Recent advances in the global theory of constant mean curvature surfaces

Rafe Mazzeo *

Abstract

The theory of complete surfaces of (nonzero) constant mean curvature in $\mathbb{R}^3$ has progressed markedly in the last decade. This paper surveys a number of these developments in the setting of Alexandrov embedded surfaces; the focus is on gluing constructions and moduli space theory, and the analytic techniques on which these results depend. The last section contains some new results about smoothing the moduli space and about CMC surfaces in asymptotically Euclidean manifolds.

1 Introduction

The study of global minimal and constant mean curvature (CMC) surfaces in $\mathbb{R}^3$ has always occupied a central position in classical differential geometry, but the former class of surfaces has undoubtedly received more attention than the latter. While partly historical accident, this is also due to the classical Weierstraß representation formula for minimal surfaces, through which many interesting examples of minimal surfaces were known. In contrast, the most famous results about CMC surfaces prior to 1980 were the rigidity theorems of Hopf and Alexandrov, which state that the only compact CMC surfaces in $\mathbb{R}^3$ which are immersed and genus 0, respectively embedded of arbitrary genus, are spheres; in some sense this is a negative result. There has been an explosive development in both these fields during the past twenty years, centering on many new examples and methods of construction, and also including various classification and structure theorems. The recent volume [4] contains up-to-the-minute accounts of many of these developments, but mostly focussing on minimal surfaces. My goal here is to survey some

*Department of Mathematics, Stanford University, Stanford, CA 94305, mazzeo@math.stanford.edu; supported by NSF Grant # DMS 0204730
of the advances in the global study of CMC surfaces from my own biased perspective.

Let Σ be a surface in \( \mathbb{R}^3 \). Its mean curvature \( H \) at a point \( p \) is the sum of the two principal curvatures (the customary factor of \( 1/2 \) is omitted), and Σ has constant mean curvature if this quantity is constant. In this case we can normalize so that either \( H \equiv 0 \) or \( H \equiv 1 \). We shall restrict attention exclusively to the second case. In addition, we shall only be concerned with complete, properly immersed surfaces of finite topology. There is a rich theory for the Plateau problem for minimal and CMC surfaces, cf. [35], [2], but we shall not discuss this at all; in particular, we shall understand all surfaces below to be complete, unless otherwise stated. On the other hand, unlike the situation for minimal surfaces, cf. [6] for example, the subject of CMC surfaces with infinite topology is mostly unexplored.

The only complete CMC surfaces known classically are the sphere, the cylinder, and the one-parameter family of rotationally invariant Delaunay surfaces, which we describe in the next section. (We should also mention a method, due originally to Bonnet but most clearly enunciated by Lawson, for associating a ‘cousin’ CMC surface in \( \mathbb{R}^3 \) to a minimal surface in \( S^3 \).) The first modern breakthrough was Wente’s discovery, detailed in his 1983 paper [36], of immersed CMC tori. Not long afterwards, Kapouleas used transcendental PDE methods to construct many new CMC surfaces, including compact ones with arbitrary genus [13], [14], and noncompact ones with finitely many ends [12]. Große-Brauckman [9] subsequently constructed certain of these noncompact surfaces with large discrete symmetry groups using more classical methods based on Schwarz reflection.

The theory has developed in two fairly distinct directions. From the techniques used for Wente’s construction ultimately has emerged the DPW (Dorfmeister-Pedit-Wu) method, which draws on the theory of integrable systems, and serves as a replacement for the Weierstraß representation; on the other hand, Kapouleas’ construction has engendered many new PDE approaches to the problem. These theories have distinct flavors, and in some senses illuminate quite different classes of surfaces. For example, the former method seems to be more successful in describing general immersed CMC surfaces, and is closely related to many of the computer experiments and simulations of CMC surfaces, cf. [15], while the latter has been particularly good in describing CMC surfaces which are ‘nearly embedded’. We specialize once again, and for the last time, to the second PDE set of methods.

It turns out that requiring CMC surfaces to be embedded is too restrictive, and in some sense even geometrically unnatural. Thus, we shall relax this hypothesis and require instead that our CMC surfaces be Alexan-
To define this, first recall that a complete CMC surface $\Sigma$ with finite topology is diffeomorphic to a punctured Riemann surface $\Sigma \setminus \{p_1, \ldots, p_k\}$; we shall always assume that $\Sigma$ and $\Sigma$ are oriented. Writing $\Sigma = \partial Y^3$, then we say that $\Sigma$ is Alexandrov embedded if the immersion $\Sigma \hookrightarrow \mathbb{R}^3$ extends to an immersion $Y \hookrightarrow \mathbb{R}^3$. This condition is naturally suited to the technique of Alexandrov reflection, cf. [18].

The basic questions we shall address are whether there are any Alexandrov embedded CMC surfaces, and if so, how might we go about constructing them and describing the totality of them. The next section discusses the main elementary examples and the basic structure theory of complete Alexandrov embedded CMC surfaces, and lays out the framework for the subsequent development. In §3 we describe a variety of analytic constructions, centered on the basic technique of Cauchy data matching. This is followed, in §4, by the ramifications of these constructions for the moduli space theory. Finally, in §5, we generalize this theory to CMC surfaces in arbitrary asymptotically Euclidean 3-manifolds and suggest some possible directions of future research.

This survey has some overlap with one given in the recent paper by Kusner [19], which I recommend strongly to the interested reader; his emphasis is rather different than mine and I hope that these two papers complement one another in a useful way. I should also mention that the forthcoming thesis of M. Jleli at Université Paris XII contains the analogues of essentially all of the results here for CMC hypersurfaces in $\mathbb{R}^n$.

My papers in this area have been written in various collaborations with Rob Kusner, Frank Pacard, Dan Pollack and Jesse Ratzkin. I owe special thanks to Frank Pacard for all he has taught me over the years! Kusner, Pacard and Pollack each read and gave me many helpful comments on a preliminary draft of this paper, but I happily take credit for the remaining errors and solecisms. As described in a previous survey [27], most of the results in this theory have parallels for solutions of the singular Yamabe problem with isolated singularities. In particular, the general moduli space results in §4 originated in my work with Pollack and Uhlenbeck [20] on deformations of solutions of the Yamabe equation with isolated singularities on the sphere. I am also grateful to Nick Korevaar and Rick Schoen for many illuminating discussions. I thank the Mittag-Leffler Institute and Universita di Roma “La Sapienza” for their hospitality while this paper was written. Finally, this paper is based on a talk given at the conference in honour of H. Brezis and F. Browder at Rutgers University in October 2001, and I would like to acknowledge that my first exposure to CMC surfaces was in a course taught by Professor Brezis at MIT in the early 1980’s.
2 Basic properties

In this section we introduce most of the basic definitions. Here and in the remainder of this paper we denote by $\mathcal{M}_{g,k}$ the set of all proper Alexandrov embedded constant mean curvature 1 surfaces in $\mathbb{R}^3$ of genus $g$ with $k$ ends, and when we say that a surface $\Sigma$ is CMC, we mean that it lies in one of these spaces. We begin by discussing the most elementary examples of complete noncompact CMC surfaces, the Delaunay surfaces. This is followed by a description of the basic structure theory of (Alexandrov embedded) CMC surfaces. We go on to discuss the Jacobi operator and its mapping properties, and the fundamental notion of nondegeneracy of a CMC surface.

2.1 Delaunay surfaces

The CMC surface with the largest symmetry group is, of course, the sphere, but the most familiar one after that is the cylinder. The sphere is the only surface (CMC or not) with full rotational symmetry, and by the classical results of Hopf and Alexandrov mentioned earlier, it is the only compact Alexandrov embedded CMC surface; on the other hand, there is an interesting family of axially symmetric CMC surfaces discovered by Delaunay \cite{7}, in which the cylinder lies as an extreme element. To describe these, consider the cylindrical graph $(t, \theta) \to (\rho(t) \cos \theta, \rho(t) \sin \theta, t)$. The CMC equation for this graph is an ODE,

$$
\rho_{tt} - \frac{1}{\rho}(1 + \rho_t^2) + (1 + \rho_t^2)^{3/2} = 0.
$$

(1)

All positive solutions of this equation are periodic; this follows from the facts that the function $H(\rho, \rho_t) = \rho^2 - 2\rho(1 + \rho_t^2)^{-1/2}$ is an integral of motion of this ODE and has compact level curves in $\{\rho > 0, H < 0\}$. (Kusner’s balancing formula, described below, gives a nice geometric interpretation of the fact that $H$ is constant on solutions.) For any $0 < \epsilon < 1$ let $\rho_\epsilon(t)$ be the solution which attains the minimum value $\epsilon$, and normalize by translating the independent variable so that $\rho_\epsilon(0) = \epsilon$. Then $\epsilon \leq \rho_\epsilon(t) \leq 2 - \epsilon$ for all $t$, and since $\rho_\epsilon$ remains positive, the surfaces $D_\epsilon$ which are the cylindrical graphs of these functions are embedded; they are called Delaunay unduloids.

There are two limiting cases: in the first, $\epsilon = 1$ and $D_1$ is the cylinder of radius 1; the other occurs when $\epsilon \to 0$, and then $D_\epsilon$ converges to an infinite array of mutually tangent spheres of radius 2 with centers along the $z$-axis. The family $D_\epsilon$ interpolates between these two extremes. The number $\epsilon$ measures the size of the ‘neck’ region.
The Delaunay family continues past the singular limit at $\epsilon = 0$ to a family of immersed (not Alexandrov embedded!) CMC surfaces known as the Delaunay nodoids. These will only play a minor role in this survey.

There is an alternate, conformal, parametrization of the $D_\epsilon$ which is more convenient for most calculations. This isothermal parametrization involves two functions $\sigma$ and $\kappa$: $\sigma$ is the unique smooth nonconstant solution to the initial value problem

$$\left(\frac{d\sigma}{ds}\right)^2 + \tau^2 \cosh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) < 0,$$

while $\kappa(s)$ is determined by

$$\frac{d\kappa}{ds} = \tau^2 e^\sigma \cosh \sigma, \quad \kappa(0) = 0.$$

Setting $t = \kappa(s)$ and $\rho(\kappa(s)) = \tau e^{\sigma(s)}$, and determining $\tau$ by $\epsilon = 1 - \sqrt{1 - \tau^2}$, then $D_\epsilon$ is parametrized by

$$X_\epsilon : \mathbb{R} \times S^1 \ni (s, \theta) \mapsto \left(\tau e^{\sigma(s)} \cos \theta, \tau e^{\sigma(s)} \sin \theta, \kappa(s)\right).$$

The metric coefficients in this coordinate system are $g_{ss} = g_{\theta\theta} = \tau^2 e^{2\sigma}$, and $g_{s\theta} = g_{\theta s} = 0$.

While this definition may seem ad hoc, it is motivated by a systematic line of reasoning in the theory of integrable systems; 

2.2 Structure theory

There are several important facts about the global structure of Alexandrov-embedded CMC surfaces of finite topology, and we review these now.

The story begins with an important theorem of Meeks [28], which states that any proper Alexandrov embedded end of a CMC surface is cylindrically bounded, i.e. contained in some tube of radius $R > 0$ around a ray. This result reflects the ‘smallness at infinity’ of the ambient Euclidean space and fails definitively, for example, in hyperbolic space. (Simple counterexamples are the equidistant surfaces around a totally geodesic copy of $\mathbb{H}^2 \subset \mathbb{H}^3$.) Meeks also proves that there are no one-ended CMC surfaces, i.e. $\mathcal{M}_{g,1} = \emptyset$.

Building on this, Korevaar, Kusner and Solomon [18] obtained a much sharper picture of the behaviour of CMC ends. More precisely, for any such end $E$ there is a half Delaunay unduloid $D_\epsilon^+ = D_\epsilon \cap \{z \geq 0\}$ and a
rigid motion \( A \) of \( \mathbb{R}^3 \) such that \( E \) converges exponentially to \( A(D_{\epsilon}^+) \). More precisely, if \( \nu \) is the outward unit normal to \( A(D_{\epsilon}^+) \), then any surface \( C^1 \) close to this Delaunay end can be written as a normal graph, i.e. as the image of

\[
A(D_{\epsilon}^+) \ni p \longrightarrow p + \phi(p) \nu(p).
\]

Then \( E \) is the normal graph of a function \( \phi \in \mathcal{C}^\infty(A(D_{\epsilon}^+)) \) which decays exponentially at infinity. The proof ultimately devolves to the exponential decay of Jacobi fields (see below) on a Delaunay surface and an adaptation of a beautiful ‘approximation improvement’ lemma due to L. Simon. Using Alexandrov reflection they prove that \( \mathcal{M}_{g,2} = \emptyset \) if \( g \geq 1 \), while \( \mathcal{M}_{0,2} \) contains only the Delaunay surfaces.

Amidst the many corollaries of this result is the fact that we can associate an asymptotic necksize parameter \( \epsilon \) to any end of \( \Sigma \in \mathcal{M}_{g,k} \). There are other ways to measure this necksize. For example, there is a ‘balancing formula’ for CMC surfaces, discovered by Kusner, which states the following. Let \( \Gamma \) be a simple closed curve on \( \Sigma \) and \( C \) an immersed surface in \( \mathbb{R}^3 \) with \( \partial C = \Gamma \); suppose that \( n \) and \( \nu \) are the unit normals to \( \Gamma \) in \( T\Sigma \) and \( C \), respectively. (Thus \( n \) is orthogonal to the unit normal to \( \Sigma \) in \( \mathbb{R}^3 \).) Then, with an appropriate choice of orientations of these normals, the quantity

\[
\frac{1}{\pi} \left( \int_{\Gamma} n \, d\sigma - \int_{C} \nu \, dA \right)
\]

depends only on the homology class of \( \Gamma \) in \( \Sigma \). Thus, for example, if \( \Gamma \) lies in an end \( E \) of \( \Sigma \) and represents a generator of \( H_1(E) \), then it is homologous to the curve \( \Gamma_{s} \) which is the normal graph in \( E \) of the circle \( \theta \to X_{s}(\theta) \) in the model Delaunay end, for any \( s \geq s_0 \). Using the exponential decay of \( E \) to this Delaunay end, this quantity can be computed explicitly, and in fact just equals \( \epsilon(2 - \epsilon)\vec{v} \), where \( \vec{v} \) is the unit vector in the direction of the Delaunay axis. This is called a balancing formula because if \( \Gamma_1, \ldots, \Gamma_k \) encircle the \( k \) ends of \( \Sigma \in \mathcal{M}_{g,k} \), and are oriented so that their sum is null homologous, then the sum of the corresponding force vectors must vanish.

In a subsequent paper [17], these authors and Meeks establish an analog of this asymptotics result for proper Alexandrov embedded CMC surfaces in the hyperbolic space \( \mathbb{H}^3 \), in the case that the mean curvature \( H > 2 \). We note that CMC (hypersurface) theory in \( \mathbb{H}^n \) is quite different in the three cases, \( H > n - 1 \), \( H < n - 1 \) and \( H = n - 1 \), the latter having the most similarities with minimal surface theory (via the cousin correspondence mentioned earlier), and the first having the most similarities with CMC theory in \( \mathbb{R}^n \). [17] also gives Delaunay asymptotics for ends of CMC hypersurfaces.
in $\mathbb{H}^n$, but only under the a priori assumption of cylindrical boundedness. There is a growing literature on CMC surfaces in hyperbolic space, cf. \cite{3}, \cite{31}, \cite{8}, and the work of H. Rosenberg and his collaborators. It is quite likely that the results of \cite{18} and \cite{17} remain true for CMC surfaces in manifolds which are only asymptotically Euclidean or hyperbolic, and indeed, the proofs from these papers may require only moderate alterations. We come back to this point in \S 5.

These asymptotics results definitely require Alexandrov-embeddness, or something near to it. For example, the Delaunay family $D_\epsilon$ continues beyond $\epsilon = 0$ to a family of immersed (not Alexandrov embedded!) rotationally invariant CMC surfaces called nodoids, and Pacard and I show \cite{22} that these become increasingly unstable as $\epsilon \to -\infty$: there are infinitely many values $\epsilon_j \to -\infty$ where new CMC surfaces which are not rotationally symmetric bifurcate away from this nodoid family. This bifurcation bears some relationship to the classical Rayleigh instability of the cylinder. It only occurs when $\epsilon \leq \epsilon_0 < 0$, and it is possible that there may be some global structure theory, akin to that described above, for properly immersed CMC surfaces for which all ends have force vector corresponding to some Delaunay nodoid $D_\epsilon$ with $\epsilon_0 < \epsilon < 0$. We leave this as an interesting unexplored direction.

One can interpret this asymptotics theorem by thinking of $\Sigma \in \mathcal{M}_{g,k}$ as decomposing into a large compact piece, about which we know very little a priori, with $k$ asymptotically Delaunay ends emerging from it. All of the gluing constructions described in \S 3 involve “filling in the black box in the middle” in different ways. Kapouleas’ original construction builds CMC surfaces around (suitably balanced) simplicial graphs where $k$ of the edges are rays tending to infinity. The finite edges and the rays are replaced by finite and semi-infinite segments of Delaunay surfaces, respectively, and the vertices become spheres. (Balancing here means that there are force vectors associated with each edge and these must cancel at each vertex. An additional condition regarding the flexibility of this arrangement is also needed.) As we discuss later, there are various other geometric possibilities for the vertices, but Korevaar and Kusner \cite{16} show that this picture is qualitatively correct in general. For any $\Sigma \in \mathcal{M}_{g,k}$, there is a simplicial graph, with explicit bounds on the numbers of edges and vertices in terms of $g$ and $k$, and a (fairly large) tubular neighbourhood around it which contains $\Sigma$. 

7
2.3 The Jacobi operator: its mapping properties, nullspace and nondegeneracy

If $\Sigma$ is a complete Alexandrov embedded CMC surface with finite topology, then we can consider all surfaces which are $C^2$ close to $\Sigma$ as normal graphs, i.e. for any $\phi \in C^2(\Sigma)$, sufficiently small, we set

$$\Sigma_\phi = \{ x + \phi(x)\nu(x) : x \in \Sigma \}$$

(3)

where $\nu(x)$ is the unit normal to $\Sigma$ at $x$. The equation which determines that $\Sigma_\phi$ also has CMC equal to 1 is a quasilinear elliptic equation for $\phi$. Its linearization at $\phi = 0$ is called the Jacobi operator $L$ (or $L_\Sigma$). It is well known that

$$L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2,$$

where $A_\Sigma$ is the second fundamental form. Solutions of $L\phi = 0$ are called Jacobi fields. If $u_t$ is a 1-parameter family of functions such that all of the surfaces $\Sigma_{u_t}$ are all CMC (with the same mean curvature), then $\phi = du_t/dt|_{t=0}$ is a Jacobi field on $\Sigma$. In general, Jacobi fields need only correspond to deformations of $\Sigma$ which are CMC to second order.

One very important consequence of the structure theory described in the last subsection is that because CMC surfaces have very well-controlled geometry at infinity, the behaviour of their Jacobi operators is governed by the behaviour of Jacobi operators on (half) Delaunay surfaces. Indeed, as we discuss in a moment, the asymptotics of any (tempered) Jacobi field on $\Sigma$ are determined by the asymptotics of tempered Jacobi fields on the model Delaunay ends. More generally, mapping properties for $L_\Sigma$ may be deduced from those for the Jacobi operators $L_\varepsilon$ on $D_\varepsilon$. This motivates a closer study of these model operators $L_\varepsilon$.

In cylindrical coordinates the expression for $L_\varepsilon$ is quite complicated, but using the isothermal parametrization, the Jacobi operator $L_\varepsilon$ on $D_\varepsilon$ takes the much simpler form

$$L_\varepsilon = \frac{1}{\tau^2 e^{2\sigma}} \left( \partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma) \right).$$

Since $\tau^2 e^{2\sigma}$ is bounded away from zero and infinity, it is sufficient to analyze the operator

$$\partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma(s)),$$

and this is what we do in practice. This analysis relies on the rotational invariance of this operator in $\theta$ and its periodicity in $s$. Thus we can simultaneously reduce by Fourier series in the $S^1$ factor and use an adapted form
of Floquet theory to handle the resulting ODEs with periodic coefficients. With these techniques we are able to determine that $L_\epsilon$ is Fredholm on certain natural exponentially weighted function spaces, and to characterize all global tempered solutions. We return to this last point shortly. The analysis of $L_\Sigma$ itself is accomplished by patching the parametrices for each of these model problems, corresponding to each of the ends $E_j$ of $\Sigma$, together with an interior parametrix. These arguments are presented in detail (for a slightly different geometric problem) in [26], cf. also [24].

We shall discuss Jacobi fields on $\Sigma$ in more detail in §4. For the present we note that one may generate a distinguished set of Jacobi fields on any CMC surface using rigid motions of the ambient space. The corresponding ‘geometric Jacobi fields’ thus correspond to infinitesimal translations and rotations of $\Sigma$ in $\mathbb{R}^3$. For a Delaunay surface $D_\epsilon$, the three independent translations induce three Jacobi fields, and the two infinitesimal rotations of the axis of symmetry induce two more. $D_\epsilon$ has a final Jacobi field, induced by change of the necksize parameter $\epsilon$. (The case $\epsilon = 1$ is handled slightly differently.) These all have the form $e^{i\ell \theta} \psi(s)$, where $\ell = 0, \pm 1$ (or rather, are real-valued combinations of these); for the Jacobi fields coming from translations, $\psi(s)$ is periodic, while for each of the others it grows linearly as $s \to \pm \infty$. These six Jacobi fields are the only global solutions of $L_\epsilon \phi = 0$ on $D_\epsilon$ which do not grow exponentially on one end or the other. In fact, there are solutions of the form $e^{i\ell \theta} \psi(s)$ for any $\ell \in \mathbb{Z}$, $|\ell| \geq 2$, but for each of these, $\psi(s + S_\epsilon) = e^{\gamma_\ell} \psi(s)$, where $S_\epsilon$ is the period of $\sigma$ and where $\gamma_\ell \in \mathbb{R}$, $\gamma_\ell \neq 0$. In particular, these other solutions have precise exponential rates of growth or decay. This set of exponents constitutes a discrete set $\Lambda_\epsilon$, called the indicial set of $L_\epsilon$.

As for the mapping properties of $L_\Sigma$, one’s first temptation might be to let this operator act on $L^2(\Sigma)$. However, this mapping does not have closed range. In fact, the parametrix method sketched above shows that the spectrum of $L_\Sigma$ on $L^2$ is comprised of bands of continuous spectrum and locally finite discrete spectrum. Unfortunately, 0 is always on the end of one of these bands of continuous spectrum. In the language of scattering theory, we say that it is a threshold value. To see this, recall from above that $\Sigma$ always admits nontrivial global bounded and linearly growing Jacobi fields on $\Sigma$, corresponding to translations and rotations in $\mathbb{R}^3$. This is already enough to imply that 0 is in the continuous spectrum by Weyl’s criterion; a closer examination of the Floquet theory, together with the existence of linearly growing (as opposed to bounded, periodic) Jacobi fields, shows that 0 is a threshold value.

In any case, one must look elsewhere to realize $L_\Sigma$ as an operator with
closed range. As indicated earlier, this may be done using spaces with exponential weights. We can use either Sobolev or Hölder spaces, and each has its advantages, but we shall use the latter. Decompose $\Sigma$ into the union of a compact piece, $K$, and a finite number of ends, $E_1, \ldots, E_k$. Fix isothermal coordinates $(s_j, \theta_j)$ on each end.

**Definition 1.** For $\ell \in \mathbb{N}$, $0 < \alpha < 1$ and $\mu \in \mathbb{R}$, we define $C^\ell_{\mu, \alpha}(\Sigma)$ to be the set of all functions $u \in C^\ell_{\mu}(K)$ and on each end $E_j$ satisfy

$$\sup_{s_j \geq 0} e^{-\mu s_j} \|w\|_{C^{\ell,\alpha}([s_j, s_j+1] \times S^1)} < \infty.$$ 

It is immediate that

$$L : C^{\ell+2,\alpha}_{\mu}(\Sigma) \longrightarrow C^{\ell,\alpha}_{\mu}(\Sigma)$$

is a bounded mapping for any $\mu \in \mathbb{R}$. However, if $\epsilon_j$ is the asymptotic necksize of the end $E_j$, then it is not hard to see that whenever $\mu \in \Lambda_{\epsilon_j}$, (4) does not have closed range. However, this is the only bad case:

**Proposition 1.** Let $\Lambda = \Lambda_{\Sigma} = \bigcup_{j=1}^k \Lambda_{\epsilon_j}$. Then if $\mu \notin \Lambda$, the mapping (4) is Fredholm. When $\mu \notin \Lambda$ is sufficiently large, this mapping is surjective, while if $\mu$ is sufficiently negative, this mapping is injective.

We conclude this subsection by stating the fundamental

**Definition 2.** The CMC surface $\Sigma$ is said to be nondegenerate if it admits no nontrivial global Jacobi fields which decay along all of the ends $E_j$, or equivalently, if (4) is injective for any $\mu < 0$.

The equivalence of the two versions of this definition is not immediate, since one could imagine the existence of Jacobi fields which decay polynomially rather than exponentially, but is true nonetheless. A duality argument then gives that if $\Sigma$ is nondegenerate, (4) is surjective when $\mu > 0$, $\mu \notin \Lambda$. There is an important refinement of this surjectivity. Choose a cutoff function $\chi(s)$ which equals 1 for $s \geq 1$ and 0 for $s \leq 0$, and consider the $6k$-dimensional vector space $W_{\Sigma}$ spanned by the elements

$$\Phi_{ij} = \chi(s_j) \phi_{i,\epsilon_j}(s_j, \theta_j), \ j = 1, \ldots, k, \ i = 1, \ldots, 6,$$

where $\{\phi_{i,\epsilon_j}\}$ are the 6 geometric Jacobi fields on $D_{\epsilon_j}$. Thus we have transplanted these special Jacobi fields to each end of $\Sigma$. Because of the exponential decay of the ends to their Delaunay models, we have

$$L_{\Sigma} \phi = O(e^{-c s_j}) \quad \forall \phi \in W_{\Sigma}, \ j = 1, \ldots, k.$$
Here \( c = \inf \{ \gamma : 0 < \gamma \in \Lambda_\Sigma \} > 0 \). Hence for \( \mu > 0 \) sufficiently small,

\[
L_\Sigma : W_\Sigma \oplus C^{k+2,\alpha}_- (\Sigma) \longrightarrow C^{k,\alpha}_- (\Sigma)
\]  

(5)

is well defined.

**Proposition 2.** If \( \Sigma \) is nondegenerate and \( 0 < \mu < \mu_0 = \inf \{|\lambda| : \lambda \in \Lambda_\Sigma \} \), then the mapping (3) is surjective.

We call \( W_\Sigma \) the deficiency subspace of \( \Sigma \). It also plays a role in the regularity of Jacobi fields.

**Proposition 3.** Let \( \phi \) be any Jacobi field with at most polynomial growth on \( \Sigma \). Then there exist functions \( \Phi \in W_\Sigma \) and \( \psi \in C^{2,\alpha}_- (\Sigma), \mu > 0 \), such that \( \phi = \Phi + \psi \).

The nonlinear mean curvature operator \( N \) does not preserve any of the spaces \( C^{k,\alpha}_\mu (\Sigma), \mu > 0 \), because the nonlinearity amplifies the exponential increase. On the other hand, while this nonlinear operator is defined on the spaces of exponentially decreasing functions \( C^{k,\alpha}_{-\mu} \), its linearization always has a cokernel there. We can extend the definition of \( N \) to \( W_\Sigma \) as follows: any \( \Phi \in W_\Sigma \) corresponds to a one-parameter family of CMC deformations of the model Delaunay surface for that end, and so we can ‘integrate’ \( \Phi \) by transplanting this to a one-parameter family of asymptotically CMC deformations localized to each of the ends of \( \Sigma \). For example, if \( \Phi \) corresponds to the rotation of a Delaunay surface, then we rotate the corresponding end, using some cutoff to join this to the identity on the rest of \( \Sigma \). The constant mean curvature condition is destroyed only at the pivoting locus which is contained in a compact set. (The definition is slightly more complicated when \( \Phi \) includes a component requiring a change of Delaunay parameter.) More generally, \( N(\Phi) \) is just the mean curvature of this new surface, and this is either compactly supported or exponentially decreasing, so that

\[
N : W_\Sigma \oplus C^{k+2,\alpha}_{-\mu} (\Sigma) \longrightarrow C^{k,\alpha}_{-\mu} (\Sigma)
\]  

(6)

is well defined. When \( \Sigma \) is nondegenerate, this map has surjective linearization. Thus the addition of \( W_\Sigma \) provides an intermediate space where the conflicting requirements of well-definedness and surjectivity are balanced.

We conclude this section by noting that there is a more general way to define nondegeneracy. The crucial feature in the discussion above is that we were able to extend the definition of \( N \) to \( W_\Sigma \) because elements of the deficiency space correspond, at least asymptotically, to actual CMC deformations. Another way to say this is that every element of \( W_\Sigma \) asymptotically
integrates to a one-parameter family of asymptotically CMC surfaces. Thus the ‘correct’ definition of nondegeneracy should be to require that all tempered Jacobi fields on Σ are similarly asymptotically integrable. The point here is that one should be able to use explicit geometric deformations of Σ to compensate for the cokernel of \( L_\Sigma \). As a very simple but important example, the sphere \( S^2 \) is degenerate in the more restricted sense; it has a three-dimensional space of Jacobi fields, which are the restrictions of the linear coordinates on \( \mathbb{R}^3 \) to \( S^2 \). However, these Jacobi fields correspond to translations of the sphere, and it is precisely this which allowed Kapouleas to use them in his gluing construction.

The possible existence of degenerate CMC surfaces creates a lot of complications throughout this whole theory. However, even amongst immersed surfaces, there are no known examples which are degenerate in the sense of this broader definition. Indeed the only known examples of degenerate surfaces in the narrower sense are the sphere (or any compact immersed CMC surface) and certain immersed ‘bubbleton’ solutions which are globally cylindrically bounded and asymptotically cylindrical, so that translation along the cylinder’s axis provides a decaying, but nevertheless ‘integrable’, Jacobi field. A major problem is to determine whether there ever exists degenerate CMC surfaces in the broader sense.

### 3 Gluing constructions

Up to this point the only examples of complete Alexandrov embedded CMC surfaces of finite topology we have seen are the Delaunay unduloids. In this section we discuss a number of closely related gluing constructions; taken together these show that CMC surfaces are quite malleable and exist in great profusion. As with all gluing constructions, the rough idea is that one starts with ‘building blocks’, pieces of surfaces which already are, or are near to, simpler CMC surfaces, and then pieces them together to form complete CMC surfaces. We use here three types of building blocks: Delaunay surfaces (or finite or semi-infinite segments of them), spheres and complete Alexandrov-embedded minimal surfaces with finite total curvature. It is necessary that all components be nondegenerate in the extended sense. Amongst the motivations for developing new analytic methods for these sorts of problems is that one wishes to show that the glued surfaces are again nondegenerate; this is important in the moduli space theory and also has the important consequence that these constructions can be iterated. In addition, until the advent of newer methods developed after Kapouleas’ work, it was unclear
whether there existed any nondegenerate surfaces beyond the unduloids.

I shall first give a rough overview of the type of argument needed here
(or indeed any singular perturbation problem). After describing Kapouleas'
construction I list the more general gluing constructions now known to work,
and conclude by describing idea of Cauchy data matching which can be
used to prove any of these. All of these surfaces have very small necksizes,
or otherwise nearly degenerate (i.e. near to the boundary of Teichmüller
space) conformal structures. It is an important philosophical point that
all of the ‘constructible’ CMC surfaces are very nearly degenerate. Glu-
ing constructions provide parametrizations of ends of the moduli spaces.
The (presumably more commonplace) surfaces with larger necksizes are not
‘reachable’ by direct methods.

Thus, again at a rough level, these components are arranged, by rotating
and translating them in space, so that they form a configuration which is
an approximate solution. Such a configuration is either CMC or approxi-
mately CMC everywhere. There is always a parameter \( \eta \) involved, so that as
\( \eta \downarrow 0 \), the discrepancy of the configuration from being exactly CMC dimin-
ishes. Unfortunately, in this same limit, the geometry of the configuration
degenerates. The main step involves using the Jacobi operator to perturb
the approximate configuration to an exact CMC surface, but again this is an
operator with ‘degenerating coefficients’. Hence we must not only show that
its Jacobi operator is surjective as in (5) when \( \eta \) is small, but also estimate
the growth of the norm of a suitable right inverse as \( \eta \downarrow 0 \). The game is
to show that this norm does not blow up faster than the inverse of the size
of the error term. There are many well-known constructions which follow
these outlines, including Taubes’ famous instanton patching, etc., and there
are many different methods of carrying out the details. I shall sketch one,
initially developed in [25], based on Cauchy data matching; this seems to be
the most efficient and involves the fewest hard estimates.

The first construction of this type for CMC surfaces was accomplished
by Kapouleas [12], inspired by a similar construction by Schoen [34] for the
singular Yamabe problem. This establishes the existence of many complete
CMC surfaces, but his method does not give a sufficiently fine understanding
of their geometry; in particular, it seems very difficult to tell whether any of
the surfaces he constructs are nondegenerate. (They are, in fact, nondegen-
erate, as follows from the alternate construction v) below.) I have already
sketched his method in §2: one begins with a simplicial graph in \( \mathbb{R}^3 \) with \( k \)
of the edges semi-infinite rays, and where each edge is labelled with a force
vector. This graph must satisfy a certain balancing and flexibility condition.
The approximate CMC surface is assembled by substituting for each vertex
We now list and give brief descriptions of various geometric operations which, it has more recently been proved, may all be done in the (nondegenerate) CMC category. Afterwards we sketch a representative proof.

i) Adding Delaunay ends [24]: If $\Sigma$ is any nondegenerate CMC surface and $p \in \Sigma$, then we may attach a half Delaunay surface $D^+_{\epsilon}$ with small necksize $\epsilon$ to $\Sigma$ at $p$. The axis of $D^+_{\epsilon}$ is directed in the outward normal direction.

ii) Connected sums [23]: Let $\Sigma_1$ and $\Sigma_2$ be nondegenerate CMC surfaces and choose points $p_j \in \Sigma_j$. Rotate and translate these surfaces so that the tangent spaces coincide at these points, but with opposite orientation. Then there exists a new CMC surface $\Sigma_1 \#_\epsilon \Sigma_2$ obtained by ‘bridging’ these surfaces with a small Delaunay neck near these points. (One can also insert any finite Delaunay segment instead of a single neck.) To show that the resulting surface is nondegenerate requires minor restrictions concerning the locations of the points.

iii) Attaching Delaunay ends to minimal $k$-noids [21]: Let $\Sigma$ be a complete nondegenerate minimal surface of finite total curvature with $k$ ends (a so-called minimal $k$-noid), all of which are asymptotic to catenoids. Form a new surface with boundary $\Sigma_\epsilon$ by truncating the ends (at distance proportional to $-1/\log \epsilon$ on the catenoid) and then rescaling by a factor of $\epsilon$. Then there is a CMC surface with $k$ ends obtained by attaching half-Delaunay surfaces $D^+_{\epsilon a_j}$ to these truncated catenoidal ends. The necksize parameters are $\epsilon a_j + O(\epsilon^2)$, where $(a_1, \ldots, a_k)$ is a vector of relative dilation factors for the catenoids modelling the ends of $\Sigma$. Since many topologically complicated nondegenerate minimal $k$-noids are known to exist, this gives similarly complex CMC surfaces. It may seem that being minimal is very far away from having constant mean curvature 1, but this is where the scaling by $\epsilon$ becomes important. For, rescaling the ensemble by $1/\epsilon$, the construction is equivalent
to attaching (albeit on a very large scale) Delaunay ends with mean curvature $\epsilon$ to a minimal surface, which seems more plausible.

iv) End-to-end gluing \[29\]: Let $\Sigma_1$ and $\Sigma_2$ be two nondegenerate CMC surfaces, each of which has an end with the same asymptotic neck-size parameter (not necessarily small). Then we may translate and rotate these surfaces so that the axes of these particular ends are on the same line, but oppositely oriented. Truncating these ends very far out, it is possible to attach the surfaces to obtain a new CMC surface with a very long approximately unduloidal tube. This requires a small strengthening of the nondegeneracy condition. Notice that while these surfaces do not necessarily have small necksizes, they contain embedded essential annuli with large conformal modulus, hence still have nearly degenerate conformal structures.

v) Kapouleas-type constructions \[25\]: One may attach half Delaunay surfaces and finite Delaunay segments, arrayed along the rays and edges of a suitable simplicial graph, with spheres at the vertices, to obtain a CMC surface which is contained in a tubular neighbourhood of this graph.

vi) Attaching Delaunay cross-pieces \[25\]: Suppose $\Sigma$ is a nondegenerate CMC surface and $p_1, p_2 \in \Sigma$. Suppose that the tangent planes at these points are parallel and oppositely oriented (so that the outward normal of $T_{p_1}\Sigma$ points towards the outward normal for $T_{p_2}\Sigma$), and the distance $|p_1 - p_2|$ is an even integer (hence essentially a multiple of a Delaunay period for $\epsilon$ small). Then, subject to a flexibility condition on this configuration as well as a slight strengthening of the nondegeneracy condition as in iv), one may attach a Delaunay segment along the axis connecting these points.

For all of these constructions, with minor provisos concerning the locations of the points where the gluings are done, the resulting surfaces are nondegenerate, and hence these operations may be performed iteratively and in various combinations. I have also omitted a detailed description of the slightly strengthened form of nondegeneracy which is needed in iv) and vi). In any case, using this, it is possible to show, cf. \[24\], \[25\], that there exist nondegenerate elements in $\mathcal{M}_{g,k}$ for any $k \geq 3$ and $g \geq 0$.

As promised, we sketch the proof of i). Thus let $\Sigma$ be a nondegenerate CMC surface and fix $p \in \Sigma$. Excise a small ball $B_\zeta(p)$, and denote by $\Sigma(\zeta)$ the surface with boundary, $\Sigma \setminus B_\zeta(p)$. Now let $D_\epsilon'$ be a half Delaunay surface,
truncated near, but not quite at the neck. More specifically, recalling the radial function \( \rho(t) \) in the original parametrization of \( D_\epsilon \), which attains its minimum value of \( \epsilon \) at \( t = 0 \), we let \( D'_\epsilon \) denote that portion of the Delaunay surface corresponding to \( t \geq t_\epsilon \) for some small \( t_\epsilon < 0 \). We now have two surfaces with boundary, and the naive hope is that if we have truncated them at the correct radii, they should fit together passably well. This is not quite true without some minor modifications. To remedy this, we replace \( \Sigma(\zeta) \) by a normal graph over it, \( \Sigma(\zeta)' = \Sigma(\zeta)_G \), where \( G \) is the Green function for the Jacobi operator with pole at \( p \) (this exists because of the nondegeneracy assumption!). The radius \( \zeta \) is chosen so that the mean curvature of this surface is still bounded. The reason for doing this is that the shape of the graph of the Green function, hence of this new surface, is approximately logarithmic, and this matches the neck region of the Delaunay surface. At any rate, we have now obtained two surfaces depending on a parameter which almost match at their boundaries; for convenience we relabel them as \( \Sigma_1^\epsilon \) and \( \Sigma_2^\epsilon \), respectively (so the first corresponds to the original surface \( \Sigma \), and the second to \( D_+^+ \)).

The second step of the construction is to consider over each of these surfaces separately the space of all nearby CMC surfaces \((\Sigma_j^\epsilon)_\phi \) which are written as normal perturbations. Here \( \phi \in C^{2,\alpha}(\Sigma_j^\epsilon) \) is small. This is an infinite dimensional space. For \( \Sigma_1^\epsilon \) it is parametrized by arbitrary (small) boundary data \( \psi \in C^{2,\alpha}(\partial \Sigma_1^\epsilon) \). For \( \Sigma_2^\epsilon \) almost the same is true, except that not every function \( \psi \) on the boundary of a half Delaunay surface corresponds to a CMC normal perturbation. The functions for which this fails lie in the span of the cross-sectional eigenmodes \( e^{i\ell \theta}, \ell = 0, \pm 1 \). What saves the day is that these functions are precisely the boundary values of the explicit geometric Jacobi fields on \( \Sigma_2^\epsilon \), and correspond to CMC surfaces which are rotations or translations of the original Delaunay surface. Thus we can also regard all small boundary values on \( \partial \Sigma_2^\epsilon \) as corresponding to CMC deformations of \( \Sigma_2^\epsilon \). The analysis required here is quite simple, and reduces to a straightforward contraction mapping argument.

The final step is to consider the set of all Cauchy data of these infinite dimensional spaces of CMC deformations of the summands. The Cauchy data of a normal perturbation \((\Sigma_j^\epsilon)_\phi \), by definition, is the pair of functions \((\psi_1, \psi_2) \), where \( \psi_1 \) is the restriction of \( \phi \) to the boundary and \( \psi_2 = \partial_\nu \phi \) is its normal derivative. The whole point of this argument is that if we can show that these Cauchy data subspaces intersect, then that point of intersection corresponds to CMC deformations of each surface such that these perturbed surfaces match up to second order along the interface. By ellip-
tic regularity for the mean curvature equation, these surfaces must actually fit together smoothly, and we are done. The argument that these infinite dimensional submanifolds must intersect is also not too difficult. The tangent space of either submanifold at \( \phi = 0 \) is the graph of the Dirichlet-to-Neumann operator for \( L_{\Sigma_j} \). One shows that the sum of the graphs of these two Dirichlet-to-Neumann operators spans the whole space when \( \epsilon \) is small enough (by verifying that the same is true for the operators obtained in the limit at \( \epsilon = 0 \)). Finally, the nonlinear Cauchy data submanifolds are graphs off of these linear subspaces, and some estimates are required to see that the neighbourhood in which these graphical representations are valid is large enough to contain the putative point of intersection.

The proofs of the remaining constructions ii) - vi) may all be done similarly. We note that in v) spheres are being used as summands, and these are only nondegenerate in the broader sense discussed at the very end of §2.

4 Moduli space theory

We have defined the spaces \( \mathcal{M}_{g,k} \) as the set of all Alexandrov embedded complete CMC surfaces of genus \( g \) with \( k \) ends. Since we do not mod out by rigid motions of the ambient space or internal isometries, this might reasonably be called the premoduli space, but we shall simply call it the moduli space of CMC surfaces with given \( g \) and \( k \).

We list a few facts about these moduli spaces which are well-known, or which follow immediately from the material in §2 and 3:

- \( \mathcal{M}_{g,1} \) is empty for all \( g \geq 0 \), \([28]\).
- \( \mathcal{M}_{g,2} \) is empty when \( g \geq 1 \). \( \mathcal{M}_{0,2} \) is equal to the set of all rotations and translations of Delaunay surfaces. These follow from Alexandrov reflection arguments, \([18]\).
- \( \mathcal{M}_{g,k} \) is nonempty for every \( g \geq 0 \) and \( k \geq 3 \). This was essentially proved by Kapouleas \([12]\), but also follows from the gluing constructions of §3. We mention some special cases. Define \( \mathcal{M}_{g,k} = \{ (\Sigma, p) : \Sigma \in \mathcal{M}_{g,k}, p \in \Sigma \} \). Then there are continuous mappings

\[
\begin{align*}
\mathcal{M}_{g,k} &\to \mathcal{M}_{g,k+1} \quad \text{(from i)),} \\
\mathcal{M}_{g_1,k_1} \times \mathcal{M}_{g_2,k_2} &\to \mathcal{M}_{g_1+g_2,k_1+k_2} \quad \text{(from ii)),} \\
\mathcal{M}_{g,k} &\to \mathcal{M}_{2g,2k-2} \quad \text{(from iv)).}
\end{align*}
\]

This final map, which comes from applying the end-to-end construction iv) to two copies of the same surface, thus doubling it across an
end, is only defined on some open set in the moduli space where some nondegeneracy condition is satisfied. We mention finally that there are many known minimal $k$-noids of high genus, as catalogued in \[24\], and for each such $g, k$ we obtain by construction iii) an element of $\mathcal{M}_{g,k}$.

The general structure of these moduli spaces is contained in the

**Theorem 1** (\[20\]). For each $g, k$, the moduli space $\mathcal{M}_{g,k}$ is a finite dimensional real analytic variety in a neighbourhood of each of its points. If $\Sigma \in \mathcal{M}_{g,k}$ is nondegenerate, then there exists a neighbourhood $\Sigma \in \mathcal{U} \subset \mathcal{M}_{g,k}$ which is a real analytic manifold of dimension $3k$.

The first assertion in this theorem means that if $\Sigma \in \mathcal{M}_{g,k}$, there is an open neighbourhood $\mathcal{V}$ of $\Sigma$ in the space of all surfaces near to $\Sigma$ (in the topology of an appropriate separable Banach space), and a real analytic diffeomorphism $\Phi : \mathcal{V} \to \mathcal{V}'$ such that $\Phi(\mathcal{V} \cap \mathcal{M}_{g,k})$ lies in a finite dimensional linear subspace $B$; furthermore, there exists a real analytic function $F$ defined on $B \cap \mathcal{V}'$ such that $\Phi(\mathcal{M}_{g,k} \cap \mathcal{V}) = F^{-1}(0)$. Notice that the dimension of $B$ may depend on $\Sigma$, and in general it is unclear if it is bounded as $\Sigma$ varies over the moduli space.

When $\Sigma$ is nondegenerate, the proof is a straightforward application of Proposition 2 and the implicit function theorem. To prove the more general assertion, one uses Proposition 1 and the ‘Kuranishi trick’ (Ljapunov-Schmidt reduction).

Note that the generic dimension of this moduli space only depends on the number of ends, but not on the genus $g$.

This result illuminates the importance of nondegeneracy in the moduli space theory. Some consequences of this theorem are that $\mathcal{M}_{g,k}$ is a real analytic stratified space, or more precisely, admits a locally finite decomposition into real analytic strata. If $\Sigma$ is a nondegenerate element in some stratum, then that stratum is maximal, i.e. it is not contained in the closure of any other stratum, and has the predicted dimension $3k$. Unfortunately, this theorem says nothing about the structure of the moduli space near its boundary, nor does it limit the number of components it might have.

We explain the dimension $3k$ which appears here. Supposing that $\Sigma$ is nondegenerate, the implicit function theorem applies to \([\mathcal{F}]\) since its linearization \([\mathcal{F}]\) is surjective. Hence a neighbourhood of $\Sigma$ in $\mathcal{M}_{g,k}$ has a real analytic coordinate chart parametrized by a ball in the nullspace of \([\mathcal{F}]\), and so we must show that this nullspace is $3k$-dimensional. Although not stated explicitly in §2, this nullspace is precisely the same as the nullspace of $L_{\Sigma}$ on $C^{2,\alpha}_{\mu}(\Sigma)$ for small $\mu > 0$. This mapping is surjective, by nondegeneracy, and
the usual integration by parts argument shows that its nullspace is identified with the cokernel of $L_\Sigma$ on $C^{2,\alpha}_{-\mu}$. On the other hand,

$$6k = \dim W_\Sigma = \dim \ker L_\Sigma|_{C^{2,\alpha}_\mu} + \dim \coker L_\Sigma|_{C^{2,\alpha}_\mu}.$$

These facts together show that the dimension of the nullspace is $3k$, as claimed. In general, however, and as an alternate derivation of this formula, we have that

$$\ker L_\Sigma|_{C^{2,\alpha}_\mu} = \coker L_\Sigma|_{C^{2,\alpha}_{-\mu}}, \quad \coker L_\Sigma|_{C^{2,\alpha}_\mu} = \ker L_\Sigma|_{C^{2,\alpha}_{-\mu}},$$

where each equality here connotes that the left and right sides are identified by the integration pairing. Hence

$$\text{ind} \left( L_\Sigma : C^{2,\alpha}_\mu(\Sigma) \to C^0_{\mu}(\Sigma) \right) = \frac{1}{2} \left( \text{ind} \left( L_\Sigma : C^{2,\alpha}_\mu(\Sigma) \to C^0_{\mu}(\Sigma) \right) - \text{ind} \left( L_\Sigma : C^{2,\alpha}_{-\mu}(\Sigma) \to C^0_{-\mu}(\Sigma) \right) \right).$$

This last quantity can be computed by a relative index theorem [20]: the answer can be computed locally at infinity. The contribution from each end is $3$, and the full relative index is the sum of these contributions over all $k$ ends.

For any $\Sigma \in \mathcal{M}_{g,k}$, Proposition 3 gives a map

$$\iota : \ker L_\Sigma|_{C^{2,\alpha}_\mu} \to W_\Sigma;$$

each Jacobi field $\phi$ gets mapped to a $6k$-tuple consisting of the components of each of the $6$ model geometric Jacobi fields on each of the $k$ ends. If $\Sigma \in \mathcal{M}_{g,k}$ is nondegenerate, this map is injective, and hence we can regard $T_\Sigma \mathcal{M}_{g,k} = \ker L_\Sigma$ as a subspace of $W_\Sigma$.

**Theorem 2 ([20]).** There is a natural symplectic structure on $W_\Sigma$ such that when $\Sigma$ is nondegenerate, the $3k$-dimensional subspace $\iota(\ker L_\Sigma) \subset W_\Sigma$ is a Lagrangian subspace.

This result is somewhat at odds with the geometric structures which exist on other standard moduli spaces. In particular, there does not seem to be a canonical Riemannian metric on these moduli spaces, and it remains unclear what this Lagrangian structure means, or how it may be used. Kusner [19] has ventured some interesting speculations about its interpretation.

At almost the same time these theorems were proved, Perez and Ros [32] discovered the analogous result for certain moduli spaces of complete
minimal surfaces of finite total curvature. Their techniques were rooted in the underlying Riemann surface theory, in particular the Weierstraß representation.

In the past few years, some new advances have been made concerning these moduli spaces. The first is a striking result of Große-Brauckman, Kusner, and Sullivan.

**Theorem 3** ([10], [11]). Let $E_3$ be the group of rigid motions of $\mathbb{R}^3$. Then $\mathcal{M}_{0,3}/E_3$ is homeomorphic to the 3-ball.

The argument to prove this relies on a beautiful mix of a classical geometric construction relating CMC surfaces in $\mathbb{R}^3$ with minimal surfaces in $S^3$ and (Dennis Sullivan’s) $\mathbb{Z}_2$ degree theory for proper mappings in the real analytic category. This theorem requires the existence of a nondegenerate element in $\mathcal{M}_{0,3}$, which is known from [21]. Notice that the dimension count is correct: $\dim E_3 = 6$ and so the (pre)moduli space is $6 + 3 = 9$ dimensional, which agrees with the $3 \cdot 3$ dimensions predicted by Theorem 1.

It is not known whether the homeomorphism in Theorem 3 is a diffeomorphism, nor whether can exist any triunduloids, i.e. elements of $\mathcal{M}_{0,3}$, which are degenerate. This latter question is quite important, and if answered in the negative, would have several interesting implications.

To phrase the final results, we let $\mathcal{R}_{g,k}$ be the Riemann moduli space for surfaces of genus $g$ with $k$ punctures, and define the ‘forgetful map’:

$$\mathcal{F} : \mathcal{M}_{g,k} \to \mathcal{R}_{g,k}.$$ 

Thus, to any CMC surface $\Sigma$ we associate its marked conformal completion $[\Sigma, p_1, \cdots, p_k]$. Thus punctured neighbourhoods around these points $p_j$ correspond to (slightly perturbed) half Delaunay surfaces. We also let $\mathcal{P}_{g,k} = \mathcal{R}_{g,k} \times \mathbb{R}^k$ and define an enhanced forgetful map

$$\tilde{\mathcal{F}} : \mathcal{M}_{g,k} \to \mathcal{P}_{g,k},$$

$$\tilde{\mathcal{F}}(\Sigma) = ([\Sigma, p_1, \cdots, p_k], \epsilon_1, \cdots, \epsilon_k).$$

In other words, this map also records the asymptotic necksizes of each end. By generalizing the arguments in [18], Kusner has recently proved the very useful

**Theorem 4** ([19]). Up to rigid motions, this enhanced forgetful map $\tilde{\mathcal{F}}$ is proper.

This result implies that if $\Sigma_j$ is any sequence of elements in $\mathcal{M}_{g,k}$, then one of the following three possibilities must hold:
• There exists a sequence of rigid motions $T_j$ such that $T_j(\Sigma_j)$ converges to an element $\Sigma \in \mathcal{M}_{g,k}$,

• The marked conformal structures of the sequence $([\Sigma_j], p_{j,1}, \ldots, p_{j,k})$ must degenerate in $\mathcal{R}_{g,k}$, or

• At least one of the necksizes $\epsilon_{j,\ell}$ tends to zero as $j \to \infty$.

This gives a very clean and intuitive picture of the modes of possible degeneration of sequences of CMC surfaces. Kusner also shows that the $\mathbb{Z}_2$ degree of this map is zero, so there an even number of preimages of any regular value.

A topic which warrants much closer examination is the structure of the moduli spaces $\mathcal{M}_{g,k}$ near their ends. More specifically, it would be very interesting to prove that $\mathcal{M}_{g,k}$ has a tractable, e.g. (semi)analytic, compactification, presumably obtained by adding moduli spaces $\mathcal{M}_{g',k'}$ with $g' \leq g$, $k' \leq k$, with at least one of the inequalities strict.

It is also natural to try to characterize the image of the unenhanced forgetful map $\mathcal{F}$. One motivation for this is that the Teichmüller space $\mathcal{R}_{g,k}$ is known to have fairly complicated topology (its fundamental group is the Artin braid group on $k$ generators of the topological surface $\Sigma$), and so if the image of $\mathcal{F}$ is relatively large, then $\mathcal{M}_{g,k}$ itself must have lots of topology.

**Theorem 5 (24).** For all $g \geq 0$ and $k \geq 3$, the mapping $\mathcal{F}$ is real analytic, and its image in $\mathcal{R}_{g,k}$ is a semianalytic (hence stratified) variety $I_{g,k}$. When $g = 0$, then for every $k \geq 3$, $\mathcal{F}$ is surjective. Likewise, if $g > 0$, then the codimension of the principal stratum of $I_{g,k}$ in $\mathcal{R}_{g,k}$ is uniformly bounded as $k \to \infty$.

The main tool in the proof is the half Delaunay addition construction from §3; the proof of the first analyticity assertion reduces to showing that the uniformizing map which takes $\Sigma$ to the unique conformally related constant negative curvature surface is real analytic. The surjectivity statement when $g = 0$ was also obtained by Kusner [14], and a somewhat weaker statement can also be deduced from Ratzkin’s end-to-end gluing construction [29].

There are many interesting questions raised by these theorems. Kusner [19] lists several important ones concerning the finer structure of the map $\mathcal{F}$, and in particular concerning the precise number of points in the preimages of regular values.
5 CMC surfaces in other asymptotically Euclidean 3-manifolds

Let \((X, h)\) be a complete three dimensional manifold with asymptotically Euclidean ends. Recall that this means that the following hypotheses hold: there exists a compact set \(K \subset X\) such that \(X \setminus K\) is a disjoint union of ends, \(F_1, \ldots, F_\ell\), each of which is equipped with a diffeomorphism \(\Psi_j : \mathbb{R}^3 \setminus B(0, R_j) \to F_j\) such that the metric \(h\) on \(X\) pulls back via \(\Psi_j^*\) to a metric on the exterior region in \(\mathbb{R}^3\) of the form \(\delta + O(r^{-1})\), with corresponding decay for its derivatives.

We shall consider the space \(\mathcal{M}_{g,k}(X, h)\) consisting of all proper Alexandrov embedded CMC surfaces in \(X\) of genus \(g\) with \(k\) ends. If \(X\) has more than one end, then we could also consider a decomposition into moduli spaces which explicitly label which end of \(\Sigma\) tends to infinity in which end of \(X\), but we shall not introduce special notation for this. As noted in §2, the asymptotics result of [18] most likely extends to this setting, but since this is not worked out anywhere, we circumvent this by assuming that this moduli space contains only CMC surfaces which are asymptotically Delaunay.

It is straightforward to check that the main results about the structure of the moduli space for CMC surfaces in \(\mathbb{R}^3\) persist in this more general situation. In particular, we still have that for any \((X, h)\), \(\mathcal{M}_{g,k}(X, h)\) is a locally real analytic variety, and near nondegenerate points it has dimension \(3k\). Note that we do not need to assume that \(X\) or \(h\) are real analytic because this theorem relies on the implicit function theorem in some function space on \(X\), and the mean curvature functional depends real analytically on the metric and its derivatives.

Consider first the case where \(X = \mathbb{R}^3\) and \(h\) is a small (decaying) perturbation of the standard metric \(\delta\). We prove below that \(\mathcal{M}_{g,k}(\mathbb{R}^3, h)\) is a small perturbation of the ‘standard’ moduli space \(\mathcal{M}_{g,k}(\mathbb{R}^3, \delta)\), as well as a standard ‘generic regularity’ theorem, that this perturbed moduli space is smooth for generic \(h\); in other words, for most small alterations of the ambient metric there are no degenerate CMC surfaces.

As an aside, I have already noted that no degenerate CMC surfaces are known to exist, and it seems very unlikely that it will be possible to construct one directly. If they exist at all, I suspect they will only be found by indirect means, by showing that the projection mapping \(\Pi\) of the big moduli space \(\mathcal{S}\) defined below must have fold singularities. It is completely unclear whether this happens for the standard (or any!) metric on \(\mathbb{R}^3\).

Let \(Z \equiv S^2_{-1}(\mathbb{R}^3)\) denote the space of all symmetric 2-tensors with coeffi-
cients in the weighted Hölder space $C^{2,\alpha}_{-1}(\mathbb{R}^3)$. Thus if $x$ is the standard Cartesian coordinate chart on $\mathbb{R}^3$, then any $\gamma \in Z$ satisfies $|\partial_x^\beta \gamma_{ij}| \leq C|x|^{-1-|\beta|}$ as $|x| \to \infty$, $|\beta| \leq 2$, and with the Hölder quotients of the second derivatives decaying like $|x|^{-3-\alpha}$. Let $U$ be an open ball in this space such that if $\gamma \in U$, then $\delta + \gamma$ is everywhere positive definite, hence is an asymptotically Euclidean metric on $\mathbb{R}^3$.

**Theorem 6.** There exists a dense subset $U' \subset U$ such that when $\gamma \in U'$, $\mathcal{M}_{g,k}(\mathbb{R}^3, \delta + \gamma)$ is a smooth analytic manifold of dimension $3k$.

**Proof.** Consider the Banach manifold $B$ of all Alexandrov embedded $C^{2,\alpha}$ surfaces in $\mathbb{R}^3$ of genus $g$ with $k$ asymptotically Delaunay ends. A coordinate chart on this manifold near any point $\Sigma$ is given by a small ball in $W_{\Sigma} \oplus C^{2,\alpha}_{-\mu}(\Sigma)$. Thus $w \in W_{\Sigma}$ induces a change of the ends, either by rigid motion or change of necksize, and $u \in C^{2,\alpha}_{-\mu}(\Sigma)$ gives a normal perturbation. We write the resulting surfaces as $\Sigma_{w,u}$.

The first step is to show that the set

$$S = \{(\Sigma, \gamma) \in B \times Z : \Sigma \in \mathcal{M}_{g,k}(\mathbb{R}^3, \delta + \gamma)\}$$

has the structure of a Banach submanifold near $\gamma = 0$. This is a straightforward application of the implicit function theorem. Fix $\Sigma \in \mathcal{M}_{g,k}$ and a coordinate chart $V$ on $B$, and define the functional $N : V \times Z \to C^{0,\alpha}_{-\mu}(\Sigma)$ by setting $N(w, u, \gamma)$ equal to the mean curvature function on $\Sigma_{w,u}$ with respect to the metric $\delta + \gamma$. The differential of this mapping $D_{12}N$ in the first two slots, at $(w, u) = (0, 0)$, is the Jacobi operator $L_{\Sigma}$, and by Proposition 2 this operator has closed range of finite codimension. We must show that the full differential $DN$ at $(0, 0, 0)$ is actually surjective. Granting this for the moment, we then obtain an analytic mapping $\Psi$ defined in a neighbourhood of $0$ in $Z$ to $V$ such that all points in $S$ near to $\Sigma$ are of the form $(\Psi(\gamma), \gamma)$.

To prove the claim about surjectivity, we must show that

$$\text{ran } (L_{\Sigma}) + \text{ran } (D_{3}N|_{\Psi}) = C^{0,\alpha}_{-\mu}(\Sigma).$$

It is necessary to reinterpret this slightly. First, the mapping (4) is the restriction of the mapping (3), i.e. from $C^{2,\alpha}_{-\mu}(\Sigma)$ to $W_{\Sigma} \oplus C^{2,\alpha}_{-\mu}(\Sigma)$. By Proposition 3, the nullspaces of these mappings are the same, hence their cokernels also agree. As in §4, duality considerations show that

$$C^{0,\alpha}_{-\mu} = \text{ran } (L_{\Sigma}|_{C^{2,\alpha}_{-\mu}}) \oplus \ker L_{\Sigma}|_{C^{2,\alpha}_{-\mu}}.$$
Hence we must show that there does not exist any $\psi \in C^{2,\alpha}(\Sigma)$ such that $L_\Sigma \psi = 0$ and $\langle \psi, D_3 N(\gamma) \rangle = 0$ for all $\gamma \in Z$. In the end, we show that when $\gamma$ ranges over all $C^{2,\alpha}$ symmetric 2-tensors supported in some small ball $\mathcal{W}$ in $\Sigma$, the functions $D_3 N(\gamma)$ fill out $C^{0,\alpha}(\mathcal{W})$, and this will prove the claim since if the orthogonality condition were to hold, then $\psi$ would have to vanish in this set $\mathcal{W}$, which is impossible since it is in the nullspace of $L_\Sigma$.

By dilating the space substantially and using an approximation argument, it will be enough to consider the simpler problem where $\Sigma = \mathbb{R}^2 \subset \mathbb{R}^3$, and so we must compute the linearization $D_3 N$ which measures how the mean curvature of the flat plane $\{(x_1, x_2, 0)\}$ in $\mathbb{R}^3$ changes as we vary the metric. This computation is not too horrible, fortunately, and we obtain that

$$D_3 N(\gamma) = (\text{div } \gamma)_3 + \frac{1}{\partial x_3} \left( \text{tr } \gamma \right);$$

here $\gamma = (\gamma_{ij})$ and the divergence and trace are computed only in the $(x_1, x_2)$ directions. The subscript 3 in the first term on the right denotes the third component of div $\gamma$, i.e. $\gamma_{3; i}$. Examining this expression, it is clear that we can specify it arbitrarily in the bounded set $\mathcal{W}$. This proves the claim.

Consider the projections $\Pi_1 : \mathcal{B} \times Z \to \mathcal{B}$ and $\Pi_2 : \mathcal{B} \times Z \to Z$. The restriction of $\Pi_2$ to $\mathcal{S}$ has surjective differential at $\Sigma'$ if and only $\Sigma'$ is non-degenerate with respect to the metric $\delta + \gamma$, $\gamma = \Pi_2(\Sigma')$. Hence by the Sard-Smale theorem, if $K \subset \mathcal{B}$ is a compact set containing $\Sigma$, then there is an open dense set $\mathcal{U}_K \subset Z$ of metrics near $\delta$ such that every surface in $\Pi_1^{-1}(K) \cap \mathcal{S} \cap \Pi_2^{-1}(\mathcal{U}_K)$ is nondegenerate (with respect to the appropriate metric).

To globalize this, cover $\mathcal{M}_{g,k}$ by a countable union of compact sets $\mathcal{K}_j$ and choose open dense neighbourhoods $\mathcal{U}_j$ in $Z$ as above. Then the intersection of these sets $\mathcal{U}_j$ is still dense, by Baire’s theorem, and has the desired property.

It is most likely possible to modify this proof to show something stronger and much more useful. Let $\mathcal{D}$ denote the set of ‘bad’ metrics, i.e. those for which the perturbed moduli space contains degenerate elements, then I expect it is true that $\mathcal{D}$ decomposes into a union of $\mathcal{D}' \cup \mathcal{D}''$ where $\mathcal{D}'$ is a codimension $2k$ submanifold in $\mathcal{U}$ and $\mathcal{D}''$ is quantitatively smaller. (In fact, probably $\mathcal{D}$ has a stratified structure with only strata of bounded codimension.) The reason for this conjecture is as follows. We may view this process of destroying degenerate surfaces as a problem in eigenvalue perturbation theory. As explained in §2, the essential spectrum of $L_\Sigma$ is a
locally finite union of intervals $I_j$ and 0 always lies on the left endpoint of one of these intervals. In addition, there is always a small interval $(-\epsilon, 0)$ which is disjoint from the spectrum; here $\epsilon$ depends on $\Sigma$. $\Sigma$ is degenerate if and only if 0 is also in the point spectrum. There is a good procedure for tracking what happens to this $L^2$ eigenvalue as $\Sigma'$ varies in $B$. To explain it, consider (for any fixed $\Sigma'$) the resolvent $R(\lambda) = \frac{1}{-L\Sigma' - \lambda^2}$; if we restrict $\lambda^2$ to lie in $B(0, \epsilon) \setminus [0, \epsilon)$, or equivalently, if $\lambda$ lies in the half-ball $B(0, \sqrt{\epsilon}) \cap \{\text{Im}\lambda < 0\}$, which is contained in the resolvent set of $-L\Sigma'$, then $R(\lambda)$ is bounded on $L^2(\Sigma')$. Using the machinery developed in [26] it is quite straightforward to show that $R(\lambda)$ continues meromorphically to some neighbourhood of 0 in either the complex plane or the logarithmic complex plane (the latter being necessary if 0 is a nontrivial ramification point in this continuation). This continued resolvent has at most a double pole at the origin, and the coefficient of $1/\lambda^2$ is the projection onto the space of $L^2$ Jacobi fields. Thus we are interested in the dependence of this meromorphic structure as $\Sigma'$ varies. If we were starting with some eigenvalue $\lambda_0^2 > 0$ (i.e. inside the continuous spectrum, but away from the ‘threshold value’ 0), then $R(\lambda)$ has only a simple pole at $\lambda_0$. As we deform $\Sigma$, this pole could move either to the left or right on the real axis or upwards, into the so-called nonphysical half-plane (where it would become a resonance, or scattering pole). This is predicted by ‘Fermi’s golden rule’ [33]. Agmon and Herbst have studied persistence of embedded eigenvalues and have shown in as yet unpublished work, but cf. also [4], that for non-threshold eigenvalues, the set of perturbations which leave the eigenvalue fixed is a submanifold of codimension equal to the multiplicity of the continuous spectrum, which in this case equals $2k$. Unfortunately, their proof does not carry over in an obvious way to our case, where the pole is double and occurs at the ramification point. I hope that this will be possible, and if it is, full details will be given elsewhere.

This somewhat arcane issue is interesting for the following reason. We have shown that for most $h$ near $\delta$, $\mathcal{M}_{g,k}(\mathbb{R}^3, h)$ is a smooth (in fact, analytic) manifold of dimension $3k$. However, the topology of this manifold may depend on $h$. It would be quite interesting if there were a canonical smoothing of the moduli space $\mathcal{M}_{g,k}(\mathbb{R}^3, \delta)$, and to show this it would be sufficient to know that the set $\mathcal{D}$ of bad metrics has codimension at least two, so that there are no ‘wall-crossing’ transitions. This is precisely the point of the preceding discussion. We note that this discussion carries over to the more general setting of the moduli space of CMC surfaces in general asymptotically Euclidean manifolds $(X, h)$, and so we might then expect to get canonical smooth moduli spaces which depend only on $X$, but not on $h$!
A case of particular interest is when $X = \overline{X} \setminus \{p_1, \ldots, p_\ell\}$, the complement of an ordered finite set of points in a smooth compact manifold, endowed with a metric which is asymptotically Euclidean near these punctures. We would then obtain a canonical object $\mathcal{M}_{g, k}(\overline{X}, \{p_1, \ldots, p_\ell\})$ associated only to the smooth structure of $\overline{X}$ and the isotopy class of the marked set of points. As ventured earlier, one might hope that this CMC moduli space could reflect some of the topology of this data.

We conclude by discussing a final example: if $p \in \overline{X}$, and $h$ is an asymptotically Euclidean metric on $X = \overline{X} \setminus \{p\}$, then $\mathcal{M}_{0, 2}(X, \epsilon^{-2}h)$ contains many nontrivial elements when $\epsilon$ is small enough. The construction is quite simple. Let $\gamma$ be a bi-infinite geodesic which is properly embedded in $X$; thus both ends converge to $\infty$ in the punctured ball around $p$. Choose Fermi coordinates $(s, r, \theta) \in \mathbb{R} \times (0, 1) \times S^1$ around $\gamma$ corresponding to the metric $\epsilon^{-2}h$ and use these to define the approximate Delaunay surface $r = \rho_\epsilon(s)$. The mean curvature of this is approximately 1, but is not exactly constant. A straightforward perturbation argument shows that there is a small perturbation of this surface which is CMC; we omit the details. Notice that there may be many nonisotopic geodesics, depending on the topology of $X$, and hence there may be many components in this moduli space. In fact, one expects that the components are in one-to-one correspondence with $\pi_1(X, p)$.

Once these Delaunay-type surfaces have been constructed, it is also not difficult to carry out the analogue of the construction i), of adding Delaunay ends, and most of the other constructions too.

While it is far from clear that this extension of the CMC moduli space theory will have any uses, these spaces are certainly natural geometric objects and I suspect that there is a lot more structure which has yet to be uncovered.

References

[1] A.D. Alexandrov, *A characteristic property of spheres*, Ann. Mat. Pura Appl. **58** (1962) 303-315.

[2] H. Brezis and J.-M. Coron, *Multiple solutions of $H$-systems and Rellich’s conjecture*, Comm. Pure Appl. Math. **37** (1984), No. 2, 149-187.

[3] R. Bryant, *Surfaces of mean curvature one in hyperbolic space*, in *Théorie des variétés minimales et applications* (Palaiseau, 1983-1984), Astérisque **154-155** (1987) 321-347.
[4] Proceedings of the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces. To appear.

[5] J. Cruz-Sampedro, I. Herbst and R. Martinez-Avendaño, Perturbations of the Wigner-von Neumann potential leaving the embedded eigenvalue fixed, Preprint (2001).

[6] P. Collin, R. Kusner, W. Meeks III and H. Rosenberg, The topology, geometry and conformal structure of properly embedded minimal surfaces. Preprint (2001).

[7] C. Delaunay, Sur la surface de revolution dont la courbure moyenne est constante, Jour. de Mathématiques, 6 (1841) 309-320.

[8] G. Gregori and R. Mazzeo, Constant mean curvature surfaces in hyperbolic space with prescribed asymptotic behaviour at infinity, in preparation.

[9] K. Große-Brauckmann, New surfaces of constant mean curvature, Math. Z. 214 (1993) 527-565.

[10] K. Große-Brauckmann, R. Kusner and J. Sullivan, Constant mean curvature surfaces with three ends, Proc. Nat. Acad. Sci. USA 97 (2000) 14067-14068.

[11] K. Große-Brauckmann, R. Kusner and J. Sullivan, Triunduloids: embedded constant mean curvatures surfaces with three ends and genus zero, [ArXiv:math.DG/0102183](http://arxiv.org/abs/math.DG/0102183).

[12] N. Kapouleas, Complete constant mean curvature surfaces in Euclidean three space, Ann. of Math. (2) 131 (1990), 239-330.

[13] N. Kapouleas Compact constant mean curvature surfaces in Euclidean three-space, J. Diff. Geom. 33 (1991), 683–715.

[14] N. Kapouleas Constant mean curvature surfaces constructed by fusing Wente tori, Invent. Math. 119 (1995), 443–518.

[15] M. Kilian, I. McIntosh, N. Schmitt, New constant mean curvature surfaces, Experiment. Math. 9 (2000) 595-611.

[16] N. Korevaar and R. Kusner The global structure of constant mean curvature surfaces, Invent. Math. 114 (1993) 311-332.
[17] N. Korevaar, R. Kusner, W. Meeks III and B. Solomon, *Constant mean curvature surfaces in hyperbolic space*, Amer. Jour. Math. 114 (1992), 1-43.

[18] N. Korevaar, R. Kusner and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geometry 30 (1989) 465–503

[19] R. Kusner, *Conformal structures and necksizes of embedded constant mean curvature surfaces*, in [1]

[20] R. Kusner, R. Mazzeo and D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. 6 (1996) 120–137.

[21] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. 9 No. 1 (2001) 169–237.

[22] R. Mazzeo and F. Pacard, *Bifurcating nodoids*. To appear in *Proceedings of the Singapore International Symposium on Geometry and Topology*, Amer. Math. Soc. Contemporary Math Series.

[23] R. Mazzeo, F. Pacard and D. Pollack, *Connected Sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001), 115–165.

[24] R. Mazzeo, F. Pacard and D. Pollack, *The conformal theory of Alexandrov embedded constant mean curvature surfaces in $\mathbb{R}^3$*, in [4].

[25] R. Mazzeo, F. Pacard, D. Pollack and J. Ratzkin, In Preparation.

[26] R. Mazzeo, D. Pollack and K. Uhlenbeck, *Moduli spaces of singular Yamabe metrics*, J. Amer. Math. Soc. 9 (1996), no. 2, 303–344.

[27] R. Mazzeo and D. Pollack, *Gluing and moduli for some noncompact geometric problems*, in *Geometric Theory of Singular Phenomena in Partial Differential Equations*, Symposia Mathematica Vol XXXVIII, Cambridge Univ. Press (1998) 17-51.

[28] W. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Differential Geom. 27 (1988), no. 3, 539–552.

[29] J. Ratzkin, *An end-to-end gluing construction for surfaces of constant mean curvature*, PhD Thesis, University of Washington (2001).
[30] R. Osserman, *A survey of minimal surfaces* Van Nostrand Reinhold, New York (1969).

[31] F. Pacard and F. Pimentel, *Attaching handles to Bryant surfaces*, arXiv:DG/0112224.

[32] J. Perez and A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. J. 45 (1996) 177-204.

[33] M. Reed and B. Simon, *Methods of Mathematical Physics, Vol. IV*, Academic Press, New York.

[34] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure and Appl. Math. XLI (1988) 317-392.

[35] M. Struwe, *Plateau’s problem and the calculus of variations*, Mathematical Notes 35, Princeton University Press, Princeton, NJ (1988).

[36] H. Wente, *Counterexamples to a conjecture of H. Hopf*, Pacific J. Math 121 (1986) 193-243.