Initialization and the quasi-geostrophic slow manifold

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Abstract

Atmospheric dynamics span a range of time-scales. The projection of measured data to a slow manifold, $\mathcal{M}$, removes fast gravity waves from the initial state for numerical simulations of the atmosphere. We explore further the slow manifold for a simple atmospheric model introduced by Lorenz and anticipate that our results will relevant to the vastly more detailed dynamics of atmospheres and oceans.

Within the dynamics of the Lorenz model, we make clear the relation between a slow manifold $\mathcal{M}$ and the “slowest invariant manifold” (SIM), which was constructed by Lorenz in order to avoid the divergence of approximation schemes for $\mathcal{M}$. These manifolds are shown to be identical to within exponentially small terms, and so the SIM in fact shares the asymptotic nature of $\mathcal{M}$.

We also investigate the issue of balancing initial data in order to remove gravity waves. This is a question of how to compute an “initialized” point on $\mathcal{M}$ whose subsequent evolution matches that from the measured initial data that in general lie off $\mathcal{M}$. We propose a choice based on the intuitive idea that the initialization procedure should not significantly alter the forecast. Numerical results demonstrate the utility of our initialization scheme.

The normal form for Lorenz’ atmospheric model shows clearly how to separate the dynamics of the different atmospheric waves. However, its construction demonstrates that any initialization procedure

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must eventually alter the forecast—the time-scale of the divergence between the initialized and the uninitialized solutions is inevitable and is inversely proportional to the square of initial level of gravity-wave activity.

1 Introduction

Wave motion in the atmosphere has a wide range of periods: large-scale motions are dominated by quasi-geostrophic Rossby waves, which have a
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period of many days; faster inertial-gravity waves, with a period of up to a few hours, are frequently negligible in the large (Gill [12, p582]). A significant problem for numerical weather prediction is that an initial measured state of the atmosphere contains noise which causes unphysical large-amplitude gravity waves to arise in numerical solutions (Gill [12, p242]). Consequently, one is forced to take a very small time step in the numerical integration (e.g. Houghton [16, p169]). This is despite the generally recognized feature that much of the troposphere and lower stratosphere is quasi-geostrophic and so dominated by Rossby wave activity (Lorenz [22]).

To overcome this problem of unphysical gravity-wave activity, initial data must be “balanced”, that is, adjusted so that subsequent gravity-wave activity is negligible (see e.g. Houghton [16, p168], or Gill [12, p245]). A straightforward balancing procedure is to project the initial data onto the normal modes of the system, which depends on the details of the numerical procedure used for forecasting (Errico [11]); the balancing is achieved by setting the gravity wave modes to zero. Such a procedure proves unsuccessful because gravity waves immediately appear in the numerical simulations—the nonlinear aspects of atmospheric dynamics are large enough to render a linear procedure ineffective. Various more sophisticated initialization schemes exist (see Daley [10], and references therein), many of which can be interpreted as projecting the initial data onto a “slow manifold” (Leith [19]; Lorenz [21]), $\mathcal{M}$. On such a slow manifold the gravity wave variables are given as functions of the Rossby wave amplitudes, and there are no rapid oscillations.

Lorenz [21, 22] and Lorenz & Krishnamurthy [20] (henceforth referred to as L80, L86 and LK87, respectively) have proposed a series of low-dimensional models of the atmosphere and find several difficulties with the practical computation of a slow manifold: that various approximation schemes for $\mathcal{M}$ are divergent; that fast gravity wave oscillations apparently must occur in some regions of $\mathcal{M}$; and that resonances between gravity waves and Rossby waves induce singularities in the manifold. Such complex dynamics of such simple models is further explored by Camassa [5] who proves, for example, the existence of chaotic dynamics. We concentrate here on one of these models (L86), in which, according to the linearized governing equations, neither Rossby waves (which we denote by $x$) nor gravity waves ($y$) are subject to damping. A three-mode Rossby-wave complex is coupled to the two gravity-wave modes through nonlinear interactions. The Rossby waves have a much longer period than the gravity waves, and in fact the period of the Rossby waves in the model becomes infinite as their amplitude tends to zero.

We emphasize that we analyze only the low-dimensional model of L86. The applications of the ideas presented herein to realistic atmospheric equa-
tions will involve much more complicating detail. Nonetheless, we anticipate no great difficulty in extending the analysis to more physical dynamics. The procedures utilized in this paper to construct slow manifolds and to appropriately initialize data have already been generalized successfully from low-dimensional toys to high-dimensional physical problems (Mercer & Roberts [23, 24]; Roberts [28]; and Watt et al [34]). The corresponding analysis of actual atmospheric models is left to further work.

The absence of damping in the model of L86 is a source of difficulties that do not arise when the waves are slightly damped, as LK87 considered. One might expect therefore that since atmospheric waves clearly are damped, the less troublesome case would be the most profitable to explore. Further, the limit as the damping $a \to 0$ is singular, and theoretically the nature of $\mathcal{M}$ is different in the two cases $a = 0$ and $a \neq 0$. It might seem perverse, therefore, to tackle the undamped case. However, a closer examination reveals that the mathematical difficulties do not disappear when damping is introduced; they are merely obscured. For if $a$ is assumed to be small then inverse powers of $a$ occur in certain calculations, and these are a precursor of the problems that arise when $a = 0$. By examining the undamped case (as does Camassa [5]) we confront head-on the fundamental difficulties which result from resonances between waves of different timescales.

We view a rational balancing procedure as having two stages. First, the functional form of the slow manifold must be determined—usually in the form $y = h(x)$. Once this is accomplished a closed set of low-dimensional evolution equations for the slow variables follows. This reduced set could be integrated forwards in time to make forecasts, although in practice the full system is integrated from the balanced initial data. Secondly, the appropriate initial values for the slow variables on $\mathcal{M}$ must be calculated from the full set of initial values. Once the slow variables $x$ are known, the appropriate fast variables are then given by $y = h(x)$. The first stage has received most attention (Baer & Tribbia [3]; Leith [19]), while the second seems relatively ignored in the meteorological literature, although the appropriateness of some different projections, according to the quality of one’s data measurements, has been discussed by Daley [9, 10]. In Section 2 we propose a different criterion than those previously considered for the selection of an initial point on the slow manifold: that the subsequent evolution corresponds as closely as possible to that of the full initial-value problem, but with gravity waves absent. (We shall make the notion of “corresponding” evolution more precise later.) A similar choice is known in physics, where fast variables are eliminated (Haake & Lewenstein [15]; van Kampen [30]), and seems to have an obvious desirability in numerical weather forecasting:
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the adjustment of the initial data should not alter the forecast. Many of the approximation schemes to compute the slow manifold implicitly assume that only the fast variables require adjustment, although Daley illustrates how this is not the case for his “optimal” projection schemes. For our proposed projection criterion also, it turns out that both fast and slow variables must be altered.

In Section 2 we briefly describe the formal computation of a slow manifold, $\mathcal{M}$, for L86, together with the choice of appropriate initial conditions on $\mathcal{M}$. In Section 3 we discuss the divergence of the power series for $\mathcal{M}$, and consider Lorenz’ computation of an alternative slow manifold, the “slowest invariant manifold” (SIM). We find that the SIM and $\mathcal{M}$ differ by terms smaller than any power of $x$ as $x \to 0$, so that the SIM has a divergent power series. A normal form transformation is presented in Section 4, where the model L86 is written as the slow evolution of five new variables; the five variables include the amplitude and phase of the gravity waves. The normal form for L86 has dynamical behavior that differs from that of L86, albeit by an exponentially small amount. Although these differences are quantitatively asymptotically small, we find that some solutions are qualitatively affected by them. Furthermore, the normal form shows that the evolution of the slow modes cannot be entirely decoupled from the fast modes. Thus there must be inevitable discrepancies in the long-term evolution of the slow waves, of the order of the square of the fast waves, between the balanced and unbalanced simulations.

Finally, in Section 5 we review the notions of a fuzzy manifold, and the implications of the present work for the initialization of numerical weather forecasting schemes.

2 Quasi-geostrophy as a subcentre manifold

2.1 Computation of a slow manifold for L86

In the model of L86, the slow “Rossby wave” variables $\mathbf{x} = (U, V, W)$ and the fast “gravity wave” variables $\mathbf{y} = (X, Z)$ evolve according to

\[
\begin{aligned}
\dot{U} &= -VW + bVZ \\
\dot{V} &= UW - bUZ \\
\dot{W} &= -UV \\
\dot{X} &= -Z \\
\dot{Z} &= X + bUV,
\end{aligned}
\] (1)

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where the over-dot denotes differentiation with respect to time, $b$ is a coupling parameter, and the superscript $T$ denotes the transpose of a row vector. Observe that when these equations are linearized about the zero equilibrium the Rossby wave variables remain constant, while the gravity wave variables oscillate sinusoidally with a period that has been normalized to $2\pi$.

When $b = 0$ the Rossby waves and the gravity waves are uncoupled: the gravity waves oscillate sinusoidally with period $2\pi$; while the Rossby waves, given by Jacobian elliptic functions, oscillate with a period inversely proportional to their initial amplitude. For small-amplitude motions the Rossby waves have a much longer period than the gravity waves—this remains superficially true when coupling is restored ($b \neq 0$).

General solutions of (1) have gravity-wave components; we should like to adjust a given initial condition for (1), called balancing, so that gravity waves do not develop, while maintaining essentially the same evolution of the physically significant Rossby waves. That is, we propose the principle that a long-term forecast should be unaffected by the initialization.

A first attempt to balance initial data might be projection onto the “geostrophic manifold”, given by $y = 0$, that is, $X = Z = 0$. This manifold is certainly free of gravity waves, but is not invariant under (1); if we set $X = Z = 0$ initially, they do not remain zero. A more sophisticated approach to balancing is to seek an invariant “quasi-geostrophic manifold”, or slow manifold, $\mathcal{M}$, on which solutions of (1) evolve slowly. Instead of seeking $y = 0$, we allow the fast variables to be functions of the slow variables, $y = h(x)$ (Leith [19]). The slow manifold is a so-called subcentre manifold (Kelley [18]). Little is known about the conditions for the existence of subcentre manifolds; indeed resonances plague attempts to construct such manifolds (Sijbrand [29]). In this section we proceed with a formal construction of $\mathcal{M}$, leaving the issues of its existence and uniqueness until later sections.

The details of the calculation of $\mathcal{M}$ are given by Roberts [26]. By substituting the ansatz $y = h(x)$ into (1), and applying the chain rule $\dot{y} = \partial h / \partial x \dot{x}$, we obtain a quasi-linear partial differential equation for $h(x)$ (Carr [6]). Since this PDE cannot be solved exactly we expand $h(x)$ as a power series in the slow variables $x$, and find

$$
X = -bUV + \mathcal{O}(|x|^4), \quad Z = b(U^2 - V^2)W + \mathcal{O}(|x|^5), \quad (2)
$$

as $|x| \to 0$. Therefore on $\mathcal{M}$ the Rossby waves evolve according to the following (slow) amplitude equations obtained by substituting (2) into (1)

$$
\dot{U} = -VW \left[ 1 - b(U^2 - V^2) \right] + \mathcal{O}(|x|^6)
$$
\[ \dot{V} = UW \left[ 1 - b(U^2 - V^2) \right] + \mathcal{O}(|x|^6) \quad (3) \]
\[ \dot{W} = -UV. \]

The procedure we have used to compute \( h(x) \), that is, to find \( X(U,V,W) \) and \( Z(U,V,W) \), is mechanical, and has also been described for the linearly damped model L80 by Vautard and Legras [32]. It is equivalent to the balancing scheme of Baer & Tribbia [3].

### 2.2 Initialization: projection of initial conditions

We must supplement our computation of the functional form of the slow manifold \( y = h(x) \) by deriving appropriate initial conditions for \( x \) on \( \mathcal{M} \). By “appropriate” we mean that the subsequent evolution on \( \mathcal{M} \) faithfully follows the behavior of the full system (1) from the original initial conditions. In general it is not sufficient to adjust only the amplitudes of the gravity waves, that is, to map the initial point \((x^*, y^*)\) of the full system to the point \((x^*, h(x^*))\) on \( \mathcal{M} \): if the full system and the slow-manifold model are to have the same future behavior from their respective initial conditions, the initial values of the Rossby wave variables also must be adjusted. The discrepancy between the initial condition for the slow variables in the full initial-value problem and that on the slow manifold is known as “initial slip” (Grad [13]), by analogy with the slip allowed for an inviscid fluid at a boundary. We now proceed to calculate this “initial slip”.

The appropriate choice of an initial point on \( \mathcal{M} \) has a straightforward derivation (Roberts [26]) when \( \mathcal{M} \) attracts neighboring solutions exponentially. In that case, the choice may be made so that the solution from the original initial condition, and that from the adjusted initial condition approach one another exponentially quickly as \( t \to \infty \). However, in the present case the slow manifold does not attract neighboring trajectories, but instead acts as a centre for their gravity-wave oscillations—a solution initially off \( \mathcal{M} \) oscillates about \( \mathcal{M} \) perpetually. We therefore aim to choose the initial point on \( \mathcal{M} \) so that its subsequent evolution maintains its relationship with the full uninitialized solution for all time.

The procedure we describe below is correct to leading order in the distance, \( r \), of the initial point \((x^*, y^*)\) from \( \mathcal{M} \). In Section 4 we shall show that in general it is not possible to improve the choice of initialized point. For example, we expected to be able to incorporate corrections of \( \mathcal{O}(r^2) \); however, we find that this cannot be done and so solutions on and off \( \mathcal{M} \) necessarily drift apart after a time of \( \mathcal{O}(r^{-2}) \).
In order to explain our procedure we begin by briefly considering the simple projection
\[(x^*, y^*) \mapsto (x^*, h(x^*)),\] (4)
used implicitly by Baer & Tribbia [3]. The adjustment of initial data, by the displacement \((0, h(x^*) - y^*)\), lies in the “gravity-wave” space spanned by the vectors \((0, 0, 0, 1, 0)\) and \((0, 0, 0, 0, 1)\): only the gravity-wave variables are altered. We call the solution obtained by projecting initial data in this way BT. However, for the projection we propose here, the two vectors that span the appropriate direction for the projection of initial conditions depend on \(x^*\) (Roberts [26]), and in general the Rossby-wave variables are also changed by the initialization process. The simple projection (4) is appropriate only in the special case when \(x^* = 0\), where we may observe that \(x(t) = 0\) for all \(t > 0\), according to both the original system (1) and the slow model (3).

A more informed projection scheme must be based upon the evolution near the slow manifold (Roberts [26]). If \(U(t)\) is a solution of (1) on \(M\), and if \(\epsilon(t) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)\) is an infinitesimal displacement from \(U\) then \(\epsilon\) evolves, according to (1), as
\[\dot{\epsilon} = (L + N_1) \epsilon,\] (5)
where \(L\epsilon = (0, 0, 0, -\epsilon_5, \epsilon_4)\), and \(N_1\) is the Fréchet derivative of the vector of nonlinear terms \(N(x) = (-VW + bVZ, UW - bUZ, -UV, 0, bUV)\) evaluated on \(M\) (see Roberts [26]). For any slow solution on \(M\), we then seek those neighboring solutions that evolve with the solution on \(M\) plus a fast oscillation. It is just these neighboring solutions which should be projected onto the particular slow solution on \(M\) in order to maintain the long-term forecast. We wish to calculate only the projection space spanned by \(\epsilon_1\) and \(\epsilon_2\), and not their magnitudes nor their individual directions. The argument is elaborated in detail in Section 4 of the paper by Roberts [26]. The result to low-order is that the projection onto \(M\) should be made along the position dependent planes spanned by vectors \(e_1, e_2\), where
\[
e_1 = \begin{bmatrix} -bV \\ bU \\ 0 \\ 0 \\ 1 \end{bmatrix} + O(\lvert x \rvert^3), \quad e_2 = \begin{bmatrix} 0 \\ 0 \\ -b(U^2 - V^2) \\ 0 \\ 1 \end{bmatrix} + O(\lvert x \rvert^3). \] (6)

We shall call the solution obtained after projecting initial data along these vectors R.

In Figure 1 we show some typical results for solutions BT and R, and compare them with solutions of (1) from uninitialized initial data. We have
chosen different initial conditions for the two graphs; these are representative of a large number of numerical runs. In both cases, \( R \) is clearly closer to the uninitialized solution than is \( BT \). In the first, \( BT \) and \( R \) are both phase-shifted with respect to the original solution, but \( R \) less so than \( BT \). Further, \( BT \) goes the “wrong way” round a fixed point of (1) at time \( t \approx 50 \), and as a result the slow variable \( W \) takes the wrong sign. In the second case, \( R \) and the original solution are indistinguishable on the figure (although they slowly diverge at later times); at times \( t \approx 75, 150 \) the original solution has gravity-wave wiggles that the projected solutions do not have (because they both lie close to the slow manifold). Here, \( R \) is strikingly better than \( BT \).

### 2.3 Incorporation of forcing

The centre manifold formalism allows one to compute the effects of a forcing of the full system (1) on the dynamics of the slow manifold (or, for that matter, the damped, forced system of \( LK87 \)). This computation is described to leading order in the amplitude of the forcing by Cox & Roberts [7], and shows that the forcing not only changes the evolution of the slow “Rossby”
wave variables, but also changes the shape and location of a slow manifold \( \mathcal{M} \). For example, if we add a constant forcing \( \mathbf{F} = (F_U, F_V, F_W, F_X, F_Z) \) to the right-hand side of (1) then the slow manifold becomes, to leading order in \(|\mathbf{F}|\),

\[
X = -bUV - F_Z + (U^2 - V^2)F_W - 2W(VF_V - UF_U) \\
+ \mathcal{O} \left(|\mathbf{x}|^4, |\mathbf{F}||\mathbf{x}|^3\right) \tag{7}
\]

\[
Z = bW(U^2 - V^2) + F_X + b(UF_V + VF_U) - b^2F_X(U^2 - V^2) \\
+ \mathcal{O} \left(|\mathbf{x}|^5, |\mathbf{F}||\mathbf{x}|^3\right). \tag{8}
\]

We have previously shown (Cox & Roberts [7]) that when forcing is present, it is important to project initial conditions onto a slow manifold given by (7–8), rather than the unperturbed slow manifold (2), in order to eliminate gravity waves.

3 Non-uniqueness of a slow manifold

3.1 Divergence of series

Lorenz [22] demonstrates that several approximation schemes aimed at constructing a slow manifold yield divergent power series. Two of these schemes (Baer & Tribbia [3]; Vautard & Legras [32]) are equivalent to the construction of a subcentre manifold (Roberts [26]) that we have just described. Not surprisingly then, the expansions (2) are found to be divergent.

Such divergence is not necessarily a computational disaster. Techniques such as Padé summation or the Shanks transform not only improve the rate of convergence of a series, they may also produce a converged “sum” of a divergent series—the Stieltjes series is a traditional example (see Section 8.3 in Bender & Orszag [4]). The use of such convergence-acceleration techniques in fluid dynamics has been reviewed by Van Dyke [31]. Here the series expansion (2) for \( \mathcal{M} \) is in terms of the three slow variables \( U, V \) and \( W \). However, the acceleration of convergence of such multi-variate expansions is less well understood than that for expansions in a single variable. Perhaps the best current technique is to use the multi-variate Padé transform proposed by Cuyt [8], but our preliminary investigations are so far inconclusive. At the very least, low-order computations of the shape of a slow manifold can be summed to within an exponentially small error using an “optimal truncation” of the series (Bender & Orszag [4, pp.94–100,122–123]).
3.2 Construction of the SIM

Lorenz (L86) describes an alternative construction of a slow manifold, called the “slowest invariant manifold” (SIM), for which convergence is not a problem. The calculation proceeds by generating a family of periodic solutions to (1), using a numerical shooting method to determine the correct initial value of one of the variables in order to ensure periodicity. This family forms an invariant manifold, the SIM, and the central issue is whether such an object is free of significant gravity-wave activity. The construction is heavily dependent on the symmetries of (1).

Lorenz’ method of generating the periodic orbits is to consider solutions for which both $V^* \equiv V(0)$ and $X^* \equiv X(0)$ are initially zero. He then takes initial values $W^* > U^* > 0$ and treats the remaining initial condition, $Z^*$, as a parameter to be chosen so that at the first zero-crossing of $U$, when $t = T$ say, $X$ vanishes too. A consequence of the simultaneous vanishing of $U(t)$ and $X(T)$, together with the symmetries of (1) (see L86), is that the constructed solution is periodic, with period $4T$. For a fixed ratio $m^{1/2} \equiv U^*/W^*$, Lorenz computes a one-parameter family of periodic orbits, as $W^*$ is varied. The curve $Z = Z^*(W^*)$ is then a section through the SIM, and the entire SIM may in principle be computed numerically by varying the ratio $U^*/W^*$. Lorenz finds that the SIM contains an infinite number of singularities, which are increasingly closely packed as $W^* \to 0$. He concludes that “there is no unequivocally slow manifold” because his candidate slow manifold, the SIM, contains regions of high gravity-wave activity close to the singularities.

The central result of this section will be that Lorenz’ SIM and the slow manifold $\mathcal{M}$ computed in the previous section are exponentially close, a concept we elaborate upon below. Furthermore, a slow manifold is not unique: indeed, there are infinitely many pretenders to that title. Such nonuniqueness arises frequently in the construction of low-dimensional models of dynamical systems (Roberts [27]), and each has the same (divergent) power-series expansion.

Lorenz shows how the singularities in the SIM arise from resonances between the slow Rossby waves and the fast gravity waves, by considering the uncoupled system (1) with $b = 0$. Then the section through the SIM in the $(W^*, Z^*)$-plane consists of the horizontal line $Z^* = 0$, on which there is no gravity-wave activity, together with the vertical lines $W^* = K/(k\pi)$, for integers $k$, on which the period of the Rossby waves is a multiple of the gravity-wave period. The singularities of the SIM when $b \neq 0$ come from a perturbation of the structure of the uncoupled system. Lorenz noted that the singular curve $Z = Z^*(W^*)$ looked like $Z = a \exp(1/W^*)\cot(K/W^*)$. 

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By computing the analytic form of $Z^*$, up to terms of $O(b)$, we now show that this expression for $Z^*$ is qualitatively correct, and that the singularities are indeed exponentially weak as $W^* \to 0$. To do so we expand the variables as power series in the coupling parameter $b$, so that

$$U \sim \sum_{j=0}^{\infty} b^j U_j,$$

and so on. (LK87 make a similar expansion for the variables, although with a different purpose.) We take initial conditions

$$U(0) = U^*, \quad V(0) = 0, \quad W(0) = W^*, \quad X(0) = 0, \quad Z(0) = Z^* \sim \sum_{j=0}^{\infty} b^j Z_j^*, $$

where $U^*$ and $W^*$ are fixed independently of $b$, and we let the time of the first zero-crossing of $U(t)$ be at $t = T = T_0 + bT_1 + \cdots$. Recall that in order to find periodic solutions of (1), our aim is to find $T$ and $Z^*$ so that

$$U(T) = X(T) = 0. \quad (9)$$

At leading order in $b$ the variables satisfy

$$\begin{align*}
\dot{U}_0 &= -V_0 W_0 \\
\dot{V}_0 &= U_0 W_0 \\
\dot{W}_0 &= -U_0 V_0 \\
\dot{X}_0 &= -Z_0 \\
\dot{Z}_0 &= X_0.
\end{align*} \quad (10)$$

The solution (L86) is

$$\begin{align*}
U_0 &= U^* \text{cn}(W^* t), \quad V_0 = U^* \text{sn}(W^* t), \quad W_0 = W^* \text{dn}(W^* t), \\
X_0 &= -Z_0^* \sin t, \quad Z_0 = Z_0^* \cos t,
\end{align*}$$

where cn, sn, dn are Jacobian elliptic functions (Abramowitz and Stegun, Chapter 16) with parameter $m = (U^*/W^*)^2$.

Expanding (9) in powers of $b$, we find at leading order that $T_0$ satisfies $U_0(T_0) = 0$. The first zero-crossing of $U_0(t)$ occurs when $t = T_0 = K/W^*$, where

$$K = K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}^{1/2} \quad (11)$$
is the complete elliptic integral of the first kind. Thus $X_0(T_0) = -Z^*_0 \sin(K/W^*)$, which vanishes if either $Z^*_0 = 0$ or if $W^* = K/(k\pi)$, for some integer $k$. This is Lorenz’ result for $b = 0$. The line $Z^*_0 = 0$ is the flat approximation to the SIM, and the vertical lines $W^* = K/(k\pi)$ indicate resonances between the Rossby waves and the gravity waves. We take $Z^*_0 = 0$, so that $X_0 = Z_0 = 0$, and now consider the terms of $O(b)$ in the expansion of the SIM. These satisfy

\begin{align*}
\dot{U}_1 &= -V_0 W_1 - V_1 W_0 \\
\dot{V}_1 &= U_0 W_1 + U_1 W_0 \\
\dot{W}_1 &= -U_0 V_1 - U_1 V_0 \\
\dot{X}_1 &= -Z_1 \\
\dot{Z}_1 &= X_1 + U_0 V_0,
\end{align*}

subject to

$$U_1(0) = V_1(0) = W_1(0) = X_1(0) = 0, \quad Z_1(0) = Z^*_1.$$  

We see immediately that $U_1 = V_1 = W_1 = X_1 = 0$ (and indeed, $U_{2n+1} = V_{2n+1} = W_{2n+1} = X_{2n} = Z_{2n} = 0$ for all $n$). However, the equations for the gravity waves yield

$$X_1(t) = -Z^*_1 \sin t - U^* \int_0^t \sin(t - \tau) \cn(W^* \tau) \sn(W^* \tau) \, d\tau,$$

and $Z_1(t) = -\dot{X}_1(t)$ from (12). Now we examine (9) at $O(b)$, which yields

$$T_1 \dot{U}_0(T_0) = 0, \quad X_1(T_0) = 0.$$

It follows that $T_1 = 0$, and upon substituting (13) into the second condition, we must choose $Z^*_1$ so that

$$Z^*_1 \sin T_0 = -U^* \int_0^{T_0} \sin(T_0 - \tau) \cn(W^* \tau) \sn(W^* \tau) \, d\tau.$$

By contour integration (see the appendix for details) we find

$$Z^*_1 = \frac{\pi}{2} \sech \left( \frac{K'}{W^*} \right) \cot \left( \frac{K}{W^*} \right) + W^* F(W^*),$$

as shown in Figure 2, where $K' = K(1 - m)$ and where $F(W^*)$ has the

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1 This result may also be derived by considering the terms of $O(b)$ as a forcing of the unperturbed system, as in Cox & Roberts [7].
asymptotic series

\[
F(W^*) \sim \sum_{n=1}^{\infty} \text{dc}^{(2n)}(0|1-m)W^{*2n}
\sim mW^{*2} + m(m + 4)W^{*4} + m(m^2 + 44m + 16)W^{*6}
+ m(m^3 + 408m^2 + 912m + 64)W^{*8} + \mathcal{O}(W^{*10}).
\]  

(16)

Here \(\text{dc}^{(2n)}\) is the \(2n\)-th derivative of the Jacobian elliptic function \(\text{dc}\).

Observe that through the factor of \(\cot(K/W^*)\) the behavior of \(Z_1^*\) is singular for \(W^*\) near \(K/(k\pi)\), but there is an exponentially small factor, \(\text{sech}(K'/W^*)\), multiplying this singular term; as \(W^* \to 0\) the singularity rapidly becomes weaker. Our approximation of the sim to \(\mathcal{O}(b)\), given by (15), agrees very well with the sim computed numerically by Lorenz.
for $b = 0.5$. He finds $Z^* \approx 0.4W^*^3$ for small $W^*$ for an initial condition corresponding to $m = 0.81$, while our equivalent result is $Z^* \approx mbW^*^3 = 0.405W^*^3$.

### 3.3 Divergent power series and periodic solutions on the SIM and on $\mathcal{M}$

We now make a few observations about the expression (15) for the section through the SIM. Undoubtedly corrections will be made to $Z^*$ at orders $b^3$, $b^5$, $\ldots$, but for our present purposes we assume that the essential features of the curve $Z^*(W^*)$ are captured by (15) alone.

Firstly, we recall the result that, in general, centre manifolds (and sub-centre manifolds, as are at issue here) are not unique. A given system may have an infinite number of centre manifolds, each differing from any other by exponentially small terms (see Carr [6]; and Roberts [25, §2.1]). Each has an identical power-series representation. Therefore the existence of more than one slow manifold ($\mathcal{M}$, and the SIM, and infinitely many others) for the atmospheric model L86 should come as no surprise. The notable feature of the exponentially small difference between $\mathcal{M}$ and the SIM is its multiplication by a term with infinitely many algebraic singularities, $\cot(K/W^*)$. This frustrating feature results from resonances between the fast and the slow waves that arise in attempting to construct a subcentre manifold, rather than a centre manifold (see Sijbrand [29]). Frequently, when one computes the power series for a centre manifold, it is divergent, even though a centre manifold contains no singularities similar to those of a subcentre manifold. We wish, then, to emphasize that the existence of singularities in the SIM and the divergent power series for any slow manifold are separate issues.

Secondly, (15) indicates that the SIM and the slow manifold $\mathcal{M}$ differ by terms that are exponentially small as $W^* \rightarrow 0$. To see this we use the fact that we have chosen $V^* = 0$ in our search for periodic solutions of (1), and so according to the slow-manifold model, $Z^* = Z(U^*, 0, W^*)$. Using Roberts’ [26] expansion of $Z(U, V, W)$, (2), we find

$$Z^* = bW^*U^*^2 + bW^*U^*^2(4W^*^2 + U^*^2) + \cdots + \mathcal{O}(b^2).$$

This agrees with the result (15) when the exponentially small term is ignored, since $U^*^2 = mW^*^2$. Indeed, further investigation reveals agreement between $W^*F(W^*)$ and $Z(U^*, 0, W^*)$ at all orders in $|(U^*, W^*)|$. Thirdly, let us examine more closely the result of using the slow-manifold model (2–3) to approximate the SIM of (1). We begin by noting that the
construction of $\mathcal{M}$ yields expressions for $X(U,V,W)$ and $Z(U,V,W)$ of the form
\[
X = UV \tilde{f}(U,V,W), \quad Z = bW \tilde{g}(U,V,W).
\]
It follows that a zero-crossing of either $U$ or $V$ necessarily implies a zero-crossing of $X$. Further, the evolution of the slow variables $U$, $V$, $W$ on $\mathcal{M}$ is governed by equations of the form
\[
\dot{U} = -VW(1 - b^2 \tilde{g}), \quad \dot{V} = UW(1 - b^2 \tilde{g}) \quad (17) \\
\dot{W} = -UV,
\]
so that $U^2 + V^2$ is invariant for this model, just as it is for the original system (L86). Consequently, the dynamics of (17) are two-dimensional and so its solutions are in general periodic. Since, by definition, the SIM is composed of the periodic solutions of (1), it might therefore appear that $\mathcal{M}$ is the SIM of (1). However, we know from our preceding remark that $\mathcal{M}$ and the SIM of (1) are not the same—they differ by exponentially small terms. This discrepancy is resolved by concluding that a slow manifold $\mathcal{M}$, constructed by the functional relationship $y = h(x)$, cannot be exactly invariant under evolution according to (1)—there will always remain exponentially small discrepancies.

The SIM was proposed because it can be computed directly without recourse to (divergent) power series. However, the power series for the SIM is the same as for $\mathcal{M}$, and is divergent. We emphasize that the divergence of its power series does not mean that the SIM does not exist, nor does it mean that the SIM is ill-defined, or “fuzzy”. The utility of the divergent series is illustrated by the excellent agreement between an approximation to $Z^*(W^*)$ based on a small number of terms for $\mathcal{M}$, and the numerical result for the behavior of $Z^*(W^*)$ as $W^* \to 0$ on the SIM. For divergent series, this agreement is good for small $W^*$ and for a fixed number of terms. In the next section we shall illustrate some of the dynamical consequences of the differences between $\mathcal{M}$ and the SIM. For most solutions the differences are slight, and result in small quantitative changes. In a practical computation, we are required to select one of the many possible slow manifolds. We consider two strong advantages of $\mathcal{M}$ over the SIM to be the possibility of computing appropriately the projection of initial conditions onto $\mathcal{M}$, and the explicit algebraic form $y = h(x)$ of $\mathcal{M}$, compared with the construction of the SIM through an ensemble of numerical solutions.
4 Normal form for L86

In Section 2 we described the construction of a slow manifold, \( \mathcal{M} \), which approximates a subset of the possible solutions of (1), and on which the fast “gravity” wave variables are slaved to the amplitudes of the slow “Rossby” waves. More general solutions of (1), with significant independent fast wave activity, involve oscillations centred on \( \mathcal{M} \). In this section we rewrite (1) so that all solutions are captured, not just the slow solutions on \( \mathcal{M} \), but where the new governing equations describe the evolution of five new slowly-varying quantities. This is a normal form calculation (see Guckenheimer & Holmes \[14\]; Arnold \[2\]), which has been sketched for the forced, damped system of LK87 by Jacobs \[17\]. In that case the variables \( U, V, W, X, Z \) apparently may be written as convergent power series in the new slow variables: for the model L86, where there is no damping, we shall see that the equivalent series are divergent. Jacobs notes that in the new variables a slow manifold takes a particularly simple form. We shall also see from the normal form calculation that there is a limit on our ability to initialize data in such a way that the long-time dynamics of the physically dominant slow “Rossby” waves are unaffected by the balancing procedure. That is, there is a limit on the agreement between the unbalanced and the balanced systems—the two forecasts inevitably diverge slowly.

We begin by noting that for the uncoupled system, (1) with \( b = 0 \), we can identify five slowly-varying quantities: \( U, V, W, R \) and \( \dot{\Theta} \), where \( X = R \cos \Theta \) and \( Z = R \sin \Theta \). (In fact, when \( b = 0 \) it follows that \( R = 0 \) and \( \dot{\Theta} = 1 \).) Our aim now is to find for the coupled system, (1) with \( b \neq 0 \), five equivalent slowly varying quantities. We do this by making successive nonlinear changes of variables: to leading order in the calculation the slowly varying quantities will be just \( U, V, W, R \) and \( \dot{\Theta} \). We shall perform algebraic manipulations on the system of equations, but there is no reduction in the dimension of the system as there was in computing a slow manifold \( \mathcal{M} \); we expect the normal form to capture all solutions of (1). This point is the essential difference between the simplifying tools of invariant manifold theory and normal form theory. The first aims to reduce the dimension of a nonlinear dynamical system, while the second transforms the system to a canonical form.

To make the normal form transformation we seek to write (1) in terms of slowly varying variables \( u, v, w, r \) and \( \dot{\theta} \). Linearly, these will be identified with the original variables \( U, V, W, R \) and \( \dot{\Theta} \), respectively. Thus the evolution of \( u, v \) and \( w \) will describe predominantly the dynamics of the slow waves, while \( r \) and \( \theta \) represent the amplitude and phase of the fast waves. We
shall expand the “physical” variables \((U, V, W, X, Z)\) as power series in the new variables \(u = (u, v, w, x, z)\), where the two variables \(x = r\cos\theta\) and \(z = r\sin\theta\) are closely related to the original fast wave variables \(X\) and \(Z\). This power-series expansion introduces exponentially small errors; the new evolutionary system for \(u\) will have solutions that differ from solutions of (1) by terms smaller than any power of \(|u|\) as \(|u| \to 0\). Further, if for a practical computation we truncate the transformation at a finite number of terms in the power series, then larger, algebraic errors are introduced.

Recall that \(x = (U, V, W)\) and \(y = (X, Z)\) so that (1) may be written as the following pair of equations,

\[
\begin{align*}
\dot{x} &= f(x, y) \quad (18) \\
\dot{y} &= By + g(x),
\end{align*}
\]

where

\[
f(x, y) = (-VW + bVZ, UW - bUZ, -UV),
\]

\[
B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad g(x) = (0, bUV).
\]

We make a change of variables,

\[
x = \chi + F(\chi, \eta), \quad y = \eta + G(\chi, \eta),
\]

(20)

where we identify \(\chi = (u, v, w)\) and \(\eta = (x, z)\). Here \(F\) and \(G\) are \(O(|(\chi, \eta)|^2)\) as \(|(\chi, \eta)| \to 0\), so that linearly \(x \sim \chi\) and \(y \sim \eta\). Substitution of (20) into (18–19) gives the following equations for the evolution of the new variables:

\[
\begin{align*}
\dot{\chi} + \frac{\partial F}{\partial \chi} \dot{\chi} + \frac{\partial F}{\partial \eta} \dot{\eta} &= f(\chi + F, \eta + G) \\
\dot{\eta} + \frac{\partial G}{\partial \chi} \dot{\chi} + \frac{\partial G}{\partial \eta} \dot{\eta} &= B\eta + BG + g(\chi + F).
\end{align*}
\]

(21)

(22)

We now assume that the nonlinear terms and the new variables’ evolution may be written as the power series

\[
F \sim \sum_{n=2}^{\infty} F_n, \quad G \sim \sum_{n=2}^{\infty} G_n,
\]

(23)

\[
\dot{\chi} \sim \sum_{n=2}^{\infty} R_n(\chi, \eta), \quad \dot{\eta} \sim B\eta + \sum_{n=2}^{\infty} S_n(\chi, \eta),
\]

(24)

where \(F_n, G_n, R_n, S_n = O(|(\chi, \eta)|^n)\). We shall solve (21–22) at successive orders in \(|(\chi, \eta)|\); at each order our aim is to choose the four quantities \(F_n, \ldots, S_n\).
\(G_n, R_n, S_n\) as simply as possible. We shall see below what the term “simple” actually means. It turns out to be possible at each order to require that \(\dot{\chi}, \dot{r}\) and \(\dot{\theta}\) are slowly varying, that is, that they are independent of the phase of the fast waves, \(\theta\).

For use in the calculation below, we note that by the chain rule
\[
\frac{\partial a}{\partial \theta} = \frac{\partial a}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial a}{\partial z} \frac{\partial z}{\partial \theta} = -\frac{\partial a}{\partial z} + \frac{\partial a}{\partial x} B\eta,
\]
for any function \(a(u, v, w, x, z)\). As an example of some of the algebraic details, consider the equation governing the slow wave components \(R_2\) and \(F_2\) which is, from (21–24),
\[
R_2 = f_2 - \frac{\partial F_2}{\partial \eta} B\eta = f_2 - \frac{\partial F_2}{\partial \theta},
\]
where
\[
f_2 = (-v(w - bz), u(w - bz), -uv).
\]
Note that \(f_2\) depends on \(\theta\) through \(z\). We now choose \(R_2\) to be independent of \(\theta\) and such that secular growth in \(F_2\) is avoided. This requires
\[
R_2 = (-vw, uw, -uv),
\]
in which case
\[
\frac{\partial F_2}{\partial \theta} = (bvz, -buz, 0).
\]  (25)
Now we choose \(F_2\) as simply as possible, that is, \(F_2 = (-bvx, bux, 0)\). We can add to this expression for \(F_2\) an arbitrary function of \(u, v, w\) and \(r\), while still satisfying (25), but for simplicity we decide that \(\theta\)-averages are to vanish. A consequence of this decision is that after averaging over the gravity waves \(u, v\) and \(w\) reduce to \(U, V\) and \(W\) respectively.

For general \(n > 2\),
\[
R_n = f_n - \sum_{m=2}^{n-1} \frac{\partial F_m}{\partial \chi} R_{n+1-m} - \sum_{m=2}^{n-1} \frac{\partial F_m}{\partial \eta} S_{n+1-m} - \frac{\partial F_n}{\partial \theta}
\equiv C_n - \frac{\partial F_n}{\partial \theta},
\]
where \(f_n\) denotes terms of order \(|u|^n\) in the expansion of \(f(x, y)\) (and \(g_n\) is defined similarly from the expansion of \(g\)), and \(C_n\) is a multi-nomial in \(x = r \cos \theta\) and \(z = r \sin \theta\) which involves quantities known from the previous calculation of \(R_m, F_m, S_m, G_m\) for \(m = 2, \ldots, n - 1\). We choose \(R_n\) to be
independent of $\theta$, in fact to be $\bar{C}_n$, where the overbar denotes the mean with respect to $\theta$, so that no secular terms arise in $F_n$. Then we choose $F_n$ as simply as possible (that is, so that $\bar{F}_n = 0$). Note that $R_n$ is uniquely defined if it is to be independent of $\theta$, and if no secular terms are to arise in $F_n$. The term $F_n$ is unique only once we have specified $F_n$. Altering the value of this average simply alters the relationship between the slow co-ordinates $\chi$ and the original variables by an amount of order $|u|^n$.

Now we consider the equation for the fast wave components $S_n$ and $G_n$, for $n \geq 2$,

$$S_n - BG_n + \frac{\partial G_n}{\partial \theta} = g_n - \sum_{m=2}^{n-1} \frac{\partial G_m}{\partial \chi} R_{n+1-m} - \sum_{m=2}^{n-1} \frac{\partial G_m}{\partial \eta} S_{n+1-m}$$

$$\equiv D_n.$$  \hspace{1cm} (26)

We note that $D_n$ can be written as a multi-nomial in $r \cos \theta$ and $r \sin \theta$ which involves quantities calculated at previous orders. If we now compute $B(26) + \partial(26)/\partial \theta$ we obtain

$$\left(1 + \frac{\partial^2}{\partial \theta^2}\right) G_n = \left(B + \frac{\partial}{\partial \theta}\right) (D_n - S_n),$$  \hspace{1cm} (27)

where we have used the identity $B^2 + I = 0$. Our aim now is to make $\dot{r}$ and $\dot{\theta}$ slowly-varying by requiring that they be independent of $\theta$, that is,

$$\dot{r} = r \rho(\chi, r^2)$$

$$\dot{\theta} = \psi(\chi, r^2).$$

But $r \dot{r} = x \dot{x} + z \dot{z}$ and $r^2 \dot{\theta} = x \dot{z} - z \dot{x}$, and so

$$\begin{bmatrix} x & z \\ -z & x \end{bmatrix} \dot{\eta} = \begin{bmatrix} x & z \\ -z & x \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \rho \\ \psi \end{bmatrix} r^2.$$

Thus

$$\dot{\eta} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x & -z \\ z & x \end{bmatrix} \begin{bmatrix} \rho \\ \psi \end{bmatrix} \equiv M \begin{bmatrix} \rho \\ \psi \end{bmatrix}. \hspace{1cm} (28)$$

Jacobs [17, Appendix B] has termed (28) a “preferred choice” for the form of $\dot{\eta}$: at this point it is clear, though, that (28) is a necessary condition for $\dot{r}$ and $\dot{\theta}$ to be slowly-varying.

Equation (28) implies that $S_n$ is of the form $\tilde{S}_n(\chi, r^2) \eta$, where $\tilde{S}_n$ is a $2 \times 2$ matrix. In (27) we choose $G_n$ to match all terms in the right-hand side, except the components of $r \cos \theta$ and $r \sin \theta$ in $D_n$ (for which $1 + \partial^2/\partial \theta^2 = 0$),

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which must be removed by $S_n$. So let us now consider these terms in (26). We know from (28) that $S_n$ is of the form

$$S_n = M \begin{bmatrix} \rho_n(\chi, r^2) \\ \psi_n(\chi, r^2) \end{bmatrix}.$$  

It is also straightforward to confirm that if the components of $G_n$ proportional to $x = r \cos \theta$ and $z = r \sin \theta$ are

$$G_{n}^{(1)} = \begin{bmatrix} \gamma_1 x + \gamma_2 z \\ \delta_1 x + \delta_2 z \end{bmatrix},$$

where the $\gamma_1$, $\gamma_2$, $\delta_1$ and $\delta_2$ are functions of $\chi$ and $r^2$, then

$$-BG_{n}^{(1)} + \frac{\partial G_{n}^{(1)}}{\partial \theta} = \begin{bmatrix} x & z \\ -z & x \end{bmatrix} \begin{bmatrix} \gamma_2 + \delta_1 \\ \delta_2 - \gamma_1 \end{bmatrix} \equiv N \begin{bmatrix} \gamma_2 + \delta_1 \\ \delta_2 - \gamma_1 \end{bmatrix}.$$

We note also that the terms in $D_n$ proportional to $r \cos \theta$ and $r \sin \theta$ may in all generality be written as $Ma + Nb$, where $a$, $b$ are vector functions of $\chi$ and $r^2$. This decomposition is unique, and therefore $(\rho_n, \psi_n) = a$ and $(\gamma_2 + \delta_1, \delta_2 - \gamma_1) = b$. Thus we have specified $S_n = a$ uniquely, although $G_n$ retains two undetermined coefficients. The two degrees of freedom left at our disposal correspond to redefining $r$ and $\theta$ by amounts of order $|u|^n$.

### 4.1 Discussion of the normal form

We have used the algebraic programming system REDUCE to implement the procedure described above to compute the normal form, which gives as the first few terms of the expansion

$$U \sim u - bux + \frac{1}{4} b^2 u (z^2 - x^2)$$
$$V \sim v + bux + \frac{1}{4} b^2 v (z^2 - x^2)$$
$$W \sim w + bz(u^2 - v^2)$$
$$X \sim x - buv + \frac{1}{2} b^2 x (v^2 - u^2)$$
$$Z \sim z + bw(u^2 - v^2).$$

Despite the apparent differences, this normal form is equivalent to that of Jacobs [17], with his damping and forcing set to zero, because we have chosen our variables $\chi$, $\eta$ differently.

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We now consider the structure of the normal form equations, and try to construct their sim. A careful examination of the co-ordinate transformation (20) at each order $n$, approximated above, reveals it be of the form

$$
U = u + uA_1 - bvA_2 \\
V = v + vA_1 + buA_2 \\
W = w + A_3 \\
X = x + X(u,v,w) + A_4 \\
Z = z + Z(u,v,w) + A_5,
$$

where $(X(u,v,w), Z(u,v,w))$ is the slow manifold $M$ computed in Section 2, and $A_j = A_j(u,v,w,x,z) = O(r)$ as $r \to 0$. The evolution of the slow variables is given by expressions of the form

$$
\dot{u} = -vw(1 - B_1(u^2, v^2, w^2, r^2)) \\
\dot{v} = uw(1 - B_1(v^2, u^2, w^2, r^2)) \\
\dot{w} = -uv(1 - r^2B_2(u^2, v^2, w^2, r^2)) \\
\dot{r} = ruvwC(u^2, v^2, w^2, r^2) \\
\dot{\theta} = 1 + D(u,v,w,r^2).
$$

The equations that govern the evolution of the four variables $(u,v,w,r)$ representing amplitudes in (31) are independent of the fast gravity-wave phase variable, $\theta$. We therefore examine first the four-dimensional system (31a–d) that results from ignoring the $\dot{\theta}$-equation, (31e). There are two invariants of the system (1), each independent of the gravity-wave phase: $U^2 + V^2$ and $V^2 + W^2 + X^2 + Z^2$ (L86). Similarly, these quantities are invariant under evolution of the normal form equations (31a–d), which therefore have two-dimensional dynamics. Solutions are therefore in general periodic. Re-introducing $\theta$-variations, we see that solutions of the normal form (31) are in general quasiperiodic, with one frequency $\omega_1$ arising from (31a–d), and a second $\omega_2$ from (31e). Singly periodic solutions occur if $\omega_1$ and $\omega_2$ are rationally related, or if $r = 0$; these solutions form the sim of the normal form, which we construct below. The original system (1), however, not only has periodic and quasiperiodic solutions, but also has aperiodic solutions (LK87). Thus the normal form and (1), though quantitatively nearly identical, have qualitatively different dynamics.

A further indication of the differences between solutions of (31) and (1) is the existence of some heteroclinic orbits in the former which are absent in the latter. To see this, we follow LK87 in considering the Hadley solutions
$P_V(F)$ of (1): namely the fixed points $(U, V, W, X, Z) = (0, F, 0, 0, 0)$. (The same considerations will apply, with appropriate changes, to the solutions $P_U(F)$: $(U, V, W, X, Z) = (F, 0, 0, 0, 0)$.) Computations by LK87 indicate that gravity waves arise on the unstable manifold $U_L(F)$ of each Hadley solution $P_V(F)$, except at isolated values of $F$. In these exceptional cases, $U_L(F)$ is heteroclinic to $P_V(F)$ and $P_V(-F)$, that is, $U_L(F)$ is asymptotic to $P_V(F)$ and $P_V(-F)$ as $t \to -\infty$ and $t \to \infty$, respectively. The near-heteroclinic behavior associated with almost all Hadley solutions causes solutions of (1) that pass through a neighborhood of the Hadley solutions to be aperiodic: solutions that remain away from the Hadley solutions are in general quasiperiodic, except on the SIM, where they are periodic. However, for the system (31a–d) the existence of two invariants and the symmetries of the system in any of the $(u, v, w)$ co-ordinate planes constrain all Hadley solutions $P_V(F)$ to be connected to their opposites $P_V(-F)$ by heteroclinic orbits. No aperiodic solutions of the normal form (31) exist. In summary, the dynamics of (31) may differ qualitatively from the dynamics of (1), particularly for solutions that pass close by the Hadley solutions. The difference is due to the essentially two-dimensional nature of the normal form which occurs when $\theta$ is decoupled, compared with the essentially three-dimensional nature of (1). The differences in the behavior of solutions of (1) and (31) that do not pass close to the Hadley solutions is small.

A final illustration of the differences between solutions of (31) and those of (1) is given by the periodic solutions, which form the SIM. The SIM of (1) was described in Section 3—to construct the SIM of (31) we first note that if we set $r = 0$ at time $t = 0$ then by (31d) $r = 0$ for all time, so the plane $r = 0$ is an invariant slow manifold of (31). Further, solutions with $r = 0$ are in general periodic (because then the variations in $\theta$ are irrelevant when the solution $(U, V, W, X, Z)$ is reconstructed from (30)). Setting $r = 0$ in (30), we see that the invariant manifold $r = 0$ is precisely $\mathcal{M}$, as calculated in Section 2. Some other periodic solutions of (31) occur for non-zero initial values of $r$—however, these have significant gravity-wave activity and they exist when the gravity waves happen to be exact harmonics of the Rossby waves. They give rise to manifolds of resonant solutions that intersect $\mathcal{M}$, akin to the lines $W^* = K/(k\pi)$ for the uncoupled system (1) with $b = 0$. The slowest invariant manifold of the normal form (31) is therefore $\mathcal{M}$, and has no singularities. However, there are infinitely many “resonant” branches, which intersect $\mathcal{M}$ as shown in Figure 3.

The relation between the normal form calculation of this section and the calculation of the slow manifold described in Section 2 is most easily seen if we assume that the new variables $\chi$ are chosen, as we have done, by setting
Figure 3: The slowest invariant manifold (\textit{sim}) of the normal form, equations (29) and (31) computed to fourth order.

\[ \mathbf{F} = 0. \] Then the transformation of the Rossby wave amplitudes is of the form

\[ x = \chi + F(\chi, \eta) = \chi + r \tilde{F}(\chi, \eta), \]

so that by setting \( r = 0 \) (that is, \( F = \eta = 0 \)) the new slow variables and the old become identical. To see the significance of setting \( r = 0 \) we consider (21–22), which become

\[
\dot{x} = f(x, h(x)) \\
\partial h \dot{x} = B h(x) + g(x),
\]

where \( h(x) = G(x, 0) \). These are precisely the equations that govern \( \dot{x} \) and \( y = h(x) \) for an invariant slow manifold (Carr [6]). So by setting \( \eta = 0 \)
in (20) we recover $\mathcal{M}$ (implicitly, through $y = h(x)$). As a consequence, we expect the normal form sums in (23) and (24) to be divergent.

There is a strong parallel between the method of normal forms and the method of averaging. (The latter method was suggested for the geostrophic approximation by Bill Dewar in private communication.) An alternative approach to the approximation of solutions of (1) is to assume that the gravity wave oscillations occur on a separate, faster time-scale than the evolution of the other variables, which are assumed to vary on the slow time-scale $\tau$. We then replace the operator $d/dt$ in (1) by the operator $d/d\tau + d/d\theta$, and write each dependent variable in (1) as a function of the new variables $u(\tau)$, $v(\tau)$, $w(\tau)$, $a(\tau)$ and the fast time $\theta$. The notation for the new variables is as for the normal form calculation, except that the gravity waves now have a slowly-varying complex amplitude $a(\tau)$. The two-time approach proceeds with similar algebraic steps as the normal form calculation, in particular at each stage an average over the fast time-scale $\theta$ is taken in order to find terms in the slow evolution equations.

### 4.2 Projection of initial conditions and the normal form transformation

If the slow manifold had been a centre manifold, characterized by an exponential approach of nearby trajectories to $\mathcal{M}$, then a rational principle can be used to compute the balanced initial data (Roberts [26]). The principle is that the long-term evolution of the original full system from its initial data must approach exponentially quickly the slow evolution on the centre manifold from the balanced initial data. This principle leads to an analysis whose framework is identical to that given in Subsection 2.1. However, the balanced initial data can be found immediately from the normal form of the equations, as we now demonstrate. Continuing the discussion for the case when $\mathcal{M}$ is a centre manifold, we find that a normal form transformation (20) can be chosen so that the new variables evolve according to equations of the form

\[
\begin{align*}
\dot{\chi} &= R(\chi), \\
\dot{\eta} &= B\eta + S(\chi, \eta) = \left(B + \tilde{S}(\chi, r^2)\right)\eta.
\end{align*}
\]

Thus $\eta = 0$ describes the centre manifold of the long-term evolution and, because the eigenvalues of $B$ are negative, $\eta$ decays exponentially quickly to

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0 (as is the case when the fast gravity waves are damped). Thus (20) gives the following parametric description of the centre manifold in the original variables

\[
x = \chi + F(\chi, 0), \quad y = G(\chi, 0).
\]

However, from equation (32) we see that the evolution of the slow variables \(\chi\) is unaffected by the fast variables \(\eta\). Thus all solutions starting from points \((\chi, \eta)\) with the same value of \(\chi\) have precisely the same \(\chi\) evolution, and because they all asymptote exponentially quickly to the centre manifold, so they all have the same long-term evolution. The hyper-surfaces in \((x, y)\)-space that are described as \(\eta\) is varied in (20) with fixed \(\chi\) form the isochronic manifolds introduced by Roberts [26]. Hence, the initial data \((\chi, \eta)\) are balanced to the centre manifold simply by setting \(\eta = 0\) (as in a linear analysis).

In terms of the original variables, the initial values \((x_0, y_0)\) are balanced by finding the corresponding normal form variables \((\chi_0, \eta_0)\) from (20), and then using

\[
x'_0 = \chi_0 + F(\chi_0, 0), \quad y'_0 = G(\chi_0, 0)
\]

as the appropriate initial conditions. From \((x'_0, y'_0)\) the evolution is slow and has precisely the same long-term dynamics as from \((x_0, y_0)\).

However, the “quasi-geostrophic” slow manifold \(\mathcal{M}\) is a subcentre manifold; it does not attract neighboring solutions exponentially, and the arguments above do not directly apply. The reason is that for a subcentre manifold the influence of the fast waves cannot be removed entirely from the evolution of the slow variables. This may be seen in (31) where the functions \(B_1\) and \(B_2\), which govern the evolution of the slow waves, also depend upon the amplitude of the fast waves, \(r\): for a subcentre manifold (32) must be replaced, in general, by an equation of the form

\[
\dot{\chi} = R(\chi, r^2).
\]

That is, the evolution of the slow variables cannot be entirely decoupled from the fast variables. Thus a simulation with both Rossby waves and gravity waves present need not be equivalent, over a long time, to any of the possible solutions with purely slow Rossby waves.

This feature of subcentre manifolds is displayed in the simple system

\[
\dot{u} = z^2, \quad \dot{x} = -z, \quad \dot{z} = x.
\]

These equations have the exact normal form, upon the transformation \(\chi = u + xz/2, x = r \cos \theta\) and \(z = r \sin \theta\),

\[
\dot{\chi} = r^2/2, \quad \dot{r} = 0, \quad \dot{\theta} = 1.
\]

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Here the slow manifold is simply $x = z = 0$ ($r = 0$), on which $\chi$ evolves according to $\dot{\chi} = 0$. Thus the long-term evolution on $\mathcal{M}$ is trivial: $\chi$ is constant. However, if there are any fast waves present, $r \neq 0$, then $\chi$ drifts according to $\dot{\chi} = r^2/2$, but there is no corresponding solution for $\chi$ on $\mathcal{M}$. The same is true for the normal form (31) of the model L86—at fourth order in $|\chi|$ we cannot avoid introducing gravity-wave terms proportional to $r^2$ into the evolution equations for the slow Rossby waves. Therefore solutions on and off the slow manifold $\mathcal{M}$ cannot share the same values of $u$, $v$ and $w$ for all time; there are inevitable discrepancies which become significant on a timescale of $\mathcal{O}(r^{-2})$.

It is now clear that the balancing procedure, the projection of initial conditions described in Section 2, which is linear in the fast gravity wave amplitude $r$, cannot be improved to be correct to $\mathcal{O}(r^2)$.

5 Conclusions

A significant part of our understanding of atmospheric dynamics rests on the concepts of quasi-geostrophy (Gill [12, Chapt. 7]). The fundamental concept is the separation of the dynamics into waves of two time-scales: the slowly evolving Rossby waves and the quickly evolving gravity waves. We have described how a slow manifold, formally composed of the ensemble of slow Rossby waves, may formally be written in the form $y = h(x)$, with $h$ developed as a power series in the slow variables $x$. This is similar to the scheme proposed by Baer & Tribbia [3], and is equivalent to that of Vautard & Legras [32]. As L86 demonstrates, the power series for $h$ is divergent, so that successive approximation schemes at first appear to converge to a slow manifold, but after the inclusion of sufficiently many terms they begin to diverge. Nevertheless, the divergence of its power series does not indicate the non-existence of a slow manifold. Each successive approximation to $\mathcal{M}$ captures the dynamics of L86 to within a higher power of $x$, but in a smaller radius around the origin. Unfortunately for our purposes, the dynamics on $\mathcal{M}$ (determined exactly, but not from its asymptotic series) differ from those of L86 by an amount that is smaller than any power of $x$ as $x \to 0$. The dynamics of the closed set of evolution equations (3) for the slow variables on $\mathcal{M}$ are genuinely slow: no gravity waves develop. But $\mathcal{M}$ is not quite invariant, so the apparent slow dynamics on $\mathcal{M}$ do not reflect genuine slow dynamics in L86. In particular $\mathcal{M}$ appears to be filled with periodic solutions, whereas the full system has a manifold of periodic solutions (Lorenz’ sim) which is different from $\mathcal{M}$ and which also contains singularities.
The issue of how to balance given data measurements by projecting them onto a slow manifold is a complicated one for realistic situations—the most appropriate projection may depend on the relative quality of the various measurements, and on the influence of different measurements on the projected initial point (Daley [9, 10]). We have described a method of projection that is certainly well-suited to models where the initial data are known exactly, such as L86. The criterion we have proposed for selecting an initial point on $\mathcal{M}$ seems appropriate for numerical weather prediction: namely, that the behavior of the balanced system should correspond for all time to that of the unbalanced system (just without the fast gravity-wave activity). This ensures that the essential features of the forecast are unchanged by the initialization process. However, we have also shown that due to resonances, as exhibited by the normal form transformation (Arnold [2]), the initialization can be carried out only to first order in the fast wave amplitude $^3$. We have illustrated the practical utility of our proposed initialization procedure with numerical integrations (Figure 1) that show the superior accuracy of the forecast from appropriately initialized data.

Certain difficulties associated with computing a slow manifold have led researchers to question the existence of such a manifold, or to label the concept “fuzzy”. A slow manifold for L86 is a subcentre manifold (Kelley [18]; Sijbrand [29]), and is not unique: just as for all low-dimensional models (Roberts [27]) there are infinitely many slow manifolds in which the fast variables are specified in terms of the slow variables. It is this non-uniqueness that accounts for the “fuzziness” in the concept of the slow manifold: the partial differential equation that governs a slow manifold for a given problem is perfectly well-defined, as is each manifold. It is just that the construction of low-dimensional models has infinitely many solutions, differing by exponentially small terms. It is possible to select one distinguished slow manifold by assuming that the fast variables may be written as power series in the slow variables. However, often, as is the case for L86, the power series for the slow manifold proves to be divergent, so that the series is asymptotic. As an alternative, the numerical construction of a slow manifold (Lorenz’ sim) was conceived to avoid the divergence problem. The sim is defined as containing all the periodic solutions. However, in the light of our previous comment that slow manifolds differ by exponentially small terms, the sim must share the divergent power series of all slow manifolds for L86.

$^3$The resonances that force certain terms to occur in the evolution equations for the normal form variables are a consequence of the linear dynamics (Arnold [2]), and are quite distinct to the resonances between the periods of the (fully nonlinear) Rossby and gravity waves that induce singularities in the slowest invariant manifold.
does not imply that the SIM does not exist, nor does it suggest that the SIM is somehow “fuzzy”. The SIM exists; it just has a divergent power series. The asymptotic series of all slow manifolds for a given system are identical—the slow manifolds differ by sub-dominant terms (Bender and Orszag [4]).

We contrast our point of view with that presented by Warn & Menard [33], relating to the slow manifold for the model L80. They argue that a slow manifold must attract neighboring solutions in order to be useful in applications. Since almost every numerical solution of L80 involves some level of fast “gravity” wave activity, they conclude that an attracting invariant slow manifold does not exist. However, it is important to recognize that a slow manifold need not be attracting in order to serve as a useful centre for the dynamics of the system. The subcentre manifold we have discussed for L86 does not attract neighboring solutions, yet it has dynamics that approximately correspond to the full initial-value problem, except that the fast gravity waves are absent. Warn and Menard argue that the slow manifold should be replaced by a more general “fuzzy” balanced set because their scheme to compute a slow manifold encounters asymptotic series, and because in their approximations to the slow manifold, gravity wave activity is observed. We have argued that a slow manifold is necessarily “fuzzy” because of its exponential non-uniqueness, which is inherent to invariant manifolds of the centre-manifold family. Each individual invariant manifold is well-defined. The “fuzziness” of the set of manifolds is an entirely separate issue from that of the divergence of the power series for each slow manifold. As Lorenz has illustrated for the SIM, one may compute a slow manifold numerically, even when it has a divergent series.

We cite two other examples of common and useful approximations that correspond to the use of a subcentre manifold, even though such a manifold does not attract neighboring solutions. The first is the incompressible approximation in fluid mechanics, where the fast variables represent sound waves and are neglected to leave the slow evolution of incompressible flow. The second is beam theory in elasticity, where rapid internal elastic vibrations are the fast variables and where large scale deformations are the slow variables (Roberts [28]). The fast “ringing” modes are eliminated in the traditional approximations of beam theory. The concept of a subcentre manifold enables one rationally to extend these traditional approximations to higher order, to derive appropriate initial and boundary conditions for the model, and to treat forcing appropriately.
A Contour integration

In order to compute \( Z^*_1 \), we wish to evaluate the integral

\[
I = U^*2 \int_0^{T_0} \sin(T_0 - \tau) \ cn(W^*\tau) \ sn(W^*\tau) \ d\tau,
\]

which appears in (14). We first expand the trigonometric function, then change variables by setting \( t = W^*\tau \), so that

\[
I = mW^* (I_c \sin T_0 - I_s \cos T_0),
\]  

(34)

where

\[
I_c = \int_0^K \cos \frac{t}{W^*} \sn(t) \cn(t) \ dt,
\]

and

\[
I_s = \int_0^K \sin \frac{t}{W^*} \sn(t) \cn(t) \ dt.
\]

To evaluate \( I_s \), we consider the contour integral

\[
I_\Gamma = \int_{\Gamma} S(t; W^*) \ dt,
\]  

(35)

where

\[
S(t; W^*) = e^{i t/W^*} \sn(t) \cn(t),
\]

and where \( \Gamma \) is the rectangle with vertices at \(-K, K, K + 2iK', -K + 2iK'\) as shown in Figure 4. We denote the straight line segments joining these vertices from \(-K\) counterclockwise by \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \), respectively, with corresponding integrals \( I_1, I_2, I_3, I_4 \). Then

\[
I_1 = \int_{-K}^K e^{i t/W^*} \sn(t) \cn(t) \ dt = 2iI_s.
\]

The integral along the parallel side of \( \Gamma \) is

\[
I_3 = \int_K^{-K} e^{i(2iK' + t)/W^*} \sn(2iK' + t) \cn(2iK' + t) \ dt = e^{-2K'/W^*}I_1.
\]

The other integrals along the vertical sides of the rectangle give

\[
I_2 = \int_0^{2K'} e^{i(K + it)/W^*} \sn(K + it) \cn(K + it) \ dt
\]

\[
= m_1^{1/2} e^{iK/W^*} \int_0^{2K'} e^{-t/W^*} nd(t|m_1) sd(t|m_1) \ dt,
\]
A Contour integration

\[ \Gamma \]

\[ \text{pole} \]

\[ \Gamma_1 \]

\[ \Gamma_2 \]

\[ \Gamma_3 \]

\[ \Gamma_4 \]

\[ -K + 2iK' \]

\[ K + 2iK' \]

\[ -K \]

\[ 0 \]

\[ K \]

\[ \text{Re} \]

\[ \text{Im} \]

\[ K' \]

\[ iK' \]

Figure 4: Paths of integration in the complex plane.

where \( m_1 = 1 - m \), and

\[ I_4 = -I_2^* \]

The contour \( \Gamma \) encloses a double pole of \( S(t; W^*) \) at \( t = iK' \), and so

\[ I_\Gamma = I_2 - I_2^* + 2i(1 + e^{-2K'/W^*})I_s = 2\pi i \text{Residue}(S(t; W^*); t = iK'). \]

To compute the residue we expand \( S(t; W^*) \) near the pole to give

\[ S(t; W^*) = e^{-K'/W^*} \left( 1 + i \frac{t - iK'}{W^*} \right) \left( \frac{m_1^{-1/2}}{t - iK'} \right) \left( \frac{-im_1^{-1/2}}{t - iK'} \right) + O(1), \]

where the singular behavior of the Jacobian elliptic functions is (Table 16.7
of Abramowitz and Stegun [1])

\[ \text{sn}(t) \sim \frac{m_1^{-1/2}}{t - iK'}, \quad \text{cn}(t) \sim \frac{-im_1^{-1/2}}{t - iK'} \quad \text{as} \quad t \rightarrow iK'. \]

Thus

\[ \text{Residue}(S(t; W^*); t = iK') = \frac{1}{mW^*}e^{-K'/W^*}. \]

Therefore

\[ I_s = \frac{\pi}{2mW^*} \text{sech} \frac{K'}{W^*} - \frac{m_1^{1/2}}{1 + e^{-2K'/W^*}} \int_0^{2K'} e^{-t/W^*} \text{nd}(t|m_1) \text{sd}(t|m_1) \, dt \sin T_0. \]

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Through successive integrations by parts we find that the integral in this expression has the asymptotic series

\[
\int_0^{2K'} e^{-t/W^*} nd(t) \, dt = \int_0^{2K'} e^{-t/W^*} \left( -\frac{1}{m} \frac{d}{dt} cd(t) \right) \, dt \\
\sim -\frac{1}{m} \left( 1 + e^{-2K'W^*} \right) \sum_{n=1}^{\infty} W^{*2n} cd^{(2n)}(0).
\]

Thus we may write

\[
I_s \sim \frac{\pi}{2mW^*} \sech \frac{K'}{W^*} + \frac{m_1^{1/2}}{m} \sum_{n=1}^{\infty} W^{*2n} cd^{(2n)}(0) \sin T_0.
\]

Now we note that the integral \(I_c\) has the asymptotic series given by

\[
I_c = \int_0^K \cos \frac{t}{W^*} \sn(t) \cn(t) \, dt \\
= \int_0^K \cos \frac{t}{W^*} \left( -\frac{1}{m} \frac{d}{dt} dn(t) \right) \, dt \\
\sim -\frac{1}{m} \sum_{n=1}^{\infty} W^{*2n} (-1)^{n-1} \left( dn^{(2n)}(K) \cos T_0 - dn^{(2n)}(0) \right)
\]

and that

\[dn^{(2n)}(K) = m_1^{1/2} nd^{(2n)}(0) = (-1)^n m_1^{1/2} cd^{(2n)}(0)\]

because \(nd(it|m) = cd(t|m_1)\). Therefore, by substituting the expressions we have calculated for \(I_s\) and \(I_c\) into (34), we find

\[
I \sim m W^* \left[ \left( \frac{m_1^{1/2}}{m} \sum_{n=1}^{\infty} W^{*2n} cd^{(2n)}(0) \cos T_0 - \frac{1}{m} \sum_{n=1}^{\infty} W^{*2n} dc^{(2n)}(0) \right) \sin T_0 \right. \\
- \left. \left[ \frac{\pi}{2mW^*} \sech \frac{K'}{W^*} + \frac{m_1^{1/2}}{m} \sum_{n=1}^{\infty} W^{*2n} cd^{(2n)}(0) \sin T_0 \right] \cos T_0 \right] \\
\sim W^* \left[ \left( -\frac{1}{m} \sum_{n=1}^{\infty} W^{*2n} dc^{(2n)}(0) \right) \sin T_0 - \frac{\pi}{2mW^*} \sech \frac{K'}{W^*} \cos T_0 \right].
\]

This is the expression we desire for \(I\). The sum that appears is divergent, because the Taylor expansion

\[
dc(W^*) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} W^{*2n} dc^{(2n)}(0)
\]
has a finite radius of convergence (equal to $K$). It follows now from (14) that

$$Z_1^* = -I/\sin T_0$$

$$= \sum_{n=1}^{\infty} W^{*2n+1} dc^{(2n)}(0) + \frac{\pi}{2} \text{sech} \frac{K'}{W^*} \cot T_0.$$

References

[1] M. Abramowitz and I.A. Stegun, editors. *Handbook of mathematical functions*. Dover, 1965. 31

[2] V.I. Arnold. *Geometrical methods in the theory of ordinary differential equations*. Springer, 1982. 17, 28

[3] F. Baer and J.J. Tribbia. On complete filtering of gravity modes through nonlinear initialization. *Mon. Wea. Rev.*, 105:1536–1539, 1977. 4, 7, 8, 10, 27

[4] C.M. Bender and S.A. Orszag. *Advanced mathematical methods for scientists and engineers*. McGraw-Hill, 1981. 10, 29

[5] R. Camassa. On the geometry of an atmospheric slow manifold. *Physica D*, 84:357–397, 1995. 3, 4

[6] J. Carr. *Applications of centre manifold theory*, volume 35 of *Applied Math Sci*. Springer-Verlag, 1981. 6, 15, 24

[7] S.M. Cox and A.J. Roberts. Centre manifolds of forced dynamical systems. *J. Austral. Math. Soc. B*, 32:401–436, 1991. 9, 10, 13, 25

[8] A. Cuyt. Padé approximants for operators: theory and application. *Lect. Notes Mathematics*, 1065, 1984. 10

[9] R. Daley. On the optimal specification of the initial state for deterministic forecasting. *Mon. Wea. Rev.*, 108:1719–1735, 1980. 4, 28

[10] R. Daley. Normal mode initialization. *Rev. Geophys. Space Phys.*, 19:450–468, 1981. 3, 4, 28

[11] R.M. Errico. Theory and application of nonlinear normal mode initialization. Technical report, NCAR note TN–344+IA, 1991. 3

[12] A.E. Gill. *Atmosphere-Ocean Dynamics*. Academic Press, 1982. 3, 27

AJ Roberts, December 4, 2021
[13] H. Grad. Asymptotic theory of the Boltzmann equation. *Phys. Fluids*, 6:147–181, 1963.

[14] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer-Verlag, 1983.

[15] F. Haake and M. Lewenstein. Adiabatic drag and initial slip in random processes. *Phys. Rev. A*, 28:3060–3612, 1983.

[16] J.T. Houghton. *The physics of atmospheres*. CUP, 2nd edition, 1989.

[17] S.J. Jacobs. Existence of a slow manifold on a model system of equations. *J. Atmos. Sci.*, 48:893–901, 1991.

[18] A. Kelley. On the Liapunov subcenter manifold. *J. Math. Anal. Appl.*, 18:472–478, 1967.

[19] C.E. Leith. Nonlinear normal mode initialisation and quasi-geostrophic theory. *J. Atmos. Sci.*, 37:958–968, 1980.

[20] E. Lorenz and Krishnamurty. On the non-existence of a slow manifold. *J. Atmos. Sci.*, 44:2940–2950, 1987.

[21] E.N. Lorenz. Attractor sets and quasi-geostrophic equilibrium. *J. Atmos. Sci.*, 37:1685–1699, 1980.

[22] E.N. Lorenz. On the existence of a slow manifold. *J. Atmos. Sci.*, 43:1547–1557, 1986.

[23] G.N. Mercer and A.J. Roberts. A centre manifold description of contaminant dispersion in channels with varying flow properties. *SIAM J. Appl. Math.*, 50:1547–1565, 1990.

[24] G.N. Mercer and A.J. Roberts. A complete model of shear dispersion in pipes. *Jap. J. Indust. Appl. Math.*, 11:499–521, 1994.

[25] A.J. Roberts. Simple examples of the derivation of amplitude equations for systems of equations possessing bifurcations. *J. Austral. Math. Soc. B*, 27:48–65, 1985.

[26] A.J. Roberts. Appropriate initial conditions for asymptotic descriptions of the long term evolution of dynamical systems. *J. Austral. Math. Soc. B*, 31:48–75, 1989.
A.J. Roberts. The utility of an invariant manifold description of the evolution of a dynamical system. *SIAM J. Math. Anal.*, 20:1447–1458, 1989. 11, 28

A.J. Roberts. The invariant manifold of beam deformations. part 1: the simple circular rod. *J. Elas.*, 30:1–54, 1993. 4, 29

J. Sijbrand. Properties of centre manifolds. *Trans. Amer. Math. Soc.*, 289:431–469, 1985. 6, 15, 28

N.G. van Kampen. Elimination of fast variables. *Physics Reports*, 124:69–160, 1985. 4

M. Vandyke. Computer-extended series. *Annu. Rev. Fluid Mech.*, 16:287–310, 1984. 10

R. Vautard and B. Legras. Invariant manifolds, quasi-geostrophy and initialisation. *J. Atmos. Sci.*, 43:565–584, 1986. 7, 10, 27

T. Warn and R. Menard. Nonlinear balance and gravity-inertial saturation in a simple atmospheric model. *Tellus*, 38A:285–294, 1986. 29

S.D. Watt, A.J. Roberts, and R.O. Weber. Dimensional reduction of a bushfire model. *Mathl. Comput. Modelling*, 21(9):79–83, 1995. 4