On Mochizuki’s idea of Anabelomorphy and its applications

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Lord of the rings Jean-Marc Fontaine

In Memoriam

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1 A protracted introduction

Shinichi Mochizuki has introduced many fundamental ideas in [Moc12e, Moc13, Moc15] and [Moc12a, Moc12b, Moc12c, Moc12d], amongst one of them is the foundational notion, which I have dubbed anabelomorphy (pronounced as anabel-o-morphy). I coined the term anabelomorphy as a concise way of expressing “Mochizuki’s anabelian way of changing ground field, rings etc.” For a precise definition of anabelomorphy see Section 2. For a picturesque way of thinking about anabelomorphy see 1.6. The notion of anabelomorphy is firmly grounded in a well-known theorem of Mochizuki [Moc97] which asserts that a $p$-adic field is determined by its absolute Galois group equipped with its (upper numbering) ramification filtration. However as there exist many non-isomorphic $p$-adic fields with topologically isomorphic Galois groups, the ramification filtration itself is fluid (for more on the notion of anabelian fluidity of ring structures see [Jos19b]) i.e. a variable and this variable controls amongst many other things, the additive structure of a $p$-adic local field, and Mochizuki’s remarkable suggestion is that this fluidity can and should be exploited in proving theorems in number theory.

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Ryōkan Daigu [AH96]
Let me remark that while this paper is inspired by Mochizuki’s work, this paper is neither intended as a survey nor does it claim to be a survey of Mochizuki’s work. I have freely built on a number of Mochizuki’s ideas and also introduced many of my own and proved a number of new results here. Nevertheless readers will find a number of important results proved here which clarify Mochizuki’s view that anabelomorphy provides a way of understanding Szpiro’s Conjecture. For readers who wish to pursue that topic, but who find the extant literature intimidating, I can do no better than point them to forthcoming papers [DHb], [DHa] for a concrete, succinct and an accessible account of many of the important ideas of [Moc12a, Moc12b, Moc12c, Moc12d].

1.1 What is Anabelomorphy?

The term anabelomorphy arises from the notion of an anabelomorphism (see Section 2) of $p$-adic fields: Two $p$-adic fields $L, K$ are anabelomorphic if there exists an anabelomorphism $L \rightsquigarrow K$ between them i.e. if there exists a topological isomorphism of the absolute Galois groups $G_L \simeq G_K$ (this terminology generalizes readily to higher dimensions by replacing the Galois group by the algebraic fundamental groups). Note that if $L, K$ are isomorphic $p$-adic fields, then $L \rightsquigarrow K$ as both certainly have isomorphic absolute Galois groups. A quantity (resp. a property, an algebraic structure) associated with a $p$-adic field is said to be amphoric if it depends only on the anabelomorphism class of $K$ i.e. if two $p$-adic fields $K, L$ in the same anabelomorphism class have the same quantity (resp. same property, isomorphic algebraic structures). The amphora of $K$ (more precisely of the topological group $G_K$) is the collection of all quantities, properties, algebraic structures associated with $K$ which depend only on the anabelomorphism class of $K$. For example if $K$ is a $p$-adic local field, then the residue characteristic $p$ of $K$ is amphoric, unramifiedness of $K/\mathbb{Q}_p$ is an amphoric property, and $K^*$ is an amphoric topological group. Reader should consult [Hos17] for proofs of all of these assertions as well as a longer list of amphoric properties (Hoshi has also found a number of new amphoric and unamphoric quantities—see [Hos13], [Hos14], [Hos18]). In this paper I also discuss several quantities associated to $K$ which are not amphoric and which do not appear in the existing literature on anabelian geometry.

Mochizuki’s remarkable observation (beginning with [Moc12c, Moc13, Moc15] and continuing in [Moc12a, Moc12b, Moc12c, Moc12d]) is that Anabelomorphy provides, in many situations, the possibility of changing ground fields of algebraic objects without the conventional passage to base extensions. This paper
provides many important examples of this. I mention a few here: Theorem 12.2, Theorem 14.1, Theorem 14.2 and Theorem 13.1. The table of contents of this paper will provide the readers with other results proved here. Mochizuki’s work shows that anabelomorphy complements the notion of base change in algebraic geometry and should be treated as such. As I realized in the course of writing of [Jos19a, Jos19b] that perfectoid algebraic geometry also provides an example of anabelomorphy. This point is discussed in Section 26.

This paper is written from the point of view of Galois representations which I believe will make it easier to appreciate and use Mochizuki’s ideas in algebraic geometry. Since there are many different results proved here, I have followed the following convention: titles of sections indicate the main results proved in that section. In Sections 2–13 (whose titles provide a clear indication of what these sections are about) of the paper I provide many results which can be immediately applied in the contexts of Galois representations and also provide explicit examples of anabelomorphy from the point of view of Galois representations, especially as Mochizuki’s work has not explicitly focused on this aspect and also importantly highlight the quantities of arithmetic interest which can be altered by anabelomorphy (notably the Swan conductor, the different and the discriminant). This is (implicitly) treated from a different viewpoint in [Moc12a, Moc12b, Moc12c, Moc12d].

1.2 Local anabelomorphy and Galois Theoretic Surgery on Number Fields

Anabelomorphy, as I demonstrate here, allows us to perform Galois-theoretic surgery on number fields (see Theorem 16.3, Theorem 16.5). That such a surgery is possible is an idea which is implicit in Mochizuki’s work—especially [Moc12e, Moc13, Moc15], but I provide a direct method to achieve this.

To understand how to perform Galois Theoretic Surgery on Number fields, it will be useful to begin with the following fundamental example (this example is due to [JR79], [Yam76] also had found examples of this type earlier). I studied both, but decided that ultimately the example of [JR79] is easier to work with (for computational purposes). Let me note that the [JR79] example has also been studied by [Hos13, Theorem G(ii)] in a different context (I thank Hoshi for providing me this reference).

Let $K = \mathbb{Q}(\zeta_p, \sqrt{p})$ and $K' = \mathbb{Q}(\zeta_p, \sqrt{1 + p})$ (for an odd prime $p$). I encountered this example when I began studying anabelian geometry in Kyoto in Spring
2018 and its study has provided many fundamental insights on Mochizuki’s ideas. These two number fields are totally ramified at the unique prime lying over $p$. The completions of these fields at the unique primes are respectively $\mathbb{Q}_p(\zeta_p, \sqrt{p})$ and $\mathbb{Q}_p(\zeta_p, \sqrt[1+p]{p})$. It was established in [JR79] that the absolute Galois groups of these two fields are isomorphic but the fields themselves are not isomorphic. In the terminology introduced here these two $p$-adic fields anabelomorphic (but not isomorphic i.e. they are strictly anabelomorphic). This example shows that there exist number fields with distinct (i.e. non-isomorphic) additive structures such that at a finite number of corresponding places the completions have isomorphic Galois groups. This leads to the following definition (see Section 16): two number fields $K, K'$ are anabelomorphically connected along corresponding finite sets of primes $v_i$ of $K$ and $w_i$ of $K'$ if $G_{K_{v_i}} \simeq G_{K'_{w_i}}$ (i.e. the local fields $K_{v_i} \leftrightarrow K'_{w_i}$ are anabelomorphic).

That such number fields exist is an implicit byproduct of Mochizuki’s remarkable Belyian Reconstruction Machine [Moc15, Theorem 1.9] and that this is one of the remarkable ideas which underlies his works [Moc12a, Moc12b, Moc12c, Moc12d].

But it remained a challenge to construct such number fields directly (i.e. without resorting to considering auxiliary curves over $p$-adic fields). Secondly the Belyian Reconstruction Machine [Moc15, Theorem 1.9] provides such fields one place at a time. It seemed to me that there might exist a direct way of constructing such fields with considerable local freedom at several different primes (simultaneously) and that this would have non-trivial consequences.

In this paper I resolve both of these problems (amongst many other new results which are described below). My Theorem 16.3 (case of one prime) and the more general Theorem 16.5 (the case of several primes) provides a direct construction. I do not use Mochizuki’s Belyian Reconstruction Machine in my proofs but constructivity of Mochizuki’s proof has never been far from my mind.

1.3 How to incorporate local anabelomorphic changes into global geometry?

Now let me come to another aspect of Mochizuki’s Belyian Reconstruction Machine [Moc15, Theorem 1.9] whose significance has been missed. The Belyian Reconstruction Machine has two fundamental operating modes, the recognition mode and the construction mode (in Mochizuki’s parlance the bi-anabelian mode and the mono-anabelian mode). The first mode is the more familiar mode which
identifies a curve by its algebraic fundamental group (amongst all such curves). On the other hand, the construction mode, which is an important discovery due to Mochizuki, provides a construction of a curve from its algebraic fundamental group.

As I realized, the true significance of Belyian Reconstruction Machine is that the construction mode allows one to incorporate local anabelomorphic changes into global arithmetic and geometry. The Belyian Reconstruction Machine incorporates Belyi’s Theorem more specifically a Belyi function, the machine allows us to propagate local anabelomorphic changes into global (i.e. over a number field) changes in arithmetic (and hence in geometry).

Let me take a moment to explain Mochizuki’s method of incorporating local changes into global arithmetic and geometry: let $X/M$ be a geometrically connected, smooth curve (of strict Belyi type) over a number field $M$ and let $M \hookrightarrow K$ be a dense embedding of $M$ into a $p$-adic field $K$ and $\pi_1(X/K) \to G_K$ be its Mochizukiod i.e. the algebraic (or tempered) fundamental group equipped with its tautological surjection to $G_K$. Choose an anabelomorphism $K \rightsquigarrow L$. Then one has a composite morphism of profinite groups $\pi_1(X/K) \to G_K \to G_L$ which provides two Mochizukoids $\Pi = \pi_1(X/K) \to G_L$. This Mochizukiod $\Pi \to G_L$ is isomorphic (topologically) to the Mochizukiod $\pi_1(X/K) \to G_K$. Starting with $\Pi \to G_L$, the Belyian Reconstruction Machine provides another curve $Y/M'$ (in general) defined over a number field $M'$ equipped with a dense embedding $M' \hookrightarrow L$ with the property that the Mochizukios of $X/K$ and $Y/L$ are isomorphic! Moreover $M, M'$ are anabelomorphically connected at their respective primes corresponding to the embeddings $M \hookrightarrow K$ and $M' \to L$! In this way local anabelomorphic changes are incorporated to provide global changes.

On the other hand since Richard Taylor’s work on the Fontaine-Mazur Conjecture (and on potential modularity) (see [Tay02]) I had been aware of another method of incorporating local changes into global arithmetic. In loc. cit. (and subsequent works) local changes are incorporated into global geometry by means of Moret-Bailly’s Theorem ([MB89]).

I realized that in fact Moret-Bailly’s Theorem should be viewed as arising from a trivial case of Anabelomorphy: in the context of this theorem (see loc. cit.), for a place $v$ of $K$ which splits completely into places $w_1, \ldots, w_n$ of $K'$, the completions $K_v \simeq K'_{w_i}$, so one has an isomorphism $X(K_v) \simeq X(K'_{w_i})$ and note that as $K_v \simeq K_{w_i}$ are isomorphic hence in particular these fields are (trivially) anabelomorphic $K_v \rightsquigarrow K'_{w_i}$.

So it occurred to me that perhaps there might exist an anabelomorphic version of Moret-Bailly’s Theorem which will provide another way of incorporating lo-
cal anabelomorphic changes into global arithmetic and geometry. This led me to Theorem 19.1 which should be understood as an anabelomorphic version of Moret-Bailly’s Theorem (for $\mathbb{P}^1 - \{0, 1, \infty\}$) about density of global points in $p$-adic topologies (for the special case of anabelomorphically connected fields). Thus I expect that there is a general anabelomorphic theorem (i.e. for any reasonable variety $X/K$ over a number field) of which Moret-Bailly’s Theorem is a special case—see Theorem 25.1 and Conjecture 25.3.

Theorem 19.1 is adequate to prove the Anabelomorphic Connectivity Theorem for Elliptic Curves (see Theorem 20.1). I show that if $E/K$ is an elliptic curve and $(K, \{v_1, \ldots, v_n\}) \sim (K', \{w_1, \ldots, w_n\})$ is an anabelomorphically connected field such that $E$ has semi-stable reduction at $v_1, \ldots, v_n$ then there exists an elliptic curve $E'/K'$ with $\text{ord}_{v_i}(j_E) = \text{ord}_{w_i}(j_{E'})$ and with potentially good reduction at all other non-archimedean primes of $K'$. This result should be considered as a prototype for understanding the sort of constructions envisaged by Mochizuki in his work on the abc-conjecture through Mochizuki’s works [Moc12e, Moc13, Moc15], and [Moc12a, Moc12b, Moc12c, Moc12d].

At any rate these results show that anabelomorphic connectivity theorems provide a way of patching local anabelomorphic modifications of a number field into global geometric data. While this remarkable idea was first discovered and studied extensively by Mochizuki in his work on anabelian geometry especially [Moc12e, Moc13, Moc15], [Moc12a, Moc12b, Moc12c, Moc12d]. The results I prove here do not make use of the central results of loc. cit. but as mentioned earlier, constructivity of Mochizuki’s results has never been far from my mind. I believe that Theorem 16.5, Theorem 20.1 and their refinements will provide yet another concrete way of exploiting Mochizuki’s ideas in more conventional algebraic geometry over number fields.

### 1.4 Role of reducible, ordinary representations and the $L$-invariant

Let me mention two other results proved here which are of general interest: in Mochizuki’s work as well as Richard Taylor’s work (the case of trivial anabelomorphy) crucial role is played by what I have called the Ordinary Synchronization Theorem (see Theorem 17.1). This says that the passage to anabelomorphically connected number field preserves the spaces of two dimensional, reducible and semi-stable representations of Hodge-Tate weights $\{0, 1\}$ at the anabelomorphically connected primes on either side: this arises from the fact (which is implicit in [Moc97]) that if $K \sim L$ is an anabelomorphism of $p$-adic local fields then one
has an isomorphism of $\mathbb{Q}_p$-vector spaces

$$\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1)) \simeq \text{Ext}^1_{G_L}(\mathbb{Q}_p(0), \mathbb{Q}_p(1)),$$

i.e. $\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ is an amphoric $\mathbb{Q}_p$-vector space. These spaces also appear in [Moc12e, Moc13, Moc15] and [Moc12a, Moc12b, Moc12c, Moc12d] (but they are not recognized in this form in loc. cit.) and it suffices to note that these spaces can also be described by means of Kummer Theory and in particular by [PR94a], [Nek93]) one has a description of this space in terms of two coordinates:

$$(\log_K(q), \text{ord}_K(q))$$

(the choice of the branch of $\log_K$ is given by $\log_K(p) = 0$).

So the fact that the $\mathcal{L}$-invariant (which given, for a Tate elliptic curve with Tate parameter $q$, by $\mathcal{L} = \frac{\log_K(q)}{\text{ord}_K(q)}$) is unamphoric is an elementary but important consequence (see Theorem 15.1). The second important observation is that the Fontaine space

$$H^1_f(G_K, \mathbb{Q}_p(1)) \subset H^1(G_K, \mathbb{Q}_p(1)) \simeq \text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$$

consisting of ordinary crystalline two dimensional representations of $G_K$ is also amphoric. This space has two different descriptions: intrinsically in terms of Kummer Theory of $\mathcal{O}^*_K$ (which is manifestly an amphoric description) and a second description in terms of a single coordinate, viz., $\log_K$.

In Theorem 18.4 and Theorem 18.5, I provide the automorphic analogs of the ordinary synchronization theorem. This says that under anabelomorphy $K \leadsto L$, principal series automorphic representations on $\text{GL}_n(K)$ and $\text{GL}_n(L)$ can be synchronized (for $n \geq 1$). For $p > 2$ and $n = 2$ (all $p$ if $n = 1$) I prove that all irreducible admissible representations on $\text{GL}_2(K)$ can be synchronized and that this synchronization respects the local Langlands correspondence. $L$-functions are amphoric. The local Langlands correspondence is compatible with these synchronizations on either side and is in suitable sense amphoric.

In Theorem 24.6, which should be thought of as the Ordinary Synchronization Theorem at archimedean primes, I provide the archimedean analog of Mochizuki’s theory of étale theta functions [Moc09] (which deals with non-archimedean primes of semi-stable reduction). This approach is quite different from Mochizuki’s treatment of archimedean primes, notably I construct a variation of mixed Hodge structures (in particular an indigenous bundle) on the punctured elliptic curve with unipotent monodromy at the origin (of the elliptic curve). I believe that my approach provides a certain aesthetic symmetry by bringing the theory at archimedean primes on par with the theory at semi-stable primes.
In Theorem 22.1 I show that for an elliptic curve $E$ over a $p$-adic field, all the four quantities: the exponent of the discriminant, the exponent of the conductor, the Kodaira Symbol and the Tamagawa Number are unamphoric.

In Section 28 I show that $p$-adic differential equations (in the sense of [And02]) on a geometrically connected, smooth, quasi-projective and anabelomorphic varieties can also be synchronized under anabelomorphy. This should be thought of as “gluing $p$-adic differential equations by their monodromy.” In particular the Riemann-Hilbert Correspondence of [And02] is amphoric.

### 1.5 Two well-known theorems are examples of anabelomorphy

The first theorem (discussed in Section 27) is the famous theorem of Fontaine-Colmez (see [CF00]) (Fontaine’s Conjecture: “weakly admissible $\iff$ admissible”). My remark is that the proof of this theorem is an example of Anabelomorphy carried out on the $p$-adic Hodge side (i.e. on the filtered $(\phi, N)$-modules).

The second theorem (discussed in Section 26) is due to Scholze (see [Sch12a]). My observation is that the theory of perfectoid spaces provides yet another example of anabelomorphy and in particular one should view the proof, in loc. cit., of the “weight monodromy conjecture” as an example of anabelomorphy. Notably that this proof replaces a hypersurface (for which one wants to prove weight-monodromy theorem) by an anabelomorphic hypersurface for which the sought result is already established by other means.

### 1.6 A picturesque way of thinking about anabelomorphy

One could think of anabelomorphy in the following picturesque way:

One has two parallel universes (in the sense of physics) of geometry/arithmetic over $p$-adic fields $K$ and $L$ respectively. If $K, L$ are anabelomorphic (i.e. $K \leftrightarrow L$) then there is a worm-hole or a conduit through which one can funnel arithmetic/geometric information in the $K$-universe to the $L$-universe through the choice of an isomorphism of Galois groups $G_K \simeq G_L$, which serves as a wormhole. Information is transfered by means of amphoric quantities, properties and alg. structures. The $K$ and $L$ universes themselves follow different laws (of algebra) as addition has different meaning in the two anabelomorphic fields $K, L$ (in general). As one might expect, some information appears unscathed on the other side, while some is altered by its passage through the wormhole. Readers will find ample evidence of this information funneling throughout this paper (and also
in [Moc12e, Moc13, Moc15] and [Moc12a, Moc12b, Moc12c, Moc12d] which lay the foundations to it).

1.7 Summary

I hope that these results will convince the readers that Mochizuki’s idea of anabelomorphy is a fundamental new tool in number theory with many potential applications (one of which is Mochizuki’s work on the \(abc\)-conjecture). Especially it should be clear to the readers, after reading this paper, that assimilating Mochizuki’s idea of anabelomorphy and anabelomorphic connectivity into the theory of Galois representations will have ramifications for arithmetic. I have considered anabelomorphy for number fields but interpolating between the number field case and my observation that perfectoid algebraic geometry is a form of anabelomorphy, it seems reasonable to imagine that anabelomorphy of higher dimensional fields will have applications to birational geometry. So I believe that the story of anabelomorphy is still at its very beginnings . . .

1.8 Acknowledgements

Some time in 1994–1995, I was introduced to Jean-Marc Fontaine when he arrived at the Tata Institute (Mumbai) to teach a course on the theory of Fontaine rings and \(p\)-adic Hodge theory. At that time there were few practitioners of this subject outside of Bonn (MPI), Chicago, Harvard, Kyoto (RIMS), Paris (ORSAY), Princeton and Tokyo and I was fortunate enough to learn \(p\)-adic Hodge theory directly from him. In the coming years, Fontaine arranged my stays in Paris (1996, 1997, and 2003) which provided me an opportunity to learn \(p\)-adic Hodge Theory from him while he (and a few others) were engaged in creating it. Influence of Fontaine’s ideas on this paper should be obvious. I dedicate this paper to the memory of Jean-Marc Fontaine. The Tata Institute of Fundamental Research, notably M. S. Raghunathan (who, I believe, headed the Indo-French Mathematics Program at the time), was instrumental in providing me with this opportunity of working with Fontaine and it is a pleasure to acknowledge the support provided by M. S. Raghunathan and the Institute.

The reflections recorded here have their origin in the survey [Jos19a] of [Moc12e, Moc13, Moc15] and [Moc12a, Moc12b, Moc12c, Moc12d] which I began writing during my stay in Kyoto (Spring 2018) and I am profoundly grateful to Shinichi Mochizuki for many conversations and correspondence on his results documented in loc. cit. as well as topics treated in this paper. He also patiently answered all my
inane questions, and hosted my stay at RIMS. Mochizuki’s deep and profoundly original ideas continue to be a source of inspiration and his influence on my ideas should be obvious. Prior to my arrival in Kyoto, I had not studied anabelian geometry in any serious way (and I claim no expertise in the subject even today), and I would like to especially thank Yuichiro Hoshi for patiently answering many inane and elementary questions on anabelian geometry and for teaching me many important results in the subject. Hoshi also made a number of suggestions and corrections which have improved this manuscript. Support and hospitality from RIMS (Kyoto) is gratefully acknowledged.

Thanks are also due to Machiel van Frankenhuijsen for many conversations on the abc-conjecture and Mochizuki’s Anabelian Reconstruction Yoga and also to Taylor Dupuy for innumerable conversations around many topics treated here and for providing versions of his manuscripts [DHb], [DHa]. While the reflections recorded here began with my stay in Kyoto, after my recent visit to University of Vermont and my conversations with Taylor Dupuy and Anton Hilado, I decided to write up a brief two/three page note containing remarks recorded in Sections 2–5, 7–11, as I felt these remarks would be useful to others. Taylor patiently listened to many of my unformed ideas and also carefully read several early versions of this manuscript and provided number of suggestions and improvements for which I am extremely grateful. Eventually I decided to include here some ideas which I had recorded in the context of my survey [Jos19a] or appeared in a condensed form in loc. cit. This exercise also led to a substantially longer manuscript than I had originally planned.

I also thank Tim Holzschuh for a careful reading of the manuscript and pointing out many typos.

2 Anabelomorphy, Amphoras and Amphoric quantities

Let \( p \) be a fixed prime number. Occassionally I will write \( \ell \) for an arbitrary prime number not equal to \( p \). Let \( K \) be a \( p \)-adic field, \( \bar{K} \) an algebraic closure of \( K \), \( G_K = \text{Gal}(\bar{K}/K) \) be its absolute galois group considered as a topological group, \( I_K \subset G_K \) (resp. \( P_K \subset G_K \)) the inertia (resp. wild inertia) subgroup of \( G_K \). Let \( K, L \) be two \( p \)-adic fields. I will say two \( p \)-adic fields \( K, L \) are anabelomorphic or anabelomorphs (or anabelomorphs of each other) if their absolute Galois groups are topologically isomorphic \( G_K \simeq G_L \). I will write \( K \rightsquigarrow L \) if \( K, L \) are anabe-
lomorphic and $\alpha : K \leftrightarrow L$ will mean a specific isomorphism $\alpha : G_K \to G_L$ of topological groups. Obviously if $L \leftrightarrow L'$ and $L' \leftrightarrow L''$ then $L \leftrightarrow L''$. So anabelomorphism is an equivalence relation on $p$-adic fields. The collection of all fields $L$ which are anabelomorphic to $K$ will be called the anabelomorphism class of $K$. I will say that $K$ is strictly anabelomorphic to $L$ or that $K \leftrightarrow L$ is a strict anabelomorphism if $K \leftrightarrow L$ but $K$ is not isomorphic to $L$.

For a fixed $p$-adic field $K$, a field $L$ anabelomorphic to $K$ will be simply referred to as an unjilt of $K$ (or that $L$ is an unjilt of $K$) and $K$ will be called a jilt of $L$. In particular the collection of allunjilts of $K$ is the anabelomorphism class of $K$.

A quantity or an algebraic structure or a property of $K$ is said to be an amphoric quantity (resp. amphoric algebraic structure, amphoric property) if this quantity (resp. alg. structure or property) depends only on the isomorphism class of the topological group $G_K$ of $p$-adic local fields, more precisely: if $\alpha : G_K \simeq G_L$ is an isomorphism of topological groups of $p$-adic fields then $\alpha$ takes the quantity (resp. algebraic structure, property) for $K$ to the corresponding quantity (resp. alg. structure, property) of $L$. A fundamental fact of anabelian geometry is that for a $p$-adic field $K$, $K^*$ is amphoric i.e. let $\alpha : K \leftrightarrow L$ be an abelomorphism of $p$-adic local fields. Then one has an isomorphism $\alpha : K^* \to L^*$ of topological groups (see [Hos17]).

This is the definition of [Moc97]. Evidently amphoric quantities (alg. structures, properties) are determined by the topological isomorphism class of the galois group of a $p$-adic local field. The collection of all amphoric quantites, algebraic structures or properties of $G_K$ is called the amphora of the topological group $G_K$. If a quantity (resp. alg. structure, property) of a $p$-adic field $K$ which is not amphoric will simply be said to be unamphoric or not amphoric.

In other words a quantity (resp. an algebraic structure, property) determined by the anabelomorphism class of a field is amphoric otherwise it is unamphoric.

For example the prime number $p$, the degree of $K/Q_p$, the ramification index of $K/Q_p$, the inertia subgroup $I_K$ and the wild inertia subgroup $P_K \subset I_K$ of $G_K$, are amphoric (see [Moc97], [Hos17]) but the ramification filtration is unamphoric.

Since I am thinking of applications of Mochizuki’s ideas, it would be useful to allow some additional generality. Consider an auxiliary topological field $E$ which will serve as a coefficient field for representations of $G_K$. Let $V/E$ be a finite dimensional $E$-vector space (as a topological vector space). Let $\rho : G_K \to \text{GL}(V)$ be a continuous representation of $G_K$. I will say that a quantity or an algebraic structure or a property of the triple $(G_K, \rho, V)$ is amphoric if it is determined by the anabelomorphism class of $K$.

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The definition of anabelomorphy of fields readily extends to smooth varieties of higher dimensions. Let $X/K$ and $Y/L$ be two geometrically connected, smooth quasi-projective varieties over field $K$ and $L$. Then I say that $X/K$ and $Y/L$ are anabelomorphic varieties (resp. tempered anabelomorphic) if there exists an isomorphism of their algebraic fundamental groups i.e. $\pi_1(X/K) \simeq \pi_1(Y/L)$ (resp. tempered fundamental groups). By [Moc12e, Corollary 2.8(ii)] it follows that if two varieties $X/K$ and $Y/L$ are anabelomorphic then $K \leftrightarrow L$. Anabelomorphism (resp. tempered anabelomorphism) also defines an equivalence relation on smooth varieties over $p$-adic fields. For a simple way of producing anabelomorphic varieties see Theorem 21.1. Of particular importance in diophantine applications is Corollary 21.2 which asserts that if $K \leftrightarrow L$ then $\mathbb{P}^1 - \{0, 1728, \infty\}/K$ and $\mathbb{P}^1 - \{0, 1728, \infty\}/L$ are anabelomorphic.

Evidently isomorphic varieties over a $p$-adic field are anabelomorphic (over that field). But the converse is not true: anabelomorphic varieties need not be isomorphic. A simple example to keep in mind is this: let $K \leftrightarrow L$ be anabelomorphic $p$-adic fields and let $X = \mathbb{P}^n_K$ and $Y = \mathbb{P}^n_L$. Then $X/K$ and $Y/L$ are anabelomorphic. If $K, L$ are not isomorphic as fields then these varieties are not isomorphic as schemes (but are anabelomorphic). For a more detailed discussion of anabelomorphic varieties see Section 21.

**Remark 2.1.** A common misconception which appears to exists is that Mochizuki’s proof of Grothendieck Conjecture for smooth, hyperbolic curves over $p$-adic fields implies that two anabelomorphic, hyperbolic curves over $p$-adic fields are isomorphic as schemes. *Unfortunately this is not true if the fields are strictly anabelomorphic (i.e. anabelomorphic but not isomorphic).* Here is a proof. Suppose that $X/K, Y/L$ are two smooth, anabelomorphic, hyperbolic curves over strictly anabelomorphic $p$-adic fields $K, L$ are isomorphic as schemes then this implies that there is an isomorphism of their generic points $K(X) \simeq L(Y)$ which implies that their constant fields are isomorphic which contradicts our assumption that $K, L$ are strictly anabelomorphic. Alternately one can use [Moc06, Theorem 6.4] to arrive at the same conclusion since any isomorphism of schemes in particular provides a dominant morphism $X_K \to X_L$ and Mochizuki’s theorem says that dominant morphisms $X_K \to X_L$ are in bijection with field embeddings $L \hookrightarrow K$. So $L \hookrightarrow K$ is a subfield and since role of $K, L$ is symmetric this implies $K \simeq L$ which contradicts our assumption that $K \leftrightarrow L$ is strict.
3 Five fundamental theorems of Anabelomorphy

For the reader’s convenience I provide here five fundamental theorems of anabelian geometry upon which anabelomorphy rests. I have organized the results in a logical manner (as opposed to a chronological order). The first fundamental theorem is due to Neukirch and Uchida.

**Theorem 3.1 (First Fundamental Theorem of Anabelomorphy).** If $K$, $L$ and number fields then $K \sim L$ if and only if $K \cong L$.

Modern proof of this result can be found in Hoshi’s excellent paper [Hos15]. The second fundamental theorem is due to Mochizuki [Moc97].

**Theorem 3.2 (Second Fundamental Theorem of Anabelomorphy).** If $K$, $L$ are $p$-adic fields then $K \simeq L$ if and only if there is a topological isomorphism of their Galois groups equipped with the respective (upper numbering) inertia filtration i.e. $(G_K, G_K^\bullet) \simeq (G_L, G_L^\bullet)$

The following theorem is a combination of many different results proved by (Neukirch, Uchida, Jarden-Ritter, Mochizuki) in different time periods.

**Theorem 3.3 (Third Fundamental Theorem of Anabelomorphy).** Let $K$ be a $p$-adic field. Then

1. The residue characteristic $p$ of $K$ is amphoric.
2. The degree $[K : \mathbb{Q}_p]$ and $e_K$ the absolute ramification index are amphoric.
3. The topological groups $K^\ast$ and $\mathcal{O}_K^\ast$ are amphoric topological groups.
4. The inertia subgroup $I_K$ and the wild inertia subgroup $P_K$ are amphoric.
5. The $p$-adic cyclotomic character $\chi_p : G_K \to \mathbb{Z}_p^\ast$ is amphoric.

For modern proof of the first three assertions see [Hos17] for the last assertion see [Moc97]. Hoshi’s paper also provides a longer list of amphoric quantities, properties and alg. structures.

The next assertion is the Jarden-Ritter Theorem [JR79]. This provides a way of deciding if two fields are anabelomorphic or not in most important cases.

**Theorem 3.4 (Fourth Fundamental Theorem of Anabelomorphy).** Let $K, L$ be $p$-adic fields with $\zeta_p \in K$ and both $K, L$ contained in $\overline{\mathbb{Q}}_p$. Write $K \supseteq K^0 \supseteq \mathbb{Q}_p$ (resp. $L \supseteq L^0 \supseteq \mathbb{Q}_p$) be the maximal abelian subfield contained in $K$. Then the following are equivalent:
Theorem 3.5 (Fifth Fundamental Theorem of Anabelomorphy). Let $K$ be a $p$-adic field and let $I_K \subseteq G_K$ (resp. $P_K \subseteq G_K$) be the inertia subgroup (resp. the wild inertia subgroup). Then $I_K$ and $P_K$ are topological characteristic subgroups of $G_K$ (i.e. invariant under all topological automorphisms of $G_K$).

For proofs see [Moc97] [Hos17].

4 Unramifiedness and tame ramifiedness of a local Galois representation are amphoric

Let $K$ be a $p$-adic field. In this section I consider continuous $G_K$ representations with values in some finite dimensional vector space over some coefficient field $E$ which will be a finite extension of one the following fields: $\mathbb{Q}_\ell, \mathbb{Q}_p$ or a finite field $\mathbb{F}_p$. All representations will be assumed to be continuous (with the discrete topology on $V$ if $E$ is a finite field) without further mention.

Let $\rho : G_K \to GL(V)$ be a representation of $G_K$. Let $\alpha : K \leftrightarrow L$ be an anabelomorphism. Then as $\alpha : G_K \simeq G_L$, so any $G_K$-representation gives rise to a $G_L$-representation by composing with $\alpha^{-1} : G_L \to G_K$ and conversely any $G_L$-representation gives rise to a $G_K$ representation by composing with $\alpha : G_K \to G_L$. One sees immediately that this isomorphism induces an equivalence between categories of finite dimensional continuous representations. In particular the category of $G_K$-representations is amphoric.

Now suppose $W \subset V$ is a $G_K$-stable subspace. Let $\alpha : G_L \to G_K$ be an anabelomorphism. Then $W$ is also a $G_L$ stable subspace of $V$. This is clear as $\rho(\alpha(g))(W) \subseteq \rho(W) \subseteq W$ for all $g \in G_L$. In particular if $\rho$ is a reducible representation of $G_K$ then so is the associated $G_L$-representation. Conversely any reducible $G_L$-representation provides a reducible $G_K$ representation. This discussion is summarized in the following elementary but useful result:

Proposition 4.1. Let $K$ be a $p$-adic field and let $E$ be a coefficient field.

(1) The category of finite dimensional $E$-representations is amphoric.
(2) Irreducibility of a $G_K$-representation is an amphoric property.

Recall that a Galois representation $\rho : G_K \to GL(V)$ is said to be an unramified representation (resp. tamely ramified) if $\rho(I_K) = \{1\}$ (resp. $\rho(P_K) = \{1\}$.

Recall that $\rho : G_K \to GL(V)$ is unramified (resp. tamely ramified) if the image $\rho(I_K) = 1$ (resp. $\rho(P_K) = 1$).

**Theorem 4.2.** Let $K$ be a $p$-adic local field. Unramifiedness (resp. tame ramifiedness) of $\rho : G_K \to GL(V)$ are amphoric properties.

*Proof.* This is clear from the definition of unramifiedness (resp. tame ramifiedness) and the fact that $I_K$ (resp. $P_K$) are amphoric (see [Hos17, Proposition 3.6]).

5 Ordinarity of a local Galois representation is amphoric

Let me note that Mochizuki (in [Moc12a, Moc12b, Moc12c, Moc12d]) considered ordinary representations arising from Tate elliptic curves. In [Hos18] Hoshi considered proper, hyperbolic curves with good ordinary reduction and the standard representation associated with the first étale cohomology of this curve. My observation (recorded here) which includes both the $\ell$-adic and the $p$-adic cases is that the general case is not any more difficult and of fundamental importance in many applications.

Let $\rho : G_K \to GL(V)$ be a continuous $E$-representation of $G_K$ with $E \supseteq \mathbb{Q}_\ell$ a finite extension of $\mathbb{Q}_\ell$ (and $\ell \neq p$). Then $(\rho, V)$ is said to be an ordinary representation of $G_K$ if the image $\rho(I_K)$ of the inertia subgroup of $G_K$ is unipotent.

**Theorem 5.1.** Ordinarity is an amphoric property of $\rho : G_K \to GL(V)$.

*Proof.* Let $\rho : G_K \to GL(V)$ be a continuous Galois representation on $G_K$ on a finite dimensional $E$ vector space with $E/\mathbb{Q}_\ell$ a finite extension. Let $L$ be a $p$-adic field with an isomorphism $\alpha : G_L \simeq G_K$. By Theorem 3.5 [Moc97] or [Hos17, Proposition 3.6] the inertia (resp. wild inertia) subgroups are amphoric. Then $\rho(\alpha(I_L)) \subset \rho(I_K)$ so the image of $I_L$ is also unipotent.

Now before I discuss the $p$-adic case, let me recall that it was shown in [Moc97] that for any $p$-adic field $K$, the $p$-adic cyclotomic character of $G_K$ is amphoric. Let $\chi_p : G_K \to \mathbb{Z}_p^*$ be a $p$-adic cyclotomic character. Recall from [PR94a] that
a \( p \)-adic representation \( \rho : G_K \to \text{GL}(V) \) with \( V \) a finite dimensional \( \mathbb{Q}_p \)-vector space is said to be an ordinary \( p \)-adic representation of \( G_K \) if there exist \( G_K \)-stable filtration \( \{V_i\} \) on \( V \) consisting of \( \mathbb{Q}_p \)-subspaces of \( V \) such that the action of \( I_K \) on \( \text{gr}_i(V) \) is given by \( \chi_p^i \) (as \( G_K \)-representations).

**Theorem 5.2.** Ordinarity of a \( p \)-adic representation \( \rho : G_K \to \text{GL}(V) \) is an amphoric property.

**Proof.** It is immediate from the Prop. 4.1 that the filtration \( V_i \) is also \( G_L \)-stable. By Theorem 3.4, \( \chi_p \) (and hence its powers) are amphoric. By definition for any \( v \in V_i \) and any \( g \in I_K \),

\[
\rho(g)(v) = \chi_p^i(g)v + V_{i+1}.
\]

Now given an isomorphism \( \alpha : G_L \to G_K \), one has for all \( g \in G_L \)

\[
\rho(\alpha(g))(v) = \chi_p^i(\alpha(g))v + V_{i+1}.
\]

This condition is thus determined solely by the isomorphism class of \( G_K \). \( \square \)

Two dimensional ordinary (reducible) \( p \)-adic representations play an important role in [Moc12a, Moc12b, Moc12c, Moc12d] (not merely because the arise from Tate elliptic curves) and I will return to this topic in Section 17 and especially Theorem 17.1.

Theorem 5.2 should be contrasted with the following result which combines results of [Moc12e, Hos13, Hos18]:

**Theorem 5.3.** (1) Let \( \alpha : K \leftrightarrow L \) be an anabelomorphism of \( p \)-adic fields. Then the following conditions are equivalent

(a) For every Hodge-Tate representation \( \rho : G_K \to \text{GL}(V) \), the composite \( \rho \circ \alpha \) is Hodge-Tate representation of \( G_L \).

(b) \( K \cong L \).

(2) There exists a prime \( p \) and a \( p \)-adic local field \( K \) and an automorphism \( \alpha : G_K \to G_K \) and a crystalline representation \( \rho : G_K \to \text{GL}(V) \) such that \( \rho \circ \alpha : G_K \to \text{GL}(V) \) is not crystalline. In other words, in general crystalline-ness is not amphoric property of \( \rho : G_K \to \text{GL}(V) \).

(3) In particular being crystalline, semi-stable or de Rham is not an amphoric property of a general \( p \)-adic representation.

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6 \( \Phi_{\text{Sen}} \) is unamphoric

Let me begin with a somewhat elementary, but surprising result which is still true (despite of the above results of Mochizuki and Hoshi). This result is surprising because of Mochizuki’s Theorem (see [Moc97]) which says that the \( p \)-adic completion \( \hat{K} \) is unamphoric. For \( \hat{K} \)-admissible representations see [Fon94].

**Theorem 6.1.** Let \( K \) be a \( p \)-adic field and let \( \alpha : L \leftrightarrow K \) be an anabelomorphism. Let \( \rho : G_K \to \text{GL}(V) \) be a \( p \)-adic representation.

1. Then \( V \) is \( \hat{K} \)-admissible if and only if \( \rho \circ \alpha \) is \( \hat{L} \)-admissible.

2. In particular \( V \) is pure of Hodge-Tate weight \( m \) as a \( G_K \)-module if and only if \( V \) is pure of Hodge-Tate weight \( m \) as a \( G_L \)-module.

**Proof.** A well-known theorem of Shankar Sen [Sen80] or [Fon94, Proposition 3.2], \( V \) is \( \hat{K} \)-admissible if and only if the image of inertia is \( \rho(I_K) \) is finite. By the Third Fundamental Theorem of Anabelomorphy (Theorem 3.3) if \( \rho(I_K) \) is finite then so is \( \rho(\alpha(I_L)) \). So the assertion follows.

Twisting \( V \) by \( \chi_p^{-m} \), one can assume that \( V \) is Hodge-Tate of weight zero as a \( G_K \)-representation. Then by Shankar Sen’s Theorem referred to earlier, image of \( I_K \) under \( \rho \) is finite. Hence the image of \( I_L \) under \( \rho \circ \alpha \) is finite. This proves the assertion. \( \square \)

Now let me prove the following elementary reformulation of Mochizuki’s Theorem [Moc12e, Theorem 3.5(ii)] which asserts that the property of being Hodge-Tate representation is unamphoric. My point is that my formulation (given below) shows more precisely why this happens. Let me set up some notation. Let \( K \) be a \( p \)-adic field and let \( H_K \subset G_K \) be the kernel of the cyclotomic character \( \chi_p : G_K \to \mathbb{Z}_p^* \). Let \( K_\infty = \overline{K}^{H_K} \) be the fixed field of \( H_K \). Let \( \alpha : L \leftrightarrow K \) be an anabelomorphism. Let \( H_L \subset G_L \) be the kernel of the cyclotomic character \( \chi_p : G_L \to \mathbb{Z}_p^* \) (note the conflation of notation made possible by the amphoricity of the cyclotomic character). Let \( L_\infty = \overline{L}^{H_L} \) be the fixed field of \( H_L \). By the amphoricity of the cyclotomic character one has an isomorphism \( H_L \simeq H_K \) and hence also of the quotients \( G_K/H_K \simeq G_L/H_L \). Hence one sees that one has an anabelomorphism \( L_\infty \leftrightarrow K_\infty \). Consider a \( p \)-adic representation \( \rho : G_K \to \text{GL}(V) \) of \( G_K \). By a fundamental theorem of [Sen80, Theorem 4], there exists an endomorphism \( \Phi_{\text{Sen}} \in \text{End}((V \otimes \hat{K})^{H_K}) \) of the \( K_\infty \)-vector space \( (V \otimes \hat{K})^{H_K} \). Another theorem of loc. cit (see [Sen80, Corollary of Theorem 6])
asserts that $G_K$-representation $V$ is Hodge-Tate if and only if $\Phi_{\text{Sen}}$ is semi-simple and eigenvalues of $\Phi_{\text{Sen}}$ are integers. Let me note that by [Sen80, Theorem 5] one can always find a basis of $(V \otimes \hat{K})^H_K$ such $\Phi_{\text{Sen}}$ is given by a matrix with coefficients in $K$. Let $\Phi_{\text{Sen}}^\alpha$ be the endomorphism of the $L_\infty$-vector space $(V \otimes \hat{L})^H_L$ (considering $V$ as a $G_L$-representation through $\alpha$).

**Theorem 6.2.** Let $K$ be a $p$-adic field. Let $\rho : G_K \to \text{GL}(V)$ be a $p$-adic representation of $G_K$. Then $\Phi_{\text{Sen}}$ is unamphoric. If $\alpha : L \hookrightarrow K$ is an anabelomorphic then $\rho \circ \alpha : G_L \to \text{GL}(V)$ is Hodge-Tate if and only if $\Phi_{\text{Sen}}^\alpha$ is semi-simple and has integer eigenvalues.

One way to understand this result is to say that $\Phi_{\text{Sen}}$ is an invariant of $\rho : G_K \to \text{GL}(V)$ which depends on the additive structure of $K$.

## 7 Artin and Swan Conductor of a local Galois representation are not amphoric

(For additional results on this topic see Section 23, for consequences of this in the context of elliptic curves see Section 22 For Artin and Swan conductors see [Ser79], [Kat88, Chapter 1]. Coefficient field of our $G_K$ representations will be a finite extension $E/\mathbb{Q}_l$ with $l \neq p$. The Artin conductor (resp. the Swan conductor) of an unramified (resp. tamely ramified) representation are zero. So one must assume that the wild inertia subgroup acts non-trivially for these conductors to be non-zero. The theorem is the following:

**Theorem 7.1.** Let $\rho : G_K \to \text{GL}(V)$ be a local Galois representation with $E = \mathbb{Q}_l$ such that the image of $P_K$ is non-trivial. Then the Artin and the Swan conductors of $\rho : G_K \to \text{GL}(V)$ are not amphoric.

**Proof.** It is enough to prove that the Swan conductor is not amphoric. This is clear as the Swan conductor is given in terms of the inertia filtration. Since $G_K \simeq G_L$ is not an isomorphism of filtered groups (by [Moc97]) so the Artin and the Swan conductors of the $G_L$ representation $G_L \to G_K \to \text{GL}(V)$ is not the same as that of the $G_K$-representation in general. To see the explicit dependence of the Artin (resp. Swan) conductors on the inertia filtration see [Ser79], [Kat88][Chap 1]. To illustrate my remark it is enough to give an example. Let $K_1 = \mathbb{Q}_p(\zeta_p, \sqrt{1+p})$ and $K_2 = \mathbb{Q}_p(\zeta_p, \sqrt{p})$. Then $\text{Gal}(\overline{K_1}/\mathbb{Q}_p) \simeq \mathbb{Z}/p \rtimes (\mathbb{Z}/p)^*$ and $\text{Gal}(\overline{K_2}/\mathbb{Q}_p) \simeq \mathbb{Z}/p \rtimes (\mathbb{Z}/p)^*$. By the character table for this finite group (see [Viv04, Theorem 3.7]), there is a
unique irreducible character $\chi$ of dimension $p - 1$. Let $f_i(\chi)$ for $i = 1, 2$ denote the exponent of the Artin conductor of $\chi$. Then by [Viv04, Cor. 5.14 and 6.12] one has

$$f_1(\chi) = p \quad (7.2)$$
$$f_2(\chi) = 2p - 1. \quad (7.3)$$

Evidently $f_1(\chi) \neq f_2(\chi)$.

The following two results are fundamental for many applications of anabelomorphy.

**Theorem 7.4.** Let $\rho : G_K \to \text{GL}(V)$ be an $\ell$-adic representation of $G_K$. Then there exists a unique, smallest integer $x \geq 0$ such that in the anabelomorphism class of $K$, there exists an $L \hookrightarrow K$ such that the Swan conductor of the $G_L$-representation $G_L \to G_K \to \text{GL}(V)$ is $x$.

**Proof.** By [Kat88, Prop. 1.9] or [Ser79, ], the Swan conductor is an integer $\geq 0$. So one can pick an $L \hookrightarrow K$ to such that the Swan conductor is minimal.

**Corollary 7.5** (Anabelomorphic Level Lowering). In the anabelomorphism class of a $p$-adic field $K$, there exists an anabelomorphism $\alpha : L \hookrightarrow K$ such that for any $\mathbb{Q}_\ell$-adic or an $\mathbb{F}_\ell$-representation $\rho : G_K \to \text{GL}(V)$, the $G_L$-representation $\rho \circ \alpha : G_L \to G_K \to \text{GL}(V)$ has the smallest Artin conductor.

## 8 Peu and Tres ramifiedness are not amphiomorphic properties

For more on the notion of peu and tres ramifiée extensions readers should consult [Ser87, Section 2.4], [Edi92], here I recall the definitions. Let $K$ be a $p$-adic field and $K \supseteq K^t \supseteq K^{nr} \supseteq \mathbb{Q}_p$ be the maximal tamely ramified (resp. maximal unramified) subextension of $K/\mathbb{Q}_p$ such that $K = K^{nr}(\sqrt{x_1}, \cdots, \sqrt{x_m})$ with $x_i \in K^{nr} - (K^{nr})^p$ for all $i$. Then $K$ is **peu ramifiée** if $v_{K^{nr}}(x_i) = 0 \mod p$ for all $i$, otherwise $K$ is **tres ramifiée**. Recall from [Edi92, Prop 8.2] that $\bar{\rho} : G_K \to \text{GL}_n(\overline{\mathbb{F}}_q)$ is peu ramifiée (i.e. the fixed field of its kernel is peu ramifiée) if and only $\bar{\rho}$ arises from a finite flat group scheme over $\mathcal{O}_K$ (the ring of integers of $K$).

**Theorem 8.1.** The property of being peu ramifiée (resp. being tres ramifiée) extension (resp. representation) is not amphiomorphic.
Proof. It will be sufficient to prove that there exist $p$-adic fields $K \hookrightarrow L$ such that $K/\mathbb{Q}_p$ is peu ramifiée and $L/\mathbb{Q}_p$ is tres ramifiée. Let $K = \mathbb{Q}_p(\zeta_p, \sqrt[p]{p})$ and $L = \mathbb{Q}_p(\zeta_p, \sqrt[p]{1+p})$. Then by [JR79] or by (Lemma 9.4) one has $G_K \simeq G_L$ and by definition of [Ser87, Section 2.4], $K$ is tres ramifiée and $L$ is peu ramifiée. Hence the claim.

Combining this with [Edi92, Prop 8.2] one gets:

**Corollary 8.2.** Finite flatness of a $G_K$-representation (into $\text{GL}(V)$ with $V$ a finite dimensional $\mathbb{F}_q$-vector space) is not an amphoric property.

Reader should contrast the above corollary with Theorem 13.1.

### 9 Discriminant and Different of a $p$-adic field are unamphoric

For definition of the *different* and the *discriminant* of a $p$-adic field see [Ser79]. The following result is fundamental for many diophantine applications.

**Theorem 9.1.** The different and the discriminant of a finite Galois extension $K/\mathbb{Q}_p$ are unamphoric.

**Proof.** By Theorem [Ser79, Chap III, Prop 6] it is sufficient to prove the different is not amphoric. By Theorem [Ser79, Chap IV, Prop 4] the different depends on the ramification filtration for $K/\mathbb{Q}_p$. So in general there exist anabelomorphs $K, L$ with distinct different and discriminants. Here is an explicit family of examples.

Let $r \geq 1$ be an integer, $p$ an odd prime and let $K_r = \mathbb{Q}_p(\zeta_{p^r}, \sqrt[p^r]{p})$ so $F_r \subset K_r$ and let $L_r = \mathbb{Q}_p(\zeta_{p^r}, \sqrt[p^r]{1+p})$ by Lemma 9.4 below one has $K_r \hookrightarrow L_r$ and hence one has $G_{L_r} \simeq G_{K_r}$. But $K_r$ and $L_r$ are not isomorphic fields so by [Moc97] they have distinct inertia filtrations. I claim that they have distinct different and discriminants. More precisely one has the following formulae for the discriminants of $K_r/\mathbb{Q}_p$ (resp. $L_r/\mathbb{Q}_p$) [Viv04, Theorem 5.15 and 6.13].

\[
v_p(\delta(K_r/\mathbb{Q}_p)) = rp^{2r-1}(p-1) + p \left( \frac{p^{2r} - 1}{p + 1} \right) - p \left( \frac{p^{2r-3} + 1}{p + 1} \right), \quad (9.2)\]

\[
v_p(\delta(L_r/\mathbb{Q}_p)) = p^r \left( r \cdot p^r - (r+1) \cdot p^{r-1} \right) + 2 \left( \frac{p^{2r} - 1}{p + 1} \right). \quad (9.3)\]

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In particular for \( r = 1 \) these are equal to \( 2p(p - 1) + 1 \) and \( p^2 - 2 \) respectively and evidently \( 2p(p - 1) + 1 \neq p^2 - 2 \) for any odd prime \( p \). This proves the assertion.

**Lemma 9.4.** Let \( r \geq 1 \) be any integer and \( p \) any odd prime. Let \( F_r = \mathbb{Q}_p(\zeta_{p^r}) \) and let \( K_r = \mathbb{Q}_p(\zeta_{p^r}, \sqrt[p^r]{p}) \) and let \( L_r = \mathbb{Q}_p(\zeta_{p^r}, \sqrt[p^r]{1 + p}) \). Then one has

\[
G_{L_r} \simeq G_{K_r}, \text{ equivalently } K_r \leftrightarrow L_r \text{ equivalently } K_r \text{ and } L_r \text{ are anabelomorphic.}
\]

**Proof.** Both fields contain \( F_r = \mathbb{Q}_p(\zeta_{p^r}) \) and by elementary Galois theory and Kummer theory one checks that \( F_r \subset K_r \) and \( F_r \subset L_r \) is the maximal abelian subfield of both \( K_r, L_r \) and both \( K_r, L_r \) have the same degree over \( \mathbb{Q}_p \). The Jarden-Ritter Theorem [JR79] says in this situation that the absolute Galois groups of \( K_r, L_r \) are isomorphic i.e. \( K_r \leftrightarrow L_r \). Hence the claim.

**Remark 9.5.** Let me remark that one could say that the abc-conjecture and Szpiro’s conjecture are about measuring (log)-discriminants and log-conductors of fields (see [Moc10]). The above results show that anabelomorphy renders these quantities fluid. The subtle problem of rendering the changes in these quantities measurable is at the heart of [Moc12a, Moc12b, Moc12c, Moc12d]. Also see [DHb], [DHa] for a discussion of this for computations of [Moc12d].

Let me set up some notation for my next result. For a \( p \)-adic field \( K/\mathbb{Q}_p \) write \( \mathfrak{d}(K/\mathbb{Q}_p) \) for the different of \( K/\mathbb{Q}_p \). This is an ideal of \( \mathcal{O}_K \). Valuation on \( \mathcal{O}_K \) is normalized so that \( v_K(\pi) = 1 \) for any uniformizer \( \pi \) of \( \mathcal{O}_K \). In contrast to the fact that different and discriminants are unamphoric, one has the following elementary but useful bound which is due to [Moc12d, Prop. 1.3] ; also see [DHa] (though this not stated in this form in loc. cit.).

**Theorem 9.6 (Different Bound).** Let \( K \) be a \( p \)-adic field. Then there exists an absolute constant \( A = A(K) \geq 0 \) determined by the anabelomorphism class of \( K \) such that for all \( L \leftrightarrow K \) one has

\[
v_L(\mathfrak{d}(L/\mathbb{Q}_p)) \leq A.
\]

**Proof.** Let \( L \leftrightarrow K \). Let \( n = [L : \mathbb{Q}_p], \), \( f = f(L/\mathbb{Q}_p) \) be the residue field degree for \( L/\mathbb{Q}_p \) and \( e = e(L/\mathbb{Q}_p) \) be the absolute ramification index. Then it is well-known, (see [Art06]) that one has

\[
v_L(\mathfrak{d}(L/\mathbb{Q}_p)) \leq e - 1 + \frac{n}{f}.
\]
So it suffices to remark that $n, f, e$ are amphoric quantities and hence depend only on the anabelomorphism class of $L$ equivalently on the anabelomorphism class of $K$. So now take

$$A(K) = \sup_{L \sim K} (v_L(\delta(L/\mathbb{Q}_p))) \leq e - 1 + \frac{n}{f}.$$ 

Hence the assertion. \qed

10 Frobenius elements are amphoric

Let $K$ be a $p$-adic field and let $L$ be an unjilt of $K$ (i.e. $L \sim K$). One has the following result of Uchida from [JR79, Lemma 3]:

**Theorem 10.1** (Uchida). Let $K$ be a $p$-adic field and let $L$ be an unjilt of $K$. If $\sigma \in G_K$ is a Frobenius element for $K$. Then for any topological isomorphism $\alpha : G_K \to G_L$, $\alpha(\sigma)$ is a Frobenius element for $L$.

This has the following important corollary.

**Theorem 10.2.** Let $K$ be a $p$-adic field and let $\rho : G_K \to \text{GL}(V)$ be a finite dimensional continuous representation of $G_K$ in an $E$-vector space with $E/\mathbb{Q}_\ell$ a finite extension and $\ell \neq p$. Then the characteristic polynomial of Frobenius ($= \det(1 - T\rho(Frob_p))$) is amphoric. In particular $L$-functions of local Galois representations are amphoric.

**Proof.** This is clear from the previous result. \qed

11 Monoradicality is amphoric

Let $K$ be a $p$-adic field. An extension $M/K$ is a monoradical extension if it is of the form $M = K(\sqrt[n]{x})$ for some $x \in K$ and in this case $x$ is a generator of $M/K$. The following is proved in [JR79].

**Theorem 11.1.** Monoradicality is amphoric and hence in particular the degree of any monoradical extension is amphoric.
12 Anabelomorphy for Tate curves and Abelian varieties with multiplicative reduction

Let me begin with the simpler example of a Tate curve over a $p$-adic field $K$. By a Tate elliptic curve I will mean an elliptic curve with split multiplicative reduction over a $p$-adic field $K$. This was an example I worked out in Spring 2018 in Kyoto and in my opinion is an important example to understand. My result is weaker than Mochizuki’s result which provides a control on the number field if the Tate curve begins its life over a number field and his approach requires working with a punctured Tate curve (i.e. non-proper). But I believe that the projective case is still quite useful and instructive. For Tate curves, readers should consult [Sil94].

By Tate’s theorem [Sil94] a Tate elliptic curve over $K$ corresponds to the data of a discrete cyclic subgroup $q_K^Z \subset K^*$. The equation of the Tate curve is then given by

$$y^2 + xy = x^3 + a_4(q_K)x + a_6(q_K),$$

with explicitly given convergent power series $a_4(q_K), a_6(q_K)$ in $q_K$.

The main theorem is the following:

**Theorem 12.1.** Let $K$ be a $p$-adic field and let $E/K$ be a Tate elliptic curve over $K$ with Tate parameter $q_K \in K^*$. Let $L$ be a $p$-adic field anabelomorphic to $K$ with an isomorphism $\alpha : G_K \simeq G_L$. Then there exists a Tate elliptic curve $E'/L$ with Tate parameter $q_L$ and an isomorphism of topological abelian groups

$$E(K) \simeq E'(L),$$

given by the isomorphism $\alpha : G_K \rightarrow G_L$. The elliptic curve $E'/L$ is given by Tate’s equation

$$y^2 + xy = x^3 + a_4(q_L)x + a_6(q_L).$$

**Proof.** The anabelomorphism $\alpha : K \leftrightarrow L$ provides an isomorphism $\alpha : G_K \simeq G_L$ which provides, by the third fundamental theorem of anabelomorphy 3.3, an isomorphism of topological groups $\alpha : K^* \simeq L^*$. Let $q_L = \alpha(q_K)$. The map $\alpha$ preserves the valuation of $q_K$ and hence $|q_L| < 1$ and so by Tate’s Theorem one gets the Tate elliptic curve $E'/L$. The composite $K^* \rightarrow L^* \rightarrow L^*/q_L^Z$ provides the isomorphism of topological groups $E(K) \simeq E'(L)$. This proves the assertion.

This argument extends readily to abelian varieties with multiplicative reduction via the uniformization theorem of [Mum72].
Theorem 12.2. Let $K$ be a $p$-adic field and let $A/K$ be an abelian variety of dimension $g \geq 1$ given by a lattice $\Lambda_A \subset (K^*)^g = K^* \times \cdots \times K^*$ in a $g$-dimensional torus given by $K$. Let $\alpha : K \leftrightarrow L$ be an anabelomorphism of $p$-adic fields. Then there exists an abelian variety $A'/L$ and a topological isomorphism of groups $A(K) \simeq A'(L)$. If $A/K$ is polarized then so is $A'/L$.

Proof. The lattice $\Lambda_K$ provides a lattice $\Lambda_L$ in $L^* \times \cdots \times L^*$ using the isomorphism $K^* \times \cdots \times K^* \simeq L^* \times \cdots \times L^*$ induced by $K^* \simeq L^*$ induced by our anabelomorphism $\alpha : K \leftrightarrow L$. The rest is immediate from the Mumford-Tate uniformization theorem. The polarization on $A'$ (given one on $A$) is left as an exercise. \qed

The following corollary is immediate:

Corollary 12.3. In the notation of the above theorem, one has an isomorphism of groups:

$$\pi_1(A/K) \simeq \pi_1(A'/L),$$

in other words $A/K$ and $A'/L$ are anabelomorphic abelian varieties.

Proof. Let $g = \dim(A)$. An étale cover of $A/K$ is an abelian variety with multiplicative reduction $B/K'$ over some finite extension $K'/K$. The covering map provides an injective homomorphism of discrete subgroups $\Lambda_A \to \Lambda_B \subset (K^*)^g$ corresponding to the étale covering $B \to A$. Since $K \leftrightarrow L$, any finite extension $K'$ of $K$ gives a finite extension $L'/L$, this correspondence is given as follows $K'$ corresponds to an open subgroup $H \subset G_K$ and the isomorphism $G_K \to G_L$ provides an open subgroup $H'$ which is isomorphic image of $H$ under this isomorphism and $L'$ is the fixed field of $H'$. Hence one has in particular that $K' \leftrightarrow L'$. Now construct an étale cover of $A'$ over $L'$ by transferring the data of the covering $\Lambda_A \leftrightarrow \Lambda_B$ (which is an inclusion of discrete subgroups of finite index) to $(L^*)^n \leftrightarrow (L'^*)^n$. By Mumford’s construction this gives a covering $B' \to A'$. The correspondence $B/K' \leftrightarrow B'/L'$ provides the required correspondence between étale coverings of $A/K$ and étale coverings of $A'/L$. This argument can be reversed. Starting with a covering of $A'$ one can arrive at a covering of $A$. Hence the result follows. \qed

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13 Anabelomorphy of group-schemes of type \((p, p, \cdots, p)\) over \(p\)-adic fields

Let \(K\) be a \(p\)-adic field and \(\mathcal{G}/K\) be a commutative, finite flat group scheme of order \(p^r\) and type \((p, p, \cdots, p)\) over \(K\). Let \(L\) be an anabelomorph of \(K\). Then the following shows that there is a commutative finite flat group scheme \(H\) of type \((p, p, \cdots, p)\) over \(L\) which is obtained from \(\mathcal{G}\). By [Ray74] the group scheme \(\mathcal{G}\) is given by a system of equations

\[ X_i^p = \delta_i X_{i+1}, \quad \delta_i \in K^*, \text{ for all } i, i + 1 \in \mathbb{Z}/r\mathbb{Z}. \]

Fix an anabelomorphism \(\alpha : G_L \to G_K\). This induces isomorphism of topological groups \(L^* \to K^*\). Let \(\tau_i \in L^*\), for all \(i \in \mathbb{Z}/r\mathbb{Z}\) be the inverse image of \(\delta_i\) under this isomorphism. Then

\[ Y_i^p = \tau_i Y_{i+1} \text{ for all } i, i + 1 \in \mathbb{Z}/r\mathbb{Z}, \]

provides a finite flat group scheme \(\mathcal{H}/L\) of type \((p, p, \cdots, p)\). Conversely starting with a group scheme of this type over \(L\), one can use an anabelomorphism between \(L, K\) to construct a group scheme over \(K\). Thus one has

**Theorem 13.1.** An anabelomorphism of fields \(L, K\) sets up a bijection between commutative finite group flat schemes of type \((p, p, \cdots, p)\) over \(L\) and \(K\) respectively. This bijection does not preserve finite flat group schemes on either side.

14 Anabelomorphy for Projective Spaces and Affine spaces

For a \(p\)-adic field \(K\) write \(\mathcal{O}_K^\times = \mathcal{O}_K - \{0\}\) for the multiplicative monoid of non-zero elements of \(\mathcal{O}_K\) and write \(\mathcal{O}_K^\times = \mathcal{O}_K^\times \cup \{0\}\) considered as a multiplicative monoid with the law \(x \cdot 0 = 0 \cdot x = 0\) for any \(x \in \mathcal{O}_K^\times\). Let \(K^* = K - \{0\}\) be the multiplicative monoid of non-zero elements of \(K\) and \(K^\times = K^* \cup \{0\}\) as a multiplicative monoid (with the obvious structure). As one learns from [Hos17], if \(K\) is a \(p\)-adic field then all of these monoids are amphoric.

Anabelomorphy of projective spaces is slightly weaker than the result for abelian varieties proved above. The main theorem is the following:
Theorem 14.1. Let $\alpha : K \leftrightarrow L$ be an anabelomorphism of $p$-adic fields. Let $n \geq 1$ be an integer. Then $\alpha : K \leftrightarrow L$ induces a natural bijection of sets

$\alpha : P^n(K) \rightarrow P^n(L)$

which is a homeomorphism with respect to the respective $p$-adic topologies when restricted to subsets $U_i(K)$ (resp. $U_i(L)$) (where $0 \leq i \leq n$) and

$U_i = \{(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) : x_j \neq 0, \forall j \neq i\} \subset P^n.$

Proof. First let me prove that $P^n(K)$ can be reconstructed (as a set) purely group theoretically from $G_K$. Since the monoid $K^\otimes, K^*$ are amphoric (Theorem 3.3), one can construct $K^\otimes_{n+1} = K^\otimes \times \cdots \times K^\otimes$ as a set, with $n + 1$ factors (and even as a multiplicative monoid with the obvious product monoid structure) and hence also construct $K^\otimes_{n+1} - \{(0, \cdots, 0)\}$ purely from $G_K$. Now the equivalence relation defining $P^n(K)$ as a set is given purely in terms of the multiplicative monoid structure of $K^*$ and $K^\otimes$. More precisely for any $(x_i) \in K^\otimes_{n+1} - \{(0, \cdots, 0)\}$, the set $\{\lambda x_i : \lambda \in K^*\}$ can be constructed purely from $K^*$ and $K^\otimes$. Hence one can construct the equivalence relation on $K^\otimes_{n+1}$ from $G_K$. In other words the equivalence relation is also amphoric and the quotient by the equivalence relation is also amphoric. So as a set $P^n(K)$ is amphoric.

Now suppose $\alpha : K \leftrightarrow L$ is an anabelomorphism of $p$-adic fields. Then $P^n(K)$ and $P^n(L)$ are described purely in terms of $K^*, L^*, K^\otimes, L^\otimes$. Moreover one has natural isomorphisms of monoids $K^* \simeq L^*$ which extends to an isomorphism $K^\otimes \simeq L^\otimes$. So one has a natural bijection sets

$K^\otimes_{n+1} - \{(0, \cdots, 0)\} = L^\otimes_{n+1} - \{(0, \cdots, 0)\}$

induced by $\alpha : K \leftrightarrow L$. The equivalence relation on each of these sets is described in terms of isomorphic monoids $K^* \simeq L^*$. Hence equivalent pairs $x, y$ of points on the left are mapped into equivalent elements as $\alpha$ induces an isomorphism of the monoids. This makes no use of the ring structure of $K, L$. That this map is a homeomorphism of topological spaces is immediate from the fact that $K$ (resp. $L$) points of the stated open subset are of the form $K^* n$ (resp. $L^* n$) and these two subsets are homeomorphic with their respective topologies. Hence the result follows.

The following corollary is an immediate consequence of the proof of Theorem 14.1.

\[ \square \]
Corollary 14.2. Let $\alpha : K \leftrightarrow L$ be an anabelomorphism of $p$-adic fields. Let $n \geq 1$ be an integer. Then $\alpha : K \leftrightarrow L$ induces a natural bijection of sets

$$\alpha : \mathbb{A}^n(K) \rightarrow \mathbb{A}^n(L)$$

which is a homeomorphism with respect to the respective $p$-adic topologies when restricted to the Zariski dense subset $U$ where

$$U = \{(x_1, \ldots, x_n) : x_j \neq 0, \forall j \} \subset \mathbb{A}^n.$$

Remark 14.3. Let me remark that if $K \leftrightarrow L$ are anabelomorphic $p$-adic fields. Then one has a trivial isomorphism of algebraic fundamental groups

$$\pi_1(\mathbb{P}^n/K) \simeq G_K \simeq G_L \simeq \pi_1(\mathbb{P}^n/L).$$

So one can consider $\mathbb{P}^n/K$ and $\mathbb{P}^n/L$ as trivially anabelomorphic varieties. Similarly for $\mathbb{A}^n/K$ and $\mathbb{A}^n/L$.

15 The $\mathfrak{L}$-invariant is unamphoric

Let $K$ be a $p$-adic field and let $V$ be a two dimensional ordinary (= reducible, semi-stable) representation of $G_K$ with coefficients in $\mathbb{Q}_p$ such that one has an exact sequence

$$0 \rightarrow \mathbb{Q}_p(1) \rightarrow V \rightarrow \mathbb{Q}_p(0) \rightarrow 0.$$ 

Then one has an invariant, defined by [Gre94], Fontaine, and others (see [Col10] for all the definitions and their equivalence), called the $\mathfrak{L}$-invariant of $V$, denoted $\mathfrak{L}(V)$, which plays a central role in the theory of $p$-adic $L$-function of $V$ and related representations of $G_K$. One of the simplest, but important, consequences of anabelomorphy is the following:

Theorem 15.1. Let $K$ be a $p$-adic field. Let $V$ be as above. Then the $\mathfrak{L}$-invariant, $\mathfrak{L}(V)$, of $V$ is unamphoric.

For a more detailed discussion of $D_{dR}(V)$ for ordinary representations see Theorem 27.1.

Proof. The representation $V$ is an extension whose class lives in

$$\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1)) = H^1(G_K, \mathbb{Q}_p(1)).$$
and this $\mathbb{Q}_p$-vector space (of dimension $[K : \mathbb{Q}_p] + 1$) is also described naturally by means of Kummer theory, I will write $q_V$ for this extension class. By [Nek93], [Col10], [PR94a] the space $H^1(G_K, \mathbb{Q}_p(1))$ is described by two natural coordinates $(\log_K(q_V), \text{ord}_K(q_V))$ where $\log_K$ is the $p$-adic logarithm (with $\log_K(p) = 0$). From [Nek93], [Col10] one sees that 

$$L(V) = \frac{\log_K(q_V)}{\text{ord}_K(q_V)}.$$ 

The assertion follows from the fact the $\log_K(u)$ for a unit $u \in \mathcal{O}_K^*$ is an unamphoric quantity (in general) as the additive structure of the field $K$ which comes into play here through the use of the $p$-adic logarithm is not an amphoric quantity: two anabelomorphic fields $K \sim_{\text{an}} L$ may not be isomorphic as fields because their additive structures may not be isomorphic.

This has the following important consequence:

**Theorem 15.2.** Let $V \in \text{Ext}^1_{\mathcal{G}_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$. Then the Hodge filtration on $D_{dR}(V)$ is unamphoric.

**Proof.** From [Col10] one sees that $\mathcal{L}(V)$ controls the Hodge filtration on the filtered $(\phi, N)$-module $D_{dR}(V)$. Therefore one deduces that anabelomorphy changes the $p$-adic Hodge filtration. See Section 27 for additional comments on this.

### 16 Anabelomorphic Connectivity Theorem for Number Fields

Note that by the Neukirch-Uchida Theorem the Galois group of a number field determines the number field. On the other hand Galois group of a local field does not determine the local field completely. Mochizuki’s idea of anabelomorphism raises the possibility of anabelomorphically modifying a number field at a finite number of places to create another number field which is anabelomorphically glued to the original one at a finite number of places. I will illustrate this idea of local anabelomorphic modifications of global number fields with an explicit collection of examples and a fundamental connectivity theorem which I discovered in my attempts to understand them.

Consider the following example (taken from [JR79]; for similar example see [Yam76]). Let $p$ be an odd prime and let $K = \mathbb{Q}(\zeta_p, \sqrt[p]{p})$ and let $K' = \mathbb{Q}(\zeta, \sqrt[1+p]{p})$. 

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Then both of these extensions are totally ramified at \( p \). Moreover at the unique primes of \( K \) (resp. \( K' \)) lying over \( p \) the completion of \( K \) (resp. \( K' \)) is given by \( \mathbb{Q}_p(\zeta_p, \sqrt[p]{p}) \) (resp. \( \mathbb{Q}_p(\zeta, \sqrt{1+p}) \)). As has been established by Jarden-Ritter (see [JR79]) one has an anabelomorphism of \( p \)-adic fields

\[
\mathbb{Q}_p(\zeta_p, \sqrt[p]{p}) \leftrightarrow \mathbb{Q}_p(\zeta, \sqrt{1+p}).
\]

So one has two non-isomorphic number fields which have anabelomorphic completions at their unique non-archimedean ramified primes.

To convince the readers that the above example is not an isolated example, let me prove that there are infinitely many anabelomorphically connected pairs of number fields. Let \( p \) be any odd prime. Let \( r \geq 1 \) be an integer. Let \( K_r = \mathbb{Q}(\zeta_p^r, \sqrt[p^r]{p}) \), \( K'_r = \mathbb{Q}(\zeta_p^r, \sqrt[p^r]{1+p}) \). These are totally ramified at \( p \) (see [Viv04]). Let \( p \) (resp. \( p' \)) be the unique prime of \( K_r \) prime lying over \( p \) in \( K_r \) (resp. the unique prime of \( K'_r \) lying over \( p \) in \( K'_r \)). The completions of \( K_r \) (resp. \( K'_r \)) with respect to these unique primes are \( K_{r,p} = \mathbb{Q}_p(\zeta_p^r, \sqrt[p^r]{p}) \) and \( K'_{r,p'} = \mathbb{Q}_p(\zeta_p^r, \sqrt[p^r]{1+p}) \) respectively. By Lemma 9.4 one has an isomorphism of the local Galois groups

\[
K_{r,p} \leftrightarrow K'_{r,p'}.
\]

The above example leads to the following definition.

I say that two number fields \( K, K' \) are anabelomorphically connected along \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \), and write this as

\[
(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{w_1, \ldots, w_n\}),
\]

if there exist non-archimedean places \( v_1, \ldots, v_n \) of \( K \) (resp. non-archimedean places \( w_1, \ldots, w_n \) of \( K' \)) and for each \( i = 1, \ldots, n \) an anabelomorphism \( K_{v_i} \leftrightarrow K'_{w_i} \) and for each \( i \) the inclusion \( K'_{w_i} \hookrightarrow K'_{w_i} \) is dense.

In particular the number fields \( K_r = \mathbb{Q}(\zeta_p^r, \sqrt[p^r]{p}), K'_r = \mathbb{Q}(\zeta_p^r, \sqrt[p^r]{1+p}) \) (and the unique primes \( p_r, p'_r \) be the lying over \( p \) in \( K_r, K'_r \)) are anabelomorphically connected along \( p_r \) and \( p'_r \):

\[
(K_r, \{p_r\}) \leftrightarrow (K'_r, \{p'_r\}).
\]

**Remark 16.1.** By the formulae for the discriminants of \( K_r, K'_r \) (see 9.4), one sees that the differents (and hence the discriminants) of anabelomorphically connected fields differ in general. This is a fundamental way in which local anabelomorphic modifications change global arithmetic data. This is a crucial point in [Moc12a, Moc12b, Moc12c, Moc12d].

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So this suggests that one should study anabelomorphically connected number fields in detail. Before one can study such fields, one should find a general way of constructing them. As mentioned in the Introduction, anabelomorphically connected number fields are (implicit) by-products of Mochizuki’s Belyian Reconstruction Machine [Moc15, Theorem 1.9]. I realized that one should look for a direct construction of such fields as this would have many arithmetic consequences. This is accomplished in Theorem 16.3 and the more general Theorem 16.5 which provide a systematic way of producing examples of anabelomorphically connected fields starting with a given number field.

In what follows I will say that a number field \( M \) is dense in a \( p \)-adic field \( L \) if there exists a place \( v \) of \( M \) such that the completion \( M_v \) of \( M \) at \( v \) is \( L \) (i.e. \( M_v = L \)).

I begin with the following easy lemma.

**Lemma 16.2.** Let \( L \) be \( p \)-adic field. Then there exists a number field \( M \subset L \) which is dense in \( L \).

**Proof.** This is a well-known consequence of Krasner’s Lemma (see [Kob94]). Let \( L = \mathbb{Q}_p(x) \) where \( p \) is the residue characteristic of \( L \). Let \( f \in \mathbb{Q}_p[X] \) be the minimal polynomial of \( x \). If \( f(X) \in \mathbb{Q}[X] \) then \( x \) is algebraic and clearly \( M = \mathbb{Q}(x) \) is the dense number field one seeks. If this is not the case then choose \( g(X) \in \mathbb{Q}[X] \) sufficiently close to \( f(X) \) in \( \mathbb{Q}_p[X] \). Then by Krasner’s Lemma (see loc. cit.) \( g \) is irreducible and if \( x_0 \) is a root of \( g(X) \) then \( \mathbb{Q}_p(x_0) = \mathbb{Q}_p(x) \) and hence \( M = \mathbb{Q}(x_0) \) is dense in \( L \). \( \square \)

Theorem 16.3 is a prototype of the more general result proved later (Theorem 16.5) and is included here for the convenience of the readers as it illustrates the main points of the general result.

**Theorem 16.3** (Anabelomorphic Connectivity Theorem). Let \( K \) be a number field and let \( v \) be a non-archimedean place of \( K \). Let \( L \) be a local field anabelomorphic to \( K_v \). Then there exists a number field \( K' \) and non-archimedean place \( w \) of \( K' \) such that

\[ K'_w \simeq L \leftrightarrow K_v. \]

In particular \( K'_w \) is anabelomorphic to \( K_v \). Equivalently \((K, \{v\}) \leftrightarrow (K', \{w\})\).

**Proof.** By the lemma, there exists a number field \( K' \subset L \) dense in \( L \). Let \( w \) be the place corresponding to the dense embedding \( K' \hookrightarrow L \) (i.e. \( K'_w = L \)). Then \( K'_w \simeq L \leftrightarrow K_v \) and hence \( K'_w \leftrightarrow K_v \) and hence \((K, \{v\}) \leftrightarrow (K', \{w\})\). Thus the assertion follows. \( \square \)
Now let us move to the general case of connectivity along several primes simultaneously. From the point of view of applications of Mochizuki’s ideas this case is fundamental.

I will use the following (non-standard) terminology: a non-archimedean local field is a finite extension of \( \mathbb{Q}_p \) for some (unspecified) prime \( p \). I say that an arbitrary, finite set of non-archimedean local fields \( \{L_1, \ldots, L_n\} \) (not all distinct and not all necessarily of the same residue characteristic) is a cohesive set of non-archimedean local fields if there exists a number field \( M \) and for every \( i \), a dense inclusion \( M \hookrightarrow L_i \) such that the induced valuations on \( M \) are all inequivalent.

**Lemma 16.4** (Potential Cohesivity Lemma). For every finite set \( \{L_1, \ldots, L_n\} \) of non-archimedean local fields (not necessarily of the same residue characteristic) there exist finite extensions \( L'_1/L_1, \ldots, L'_n/L_n \) such that \( \{L'_1, \ldots, L'_n\} \) is a cohesive system of non-archimedean local fields.

**Proof.** By Lemma 16.2 the result is true for \( n = 1 \) with \( L'_1 = L_1 \). The general case will be proved by induction on \( n \). Suppose that the result has been established for the case of \( n - 1 \) fields. So for every set \( L_1, \ldots, L_{n-1} \) of non-archimedean fields there exists finite extensions \( L'_1, \ldots, L'_{n-1} \) of non-archimedean fields and a number field \( M \in L'_i \) which is dense inclusion for \( i = 1, \ldots, n - 1 \) and the valuations induced on \( M \) are all inequivalent. Choose \( \alpha \in M \) such that \( \mathbb{Q}(\alpha) = M \).

Now suppose that \( p \) is the residue characteristic of \( L_n \) and \( L_n = \mathbb{Q}_p(x_n) \). By Lemma 16.2 there exists a number field dense in \( L_n \). By Krasner’s Lemma one can choose \( \beta \in L_n \) to be algebraic and sufficiently close to \( x_n \) such that \( L_n = \mathbb{Q}_p(\beta) = \mathbb{Q}_p(x_n) \). Now consider the finite extensions \( L'_n = L_n(\alpha) \) and \( L''_i = L'_i(\beta) \) for \( i = 1, \ldots, n - 1 \). Then \( \mathbb{Q}(\alpha, \beta) \subset L''_i \) for \( i = 1, \ldots, n - 1 \) and \( \mathbb{Q}(\alpha, \beta) \subset L'_n \). Write \( L''_n = L''_n \) (for symmetry of notation). Then one sees that there exists a common number field \( M \) contained in all of \( L''_i \). If \( M \) is not dense in each of \( L''_i \) one can extend \( L''_i \) further to achieve density. Similarly if the induced valuations on \( M \) are not all inequivalent, one can extend \( L''_i \) further to achieve this as well.

Let me explain how the last two steps are carried out. To avoid notational chaos, I will prove the assertion for \( n = 2 \). So the situation is that one has two non-archimedean fields \( L_1, L_2 \) and a common number field \( M \) contained in both of them. There are two possibilities residue characteristics of \( L_1, L_2 \) are equal or they are not equal. First assume that the residue characteristics are equal (say equal to \( p \)). Then \( L_1, L_2 \) are both finite extensions of \( \mathbb{Q}_p \) and so there exists a finite extension \( L \) containing both of them as subfields. Pick such an \( L \). Then there is
a number field $M'$ dense in $L$. Now choose a number field $F$, with $[F : \mathbb{Q}] > 1$, which is totally split at $p$ and such that $M', F$ are linearly disjoint over $\mathbb{Q}$. Then let $M'' = MF \hookrightarrow L$ and since $F$ is completely split there exist two primes $v_1 \neq v_2$ of $M''$ lying over $p$ such that $M''_{v_1} = L$ and $M''_{v_2} = L$. Thus the system $L_1 = L, L_2 = L$ is now cohesive as $M'' \hookrightarrow L_1 = L$ and $M'' \hookrightarrow L_2 = L$ are dense inclusions corresponding to distinct primes of $M''$.

So now assume $L_1, L_2$ have distinct residue characteristics and $M$ is a number field contained in both of them. If $v_1$ (resp. $v_2$) is the prime of $M$ corresponding to the inclusion $M \hookrightarrow L_1$ (resp. $M \hookrightarrow L_2$), then $M_{v_1} \hookrightarrow L_1$ and $M_{v_2} \hookrightarrow L_2$ are finite extensions of non-archimedean fields. One proceeds by descending induction on $[L_1, M_{v_1}], [L_2, M_{v_2}]$. By the primitive element theorem there exists an $x_1 \in L_1$ (resp. $x_2 \in L_2$) such that $L_1 = M_{v_1}(x_1)$ (resp. $L_2 = M_{v_2}(x_2)$). Choose an irreducible polynomial $f \in M[X]$ which is sufficiently close to the minimal polynomials of $x_1$ (resp. $x_2$) in $L_1[X]$ and $L_2[X]$ respectively. Then $f$ has a root in both $L_1, L_2$ (by Krasner’s Lemma) and the field $M' = M[X]/(f)$ embeds in both $L_1, L_2$ and if $v'_1$ (resp. $v'_2$) is the prime over $v_1$ (resp. $v_2$) corresponding to the inclusion $M' \hookrightarrow L_1$ and $M' \hookrightarrow L_2$ are dense inclusions of $M'$ in $M'_{v'_1} \subset L_1$ (resp. $M$ in $M'_{v'_2} \hookrightarrow L_2$) and $[L_1, M'_{v'_1}] < [L_1, M_{v_1}]$ and similarly for $L_2$. Thus by enlarging $M$ in this fashion one is eventually led to a cohesive system as claimed. 

Now I can state and prove the general anabelomorphic connectivity theorem for number fields.

**Theorem 16.5** (Anabelomorphic Connectivity Theorem II). Let $K$ be a number field. Let $v_1, \ldots, v_n$ be a finite set of non-archimedean places of $K$. Let $\alpha_i : K_{v_i} \leftrightarrow L_i$ be arbitrary anabelomorphisms with non-archimedean local fields $L_1, \ldots, L_n$. Then there exist

1. finite extensions $L'_i/L_i$ (for all $i$) and a dense embedding of a number field $M \subset L'_i$ and places $w_1, \ldots, w_n$ of $M$ induced by the embeddings $M' \hookrightarrow L'_i$ (i.e. the collection $\{L'_i\}$ of non-archimedean fields is cohesive) and

2. a finite extension $K'/K$ and places $u_1, \ldots, u_n$ of $K'$ lying over the places $v_i$ of $K$ (for all $i$) together with anabelomorphisms $K'_{u_i} \leftrightarrow L'_i$.

3. Equivalently $(K', \{u_1, \ldots, u_n\}) \leftrightarrow (M', \{w_1, \ldots, w_n\})$ and $u_i|v_i$ for all $i = 1, \ldots, n$. 

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In particular given any number field $K$ and a collection of non-archimedean places of $K$. There exists a finite extension $K'/K$ and a number field $M'$ which is anabelomorphically connected to $K'$ along some place of $K'$ lying over each of places $v_1, \ldots, v_n$ of $K$.

**Proof.** By the Cohesivity Lemma (Lemma 16.4) one can replace $L_1, \ldots, L_n$ by a cohesive collection $L'_1, \ldots, L'_n$ with $L'_i/L_i$ finite extensions and a number field $M' \subset L'_i$ dense in each $L'_i$ and such that the induced valuations on $M'$ are all inequivalent. The finite extensions $L'_i/L_i$ provide open subgroups $H'_i \subset G_{L_i}$ of $G_{L_i}$. Since one has anabelomorphisms $\alpha_i : K_{v_i} \leftrightarrow L_i$, let $H_i = \alpha^{-1}(H'_i)$ be the inverse image of $H'_i$ in $G_{v_i}$. By continuity of $\alpha_i$, $H'_i$ are open subgroups of $G_{v_i}$. Let $G' \subset G_K$ be the open subgroup of $G_K$ generated by the decomposition groups of all primes except $v_1, \ldots, v_n$ and the open subgroups $H_i$ for $i = 1, \ldots, n$. Let $K'$ be the fixed field of $G'$ (in our fixed algebraic closure of $K$). Let $u_i$ be the unique place of $K'$ lying over the $v_i$ such that $G_{K'_u_i} \simeq H_i$. Then by construction

$$G_{K'_u_i} \simeq H_i \simeq H'_i \simeq G_{L'_i} \simeq G_{M_{wi}}$$

and hence one has established that

$$(K', \{u_1, \ldots, u_n\}) \leftrightarrow (M', \{v_1, \ldots, v_n\})$$

Now let me prove some Theorems which will be useful in applying these results to arithmetic problems (such as those envisaged in [Moc12a, Moc12b, Moc12c, Moc12d]). I will begin with some preparatory lemmas which are well-known but difficult to find in the form I will need here.

**Lemma 16.6.** Let $L$ be a $p$-adic field. Then there exist infinitely many number fields $M$ with a dense embedding $M \hookrightarrow L$.

**Proof.** This is easy to prove and is left as an exercise!  

**Theorem 16.7.** Let $K$ be a number field and let $v$ be a non-archimedean place of $K$. Then there exist infinitely many anabelomorphically connected number fields $(K, \{v\}) \leftrightarrow (K', \{w\})$. If $K_p \leftrightarrow L$ is a strict anabelomorphism then $(K, \{p\}) \leftrightarrow (K', \{q\})$ are strictly anabelomorphically connected fields.
Proof. By Lemma 16.6 there exist infinitely many number fields $K'$ with a dense embedding $K' \hookrightarrow L$.

Let $K'$ be a number field which is dense in $L$ and let $q$ be the prime of $K'$ corresponding to the embedding $K' \hookrightarrow L$. Then one has anabelomorphically connected fields $(K, \{p\}) \leftrightarrow (K', \{q\})$. \qed

**Theorem 16.8.** Let $K$ be a number field and let $p$ be a prime of $K$ lying over $p$. Then there exist infinitely many anabelomorphically connected fields $(K, \{p\}) \leftrightarrow (M, \{q\})$ such that $\deg(M/\mathbb{Q}) \rightarrow \infty$. If $K_p$ is strictly anabelomorphic to $L$ then $(K, \{p\}) \leftrightarrow (M, \{q\})$ is a strict anabelomorphic connectivity.

Proof. Let us suppose that there is an anabelomorphically connected number field (with $(K, \{p\}) \leftrightarrow (M, \{q\})$) with $\deg(M/\mathbb{Q})$ maximal amongst all such fields. Let $p$ be the prime lying below $q$ (and hence also $p$) in $\mathbb{Z}$. Choose a quadratic field $F$ which is completely split at $p$ and which is also totally split at any prime ramifying in $M$. Then $F \cap M$ has no primes of ramification and by the Hermite-Minkowski Theorem, $F \cap M = \mathbb{Q}$. Let $M' = FM$. Then by construction $M'/M$ is totally split at $q$. Let $q'$ be a prime of $M'$ lying over $q$ of $M$. Then $M'_{q'} \simeq M_q$ and so one has anabelomorphisms $K_p \leftrightarrow M_q \leftrightarrow M'_{q'}$ and hence $(K, \{p\}) \leftrightarrow (M', \{q'\})$ and $\deg(M') \succ \deg(M)$ which contradicts the maximality of $\deg(M)$. \qed

**Remark 16.9** (Transition formula for uniformizers). As has been pointed out earlier strictly anabelomorphic fields have distinct additive structures. So one should expect that uniformizing elements which are related to the respective additive structures. Since anabelomorphic fields may even be linearly disjoint over $\mathbb{Q}_p$, so comparing uniformizers of anabelomorphic fields directly is impossible. However one can easily establish relations in any common overfield containing both the fields. It is, in fact, possible to find relationships between uniformizers (of our anabelomorphic fields) in an overfield which resemble “gluing” or “transition formulae” in algebraic geometry. Let me illustrate this by means of an explicit example using the number fields $K = \mathbb{Q}(\zeta_p, \sqrt[p]{p})$ and $K' = \mathbb{Q}(\zeta, \sqrt[1+p]{p})$ considered above. This is based on the computation of uniformizers in [Viv04, Lemma 5.7, Lemma 6.4]. Let

\[
\omega_1 = \frac{1 - \zeta_p}{\sqrt[p]{p}} \tag{16.10}
\]

\[
\omega_2 = \frac{1 - \zeta_p}{(1 + p - \sqrt[1+p]{p})} \tag{16.11}
\]
Note that the right hand sides of both are algebraic numbers and hence so are \( \omega_1, \omega_2 \). By loc. cit., \( \omega_1 \) (resp. \( \omega_2 \)) is a uniformizer for \( K_p/Q_p(\zeta_p) \) (resp. \( K'_p/Q_p(\zeta_p) \)).

Now let me show how this provides a transition formula for uniformizers. Consider \( K, K' \) as being embedded in a common algebraic field extension (such as \( \bar{\mathbb{Q}} \)). One has

\[
\omega_1 \cdot \sqrt{p} = 1 - \zeta_p \quad (16.12)
\]

\[
\omega_2 \cdot \left(1 + p - \sqrt[3]{1 + p}\right) = 1 - \zeta_p. \quad (16.13)
\]

Hence by eliminating \( 1 - \zeta_p \), one arrives at the transition formula for uniformizers of the two anabelomorphic fields \( K_p \leftrightarrow K'_p \):

\[
\omega_1 \cdot \sqrt{p} = \omega_2 \cdot \left(1 + p - \sqrt[3]{1 + p}\right),
\]

or

\[
\omega_1 = \left(\frac{1 + p - \sqrt[3]{1 + p}}{\sqrt{p}}\right) \cdot \omega_2,
\]

equivalently

\[
\omega_2 = \left(\frac{\sqrt{p}}{1 + p - \sqrt[3]{1 + p}}\right) \cdot \omega_1.
\]

Note that this transition formula cannot be seen at the level of \( K, K' \) (or even \( K_p, K'_p \)) but requires passage to a common overfield where comparisons can be made. The point to remember is that anabelomorphic fields can have distinct additive structures and changes in the additive structure also affects uniformizing elements.

### 17 The Ordinary Synchronization Theorem

A fundamental result discovered by Mochizuki (see [Moc12e, Moc13, Moc15]) is the Synchronization of Geometric Cyclotomes. This plays a fundamental role in [Moc12a, Moc12b, Moc12c, Moc12d]. For a catalog of synchronizations in [Moc12a, Moc12b, Moc12c, Moc12d] see [DHb]. The elementary result given below is inspired by Mochizuki’s result and is quite fundamental (despite the simplicity of its proof) in applications of anabelomorphy to Galois representations. This result asserts that two anabelomorphically connected number fields see the “same” ordinary two dimensional local Galois representations at primes on either side which are related through anabelomorphy. The theorem is the following:
Theorem 17.1 (The Ordinary Synchronization Theorem). Let \((K, \{v_1, \ldots, v_n\}) \hookrightarrow (K', \{w_1, \ldots, w_n\})\) be a pair of anabelomorphically connected number fields. Then one has for all primes \(\ell\) (including \(p\)) and for all \(i\), an isomorphism of \(\mathbb{Q}_\ell\)-vector spaces

\[
\text{Ext}^1_{G_{v_i}}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)) \simeq \text{Ext}^1_{G_{w_i}}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)).
\]

This theorem is immediate from the following Lemma.

Lemma 17.2. Let \(K \hookrightarrow L\) be two anabelomorphic \(p\)-adic fields. Then one has an isomorphism of \(\mathbb{Q}_\ell\)-vector spaces

\[
\text{Ext}^1_{G_K}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)) \simeq \text{Ext}^1_{G_L}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)).
\]

Proof. Let \(\alpha : K \hookrightarrow L\) be an anabelomorphism. It is standard that one has

\[
\text{Ext}^1_{G_K}(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(1)) \simeq H^1(G_K, \mathbb{Q}_\ell(1))
\]

(continuous cohomology group). By Kummer Theory one knows that \(H^1(G_K, \mathbb{Q}_\ell(1))\) can be described as \((\text{proj lim} K^*/K^*\ell^n) \otimes \mathbb{Q}_\ell\) and one has a similar description for \(G_L\). Then one has isomorphisms of topological groups \(K^* \to L^*\) and hence also of subgroups \((K^*)^{\ell^n} \simeq (L^*)^{\ell^n}\) compatible with their respective inclusions in \(K^*\) (resp. \(L^*\)). Hence one has isomorphism of groups

\[
\frac{K^*}{(K^*)^{\ell^n}} \simeq \frac{L^*}{(L^*)^{\ell^n}}.
\]

This also compatible with projections to similar groups for \(\ell^{n-1}\). Thus one has an isomorphism of the inverse limits and hence in particular on tensoring with \(\mathbb{Q}_\ell\).

Alternately one can simply invoke the fact that \(\mathbb{Q}_\ell(1)\) is an amphoric \(G_K \simeq G_L\) module. The two cohomologies depends only on the topology of \(G_K \simeq G_L\). So the claim is obvious.

This proof leads to the following (which is useful in many applications)

Lemma 17.3 (Bootstrapping Lemma). Let \(V\) be an amphoric \(G_K\)-module (i.e. an abelian topological group or a topological \(\mathbb{Z}_\ell\)-module with a continuous action of \(G_K\) which is determined by the anabelomorphism class of \(K\)). Then \(H^1(G_K, V)\) is amphoric.

Proof. The proof is clear: continuous \(G_K\)-cohomology is determined by the topology of \(G\) and by the topological isomorphism class of \(V\). As \(V\) is amphoric the result follows.
The following theorem is key in [Moc12a, Moc12b, Moc12c, Moc12d], but I think that this formulation illustrates an important point which is not stressed in loc. cit. where it occurs in the guise of the amphoricity of log-shell tensored with $\mathbb{Q}_p$ (for the log-shell see [Hos17], [DHb]). Let $K$ be a $p$-adic field. Let

$$H_f^1(G_K, \mathbb{Q}_p(1)) \subset H^1(G_K, \mathbb{Q}_p(1))$$

be the (Fontaine) subspace of (ordinary) crystalline two dimensional $G_K$-representations in $\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$.

**Theorem 17.4.** Let $K \hookrightarrow L$ be a pair of anabelomorphic $p$-adic fields. Then one has an isomorphism of $\mathbb{Q}_p$-vector spaces

$$H_f^1(G_K, \mathbb{Q}_p(1)) \simeq H_f^1(G_L, \mathbb{Q}_p(1)).$$

In other words the space $H_f^1(G_K, \mathbb{Q}_p(1))$, of crystalline-ordinary two dimensional $\mathbb{Q}_p$-representations of the form $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p(0) \to 0$, of $G_K$ is amphoric!

**Remark 17.5.** For readers of [Moc12e, Moc13, Moc15] and [Moc12a, Moc12b, Moc12c, Moc12d] let me remark that $H_f^1(G_K, \mathbb{Q}_p(1))$ is the log-shell tensored with $\mathbb{Q}_p$ (for an excellent and accessible discussion of log-shells, [Hos19], [DHb], [DHa] and more sophisticated readers can see [Moc15]). This is because one has an isomorphism of finite dimensional $\mathbb{Q}_p$-vector spaces

$$H_f^1(G_K, \mathbb{Q}_p(1)) \simeq \left( \lim_{\leftarrow n} \mathcal{O}_K^*/\mathcal{O}_K^* p^n \right) \otimes_{\mathbb{Q}_p} \simeq U_K \otimes \mathbb{Q}_p,$$

where $U_K$ is the group of units $u \in \mathcal{O}_K^*$ congruent to 1 modulo the maximal ideal. One could say that the space $H_f^1(G_K, \mathbb{Q}_p(1))$ of two dimensional ordinary crystalline $G_K$-representations plays a role in [Moc12a, Moc12b, Moc12c, Moc12d].

**Corollary 17.6.** Let $K$ be a $p$-adic field. Then the natural representation of $\text{Out}(G_K)$ on $H^1(G_K, \mathbb{Q}_p(1))$ is reducible but possibly non-semi-simple.

**Proof.** The subspace $H_f^1(G_K, \mathbb{Q}_p(1)) \subset H^1(G_K, \mathbb{Q}_p(1))$ are amphoric spaces so stable under the action of outer automorphisms of $G_K$ and so reducibility is immediate. □

**Remark 17.7.** This corollary is also found in [Moc15, Theorem 1.10] (but perhaps stated differently). I do not know how to prove non-semisimplicity but non semisimplicity is responsible for the phenomenon observed in [Moc12a, Moc12b, Moc12c, Moc12d] namely, the mixing of “Frobenius-like portions” and the “étale like portions.” Note that in [Moc12a, Moc12b, Moc12c, Moc12d] there are other ways such a mixing occurs.
18 Automorphic Synchronization Theorems: Anabelomorphy and the local Langlands correspondence

This section is independent of the rest of the paper. I will assume that readers are familiar with the basic theory of automorphic representations at least for $\text{GL}_n$ though the main result proved here is for $\text{GL}_2$. The representations in this section will be smooth, complex valued representations of $\text{GL}_n(K)$. There is an automorphic analog of the Ordinary Synchronization Theorem (Theorem 17.1) which says that one can use an anabelomorphism $K \leftrightarrow L$ to synchronize automorphic representations of $\text{GL}_n(K)$ and $\text{GL}_n(L)$. Note that topological groups $\text{GL}_n(K)$ and $\text{GL}_n(L)$ are not topologically homeomorphic (except for $n = 1$). I prove the automorphic synchronization theorem for principal series for $\text{GL}_n(K)$ for any $n \geq 1$ and also for all irreducible admissible representations of $\text{GL}_2(K)$ for $p$ an odd prime. These restrictions are mostly to keep the section short, I expect that by the time this paper is completed, a complete result for all $n$ and all $p$ will be available.

The following two lemmas will be used in the subsequent discussion.

**Lemma 18.1.** Let $K$ be a $p$-adic field. Let $q$ be the cardinality of the residue field of $K$. Then

1. The homomorphism $\ord_K : K^* \to \mathbb{Z}$ given by $x \mapsto \ord_K(x)$ is amphoric.
2. The homomorphism $\| - \| : K^* \to \mathbb{R}^*$ defined by $\| x \| = q^{\ord_K(x)}$ is amphoric.

**Proof.** By [JR79] $q$ is amphoric. It is clear that the second assertion follows from the first. So it is sufficient to prove the first assertion. This is done in [Hos17].

**Lemma 18.2.** Let $K$ be a $p$-adic field and let $\alpha : K \leftrightarrow L$ be an anabelomorphism. Let $W_K$ (resp. $W_L$) be the Weil group of $K$ (resp. $L$) and let $W'_K$ (resp. $W'_L$) be the Weil-Deligne group of $K$ (resp. $L$). Then one has topological isomorphisms

1. $W_K \simeq W_L$, and
2. $W'_K \simeq W'_L$

which maps Frobenius element of $W_K$ to $W_L$ (and resp. for Weil-Deligne groups).
Proof. The anabelomorphism \( \alpha : K \leftrightarrow L \) gives an isomorphism \( \alpha : G_K \to G_L \).
The cardinality \( q \) of the residue field of \( K \) is amphoric (see [Hos17]). Let \( \mathbb{F}_q \) be the residue field of \( K \) (and hence of \( L \)). Let \( \text{Frob}_K \in G_K \) be a Frobenius element for \( K \) and let \( \text{Frob}_L = \alpha(\text{Frob}_K) \) be the Frobenius element of \( L \) corresponding to the Frobenius element of \( G_K \). Then \( W_K \) is the fibre-product of the two arrows

\[
\begin{array}{c}
\text{Frob}_K^Z \\
\downarrow \\
G_K \\
\rightarrow \\
\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q).
\end{array}
\]

Since \( G_K \simeq G_L \) it is now clear that \( W_K \simeq W_L \) and this takes preserves Frobenius elements. The assertion for Weil-Deligne groups is immediate from this and by the existence of \( \| - \| : K^* \to \mathbb{R}^* \) given by the previous lemma and the definition of the Weil-Deligne group.

Assume that \( K, L \) are \( p \)-adic fields and \( \alpha : K \leftrightarrow L \) is an anabelomorphism. Let me first show how to setup a bijective correspondence between principal series representations of \( \text{GL}_n(K) \) and \( \text{GL}_n(L) \).

Let me begin with the \( n = 1 \) case. In this case \( \text{GL}_1(K) = K^* \) and is \( \alpha : L \leftrightarrow K \) an anabelomorphism then \( \alpha \) provides an isomorphism \( \alpha : L^* \to K^* \). Hence any character \( \chi : \text{GL}_1(K) \to \mathbb{C}^* \) provides a character \( \chi \circ \alpha : L^* \to \mathbb{C}^* \) and conversely. Thus one obtains a bijection between admissible \( \text{GL}_1(K) \) representations and admissible \( \text{GL}_1(L) \) representations which is given by \( \chi \mapsto \chi \circ \alpha \). The local Langlands correspondence sets up a bijection between admissible representations of \( \text{GL}_1(K) \) and one dimensional representations of the Weil-Deligne group with \( N = 0 \) and hence of Weil group \( W_K \) of appropriate sort hence a representation of the Weil group \( W_L \) of the appropriate sort and conversely.

Since the local Langlands Correspondence match \( L \)-functions on either side and on the Galois side I have established (Theorem 10.2) that \( L \)-functions on the Galois side are amphoric, so it follows that automorphic \( L \)-functions are amphoric. Since conductors of characters are not amphoric (see Theorem 7.1), it follows that under anabelomorphy conductors are not amphoric. Moreover \( \varepsilon \)-factors require a choice of additive character and hence neither the conductor nor the \( \varepsilon \)-factors are amphoric (as both are dependent on the inertia filtration via its control of the additive structure–for example). Hence one has proved that

**Theorem 18.3.** Let \( K \) be a \( p \)-adic field and let \( \alpha : L \leftrightarrow K \) be an anabelomorphism of \( p \)-adic fields. Then \( \chi \mapsto \chi \circ \alpha \) sets up a bijection between admissible
representations of $GL_1(K)$ and admissible representations of $GL_1(L)$. This correspondence is compatible with the local Langlands correspondence on either side. $L$-functions of irreducible admissible representations are amphoric. The conductor and $\varepsilon$-factors are not amphoric.

Now let me discuss the $GL_n(K)$ for $n \geq 2$. The datum required to give a principal series representations of $GL_n(K)$ consists of an $n$-tuple of continuous characters $(\chi_1, \ldots, \chi_n)$ of $K^*$ with values in $\mathbb{C}^*$. The associated principal series representation is denoted by $\pi(\chi_1, \ldots, \chi_n)$ and every principal series representation is of this type.

The following theorem should be considered as the local automorphic analog of the ordinary synchronization theorem (Theorem 17.1). The first main theorem of the section is the following.

**Theorem 18.4 (Automorphic Ordinary Synchronization Theorem).** Let $\alpha : K \leftrightarrow L$ be an anabelomorphism of $p$-adic fields. Then there is a natural bijection between principal series representations of $GL_n(K)$ and principal series representations of $GL_n(L)$ which is given by

$$
\pi(\chi_1, \ldots, \chi_n) \mapsto \pi(\chi_1 \circ \alpha, \ldots, \chi_n \circ \alpha).
$$

This correspondence takes irreducible principal series representations to irreducible principal series representations.

**Proof.** The correspondence $(\chi_1, \ldots, \chi_n) \mapsto (\chi_1 \circ \alpha, \ldots, \chi_n \circ \alpha)$ sets up a bijection between $n$-tuples of continuous characters of $K^* \to \mathbb{C}^*$ and $L^* \to \mathbb{C}^*$. Every principal series representation $\pi$ of $GL_n(K)$ is of the form $\pi = \pi(\chi_1, \ldots, \chi_n)$ (similarly for $GL_n(L)$) so the assertion is immediate.

Now to prove that an irreducible principal series is mapped to an irreducible one it is sufficient to note that if $\chi_i \times \chi_j = \| - \|^{\pm 1}$ then so is $\chi_i \circ \alpha \cdot \chi_j \circ \alpha = \| - \|^{\pm 1} \circ \alpha$. So if $\pi$ is irreducible, it remains so under this correspondence. \qed

Now suppose $n = 2$ and $p \geq 3$ (i.e. $p$ is an odd prime). Then one knows that every supercuspidal representation $\pi$ of $GL_2(K)$ arise, up to twisting by one dimensional characters, by base change $\pi = BC(K_1/K, \chi)$ where $K_1 \supseteq K$ is a quadratic extension, $\chi : K_1^* \to \mathbb{C}^*$ is a character such that if $\tau \in Gal(K_1/K)$ is the unique non-trivial element then $\chi^\tau \neq \chi$.

Now suppose $\alpha : L \leftrightarrow K$. Then there exists a unique quadratic field $L_1/L$ such that $K_1 \leftrightarrow L_1$ and $Gal(\bar{K}/K_1) \subset G_K$ is the open subgroup of index two corresponding to $K_1/K$ and $G_{L_1} \subseteq G_L$ is the corresponding open subgroup under
α. A character $\chi : K_1^* \to \mathbb{C}^*$ provides a character $L_1^* \to \mathbb{C}^*$ by composing with $\alpha : L_1^* \to K_1^*$ and if $\tau' : \text{Gal}(L_1/L)$ is the unique non-trivial element then evidently $(\chi \circ \alpha)^{\tau'} \neq \chi \circ \alpha$. Hence one obtains a supercuspidal representation $\pi' = BC(L_1/L, \chi')$ where $\chi' = \chi \circ \alpha$. Thus under anabelomorphy $BC(K_1/K, \chi) \mapsto BC(L_1/L, \chi \circ \alpha)$. This procedure is symmetrical in $L$ and $K$ so this establishes a bijection between supercuspidal representations under both the sides.

Finally note that from my discussion of the principal series correspondence one sees that Steinberg representation of $GL_2(K)$ corresponding to the irreducible sub (resp. quotient) of $\pi(1, ||-||)$ (resp. $\pi(1, ||-||^{-1})$) is mapped to the corresponding object of $GL_2(L)$.

Moreover up to twisting by one dimensional characters every irreducible admissible presentation is one the three types: irreducible principal series, Steinberg and supercuspidal. And any twist of an irreducible admissible representation of $GL_2(K)$ is mapped to the corresponding twist of the appropriate irreducible admissible representation. Hence the following is established:

**Theorem 18.5 (Automorphic Synchronization Theorem).** Let $p$ be an odd prime and let $L \leftrightarrow K$ be an anabelomorphism of $p$-adic fields. Then this anabelomorphism induces a bijection between irreducible admissible representations of $GL_2(K)$ and $GL_2(L)$. This correspondence takes (twists of) irreducible principal series to irreducible principal series, Steinberg to Steinberg and supercuspidal to supercuspidal representations.

The local Langlands correspondence is a bijection between complex, semi-simple representations of Weil-Deligne group $W_K'$ and irreducible, admissible representations of $GL_2(K)$. The correspondence maps an irreducible principal series $\pi(\chi_1, \chi_2) \mapsto \chi_1 \oplus \chi_2$ ($\chi_i$ are considered as characters of the Weil-Deligne group via the Artin map). The Steinberg representation maps to the special representation $sp(2)$ of the Weil-Deligne group. A supercuspidal representation $BC(K_1/K, \chi)$ is mapped to the irreducible Weil-Deligne representation which is obtained by induction of $\chi$ from $W_{K_1}$ to $W_K$.

Now given an anabelomorphism $\alpha : L \leftrightarrow K$ and a Weil-Deligne representation $\rho : W_K' \to GL(V)$, one can associate to it the Weil-Deligne representation $\rho \circ \alpha : W_L' \to GL(V)$. This evidently takes semi-simple representations to semi-simple representations and by construction this is compatible with the local Langlands correspondence on both the sides.

Note that Artin conductors of representations on both the sides of the local Langlands correspondence are dependent on the ramification filtration and hence conductors are unamphoric (Theorem 7.1). The epsilon factors depend on additive
structures (for example the data of an epsilon factor requires an additive character) and so epsilon factors are manifestly not amphoric. Thus one has proved:

**Theorem 18.6** (Compatibility of the local Langlands Correspondence). *Let $p$ be an odd prime and let $L \leftrightarrow K$ be anabelomorphic $p$-adic fields. Then the local Langlands correspondence for $GL_2(K)$ is compatible with the automorphic synchronization provided by Theorem 18.5. $L$-functions are amphoric but the conductors of Weil-Deligne representations and irreducible, admissible representations are not amphoric. Moreover epsilon factors are not amphoric.*

**Remark 18.7.** I expect the above results to be true for $p = 2$, but their proofs will be a little more involved as there are many more representations to deal with.

**Remark 18.8.** Let $K$ be a $p$-adic field and let $K \leftrightarrow L$ be an anabelomorphism of $p$-adic fields. Then by [Hos17] the Brauer group $Br(K)$ is amphoric and hence given a division algebra $D_K$ over $K$, there exists a division algebra $D_L$ which corresponds to $D_K$. It seems reasonable to expect that, at least for the case where $D_K$ (and hence $D_L$) is a quaternion division algebra, there exists synchronization of admissible representations of $D_K^*$ and $D_L^*$ which is compatible with the above constructions.

### 19 Anabelomorphic Density Theorem

Let me now illustrate fundamental arithmetic consequences of the anabelomorphic connectivity theorems (Theorems 16.3, 16.5). Let me begin with the following elementary but important result which should be considered to be the anabelomorphic analog of Moret-Baily’s Theorem [MB89]. At the moment I do not know how to prove the full version of this theorem but already the version I prove below is enough to provide applications to elliptic curves. Let

$$U = \mathbb{P}^1 - \{0, 1, \infty\},$$

then for any field $L$, $U(L) = L^* - \{1\}$ (see Theorem 25.1 for a general result). If we have an anabelomorphism $L \leftrightarrow K$ then one has an isomorphism $L^* \to K^*$ of topological groups and hence an isomorphism topological spaces (with the respective $p$-adic topologies)

$$U(L) = L^* - \{1\} \simeq U(K) = K^* - \{1\}.$$
**Theorem 19.1** (Anabelomorphic Density Theorem). Let $U = \mathbb{P}^1 - \{0, 1, \infty\}$. Let $K$ be a number field. Let $(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{w_1, \ldots, w_n\})$ be an anabelomorphically connected number field. Then the inclusion

$$U(K') \subset \prod_i U(K'_{w_i}) \simeq \prod_i U(K_{v_i})$$

is dense for the $p$-adic topology on the right and hence also Zariski dense.

**Proof.** The proof is clear from the definition and the fact that the weak Approximation Theorem [PR94b, Chap 7, Prop. 7.1] holds for $\mathbb{P}^1$ and hence also for its Zariski open subsets. \hfill \Box

### 20 Anabelomorphic Connectivity Theorem for Elliptic Curves

To understand arithmetic consequences of the above theorem, fix an identification of schemes

$$U = \mathbb{P}^1 - \{0, 1, \infty\} \simeq \mathbb{P}^1 - \{0, 1728, \infty\}.$$

Then for any field $L$, $U(L) = L^* - \{1\}$ and composite mapping

$$L^* - \{1\} = U(L) \to \mathbb{P}^1 - \{0, 1, \infty\} \simeq \mathbb{P}^1 - \{0, 1728, \infty\}$$

allows to view the open subset $U(L)$ as $j$-invariants of elliptic curves over $L$ except for $j = 0, 1728$. If we have an anabelomorphism $L \leftrightarrow K$ then one has an isomorphism of topological spaces (with the respective $p$-adic topologies)

$$U(L) = L^* - \{1\} \simeq U(K) = K^* - \{1\}.$$

**Theorem 20.1** (Anabelomorphically Connectivity Theorem for Elliptic Curves). Let

$$(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{w_1, \ldots, w_n\})$$

be an anabelomorphically connected pair of number fields. Let $E/K$ be an elliptic curve over a number field $K$ with $j_E \neq 0, 1728$. Then there exists an elliptic curve $E'/K'$ such that

1. For all $i$ one has $\text{ord}_{v_i}(j_E) = \text{ord}_{w_i}(j_{E'}).$
(2) The $j$-invariant $j_{E'}$ of $E'$ is integral at all non-archimedean places of $K'$ except $w_1, \ldots, w_n$.

(3) In particular if $E$ has semi-stable reduction at $v_i$ then $E'$ has semistable reduction at $w_i$ and one has for the Tate parameters $v(q_{E,v_i}) = v(q_{E',w_i})$.

(4) $E'/K'$ has potential good reduction at all non-archimedean primes of $K'$ except at $w_1, \ldots, w_n$.

Proof. Let $j = j_E$ be the $j$-invariant of $E/K$. At any place $v$ of semi-stable reduction one has $v(j) < 0$. Let $\alpha_i : K_{v_i} \leftrightarrow K'_{w_i}$ be the given anabelomorphisms. Let $j_i = \alpha_i(j) \in K'^*_{w_i}$. Then by the Theorem 19.1 one sees that

$$U(K') \hookrightarrow \prod_i U(K_{v_i}) = \prod_i K^*_{v_i} - \{1\} \simeq \prod_i U(K'_{w_i}) = \prod_i K'^*_{w_i} - \{1\}$$

is dense. Hence there exists a $j' \in K'$ which is sufficiently close to each of the $j_i$ and is $w$-integral for all other non-archimedean valuations $w$ of $K'$.

By the well-known theorem of Tate [Sil86] there exists an elliptic curve $E'/K'$ with $j$-invariant $j'$. By construction $j_{E'} = j'$ is sufficiently close to $j_i$ for each $i$ and as $E/K$ has semi-stable reduction at each $v_i$ the valuation of $j_E$ at each $v_i$ is negative. Moreover for other non-archimedean valuations $w$ of $K'$, $j'$ is $w$-integral by construction and so $E'$ has potential good reduction at such $w$.

As $j'$ is sufficiently close to $j_i$ and the anabelomorphism $K_{v_i} \leftrightarrow K'_{w_i}$ preserves valuations on both the sides, the other assertions follow from the relationship between $j$-invariants and Tate parameters at primes of semi-stable reduction.

A particularly useful consequence of this is the following:

Corollary 20.2. Let $E/F$ be an elliptic curve with at least one prime of potentially semi-stable non-smooth reduction. Then there exists a pair of anabelomorphically connected number fields $(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{v'_1, \ldots, v'_n\})$ such that

(1) $F \subseteq K$ is a finite extension

(2) and $E_K = E \times_F K$ has semi-stable reduction,

(3) $v_1, \ldots, v_n$ is the set of primes of semi-stable reduction of $E/K$. 

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21 Anabelomorphy and Fundamental Groups

Now let me provide basic example of anabelomorphic varieties so that the results of this paper can be placed in the context of Mochizuki’s approach via fundamental groups. The typical example of interest for us is the following. Suppose $F, K, L$ are $p$-adic fields with inclusion $F \hookrightarrow K$ and $F \hookrightarrow L$ of $p$-adic fields such that $K, L$ are linearly disjoint over $F$ (note that $K/F$ and $L/F$ are finite extensions as $F, K, L$ are all $p$-adic fields) and an anabelomorphism $K \hookrightarrow L$. Let $X/F$ be a geometrically connected scheme of finite type over $F$. In this situation one has the fibre products $X_K = X \times_F K$ and $X_L = X \times_F L$.

Theorem 21.1. Given a $p$-adic field $F$ and $p$-adic field extensions $K/F$ and $L/F$ which are linearly disjoint over $F$ and a geometrically connected scheme of finite type $X/F$ and an anabelomorphism of $p$-adic fields $K \hookrightarrow L$ (i.e. one has a diagram of field inclusions and anabelomorphisms)

\[ K \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow L \]

Then there is an isomorphism of fundamental groups $\pi_1(X_K/K) \cong \pi_1(X_L/L)$.
In other words $X_K$ and $X_L$ are anabelomorphic schemes.

Proof. The fundamental group $\pi_1(X_K/K)$ (resp. $\pi_1(X_L/L)$) is the fibre-product of the arrows

\[ \pi_1(X/F) \]

\[ G_K \longleftarrow G_F \]

and similarly $\pi_1(X_L/L)$ is the fibre product of $\pi_1(X/F)$ and $G_L \hookrightarrow G_F$. Since $G_L \cong G_K$ the claim follows.

An important corollary (well-known to every one at RIMS) is this:

Corollary 21.2. Let $K \hookrightarrow L$ be two anabelomorphic $p$-adic fields. Then one has

$\pi_1(\mathbb{P}^1 - \{0, 1728, \infty\}/K) \cong \pi_1(\mathbb{P}^1 - \{0, 1728, \infty\}/L),$

in particular $\mathbb{P}^1 - \{0, 1728, \infty\}/K$ and $\mathbb{P}^1 - \{0, 1728, \infty\}/L$ are anabelomorphic.
Remark 21.3. A simple example of this type is the following. Consider the diagram in which all upward/downward arrows are inclusions of fields (dense if going from number field to $p$-adic field), horizontal arrows are a strict anabelomorphic connectivity and strict anabelomorphism respectively.

\[
(Q(ζ_\mathcal{P}), \sqrt{p}), p) \xrightarrow{\text{str. anabel. conn.}} (Q(ζ_\mathcal{P}, \sqrt{1+p}), p')
\]

\[
Q_p(ζ_\mathcal{P}, \sqrt{p}) \xrightarrow{\text{str. anabelomorphism}} Q_p(ζ_\mathcal{P}, \sqrt{1+p})
\]

\[
Q_p(ζ_\mathcal{P}) \xrightarrow{\text{str. anabelomorphism}} Q_p(ζ_\mathcal{P}, \sqrt{1+p})
\]

\[
Q(ζ_\mathcal{P}) \xrightarrow{\text{str. anabelomorphism}} Q(ζ_\mathcal{P})
\]

In [Moc12a, Moc12b, Moc12c, Moc12d] arithmetic information travels along anabelomorphisms in the following sense. In the context of the present example, say $E/Q(ζ_\mathcal{P})$ is an elliptic curve with semi-stable reduction (say $E : y^2 = x(x - 1)(x - ζ_p)$) at the prime lying over $p$. This curve is, of course, visible as an elliptic curve over all the fields in the diagram.

Now suppose one has a torsion point on $E$ over $Q(ζ_\mathcal{P}, \sqrt{p})$ (and hence over $Q_p(ζ_\mathcal{P}, \sqrt{p})$) then this sort of information can be pushed along the arrow of (strict) anabelomorphy to $E$ over $Q_p(ζ_\mathcal{P}, \sqrt{1+p})$ (for instance by the Ordinary Synchronization Theorem (Theorem 17.1)). However there is no algebro-geometric way of transferring this information between these two fields directly as the fields linked by anabelomorphy in the above diagram are in fact linearly disjoint over $Q(ζ_\mathcal{P})$ (resp. $Q_p(ζ_\mathcal{P})$).

As I have shown in Theorem 16.8 and Corollary 20.2 there are infinitely many anabelomorphically connected number fields containing $Q(ζ_\mathcal{P})$ with (strict) anabelomorphic connectivity at primes lying over $p$.

Moreover Theorem 21.1 says that if $X = E - \{O\}$ is considered as a curve over $Q_p(ζ_\mathcal{P})$ then

\[
π_1(X/Q_p(ζ_\mathcal{P}, \sqrt{p})) ≃ π_1(X/Q_p(ζ_\mathcal{P}, \sqrt{1+p})).
\]

Mochizuki’s Belyian Reconstruction Machine [Moc15, Theorem 1.9] (which applies as $X$ is hyperbolic and strict Belyi type) keeps a track of the number field of definition $Q(ζ_\mathcal{P})$ of $X$ as one travels along the arrows of anabelomorphy.
22 Anabelomorphy of Elliptic Curves II: Kodaira Symbol, Exponent of Conductor and Tamagawa Number are all unamphoric

Now let me explain a fundamental consequence of the fact (proved in Theorem 7.1) that the Artin and the Swan conductor of a Galois representation are unamphoric in the context of elliptic curves. Let $K$ be a $p$-adic field and let $\ell \neq p$ be a prime and let $E/\mathbb{Q}_\ell$ be a finite extension which will serve as a coefficient field for $G_K$-representations.

It should be remarked that Tate’s algorithm [Sil94, Chapter IV, 9.4] for determining the special fiber of an elliptic curve over a $p$-adic field is completely dependent on the additive structure of the field–especially steps 6 and beyond are strongly dependent on the additive structure of the field.

The purpose of this section is to prove the following:

**Theorem 22.1.** Let $E/F$ be an elliptic curve over a $p$-adic field and let $K \rightsquigarrow L$ be anabelomorphic $p$-adic fields with $K \supseteq F$ and $L \supseteq F$. Then

1. $E_L = E \times_F L$ and $E_K = E \times_F K$ are anabelomorphic elliptic curves over the respective fields.
2. $E_K$ has potential good reduction if and only if $E_L$ has potential good reduction.
3. $E_K$ has semi-stable reduction if and only if $E_L$ has semi-stable reduction.
4. Assume $E_K$ (and hence $E_L$) has semi-stable reduction. Then
   (a) The Kodaira Symbol of $E_K$ is the same as the Kodaira Symbol of $E_L$.
   (b) The exponent of the conductor of $E_K$ and $E_L$ are the same.
5. In general the following quantities are unamphoric.
   (a) The valuation of the discriminant,
   (b) the exponent of conductor of $E_K$.
   (c) The Kodaira Symbol of $E_K$, and
   (d) the Tamagawa number of $E_K$. 

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6) In particular the number of irreducible components of the special fibre of 
$E_K$ is not amphoric.

7) In particular amongst all $E_L$ with $L \rightsquigarrow K$, there is one $L$ for which 
$\text{ord}_L(\Delta_{E_L})$ is minimal.

Proof. The first assertion is immediate from Theorem 21.1. The next two assertions are special cases of the a more general result due to Mochizuki (see [Moc12e, 
Theorem 2.14(ii)]–which is more general than what is asserted here). Mochizuki also proved (in loc. cit.) the assertion about Kodaira Symbols in the semi-stable 
case as in this case Kodaira symbol is $I_n$ where $n$ is the number of irreducible 
components of the special fibre. I provide direct proofs in the above special case 
for the reader’s convenience.

Let $j = j_E$ be the $j$-invariant of $E$. Then by [Sil86, Chap VII, Prop 5.5] $E_K$ 
has potential good reduction if and only if $\text{ord}_K(j) \geq 0$. If $j = 0$ then $j$-invariant 
is integral in both $K$ and $L$ (because it is already so in $F$). So assume $j \neq 0$. 
Since $\text{ord}_K$ is an amphoric function, one has $\text{ord}_L(j) = \text{ord}_K(j)$ so the assertion 
follows. In this case the Kodaira Symbol of $E_K$ is $I_n$ where $n = -\text{ord}_K(j)$ and 
hence is uniquely determined by amphoricity of $\text{ord}_K$. The exponent of conductor 
in the semi-stable case is always one equal to one. Hence the Kodaira Symbol and 
the exponent of the conductor of $E_K$ are the same as those of $E_L$.

So it remains to prove the other assertions. To prove these assertions it suffices 
to give examples. Let me remark that these examples also show that the hypothesis 
of stable reduction in [Moc12e, Theorem 2.14(ii)] cannot be relaxed. The last 
assertion is immediate from the penultimate one as the Kodaira Symbol of $E_K$ 
also encodes the number of irreducible components of the special fibre. Let $F = 
\mathbb{Q}_3(\zeta_9)$, let $K = F(\sqrt[9]{3})$ and $L = F(\sqrt[9]{2})$. Then $K \rightsquigarrow L$ as can be easily 
checked. Both of these field have degree 

$$[K : \mathbb{Q}_3] = [L : \mathbb{Q}_3] = 54.$$

Let $E : y^2 = x^3 + 3x^2 + 9$ and $E_K$ and $E_L$ be as above. Let $\Delta$ be the minimal 
discriminant (over the relevant field), $f$ be the exponent of the conductor, the list 
of Kodaira Symbols and the definition of the Tamagawa number are in [Sil94]. 
The following table shows the values for $E_K$ and $E_L$. 

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Here is another example; let \( E : y^2 = x^3 + 3x^2 + 3 \) and let \( K, L, E_K, E_L \) be as above. Then one has

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Curve} & v(\Delta) & f & \text{Kodaira Symbol} & \text{Tamagawa Number} \\
\hline
E_K & 6 & 4 & IV & 1 \\
E_L & 6 & 2 & I_0^* & 4 \\
\hline
\end{array}
\]

Here are two more random examples where all the four quantities are simultaneously different.

Let

\[
K = \mathbb{Q}(\zeta_9) \quad L_1 = K(\sqrt{3}) \quad L_2 = K(\sqrt{4}),
\]

and let

\[
E : y^2 = x^3 + (\zeta_9^5 + 8\zeta_9^4 - \zeta_9^3 + \zeta_9^2 - 2\zeta_9 - 11)x + (-408\zeta_9^5 - 6\zeta_9^4 + 201\zeta_9^3 + 37\zeta_9^2 - 38\zeta_9 + 1348).
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Curve} & v(\Delta) & f & \text{Kodaira Symbol} & \text{Tamagawa Number} \\
\hline
E_K & 12 & 6 & IV^* & 3 \\
E_L & 12 & 10 & IV & 1 \\
\hline
\end{array}
\]

\[
E : y^2 = x^3 + (2\zeta_9^5 + 8\zeta_9^4 + \zeta_9^3 - \zeta_9^2 + 2\zeta_9 + 5)x + (869\zeta_9^5 + 159\zeta_9^4 - 47\zeta_9^3 - 125\zeta_9^2 + 354\zeta_9 + 713).
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Curve} & v(\Delta) & f & \text{Kodaira Symbol} & \text{Tamagawa Number} \\
\hline
E_K & 15 & 15 & II & 1 \\
E_L & 39 & 37 & IV & 3 \\
\hline
\end{array}
\]

For the same fields \( K, L_1, L_2 \) as in the previous example and for the curve
Now let me provide two examples for $p = 2$. Again these are examples (taken from my data) where all the four quantities are simultaneously different. Let $F = \mathbb{Q}_2(\zeta_{16})$, $K = F(\sqrt[4]{\zeta_8 - 1}, \sqrt[8]{\zeta_8 - 1})$, $L = F(\sqrt[8]{\zeta_4 - 1})$ (these fields are considered in [JR79]). By loc. cit. $K$ and $L$ are anabelomorphic of degree $n = 32$ and totally ramified extensions of $\mathbb{Q}_2$.

$E : y^2 = x^3 + (-2\zeta_8^7 - 2\zeta_8^6 + 2\zeta_8^5 - 2\zeta_8^4 - 2\zeta_8^3 + 4\zeta_8^2 + 6\zeta_{16} + 30)x$
$+ (32\zeta_8^7 - 76\zeta_8^6 - 8\zeta_8^5 + 32\zeta_8^4 - 24\zeta_8^3 - 20\zeta_8^2 + 16\zeta_8 - 28).$

Then

| Curve | $v(\Delta)$ | $f$ | Kodaira Symbol | Tamagawa Number |
|-------|-------------|-----|----------------|-----------------|
| $E_K$ | 15          | 9   | $IV^*$         | 3               |
| $E_L$ | 27          | 19  | $II^*$         | 1               |

$E : y^2 = x^3 + (-2\zeta_8^6 - 2\zeta_8^4 + 4\zeta_8^2 + 2)x$
$+ (28\zeta_8^6 - 40\zeta_8^5 - 24\zeta_8^4 + 8\zeta_8^3 + 16\zeta_8^2 - 40\zeta_8 + 60).$

Then

| Curve | $v(\Delta)$ | $f$ | Kodaira Symbol | Tamagawa Number |
|-------|-------------|-----|----------------|-----------------|
| $E_K$ | 64          | 60  | $I_0^*$        | 2               |
| $E_L$ | 52          | 52  | $II$           | 1               |

These computations were carried out using SageMath [S+17].
23  Artin Conductors, Swan Conductors and Discriminants II

The results established in this section provide a complement to the results of the previous section and the earlier section on Swan Conductors.

Let $F$ be a $p$-adic field and let $X/F$ be a geometrically connected, smooth quasi-projective variety over $F$. Let $K \hookrightarrow L$ be anabelomorphic $p$-adic fields containing $F$. Write $X_K = X \times_F K$ and $X_L = X \times_F L$.

Lemma 23.1. Let $K \hookrightarrow L$ be an anabelomorphism of $p$-adic fields. Let $K^{nr}$ (resp. $L^{nr}$) be the maximal unramified extension of $K$ (resp. $L$). Then

$K^{nr} \hookrightarrow L^{nr}$.

Proof. Since the inertia subgroup $I_K$ is amphoric by Theorem 3.2 and by the fact that $K^{nr}$ is the fixed field of $I_K$, the result is obvious. $\square$

For geometric applications it is convenient to work with a strictly Henselian ring. As Artin and Swan conductors are unaffected by passage to unramified extensions, this passage to strictly Henselian rings is harmless. In particular one can work over $K^{nr}$. By the above lemma, $K^{nr} \hookrightarrow L^{nr}$ and hence one can affect the passage to a strictly Henselian ring without affecting anabelomorphic data.

If $X/K$ is a geometrically connected, smooth, proper variety and $X_{\bar{\eta}}$ (resp. $X_s$) is the geometric generic fibre (resp. special fibre) of a regular, proper model then one has a discriminant $\Delta_{X/K}$ defined as in [Sai88]. This coincides with the usual discriminant if $X/K$ is an elliptic curve and provides a discriminant which coincides with the usual discriminant of the elliptic curve. The main theorem of loc. cit. asserts that if $X/K$ is a curve then by loc. cit. one has

$$-\text{ord}_{K}(\Delta_{X/K}) = \text{Artin}(X/K) = \chi_{\text{et}}(X_{\bar{\eta}}) - \chi_{\text{et}}(X_s) + \text{Swan}(H^1_{\text{et}}(X, \mathbb{Q}_\ell)).$$

Theorem 23.2. Let $F$ be a $p$-adic field and let $X/F$ be a geometrically connected, smooth proper variety over $F$. Let $K \hookrightarrow L$ be anabelomorphic $p$-adic fields containing $F$. Write $X_K = X \times_F K$ and $X_L = X \times_F L$. Let $\ell \neq p$ be a prime.

(1) $X_K$ and $X_L$ are anabelomorphic,

(2) $\sum_{i\geq0}(-1)^i\text{Swan}(H^i_{\text{et}}(X, \mathbb{Q}_\ell))$ is unamphoric.

(3) Suppose $X/K$ is one dimensional i.e. a curve. Then $\text{ord}_K(\Delta_{X/K})$ is unamphoric.
(4) In particular if $X/K$ is an elliptic curve then $\text{ord}_K(\Delta_{X/K})$ is unamphoric.

Proof. The first assertion is immediate from 21.1 (and included here only for convenience). In Theorem 7.1, I have shown that Artin and Swan conductors are unamphoric. The last two assertion are immediate from the second by [Sai88]. To prove the second, let me prove a more refined claim here. Let $\rho : G_K \to \text{GL}(V)$ be a $G_K$-representation in a finite dimension $\mathbb{Q}_\ell$ vector space (with $\ell \neq p$ a prime) such that the image of the wild inertia subgroup $P_K$ is finite.

I claim in fact that the breaks in the break-decomposition of $V$ (see [Kat88, Lemma 1.5]) are unamphoric. If $K \rightsquigarrow L$ is a strict anabelomorphism then the $G_K$ and $G_L$ have distinct ramification filtrations and the proof of the break-decomposition shows that the break-decomposition is dependent on the ramification filtration. Hence the break-decomposition itself is unamphoric in general. Hence the Swan conductor which is a measure of the breaks in the break-decomposition is unamphoric.

The following records the unamphorticity of the break-decomposition:

**Corollary 23.3.** Let $K$ be a $p$-adic field and let $\rho : G_K \to \text{GL}(V)$ be a continuous representation of $G_K$ in a $\mathbb{Q}_\ell$-vector space $V$ such that the wild inertia subgroup $P_K$ operates through a finite quotient. Then the break-decomposition of $V$ is unamphoric. In particular the breaks in the break-decomposition are unamphoric.

## 24 The theory of Tate curves at archimedean primes and $\Theta$-functions at archimedean primes

In [Moc12e, Moc13, Moc15] and especially in [Moc12a, Moc12b, Moc12c, Moc12d] the theory of elliptic curves at archimedean primes poses some difficulty (this also discussed in [DHb], [DHa]). The reason is this: on one hand any pure $\mathbb{Q}$-Hodge structure is semi-simple on the other hand there are no one dimensional $\mathbb{Q}$-Hodge structures of weight one, and so the Hodge structure of an elliptic curve is indecomposable as a $\mathbb{Q}$-Hodge structure. This is in contrast to the situation at the non-archimedean primes of semi-stable reduction (where the Galois representation is in fact reducible). I want to explain how to circumvent this difficulty and provide a description parallel to Theorem 17.1 at infinity. One should think of Theorem 24.6 (see below) as the *Ordinary Synchronization Theorem at Infinity*. The theory of this section, especially Theorem 24.11 should also be compared with [Moc09]
where Mochizuki constructs Galois cohomology classes (in $H^1(G_K, \mathbb{Q}_p(1))$) corresponding to $\Theta$-functions on an elliptic curve.

For the Diophantine applications which Mochizuki considers in [Moc12a, Moc12b, Moc12c, Moc12d], let $K$ be a number field which one typically assumes to have no real embeddings. Let $E/K$ be an elliptic curve and assume that the Faltings height $h(E)$ of $E$ is large. By the known facts about Faltings height, $h(E) \gg 0$ corresponds to $h(j_E) \gg 0$ and equivalently this means that the Schottky (uniformization) parameter $q_E = e^{2\pi i E}$ of $E$ is small.

Schottky uniformization of elliptic curves says that one has an isomorphism of complex abelian manifolds

$$\mathbb{C}^*/q_E^\mathbb{Z} \xrightarrow{\sim} E(\mathbb{C})$$

at infinity (let me remind the readers that Tate’s Theory of $p$-adic uniformization of elliptic curves is modeled on Schottky uniformization of elliptic curves). So the theory of elliptic curves of large Faltings height corresponds to the theory of complex tori with a small Schottky parameter. To describe this in parallel with the Theory of Tate curves at non-archimedean primes, let me begin by recalling the following well-known fact from mixed Hodge Theory ([Car87], [Del97])

**Lemma 24.1.** One has an isomorphism of abelian groups:

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathbb{C}^*.$$

In particular the Schottky parameter $q_E \in \mathbb{C}^*$ provides a unique mixed Hodge structure

$$H = H(E) \in \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathbb{C}^*$$

(not to be confused with the usual Hodge structure $H^1(E, \mathbb{Z})$ which is of weight one. The mixed Hodge structure $H(E)$ comes equipped with a weight filtration and unipotent monodromy (see [Del97]). In particular let me recall the formula from [Del97]:

\begin{align}
H_C &= \mathbb{C}e_0 \oplus \mathbb{C}e_1, \\
W_{-2} \subset H &= \mathbb{C}e_1, \\
F^0 \subset H &= \mathbb{C}e_0, \\
H_Z &= 2\pi i \mathbb{Z}e_0 \oplus \mathbb{Z}(e_0 + \log(q)e_1) \subset H_C.
\end{align}

The mapping $\mathbb{Z}(1) \to H_Z$ is given by $2\pi i \mapsto 2\pi i e_1$ and $H_Z \to \mathbb{Z}(0)$ is given by $e_0 \mapsto 1$. Then one has an exact sequence of mixed Hodge structures

$$0 \to \mathbb{Z}(1) \to H \to \mathbb{Z}(0) \to 0,$$
whose class in $\text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1))$ is given by $q \in \mathbb{C}^*$.

Now let $u = e^{2\pi iz}$ with $z \in \mathbb{C}$ and let $\Theta_E = \Theta(q, 0)$ where $\Theta(q, z) = 1 + O(q)$ is a suitable Jacobi Theta function on $E/\mathbb{C}$. For $0 < |q| \ll 1$, $\Theta_E \in \mathbb{C}^*$ and hence provides us a mixed Hodge structure $H_{E, \Theta} \in \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1))$ given by $\Theta_E \in \mathbb{C}^*$.

Thus I have proved the following:

**Theorem 24.6.** Let $E/\mathbb{C}$ be an elliptic curve with Schottky parameter $q = q_E$ such that $0 < |q| \ll 1$. Then

1. there is mixed Hodge structure $H = H(E) \in \text{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(1)) \cong \mathbb{C}^*$ whose extension class corresponds to $q \in \mathbb{C}^*$, and

2. there is a mixed Hodge structure $H_{\Theta} = H(E, \Theta) \in \text{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(1))$ whose extension class corresponds to the $\theta$-value $\Theta_E = \Theta(q, 0) \in \mathbb{C}^*$.

**Remark 24.7.** The mixed Hodge structures $H(E), H(E, \Theta)$ correspond, at a prime $v$ of semi-stable reduction, to the $G_v$-modules $H^1(E, \mathbb{Q}_p)$ and the $\Theta$-value class constructed by Mochizuki in [Moc09], [Moc15].

**Remark 24.8.** Comparing the definition above of $H$ and with the formula of Fontaine for $L$-invariant, I define the $L$-invariant $\mathcal{L}_\infty(H) = \frac{\log(q)}{2\pi i}$. If $q = e^{2\pi i \tau}$ then $\mathcal{L}_\infty(H) = \tau$! So $\tau$ is the $L$-invariant of the elliptic curve at archimedean primes and anabelomorphy changes the $L$-invariant at all the places.

Let me remark that the construction given above can be extended to provide results over a geometric base scheme (see [Del97]). For example let $E/\mathbb{C}$ be an elliptic curve and let $X = E - \{O\}$. Let $f \in \mathcal{O}_X^*$ be a meromorphic function on $E$ which is an invertible function on $X$. More generally one can consider any open subset $U$ of $E$ and consider $f \in \mathcal{O}_U^*$ i.e. an invertible function on $U$. Then there exists a variation of mixed Hodge structures (over $U$) $H(E, f) \in \text{Ext}^1_{V-MHS}(\mathbb{Z}(0), \mathbb{Z}(1))$ such that under the natural identification

$$\text{Ext}^1_{V-MHS}(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathcal{O}_U^*$$

the extension class corresponding to $H(E, f)$ is equal to $f \in \mathcal{O}_U^*$. The mixed Hodge structure $H(E, f)$ is constructed as follows (see [Del97]). Let $V = \mathcal{O}_U e_1 + \mathcal{O}_U e_2$ be a locally free $\mathcal{O}_U$ module with basis $e_1, e_2$. The connection $\nabla$ (with logpoles at $O$) on $V$ is defined by

$$\nabla = d + \begin{pmatrix} 0 & 0 \\ -\frac{df}{f} & 0 \end{pmatrix}. $$

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The rest of the data required to define a variation of mixed Hodge structures is defined by the formulae above. Let me remark that the triple \((V, \nabla, \text{Fil}(V))\) consisting of the bundle \(V\) together with the connection \(\nabla\) and the Hodge filtration is the data of an indigenous bundle (equivalently a rank two oper) on \(U\).

So one can apply this consideration to a chosen \(f\) such as a theta function on \(E\) which does not vanish on the open set \(X = E - \{O\}\). By the theory of theta functions, up to scaling by a constant, there is a unique function with this property, denoted by \(\Theta(q, z)\). Note that a theta function is, strictly speaking, not a function on the curve as it is quasi-periodic. But by [WW96], the logarithmic derivative of any of the four standard theta functions with periods \(\{1, \tau\}\) satisfies

\[
\frac{\theta(q, z + 1)'}{\theta(q, z + 1)} = \frac{\theta(q, z)'}{\theta(q, z)}
\]  

\[
\frac{\theta(q, z + \tau)'}{\theta(q, z + \tau)} = -2\pi i + \frac{\theta(q, z)'}{\theta(q, z)}.
\]  

More precisely, there is a vector bundle on \(E\) of rank two and a connection on \(E\), with log-poles at \(O\), which on the universal cover \(\mathbb{C}\) of \(E\) is given by the connection matrix as above with \(f = \Theta(q, z)\). At any rate the connection defined by the above formula on \(\mathbb{C}\) descends to \(E\) (with log-poles at \(O\)). Hence one has proved that

**Theorem 24.11.** Let \(E/\mathbb{C}\) be an elliptic curve with Schottky parameter \(q = q_E\) such that \(0 < |q| \ll 1\) and let \(X = E - \{O\}\). Let \(\Theta(q, z)\) be a Theta function on \(E\) which does not vanish on \(X\) and normalized so that \(\Theta(q, z) = 1 + O(q)\). Then there is a variation of mixed Hodge structures over \(X\), denoted

\[H = H(E, \Theta(q, z)) \in \text{Ext}^1_{\mathcal{V} \cdot \text{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1)) \simeq \mathcal{O}_X^\ast\]

such that the extension class of \(H(E, f)\) corresponds to \(\Theta(q, z) \in \mathcal{O}_X^\ast\) (here \(\mathcal{O}_X^\ast\) is the group of holomorphic functions which are invertible on \(X\)). This class is compatible with the class constructed above.

**Remark 24.12.** Let me remark that this construction is anabelomorphic: Mochizuki’s Anabelian Yoga reconstructs monoids such as \(\mathcal{O}_X^\ast\) from the étale fundamental group of a punctured elliptic curve! Moreover under what Mochizuki calls “Auto-Holomorphic Symmetries” (at archimedean primes) the class constructed above transforms by the rules of transformations of theta functions (i.e. modular forms of half integer weight).
Remark 24.13. Let $X$ be a non-proper hyperbolic Riemann surface, $\pi_1^{\text{top}}(X)$ its topological fundamental group. Let $\mathbb{Q}[\pi_1^{\text{top}}(X)]$ be the group ring and $J \subset \mathbb{Q}[\pi_1^{\text{top}}(X)]$ be the augmentation ideal. By well-known results for each $r \geq 1$, $\mathbb{Q}[\pi_1^{\text{top}}(X)]/J^{r+1}$ carries a mixed Hodge structure and good unipotent variations of mixed Hodge structures on $X$ of nilpotence $\leq 2$ with values in $\mathbb{Q}$ (or a real field) arise precisely from finite dimensional representations $V$ of $\pi_1^{\text{top}}(X)$ such that the natural map $\mathbb{Q}[\pi_1^{\text{top}}(X)]/J^3 \to \text{End}(V)$ is a morphism of mixed Hodge structures ([HZ87, Theorem ]). The Yoneda ext-group in the category of good unipotent variations of mixed Hodge structure on $X$, denoted $\text{Ext}_{\pi_1}^{1}(\mathbb{Q}(0), \mathbb{Q}(1))$ is also described, by [CH89, Theorem 12.1] in an essentially anabelomorphic way as follows:

$$H^1_{\text{ef}}(X, \mathbb{Q}(1)) \simeq H^1_{\text{ef}}(\pi_1^{\text{top}}(X), \mathbb{Q}(1)) \simeq \text{Ext}_{\pi_1}^{1}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathcal{O}_X^* \otimes \mathbb{Q}.$$ 

The cohomology $H^1_{\text{ef}}$ is Beilinson’s Absolute Hodge Cohomology. The middle isomorphism makes it clear that the group of extensions on the right is anabelomorphic (in the complex analytic space $X$).

25 Anabelomorphic Density Theorem II

Now let me prove the anabelomorphic density theorem. This section is a bit technical and skipped in the initial reading and is certainly independent of the rest of the paper. In this section by “assume Grothendieck’s section conjecture holds for $X/K$” I will mean that $X(K) \hookrightarrow \text{Sect}(G_K, \pi_1(X/K))$ and that there is some characterization of the image of this set, with some reasonable functoriality in $X/K$. I will simply write $\text{Hom}(G_K, \pi_1(X/K))$ to means the subset of sections $\text{Sect}(G_K, \pi_1(X/K))$ which are characterized as arising from $X(K)$.

Let me emphasize that the only evidence for such an expectation at the moment except Mochizuki’s Theorem that for a hyperbolic curve $X/K$, $X(K)$ injects into the set of sections of $\pi_1(X/K)$ and Theorem 12.2 and Theorem 14.1.

Recall that two varieties $X/K, Y/L$ (both are assumed to be geometrically connected, smooth and quasi-projective) over $p$-adic fields $K, L$ are said to be anabelomorphic if $\pi_1(X/K) \simeq \pi_1(Y/L)$ is an isomorphism of topological groups. As has been pointed out earlier, this condition implies that $K \leftrightarrow L$.

Let me extend the notion of anabelomorphically connected number fields slightly. I will write

$$(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{v'_{1,1}, \ldots, v'_{1,r_1}; \ldots; v'_{n,1}, \ldots, v'_{n,r_n}\})$$
and say that $K, K'$ are \textit{anabelomorphically connected} along non-archimedean places $v_1, \ldots, v_n$ of $K$ and non-archimedean places $v'_{1,1}, \ldots, v'_{1,r_1}; \ldots; v'_{n,1}, \ldots, v'_{n,r_n}$ of $K'$ if

$$K_{v_i} \leftrightarrow K'_{v_{i,j}}$$

for each $i$, one has $K_{v_i} \leftrightarrow K'_{v_{i,j}}$, for all $1 \leq j \leq r_i$.

Clearly this extends the notion introduced previously by allowing several primes of $K'$ lying over a place of $K$.

\textbf{Theorem 25.1.} Let $K$ be a number field and let $v_1, \ldots, v_n$ be a finite set of non-archimedean places of $K$. Let $(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{v'_{1,1}, \ldots, v'_{n,n}\})$ be anabelomorphically connected number field. Let $X/K$ (resp. $Y/K'$) be a geometrically connected, smooth, quasi-projective variety over $K$ (resp. $K'$). Suppose the following conditions are met:

1. $X/K_{v_i}$ and $Y/K'_{v'_{i}}$ are anabelomorphic varieties for $1 \leq i \leq n$, and
2. $X(K_{v_i}) \neq \emptyset$ for all $1 \leq i \leq n$, and
3. Grothendieck’s section conjecture holds for $X/K_{v_i}$ and $Y/K'_{v'_{i}}$, and
4. suppose that one is given a non-empty open subset (in the $v_i$-adic topology)
   $$U_i \subseteq X(K_{v_i}).$$

Then there exists a finite extension $K''/K'$ and places $v''_{1,1}, \ldots, v''_{1,r_1}; \ldots; v''_{n,1}, v''_{n,r_n}$ of $K''$

1. such that one has the anabelomorphic connectivity chain
   $$(K, \{v_1, \ldots, v_n\}) \leftrightarrow (K', \{v'_{1,1}, \ldots, v'_{n,n}\}) \leftrightarrow (K'', \{v''_{1,1}, \ldots, v''_{1,r_1}; \ldots; v''_{n,1}, v''_{n,r_n}\})$$
2. and, for all corresponding primes in the above connectivity chain, bijections
   $$Y(K''_{v''_{i,j}}) \simeq Y(K'_{v'_i}) \simeq X(K_{v_i})$$
3. and a point $y \in Y(K'')$ whose image in $Y(K''_{v''_{i,j}}) \simeq Y(K'_{v'_i}) \simeq X(K_{v_i})$ (for all $i, j$) is contained in $U_i$.

\textbf{Proof.} The proof will use Lemma 25.2 given below. By the hypothesis that $X/K_{v_i}, Y/K'_{v'_{i}}$ are anabelomorphic, one has by Lemma 25.2, that for each $i$, there is a natural bijection of sets

$$X(K_{v_i}) \simeq Y(K'_{v'_{i}}).$$
and hence the latter sets are non-empty because of our hypothesis.

Now the usual Moret-Bailly Theorem [MB89] can be applied to $Y/K'$ with $S = \{v'_1, \ldots, v'_n\}$ so there exists a finite extension $K''/K'$ which is totally split at all the primes $v'_i$ into primes $v''_{i,j}$ with $j = 1, \ldots, r_i = [K'' : K']$ and hence for each $i$ one has isomorphisms $K'_{v'_i} \simeq K''_{v''_{i,j}}$ (for all $j$) and hence for each $i$ one has $K'_{v'_i} \rightsquigarrow K''_{v''_{i,j}}$ (for all $j$) and hence one has the stated anabelomorphic connectivity. The remaining conclusions are consequences of the usual Moret-Bailly Theorem.

Let me begin the following lemma which explains the role of Grothendieck’s section conjecture in this context. Note that in the context of the usual Moret-Bailly Theorem, the fields $K, L$ are isomorphic so one may take $Y = X$ and the section conjecture hypothesis is unnecessary. So anabelomorphy really underlies the sort of phenomena which lie at heart of [MB89].

**Lemma 25.2.** Let us suppose that $X/K$ and $Y/L$ are two geometrically connected, smooth, quasi-projective anabelomorphic varieties over $p$-adic fields $K, L$ (i.e. $\pi_1(X/K) \simeq \pi_1(Y/L)$ is an isomorphism of topological groups and in particular one has $K \rightsquigarrow L$). Assume that Grothendieck’s Section Conjecture holds for $X/K$ and $Y/L$. Then one has a natural bijection of sets

$$X(K) \simeq Y(L),$$

and in particular if $X(K) \neq \emptyset$ then $Y(L) \neq \emptyset$.

**Proof.** Let me remark that if $X/K$ and $Y/L$ are two varieties over anabelomorphic fields $K \rightsquigarrow L$ such that $\pi_1(X/K) \simeq \pi_1(Y/L)$ i.e. $X/K, Y/L$ are anabelomorphic varieties

then Grothendieck’s section conjecture, which asserts that

$$X(K) \rightarrow \text{Hom}(G_K, \pi(X/K))$$

is a bijection of sets, implies that there is a natural bijection of sets

$$X(K) \simeq \text{Hom}(G_K, \pi_1(X/K)) \rightsquigarrow \text{Hom}(G_L, \pi_1(Y/L)) \simeq Y(L),$$

and now the last assertion is obvious.

Note that Grothendieck’s Section Conjecture is difficult. The following conjecture is adequate for most arithmetic applications.
Conjecture 25.3. Let $F$ be a $p$-adic field and let $X/F$ be a geometrically connected, smooth, quasi-projective variety over $F$. Let $K \leftrightarrow L$ be two anabelomorphic $p$-adic fields containing $F$. Let $X_K = X \times_F K$ (resp. $X_L = X \times_F L$). Then

(1) There exists a finite extension $K'/K$ and $L'/L$ and an anabelomorphism $K' \leftrightarrow L'$ such that there is a natural bijection of sets $X_{K'}(K') \rightarrow X_{L'}(L')$.

(2) There is a Zariski dense open subset $U \subset X/F$ such that the induced mapping $U_{K'}(K') \rightarrow U_{L'}(L')$ is a homeomorphism of topological spaces with respective topologies on either side.

26 Perfectoid algebraic geometry as an example of anabelomorphy

Now let me record the following observation which I made in the course of writing [Jos19a] and [Jos19b]. A detailed treatment of assertions of this section will be provided in [DJ] where we establish many results in parallel with classical anabelian geometry.

Let $K$ be a complete perfectoid field of characteristic zero. Let $K^\flat$ be its tilt. Let $L$ be another perfectoid field with $L^\flat \simeq K^\flat$. In perfectoid algebraic geometry $K, L$ are called an untilts of $K^\flat$. For example the $p$-adic completions $K, L$ respectively of $\mathbb{Q}_p((\sqrt[p]{t}))$ and $\mathbb{Q}_p((\sqrt[p]{t}))$ are both perfectoid fields with $L^\flat \simeq K^\flat = \mathbb{F}_p((t^{1/p\infty}))$. The theory of perfectoid fields shows that

$$G_L \simeq G_{L^\flat} \simeq G_{K^\flat} \simeq G_K.$$ 

This is in fact compatible with the inertia filtration on all the groups. Thus one sees that filtered group $G_K$ does not identify our perfectoid field $K$ uniquely.

In particular this suggests that the filtered absolute Galois group of a perfectoid field of characteristic zero has non-trivial outer automorphisms which does not respect the ring structure of $K$.

This is the perfectoid analog of the fact that the absolute Galois group $G_K$ of a $p$-adic field $K$ has automorphisms which do not preserve the ring structure of $K$. Now let me explain that the main theorem of [Sch12b] provides the perfectoid analog of anabelomorphy (in all dimensions).

Suppose that $K$ is a complete perfectoid field of characteristic zero. Let $X/K$ be a perfectoid variety over $K$, which I assume to be reasonable, to avoid inane
pathologies. Let $\pi_1(X/K)$ be its étale site. Let $X^\flat/K^\flat$ be its tilt. Then the main theorem of [Sch12b] asserts that

**Theorem 26.1.** The tilting functor provides an equivalence of categories $\pi_1(X/K) \to \pi_1(X^\flat/K^\flat)$. If $L$ is any untilt of $K^\flat$ and $Y/L$ is any perfectoid variety with tilt $Y^\flat/L^\flat \simeq X^\flat/K^\flat$. Then one has $\pi_1(X/K) \simeq \pi_1(Y/L)$ and in particular $X/K$ and $Y/L$ are perfectoid anabelomorphs of each other.

In particular one says that $X/K$ and $Y/L$ are anabelomorphic perfectoid varieties over anabelomorphic perfectoid fields $K \leftrightarrow L$. Thus one can envisage proving theorems about $X/K$ by picking an anabelomorphic variety in the anabelomorphism class which is better adapted to the properties (of $X/K$) which one wishes to study. In some sense Scholze’s proof of the weight monodromy conjecture does precisely this: Scholze replaces the original hypersurface by a (perfectoid) anabelomorphic hypersurface for which the conjecture can be established by other means.

### 27 The proof of the Fontaine-Colmez Theorem as an example of anabelomorphy on the Hodge side

Let me provide an important example of Anabelomorphy which has played a crucial role in the theory of Galois representations. The Colmez-Fontaine Theorem which was conjectured by Jean-Marc Fontaine which asserts that “every weakly admissible filtered $(\phi, N)$ module is an admissible filtered $(\phi, N)$ module” and proved by Fontaine and Colmez in [CF00]. The proof proceeds by changing the Hodge filtration on a filtered $(\phi, N)$-module.

*This should be viewed as an example of anabelomorphy but carried out on the $p$-adic Hodge structure.*

The idea of [CF00] is to replace the original Hodge filtration (which may make the module possibly inadmissible) by a new Hodge filtration so that the new module becomes admissible i.e. arises from a Galois representation. So in this situation the $p$-adic Hodge filtration is considered mobile while other structures remain fixed. This allows one to keep the $p$-adic field $K$ fixed. Let me remark that by Theorem 15.1 one knows that the $\mathcal{L}$-invariant of an elliptic curve over a $p$-adic field is unamphoric together with the fact that $\mathcal{L}$-invariant is related to the filtration of the $(\phi, N)$-module (see [Maz94]). So the filtration is moving in some sense but the space on which the filtration is defined is also moving because the Hodge filtration for the $G_K$-module $V$ lives in the $K$-vector space $D_{st}(V)$, while the Hodge
filtration for the $G_L$-module $V$ lives in an $L$-vector space. As Mochizuki noted in his e-mail to me “it remains a significant challenge to find containers where the $K$-vector space $D_{dR}(\rho, V)$ and $L$-vector space $D_{dR}(\rho \circ \alpha, V)$ can be compared.” My observation recorded below says that in fact there is a natural way to compare these spaces under a reasonable assumption.

Let $K$ be a $p$-adic field and let $\alpha : L \hookrightarrow K$ be an anabelomorphism of $p$-adic fields. Consider $\rho : G_K \to \text{GL}(V)$ of $G_K$. Suppose that $V$ is a de Rham representation in the sense of [Fon94]. As was proved in [Hos13] $\rho \circ \alpha : G_L \to \text{GL}(V)$ need not be de Rham. Suppose $V$ is ordinary. Then by [PR94a], $V$ is then semi-stable and hence also de Rham. By Theorem 5.2 one deduces that the $G_L$-representation $\rho \circ \alpha : G_L \to \text{GL}(V)$ is also ordinary and hence also de Rham. Let me write $D_{dR}(\rho, V)$ for the $K$-vector space associated to the de Rham representation $\rho : G_K \to \text{GL}(V)$ of $G_K$ and write $D_{dR}(\rho \circ \alpha, V)$ for the $L$-vector space associated to the de Rham representation $\rho \circ \alpha : G_L \to \text{GL}(V)$ of $G_L$.

**Theorem 27.1.** Let $K$ be a $p$-adic field, let $\alpha : L \hookrightarrow K$ be an anabelomorphism of $p$-adic fields. Let $\rho : G_K \to \text{GL}(V)$ be a de Rham representation of $G_K$ such that $\rho \circ \alpha : G_L \to \text{GL}(V)$ is also de Rham (for example $V$ is ordinary). Then for all sufficiently large integers $k > 0$, there is a natural isomorphism of $\mathbb{Q}_p$-vector spaces

$$D_{dR}(\rho, V(k)) \simeq D_{dR}(\rho \circ \alpha, V(k)).$$

**Remark 27.2.** Note that the Hodge filtration on the $K$-vector space $D_{dR}(V(k))$ is up to shifting, the filtration on the $K$-vector space $D_{dR}(V)$. So the twisting does not affect the underlying $K$-vector space in any serious way. In particular these two $K$-vector spaces are naturally isomorphic.

**Proof.** Let $G_L$ act on $V$ through the isomorphism $\alpha$. So $V$ is also a $G_L$-module. Then as $G_K \simeq G_L$, one has an isomorphism of $\mathbb{Q}_p$-vector spaces (given by $\alpha$):

$$H^1(G_K, V) \simeq H^1(G_L, V).$$

By a fundamental observation of [BK90] there is a natural mapping, called the Bloch-Kato exponential,

$$\exp_{BK} : D_{dR}(\rho, V(k)) \to H^1(G_K, V(k))$$

which is an isomorphism for all sufficiently large $k > 0$. There is a similar isomorphism for $L$ since $V$ is also a de Rham representation of $G_L$ through $\alpha$. Now putting all this together the isomorphism in the theorem is obvious. 

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Corollary 27.3. Let $K$ be a $p$-adic field. Let $\rho : G_K \to \text{GL}(V)$ be a $p$-adic representation of $G_K$. Then ordinarity is an amphoric property of $V$ and when $V$ is ordinary $D_{dR}(V)$ is an amphoric $\mathbb{Q}_p$-vector space.

28 Anabelomorphy for $p$-adic differential equations

This section is independent of the rest of the paper. A reference for this material contained in this section is [And02]. Here I provide a synchronization theorem for $p$-adic differential equations in the sense of [And02, Chap. III, Section 3]. Let $X/K$ be a geometrically connected, smooth, quasi-projective variety over a $p$-adic field $K$. Let $X^{an}/K$ denote the strictly analytic Berkovitch space associated to $X/K$. By the Riemann-Hilbert Correspondence, I mean [And02, Chapter III, Theorem 3.4.6].

Theorem 28.1 (Synchronization of $p$-adic differential equations). Let $X/K$ and $Y/L$ be two geometrically connected, smooth, quasi-projective varieties over $p$-adic fields $K$ and $L$. Assume that $X^{an}/K$ and $Y^{an}/L$ are anabelomorphic strictly analytic spaces (i.e. $\alpha : \pi_1(X^{an}/K) \simeq \pi_1(Y^{an}/L)$ (where the fundamental groups with respect to a $K$-rational (resp. an $L$-rational) base point)). Then there exists a natural bijection $\alpha$ between $p$-adic differentials on $X^{an}/K$ and $Y^{an}/L$ which associates to a $p$-adic differential equation $(V, \nabla)$ on $X^{an}/K$, a $p$-adic differential equation $Y^{an}/L$ such that the associated (discrete) monodromy representation of $\pi_1(Y^{an}/L)$ is given composing with $\alpha^{-1} : \pi_1(Y^{an}/L) \xrightarrow{\sim} \pi_1(X^{an}/K)$.

Corollary 28.2 (Anabelomorphy of $p$-adic differential equations). Let $X/F$ be a geometrically connected, smooth, quasi-projective variety over a $p$-adic field $F$. Let $K \leftrightarrow L$ be anabelomorphic $p$-adic fields containing $F$. Then given any $\alpha : K \to L$, there exists a natural bijection between $p$-adic differentials on $X^{an}_K$ and on $X^{an}_L$ such that the monodromy representations on either side are obtained by composing with the anabelomorphism $\alpha : X^{an}_K \leftrightarrow X^{an}_L$ given by Theorem 21.1. This construction is compatible with the Riemann-Hilbert Correspondence. Hence the Riemann-Hilbert Correspondence is amphoric.

One should think of this as gluing of $p$-adic differential equations by their monodromy representations.

Since it is well-known that the analog in theory of differential equations of that the local index of irregularity is the Swan conductor of a Galois representation, the following conjecture is natural given my earlier results on the Swan conductor:
Conjecture 28.3 (Index of Irregularity is Unamphoric). In the situation of the above corollary, assume that $X/K$ is a curve (i.e. $\dim(X) = 1$), then the index of irregularity of a $p$-adic differential equation $(V, \nabla)$ on $X/K$ is unamphoric. More generally, the irregularity module of the differential equation $(V, \nabla)$ is unamphoric ($X$ need not be a curve for this).

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