The unified sub-equation method and its applications to conformable space-time fractional fourth-order Pochhammer-Chree equation

Abstract

In this article, we apply the unified sub-equation method proposed by Lu Bin and Zhang Hong Qing to construct many new Jacobi elliptic function solutions, solitons and other solutions for the conformable space-time fractional fourth-order Pochhammer-Chree equation. This method is direct and more powerful than the projective Riccati equation method, and developed by Yan. The solutions and other solutions of this equation can be found from the Jacobi elliptic solutions when its modulus or respectively. Comparing our new results with the well-known results is given.

Keywords: unified sub-equation method, jacobi elliptic function solutions, dark, singular and bright solitons, periodic solutions, the conformable space-time fractional fourth-order pochhammer-chree equation

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Introduction

When the nonlinear partial differential equations (PDEs) are analyzed, one of the most important equation is the construction of the exact solutions of those equations. Searching for the exact solutions of those equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, geochemistry, thermodynamics, soli mechanics, civil engineering, and non-Newtonian fluids to the natural phenomena. The solitons and other solutions of this equation can be found from the Jacobi elliptic solutions when its modulus or respectively. Comparing our new results with the well-known results is given.

The objective of this article is to apply a unified sub-equation method combined with the conformable space-time fractional derivatives for finding many new Jacobi elliptic function solutions, solitons and other solutions of the following nonlinear conformable space-time fractional fourth-order Pochhammer-Chree equation:

\[
\frac{\partial^{2n} u}{\partial t^{2n}} - \frac{\partial^{2n} u}{\partial x^{2n}} \left( \frac{\partial^{2n+\beta} u}{\partial x^{2n+\beta}} \right) - \frac{\partial^{2n+\beta}}{\partial x^{2n+\beta}} \left( \alpha u + \beta u^{n+1} + \gamma u^{2n+1} \right) = 0, \quad n \geq 1,
\]

(1.1)

Where \(0 < \alpha, \beta \leq 1\) and \(u(x, t)\) is a real function, while \(\alpha_1, \beta_1\) and \(\gamma_1\) are arbitrary constants. Equation (1.1) represents a nonlinear model of longitudinal wave propagation of elastic rods. Here the exponent \(n \geq 1\) is the power law nonlinearity parameter. When \(\alpha = \beta = 1\), Equation (1.1) has been discussed in \(^{26}\) using the generalized projective Riccati equation method, in\(^{35}\) using the extended \((G'/G)\) -expansion method, in\(^{40}\) using the \((G'/G)\) -expansion method, in\(^{41}\) using the tanh-coth and the sine-cosine methods, and in\(^{42}\) using the exp-function method.

This article is organized as follows: In Section 2, the description of conformable fractional derivative is given. In Section 3, the description of the unified sub-equation method combined with the conformable space-time fractional derivatives is obtained. In Section 4, we apply this method to the conformable space-time fractional fourth-order Pochammer-Chree equation (1.1). In Section 5, we present the graphical representations for some solutions of Equation (1.1). In Section 6, conclusions are obtained. To the best of our knowledge, Equation (1.1) has not been previously considered in literature using the method of Section 3.

Description of the conformable fractional derivative

Khalil et al.\(^{31}\) introduced a novel definition of fractional derivative named the conformable fractional derivative, which can rectify the deficiencies of the other definitions.

**Definition1:** Suppose \(f : [0, \infty) \rightarrow \mathbb{R}\) is a function. Then, the conformable fractional derivative of \(f\) of order \(\alpha\) is defined as
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For all \( t > 0 \) and \( \alpha \in (0,1) \), several properties of the conformable fractional derivative are given as in [1-3] and can be expressed as

\[
T_{\alpha}(f(t)) = \lim_{\tau \to 0} \frac{f(t + \tau t^{1-\alpha}) - f(t)}{\tau}.
\]  

(2.1)

Step 1: We assume that Equation (3.3) has the formal solution:

\[
\eta(x) = a_0 + \sum_{i=1}^{N} f^{-1}(x) \left[ q_i f(x) + b_i g(x) \right].
\]  

(3.4)

Where \( q_i, b_i, c_i \) \( (i = 1, \ldots, N) \) are constants to be determined later, \( f(x) \) and \( g(x) \) satisfy the auxiliary ODEs:

\[
\frac{d}{d\xi} f(x) = f(x) g(x),
\]  

(3.5)

\[
\frac{d}{d\xi} g(x) = q + r f(x) + r f^2(x).
\]  

(3.6)

Where \( q, r, c \) are constants.

Step 2: We assume that Equation (3.3) has the formal solution:

\[
\eta(x) = a_0 + \sum_{i=1}^{N} f^{-1}(x) \left[ q_i f(x) + b_i g(x) \right].
\]  

(3.4)

Where \( q_i, b_i, c_i \) \( (i = 1, \ldots, N) \) are constants to be determined later, \( f(x) \) and \( g(x) \) satisfy the auxiliary ODEs:

\[
\frac{d}{d\xi} f(x) = f(x) g(x),
\]  

(3.5)

\[
\frac{d}{d\xi} g(x) = q + r f(x) + r f^2(x).
\]  

(3.6)

Where \( q, r, c \) are constants.

Step 3: We determine the positive integer \( s \) in (3.4) by using the homogeneous balance between the highest order derivatives and the nonlinear terms in Equation (3.3). More precisely, we define the degree of \( u(\xi) \) as \( D[u(\xi)] = N \), which gives rise to the degree of other expressions as follows:

\[
D \left[ u^{\xi \alpha} \left( \frac{d^{\xi x_1} u(\xi)}{d\xi^{x_1}} \right)^{q_i} \right] = N P_i + s_i (q_i + N).
\]  

(3.8)

From (3.8) we can get the value of \( N \) in (3.4). In some nonlinear equations, the balance number \( N \) is not a positive integer. In this case, we make the following transformations:

When \( N = \frac{q_i}{P_i} \), where \( \frac{q_i}{P_i} \) is a fraction in the lowest terms, we let

\[
\eta(x) = \left[ f(x) \right]^{q_i/P_i},
\]  

(3.9)

When \( N \) is a negative number, we let

\[
\eta(x) = \left[ f(x) \right]^N,
\]  

(3.10)

And substitute (3.9) or (3.10) into Equation (3.3) to get a new equation in terms of the function \( f(x) \) with a positive integer balance number.

Step 4: We substitute (3.4) along with (3.5)-(3.7) into Equation (3.3) and collect all terms of the same order of \( f(x) \) \( g(x) \) \( (i, j = 0, 1, \ldots, N) \) and set them to zero, yield a set of algebraic equations which can be solved by using the Maple or Mathematical to find \( a_i, b_i, c_i, q, r, c \).
Step 5: It is well-known that (3.5), (3.6) have the following Jacobi elliptic function solutions:

(1) If \( q = (1 + m^2) \), \( r = -2m^2 \), \( c = -1 \), then
\[
f_1(\xi) = \frac{1}{sn(\xi, m)}, \quad g_1(\xi) = -\frac{cn(\xi, m)dn(\xi, m)}{sn(\xi, m)}.
\]

(2) If \( q = (1 - 2m^2) \), \( r = 2m^2 \), \( c = \left(\frac{3}{4} - 1\right) \), then
\[
f_2(\xi) = \frac{1}{cn(\xi, m)}, \quad g_2(\xi) = \frac{sn(\xi, m)dn(\xi, m)}{cn(\xi, m)}.
\]

(3) If \( q = (-2 + m^2) \), \( r = 2 \), \( c = (1 - m^2) \), then
\[
f_3(\xi) = \frac{1}{dn(\xi, m)}, \quad g_3(\xi) = -\frac{m^2sn(\xi, m)cn(\xi, m)}{dn(\xi, m)}.
\]

(4) If \( q = (1 + m^2) \), \( r = -2 \), \( c = -m^2 \), then
\[
f_4(\xi) = sn(\xi, m), \quad g_4(\xi) = \frac{cn(\xi, m)dn(\xi, m)}{sn(\xi, m)}.
\]

(5) If \( q = (1 - 2m^2) \), \( r = (-2 + 2m^2) \), \( c = m^2 \), then
\[
f_5(\xi) = cn(\xi, m), \quad g_5(\xi) = -\frac{sn(\xi, m)dn(\xi, m)}{cn(\xi, m)}.
\]

(6) If \( q = (-2 + m^2) \), \( r = (2 - 2m^2) \), \( c = 1 \), then
\[
f_6(\xi) = dn(\xi, m), \quad g_6(\xi) = -\frac{m^2sn(\xi, m)cn(\xi, m)}{dn(\xi, m)}.
\]

(7) If \( q = (-2 + m^2) \), \( r = (-2 + 2m^2) \), \( c = -1 \), then
\[
f_7(\xi) = \frac{cn(\xi, m)}{sn(\xi, m)}, \quad g_7(\xi) = -\frac{dn(\xi, m)}{sn(\xi, m)}.
\]

(8) If \( q = (1 - 2m^2) \), \( r = (2m^2 - 2m^4) \), \( c = -1 \), then
\[
f_8(\xi) = \frac{dn(\xi, m)}{sn(\xi, m)}, \quad g_8(\xi) = -\frac{cn(\xi, m)}{sn(\xi, m)dn(\xi, m)}.
\]

(9) If \( q = (-2 + m^2) \), \( r = -2 \), \( c = (-1 + m^2) \), then
\[
f_9(\xi) = \frac{sn(\xi, m)}{cn(\xi, m)}, \quad g_9(\xi) = \frac{dn(\xi, m)}{sn(\xi, m)cn(\xi, m)}.
\]

(10) If \( q = (1 + m^2) \), \( r = -2m^2 \), \( c = -1 \), then
\[
f_{10}(\xi) = \frac{dn(\xi, m)}{cn(\xi, m)}, \quad g_{10}(\xi) = \frac{cn(\xi, m)dn(\xi, m)}{sn(\xi, m)}.
\]

(11) If \( q = (1 - 2m^2) \), \( r = -2 \), \( c = (m^2 - m^4) \), then
\[
f_{11}(\xi) = \frac{sn(\xi, m)}{cn(\xi, m)}, \quad g_{11}(\xi) = \frac{cn(\xi, m)sn(\xi, m)}{dn(\xi, m)}.
\]

(12) If \( q = \frac{1}{2}(-1 + 2m^2) \), \( r = -\frac{1}{2} \), \( c = -\frac{1}{4} \), then
\[
f_{12}(\xi) = \frac{cn(\xi, m) + 1}{sn(\xi, m)}, \quad g_{12}(\xi) = \frac{1}{2}dc(\xi, m).
\]

(13) If \( q = \frac{1}{2}(m^2 + 1) \), \( r = \frac{1}{2}(1 - m^2) \), \( c = -\frac{1}{4}(1 - m^2) \), then
\[
f_{13}(\xi) = \frac{sn(\xi, m)}{msn(\xi, m) + 1}, \quad g_{13}(\xi) = \frac{1}{2}msn(\xi, m).
\]

(14) If \( q = -\frac{1}{2}(1 + m^2) \), \( r = \frac{1}{2}(m^2 - 1)^2 \), \( c = -\frac{1}{4} \), then
\[
f_{14}(\xi) = \frac{msn(\xi, m) + 1}{msn(\xi, m) + 1}, \quad g_{14}(\xi) = \frac{1}{2}dc(\xi, m).
\]

(15) If \( q = -\frac{1}{2}(1 + m^2) \), \( r = -\frac{1}{2}(m^2 - 1)^2 \), \( c = -\frac{1}{4} \), then
\[
f_{15}(\xi) = \frac{msn(\xi, m) + 1}{sn(\xi, m)}, \quad g_{15}(\xi) = \frac{1}{2}msn(\xi, m).
\]

(16) If \( q = (m^2 - 6m + 1) \), \( r = -2 \), \( c = 4m - (m - 1)^2 \), then
\[
f_{16}(\xi) = \frac{msn(\xi, m)}{msn(\xi, m) + 1}, \quad g_{16}(\xi) = \frac{cn(\xi, m)dn(\xi, m)}{mn^2(\xi, m) + 1}.
\]

(17) If \( q = (m^2 + 6m + 1) \), \( r = -2 \), \( c = -4m, (m + 1)^2 \), then
\[
f_{17}(\xi) = \frac{msn(\xi, m)}{msn(\xi, m) + 1}, \quad g_{17}(\xi) = \frac{cn(\xi, m)dn(\xi, m)}{mn^2(\xi, m) + 1}.
\]

(18) If \( q = -\frac{1}{2}(1 + m^2) \), \( r = 1 \), \( c = \frac{1}{4}(m^2 - 1) \), then
\[
f_{18}(\xi) = \frac{cn(\xi, m)}{sn(\xi, m) + 1}, \quad g_{18}(\xi) = \frac{1}{2}dc(\xi, m).
\]

(19) If \( q = \frac{1}{2}(2 - m^2) \), \( r = -\frac{m^4}{2}, c = -\frac{1}{4} \), then
\[
f_{19}(\xi) = \frac{cn(\xi, m) + 1}{sn(\xi, m)}, \quad g_{19}(\xi) = \frac{1}{2}dc(\xi, m).
\]

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\[ f_{14}(\xi) = \frac{dn(\xi, m) \pm 1}{sn(\xi, m)}, \quad g_{14}(\xi) = \mp cs(\xi, m). \]

(20) If \( q = \frac{1}{2} \left(2m^2 - 1\right) \), \( r = -\frac{1}{2} \), \( c = -\frac{1}{4} \), then

\[ f_{20}(\xi) = \frac{sn(\xi, m)}{1 \pm cn(\xi, m)}, \quad g_{20}(\xi) = \pm ds(\xi, m). \]

(21) If \( q = -\frac{1}{2} \left(1 + m^2\right) \), \( r = -\frac{1}{2} \), \( c = -\frac{1}{4} \left(m^2 - 1\right)^2 \), then

\[ f_{21}(\xi) = \frac{sn(\xi, m)}{cn(\xi, m) \pm dn(\xi, m)}, \quad g_{21}(\xi) = \pm ns(\xi, m). \]

Where \( sn(\xi, m) \), \( cn(\xi, m) \) and \( dn(\xi, m) \) are Jacobi elliptic sine function, Jacobi elliptic cosine function, Jacobi elliptic function of the third kind respectively, and \( m \) denotes the modulus of Jacobi elliptic functions, where \( 0 \leq m \leq 1 \). It is well-known\(^{54-56} \) that the Jacobi-elliptic functions satisfy the following relations:

\[
\begin{align*}
\frac{1}{2} (\xi) &= 1 - sn^2(\xi, m), \quad dn'(\xi, m) = 1 - m^2 sn^2(\xi, m), \\
\frac{1}{m} (\xi, m) &= cn(\xi, m)dn(\xi, m), \quad cn'(\xi, m) = -sn(\xi, m)dn(\xi, m), \\
\frac{1}{m} (\xi, m) &= -m^2 sn(\xi, m)cn(\xi, m), \\
\frac{1}{m} (\xi, m) &= \frac{1}{sn(\xi, m)}, \quad \frac{1}{cn(\xi, m)} = \frac{1}{nd(\xi, m)}, \quad \frac{1}{sn(\xi, m)} = \frac{1}{dn(\xi, m)}, \\
\frac{1}{m} (\xi, m) &= \frac{sn(\xi, m)}{cn(\xi, m)}, \quad \frac{dn(\xi, m)}{sn(\xi, m)} = \frac{cn(\xi, m)}{sn(\xi, m)}, \\
\frac{1}{m} (\xi, m) &= \frac{dn(\xi, m)}{sn(\xi, m)}, \quad \frac{dc(\xi, m)}{dn(\xi, m)} = \frac{dn(\xi, m)}{sn(\xi, m)}, \quad \frac{dc(\xi, m)}{sn(\xi, m)} = \frac{dn(\xi, m)}{sn(\xi, m)}, \\
\frac{1}{m} (\xi, m) &= \frac{1}{sn(\xi, m)}.
\end{align*}
\]

The Jacobi elliptic functions degenerate into hyperbolic functions when \( m \to 1 \) as follows:

\[
\begin{align*}
\frac{1}{m} (\xi, 1) &= \tanh(\xi), \quad \frac{1}{m} (\xi, 1) = \coth(\xi), \quad \frac{1}{m} (\xi, 1) = \coth(\xi), \\
\frac{1}{m} (\xi, 1) &= \frac{1}{m} (\xi, 1) = \frac{1}{m} (\xi, 1) = 1, \\
\frac{1}{m} (\xi, 1) &= \frac{1}{m} (\xi, 1) = \frac{1}{m} (\xi, 1) = \frac{1}{m} (\xi, 1) = 1, \\
\frac{1}{m} (\xi, 1) &= \frac{1}{m} (\xi, 1) = \frac{1}{m} (\xi, 1) = \frac{1}{m} (\xi, 1) = 1.
\end{align*}
\]

And into trigonometric functions when \( m \to 0 \) as follows:

\[
\begin{align*}
\frac{1}{m} (\xi, 0) &= \sin(\xi), \quad \frac{1}{m} (\xi, 0) = \cos(\xi), \\
\frac{1}{m} (\xi, 0) &= \frac{1}{m} (\xi, 0) = 1, \\
\frac{1}{m} (\xi, 0) &= \frac{1}{m} (\xi, 0) = 1, \\
\frac{1}{m} (\xi, 0) &= \frac{1}{m} (\xi, 0) = 1, \\
\frac{1}{m} (\xi, 0) &= \frac{1}{m} (\xi, 0) = 1.
\end{align*}
\]

The constants of integration, we get

\[
\left(\frac{1}{m} - \frac{1}{m} \right) u - \frac{1}{m} u + \frac{1}{m} v - \frac{1}{m} v = 0, \quad (3.3)
\]

Balancing \( u \) with \( u^{2m+1} \), we get \( N = \frac{1}{m} \). According to (3.9) we use the transformation:

\[
\frac{1}{m} (\xi) = \frac{1}{m} (\xi), \quad (4.1)
\]

Where \( \frac{1}{m} (\xi) \) is a new function of \( \xi \), to reduce Equation (3.3) into the new ODE:

\[
\left(\frac{1}{m} - \frac{1}{m} \right) v - \frac{1}{m} u^{2m} - \frac{1}{m} v^{2m+1} = 0, \quad (4.2)
\]

Balancing \( UV \) with \( v^{4} \) in Equation (4.5), we get \( N = 1 \). According to the form (3.4), Equation (4.5) has the formal solution:

\[
\left(\frac{1}{m} - \frac{1}{m} \right) y - \frac{1}{m} y^{2} - \frac{1}{m} y^{2m} = 0, \quad (4.3)
\]
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Equation (4.5) and collecting all terms of the same order of \( f'(\xi) \), \( g'(\xi) \), \((i,j=0,1,2,...)\) and setting them to zero, we have the following algebraic equations:

\[
\begin{align*}
\left\{ f'(\xi): \right. & 6n^2 \gamma, \alpha, q, b c + c_i, a_i - c_i, b_i c - c_i, b_i c + n^2 c_i, a_i c - n^2, \gamma, \alpha, a_i c - n^2, \gamma, b_i c - c_i, b_i c = 0, \\
\left. f''(\xi): \right. & c_i, r b_i + c_i, n r, b_i + n^2, \gamma, r, b_i = 0, \\
\left. f'''(\xi): \right. & -n^2, \beta, a_i, b_i^3 + 3n^2, \alpha, a_i b_i^2 c + 2n^2, q, a_i q b_i c - 12n^2, \gamma, q, a_i q b_i c = 0, \\
\left. f''''(\xi): \right. & c_i, a_i q - n^2, c_i, a_i q c + 6n^2, \gamma, a_i q c - n^2, a_i q c - n^2, \gamma, a_i q c - c_i, a_i q c = 0, \\
\left. f'''''(\xi): \right. & -2n^2, \gamma, b_i q c + 6n^2, \gamma, b_i q c - 2n^2, a_i q c - 2n^2, \gamma, a_i q c + c_i, a_i q c = 0, \\
\left. g(\xi) \right. & 3\beta, a_i q b_i - 4n^2, \gamma, a_i q b_i + n^2 c_i, a_i q b_i + 12n^2, \gamma, a_i q b_i = 0, \\
\left. f''''(\xi): \right. & 3n^2, \beta, a_i q b_i - 2n^2, \gamma, a_i q b_i = 0, \\
\left. f''''''(\xi): \right. & 2c_i, r b_i - 2n^2, c_i, a_i q c - n^2, a_i q c r - 2n^2, \gamma, b_i q c - 6n^2, \gamma, b_i q c = 0, \\
\left. f''''''(\xi): \right. & 2c_i, a_i q - 4n^2, a_i q c - 8n^2, \gamma, a_i q c - 2n^2, \gamma, a_i q c - c_i, a_i q c = 0, \\
\left. g(\xi) \right. & 2c_i, a_i q b_i + n^2 b_i q - 2n^2, \gamma, a_i q b_i = 4n^2, \gamma, a_i q b_i + 4n^2, \gamma, a_i q b_i q - 3n^2, \beta, a_i q b_i = 0, \\
\left. f(\xi) \right. & 2n^2, \gamma, a_i q b_i + 2n^2 c_i, a_i q b_i + 4n^2, \gamma, a_i q b_i q - 6n^2, \beta, a_i q b_i b_i - 12n^2, \gamma, a_i q b_i + n^2 c_i, a_i q b_i = 0, \\
\left. f''''(\xi): \right. & 2c_i, a_i q b_i + 2n^2, \gamma, a_i q b_i - c_i, a_i q b_i = 0.
\end{align*}
\]

According to Step 5 of Section 3, we have the following results:

**Result 1:** If we substitute \( q = (1 + m^2) \), \( r = -2m^2 \), \( c = -1 \) into the algebraic equations (4.7) and using Maple, we have

\[
\begin{align*}
\begin{cases}
m = m, n = 1, \beta_1 = 0, a_0 = 0, a_1 = 0, b_1 = \\
\frac{2a_1}{\gamma_1(1 + m^2)}, c_1 = \frac{-a_1}{(1 + m^2)}, a_0 < 0, \gamma_1 < 0
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
\begin{cases}
m = m, n = 2, \alpha_1 = \beta_1 = 0, \gamma_1 = \frac{15}{64} \frac{a_1}{\gamma_1}, a_0 = 0, a_1 = 0, b_1 = \\
\frac{3 \beta_1}{8 \gamma_1}, c_1 = \frac{-3 \beta_1}{4 m}, a_0 < 0, \gamma_1 < 0
\end{cases}
\end{align*}
\]

Substituting (4.8) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x, t) = \sqrt{\frac{2a_1}{\gamma_1}} \left[ \cot(\frac{x}{\beta} - \sqrt{\frac{2a_1}{\gamma_1}}, t^\alpha) \right],
\]

while if \( m \to 1 \), then we have the solitary wave solution:

\[
u(x, t) = \sqrt{\frac{2a_1}{3 \gamma_1}} \left[ \coth(\frac{x}{\beta} - \sqrt{\frac{2a_1}{3 \gamma_1}}, t^\alpha) - \tanh(\frac{x}{\beta} - \sqrt{\frac{2a_1}{3 \gamma_1}}, t^\alpha) \right].
\]

Substituting (4.9) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x, t) = \frac{-3 \beta_1}{8 \gamma_1} \left[ \frac{1}{1 - \frac{1}{\gamma_1}}, \frac{1}{\gamma_1}, \frac{1}{\gamma_1}, \frac{1}{\gamma_1} \right]^{\frac{1}{2}}, \gamma_1 < 0, \beta_1 > 0
\]

where \( \xi = \frac{x}{\beta} - \frac{1}{4 \gamma_1}, a \). If \( m \to 1 \), then we have the singular soliton solution:

\[
u(x, t) = \frac{-3 \beta_1}{8 \gamma_1} \left[ 1 - \cot(\frac{x}{\beta} - \frac{1}{4 \gamma_1}, \frac{1}{\gamma_1}, \frac{1}{\gamma_1}, \frac{1}{\gamma_1}) \right]^{\frac{1}{2}}.
\]

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Result 2: If we substitute \( q = (1 - 2m^2), r = 2m^2, c = (m^2 - 1) \) into the algebraic equations (4.7) and use the Maple, we have

\[
m = m, n = 2, a_0 = \frac{-3 \beta_1}{8 \gamma_1}, a_1 = \frac{3 \beta_1}{8 \gamma_1}, c_1 = \frac{\beta_1}{4 \sqrt{\gamma_1 (m^2 - 1)}}, c_2 = \frac{3 \beta_1^2 (4m^2 - 1)}{64 \gamma_1 (m^2 - 1)}, b_1 = 0, \gamma_1 (m^2 - 1) > 0.
\]  

(4.15)

Substituting (4.15) into (4.4), (4.6), we have the periodic solution:

\[
u(x,t) = \left[ -\frac{3 \beta_1}{8 \gamma_1} \left( \frac{1}{1 - \frac{1}{\csc^2 (\xi, m)}} \right) \right]^{\frac{1}{2}}, \gamma_1 > 0, \beta_1 < 0.
\]  

(4.16)

where \( \xi = \frac{x^\beta}{\beta} - \left( \frac{-\alpha_1}{4 \sqrt{2m^2 - 5}} \right) \frac{t^a}{\alpha} \). If \( m \to 0 \), then we have the dark soliton solution:

\[
u(x,t) = \sqrt{\frac{-2 \alpha_1}{3 \gamma_1}} \left[ \tanh \left( \frac{x^\beta}{\beta} - \frac{\alpha_1}{3} \frac{t^a}{\alpha} \right) \right].
\]  

(4.19)

Result 3: If we substitute \( q = (2 + 2m^2), r = 2, c = (1 - m^2) \) into the algebraic equations (4.7) and use the Maple, we have

\[
u(x,t) = \left[ -\frac{3 \beta_1}{8 \gamma_1} \left( 1 + \frac{1}{\sec (\xi, m)} \right) \right]^{\frac{1}{2}}, \gamma_1 > 0, \beta_1 > 0
\]  

(4.22)

where \( \xi = \frac{x^\beta}{\beta} - \left( \frac{-\alpha_1}{4 \sqrt{m^2 - 5}} \right) \frac{t^a}{\alpha} \). If \( m \to 1 \), then we have the dark soliton solution:

\[
u(x,t) = \left[ -\frac{3 \beta_1}{8 \gamma_1} \left( 1 + \tan \left( \frac{x^\beta}{\beta} - \frac{\alpha_1}{4} \frac{t^a}{\alpha} \right) \right) \right]^{\frac{1}{2}}.
\]  

(4.23)

Result 4: If we substitute \( q = (1 + m^2), r = -2, c = -m^2 \) into the algebraic equations (4.7) and use the Maple, we have

\[
m = m, n = 2, a_0 = \frac{-3 \beta_1}{8 \gamma_1}, b_1 = 0, a_1 = \frac{-3 \beta_1}{8 \gamma_1}, \alpha = \frac{15 \beta_1^2 (m^2 - 1)}{64 \gamma_1 m^2}, c_1 = \frac{\beta_1}{4 m \gamma_1}, \gamma_1 (m^2 - 1) < 0
\]  

(4.21)

Substituting (4.21) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x,t) = \sqrt{\frac{-2 \alpha_1}{3 \gamma_1}} \left[ \tanh \left( \frac{x^\beta}{\beta} - \frac{\alpha_1}{3} \frac{t^a}{\alpha} \right) \right].
\]  

(4.20)

Result 5: If we substitute \( q = (1 - 2m^2), r = (-2 + 2m^2), c = m^2 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
n & = 1, a_0 = \frac{-\beta_1}{3 \gamma_1}, a_1 = 0, b_1 = -\frac{\beta_1}{3 \gamma_1 (2m^2 - 1)}, c_1 = (3 \gamma_1 m^2 - 5), a_1 = \frac{\beta_1}{18 \gamma_1 (2m^2 - 1)}, \gamma_1 (2m^2 - 1) < 0
\end{align*}
\]  

(4.24)

Substituting (4.24) into (4.4), (4.6), we have the Jacobi elliptic solutions:

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\[ u(x,t) = \frac{-\beta_1}{3\gamma_1} \left[ 1 - \frac{1}{\sqrt{2m^2 - 1}} \left( \frac{\sin(\xi, m) d\xi}{\cos(\xi, m)} \right) \right] \]

where \( \xi = \frac{x^\beta}{\beta} - \frac{\beta_1}{3\sqrt{2m^2 - 1}} \left( \frac{1}{\gamma_1} \right) \), \( \gamma_1 > 0 \) or \( \gamma_1 < 0 \). If \( m \to 1 \), then we have the dark soliton solution:

\[ u(x,t) = \frac{-\beta_1}{3\gamma_1} \left[ 1 - \tan \left( \frac{x^\beta}{\beta} - \frac{\beta_1}{3\sqrt{2m^2 - 1}} \left( \frac{1}{\gamma_1} \right) \right) \right], \quad \gamma_1 < 0. \]

**Result 6:** If we substitute \( q = (-2+m^2) \), \( r = (2-2m^2) \), \( c = 1 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
m = m, n = 2, a_0 & = \frac{3\beta_1^2}{64\gamma_1} m^2 + 8 \gamma_1, a_1 = -\frac{3\beta_1}{8\gamma_1}, a_2 = 0, c_1 = \frac{\beta_1^2}{4\gamma_1}, \gamma_1 > 0 \\
\end{align*}
\]

Substituting (4.32) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \frac{1}{\left( \frac{3\beta_1}{8\gamma_1} \right)^{1/2}} d\xi, \quad \gamma_1 > 0, \beta_1 < 0. \]

**Result 7:** If we substitute \( q = (-2+m^2) \), \( r = (2+2m^2) \), \( c = -1 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
m = m, n = 2, a_0 & = \frac{3\beta_1}{8m^2\gamma_1}, a_1 = \frac{3\beta_1}{4m^2\gamma_1} \left( m^2 - 2 + 2\sqrt{1 - m^2} \right), a_2 = 0, \gamma_1 > 0. \\
\end{align*}
\]

Substituting (4.30) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \frac{1}{\sqrt{\gamma_1}} \left( \frac{\sin(\xi, m)}{\sin(\xi, m)} \right), \quad \gamma_1 < 0. \]

where \( \xi = \frac{x^\beta}{\beta} - \frac{\beta_1}{4\gamma_1} \left( \frac{1}{\gamma_1} \right) \). If \( m \to 0 \), then we have the periodic solution:

\[ u(x,t) = \frac{1}{\sqrt{\gamma_1}} \cos \left( \frac{x^\beta}{\beta} - \frac{\beta_1}{2\gamma_1} \right), \quad \gamma_1 < 0. \]

**Result 8:** If we substitute \( q = (1-2m^2) \), \( r = (2m^2 - 2m^4) \), \( c = -1 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
m = m, n = 1, a_0 = 0, a_1 = a_2 = \frac{2\alpha_1}{\gamma_1} \left( \frac{1}{\gamma_1} \right) \left( \frac{1}{2m^2 - 5} \right) \left( \frac{1}{\gamma_1} \right), \gamma_1 > 0. \\
\end{align*}
\]

Substituting (4.33) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \frac{1}{\sqrt{\gamma_1}} \left( \frac{\cos(\xi, m)}{\cos(\xi, m)} \right), \quad \gamma_1 > 0. \]

**Result 9:** If we substitute \( q = (-2+m^2) \), \( r = -2 \), \( c = (-1+m^2) \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
m = m, n = 1, a_0 = 0, a_1 = a_2 = \frac{-\alpha_1}{\gamma_1} \left( \frac{1}{\gamma_1} \right) \left( \frac{1}{2m^2 - 5} \right) \left( \frac{1}{\gamma_1} \right), \gamma_1 > 0. \\
\end{align*}
\]

Substituting (4.34) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \frac{1}{\sqrt{\gamma_1}} \left( \frac{\csc(\xi, m)}{\csc(\xi, m)} \right), \quad \gamma_1 > 0. \]

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\[ u(x,t) = \sqrt{2\alpha_1 \gamma_1} \left( \frac{\alpha_1}{2m^2-5} \right) \left( \frac{dn(\xi,m)}{sn(\xi,m)cn(\xi,m)} \right), \quad \alpha_1 > 0, \gamma_1 < 0 \]  \hspace{1cm} (4.36)

where \( \xi = \frac{x^\beta}{\beta} \left( \frac{-\alpha_1}{\sqrt{2m^2-5}} \right)^{\frac{1}{\alpha}} \). If \( m \rightarrow 1 \), then we have the singular soliton solution:

\[ u(x,t) = \frac{-2\alpha_1}{3\gamma_1} \left( \frac{coth(x^\beta/\sqrt{3}) - c}{\alpha} \right), \quad (4.37) \]

while if \( m \rightarrow 0 \), then we have the periodic solution:

\[ u(x,t) = \frac{-2\alpha_1}{3\gamma_1} \left( \frac{coth(x^\beta/\sqrt{3}) - c}{\alpha} \right), \quad (4.38) \]

Result 10: If we substitute \( \alpha = 1-2m^2 \), \( c = 1-2m^4 \) into the algebraic equations (4.7) and use the Maple, we have

\[ m=m,n=2,\alpha_1 = -\frac{3\beta_1}{8\gamma_1}, \beta_1 = -\frac{3\beta_1}{8\gamma_1}, h_0 = 0, \gamma_1 = -\frac{3}{4\gamma_1} \gamma_1 < 0 \] \hspace{1cm} (4.39)

Substituting (4.39) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \left( \frac{3\beta_1}{8\gamma_1} + \frac{1}{4\gamma_1} \right) \left( \frac{dn(\xi,m)}{cn(\xi,m)} \right), \quad (4.40) \]

where \( \xi = x^\beta/\sqrt{3} \left( \frac{-\alpha_1}{\sqrt{2m^2-1}} \right)^{\frac{1}{\alpha}} \). If \( m \rightarrow 0 \), then we have the periodic solution:

\[ u(x,t) = \left( \frac{3\beta_1}{8\gamma_1} + \frac{1}{4\gamma_1} \right) \left( \frac{dn(\xi,m)}{cn(\xi,m)} \right), \quad (4.41) \]

Result 11: If we substitute \( \alpha = 1-2m^2 \), \( c = 1-2m^4 \) into the algebraic equations (4.7) and use the Maple, we have

\[ m=m,n=1,\alpha_0 = 0, h_0 = 0, h_1 = -\frac{2\alpha_1}{\gamma_1 (4m^2-1)}, \gamma_1 > 0 \] \hspace{1cm} (4.42)

Substituting (4.42) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \left[ \frac{2\alpha_1}{\gamma_1 (4m^2-1)} \right] \left( \frac{dn(\xi,m)}{cn(\xi,m)sn(\xi,m)} \right), \quad (4.43) \]

If \( m \rightarrow 0 \), then we have the same periodic solution (4.11), while if \( m \rightarrow 1 \), then we have the same singular soliton solution (4.37).

Result 12: If we substitute \( q = -1-2m^2 \), \( r = -1 \), \( c = -1 \) into the algebraic equations (4.7) and use the Maple, we have:

\[ m=m,n=1,\alpha_0 = 0, h_0 = 0, \gamma_1 = \frac{-\alpha_1}{\gamma_1 (m^2-1)}, \gamma_1 < 0 \] \hspace{1cm} (4.44)

Substituting (4.44) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \left( \frac{\alpha_1}{\gamma_1 (m^2-1)} \right) \left( \frac{dn(\xi,m)}{sn(\xi,m)cn(\xi,m)} \right), \quad (4.45) \]

where \( \xi = x^\beta/\sqrt{2m^2-1} \left( \frac{-\alpha_1}{\sqrt{2m^2-1}} \right)^{\frac{1}{\alpha}} \). If \( m \rightarrow 0 \), then we have the same periodic solution (4.34).

Result 13: If we substitute \( q = -1 \left( m^2+1 \right), r = -1 \left( m^2-1 \right), c = -1 \) into the algebraic equations (4.7) and use the Maple, we have:

\[ m=m,n=1,\alpha_0 = 0, h_0 = 0, \gamma_1 = \frac{-\alpha_1}{\gamma_1 (m^2-1)}, \gamma_1 < 0 \] \hspace{1cm} (4.46)

Substituting (4.46) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \left( \frac{\alpha_1}{\gamma_1 (m^2-1)} \right) \left( \frac{dn(\xi,m)}{sn(\xi,m)cn(\xi,m)} \right), \quad (4.47) \]

where \( \xi = x^\beta/\sqrt{2m^2-1} \left( \frac{-\alpha_1}{\sqrt{2m^2-1}} \right)^{\frac{1}{\alpha}} \). If \( m \rightarrow 0 \), then we have the same periodic solution (4.34).

Result 14: If we substitute \( q = -1 \left( m^2+1 \right), r = -1 \left( m^2-1 \right), c = -1 \) into the algebraic equations (4.7) and use the Maple, we have:

\[ m=m,n=1,\alpha_0 = 0, h_0 = 0, \gamma_1 = \frac{-\alpha_1}{\gamma_1 (m^2-1)}, \gamma_1 < 0 \] \hspace{1cm} (4.48)

Substituting (4.48) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = \left( \frac{\alpha_1}{\gamma_1 (m^2-1)} \right) \left( \frac{dn(\xi,m)}{sn(\xi,m)cn(\xi,m)} \right), \quad (4.49) \]
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where \( \xi = \frac{x^\beta}{\beta} \sqrt{\frac{-2\alpha_i}{\alpha}} \). If \( m \to 1 \), then we have the same dark soliton solution (4.26).

**Result 15:** If we substitute \( q = -\frac{1}{2}(1 + m^2) \), \( r = -\frac{1}{2}(m^2 - 1)^2 \), \( c = -\frac{1}{4} \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
\begin{cases}
m = m, n = 2, \alpha_i = \frac{-3\beta_i^2}{64\gamma_i}, a_0 = \frac{-3\beta_i}{8\gamma_i}, a_1 = 0, \\
b_1 = \frac{-3\beta_i}{8\gamma_i}, c_1 = \frac{\beta_i}{4\gamma_i}, \gamma_i < 0
\end{cases}
\end{align*}
\]

Substituting (4.50) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x,t) = \frac{-3\beta_i}{8\gamma_i} \left[ 2 + \frac{3\alpha_i}{\alpha} \right]^{\frac{1}{2}} \right], \beta_i > 0, \gamma_i < 0.
\]

while if \( m \to 0 \), then we have the periodic solutions.

\[
u(x,t) = \frac{-3\beta_i}{8\gamma_i} \left[ 2 \csc^2 \left( \frac{x^\beta}{\beta} - \frac{3\alpha_i}{\alpha} \right) \right]^{\frac{1}{2}} \right], \beta_i > 0, \gamma_i < 0.
\]

**Result 16:** If we substitute \( q = (m^2 - 6m + 1) \), \( r = -2 \), \( c = 4m \) \((m - 1)^2 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
\begin{cases}
m = m, n = 1, \beta_i = 0, a_0 = 0, a_1 = 0, b_1 = \frac{2\alpha_i}{\gamma_i} \left( \frac{3}{2} \right) \left[ 2m^2 - 12m + 1 \right], \\
, c_1 = \frac{-3\alpha_i}{\gamma_i} \left( \frac{3}{2} \right) \left[ 2m^2 - 12m + 1 \right] < 0, \gamma_i < 0
\end{cases}
\end{align*}
\]

Substituting (4.54) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x,t) = \frac{2\alpha_i}{\gamma_i} \left( \frac{3}{2} \right) \left[ 2m^2 - 12m + 1 \right] \left[ \csc(\xi,m) \right] \left[ \sin^2(\xi,m) + 1 \right]^{-\frac{3}{2}} \right], \beta_i > 0, \gamma_i < 0.
\]

**Result 17:** If we substitute \( q = (m^2 - 6m + 1) \), \( r = -2 \), \( c = -4(m + 1)^2 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
\begin{cases}
m = m, n = 2, a_0 = \frac{-3\beta_i}{8\gamma_i}, a_1 = \frac{-3\beta_i}{8\gamma_i}, b_1 = 0, a_i = \frac{-3\beta_i^2}{256m\gamma_i}, \\
c_1 = \frac{\beta_i}{8\gamma_i}, \gamma_i < 0
\end{cases}
\end{align*}
\]

Substituting (4.57) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[
u(x,t) = \frac{-3\beta_i}{8\gamma_i} \left[ 1 + \frac{(m + 1)\sin(\xi,m)}{\sin^2(\xi,m) + 1} \right]^{\frac{1}{2}} \left[ 2 + \frac{3\alpha_i}{\alpha} \right]^{\frac{1}{2}} \left[ \csc(\xi,m) \right]^{\frac{1}{2}}, \beta_i > 0, \gamma_i < 0.
\]

where \( \xi = \frac{x^\beta}{\beta} - \frac{3\alpha_i}{\alpha} \). If \( m \to 1 \), then we have the solitary wave solution:

\[
u(x,t) = \frac{-3\beta_i}{8\gamma_i} \left[ 2 \left[ 1 + \frac{3\alpha_i}{\alpha} \right] \right]^{\frac{1}{2}} \left[ \csc(\xi,m) \right]^{\frac{1}{2}}, \beta_i > 0, \gamma_i < 0.
\]

**Result 18:** If we substitute \( q = -\frac{1}{2}(1 + m^2) \), \( r = -\frac{1}{2}(m^2 - 1)^2 \) into the algebraic equations (4.7) and use the Maple, we have

\[
\begin{align*}
\begin{cases}
m = m, n = 2, a_0 = \frac{-3\beta_i}{8\gamma_i}, b_i = \frac{-3\beta_i}{8\gamma_i}, a_i = 0, \\
, c_1 = \frac{\beta_i}{4\gamma_i}, \gamma_i < 0
\end{cases}
\end{align*}
\]

Substituting (4.60) into (4.4), (4.6), we have the Jacobi elliptic solutions:
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where \( \xi = \frac{x^\beta}{\beta} - \left( \frac{\beta}{4} \sqrt{\frac{3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha} \). If \( m \to 0 \), then we have the periodic solutions (4.17) and (4.41) respectively.

Result 19: If we substitute \( q = -\frac{1}{2} (2 - m^2) \), \( r = -\frac{1}{4} m^4 \), \( c = -\frac{1}{4} \) into the algebraic equations (4.7) and use the Maple, we have

Substituting (4.62) into (4.4), (4.6), we have the Jacobi elliptic solutions:

where \( \xi = \frac{x^\beta}{\beta} - \left( \frac{\beta}{2} \sqrt{\frac{3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha} \). If \( m \to 1 \), then we have the solitary wave solutions:

Result 21: If we substitute \( q = -\frac{1}{2} (1 + m^2) \), \( r = -\frac{1}{2} \), \( c = -\frac{1}{4} (m^2 - 1)^2 \) into the algebraic equations (4.7) and use the Maple, we have

while if \( m \to 0 \), then we have the periodic solution:

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Substituting (4.69) into (4.4), (4.6), we have the Jacobi elliptic solutions:

\[ u(x,t) = -\frac{\beta_1}{3\gamma_1} \left[ 1 \pm 2\sqrt{\frac{2}{1 + m^2}} \right] \sinh(\xi, m), \quad (4.70) \]

where \( \xi = \frac{x^\beta}{\beta} - \frac{\beta_1}{3\gamma_1} \left( \frac{1 - e^\alpha}{2} \right) \). If \( m \to 1 \), then we have the singular soliton solution:

\[ u(x,t) = -\frac{\beta_1}{3\gamma_1} \left[ 1 \pm \coth \left( \frac{x^\beta}{\beta} - \frac{\beta_1}{3\gamma_1} \left( \frac{1 - e^\alpha}{2} \right) \right) \right], \quad (4.71) \]

while if \( m \to 0 \), then we have the periodic solution:

\[ u(x,t) = -\frac{\beta_1}{3\gamma_1} \left[ 1 \pm \sqrt{2} \csc \left( \frac{x^\beta}{\beta} - \frac{\beta_1}{3\gamma_1} \left( \frac{1 - e^\alpha}{2} \right) \right) \right], \quad (4.72) \]

The graphical representations of some solutions

In this section, we present some graphs of the solitons and other solutions of Eq. (1.1). Let us now examine Figures (1 - 12) as it illustrates some of our solutions obtained in this article. To this aim, we select some special values of the parameters obtained for example, in some of the solutions of (4.10), (4.12), (4.22), (4.23), (4.28), (4.37), (4.41), (4.56), (4.58), (4.64), (4.67) and (4.72) of the conformable space-time fractional fourth-order Pochhammer-Chree equation (1.1). For more convenience the graphical representations of these solutions are shown in the following figures.

From the above Figures, one can see that the obtained solutions possess the Jacobi elliptic solutions, the conformable fractional solitary wave solution, the Jacobi elliptic conformable fractional solution and the solitary wave solutions. Also, these Figures expressing the behaviour of these solutions which give some perspective readers how the behaviour solutions are produced.

Conclusion

We have derived many Jacobi elliptic function solutions, the solitary wave solutions, singular solitary wave solutions and the trigonometric function solutions of the conformable space-time fractional fourth-order Pochhammer-Chree equation (1.1) using the unified sub-equation method combined with the conformable space-time fractional derivatives described in Sec. 3. On comparing our results in this article with that obtained in Refs. 27-30 using different methods, we conclude that the Jacobi elliptic solutions obtained in our article are new, while some solitary wave solutions, singular solitary wave solutions and the trigonometric function solutions obtained in our article are equivalent to that obtained in Refs. 27-30. From these discussions, we conclude that the proposed method of Sec.3, is direct, concise and effective powerful mathematical tools for obtaining the exact solutions of other nonlinear evolution equations. Finally, our results in this article have been checked using the Maple by putting them back into the original equation (1.1).

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Conflict of interest

The author declares that there is no conflict of interest.

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