Homotopic Convex Transformation: A New Method to Smooth the Landscape of the Traveling Salesman Problem

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Abstract—This paper proposes a novel landscape smoothing method for the symmetric Traveling Salesman Problem (TSP). We first define the homotopic convex (HC) transformation of a TSP as a convex combination of a well-constructed simple TSP and the original TSP. We observe that controlled by the coefficient of the convex combination, (i) the landscape of the HC transformed TSP is smoothed in terms that its number of local optima is reduced compared to the original TSP; (ii) the fitness distance correlation of the HC transformed TSP is increased. We then propose an iterative algorithmic framework in which the proposed HC transformation is combined with a heuristic TSP solver. It works as an escaping scheme from local optima for improving the global search ability of the combined heuristic. A case study with the 3-Opt local search as the heuristic solver shows that the resultant algorithm significantly outperforms iterated local search and two other smoothing-based TSP heuristic solvers on most of commonly-used test instances.

Index Terms—Homotopic convex transformation, traveling salesman problem, landscape smoothing, local search, combinatorial optimization

I. INTRODUCTION

As a well-known N\textsuperscript{P}-hard combinatorial optimization problem, the Traveling Salesman Problem (TSP) has been extensively studied for years. Algorithms, including both exact and metaheuristics, have been developed in a number of literatures. It’s been well acknowledged that exact algorithms are only able to deal with small size TSPs. Metaheuristics, although cannot guarantee to find the global optimum, can provide nearly globally optimal solutions within tolerable time for medium and large size TSPs.

The hardness of TSP is due to the ruggedness and irregularity of its fitness landscape. It is extremely hard for a search-based meta-heuristic to find the global optimum of an TSP when there are innumerable local optima in the landscape.

In this paper, we propose a novel method to smooth the landscape of TSP. For a TSP with \( n \) cities, we first constructs a 2D Euclidean TSP in which the \( n \) cities lie on the convex hull of the TSP graph and the order of the cities follows the order they appear in a known optimum of the original TSP. The generated TSP is named the convex-hull TSP. It can be proved that the fitness landscape of the convex-hull TSP is unimodal, which means that a search process started from any initial solution will always end in the global optimum of the convex-hull TSP. After getting the convex-hull TSP, the original TSP is transformed by a convex combination of the convex-hull TSP and the original TSP with coefficient \( \lambda \in [0,1] \). When \( \lambda = 0 \), the transformed TSP is just the original TSP, while \( \lambda = 1 \) means the convex-hull TSP. This actually defines a homotopic transformation from the original TSP to the transformed TSP. We thus call the proposed transformation as the homotopic convex (HC) transformation.

To investigate the characteristics of the HC transformation, we conduct systemic experiments on 12 TSP instances from the well-known TSPLIB. In our experiment, the ruggedness of the TSP landscape is measured by three metrics, including local optimum density, escaping rate, and fitness distance correlation (FDC). These metrics are computed from the results of 1000 runs of Iterated Local Search (ILS) with 3-Opt local search and double bridge perturbation. In our experiments, different values of \( \lambda \) are tested and their influence to the smoothing effect is analyzed.

We observe from the experiments that the HC transformation exhibits the following three features. First, the landscape of the transformed TSP has less number of local optima than the original TSP, which implies a smoother TSP landscape. Second, in the transformed TSP, the FDC is higher than that of the original TSP, which implies that a search algorithm can find a path to the global optimum by observing the fitness change of the encountered local optima. Third, the strength of smoothing can be controlled by the coefficient \( \lambda \). In addition, we observe that the runtime for the ILS to find the global optimum is significantly reduced after the transformation. The observations imply that the transformed TSP has a reduced hardness, which is in accord with the intuition that a smoother landscape means an easier TSP.

We thereby make use of these features to propose a general iterative algorithmic framework. In the framework, the HC transformation is combined with a TSP heuristic. During the search, the heuristic works on the transformed TSP, while the transformed TSP is iteratively updated by the newly-found best solution with respect to the original TSP. As a case study, we embed the 3-Opt local search heuristic into the proposed framework and the instantiated algorithm is called Landscape Smoothing iterated Local Search (LSiLS). In the experimental study, LSiLS is compared against the original ILS and two existing landscape smoothing based algorithms on seven TSP instances. Comparison results show that LSiLS significantly outperforms the counterparts on most of the test...
instances. Note that we do not claim that the proposed LSiLS is better than the state-of-the-art TSP algorithms. Our goal is to show the potential of the HC transformation on improving the existing TSP algorithms.

The rest of this paper is organized as follows. Section III presents the related work. In Section III we present the definition of the HC transformation and the experimental studies on the investigation of the characteristics of the HC transformation. In Section IV the framework and the LSiLS is presented. Experimental results against its counterparts are also provided in this section. Section V concludes the paper.

II. RELATED WORK

Given a set of cities and cost between every pair of cities, the TSP is to find the most cost-effective tour that visits every city exactly once and returns to the starting city. It is \textit{NP}-hard and one of the most widely used tested in combinatorial optimization. The formal definition of the TSP can be stated as follows. Let \( G = (V, E) \) be a fully connected graph with cities as vertexes, where \( V \) is the vertex set and \( E \) the edge set. Denote \( c_{ij} > 0 \) the cost of the edge between vertex \( i \) and vertex \( j \), the objective function of a TSP is defined as

\[
\text{minimize } f(x) = c_{x(n)}x(1) + \sum_{i=1}^{n-1} c_{x(i), x(i+1)},
\]

subject to \( x = (x(1), x(2), \ldots, x(n)) \in \mathcal{P}_n \), where \( f : \mathcal{P}_n \rightarrow \mathbb{R} \) is the objective function and \( \mathcal{P}_n \) is the permutation space of \( \{1, 2, \ldots, n\} \). In this paper we focus on the symmetric TSPs, i.e., \( c_{ij} = c_{ji} \) for all \( i, j \in \{1, 2, \ldots, n\} \).

The hardness of a TSP is usually characterized by its fitness landscape. Formally, the fitness landscape of a combinatorial optimization problem is defined by \( (S, f, d) \), where \( S \) is the solution space, \( f : S \rightarrow \mathbb{R} \) is the objective function (fitness) and \( d \) is a distance measure on \( S \). For the TSP, the distance between any two solutions \( x_1 \) and \( x_2 \) is defined as the number of edges that one solution differ from the other:

\[
d(x_1, x_2) = |\{e \in E | e \in x_1 \land e \notin x_2\}|
\]

where \( e \in x \) means edge \( e \) is in the solution \( x \).

A great amount of efforts have been made on the TSP fitness landscape analysis. Among these studies, Stadler and Schnabl [4] investigated the landscape of symmetric and asymmetric TSPs by random walk techniques. Boese [5] first observed that there is a strong correlation between the distance of a solution to a global optimum and its cost when he investigated the landscape of a specific TSP instance att532. He called this phenomenon the \textit{big valley structure}. The existence of the big valley structure is confirmed in [6] and [7].

In the work of Fonlupt et al. [8], it is found that the TSP landscape defined by 2-Opt-move has a higher fitness distance correlation (FDC) value than the TSP landscape defined by city-swap. Merz and Freisleben [9] studied the ruggedness and the FDC of the TSP landscape using the 3-Opt local search and the Lin-Kernighan (LK) local search [11]. They observed that local optima are frequently close to each other and also close to the global optimum. Tataraynani and Prügel-Bennett [10, 11] also used the 3-Opt local search as a tool to analyze the characteristics of the fitness landscape of 11 different classes of TSP. They found that the high-quality local optima are more likely to be found by a local search process than low-quality local optima and the probability of finding a global optimum decreases exponentially with increasing problem size. They also found that Euclidean TSPs have relatively high FDC values, and the value is relatively low for random TSPs. For more detailed survey, interested readers please see [12, 13] and [14].

Existing TSP heuristics fall basically into two categories. One kind is constructive based where a solution to a TSP is constructed gradually from partial to whole. Examples of such including ant colony optimization [15], greedy randomized adaptive search procedure [16], and many others. The other kind is local search based. Local search first defines a neighborhood structure on the solution space of TSP. Starting from an initial solution, local search iteratively explores the neighborhood of the current solution for a better solution. Local search can only guarantee to find local optimum. The most widely-used local search for TSP is the k-Opt local search \((k \geq 2)\), in which the neighborhood structure is defined by edge exchange. In each descent move of a k-Opt local search, \( k \) edges are replaced by another \( k \) edges. The well-known 2-Opt local search, 3-Opt local search and Lin-Kernighan (LK) local search [11] all belong to this class.

To escape from local optima, existing global search techniques for TSP are usually built upon local search through criterion, solution or problem perturbation, and landscape smoothing. In criterion perturbation, rather than only accepting better solution in the neighborhood, a worse solution could be accepted in probability (e.g. Simulated Annealing [17, 18]) or if it prevents from returning to previously visited areas (e.g. Tabu Search [19]). In the solution perturbation methods, the current local optimum is perturbed to get a new solution and the local search process restarts from that new solution. The well-known Iterated Local Search (ILS) [20] belongs to this category.

The problem perturbation methods modify the problem itself so that the current local optimum is no longer locally optimal in the modified problem. In the Guided Local Search (GLS) proposed by Voudouris and Tsang [21], the original TSP is changed by adding penalties on some selected edges. Whenever GLS is trapped in a local optimum, edges from the local optimum are evaluated and penalized so that the local optimum is no longer locally optimal in the next round of local search. Shi et al. [22] improved GLS based on the big valley assumption of the TSP. In the improved GLS, the algorithm maintains an elite solution and the edges in the elite solution will be protected from the penalization. Walshaw [23] introduced a multilevel refinement approach to simplify the TSP. In each step, several selected cities are matched and the edges between them is fixed so that the problem is coarsened to an easier problem. In the backbone-guided local search proposed by Zhang and Looks [24], the TSP edge cost is reduced proportionally to the frequency that the edge appears in the historical local optima set.

Very few global search approaches have been proposed based on landscape smoothing. Gu and Huang [25] propose
to smooth the landscape of a TSP by edge cost manipulation. They normalize and transform the edge cost to

$$c'_{i,j}(\alpha) = \begin{cases} \bar{c} + (c_{i,j} - \bar{c})^\alpha & \text{if } c_{i,j} \geq \bar{c}, \\ \bar{c} - (\bar{c} - c_{i,j})^\alpha & \text{if } c_{i,j} < \bar{c}, \end{cases}$$

(3)

where $\bar{c}$ is the average cost of all edges and $\alpha \geq 1$ is the smoothing factor. When $\alpha = 1$, it is just the original TSP.

On the other hand, note that $\lim_{\alpha \to \infty} c'_{i,j} = \bar{c}$. That is, with increasing $\alpha$, all edge costs approach a fixed value $\bar{c}$. This implies that all solutions will have the same fitness, i.e., the TSP landscape is smoothed to a plane, as illustrated in Fig. 1. Gu and Huang did not carry out systemic experiments to investigate the smoothing effect of their approach to the fitness landscape. It was studied by Coy et al. [26], who claimed that the smoothing method can help local search escape from poor local optima by moving uphill occasionally and by not taking a downhill move occasionally. Further study by Coy et al. [27] shows that a sequential smoothing algorithm with alternated convex and concave smoothing function can achieve satisfactory performance. Hasegawa and Hiramatsu [28] combines the Memetic Algorithm (MA) with Gu and Huang’s smoothing method and show that it is beneficial to apply the smoothing method in MA.

III. HOMOTOPIC CONVEX TRANSFORMATION

In this section, we propose a new landscape smoothing method called the Homotopic Convex (HC) Transformation. It transforms the original TSP by combining it with a well-designed unimodal TSP named as the convex-hull TSP. In the following, we first define the convex-hull TSP and the HC transformation. The effects of the HC transformation to landscape smoothing are illustrated afterwards.

A. Convex-Hull TSP

Although TSP is NP-hard, some TSPs are easy to solve. For example, a 2D Euclidean TSP with all of the cities lie on the convex hull of the TSP graph, which we name it as convex-hull TSP, is easy for all local search based algorithms [29]. Fig. 2(a) shows an example of the convex-hull TSP and its global optimum. The only global optimum of a convex-hull TSP is the convex hull itself.

Here we prove that the convex-hull TSP is unimodal for any k-Opt local search, i.e., the TSP tour formed by the edges on the convex hull is the only k-optimal tour [30] of such TSP:

Definition. A tour is said to be $k$-optimal if it is impossible to obtain a tour with smaller cost by replacing any $k$ of its edges by any other set of $k$ edges.

Theorem 1. (\cite{29}) Denote $C_k$ the set of all $k$-optimal tours, we have $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n$. In other words, a $k$-optimal tour is also a $k'$-optimal tour for $k' < k$.

Theorem 2. For a convex-hull TSP $f_c$, let $x_c$ denote the convex hull tour, then $x_c$ is the only $k$-optimal tour in the solution space of $f_c$ for any $k \in \{2, 3, \ldots, n\}$. In other words, the convex-hull TSP is a unimodal to $k$-Opt local search.

Proof. It is sure that $x_c$ is $k$-optimal for any $k \in \{2, 3, \ldots, n\}$. We here prove the uniqueness of $x_c$ by contradiction. Assume there is another $k$-optimal tour $x_o$ for $f_c$. By Theorem 1 we known that $x_o$ is also 2-optimal. Since $x_o \neq x_c$, there are at least two edges in $x_o$ that cross each other. According to the triangle inequality, it is always possible to replace the crossed edges with non-interacting edges which are less cost. This indicates that $x_o$ is not 2-optimal, which contradict the assumption.

B. Definition of the HC Transformation

For a given TSP $f_o$ with a known local or global optimum, the HC transformation first constructs a convex-hull TSP, denoted by $f_c$, and makes sure that $f_c$ has the same optimum as $f_o$. Assume that the original TSP $f_o$ has $n$ cities, to construct the corresponding convex-hull TSP $f_c$, the HC transformation simply let $n$ cities uniformly distribute on a circle following the order they appear in the optimum of $f_o$. To level the scale of $f_c$ and $f_o$, the city interval on the circle of $f_c$ is set to be

$$\text{city interval} = \frac{1}{n} \sum_{i=1}^{n} c_{i,m(i)},$$

where $c_{i,m(i)}$ denotes the edge cost in the original TSP $f_o$ and $m(i)$ denotes the index of the nearest city to city $i$ in $f_o$, i.e.,

$$m(i) = \arg\min_{j \in \{1, \ldots, n\}, j \neq i} c_{i,j}.$$

Once all the cities are arranged on the circle, the distance, denoted as $c_{i,j}$, between each city pair $(i, j)$ is computed and considered as the edge cost in the convex-hull TSP. Fig. 3(b) shows an example convex-hull TSP (and its global optimum) constructed for the TSP instance eil51 (Fig. 3(a)).
The objective function of the transformed TSP, denoted as $f$, is calculated by

$$g(x) = (1 - \lambda) f_o(x) + \lambda f_c(x).$$

The mechanism of the HC transformation is illustrated in Fig. 4. Comparing Fig. 4(a) with Fig. 4(b), it is seen that in the TSP smoothing approach proposed in [25], with $\alpha \to \infty$, the landscape becomes flat. While in the HC transformation, with $\lambda \to 1$, the landscape becomes unimodal. The key difference is that the optimum can be preserved after smoothing by the HC transformation.

To illustrate the effects of the HC transformation on TSP landscape smoothing, in the following we conduct systemic experiments to study the changes of the TSP landscape after transformation. In the experiments, we execute ILS on the transformed TSP and use the following four metrics to characterize the landscape:

- **Local Optimum Density**: It is the number of local optima encountered by an ILS process per 100 moves. Here one move means the local search algorithm moves from the current solution to a new solution.
- **Escaping Rate**: It is the success rate that a new local optimum is reached by ILS starting from a new solution obtained by perturbing the current local optimum.
- **Fitness Distance Correlation (FDC)**: To calculate FDC, 1000 local optima (denoted as $x_{LO}$) obtained by ILS are randomly selected. Their function value $f(x_{LO})$ and their distance to the nearest global optimum $d_{opt}$ are counted. Then FDC is defined as

$$\text{FDC}(f(x_{LO}), d_{opt}) = \frac{\text{cov}(f(x_{LO}), d_{opt})}{\sigma(f(x_{LO})) \sigma(d_{opt})}$$

where $\text{cov}(\cdot)$ denotes the covariance and $\sigma(\cdot)$ denotes the standard deviation.
- **Runtime**: It is the average runtime (CPU time) for ILS to find the global optimum.

The first two metrics, local optimum density and escaping rate, measure the ruggedness of TSP. A lower local optimum density means a smoother TSP landscape. It is intuitive that the escaping rate is negatively correlated to the average diameter of the attractive basin of local optima. A large size of the attractive basin implies a few local optima on the TSP landscape. Hence a lower escaping rate means a smoother TSP landscape. The third metric, FDC, measures the overall trend of the TSP landscape. A higher FDC means a more regular and easier TSP landscape. The last metric, runtime, directly reflects the hardness of TSP to ILS.

In the experiments, ILS is used to sample local optima on the TSP landscape. As shown in Alg. 1, ILS iteratively executes a local search procedure and a perturbation procedure till the stopping criterion is met. In our experiments, the 3-Opt local search and the double bridge perturbation (Fig. 5) are selected as the local search method and the perturbation method of ILS. In Alg. 1 LocalSearch($x_j$, $x_{best}$) takes $x_j$ and the current best $x_{best}$ as input, and returns a local optimum $x_{j+1}$ and a new $x_{best} = \arg \min \{ f(x_{j+1}), f(x_{best}) \}$.

**Algorithm 1**: Iterated Local Search

1. $x_0 \leftarrow$ random or heuristically generated solution;
2. $x_0 \leftarrow$ LocalSearch($x_0$);
3. $j \leftarrow 0$;
4. **while** stopping criterion is not met **do**
5. \[ x_j' \leftarrow \text{Perturbation}(x_j); \]
6. \[ \{x_{j+1}, x_{best}\} \leftarrow \text{LocalSearch}(x_j', x_{best}); \]
7. $j \leftarrow j + 1$;
8. return the historical best solution $x_{best}$.
In our experiments, we select 12 instances from the TSPLIB as the test instances. The test instances and the corresponding information are shown in Table I. The third column shows the edge cost type of each test instance, in which \( EUC_{2D} \) means the edge weights are Euclidean distances in a 2D space, \( GEO \) means the edge weights are geographical distances and \( EXPLICIT \) means the edge weights are listed explicitly in the TSP description file.

| Instance | Size | Edge type |
|----------|------|-----------|
| elli51   | 51   | EUC_{2D}  |
| berlin52 | 52   | EUC_{2D}  |
| s70      | 70   | EUC_{2D}  |
| pr76     | 76   | EUC_{2D}  |
| rat99    | 99   | EUC_{2D}  |
| rd100    | 100  | EUC_{2D}  |
| chi130   | 130  | EUC_{2D}  |
| kroA150  | 150  | EUC_{2D}  |
| gr96     | 96   | GEO       |
| brazil58 | 58   | EXPLICIT  |
| gr120    | 120  | EXPLICIT  |
| si175    | 175  | EXPLICIT  |

### C. The effects of the HC transformation with global optimum

To study, we first use LKH [32] to obtain the global optima of the generated transformed TSPs. On each transformed TSP, we run LKH 100 times and record the best solution as the global optimum. Since LKH is one of the state-of-the-art algorithms for the TSP and the size of the transformed TSPs is relatively small, we believe the TSP solutions found by LKH are indeed the global optima of the transformed TSPs.

On each transformed TSP, 1000 runs of ILS are executed from random solutions and stop only when the globally optimal cost (addressed by LKH) is reached. We record all the local optima ever encountered by ILS and the other relating information in the entire search history. Then we compute the four metrics for each transformed TSP, respectively.

**Table I**

**Test Instances**

| Instance | Size | Edge type |
|----------|------|-----------|
| elli51   | 51   | EUC_{2D}  |
| berlin52 | 52   | EUC_{2D}  |
| s70      | 70   | EUC_{2D}  |
| pr76     | 76   | EUC_{2D}  |
| rat99    | 99   | EUC_{2D}  |
| rd100    | 100  | EUC_{2D}  |
| chi130   | 130  | EUC_{2D}  |
| kroA150  | 150  | EUC_{2D}  |
| gr96     | 96   | GEO       |
| brazil58 | 58   | EXPLICIT  |
| gr120    | 120  | EXPLICIT  |
| si175    | 175  | EXPLICIT  |

In practice, the global optimum is not known. We therefore investigate the effects of the HC transformation when a local optimum is used to construct the convex-Hull TSP. The same test instances in Table I are used. For each instance, we run a round of 3-Opt local search to obtain a local optimum \( x_{LO} \). Then we use this local optimum to construct the convex-hull TSP and transform the original TSP based on the constructed
Fig. 6. The local optimum density of the transformed TSPs with different \( \lambda \) values. The HC transformation is based on the global optimum of the original TSP.

Fig. 7. The escaping rate of ILS on the transformed TSPs with different \( \lambda \) values. The HC transformation is based on the global optimum of the original TSP.

Fig. 8. The fitness distance correlation of the transformed TSPs with different \( \lambda \) values. The HC transformation is based on the global optimum of the original TSP.
optimum of the original TSP. In Fig. 11 we can see that, on five instances the curve of the escaping rate is not very smooth. On four instances (st170, pr76, rd100 and kroA150), the escaping rate first increases a little then starts to decrease. Hence, we can state that on 9 of the 12 test instances, the escaping rate is approximately negatively related to $\lambda$. Through comparing Fig. 10 and Fig. 11, with Fig. 6 and Fig. 7 we conclude that, on most of the test instances, the HC transformation based on local optimum can achieve approximately the same smoothing effect as the HC transformation based on global optimum.

Fig. 12 shows the FDC versus $\lambda$ when the HC transformation is based on local optimum. From Fig. 12 we can see that, on four instances (eil51, kroA150, gr96, and brazil58), there is a general trend that the FDC increases as $\lambda$ increases, in spite of the fact that on some instances the FDC curve is not very smooth. On four instances (st70, rd100, ch130 and gr120), the FDC first decreases a little then starts to increase. On rat99, although the lowest FDC appears when $\lambda = 0.05$, but in general the FDC is increase as $\lambda$ increases. Hence, we can state that on 9 of the 12 test instances, the FDC is approximately positively related to $\lambda$. Through comparing Fig. 12 to Fig. 8 we can conclude that, on most of the test instances, the HC transformation based on local optimum can achieve approximately the same FDC-increasing effect as the HC transformation based on global optimum.

Fig. 13 shows the runtime for ILS to find the global optimum when the HC transformation is based on local optimum. From Fig. 13 we can see that, on five instances (eil51, rat99, ch130, gr120 and si175), the runtime decreases as $\lambda$ increases, but in general the FDC is increase as $\lambda$ increases. Hence, we can state that on 10 of the 12 test instances, the FDC decreases. Hence we can state that on 9 of the 12 test instances, the escaping rate is approximately negatively related to $\lambda$. Through comparing Fig. 10 and Fig. 11 with Fig. 6 and Fig. 7 we conclude that, on most of the test instances, the HC transformation based on local optimum can achieve approximately the same smoothing effect as the HC transformation based on global optimum.

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Base on the above observations, we may conclude that, in most test instances, the HC transformation based on local optimum can archive approximately the same effects as the

![Graphs and Tables](image-url)

**Table II**

| Instance | $f(x_{opt})$ | $f(x_{LO})$ | excess (%) |
|----------|--------------|--------------|------------|
| eil51    | 426          | 428          | 0.4695     |
| berlin52 | 7542         | 7902         | 4.7733     |
| st70     | 675          | 684          | 1.3333     |
| pr76     | 108159       | 109190       | 0.9532     |
| rat99    | 1211         | 1217         | 0.4955     |
| rd100    | 7910         | 8054         | 1.8205     |
| ch130    | 6110         | 6363         | 4.1408     |
| kroA150  | 26524        | 27040        | 1.9454     |
| gr96     | 55209        | 55507        | 0.5398     |
| brazil58 | 25395        | 25643        | 0.9766     |
| gr120    | 6942         | 7066         | 1.7862     |
| si175    | 21407        | 21485        | 0.3644     |

where $x_{opt}$ is the global optimum. The other experimental settings are the same as those in Section III-B.

The local optimum density of different transformed TSPs are shown in Fig. 14. From Fig. 10 we can see that, on seven instances (eil51, st70, pr76, rat99, rd100, gr96 and si175) the local optimum density decreases as $\lambda$ increases. On four instances (berlin52, ch130, kroA150 and gr120), the local optimum density first increases a little then starts to decrease. Hence, we can state that on 11 of the 12 test instances, the local optimum density is approximately negatively related to $\lambda$. As for the escaping rate, the experimental results are shown in Fig. 11. From Fig. 11 we can see that, on five instances (eil51, rat99, ch130, gr96 and gr120), in general the escaping rate decreases as $\lambda$ increases in spite of the fact that on some instances the curve of the escaping rate is not very smooth. On four instances (st170, pr76, rd100 and kroA150), the escaping rate first increases a little then starts to decrease. Hence, we state that on 9 of the 12 test instances, the escaping rate is approximately negatively related to $\lambda$. Through comparing Fig. 10 and Fig. 11 with Fig. 6 and Fig. 7 we conclude that, on most of the test instances, the HC transformation based on local optimum can achieve approximately the same smoothing effect as the HC transformation based on global optimum.
Fig. 10. The local optimum density of the transformed TSPs with different λ values. The HC transformation is based on a local optimum of the original TSP.

Fig. 11. The escaping rate of ILS on the transformed TSPs with different λ values. The HC transformation is based on a local optimum of the original TSP.

Fig. 12. The fitness distance correlation of the transformed TSPs with different λ values. The HC transformation is based on a local optimum of the original TSP.
the HC transformation based on global optimum.

### E. Influence of different local optima

In the previous experiment, on some test instances, especially on berlin52, the HC transformation based on local optimum does not get an ideal smoothing and FDC-increasing effect as the HC transformation based on global optimum. Recall that in Table II, the local optimum used to construct the convex-hull TSP for berlin52 has a relatively low fitness compared to the local optima of other instances. Is this the reason that we do not get a good smoothing performance on berlin52? To answer, we conduct the following experiment.

For berlin52, besides the local optimum used in the previous experiment, we select another three local optima with higher or lower function values. The function values of the global optimum and the selected local optima of berlin52 are listed in Table III. In Table III, the local optima are sorted based on their function values and LO3 is the local optimum used in the previous experiment (marked by underline).

Based on the new selected local optima, the same HC transformation experiment is conducted on berlin52. The results on the four metrics against different \( \lambda \) values are shown in Figs. 14, 15, 16 and 17, respectively. From the results we can see that, when the selected local optimum has a relatively high fitness, e.g. LO1 and LO2, the general tendency of the curves of the four metrics achieved by the local optimum is similar to the tendency of the curves of the four metrics achieved by the global optimum, while when the selected local optimum has a relatively poor fitness, e.g. LO3 and LO4, the difference between the curves achieved by local optimum and global optimum is very obvious. This means that the HC transformation based on a high-quality local optimum can achieve approximately the same transformation performance as the HC transformation based on the global optimum. Hence, we conclude that a high quality local optimum can be used as a good estimate of the global optimum for the proposed HC transformation. In other words, the HC transformation with a high quality local optimum can still smooth the TSP landscape as with the global optimum.

### IV. Landscape smoothing iterated local search

We claim that the proposed HC transformation can be used to improve the global search ability of existing TSP heuristics. To justify, in this section, we first propose a general framework in which the HC transformation is combined with a local search for TSP. Then an instantiation algorithm of the framework with the 3-Opt local search and double bridge perturbation is proposed and compared against some known TSP smoothing algorithms. Note here that our objective is not to propose a state-of-the-art algorithm for the TSP, but to show that the proposed HC transformation can be used to improve existing TSP algorithms.

#### A. The general framework

The general framework is summarized in Alg. 2. Similar to solution perturbation based algorithms, the framework iteratively executes a local search procedure and a perturbation procedure. The key is that the local search is performed on the transformed TSP \( g = (1-\lambda)f_o + \lambda f_c \). A local optimum, denoted as \( x_0 \), is first obtained by any local search algorithm (line 3) from a random initial solution \( x_{init} \) (line 2). The algorithm then repeats the following procedure until stop. At each iteration, the current best solution with respect to \( f_o \), denoted as \( x_{f_o, best} \), is used to construct a convex-hull TSP.
Landscape Smoothing iterated Local Search (LSiLS).

As the perturbation operator. The resultant algorithm is named the local search operator and the double bridge perturbation as the perturbation operator. The resultant algorithm is named Landscape Smoothing iterated Local Search (LSiLS).

In the experimental study, we chose the 3-Opt local search as the local search operator and the double bridge perturbation as the perturbation operator. The resultant algorithm is named Landscape Smoothing iterated Local Search (LSiLS).

\[ x_{\text{init}} \leftarrow \text{random or heuristically generated solution}; \]
\[ x_0 \leftarrow \text{LocalSearch}(x_{\text{init}} | f_o); \]
\[ x_{f_o, \text{best}} \leftarrow x_0; \]
\[ j \leftarrow 0; \]
\[ \text{while stopping-criterion is not met do} \]
\[ \text{Construct convex-hull TSP } f_c \text{ based on } x_{f_o, \text{best}}; \]
\[ g \leftarrow (1 - \lambda)f_o + \lambda f_c; \]
\[ x'_j \leftarrow \text{Perturbation}(x_j); \]
\[ \{x_{j+1}, x_{f_o, \text{best}}\} \leftarrow \text{LocalSearch}(x'_j, x_{f_o, \text{best}} | g); \]
\[ j \leftarrow j + 1; \]
\[ \lambda \leftarrow \text{Update}(\lambda); \]
\[ \text{return } x_{f_o, \text{best}}. \]
TABLE IV

| Function Evaluation Period (×10^7) | 0 → 1 | 1 → 2 | 2 → 3 | 3 → 4 | 4 → 5 | 5 → 6 | 6 → 7 | 7 → 8 | 8 → 9 | 9 → 10 |
|------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Setting1                           | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  | 0.02  |
| Setting2                           | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  | 0.04  |
| Setting3                           | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  | 0.06  |
| Setting4                           | 0.00  | 0.00  | 0.01  | 0.01  | 0.01  | 0.02  | 0.02  | 0.03  | 0.04  | 0.04  |
| Setting5                           | 0.00  | 0.01  | 0.02  | 0.03  | 0.04  | 0.05  | 0.06  | 0.07  | 0.08  | 0.09  |

B. Performance Comparison

The developed LSiLS is compared against ILS, the smoothing algorithm proposed by Gu and Huang [25] (denoted as GH) and the sequential smoothing algorithm proposed by Coy et al. [27] (denoted as SSA).

Seven instances are chosen from the TSPLIB as the test instances, including rd400, p654, u724, pch1173, r1304, vm1748 and u1817. Each algorithm is executed 100 runs on each test instance from different random initial solutions. The stopping criterion of each run is $10^6$ function evaluations. For LSiLS, we test five different settings of $\lambda$. In the first three a constant value of $\lambda$ is applied during the search and in the last two settings, $\lambda = 0$ at the beginning and increases during the search. The detailed settings of $\lambda$ in the LSiLS implementations are shown in Table IV. For the implementations of GH and SSA, the parameter settings are the same as in [27]. That is, for GH the smoothing factor $\alpha = 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ in the first 6 local search rounds and for SSA $\alpha = 7 \rightarrow 5 \rightarrow 3 \rightarrow 1$ in the first 4 local search rounds. In both GH and SSA, if $\alpha$ has dropped to 1 and the function evaluation budget has not been run out, the algorithm will execute ILS on the original TSP landscape (i.e. $\alpha = 1$) until the function evaluation budget is run out. We use the 3-Opt local search and the double bridge perturbation in the implementations of ILS, LSiLS, GH and SSA. In addition, the 3-Opt local search implementation is speeded up by the techniques of near neighbor search and don’t look bits [33]. The other experimental settings are the same to the settings in the previous section.

Fig. 18 shows the mean best excess achieved by different smoothing algorithms against time. From Fig. 18 we can see that, LSiLS with Setting3 (i.e. $\lambda$ equals to constant 0.06) has the shortest CPU time on six of the seven test instance, and on large instances, the runtime of LSiLS with increasing $\lambda$ value (i.e. Setting4 and Setting5) is lower than that of GH and SSA. This means that LSiLS with increasing $\lambda$ value can find better solutions and requires less time than those of GH and SSA in most cases.

Table V shows the mean CPU time used by each algorithm in the experiments. From Table V we can see that, LSiLS with Setting3 (i.e. $\lambda$ equals to constant 0.06) has the shortest CPU time on six of the seven test instance, and on large instances, the runtime of LSiLS with increasing $\lambda$ value (i.e. Setting4 and Setting5) is lower than that of GH and SSA. This means that LSiLS with increasing $\lambda$ value can find better solutions and requires less time than those of GH and SSA in most cases.

V. CONCLUSIONS

In this paper, a new TSP landscape smoothing method called the homotopic convex (HC) transformation was proposed. The HC transformation is defined as the convex combination of the original TSP and a convex-hull TSP. For a given TSP, the convex-hull TSP is constructed based on a known optimum. We proved that the convex-hull TSP is unimodal for any $k$-Opt local search ($k \geq 2$). Empirical study showed that controlled by the convex coefficient, the proposed HC transformation can smooth the landscape in terms that in the transformed TSP, the number of local optima is reduced, and the fitness distance correlation of the TSP landscape is increased. Based on the observations, we proposed an landscape smoothing based iterative algorithmic framework, in which the HC transformation is combined with a local search algorithm. An instantiation of the algorithmic framework with 3-Opt local search, named Landscape Smoothing iterated Local Search (LSiLS), was studied on some widely-used TSP test instances. The experimental results showed that, LSiLS significantly outperforms ILS and two existing TSP smoothing algorithms on most test instances, which implies that the HC transformation can be used as a way to improve the global search ability of local search.

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Fig. 18. Comparison result between ILS, LSiLS, GH and SSA

| Instance | ILS   | GH    | SSA   | LSiLS Setting1 | LSiLS Setting2 | LSiLS Setting3 | LSiLS Setting4 | LSiLS Setting5 |
|----------|-------|-------|-------|----------------|----------------|----------------|----------------|----------------|
| rd400    | 48.04s| 81.37s| 68.88s| 42.75s         | 34.58s         | 28.55s         | 79.98s         | 65.87s         |
| p654     | **7.96s** | 13.57s| 10.85s| 8.11s          | 8.14s          | 8.36s          | 16.38s         | 16.32s         |
| u724     | 76.40s| 138.50s| 119.41s| 60.86s        | 41.52s         | 28.71s         | 115.37s         | 85.00s         |
| pcb1173  | 119.94s | 229.56s| 201.21s| 56.18s        | 26.49s         | 18.59s         | 117.19s         | 80.67s         |
| rl1304   | 73.89s| 165.36s| 127.96s| 61.47s        | 43.28s         | 31.10s         | 109.97s          | 85.77s         |
| vm1748   | 116.19s | 229.42s| 196.85s| 78.50s        | 49.04s         | **31.64s**     | 151.44s         | 103.60s        |
| u1817    | 123.23s | 230.96s| 212.64s| 53.15s        | 24.99s         | **17.79s**     | 122.85s         | 79.61s         |

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