Morley Finite Element Method for the Eigenvalues of the Biharmonic Operator

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Abstract

This paper studies the nonconforming Morley finite element approximation of the eigenvalues of the biharmonic operator. A new $C^1$ conforming companion operator leads to an $L^2$ error estimate for the Morley finite element method which directly compares the $L^2$ error with the error in the energy norm and, hence, can dispense with any additional regularity assumptions. Furthermore, the paper presents new eigenvalue error estimates for nonconforming finite elements that bound the error of (possibly multiple or clustered) eigenvalues by the approximation error of the computed invariant subspace. An application is the proof of optimal convergence rates for the adaptive Morley finite element method for eigenvalue clusters.

Keywords

eigenvalue problem, eigenvalue cluster, Kirchhoff plate, biharmonic, Morley, adaptive finite element method

AMS subject classifications

65M12, 65M60, 65N25

1 Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be an open bounded Lipschitz domain with polygonal boundary $\partial \Omega$ and outer unit normal $\nu$. The boundary is decomposed into mutually disjoint parts

$\partial \Omega = \Gamma_C \cup \Gamma_S \cup \Gamma_F$

such that $\Gamma_C$ and $\Gamma_C \cup \Gamma_S$ are closed sets. The vector space of admissible functions reads as

$V := \{v \in H^2(\Omega) \mid v|_{\Gamma_C \cup \Gamma_S} = 0 \text{ and } (\partial v / \partial \nu)|_{\Gamma_C} = 0\}.$

The biharmonic eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ with

$$(D^2 u, D^2 v)_{L^2(\Omega)} = \lambda (u, v)_{L^2(\Omega)} \quad \text{for all } v \in V.$$ (1.1)

In the Kirchhoff-Love plate model [41], the problem (1.1) describes the vibrations of a thin elastic plate subject to clamped ($\Gamma_C$), simply supported ($\Gamma_S$) or free ($\Gamma_F$) boundary conditions. Nonconforming finite element discretisations of (1.1) appear attractive because they circumvent the use of complicated $C^1$ conforming FEMs [19]. The nonconforming Morley finite element based on piecewise quadratic polynomials can furthermore be employed for the computation of lower eigenvalue bounds [12]. For the linear biharmonic problem, the adaptive Morley FEM has been proven to produce optimal convergence rates [29] [14].

A priori error estimates for the Morley finite element discretisation of eigenvalue problems can be found in [36]. In the a posteriori error analysis, in particular for the analysis of adaptive algorithms, the $L^2$ error of the eigenfunction approximation can be viewed as a perturbation of the right-hand side. Indeed, for conforming finite elements, the higher-order
$L^2$ error control follows from the Aubin-Nitsche duality technique [40]. This argument fails to hold in its original form in the case of nonconforming finite elements. In order to obtain error estimates in the $L^2$ norm that do not require additional assumptions on the regularity of the solution, the works [13, 34] introduced (for the Crouzeix-Raviart discretisation of second-order problems) certain conforming companion operators that allow the proof of such $L^2$ estimates. This paper introduces a corresponding operator for the Morley finite element. This operator leads to a new $L^2$ error estimate for the Morley finite element without any additional regularity assumption. This is of particular interest in the case of non-clamped boundary conditions where, in general, the exact solution is expected to belong to $H^2(\Omega) \setminus H^{5/2}(\Omega)$.

Practical adaptive algorithms for multiple eigenvalues [20] or eigenvalue clusters [25, 24] are based on a posteriori error estimators that involve the sum of the residuals of all discrete eigenfunctions of interest. Let $\lambda_{n+1} \leq \cdots \leq \lambda_{n+N}$ be the eigenvalue cluster of interest with discrete approximations $\lambda_{\ell,n+1} \leq \cdots \leq \lambda_{\ell,n+N}$ computed by the Morley FEM. These error estimators bound the distance of the exact invariant subspace of the corresponding eigenfunctions $W = \text{span}\{u_{n+1}, \ldots, u_{n+N}\}$ and the invariant subspace of discrete eigenfunctions $W_\ell = \text{span}\{u_{\ell,n+1}, \ldots, u_{\ell,n+N}\}$. For conforming finite elements, the results of [31] show that this distance acts as an upper bound of the eigenvalue error. This result, however, does not directly apply to nonconforming finite element methods. A generalisation for the Crouzeix-Raviart FEM for the eigenvalues of the Laplacian is given in [6] where it is used that the nonconforming finite element space has an $H^1$-conforming subspace. The Morley finite element does not satisfy a corresponding condition; this paper develops a new technique which allows the proof of eigenvalue error estimates of the form

$$|\lambda_j - \lambda_{\ell,j}|/\max\{\lambda_j, \lambda_{\ell,j}\} \leq C \sin^2_{\alpha, NC}(W, W_\ell).$$

The constant $C$ and its dependence on the eigenvalue cluster will be quantified more precisely. The angles are measured in the discrete energy scalar product ($L^2$ product of the piecewise Hessians). The main idea is to study an auxiliary eigenvalue problem in the sum $\tilde{V}_\ell := V + V_\ell$ of the continuous space $V$ and the discrete space $V_\ell$. The arguments in the proof rely on a careful analysis of the Morley interpolation operator and the conforming companion operator.

As an application, the paper presents optimal convergence rates of the adaptive Morley FEM for eigenvalue clusters. The proofs follow the methodology of [17, 38] which has already been applied in [21, 16, 13] for simple eigenvalues, in [20] for multiple eigenvalues, and in [25, 24] for clustered eigenvalues.

The remaining parts of this paper are organised as follows. Section 2 introduces the necessary notation on triangulations and data structures, it proves new error estimates for the Morley interpolation operator, and it presents a new conforming companion operator. Section 3 is devoted to the discretisation of the biharmonic eigenvalue problem and derives new $L^2$ error estimates and new error estimates for the eigenvalues whose proof is based on a new methodology. Section 4 applies the new results to the adaptive finite element method for clustered eigenvalues and proves its optimal convergence rates.

Throughout the paper standard notation on Lebesgue and Sobolev spaces is employed. The integral mean is denoted by $f$. The bullet • denotes the identity. For any smooth function $f : \Omega \to \mathbb{R}$ the Curl reads as $\text{Curl } f := (-\partial f/\partial x_2, \partial f/\partial x_1)$. For a sufficiently smooth vector field $\beta : \Omega \to \mathbb{R}^2$, define

$$\text{Curl } \beta := \left(\begin{array}{cc} -\partial \beta_1/\partial x_2 & \partial \beta_1/\partial x_1 \\ -\partial \beta_2/\partial x_2 & \partial \beta_2/\partial x_1 \end{array}\right).$$

The symmetric part of a matrix $X$ is denoted by $\text{sym}(X)$ and the space of symmetric $2 \times 2$ matrices is denoted by $\mathbb{S}$. The notation $a \lesssim b$ abbreviates $a \leq Cb$ for a positive generic.
constant $C$ that may depend on the domain $\Omega$ and the initial triangulation $\mathcal{T}_0$ but not on
the mesh-size or the eigenvalue cluster of interest. The notation $a \approx b$ stands for $a \lesssim b \lesssim a$.

2 The Morley Finite Element Space

This section introduces the necessary notation and data structures in Subsection 2.1 and
proves some new results for the Morley finite element in the remaining subsections.

2.1 Notation and Data Structures

**Triangulations.** Let $\mathcal{T}_0$ be a regular triangulation of $\Omega$, i.e., $\cup \mathcal{T}_0 = \Omega$ and any two distinct
elements of $\mathcal{T}_0$ are either disjoint or their intersection is exactly one common vertex or exactly
one common edge. Throughout this paper, any regular triangulation of $\Omega$ is assumed to be
admissible in the sense that it is regular and a refinement of some initial triangulation
$\mathcal{T}_0$ created by newest-vertex bisection with proper initialisation of the refinement edges
[4, 39]. The set of all admissible refinements is denoted by $\mathcal{T}$. The restriction to this class
of triangulations is not essential in Sections 2–3, but is made to ease notation in view of
the adaptive algorithms studied in Section 4. Given a triangulation $\mathcal{T}_T \in \mathcal{T}$, the piecewise
constant mesh-size function $h_T := h_\mathcal{T}$ is defined by $h_T|_T := \text{meas}(T)^{1/2}$ for any
triangle $T \in \mathcal{T}_T$. For all regular triangulations $\mathcal{T}_T \in \mathcal{T}$ of $\Omega$, it is assumed that the relative
interior of each boundary edge is contained in one of the parts $\Gamma_C$, $\Gamma_S$, or $\Gamma_F$ (in fact, this
is only a condition on $\mathcal{T}_0$).

**Edges.** The set of edges of a triangle $T$ is denoted by $\mathcal{F}(T)$. The edges of $\mathcal{T}_T$ read as $\mathcal{F}_T := \mathcal{F}(\mathcal{T}_T) := \cup_{T \in \mathcal{T}_T} \mathcal{F}(T)$. The edges that belong to the boundary read $\mathcal{F}_T(\partial \Omega)$ and the interior
edges read $\mathcal{F}_T(\Omega) := \mathcal{F}_T \setminus \mathcal{F}_T(\partial \Omega)$. Let $\Gamma \subseteq \partial \Omega$ be a subset of the boundary $\partial \Omega$. The boundary
edges that belong to $\Gamma$ are denoted by $\mathcal{F}_T(\Gamma) := \{ F \in \mathcal{F}_T \mid |H^1(\Omega \cap \Gamma) > 0 \}$, where $H^1$ is the
one-dimensional Hausdorff measure. Furthermore, define $\mathcal{F}_T(\Omega \cup \Gamma) := \mathcal{F}_T(\Omega) \cup \mathcal{F}_T(\Gamma)$. For
any edge $F \in \mathcal{F}_T$, the edge patch is defined as $\omega_F := \text{int}(\cup \{ T \in \mathcal{T}_T \mid F \in \mathcal{F}(T) \})$. Given any
vertex of $\mathcal{T}_T$, the set of edges that share $z$ is denoted by $\mathcal{F}_T(z) := \{ F \in \mathcal{F}_T \mid z \in F \}$. The
length of an edge $F$ reads $h_F$.

**Vertices.** The set of vertices of a triangle $T$ is denoted by $N(T)$. Define $N_T := N(\mathcal{T}_T) := \cup_{T \in \mathcal{T}_T} N(T)$ as the set of vertices of $\mathcal{T}_T$. The set of vertices that belong to some subset $\omega \subseteq \Omega$
are denoted by $N_T(\omega) := N_T \cap \omega$.

**Normal and tangent vectors.** Let every edge $F \in \mathcal{F}_T$ be equipped with a fixed normal
vector $\nu_F$. If $F \in \mathcal{F}_T(\partial \Omega)$ belongs to the boundary, $\nu_F := \nu$ is chosen to point outwards
$\Omega$. Let for any edge $F \in \mathcal{F}_T$ with normal vector $\nu_F = (\nu_F(1); \nu_F(2))$ the tangent vector be
defined as $\tau_F := (-\nu_F(2); \nu_F(1))$ and denote by $\tau := (-\nu(2); \nu(1))$ the tangent vector of $\partial \Omega$.

**Jumps.** Given $F \in \mathcal{F}_T(\Omega)$, $F = \partial T^+_T \cap \partial T^-_T$ shared by two triangles $(T^+_T, T^-_T) \in \mathcal{T}_T^2$, and a
piecewise (possibly vector-valued) smooth function $v$, define the jump of $v$ across $F$ by

$$[v]|_F := v|_{T^+_T} - v|_{T^-_T}.$$  

For edges $F \subseteq \partial \Omega$ on the boundary, $[v]|_F := v|_F$ denotes the trace.

**Piecewise polynomials and oscillations.** The set of polynomials of degree $\leq k$ over
a subset $\omega \subseteq \Omega$ is denoted by $\mathcal{P}_k(\omega)$. The set of piecewise polynomial functions of degree
\[ \leq k \] with respect to \( \mathcal{T}_f \) is denoted by \( \mathcal{P}_k(\mathcal{T}_f) \). The \( L^2 \) projection onto \( \mathcal{P}_k(\mathcal{T}_f) \) is denoted by \( \Pi^k_{\mathcal{T}_f} \equiv \Pi^k_\mathcal{T} \). The \( k \)-th order oscillations of a given function \( f \in L^2(\Omega) \) is defined as

\[
\text{osc}_k(f, \mathcal{T}_f) := \|h^2_k(1 - \Pi^k_\mathcal{T})f\|_{L^2(\Omega)}.
\]

**Piecewise action of differential operators.** The piecewise action of a differential operator is indicated by the subscript NC, i.e., the piecewise versions of \( D \) and \( D^2 \) read as \( D_{NC} \equiv D_{NC(\mathcal{T})} \) and \( D_{NC}^2 \equiv D_{NC(\mathcal{T})}^2 \), e.g., \( (D_{NC}v)|_T = D(v|_T) \) for any \( T \in \mathcal{T}_f \). The dependence on \( \mathcal{T}_f \) in this notation is dropped whenever there is no risk of confusion.

**Functional setting.** The vector space of admissible functions reads as

\[
V := \left\{ v \in H^2(\Omega) \mid v|_{\Gamma_C \cup \Gamma_S} = 0 \text{ and } (\partial v/\partial \nu)|_{\Gamma_C} = 0 \right\}.
\]

Define the bilinear form

\[
a(v, w) := (D^2v, D^2w)_{L^2(\Omega)} \quad \text{for all } (v, w) \in V^2
\]

with induced seminorm \( \|\cdot\| := a(\cdot, \cdot)^{1/2} \) and \( b(\cdot, \cdot) := (\cdot, \cdot)_{L^2(\Omega)} \) with induced norm \( \|\cdot\| \).

Throughout this paper it is assumed that the only affine function in \( \mathcal{P}_1(\Omega) \) is zero, i.e., \( V \cap \mathcal{P}_1(\Omega) = \{0\} \). Hence, \( a \) is a scalar product on \( V \) with norm \( \|\cdot\| \).

The Morley finite element space reads as

\[
V_\ell := \left\{ v \in \mathcal{P}_2(\mathcal{T}_\ell) \mid \begin{array}{l}
v \text{ is continuous at } N_\ell(\Omega) \text{ and vanishes at } N_\ell(\Gamma_C \cup \Gamma_S); \\
D_{NC}v \text{ is continuous at the interior edges' midpoints} \\
\text{and vanishes at the midpoints of the edges of } \Gamma_C
\end{array} \right\}.
\]

On each triangle the local degrees of freedom are the evaluation of the function at each vertex and the evaluation of the normal derivative at the edges’ midpoints. See Figure 1a for an illustration.

The discrete version of the energy scalar product reads as

\[
a_{NC}(v, w) := (D_{NC}^2v, D_{NC}^2w)_{L^2(\Omega)} \quad \text{for all } (v, w) \in (V + V_\ell)^2
\]

with induced discrete energy norm \( \|\cdot\|_{NC} := a_{NC}(\cdot, \cdot)^{1/2} \). Indeed, the assumption \( V \cap \mathcal{P}_1(\Omega) = \{0\} \) implies \( V_\ell \cap \mathcal{P}_1(\Omega) = \{0\} \). Hence, \( a_{NC}(\cdot, \cdot) \) defines a scalar product on \( V_\ell \) (as shown in Corollary 2.3 the ellipticity is even uniform in the mesh parameter).

**Principal angles between subspaces.** For finite-dimensional subspaces \( X \subseteq V + V_\ell \) and \( Y \subseteq V + V_\ell \) the sine of the largest principal angle from \( X \) to \( Y \) is denoted by

\[
\sin_{a,NC} \angle (X, Y) = \sup_{x \in X} \inf_{y \in Y} \frac{\|x - y\|_{NC}}{\|x\|_{NC} \cdot \|y\|_{NC}}.
\]

It is well known [30, Thm. 6.34 in Chapter 1, §6] that in the case of \( \dim(X) = \dim(Y) < \infty \) it holds that

\[
\sin_{a,NC} \angle (X, Y) = \sin_{a,NC} \angle (Y, X)
\]

as well as

\[
\sin_{a,NC} \angle (X, Y) \leq \sin_{a,NC} \angle (X, Z) + \sin_{a,NC} \angle (Z, Y)
\]

for any subspace \( Z \subseteq V + V_\ell \) with \( \dim(X) = \dim(Y) = \dim(Z) < \infty \).
Proposition 2.1 estimates for the Morley interpolation operator.
A piecewise integration by parts proves the projection property for the Hessian

$$\Pi_T^2 D_{sc}^2 = D_{sc}^2 J_t.$$ (2.3)

2.2 Morley Interpolation Operator

Let $\mathcal{T}_{t+m}$ be any admissible refinement of $\mathcal{T}_t$. The Morley interpolation operator $J_t : V + V_{t+m} \to V_t$ is defined via

$$(J_t v)(z) = v(z)$$

for any $z \in \mathcal{N}_t$ and any $v \in V + V_{t+m}$.

A piecewise integration by parts proves the projection property for the Hessian

$${\Pi}_T^2 D_{sc}^2 = D_{sc}^2 J_t.$$ (2.3)

The following generalisation of the trace inequality [11, 22] is necessary for proving error estimates for the Morley interpolation operator.

**Proposition 2.1** (discrete trace inequality). Let $T \in \mathcal{T}_t$ be a triangle and $K$ be a regular triangulation of $T$ and let $G \in \mathcal{T}(T)$ be an edge of $T$. Any piecewise (with respect to $K$) smooth function $f$ satisfies the discrete trace inequality

$$\|f\|_{L^2(G)} \lesssim h_T^{-1/2} \|f\|_{L^2(T)} + h_T^{1/2} \|D_{sc} f\|_{L^2(T)} + h_T^{1/2} \sqrt{\sum_{F \in \mathcal{T}(K) \cap \partial T} h_F^{-1} \|f\|_{L^2(F)}}.$$ (2.4)

**Proof.** Denote by $P_G$ the vertex of $T$ opposite to $G$. A piecewise integration by parts proves the discrete trace identity

$$\frac{1}{2} \int_T (\bullet - P_G) \cdot D_{sc} f \, dx = - \int_T f \, dx + \text{dist}(P_G, G) \int_G f \, ds + \sum_{F \in \partial \mathcal{T}(K)} \int_F (\bullet - P_G) \cdot \nu_F |f|_F \, ds.$$

The application of this identity to the function $f^2$ together with elementary algebraic manipulations and $\text{dist}(P_G, G) \leq \text{diam}(T) \lesssim h_T$ result in

$$\|f\|_{L^2(G)}^2 \lesssim \int_T D_{sc} (f^2) \, dx \left[ + h_T^{-1} \|f\|_{L^2(T)}^2 + h_T^{-1} \sum_{F \in \partial \mathcal{T}(K)} \int_F (\bullet - P_G) \cdot \nu_F |f^2|_F \, ds. \right.$$ (2.4)

The Young inequality shows that the first term on the right-hand side can be controlled as

$$\left| \int_T D_{sc}(f^2) \, dx \right| = \left| \int_T 2f D_{sc} f \, dx \right| \leq 2h_T^{-1/2} \|f\|_{L^2(T)} h_T^{1/2} \|D_{sc} f\|_{L^2(T)} \leq h_T^{-1/2} \|f\|_{L^2(T)}^2 + h_T \|D_{sc} f\|_{L^2(T)}^2.$$

It remains to bound the third term on the right-hand side of (2.4). Let $F \in \mathcal{T}(K)$ be an interior edge shared by two triangles $K_+$ and $K_-$ such that $F = K_+ \cap K_-$. Denote $f_f := f|_K$, and $f_{-f} := f|_{K_-}$. A direct calculation proves for the jump of $f^2$ across $F$ that

$$[f^2]_F = [f]_F (f_f + f_{-f}).$$
Thus, the Cauchy and triangle inequalities followed by the Young inequality prove
\[
\int_{F} (\bullet - P_G) \cdot \nu_F [f^2]_F ds \\
\leq \text{diam}(T) h_T^{-1} h_F^{1/2} \| f_F \|_{L^2(F)} h_T^{1/2} h_T^{-1/2} (\| f_+ \|_{L^2(F)} + \| f_- \|_{L^2(F)}) \\
\leq \text{diam}(T) \left( h_T^{-1} h_T (\| f_+ \|_{L^2(F)} + h_F h_T^{-1} (\| f_+ \|_{L^2(F)} + \| f_- \|_{L^2(F)})^2 \right).
\]

The trace inequality \cite{11,22} and an inverse estimate \cite{8} applied to the edge patch \(\Omega_F\) prove that
\[
h_T h_T^{-1} (\| f_+ \|_{L^2(F)} + \| f_- \|_{L^2(F)})^2 \lesssim h_T^{-1} \| f \|_{L^2(\Omega_F)}^2.
\]

The foregoing two displayed inequalities, the finite overlap of the edge patches and the shape regularity prove
\[
h_T^{-1} \sum_{F \in \mathcal{F}(K)} \int_{F} (\bullet - P_G) \cdot \nu_F [f^2]_F ds \lesssim h_T^{-1} \| f \|_{L^2(T)}^2 + h_T \sum_{F \in \mathcal{F}(K)} \int_{F} \nu_F [f^2]_F ds, \quad (2.5)
\]

The combination of the above estimates concludes the proof. \(\square\)

**Remark 2.2.** In Proposition 2.4, the ratio \(h_T/h_F\) is not required to be uniformly bounded.

The next proposition provides an error estimate for the Morley interpolation operator. In contrast to the estimate from \cite{12} with an explicit constant for the Morley interpolation when applied to an \(H^2\) function, the following result gives an estimate for more general piecewise smooth functions.

**Proposition 2.3** (Error estimate for the Morley interpolation). Let \(T \in \mathcal{T}_t\) be a triangle, and let \(\mathcal{T}_{\ell+m}\) be a regular triangulation of \(T\). Any \(v_{\ell+m} \in V + V_{\ell+m}\) and its interpolation \(I_{\ell}v_{\ell+m}\) satisfy
\[
\| h_T^{-2} (1 - I_{\ell})v_{\ell+m} \|_{L^2(T)} + h_T^{-1} D_{\text{NC}} (1 - I_{\ell})v_{\ell+m} \|_{L^2(T)} \lesssim D_{\text{NC}}^2 (1 - I_{\ell})v_{\ell+m} \|_{L^2(T)}. \quad (2.5)
\]

**Remark 2.4.** Error estimates of this type are stated and utilized in \cite{29} with a proof based on equivalence of norms. To make the constant in the estimate more transparent, a new proof is given here. It shall be pointed out that the constant in the assertion of Proposition 2.3 does not depend on the triangulation \(\mathcal{T}_{\ell+m}\).

**Proof of Proposition 2.3**. Let, without loss of generality, \(v_{\ell+m} \in H^3(\text{int}(T)) + V_{\ell+m}\) (the general case then follows with a density argument). The discrete Friedrichs inequality \cite{8} Thm. 10.6.12] together with a scaling argument and the fact that \(I_{\ell}v_{\ell+m}\) is continuous on \(T\) yield that
\[
\left\| (1 - I_{\ell})v_{\ell+m} \right\|_{L^2(T)}^2 \lesssim \int_{\partial T} (1 - I_{\ell})v_{\ell+m} ds + h_T^{-2} \sum_{F \in \mathcal{F}(\mathcal{T}_{\ell+m})} h_F^{-1} \| v_{\ell+m} \|_{L^2(F)}^2 \|
\]
\[
+ h_T D_{\text{NC}} (1 - I_{\ell})v_{\ell+m} \|_{L^2(T)}.
\]

For any edge \(G \in \mathcal{F}(T)\), the Hölder and Friedrichs inequalities prove that
\[
\left\| \int_{G} (1 - I_{\ell})v_{\ell+m} ds \right\| \lesssim h_G^{1/2} \| (1 - I_{\ell})v_{\ell+m} \|_{L^2(G)} \lesssim h_G^{3/2} \| \partial (1 - I_{\ell})v_{\ell+m}/\partial \nu_G \|_{L^2(G)}.
\]
(Note that \(v_{t+m}\) is differentiable and continuous along \(G\).) The discrete trace inequality from Proposition 2.1 proves that this is controlled by some constant times
\[
h_T \|D_{\text{NC}}(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)} + h_T^2 \|D_{\text{NC}}^2(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)}
\]
\[
+ h_T^2 \sum_{F \in \mathcal{F}(\mathcal{T}_{t+m})} h_T^{-1} \|[D_{\text{NC}} v_{t+m}]_F\|_{L^2(F)}.
\]
For any face \(F \in \mathcal{F}(\mathcal{T}_{t+m})\) with \(F \subseteq \partial T\), the Friedrichs and Poincaré inequality prove that
\[
h_F^{-1} \|[v_{t+m}]_F\|_{L^2(F)} \leq h_F \|[D_{\text{NC}} v_{t+m}]_F \tau_F\|_{L^2(F)} \leq h_F^3 \|[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)}.
\]
Altogether,
\[
\|(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)} \leq h_T \|D_{\text{NC}}(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)} + h_T^2 \|D_{\text{NC}}^2(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)}
\]
\[
+ h_T^2 \sum_{F \in \mathcal{F}(\mathcal{T}_{t+m})} h_F \|[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)}.
\]
The discrete Friedrichs inequality [3] Thm. 10.6.12] together with a scaling argument imply
\[
h_T \|D_{\text{NC}}(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)} \leq h_T^2 \|D_{\text{NC}}^2(1-\mathbb{J}_\ell) v_{t+m}\|_{L^2(T)}.
\]
For the estimate of the jump terms let \(F = \text{conv}\{z_1, z_2\} \in \mathcal{F}(\mathcal{T}_{t+m})\) be the convex hull of the vertices \(z_1, z_2\) such that \(F\) is an interior edge and denote, for \(j \in \{1, 2\}\), by \(\varphi_j \in \mathcal{P}_1(\mathcal{T}_{t+m})\) the piecewise affine function with \(\varphi_j(z_j) = 1\) and \(\varphi_j(y) = 0\) for all \(y \in \mathcal{N}(\mathcal{T}_{t+m}) \setminus \{z_j\}\). The piecewise quadratic edge-bubble function \(b_F := 6\varphi_1\varphi_2 \in H^1_\text{curl}(\omega_F)\) satisfies
\[
\|b_F\|_{L^\infty(T)} = 3/2 \quad \text{and} \quad \int_F b_F \, ds = h_F.
\]
Define \(\psi_F := (b_F[D_{\text{NC}}^2 v_{t+m}]_F \tau_F) \in H^1_\text{curl}(\omega_F; \mathbb{R}^2)\). Since \([D_{\text{NC}}^2 v_{t+m}]_F\) is constant along \(F\), it follows that
\[
\|[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)} = \|b_F^{1/2}[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)}.
\]
For any \(v \in H^2(\omega_F)\), an integration by parts and the \(L^2\)-orthogonality of \(\text{Curl}\, \psi_F\) on \(D^2 v\) reveal that
\[
\|b_F^{1/2}[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)} = \int_F \{(D_{\text{NC}}^2 v_{t+m})_F \tau_F\} \cdot \psi_F \, ds = (D_{\text{NC}}^2 (v_{t+m} - v), \text{Curl}\, \psi_F)_{L^2(\omega_F)}.
\]
The Cauchy and inverse inequalities prove that this is bounded by
\[
\|[D_{\text{NC}}^2 (v_{t+m} - v)]_F \text{Curl}\, \psi_F\|_{L^2(\omega_F)} \leq \|D_{\text{NC}}^2 (v_{t+m} - v)\|_{L^2(\omega_F)} \|[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|.
\]
This implies
\[
h_F \|[D_{\text{NC}}^2 v_{t+m}]_F \tau_F\|_{L^2(F)} \leq \min_{v \in H^2(\text{int}(T))} \|D_{\text{NC}}^2 (v_{t+m} - v)\|_{L^2(\omega_F)}^2.
\]
The sum over all interior edges of \(\mathcal{F}(\mathcal{T}_{t+m})\) and the finite overlap of edge-patches prove the result.

### 2.3 Conforming Companion Operator

This subsection is devoted to the design of a new conforming companion operator. In contrast to the operators introduced in [13, 34], \(H^2\) conformity is required. Compared to certain averaging operators that can be found in the literature [7, 27], the proposed
companion operator has additional conservation properties for the integral mean and the integral mean of the Hessian. A similar approach has been independently developed in \[33\]. In contrast to that work, the operator presented here satisfies an additional best-approximation property.

The Hsieh-Clough-Tocher (HCT) finite element \[19\] enters the design of a conforming companion operator. Let any \( T \in \mathcal{T} \) be decomposed into three sub-triangles as depicted in Figure 1b, where the vertex shared by the three sub-triangles is the midpoint \( \text{mid}(T) \). Given this triangulation \( \mathcal{K}_T(T) \) of \( T \), let

\[
V_{\text{HCT}}(\mathcal{T}_T) := \{ v \in V \mid v|_T \in \mathcal{P}_d(\mathcal{K}_T(T)) \text{ for all } T \in \mathcal{T}_T \}.
\]

The local degrees of freedom on each triangle \( T \) are the nodal values of the function and its derivative and the value of the normal derivative at the midpoints of the edges of \( T \) in Figure 1b.

Such conforming finite elements turn out to be useful for the theoretical analysis. The following proposition presents a simple averaging operator, similar to that of \[7, 27\], for the case of more general boundary conditions.

**Proposition 2.5 (HCT enrichment).** There exists an operator \( A : V_T \to V_{\text{HCT}}(\mathcal{T}_T) \) such that any \( v_T \in V_T \) satisfies

\[
\| h_T^2(v_T - Av_T) \|_{L^2(\Omega)}^2 \lesssim \sum_{F \in \mathcal{T}_T(\mathcal{K}_T(T))} h_F \| [D^2 v_T]_{F \mathcal{T} F} \|_{L^2(F)} + \sum_{F \in \mathcal{T}_T(\Gamma_F)} h_F \| \tau_F \cdot [D^2 v_T]_{F \mathcal{T} F} \|_{L^2(F)}^2
\]

\[
\lesssim \min_{v \in V} \| D^2_{\text{NC}}(v_T - v) \|_{L^2(\Omega)}.
\]

**Proof.** Given \( v_T \in V_T \), define \( Av_T \in V_{\text{HCT}}(\mathcal{T}_T) \) by setting the degrees of freedom as follows

\[
(v_T - Av_T)(z) = 0 \quad \text{for all } z \in N_T,
\]

\[
\frac{\partial (v_T - Av_T)}{\partial \nu F}(\text{mid}(F)) = 0 \quad \text{for all } F \in \mathcal{T}_F,
\]

\[
D(Av_T)(z) = \text{card}(\mathcal{T}_F(z))^{-1} \sum_{T \in \mathcal{T}_F(z)} (Dv_T|_T)(z) \quad \text{for all } z \in N_T(\Omega \cup \Gamma_F).
\]

In other words, the degrees of freedom are defined by averaging. For the remaining vertices on the boundary, set

\[
D(Av_T)(z) = 0 \quad \text{for all } z \in N_T(\Gamma_S) \text{ with angle } \neq \pi \text{ and all } z \in N_T(\Gamma_C)
\]

and, for all \( z \in N_T(\Gamma_C) \) with angle = \( \pi \),

\[
\frac{\partial Av_T}{\partial \tau}(z) = 0 \quad \text{and} \quad \frac{\partial Av_T}{\partial \nu}(z) = (\text{card}(\mathcal{T}_F(z)))^{-1} \sum_{F \in \{F_+, F_+\}} \frac{\partial v_T}{\partial \nu}(z)\big|_F
\]

where \( (F_+, F_-) \in (\mathcal{T}_F(\Gamma_S))^2 \) are the two boundary edges sharing \( z \). Note that, for corners of the domain \( \Omega \) with angle \( \neq \pi \), the simply supported boundary condition implies that the full derivatives vanish at \( z \).

The remaining part of the proof is devoted to the error estimate for \( A \). For a multi-index \( \alpha \) of length \( |\alpha| = 1 \) and any vertex \( z \in N_T \), let \( \psi_{z, \alpha} \) denote the nodal basis function of \( V_{\text{HCT}}(\mathcal{T}_T) \) with \( (\partial \psi_{z, \alpha}/\partial x^\alpha)(z) = 1 \) that vanishes for the remaining degrees of freedom of the HCT finite element. Since the HCT finite element is a finite element in the sense of \[19\], for any \( T \in \mathcal{T}_T \) the function \( v_T|_T \in \mathcal{P}_2(T) \) can be represented by means of the local HCT basis functions. By definition of \( A \), the difference \( v_T - Av_T \) can be represented as follows

\[
\| h_T^2(v_T - Av_T) \|_{L^2(\Omega)}^2 = \sum_{T \in \mathcal{T}_T} \left( h_T^2 \sum_{z \in N(T)} \sum_{|\alpha|=1} \left| \frac{\partial^{|\alpha|} (v_T - Av_T)}{\partial x^\alpha}(z) \psi_{z, \alpha} \right|^2 \right)_{L^2(T)}.
\]
For any \( T \in \mathcal{T}_h \), the scaling of the basis functions \[9\] Thm. 6.3.1, p. 344 reads as
\[
\| h_T^{-2} \psi_{z, \alpha} \|_{L^2(T)} \lesssim 1 \quad \text{for } |\alpha| = 1.
\]
Thus, the triangle inequality implies that
\[
\| h_T^{-2} (v T - A v_T) \|_{L^2(T)}^2 \lesssim \sum_{T \in \mathcal{T}_h} \sum_{z \in N(T)} |D(v T - A v_T)(z)|^2.
\]
The triangle inequality and equivalence of seminorms prove, for any vertex \( z \in N(T) \), that
\[
|D(v T - A v_T)(z)|^2 \lesssim \sum_{F \in \mathcal{F}_h(z) \cap \mathcal{F}_h(T)} [D_{nc} v_T(z)]^2 \lesssim \sum_{F \in \mathcal{F}_h(z) \cap \mathcal{F}_h(T)} h^{-1} F \|[D_{nc} v_T]^2\|_{L^2(F)}^2. \quad (2.6)
\]
For any vertex \( z \in N(T) \) and any triangle \( T \) with \( z \in T \) the definition of \( A \) implies
\[
|(D_{nc} v_T - A v_T)(z)| = |D v_T(z)|.
\]
Any vertex \( z \in N(T) \) and any triangle \( T \) with \( z \in T \) satisfy
\[
|\partial (v_T - A v_T) / \partial \tau(z) | = |(\partial v_T / \partial \tau)(z) |
\]
and, as in (2.6), it follows in the case that the angle at \( z \) equals \( \pi \), that
\[
|(\partial (v_T - A v_T) / \partial \tau)(z) | \lesssim \sum_{F \in \mathcal{F}_h(z) \cap \mathcal{F}_h(T)} |\partial v_T / \partial \tau|_F(z) |.
\]
Equivalence of norms and Poincaré inequalities along \( F \in \mathcal{F}_h \) prove
\[
|\partial v_T / \partial \tau F|_F(z) \lesssim h^{-1/2} F \|[\partial v_T / \partial \tau F]|_F \|_{L^2(F)} \lesssim h^{-1/2} F \|[\partial v_T / \partial \tau F]|_F \|_{L^2(F)} \lesssim h^{-1/2} F \|[\partial v_T / \partial \tau F]|_F \|_{L^2(F)} \lesssim h^{-1/2} F \|[\partial v_T / \partial \tau F]|_F \|_{L^2(F)}.
\]
This proves the first inequality of the proposition.

The proof of the efficiency estimate can be carried out by using the bubble function technique from the proof of Proposition 2.3. \( \square \)

**Proposition 2.6** (companion operator). For any \( v_T \in V_T \) there exists some \( \psi v_T \in V \) such that \( v_T - \psi v_T \) and its second-order partial derivatives are \( L^2 \)-orthogonal on the space \( \mathcal{P}_0(\mathcal{T}_h) \) of piecewise constants,
\[
\Pi_0^0(v_T - \psi v_T) = 0 \quad \text{and} \quad \Pi_0^0(D_{nc}^2(v_T - \psi v_T)) = 0. \quad (2.7)
\]
It satisfies the approximation and stability property
\[
\| h_T^{-2} (v_T - \psi v_T) \|_{L^2(\Omega)} + \| h_T^{-1} D_{nc} (v_T - \psi v_T) \|_{L^2(\Omega)} + \| D_{nc}^2 (v_T - \psi v_T) \|_{L^2(\Omega)} \lesssim \min_{v \in V} \| D_{nc}^2 (v_T - v) \|_{L^2(\Omega)}. \quad (2.8)
\]

**Proof.** The design follows in three steps.

**Step 1.** Proposition 2.3 and inverse estimates \[8\] prove for the operator \( A \) that
\[
\| h_T^{-2} (v_T - A v_T) \|_{L^2(\Omega)} + \| h_T^{-1} D_{nc} (v_T - A v_T) \|_{L^2(\Omega)} + \| D_{nc}^2 (v_T - A v_T) \|_{L^2(\Omega)} \lesssim \min_{v \in V} \| D_{nc}^2 (v_T - v) \|_{L^2(\Omega)}.
\]

**Step 2.** Let \( T = \text{conv} \{ z_1, z_2, z_3 \} \) be a triangle of \( \mathcal{T}_h \) and let \( F \in \mathcal{T}(T) \) with \( F = \text{conv} \{ z_1, z_2 \} \) and denote the continuous nodal \( \mathcal{P}_1 \) basis functions by \( \varphi_1, \varphi_2, \varphi_3 \in \mathcal{P}_1(\mathcal{T}_h) \cap \mathcal{P}_1(F) \)
$H^1(\Omega)$. Let $\nu_T$ denote the outward pointing unit normal of $T$ and define the function $\zeta_{F,T}$ by

$$
\zeta_{F,T} := 30(\nu_T \cdot \nu_F) \text{dist}(z_3, F)\varphi_1^2\varphi_2^2\varphi_3.
$$

For any $F \in \mathcal{T}_\ell$, the function

$$
\zeta_F := \begin{cases} 
\zeta_{F,K} & \text{on triangles } K \in \mathcal{T}_\ell \text{ with } F \in \mathcal{T}(K), \\
0 & \text{otherwise}
\end{cases}
$$

satisfies $\zeta_F \in H^2(\Omega)$ and $\text{supp}(\zeta_F) = \overline{F_T}$ as well as $\int_F \frac{\partial \zeta_F}{\partial \nu_F} \, dx = 1$. For the proof that $\zeta_F$ is continuously differentiable across interior edges $F$, note that any adjacent triangle $T$ satisfies $D\varphi_3|_T = (\text{dist}(z_3, F))^{-1}\nu_T$ as well as

$$(D\zeta,F) \nu_F = 30(\nu_T \cdot \nu_F) \text{dist}(z_3, F)\varphi_1^2\varphi_2^2(D\varphi_3\nu_F) = 30\varphi_1^2\varphi_2^2.$$

Hence, $\zeta_F \in H^2(\Omega)$.

If $F \in \mathcal{T}_T(\Omega)$, it holds that $\zeta_F \in H^2_0(\omega_F)$. Define the operator $\tilde{A} : V_\ell \to V$ which acts as

$$
\tilde{A}v_\ell := A v_\ell + \sum_{F \in \mathcal{T}(\Omega) \cap \mathcal{T}_T} \left( \int_F \frac{\partial (v_\ell - A v_\ell)}{\partial \nu_F} \, ds \right) \zeta_F.
$$

An immediate consequence of this choice reads as

$$
\int_F \frac{\partial \tilde{A}v_\ell}{\partial \nu_F} \, ds = \int_F \frac{\partial v_\ell}{\partial \nu_F} \, ds \quad \text{for all } F \in \mathcal{T}_\ell.
$$

An integration by parts shows the integral mean property of the Hessian $\Pi^o D^2 \tilde{A} = D^2_{NC}$. The scaling $\|\zeta_F\|_{L^2(T)} \lesssim h_T^{-\frac{1}{2}}$ and the trace inequality \cite{11,22} prove, for any $T \in \mathcal{T}_\ell$, that

$$
\begin{align*}
&h_T^{-2} \left\| \sum_{F \in \mathcal{T}(T)} \left( \int_F \frac{\partial (v_\ell - A v_\ell)}{\partial \nu_F} \, ds \right) \zeta_F \right\|_{L^2(T)} \\
&\quad \lesssim \sum_{F \in \mathcal{T}(T)} \left\| \int_F \frac{\partial (v_\ell - A v_\ell)}{\partial \nu_F} \, ds \right\| \\
&\quad \lesssim h_T^{-1} \|D_{NC}(v_\ell - A v_\ell)\|_{L^2(T)} + \|D^2_{NC}(v_\ell - A v_\ell)\|_{L^2(T)}.
\end{align*}
$$

This together with the first step of the proof and inverse estimates \cite{8} show that

$$
\begin{align*}
\|h_T^{-2}(v_\ell - \tilde{A}v_\ell)\|_{L^2(\Omega)} + h_T^{-1}D_{NC}(v_\ell - \tilde{A}v_\ell)\|_{L^2(\Omega)} + \|D^2_{NC}(v_\ell - \tilde{A}v_\ell)\|_{L^2(\Omega)} \\
\quad \lesssim \min_{v \in V} \|D^2_{NC}(v_\ell - v)\|_{L^2(\Omega)}.
\end{align*}
$$

\textbf{Step 3.} On any triangle $T = \text{conv}\{z_1, z_2, z_3\}$ with nodal basis functions $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{P}_0(T)$, the volume bubble function is defined as

$$
\tilde{b}_T := 2520\varphi_1^2\varphi_2^2\varphi_3^2 \in H^2_0(\text{int}(T))
$$

and satisfies $\int_T \tilde{b}_T \, dx = 1$. Define

$$
\mathcal{C}v_\ell := \tilde{A}v_\ell + \sum_{F \in \mathcal{T}_T} \left( \int_T (v_\ell - \tilde{A}v_\ell) \, dx \right) \tilde{b}_T.
$$

The difference $v_\ell - \mathcal{C}v_\ell$ is $L^2$ orthogonal to all piecewise constant functions. Since $\tilde{b}_T$ vanishes on $F \in \mathcal{T}_\ell$, $\mathcal{C}$ enjoys the integral mean property $\Pi^o D^2 \mathcal{C} = D^2_{NC}$. The fact that $\|\tilde{b}_T\|_{L^\infty(T)} \lesssim 1$ and the H"older inequality prove

$$
\left\| \int_T (v_\ell - \tilde{A}v_\ell) \, dx \tilde{b}_T \right\|_{L^2(T)} \lesssim \|v_\ell - \tilde{A}v_\ell\|_{L^2(T)}.
$$

Hence, the triangle inequality, (2.9) and inverse estimates prove the claimed error estimate for $\mathcal{C}$. 

\hfill \Box
Remark 2.7. The operator $\mathcal{E}$ maps into a discrete space, namely the sum of $V_{HCT}(T_\ell)$ and $P_0(T_\ell)$.

Corollary 2.8 (discrete Poincaré-Friedrichs inequality for Morley functions). There exists a positive constant $C_{DF}$ such that any $v_\ell \in V_\ell$ satisfies

$$
\|v_\ell\| \leq C_{DF} \text{diam}(\Omega)^2 \|v_\ell\|_{\text{NC}}.
$$

Proof. The proof follows from the triangle inequality

$$
\|v_\ell\| \leq \|v_\ell - \mathcal{E}v_\ell\| + \|\mathcal{E}v_\ell\|.
$$

The first term on the right-hand side can be bounded via (2.8) while the second term for $\mathcal{E}v_\ell \in V$ is controlled by a Poincaré-Friedrichs-type estimate and the stability of the operator $\mathcal{E}$.

2.4 $L^2$ Error Estimate for the Morley FEM

This section presents $L^2$ and best-approximation error estimates for the Morley finite element discretisation of the linear biharmonic equation. The companion operator from Subsection 2.3 allows the proof of an $L^2$ error estimate for possibly singular solutions of the biharmonic equation. Given $f \in L^2(\Omega)$, the weak formulation seeks $u \in V$ such that

$$
a(u,v) = b(f,v) \quad \text{for all } v \in V. \tag{2.10}
$$

Throughout this paper, $0 < s \leq 1$ indicates the elliptic regularity of the solution to (2.10) in the sense that $\|u\|_{H^{2+s}(\Omega)} \leq C(s) \|f\|_{L^2(\Omega)}$.

The Morley finite element discretisation of (2.10) seeks $u_\ell \in V_\ell$ such that

$$
a_{\text{NC}}(u_\ell,v_\ell) = b(f,v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{2.11}
$$

The following best-approximation is a refined version of a result of [27]. An alternative proof of the version stated here is given in [33].

Proposition 2.9 (best-approximation result). The exact solution $u$ of (2.10) and the discrete solution $u_\ell$ of (2.11) satisfy

$$
\|u - u_\ell\|_{\text{NC}} \lesssim \|(1 - \Pi_0^2) D^2 u\|_{L^2(\Omega)} + \text{osc}_2(f, T_\ell).
$$

Proof. The projection property (2.7) of the interpolation operator $I_\ell$ and the Pythagoras theorem show that

$$
\|u - u_\ell\|_{\text{NC}}^2 = \|u_\ell - I_\ell u\|_{\text{NC}}^2 + \|u - I_\ell u\|_{\text{NC}}^2.
$$

Since $\|u - I_\ell u\|_{\text{NC}} = \|(1 - \Pi_0^2) D^2 u\|$, it remains to estimate the first term on the right-hand side. Set $\varphi_\ell := u_\ell - I_\ell u$. The properties of the companion operator from Proposition 2.6 show that

$$
\|u_\ell - I_\ell u\|_{\text{NC}}^2 = a_{\text{NC}}(u_\ell - u, \varphi_\ell) = b(f, \varphi_\ell) + (1 - \Pi_0^2) D^2 u, D_{\text{NC}}^2(\mathcal{E} - 1) \varphi_\ell)_{L^2(\Omega)}.
$$

The approximation and stability properties (2.8) show that this is bounded by

$$
(\|h_\ell^2 f\| + \|(1 - \Pi_0^2) D^2 u\|) \|\varphi_\ell\|_{\text{NC}}.
$$

The efficiency $\|h_\ell^2 f\| \lesssim \|(1 - \Pi_0^2) D^2 u\| + \text{osc}_2(f, T_\ell)$ follows from the arguments of Verfürth [12], see, e.g., [24 Prop. 3.1]. This concludes the proof. \qed
Error estimates for the Morley FEM in the $L^2$ norm are well-established [32] for the case of a smooth solution $u \in V \cap H^3(\Omega)$. The smoothness enters the classical proofs in that traces of certain second-order derivatives are assumed to exist. This smoothness assumption is satisfied for the purely clamped case $\partial \Omega = \Gamma_C$ where it is known [33, 35] that $u \in H^{5/2+\varepsilon}$ for some $\varepsilon > 0$. For the more general boundary conditions considered here, this smoothness assumption is not satisfied in general. The new companion operator $C$ from Proposition 2.6 allows the proof of an $L^2$ error estimate for any $u \in V$.

**Proposition 2.10** ($L^2$ control for the linear problem). The exact solution $u$ of (2.11) and the discrete solution $u_\ell$ of (2.10) satisfy
\[
\|u - u_\ell\| \lesssim \|h_0\|_\infty (\|u - u_\ell\|_{NC} + \text{osc}_2(f, \mathcal{T}_\ell)).
\]

*Proof.* Let $e := u - u_\ell$ and let $z \in V$ denote the solution of
\[
a(z, v) = b(e, v) \quad \text{for all } v \in V.
\]

Since $\Pi^0_\ell(u_\ell - C u_\ell) = 0$ by Proposition 2.6, it holds that
\[
\|e\|^2 = b(C u_\ell - u_\ell, e) + b(e, u - C u_\ell)
= b(C u_\ell - u_\ell, (1 - \Pi^0_\ell)e) + a(z, u - C u_\ell).
\] (2.12)

Piecewise Poincaré inequalities, the discrete Friedrichs inequality [8, Thm. 10.6.12], and (2.8) lead to
\[
b(C u_\ell - u_\ell, (1 - \Pi^0_\ell)e) \lesssim \|h_0\|_\infty^2 \|e\|^2_{NC}.
\]

The second term of the right-hand side in (2.12) satisfies
\[
a(z, u - C u_\ell) = a_{NC}(z, u - u_\ell) + a_{NC}(z, u_\ell - C u_\ell).
\] (2.13)

The projection property (2.6) of $\mathcal{I}_\ell$, the problems (2.10) and (2.11), the Cauchy inequality and the approximation and stability properties (2.5) prove for the first term of the right-hand side in (2.12) that
\[
a_{NC}(z, u - u_\ell) = b(f, z - \mathcal{I}_\ell z) \lesssim \|h_0^2 f\|_{L^2(\Omega)} \|(1 - \Pi^0_\ell)D^2 z\|_{L^2(\Omega)}.
\]

The integral mean property (2.7) of $C$ and the approximation and stability properties (2.8) prove for the second term of (2.13) that
\[
a_{NC}(z, u_\ell - C u_\ell) = a_{NC}(z - \mathcal{I}_\ell z, u_\ell - C u_\ell) \lesssim \|u - u_\ell\|_{NC} \|(1 - \Pi^0_\ell)D^2 z\|_{L^2(\Omega)}.
\]

The regularity estimates of [33, 26] and the stability of the problem (2.10) prove that
\[
\|(1 - \Pi^0_\ell)D^2 z\|_{L^2(\Omega)} \lesssim \|h_0\|_{\infty}^s \|z\|_{H^{2+\varepsilon}(\Omega)} \lesssim \|h_0\|_{\infty}^s \|e\|_{L^2(\Omega)}.
\]

Efficiency estimates in the spirit of [12] show that
\[
\|h_0^2 f\|_{L^2(\Omega)} \lesssim \|u - u_\ell\|_{NC} + \text{osc}_2(f, \mathcal{T}_\ell).
\]

The combination of the foregoing estimates concludes the proof. \qed

3 Morley FEM for the Biharmonic Eigenvalue Problem

This section is devoted to the Morley finite element discretisation of the biharmonic eigenvalue problem. Subsection 3.1 describes an abstract framework for the discretisation of selfadjoint eigenproblems. Subsection 3.2 presents the finite element method along with a new $L^2$ error estimate. Error estimates for the eigenfunctions are given in Subsection 3.3–3.4.
3.1 Abstract Approximation of Eigenvalue Clusters

Let \((H, a(\cdot, \cdot))\) be a separable Hilbert space over \(\mathbb{R}\) with induced norm \(\|\cdot\|_a\) and let \(b(\cdot, \cdot)\) be a scalar product on \(H\) with induced norm \(\|\cdot\|_b\) such that the embedding \((H, \|\cdot\|_a) \hookrightarrow (H, \|\cdot\|_b)\) is compact. In the applications of this paper, \(a\) and \(b\) are the bilinear forms defined in Subsection 2.1 and, hence, no notational distinction is made for the possibly more general bilinear forms \(a, b\) in this subsection. Consider the following eigenvalue problem: Find eigenpairs \((\lambda, u) \in \mathbb{R} \times H\) with \(\|u\|_b = 1\) such that

\[
a(u, v) = \lambda b(u, v) \quad \text{for all } v \in H.
\]

(3.1)

It is well known from the spectral theory of selfadjoint compact operators [18] [30] that the eigenvalue problem (3.1) has countably many eigenvalues, which are real and positive with \(+\infty\) as only possible accumulation point. Suppose that the eigenvalues are enumerated as

\[0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots\]

and let \((u_1, u_2, u_3, \ldots)\) be some \(b\)-orthonormal system of corresponding eigenfunctions. For any \(j \in \mathbb{N}\), the eigenspace corresponding to \(\lambda_j\) is defined as

\[E(\lambda_j) := \{u \in H \mid (\lambda_j, u) \text{ satisfies (3.1)}\} = \text{span}\{u_k \mid k \in \mathbb{N} \text{ and } \lambda_k = \lambda_j\}.
\]

In the present case of an eigenvalue problem of (the inverse of) a compact operator, the spaces \(E(\lambda_j)\) have finite dimension. The discretisation of (3.1) is based on a family (over a countable index set \(I\)) of separable (not necessarily finite-dimensional) Hilbert spaces \(H_\ell\) with scalar products \(a_{\text{NC}}(\cdot, \cdot)\) and \(b_{\text{NC}}(\cdot, \cdot)\) on \(H + H_\ell\) with induced norms \(\|\cdot\|_{a,\text{NC}}\) and \(\|\cdot\|_{b,\text{NC}}\) such that \(a_{\text{NC}}\) and \(b_{\text{NC}}\) coincide with \(a\) and \(b\) when restricted to \(H\)

\[a_{\text{NC}}|_{H \times H} = a \quad \text{and} \quad b_{\text{NC}}|_{H \times H} = b.
\]

The discrete eigenvalue problem seeks eigenpairs \((\lambda_\ell, u_\ell) \in \mathbb{R} \times H_\ell\) with \(\|u_\ell\|_{b,\text{NC}} = 1\) such that

\[a_{\text{NC}}(u_\ell, v_\ell) = \lambda b_{\text{NC}}(u_\ell, v_\ell) \quad \text{for all } v_\ell \in H_\ell.
\]

(3.2)

The discrete eigenvalues can be enumerated as

\[0 < \lambda_{\ell,1} \leq \lambda_{\ell,2} \leq \lambda_{\ell,3} \ldots\]

with corresponding \(b_{\text{NC}}\)-orthonormal eigenfunctions \((u_{\ell,1}, u_{\ell,2}, u_{\ell,3}, \ldots)\). For a cluster of eigenvalues \(\lambda_{n+1}, \ldots, \lambda_{n+N}\) of length \(N \in \mathbb{N}\), define the index set \(J := \{n+1, \ldots, n+N\}\) and the spaces

\[W := \text{span}\{u_j \mid j \in J\} \quad \text{and} \quad W_\ell := \text{span}\{u_{\ell,j} \mid j \in J\}.
\]

The eigenspaces \(E(\lambda_j)\) may differ for different \(j \in J\).

Assume that the cluster is contained in a compact interval \([A, B]\) in the sense that

\[\{\lambda_j \mid j \in J\} \cup \{\lambda_{\ell,j} \mid \ell \in I, j \in J\} \subseteq [A, B].\]

This implies

\[\sup_{\ell \in I} \max_{(k,l) \in J^2} \max \left\{\lambda_{k,l}^{\ell,j} \lambda_{\ell,j}^{-1} \lambda_{k,l}^{-1}\right\} \leq B/A.
\]

(3.3)

Recall that \(\dim(H_\ell) \in \mathbb{N} \cup \{\infty\}\) and let \(J^C := \{1, \ldots, \dim(H_\ell)\} \setminus J\) denote the complement of \(J\). Assume that the cluster is separated from the remaining part of the spectrum in the sense that there exists a separation bound

\[M_J := \sup_{\ell \in I} \sup_{j \in J^C} \max_{k \in J} \frac{\lambda_k}{|\lambda_{\ell,j} - \lambda_k|} < \infty.
\]
In particular, this assumption requires that the definition of the cluster \( J \) does not split a multiple eigenvalue. Given \( f \in H \), let \( u \in H \) denote the unique solution to the linear problem
\[ a(u, v) = b(f, v) \quad \text{for all} \quad v \in H. \]
The quasi-Ritz projection \( R_\ell u \in H_\ell \) is defined as the unique solution to
\[ a_{NC}(R_\ell u, v_\ell) = b_{NC}(f, v_\ell) \quad \text{for all} \quad v_\ell \in H_\ell. \]
Let \( P_\ell \) denote the \( b_{NC} \)-orthogonal projection onto \( W_\ell \) and define
\[ \Lambda_\ell := P_\ell \circ R_\ell. \tag{3.4} \]
For any eigenfunction \( u \in W \), the function \( \Lambda_\ell u \in W_\ell \) is regarded as its approximation. This approximation does not depend on the basis of \( W_\ell \). Notice that \( \Lambda_\ell u \) is neither computable without knowledge of \( u \) nor necessarily an eigenfunction.

The following result is essentially contained in the book of [10] and in [15] for a conforming finite element discretisations. The version stated here is proven in [24].

**Proposition 3.1.** Any eigenpair \((\lambda, u) \in \mathbb{R} \times W \) of (3.1) with \( \|u\|_{b} = 1 \) satisfies
\[ \|R_\ell u - \Lambda_\ell u\|_{b, NC} \leq M_\ell \|u - R_\ell u\|_{b, NC} \quad \text{and} \quad \|u - P_\ell u\|_{b, NC} \leq \|u - \Lambda_\ell u\|_{b, NC} \leq (M_\ell + 1) \|u - R_\ell u\|_{b, NC}. \]

**Proof.** See [24].

The following algebraic identity applies frequently in the analysis. It states the important property that, although \( \Lambda_\ell u \) is no eigenfunction in general, \( \Lambda_\ell u \) satisfies an equation that is similar to an eigenfunction property.

**Lemma 3.2.** Any eigenpair \((\lambda, u) \in \mathbb{R} \times H \) of (3.1) satisfies
\[ a_{NC}(\Lambda_\ell u, v_\ell) = \lambda b_{NC}(P_\ell u, v_\ell) \quad \text{for all} \quad v_\ell \in H_\ell. \]
In other words, \( R_\ell \) and \( P_\ell \) commute, \( P_\ell \circ R_\ell = R_\ell \circ P_\ell \).

**Proof.** The proof is given in [25, Lemma 2.2].

The following theorem of [31] gives an abstract eigenvalue error estimate in case \( H_\ell \subseteq H \).

**Theorem 3.3** (Corollary 3.4 of [31]). Suppose \( H_\ell \subseteq H \) and let, for \( p \in \mathbb{N} \), \( \lambda_p \) be an eigenvalue of (3.1) with multiplicity \( q \in \mathbb{N} \), so that
\[ \lambda_{p-1} < \lambda_p = \cdots = \lambda_{p+q-1} < \lambda_{p+q} \]
(with the convention \( \lambda_0 := 0 \)) and suppose that
\[ \min_{j=1, \ldots, p-1} |\lambda_{\ell, j} - \lambda_p| \neq 0. \]
Let \( T : H \to H \) denote the solution operator of the associated linear problem, i.e., for given \( f \in H \), \( Tf \in H \) solves
\[ a(Tf, v) = b(f, v) \quad \text{for all} \quad v \in H. \]
Then, for any \( k \in \{p, \ldots, p + q - 1\} \), the following estimate holds
\[ \frac{\lambda_{\ell, k} - \lambda_p}{\lambda_{\ell, k}} \leq \left(1 + \max_{j=1, \ldots, p-1} \frac{\lambda_{\ell, j} - \lambda_p}{\lambda_{\ell, j}} \sup_{f \in \text{span}(u_1, \ldots, u_{p-1})} \frac{1}{\|f\|_{a, 1}} \right) \sup_{u \in E(\lambda_p), v \in H_\ell} \inf_{\|u\|_{a, 1} = 1} \|u - v\|_{a, 1}^2 \]
where the maximum and supremum in the parentheses are 0 for \( p = 1 \).
Remark 3.4. In this paper, the first supremum will usually be estimated through (a power of) some Friedrichs-type constant although it can be seen that in case of a finite element space $V_t$ this quantity even decays as a certain power of the maximum mesh-size.

Remark 3.5. In [31] the result of Theorem 3.3 is stated for a finite-dimensional space $H_\ell$, but it is valid even if $H_\ell$ has infinite dimension. Only the finite dimension of the eigenspaces is required. One way to see this is to trace carefully the arguments in the proof of [31]. For the reader’s convenience, another argument is given here that reduces the stated result for $\dim H_\ell = \infty$ to the finite-dimensional case. To this end, consider the finite-dimensional subspace

$$\tilde{H}_\ell := \text{span}\{u_{\ell,1}, \ldots, u_{\ell,p+q-1}, R_\ell u_{p}, \ldots, R_\ell u_{p+q-1}, R_\ell T u_{\ell,p}, \ldots, R_\ell T u_{\ell,p-1}\} \subseteq H_\ell.$$ 

The finite-dimensional space $\tilde{H}_\ell$ is constructed in such a way that the first $p+q-1$ eigenvalues $\lambda_{\ell,1}, \ldots, \lambda_{\ell,p+q-1}$ that are relevant for the statement of Theorem 3.3 are attained in $\tilde{H}_\ell$ and similarly all further quantities in the estimate are attained in this finite-dimensional space. For instance,

$$\sup_{u \in E(\lambda_p)} \inf_{v \in H_\ell} \|u - v\|^2_u = \sup_{u \in E(\lambda_p)} \|u - R_\ell u\|^2_u = \sup_{u \in \text{span}\{u_{p+q}, \ldots, u_{p+q-1}\}} \|u - R_\ell u\|^2_u$$

is realised in $\tilde{H}$. Theorem 3.3 can be employed for $\tilde{H}_\ell$ in its original version and is thereby also valid for $H_\ell$ because the claimed inequality is the same.

Remark 3.6. The conformity assumption $H_\ell \subseteq H$ is essential for the proof of Theorem 3.3 and the result may be not true in general for nonconforming approximations where $H_\ell \not\subseteq H$. Subsection 3.4 will apply Theorem 3.3 to a modified setting.

Remark 3.7. In Subsection 3.4 below, Theorem 3.3 will be applied to the case that $H_\ell := V \subseteq V_t := V + V_\ell := H$ where $V_\ell$ is a nonconforming finite element space and $V$ itself is a subspace of the enhanced space $V_t$.

Remark 3.8 (normalisation). The eigenvalue problems in this paper are based on the normalisation $\|\cdot\|_{b,\text{NC}} = 1$ and typically approximation quantities like

$$\sup_{v \in W} \inf_{v \in W_\ell} \|w - v\|^2_{a,\text{NC}}$$

arise in the analysis. To see that this quantity essentially describes the angle $\sin^2_{a,\text{NC}} \angle(W, W_\ell)$ up to some scaling, consider the expansion of $w$ in terms of the eigenfunctions of $W$. Then the eigenvalue problem implies

$$\sin^2_{a,\text{NC}} \angle(W, W_\ell) = \sup_{w \in W \setminus \{0\}} \frac{\inf_{v \in W_\ell} \|w - v\|^2_{a,\text{NC}}}{\|w\|^2_{b,\text{NC}}} \left( \frac{\|w\|^2_{b,\text{NC}}}{\|w\|^2_{a,\text{NC}}} \right)$$

$$\leq \frac{1}{\lambda_{n+1}} \sup_{u \in W} \inf_{v \in W_\ell} \frac{\|w - v\|^2_{a,\text{NC}}}{\|w\|^2_{a,\text{NC}}}$$

$$\leq \frac{\lambda_{n+1}}{\lambda_{n+1}} \sin^2_{a,\text{NC}} \angle(W, W_\ell) \leq \frac{B}{A} \sin^2_{a,\text{NC}} \angle(W, W_\ell).$$

This means that the error quantities are comparable up to a factor described by the ratio of the cluster bounds.
3.2 Morley FEM Discretisation for the Eigenvalue Problem

The weak form of the biharmonic eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times V$ with $\|u\| = 1$ such that
\[ a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V. \]  

The Morley finite element discretisation of problem (3.5) seeks $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $\|u_\ell\| = 1$ such that
\[ a_{nc}(u_\ell, v_\ell) = \lambda b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \]  

Recall the notation from Subsection 3.1 for $H = V$ and $H_\ell = V_\ell$ and the exact and discrete eigenvalues
\[ 0 < \lambda_1 \leq \lambda_2 \leq \ldots \quad \text{and} \quad 0 < \lambda_{\ell,1} \leq \cdots \leq \lambda_{\ell, \dim(V_\ell)} \]
and their corresponding $b$-orthonormal systems of eigenfunctions
\[ (u_1, u_2, u_3, \ldots) \quad \text{and} \quad (u_{\ell,1}, u_{\ell,2}, \ldots, u_{\ell, \dim(V_\ell)}). \]

The eigenvalue cluster is described by the index set $J := \{n + 1, \ldots, n + N\}$ and the spaces $W := \text{span}\{u_j \mid j \in J\}$ and $W_\ell := \text{span}\{u_{\ell,j} \mid j \in J\}$. The cluster is contained in the interval $[A, B]$. Furthermore, the following separation condition is assumed (cf. Subsection 3.1).
\[ M_J := \sup_{\ell \in J} \sup_{j \in J} \max_{k \in J} \frac{\lambda_k}{|\lambda_{\ell,j} - \lambda_k|} < \infty. \]  

**Proposition 3.9** ($L^2$ control). Provided $\|h_0\|_\infty \ll 1$, any eigenpair $(\lambda, u) \in \mathbb{R} \times W$ of (3.5) with $\|u\| = 1$ satisfies for some constant $C_L^2$ that
\[ \|u - P_\ell u\| \leq \|u - \Lambda_\ell u\| \leq C_L^2 (1 + M_J) \|h_0\|_\infty \|u - \Lambda_\ell u\|_{nc}. \]

**Proof.** The combination of Proposition 3.1 with Proposition 2.9 and Proposition 2.9 leads to
\[ \|u - \Lambda_\ell u\| \lesssim (1 + M_J) \|h_0\|_\infty (\|u - \Lambda_\ell u\|_{nc} + \text{osc2}(\lambda u, T_\ell)). \]
Provided $\|h_0\|_\infty \ll 1$, the oscillation term can be absorbed. \qed

The following proposition is based on the comparison result from Proposition 2.9 and states a best-approximation property for $\Lambda_\ell u$.

**Proposition 3.10** (best-approximation result). Provided $\|h_0\|_\infty \ll 1$, any eigenfunction $u \in W$ of (3.5) with $\|u\| = 1$ satisfies
\[ \|u - \Lambda_\ell u\|_{nc} \lesssim \|(1 - \Pi^D_{\ell})D^2 u\|_{L^2(\Omega)}. \]

**Proof.** Recall that the quasi-Ritz projection $R_\ell u$ solves (2.11) with right-hand side $f = \lambda u$. The triangle inequality proves
\[ \|u - \Lambda_\ell u\|_{nc} \leq \|u - R_\ell u\|_{nc} + \|R_\ell u - \Lambda_\ell u\|_{nc}. \]
Set $\varphi_\ell := R_\ell u - \Lambda_\ell u$. The definition of $R_\ell$ and the discrete problem (cf. Lemma 3.2) prove that
\[ \|R_\ell u - \Lambda_\ell u\|_{nc}^2 = a_{nc}(R_\ell u - \Lambda_\ell u, \varphi_\ell) = \lambda(b(u - P_\ell u, \varphi_\ell)). \]
Hence, the Cauchy and discrete Friedrichs inequalities (Corollary 2.8) and the $L^2$ control from Proposition 3.9 prove that
\[ \|R_\ell u - \Lambda_\ell u\|_{nc} \lesssim \lambda (1 + M_J) \|h_0\|_\infty \|u - \Lambda_\ell u\|_{nc}. \]
The combination of the foregoing estimates with Proposition 2.9 results in
\[ \|u - \Lambda_\ell u\|_{nc} \lesssim \|(1 - \Pi^D_{\ell})D^2 u\|_{L^2(\Omega)} + \lambda(1 + M_J) \|h_0\|_\infty \|u - \Lambda_\ell u\|_{nc} + \text{osc2}(\lambda u, T_\ell). \]
If $\|h_0\|_\infty \ll 1$ is sufficiently small, the higher-order terms on the right-hand side can be absorbed. \qed
3.3 A Nonstandard Quasi-Ritz Projection

This subsection introduces the setting which is necessary for the eigenvalue estimates of Subsection 3.4.

Define $V_\ell := V + V_\ell$ as the sum of the continuous and the discrete space. Given $f \in V$, let $u \in V$ denote the solution to (2.10), namely

$$a(u,v) = b(f,v) \quad \text{for all } v \in V.$$  

The quasi-Ritz projection $\hat{R}_\ell u \in \hat{V}_\ell$ is defined as the solution of

$$a_{NC}(\hat{R}_\ell u, \hat{v}_\ell) = b(f, \hat{v}_\ell) \quad \text{for all } \hat{v}_\ell \in \hat{V}_\ell.$$  

**Remark 3.11.** This definition corresponds to the definition of $R_\ell$ of Subsection 3.2 with $H_\ell$ replaced by $\hat{V}_\ell$. It should be emphasised that in the present case there is an inclusion $V \subseteq \hat{V}_\ell$. This is an admissible choice in the framework of Subsection 3.1.

This setting leads to a new view on nonconforming finite element schemes in the following sense: Both $V$ and $V_\ell$ are subspaces of the space $\hat{V}_\ell$ and the solutions $u \in V$ and $u_\ell \in V_\ell$ of (2.10) and (2.11) are “conforming approximations” of $R_\ell u$. To the best of the author’s knowledge, this is a new approach to nonconforming finite elements that has not been studied in the existing literature.

It is crucial that the nonconforming interpolation operator $I_\ell$ is defined on $\hat{V}_\ell$ as well as $\hat{V}_{\ell+m} = V + V_{\ell+m}$ with respect to a refined triangulation $\mathcal{J}_{\ell+m}$. This operator and the conforming companion operator $\mathcal{C}$ from Proposition 2.6 establish suitable connections between the spaces $V$, $V_\ell$, $\hat{V}_\ell$, $\hat{V}_{\ell+m}$ and $\hat{V}_{\ell+m}$. Those two operators displayed in Figure 2 are the core of the analysis of $\hat{R}_\ell$ which is essential to derive eigenvalue error estimates.

The following proposition gives an $L^2$ error estimate for the quasi-Ritz projection $\hat{R}_\ell$.

**Proposition 3.12** ($L^2$ error estimate for $\hat{R}_\ell$). Let $u \in V$ solve the linear problem (2.10) with right-hand side $f \in V$. Then, $\hat{R}_\ell u$ satisfies the following $L^2$ error estimate

$$\|u - \hat{R}_\ell u\| \lesssim \|h_\ell\| \|u - \hat{R}_\ell u\|_{NC}.$$  

**Remark 3.13.** The conformity $V \subseteq \hat{V}_\ell$ shows that $u$ is the $a_{NC}$-orthogonal projection of $\hat{R}_\ell u$ onto $V$. Therefore, one may think of using a standard duality argument for the proof of the $L^2$ error control. Indeed, this procedure can be applied, but it will not immediately lead to a right-hand side that is explicit in the mesh-size $\|h_\ell\|_{\infty}$. Therefore, the proof of Proposition 3.12 employs a different technique based on the operators $\mathcal{J}_\ell$ and $\mathcal{C}$ to obtain an estimate in terms of $\|h_\ell\|_{\infty}$.

**Proof of Proposition 3.12** Set $\hat{e} := u - \hat{R}_\ell u$ and let $z \in V$ denote the solution to

$$a(z,w) = b(\hat{e},w) \quad \text{for all } w \in V.$$  

Figure 2: Mappings between the spaces $\hat{V}_\ell, \hat{V}_{\ell+m}, V, V_\ell$ and $V_{\ell+m}; \iota$ is the inclusion.
Proposition 3.14.

Let \( u \in V \) solve (2.10) with right-hand side \( f \in V \). Then the quasi-Ritz projection \( \hat{R}_\ell u \) satisfies

\[
\|u - \hat{R}_\ell u\|_{NC} \lesssim \|(1 - \Pi^0)D^2u\|_{L^2(\Omega)} + \text{osc}_2(f, \mathcal{T}_\ell).
\]

Proof. The triangle inequality shows for the nonconforming interpolation operator \( I_\ell \) that

\[
\|u - \hat{R}_\ell u\|_{NC} \leq \|\hat{R}_\ell u - I_\ell u\|_{NC} + \|u - I_\ell u\|_{NC}.
\]

Since \( \|u - I_\ell u\|_{NC} = \|(1 - \Pi^0)D^2u\| \) by the projection property (2.5), it remains to estimate the first term on the right-hand side. Set \( \hat{\varphi}_\ell := \hat{R}_\ell u - I_\ell u \). The definition of \( \hat{R}_\ell \), the projection property (2.5) and the properties of the companion operator from Proposition 2.9 yield

\[
\|\hat{R}_\ell u - I_\ell u\|_{NC}^2 = a_{NC}(\hat{R}_\ell u - I_\ell u, \hat{\varphi}_\ell) = b(f, \hat{\varphi}_\ell) = a_{NC}(u, \mathcal{T}_\ell \hat{\varphi}_\ell) = b(f, \hat{\varphi}_\ell - \mathcal{C}_{I_\ell} \hat{\varphi}_\ell) - a_{NC}(u, (1 - \mathcal{C})\mathcal{T}_\ell \hat{\varphi}_\ell).
\]

The triangle inequality and the approximation and stability properties (2.5) and (2.8) show for the first term that

\[
b(f, \hat{\varphi}_\ell - \mathcal{C}_{I_\ell} \hat{\varphi}_\ell) \lesssim \|h^2 f\| \|\hat{\varphi}_\ell\|_{NC}.
\]

The known efficiency

\[
\|h^2 f\| \lesssim \|(1 - \Pi^0)D^2u\| + \text{osc}_2(f, \mathcal{T}_\ell)
\]

follows from the arguments of Verfürth [42].

The projection property (2.7) of \( \mathcal{C} \) and (2.8) reveal

\[
a_{NC}(u, (1 - \mathcal{C})\mathcal{T}_\ell \hat{\varphi}_\ell) = ((1 - \Pi^0)D^2u, D_{NC}^2(1 - \mathcal{C})\mathcal{T}_\ell \hat{\varphi}_\ell)_{L^2(\Omega)}.
\]

This and the stability properties (2.5) and (2.8) conclude the proof.

The next proposition states that the error \( u - \hat{R}_\ell u \) in the energy norm is comparable with the best-approximation of \( Du \) by piecewise constants.

Proposition 3.14 (comparison for \( \hat{R}_\ell \)). Let \( u \in V \) solve (2.10) with right-hand side \( f \in V \). Then the quasi-Ritz projection \( \hat{R}_\ell u \) satisfies

\[
\|u - \hat{R}_\ell u\|_{\text{osc}} \lesssim \|(1 - \Pi^0)D^2u\|_{L^2(\Omega)} + \text{osc}_2(f, \mathcal{T}_\ell).
\]

Proof. The triangle inequality shows for the nonconforming interpolation operator \( I_\ell \) that

\[
\|u - \hat{R}_\ell u\|_{\text{osc}} \leq \|\hat{R}_\ell u - I_\ell u\|_{\text{osc}} + \|u - I_\ell u\|_{\text{osc}}.
\]

With the companion operator \( \mathcal{C} \) from Proposition 2.6 and the interpolation operator \( I_\ell \), it follows that

\[
\|\mathcal{C} e\|^2 = b((1 - \mathcal{C})I_\ell e, \mathcal{C} e) + b((1 - I_\ell)\mathcal{C} e, e) + b(\mathcal{C} I_\ell e, e).
\]

The Cauchy inequality and the error estimates (2.5) and (2.8) bound the first two terms on the right-hand side as

\[
b((1 - \mathcal{C})I_\ell e, \mathcal{C} e) + b((1 - I_\ell)\mathcal{C} e, e) \lesssim \|h_0\|_{\text{osc}} \|\mathcal{C} e\|_{NC} \|e\|.
\]

Since \( a(z, \mathcal{C} e) = a(\mathcal{C} e, z) = a(u - \hat{R}_\ell u, z) = 0 \) by the definition of \( \hat{R}_\ell \), the remaining term of (3.8) satisfies

\[
b(\mathcal{C} I_\ell e, e) = a(z, \mathcal{C} e) = a_n(z, (I_\ell - 1)e) + a_{nc}(z, (\mathcal{C} - 1)\mathcal{C} e).
\]

The projection properties (2.5) and (2.7) imply that \( D_{nc}^2(I_\ell - 1)\mathcal{C} e \) as well as \( D_{nc}^2(\mathcal{C} - 1)\mathcal{C} e \) are \( L^2 \)-orthogonal onto piecewise constants. This and the elliptic regularity show that

\[
a_{nc}(z, (I_\ell - 1)e) + a_{nc}(z, (\mathcal{C} - 1)\mathcal{C} e) = \|(1 - \mathcal{C}^0)D^2z, D_{nc}^2(I_\ell - 1)\mathcal{C} e\|_{L^2(\Omega)} + \|(1 - \Pi^0)D^2z, D_{nc}^2(\mathcal{C} - 1)\mathcal{C} e\|_{L^2(\Omega)}
\]

\[
\lesssim \|h_0\|_{\text{osc}} \|z\|_{H^{2+\alpha}(\Omega)} \|\mathcal{C} e\|_{NC} \lesssim \|h_0\|_{\text{osc}} \|e\|_{NC}.
\]

The combination of the above estimates concludes the proof. \( \square \)
3.4 Eigenvalue Error Estimates

This section extends the results of the foregoing subsection to eigenvalue problems. This leads to eigenvalue error estimates for the Morley finite element method.

Note that \( \hat{V} \) equipped with the scalar product \( a_{\text{sc}} \) is a Hilbert space. The space \( \hat{V} \) is a subspace of the finite product \( H^2(\mathcal{T}_\ell) := \prod_{T \in \mathcal{T}_\ell} H^2(\text{int}(T)) \) and the embedding \( (\hat{V}, \| \cdot \|_{\text{sc}}) \to (L^2(\Omega), \| \cdot \|) \) is compact for a fixed triangulation \( \mathcal{T}_\ell \) (for more details on such broken Sobolev spaces see [9]). Hence, the eigenvalue problem

\[
 a_{\text{sc}}(\hat{u}_\ell, \hat{v}_\ell) = \lambda_\ell b(\hat{u}_\ell, \hat{v}_\ell) \quad \text{for all } \hat{v}_\ell \in \hat{V}_\ell
\]  

has a countable and discrete spectrum

\[
 0 < \hat{\lambda}_{\ell,1} \leq \hat{\lambda}_{\ell,2} \leq \cdots
\]

with corresponding \( b \)-orthonormal eigenfunctions \((\hat{u}_{\ell,1}, \hat{u}_{\ell,2}, \ldots)\). For an eigenvalue cluster described by the index set \( J = \{ n + 1, \ldots, n + N \} \), the set \( \hat{W}_\ell := \text{span}\{\hat{u}_{\ell,j} \mid j \in J\} \) describes the corresponding invariant subspace with the \( L^2 \)-projection \( \hat{P}_\ell \) onto \( \hat{W}_\ell \) and let \( \hat{\Lambda}_\ell := \hat{P}_\ell \circ \hat{R}_\ell \).

The eigenvalue problem (3.9) is related to the (inverse of) a compact operator for each triangulation \( \mathcal{T}_\ell \). The first important observation is that the spectrum is robust under mesh-refinement.

**Proposition 3.15.** Let \( (\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0} \) be a sequence of nested triangulations with \( \| h_0 \|_\infty \ll 1 \). Then any \( j \in \mathbb{N} \) and the constant \( C \) from the estimate in (2.5) satisfy

\[
\frac{\lambda_{\ell,j}}{1 + C \| h_\ell \|_\infty \lambda_{\ell,j}} \leq \hat{\lambda}_{\ell,j} \lambda_{\ell,j}.
\]  

In particular, if \( \| h_\ell \|_\infty \to 0 \) as \( \ell \to \infty \), one has convergence \( \hat{\lambda}_{\ell,j} \to \lambda_j \).

**Proof.** The min-max principle [13] shows, for any \( j \in \mathbb{N} \), that

\[
\hat{\lambda}_{\ell,j} \leq \min \{ \lambda_j, \lambda_{\ell,j} \}.
\]

An application of the methodology of [12] Thms. 1–2] yields the lower eigenvalue bound in case that \( \| h_\ell \|_\infty \) is sufficiently small

\[
\frac{\lambda_{\ell,j}}{1 + C \| h_\ell \|_\infty \lambda_{\ell,j}} \leq \hat{\lambda}_{\ell,j}
\]  

for some constant \( C \approx 1 \). In fact, the arguments from [12] can be applied in this modified setting because the Morley interpolation operator \( \mathcal{I}_\ell \) is defined for functions in \( \hat{V}_\ell \) and satisfies the projection property (2.5) and the approximation and stability property (2.6).

Altogether one has the two-sided estimate (3.10). This implies the convergence \( |\lambda_{\ell,j} - \lambda_{\ell,j}| \to 0 \) as \( \ell \to \infty \). The triangle inequality and the a priori estimates of [12] prove \( \hat{\lambda}_{\ell,j} \to \lambda_j \).

The robustness implies the following separation bound.

**Corollary 3.16.** Provided \( \| h_0 \|_\infty \ll 1 \), there exists a separation constant for the cluster \( J \) in the sense that

\[
\bar{M}_J := \sup_{\mathcal{T}_\ell \in \mathcal{T}, j \in J} \max \max \left\{ \frac{\hat{\lambda}_{\ell,k}}{|\lambda_j - \hat{\lambda}_{\ell,k}|}, \frac{\hat{\lambda}_{\ell,\ell}}{|\lambda_{\ell,j} - \hat{\lambda}_{\ell,\ell}|}, \frac{\lambda_k}{|\lambda_j - \lambda_k|}, \frac{\lambda_{\ell,j}}{|\lambda_{\ell,j} - \lambda_k|} \right\} < \infty.
\]  

This formula uses the convention \( \lambda_{\ell,j} := \lambda_{\ell,\dim(V_\ell)} \) for \( j > \dim(V_\ell) \).
Remark 3.17. The separation condition \((3.11)\) implies \((3.7)\) with \(M_J \leq \tilde{M}_J\).

This separation constant allows the use of the framework of Subsection \((3.1)\) where the space \(V\) is approximated by \(\tilde{V}_\ell\).

**Proposition 3.18** \((L^2)\) error estimate for \(\tilde{\Lambda}_\ell\). Provided \(\|h_0\|_{\infty} \ll 1\), any eigenpair \((\lambda, u) \in \mathbb{R} \times W\) of \((3.30)\) with \(\|u\| = 1\) satisfies
\[
\|u - \Lambda_\ell u\| + \|u - \tilde{\Lambda}_\ell u\| \lesssim (1 + \tilde{M}_J)\|h_0\|_{\infty}^2\|(1 - \Pi^0_\ell)D^2u\|.
\]

**Proof.** An immediate consequence of Proposition \((3.1)\) (where \(H_\ell\) is replaced by \(\tilde{V}_\ell\) and \(\Lambda_\ell\) is replaced by \(\tilde{\Lambda}_\ell\)) and Proposition \((3.14)\) reads
\[
\|u - \tilde{\Lambda}_\ell u\| \leq (1 + \tilde{M}_J)\|u - \tilde{\Lambda}_\ell u\| \lesssim (1 + \tilde{M}_J)\|h_0\|_{\infty}^2\|(1 - \Pi^0_\ell)D^2u\|.\]

Proposition \((3.9)\) the best approximation result of Proposition \((3.10)\) and \(M_J \leq \tilde{M}_J\) imply
\[
\|u - \Lambda_\ell u\| \leq C_{L^2}(1 + \tilde{M}_J)\|h_0\|_{\infty}^2\|(1 - \Pi^0_\ell)D^2u\|.
\]

The sum of the preceding two displayed formulas concludes the proof: Since \(\|h_0\|_{\infty} \ll 1\), the oscillation term \(\text{osc}_2(\lambda u, T_\ell)\) can be absorbed. \(\square\)

The next result states that the error of the eigenfunction approximation \(\tilde{\Lambda}_\ell u\) in \(\tilde{V}_\ell\) is comparable with the best-approximation of the Hessian by piecewise constants.

**Proposition 3.19** \((\text{comparison result for } \tilde{\Lambda}_\ell)\). Provided \(\|h_0\|_{\infty} \ll 1\), any eigenpair \((\lambda, u) \in \mathbb{R} \times W\) of \((3.30)\) with \(\|u\| = 1\) satisfies
\[
\|(1 - \tilde{\Lambda}_\ell)u\|_{\text{SC}} \lesssim \|(1 - \Pi^0_\ell)D^2u\|.
\]

**Proof.** The triangle inequality gives
\[
\|(1 - \tilde{\Lambda}_\ell)u\|_{\text{SC}} \leq \|(1 - \tilde{\Lambda}_\ell)u\|_{\text{SC}} + \|(\tilde{\Lambda}_\ell - \tilde{\Lambda}_\ell)u\|_{\text{SC}}.
\]

Proposition \((3.14)\) implies that the first term on the right-hand side is controlled by \(\|(1 - \Pi^0_\ell)D^2u\|\). Set \(\tilde{\varphi}_\ell := (\tilde{\Lambda}_\ell - \tilde{\Lambda}_\ell)u\). The definition of \(\tilde{\Lambda}_\ell\) (note that the right-hand side is \(f := \lambda u\)) and Lemma \((3.2)\) (with \(H_\ell\) replaced by \(\tilde{V}_\ell\)) lead to
\[
\|(\tilde{\Lambda}_\ell - \tilde{\Lambda}_\ell)u\|_{\text{SC}} \leq \text{osc}_2((\tilde{\Lambda}_\ell - \tilde{\Lambda}_\ell)u, \tilde{\varphi}_\ell) = \lambda(b(u - \tilde{\varphi}_\ell, \tilde{\varphi}_\ell) \leq \lambda\|u - \tilde{\varphi}_\ell\| \|	ilde{\varphi}_\ell\|.
\]

The discrete Friedrichs inequality (Corollary \((2.8)\)) shows that \(\|\tilde{\varphi}_\ell\| \lesssim \|	ilde{\varphi}_\ell\|_{\text{SC}}\). The \(L^2\) error estimate from Proposition \((3.18)\) concludes the proof. Indeed, the resulting higher-order term \(\|(1 + \tilde{M}_J)\|h_0\|_{\infty}^2\|(1 - \tilde{\Lambda}_\ell)u\|_{\text{SC}}\) can be absorbed for \(\|h_0\|_{\infty} \ll 1\). \(\square\)

The tools developed in this section lead to the following eigenvalue error estimate.

**Theorem 3.20** \((\text{eigenvalue error estimates})\). Provided \(\|h_0\|_{\infty} \ll 1\), it holds that
\[
\max_{j \in J} \frac{|\lambda_j - \tilde{\lambda}_\ell|}{\max\{\lambda_j, \tilde{\lambda}_\ell\}} \lesssim (1 + \tilde{M}_J^2B^2)\sin^2(\omega_{\text{NC}}) \lesssim (W, W_\ell)
\]
\[
\lesssim (1 + \tilde{M}_J^2B^2) \sup_{u \in W, \|u\|_{\text{SC}} = 1} \|(1 - \Pi^0_\ell)D^2u\|_{L^2(\Omega)}^2.
\]

The proof of Theorem \((3.20)\) requires the following Lemma with the constant \(C_{dF}\) from the discrete Friedrichs inequality of Corollary \((2.8)\).
Lemma 3.21. The separation condition (3.11) from Corollary 3.16 implies
\[
\max_{j \in J} \frac{|\lambda_j - \lambda_{\ell,j}|}{\max\{\lambda_j, \lambda_{\ell,j}\}} \leq 2(1 + \overline{M}^2 B^2 C_{dF}^4) \left( \sin^2_{a,NC} \angle (W, \overline{W}_\ell) + \sin^2_{a,NC} \angle (W, W_\ell) \right).
\]

Proof. Notice that, in contrast to the case of conforming finite element methods, the sign of \(\lambda_j - \lambda_{\ell,j}\) is not known in the present case of nonconforming methods.

The min-max principle and Theorem 3.3 (where \(H\) is replaced by \(\overline{V}_\ell\) and \(H_\ell\) is replaced by \(V\)) prove
\[
\lambda_j - \lambda_{\ell,j} \leq \lambda_j - \hat{\lambda}_{\ell,j} \leq \lambda_j(1 + \overline{M}^2 B^2 C_{dF}^4) \sin^2_{a,NC} \angle (W, W_\ell).
\]  (3.12)

Here, Theorem 3.3 has been applied to the case that the eigenvalues in \(V\) are Ritz values of the eigenvalues in \(\overline{V}_\ell\). Notice carefully that Theorem 3.3 does not require a finite dimension of the “approximating” subspace (in this case \(V\)) as pointed out in Remark 3.9.

Since the eigenvalue cluster \(J\) is finite and, therefore, the spaces \(\overline{W}_\ell\) and \(W_\ell\) have equal finite dimension, the identity (2.24) implies that
\[
\sin^2_{a,NC} \angle (W, W_\ell) = \sin^2_{a,NC} \angle (W, \overline{W}_\ell).
\]

In order to bound the modulus \(|\lambda_j - \hat{\lambda}_{\ell,j}|\), consider also the reverse sign. Notice that the nonconforming finite element space \(V_\ell\) acts as a conforming subspace of \(\overline{V}_\ell\). The min-max principle and Theorem 3.3 (where \(H\) is replaced by \(\overline{V}_\ell\)) then prove
\[
\lambda_{\ell,j} - \lambda_{\ell,j} \leq \lambda_{\ell,j} - \hat{\lambda}_{\ell,j} \leq \lambda_{\ell,j}(1 + \overline{M}^2 B^2 C_{dF}^4) \sin^2_{a,NC} \angle (W, \overline{W}_\ell).
\]

The formulas (2.21)–(2.22) imply
\[
\sin^2_{a,NC} \angle (W, W_\ell)/2 \leq \sin^2_{a,NC} \angle (W, \overline{W}_\ell) + \sin^2_{a,NC} \angle (W, W_\ell) = \sin^2_{a,NC} \angle (W, \overline{W}_\ell) + \sin^2_{a,NC} \angle (W, W_\ell).
\]

Proof of Theorem 3.20. For any \(j \in J\), Lemma 3.21 implies
\[
\max\{\lambda_j, \lambda_{\ell,j}\} \leq 2(1 + \overline{M}^2 B^2 C_{dF}^4) \left( \sin^2_{a,NC} \angle (W, \overline{W}_\ell) + \sin^2_{a,NC} \angle (W, W_\ell) \right).
\]

Proposition 3.19 shows
\[
\sin^2_{a,NC} \angle (W, \overline{W}_\ell) \leq \sin^2_{a,NC} \angle (W, W_\ell).
\]

This proves the first stated inequality. The second inequality follows from Proposition 3.10.

Remark 3.22. Similar eigenvalue error estimates can be proven for the nonconforming \(P_1\) finite element method for the eigenvalues of the Laplacian or the Stokes operator with the operators described in [24]. The error estimates of [10] for the eigenvalues of the Laplacian are based on a different methodology. The authors make use of a conforming \(P_1\) subspace which makes a generalization to the Stokes or the biharmonic eigenvalue problem appear difficult. On the other hand, they require less restrictions on the initial mesh-size.

4 Adaptive Finite Element Method

As an application of the \(L^2\) and eigenvalue error estimates developed in the foregoing sections, this section presents optimal convergence rates for the adaptive Morley FEM for eigenvalue clusters.
4.1 Adaptive Algorithm and Optimal Convergence Rates

This subsection introduces the adaptive algorithm and states the optimality result.

For any triangle $T \in \mathcal{T}_\ell$, the explicit residual-based error estimator consists of the sum of the residuals of the computed discrete eigenfunctions $(u_{\ell,j})_{j \in J}$,

$$
\eta_\ell^2(T) := \sum_{j \in J} \left( h_T^2 \| \lambda_{\ell,j} u_{\ell,j} \|^2_{L^2(T)} + \sum_{F \in \mathcal{F}(T) \cap \mathcal{F}(\Omega \setminus \Gamma_S)} h_T \| [D^2_{nc} u_{\ell,j}] F_T \|_{L^2(F)}^2 \right) + \sum_{F \in \mathcal{F}(T) \cap \mathcal{F}(\Gamma_S)} h_T \| ([D^2_{nc} u_{\ell,j}] F_T F_T)^\top \cdot \tau_F \|_{L^2(F)}^2).
$$

Let, for any subset $\mathcal{K} \subseteq \mathcal{T}$,

$$
\eta_\ell^2(\mathcal{K}) := \sum_{T \in \mathcal{K}} \eta_\ell^2(T).
$$

This type of error estimator was introduced by [2] [3] and [28] for linear problems. The methodology to consider the sum of the residuals of the computed eigenfunctions was first employed in [20] for the case of a multiple eigenvalue.

The adaptive algorithm is driven by this computable error estimator and runs the following loop.

**Algorithm 4.1** (AFEM for the biharmonic eigenvalue problem).

**Input:** Initial triangulation $\mathcal{T}_0$, bulk parameter $0 < \theta < 1$.

**for** $\ell = 0, 1, 2, \ldots$

Solve. Compute discrete eigenpairs $(\lambda_{\ell,j}, u_{\ell,j})_{j \in J}$ of (3.6) with respect to $\mathcal{T}_\ell$.

Estimate. Compute local contributions of the error estimator $(\eta_\ell^2(T))_{T \in \mathcal{T}_\ell}$.

Mark. Choose a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \eta_\ell^2(\mathcal{M}_\ell) \leq \eta_\ell^2(\mathcal{M}_\ell)$.

Refine. Generate $\mathcal{T}_{\ell+1}$ from $\mathcal{T}_\ell$ and $\mathcal{M}_\ell$ with newest-vertex bisection [4, 39].

**end for**

**Output:** Triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $((\lambda_{\ell,j}, u_{\ell,j})_{j \in J})_\ell$.

Let, for any $m \in \mathbb{N}$, the set of triangulations in $\mathcal{T}$ whose cardinality differs from that of $\mathcal{T}_0$ by $m$ or less be denoted by

$$
\mathcal{T}(m) := \{ \mathcal{T} \in \mathcal{T} \mid \text{card}(\mathcal{T}) - \text{card}(\mathcal{T}_0) \leq m \}.
$$

Define the seminorm

$$
|u|_{A_\sigma} := \sup_{m \in \mathbb{N}} m^\sigma \inf_{\tau \in \mathcal{T}(m)} \|(1 - \Pi^0_\tau) D^2 u\|
$$

and the approximation class

$$
A_\sigma := \left\{ v \in V \mid |v|_{A_\sigma} < \infty \right\}.
$$

The set $A_\sigma$ does not depend on the finite element method and instead concerns the approximability of the Hessian by piecewise constant functions. The following alternative set, also referred to as approximation class, is employed in the analysis of the optimal convergence rates

$$
A_\sigma^{\text{Morley}} := \left\{ u \in V \mid |u|_{A_\sigma^{\text{Morley}}} < \infty \right\}
$$

for

$$
|u|_{A_\sigma^{\text{Morley}}} := \sup_{m \in \mathbb{N}} m^\sigma \inf_{\tau \in \mathcal{T}(m)} \| u - \Lambda_{\sigma} u \|.
$$

Proposition 3.10 establishes the equivalence of those two approximation classes in the sense that any eigenfunction $u \in W$ satisfies $u \in A_\sigma$ if and only if $u \in A_\sigma^{\text{Morley}}$. The following theorem states optimality of Algorithm 4.1. The proof will be outlined throughout the remaining parts of this section.
Theorem 4.2 (optimal convergence rates). Let \( \Omega \) be simply-connected. Provided the bulk parameter \( \theta \ll 1 \) and the initial mesh-size \( |h_0|_\infty \ll 1 \) are sufficiently small, Algorithm 4.1 computes triangulations \((T_\ell)\) and discrete eigenpairs \((\lambda_{\ell,j},\mathbf{u}_{\ell,j})_{j\in J}\) with optimal rate of convergence in the sense that, for some constant \( C_{\text{opt}} \),

\[
\sup_{\ell \in \mathbb{N}} \left( \text{card}(T_\ell) - \text{card}(T_0) \right)^{\sigma} \left( \sum_{j \in J} \| u_j - \Lambda_{\ell} u_j \|_{\hat{H}^1_{\text{NC}}} \right)^{1/2} \leq C_{\text{opt}} \left( \sum_{j \in J} | u_j |_{\hat{A}_{\text{Morley}}}^2 \right)^{1/2}.
\]

Remark 4.4. In other words, the functions of \( \mathfrak{X}(T_\ell) \) satisfy that \( \partial (\psi \cdot \nu) / \partial \tau = 0 \) on \( \Gamma_S \cup \Gamma_F \) and \( (D\psi \tau) \cdot \tau \) is constant on each connectivity component of \( \Gamma_F \). The definition of \( \mathfrak{X}(T_\ell) \) above is stated in such a way that one can see that this defines \( \text{card}(T_\ell(\Gamma_S \cup \Gamma_F)) \) and \( \text{card}(N_\ell(\Gamma_F)) \) linearly independent constraints on \( P_1(\mathbb{R}^2) \cap \hat{H}^1(\Omega; \mathbb{R}^2) \). Recall that \( \Gamma_C \) and \( \Gamma_C \cup \Gamma_S \) are assumed to be closed sets and, thus, \( N_\ell(\Gamma_F) \) contains exactly those vertices that are shared by two edges of \( \Gamma_F \).

Theorem 4.5 (discrete Helmholtz decomposition for piecewise constant symmetric tensor fields). Let \( \Omega \) be simply-connected. Given any piecewise constant symmetric tensor field \( \sigma_\ell \in \mathfrak{P}_0(\Omega; \mathbb{S}) \), there exist unique \( \phi_\ell \in V_\ell \) and \( \psi_\ell \in \mathfrak{X}(T_\ell) \) such that

\[
\sigma_\ell = D^2_{\text{NC}} \phi_\ell + \text{sym Curl} \psi_\ell.
\]  

The decomposition is \( L^2 \) orthogonal and the functions \( \phi_\ell, \psi_\ell, \sigma_\ell \) from (4.1) satisfy

\[
\| D^2_{\text{NC}} \phi_\ell \|_{L^2(\Omega)} + \| \text{Curl} \psi_\ell \|_{L^2(\Omega)} \lesssim \| \sigma_\ell \|_{L^2(\Omega)}.
\]
The proof of this formula follows from the well-known Euler formulae (for two space dimensions and simply-connected domains; the proof follows from mathematical induction)

\[ \text{card}(\mathcal{N}_\ell) + \text{card}(\mathcal{T}_\ell) = 1 + \text{card}(\mathcal{I}_\ell) \quad \text{and} \quad 2 \text{card}(\mathcal{T}_\ell) + 1 = \text{card}(\mathcal{N}_\ell) + \text{card}(\mathcal{I}_\ell(\Omega)). \]

The proof of the stability [4.2] is proven in [14] Lemma 3.3.

The remaining parts of this subsection prove the discrete reliability for a theoretical error estimator. The idea to include such a non-computable quantity in the analysis of adaptive algorithms was first introduced in [20] in the context of multiple eigenvalues. The theoretical error estimator does not depend on the choice of the discrete eigenfunctions. Given an eigenpair \((\lambda, u)\), the error estimator is defined, for any \(T \in \mathcal{T}_\ell\), as

\[
\mu^2(T, \lambda, u) := \sum_{j \in J} \left( h_T^2 \|\lambda P_T u\|_{L^2(T)}^2 + \sum_{F \in \mathcal{T}(T) \cap \mathcal{T}_\ell(\Omega \cup \Gamma_C)} h_T \|[(D^{2}_{\text{sym}} \Lambda u)]_{F \cap F_T} \|^2_{L^2(F)} \right) + \sum_{F \in \mathcal{T}(T) \cap \mathcal{T}_\ell(\Gamma_S)} h_T \|[(D^{2}_{\text{sym}} \Lambda u)]_{F \cap F_T} \cdot \tau_F \|^2_{L^2(F)}.
\]

Define, for any subset \(\mathcal{K} \subseteq \mathcal{T}_\ell\),

\[
\mu^2(\mathcal{K}, \lambda_j, u_j) := \sum_{T \in \mathcal{K}} \mu^2(T, \lambda_j, u_j) \quad \text{and} \quad \mu^2(\mathcal{K}) := \sum_{j \in J} \mu^2(T, \lambda_j, u_j).
\]

The following shorthand notation for higher-order terms with respect to an eigenpair \((\lambda, u) \in \mathbb{R} \times W\) of (3.5) is employed throughout this section

\[
r_{\ell,m} := \|h_0\|_{\infty}\lambda(1 + M_J)C_{L^2} \sqrt{\|u - \Lambda u\|^2 + \|u - \Lambda_{\ell+m} u\|^2}.
\]

The following Lemma carefully explores the properties of the quasi-interpolation of [37].

**Lemma 4.6** (Scott-Zhang quasi-interpolation). Let \(\mathcal{T}_{\ell+m}\) be a refinement of \(\mathcal{T}_\ell\) and let \(\psi_{\ell+m} \in \mathcal{P}_1(\mathcal{T}_{\ell+m}; \mathbb{R}^2) \cap H^1(\Omega; \mathbb{R}^2)\) be such that \((D\psi_{\ell+m}) \cdot \nu = 0\) on \(\Gamma_S \cup \Gamma_F\) and \((D\psi_{\ell+m}) \cdot \nu \) is constant on each connectivity component of \(\Gamma_F\). Then there exists \(\psi_{\ell} \in \mathcal{P}_1(\mathcal{T}_\ell; \mathbb{R}^2) \cap H^1(\Omega; \mathbb{R}^2)\) with the property that \(\psi_{\ell,F} = \psi_{\ell+m}|_F\) for all edges \(F \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}\). Moreover, the function \(\psi_{\ell}\) can be chosen in such a way that it preserves the boundary conditions in the sense that \((D\psi_{\ell}) \cdot \nu = 0\) on \(\Gamma_S \cup \Gamma_F\) and \((D\psi_{\ell}) \cdot \nu \) is constant on each connectivity component of \(\Gamma_F\). This quasi-interpolation satisfies the approximation and stability estimate

\[
\|h^{-1}_{\ell}(\psi_{\ell+m} - \psi_{\ell})\|_{L^2(\Omega)} + \|D(\psi_{\ell+m} - \psi_{\ell})\|_{L^2(\Omega)} \lesssim \|D\psi_{\ell+m}\|_{L^2(\Omega)}.
\]

**Remark 4.7.** The quasi-interpolation of Lemma 4.6 preserves the boundary conditions imposed on the space \(\mathcal{X}(\mathcal{T}_{\ell+m})\) for any refinement \(\mathcal{T}_{\ell+m}\).

**Proof of Lemma 4.6.** The methodology of [37] assigns to each vertex \(z \in \mathcal{N}_\ell\) some edge \(F_z \in \mathcal{T}_\ell\). The choice assigns, whenever possible, to a vertex \(z \in \mathcal{N}_\ell\) an edge \(F_z \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}\). For vertices \(z \in \mathcal{T}_F\) that touch the free boundary, choose \(F_z \in \mathcal{T}_F(\Gamma_F)\) if this does not contradict a possible choice of \(F_z \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}\). Let, for any edge \(F_z \in \mathcal{T}_\ell\), \(\Phi_z \in L^2(F_z)\) denote the Riesz representation of the point evaluation \(\delta_z\) at \(z\) in the space \(\mathcal{X}(F_z)\).
For vertices that touch the simply supported part of the boundary but not the free part \( z \in \Gamma_S \setminus \Gamma_F \) and that do not belong to any edge of \( \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1} \), denote the adjacent boundary edges by \((F_1, F_2) \in \mathcal{T}_F^2\) and define

\[
\nu_{F_1} \cdot \psi_{\ell}(z) = \int_{F_1} \Phi_z \nu_{F_1} \cdot \psi_{\ell+1} \, ds \quad \text{and} \quad \nu_{F_2} \cdot \psi_{\ell}(z) = \int_{F_2} \Phi_z \nu_{F_2} \cdot \psi_{\ell+1} \, ds.
\]

If the angle between \( F_1 \) and \( F_2 \) equals \( \pi \), then \( \nu_{F_1} = \nu_{F_2} \) and this definition is valid for edges by \((F_1, F_2) \in \mathcal{T}_F^2\). In this case set \( \tau_{F_1} \cdot \psi_{\ell}(z) = \int_{F_1} \Phi_z \tau_{F_1} \cdot \psi_{\ell+1} \, ds \). For all remaining vertices \( z \) of \( \mathcal{T}_\ell \), define

\[
\psi_{\ell}(z) \cdot e_j := \int_{F_2} \Phi_z \psi_{\ell+1} \cdot e_j \, ds \quad \text{for the unit vectors} \quad e_j \in \{(1; 0), (0; 1)\}.
\]

This definition of \( \psi_{\ell} \) is an admissible choice in the setting of [37]. In particular, \( \psi_{\ell} \) coincides with \( \psi_{\ell+1} \) on edges of \( \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1} \). The error estimate follows from the theory in [37].

It remains to show the claimed boundary conditions. Recall that \( \psi_{\ell+1} \) satisfies \( (D\psi_{\ell+1} \tau) \cdot \nu = 0 \) on \( \Gamma_S \cup \Gamma_F \) and \( (D\psi_{\ell+1} \tau) \cdot \tau \) is constant on each connectivity component of \( \Gamma_F \). In particular, this implies that \( \psi_{\ell+1} \cdot \nu \) is constant along each straight part of \( \Gamma_S \cup \Gamma_F \) and that \( \psi_{\ell+1} \cdot \tau \) is affine along each straight part of \( \Gamma_F \). Therefore, the above assignment of the nodal values interpolates \( \psi_{\ell+1} \cdot \nu \) along \( \Gamma_S \cup \Gamma_F \) and \( \psi_{\ell+1} \cdot \tau \) along \( \Gamma_F \) exactly and so these boundary conditions are valid for \( \psi_{\ell} \).

The next proposition states the discrete reliability. The idea to prove such type of result by means of a discrete Helmholtz decomposition was first employed in [11] for the Poisson equation.

**Proposition 4.8** (discrete reliability). There exists a constant \( C_{\text{disc}} \approx 1 \) such that, for \( \|h_0\|_\infty \ll 1 \), any admissible refinement \( \mathcal{T}_{\ell+1} \in \mathcal{T}(\mathcal{T}_\ell) \) of \( \mathcal{T}_\ell \in \mathcal{T} \) and any eigenpair \((\lambda, u) \in \mathbb{R} \times W\) of (3.3) with \( \|u\| = 1 \) and \( r_{\ell+1} \) from (4.3) satisfy

\[
2\| (\lambda_{\ell+1} - \lambda_{\ell}) u \|_{H^2} \leq C_{\text{disc}} (\| u \|_{L^2} \| \nabla \psi_{\ell+1} \|_{L^2} + r_{\ell+1}^2).
\]

**Proof.** The discrete Helmholtz decomposition from Theorem 4.5 leads to \( \phi_{\ell+1} \in \mathcal{V}_{\ell+1} \) and \( \psi_{\ell+1} \in \mathcal{X}(\mathcal{T}_{\ell+1}) \) such that

\[
D_{\text{disc}}^2 ((\lambda_{\ell+1} - \lambda_{\ell}) u) = D_{\text{disc}}^2 \phi_{\ell+1} + \text{sym \ Curl} \psi_{\ell+1}.
\]

The orthogonality of the decomposition proves

\[
\| (\lambda_{\ell+1} - \lambda_{\ell}) u \|_{H^2}^2 = a_{\text{disc}} ((\lambda_{\ell+1} - \lambda_{\ell}) u, \phi_{\ell+1}) - (D_{\text{disc}}^2 \lambda_{\ell} u, \text{Curl} \psi_{\ell+1})_{L^2(\Omega)}.
\]

The projection property of the Morley interpolation operator (2.2), Lemma 3.2, the \( L^2 \) control of Proposition 3.10 and the approximation and stability property (2.5) prove for the first term of (4.4) that

\[
a_{\text{disc}} ((\lambda_{\ell+1} - \lambda_{\ell}) u, \phi_{\ell+1}) = \lambda b ((P_{\ell+1} - P_{\ell}) u, \phi_{\ell+1}) + \lambda b (P_{\ell} u, (1 - \mathcal{T}_{\ell}) \phi_{\ell+1})
\]

\[
\lesssim \| r_{\ell+1} \|_{L^2(\Omega)} + \| h_{\ell+1} \|_{L^2(\Omega)} \| \phi_{\ell+1} \|_{H^2} \|
\]

Let \( \psi_{\ell} \in \mathcal{P}(\mathcal{T}_{\ell}; \mathbb{R}^2) \cap H^1(\Omega; \mathbb{R}^2) \) denote the quasi-interpolation from Lemma 4.6. The function \( \psi_{\ell} \) preserves those boundary conditions of \( \psi_{\ell+1} \) that are necessary to guarantee that \( \text{Curl} \psi_{\ell} \) and \( D_{\text{disc}}^2 \lambda_{\ell} u \) are \( L^2 \)-orthogonal. Hence, an integration by parts shows for the second term of (4.4) that

\[
(D_{\text{disc}}^2 \lambda_{\ell} u, \text{Curl} \psi_{\ell+1})_{L^2(\Omega)} = \sum_{F \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}} \int_F [(D_{\text{disc}}^2 \lambda_{\ell} u)_{|F^T}] \cdot (\psi_{\ell+1} - \psi_{\ell}) \, ds.
\]

The boundary conditions of \( \psi_{\ell+1} \) and \( \psi_{\ell} \) plus Cauchy and trace inequalities and the approximation and stability properties of the Scott-Zhang quasi-interpolation prove that this
is bounded by \( \|D\psi_{\ell+m}\|_{L^2(\Omega)} \) times

\[
\left( \sum_{T \in T_\ell \setminus T_{\ell+m}} \sum_{T \in T(T)} h_F \|D_{NC}^2 \Lambda_T u\|_{\mathcal{F}^2}^2 + \sum_{F \in \mathcal{F}(T)} h_F \|\tau_T^F\|_{L^2(F)}^2 \right)^{1/2}.
\]

The combination of the foregoing estimates and the stability \( \Theta_2 \) conclude the proof. \( \square \)

The following reliability and efficiency are an immediate consequence of the discrete reliability and a priori convergence results (e.g., Proposition 3.10).

**Corollary 4.9** (reliability and efficiency). Provided \( \|h_0\|_\infty \ll 1 \), it holds that

\[
\|u - \Lambda u\|_{NC}^2 \lesssim \mu_\ell^2(\mathcal{F}_\ell, \lambda, u) \lesssim \|u - \Lambda \eta u\|_{NC}^2.
\]

\( \square \)

### 4.3 Proof of Optimal Convergence Rates

The proof of the discrete reliability is the main step in proving optimal convergence rates for Algorithm 4.1. Proofs for optimal convergence rates of the Dörfler marking strategy [23] are mainly based on the ideas of [38, 17] and were recently unified in the axiomatic framework of [10]. Hence, the remaining arguments are not carried out in detail here but only sketched with references to similar proofs in the literature.

The quasi-orthogonality for the Morley FEM was first proven by [29] in the context of the linear biharmonic problem. The following result is an extension to the case of eigenvalue problems.

**Proposition 4.10** (quasi-orthogonality). Under the hypothesis \( \|h_0\|_\infty \ll 1 \) there exists a constant \( C_{qo} \) such that any eigenpair \( (\lambda, u) \in \mathbb{R} \times W \) of \( 3.15 \) with \( \|u\| = 1 \), any \( \mathcal{F}_\ell \in \mathcal{T} \) and any admissible refinement \( T_{\ell+m} \in \mathcal{T}(\mathcal{F}_\ell) \) satisfy

\[
|2a_{NC}(u - \Lambda_{\ell+m} u, \Lambda_{\ell+m} u - \Lambda \eta u)| \lesssim C_{qo}(\|h_2^2 \Lambda P u\|_{L^2(\cup T_{\ell+m} \setminus \mathcal{T}_{\ell+m})} + \eta_{\ell,m}) \|u - \Lambda_{\ell+m} u\|_{NC}.
\]

**Proof.** The properties of the operator \( \mathcal{I}_\ell \) of Section 2 together with the arguments of [29] and [24] lead to the proof. In particular the constant of Proposition 2.3 (which is independent of \( T_{\ell+m} \)) enters the analysis. The details are omitted. \( \square \)

The following result states an equivalence of the theoretical error estimator \( \mu_\ell \) with the practical error estimator \( \eta_\ell \).

**Proposition 4.11** (bulk criterion). Suppose that \( \|h_0\|_\infty \ll 1 \) satisfies (3.7) and

\[
\epsilon := \max_{j \in J} \|u_j - \Lambda \eta u_j\|_{NC} \leq 1 + 1/(2N) - 1 \quad \text{for all } \mathcal{F}_\ell \in \mathcal{T}.
\]

Then, for any \( T \in \mathcal{T}_\ell \), the error estimator contributions can be compared as follows

\[
N^{-1} \sum_{j \in J} \mu_\ell^2(T, \lambda_j, u_j) \leq (B/A)^2 \epsilon_0^2(T) \leq (B/A)^4 (2N + 4N^2) \sum_{j \in J} \mu_\ell^2(T, \lambda_j, u_j).
\]

Therefore, \( \mu_\ell(M_\ell) := N^{-1} \sum_{T \in M_\ell} \sum_{j \in J} \mu_\ell^2(T, \lambda_j, u_j) \) satisfies the bulk criterion

\[
\tilde{\theta}_\ell(\mathcal{F}_\ell) \leq \mu_\ell(M_\ell)
\]

for the modified bulk parameter

\[
\tilde{\theta} := ((B/A)^4 (2N^2 + 4N^3))^{-1} \theta < 1.
\]
Proof. The proof follows from Lemma 5.1 and Proposition 5.2 of [25].

**Proposition 4.12** (error estimator reduction for $\mu_\ell$). Provided $\|h_0\|_\infty \ll 1$, there exist constants $0 < \rho_1 < 1$ and $0 < K < \infty$ such that $T_\ell$ and its one-level refinement $T_{\ell+1}$ generated by Algorithm (4.4) and any eigenfunction $u \in W$ with $\|u\| = 1$ and eigenvalue $\lambda$ satisfy (with $r_{\ell,1}$ from (4.3)) that

$$\mu_{\ell+1}^2(T_{\ell+1}, \lambda, u) \leq \rho_1 \mu_\ell^2(T_\ell, \lambda, u) + K \left( \|\Lambda_{\ell+1} u - \Lambda_{\ell} u\|_W^2 + \|h_0\|_\infty^4 r_{\ell,1}^2 \right).$$

**Proof.** The proof is analogous to the proof of Proposition 3.9. The details are omitted. The proof of Theorem 4.2 follows the lines of [24, Sect. 5.5] and is almost identical. Therefore, the details are omitted here.

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