Finite $p$-groups with a Frobenius group of automorphisms whose kernel is a cyclic $p$-group

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to Victor Danilovich Mazurov on the occasion of his 70th birthday

Abstract

Suppose that a finite $p$-group $P$ admits a Frobenius group of automorphisms $FH$ with kernel $F$ that is a cyclic $p$-group and with complement $H$. It is proved that if the fixed-point subgroup $C_P(H)$ of the complement is nilpotent of class $c$, then $P$ has a characteristic subgroup of index bounded in terms of $c, |C_P(F)|$, and $|F|$ whose nilpotency class is bounded in terms of $c$ and $|H|$ only. Examples show that the condition of $F$ being cyclic is essential. The proof is based on a Lie ring method and a theorem of the authors and P. Shumyatsky about Lie rings with a metacyclic Frobenius group of automorphisms $FH$. It is also proved that $P$ has a characteristic subgroup of $(|C_P(F)|, |F|)$-bounded index whose order and rank are bounded in terms of $|H|$ and the order and rank of $C_P(H)$, respectively, and whose exponent is bounded in terms of the exponent of $C_P(H)$.

Key words. finite $p$-group, Frobenius group, automorphism, nilpotency class, Lie ring

1 Introduction

It has long been known that results on ‘semisimple’ fixed-point-free automorphisms of nilpotent groups and Lie rings can be applied for studying ‘unipotent’ $p$-automorphisms of finite $p$-groups. Alperin [1] was the first to use Higman’s theorem on Lie rings and nilpotent groups with a fixed-point-free automorphism of prime order $p$ in the study of a finite $p$-group $P$ with an automorphism $\varphi$ of order $p$. Namely, Alperin [1] proved that the

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derived length of $P$ is bounded in terms of the number of fixed points $p^m = |C_P(\varphi)|$. Later the first author [9] improved the argument to obtain a subgroup of $P$ of $(p, m)$-bounded index and of $p$-bounded nilpotency class, and the second author [18] noted that this class can be bounded by $h(p)$, where $h(p)$ is Higman’s function bounding the nilpotency class of a Lie ring or a nilpotent group with a fixed-point-free automorphism of order $p$. Henceforth we write for brevity, say, “$(a, b, \ldots)$-bounded” for “bounded above by some function depending only on $a, b, \ldots$”. Further strong results on $p$-automorphisms of finite $p$-groups were obtained by Klingen [16], McKay [22], Shalev [25], Medvedev [23, 24], Jaikin-Zapirain [6], Shalev and Zelmanov [20] giving subgroups of bounded index and of bounded derived length or nilpotency class. The proofs of most of these ‘unipotent’ results were also based on the ‘semisimple’ theorems of Higman [4], Kreknin [7], Kreknin and Kostrikin [8] on fixed-point-free automorphisms of Lie rings.

In the present paper ‘unipotent’ theorems are derived from the recent ‘semisimple’ results of the authors and Shumyatsky [15, 20] about groups $G$ (and Lie rings $L$) admitting a Frobenius group $FH$ of automorphisms with kernel $F$ and complement $H$. The results concern the connection between the nilpotency class, order, rank, and exponent of $G$ and the corresponding parameters of $C_G(H)$. The more difficult of these results is about the nilpotency class, and its proof is based on the corresponding Lie ring theorem. Namely, it was proved in [15] that if the kernel $F$ is cyclic and acts on a Lie ring $L$ fixed-point-freely, $C_L(F) = 0$, and the fixed-point subring $C_L(H)$ of the complement is nilpotent of class $c$, then $L$ is nilpotent of $(c, |H|)$-bounded class (under certain assumptions on the additive group of $L$, which are satisfied in many important cases, like $L$ being an algebra over a field, or being finite). Note that examples show that the condition of $F$ being cyclic is essential. This Lie ring result also implied a similar result for a finite group $G$ with a Frobenius group $FH$ of automorphisms with cyclic fixed-point-free kernel $F$ such that $C_G(H)$ is nilpotent of class $c$, with reduction to nilpotent case provided by classification and representation theory arguments. The fixed-point-free action of $F$ alone was known to imply nice properties of the Lie ring (solubility of $|F|$-bounded derived length by Kreknin’s theorem [7]) and of the group (solubility and well-known bounds for the Fitting height due to Thompson [27], Kurzweil [17], Turull [28], and others — although an analogue of Kreknin’s theorem is still an open problem for groups). But the conclusions of the results in [15] are in a sense much stronger, due to the combination of the hypotheses on fixed points of $F$ and $H$, either of which on its own is insufficient.

We now state the ‘unipotent’ version of the nilpotency class result as in [15].

**Theorem 1.1.** Suppose that a finite $p$-group $P$ admits a Frobenius group $FH$ of automorphisms with cyclic kernel $F$ of order $p^k$. Let $c$ be the nilpotency class of the fixed-point subgroup $C_P(H)$ of the complement. Then $P$ has a characteristic subgroup of index bounded in terms of $c$, $|F|$, and $|C_P(F)|$ whose nilpotency class is bounded in terms of $c$ and $|H|$ only.

The proof is quite similar to the proofs of the aforementioned results of Alperin [1] and Khukhro [9], with the Lie ring theorem in [15] taking over the role of the Higman–Kreknin–Kostrikin theorem. However, first a certain combinatorial corollary of that Lie ring theorem has to be derived (Proposition 2.2). Example 3.6 shows that the condition of the kernel $F$ being cyclic in Theorem 1.1 is essential.
We now state the unipotent versions of the rank, order, and exponent results in [15]. (By the rank we mean the minimum number $r$ such that every subgroup can be generated by $r$ elements.)

**Theorem 1.2.** Suppose that a finite $p$-group $P$ admits a Frobenius group $FH$ of automorphisms with cyclic kernel $F$ of order $p^k$. Then $P$ has a characteristic subgroup $Q$ of index bounded in terms of $|F|$ and $|C_P(F)|$ such that

(a) the order of $Q$ is at most $|C_P(H)|^{|H|};$

(b) the rank of $Q$ is at most $r|H|$, where $r$ is the rank of $C_P(H);$ 

(c) the exponent of $Q$ is at most $p^{2e}$, where $p^e$ is the exponent of $C_P(H).$

Note that the estimates for the order and rank are best-possible, and for the exponent close to being best-possible (and independent of $|FH|$). The proof is facilitated by a straightforward reduction to powerful $p$-groups. Then certain versions of the ‘free $H$-module arguments’ are applied to abelian $FH$-invariant sections. If a finite group $G$ admits a Frobenius group of automorphisms $FH$ with complement $H$ and with kernel $F$ acting fixed-point-freely, then every elementary abelian $FH$-invariant section of $G$ is a free $kH$-module (for various prime fields $k$). This is exactly what provides a motivation for seeking results bounding various parameters of $G$ in terms of those of $C_P(H)$ and $|H|$. In the ‘semisimple’ situation this fact is a basis of the results on the order and rank in [15]. The exponent result in [15] is more difficult, but in our unipotent situation a simpler argument can be used based on powerful $p$-groups to produce a much better result, with the estimate for the exponent depending only on the exponent of $C_P(H)$.

It should be mentioned that the ‘semisimple’ results on the order and rank in [15] do not assume the kernel to be cyclic. What the ‘unipotent’ analogue of these results for non-cyclic kernel should be is unclear at the moment. The results of the present paper can be regarded as generalizations of the results of [15], where the kernel $F$ acts on $G$ fixed-point-freely, to the case of ‘almost fixed-point-free’ kernel. It is natural to expect that similar restrictions, in terms of the complement $H$ and its fixed points $C_G(H)$, should hold for a subgroup of index bounded in terms of $|C_G(F)|$ and other parameters: ‘almost fixed-point-free’ action of $F$ implying that $G$ is ‘almost’ as good as when $F$ acts fixed-point-freely. In the coprime ‘semisimple’ situation such restrictions were recently obtained in [13] for the order and rank of $G$, and in [14] and [19] for the nilpotency class. For the moment it is unclear how to combine these semisimple and unipotent results in a general setting, without assumptions on the orders of $G$ and $FH$; note that the results in [15] for the fixed-point-free kernel were free of such assumptions.

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## 2 Lie ring technique

First we recall some definitions and notation. Products in a Lie ring are called commutators. The Lie subring generated by a subset $S$ is denoted by $\langle S \rangle$ and the ideal by $\text{id} \langle S \rangle$.

Terms of the lower central series of a Lie ring $L$ are defined by induction: $\gamma_1(L) = L$; $\gamma_{i+1}(L) = [\gamma_i(L), L]$. By definition a Lie ring $L$ is nilpotent of class $h$ if $\gamma_{h+1}(L) = 0$. 

3
A simple commutator $[a_1, a_2, \ldots, a_s]$ of weight (length) $s$ is by definition the commutator $[[[a_1, a_2], a_3], \ldots, a_s]$.

Let $A$ be an additively written abelian group. A Lie ring $L$ is $A$-graded if

$$L = \bigoplus_{a \in A} L_a \quad \text{and} \quad [L_a, L_b] \subseteq L_{a+b}, \ a, b \in A,$$

where the grading components $L_a$ are additive subgroups of $L$. Elements of the $L_a$ are called homogeneous (with respect to this grading), and commutators in homogeneous elements homogeneous commutators. An additive subgroup $H$ of $L$ is said to be homogeneous if $H = \bigoplus_{a} (H \cap L_a)$; then we set $H_a = H \cap L_a$. Obviously, any subring or an ideal generated by homogeneous additive subgroups is homogeneous. A homogeneous subring and the quotient ring by a homogeneous ideal can be regarded as $A$-graded rings with the induced gradings.

Suppose that a Lie ring $L$ admits a Frobenius group of automorphisms $FH$ with cyclic kernel $F = \langle \varphi \rangle$ of order $n$. Let $\omega$ be a primitive $n$-th root of unity. We extend the ground ring by $\omega$ and denote by $\bar{L}$ the ring $L \otimes \mathbb{Z} [\omega]$. Then $\varphi$ naturally acts on $\bar{L}$ and, in particular, $C_{\bar{L}}(\varphi) = C_{L}(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$.

**Definition.** We define $\varphi$-components $L_k$ for $k = 0, 1, \ldots, n - 1$ as the ‘eigensubspaces’

$$L_k = \{ a \in \bar{L} \mid a^\varphi = \omega^k a \}.$$

It is well known that $n \bar{L} \subseteq L_0 + L_1 + \cdots + L_{n-1}$ (see, for example, [4 Ch. 10]). This decomposition resembles a $(\mathbb{Z}/n\mathbb{Z})$-grading because of the inclusions $[L_s, L_t] \subseteq L_{s+t \pmod{n}}$, but the sum of $\varphi$-components is not direct in general.

**Definition.** We refer to commutators in elements of $\varphi$-components as being $\varphi$-homogeneous.

**Index Convention.** Henceforth a small letter with index $i$ denotes an element of the $\varphi$-component $L_i$, so that the index only indicates the $\varphi$-component to which this element belongs: $x_i \in L_i$. To lighten the notation we will not use numbering indices for elements in $L_j$, so that different elements can be denoted by the same symbol when it only matters to which $\varphi$-component these elements belong. For example, $x_1$ and $x_1$ can be different elements of $L_1$, so that $[x_1, x_1]$ can be a nonzero element of $L_2$. These indices will be considered modulo $n$; for example, $a_{-i} \in L_{-i} = L_{n-i}$.

Note that under the Index Convention a $\varphi$-homogeneous commutator belongs to the $\varphi$-component $L_s$, where $s$ is the sum modulo $n$ of the indices of all the elements occurring in this commutator.

Since the kernel $F$ of the Frobenius group $FH$ is cyclic, the complement $H$ is also cyclic. Let $H = \langle h \rangle$ be of order $q$ and $\varphi^h = \varphi^r$ for some $1 \leq r \leq n - 1$. Then $r$ is a primitive $q$-th root of unity in the ring $\mathbb{Z}/n\mathbb{Z}$.

The group $H$ permutes the $\varphi$-components $L_i$ as follows: $L_i^h = L_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Indeed, if $x_i \in L_i$, then $(x_i^h)^r = x_i^{hr^{-1}h} = (x_i^r)^h = \omega^{ir} x_i^h$, so that $L_i^h \subseteq L_{ri}$; the reverse inclusion is obtained by applying the same argument to $h^{-1}$.

**Notation.** In what follows, for a given $u_k \in L_k$ we denote the element $u_k^h$ by $u_{r(k)}$ under the Index Convention, since $L_k^h = L_{r(k)}$. We denote the $H$-orbit of an element $x_i$ by $O(x_i) = \{ x_i, x_{ri}, \ldots, x_{rin-1} \}$. 

4
Combinatorial theorem. We are going to prove a combinatorial consequence of the Makarenko–Khukhro–Shumyatsky theorem in [15], which we state in a somewhat different form, in terms of \((\mathbb{Z}/n\mathbb{Z})\)-graded Lie rings with a cyclic group of automorphisms \(H\).

**Theorem 2.1 (\([15\text{, Theorem 5.5 (b)]}\))**. Let \(M = \bigoplus_{i=0}^{n-1} M_i\) be a \((\mathbb{Z}/n\mathbb{Z})\)-graded Lie ring with grading components \(M_i\) that are additive subgroups satisfying the inclusions \([M_i, M_j] \subseteq M_{i+j \pmod{n}}\). Suppose \(M\) admits a finite cyclic group of automorphisms \(H = \langle h \rangle\) of order \(q\) such that \(M_i^h = M_{ri}\) for some element \(r \in \mathbb{Z}/n\mathbb{Z}\) having multiplicative order \(q\). If \(M_0 = 0\) and \(C_M(H)\) is nilpotent of class \(c\), then for some functions \(u = u(c, q)\) and \(f = f(c, q)\) depending only on \(c\) and \(q\), the Lie subring \(n^u M\) is nilpotent of class \(f-1\), that is, \(\gamma_f(n^u M) = n^u \gamma_f(M) = 0\).

The corresponding theorems in [15] were stated about Lie rings admitting a Frobenius group \(FH\) of automorphisms with cyclic kernel \(F = \langle \varphi \rangle\) of order \(n\). After extension of the ground ring, the \(\varphi\)-components behave like components of a \((\mathbb{Z}/n\mathbb{Z})\)-grading, as we saw above. In fact, the proofs in [15] only used the ‘grading’ properties of the \(\varphi\)-components, so that Theorem 2.1 was actually proved therein. The following proposition is a combinatorial consequence of this theorem.

**Proposition 2.2.** Let \(f = f(c, q)\), \(u = u(c, q)\) be the functions in Theorem 2.1. Suppose that a Lie ring \(L\) admits a Frobenius group of automorphisms \(FH\) with cyclic kernel \(F = \langle \varphi \rangle\) of order \(n\) and with complement \(H\) of order \(q\) such that the fixed-point subring \(C_L(H)\) of the complement is nilpotent of class \(c\). Then for the \((c, q)\)-bounded number \(w = (u+1)f\) the \(n^w\)-th multiple \(n^w[x_{i_1}, x_{i_2}, \ldots, x_{i_f}]\) of every simple \(\varphi\)-homogeneous commutator in \(\tilde{L} = L \otimes \mathbb{Z}[w]\) of weight \(f\) with non-zero indices can be represented as a linear combination of \(\varphi\)-homogeneous commutators of the same weight \(f\) in elements of the union of \(H\)-orbits \(\bigcup_{i=1}^f O(x_{i_i})\) each of which contains a subcommutator with zero sum of indices modulo \(n\).

**Remark 2.3.** Similar combinatorial propositions were also proved for Lie algebras in [19] and for Lie rings whose ground ring contains the inverse of \(n\) in [14].

**Proof.** The idea of the proof is application of Theorem 2.1 to a free Lie ring with operators \(FH\). Given arbitrary (not necessarily distinct) non-zero elements \(i_1, i_2, \ldots, i_f \in \mathbb{Z}/n\mathbb{Z}\), we consider a free Lie ring \(K\) over \(R\) with \(qf\) free generators in the set

\[
Y = \{ [y_{i_1}, y_{r_{i_1}^{-1}i_1}], \ldots, y_{i_f}, y_{r_{i_f}^{-1}i_f}] \},
\]

where indices are formally assigned and regarded modulo \(n\) and the subsets \(O(y_{i_i}) = \{ y_{i_i}, y_{r_{i_i}^{-1}i_i} \}\) are disjoint. Here, as in the Index Convention, we do not use numbering indices, that is, all elements \(y_{r_{i_i}^{-1}i_i}\) are by definition different free generators, even if indices coincide. (The Index Convention will come into force in a moment.) For every \(i = 0, 1, \ldots, n-1\) we define the additive subgroup \(K_i\) generated by all commutators in the generators \(y_{j_i}\) in which the sum of indices of all entries is equal to \(i\) modulo \(n\). Then \(K = K_0 \oplus K_1 \oplus \cdots \oplus K_{n-1}\). It is also obvious that \([K_i, K_j] \subseteq K_{i+j \pmod{n}}\); therefore this is a \((\mathbb{Z}/n\mathbb{Z})\)-grading. The Lie ring \(K\) also has the natural \(\mathbb{N}\)-grading \(K = G_1(Y) \oplus G_2(Y) \oplus \cdots\).
with respect to the generating set \( Y \), where \( G_i(Y) \) is the additive subgroup generated by all commutators of weight \( i \) in elements of \( Y \).

We define an action of the Frobenius group \( FH \) on \( K \) by setting \( k^h = \omega^h k_i \) for \( k_i \in K_i \) and extending this action to \( K \) by linearity. An action of \( H \) is defined on the generating set \( Y \) as a cyclic permutation of elements in each subset \( O(y_i) \) by the rule \( (y_{r+i})^h = y_{r+1+i} \) for \( k = 0, \ldots, q - 2 \) and \( (y_{r+i+1})^h = y_i \). Then \( O(y_i) \) becomes the \( H \)-orbit of the element \( y_i \).

Clearly, \( H \) permutes the components \( K_i \) by the rule \( K_i^h = K_i \) for all \( i \in \mathbb{Z}/n\mathbb{Z} \).

Let \( J = \text{id}(K_0) \) be the ideal generated by the \( \varphi \)-component \( K_0 \). Clearly, the ideal \( J \) consists of linear combinations of commutators in elements of \( Y \) each of which contains a subcommutator with zero sum of indices modulo \( n \). The ideal \( J \) is generated by homogeneous elements with respect to the gradings \( K = \bigoplus_i G_i(Y) \) and \( K = \bigoplus_{i=0}^{n-1} K_i \) and therefore is homogeneous with respect to both gradings. Note also that the ideal \( J \) is obviously \( FH \)-invariant.

Let \( I = \text{id}(\gamma_{c+1}(C_K(H)))^F \) be the smallest \( F \)-invariant ideal containing the subring \( \gamma_{c+1}(C_K(H)) \). The ideal \( I \) is obviously homogeneous with respect to the grading \( K = \bigoplus_i G_i(Y) \) and is \( FH \)-invariant. The fact that the ideal \( I \) is \( F \)-invariant, implies that \( ni \subseteq I_0 \oplus \cdots \oplus I_{n-1} \), where \( I_k = I \cap K_k \) for \( k = 0, 1, \ldots, n - 1 \). Indeed, for \( z \in I \), for every \( i \), \( 0, \ldots, n - 1 \) we have \( z_i := \sum_{s=0}^{n-1} \omega^{-is} z_{r'} \in K_i \) and \( nz = \sum_{i=0}^{n-1} z_i \). We denote \( \hat{I} = I_0 \oplus \cdots \oplus I_{n-1} \). This is an ideal of \( K \), which is homogeneous with respect to both gradings \( K = \bigoplus_i G_i(Y) \) and \( K = \bigoplus_{i=0}^{n-1} K_i \). It is also \( FH \)-invariant, since \( I \) is \( FH \)-invariant and the components \( K_i \) are permuted by \( FH \).

Consider the quotient Lie ring \( N = K/(J + \hat{I}) \). Since the ideals \( J \) and \( \hat{I} \) are homogeneous with respect to the gradings \( K = \bigoplus_i G_i(Y) \) and \( K = \bigoplus_{i=0}^{n-1} K_i \), the quotient ring \( N \) has the corresponding induced gradings. We use indices to denote the components \( N_i \) of the \((\mathbb{Z}/n\mathbb{Z})\)-grading induced by \( K = \bigoplus_{i=0}^{n-1} K_i \). Note that \( N_0 = 0 \) by the construction of \( J \).

The group \( H \) permutes the grading components of \( N = N_0 \oplus \cdots \oplus N_{n-1} \) with regular orbits of length \( q \). Therefore elements of \( C_N(H) \) have the form \( a + \omega^h a + \cdots + \omega^{hr} a \). Hence \( C_N(H) \) is contained in the image of \( C_K(H) \) in \( N = K/(J + \hat{I}) \) and therefore \( \gamma_{c+1}(C_N(H)) \) is contained in the image of the ideal \( I \) by its construction. Then \( n\gamma_{c+1}(C_N(H)) = 0 \) since \( nI \subseteq \hat{I} \).

The group \( H \) also permutes the \((\mathbb{Z}/n\mathbb{Z})\)-grading components of \( M := nN = \bigoplus_{i=0}^{n-1} M_i \), where \( M_i = nN_i \), with regular orbits of length \( q \). Therefore, \( C_M(H) = nC_N(H) \) and \( \gamma_{c+1}(C_M(H)) = \gamma_{c+1}(nC_N(H)) = n^{c+1}\gamma_{c+1}(C_N(H)) = 0 \).

Since \( N_0 = 0 \), we also have \( M_0 = 0 \).

By Theorem 221 for some \((c, q)\)-bounded function \( u = u(c, q) \) the Lie ring \( n^u M \) is nilpotent of \((c, q)\)-bounded class \( f - 1 = f(c, q) - 1 \). Consequently,

\[
n^{u+1}f[y_{i_1}, y_{i_2}, \ldots, y_{i_f}] = [n^{u+1}y_{i_1}, n^{u+1}y_{i_2}, \ldots, n^{u+1}y_{i_f}] \in J + \hat{I}.
\]

Note that we should take the factors \( n^{u+1} \) because the elements \( y_{i_*} \in K \) may not belong to the preimage of \( M = nN \). Since both ideals \( J \) and \( \hat{I} \) are homogeneous with respect to the grading \( K = \bigoplus_i G_i(Y) \), this means that the left-hand side is equal modulo the ideal \( \hat{I} \) to a linear combination of commutators of the same weight \( f \) in elements of \( Y \) each of which contains a subcommutator with zero sum of indices modulo \( n \).
Now suppose that $L$ is an arbitrary Lie ring satisfying the hypothesis of Proposition 2.2 and let $\tilde{L} = L \otimes \mathbb{Z}[\omega]$. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_f}$ be arbitrary $\varphi$-homogeneous elements of $L$. We define the homomorphism $\delta$ from the free Lie ring $K$ into $\tilde{L}$ extending the mapping

$$y_{s, i_s} \rightarrow x_{i_s}^{k_s} \quad \text{for} \quad s = 1, \ldots, f \quad \text{and} \quad k = 0, 1, \ldots, q - 1.$$ 

It is easy to see that $\delta$ commutes with the action of $FH$ on $K$ and $\tilde{L}$. Therefore $\delta(O(y_{i_s})) = O(x_{i_s})$ and $\delta(I) = 0$, since $\gamma_{c+1}(C_L(H)) = 0$ and $\delta(C_K(H)) \subseteq C_{\tilde{L}}(H)$. We now apply $\delta$ to the representation of $n^{(a+1)/[y_{i_1}, y_{i_2}, \ldots, y_{i_f}]}$ constructed above. Since $\delta(I) \subseteq \delta(I) = 0$, as the image we obtain a required representation of $n^{(a+1)/[x_{i_1}, x_{i_2}, \ldots, x_{i_f}]}$ as a linear combination of commutators of weight $f$ in elements of the set $\delta(Y) = \bigcup_{s=1}^{f} O(x_{i_s})$ each of which has a subcommutator with zero sum of indices modulo $n$. \qed



### 3 Nilpotency class

We begin with two lemmas that are well-known in folklore. Induced automorphisms of invariant subgroups and sections are denoted by the same letters. Fixed-point subgroups are denoted as centralizers in the natural semidirect products.

**Lemma 3.1** (see, e. g., [11] Theorem 1.6.1). If $\alpha$ is an automorphism of a finite group $G$ and $N$ is an $\alpha$-invariant subgroup of $G$, then $|C_{G/N}(\alpha)| \leq |C_G(\alpha)|$. \hfill \qed

**Lemma 3.2** (see, e. g., [11] Theorem 1.6.2). If $\alpha$ is an automorphism of a finite group $G$ and $N$ is an $\alpha$-invariant subgroup of $G$ such that $(|N|, |\alpha|) = 1$, then $C_{G/N}(\alpha) = C_G(\alpha)N/N$. \hfill \qed

**Lemma 3.3** (see, e. g., [11] Corollary 1.7.4). If $\varphi$ is an automorphism of order $p^k$ of a finite abelian $p$-group $A$ and $|C_A(\varphi)| = p^t$, then the rank of $A$ is at most $sp^k$. \hfill \qed

The following lemma is a well-known consequence of the theory of powerful $p$-groups [21].

**Lemma 3.4** (see, e. g., [12] Corollary 11.21). If a finite $p$-group $P$ has rank $r$ and exponent $p^e$, then $|P|$ is $(p, r, e)$-bounded.

**Proof of Theorem 1.1**. Recall that $P$ is a finite $p$-group admitting a Frobenius group $FH$ of automorphisms with cyclic kernel $F = \langle \varphi \rangle$ of order $p^k$ and complement $H$ of order $q$. Let $p^m = |C_P(F)|$ and let $C_P(H)$ be nilpotent of class $c$. We need to find a characteristic subgroup of $(p, k, m)$-bounded index and of $(c, q)$-bounded nilpotency class.

Consider the associated Lie ring $L(P) = \bigoplus_{i} \gamma_i(P)/\gamma_{i+1}(P)$, where $\gamma_i$ denotes the $i$th term of the lower central series (see, e. g., §3.2 in [11]). Extend the ground ring by a $p^k$-th primitive root of unity $\omega$ setting $L = L(P) \otimes \mathbb{Z}[\omega]$ and regarding $L(P)$ as $L(P) \otimes 1$. The group $FH$ naturally acts on $L$. We define the $\varphi$-components as in §2 (with $n = p^k$); recall that $p^k L \subseteq L_0 + L_1 + \cdots + L_{p^k-1}$. Since any $\varphi$-homogeneous commutator with zero sum of indices modulo $p^k$ belongs to $L_0$, by Proposition 2.2 we obtain

$$p^{k(f+w)} \gamma_f(L) = p^{kw} \gamma_f(p^k L) \subseteq p^{kw} \gamma_f(L_0 + L_1 + \cdots + L_{p^k-1}) \subseteq id\langle L_0 \rangle$$
for the functions $f = f(c, q)$, $w = w(c, q)$ in that proposition. Since $L_0 = C_{L(P)}(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and $p^n C_{L(P)}(\varphi) = 0$ by Lemma 3.4 and the Lagrange theorem, we obtain

$$p^{k(f+w)+m} \gamma_f(L) \subseteq p^m \langle L_0 \rangle = 0.$$  

In particular, $p^{k(f+w)+m} \gamma_f(L(P)) = 0$. In terms of the group $P$ this means that the factors $\gamma_i(P)/\gamma_{i+1}(P)$ have exponent dividing $p^{k(f+w)+m}$ for all $i \geq f$.

By Lemmas 3.1 and 3.3, the rank of every factor $\gamma_i(P)/\gamma_{i+1}(P)$ is at most $mp^k$. Together with the bound for the exponent, this gives a bound for the order, which we state as a lemma.

**Lemma 3.5.** Suppose that $P$ is a finite $p$-group admitting a Frobenius group $FH$ of automorphisms with cyclic kernel $F = \langle \varphi \rangle$ of order $p^k$ and complement $H$ of order $q$. Let $p^m = |C_P(F)|$ and let $C_P(H)$ be nilpotent of class $c$. Then $|\gamma_i(P)/\gamma_{i+1}(P)| \leq p^{(k_f+kw+m)mp^k}$ for all $i \geq f$, where $f = f(c, q)$ and $w = w(c, q)$ are the functions in Proposition 2.2.

Lemma 3.5 can be applied to any $FH$-invariant subgroup $Q$ of $P$. In particular, we choose $Q = \gamma_{u+1}(P(\varphi))$, where $U = (k f + kw + m)mp^k$. Clearly, $Q \leq P$, so that $|C_Q(\varphi)| \leq p^m$. By Lemma 3.5, $|\gamma_i(Q)/\gamma_{i+1}(Q)| \leq p^U$ for all $i \geq f$. On the other hand, by the well-known theorem of P. Hall [3, Theorem 2.56] we have $|\gamma_i(Q)/\gamma_{i+1}(Q)| \geq p^U+1$ if $\gamma_{i+1}(Q) \neq 1$. To avoid a contradiction we must conclude that $\gamma_f(Q) = 1$. Thus, $Q$ is nilpotent of $(c, q)$-bounded class.

The automorphism $\varphi$ acts trivially on the factors of the lower central series of $P(\varphi)$. Since $|C_{P(\varphi)}(\varphi)| = p^{m+k}$, by Lemma 3.1 the orders of all these factors are at most $p^{m+k}$. Since the quotient $P(\varphi)/Q$ is nilpotent of class $U$ by construction, its order is at most $p^{(m+k)U} = p^{(m+k)(kf+kw+m)mp^k}$, which is a $(p, k, m, c)$-bounded number. Thus, $Q$ has $(p, k, m, c)$-bounded index in $P$ and $(c, q)$-bounded nilpotency class. The subgroup $Q$ contains a characteristic subgroup $P^{op}$ for some $(p, k, m, c)$-bounded number $e$. Since the rank of $P$ is $(p, k, m, c)$-bounded, the index of $P^{op}$ in $P$ is also $(p, k, m, c)$-bounded by Lemma 3.3.

We now produce an example showing that the condition of the kernel being cyclic in Theorem 1.1 is essential.

**Example 3.6.** Let $L$ be a Lie ring whose additive group is the direct sum of three copies of $\mathbb{Z}_2$, the group of 2-adic integers, with generators $e_1, e_2, e_3$ as a $\mathbb{Z}_2$-module, and let the structure constants of $L$ be $[e_1, e_2] = 4e_3$, $[e_2, e_3] = 4e_1$, $[e_3, e_1] = 4e_2$. A Frobenius group $FH$ of order 12 acts on $L$ as follows: $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_j) = e_i$ and $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1(\text{mod} 3)}$. Since $L$ is a powerful Lie $\mathbb{Z}_2$-algebra, by [2, Theorem 9.8] the Baker–Campbell–Hausdorff formula defines the structure of a uniformly powerful pro-$2$-group $P$ on the same set $L$. For any positive integer $n$, the quotient of $P$ by $P^{2n} = 2^nL$ is a finite 2-group $T$. The induced action of $FH$ on $T$ is such that $|C_T(F)| = 8$ and $C_T(H)$ is cyclic, while the derived length of $T$ is about $\log_2 n$. 

8
4 Order, rank, and exponent

Suppose that a finite abelian group $V$ admits a Frobenius group of automorphisms $FH$ with cyclic kernel $F = \langle \varphi \rangle$ of order $n$. We can extend the ground ring by a primitive $n$-th root of unity $\omega$ forming $W = V \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ and define the natural action of the group $FH$ on $W$. As a $\mathbb{Z}$-module (abelian group), $\mathbb{Z}[\omega] = \bigoplus_{i=0}^{E(n)-1} \omega^i \mathbb{Z}$, where $E(n)$ is the Euler function. Hence,

$$W = \bigoplus_{i=0}^{E(n)-1} V \otimes \omega^i \mathbb{Z},$$

so that $|W| = |V|^{E(n)}$. Similarly, $C_W(\varphi) = \bigoplus_{i=0}^{E(n)-1} C_V(\varphi) \otimes \omega^i \mathbb{Z}$, so that $|C_W(\varphi)| = |C_V(\varphi)|^{E(n)}$.

As in §2 for $\tilde{L}$, we define $\varphi$-components $W_k$ for $k = 0, 1, \ldots, n-1$ as the ‘eigensubspaces’

$$W_k = \{ a \in W \mid a^r = \omega^k a \}.$$

Recall that $W$ is an ‘almost direct sum’ of the $W_i$: namely,

$$nW \subseteq W_0 + W_1 + \cdots + W_{n-1}$$

and

$$\text{if } w_0 + w_1 + \cdots + w_{n-1} = 0 \text{ for } w_i \in W_i, \text{ then } nw_i = 0 \text{ for all } i.$$  

As in §2 we refer to elements of $\varphi$-components as being $\varphi$-homogeneous, and apply the Index Convention using lower indices of small Latin letters to only indicate the $\varphi$-component containing this element.

As before, since the kernel $F$ of the Frobenius group $FH$ is cyclic, the complement $H$ is also cyclic, $H = \langle h \rangle$, say, of order $q$, and $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n - 1$, which is a primitive $q$-th root of unity in $\mathbb{Z}/n\mathbb{Z}$. The group $H$ permutes the $\varphi$-components $W_i$ by the rule $W_i^h = W_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. For $u_k \in W_k$ we denote $u_k^h$ by $u_{ri,k}$ under the Index Convention.

From now on we assume in addition that $V$ is an abelian $FH$-invariant section of the $p$-group $P$ in Theorem 1.2. Recall that $|\varphi| = n = p^k$ and $|C_P(\varphi)| = p^m$.

**Lemma 4.1.** There is a characteristic subgroup $U$ of $V$ such that $|U|$ is $(p, k, m)$-bounded and

(a) $|V/U| \leq |C_V(H)|^{|H|};$

(b) the rank of $V/U$ is at most $r|H|$, where $r$ is the rank of $C_P(H);$  

(c) the exponent of $V/U$ is at most $p^r$, where $p^r$ is the exponent of $C_P(H).$

**Proof.** The group $H$ acts on the set of $\varphi$-components $W_i$ with one single-element orbit $\{W_0\}$ and $(p^k - 1)/q$ regular orbits. We choose one element in every regular $H$-orbit and let $Y = \sum_{i=1}^{(p^k - 1)/q} W_{i_j}$ be the sum of these chosen $\varphi$-components. The mapping $\vartheta: y \to y + y^h + \cdots + y^{h^{r-1}}$ is a homomorphism of the abelian group $Y$ into $C_W(H)$. We claim that $p^k \text{Ker } \vartheta = 0$. Indeed, if $y \in \text{Ker } \vartheta$ is written as $y = \sum_{j=1}^{(p^k - 1)/q} y_{i_j}$ for $y_{i_j} \in W_{i_j}$, then $\vartheta(y)$ is equal to $y$ plus a linear combination of elements of $\varphi$-components $W_{ri_{i_j}}$ with all the indices $r^{i_{i_j}}$ being different from the indices $i_1, \ldots, i_{(p^k - 1)/q}$. Therefore the equation
\( \vartheta(y) = 0 \) implies \( p^k y_i = 0 \) by (3), so that \( p^k y = 0 \). Clearly, \(|Y/\text{Ker } \vartheta| \leq |C_W(H)|\), the rank of \( Y/\text{Ker } \vartheta \) is at most the rank of \( C_W(H) \), and the exponent of \( Y/\text{Ker } \vartheta \) is at most the exponent of \( C_W(H) \).

Let \( p^t \) be the maximum of \( p^k \) and the exponent of \( W_0 \), which is a \((p, k, m)\)-bounded number. Then \( \Omega_f(W) \geq W_0 + \text{Ker } \vartheta \) (where we use the standard notation \( \Omega \) for the subgroup generated by all elements of order dividing \( p^t \)). Since

\[
p^k W \leq W_0 + W_1 + \cdots + W_{p^k-1} = W_0 + Y + Y^h + \cdots + Y^{h^t-1},
\]

we obtain the following.

**Lemma 4.2.** The image of \( p^k W \) in \( W/\Omega_f(W) \) is contained in the image of \( Y + Y^h + \cdots + Y^{h^t-1} \) in \( W/\Omega_f(W) \), and the image of \( Y \) is a homomorphic image of \( Y/\text{Ker } \vartheta \).

We claim that \( U = \Omega_{f+k}(V) \) is the required characteristic subgroup. The rank of the abelian group \( V \) is at most \( mp^k \) by Lemmas 3.1 and 3.3. Hence \( \Omega_{f+k}(V) \) being of bounded exponent has \((p, k, m)\)-bounded order. We now verify that parts (a), (b), (c) are satisfied.

(a) In the abelian \( p \)-group \( W \) the order of the image of \( p^k W \) in \( W/\Omega_f(W) \) is equal to \( |W/\Omega_{f+k}(W)| \). Therefore Lemma 4.2 and the fact that \(|Y/\text{Ker } \vartheta| \leq |C_W(H)|\) imply

\[
|W/\Omega_{f+k}| \leq |Y/\text{Ker } \vartheta|^{[H]} \leq |C_W(H)|^{[H]}, \tag{4}
\]

Clearly, \( \Omega_{f+k}(W) = \Omega_{f+k}(V) \otimes \mathbb{Z}[\omega] \) and therefore \( |\Omega_{f+k}(W)| = |\Omega_{f+k}(V)|^{E(p^k)} \). Since \( |W| = |V|^{E(p^k)} \) and \( |C_W(\varphi)| = |C_V(\varphi)|^{E(p^k)} \), taking the \( E(p^k) \)-th root of both sides of (4) gives \( |V/\Omega_{f+k}(V)| \leq |C_V(H)|^{[H]} \).

(b) Similarly, the rank of the image of \( p^k W \) in \( W/\Omega_f(W) \) is equal to the rank of \( W/\Omega_{f+k} \). By Lemma 4.2 we obtain that the rank of \( W/\Omega_{f+k}(W) \) is at most \( |H| \) times the rank of \( Y/\text{Ker } \vartheta \), which in turn is at most the rank of \( C_W(H) \). Since the ranks are multiplied by \( E(p^k) \) when passing from \( V \) to \( W \), we obtain that the rank of \( V/\Omega_{f+k}(V) \) is at most \( |H| \) times the rank of \( C_V(H) \), which in turn does not exceed \( r \), the rank of \( C_P(H) \), because \( C_P(H) \) covers \( C_V(H) \) by Lemma 3.2 since the action of \( H \) is coprime.

(c) Finally, the exponent of the image of \( p^k W \) in \( W/\Omega_f(W) \) is equal to the exponent of \( W/\Omega_{f+k} \). By Lemma 4.2 we obtain that the exponent of \( W/\Omega_{f+k}(W) \) does not exceed the exponent of \( Y/\text{Ker } \vartheta \) which is at most the exponent of \( C_W(H) \) and, consequently, that of \( C_V(H) \). Since the action of \( H \) is coprime, by Lemma 3.2 the exponent of \( C_V(H) \) (and therefore the exponent of \( W/\Omega_{f+k}(W) \) as well) is at most \( p^r \), the exponent of \( C_P(H) \).

**Proof of Theorem 1.2** Recall that \( P \) is a finite \( p \)-group admitting Frobenius group \( FH \) of automorphisms with cyclic kernel \( F \) of order \( p^k \) with \( p^m = |C_P(F)| \) fixed points of the kernel. Let \( p^s = |C_P(H)| \), let \( r \) be the rank of \( C_P(H) \), and \( p^e \) the exponent of \( C_P(H) \). We need to find a characteristic subgroup \( Q \) of \((p, k, m)\)-bounded index with required bounds for the order, rank, and exponent. We can of course find such a subgroup separately for each of these parameters and then take the intersection.

By Lemmas 3.1 and 3.3 the rank of \( P \) is at most \( mp^k \). Hence \( P \) has a characteristic powerful subgroup of \((p, k, m)\)-bounded index by [21, Theorem 1.14]. Therefore we can assume \( P \) to be powerful from the outset.

By [10] (see also [12, Theorem 12.15]), the group \( P \) has a characteristic subgroup \( P_1 \) of \((p, k, m)\)-bounded index that is soluble of \( p^k \)-bounded derived length at most \( 2K(p^k) \)
(where $K$ is Kreknin’s function bounding the derived length of a Lie ring with a fixed-point-free automorphism of order $p^k$). Let $\mathcal{D}$ be the set of factors of the derived series of $P_1$. For any $V \in \mathcal{D}$, we have, by Lemma 3.1, that $|V| \leq p^g |C_V(H)|^{|H|}$ for some $(p, k, m)$-bounded number $g = g(p, k, m)$. Then

$$|P_1| = \prod_{V \in \mathcal{D}} |V| \leq p^{2gK(p^k)} \prod_{V \in \mathcal{D}} |C_V(H)|^{|H|} = p^{2gK(p^k)} |C_{P_1}(H)|^{|H|}$$

by Lemma 3.2 since the action of $H$ is coprime. Since the rank of the powerful $p$-group $P$ is at most $mp^k$, by taking the $(p, k, m)$-bounded power $P_1^{f(p, k, m)}$ with $f(p, k, m) = p^{2gK(p^k)}$ we obtain a characteristic subgroup which has $(p, k, m)$-bounded index by Lemma 3.4.

The order of $P_1^{f(p, k, m)}$ is at most $|C_{P}(H)|^{|H|}$. Indeed, either the exponent of $P_1$ is at most $f(p, k, m)$ and then $P_1^{f(p, k, m)} = 1$, or the exponent of $P_1$ is greater than $f(p, k, m)$ and then $|P_1 : P_1^{f(p, k, m)}| \geq f(p, q, m)$, whence $|P_1^{f(p, k, m)}| \leq |C_{P_1}(H)|^{|H|} \leq |C_{P}(H)|^{|H|}$.

The powerful $p$-group $P$ has a series

$$P > P^{p^{k_1}} > P^{p^{k_2}} > \cdots > 1$$

with uniformly powerful factors of strictly decreasing ranks. For every factor $S$ of this series having exponent, say, $p^t$, its subgroup $V = S^{p^{(t+1)/2}}$ is abelian. By Lemma 3.1, the subgroup $V$ has a characteristic subgroup $U$ of $(p, k, m)$-bounded index such that the rank of $V/U$ is at most $r|H|$. The rank of $V$ is equal to the rank of $S$ and $V$ is generated by elements of order $p^{t/2}$. If the rank of $S$ is greater than the rank of $U$, then there exists an element of order $p^t$ that belongs to $U$ and thus belongs to $(p, k, m)$-bounded. Therefore the rank of $S$ can be higher than $r|H|$ only if the exponent of $S$ is $(p, k, m)$-bounded. Since the rank of $P$ is at most $mp^k$, all the factors in (5) of rank higher than $r|H|$ combine in a quotient $P/P^{p^{k_0}}$ of $(p, k, m)$-bounded order; then $P^{p^{k_0}}$ is the required characteristic subgroup of $(p, k, m)$-bounded index and of rank at most $r|H|$.

Let $p^r$ be the exponent of $P$. Since in the powerful group $P$ the series $P > P^p > P^{p^2} > P^{p^3} > \cdots$ is central, the subgroup $P^{p^{(r+1)/2}}$ is abelian. By Lemma 3.1, the exponent of $P^{p^{(r+1)/2}}$ is at most $p^{r+f}$ for some $(p, k, m)$-bounded number $f$. Hence the exponent of $P$ is at most $p^{2r+g}$ for some $(p, k, m)$-bounded number $g = g(p, k, m)$. Since the rank of $P$ is at most $mp^k$, by Lemma 3.4 the characteristic subgroup $P^{p^{r+g}}$ has $(p, k, m)$-bounded index and exponent at most $p^{2r}$. 

\begin{flushright}
$\square$
\end{flushright}

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