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Experiments on Corn Pressure in Silo Cells – Translation and Comment of Janssen’s Paper from 1895

Abstract  The German engineer H.A. Janssen gave one of the first accounts of the often peculiar behavior of granular material in a paper published in German in 1895. From simple experiments with corn he inferred the saturation of pressure with height in a granular system. Subsequently, Janssen derived the equivalent of the barometric formula for granular material from the main assumption that the walls carry part of the weight. The following is a translation of this article. The wording is chosen as close as possible to the original. While drawings are copied from the original, figures displaying data are redone for better readability. The translation is complemented by some bibliographical notes and an assessment of earlier work, wherein Hagen predicted the saturation of pressure with depth in 1852, and Huber-Burnand demonstrated that saturation qualitatively as early as in 1829. We conclude with a brief discussion of more recent developments resting on Janssen’s work.

During the last decades the export of corn from the corn countries of the world to the culture states of Europe has made extraordinary progress. Considering the length of the route of transport, this development could only have been achieved by low production costs, be it due to low wages, be it due to extensive application of mechanical equipment both for tillage and for harvesting. Further, consider improvements of corn depositories and the means of corn transportation. Predominantly in the United States of North America storage silos have proven themselves to be beneficial. Constructed on a large scale, these silos are designed to receive the corn arriving in railway carriages, to store it, and to dispense it to river and sea vessels [2]. The extensive mechanical equipment for stocking the fruit, weighing it, and loading it onto ships is highly efficient. Some realizations are capable of handling several hundreds of tons per hour. The interior of such a storage silo – called an elevator in North America – consists of a number of vertical tubular cavities serving to receive the corn from street-level to the attic. The profile of these tubes or cells has been exclusively rectangular with common walls from wooden shafts and bottom casing. Recently the walls are also produced from iron-strengthened brickwork, and in six-fold profiles like in honeycombs [2]. The grain is introduced through the upper end of the cell through a hatch. For discharge, stock transfer, or embarkation, the bottoms of the cells provide close-able openings. The dimensions of the cells vary considerably; the largest ones have a capacity of up to around 250 t, in which case the height of the corn reaches around 25 m. It is obvious that the enormous content of the cell must exert considerable pressure on the side walls and the bottom. While various construction guidebooks indicate the necessity for suf-
sufficient anchoring of the cell walls, nowhere is anything in particular given for the determination of the corn pressure against the side walls. The formulas valid for the pressure in liquids are not applicable for corn as the friction among the grains influences the pressure transmission considerably. Likewise, the formulas customary for the calculation of the pressure against feed walls are alone for the reason not applicable for the present case, because the latter are derived for straight walls, while the content of a silo cell is surrounded by vertical walls from all sides. To my knowledge, the only publication about experiments to determine the corn pressure in silo cells were done in the *Engineering* on October 27th, 1882 [3]. These experiments establish that the pressure of corn in a silo cell against the support grows with increasing depth up to a certain value, but does not grow further at larger depth. The largest emerging pressure on the area depends on the profile of the cell, and is supposedly proportional to the diameter of the circles inscribed in the profile.

Since it appeared valuable to me to gain insight about the amount of the pressure increase in the silo cell from the surface down to arbitrary depth, I performed detailed experiments, whose results shall be reproduced here.

| Table 1 Runs 1–3 |
|------------------|
| run 1, cell 1    | run 2, cell 2    | run 3, cell 2    |
| area 20·20 cm    | area 30·30 cm    | area 30·30 cm    |
| content: wheat  | content: wheat  | content: wheat  |
| spe. weight $\gamma = 0.8$ | spe. weight $\gamma = 0.8$ | spe. weight $\gamma = 0.8$ |
| bottom pressure  | corn content kg  | bottom pressure  | corn content kg  | bottom pressure  | corn content kg  |
| kg               | kg               | kg               | kg               | kg               | kg               |
| 2.0              | 2.53             | 12.5             | 18.3             | 12.5             | 19.5             |
| 2.5              | 3.5              | 14.0             | 22.25            | 15.0             | 27.0             |
| 2.7              | 3.9              | 16.5             | 31.8             | 17.5             | 36.5             |
| 4.0              | 6.0              | 18.0             | 38.1             | 19.5             | 49.0             |
| 4.2              | 7.1              | 19.0             | 44.4             | 21.0             | 69.7             |
| 4.4              | 8.0              | 21.0             | 65.0             | 22.0             | 90.0             |
| 4.6              | 9.2              | 22.0             | 78.5             | 23.0             | 180.0 $^2$       |
| 5.0              | 10.0             | 22.5             | 90.0             |                 |                  |
| 5.5              | 12.0             | 23.0             | 180.0 $^1$       |                 |                  |
| 6.0              | 16.0             |                 |                  |                 |                  |
| 6.2              | 17.9             |                 |                  |                 |                  |
| 6.4              | 20.0             |                 |                  |                 |                  |
| 6.6              | 21.8             |                 |                  |                 |                  |
| 6.8              | 24.0             |                 |                  |                 |                  |
| 7.0              | 26.0             |                 |                  |                 |                  |
| 7.2              | 28.0             |                 |                  |                 |                  |
| 7.4              | 36.0             |                 |                  |                 |                  |
| 7.5              | 50.0             |                 |                  |                 |                  |
| 7.6              | 62.0             |                 |                  |                 |                  |

Four wooden sample cells were produced with quadratic profile and side lengths of 20, 30, 40, and 60 cm. During the time of the experiment the respective cell is mounted on four screws $S$, Fig. 1 and 2. The lower end of the tube is secluded by a well-fitting movable base. This bottom of the cell rests on a decimal balance that is counterbalanced by weights on the plate $G$ before the experiment. In this center position a foothold is placed underneath the plate $G$ and $G$ is loaded with an additional piece of weight. The corn is weighed in hollow vessels before being poured into the apparatus. Now the cell is filled up to the point where plate $G$ with the applied weights starts moving upwards from its center position. Weighing the remaining corn in the filling container, one can determine accurately the content in the cell that is causing the now known pressure at the bottom. After placing an additional weight on the plate $G$, the apparatus is lifted with the screws $S$ to the point where the balance is in equilibrium again. Now the experiment is continued. In this way it was possible to determine the bottom pressure for various filling heights while filling the cell only once.

The results are displayed graphically in Figs. 4 to 7. The smooth path of the observed pressure curve for the runs 2 to 6 lets us conclude that substantial observational errors have not occurred. Run number 1 – which was conducted first – displays relatively large observational errors, because the observers became more proficient in the handling of the instruments only during the course of the experiments.

Were the content of the cell to consist of a liquid, the bottom pressure would be equal to the weight of the cell’s content. The experiments with corn, however, yield a much smaller bottom pressure. This fact can be related to the fric-
### Table 2 Runs 4–6

|       | run 4, cell 3 | run 5, cell 4 | run 6, cell 4 |
|-------|--------------|--------------|--------------|
| area  | 40 · 40 cm   | 60 · 60 cm   | 60 · 60 cm   |
| content: | wheat      | wheat       | wheat       |
| spec. weight: $\gamma = 0.8$ |       | $\gamma = 0.8$ | $\gamma = 0.8$ |
| bottom pressure [kg] | corn content [kg] | bottom pressure [kg] | corn content [kg] | bottom pressure [kg] | corn content [kg] |
| 30    | 40.0          | 80           | 100          |
| 35    | 52.5          | 120          | 150          |
| 40    | 66.0          | 140          | 170          |
| 45    | 85.0          | 155          | 180          |
| 50    | 106.8         | 165          | 185          |
| 55    | 138.2         | 170          | 190          |
| 58    | 165.0         | 175          | 190          |
| 60    | 192.0         | 180          | 200          |
| 63    | 384.0 $^3$    | 185          | 200          |

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**Figs. 4–9**

- **Fig. 4 Run 1.**
- **Fig. 5 Runs 2 and 3.**
- **Fig. 6 Run 4.**
- **Fig. 7 Runs 5 and 6.**
- **Fig. 8**
- **Fig. 9**

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The investigations showed that the friction between corn and cell walls becomes so large at increasing depth that a pressure increase is no longer noticeable. Hence, the friction between corn and cell walls needs to be equal to the weight of the enclosed layers of corn. The magnitude of the corn pressure exerted on the encompassing walls in this case – the pressure apparently reaches its highest value – can be calculated in the following way. Let $P_{r,\text{max}}$ be the largest pressure of the corn against the wall, $f$ be the friction coefficient between corn and cell walls, $s$ be the side of the quadratic cell profile, $dh$ be the height of a corn layer, $\gamma$ be the specific weight of the corn, then Fig. 8 yields the equation

$$P_{r,\text{max}} = f 4s dh = \gamma s^2 dh,$$

$$P_{r,\text{max}} = \frac{\gamma s^2}{4f}.$$  \hspace{1cm} (1)

Assuming a constant coefficient of friction $f$ for variable pressures, the maximum pressure exerted on the cell walls by the corn for various cell widths of quadratic profile is proportional to the side length of the cell profile. To determine the value of $f$, the following experiments were performed:

A flat container, Fig. 9, was filled with wheat and covered with a wooden plate. The plate could be pulled across the corn with a string that was connected to a spring balance $F$. By putting weights on top the pressure between plate and corn could be regulated precisely. The result of these experiments is given in the following tables.

The investigations No 10 and 11 were done in the same way on different days. The difference of the friction coef-
coefficients determined this way – there is a deviation of 8% – should be ascribed to different humidity. The maximum value for $f$ resulted from run 8 and was 0.346, the minimum value from runs 10 and 12 was 0.302.

These investigations where the pressure was gradually increased by a factor of seven show that the value for the friction coefficient is not significantly influenced thereby; therefore, $f$ can be assumed constant with sufficient accuracy for different filling heights. Provided that the horizontal pressure of the corn is proportional to the vertical pressure, the latter can be calculated in the following way:

Let $P =$ total pressure of the corn on the cell floor, $p =$ vertical pressure of the corn, $p_r =$ horizontal pressure of the corn, $f =$ friction coefficient between corn and cell wall, $K = \frac{p_r}{p}$; $pK = p_r f$, $s =$ side length of the quadratic cell profile, $u =$ circumference of the cell $= 4s$, $F =$ area of the profile $= s^2$, $x =$ filling height of the corn in the cell, $\gamma =$ specific weight of the content, $e =$ base of the natural logarithm;

$$F(p + dp - p) = \gamma F dx - f p u dx$$

$$dp = \gamma dx - \frac{u}{F} dx$$

$$\frac{dp}{\gamma(1 - \frac{K u p}{F})} = dx$$

$$-F \ln \left[1 - \frac{K u p}{F} \right] = K u(x - x_0)$$

$x = 0; p = 0$

$$\ln \left[1 - \frac{K u p}{F} \right] = \frac{K u x}{F}$$

$$1 - e^{-\frac{K u p}{F}} = \frac{K u p}{F}$$

$$1 - e^{-\frac{8}{4K}} = \frac{4K}{s^2}$$

$$p = \frac{s^2}{4K} (1 - e^{-4K \frac{p}{u}})$$

(2)

$$p = \frac{s^2}{4K} (1 - e^{-4K \frac{p}{u}})$$

(3)

In the latter two equations only $K$ is unknown, which can be determined from the experiments 1 to 6. In run 4 the largest total bottom pressure was found at 63 kg or 3.94 kp per 1 qdm. The friction at the circumference in this case for a layer of 1 dm is $16\gamma = 16 \cdot 0.8 = 12.8 \text{ kg} = p_r f u$;

$$p_r f = \frac{12.8}{u} = \frac{12.8}{16} = 0.8; \quad K = \frac{p_r f}{p} = \frac{0.8}{3.94} = 0.203.$$  

For simplification I set $K = 0.2; \gamma = 0.8$. Then Eqs. (2) and (3) assume the following form

$$p = s \left(1 - e^{-0.8s} \right)$$

(2a)

$$p = s \left(1 - e^{-0.8s} \right)$$

(3a)

Figure 11 displays the results of run 4 and the calculated corn pressures according to Eq. (3a). Better agreement could have been achieved by assuming a larger value of $K$, considering the pressure measurements for smaller filling heights, since it cannot be ruled out that at the observation of the

largest pressure an observational error has occurred. In the same way as described above, $K$ is determined for the other runs, hence:

- run 1: $K = 0.211$
- run 2, 3: $K = 0.235$
- run 4: $K = 0.203$
- run 6: $K = 0.227$

The deviations can be explained by a slight variability of $f$. In runs 2 and 3, $f$ should have been larger by 15.5% compared to run 4. The experiments for the determination of the value of the friction coefficient result in (according to runs 7 to 12) $f = 0.302$ to 0.346, or 14.1% deviation. For the calculation of the side pressure against the cell wall one can therefore assume that the value of $f$ was around 0.346 in run 2 and 3 and around 0.302 in run 4. One gets from $p_s = Kp$ for run 2 and 3: $p_s = \frac{0.235}{0.346}p = 0.68p$, and for run 4: $p_s = \frac{0.203}{0.302}p = 0.675p$, or rounded $p_s = 0.7p$

Hereby the mean pressure against a side wall of a cell of quadratic profile is determined. However, we can safely assume that near the corners of the profile the pressure is lower than this mean value, while it is higher in the center part of the walls. It is shown in Fig. 12 how the pressure is transmitted to the side walls. Thereby it was assumed that the distribution of the corn pressure originates radially from the center of the cell and exerts a pressure against the wall of $p \sin \alpha$, with $\alpha$ being the angle under which the pressure ray hits the wall. The maximum pressure against the wall is reached in the middle of the cell wall with around 1.15 of the mean pressure, or $= 1.15 \cdot 0.7 \cdot p \approx 0.8p$. For the calculation of the wall thickness the ansatz of uniform loading can be made with sufficient accuracy

$$ p_s = 0.75p = s \cdot 0.75 \left(1 - e^{-0.8s}\right). \quad (4a) $$

Even if the above experimental and calculated results are sufficient to examine silo bottoms and silo walls for their necessary stiffness, a simpler way of calculation might be desirable. We are on the safe side for the usual kinds of corn if we assume wheat as cell content with a specific weight of 0.80, set $K = 0.20$, and calculate the corn pressure at the bottom according to formulas (2a) and (3a). From the graph in Fig. 13 one can obtain the value for the brackets in Eqs. (2a) and (3a) $= \left(1 - e^{-0.8s}\right) = Z$ for arbitrary $\frac{x}{s}$.

From the just determined vertical pressures one could get the respective side pressures – which are decisive for the dimensioning of the cell walls – by multiplying by 0.75; for greater convenience the value of $w = 0.75 \left(1 - e^{-0.8s}\right)$ can be obtained from Fig. 14 for arbitrary $\frac{x}{s}$.

Originally it was the intention of the author to determine the side pressure of the corn directly in the experiments.
Example 2. Calculation of the side pressure of a silo cell of

Table 6 Example 2. Calculation of the side pressure of a silo cell of

The Pressure of Stored Grain, and can be seen as an earlier account of the saturation effect observed in these silo experiments. This work is clearly acknowledged by Janssen, so it may seem unjustified to call this saturation effect alone the Janssen effect.

There exists an earlier partial translation of Janssen’s paper in [4], On the Pressure of Grain in Silos, which is listed there under the category of Foreign Abstracts and contains a short summary of the article by an editor. According to this editor, the saturation of the pressure “[…] has long been recognised”. While Janssen’s experiments are described in somewhat more detail, the Eqs. (2) and (3) are only cited without derivation and are indeed copied wrongly from the original.

In fact, one can infer from earlier work that is not cited by Janssen, that the saturation of the pressure with depth was already more or less well known in Janssen’s time. Janssen’s work is anticipated in 1852 by a paper of G.H.L. Hagen [5], who became famous for his work in fluid dynamics, and an even earlier investigation by Huber-Burnand in 1829 [6]. Hagen imagined a cylinder of radius \( r \) and height \( h \) inside a container filled with sand of a specific weight \( \gamma \). The pressure exerted on the bottom area \( \pi r^2 \) by the weight of the sand-container is diminished by the friction experienced by that cylinder from the sand surrounding it. Assuming isotropic horizontal pressure throughout the system, Hagen derives the formula

\[
p(h) \propto r^2 \pi \gamma h - 2r \pi \gamma rh^2
\]

with some friction coefficient \( l \). He discards the decreasing branch of the solution and predicts the saturation of the pressure with depth after the height reaches the value \( h = r/(4l) \). Using the latter relation, Hagen subsequently tests his predictions in an apparatus quite similar to Janssen’s setup. He determines values for \( l \) between 0.154 and 0.22, but he does not check if the pressure below the maximum follows Eq. (4) which predicts a steeper increase than Janssen’s Eq. (2). Hagen rather continues to analyze in detail the flow through an opening of his container and finds that the mass flow is proportional to some effective radius to the power 5/2, cf. Ref. [7, sec. 10.2]. For both flow and pressure measurements, Hagen mentions earlier work by Huber-Burnand without giving any reference. The appropriate reference is most likely Ref. [6] which discusses largely qualitative results of experiments with sand, peas, and – an egg. The egg is placed in a box, covered with several inches of sand, and loaded with a weight of 25 kg; the egg demonstrates – by not breaking, of course – that only a small fraction of the pressure from the top actually reaches the bottom.

To this end, one of the sample cells was equipped with a side lid that was pressed on the cell by suitable weights by an angle lever. The pressure necessary to open the lid could then be calculated. But while conducting the experiment the opening of the lid happened so slowly that accurate results could not be obtained. The same occurred with the attempt to measure the bottom pressure at various points on the bottom via small lids. In my opinion this feature can be traced back to the fact that arching occurs as the profile of the corn column narrows, which significantly influences the pressure transmission for smaller profiles. The attempt to equip the entire experiment from top to bottom with a movable cell wall was discarded because substantial inconveniences would have been incurred. Also the calculation method described above led to the same goal.

Besides wheat, further investigations were conducted with rye and maize, whose results are in agreement with the experiments described above. With rye – with a specific weight of 0.75 – the pressures fell short of the ones for wheat by around 20%. Maize at the same specific weight as wheat (\( \gamma = 0.80 \)) produced a larger bottom pressure by 22% due to its smoother grain surface. For a silo cell that is supposed to contain maize, the stiffness of the cell walls and the bottom must be increased by 22%.

The experiments described above – from which the calculations for arbitrary cell sizes are deduced – are performed in small apparatuses only, because the creation of larger sample cells is connected with non-negligible costs. Regardless, they should be clarifying in some aspect, and it is hoped that they find confirmation in experiments on larger scale like the ones announced by a well-known mill-building firm. I shall follow up with a further communication on that subject at that point.

| Table 5 | Table 6 | Example 1. Calculation of the bottom pressure for a cell of 4 × 4 m base: \( s = 4 \). Z from Fig. 13, \( p = sZ \). \( P = sZ \). |
|---|---|---|
| filling \( x/m \) | \( x/s \) | \( Z \) | \( p \) | \( P \) |
| 1 | 0.25 | 0.18 | 0.72 | 11.5 |
| 2 | 0.50 | 0.33 | 1.32 | 21.1 |
| 3 | 0.75 | 0.45 | 1.80 | 28.8 |
| 4 | 1.00 | 0.55 | 2.20 | 35.2 |
| 5 | 1.25 | 0.63 | 2.52 | 40.3 |
| 6 | 1.50 | 0.70 | 2.80 | 44.7 |
| 8 | 2.00 | 0.80 | 3.20 | 51.2 |
| 10 | 2.50 | 0.86 | 3.44 | 55.0 |
| 12 | 3.00 | 0.91 | 3.64 | 58.2 |
| 14 | 3.50 | 0.94 | 3.76 | 60.1 |

| Table 6 | Example 2. Calculation of the side pressure of a silo cell of 3 × 3 m base: \( s = 3 \). \( w \) after Fig. 14, \( p_s = sw \). |
|---|---|---|
| filling \( x/m \) | \( x/s \) | \( w \) | \( p_s \) [t/m²] |
| 1.5 | 0.50 | 0.245 | 0.735 |
| 3.0 | 1.00 | 0.410 | 1.23 |
| 4.5 | 1.50 | 0.530 | 1.59 |
| 6.0 | 2.00 | 0.595 | 1.79 |
| 8.0 | 2.67 | 0.655 | 1.97 |
| 10.0 | 3.33 | 0.690 | 2.07 |
| 12.0 | 4.00 | 0.720 | 2.16 |
Impact of Janssen’s paper

The journal where Janssen published his results can still be found today [8] – it is still a publication of the Verein Deutscher Ingenieure – the Association of German Engineers. Figure 15 shows the citations of Janssen’s paper [1] in the years since 1978. As of September 1st 2005, the article was cited 375 times with only 40 citations recorded for the time 1977 and earlier [9]. This demonstrates growing interest in granular systems in general and in one of its earliest papers in particular.

It is interesting to note that Janssen’s formula (2) is usually derived for the more symmetric cylindrical geometry [7, sec. 5.2] and that more involved profiles require some averaging of the wall stress along the perimeter, which can be done explicitly, cf. [7, sec. 5.5], or implicitly as in [10, sec. 3.1.4]. In any case, the profile does only change the prefactors in Eq. (2) but the overall behavior remains the same. Since he was experimenting with a square profile, Janssen was well aware of that kind of complication as seen in his more involved argument connected to Fig. 12. It was also found empirically that the saturation pressure was proportional to the diameter of the circle inscribed into the profile by Roberts [3]. Janssen reports this earlier finding, and Fig. 12 may suggest that he considered it essentially correct.

Janssen’s model rests on a number of approximations that were scrutinized later on: First, Eq. (1) amounts to full mobilization of friction at the walls. While that is not true in general, it might well hold to a good degree for Janssen’s experimental protocol where the walls of the container were shifted upwards by the screws \( S \) in Figs. 1 and 2 after each step before filling in more grain. Second, the vertical and horizontal stresses are introduced as principal stresses which is clearly inconsistent with the existence of a finite friction at the walls. These principal stresses – horizontal and vertical pressures \( p_h \) and \( p_v \) respectively – are then related by a constitutive law, \( pK = pf \), which is not motivated any further. Third, a uniform distribution of stresses is assumed for any slice of the silo, cf. Fig. 10. Taking the constitutive law for granted, this translates into a finite shear stress in the center of the silo which must be ruled out by symmetry. All three of these objections can be overcome by more refined continuum theories [7], and the result in Eq. (2) remains valid with appropriate changes to the prefactors. There are, however, more serious problems when considering relatively large overloads on top of the grains in the silo [7, sec. 5.5].

If one probes in more detail the granular structure within the silo and especially when considering the dynamics of the granular material, Janssen’s approach no longer yields satisfactory answers [10]. That said, it is quite remarkable that Janssen already postulates that it is arching within the granular assembly what prevented him from measuring the bottom pressure at various points.

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