Abstract. We introduce a new ‘geometric realization’ of an (abstract) simplicial complex, inspired by probability theory. This space (and its completion) is a metric space, which has the right (weak) homotopy type, and which can be compared with the usual geometric realization through a natural map, which has probabilistic meaning: it associates to a random variable its probability mass function. This ‘probability law’ function is proved to be a (Serre) fibration and a (weak) homotopy equivalence.

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1. Introduction and main results

In this paper we consider a new ‘geometric realization’ of an (abstract) simplicial complex, inspired by probability theory. This space is a metric space, which has the right (weak) homotopy type, and can be compared with the usual geometric realization through a map, which is very natural in probabilistic terms: it associates to a random variable its probability mass function. This ‘probability law’ function is proved to be a (Serre) fibration and a (weak) homotopy equivalence. This construction passes to the completion, and has nice functorial properties.

We specify the details now. Let $S$ be a set, and $\mathcal{P}_f(S)$ the set of its finite subsets. We set $\mathcal{P}_f^*(S) = \mathcal{P}_f(S) \setminus \{\emptyset\}$. Recall that an (abstract) simplicial complex is a collection of subsets $\mathcal{K} \subset \mathcal{P}_f^*(S)$ with the property that, for all $X \in \mathcal{K}$ and $Y \in \mathcal{P}_f^*(S)$, $Y \subseteq X \Rightarrow Y \in \mathcal{K}$. The elements of $\mathcal{K}$ are called its faces, and the vertices of $\mathcal{K}$ are the union of the elements of $\mathcal{K}$.

We endow $S$ with the discrete metric of diameter 1, and with the Borel $\sigma$-algebra associated to this topology. We let $\Omega$ denote a nonatomic standard probability space with measure $\lambda$. Recall that all such probability spaces are isomorphic and can be identified in particular with any hypercube $[0, 1]^n$, $n \geq 1$ endowed with the Lebesgue measure. We define $L(\Omega, S)$ as the set of random variables $\Omega \to S$, that is the set of measurable maps $\Omega \to S$ modulo the equivalence relation $f \equiv g$ if $f$ and $g$ agree almost everywhere, that is $\lambda(\{x; f(x) \neq g(x)\}) = 0$. We consider it as a metric space, endowed with the metric

$$d(f, g) = \int_{\Omega} d(f(t), g(t))dt = \lambda(\{x \in \Omega; f(x) \neq g(x)\}).$$
We define $L(\Omega, \mathcal{K})$ as the subset of $L(\Omega, S)$ made of the (equivalence classes of) measurable maps $f: \Omega \to S$ such that $\{s \in S \mid \lambda(f^{-1}((s))) > 0\} \in \mathcal{K}$.

Recall that the (usual) ‘geometric’ realization of $\mathcal{K}$ is defined as

$$|\mathcal{K}| = \{t: S \to [0, 1] \mid \{s \in S \mid t_s > 0\} \in \mathcal{K} \& \sum_{s \in S} t_s = 1\}$$

and that its topology is given by the direct limit of the $[0, 1]^A$ for $A \in \mathcal{P}(S)$. There is a natural map $L(\Omega, \mathcal{K}) \to |\mathcal{K}|$ which associates to $f: \Omega \to \mathcal{K}$ the element $t: S \to [0, 1]$ defined by $t_s = \lambda(f^{-1}((s)))$. In probabilistic terms, it associates to the random variable $f$ its probability law, or probability mass function. We denote $|\mathcal{K}|_1$ the same set as $|\mathcal{K}|$, but with the topology defined by the metric $|a - b| = \sum_{s \in S} |a(s) - b(s)|$. We denote $|\mathcal{K}|_1$ its completion as a metric space.

It is easily checked that, unless $S$ is finite, $L(\Omega, \mathcal{K})$ is not in general closed in $L(\Omega, S)$, and therefore not complete. We denote $\bar{L}(\Omega, \mathcal{K})$ its closure inside $L(\Omega, S)$. The ‘probability law’ map $\Psi: L(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$ is actually continuous, and can be extended to a map $\bar{\Psi}: \bar{L}(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$. Kane’s theorem about the contractibility of $\text{Aut}(\Omega)$ (see [2]) easily implies that these maps have contractible fibers. The goal of this note is to specify the homotopy-theoretic features of them. We get the following results.

**Theorem 1.1.**

1. The map $L(\Omega, \mathcal{K}) \to \mathcal{T}(\Omega, \mathcal{K})$ is a weak homotopy equivalence.
2. The ‘probability law’ map $L(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$ is a Serre fibration and a weak homotopy equivalence. It admits a continuous global section.
3. The ‘probability law’ map $\bar{L}(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$ is a Serre fibration and a weak homotopy equivalence. It admits a continuous global section.
4. $L(\Omega, \mathcal{K})$ and $\bar{L}(\Omega, \mathcal{K})$ have the same weak homotopy type as the ‘geometric realization’ $|\mathcal{K}|$ of $\mathcal{K}$.

In other terms, in the commutative diagram below, the vertical maps are Serre fibrations, and all the maps involved are weak homotopy equivalences, the map $|\mathcal{K}| \to |\mathcal{K}|_1$ being in addition a strong homotopy equivalence. Actually, when $\mathcal{K}$ is finite, we prove that all the maps are strong homotopy equivalences (see theorems 4, 6).

$$
\begin{array}{ccc}
L(\Omega, \mathcal{K}) & \xrightarrow{\Psi_k} & \bar{L}(\Omega, \mathcal{K}) \\
\psi_k \downarrow & & \downarrow \psi_k \\
|\mathcal{K}| & \xrightarrow{\iota} & |\mathcal{K}|_1 \\
\end{array}
$$

We now comment on the functorial properties of this construction. By definition, a morphism $\varphi: \mathcal{K}_1 \to \mathcal{K}_2$ between simplicial complexes is a map from the set $\bigcup \mathcal{K}_1$ of vertices of $\mathcal{K}_1$ to the set of vertices of $\mathcal{K}_2$ with the property that $\forall F \in \mathcal{K}_1 \varphi(F) \in \mathcal{K}_2$. We denote $\text{Simp}$ the corresponding category of simplicial complexes. For such an abstract simplicial complex $\mathcal{K}$, our space $L(\Omega, \mathcal{K})$ as for ambient space $L(\Omega, S)$ with $S = \bigcup \mathcal{K}$ the set of vertices of $\mathcal{K}$.

Let $\text{Set}$ denote the category of sets and $\text{Met}_1$ denote the full subcategory of the category metric spaces and contracting maps made of the spaces of diameter at most 1. Let $\text{CMet}_1$ be the full subcategory of $\text{Met}_1$ made of complete metric spaces. There is a completion functor $\text{Comp}: \text{Met}_1 \to \text{CMet}_1$ which associates to each metric space its completion. Then $L(\Omega, \bullet): X \leadsto L(\Omega, X)$ defines a functor $\text{Set} \to \text{CMet}_1$ (see [3]). It can be decomposed as $L(\Omega, \bullet) = \text{Comp} \circ L(\Omega, \bullet)$ where $L(\Omega, S)$ is the subspace of $L(\Omega, S)$ made of the (equivalence classes of) functions $f: \Omega \to S$ of essentially finite image, that is such that there exists $S_0 \subset S$ finite such that $\sum_{s \in S_0} \lambda(f^{-1}((s))) = 1$.

We prove in section 2.4 below that our simplicial constructions have similar functorial properties, which can be summed up as follows.

**Proposition 1.2.** $L(\Omega, \bullet)$ and $\bar{L}(\Omega, \bullet)$ define functors $\text{Simp} \to \text{Met}_1$ and $\text{Simp} \to \text{CMet}_1$, with the property that $\bar{L}(\Omega, \bullet) = \text{Comp} \circ L(\Omega, \bullet)$. 
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2. Simplicial properties and completion

In this section we prove part (1) of theorem \[1\]. We start by proving the functorial properties stated in the introduction.

2.1. Functorial properties. We denote, as in the previous section, $\bar{L}(\Omega, K)$ the closure of $L(\Omega, K)$ inside $L(\Omega, S)$. As a closed subset of a complete metric space, it is a complete metric space. For any $f \in L(\Omega, S)$, we denote

$$f(\Omega) = \{ s \in S \mid \lambda(f^{-1}(\{s\})) > 0 \}$$

the essential image of an arbitrary measurable map $\Omega \to S$ representing $f$.

Lemma 2.1. Let $f \in L(\Omega, S)$. Then $f \in \bar{L}(\Omega, K)$ if and only if every nonempty finite subset of $f(\Omega)$ belongs to $K$.

Proof. Assume $f \in \bar{L}(\Omega, K)$ and let $F \subset f(\Omega)$ be a nonempty finite subset as in the statement. We set $m = \min\{ \lambda(f^{-1}(\{s\})) \mid s \in F \}$. We have $m > 0$. Since $f \in \bar{L}(\Omega, K)$, there exists $f_0 \in L(\Omega, K)$ such that $d(f, f_0) < m$. We then have $F \subset f_0(\Omega)$. Indeed, there would otherwise exist $s \in F \setminus f_0(\Omega)$, and then $d(f, f_0) \geq \lambda(f^{-1}(\{s\})) > m$, a contradiction. From this we get $F \subset K$. Conversely, assume that every nonempty finite subset of $f(\Omega)$ belongs to $K$. From [3] proposition 3.3 we know that $f(\Omega) \subset S$ is countable. If $f(\Omega)$ is finite we have $f(\Omega) \in K$ by assumption and $f \in L(\Omega, K)$. Otherwise, let us fix a bijection $N \to f(\Omega)$, $n \mapsto x_n$ and define $f_n \in L(\Omega, S)$ by $f_n(t) = f(t)$ if $t \in \{ x_0, \ldots, x_n \}$, and $f_n(t) = x_0$ otherwise. Clearly $f_n(\Omega) \subset f(\Omega)$ is nonempty finite hence belongs to $K$, and $f_n \in L(\Omega, K)$. On the other hand, $d(f_n, f) \leq \sum_{k>n} \lambda(f^{-1}(\{x_k\})) \to 0$, hence $f \in \bar{L}(\Omega, K)$ and this proves the claim. \[\square\]

We prove that, as announced in the introduction, $\bar{L}(\Omega, \bullet)$ provides a functor $\text{Simp} \to \text{CMet}_1$ that can be decomposed as $\text{Comp} \circ L(\Omega, \bullet)$, where $L(\Omega, \bullet)$ is itself a functor $\text{Simp} \to \text{CMet}_1$.

Let $\varphi \in \text{Hom}_\text{Simp}(K_1, K_2)$ that is $\varphi : \bigcup K_1 \to \bigcup K_2$ such that $\varphi(F) \in K_2$ for all $F \in K_1$. If $f \in L(\Omega, K_1)$, then $g = L(\Omega, \varphi)(f) = \varphi \circ f$ is a measurable map and $g(\Omega) = \varphi(f(\Omega))$. Since $f(\Omega) \in K_1$ and $\varphi$ is simplicial we get that $\varphi(f(\Omega)) \in K_2$ hence $g \in L(\Omega, K_2)$. From this one gets immediately that $L(\Omega, \bullet)$ indeed defines a functor $\text{Simp} \to \text{Met}_1$.

Similarly, if $f \in \bar{L}(\Omega, K_1)$ and $g = \varphi \circ f = L(\Omega, \varphi)(f) \in L(\Omega, S)$, then again $g(\Omega) = \varphi(f(\Omega))$. But, for any finite set $F \subset g(\Omega) = \varphi(f(\Omega))$ there exists $F' \subset f(\Omega)$ finite and with the property that $F = \varphi(F')$. Now $f \in \bar{L}(\Omega, K_1)$ implies $F' \in K_1$, by lemma \[2.4\] hence $F \in K_2$ because $\varphi$ is a simplicial morphism. By lemma \[2.1\] one gets $g \in \bar{L}(\Omega, K_2)$, hence $\bar{L}(\Omega, \bullet)$ defines a functor $\text{Simp} \to \text{CMet}_1$.

We checks immediately that $L(\Omega, \bullet) = \text{Comp} \circ L(\Omega, \bullet)$, and this proves proposition \[1.2\].

2.2. Technical preliminaries. We denote by 2 in the notation $L(\Omega, 2)$ a set with two elements. When needed, we will also assume that this set is pointed, that is contains a special point called 0, so that $f \in L(\Omega, 2)$ can be identified with $\{ t \in \Omega ; f(t) \neq 0 \}$, up to a set of measure 0. Note that these conventions agree with the set-theoretic definition of 2 = \{0, 1\} = \{0, \{0\}\}.

Lemma 2.2. Let $F$ be a set. The map $f \mapsto \{ t \in \Omega ; f(t) \notin F \}$ is uniformly continuous $L(\Omega, S) \to L(\Omega, 2)$, and even contracting.

Proof. Let $f_1, f_2 \in L(\Omega, S)$, and $\Psi : L(\Omega, S) \to L(\Omega, 2)$ the map defined by the statement. Then $\Psi(f_1)(t) \neq \Psi(f_2)(t) \Rightarrow f_1(t) \neq f_2(t)$, hence $d(\Psi(f_1)(t), \Psi(f_2)(t)) \leq d(f_1(t), f_2(t))$ for all $t \in \Omega$ and finally $d(\Psi(f_1), \Psi(f_2)) \leq d(f_1, f_2)$, whence $\Psi$ is contracting and uniformly continuous. \[\square\]

Lemma 2.3. Let $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$. Then

$$\lambda([a, b] \cap [c, d]) \leq |a - c| + |b - d|.$$ 

Proof. There are six possible relative positions of $c$ with respect to $a \leq b$ to consider, which are depicted as follows.
In three of them, namely \(a \leq b \leq c \leq d\), \(c \leq d \leq a \leq b\), and \(c \leq a \leq b \leq d\), we have \(\lambda([a, b]\{c, d]\) = 0. In case \(c \leq a \leq d \leq b\), we have \(\lambda([a, b]\{c, d]\) = \lambda([d, b]) = |b - d| \leq |a - c| + |b - d|\). In case \(a \leq c \leq b \leq d\), we have \(\lambda([a, b]\{c, d]\) = \lambda([a, c]) = |a - c| \leq |a - c| + |b - d|\). Finally, when \(a \leq c \leq d \leq b\), we have \(\lambda([a, b]\{c, d]\) = \lambda([a, c] \cup [d, b]) = |a - c| + |b - d|\), and this proves the claim.

\[\square\]

**Lemma 2.4.** Let \(\Delta^r = \{\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}_+^r \mid \alpha_1 + \cdots + \alpha_r = 1\}\) denote the \(r\)-dimensional simplex. The map \(\Delta^r \to L(\Omega, \{1, \ldots, r\})\) defined by \(\alpha \mapsto f_\alpha\) where \(f_\alpha(t) = i\) if \(t \in [\alpha_1 + \cdots + \alpha_{i-1}, \alpha_1 + \cdots + \alpha_i]\) is continuous. More precisely it is \(2r\)-Lipschitz if \(\Delta^r\) is equipped with the metric 
\[
d(\alpha, \alpha') = \sum_i |\alpha_i - \alpha'_i|.
\]

**Proof.** We fix an identification \(\Omega \simeq [0,1]\). Let \(\alpha, \alpha' \in \Delta^r\). We denote \(\beta_i = \alpha_1 + \cdots + \alpha_i, \beta_0 = 0\), and we similarly define the \(\beta'_i\). We have \(\beta_i - \beta_{i-1} = \alpha_i\) hence \(|\beta'_i - \beta_i| \leq \sum_{k \leq i} |\alpha'_k - \alpha_k|\) and finally \(\sum_i |\beta'_i - \beta_i| \leq r \sum_i |\alpha'_i - \alpha_i|\). Now, for \(t \in [\beta_i, \beta_{i+1}]\) we have \(f^* t = f^* (t)\) unless \(t \notin [\beta'_i, \beta'_{i+1}].\) From this and lemma 2.3 we get 
\[
d(f^* \alpha, f^* \alpha') \leq \sum_{i=1}^r \lambda (|\beta_i - \beta_{i+1}| + |\beta'_i - \beta'_{i+1}|) \leq 2 \sum_{i=1}^r |\beta_i - \beta_{i+1}| \leq 2r \sum_{i=1}^r |\alpha_i - \alpha'_i|
\]
and this proves the claim.

\[\square\]

**Lemma 2.5.** Let \(\mathcal{K}\) be a simplicial complex and \(X\) a topological space, and \(A \subset X\). If \(\gamma_0, \gamma_1 : X \to \tilde{L}(\Omega, \mathcal{K})\) are two continuous maps such that \(\forall x \in X\) \(\gamma_0(x)(\Omega) \subset \gamma_1(x)(\Omega)\), and \(\gamma_0|_A = \gamma_1|_A\), then \(\gamma_0\) and \(\gamma_1\) are homotopic relative to \(A\). Moreover, if \(\gamma_0\) and \(\gamma_1\) take value inside \(L(\Omega, \mathcal{K})\), then the homotopy takes values inside \(L(\Omega, \mathcal{K})\).

**Proof.** We fix an identification \(\Omega \simeq [0,1]\). We define \(H : [0,1] \times X \to L(\Omega, S)\) by \(H(u, x)(t) = \gamma_0(x)(t)\) if \(t \geq u\) and \(H(u, x)(t) = \gamma_1(x)(t)\) if \(t < u\). We have \(H(0, \bullet) = \gamma_0\) and \(H(1, \bullet) = \gamma_1\).

We first check that \(H\) is indeed a (set-theoretic) map \([0,1] \times X \to \tilde{L}(\Omega, \mathcal{K})\). For all \(u \in [0,1]\) and \(x \in X\) we have \(H(u, x)(\Omega) \subset \gamma_0(x)(\Omega) \cup \gamma_1(x)(\Omega) = \gamma_1(x)(\Omega)\). Therefore \(H(u, x)(\Omega) \subset \mathcal{K}\) if \(\gamma_1(x) \in L(\Omega, \mathcal{K})\), and all nonempty finite subsets of \(H(u, x)(\Omega) \subset \gamma_1(x)(\Omega)\) belong to \(\mathcal{K}\) if \(\gamma_1(x) \in \tilde{L}(\Omega, \mathcal{K})\). From this, by lemma 2.3 we get that \(H\) takes values inside \(\tilde{L}(\Omega, \mathcal{K})\), and even inside \(L(\Omega, \mathcal{K})\) if \(\gamma_1 : X \to L(\Omega, \mathcal{K})\).

Now, we check that \(H\) is continuous over \([0,1] \times X\). We have \(d(H(u, x), H(v, x)) \leq |u - v|\) for all \(u, v \in [0,1]\) and, for all \(x, y \in X\) and \(u \in [0,1]\), we have 
\[
d(H(u, x), H(u, y)) = \int_0^u d(\gamma_1(x)(t), \gamma_1(y)(t))dt + \int_u^1 d(\gamma_0(x)(t), \gamma_0(y)(t))dt
\]
\[
\leq \int_0^1 d(\gamma_1(x)(t), \gamma_1(y)(t))dt + \int_0^1 d(\gamma_0(x)(t), \gamma_0(y)(t))dt = d(\gamma_0(x), \gamma_0(y)) + d(\gamma_0(x), \gamma_0(y))
\]
from which we get \(d(H(u, x), H(v, y)) \leq |u - v| + d(\gamma_1(x), \gamma_1(y)) + d(\gamma_0(x), \gamma_0(y))\) for all \(x, y \in X\) and \(u, v \in [0,1]\). For any given \((u, x) \in [0,1] \times X\) this proves that \(H\) is continuous at \((u, x)\). Indeed, given \(\varepsilon > 0\), from the continuity of \(\gamma_0, \gamma_1\) we get that, for some open neighborhood \(V\)
Proof. Let $\beta \subset L$ be an open subset of $L$. But $c$ is empty) subset such that $p(\beta) = c$ for all $c \in C_0$. Moreover, $p(\beta) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.

Proposition 2.6. Let $C$ be a compact subspace of $[L(\Omega, K)]$ and $C_0 \subset C \cap L(\Omega, K)$ a (possibly empty) subset such that $\bigcup_{c \in C_0} c(\Omega)$ is finite. Then there exists a continuous map $p : C \to L(\Omega, K)$ such that $p(c) = c$ for all $c \in C_0$. Moreover, $p(\beta) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.

Proof. For any $s \in S$ and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ we denote $O_{s,n} = \{ f \in L(\Omega, S) \mid \lambda(f^{-1}(\{s\})) > 1/n \}$. It is an open subset of $L(\Omega, S)$, hence $C_{s,n} = C \cap O_{s,n}$ is an open subset of $C$. Now, for every $c \in C$ there exists $s \in S$ such that $\lambda(c^{-1}(\{s\})) > 0$ hence $c \in C_{s,n}$ for some $n$. Then $C$ is compact and covered by the $C_{s,n}$ hence there exists $s_1, \ldots, s_r \in S$ and $n_1, \ldots, n_r \in \mathbb{N}^*$ such that $C \subset \bigcup_{i=1}^r O_{s_i,n_i}$. Up to replacing the $n_i$’s by their maximum, we may suppose $n_1 = \cdots = n_r = n_0$. Let then $F' = \bigcup_{c \in C_0} c(\Omega) \in \mathcal{E}$. We set $F = \{ s_1, \ldots, s_r \} \cup F'$. For any $i \in \{ 1, \ldots, r \}$ we set $O_i = O_{s_i,n_0}$.

For any $c \in C$, we set $\Omega_c = \{ t \in [0,1] \mid c(t) \notin F \}$, and

$$\alpha_i(c) = \frac{d(c, c^\circ O_i)}{\sum_j d(c, c^\circ O_j)}$$

and $\beta_i(c) = \sum_{k \leq i} \alpha_k(c)$, where $c^\circ X$ denotes the complement of $X$. These define continuous maps $C \to \mathbb{R}_+$. We fix an identification $\Omega \simeq [0,1]$, so that intervals make sense inside $\Omega$. We then set

$$p(c)(t) = \begin{cases} c(t) & \text{if } c(t) \in F, \text{ i.e. } t \notin \Omega_c \\ s_i & \text{if } t \in \Omega_c \cap [\beta_i(c), \beta_i(c)] \end{cases}$$

Let $s \in S$ and $c \in C$ and $c^s, s = 1,2$ the corresponding $r$-tuples $(c_1^s, \ldots, c_r^s) \in \Delta^r$ given by $\alpha_i^s = \alpha_i(c^s)$. When $t \notin \Omega_{c_1} \cup \Omega_{c_2}$ we have $p(c_1)(t) = p(c_2)(t)$, hence

$$d(p(c_1), p(c_2)) = \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t))dt \leq \lambda(\Omega_{c_1} \Delta \Omega_{c_2}) + \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t))dt$$

and we know $\lambda(\Omega_{c_1} \Delta \Omega_{c_2}) \leq d(c_1, c_2)$ because of lemma 2.2. Therefore we only need to check that the term $\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t))dt$ is continuous. But, by lemma 2.3 we have

$$\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t))dt = \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(f_{c_1^0}(t), f_{c_2^0}(t))dt \leq d(f_{c_1^0}, f_{c_2^0}) \leq 2|\alpha^1 - \alpha^2|$$

whence the continuity, by continuity of $c \to \Omega_c$.

We must now check that $p$ takes values inside $L(\Omega, K)$. Let $c \in C$. We know that $p(c)(\Omega) \subset F$ is finite, and

$$p(c)(\Omega) \subset c(\Omega) \cup \{ s_i ; c \in O_i \}.$$
Proposition 2.8. Let $C$ be a compact space, and $x_0 \in C$. For any simplicial complex $K$, and any continuous map $\gamma : C \to L(\Omega, K)$, there exists a continuous map $\hat{\gamma} : (C, x_0) \to (L(\Omega, K), \gamma(x_0))$ which is homotopic to $\gamma$ relative to $\{(x_0), \{\gamma(x_0)\}\}$, and such that $\bigcup_{x \in C} \hat{\gamma}(x)(\Omega)$ is finite.

Proof. Let $C' = \gamma(C) \subset L(\Omega, K)$. It is compact, hence applying corollary 2.7 to it and to $\{c_i\} = \{\gamma(x_0)\}$ we get a continuous map $\rho : C' \to L(\Omega, K)$ such that $\bigcup_{c \in C'} \rho(c)(\Omega)$ is finite, and $\rho(c)(\Omega) \subset \gamma(c)(\Omega)$ for all $c \in C'$. Therefore, letting $\hat{\gamma} = \rho \circ \gamma : C \to L(\Omega, K)$, we get that $\bigcup_{x \in C} \hat{\gamma}(x)(\Omega)$ is finite. Since $\hat{\gamma}(x)(\Omega) \subset \gamma(x)(\Omega)$ for all $x \in C$, we get from lemma 2.5 that $\gamma$ and $\hat{\gamma}$ are homotopic, hence the conclusion.

□

Proposition 2.9. Let $C$ be a compact space (and $x_0 \in C$), $K$ a simplicial complex, and a pair of continuous maps $\gamma_0, \gamma_1 : C \to L(\Omega, K)$ (with $\gamma_0(x_0) = \gamma_1(x_0)$). If $\gamma_0$ and $\gamma_1$ are homotopic as maps in $L(\Omega, K)$ (relative to $\{(x_0), \{\gamma_0(x_0)\}\}$), then they are homotopic inside $L(\Omega, K)$ (relative to $\{(x_0), \{\gamma_0(x_0)\}\}$).

Proof. After proposition 2.8 there exists $\hat{\gamma}_0, \hat{\gamma}_1 : C \to L(\Omega, K)$ such that $\hat{\gamma}_i$ is homotopic to $\gamma_i$ with the property that $\bigcup_{x \in C} \hat{\gamma}_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Without loss of generality, one can therefore assume that $\bigcup_{x \in C} \gamma_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Let $H : C \times [0, 1] \to L(\Omega, K)$ be a homotopy between $\gamma_0$ and $\gamma_1$. Let $C' = H(C \times [0, 1])$ and $C_0 = \gamma_0(C) \cup \gamma_1(C)$. These are two compact spaces which satisfy the assumptions of proposition 2.6. If $p : C' \to L(\Omega, K)$ is the continuous map afforded by this proposition, then $\hat{H} = p \circ H$ provides a homotopy between $\gamma_0$ and $\gamma_1$ inside $L(\Omega, K)$. The ‘relative’ version of the statement is proved similarly.

□

In particular, when $C$ is equal to the $n$-sphere $S^n$, this proves that the natural map $[S^n, L(\Omega, K)]_* \to [S^n, L(\Omega, K)]_*$, between sets of pointed homotopy classes is injective. In order to prove theorem 1.1(1), we need to prove that it is surjective. Let us consider a continuous map $\gamma : S^n \to L(\Omega, K)$ and set $C = \gamma(S^n)$. It is a compact subspace of $L(\Omega, K)$. Applying proposition 2.6 with $C_0 = \emptyset$ we get $p : C \to L(\Omega, K)$ such that $p(c)(\Omega) \subset c(\Omega)$ for any $c \in C$. Let then $\hat{\gamma} = p \circ \gamma : S^n \to L(\Omega, K)$. From lemma 2.5 we deduce that $\hat{\gamma}$ and $\gamma$ are homotopic inside $L(\Omega, K)$, and this concludes the proof of part (1) of theorem 1.1.

3. Homotopies inside $L(\Omega, \{0, 1\})$

In this section we denote $L(2) = L(\Omega, 2) = L(\Omega, \{0, 1\})$, with $d(0, 1) = 1$. Since we are going to use Lipschitz properties of maps, we specify our conventions on metrics. When $(X, d_X)$ and $(Y, d_Y)$ are two metric spaces, we endow $X \times Y$ with the metric $d_X + d_Y$, and the space $C^0([0, 1], X)$ of continuous maps $[0, 1] \to X$ with the metric of uniform convergence $d(\alpha, \beta) = \|\alpha - \beta\|_{\infty} = \sup_{t \in [1]} |\alpha(t) - \beta(t)|$. Recall that the topology on $C^0([0, 1], X)$ induced by this metric is the compact-open topology. For short we set $C^0(X) = C^0([0, 1], X)$.

Identifying $L(2) = L(\Omega, 2)$ with the space of measurable subsets of $\Omega$ (modulo subsets of measure 0) endowed with the metric $d(E, F) = \lambda(E \Delta F)$, where $\Delta$ is the symmetric difference operator, we have the following lemma. This lemma can be viewed as providing a continuous parametrization by arc-length of natural geodesics inside the metric space $L(2)$.

Lemma 3.1. The exists a continuous map $g : L(2) \times [0, 1] \to L(2)$ such that $g(A, 0) = A$, $\lambda(g(A, u)) = \lambda(A)(1-u)$ and $g(A, u) \supset g(A, v)$ for all $A$ and $u \leq v$. Moreover, it satisfies

$$\lambda(g(E, u) \Delta g(F, v)) \leq 4 \lambda(E \Delta F) + |v - u|$$

for all $E, F \in L(2)$ and $u, v \in [0, 1]$.

Proof. We fix an identification $\Omega \simeq [0, 1]$. For $E \in L(2) \setminus \{\emptyset\}$ we define $g_E(t) = \lambda([t, 1] \cap E)/\lambda(E)$. The map $g_E$ is obviously (weakly) decreasing and continuous $[0, 1] \to [0, 1]$, with $g_E(0) = 1$ and $g_E(1) = 0$. It is therefore surjective, and we can define a (weakly) decreasing map $g_E^{-1} : [0, 1] \to [0, 1]$ by $g_E^{-1}(u) = \inf g_E^{-1}^{-1}(\{u\})$. Since $g_E$ is continuous, we have $g_E^{-1}(\Omega) \subset \Omega$.

One defines $g(E, u) = E \cap g_E^{-1}(1-u)$ if $\lambda(E) \neq 0$, and $g(0, u) = \emptyset$. We have $\lambda(g(E, u)) = \lambda(E \cap [g_E^{-1}(1-u), 1]) = g_E^{-1}(1-u)\lambda(E) = (1-u)\lambda(E)$ when $\lambda(E) \neq 0$, and $\lambda(g(0, u)) = 0 = \lambda(E)(1-u)$ if $\lambda(E) = 0$. It is clear that $g(E, u) \subset g(E, v)$ for all $u \leq v$.
Moreover, clearly \( g(E, 0) = E \) since \( E \cap [\psi_E(1), 1] \subset E \) and \( \lambda(E \cap [\psi_E(1), 1]) = \varphi_E(\psi_E(1))\lambda(E) = \lambda(E) \). It remains to prove that \( g \) is continuous.

Let \( E, F \in L(2) \) and \( u, v \in [0, 1] \). We first assume \( \lambda(E)\lambda(F) > 0 \). Without loss of generality we can assume \( \psi_E(1 - u) \leq \psi_F(1 - v) \). Then \([\psi_E(1 - u), 1] \supset [\psi_F(1 - v), 1]\), and \( g(E, u)\Delta g(F, v) \) can be decomposed as

\[
((E \setminus F) \cap [\psi_E(1 - u), 1]) \cup ([F \setminus E] \cap [\psi_F(1 - v), 1]) \cup ((E \cap F) \cap [\psi_E(1 - u), 1]) \cup ([F \cap E] \cap [\psi_F(1 - v), 1]).
\]

Since the first two pieces are included inside \( E \Delta F \), we get

\[
\lambda(g(E, u)\Delta g(F, v)) \leq \lambda(E \Delta F) + \lambda((E \cap F) \cap [\psi_E(1 - u), 1] \cup ([F \cap E] \cap [\psi_F(1 - v), 1])\]

hence

\[
\lambda((E \cap F) \cap [\psi_E(1 - u), 1]) = \lambda(E \cap F \cap [\psi_E(1 - u), 1]) - \lambda(E \cap F \cap [\psi_F(1 - v), 1]) \leq (1 - u)\lambda(E) - \lambda(E \cap F \cap [\psi_F(1 - v), 1])
\]

Now, since \( F = (E \cap F) \cup (F \setminus E) \), we have \( F \cap [\psi_F(1 - v), 1] = ((E \cap F) \cap [\psi_F(1 - v), 1]) \cup ((F \setminus E) \cap [\psi_F(1 - v), 1]) \) hence

\[
(1 - v)\lambda(F) = \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda((F \setminus E) \cap [\psi_F(1 - v), 1]) \leq \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda(F \setminus E) \leq \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda(F \Delta E).
\]

It follows that

\[
\lambda((E \cap F) \cap [\psi_F(1 - v), 1]) \leq \lambda(F \Delta E) - (1 - v)\lambda(F)
\]

and finally

\[
\lambda(g(E, u)\Delta g(F, v)) \leq 2\lambda(E \Delta F) + (1 - u)\lambda(E) - (1 - v)\lambda(F)
\]

Therefore we get the inequality \( \lambda(g(E, u)\Delta g(F, v)) \leq 4\lambda(E \Delta F) + |v - u| \), that we readily check to hold also when \( \lambda(E)\lambda(F) = 0 \). This proves that \( g \) is continuous, whence the claim.

We provide a 2-dimensional illustration, with \( \Omega = [0, 1]^2 \). The map constructed in the proof depends on an identification \( [0, 1]^2 \simeq [0, 1] \) (up to a set of measure 0). An explicit one is given by the binary-digit identification

\[
0, \varepsilon_1\varepsilon_2\varepsilon_3 \cdots \mapsto (0, \varepsilon_1\varepsilon_3\varepsilon_5 \ldots , 0, \varepsilon_2\varepsilon_4\varepsilon_6 \ldots )
\]

with the \( \varepsilon_i \in \{0, 1\} \). Then, when \( A \) is some (blue) rectangle, the map \( u \mapsto g(A, u) \) looks as follows.

The above lemma is actually all what is needed to prove theorem \[\text{I.1}\] in the case of binary random variables, that is \( S = \{0, 1\} \), as we will illustrate later (see corollary \[\text{I.5}\]). In the general case however, we shall need a more powerful homotopy, provided by proposition \[\text{3.6}\] below. The next lemmas are preliminary technical steps in view of its proof.

**Lemma 3.2.** The map \( C^0(L(2)) \times L(2) \to C^0([0, 1]) \) defined by \( (E_\bullet, A) \to \alpha \) where \( \alpha(u) = \lambda(E_u \cap A) \), is 1-Lipschitz.

**Proof.** Let \( \alpha, \beta \) denote the images of \((E_\bullet, A)\) and \((F_\bullet, B)\), respectively. Then, for all \( u \in I \), we have

\[
|\alpha(u) - \beta(u)| = |\lambda(E_u \cap A) - \lambda(F_u \cap B)| \leq \lambda((E_u \cap A) \Delta (F_u \cap B))
\]
From the general set-theoretic inequality \((X \cap A) \Delta (Y \cap B) \subseteq (X \Delta Y) \cup (A \Delta B)\) one gets
\[
\lambda ((E_u \cap A) \Delta (F_u \cap B)) \leq \lambda (E_u \Delta F_u) + \lambda (A \Delta B),
\]
hence \(\|\alpha - \beta\|_\infty \leq \sup_u \lambda (E_u \Delta F_u) + \lambda (A \Delta B)\) and this proves the claim.

**Lemma 3.3.** A map \(\Phi_- : C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))\) is defined as follows. To \((a, E_*, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))\) one associates the map
\[
\Phi_-(a, E_*, A) : u \mapsto g \left( E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u), \alpha(u))}{\alpha(u)} \right)
\]
if \(a(u) \neq 0\), and otherwise \(u \mapsto \emptyset\), where \(\alpha(u) = \lambda(A \cap E_u)\). Then, the map \(\Phi_-\) is continuous.

**Proof.** Let us fix \((a, E_*, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2)\), and let \(\varepsilon > 0\). Consider \(\hat{m} : [0,1] \times [\varepsilon/12,1] \to [0,1]\) be defined by \(\hat{m}(x,y) = \min(x,y)/y\). It is clearly continuous on the compact space \([0,1] \times [\varepsilon/12,1]\), hence uniformly continuous, hence there exists \(\eta > 0\) such that \(\max(|x_1 - x_2|, |y_1 - y_2|) < \eta \Rightarrow |\hat{m}(x_1, y_1) - \hat{m}(x_2, y_2)| < \varepsilon/6\). Clearly one can assume \(\eta \leq \varepsilon/6\) as well.

Let us then consider \((b, F_*, B) \in C^0([0,1]) \times C^0(L(2)) \times L(2)\) such that \(\|a-b\|_\infty + \sup_u \lambda (E_u \Delta F_u) + \lambda (A \Delta B) \leq \eta\). From lemma 6.2 we get \(\|\alpha - \beta\|_\infty \leq \eta\). Let us consider \(I_0 = \{u \in [0,1] \mid \alpha(u) \leq \varepsilon/3\}\). We have by definition \(\alpha(0,1) \setminus I_0 \subseteq \varepsilon/3, 1 \subseteq [\varepsilon/12,1]\) and, since \(\|\alpha - \beta\|_\infty \leq \varepsilon/6\), we have \(\beta([0,1] \setminus I_0) \subseteq \varepsilon/6, 1 \subseteq [\varepsilon/12,1]\). Moreover, since
\[
|a(u)\lambda(E_u) - b(u)\lambda(F_u)| \leq |a(u) - b(u)|\lambda(E_u) + b(u)|\lambda(E_u) - \lambda(F_u)| \leq |a(u) - b(u)| + \lambda(E_u \Delta F_u) \leq \eta
\]
we get that, for all \(u \not\in I_0\), we have \(|\hat{m}(a(u)\lambda(E_u)), \alpha(u)) - \hat{m}(b(u)\lambda(F_u)), \beta(u))| < \varepsilon/6\). Moreover, since in particular \(\alpha(u)\beta(u) \neq 0\), we get from the general inequality \(\lambda(g(X, x)\Delta g(Y, y)) \leq 4\lambda(X \Delta Y) + |x-y|\) of lemma 5.1 that, for all \(u \not\in I_0\),
\[
d(\Phi_-(a, E_*, A)(u), \Phi_-(b, F_*, B)(u)) \leq 4\lambda((E_u \cap A) \Delta (F_u \cap B)) + |\hat{m}(a(u)\lambda(E_u), \alpha(u)) - \hat{m}(b(u)\lambda(F_u), \beta(u))| \leq 4\lambda(E_u \Delta F_u) + \lambda(A \Delta B) + \varepsilon/6 \leq 4\varepsilon/6 + \varepsilon/6 < \varepsilon
\]
Now, if \(u \in I_0\), then \(\Phi_-(a, E_*, A)(u) \subseteq E_u \cap A\) hence \(\lambda(\Phi_-(a, E_*, A)(u)) \leq \lambda(E_u \cap A) = \alpha(u) \leq \varepsilon/3\) and \(\lambda(\Phi_-(b, F_*, B)(u)) \leq \lambda(F_u \cap B) = \beta(u) \leq \varepsilon/3 + \varepsilon/6 = \varepsilon/2\). Whence
\[
d(\Phi_-(a, E_*, A)(u), \Phi_-(b, F_*, B)(u)) \leq \lambda(\Phi_-(a, E_*, A)(u)) + \lambda(\Phi_-(b, F_*, B)(u)) \leq 5\varepsilon/6 < \varepsilon.
\]
It follows that \(d(\Phi_-(a, E_*, A), \Phi_-(b, F_*, B)) \leq \varepsilon\) and \(\Phi_-\) is continuous at \((a, E_*, A)\), which proves the claim.

We use the convention \(g(X, t) = X\) for \(t \leq 0\) and \(g(X, t) = \emptyset\) for \(t > 1\), so that \(g\) is extended to a continuous map \(L(2) \times \mathbb{R} \to L(2)\). The notation \(^cA\) denotes the complement inside \(\Omega\) of the set \(A\), identified with an element of \(L(\Omega, 2)\).

**Lemma 3.4.** A map \(\Phi_+ : C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))\) is defined as follows. To \((a, E_*, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))\) one associates the map
\[
\Phi_+(a, E_*, A) : u \mapsto g \left( E_u \cap (^cA), 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda(E_u) - \alpha(u)} \right)
\]
if \(\alpha(u) \neq \lambda(E_u)\), and otherwise \(u \mapsto \emptyset\), where \(\alpha(u) = \lambda(A \cap E_u)\). Then, the map \(\Phi_+\) is continuous.

The proof is similar to the one of the previous lemma, and left to the reader.

**Lemma 3.5.** The map \((f, g) \mapsto (t \mapsto f(t) \cup g(t))\) is continuous \(C^0(L(2))^2 \to C^0(L(2))\), and even \(1\)-Lipschitz.
Proof. The map \((X,Y) \mapsto X \cup Y\) is 1-Lipschitz because of the general set-theoretic fact \((X_1 \cup Y_1) \Delta (X_2 \cup Y_2) \subseteq (X_1 \Delta X_2) \cup (Y_1 \Delta Y_2)\) from which we deduce \(\lambda((X_1 \cup Y_1) \Delta (X_2 \cup Y_2)) \leq \lambda(X_1 \Delta X_2) + \lambda(Y_1 \Delta Y_2)\), which proves that \((X,Y) \mapsto X \cup Y\) is 1-Lipschitz \(L(2)^2 \to L(2)\). It follows that the induced map \(C^0(L(2)^2) = C^0(L(2))^2 \to C^0(L(2))\) is 1-Lipschitz and thus continuous, too. □

The following proposition informally says that, when \(E_* \in C^0(L(2))\) is a path inside \(L(2)\) with \(A \subseteq E_0\), then we can find another path \(\Phi_* \in C^0(L(2))\) such that \(\Phi_u \subseteq E_0\) for all \(u\), and the ration \(\lambda(\Phi_1)/\lambda(E_0)\) follows any previously specified variation starting at \(\lambda(A)\) and, moreover, that this can be done continuously.

**Proposition 3.6.** There exists a continuous map \(\Phi : C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))\) with the property that, for all \((a, E_*, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2)\) such that \(A \subseteq E_0\) and \(a(0)\lambda(E_0) = \lambda(A)\), we have \(\Phi(a, E_*, A)(0) = A\) and, for all \(u \in [0,1]\), \(\Phi(a, E_*, A)(u) \subseteq E_u\) and \(\lambda(\Phi(a, E_*, A)(u)) = a(u)\lambda(E_u)\).

**Proof.** We define \(\Phi(a, E_*, A)(u) = \Phi(a, E_*, A)(u) \cup \Phi_+(a, E_*, A)(u)\). By combining lemmas \(3.3\) and \(3.4\) we get that \(\Phi\) is continuous. Moreover, \(\Phi(a, E_*, A)(u) \subseteq E_u \cap A\) and \(\Phi_+(a, E_*, A)(u) \subseteq E_u \cap (\bar{C} A)\) hence \(\Phi(a, E_*, A)(u) = \Phi(a, E_*, A)(u) \cup \Phi_+(a, E_*, A)(u) \subseteq E_u\), with \(\lambda(\Phi(a, E_*, A)(u)) = \lambda(\Phi_-(a, E_*, A)(u)) + \lambda(\Phi_+(a, E_*, A)(u))\). Letting \(\alpha(u) = \lambda(E_u \cap A)\), again by lemmas \(3.3\) and \(3.4\) we get

\[
\lambda(\Phi_-(a, E_*, A)(u)) = \lambda \left( g \left( E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u), \alpha(u))}{\alpha(u)} \right) \right) = \min(a(u)\lambda(E_u), \alpha(u))
\]

and, since \(\lambda(E_u) - \alpha(u) = \lambda(E_u) - \lambda(A \cap E_u) = \lambda(\bar{C} A) \cap E_u\),

\[
\lambda(\Phi_+(a, E_*, A)(u)) = \lambda \left( g \left( E_u \cap (\bar{C} A), 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda(\bar{C} A) \cap E_u} \right) \right) = \max(0, a(u)\lambda(E_u) - \alpha(u)).
\]

Therefore we get \(\lambda(\Phi(a, E_*, A)(u)) = \max(0, a(u)\lambda(E_u) - \alpha(u)) + \min(a(u)\lambda(E_u), \alpha(u)) = a(u)\lambda(E_u)\) for all \(u \in [0,1]\). Finally, since \(A \subseteq E_0\) and \(\alpha(0) = \lambda(E_0 \cap A) = \lambda(A) = \lambda(E_0)a(0)\), we get that \(\Phi(a, E_*, A)(0) = g(E_0 \cap A, 0) \cup g(E_0 \cap (\bar{C} A), 1) = A \cup \emptyset = A\), and this proves the claim. □

As before, we provide an illustration, when \(A \subseteq \Omega\) is the same (blue) rectangle, and \(E_*\) associates continuously to any \(u \in [0,1]\) some rectangle, whose boundary is dashed and in red. In this example, the map \(a\) is taken to be affine, from \(\lambda(A)/\lambda(E_0)\) to 0. The first row depicts the map \(u \mapsto E_u\), and the second row superposes it with the map \(u \mapsto \Phi(a, E_*, A)(u)\), depicted in blue.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| u ~ 0.0 | u ~ 0.1 | u ~ 0.2 | u ~ 0.3 | u ~ 0.4 | u ~ 0.5 | u ~ 0.6 |
| u ~ 0.7 | u ~ 0.8 | u ~ 0.9 |

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| u ~ 0.0 | u ~ 0.1 | u ~ 0.2 | u ~ 0.3 | u ~ 0.4 | u ~ 0.5 | u ~ 0.6 |
| u ~ 0.7 | u ~ 0.8 | u ~ 0.9 |

**4. Probability law**

**4.1. The law maps.** Recall from [5] that the weak (or coherent) topology on \(|\mathcal{K}|\) is the topology such that \(U\) is open in \(|\mathcal{K}|\) iff \(U \cap |F|\) is open for every \(F \in \mathcal{K}\), where \(|F| = \{\alpha : F \to [0,1] | \sum_{s \in F} \alpha(s) = 1\}\) is given the topology induced from the product topology of \([0,1]^F\). For each \(p \geq 1\), we can put a metric topology on the same set, in order to define a metric space \(|\mathcal{K}|_{d_p}\) by the metric \(d_p(\alpha, \beta) = \sqrt{\sum_{s \in S} |\alpha(s) - \beta(s)|^p}\). The map \(|\mathcal{K}| \to |\mathcal{K}|_{d_p}\) is continuous, and it is an homeomorphism iff \(|\mathcal{K}|\) is metrizable iff it is satisfies the first axiom of countability, iff \(\mathcal{K}\) is locally finite (see [5] p. 119 ch. 3 sec. 2 theorem 8 for the case \(p = 2\), but the proof works for \(p \neq 2\) as well).
For $\alpha : S \to [0, 1]$, we denote the support of $\alpha$ by $\text{supp}(\alpha) = \{ s \in S \mid \alpha(s) \neq 0 \}$. We let $\Psi_0 : L(\Omega, \mathcal{K}) \to |\mathcal{K}|$ be defined by associating to a random variable $f \in L(\Omega, \mathcal{K})$ its probability law $s \mapsto \lambda(f^{-1}(\{s\}))$.

4.2. **Non-continuity of $\Psi_0$.** We first prove that $\Psi_0$ is not continuous in general, by providing an example. Let us consider $S = \mathbb{N} = \mathbb{Z}_{\geq 0}$, and $\mathcal{K} = \mathcal{P}_1^*(\mathbb{N})$. We introduce

$$U = \left\{ \alpha \in |\mathcal{K}| \mid \forall s \neq 0 \; \alpha(s) < \frac{1}{\#\text{supp}(\alpha)} \right\}.$$  

We note that $U$ is open in $|\mathcal{K}|$. Indeed, if $F \in \mathcal{K}$ we have

$$U \cap F = \left\{ \alpha : F \to [0, 1] \mid \sum_{s \in F} \alpha(s) = 1 \; \& \; \forall s \neq 0 \; \alpha(s) < \frac{1}{\#\text{supp}(\alpha)} \right\}$$

which is equal to

$$\bigcup_{G \subset F \setminus \{0\}} \left\{ \alpha : G \to [0, 1] \mid \alpha(0) + \sum_{s \in G} \alpha(s) = 1 \; \& \; \forall s \in G \; 0 < \alpha(s) < \frac{1}{\#G + 1} \right\}$$

and it is open as the union of a finite collection of open sets. Now consider $\Psi_0^{-1}(U)$, and let $f_0 \in L(\Omega, \mathcal{K})$ be the constant map $t \mapsto 0$. Clearly $\alpha_0 = \Psi_0(f_0)$ is the map $0 \mapsto 1$, $k \mapsto 0$ for $k \geq 1$, and $\alpha_0 \in U$. If $\Psi_0^{-1}(U)$ is open, there exists $\varepsilon > 0$ such that it contains the open ball centered at $f_0$ with radius $\varepsilon$. Let $n$ such that $1/n < \varepsilon/3$, and define $f \in L([0, 1], \mathcal{K})$ by $f(t) = 0$ for $t \in [0, 1 - 2/n]$, $f(t) = k$ for $t \in [1 - 2/n + k - 1/n^2, 1 - 2/n + k/n^2]$ and $1 \leq k \leq n^2$, and finally $f(t) = n^2 + 1$ for $t \in [1 - \frac{1}{n^2}, 1]$. The graph of $f$ for $n = 3$ is depicted below.

We have $d(f, f_0) = 2/n < 2\varepsilon/3 < \varepsilon$ hence we should have $\alpha = \Psi_0(f) \in U$. But the support of $\alpha$ has cardinality $n^2 + 2$, and $\alpha(n^2 + 1) = 1/n > 1/(n^2 + 2)$, contradicting $\alpha \in U$. This proves that $\Psi_0$ is not continuous.

4.3. **Continuity of $\Psi$ and existence of global sections.** For short, we now denote $|\mathcal{K}|_p = |\mathcal{K}|_{d_p}$. We consider the same ‘law’ map $\Psi : L(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$. We prove that it is uniformly continuous (and actually 2-Lipschitz). Indeed, if $f, g \in L(\Omega, \mathcal{K})$, and $\alpha = \Psi(f), \beta = \Psi(g)$, then

$$d_1(\alpha, \beta) = \sum_{s \in S} |\alpha(s) - \beta(s)| = \sum_{s \in S} |\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))|$$

and $|\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))| \leq \lambda(f^{-1}(s) \Delta g^{-1}(s))$. But $f^{-1}(s) \Delta g^{-1}(s) = \{ t \in f^{-1}(s) \mid f(t) \neq g(t) \}$, whence

$$d_1(\alpha, \beta) \leq \sum_{s \in S} \int_{f^{-1}(s)} d(f(t), g(t))dt + \sum_{s \in S} \int_{g^{-1}(s)} d(f(t), g(t))dt = 2 \int_{\Omega} d(f(t), g(t))dt$$

whence $d_1(\alpha, \beta) \leq 2d(f, g)$. It follows that it induces a continuous map $\overline{L}(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$, where

$$|\mathcal{K}|_1 = \{ \alpha : S \to [0, 1] \mid \mathcal{P}_1^*(\text{supp}(\alpha)) \subset \mathcal{K} \; \& \; \sum_{s \in S} \alpha(s) = 1 \}$$
endowed with the metric \( d(\alpha, \beta) = \sum_{s \in S} |\alpha(s) - \beta(s)| \) is the completion of \(|K|_1\). This map associates to \( f \in \bar{L}(\Omega, K) \) the map \( \alpha(s) = \lambda(f^{-1}(s)) \).

The fact that \(|K|_1\) has the same homotopy type than \(|K|\) has originally been proved by Dowker in [1] in a more general context, and another proof was subsequently provided by Milnor in [2].

It is clear that every mass distribution on the discrete set \( S \) is realizable by some random variable. We first show that it is possible to do this continuously. In topological terms, this proves the following statement.

**Proposition 4.1.** The maps \( \Psi \) and \( \overline{\Psi} \) admit global (continuous) sections.

**Proof.** We fix some (total) ordering \( \leq \) on \( S \) and some identification \( \Omega \simeq [0,1] \). We define \( \sigma : |K|_1 \to \bar{L}(\Omega, K) \) as follows. For any \( \alpha \in |K|_1 \), \( S_\alpha = \text{supp}(\alpha) \subset S \) is countable. Let \( A_\pm : S \to \mathbb{R}_+ \) denote the associated cumulative mass functions \( A_+(s) = \sum_{u \leq s} \alpha(u) \) and \( A_-(s) = \sum_{u < s} \alpha(u) \).

They induce increasing injections \( (S_\alpha, \leq) \to [0,1] \). The map \( \sigma(\alpha) \) is defined by \( \sigma(\alpha)(t) = a \) if \( A_-(a) \leq t < A_+(a) \). We have \( \sigma(\alpha)(\Omega) = S_\alpha \). Since \( \alpha \in |K|_1 \) every non-empty finite subset of \( S_\alpha \) belongs to \( K \) hence \( \sigma(\alpha) \in \bar{L}(\Omega, K) \), and \( \sigma(\alpha) \in L(\Omega, K) \) as soon as \( \alpha \in |K|_1 \).

Clearly \( \Psi \circ \sigma \) is the identity. We prove now that \( \sigma \) is continuous at any \( \alpha \in \bar{L}(\Omega, K) \). Let \( \varepsilon > 0 \). There exists \( S_\eta^0 \subset S_\alpha \) finite (and non-empty) such that \( \sum_{s \in S_\alpha \setminus S_\eta^0} \alpha(s) \leq \varepsilon/3 \). Let \( n = |S_\eta^0| > 0 \). We set \( \eta = \varepsilon/3n \). Let \( \beta \in |K|_1 \) with \(|\alpha - \beta|_1 \leq \eta \), and set \( B_+(s) = \sum_{u \leq s} \beta(u) \) and \( B_-(s) = \sum_{u < s} \beta(u) \). We have

\[
d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon/3 + \sum_{a \in S_\eta^0} \int_{A_-(a)}^{A_+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt
\]

Now note that \(|A_\pm(a) - B_\pm(a)| \leq |\alpha - \beta|_1 \leq \varepsilon/3n \) for each \( a \in S_\eta^0 \) hence

\[
\int_{A_-(a)}^{A_+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt \leq \frac{2\varepsilon}{3n} + \int_{\min(A_+(a), B_+(a))}^{\max(A_-(a), B_-(a))} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt = \frac{2\varepsilon}{3n}
\]

since \( \sigma(\alpha)(t) = \sigma(\beta)(t) \) for each \( t \in [\min(A_-(a), B_-(a)), \max(A_+(a), B_+(a))] \), and this yields \( d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon \). This proves that \( \sigma \) is continuous at any \( \alpha \in \bar{L}(\Omega, K) \). Therefore \( \sigma \) provides a continuous global section of \( \Psi \), which obviously restricts to a continuous global section of \( \overline{\Psi} \). \( \square \)

### 4.4. Homotopy lifting properties

Let \( \Psi_K : L(\Omega, K) \to |K|_1 \) and \( \overline{\Psi}_K : \bar{L}(\Omega, K) \to |K|_1 \) denote the law map. If \( \alpha \) is a cardinal, we let \( \Psi_\alpha \) (resp. \( \overline{\Psi}_\alpha \)) denote the map associated to the simplicial complex \( P^*_\alpha \). Recall that a continuous map \( p : E \to B \) is said to have the homotopy lifting property (HLP) with respect to some topological space \( X \) if, for any (continuous) maps \( H : X \times [0,1] \to B \) and \( h : X \to E \) such that \( p \circ h = H(\bullet, 0) \), there exists a map \( \tilde{H} : X \times [0,1] \to E \) such that \( p \circ \tilde{H} = H \).

A Hurewicz fibration is a map having the HLP w.r.t. arbitrary topological spaces. A Serre fibration is a map having the HLP w.r.t. to all \( n \)-spheres, and this is equivalent to having the HLP w.r.t. to any CW-complex.

**Lemma 4.2.** If \( \Psi_\alpha \) (resp. \( \overline{\Psi}_\alpha \)) has the HLP w.r.t. the space \( X \), then the map \( \Psi_K \) (resp. \( \overline{\Psi}_K \)) has the HLP w.r.t. the space \( X \) for every simplicial complex whose vertex set has cardinality \( \alpha \).
Proof. This is a straightforward consequence of the fact that, by definition, the following natural square diagram are cartesian, where $S = \bigcup \mathcal{K}$ is the vertex set of $\mathcal{K}$.

\[
\begin{array}{ccc}
L(\Omega, \mathcal{K}) & \xrightarrow{\mathcal{L}} & L(\Omega, S) \\
|\mathcal{K}|_1 & \xrightarrow{\mathcal{P}_1^*(S)_1} & |\mathcal{P}_1^*(S)|_1 \\
\end{array}
\]

Notice that the following lemma applies in particular to every compact metrizable space (e.g. the $n$-spheres). Recall that $\aleph_0$ denotes the cardinality of $\mathbb{N}$.

**Lemma 4.3.** Let $X$ be a separable space. If $\Psi_{\aleph_0}$ (resp. $\tilde{\Psi}_{\aleph_0}$) has the HLP w.r.t. the space $X$ then, for every infinite cardinal $\gamma$, the map $\Psi_\gamma$ (resp. $\Psi_\gamma$) has the HLP w.r.t. the space $X$.

**Proof.** Let $S$ be a set of cardinality $\gamma$, $H : X \times [0, 1] \to |\mathcal{P}_1^*(S)|_1$ (resp. $\tilde{H} : X \times [0, 1] \to |\mathcal{P}_1^*(S)|_1$) and $h : X \to L(\Omega, S)$ (resp. $\tilde{h} : X \to L(\Omega, S)$) be continuous maps such that $\Psi_S \circ h = H(\cdot, 0)$ (resp. $\tilde{\Psi}_S \circ \tilde{h} = \tilde{H}(\cdot, 0)$). Since $X$ is separable, $X \times [0, 1]$ is also separable and so are $H(X \times [0, 1])$ and $\tilde{H}(X \times [0, 1])$. Let $(x_n)_{n \in \mathbb{N}}$ be a dense sequence of elements of $H(X \times [0, 1])$ (resp. $\tilde{H}(X \times [0, 1])$). Each $	ext{supp}(x_n) \subset S$ is countable, and therefore so is $D = \bigcup_n \text{supp}(x_n)$.

We first claim that, for any $\alpha \in H(X \times [0, 1])$ (resp. $\alpha \in \tilde{H}(X \times [0, 1])$) we have $	ext{supp}(\alpha) \subset D$. Indeed, if $\alpha(s_0) \neq 0$ for some $s_0 \notin D$, then there exists $x_n$ such that $d(x_n, \alpha) < \alpha(s_0)$. But since $d(x_n, \alpha) = \sum_{s \in \mathcal{S}} |\alpha(s) - x_n(s)|$, this condition implies $x_n(s_0) \neq 0$, contradicting $	ext{supp}(x_n) \subset D$. Therefore $	ext{supp}(\alpha) \subset D$ for all $\alpha \in H(X \times [0, 1])$ (resp. $\alpha \in \tilde{H}(X \times [0, 1])$), and $H$ (resp. $\tilde{H}$) factorizes through a map $H_D : X \times [0, 1] \to |\mathcal{P}_1^*(D)|_1$ (resp. $\tilde{H}_D : X \times [0, 1] \to |\mathcal{P}_1^*(D)|_1$) and the natural inclusion $|\mathcal{P}_1^*(D)|_1 \subset |\mathcal{P}_1^*(S)|_1$ (resp. $|\mathcal{P}_1^*(D)|_1 \subset |\mathcal{P}_1^*(S)|_1$).

Notice that this implies that $h$ (resp. $\tilde{h}$) takes values in $L(\Omega, D)$ (resp. $L(\Omega, D)$), too. By assumption, there exists $\tilde{H}_D : X \times [0, 1] \to L(\Omega, D)$ (resp. $\tilde{H}_D : X \times [0, 1] \to L(\Omega, D)$) such that $\Psi_D \circ \tilde{H}_D = H_D$ and with $\tilde{H}_D(\cdot, 0) = h$ (respectively, $\tilde{H}_D(\cdot, 0) = \tilde{h}$). Composing $\tilde{H}_D$ (resp. $\tilde{H}_D$) with the natural injection $L(\Omega, D) \rightarrow L(\Omega, S)$ (resp. $L(\Omega, D) \rightarrow L(\Omega, S)$) we get the lifting $H$ (resp. $\tilde{H}$) we want, and this proves the claim.

\[
\begin{array}{ccc}
L(\Omega, D) & \xrightarrow{\mathcal{L}} & L(\Omega, S) \\
|\mathcal{P}_1^*(D)|_1 & \xrightarrow{\mathcal{P}_1^*(S)_1} & |\mathcal{P}_1^*(S)|_1 \\
\end{array}
\]

**Proposition 4.4.** Let $X$ be a topological space and $\gamma$ a countable cardinal. Then $\Psi_\gamma$ has the HLP property w.r.t. $X$ as soon as $\gamma$ is finite or $X$ is compact. Moreover $\Psi_\gamma$ has the HLP property w.r.t. $X$ as soon as $X$ is compact.

**Proof.** Let $X$ be an arbitrary topological space. Our cardinal $\gamma$ is the cardinal of some initial segment $S \subset \mathbb{N} = \mathbb{Z}_{\geq 0}$ that is, either $S = [0, m]$ for some $m$, or $S = \mathbb{N}$. Let $H : X \times [0, 1] \to |\mathcal{P}_1^*(S)|_1$ and $h : X \to L(\Omega, S)$ such that $H(\cdot, 0) = \Psi_S \circ h$. For $(x, u) \in X \times [0, 1]$, the element $H(x, u) \in |\mathcal{P}_1^*(S)|_1$ is of the form $(H(x, u)_s)_{s \in \mathcal{S}}$, with $\sum_{s \in \mathcal{S}} H(x, u)_s = 1$. Since, for each $s \in S$, the map $|\mathcal{P}_1^*(S)|_1 \rightarrow [0, 1]$ given by $\alpha \mapsto \alpha(s)$ is 1-Lipschitz, the composite map $(x, u) \mapsto H(x, u)_s$ defines a continuous map $X \times [0, 1] \rightarrow [0, 1]$.

Let us choose $x \in X$. We set, with the convention $0/0 = 0$,

\[
a_n(x, u) = \frac{H(x, u)_n}{1 - \sum_{k < n} H(x, u)_k} \in [0, 1], \quad A_n(x) = h(x)^{-1}(\{n\}) \in L(2)
\]
and we construct recursively, for each \( n \in \mathbb{N} \),

- maps \( \Omega_n(x, \bullet) : [0, 1] \to L(2) \)
- maps \( E^{(n)}_{x, \bullet} : [0, 1] \to L(2) \)

by letting

\[
E^{(n)}_{x, u} = \Omega \setminus \bigcup_{k < n} \Omega_k(x, u), \quad \Omega_n(x, u) = \Phi(a_n(x, \bullet), E^{(n)}_{x, \bullet}, A_n(x))(u)
\]

where \( \Phi \) is the map afforded by proposition 3.3. In order for this to be defined at any given \( n \), one needs to check that \( A_n(x) \subset E^{(n)}_{x, 0} \) and \( a_n(x, 0)\lambda(E^{(n)}_{x, 0}) = \lambda(A_n(x)) \). This is easily checked by induction because, if \( \Omega_k, E^{(k)} \) are defined for \( k < n \), then

\[
\Omega_k(x, 0) = \Phi \left( a_n(x, \bullet), E^{(n)}_{x, \bullet}, A_n(x) \right)(0) = A_n(x) = h(x)^{-1}(\{n\})
\]

hence

\[
E^{(n)}_{x, 0} = \Omega \setminus \bigcup_{k < n} A_k(x) = h(x)^{-1}(S \setminus [0, n]) \cup h(x)^{-1}(\{n\}) = A_n(x)
\]

and moreover \( \lambda(A_n(x)) = \lambda(h(x)^{-1}(\{n\})) = H(x, 0)_n = a(x, 0)\lambda(E^{(n)}_{x, 0}) \). Therefore these maps are well-defined.

From their definitions and the properties of \( \Phi \) one gets immediately by induction that

\[
a_n(x, u)\lambda(E^{(n)}_{x, u}) = H(x, u)_n = \lambda(\Omega_n(x, u))
\]

for all \((x, u) \in X \times [0, 1]\).

For a given \((x, u)\), the sets \( \Omega_n(x, u) \) are essentially disjoint, since \( \Omega_n(x, u) \subset E^{(n)}_{x, u} = \Omega \setminus \bigcup_{k < n} \Omega_k(x, u) \), and moreover \( \bigcup_n \Omega_n(x, u) = \Omega \) since \( \sum_n \lambda(\Omega_n(x, u)) = \sum_n H(x, u)_n = 1 \). Therefore, we can define a map \( \tilde{H} : X \times [0, 1] \to L_\ell(S) \) by setting \( \tilde{H}(x, u)(t) = n \) if \( t \in \Omega_n(x, u) \). Clearly \( (\Psi_S \circ \tilde{H}(x, u))_n = \lambda(\Omega_n(x, u)) = H(x, u)_n \) for all \( n \), hence \( \Psi_S \circ \tilde{H} = H \). Moreover \( \tilde{H}(x, 0)_n = \Omega_n(x, 0) = A_n(x) = h(x)^{-1}(\{n\}) \) hence \( \tilde{H}(x, 0) = h(x) \) for all \( x \in X \).

Therefore it only remains to prove that \( \tilde{H} : X \times [0, 1] \to L_\ell(\Omega, S) \) is continuous.

Let us define the auxiliary maps \( \tilde{H}_n : X \times [0, 1] \to L(\Omega, \{0, \ldots, n\}) \) by \( \tilde{H}_n(x, u)(t) = \tilde{H}(x, u)(t) \).

If \( \tilde{H}(x, u)(t) < n \) and \( \tilde{H}_n(x, u)(t) = n \) if \( \tilde{H}(x, u)(t) \geq n \) then \( \tilde{H}_n(x, u)(t) = \min(n, \tilde{H}(x, u)(t)) \).

We first prove that each \( \tilde{H}_n \) is continuous. Let \((x_0, u_0), (x, u) \in X \times [0, 1] \). We have

\[
d(\tilde{H}_n(x, u), \tilde{H}_n(x_0, u_0)) = \sum_{k=0}^n \int_{\Omega_k(x_0, u_0)} d((\tilde{H}_n(x, u)(t), \tilde{H}_n(x_0, u_0)(t))dt
\]

and hence

\[
d(\tilde{H}_n(x, u), \tilde{H}_n(x_0, u_0)) \leq \sum_{k=0}^n \lambda(\Omega_k(x_0, u_0) \setminus \Omega_k(x, u)) \leq \sum_{k=0}^n \lambda(\Omega_k(x_0, u_0)) \Delta \Omega_k(x, u)
\]

and therefore it remains to prove that the maps \((x, u) \mapsto \Omega_n(x, u) \) are continuous for each \( n \in \mathbb{N} \).

We thus want to prove that \( \Omega_n(\bullet, \bullet) \in C^0(X \times [0, 1], L(2)) \), which we identify with the space \( C^0(X, C^0([0, 1], L(2))) = C^0(X, C^0(L(2))) \) since \([0, 1] \) is (locally) compact. Recall that \( \Phi \) is continuous \( C^0([0, 1]) \times C^0(L(2)) \times C^0(L(2)) \). Moreover, for arbitrary spaces \( Y, Z \) and a map \( g \in C^0(Y, Z) \), the induced map \( C^0(X, Y) \to C^0(X, Z) \) is continuous. Letting \( Y = C^0([0, 1]) \times C^0(L(2)) \times L(2) \) and \( Z = C^0(L(2)) \), we deduce from \( \Phi : Y \to Z \) a continuous map \( \Phi : C^0(X, Y) \to C^0(X, Z) \), that is

\[
\Phi : C^0(X, C^0([0, 1]) \times C^0(L(2))) \times C^0(L(2)) \to C^0(X, C^0(L(2)))
\]

By induction and because the maps \( a_n, A_n \) are clearly continuous for any \( n \), we get that all the maps involved are continuous, through the recursive identities

- \( \Omega_n = \Phi(\Omega_n, E^{(n)}_{x, \bullet}, A_n(\bullet)) \)
\[ E^{(n)}_{x,u} = \Omega \setminus \bigcup_{k<n} \Omega_k(x,u) \]
and this proves the continuity of \( \tilde{H}_n \).

If \( S \) is finite this proves that \( \tilde{H} \) is continuous, because \( \tilde{H} = \tilde{H}_n \) for \( n \) large enough in this case. Let us now assume that \( S = N \) and \( X \) is compact. We want to prove that the sequence \( \tilde{H}_n \) converges uniformly to \( \tilde{H} \). Since each \( \tilde{H}_n \) is continuous this will prove that \( \tilde{H} \) is continuous. Let \( \varepsilon > 0 \). Let \( U_n = \{(x,u) \in X \times [0,1] \mid \sum_{k \leq n} H(x,u)_k > 1 - \varepsilon \} \). Since \( H \) is continuous this defines a collection of open subsets in the compact space \( X \times [0,1] \), and since \( \sum_{k \leq n} H(x,u)_k \to 1 \) when \( n \to \infty \) for any \((x,u) \in X \times [0,1]\), this collection is an open covering of \( X \times [0,1] \). By compactness, and because this collection is a filtration, we have \( X \times [0,1] = \bigcup_{n=0}^{\infty} U_n \) for some \( n_0 \in N \). But then, for any \((x,u) \in X \times [0,1]\) and \( n \geq n_0 \) we have

\[ d(\tilde{H}_n(x,u), \tilde{H}(x,u)) = \lambda \left( \bigcup_{k>n} \Omega_k(x,u) \right) = \sum_{k>n} H(x,u)_k \leq \varepsilon \]
and this proves the claim.

Since it is far simpler in this case, we provide an alternative proof for the case of binary random variables.

**Corollary 4.5.** The map \( \Psi_2 = \Psi_{\{0,1\}} \) is a Hurewicz fibration.

**Proof.** (alternative proof) Let \( X \) be a space, and \( H : X \times [0,1] \to |P^*_2(2)|_1 \) and \( h : X \to L(\Omega, 2) \) such that \( H(\bullet, 0) = \Psi_2 \circ h \). Note that \( |P^*_2(2)|_1 = \{ \alpha : [0,1] \to R_+ \mid \alpha(0) + \alpha(1) = 1 \} \) is isometric to \([0,1]\) through the isometry \( j : \alpha \mapsto \alpha(1) \), where the metric on \([0,1]\) is the Euclidean one. If \( \alpha = \Psi_2 \circ h(x) \), we have \( \alpha(0) = 1 - \lambda(h(x)) \), \( j(\Psi_2(h(x))) = \alpha(1) = \lambda(h(x)) \).

Using the map \( g \) of lemma \( 3.1 \), we note that \( \lambda(c g(cA,u)) = u(1-u) \lambda(A) = u \lambda(\Omega) + (1-u) \lambda(A) \) and we define, for \( A \in L(2) \) and \( a \in [0,1] \),

- \( g(A, a) = g(A, 1 - a/\lambda(A)) \) if \( a < \lambda(A) \),
- \( g(A, \lambda(A)) = A \),
- \( g(A, a) = g(cA, (a - \lambda(A))/(1 - \lambda(A))) \) if \( a > \lambda(A) \).

We prove that \( \tilde{g} : L(2) \times [0,1] \to L(2) \) is continuous at each \((A_0, a_0) \in L(2)\). The case \( a_0 \neq \lambda(A_0) \) is clear from the continuity of \( g \), as there is an open neighborhood of \((A_0, a_0)\) on which \( a - \lambda(A) \) has constant sign. Thus we can assume \( a_0 = \lambda(A_0) \). Then

\[ d(\tilde{g}(A, a), \tilde{g}(A_0, a_0)) = d(\tilde{g}(A, a), A_0) \leq d(\tilde{g}(A, a), A) + d(A, A_0) \]

But, if \( a < \lambda(A) \) we have by the inequality of lemma \( 3.1 \)

\[ d(\tilde{g}(A, a), A) = d \left( g \left( A, 1 - \frac{a}{\lambda(A)} \right), g(A, 0) \right) \leq \left| 1 - \frac{a}{\lambda(A)} \right| \]
and, if \( a > \lambda(A) \), we have, noticing that \( A \rightarrow cA \) is an isometry of \( L(2) \) (as \( A \Delta B = \langle cA \rangle \Delta \langle cB \rangle \)),

\[ d(\tilde{g}(A, a), A) = d \left( c g(cA, \frac{a - \lambda(A)}{1 - \lambda(A)}), A \right) = d \left( g \left( cA, \frac{a - \lambda(A)}{1 - \lambda(A)} \right), cA \right) \leq \left| \frac{a - \lambda(A)}{1 - \lambda(A)} \right| \]
which altogether imply

\[ d(\tilde{g}(A, a), \tilde{g}(A_0, a_0)) \leq d(A, A_0) + \max \left( \left| 1 - \frac{a}{\lambda(A)} \right|, \left| \frac{a - \lambda(A)}{1 - \lambda(A)} \right| \right) \]

Since the RHS is continuous with value 0 at \((A_0, a_0)\) with \( a_0 = \lambda(A_0) \), this proves the continuity of \( g \).

It is readily checked that \( \lambda(\tilde{g}(A, a)) = a \) for all \( A, a \). We then define \( \tilde{H} : X \times [0,1] \to L(\Omega, 2) \) by \( \tilde{H}(x,u) = \tilde{g}(h(x), j(H(x,u))) \). We have \( \lambda(\tilde{H}(x,u)) = j(H(x,u)) \) hence \( \Psi_2 \circ \tilde{H} = H \), and \( \tilde{H}(x,0) = h(x) \) for all \( x \in X \), therefore \( \tilde{H} \) provides the lifting we want. \( \square \)

Altogether, these statements imply the following result, which completes the proof of theorem.
Theorem 4.6. For an arbitrary simplicial complex $\mathcal{K}$, the maps $\Psi_\mathcal{K}$ and $\overline{\Psi}_\mathcal{K}$ are Serre fibrations and weak homotopy equivalences. If $\mathcal{K}$ is finite, then $\Psi_\mathcal{K}$ and $\overline{\Psi}_\mathcal{K}$ are Hurewicz fibrations and homotopy equivalences.

Proof. Let $\mathcal{K}$ be an arbitrary simplicial complex. We first prove that $\Psi_\mathcal{K}$ and $\overline{\Psi}_\mathcal{K}$ are Serre fibrations. By lemmas 4.2 and 4.3 and since the $n$-spheres are separable spaces, we can restrict ourselves to proving the same statement for $\Psi_\gamma$ and $\overline{\Psi}_\gamma$ when $\gamma \leq \aleph_0$, and this is true in this case because the $n$-spheres are compact, by proposition 4.4. Let us now choose $\{x_0\} \in \mathcal{K}$, and define $\tilde{x}_0 : S \to [0, 1]$ to be given by $x_0 \mapsto 1$ and $x \mapsto 0$ if $x \neq x_0$. Then $\tilde{x}_0 \in |\mathcal{K}|_1$, and the fiber above it $\Psi_\mathcal{K}^{-1}(\{\tilde{x}_0\}) = \overline{\Psi}_\mathcal{K}^{-1}(\{\tilde{x}_0\})$ is a point. Since these two maps are Serre fibrations this implies that they are weak homotopy equivalences.

We now prove the second part of the statement. If $\mathcal{K}$ is a finite simplicial complex, by lemma 4.2 and proposition 4.4 we get that $\Psi_\mathcal{K}$ and $\overline{\Psi}_\mathcal{K}$ are Hurewicz fibrations. Picking again some $\{x_0\} \in \mathcal{K}$ we get that the fiber above some point is itself a point, whence the homotopy fiber of these Hurewicz fibrations is contractible and they are homotopy equivalences. This proves the claim.

□

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