FERMIONS IN THE LOWEST LANDAU LEVEL:
Bosonization, $W_\infty$ Algebra, Droplets, Chiral Bosons

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ABSTRACT

We present field theoretical descriptions of massless (2+1) dimensional nonrelativistic fermions in an external magnetic field, in terms of a fermionic and bosonic second quantized language. An infinite dimensional algebra, $W_\infty$, appears as the algebra of unitary transformations which preserve the lowest Landau level condition and the particle number. In the droplet approximation it reduces to the algebra of area-preserving diffeomorphisms, which is responsible for the existence of a universal chiral boson Lagrangian independent of the electrostatic potential. We argue that the bosonic droplet approximation is the strong magnetic field limit of the fermionic theory. The relation to the $c = 1$ string model is discussed.
Introduction

The study of nonrelativistic fermions in the presence of electromagnetic field in (2+1) dimensions is obviously important in condensed matter problems, such as the quantum Hall effect (QHE)[1]. Such systems have a further connection with (1+1) dimensional problems, such as the c = 1 string model. In a previous paper [2] we showed that a system of massless (2+1) dimensional charged nonrelativistic fermions in a uniform magnetic field is equivalent to a system of (1+1) dimensional nonrelativistic fermions, which in an appropriate external potential, is known to describe the c = 1 string model [3,4]. In the problem of QHE the applied electrostatic potential is absent except at the edge of the system but the fermions (electrons) are mutually interacting through Coulomb force. On the other hand in the c = 1 string model the mutual interaction is absent but the fermions are in a static potential \( A_0 = \frac{1}{2}(y^2 - x^2) - \mu \) in the double scaling limit [5].

In [2], using a classical hydrodynamic formulation, we arrived at the droplet picture, influenced by the Thomas-Fermi type classical model [6,7,8]. We established that the dynamics of droplets is described by an effective collective field Lagrangian [9,10]. The results derived are entirely consistent with the theory of edge excitations in the quantum Hall effect [11].

In this paper we describe the various fermionic and bosonic descriptions of two-dimensional massless nonrelativistic fermions in an external magnetic field and discuss the relation between them.

We further study the symmetries of the original system and how they manifest themselves in both the fermionic and bosonic field theoretic description. In particular we find the existence of an infinite dimensional algebra, \( W_\infty \) algebra [12,13], which was noticed in the study of QHE [14] and independently realized and studied extensively in the framework of c = 1 string model [8,15,16]. In both the fermionic and bosonic description we find that the \( W_\infty \) algebra is the algebra of unitary transformations which preserve the lowest Landau level condition and the particle number.
In the droplet approximation the $W_\infty$ algebra is reduced to the algebra of area-preserving diffeomorphisms. As a consequence we find that the dynamics of the droplets is described by a universal chiral boson Lagrangian independent of the details of the electrostatic potential.

Finally we discuss the relation of our results to the $c = 1$ string model.

**Field Theory of Massless Fermions in External Magnetic Field**

The many-body Hamiltonian of a system of massive (mass $m$) fermions in electromagnetic field in two space dimensions is given by

$$H = \frac{1}{2m} \sum_{a=1}^{N} (\Pi_a)^2 + \sum_a A_0(x_a)$$

$$= \frac{1}{2m} \sum_{a=1}^{N} (\Pi_a^x + i\Pi_a^y)(\Pi_a^x - i\Pi_a^y) + \sum_a A_0(x_a) + \frac{B}{2m} N, \quad (1)$$

where

$$\Pi^i = p_i - A^i(x) = -i \frac{\partial}{\partial x^i} - A^i(x), \quad i = x, y \quad (2)$$

and $B$ is a uniform external magnetic field defined by $\vec{\nabla} \times \vec{A} = -B$. In what follows we shall omit the last constant term in (1). The corresponding Schrödinger wave function $\Psi(x_1, x_2, \cdots x_N)$ is a totally antisymmetric function of $x$'s. As is well known, the fermions occupy discrete, degenerate Landau levels, uniformly separated by an energy gap $E_c = \frac{B}{m}$ (we have set $\hbar = c = e = 1$).

The massless limit of the above system restricts the fermions to the lowest Landau level for which the wavefunction obeys

$$(\Pi_a^x - i\Pi_a^y)\Psi(x_1, x_2, \cdots x_N) = 0 \quad (3)$$

In the symmetric gauge, where $A^x = By/2$, $A^y = -Bx/2$, the lowest Landau level

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condition is written as
\[(\partial_{z_a} + \frac{z_a}{2})\Psi(x_1, x_2, \cdots x_N) = 0\] (4)
where \(z_a = \sqrt{\frac{B}{2}}(x_a + iy_a)\), \(\bar{z}_a = \sqrt{\frac{B}{2}}(x_a - iy_a)\). The solution to (4) is given by
\[\Psi(x_1, x_2, \cdots x_N) = f(\bar{z}) \exp\left(-\frac{1}{2} \sum_{a=1}^{N} |z_a|^2\right)\] (5)
where \(f(\bar{z})\) is a polynomial in the variables \(\bar{z}_a\).

The projection of the Hamiltonian \(H = A_0(\hat{z}_a, \hat{\bar{z}}_a)^*\) onto the lowest Landau level is given by
\[H = \sum_a \hat{\dagger} A_0(\frac{\partial}{\partial \bar{z}_a} + \frac{z_a}{2}, \bar{z}_a)\hat{\dagger}\] (6)
where \(\hat{\dagger} \hat{\dagger}\) denotes the ordering where all the derivatives are on the left [17].

One can bosonize this system by using a singular gauge transformation [18].
\[\Psi(x_1, x_2, \cdots x_N) = e^{i \sum_{a>b} Im \ln(\bar{z}_a - \bar{z}_b)} \Phi(x_1, x_2, \cdots x_N)\] (7)
where \(\Phi(x_1, x_2, \cdots x_N)\) is a symmetric wave function. As a result we obtain, for the massless case, an effective bosonic Hamiltonian which is of the form
\[H = \sum_a \hat{\dagger} A_0(\frac{\partial}{\partial \bar{z}_a} + \frac{z_a}{2} + \frac{1}{2} \sum_{b\neq a} \frac{1}{\bar{z}_a - \bar{z}_b}, \bar{z}_a)\hat{\dagger}\] (8)
The bosonized analog of the lowest Landau level condition (4) takes the form
\[\left(\frac{\partial}{\partial \bar{z}_a} + \frac{z_a}{2} - \frac{1}{2} \sum_{b\neq a} \frac{1}{\bar{z}_a - \bar{z}_b}\right)\Phi(x_1, x_2, \cdots x_N) = 0\] (9)

The massless quantum mechanical system given in (3)-(9) can be further expressed in a second quantized language. There are two equivalent field theory descriptions for the fermionic representation (3)-(6).

\* We consistently use \(f(z, \bar{z})\) for \(f(x, y)|_{x = \sqrt{\frac{B}{2}(x+y)}} y = \frac{1}{2} \sqrt{\frac{B}{2}(x-y)}\)
The first description (“unreduced”) is to introduce the standard fermion creation and annihilation operators

\[
\{ \hat{\psi}(x), \hat{\psi}^\dagger(x') \} = \delta(x - x')
\]  

and define

\[
\Psi(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \langle 0 | \hat{\psi}(x_1) \hat{\psi}(x_2) \cdots \hat{\psi}(x_N) | \Psi \rangle
\]  

The lowest Landau level condition (4) and the Hamiltonian (6) are derived respectively from

\[
(\partial_z + \frac{\bar{z}}{2}) \hat{\psi}(x) | \Psi \rangle = 0
\]  

\[
H = \int dx \hat{\psi}^\dagger(x) A_0(\partial_{\bar{z}} + \frac{\bar{z}}{2}) \hat{\psi}(x)
\]  

\[
\doteq \int dx \hat{\psi}^\dagger(x) A_0(x) \hat{\psi}(x)
\]  

where \( \doteq \) is an equality up to an operator which vanishes when it acts on \( | \Psi \rangle \) from the right.

The second description (“reduced”) is to impose the operator constraint

\[
\left( \partial_z + \frac{1}{2} \bar{z} \right) \hat{\psi}(x, t) = 0
\]  

which again produces the lowest Landau level condition (4). A general solution of (14) has the form

\[
\hat{\psi}(x, t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} \hat{C}_n(t)
\]  

where the modes \( \hat{C}_n \) satisfy the usual anticommutation relations \( \{ \hat{C}_n, \hat{C}_m^\dagger \} = \delta_{nm} \). The constrained \( \hat{\psi} \) operators no longer satisfy the commutation relation (10). In terms of the constrained operators the Hamiltonian is given by (13) and is exactly equal to the expression in the second line of (13).
It can be easily shown that these two descriptions are equivalent. We only remark that they can be considered as two different methods of quantizing the following classical Lagrangian

\[ \mathcal{L} = \bar{\psi}(x, t) \left( i \frac{\partial}{\partial t} - A_0(x) \right) \psi(x, t) \]  

(16)

with the constraint \((\partial_z + \frac{1}{2} \bar{z}) \psi(x, t) = 0\). The reduced method amounts to solving the constraint explicitly, substituting the solution into the Lagrangian and developing a canonical quantization for the unconstrained variables.

Similarly the bosonized massless quantum mechanical system (7)-(8) is expressed, in a second quantized language, in terms of a Hamiltonian given by

\[ H = \int dx \hat{\phi}^\dagger(x) \frac{\partial}{\partial z} \bar{z} + \frac{1}{2} \int dx A_0(x) \hat{\phi}^\dagger(x) \bar{z} \hat{\phi}(x) \]

\[ = \int dx \hat{\phi}^\dagger(x) A_0(x) \hat{\phi}(x) \]  

(17)

where \(\hat{\phi}\) is a bosonic field operator satisfying the usual commutation relations \([\hat{\phi}(x), \hat{\phi}^\dagger(x')] = \delta(x - x')\) and \(\hat{\rho}(x) \equiv \hat{\phi}^\dagger(x) \hat{\phi}(x)\). The analog of the lowest Landau level condition is of the form

\[ \left( \frac{\partial}{\partial z} + \frac{\bar{z}}{2} - \frac{1}{2} \int \frac{\hat{\rho}(x')}{\bar{z} - \bar{z}'} \right) \hat{\phi}(x, t) |\Phi\rangle = 0 \]  

(18)

This is the bosonic analog of the unreduced description.

We should remark here that in the bosonic formalism the lowest Landau level condition (18) cannot be straightforwardly reduced to an operator equation because the constraint equation is nonlinear in \(\hat{\phi}\).

If one ignores the subtlety of operator ordering, namely the order \(\hbar\) corrections, it is possible to describe the system in terms of a classical Lagrangian of the same
form as in (16), where $\psi$ is replaced by $\phi$, with the constraint

$$
\left( \frac{\partial}{\partial z} + \frac{\dot{z}}{2} - \frac{1}{2} \int \frac{\rho(x')}{z - z'} \right) \phi(x, t) = 0 \quad (19)
$$

In terms of the reduced formalism one would solve the classical constraint equation (19) and then develop a canonical formalism for the unconstrained variables. A priori this may not be equivalent to the bosonic representation described by (17)-(18) at the quantum mechanical level, but presumably a proper choice of operator ordering in terms of the unconstrained variables can be made so that the fully quantum mechanical bosonic representation is recovered.

In [2] the bosonic problem was further formulated in terms of hydrodynamic variables

$$
\phi = \sqrt{\rho} e^{i\theta} \quad (20)
$$

It was shown that the corresponding Lagrangian has the form

$$
L = \int d^2x d^2x' \rho(x) 2\pi G(x - x') \dot{\rho}_V(x') - \int d^2x \rho(x) A_0 \quad (21)
$$

while the classical lowest Landau level condition is written as

$$
B + \frac{1}{2} \nabla^2 \ln \rho - 2\pi \rho V - 2\pi \rho = 0 \quad (22)
$$

where $\rho_V(x)$ denotes the vortex distribution function defined by

$$
-2\pi \rho_V(x) = \epsilon^{ij} \partial_i \partial_j \theta(x) \quad (23)
$$

and $G$ is the Green’s function satisfying

$$
\epsilon^{ij} \partial_i \partial_j G(x - x') = \delta^2(x - x'), \quad \nabla^2 G = 0. \quad (24)
$$

One can show that (22) provides a way of writing (19) in terms of the hydrodynamic variables (20).
Since the full quantum mechanical constraint (18) is rather difficult to implement, our approach in [2] was to treat (22) as a classical constraint equation, solve it and then develop a canonical formalism for the unconstrained variables.

Two-Dimensional Fermions and $W_\infty$ Algebra

In terms of the standard coherent state representation:

$$|z\rangle = e^{z\hat{a}^\dagger}|0\rangle, \quad \int d^2 z \ e^{-|z|^2} |z\rangle \langle z| = 1, \text{ etc.}$$

(25)

the constrained operator $\hat{\psi}$ in (15) can be written as

$$\hat{\psi}(x,y) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \sum_n \langle z|n\rangle \hat{C}_n$$

(26)

Next let us consider a unitary transformation in the space of $\hat{C}_n$:

$$\hat{C}_n' = u_{nm} \hat{C}_m = \langle n|\hat{u}|m\rangle \hat{C}_m$$

(27)

An infinitesimal transformation is generated by a hermitian operator which we write as $\xi(\hat{a}, \hat{a}^\dagger)\hat{\xi}$ with the anti-normal order symbol, where $\xi$ is a real function when $\hat{a}$ and $\hat{a}^\dagger$ are replaced by $z$ and $\bar{z}$ respectively. Then using (26) we obtain the following infinitesimal transformation for $\hat{\psi}$:

$$\delta \hat{\psi}(x,y) = i \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \hat{\xi}(\partial_{\bar{z}}, \bar{z}) \hat{\xi} \sum_n \langle z|n\rangle \hat{C}_n = i \hat{\xi}(\partial_{\bar{z}} + \frac{1}{2}z, \bar{z}) \hat{\psi}(x,y)$$

(28)

where $\hat{\xi}$ $\hat{\xi}$ indicates that the derivatives are placed on the left of $z$ and $\bar{z}$. The charge density $\hat{\rho} = \hat{\psi}^\dagger \hat{\psi}$ is transformed as

$$\delta \hat{\rho}(x,y) = i \left( \hat{\xi}(\partial_{\bar{z}} + z, \bar{z}) \hat{\xi} - \hat{\xi}(z, \partial_{\bar{z}} + \bar{z}) \right) \hat{\rho}(x,y)$$

(29)

As a result we find, by partially integrating (29), that the total fermion number
remains invariant as it should be:

$$\int dxdy\delta \rho(x, y) = 0$$  \hfill (30)

Since $\partial z + \bar{z}/2$ commutes with $\partial \bar{z} + z/2$ the transformation (28) can be considered as the most general linear infinitesimal transformation which preserves the lowest Landau level condition (14). It is straightforward to find that the generator of this transformation is given by

$$\hat{\rho}[\xi] \equiv \int dxdy\xi(x, y)\hat{\rho}(x, y) = \int dxdy\hat{\psi}^\dagger(x, y)\xi(\partial z + \frac{1}{2}z, \bar{z})\hat{\psi}(x, y)$$  \hfill (31)

Using (15) and the anticommutation relation for $\hat{C}$’s we find that the algebra satisfied by the generators $\hat{\rho}[\xi]$ is given by

$$[\hat{\rho}[\xi_1], \hat{\rho}[\xi_2]] = \frac{i}{B}\hat{\rho}[\{\xi_1, \xi_2\}]$$  \hfill (32)

where

$$\{\xi_1, \xi_2\} = iB\sum_{n=1}^{\infty} \frac{(-)^n}{n!} (\partial^n_z \xi_1 \partial^n_{\bar{z}} \xi_2 - \partial^n_{\bar{z}} \xi_1 \partial^n_z \xi_2)$$  \hfill (33)

By choosing $\xi(x, y) = \exp ipx + qy$ we obtain the commutation relation of the fermion density in the momentum space:

$$[\hat{\rho}(p, q), \hat{\rho}(p', q')] = -2i \sin\left(\frac{pq' - qp'}{2B}\right)e^{i\frac{pq' + qp'}{2B}} \hat{\rho}(p + p', q + q')$$  \hfill (34)

where $\hat{\rho}(p, q) = \int dxdye^{ipx}e^{iqy}\hat{\rho}(x, y) \equiv \hat{\rho}[\exp ipx + qy]$. And also by choosing $\xi(z, \bar{z}) = z^l \bar{z}^m$ we obtain

$$[\hat{\rho}_{rs}, \hat{\rho}_{lm}] = \sum_{n=1}^{\min(l, s)} \frac{(-)^n}{n!} \frac{l!s!}{(l-n)!(s-n)!} \hat{\rho}_{r+l-n, s+m-n} - (s \leftrightarrow m, l \leftrightarrow r)$$  \hfill (35)

where $\hat{\rho}_{lm} = \int dxdy z^l \bar{z}^m \hat{\rho} \equiv \hat{\rho}[z^l \bar{z}^m]$, which is a coefficient of the power series expansion of $\hat{\rho}(p, q)$. The Lie algebra (32) and its representations (34) and (35)
in the specific bases are manifestations of the $W_\infty$ algebra [12], which in this case is the algebra of $U(\infty)$. It corresponds to unitary transformations which preserve the lowest Landau level condition and the fermion number.

The Hamiltonian, given by $H = \int A_0 \hat{\rho}$, is also an element of $W_\infty$. Using the Heisenberg equation of motion and the commutation relation (32) we obtain the equation of motion for the fermion density projected on the lowest Landau level:

$$
\partial_t \hat{\rho}(x, y, t) = i \sum_{n=1}^{\infty} \frac{1}{n!} [\partial^n_z (\hat{\rho} \partial^n_A - \partial^n_z (\hat{\rho} \partial^n_A)]
$$

We now turn to the bosonic formalism (17)-(18). Let us look for a unitary transformation on the state $|\Phi\rangle$ which preserves (18). This is generated by an operator $\mathcal{O}$ such that

$$
[(\partial_z + \bar{z}) - \frac{1}{2} \int d\bar{x}' \frac{\hat{\rho}(\bar{x}')}{\bar{z} - \bar{z}'}) \hat{\phi}(x), \mathcal{O}|\Phi\rangle = 0
$$

We identify the operator $\mathcal{O}$ to be

$$
\mathcal{O}[\xi] = \int d\bar{x} \hat{\phi}^\dagger(\bar{x}) \xi (\partial_z + \bar{z}) + \frac{1}{2} \int d\bar{x}' \frac{\rho(\bar{x}')}{\bar{z} - \bar{z}'} \hat{\phi}(x)
$$

One can now show using (37) that the operator $\mathcal{O}[\xi]$ generates the $W_\infty$ algebra on the space of states satisfying the lowest Landau level condition, namely

$$
\langle \Phi | [\mathcal{O}[\xi_1], \mathcal{O}[\xi_2]] = \langle \Phi | \frac{i}{B} \mathcal{O}[\{\{\xi_1, \xi_2\}\}]
$$

where $\{\{\xi_1, \xi_2\}\}$ is defined as in (33). The Hamiltonian (17) is again an element of $W_\infty$.

We find that in both the fermionic and bosonic formulation the $W_\infty$-algebra emerges as the algebra of unitary transformations of physical states.
Droplet Approximation, Chiral Bosons and $w_\infty$ algebra

The droplet approximation considered in [2] is based on the hydrodynamic formulation obtained in (21)-(24). In this case an approximate solution to the lowest Landau level equation (22) is considered, where the particle density is maximum, $\rho = B/2\pi$, inside and zero outside a boundary in space and correspondingly the vortex density is maximum, $\rho_V = B/2\pi$, outside and zero inside the boundary. The only dynamical variable in this picture is the one parametrizing the boundary. The classical ground state configuration corresponds to a boundary specified by the curve $A_0(x, y) = 0$, while excited states of the system correspond to deformations of this boundary.

The above choice for $\rho$ and $\rho_V$ satisfies the condition (which can be considered as the lowest Landau level condition in the droplet approximation)

$$2\pi \rho + 2\pi \rho_V - B = 0, \quad \rho \rho_V = 0 \tag{40}$$

and is such that the energy $\int d^2xA_0\rho$ stays in the neighborhood of minimum, namely the particle density is maximum in the region of negative $A_0$ (we assume that this is the region inside the boundary) and minimum in the region of positive $A_0$ (outside the boundary).

In order to specify the boundary we choose a coordinate system $r, s$, where

$$r = A_0(x, y), \quad s = S(x, y), \quad \text{such that } dx dy = dr ds \tag{41}$$

The transformation (41) is a transformation that preserves the local area and satisfies $\{A_0, S\}_{PB} \equiv \partial_x A_0 \partial_y S - \partial_x S \partial_y A_0 = 1$. Assuming that the boundary is closed, we parametrize $\rho$ and $\rho_V$ as

$$\rho(x, y, t) = \frac{B}{2\pi} \theta(r(s, t) - r), \quad \rho_V(x, y, t) = \frac{B}{2\pi} \theta(r - r(s, t)) \tag{42}$$

and regard $r(s, t)$ as a set of dynamical variables. The above ansatz solves explicitly the constraint equation (40). We restrict $r(s, t)$ to be a singlevalued function of $s$. 

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Then the Lagrangian of the system, eq. (21), is given by

\[ L = \frac{B^2}{8\pi} \oint ds \oint ds' r(s) \epsilon(s - s') \dot{r}(s') - \frac{B}{4\pi} \oint ds r^2(s) \]

(43)

After rescaling the field, namely \( r(s, t) \rightarrow \sqrt{2\pi}r(Bs, t) \), it reduces to the chiral boson Lagrangian \(^*\) [19,20]. The equation of motion for the field \( r(s, t) \) is

\[ (\partial_t - \frac{1}{B} \partial_s) r(s, t) = 0 \]

(44)

Based on (43) we quantize the system and we study the corresponding quantum collective motion of the original fermions as the quantum mechanical droplet motion. The canonical commutation relation for the chiral boson field is

\[ [\hat{r}(s, t), \hat{r}(s', t)] = \frac{2\pi i}{B^2} \delta'(s - s') \]

(45)

Using (43) and (45) we can show that the density operator \( \hat{\rho} \) satisfies the following commutation relation:

\[ [\hat{\rho}[\xi_1], \hat{\rho}[\xi_2]] = \frac{i}{B} \hat{\rho}[\{\xi_1, \xi_2\} P_B] \]

(46)

where

\[ \hat{\rho}[\xi] = \int dx dy \xi(x, y) \hat{\rho}(x, y) \quad \text{and} \quad \{\xi_1, \xi_2\} P_B = \epsilon_{ij} \partial_i \xi_1 \partial_j \xi_2 \]

(47)

and \( \xi(x, y) \) is a real function. This is the algebra of area-preserving diffeomorphisms, which is also called classical \( \omega_{\infty} \) algebra. Thus the \( \hat{\rho} \)'s generate the area-preserving diffeomorphisms as quantum mechanical unitary transformations. It is

\(^*\) We should remark here that the transformation (41) may have singular points. This is so if the inverse transformation is multivalued. If this is the case there exist multi-boundaries and we have to introduce multi-chiral boson fields, one for each boundary. We ignore this complication in this paper.
obvious that these transformations preserve the particle number, or in other words
the area of the droplet since

$$\delta_\xi \hat{\rho}(x, y) = \frac{1}{B} \epsilon_{ij} \partial_i \xi \partial_j \hat{\rho}(x, y)$$  \hspace{1cm} (48)

Since the Hamiltonian of the system is given by

$$H = \int dxdy A_0(x, y) \hat{\rho}(x, y) \equiv \hat{\rho}[A_0]$$  \hspace{1cm} (49)

it is an element of the algebra. Therefore all the systems with $A_0$'s related by
area-preserving transformations are unitarily equivalent.

Let us now discuss the relationship between the full fermionic theory and the
droplet approximation we used for the corresponding bosonic theory.

Let us first compare the algebras (32)-(33) and (46)-(47). Since $z$ is propor-
tional to $\sqrt{B}$, in the large $B$ limit one may neglect the higher derivative terms in
(33). In this limit the $W_\infty$ algebra is reduced to the classical $w_\infty$ algebra:

$$\{ \xi_1, \xi_2 \} \rightarrow \{ \xi_1, \xi_2 \}_{PB}$$  \hspace{1cm} (50)

More generally we can say that the droplet approximation is the large $B$ limit of
the hydrodynamic formulation described by (21)-(24), which is equivalent to the
original fermion theory at least semiclassically. In order to justify this let us go
back to the lowest Landau level condition as expressed in (22). This is the equation
for vortices [21]. $B$ is the only dimensional parameter $B \sim 1/l^2$. If we were to
solve (22) the existence of the term $\nabla^2 \ln \rho$ would result to the softening of the
step function ansatz for $\rho(x, y)$ producing a thick boundary, where the thickness is
necessarily of order $\sim 1/\sqrt{B}$. In that sense the droplet approximation, where the
boundary is sharp, is valid only when $1/\sqrt{B}$ is much smaller than the size of the
separation of the boundaries of the droplet which is determined by $A_0$. 

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Relation to $c = 1$ string theory

As we have mentioned earlier, the system of two-dimensional nonrelativistic massless fermions in a uniform magnetic field is equivalent to one-dimensional nonrelativistic fermions. If we further choose the electrostatic potential to be of the form $A_0 = \frac{1}{2}(y^2 - x^2) - \mu$, it describes the $c = 1$ string model [5]. Therefore it is of no surprise that a similar algebraic structure, like the $W_\infty$ algebra, emerges in both the (2+1) dimensional fermionic system and the (1+1) dimensional one which describes the $c = 1$ string model.

In a series of papers [8,16] the field $W(p,q,t)$ was introduced, which is a bilinear in terms of one-dimensional fermions and carries the $W$ algebra. In our case it is the two-dimensional fermion density that carries the $W$ algebra. Due to the lowest Landau level condition one can write the two-dimensional fermion density in the momentum space as

$$\hat{\rho}(p,q) = \sum_{nl} C_l^\dagger C_n \langle l | \exp \frac{i(p - iq)\hat{a}}{\sqrt{2B}} \exp \frac{i(p + iq)\hat{a}^\dagger}{\sqrt{2B}} | n \rangle$$

Using this we find that there is a relationship between the one-dimensional bilinear $W$ and our two-dimensional fermion density.

$$\hat{\rho}(p,q,t) = 2 \exp(-\frac{p^2 + q^2}{4B})W(p,-q,t)$$

where $B$ plays the role of $\hbar^{-2}$ in the definition of $W$. This relation is further extended in deriving the equation of motion and Ward-identities for the correlators of the bilinears.

Let us now discuss the relation between the droplet approximation and the $c = 1$ string model. We argued in the previous section that the droplet approximation is valid only when $1/\sqrt{B}$ is much smaller than the characteristic length introduced by $A_0$. Since the potential appropriate for $c = 1$ string model is of the form $A_0 = \frac{1}{2}(y^2 - x^2) - \mu$, the droplet approximation makes sense when $1/\mu \ll B$,
which corresponds to a weak string coupling ($\mu \to \infty$). In order to describe the theory away from the weak coupling region, one should solve the constraint (22). In that case the boundary becomes blurry and the droplet picture disappears as $\mu \sim 1/B$.

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