Uncertainties of hadronic scalar decay calculations

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Dispersion relations—a corollary of unitarity and analyticity—are among the few methods which can be employed to study low energy processes in QCD. In this work we critically revisit the calculation of decay widths of a hypothetical light scalar boson into pions and kaons. Strong interactions of the mesons in the final state affect these decays significantly. We argue that applications of the dispersion relations to the calculation of these decay widths, which are present in the literature, rely on an uncontrollable approximation of the reduced S-matrix. When a limited number of initial and final decay channels are considered, in general neither a unitary S-matrix nor its version corrected by an inelasticity factor are justified. The scalar form factor calculation provides a transparent example to illustrate this statement. Therefore, the results of this calculation, which are important for theoretical, cosmological and experimental applications, must be treated with caution. Our work, therefore, calls for a breakthrough in the understanding of the dispersion relations of realistic many-channel systems.

I. INTRODUCTION

Light scalar bosons, singlet with respect to the gauge group of the Standard Model of particle physics (SM), appear in many phenomenological models, see e.g. [1–3]. They became a part of physics programs of current and planned experiments (see, e.g. the SHiP [4], MATHUSLA [5] and FASER [6] physics papers). From the perspectives of direct experimental search, knowing the decay widths and branching ratios of such bosons is crucial since they determine the lifetimes (decay length) and detection signatures respectively.

The decay width (and branching ratios consequently) of the light scalar boson with mass around 1 GeV suffers from the uncertainties originating from a lack of understanding of its hadronic decays [7–10]. One of the methods widely applied with the aim of reducing these uncertainties is the dispersive approach. In particular it was applied to study the hadronic decay channels of light scalar bosons in relation to the phenomenology of a Higgs boson \( H \), see e.g. [11–17]. Even though there was some controversy in the literature, see, e.g. discussion in [18], the method of [17] seems to be the most advanced and accurate and has been widely adopted to estimate the hadronic decay rates of GeV-scale exotic scalars [7, 19, 20]. However, the question of reliability of the dispersive approach is not studied in detail.

The reason to question the reliability of the approach is the following. The dispersion relations, based on analyticity, which is in turn a consequence of causality, allow for reconstructing an amplitude \( A \) from its imaginary part,

\[
A(s) = \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} A(s')}{s' - s - i\epsilon}.
\]

The amplitude of the decay of the scalar boson \( S \to XX \), \( X = \pi, K, \ldots \) is proportional to a linear combination of form factors \( F(s) = \langle X(q)X(q')|j|0 \rangle, \) \( s = (q+q')^2 \), where \( j \) is the scalar current, i.e. \( j = \sum \bar{q}q \). The Watson theorem [21], which is based on unitarity, states that the phase of the form factor \( \alpha \) equals the phase shift \( \delta \) of the scattering amplitude \( XX \to XX \). This observation transforms (1) into an integral equation determining the form factor \( F \) for a given function \( \delta \),

\[
\text{Re} F(s) = \frac{1}{\pi} P \int_0^\infty ds' \frac{\text{Re} F(s') \tan(\delta)}{s' - s},
\]

where symbol \( P \) indicates the principal value integral and the real and imaginary parts of \( F \) are related by the phase \( \alpha \). In order to determine the form factor \( F \) one needs to know the function \( \delta(s) \) up to infinite energies. Meanwhile, in practice, the phase shifts \( \delta \) are measured only in the limited energy intervals.

How sensitive is the approach to the behaviour of \( \delta \) at large energies? Naively one can expect that the effect of small variations of the phase \( \delta \) at large \( s' \) is suppressed by the factor \( 1/(s' - s) \). However, the variations of the phase are not necessarily small. For instance, if a resonance is present at some energy \( \sqrt{s} \), the phase shift \( \delta \) increases by
π around \( s_\tau \). In this case the solution of Eq. (2) exhibits a pronounced peak.

This is, however, not the only problem. Indeed, the equation (2) was obtained under the assumption of validity of the Watson theorem. However, the theorem is formulated for a single reaction channel (e.g. \( \pi \pi \to \pi \pi \)). In the presence of inelastic channels (such as \( \pi \pi \to K\bar{K} \)) it should be modified to include these extra channels. The single equation (2) becomes a system of coupled singular integral equations. Most of the analyses up to date were restricted to the two channel case. So another question is how the inelastic channels, which are not accounted for in the two channel consideration, modify the result. We argue here that the answer to this question is non-trivial, and that the extra channels cannot be neglected in general case.

This work is organized as follows. Section II contains an overview of the calculation of the scalar decay rates into hadrons. We define the main objects of interests, namely the form factors in Section II A. The dispersion relations and unitarity are discussed in Section II B. In Section III we describe the reduction of an \( \mathcal{M} \) channel system to an \( \mathcal{R} \) channel system with \( \mathcal{R} < \mathcal{M} \). We argue that the unitary \( \mathcal{R} \) channel system cannot properly describe the behaviour of the form factors, which can only be calculated in the full \( \mathcal{M} \) channel system. We illustrate this statement on the example of \( \mathcal{M} = 2 \to \mathcal{R} = 1 \) reduction. This demonstration implies that within any treatments suggested so far the uncertainties in the hadronic channels remain untamed and may reach a factor of order ten, contrary to the recent report [19]. We discuss the implications of our statement in Section IV.

II. REVIEW OF THE FORMALISM

In this Section we define the Lagrangian of the model and introduce the form factors following the notations of Ref. [17]. At low energies, where the Chiral Perturbative Theory (ChPT) is reliable, it can be used to calculate these form factors. At higher energies the dispersive approach is employed. We present a brief review of the dispersion methods and sketch the analysis of Ref. [17].

A. Scalar form factors

Imagine a light scalar \( S \) mixed with the SM Higgs scalar. If the mass of \( S \) is well below the electroweak scale the Higgs boson can be integrated out. The renormalizable low energy interaction Lagrangian then reads

\[
\mathcal{L}_{\text{int}} = - \sum_f \frac{g_f m_f}{v} \bar{\psi} f \psi f S,
\]

where the sum runs over all SM fermions \( f \) with masses \( m_f, v \approx 246 \text{ GeV} \), and the effective coupling \( g_f \) originates from the mixing of \( S \) with the SM Higgs.

Lagrangian (3) controls decay channels of the scalar \( S \). While the leptonic decay rates can be readily calculated, this is not the case for the hadronic channels, if the scalar is at the GeV-scale. The reason is that final state interactions of the strongly interacting particles alter the result significantly. Recall, the amplitude of the process \( S \to XX, X = \pi, K, \ldots \) is proportional to a linear combination of the form factors \( F(s) = \langle X(q)X(q')|j(0) \rangle, s = (q + q')^2 \), where \( j \) is the scalar current, i.e. \( j = \sum_{\text{quarks}} \bar{q}q \). The contribution of light quarks can be addressed directly, whereas heavy quarks can be accounted for using the trick by Shifman, Vainshtein, and Zakharov [22]. In short, the scalar currents of heavy quarks are related to those of light quarks via the trace of energy-momentum tensor \( \Theta_{\mu} \).

In Ref. [17] only the decay channels to pions and kaons have been considered. Below we refer to this as the two channel case. Analogously, in the single channel case only pions are considered.

The independent form factors responsible for the decay of \( S \) in the two channel case are

\[
\langle \pi^i(p)\pi^j(p')|\Theta_{\mu}(0) \rangle = \Theta_{\mu}(s)\delta^{ij}, \quad (4a)
\]

\[
\langle \pi^i(p)\pi^j(p')|m_{\bar{u}u} + m_{\bar{d}d}|0 \rangle = \Gamma_{\pi}(s)\delta^{ij}, \quad (4b)
\]

\[
\langle \pi^i(p)\pi^j(p')|m_{\bar{s}s}|0 \rangle = \Delta_{\pi}(s)\delta^{ij}, \quad (4c)
\]

where \( s = (p + p')^2 \). Kaon form factors are defined analogously. The decay width \( \Gamma \) is proportional [17] to the square of

\[
G_\pi(s) = \frac{2}{9} \Theta_{\pi}(s) + \frac{7}{9} \left( \Gamma_{\pi}(s) + \Delta_{\pi}(s) \right). \quad (5)
\]

An analogous combination is defined for kaons.

At sufficiently low energies, the values of the form factors defined above can be computed using the ChPT. To the leading order in momentum (\( \mathcal{L}_2 \) order) one gets [17]

\[
\Theta_{\pi}(s) = s + 2m_\pi^2, \quad \Theta_{K}(s) = s + 2m_K^2, \quad (6a)
\]

\[
\Gamma_{\pi}(s) = m_\pi^2, \quad \Gamma_{K}(s) = \frac{1}{2} m_K^2, \quad (6b)
\]

\[
\Delta_{\pi}(s) = 0, \quad \Delta_{K}(s) = m_K^2 - \frac{1}{2} m_\pi^2. \quad (6c)
\]

In the next subsection we describe how to determine the form factors (4) at larger \( s \).

B. Dispersion relations and unitarity

In this subsection we discuss how unitarity and analyticity can be used to reconstruct the form factors from the scattering data.

The scattering of asymptotic states, such as pions, kaons, eta mesons, etc., is described by an unitary S-matrix. The standard definition of the T-matrix is

\[
S_{ij} = \delta_{ij} + 2i \sqrt{\sigma_i \sigma_j} T_{ij}, \quad (7)
\]
The fundamental quark interactions for each channel $i$ correspond to the form factors $F_i(s)$. In the two channel case, with $i = 1, 2$ denoting pions and kaons respectively (so that they match with Eqs. (4) and (6)), these are

$$F_1(s) = \langle 0|\hat{X}_0|\pi\bar{\pi}\rangle, \quad F_2(s) = \langle 0|\hat{X}_0|K\bar{K}\rangle,$$

where $\hat{X}_0$ is a scalar operator of zero isospin, e.g. $m_u\bar{u}u + m_d\bar{d}d$, see Eq. (4). For the form factors of this type, one can derive the unitarity conditions

$$\text{Im} F_i = \sum_{j=1}^n T_{ij}^* \sigma_j F_j(s).$$

Note that the conditions (10) are valid for the general case when $n$ channels are present.

The form factor can be reconstructed from its imaginary part using (1). This leads to a set of $n$ coupled integral equations with singular kernels known as the Muskhelishvili–Omnès equations

$$F_i(s) = \frac{1}{\pi} \sum_{j=1}^n \int_{4m_i^2}^{\infty} \frac{ds'}{s'-s} T_{ij}(s') \sigma_j F_j(s'), \quad i = 1 \ldots n.$$  

One can see immediately that Eqs. (11) require the off-diagonal elements of $T$ in the unphysical region (that is below the corresponding threshold).

The single channel case is an important basic situation. If only one final state is considered, the unitary scattering matrix is fully determined by the phase shift $\delta$, $S = \exp(2i\delta(s))$, and Eq. (10) simplifies to

$$\text{Im} F(s) = -\left(e^{-2i\delta} - 1\right) F(s).$$

The form factor is a real analytic function of $s$ in the plane with a cut along the real axis starting from the $4m_i^2$ threshold. For the real values of $s$ we define $F(s) \equiv F(s + i\epsilon)$. Since $F(s^*) = F^*(s)$, the imaginary part of $F$ is related to the discontinuity as $\text{Im} F(s + i\epsilon) = 2i(F(s + i\epsilon) - F(s - i\epsilon))$. Therefore, eq (12) can be rewritten as

$$F(s + i\epsilon) = F(s - i\epsilon)e^{2i\delta}.$$  

Thus the phase of the form factor is equal to the phase shift of elastic scattering amplitude. This relation is known as the Watson theorem [21].

Noticing that the factor $T^*\sigma = (\exp(2i\delta) - 1)/2i$ in Eq. (10) can be rewritten as $(\exp(2i\delta) - 1)/2i = \exp(i\delta)\sin\delta$, we get the standard form of the Omnès equation (we present here a once subtracted form)

$$F(z) = F(0) + s \int_{4m_i^2}^{\infty} \frac{ds}{s(s-z)} e^{i\delta(s)} \sin\delta(s) F^*(s).$$  

The single channel equation (14) has an analytic solution [23]

$$F(z) = F(0) \exp\left\{\frac{z}{\pi} \int_{4m_i^2}^{\infty} ds \frac{\delta(s)}{s(s-z)}\right\}.$$  

In fact, a general solution is given by

$$F(z) = P(z)\Omega(z),$$

where $P(z)$ is a polynomial in $z$ and the Omnès function reads

$$\Omega(z) = \exp\left\{\frac{z}{\pi} \int_{4m_i^2}^{\infty} ds \frac{\delta(s)}{s(s-z)}\right\}.$$  

The general analytic solution to the system (11) for $n > 1$ cannot be obtained in a closed form. However, it is known that the system has $n$ independent (fundamental) solutions [24]. Let us denote these solutions as $F^{(j)}_i$, where $i$ indicates the final state, i.e. $i = \pi, K, \ldots$, whilst $j$ enumerates the independent solutions. One can combine $n$ fundamental solutions $F^{(j)}_i$ into a matrix. Let us denote the determinant of this matrix as $\bar{F}$. From (7) and (10) one can find that

$$\bar{F} = \det(S)\bar{F}^\ast.$$  

Unitarity of the S-matrix implies that $\det S = \exp(2i\Delta)$, where $\Delta$ is the sum of the phase shifts. Therefore the relation (18) has precisely the form of the single channel equation (13). Solution (17) allows for performing a cross check of the numerical solution of the $n$ channel case. Moreover, the analytical solution allows one to determine the behaviour of the $\bar{F}$ at infinity. It is controlled by the value of $\Delta(\infty)$ and reads

$$\bar{F}(s) \sim s^{-\Delta(\infty)/\pi}.$$  

C. Two channel case

In this Section we limit ourselves to the two channel case, that is, only pions and kaons are considered. For the sake of simplicity, we omit all the Clebsch–Gordan coefficients. The standard parametrisation of the $2 \times 2$ unitary S-matrix is

$$S = \begin{pmatrix} 1 & \frac{\eta e^{2i\delta_\pi}}{(1 - \eta^2)^{1/2}} e^{i(\delta_\pi + \delta_K)} e^{i(\delta_\bar{\pi} + \delta_{\bar{K}})} \\ \frac{\eta e^{2i\delta_K}}{(1 - \eta^2)^{1/2}} e^{i(\delta_\pi + \delta_K)} & 1 \end{pmatrix}.$$  

where the phases $\delta_\pi$, $\delta_K$ and the inelasticity $\eta$ are functions of $s$ which should be obtained from experimental data. As we have mentioned above, the off-diagonal elements of $T(s)$ are needed both in the physical and unphysical regions. Therefore, analytic parametrisations allowing for extrapolation of $T_{21}$ below $s = 4m_{\bar{K}}^2$ are needed. There are several empiric parametrisations of this type [25–27].
As we have already mentioned, there are no known solutions for \( n > 1 \) channels, so the system (11) should be solved numerically. This is a challenging task since one has to deal with singular kernels. The principal value integrals should be treated carefully. There are two different approaches to the problem, the iterative one of Ref. [17] and the one of Ref. [28], which is based on the singular value decomposition. Our numerical studies is based on the latter method.

One can construct two linearly independent solutions \( C_i \) and \( D_i \) to the system (11) as

\[
C_i(s)|_{s=0} = \delta_{i1}, \quad D_i(s)|_{s=0} = \delta_{i2}. \tag{21}
\]

These solutions are combined to get the form factors (4a), (4b) and (4c). The coefficients in the linear combinations are fixed using the ChPT results (6a), (6b) and (6c) as

\[
\Gamma_+(s) = \Gamma_+(0)C_1(s) + \Gamma_+(0)D_1(s), \quad \Gamma_K(s) = \Gamma_+(0)C_2(s) + \Gamma_+(0)D_2(s), \tag{22}
\]

and analogously for other form factors [17]. In principle, Eqs. (22) define the solution for the form factors at any values \( s \).

Let us note in passing that it has not been rigorously demonstrated that the methods of Refs. [17] and [28] guarantee the proper behaviour of the solutions \( C_i \) and \( D_i \) at infinity [24] provided that only (21) are required.

### III. REDUCING THE NUMBER OF CHANNELS

In this Section we consider implications of the restriction to a certain number of channels.

#### A. General discussion

Suppose that the system can be fully described by a \( \mathcal{M} \) channel S-matrix (where \( \mathcal{M} \) can be infinite). Of course, exact solution for such a system for large \( \mathcal{M} \) is impossible. Therefore, in real world applications the number of channels is limited. Let us consider \( \mathcal{R} < \mathcal{M} \) channels. In order to make the discussion somewhat more transparent we write down the system of equations explicitly

\[
F_1(s) = \sum_{j=1}^{\mathcal{R}} \int \frac{ds'}{\pi} \frac{T_{1j}(s')\sigma_j(s')}{s-s'} F_j(s') + \left\{ \sum_{j=\mathcal{R}+1}^{\mathcal{M}} \int \frac{ds'}{\pi} \frac{T_{1j}(s')\sigma_j(s')}{s-s'} F_j(s') \right\}
\]

\[
F_2(s) = \sum_{j=1}^{\mathcal{R}} \int \frac{ds'}{\pi} \frac{T_{2j}(s')\sigma_j(s')}{s-s'} F_j(s') + \left\{ \sum_{j=\mathcal{R}+1}^{\mathcal{M}} \int \frac{ds'}{\pi} \frac{T_{2j}(s')\sigma_j(s')}{s-s'} F_j(s') \right\}
\]

\[
F_\mathcal{R}(s) = \sum_{j=1}^{\mathcal{R}} \int \frac{ds'}{\pi} \frac{T_{\mathcal{R}j}(s')\sigma_j(s')}{s-s'} F_j(s') + \left\{ \sum_{j=\mathcal{R}+1}^{\mathcal{M}} \int \frac{ds'}{\pi} \frac{T_{\mathcal{R}j}(s')\sigma_j(s')}{s-s'} F_j(s') \right\}
\]

The terms in brackets depend on the form-factors with \( j > \mathcal{R} \) and can only be found from the full system of equations. The only generic statement which can be done is that these undetermined terms are suppressed as \((4m_{\mathcal{R}}^2-s)/(4m_{\mathcal{R}+j}^2-s)\) which follows from the fact that \( \sigma_j(s)|_{s<4m_j^2} = 0 \), see Eq. 8. In particular, \( \eta \eta \) channel is suppressed rather weakly compared to the KK channel, \((m_K/m_\eta)^2 \approx 0.8\). Therefore, the corrections originated from the terms in brackets are set equal to zero. Validity of this assumption may be justified only far below the thresholds of the neglected channels (or there is a particular reason to believe that some channel has suppressed T-matrix elements, like the phase space suppression of the 4\( \pi \) channel at low energies).

Usual assumption for the reduction to \( \mathcal{R} \) channels is based on the observation that below the \( \mathcal{R}+1 \) threshold the \( \mathcal{R} \times \mathcal{R} \) S-matrix is unitary. However, as we see from (23) the contribution of the high energy channels do not disappear below their threshold, making this reduction impossible.\(^1\) Below we illustrate this statement with the simplest example.

#### B. Example of 2 to 1 reduction

As we have stressed above, there is no control of an error arising from the reduction of channels. The error can be obtained by the direct comparison of the solutions of the system with \( \mathcal{M} \) and \( \mathcal{R} \) channels. We perform such a comparison taking \( \mathcal{M} = 2 \) and \( \mathcal{R} = 1 \).

The two channel case is solved numerically using the method based on Ref. [28]. We use the parametriza-

\(^1\) There is a recent work [29] which offers a parametrization for the strange scalar form factor \( \Delta_s \) that is claimed to be valid up to 2 GeV. It is tempting to check if this approach can be generalized to other form factors.
tion of the T-matrix by Truong and Willey [16]. This parametrisation admits an analytical solution and thus allows us to cross check the numerical results.

In the single channel case, the solution is given by Eq. (15), where the phase $\delta(s)$ is obtained as a phase of $S_{11}$, see Eq. (20). For definiteness we consider $\Gamma_\pi$. The solution of the single channel equation is shown in Fig. 1. The phase $\delta$ approaches $2\pi$ at large $s$. Therefore, according to (19), the solution decreases as $s^{-2}$.

In the two channel case, there are two independent solutions, $F_{11}(s)$ and $F_{12}(s)$. For the parametrisation [16] the solutions can be constructed directly as $F_{ij} = T_{ij}/\sqrt{s}$. The form factor $\Gamma_\pi$ is a linear combination of type $C_{11}F_{11} + C_{12}F_{12}$. At large $s$ we have $F_{11} \sim s^{-1}$, whilst $F_{12} \sim s^{-2}$. Therefore we set $C_{11} = 0$ and $C_{12} = m_\pi^2/F_{12}(4m_K^2)$. The absolute value of the form factor $\Gamma_\pi$ in the two channel approximation is shown in Fig. 1.

As one observes from Fig. 1, the single channel form factor is much larger around 1 GeV. The partial decay width is proportional to the second power of the form factor, therefore we show the square of the ratio of the form factors as a function of $s$. It is clear that the result is overestimated in the single channel case. The horizontal line shows the position of 1.

Nevertheless, our general conclusion is that the naive reduction from two channels to a single channel does not work.

The example above demonstrates that the reduction of a $2 \times 2$ unitary matrix (20) to a $1 \times 1$ unitary matrix $\exp(2i\delta)$ leads to significant errors. Note that in the two channel case $S_{11} = \eta \exp(2i\delta)$, where the inelasticity parameter $\eta(s)$ is equal to 1 below the first inelastic threshold $4m_K^2$ and is less than 1 above.

In fact, it is possible to account for inelasticity $\eta(s)$ above the kaon threshold in a better way. Let us consider $S_{11} = \eta \exp(2i\delta)$ as a single channel scattering matrix. It can be rewritten in yet another form

$$S_{inel} = \eta \exp(2i\delta) = e^{2i(\delta + \frac{1}{2} \log |s|)} = e^{2i\delta_c},$$

(24)

where we have introduced the complex phase

$$\delta_c = \delta + i\psi, \quad \psi(s) = \left[-\frac{1}{2} \log \eta(s)\right].$$

(25)

This S-matrix is no longer unitary and it is not legitimate to apply the arguments based on unitarity in the way it has been done in Section II. However, it is still possible (see, e.g. [34]) to relate the form factor and the scattering amplitude in the case of a complex phase shift using the trick of Refs. [35, 36]. This leads to replacement of the phase shift $\delta$ in the equations with the effective phase $\phi$ given by [34]

$$\tan \phi = \frac{e^{-2\psi} \sin 2\delta}{1 + e^{-2\psi} \cos 2\delta}.$$  

(26)

The Omnès solution (15) with this phase accounts for inelasticity. It is quite remarkable that if a resonance—

which manifests itself as a jump of the phase shift $\delta$ by $\pi$—occurs in the inelastic region, the phase (26) doesn’t change by $\pi$. This is precisely the reason why the relative position of the resonance and the kaon threshold is so important. In our example the resonance is located

2 As was pointed in Ref. [17], the sign of the parameter $\lambda$ in the parametrisation must be flipped in order to fit the experimental data, so we use $\lambda = -0.1$. 

FIG. 1. Absolute values of the pion form factors as functions of $s$. The solid line represents $|\Gamma_\pi(s)|$ obtained in the single channel approximation (Eq. (15)) and normalized at $s = 4m_\pi^2$ according to (6b). The dotted line is the analogous form factor in the two channel approximation.

FIG. 2. The square of the ratio of the absolute values of the form factors as a function of $s$. It is clear that the result is overestimated in the single channel case. The horizontal line shows the position of 1.
before the onset of inelasticity and the form factor calculated with the phase \( \delta \) does not differ very much from the one calculated with \( \delta \), see Fig. 3. Therefore, even the improved reduction, which attempts to imitate heavy channels by inelasticity of the reduced S-matrix, does not resolve the problem.

One can argue that in some other parametrisation, e.g. the one used in Ref. [30], the procedure of introducing the complex phase could lead to a resonable agreement between the single channel and two channel approaches. This is indeed the case for the form factor \( \Gamma_\pi \). However, this cannot be achieved simultaneously for all form factors, in particular for \( \Delta_\pi \), which contributes to the decay amplitude in the same way as \( \Gamma_\pi \), see Eq. (5).

C. Scalar radius of the pion

At the same time, there are successful predictions that can be obtained from a reduced 2 channel approach.

The pion form factor \( \Gamma_\pi \) has been considered, e.g., in Refs. [28, 30–33, 37, 38] in the context of the scalar radius of the pion \( \langle r_s^2 \rangle_\pi \).

\[
\Gamma_\pi(z) = \Gamma_\pi(0) \left( 1 + \frac{1}{6} \langle r_s^2 \rangle_\pi z + O(z^2) \right). \tag{27}
\]

The scalar radius is related to the coupling constant \( \ell_4 \) [39] in ChPT and therefore is of considerable interest. The result obtained using the dispersive approach in the two channel approximation [30] is in agreement with lattice simulations [40] and experimental measurements [37].

The possible explanation of this good agreement, in spite of the problems described in the previous Section, is the following. The scalar radius (27) measures the slope of the form factor at \( s = 0 \). The form factor can always be represented in the form (15) where now \( \delta \) is the phase of the form factor, not the scattering phase. Then, from (27) and (15) one can get the scalar radius by integrating \( \delta \) over \( s \). The values of \( \delta \) at large \( s \) are suppressed by \( s^2 \), so the scalar radius is rather insensitive to the dynamics at high energies. Nevertheless, there was an active discussion in the literature [30–32, 38, 41] about the importance of the two channel approach.

IV. DISCUSSION

In this work we have investigated the impact of the final state interaction on the width of a light Higgs-like scalar. The decay width to mesons is proportional [17] to \( |G_i(s)|^2 \), where \( G_i(s) \) is the linear combination (5) of three form factors defined in Eq. (4) and \( i = \pi, K, \ldots \). Most of the analyses\(^3\) have been limited to the two channel case, namely, \( i = \pi, K \). Though all data relevant for the two channels (two scattering phases and inelasticity) have been encoded in a number of phenomenological parametrizations [25–27, 42], we argue that there is an intrinsic uncertainty related to neglected channels. This can be most easily inferred from Eqs. (23). There is a contribution from neglected channels which can be understood as an exchange of heavier states in the loops and which is simply dropped in the two channel analysis. This contribution is suppressed for quantities defined at small \( s \), such as scalar radius of the pion, but becomes more and more important once one goes away from \( s = 0 \). The decay width of a light scalar boson \( S \) to hadrons as a function of \( m_S \) is extremely sensitive to the effects of higher channels.

There is yet another source of uncertainty in the application of dispersion approach to the calculation of the form factors. This uncertainty stems from the fact that the off-diagonal elements of the T-matrix are needed in the unphysical region. As we have discussed in section II, \( T(s) \) cannot be directly extracted from the experimental data and therefore is defined as an approximation, which depends on a particular analytic parametrisation of the S-matrix.

To summarize, the method of Ref. [17] should be applied with great caution. Moreover, as we have demonstrated, the present form of the reduction of the number of channels leads to significant uncertainties which are beyond theoretical control. We believe that this fact should be clarified when exclusion plots or projected sensitivities for the hypothetical light scalars are presented.

\(^3\) The only exception is the work by Moussallam [28], where four pion final state has been effectively accounted for using the parametrisation of Refs. [26, 27]. However, he found that the available experimental data are not enough to fix the three channel parametrisation, and different choises of the model parameters lead to significantly different results.
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