Optimal Input Placement in Lattice Graphs *

Isaac Klickstein a, Francesco Sorrentino a

aDepartment of Mechanical Engineering, 1 University of New Mexico, Albuquerque, NM 87131

Abstract

The control of dynamical, networked systems continues to receive much attention across the engineering and scientific research fields. Of particular interest is the proper way to determine which nodes of the network should receive external control inputs in order to effectively and efficiently control portions of the network. Published methods to accomplish this task either find a minimal set of driver nodes to guarantee controllability or a larger set of driver nodes which optimizes some control metric. Here, we investigate the control of lattice systems which provides analytical insight into the relationship between network structure and controllability. First we derive a closed form expression for the individual elements of the controllability Gramian of infinite lattice systems. Second, we focus on nearest neighbor lattices for which the distance between nodes appears in the expression for the controllability Gramian. We show that common control energy metrics scale exponentially with respect to the maximum distance between a driver node and a target node.

Key words: Networks; Linear optimal control; Lattices; Lyapunov equation; Discrete Fourier transforms.

1 Introduction

The control of dynamical networks continues to receive much attention throughout the engineering literature [10,4,32,7]. Typically, the problems explored consist of either finding a minimal set of driver nodes (structural controllability [8,11], exact controllability [29], among others [3,15,30]) or determining the particular control inputs (minimum energy [25,26], LQR [9,7], proportional [28,20], and others). The minimum energy control strategy is often investigated as it lower bounds the $L^2$ norm of any other control strategy that performs the same control action (initial condition to final condition). The minimum energy control strategy can be characterized by the controllability Gramian, a symmetric positive semi-definite matrix that is a function of the weighted adjacency matrix of the underlying network and the distribution of inputs into the system.

It has been shown that the minimal set of driver nodes which ensures controllability, may not be numerically feasible [19], that is, the resulting control input requires the inversion of a very ill-conditioned matrix. To compensate, additional driver nodes may be added to the minimal set either randomly [25,26,7] or according to some heuristic [2]. Alternatively, it has been shown that optimizing the placement of driver nodes with respect to a variety of submodular control energy metrics is NP-hard [14,22] which inspired the development of greedy approximation algorithms [17,16,21,22]. These algorithms require many high accuracy computations of controllability Gramians along with their inverse or determinant to make the necessary decisions in the greedy algorithm which is computationally expensive even when the graph is sparse [1].

Our first contribution is a closed form expression for each entry of the controllability Gramian for arbitrary infinite lattice networks. This framework allows one to investigate the role that various connectivity patterns have on the control energy.

Our second contribution finds the exponential decay of an entry in the Gramian with respect to the distance between the target nodes and the driver nodes in a nearest neighbor lattice graph. Using this decay rate, and the Cauchy interlacing theorem, these control metrics in the nearest neighbor lattice graph are lower bounded by a function of the distance between a driver node and the target node furthest away. This exponential scaling we derive has previously been observed numerically [2,24] for the finite chain graph where the authors demonstrated the exponential scaling holds for graphs of a general topology. Taken together, driver node placement algorithms that minimize the distance between driver nodes and target nodes by using a discrete location formulation [12] may be a computationally cheaper alternative to the currently available greedy approximation algorithms to choose a driver node set.
The remainder of the paper is as follows. In section 2 we present the derivation of both the time-varying and the steady state controllability Gramian of a general lattice network. In section 3 we specialize the results in section 2 to nearest neighbor lattices. In section 4 we discuss the implications of our methodology to developing driver node placement algorithms.

2 Preliminaries

We first define a lattice graph in general, then write the linear dynamics that governs the states of each node. With the given definitions, we then derive exact formulas for the elements of the controllability Gramian, which contains information about the minimum control energy, or effort, required to perform any required control action.

2.1 Lattice Graphs

Here we define lattice graphs and introduce some notational simplifications which we use in the following derivations. Note that while typically in the graph theory literature nodes of a graph are labeled with a single positive integer, here we define node labels as vectors of integers.

Definition 1 (Lattice Graphs) A lattice graph \( L = (V, E) \) consists of an infinite set of nodes \( V = \{v_i | i \in \mathbb{Z}^d \} \) where \( i \) is node \( v_i \) ’s index and a set of edges \( E \subset V \times V \). The lattice graph can be completely described by three properties.

1. The dimension of the lattice \( d \) is the dimension of the indices of the nodes \( i \in \mathbb{Z}^d \), called lattice sites. There is a node \( v_i \) at every lattice site.
2. The coupling of the lattice is described by a set \( N \subset \mathbb{Z}^d \) such that,

\[
(v_{i+n}, v_i) \in E, \ \forall n \in N, \ \forall i \in \mathbb{Z}^d \tag{1}
\]

and \( (v_{i+n}, v_i) \notin E \) for \( n \notin N, \ \forall i \in \mathbb{Z}^d \). The number of incoming edges of every node \( v_i \in V \) is \(|N|\). Note that a lattice in general is directed.
3. There is a function \( \psi : N \mapsto \mathbb{R} \) that describes the edge weights. This function implies that the weight of edge \( (v_{i+n}, v_i) \) is equal to the weight of edge \( (v_{j+n}, v_j) \) for all \( i, j \in \mathbb{Z}^d \), i.e., it is independent of the node index. The weights may be positive or negative.

Two examples of lattices are shown in Fig. 1. The first example in Fig. 1(A) is a one-dimensional nearest neighbor lattice. Here, each node \( v_i \) has \(|N| = 3 \) neighbors; \( v_{i-1}, v_{i+1}, \) and the node itself, \( v_i \) for every \( i \in \mathbb{Z} \). The second example in Fig. 1(B) is a two-dimensional lattice with directed edges listed in the caption. We emphasize that the remaining results in this section are applicable to any lattice of arbitrary dimension and connectivity pattern that can be described by Definition 1.

We also find it useful to define some additional operations on integer vectors in order to maintain clarity in some of the following equations.

Definition 2 (Operations on Integer Tuples)

- Let the product of two indices be defined elementwise, that is, the \( k \)th element of the product \( (ij)_{k} = ik_{j} \).
- Let the Dirac delta function between two indices be defined as \( \delta_{ij} = \prod_{k=1}^{d} \delta_{ik_{j}} \) where \( \delta_{ik_{j}} = 1 \) if \( i_{k} = j_{k} \) and \( \delta_{ik_{j}} = 0 \) otherwise.
- Let the exponential of an index be \( e_{ai} = \prod_{k=1}^{d} e_{aik} \) where \( a \) is a complex coefficient.
- Let the integral with respect to an integer vector be defined as,

\[
\int_{a}^{b} f(i) di = \int_{a}^{b} \cdots \int_{a}^{b} f(i) di_1 \ldots di_d \tag{2}
\]

2.2 Linear Dynamics

Each node \( v_i \) in our lattice \( L \) is endowed with a time-varying state \( x_i(t) \in \mathbb{R} \). These time-varying states may represent position or velocity for formation-type problems [27], or transfer rates in routing problems [31], or average excitement in neuronal networks [23], and many other network applications. Before presenting the dynamical system that describes the time evolution of the states of each node, we will introduce two finite subsets of the nodes in the lattice.

Definition 3 (Driver Nodes) Let \( D \subset V \) be the set of driver nodes of cardinality \(|D| = n_d \). If node \( v_i \in D \), then node \( v_i \) receives an external control input \( u_i(t) \) which we...
are free to define. We assume that no control input can be connected to more than a single node.

**Definition 4 (Target Nodes)** Let \( T \subset \mathcal{V} \) be the set of target nodes of cardinality \( |T| = n_t \). If \( v_t \in T \), then there is a desired value for the state \( x_i(t_f) = x_{i,f} \) which is chosen before applying the control action.

**Remark 5** Note that we will assume both \( D \) and \( T \) are finite subsets of \( \mathcal{V} \) and they may overlap, that is, it is possible for a node \( v_i \in D \cup T \).

The linear differential equation that governs the time evolution of the each state is,

\[
\dot{x}_i(t) = \sum_{n \in \mathcal{N}} \psi(n)x_{i+n}(t) + \sum_{v_a \in \mathcal{D}} \delta_{i,a}u_a(t). \tag{3}
\]

While here we consider scalar dynamics (\( x_i(t) \in \mathcal{R} \) and \( \psi(n) \in \mathcal{R} \)), the extension to multi-dimensional states, \( x_i(t) \in \mathcal{R}^n_s \) and \( \psi(n) \in \mathcal{R}^n_s \times \mathcal{R}_s \) is straightforward. Note that the time evolution of the states of each node \( v_i \) are a function of its immediate neighbors \( v_{i+n} \) for all \( n \in \mathcal{N} \) and possibly a driver node if \( v_i \in D \). Each node has an initial condition \( x_i(0) = x_{i,0} \) for all \( v_i \in \mathcal{V} \), and the target nodes have a final condition \( x_i(t_f) = x_{i,f} \) for all \( v_i \in T \). We would like to achieve this final condition with minimal control energy.

\[
\min J = \frac{1}{2} \int_0^{t_f} \sum_{v_a \in \mathcal{D}} u_a^2(\tau)d\tau
\]

s.t.

\[
\dot{x}_i(t) = \sum_{n \in \mathcal{N}} \psi(n)x_{i+n}(t) + \sum_{v_a \in \mathcal{D}} \delta_{i,a}u_a(t), \quad v_i \in \mathcal{V}
\]

\[
x_i(0) = x_{i,0}, \quad v_i \in \mathcal{V}, \quad x_i(t_f) = x_{i,f}, \quad v_i \in T \tag{4}
\]

The optimal cost of this optimal control problem can be written \([7]\),

\[
J^* = \frac{1}{2} h^T \hat{W}^{-1} b \tag{5}
\]

The vector \( b \) has entries representing the difference between the prescribed final condition \( x_{i,f} \) and what the state would be without any external control input. The matrix \( \hat{W} \) is the output controllability Gramian \([7]\) which is a principal submatrix of the controllability Gramian corresponding to the indices of the target nodes. The elements of the controllability Gramian obey the Lyapunov equation,

\[
\hat{W}_{ij}(t) = \sum_{n \in \mathcal{N}} \psi(n)\hat{W}_{i+n,j}(t) + \sum_{n \in \mathcal{N}} \psi(n)\hat{W}_{i,j+n}(t)
+\sum_{v_a \in \mathcal{D}} \delta_{i,a}\delta_{j,a}. \tag{6}
\]

The solution of Eq. (6) for systems described by lattice graphs is derived by applying the 2d-dimension discrete time Fourier transform.

**Theorem 6** Let \( \mathcal{I} = \sqrt{-1} \), the unit complex value, and let,

\[
\phi(k) = \sum_{n \in \mathcal{N}} \psi(n)e^{-\mathcal{I}nk} \tag{7}
\]

where we call \( \phi(\cdot) : \mathbb{Z}^d \rightarrow \mathcal{C} \) the lattice function for lattice \( \mathcal{L} \). The time evolution of the controllability Gramian entries for the infinite lattice is,

\[
\hat{W}_{ij}(t) = \frac{1}{(2\pi)^{2d}} \sum_{v_a \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}(i-a)i}e^{-\mathcal{I}(j-a)j} \times \theta(t, i, j)d\hat{d} \hat{j} \tag{8}
\]

where the time varying portion,

\[
\theta(t, i, j) = \exp \left[ \frac{\left( \phi(i) + \phi(j) \right)t}{\phi(i) + \phi(j)} - 1 \right] \tag{9}
\]

**Proof.** First, we state two facts:

- Let \( a \) be a complex number. Then \( a \int_0^t e^{at}dt = e^{at} - 1 \).
- Let \( m \) be an integer. An important integral that appears is \( \int_{-\pi}^{\pi} e^{-\mathcal{I}ms}dx = 2\pi \delta_{m,0} \)

From Eq. (8), it is simple to show that,

\[
\sum_{n \in \mathcal{N}} \psi(n)W_{i+n,j} = \frac{1}{(2\pi)^{2d}} \sum_{v_a \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi(i)
\times e^{-\mathcal{I}(i-a)i}e^{-\mathcal{I}(j-a)j}\theta(t, i, j)d\hat{d} \hat{j} \tag{10}
\]

Applying Eq. (10) and the two facts stated above to Eq. (6), we can complete the proof,

\[
\hat{W}_{ij}(t) = \frac{1}{(2\pi)^{2d}} \sum_{v_a \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}(i-a)i}e^{-\mathcal{I}(j-a)j} \times \left( e^{(\phi(i)+\phi(j))t} - 1 \right) d\hat{d} \hat{j} + \sum_{v_a \in \mathcal{D}} \delta_{i,a}\delta_{j,a}
= \hat{W}_{ij}(t) - \frac{1}{(2\pi)^{2d}} \sum_{v_a \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}(i-a)i} \times e^{-\mathcal{I}(j-a)j}d\hat{d} \hat{j} + \sum_{v_a \in \mathcal{D}} \delta_{i,a}\delta_{j,a}
= \hat{W}_{ij}(t) - \sum_{v_a \in \mathcal{D}} \prod_{k=1}^{d} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}(j-k)j}d\hat{j} \right] \times \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}(k-a)k}d\hat{k} \right] + \sum_{v_a \in \mathcal{D}} \delta_{i,a}\delta_{j,a}
= \hat{W}_{ij}(t) - \sum_{v_a \in \mathcal{D}} \delta_{i,a}\delta_{j,a} = \hat{W}_{ij}(t) \tag{11}
\]
Remark 7 Of particular interest to us is the case when \( \phi(k) < 0 \) because then there exists a steady state solution to Eq. (6).

\[
W_{ij} = \lim_{t \to \infty} W_{ij}(t) = -\frac{1}{(2\pi)^{2d}} \sum_{v \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-i(\omega-\alpha)} \phi(\hat{i}) + \phi(\hat{j})}{\phi(\hat{i}) + \phi(\hat{j})} d\hat{i}d\hat{j}.
\]  

(12)

When the dynamical system is stable, the minimum energy expression in Eq. (5) approaches a constant value. Note that \( \phi(k) < 0 \) implies \( 0 \in N^* \) and \( \psi(0) < \sum_{n \in N, n \neq 0} \psi(n) \), i.e., there exists a large enough, and negative, self-loop at each node.

From Eq. (6), we know \( W_{ij}(t) \) is a real number and so the complex portion of both Eq. (8) and (12) must be zero. With this in mind, Eq. (8) can be rewritten as,

\[
W_{ij}(t) = \frac{1}{(2\pi)^{2d}} \sum_{v \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[ \frac{(\alpha_n + \beta_n\omega)(r(t) - 1)}{\sigma^2 + \omega^2} \right] \delta\hat{i}\delta\hat{j},
\]  

(13)

where the functions are,

\[
\begin{align*}
\sigma &= \sum_{n \in N^*} \psi(n) \left[ \cos \left( \sum_{k=1}^{d} n_k \hat{i}_k \right) + \cos \left( \sum_{k=1}^{d} n_k \hat{j}_k \right) \right], \\
\omega &= \sum_{n \in N^*} \psi(n) \left[ \sin \left( \sum_{k=1}^{d} n_k \hat{i}_k \right) + \sin \left( \sum_{k=1}^{d} n_k \hat{j}_k \right) \right], \\
\alpha_n &= \cos \left( \sum_{k=1}^{d} (i_k - a_k) \hat{i}_k + (j_k - a_k) \hat{j}_k \right), \\
\beta_n &= \sin \left( \sum_{k=1}^{d} (i_k - a_k) \hat{i}_k + (j_k - a_k) \hat{j}_k \right).
\end{align*}
\]

(14)

Similarly, Eq. (12) can be rewritten as,

\[
W_{ij} = -\frac{1}{(2\pi)^{2d}} \sum_{v \in \mathcal{D}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\alpha_n + \beta_n\omega}{\sigma^2 + \omega^2} d\hat{i}d\hat{j}.
\]  

(15)

The expressions in Eqs. (13) and (15) lend themselves to multi-dimensional numerical integration as the imaginary portion has been removed.

3 Nearest Neighbor Lattice

The one-dimensional \((d = 1)\), nearest neighbor \(\{0, 1, -1\}\) lattice (NNOD lattice) has the property that the shortest path between nodes \(v_i\) and \(v_j\) is \(|i - j|\). An example of this lattice is shown in Fig. 1(A). Let the weight be \(\psi(1) = \psi(-1) = s > 0\) and \(\psi(0) = p < -2s\). The lattice function of the NNOD lattice is \(\phi(i) = p + 2s \cos i < 0\). As the Gramian entries are linear with respect to the contribution of each of the driver nodes, we examine each driver node separately so we assume \(n_d = 1\). We shift the indices \(i' = i - a\) and \(j' = j - a\), to place the driver node at \(a' = 0\). With this offset in mind, let \(G_{i',j'} = W_{i,j}\) represent the shifted Gramian entries. Equation (15) for the NNOD lattice can be written as,

\[
G_{i',j'} = -\frac{1}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \frac{\cos(i'\hat{i}) \cos(j'\hat{j})}{2p + 2s \cos i + 2s \cos j} d\hat{i}d\hat{j}.
\]  

(16)

An analogous version of the double integral in Eq. (16) has been derived separately in the context of solving the discretized Hehnholz equation [6,13].

Lemma 8 (Recursion for Diagonal Values [13]) Let \(\alpha = \frac{p^2}{(2s)^2} - 1\), which, if the system is stable, corresponds to a value of \(\alpha > 1\). The diagonal values, \(G_{i',i'}\), can be found from the recursion relation,

\[
G_{i'+1,i'+1} = \frac{4i'}{2i' + 1} \alpha G_{i',i'} - \frac{2i' - 1}{2i' + 1} G_{i'-1,i'-1}
\]  

(17)

with initial values,

\[
\begin{align*}
G_{0,0} &= \frac{1}{\pi|p|} K \left( \frac{4s^2}{p^2} \right), \\
G_{1,1} &= \left( \frac{|p|}{2\pi s^2} - \frac{1}{\pi|p|} \right) K \left( \frac{4s^2}{p^2} \right) - \frac{|p|}{2\pi s^2} E \left( \frac{4s^2}{p^2} \right)
\end{align*}
\]

(18)

where \(K(\cdot)\) and \(E(\cdot)\) are the first and second complete elliptic integrals, respectively.

Corollary 9 Let \(z_{i'} = G_{i',i'}/G_{i'-1,i'-1}\) for \(i' > 0\) represent the instantaneous rate of decay along the diagonal. The recursion in Eq. (17) can be rewritten in terms of \(z_{i'}\).

\[
z_{i'+1}z_{i'} = \frac{4i'}{2i' + 1} \alpha z_{i'} - \frac{2i' - 1}{2i' + 1}
\]  

(19)

For large \(i'\), this yields the approximate solution,

\[
\tilde{z} = \alpha - \sqrt{\alpha^2 - 1},
\]  

(20)

so that the asymptotic rate of decay of the diagonal elements is exponential,

\[
G_{i',i'} \sim (\tilde{z})^{i'}.
\]  

(21)

In Fig. 2 we show that for both large and small values of \(\alpha\), Eq. (21) does a satisfactory job approximating \(G_{i',i'}\) even for small values of \(i'\).
3.1 Control Metrics

Two particularly useful control metrics are the trace of the inverse of the output controllability Gramian and the logarithm of the determinant of the output Gramian [17]. We assume the system is output controllable, so that \( W \) is positive definite and order the eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). As we will see next, with knowledge of the diagonal elements, we can lower bound the control metrics using the Cauchy interlacing theorem [5].

**Theorem 10 (Cauchy Interlacing Theorem [5])**

Let \( X \) and \( Y \) be two symmetric matrices of size \( n \) and \( m \) respectively, \( n > m \), such that \( Y \) is a principal submatrix of \( X \). Order their eigenvalues such that \( \lambda_i(X) \leq \lambda_{i+1}(X) \) and \( \lambda_i(Y) \leq \lambda_{i+1}(Y) \). Then, each eigenvalue of \( X \) can be bounded by,

\[
\lambda_i(Y) \leq \lambda_i(X) \leq \lambda_{n-m+i}(Y), \quad i = 1, \ldots, m
\]

Also helpful is the Gerschgorin disc theorem [5] as it provides a way to compute an upper bound to the largest eigenvalue of \( W \) for any given choice of the driver nodes.

**Theorem 11 (Gerschgorin Disc Theorem [5])**

Let \( A = \{ A_{i,j} \} \) be an \( n \times n \) square matrix. The eigenvalues of \( A \), \( \lambda(A) \), lie in the union of the \( n \) discs \( D_i \), \( i = 1, \ldots, n \), in the complex plane, each centered at \( A_{i,i} \), with radius \( C_i = \sum_{j=1,j\neq i}^{n} A_{i,j} \). The largest eigenvalue of the matrix \( A \) can thus be upper bounded by,

\[
\lambda_{\text{max}}(A) \leq \max_{1<i<n} \sum_{j=1}^{n} A_{i,j}
\]

**Corollary 12** It is not difficult to show from Eqs. (16) and (18) that \( G_{0,0} > G_{i',j'} \) for \( |i'| + |j'| > 0 \). Then, the largest eigenvalue of \( W \) for any choice of the driver node, \( v_a \), can be upper bounded by,

\[
\lambda_{\text{max}}(W) \leq \max_{v_i \in T} \sum_{v_j \in T} G_{i',j'} \leq n_t G_{0,0}
\]

where we use the shift \( i' = i - a \) and \( j' = j - a \).

3.1.1 Inverse of the Trace

Let \( B = \{ \mathbf{y} \in \mathbb{R}^n : ||\mathbf{y}|| = 1 \} \) be the \( n_t \)-dimensional ball centered at the origin. The average minimum energy required to reach a point \( \mathbf{b} \), located on the unit hypersphere, from the origin, is found by integrating over the \( B \) [18,17].

\[
\int_B \mathbf{y}^T \bar{W}^{-1} \mathbf{y} \, d\mathbf{y} = \frac{1}{n_t} \text{Tr}(\bar{W}^{-1})
\]

Let \( n_t > 1 \) and \( v_{i'} \in T \) such that \( \ell' = \max_{v_{i'} \in T} \), i.e., the index of the node furthest from the single driver node. Using the Cauchy interlacing theorem which states that \( \lambda_1(\bar{W}) < G_{i',i'} \), we can lower bound the trace of the inverse of the output controllability Gramian.

\[
\text{Tr}(\bar{W}^{-1}) \geq \frac{1}{\lambda_1(\bar{W})} \geq \frac{1}{G_{i',i'}} \sim \tilde{z}^{-\ell'}
\]

From the above expression we see that, as \( 0 < \tilde{z} < 1 \), by reducing \( \ell' \), we can reduce the lower bound. An example of the lower bound in Eq. (26) is shown in Fig. 3(A) for a NNOD lattice with edge weights \( p = 3 \) and \( s = 1 \). Namely, the figure shows that by reducing \( \ell' \) one can exponentially decrease the trace of the inverse output Controllability Gramian. This indicates that a convenient choice of an alternate driver node is the one that reduces the distance to the furthest target node.
3.1.2 Volume of the Controllability Ellipsoid

The controllability ellipsoid is defined as the set of vectors $y$ such that,

$$ S = \{ y \in \mathbb{R}^n | y^T \bar{W}^{-1} y \leq 1 \} \quad (27) $$

The volume of $S$ [17] is

$$ V(\bar{W}) = \frac{\pi^{n_t/2}}{\Gamma(n_t/2 + 1)} (\det \bar{W})^{1/n_t} \quad (28) $$

Note that $c V(\bar{W}) = V(c \bar{W})$, where, from Gershgorin theorem we see that by setting $c = \frac{1}{\max_{i,j} |c_{ij}|}$, $\lambda_{n_t} (c \bar{W}) < 1$.

Minimizing $-\log \det(c \bar{W})$ can be interpreted as maximizing the volume of the reachable subspace for any value $J^*$.

$$ -\log \det(c \bar{W}) \geq -\log(\lambda_1(c \bar{W})) \geq -\log(c \Gamma_{G_V,c}) \sim -\log(\bar{z}) \ell' \quad (29) $$

As $0 < \bar{z} < 1$, $-\log(\bar{z}) > 0$, and so the lower bound of $-\log \deg(c \bar{W})$ scales with $\ell'$. An example of the lower bound in Eq. (29) is shown in Fig. 3(B). Namely, the figure shows that the bound holds tightly for $n_t = 1$ but becomes more conservative as the number of targets increases. By selecting a driver node such that $\ell'$ is minimized the log $\det(\bar{W})$ may exponentially decrease the minimum control energy.

4 Discussion and Conclusion

We have derived the exact expressions of the controllability Gramian for networked systems where the underlying topology is a lattice. We have specialized the results to NNOD lattices so that the length of the shortest path between two nodes appears in the expression for the Gramian entries. Our analytical expressions are in agreement with previously reported observations [2,24], that control energy metrics scale exponentially with respect to distance metrics for finite graphs with general topology. Current research into driver node selection methods require the computation of many controllability Gramians (and their log determinant or inverse trace) which becomes prohibitively expensive for large networks in terms of both computation and storage. Instead, the positive correlation between distance metrics with respect to the driver nodes and target nodes and the energy metrics discussed suggests that one may construct a heuristic driver node selection method based purely on the topology of the underlying graph. An exhaustive numerical comparison between the greedy approximation algorithms [17,22] and heuristic algorithms that minimize a distance metric between the sets of driver nodes and target nodes is forthcoming in a future publication.

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References

[1] Peter Benner and Jens Saak. Numerical solution of large and sparse continuous time algebraic riccati and lyapunov equations: a state of the art survey. GAMM-Mitteilungen, 36(1):32–52, 2013.
[2] Yu-Zhong Chen, Le-Zhi Wang, Wen-Xu Wang, and Ying-Cheng Lai. Energy scaling and reduction in controlling complex networks. Royal Society open science, 3(4):160064, 2016.
[3] Noah J. Cowan, Erick J. Chastain, Daril A. Vilhena, James S. Freudenberg, and Carl T. Bergstrom. Nodal dynamics, not degree distributions, determine the structural controllability of complex networks. PloS one, 7(6):e38398, 2012.
[4] Jianxi Gao, Yang-Yu Liu, Raissa M. D’souza, and Albert-László Barabási. Target control of complex networks. Nature communications, 5:5415, 2014.
[5] Gene H. Golub and Charles F. Van Loan. Matrix computations, volume 3. JHU Press, 2012.
[6] Shigetoshi Katsura, Tohru Morita, Sakari Inawashiro, Tsuyoshi Horiguchi, and Yoshihiko Abe. Lattice green’s function. introduction. Journal of Mathematical Physics, 12(5):892–895, 1971.
[7] Isaac Klickstein, Afroza Shirin, and Francesco Sorrentino. Energy scaling of targeted optimal control of complex networks. Nature communications, 8:15415, 2017.
[8] Chung-Ta Lin. Structural controllability. IEEE Transactions on Automatic Control, 19(3):201–208, 1974.
[9] Fu Lin, Makan Fardad, and Mihailo R. Jovanovic. Augmented lagrangian approach to design of structured optimal state feedback gains. IEEE Transactions on Automatic Control, 56(12):2923–2929, 2011.
[10] Yang-Yu Liu and Albert-László Barabási. Control principles of complex systems. Reviews of Modern Physics, 88(3):035006, 2016.
[11] Yang-Yu Liu, Jean-Jacques Slotine, and Albert-László Barabási. Controllability of complex networks. Nature, 473(7346):167, 2011.
[12] Pitu B. Mirchandani and Richard L. Francis. Discrete location theory. John Wiley & Sons, Inc, 1990.
[13] Tohru Morita. Useful procedure for computing the lattice green’s function-square, tetragonal, and bcc lattices. Journal of mathematical physics, 12(8):1744–1747, 1971.
[14] Alex Oshevsky. Minimal controllability problems. IEEE Transactions on Control of Network Systems, 1(3):249–258, 2014.
[15] Sérgio Pequito, Guilherme Ramos, Souymya Kar, A. Pedro Aguiar, and Jaime Ramos. The robust minimal controllability problem. Automatica, 82:261–268, 2017.
[16] Tyler Summers. Actuator placement in networks using optimal control performance metrics. In Decision and Control (CDC), 2016 IEEE 55th Conference on, pages 2703–2708. IEEE, 2016.
[17] Tyler H. Summers, Fabrizio L. Cortesi, and John Lygeros. On submodularity and controllability in complex dynamical networks. *IEEE Transactions on Control of Network Systems*, 3(1):91–101, 2016.

[18] Tyler H. Summers and John Lygeros. Optimal sensor and actuator placement in complex dynamical networks. *IFAC Proceedings Volumes*, 47(3):3784–3789, 2014.

[19] Jie Sun and Adilson E. Motter. Controllability transition and nonlocality in network control. *Physical review letters*, 110(20):208701, 2013.

[20] Yang Tang, Feng Qian, Huijun Gao, and Jürgen Kurths. Synchronization in complex networks and its application—a survey of recent advances and challenges. *Annual Reviews in Control*, 38(2):184–198, 2014.

[21] Vasileios Tzoumas, Mohammad Amin Rahimian, George J. Pappas, and Ali Jadbabaie. Minimal actuator placement with optimal control constraints. In *American Control Conference (ACC)*, 2015, pages 2081–2086. IEEE, 2015.

[22] Vasileios Tzoumas, Mohammad Amin Rahimian, George J. Pappas, and Ali Jadbabaie. Minimal actuator placement with bounds on control effort. *IEEE Transactions on Control of Network Systems*, 3(1):67–78, 2016.

[23] Jin-Liang Wang, Huai-Ning Wu, and Tingwen Huang. Passivity-based synchronization of a class of complex dynamical networks with time-varying delay. *Automatica*, 56:105–112, 2015.

[24] Le-Zhi Wang, Yu-Zhong Chen, Wen-Xu Wang, and Ying-Cheng Lai. Physical controllability of complex networks. *Scientific reports*, 7:40198, 2017.

[25] Gang Yan, Jie Ren, Ying-Cheng Lai, Choy-Heng Lai, and Baowen Li. Controlling complex networks: How much energy is needed? *Physical review letters*, 108(21):218703, 2012.

[26] Gang Yan, Georgios Tsitsiklis, Baruch Barzel, Jean-Jacques Slotine, Yang-Yu Liu, and Albert-László Barabási. Spectrum of controlling and observing complex networks. *Nature Physics*, 11(9):779, 2015.

[27] Wenwu Yu, Guanrong Chen, and Ming Cao. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 46(6):1089–1095, 2010.

[28] Wenwu Yu, Guanrong Chen, and Jinhu Lü. On pinning synchronization of complex dynamical networks. *Automatica*, 45(2):429–435, 2009.

[29] Zhengzhong Yuan, Chen Zhao, Zengru Di, Wen-Xu Wang, and Ying-Cheng Lai. Exact controllability of complex networks. *Nature communications*, 4:2447, 2013.

[30] Xizhe Zhang, Huaizhen Wang, and Tianyang Lv. Efficient target control of complex networks based on preferential matching. *PloS one*, 12(4):e0175375, 2017.

[31] Liang Zhao, Ying-Cheng Lai, Kwangho Park, and Nong Ye. Onset of traffic congestion in complex networks. *Physical Review E*, 71(2):026125, 2005.

[32] Tong Zhou. On the controllability and observability of networked dynamic systems. *Automatica*, 52:63–75, 2015.