Braided equivariant crossed modules and cohomology of \( \Gamma \)-modules

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Abstract

If \( \Gamma \) is a group, then braided \( \Gamma \)-crossed modules are classified by braided strict \( \Gamma \)-graded categorial groups. The Schreier theory obtained for \( \Gamma \)-module extensions of the type of an abelian \( \Gamma \)-crossed module is a generalization of the theory of \( \Gamma \)-module extensions.

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1 Introduction

Crossed modules have been used widely, and in various contexts, since their definition by Whitehead [21] in his investigation of the algebraic structure of second relative homotopy groups. Brown and Spencer [4] showed that crossed modules are determined by \( G \)-groupoids (or strict categorial groups), and hence crossed modules can be studied by the theory of category. Thereafter, Joyal and Street [14] extended the result in [4] for braided crossed modules and braided strict categorial groups. A braided strict categorial group is a braided categorial group in which the unit, associativity constraints are strict and every object is invertible (\( x \otimes y = 1 = y \otimes x \)).

A brief summary of researches related to crossed modules was given in [6] by Carrasco et al. Results on the category of abelian crossed modules appeared in this work. Previously, the notion of abelian crossed module was characterized by that of the center of a crossed module in the paper of Norrie [19].

In [12], Fröhlich and Wall introduced the notion of graded categorical group. Thereafter, Cegarra and Khmaladze constructed the abelian (symmetric) cohomology of \( \Gamma \)-modules which was applied on the classification for braided (symmetric) \( \Gamma \)-graded categorical groups in [9] (8).

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The purpose of this paper is to study kinds of crossed modules which are defined by braided strict \( \Gamma \)-graded categorical groups. This result is an extension of the result of Joyal and Street mentioned above. After this introductory Section 1, Section 2 is devoted to recalling some necessary fundamental notions and results of braided (symmetric) graded categorical groups and factor sets of braided graded categorical groups. In Section 3 we show that the category \( \text{BrGr}^* \) of braided strict categorical groups and regular symmetric monoidal functors is equivalent to the category \( \text{BrCross} \) of braided crossed modules (Theorem 3.6). Each morphism in the category \( \text{BrCross} \) consists of a homomorphism \((f_1, f_0) : M \to M'\) of braided crossed modules and an element of the group of abelian 2-cocycles \(Z^2_{ab}(\pi_0M, \pi_1M')\) in the sense of [11]. This result is a continuation of the result in [14] (Remark 3.1). It is obtained as a consequence of Classification Theorem 4.10.

In Section 4 we extend the result in Section 3 to graded structures by introducing the notions of braided \( \Gamma \)-crossed module and braided strict \( \Gamma \)-graded categorical group to classify braided \( \Gamma \)-crossed modules (see [15]). Theorem 4.10 states that the category \( \Gamma \text{BrGr}^* \) of braided strict \( \Gamma \)-graded categorical groups and regular \( \Gamma \)-graded symmetric monoidal functors is equivalent to the category \( \Gamma \text{BrCross} \) of braided \( \Gamma \)-crossed modules. Each morphism in the category \( \Gamma \text{BrCross} \) consists of a homomorphism \((f_1, f_0) : M \to M'\) of braided \( \Gamma \)-crossed modules and an element of the group of symmetric 2-cocycles \(Z^2_{\Gamma,s}(\pi_0M, \pi_1M')\) in the sense of [9].

The problem of group extensions of the type of a crossed module has been mentioned in [19, 10, 3]. In Section 5 we show a treatment of the similar problem for \( \Gamma \)-module extensions of the type of an abelian \( \Gamma \)-crossed module. The Schreier theory for such extensions (Theorem 5.3) is presented by means of graded symmetric monoidal functors, and therefore we obtain the classification theorem of \( \Gamma \)-module extensions of the type of an abelian \( \Gamma \)-crossed module (Theorem 5.4).

The case of (non-braided) \( \Gamma \)-crossed modules is studied by Quang and Cuc in [17]. The results generalizes both the theory of group extensions of the type of a crossed module and the one of equivariant group extensions.

2 Preliminaries

2.1 Braided (symmetric) graded categorical groups

Let \( \Gamma \) be a fixed group, which we regard as a category with exactly one object, say \(*\), where the morphisms are the members of \( \Gamma \) and the composition law is the group operation. A grading on a category \( G \) is then a functor, say \( gr : G \to \Gamma \). For any morphism \( u \) in \( G \) with \( gr(u) = \sigma \), we refer to \( \sigma \) as the grade of \( u \). The grading \( gr \) is said to be stable if for any \( X \in \text{Ob}G \) and any \( \sigma \in \Gamma \) there exists an isomorphism \( u \) in \( G \) with domain \( X \) such that \( gr(u) = \sigma \).
A braided Γ-graded monoidal category \[ \mathcal{G} = (\mathbb{G}, gr, \otimes, I, a, r, l, c) \] consists of a category \( \mathbb{G} \), a stable grading \( gr : \mathbb{G} \to \Gamma \), graded functors \( \otimes : \mathbb{G} \times_{\Gamma} \mathbb{G} \to \mathbb{G} \) and \( I : \Gamma \to \mathbb{G} \), and graded natural equivalences defined by isomorphisms of grade 1 \( a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \), \( l_X : I \otimes X \xrightarrow{\sim} X, r_X : X \otimes I \xrightarrow{\sim} X \) and \( c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X \) satisfying the following coherence conditions:

\[
\begin{align*}
a_{X,Y,Z \otimes T}a_{X \otimes Y,Z,T} & = (id_X \otimes a_{Y,Z,T})a_{X,Y \otimes Z,T}(a_{X,Y,Z} \otimes id_T), \\
(id_X \otimes l_Y)a_{X,I,Y} & = r_X \otimes id_Y, \\
(id_Y \otimes c_{X,Z})a_{y,X,Z}(c_{X,Y} \otimes id_Z) & = a_{Y,Z,X}c_{X,Y \otimes Z}a_{X,Y,Z}, \quad (1) \\
(c_{X,Z} \otimes id_Y)a_{X,Z,Y}^{-1}(id_X \otimes c_{Y,Z}) & = a_{Z,X,Y}^{-1}c_{X,Y \otimes Z}a_{X,Y,Z}^{-1}. \quad (2)
\end{align*}
\]

A braided Γ-graded categorical group \([9]\) is a braided Γ-graded monoidal groupoid such that, for any object \( X \), there is an object \( X' \) with an arrow \( X \times X' \to 1 \) of grade 1. If the braiding \( c \) is a symmetric constraint, that is, it satisfies the condition \( c_{X,Y} \circ c_{X,Y} = id_{X \otimes Y} \) (in this case the relation \([2]\) coincides with the relation \([1]\)), then \( \mathbb{G} \) is called a symmetric Γ-graded categorical group or a graded Picard category \([8]\). Then the subcategory \( \text{Ker} \mathbb{G} \) (whose objects are the objects of \( \mathbb{G} \) and morphisms are the morphisms of grade 1 in \( \mathbb{G} \)) is a braided categorical group (a Picard category, respectively).

Let \( (\mathbb{G}, gr) \) and \( (\mathbb{G}', gr') \) be two (braided symmetric) Γ-graded categorical groups. A graded symmetric monoidal functor from \( (\mathbb{G}, gr) \) to \( (\mathbb{G}', gr') \) is a triple \((F, \tilde{F}, F_*)\), where \( F : (\mathbb{G}, gr) \to (\mathbb{G}', gr') \) is a Γ-graded functor, \( \tilde{F}_{X,Y} : FX \otimes FY \to F(X \otimes Y) \) are natural isomorphisms of grade 1 and \( F_* : I' \to FI \) is an isomorphism of grade 1, such that the following coherence conditions hold:

\[
\begin{align*}
\tilde{F}_{X,Y \otimes Z}(id_{FX} \otimes \tilde{F}_{Y,Z})a_{FX,FY,FZ} & = F(a_{X,Y,Z})(\tilde{F}_{X,Y} \otimes id_{FZ}), \\
F(r_X)\tilde{F}_{X,I}(id_{FX} \otimes F_*) & = r_{FX}, \quad F(l_X)\tilde{F}_{I,X}(F_* \otimes id_{FX}) = l_{FX}, \\
\tilde{F}_{Y,X}c_{FX,FY} & = F(c_{X,Y})\tilde{F}_{X,Y}.
\end{align*}
\]

Let \((F, \tilde{F}, F_*), (F', \tilde{F}', F'_*)\) be two graded symmetric monoidal functors. A graded symmetric monoidal natural equivalence \( \theta : F \xrightarrow{\sim} F' \) is a graded natural equivalence such that, for all objects \( X, Y \) of \( \mathbb{G} \), the following coherence conditions hold

\[
\tilde{F}_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X \otimes Y}\tilde{F}_{X,Y}, \quad \theta_IF_* = F'_*, \quad (3)
\]

that is, a monoidal natural equivalence.
2.2 Braided (symmetric) graded categorical groups of type 
\((M, N)\) and the theory of obstructions

Let \(G\) be a braided \(\Gamma\)-graded categorical group. We write 
\[M = \pi_0(\ker G) = \pi_0\Gamma G\] 
for the abelian group of 1-isomorphism classes of the objects in \(G\)
and 
\[N = \pi_1(\ker G) = \pi_1\Gamma G\] 
for the abelian group of 1-automorphisms of the unit object of \(G\). Then \(G\) 
duces \(\Gamma\)-module structures on \(M, N\) and a
normalized 3-cocycle \(h \in Z^3_{\ab}(M, N)\) in the sense of \([9]\). From these data,
the authors of \([9]\) constructed a braided \(\Gamma\)-graded categorical group, denoted 
by \(G(h)\) (or \(\int_\Gamma(M, N, h)\)), which is equivalent to \(G\). Below, we briefly recall 
this construction.

The objects of \(G(h)\) are the elements \(s \in M\) and their arrows are pairs 
\((a, \sigma) : r \to s\) consisting of an element \(a \in N\) and an element \(\sigma \in \Gamma\) with 
\(\sigma r = s\).

The composition of two morphisms 
\((r \xrightarrow{(a, \sigma)} s \xrightarrow{(b, \tau)} t)\) is defined by 
\[(b, \tau) \circ (a, \sigma) = (b + \tau a + h(r, \tau, \sigma), \tau \sigma).\]

The graded tensor product is defined by 
\[(r \xrightarrow{(a, \sigma)} s) \otimes (r' \xrightarrow{(b, \sigma')} s') = (rr' \xrightarrow{(a+b+h(r, r', \sigma'), \sigma)} ss').\]

The unit constraints are strict, \(I_s = (0, 1) = r_s : s \to s\). The associativity
and braiding constraints are, respectively, given by 
\[a_{r,s,t} = (h(r, s, t), 1) : (rs)t \to r(st),\]
\[c_{r,s} = (h(r, s), 1) : rs \to sr.\]

The stable \(\Gamma\)-grading is defined by \(gr(a, \sigma) = \sigma\). The unit graded functor 
\(I : \Gamma \to G(h)\) is defined by 
\[I(* \xrightarrow{0} *) = (1 \xrightarrow{(0, \sigma)} 1).\]

We call \(G(h)\) a reduced braided \(\Gamma\)-graded categorical group of \(G\). In the case
when \(G\) is a \(\Gamma\)-graded Picard category, then \(h \in Z^3_{\ab}(\Gamma, \omega)\) in the sense of \([8]\) and
\(G(h)\) is a \(\Gamma\)-graded Picard category.

Let \(G, G'\) be \(\Gamma\)-graded Picard categories, and let \(G(h) = \int_\Gamma(M, N, h),\)
\(G'(h') = \int_\Gamma(M', N', h')\) be their reduced \(\Gamma\)-graded Picard categories, respectively. 
A graded functor \(F : G(h) \to G'(h')\) is said to be of type \((\varphi, f)\) if 

\[F(s) = \varphi(s),\quad F(a, \sigma) = (f(a), \sigma),\quad s \in M,\ a \in N,\ \sigma \in \Gamma,\]

where \(\varphi : M \to M',\ f : N \to N'\) are homomorphisms of \(\Gamma\)-modules. Then the function 
\[k = \varphi^*h' - f_*h\]
is called an obstruction of the functor \(F\).

Based on the results on monoidal functors of type \((\varphi, f)\) presented in 
\([18]\), we obtain the following results with some appropriate modifications.
Proposition 2.1. Let $G, G'$ be braided $\Gamma$-graded categorical groups, and let $G(h), G'(h')$ be their reduced braided $\Gamma$-graded categorical groups, respectively.

i) Any graded symmetric monoidal functor $(F, \tilde{F}): G \to G'$ induces a graded symmetric monoidal functor $G(h) \to G'(h')$ of type $(\varphi, f)$.

ii) Any graded symmetric monoidal functor $G(h) \to G'(h')$ is a graded functor of type $(\varphi, f)$.

Proposition 2.2 ([8], Theorem 3.9). The graded functor $(F, \tilde{F}): G(h) \to G'(h')$ of type $(\varphi, f)$ is realizable, that is, there are isomorphisms $\tilde{F}_{x,y}$ so that $(F, \tilde{F})$ is a graded symmetric monoidal functor, if and only if its obstruction $k$ vanishes in $H^3_{\Gamma, s}(M, N')$. Then, there is a bijection

$$\text{Hom}_{(\varphi, f)}[G(h), G'(h')] \leftrightarrow H^2_{\Gamma, s}(M, N'),$$

where $\text{Hom}_{(\varphi, f)}[G(h), G'(h')]$ denotes the set of homotopy classes of graded symmetric monoidal functors of type $(\varphi, f)$ from $G(h)$ to $G'(h')$.

Note that $H^2_{\Gamma, s}(M, N') = H^2_{\Gamma, ab}(M, N')$.

2.3 Factor sets in braided graded categorical groups

According to the definition of a factor set with coefficients in a monoidal category [7], we now establish the following terminology.

Definition 2.3. A symmetric factor set $F$ on $\Gamma$ with coefficients in a braided categorical group $G$ (or a pseudofunctor from $\Gamma$ to the category of braided categorical groups in the sense of Grothendieck [13]) consists of a family of symmetric monoidal auto-equivalences $F^\sigma: G \to G, \sigma \in \Gamma$, and isomorphisms between symmetric monoidal functors $\theta^\sigma,\tau: F^\sigma F^\tau \to F^{\sigma \tau}, \sigma, \tau \in \Gamma$ satisfying the conditions:

i) $F^1 = id_G$,

ii) $\theta^{1,\sigma} = id_{F^\sigma} = \theta^{\sigma,1}, \sigma \in \Gamma$,

iii) for all $\sigma, \tau, \gamma \in \Gamma$, the following diagram commutes

$$\begin{array}{ccc}
F^\sigma F^\tau F^\gamma & \xrightarrow{\theta^{\sigma,\tau} F^\gamma} & F^{\sigma \tau} F^\gamma \\
F^\sigma \theta^{\tau,\gamma} \downarrow & & \downarrow \theta^{\sigma \tau,\gamma} \\
F^\sigma F^{\tau \gamma} & \xrightarrow{\theta^{\sigma,\tau \gamma}} & F^{\sigma \tau \gamma}.
\end{array}$$

We write $F = (G, F^\sigma, \theta^{\sigma,\tau})$, or simply $(F, \theta)$.

The following lemma comes from an analogous result on graded monoidal categories [7] or a part of Theorem 1.2 [20]. We sketch the proof since we need some of its details.

Lemma 2.4. Any braided $\Gamma$-graded categorical group $(G, gr)$ determines a symmetric factor set $F$ on $\Gamma$ with coefficients in a braided categorical group $\text{Ker} G$. 
Proof. For each $\sigma \in \Gamma$, we construct a symmetric monoidal functor $F^\sigma = (F^\sigma, \widetilde{F}^\sigma) : \text{Ker } G \to \text{Ker } G$ as follows. For any $X \in \text{Ker } G$, since the grading $gr$ is stable, there is an isomorphism $\Upsilon^\sigma_X : X \sim F^\sigma X$, where $F^\sigma X \in \text{Ker } G$, and $gr(\Upsilon^\sigma_X) = \sigma$. In particular, if $\sigma = 1$ we set $F^1 X = X$ and $\Upsilon^1_X = \text{id}_X$.

For any morphism $f : X \to Y$ of grade 1 in $\text{Ker } G$, a morphism $\widetilde{F}^\sigma(f)$ in $\text{Ker } G$ is determined by

$$\widetilde{F}^\sigma(f) = \Upsilon^\sigma_Y \circ f \circ (\Upsilon^\sigma_X)^{-1}.$$ 

Natural isomorphisms $\widetilde{F}^\sigma_{X,Y} : F^\sigma X \otimes F^\sigma Y \sim F^\sigma (X \otimes Y)$ are determined by

$$\widetilde{F}^\sigma_{X,Y} = (\Upsilon^\sigma_X \otimes \Upsilon^\sigma_Y) \circ (\Upsilon^\sigma_{X \otimes Y})^{-1}.$$ 

Moreover, for any pair $\sigma, \tau \in \Gamma$, there is an isomorphism between monoidal functors $\theta^{\sigma,\tau} : F^\sigma F^\tau \sim F^{\sigma \tau}$, where $\theta^{1,\sigma} = \text{id}_{F^\sigma} = \theta^{\sigma,1}$, which is determined by

$$\theta^{\sigma,\tau}_X = \Upsilon^{\sigma \tau}_X \circ \Upsilon^\tau_X \circ (\Upsilon^\sigma_X)^{-1},$$ 

for all $X \in \text{Ob } G$.

The pair $(F, \theta)$ determined above is a symmetric factor set. 

3 Braided crossed modules

We first recall that a crossed module \([21]\) $(B, D, d, \vartheta)$ consists of groups $B, D$, group homomorphisms $d : B \to D$, $\vartheta : D \to \text{Aut } B$ satisfying

- $C_1$. $\vartheta d = \mu$,
- $C_2$. $d(\vartheta_x(b)) = \mu_x(d(b))$, $x \in D, b \in B$,

where $\mu_x$ is an inner automorphism given by conjugation with $x$.

In this paper, the crossed module $(B, D, d, \vartheta)$ is sometimes denoted by $B \overset{d}{\to} D$, or by $d : B \to D$. For convenience, we write the addition for the operation in $B$ and the multiplication for that in $D$.

The notion of braided crossed module over a groupoid was originally introduced by Brown and Gilbert in \([2]\). Later, the notion of braided crossed module over groups appeared in the work of Joyal and Street \([14]\) (Remark 3.1).

Definition 3.1 \([14]\). A braided crossed module $\mathcal{M}$ is a crossed module $(B, D, d, \vartheta)$ together with a map $\eta : D \times D \to B$ satisfying the following conditions:

- $C_3$. $\eta(x, yz) = \eta(x, y) + \vartheta_y \eta(x, z)$,
- $C_4$. $\eta(xy, z) = \vartheta_x \eta(y, z) + \eta(x, z)$,
- $C_5$. $d\eta(x, y) = xyx^{-1}y^{-1}$,
- $C_6$. $\eta(d(b), x) + \vartheta_x b = b$,
- $C_7$. $\eta(x, d(b)) + b = \vartheta_x b$,

where $b \in B$, $x, y, z \in D$. 


A braided crossed module is called a symmetric crossed module (see Aldrovandi and Noohi [1]) if \( \eta(x, y) + \eta(y, x) = 0 \) for all \( x, y \in D \). In this case, the conditions \( C_3 \) and \( C_4 \) coincide, the conditions \( C_6 \) and \( C_7 \) coincide.

The following properties follow from the definition of a braided crossed module.

**Proposition 3.2.** Let \( \mathcal{M} \) be a braided crossed module.

i) \( \eta(x, 1) = \eta(1, y) = 0 \).

ii) \( \ker d \) is a subgroup of \( Z(B) \).

iii) \( \text{Coker } d \) is an abelian group.

iv) The homomorphism \( \vartheta \) induces the identity on \( \ker d \), and hence the action of \( \text{Coker } d \) on \( \ker d \), given by

\[
sa = \vartheta_x(a), \quad a \in \ker d, \quad x \in s \in \text{Coker } d,
\]

is trivial.

The abelian groups \( \ker d \) and \( \text{Coker } d \) are also denoted by \( \pi_1 \mathcal{M} \) and \( \pi_0 \mathcal{M} \), respectively.

**Example 3.3.** Let \( N \) be a normal subgroup of a group \( G \) so that the quotient group \( G/N \) is abelian, in other words, let \( N \) be a normal subgroup in \( G \) which contains the derived group (or the commutator subgroup) of \( G \). Then, \( (N, G, i, \mu, [\cdot, \cdot]) \) is a braided crossed module, where \( i : N \to G \) is an inclusion, \( \mu : G \to \text{Aut } N \) is defined by conjugation and \( \eta : G \times G \to N, \eta(x, y) = [x, y] = x y x^{-1} y^{-1} \).

According to Joyal and Street [14], each braided crossed module is determined by a braided strict categorical group. We now classify the category of braided crossed modules.

**Definition 3.4.** A homomorphism of braided crossed modules \((B, D, d, \vartheta, \eta)\) and \((B', D', d', \vartheta', \eta')\) consists of group homomorphisms \( f_1 : B \to B' \), \( f_0 : D \to D' \) such that:

\[
\begin{align*}
H_1. & \quad f_0 d = d' f_1, \\
H_2. & \quad f_1(\vartheta_x b) = \vartheta'_x(f_0(x)) f_1(b), \\
H_3. & \quad f_1(\eta(x, y)) = \eta'(f_0(x), f_0(y)),
\end{align*}
\]

for all \( x, y \in D, b \in B \).

Therefore, a homomorphism of braided crossed modules is that of crossed modules which satisfies \( H_3 \).

We determine the category

\[
\text{BrCross}
\]

whose objects are braided crossed modules and whose morphisms are triples \((f_1, f_0, \varphi)\), where \( (f_1, f_0) : (B \xrightarrow{d} D) \to (B' \xrightarrow{d'} D') \) is a homomorphism.
of braided crossed modules and \( \varphi \in Z^2_{ab}(\text{Coker} \ d, \text{Ker} \ d') \). The composition with the morphism \( (f'_1, f'_0, \varphi') : (B' \xrightarrow{d'} D') \to (B'' \xrightarrow{d''} D'') \) is given by

\[
(f'_1, f'_0, \varphi') \circ (f_1, f_0, \varphi) = (f'_1 f_1, f'_0 f_0, (f'_1)^* \varphi + (f_0)^* \varphi').
\]

(4)

**Definition 3.5.** A symmetric monoidal functor \((F, \tilde{F}) : G \to G'\) is termed regular if

\[
B_1. F(x) \otimes F(y) = F(x \otimes y),
B_2. F(b) \otimes F(c) = F(b \otimes c),
B_3. \tilde{F}_{x,y} = \tilde{F}_{y,x},
\]

for \(x, y \in \text{Ob}G, \ b, c \in \text{Mor}G\).

Denote by

\[\text{BrGr}^*\]

the category of braided strict categorical groups and regular symmetric monoidal functors and denote by \(p : D \to \text{Coker} \ d\) a canonical projection, we obtain the following classification result.

**Theorem 3.6 (Classification Theorem).** There exists an equivalence

\[
\Phi : \text{BrCross} \to \text{BrGr}^*,
\]

\[
B \to D \mapsto G_{B \to D}
\]

\[
(f_1, f_0, \varphi) \mapsto (F, \tilde{F})
\]

where \(F(x) = f_0(x), F(b) = f_1(b), \tilde{F}_{x,y} = \varphi(px, py), \) for \(x, y \in D, b \in B\).

**Proof.** The proof of this theorem is a particular case of Theorem 4.10 in the next section. \(\square\)

**Remark 3.7.** Denote by \(\text{BrCross}\) the subcategory of \(\text{BrCross}\) whose morphisms are homomorphisms of braided crossed modules \(\varphi = 0\) and denote by \(\text{BrGr}^*\) the subcategory of \(\text{BrGr}^*\) whose morphisms are strict monoidal functors \(\tilde{F} = \text{id}\). Then, these two categories are equivalent via \(\Phi\).

### 4 Braided \(\Gamma\)-crossed modules

The main objective of this section is to classify braided \(\Gamma\)-crossed modules by means of strict braided graded categorical groups. First, observe that if \(B\) is a \(\Gamma\)-group, the group \(\text{Aut} B\) of all automorphisms of \(B\) is also a \(\Gamma\)-group under the action

\[
(\sigma f)(b) = \sigma(f(\sigma^{-1}b)), \ b \in B, \ f \in \text{Aut} B, \ \sigma \in \Gamma.
\]

Then, the map \(\mu : B \to \text{Aut} B, b \mapsto \mu_b \) (\(\mu_b\) is an inner automorphism of \(B\) given by conjugation with \(b\)) is a homomorphism of \(\Gamma\)-groups.
Definition 4.1. Let $B$ and $D$ be $\Gamma$-groups. A \textit{braided (symmetric) $\Gamma$-crossed module}, is a braided (symmetric) crossed module $\mathcal{M} = (B, D, d, \vartheta, \eta)$ in which $d : B \to D$, $\vartheta : D \to \text{Aut}B$ are $\Gamma$-group homomorphisms satisfying the following conditions:

1. $\sigma(\vartheta_x(b)) = \vartheta_{\sigma x}(\sigma b)$,
2. $\sigma \eta(x, y) = \eta(\sigma x, \sigma y)$,

where $\sigma \in \Gamma$, $x, y \in D$ and $b \in B$.

Braided (symmetric) $\Gamma$-crossed modules are also called braided (symmetric) equivariant crossed modules by Noohi [15].

The following properties are implied from the definition of a braided $\Gamma$-crossed module.

Proposition 4.2. Let $\mathcal{M}$ be a braided $\Gamma$-crossed module.

i) $\text{Ker} \ d$ is a $\Gamma$-submodule of $\text{Z}(B)$.

ii) $\text{Coker} \ d$ is a $\Gamma$-module under the action

$\sigma s = [\sigma x]$, $x \in s \in \text{Coker} \ d$, $\sigma \in \Gamma$.

Example 4.3. In Example 3.3, if $G$ and $N$ are $\Gamma$-groups, then $\mathcal{G} = (\mathcal{N}, G, i, \mu, [\cdot, \cdot])$ is a braided $\Gamma$-crossed module.

Example 4.4. Let $d : B \to D$ be a morphism of $\Gamma$-module and let $D$ act trivially on $B$. Let $\eta : D \times D \to \text{Kerd}$ be a biadditive function satisfies $\Gamma_2$

and

$\eta|_{\text{Im}d \times D} = 0 = \eta|_{D \times \text{Im}d}$.

Then, $(B, D, d, 0, \eta)$ is a braided $\Gamma$-crossed module.

We now show that braided $\Gamma$-crossed modules are determined by braided strict $\Gamma$-graded categorical groups. First, we say that a symmetric factor set $(F, \theta)$ on $\Gamma$ with coefficients in a braided categorical group $G$ is \textit{regular} if $F^\sigma$ is a regular symmetric monoidal functor and $\theta^\sigma, \tau = id$, for all $\sigma, \tau \in \Gamma$.

Definition 4.5. A braided $\Gamma$-graded categorical group $(\mathbb{G}, gr)$ is called \textit{strict} if

i) $\text{Ker} \mathbb{G}$ is a braided strict categorical group,

ii) $\mathbb{G}$ induces a regular symmetric factor set $(F, \theta)$ on $\Gamma$ with coefficients in $\text{Ker} \mathbb{G}$.

Equivalently, a braided $\Gamma$-graded categorical group $(\mathbb{G}, gr)$ is \textit{strict} if it is a $\Gamma$-graded extension of a braided strict categorical group by a regular symmetric factor set.

● Constructing the braided strict $\Gamma$-graded categorical group $\mathbb{G} = \mathbb{G}_\mathcal{M}$ associated to a braided $\Gamma$-crossed module $\mathcal{M} = (B, D, d, \vartheta, \eta)$.
Objects of \( \mathcal{G} \) are the elements of the group \( D \), a \( \sigma \)-morphism \( x \to y \) is a pair \( (b, \sigma) \), where \( b \in B, \sigma \in \Gamma \) such that \( \sigma x = d(b)y \). The composition of two morphisms is given by

\[
(x \xrightarrow{(b, \sigma)} y \xrightarrow{(c, \tau)} z) = (x \xrightarrow{(\tau b + c, \tau \sigma)} z).
\]  

Since \( B \) is a \( \Gamma \)-group, the composition is associative and unitary.

For each morphism \( (b, \sigma) \) in \( \mathcal{G} \), we have

\[
(b, \sigma)^{-1} = (-\sigma^{-1} b, \sigma^{-1}),
\]

and hence \( \mathcal{G} \) is a groupoid.

The tensor operation on objects is given by the addition in the group \( D \) and, for two morphisms \( (x \xrightarrow{(b, \sigma)} y), (x' \xrightarrow{(c, \tau)} y') \) in \( \mathcal{G} \), we define

\[
(x \xrightarrow{(b, \sigma)} y) \otimes (x' \xrightarrow{(c, \tau)} y') = (xx' \xrightarrow{(b + \theta y, c, \tau)} yy').
\]  

The functoriality of the tensor operation is implied from the compatibility of the action \( \theta \) with the \( \Gamma \)-action and from the conditions in the definition of a braided \( \Gamma \)-crossed module.

Associativity and unit constraints of the tensor operation are strict. The braiding constraint \( \mathbf{c} \) is defined by

\[
\mathbf{c}_{x,y} = (\eta(x, y), 1) : xy \to yx.
\]

By the relation \( C_5 \), \( \mathbf{c}_{x,y} \) is actually a morphism in \( \mathcal{G} \). Due to the conditions \( C_3, C_4 \), the braiding constraint \( \mathbf{c} \) is compatible with the associativity constraint \( \mathbf{a} \). The naturality of \( \mathbf{c} \) follows from the conditions \( \Gamma_2, C_1, C_3, C_4, C_6, C_7 \).

The \( \Gamma \)-grading \( gr : \mathcal{G} \to \Gamma \) is given by

\[
(b, \sigma) \mapsto \sigma.
\]

The unit graded functor \( I : \Gamma \to \mathcal{G} \) is defined by

\[
I(\ast \xrightarrow{\sigma} \ast) = (1 \xrightarrow{(0, \sigma)} 1).
\]

Since \( \text{Ob}\mathcal{G} = D \) is a group and \( x \otimes y = xy \), every object of \( \mathcal{G} \) is invertible, and hence \( \text{Ker}\mathcal{G} \) is a braided strict categorical group.

We now show that \( \mathcal{G} \) induces a regular symmetric factor set \( (F, \theta) \) on \( \Gamma \) with coefficients in \( \text{Ker}\mathcal{G} \). For any \( x \in D, \sigma \in \Gamma \), we set \( F^\sigma(x) = \sigma x, \ Y^\sigma_x = (x \xrightarrow{(0, \sigma)} \sigma x) \). Then, according to the proof of Lemma 2.4, we have \( F^\sigma(b, 1) = (\sigma b, 1) \) and \( \theta^\sigma.\tau = id \). From the braided \( \Gamma \)-crossed module structure of \( \mathcal{M} \), it follows that \( F^\sigma \) is a regular symmetric monoidal functor on \( \text{Ker}\mathcal{G} \).

- Constructing the braided \( \Gamma \)-crossed module associated to a braided strict \( \Gamma \)-graded categorical group \( \mathcal{G} \).
Set
\[ D = \text{Ob}\, \mathbb{G}, \quad B = \{ x \xrightarrow{b} 1 \mid x \in D, \; gr(b) = 1 \}. \]

The operations in \( D \) and \( B \) are given by
\[ xy = x \otimes y, \quad b + c = b \otimes c, \]
respectively. Then \( D \) becomes a group in which the unity is 1 and the inverse of \( x \) is \( x^{-1} \) \((x \otimes x^{-1} = 1)\), \( B \) is group in which the zero element is the morphism \((1 \xrightarrow{id} 1)\) and the inverse of \((x \xrightarrow{b} 1)\) is the morphism \((x^{-1} \xrightarrow{b} 1)(b \otimes b = id_1)\). Since \( \mathbb{G} \) has a regular symmetric factor set \((F, \theta)\), \( D \) and \( B \) are \( \Gamma \)-groups under the actions
\[
\sigma x = F^\sigma(x), \quad x \in D, \sigma \in \Gamma, \\
\sigma b = F^\sigma(b), \quad b \in B,
\]
respectively. The correspondences \( d : B \to D \) and \( \vartheta : D \to \text{Aut} B \) are, respectively, given by
\[
d(x \xrightarrow{b} 1) = x, \\
\vartheta_y(x \xrightarrow{b} 1) = (yx^{-1} \xrightarrow{id_y + b + id_y^{-1}} 1).
\]
Since \( B \) and \( D \) are \( \Gamma \)-groups, \( d \) and \( \vartheta \) are \( \Gamma \)-group homomorphisms.

The map \( \eta : D \times D \to B \) is defined by
\[
\eta(x, y) = c_{x,y} \otimes id_{x^{-1}} \otimes id_{y^{-1}} : xy^{-1}y^{-1} \to 1.
\]

Now we will classify the category of braided \( \Gamma \)-crossed modules.

**Definition 4.6.** A homomorphism \( \mathcal{M} \to \mathcal{M}' \) of braided \( \Gamma \)-crossed modules is a homomorphism \((f_1, f_0)\) of braided crossed modules, where \( f_1, f_0 \) are \( \Gamma \)-group homomorphisms.

**Remark on notations.** Each morphism \( x \xrightarrow{(b, \sigma)} y \) in \( \mathbb{G}_\mathcal{M} \) is written in the form
\[
x \xrightarrow{(0, \sigma)} \sigma x \xrightarrow{(b, 1)} y,
\]
and then each graded symmetric monoidal functor \((F, \tilde{F}) : \mathbb{G}_\mathcal{M} \to \mathbb{G}_{\mathcal{M}'} \) defines a function \( f : D^2 \cup (D \times \Gamma) \to B' \) by
\[
(f(x, y), 1) = \tilde{F}_{x,y}, \quad (f(x, \sigma), \sigma) = F(x \xrightarrow{(0, \sigma)} \sigma x).
\]

**Lemma 4.7.** Let \((f_1, f_0) : \mathcal{M} \to \mathcal{M}'\) be a homomorphism of braided \( \Gamma \)-crossed modules. Then there is a graded symmetric monoidal functor \((F, \tilde{F}) : \mathbb{G}_\mathcal{M} \to \mathbb{G}_{\mathcal{M}'}\) such that \( F(x) = f_0(x) \), \( F(b, 1) = (f_1(b), 1) \), if and only if \( f = p^*(\varphi) \), where \( \varphi \in Z^2_{\Gamma,s}(\text{Coker} \, d, \text{Ker} \, d') \), and \( p : D \to \text{Coker} \, d \) is a canonical projection.
Proof. Since $f_0$ is a homomorphism and $Fx = f_0(x)$, $\tilde{F}_{x,y} : FxFy \rightarrow F(xy)$ is a morphism of grade 1 in $G'$ if and only if $df(x, y) = 1'$, or $f(x, y) \in \text{Ker } d' \subset Z(B')$.

Also, since $f_0$ is a $\Gamma$-homomorphism, $Fx \xrightarrow{(f(x, \sigma), \sigma)} F\sigma x$ is a morphism of grade $\sigma$ in $G'$ if and only if $df(x, \sigma) = 1'$, or $f(x, \sigma) \in \text{Ker } d' \subset Z(B')$.

- The condition so that $F$ preserves the composition of two morphisms.

Since $f_1$ is a group homomorphism, $F$ preserves the composition of two morphisms in terms of $(0, \sigma)$,

$$(x \xrightarrow{(0, \sigma)} y \xrightarrow{(0, \tau)} z),$$

if and only if

$$\tau f(x, \sigma) + f(\sigma x, \tau) = f(x, \tau \sigma). \quad (8)$$

- The condition so that $\tilde{F}_{x,y}$ is natural.

For morphisms of grade 1, we consider the diagram

$$
\begin{array}{ccc}
F(x)F(y) & \xrightarrow{\tilde{F}_{x,y}} & F(xy) \\
F(b,1) \otimes F(c,1) & \downarrow & F[(b,1) \otimes (c,1)] \\
F(x')F(y') & \xrightarrow{\tilde{F}_{x',y'}} & F(x'y')
\end{array}
$$

Since $f_1$, $f_0$ are homomorphisms satisfying the condition $H_2$,

$$F(b,1) \otimes F(c,1) = F[(b,1) \otimes (c,1)].$$

Then since $f(x, y)$, $f(x', y') \in Z(B')$, the above diagram commutes if and only if

$$f(x, y) = f(x', y'),$$

for $x = d(b)x'$, $y = d(c)y'$. Thus, $\tilde{F}$ defines a function $\varphi : \text{Coker}^2 d \rightarrow \text{Ker } d'$,

$$\varphi(r, s) = f(x, y), \quad r = px, s = py.$$

- For morphisms in terms of $(0, \sigma)$, we consider a diagram

$$
\begin{array}{ccc}
F(x)F(y) & \xrightarrow{\tilde{F}_{x,y}} & F(xy) \\
F(0,\sigma) \otimes F(0,\sigma) & \downarrow & F[(0,\sigma) \otimes (0,\sigma)] \\
F(\sigma x)F(\sigma y) & \xrightarrow{\tilde{F}_{\sigma x,\sigma y}} & F(\sigma x)(\sigma y) = F\sigma(xy).
\end{array}
$$

According to Proposition 4.2, the above diagram commutes if and only if

$$\sigma f(x, y) + f(xy, \sigma) = f(x, \sigma) + f(y, \tau) + f(\sigma x, \sigma y). \quad (9)$$

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• Since the following square commutes

\[
\begin{array}{ccc}
F_x & \xrightarrow{(f(x,\sigma),\sigma)} & F_{\sigma x} \\
F(b,1) & \downarrow & F(\sigma b,1) \\
F_y & \xrightarrow{(f(y,\sigma),\sigma)} & F_{\sigma y}
\end{array}
\]

and \( f_1 \) is a \( \Gamma \)-group homomorphism, we have \( f(x,\sigma) = f(y,\sigma) \), for \( x = d(b)y \).

This determines a function \( \varphi : \text{Coker} \ d \times \Gamma \to \text{Ker} \ d' \),

\[
\varphi(r,\sigma) = f(x,\sigma), \ r = px.
\]

Therefore, we obtain a function \( \varphi : \text{Coker} \ 2 \cup \text{Coker} \ d \times \Gamma \to \text{Ker} \ d' \).

The function \( \varphi \) is normalized in the sense that

\[
\varphi(1,r) = \varphi(s,1) = 0 = \varphi(s,1_\Gamma).
\]

The first two equalities follow from the property \( F(1) = 1' \) and the compatibility of \( (F, \tilde{F}) \) with unit constraints. The final equality holds owing to \( f(x,1_\Gamma) = 0 \) (following from the relation (8)).

By Proposition 4.2, the compatibility of \( (F, \tilde{F}) \) with associativity constraints is equivalent to

\[
f(y,z) + f(x,yz) = f(x,y) + f(xy,z).
\] (10)

The compatibility of \( (F, \tilde{F}) \) with the braiding constraints implies

\[
f(x,y) + f_1(\eta(x,y)) = \eta'(f_0(x),f_0(y)) + f(y,x).
\]

By the fact that \( f(x,y) \in \text{Ker} \ d' \subset Z(B') \) and by the condition \( H_3 \), one has

\[
f(x,y) = f(y,x).
\] (11)

From the relations (8)–(11), it follows that \( \varphi \in Z^2_{\Gamma,s}(\text{Coker} \ d, \text{Ker} \ d'). \)

We define the category \( \text{rBrCross} \)

whose objects are braided \( \Gamma \)-crossed modules and whose morphisms are triples

\( (f_1,f_0,\varphi) \), where \( (f_1,f_0) : (B \xrightarrow{d} D) \to (B' \xrightarrow{d'} D') \) is a homomorphism of braided \( \Gamma \)-crossed modules, and \( \varphi \in Z^2_{\Gamma,s}(\text{Coker} \ d, \text{Ker} \ d') \). The composition is given by (11).

Note that a braided strict \( \Gamma \)-graded categorical group \( \mathcal{G} \) induces \( \Gamma \)-actions on the group \( D \) of objects and on the group \( B \) of morphisms of grade 1, we state the following definition.
Definition 4.8. A graded symmetric monoidal functor \((F, \tilde{F}) : \mathcal{G} \to \mathcal{G}'\) is termed *regular* if

\begin{align*}
B_1 & : F(x) \otimes F(y) = F(x \otimes y), \\
B_2 & : F(b) \otimes F(c) = F(b \otimes c), \\
B_3 & : \tilde{F}_{x,y} = \tilde{F}_{y,x}, \\
B_4 & : F(\sigma x) = \sigma F(x), \\
B_5 & : F(\sigma b) = \sigma F(b),
\end{align*}

where \(x, y \in \text{Ob} \mathcal{G}, b, c \) are morphisms of grade 1 in \(\mathcal{G}\), \(\sigma \in \Gamma\).

The graded symmetric monoidal functor mentioned in Lemma 4.7 is regular.

Lemma 4.9. Let \(\mathcal{G}, \mathcal{G}'\) be corresponding braided strict \(\Gamma\)-graded categorical groups associated to braided \(\Gamma\)-crossed modules \(\mathcal{M}, \mathcal{M}'\), and let \((F, \tilde{F}) : \mathcal{G} \to \mathcal{G}'\) be a regular graded symmetric monoidal functor. Then, the triple \((f_1, f_0, \varphi)\), where

i) \(f_0(x) = F(x), \ (f_1(b), 1) = F(b, 1), \ \sigma \in \Gamma, b \in B, x \in D\),

ii) \(p^* \varphi = f\), where \(f\) is defined by (7),

is a morphism in the category \(\Gamma \text{BrCross}\).

Proof. Due to the conditions \(B_1\) and \(B_4\), \(f_0\) is a \(\Gamma\)-group homomorphism. By the assumption that \(F\) preserves the composition of two morphisms of grade 1 and by the condition \(B_5\), \(f_1\) is a \(\Gamma\)-group homomorphism. Any \(b \in B\) can be considered as a morphism \((db \to 1)\) in \(\mathcal{G}\), and hence \((f_0(db) \to f_1(b, 1)')\) is a morphism in \(\mathcal{G}'\), that is, the relation \(H_1\) holds. The relation \(H_2\) follows from the condition \(B_2\) and the homomorphism property of \(f_1\).

According to the proof of Lemma 4.7, the compatibility of \((F, \tilde{F})\) with braiding constraints and the condition \(B_3\) lead to the relation \(H_3\). So, \((f_1, f_0)\) is a homomorphism of braided crossed \(\Gamma\)-modules. Thus, by Lemma 4.7 the function \(f\) determines a function \(\varphi \in Z^2_{\Gamma,d}(\text{Coker} d, \text{Ker} d')\) such that \(f = p^* \varphi\), where \(p : D \to \text{Coker} d\) is a canonical projection. Therefore, \((f_1, f_0, \varphi)\) is a morphism in \(\Gamma \text{BrCross}\).

Denote by

\[\Gamma \text{BrGr}^*\]

the category of braided strict \(\Gamma\)-graded categorical groups and regular graded symmetric monoidal functors, we obtain the following result which is an extension of Theorem 3.6.

Theorem 4.10 (Classification Theorem). There exists an equivalence

\[\Phi : \Gamma \text{BrCross} \to \Gamma \text{BrGr}^*,\]

\[B \to D \iff \mathcal{G}_{B \to D}\]

\[(f_1, f_0, \varphi) \iff (F, \tilde{F})\]
where $F(x) = f_0(x)$, $F(b,1) = (f_1(b),1)$, $F(x \mapsto \sigma x) = (\varphi(px,\sigma),\sigma)$, $\tilde{F}_{x,y}((\varphi(px,py),1)$, for $x \in D, b \in B, \sigma \in \Gamma$.

**Proof.** Suppose that $\mathbb{G}, \mathbb{G}'$ are braided strict $\Gamma$-graded categorical groups associated to braided $\Gamma$-crossed modules $B \to D, B' \to D'$, respectively. By Lemma 4.7 the correspondence $(f_1, f_0, \varphi) \mapsto (F, \tilde{F})$ determines an injection on the homsets

$$\Phi : \text{Hom}_{\Gamma \text{BrCross}}(B \to D, B' \to D') \to \text{Hom}_{\Gamma \text{BrGr}^*}(\mathbb{G}, \mathbb{G}').$$

According to Lemma 4.9 $\Phi$ is surjective.

If $\mathbb{G}$ is a braided strict $\Gamma$-graded categorical group, and $\mathcal{M}_\mathbb{G}$ is its associated braided $\Gamma$-crossed module, then $\Phi(\mathcal{M}_\mathbb{G}) = \mathbb{G}$ (not only isomorphic). Therefore, $\Phi$ is an equivalence.

**Remark 4.11.** In the above theorem, if $B \to D$ is a symmetric $\Gamma$-crossed module, then $\mathbb{G}_{B \to D}$ is a symmetric strict $\Gamma$-graded categorical group. Let $\Gamma \text{SymCross}$ denote the full subcategory of the category $\Gamma \text{BrCross}$ whose objects are symmetric crossed $\Gamma$-modules, and let $\Gamma \text{PiGr}^*$ denote the full subcategory of the category $\Gamma \text{BrGr}^*$ whose objects are symmetric strict $\Gamma$-graded categorical groups. Then these two subcategories are equivalent and the following diagram commutes

$$\begin{array}{ccc}
\Gamma \text{SymCross} & \xrightarrow{\Phi} & \Gamma \text{PiGr}^* \\
J \downarrow & & \downarrow J^* \\
\Gamma \text{BrCross} & \xrightarrow{\Phi} & \Gamma \text{BrGr}^*,
\end{array}$$

where $J, J^*$ are full embedding functors.

**Remark 4.12.** When $\Gamma = 1$ is a trivial group, then the categories $\Gamma \text{BrCross}$ and $\Gamma \text{BrGr}^*$ are the categories $\text{BrCross}$ and $\text{BrGr}^*$, respectively. Therefore, we obtain Theorem 3.6.

### 5 Classification of $\Gamma$-module extensions of the type of an abelian $\Gamma$-crossed module

In this section, we present the theory of $\Gamma$-module extension of the type of an abelian $\Gamma$-crossed modules, which is analogous to the theory of group extension of the type of a crossed module [19, 10, 3].

In [6], if $d : B \to D$ is a homomorphism of abelian groups and $D$ acts trivially on $B$, then $(B,D,d,0)$ is called an abelian crossed module. Let us note that any abelian crossed module is defined by a strict Picard category,
that is, a symmetric categorical group in which \( a = id, c = id, l = id = r \) and for each object \( x \), there is an object \( y \) such that \( x \otimes y = 1 \).

By an abelian \( \Gamma \)-crossed module, we shall mean a braided \( \Gamma \)-crossed module 
\[
(B, D, d, \vartheta, \eta) \text{ that } \vartheta = 0, \eta = 0. \text { Then } d \text { is a homomorphism of } \Gamma\text{-modules.}
\]

According to the construction in Section 4, each abelian \( \Gamma \)-crossed module \( M = (B, D, d) \) defines a \( \Gamma \)-graded category \( G_M \) whose \( \text{Ker} G \) is a strict Picard category. In this case, we say that \( G_M \) is a strict \( \Gamma \)-graded Picard category. A homomorphism \((f_1, f_0) : (B, D, d) \to (B', D', d')\) of abelian \( \Gamma \)-crossed modules consists of \( \Gamma \)-module homomorphisms \( f_1 : B \to B' \) and \( f_0 : D \to D' \) such that 
\[
f_0d = d'f_1.
\]

Note that in this section, since \( B \) and \( D \) are abelian groups, we write + for the operations on \( B, D \).

**Definition 5.1.** Let \( M = (B, D, d) \) be an abelian \( \Gamma \)-crossed module, and let \( Q \) be a \( \Gamma \)-module. A \( \Gamma \)-module extension of \( B \) by \( Q \) of type \( M \), denoted by \( E_{d,Q} \), is a short exact sequence of \( \Gamma \)-module homomorphisms,
\[
E : 0 \to B \stackrel{j}{\to} E \stackrel{p}{\to} Q \to 0,
\]
and a homomorphism of abelian \( \Gamma \)-crossed modules \((id, \varepsilon) : (B \to E) \to (B \to D)\).

Two extensions \( E_{d,Q} \) and \( E'_{d,Q} \) are said to be equivalent if the following diagram commutes
\[
\begin{array}{ccc}
E : 0 & \to & B \downarrow \alpha \\
& \nearrow & \\
E' : 0 & \to & B' \downarrow \varepsilon'
\end{array}
\]
and \( \varepsilon'\alpha = \varepsilon \). Obviously, \( \alpha \) is an isomorphism of \( \Gamma \)-modules.

Each extension \( E_{d,Q} \) induces a \( \Gamma \)-module homomorphism \( \psi : Q \to \text{Coker } d \) such that \( \psi p = q\varepsilon \), where \( q : D \to \text{Coker } d \) is a canonical projection. Moreover, \( \psi \) is dependent only on the equivalence class of the extension \( E_{d,Q} \), and then we say that \( E_{d,Q} \) induces \( \psi \). The set of equivalence classes of extensions \( E_{d,Q} \) inducing \( \psi : Q \to \text{Coker } d \) is denoted by 
\[
\text{Ext}^M_{\Gamma}(Q, B, \psi).
\]

Now, in order to study this set we apply the obstruction theory for graded symmetric monoidal functors between strict \( \Gamma \)-graded Picard categories \( \text{Dis}_{\Gamma,s} Q \) and \( \mathbb{G}_{B \to D} \), where the discrete \( \Gamma \)-graded Picard category \( \text{Dis}_{\Gamma,s} Q \) is defined by (see Subsection 2.2)
\[
\text{Dis}_{\Gamma,s} Q = \int_{\Gamma}(Q, 0, 0).
\]
It is just the strict $\Gamma$-graded Picard category associated to the abelian $\Gamma$-crossed module $(0, Q, 0)$ (see Section 4).

**Lemma 5.2.** Let $\mathcal{M} = (B, D, d)$ be an abelian $\Gamma$-crossed module, $Q$ be a $\Gamma$-module and $\psi : Q \to \text{Coker } d$ be a $\Gamma$-module homomorphism. Then for each graded symmetric monoidal functor $(F, \tilde{F}) : \text{Dis}_{\Gamma,s} Q \to \mathcal{G}_\mathcal{M}$ which satisfies $F(0) = 0$ and induces the pair of $\Gamma$-module homomorphisms $(\psi, 0) : (Q, 0) \to (\text{Coker } d, \text{Ker } d)$, there exists an extension $\mathcal{E}_{d,Q}$ inducing $\psi$.

Such an extension $\mathcal{E}_{d,Q}$ is called associated to the graded symmetric monoidal functor $(F, \tilde{F})$.

**Proof.** Suppose that $(F, \tilde{F}) : \text{Dis}_{\Gamma,s} Q \to \mathcal{G}_\mathcal{M}$ is a graded symmetric monoidal functor. Then $(F, \tilde{F})$ determines a function $f : Q^2 \cup (Q \times \Gamma) \to B$ by (7),

$$f(u, v, 1) = \tilde{F}_{u,v}, \quad f(u, \sigma, \sigma) = F(u \xrightarrow{(0,\sigma)} \sigma u).$$

The function $f$ is “normalized” in the sense that

$$f(u, 1_\Gamma) = 0, f(u, 0) = 0 = f(0, v).$$

Since $F$ preserves the identity morphism, one has the first equality. The later equalities follow from the assumption $F(0) = 0$ and the compatibility of $(F, \tilde{F})$ with unit constraints. It follows from the definition of a morphism in $\mathcal{G}$ that

$$\sigma F(u) = df(u, \sigma) + F(\sigma u), \quad (12)$$
$$F(u) + F(v) = df(u, v) + F(u + v). \quad (13)$$

The function $f$ defined as above is just a 2-cocycle in $Z^2_{\Gamma,s}(Q, B)$.

From the 2-cocycle $f$, we construct an exact sequence of $\Gamma$-modules

$$\mathcal{E}_F : 0 \to B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \to 0,$$

where $E_0$ is the crossed product extension $B \times_f Q$ and $j_0(b) = (b, 1)$, $p_0(b, u) = u$, for $b \in B, u \in Q$. The $\Gamma$-module structure of $E_0$ is given by

$$(b, u) + (c, v) = (b + c + f(u, v), u + v),$$
$$\sigma(b, u) = (\sigma b + f(u, \sigma), \sigma u).$$

Now we determine $\Gamma$-module homomorphism $\varepsilon : E_0 \to D$. By the assumption, $(F, \tilde{F})$ induces a $\Gamma$-module homomorphism $\psi : Q \to \text{Coker } d$ by $\psi(u) = [Fu] \in \text{Coker } d$. Thus, the element $Fu$ is a representative of $\text{Coker } d$ in $D$. Then for $(b, u) \in E_0$, we set

$$\varepsilon(b, u) = db + Fu. \quad (14)$$
Therefore, $\varepsilon$ is a $\Gamma$-module homomorphism thanks to the relations (12) and (13). It is easy to see that $\varepsilon \circ j_0 = d$. Further, this extension induces $\Gamma$-module homomorphism $\psi : Q \to \text{Coker } d$, since

$$
\varepsilon(b, u) = q(db + Fv) = q(Fv) = \psi(u) = \psi p_0(b, u),
$$

for all $u \in Q$. \hfill \Box

**Theorem 5.3** (Schreier Theory for $\Gamma$-module extensions of the type of an abelian $\Gamma$-crossed module). Let $\mathcal{M} = (B, D, d)$ be an abelian $\Gamma$-crossed module, and let $\psi : Q \to \text{Coker } d$ be a $\Gamma$-module homomorphism. There exists a bijection

$$
\Omega : \text{Hom}_{[\psi, 0]}[\text{Dis}_{\Gamma,s} Q, G_M] \to \text{Ext}_{Z[\Gamma]}^M (Q, B, \psi).
$$

**Proof.** Step 1: Graded symmetric monoidal functors $(F, \tilde{F})$, $(F', \tilde{F}')$ are homotopic if and only if the corresponding associated extensions $E_{d,Q}, E'_{d,Q}$ are equivalent.

Suppose that $F, F' : \text{Dis}_{\Gamma,s} Q \to G_M$ are homotopic by a homotopy $\alpha : F \to F'$. Then, there is a function $g : Q \to B$ such that $\alpha_u = (g(u), 1)$, that is,

$$
F(u) = dg(u) + F'(u).
$$

(15)

The naturality and the coherence condition (3) of the homotopy $\alpha$ lead to $g(0) = 0$ and

$$
f(u, \sigma) + g(\sigma u) = \sigma g(u) + f'(u, \sigma),
$$

(16)

$$
f(u, v) + g(u + v) = g(u) + g(v) + f'(u, v).
$$

(17)

According to Lemma 5.2, there exist the extensions $E_{d,Q}$ and $E'_{d,Q}$ associated to $F$ and $F'$, respectively. Then, thanks to the relations (16) and (17), the map

$$
\alpha^* : E_F \to E'_{F'}, \quad (b, u) \mapsto (b + g(u), u)
$$

is a homomorphism of $\Gamma$-modules. Further, $\alpha^*$ is an isomorphism. The equality $\varepsilon' \alpha^* = \varepsilon$ is implied from the relations (14) and (15):

$$
\varepsilon'(b, u) = \varepsilon'(b + g(u), u) = d(b + g(u)) + F'u = d(b) + d(g(u)) + F'u = d(b) + Fu = \varepsilon(b, u).
$$

Therefore, two extensions $E_{d,Q}$ and $E'_{d,Q}$ are equivalent.

Now, suppose that $E_{d,Q}$ and $E'_{d,Q}$ are two extensions associated to $(F, \tilde{F})$ and $(F', \tilde{F}')$, respectively. If $\alpha^* : E_F \to E'_{F'}$ is an equivalence of these extensions, then it is straightforward to see that

$$
\alpha^*(b, u) = (b + g(u), u),
$$

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where \( g : Q \to B \) is a function with \( g(0) = 0 \). By retracing our steps, \( \alpha_u = (g(u), 1) \) is a homotopy between \((F, \tilde{F})\) and \((F', \tilde{F}')\).

**Step 2: \( \Omega \) is surjective.**

Assume that \( E = E_{d,Q} \) is an extension of type \( \mathcal{M} \). We prove that \( E \) defines a graded symmetric monoidal functor \((F, \tilde{F}) : \text{Dis}_{\Gamma,s} Q \to \mathcal{G}_M\). For any \( u \in Q \), choose a representative \( e_u \in E \) such that \( p(e_u) = u, e_0 = 0 \). Each element of \( E \) can be represented uniquely as \( b + e_u \) for \( b \in B, u \in Q \).

The representatives \( \{e_u\} \) induce a normalized function \( f : \mathbb{Z}^2 \cup (Q \times \Gamma) \to B \) by

\[
e_u + e_v = f(u, v) + e_{u+v},
\]

\[
\sigma e_u = f(u, \sigma) + e_{\sigma u}.
\]

Now, we construct a graded symmetric monoidal functor \((F, \tilde{F}) : \text{Dis}_{\Gamma,s} Q \to \mathcal{G}_M\) as follows. Since \( \psi(u) = \psi\varepsilon(e_u) = q\varepsilon(e_u), \varepsilon(e_u) \) is a representative of \( \psi(u) \) in \( D \). Thus, we set

\[
F(u) = \varepsilon(e_u), \quad F(u \xmapsto{\sigma} \sigma u) = (f(u, \sigma), \sigma), \quad \tilde{F}_{u,v} = (f(u, v), 1).
\]

The relations (18) and (19) show that \( F(\sigma) \) and \( \tilde{F}_{u,v} \) are morphisms in \( \mathcal{G} \), respectively. The associativity and commutativity laws and the \( \Gamma \)-group property of \( B \) show that \( f \in \mathbb{Z}^3_{\Gamma,s}(Q, B) \), and hence \((F, \tilde{F})\) is a graded symmetric monoidal functor of type \((\psi, 0)\).

Let \( \mathcal{G} \) be a \( \Gamma \)-graded Picard category associated to an abelian \( \Gamma \)-crossed module \( B \xrightarrow{d} D \). Since \( \pi_0 \mathcal{G} = \text{Coker} d \) and \( \pi_1 \mathcal{G} = \text{Ker} d \), it follows from Subsection 2.2 that the reduced \( \Gamma \)-graded Picard category \( \mathcal{G}(h) \) of \( \mathcal{G} \) is of the form

\[
\mathcal{G}(h) = \int_{\Gamma} (\text{Coker} d, \text{Ker} d, h), \quad h \in \mathbb{Z}^3_{\Gamma,s}(\text{Coker} d, \text{Ker} d).
\]

Then \( \Gamma \)-module homomorphism \( \psi : Q \to \text{Coker} d \) induces an obstruction

\[
\psi^* h \in \mathbb{Z}^3_{\Gamma,s}(Q, \text{Ker} d).
\]

We now use this notion of obstruction to state and prove the following theorem.

**Theorem 5.4.** Let \( \mathcal{M} = (B, D, d) \) be an abelian \( \Gamma \)-crossed module, and let \( \psi : Q \to \text{Coker} d \) be a homomorphism of \( \Gamma \)-modules. Then, the vanishing of \( \psi^* h \) in \( H^2_{\Gamma,s}(Q, \text{Ker} d) \) is necessary and sufficient for there to exist an extension \( \mathcal{E}_{d,Q} \) of type \( \mathcal{M} \) inducing \( \psi \). Further, if \( \psi^* h \) vanishes, then the set of equivalence classes of such extensions is bijective with \( H^2_{\Gamma,s}(Q, \text{Ker} d) \).
Proof. By the assumption $\overline{\psi^*h} = 0$, it follows from Proposition 2.2 that there is a graded symmetric monoidal functor $(\Psi, \tilde{\Psi}) : \text{Dis}_\Gamma, s Q \rightarrow \mathbb{G}(h)$. Then the composition of $(\Psi, \tilde{\Psi})$ and the canonical graded symmetric monoidal functor $(H, \tilde{H}) : \mathbb{G}(h) \rightarrow \mathbb{G}$ is a graded symmetric monoidal functor $(F, \tilde{F}) : \text{Dis}_\Gamma, s Q \rightarrow \mathbb{G}$, and hence by Lemma 5.2 we obtain an associated extension $\mathcal{E}_{d,Q}$.

Conversely, suppose that

$$
\mathcal{E} : 0 \rightarrow B \xrightarrow{j} E \xrightarrow{p} Q \rightarrow 0
$$

is a $\Gamma$-module extension of type $\mathcal{M}$ inducing $\psi$. Let $\mathcal{G}'$ be a strict $\Gamma$-graded Picard category associated to the abelian $\Gamma$-crossed module $(B, E, j)$. Then, according to Proposition 4.7 there is a graded symmetric monoidal functor $F : \mathcal{G}' \rightarrow \mathbb{G}$. Since the reduced $\Gamma$-graded Picard category of $\mathcal{G}'$ is $\text{Dis}_\Gamma, s Q$, it follows from Proposition 2.1 that $F$ induces a graded symmetric monoidal functor of type $(\psi, 0)$ from $(Q, 0, 0)$ to $(\text{Coker} \, d, \text{Ker} \, d, h)$. Now, thanks to Proposition 2.2 the obstruction of the pair $(\psi, 0)$ vanishes in $H^3_{\Gamma, s}(Q, \text{Ker} \, d)$, i.e., $\overline{\psi^*h} = 0$.

The final assertion of Theorem 5.4 is obtained from Theorem 5.3. First, there is a natural bijection

$$
\text{Hom}[\text{Dis}_\Gamma, s Q, \mathbb{G}] \leftrightarrow \text{Hom} \text{Dis}_\Gamma, s Q, \mathbb{G}(h)].
$$

Then, since $\pi_0(\text{Dis}_\Gamma, s Q) = Q, \pi_1 G(h) = \text{Ker} \, d$, the bijection

$$
\text{Ext}^M_{\mathcal{Z}_{\Gamma}}(Q, B, \psi) \leftrightarrow H^2_{\Gamma, s}(Q, \text{Ker} \, d)
$$

follows from Theorem 5.3 and Proposition 2.2.

We now consider the special case when $\mathcal{M} = (B, \text{Aut} \, B, 0)$ is an abelian $\Gamma$-crossed module. Then, each $\Gamma$-module extension of type $\mathcal{M}$ inducing $\psi : Q \rightarrow \text{Aut} \, B$ is just an extension of $\Gamma$-modules,

$$
0 \rightarrow B \rightarrow E \rightarrow Q \rightarrow 0,
$$

inducing $\psi$. Thus, Theorem 5.4 leads to the following consequence.

**Corollary 5.5** ([8], Theorem 2.4). Let $B, Q$ be $\Gamma$-modules, and let $\psi : Q \rightarrow \text{Aut} \, B$ be a $\Gamma$-module homomorphism. Then, there is an obstruction class $\overline{k} \in H^3_{\Gamma, s}(Q, B)$ whose vanishing is necessary and sufficient for there to exist a $\Gamma$-module extension of $B$ by $Q$ inducing $\psi$. Further, if $\overline{k}$ vanishes, then there exists a bijection

$$
\text{Ext}^M_{\mathcal{Z}_{\Gamma}}(Q, B, \psi) \leftrightarrow H^2_{\Gamma, s}(Q, B).
$$

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