Design of cylindrical composite panels with mild camber for biaxial compression taking into account geometrical non-linearity with uniform heating and rigid support

O V Mitrofanov and M V Klesareva
Moscow Aviation Institute (National Research University), 4 Volokolamskoye Highway, Moscow 125993, Russia

e-mail: Oleg1Mitrofanov@yandex.ru

Abstract. Based on geometrically non-linear task analytic solution for cylindrical composite panel with mild camber method of best possible design for post-buckling state with biaxial compression is suggested. Uniform heating and panel all-boundary rigid support are taken into account in basic correlations.

1. Introduction
Let us consider fuselage panel best possible thickness definition with acceptable post-buckling behavior in case of loads close to ultimate level. It should be noted that with loading below limit level buckling of the considered panel is not acceptable. Let us assume that the investigation object is cylindrical composite panel with mild camber under biaxial compression with all-boundary rigid support. Uniform heating influence on the panel which can be for example a fuselage skin fragment should also be taken into account. Task of design based on post-buckling state with compression taking into account uniform thermal effect \( t \) is of practical interest.

2. Main correlations
Let us use correlations of composite thin-walled structures theory by V. Vasiliev [1]. Let us put down physical equations for composite multiple sandwich taking into account thermal effect:

\[
\begin{align*}
\varepsilon_x & = \frac{p_x - \mu_{xy} y}{E_x} - \alpha_x t, \quad \varepsilon_y = \frac{p_y - \mu_{xy} x}{E_y} - \alpha_y t \\
\end{align*}
\]

with \( \alpha_x, \alpha_y \) being effective coefficients of linear thermal expansion

\[
\alpha_x = \frac{B_1 B_{22} - B_2 B_{12}}{B_1 B_{22} - B_{12}^2}, \quad \alpha_y = \frac{B_2 B_{11} - B_1 B_{12}}{B_1 B_{22} - B_{12}^2}, \quad B_{ip} = \sum_{i=1}^{k} b_{i}^{p}, \quad p = 1, 2,
\]

\[
b_{11}^{i} = E_1 \left( \alpha_1^{i} + \mu_{12} \alpha_2^{i} \right) \cos^2 \phi_i + E_2 \left( \alpha_2^{i} + \mu_{21} \alpha_1^{i} \right) \sin^2 \phi_i,
\]
\begin{equation}
\begin{split}
b_{l_2} &= E_1 \left( \alpha_1 + \mu_1 \alpha_2 \right) \sin^2 \phi_i + E_2 \left( \alpha_2' + \mu_2' \alpha_1' \right) \cos^2 \phi_i.
\end{split}
\end{equation}

Considering main correlations of orthotropic body elasticity theory for plane stress condition and solving them in stress following can be derived for the \textit{i}-th layer with fiber angle \( \phi_i \):

\begin{equation}
\sigma_1^{(i)} = E_1^{(i)} \left[ \left[ \frac{1}{E_x} \left( \cos^2 \phi_i + \mu_1 \cos^2 \phi_i \right) - \frac{\mu_{xy}}{E_x} \left( \sin^2 \phi_i + \mu_1 \sin^2 \phi_i \right) \right] \sigma_x + \left[ -\frac{\mu_{xy}}{E_y} \left( \cos^2 \phi_i + \mu_1 \cos^2 \phi_i \right) \right] \sigma_y + \frac{1}{E_y} \left( \sin^2 \phi_i + \mu_1 \sin^2 \phi_i \right) \tau_{xy} + \left[ \sigma_x \left( \cos^2 \phi_i + \mu_1 \cos^2 \phi_i \right) \right] \cdot i \right],
\end{equation}

\begin{equation}
\sigma_2^{(i)} = E_2^{(i)} \left[ \left[ \frac{1}{E_x} \left( \sin^2 \phi_i + \mu_2 \cos^2 \phi_i \right) - \frac{\mu_{xy}}{E_x} \left( \cos^2 \phi_i + \mu_2 \cos^2 \phi_i \right) \right] \sigma_x + \left[ -\frac{\mu_{xy}}{E_y} \left( \sin^2 \phi_i + \mu_2 \sin^2 \phi_i \right) \right] \sigma_y + \frac{1}{G_{xy}} \left( \sin^2 \phi_i + \mu_2 \sin^2 \phi_i \right) \tau_{xy} + \left[ \sigma_x \left( \sin^2 \phi_i + \mu_2 \sin^2 \phi_i \right) \right] \cdot i \right],
\end{equation}

\begin{equation}
\tau_{l_2}^{(i)} = G_{l_2}^{(i)} \left[ \sin 2 \phi_i \left( \frac{1}{E_x} \frac{\mu_{xy}}{E_y} \right) \sigma_x + \sin 2 \phi_i \left( \frac{1}{E_x} \frac{\mu_{xy}}{E_y} \right) \sigma_y + \cos 2 \phi_i \left( \frac{1}{G_{xy}} \tau_{xy} \right) \sin 2 \phi_i \left( \phi_y - \alpha_x \right) \cdot i \right],
\end{equation}

with \( \sigma_1^{(i)} \left( E_1^{(i)} \right) \) being normal stress (elasticity modulus) along fibers in the layer with fiber angle \( \phi = i \).

Method of thin-walled structures design based on post-buckling state is shown in reference [2]. Let us write non-linear equation of strain compatibility of the following form taking into account cylindrical panel mild camber

\begin{equation}
L_1(F) - L_2(W) = 0,
\end{equation}

with \( F \) being stress function, \( L_m \) being functional

\begin{equation}
L_1(F) = \frac{1}{E_y} \frac{\partial^4 F}{\partial x^4} + \left( \frac{1}{G_{xy}} \frac{2\mu_{xy}}{E_x} \right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 F}{\partial y^4}, \quad L_2(W) = \left( \frac{\partial^2 W}{\partial x^2} \right)^2 - \left( \frac{\partial^2 W}{\partial x^2 \partial y} \right) \left( \frac{\partial^2 W}{\partial y^2} \right) - \frac{1}{R} \frac{\partial^2 W}{\partial x^4}.
\end{equation}

Karman non-linear equation of type [2]

\begin{equation}
L_3(F,W) - L_4(W) = 0
\end{equation}
with \( L_3(F, W) = \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 W}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial W}{\partial x} + R \frac{\partial^2 F}{\partial x^2} \).

\[
L_q(W) = \frac{1}{\delta} \left[ D_x \frac{\partial^4 W}{\partial x^4} + 2D_3 \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} \right] \]

Equations (2) and (3) are basic geometrically non-linear correlations of composite panel post-buckling behavior task.

3. Non-linear task solution and panels design method

Let us represent panel flexure as

\[
W = f \sin^2 \alpha x \cdot \sin^2 \beta y, \quad (4)
\]

with \( \alpha = \pi m / a, \beta = \pi n / b \) being wave generation parameters.

Substituting flexure (4) into equations (2) stress function of following form can be obtained after some transformations and coefficients definition:

\[
F = f^2 \delta \left[ A_1 \cos 2\alpha x - A_2 \cos 4\alpha + A_3 \cos 2\beta y - A_4 \cos 4\beta y - A_5 \cos 2\alpha x \cos 2\beta y + A_6 \cos 4\alpha \cos 2\beta y + A_7 \cos 2\alpha x \cos 4\beta y \right] + \frac{f \delta}{R} \left\{ -A_8 \cos 2\alpha x + A_9 \cos 2\alpha x \cos 2\beta y \right\} + \frac{N_x y^2}{2} + \frac{N_y x^2}{2}.
\]

With following designations introduced:

\[
A_1 = E_y \frac{\beta^2}{32\alpha^2}, \quad A_2 = \frac{E_y \beta^2}{512 \alpha^2}, \quad A_3 = E_x \frac{\alpha^2}{32\beta^2}, \quad A_4 = E_x \frac{\alpha^2}{512 \beta^2},
\]

\[
A_5 = 2 \frac{\alpha^2 \beta^2}{G_{ab}}, \quad A_6 = \frac{\alpha^2 \beta^2}{64 G_{4a}}, \quad A_7 = \frac{\alpha^2 \beta^2}{64 G_{4b}}, \quad A_8 = \frac{E_y}{16 \alpha^2}, \quad A_9 = \frac{1}{16 \alpha^2 G_{ab}},
\]

\[
G_{ab} = \frac{\alpha^4}{E_y} \left( \frac{1}{G_{xy}} - \frac{2 \mu_{xy}}{E_x} \right) \alpha^2 \beta^2 + \frac{\beta^4}{E_x}, \quad G_{4a} = \frac{16 \alpha^4}{E_y} \left( \frac{1}{G_{xy}} - \frac{2 \mu_{xy}}{E_x} \right) 4 \alpha^2 \beta^2 + \frac{\beta^4}{E_x},
\]

\[
G_{4b} = \frac{\alpha^4}{E_y} \left( \frac{1}{G_{xy}} - \frac{2 \mu_{xy}}{E_x} \right) 4 \alpha^2 \beta^2 + 16 \beta^4.
\]

Task solution by Bubnov-Galerkin method is reduced to following equation:

\[
\delta^3 D + f^2 \delta E_m + \delta E_m + f \delta E_m - \frac{N_x \alpha^2}{2} - \frac{N_y \beta^2}{2} = 0
\]

with \( \bar{D}_{mn} = \frac{1}{4} \left[ 3 \bar{D}_x \alpha^4 + 2 \bar{D}_3 \alpha^2 \beta^2 + 3 \bar{D}_y \beta^4 \right], \quad \bar{D}_k = \frac{D_k}{\delta^3}, \quad \bar{E}_m = \frac{1}{128} \left[ \frac{30}{16} E_x \alpha^4 + \frac{4 \alpha^6 \beta^2}{G_{ab}} - \frac{\alpha^4 \beta^4}{G_{4a}} + \frac{\alpha^4 \beta^4}{G_{4b}} + \frac{17}{8} E_x \beta^4 \right], \quad \bar{E}_m = \frac{1}{32 R^2} \left[ E_y + \frac{\alpha^4}{2 G_{ab}} \right], \quad \bar{E}_m = \frac{\beta^2}{4 R} \left[ \frac{E_y}{16} - \frac{9 \alpha^4}{64 G_{ab}} \right].
When considering small flexures in longitudinally compressed cylindrical panel stability task solution following equation in compression force can be obtained:

\[
N_x = \frac{\delta^3}{\alpha^2} \left[ 4D_x \alpha^4 + \frac{8}{3} D_3 \alpha^2 \beta^2 + 4D_4 \beta^4 \right] + \frac{\delta}{6R^2} \left( \frac{E_y}{\alpha^2} + \frac{\alpha^2}{2} \left( \frac{\alpha^4}{E_y} + \frac{1}{G_{xy}} \frac{2\mu_{xy}}{E_x} \right) \alpha^2 \beta^2 + \beta^4 \right). \tag{6}
\]

It should be noted that wave generation critical parameters \( m_{cr} \) and \( n_{cr} \) are defined from correlations \( \partial N_x / \partial m = 0 \) and \( \partial N_x / \partial n = 0 \). Besides equation for combined loading similar to equation (6) can be obtained with fixed correlation of considered forces.

Let us write further formulas for membrane stress in middle surface:

\[
\sigma_x = -f^2 S_1 - \frac{f}{R} S_2 - \frac{N_x}{\delta}, \tag{7}
\]

\[
\sigma_y = -f^2 S_3 - \frac{f}{R} S_4 - \frac{N_y}{\delta}, \tag{8}
\]

With following functions designated:

\[
S_1 = -4 \left\{ \beta^2 A_3 \cos 2\beta_y x - 4\beta^2 A_4 \cos 4\beta \cos 2\beta_y + \beta^2 A_5 \cos 2\alpha x \cos 2\beta_y + 4\beta^2 A_7 \cos 2\alpha x \cos 4\beta \right\},
\]

\[
S_2 = 4\beta^2 A_4 \cos 2\alpha x \cos 2\beta_y,
\]

\[
S_3 = -4 \left\{ \alpha^2 A_4 \cos 2\alpha x - 4\alpha^2 A_2 \cos 4\alpha - \alpha^2 A_5 \cos 2\alpha x \cos 2\beta_y + 4\alpha^2 A_7 \cos 2\alpha x \cos 2\beta_y + \alpha^2 A_7 \cos 2\alpha x \cos 4\beta \right\},
\]

\[
S_4 = -4\alpha^2 A_6 \cos 2\alpha x + 4\alpha^2 A_4 \cos 2\alpha x \cos 2\beta y.
\]

Let us state a remark concerning functions \( S(x, y) \). It should be noted that above mentioned functions are periodic and for definition of \( \xi \) coordinates in which maximum stress is observed in general parametric studies are needed. It should also be noted that surfaces \( S(x, y) \) do not depend on panel thickness and are defined by stiffness parameters (of composite layout) correlations and panel geometrical parameters ratios.

Let us transform equation (1) for stress along fibers in standard layout \( 0^\circ/\pm 45^\circ/90^\circ \). We obtain following:

\[
\sigma_1^{(0)} = E_1 \left( \sigma_x \left( 1 - \mu_{12} \mu_{xy} \right) + \sigma_y \left( \mu_{12} - \mu_{xy} \right) \right) + E_1 \left( \alpha_x + \alpha_y \mu_{12} - \left( \alpha_1 + \mu_{12} \right) \alpha_2 \right) \cdot t, \tag{9}
\]

\[
\sigma_1^{(90)} = E_1 \left( \sigma_x \left( 1 - \mu_{12} \mu_{xy} \right) + \sigma_y \left( \mu_{12} - \mu_{xy} \right) \right) + E_1 \left( \alpha_x \mu_{12} + \alpha_y \mu_{12} - \left( \alpha_1 + \mu_{12} \right) \alpha_2 \right) \cdot t \tag{10}
\]
Shown equations (9)–(11) can be used for composite package stress limit definition according to the first structure theory for combined loading with acting stress $\sigma_x$ and $\sigma_y$. In this case with stress along fibers tendency to ultimate stress limit $\left(\sigma^{(i)}_1 \rightarrow \sigma^{(i)}_1\right)$ irreducible combined load can be defined.

Assuming that possible reason of standard composite package $0^\circ/\pm 45^\circ/90^\circ$ failure is failure of layer fibers with reinforcement $\varphi = 0^\circ$. Then substituting equations (7)–(8) which describe post-buckling behavior into equation (9) we derive

$$\bar{\sigma}_{1}^{(0)} = \overline{E}_{1}^{(0)} \frac{1-\mu_{12}^{(0)} \mu_{yx}}{E_{x}} \left(-f^{2} S_{1} - \frac{f}{R} S_{2} - \frac{T_{x}}{\delta}\right) + \overline{E}_{1}^{(0)} \frac{\mu_{12}^{(0)} - \mu_{yx}}{E_{y}} \left(-f^{2} S_{3} - \frac{f}{R} S_{4} - \frac{T_{y}}{\delta}\right) + \overline{E}_{1} \left\{\alpha_{x} + \alpha_{y} \mu_{12}^{(0)} \right\} \cdot t.$$  

From which with given loads per unit $T_{x} = \sigma_{x} \delta$ and $T_{y} = \sigma_{y} \delta$ following is derived

$$f = -B + \sqrt{B^{2} - 4AC} \overline{E}_{1}$$  

with

$$A = \overline{E}_{1}^{(0)} \frac{1-\mu_{12}^{(0)} \mu_{yx}}{E_{x}} S_{1} + \overline{E}_{1}^{(0)} \frac{\mu_{12}^{(0)} - \mu_{yx}}{E_{y}} S_{3}, \quad B = \overline{E}_{1}^{(0)} \frac{1-\mu_{12}^{(0)} \mu_{yx}}{R E_{x}} S_{2} + \overline{E}_{1}^{(0)} \frac{\mu_{12}^{(0)} - \mu_{yx}}{R E_{y}} S_{4},$$

$$C = \overline{E}_{1}^{(0)} \frac{1-\mu_{12}^{(0)} \mu_{yx}}{E_{x}} \frac{T_{x}}{\delta} + \overline{E}_{1}^{(0)} \frac{\mu_{12}^{(0)} - \mu_{yx}}{E_{y}} \frac{T_{y}}{\delta} + \left\{\alpha_{x} + \alpha_{y} \mu_{12}^{(0)} \right\} \cdot t.$$  

Further substituting (12) into (5) let us put down equation defining cylindrical panel composite package minimal thickness in case of breaking composite layer fibers with longitudinal reinforcement

$$\delta^{3} D_{1} + \left(-B + \sqrt{B^{2} - 4AC} \overline{E}_{1} \right)^{2} \delta E_{m1} + \delta E_{m2} - \sqrt{-B + \sqrt{B^{2} - 4AC}} \delta E_{m3} - \frac{N_{x} \alpha^{2}}{2} - \frac{N_{y} \beta^{2}}{2} = 0.$$  

Applying transformations with possible fibers breaking in layers with reinforcement $\varphi = \pm 45^\circ$ and $\varphi = 90^\circ$ similar non-linear equations in thickness can be obtained providing fibers breakage in a layer with one of the above-mentioned fiber angles according to the 1-st structure theory. Thus cylindrical composite panels design method based on post-buckling state with biaxial compression taking into account thermal affect is reduced to following. At first thicknesses are calculated using equations of type (13) with possible failure reasons being fiber breakage with reinforcement $0^\circ$, $\pm 45^\circ$ and $90^\circ$. Further maximum value which is the target thickness $\delta$ post-buck of cylindrical composite panel is chosen.

It should be noted that shown method can be appended with consideration of stability limitations. That is loads used for suggested method should be ultimate ones. Generally for composite structures buckling is not acceptable within limit loading level. In this case with given load value and known
composite layout from equation (6) panel thickness $\delta_{\text{stab}}$ can be defined and further maximum thickness value should be chosen ($\delta_{\text{stab}}, \delta_{\text{post-buck}}$).

4. Conclusion
Shown design method for cylindrical composite panels with mild camber based on post-buckling state with biaxial compression taking into account uniform heating and rigid support can be used for aircraft structures design with limitations in stability and load-bearing capacity.

References
[1] Vasiliev V 1988 Composite Structures Mechanics (Moscow: Mashinostroenie)
[2] Mitrofanov O 2003 Engineer Composite Load-Bearing Panels Design Methods (Moscow: Sputnik+)