Algebraic geometry

Connections and restrictions to curves

Connexions et restrictions aux courbes

Indranil Biswas\textsuperscript{a,b}, Sudarshan Gurjarc\textsuperscript{c}

\textsuperscript{a} School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
\textsuperscript{b} Mathematics Department, EISTI–University Paris-Seine, Avenue du parc, 95000, Cergy-Pontoise, France
\textsuperscript{c} Department of Mathematics, Indian Institute of Technology, Mumbai 400076, India

\section{Abstract}

We construct a vector bundle $E$ on a smooth complex projective surface $X$ with the property that the restriction of $E$ to any smooth closed curve in $X$ admits an algebraic connection while $E$ does not admit any algebraic connection.

\section{Résumé}

Nous construisons un fibré vectoriel $E$ sur une surface complexe lisse $X$ tel que la restriction de $E$ à toute courbe lisse fermée contenue dans $X$ admet une connexion algébrique, sans que $E$ lui-même admette une telle connexion algébrique.

1. Introduction

Let $X$ be an irreducible smooth complex projective variety with cotangent bundle $\Omega^1_X$ and $E$ a vector bundle on $X$. The coherent sheaf of local sections of $E$ will also be denoted by $E$. A connection on $E$ is a $k$-linear homomorphism of sheaves $D : E \rightarrow E \otimes \Omega^1_X$ satisfying the Leibniz identity, which says that $D(f s) = f D(s) + s \otimes df$, where $s$ is a local section of $E$ and $f$ is a locally defined regular function.

Consider the sheaf of differential operators $\text{Diff}_X^i(E, E)$, of order $i$ on $E$, and the associated symbol homomorphism $\sigma : \text{Diff}_X^1(E, E) \rightarrow \text{End}(E) \otimes TX$. The inverse image

\[ \text{At}(E) := \sigma^{-1}(\text{Id}_E \otimes TX) \]

is the Atiyah bundle for $E$. The resulting short exact sequence

\[ 0 \rightarrow \text{Diff}_X^0(E, E) = \text{End}(E) \rightarrow \text{At}(E) \xrightarrow{\sigma} TX \rightarrow 0 \]  \hspace{1cm} (11)
is called the Atiyah exact sequence for $E$. A connection on $E$ is a splitting of \((1.1)\). We refer the reader to [1] for the details; in particular, see [1, p. 187, Theorem 1] and [1, p. 194, Proposition 9].

When $X$ is a complex curve, Weil and Atiyah proved the following \([13],[1]\):

A vector bundle $V$ on an irreducible smooth projective curve defined over $C$ admits a connection if and only if the degree of each indecomposable component of $V$ is zero.

This was first proved in \([13]\); see also [6, p. 69, \textsc{Théorème de Weil}] for an exposition of it. The above criterion also follows from [1, p. 188, Theorem 2], [1, p. 201, Theorem 8] and [1, Theorem 10].

A semistable vector bundle $V$ on a smooth complex projective variety $X$ admits a connection if all the rational Chern classes of $E$ vanish \([12, p. 40, \text{Corollary 3.10}]\). On the other hand, a vector bundle $W$ on $X$ is semistable if and only if the restriction of $W$ to a general complete intersection curve, which is an intersection of hyperplanes of sufficiently large degrees, is semistable \([5, p. 637, \text{Theorem 1.2}],[11, p. 221, \text{Theorem 6.1}]\). On the other hand, any vector bundle $E$ whose restriction to every curve is semistable actually satisfies very strong conditions \([3]\); for example, if $X$ is simply connected, then $E$ must be of the form $E^\text{Gr}$ for some line bundle $L$.

The following is a natural question to ask.

**Question 1.1.** Let $E$ be a vector bundle on $X$ such that, for every smooth closed curve $C \subset X$, the restriction $E|_C$ admits a connection. Does $E$ admit a connection?

Our aim is to show that, in general, the above vector bundle $E$ does not admit a connection.

To produce an example of such a vector bundle, we construct a smooth complex projective surface $X$ with $\text{Pic}(X) = \mathbb{Z}$ such that $X$ admits an ample line bundle $L_0$ with $H^1(X, L_0) \neq 0$. Since $\text{Pic}(X) = \mathbb{Z}$, the ample line bundles on $X$ are naturally parametrized by positive integers. Let $L$ be the smallest ample line bundle (with respect to this parametrization) with the property that $H^1(X, L) \neq 0$. Let $E$ be a nontrivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0.$$  

We prove that the vector bundle $\text{End}(E)$ has the property that the restriction of it to every smooth closed curve in $X$ admits a connection, while $\text{End}(E)$ does not admit a connection; see Theorem 3.1.

A surface $X$ of the above type is constructed by taking a hyper-Kähler 4-fold $X'$ with $\text{Pic}(X') = \mathbb{Z}$. Let $Y \subset X'$ be a smooth ample hypersurface such that $H^j(X', \mathcal{O}_{X'}(Y)) = 0$ for $j = 1, 2$, and let $Z$ be a very general ample hypersurface of $X'$ such that $H^j(X', \mathcal{O}_{X'}(Z)) = 0$ for $j = 1, 2$ and $H^2(X', \mathcal{O}_{X'}(Z - Y)) = 0$. Now take the surface $X$ to be the intersection $Y \cap Z$.

**2. Construction of a surface**

We will construct a smooth complex projective surface $S$ with Picard group $\mathbb{Z}$ that has an ample line bundle $L$ with $H^1(S, L) \neq 0$.

Let $X$ be a hyper-Kähler 4-fold with Picard group $\mathbb{Z}$. For example, a sufficiently general deformation of $\text{Hilb}^2(M)$, where $M$ is a polarized $K3$ surface, will have this property. Let $Y \subset X$ be a smooth ample hypersurface. Note that the vanishing theorem of Kodaira says that

$$H^j(X, \mathcal{O}_X(Y)) = 0$$

for all $j > 0$, because $K_X$ is trivial \([10]\). Let $Z$ be a very general ample hypersurface of $X$ such that both the line bundles $\mathcal{O}_X(Z)$ and $\mathcal{O}_X(Z - Y)$ are ample. In view of the vanishing theorem of Kodaira, the ampleness of $\mathcal{O}_X(Z)$ implies that

$$H^j(X, \mathcal{O}_X(Z)) = 0$$

for all $j > 0$, while that of $\mathcal{O}_X(Z - Y)$ implies that

$$H^j(X, \mathcal{O}_X(Z - Y)) = 0$$

for all $j > 0$. Let

$$\iota : S := Y \cap Z \hookrightarrow X$$

be the intersection and

$$L := \mathcal{O}_X(Y)|_S$$

the restriction of it. Note that $L$ is ample.

Let $I := \mathcal{O}_X(-S) \subset \mathcal{O}_X$ be the ideal sheaf for $S$. Tensoring the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_S \longrightarrow 0$$
by $\mathcal{O}_X(Y)$, we get an exact sequence
\[ 0 \rightarrow \mathcal{I}(Y) \rightarrow \mathcal{O}_X(Y) \rightarrow \iota_* L \rightarrow 0. \] (2.4)
The natural inclusion of $\mathcal{O}_X(-Z)$ in $\mathcal{O}_X$ and $\mathcal{O}_X(Y-Z)$ together produce an inclusion of $\mathcal{O}_X(-Z)$ in $\mathcal{O}_X \oplus \mathcal{O}_X(Y-Z)$. Consequently, we have an exact sequence
\[ 0 \rightarrow \mathcal{O}_X(-Z) \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X(Y-Z) \rightarrow \mathcal{I}(Y) \rightarrow 0. \] (2.5)
In view of (2.1), the connecting homomorphism
\[ H^1(S, L) \rightarrow H^2(X, \mathcal{I}(Y)) \] (2.6)
in the long exact sequence of cohomologies associated with (2.4) is an isomorphism.
Since the canonical line bundle of $X$ is trivial, Serre’s duality gives:
\[ H^{2+j}(X, \mathcal{O}_X(-Z))^* = H^{2-j}(X, \mathcal{O}_X(Z)). \]
So using (2.2), we conclude that the left-hand side vanishes for $j = 0, 1$. Again, by Serre’s duality,
\[ H^2(X, \mathcal{O}_X(Y-Z))^* = H^2(X, \mathcal{O}_X(Z-Y)) = 0 \] (see (2.3)).
Thus, in the long exact sequence of cohomologies associated with (2.5), we have
\[ H^2(X, \mathcal{O}_X(-Z)) = 0 = H^{2+j}(X, \mathcal{O}_X(-Z)), \text{ and } H^2(X, \mathcal{O}_X(Y-Z)) = 0. \]
Hence this long exact sequence of cohomologies associated with (2.5) gives an isomorphism
\[ H^2(X, \mathcal{O}_X) \sim H^2(X, \mathcal{I}(Y)); \]
so combining this with the isomorphism in (2.6), it now follows that $H^1(S, L)$ is isomorphic to $H^2(X, \mathcal{O}_X)$. We have $\dim H^2(X, \mathcal{O}_X) = 1$, so
\[ \dim H^1(S, L) = 1. \] (2.7)
By the Grothendieck–Lefschetz hyperplane theorem for Picard’s group, the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism [7, Exposé XII]; in fact, a weaker version given in [8, Chapter IV, p. 179, Corollary 3.2] suffices for our purpose. By the generalized Noether–Lefschetz theorem (see [9, p. 121, Theorem 5.1]), the restriction map $\text{Pic}(Y) \rightarrow \text{Pic}(S)$ is also an isomorphism. Thus $\text{Pic}(S)$ is isomorphic to $\mathbb{Z}$. Combining this with (2.7), it follows that the surface $S$ has the desired properties.

3. Question 1.1 in special cases

In this section, we will first use the construction in Section 2 to show that Question 1.1 in the introduction has a negative answer in general. Then we will show that, in some particular cases, the answer is affirmative.

3.1. Example with a negative answer

We will construct a smooth projective surface $X$ and a vector bundle $E$ on it that does not admit any connection, while the restriction of $E$ to every smooth curve in $X$ admits a connection.
Let $X$ be a smooth complex projective surface with $\text{Pic}(X) = \mathbb{Z}$ that admits an ample line bundle $L$ with $H^1(X, L) \neq 0$; we saw in Section 2 that such a surface exists. Let $\mathcal{O}_X(1)$ denote the ample generator of $\text{Pic}(X)$. Then $L = \mathcal{O}_X(r) = \mathcal{O}_X(1)^{\otimes r}$ with $r$ positive. We choose $L$ with the smallest possible $r$. Since $\text{Pic}(X) = \mathbb{Z}$, we have $H^1(X, \mathcal{O}_X) = 0$ because $H^1(X, \mathcal{O}_X) = 0$ is the (abelian) Lie algebra of the Lie group $\text{Pic}(X)$. On the other hand, the Kodaira vanishing theorem says that $H^1(X, \mathcal{O}_X(-k)) = 0$ for all $k > 0$. Therefore, it follows that
\[ H^1(X, L \otimes \mathcal{O}_X(-d)) = 0, \forall \; d > 0. \] (3.1)
Let
\[ 0 \rightarrow L \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0 \] (3.2)
be the non-split extension corresponding to a non-zero element in $H^1(X, L)$. 
Theorem 3.1. The vector bundle $\text{End}(E) = E \otimes E^*$ in (3.2) has the property that the restriction of it to every smooth closed curve in $X$ admits a connection. The vector bundle $\text{End}(E)$ does not admit a connection.

Proof. Take any smooth closed curve $C \subset X$. So $C \in |O_X(d)|$ with $d$ positive. Consider the restriction homomorphism $H^1(X, L) \rightarrow H^1(C, L|_C)$. Using the long exact sequence of cohomologies associated with

$$0 \rightarrow L \otimes O_X(-d) \rightarrow L \rightarrow L|_C \rightarrow 0$$

we conclude that its kernel is $H^1(X, L \otimes O_X(-d))$, which is zero by (3.1). In particular, the extension class for (3.2) has a nonzero image in $H^1(C, L|_C)$. Therefore, the restriction of the exact sequence (3.2) to $C$ does not split.

We will show that $E|_C$ is indecomposable.

Assume that $E|_C = L_1 \oplus L_2$ with degree$(L_1) \geq$ degree$(L_2)$. Since degree$(E|_C) = \text{degree}(L|_C) > 0 = \text{degree}(O_C)$, the composition

$$L_1 \rightarrow E|_C \rightarrow O_C$$

is the zero homomorphism. Hence $L_1$ coincides with the subbundle $L|_C \subset E|_C$. This contradicts the earlier observation that the restriction of the exact sequence (3.2) to $C$ does not split. Hence, we conclude that $E|_C$ is indecomposable.

Consider the projective bundle $\mathbb{P}(E|_C) \rightarrow C$. Let $E_{\text{PGL}(2)} \rightarrow C$ be the principal $\text{PGL}(2, \mathbb{C})$-bundle corresponding to it. Since $E$ is indecomposable, it follows that $E_{\text{PGL}(2)}$ admits an algebraic connection [2, p. 342, Theorem 411]. The vector bundle $\text{End}(E|_C) \rightarrow C$ is associated with $E_{\text{PGL}(2)}$ for the adjoint action of $\text{PGL}(2, \mathbb{C})$ on $\text{End}_{\mathbb{C}}(\mathbb{C}^2) = M(2, \mathbb{C})$. Therefore, a connection on $E_{\text{PGL}(2)}$ induces a connection on the vector bundle $\text{End}(E|_C)$. Hence, we conclude that $\text{End}(E|_C) = \text{End}(E)|_C$ admits an algebraic connection.

On the other hand, $c_2(\text{End}(E)) = -c_1(L)^2 \neq 0$. This implies that the vector bundle $E$ on $X$ does not admit a connection [1, Theorem 4].

3.2. Special cases with positive answer

Let $S$ be a smooth complex projective curve, $X$ a smooth complex projective variety and $p : X \rightarrow S$ a smooth surjective morphism such that every fiber of $p$ is rationally connected. Assume that there is a smooth closed curve $\widetilde{S} \subset X$ such that the restriction

$$p|_{\widetilde{S}} : \widetilde{S} \rightarrow S$$

is an étale morphism.

Lemma 3.2. Let $E$ be a vector bundle on $X$ whose restriction to every smooth curve on $X$ admits a connection. Then $E$ admits a connection.

Proof. Let $Y$ be a smooth complex projective rationally connected variety and $V$ a vector bundle on $Y$, such that for every smooth rational curve $\mathbb{CP}^1 \rightarrow Y$ the restriction $r^*V$ has a connection. Any connection on a curve is flat, and $\mathbb{CP}^1$ is simply connected, so the above vector bundle $r^*V$ is trivial. This implies that the vector bundle $V$ is trivial [4, Proposition 1.2].

From the above observation, it follows that $E = p^*p_*E$. Therefore, it suffices to show that $p_*E$ admits a connection. Now, by the given condition, the vector bundle $(p|_{\widetilde{S}})^*p_*E = E|_{\widetilde{S}}$ admits a connection. Fix a connection $D$ on $E|_{\widetilde{S}}$. Averaging $D$ over the fibers of $p$, we get a connection on $p_*E$. This completes the proof.

Acknowledgements

We are very grateful to Jason Starr for his generous help. We thank the referee heartily for going through the paper carefully and providing comments to improve the exposition. The first-named author is supported by a J.C. Bose Fellowship.

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