Modified Affine Hecke Algebras and Drinfeldians of Type A

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Abstract

We introduce a modified affine Hecke algebra by a singular transformation of the usual affine Hecke algebra $\hat{H}_q(l)$ of type $A_{l-1}$. The modified affine Hecke algebra $\hat{H}_{q\eta}(l)$ ($\hat{H}_{q\eta}^+(l)$) depends on two deformation parameters $q$ and $\eta$. When the parameter $\eta$ is equal to zero the algebra $\hat{H}_{q\eta=0}(l)$ coincides with $\hat{H}_q(l)$, if the parameter $q$ goes to 1 the algebra $\hat{H}_{q\eta=1}(l)$ is isomorphic to the degenerate affine Hecke algebra $\Lambda_\eta(l)$ introduced by Drinfeld. We construct a functor $F_{q\eta}$ from a category of representations of $\hat{H}_{q\eta}^+(l)$ into a category of representations of Drinfeldian $D_{q\eta}(sl(n+1))$ which has been introduced by the first author. This functor depends on two continuous deformation parameters $q$ and $\eta$. If the parameter $\eta$ is equal to zero then the functor $F_{q\eta=0}$ coincides with the duality functor constructed by Chari and Pressley for the affine Hecke algebra $\hat{H}_q^+(l)$ and the quantum affine algebra $U_q(sl(n+1)[u])$. When the parameter $q$ goes to 1 the functor $F_{q=1\eta}$ coincides with Drinfeld’s functor for the degenerate affine Hecke algebra $\Lambda_q(l)$ and the Yangian $Y_q(sl(n+1))$.

1 Introduction

One of the most remarkable results of the classical representation theory is the Frobenius-Schur duality between the finite-dimensional irreducible representations of the general or special linear groups and symmetric groups. The duality means that any finite-dimensional irreducible representation of the Lie algebra $g$ (or its universal enveloping algebra $U(g)$, where $g = gl(n+1)$ or $sl(n+1) \simeq A_n$, can be obtained by decomposing of the $l$-fold tensor product of the fundamental...

\textsuperscript{1}The talk given by V.N. Tolstoy
(natural) representation \( V = \mathbb{C}^{n+1} \) with respect to the action of the symmetric group \( S(l) \) (or its group algebra \( \mathbb{C}[S(l)] \)).

After discovery of the quantum groups [1, 4, 10], Jimbo [7] proved the q-analogue of the Frobenius-Schur duality replacing \( U(g) \) by \( U_q(g) \) and \( \mathbb{C}[S(l)] \) by its q-analogue \( H_q(l) \), the Hecke algebra of type \( A_{l-1} \). Slightly earlier in 1985, Drinfeld [3] discovered an analogue of the Frobenius-Schur theory for the Yangian \( Y_q(sl(n+1)) \) and the degenerate affine Hecke algebra \( \Lambda_q(l) \). Later, Chari and Pressley [2] proved the q-analogue of the duality for the quantum affine algebra \( U_q(sl(n+1)) \) and the affine Hecke algebra \( H_q(l) \).

In this paper, we extend the results of Drinfeld and Chari-Pressley to the case of the Drinfeldian \( D_q(sl(n+1)) \) [12] - [14] which is the rational-trigonometric deformation of the universal enveloping algebra of the loop algebra \( sl(n+1)[u] \). In this case, the role of \( H_q(l) \) is played by the modified affine Hecke algebra \( H_{qη}^+(l) \) which we obtain by a singular transformation of the affine Hecke \( H_q(l) \). Our functor \( \mathcal{F}_q \) from a category of representations of \( H_{qη}^+(l) \) in a category of those of the Drinfeldian \( D_q(sl(n+1)) \) depends on two continuous deformation parameters \( q \) and \( η \).

If the parameter \( η \) is equal to zero then the functor \( \mathcal{F}_{q=0} \) coincides with the duality functor constructed by Chari and Pressley [2] for the affine Hecke algebra \( H_q^+(l) \) and the quantum affine algebra \( U_q(sl(n+1)[u]) \). When the parameter \( q \) goes to 1 the functor \( \mathcal{F}_{q=1} \) coincides with Drinfeld’s functor for the degenerate affine Hecke algebra \( \Lambda_q(l) \) and the Yangian \( Y_q(sl(n+1)) \) [3].

2 Affine Hecke and modified affine Hecke algebras

We start from the definition of the affine Hecke algebra [1, 3, 10].

**Definition 2.1** The affine Hecke algebra \( \hat{H}_q(l) := \hat{H}_q(A_{l-1}) \) of type \( A_{l-1} \) is an associative algebra over \( \mathbb{C}[q,q^{-1}] \), generated by the elements \( \sigma_1^\pm1, σ_2^\pm1, . . . , σ_{l-1}^\pm1, z_1^\pm1, z_2^\pm1, . . . , z_l^\pm1 \) with the following defining relations:

\[
\sigma_i σ_i^{-1} = \sigma_i^{-1} σ_i = 1,  \\
σ_i - σ_i^{-1} = (q - q^{-1}),  \\
σ_i σ_{i+1} σ_i = σ_{i+1} σ_i σ_{i+1},  \\
σ_i σ_j = σ_j σ_i \quad \text{if } |i - j| > 1,  \\
z_j z_j^{-1} = z_j^{-1} z_j = 1,  \\
z_j z_k = z_k z_j,  \\
σ_i z_j = z_j σ_i \quad \text{if } j \neq i \text{ or } i + 1,  \\
σ_i z_i = z_i z_i^{-1}.  
\]

An associative algebra generated by the elements \( σ_i^\pm1, i \in \{1, 2, . . . , l - 1\} \), with the defining relations (2.1)-(2.3) is called the Hecke algebra \( H_q(l) := H_q(A_{l-1}) \).

Sometimes it is useful to use the last relation (2.8) in another forms. Namely applying the relation (2.2) one obtains

\[
σ_i z_i - z_{i+1} σ_i = (q^{-1} - q) z_{i+1}  
\]

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or
\[ z_i \sigma_i - \sigma_i z_{i+1} = (q^{-1} - q) z_{i+1} . \] (2.10)

The permutation relations for the inverse powers of the generators \( z_i \) looks like
\[
\begin{align*}
    z_i^{-1} \sigma_i - \sigma_i z_i^{-1} &= (q^{-1} - q) z_i^{-1}, \\
    \sigma_i z_i^{-1} - z_i^{-1} \sigma_i &= (q^{-1} - q) z_i^{-1}.
\end{align*}
\] (2.11)

Using the relations (2.9)-(2.11) and (2.7) it is easy to see that any polynomial of the elements \( \sigma_i^{\pm 1} \) \((i = 1, \ldots, l - 1)\), and \( z_j^{\pm 1} \) \((j = 1, \ldots, l)\) may be put in order such that all elements \( \sigma_i^{\pm 1} \) are located from the left-hand side (or from the right-hand side) of the elements \( z_j^{\pm 1} \), i.e. any polynomials of \( \sigma_i^{\pm 1} \) and \( z_j^{\pm 1} \) is represented as a sum of the monomials of the type
\[
    z_1^{n_1} z_2^{n_2} \cdots z_i^{n_i} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \quad \text{or} \quad \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} z_1^{n_1} z_2^{n_2} \cdots z_i^{n_i}, \quad n_i \in \mathbb{Z},
\] (2.12)
where among the elements \( \sigma_{i_j} \) can be equal. This result is reformulated as the following proposition.

**Proposition 2.1** There is an isomorphism of the vector spaces \( \tilde{H}_q(l) \) and \( \mathbb{C}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}] \otimes H_q(l) \) (or \( H_q(l) \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}] \)):
\[
    \tilde{H}_q(l) \simeq \mathbb{C}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}] \otimes H_q(l) \quad \text{(or } \hat{H}_q(l) \simeq H_q(l) \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}] \text{)}.
\] (2.13)

The subalgebra \( \hat{H}_q^{\pm}(l) \subset \hat{H}_q(l) \), which is generated by \( H_q(l) \) and the elements \( z_1, z_2, \ldots, z_l \) will be also called the affine Hecke algebra.

The affine Hecke \( \hat{H}_q^{\pm}(l) \) (and also \( H_q(l) \)) does not contain any singular elements at \( q \to 1 \) and
\[
    \lim_{q \to 1} \hat{H}_q(l) \simeq \hat{\Sigma}(l), \quad \text{and} \quad \lim_{q \to 1} \hat{H}_q^{\pm}(l) \simeq \hat{\Sigma}^{\pm}(l),
\] (2.14)
where by \( \hat{\Sigma}(l) \) (\( \hat{\Sigma}^{\pm}(l) \)) we denote the affine symmetric group algebra generated by the group algebra of the symmetric group \( \mathbb{C}[S(l)] \) and the affine elements \( z_1^{\pm}, z_2^{\pm}, \ldots, z_l^{\pm} \) \((z_1, z_2, \ldots, z_l)\) with the defining relation (2.10)-(2.8) for \( q = 1 \).

Now we introduce a modified the affine Hecke algebra by the singular translation of the affine elements \( z_j \):
\[
    u_j = z_j + \frac{\eta}{q - q^{-1}} \quad \text{for } j = 1, 2, \ldots, l .
\] (2.15)

This transformation changes only the last relation (2.8) from the set (2.1)- (2.8), which takes now the form
\[
    \sigma_i u_i = u_{i+1} \sigma_i^{-1} + \eta .
\] (2.16)
A remarkable fact is that while the transformation (2.17) contains terms which are singular, in the classical limit \( q \to 1 \), the permutation relations (2.16) for the newly defined generators \( u_i \) do not. So we have:

**Definition 2.2** The modified affine Hecke algebra \( \hat{H}_q^{\pm}(l) = \hat{H}_q^{\pm}(A_{l-1}) \) of type \( A_{l-1} \) is an associative algebra over \( \mathbb{C}[q, q^{-1}, \eta] \) generated by the elements \( \sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_l^{\pm 1} \), and \( u_1, u_2, \ldots, u_l \).
with the following defining relations:

\[
\begin{align*}
\sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1, & (2.17) \\
\sigma_i - \sigma_i^{-1} &= (q - q^{-1}), & (2.18) \\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & (2.19) \\
\sigma_i\sigma_j &= \sigma_j\sigma_i & \text{if } |i - j| > 1, & (2.20) \\
u_iu_k &= u_ku_j, & (2.21) \\
\sigma_iu_j &= u_j\sigma_i & \text{if } j \neq i \text{ or } i + 1, & (2.22) \\
\sigma_iu_i &= u_{i+1}\sigma_i^{-1} + \eta. & (2.23)
\end{align*}
\]

The "\(\eta\) - analog" of the relations (2.9), (2.10) now looks like

\[
\begin{align*}
\sigma_iu_i - u_{i+1}\sigma_i &= (q^{-1} - q)u_{i+1} + \eta, & (2.24)
\end{align*}
\]

It is obvious that the statement of the Proposition 2.1 remains valid for the modified affine Hecke algebra.

One can extend the algebra \(\hat{H}_{q\eta}^+(l)\) adding generators \(u_j^{-1}\) inverse to the elements \(u_j: u_ju_j^{-1} = u_ju_j^{-1} = 1\). In this way one obtains the total modified affine Hecke algebra \(\hat{H}_{q\eta}(l)\). However in the present paper we need only the subalgebra \(\hat{H}_{q\eta}^+(l) \subset \hat{H}_{q\eta}^+(l)\).

The algebra \(\hat{H}_{q\eta}^+(l)\) is a two-parameter \((q, \eta)\)-deformation of \(\hat{\Sigma}^+(l)\). However it is easy to see that the modified affine Hecke algebra \(\hat{H}_{q\eta}^+(l)\) is essentially independent of the parameter \(\eta\), provided that \(\eta \neq 0\). In fact, if \(\eta \neq 0\) and \(\eta' \neq 0\) the map \(\hat{H}_{q\eta}^+(l) \rightarrow \hat{H}_{q\eta'}^+(l)\) given by \(\sigma_i \mapsto \sigma_i\), \(\eta^{-1}u_j \mapsto \eta'^{-1}u_j\) is clearly an isomorphism of these algebras. Thus one might as well take \(\eta = 1\), however we keep the parameter \(\eta\) for visualization.

It is obvious that \(\hat{H}_{q\eta=0}^+(l) = \hat{H}_q^+(l)\). On the other hand, in the limit \(q \rightarrow 1\) the modified affine Hecke algebra goes into the degenerate affine Hecke algebra \(\Lambda_\eta(l)\) constructed by Drinfeld in 1985 2. The relations between the modified affine Hecke algebra \(\hat{H}_{q\eta}^+(l)\) and the algebras \(\hat{H}_q^+(l), \Lambda_\eta(l), \hat{\Sigma}^+(l)\) (and also their subalgebras) are shown in the picture:

\[
\begin{align*}
H_q(l) &\subset \hat{H}_{q\eta}^+(l) &\eta \rightarrow 0 &\hat{H}_q^+(l) \supset H_q(l) \\
\downarrow q \rightarrow 1 &\downarrow q \rightarrow 1
\end{align*}
\]

Fig.1. A diagram of the limit algebras of the modified affine Hecke algebra \(\hat{H}_{q\eta}^+(l)\) and their subalgebras. The arrows show passages to the limits.

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\(^2\)This algebra was also obtained by Drinfeld from the affine Hecke algebra \(\hat{H}_q^+(l)\) by letting \(q \rightarrow 1\) in a certain non-trivial fashion.
3 Drinfeldian and Yangian of type $A_n$

First we recall the defining relations of the q-quantized universal enveloping algebra $U_q(sl(n+1))$ ($sl(n+1) := sl(n+1, \mathbb{C}) \simeq A_n$) and construction of its Cartan-Weyl basis.

Let $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots of $sl(n+1)$ endowed with the following scalar product: $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$, $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_{i+1} \rangle = -1$, $\langle \alpha_i, \alpha_j \rangle = 0$ ($|i-j| > 1$). The corresponding Dynkin diagram is presented on the picture:

$$\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-1} \quad \alpha_n$$  \hspace{1cm} (3.1)

Fig.3. Dynkin diagram of the Lie algebra $sl(n+1)$.

The quantum algebra $U_q(sl(n+1))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}, e_{\pm \alpha_i}$ ($i = 1, 2, \ldots, n$) with the defining relations:

$$q^{h_{\alpha_i}}q^{-h_{\alpha_i}} = q^{-h_{\alpha_i}}q^{h_{\alpha_i}} = 1,$$

$$q^{h_{\alpha_i}}q^{h_{\alpha_j}} = q^{h_{\alpha_j}}q^{h_{\alpha_i}},$$

$$q^{h_{\alpha_i}}e_{\pm \alpha_j}q^{-h_{\alpha_i}} = q^{\pm (\alpha_i, \alpha_j)}e_{\pm \alpha_j},$$

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} [h_{\alpha_i}]_q$$

$$[e_{\pm \alpha_i}, e_{\pm \alpha_j}] = 0 \quad (|i-j| \geq 2),$$

$$[[e_{\pm \alpha_i}, e_{\pm \alpha_j}]_{q^{\pm \alpha_i}q^{\pm \alpha_j}}]_q = 0 \quad (|i-j| = 1),$$

where $[h]_q := (q^h - q^{-h})/(q - q^{-1})$ is standard notation for the "q-number" and $[\cdot, \cdot]_q$ is the q-commutator:

$$[e_\beta, e_\gamma]_q := e_\beta e_\gamma - q^{(\beta, \gamma)} e_\gamma e_\beta.$$  \hspace{1cm} (3.3)

The Hopf structure on $U_q(sl(n+1))$ is given by the following formulas for a comultiplication $\Delta_q$, an antipode $S_q$, and a co-unit $\varepsilon_q$:

$$\Delta_q(q^{\pm h_{\alpha_i}}) = q^{\pm h_{\alpha_i}} \otimes q^{\pm h_{\alpha_i}},$$

$$\Delta_q(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + q^{-h_{\alpha_i}} \otimes e_{\alpha_i},$$

$$\Delta_q(e_{-\alpha_i}) = e_{-\alpha_i} \otimes q^{h_{\alpha_i}} + 1 \otimes e_{-\alpha_i};$$

$$S_q(q^{\pm h_{\alpha_i}}) = q^{\mp h_{\alpha_i}},$$

$$S_q(e_{\alpha_i}) = -q^{h_{\alpha_i}}e_{\alpha_i},$$

$$S_q(e_{-\alpha_i}) = -e_{-\alpha_i}q^{-h_{\alpha_i}};$$

$$\varepsilon_q(q^{\pm h_{\alpha_i}}) = 1,$$

$$\varepsilon_q(e_{\pm \alpha_i}) = 0.$$  \hspace{1cm} (3.4)

(3.5)
Below we shall also use another basis in the Cartan subalgebra of the Lie algebra $sl(n+1)$. Namely we set
\[
\begin{align*}
e_{11} &= \frac{1}{n+1} (n h_{\alpha_1} + (n-1) h_{\alpha_2} + \cdots + 2h_{\alpha_{n-1}} + h_{\alpha_n} + N) , \\
e_{22} &= \frac{1}{n+1} (n h_{\alpha_1} + (n-1) h_{\alpha_2} + \cdots + 2h_{\alpha_{n-1}} + h_{\alpha_n} + N) - h_{\alpha_1} , \\
\vdots & \quad \vdots \\
e_{ii} &= \frac{1}{n+1} (n h_{\alpha_1} + (n-1) h_{\alpha_2} + \cdots + 2h_{\alpha_{n-1}} + h_{\alpha_n} + N) - \sum_{k=1}^{i-1} h_{\alpha_k} , \\
\vdots & \quad \vdots \\
e_{n+1n+1} &= \frac{1}{n+1} (-h_{\alpha_1} - 2h_{\alpha_2} - \cdots -(n-1)h_{\alpha_{n-1}} - nh_{\alpha_n} + N) .
\end{align*}
\] (3.7)

Here $N$ is a central element of $g$ (and also of $U_q(g)$), which is equal to 0 for the case $g = sl(n+1)$ and $N \neq 0$ for $g = gl(n+1)$. It is easy to see that
\[
\begin{align*}
h_{\alpha_i} &= e_{ii} - e_{i+1i+1} \quad (i = 1, \ldots, n) , \\
N &= e_{11} + e_{22} + \cdots + e_{n+1n+1} .
\end{align*}
\] (3.8)

A dual basis to the elements $e_{ii}$ $(i = 1, 2, \ldots, n+1)$ will be denoted by $\epsilon_i$ $(i = 1, 2, \ldots, n+1)$: $\epsilon_i(e_{jj}) = (\epsilon_i, \epsilon_j) = \delta_{ij}$. In the terms of $\epsilon_i$ the positive root system $\Delta_+$ of $sl(n+1)$ is presented as follows
\[
\Delta_+ = \{ \epsilon_i - \epsilon_j | 1 \leq i < j \leq n + 1 \} ,
\] (3.9)

where $\epsilon_i - \epsilon_{i+1}$ are the simple roots:
\[
\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i = 1, 2, \ldots, n) .
\] (3.10)

The root $\theta := \epsilon_1 - \epsilon_{n+1}$ is maximal one:
\[
\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_n .
\] (3.11)

For the root vectors $e_{\epsilon_i - \epsilon_j}$ $(i \neq j)$ the standard notations are also used
\[
\begin{align*}
e_{ij} &:= e_{\epsilon_i - \epsilon_j} , \\
e_{ji} &:= e_{\epsilon_j - \epsilon_i} \quad (1 \leq i < j \leq n + 1)
\end{align*}
\] (3.12)

In particular, $e_{ii+1}$, $e_{i+1i}$ are the Chevalley elements: $\epsilon_{i\!+\!1i} = e_{\alpha_i}$, $e_{i\!+\!1i} = e_{-\alpha_i}$ $(i = 1, \ldots, n)$.

For construction of the composite root vectors $e_{ij}$ $(j \neq i \pm 1)$ we fix the following normal ordering of the positive root system $\Delta_+$ (see [1], [3])
\[
(\epsilon_1 - \epsilon_2), (\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3), \ldots, (\epsilon_1 - \epsilon_i, \ldots, \epsilon_{i-1} - \epsilon_1), \ldots, (\epsilon_1 - \epsilon_{n+1}, \ldots, \epsilon_n - \epsilon_{n+1}) .
\] (3.13)

According to with this ordering we set
\[
\begin{align*}
e_{ij} &:= [e_{ik}, e_{kj}]_{q^{-1}} , \\
e_{ji} &:= [e_{jk}, e_{ki}]_q \quad (1 \leq i < k < j \leq n + 1)
\end{align*}
\] (3.14)

It should be stressed that the structure of the composite root vectors (3.14) is independent of choice of the index $k$ in the r.h.s. of the definition (3.14). In particular one has
\[
\begin{align*}
e_{ij} &:= [e_{i\!+\!1i}, e_{i\!+\!1j}]_{q^{-1}} = [e_{ij\!-\!1}, e_{j\!-\!1j}]_{q^{-1}} \quad (1 \leq i < j \leq n + 1) , \\
e_{ji} &:= [e_{j\!+\!1i}, e_{i\!+\!1j}]_q = [e_{jj\!-\!1}, e_{j\!-\!1j}]_q \quad (1 \leq i < j \leq n + 1)
\end{align*}
\] (3.15)
General properties of the Cartan-Weyl basis \( \{e_{ij}\} \) can be found in [11, 8, 4].

As it was noted in [12] the Dynkin diagrams of the non-twisted affine algebras can be also used for classification of the Drinfeldians and the Yangians. In the case of \( \mathfrak{sl}(n+1) \), the Dynkin diagram of the corresponding affine Lie algebra \( \widetilde{\mathfrak{sl}}(n+1) \) is presented by the picture:

\[
\begin{align*}
\delta - \theta & \\
& \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-1} \quad \alpha_n
\end{align*}
\]

Fig. 3. Dynkin diagram of the affine Lie algebra \( \widetilde{\mathfrak{sl}}(n+1) \).

A general definition of the Drinfeldian \( D_q(g) \) corresponding to a simple Lie algebra \( g \) is given in [14, 3, 14]. The defining relations for generators of \( D_q(g) \) presented in [12, 13, 14] depend explicitly on the choice of an element \( e_{-\theta} \in U_q(g) \) of the weight \(-\theta\), such that \( q > \lim_{q \to 1} e_{-\theta} \neq 0 \). Here we present specification of that general definition to the case of \( g = \mathfrak{sl}(n+1) \) and set

\[
\tilde{e}_{-\theta} = q^{\frac{1}{2}}, \quad \tilde{e}_{i+1} = e_{n+1} e_{i+1} + e_{i+1} e_{n+1}.
\]

After some calculations we obtain the following result.

**Proposition 3.1** The Drinfeldian \( D_q^+(\mathfrak{sl}(n+1)) \) \((n > 1)\) is generated (as a unital associative algebra over \( \mathbb{C}[\log q, \eta] \)) by the algebra \( U_q(\mathfrak{sl}(n+1)) \) and the elements \( \xi_{\delta - \theta}, q^{\pm h} \) with the relations:

\[
q^{\pm h} \text{ everything} = \text{ everything } q^{\pm h}, \quad (3.18)
\]

\[
q^{\xi_{11}} \xi_{\delta - \theta} = q^{-1} \xi_{\delta - \theta} q^{\xi_{11}}, \quad (3.19)
\]

\[
q^{\xi_{ii}} \xi_{\delta - \theta} = \xi_{\delta - \theta} q^{\xi_{ii}} \quad \text{for } i = 2, 3, \ldots, n, \quad (3.20)
\]

\[
q^{\xi_{n+1,i+1}} \xi_{\delta - \theta} = q^{\xi_{\delta - \theta} q^{\xi_{n+1,i+1}}}, \quad (3.21)
\]

\[
[x_{\delta - \theta}, e_{i+1}] = 0 \quad \text{for } i = 2, 3, \ldots, n - 1, \quad (3.22)
\]

\[
[x_{\delta - \theta}, e_{i+1}] = 0 \quad \text{for } i = 2, 3, \ldots, n - 1, \quad (3.23)
\]

\[
[[\xi_{\delta - \theta}, e_{n+1}], e_{n+1}] = 0, \quad (3.24)
\]

\[
[[\xi_{\delta - \theta}, e_{n+1}], e_{n+1}] = 0, \quad (3.25)
\]

\[
[x_{\delta - \theta}, e_{n+1}] = q^{-2} \xi_{\delta - \theta} (q^{x_{\delta - \theta}} - e_{n+1} e_{n+1} x_{\delta - \theta} e_{n+1}) \quad (3.26)
\]

\[
[[\xi_{\delta - \theta}, e_{n+1}], e_{n+1}] = q^{-2} \xi_{\delta - \theta} (q^{x_{\delta - \theta}} - e_{n+1} e_{n+1} x_{\delta - \theta} e_{n+1}) \quad (3.27)
\]

The Hopf structure of \( D_q^+(\mathfrak{sl}(n+1)) \) is defined by the formulas (3.24)-(3.27) for \( U_q(\mathfrak{sl}(n+1)) \) (i.e. \( \Delta_q(x) = \Delta_q(x), S_q(x) = S_q(x) \) for \( x \in U_q(g) \)) and \( \Delta_q(q^{\pm h}) = q^{\pm h} \otimes q^{\pm h}, S_q(q^{\pm h}) = q^{\mp h} \).
The comultiplication and the antipode of $\xi_{\delta-\theta}$ are given by

$$
\Delta_{q\eta}(\xi_{\delta-\theta}) = \xi_{\delta-\theta} \otimes 1 + q^{e_{\delta+1}+e_{\eta+1}+1} \otimes \xi_{\delta-\theta} + \eta\left(e_{\eta+1}q^{e_{\eta+1}} \otimes e_{\delta+1}\right)
$$

(3.28)

$$
S_{q\eta}(\xi_{\delta-\theta}) = -q^{h_{\delta}-e_{\delta+1}+e_{\eta+1}+1}\xi_{\delta-\theta} + \eta\left[h_{\delta} + e_{\delta+1} + e_{\eta+1} + 1\right]q^{h_{\delta} - e_{\delta+1} + e_{\eta+1} + 1}e_{\eta+1} + \eta \sum_{k=1}^{n} q^{-k}\left(q - q^{-1}\right)^{k-1}
$$

(3.29)

It is not difficult to check that the substitution $\xi_{\delta-\theta} = q^{e_{\delta+1}+e_{\eta+1}+1}e_{\eta+1}$ satisfies the relations (3.18)-(3.29), i.e. there is a simple homomorphism $D_{q\eta}(sl(n+1)) \to U_q(sl(n+1))$. Moreover the both sides of the relations (3.26) and (3.27) are equal to zero independently. Therefore we can construct a “evaluation representation” $\rho_{ev}$ of $D_{q\eta}(sl(n+1))$ in $U_q(sl(n+1)) \otimes \mathbb{C}[u]$ as follows

$$
\rho_{ev}(q^{h_{\delta}}) = 1, \quad \rho_{ev}(\xi_{\delta-\theta}) = uq^{e_{\delta+1}+e_{\eta+1}+1}e_{\eta+1},
$$

$$
\rho_{ev}(q^{h_{\eta}}) = q^{h_{\eta}}, \quad \rho_{ev}(e_{\pm\alpha_i}) = e_{\pm\alpha_i} \quad (1 \leq i \leq n).
$$

(3.30)

We denote by $D_{q\eta}(sl(n+1))$ the Drinfeldian $D_{q\eta}^{\prime}(sl(n+1))$ with the central element $h_{\delta} = 0$. It is obvious that

$$
D_{q=0}(sl(n+1)) \simeq U_q(sl(n+1)[u])
$$

(3.31)

as Hopf algebras. If $q \to 1$ then the limit Hopf algebra $D_{q=1}\eta(sl(n+1))$ (and also $D_{q=1}\eta(sl(n+1))$ is isomorphic to the Yangian $Y_{\eta}(sl(n+1))$ $(Y_{\eta}^{\prime}(sl(n+1))$ with $h_{\delta} \neq 0)$ [12]:

$$
D_{q=1}(sl(n+1)) \simeq Y_{\eta}(sl(n+1)).
$$

(3.32)

By setting $q = 1$ in (3.18)-(3.29), we obtain the defining relations of the Yangian $Y_{\eta}^{\prime}(sl(n+1))$ and its Hopf structure in the Chevalley basis. This result is formulated as the proposition.

**Proposition 3.2** The Yangian $Y_{\eta}^{\prime}(sl(n+1))$ ($n > 1$) is generated (as an unital associative algebra over $\mathbb{C}[\eta]$) by the algebra $U(sl(n+1))$ and the elements $\xi_{\delta-\theta}, h_{\delta}$ with the relations:

$$
[h_{\delta}, \text{everything}] = 0,
$$

(3.33)

$$
[e_{11}, \xi_{\delta-\theta}] = -\xi_{\delta-\theta},
$$

(3.34)

$$
[e_{\eta+1}, \xi_{\delta-\theta}] = \xi_{\delta-\theta},
$$

(3.35)

$$
[e_{ii}, \xi_{\delta-\theta}] = 0 \quad \text{for } i = 2, 3, \ldots, n,
$$

(3.36)

$$
[\xi_{\delta-\theta}, e_{ii+1}] = 0 \quad \text{for } i = 2, 3, \ldots, n - 1,
$$

(3.37)

$$
[e_{ii+1}, \xi_{\delta-\theta}] = 0 \quad \text{for } i = 2, 3, \ldots, n - 1,
$$

(3.38)

$$
[e_{12}[e_{12}, \xi_{\delta-\theta}]] = 0,
$$

(3.39)

$$
[[\xi_{\delta-\theta}, e_{n+1}], e_{n+1}] = 0,
$$

(3.40)

$$
[[e_{12}, \xi_{\delta-\theta}], \xi_{\delta-\theta}] = \eta\left(e_{12}, e_{n+1}\right)\xi_{\delta-\theta} - e_{n+1}\left[e_{12}, \xi_{\delta-\theta}\right],
$$

(3.41)

$$
[[\xi_{\delta-\theta}[\xi_{\delta-\theta}, e_{n+1}]] = \eta\left([e_{n+1}, e_{n+1}]\xi_{\delta-\theta} - e_{n+1}\left[\xi_{\delta-\theta}, e_{n+1}\right]\right).
$$

(3.42)
\textit{The Hopf structure of the Yangian is trivial for } U(sl(n+1)) \oplus \mathbb{C}h_\delta \subset Y_q'(sl(n+1)) \textit{ (i.e. } \Delta_q(x) = x \otimes 1 + 1 \otimes x, S_q(x) = -x \textit{ for } x \in \mathfrak{sl}(n+1) \oplus \mathbb{C}h_\delta \textit{ and it is not trivial for the element } \xi_{\delta-\theta} :\)

\[
\Delta_q(\xi_{\delta-\theta}) = \xi_{\delta-\theta} \otimes 1 + 1 \otimes \xi_{\delta-\theta} + \eta \left( \frac{1}{2} h_\delta \otimes e_{n+1} + \sum_{i=1}^{n+1} e_{n+i} \otimes e_{i1} \right),
\]

(3.43)

\[
S_q(\xi_{\delta-\theta}) = -\xi_{\delta-\theta} + \eta \left( \frac{1}{2} h_\delta e_{n+1} + \sum_{i=1}^{n+1} e_{n+i} e_{i1} \right).
\]

(3.44)

An analog of the diagram (2.23) for the Drinfeldian \( D_q \) is presented by the picture:

\[
\begin{align*}
\xymatrix{ U_q(sl(n+1)) & U_q(sl(n+1)[u]) \ar[ll]_{\eta \rightarrow 0} \\
U(sl(n+1)) & U(sl(n+1)[u]) \ar[ll]_{\eta \rightarrow 0} \\
q \rightarrow 1 & q \rightarrow 1 
}\end{align*}
\]

(3.45)

\textit{Fig.4. A diagram of the limit Hopf algebras of the Drinfeldian } \( D_q \) \textit{ and their subalgebras. The arrows show passages to the limits.}

\section{Duality between } \( D_q \) \textit{ and } \( \hat{H}_q^+(l) \)

Let \( V \) be the natural \((n+1)\)-dimensional representation of the quantum algebra \( U_q(sl(n+1)) \) with basis \( \{ v_1, v_2, \ldots, v_{n+1} \} \) on which the action of \( U_q(sl(n+1)) \) is given by

\[
e_{i-1}v_k = \delta_{ik}v_{k+1},
\]

\[
e_{i+1}v_k = \delta_{ik}v_{k-1},
\]

\[
q^{s}e_{ii}v_k = q^{\delta_{ik}}v_k.
\]

(4.1)

Let \( T : V \otimes V \rightarrow V \otimes V \) be a linear map given by

\[
T(v_r \otimes v_s) = \begin{cases} 
qv_r \otimes v_s & \text{if } r = s, \\
v_s \otimes v_r & \text{if } r \leq s, \\
v_s \otimes v_r + (q - q^{-1})v_r \otimes v_s & \text{if } r \geq s.
\end{cases}
\]

(4.2)

It is not difficult to check that the elements \( \sigma_i \in \text{End}_{\mathbb{C}}(V^{\otimes l}) \) which act as \( T \) on \( i^{\text{th}} \) and \((i+1)^{\text{th}} \) factors of the tensor product, and as the identity on the other factors, for \( i = 1, 2, \ldots, l \) define the representation of the Hecke algebra \( H_q(l) \) on \( V^{\otimes l} \).

We say that a representation of \( D_q \) has a level \( l \) if its restriction to \( U_q(sl(n+1)) \) is sum of representations each of which occurs in \( V^{\otimes l} \). Now we announce the main result.

\textbf{Theorem 4.1} \textit{(i) Let } \( M \) \textit{ be a finite-dimensional right } \( \hat{H}_q^+(l) \)-module and we set } \( W_M = M \otimes_{H_q(l)} V^{\otimes l} \). \textit{Then there exists a homomorphism } \( \pi : D_q(sl(n+1)) \rightarrow \text{End}_{\mathbb{C}}W_M \) \textit{ such that}

\[
\pi(x)(m \otimes v) = m \otimes \Delta_q^{(l)}(x)v \quad \text{for } x \in U_q(sl(n+1)),
\]

(4.3)

\[
\pi(\xi_{\delta-\theta})(m \otimes v) = m \otimes \left( \Delta_q^{(l)}(\xi_{\delta-\theta}|_{\xi_{\delta-\theta} = u_i}) \right)v
\]

(4.4)
for $m \in M$, $v \in V^\otimes l$. For $l \leq n$ the functor $F_{q\eta}(M) : M \to W_M$ is an equivalence between the category of finite-dimensional right $\hat{H}^+_{q\eta}(l)$-modules and the category of finite-dimensional left $D_{q\eta}(sl(n+1))$-modules of level $l$.

(ii) For $\eta = 0$ the functor $F_{q\eta=0}(M)$ is an equivalence between the category of finite-dimensional right $\hat{H}^+_\eta(l)$-modules and the category of finite-dimensional left $U_q(sl(n+1))$-modules of level $l \leq n$.

(iii) For $q \to 0$ the functor $F_{q\eta=1}(M)$ is an equivalence between the category of finite-dimensional right $\Lambda_\eta(l)$-modules and the category of finite-dimensional left $Y_\eta(sl(n+1))$-modules of level $l \leq n$.

Here $\Delta^{(l)}_{q\eta}$ is the $l$-fold coproduct

$$\Delta^{(l)}_{q\eta} : D_{q\eta}(sl(n+1)) \to D_{q\eta}(sl(n+1)) \otimes \cdots \otimes D_{q\eta}(sl(n+1)) \quad (l - \text{fold}).$$

(4.5)

In particular

$$\Delta^{(2)}_{q\eta}(\cdot) = \Delta_{q\eta}(\cdot)$$

(4.6)

The symbol $i_{\xi_{\delta - \theta}} = u_i$ in (4.4) means that the $i$-th component of the affine element $\xi_{\delta - \theta}$ in the $l$-fold coproduct $\Delta^{(l)}_{q\eta}(\xi_{\delta - \theta})$ has to replace by the affine Hecke element $u_i$.

The proof of the part (i) of Theorem 4.1 is analogous to the proof of the duality theorem between the affine Hecke algebra $\hat{H}_{q\eta}(l)$ and the quantum affine algebra $U_q(sl(n+1))$ (see [3]). The parts (ii) and (iii) are proven by direct comparison of $F_{q\eta=0}(M)$ and $F_{q=1}(M)$ with the Chari-Pressley’s and Drinfeld’s functors [2], [3].

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References

[1] Bernshtein I.N. and Zelevinskii A.V., Representation of the group $GL(n,F)$, where $F$ is local Archimedean field, Usp.Math. Nauk, 31 (1976), 5-70.

[2] Chari V. and Pressley A., Quantum Affine Algebras and Affine Hecke Algebras, e-print math.QA/9501003 (1995).

[3] Chari V. and Pressley A., A guide to quantum groups, Cambridge University Press, 1994.

[4] Drinfeld V.G., Hopf algebras and quantum Yang-Baxter equation, Soviet Math. Dokl., 283 (1985), 1060-1064.

[5] Drinfeld V.G., Degenerate affine Hecke algebras and Yangians, Func. Anal. Appl., 20 (1986), 62-64.
[6] Jimbo M., A q-difference analogue of $U(g)$ and Yang-Baxter equation, *Lett. Math. Phys.*, **10** (1985), 63-69.

[7] Jimbo M., A q-analogue of $U_q(gl(n+1))$, Hecke algebras and Yang-Baxter equation, *Lett. Math. Phys.*, **11** (1986), 247-252.

[8] Khoroshkin S.M. and Tolstoy V.N., Universal R-matrix for quantized (super)algebras, *Commun. Math. Phys.*, **141** (1991), 599-617.

[9] Khoroshkin S.M., and Tolstoy V.N., Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan-Weyl realizations for quantum affine algebras, *Preprint MPIM Bonn (Germany), MPI/94-23* (1994), 29p.; *e-print* hep-th/9404037 (1994).

[10] Rogavski J.D., On modules over the Hecke algebras of p-adic group, *Invent. Math.*, **79** (1985), 443-465.

[11] Tolstoy V.N., Extremal projectors for quantized Kac-Moody superalgebras and some of their applications, *Lectures Notes in Physics* **370** (1990), 118-125.

[12] Tolstoy V.N., Connection between Yangians and Quantum Affine Algebras, Proceedings of the X-th Max Born Symposium, (Wroclaw, 1996, eds: J. Lukierski, M.Mozrzymas). PWN - Polish Scientific Publishers - Warszawa (1997), 99-117.

[13] Tolstoy V.N., Two-parameter deformations of loop algebras and superalgebras, Proc. of the 5-th Wigner Symposium, (Vienna, 1997, eds: P.Kasperkovitz, D.Grau). World Scientific, Singapore-New Jersey-London-Hong Kong (1998), 25-27; *e-print* math.QA/9712028 (1997).

[14] Tolstoy V.N., Drinfeldians, Proc. vol. *"Lie Theory and its Application in Physics II"* (eds: H.-D. Doebner, V.K. Dobrev and J. Hilgert, World Scientific, Singapore, 1998, 981-02-3539-9, pp. 325-337; *e-print* math.QA/9803008 (1998).