On the large-time asymptotics of the defocusing mKdV equation with step-like initial data

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Abstract

It is concerned with the large-time asymptotics of the Cauchy problem of the defocusing modified Korteweg-de Vries (mKdV) equation with step-like initial data subject to compact perturbations, that is,

\[ q_0(x) - q_{0u}(x) = 0, \text{ for } |x| > N \]

with some positive \( N \), where

\[ q_{0u}(x) = \begin{cases} 
  c_l, & x \leq 0, \\
  c_r, & x > 0, 
\end{cases} \]

and \( c_l > c_r > 0 \). It follows from the standard direct and inverse scattering theory that an RH characterization for the step-like problem is constructed. By performing the nonlinear steepest descent analysis, we mainly derive the large-time asymptotics in the each of four asymptotic zones in the \((x, t)\)-half plane.

Keywords: Defocusing mKdV equation, Step-like initial data, RH problem, Nonlinear steepest descent analysis, Large-time asymptotics.

Mathematics Subject Classification: 35Q51; 35Q15; 35C20; 37K15; 37K40.

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1 Introduction and statement of results

In the present work, we study the Cauchy problem for the defocusing modified Korteweg-de Vries (mKdV) equation

\[ q_t(x, t) - 6q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]  

\[ q(x, 0) = q_0(x), \]  

where

\[ q_0(x) \to c_r \text{ as } x \to +\infty \quad \text{and} \quad q_0(x) \to c_l \text{ as } x \to -\infty \]  

with real constants $c_l > c_r > 0$. We are concerned with the large-time asymptotics of the solution of (1.1) with the boundary condition (1.3).

Throughout the text, we will assume that the initial data $q_0(x)$ equals identically to the “background wave” outside a compact set, i.e.,

**Assumption 1.1.** The initial data $q_0(x)$ admits a compact perturbation of the shock initial data $q_{0c}(x)$ defined as

\[ q_{0c}(x) = \begin{cases} 
  c_l, & x \leq 0, \\
  c_r, & x > 0,
\end{cases} \]  

that is, $q_0(x) - q_{0c}(x) = 0$ for $|x| > N$ with some positive $N$.

Assumption 1.1 is required to avoid the technical procedures associated to the introduction of analytical approximations or $\tilde{\partial}$ continuous extensions for the reflection coefficient by performing the steepest analysis. This assumption is exhibited purely for convenience, and it does not affect the structure of the final asymptotic formulas.

Due to the symmetries $q \rightarrow -q$, $x \rightarrow -x$ as well as $t \rightarrow -t$ in (1.1), it is sufficient to focus on the scenario where $c_l \geq |c_r|$. When $c_r = 0$, the case has been studied in focusing mKdV equation (cf. [16]), and we omit the simple case for defocusing mKdV equation in the present paper. Then, it is noticed that the case $c_l > c_r > 0$ and the case $c_l > 0 > c_r > -c_l$ lead to quite different asymptotic analysis for focusing mKdV equation (cf. [11]) though; however, their analysis are quite similar in defocusing mKdV equation. As the consequence, we restrict $c_l > c_r > 0$ through the context.

The defocusing mKdV equation (1.1) serves as a canonical model in the area of mathematical physics to describe varieties of nonlinear phenomenon, such as acoustic wave and phonons in a certain anharmonic lattice [25, 32], Alfvén wave in a cold collision-free plasma [14, 12]. Due to the aforementioned physically meaningful phenomenon, the mKdV equation has attracted great interests, and significant progresses, especially in the field of large-time asymptotics, have been achieved over few past decades. In 1981, Segur and Ablowitz extended IST technique to derive the leading asymptotics for the solution of the mKdV equation, including full information
on the phase \[1\]. By deforming contours to reduce the original oscillatory RH problem to a solvable model, Deift and Zhou developed rigorous analytical method to present the long-time asymptotic representation of the solution for defocusing mKdV equation with initial data belonging Schwartz class \([6]\). It is also worthwhile to see that Lenells constructed a RH problem with Carleson jump contours, and derive the long-time asymptotics for the defocusing mKdV equation for initial data with limited decay and regularity \([20]\). Recently, Chen and Liu extended the asymptotics to the solutions for defocusing mKdV equation \([4]\) with initial data in lower regularity spaces. The works listed above mainly address the large-time asymptotics under the zero background \(q_0(x) \to 0\) as \(x \to \pm \infty\). Besides the zero background, the large-time asymptotics of mKdV equation \([13]\) with symmetric boundary \(q_0(x) \to \pm 1, x \to \pm \infty\) are investigated in \([31, 33]\) via the \(\bar{\partial}\) steepest descent method \([9]\).

It’s the aim of the present work to fulfill the asymptotics of defocusing mKdV equation with the step-like initial data (asymmetric boundary). Indeed, Cauchy problems for nonlinear integrable systems with step-like initial data have a long history and origin from the pioneering work of Gurevich and Pitaevsky for KdV equation \([12]\). Since RH technique became a powerful tool to investigate Cauchy problem for integrable equations, amount of step-like initial-value problems as follows were studied. Refer to Boutet de Monvel, Lenells and Shepelsky as well as their co-authors’ work \([22–24]\) on the focusing nonlinear Schrödinger (NLS) equation; refer \([11, 16–19]\) for the focusing mKdV equation; refer \([26–28, 30]\) for some nonlocal integrable PDEs. The aforementioned results are almost devoted to Cauchy problem associated with non-self-adjoint Lax operator. Of particular interest to us are the works \([10, 13]\), where they exhibited detailed formulas for the long-time asymptotics and small dispersion asymptotics for the defocusing NLS equation which corresponds to self-adjoint Lax operator, respectively.

In the present work, we perform the nonlinear steepest descent analysis to exhibit four asymptotic zones in the \((x,t)\)-half plane, and derive the large-time asymptotics in the each of these identified sectors, which particularly reveals rich mathematical structures of the defocusing mKdV equation. Our results are listed below.

**Main Results.** Large-time asymptotics of \(q(x,t)\) shows qualitatively different behaviors in different regions of the \((x,t)\)-half plane. More precisely, these regions are given by the following definition.

**Definition 1.1.** For the constant \(c_l\) and \(c_r\), we define

- **Left field:** \(\mathcal{R}_I := \{(x,t) | \xi < -\frac{c_l^2}{2}\}\),
- **Slowly varied region:** \(\mathcal{R}_{II} := \{(x,t) | -\frac{c_l^2}{2} < \xi < -\frac{c_r^2}{2}\}\),
- **Central field:** \(\mathcal{R}_{III} := \{(x,t) | -\frac{c_r^2}{2} < \xi < \frac{c_r^2}{2}\}\),
- **Right field:** \(\mathcal{R}_{IV} := \{(x,t) | \xi > \frac{c_r^2}{2}\}\),

where \(\xi = x/(12t)\); see Figure [7] for an illustration.

Large-time asymptotics of \(q(x,t)\) in these zones are main results of the present work.

**Theorem 1.1.** Let \(q(x,t)\) be the global solution of the Cauchy problem \([1.1] - [1.3]\) for the defocusing mKdV equation over the real line under the Assumption \([4.1]\) and denote by \(r(k)\) the reflection coefficient. As \(t \to +\infty\), we have the following asymptotics of \(q(x,t)\) in the regions \(\mathcal{R}_I - \mathcal{R}_{IV}\) given in Definition \([4.1]\)

(a) For \(\xi \in \mathcal{R}_I\), we have

\[
q(x,t) = D_{I,\infty}^2(\xi) \left( c_l + t^{-1/2} f_I(\xi) \right) + O(t^{-1}),
\]

(1.5)
where

\[
f_I(\xi) = \frac{\sqrt{3}}{12} \left( -\xi - \frac{c^2}{2} \right) - \xi + \frac{c^2}{2} \left[ \hat{\beta}_{12}^{(\eta)} + \hat{\beta}_{12}^{(-\eta)} - \frac{\nu(\eta)}{\beta_{12}^{(\eta)}} - \frac{\nu(\eta)}{\beta_{12}^{(-\eta)}} + \left( -\xi + \frac{c^2}{2} \right)^\frac{1}{2} \left( \hat{\beta}_{12}^{(\eta)} + \hat{\beta}_{12}^{(-\eta)} \frac{\nu(\eta)}{\beta_{12}^{(\eta)}} + \frac{\nu(\eta)}{\beta_{12}^{(-\eta)}} \right) \right]. \quad (1.6)
\]

\(D_{1,\infty}\) is given by \(3.8\) with \(D_I(\xi; k)\) defined in \(3.6\). \(\hat{\beta}_{12}^{(\pm \eta)}\) and \(\hat{\beta}_{21}^{(\pm \eta)}\) are shown in \(3.35\) and \(3.40\) with \(\nu(\eta) = -\frac{1}{2\pi} \log(1 - |r(\eta)|^2)\) as well as \(\eta_I(\xi) := \sqrt{-\xi + c_f^2/2}\).

(b) For \(\xi \in \mathcal{R}_{II}\), we have

\[
q(x, t) = D_{II,\infty}^{-1}(\xi) \left( \sqrt{\frac{x}{6l}} + t^{-1} f_{II}(\xi) \right) + O(t^{-2}), \quad (1.7)
\]

where

\[
f_{II}(\xi) := \frac{\sqrt{2}}{16} \left( D_{II,0} \right)^{-1/2} \left( \sqrt{-\xi/6} \right) - D_{II,0}^{-1/2} \left( \sqrt{-\xi/6} \right)^2
\]

\[
- \left( D_{II,0} \right)^{1/2} \left( \sqrt{-\xi/6} \right) - 1
\]

\[
\frac{(D_{II,0})^{1/2} \left( \sqrt{-\xi/6} \right) - D_{II,0}^{-1/2} \left( \sqrt{-\xi/6} \right)^2}{\eta^4 \sqrt{-\xi/6}}
\]

\[i \bar{s}_1 + i \bar{t}_1 \eta^2 \]

\[\eta^4 \sqrt{-\xi/6}
\]

\[\left( \frac{x}{6l} \right)^2 + O(t^{-2}) \]

(1.8)

The quantities \(D_{II,\infty}\) is defined in \(1.1\), \(\bar{D}_{II,0}(\eta) = \lim_{k \to \eta}(k - \eta)^{-1/4}D_{II}(k)\) with \(D_{II}(k)\) given in \(1.5\). Besides, \(\eta(\xi) = \sqrt{-\xi/6}, \bar{s}_1 = \frac{\Gamma(2)}{64 \Gamma(4)}, \nu_1 = -(7s_1)/5, \) where \(\Gamma(\cdot)\) is the gamma function.

(c) For \(\xi \in \mathcal{R}_{III}\), we have

\[
q(x, t) = c_r + t^{-1} f_{III}(\xi) + O(t^{-2}), \quad (1.9)
\]

where

\[
f_{III}(\xi) := \sqrt{2} \frac{\left( r^{-\frac{1}{2}} \left( \sqrt{-\xi/12 + c_f^2/2} \right) - r^{-\frac{3}{2}} \left( \sqrt{-\xi/12 + c_f^2/2} \right)^2 \right)}{r^{-1} \left( \sqrt{-\xi/12 + c_f^2/2} \right) - 1}
\]

\[
- \frac{\left( r^{-\frac{1}{2}} \left( \sqrt{-\xi/12 + c_f^2/2} \right) - r^{-\frac{3}{2}} \left( \sqrt{-\xi/12 + c_f^2/2} \right)^2 \right)}{r^{-1} \left( \sqrt{-\xi/12 + c_f^2/2} \right) - 1}
\]

\[
\frac{i \bar{s}_1 + i \bar{t}_1 \eta^2}{\eta^3 \sqrt{-\xi/12 + c_f^2/2}}
\]

(1.10)

with \(\eta(\xi) := \sqrt{-\xi + c_f^2/2}. \) \(s_1\) and \(\nu_1\) are given as the same in the part (b).
(d) For $\xi \in \mathcal{R}_{IV}$, we have
\[
q(x,t) = c_r + \mathcal{O}(t^{-1/2} e^{-16t\xi^{3/2}}).
\]

In the left-most region $\mathcal{R}_I$, the leading term in the formula (1.11) is presented by the constant $c_l$ multiplied by a slowly varying factor which tends to 1 as $\xi \to -\infty$, in compatible with the boundary condition (1.13) as $x < 0$. The asymptotic solution in this region admits the sub-leading term as the form $t^{-1/2}f_I$, which is constructed by parabolic cylinder parametrix.

For $\xi \in \mathcal{R}_{II}$, a slowly varied region occurs, which bridges the left field and central region. The leading term is formed by the product of slowly varied term $\sqrt{-t}$ and an integration factor, and the sub-leading term $t^{-1/2}f_{II}$ is constructed by the Airy parametrix. As we expect, when $\xi$ tends to $-c_I^2/2$, the leading term of (1.11) matches the leading asymptotics in the (1.5); and when $\xi$ tends to $-c_I^2/2$, the leading term of (1.7) matches the leading term of (1.11). The asymptotics in the central region $\mathcal{R}_{III}$ and right field $\mathcal{R}_{IV}$ admit the same leading term.

Matching of the Leading Term. The asymptotic formulas (1.15) and (1.17) match formally to leading order on the critical line $\xi = -c_I^2/2$ where $\mathcal{R}_I$ and $\mathcal{R}_{II}$ meet. Indeed, as $\xi \to -c_I^2/2$, it follows from (1.13), (1.7) (5.8) as well as (1.17) that
\[
D_{I,\infty}^{-2}(\xi)_{\xi = -c_I^2/2} = D_{II,\infty}^{-2}(\xi)_{\xi = -c_I^2/2} \equiv \frac{1}{\sqrt{2\pi}} \left( \int_{-c_I}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \right) c_l,
\]
which coincide with the leading term of each other.

Similarly, as $\xi \to -c_I^2/2$, it is also readily verified that
\[
D_{II,\infty}^{-2}(\xi)_{\xi = -c_I^2/2} \equiv c_r\left|_{\xi = -c_I^2/2} \right. = c_r.
\]
As for the matching on the $\xi = c_I^2/2$, it is obvious.

Comparison between focusing and defocusing cases. As mentioned above, most studies of the long-time asymptotics for mKdV equation with step-like initial functions focus on the focusing scenario. The analysis of long-time asymptotics for the focusing mKdV equation, especially when the initial functions go to nonzero constants at infinity, namely,
\[
q_0(x) \to \begin{cases} 
c_l, & x \to -\infty, 
c_r, & x \to +\infty, 
\end{cases}
\]
has been addressed in the several studies, including but not limited to the references [11, 18, 19]. The asymptotic zones $\mathcal{R}_I, \mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ of the present work are analogues of the left-most region, central region and right-most region in [18, 19], respectively. Nonetheless, our analysis diverges in certain aspects from these references:

- Notably, our study introduces an additional region, $\mathcal{R}_{III}$. The leading-order asymptotic expressions for regions $\mathcal{R}_{III}$ and $\mathcal{R}_{IV}$ are identical; however, the error bounds stated in (1.9) and (1.11) differ. Therefore, it prompts us to consider them as a distinct zone.

- Furthermore, the central region in [19] is characterized by hyper-elliptic wave behavior, where the asymptotics are described via theta functions related to a genus 2 Riemann surface. In contrast, our current work, which addresses the defocusing scenario, identifies both middle sectors ($\mathcal{R}_{II}$ and $\mathcal{R}_{III}$) as genus 0 sectors, marking a significant departure from the genus-2 classification observed in the focusing case.

Outline of This Paper. The rest of this paper is organized as follows. In Section 2, we follow the standard direct and inverse scattering theory to characterize an RH problem for the Cauchy problem (1.1)–(1.3) under the Assumption 1.1 which serves as the starting point of performing nonlinear steepest descent technique. In Section 3, we correspondingly prove the parts (a)–(d) stated in the main Theorem 1.1.
Throughout this paper, the following notations will be used.

- As usual, the classical Pauli matrices $\{\sigma_j\}_{j=1,2,3}$ are defined by
  
  $$
  \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
  $$

  (1.14)

  For a $2 \times 2$ matrix $A$, we also define
  
  $$
  e^{\sigma_j}A := e^{\sigma_j}Ae^{-\sigma_j}, \quad j = 1, 2, 3.
  $$

- For a complex-valued function $f(z)$, we use $f^*(z) := \overline{f(z)}$, $z \in \mathbb{C}$, to denote its Schwartz conjugation.

- For a region $U \subseteq \mathbb{C}$, we use $U^*$ to denote the conjugated region of $U$.

- For any smooth oriented curve $\Sigma$, the Cauchy operator $C$ on $\Sigma$ is defined by
  
  $$
  Cf(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Sigma.
  $$

  (1.15)

  Given a function $f \in L^p(\Sigma)$, $1 \leq p < \infty$,
  
  $$
  C_\pm f(z) := \lim_{z' \to z \in \Sigma} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - z'} d\zeta
  $$

  stands for the positive / negative (according to the orientation of $\Sigma$) non-tangential boundary value of $Cf$. It is also used to adopting $f_\pm(z)$ to represent the non-tangential limits from the positive / negative side respectively, i.e., $f_\pm(z) = \lim_{\text{positive/negative side} \ni \zeta \to z} f(\zeta)$.

- If $A$ is a matrix, then $(A)_{ij}$ stands for its $(i,j)$-th entry, and $[A]_j$ represents the $j$-th column.

2 Preliminaries

2.1 Jost solutions of Lax equations

As the member of AKNS hierarchy [2], it is followed that the Lax pair of mKdV equation (1.1) is given by

$$
\begin{align*}
\Phi_x + ik\sigma_3 \Phi &= Q\Phi, \\
\Phi_t + 4ik^3\sigma_3 \Phi &= V\Phi,
\end{align*}
$$

(2.1)

where $\Phi(x,t;k)$ is a $2 \times 2$ matrix-valued function with the spectral parameter $k \in \mathbb{C}$. Here, $Q(x,t)$ and $V(x,t)$ are some matrices associated to the potential function $q(x,t)$. Specifically,

$$
Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ q(x,t) & 0 \end{pmatrix},
$$

(2.2)

$$
V(x,t) = 4kQ(x,t) + 2ik\sigma_3 (Q_x(x,t) - Q^2(x,t)) + 2Q^3(x,t) - Q_{xx}(x,t).
$$

(2.3)

The compatible condition $\Phi_{xt} = \Phi_{tx}$ is equivalent to the defocusing mKdV equation (1.1). For $j \in \{r,l\}$, it follows from the substitution $q(x,t) = c_j$ in (2.2)–(2.3) that the Lax equations (2.1) admit explicit solutions

$$
\Phi_j^p(x,t;k) = \Delta_j(k)e^{-i(X_j(k)x + \Omega_j(k)t)\sigma_3}.
$$

(2.4)

Here, for $j \in \{r,l\}$,

$$
X_j(k) = \sqrt{k^2 - c_j^2}, \quad \Omega_j(k) = 2(2k^2 + c_j^2)X_j(k),
$$

(2.5)
and

\[
\Delta_j(k) = \frac{1}{2} \begin{pmatrix}
\chi_j(k) + \chi_j^{-1}(k) \\
-i (\chi_j(k) - \chi_j^{-1}(k)) \\
i (\chi_j(k) - \chi_j^{-1}(k)) \\
\chi_j(k) + \chi_j^{-1}(k)
\end{pmatrix}.
\]  

(2.6)

The function \( \chi_j(k) \) is defined by

\[
\chi_j(k) : \mathbb{C}\backslash [-c_j, c_j] \to \mathbb{C}, \quad \chi_j(k) = \left( \frac{k - c_j}{k + c_j} \right)\]

(2.7)

with the branch being chosen such that \( \chi_j(k) = 1 + \mathcal{O}(k^{-1}) \) as \( k \to \infty \).

For \( j \in \{r, l\} \), we denote \( \Phi_j^p(x; k) := \Phi_j^p(x, 0; k) \), where \( \Phi_j^p(x, 0; k) \) is defined in (2.4) for \( t = 0 \). To proceed, we consider the Lax pair (2.1) for \( t = 0 \) and define the Jost functions \( \Phi_j(x; k) := \Phi_j(x, 0; k) \), which satisfy (2.4), and admit the following asymptotic conditions

\[
\Phi_j(x; k) = \Phi_j^p(x; k) (I + o(1)), \quad x \to -\infty, \quad k \in \mathbb{R},
\]

(2.8)

\[
\Phi_j(x; k) = \Phi_j^p(x; k) (I + o(1)), \quad x \to +\infty, \quad k \in \mathbb{R}.
\]

(2.9)

For \( j \in \{r, l\} \), the Jost functions can be expressed in terms of the solutions of the Volterra integral equation in what follows

\[
\Phi_j(x; k) = \Phi_j^p(x; k) + \int_{-\infty}^{x} K^p_j(x, y) \Phi_j^p(y; k) \, dy,
\]

(2.10)

where \( \infty_r := +\infty \), \( \infty_l := -\infty \). The kernel \( K^p_j(x, y) \) is independent of the parameter \( k \), and its existence and other properties are studied, for example, in [11, 21].

Some basic properties of matrix-valued functions \( \Phi_r \) and \( \Phi_l \) are collected in the following proposition whose proof is standard.

**Proposition 2.1.** Assuming the initial data \( q_0(x) \) satisfies the Assumption 1.1, then the Jost solutions \( \Phi_r \) and \( \Phi_l \) defined in (2.10) have the following properties for \( j \in \{r, l\} \):

(a) For each \( x \in \mathbb{R} \), denoting the \([\Phi_j]_1 \) be the \( l \)-th (\( l = 1, 2 \)) column of \( \Phi_j \) (\( j \in \{r, l\} \)), then

\([\Phi_j]_1(x; k) \) is holomorphic for \( k \in \mathbb{C}^- \) and has continuous extension to \( \overline{\mathbb{C}} \backslash \{-c_r, c_r\} \),

\([\Phi_j]_2(x; k) \) is holomorphic for \( k \in \mathbb{C}^+ \) and has continuous extension to \( \overline{\mathbb{C}} \backslash \{-c_r, c_r\} \),

\([\Phi_j]_1(x; k) \) is holomorphic for \( k \in \mathbb{C}^+ \) and has continuous extension to \( \overline{\mathbb{C}} \backslash \{-c_l, c_l\} \),

\([\Phi_j]_2(x; k) \) is holomorphic for \( k \in \mathbb{C}^- \) and has continuous extension to \( \overline{\mathbb{C}} \backslash \{-c_l, c_l\} \).

(b) \( \det \Phi_j(x; k) = 1, \ k \in \mathbb{R} \backslash [-c_j, c_j] \).

(c) As \( k \to \infty \), we have

\[
([\Phi_j(x; k)]_1, [\Phi_r(x; k)]_2) e^{ikx\sigma_3} = I + \mathcal{O}(k^{-1}), \quad k \in \mathbb{C}^+,
\]

(2.11)

\[
([\Phi_r(x; k)]_1, [\Phi_l(x; k)]_2) e^{ikx\sigma_3} = I + \mathcal{O}(k^{-1}), \quad k \in \mathbb{C}^-.
\]

(2.12)

(d) \( \Phi_j(x; k) \) admits the symmetries

\[
\sigma_1 \Phi_j(x; k) \sigma_1 = \Phi_j(x; k),
\]

(2.13)

\[
[\Phi_j]_1(x; k) = \sigma_1 [\Phi_j]_2(x; -k),
\]

(2.14)

\[
\Phi_j(x; -k) = \Phi_j(x; k).
\]

(2.15)

(e) For \( k \in (-c_j, c_j) \), we have

\[
\Phi_j^+ (k) = \Phi_j^-(k) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(2.16)

(f) As \( k \to \pm c_j \), we have \( \Phi_j(k) \sim (k \pm c_j)^{-\frac{1}{2}} \).
The matrices $\Phi_r(x; k)$ and $\Phi_t(x; k)$ are the solutions of the Lax pair \((2.4)\). Hence they are linearly dependent, and there exists a scattering matrix $S(k)$ which is dependent of $x$ such that

$$S(k) = \Phi_r^{-1}(x; k)\Phi_t(x; k).$$ \hspace{1cm} (2.17)

Owing to the symmetries of the $x$-part in \((2.4)\), the scattering matrix $S(k)$ admits the following structure

$$S(k) = \begin{pmatrix} a(k) & b^*(k) \\ b(k) & a^*(k) \end{pmatrix},$$ \hspace{1cm} (2.18)

where the scattering coefficients $a(k), b(k)$ are given by

$$a(k) = \det ([\Phi_t]_1, [\Phi_r]_2),$$ \hspace{1cm} (2.19)

$$b(k) = \det ([\Phi_r]_1, [\Phi_t]_1).$$ \hspace{1cm} (2.20)

To proceed, we define the reflection coefficient

$$r(k) := \frac{b(k)}{a(k)},$$ \hspace{1cm} (2.21)

and summarize the properties of the functions $a(k), b(k)$ and $r(k)$ below.

**Proposition 2.2.** Spectral functions $a(k), b(k)$ and reflection coefficient $r(k)$ satisfy the following properties:

(a) $a(k)$ is holomorphic for $k \in \mathbb{C}^+$, and it could be continuously extended up to the boundary $\mathbb{R} \setminus \{-c_l, -c_r, c_r, c_l\}$. As $k \to \pm c_j, \text{ for } j \in \{r, l\}$, we have $a(k) \sim (k \pm c_j)^{-1/4}$, and for $k \in \mathbb{C}^+$, as $k \to \infty$, we have $a(k) = 1 + \mathcal{O}(k^{-1})$.

*The spectral function $b(k)$ is defined for $k \in \mathbb{R} \setminus \{-c_l, -c_r, c_r, c_l\}$, and the reflection coefficient $r$ is defined for $k \in \mathbb{R} \setminus \{-c_l, -c_r, c_r, c_l\}$. Moreover, under the Assumption \[2.1\], the function $a(k)$ and $b(k)$ have an (sectionally) analytical extension in $\mathbb{C}$, thus the function $r(k)$ is sectionally holomorphic for $k \in \mathbb{C}$.

(b) $a(k)$ has no zeros in the complex plane $\mathbb{C}$.

(c) In their domains of definition, we have

$$a(k) = a(-k), \quad b(k) = b(-k), \quad r(k) = r(-k).$$ \hspace{1cm} (2.22)

(d) $a(k), b(k)$ satisfy the jump relations

$$a_+(k) = a_-(k), \quad b_+(k) = -b_-(k), \quad k \in (-c_r, c_r),$$ \hspace{1cm} (2.23)

$$a_+(k) = b_+(k), \quad b_+(k) = a_-(k), \quad k \in (-c_l, -c_r) \cup (c_r, c_l),$$ \hspace{1cm} (2.24)

$$a_+(k) = a_-(k), \quad b_+(k) = b_-(k), \quad k \in (-\infty, -c_l) \cup (c_l, +\infty).$$ \hspace{1cm} (2.25)

(e) $r(k)$ satisfies the jump relations

$$r_+(k) = -r_-(k), \quad k \in (-c_r, c_r),$$ \hspace{1cm} (2.26)

$$r_+(k) = \frac{1}{r_-(k)}, \quad k \in (-c_l, -c_r) \cup (c_r, c_l),$$ \hspace{1cm} (2.27)

$$r_+(k) = r_-(k), \quad k \in (-\infty, -c_l) \cup (c_l, +\infty),$$ \hspace{1cm} (2.28)

and it can be readily seen that $|r_\pm(k)| = 1$ for $k \in (-c_l, -c_r) \cup (c_r, c_l)$ and $|r(k)| < 1$ for $k$ belongs to the other intervals on $\mathbb{R}$.

**Proof.** Item (a). The analytical properties of $a(k), b(k)$ follow from the analytical properties of Jost functions given in Proposition \[2.1\] and the formulas \((2.19)-(2.20)\). It follows from the item (f) stated in Proposition \[2.1\]...
that \(a(k)\) and \(b(k)\) have at most four-root singularities at the branch points ±\(c_j\) for \(j \in \{r, l\}\). The large \(k\) asymptotics of \(a(k)\) is straightforward obtained by calculation.

Under the Assumption \([14]\), it ensures that eigenfunctions \(\Phi_j(x; k), j \in \{r, l\}\) and thus \(r(k)\) are sectionally holomorphic in \(\mathbb{C}\) \([27, \text{Section 3.2}].\)

**Item (b).** It follows from the similar manners given in \([10, \text{Proposition 3.3}]\) and the properties of self-adjoint Lax operator \(L := i\sigma_3\partial_x - i\sigma_3Q_0\) that \(a(k)\) has no zeros in \(\mathbb{C}\).

**Item (c).** It follows from the symmetries of Jost functions given in item (d) of Proposition \([2.1]\).

**Item (d).** It follows in a straightforward way of item (d).

**Item (e).** Which implies that \(\Phi_j(x; k), j \in \{r, l\}\) and \(s(k)\) are constant with respect to time is due to our choice of normalization of the Jost functions in \((2.31)\) as well and \((2.32)\) that

\[
S_+^{(k)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{S}_-^{(k)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \in (-c_r, c_r),
\]

(2.29)

and

\[
S_+^{(k)} = \mathcal{S}_-^{(k)}, \quad k \in (-c_l, -c_r) \cup (c_r, c_l).
\]

Then it is accomplished via considering the entries of the above relations.

**Remark 2.1.** Notice that similarly to the case of the NLS equation, assuming that \(\int_{-\infty}^{0} |q_0(x) - c_1| \, dx < \infty\) and \(\int_{0}^{\infty} |q_0(x) - c_1| \, dx < \infty\), the reflection coefficient \(r(k)\) are defined, in general, for \(k \in \mathbb{R}\) only. In the large-time analysis of the RH problem, it is advantageous to have \(r(k)\) continued, as meromorphic functions, into \(\mathbb{C}\); then this will allow us to proceed with appropriate RH problem deformations below. Otherwise \(r(k)\) have to be taken in such ways that as Deift-Zhou originally did \([6]\) or \(\bar{\partial}\)-generalization \([4]\). For clarity’s sake, we therefore assume that the initial data \(q_0(x)\) admit a compact perturbation of the pure step initial data \(q_{0c}(x)\) defined in \([14]\), which guarantees the item (a) of the Proposition \([2.2]\).

### 2.2 RH characterization of the mKdV equation

Aim to constructing a basic RH problem for large-time asymptotic analysis, it is required to proceed the time evolution of scattering data. Assuming that the solution \(q(x, t)\) of Cauchy problem \((1.1)-(1.3)\) exists for \(t \geq 0\), it is followed that

\[
\Phi_l(x, t; k) = \Phi_l^0(x, t; k) (I + o(1)), \quad x \to -\infty, \quad k \in \mathbb{R},
\]

(2.31)

\[
\Phi_r(x, t; k) = \Phi_r^0(x, t; k) (I + o(1)), \quad x \to +\infty, \quad k \in \mathbb{R}.
\]

(2.32)

with \(\Phi_l^0(x, t; k), j \in \{r, l\}\) are similarly defined as \(\Phi_l^0(x; k)\).

Since, from \((2.31)\) and \((2.32)\), \(\Phi_l\) and \(\Phi_r\) are defined as simultaneous solutions of both parts of the Lax pair \([2.1]\), it is followed the linearly dependent relation

\[
\Phi_l(x, t; k) = \Phi_r(x, t; k) S(t; k), \quad k \in \mathbb{R} \setminus \{c_r, c_l\}.
\]

(2.33)

On the account of the \(t\)-part of Lax pair \([2.1]\), it is obtained that

\[
\frac{\partial S(t; k)}{\partial t} \equiv 0,
\]

(2.34)

which implies that \(a'(t; k) = 0\) and \(b'(t; k) = 0\) with \(\cdot' := \frac{\partial}{\partial t}\). The claim shows that the scattering data \(a\) and \(b\) are constant with respect to time is due to our choice of normalization of the Jost functions in \((2.31)\) as well as \((2.32)\).

Now we are ready to use the Jost functions to construct the piecewise analytical matrix-valued function as follows

\[
M(x, t; k) := \begin{cases} \left( \frac{\Phi_l(x, t; k)}{a(k)}, \frac{[\Phi_r(x, t; k)]_1}{a^r(k)} \right) e^{it\theta(x; k)s_1}, & k \in \mathbb{C}^+, \\
\left( \frac{\Phi_r(x, t; k)}{a'(k)}, \frac{[\Phi_l(x, t; k)]_2}{a^r(k)} \right) e^{it\theta(x; k)s_1}, & k \in \mathbb{C}^-,
\end{cases}
\]

(2.35)
where \( \theta(\xi; k) = 4k^3 + 12k\xi \) and \( \xi = x/(12t) \).

The matrix-valued function \( M(k) := M(x, t; k) \) satisfies the RH problem below.

**RH problem 2.1.**

- **\( M(k) \) is holomorphic for \( k \in \mathbb{C} \setminus \mathbb{R} \).**
- For \( k \in \mathbb{R} \), we have
  \[
  M_+(k) = M_-(k)V(k),
  \]
  where
  \[
  V(k) = \begin{cases}
  \frac{1-r^+e^{-2it\theta}}{r e^{2it\theta}} & k \in (0, -c_t) \cup (c_t, +\infty), \\
  0 & k \in (-c_t, -c_r) \cup (c_r, c_t), \\
  \frac{e^{2it\theta}}{e^{-2it\theta}} & k \in (-c_r, c_r).
  \end{cases}
  \]
- As \( k \to \infty \) in \( \mathbb{C} \setminus \mathbb{R} \), we have \( M(k) = I + \mathcal{O}(k^{-1}) \).
- As \( k \to \pm c_r \), we have
  \[
  M(k) = \mathcal{O}\left(\frac{(k \mp c_r)^{\pm \frac{1}{2}}}{(k \mp c_r)^{\mp \frac{1}{2}}}, k \in \mathbb{C}^+, \right.
  \]
  \[
  M(k) = \mathcal{O}\left(\frac{(k \mp c_r)^{\pm \frac{1}{2}}}{(k \mp c_r)^{\mp \frac{1}{2}}}, k \in \mathbb{C}^-. \right)
  \]

It follows from the item (d) of Proposition 2.1 and item (c) of Proposition 2.2 that the solution \( M(k) \) of the RH problem automatically satisfies the following symmetries:

\[
M(k) = \sigma_1 M^*(k) \sigma_1 = M(-k) = \sigma_1 M(-k) \sigma_1.
\]

Furthermore, one can prove that the solution of the Cauchy problem (1.1)-(1.3) exists and could be reconstructed by the formula in what follows:

\[
q(x, t) = 2i \lim_{k \to \infty} \left( kM(x, t; k) \right)_{12}, \quad (x, t) \in \mathbb{R} \times [0, +\infty).
\]

**Possible factorizations of jump matrix**

A crucial step of performing the steepest descent analysis is to open lenses, which is aided by two well known factorizations of the jump matrix \( V(k) \) defined in (2.37). Possible factorizations utilized throughout the context are listed as follows.

For \( k \in (-\infty, -c_t) \cup (c_t, +\infty) \),

\[
\begin{pmatrix}
1 - r^+e^{-2it\theta} & -r^+e^{-2it\theta} \\
r e^{2it\theta} & 1
\end{pmatrix} = \begin{pmatrix}
1 & -r^+e^{-2it\theta} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r e^{2it\theta} & 1
\end{pmatrix},
\]

\[
= \begin{pmatrix}
1 & 0 \\
r e^{2it\theta} & 1
\end{pmatrix} \begin{pmatrix}
(1 - r^+)^{\sigma_3} & 0 \\
0 & 1/(1 - r^{-1})
\end{pmatrix}.
\]

For \( k \in (-c_t, -c_r) \cup (c_r, c_t) \), where \( r_+ = \frac{1}{r_-} \),

\[
\begin{pmatrix}
0 & -r^+e^{-2it\theta} \\
r_+e^{2it\theta} & 1
\end{pmatrix} = \begin{pmatrix}
1 & -r^+e^{-2it\theta} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r_+e^{2it\theta} & 1
\end{pmatrix},
\]

\[
= \begin{pmatrix}
1 & 0 \\
r_+e^{2it\theta} & 1
\end{pmatrix} \begin{pmatrix}
0 & -r^+e^{-2it\theta} \\
1/(1 - r_+^{2it\theta}) & 0
\end{pmatrix} \begin{pmatrix}
1 & -r^+e^{-2it\theta} \\
1/(1 - r_+^{2it\theta}) & 1
\end{pmatrix}.
\]

For \( k \in (-c_r, c_r) \), where \( r_+ = -r_-^* \),

\[
\begin{pmatrix}
0 & -e^{-2it\theta} \\
e^{2it\theta} & 0
\end{pmatrix} = \begin{pmatrix}
1 & -r_-^*e^{-2it\theta} \\
e^{2it\theta} & 0
\end{pmatrix} \begin{pmatrix}
0 & -e^{-2it\theta} \\
1/(1 - r_-^{2it\theta}) & 0
\end{pmatrix} \begin{pmatrix}
1 & -r_-^*e^{-2it\theta} \\
1/(1 - r_-^{2it\theta}) & 1
\end{pmatrix},
\]

\[
= \begin{pmatrix}
1 & 0 \\
e^{2it\theta} & 1
\end{pmatrix} \begin{pmatrix}
0 & -e^{-2it\theta} \\
1/(1 - r_-^{2it\theta}) & 0
\end{pmatrix} \begin{pmatrix}
1 & -r_-^*e^{-2it\theta} \\
1/(1 - r_-^{2it\theta}) & 1
\end{pmatrix}.
\]
3 Asymptotic analysis of the RH problem for $M$ in $\mathcal{R}_I$

3.1 First transformation: $M \to M^{(1)}$

The $g$-function

For $\xi \in \mathcal{R}_I$, we introduce

$$g_I(\xi; k) := (4k^2 + 12\xi + 2c_I^2)X_I(k)$$ (3.1)

with $X_I(k) = \sqrt{k^2 - c_I^2}$ given in (2.5). The branch of the square root is chosen such that $X_I(k) = k + \mathcal{O}(k^{-1})$ as $k \to \infty$. It’s readily verified that the following properties for $g_I$ defined in (3.1) hold true.

**Proposition 3.1.** The $g_I$ function defined in (3.1) satisfies the following properties:

- $g_I(\xi; k)$ is holomorphic for $k \in \mathbb{C} \setminus [-c_I, c_I]$.
- As $k \to \infty$ in $\mathbb{C} \setminus [-c_I, c_I]$, we have $g_I(\xi; k) = \theta(\xi; k) + \mathcal{O}(k^{-1})$.
- For $k \in (-c_I, c_I)$, $g_{I,+}(\xi; k) + g_{I,-}(\xi; k) = 0$.

It’s readily seen that the $k$-derivative of $g_I(k)$ is given by

$$g'_I(k) = \frac{12k(k - \eta_I(\xi))(k + \eta_I(\xi))}{X_I(k)}, \quad \eta_I(\xi) = \sqrt{-\xi + \frac{c_I^2}{2}} \in (c_I, +\infty).$$ (3.2)

The signature table for $\text{Im} \ g_I$ is illustrated in Figure 2, where “+” represents that $\text{Im} \ g_I > 0$, and “−” represents $\text{Im} \ g_I < 0$.

![Signature table of the function Im $g_I(\xi; k)$ for $\xi \in \mathcal{R}_I$.](image)

**RH problem for $M^{(1)}$**

By the $g_I$ function defined in (3.1), we define a new matrix-valued function $M^{(1)}(k) := M^{(1)}(x, t; k)$ by

$$M^{(1)}(x, t; k) := M(x, t, k) e^{i\alpha g_I(\xi; k) - \theta(\xi; k)\sigma_3}.$$ (3.3)

Then RH problem for $M^{(1)}$ read as follows:

**RH problem 3.1.**

- $M^{(1)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \mathbb{R}$.
- For $k \in \mathbb{R}$, we have

$$M^{(1)}_+(k) = M^{(1)}_-(-k) V^{(1)}(k), \quad k \in \mathbb{R},$$ (3.4)

where

$$V^{(1)}(k) = \begin{cases} 
\begin{pmatrix} 1 - r^+ & -r^+ e^{-2i\alpha g_I} \\
re^{2i\alpha g_I} & 1 \end{pmatrix}, & k \in (-\infty, -c_I) \cup (c_I, +\infty), \\
\begin{pmatrix} 0 & -r^+ \\
r^+ e^{-2i\alpha g_I} & 1 \end{pmatrix}, & k \in (-c_I, -c_r) \cup (c_r, c_I), \\
\begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}, & k \in (-c_r, c_r).
\end{cases}$$ (3.5)
As $k \to \infty$ in $C \setminus R$, we have $M^{(1)}(k) = I + O(k^{-1})$.

$M^{(1)}(k)$ admits the same singular behavior as $M(k)$ at branch points $\pm c_r$.

### 3.2 Second transformation: $M^{(1)} \to M^{(2)}$

It’s the aim of the second transformation to serve as a preparation for the subsequent one via introducing an auxiliary $D$ function, which will proceed us to open the rays $(-\infty, -\eta)$ and $(\eta, +\infty)$.

**The $D$ function**

Let us define an auxiliary function

$$D_I(\xi; k) := \exp \left\{ \frac{X_I(k)}{2\pi i} \left[ \left( \int_{-\eta}^{-c_I} + \int_{c_I}^{\eta} \right) \log(1 - r(s) r^*(s)) \frac{ds}{X_I(s)(s - k)} + \left( \int_{-c_I}^{-c_r} + \int_{c_r}^{c_I} \right) \log r_+(s) \frac{ds}{X_{I+}(s)(s - k)} \right] \right\}. \quad (3.6)$$

The properties of $D_I$ function are collected as follows.

**Proposition 3.2.** The $D_I$ function defined in (3.6) satisfies the following properties for $\xi \in \mathcal{R}_I$:

(a) $D_I(k)$ is holomorphic for $C \setminus ([-\eta, -c_I] \cup [c_I, \eta])$.

(b) $D_I(k)$ satisfies the jump relation:

$$D_{I,+}(k) = D_{I,-}(k)(1 - rr^*), \quad k \in (-\eta, -c_I) \cup (c_I, \eta),$$
$$D_{I,+}(k)D_{I,-}(k) = r_+(k), \quad k \in (-c_I, -c_r) \cup (c_r, c_I),$$
$$D_{I,+}(k)D_{I,-}(k) = 1, \quad k \in (-c_r, c_r). \quad (3.7)$$

(c) $D_I(k)$ admits the symmetry $D_I^{-1}(k) = \overline{D_I(k)}$ for $k \in C \setminus ([-\eta, -c_I] \cup (c_I, \eta))$.

(d) As $k \to \infty$, $D_I(\xi; k) = D_{I,\infty}(\xi) + O(k^{-1})$ where

$$D_{I,\infty}(\xi) := \exp \left\{ \frac{1}{2\pi i} \left[ \left( \int_{-\eta}^{-c_I} + \int_{c_I}^{\eta} \right) \log(1 - r(s) r^*(s)) \frac{ds}{X_I(s)} + \left( \int_{-c_I}^{-c_r} + \int_{c_r}^{c_I} \right) \log r_+(s) \frac{ds}{X_{I+}(s)} \right] \right\}, \quad (3.8)$$

with $|D_{I,\infty}(\xi)| = 1$.

(e) $D_{I,\infty}(\xi)$ holds that $\lim_{\xi \to -\infty} D_{I,\infty}^{-2}(\xi) = 1$.

(f) $D_I(k)$ shows the following singular behavior at each endpoint

$$D_I(k) = (k \mp c_j)^{\nu(k)} \lim_{k \to \pm c_j} D_{I,0}(k), \quad k \in \{r, l\},$$
$$D_I(k) = (k \mp \eta)^{\nu(\pm \eta)} D_{I,b}(k), \quad k \to \pm \eta, \quad (3.9)$$

where $\nu(k) = -\frac{1}{2\pi} \log(1 - r(k)r^*(k))$, and $D_{I,0}(k)$ and $D_{I,b}(k)$ are bounded functions taking a definite limit as $k$ approaches each singular point non-tangentially.

**Proof.** Item (a). It follows from the general properties of Cauchy type integration transformation that property (a) is valid.

Item (b). The property (b) is an immediate consequence by using the well-known Plemelj formulae.

Item (c). It can be straightforward verified.

Item (d). As $k \to \infty$, the asymptotic expansion of $D_I(\xi; k)$ is obtained by straightforward calculation.

Item (e). It follows from $|r(k)| = 1$ for $k \in (-c_I, -c_r) \cup (c_r, c_I)$ as well as the definition of $D_{I,\infty}(\xi)$ defined in (3.8) that

$$D_{I,\infty}(\xi) = \exp \left\{ \frac{1}{2\pi i} \left[ \left( \int_{-\eta}^{-c_I} + \int_{c_I}^{\eta} \right) \log(1 - r(s) r^*(s)) \frac{ds}{X_I(s)} + \left( \int_{-c_I}^{-c_r} + \int_{c_r}^{c_I} \right) \frac{i \arg r_+(s)}{X_{I+}(s)} \frac{ds}{ds} \right] \right\}. \quad (3.10)$$

Owing to the decay property of $r$, it shows that

$$\lim_{\xi \to -\infty} D_{I,\infty}(\xi) = e^{-\frac{i}{2\pi} C}, \quad (3.11)$$
where the constant $C$ is given by

$$C = \left( \int_{-\infty}^{-c_I} + \int_{c_I}^{\infty} \right) \frac{\log(1 - |r(s)|^2)}{X_I(s)} ds + \left( \int_{-c_I}^{0} + \int_{0}^{c_I} \right) \frac{i \arg r_+(s)}{X_{I+}(s)} ds. \quad (3.12)$$

It follows from straightforward calculation that $C \in 2\pi^2(1 + 2\mathbb{Z})$, which points to this item.

Item (f). It follows in a straight way of the \((3.6)\). \qed

**RH problem for $M^{(2)}$**

Let us define a new matrix-valued function $M^{(2)}(k) := M^{(2)}(x, t; k)$ by

$$M^{(2)}(k) = D_{I,\infty}^{\sigma_2}(\xi)M^{(1)}(k)D_{I}^{-\sigma_2}(k). \quad (3.13)$$

Then RH problem for $M^{(2)}$ reads as follows:

**RH problem 3.2.**

- $M^{(2)}(k)$ is holomorphic for $k \in \mathbb{C}\setminus\mathbb{R}$.
- For $k \in \mathbb{R}$, we have
  $$M^{(2)}_+(k) = M^{(2)}_-(k)V^{(2)}(k), \quad (3.14)$$

  where

  $$V^{(2)}(k) = \begin{cases}
  \begin{pmatrix} 1 - rr^* & -D^2_{I} e^{-2it_1} \\ D_{I}^{-2} e^{2it_1} & 1 \end{pmatrix}, & k \in (-\infty, -\eta_1) \cup (\eta_1, +\infty), \\
  \begin{pmatrix} 1 & -D^2_{I} e^{-2it_1} \\ 1 & 1 - rr^* \end{pmatrix}, & k \in (-\eta_1, -c_I) \cup (c_I, \eta_1), \\
  \begin{pmatrix} 0 & -1 \\ 1 & D_{I}^{-1} e^{-2it_1} \end{pmatrix}, & k \in (-c_I, -c_\tau) \cup (c_\tau, c_I), \\
  \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & k \in (-c_\tau, c_\tau). 
  \end{cases} \quad (3.15)$$

- As $k \to \infty$ in $\mathbb{C}\setminus\mathbb{R}$, we have $M^{(2)}(k) = I + \mathcal{O}(k^{-1})$.
- As $k \to \pm c_I$, we have $M^{(2)}(k) = \mathcal{O}\left((k \mp c_I)^{-\frac{1}{2}}\right)$.

**3.3 Third transformation: $M^{(2)} \to M^{(3)}$**

The aim of the third transformations is to open lenses such that some jump matrices which on the deformed contours of the newly generated RH problem could exponentially decay as $t \to \infty$.

Now we are ready to define $M^{(3)} := M^{(3)}(x, t; k)$ by

$$M^{(3)}(k) = M^{(2)}(k)D_{I}^{\sigma_2}(k)G(k)D_{I}^{-\sigma_2}(k), \quad (3.16)$$

where

$$G(k) := \begin{cases}
  \begin{pmatrix} 1 & 0 \\ -r e^{-2it_1} & 1 \end{pmatrix}, & k \in U_1^{(3)}, \\
  \begin{pmatrix} 1 & 0 \\ -r^* e^{-2it_1} & 1 \end{pmatrix}, & k \in U_1^{(3)*}, \\
  \begin{pmatrix} 1 & 0 \\ \frac{1}{1 - rr} e^{-2it_1} & 1 \end{pmatrix}, & k \in U_2^{(3)}, \\
  \begin{pmatrix} 1 & 0 \\ \frac{1}{1 - rr} e^{2it_1} & 1 \end{pmatrix}, & k \in U_2^{(3)*}, \\
  I, & \text{elsewhere}, 
  \end{cases} \quad (3.17)$$

Here, the domains $U_j^{(3)}$, $j = 1, 2, 3$ are illustrated in Figure [3].

RH problem for $M^{(3)}$ reads as follows:
RH problem 3.3.

- $M^{(3)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \Gamma^{(3)}$, where $\Gamma^{(3)} := \bigcup_{j=1}^{4} (\Gamma_j \cup \Gamma_j^*) \cup [-c_l, c_l]$; see Figure 3 for an illustration.

- For $k \in \Gamma^{(3)}$, we have

$$M^{(3)}_{+}(k) = M^{(3)}_{-}(k)V^{(3)}(k),$$

where

$$V^{(3)}(k) = \begin{cases}
    \begin{pmatrix}
        1 & 0 \\
        -D^{-2}_{I} e^{2\imath g t} & 1
    \end{pmatrix}, & k \in \Gamma_{1}^{(3)} \cup \Gamma_{4}^{(3)};
    \\
    \begin{pmatrix}
        1 & 0 \\
        -D^{-2}_{I} e^{-2\imath g t} & 1
    \end{pmatrix}, & k \in \Gamma_{1}^{(3)*} \cup \Gamma_{4}^{(3)*};
    \\
    \begin{pmatrix}
        1 & 0 \\
        -D^{-2}_{I} e^{2\imath g t} & 1
    \end{pmatrix}, & k \in \Gamma_{2}^{(3)} \cup \Gamma_{3}^{(3)};
    \\
    \begin{pmatrix}
        0 & -1 \\
        1 & 0
    \end{pmatrix}, & k \in (-c_l, c_l).
\end{cases}$$

- As $k \to \infty$ in $\mathbb{C} \setminus \Gamma^{(3)}$, we have $M^{(3)}(k) = I + \mathcal{O}(k^{-1}).$

- As $k \to \pm c_l$, we have $M^{(3)}(k) = \mathcal{O}\left((k \mp c_l)^{-1/4}\right)$.

**Remark 3.1.** To obtain the $V^{(3)}$ for $k \in (-c_l, c_l)$, we used the facts that $D_{I,+}D_{I,-} = r_+,$ $r_+ r_- = 1$ for $k \in (-c_l, -c_r) \cup (c_r, c_l); D_{I,+}D_{I,-} = 1, r_+ = -r_-^*$ for $k \in (-c_r, c_r)$ as well as $g_{I,+} + g_{I,-} = 0$ for $k \in (-c_l, c_l).

![Figure 3: The jump contours of RH problem for $M^{(3)}$ when $\xi \in \mathcal{R}_I$.]

### 3.4 Analysis of RH problem for $M^{(3)}$

Note that $V^{(3)} \to I$ as $t \to +\infty$ on the contours $\Gamma_{j}^{(3)} \cup \Gamma_{j}^{(3)*}$ for $j = 1, \ldots, 4$, it follows that $M^{(3)}$ is approximated, to the leading order, by the global parametrix $M^{(\infty)}$ given below. The sub-leading contribution stems from the local behavior near the saddle points $\pm \eta_l$, which is well approximated by the parabolic cylinder parametrix.

**Global parametrix**

As $t$ large enough, the jump matrix $V^{(3)}$ approaches

$$V^{(\infty)} = \begin{pmatrix}
    0 & -1 \\
    1 & 0
\end{pmatrix}, \quad k \in (-c_l, c_l).$$

For $k \in \mathbb{C} \setminus [-c_l, c_l]$, $V^{(3)} \to I$ as $t \to \infty$. Then it is naturally established the following parametrix for $M^{(\infty)}$. 

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RH problem 3.4.

- $M^{(\infty)}(k)$ is holomorphic for $k \in \mathbb{C}\setminus[-c_l,c_l]$.
- For $k \in (-c_l,c_l)$, we have
  \[
  M^{(\infty)}_+ = M^{(\infty)}_- \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
  \]
  (3.21)
- As $k \to \infty$ in $\mathbb{C}\setminus[-c_l,c_l]$, we have $M^{(\infty)} = I + \mathcal{O}(k^{-1})$.
- As $k \to \pm c_l$, $M^{(\infty)} = \mathcal{O}((k \mp c_l)^{-1/4})$.

Then the unique solution of $M^{(\infty)}$ is given by
\[
M^{(\infty)} = \Delta_l(k),
\]
(3.22)
with $\Delta_l(k)$ defined by (2.6) for $j = l$.

Local parametrices near $\pm \eta_l$

Let
\[
U^{(r)} = \{ k : |k - \eta_l| < \varrho \}, \quad U^{(l)} = \{ k : |k + \eta_l| < \varrho \},
\]
(3.23)
be two small disks around $\eta_l$ and $-\eta_l$, respectively, where
\[
\varrho < \frac{1}{3} \min \{ |\eta_l - c_l|, |\eta_l + c_l|, |\eta_l| \}.
\]
(3.24)
For $\ell \in \{r, l\}$, we intend to solve the following local RH problem for $M^{(\ell)}$.

RH problem 3.5.

- $M^{(\ell)}(k)$ is holomorphic for $k \in \mathbb{C}\setminus\Gamma^{(\ell)}$, where
  \[
  \Gamma^{(\ell)} := U^{(\ell)} \cap \Gamma^{(3)};
  \]
  (3.25)
  see Figure 4 for an illustration.
- For $k \in \Gamma^{(\ell)}$, we have
  \[
  M^{(\ell)}_+(k) = M^{(\ell)}_-(k)V^{(\ell)}(k),
  \]
  (3.26)
  where $V^{(\ell)}(k)$ is defined as (3.19).
- As $k \to \infty$ in $\mathbb{C}\setminus\Gamma^{(\ell)}$, we have $M^{(\ell)}(k) = I + \mathcal{O}(k^{-1})$.

Figure 4: The local jump contours of RH problem for $M^{(r)}$ (right) and $M^{(l)}$ (left) for $\xi \in \mathcal{R}_1$. 

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The $M^{(r)}$ can be constructed from the parabolic cylinder parametrix shown in Appendix A in a standard manner. To do the first, we make the change of variable $k \to \zeta_r$, 
\begin{equation}
g_I(k) = g_I(\eta) + \frac{1}{4t} \zeta^2_r, \tag{3.27}
\end{equation}
\begin{equation}
\zeta_r = \sqrt{4t(g_I(k) - g_I(\eta))} = t^{1/2} \sqrt{2g_I''(\eta)}(k - \eta) \left(1 + \mathcal{O}(k - \eta)\right), \tag{3.28}
\end{equation}
\begin{equation}
k = \eta + t^{-1/2} \frac{1}{\sqrt{2g_I''(\eta)}} \zeta_r \left(1 + \mathcal{O}(\zeta_r)\right). \tag{3.29}
\end{equation}

Here $g_I''(\eta) = \frac{24\left(-\xi + \frac{c^2}{2}\right)}{\sqrt{-\xi - \frac{c^2}{2}}} > 0$ for $\xi < -\frac{c^2}{2}$; therefore $tg_I''(\eta) > 0$.

Under the change of variable and the singular behavior at $\eta$ (see (3.9)), it is followed that the RH problem for $M^{(r)}$ could be approximated by the RH problem $\hat{M}^{(r)}$ listed as follows

**RH problem 3.6.**

- $\hat{M}^{(r)}(\zeta_r)$ is holomorphic for $\zeta_r \in \mathbb{C} \setminus \Gamma^{(r)}$, where $\Gamma^{(r)}$ is defined in (3.28).

- For $\zeta_r \in \Gamma^{(r)}$, we have
  \begin{equation}
  \hat{M}^{(r)}(\zeta_r) = \hat{M}^{(r)}_+(\zeta_r)\hat{V}^{(r)}(\zeta_r), \tag{3.30}
  \end{equation}

where
\begin{equation}
\hat{V}^{(r)}(k) = e^{-itg_I(\eta)\sigma_3} D_0(\eta)\hat{\zeta}_r^{\sigma_3}\hat{\zeta}_r^{\sigma_3} e^{-\frac{i}{4t}\hat{\zeta}_r^2} = \begin{pmatrix} 1 & 0 \\ \frac{\eta}{2} & 1 \end{pmatrix}, \quad \zeta_r \in \Gamma_1^{(r)},
\end{equation}
\begin{equation}
\begin{pmatrix} 1 & \frac{\eta}{2} \\ 0 & 1 \end{pmatrix}, \quad \zeta_r \in \Gamma_1^{(r)^*},
\end{equation}
\begin{equation}
\begin{pmatrix} 1 & 0 \\ \frac{\eta}{2} & 1 \end{pmatrix}, \quad \zeta_r \in \Gamma_2^{(r)},
\end{equation}
\begin{equation}
\begin{pmatrix} 1 & 0 \\ \frac{\eta}{2} & 1 \end{pmatrix}, \quad \zeta_r \in \Gamma_2^{(r)^*}.
\end{equation}

- As $\zeta_r \to \infty$ in $\mathbb{C} \setminus \Gamma^{(r)}$, we have $\hat{M}^{(r)}(\zeta_r) = I + \mathcal{O}(\zeta_r^{-1})$.

According to the small norm theory [8], we have
\begin{equation}
M^{(r)} = \hat{M}^{(r)} + \mathcal{O}(t^{-1}) \tag{3.32}
\end{equation}
for positive $t$ large enough.

Then it is followed that the localized RH problem for $\hat{M}^{(r)}$ is given by
\begin{equation}
\hat{M}^{(r)} = \Delta_I(k) \cdot e^{-itg_I(\eta)\sigma_3} D_0(\eta)^{\sigma_3} M^{(PC)}(\zeta_r(k); r(\eta)), \tag{3.33}
\end{equation}
where $M^{(PC)}$ is the solution of RH problem A.1.

Moreover, it follows from (3.32), (3.33) and (A.24) that as $t$ large enough,
\begin{equation}
M^{(r)} = \Delta_I \cdot \begin{pmatrix} 1 & -i \hat{\beta}_1^{(r)}(n) \\ i \hat{\beta}_2^{(r)}(n) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{O}(t^{-1}) \tag{3.34}
\end{equation}
with
\begin{equation}
\hat{\beta}_1^{(r)}(n) = \sqrt{\frac{2\pi e^{\frac{c^2}{4}} e^{-\frac{c^2}{2} n}}{r(\eta) \Gamma(-i\nu(\eta))}} D_{I,\nu}(\eta) e^{-2itg_I(\eta)}, \quad \hat{\beta}_2^{(r)}(n) = \nu(\eta)/\hat{\beta}_1^{(r)}(n), \tag{3.35}
\end{equation}
where $D_{I,\nu}(\eta) = \lim_{\kappa \to \infty} (k - \eta)^{-i\nu(k)} D_I(k)$.

Finally, the RH problem for $M^{(l)}$ can be solved in a similar manner. Indeed, it can be readily seen that $g_I(\eta) = g_I(-\eta), g_I''(\eta) = g_I''(-\eta) > 0$; thus we make the change of variable $k \to \zeta_l$:
\begin{equation}
g_I(k) = g_I(\eta) + \frac{1}{4t} \zeta^2_l, \tag{3.36}
\end{equation}
\begin{equation}
\zeta_l = \sqrt{4t(g_I(k) - g_I(\eta))} = t^{1/2} \sqrt{2g_I''(\eta)}(k + \eta) \left(1 + \mathcal{O}(k + \eta)\right), \tag{3.37}
\end{equation}
\begin{equation}
k = -\eta + t^{-1/2} \frac{1}{\sqrt{2g_I''(\eta)}} \zeta_l \left(1 + \mathcal{O}(\zeta_l)\right). \tag{3.38}
\end{equation}
For later use, we give an analogue of (3.34) as the end of this subsection. As \( t \to +\infty \),

\[
M^{(t)} = \Delta_t \left( I + t^{-\frac{1}{2}} \right) \begin{pmatrix}
1 & 0 \\
-\frac{1}{\sqrt{2} \pi} \Gamma(-\eta) & -\frac{i \hat{\beta}\eta}{\pi}
\end{pmatrix} e^{\frac{-i \hat{\beta}\eta}{\pi} t} + \mathcal{O}(t^{-\frac{1}{2}}),
\]

(3.39)

with

\[
\hat{\beta}\eta = \sqrt{2 \pi} e^{i \frac{\pi}{4}} e^{-\frac{\pi}{4} \eta},
\]

(3.40)

where \( D_I,\nu(\eta) = \lim_{k \to -\infty} \left( (k + \eta)^{-\nu(k)} D_I(k) \right) \). Here we additionally use the fact \( r(z) = \overline{r(-\bar{z})} \), thus \( r(-\eta) = \overline{r(\eta)} \) and \( \nu(\eta) = \nu(-\eta) \).

**3.5 Small norm RH problem for \( M^{(err)} \)**

Define

\[
M^{(err)}(x,t;k) := \begin{cases}
M^{(3)}(x,t;k) \left( M^{(\infty)}(x,t;k) \right)^{-1}, & k \in \mathbb{C} \setminus \left( U^{(r)} \cup U^{(l)} \right) \\
M^{(3)}(x,t;k) \left( M^{(r)}(x,t;k) \right)^{-1}, & k \in U^{(r)} \\
M^{(3)}(x,t;k) \left( M^{(l)}(x,t;k) \right)^{-1}, & k \in U^{(l)}.
\end{cases}
\]

(3.41)

It’s then readily seen that \( M^{(err)} \) satisfies the following RH problem.

**RH problem 3.7.**

- **\( M^{(err)} \) is holomorphic for \( k \in \mathbb{C} \setminus \Gamma^{(err)} \), where**

\[
\Gamma^{(err)} := \partial U^{(r)} \cup \partial U^{(l)} \cup \left( \Gamma^{(3)} \setminus \left( U^{(r)} \cup U^{(l)} \right) \right);
\]

(3.42)

see Figure 2 for an illustration.

- **For \( k \in \Gamma^{(err)} \), we have**

\[
M^{(err)}_+(k) = M^{(err)}(k)V^{(err)}(k),
\]

(3.43)

where

\[
V^{(err)}(k) = \begin{cases}
M^{(\infty)}(k)V^{(3)}(k)M^{(\infty)}(k)^{-1}, & k \in \Gamma^{(3)} \setminus \left( U^{(r)} \cup U^{(l)} \right) \\
M^{(r)}(k)M^{(\infty)}(k)^{-1}, & k \in \partial U^{(r)} \\
M^{(l)}(k)M^{(\infty)}(k)^{-1}, & k \in \partial U^{(l)}.
\end{cases}
\]

(3.44)

- **As \( k \to \infty \) in \( k \in \mathbb{C} \setminus \Gamma^{(err)} \), we have** \( M^{(err)}(k) = I + \mathcal{O}(k^{-1}) \).

- **As \( k \to \pm \epsilon_1 \), we have** \( M^{(err)}(k) = \mathcal{O}(1) \).

![Diagram](image)

Figure 5: The jump contour \( \Gamma^{(err)} \) of RH problem for \( M^{(err)} \) for \( \xi \in \mathcal{R}_I \).

A simple calculation shows that for \( p = 1, 2, \infty \),

\[
\|V^{(err)} - I\|_{L^p} = \begin{cases}
\mathcal{O}(e^{-\epsilon_1}), & k \in \Gamma^{(err)} \setminus \left( U^{(r)} \cup U^{(l)} \right), \\
\mathcal{O}(t^{-1/2}), & k \in \partial U^{(r)} \cup \partial U^{(l)}.
\end{cases}
\]

(3.45)
with some positive constant $c$.

It then follows from the small norm RH problem theory $[5]$ that there exists a unique solution to RH problem (3.7) for large positive $t$. Moreover, according to [3], we have

$$M^{(err)}(k) = I + \frac{1}{2\pi i} \int_{\Gamma^{(err)}} \frac{\mu(s)(V^{(err)}(s) - I)}{s - k} \, ds,$$  \hspace{1cm} (3.46)

where $\mu \in I + L^2(\Gamma^{(err)})$ is the unique solution of the Fredholm type equation

$$\mu = I + C_{(err)} \mu.$$  \hspace{1cm} (3.47)

Here, $C_{(err)} : L^2(\Gamma^{(err)}) \to L^2(\Gamma^{(err)})$ is an integral operator defined by $C_{(err)} f(z) = C_-(f(V^{(err)}(z) - I))$ with $C_-$ being the Cauchy projection operator on $\Gamma^{(err)}$. Thus,

$$\|C_{(err)}\| \leq \|C_-\|_{L^2 \to L^2} \|V^{(err)} - I\|_{L^2} = \mathcal{O}(t^{-1/2}),$$  \hspace{1cm} (3.48)

which implies that $I - C_{(err)}$ is invertible for sufficient large $t$, and $\mu$ exists uniquely with

$$\|\mu - I\|_{L^2(\Gamma^{(err)})} = \mathcal{O}(t^{-1/2}).$$  \hspace{1cm} (3.49)

For later use, we conclude this section with behaviors of $M^{(err)}$ at $k = \infty$. By (3.46), it follows that

$$M^{(err)}(k) = I + \frac{M^{(err)}_1}{k} + \mathcal{O}(k^{-2}), \quad k \to \infty,$$  \hspace{1cm} (3.50)

where

$$M^{(err)}_1 = -\frac{1}{2\pi i} \int_{\Gamma^{(err)}} \mu(s)(V^{(err)}(s) - I) \, ds.$$  \hspace{1cm} (3.51)

**Proposition 3.3.** With $M^{(err)}_1$ defined in (3.51), we have, as $t \to +\infty$,

$$M^{(err)}_1 = \begin{cases} \star & -it^{-\frac{1}{2}} \frac{\sqrt{\bar{\beta}}}{\pi} \left( \frac{\xi - \frac{\bar{\beta}}{2}}{\sqrt{\pi^2 + \bar{\beta}^2}} \left[ \frac{\beta^{(n)}_{12}}{I_{12}} + \frac{\beta^{(-m)}_{12}}{I_{12}} \right] \right) \\ \star & + \mathcal{O}(t^{-1}) \end{cases},$$  \hspace{1cm} (3.52)

where $\star$ represents the terms that we are not concerned with, and $\hat{\beta}_{12}^{(n)}$, $\hat{\beta}_{12}^{(-m)}$ are defined in (3.39) and (3.40) respectively.

**Proof.** Let us divide $M^{(err)}_1$ into three parts by

$$I_1 := -\frac{1}{2\pi i} \int_{\Gamma^{(err)}} (\mu(s) - I) \left( V^{(err)}(s) - I \right) \, ds,$$  \hspace{1cm} (3.53)

$$I_2 := -\frac{1}{2\pi i} \int_{\Gamma^{(err)} \setminus \partial U(t)} \left( V^{(err)}(s) - I \right) \, ds,$$  \hspace{1cm} (3.54)

$$I_3 := -\frac{1}{2\pi i} \int_{\partial U(t) \setminus \partial U(k)} \left( V^{(err)}(s) - I \right) \, ds.$$  \hspace{1cm} (3.55)

It follows from (3.35) and (3.48) that $I_1 = \mathcal{O}(t^{-1})$ and $I_2 = \mathcal{O}(e^{-ct})$ with some positive constant $c$.

Following from (2.22, 3.34, 3.39) and (3.44), it is seen that

$$I_3 = -\frac{1}{2\pi i} \int_{\partial U(t)} \frac{t^{-1/2}}{\sqrt{2g_I'(\eta)}} \Delta_I(s) \begin{pmatrix} 0 & -i \hat{\beta}_{21}^{(m)} \\ i \hat{\beta}_{21}^{(-m)} & 0 \end{pmatrix} \Delta_I^{-1}(s) \, ds$$

$$-\frac{1}{2\pi i} \int_{\partial U(t)} \frac{1}{\sqrt{2g_I'(\eta)}} \Delta_I(s) \begin{pmatrix} 0 & -i \hat{\beta}_{21}^{(m)} \\ i \hat{\beta}_{21}^{(-m)} & 0 \end{pmatrix} \Delta_I^{-1}(s) \, ds + \mathcal{O}(t^{-1})$$

$$= \frac{t^{-1/2}}{\sqrt{2g_I'(\eta)}} \Delta_I(\eta_I) \begin{pmatrix} 0 & -i \hat{\beta}_{21}^{(m)} \\ i \hat{\beta}_{21}^{(-m)} & 0 \end{pmatrix} \Delta_I^{-1}(\eta_I) + \frac{t^{-1/2}}{\sqrt{2g_I'(\eta)}} \Delta_I(\eta_I) \begin{pmatrix} 0 & -i \hat{\beta}_{21}^{(-m)} \\ i \hat{\beta}_{21}^{(-m)} & 0 \end{pmatrix} \Delta_I^{-1}(\eta_I) + \mathcal{O}(t^{-1}),$$  \hspace{1cm} (3.56)
where we apply the residue theorem and \( g''_l(\eta_l) = g''_l(-\eta_l) \) for the second equality.

For \( \xi < -\frac{1}{2} \), then \( \eta_l > c_l \), it follows from (2.7) that \( \chi^{-1}_l(\eta_l) = \chi_l(-\eta_l) \). As a consequence, it is thereby shown that \( \Delta_l^{-1}(\eta_l) = \Delta_l(-\eta_l) \) by recalling \( \det \Delta_l = 1 \).

With the above claims, it is followed that

\[
I_3 = \frac{t^{-1/2}}{\sqrt{2g'_I(\eta_l)}} \left[ \Delta_l(\eta_l) \begin{pmatrix} 0 & -i\beta_{12}^{(n)} \\ i\beta_{21}^{(n)} & 0 \end{pmatrix} \Delta_l^{-1}(\eta_l) + \Delta_l^{-1}(\eta_l) \begin{pmatrix} 0 & -i\beta_{12}^{(-n)} \\ i\beta_{21}^{(-n)} & 0 \end{pmatrix} \Delta_l(\eta_l) \right] + O(t^{-1})
\]

\[
= \frac{t^{-1/2}}{\sqrt{2g'_I(\eta_l)}} \left[ \left( \chi_l^2(\eta_l) + \chi_l^{-2}(\eta_l) \right) \left( \beta_{12}^{(n)} - \beta_{12}^{(-n)} - \beta_{21}^{(n)} - \beta_{21}^{(-n)} \right) \right] + O(t^{-1}).
\]

Recalling the definition of \( g_l, \chi_l, \beta_{12}^{(n)} \) and \( \beta_{21}^{(n)} \), we finally obtain (3.52). □

### 3.6 Proof of the part (a) of Theorem 1.1

By tracing back the transformations (3.3), (3.6), (3.10) and (3.11), we conclude, for \( k \in \mathbb{C} \setminus \Gamma^{(3)} \),

\[
M(k) = D_{l,\infty}^{-\sigma_3}(\xi)M^{(err)}M^{(\infty)}D_{l}^{-\sigma_3}(k)e^{-it(g_l-\theta)\sigma_3},
\]

(3.58)

where \( D_{l,\infty}, M^{(err)}, M^{(\infty)} \) are defined in (3.3), (3.41) and (3.22) respectively.

From the reconstruction formula stated in (2.41), we then obtain that

\[
g(x,t) = 2iD_{l,\infty}^{-2}(\xi) \left[ (M^{(err)})_{12} + \lim_{k \to \infty} k (\Delta_l(k))_{12} \right].
\]

(3.59)

Together with (2.6) and (3.52), it is accomplished the part (a) of Theorem 1.1

### 4 Asymptotic analysis of the RH problem for \( M \) in \( \mathcal{R}_{II} \)

At the beginning of this section, it is indicated that we adopt the same notations (such as \( M^{(1)}, M^{(2)}, M^{(err)} \)) as those used in the previous section, and we believe this will not lead to any confusion.

#### 4.1 First transformation: \( M \to M^{(1)} \)

The \( g \)-function

For \( \xi \in \mathcal{R}_{II} \), we introduce

\[
g_{II}(\xi;k) := 4X^3(\eta_l), \quad X_{\eta}(\xi;k) := \sqrt{k^2 - \eta^2}(\xi),
\]

(4.1)

with \( \eta(\xi) := \sqrt{2\xi} \in (c_r, c_l) \). The branch of the square root is chosen such that \( X_\eta(k) = k + O(k^{-1}) \) as \( k \to \infty \).

It’s readily verified that the following properties for \( g_{II} \) defined in (4.1) hold true.

**Proposition 4.1.** The \( g_{II} \) function defined in (4.1) satisfies the following properties:

- \( g_{II}(\xi;k) \) is holomorphic for \( k \in \mathbb{C} \setminus [-\eta(\xi), \eta(\xi)] \).
- For \( k \in (-\eta, \eta) \), \( g_{II,+}(\xi;k) + g_{II,-}(\xi;k) = 0 \).
- As \( k \to \infty \) in \( \mathbb{C} \setminus [-\eta(\xi), \eta(\xi)] \), we have \( g_{II}(\xi;k) = \theta(\xi;k) + O(k^{-1}) \).
- As \( k \to \pm \eta \), \( g_{II}(\xi;k) = O((k \mp \eta)^{3/2}) \).

The \( k \)-derivative of \( g \) function can be expressed by

\[
g'_{II}(k) = 12k\sqrt{k^2 - \eta^2}.
\]

The signature table for \( \text{Im } g_{II} \) is illustrated in Figure [6].
4.2 Second transformation: \( M^{(1)} \rightarrow M^{(2)} \)

The \( D \) function

Similar to (3.6), we define an auxiliary function

\[
D_{II}(\xi; k) := \exp \left\{ \frac{X_\delta(k)}{2\pi i} \left( \int_{-\eta}^{-c_r} + \int_{c_r}^{\eta} \frac{\log r_+(s)}{X_{\eta+}(s)(s-k)} \, ds \right) \right\}. \tag{4.5}
\]

It follows the next proposition.

**Proposition 4.2.** The \( D_{II} \) function defined in (4.5) satisfies the following properties for \( \xi \in \mathcal{R}_{II} \):

(a) \( D_{II}(k) \) is holomorphic for \( k \in \mathbb{C} \setminus [-\eta, \eta] \).

(b) \( D_{II}(k) \) satisfies the jump relation:

\[
\begin{align*}
D_{II,+}(k)D_{II,-}(k) &= r_+(k), \quad k \in (-\eta, -c_r) \cup (c_r, \eta), \\
D_{II,+}(k)D_{II,-}(k) &= 1, \quad k \in (-c_r, c_r), \\
D_{II,+}(k) &= D_{II,-}(k), \quad \text{elsewhere.}
\end{align*} \tag{4.6}
\]
(c) As \( k \to \infty \) in \( \mathbb{C} \setminus [-\eta, \eta] \), 
\[ D_{II}(\xi;k) = D_{II,\infty}(\xi) + \mathcal{O}(k^{-1}) \]
where
\[
D_{II,\infty}(\xi) := \exp \left\{ -\frac{1}{2\pi i} \left( \int_{-\eta}^{c_r} + \int_{c_r}^{\eta} \right) \frac{\log r_+(s)}{X_{\eta+}(s)(s-k)} \, ds \right\}. \tag{4.7}
\]

(d) \( D_{II}(k) \) shows the following singular behavior at each endpoint
\[
D_{II}(k) = (k \mp p)\text{sgn Im} \, k \, D_{II,0}(k), \quad k \to \pm p, \quad p \in \{ c_r, c_l, \eta \}. \tag{4.8}
\]
where \( D_{II,0}(k) \) is a bounded function taking a definite limit as \( k \) approaches each singular point non-tangentially.

Proof. The similar manners stated in the proof of Proposition 3.2 could be also applied for this proposition. \( \square \)

**RH problem for \( M^{(2)} \)**

Let us define a new matrix-valued function \( M^{(2)}(k) := M^{(2)}(x,t,k) \) by
\[
M^{(2)}(k) = D_{II,\infty}^{(2)}(\xi)M^{(1)}(k)D_{II}^{(2)}(k). \tag{4.9}
\]

Then RH conditions for \( M^{(2)} \) are listed as follows:

**RH problem 4.2.**

- \( M^{(2)}(k) \) is holomorphic for \( k \in \mathbb{C} \setminus \mathbb{R} \).
- For \( k \in \mathbb{R} \), we have
\[
M^{(2)}_{-}(k) = M^{(2)}_{-}(k)V^{(2)}(k), \tag{4.10}
\]
where
\[
V^{(2)}(k) = \begin{cases} 
\begin{pmatrix} 1 - rr^* & -D_{II}^{(2)}r^*e^{-2itg_{II}} \\ D_{II}^{(2)}r^*e^{2itg_{II}} & 1 \end{pmatrix}, & k \in (-\infty, -c_l) \cup (c_l, +\infty), \\
\begin{pmatrix} 0 & -D_{II}^{(2)}r^*e^{-2itg_{II}} \\ D_{II}^{(2)}r^*e^{2itg_{II}} & 1 \end{pmatrix}, & k \in (-c_l, -\eta) \cup (\eta, c_l), \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \in (-\eta, -c_r) \cup (c_r, \eta), \\
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & k \in (-c_r, c_r).
\end{cases} \tag{4.11}
\]
- As \( k \to \infty \) in \( \mathbb{C} \setminus \mathbb{R} \), we have \( M^{(2)}(k) = I + \mathcal{O}(k^{-1}) \).
- As \( k \to \pm \eta \), we have \( M^{(2)}(k) = \mathcal{O}\left((k \mp \eta)^{-\frac{1}{2}}\right) \).

To obtain the \( V^{(2)} \) for \( k \in (-\eta, -c_r) \cup (c_r, \eta) \), we used the fact that \( D_{II,+}D_{II,-} = r_+ r^*_+ = 1 \) for \( k \in (-\eta, -c_r) \cup (c_r, \eta) \).

**4.3 Third transformation: \( M^{(2)} \to M^{(3)} \)**

Similar to (3.10), we define
\[
M^{(3)}(k) = M^{(2)}(k)D_{II}^{(2)}(k)G(k)D_{II}^{(-2)}(k), \tag{4.12}
\]
where
\[
G := \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -re^{2itg_{II}} & 1 \end{pmatrix}, & k \in U_{1}^{(3)} \cup U_{2}^{(3)}, \\
\begin{pmatrix} 1 & -r^*e^{-2itg_{II}} \\ 0 & 1 \end{pmatrix}, & k \in U_{1}^{(3)*} \cup U_{2}^{(3)*}, \\
I, & \text{elsewhere.}
\end{cases} \tag{4.13}
\]

Then RH problem for \( M^{(3)} \) reads as follows:
RH problem 4.3.

- $M^{(3)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \Gamma^{(3)}$, where $\Gamma^{(3)} := \cup_{j=1}^{2}(\Gamma^{(3)}_j \cup \Gamma^{(3)*}_j) \cup [-\eta, \eta]$; see Figure 7 for an illustration.

- For $k \in \Gamma^{(3)}$, we have

\[
M^{(3)}_+(k) = M^{(3)}_-(k)V^{(3)}(k),
\]

with

\[
V^{(3)}(k) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-D_{11}^{-2}r e^{2it\theta_{11}} & 1 \\
0 & -D_{11}^{-2}r e^{-2it\theta_{11}} \\
1 & 0 
\end{pmatrix}, & k \in \Gamma^{(3)}_1 \cup \Gamma^{(3)}_2, \\
\begin{pmatrix} 1 & 0 \\
-D_{11}^{-2}r^{-e^{-2it\theta_{11}}+} & 1 \\
0 & -1 \\
1 & 0 
\end{pmatrix}, & k \in (\theta_1, -c_r) \cup (c_r, \theta), \\
\begin{pmatrix} 0 & 1 \\
1 & 0 
\end{pmatrix}, & k \in (-c_r, c_r).
\end{cases}
\]  

(4.15)

- As $k \to \infty$ in $\mathbb{C} \setminus \Gamma^{(3)}$, we have $M^{(3)}(k) = I + O(k^{-1})$.

- As $k \to \pm \eta$, we have $M^{(3)}(k) = O((k \mp \eta)^{-1/4})$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure7.png}
\caption{The jump contours of RH problem for $M^{(3)}$ when $\xi \in \mathcal{R}_{11}$.}
\end{figure}

4.4 Analysis of RH problem for $M^{(3)}$

It’s also noted that $V^{(3)} \to I$ as $t \to +\infty$ on the contours $\Gamma^{(3)}_j \cup \Gamma^{(3)*}_j$ for $j = 1, 2$. Thus it follows that $M^{(3)}$ is approximated, to the leading order, by the global parametrix $M^{(\infty)}$ given below. The sub-leading contribution stems from the local behavior near the saddle points $\pm \eta$, which is well approximated by the Airy parametrix.

Global Parametrix

As $t$ large enough, the jump matrix $V^{(3)}$ approaches

\[
V^{(\infty)}(k) = \begin{pmatrix} 0 & -1 \\
1 & 0 
\end{pmatrix}, \quad k \in (-\eta, \eta).
\]

(4.16)

For $k \in \mathbb{C} \setminus [-\eta, \eta]$, $V^{(3)} \to I$ as $t \to \infty$. Following the manners of constructing RH problem 4.3, we similarly obtain:

RH problem 4.4.

- $M^{(\infty)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus [-\eta, \eta]$.

- For $k \in (-\eta, \eta)$, we have

\[
M^{(\infty)}_+ = M^{(\infty)}_- \begin{pmatrix} 0 & -1 \\
1 & 0 
\end{pmatrix}.
\]

(4.17)
As $k \to \infty$ in $\mathbb{C} \setminus [-\eta, \eta]$, we have $M^{(\infty)}(k) = I + \mathcal{O}(k^{-1})$.

As $k \to \pm \eta$, $M^{(\infty)}(k) = \mathcal{O}((k \mp \eta)^{-1/4})$.

Then the unique solution of $M^{(\infty)}$ is given by

$$M^{(\infty)}(k) = \Delta_{\eta}(k) := \frac{1}{2} \begin{pmatrix} \chi_{\eta}(k) + \chi^{-1}_{\eta}(k) & i \left( \chi_{\eta}(k) - \chi^{-1}_{\eta}(k) \right) \\ -i \left( \chi_{\eta}(k) - \chi^{-1}_{\eta}(k) \right) & \chi_{\eta}(k) + \chi^{-1}_{\eta}(k) \end{pmatrix},$$ (4.18)

with

$$\chi_{\eta}(k) = \left( \frac{k - \eta}{k + \eta} \right)^{\frac{1}{2}} = \left( \frac{k - \sqrt{-2\xi}}{k + \sqrt{-2\xi}} \right)^{\frac{1}{2}}.$$ (4.19)

For later use, it is required that, as $k \to \eta$,

$$M^{(\infty)} = \frac{(2\eta)^{\frac{1}{2}}}{2(k - \eta)^{1/4}} \left[ \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{(k - \eta)^{\frac{1}{2}}}{(2\eta)^{\frac{1}{2}}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + \frac{k - \eta}{8\eta} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{(k - \eta)^{\frac{3}{2}}}{4(2\eta)^{3/2}} \begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix} + \mathcal{O}\left((k - \eta)^{2}\right) \right].$$ (4.20)

Local parametrices near $\pm \eta$

Let

$$U^{(r)} = \{ k : |k - \eta| < \varrho \}, \quad U^{(l)} = \{ k : |k + \eta| < \varrho \},$$ (4.21)

be two small disks around $\eta$ and $-\eta$, respectively, where

$$\varrho < \frac{1}{3} \min \{ |\eta - c_{l}|, |\eta + c_{l}|, |\eta| \}.$$ (4.22)

For $\ell \in \{r, l\}$, we intend to solve the following local RH problem for $M^{(\ell)}$.

**RH problem 4.5.**

- $M^{(\ell)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \Gamma^{(\ell)}$, where
  $$\Gamma^{(\ell)} := U^{(\ell)} \cap \Gamma^{(3)};$$ (4.23)

  see Figure 8 for an illustration.

- For $k \in \Gamma^{(\ell)}$, we have
  $$M^{(\ell)}_{+}(k) = M^{(\ell)}_{-}(k)V^{(\ell)}(k),$$ (4.24)

  and where $V^{(\ell)}(k)$ is defined as (4.15).

- As $k \to \infty$ in $\mathbb{C} \setminus \Gamma^{(\ell)}$, we have $M^{(\ell)}(k) = I + \mathcal{O}(k^{-1})$.

![Figure 8: The local jump contours of RH problem for $M^{(\ell)}$ (right) and $M^{(\ell)}$ (left) for $\xi \in \mathcal{R}_{II}$.](image)
By the definition of $g_{II}$ function (4.11), we define the fractional power $g_{II}(\xi;k)^{3/2}$ for $k \in (-\infty, \eta]$ with the branch fixed by the requirement that $g_{II}^{-3/2} > 0$ for $k > \eta$. Introduce
\begin{equation}
 f(k) := -\left(\frac{3}{2} g_{II}(k)\right)^{\frac{1}{2}}, \quad k \in \mathbb{C}\setminus(-\infty, \eta].
 \end{equation}
As $k \to \eta$, it follows that
\begin{equation}
 f(\xi;k) = -2 \cdot 6^{\frac{2}{3}}\eta (k-\eta) \left(1 + \frac{k-\eta}{2\eta} + O((k-\eta)^2)\right).
 \end{equation}
Introduce a new scaled variable
\begin{equation}
 \zeta_r(\xi;k) = i^{2/3} f(\xi;k).
 \end{equation}
Then
\begin{equation}
 \frac{4}{3} \xi^2 = \begin{cases} 
 2i t g_{II}(k), & \text{Im } k > 0, \\
 -2i t g_{II}(k), & \text{Im } k < 0,
\end{cases}
\end{equation}
where the cut $(-)^{3/2}$ runs along $\mathbb{R}^-$ in the Airy parametrix stated in Appendix B.

Under the change of variable (4.28), we can solve the $M^{(r)}$ by using the Airy parametrix exhibited in Appendix B in a standard way. More precisely, define the local parametrix $M^{(r)}$ by
\begin{equation}
 M^{(r)}(x,t;k) := P^{(r)}(\xi;k) \psi^{(Ai)}(\zeta_r(\xi;k)) Q^{(r)}(\xi;k),
 \end{equation}
where $P^{(r)}$ is the matching factor given by
\begin{equation}
 P^{(r)}(k) := \begin{cases} 
 M^{(\infty)} Q^{(r)}(k) = \left(\psi^{(Ai)}_0\right)^{-1} \zeta_r^{\frac{1}{4}}, & k \in \mathbb{C}^+ \cap U^{(r)}, \\
 M^{(\infty)} Q^{(r)}(k) = \left(\psi^{(Ai)}_0\right)^{-1} \zeta_r^{\frac{1}{4}}, & k \in \mathbb{C}^- \cap U^{(r)}.
\end{cases}
\end{equation}
where $M^{(\infty)}$ is given by (4.18), $(\psi^{(Ai)}_0)^{-1}$ is defined in (4.10) and
\begin{equation}
 Q^{(r)}(k) := \begin{cases} 
 Q^{(r)}_1 = \sigma_1 r^{-\frac{3}{2}} D_{II}^{-\sigma_1} \sigma_3, & k \in \mathbb{C}^+ \cap U^{(r)}, \\
 Q^{(r)}_2 = \sigma_1 (r^{\frac{3}{2}} D_{II}^{-\sigma_1} \sigma_3, & k \in \mathbb{C}^- \cap U^{(r)}.
\end{cases}
\end{equation}
Under the definitions above, an equivalent form of $P^{(r)}(k)$ is given by
\begin{equation}
 P^{(r)}(\xi;k) := \begin{cases} 
 e^{\frac{\pi}{\sqrt{3}} \sqrt{\pi} M^{(\infty)} \sigma_3 \sigma_1 \sigma_3 \left(1 - \frac{1}{1}\right)} \zeta_r^{\frac{1}{2}}, & k \in \mathbb{C}^+ \cap U^{(r)}, \\
 e^{\frac{\pi}{\sqrt{3}} \sqrt{\pi} M^{(\infty)} \sigma_3 \sigma_1 \sigma_3 \left(1 - \frac{1}{1}\right)} \zeta_r^{\frac{1}{2}}, & k \in \mathbb{C}^- \cap U^{(r)},
\end{cases}
\end{equation}
which is convenient for use.

For later use, it follows from (4.8), (4.20), (4.26) and (4.32) that as $k \to \eta$,
\begin{equation}
 P^{(r)}(\xi;k) = e^{\frac{\pi}{\sqrt{3}} \sqrt{\pi} (2\eta)^{\frac{3}{4}}} \begin{pmatrix} 
 i \left(D_{II,0} \sigma^{-\frac{3}{2}}(\eta) - D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)\right) (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}} \\
 i \left(D_{II,0} \sigma^{-\frac{3}{2}}(\eta) + D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)\right) (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}}
\end{pmatrix} D_{II,0}^{1} \sigma^{-\frac{3}{2}}(\eta)^{\frac{1}{2}} - D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)^{\frac{1}{2}} (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}}
\end{equation}
where $D_{II,0} := D_{II,0}(\eta) = \lim_{k \to \eta}(\gamma(k) - \eta)^{-1/4} D_{II}(k)$.

One can verify that $P^{(r)}$ is holomorphic for $k \in U^{(r)}$ and $M^{(\infty)} M^{(r)^{-1}} \to I$ for $k \in \partial U^{(r)}$ as $t \to \infty$ with the aid of Proposition (4.2), (4.18) and (4.32).

Finally, the RH problem for $M^{(l)}$ can be solved in similar manners by defining a new holomorphic factor $P^{(l)}(\xi;k)$, which is similar to (4.30) with some modifications. Here, we only show that, as $k \to -\eta$,
\begin{equation}
 P^{(l)}(k) = e^{\frac{\pi}{\sqrt{3}} \sqrt{\pi}(2\eta)^{\frac{3}{4}}} \begin{pmatrix} 
 \left(D_{II,0} \sigma^{-\frac{3}{2}}(\eta) - D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)\right) (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}} \\
 \left(D_{II,0} \sigma^{-\frac{3}{2}}(\eta) + D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)\right) (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}}
\end{pmatrix} D_{II,0}^{1} \sigma^{-\frac{3}{2}}(\eta)^{\frac{1}{2}} - D_{II,0}^{-1} \sigma^{-\frac{3}{2}}(\eta)^{\frac{1}{2}} (2\eta)^{\frac{3}{4}} (6\eta)^{\frac{1}{4}}
\end{equation}
where $D_{II,0}$ is also the quantity in $P^{(r)}(\xi;k)$ due to the symmetry of $D_{II}$, and we additionally use the symmetric relation $r(\eta) = \bar{r}(\eta)$ for $\eta \in \mathbb{R}$.
4.5 Small norm RH problem for $M^{(err)}$

Define

$$M^{(err)}(x, t; k) := \begin{cases} M^{(3)}(x, t; k) \left( M^{(\infty)}(x, t; k) \right)^{-1}, & k \in \mathbb{C} \setminus \left( U^{(r)} \cup U^{(l)} \right), \\ M^{(r)}(x, t; k) \left( M^{(\infty)}(x, t; k) \right)^{-1}, & k \in U^{(r)}, \\ M^{(l)}(x, t; k) \left( M^{(\infty)}(x, t; k) \right)^{-1}, & k \in U^{(l)}. \end{cases} \quad (4.35)$$

It’s readily seen that $M^{(err)}$ satisfies the following RH problem.

**RH problem 4.6.**

- $M^{(err)}$ is holomorphic for $k \in \mathbb{C} \setminus \Gamma^{(err)}$, where
  $$\Gamma^{(err)} := \partial U^{(r)} \cup \partial U^{(l)} \cup \left( \Gamma^{(3)} \setminus \left( U^{(r)} \cup U^{(l)} \right) \right);$$
  see Figure 9 for an illustration.

- For $k \in \Gamma^{(err)}$, we have
  $$M^{(err)}_{+}(k) = M^{(err)}_{-}(k)V^{(err)}(k),$$
  where
  $$V^{(err)}(k) = \begin{cases} M^{(\infty)}(k)V^{(3)}(k)M^{(\infty)}(k)^{-1}, & k \in \Gamma^{(3)} \setminus \left( U^{(r)} \cup U^{(l)} \right), \\ M^{(r)}(k)M^{(\infty)}(k)^{-1}, & k \in \partial U^{(r)}, \\ M^{(l)}(k)M^{(\infty)}(k)^{-1}, & k \in \partial U^{(l)}. \end{cases} \quad (4.38)$$

- As $k \to \infty$ in $k \in \mathbb{C} \setminus \Gamma^{(err)}$, we have $M^{(err)}(k) = I + \mathcal{O}(k^{-1})$.

- As $k \to \pm \eta$, we have $M^{(err)}(k) = \mathcal{O}(1)$.

![Diagram](image-url)

Figure 9: The jump contour $\Gamma^{(err)}$ of RH problem for $M^{(err)}$ for $\xi \in \mathcal{R}_{11}$.

A simple calculation shows that for $p = 1, 2, \infty$,

$$\|V^{(err)} - I\|_{L^p} = \begin{cases} \mathcal{O}(e^{-ct}), & k \in \Gamma^{(err)} \setminus \left( U^{(r)} \cup U^{(l)} \right), \\ \mathcal{O}(t^{-1}), & k \in \partial U^{(r)} \cup \partial U^{(l)}, \end{cases} \quad (4.39)$$

with some positive constant $c$. Indeed, for $k \in \Gamma^{(3)} \setminus \left( U^{(r)} \cup U^{(l)} \right)$, it follows from (4.38) and (4.18) that

$$\|V^{(err)} - I\|_{L^p} = \|\Delta_\eta[V^{(3)} - I] \Delta_\eta^{-1}\| = \mathcal{O}(e^{-ct})$$

with some constant $c$. For $k \in \partial U^{(r)}$ (or $\partial U^{(l)}$), we obtain from (4.38), (4.29) and (4.18) that

$$|V^{(err)} - I| = |\Delta_\eta Q^{(r)}(\Psi^{(1)}_{0})^{-1} \sigma_\eta A^{(1)} Q^{(r)}(\Psi^{(1)}_{0})^{-1} - I| = |\Delta_\eta Q^{(r)}(\Psi^{(1)}_{0})^{-1} \sigma_\eta A^{(1)} - I) Q^{(r)}(\Delta_\eta^{-1}}|$$

$$\leq \mathcal{O}(t^{-1}),$$

(4.41)
It then follows from the small norm RH problem theory \[8\] that there exists a unique solution to RH problem 4.6 for large positive \(t\). Furthermore, according to Beals-Coifman theory [3] again, the solution for \(M^{(\text{err})}\) can be given by
\[
M^{(\text{err})}(k) = I + \frac{1}{2\pi i} \int_{\Gamma_{\text{err}}} \frac{\mu(s) (V^{(\text{err})}(s) - I)}{s - k} \, ds,
\]
where \(\mu \in I + L^2(\Gamma^{(\text{err})})\) is the unique solution of (3.44).

Analogous to the estimate (3.38)
\[
\|C^{(\text{err})}\| \leq \|C^{-}\|_{L^2 \to L^2} \|V^{(\text{err})} - I\|_{L^2} \lesssim O(t^{-1}),
\]
which implies that \(I - C^{(\text{err})}\) is invertible for large positive \(t\) in this case.

The next proposition is an analogue of Proposition 3.3 with \(M_1^{(\text{err})}\) defined in (3.51).

**Proposition 4.3.** With \(M_1^{(\text{err})}\) defined in (3.51), we have, as \(t \to +\infty\),
\[
M_1^{(\text{err})} = \text{* } \frac{3\sqrt{2}}{6} \left( \frac{D_{11,6}^{-1/2} (\sqrt{x^2/6} - D_{11,6}^{-1/2} (\sqrt{x^2/6}))^2}{D_{11,6}^{-1/2} (\sqrt{x^2/6} - D_{11,6}^{-1/2} (\sqrt{x^2/6}))^2} \text{* } \right)_{\nu_1 + \nu_2 t^2/n^2} + O(t^{-2}),
\]
with \(\eta = \sqrt{2} t\), \(s_1 = \frac{\nu(\eta)}{5 + \nu(\eta)}\), \(\nu_1 = -\frac{2}{5} s_1\).

**Proof.** Let us divide \(M_1^{(\text{err})}\) into three parts by
\[
I_1 := -\frac{1}{2\pi i} \int_{\Gamma_{\text{err}}} \frac{\mu(s) - I}{V^{(\text{err})}(s) - I} \, ds,
\]
\[
I_2 := -\frac{1}{2\pi i} \int_{\Gamma_{\text{err}} \setminus \{U^{(r)} \cup U^{(l)}\}} \frac{V^{(\text{err})}(s) - I}{V^{(\text{err})}(s) - I} \, ds,
\]
\[
I_3 := -\frac{1}{2\pi i} \int_{\partial U^{(r)} \cup \partial U^{(l)}} \frac{V^{(\text{err})}(s) - I}{V^{(\text{err})}(s) - I} \, ds.
\]

Analogous to Proposition 3.3 the main contribution for \(M_1^{(\text{err})}\) stems from the \(I_3\), and the error bounds stems from \(I_1\) with \(I_1 = O(t^{-1})\) by using (3.33).

To estimate \(I_3\), let us divide \(I_3 := I_3^{(r)} + I_3^{(l)}\), where \(I_3^{(r)}\) and \(I_3^{(l)}\) are two contour integrations along \(\partial U^{(r)}\) and \(\partial U^{(l)}\) respectively. Detailed analysis is given for \(I_3^{(r)}\) below, and the similar manners could be applied to \(I_3^{(l)}\).

To proceed, let us define
\[
T_j^{(r)}(\xi; k) := f^{-3j/2} \Delta_\eta Q^{(r)}_1^{-1} K_j Q^{(r)}_1 \Delta_\eta^{-1}, \quad j \geq 1,
\]
where \(f, \Delta_\eta, K_j\) and \(Q^{(r)}\) are defined in (4.23), (1.18), (3.12) and (4.31), respectively. From the definition of (4.33), it’s readily verified that the expansion \(\sum_{j=1}^{\infty} t^{-j} T_j^{(r)}\) converges absolutely for large positive \(t\) under the condition that \(\xi \in \mathcal{R}_{\mathcal{I}}\) and \(k \in \partial U^{(r)}\). It then follows from (4.33) that
\[
I_3^{(r)} = -\frac{1}{2\pi i} \int_{\partial U^{(r)}} \left( \sum_{j=1}^{\infty} T_j^{(r)}(\xi; k) d\nu \right) \, ds = t^{-1} \text{Res}_{k=\eta} T_1^{(r)} + O(t^{-2}).
\]
It remains to estimate the \(\text{Res}_{k=\eta} T_1^{(r)}\).

With the expression (B.12), we rewrite
\[
T_1^{(r)} = P^{(r)}(x, 1; k) f^{-2} \Psi_0^{-1}(\Lambda) K \frac{\Lambda}{f^{3/2}} \Psi_0^{-1}(\Lambda) f^{2} P^{(r)}(x, 1; k)^{-1},
\]
where \(P^{(r)}(x, 1; k)\) is the holomorphic factor near \(\eta\) and defined by
\[
P^{(r)}(x, 1; k) := \Delta_\eta(x, 1; k) Q^{(r)}(x, 1; k)^{-1} \left( \Psi_0^{-1}(\Lambda) \right)^{-1} \left( f(x, 1; k) \right)^{2}\).
\]
On the account of (4.26), we obtain that, for \( k \to \eta \)
\[
\begin{align*}
  f^{1/4} &= e^{-\frac{x}{2} + \frac{3}{2} \eta^2 (k-\eta) \frac{k}{2}} \left( 1 + \frac{k-\eta}{8\eta} + O\left( (k-\eta)^2 \right) \right), \\
  f^{3/2} &= e^\frac{x}{2} \frac{2 + \eta^2}{\eta^2} (k-\eta) \frac{k}{2} \left( 1 + \frac{3k-\eta}{4\eta} + O\left( (k-\eta)^2 \right) \right).
\end{align*}
\] (4.52)

Then, as \( k \to \eta \), it is followed that the middle factor of \( T_1^{(r)} \) defined in (4.30) is given by
\[
\begin{align*}
  f^{-\tfrac{x}{2}} T_0^{(A)} \frac{K_1}{f^{1/2}} \left( T_0^{(A)} \right)^{-1} f^\tfrac{x}{2} &= \frac{3}{2} \left( \begin{array}{cc} 0 & s_1 f^{-2} \\ -\nu_1 f^{-1} & 0 \end{array} \right) \\
  &= \frac{3}{2} \left( \begin{array}{cc} 0 & s_1 \left( \frac{1}{(1296)^{1/3} \eta^2 (k-\eta)^2} - \frac{4}{(1296)^{1/3} \eta^2 (k-\eta)^2} \right) \right) + O(1).
\end{align*}
\] (4.53)

The rest mission is to calculate the quantities \( P^{(r)}(x; 1; \eta) \) and the \( k \)-derivative of \( P^{(r)}(x; 1; \eta) \). Indeed, the former one equals \( (4.33) \) for fixed \( t = 1 \); the later one does not determine the Res\(_{k=\eta} T_1^{(r)} \) by observing that the term \((k-\eta)^{3/2}\) in \( (4.32) \).

Substitute (4.33), (4.52) into (4.50), we obtain that
\[
\text{Res}_{k=\eta} T_1^{(r)} = \left( \begin{array}{c} 3_{96} \sqrt{x/6} \left( \begin{array}{c} D_{11,0} r^{-1/2} \left( \sqrt{x/6} \right) - D_{11,0}^2 r^{1/2} \left( \sqrt{x/6} \right) \end{array} \right) \frac{s_1 + x/\nu_1 \eta^2}{\eta^3 \sqrt{x/6}} \right).
\] (4.54)

Analogous to estimate \( I_1^{(r)} \), we have
\[
I_1^{(l)} = t^{-1} \text{Res}_{k=-\eta} T_1^{(l)} + O(t^{-2}),
\] (4.55)
where \( T_1^{(l)} \) is similarly defined as \( T_1^{(r)} \) with some modifications. Following the steps of calculating Res\(_{k=\eta} T_1^{(r)} \), we also obtain
\[
\text{Res}_{k=-\eta} T_1^{(l)} = \left( \begin{array}{c} -3_{96} \sqrt{x/6} \left( \begin{array}{c} D_{11,0} r^{-1/2} \left( \sqrt{x/6} \right) - D_{11,0}^2 r^{1/2} \left( \sqrt{x/6} \right) \end{array} \right) \frac{s_1 + x/\nu_1 \eta^2}{\eta^3 \sqrt{x/6}} \right).
\] (4.56)

On the account of (4.49), (4.51), (4.55) and (4.56), it arrives at the end of the proof.

4.6 Proof of the part (b) of Theorem 1.1

By tracing back the transformations (4.2), (4.3), (4.12) and (4.33), we conclude, for \( k \in \mathbb{C} \setminus \Gamma(3) \),
\[
M(k) = D_{II,\infty}^{\sigma_3}(\xi) M^{(cr)}(\xi) D_{II,\infty}^{\sigma_3}(k) e^{-i(tg_{II,\theta})(\xi)}
\] (4.57)
where \( D_{II,\infty}, M^{(cr)}, M^{(\infty)} \) are defined in (4.7), (4.35) and (4.18) respectively.

Together with the reconstruction formula stated in (2.41), it is accomplished that
\[
q(x, t) = 2i D_{II,\infty}^2(\xi) \left[ (M_1^{(cr)})_{12} + \lim_{k \to \infty} k (\Delta q(k))_{12} \right].
\] (4.58)

On the account of (4.18) and (4.44), we obtain the part (b) of Theorem 1.1

5 Asymptotic analysis of the RH problem for \( M \) in \( R_{III} \)

5.1 First transformation: \( M \to M^{(1)} \)

The \( g \)-function

For \( \xi \in \mathcal{R}_{III} \), we introduce
\[
g_{III}(\xi; k) := (4k^2 + 12\xi + 2c_2^2) X_r(k), \quad X_r(k) = \sqrt{k^2 - c_2^2},
\] (5.1)
where the branch of the square root being chosen such that \( X_r(k) = k + O(k^{-1}) \) as \( k \to \infty \). It’s readily verified that the following properties for \( g_{III} \) defined in (5.1) hold true.
Proposition 5.1. The $g_{III}$ function defined in (5.1) satisfies the following properties:

- $g_{III}(\xi; k)$ is holomorphic for $k \in \mathbb{C} \setminus [-c_r, c_r]$.
- As $k \rightarrow \infty$ in $\mathbb{C} \setminus [-c_r, c_r]$, we have $g_{III}(\xi; k) = \theta(\xi; k) + \mathcal{O}(k^{-1})$.
- For $k \in (-c_r, c_r)$, $g_{III,+}(\xi; k) + g_{III,-}(\xi; k) = 0$.

It’s readily seen that the $k$-derivative of $g_{III}(k)$ is given by

$$g'_{III}(k) = 12k - \eta_r(\xi) (k + \eta_r(\xi)) \eta_r(k), \quad \eta_r(\xi) = \sqrt{-\xi + \frac{\varepsilon^2}{2}} \in (0, c_r).$$  

(5.2)

The signature table for $\text{Im}g_{III}$ is illustrated in Figure 10.

\text{Figure 10: Signature table of the function $\text{Im}g_{III}(\xi; k)$ for $\xi \in \mathcal{R}_{III}$.}

RH problem for $M^{(1)}$

With the aid of $g_{III}$ function defined in (5.1), we define a new matrix-valued function $M^{(1)}(k) := M^{(1)}(x, t; k)$ by

$$M^{(1)}(x, t; k) := M(x, t; k)e^{itg_{III}(\xi; k) - \theta(\xi; k))\sigma_3}.$$  

(5.3)

Then RH problem for $M^{(1)}$ reads as follows:

RH problem 5.1.

- $M^{(1)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \mathbb{R}$.
- For $k \in \mathbb{R}$, we have

$$M^{(1)}_+(k) = M^{(1)}_-(k)V^{(1)}(k), \quad k \in \mathbb{R},$$  

(5.4)

where

$$V^{(1)}(k) = \begin{cases} 
\begin{pmatrix}
1 & -r^*_e^{2itg_{III}} \\
r^*_e^{2itg_{III}} & 1
\end{pmatrix}, & k \in (-\infty, -c_t) \cup (c_t, +\infty), \\
\begin{pmatrix}
0 & -r^*_e^{2itg_{III}} \\
r^*_e^{2itg_{III}} & 1
\end{pmatrix}, & k \in (-c_t, -c_r) \cup (c_r, c_t), \\
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, & k \in (-c_r, c_r).
\end{cases}$$  

(5.5)

- As $k \rightarrow \infty$ in $\mathbb{C} \setminus \mathbb{R}$, we have $M^{(1)}(k) = I + \mathcal{O}(k^{-1})$.
- $M^{(1)}(k)$ admits the same singular behavior as $M(k)$ at branch points $\pm c_r$. 

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5.2 Second transformation: $M^{(1)} \to M^{(2)}$

Differing from the previous sections, it does not require an auxiliary function $D$ to open lenses for this case. Let us define a new matrix-valued function $M^{(2)}(x,t;k)$ by

$$M^{(2)}(k) = M^{(1)}(k)G(k), \quad (5.6)$$

where

$$G := \begin{cases} 
1 & k \in U^{(2)}_{1} \cup U^{(2)}_{2}, \\
1 - re^{2it\theta} & k \in U^{(2)*}_{1} \cup U^{(2)*}_{2}, \\
I & \text{elsewhere}. 
\end{cases} \quad (5.7)$$

RH conditions for $M^{(2)}$ are listed as follows.

**RH problem 5.2.**

- $M^{(2)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus \Gamma^{(2)}$, where $\Gamma^{(2)} := \bigcup_{j=1}^{2} (\Gamma^{(2)}_{j} \cup \Gamma^{(2)*}_{j}) \cup [-c_{r}, c_{r}]$; see Figure 11 for an illustration.

- For $k \in \Gamma^{(2)}$, we have

$$M^{(2)}_{+}(k) = M^{(2)}_{-}(k)V^{(2)}(k), \quad (5.8)$$

where

$$V^{(2)}(k) = \begin{cases} 
1 & k \in \Gamma^{(2)}_{+} \cup \Gamma^{(2)}_{-}, \\
1 - r^{*}e^{-2it\theta} & k \in \Gamma^{(2)*}_{1} \cup \Gamma^{(2)*}_{2}, \\
0 & k \in (-c_{r}, c_{r}). 
\end{cases} \quad (5.9)$$

- As $k \to \infty$ in $\mathbb{C} \setminus \Gamma^{(2)}$, we have $M^{(2)}(k) = I + O(k^{-1})$.

- As $k \to \pm c_{r}$, $M^{(2)}(k) = O((k \mp c_{r})^{-1/4})$.

![Figure 11: The jump contours of RH problem for $M^{(2)}$ when $\xi \in \mathcal{R}_{III}$.](image)

5.3 Analysis of RH problem for $M^{(2)}$

Note that $V^{(2)} \to I$ as $t \to +\infty$ on the contours $\Gamma^{(2)}_{j} \cup \Gamma^{(2)*}_{j}$ for $j = 1, 2$, it follows that $M^{(2)}$ is approximated, to the leading order, by the global parametrix $M^{(\infty)}$ given below. The sub-leading contribution stems from the local behavior near the branch points $\pm c_{r}$, which is also well approximated by the Airy parametrix.
Global parametrix

As $t$ large enough, the jump matrix $V^{(2)}$ approaches

$$V^{(\infty)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \in (-c_r, c_r).$$  \hfill (5.10)

For $k \in \mathbb{C} \setminus [-c_r, c_r]$, $V^{(2)} \to I$ as $t \to \infty$. Then it is naturally established the following parametrix for $M^{(\infty)}$.

**RH problem 5.3.**

- $M^{(\infty)}(k)$ is holomorphic for $k \in \mathbb{C} \setminus [-c_r, c_r]$.

- For $k \in (-c_r, c_r)$, we have
  $$M^{(\infty)}_{+} = M^{(\infty)}_{-} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  \hfill (5.11)

- As $k \to \infty$ in $\mathbb{C} \setminus [-c_r, c_r]$, we have $M^{(\infty)} = I + O(k^{-1})$.

- As $k \to \pm c_r$, $M^{(\infty)} = O((k \mp c_r)^{-1/4})$.

Then the unique solution of $M^{(\infty)}$ is given by

$$M^{(\infty)} = \Delta_r(k),$$  \hfill (5.12)

with $\Delta_r(k)$ defined by (2.10) for the subscript $j$ being chosen to $r$.

**Local parametrix near $\pm c_r$**

In each neighborhood of $k = \pm c_r$, we still need to construct a local parametrix $M^{(\ell)}$ for $\ell \in \{r, l\}$. These parametrices can be built with of Airy functions as last section. Thus the details are omitted here.

Finally, define the small norm RH problem

$$M^{(err)}(x, t; k) := \begin{cases} M^{(2)}(x, t; k) \left( M^{(\infty)}(x, t; k) \right)^{-1}, & k \in \mathbb{C} \setminus \left( U^{(r)} \cup U^{(l)} \right), \\ M^{(2)}(x, t; k) \left( M^{(r)}(x, t; k) \right)^{-1}, & k \in U^{(r)}, \\ M^{(2)}(x, t; k) \left( M^{(l)}(x, t; k) \right)^{-1}, & k \in U^{(l)}. \end{cases}$$  \hfill (5.13)

Following the analogous manners in the Subsection 4.5 one can exhibit a following proposition, which is similar to Proposition 4.3.

**Proposition 5.2.** With $M^{(err)}$ defined in (5.11), we have, as $t \to +\infty$,

$$M^{(err)}_1(t) = \begin{cases} 3\sqrt{2} & \frac{3\sqrt{2}}{96} \left( \frac{r^{-\frac{1}{2}} \left( \sqrt{-x/12 + c_r^2/2} \right)^2 - r^{\frac{1}{2}} \left( \sqrt{-x/12 + c_r^2/2} \right)^2}{r^{-1} \left( \sqrt{-x/12 + c_r^2/2} \right)^{-1}} \right) \frac{s_1 + \nu_1 \eta^2}{\eta^2 \sqrt{-x/12 + c_r^2/2}}, & k \in \mathbb{C} \setminus \Gamma^{(2)}, \\ + O(t^{-2}), \end{cases}$$

with $\eta_r = \sqrt{-\xi + c_r^2/2}$, $s_1 = \frac{r(\xi)}{54 \Gamma(\frac{1}{3})}$, $\nu_1 = -\frac{7}{5} s_1$.

5.4 Proof of the part (c) of Theorem 1.1

By tracing back the transformations (5.3), (5.6), and (5.13), we conclude, for $k \in \mathbb{C} \setminus \Gamma^{(2)}$,

$$M(k) = M^{(err)} M^{(\infty)} e^{-it \theta_1} e^{-\bar{\theta}_3}.$$  \hfill (5.15)

Together with the reconstruction formula stated in (2.41), it is accomplished that

$$q(x, t) = 2i \left[ \left( M^{(err)}_1 \right)_{12} + \lim_{k \to \infty} k \left( \Delta_r(k) \right)_{12} \right].$$  \hfill (5.16)

On the account of (5.12) and (5.14), we obtain the part (c) of Theorem 1.1.
6 Asymptotic analysis of the RH problem for $M$ in $\mathcal{R}_{IV}$

To investigate large-time asymptotics of the Cauchy problem (1.1) – (1.3) in the zone $\mathcal{R}_{IV}$, one should notice the original phase function $\theta(\xi; k) = 4k^3 + 12k\xi$ with two pure imaginary saddle points $\pm i\sqrt{\xi}$ for $\xi > 0$. The signature table of the function

$$\text{Im} \theta(\xi; k) = 4k^2(3k^2_1 - k^2_2 + 3\xi), \quad k_1 = \text{Re} k, \quad k_2 = \text{Im} k.$$ 

is illustrated in Figure 12.

Define two contours $\Gamma_1 = k_1 + i\sqrt{\xi}, \Gamma^*_1 = k_1 - i\sqrt{\xi}$, which are parallel to $\mathbb{R}$ and two domains $U_1 = \{k : 0 < \text{Im} k < i\sqrt{\xi}\}, \quad U^*_1 = \{k : -i\sqrt{\xi} < \text{Im} k < 0\};$ see Fig 13 for an illustration.

We are now ready to define the following transformation

$$M^{(1)}(x, t; k) := M(x, t; k)G(x, t; k), \quad (6.1)$$

where

$$G := \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
-rc^{2it\theta} & 1 
\end{array} \right), & k \in U_1, \\
\left( \begin{array}{cc} 1 & -r^*e^{-2it\theta} \\
0 & 1 
\end{array} \right), & k \in U^*_1, \\
I, & \text{elsewhere.} 
\end{cases} \quad (6.2)$$

Then $M^{(1)}$ satisfies the following RH problem:

RH problem 6.1.

- $M^{(1)}(k)$ is holomorphic for $k \in \mathbb{C}\setminus([-c_r, c_r] \cup \Gamma_1 \cup \Gamma^*_1)$; see Figure 12 for an illustration.

- For $k \in \mathbb{R} \cup \Gamma_1 \cup \Gamma^*_1$, we have

$$M^{(1)}_{-}(k) = M^{(1)}_{+}(k)V^{(1)}(k), \quad (6.3)$$

where

$$V^{(1)}(k) = \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
rc^{2it\theta} & 1 
\end{array} \right), & k \in \Gamma_1, \\
\left( \begin{array}{cc} 1 & -r^*e^{-2it\theta} \\
0 & 1 
\end{array} \right), & k \in \Gamma^*_1, \\
\left( \begin{array}{cc} 0 & -1 \\
1 & 0 
\end{array} \right), & k \in (-c_r, c_r). 
\end{cases} \quad (6.4)$$

- As $k \to \infty$ in $\mathbb{C}\setminus(\mathbb{R} \cup \Gamma_1 \cup \Gamma^*_1)$, $M^{(1)}(k) = I + \mathcal{O}(k^{-1})$. 

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Remark 6.1. The jump condition on \( \mathbb{R} \) should be analyzed in terms of \( k \in (-\infty, -c_1) \cup (c_1, +\infty), \) \( k \in (-c_1, -c_r) \cup (c_r, c_1) \) and \( k \in (-c_r, c_r) \) respectively. For \( k \in (-c_1, -c_r) \cup (c_r, c_1), \) we indeed use the relation \( r_+ r_- = 1 \) to obtain that \( V^{(1)} = I. \)

![Figure 13: Jump contours of RH problem for \( M^{(1)} \) when \( \xi \in \mathbb{R}_IV. \)](image)

As \( t \to \infty, \) it can be obtained that

\[
M^{(1)} \to \Delta_r(k), \quad |V^{(1)}(k) - I| = \mathcal{O}\left(e^{-8\pi k (3k_2^2 - k_2^2 + 3\xi)}\right). \tag{6.5}
\]

Notice that \( k_2 = i\sqrt{\xi}, \) we have \( |V^{(1)}(k) - I| = \mathcal{O}\left(e^{-16\pi \sqrt{\xi}/2}\right). \) Using the reconstruction formula (2.41), we have

\[
q(x,t) = c_r + \mathcal{O}(t^{-1/2}e^{-16\pi \sqrt{\xi}/2}),
\]

which is the part (d) of Theorem 1.1.

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### A Parabolic cylinder parametrix

Find a matrix-valued function \( M^{(PC)}(\xi) := M^{(PC)}(\xi; \kappa), \) and the solution is characterized by the following RH problem:

**RH problem A.1.**

- \( M^{(PC)} \) is holomorphic for \( \xi \in \mathbb{C} \setminus \Gamma^{(PC)}, \) where

\[
\Gamma^{(PC)} := \left\{ \Re e^{\pm i \pi/4} \right\} \cup \left\{ \Re e^{\pm i \pi/2} \right\}. \tag{A.1}
\]

*See Figure A for an illustration.*

- For \( \xi \in \Gamma^{(PC)}, \) we have

\[
M^{(PC)}_+(\xi) = M^{(PC)}_-^{(PC)}(\xi) V^{(PC)}(\xi), \tag{A.2}
\]

where

\[
V^{(PC)}(\xi) = \begin{cases}
\zeta^{\nu} e^{-\frac{i\pi}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}, & \xi \in \Re e^{i \pi/4}, \\
\zeta^{\nu} e^{-\frac{i\pi}{2} \sigma_3} \begin{pmatrix} 1 & -\kappa^* \\ 0 & 1 \end{pmatrix}, & \xi \in \Re e^{-i \pi/4}, \\
\zeta^{\nu} e^{-\frac{i\pi}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ -\kappa & \pm 1 \end{pmatrix}, & \xi \in \Re e^{-\pi/4}, \\
\zeta^{\nu} e^{-\frac{i\pi}{2} \sigma_3} \begin{pmatrix} 1 & -\kappa^* \\ 0 & 1 \end{pmatrix}, & \xi \in \Re e^{3\pi/4},
\end{cases} \tag{A.3}
\]

and \( \nu = \nu(\eta). \)
Figure A1: The jump contours of RH problem for $M^{(PC)}$.

- As $\zeta \to \infty$ in $\mathbb{C} \setminus \Gamma^{(PC)}$, we have $M^{(PC)}(\zeta) = I + M_1^{(PC)} \zeta^{-1} + O(\zeta^{-2})$.

The RHP $M^{(PC)}(\zeta)$ has an explicit solution, which is associated to the Webber equation $(\frac{d^2}{d\zeta^2} + (\frac{1}{4} - \frac{z^2}{2} + a)) D_\alpha(z) = 0$. Taking the transformation $M^{(PC)} = \psi(\zeta) \mathcal{P} \zeta^{-i\nu \sigma_3} e^{\frac{i\pi}{2} \sigma_3}$, \(A.4\)

where

\[
\mathcal{P}(\xi) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix}, & \text{arg} \zeta \in (0, \pi/4), \\
\begin{pmatrix} 1 & -\kappa^* \\ 0 & 1 \end{pmatrix}, & \text{arg} \zeta \in (-\pi/4, 0), \\
\begin{pmatrix} 1 & 0 \\ \frac{\kappa}{1-\kappa^*} & 1 \end{pmatrix}, & \text{arg} \zeta \in (-3\pi/4, -\pi), \\
\begin{pmatrix} 1 & \kappa^* \\ 0 & 1 \end{pmatrix}, & \text{arg} \zeta \in (3\pi/4, \pi), \\
I, & \text{else}
\end{cases}
\]

\(A.5\)

The RH problem for $\psi$ reads as follows:

**RH problem A.2.**

- $\psi$ is holomorphic for $\zeta \in \mathbb{C} \setminus \mathbb{R}$.

- Due to the branch cut along $\mathbb{R}^-$, $\psi(\zeta)$ takes continuous boundary values $\psi_{\pm}$ on $\mathbb{R}$ and

\[
\psi_+(\zeta) = \psi_-(\zeta) V^{(\psi)}, \quad \zeta \in \mathbb{R},
\]

where

\[
V^{(\psi)} = \begin{pmatrix} 1 - \kappa \kappa^* & -\kappa^* \\ \kappa & 1 \end{pmatrix}.
\]

\(A.7\)

- As $\zeta \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$, we have

\[
\psi = \zeta^{i\nu \sigma_3} e^{-\frac{i\pi}{2} \sigma_3} \left( I + M_1^{(PC)} \zeta^{-1} + O(\zeta^{-2}) \right).
\]

\(A.8\)

Differentiating \(A.6\) with respect to $\zeta$, and combining $\frac{i\kappa}{2} \sigma_3 \psi_+ = \frac{i\kappa}{2} \sigma_3 \psi - V^{(\psi)}$, we obtain

\[
\left( \frac{d\psi}{d\zeta} + \frac{i\kappa}{2} \sigma_3 \psi \right)_+ = \left( \frac{d\psi}{d\zeta} + \frac{i\kappa}{2} \sigma_3 \psi \right)_- - V^{(\psi)}.
\]

\(A.9\)

It’s not difficult to verify the matrix function $\left( \frac{d\psi}{d\zeta} + \frac{i\kappa}{2} \sigma_3 \psi \right)^{-1}$ has no jump along the real axis and is an entire function with respect to $\zeta$. Combining \(A.4\), it is straightforward calculated that

\[
\left( \frac{d\psi}{d\zeta} + \frac{i\kappa}{2} \sigma_3 \psi \right)^{-1} = \left[ \frac{dM^{(PC)}}{d\zeta} + M^{(PC)} i\nu \sigma_3 \right] \left( M^{(PC)} \right)^{-1} + \frac{i\kappa}{2} \left[ \sigma_3, M^{(PC)} \right] \left( M^{(PC)} \right)^{-1},
\]

\(A.10\)
The first term in the R.H.S of (A.10) tends to zero as $\zeta \to \infty$. We use $M^{(PC)} = I + M_1^{(PC)} \zeta^{-1} + \mathcal{O}(\zeta^{-2})$ as well as Liouville theorem to obtain that there exists a constant matrix $\beta^{mat}$ such that

$$
\beta^{mat} := \begin{pmatrix}
0 & \beta^{(q)}_{12} \\
\beta^{(q)}_{21} & 0
\end{pmatrix} = \frac{i}{2} \left[ \sigma_3, M_1^{(PC)} \right] = \begin{pmatrix}
0 & i[M_1^{(PC)}]_{21} \\
-i[M_1^{(PC)}]_{12} & 0
\end{pmatrix},
$$

(A.11)

which implies that $[M_1^{(PC)}]_{12} = -i\beta^{(q)}_{12}$, $[M_1^{(PC)}]_{21} = i\beta^{(q)}_{21}$. Using Liouville theorem again, we have

$$
\frac{d\psi}{d\zeta} + \frac{i}{2} \sigma_3 \psi = \beta^{mat} \psi.
$$

(A.12)

We rewrite the above equality to the following ODE systems

$$
\frac{d\psi_{11}}{d\zeta} + \frac{i}{2} \psi_{11} = \beta^{(q)}_{12} \psi_{21},
$$

(A.13)

$$
\frac{d\psi_{21}}{d\zeta} - \frac{i}{2} \psi_{21} = \beta^{(q)}_{21} \psi_{11},
$$

(A.14)

as well as

$$
\frac{d\psi_{12}}{d\zeta} + \frac{i}{2} \psi_{12} = \beta^{(q)}_{12} \psi_{22},
$$

(A.15)

$$
\frac{d\psi_{22}}{d\zeta} - \frac{i}{2} \psi_{22} = \beta^{(q)}_{21} \psi_{12}.
$$

(A.16)

From (A.13) to (A.16), we can solve that

$$
\frac{d^2\psi_{11}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta^{(q)}_{12} \beta^{(q)}_{21} \right) \psi_{11} = 0,
$$

(A.17)

$$
\frac{d^2\psi_{12}}{d\zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta^{(q)}_{12} \beta^{(q)}_{21} \right) \psi_{12} = 0.
$$

(A.18)

We set $\nu = \beta^{(q)}_{12} \beta^{(q)}_{21}$. For $\psi_{11}$, $\text{Im} \zeta > 0$ we introduce the new variable $\bar{\eta} = \zeta e^{-\frac{\pi}{4\nu}}$, and the first equation of (A.17) becomes

$$
\frac{d^2\bar{\eta}}{d\eta^2} + \left( \frac{1}{2} \eta^2 + \frac{\pi}{4} + i\nu \right) \bar{\eta} = 0.
$$

(A.19)

For $\zeta \in \mathbb{C}^+$, $0 < \text{Arg} \zeta < \pi$, $-\frac{3\pi}{4} < \text{Arg} \bar{\eta} < \frac{\pi}{4}$. We have $\psi_{11} = e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) \sim \zeta^{\nu} e^{-\frac{\pi}{4} i \zeta}$ corresponding to the $(1,1)$-entry of (A.8). To save the space, we present the other results for $\psi$ below.

The unique solution to is when $\zeta \in \mathbb{C}^+$,

$$
\psi(\zeta) = \begin{pmatrix}
\frac{\nu(\eta)}{\beta^{(q)}_{12}} e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) & -\frac{\nu(\eta)}{\beta^{(q)}_{21}} e^{-\frac{\pi}{4} \nu(\eta-1)} D_{\nu(\eta)-1}(e^{-\frac{\pi}{4} i \zeta}) \\
\frac{\nu(\eta)}{\beta^{(q)}_{21}} e^{-\frac{\pi}{4} \nu(\eta+1)} D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) & \frac{\nu(\eta)}{\beta^{(q)}_{12}} e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)-1}(e^{-\frac{\pi}{4} i \zeta})
\end{pmatrix},
$$

(A.20)

when $\zeta \in \mathbb{C}^-$,

$$
\psi(\zeta) = \begin{pmatrix}
\frac{\nu(\eta)}{\beta^{(q)}_{12}} e^{-\frac{\pi}{4} \nu(\eta)+1} D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta}) & -\frac{\nu(\eta)}{\beta^{(q)}_{21}} e^{-\frac{\pi}{4} \nu(\eta)-1} D_{\nu(\eta)-1}(e^{\frac{\pi}{4} i \zeta}) \\
\frac{\nu(\eta)}{\beta^{(q)}_{21}} e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)+1}(e^{\frac{\pi}{4} i \zeta}) & \frac{\nu(\eta)}{\beta^{(q)}_{12}} e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta})
\end{pmatrix},
$$

(A.21)

which is derived in [A.4, Section 4].

From (A.4), we know that $(\psi_-)^{-1} \psi_+ = V^{(\zeta)}$ and

$$
\kappa = \psi_{-11} \psi_{21} - \psi_{-21} \psi_{11} = e^{\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta}) \cdot e^{-\frac{3\pi}{4} \nu(\eta)} \left[ \partial_\eta (D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta})) + \frac{i}{2} D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) \right]
$$

$$
- e^{-\frac{\pi}{4} \nu(\eta)} D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) \cdot e^{\frac{\pi}{4} \nu(\eta)} \left[ \partial_\eta (D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta})) + \frac{i}{2} D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta}) \right]
$$

$$
eq e^{\frac{\pi}{4} \nu(\eta)} \frac{\nu(\eta)}{\beta^{(q)}_{12}} \mathcal{W} \left( D_{\nu(\eta)}(e^{\frac{\pi}{4} i \zeta}), D_{\nu(\eta)}(e^{-\frac{\pi}{4} i \zeta}) \right)
$$

$$
eq e^{-\frac{\pi}{4} \nu(\eta)} \frac{\nu(\eta)}{\beta^{(q)}_{12}} \sqrt{2 \pi e^{\frac{\pi}{4} i}} \frac{1}{\Gamma(-\nu(\eta))},
$$

(A.22)
where
\[ \beta^{(n)}_{12} = \frac{\sqrt{2\pi e^{\frac{2\pi}{n}} e^{-\frac{\pi i}{n}}}}{\kappa \Gamma(-i\nu(\eta))}, \quad \beta^{(n)}_{12} \beta^{(n)}_{21} = \nu(\eta). \] (A.23)

Finally we have, as \( \zeta \to \infty \),
\[ M^{(PC)} = I + \frac{1}{\zeta} \left( \begin{array}{cc} 0 & -i\beta^{(n)}_{12} \\ i\beta^{(n)}_{21} & 0 \end{array} \right) + O(\zeta^{-2}). \] (A.24)

B Airy parametrix

Fix the four rays on the complex plane
\[ \Gamma^{(Ai)}_1 = \{ \zeta \in \mathbb{C} | \arg \zeta = 2\pi/3 \}, \quad \Gamma^{(Ai)}_2 = \{ \zeta \in \mathbb{C} | \arg \zeta = \pi \}, \]
\[ \Gamma^{(Ai)}_3 = \{ \zeta \in \mathbb{C} | \arg \zeta = -2\pi/3 \}, \quad \Gamma^{(Ai)}_4 = \{ \zeta \in \mathbb{C} | \arg \zeta = 0 \}, \] (B.1)

all oriented from left-to-right. Then we define the sectors
\[ S_1 = \{ \zeta \in \mathbb{C} | \arg \zeta \in (0, 2\pi/3) \}, \quad S_2 = \{ \zeta \in \mathbb{C} | \arg \zeta \in (2\pi/3, \pi) \}, \]
and \( S_3, S_4 \) are the conjugative sectors of \( S_2, S_1 \) respectively; see Figure B2 for an illustration.

![Figure B2: The contours of the model Airy RH problem.](image)

Find a \( 2 \times 2 \) matrix-valued function \( \Psi^{(Ai)}(\zeta) \) be the solution of the following RH problem:

**RH problem B.1.**

- \( \Psi^{(Ai)} \) is holomorphic for \( \zeta \in \mathbb{C} \setminus \Gamma^{(Ai)} \), where \( \Gamma^{(Ai)} = \bigcup_{j=1}^{4} \Gamma^{(Ai)}_j \).
- For \( \zeta \in \Gamma^{(Ai)} \), we have
  \[ \Psi^{(Ai)}_+(\zeta) = \Psi^{(Ai)}_-(\zeta)V^{(Ai)}(\zeta), \] (B.3)
  where
  \[ V^{(Ai)}(\zeta) = \begin{cases} \left( \begin{array}{cc} 1 & 0 \\ e^{\frac{2\pi}{3}\zeta} & 1 \end{array} \right), & \zeta \in \Gamma^{(Ai)}_1 \cup \Gamma^{(Ai)}_3, \\ \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & \zeta \in \Gamma^{(Ai)}_2, \\ \left( \begin{array}{cc} 1 & e^{-\frac{2\pi}{3}\zeta} \\ 0 & 1 \end{array} \right), & \zeta \in \Gamma^{(Ai)}_4. \end{cases} \] (B.4)
- As \( \zeta \to \infty \) in \( \mathbb{C} \setminus \Gamma^{(Ai)} \), we have \( \Psi^{(Ai)}(\zeta) = I + O(\zeta^{-1}) \).

The explicit form of \( \Psi^{(Ai)} \) is given by
\[ \Psi^{(Ai)}(\zeta) = \begin{cases} A(\zeta) e^{\frac{2\pi}{3}\zeta}, & \zeta \in S_1, \\ A(\zeta) \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) e^{\frac{2\pi}{3}\zeta}, & \zeta \in S_2, \\ A(\zeta) \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) e^{\frac{2\pi}{3}\zeta}, & \zeta \in S_3, \\ A(\zeta) e^{\frac{2\pi}{3}\zeta}, & \zeta \in S_4, \end{cases} \] (B.5)
where
\[
\mathcal{A}(\zeta) = \begin{pmatrix}
\text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\
\text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta)
\end{pmatrix} e^{-\frac{i\pi}{3}\sigma_3}, \quad \zeta \in \mathbb{C}^+,
\end{equation}
\] (B.6)
with \(\omega = e^{2\pi i/3}\).

\[\Psi^{(Ai)}(\zeta)\] admits the full asymptotic behavior as \(\zeta \to \infty\):
\[
\Psi^{(Ai)}(\zeta) \sim e^{\frac{\pi i}{2\sqrt{3}}\zeta} \sum_{j=0}^{\infty} \left( \frac{(-1)^j s_j}{\zeta} \right)^{-j} e^{-\frac{\sqrt{3}}{2} \zeta} \left( \frac{2}{3} \right)^{-j}, \quad \zeta \to \infty,
\]
where
\[
s_0 = \nu_0 = 1, \quad s_j = \frac{\Gamma(3j + \frac{1}{2})}{54j^2 \Gamma(j + \frac{3}{2})}, \quad \nu_j = \frac{6j + 1}{1 - 6j^2} s_j, \quad j \geq 1.
\]
(B.7)

For the convenience of our use, we rewrite (B.7) to
\[
\Psi^{(Ai)}(\zeta) \sim \zeta^{-\frac{\pi i}{2\sqrt{3}}} \Psi^{(Ai)}_0 \left( I + \sum_{j=1}^{\infty} \frac{\psi^{(Ai)}_0}{\zeta^{3j/2}} \right), \quad \zeta \to \infty,
\]
where
\[
\psi^{(Ai)}_0 = e^{\frac{\pi i}{2\sqrt{3}}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\sqrt{3}}{2} \zeta} = e^{-\frac{\sqrt{3}}{2} \zeta} e^{\frac{\sqrt{3}}{2} \zeta} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\]
(B.10)
\[
\psi^{(Ai)}_j := e^{\frac{\pi i}{2\sqrt{3}}} \begin{pmatrix} (-1)^j s_j \\ (-1)^j \nu_j \end{pmatrix} e^{-\frac{\sqrt{3}}{2} \zeta}.
\]
(B.11)

Let
\[
\mathcal{K}_j := \left( \psi^{(Ai)}_0 \right)^{-1} \left( \frac{2}{3} \right)^{-j} \mathcal{R}_j \left( \psi^{(Ai)}_j \left( \frac{2}{3} \right)^{-j} \right) = \left( \frac{3}{2} \right)^{j \frac{1}{3}} e^{-\frac{\sqrt{3}}{2} \zeta} \begin{pmatrix}
(-1)^j (s_j - \nu_j) & s_j - \nu_j \\
(-1)^j (s_j + \nu_j) & s_j + \nu_j
\end{pmatrix} e^{-\frac{\sqrt{3}}{2} \zeta}.
\]
(B.12)

Thus as \(\zeta \to \infty\), we obtain the closed asymptotic form
\[
\Psi^{(Ai)}(\zeta) \sim \zeta^{-\frac{\pi i}{2\sqrt{3}}} \Psi^{(Ai)}_0 \left( I + \sum_{j=1}^{\infty} \frac{\mathcal{K}_j}{\zeta^{3j/2}} \right),
\]
(B.13)

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