Diffusive Shock Acceleration in $N$ Dimensions

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Abstract

Collisionless shocks are often studied in two spatial dimensions (2D) to gain insights into the 3D case. We analyze diffusive shock acceleration for an arbitrary number $N \in \mathbb{N}$ of dimensions. For a nonrelativistic shock of compression ratio $\mathcal{R}$, the spectral index of the accelerated particles is $s_E = 1 + N/\mathcal{R} - 1$; this curiously yields, for any $N$, the familiar $s_E = 2$ (i.e., equal energy per logarithmic particle energy bin) for a strong shock in a monatomic gas. A precise relation between $s_E$ and the anisotropy along an arbitrary relativistic shock is derived and is used to obtain an analytic expression for $s_E$ in the case of isotropic angular diffusion, affirming an analogous result in 3D. In particular, this approach yields $s_E = (1 + \sqrt{13})/2 \simeq 2.30$ in the ultrarelativistic shock limit for $N = 2$, and $s_E (N \to \infty) = 2$ for any strong shock. The angular eigenfunctions of the isotropic-diffusion transport equation reduce in 2D to elliptic cosine functions, providing a rigorous solution to the problem; the first function upstream already yields a remarkably accurate approximation. We show how these and additional results can be used to promote the study of shocks in 3D.

Unified Astronomy Thesaurus concepts: High energy astrophysics (739); Shocks (2086); Magnetic fields (994); Cosmic rays (329)

1. Introduction

Collisionless shocks are known to accelerate charged particles to ultrarelativistic energies in a wide range of astronomical systems. Diffusive shock acceleration (DSA) is a first-order Fermi mechanism (Fermi 1949; Bell 1978), thought to be responsible for this process. DSA can explain, under certain assumptions, the power-law spectra of high-energy particles inferred in various astrophysical phenomena; for reviews, see Drury (1983), Blandford & Eichler (1987), and Sironi et al. (2015). While most DSA studies focus on $N = 3$ spatial dimensions (3D; but see the 1D analysis of Keshet 2017, discussed below), valuable insights and further analytic progress is possible in other dimensions, in particular low $N$ where the problem becomes simpler and more easily tractable computationally by ab initio simulations.

DSA involves energetic particles, scattered by electromagnetic modes, bouncing between the upstream and downstream sides of a shock, gradually gaining energy in each cycle. The process is still not understood from first principles. A self-consistent model would need to simultaneously account for the injection and acceleration of the particles, their scattering off electromagnetic irregularities, and the formation of these irregularities by the bulk flow and by the accelerated particles themselves. One approach to the problem is the test-particle approximation, evolving the particle distribution function (PDF) $f$ by adopting some ansatz for the scattering mechanism and neglecting the backreaction of the accelerated particles on the shock and on the scattering medium.

This approach was used to derive the energy spectral index,

$$s_E \equiv -\frac{d \ln n(E)}{d \ln E} = 1 - N - \frac{d \ln f}{d \ln E}$$

(1)

where $n(E)$ is the specific particle density. In nonrelativistic shocks in 3D (Axford et al. 1977; Krymskii 1977; Bell 1978; Blandford & Ostriker 1978),

$$s_E \simeq \frac{\mathcal{R} + 2}{\mathcal{R} - 1}$$

depends only on the shock compression ratio $\mathcal{R}$, although this result does not necessarily apply when scattering is highly anisotropic (Keshet et al. 2020).

For sufficiently isotropic scattering around a strong shock in an ideal monatomic gas, $\mathcal{R} \to 4$ then implies that $s_E \simeq 2$. While this approach does not address possible nonlinear effects on the shock and the scattering modes (see Blandford & Eichler 1987; Malkov & Drury 2001; Ellison et al. 2016, for reviews), it is consistent with a wide range of observations. The flat, $s_E \to 2$ spectrum is peculiar in its logarithmic energy divergence, and one may ask if it is a coincidence that this spectrum appears to be most relevant in nature. It is interesting to generalize the result to dimensions other than three and to ask, for example, whether the emerging flat spectrum is general or unique to a monatomic gas in 3D.

Relativistic shocks raise additional questions, which can benefit from an analysis with a different number of dimensions. DSA is more complicated in a relativistic shock, as the PDF can no longer be approximated as isotropic. Assuming small-angle scattering, parameterized by an angular diffusion function $\Delta$, analyses of DSA in relativistic shocks by numerical (e.g., Bednarcz & Ostrowski 1998; Achterberg et al. 2001), semianalytic (Kirk & Schneider 1987; Heavens & Drury 1988; Kirk et al. 2000; Keshet 2006), and analytic (Keshet & Waxman 2005, henceforth KW05) methods found a spectral index $s_E \simeq 22/9 \simeq 2.22$ in the ultrarelativistic shock limit for isotropic diffusion.

This result broadly agrees with observations of systems associated with nonmagnetized relativistic shocks, such as $\gamma$-ray burst (GRB) afterglows (e.g., Waxman 2006, and references therein) and possibly also jets in BL Lac objects. An analysis of ~300 GRB afterglows found $s_E \simeq 2.25$ as most likely, but with a broad, $s_E = 2.36 \pm 0.59$, Gaussian

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distribution (Curran et al. 2010), probably due a long tail of soft-spectrum GRBs (Ryan et al. 2015). Focusing only on short GRB afterglows, a distribution of $s_{E} = 2.43^{+0.28}_{-0.28}$ was found from a sample of 38 such events (Fong et al. 2015). Similarly, jets in BL Lac objects, in which the polarization pattern is consistent with shock-generated magnetic fields, show $s_{E} = 2.28 \pm 0.06$ (Hovatta et al. 2014). It is thought that magnetized relativistic shocks cannot efficiently accelerate particles (e.g., Kirk & Heavens 1989; Begelman & Kirk 1990; Ballard & Heavens 1991; Ostrowski & Bednarcz 2002; Sironi & Spitkovsky 2009) although in the extreme limit of pulsar wind nebulae (PWN), the highly magnetized termination shock is thought to accelerate an extremely hard, $s_{E} \approx 1$ spectrum, possibly through DSA (e.g., Fleishman & Bietenholz 2007; O. Arad et al. 2020, in preparation, henceforth A20). Insights obtained from DSA in other dimensions may shed light on these phenomena.

The problem of DSA in relativistic shocks has not been rigorously solved, and the dependence of $s_{E}$ on the diffusion function is not yet entirely clear. For instance, a precise relation exists between the spectrum and the particle anisotropy at the shock front; for isotropic diffusion in 3D, this leads to (KW05)

$$s_{E} \approx \beta_{d} + 2\beta_{d} - 2\beta_{d} \beta_{d} + \beta_{d}^{3} \beta_{d} - \beta_{d},$$

(3)

where $\beta$ is the fluid velocity with respect to the shock, normalized to the speed of light $c$, and upstream (downstream) quantities are labeled with a subscript $u$ (subscript $d$), written henceforth only when necessary. This result is quite sensitive to the angular form of the diffusion function (Keshet 2006, and A20), although, interestingly, not to local feedback from the relativistic particles (Nagar & Keshet 2019). It would be useful to generalize Equation (3), in particular to 2D, where ab initio simulations are increasingly capable of resolving particle acceleration.

In this study, we generalize the test-particle analysis of DSA to $N = 3$ dimensions. As the preceding discussion indicates, analyzing such DSA is useful for several reasons. First, low-dimensional studies are often essential due to the complexity of the 3D case. Indeed, as resolving a 3D collisionless shock is at present prohibitively expensive computationally, much of the progress in the field has relied on the analysis of 1D or 2D systems. The $N = 2$ case is especially important, as 2D shocks manifest key processes relevant to 3D, yet can be substantially evolved numerically (e.g., Spitkovsky 2008a; Keshet et al. 2009; Martins et al. 2009; Sironi & Spitkovsky 2009; Sironi et al. 2013; Caprioli et al. 2014, 2017). Indeed, 3D experiments are often interpreted based on 2D simulations (e.g., Takabe et al. 2008; Kuramitsu et al. 2011; Liu et al. 2011; Haberberger et al. 2012). Second, $N = 3$ studies provide valuable insights which are inherently inaccessible in a 3D framework. For example, DSA in 1D is uniquely independent of the scattering function, so its curious behavior in the ultrarelativistic limit may be indicative of 3D shocks, in which the scattering function is important but poorly constrained (Keshet 2017). As another example, we show that the flat spectrum ($s_{E} = 2$ for an ideal, monatomic gas) in the nonrelativistic, strong shock limit is independent of $N$, suggesting that the corresponding logarithmic energy convergence is not coincidental. Third, low-dimensional analyses may be effectively applicable to some physical systems. For instance, in a strongly magnetized parallel shock, magnetic confinement can render the system effectively 1D. Fourth, experimental work has recently managed to effectively realize 2D shocks, in systems such as gas tubes (e.g., Skews et al. 2015) and shallow-water analogs (e.g., Foglizzo et al. 2012). Finally, our $N = 3$ results can be used for code development and verification, and for pedagogical purposes.

The paper is organized as follows. In Section 2, we outline the DSA problem in $N$ dimensions. The problem is solved for a nonrelativistic shocks in Section 3. We specify to $N = 2$ in Section 4, calculating the spectrum and PDF for an arbitrarily relativistic shock in several analytic and semianalytic methods, with an emphasis on the ultrarelativistic shock limit. The analysis of relativistic shocks is generalized to $N$ dimensions in Section 5. Our results are summarized and discussed in Section 6. In Appendix A, we derive the transport equation in 2D. In Appendix B, we derive the Maxwell–Jüttner (also known as the Jüttner–Synge, or JS) distribution for an arbitrarily relativistic gas in 2D, subsequently used in Appendix C to derive the 2D JS equation of state (EOS) and the corresponding shock jump conditions. Appendix D details the convergence properties of our results and our error estimation.

2. DSA in $N$ Dimensions

In this section, we present the DSA problem in a general setting with $N$ spatial dimensions. In Section 2.1, we discuss the $N$-dimensional shock jump conditions. We lay out the $N$-dimensional DSA setup in Section 2.2.

2.1. Shock Jump Conditions

In a nonrelativistic fluid, the adiabatic index of an ideal gas is given by $\Gamma_{ad} = 1 + 2\nu^{-1}$ (e.g., Ryden 2016), where $\nu$ is the effective number of particle degrees of freedom. The Rankine–Hugoniot jump conditions (e.g., Landau & Lifshits 1959) hold in any $N$, so the compression ratio of a strong nonrelativistic shock is given by

$$R \equiv \frac{\beta_{u}}{\beta_{d}} \to \frac{\Gamma_{ad} + 1}{\Gamma_{ad} - 1} = 1 + \nu.$$

(4)

In an ultrarelativistic fluid, $\Gamma_{ad} = 1 + \nu^{-1}$, so here, for a strong shock (e.g., Keshet 2017),

$$R \equiv \frac{\beta_{u}}{\beta_{d}} \to \frac{\epsilon_{u} + P_{u}}{\epsilon_{d} + P_{d}} \approx \frac{\epsilon_{d}}{P_{d}} \to \frac{1}{\Gamma_{ad} - 1} = \nu,$$

(5)

where $\epsilon$ is the internal energy density and $P$ is the pressure.

In a relativistic fluid, the adiabatic index is typically assumed to vary smoothly between the above nonrelativistic and ultrarelativistic limits, according to the JS EOS (Jüttner 1911; Synge 1957). The phase-space particle distribution in such a fluid, known as the JS distribution, can be derived by minimizing the free energy and is used to infer the JS EOS. Here, we focus on the 2D case, deriving the 2D JS distribution in Appendix B, and the respective JS EOS and shock jump conditions in Appendix C.

Summarizing the results of Appendix C, the 2D JS EOS can be written in terms of the dimensionless inverse temperature,
\( \zeta = mc^2/(k_B T) \), in the form

\[
\Gamma_{ad} = 2 - \frac{1}{2 + \zeta} \approx \begin{cases} 
2 - \frac{1}{\zeta} + O(\zeta^{-2}) & \zeta \gg 1; \\
\frac{3}{2} + \frac{\zeta}{4} + O(\zeta^2) & \zeta \ll 1,
\end{cases}
\]

where \( m \) is the particle mass, \( T \) is the plasma temperature, and \( k_B \) is the Boltzmann constant. The nonrelativistic and ultrarelativistic limits in Equation (6) agree with the respective limits in Equations (4)–(5), where \( \nu = 2 \) for a monatomic gas in 2D. Then, in the case of a strong shock in 2D, the EOS, along with the conservation of mass, momentum, and energy fluxes across the shock, yields the jump condition

\[
1 + \frac{\zeta_d + 2}{(\zeta_d + 1)\zeta_d} = (1 - \beta_a\beta_c)\gamma_a\gamma_c,
\]

where \( \zeta_d \) is derived as a function of \( \beta_a \) as the positive root of a seventh-order polynomial provided in Equation (C20). Here and henceforth, \( \gamma \equiv (1 - \beta^2)^{-1/2} \) is the fluid Lorentz factor. The resulting downstream adiabatic index and shock compression ratio \( R \) are presented, as a function of \( \beta_a \), in Figure 1.

### 2.2. DSA Setup

Consider DSA in \( N \geq 2 \) dimensions. We work, as in most analytic studies, in the test-particle approximation. More precisely, as the scattering function in a relativistic shock (and sometimes even in a nonrelativistic shock; see Keshet et al. 2020) is not rigorously known, we simply work with a prescribed scattering function. We avoid the injection problem, assuming that particles are injected at the shock front with sufficiently high energies to easily cross the shock.

Let \( z_s \) be the oriented distance from the shock, in the shock frame (we henceforth omit the subscript \( s \) unless necessary), such that the flow is in the positive \( z \) direction and the shock is at \( z = 0 \). We assume that by averaging over constant \( z \) planes, parallel to the shock front, the resulting, reduced PDF, \( f(z, p) \), is time independent, where \( p \) is the particle momentum.

Further assuming that plane-averaging leaves no preferred direction in the system, we arrive at the reduced Lorentz-invariant, steady-state PDF \( f(z, p, \mu) \). Here, \( \text{arc}(\mu) \) is the angle between momentum and flow directions. In this mixed-frame, three-dimensional phase space, \( z \) is measured in the shock frame, whereas \( p \) and \(-1 \leq \mu \leq 1 \) are measured in the fluid frame. Note that unlike the polar angle \( \theta \equiv \text{arc}(\mu) \) in \( N \geq 2 \) dimensions, the azimuthal angle \( \phi \equiv \text{arc}(\mu) \) for \( N = 2 \) is periodic.

Under the above assumptions, the particle momentum \( p \) is much larger than any momentum scale in the system. The lack of a characteristic scale then implies a power-law spectrum, so the PDF may be written as \( f = q(z, \mu)\mu^{-\nu} \), reducing the problem to determining the constant \( s_p \) and the function \( q(z, \mu) \). The momentum spectral index \( s_p \) is related to the energy spectral index \( s_E \) by

\[
s_p = s_E + N - 1.
\]

Figure 2 illustrates the reduced PDF \( q(x, \mu) \) of accelerated particles around a strong relativistic shock with covariant velocity \( \gamma_p/\beta_c = 10 \), with the JS EOS, in 2D. Here, \( x \) is defined by \( z \approx \text{arctanh} \( \xi \) \), such that negative (positive) \( \xi \) correspond to the upstream (downstream) region, and \( \xi = 0 \) is the shock front. The PDF is shown based on a numerical code (orange surface; Nagar & Keshet 2019) and on \( N = 10 \) upstream eigenfunctions derived in Section 4.3 (blue disks intersecting the surface). The normalization (unit integral of \( q_\mu \) over \( \mu \)) is arbitrary.

and on our upstream eigenfunction expansion (see Section 4.3; blue disks intersecting the surface, shown in the upstream only). The unbound coordinate \( z \) is mapped onto a compact coordinate, \(-1 < \xi \equiv \text{tanh}(D\gamma^2z/c) < 1 \), so the figure includes both the far upstream (\( \xi \rightarrow -1 \)) and far downstream (\( \xi \rightarrow +1 \)).

### 3. Nonrelativistic Shocks in \( N \) Dimensions

It is interesting to generalize the classical spectrum (2) of particles accelerated in a nonrelativistic shock for an arbitrary dimension \( N \geq 2 \). One way to do so is, in analogy with Krymskii (1977), by generalizing the steady-state Fokker–Planck equation (Parker 1965).

\[
\nabla (N/\beta) = \frac{1}{c} \nabla \cdot [D N \nabla N] + \frac{1}{N} \partial_t (\nabla p) \nabla \cdot \beta, \quad (9)
\]
to \(N\) dimensions, where \(D(x, p)\) is the spatial diffusion coefficient,

\[
N \equiv p^{N-1} \int f(x, p) d\Omega
\]  

(10)

is the specific (per unit momentum) number density of the accelerated particles, and \(d\Omega = \sin^{N-2}(\theta_1)\sin^{N-3}(\theta_2)\ldots \sin(\theta_N-2)d\theta_1d\theta_2\ldots d\theta_N d\phi\) is the solid angle interval in \(N\) dimensions. Here, \(\theta_1 \in [0, \pi]\) are polar angles, and \(\phi \in [0, 2\pi]\) is an azimuthal angle.

Equation (9) includes, in addition to convection and diffusion terms, also a term accounting for the \(p \propto \rho^{1/N}\) particle momentum boost, with \(\rho\) being the mass density of the plasma, due to shock compression,

\[
\frac{dN}{dt}
\]

\[
= \frac{d}{dt} \left[ -\partial_p n(>p) \right]_{\text{ad}} = -\partial_p \left[ \frac{dn(>p)}{dt} \right]_{\text{ad}}
\]

\[
= -\partial_p \frac{dp}{dt} N = \frac{1}{N} \partial_p \left( \rho N \right) \nabla \cdot \beta,
\]

(11)

mediated in the DSA picture by magnetic structures. Here, \(n(>p) \equiv \int_{p}^{\infty} N dp\) is the number density of particles with momentum larger than \(p\), and in the last equality of Equation (11), we used the continuity equation,

\[
\frac{d\rho}{dt} = -c\rho \nabla \cdot \beta.
\]

(12)

Using the boundary condition of no energetic particles reaching infinitely far upstream, the integration of Equation (9) yields

\[
N_\beta = \frac{1}{c} D \partial_\gamma N - \frac{\beta_d - \beta_d}{N} H(z) \partial_p (\rho N)\big|_{z=0},
\]

(13)

where \(H\) is the Heaviside step function. The solution \(N\) to this equation decays exponentially upstream and is uniform downstream, imposing the requirement

\[
N_d = -\frac{R - 1}{N} \partial_p (p N_d)\big|_{z=0}.
\]

(14)

The implied energy spectral index,

\[
s_E = 1 + \frac{N}{R - 1},
\]

(15)

is then a function of \(N\) and \(R\) alone. For a strong shock in a monatomic ideal gas, \(R = 1 + N\), and so Equation (15) curiously yields \(s_E = 2\), regardless of the dimension.

It is useful, here and in anticipation of Section 4.4, to introduce an alternative method for deriving the spectrum, by considering the fractional energy gain \(g\) and return probability \(P_{\text{ret}}\) of a particle undergoing a Fermi cycle (Fermi 1949), crossing the shock back and forth. We thus generalize the computation of Bell (1978) to \(N\) dimensions, deriving the spectral index as

\[
s_E \approx 1 - \frac{\ln(P_{\text{ret}})}{\ln(1 + g)}.
\]

(16)

Here, we define angular brackets

\[
\langle \ldots \rangle = \frac{\int \ldots \, dj}{\int dj}
\]

(17)

as averaging over the flux crossing the shock,

\[
dj = (\mu + \beta) q(0, \mu) d\Omega.
\]

(18)

Consider a relativistic particle in a nonrelativistic flow, crossing the shock front to the upstream region at some angle \(-1 \leq \mu \leq -\beta\), and subsequently, after some scattering, crossing back downstream at an angle \(-\beta \leq \mu \leq 1\). We choose \(\mu_u\) and \(\mu_d\) in the downstream frame, although this choice does not affect the energy gain in the nonrelativistic shock regime. Neglecting correlations between \(\mu_u\) and \(\mu_d\), (a discussion of such correlations, see A20), the flux-averaged fractional energy gain in the downstream frame is

\[
\langle g \rangle = \frac{E_{i+1}}{E_i} - 1 = \langle \beta(\mu_u - \mu_d) \rangle + O(\beta^2),
\]

(19)

where \(E_i\) is the particle energy in the \(i\)th cycle and \(\beta_i \equiv (\beta_d - \beta_d)/(1 - \beta_d \beta_u) \approx (R - 1)\beta_d\) is the relative velocity between the upstream and downstream frames. We note that the fractional energy gain may similarly be calculated in the upstream frame, with the advantage of diminished correlations between \(\mu_u\) and \(\mu_u\); this is utilized in the relativistic shock analysis of Section 4.4.

For \(N > 1\), averaging \(\mu_u\) and \(\mu_d\) over the flux element of Equation (18) and assuming an approximately isotropic PDF, \(q(0, \mu) \propto 1 + O(\beta)\), Equation (19) yields

\[
\langle g \rangle = \beta_\gamma \sqrt{\pi} \Gamma \left(1 + \frac{N}{2}\right) \Gamma \left(1 + \frac{N}{2}\right) + O(\beta^2),
\]

(20)

where \(\Gamma(\chi)\) is the gamma function.

The probability \(P_{\text{ret}}\) of a particle crossing the shock downstream to return upstream may be found from the particle flux crossing the shock in each direction. Assuming that the downstream PDF is isotropic up to second-order corrections,
Equation (23) becomes
\[ (\mu + \beta) \partial_\theta \omega (\tau, \phi) = \partial_\theta [D(\phi) \partial_\theta \omega], \]  
(24)
where we defined \( \tau \equiv (c \gamma)^{-1} \int_0^\infty D_2(z, \rho) dz \) as the shock-frame optical depth. In the case of isotropic diffusion, \( D = \text{const} \), Equation (24) can be solved by separation of variables (e.g., Kirk et al. 2000). The angular functions in 2D are then given by the periodic Mathieu functions, also known as elliptic cosine functions, utilized in Section 4.3.

4.2. Analytic Spectrum–Anisotropy Connection

Following KW05 and using similar notations, we exploit the stationary nature of the PDF at shock-grazing angles, where \( \mu + \beta = 0 \). We expand the PDF and diffusion function in each frame about the grazing angle,
\[ q(\tau, \phi) = a_0(\tau) + a_1(\tau) (\mu + \beta) + a_2(\tau) (\mu + \beta)^2 + \ldots \]  
(25)
and
\[ D(\phi) = d_0 + d_1 (\mu + \beta) + d_2 (\mu + \beta)^2 + \ldots \]  
(26)
The transport Equation (24) then implies the precise relation
\[ \frac{2a_2}{a_1} + \frac{d_1}{d_0} + \beta \gamma^2 = 0, \]  
(27)
Notice that here we assumed that the PDF is an analytic function near the grazing angle.

Next, we use continuity across the shock to relate the upstream and downstream expansions of \( q \). To first order, this yields
\[ r_u + s_p \beta_u = r_d + s_p \beta_d, \]  
(28)
where
\[ r \equiv \gamma^{-2} \frac{a_1(\tau = 0)}{a_0(\tau = 0)} \]  
(29)
is a measure of the PDF anisotropy along the shock. Using Equation (27), the second-order expansion of continuity across the shock then provides a precise relation between the particle spectrum and the anisotropy along the shock front,
\[ s_p (s_p + 1) \| \beta^2 \|^2 + (2 s_p + 1) \| \beta \| = \| r d \|, \]  
(30)
where we defined the norm \( \| \ldots \| \equiv (\ldots)_{\mu} - (\ldots)_{\lambda} \) as an operator taking the difference across the shock. Here, \( d \equiv \gamma^{-2} d \lambda/d_0 \) is a measure of the diffusion function anisotropy near the grazing angle. For isotropic diffusion, \( d = 0 \), and the right-hand side of the equation vanishes.

Combining Equations (28) and (30), we may now derive \( s_p \) as a function of the anisotropy parameter \( r \) in any frame. For example, in terms of \( r_u \),
\[ s_p = \frac{r_u + \beta_u + d_u}{\sqrt{\left(r_u + \frac{\beta_u + d_u}{2}\right)^2 + r_u d - \beta}}. \]  
(31)
For isotropic diffusion, \( d = 0 \), and Equation (31) simplifies to
\[ s_p^{(\text{iso})} = -\frac{r_u + \frac{\beta_u}{2}}{\beta_u - \beta}, \]  
(32)
Analogous expressions for $s_p$ as a function of $r_d$ are obtained by interchanging subscripts $u \leftrightarrow d$ and reversing the sign of the $\| \ldots \|$ operator in Equations (31) and (32).

One can also express the spectrum as a function of the shock-frame grazing anisotropy, which we define as $r_\gamma \equiv a_1(\gamma)/a_0(\gamma)$. Expanding $q$ around the shock-frame grazing angle, $\mu_\gamma = 0$, and using continuity across the shock to relate $r_\gamma$ and, say, $r_u$, one finds

$$r_\gamma = r_u + s_p \beta_\gamma.$$  

Plugging this result into Equations (28) and (30) yields

$$s_p = \frac{r_u + \| \beta_d \|}{\| \beta_u \|} \sqrt{(r_u + \| \beta_d \|)^2 + r_u b \left( 1 - \| \frac{\beta_\gamma}{\beta_u} \| \right)},$$  

where $b \equiv \beta_u + \beta_d$. For isotropic diffusion, Equation (34) simplifies to

$$s_p^{(iso)} = \frac{r_u}{\beta_u + \beta_d} \left( 1 \pm \sqrt{1 + \frac{\beta_u + \beta_d}{r_u}} \right).$$  

Equations (31) and (34) provide a powerful, precise connection between the spectrum and the anisotropy of shock-grazing particles. These results do not rely on the test-particle approximation and are valid for any small-angle scattering described by $D$. The 3D analog of this spectrum–anisotropy connection was derived in KW05.

The spectrum–anisotropy relation may be used to estimate the spectrum, if one can constrain the grazing anisotropy parameter $r_\gamma$. One useful constraint arises from the limit of infinite compressibility, $\Gamma_d \to 1$, where $\beta_d = 0$. Here, the escape probability vanishes, so Equation (16) implies a spectral index $s_{\pm} \to 1$ (KW05). Another constraint is obtained in the nonrelativistic shock limit, studied in Section 3. In 2D, Equation (15) yields $s_{\pm} = (R + 1)/(R - 1)$ in this limit, so in a strong nonrelativistic shock in 2D one may infer, for example, that $r_d = [2\beta_u(2\beta_u - \beta_d) - d_u d_d]/(5\beta_u - \beta_d + 4\beta_u - d_u)$.

Next, we derive an expression for the spectrum in the special case of isotropic diffusion, following the method of KW05. We focus on the downstream frame, where, unlike in the upstream, the anisotropy does not become very strong even in the ultrarelativistic shock limit. Combining Equations (15), (28), and (30), we rewrite Equations (28)–(30) as

$$r_d = \frac{s_p^2(\beta_u - \beta_d) - s_p \beta_d}{2s_p + 1}.$$  

In the nonrelativistic shock limit, we may now quantify the downstream anisotropy as

$$a_1^{(d)} = \frac{2s_p(2\beta_u - \beta_d)}{5\beta_u - \beta_d}. $$  

Equation (37) holds not only in the nonrelativistic shock limit, but also for any $\beta_u$ in the infinite compressibility limit where $\beta_d = 0$. Following KW05, we extrapolate the result for arbitrary $[\beta_u, \beta_d]$; the reasoning for this extrapolation is further discussed in Section 6. Plugging Equation (37) into the downstream version of Equation (32) finally yields

$$s_p^{(iso)} = \frac{r_u + \frac{\beta_u}{2}}{\frac{\beta_u}{2} + r_u \beta_u + \frac{\beta_d^2}{4}} \beta_d - \beta_d \left( \frac{1}{2} + \frac{2R^2}{\sqrt{5}R - 1} + \frac{1}{2} \frac{2R}{\sqrt{5}R - 1} + \frac{4R^2}{\sqrt{5}R - 1} \right)^2.$$  

Resembling the 3D case, we find that also in 2D, the result (Equation (38)) of the above extrapolation is consistent with the spectrum computed in other numerical or semianalytical methods for an arbitrarily relativistic shock and any EOS. Figure 3 shows that our analytic estimate is in excellent agreement with the upstream eigenfunction expansion presented in Section 4.3, in both nonrelativistic and ultrarelativistic limits. In particular, in the ultrarelativistic shock limit, $\beta_u \to 1$ and $\beta_d \to 1/2$, Equation (38) implies that

$$s_p^{(iso)}(\gamma_u \to \infty) \to \frac{3 + \sqrt{13}}{2} \approx 3.303,$$  

consistent within 0.2% with the eigenfunction method. Interestingly, in the transrelativistic regime, $\gamma \beta \approx 1$, there is a slight, $\sim 1\%$ deviation between the two methods.

4.3. Upstream Eigenfunction Expansion

Next, we focus on the case of isotropic diffusion, $D = \text{const.}$, where the problem becomes largely analytically tractable. Here, the transport Equation (24) can be directly solved by expanding the PDF in upstream eigenfunctions, in a method parallel to that applied by Kirk et al. (2000) for the three-dimensional case. An advantage of working in two dimensions is that the eigenfunctions reduce to the well-known elliptic cosine functions, as we show below.

Separating the PDF variables, let

$$q(\tau, \phi) \equiv T(\tau) \Phi(\phi).$$  

Plugging this into Equation (24), we obtain two separate equations, connected by an eigenvalue $\Lambda$ which we define such that

$$T' (\tau) = \frac{\Lambda}{2} T (\tau)$$  

and

$$\Phi''(\phi) = \frac{\Lambda}{2} (\beta + \mu) \Phi(\phi).$$  

The spatial Equation (41) indicates an exponential spatial dependence,

$$T(\tau) \propto e^{\pm \sqrt{\Lambda} \tau},$$  

where the boundary conditions dictate that $\Lambda > 0$ upstream and $\Lambda < 0$ downstream. The solution to the angular Equation (42) under our assumption of axisymmetry, $q(\tau, -\phi) = q(\tau, \phi)$, is given by

$$\Phi(\phi) \propto C \left( -2\beta \Lambda, \frac{\phi}{2} \right).$$  

where $C(\alpha, \Lambda, \chi)$ are the Mathieu cosine functions (see, e.g., McLachlan 1951), defined as the solutions of the Mathieu
that are even in $x$, namely $C(a, \Lambda, x) = C(a, \Lambda, -x)$.

Next, we impose $2\pi$ periodicity in $\phi$, which corresponds to $1\pi$ periodicity in $x$. In general, for a given $\Lambda$, $C(a, \Lambda, x)$ becomes periodic in $x$ only for a discrete, infinite set of so-called characteristic values $a = a_r(\Lambda)$, which are the roots of a continued fraction equation (Incé 1927), which we write as

$$\frac{1}{(a - 2\Lambda \cos(2\chi))} = \frac{\infty}{\phi \cdots}$$

(Mathieu functions with $1\pi$ periodicity correspond to $a_r(\Lambda)$ characteristic values with an even index, $r = 2j$, where $j = \{0, 1, 2 \ldots \} \geq 0$. Here, $r$ is an index, not to be confused with the anisotropy parameter (which we defined in Section 4.2 and do not use in the current section).

In Equation (44), $a = -2\Lambda = a_r(\Lambda)$, which we can now solve for $\Lambda$. For each $j$, we find two such solutions, denoted $\Lambda_{ij}(\beta)$, namely

$$-2\beta\Lambda_{ij} = a_{ij}(\Lambda_{ij}).$$

For $j > 0$, the solutions satisfy $\Lambda_j > 0$ and $\Lambda_{j} < 0$. For $j = 0$, there is one positive solution, denoted $\Lambda_{00}$, and one trivial solution, denoted $\Lambda_{00} = 0$. This behavior is guaranteed by the Sturm–Liouville theory, noting that the even-parity function $C(a_j, \Lambda, x)$ has $2j$ zeros in the interval $0 < x < \pi$ (McLachlan 1951, Section 2.13) and are denoted $c_{2j}(x, \Lambda)$. Our $1\pi$-periodic functions can therefore be written as

$$\Phi_{ij}(\phi) = c_{2j}(\phi \frac{a_{ij}}{2}, \Lambda_{ij}).$$

The eigenfunctions obey the orthogonality relation

$$\int_0^\pi (\mu + \beta)\Phi_i(\phi)\Phi_j(\phi)d\phi = w_i(\beta)\delta_{ij},$$

where $w_i(\beta)$ are constant weights. This result can be verified by comparing Equation (49) after applying Equation (42) to either $\Phi_i$ or $\Phi_j$, and integrating by parts. Limiting expressions for all eigenvalues and eigenfunctions in the $\gamma \gg 1$ limit are discussed in Section 4.4. In the limit $\Lambda \to 0$, $\Phi_{ij}$ reduces to $\cos(2\phi)$.

The upstream PDF can now be written in the form

$$f_u = \gamma u_0^{\gamma - p} \sum_{i=0}^\infty \frac{\kappa_i}{\phi_u, \Lambda_i} e^{L_i \tau/2},$$

where $\kappa_i$ are constants. An analogous expansion can be carried out downstream, if necessary. As $\Phi_{ij}$ is not composed of any $j \geq 0_+$ eigenfunctions, the orthogonality relation (49) implies that

$$\int_0^\pi (\mu + \beta)\Phi_{ij}(\phi)d\phi = 0 \quad \forall j \geq 0_+.$$
4.4. Ultrarelativistic Limit

In the ultrarelativistic limit, the upstream-frame eigenvalues become large, \( \Lambda \gg 1 \), and limiting expressions of all eigenvalues and eigenfunctions may be found in Ogilvie & Daalhuis (2015). Simplifying their expressions, in this limit the upstream eigenvalues are given by

\[
\Lambda_j = 4\gamma^4(1 + 4j)^2 + O(\gamma^2),
\]

and the upstream eigenfunctions become, in terms of \( y \equiv (1 + \mu)\gamma^2 \),

\[
\begin{align*}
\text{ce}_{2j}(y) & \approx e^{\sqrt{\frac{2}{\gamma^2}}y}F_1\left(-j; \frac{1}{2}; \frac{\sqrt{\frac{2}{\gamma^2}}y}{\gamma^2}\right) = e^{\sqrt{\frac{2}{\gamma^2}}y} \\
& \times \left[1 - j + \sum_{k=0}^{j} \left(\frac{2\sqrt{\frac{2}{\gamma^2}}}{\gamma^2}y\right)^k \prod_{i=2}^{j} \frac{j - i + 1}{i(2i - 1)} \right],
\end{align*}
\]

up to an inconsequential normalization. Here, \( _1F_1(a; b; z) \) is the confluent hypergeometric function of the first kind. In particular, using the first eigenfunction only, we may approximate

\[
q_0(0, \phi_0) \propto \text{ce}_{0}\left(\frac{\phi_0}{2}, \Lambda_{00}^{(u)} \approx 4\gamma^2\right) \propto e^{-(1+\mu)\gamma^2}.
\]

The approximation (58) may also be derived by an asymptotic analysis of the angular transport Equation (42) in the upstream frame (resembling Kirk & Schneider 1989, but note the factor of 2 difference between our definitions of \( \{\Lambda, y\} \) and theirs). After a change of variables, Equation (42) becomes

\[
2y\left(1 - \frac{y}{2\gamma^2}\right)\Phi''(y) + \left(1 - \frac{y}{\gamma^2}\right)\Phi'(y) = \frac{\Lambda}{\gamma^4}(2y - 1)\Phi(y).
\]

Taking into account that in the region \( \mu > -\beta \) the eigenfunctions are exponentially small, we focus on the regime \(-1 \leq \mu \leq -\beta\). Noting that \( y/\gamma^2 = 1 + \mu \ll 1 \), we neglect such terms. Additionally, we plug in Equation (57), to yield

\[
2y\Phi''(y) + \Phi'(y) = (1 + 4j)^2(2y - 1)\Phi(y).
\]

Bounded solutions to Equation (61) are given by

\[
\Phi(y) = e^{-y(1+4j)}\sqrt{y} U\left[\frac{1}{2}(1 - 2j), \frac{3}{2}, 2(4j + 1)y\right],
\]

where \( U(a, b, z) \) is the confluent hypergeometric function of the second kind. For integer \( j \geq 0 \), this is equivalent to Equation (58) up to a normalization.
The above approximations are useful because it is exceedingly difficult to compute the exact eigenfunctions as one approaches $\beta \to 1$. Indeed, these approximations are used for the highly relativistic shocks shown in Figure 3 (right of the vertical line). For $\gamma_0\beta_u \to \infty$, we obtain the asymptotic spectrum

$$s_E = 2.2985 \pm 0.0001,$$  \hspace{1cm} (63)

estimated by extrapolating the data in the figure to $\gamma_0\beta_u \to \infty$, with weights given by the convergence tests. If we use only the first eigenfunction, this approximation (59) with the single-eigenfunction overlap (55) yields

$$s_E = 2.2988 \pm 0.0001.$$  \hspace{1cm} (64)

One can use the approximate first eigenfunction upstream to infer the spectrum in the ultrarelativistic shock limit, even without computing the precise elliptic cosine functions. This can be carried out using the single-eigenfunction overlap (55), if the first eigenfunction downstream $\Phi^{(d)}_0$ can also be approximated. As this function is positive definite and the overlap region is near $\mu = -1$, we may approximate

$$\Phi^{(d)}_{0+} \propto \exp[-c_1(1 + \mu)^2 - c_2(1 + \mu)^3 - \ldots - c_n(1 + \mu)^n]$$  \hspace{1cm} (65)

with finite $n$ terms. Plugging this into the transport Equation (42) fixes the coefficients $c_1 = (1 - \beta_d)\Lambda/2$, $c_2 = [1(1 - \beta_d)^2\Lambda - \beta_d]\Lambda/12$, etc.; the value of $\Lambda$ is then fixed by orthogonality with $\Phi^{(d)}_{0-}$, i.e.,

$$\int_0^\pi (\mu_+ + \beta_d)\Phi^{(d)}_{0+}d\beta_d = 0.$$  \hspace{1cm} (66)

This procedure converges rapidly; taking $n = 2$ already gives $s_E \simeq 2.38$.

A simpler but less accurate method is to compute the return probability and the mean energy gain directly from the approximate first eigenfunction upstream and then use Equation (16) to determine the spectrum. With the approximation (59), the energy gain, best computed in the upstream frame (A20), is

$$\langle 1 + g \rangle = \frac{\int_{d_0^+}^{1 + \beta_0 \mu} d_0^+ \mu d_0^- d_0^+}{\int_{d_0^+}^{1 + \beta_0 \mu} d_0^+ d_0^-} \simeq 2.31,$$  \hspace{1cm} (67)

where the index “+” indicates forward (i.e., toward downstream) directions, $-\beta \leq \mu \leq 1$, and the index “−” indicates backward (toward upstream) directions, $-1 \leq \mu \leq -\beta$. The return probability is given by

$$P_{ret} = \frac{\int_{d_0^+}^{1 + \beta_0 \mu} d_0^-}{\int_{d_0^+}^{1 + \beta_0 \mu} d_0^+} \simeq 0.33,$$  \hspace{1cm} (68)

where we used $s_E = 2.30$ to obtain a numerical estimate. If, instead, we leave $s_E$ undetermined, Equation (16) yields a rather crude approximation, $s_E \simeq 2.07$.

5. Relativistic Shocks in $N \geq 3$ Dimensions

The preceding analysis can be generalized for arbitrary $N \geq 3$ dimensions. The same assumptions leading to Equation (23) in 2D yield the $N \geq 3$ transport equation

$$(\mu + \beta)\partial_\mu q(\tau, \mu) = \frac{\partial_\mu[(1 - \mu^2)^{N-1}D(\mu)\partial_\mu q]}{(1 - \mu^2)^{N-1}}.$$  \hspace{1cm} (69)

The spectrum–anisotropy connection (31) generalizes to

$$s_p = \frac{v + \sqrt{v^2 - r_d(d_d - d_u - (N - 3)(\beta_d - \beta_0))}}{\beta_d - \beta_0},$$  \hspace{1cm} (70)

where we defined $v \equiv [(N - 2)\beta_d + \beta_d + d_u + 2r_d]/2$ and used the downstream anisotropy parameter $r_d$ as defined in Section 4.2.

For isotropic diffusion, the downstream anisotropy of a nonrelativistic shock is therefore given, for any $N \geq 2$, by

$$a_1^{(d)} = \frac{N\beta_d(2\beta_d - \beta_0)}{(N + 3)\beta_d + (N - 3)\beta_0}.$$  \hspace{1cm} (71)

Invoking the same assumptions leading to the spectrum, Equation (38), now gives

$$s_E^{(iso)} \simeq \frac{v + \sqrt{v^2 + (R - 1)[\omega(N + 1) + (N - R)(N - 1)]}}{R - 1}$$  \hspace{1cm} (72)

for any $N \geq 2$. Here, $v \equiv N + \omega - (N - 1)/2$ and $\omega \equiv \gamma_d^{-2}N/R(2R - 1)/(N + 3)R + N - 3$. Interestingly, as $N$ becomes large, the spectrum approaches $s_E \to 1 + N/R + O(N^{-2})$, asymptotically giving the familiar $s_E \to 2$ for an arbitrarily relativistic, strong shock.

The angular component of the transport equation becomes, for $N \geq 3$,

$$(1 - \mu^2)\Phi''(\mu) - (N - 1)\mu\Phi' = (\mu + \beta)\frac{\Lambda}{2}\Phi.$$  \hspace{1cm} (73)

In the upstream of an ultrarelativistic shock, we may approximate Equation (73) around $\mu = -1$ (as in Section 4.4) to find

$$\Phi_j(y) \simeq e^{-(N-1)j\gamma^2y}P_{F_1}\left[-j, \frac{N - 1}{2}, 2(N - 1 + 4j)\gamma^2\right].$$  \hspace{1cm} (74)

with the eigenvalues

$$\Lambda_j = 4\gamma^4(4j + N - 1)^2 + O(\gamma^2),$$  \hspace{1cm} (75)

derived by assuming that

$$\Phi_j = e^{-\frac{\gamma\sqrt{N}}{2\gamma^2}}\sum_{n=0}^{j} \alpha_n y^n$$  \hspace{1cm} (76)

(c.f. Kirk & Schneider 1989, Equation (A5); notice the factor of 2 difference in the definition of $\Lambda_j$). The first upstream eigenfunction is therefore

$$\Phi_0 \simeq e^{-(N-1)(1+\mu)^2\gamma^2}$$  \hspace{1cm} (77)

for an ultrarelativistic shock in any $N \geq 2$.

Given the first upstream eigenfunction, one can directly estimate the spectrum in the methods outlined in Section 4.4, as we demonstrate for $N = 3$. Approximating the first downstream eigenfunction as in Equation (65), we rapidly converge on $s_E \simeq 2.23$; taking a single term $(n = 1)$ in this equation already yields $s_E \simeq 2.29$. One could derive the spectrum from
the first eigenfunction using Equation (16) instead. For \( N = 3 \),
the energy gain in Equation (67) can be computed analytically,
\[
(1 + g) = 5(1 - e + 3e^3[E_1^E(-3) - E_1^E(-2)]) \approx 2.21,
\]
where \( E_1 \) is the exponential integral. The return probability can be
computed as in Equation (68), giving \( P_{ext} = 1 + [\Gamma(\xi_G + 1, 2)
- 3\Gamma(\xi_G, 2)]/3\Gamma(\xi_E, 3) - \Gamma(\xi_G + 1, 3) \approx 0.38 \), the latter
estimate obtained by assuming \( \xi_E = 2.23 \). Alternatively,
solving Equation (68) for \( \xi_E \) gives the approximate \( \xi_E \approx 2.26 \).

6. Summary and Discussion

We generalize the analysis of DSA in collisionless shocks to
an arbitrary number \( N \) of spatial dimensions in order to
facilitate the understanding of shock studies in 2D and
in search for insights into the 3D case. The problem, illustrated in
Figure 2, is solved in the test-particle, small-angle scattering
approximation, first for nonrelativistic shocks (Section 3) and,
with additional assumptions, also for relativistic shocks (in
Sections 4 and 5).

Curiously, for any \( N \), we recover the familiar, flat spectral
index \( \xi_E = 2 \) (see Equation (15)), in which energy diverges
only logarithmically, for a nonrelativistic strong shock in a
monatomic gas. The same result is obtained in the \( N \rightarrow \infty 
\)
limit also for a relativistic strong shock, at least when scattering
is isotropic. These results highlight the important role of the flat
spectrum, which tends to emerge observationally even in the
presence of nonlinear effects which naively may have distorted
it. A similar conclusion was pointed out (Keshet 2017) based
on both nonrelativistic and ultrarelativistic shocks in 1D. It is
interesting to mention, in this context, that a flat spectrum
naturally arises in 3D if the microphysical plasma configuration
is assumed to be self-similar (Katz et al. 2007).

Numerical, in particular ab initio kinetic, simulations of
collisionless shocks in 2D play an important role in the
theoretical study of the less accessible 3D problem. We devote
special attention (in Section 4) to relativistic shocks in 2D; the
results are subsequently generalized for \( N \geq 3 \) (in Section 5).
In particular, an exact relation is derived between the spectral
index and the shock-grazing anisotropy parameter \( \alpha_{1,0} \),
generalizing a 3D result (KW05) to \( N = 2 \) (Equation (31))
and to \( N \geq 3 \) (Equation (70)). The \( N \geq 3 \) analysis is mainly
useful for the \( N = 3 \) case. The nature of DSA in \( N \geq 3 \) is not
well known, and our simplifying assumptions may be
unrealistic. In particular, reducing the dimensionality of such
a problem by averaging over planes parallel to the shock may
be unjustified.

In the case of isotropic scattering, the problem of DSA in a
relativistic shock can be solved using a rapidly converging
expansion in upstream eigenfunctions, as shown with numerically
computed eigenfunctions in 3D (Kirk et al. 2000). In 2D,
the angular eigenfunctions of the transport Equation (24)
reduce to the elliptic cosine functions, so the rapidly
converging expansion, Equation (50), involves familiar special
functions and becomes more transparent.

As in 3D, the first upstream eigenfunction is found to
provide an excellent approximation for the angular PDF
(Equation (54)), and it alone provides an accurate estimate of
the spectrum for an arbitrary shock; see Figures 3–5. We show
(in Section 5) how the spectrum can be derived directly from
this eigenfunction or its approximation, Equation (77).

For isotropic scattering, the spectrum–anisotropy relation
can be used to infer an analytic expression for the spectrum, for
\( N = 2 \) (Equation (38)) and more generally for any \( N \geq 2 \)
(Equation (72)), in a method previously invoked in 3D
by KW05. This method relies on two approximations, which
appear to be precise or at least very accurate: assuming an
analytic behavior of the PDF near the grazing angle, and
extrapolating the downstream-frame anisotropy \( \alpha_{1,0} \),
and more generally \( \alpha_{1,0} \), of nonrelativistic shocks (see Equation (71))
to the relativistic shock regime. The latter extrapolation is particularly interesting:
while Equation (71) is precise when \( \beta_\alpha \) and \( \beta_d \) are small, as well as for any \( \beta_a \) in the \( \beta_a \rightarrow 0 \) limit, it is not a priori clear
that it should remain accurate for large \( \beta_d \).

One can show, however, that the deviation \( A(\beta_a, \beta_d) \) of
\( \alpha_{1,0} \) from its extrapolated value is not large. Consider, for
example, ultrarelativistic shocks in 3D, and approximate the
PDF using the first upstream eigenfunction, so Equations (3)
and (77) give \( A = (\beta_\alpha)/2(1 - \beta_\alpha)/(1 + \beta_\alpha) \).
While this approximation is inaccurate, it yields \( A = 0 \) in both \( \beta_\alpha = 0 \)
and \( \beta_\alpha = 1 \) limits, and a maximal \( \sim 11\% \) deviation found
at \( \beta_\alpha = 1/2 \). (Note that while the \( \beta_\alpha > 1/3 \) regime invoked
here is not physical, our analysis remains valid for any \( 0 \leq \beta_\alpha < \beta_d < 1 \)).

The limit where both \( \beta_\alpha \rightarrow 1 \) and \( \beta_a < \beta_a \rightarrow 1 \)
and \( \beta_\alpha \rightarrow 1 \) is particularly useful, because in the \( \{\beta_\alpha, \beta_d\} \) phase space, it is
diagonally opposite to the nonrelativistic case, and because
one may evaluate analytically the overlap between the first
eigenfunctions upstream and downstream by invoking the
ultrarelativistic approximation in both frames; this yields
(\( \beta_\alpha - \beta_\alpha \) / \( \beta_\alpha \) ) \( \approx 2 \), in agreement with Equation (3),
suggesting that the extrapolation is exact. As another example, notice that the
diverging, \( s \rightarrow \infty \) spectral index expected when \( \beta_a \rightarrow \beta_d \),
so that the shock weakens and disappears, implies that \( A \) cannot
become too negative, \( A(\beta, \beta) > -(3/2 - \beta_d^2)/(1 - \beta^2) \)
for any \( 0 \leq \beta < 1 \). Moreover, requiring in this limit that the
upstream phase space, it is

The above arguments can be directly generalized for 2D,
where the elementary nature of the eigenfunctions renders it
easier to test the extrapolation also inside the boundary, as
illustrated by Figure 5. For example, in the \( \beta_\alpha \rightarrow 1 \) and
\( \beta_\alpha < \beta_a \rightarrow 1 \) limit, we obtain \( \langle \beta_a - \beta_\alpha \rangle s \approx 1 \), in agreement
with Equation (38), suggesting that the extrapolation here too is
exact. The downstream anisotropy associated with the first
upstream eigenfunction can be directly evaluated in 2D for any
shock; its deviation from Equation (71) is less than 35\% for any
\( 0 \leq \beta_\alpha < 0.5 < \beta_a \). However, the anisotropy inferred from
the first-eigenfunction approximation is inaccurate even at small
\( \beta_a \). Better convergence is obtained by considering the spectrum
computed in the eigenfunction method, as shown in Figure 5
using the single-eigenfunction overlap (55), indicating that the
extrapolation is accurate to better than a percent throughout the
physical regime. The extrapolation is likely to be accurate also
at higher \( \beta_a \), where the eigenfunction method appears to be less
adequate.

More important, however, than the above indications in
support of the extrapolation is its success in accounting for the
spectrum computed using alternative numerical or semianalytic
methods for an arbitrarily relativistic shock and any EOS, as
shown for 3D in KW05. The success of this approach in
accounting for the spectrum also in 2D, as we demonstrate in
Figure 3, further supports the extrapolation and its validity in both 2D and 3D. Although the extrapolated downstream anisotropy, Equation (71), in particular $a_1/a_0 = \beta_a - \beta_d/2$ in 3D, was not yet justified analytically, we conclude that it may be safely used as a robust diagnostic in shock studies.

Ab initio particle-in-cell simulations of highly relativistic shocks in 2D have been able to resolve the onset of particle acceleration, giving rise to nonthermal tails which are consistent with spectral indices in the range $\gamma_k \approx 2.3-2.5$ (Spitkovsky 2008b; Sironi et al. 2013), somewhat softer than the $\gamma_k \approx 2.22$ anticipated in 3D. For isotropic scattering, in the ultrarelativistic shock limit, we find (see Equation (39)) that $x_k \rightarrow (1 + \sqrt{13})/2 \approx 2.303$, possibly accounting for this discrepancy. Future kinetic simulations in 2D, expected to resolve the spectrum much more accurately, could be compared more carefully to our results.

We find that in 2D, as in 3D, the spectrum in the ultrarelativistic shock limit does not depend on the equation of state. This differs markedly from the 1D case (see Keshet 2017), which thus appears to be the exception.

Our results are useful as a tool for validating numerical simulations in any dimension. In particular, we present three different methods to infer the spectral index even from an approximate PDF: (i) through the spectrum–grazing anisotropy connection, (ii) by approximating the first downstream eigenfunction, and (ii) through the energy gain and escape probability. The usefulness of these methods is demonstrated for $N = 2$ in Section 4.4 and for $N \geq 3$ in Section 5.

Our results are relevant for simulations with an equal number of spatial and momentum dimensions, e.g., 2D2V PIC simulations with two spatial and two momentum dimensions. The results are also relevant to simulations with more momentum than spatial dimensions, provided that the motion of the accelerated particles is effectively confined to the simulated space, for example, in 2D3V simulations with a planar inflow and a high-energy particle scattering function that is approximately confined to the simulated plane. One can consider more general cases, in which particles appreciably scatter outside the simulated space, but such a study would typically require a more sophisticated treatment, incorporating subrelativistic velocities in the analyzed space.

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Appendix

We consider some technical aspects of the DSA problem in 2D2V, i.e., for two spatial and two momentum dimensions.

Appendix A

Transport Equation in 2D

We consider an infinite, linear (2D version of planar) shock at $z = 0$, with flow in the positive $z$ direction. Relativistic particles with momentum $p$ are scattered by electromagnetic modes moving with flow on both sides of the shock. Their PDF $f$ obeys the Fokker–Planck equation, written in the fluid frame as (e.g., Blandford & Eichler 1987),

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = \frac{\partial}{\partial p} \left( D_{pp} \frac{\partial f}{\partial p} \right). \quad (A1)$$

Here, $v$ is the particle velocity, $\phi$ is the direction of its momentum with respect to the $z$-axis, and $D_{pp}$ is the momentum-space diffusion tensor. The diffusion tensor is defined as

$$D_{pp} \equiv \left( \begin{array}{cc} D_{pp} & p D_{p\phi} \\ p D_{p\phi} & p^2 D_{\phi\phi} \end{array} \right),$$

$$D_{kl} \equiv \left( \frac{\Delta k \Delta l}{2 \Delta t} \right) \equiv \frac{1}{2 \Delta t} \int \Delta k \Delta l \psi(p, \Delta p) d(\Delta p), \quad (A2)$$

where $\psi(p, \Delta p) d(\Delta p)$ is the element of probability of a particle changing its momentum from $p$ to $p + \Delta p$ in time $\Delta t$, and indices $l$ and $k$ represent $p$ or $\phi$.

Assuming that $f(z, p, \phi)$ depends spatially only on the distance from the shock front and approximating the velocity of the accelerated particles by $c$, the second term on the left-hand side of Equation (A1) becomes $c \mu \frac{\partial f}{\partial \phi}$. Assuming elastic scattering in the fluid frame, momentum-space diffusion has contributions only from the $D_{\phi\phi}$ component. Equation (A1) thus becomes

$$\frac{\partial f}{\partial t} + c \mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial p} \left( \begin{array}{cc} 0 & 0 \\ 0 & p^2 D_{\phi\phi} \end{array} \right) \left( \begin{array}{c} \frac{\partial f}{\partial p} \\ \frac{1}{p} \frac{\partial f}{\partial \phi} \end{array} \right) \right). \quad (A3)$$

Assuming a steady state in the shock frame and switching to a mixed coordinate system where $z$ is measured in the shock frame, we obtain

$$c \gamma (\beta + \mu) \frac{\partial f}{\partial z} = \frac{\partial}{\partial \phi} \left( D_{\phi\phi} \frac{\partial f}{\partial \phi} \right). \quad (A4)$$

where the subscript $s$ refers to shock-frame variables.

Appendix B

Maxwell–Jüttner Distribution in 2D

The PDF of an arbitrarily relativistic ideal gas, known as the Maxwell–Jüttner or JS distribution, is given (in any dimension) by $f(x^\mu, p^\mu) = A e^{-(\Theta x^\mu p^\mu)}$ (e.g., Groot et al. 1980; Cercignani & Kremer 2012), where $A$ is a temperature-dependent normalization, $u_\parallel$ is the covariant velocity (three-vector in 2D, henceforth) of the system, $x^\mu$ is the position, $p^\mu$ is the momentum, $\Theta \equiv (k_B T)^{-1}$ is the thermodynamic inverse temperature, $k_B$ is the Boltzmann constant, and $T$ is the temperature in Kelvin. We derive $f$ for the 2D case below. In Appendix C, we derive the corresponding EOS, which is found to be simpler than in the 1D or 3D cases, and the resulting jump conditions across a strong shock.

We consider a flat spacetime of $N = 2$ spatial dimensions, governed by the metric $g^\mu_\nu$ with the sign convention of $\{+, -,-\}$. The distribution function for an arbitrarily relativistic gas can be derived by minimizing the free energy of the
system for a conserved particle number (e.g., Hakim 2011). The free energy density is given by

\[ F = \epsilon - k_B T \sigma, \]  

(B1)

where \( \epsilon \) is the internal energy density and \( \sigma \) is the entropy density. We may equivalently minimize \( F \) for a conserved particle number density \( n \),

\[ \delta n = 0, \quad \delta F = 0. \]  

(B2)

Define

\[ \sigma \equiv \frac{u_\mu S^\mu}{c^2}, \]  

(B3)

\[ n \equiv \frac{1}{c^2} \int u_\nu p^n f(p^\mu) \frac{d^4p}{p_0}, \]  

(B4)

\[ \epsilon \equiv \frac{1}{c^2} \int \frac{(u_\nu p^n)^2}{m^2} f(p^\mu) \frac{d^4p}{p_0}, \]  

(B5)

and the entropy current

\[ S^\mu \equiv - \int \frac{p^\mu}{m} f(p^\nu) \ln f(p^\mu) \frac{d^4p}{p_0} mc, \]  

(B6)

where \( (d^4p/p_0)mc \) is the Lorentz-invariant momentum volume element. Here, \( u_\mu \) is the average velocity of the fluid, not to be confused with the individual velocity of each particle (which we denote \( p^\mu/m \)). Henceforth, we adopt \( N = 2 \).

Plugging these definitions into Equations (B2) and (B1) yields

\[ \delta (\Theta F) = \delta \int [\Theta (u_\mu p^n)^2 + \ln f(p^\mu)] u_\mu p^n f(p^\mu) \frac{d^4p}{c p_0} = 0 \]  

(B7)

and

\[ \delta n = \delta \int \frac{1}{c} u_\mu p^n f(p^\mu) \frac{d^4p}{p_0} = 0. \]  

(B8)

Introducing a Lagrange multiplier \( L \) gives

\[ \delta \int \frac{1}{c} [\Theta (u_\mu p^n) + \ln f(p^\mu) + L] u_\mu p^n f(p^\mu) \frac{d^4p}{p_0} = 0, \]  

(B9)

from which

\[ f(p^\mu) = Ae^{-(\Theta u_\mu p^n)}, \]  

(B10)

where \( A \) is a normalization factor.

**Appendix C**

**Jüttner–Synge Equation of State and Shock Jump Conditions in 2D**

To determine the normalization factor \( A \), define the generating function

\[ \Pi \equiv \int e^{-(\Theta u_\mu p^n)} \frac{d^4p}{p_0} mc, \]  

(C1)

such that

\[ n = - \frac{A}{mc^2} \frac{\partial}{\partial \Theta} \Pi. \]  

(C2)

In the fluid frame, \( u_0 = c \) and \( u_{1,2} = 0 \), so in relativistic polar coordinates

\[ \begin{bmatrix} p_0 \\
\rho_1 \\
\rho_2 \end{bmatrix} = m c \begin{bmatrix} \cosh \chi \\
\sinh \chi \cos \phi \\
\sinh \chi \sin \phi \end{bmatrix}, \]  

(C3)

where \( \chi \) is the rapidity and \( \phi \) is the angle with respect to the \( x \)-axis. Equation (C1) then becomes

\[ \Pi = \int e^{-\Theta mc^2} \cosh \chi \sinh \chi \frac{d\chi}{d\phi} = 2 \pi m e^{-\Theta mc^2}, \]  

(C4)

so Equation (C2) yields

\[ A = \frac{n(\Theta c)^2}{2 \pi (1 + \Theta mc^2)} e^{-\Theta mc^2}. \]  

(C5)

Next, consider the energy-momentum tensor

\[ T^\mu_\nu = \int \frac{p^\mu p^\nu}{m} f(x, p) \frac{d^4p}{p_0} mc = \frac{A}{m} \frac{\partial^2 \Pi}{\partial u^\mu \partial u^\nu}. \]  

(C6)

Using the coordinate transformation \( \tilde{u}^\mu \equiv \Theta u^\mu \),

\[ T^\mu_\nu = \frac{A}{m} \frac{\partial^2 \Pi}{\partial \tilde{u}^\mu \partial \tilde{u}^\nu} = \frac{3 + 3\Theta mc^2 + (\Theta mc^2)^2 n}{1 + \Theta mc^2} \frac{n}{\Theta} \tilde{u}^\mu \tilde{u}^\nu - \frac{n}{\Theta} \eta^\mu_\nu, \]  

(C7)

A comparison with the perfect-fluid energy-momentum tensor,

\[ T^\mu_\nu = (\epsilon + P) u^\mu u^\nu - P \eta^\mu_\nu, \]  

(C8)

where \( P \) is the pressure, yields the EOS

\[ \epsilon + P = \frac{3 + 3\Theta mc^2 + (\Theta mc^2)^2 n}{1 + \Theta mc^2} \frac{n}{\Theta}; \]  

(C9)

\[ P = \frac{n}{\Theta}, \]  

(C10)

the latter being the ideal gas law.

The shock jump conditions can now be determined (as in Kirk & Duffy 1999) from energy-momentum conservation in the absence of external forces,

\[ \nabla T^\mu_\nu = 0, \]  

(C11)

In the fluid’s rest frame, Equation (C11) becomes

\[ \gamma u^a u^b \beta_a = \gamma u^a u^b \beta^c_0; \]  

(C12)

\[ \gamma^2 w^a \beta^2_a + P_a = \gamma^2 w^a \beta^2_d + P_d; \]  

(C13)

and

\[ \gamma^2 w^a \beta_a = \gamma^2 w^a \beta_d. \]  

(C14)

where \( \rho \equiv mn \) is the mass density and \( w \equiv P + \epsilon \) is the proper enthalpy density.

Assuming a cold upstream, we neglect the upstream pressure and internal energy, so \( w_u/\rho u c^2 \approx 1 \), and Equations (C12)–(C14) give \( \epsilon_u = \gamma_u \rho u c^2 \). Introducing the adiabatic index \( \Gamma_{ad} \) through

\[ P = (\Gamma_{ad} - 1)(\epsilon - \rho c^2) \]  

(C15)
then leads to
\[
\frac{w_d}{\rho_d c^2} = (\gamma_r - 1)\Gamma_{ad} + 1
\] (C16)
and the jump condition (Blandford & McKee 1976)
\[
\gamma_r^2 = \frac{(\frac{w_d}{\rho_d c^2})^2 (\gamma_r + 1)}{(\gamma_r - 1)(2 - \Gamma_{ad})\Gamma_{ad} + 2}.
\] (C17)
Finally, the above yields
\[
\Gamma_{ad} = 1 + \frac{1 + \zeta}{2 + \zeta}
\] (C18)
and, assuming a strong shock,
\[
\gamma_r = \frac{\epsilon_d}{\rho_d c^2} = 1 + \frac{\zeta_d + 2}{(1 + \zeta_d)\zeta_d}.
\] (C19)
Equations (C18)–(C19) can be plugged into Equations (C16) and (C17), obtaining \(\zeta_d\) as a function of \(\beta_u\) as the positive root of the polynomial
\[
\beta_u^2(2\zeta_d^3 + 7\zeta_d^2 + 8\zeta_d + 4)(\zeta_d^2 + 3\zeta_d + 3)^2 = (3\zeta_d^3 + 10\zeta_d^2 + 12\zeta_d + 6)^2.
\] (C20)
Equation (C19) then gives
\[
1 + \frac{\zeta_d + 2}{(1 + \zeta_d)\zeta_d} = (1 - \beta_d \beta_u)\gamma_d \gamma_u.
\] (C21)
which, given \(\beta_u\), can be solved for \(\beta_d\).

**Appendix D Convergence and Errors**

We demonstrate the convergence of the expansion in upstream eigenfunctions by varying the number \(M\) of terms in the expansion and in the numerical integration resolution. The spectral index, extrapolated to infinite resolution, is shown in Figures D1 and D2 as a function of \(M^{-1}\), for the exact elliptic cosine functions with \(\gamma_u \beta_u = 1\) and for the ultrarelativistic shock approximation, Equation (58), with \(\gamma_u \beta_u = 100\), respectively. The errors bars presented in Figure 3 are estimated by extrapolating the data to \(M \to \infty\).
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