PATH CONNECTED COMPONENTS OF THE SPACE OF VOLTERRA-TYPE INTEGRAL OPERATORS

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Abstract. We study the topological structure of the space of Volterra-type integral operators on Fock spaces endowed with the operator norm. We proved that the space has the same connected and path connected components which is the set of all compact operators acting on the Fock spaces. We also obtained a characterization of isolated points of the space of the operators and showed that there exists no essentially isolated Volterra-type integral operator.

1. INTRODUCTION

The theory of Volterra-type integral operators have been a subject of high interest during the past three decades. The operators have been studied quite extensively on various functional spaces over several domains. Most of the studies made aimed at describing their operator-theoretic properties in terms of the function-theoretic properties of the symbols inducing them. See for example [1, 2, 3, 4, 7, 6, 11, 12, 13] and the related references therein. In contrast, there have not been much effort to understand the topological structure of the space of the operators. Recently, the compact difference structure of the operators have been described in [9]. In this note, we continue that line of research and describe the path connected components and the connected components of the space of the operators acting on Fock spaces.

We recall that if $g$ is a holomorphic function over a given domain, the Volterra-type integral operator $V_g$ is defined by

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$ 

Let $\mathbb{C}$ be the complex plane and $0 < p \leq \infty$. Then the classical Fock spaces $\mathcal{F}_p$ consist of entire functions $f$ for which

$$\|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{2}{p}|z|^2}dA(z) < \infty, \quad 0 < p < \infty \quad \text{and}$$

$$\|f\|_\infty = \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{1}{2}|z|^2} < \infty, \quad p = \infty,$$

where $dA$ denotes the Lebesgue area measure. In particular, the space $\mathcal{F}_2$ is a reproducing kernel Hilbert space with kernel and normalized reproducing kernel

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functions given by the explicit formulas

\[ K_w(z) = e^{\overline{w}z} \quad \text{and} \quad k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}. \]

Furthermore, a simple computation shows that the kernel function \( K_w \) belongs to all the Fock space \( \mathcal{F}_p \) with norms

\[ \|K_w\|_p = e^{\frac{1}{2}|w|^2} \quad (1.1) \]

for all \( w \in \mathbb{C} \) and \( 0 < p \leq \infty \).

A crucial ingredient to prove our results is the Littlewood-Paley type estimate for Fock spaces. For \( 0 < p < \infty \), the estimate was proved in [3] and reads

\[ \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z) \quad (1.2) \]

for all \( f \) in \( \mathcal{F}_p \). The analogous result for \( p = \infty \) follows from [8] and becomes

\[ \|f\|_{\infty} \simeq |f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)|}{1 + |z|} e^{-\frac{1}{2}|z|^2}. \quad (1.3) \]

We close this section with a word on notation. We denote by \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \) the space of all bounded Volterra-type integral operators \( V_g : \mathcal{F}_p \to \mathcal{F}_q \) equipped with the operator norm topology unless otherwise specified.

2. The main results

We start this section stating our first main result on path connected components of the space \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \).

**Theorem 2.1.** Let \( 0 < p, q \leq \infty \) and \( V_g : \mathcal{F}_p \to \mathcal{F}_q \) be a compact operator. Then \( V_g \) and \( V_{g(0)} \) belong to the same path connected components of the space \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \).

**Proof.** Since \( V_g \) is compact, by Corollary 2 of [7] and Theorem 1.1 of [8], the symbol \( g \) has affine form \( g(z) = az + b \). If \( a = 0 \), then \( V_g = V_{g(0)} = 0 \) and the assertion of the result holds trivially. Thus, we assume that \( a \neq 0 \) and consider a sequence of scaling functions \( g_t : [0,1] \to \mathbb{C}, \quad g_t(z) = g(tz) \). Then, \( V_{g_t} : \mathcal{F}_p \to \mathcal{F}_q \) is compact for all \( t \) and satisfies

\[ V_g = V_{g_t} \quad \text{and} \quad V_{g(0)} = V_{g_0}. \]

We define an operator \( T : [0,1] \to \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \) by \( T(x) = V_{g_x} \). Then, to prove our results it suffices to show that for every \( x \) in \( [0,1] \)

\[ \lim_{t \to x} \|V_{g_t} - V_{g_x}\|_q = 0. \]

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1The notation \( U(z) \lesssim V(z) \) (or equivalently \( V(z) \gtrsim U(z) \)) means that there is a constant \( C \) such that \( U(z) \leq CV(z) \) holds for all \( z \) in the set of a question. We write \( U(z) \asymp V(z) \) if both \( U(z) \lesssim V(z) \) and \( V(z) \lesssim U(z) \).
For the case when \(0 < p, q < \infty\), applying the Littlewood–Paley type estimate (1.2) and linearity of the integral we have

\[
\|V_{g}\|_{q} \leq \int_{\mathbb{C}} \left| \frac{g_{1}(z) - g_{2}(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z)
\]

\[
= |a| \int_{\mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z)
\]

\[
= |a|^{q} |t - x|^{q} \int_{\mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z). \quad (2.1)
\]

We now consider the case when \(p \leq q < \infty\). For this, we observe that

\[
\int_{\mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z) \leq \int_{\mathbb{C}} |f(z)|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z) \simeq \|f\|_{q}^{q} \|f\|_{p}^{q}, \quad (2.2)
\]

where the last inequality follows from the inclusion \(F_{p} \subseteq F_{q}\) whenever \(p \leq q\).

On the other hand, if \(p \leq q = \infty\), then applying (1.3)

\[
\|V_{g}\|_{\infty} \simeq \sup_{z \in \mathbb{C}} \left| \frac{g_{1}(z) - g_{2}(z)}{1 + |z|} \right| e^{-\frac{\|z\|^{2}}{2}}
\]

\[
= |a| \sup_{z \in \mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right| e^{-\frac{\|z\|^{2}}{2}} \leq |a| |t - x| \|f\|_{\infty} \leq |a| |t - x| \|f\|_{p}. \quad (2.3)
\]

If \(q < p < \infty\), then applying Hölder’s inequality, the integral in (2.1) is further estimated by

\[
\int_{\mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z) \leq \left( \int_{\mathbb{C}} |f(z)|^{p} e^{-\frac{\|z\|^{2}}{2}} dA(z) \right)^{\frac{q}{p}} \times \left( \int_{\mathbb{C}} \frac{dA(z)}{(1 + |z|)^{\frac{p}{q}}} \right)^{\frac{q - p}{q}} \simeq \|f\|_{p}^{q} \left( \int_{\mathbb{C}} \frac{dA(z)}{(1 + |z|)^{\frac{p}{q}}} \right)^{\frac{q - p}{q}}. \quad (2.4)
\]

Since \(V_{g}\) is compact, by Theorem 1.1 of [10], it holds that \(q > 2p/(p + 2)\) and hence the last integral in (2.4) is finite.

It remains to show when \(p = \infty\) and \(q < \infty\). To this end, using (2.1),

\[
\|V_{g}\|_{q} \simeq |a|^{q} |t - x|^{q} \int_{\mathbb{C}} \left| \frac{f(z)}{1 + |z|} \right|^{q} e^{-\frac{\|z\|^{2}}{2}} dA(z)
\]

\[
\leq |a|^{q} |t - x|^{q} \|f\|_{\infty} \int_{\mathbb{C}} \frac{dA(z)}{(1 + |z|)^{q}} \lesssim |a|^{q} |t - x|^{q} \|f\|_{\infty}, \quad (2.5)
\]

where the last estimate is possible since \(V_{g} : F_{\infty} \to F_{q}\) is bounded, by Theorem 1.1 of [8], \(q\) has to be bigger than 2, and hence the last integral above converges.

From the series of estimates in (2.1), (2.2), (2.3), (2.4), and (2.5) we arrive at

\[
\|V_{g} - V_{g}\|_{q} \lesssim |t - x| \|f\|_{p}
\]

for all \(0 < p, q \leq \infty\) from which we deduce that the assertion of the theorem holds.

\(\square\)
An immediate consequence of the above main result is that the set of compact Volterra-type integral operators acting between Fock spaces is path connected. Indeed, if \( V_{g_1} \) and \( V_{g_2} \) are two compact Volterra-type integral operators, then \( g_1(z) = az + b \) and \( g_2(z) = cz + d \). By Theorem 2.1, it also follows that \( V_{g_1} \) and \( V_{g_1(0)} \) belong to the same path connected component. The same holds true for \( V_{g_2} \) and \( V_{g_2(0)} \). But \( V_{g_1(0)} = V_{g_2(0)} \) is the zero operator and hence the assertion. We record this observation as a corollary below for the sake of easier further referencing.

**Corollary 2.2.** Let \( 0 < p, q \leq \infty \). Then the set of all compact operators \( V_g : \mathcal{F}_p \to \mathcal{F}_q \) is a path connected component of the space \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \).

If \( 0 < q < p \leq \infty \), it is shown in [10] and [8] that the operator \( V_g : \mathcal{F}_p \to \mathcal{F}_q \) is bounded if and only if it is compact. An immediate consequence of this fact and Theorem 2.2 above is that the whole space \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \) is path connected when \( 0 < q < p \leq \infty \). It also means that there exists no isolated (singleton component) bounded Volterra-type integral operator in this case. For the case when \( 0 < p \leq q < \infty \), our next main result characterizes the isolated Volterra-type integral operators in \( V_g(\mathcal{F}_p, \mathcal{F}_q) \).

**Theorem 2.3.** Let \( 0 < p \leq q \leq \infty \) and \( V_g : \mathcal{F}_p \to \mathcal{F}_q \) be a bounded operator. Then the following statements are equivalent:

(i) \( V_g \) is isolated in \( \mathcal{V}(\mathcal{F}_p, \mathcal{F}_q) \);

(ii) \( V_g \) is not compact, and hence \( g(z) = az^2 + bz + c \) with \( a \neq 0 \);

**Proof.** The assertion (i) implies (ii) follows easily by an application of Corollary 2.2. On the other hand, if (ii) holds, by Theorem 2 of [7], \( g \) has the form \( g(z) = az^2 + bz + c \) and \( a \neq 0 \). Then to conclude (i), it suffices to show that there exists a positive number \( m \) such that

\[
\|V_g - V_{g_1}\| \geq m
\]

for all complex polynomials of the form \( g_1(z) = dz^2 + ez + f \) with either \( a \neq d \) or \( b \neq e \). In view of this, when \( p \leq q < \infty \), we have

\[
\|V_g - V_{g_1}\|^q \geq \|V_gk_w - V_{g_1k_w}\|^q \geq C \int_{\mathbb{C}} \frac{|g'(z) - g_1'(z)|^q|k_w(z)|^q}{(1 + |z|)^q} e^{-\frac{q}{2}|z|^2} dA(z)
\]

\[
= C \int_{\mathbb{C}} \frac{2z(a - d) + b - e|^b|k_w(z)|^q}{(1 + |z|)^q} e^{-\frac{q}{2}|z|^2} dA(z)
\]

We first assume that \( a \neq d \) and further estimate the last integral from below as

\[
C \int_{\mathbb{C}} \frac{2z(a - d) + b - e|^b|k_w(z)|^q}{(1 + |z|)^q} e^{-\frac{q}{2}|z|^2} dA(z) \geq C_1 \int_{\mathbb{C}} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \geq C_1 \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z),
\]

for all \( w \in \mathbb{C} \) where \( D(w,1) \) is a disc of center \( w \) and radius 1 in the complex plane, and \( C \) and \( C_1 \) are some positive constants. Since \( |k_w|^q \) is subharmonic, it
follows that
\[ C_1 \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \geq C_2 |k_w(w)|^q e^{-\frac{q}{2}|w|^2} = C_2 > 0, \]
where we use (1.1) for the last equality. Then, we may set \( m = C_2 \) to claim the assertion in this case.

If \( a = d \), then by hypothesis, \( b \neq e \). Furthermore, for each \( z \in D(w,1) \), it holds that \( 1 + |z| \simeq 1 + |w| \). Using this and arguing as above we have
\[
C \int_{\mathbb{C}} \frac{|b - e|^q |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z)}{(1 + |z|)^q} \geq \frac{C_3}{(1 + |w|)^q} \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z)
\]
\[
\geq \frac{C_4}{(1 + |w|)^q} |k_w(w)|^q e^{-\frac{q}{2}|w|^2} = \frac{C_4}{(1 + |w|)^q}
\]
for all \( w \in \mathbb{C} \). In particular setting \( w = 0 \) and hence \( C_4 = m \), we arrive at the desired assertion.

For the case when \( p \leq q = \infty \), applying (1.3) and arguing as above we obtain
\[
\|V_g - V_{g_1}\| \geq \|V_g k_w - V_{g_1} k_w\|_{\infty} \geq C \sup_{z \in \mathbb{C}} \frac{|g'(z) - g_1'(z)| |k_w(z)|}{1 + |z|} e^{-\frac{q}{2}|z|^2}
\]
\[
\geq C \frac{|g'(z) - g_1'(z)||k_w(z)|}{1 + |z|} e^{-\frac{q}{2}|z|^2} \geq C \frac{2w(a - d) + b - e}{1 + |w|}
\]
for all \( w \in \mathbb{C} \). In particular, setting \( w = 0 \) and \( m = C|b - e| \) we arrive at our conclusion whenever \( b \neq e \). If \( b = e \), then \( a \neq d \) and letting \( |w| \to \infty \), we observe that \( \|V_g - V_{g_1}\| \geq 2C|a - d| = m \). \( \square \)

Interesting consequence of this result and Corollary 2.2 is that the space \( V_g(F_p,F_q) \) has the same connected and path connected components which is just the set of all compact operators acting between the Fock spaces.

2.1. Essentially connected. A natural question following Theorem 2.3 is whether every isolated Volterra-type integral operator in \( V(F_p,F_q) \) is still isolated under the essential norm topology which is weaker than the topology induced by the operator norm. The main result of this section shows this is not the case. In deed, we will show that there exists no essentially isolated point in the space \( V(F_p,F_q) \) when endowed with the essential norm.

We recall that for Banach spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), the essential norm \( \|T\|_e \) of a bounded linear operator \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) is defined by
\[
\|T\|_e = \inf_K \{ \|T - K\| ; \ K : \mathcal{H}_1 \to \mathcal{H}_2 \ \text{is a compact operator} \}.
\]
In particular, the operator \( T \) is compact if and only if its essential norm is zero.

Let us first prove the following proposition which is interest of its own, and provides a useful and interesting estimate for the essential norm of the Volterra-type integral operators on Fock spaces.

**Proposition 2.4.** Let \( 1 \leq p \leq q \leq \infty \) and \( V_g : F_p \to F_q \) be a bounded operator. That is \( g(z) = az^2 + bz + c \). Then
\[
\|V_g\|_e \simeq |a|.
\] (2.6)
Proof. If \( q < \infty \), then by a particular case of Theorem 3 in [7], we have

\[
\|V_g\|_e^q \simeq \limsup_{|w| \to \infty} \int_{D(w,1)} |g'(z)|^q |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z). \tag{2.7}
\]

We proceed to estimate the integral in (2.7) from both above and below to arrive at (2.6). First observe that

\[
\int_{D(w,1)} |g'(z)|^q |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z) \geq \int_{D(w,1)} \frac{|g'(z)|^q |k_w(z)|^q}{(1 + |z|)^q} e^{-\frac{q}{2} |z|^2} dA(z) \geq |a|^q \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z)
\]

\[
\int_{D(w,1)} |g'(z)|^q |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z) \geq \frac{1}{(1 + |w|)^q} \int_{D(w,1)} |g'(z)|^q |k_w(z)|^q e^{-\frac{q}{2} |w|^2} = \frac{|g'(w)|^q}{(1 + |w|)^q} = \frac{|2aw + b|^q}{(1 + |w|)^q}.
\]

It follows from this and (2.7) that

\[
\|V_g\|_e^q \simeq \limsup_{|w| \to \infty} \frac{2aw + b|^q}{(1 + |w|)^q} \simeq |a|^q
\]

and one side of the inequality in (2.6) follows. To prove the other side, using the inequality \(|a + b|^q \leq 2^q(|a|^q + |b|^q)\) and (1.1), we estimate the integral in (2.7) also have

\[
\int_{D(w,1)} |g'(z)|^q |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z) \leq |a|^q \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2} |z|^2} dA(z)
\]

\[
+ |b|^q \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q}{2} |w|^2} dA(z) \simeq |a|^q + \frac{|b|^q}{(1 + |w|)^q}
\]

from which we deduce the other side inequality

\[
\|V_g\|_e^q \leq \limsup_{|w| \to \infty} |a|^q + \limsup_{|w| \to \infty} \frac{|b|^q}{(1 + |w|)^q} = |a|^q. \tag{2.9}
\]

On the other hand, if \( q = \infty \), using the fact that \( k_w \) is a weakly null sequence and eventually (1.1), we estimate the essential norm from below by

\[
\|V_g\|_e \geq \limsup_{|w| \to \infty} \|V_g k_w\|_\infty \simeq \limsup_{|w| \to \infty} \sup_{z \in D(w,1)} \frac{|g'(z)| |k_w(z)| e^{-\frac{1}{2} |z|^2}}{1 + |z|} \geq \limsup_{|w| \to \infty} \frac{|g'(w)| |k_w(w)| e^{-\frac{1}{2} |w|^2}}{1 + |w|} = \limsup_{|w| \to \infty} \frac{|aw + b|}{1 + |w|} \simeq |a|.
\]

To prove the upper estimates in (2.6), we consider maps \( \psi_k : \mathbb{C} \to \mathbb{C}, \ \psi_k(z) = \frac{k}{k+1} z \) for each \( k \in \mathbb{N} \). Then \( C_{\psi_k} : \mathcal{F}_p \to \mathcal{F}_\infty \) constitutes a sequence of compact composition operators for all \( p \geq 1 \). On the other hand, if \( V_g \) is bounded, then
\[ V_g(C_{\psi_k}) : F_p \to F_\infty \] also constitutes a sequence of compact operators. Using this,
\[
\|V_g\|_e \leq \|V_g - V_g(C_{\psi_k})\| = \sup_{\|f\|_p \leq 1} \| (V_g - V_g(C_{\psi_k})) f \|_\infty
\]
\[
\simeq \sup_{\|f\|_p \leq 1} \sup_{z \in \mathbb{C}} \frac{|g'(z)| |f(z) - f(\psi_k(z))|}{1 + |z|} e^{-\frac{1}{2}|z|^2}.
\]
The supremum above is comparable to the quantity
\[
\sup_{\|f\|_p \leq 1} \sup_{|z| > r} \frac{|g'(z)|}{1 + |z|} |f(z) - f(\psi_k(z))| e^{-\frac{1}{2}|z|^2}
\]
\[
+ \sup_{\|f\|_p \leq 1} \sup_{|z| \leq r} \frac{|g'(z)|}{1 + |z|} |f(z) - f(\psi_k(z))| e^{-\frac{1}{2}|z|^2}
\]
for a certain fixed positive number \( r \). Then, the first summand above is bounded by
\[
\sup_{|z| > r} \left( \frac{|g'(z)|}{1 + \psi'(z)} \right) \sup_{\|f\|_p \leq 1} \sup_{|z| > r} \left( |f(z) - f(\psi_k(z))| e^{-\frac{1}{2}|z|^2} \right)
\]
\[
\leq \sup_{|z| > r} \left( \frac{|g'(z)|}{1 + |z|} \right) \sup_{\|f\|_\infty} \|f\|_\infty \leq \sup_{|z| > r} \left( \frac{|g'(z)|}{1 + |z|} \right) \sup_{\|f\|_p} \|f\|_p
\]
\[
\leq \sup_{|z| > r} \frac{|az + b|}{1 + |z|}. \quad (2.11)
\]
As for the second summand in (2.10), we observe that by integrating the function \( f' \) along the radial segment \( \left[ \frac{k^2}{k+1} z, z \right] \) we find
\[
|f(z) - f\left( \frac{k}{k+1} z \right)| \leq \frac{|z||f'(z^*)|}{k+1}
\]
for some \( z^* \) in the radial segment \( \left[ \frac{k^2}{k+1} z, z \right] \). By Cauchy estimate’s for \( f' \),
\[
|f'(z^*)| \leq \frac{1}{r} \max_{|z| = 2r} |f(z)|,
\]
and hence
\[
|f(z) - f\left( \frac{k}{k+1} z \right)| \leq \frac{|z|}{r(k+1)} \max_{|z| = 2r} |f(z)|.\]
The above estimates ensure that
\[
\frac{|g'(z)|}{1 + |z|} |f(z) - f\left( \frac{k}{k+1} z \right)| e^{-\frac{1}{2}|z|^2} \leq \frac{|z|}{r(k+1)} \max_{z \in \mathbb{C}} \left( \frac{|az + b|}{1 + |z|} e^{-\frac{1}{2}|z|^2} \right) \max_{|z| = 2r} |f(z)|
\]
\[
\simeq \frac{|z|}{r(k+1)} \max_{|z| = 2r} |f(z)|.
\]
Using the fact that \( |f(z)| e^{-|z|^2/2} \leq \|f\|_\infty \) we further estimate
\[
\max_{|z| = 2r} |f(z)| \leq \max_{|z| = 2r} e^{\frac{1}{2}|z|^2} \|f\|_\infty \leq e^{2r^2} \|f\|_p.
\]
Now combining all the above estimates, we find that the second piece of the sum in (2.10) is bounded by
\[
\sup_{\|f\|_p \leq 1} \sup_{|z| \leq r} \left| f(z) - f(\psi_k(z)) \right| e^{-\frac{\nu}{2}|z|^2} \lesssim \frac{e^{2\nu^2}}{k+1} \to 0 \text{ as } k \to \infty,
\]
from which, (2.11), and since \(r\) is arbitrary, we deduce
\[
\|V_g\|_e \lesssim \sup_{|z|>r} \frac{|g'(z)|}{1+|z|} = \limsup_{|z| \to \infty} \frac{|az+b|}{1+|z|} \simeq |a|.
\]
\[\square\]

We may now state the main result of this section on essentially isolated points.

**Theorem 2.5.** Let \(1 \leq p \leq q \leq \infty\). Then there exists no essentially isolated Volterra-type integral operator in the space \(V(F_p,F_q)\).

**Proof.** By Corollary 2.2, all compact Volterra-type integral operators are connected. This together with the fact that the essential norm topology is weaker than the operator norm topology, no compact operator is essentially isolated. Thus, we consider an operator \(V_g \in V(F_p,F_q)\), and assume that it is isolated in the operator norm topology. Then by Theorem 2.3, \(V_g\) is not compact and hence \(g(z) = az^2 + bz + c\) with \(a \neq 0\). We plan to show that \(V_g\) is not essentially isolated. That is every open ball with positive radius and center \(V_g\) is not contained in the singleton set \(\{V_g\}\). If \(V_{g_1}\) with \(g_1(z) = a_1z^2 + b_1z + c_1\) belongs to \(V(F_p,F_q)\), then by applying linearity of the integral and Proposition 2.4,
\[
\|V_{g_1} - V_g\|_e = \|V_{g-g_1}\|_e \simeq |a-a_1|
\]
which gives an estimate for the essential norm of the difference of two Volterra-type integral operators. Now choosing \(g_1(z) = a_1z^2 + b_1z + c_1\) in such a way that \(a = a_1\) and \(V_{g_1} \neq V_g\), we observe that \(g_1\) belongs to every ball of center \(V_g\) and positive radius. But \(V_{g_1}\) does not belong to \(\{V_g\}\) and completes the proof. \(\square\)

For each \(a \in \mathbb{C}\), we set
\[
V_a = \left\{ V_{ga} \in V(F_p,F_q) : g_a(z) = az^2 + bz + c, \; b, c \in \mathbb{C} \right\}.
\]
Then if \(a = 0, V_0\) corresponds to the set of compact operators in \(V(F_p,F_q)\) which is path connected by Corollary 2.2. If \(a \neq 0\), then \(V_{ga}\) is not essentially isolated by Theorem 2.5. As a consequence of all these, we get the following.

**Corollary 2.6.** Let \(1 \leq p \leq q \leq \infty\). Then \(V(F_p,F_q)\) has the following essentially path connected components:
\[
V(F_p,F_q) = \bigcup_{a \in \mathbb{C}} V_a.
\]
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