INTEGRATION WITH RESPECT TO THE HAAR MEASURE ON UNITARY, ORTHOGONAL AND SYMPLECTIC GROUP

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ABSTRACT. We revisit the work of the first named author and using simpler algebraic arguments we calculate integrals of polynomial functions with respect to the Haar measure on the unitary group \( U(d) \). The previous result provided exact formulas only for \( 2d \) bigger than the degree of the integrated polynomial and we show that these formulas remain valid for all values of \( d \). Also, we consider the integrals of polynomial functions on the orthogonal group \( O(d) \) and the symplectic group \( Sp(d) \). We obtain an exact character expansion and the asymptotic behavior for large \( d \). Thus we can show the asymptotic freeness of Haar-distributed orthogonal and symplectic random matrices, as well as the convergence of integrals of the Itzykson–Zuber type.

1. Introduction

Let \( G \subset \text{End}(\mathbb{C}^d) \) be a compact Lie group viewed as a group of matrices. The matrix structure provides a very natural coordinate system on \( G \); in particular we are interested in the family of functions \( e_{ij} : G \to \mathbb{C} \) defined by \( e_{ij} : M_d(\mathbb{C}) \ni m \mapsto m_{ij} \) which to a matrix assign one of its entries. We call polynomials in \((e_{ij})\) polynomial functions on \( G \). In this article we are interested in the integrals of polynomial functions in \((e_{ij}, e_{ij}^*)\) on compact Lie groups with respect to the Haar measure on \( G \), i.e. the integrals of the form

\[
\int_G \mathcal{U}_{i_1j_1} \cdots \mathcal{U}_{i_nj_n} \overline{\mathcal{U}}_{i'_1j'_1} \cdots \overline{\mathcal{U}}_{i'_n,j'_n} \, d\mathcal{U}.
\]

For simplicity, such integrals will be called moments of the group \( G \).

If we consider a matrix-valued random variable \( \mathcal{U} \) the distribution of which is the Haar measure on \( G \) then the integrals of the form (1) have a natural interpretation as certain moments of entries of \( \mathcal{U} \) and they appear very naturally in the random matrix theory. The reason for this is that quite many random matrix ensembles \( X \) are invariant with respect to the conjugation by elements of the group \( G \) and therefore can be written as \( X = \mathcal{U}X'\mathcal{U}^{-1} \),

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where $U$ and $X'$ are independent matrix-valued random variables and the distribution of $U$ is the Haar measure on $G$. As a result, the expressions similar to

\[(2) \quad E \text{Tr}(X_1 U^{s_1} X_2 U^{s_2} \cdots X_n U^{s_n})\]

are quite common in the random matrix theory, where $s_1, \ldots, s_n \in \{1, \star\}$ and $X_1, \ldots, X_n$ are some matrix-valued random matrices independent from $U$. It is easy to see that the calculation of (2) can be easily reduced to the calculation of (1). In the random matrix theory we are quite often interested not in the exact value of the expression of type (2) but in its asymptotic behavior if $d$ tends to infinity. The results of this type were obtained for the first time by Weingarten [Wei78].

In this article we are interested in the case when $G \subset M_d(\mathbb{C})$ belongs to one of the series of the classical Lie groups, i.e. $G$ is either the unitary group $U(d)$ or the orthogonal group $O(d)$ or the symplectic group $Sp(d/2)$, where in the latter case we assume that $d$ is even. Firstly, we revisit a part of the work of the first named author [Col03] and compute with a new convolution formula the moments of the unitary group. This formula gives a new combinatorial insight into the relation between free probability and asymptotics of moments of the unitary group. Then, we make use of other features of invariant theory to give an explicit integration formula on the orthogonal and symplectic groups and to compute asymptotics in the latter case. This allows us to prove a new convergence result for a large family of matrix integrals. Our main tool is the Schur–Weyl duality for the unitary group and its analogues for the orthogonal and symplectic groups.

2. Integration over unitary groups

2.1. Schur–Weyl duality for unitary groups. We recall a couple of notations and standard facts. A non-increasing sequence of nonnegative integers $\lambda = (\lambda_1, \ldots)$ is said to be a partition of the integer $n$ (abbreviated by $\lambda \vdash n$) if $\sum_1 \lambda_i = n$. We denote by $l(\lambda)$ its length, i.e. the largest index $i$ for which $\lambda_i$ is non-zero.

There is a canonical way to parameterize all irreducible polynomial representations $\rho^\lambda_{U(d)} : U(d) \to \text{End} V^\lambda_{U(d)}$ of the compact unitary group $U(d)$ by partitions $\lambda$ such that $l(\lambda) \leq d$. The character of this representation evaluated on the torus is the Schur polynomial $s_{\lambda,d}$ (see [Ful97]). By $s_{\lambda,d}(x)$ we shall understand $s_{\lambda,d}(x, \ldots, x)$ with $d$ copies of $x$. In particular, $s_{\lambda,d}(1)$ is the dimension of the representation $V^\lambda_{U(d)}$ of $U(d)$.

The group algebra $\mathbb{C}[S_n]$ of the symmetric group $S_n$ is semi-simple. It is endowed with its canonical basis $\{\delta_\sigma\}_{\sigma \in S_n}$. The irreducible representations
The following isomorphism holds:

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} \text{End} V_{S_n}^\lambda.$$  

For any $\lambda \vdash n$, let $p^\lambda = \frac{\chi^\lambda(e)}{n!} \lambda \in \mathbb{C}[S_n]$ be the minimal central projector onto $\text{End} V_{S_n}^\lambda$. We define for future use the algebra

$$C_d[S_n] = \left( \sum_{\lambda \vdash n, l(\lambda) \leq d} p^\lambda \right) \mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n, l(\lambda) \leq d} \text{End} V_{S_n}^\lambda.$$  

Consider the representation $\rho^d_{S_n}$ of $S_n$ on $(\mathbb{C}^d)^{\otimes n}$, where

$$\rho^d_{S_n}(\pi) : v_1 \otimes \cdots \otimes v_n \mapsto v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}$$  

given by natural permutation of elementary tensors. We consider also the representation $\rho^n_{U(d)}$ of $U(d)$ on $(\mathbb{C}^d)^{\otimes n}$, where

$$\rho^n_{U(d)}(U) : v_1 \otimes \cdots \otimes v_n \mapsto U(v_1) \otimes \cdots \otimes U(v_n)$$  
is the diagonal action. Since the representations $\rho^d_{S_n}$ and $\rho^n_{U(d)}$ commute, we obtain a representation $\rho_{S_n \times U(d)}$ of $S_n \times U(d)$ on $(\mathbb{C}^d)^{\otimes n}$.

**Theorem 2.1** (Schur–Weyl duality for unitary groups [Wey39]). The action of $S_n \times U(d)$ is multiplicity free, i.e. no irreducible representation of $S_n \times U(d)$ occurs more than once in $\rho_{S_n \times U(d)}$. The decomposition of $\rho_{S_n \times U(d)}$ into irreducible components is given by

$$\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash n, l(\lambda) \leq d} V_{S_n}^\lambda \otimes V_{U(d)}^\lambda,$$

where $S_n \times U(d)$ acts by $\rho^\lambda_{S_n} \otimes \rho^\lambda_{U(d)}$ on the summand corresponding to $\lambda$.

We shall consider the inclusion of algebras

$$\rho^d_{S_n} (C_d[S_n]) \subseteq \text{End}(\mathbb{C}^d)^{\otimes n}.$$  

Equations (4) and (5) show that $\rho^d_{S_n}$ is injective when restricted to $C_d[S_n]$ and for this reason we shall omit $\rho^d_{S_n}$ whenever convenient and consider $C_d[S_n]$ as sitting inside $\text{End}(\mathbb{C}^d)^{\otimes n}$. Conversely, we can identify every element of the image $\rho^d_{S_n}(C_d[S_n]) \subseteq \text{End}(\mathbb{C}^d)^{\otimes n}$ with the unique corresponding element of the group algebra $C_d[S_n]$. 

$$\rho^\lambda_{S_n} : S_n \to \text{End} V_{S_n}^\lambda$$ are canonically labelled by $\lambda \vdash n$ via the Schur functor (see [Ful97] as well); we denote the corresponding characters by $\chi^\lambda$. 

The following isomorphism holds:

$$C_d[S_n] \cong \bigoplus_{\lambda \vdash n} \text{End} V_{S_n}^\lambda.$$
2.2. **Conditional expectation.** For $A \in \text{End}(\mathbb{C}^d \otimes n)$ we define

\begin{equation}
E(A) = \int_{U(d)} U \otimes n A (U^{-1}) \otimes n \, dU,
\end{equation}

where the integration is taken with respect to the Haar measure on the compact group $U(d)$.

We recall that for an algebra inclusion $M \subset N$, a conditional expectation is a $M$-bimodule map $E : N \to M$ such that $E(1_N) = 1_M$.

**Proposition 2.2.** $E$ defined in (6) is a conditional expectation of $\text{End}(\mathbb{C}^d \otimes n)$ onto $\mathbb{C}_d[S_n]$. We regard $\text{End}(\mathbb{C}^d \otimes n)$ as an Euclidean space with a scalar product $\langle A, B \rangle = \text{Tr} A^* B$. Then $E$ is an orthogonal projection onto $\rho_d^{S_n}(\mathbb{C}_d[S_n])$. Moreover, it is compatible with the trace in the sense that

$$\text{Tr} \circ E = \text{Tr}.$$  

**Proof.** Since Haar measure is a probability measure invariant with respect to the left and right multiplication therefore $E(A)$ commutes with the action of the unitary group $U(d)$ for every $A \in \text{End}(\mathbb{C}^d \otimes n)$. Theorem 2.1 shows that $E(A) \in \mathbb{C}_d[S_n]$ and that the range of $E$ is exactly $\mathbb{C}_d[S_n]$. Since $\langle E(A), E(B) \rangle = \langle E(A), B \rangle$ it follows that $E$ is an orthogonal projection. The other statements of the Proposition can be easily checked directly. $\square$

For $A \in \text{End}(\mathbb{C}^d \otimes n)$ we set

\begin{equation}
\Phi(A) = \sum_{\sigma \in S_n} \text{Tr} (A \rho_{S_n}^d (\sigma^{-1})) \delta_\sigma \in \mathbb{C}(S_n).
\end{equation}

**Proposition 2.3.** $\Phi$ fulfills the following properties:

1. $\Phi$ is a $\mathbb{C}[S_n] - \mathbb{C}[S_n]$ bimodule morphism in the sense that

$$\Phi(A \rho_{S_n}^d (\sigma)) = \Phi(A) \sigma,$$

$$\Phi(\rho_{S_n}^d (\sigma) A) = \sigma \Phi(A);$$

2. $\Phi(\text{Id})$ coincides with the character of $\rho_{S_n}^d$ hence it is equal to

\begin{equation}
\Phi(\text{Id}) = n! \sum_{\lambda \vdash n} \frac{s_{\lambda,d}(1)}{\chi^\lambda(e)} p^\lambda
\end{equation}

and is an invertible element of $\mathbb{C}_d[S_n]$; its inverse will be called Weingarten function and is equal to

\begin{equation}
W_g = \frac{1}{(n!)^2} \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{\chi^\lambda(e)^2}{s_{\lambda,d}(1)} \chi^\lambda
\end{equation}

3. the relation between $\Phi(A)$ and $E(A)$ is explicitly given by

$$\Phi(A) = E(A) \Phi(\text{Id});$$
4. the range of $\Phi$ is equal to $\mathbb{C}_d[S_n]$;
5. in $\mathbb{C}_d[S_n]$, the following holds true

\[ \Phi(A \circ E(B)) = \Phi(A) \Phi(B) \Phi(\text{Id})^{-1}. \]

**Proof.** Points 1 and 2 are immediate. Point 1 implies
\[ \Phi(A) = \Phi(E(A)) = \Phi(\text{Id} \circ E(A)) = \Phi(\text{Id}) \circ E(A), \]
which proves point 3. Point 4 follows from point 3 and point 2. Point 5 follows from points 1 and 3. \qed

**Corollary 2.4.** Let $n$ be a positive integer and $i = (i_1, \ldots, i_n)$, $i' = (i'_1, \ldots, i'_n)$, $j = (j_1, \ldots, j_n)$, $j' = (j'_1, \ldots, j'_n)$ be $n$-tuples of positive integers. Then

\[ \int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} U_{i'_1 j'_1} \cdots U_{i'_n j'_n} dU = \sum_{\sigma, \tau \in S_n} \delta_{i_1, i'_{\omega(1)}} \cdots \delta_{i_n, i'_{\omega(n)}} \delta_{j_1, j'_{\tau(1)}} \cdots \delta_{j_n, j'_{\tau(n)}} W_g(\tau^{-1}). \]

If $n \neq n'$ then

\[ \int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} U_{i'_1 j'_1} \cdots U_{i'_n j'_n} dU = 0. \]

**Proof.** In order to show (11) it is enough to take appropriate $A$ and $B$ in $M_d(\mathbb{C})^\otimes n$ and take the value of both sides of (10) in $e \in S_n$.

For every $u \in \mathbb{C}$ such that $|u| = 1$ the map $U(d) \ni U \mapsto uU \in U(d)$ is measure preserving therefore

\[ \int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} U_{i'_1 j'_1} \cdots U_{i'_n j'_n} dU = \int_{U(d)} uU_{i_1 j_1} \cdots uU_{i_n j_n} uU_{i'_1 j'_1} \cdots uU_{i'_n j'_n} dU \]

and (12) follows. \qed

The above result was obtained by the first named author [Col03] under the assumption $n \geq d$. As we shall see, this assumption is not necessary.

For $n \geq d$ the formula (9) takes the simpler form

\[ W_g = \frac{1}{(n!)^2} \sum_{\lambda \vdash n} \frac{\chi_{\lambda}(e)^2}{s_{\lambda,d}(1)} \chi_{\lambda}, \]

with no restrictions on the length of $\lambda$. The right-hand side is a rational function of $d$ and hence we may consider it for any $d \in \mathbb{C}$. However, the polynomial $d \mapsto s_{\lambda,d}(1)$ has zeros in integer points $-l(\lambda), -l(\lambda) + \ldots$.
1, \ldots, l(\lambda) - 1, l(\lambda) and hence the right-hand side of (13) has poles in points 
\(-n, -n + 1, \ldots, n - 1, n\) and therefore is not well-defined on the whole \(\mathbb{C}\).

Nevertheless, even for the case \(d < n\), let us plug this incorrect value (13) into (11). In this way the right-hand side of (11) becomes a rational function in \(d\). We claim that for every \(d \in \mathbb{N}\) for which the left-hand side of (11) makes sense (i.e. if \(i_1, \ldots, i_n, i'_1, \ldots, i'_n, j_1, \ldots, j_n, j'_1, \ldots, j'_n \in \{1, \ldots, d\}\)), the right-hand side also makes sense (possibly after some cancellations of poles) and is equal to the left-hand side of (11). Indeed, let us view the product \(\Phi(A)\Phi(B) Wg\) as an element of \(\mathbb{C}[S_n]\) with rational coefficients in \(d\). For the choice of \(A, B \in M_d(\mathbb{C}) \otimes n\) used in the proof of Corollary 2.4 we must have \(\Phi(A), \Phi(B) \in \mathbb{C}_d[S_n]\) therefore the product \(\Phi(A)\Phi(B) Wg\) is an element of \(\mathbb{C}_d[S_n]\) with rational coefficients in \(d\). Since (9) and (13) regarded as elements of \(\mathbb{C}[S_n]\) with rational coefficients in \(d\) coincide on \(\mathbb{C}_d[S_n]\) hence our claim holds true.

We summarize the above discussion in the following proposition.

**Proposition 2.5.** For fixed values of the indices \(i, j, i', j'\) the integral

\[
\int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} \frac{U_{i'_1 j'_1}}{U_{i'_1 j'_1}} \cdots \frac{U_{i'_n j'_n}}{U_{i'_n j'_n}} dU
\]

is a rational function of \(d\).

Furthermore, the equation (11) remains true (possibly after some cancellations of poles) if we replace the correct value (9) of Weingarten function by (13).

**Example.** Corollary 2.4 implies that for \(d \geq 2\)

\[
\int_{U(d)} |U_{11}|^2 dU = \int_{U(d)} U_{11} U_{11} U_{11}^* U_{11}^* dU = 2 \, Wg\left(\begin{array}{c} 1 \\ 1 \end{array} \right) + 2 \, Wg\left(\begin{array}{c} 1 \\ 2 \end{array} \right) = 2 - \frac{1}{d^2 - 1} + 2 - \frac{1}{d(d^2 - 1)},
\]

where the values of the Weingarten function were computed by (13) and where \(\left(\sigma(1) \cdots \sigma(n)\right)\) denotes the permutation \(\sigma\). The right-hand side appears to make no sense for \(d = 1\), nevertheless after algebraic simplifications we obtain

\[
\int_{U(d)} |U_{11}|^2 dU = \frac{2}{d(d+1)}
\]

which is a correct value for all \(d \geq 1\).

2.3. **Asymptotics of the Weingarten function.** In this section we compute the first order asymptotic of the Weingarten function for large values of \(d\).
Consider the algebra $\mathbb{C}[S_n][[d^{-1}]]$ of functions on $S_n$ valued in formal power series in $d^{-1}$ and the vector space

$$\mathcal{A} = \text{Vect} \{ \alpha \delta_\sigma : \alpha = O(d^{-|\alpha|}) \text{ and } \alpha d^{|\alpha|} \text{ is a power series in } d^{-2} \},$$

where $|\sigma|$ denotes the minimal number of factors necessary to write $\sigma$ as a product of transpositions. By the triangle inequality $|\sigma_1| + |\sigma_2| \geq |\sigma_1 \sigma_2|$ and the parity property $(-1)^{|\sigma_1|}(-1)^{|\sigma_2|} = (-1)^{|\sigma_1 \sigma_2|}$, $\mathcal{A}$ turns out to be a unital subalgebra of $\mathbb{C}[S_n][[d^{-1}]]$.

It is easy to check that $d^{-n}\Phi(Id) \in \mathcal{A}$. Since $d^{-n}\Phi(Id) = \delta_e + O(d^{-1})$ therefore its inverse $d^n W_g = \sum_i (1-d^{-n}\Phi(Id))^i$ makes sense as a formal power series in $d^{-1}$. The following proposition follows immediately.

**Proposition 2.6.** $d^n W_g \in \mathcal{A}$. Equivalently, for any $\sigma \in S_n$, $W_g(\sigma) = O(d^{-n-|\sigma|})$.

In order to find a more precise asymptotic expansion we consider the two-sided ideal $I$ in $\mathcal{A}$ generated by $d^{-2}\delta_e$. It is easy to check that the quotient algebra $\mathcal{A}/I$ regarded as a vector space is spanned by vectors $d^{-|\sigma|}\delta_\sigma$. The products of these elements are given by

$$(d^{-|\sigma|}\delta_\sigma)(d^{-|\rho|}\delta_\rho) = \begin{cases} d^{-|\sigma\rho|}\delta_{\sigma\rho} & \text{if } |\sigma\rho| = |\sigma| + |\rho|, \\ 0 & \text{if } |\sigma\rho| < |\sigma| + |\rho|. \end{cases}$$

Biane [Bia97] considered an algebra which as a vector space is equal to $\mathbb{C}[S_n]$ with the multiplication

$$\delta_\sigma \ast \delta_\rho = \begin{cases} \delta_{\sigma\rho} & \text{if } |\sigma\rho| = |\sigma| + |\rho|, \\ 0 & \text{if } |\sigma\rho| < |\sigma| + |\rho|. \end{cases}$$

One can easily see now that $d^{-|\rho|}\delta_\rho \mapsto \delta_\rho$ provides an isomorphism of $\mathcal{A}/I$ and Biane algebra. Under this isomorphism $d^{-n}\Phi(Id)$ is mapped into $\zeta = \sum_{\sigma \in S_n} \delta_\sigma$. The inverse of $\zeta$ in Biane algebra is called Möbius function and is given explicitly by

$$\text{Moeb}(\sigma) = \prod_{1 \leq i \leq k} c_{|C_i|-1} (-1)^{|C_i|-1},$$

where $\sigma$ is a permutation with a cycle decomposition $\sigma = C_1 \cdots C_k$ and

$$c_n = \frac{(2n)!}{n!(n+1)!}$$

is the Catalan number.

**Corollary 2.7.** $d^{n+|\sigma|} W_g(\sigma) = \text{Moeb}(\sigma) + O(d^{-2})$.
3. INTEGRATION OVER ORTHOGONAL GROUPS

3.1. Schur–Weyl duality for orthogonal groups.

3.1.1. Brauer algebras. We consider the group of orthogonal matrices

\[ O(d) = \{ M \in \text{GL}(d), M^{-1} = M^t = M^* \} . \]

Its invariant theory has first been studied by R. Brauer [Bra37] who introduced a family of algebras, nowadays called Brauer algebras. These algebras have been at the center of many investigations (see [BW89, Gro99] and the references therein). Some actions of these algebras lead to an analogue of the Schur–Weyl duality in the case of the orthogonal group and symplectic groups and for this reason they are very useful for our purposes.

Consider \( 2n \) vertices arranged in two rows: the upper one with \( n \) vertices denoted by \( U_1, \ldots, U_n \) and the bottom row with \( n \) vertices denoted by \( B_1, \ldots, B_n \).

![Figure 1. Example of an element of \( P_{20} \)](image)

We regard \( S_{2n} \) as a group of permutations of the set of vertices and denote by \( P_{2n} \) the set of all pairings of this set. An example of such a pairing is presented on Figure 1. We can view \( P_{2n} \) as a set of permutations \( \sigma \in S_{2n} \) such that \( \sigma^2 = e \) and \( \sigma \) has no fixpoints. We will consider the action \( \rho_{S_{2n}} \) of \( S_{2n} \) on \( P_{2n} \) by conjugation under the embedding \( P_{2n} \subset S_{2n} \) described above. By \( \mathbb{C}[P_{2n}] \) we denote the linear space spanned by \( P_{2n} \). We equip this linear space with a bilinear symmetric form \( \langle \cdot, \cdot \rangle \) by requirement that elements of \( P_{2n} \) form an orthonormal basis. The embedding \( P_{2n} \subset S_{2n} \) extends linearly to the inclusion of \( S_{2n} \)-modules \( \mathbb{C}[P_{2n}] \subset \mathbb{C}[S_{2n}] \) and the scalar product can be described as

\[ \langle a, b \rangle = \frac{\chi_{\text{reg}}(ab^*)}{\chi_{\text{reg}}(e)} , \]

where \( \chi_{\text{reg}} \) denotes the character of the left regular representation.

The Brauer algebra \( B(d, n) \) regarded as a vector space is isomorphic to \( \mathbb{C}(P_{2n}) \). The multiplication in the algebra \( B(d, n) \) depends on the parameter \( d \), but in this article we will not use the multiplicative structure of the Brauer algebra.
3.1.2. **Canonical representation of the Brauer algebra.** By $\langle \cdot, \cdot \rangle$ we denote the canonical bilinear symmetric forms on $\mathbb{C}^d$ and on $(\mathbb{C}^d)^{\otimes n}$. The canonical representation $\rho_B$ of the Brauer algebra $B(d, n)$ on $(\mathbb{C}^d)^{\otimes n}$ is defined as follows: in order to compute $\langle u_1 \otimes \cdots \otimes u_n, \rho_B(p)[b_1 \otimes \cdots \otimes b_n] \rangle$, where $p \in P_{2n}$ and $u_1, \ldots, u_n, b_1, \ldots, b_n \in \mathbb{C}^d$ we assign to the upper vertices of $p$ vectors $u_1, \ldots, u_n$ and to bottom vertices vectors $b_1, \ldots, b_n$. The value of $\langle u_1 \otimes \cdots \otimes u_n, \rho_B(p)[b_1 \otimes \cdots \otimes b_n] \rangle$ is defined to be a product of the scalar products of vectors assigned to vertices joined by the same line. For example, for the diagram $p$ from Figure 1 we obtain:

$$
\langle u_1 \otimes \cdots \otimes u_{10}, \rho_B(p)[b_1 \otimes \cdots \otimes b_{10}] \rangle = \langle u_1, u_3 \rangle \langle u_2, u_4 \rangle \times \langle u_5, b_7 \rangle \langle u_6, u_{10} \rangle \langle u_7, u_9 \rangle \langle u_8, b_8 \rangle \langle b_1, b_3 \rangle \langle b_2, b_5 \rangle \langle b_4, b_6 \rangle \langle b_9, b_{10} \rangle.
$$

In the above construction we used implicitly the isomorphism of vector spaces

$$
\text{End}(\mathbb{C}^d)^{\otimes n} = \bigotimes_{i \in \{U_1, \ldots, U_n, B_1, \ldots, B_n\}} \mathbb{C}^d.
$$

We will consider the action of $S_{2n}$ on $\text{End}(\mathbb{C}^d)^{\otimes n}$ by permutation of factors on the right-hand side of (16).

We consider the representation $\rho_{O(d)}^n$ of $O(d)$ on $(\mathbb{C}^d)^{\otimes n}$, where

$$
\rho_{O(d)}^n(O) : v_1 \otimes \cdots \otimes v_n \mapsto O(v_1) \otimes \cdots \otimes O(v_n)
$$

is the diagonal action.

**Theorem 3.1** (Schur–Weyl duality for orthogonal groups [Bra37, Wen88]). The commutant of $\rho_{O(d)}^n$ is equal to $\rho_B(\mathbb{C}[P_{2n}])$. Furthermore if $d \geq n$ then $\rho_B$ is injective.

### 3.2. Integration formula.

#### 3.2.1. For $A \in \text{End}(\mathbb{C}^d)^{\otimes n}$ we define

$$
E(A) = \int_{O(d)} O^{\otimes n} A(O^t)^{\otimes n} \, dO.
$$

**Proposition 3.2.** $E$ is a conditional expectation of $\text{End}(\mathbb{C}^d)^{\otimes n}$ into $\rho_B(\mathbb{C}[P_{2n}])$, in particular it satisfies $E^2 = E$. We regard $\text{End}(\mathbb{C}^d)^{\otimes n}$ as a Euclidean space with a scalar product $\langle A, B \rangle = \text{Tr} AB^*$. Then $E$ is an orthogonal projection onto $\rho_B(\mathbb{C}[P_{2n}])$. It is compatible with the trace in the sense that

$$
\text{Tr} \circ E = \text{Tr}.
$$

**Proof.** Proof is analogous to the proof of Proposition 2.2 but instead of Theorem 2.1 we use Theorem 3.1. □
For $A \in \text{End}(\mathbb{C}^d)^{\otimes n}$ we set

$$\Phi(A) = \sum_{p \in P_{2n}} p \text{Tr}(\rho_B(p)^t A) \in \mathbb{C}[P_{2n}]$$

By the representation $\rho_B$ every element of $\mathbb{C}(P_{2n})$ can be viewed as an element of $\text{End}(\mathbb{C}^d)^{\otimes n}$ and therefore we can consider the linear map

$$\tilde{\Phi} = \Phi \circ \rho_B : \mathbb{C}(P_{2n}) \to \mathbb{C}(P_{2n}).$$

The matrix of the operator $\tilde{\Phi}$ coincides with the Gramm matrix of the set of vectors $\rho_B(p) \in \text{End}(\mathbb{C}^d)^{\otimes n}$ indexed by $p \in P_{2n}$. We denote by $W_g$ the inverse of $\tilde{\Phi}$. We postpone the problem if this inverse exists to Proposition 3.10.

We denote by $\Pi_{p_1,p_2}$ the partition induced by the action of the group generated by $p_1, p_2$.

**Proposition 3.3.** $\rho_B$, $E$, $\Phi$ are morphisms of $S_{2n}$-spaces. As a consequence, $\langle p_1, W_g p_2 \rangle$ depends only on the conjugacy class of $p_1 p_2$.

*Proof.* The proof of this proposition is straightforward. $\square$

By a change of labels we can view $P_{2n}$ as the set of pairings of the set $\{1,\ldots,2n\}$. We do not care about the choice of the way in which labels $\{U_1,\ldots,U_n,B_1,\ldots,B_n\}$ are replaced by $\{1,\ldots,2n\}$. For a tuple of indices $i = (i_1,\ldots,i_{2n})$, where $i_1,\ldots,i_{2n} \in \{1,\ldots,d\}$ and a pairing $p \in P_{2n}$ we set $\delta^p_i = 1$ if for each pair $a,b \in \{1,\ldots,2n\}$ connected by $p$ we have $i_a = i_b$; otherwise we set $\delta^p_i = 0$.

**Corollary 3.4.** The following formulas hold true:

$$E = \rho_B \circ W_g \circ \Phi,$$

$$\text{Tr} A E(B) = \sum_{p_1,p_2 \in P_{2n}} \text{Tr} \left( A \rho_B(p_1) \right) \text{Tr} \left( \rho_B(p_2)^t B \right) \langle p_1, W_g p_2 \rangle.$$

For every choice of $u_1,\ldots,u_{2n},v_1,\ldots,v_{2n}$ we have

$$\int_{O(d)} \langle u_1, Ov_1 \rangle \cdots \langle u_{2n}, Ov_{2n} \rangle \ dO =$$

$$\sum_{p_1,p_2 \in P_{2n}} \langle u_1 \otimes \cdots \otimes u_n, \rho_B(p_1) u_{n+1} \otimes \cdots \otimes u_{2n} \rangle \times$$

$$\langle v_1 \otimes \cdots \otimes v_n, \rho_B(p_2) v_{n+1} \otimes \cdots \otimes v_{2n} \rangle \langle p_1, W_g p_2 \rangle.$$
In particular, for every choice of indices \( i = (i_1, \ldots, i_{2n}), j = (j_1, \ldots, j_{2n}) \)

\[
(21) \quad \int_{O(d)} O_{i_1 j_1} \cdots O_{i_{2n} j_{2n}} \, dO = \sum_{p_1, p_2 \in \mathcal{P}_{2n}} \delta_{i_1}^{p_1} \delta_{j_1}^{p_2} \langle p_1, W_g p_2 \rangle.
\]

The moments of an odd number of factors vanish:

\[
(22) \quad \int_{O(d)} O_{i_1 j_1} \cdots O_{i_{2n+1} j_{2n+1}} \, dO = 0.
\]

**Proof.** It is enough to take appropriate matrices in the canonical basis to establish this result.

The map \( O(d) : O \mapsto -O \in O(d) \) preserves the Haar measure therefore

\[
\int_{O(d)} O_{i_1 j_1} \cdots O_{i_{2n+1} j_{2n+1}} \, dO = \int_{O(d)} (-O_{i_1 j_1}) \cdots (-O_{i_{2n+1} j_{2n+1}}) \, dO
\]

which shows (22). \( \square \)

Therefore \( W_g \) appears to be of fundamental importance in the computation of moments of the orthogonal group, and it is of theoretical importance to give a closed formula for it. We shall do this in the following.

3.2.2. *An abstract formula for the orthogonal Weingarten function.* Let \( \text{Id} \in \mathcal{P}_{2n} \) be any fixed pairing; to have a concrete example let us say that \( \text{Id} \) is the identity of the Brauer algebra, i.e. the pairing which connects the pairs of vertices \( U_i, B_i \) with each \( 1 \leq i \leq n \).

\[
\begin{array}{cccccccccc}
U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & \cdots & U_n \\
\hline
\text{Id} = & & & & & & & & & \\
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & \cdots & B_n \\
\end{array}
\]

**Figure 2.** Identity in Brauer algebra.

From now on we fix an inclusion of the hyperoctahedral group \( O_n \) into \( S_{2n} \) by considering \( O_n \) as the global stabilizer of \( \text{Id} \) under the action of \( S_{2n} \).

We equip the set \( P_{2n} \) of pairings with a metric \( l \) by setting

\[
l(p_1, p_2) = \frac{|p_1 p_2|}{2},
\]

where pairings \( p_1, p_2 \) are regarded on the right-hand side as elements of \( S_{2n} \).
Lemma 3.5. If $p_1, p_2 \in \mathcal{P}_{2n}$ then

\begin{equation}
\text{Tr} \rho_B(p_1) \rho_B(p_2)^t = d^{n-l(p_1, p_2)}.
\end{equation}

Furthermore, $l(p_1, p_2)$ is an integer number.

Each right class $\pi O_n$ of $S_{2n}/O_n$ is uniquely determined by its action $\pi(\text{Id})$ on the identity diagram hence the right classes $S_{2n}/O_n$ are in one-to-one correspondence with the elements of $\mathcal{P}_{2n}$.

We set $|\pi O_n| = \min_{\sigma \in \pi O_n} |\sigma|$. Then

\begin{equation}
\langle \tilde{\Phi}(\pi(\text{Id})), \text{Id} \rangle = d^{n-|\pi O_n|}.
\end{equation}

Let a left and right class $O_n \rho O_n$ be fixed. The value of $|\pi O_n|$ does not depend on the choice of $\pi \in O_n \rho O_n$ therefore the definition $|O_n \rho O_n| = |\rho O_n|$ makes sense.

Proof. Let $e_1, \ldots, e_d$ be the orthogonal basis of $\mathbb{C}^d$; then

\begin{equation}
\text{Tr} \rho_B(p_1) \rho_B(p_2)^t = \sum_{1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d} \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \rho_B(p_1)(e_{i_1} \otimes \cdots \otimes e_{i_n}) \rangle \times \langle e_{j_1} \otimes \cdots \otimes e_{j_n}, \rho_B(p_2)(e_{j_1} \otimes \cdots \otimes e_{j_n}) \rangle.
\end{equation}

To every upper vertex $U_k$ (respectively, bottom vertex $B_k$) we assign the appropriate index $i_k$ (respectively, $j_k$). From the very definition of $\rho_B$, the right-hand side is equal to 1 if the indices corresponding to each pair of vertices connected by $p_1$ or $p_2$ are equal; otherwise the right-hand side is equal to 0. It follows that

\begin{equation}
\text{Tr} \rho_B(p_1) \rho_B(p_2)^t = d^k \text{number of connected components of the graph depicting } p_1 \text{ and } p_2.
\end{equation}

We observe that each connected component of the graph depicting $p_1$ and $p_2$ corresponds to a pair of orbits of the permutation $p_1 p_2$. The number of orbits of $p_1 p_2$ is equal to $2n - |p_1 p_2|$ which finishes the proof of the first part.

The above considerations imply that

\begin{equation}
\langle \tilde{\Phi}(\pi(\text{Id})), \text{Id} \rangle = d^{n-\frac{1}{2} |\pi \text{Id} \pi^{-1} \text{Id}|}.
\end{equation}

Let $\sigma \in \pi O_n$. Since

\begin{equation}
|\pi \text{Id} \pi^{-1} \text{Id}| = |\sigma \text{Id} \sigma^{-1} \text{Id}| \leq |\sigma| + |\text{Id} \sigma^{-1} \text{Id}| = 2|\sigma|
\end{equation}

therefore

\begin{equation}
|\pi \text{Id} \pi^{-1} \text{Id}| \leq 2 |\pi O_n|.
\end{equation}

We can decompose the set of vertices $\{U_1, \ldots, U_n, B_1, \ldots, B_n\}$ into two classes in such a way that the graph depicting pairings $\pi(\text{Id})$ and $\text{Id}$ is bipartite, or—in other words—each of the pairings $\pi(\text{Id}), \text{Id}$ regarded a permutation maps these two classes into each other. We leave it to the reader
to check that there exists a unique permutation \( \sigma \in \pi \) which is equal to identity on the first of these classes. It follows

\[ |\pi \text{ Id} \pi^{-1} \text{ Id}| = 2 |\pi| \]

which shows that

\[ |\pi \text{ Id} \pi^{-1} \text{ Id}| \geq 2 |\pi \text{ O}_n|. \]

Let \( \sigma \in \text{ O}_n \). Then

\[ |\pi \text{ Id} \pi^{-1} \text{ Id}| = |\pi \text{ Id} \pi^{-1} \sigma^{-1} \text{ Id} \sigma| = |\sigma \pi \text{ Id} \pi^{-1} \sigma^{-1} \text{ Id}| \]

therefore \( |\pi \text{ O}_n| = |\sigma \pi \text{ O}_n| \) finishes the proof.

\[ \square \]

**Lemma 3.6.** The sum of dimensions of representations of \( S_{2n} \) of shape \( 2y_1 \geq 2y_2 \geq \ldots \), where \( y_1 + y_2 + \ldots = n \) equals the cardinality of \( \text{ P}_{2n} \).

**Proof.** The Robinson–Schensted–Knuth algorithm provides a bijection between permutations and pairs \( (P, Q) \) of standard Young tableaux of the same shape. Furthermore if \( \sigma \mapsto (P, Q) \) then \( \sigma^{-1} \mapsto (Q, P) \); it follows that the RSK algorithm is a bijection between involutions \( \sigma = \sigma^{-1} \) and standard Young tableaux.

It is easy to show that for any idempotent without fixed point, the RSK algorithm which gives a pair of tableaux \( (P, Q) \) of same shape satisfies the additional property that \( P = Q \). Furthermore, implementing the reverse of RSK algorithm (see [Ful97]) shows that the tableaux must have the shape prescribed in the Lemma, and that any such tableau gives rise to an idempotent without fixed point.

\[ \square \]

**Proposition 3.7.** The space \( \text{ C}(\text{ P}_{2n}) \) splits under the action of \( S_{2n} \) as a direct sum of representations associated to Young diagrams of the shape \( 2y_1 \geq 2y_2 \geq \ldots \), where \( y_1 + \ldots + y_q = n \), hence the action is multiplicity–free.

**Proof.** Following Fulton [Ful97], let us consider a diagram of shape \( 2y_1 \geq 2y_2 \geq \ldots \) and consider its row numbering Young tableau. Let \( C \) be the column invariant subgroup of \( S_{2n} \) and \( L \) the line invariant subgroup; both these groups are isomorphic to a product of symmetric group. We consider the projection operator \( p_C \) associated to the trivial representation of \( C \) and the projection operator \( p_L \) associated to the alternate representation of \( L \). One can see geometrically that these two operators commute and that the partition \( (1, 2)(3, 4), \ldots, (2n - 1, 2n) \) is not in the kernel of \( p_C \circ p_L \).

The dimension argument of Lemma concludes the proof and shows uniqueness of the occurrence of any representation of shape \( 2y_1 \geq 2y_2 \geq \ldots \).
Proposition 3.8. The eigenspaces of $\tilde{\Phi}$ are indexed by Young diagrams $\lambda$ with the shape $2l_1 \geq 2l_2 \geq \ldots$. The corresponding eigenvalue is given by

$$z_\lambda = \frac{\sum_{\pi \in O_n \backslash S_{2n}/O_n} d^{n-|\pi|} \sum_{\sigma \in \pi} \chi^\lambda(\sigma)}{\sum_{\sigma \in O_n} \chi^\lambda(\sigma)}$$

and the corresponding eigenspace is equal to the image of $\rho_{S_{2n}}(p^\lambda)$.

Proof. $\tilde{\Phi}$ is a morphism of $S_{2n}$–spaces by Proposition 3.3, hence Proposition 3.7 gives the classification of the eigenspaces of $\tilde{\Phi}$. Let $\lambda$ be as in Proposition 3.7; then the element $\rho_{S_{2n}}(p^\lambda)(\text{Id})$ is non-zero and belongs to an irreducible submodule of $\mathbb{C}(P_{2n})$ thus it satisfies

$$\tilde{\Phi}(\rho_{S_{2n}}(p^\lambda)(\text{Id})) = z_\lambda \rho_{S_{2n}}(p^\lambda)(\text{Id}).$$

We have therefore by bilinearity

$$\langle \tilde{\Phi} \rho_{S_{2n}}(p^\lambda)(\text{Id}), \text{Id} \rangle = z_\lambda \langle \rho_{S_{2n}}(p^\lambda)(\text{Id}), \text{Id} \rangle = z_\lambda \sum_{\sigma \in O_n} p^\lambda(\sigma).$$

Lemma 3.5 can be used to evaluate the left–hand side of (26). Since the left–hand side of (26) is non–zero for sufficiently big $d$, hence also the right–hand side is non–zero and the division makes sense.

□

Theorem 3.9. The Weingarten function is given by

$$W_g = \sum_\lambda \frac{1}{z_\lambda} \rho_{S_{2n}}(p^\lambda),$$

where the sum runs over diagrams $\lambda$ with a shape prescribed in Proposition 3.7 and $z_\lambda$ was defined in Equation (25).

In particular,

$$\langle p_1, W_g p_2 \rangle = \sum_\lambda \frac{1}{z_\lambda (2n)!} \chi^\text{reg}\{ \rho_{S_{2n}}(p^\lambda)(p_1) \cdot \rho_{S_{2n}}(p^\lambda)(p_2) \}$$

where $\rho_{S_{2n}}(p^\lambda)(p_1)$ are considered as elements of $\mathbb{C}[S_{2n}]$, $\cdot$ is the multiplication in $\mathbb{C}[S_{2n}]$.

Proof. The first point follows from the above discussion and for the second it is enough to observe that $\langle p_1, p_2 \rangle = \frac{1}{(2n)!} \chi^\text{reg}(p_1 p_2^\dagger)$. □

Observe that Equation (28) is a closed formula for $\langle p_1, W_g p_2 \rangle$ as a (rational) function of the parameter dimension $d$, expressed in terms of the characters of the symmetric group (though complicated when fully expanded - in which case the expressions of $p_\lambda$ and $z_\lambda$ should be taken in consideration and inconvenient to implement on a computer).
A priori, Corollary 3.4 is valid only for $d \geq n$ since in this case $\rho_B$ is injective and therefore $\Phi$ is invertible; otherwise the Weingarten function does not exist. The following result deals also with the cases $d < n$.

**Proposition 3.10.** Corollary 3.4 remains true for all values of $d$ and $n$ if the following definition of the Weingarten function is used:

\[
Wg = \sum_{\lambda} \frac{1}{z_\lambda} \rho_{S_{2n}}(p^\lambda),
\]

where the sum is taken over all diagrams $\lambda$ with a shape prescribed in Proposition 3.7 for which $z_\lambda \neq 0$.

**Proof.** Since $E$ is an orthogonal projection, it is enough to check the validity of (18) on the range of $\rho_B$. We denote by $V \subseteq \mathbb{C}(P_{2n})$ the span of the images of $\rho_{S_{2n}}(p^\lambda)$ for which $z_\lambda \neq 0$; the range of $\rho_B$ is equal to $\rho_B(V)$ hence it is enough to show that

\[
E \circ \rho_B = \rho_B \circ Wg \circ \Phi
\]

holds true on $V$. The latter equality is obvious since the inverse of $\Phi : V \rightarrow V$ is equal to $Wg$ given by (29).

We can treat $\Phi_d : \mathbb{C}(P_{2n}) \rightarrow \mathbb{C}(P_{2n})$ as a matrix the entries of which are polynomials in $d$ and therefore its inverse $Wg_d : \mathbb{C}(P_{2n}) \rightarrow \mathbb{C}(P_{2n})$ makes sense as a matrix the entries of which are rational functions of $d \in \mathbb{C}$; therefore $Wg_d$ is well-defined for all $d \in \mathbb{C}$ except for a finite set; it is explicitly given by (27). For fixed $A, B \in M_{d_0}(\mathbb{C})^\otimes n$ let us plug this (incorrect for $d_0 < n$) value of $Wg_d$ into (19); the right-hand side becomes a rational function of $d$ and we claim that after the cancellation of poles it has a limit $d \rightarrow d_0$ which is indeed equal to the left-hand side of (19). In other words, we claim that

\[
E = \lim_{d \rightarrow d_0} \rho_B \circ Wg_d \circ \Phi.
\]

It is indeed the case since for every $d \in \mathbb{C}$ the value of $Wg_d \left(\Phi(A)\right)$ is the same no matter if we use (27) or (29).

We summarize the above discussion in the following proposition.

**Proposition 3.11.** Corollary 3.4 remains true for all values of $d$ and $n$ if the Weingarten function is regarded as a rational function computed in (27); possibly after some cancellation of poles.

### 3.3. Asymptotics of Weingarten function.

For pairings $p_1, p_2 \in P_{2n}$ let $2n_1, 2n_2, \ldots$ denote the numbers of elements in the orbits of the action of $\{p_1, p_2\}$. We define the Möbius function

\[
\text{Moeb}(p_1, p_2) = \prod_i (-1)^{n_i-1}c_{n_i-1},
\]
where $c_n$ is the Catalan number defined in (14).

**Lemma 3.12.** For every $p \in P_{2n}$ and $|d|$ sufficiently large we have

$$W_g(p) = d^{-n} \sum_{k \geq 0} \sum_{p = p_0 \ldots p_k \atop p_i \neq p_{i+1} \text{ for } i \in \{0, 1, \ldots, k-1\}} (-1)^k d^{-\rho(p_0, p_1) - \cdots - \rho(p_{n-1}, p_n)} p_n.$$ 

**Proof.** It is enough to observe

$$d^{-n} \Phi(p) = p + \sum_{p' \neq p} d^{-\rho(p, p')} p'$$

and use the power series expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for the operator $d^{-n} \Phi$.

**Theorem 3.13.** The leading term of the Weingarten function is given by

$$(31) \quad \langle p, W_g p' \rangle = d^{-n-\rho(p, p')} \text{Moeb}(p, p') + O(d^{-n-\rho(p, p')-1}).$$

**Proof.** Lemma [312] implies that we need to find explicitly all tuples of pairings $p_0, \ldots, p_k$ such that $p_0 = p$, $p_k = p'$ which fulfill $p_i \neq p_{i+1}$ for $i \in \{0, \ldots, k-1\}$ and $\rho(p_0, p_1) + \cdots + \rho(p_{k-1}, p_k) = \rho(p_0, p_k)$.

For every such tuple the triangle inequality implies that $\rho(p_0, p_i) + \rho(p_i, p_k) = \rho(p_0, p_k)$, or equivalently, $|p_0 p_i| + |p_i p_k| = |p_0 p_k| = |(p_0 p_i) (p_i p_k)|$. The latter condition implies that every orbit of $p_0 p_i \in S_{2n}$ must be a subset of one of the orbits of $p_0 p_k$ [Bia97, Bia98]. Therefore pairing $p_i$ cannot connect vertices which belong to different connected components of the graph spanned by $p_0$ and $p_k$. It follows that it is enough to consider the case if the graph spanned by $p_0$ and $p_k$ is connected.

Suppose that the graph spanned by $p_0$ and $p_k$ is connected. It follows that the permutation $p_0 p_k$ consists of two $n$-cycles, we denote one of them by $\pi$. Since every orbit of $p_0 p_i$ is a subset of one of the orbits of $p_0 p_k$ therefore it makes sense to consider the restriction $\rho_i$ of $p_0 p_i$ to the support of $\pi$.

Observe that knowing $\rho_i$ we can reconstruct the pairing $p_i$ by the formula

$$p_i(s) = \begin{cases} p_0 p_i(s) & \text{if } \rho_i(s) \text{ is defined}, \\ \rho_i^{-1} p_0(s) & \text{otherwise}. \end{cases}$$

It follows that the solutions of the equation $\rho(p_0, p_i) + \rho(p_i, p_k) = \rho(p_0, p_k)$ can be identified with the solutions of the equation $|\rho| + |\rho^{-1} \pi| = |\pi|$.

Now one can easily see that the tuples of pairings $p_1, \ldots, p_{k-1}$ which fulfill $p_i \neq p_{i+1}$ for $i \in \{0, \ldots, k-1\}$ and $\rho(p_0, p_1) + \cdots + \rho(p_{k-1}, p_k) = \rho(p_0, p_k)$ are in one-to-one correspondence with tuples of permutations $\rho_1, \ldots, \rho_{k-1}$ such that $\rho_i \neq \rho_{i+1}$ and $|\rho_0 \rho_1^{-1}| + \cdots + |\rho_{k-1} \rho_k^{-1}| = |\rho_0 \rho_k^{-1}|$, where $\rho_0$ is
the identity permutation and $\rho_k = \pi$. The results of Biane [Bia97] finish
the proof.

3.4. Cumulants. Recall that in the work of the first named author [Col03] the asymptotics of cumulants of unitary Weingarten functions have been obtained (Theorem 2.15). The purpose of this section is to establish the counterpart of this result for orthogonal $W_g$ functions.

As we see by Proposition 3.3, the function $W_g$ can be labelled by $W_g(\lambda, d)$ where $\lambda \vdash n$ is a partition of the number $n$. It will be more convenient to define in the obvious way $W_g(\pi, d)$ where $\pi$ is a partition of the interval $[1, n]$. For partitions $\Pi, \Pi'$ of $[1, n]$ such that $\pi \leq \Pi \leq \Pi'$, it is of fundamental importance to have a good understanding of relative cumulants $C_{\pi, \Pi, \Pi'}$ of $W_g$ defined implicitly by the relation

$$W_g(\Pi', d) = \sum_{\Pi \leq \Pi'' \leq \Pi'} C_{\pi, \Pi, \Pi''}$$

whenever $\Pi' \geq \Pi$, with $W_g(\Pi) = \prod_k W_g(\pi_k | V_k)$ if one denotes $\Pi = \{V_1, \ldots, V_k\}$.

Lemma 3.14. The relative cumulant is given for $d$ large enough, by

$$C_{\pi, \Pi, \Pi'} = (-1)^k d^{-n} \sum_{p_0, p_1, \ldots, p_k} \sum_{\sup(\Pi, \pi_1, \ldots, \pi_k) = \Pi'} (-1)^k d^{-l(p_0, p_1) - \cdots - l(p_{n-1}, p_n)}$$

The leading order of the series of $C_{\pi, \Pi, \Pi'}$ is therefore the number of $k$-tuples $(\pi_1, \ldots, \pi_k)$ of elements of $P_{2n}$ such that $l(\pi, \pi_1) + l(\pi_1, \pi_2) + \cdots + l(\pi_k, \text{Id}) = n + l(\pi, \text{Id}) - 2(\#\text{blocks}(\Pi') - \#\text{blocks}(\Pi))$ together with the requirement that the $sup(\Pi, \pi_1, \ldots, \pi_k) = \Pi'$

Proof. For the first point, it is enough to check that this equation satisfies the moment-cumulant Equation. Asymptotics of the leading order is elementary. For a less direct approach, see also [Col03].

In order to compute the leading order, it is enough to compute the number of $k$-tuples $(\pi_1, \ldots, \pi_k)$ of elements of $P_{2n}$ such that $d(\pi, \pi_1) + d(\pi_1, \pi_2) + \cdots + d(\pi_k, \text{Id}) = n + l(\pi, \text{Id}) - 2(\#\text{blocks}(\Pi') - \#\text{blocks}(\Pi))$ together with the requirement that the $sup(\pi, \pi_1, \ldots, \pi_k) = 1_n$. We call $B[\pi, k]$ this number.

Denote by $\tau_1, \ldots, \tau_n$ the disjoint transpositions generating the pairing $\text{Id} \in P_{2n}$, and $G$ be the subgroup of $S_{2n}$ generated by these transpositions. This group has the structure of $(\mathbb{Z}/2\mathbb{Z})^n$. 
The symmetric group $S_n$ can be regarded as a subset of $P_{2n}$ when we identify permutation $\sigma$ with a pairing which connects the upper vertex $U_i$ with the bottom vertex $B_{\sigma(i)}$ for all values of $1 \leq i \leq n$. We say that pairings which can be obtained by this construction are permutation-like. The group $G$ acts on $P_{2n}$ by conjugations and one checks easily that in any orbit under the action of $G$ there exist at least one permutation-like element. Moreover, two permutation-like element in a same orbit are conjugate to each other when regarded as elements of $S_n$. More precisely, each orbit has $2^l$ elements, where $l$ is the number of cycles with at least 3 elements in $S_n$.

Fix $\pi \in P_{2n}$ and call $k$ the number of its connected components (i.e. the number of cycles -including trivial cycles (two-element orbits) and transpositions (four-elements orbits) of an associated permutation-like element).

Let $\pi \in S_n$ be one image $\pi$. Consider the number of $k$-tuples $(\sigma_1, \ldots, \sigma_k)$ of permutations of $S_n$ such that $\sigma_1 \ldots \sigma_k \sigma = e$, the group generated by $\sigma_1, \ldots, \sigma_k$ acts transitively on $[1, n]$ and $|\sigma| + |\sigma_1| + \ldots + |\sigma_k| = 2n - 2$. This number has already been computed in [BMS00] and it is

$$A[\sigma, k] = k \frac{(nk - n - 1)!}{(nk - 2n + |\sigma| + 2)!} \prod_{i \geq 1} \left[ i \binom{ki - 1}{i} \right]^{d_i}$$

where $d_i$ denotes the number of cycles with $i$ elements of $\sigma$.

**Proposition 3.15.** $B[\pi, k] = 2^{k-1} A[\sigma, k]$.

**Proof.** The group $G$ acts by conjugation on $k$-tuples $(\pi_1, \ldots, \pi_k)$ arising in the counting of $B[\pi, k]$. Choose one element of $G$ that turns $\pi$ into permutation like. Introduce the group $G'$ generated by $\tau_{i_1} \tau_{i_2} \tau_{i_3} \ldots$ where $\tau_{j_1} \tau_{j_2} \tau_{j_3} \ldots \tau_{j_{k,j}}$ correspond to elements of the $j$-th cycle of $\pi$. This group has the structure of $(\mathbb{Z}/2\mathbb{Z})^k$ and acts by restriction of $G$ on the $k$-tuples $(\pi_1, \ldots, \pi_k)$. One checks that for any $k$-tuple $(\pi_1, \ldots, \pi_k)$ satisfying length conditions, there exists two and only two elements of $G'$ such that their action turns all $k$-tuples into permutation like elements. \qed

**Theorem 3.16.** $C_{\pi, \Pi, \Pi'}$ is a rational fraction of order $d_{n-1}(\Pi, \Pi') - 2(\text{blocks}(\Pi') - \text{blocks}(\Pi))$ whose leading term is given by $\gamma_{\pi, \Pi, \Pi'}$. Assume that $\pi$ has $d_i$ cycles of length $i - 1$. Then

$$(32) \quad \gamma_{\pi, \Pi, \Pi'} = (-1)^{|\Pi|} \frac{2^{2q} - 2|\Pi| - 1(3q - 3 - |\Pi|)!}{(2q)!} \prod_{i=1}^{d_i} \left( \frac{(2i - 1)!}{(i - 1)!^2} \right)^{d_i}$$

**Proof.** The proof is exactly the same as that of Theorem 2.15 in [Col03]. It is enough, in Equation (2.56), to replace $A[\sigma, k]$ by $B[\pi, k]$ \qed
4. INTEGRATION OVER SYMPLECTIC GROUPS

Let $e_1, \ldots, e_d, f_1, \ldots, f_d$ be an orthonormal basis of $\mathbb{C}^{2d}$. We refer to this basis as the canonical basis. Consider the bilinear antisymmetric form $\langle \cdot, \cdot \rangle$ such that

$$
\langle e_i, f_j \rangle = \delta_{i,j}, \quad \langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0
$$

The symplectic group $\text{Sp}(d)$ is the set of unitary matrices of $M_{2d}(\mathbb{C})$ preserving $\langle \cdot, \cdot \rangle$. Also by $\langle \cdot, \cdot \rangle$ we denote the bilinear form on $(\mathbb{C}^{2d})^\otimes n$ given by the canonical tensor product of forms $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{2d}$. This form is symmetric if $n$ is even and antisymmetric if $n$ is odd.

The Brauer algebra $B(-d, n)$ admits a natural action onto the space $(\mathbb{C}^{2d})^\otimes n$ given in the same way as in Section 3.1.2 with the difference that $\langle \cdot, \cdot \rangle$ should be understood as in Equation (33).

The most of the results from the section 3 remain true also for the symplectic case. Below we present briefly which changes are necessary.

**Theorem 4.1** (Schur–Weyl duality for symplectic groups [Bra37, Wen88, BW89]). The commutant of $\rho_{\text{Sp}(d)}(\text{Sp}(d))$ is equal to $\rho_{B}(\mathbb{C}[P_{2n}])$. Furthermore if $d \geq n$ then $\rho_{B}$ is injective.

For $A \in \text{End}(\mathbb{C}^{2d})^\otimes n$ we set

$$
\mathbb{E}(A) = \int_{\text{Sp}(d)} O^\otimes n A(O^t)^\otimes n \, dO
$$

and define $\Phi(A)$ as in (17). All results of Section 3 remain true with the only difference that the value of $d$ in all formulas should be replaced by $(-d)$.

As for the cumulants, $\gamma_{\pi, \pi, 1_n}$ should be replaced by $(-1)^{k+1} \gamma_{\pi, \pi, 1_n}$ where $k$ is the number of blocks of $\pi$.

5. EXPECTATION OF PRODUCT OF RANDOM MATRICES AND FREE PROBABILITY

This section is rather sketchy since it follows very closely the work of the first–named author [Col03].

5.1. **Asymptotic freeness for orthogonal matrices.** Let $n$ be an integer. We consider the following enumeration of $8n$ integers: $1, \ldots, 4n, \overline{1}, \ldots, \overline{4n}$. Consider $\mathcal{T}$ the subset of $B_{8n}$ such that any pairing links an $i$ with a $\overline{j}$. This set is isomorphic to $S_{4n}$. Call $\mathcal{Z}$ the element of $B_{8n}$ linking $2i - 1$ to $2i$ and $2j - 1$ to $2j$, and $\mathcal{S}$ the subset of $B_{8n}$ such that elements link $2i - 1$ to $2i$ and an odd (resp. even) $j$ to an odd (resp. even) $k$. 
Let $A^{(1)}, \ldots, A^{(2n)}$ be (constant) matrices in $M_d(\mathbb{C})$. For $\tau \in B_{4n}$, and $B$ a random matrix, define
\begin{equation}
\text{tr}(A^{(1)}, \ldots, A^{(2n)}, B, \tau) = d^{-\text{loops}(\Xi, \tau)}E\left( \sum_{k_1, \ldots, k_{4n}, l_1, \ldots, l_{4n}} \prod_{i=1}^{n} B_{k_{2i-1}, k_{2i}} A^{(i)}_{k_{2i-1}, k_{2i}} \delta_{\tau} \right)
\end{equation}
where $\delta_{\tau} = 1$ if for all pair $(i, j)$ of $\tau$, $k_i = k_j$, and 0 else. This expression is obviously a product of normalized traces of $\{B, B^t\}$ alternating with $\{A^{(i)}, A^{(i)t}\}$

Let $\tau \in T$ and $\sigma \in S$. Define
\begin{equation}
\text{tr}(A^{(1)}, \ldots, A^{(2n)}, \tau, \sigma) = d^{-\text{loops}(\sigma, \tau)}E\left( \sum_{k_1, \ldots, k_{2n}, l_1, \ldots, l_{2n}} \prod_{i=1}^{n} A^{(i)}_{k_{2i-1}, k_{2i}} \delta_{\tau} \delta_{\sigma} \right)
\end{equation}
As in Equation (34) this expression is obviously a product of normalized traces of $\{A^{(i)}, A^{(i)t}\}$.

Let $O$ be a random orthogonal Haar distributed matrix in $M_d(\mathbb{C})$. One establishes easily
\begin{equation}
\text{tr}(A^{(1)}, \ldots, A^{(2n)}, O, \tau) = \sum_{\sigma \in S} \text{tr}(A^{(1)}, \ldots, A^{(2n)}, \tau, \sigma) \tilde{W}_g(\sigma, \Xi) d^{l(\Xi, \tau) - l(\Xi, \sigma) - l(\sigma, \tau)}
\end{equation}
where $\tilde{W}_g$ is the asymptotic normalized Wg function restricted on the set $\{1, \ldots, 4n\}$. From this we obtain

**Lemma 5.1.** In Equation (36), assuming that $\{A^{(i)}, A^{(i)t}\}$ admits a joint limit distribution with respect to the normalized trace $\text{tr}$ on $M_d(\mathbb{C})$, any term on the right hand side has asymptotic order $\leq 0$. In case $l(\Xi, \tau) - l(\Xi, \sigma) - l(\sigma, \tau) = 0$, at least two factors of $\text{tr}(A^{(1)}, \ldots, A^{(2n)}, \tau, \sigma)$ have to be of the kind $\text{tr}(A^{(i)})$. In addition, at least two of the such indices $i$ are such that neither the pattern "...OA^{(i)}O*..." nor "...O*A^{(i)}O..." occurs in the cycle decomposition.

**Proof.** The first point is an obvious consequence on triangle inequality. In the case $l(\Xi, \tau) - l(\Xi, \sigma) - l(\sigma, \tau) = 0$, observe that since $l(\Xi, \sigma) \geq n$, one has to have $l(\sigma, \tau) \leq 3n - 1$. The remaining assertions are an easy adaptation of [Col03], Proposition 3.3. (note that $l(\sigma, \tau) \geq 2n$ according to the definition of $T$ and $S$ and the proof follows by an easy graphical interpretation and the description of geodesic given in proof of [Col03], Theorem 3.13) \hfill \Box

From this we deduce:
Theorem 5.2. Let $O_1, O_2, \ldots$ be independent copies of orthogonal ensembles. And $W$ a set of matrices such that the set $(W, W^t)$ admits a limit distribution. Then $W, \{O_1, O_1^*\}, \{O_2, O_2^*\}, \ldots$ are asymptotically free. This convergence holds almost surely.

Proof. Asymptotic freeness is an immediate application of definition of freeness together with the previous Lemma and asymptotic multiplicativity of $Wg$ function established at Theorem 3.13

The proof of almost sure convergence is a consequence on the computation of cumulants of $Wg$ function in Theorem 3.16 together with an application of Chebyshev inequality and Borel-Cantelli lemma (see [Co03], Theorem 3.7 for details).

Remark. We would like to draw the attention of the reader on the fact that the situation is not as general as for the unitary case. For example, in the unitary case, the matrix family $(2^d E_{i,i+1}, (O, O^*)) \in M_d(\mathbb{C})$ admits an asymptotic joint law whereas this is not true in the orthogonal case. One way of getting around this problem is to assume that matrices are bounded. An other option is to modify the joint law assumption by enlarging the family $W$ to $W, W^t$ as we do in the previous Theorem. It is also possible to write down a necessary and sufficient relation from Equation (36) but to our knowledge, there is no mathematical need for this at this point.

5.2. Orthogonal matrix integral. In this section we deal with orthogonal matrix integrals, and in particular with the orthogonal Itzykson-Zuber integral. For unitary matrix integrals many tools are available and this paper together with [Co03] just provide a complementary mathematical approach. However, interestingly enough, it seems that up to now there were no systematic tools for the study of non-unitary (i.e. orthogonal, symplectic) matrix integrals. One bright side of our approach is to provide such a tool and therefore new formulae to theoretical physics.

Theorem 5.3. Let $W$ be a family of matrices such that the family $W, W^t$ admits a limit joint distribution. Let $O_1, \ldots, O_k$ be independent Haar distributed unitary (resp. orthogonal or symplectic) matrices. Let $(P_{i,j})_{1 \leq i, j \leq k}$ and $(Q_{i,j})_{1 \leq i, j \leq k}$ be two families of noncommutative polynomials in $O_1, O_1^*, \ldots, O_k, O_k^*$ and $W$. Let $A_d$ be the random variable $\sum_{k=1}^k \prod_{j=1}^k \text{tr} P_{i,j}(O, O^*, W)$ and $B_d$ the variable $\sum_{k=1}^k \prod_{j=1}^k \text{tr} Q_{i,j}(O, O^*, W)$, where $\text{tr} x = \frac{1}{d} \text{Tr} x$ for $x \in M_d(\mathbb{C})(\mathbb{C})$ denotes the normalized trace.

- (i) For each $d$, the analytic function $z \rightarrow d^{-2} \log \mathbb{E} \exp\{zd^2 A_d\} = \sum_{n \geq 1} a_{d,n} z^n$
is such that for all \( n \) the limit \( \lim_{d \to \infty} a_{d,n} \) exists and is finite. It depends only on the limit distribution of \( W \) and on the polynomials \( P_{i,j} \).

• (ii) For each \( d \), the analytic function

\[
z \mapsto \frac{\mathbb{E} \exp(zB_d + zd^2A_d)}{\mathbb{E} \exp(zd^2A_d)} = 1 + \sum_{n \geq 1} b_{d,n}z^n
\]

is such that for all \( n \) the limit \( \lim_{d \to \infty} b_{d,n} \) exists and is finite. It depends only on the limit distribution of \( W \) and on the polynomials \( P_{i,j} \) and \( Q_{i,j} \).

Proof. This is a straightforward application of Theorem 3.16. See Theorem 4.1 of \([\text{Co03}]\) for details.

As a further illustration of our results on the asymptotics of cumulants, we state the asymptotics of

\[
d^{-2}C_n(d \operatorname{Tr} A_d O B_d O^*)
\]

This number is also known as the coefficient of the series of the orthogonal Itzykson-Zuber integral. Observe that if \( A_d, B_d \) are real symmetric, the Harish-Chandra formula applies and yields a formula for finite dimensional IZ integral provided that the eigenvalues of \( A_d \) and \( B_d \) have no multiplicity. Without these assumptions, there is no formula to our knowledge. However, interesting results have been obtained in \([\text{BH03}]\) (see also references therein) about asymptotics of symplectic Harisch-Chandra integrals and the two results would deserve to be compared.

The asymptotic convergence of \( d^{-2}C_n(d \operatorname{Tr} A_d O B_d O^*) \) provided that \( A_d, A_d^t, B_d, B_d^t \) admit a joint limit distribution is already granted by Theorem 5.3. Let \( G_n \) be the set of (not-necessarily connected) planar graphs (such that any connected component is drawn on a distinct sphere) with \( n \) edges together with the following data and conditions:

i each face has an even number of edges,

ii the edges are labelled from 1 to \( n \),

iii there is a bicolouring in white and black of the vertices such that each black vertex has only white neighbors and vice versa.

To each such graph \( g \in G_n \) we associate the permutations \( \sigma(g) \) (resp. \( \tau(g) \)) of \( S_n \) defined by turning clockwise (resp. counterclockwise) around the white (resp. black) vertices and the function

\[
\text{Moeb}(g) = \gamma_{\tau \sigma^{-1}, \Pi_{r} \vee \Pi_{\sigma}, q + |\tau \sigma^{-1}| + 2(C(\Pi_{r} \vee \Pi_{\sigma}) - 1)}.
\]

For this definition to make sense in the orthogonal framework, we chose an embedding of \( S_n \) into \( B_{2n} \) by partitioning \([1, 2n]\) into two sets \( V_1 \) and \( V_2 \) of
In a permutation $\sigma$, we associate an element of $B_{2n}$, pairing the $i^{th}$ element of $V_1$ to the $\sigma(i)^{th}$ element of $V_2$.

For example in the picture,

$$\sigma = (1 \ 13 \ 2)(3 \ 5 \ 4)(6 \ 7)(8 \ 9 \ 10)(11 \ 12)(16 \ 17)(14 \ 15)$$

$$\tau = (5 \ 6)(7 \ 8)(10 \ 11)(2 \ 3 \ 9)(12 \ 13)(14 \ 17)(15 \ 16)$$

$$\tau \sigma^{-1} = (1 \ 3)(5 \ 9 \ 7)(6 \ 8 \ 11 \ 13 \ 4)(2 \ 12 \ 10)(17 \ 15)(14 \ 16)$$

Two graphs are said to be equivalent if there is a positive oriented diffeomorphism of the plane transforming one to the other and respecting the coloring of the vertices and the labelling of the edges. We call $\sim$ this equivalence relation. For a permutation $\sigma \in S_n$, we call $\langle X \rangle_{\sigma}$ the amount

$$\prod_{i=1}^{k} \text{tr} X^i$$

if $\sigma$ splits into orbits containing $l_1, \ldots, l_k$ elements.

**Theorem 5.4.** If $X_d, Y_d, X^l_d, Y^l_d$ admit a joint limit distribution, one has

$$\lim_{d \to \infty} d^{-2} C_n(d^2 A) = \sum_{g \in G_d / \sim} \langle X \rangle_{\tau(g)} \langle Y \rangle_{\sigma(g)} \text{Moeb}(g)$$

We omit this proof, for it is almost the same as that of Theorem 4.3 of [Col03]. Observe that the asymptotic result only depends on traces of polynomials in $X_d$ and traces of polynomials in $Y_d$. Mixed patterns (involving traces of a non-commutative polynomial in the four variables $X_d, Y_d, X^l_d, Y^l_d$) do not occur in the limit. However we need a control on the joint moments. In other words, the same diagrams appear as in the unitary case. The only difference is that the orthogonal function Moeb is the unitary one times $2^{\#\text{connected components} - 1}$. 
Theorem 5.5. Let $X_d$ be a rank one projection and assume that $(Y_d, Y^t_d)$ has a limit joint distribution whose first marginal is $\mu$.

\[(38) \lim_{d} d^{-1} \cdot C_n(d \text{Tr}(X_d O Y_d O^*) = (n - 1)!k_n(\mu)\]

In other words, the coefficients of $z \rightarrow d^{-1} \log E e^{d \text{Tr}(X_d O Y_d O^*)}$ converge pointwise to those of the primitive of $R$-transform of $\mu$.

The proof goes along the same lines as Theorem 4.7 of [Col03], therefore we omit it. Observe that this result is exactly the same as for the unitary case, except that we need an extra control on the joint moments of $X_d, Y_d, X^t_d, Y^t_d$.

5.3. Orthogonal replaced by symplectic. The statement when replacing orthogonal matrices by symplectic should be replaced in the following way: if $P$ is the unitary such that $P O^T P = O^*$, then $(X_d, X^t_d)$ (resp. $(Y_d, Y^t_d)$) should be replaced by $(X_{2d}, PX^t_{2d} P)$ (resp. $(Y_{2d}, PY^t_{2d} P)$). Theorem 5.3 remains true. Theorem 5.4 as well (one only needs to modify accordingly the definition of Moeb).

In Theorem 5.5 $\mu$ should be replaced by $-\mu$.

6. Examples of $Wg$ Function

We present below the values of the Weingarten function computed for the orthogonal group $O_d$. In order to obtain the appropriate results for the symplectic group $Sp_d$ one should replace in the formulas $d$ by $-d$. These formulae have been obtained directly from the definition of $Wg$, without the help of formula (28). Observe that relative cumulants that can be obtained from these value yield asymptotics predicted by Theorem 3.16 formula (32).
\[ W_g([1]) = d^{-1}, \]
\[ W_g([1, 1]) = \frac{d + 1}{d(d - 1)(d + 2)}, \]
\[ W_g([2]) = \frac{-1}{d(d - 1)(d + 2)}, \]
\[ W_g([1, 1, 1]) = \frac{d^2 + 3d - 2}{d(d - 1)(d - 2)(d + 2)(d + 4)}, \]
\[ W_g([2, 1]) = \frac{-1}{d(d - 1)(d - 2)(d + 4)}, \]
\[ W_g([3]) = \frac{2}{d(d - 1)(d - 2)(d + 2)(d + 4)}, \]
\[ W_g([4]) = \frac{-5d - 6}{d(d + 1)(d + 2)(d + 4)(d + 6)(d - 1)(d - 2)(d - 3)}, \]
\[ W_g([3, 1]) = \frac{2d + 8}{(d + 1)(d + 2)(d + 4)(d + 6)(d - 1)(d - 2)(d - 3)}, \]
\[ W_g([2, 2]) = \frac{d^2 + 5d + 18}{d(d + 1)(d + 2)(d + 4)(d + 6)(d - 1)(d - 2)(d - 3)}, \]
\[ W_g([2, 1, 1]) = \frac{-d^3 - 6d^2 - 3d + 6}{d(d + 1)(d + 2)(d + 4)(d + 6)(d - 1)(d - 2)(d - 3)}, \]
\[ W_g([1, 1, 1, 1]) = \frac{d^4 + 7d^3 + d^2 - 35d - 6}{d(d + 1)(d + 2)(d + 4)(d + 6)(d - 1)(d - 2)(d - 3)}. \]

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