have closed range. The following are equivalent:

1. \((\mathcal{G}_{U,W,W})^* = \mathcal{G}_{L^{-1}_W U^*, L^{-1}_W W, W}\).
2. The operators \( \phi_{WW} T_W^* L_W^{-1} U^* \) and \( T_W^* U \) have the same range as \( \phi_{WW} T_W^* W \) and \( T_W (L_W^{-1})^* U \), respectively, for all \( i \in I \).
3. \( \phi_{WW} T_W^* L_W^{-1} U^* = T_W^* S_W^{-1} U^* \) and \( \phi_{WW} T_W^* U = T_W^* S_W^{-1} U \).

4. **Stability of \( U \)-Cross Gram Matrices of Operators**

In this section, we state a general stability for the invertibility of \( U \)-fusion cross Gram matrices, compared to the results on the invertibility of multipliers [48].

**Theorem 4.1.** Let \( W = \{(W_i, v_i)\}_{i \in I} \) be a Bessel fusion sequence and \( U_1, U_2 \in B(\mathcal{H}) \) with \( \|U_1 - U_2\| < \mu \). Also, let \( V = \{(V_i, v_i)\}_{i \in I} \) and \( Z = \{(Z_i, v_i)\}_{i \in I} \) be fusion frames on \( \mathcal{H} \) such that \( \mathcal{G}_{U_1, W, V} \) is invertible and

\[
\left( \sum_{i \in I} v_i^2 \|\pi_Z f - \pi_V f \|^2 \right)^{\frac{1}{2}} \leq \lambda_1 \left( \sum_{i \in I} v_i^2 \|\pi_Z f \|^2 \right)^{\frac{1}{2}} + \lambda_2 \left( \sum_{i \in I} v_i^2 \|\pi_V f \|^2 \right)^{\frac{1}{2}} + \epsilon \|f\|,
\]

for all \( f \in \mathcal{H} \), in addition

\[
\mu + \left( \lambda_1 + \lambda_2 + \frac{\epsilon}{\sqrt{B}} \right) \left( \sqrt{B} \|S_Z^{-1}\| \left( \sum_{i \in I} |v_i|^2 \right)^{\frac{1}{2}} \|U_2\| + \|U_2\| \right) < \frac{\|G_{U_1, V, W}^{-1}\|^{-1}}{B \|S_Z^{-1}\|},
\]

where \( \epsilon > 0 \), \( B = \max \{B_W, B_V, B_Z\} \) and \( \sum_{i \in I} |v_i|^2 < \infty \). Then \( \mathcal{G}_{U_2, W, Z} \) is also invertible.

**Proof.** First, note that

\[
\| (S_V - S_Z) f \| = \left\| \sum_{i \in I} v_i^2 (\pi_V f - \pi_Z f) \right\| \\
\leq \left( \sum_{i \in I} |v_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in I} v_i^2 \| (\pi_V f - \pi_Z f) \|^2 \right)^{\frac{1}{2}},
\]

for all \( f \in \mathcal{H} \). Therefore,

\[
\| (S_V^{-1} - S_Z^{-1}) f \| = \| S_V^{-1} (S_V - S_Z) S_Z^{-1} f \| \\
\leq \| S_V^{-1} \| \| (S_V - S_Z) S_Z^{-1} f \| \\
\leq \| S_V^{-1} \| \left( \sum_{i \in I} |v_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in I} v_i^2 \| (\pi_V f - \pi_Z f) \|^2 \right)^{\frac{1}{2}},
\]

\[
\leq \| S_V^{-1} \| \left( \sum_{i \in I} |v_i|^2 \right)^{\frac{1}{2}}.
\]
Finally, applying (8), we obtain

\[
\left( \sum_{i \in I} v_i^2 \left\| (S_{V}^{-1} \pi_{V_i} - S_{Z}^{-1} \pi_{Z_i}) f \right\|^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{i \in I} v_i^2 \left( \left\| (S_{V}^{-1} \pi_{V_i} - S_{V}^{-1} \pi_{Z_i}) f \right\| + \left\| (S_{V}^{-1} \pi_{Z_i} - S_{Z}^{-1} \pi_{Z_i}) f \right\|^2 \right) \right)^{\frac{1}{2}} \\
\leq \left( \sum_{i \in I} v_i^2 \left( \left\| S_{V}^{-1} \right\| \left\| \pi_{V_i} f - \pi_{Z_i} f \right\| + \left\| S_{V}^{-1} - S_{Z}^{-1} \right\| \left\| \pi_{Z_i} f \right\|^2 \right) \right)^{\frac{1}{2}} \\
\leq \left[ \left( \sum_{i \in I} v_i^2 \left\| S_{V}^{-1} \right\|^2 \left\| \pi_{V_i} f - \pi_{Z_i} f \right\|^2 \right)^{\frac{1}{2}} + \left\| S_{V}^{-1} - S_{Z}^{-1} \right\| \left( \sum_{i \in I} v_i^2 \left\| \pi_{Z_i} f \right\|^2 \right)^{\frac{1}{2}} \right] \\
\leq \left\| S_{V}^{-1} \right\| \left( \sqrt{\lambda_1 B_{Z}} + \sqrt{\lambda_2 B_{V}} + \epsilon \right) \left\| f \right\| \\
+ \left\| S_{V}^{-1} \right\| \left\| S_{Z}^{-1} \right\| \sqrt{B_{Z}} \left( \sum_{i \in I} v_i^2 \right)^{\frac{1}{2}} \left( \sqrt{\lambda_1 B_{Z}} + \sqrt{\lambda_2 B_{V}} + \epsilon \right) \left\| f \right\| \\
\leq \left\| S_{V}^{-1} \right\| \left( \sqrt{\lambda_1 B_{Z}} + \sqrt{\lambda_2 B_{V}} + \epsilon \right) \left( 1 + \left\| S_{Z}^{-1} \right\| \sqrt{B_{Z}} \left( \sum_{i \in I} v_i^2 \right)^{\frac{1}{2}} \right) \left\| f \right\|.
\]

Finally, applying (8), we obtain

\[
\left\| G_{U_1,W,V} - G_{U_2,W,Z} \right\| \\
= \left\| \phi_{W,V} T_{V} U_1 T_W - \phi_{W,Z} T_{Z} U_2 T_W \right\| \\
\leq \left\| \phi_{W,V} T_{V} U_1 T_W - \phi_{W,Z} T_{Z} U_2 T_W \right\| + \left\| \phi_{W,V} T_{V} U_2 T_W - \phi_{W,Z} T_{Z} U_2 T_W \right\| \\
\leq \left\| S_{V}^{-1} \right\| \sqrt{B_{W} B_{V}} \left\| U_1 - U_2 \right\| + \left( \sum_{i \in I} v_i^2 \left\| (S_{V}^{-1} \pi_{V_i} - S_{Z}^{-1} \pi_{Z_i}) U_2 T_W \right\|^2 \right)^{\frac{1}{2}} \\
\leq \left\| S_{V}^{-1} \right\| \sqrt{B_{W} B_{V}} + \left( \sum_{i \in I} v_i^2 \left\| (S_{V}^{-1} \pi_{V_i} - S_{Z}^{-1} \pi_{Z_i}) \right\|^2 \right)^{\frac{1}{2}} \left\| U_2 \right\| \sqrt{B_{W}} \\
\leq \left\| S_{V}^{-1} \right\| \sqrt{B_{W} B_{V}} + \left( \sum_{i \in I} v_i^2 \left\| (S_{V}^{-1} \pi_{V_i} - S_{Z}^{-1} \pi_{Z_i}) \right\|^2 \right)^{\frac{1}{2}} \left\| U_2 \right\| \sqrt{B_{W}} \\
+ \left\| S_{V}^{-1} \right\| \left( \sqrt{\lambda_1 B_{Z}} + \sqrt{\lambda_2 B_{V}} + \epsilon \right) \left( 1 + \left\| S_{Z}^{-1} \right\| \sqrt{B_{Z}} \left( \sum_{i \in I} v_i^2 \right)^{\frac{1}{2}} \right) \left\| U_2 \right\| \sqrt{B_{W}} \\
< \left\| G_{U_1,W,V} \right\|^{-1}.
\]

Therefore, \( G_{U_2,W,Z} \) is also invertible by Proposition 2.7. □
Remark 4.2. It is worthwhile to mention that if we consider in Theorem 4.1

1. the perturbation condition

$$\| \pi Z_i f - \pi V_i f \| \leq \lambda_1 v_i \| \pi Z_i f \| + \lambda_2 v_i \| \pi V_i f \| + \epsilon \| f \|,$$

we can replace the assumption $\sum_{i \in I} |v_i|^2 < \infty$ by bounded weights. Hence, uniform fusion frames satisfy a slightly different version of this theorem.

2. the condition

$$\left( \sum_{i \in I} \| \pi Z_i f - \pi V_i f \|^2 \right)^{\frac{1}{2}} \leq \lambda_1 \left( \sum_{i \in I} v_i^2 \| \pi Z_i f \|^2 \right)^{\frac{1}{2}} + \lambda_2 \left( \sum_{i \in I} v_i^2 \| \pi V_i f \|^2 \right)^{\frac{1}{2}} + \epsilon \| f \|,$$

instead of (3.4), we get the same result substituting $\sum_{i \in I} |v_i|^4$ for $\sum_{i \in I} |v_i|^2$.

As a consequence, we obtain the following result.

Corollary 4.3. Let $W = \{(W_i, v_i)\}_{i \in I}$ be a fusion Riesz basis with bounds $A_W$ and $B_W$ and $Z = \{(Z_i, v_i)\}_{i \in I}$ a fusion frame in $H$ such that (7) holds. Also, $U \in B(H)$ with $\|U - I_H\| < \mu$. If

$$\mu + \left( \lambda_1 + \lambda_2 + \frac{\epsilon}{\sqrt{B}} \right) \left( \sqrt{B} \left\| S_Z^{-1} \right\| \left( \sum_{i \in I} |v_i|^2 \right)^{\frac{1}{2}} \| U \| + \| U \| \right) < \frac{A_W}{B \left\| S_W^{-1} \right\|},$$

where $\epsilon > 0$ and $B = \max\{B_W, B_Z\}$ and $\sum_{i \in I} |v_i|^2 < \infty$. Then $G_{U, W, Z}$ is also invertible.

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