A RIGOROUS REAL TIME FEYNMAN PATH INTEGRAL

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Abstract. Using improper Riemann integrals, we will formulate a rigorous version
of the real-time, time-sliced Feynman path integral for the $L^2$ transition probability
amplitude. We will do this for nonvector potential Hamiltonians with potential which
has at most a finite number of discontinuities and singularities. We will also provide
a Nonstandard Analysis version of our formulation.

1. Introduction and Notations. In this paper, we will formulate a rigorous
version of the real-time, time-sliced Feynman path integral for the $L^2$ transition probability
amplitude

\[ \langle \phi^*, \exp \left( \frac{-i t \bar{H}}{\hbar} \right) \psi \rangle_{L^2} = \int_{\mathbb{R}^n} \phi(\bar{x}) \left[ \exp \left( \frac{-i t \bar{H}}{\hbar} \right) \psi \right](\bar{x}) \, d\bar{x}, \]

where $\phi, \psi \in L^2$, $\bar{H} = \frac{-\hbar^2}{2m} \Delta + V(\bar{x})$ is essentially self-adjoint, $\bar{H}$ is the closure of $H$,
and $\phi, \psi, V$ each carries at most a finite number of singularities and discontinuities.

In flavor of physics literature, we will formulate the Feynman path integral with
improper Riemann integrals. In hope that with further research we can formulate
a rigorous polygonal path integral, we will also provide a Nonstandard Analysis
version of the Feynman path integral. Using Nonstandard Analysis is not essential
to our formulation, and the idea of using Nonstandard Analysis on the Feynman
path integral is not a new concept. For readers interested in Nonstandard Analysis,
and its applications to Feynman path integrals, see [1], [10], [13], [19], [22], and
references within. We will assume that the reader is familiar with Nonstandard Analysis.

In physics, the Feynman path integral is formulated on the propagator and it is
formally given by (see [11], [14], and [21])

\[ K_t(\bar{x}, \bar{x}_0) = \lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^{(k-1)n}} \exp \left[ \frac{\imath \epsilon}{\hbar} S_k(\bar{x}, ... \bar{x}_0) \right] \, d\bar{x}_1 \cdots d\bar{x}_{k-1}, \]

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where

\[(1.3)\]
\[w_{n,k} = \left(\frac{m}{2i\pi\hbar}\right)^k, \quad \epsilon = \frac{t}{k}.
\]
\[S_k (\vec{x} = \vec{x}_k, \ldots, \vec{x}_0) = \sum_{j=1}^{k} \left[ \frac{m}{2} \left( \frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon} \right)^2 - V (\vec{x}_j) \right],
\]
and all integrals are improper Riemann integrals.

In mathematics, there is a rigorous time-sliced Feynman path integral for the wave function (see [5] and [18])

\[(1.4)\]
\[\exp\left( -\frac{it\bar{H}}{\hbar} \right) \psi (\vec{x}) = \lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^{kn}} \exp \left[ \frac{i\epsilon}{\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_0) \right] \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_{k-1}.
\]

where the integrals in (1.4) are improper Lebesgue integrals and their convergence is in the \(L^2\) norm. Other popular rigorous versions of the Feynman path integral are the Wiener integral (see [7], [8], [12], [16], and [18]), generalization of Fresnel integrals (see [2]), and Henstock integrals (see [15]). For a more detailed exposition and further references, see [2] and [3].

Our main concern in this paper is to provide a rigorous version of (1.2) for the transition probability amplitude given in (1.1) by using (1.4). We will show that for any essentially self-adjoint Hamiltonian with potential that carries a finite number of singularities and discontinuities and for any \(\phi, \psi \in L^2\) which also has a finite number of singularities and discontinuities the following holds

\[(1.5)\]
\[\int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \exp \left( -\frac{it\bar{H}}{\hbar} \right) \psi (\vec{x}) \right] (\vec{x}) \, d\vec{x} = \lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^{(k+1)n}} \phi (\vec{x}_k) \exp \left[ \frac{i\epsilon}{\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_0) \right] \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_k.
\]

In the last line of (1.5), the integral is an improper Riemann integral over \(\mathbb{R}^{(k+1)n}\).

A trivial application of Nonstandard Analysis on the \(k\) limit in (1.5) yields

\[(1.6)\]
\[\int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \exp \left( -\frac{it\bar{H}}{\hbar} \right) \psi (\vec{x}) \right] (\vec{x}) \, d\vec{x} = \text{st} \left\{ w_{n,\omega} \int_{\mathbb{R}^{(\omega+1)n}} \phi (\vec{x}_\omega) \exp \left[ \frac{i\epsilon}{\hbar} S_\omega (\vec{x}_\omega, \ldots, \vec{x}_0) \right] \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_\omega \right\},
\]

where the integral in the last line of (1.6) is a \(\ast\)-transformed improper Riemann integral over \(\mathbb{R}^{(\omega+1)n}\), and \(\omega \in \mathbb{N} - \mathbb{N}\).

The main idea in the proof of (1.5) is the following. For simplicity, suppose \(f (x) \in L^2 (\mathbb{R}), g (x, y) \in L^2 (\mathbb{R} \times \mathbb{R})\) are such that they are bounded and continuous. Further, suppose that both

\[(1.7)\]
\[h (x) = \int_{-a}^{b} g (x, y) \, dy,
\]
\[p (x) = \text{l.i.m}_{a,b \to \infty} \int_{-a}^{b} g (x, y) \, dy.
\]
are in $L^2(\mathbb{R})$ as a function of $x$. In (1.7), we take the integral to be Lebesgue integrals and the limits are taken independent of each other. Notice that for $p(x)$, we can interpret the integral as an improper Lebesgue integral with convergence in the $L^2$ topology. Let us denote $\chi_{[c,d]}$ to be the characteristic function on $[c,d]$. Schwarz’s inequality then implies

$$\left| \int_{\mathbb{R}} f(x) p(x) - \int_{-c}^{d} \int_{-a}^{b} f(x, y) \, dx \, dy \right| \leq ||f||_2 ||p - h||_2 + ||f - \chi_{[-c,d]} p||_2 ||h||_2 \to 0. \tag{1.8}$$

Thus, we can write

$$\int_{\mathbb{R}} f(x) p(x) = \lim_{a, b, c, d \to \infty} \int_{-c}^{d} \int_{-a}^{b} f(x, y) \, dx \, dy, \tag{1.9}$$

where the limits are all taken independent of each other. Since $f$ and $g$ are bounded and continuous, the Lebesgue integral over $[-a, b] \times [-c, d]$ in (1.9) can be replaced by a Riemann integral. Since the limits are taken independent of each other, we can then interpret the right hand-side of (1.9) as an improper Riemann integral. If $f$ and $g$ carry singularities and discontinuities, care must be taken in the region of integration so that the replacement of Lebesgue integral with Riemann integrals can be done.

We now set some notations to deal with $n$-dimensional integrations, singularities and discontinuities. Let $k \in \mathbb{N}$ and $0 \leq l \leq k$. We will denote the interior of the $l$th box by

$$A^l = (-a^l_1, b^l_1) \times \cdots \times (-a^l_n, b^l_n), \tag{1.10}$$

for positive and large $a$’s and $b$’s. Let $K = \{ \vec{y}_1 \ldots \vec{y}_p \}$ be the set of discontinuous and singular points of $\phi, \psi$ and $V$. For each $\vec{y}_q = (y^q_1, \ldots, y^q_n) \in K$, denote the $l$th box centered at $\vec{y}_q$ by

$$B^l_q = (y^q_1 - \frac{1}{c^q_l}, y^q_1 + \frac{1}{c^q_l}) \times \cdots \times (y^q_n - \frac{1}{d^q_l}, y^q_n + \frac{1}{d^q_l}), \tag{1.11}$$

for positive and large $c$’s and $d$’s. Let

$$C^l = A^l - \left\{ \bigcup_{q=1}^{p} B^l_q \right\}. \tag{1.12}$$

For arbitrary large $a$’s, $b$’s, $c$’s, and $d$’s, $C^l$ is a box which encloses the set $K$ and at each point of $K$, a small box centered at that point is taken out. Associated with $C^l$ is a set of indices

$$\{j_l\} = \{a^l_1, \ldots, a^l_n, b^l_1, \ldots, b^l_n, c^{1,l}_1, \ldots, c^{1,l}_n, \ldots, c^{p,l}_1, \ldots, c^{p,l}_n, d^{1,l}_1, \ldots, d^{1,l}_n, \ldots, d^{p,l}_1, \ldots, d^{p,l}_n \}.$$
We will denote by \( \{j_l\} \to \infty \) to mean
\[
a_1^l, \ldots, a_n^l, b_1^l, \ldots, b_n^l, c_1^l, \ldots, c_n^l, \ldots, e_1^p, \ldots, e_n^p, \ldots,
\]
where all indices goes to infinity independent of each other. Notice that as \( \{j_l\} \to \infty \), we recover \( \mathbb{R}^n \) a.e. from \( C^l \). We will denote by \( \chi_{\{j_l\}} \) the characteristic function on \( C^l \). Lastly, let us write
\[
D\{j_k^0\} = C^0 \times \cdots \times C^k.
\]
Associated with \( D\{j_k^0\} \) is a set of indices
\[
\{J_k^0\} = \bigcup_{l=0}^k \{j_l\},
\]
and as before, we will use the notation \( \{J_k^0\} \to \infty \) to mean
\[
\{j_0\} \to \infty, \ldots, \{j_k\} \to \infty,
\]
where the indices are taken to infinity independent of each other.

From here on, we will assume that \( \phi, \psi \in L^2 \) and \( V \) are such that they have at most a finite number of singularities and discontinuities and the set of those points are denoted as \( K = \{\vec{y}_1 \ldots \vec{y}_p\} \). Finally, we will denote by \( \int_O \) to be Riemann or improper Riemann integration over the region \( O \) and \( \int_\mathcal{O} \) to be Lebesgue integration over the region \( O \).

2. Feynman Path Integrals. The standard derivation of (1.4) is via the Trotter product formula (see [5], [16], and [18]) which says that for any essentially self-adjoint \( H = H_0 + V \) with \( H_0 = -\frac{\hbar^2}{2m} \Delta \) and any \( \psi \in L^2 \),
\[
(2.1) \quad \exp \left( -\frac{itH}{\hbar} \right) \psi = \text{l.i.m.}_{k \to \infty} \left\{ \exp \left( -\frac{itV}{\hbar k} \right) \exp \left( -\frac{itH_0}{\hbar k} \right) \right\}^k \psi.
\]

Thus, we have the following

**Lemma 2.1.** Suppose \( H = H_0 + V \) is essentially self-adjoint. Let \( \psi, \phi \in L^2 \), then
\[
(2.2) \quad \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( -\frac{itH}{\hbar} \right) \psi \right](\vec{x}) \, d\vec{x} = \lim_{k \to \infty} \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \left\{ \exp \left( -\frac{itV}{\hbar k} \right) \exp \left( -\frac{itH_0}{\hbar k} \right) \right\}^k \psi \right](\vec{x}) \, d\vec{x}.
\]
Proof. The proof is just an application of (2.1) and Schwarz’s inequality. □

It is well known that (see [5] and [18]) for
\[ \nu \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \]
(2.3)
\[ \exp \left( -\frac{itH_0}{\hbar} \right) \nu (\vec{x}_1) = \left( \frac{m}{2\pi \hbar} \right)^\frac{n}{2} \int_{\mathbb{R}^n} \exp \left[ \frac{imc}{2\hbar} \left( \frac{\vec{x}_1 - \vec{x}_0}{\epsilon} \right)^2 \right] \nu (\vec{x}_0) \, d\vec{x}_0, \]
and that the operator \( \exp \left( -\frac{itH_0}{\hbar} \right) \) is unitary. Thus, we can write
(2.4)
\[ \begin{align*}
\exp \left( -\frac{itH_0}{\hbar} \right) \psi (\vec{x}_1) &= \left[ \exp \left( -\frac{itH_0}{\hbar} \right) \left( \lim_{j_0 \to \infty} \chi_{\{j_0\}} \psi \right) \right] (\vec{x}_1) = \\
&= \lim_{j_0 \to \infty} \exp \left( -\frac{itH_0}{\hbar} \right) \chi_{\{j_0\}} \psi (\vec{x}_1) = \\
&= \lim_{j_0 \to \infty} \left( \frac{m}{2\pi \hbar} \right)^\frac{n}{2} \int_{C_0} \exp \left[ \frac{imc}{2\hbar} \left( \frac{\vec{x}_1 - \vec{x}_0}{\epsilon} \right)^2 \right] \psi (\vec{x}_0) \, d\vec{x}_0. 
\end{align*} \]

Notice that by construction of the region \( C^0 \), \( \psi \) is bounded and continuous on \( C^0 \), hence the Lebesgue integral in the last line of (2.4) can be replaced by a Riemann integral.

For notation convenience, we will denote
(2.5)
\[ \rho (\vec{x}_k, \{J_0^{k-1}\}) = \\
\int_{D_{\{J_0^{k-1}\}}} \exp \left[ \frac{it}{(k+1)\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_o) \right] \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_{k-1}, \]
\[ T = \exp \left( -\frac{itV}{k\hbar} \right) \exp \left( -\frac{itH_0}{k\hbar} \right), \quad T^k = \left\{ \exp \left( -\frac{itV}{k\hbar} \right) \exp \left( -\frac{itH_0}{k\hbar} \right) \right\}^k. \]

Lemma 2.2. Suppose \( H = H_0 + V \) is essentially self-adjoint. Let \( \psi \in L^2 \), then for \( k \in \mathbb{N} \) the following holds
(2.6)
\[ \begin{align*}
\left\{ \exp \left( -\frac{itV}{k\hbar} \right) \exp \left( -\frac{itH_0}{k\hbar} \right) \right\}^k \psi &= \\
&= \lim_{j_0 \to \infty} \int_{D_{\{J_0^{k-1}\}}} \exp \left[ \frac{it}{\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_o) \right] \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_{k-1}. 
\end{align*} \]

Proof. We will proof (2.6) by induction. Suppose \( k = 2 \), then (2.4) implies
(2.7)
\[ T^2 \psi = T \left\{ \lim_{j_0 \to \infty} \rho (\vec{x}_1, \{J_0^0\}) \right\} = \\
\exp \left( -\frac{itV}{2\hbar} \right) \exp \left( -\frac{itH_0}{2\hbar} \right) \left\{ \lim_{j_1 \to \infty} \chi_{\{j_1\}} \left[ \lim_{j_0 \to \infty} \rho (\vec{x}_1, \{J_0^0\}) \right] \right\}. \]
Since multiplication by a characteristic function, \( \exp \left( \frac{-itV}{2\hbar} \right) \) and \( \exp \left( \frac{-itH_0}{2\hbar} \right) \) are all continuous operators from \( L^2 \) to \( L^2 \), we can take the \( L^2 \) limits in (2.7) outside of the operators and we can do this in any order we wish. Hence, (2.6) is true for \( k = 2 \). Assuming (2.6) to be true for \( k \), then

\[
T^{k+1}\psi = \exp \left( \frac{-itV}{(k+1)\hbar} \right) \times \exp \left( \frac{-itH_0}{(k+1)\hbar} \right) \left\{ \lim_{\{J_k\} \to \infty} \chi_{\{J_k\}} \rho(\vec{x}_k, \{J_0^{k-1}\}) \right\}.
\]

By the same reasoning as for the case of \( k = 2 \), we can take all the \( L^2 \) limits in (2.8) outside of the operators and we can do this in any order we wish. Hence, (2.6) is true for all \( k \in \mathbb{N} \).

**Proposition 2.3.** Suppose \( H = H_0 + V \) is essentially self-adjoint. Let \( \psi, \phi \in L^2 \), then for all \( k \in \mathbb{N} \) the following is true

\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left\{ \exp \left( \frac{-itV}{k\hbar} \right) \exp \left( \frac{-itH_0}{k\hbar} \right) \right\}^k \psi(\vec{x}) \, d\vec{x} = w_{n,k} \int_{\mathbb{R}^n} \phi(\vec{x}_k) \exp \left[ \frac{i\epsilon}{\hbar} S_k(\vec{x}_k, \ldots, \vec{x}_0) \right] \psi(\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_k.
\]

**Proof.** Lemma 2.2 implies that

\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left\{ \exp \left( \frac{-itV}{k\hbar} \right) \exp \left( \frac{-itH_0}{k\hbar} \right) \right\}^k \psi(\vec{x}) \, d\vec{x} = w_{n,k} \int_{\mathbb{R}^n} \left\{ \lim_{\{J_k\} \to \infty} \chi_{\{J_k\}} \right\} \left\{ \lim_{\{J_0^{k-1}\} \to \infty} w_{n,k} \int_{D_{\{J_0^{k-1}\}}} \exp \left[ \frac{i\epsilon}{\hbar} S_k(\vec{x}_k, \ldots, \vec{x}_0) \right] \psi(\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_{k-1} \right\} d\vec{x}_k.
\]

We now apply the idea in (1.8) and (1.9). Since all limits in (2.10) are taken independent of each other, we can use Schwarz’s inequality and take all the \( L^2 \) limits outside of the integral as pointwise limits. Thus,

\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left\{ \exp \left( \frac{-itV}{k\hbar} \right) \exp \left( \frac{-itH_0}{k\hbar} \right) \right\}^k \psi(\vec{x}) \, d\vec{x} = w_{n,k} \lim_{\{J_k\} \to \infty} \int_{D_{\{J_0^{k-1}\}}} \phi(\vec{x}_k) \exp \left[ \frac{i\epsilon}{\hbar} S_k(\vec{x}_k, \ldots, \vec{x}_0) \right] \psi(\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_k.
\]
By construction of \( D \{ J_0 \} \), the integrand \( \phi (\vec{x}_k) \exp \left( \frac{it}{\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_0) \right) \psi (\vec{x}_0) \) in (2.12) is a bounded and continuous function on \( D \{ J_0 \} \). Hence, we can replace the Lebesgue integrals in (2.12) by Riemann integrals. Since all limits in (2.12) are taken independent of each other, we can interpret (2.12) as an improper Riemann integral.

We are now ready to proof (1.5).

**Theorem 2.4.** Suppose \( H = H_0 + V \) is essentially self-adjoint. Let \( \psi, \phi \in L^2 \). Furthermore, suppose that \( \psi, \phi, \) and \( V \) has at most a finite number of singularities and discontinuities. With our previously defined notations, the following is true

\[
\int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \exp \left( \frac{-it\tilde{H}}{\hbar} \right) \psi \right] (\vec{x}) \, d\vec{x} = \lim_{k \to \infty} w_{n,k} \int_{r_{\mathbb{R}(k+1)^n}} \phi (\vec{x}_k) \exp \left( \frac{it}{\hbar} S_k (\vec{x}_k, \ldots, \vec{x}_0) \right) \psi (\vec{x}_0) \, d\vec{x}_0 \ldots d\vec{x}_k.
\]

*Proof.* Follows from lemma 2.1 and proposition 2.3. \( \square \)

### 3. Nonstandard Feynman Path Integrals

A trivial application of Nonstandard Analysis on the \( K \) limit in (2.2) will produce (1.6). It is our hope that with further research, a rigorous nonstandard polygonal path integral can be formulated.

**Theorem 3.1.** Suppose \( H = H_0 + V \) is essentially self-adjoint. Let \( \psi, \phi \in L^2 \). Furthermore, suppose that \( \psi, \phi, \) and \( V \) has at most a finite number of singularities and discontinuities. With our previously defined notations, the following is true

\[
\int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \exp \left( \frac{-it\tilde{H}}{\hbar} \right) \psi \right] (\vec{x}) \, d\vec{x} = \text{st} \left\{ \int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \left[ \exp \left( \frac{-itV}{\omega \hbar} \right) \exp \left( \frac{-itH_0}{\omega \hbar} \right) \right] \right] (\vec{x}) \, d\vec{x} \right\}.
\]

where the integral in the last line of (3.1) is a *)-transformed improper Riemann integral over \( \mathbb{R}^{(\omega+1)n} \), and \( \omega \in *\mathbb{N} - \mathbb{N} \).

*Proof.* The nonstandard equivalent of lemma 2.1 is that for any \( \omega \in *\mathbb{N} - \mathbb{N} \),

\[
\int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \exp \left( \frac{-it\tilde{H}}{\hbar} \right) \psi \right] (\vec{x}) \, d\vec{x} = \text{st} \left\{ \int_{\mathbb{R}^n} \phi (\vec{x}) \left[ \left[ \exp \left( \frac{-itV}{\omega \hbar} \right) \exp \left( \frac{-itH_0}{\omega \hbar} \right) \right] \right] (\vec{x}) \, d\vec{x} \right\}.
\]

After *)-transforming proposition 2.3, Equation (3.1) follows from (3.2). \( \square \)
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