Phase Transitions in Ferromagnetic Ising Models with spatially dependent magnetic fields

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Abstract

In this paper we study the nearest neighbor Ising model with ferromagnetic interactions in the presence of a space dependent magnetic field which vanishes as $|x|^{-\alpha}$, $\alpha > 0$, as $|x| \to \infty$. We prove that in dimensions $d \geq 2$ for all $\beta$ large enough if $\alpha > 1$ there is a phase transition while if $\alpha < 1$ there is a unique DLR state.

1 Introduction

The Ising Model is one of the most studied subjects in Statistical Physics and will complete a century in a few years.\footnote{Wilhelm Lenz introduced the model in 1920.} The literature about ferromagnetic Ising models on $\mathbb{Z}^d$, $d \geq 2$, is mainly focused on cases where the external magnetic field is constant. We will study ferromagnetic nearest neighbor hamiltonians of the form

$$H^w_\Lambda(\sigma) = -J \sum_{|x-y|=1, x,y\in\Lambda} \sigma(x)\sigma(y) - \sum_{x\in\Lambda} h(x)\sigma(x) - J \sum_{|x-y|=1, x\in\Lambda, y\notin\Lambda} \sigma(x)w(y)$$  \hspace{1cm} (1)

where $\Lambda$ is any finite subset of $\mathbb{Z}^d$, $\sigma \in \{-1,1\}^\Lambda$ is a spin configuration in $\Lambda$, $w \in \{-1,1\}^{\Lambda^c}$ a boundary condition and $J > 0$ the interaction strength.
When the magnetic field \( h(\cdot) \) is constant, that is \( h(x) = h \) for all \( x \in \mathbb{Z}^d \) and \( h = 0 \), then the classical Peierls’ argument guarantees the existence of a phase transition. If instead \( h \neq 0 \) at all temperatures there is a unique DRL measure, as it follows from the Lee-Yang Theory and GHS inequalities. The absence of phase transitions comes from the differentiability of the free energy with respect to the parameter \( h \).

Alternating signs fields on the lattice \( \mathbb{Z}^2 \) are considered in [14], constant fields on semi-infinite lattices are studied in [2, 11]. The magnetic field in all these models has some spatial symmetry. The challenging case of i.i.d. random magnetic fields on \( \mathbb{Z}^d \) with zero mean has been studied in [1, 4, 6, 7, 8] and the case with positive mean in [10]. Some deterministic and not spatially symmetric fields have been considered in [3].

In this paper we study the hamiltonian (1) in \( \mathbb{Z}^d \), \( d \geq 2 \), with a non negative, space dependent magnetic field \( h(\cdot) \) of the form

\[
h(x) = \frac{h^*}{|x|^\alpha}, \quad \alpha > 0, h^* > 0
\]  

where if \( x = (x_1, \ldots, x_d) \) then \( |x| = \sum_{i=1}^{d} |x_i| \). Calling \( Z^w_{\beta, h(\cdot), \Lambda} \) the corresponding partition function one can easily check that (along van Hove sequences)

\[
\lim_{\Lambda \to \mathbb{Z}^d} \frac{\log Z^w_{\beta, h(\cdot), \Lambda}}{\beta |\Lambda|} = p_\beta
\]

independently of the boundary conditions \( w \). The limit \( p_\beta \) is equal to the thermodynamic pressure without magnetic fields (i.e. \( h^* = 0 \)). This indicates that the presence of \( h(\cdot) \) does not change the thermodynamics thus suggesting that a phase transition may occur for \( \beta \) large, just as when the magnetic field is absent. However surface effects are relevant in the analysis of phase transitions and indeed we shall prove in Theorem 5 that when \( \alpha < 1 \) there is a unique DLR measure, while when \( \alpha > 1 \) there is a phase transition for \( \beta \) large enough, see Theorem 1.

The existence of phase transitions at \( \alpha > 1 \) is based on the validity of the Peierls bounds for contours. The proof of uniqueness when \( \alpha < 1 \) at low temperatures is more involved and it is based on an iterative scheme introduced in [5]. For \( \alpha = 1 \) we have partial results but not a complete characterization.

## 2 Existence of phase transitions

In this section we shall prove:

**Theorem 1.** Let \( h(\cdot) \) be as in (2) with \( \alpha > 1 \). Then for \( \beta \) large enough there is a phase transition, namely the plus and minus Gibbs measures \( \mu^\pm_{\beta, h(\cdot), \Lambda} \) converge weakly as \( \Lambda \to \mathbb{Z}^d \) to mutually distinct DLR measures.

As we shall see the result extends to \( \alpha = 1 \) under the additional assumption that \( h^* \) is small enough and to non negative magnetic fields which are “local perturbations” of (2) (by this we mean that the \( L^1 \) norm of the difference is finite). We shall prove the theorem using the Peierls’ argument which holds for magnetic fields which satisfy (3) below.
Lemma 2. Let \( h(\cdot) \) be a non negative magnetic field such that

\[
J|\partial \Delta| > 2 \sum_{x \in \Delta} h(x)
\]

for all finite regions \( \Delta \subset \mathbb{Z}^d \). (\( \partial \Delta \) the bonds from \( \Delta \) to \( \Delta^c \)). Then for all \( \beta \) large enough there is a phase transition.

Proof. We shall use (3) to prove the validity of the Peierls bounds. Then, by standard arguments, the weak limits of Gibbs measures with plus and minus boundary conditions are DLR measures \( \mu_{\beta,h}^\pm \) with disjoint supports (i.e. they are mutually singular). We thus have a phase transition and the lemma will be proved.

Proof of the Peierls bounds. Let \( \gamma \) be a contour and \( I(\gamma) \) the interior of \( \gamma \), i.e. the points which are connected to \( \infty \) only via paths which cross \( \gamma \). Suppose \( \gamma \) is a minus contour and call \( \partial \gamma \) the sites in \( I(\gamma) \) which are connected to \( I(\gamma)^c \). Denote by \( \mathcal{Z}_{I(\gamma),h(\cdot)}(\sigma_{I(\gamma)}(x) = 1, x \in \partial \gamma) \) the partition function in \( I(\gamma) \) with magnetic field \( h(\cdot) \), minus boundary conditions and with the constraint that \( \sigma_{I(\gamma)}(x) = 1 \) for all \( x \in \partial \gamma \), i.e. the sites in \( I(\gamma) \) connected to \( I(\gamma)^c \). Then

\[
\mathcal{Z}_{I(\gamma),h(\cdot)}^-(\sigma_{I(\gamma)}(x) = 1, x \in \partial \gamma) \leq e^{\beta \sum_{x \in I(\gamma)} h x} \mathcal{Z}_{I(\gamma),h(\cdot); h \equiv 0}^-(\sigma_{I(\gamma)}(x) = 1, x \in \partial \gamma) \\
\leq e^{-2\beta J|\partial \gamma|} e^{\beta \sum_{x \in I(\gamma)} h x} \mathcal{Z}_{I(\gamma),h(\cdot); h \equiv 0}^- (\sigma_{I(\gamma)}(x) = -1, x \in \partial \gamma) \\
\leq e^{-2\beta J|\partial \gamma|} e^{\beta \sum_{x \in I(\gamma)} h x} \mathcal{Z}_{I(\gamma),h(\cdot)}^-(\sigma_{I(\gamma)}(x) = -1, x \in \partial \gamma).
\]

Thus by (3) the weight of the contour \( \gamma \) is bounded by

\[
\frac{\mathcal{Z}_{I(\gamma),h(\cdot)}^-(\sigma_{I(\gamma)}(x) = 1, x \in \partial \gamma)}{\mathcal{Z}_{I(\gamma),h(\cdot)}^-(\sigma_{I(\gamma)}(x) = -1, x \in \partial \gamma)} \leq e^{-\beta J|\partial \gamma|}. \tag{4}
\]

Same bound hold for the plus contours hence the Peierls bounds are proved. \( \square \)

The proof of Theorem 1 will be obtained by reducing to magnetic fields for which (3) is satisfied, a task that will be achieved via a few lemmas where we shall extensively use the Isoperimetric Inequality: for any finite \( \Delta \subset \mathbb{Z}^d \) (\( d \geq 2 \))

\[
|\Delta|^\frac{d-1}{d} \leq \frac{|\partial \Delta|}{2d}.
\]

Lemma 3. Let \( h(\cdot) \) be as in (2) with \( \alpha > 1 \). Then there is \( C \equiv C(h^*, \alpha, d, J) > 0 \) so that (3) holds for all finite regions \( \Delta \) such that \( |\Delta| > C \).

Proof. Since \( h(x) \) is a decreasing function of \( |x| \) for \( |x| > 0 \), calling \( B(0,R) := \{ x : |x| \leq R \} \) we have

\[
\sum_{x \in \Delta} h(x) \leq \sum_{x \in B(0,R)} h(x), \quad \text{for } R \text{ such that } |B(0,R)| \geq |\Delta| + 1
\]

(+1 because \( h(0) = 0 \)). We claim that the condition \( |B(0,R)| \geq |\Delta| + 1 \) is satisfied if

\[
R = \text{smallest integer } \geq c|\partial \Delta|^{\frac{1}{d-1}} \tag{5}
\]
with $c$ large enough. In fact, recalling that $|\partial B(0, n)| = 2d \cdot n^{d-1}$, we have $|B(0, R)| \geq aR^d$, $a > 0$ small enough, hence using the isoperimetric inequality
\[ |B(0, R)| \geq aR^d \geq ac^d |\partial B(0, n)| \geq ac^d (2d) \frac{d}{d-1} |\Delta| \geq |\Delta| + 1 \]
for $c$ large enough.

Thus the lemma will be proved once we show that
\[
\lim_{R \to \infty} \frac{1}{R^{d-1}} \sum_{|x| \leq R} h(x) = 0.
\]
Recalling that $|\partial B(0, n)| = 2d \cdot n^{d-1}$ this is implied by
\[
\lim_{R \to \infty} \sum_{n=1}^{R} n^{d-1} \frac{1}{R^{d-1} n^\alpha} = 0
\]
whose validity follows from the Lebesgue dominated convergence theorem. The lemma is thus proved.

Observe that when $\alpha = 1$ and $h^*$ is small enough then (3) holds again for all finite regions $\Delta$ large enough. The proof is analogous except at the end as we only have
\[
\limsup_{R \to \infty} \frac{1}{R^{d-1}} \sum_{|x| \leq R} \frac{1}{|x|^\alpha} \leq c
\]

**Lemma 4.** Let $h(\cdot)$ be as in (2) with $\alpha > 1$, then there is $R$ so that (3) holds for all finite $\Delta$ when the magnetic field is $h$:
\[
\hat{h}(x) = \begin{cases} 0 & \text{if } |x| \leq R \\ h(x) & \text{if } |x| > R \end{cases}
\]

**Proof.** Suppose $|\Delta| > C$, $C$ the constant in Lemma 3 then
\[
2 \sum_{x \in \Delta} \hat{h}(x) \leq 2 \sum_{x \in \Delta} h(x) \leq J|\partial \Delta|
\]
Suppose next $|\Delta| \leq C$, then by the Isoperimetric Inequality,
\[
\sum_{x \in \Delta} \hat{h}(x) = \sum_{x \in \Delta; |x| > R} \hat{h}(x)
\]
\[
\leq \frac{h^* |\Delta|}{R^\alpha} \leq \frac{h^* |\partial \Delta|}{R^\alpha (2d) \pi^{d-1}} \leq \frac{h^* C \frac{1}{\pi^{d-1}} |\partial \Delta|}{R^\alpha (2d) \pi^{d-1}}
\]
which is $\leq J|\partial \Delta|$ for $R$ sufficiently large.

**Proof of Theorem 1.** Let $h(\cdot)$ be as in (2) with $\alpha > 1$. By Lemma 2 and 4 for $\beta$ large enough there is a phase transition for the system with magnetic field $\hat{h}(\cdot)$, let
the corresponding DLR measures obtained as limit of the Gibbs measures with plus respectively minus boundary conditions. Call \( \phi(x) := h(x) - \hat{h}(x) = 1_{|x|<R}h(x) \) and define the probability measures
\[
d\nu_{\pm}^{\beta, h(\cdot)}(\sigma) := C_{\pm} e^{\beta \sum \phi(x) \sigma(x)} d\mu_{\pm, \hat{h}(\cdot)}(\sigma)
\]
\((C_{\pm} the normalization constants). They are DLR measures with magnetic field \( h(\cdot) \) and they are absolutely continuous w.r.t. \( \mu_{\pm, \hat{h}(\cdot)} \). Hence they also have disjoint supports and are therefore distinct. Theorem 1 is proved.

3 Restricted ensembles and contour partition functions

We fix hereafter
\[
h(x) = \frac{h^*}{|x|^\alpha}, \quad x \neq 0, \quad h^* > 0, \quad \alpha \in (0, 1)
\]
and we shall prove that

**Theorem 5.** Let \( h(\cdot) \) as in 7, then for any \( \beta \) large enough there is a unique DLR measure.

In this section we shall prove some crucial estimates which will be used in the next section to prove Theorem 5 but which have an interest in their own right. Observe that when \( h(\cdot) \) is given by (7) the condition (3) may fail for some \( \Delta \) for instance a large ball centered at the origin.

With this in mind we classify the contours \( \gamma \) by saying that \( \gamma \) is “slim” if
\[
J|\gamma| > 2 \sum_{x \in I(\gamma)} h(x)
\]
where \( I(\gamma) \) is the interior of \( \gamma \), namely the union of all sites \( x \) such that if a path connects \( x \) to infinity then necessarily it crosses \( \gamma \). We call “fat” the contours which do not satisfy (8). Following Pirogov-Sinai we then introduce plus-minus restricted ensembles where spin configurations are restricted in such a way that there are only slim contours. We thus define for any bounded region \( \Lambda \) the plus-minus restricted partition functions
\[
Z_{\Lambda}^{\pm, \text{slim}} := \sum_{\sigma_\Lambda: \text{all contours are slim}} e^{-\beta H(\sigma_\Lambda |\pm 1_{\Lambda^c})}.
\]

Obviously the pressures in the plus and minus ensembles are equal but the Pirogov-Sinai theory requires for the existence of a phase transition finer conditions on the finite volume corrections to the pressure namely that the latter differs from the limit pressure by a surface term. In our case the correction is larger than a surface term because \( \alpha < 1 \) as shown by the following:

**Theorem 6.** For any \( \beta \) large enough there are positive constants \( c_1 \) and \( c_2 \) so that
\[
Z_{\Lambda}^{\pm, \text{slim}} \leq c_1 e^{-\beta c_2 \sum_{x \in \Lambda} h(x)} Z_{\Lambda}^{\pm, \text{slim}}.
\]
Proof. By repeating the proof of Theorem 1 and denoting by $E_{-\text{slim}}^{-\Lambda}$ the expectation w.r.t. the Gibbs measure in the minus restricted ensemble, we have for any $x \in \Lambda$:

$$E_{-\text{slim}}^{-\Lambda}(\sigma(x)) \leq -1 + \sum_{\gamma: J(\gamma) \geq 0} e^{-\beta J(\gamma)} = -m^*, \ m^* > 0 \quad (11)$$

for $\beta$ large enough. Then

$$\mu_{-\text{slim}, \beta}(\sum_{x \in \Lambda} h(x) \sigma(x) \leq -m^*) \geq \frac{m^*}{2 - m^*} \quad (12)$$

To prove (12) let $X$ be a random variable with values in $[-1, 1]$ and $P$ its law. Suppose that $E(X) \leq -m^*$ and call $p := P[X \leq -m^*/2]$, then

$$-m^* \geq -1(1 - p) - \frac{m^*}{2}p, \ (1 - \frac{m^*}{2})p \leq (1 - m^*)$$

hence (12).

Calling $Z_{-\text{slim}}^{-\Lambda}(A)$ the partition function with the constraint $A$, we can rewrite (12) as:

$$Z_{-\text{slim}}^{-\Lambda} \leq \frac{2 - m^*}{m^*}Z_{-\text{slim}}^{-\Lambda} \left( \sum_{x \in \Lambda} h(x) \sigma(x) \leq -m^* \right)$$

$$Z_{-\text{slim}}^{-\Lambda} \leq \frac{2 - m^*}{m^*}e^{-\beta \sum_{x \in \Lambda} h(x)} Z_{-\text{slim}}^{-\Lambda, \beta, \Lambda}$$

$$= \frac{2 - m^*}{m^*}e^{-\beta \sum_{x \in \Lambda} h(x)} Z_{-\text{slim}}^{+, \Lambda}$$

By repeating the previous argument we get

$$Z_{+, \text{slim}}^{+, \Lambda} \leq \frac{2 - m^*}{m^*}e^{-\beta \sum_{x \in \Lambda} h(x)} Z_{+, \text{slim}}^{+, \Lambda}$$

where $Z_{+, \text{slim}}^{+, \Lambda}$ is the partition function with the contribution of the magnetic field $h(\cdot)$. This concludes the proof of the theorem. \hfill \square

In the next section we shall use a corollary of Theorem 6 that we state after introducing some notation. The geometry is as follows:

$\Lambda$ is a cube with center the origin, $\Delta$ a subset of $\Lambda$ and $K$ a subset of $\Delta$ which is union of disjoint connected set $K_i$ where for each $i$ the complement $\bar{K}_i$ of $K_i$ has a unique maximally connected component (i.e. there are no “holes” in $K_i$). We also suppose that each $K_i$ is fat and that $\delta_{\text{out}} K \subset \Delta$ where: given a set $A$ we denote by $\delta_{\text{out}} A$ the set of all $x \in \bar{A}$ which are connected to $A$ and by $\delta_{\text{in}} A$ the set of all $x \in A$ which are connected to $\bar{A}$.

With $\Lambda$, $\Delta$ and $K$ as above we denote by $\mathcal{X}_{\Lambda, \Delta, K}$ the set of all configuration $\sigma_{\Lambda}$ which have the following properties.

- $\sigma_{\Lambda} = -1$ on $\delta_{\text{out}} \Delta$, $\sigma_{\Lambda} = -1$ on $D^- \subset \delta_{\text{out}} \Delta$ and $\sigma_{\Lambda} = +1$ on $D^+ \subset \delta_{\text{out}} \Delta \setminus D^-$. 
- $\sigma_{\Lambda} = -1$ on $\delta_{\text{out}} K$ and $\sigma_{\Lambda} = +1$ on $\delta_{\text{in}} K$.

We denote by $Z_{\Lambda}^\omega(\mathcal{X}_{\Lambda, \Delta, K})$ the partition function in $\Lambda$ with constraint $\mathcal{X}_{\Lambda, \Delta, K}$ and boundary conditions $\omega$. Then:
Corollary 1. Under the same assumptions of Theorem 4

\[ Z^\omega(\Lambda, \Delta, K) \leq c_1 e^{-\beta c_2 \sum_{x \in \Delta \setminus K} h(x)} e^{-2\beta J|\delta_n(K)|} e^{-2\beta J|\delta_{\text{out}}(\Delta)|+4\beta J|D_{-}|} Z^\omega_\Lambda \]  

(13)

In the applications of the next section the connected components of \( \Delta \) should intersect some given set and this will enable to control the sum over \( \Delta \) via the bound \( e^{-2\beta J|\delta_{\text{out}}(\Delta)|} \). The sum over \( K \) is instead controlled as follows. We introduce the fat-contours partition function on the whole \( \mathbb{Z}^d \) as

\[ Z^{\text{fat}} := \sum_{n=0}^{\infty} \sum_{\gamma_1, \ldots, \gamma_n} e^{-\beta J \sum |\gamma_i|} \]  

(14)

where the sum * refers to a sum over only fat contours such that \( I(\gamma_i) \cap I(\gamma_j) = \emptyset \) for all \( i \neq j \).

**Theorem 7.** For any \( \beta \) large enough there is a positive constant \( c_3 \) so that

\[ Z^{\text{fat}} \leq c_3 \]  

(15)

**Proof.** We order the points of \( \mathbb{Z}^d \) in a way which respects the distance from the origin and given a contour \( \gamma \) we denote by \( X(\gamma) \) the minimal point in \( \gamma \) with the given order. By the definition of fat contours and supposing \( X(\gamma) \neq 0 \),

\[ J|\gamma| \leq 2 \sum_{x \in I(\gamma)} h(x) \leq \frac{2h^*}{|X(\gamma)|^\alpha |I(\gamma)|} \leq \frac{2h^* C_p}{|X(\gamma)|^\alpha |\gamma|^\frac{d-1}{d}} \]

where \( C_p \) is the iso-perimetric constant. Hence

\[ |\gamma| \geq \left( \frac{J}{2C_p} \right)^{d-1} |X(\gamma)|^{\alpha(d-1)}, \quad X(\gamma) \neq 0 \]  

(16)

We write

\[ Z^{\text{fat}} = \sum_n \sum_{x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_n} \prod_{i=1}^n 1_{X(\gamma_i) = x_i} e^{-\beta J|\gamma_i|} \]

\[ \leq \prod_{x \in \mathbb{Z}^d} \left( 1 + \sum_{\gamma \text{ fat}: X(\gamma) = x} e^{-\beta J|\gamma|} \right) \]

\[ = (1 + \sum_{\gamma \text{ fat}: X(\gamma) = 0} e^{-\beta J|\gamma|}) \prod_{x \neq 0} \left( 1 + \sum_{\gamma \text{ fat}: X(\gamma) = x} e^{-\beta J|\gamma|} \right) \]

which using (10) proves (15). \( \square \)

Before moving to the next section with the proof of Theorem 5 we point out that by the Dobrushin’s Uniqueness Theorem there is a unique DRL state also at high temperatures and since the system is ferromagnetic, uniqueness may be expected to hold at all temperatures. However the proof of such a statement when the external field is zero do not seem to extend easily to our case, see [9] and [13].
4 Uniqueness at low temperatures

In this section we prove Theorem 5. For any positive integer $n$ we denote by $\Lambda_n$ the cube with center the origin and side $2n + 1$. We fix a positive integer $L$, eventually $L \to \infty$, and arbitrarily the spins outside $\Lambda_L$, denoting by $\mu_L$ the Gibbs measure on $\{-1,1\}^{\Lambda_L}$ with the given boundary conditions and external magnetic field as in (7).

Definitions.

- Given $\sigma_{\Lambda_L}$, $\Delta \subset \Lambda_L$, $B : B \cap \Delta = \emptyset$ and $x \in X$ we say that $x$ is connected to $B$ in $\Delta$ if there is $X \subset \Delta$ such that: $x \in \Delta$, $X$ is connected to $B$ and $\sigma_{\Lambda_L} \equiv -1$ on $X$.

- $C_L$ denotes the random set of sites $x \in \Lambda_L$ which are connected to $\Lambda_L + 1 \setminus \Lambda_L$.

- If $C$ is a set we denote by $\delta_{\text{out}}(C)$ the sites in the complement $\bar{C}$ of $C$ which are connected to $C$.

Suppose $C_L = C$ then the spins in $\delta_{\text{out}}(C) \cap \Lambda_L$ are all equal to $+1$. Moreover if we change the configuration $\sigma_{\Lambda_L}$ leaving unchanged the spins in $C \cup \delta_{\text{out}}(C)$ we still have $C_L = C$. Thus the spins in $\Lambda_L \setminus (C \cup \delta_{\text{out}}(C))$ are distributed with Gibbs measure with plus boundary conditions. We shall prove that there exists $b^* < 1$ so that

$$\lim_{L \to \infty} \mu_L \left[ C_L \cap \Lambda_L (1 - b^*) = \emptyset \right] = 1$$

which then proves that $\mu_L$ converges weakly to the weak limit of Gibbs measures with plus boundary conditions. By standard arguments this yields Theorem 5 so that we are reduced to the proof of (17). The proof of (17) uses an iterative argument introduced in [5].

**Definition.** Given $k \leq L$ and $M \subset \Lambda_{k+1} \setminus \Lambda_k$ we define $C_{k,M}(\sigma_{\Lambda_L})$ as the set of all $x \in \Lambda_k$ which are connected to $M$ in $\Lambda_k$. In particular $C_{L,M} = C_L$ if $M = \Lambda_{L+1} \setminus \Lambda_L$.

It readily follows from the definitions that for $k < L$:

$$C_L \cap \Lambda_k = C_{k,M} \text{ if } M = M_k = C_L \cap (\Lambda_{k+1} \setminus \Lambda_k).$$

The heuristic idea of the proof of Theorem 5 goes as follows. Suppose that $M_k = L^a$, $a > 0$, $k_0$ a fraction of $L$. Let $0 < a' < a$, fix a constant $b < 1$ suitably small and distinguish two cases:

$$|M_k| \leq L^{a'} \quad \text{for some } k \in [k_0 - bL, k_0)$$

and the complement where

$$|M_k| > L^{a'} \quad \text{for all } k \in [k_0 - bL, k_0)$$

In the latter set $C_{k_0}$ has cardinality larger than $bLL^{a'}$. By using Corollary 1 we gain by changing the minuses into pluses by a term due to the interaction with the pluses around $C_{k_0}$ (which will be used to control entropy), but we loose with the interaction with the spins in $M_{k_0}$ which are minuses. This is proportional to $L^a$ and should be balanced by the gain due to the magnetic field which is proportional to $bLL^{a'}L^{-\alpha}$. Thus if

$$L^{1+a'-\alpha} > L^a$$
with probability going to 1 as \( L \to \infty \) we can reduce to the case \( |\mathcal{M}_k| \leq L^{a'} \) for some \( k \in [k_0 - bL, k_0) \). We can satisfy the previous inequality with \( a' = a - \frac{1}{2} \) and then iterate the argument to prove that after finitely many steps we get for some \( k, \mathcal{M}_k = \emptyset \) and thus conclude the proof.

With this in mind we introduce the sequence \( a_n, n \geq 0 \), by setting

\[
a_0 = d - 1, \quad a_{n+1} = a_n - \frac{1 - \alpha}{2}
\]

and call \( n^* \) the largest integer such that \( a_{n^*} \geq 0 \). Let \( s_0 = L \) and for \( 1 \leq n \leq n^* \) let

\[
s_n \quad \text{the largest } k \text{ smaller than } s_{n-1} \text{ such that } |\mathcal{M}_k| \leq L^{a_n}
\]

setting \( s_n = 0 \) if either \( s_{n-1} = 0 \) or \( k \) in (20) does not exist. We then define \( s_{n^*+1} \) as

\[
s_{n^*+1} \quad \text{is the largest } k \text{ smaller than } s_{n^*} \text{ such that } |\mathcal{M}_k| = 0
\]

Let \( b > 0 \) be such that

\[
bn^* < \frac{1}{100}
\]

Then \( \mathcal{C}_L \cap \Lambda_{L^{1-b^*}} = \emptyset \) in the set

\[
\mathcal{G} := \bigcap_{1 \leq n \leq n^*+1} \{ s_{n-1} - s_n \leq bL \}
\]

provided \( b^* > 1/2 \) so that (17) will follow once we prove that

\[
\lim_{L \to \infty} \mu_L[\mathcal{G}] = 1.
\]

We shall prove that for any \( 1 \leq p \leq n^* + 1 \)

\[
\lim_{L \to \infty} \mu_L[s_{p+1} < s_p - bL ; s_p \geq L - pbL] = 0
\]

which yields (24).

We write \( \mu_L[s_p < s_{p-1} - bL ; s_{p-1} \geq L - (p-1)bL] \) as the ratio of two partition functions and in the sequel we study the partition function in the numerator, that we call simply \( Z \). We have

\[
Z \leq \sum_{L \geq k \geq L-pbL} \sum_{M \in \Lambda_{k+1} \setminus \Lambda_k, |M| \leq L^{a_p}} Z_{\Lambda_L}(\mathcal{C}_{k,M} \geq bL^{1+a_{p+1}})
\]

\( \mathcal{C}_{k,M} \) can be decomposed into maximally connected components, each one of them is a connected set whose complement has an unbounded maximally connected components and maybe several maximally connected finite components. The latter are distinguished into fat and slim and we call \( \bar{\mathcal{C}}_{k,M}^{\text{fat}} \) and \( \bar{\mathcal{C}}_{k,M}^{\text{slim}} \) the union of all the fat, respectively slim ones. We then have

\[
Z_{\Lambda_L}(\mathcal{C}_{k,M} \geq bL^{1+a_{p+1}}) \leq \sum_{\Delta, K: K \subset \Delta, |\Delta \setminus K| \geq bL^{1+a_{p+1}}} \sum_{\mathcal{C}_{k,M}^{\text{fat}} = K} Z_{\Lambda_L}(\mathcal{C}_{k,M} \geq bL^{1+a_{p+1}})
\]

\( \mathcal{C}_{k,M} \cup K \cup \bar{\mathcal{C}}_{k,M}^{\text{slim}} = \Delta \).
where the * sum means that $\Delta$ should be in the range of $C_{k,M} \cup \bar{C}_{fat} \cup \bar{C}_{slim}$ and $K$ in the range of $\bar{C}_{fat}^k$.

We are now in setup of Corollary 1 and Theorem 7 which yield

$$Z_{tL} \left( |C_{k,M}| \geq bL^{1+\alpha p+1} \right) \leq \sum_{\Delta}^* c_1 e^{-\beta c_2 bL^{1+\alpha p+1}h^*L^{-\alpha}} e^{-2\beta J|\delta_{out}(\Delta)|+4\beta J|M|} c_3 Z_{tL}$$

(28)

We then get from (26)

$$Z_{tL} \leq c_1 c_3 e^{-\beta c_2 bL^{1+\alpha p+1}} \sum_{L \geq k \geq L-pbL} M \in \Lambda_{k+1} \setminus \Lambda_k, |M| \leq L^{\alpha p} \sum_{\Delta}^* e^{-2\beta J|\delta_{out}(\Delta)|}$$

(29)

Since the sum is over $\Delta$ which are in the range of $C_{k,M} \cup \bar{C}_{fat} \cup \bar{C}_{slim}$, $\Delta$ is the union of a finite number of disjoint connected sets (without “holes”, see Section 3), say $\Delta_1, .., \Delta_n$, such that $\delta_{out}(\Delta_i)$ is a * connected set which intersects $M$. Thus $n \leq |M|$ and we can bound the * sum over $\Delta$ by summing over $n \leq |M|$ disjoint * connected sets which intersect $M$. Hence

$$\sum_{\Delta}^* e^{-2\beta J|\delta_{out}(\Delta)|} \leq \sum_{n=1}^{|M|} \frac{M!}{n!(M-n)!} e^{-\beta c_4 n} \leq \left(1 + e^{-\beta c_4}\right)^{|M|}$$

(30)

where $c_4$ is such that

$$e^{-\beta c_4} \geq \sum_{D \geq 0, D*connected} e^{-2\beta J|D|}$$

(31) holds for $\beta$ large enough, see for instance Lemma 3.1.2.4 in [5]. Then recalling (29)

$$\frac{Z}{Z_{tL}} \leq c_1 c_3 e^{-\beta c_2 bL^{1+\alpha p+1} - 4\beta J L^{\alpha p}} \left(1 + e^{-\beta c_4}\right)^{L^{\alpha p}} L e^{c_5 L^{\alpha p} \log L}$$

which recalling the definition of $a_n$ proves that

$$\mu_L \left[ s_{p+1} < s_p - bL \right] \leq c_6 e^{-\beta c_2 bL^{1+\alpha p+1} - \alpha}$$

(32)

thus proving (25) and hence (24).

5 Concluding remarks

We have proved that when the magnetic field is given by (7) for all $\beta$ large enough there is a phase transition when $\alpha > 1$ while, if $\alpha < 1$, there is a unique DLR state. It seems plausible that uniqueness extends to all $\beta$ but we do not have a proof. Using the random cluster representation uniqueness is related to the absence of percolation (see [9]), perhaps this can be useful to deal with this question. When $\alpha = 1$ and $h^*$ small enough the proof of Section 2 applies and we thus have a phase transition. However,
our proof of uniqueness does not extend to the case \( \alpha = 1 \) no matter how large is \( h^* \) and a different approach should be used maybe related to an extension of Minlos-Sinai or the Wulff shape problem.

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