SINGULAR ELLIPTIC PROBLEMS WITH LACK OF COMPACTNESS

Marius GHERGU and Vicențiu RĂDULESCU
Department of Mathematics, University of Craiova, 200585 Craiova, Romania

Abstract. We consider the following nonlinear singular elliptic equation

\[-\text{div}(|x|^{-2a}|x|^{-b}|x|^{p-2}u + \lambda g(x) \in \mathbb{R}^N,\]

where \(g\) belongs to an appropriate weighted Sobolev space, and \(p\) denotes the Caffarelli-Kohn-Nirenberg critical exponent associated to \(a, b,\) and \(N.\) Under some natural assumptions on the positive potential \(K(x)\) we establish the existence of some \(\lambda > 0\) such that the above problem has at least two distinct solutions provided that \(\lambda \in (0, \lambda_0).\) The proof relies on Ekeland’s Variational Principle and on the Mountain Pass Theorem without the Palais-Smale condition, combined with a weighted variant of the Brezis-Lieb Lemma.

Key words: singular elliptic equation, perturbation, multiple solutions, singular minimization problem, critical point, weighted Sobolev space.

2000 Mathematics Subject Classification: 35B20, 35B33, 35J20, 35J70, 47J20, 58E05.

1 Introduction and the main result

Many papers have been devoted in the last decades to the study of several questions concerning degenerate elliptic problems. We start with the following example

\[\left\{ \begin{array}{ll}
\text{div}(a(x)\nabla u) + f(u) = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.\]

where \(\Omega\) is an arbitrary domain in \(\mathbb{R}^N (N \geq 1),\) and \(a\) is a nonnegative function that may have “essential” zeroes at some points or even may be unbounded. The continuous function \(g\) satisfies \(f(0) = 0\) and \(tf(t)\) behaves like \(|t|^p\) as \(|t| \to \infty\), with \(2 < p < 2^*\), where \(2^*\) denotes the critical Sobolev exponent. Notice that equations of this type come from the consideration of standing waves in anisotropic Schrödinger equations (see [2, 20, 21, 25]). Equations like (1) are also introduced as models for several physical phenomena related to equilibrium of anisotropic media which possibly are somewhere “perfect” insulators or “perfect” conductors (see [10], p. 79). Problem (1) has also some interest in the framework of optimization and \(G\)-convergence (see, e.g., [14] and the references therein).

Classical results (see [1, 17]) ensure the existence and the multiplicity of positive or nodal solutions for problem (1), provided that the differential operator \(Tu := \text{div}(a(x)\nabla u)\) is uniformly elliptic. Several difficulties occur both in the degenerate case (if \(\inf_{\Omega} a = 0\)) and in the singular case (if \(\sup_{\Omega} a = +\infty\)). In these situations the classical methods fail to be applied directly so that

*Correspondence address: Vicențiu Rădulescu, Department of Mathematics, University of Craiova, 200585 Craiova, Romania, fax: +40-251.41.16.88. E-mail: radulescu@inf.ucv.ro
the existence and the multiplicity results (which hold in the nondegenerate case) may become a delicate matter that is closely related to some phenomena due to the degenerate character of the differential equation. These problems have been intensively studied starting with the pioneering paper by Murthy and Stampacchia [15] (we also refer to [3] [13] [16], as well as to the monograph [22]).

A natural question that arises in concrete applications is to see what happens if these elliptic (degenerate or nondegenerate) problems are affected by a certain perturbation. It is worth pointing out here that the idea of using perturbation methods in the treatment of nonlinear boundary value problems was introduced by Struwe [23]. Recently, many authors have been interested in this kind of perturbation problems involving both critical and sub- or super-critical Sobolev exponent (see, e.g., [9] [18] [24]).

Our aim in this paper is to study the following degenerate perturbed problem

\[-\text{div}(|x|^{-2a} \nabla u) = K(x)|x|^{-bp}|u|^{p-2}u + \lambda g(x) \quad \text{in} \quad \mathbb{R}^N,\]

where

\[
\begin{cases}
\text{for } N \geq 3 : & -\infty < a < \frac{N-2}{2}, \quad a < b < a+1, \quad \text{and} \quad p = \frac{2N}{N-2+2(b-a)}; \\
\text{for } N = 2 : & -\infty < a < 0, \quad a < b < a+1, \quad \text{and} \quad p = \frac{2}{b-a}; \\
\text{for } N = 1 : & -\infty < a < -\frac{1}{2}, \quad a+\frac{1}{2} < b < a+1, \quad \text{and} \quad p = \frac{2}{-1+2(b-a)}. 
\end{cases}
\]

Equation (2) contains the critical Caffarelli-Kohn-Nirenberg exponent \( p \), defined as in (3). In this critical case, some concentration phenomena may occur, due to the action of the noncompact group of dilations in \( \mathbb{R}^N \). The lack of compactness of problem (2) is also given by the fact that we are looking for entire solutions, that is, solutions defined on the whole space.

The reason for which we choose the parameters \( a, b, \) and \( p \) to satisfy the assumption (3) has to do with the following inequality, due to Caffarelli, Kohn, and Nirenberg (see [6]):

\[\left( \int_{\mathbb{R}^N} |x|^{-bp} |u|^p dx \right)^{1/p} \leq C_{a,b} \left( \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx \right)^{1/2},\]

for all \( u \in C_0^\infty(\mathbb{R}^N) \), where \( a, b, \) and \( p \) satisfy the condition (3). We point out that the inequality also holds true for \( b = a + 1 \) (if \( N \geq 1 \)) and \( b = a \) (if \( N \geq 3 \)) but, in these cases, the best Sobolev constant \( C_{a,b} \) in (4) is never achieved (see [7] for details). The Caffarelli–Kohn–Nirenberg inequality also contains as particular cases the classical Sobolev inequality (if \( a = b = 0 \)) and the Hardy inequality (if \( a = 0 \) and \( b = 1 \)); we refer to [4] [11] [19] for further details.

The extremal functions for (4) are ground state solutions of the singular Euler equation

\[-\text{div}(|x|^{-2a} \nabla u) = |x|^{-bp} |u|^{p-2}u, \quad \text{in} \quad \mathbb{R}^N.\]

This equation has been recently studied (see [7] [26]) in connection with a complete understanding of the best constants, the qualitative properties of extremal functions, the existence (or nonexistence) of minimizers, and the symmetry properties of minimizers.

The function \( K \) is assumed to fulfill

\[(K1) \quad K \in L^\infty(\mathbb{R}^N),\]

\[(K2) \quad \text{esslim}_{|x| \to 0} K(x) = \text{esslim}_{|x| \to \infty} K(x) = K_0 \in (0, \infty) \quad \text{and} \quad K(x) \geq K_0 \quad \text{a.e. in} \quad \mathbb{R}^N,\]

\[(K3) \quad \text{meas} \left( \{ x \in \mathbb{R}^N : K(x) > K_0 \} \right) > 0.\]
Many authors have made contributions in the study of this problem, especially for the case $\lambda = 0$. The Palais-Smale condition (PS) plays a central role when variational methods are applied in the study of problem (2). In this paper, we establish the existence and the multiplicity of nontrivial solutions of (2) with $\lambda > 0$ sufficiently small, in a case where the condition (PS) is not assumed even for $\lambda = 0$. More precisely, we will show that there exists at least two weak solutions of (2) for $g \neq 0$ in an appropriate weighted Sobolev space and $\lambda > 0$ small enough. Our proof relies on Ekeland’s Variational Principle [12] and on the Mountain Pass Theorem without the Palais-Smale condition (in the sense of Brezis and Nirenberg, see [5]), combined with a weighted variant of the Brezis-Lieb Lemma [3].

The natural functional space to study problem (2) is $H^1_0(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left( \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, dx \right)^{1/2}. \tag{5}$$

It turns out that $H^1_0(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1_0(\mathbb{R}^N).$$

It follows that (4) holds for all $u \in H^1_0(\mathbb{R}^N)$. According to [7] we have

$$H^1_0(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N \setminus \{0\}) \|\cdot\|}, \tag{6}$$

where $\|\cdot\|$ is given by (5). Let $\|\cdot\|_{-1}$ denote the norm in the dual space $H^{-1}_0(\mathbb{R}^N)$ of $H^1_0(\mathbb{R}^N)$.

Throughout this paper we suppose that $g \in H^{-1}_0(\mathbb{R}^N) \setminus \{0\}$.

For an arbitrary open set $\Omega \subset \mathbb{R}^N$ let $L^p_0(\Omega)$ be the space of all measurable real functions $u$ defined on $\Omega$ such that $\int_\Omega |x|^{-bp} |u|^p \, dx$ is finite. By (1) it follows that the weighted Sobolev space $H^1_0(\Omega)$ is continuously embedded in $L^p_0(\Omega)$.

**Definition 1.** We say that a function $u \in H^1_0(\mathbb{R}^N)$ is a weak solution of problem (2) if

$$\int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} K(x) |x|^{-bp} |u|^{p-2} u v \, dx - \lambda \int_{\mathbb{R}^N} g(x) v \, dx = 0,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Obviously, the solutions of problem (2) correspond to critical points of the energy functional

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} K(x) |x|^{-bp} |u|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x) u \, dx,$$

where $u \in H^1_0(\mathbb{R}^N)$.

Our main result is the following.

**Theorem 1.** Suppose that assumptions (K1), (K2), (K3) are fulfilled and fix $g \in H^{-1}_0(\mathbb{R}^N) \setminus \{0\}$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, problem (2) has at least two solutions.

Since the embedding $H^1_0(\mathbb{R}^N) \hookrightarrow L^p_0(\mathbb{R}^N)$ is not compact, the energy functional $J_\lambda$ fails to satisfy the (PS) condition. Such a failure brings about difficulty in applying a variational approach to equation (2). Furthermore, since $g \neq 0$, then 0 is no longer a trivial solution of problem (2) and, therefore, the Mountain Pass Theorem cannot be applied directly. Using some ideas developed in [21], we obtain the first solution by applying Ekeland’s Variational Principle. Then, the Mountain
Pass Theorem without the Palais-Smale condition yields a bounded Palais-Smale sequence whose weak limit is a critical point of $J_\lambda$. The proof is concluded by showing that these two solutions are distinct because they realize different energy levels.

The paper is organized as follows. In Section 2 we give some technical results which allow us to give a variational approach of our main result that we prove in Section 3. We point out that since the perturbation term $g$ is not assumed to be non-negative then we can not expect that the distinct solutions given by Theorem 1 are positive. However, if $g \geq 0$ is a non-trivial perturbation, then a straightforward argument based on the maximum principle implies that the solutions of problem 2 are positive.

**Notations.** Throughout this paper we will denote by $B_R$ the open ball in $H^1_0(\mathbb{R}^N)$ centered at origin and having radius $R > 0$. We also denote by $(\cdot, \cdot)$ the duality pairing between $H^1_0(\mathbb{R}^N)$ and $H^{-1}_0(\mathbb{R}^N)$. The notations ”$\rightharpoonup”$ and ”$\to$” stand, respectively, for the weak and for the strong convergence in an arbitrary Banach space.

## 2 Auxiliary results

Define the functionals $J_0, I : H^1_0(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u|^p \, dx,$$

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} K_0(x)|x|^{-bp} |u|^p \, dx.$$

The Caffarelli-Kohn-Nirenberg inequality (4) and the conditions (5) imply that the functionals $J_\lambda, J_0,$ and $I$ are well defined and $J_\lambda, J_0, I \in C^1(H^1_0(\mathbb{R}^N), \mathbb{R})$.

**Remark 1.** If $\Omega \subset \mathbb{R}^N$ is a smooth bounded set such that $0 \notin \overline{\Omega}$ then, by the Sobolev inequality, we have

$$\left( \int_{\Omega} |x|^{-bp} |u|^p \, dx \right)^{1/p} \leq C_1 \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} \leq C_2 \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2} \leq C_3 \left( \int_{\Omega} |x|^{-2a} |\nabla u|^2 \, dx \right)^{1/2},$$

for all $u \in H^1_0(\Omega)$. It follows that $H^1_0(\Omega)$ is compactly embedded in $L^p_0(\Omega)$.

Remark 1 implies that if $\{u_n\}$ is a sequence that converges weakly to some $u_0$ in $H^1_0(\mathbb{R}^N)$ then $\{u_n\}$ is bounded in $H^1_0(\mathbb{R}^N)$. Therefore, we can assume (up to a sequence) that

$$u_n \rightharpoonup u_0 \text{ in } L^p_{0, \text{loc}}(\mathbb{R}^N \setminus \{0\}) \text{ and } u_n \to u_0 \text{ a.e. in } \mathbb{R}^N. \quad (7)$$

**Definition 2.** Let $X$ be a Banach space, $F : X \to \mathbb{R}$ be a $C^1$–functional and $c$ be a real number. A sequence $\{u_n\} \subset X$ is called a $(PS)_c$ sequence of $F$ if $F(u_n) \to c$ and $\|F'(u_n)\|_{X^*} \to 0$.

Our first result shows that if a $(PS)_c$ sequence of $J_\lambda$ is weakly convergent then its limit is a solution of problem 2.

**Lemma 1.** Let $\{u_n\} \subset H^1_0(\mathbb{R}^N)$ be a $(PS)_c$ sequence of $J_\lambda$ for some $c \in \mathbb{R}$. Suppose that $\{u_n\}$ converges weakly to some $u_0$ in $H^1_0(\mathbb{R}^N)$. Then $u_0$ is a solution of problem 2.

**Proof.** Let $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ be an arbitrary function and set $\Omega := \text{supp } \varphi$. Since $J_\lambda'(u_n) \to 0$ in $H^{-1}_0(\mathbb{R}^N)$ we obtain $(J_\lambda'(u_n), \varphi) \to 0$ as $n \to \infty$, that is,

$$\lim_{n \to \infty} \left( \int_{\Omega} |x|^{-2a} \nabla u_n \cdot \nabla \varphi \, dx - \int_{\Omega} K(x)|x|^{-bp} |u_n|^{p-2} u_n \varphi \, dx - \lambda \int_{\Omega} g(x) \varphi \, dx \right) = 0. \quad (8)$$
By virtue of (6) we deduce that the above equality holds for all $\varphi \in \mathcal{D}$. Consequently, relations (8), (9), and (10) yield

$$\lim_{n \to \infty} \int \nabla \varphi \cdot \nabla u_n \, dx = \int \nabla \varphi \cdot \nabla u_0 \, dx. \quad (9)$$

The boundedness of $\{u_n\}$ in $H^1_0(\Omega)$ and the Caffarelli-Kohn-Nirenberg inequality imply that $\{u_n\}$ is bounded in $L^{p/(p-1)}(\Omega)$. Since $\|u_n\|_{L^{p/(p-1)}(\Omega)} \to \|u_0\|_{L^{p/(p-1)}(\Omega)}$ a.e. in $\Omega$ (which is a consequence of (6)), we deduce that $\|u_0\|_{L^{p/(p-1)}(\Omega)}$ is bounded in $\Omega$. Therefore

$$\lim_{n \to \infty} \int \nabla \varphi \cdot \nabla u_n \, dx = \int \nabla \varphi \cdot \nabla u_0 \, dx. \quad (10)$$

Consequently, relations (5), (9), and (10) yield

$$\int \nabla \varphi \cdot \nabla u_n \, dx - \int \nabla \varphi \cdot \nabla u_0 \, dx - \lambda \int g(\varphi) \, dx = 0.$$

By virtue of (6) we deduce that the above equality holds for all $\varphi \in H^1_0(\Omega)$ which means that $J'(u_0) = 0$. The proof of our lemma is now complete. $\square$

We now establish a weighted variant of the Brezis-Lieb Lemma (see [3]).

**Lemma 2.** Let $\{u_n\}$ be a sequence which is weakly convergent to $u_0$ in $H^1_0(\Omega)$. Then

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|x|^{-bp} \left( |u_n|^p - |u_n - u_0|^p \right) \, dx = \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p \, dx. \quad (11)$$

**Proof.** Using the boundedness of $\{u_n\}$ in $H^1_0(\Omega)$ and the Caffarelli-Kohn-Nirenberg inequality, it follows that the sequence $\{u_n\}$ is bounded in $L^{p/(p-1)}(\Omega)$. Let $\varepsilon > 0$ be a positive real number. By (K1) and (K2), we can choose $R_{r_\varepsilon} > r_\varepsilon > 0$ such that

$$\int_{|x| < r_\varepsilon} K(x)|x|^{-bp} |u_0|^p \, dx < \varepsilon, \quad (11)$$

and

$$\int_{|x| > R_{r_\varepsilon}} K(x)|x|^{-bp} |u_0|^p \, dx < \varepsilon. \quad (12)$$

Denote $\Omega_{\varepsilon} = \overline{B(0,R_{r_\varepsilon}) \setminus B(0,r_\varepsilon)}$. We have

$$\left| \int_{\mathbb{R}^N} K(x)|x|^{-bp} \left( |u_n|^p - |u_0|^p - |u_n - u_0|^p \right) \, dx \right| \leq \int_{\Omega_{\varepsilon}} K(x)|x|^{-bp} \left( |u_n|^p - |u_0|^p \right) \, dx + \int_{\Omega_{\varepsilon}} K(x)|x|^{-bp} |u_n - u_0|^p \, dx \quad (13)$$

By the Lagrange Mean Value Theorem we have

$$\int_{|x| < r_\varepsilon} K(x)|x|^{-bp} \left( |u_n|^p - |u_n - u_0|^p \right) \, dx = p \int_{|x| < r_\varepsilon} K(x)|x|^{-bp} \theta u_0 + (u_n - u_0)^{p-1} |u_0| \, dx, \quad (13)$$
where $0 < \theta(x) < 1$. Next, we employ the following elementary inequality: for all $s > 0$ there exists a constant $c = c(s)$ such that
\[(x + y)^s \leq c(x^s + y^s) \text{ for any } x, y \in (0, \infty).\]

Then, by Hölder’s inequality and relation (11) we deduce that
\[
\int_{|x| < r_s} K(x)|x|^{-bp} |\theta u_0 + (u_n - u_0)|^{p-1}|u_0| \, dx \\
\leq c \int_{|x| < r_s} K(x)|x|^{-bp} (|u_0|^p + |u_n - u_0|^{p-1}|u_0|) \, dx \\
= c \int_{|x| < r_s} K(x)|x|^{-bp} |u_0|^p \, dx + c \int_{|x| < r_s} K(x)|x|^{-bp} |u_n - u_0|^{p-1}|u_0| \, dx \\
\leq c \varepsilon + c \left( \int_{|x| < r_s} K(x)|x|^{-bp} |u_n - u_0|^p \, dx \right)^{1/p} \left( \int_{|x| < r_s} K(x)|x|^{-bp} |u_0|^p \, dx \right)^{1/p},
\]
where the constant $c_1$ is independent of $n$ and $\varepsilon$. Using relation (13) we have
\[
\int_{|x| < r_s} K(x)|x|^{-bp} |u_0|^p \, dx + \int_{|x| < r_s} K(x)|x|^{-bp} (|u_n|^p - |u_n - u_0|^p) \, dx \leq p \tilde{c}_1 (\varepsilon + \varepsilon^{1/p}). \tag{14}
\]
In a similar manner we obtain
\[
\int_{|x| > R_s} K(x)|x|^{-bp} |u_0|^p \, dx + \int_{|x| > R_s} K(x)|x|^{-bp} (|u_n|^p - |u_n - u_0|^p) \, dx \leq p \tilde{c}_2 (\varepsilon + \varepsilon^{1/p}). \tag{15}
\]
Since $u_n \rightharpoonup u_0$ in $H^1_a(\mathbb{R}^N)$, relation (7) yields
\[
\lim_{n \to \infty} \int_{\Omega_s} K(x)|x|^{-bp} (|u_n|^p - |u_0|^p) \, dx = 0, \\
\lim_{n \to \infty} \int_{\Omega_s} K(x)|x|^{-bp} |u_n - u_0|^p \, dx = 0. \tag{16}
\]
Now, relations (14), (15), and (16) yield
\[
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^N} K(x)|x|^{-bp} (|u_n|^p - |u_0|^p - |u_n - u_0|^p) \, dx \right| \leq (pC + 1) (\varepsilon + \varepsilon^{1/p}).
\]
Since $\varepsilon > 0$ is arbitrary, it follows that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|x|^{-bp} (|u_n|^p - |u_n - u_0|^p) \, dx = \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p \, dx.
\]
This concludes the proof. \(\square\)

**Lemma 3.** Let $\{v_n\}$ be a sequence which converges weakly to 0 in $H^1_a(\mathbb{R}^N)$. Then the following properties hold true
\[
\lim_{n \to \infty} [J_\lambda(v_n) - I(v_n)] = 0, \\
\lim_{n \to \infty} [(J'_\lambda(v_n), v_n) - (I'(v_n), v_n)] = 0.
\]
Proof. A simple computation yields
\[ J_\lambda(v_n) = I(v_n) - \frac{1}{p} \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x)v_n \, dx, \]
\[ \langle J_\lambda'(v_n), v_n \rangle = \langle I'(v_n), v_n \rangle - \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x)v_n \, dx. \]
Since \( v_n \to 0 \) in \( H^1_0(\mathbb{R}^N) \), it follows from the above equalities that it suffices to prove that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx = 0. \]  
(17)

Fix \( \varepsilon > 0 \). By our assumptions (K1) and (K2), there exists \( R_\varepsilon > r_\varepsilon > 0 \) such that
\[ |K(x) - K_0| = K(x) - K_0 < \varepsilon \quad \text{for a.e.} \quad x \in \mathbb{R}^N \setminus \Omega_\varepsilon, \]
where \( \Omega_\varepsilon = \overline{B(0, R_\varepsilon)} \setminus B(0, r_\varepsilon) \). Next, we have
\[ \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx \]
\[ = \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx + \int_{\Omega_\varepsilon} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx \]
\[ \leq \varepsilon \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |x|^{-bp} |v_n|^p \, dx + (\|K\|_\infty - K_0) \int_{\Omega_\varepsilon} |x|^{-bp} |v_n|^p \, dx \]
\[ \leq \varepsilon \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p \, dx + (\|K\|_\infty - K_0) \int_{\Omega_\varepsilon} |x|^{-bp} |v_n|^p \, dx. \]

Since \( v_n \to 0 \) in \( H^1_0(\mathbb{R}^N) \), the Caffarelli-Kohn-Nirenberg inequality implies that \( \{v_n\} \) is bounded in \( L^p(\mathbb{R}^N) \). Moreover, by (7), it follows that \( v_n \to 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \). The above relations yield
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp} |v_n|^p \, dx \leq C \varepsilon \]
for some constant \( C > 0 \) independent of \( n \) and \( \varepsilon \). Since \( \varepsilon > 0 \) was arbitrarily chosen, we conclude that (17) holds and the proof of Lemma 3 is now complete.

\begin{lemma}
There exists \( \lambda_1 > 0 \) and \( R = R(\lambda_1) > 0 \) such that for all \( \lambda \in (0, \lambda_1) \), the functional \( J_\lambda \) admits a \((PS)_{c_0, \lambda}\) sequence with \( c_{0, \lambda} = c_{0, \lambda}(R) = \inf_{u \in B_R} J_\lambda(u) \). Moreover, \( c_{0, \lambda} \) is achieved by some \( u_0 \in H^1_0(\mathbb{R}^N) \) with \( J_\lambda(u_0) = 0 \).
\end{lemma}

\textbf{Proof.} Fix \( \lambda \in (0, 1) \). For all \( u \in H^1_0(\mathbb{R}^N) \), the assumption (K1) and the Caffarelli-Kohn-Nirenberg inequality imply
\[ J_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x)u \, dx \]
\[ \geq \frac{1}{2} \|u\|^2 - \frac{\|K\|_\infty}{p} C_{a,b}^p \|u\|^p - \lambda \|g\|_{-1} \|u\|. \]
\[ \text{We now apply the inequality } \alpha \beta \leq \frac{\alpha^2 + \beta^2}{2}, \text{ for any } \alpha, \beta \geq 0. \text{ Hence} \]
\[ J_\lambda(u) \geq \frac{1 - \lambda}{2} \|u\|^2 - \frac{\|K\|_\infty}{p} C_{a,b}^p \|u\|^p - \frac{\lambda}{2} \|g\|_{-1}^2. \]  
(18)
Since \( p > 2 \) and the right side of (18) is a decreasing function on \( \lambda \), we find \( \lambda_1 > 0 \) and \( R = R(\lambda_1) > 0 \), \( \delta = \delta(\lambda_1) > 0 \) such that

\[
J_\lambda(u) \geq -\frac{\lambda}{2}\|g\|_{-1}^2, \quad \text{for all } u \in \overline{B}_R \text{ and } \lambda \in (0, \lambda_1)
\]  

(19)

and

\[
J_\lambda(u) \geq \delta > 0, \quad \text{for all } u \in \partial B_R \text{ and } \lambda \in (0, \lambda_1).
\]  

(20)

For instance, we can take

\[
\lambda_1 := \min \left\{ \frac{1}{2}, \frac{1}{2\|g\|_{-1}^2} \left( \frac{1}{2} - \frac{1}{p} \right) r_0^2 \right\},
\]

\[
r_0 := \left[ \frac{1}{2\|K\|_{\infty C_{a,b}}} \right]^{1/(p-2)}, \quad R := \left[ \frac{1 - \lambda_1}{\|K\|_{\infty C_{a,b}}} \right]^{1/(p-2)}
\]

and

\[
\delta(\lambda_1) := \frac{\lambda_1}{2}\|g\|_{-1}^2.
\]

Using now the estimate (18), we easily deduce (19) and (20).

Next, we define \( c_{0,\lambda} := c_{0,\lambda}(R) = \inf \{J_\lambda(u) : u \in \overline{B}_R \} \). We first note that \( c_{0,\lambda} \leq J_\lambda(0) = 0 \).

The set \( \overline{B}_R \) becomes a complet metric space with respect to the distance

\[
\text{dist}(u, v) = \|u - v\|, \quad \text{for any } u, v \in \overline{B}_R.
\]

The functional \( J_\lambda \) is lower semi-continuous and bounded from below on \( \overline{B}_R \). Then, by Ekeland’s Variational Principle [12] Theorem 1.1, for any positive integer \( n \) there exists \( u_n \) such that

\[
c_{0,\lambda} \leq J_\lambda(u_n) \leq c_{0,\lambda} + \frac{1}{n}
\]  

(21)

and

\[
J_\lambda(w) \geq J_\lambda(u_n) - \frac{1}{n}\|u_n - w\| \quad \text{for all } w \in \overline{B}_R.
\]  

(22)

We first show that \( \|u_n\| < R \) for \( n \) large enough. Indeed, if not, then \( \|u_n\| = R \) for infinitely many \( n \), and so (up to a subsequence) we can assume that \( \|u_n\| = R \) for all \( n \geq 1 \). It follows that \( J_\lambda(u_n) \geq \delta > 0 \). Using (21) and letting \( n \to \infty \), we have \( 0 \geq c_{0,\lambda} \geq \delta > 0 \), which is a contradiction.

We now claim that \( J'_\lambda(u_n) \to 0 \) in \( H_{-1}^{-1}(\mathbb{R}^N) \). Fix \( u \in H_{-1}^1(\mathbb{R}^N) \) with \( \|u\| = 1 \) and let \( w_n = u_n + tu \). For some fixed \( n \), we have \( \|w_n\| \leq \|u_n\| + t < R \) if \( t > 0 \) is small enough. Then relation (22) yields

\[
J_\lambda(u_n + tu) \geq J_\lambda(u_n) - \frac{t}{n}\|u\|,
\]

that is,

\[
\frac{J_\lambda(u_n + tu) - J_\lambda(u_n)}{t} \geq -\frac{1}{n}\|u\| = -\frac{1}{n}.
\]

Letting \( t \searrow 0 \) it follows that \( \langle J'_\lambda(u_n), u \rangle \geq -\frac{1}{n} \). Arguing in a similar way for \( t \not\to 0 \), we obtain

\( \langle J'_\lambda(u_n), u \rangle \leq \frac{1}{n} \). Since \( u \in H_{-1}^1(\mathbb{R}^N) \) with \( \|u\| = 1 \) has been arbitrarily chosen, we have

\[
\|J'_\lambda(u_n)\| = \sup_{u \in H_{-1}^1(\mathbb{R}^N), \|u\|=1} |\langle J'_\lambda(u_n), u \rangle| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty.
\]
We have proved the existence of a \((PS)_{c_0, \lambda}\) sequence, i.e., a sequence \(\{u_n\} \subset H^1_{a}(\mathbb{R}^N)\) with 
\[
J_{\lambda}(u_n) \rightarrow c_{0, \lambda} \quad \text{and} \quad J'_{\lambda}(u_n) \rightarrow 0 \quad \text{in} \quad H^1_{a}(\mathbb{R}^N).
\] (23)
Since \(\|u_n\| \leq R\), it follows that \(\{u_n\}\) converges weakly (up to a subsequence) in \(H^1_{a}(\mathbb{R}^N)\) to some \(u_0\). Moreover, relations (1), (23), and Remark 1 yield 
\[
J_{\lambda}(u_n) \rightarrow u_0 \quad \text{in} \quad H^1_{a}(\mathbb{R}^N), \quad u_n \rightarrow u_0 \quad \text{a.e. in} \quad \mathbb{R}^N
\] (24)
and 
\[
J'_{\lambda}(u_0) = 0.
\] (25)
Next, we prove that \(J_{\lambda}(u_0) = c_{0, \lambda}\). Using relations (24) and (25) we have
\[
o(1) = \langle J'_{\lambda}(u_0), u_0 \rangle = \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u_0|^2 dx - \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p dx - \lambda \int_{\mathbb{R}^N} g(x)u_0 dx.
\]
Therefore
\[
J_{\lambda}(u_0) = \left(1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x)u_0 dx + o(1).
\]
Hence
\[
J_{\lambda}(u_0) = \left(1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x)u_0 dx + o(1).
\]
Fatou’s Lemma and relations (24), (25) imply
\[
c_{0, \lambda} = \lim \inf_{n \to \infty} J_{\lambda}(u_n) \geq \left(1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x)u_0 dx = J_{\lambda}(u_0).
\]
Thus, \(c_{0, \lambda} \geq J_{\lambda}(u_0)\). On the other hand, since \(u_0 \in \overline{B_R}\), we deduce that \(J_{\lambda}(u_0) \geq c_{0, \lambda}\), so \(J_{\lambda}(u_0) = c_{0, \lambda}\). This concludes the proof of Lemma 1.

### 3 Proof of Theorem 1

Define
\[
S = \{u \in H^1_{a}(\mathbb{R}^N) \setminus \{0\} : \langle I'(u), u \rangle = 0\}.
\]
We claim that \(S \neq \emptyset\). For this purpose we fix \(u \in H^1_{a}(\mathbb{R}^N) \setminus \{0\}\) and set, for any \(\lambda > 0\),
\[
\Psi(\lambda) = \langle I'(\lambda u), \lambda u \rangle = \lambda^2 \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx - \lambda^p \int_{\mathbb{R}^N} K_0|u|^{-bp} |u|^p dx.
\]
Since \(p > 2\), it follows that \(\Psi(\lambda) < 0\) for \(\lambda\) large enough and \(\Psi(\lambda) > 0\) for \(\lambda\) sufficiently close to the origin. So, there exists \(\lambda > 0\) such that \(\Psi(\lambda) = 0\), that is, \(\lambda u \in S\).

**Proposition 1.** Let \(I_\infty := \inf \{ I(u) : u \in S \}\). Then there exists \(\bar{u} \in H^1_{a}(\mathbb{R}^N)\) such that
\[
I_\infty = I(\bar{u}) = \sup_{t \geq 0} I(t\bar{u}).
\] (26)

**Proof.** For some fixed \(\varphi \in H^1_{a}(\mathbb{R}^N) \setminus \{0\}\) denote
\[
f(t) = I(t\varphi) = \frac{t^2}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla \varphi|^2 dx - \frac{K_0}{p} t^p \int_{\mathbb{R}^N} |x|^{-bp} |\varphi|^p dx.
\]
We have
\[ f'(t) = t \int_{\mathbb{R}^N} |x|^{-2a} |\nabla \varphi|^2 \, dx - K_0 \, t^{p-1} \int_{\mathbb{R}^N} |x|^{-bp} |\varphi|^p \, dx. \]

Then \( f \) attains its maximum at
\[ t_0 = t_0(\varphi) := \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla \varphi|^2 \, dx}{\int_{\mathbb{R}^N} K_0 |x|^{-bp} |\varphi|^p \, dx} \right\}^{1/(p-2)}. \]

Hence
\[ f(t_0) = I(t_0\varphi) = \sup_{t \geq 0} I(t\varphi) = \left( \frac{1}{2} - \frac{1}{p} \right) \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla \varphi|^2 \, dx}{\left( \int_{\mathbb{R}^N} K_0 |x|^{-bp} |\varphi|^p \, dx \right)^{2/p}} \right\}^{p/(p-2)}. \]

It follows that
\[ \inf_{\varphi \in H_0^1(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I(t\varphi) = \left( \frac{1}{2} - \frac{1}{p} \right) [S(a, b)]^{p/(p-2)}, \tag{27} \]
where
\[ S(a, b) = \inf_{\varphi \in H_0^1(\mathbb{R}^N) \setminus \{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla \varphi|^2 \, dx}{\left( \int_{\mathbb{R}^N} K_0 |x|^{-bp} |\varphi|^p \, dx \right)^{2/p}} \right\}. \tag{28} \]

We now easily observe that for every \( u \in \mathcal{S} \) we have \( t_0(u) = 1 \), so, by (27) it follows that
\[ I(u) = \sup_{t \geq 0} I(tu) \quad \text{for all} \quad u \in \mathcal{S}. \tag{29} \]

According to [7, Theorems 1.2, 7.2, 7.6], the infimum in (28) is achieved by a function \( U \in H_0^1(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} K_0 |x|^{-bp} |U|^p \, dx = 1 \). Letting \( \bar{u} = [S(a, b)]^{1/(p-2)} U \), we see that \( \bar{u} \in \mathcal{S} \) and
\[ I(\bar{u}) = \left( \frac{1}{2} - \frac{1}{p} \right) [S(a, b)]^{p/(p-2)}. \tag{30} \]

Relations (29) and (30) yield
\[ I_\infty = \inf_{u \in \mathcal{S}} I(u) = \inf_{u \in \mathcal{S}} \sup_{t \geq 0} I(tu) \geq \inf_{u \in H_0^1(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I(tu) = \left( \frac{1}{2} - \frac{1}{p} \right) [S(a, b)]^{p/(p-2)} = I(\bar{u}), \]
which concludes our proof. \( \square \)

**Proposition 2.** Assume that \( \{u_n\} \) is a \((PS)_e\) sequence of \( J_\lambda \) which is weakly convergent in \( H^1_a(\mathbb{R}^N) \) to some \( u_0 \). Then the following alternative holds: either \( \{u_n\} \) converges strongly in \( H^1_a(\mathbb{R}^N) \), or \( c \geq J_\lambda(u_0) + I_\infty \).

**Proof.** Since \( \{u_n\} \) is a \((PS)_e\) sequence and \( u_n \rightharpoonup u_0 \) in \( H^1_a(\mathbb{R}^N) \) we have
\[ J_\lambda(u_n) = c + o(1) \quad \text{and} \quad \langle J_\lambda'(u_n), u_n \rangle = o(1). \tag{31} \]

Denote \( v_n = u_n - u_0 \). It follows that \( v_n \rightharpoonup 0 \) in \( H^1_a(\mathbb{R}^N) \) which implies
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-2a} \nabla v_n \cdot \nabla u_0 \, dx = 0, \]

From (34) it follows that
\[ \mu \] 
assume that
\[ t \] 
From the above relations we deduce the existence of some
\[ \{ \] 
and the conclusion follows. For this purpose, we denote
\[ p > 2 \]
Since \( \delta \) we gain the fact that \( H \) using the fact that \( v_n \) and \( v_n \rightarrow 0 \) in \( H^1_0(\mathbb{R}^N) \), it remains only to show that \( I(v_n) \rightarrow I_\infty + o(1) \). Taking \( t > 0 \) have
\[ \langle I'(tv_n), v_n \rangle = t^2 \int_{\mathbb{R}^N} |x|^{-2a} |\nabla v_n|^2 dx - t^p K_0 \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p dx. \]
If we prove the existence of a sequence \( \{ t_n \} \subset (0, \infty) \) with \( t_n \rightarrow 1 \) and \( \langle I'(t_nv_n), t_nv_n \rangle = 0 \), then \( t_nv_n \in S \). This implies that
\[ I(v_n) = I(t_nv_n) + \frac{1 - t_n^2}{2} \|v_n\|^2 - \frac{1 - t_n^p}{p} K_0 \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p dx \]
and the conclusion follows. For this purpose, we denote
\[ \alpha_n = \int_{\mathbb{R}^N} |x|^{-2a} |\nabla v_n|^2 dx = \|v_n\|^2 \geq 0, \]
\[ \beta_n = K_0 \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p dx \geq 0, \]
\[ \mu_n = \alpha_n - \beta_n. \]
From \( \delta_0 \) it follows that \( \mu_n = \langle I'(v_n), v_n \rangle \rightarrow 0 \) as \( n \rightarrow \infty \). If \( \mu_n = 0 \), then we take \( t_n = 1 \). We next assume that \( \mu_n \neq 0 \). Let \( \delta \in \mathbb{R} \) with \( |\delta| > 0 \) sufficiently small and \( t = 1 + \delta \). Then
\[ \langle I'(tv_n), v_n \rangle = (1 + \delta)^2 \alpha_n - (1 + \delta)^p \beta_n = (1 + \delta)^2 \alpha_n - (1 + \delta)^p (\alpha_n - \mu_n) \]
\[ = \alpha_n (2\delta - p\delta + o(\delta)) + (1 + \delta)^p \mu_n \]
\[ = \alpha_n (2 - p) \delta + o(\delta) + (1 + \delta)^p \mu_n. \]
Since \( p > 2, \alpha_n \rightarrow t^2 > 0 \) and \( \mu_n \rightarrow 0 \), for \( n \) large enough we can define
\[ \delta_n^+ = \frac{2|\mu_n|}{\alpha_n(p - 2)} \]
and
\[ \delta_n^- = -\frac{2|\mu_n|}{\alpha_n(p - 2)} \]. It follows that
\[ \delta_n^+ \searrow 0 \quad \text{and} \quad \langle I'((1 + \delta_n^+)v_n), (1 + \delta_n^+)v_n \rangle < 0, \]
\[ \delta_n^- \nearrow 0 \quad \text{and} \quad \langle I'((1 + \delta_n^-)v_n), (1 + \delta_n^-)v_n \rangle < 0. \]
From the above relations we deduce the existence of some \( t_n \in (1 + \delta_n^-, 1 + \delta_n^+) \) such that \( t_n \rightarrow 1 \) and \( \langle I'(t_nv_n), t_nv_n \rangle = 0 \). This concludes the proof. \( \square \)
We now fix $\bar{u} \in H^1_0(\mathbb{R}^N)$ such that (20) holds. Since $p > 2$, there exists $\bar{t}$ such that

$$I(t\bar{u}) < 0 \quad \text{for all} \quad t > \bar{t},$$

$$J_\lambda(t\bar{u}) < 0 \quad \text{for all} \quad t > \bar{t} \quad \text{and} \quad \lambda > 0.$$

Set

$$\mathcal{P} = \{ \gamma \in C([0,1], H^1_0(\mathbb{R}^N)) ; \gamma(0) = 0, \gamma(1) = \bar{t}\bar{u} \},$$

and

$$c_g = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} J_\lambda(u).$$

(35) Proposition 3. There exists $\lambda_0 > 0$, $R_0 = R_0(\lambda_0) > 0$, $\delta_0 = \delta_0(\lambda_0) > 0$ such that $J_\lambda |_{\partial B_{R_0}} \geq \delta_0$ and $c_g < c_{0,\lambda} + I_\infty$ for all $\lambda \in (0, \lambda_0)$, where $c_{0,\lambda} = \inf_{u \in B_{R_0}} J_\lambda(u)$.

Proof. By our hypothesis ($K3$) and the definition of $I$ we can assume that

$$J_0(t\bar{u}) < I(t\bar{u}) \quad \text{for all} \quad t > 0.$$

An elementary computation implies the existence of some $t_0 \in (0, \bar{t})$ such that

$$\sup_{t \geq 0} J_0(t\bar{u}) = J_0(t_0\bar{u}) < I(t_0\bar{u}) \leq \sup_{t \geq 0} I(t\bar{u}) = I_\infty.$$

So, we can choose $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t \geq 0} J_0(t\bar{u}) < I_\infty - \varepsilon_0.$$

(37) Set

$$\lambda_0 := \min \left\{ \frac{\varepsilon_0}{2t \| \bar{u} \|_g \| g \|_{-1}}, \frac{\varepsilon_0}{2\| g \|_{-1}} \right\}. $$

(38) Applying Lemma 4 it follows that there exists $R_0 = R_0(\lambda_0) > 0$ such that for all $\lambda \in (0, \lambda_0)$ the conclusion of Lemma 4 holds. Moreover, by virtue of its proof, there exists $\delta_0 = \delta(\lambda_0) > 0$ such that $J_\lambda |_{\partial B_{R_0}} \geq \delta_0$. Then relations (35), (37) and (39) yield

$$c_{0,\lambda} = \inf_{u \in B_{R_0}} J_\lambda(u) = -\frac{\lambda}{2} \| g \|_{-1}^2 > -\frac{\varepsilon_0}{2}, \quad \text{for all} \quad \lambda \in (0, \lambda_0).$$

(39) For $u \in \gamma_0 = \{ t\bar{u} ; 0 \leq t \leq 1 \} \in \mathcal{P}$ we have

$$|J_\lambda(u) - J_0(u)| = \lambda \left| \int_{\mathbb{R}^N} g(x) u dx \right| \leq \lambda \bar{t} \| \bar{u} \|_g \| g \|_{-1} \leq \frac{\varepsilon_0}{2}, \quad \text{for all} \quad \lambda \in (0, \lambda_0).$$

Therefore

$$J_\lambda(u) \leq J_0(u) + \frac{\varepsilon_0}{2}, \quad \text{for all} \quad \lambda \in (0, \lambda_0).$$

(40) Using relations (37), (39) and (40) we obtain

$$c_g = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} J_\lambda(u) \leq \sup_{u \in \gamma_0} J_\lambda(u) \leq \sup_{u \in \gamma_0} J_0(t\bar{u}) + \frac{\varepsilon_0}{2} < I_\infty - \frac{\varepsilon_0}{2} < I_\infty + c_{0,\lambda}.$$

This completes the proof.
Proof of Theorem 1 concluded. Consider $R_0 > 0$, $\delta_0 > 0$ given by Proposition 3. In view of its proof, we deduce that for all $\lambda \in (0, \lambda_0)$, the conclusion of Lemma 4 holds. Therefore, we obtain the existence of a solution $u_0$ of problem 2 such that $J_\lambda(u_0) = c_{0,\lambda}$.

On the other hand, applying the Mountain Pass Theorem without the Palais-Smale condition (see [5, Theorem 2.2]), it follows that there exists a $(PS)_{c_g}$ sequence $\{u_n\}$ of $J_\lambda$, that is, $J_\lambda(u_n) = c_g + o(1)$ and $J'_\lambda(u_n) \to 0$ in $H^{-1}_a(\mathbb{R}^N)$.

Therefore
\[ c_g + o(1) + \frac{1}{p} \|J'_\lambda(u_n)\|_{-1} \|u_n\| \geq J_\lambda(u_n) - \frac{1}{p} \langle J'_\lambda(u_n), u_n \rangle \]
\[ \geq \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 - \lambda \left( 1 - \frac{1}{p} \right) \|g\| \|u_n\|. \]

This above inequality shows that $\{u_n\}$ is bounded in $H^1_a(\mathbb{R}^N)$. Thus we can assume (up to a subsequence) that $u_n \rightharpoonup u_1$ in $H^1_a(\mathbb{R}^N)$. By Lemma 1 it follows that $u_1$ is a weak solution of problem 2.

We claim that $u_0 \neq u_1$. Indeed, by Proposition 2, the following alternative holds: either $u_n \to u_1$ in $H^1_a(\mathbb{R}^N)$, which gives
\[ J_\lambda(u_1) = \lim_{n \to \infty} J_\lambda(u_n) = c_g > 0 \geq c_{0,\lambda} = J_\lambda(u_0) \]
and the conclusion follows; or
\[ c_g = \lim_{n \to \infty} J_\lambda(u_n) \geq J_\lambda(u_1) + I_\infty. \]

In the last case, if we suppose that $u_1 = u_0$ then $J_\lambda(u_1) = J_\lambda(u_0) = c_{0,\lambda}$ and so $c_g \geq c_{0,\lambda} + I_\infty$, which contradicts Proposition 3. The proof of Theorem 1 is now complete. \[\square\]

References

[1] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 7 (1973), 349-381.

[2] J. Byeon and Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, *Calc. Var. Partial Differential Equations* 18 (2003), 207-219.

[3] H. Brezis and E. H. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, 88 (1983), 486-490.

[4] H. Brezis and M. Marcus, Hardy’s inequalities revisited, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 25 (1997), 217-237.

[5] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Comm. Pure Appl. Math.* 36 (1983), 486-490.

[6] L. Caffarelli, R. Kohn, and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.* 53 (1984), 259-275.

[7] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions, *Comm. Pure Appl. Math.* 54 (2001), 229-258.

[8] G. R. Cirmi and M. M. Porzio, $L^1$-solutions for some nonlinear degenerate elliptic and parabolic equations, *Ann. Mat. Pura Appl.* (4) 169 (1995), 67-86.

[9] F. Cîrstea and V. Rădulescu, Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent, *Topol. Meth. Nonlinear Anal.* 15 (2000), 281-298.
[10] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 1: Physical Origins and Classical Methods*, Springer-Verlag, Berlin Heidelberg, 1985.

[11] E. B. Davies, A review of Hardy inequalities, in *The Maz’ya anniversary collection, Vol. 2 (Rostock, 1998)*, pp. 55-67, Oper. Theory Adv. Appl., 110, Birkhäuser, Basel, 1999.

[12] I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443-473.

[13] E. Fabes, C. Kenig, and R. Serapioni, The local regularity of solutions of degenerate elliptic operators, *Comm. Partial Diff. Eqns.* **7** (1982), 77-116.

[14] B. Franchi, R. Serapioni, and F. Serra Cassano, Approximation and imbedding theorems for weighted Sobolev spaces associated to Lipschitz continuous vector fields, *Boll. Un. Mat. Ital. B* **11** (1977), 83-117.

[15] M. K. V. Murthy and G. Stampacchia, Boundary value problems for some degenerate elliptic operators, *Ann. Mat. Pura Appl.* **80** (1968), 1-122.

[16] D. Passaseo, Some concentration phenomena in degenerate semilinear elliptic problems, *Nonlinear Anal.* **24** (1995), 1011-1025.

[17] P. Rabinowitz, Variational methods for nonlinear elliptic eigenvalue problems, *Indiana Univ. Math. J.* **23** (1974), 729-745.

[18] V. Rădulescu and D. Smets, Critical singular problems on infinite cones, *Nonlinear Anal.* **54** (2003), 1153-1164.

[19] V. Rădulescu, D. Smets, and M. Willem, Hardy–Sobolev inequalities with remainder terms, *Topol. Meth. Nonlinear Anal.* **20** (2002), 145-149.

[20] B. Sirakov, Standing wave solutions of the nonlinear Schrödinger equation in $\mathbb{R}^N$, *Ann. Mat. Pura Appl. (4)* **181** (2002), 73-83.

[21] W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977), 149-162.

[22] E. W. Stredulinsky, *Weighted Inequalities and Degenerate Elliptic Partial Differential Equations*, Springer-Verlag, Berlin, New York, 1984.

[23] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, *Manuscripta Math.* **32** (1980), 335-364.

[24] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponents, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **9** (1992), 281-304.

[25] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* **153** (1993), 229-244.

[26] Z.-Q. Wang and M. Willem, Singular minimization problems, *J. Differential Equations* **161** (2000), 317-320.