HIGHER INTEGRABILITY FOR MINIMIZERS OF THE MUMFORD-SHAH FUNCTIONAL

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Abstract. We prove higher integrability for the gradient of local minimizers of the Mumford-Shah energy functional, providing a positive answer to a conjecture of De Giorgi.

1. Introduction

Free discontinuity problems are a class of variational problems which involve pairs \((u,K)\) where \(K\) is some closed set and \(u\) is a function which minimizes some energy outside \(K\). One of the most famous examples is given by the Mumford-Shah energy functional, which arises in image segmentation [10]: given an open set \(\Omega \subset \mathbb{R}^n\), for any \(K \subset \Omega\) relatively closed and \(u \in W^{1,2}(\Omega \setminus K)\), one defines the Mumford-Shah energy of \((u,K)\) in \(\Omega\) to be

\[
MS(u,K)[\Omega] := \int_{\Omega \setminus K} |\nabla u|^2 + H^{n-1}(K \cap \Omega).
\]

We say that the pair \((u,K)\) is a local minimizer for the Mumford Shah energy in \(\Omega\) if, for every ball \(B = B_\rho(x) \subset \Omega\),

\[
MS(u,K)[B] \leq MS(v,H)[B]
\]

for all pairs \((v,H)\) such that \(H \subset \Omega\) is relatively closed, \(v \in W^{1,2}(\Omega \setminus H)\), \(K \cap (\Omega \setminus B) = H \cap (\Omega \setminus B)\), and \(u = v\) almost everywhere in \((\Omega \setminus B) \setminus K\). We denote the set of local minimizers in \(\Omega\) by \(\mathcal{M}(\Omega)\).

The existence of local minimizers is by now well-known [6, 3, 2, 4]. In [5], De Giorgi formulated a series of conjectures on the properties of local minimizers. One of them states as follows [5, Conjecture 1]:

Conjecture (De Giorgi): If \((u,K)\) is a (local) minimizer of the Mumford-Shah energy inside \(\Omega\), then there exists \(\gamma \in (1,2)\) such that \(|\nabla u|^2 \in L^{\gamma}(\Omega' \setminus K)\) for all \(\Omega' \subset \subset \Omega\).

A positive answer to the above conjecture was given in [7] when \(n = 2\). The proof there strongly relies on the two-dimensional assumption, since it uses the description of minimal Caccioppoli partitions. The aim of this note is to provide a positive answer in arbitrary dimension. Since our proof avoids any compactness argument, our constants are potentially computable.

This is our main result:

Theorem 1.1. There exist dimensional constants \(\bar{C} > 0\) and \(\bar{\gamma} = \bar{\gamma}(n) > 1\) such that, for all \((u,K) \in \mathcal{M}(B_2)\),

\[
\int_{B_{1/2} \setminus K} |\nabla u|^{2\bar{\gamma}} \leq \bar{C}.
\]
By a simple covering/rescaling argument, one deduces the validity of the conjecture with \( \gamma = \bar{\gamma} \). We also remark that our result applies with trivial modifications to the “full” Mumford-Shah energy
\[
MS_g(u, K)[\Omega] := \int_{\Omega \setminus K} |\nabla u|^2 + \alpha \int_{\Omega} |u - g|^2 + \beta \mathcal{H}^{n-1}(K \cap \Omega),
\]
where \( \alpha, \beta > 0 \), and \( g \in L^2(\Omega) \cap L^\infty(\Omega) \).

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2. Preliminaries

In the next proposition we collect the main known properties of local minimizers that will be used in the sequel.

**Proposition 2.1.** There exists a dimensional constant \( C_0 \) such that for all \((u, K) \in \mathcal{M}(B_2)\), the following properties hold true.

(i) \( u \) is harmonic in \( B_2 \setminus K \).

(ii) For all \( x \in B_1 \) and all \( \varrho < 1 \)
\[
\int_{B_\varrho(x) \setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K \cap B_\varrho(x)) \leq C_0 \varrho^{n-1}.
\]

(iii) For all \( x \in K \cap B_1 \) and all \( \varrho < 1 \),
\[
\mathcal{H}^{n-1}(K \cap B_\varrho(x)) \geq \varrho^{n-1}/C_0.
\]

(iv) There is a dimensional constant \( \varepsilon(n) > 0 \) such that, for every \( \varepsilon \in (0, \varepsilon(n)) \), there exists \( C_\varepsilon > 0 \) for which the following statement holds true: For all \( x \in K \cap B_1 \) and all \( \varrho < 1 \) there exists a \( y \in B_{\varrho/2}(x) \cap K \), a unit vector \( \bar{\nu} \) and a \( C^{1,1/4} \) function \( f : \mathbb{R}^n \to \mathbb{R} \) such that
\[
K \cap B_{2\varrho/\varepsilon}(y) = [y + \text{graph}_\bar{\nu}(f)] \cap B_{2\varrho/\varepsilon}(y),
\]
where
\[
\text{graph}_\bar{\nu}(f) := \left\{ z \in \mathbb{R}^n : z \cdot \bar{\nu} = f(z - (\bar{\nu} \cdot z)z) \right\}.
\]
Moreover
\[
f(0) = 0, \quad \|\nabla f\|_\infty + \varrho^{1/4}\|\nabla f\|_{C^{1/4}} \leq C_0 \varepsilon,
\]
and
\[
\sup_{B_{\varrho/\varepsilon}(y)} \varrho|\nabla u|^2 \leq C_0 \varepsilon.
\]

**Proof.** Point (i) is easy. Point (ii) is well known and it can be proved by comparison, see [2] Lemma 7.19. Point (iii) has been proved by Carriero, De Giorgi and Leaci in [3], see also [2] Theorem 7.21. Point (iv) expresses the porosity of the set where \( K \) is not a smooth graph. This has been proved in \[3, 8, 11\], see also \[9\]. More precisely, in these papers it has been proved that for any fixed positive \( \varepsilon \) there exists a constant \( C_\varepsilon \) such that, for all \( x \in K \cap B_1 \) and \( \varrho < 1 \), there exists a point \( y \in B_{\varrho}(x) \cap K \) and a ball \( B_r(y) \subset B_\varrho(x) \), with \( r \geq 2 \varrho/C_\varepsilon \), such that
\[
\frac{1}{r^{n-1}} \int_{B_r(y)} |\nabla u(z)|^2 \, dz + \frac{1}{r^{n+1}} \inf_{\nu \in S^{n-1}} \int_{K \cap B_r(y)} |(z - y) \cdot \nu|^2 \, d\mathcal{H}^{n-1}(z) \leq \varepsilon,
\]
see [8, Theorem 1.1]. From this one applies the \( \varepsilon \)-regularity theorem, [2, Theorems 8.2 and 8.3] to deduce (2.1) and (2.3). Finally, (2.4) follows from [2, (2.5)] and classical estimates for the Neumann problem, see for instance [2, Theorem 7.53]. □

The following simple geometric lemma will be useful:

**Lemma 2.2 (A geometric lemma).** Let \( G \) be closed set such that
\[
G \cap B_2 = \text{graph}_{e_n}(f)
\]
for some Lipschitz function \( f : \mathbb{R}^{n-1} \to \mathbb{R} \) satisfying
\[
f(0) = 0 \quad \text{and} \quad \| \nabla f \|_{\infty} \leq \varepsilon.
\]

Then, provided \( \varepsilon \leq 1/12 \),
\[
\text{dist}(x, (\overline{B}_{1+\delta} \setminus B_{1+\delta}) \cap G) \leq \frac{3}{2} \delta \quad \forall \delta \in (0, 1/2), \quad x \in (\overline{B}_{1+\delta} \setminus B_1) \cap G.
\]

**Proof.** First notice that, by (2.6),
\[
\| f \|_{L^\infty(B_2)} + \| \nabla f \|_{L^\infty(B_2)} \leq 2\varepsilon.
\]

Given a point \( x = (x', f(x')) \in (\overline{B}_{1+\delta} \setminus B_1) \cap G \), set \( \alpha := \frac{1+5\delta/4}{|x|} \) and let us consider the point \( \bar{x} := (\alpha x', f(\alpha x')) \). Since \( |x| \geq 1 \) we have \( 0 < (\alpha - 1)|x| \leq 5\delta/4 \), hence
\[
|f(\alpha x') - \alpha f(x')| \leq |f(\alpha x') - f(x')| + (\alpha - 1)|f(x')| \\
\leq (\alpha - 1) \left( \| \nabla f \|_{\infty} |x'| + \| f \|_{\infty} \right) \\
\leq 2(\alpha - 1)\varepsilon |x| = \frac{5}{2}\varepsilon \delta,
\]
and
\[
|\alpha x| = 1 + \frac{5}{4} \delta.
\]
Thus, provided \( \varepsilon \leq 1/12 \) we get
\[
\left| |\bar{x}| - \left( 1 + \frac{5}{4} \delta \right) \right| \leq |\bar{x} - \alpha x| = |f(\alpha x') - \alpha f(x')| \leq \frac{5}{2} \varepsilon \delta < \frac{\delta}{4}
\]
and
\[
|\bar{x} - x| \leq |\bar{x} - \alpha x| + (\alpha - 1)|x| = |f(\alpha x') - \alpha f(x')| + (\alpha - 1)|x| \leq \frac{5}{2} \varepsilon \delta + \frac{5}{4} \delta < \frac{3}{2} \delta,
\]
which imply that \( \bar{x} \in (\overline{B}_{1+2\delta} \setminus B_{1+\delta}) \cap G \), concluding the proof. □

**Remark 2.3.** In the sequel we will apply Proposition 2.1 only with \( \varepsilon := \min\{\varepsilon(n)/C_0, 1/(12C_0)\} \), where \( \varepsilon(n) \) and \( C_0 \) are as in Proposition 2.1 and the factor 1/12 comes from Lemma 2.2. Hence, with this choice, also the constant \( C_\varepsilon \) will be dimensional.
3. Proof of Theorem 1.1

Let $M \gg 1$ to be fixed, and for $h \in \mathbb{N}$ define the following set

$$A_h := \{ x \in B_2 \setminus K \text{ such that } |\nabla u(x)|^2 \geq M^{h+1} \}.$$  

(3.1)

Notice that the sets $A_h$ depend on $M$. However, later $M$ will be fixed to be a large dimensional constant, so for notational simplicity we drop the dependence on $M$. We will use the notation $\mathcal{N}_\varepsilon(E)$ to denote the $\varepsilon$-neighbourhood of a set $E$, i.e., the set of points at distance less than $\varepsilon$ from $E$.

The idea of the proof is the following: since $u$ is harmonic outside $K$ and the integral of $|\nabla u|^2$ over a ball of radius $r$ is controlled by $r^{n-1}$ (see Proposition 2.1(ii)), it follows by elliptic regularity that $A_h$ is contained in a $M^{-h}$-neighborhood of $K$ (Lemma 3.1). However, for the set $K$ we have a porosity estimate which tells us that inside every ball of radius $\varepsilon$ there is a ball of comparable radius where $|\nabla u|^2 \leq C/\varepsilon$ (see Proposition 2.1(iv)). Hence, this implies that the size of $A_h$ is smaller than what one would get by just using that $A_h \subset \mathcal{N}_{M^{-h}}(K)$. Indeed, by induction over $h$ we can show that $A_h$ is contained in the $M^{-h}$-neighborhood of a set $K_h$ obtained from $K_{h-1}$ by removing the “good balls” where (2.4) hold (Lemma 3.3). Since the $\mathcal{H}^{n-1}$ measure of $K_h$ decays geometrically (see (3.6)), this allows us to obtain a stronger estimate on the size of $A_h$ which immediately implies the higher integrability. To make this argument rigorous we actually have to suitably localize our estimates, and for this we need to introduce some suitable sequences of radii (Lemma 3.2).

Lemma 3.1. There exists a dimensional constant $M_0$ such that for $M \geq M_0$ and all $(u, K) \in \mathcal{M}(B_2)$ and $R \leq 1$

$$A_h \cap B_{R-2M^{-h}} \subset \mathcal{N}_{M^{-h}}(K \cap \overline{B_R}) \quad \text{for all } h \in \mathbb{N}.$$  

Proof. Let $x \in A_h \cap B_{R-2M^{-h}}$, $d := \text{dist}(x, K)$, and $z \in K$ a point such that $|x - z| = d$. If $d > M^{-h}$ then

$$B_{M^{-h}}(x) \cap K = \emptyset \quad \text{and} \quad u \text{ is harmonic on } B_{M^{-h}}(x).$$

Hence, by the definition of $A_h$, the mean value property for subharmonic function\footnote{Notice that, because $u$ is harmonic, $|\nabla u|^2$ is subharmonic. Instead, when one deals with the full functional (1.2) (or if one wants to consider more general energy functionals than $\int |\nabla u|^2$), the mean value estimate has to be replaced by the one-sided Harnack inequality for subsolutions to uniformly elliptic equations, which in the case of minimizers of (1.2) reads as:}

$$M^{h+1} \leq |\nabla u(x)|^2 \leq \int_{B_{M^{-h}}(x)} |\nabla u|^2 \leq \frac{C_0}{|B|} M^h,$$

which is impossible if $M$ is large enough. Moreover, since $x \in B_{R-2M^{-h}}$ and $d \leq M^{-h}$ we see that $z \in B_R$, proving the claim. \hfill \square

Lemma 3.2 (Good radii). There are dimensional positive constants $M_1$ and $C_1$ such that for $M \geq M_1$ we can find three sequences of radii $\{R_h\}_{h \in \mathbb{N}}$, $\{S_h\}_{h \in \mathbb{N}}$ and $\{T_h\}_{h \in \mathbb{N}}$ for which the following properties hold true for every $(u, K) \in \mathcal{M}(B_2)$.

(i) $1 \geq R_h \geq S_h \geq T_h \geq R_{h+1},$
Lemma 3.3. Let $\mathcal{F}_j = \{B_{M^{-j/C_\varepsilon}}(y_i),\ y_i \in K, \ i = 1, \ldots, N_j\}, \ j = 1, \ldots, h,$ such that

(i) If $B_1^1, B_2^1 \in \bigcup_{j=1}^h \mathcal{F}_j$ are distinct balls, then $\mathcal{N}_{4M^{-(h+1)}}(B_1^1) \cap \mathcal{N}_{4M^{-(h+1)}}(B_2^1) = \emptyset.$

(ii) If $B_{M^{-j/C_\varepsilon}}(y_i) \in \mathcal{F}_j$ then there is a unit vector $\nu$ and a $C^1$ function $f : \mathbb{R}^{n-1} \to \mathbb{R},$ with $f(0) = 0$ and $\|\nabla f\|_{\infty} \leq \varepsilon,$ such that

$$K \cap B_{2M^{-j/C_\varepsilon}}(y_i) = [y_i + \text{graph}_\nu(f)] \cap B_{2M^{-j/C_\varepsilon}}(y_i) \quad \text{and} \quad \sup_{B_{2M^{-j/C_\varepsilon}}(y_i)} |\nabla u|^2 < M^{j+1}.$$  

(iii) Let $\{R_h\}_{h \in \mathbb{N}}, \ \{S_h\}_{h \in \mathbb{N}}$ and $\{T_h\}_{h \in \mathbb{N}}$ be the sequences of radii constructed in Lemma 3.2 and define

$$K_h := (K \cap \overline{B}_{S_h}) \setminus \left( \bigcup_{j=1}^h \bigcup_{B \in \mathcal{F}_j} B \right), \quad (3.2)$$

and

$$\bar{K}_h := (K \cap \overline{B}_{T_h}) \setminus \left( \bigcup_{j=1}^h \bigcup_{B \in \mathcal{F}_j} \mathcal{N}_{2M^{-(h+1)}}(B) \right) \subset K_h. \quad (3.3)$$

Proof. We set $R_1 = 1.$ Given $R_h,$ we show how to construct $S_h, T_h$ and $R_{h+1}.$ With no loss of generality (up to slightly enlarge $M$) we can assume that $\sqrt{M}/8$ is a natural number. Then we write

$$\overline{B}_{R_h} \setminus \overline{B}_{R_h^M} = \bigcup_{i=1}^\infty \overline{B}_{R_h^M - i8M^{-(h+1)}} \setminus \overline{B}_{R_h^M}.$$  

Being the annuli in the right hand side disjoints, there is at least an index $\tilde{i}$ such that

$$\mathcal{H}^{n-1}\left(K \cap \overline{B}_{R_h^M - (\tilde{i} - 1)8M^{-(h+1)}} \setminus \overline{B}_{R_h^M - \tilde{i}8M^{-(h+1)}}\right) \leq 8M^{-\frac{(h+1)}{2}}\mathcal{H}^{n-1}\left(K \cap \overline{B}_{R_h^M} \setminus \overline{B}_{R_h^M - \tilde{i}8M^{-(h+1)}}\right) \leq 8M^{-\frac{(h+1)}{2}}\mathcal{H}^{n-1}\left(K \cap \overline{B}_{1}\right) \leq C_1M^{-\frac{(h+1)}{2}},$$

where in the last inequality we have taken into account Proposition 2.1(ii). Then we set

$$S_h := R_h - (\tilde{i} - 1)8M^{-(h+1)}, \quad R_{h+1} := R_h - \tilde{i}8M^{-(h+1)}, \quad T_h := (S_h + R_{h+1})/2.$$  

Then properties (i), (ii) and (iii) trivially hold, and while (iv) follow from (ii) by choosing $M$ large enough.\hfill \square

Lemma 3.3. Let $C_0, \varepsilon, C_\varepsilon, C_1, M_1$ be as in Proposition 2.1 and Lemma 3.2, with $\varepsilon$ as in Remark 2.3. There exist dimensional constants $C_2, M_2, \eta > 0,$ with $M_2 \geq M_1,$ such that, for every $M \geq M_2,$ $(u, K) \in \mathcal{M}(B_2),$ and $h \in \mathbb{N},$ we can find $h$ families of disjoint balls $\mathcal{F}_j$ such that

(i) If $B_1^1, B_2^1 \in \bigcup_{j=1}^h \mathcal{F}_j$ are distinct balls, then $\mathcal{N}_{4M^{-(h+1)}}(B_1^1) \cap \mathcal{N}_{4M^{-(h+1)}}(B_2^1) = \emptyset.$

(ii) If $B_{M^{-j/C_\varepsilon}}(y_i) \in \mathcal{F}_j$ then there is a unit vector $\nu$ and a $C^1$ function $f : \mathbb{R}^{n-1} \to \mathbb{R},$ with $f(0) = 0$ and $\|\nabla f\|_{\infty} \leq \varepsilon,$ such that

$$K \cap B_{2M^{-j/C_\varepsilon}}(y_i) = [y_i + \text{graph}_\nu(f)] \cap B_{2M^{-j/C_\varepsilon}}(y_i) \quad \text{and} \quad \sup_{B_{2M^{-j/C_\varepsilon}}(y_i)} |\nabla u|^2 < M^{j+1}.$$  

(iii) Let $\{R_h\}_{h \in \mathbb{N}}, \ \{S_h\}_{h \in \mathbb{N}}$ and $\{T_h\}_{h \in \mathbb{N}}$ be the sequences of radii constructed in Lemma 3.2 and define

$$K_h := (K \cap \overline{B}_{S_h}) \setminus \left( \bigcup_{j=1}^h \bigcup_{B \in \mathcal{F}_j} B \right), \quad (3.2)$$

and

$$\bar{K}_h := (K \cap \overline{B}_{T_h}) \setminus \left( \bigcup_{j=1}^h \bigcup_{B \in \mathcal{F}_j} \mathcal{N}_{2M^{-(h+1)}}(B) \right) \subset K_h. \quad (3.3)$$
Then there exists a finite set of points $C_h := \{x_i\}_{i \in I_h} \subset \tilde{K}_h$ such that
\[|x_j - x_k| \geq 3M^{-(h+1)} \quad \forall j, k \in I_h, j \neq k,\] (3.4)
\[N_{M^{-(h+1)}}(K_h \cap \overline{B}_{R_{h+1}}) \subset \bigcup_{x_i \in C_h} B_{8M^{-(h+1)}}(x_i).\] (3.5)

Moreover
\[\mathcal{H}^{n-1}(K_{h+1}) \leq (1 - \eta)\mathcal{H}^{n-1}(K_h) + C_1 M^{-\frac{h+1}{2}},\] (3.6)
\[|N_{M^{-(h+1)}}(K_h \cap \overline{B}_{R_{h+1}})| \leq C_2 M^{-(h+1)}\mathcal{H}^{n-1}(K_h).\] (3.7)

(iv) Let $A_h$ be as in (3.1). Then
\[A_{h+2} \cap B_{R_{h+2}} \subset N_{M^{-(h+1)}}(K_h \cap B_{R_{h+1}}).\] (3.8)

**Proof.** We proceed by induction. For $h = 1$ we set $F_1 = \emptyset$, so that $K_1 = K \cap \overline{B}_{S_1}$ and $\tilde{K}_1 = K \cap \overline{B}_{T_1}$. We also choose $C_1$ to be a maximal family of points at distance $3M^{-2}$ from each other. Clearly (i), (ii), and (3.6) are true. The other properties can be easily obtained as in the steps below and the proof is left to the reader.

Assuming we have constructed $h$ families of balls $\{F_j\}_{j=1}^h$ as in the statement of the Lemma, we show how to construct the family $F_{h+1}$. For this, let $C_h = \{x_i\}_{i \in I_h} \subset \tilde{K}_h$ be a family of points satisfying (3.4), and let us consider the family of disjoint balls
\[G_{h+1} := \{B_{M^{-(h+1)}}(x_i)\}_{i \in I_h}.

**Step 1.** We show that
\[B_{M^{-(h+1)}}(x_i) \cap K_h = B_{M^{-(h+1)}}(x_i) \cap K \quad \forall x_i \in C_h.

Indeed, assume by contradiction there is a point $x \in B_{M^{-(h+1)}}(x_i) \cap (K \setminus K_h)$. First of all notice that, by Lemma 3.2(ii), since $x_i \in B_{T_h}$ we get that $x \in B_{S_h}$. Hence, by the definition of $K_h$, there is a ball $\tilde{B} \in F_j$, $j \leq h$, such that $x \in \tilde{B}$. But then
\[\text{dist}(x_i, \tilde{B}) \leq |x - x_i| \leq M^{-(h+1)} < 2M^{-(h+1)},\] a contradiction to the fact that $x_i \in \tilde{K}_h$.

**Step 2.** We claim that there exists a positive dimensional constant $\eta_0$ such that, if $N_h$ is the cardinality of $I_h$, then
\[N_h M^{-(h+1)(n-1)} \geq \eta_0 \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}}).\]

Indeed, by (3.5) and Proposition 2.1(ii),
\[
\mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}}) = \mathcal{H}^{n-1}\left(K_h \cap \overline{B}_{R_{h+1}} \cap \bigcup_{x_i \in C_h} B_{8M^{-(h+1)}}(x_i)\right)
\leq C_0 N_h \left(8M^{-(h+1)}\right)^{n-1} = \frac{1}{\eta_0} N_h M^{-(h+1)(n-1)},
\]
where $\eta_0 := 1/(C_0 8^{n-1})$.

**Step 3.** By Proposition 2.1(iv) and Remark 2.3, for every ball $B_{M^{-(h+1)}}(x_i) \in G_{h+1}$ there exists a ball
\[B_{M^{-(h+1)}/C_0} (y_i) \subset B_{M^{-(h+1)}}(x_i)\] (3.9)
such that
\[ \sup_{B_{2M^{-h+1}}(y_i)} | \nabla u |^2 \leq \varepsilon M^{h+1} < M^{h+2} \]
and
\[ K \cap B_{2M^{-h+1}}(y_i) = [y_i + \operatorname{graph}_\nu(f)] \cap B_{2M^{-h+1}}(y_i), \]
for some unit vector \( \nu \) and some \( C^1 \) function \( f \) such that
\[ f(0) = 0 \quad \text{and} \quad \| \nabla f \| \leq \varepsilon. \]

We define
\[ \mathcal{F}_{h+1} := \{ B_{M^{-h+1}}(y_i) \}_{i \in I_h}. \]

In this way property (ii) in the statement of the lemma is satisfied. Moreover, since the balls \( \{ B_{3M^{-h+1}/2}(x_i) \}_{i \in I_h} \) are disjoint (because \( |x_j - x_k| \geq 3M^{-h+1} \)) and do not intersect
\[ \bigcup_{j=1}^{h} \bigcup_{B \in \mathcal{F}_j} \mathcal{N}_{M^{-h+1}}(B) \]
it follows from (3.9) that also property (i) is satisfied provided we choose \( M \) sufficiently large.

We define \( K_{h+1} \) and \( \tilde{K}_{h+1} \) as in the statement of the lemma and we take \( \mathcal{C}_{h+1} = \{ x_i \}_{i \in I_{h+1}} \) a maximal sets of points in \( \tilde{K}_{h+1} \) satisfying
\[ |x_j - x_k| \geq 3M^{-(h+2)} \quad \forall j \neq k. \]

**Step 4.** The set of points \( \mathcal{C}_{h+1} \) defined in the previous step satisfies by construction (3.21). We now prove it also satisfies (3.3). For this, let \( x \in \mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}}) \) and let \( \bar{x} \in K_{h+1} \cap B_{R_{h+2}} \) be such that
\[ |x - \bar{x}| = \operatorname{dist}(x, K_{h+1} \cap \overline{B}_{R_{h+2}}) \leq M^{-(h+2)}. \]

In case \( \bar{x} \in \tilde{K}_{h+1} \), by maximality there exists a point \( x_i \in C_{h+1} \) such that \( |\bar{x} - x_i| \leq 3M^{-(h+2)} \), hence \( x \in B_{5M^{-(h+2)}}(x_i) \) and we are done. So, let us assume that
\[ \bar{x} \in (K_{h+1} \cap \overline{B}_{R_{h+2}}) \setminus \tilde{K}_{h+1}. \]

In this case, by the definition of \( K_{h+1} \) and \( \tilde{K}_{h+1} \), there exists a ball \( B \in \bigcup_{j=1}^{h+1} \mathcal{F}_j \) such that
\[ x \in K \cap \mathcal{N}_{2M^{-(h+2)}}(B) \setminus \bar{B}. \]

Thanks to property (ii) we can apply (a scaled version of) Lemma (2.2) to find a point
\[ y \in K \cap \mathcal{N}_{4M^{-(h+2)}}(B) \setminus \mathcal{N}_{2M^{-(h+2)}}(B) \]
such that
\[ |\bar{x} - y| \leq 3M^{-(h+2)}. \]

Since \( \bar{x} \in \overline{B}_{R_{h+2}} \) and \( T_{h+1} = R_{h+2} + 4M^{-(h+2)} \)
\[ y \in \left( K \cap B_{T_{h+1}} \cap \mathcal{N}_{4M^{-(h+2)}}(B) \right) \setminus \mathcal{N}_{2M^{-(h+2)}}(B) \subset \tilde{K}_{h+1}, \]
where the last inclusion follows by property (i) and the definition of \( \tilde{K}_h \). Again by maximality, there exists a point \( x_i \in \mathcal{C}_{h+1} \) such that \( |y - x_i| \leq 3M^{-(h+2)} \), hence
\[ |x_i - x| \leq |x_i - y| + |y - \bar{x}| + |\bar{x} - x| \leq 7M^{-(h+2)}, \]
which completes the proof of (3.5).
Step 5. We prove (3.6). Notice that, being the balls in $F_{h+1}$ disjoint, thanks to Step 1, the density estimates in Proposition 2.1(iii), Step 2, and choosing $\eta := \eta_0/C_n^0$ we get

$$\mathcal{H}^{n-1}(K_{h+1}) \leq \mathcal{H}^{n-1}\left(K_h \setminus \bigcup_{i \in I_h} B_{M^{-(h+1)}/C_\varepsilon}(y_i)\right)$$

$$= \mathcal{H}^{n-1}(K_h) - \sum_{i \in I_h} \mathcal{H}^{n-1}(K_h \cap B_{M^{-(h+1)}/C_\varepsilon}(y_i))$$

$$\leq \mathcal{H}^{n-1}(K_h) - \frac{\eta}{C_0} M^{-(h+1)(n-1)}$$

$$\leq \mathcal{H}^{n-1}(K_h) - \frac{\eta}{C_0} \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}})$$

$$= (1 - \eta) \mathcal{H}^{n-1}(K_h) + \eta \left[ \mathcal{H}^{n-1}(K_h) - \mathcal{H}^{n-1}(K_h \cap \overline{B}_{R_{h+1}}) \right]$$

$$\leq (1 - \eta) \mathcal{H}^{n-1}(K_h) + \mathcal{H}^{n-1}(K \cap \overline{B}_{S_h} \setminus \overline{B}_{R_{h+1}})$$

$$\leq (1 - \eta) \mathcal{H}^{n-1}(K_h) + C_1 M^{-\frac{(h+1)}{2}},$$

where in the last step we used Lemma 3.2(iii).

Step 6. We prove (3.7). By (3.5),

$$\mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}}) \subset \bigcup_{x_i \in C_{h+1}} B_{8M^{-(h+2)}}(x_i),$$

hence, denoting with $N_{h+1}$ the cardinality of $I_{h+1},$

$$|\mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}})| \leq 8^n M^{-(h+2)} N_{h+1} M^{-(h+2)(n-1)}. \tag{3.10}$$

Also, by Step 1 with $h$ replaced by $h + 1,$

$$B_{M^{-(h+2)}}(x_i) \cap K_{h+1} = B_{M^{-(h+2)}}(x_i) \cap K \quad \forall x_i \in C_{h+1},$$

hence by the density estimates in Proposition 2.1(iii),

$$M^{-(h+2)(n-1)} \leq C_0 \mathcal{H}^{n-1}\left(K_{h+1} \cap B_{M^{-(h+2)}}(x_i)\right).$$

The above equation and (3.10), together with the disjointness of the balls $\{B_{M^{-(h+2)}}(x_i)\}_{i \in I_{h+1}}$ imply

$$|\mathcal{N}_{M^{-(h+2)}}(K_{h+1} \cap \overline{B}_{R_{h+2}})| \leq C_0 8^n M^{-(h+2)} \sum_{i \in I_{h+1}} \mathcal{H}^{n-1}\left(K_{h+1} \cap B_{M^{-(h+2)}}(x_i)\right) \leq C_2 M^{-(h+2)} \mathcal{H}^{n-1}(K_{h+1}).$$

Step 7. We are left to show point (iv). Let $x \in A_{h+1} \cap B_{R_{h+3}}.$ By Lemma 3.2 $R_{h+2} - R_{h+3} \geq 8M^{-(h+3)},$ hence, by Lemma 3.1

$$A_{h+3} \cap B_{R_{h+3}} \subset \mathcal{N}_{M^{-(h+3)}}(K \cap \overline{B}_{R_{h+2}}) \subset \mathcal{N}_{M^{-(h+2)}}(K \cap \overline{B}_{R_{h+2}}),$$

Let $\bar{x} \in K \cap \overline{B}_{R_{h+2}}$ a point realizing the distance and assume by contradiction that $\bar{x} \in K \setminus K_{h+1}.$ By the definition of $K_{h+1}$ and since $R_{h+2} \leq S_{h+1},$ this means that there is a ball $\bar{B} \in \bigcup_{j=1}^{h+1} F_j$ such that $\bar{x} \in \overline{B}.$ Since $|\bar{x} - x| \leq M^{-(h+2)}$ and the radius of $\overline{B}$ is at least $M^{-(h+1)}/C_\varepsilon,$ we can choose $M$ large enough so that $x \in 2\overline{B}.$ But then, by property (ii) of the statement,

$$|\nabla u(x)|^2 < M^{h+2},$$

a contradiction to the fact that $x \in A_{h+3}.$ \qed
We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Iterating (3.6) we obtain

$$\mathcal{H}^{n-1}(K_h) \leq C_1 \sum_{i=0}^{h} (1 - \eta)^{h-i} M^{-\frac{i}{2}} \leq C_1 h \max \left\{ (1 - \eta)^h, M^{-h/2} \right\}. \quad (3.11)$$

We now fix $M := M_2$ where $M_2$ is the constant appearing in Lemma 3.3 and choose $\alpha \in (0, 1/4)$ such that $(1 - \eta) \leq M^{-2\alpha}$. In this way, since $2\alpha < 1/2$ it follows from (3.11) that

$$\mathcal{H}^{n-1}(K_h) \leq C_1 h M^{-2\alpha h}.$$

Hence, by (3.5), (3.7), and the above equation, we obtain

$$|A_{h+2} \cap B_{R_{h+2}}| \leq |\mathcal{N}_{M^{-(h+1)}}(K_h \cap B_{R_{h+1}})| \leq C_1 C_2 h M^{-h(1+2\alpha)} \quad \forall h \geq 1,$$

so Lemma 3.2(iv) and the definition of $A_h$ (see (3.1)) finally give

$$\left| \{x \in B_{1/2} \setminus K : |\nabla u|^2(x) \geq M^h \} \right| \leq C_1 C_2 M^{2+4\alpha} h M^{-h(1+2\alpha)} \quad \forall h \geq 3. \quad (3.12)$$

Since

$$\int_{B_{1/2} \setminus K} |\nabla u|^{2\gamma} = \gamma \int_0^{\infty} t^{\gamma-1} |(B_{1/2} \setminus K) \cap \{|\nabla u|^2 \geq t\}| \, dt \leq M^\gamma \sum_{h=0}^{\infty} M^{h\gamma} \left| (B_{1/2} \setminus K) \cap \{|\nabla u|^2 \geq M^h \} \right|,$$

(3.12) implies the validity of (1.1) with, for instance, $\bar{\gamma} = 1 + \alpha$. \qed
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