The anisotropic $p$-capacity and the anisotropic Minkowski inequality

Chao Xia* & Jiabin Yin

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: chaoxia@xmu.edu.cn, jiabinyin@126.com

Received February 19, 2021; accepted June 18, 2021; published online October 15, 2021

Abstract In this paper, we prove a sharp anisotropic $L^p$ Minkowski inequality involving the total $L^p$ anisotropic mean curvature and the anisotropic $p$-capacity for any bounded domains with smooth boundary in $\mathbb{R}^n$. As consequences, we obtain an anisotropic Willmore inequality, a sharp anisotropic Minkowski inequality for outward $F$-minimising sets and a sharp volumetric anisotropic Minkowski inequality. For the proof, we utilize a nonlinear potential theoretic approach which has been recently developed by Agostiniani et al. (2019).

Keywords Minkowski inequality, anisotropic mean curvature, anisotropic $p$-Laplacian, nonlinear potential theory, $p$-capacity

MSC(2020) 53C21, 31C15, 52A39, 49Q10

Citation: Xia C, Yin J B. The anisotropic $p$-capacity and the anisotropic Minkowski inequality. Sci China Math, 2022, 65: 559–582, https://doi.org/10.1007/s11425-021-1884-1

1 Introduction

The Minkowski inequality in the theory of convex geometry says that if $\Omega \subset \mathbb{R}^n$ is a convex domain with smooth boundary, then

$$\frac{1}{\omega_{n-1}} \int_{\partial \Omega} H \, d\sigma \geq \left( \frac{|\partial \Omega|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

(1.1)

with the equality holding if and only if $\Omega$ is a round ball. Here, $H$ is the mean curvature of $\partial \Omega$ with respect to the outward unit normal and $\omega_{n-1} = |S^{n-1}|$, the area of the unit $(n-1)$-sphere.

In recent decades, many mathematicians are interested in the question whether (1.1) holds for non-convex domains. By using smooth solutions to the inverse mean curvature flow, Guan and Li [20] proved that (1.1) holds for star-shaped domains with mean convex boundary. Huisken and Ilmanen [22] developed a level set weak formulation for the inverse mean curvature flow, which was adopted by Huisken [21] to prove (1.1) for outward minimising sets (see also [19]). Quite recently, a powerful and elegant approach based on nonlinear potential theory has been developed by Agostiniani et al. [1] (see also the related

*Corresponding author
papers [2, 4, 18]), which enables them to establish a sharp $L^p$ version of the Minkowski inequality involving the $p$-capacity of $\Omega$ ($1 < p < n$) for general bounded domains with smooth boundary

\[
\frac{1}{\omega_{n-1}} \int_{\partial \Omega} \left| \frac{H}{n-1} \right|^p d\sigma \geq \left[ \left( \frac{p-1}{n-p} \right)^{p-1} \frac{1}{\omega_{n-1}} \text{Cap}_p(\Omega) \right]^{\frac{n-p}{n-p-1}}.
\]  

(1.2)

Here, $\text{Cap}_p(\Omega)$ is called the $p$-capacity of $\Omega$, given by

\[
\text{Cap}_p(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p dx : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } \Omega \right\}.
\]

We remark that various geometric inequalities involving the mean curvature integral, the capacity and the surface area have been considered by Xiao [39]. In particular, (1.2) has been proved in [39, Theorem 3.1] in the case of star-shaped, mean-convex domains. As $p \to 1$ in (1.2), Agostiniani et al. [1] have shown that the limit version of (1.2) recovers (1.1) for outward minimizing sets by Huisken [21]. Moreover, they are able to obtain a volumetric version of the Minkowski inequality

\[
\frac{1}{\omega_{n-1}} \int_{\partial \Omega} \left| \frac{H}{n-1} \right|^p d\sigma \geq \left( \frac{\text{Cap}_p(\Omega)}{|\Omega|} \right)^{\frac{n-2}{n}}.
\]  

(1.3)

for any bounded domains. Previously, (1.3) has been shown by Trudinger [32] and Chang and Wang [13] (see also [26]) for bounded domains with 2-convex boundary. The idea of Agostiniani et al. [1] is based on a conformal change of the Euclidean metric and a monotone integral over level sets of $p$-capacitary potentials under the conformal metric. Actually, it turns out that the results can be obtained by working directly on the Euclidean metric rather than the conformal metric. It is worth highlighting that their approach can be used to obtain sharp geometric inequalities for hypersurfaces in manifolds with nonnegative Ricci curvature [2, 4, 18] as well as for static manifolds [3, 10, 11] (see also [12, 38]).

In this paper, we are concerned with an anisotropic version of the Minkowski inequality. Let $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a Minkowski norm on $\mathbb{R}^n$. The unit Wulff ball $W_F$ with respect to $F$ (centered at the origin) is given by

\[ W_F = \{ F^o(x) < 1 \}, \]

where $F^o$ is the dual norm of $F$. For a bounded domain $\Omega$ with smooth boundary, the anisotropic area of $\partial \Omega$ is defined by

\[ |\partial \Omega|_F = \int_{\partial \Omega} F(\nu) d\sigma. \]

The well-known Wulff theorem (see, e.g., [24, Theorem 20.8]) says that Wulff balls are the only minimizers for the anisotropic isoperimetric problem. Equivalently, the Wulff inequality holds true:

\[ |\partial \Omega|_F \geq n |W_F|^\frac{1}{n} |\Omega|^{\frac{1}{n}-\frac{1}{n}}. \]  

(1.4)

The equality in (1.4) holds if and only if $\Omega$ is a Wulff ball.

The anisotropic mean curvature $H_F$ of $\partial \Omega$ arises from the first variational formula for $|\partial \Omega|_F$. The anisotropic Minkowski inequality says that if $\Omega \subset \mathbb{R}^n$ is a convex domain with smooth boundary, then

\[ \frac{1}{\kappa_{n-1}} \int_{\partial \Omega} H_F F(\nu) d\sigma \geq \left( \frac{|\partial \Omega|_F}{\kappa_{n-1}} \right)^{\frac{n-2}{n}} \frac{n-2}{n} \]  

(1.5)

with the equality holding if and only if $\Omega$ is a Wulff ball. Here, $\kappa_{n-1} = n |W_F|$. When $F$ is the Euclidean norm $F(\xi) = |\xi|$, then (1.4) and (1.5) reduce to the classical isoperimetric inequality and the Minkowski inequality (1.1), respectively. Inequality (1.5) is in fact equivalent to the following Minkowski inequality for mixed volumes:

\[ V_{(2)}(\Omega, W_F) \geq V_{(1)}(\Omega, W_F) \frac{n-2}{n-1} |W_F|^{\frac{1}{n-1}}, \]  

(1.6)
where \( V_i(\Omega, W_F) \) is the mixed volume for two convex bodies \( \Omega \) and \( W_F \), which is given by
\[
|\Omega + tW_F| = \sum_{k=0}^{n} \binom{n}{k} t^k V(k)(\Omega, W_F).
\]

Inequality (1.6) is a special one for a class of Alexandrov-Fenchel inequalities for mixed volumes in the theory of convex bodies (see, e.g., Schneider’s celebrated book [29, Subsection 7.3] or [37, Introduction]).

Regarding (1.5), Xia [37] showed that (1.5) holds for star-shaped domains with \( F \)-mean-convex (\( H_F \geq 0 \)) boundary, by using the smooth solution to the inverse anisotropic mean curvature flow in the same spirit of Guan and Li [20]. Della Pietra et al. [15] has considered the level set formulation of the inverse anisotropic mean curvature flow in the spirit of Huisken and Ilmanen [22], and proved the existence of the weak solutions by using an \( L^p \)-type approximation. However, such an approximation seems not sufficient to prove (1.5) for outward \( F \)-minimising sets.

In this paper, we will use the method developed in [1, 4, 18] to study (1.5) for non-convex domains. For \( 1 < p < n \), the anisotropic \( p \)-capacity of \( \Omega \) is defined by
\[
\text{Cap}_{F,p}(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} F^p(\nabla u)dx : u \in C_c^\infty(\mathbb{R}^n), u \geq 1 \text{ on } \Omega \right\}.
\]

We remark that there are early considerations for basic properties of the anisotropic \( p \)-capacity (see, e.g., Maz’ya’s book [25, Subsection 2.2]). Also, the Brunn-Minkowski theory has been extended to the anisotropic \( p \)-capacity [5, 6, 34].

In the following, we generalize (1.2) to the anisotropic case.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n (n \geq 3) \) be a bounded domain with smooth boundary. Then for every \( 1 < p < n \), the following inequality holds:
\[
\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \left| \frac{H_F}{n-1} \right|^p F(\nu)d\sigma \geq \left( \frac{n-1}{p-1} \right)^{p-1} \frac{1}{\kappa_{n-1}} \text{Cap}_{F,p}(\Omega)^\frac{n-p}{n-1}.
\]

Moreover, the equality holds in (1.7) if and only if \( \Omega \) is a Wulff ball.

As a direct consequence (with \( p = n - 1 \)), we get the anisotropic Willmore inequality.

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n (n \geq 3) \) be a bounded domain with smooth boundary. Then
\[
\int_{\partial \Omega} \left| \frac{H_F}{n-1} \right|^{n-1} F(\nu)d\sigma \geq \kappa_{n-1}.
\]

Moreover, the equality holds in (1.8) if and only if \( \Omega \) is a Wulff ball.

Theorem 1.1 is a special case of Theorem 5.6. Consider the anisotropic \( p \)-capacitary potential \( u \), which satisfies
\[
\begin{cases}
\Delta_{F,p} u = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\
u = 1 & \text{on } \partial \Omega, \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]
where \( \Delta_{F,p} \) is the anisotropic \( p \)-Laplace operator (see Section 2). It is not hard to see that
\[
\text{Cap}_{F,p}(\Omega) = \int_{\mathbb{R}^n \setminus \bar{\Omega}} F^p(\nabla u)dx = \int_{\partial \Omega} F^{p-1}(\nabla u)F(\nu)d\sigma.
\]

Let \( 1 < p < n \) and \( q \geq 1 + \frac{1}{p^*} \), where \( p^* := \frac{(n-1)(p-1)}{n-p} \). Let \( \Phi_{p,q} : [1, \infty) \to \mathbb{R} \) be defined by
\[
\Phi_{p,q}(\tau) := \tau^{(q-1)p^*} \int_{\{u = 1/\tau\}} F^{p(q-1)}(\nabla u)F(\nu)d\sigma.
\]

The asymptotic behavior of \( \Phi_{p,q} \) corresponds to \( \text{Cap}_{F,p}(\Omega) \) and the key ingredient to prove Theorem 5.6 is that \( \Phi_{p,q} \) is an (essentially) monotone non-increasing function (see Sections 4 and 5).

Next, by letting \( p \to 1 \) in (1.7), we get the following extended anisotropic Minkowski inequality.
Theorem 1.3. If $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded domain with smooth boundary, then

$$\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \frac{H_F}{n-1} F(\nu) d\sigma \geq \left( \frac{1}{\kappa_{n-1}} \inf_{\Omega \subset U, \partial U \text{ smooth}} |\partial U|_F \right)^{\frac{n-2}{n}}. \quad (1.9)$$

Moreover, the equality in (1.9) is achieved by Wulff balls.

As a consequence, we are able to deduce a sharp anisotropic Minkowski inequality for outward $F$-minimising sets (see Definition 6.3) and a sharp volumetric anisotropic Minkowski inequality.

Corollary 1.4. If $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded outward $F$-minimising set with smooth boundary, then

$$\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \frac{H_F}{n-1} F(\nu) d\sigma \geq \left( \frac{|\partial \Omega|_F}{\kappa_{n-1}} \right)^{\frac{n-2}{n}}. \quad (1.10)$$

Moreover, the equality in (1.10) is achieved by Wulff balls. Conversely, if the equality holds in (1.10) for some bounded strictly outward $F$-minimising open set with smooth and strictly $F$-mean-convex boundary, then $\Omega$ is a Wulff ball.

Corollary 1.5. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded set with smooth boundary. Then

$$\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \frac{H_F}{n-1} F(\nu) d\sigma \geq \left( \frac{n|\Omega|}{\kappa_{n-1}} \right)^{\frac{n-2}{n}}. \quad (1.11)$$

Moreover, the equality holds in (1.11) if and only if $\Omega$ is a Wulff ball.

To end the introduction, we make some remarks on the strictly outward $F$-minimising hull. Let $\Omega$ be a bounded domain with smooth boundary. It is well known that the infimum

$$\inf_{\Omega \subset U, \partial U \text{ smooth}} |\partial U|_F$$

may not be achieved in general. In the setting of the set of finite perimeter, the infimum

$$\inf_{E \text{ a set of finite perimeter}} P_F(E) \quad (1.12)$$

can be achieved by the direct method in calculus of variations. Here,

$$P_F(E) = \int_{\partial^* E} F(\nu_E) d\mathcal{H}^{n-1}$$

is the anisotropic perimeter of $E$, where $\partial^* E$ is the reduced boundary of $E$ and $\nu_E$ is the measure-theoretical outward unit normal of $E$ (see, e.g., [24, Chapter 20]). There is a so-called strictly outward $F$-minimising hull corresponding to $\Omega$ which attains the infimum (see [1, 17, 22]). Similar to [17, Definition 2.12], the strictly outward $F$-minimising hull of $\Omega$ denoted by $\Omega^*$ can be reformulated as the measure-theoretical $\text{Int}(E)$ of any set $E$ that solves the minimisation problem

$$|E| = \inf_{G \in \text{SOFMBE}(\Omega)} |G|,$$

where

$$\text{SOFMBE}(\Omega) = \{ G \mid \Omega \subset G \text{ and } G \text{ is bounded and strictly outward } F\text{-minimising} \}$$

is the strictly outward $F$-minimising bounded envelope (SOFMBE) of $\Omega$. It turns out that $\Omega^*$ is the unique maximal volume solution to the least anisotropic perimeter problem with obstacle $\Omega$, up to zero-measure modification (see [17, Theorem 2.16]). (In our case, the assumption on the existence of an exhaustion of $\mathbb{R}^n$ by a sequence of strictly outward $F$-minimising sets is automatically satisfied. In fact, such an exhaustion is given by the Wulff balls.) In particular, $\Omega^*$ attains the infimum in (1.12), namely,

$$P_F(\Omega^*) = \inf_{E \text{ a set of finite perimeter}} P_F(E).$$
It is direct to see that
\[
\inf_{\Omega \subset U} |\partial U|_F \geq \inf_{\Omega \subset E} P_F(E) = P_F(\Omega^*).
\]

Thus it follows from (1.9) that
\[
\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \left| \frac{H_F}{n-1} \right| F(\nu) d\sigma \geq \left( \frac{P_F(\Omega^*)}{\kappa_{n-1}} \right)^{\frac{n-2}{n-1}}. \tag{1.13}
\]

**Remark 1.6.** In [1], the isotropic version of (1.13) has been shown. In their argument, a regularity result in [30] and an exterior smooth approximation in [28] for the strictly outward minimising hull \( \Omega^* \) have to play a role. In this paper, by showing a stronger inequality (1.9), we can avoid using the regularity and an exterior smooth approximation for the anisotropic case, which seems unavailable in the reference.

The rest of this paper is organized as follows. In Section 2, we review the basic concepts in the anisotropic setting, including the anisotropic mean curvature for level sets and the anisotropic \( p \)-Laplace operator. In particular, we prove a Kato-type identity for anisotropic \( p \)-harmonic functions. In Section 3, we make a systematic study of the anisotropic \( p \)-capacitary potential, focusing on its asymptotic behavior. In Sections 4 and 5, we establish the essential and effective monotonicity properties for the crucial function \( \Phi_{p,q}(\tau) \), and in turn, prove the \( L^p \) anisotropic Minkowski inequality, Theorem 1.1. In Section 6, we study the outward \( F \)-minimising sets, the limit of Cap\( F;p \) as \( p \to 1^+ \) and finally prove Theorem 1.3 and Corollaries 1.4 and 1.5.

## 2 Preliminaries

### 2.1 The Minkowski norm, the Wulff shape and the anisotropic area

Let \( F \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) be a Minkowski norm on \( \mathbb{R}^n \), in the sense that

(i) \( F \) is a norm in \( \mathbb{R}^n \), i.e., \( F \) is a convex, even, 1-homogeneous function satisfying \( F(x) > 0 \) when \( x \neq 0 \) and \( F(0) = 0 \);

(ii) \( F \) satisfies a uniformly elliptic condition: \( \nabla^2(\frac{1}{2} F^2) \) is positive definite in \( \mathbb{R}^n \setminus \{0\} \).

The dual norm \( F^\circ : \mathbb{R}^n \to [0, +\infty] \) of \( F \) is defined as
\[
F^\circ(x) = \sup_{\xi \neq 0} \frac{\langle \xi, x \rangle}{F(\xi)}.
\]

\( F^\circ \) is also a Minkowski norm on \( \mathbb{R}^n \).

Furthermore,
\[
F(\xi) = \sup_{x \neq 0} \frac{\langle \xi, x \rangle}{F^\circ(x)}.
\]

We remark that throughout this paper, we use conventionally \( \xi \) as the variable for \( F \) and \( x \) as the variable for \( F^\circ \).

Define
\[
\mathcal{W}_F = \{ x \in \mathbb{R}^n : F^\circ(x) < 1 \}.
\]

For the simplicity of notations, we will define \( \mathcal{W}_F = \mathcal{W} \). We call \( \mathcal{W} \) the unit Wulff ball centered at the origin and \( \partial \mathcal{W} \) the Wulff shape.

More generally, we define
\[
\mathcal{W}_r(x_0) = r \mathcal{W} + x_0,
\]
and call it the Wulff ball of radius \( r \) centered at \( x_0 \). We simply define \( \mathcal{W}_r = \mathcal{W}_r(0) \).

The following properties of \( F \) and \( F^\circ \) hold true and will be frequently used in this paper (see, e.g., [15, 36]).
Proposition 2.1. Let $F : \mathbb{R}^n \to [0, \infty)$ be a Minkowski norm. Then for any $x, \xi \in \mathbb{R}^n \setminus \{0\}$, the following hold:

1. $\langle F(\xi), \xi \rangle = F(\xi)$ and $\langle F'_{x}(x), x \rangle = F'(x)$.
2. $\sum_{j} F_{i,j}(\xi)\xi_j = 0$ for any $i = 1, \ldots, n$.
3. $F(F'_{x}(x)) = F'(F(\xi)) = 1$.
4. $F'(x)F_{x}(F'_{x}(x)) = x$ and $F(\xi)F'_{x}(F(\xi)) = \xi$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial \Omega$ and $\nu$ be its unit outward normal of $\partial \Omega$. The anisotropic area $|\partial \Omega|_F$ of $\Omega$ is defined by

$$|\partial \Omega|_F = \int_{\partial \Omega} F(\nu) d\sigma. \tag{2.1}$$

Note that when $\Omega = W$, the unit Wulff ball, one can check by the divergence theorem that

$$|\partial W|_F = \int_{\partial W} \frac{1}{|\nabla F|} d\sigma = \int_{W} \text{div}(\nabla u) dx = n|W|. \tag{2.2}$$

For notation simplicity, we define $\kappa_{n-1} = |\partial W|_F = n|W|$.

2.2 Anisotropic $p$-Laplacian

Let $u$ be twice continuously differentiable at $x \in \mathbb{R}^n$. We denote by $F_i, F_{ij}, \ldots$ the partial derivatives of $F$ and by $u_i, u_{ij}, \ldots$ the partial derivatives of $u$:

$$F_i = \frac{\partial F}{\partial \xi_i}, \quad F_{ij} = \frac{\partial^2 F}{\partial \xi_i \partial \xi_j}, \quad u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

For $x$ such that $\nabla u(x) \neq 0$, define

$$a_{ij}(\nabla u)(x) := \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left( \frac{1}{2} F^2 \right)(\nabla u(x)) = (F_iF_j + FF_{ij})(\nabla u(x)),$$

$$a_{ij,p}(\nabla u)(x) := \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left( \frac{1}{p} F^p \right)(\nabla u(x)) = F^{p-2}(a_{ij} + (p-2)F_{ij})(\nabla u(x)). \tag{2.3}$$

The anisotropic Laplacian and $p$-Laplacian of $u$ at regular points ($|\nabla u| \neq 0$) are given by

$$\Delta_F u := a_{ij}(\nabla u)u_{ij},$$

$$\Delta_{F,p} u := a_{ij,p}(\nabla u)u_{ij} = F^{p-2}(\Delta_F u + (p-2)F_{ij}u_{ij}), \tag{2.4}$$

respectively (see, e.g., [14, 33]). For notation simplicity, we also introduce

$$V(\xi) = \frac{1}{p} F^p(\xi)$$

and

$$W_{ij} = \partial_{x_j}(\partial_{\xi_i} V(\nabla u)) = a_{ik,p}(\nabla u)u_{kj}. \tag{2.5}$$

One easily sees that

$$\text{tr}(W) = \Delta_{F,p} u.$$

Let $1 \leq k \leq n$ be an integer. For an $n$-vector $\lambda = (\lambda_1, \ldots, \lambda_n)$, the $k$-th elementary symmetric function on $\lambda$ is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$
Given a real matrix \( B = (b_{ij}) \in \mathbb{R}^{n \times n} \), we denote by \( S_k(B) \) the sum of all the principal minors of \( B \) of order \( k \). In particular,

\[
S_1(B) = \text{tr}(B),
\]

the trace of \( B \), and

\[
S_n(B) = \det(B),
\]

the determinant of \( B \). When the eigenvalues \( \lambda_B \) of \( B \) are all real, it is clear that

\[
S_k(B) = \sigma_k(\lambda_B).
\]

For more properties on \( S_k(B) \), we refer to [16, 27].

For our purpose, we consider \( k = 2 \). By setting

\[
S^{ij}_2(B) = \text{tr}B^{ij} - b_{ji},
\]

we have

\[
S_2(B) = \frac{1}{2} S^{ij}_2(B) b_{ij} = \frac{1}{2} ((\text{tr}B)^2 - \text{tr}B^2).
\] (2.6)

Of particular interest for our purpose is \( S_2(W) \), where \( W \) is given by (2.5).

Since \( \text{tr}W = \Delta_{F,p}u \), it holds that

\[
S^{ij}_2(W) = \Delta_{F,p}u \delta_{ij} - a_{jk,p}(\nabla u)u_{ki}.
\] (2.7)

Moreover, in this setting, \( S^{ij}_2(W) \) is divergence free (see, e.g., [8, 14, 16]), i.e.,

\[
\partial_{x_j} S^{ij}_2(W) = 0.
\] (2.8)

It follows from (2.6) and (2.8) that if \( \Delta_{F,p}u = 0 \), then

\[
\partial_{x_j}(S^{ij}_2(W)V_{ki}(\nabla u)) = 2S_2(W) = -a_{ik,p}a_{jl,p}u_{kj}u_{il}.
\] (2.9)

2.3 The anisotropic mean curvature

We recall the concept of anisotropic mean curvature for a hypersurface in \( \mathbb{R}^n \) (see, e.g., [33, 37]).

Let \( M \) be a smooth embedded hypersurface in \( \mathbb{R}^n \) and \( \nu \) be one unit normal of \( M \). The corresponding anisotropic normal of \( M \) is defined by

\[
\nu_F = \nabla F(\nu).
\]

The anisotropic principal curvatures

\[
\kappa_F = (\kappa_1^F, \ldots, \kappa_{n-1}^F) \in \mathbb{R}^{n-1}
\]

are defined as the eigenvalues of the map

\[
d\nu_F: T_xM \to T_{\nu_F(x)}\partial W.
\]

The anisotropic mean curvature (with respect to \( \nu \)) is defined to be

\[
H_F = \sigma_1(\kappa_F) = \sum_i \kappa_i^F.
\]

The second anisotropic mean curvature (with respect to \( \nu \)) is defined to be

\[
\sigma_2(\kappa_F) = \sum_{i<j} \kappa_i^F \kappa_j^F.
\]

A well-known variational characterization for \( H_F \) is that it arises from the first variation of the anisotropic area functional \( |\partial \Omega|_F \) (see, e.g., [37, Proposition 2.1]), namely,

\[
\frac{d}{dt} \int_{\partial \Omega_t} F(\nu) d\sigma = \int_{\partial \Omega_t} H_F(Y, \nu) d\sigma,
\]
where $Y$ is the variation vector field for $\partial \Omega_t$.

In this paper, we are interested in the case where $M$ is given by a regular level set of a smooth function $u$, i.e., $M = \{u = t\}$ for some regular value $t$. For our purpose, we choose the unit normal
\[
\nu = -\frac{\nabla u}{|\nabla u|}
\]
and
\[
\nu_F = -F_\xi(\nabla u), \quad H_F = -\text{div}(F_\xi(\nabla u)).
\]
In this case, we have
\[
H_F = -\text{div}(F_\xi(\nabla u)) = -F_{ij}u_{ij}. \tag{2.10}
\]
Here, div is the Euclidean divergence (see, e.g., [16]).

From [16, Theorem 2.5], we also have
\[
\sigma_2(\kappa_F) = S_2(F_{ik}u_{kj}).
\]
It follows that
\[
\frac{n-2}{n-1}S_1^2(F_{ik}u_{kj}) - 2S_2(F_{ik}u_{kj}) = \frac{n-2}{n-1}\sigma_1^2(\kappa_F) - 2\sigma_2(\kappa_F). \tag{2.11}
\]
We remark that the right-hand side of (2.11) is exactly the square of the traceless part of the anisotropic second fundamental form, which is nonnegative.

Next, we give two propositions for the connection between an anisotropic $p$-harmonic function $u$ and the anisotropic curvatures of regular level sets of $u$. The first one is a formula for the anisotropic mean curvature of a regular level set of $u$.

**Proposition 2.2.** Let $u$ satisfy $\Delta_{F,p}u = 0$. Then the anisotropic mean curvature of a regular level set of $u$ is given by
\[
H_F = (p-1)F^{-1}F_iF_ju_{ij}. \tag{2.12}
\]
**Proof.** Note that $\Delta_{F,p}u = 0$ implies
\[
FF_{ij}u_{ij} + (p-1)F_iF_ju_{ij} = 0.
\]
The assertion follows from (2.10). $\Box$

The second is a Kato-type identity for anisotropic $p$-harmonic functions, which generalizes [18, Proposition 4.4].

**Proposition 2.3.** Let $u$ satisfy $\Delta_{F,p}u = 0$. Let $\kappa_F$ denote the anisotropic principal curvature of a regular level set of $u$. Then at any point where $F(\nabla u) \neq 0$, the following identity holds true:
\[
a_{ij}a_{kl}u_{ik}u_{jl} = F^2(\nabla u)\left[\frac{n-2}{n-1}\sigma_1^2(\kappa_F) - 2\sigma_2(\kappa_F)\right]
+ \left(1 + \frac{(p-1)^2}{n-1}\right)|\nabla(F(\nabla u))|_{a_F}^2
+ \left(1 - \frac{(p-1)^2}{n-1}\right)|\nabla^T(F(\nabla u))|_{a_F}^2, \tag{2.13}
\]
where
\[
|\nabla(F(\nabla u))|_{a_F}^2 := a_{ij}(F_ku_{ki})(F_lu_{lj}) \geq 0 \tag{2.14}
\]
and
\[
|\nabla^T(F(\nabla u))|_{a_F}^2 := FF_{ij}(F_ku_{ki})(F_lu_{lj}) \geq 0. \tag{2.15}
\]

**Remark 2.4.** $|\cdot|_{a_F}$ can be viewed as the norm with respect to the positive definite matrix $(a_{ij})$. Also by the definition, we have
\[
|\nabla(F(\nabla u))|_{a_F}^2 = |\nabla^T(F(\nabla u))|_{a_F}^2 + (F_iF_ju_{ij})^2. \tag{2.16}
\]
Proof of Proposition 2.3. The proof is inspired by [36]. Noting that \(a_{ij} = FF_{ij} + F_i F_j\), we can separate

\[
\Delta_F u = a_{ij} u_{ij} = A + B,
\]

where

\[
A = F_i F_j u_{ij} \quad \text{and} \quad B = F F_{ij} u_{ij}.
\]

Since

\[
a_{ij} a_{kl} u_{ik} u_{jl} = A^2 + 2 F F_{kl} F_j u_{ik} u_{jl} + F^2 F_{ij} F_{kl} u_{ik} u_{jl}
\]

and

\[
\frac{(A + B)^2}{n} + \frac{n}{n - 1} \left( B - \frac{n - 1}{n} A \right)^2 = A^2 + \frac{1}{n - 1} B^2,
\]

we have

\[
a_{ij} a_{kl} u_{ik} u_{jl} = \frac{(A + B)^2}{n} + \frac{n}{n - 1} \left( B - \frac{n - 1}{n} A \right)^2 - \frac{1}{n - 1} B^2 + 2 F F_{kl} F_j u_{ik} u_{jl} + F^2 F_{ij} F_{kl} u_{ik} u_{jl}
\]

\[
= \left( \frac{(p - 1)^2}{n - 1} - 1 \right) F(F_{ij} u_{ij})^2 + 2 F F_{ij} F_{kl} u_{ik} u_{jl}
\]

\[
+ 2 F F_{ij} F_{kl} u_{ik} u_{jl} - \frac{1}{n - 1} (F F_{ij} u_{ij})^2
\]

\[
= \left( \frac{(p - 1)^2}{n - 1} - 1 \right) F(F_{ij} u_{ij})^2 + 2 F F_{ij} F_{kl} u_{ik} u_{jl}
\]

\[
+ 2 F F_{ij} F_{kl} u_{ik} u_{jl} - \frac{1}{n - 1} (F F_{ij} u_{ij})^2
\]

\[
+ \left( 1 + \frac{(p - 1)^2}{n - 1} \right) |\nabla(F(\nabla u))|^2_{a_F}
\]

\[
+ \left( 1 - \frac{(p - 1)^2}{n - 1} \right) |\nabla^T(F(\nabla u))|^2_{a_F} + \left( 1 + \frac{(p - 1)^2}{n - 1} \right) |\nabla(F(\nabla u))|^2_{a_F}
\]

\[
= \left( 1 - \frac{(p - 1)^2}{n - 1} \right) |\nabla^T(F(\nabla u))|^2_{a_F} + \left( 1 + \frac{(p - 1)^2}{n - 1} \right) |\nabla(F(\nabla u))|^2_{a_F}
\]

\[
+ 2 F F_{ij} F_{kl} u_{ik} u_{jl} - \frac{1}{n - 1} (F F_{ij} u_{ij})^2,
\]

where in the last equality we have used (2.16). Finally, using (2.11), we obtain

\[
F_{ij} F_{kl} u_{ik} u_{jl} - \frac{1}{n - 1} (F_{ij} u_{ij})^2 = \frac{n - 2}{n - 1} S^2(F_{ik} u_{kj}) - 2 S_2(F_{ik} u_{kj})
\]

\[
= \frac{n - 2}{n - 1} \sigma^2_F(\kappa_F) - 2 \sigma_2(\kappa_F).
\]

This completes the proof of Proposition 2.3.
We also have some facts as follows.

**Proposition 2.5.** Let \( u \) satisfy \( \Delta_{F,p} u = 0 \). Then at any point where \( F(\nabla u) \neq 0 \), the following identities hold true:

\[
F^{2-p} a_{ijkl} F_{i} u_{ik} F_{j} u_{j} = |\nabla T(F(\nabla u))|^{2}_{a_F} + \frac{1}{p-1} H^{2}_{F} F^{2}(\nabla u),
\]

\[
a_{ijkl} u_{ik} u_{lj} F^{4-2p} = \frac{n}{n-1} H^{2}_{F} F^{2} + 2(p-1)|\nabla T(F(\nabla u))|^{2}_{a_F}
+ F^{2}(\nabla u) \left[ \frac{n-2}{n-1} \sigma^{2}_{1}(\kappa_F) - 2\sigma_{2}(\kappa_F) \right].
\]

**Proof.** Identity (2.21) follows directly from (2.3). From Proposition 2.3, we have

\[
a_{ijkl} u_{ik} u_{lj} F^{2} = (a_{ijkl} + (p-2) F_{i} F_{k})(a_{ilm} + (p-2) F_{i} F_{l}) u_{ik} u_{lj}
= a_{ijkl} u_{ik} u_{lj} + 2(p-2)|\nabla (F(\nabla u))|^{2}_{a_F} + \frac{(p-2)^{2}}{(p-1)^{2}} H^{2}_{F} F^{2}
= \frac{n}{n-1} H^{2}_{F} F^{2} + 2(p-1)|\nabla T(F(\nabla u))|^{2}_{a_F}
+ F^{2}(\nabla u) \left[ \frac{n-2}{n-1} \sigma^{2}_{1}(\kappa_F) - 2\sigma_{2}(\kappa_F) \right].
\]

By using (2.3) and (2.23), directing computation gives that

\[
a_{ijkl} u_{ik} u_{lj} F^{4-2p} = (a_{ijkl} + (p-2) F_{i} F_{k})(a_{ilm} + (p-2) F_{i} F_{l}) u_{ik} u_{lj}
= \frac{n}{n-1} H^{2}_{F} F^{2} + 2(p-1)|\nabla T(F(\nabla u))|^{2}_{a_F}
+ F^{2}(\nabla u) \left[ \frac{n-2}{n-1} \sigma^{2}_{1}(\kappa_F) - 2\sigma_{2}(\kappa_F) \right].
\]

This completes the assertion. \( \square \)

## 3 The anisotropic \( p \)-capacitary potential

In this section, we will introduce the anisotropic \( p \)-capacity and its related properties.

**Definition 3.1** (The anisotropic \( p \)-capacity). Let \( \Omega \subset \mathbb{R}^{n} \) be a bounded open set with smooth boundary. For \( p \in [1, n) \), the anisotropic \( p \)-capacity of \( \Omega \) is defined as

\[
\text{Cap}_{F,p}(\Omega) = \inf \left\{ \int_{\mathbb{R}^{n}} F^{p}(\nabla u) dx \bigg| u \in C^{\infty}_{c}(\mathbb{R}^{n}), u \geq 1 \text{ on } \Omega \right\}.
\]

(3.1)

It is easy to check that (see, e.g., [25])

\[
\text{Cap}_{F,p}(\Omega) = \inf \left\{ \int_{\mathbb{R}^{n}} F^{p}(\nabla u) dx \bigg| u \geq \chi_{\Omega}, u \in C^{\infty}_{c}(\mathbb{R}^{n}) \right\}.
\]

(3.2)

Next, we consider the case \( p \in (1, n) \). The variational structure of the above definition leads naturally to the formulation of the problem

\[
\begin{cases}
\Delta_{F,p} u = 0 & \text{in } \mathbb{R}^{n} \setminus \bar{\Omega}, \\
u = 1 & \text{on } \partial \Omega, \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]

(3.3)

and we say that a function \( u \in W^{1,p}_{\text{loc}}(U) \) is a weak solution of \( \Delta_{F,p} u = 0 \) in an open set \( U \) if

\[
\int_{U} (F^{p-1}(\nabla u) F_{\xi} (\nabla u), \nabla \psi) dx = 0
\]
for any $\psi \in C_0^\infty(U)$. A weak solution satisfying (3.3) is called the anisotropic $p$-capacitary potential associated with $\Omega$.

We collect the existence and regularity results for the anisotropic $p$-capacitary potentials in the following theorem (see, e.g., [8, 9]).

**Theorem 3.2** (The existence and regularity of anisotropic $p$-capacitary potentials). Let $\Omega$ be a bounded open set with smooth boundary, and let $1 < p < n$. Then

1. there exists a unique weak solution $u \in C^{1,\alpha}(\mathbb{R}^n \setminus \bar{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega)$ to (3.3);
2. $u$ is smooth away from $\text{Crit}(u) := \{\nabla u = 0\}$;
3. the solution $u$ fulfills

$$\text{Cap}_{F,p}(\Omega) = \int_{\mathbb{R}^n \setminus \bar{\Omega}} F^p(\nabla u)dx.$$  \hfill (3.4)

**Remark 3.3.** Note that since $\partial \Omega$ is assumed to be smooth, by the Hopf lemma for anisotropic $p$-harmonic functions, we have $\nabla u \neq 0$ in a neighborhood of $\partial \Omega$. In particular, $u$ is smooth in $\partial \Omega$. Coupled with this fact, the asymptotic expansions below imply that

$$\text{Crit}(u) = \{x \in \mathbb{R}^n \setminus \bar{\Omega} \mid Du(x) = 0\}$$

is a compact subset of $\mathbb{R}^n \setminus \bar{\Omega}$ (generically depending on $p$), and in turn that $u$ is analytic outside this set.

For $1 < p < n$, let

$$\Gamma_{F,p}(x) = \frac{p - 1}{n - p} \left(\frac{1}{\kappa_{n-1}}\right)^{\frac{1}{p-1}} F^0(x) \frac{p}{p-n}.$$  \hfill (3.35)

One can check that

$$\Delta_{F,p} \Gamma_{F,p}(x) = \delta_0 \quad \text{in } \mathbb{R}^n,$$

where $\delta_0$ is the Dirac delta function about the origin. We call $\Gamma_{F,p}$ the fundamental solution to $\Delta_{F,p}u = 0$ in $\mathbb{R}^n$ (see [35]).

**Proposition 3.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and $1 < p < n$. Let $u$ be a weak solution of $\Delta_{F,p}u = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u$ satisfies

1. $\lim_{|x| \to +\infty} \frac{u(x)}{\Gamma_{F,p}(x)} = \text{Cap}_{F,p}(\Omega)^{\frac{1}{p-1}},$
2. $\nabla u(x) = \text{Cap}_{F,p}(\Omega)^{\frac{1}{p-1}} \nabla \Gamma_{F,p}(x) + o(|x|^{-\frac{n+1}{p-1}})$ as $|x| \to +\infty$.

**Proof.** If $u$ is a solution of (3.3), it is a standard argument by using the comparison theorem to show that there exist two positive constants $C_1$ and $C_2$ such that

$$C_1 \Gamma_{F,p} \leq u \leq C_2 \Gamma_{F,p}.$$

Following the argument of [23, Theorem 1.1 and Remark 1.5] (see [35, Theorem 4.1 and Remark 4.1] for the anisotropic case), we conclude that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{|x| \to +\infty} \frac{u(x)}{\Gamma_{F,p}(x)} = \gamma, \quad \text{as } |x| \to +\infty.$$ \hfill (3.5)

$$\lim_{|x| \to +\infty} (F^0(x))^{\frac{1}{p-1}} (\nabla u - \gamma \nabla \Gamma_{F,p}) = 0.$$ \hfill (3.6)

It remains to show

$$\gamma = \text{Cap}_{F,p}(\Omega)^{\frac{1}{p-1}}.$$  \hfill (3.7)

Through the following integration by parts that holds true by the anisotropic $p$-harmonicity of $u$:

$$\int_{\partial \Omega} F^{p-1}(\nabla u)F(\nu)ds = - \lim_{R \to \infty} \int_{\partial \Omega_R} F^{p-1}(\nabla u)(F_\xi(\nabla u), \nu_{\partial \Omega_R})d\sigma,$$ \hfill (3.37)
where $\nu = -\frac{\nabla u}{|\nabla u|}$ on $\partial \Omega$ and $\nu_{\partial W_R}$ is the outward unit normal of $W_R$, we shall see in Proposition 3.5 below that the left-hand side of (3.7) is exactly $\text{Cap}_{F,p}(\Omega)$. We next compute the limit on the right-hand side of (3.7). Since
\[
\nabla u = \gamma \nabla \Gamma_{F,p} + o(|x|^{-\frac{\alpha-1}{p-1}}),
\]
we have
\[
F(\nabla u) = \gamma F(\nabla \Gamma_{F,p}) + o(|x|^{-\frac{\alpha-1}{p-1}}) = \gamma \left( \frac{1}{K_{n-1}} \right)^{\frac{1}{p-1}} F^o(x)^{-\frac{\alpha-1}{p-1}} + o(|x|^{-\frac{\alpha-1}{p-1}}).
\]
On $\partial W_R$,
\[
\nu_{\partial W_R} = \frac{\nabla F^o}{|\nabla F^o|} = -K_{n-1}^{\frac{1}{p-1}} F^o(x)^{\frac{n-1}{p-1}} \nabla \Gamma_{F,p} = -K_{n-1}^{\frac{1}{p-1}} F^o(x)^{\frac{n-1}{p-1}} \gamma^{-1} \nabla u + o(1).
\]
It follows that on $\partial W_R$,
\[
(F_{\xi}(\nabla u), \nu_{\partial W_R}) = -K_{n-1}^{\frac{1}{p-1}} F^o(x)^{\frac{n-1}{p-1}} \gamma^{-1} \frac{F(\nabla u)}{|\nabla F^o|} + o(1) = -\frac{1}{|\nabla F^o|} + o(1).
\]
Hence
\[
F^{p-1}(\nabla u) \langle F_{\xi}(\nabla u), \nu_{\partial W_R} \rangle = -\gamma^{p-1} \frac{1}{K_{n-1}} \frac{(F^o(x))^{1-n}}{|\nabla F^o|} + o(|x|^{1-n}).
\]
Combining the fact that
\[
\int_{\partial W_R} \frac{(F^o(x))^{1-n}}{|\nabla F^o|} d\sigma = K_{n-1},
\]
we deduce that
\[
\lim_{R \to \infty} \int_{\partial W_R} F^{p-1}(\nabla u) \langle F_{\xi}(\nabla u), \nu_{\partial W_R} \rangle d\sigma = -\gamma^{p-1}.
\]
Thus
\[
\gamma = \text{Cap}_{F,p}(\Omega)^{\frac{1}{p-1}},
\]
and (1) and (2) of Proposition 3.4 follow.

We need the following expression for $\text{Cap}_{F,p}(\Omega)$ in terms of an integral on $\partial \Omega$.

**Proposition 3.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and let $1 < p < n$. Then the solution $u$ to (3.3) satisfies
\[
\text{Cap}_{F,p}(\Omega) = \int_{\partial \Omega} F^{p-1}(\nabla u) F(\nu) d\sigma,
\]
where $\nu = -\frac{\nabla u}{|\nabla u|}$ on $\partial \Omega$.

**Proof.** For $\varepsilon > 0$, let $U_\varepsilon$ be the $\varepsilon$-tubular neighborhood of $\text{Crit}(u)$, namely,
\[
U_\varepsilon = \{ x \in \mathbb{R}^n \setminus \bar{\Omega} \mid \text{dist}(x, \text{Crit}(u)) < \varepsilon \},
\]
where $\text{dist}(x, \text{Crit}(u))$ is the distance of $x$ from $\text{Crit}(u)$. By the compactness of $\text{Crit}(u)$ in $\mathbb{R}^n \setminus \bar{\Omega}$, we have $U_\varepsilon \subset \{ u \geq t \}$ for $\varepsilon > 0$ and $t > 0$ small enough. Since $F(\nabla u) = 0$ on $\text{Crit}(u)$, we have from the identity (3.4) and by the monotone convergence theorem that
\[
\text{Cap}_{F,p}(\Omega) = \lim_{\varepsilon \to 0^+} \lim_{t \to 0^+} \int_{\{ u \geq t \} \setminus U_\varepsilon} F^p(\nabla u) dx.
\]
Then for $\varepsilon$ and $t$ small enough, by integration by parts,

$$\int_{\{u \geq t\} \setminus U_x} F^p(\nabla u) \, dx = \int_{\{u \geq t\} \setminus U_x} \text{div}(u F^{p-1}(\nabla u) F_{\xi}(\nabla u)) \, dx$$

$$= -t \int_{\{u = t\}} F^p(\nabla u) \, |\nabla u| \, d\sigma + \int_{\partial \Omega} F^{p-1}(\nabla u) F(\nu) \, d\sigma$$

$$+ \int_{\partial U_x} u F^{p-1}(\nabla u)(F_{\xi}(\nabla u), \nu_{\partial U_x}) \, d\sigma.$$ 

By the computation in the proof of Proposition 3.4, we see

$$\int_{\{u = t\}} F^p(\nabla u) \, |\nabla u| \, d\sigma$$

is bounded when $t \to 0^+$, which implies

$$-t \int_{\{u = t\}} F^p(\nabla u) \, |\nabla u| \, d\sigma \to 0 \quad \text{as} \quad t \to 0.$$

On the other hand, we have

$$\int_{\partial U_x} u F^{p-1}(\nabla u)(F_{\xi}(\nabla u), \nu_{\partial U_x}) \, d\sigma \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

since $F(\nabla u)$ tends to zero as $x$ approaches $\text{Crit}(u)$. The assertion follows. \qed

### 4 Essential monotonicity

In this section, we assume that $u$ is an anisotropic $p$-harmonic function, i.e., $\Delta_{F,p}u = 0$. We will establish some lemmas for later use.

Let

$$p^* := \frac{(n-1)(p-1)}{n-p}$$

and

$$\Lambda = \left\{ (p, q) \in \mathbb{R}^2 \mid 1 < p < n \text{ and } q \geq 1 + \frac{1}{p^*} \right\}.$$

For $(p, q) \in \Lambda$, we consider the function $\Phi_{p,q} : [1, +\infty) \to \mathbb{R}^n$ given by

$$\Phi_{p,q}(\tau) := \tau^{(q-1)p^*} \int_{\{u = 1/\tau\}} F^{q(p-1)}(\nabla u) F(\nu) \, d\sigma. \quad (4.1)$$

By the asymptotic behavior given in Proposition 3.4, it is direct to check the limits of $\Phi_{p,q}(\tau)$.

**Lemma 4.1.** It holds that

$$\lim_{\tau \to +\infty} \Phi_{p,q}(\tau) = \left( \frac{n-p}{p-1} \right)^{(q-1)p^*} \left( \kappa_{n-1} \right)^{\frac{(q-1)(p-1)}{n-p}} (\text{Cap}_{F,p}(\Omega))^{1 - \frac{(q-1)(p-1)}{n-p}}. \quad (4.2)$$

**Proof.** From the asymptotic behavior of $u$ in Proposition 3.4, we know that for $\varepsilon > 0$, there exists $\tau_0$ such that for $\tau > \tau_0$ and $u(x) = \frac{1}{\tau}$,

$$(1 - \varepsilon)\gamma G_{F,p} \leq u(x) \leq (1 + \varepsilon)\gamma G_{F,p},$$

$$(1 - \varepsilon)\gamma \left( \frac{1}{\kappa_{n-1}} \right)^{\frac{1}{n-1}} F^{\alpha}(x)^{-\frac{n-1}{\alpha - 1}} \leq F(\nabla u(x)) \leq (1 + \varepsilon)\gamma \left( \frac{1}{\kappa_{n-1}} \right)^{\frac{1}{n-1}} F^{\alpha}(x)^{-\frac{n-1}{\alpha - 1}}.$$
From this one deduces that on such \( \{ u(x) = \frac{1}{r} \} \)

\[
(1 - \tilde{\varepsilon}) \gamma \left( \frac{n - p}{p - 1} \right) \left( \frac{q - 1}{p - 1} \right)^p \left( \kappa_{n-1} \right)^{\frac{(q - 1)(p - 1)}{n - p}} \tau^{-(q - 1)p^*} \leq F^{(q - 1)(p - 1)}(\mathbf{\nabla} u(x))
\]

\[
\leq (1 + \tilde{\varepsilon}) \gamma \left( \frac{n - p}{p - 1} \right) \left( \frac{q - 1}{p - 1} \right)^p \left( \kappa_{n-1} \right)^{\frac{(q - 1)(p - 1)}{n - p}} \tau^{-(q - 1)p^*},
\]

where \( \tilde{\varepsilon} \) is some function on \( \varepsilon \) with the property \( \lim_{\varepsilon \to 0} \tilde{\varepsilon} = 0 \). Arguing as in Proposition 3.5, we see that

\[
\int_{\{ u = 1/r \}} F^{p-1} F(\nu) d\sigma = \text{Cap}_{F,p}(\Omega).
\]

Since

\[
\tau^{(q - 1)p^*} \int_{\{ u = 1/r \}} F^{q(p - 1)} F(\nu) d\sigma = \tau^{(q - 1)p^*} \int_{\{ u = 1/r \}} F^{(q - 1)(p - 1)}(F^{p-1} F(\nu)) d\sigma,
\]

it follows that

\[
(1 - \tilde{\varepsilon}) \left( \frac{n - p}{p - 1} \right)^{(q - 1)p^*} \left( \kappa_{n-1} \right)^{\frac{(q - 1)(p - 1)}{n - p}} (\text{Cap}_{F,p}(\Omega))^{1 - \frac{(q - 1)(p - 1)}{n - p}} \leq \tau^{(q - 1)p^*} \int_{\{ u = 1/r \}} F^{q(p - 1)} F(\nu) d\sigma
\]

\[
\leq (1 + \tilde{\varepsilon}) \left( \frac{n - p}{p - 1} \right)^{(q - 1)p^*} \left( \kappa_{n-1} \right)^{\frac{(q - 1)(p - 1)}{n - p}} (\text{Cap}_{F,p}(\Omega))^{1 - \frac{(q - 1)(p - 1)}{n - p}}.
\]

The assertion follows by letting \( \tau \to \infty \) and then \( \varepsilon \to 0 \). \( \square \)

For \( (p, q) \in \Lambda \), let us consider the vector field \( X \) with the component

\[
X^j = -(q - 1)u^{-(q - 1)p^* + 2} F^{q(p - 1) - 1} \left( a_{jk,p} F^{1-p} F_i u_{ik} - p^* \frac{F_j}{u} \right), \quad (4.3)
\]

where

\[
a_{jk,p} = F^{p-2}(a_{jk} + (p - 2) F_j F_k).
\]

Next, we have the following lemma.

**Lemma 4.2.** \( \Phi_{p,q} \) is differentiable at the regular values of \( u \) and

\[
\frac{d}{d\tau} \Phi_{p,q}(\tau) = \int_{\{ u = 1/r \}} \left( X, \frac{\mathbf{\nabla} u}{|\mathbf{\nabla} u|} \right) d\sigma = \int_{\{ u < 1/r \}} \text{div} X dx. \quad (4.4)
\]

**Proof.** By the co-area formula and the divergence theorem, we have

\[
\frac{d}{d\tau} \Phi_{p,q}(\tau) = \frac{d}{d\tau} \left\{ \int_{\{ u = 1/r \}} u^{-(q - 1)p^*} F^{q(p - 1)} F(\nu) d\sigma \right\}
\]

\[
= \frac{d}{d\tau} \left\{ \int_{\{ u < 1/r \}} \partial_{x_i} \left( u^{-(q - 1)p^*} F^{q(p - 1)} F^{p-1} F_i \right) dx \right\}
\]

\[
= \frac{d}{d\tau} \left\{ \int_{\{ u < 1/r \}} (q - 1)u^{-(q - 1)p^*} F^{q(p - 1)} \left( - p^* \frac{F}{u} + (p - 1) \frac{F_i u_{ik}}{F} \right) dx \right\}
\]

\[
= -(q - 1)\tau^{(q - 1)p^* - 2} \int_{\{ u = 1/r \}} F^{q(p - 1) - 1} \left( (p - 1) \frac{F_j u_{ik}}{F} - p^* \frac{F}{u} \right) F(\nu) d\sigma. \quad (4.5)
\]

By using the definition of \( X \) and the fact

\[
a_{jk,p} u_j = (p - 1) F^{p-2} F_j,
\]
we deduce
\[ (X, \nabla u) = - (q - 1) u^{-(q-1)p^*+2} F^q(p-1)^{-1}(p-1)F_k F_i u_{ik} - p^* \frac{F^2}{u}. \]

Then, we obtain the first equality. The second equality follows from the divergence theorem.

Finally, we compute the divergence of \( X \).

**Lemma 4.3 (Divergence of \( X \)).** For any \((p, q) \in \Lambda\), let \( u \) be a solution of (3.3) and \( X \) be the vector field defined in (4.3). Then the following identity holds at any point \( x \in \mathbb{R}^n \setminus \Omega \) such that \( F(\nabla u) \neq 0 \):

\[
\text{div} X = -(q - 1) u^{-(q-1)p^*+2} F^q(p-1)^{-1}(\nabla u) \left\{ \left[ \frac{n-2}{n-1} \sigma^2_1(\kappa_F) - 2\sigma_2(\kappa_F) \right] \right. \\
+ \left. (q(p-1) - 1) \frac{|\nabla^T(F(\nabla u))|^2_{a_F}}{F^2(\nabla u)} \right\} \leq 0.
\]

**Proof.** Using (2.7) and \( \Delta_{F,p} u = 0 \), we can write \( X \) in the following way:

\[
X^j = (q - 1) \{ u^{-(q-1)p^*+2} F^q(p-2)(p-1)^{-1} S^{ij}_{22} V_i(\nabla u) + p^* u^{-(q-1)p^*+1} F^q(p-1)(p-1)^{-1} F^p F_j \}.
\]

Separate
\[
\text{div} X = (q - 1)(I + II),
\]

where
\[
I = \partial_{x_j} (u^{-(q-1)p^*+2} F^q(p-2)(p-1)^{-1} S^{ij}_{22} V_i(\nabla u)), \\
II = p^* \partial_{x_j} (u^{-(q-1)p^*+1} F^q(p-1)(p-1)^{-1} F^p F_j).
\]

By using Proposition 2.2 and (2.9), we have the following computations:
\[
I = -(2 - (q - 1)p^*) u^{-(q-1)p^*+1} F^q(p-1)^{-1} H_F - u^{-(q-1)p^*+2} F^q(p-1)^{-3} \\
\times \left\{ ((q - 2)(p - 1) - 1) F^{2-2p} a_{k,p} F_k F_i u_{ki} u_{ij} + F^{4-2p} a_{k,p} a_{l,p} u_{ik} u_{lj} \right\}
\]

and
\[
II = (1 - (q - 1)p^*) p^* u^{-(q-1)p^*} F^q(p-1)^{-1} + (q - 1) p^* u^{-(q-1)p^*} F^q(p-1)^{-1} H_F.
\]

Next, by using (2.21) and (2.22), we have
\[
((q - 2)(p - 1) - 1) F^{2-2p} a_{k,p} F_k F_i u_{ki} u_{ij} + F^{4-2p} a_{k,p} a_{l,p} u_{ik} u_{lj} \\
= ((q - 2)(p - 1) - 1) \left( |\nabla^T(F(\nabla u))|^2_{a_F} + \frac{1}{p-1} H_F^2 F^2 \right) + \frac{n}{n-1} H_F^2 F^2 \\
+ 2(q - 1)(p - 1) |\nabla^T(F(\nabla u))|^2_{a_F} + F^2 \left[ \frac{n-2}{n-1} \sigma^2_1(\kappa_F) - 2\sigma_2(\kappa_F) \right] \\
= \left( (q - 1) - \frac{1}{p^*} \right) F^2 H_F^2 + (q(p - 1) - 1) |\nabla^T(F(\nabla u))|^2_{a_F} \\
+ F^2 \left[ \frac{n-2}{n-1} \sigma^2_1(\kappa_F) - 2\sigma_2(\kappa_F) \right].
\]

Substituting (4.9)–(4.11) into (4.8) yields (4.6).

\[ \square \]

5 Effective monotonicity and the \( L^p \) anisotropic Minkowski inequality

The aim of this section is to give a complete proof of Theorem 1.1. For this purpose, we will establish the following two effective monotonicity inequalities:
\[
\Phi_{p,q}(1) \leq 0 \quad \text{and} \quad \Phi_{p,q}(+\infty) = \lim_{\tau \to \infty} \Phi_{p,q}(\tau) \leq \Phi_{p,q}(1).
\]
5.1 The first effective inequality

Combining Lemmas 4.2 and 4.3, we see that \( \Phi'_{p,q}(s) \leq \Phi'_{p,q}(S) \) for any \( s < S \) if \( \text{Crit}(u) = \emptyset \). However, in general \( \text{Crit}(u) \neq \emptyset \). Nevertheless, we are able to provide an effective version of the considered monotonicity, showing that is actually in force, provided \( S \) is large enough and \( s \) is close to 1. This is enough to get the desired effective inequality \( \Phi'_{p,q}(1) \leq 0 \).

**Theorem 5.1** (The effective monotonicity formula—1). For \( (p, q) \in \Lambda \), let \( u \) be the solution to (3.3) and let \( 1 < \bar{s} < \bar{S} < +\infty \) be such that \( \text{Crit}(u) \subset \{ \bar{S}^{-1} < u < \bar{s}^{-1} \} \). Then for every \( 1 \leq s \leq \bar{s} \leq \bar{S} \leq S \), the inequality

\[
\Phi'_{p,q}(s) \leq \Phi'_{p,q}(S)
\]

holds true, where \( \Phi_{p,q} \) is defined in (4.1). Moreover, one has \( \Phi'_{p,q}(1) \leq 0 \).

**Remark 5.2.** The existence of \( \bar{s} \) and \( \bar{S} \) follows from Remark 3.3 and Proposition 3.4.

**Proof of Theorem 5.1.** Fix \( s \) and \( S \) such that \( 1 \leq s \leq \bar{s} \leq \bar{S} \leq S \). For given \( \varepsilon > 0 \), let \( \chi : [0, +\infty) \to \mathbb{R} \) be a smooth nonnegative cut-off function such that

\[
\begin{cases}
\chi(t) = 0 & \text{if } t < \frac{1}{2} \varepsilon, \\
\chi'(t) \geq 0 & \text{if } \frac{1}{2} \varepsilon \leq t \leq \frac{3}{2} \varepsilon, \\
\chi(t) = 1 & \text{if } t > \frac{3}{2} \varepsilon.
\end{cases}
\]

(5.2)

Define a smooth vector field

\[
\tilde{X} = \chi(u^{-(q-1)p^*} F^{(q-1)(p-1)}(\nabla u)) X.
\]

Since \( F(\nabla u) = 0 \) on \( \text{Crit}(u) \), we have \( X = 0 \) in \( \text{Crit}(u) \). By choosing \( \varepsilon \) small enough, we can make sure that

\[
\tilde{X} = X \quad \text{on } \{ u = s^{-1} \} \text{ and } \{ u = S^{-1} \},
\]

since \( \text{Crit}(u) \subset \{ \bar{S}^{-1} < u < \bar{s}^{-1} \} \subset \{ S^{-1} < u < s^{-1} \} \). Define

\[
\Theta := (X, \nabla (u^{-(q-1)p^*} F^{(q-1)(p-1)}(\nabla u)))
\]

\[
= (q-1)(p-1)u^{-(q-1)p^*} F^{(q-1)(p-1)-1} \left( X, \nabla (F(\nabla u)) - \frac{n-1}{n-p} \frac{\nabla u}{u} F \right).
\]

Then we have the following computations:

\[
\int_{\{ u = 1/s \}} \left< X, \frac{\nabla u}{|\nabla u|} \right> d\sigma - \int_{\{ u = 1/S \}} \left< X, \frac{\nabla u}{|\nabla u|} \right> d\sigma
\]

\[
= \int_{\{ S < u < 1/s \}} \text{div} \tilde{X} d\sigma = \int_{\{ S < u < 1/s \}} \chi \text{div} X d\sigma + \int_{U_{\delta}/2} \chi' \Theta d\sigma,
\]

(5.3)

where in the last identity we have used the tubular neighborhood of \( \text{Crit}(u) \) defined for every \( \delta > 0 \) as

\[
U_{\delta} = \{ u^{-(q-1)p^*} F^{(q-1)(p-1)}(\nabla u) \leq \delta \}.
\]

From Lemma 4.2, we can see that the first term on the right-hand side of (5.3) is non-positive, and we next prove that \( \Theta \leq 0 \). By using (4.3), (2.12) and (2.15), we have

\[
\Theta = -(q-1)^2(p-1)u^{-2(q-1)p^*+2} F^{(2q-1)(p-1)-2} \times \left\{ \frac{a_{jk,p} F_ik u_i F^{2-p} - p^{F F_j} u_j}{u} \right\} \left( F_t u_j - \frac{(n-1)}{n-p} \frac{F}{u} u_j \right)
\]

\[
= -(q-1)^2(p-1)u^{-2(q-1)p^*+2} F^{(2q-1)(p-1)-1} \times \left\{ \frac{a_{jk,p} F_ik u_i F^{2} F_j (n-1)^2}{F_n u} + \frac{(n-1)^2(p-1) F^2}{(n-p)^2} \frac{2p^* F_k F_i u_{ik}}{u} \right\}
\]
\[(q - 1)^2 u^{-2(q-1)p^*+2} F^{(2q-1)(p-1)-1} \times \left\{ H_F - p^* \frac{F}{u} \right\}^2 + (p - 1) \frac{\nabla^T (F \nabla u))_{\alpha \beta}^2}{F^2} \right\} \leq 0. \tag{5.4} \]

This completes the proof of (5.1).

It follows from (5.1) that for every \( S \geq \bar{S} \),

\[ \Phi'_{p,q}(1) \leq \Phi'_{p,q}(S). \]

Integrating both sides of the above inequality on an interval of the form \((\bar{S}, S)\), we obtain

\[ \Phi'_{p,q}(1)(S - \bar{S}) + \Phi_{p,q}(\bar{S}) \leq \Phi_{p,q}(S). \]

If by contradiction, \( \Phi'_{p,q}(1) > 0 \), letting \( S \to +\infty \) in the above inequality, we deduce

\[ \lim_{S \to +\infty} \Phi_{p,q}(S) \to +\infty \]

against the boundedness of \( \Phi_{p,q} \) by Lemma 4.1. \( \square \)

5.2 The second effective inequality \( \Phi_{p,q}(\infty) \leq \Phi_{p,q}(1) \)

For a given \((p, q) \in \Lambda \) and a given \( 0 < \lambda < 1 \), we consider the vector field

\[ Y_\lambda = (u^{-1} - \lambda) X - F^{q(p-1)} u^{-(q-1)p^*} \nabla F \nabla u. \tag{5.5} \]

It is convenient to observe that at a regular value of \( u \), it holds that

\[ (\tau - \lambda) \Phi'_{p,q}(\tau) - \Phi_{p,q}(\tau) = \int_{\{u = 1/\tau\}} \left\{ Y_\lambda, \frac{\nabla u}{|\nabla u|} \right\}. \tag{5.6} \]

Next, we compute the divergence of \( Y_\lambda \).

**Lemma 5.3** (Divergence of \( Y_\lambda \)). For any \((p, q) \in \Lambda \) and any \( 0 < \lambda < 1 \), let \( u \) be the solution of (3.3) and \( Y_\lambda \) be the vector field defined in (5.5). Then the following identity holds at any point \( x \in \mathbb{R}^n \setminus \bar{\Omega} \) such that \( F(\nabla u) \neq 0 \):

\[ \text{div} Y_\lambda = (u^{-1} - \lambda) \text{div} X \leq 0, \]

where \( \text{div} X \) is non-positive defined in (4.6).

**Proof.** By the definition (5.5) of \( Y_\lambda \), we obtain

\[ \text{div} Y_\lambda = (u^{-1} - \lambda) \text{div} X - u^{-2}(X, \nabla u) - \text{div}(F^{q(p-1)} u^{-(q-1)p^*} F \nabla F \nabla u)). \]

By using the definition of \( X \), we compute

\[ u^{-2}(X, \nabla u) = -(q - 1) u^{-(q-1)p^*} F^{q(p-1)-1} \left( (p - 1) F_k F_i u_{ik} - p^* F^2 \frac{F^2}{u} \right). \]

By using \( \Delta_{F,p}u = \text{div}(F^{p-1} F \nabla u)) = 0 \), we get

\[ \text{div}(F^{q(p-1)} u^{-(q-1)p^*} F \nabla u)) = (q - 1) u^{-(q-1)p^*} F^{q(p-1)-1} \left( (p - 1) F_k F_i u_{ik} - p^* F^2 \frac{F^2}{u} \right). \]

The assertion follows. \( \square \)

**Theorem 5.4** (The effective monotonicity formula—II). For any \( 1 < p < n \), let \( u \) be the solution of (3.3) and let \( 1 < \bar{s} < \bar{S} < +\infty \) be such that \( \text{Crit}(u) \subset \{ \bar{s}^{-1} < u < \bar{s}^{-1} \} \). Then for every \( 1 \leq s \leq \bar{s} \leq \bar{S} \leq S \), the inequality

\[ (s - \lambda) \Phi'_{p,q}(s) - \Phi_{p,q}(s) \leq (S - \lambda) \Phi'_{p,q}(S) - \Phi_{p,q}(S) \tag{5.7} \]

holds true. Moreover, one has \( \Phi_{p,q}(+\infty) \leq \Phi_{p,q}(1) \).
Proof. Fix $s$ and $S$ such that $1 \leq s \leq \bar{s} \leq \bar{S} \leq S$. Let $\chi : [1, +\infty) \to \mathbb{R}$ be the same smooth nonnegative cut-off function as in the proof of Theorem 5.1 so that the properties in (5.2) hold. To simplify the notation, let us also set 

$$\eta_\lambda(u) = (u^{-1} - \lambda)^{-1}.$$ 

Then let us consider the smooth vector field 

$$\tilde{Y}_\lambda = \chi(\eta_\lambda(u))u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u)Y_\lambda,$$

where $Y_\lambda$ has been defined in (5.4). Again, choosing $\varepsilon$ small enough, we can suppose $\tilde{Y}_\lambda = Y_\lambda$ on $\{u = 1/s\}$ and $\{u = 1/S\}$ with $s$ and $S$ as in the statement.

Define 

$$\Psi := \langle \nabla(\eta_\lambda(u))u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u), Y_\lambda \rangle.$$

By applying the divergence theorem to the smooth vector field $\tilde{Y}_\lambda$ on the region $\{1/S < u < 1/s\}$, we have 

$$\int_{\{u = 1/s\}} \langle \tilde{Y}_\lambda, \nabla u \rangle d\sigma - \int_{\{u = 1/S\}} \langle \tilde{Y}_\lambda, \nabla u \rangle d\sigma \int_{\{1 < u < 1\} \setminus U_{\varepsilon/2}} \chi(\eta_\lambda(u))u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u) \text{div} \tilde{Y}_\lambda dx + \int_{U_{3\varepsilon/2} \setminus U_{\varepsilon/2}} \chi(\eta_\lambda(u))u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u) d\sigma,$$

where this time the tubular neighborhood of $\text{Crit}(u)$ is defined for every $\delta > 0$ as 

$$U_\delta = \{\eta_\lambda(u)u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u) \leq \delta\}.$$

As observed in Lemma 5.3, the divergence of $Y_\lambda$ is non-positive on $\{1/S < u \leq 1/s\} \setminus U_{\varepsilon/2}$, where clearly $F(\nabla u) \neq 0$. Next, we prove that $\Psi$ is non-positive on $U_{3\varepsilon/2} \setminus U_{\varepsilon/2}$.

Notice that 

$$\nabla(\eta_\lambda(u))u^{-(q-1)p^*}F^{(q-1)(p-1)}(\nabla u)) = u^{-2(q-1)p^*}\eta_\lambda^2(u)F^{(q-1)(p-1)}(\nabla u) = u^{-2(q-1)p^*}\eta_\lambda^2(u)F^{(q-1)(p-1)}(\eta_\lambda(\nabla u)) + (q-1)(p-1)\eta_\lambda \times u^{-2(q-1)p^*}F^{(q-1)(p-1)}(\nabla u) \left(\nabla(F(\nabla u)) \frac{n-1}{n-p}F \nabla u\right).$$

By using Proposition 2.2, (5.4) and (5.8), a direct computation gives 

$$\Psi = \Theta - 2(q-1)\eta_\lambda(\eta_\lambda(u))u^{-2(q-1)p^*}F^{(2q-1)(p-1)}(\frac{n-1}{n-p}F \nabla u \ n^2) \left(\frac{n-1}{n-p}F \nabla u \ n^2\right)^2 \left(\nabla(F(\nabla u)) \frac{n-1}{n-p}F \nabla u\right)^2 \leq 0.$$ 

This completes the proof of (5.7).

It remains to show that 

$$\lim_{\tau \to +\infty} \Phi_{p,q}(\tau) \leq \Phi_{p,q}(1).$$

Applying the inequality (5.7) with $0 < \lambda < 1$, $s = 1$ and $S = \bar{S}$, we get 

$$\Phi_{p,q}(S) - \Phi_{p,q}(1) \leq (S - \lambda)\Phi_{p,q}'(S) - (1 - \lambda)\Phi_{p,q}'(1).$$
Observe now that (5.1) holds also for $\bar{S} < s < S$, because Crit$(u) \cap [\bar{S}, S] = \emptyset$. Then the same argument as in Theorem 5.1 to deduce $\Phi'_{p,q}(1) \leq 0$ gives also $\Phi'_{p,q}(s) \leq 0$ for any $s > \bar{S}$. In particular, $\Phi'_{p,q}$ is a definitely bounded monotone function, and this implies
\[
\liminf_{S \to +\infty} \Phi'_{p,q}(S) \leq 0.
\]
Hence, passing to the limit inferior as $S \to +\infty$ in the above inequality yields
\[
\lim_{S \to +\infty} \Phi_{p,q}(S) - \Phi_{p,q}(1) = -(1-\lambda)\Phi'_{p,q}(1).
\]
Letting $\lambda \to 1^-$ on the right-hand side leads to the second effective inequality $\lim_{S \to +\infty} \Phi_{p,q}(S) \leq \Phi_{p,q}(1)$. 

5.3 The $L^p$ anisotropic Minkowski inequality

We are ready to prove the following geometric inequalities between anisotropic $p$-capacity and total anisotropic mean curvatures.

**Theorem 5.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $(p, q) \in \Lambda$. Then
\[
\left( \int_{\partial \Omega} \frac{H_F}{n-1} |F(\nu)| d\sigma \right)^{\frac{4}{q}} \geq \left( \frac{p-1}{n-p} \right)^{\frac{p-1}{p-1}} \frac{\text{Cap}_{F, p}(\Omega)}{|\partial \Omega|^{\frac{1}{n-1}}}. \tag{5.9}
\]
Moreover, the equality holds in (5.9) if and only if $\Omega$ is a Wulff ball.

**Proof.** Recall the formula (4.5) and also the identity (2.10). Thus the first effective inequality
\[
\Phi'_{p,q}(1) \leq 0
\]
implies that
\[
0 \leq \int_{\partial \Omega} F^{q(p-1)}(\nabla u)(H_F - p^*F(\nabla u))F(\nu) d\sigma,
\]
and thus, by Hölder’s inequality, one gets
\[
\int_{\partial \Omega} F^{q(p-1)}(\nabla u)F(\nu) d\sigma \leq \left( \frac{n-p}{p-1} \right)^{q(p-1)} \int_{\partial \Omega} \left| \frac{H_F}{n-1} \right|^{q(p-1)} F(\nu) d\sigma. \tag{5.10}
\]
On the other hand, by Hölder’s inequality, we obtain
\[
\int_{\partial \Omega} F^{p-1} F(\nu) d\sigma \leq \left( \int_{\partial \Omega} F^{q(p-1)} F(\nu) d\sigma \right)^{\frac{1}{q}} \left( \int_{\partial \Omega} F(\nu) d\sigma \right)^{\frac{q-1}{q}}.
\]
This combining (3.8) and (5.10) yields (5.9).

Assume now that the equality holds in (1.7). Then the equality holds in (5.9), and consequently, $\Phi'_{p,q}(1) = 0$. It follows that div$X = 0$ along $\partial \Omega$. From (4.6), we see that
\[
\frac{n-2}{n-1} \sigma^2_1(\kappa_F) - 2\sigma_2(\kappa_F) = 0.
\]
Then $\partial \Omega$ is umbilical in the anisotropic sense, which yields that $\partial \Omega$ is of a Wulff shape. 

**Theorem 5.6.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $(p, q) \in \Lambda$. Then
\[
\frac{1}{\kappa_{n-1}} \int_{\partial \Omega} \frac{H_F}{n-1} |F(\nu)| d\sigma \geq \left( \frac{p-1}{n-p} \right)^{p-1} \frac{1}{\kappa_{n-1}} \text{Cap}_{F, p}(\Omega)^{1-\frac{(p-1)(p-1)}{n-p}}. \tag{5.11}
\]
The equality holds in (5.11) if and only if $\Omega$ is a Wulff ball.
Proof. The second effective inequality \( \Phi_{p,q}(\infty) \leq \Phi_{p,q}(1) \) combining the asymptotic behavior (4.2) implies that

\[
\left(\frac{n-p}{p-1}\right) \left(\frac{q-1}{p-1}\right) (\kappa_{n-1})^{\frac{(q-1)(p-1)}{n-p}} \left(\text{Cap}_{F,p}(\Omega)\right)^{1-\frac{(q-1)(p-1)}{n-p}} = \lim_{\tau \to \infty} \Phi_{p,q}(\tau) \leq \Phi_{p,q}(1) = \int_{\partial \Omega} F^{(p-1)}(\nabla u) F(\nu) d\sigma.
\]

Then by using (5.10), we obtain (5.11). The rigidity follows from the proof as that for Theorem 5.5. \( \square \)

Proofs of Theorems 1.1 and 1.2. Let \( q = p/(p-1) \) in (5.11). We obtain the anisotropic \( L^q \) Minkowski inequality (1.7) in Theorem 1.1. Let \( q = \frac{n-1}{p-1} \) in (5.11). We obtain the anisotropic Willmore inequality (1.8) in Theorem 1.2.

\section{The anisotropic Minkowski inequality for outward \( F \)-minimising sets}

Definition 6.1. Let \( E \subset \mathbb{R}^n \) be a bounded set of finite perimeter (s.f.p. for short). The anisotropic perimeter of \( E \) is defined by

\[
P_F(E) = \int_{\partial^* E} F(\nu_E) d\mathcal{H}^{n-1},
\]

where \( \partial^* E \) is the reduced boundary of \( E \) and \( \nu_E \) is the measure-theoretical outward unit normal of \( E \).

Remark 6.2. When \( E \) has smooth boundary, then

\[
P_F(E) = |\partial E|_F = \int_{\partial E} F(\nu) d\sigma.
\]

Definition 6.3 (Outward \( F \)-minimising and strictly outward \( F \)-minimising sets). Let \( E \subset \mathbb{R}^n \) be a bounded set of finite perimeter. We say that \( E \) is outward \( F \)-minimising if

\[
P_F(E) \leq P_F(G) \quad \text{for } E \subset G \subset \mathbb{R}^n.
\]

We say that \( E \) is strictly outward \( F \)-minimising if it is outward \( F \)-minimising and the equality in (6.1) holds only if \( |G \setminus E| = 0 \).

Proposition 6.4. Let \( \Omega \) be an outward \( F \)-minimising set with smooth boundary. Then

\[
\inf_{\Omega \subset U \text{ is smooth}} |\partial U|_F = |\partial \Omega|_F.
\]

Proof. It follows directly from the definition. \( \square \)

Proposition 6.5. Let \( \Omega \) be an outward \( F \)-minimising set with smooth boundary. Then \( H_F \geq 0 \).

Proof. It follows from (6.2) and the first variational formula of \( |\partial U|_F \). \( \square \)

Proposition 6.6. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with smooth boundary. Then

\[
\lim_{p \to 1^+} \text{Cap}_{F,p}(\Omega) = \text{Cap}_{F,1}(\Omega) = \inf_{\Omega \subset U \text{ is smooth}} |\partial U|_F.
\]

Proof. The second equality can be found in [25, Lemma 2.2.5] (see also [2]). For the convenience of the reader, we provide a proof here.

Step 1. It holds that

\[
\text{Cap}_{F,1}(\Omega) \geq \inf_{\Omega \subset U \text{ is smooth}} |\partial U|_F.
\]

By the co-area formula, for any \( u \in C^\infty_c(\mathbb{R}^n) \) with \( u \geq \chi_\Omega \), we have

\[
\int_{\mathbb{R}^n} F(\nabla u) dx \geq \int_0^1 |\partial \{ u > t \}|_F dt \geq \inf_{\Omega \subset U \text{ is smooth}} |\partial U|_F.
\]
In particular, taking the infimum over any such $u$, we get (6.4).

**Step 2.** It holds that

$$\text{Cap}_{F,1}(\Omega) \leq \liminf_{p \to 1^+} \text{Cap}_{F,p}(\Omega).$$

(6.5)

For every $u \in C^\infty_c(\mathbb{R}^n)$ with $u \geq \chi_\Omega$ and any $q > 0$, by the definition of $\text{Cap}_{F,1}(\Omega)$ and Hölder’s inequality, we have

$$\text{Cap}_{F,1}(\Omega) \leq \int_{\mathbb{R}^n} F(\nabla u^n)dx \leq q \left( \int_{\mathbb{R}^n} u^n \frac{(q-1)p}{n-p} dx \right)^{(p-1)/p} \left( \int_{\mathbb{R}^n} F^p(\nabla u)dx \right)^{1/p}.$$ (6.6)

Choose $q = \frac{p(n-1)}{n-p} > 1$. Then

$$\frac{(q-1)p}{p-1} = \frac{np}{n-p},$$

which is the Sobolev critical exponent of $p$. The anisotropic Sobolev inequality with the best constant [7] says that

$$\left( \int_{\mathbb{R}^n} u^n \frac{np}{n-p} dx \right)^{n/p} \leq \left( \frac{\omega_{n-1}}{\kappa_{n-1}} \right)^{\frac{1}{n}} T_{n,p} \left( \int_{\mathbb{R}^n} F^p(\nabla u)dx \right)^{\frac{1}{p}},$$

(6.7)

where $T_{n,p}$ is the best constant in the classical Sobolev inequality due to Talenti [31].

It follows from (6.6) and (6.7) that

$$\text{Cap}_{F,1}(\Omega) \leq \frac{p(n-1)}{n-p} \left( \frac{\omega_{n-1}}{\kappa_{n-1}} \right)^{\frac{1}{n}} T_{n,p} \left( \int_{\mathbb{R}^n} F^p(\nabla u)dx \right)^{\frac{n-1}{p}}.$$ (6.8)

Taking the infimum over any $u \in C^\infty_c(\mathbb{R}^n)$ with $u \geq \chi_\Omega$ in (6.8), we obtain

$$\text{Cap}_{F,1}(\Omega) \leq \frac{p(n-1)}{n-p} \left( \frac{\omega_{n-1}}{\kappa_{n-1}} \right)^{\frac{1}{n}} T_{n,p} \left( \text{Cap}_{F,p}(\Omega) \right)^{\frac{n-1}{p}}.$$ (6.9)

Passing to the limit in (6.9) as $p \to 1^+$, since $\lim_{p \to 1^+} T_{n,p} = n^{1/2} \omega_{n-1}^{1/2}$, we conclude (6.5).

**Step 3.** It holds that

$$\limsup_{p \to 1^+} \text{Cap}_{F,p}(\Omega) \leq \inf_{1 \in \Omega, U \subset \mathbb{R}^n, \partial U \text{ is smooth}} |\partial U|_F.$$ (6.10)

Let $U \subset \mathbb{R}^n$ be any open and bounded set with smooth boundary such that $\Omega \subset U$. Let $d_U$ be the distance function

$$d_U(x) = \text{dist}(x, U).$$

It is well known that $d_U$ is smooth in a neighborhood of $U$. Introduce a smooth cut-off function $\chi_\varepsilon$ satisfying

$$\begin{cases} 
\chi_\varepsilon = 1 & \text{if } t < \varepsilon, \\
-\frac{1}{\varepsilon} < \chi_\varepsilon'(t) < 0 & \text{if } \varepsilon < t < 2\varepsilon, \\
\chi_\varepsilon = 0 & \text{if } t > 2\varepsilon,
\end{cases}$$

and set $\eta_\varepsilon = \chi_\varepsilon(d_U)$. Choose $\varepsilon$ small enough so that $\eta_\varepsilon$ is smooth. Hence

$$\text{Cap}_{F,p}(\Omega) \leq \int_{\mathbb{R}^n} F^p(\nabla \eta_\varepsilon)dx$$

for any $p \geq 1$. Letting $p \to 1^+$, and using the co-area formula and the mean value theorem, we get

$$\limsup_{p \to 1^+} \text{Cap}_{F,p}(\Omega) \leq \int_{\mathbb{R}^n} F(\nabla \eta_\varepsilon)dx = \int_{\varepsilon}^{2\varepsilon} |\chi_\varepsilon'(t)| \int_{\{d_U = t\}} F(\nu)d\sigma dt.$$

holds for every
\[ (6.12) \]
and (6.11) follows. Hence the equality in (6.13) holds.

**Proof of Corollary 1.4.** The first assertion follows directly from Theorem 1.3 and Propositions 6.4 and 6.5.

Next, we consider the equality case in (1.10) for some strictly outward \( F \)-minimising set \( \Omega \) with smooth and strictly \( F \)-mean-convex boundary. Let \( \{ \partial \Omega_t \}_{t \in [0,T)} \) be the evolution of \( \partial \Omega \) under the smooth inverse anisotropic mean curvature flow for some \( T > 0 \). By [15], we know that the weak inverse anisotropic mean curvature flow starting at \( \partial \Omega \) coincides with the smooth flow \( \{ \Omega_t \}_{t \in [0,T^*)} \) for some \( 0 < T^* \leq T \). In view of [15, Proposition 3.4] (see also [22, Lemma 2.4]), \( \Omega_t \) is strictly outward \( F \)-minimising and strictly \( F \)-mean-convex for every \( t \in [0,T^*) \). Then (1.10) holds for every \( \partial \Omega_t \) with \( t \in [0,T^*) \). For \( t \in [0,T^*) \), we can define a monotonic quantity as follows:

\[ \Psi(t) = |\partial \Omega_t|_F^{-\frac{n-2}{n}} \int_{\partial \Omega_t} H_F F(\nu) d\sigma. \]

By using [37, Proposition 2.1], we have

\[ \Psi'(0) = -|\partial \Omega|_F^{-\frac{n-2}{n}} \int_{\partial \Omega} \left( \frac{1}{H_F} \right) \left( h_F - \frac{H_F}{n-1} g_F \right)^2 d\sigma \leq 0. \]

(6.11)

Here,

\[ h_F - \frac{H_F}{n-1} g_F \]

is the traceless anisotropic second fundamental form. If the strict inequality \( \Psi'(0) < 0 \) holds, it follows that

\[ \Psi(t) < \Psi(0) = (n-1) \kappa_{n-1}^{-1/(n-1)} \quad \text{for some } t \in (0,T^*), \]

which contradicts (1.10) for some outward anisotropic minimising \( \Omega_t \) with strictly anisotropic mean-convex boundary. Therefore, \( \Psi'(0) = 0 \). It follows from (6.11) that \( \partial \Omega \) is totally umbilical in the anisotropic sense, and in turn, \( \partial \Omega \) is of a Wulff shape.

**Proof of Corollary 1.5.** For any bounded set \( U \subset \mathbb{R}^n \) with smooth boundary and \( \Omega \subset U \), by the Wulff inequality (1.4), we have

\[ \frac{|\partial U|_F}{\kappa_{n-1}} \geq \left( \frac{n|U|}{\kappa_{n-1}} \right)^{\frac{n-1}{n}} \geq \left( \frac{n|\Omega|}{\kappa_{n-1}} \right)^{\frac{n-1}{n}}. \]

Thus

\[ \frac{1}{\kappa_{n-1}} \inf_{\Omega \subset U} |\partial U|_F \geq \left( \frac{n|\Omega|}{\kappa_{n-1}} \right)^{\frac{n-1}{n}}. \]

(6.13)
The inequality follows from Theorem 1.3 and (6.13). The equality classification for (1.11) follows from that of the Wulff inequality.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11871406). The authors thank Dr. Mattia Fogagnolo for attracting their attention to the recent preprint [17] and explaining the new reformation of the strictly outward minimizing hull to them. The authors also thank Professors Virginia Agostiniani, Lorenzo Mazzieri and Deping Ye for their interest.

References

1. Agostiniani V, Fogagnolo M, Mazzieri L. Minkowski inequalities via nonlinear potential theory. arXiv:1906.00322v4, 2019
2. Agostiniani V, Fogagnolo M, Mazzieri L. Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. Invent Math, 2020, 222: 1033–1101
3. Agostiniani V, Mazzieri L. On the geometry of the level sets of bounded static potentials. Comm Math Phys, 2017, 355: 261–301
4. Agostiniani V, Mazzieri L. Monotonicity formulas in potential theory. Calc Var Partial Differential Equations, 2020, 59: 6
5. Akman M, Gong J, Hineman J, et al. The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity. arXiv:1709.00447v2, 2017
6. Akman M, Lewis J, Saari O, et al. The Brunn-Minkowski inequality and a Minkowski problem for $A$-harmonic Green’s function. Adv Calc Var, 2021, 14: 247–302
7. Alvino A, Ferone V, Trombetti G, et al. Convex symmetrization and applications. Ann Inst H Poincaré Anal Non Linéaire, 1997, 14: 275–293
8. Bianchini C, Ciracolo G. Wulff shape characterizations in overdetermined anisotropic elliptic problems. Comm Partial Differential Equations, 2018, 43: 790–820
9. Bianchini C, Ciracolo G, Salani P. An overdetermined problem for the anisotropic capacity. Calc Var Partial Differential Equations, 2016, 55: 84
10. Borghi S, Mazzieri L. On the mass of static metrics with positive cosmological constant: I. Classical Quantum Gravity, 2018, 35: 125001
11. Borghi S, Mazzieri L. On the mass of static metrics with positive cosmological constant: II. Comm Math Phys, 2020, 377: 2079–2158
12. Bray H, Miao P Z. On the capacity of surfaces in manifolds with nonnegative scalar curvature. Invent Math, 2008, 172: 459–475
13. Chang S Y A, Wang Y. Inequalities for quermassintegrals on $k$-convex domains. Adv Math, 2013, 248: 335–377
14. Cianchi A, Salani P. Overdetermined anisotropic elliptic problems. Math Ann, 2009, 345: 859–881
15. Della Pietra F, Gavitone N, Xia C. Motion of level sets by inverse anisotropic mean curvature. Comm Anal Geom, 2021, in press
16. Della Pietra F, Gavitone N, Xia C. Symmetrization with respect to mixed volumes. Adv Math, 2021, 388: 107887
17. Fogagnolo M, Mazzieri L. Minimising hulls, $p$-capacity and isoperimetric inequality on complete Riemannian manifolds. arXiv:2012.09490v1, 2020
18. Fogagnolo M, Mazzieri L, Pinamonti A. Geometric aspects of $p$-capacitary potentials. Ann Inst H Poincaré Anal Non Linéaire, 2019, 36: 1151–1179
19. Freire A, Schwartz F. Mass-capacity inequalities for conformally flat manifolds with boundary. Comm Partial Differential Equations, 2014, 39: 98–119
20. Guan P F, Li J F. The quermassintegral inequalities for $k$-convex starshaped domains. Adv Math, 2009, 221: 1725–1732
21. Huisken G. An isoperimetric concept for the mass in general relativity. Oberwolfach Rep, 2006, 3: 1898–1899
22. Huisken G, Ilmanen T. The inverse mean curvature flow and the Riemannian Penrose inequality. J Differential Geom, 2001, 59: 353–437
23. Kichenassamy S, Véron L. Singular solutions of the $p$-Laplace equation. Math Ann, 1986, 275: 599–615
24. Maggi F. Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory. Cambridge Studies in Advanced Mathematics, 135. Cambridge: Cambridge University Press, 2012
25. Maz’ya V. Sobolev Spaces: With Applications to Elliptic Partial Differential Equations, 2nd ed. Grundlehren der mathematischen Wissenschaften, vol. 342. Heidelberg: Springer, 2011
26. Qiu G. A family of higher-order isoperimetric inequalities. Commun Contemp Math, 2015, 17: 1450015
27. Reilly R C. On the Hessian of a function and the curvatures of its graph. Michigan Math J, 1973, 20: 373–383
28. Schmidt T. Strict interior approximation of sets of finite perimeter and functions of bounded variation. Proc Amer Math Soc, 2015, 143: 2069–2084
29 Schneider R. Convex Bodies: The Brunn-Minkowski Theory. Cambridge: Cambridge University Press, 2013
30 Sternberg P, Ziemer W P, Williams G. $C^{1,1}$-regularity of constrained area minimizing hypersurfaces. J Differential Equations, 1991, 94: 83–94
31 Talenti G. Best constant in Sobolev inequality. Ann Mat Pura Appl (4), 1976, 110: 353–372
32 Trudinger N S. Isoperimetric inequalities for quermassintegrals. Ann Inst H Poincaré Anal Non Linéaire, 1994, 11: 411–425
33 Wang G F, Xia C. A characterization of the Wulff shape by an overdetermined anisotropic PDE. Arch Ration Mech Anal, 2011, 199: 99–115
34 Wang G F, Xia C. A Brunn-Minkowski inequality for a Finsler-Laplacian. Analysis Berlin, 2011, 31: 103–115
35 Wang G F, Xia C. Blow-up analysis of a Finsler-Liouville equation in two dimensions. J Differential Equations, 2012, 252: 1668–1700
36 Wang G F, Xia C. An optimal anisotropic Poincaré inequality for convex domains. Pacific J Math, 2012, 258: 305–325
37 Xia C. Inverse anisotropic mean curvature flow and a Minkowski type inequality. Adv Math, 2017, 315: 102–129
38 Xiao J. The $p$-harmonic capacity of an asymptotically flat 3-manifold with non-negative scalar curvature. Ann Henri Poincaré, 2016, 17: 2265–2283
39 Xiao J. $P$-capacity vs surface area. Adv Math, 2017, 308: 1318–1336