Equivariant Topological Sigma Models

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Abstract

We identify and examine a generalization of topological sigma models suitable for coupling to topological open strings. The targets are Kähler manifolds with a real structure, i.e. with an involution acting as a complex conjugation, compatible with the Kähler metric. These models satisfy axioms of what might be called “equivariant topological quantum field theory,” generalizing the axioms of topological field theory as given by Atiyah. Observables of the equivariant topological sigma models correspond to cohomological classes in an equivariant cohomology theory of the targets. Their correlation functions can be computed, leading to intersection theory on instanton moduli spaces with a natural real structure. An equivariant $\mathbb{C}P^1 \times \mathbb{C}P^1$ model is discussed in detail, and solved explicitly. Finally, we discuss the equivariant formulation of topological gravity on surfaces of unoriented open and closed string theory, and find a $\mathbb{Z}_2$ anomaly explaining some problems with the formulation of topological open string theory.

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1. Introduction

Witten’s topological quantum field theory \[1,2,3\] has convinced us recently of its intrinsic power. In particular, and quite surprisingly, topological gravity in two dimensions has been shown to be equivalent to the one matrix model \[4,5,6\]. On the other hand, matrix models \[7\] has been invented in string physics as a non-perturbative definition of string theory (working at least in toy dimensions). Due to the intimate relations of the matrix models to the Liouville theory, the equivalence of the topological two-dimensional gravity to its “physical” counterpart is an amazing example of possible tight relations between the conventional and topological versions of a physical theory.

Two dimensional topological gravity has been completely solved (perturbatively in the string coupling constant) by finding a set of recursion relations for the correlation functions \[8\]. Having analyzed pure gravity, it is natural to examine the structure of two-dimensional quantum gravity by looking for natural and still sufficiently manageable generalizations of the simplest case. Both of the above-mentioned approaches to two-dimensional quantum gravity are known to offer such natural generalizations of their own. The pure-gravity critical point of the one matrix model has found its natural generalizations in the multi-critical points of the model and in the multi-matrix models \[9\]. In the language of conventional string theory, these models describe strings in non-zero dimensions. From the two-dimensional viewpoint they correspond to two-dimensional gravity coupled to matter.

The topological approach also offers its own natural class of matter systems that can be coupled to pure topological gravity without spoiling its underlying symmetry, the so-called topological matter systems \[4,6,9\]. One particular class of topological matter with a direct geometrical interpretation are topological sigma models \[10\]; coupling of topological sigma models to topological gravity describes topological strings moving on a topological target manifold \[11,12,13\].

These two directions of generalizing the pure gravity system are closely related to each other. In particular, the multi-matrix models have been argued \[11\] to correspond to topological gravity coupled to a topological matter system. This point has been further elaborated in \[12\], where the matter system has been identified with the topological version of \(N = 2\) minimal models, and the complete set of recursion relations for the correlation functions of the theory has been found.
While the above-mentioned generalizations of the pure gravity system have been thoroughly discussed recently, and much of their structure is clarified by now, there is one interesting and manageable generalization of pure topological gravity that has not received much attention. Namely, having constructed pure gravity on closed orientable surfaces, one particularly natural and simple generalization that, hopefully, could reveal some interesting features of the theory, is pure gravity on surfaces with boundaries and crosscaps, or in other words, theory of topological open strings. Within the topological framework, this point was first addressed by Myers [13], and by Hughes and Montano [14]. The authors of [14] have considered pure topological gravity on surfaces with boundaries, and discussed recursion relations for the theory. In the matrix-model approach, open strings have been analyzed by Kazakov in the Veneziano limit and then by Kostov in the double scaling limit [15]. As for the non-orientability phenomena in matrix models, these have been analyzed thoroughly by Myers and Periwal, Harris and Martinec, and Brézin and Neuberger [16].

The purpose of this paper is to present an “equivariant” approach to topological matter on manifolds with boundaries and crosscaps. The origin of this approach can be traced back to orbifold theories of critical string theory [17], and, in particular, to the fact that open string vacua can be obtained by applying a generalized orbifold procedure to left-right symmetric closed string vacua [18, 19]. The ideas of [18, 19] have been used in open string model building [20], and allowed for the study of target duality in open string compactifications [21]. Moreover, the world-sheet orbifold approach to open strings has also proved useful in relating conformal field theory open strings to Chern-Simons-Witten theory in three dimensions [22].

Inspired by this, we might consider field theories invariant (in the orbifold sense) under a discrete symmetry group, acting possibly both on the space-time and the field manifold. We encounter analogous situations frequently in geometry, where the theories which are required to respect an additional structure of a group action are usually called “equivariant” (equivariant cohomologies, equivariant K-theory [23] etc.). As it turns out that the topological sigma models we aim to investigate represent examples of a more general notion of equivariant topological field theory defined in the Appendix below, it seems natural to call them “equivariant topological sigma models.”

In cohomological field theories [4], of which the equivariant topological sigma model is an example, the functional integral for the correlation functions of physical
observables leads naturally to the intersection theory for (co)homologies of moduli spaces (of instantons, complex curves, or whatever corresponds to relevant classical configurations of the particular Lagrangian). The equivariant approach to open strings has the advantage that it can be easily combined with the cohomological approach to the correlation functions of the theory, leading naturally to an equivariant cohomology theory on the moduli spaces, thus making the theory a relatively straightforward generalization of the non-equivariant case.

In fact, what we will really encounter in equivariant topological sigma models is a very special case of equivariant theory. Our targets are complex (Kähler) manifolds, with the orbifold group $\mathbb{Z}_2$ acting as a complex conjugation on them. Such an antiholomorphic involution does not exist necessarily on any manifold $M$, and defines what is called in algebraic geometry “a real structure” on $M$ (see e.g. [24, 25] for real algebraic geometry). The set of fixed points of the involution is called “the real part” of $M$, and denoted by $\mathbb{R}M$. Analogously, the set of all points of $M$ is denoted by $\mathbb{C}M$. (We will use this terminology and notation throughout this paper.) The couple represented by a complex manifold and an antiholomorphic involution on it then represents the algebro-geometrical definition of a real manifold. Trying to construct equivariant topological sigma models suitable for coupling to open strings, we will thus be directed to the realm of real varieties in the sense just mentioned. This is not too surprising, because surfaces with boundaries and crosscaps that emerge in open string theory are themselves typical examples of such real varieties [26, 27], and open string theory thus seems to be just a real version of (complex) closed string theory.

Heuristically, we can expect a connection between topological open strings and real algebraic geometry on purely mathematical grounds. Indeed, one of the deep results of topological sigma models is the fact that they offer exotic topological invariants of pseudo-holomorphic curves in symplectic manifolds as studied by Gromov and Floer [4, 28, 29, 30]. Mathematicians might then naturally ask: Is there a physical theory that gives invariants for anti-holomorphic involutions of Kähler manifolds? For a Kähler target, recalling the form of the Lagrangians of both the topological and the conventional sigma models, we can easily see that while holomorphic diffeomorphisms preserving the Kähler structure are symmetries of both the Lagrangians, anti-holomorphic involutions of $M$ are not; they do not represent symmetries of the topological sigma model Lagrangian even if they preserve the
Kähler structure. Indeed, the gauge-fixed topological Lagrangian is dominated by instantons in the semiclassical limit, and complex conjugations on $M$ map instantons to anti-instantons. To save the day, we have to supplement the target complex conjugation with another operation that also changes instantons to anti-instantons. A natural way to do it is to invert the complex structure on the world-sheet, a procedure that is known to lead immediately to open strings in the framework of world-sheet orbifolds. Hence, we expect invariants associated to complex conjugations of Kähler manifolds along the lines of [2, 4] to be related to topological open string theory. This might represent one more reason why to study equivariant topological sigma models.

This paper is organized as follows. Before studying two-dimensional equivariant sigma models, we discuss general equivariant topological quantum field theory (Section 2), and define it axiomatically in the Appendix. As a particular example of such an equivariant theory, we examine $\mathbb{Z}_2$-equivariant topological matter in two dimensions. These models can be considered as possible backgrounds for topological open strings. Most of the general structure of the models can be inferred from the axioms themselves, irrespective of a concrete realization of the matter system. In particular, physical states that correspond to integrating out a boundary or cross-cap can be defined, representing a very simple, topological sigma model version of the boundary and crosscap states well known in (conformal field theory of) open strings [31].

In Section 3, we specialize our discussion to ($\mathbb{Z}_2$-) equivariant topological sigma models with a Kähler target $M$. Observables are shown to be in correspondence with the cohomological classes of an equivariant cohomology theory on $M$. Functional integral representation of their correlation functions is reduced by standard arguments to integrals over the moduli spaces of (equivariant) instantons of the sigma model, and consequently, to intersection theory in equivariant cohomologies. Some particular examples (a $\mathbb{C}P^1 \times \mathbb{C}P^1$ model, $K3$ surfaces) are studied in detail.

In Section 4 we examine the coupling of equivariant topological sigma models to equivariant topological gravity representing the theory of topological open strings. First of all, we show that the boundary analogue of the puncture operator does not have any descendants, a fact that can be understood either on physical grounds or by using arguments from real algebraic geometry. More importantly, however, we demonstrate that the $\mathbb{Z}_2$ symmetry to be gauged in topological gravity is plagued by
an anomaly, which prevents us from constructing a non-trivial topological gravity on surfaces with boundaries and crosscaps, at least within the equivariant framework adopted throughout the paper.

2. Equivariant Topological Field Theory

A particularly natural way of thinking about topological field theory is the axiomatic approach proposed by Atiyah [32], following the axiomatization of 2D conformal field theory given by Segal [33]. We will closely follow reference [32] in our treatment here (see also [34]).

In the axiomatic approach, one considers the $D$-dimensional topological quantum field theory as a set of rules that associate to any oriented $(D-1)$-dimensional manifold $\Sigma$ a vector space $\mathcal{H}_\Sigma$ (the space of quantum states of the canonical quantization on $\Sigma \times \mathbb{R}$), and to any oriented $D$-dimensional manifold $Y$ with boundary $\partial Y$ a vector $\Psi_Y$ from $\mathcal{H}_{\partial Y}$ (the “vacuum state” corresponding to the functional integral on $Y$) [32], in a way compatible with general physical requirements summarized in a set of axioms. Instead of repeating these axioms in this paper here (see [32]), we present in the Appendix their equivariant generalization, i.e. their extension to the case when there is an action of a fixed (discrete) group $G$, both on space-time manifolds and on their Hamiltonian slices, which must be respected by the data that define the topological quantum field theory. The non-equivariant theory of [32] is a special case of this more general axiomatics for $G = 1$.

Although there seems to be an explicit example of an equivariant topological field theory in three dimensions [22], the main field of application and the power of the axioms is still in two dimensions, as we are now going to see.

2.1 Equivariant Topological Matter in Two Dimensions

Our main concern in this paper are equivariant field theories suitable for coupling to topological open strings, i.e. equivariant topological matter systems in $D = 2$. Open strings are known to be related to closed string theory via the $\mathbb{Z}_2$ orbifold
procedure \[LS, L9\] that acts on the world-sheet of the string as world-sheet parity, this action being possibly supplemented with an action on the target. Hence, we will specialize henceforth to $\mathbb{Z}_2$ equivariant topological theory in two dimensions, and the word “equivariant” will mean “$\mathbb{Z}_2$-equivariant” from now on. The orbifold group $\mathbb{Z}_2$ will act on world-sheets and their Hamiltonian slices by reversing their orientation, and the axioms thus require a slight, obvious modification (which we do not present explicitly here). Modding out the $\mathbb{Z}_2$-surfaces by the orbifold group will result in surfaces with boundaries and crosscaps, referred to as Klein surfaces. We will sometimes use the same symbol $\Sigma$ both for a Klein surface and for its orientable double, and call the real part of the double “the boundary” of $\Sigma$, with the hope of not causing any confusion.

Possible Hamiltonian slices of the equivariant world-sheets are disjoint unions of $\mathbb{Z}_2$-manifolds of two possible topological types, corresponding to doubles of the closed and open strings. Thus, physical states in any equivariant topological matter system are divided into two classes according to their world-sheet topology. We will denote the sector of closed string physical states as $\mathcal{H}$, and pick a basis $\mathcal{O}_1, \ldots, \mathcal{O}_N$ in it. On account of the topological symmetry, each element of $\mathcal{H}$ represents a point-like observable living in the interior of the world-sheet. Analogously, we will pick a basis in the open string Hilbert space $\tilde{\mathcal{H}}$, say $\tilde{\mathcal{O}}_1, \ldots, \tilde{\mathcal{O}}_N$, with the corresponding point-like observables living at the world-sheet boundary. To make our life simple, we will restrict ourselves throughout this paper to the topological matter systems with only bosonic observables in both sectors. The general case could be considered analogously.

Many crucial properties of topological matter can be inferred simply from the axioms themselves (\textit{cf.} the Appendix), irrespective of any possible underlying structure of the theory. In particular, the factorization axiom represents a powerful tool allowing the theory to be solved in terms of a few elementary building blocks. Most of the structure of the non-equivariant topological matter systems is known to be encoded \[4, 6\] in the operator product expansion (OPE henceforth) of its BRST invariant observables:

$$\mathcal{O}_\alpha \cdot \mathcal{O}_\beta = \sum_\gamma c_{\alpha\beta}\gamma \mathcal{O}_\gamma,$$  \hspace{1cm} (2.1)

which is valid independently of the location of the punctures the observables are

\footnote{As a rule, the objects that correspond to the open sector will be marked by $\tilde{\cdot}$, to distinguish them from those of the closed sector.}
inserted in, as a result of the topological BRST symmetry. The OPE algebra is an associative, commutative algebra, which we will assume to have an identity, $O_1$. (We will sometimes write tacitly $1$ for $O_1$.) Having known the OPE algebra, the only ingredient needed to calculate completely all genus zero amplitudes is the metric on the space of observables, given by the two point function on the sphere:

$$\eta_{\alpha\beta} = \langle O_\alpha O_\beta \rangle_0.$$  \hfill (2.2)

Indices in the closed sector are lowered and raised by $\eta_{\alpha\beta}$ and its inverse.

At genus $g$, we can compute the correlation functions in the non-equivariant theory, using factorization:

$$\langle \ldots \rangle_g = \sum_{\alpha\beta} \langle \ldots O_\alpha \rangle_{g-1} \eta^{\alpha\beta} \langle O_\beta \rangle_1,$$  \hfill (2.3)

and carrying out explicitly the functional integral over the torus with the $O_\beta$ insertion. This functional integral is equivalent to a physical operator $W$, located in the puncture:

$$W = \sum_{\alpha\beta} \eta^{\alpha\beta} \langle O_\beta \rangle_1 O_\alpha = \sum_{\alpha\beta\gamma} \eta^{\alpha\beta} c_{\beta\gamma} O_\alpha.$$  \hfill (2.4)

Genus $g$ correlation function are given by

$$\langle \ldots \rangle_g = \langle \ldots W^g \rangle_0,$$  \hfill (2.5)

thus completing the solution of the theory in terms of $c_{\alpha\beta\gamma}$ and $\eta_{\alpha\beta}$.

In equivariant topological matter theory, we can calculate any correlation function in terms of a few building blocks using similar methods as in the non-equivariant case. The first information we need is the action of the orbifold group $\mathbb{Z}_2$ on the space of observables, which splits it to even and odd subspaces. We will denote the matrices representing the $\mathbb{Z}_2$ action on the closed and open sector by $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\tilde{\alpha}\tilde{\beta}}$ respectively.

Two-point functions of open observables on the disc define a metric in the open sector:

$$\bar{\eta}_{\tilde{\alpha}\tilde{\beta}} = \langle \tilde{O}_{\tilde{\alpha}} \tilde{O}_{\tilde{\beta}} \rangle_{\text{disc}}.$$  \hfill (2.6)

Furthermore, we will need mixed two point functions of one closed and one open state:

$$\bar{\eta}_{\alpha\beta} = \langle \tilde{O}_{\tilde{\alpha}} O_\beta \rangle_{\text{disc}}.$$  \hfill (2.7)
Open sector indices will be lowered and raised by $\tilde{\eta}_{\alpha\beta}$.

The operator product expansion of two open string observables located at the same component of the world-sheet boundary is again an open string observable (compare figure (1)). Simple topological considerations show that operator product of one closed and one open state should be equal to a sum over open states only (cf. figures (2), (3)). Consequently, the OPE algebra has the structure of a semi-direct product:

\begin{equation}
\tilde{\mathcal{O}}_\alpha \cdot \tilde{\mathcal{O}}_\beta = \sum_{\tilde{\gamma}} \tilde{c}_{\alpha\beta \gamma} \tilde{\mathcal{O}}_\gamma,
\end{equation}

\begin{equation}
\mathcal{O}_\alpha \cdot \tilde{\mathcal{O}}_\beta = \sum_{\tilde{\gamma}} \tilde{d}_{\alpha\beta \gamma} \tilde{\mathcal{O}}_\gamma
\end{equation}

for some coefficients $\tilde{c}_{\alpha\beta \gamma}, \tilde{d}_{\alpha\beta \gamma}$. These coefficients are related to three point functions on the disc via

\begin{align*}
\tilde{c}_{\alpha\beta \gamma} &= \tilde{\eta}^{\gamma\delta} \left\langle \tilde{\mathcal{O}}_\alpha \tilde{\mathcal{O}}_\beta \tilde{\mathcal{O}}_\delta \right\rangle_{\text{disc}}, \\
\tilde{d}_{\alpha\beta \gamma} &= \tilde{\eta}^{\gamma\delta} \left\langle \mathcal{O}_\alpha \tilde{\mathcal{O}}_\beta \tilde{\mathcal{O}}_\delta \right\rangle_{\text{disc}}.
\end{align*}

Note that the $\tilde{d}_{\alpha\beta \gamma}$ are not independent of the structure constants we have already defined. Physically, the process of one open and one closed string state approaching each other is not elementary, and can be decomposed into two elementary processes, using factorization as in figure (3). On account of this, we arrive at the following identity:

\begin{equation}
\tilde{d}_{\alpha\beta \gamma} = \tilde{\eta}_{\alpha\sigma} \tilde{\eta}^{\sigma\rho} \tilde{c}_{\rho\beta \gamma}.
\end{equation}

The OPE algebra (2.1), (2.8) and (2.9), together with the two point functions, allow one to calculate any $n$-point function at the lowest genus, i.e. on the disc in case we have at least one open observable inside the correlator.

Higher genus correlation functions can be computed using operators that represent integrating out a boundary or a crosscap. These boundary and crosscap operators,

\begin{align*}
B &= \sum_\sigma b^\sigma \mathcal{O}_\sigma, \quad \text{(2.11)} \\
C &= \sum_\sigma c^\sigma \mathcal{O}_\sigma, \quad \text{(2.12)}
\end{align*}

\footnote{We will assume that, similarly as the closed sector, the open sector contains its vacuum state, with the corresponding operator (say $\tilde{\mathcal{O}}^{-1}$) being an identity in the OPE algebra restricted to $\tilde{\mathcal{H}}$.}
are closed string operators obtained by performing the functional integral in the theory on the world-sheet of the topology of a disc or a real projective plane, both with one hole cut out of the surface (see figure (4)). The coefficients in (2.11), (2.12) can be inferred from the explicit expression for the functional integral:

\[ b^\sigma = \sum_\beta \eta^{\sigma \beta} \langle O_\beta \rangle_{\text{disc}} = \sum_\beta \eta^{\sigma \beta} \hat{n}_{\beta \bar{\beta}} \]

\[ c^\sigma = \sum_\beta \eta^{\sigma \beta} \langle O_\beta \rangle_{\text{RP}^2}. \]

These operators represent amusingly simple analogues of the boundary and crosscap states known from critical string theory \[31\]. Making use of these operators, we can compute any correlation function on surface \( \Sigma \) with \( h \) handles, \( b \) boundaries, and \( c \) crosscaps:

\[ \langle \ldots \rangle_{(h,b,c)} = \langle \ldots \rangle \cdot W^h \cdot B^{b-1} \cdot C^c \rangle_{\text{disc}} \],

where we have assumed for simplicity that all of the open string observables are located at the same component of \( \partial \Sigma \).

Let us now notice that one important topological fact constrains significantly the structure of any equivariant topological theory in two dimensions. Indeed, it is well known that one handle and one crosscap on a surface are topologically equivalent to three crosscaps on the same surface, irrespective of the rest of the topology of the surface. We have assumed that the equivariant topological matter systems satisfy the factorization conditions of the amplitudes, a crucial axiom of the theory. As a result of the topological identity mentioned above, any surface with at least one handle and one crosscap can be factorized in many different ways. The consistency conditions of these factorizations are expressed in terms of an identity for the handle and crosscap operators:

\[ C \cdot W = C^3. \]

We will refer to this equation as the “topological identity” below.

We will now show that, as a result of the topological identity, the crosscap operator \( C \) cannot be invertible as an element of the OPE algebra on \( \mathcal{H} \) in any topological matter system with nontrivial \( \Omega \). Indeed, were the crosscap operator invertible, we would obtain by multiplying both sides of the topological identity (2.16) by \( C^{-1} \) that \( C^2 \),

\[ C^2 = \sum_{\alpha \beta \gamma} c^\beta c^\gamma c_{\beta \gamma}^\alpha \mathcal{O}_\alpha. \]
should be equal to $W$, equation (2.4). Making now use of the fact that one-point functions on the Klein bottle can be computed in two different ways that are to be equivalent to each other:

$$\langle O_\alpha \rangle_{\text{Klein bottle}} = \langle O_\alpha C^2 \rangle = \text{Tr}(O_\alpha \Omega),$$  \hspace{1cm} (2.18)

we get

$$\sum_{\beta\gamma} c_{\alpha\beta\gamma} c^\beta c^\gamma = \sum_{\delta\epsilon} c_{\alpha\delta\epsilon} \Omega_{\epsilon}^\delta.$$  \hspace{1cm} (2.19)

Substituting this equation to (2.17), we can see that $C^2 = W$ if and only if $\sum_{\beta} c_{\alpha\beta} = \sum_{\beta\gamma} c_{\alpha\beta\gamma} \Omega_{\gamma}^{\beta}$. Setting $\alpha = 1$, this leads to $\text{Tr}(1) = \text{Tr}(\Omega)$, and thus to $\Omega = 1$. Consequently, the assumption that $C$ is invertible entails triviality of the action of the orbifold group on $\mathcal{H}$.

In a theory with nontrivial $\Omega$, and thus with $C$ non-invertible, the conclusion of the previous paragraph is avoided by obtaining a weaker coordinate expression

$$\sum_{\alpha\gamma\sigma\rho} c_{\alpha\beta\gamma\sigma} c^\alpha c_{\gamma\sigma}^\rho \left( \delta^{\sigma}_{\rho} - \Omega^{\sigma}_{\rho} \right) = 0,$$  \hspace{1cm} (2.20)

which is equivalent to the topological identity, and which can well be valid with a non-trivial $\Omega$.

Using the OPE algebra, any $n$-point function can be reduced to a correlation function with at most one closed string state, and at most one open string state at each boundary component of the world-sheet. To solve the theory completely, it is useful to know the operator that corresponds to integrating out one boundary component with an open string observable, $\tilde{O}_{\tilde{\alpha}}$, inserted at it. Using factorization, we get

$$B_{\tilde{\alpha}} = \sum_{\beta\sigma} \eta^{\sigma\beta} \langle O_{\beta} \tilde{O}_{\tilde{\alpha}} \rangle_{\text{disc}} O_{\sigma}$$

$$= \sum_{\beta\sigma} \eta^{\sigma\beta} \tilde{\eta}_{\beta\tilde{\alpha}} O_{\sigma}.$$  \hspace{1cm} (2.21)

Beside the various boundary and crosscap operators, there is another object already known from the ancient dual models [35], and studied thoroughly in string field theory [36], that can be found to have a simple analogue in topological field
theory: the upsilon operator of open string – closed string transitions. Indeed, it can be easily shown that the operator

$$\Upsilon_{\alpha}^{\beta} = \sum_{\gamma} \tilde{\eta}_{\alpha\gamma} \eta^{\gamma\beta},$$

(2.22)

intertwining from $\tilde{\mathcal{H}}$ to $\mathcal{H}$, does exactly what the open string – closed string transition operator is supposed to do in the equivariant topological matter system. This operator corresponds to the functional integral on the surface of figure (5). On the other hand, operator $\Upsilon_{\alpha}^{\beta}$ representing the inverse process is obtained by raising and lowering the indices of $\Upsilon_{\alpha}^{\beta}$.

The simplicity of $\Upsilon$ reflected in equation (2.22) is one more example of the fascinating simplicity of topological field theory, and allows one to observe some simple facts about the models. For example, using the upsilon operator, equation (2.21) can be written as

$$B_{\alpha}^{\beta} = \Upsilon_{\alpha}^{\beta} \mathcal{O}_{\beta},$$

(2.23)

which is topologically obvious. Moreover, one expects some other simple identities to be valid. To mention at least one example, we have two expressions for the correlation functions on the surface with one hole and one handle,

$$\langle \mathcal{O}_{\alpha} \rangle_{(1,1,0)} = \text{Tr}( \Upsilon \Upsilon \mathcal{O}_{\alpha} ) = \langle \mathcal{O}_{\alpha} \cdot W \cdot B \rangle_{0},$$

(2.24)

where in the trace formula the hole of the surface is generated by opening the world-sheet by $\Upsilon$ and closing it again by $\Upsilon$. All of these particularities can be proved explicitly for equivariant topological sigma models, which we will now examine.
3. Equivariant Topological Sigma Models

The basic multiplet of the topological sigma model with a Kähler target $M$ is the BRST multiplet of target coordinates $X^\mu, \mu = 1, \ldots, 2m$, and their topological ghosts:

$$[Q, X^\mu] = \psi^\mu, \quad \{Q, \psi^\mu\} = 0.$$  \hspace{1cm} (3.1)

There are no gauge symmetries to be fixed in the topological sigma model beside the topological symmetry, hence the absence of secondary ghosts in this BRST algebra. Observables are thus to be constructed in a different way than in topological Yang-Mills theory; they are known to correspond to de Rham cohomology classes of the target $[2]$.

The most general Lagrangian $\mathcal{L}_t$ that respects the deformation symmetry underlying the BRST algebra (3.1), is a topological invariant $[37]$. Upon choosing a gauge fixing condition and introducing the corresponding antighost – auxiliary field BRST multiplet, we can write the Lagrangian in its gauge fixed version $[37]$:

$$\mathcal{L}_0 = \mathcal{L}_t + \{Q, \int_\Sigma \Psi\} \hspace{1cm} (3.2)$$

The conventional choice for the gauge fixing fermion leads, after solving the equations of motion for the auxiliary field, to the topological sigma model Lagrangian of $[2]$:

$$\mathcal{L}_0 = \int_\Sigma \left\{ \partial \bar{X}^I \partial X^I G_{IJ} - \frac{i}{2} \left( \rho^I_z D_z \bar{\psi}^j \partial^\bar{I} \psi^j + \rho^j_z D^\bar{I} \bar{\psi} \partial^I \psi^j \right) G_{IJ} - \frac{1}{4} \psi^I \bar{\psi}^j \rho^I_z \rho^j_z R_{IJ} \right\}, \hspace{1cm} (3.3)$$

where $\rho$ are the antighosts and $R$ is the curvature tensor of the Kähler metric.

Assume now that we are given an antiholomorphic target involution $\Omega$ that preserves the Kähler structure on $M$. With the choice of the gauge fixing fermion as in (3.3), the gauge fixed Lagrangian is invariant under simultaneous conjugation of the target and the world-sheet. Hence, we can try to construct an equivariant topological field theory satisfying the axioms of the previous section, via modding out the topological sigma model by the $\mathbb{Z}_2$ acting simultaneously on the world-sheet and the target.

As any other equivariant topological matter system in two dimensions, the equivariant topological sigma model can be indeed viewed upon as a theory of matter on
surfaces with boundaries and crosscaps. The equivariant origin of the theory leads to natural boundary conditions imposed on $X^\mu$ at $\partial \Sigma$. Insisting on the validity of the BRST algebra has the effect of fixing the boundary conditions on topological ghosts $\psi$ as well. This knowledge is sufficient for identifying observables in the equivariant topological sigma model. Upon choosing coordinates $(X^I, \bar{X}^I)$ on $M$ in which the involution act as a pairwise conjugation:

$$\Omega : X^I \to \bar{X}^I, \quad \psi^I \to \bar{\psi}^I,$$

we have at the boundary of $\Sigma$:

$$X^I|_{\partial \Sigma} = \bar{X}^I|_{\partial \Sigma}, \quad (3.5)$$
$$\psi^I|_{\partial \Sigma} = \bar{\psi}^I|_{\partial \Sigma} \quad (3.6)$$

for the allowed mappings of $\Sigma$ to $M$. Hence, in the equivariant topological sigma model the boundary of the world-sheet is constrained to be mapped into $\mathbb{R}M$ (see figure (6)), and the ghost field $\psi$ is at the boundary restricted to be (co)tangent to $\mathbb{R}M$. At $\partial \Sigma$ we thus have the restricted topological BRST algebra

$$[\bar{Q}, \bar{X}^I] = \bar{\psi}^I, \quad (3.7)$$
$$\{\bar{Q}, \bar{\psi}^I\} = 0 \quad (3.8)$$

The $m$-tuple $\bar{X}^I$ of real coordinates, defined by $\bar{X}^I \equiv X^I|_{\partial \Sigma}$, forms a coordinate system on $\mathbb{R}M$. (We have again used $\bar{}$ to distinguish objects living at the boundary from those living in the interior of the world-sheet.)

Vertex operators of open string states in critical open string theory live on $\partial \Sigma$. Analogously, open string physical observables of the equivariant topological sigma models correspond to BRST invariant operators composed of the fields that have survived at the boundary. Given a differential form $\bar{A}$ on $\mathbb{R}M$,

$$\bar{A} = \sum \bar{A}_{I_1 \ldots I_n}(\bar{X}) \, d\bar{X}^{I_1} \wedge \ldots \wedge d\bar{X}^{I_n}, \quad (3.9)$$

we construct out of $\bar{A}$ a composite operator $\bar{\mathcal{O}}_{\bar{A}}$,

$$\bar{\mathcal{O}}_{\bar{A}} = \sum \bar{A}_{I_1 \ldots I_n}(\bar{X}) \, \bar{\psi}^{I_1} \ldots \bar{\psi}^{I_n}, \quad (3.10)$$

localized in a point at $\partial \Sigma$. The BRST commutator of this observable can be inferred from (3.8), leading to

$$\{\bar{Q}, \bar{\mathcal{O}}_{\bar{A}}\} = \bar{\Omega}_{\bar{d}\bar{A}}, \quad (3.11)$$
where $d$ is the exterior derivative on $R.M$. Hence, $\tilde{O}$ is physical for $\tilde{A}$ a closed differential form on $R.M$. Reading (3.11) from the right to the left, we find exact differential forms on $R.M$ to be BRST trivial. Consequently, non-trivial physical observables in the open sector are given by de Rham cohomological classes on $R.M$.

The whole set of physical observables in the equivariant topological sigma model is thus a direct sum of the cohomologies of $CM$ and $R.M$ over the reals:

$$\mathcal{H} \oplus \tilde{\mathcal{H}} = H^*(CM, R) \oplus H^*(R.M, R).$$

For further use we will pick a basis, $O_1, \ldots, O_N, \tilde{O}_1, \ldots, \tilde{O}_N$, in $\mathcal{H} \oplus \tilde{\mathcal{H}}$.

The list of physical observables is not exhausted by these zero-form observables. Indeed, as usual in cohomological field theories [3], observables $O_\alpha, \tilde{O}_\beta$ give rise to a hierarchy of descent equations for BRST invariant observables, with the corresponding top form serving as a possible new term in the Lagrangian. While any zero-form observable $O_\alpha$ of the closed sector can be associated naturally via the descent equations a one-form and a two-form observable, any zero form observable localized at the boundary of the world-sheet generates just a one-form living at the boundary:

$$dO_\alpha = \{Q, O_{\alpha(1)}\}, \quad \tilde{d}\tilde{O}_\beta = \{\tilde{Q}, \tilde{O}_{\beta(1)}\},$$
$$dO_{\alpha(1)} = \{Q, O_{\alpha(2)}\}, \quad \tilde{d}\tilde{O}_{\beta(1)} = 0,$$ (3.13)

allowing the Lagrangian to be generalized by adding the BRST invariant top-forms to

$$\mathcal{L} = \mathcal{L}_0 + \sum_\alpha a_\alpha \int_\Sigma O_{\alpha(2)} + \sum_\beta \tilde{a}_\beta \int_{\partial\Sigma} \tilde{O}_{\beta(1)}.$$ (3.14)

Here only the invariant observables should contribute. After the functional integral for correlation functions is reduced to calculations on an instanton moduli space, the couple $(Q, \tilde{Q})$ represents a couple of exterior-derivative operators on the moduli space and on its real part respectively.

The space of observables carries a representation of the orbifold group. In the closed sector, the action of the orbifold group is induced naturally from the underlying geometry: Given a $\mathbb{Z}_2$-action on a manifold, this induces a natural $\mathbb{Z}_2$ action on the cohomology ring of this manifold,

$$\Omega^* : H_*(CM, R) \to H_*(CM, R),$$ (3.15)
via the pull-back of the cohomology classes by $\Omega$. This involution (on integer cohomologies) is an important invariant in the theory of real algebraic varieties \[24\]. The $\Omega^*$ acting on the de Rham cohomologies of $CM$ preserves the natural metric given by the intersection form (supposing the complex dimension of $M$ is even), and serves as the action of the orbifold group on the sector of closed observables of the model.

To avoid problems with topologically non-trivial closed string configurations, targets are usually required \[6\] to be simply connected, $\pi_1(CM) = 1$. For analogous reasons we should forbid open string configurations of nontrivial topology in $M$. As a result of the equivariant boundary conditions, both ends of the open string are constrained to sit in the real part of the target. Nontrivial open string topologies would thus typically emerge for $RM$ not being connected. Hence, we will limit ourselves throughout to the real structures with $\pi_0(RM) = 1$.

Under these assumptions, we will argue that all of the open string observables of the equivariant topological sigma model are even under the orbifold group (unlike their analogues in a general equivariant two-dimensional matter theory).

Heuristically, our arguments are as follows. By virtue of $\pi_0(RM) = 1$, any open string configuration is topologically trivial. Hence, classical ground states carry zero energy by Hodge-theoretical arguments \[1, 2\], and are pointlike in the target. The orbifold group action reduces on these point-like configurations to the pure target action. This target action preserves $RM$ pointwise, and the only possible nontriviality of the action of the orbifold group on the open sector reduces to the action on the vacuum, $\tilde{\Omega}|\tilde{0}\rangle = \pm|\tilde{0}\rangle$. This sign can be fixed from the OPE algebra, and all of the open states are even under $\tilde{\Omega}$.

In the case of non-equivariant topological sigma models, the OPE algebra is known to give a deformation of the cohomology ring of $M$. One may thus wonder what the classical structure of $\mathcal{H} \oplus \tilde{\mathcal{H}}$ is, of which the quantum OPE algebra can be expected to be a deformation. Notice first that there is a natural structure of an $\mathcal{H}$-module on $\mathcal{H} \oplus \tilde{\mathcal{H}}$. Indeed, to define a multiple of a cohomology class $\tilde{\omega}$ on $RM$ by a cohomology class $\omega$ on $CM$, we will pull back the cohomology class $\omega$ from $CM$ to $RM$, and make the wedge product with $\tilde{\omega}$. Equipped with this natural structure, the module (3.12) associated with the pair represented by a manifold $M$ and an involution on it, can be proved to be a (non-classical) equivariant cohomology ring
in the sense of \cite{38}. We expect this equivariant cohomology theory to be recovered in the classical limit of the OPE algebra of the equivariant topological sigma model.

Let us now consider the correlation functions, in the theory with $a_\alpha = \bar{a}_\bar{\alpha} = 0$ in the Lagrangian \eqref{3.14}. The functional integral definition of the correlation functions,

$$\langle \ldots \rangle = \int DXX \psi D\chi D\lambda \ldots e^{-\mathcal{L}_0}, \quad (3.16)$$

gets reduced by standard arguments to what might serve as the topological definition of the correlation functions. In more detail, the functional integral is dominated in the semi-classical limit by instantons, and as a result of the BRST invariance, the semi-classical result is exact. After the bosonic and fermionic determinants are cancelled against each other, the functional integral is reduced to a finite-dimensional integral over the space of instantons. Remembering now the equivariant structure of the theory, it is easy to show that relevant to this calculation are equivariant instantons, \textit{i.e.} those holomorphic mappings from the world-sheet to the target that are invariant under the simultaneous action of the orbifold group on the target and the world-sheet.

Given an equivariant instanton $\Sigma \to M$, we get the formal dimension of its component of the moduli space of equivariant instantons using an equivariant (or $KR$-theoretical \cite{23}) version of the index theorem. The dimension is equal to the number of equivariant infinitesimal deformations of the instanton, minus the number of equivariant obstructions for integrating these deformations. Without any calculation, this dimension can be determined by observing that the involution on $\Sigma$ and $M$ induces an involution on the moduli space $\mathcal{S}$ of \textit{all} instantons on $\Sigma$ which acts as a complex conjugation with respect to the natural complex structure of $\mathcal{S}$, and that the moduli space of equivariant instantons is precisely the real part $\mathcal{R}\mathcal{S}$ of $\mathcal{S}$. Then, supposing $\mathcal{R}\mathcal{S}$ is non-empty,

$$\dim_{\mathcal{R}} \mathcal{R}\mathcal{S} = \dim_{\mathcal{C}} \mathcal{C}\mathcal{S}, \quad (3.17)$$

which is a general formula valid for any complex manifold with real structure.

\footnote{Note the important point that this equivariant cohomology theory is \textit{not} the same as the $G$-equivariant cohomology theory based on the classification spaces $BG$ of $G$ (see \textit{e.g.} \cite{34} for the latter). It is interesting to see a non-classical equivariant cohomology arising naturally in a physical context.}
Using these facts, we can find a topological description of the correlation functions, analogously as in the non-equivariant case \[4, 6\]. Let

\[
\Phi : \Sigma \times S \to CM
\]

(3.18)
denote the universal instanton. Recall that \(\Phi\) maps a point \(z\) in \(\Sigma\) and an instanton \(f\) in \(S\) to \(f(z)\). It immediately follows that the real part of the universal instanton is mapped to the real part of \(M\):

\[
\Phi : \partial \Sigma \times \mathbb{R}S \to \mathbb{R}M.
\]

(3.19)

Using this fact it is obvious that, picking a point \(x\) at \(\partial \Sigma\), we can associate with any cohomology class \(\tilde{\alpha}\) on \(\mathbb{R}M\) a cohomology class \(\Phi^*\tilde{\alpha}\) on \(\mathbb{C}S\). This cohomology class can be pulled back to \(\mathbb{R}S\) by the canonical embedding \(\iota : \mathbb{R}S \to \mathbb{C}S\).

With this recipe, the correlation functions of the equivariant topological sigma model reduce to integrations of the cohomology classes on the moduli spaces of (equivariant) instantons:

\[
\langle O_{\alpha_1} \ldots O_{\alpha_n} \cdot \bar{O}_{\beta_1} \ldots \bar{O}_{\beta_s} \rangle = \int_{\mathbb{R}S} t^*\Phi^*\alpha_1 \wedge \ldots \wedge t^*\Phi^*\alpha_n \wedge \Phi^*\bar{\beta}_1 \wedge \ldots \wedge \Phi^*\bar{\beta}_s.
\]

(3.20)

This formula reduces in the classical limit, in which only the homotopically trivial instantons contribute, to

\[
\langle O_{\alpha_1} \ldots O_{\alpha_n} \cdot \bar{O}_{\beta_1} \ldots \bar{O}_{\beta_s} \rangle = \int_{\mathbb{R}M} t^*\alpha_1 \wedge \ldots \wedge t^*\alpha_n \wedge \bar{\beta}_1 \wedge \ldots \wedge \bar{\beta}_s,
\]

(3.21)

which corresponds precisely to the equivariant cohomology theory mentioned above.

The cohomological expression for the correlation functions can be translated to the dual language of intersection numbers of homology classes using Poincaré duality \[39\]. Let us denote by \(M_\alpha\) (resp. by \(\tilde{M}_\beta\)) a representant of the Poincaré dual of cohomology class \(\alpha\) on \(CM\) (resp. \(\beta\) on \(RM\)). The Poincaré duals of the cohomology classes \(t^*\Phi^*\alpha\) and \(\Phi^*\tilde{\beta}\) on the moduli spaces are (homologous to) the set of those instantons that map the point of the world-sheet in which the observable

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of $\tilde{\mathcal{O}}_\beta$ is inserted, to $M_\alpha$ or $\tilde{M}_\beta$ respectively. Denote these Poincaré duals by $L_\alpha$ and $\tilde{L}_\beta$, and note that the Poincaré dual of the pullback $\iota^*\Phi^*\alpha$ is homologous to the intersection of $L_\alpha$ with $\mathcal{R}\mathcal{S}$; we will denote this homology class on $\mathcal{R}\mathcal{S}$ by $\tilde{L}_\alpha$.

Using this machinery, we can write down the dual expression for the correlation functions:

$$\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \cdot \tilde{\mathcal{O}}_{\beta_1} \cdots \tilde{\mathcal{O}}_{\beta_s} \rangle = \#(\tilde{L}_{\alpha_1} \cap \cdots \cap \tilde{L}_{\alpha_n} \cap \tilde{L}_{\beta_1} \cap \cdots \cap \tilde{L}_{\beta_s}), \quad (3.22)$$

where the intersection number is computed in $\mathcal{R}\mathcal{S}$.

For these rather formal expressions to have sense, we must ensure some obvious facts. First, the Poincaré duality only works for oriented manifolds. While the moduli spaces $\mathcal{CS}$ are canonically oriented, their real parts can in principle be non-orientable. Orientability of moduli spaces is important, but in general hard to prove [40]. We will thus simply assume that our moduli spaces are orientable, as will be the case in the example we will consider below.

Second, aiming to mod out the non-equivariant theory by $\Omega$, we should ensure that it is a symmetry of the theory, even at quantum level; the zero-mode measure of the functional integral may be non-invariant under $\Omega$ on some components of $\mathcal{S}$, and violate the $\Omega$ symmetry by a $\mathbb{Z}_2$ anomaly. The requirement of absence of such anomalies will restrict the dimension of the target to even complex-dimensional targets, and an analogous $\mathbb{Z}_2$ anomaly will return again in Section 4 where we consider equivariant topological gravity.

More General Boundary Conditions

Up to now we have studied the simplest type of boundary conditions, namely those induced from a single $\mathbb{Z}_2$ involution on the target. In [29] and [30], Floer and Gromov studied more general classes of boundary conditions on pseudo-holomorphic curves in symplectic manifolds with almost complex structures. While Floer considered curves with different boundary components embedded into different Lagrangian submanifolds $L_i$ ($i = 1, \ldots, n$) in the target, Gromov studied the same situation with the $L_i$'s being totally real submanifolds. It would be desirable to have a quantum-field-theoretical description of these cases. Here we comment briefly on what is required in order to obtain such a description.

As for the class of boundary conditions studied by Gromov [31], they can be eas-
ily incorporated into the equivariant setting as follows. Let us first find, for each \( L_i \), an antiholomorphic involution \( \Omega_i \) which has \( L_i \) as its set of fixed points. This can be done at least locally \([31]\), in the sense that for each \( L_i \) there exists such an involution on a neighborhood of \( L_i \) in the target, provided the complex structure is integrable and \( L_i \) is real analytic in the target \([31]\). (This situation is generic enough, as any complex structure can be deformed near \( L_i \) to allow for such an involution \([30]\). For physically interesting cases, one can use BRST invariance to prove independence of physical results on the deformation, see \([1, 2]\) for an analogous situation.) These \( \Omega_i \)'s generate a discrete group (denoted by \( G \)) of automorphisms of the target. \( G \) can now be used as an orbifold group (or a group of equivariance in the sense of the Appendix) on our theory. Obviously, the resulting orbifold model allows us to study the desired pseudo-holomorphic curves with different boundary components in different Lagrangian submanifolds \( L_i \). However, there is one important technical point that prevents us from computing more. Namely, the orbifold model can be looked upon as an (orbifold) theory on \( M/G_0 \), where \( G_0 \) is the subgroup of holomorphic automorphisms contained in \( G \). Obviously, the fundamental group of \( M/G_0 \) is \( G_0 \) (with \( M \) simply connected, in accord with general assumptions of this paper). This brings us to the as yet mostly unexplored theory of topological sigma models with targets \( N \) of \( \pi_1(N) \neq 1 \). Exactly these technical reasons have led the author to the decision not to consider the more general class of equivariant topological sigma models in this paper (cf. the discussion that follows eq. (3.15)).

As for the class of boundary conditions studied by Floer \([29]\), the situation is essentially similar. Indeed, given a generic Lagrangian submanifold in the target manifold \( M \) and assuming that the symplectic structure tames the almost complex structure on \( M \) \([30]\), it can be easily shown that the Lagrangian submanifold is totally real. In this sense we end up with the situation analyzed in the previous paragraph. Still, it does not seem that the fully general case can be incorporated easily into the equivariant approach, the point being that we might fail to find, for a generic Lagrangian submanifold, an involution defined globally on the whole target. Hence, we encounter orbifolds that are only locally modelled by factors of domains in \( \mathbb{R}^n \) by discrete groups, without being globally factors of a manifold by a discrete group. This would require a generalization of the framework for equivariant theories as we present it in this paper.
3.1 Examples

The simplest example of a non-equivariant topological sigma model discussed in [4] is the topological $\mathbb{C}P^1$ model. Nevertheless, $\mathbb{C}P^1$ is not suitable as a target for the equivariant topological sigma model. Indeed, although the OPE algebra of the $\mathbb{C}P^1$ model is symmetric under any complex conjugation of the target, the metric on the space of observables (given by the intersection matrix) is not. This is indeed an example of the $\mathbb{Z}_2$ anomaly mentioned in the previous section.

To avoid the $\mathbb{Z}_2$ anomaly, we have to look for another target. Since we would like to illustrate the structure of the theory in the open sector, we want the real part of the target to be cohomologically non-trivial. Typical manifolds of nontrivial cohomology are spheres. It is easy to find a complex manifold $M$ with a real structure such that the real part $\mathbb{R}M$ is isomorphic to the $m$-sphere, $S^m$. Indeed, such a manifold can be obtained by writting the equation for $S^m$ as a submanifold in $\mathbb{R}P^{m+1}$, which in homogeneous real coordinates $\zeta_i, i=1,\ldots,m+2$ on $\mathbb{R}P^{m+1}$ reads:

$$\zeta_1^2 + \ldots + \zeta_{m+1}^2 - \zeta_{m+2}^2 = 0,$$

and then taking this equation as defined over $\mathbb{C}$. This defines a complex algebraic variety $N$ in $\mathbb{C}P^{m+1}$. Clearly, the standard real structure $\zeta_i \to \bar{\zeta}_i$ induces on it a real structure of which $S^m$ is the real part.

For $m = 2$, we get a quadric in $\mathbb{C}P^3$. As a result of the uniqueness (up to isomorphism) of the non-degenerate symmetric quadratic form on $\mathbb{C}^4$, any two smooth quadrics in $\mathbb{C}$ are isomorphic to each other [1], and consequently, to $\mathbb{C}P^1 \times \mathbb{C}P^1$. The real structure that is induced on $\mathbb{C}P^1 \times \mathbb{C}P^1$ from the involution we started with, maps the two $\mathbb{C}P^1$ factors to each other, just reversing their complex structure.

The equivariant topological sigma model with this target will be our main example in this paper. Before going to analyze its structure, let us notice that the construction of the target offers at least two hints how to construct targets for equivariant topological sigma models. First, any smooth algebraic variety in $\mathbb{C}P^m$ with real coefficients defines a real algebraic variety which may (or may not) carry a Kähler structure. Second, given any complex manifold $M$ and denoting by $\overline{M}$ the same topological manifold with the reversed complex structure, the product $M \times \overline{M}$ carries a natural real structure that maps the first factor of this product to the second one. Note that the moduli spaces of equivariant instantons are orientable in
this class of models.

The Equivariant $\mathbb{C}P^1 \times \mathbb{C}P^1$ Model

Closed sector observables of the model are in one-to-one correspondence with real cohomology classes of $\mathbb{C}P^1 \times \mathbb{C}P^1$. The cohomology groups of $\mathbb{C}P^1 \times \mathbb{C}P^1$ are spanned by a constant function 1, a volume 2-form $\omega_1$ of the first $\mathbb{C}P^1$ factor, a volume 2-form $\omega_2$ of the second $\mathbb{C}P^1$ factor, and their wedge product $\omega \equiv \omega_1 \wedge \omega_2$. The complex structure of $\mathbb{C}P^1 \times \mathbb{C}P^1$ induces on both of the $\mathbb{C}P^1$ factors a natural orientation. Our conventions are such that $\omega_1$ coincides with the orientation of the first $\mathbb{C}P^1$, while $\omega_2$ is opposite to the natural orientation of the second $\mathbb{C}P^1$. We will use symbols $O_1, O_{\omega_1}, O_{\omega_2}, O_\omega$ to denote the local observables that correspond to these cohomology classes. Moreover, it will be convenient to fix a coordinate system $Z_1, Z_2$ on the target such that the involution $\Omega$ we aim to study takes $Z_1$ to $\bar{Z}_2$ and $Z_2$ to $\bar{Z}_1$.

The intersection form on $\mathbb{C}P^1 \times \mathbb{C}P^1$ defines the metric on the space of observables, with the non-zero elements

$$\eta_{\omega} = \eta_{\omega_1 \omega_2} = 1.$$ (3.24)

To be able to study the equivariant version of this model, we will first determine the genus zero correlation functions,

$$F_{m_1m_2m} \equiv \langle O_{\omega_1} \cdots O_{\omega_1} \cdot O_{\omega_2} \cdots O_{\omega_2} \cdot O_{\omega} \cdots O_{\omega} \rangle_0,$$ (3.25)

of the non-equivariant $\mathbb{C}P^1 \times \mathbb{C}P^1$ model. First, we have two independent ghost numbers in the $\mathbb{C}P^1 \times \mathbb{C}P^1$ model, corresponding to the homotopy classes of mappings from the sphere to the two $\mathbb{C}P^1$ factors of the target. Zero modes of the functional integral must be absorbed by the net ghost number of the observables in the correlator, which gives two conditions on the dimension of the instanton moduli space that can contribute to the correlation function. Instantons of the instanton number $(k, \ell)$ are given by:

$$Z_1 = a_1 \frac{\prod_{i=1}^{k}(z - b_{1,i})}{\prod_{i=1}^{k}(z - c_{1,i})}, \quad Z_2 = a_2 \frac{\prod_{i=1}^{\ell}(z - b_{2,i})}{\prod_{i=1}^{\ell}(z - c_{2,i})}.$$ (3.26)
This component of the moduli space of instantons has (complex) dimension \(2k + 2\ell + 2\), as can be easily verified by direct counting of the independent parameters in (3.26). Each of the instanton numbers can be weighted in the correlation functions by an independent “coupling constant.” We will denote them \(\beta_1\) and \(\beta_2\).

Putting all this together, we get the following correlation functions:

\[
\mathcal{F}_{m_1, m_2 m} = \begin{cases} 
\beta_1^k \beta_2^\ell & \text{for } m_1 + m = 2k + 1 \text{ and } m_2 + m = 2\ell + 1, \\
0 & \text{otherwise.} 
\end{cases} \quad (3.27)
\]

Hence, the two point functions coincide with the intersection numbers of the corresponding homology classes, and the three point functions lead to the following OPE algebra:

\[
\begin{align*}
\mathcal{O} \cdot \mathcal{O}_{\text{anyth.}} &= \mathcal{O}_{\text{anyth.}}, \\
\mathcal{O}_{\omega_1} \cdot \mathcal{O}_{\omega_1} &= \beta_1, \\
\mathcal{O}_{\omega_2} \cdot \mathcal{O}_{\omega_2} &= \beta_2, \\
\mathcal{O}_{\omega_1} \cdot \mathcal{O}_{\omega_2} &= \mathcal{O}_{\omega}, \\
\mathcal{O}_{\omega_1} \cdot \mathcal{O}_\omega &= \beta_1 \mathcal{O}_{\omega_2}, \\
\mathcal{O}_{\omega_2} \cdot \mathcal{O}_\omega &= \beta_2 \mathcal{O}_{\omega_1}, \\
\mathcal{O}_\omega \cdot \mathcal{O}_\omega &= \beta_1 \beta_2. 
\end{align*} \quad (3.28)
\]

Thus, we can see that the OPE algebra of the \(\mathbb{C}P^1 \times \mathbb{C}P^1\) model is a two-parameter deformation of the classical de Rham cohomology ring, to which the OPE reduces in the limit \(\beta_1, \beta_2 \to 0\). In this classical limit, only the instantons homotopic to a constant mapping contribute to the partition functions. Moreover, we can see that the \(\mathbb{C}P^1 \times \mathbb{C}P^1\) theory is in the obvious sense a product of two \(\mathbb{C}P^1\) models.

Upon setting \(\beta_1 = \beta_2 = \beta\), the quantum theory is invariant under the complex conjugation that takes \(Z_1 \to \bar{Z}_2\). The complex conjugation acts on the set of closed observables as

\[
\Omega : \begin{align*}
\mathcal{O}_1 &\rightarrow \mathcal{O}_1, \\
\mathcal{O}_\omega &\rightarrow \mathcal{O}_\omega, \\
\mathcal{O}_{\omega_1} &\rightarrow \mathcal{O}_{\omega_2}, \\
\mathcal{O}_{\omega_2} &\rightarrow \mathcal{O}_{\omega_1}. 
\end{align*} \quad (3.29)
\]

We will now study the equivariant sigma model based on this involution.

Observables in the open sector correspond to cohomology classes of the real part of \(\mathbb{C}P^1 \times \mathbb{C}P^1\), which is topologically \(\mathbb{C}P^1\). These are given by a constant function,
\( \bar{1}, \) and a volume form, \( \bar{\omega}. \) The corresponding observables, \( \bar{O}_1 \) and \( \bar{O}_{\bar{\omega}}, \) are both even under the action of the orbifold group.

Consider a surface \( \Sigma \) with a fixed complex conjugation on it. This conjugation defines a real structure on the moduli space of all instantons on \( \Sigma, \) of which the space of equivariant instantons is the real part. The involution on \( \mathbb{C}S \) takes instantons of instanton number \( (k, \ell) \) to those of instanton number \( (\ell, k) \). Consequently, the real part \( \mathbb{R}S \) belongs entirely to the subspace with instanton number \( (k, k) \) for some \( k. \)

In particular, computations of the metric and OPE algebra will require to know the explicit form of the instantons on the disc:

\[
Z_1 = a \cdot \frac{\prod_{i=1}^{k}(z - b_i)}{\prod_{i=1}^{k}(z - c_i)}, \quad Z_2 = \bar{a} \cdot \frac{\prod_{i=1}^{k}(z - \bar{b}_i)}{\prod_{i=1}^{k}(z - \bar{c}_i)}.
\]

These instantons are in one-to-one correspondence with the instantons on the sphere in the non-equivariant \( \mathbb{C}P^1 \) model \[4\].

The metric on the open sector is given by

\[
\bar{\eta}_{\bar{\omega}} = 1.
\]

(To be precise, we should distinguish here and in \[3.24\] the intersection form on the equivariant de Rham cohomologies of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) from the full metric on the space of observables as given by two point functions. Indeed, despite the fact that the lowest, classical contribution to these two point functions will be given by the intersection form, we will see below that classical contributions to some two point functions can acquire “quantum corrections” from instantons of non-zero instanton number.)

The OPE in the open sector reads

\[
\bar{O}_1 \cdot \bar{O}_{\text{anyth.}} = \bar{O}_{\text{anyth.}},
\]

\[
\bar{O}_{\bar{\omega}} \cdot \bar{O}_{\bar{\omega}} = \bar{\beta} \bar{O}_1.
\]

We have denoted by \( \bar{\beta} \) the “coupling constant” that weights the contribution from the equivariant instantons of instanton number \( (k, k) \) by \( \bar{\beta}^k. \) It is easy to show, making use of the factorization axiom, that \( \bar{\beta} = \beta. \)

The mixed two point functions can be computed analogously, leading to

\[
\hat{\eta}_{1\bar{\omega}} = \hat{\eta}_{\omega\bar{1}} = \hat{\eta}_{\omega\bar{1}} = 1, \\
\hat{\eta}_{\bar{\omega}\bar{\omega}} = \beta.
\]

(3.32)
Perhaps the only surprising result here may be the $\beta$-dependence of the mixed two-point function $\hat{\eta}_\omega \tilde{\omega}$, and we will present here the computation explicitly. It can serve as a typical example of calculations in equivariant topological sigma models, and can thus be illuminating.

To calculate the two-point function

$$\langle \mathcal{O}_\omega \tilde{\mathcal{O}}_{\tilde{\omega}} \rangle_{\text{disc}} \quad (3.33)$$

that defines this element of $\hat{\eta}$, we must pull back both $\omega$ and $\tilde{\omega}$ to the real part of the instanton moduli space via the universal real instanton, and compute the integral of their wedge product over $\mathbb{R}S$. Or, in the dual language, we must count intersections of the homology cycles that are Poincaré dual to these differential forms. Although $\iota^*\omega$ is zero simply because this is a 4-form on a 2-manifold, the pullback of $\omega$ to $\mathbb{R}S$ via $\Phi \circ \iota$ can be non-zero. On dimensional grounds, the only instanton number that can give a non-zero contribution to $\langle 3.33 \rangle$, is $k = 1$. In this component of the moduli space of equivariant instantons, the Poincaré dual of $\iota^*\Phi^*\omega$ is (homologous to) the submanifold consisting of those instantons $\langle 3.30 \rangle$ with $k = 1$ that map the generic point of the world-sheet in which $\mathcal{O}_\omega$ is located, say $z = i$, to the Poincaré dual of $\omega$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$, say $(Z_1, Z_2) = (\infty, i)$. Analogously, the Poincaré dual of $\iota^*\tilde{\omega}$ is (homologous to) the set of instantons that map $z = 1$ to, say, $(Z_1, Z_2) = (0, 0)$. These two cohomology classes intersect in one point, thus giving equation $\langle 3.32 \rangle$.

The information on the two point functions can be conveniently summarized as a “metric” on the space of all observables of the model:

$$H = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \beta & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (3.34)$$

(We have used the quotation symbols here because $H$ can degenerate. Actually, this “metric” degenerates for $\beta = \frac{1}{4}$, which is, amusingly, the critical value of $\beta$ in the non-equivariant version of the model $\langle 4 \rangle$. Indeed, restoring the open string coupling constant $\lambda$ and weighting the contributions to $H$ from surfaces of the Euler characteristic $\chi$ by $\lambda^{-\chi}$, we get

$$\det H = 4\lambda^2 \beta - 1. \quad (3.35)$$

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On the other hand, the “exact” partition function \[4\] for the non-equivariant \(CP^1 \times CP^1\) model, defined as the sum of the partition functions over the Riemann surfaces of all genera properly weighted by the closed string coupling constant \(\lambda_c\), is equal to

\[
\langle 1 \rangle^{\text{exact}} = \sum_g \lambda_c^{g-1} \langle 1 \rangle_g = \frac{1}{\lambda_c} \text{Tr} \left( W^{-1} \cdot \frac{1}{1 - \lambda_c W} \right) = \frac{4}{1 - 16\lambda_c^2/\beta^2}. \tag{3.36}
\]

Comparing (3.35) and (3.36) we easily observe that they are singular for the same value of \(\beta\), assuming the usual relation between the closed string and open string coupling constant dictated by factorization, \(\lambda_c = \lambda^2\), is valid. It would be nice to know precise reasons for this “coincidence” of the singular values of \(\beta\).)

The OPE algebra is completed by calculating the mixed terms:

\[
\begin{align*}
O_1 \cdot \tilde{O}_\text{anyth.} &= \tilde{O}_\text{anyth.}, \\
O_{\omega_1} \cdot \tilde{O}_{\omega} &= \beta \tilde{O}_1, \\
O_{\omega_2} \cdot \tilde{O}_{\omega} &= \beta \tilde{O}_1, \\
O_{\omega_1} \cdot \tilde{O}_1 &= \tilde{O}_\omega, \\
O_{\omega_2} \cdot \tilde{O}_1 &= \tilde{O}_\omega, \\
O_{\omega} \cdot \tilde{O}_{\omega} &= \beta \tilde{O}_1, \\
O_{\omega} \cdot \tilde{O}_1 &= \beta \tilde{O}_\omega, \\
O_{\omega} \cdot \tilde{O}_\text{anyth.} &= \beta \tilde{O}_\omega,
\end{align*}
\tag{3.37}
\]

which can be obtained either from (3.34) using (2.10), or by the direct calculation of the corresponding three point functions.

The boundary state can be computed using \(\langle O_\alpha \rangle_{\text{disc}} = \langle O_\alpha \cdot B \rangle_0\), leading to:

\[
B = O_{\omega_1} + O_{\omega_2}. \tag{3.38}
\]

Quite analogously, the crosscap state can be computed \(\tilde{C}\)

\[
C = O_{\omega_1} + O_{\omega_2}. \tag{3.39}
\]

We can now compute the closed observables that correspond to integrating out a boundary with an operator insertion on it. Because \(\tilde{O}_1\) is invisible under correlator, \(B = B_1\). The state that corresponds to one \(\bar{\omega}\)-insertion at the boundary,

\[
B_{\bar{\omega}} = \beta O_1 + O_{\omega}. \tag{3.40}
\]

\(^2\)Note the equality of the crosscap and boundary states, which is reminiscent of a condition from modular geometry of open strings that requires equality of the massless parts of the crosscap and boundary states of the theory \([31, 22]\), necessary in order to get rid of BRST anomalies.
can be read off from \( \langle O_\alpha \bar{O}_\omega \rangle_{\text{disc}} = \langle O_\alpha \cdot B_\omega \rangle_0 \). Having known \( B \) and \( B_\omega \), the \( \Upsilon \) operator of open string – closed string transitions can be inferred from (2.23), and identities such as (2.24) can be verified by direct calculation.

Identifying now the state that corresponds to integrating out a handle:

\[
W = 4 \mathcal{O}_\omega,
\]

we have identified all the necessary ingredients to analyze the topological identity

\[
C \cdot W = C^3
\]

in the \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) model. The left hand side of equation (2.16) reads

\[
C \cdot W = (\mathcal{O}_{\omega_1} + \mathcal{O}_{\omega_2}) \cdot 4 \mathcal{O}_\omega
= 4\beta (\mathcal{O}_{\omega_1} + \mathcal{O}_{\omega_2}),
\]

while the right hand side is

\[
C^3 = (\mathcal{O}_{\omega_1} + \mathcal{O}_{\omega_2})^3
= (\mathcal{O}_{\omega_1} + \mathcal{O}_{\omega_2}) \cdot 2(\beta \mathcal{O}_1 + \mathcal{O}_\omega)
= 4\beta (\mathcal{O}_{\omega_1} + \mathcal{O}_{\omega_2}).
\]

Hence, we have explicitly proved that the topological identity

\[
C \cdot W = C^3
\]

is valid in the equivariant \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) model. This is desirable, because otherwise the amplitudes would be in clash with factorization.

**Real K3 Surfaces**

Another important class of manifolds suitable as targets for a topological sigma model are the K3 surfaces \([4, 11]\). These are defined as complex surfaces (i.e. they are of real dimension four) with the vanishing first Chern class and the first Betti number. Each K3 surface is a simply connected Kähler manifold, carrying a Ricci flat metric. Hence, K3 surfaces are Calabi-Yau manifolds in the lowest non-trivial dimension. For more details on the geometry of K3 surfaces, see e.g. \([12]\).

We now wonder whether the K3 topological sigma models can be twisted by \( \mathbb{Z}_2 \) to produce an equivariant topological sigma model. In particular, we must find out whether K3 surfaces can carry a real structure. Suppose first that there exists a real
structure on a $K3$ surface $X$. Its real part $RX$ is, if non-empty, of real dimension two. In this paper we are only interested in orientable and connected $RX$ for reasons sketched above. Consequently, we are looking for the real $K3$ surfaces with the real part an orientable, connected Riemann surface.

The problem of existence of real structures on $K3$ surfaces has recently been addressed in the literature [43, 25], leading to interesting results. While any two complex $K3$ surfaces are topologically isomorphic, general results of [25] show that there are exactly 66 distinct topological types that can be realized as the real part $RX$ of a real $K3$ surface. All of them are orientable [43, 25]. Discarding surfaces with $\pi_0(RX) \neq 1$, 12 distinct topologies of $RX$ still remain: the empty set, and surfaces with $g$ handles for $g = 0, \ldots, 10$; see [25].

The space of physical observables of the theory is now generated by the observables that correspond to the cohomology classes of $CX$ and $RX$. Denoting $X_g$ the real $K3$ surface with the real part isomorphic to the Riemann surface of genus $g$, a basis in the cohomology ring is given by a 0-form 1, 22 two-forms $\omega_i$, $i = 1, \ldots, 22$, and a four-form $\omega$ on $CX_g$ in the closed sector, whereas in the open sector we have the basis consisting of a 0-form $\tilde{1}$, $2g$ one-forms $\tilde{\omega}_j$, $j = 1, \ldots, 2g$, and a two-form $\tilde{\omega}$ on $RX_g$. The $2g$ observables corresponding to the one-forms on $RX_g$ behave as fermions. We will thus stop our discussion of the equivariant $K3$ models here, as we have decided to make our life simple, and not to consider fermionic observables in this paper.
4. Coupling to Equivariant Topological Gravity?

Equivariant topological matter systems we have defined and discussed in this paper are natural candidates for coupling to two dimensional topological gravity on surfaces with boundaries and crosscaps. In turn, it is natural to expect that such a gravity theory would fit into the equivariant framework we have advocated here, and could be conveniently formulated as an “equivariant topological gravity.” We will finish this paper by an attempt to construct such an equivariant topological gravity. However, our discussion will be rather tentative, as we have no definite answer to some crucial questions of the equivariant theory.

Topological gravity can be described in many different ways. We will use the formulation given by Verlinde and Verlinde in [5]. The basic multiplet of the topological BRST symmetry is given by the spin connection and its ghost partners:

\[
\{Q, \omega\} = \psi_0, \quad \{Q, \psi_0\} = d\gamma_0, \quad \{Q, \gamma_0\} = 0.
\] (4.1)

Bearing in mind the geometrical origin of \(\omega\) and using the assumption of BRST invariance, we fix the boundary conditions for this multiplet in the equivariant theory. Under the orbifold group action, \(\omega, \psi_0\) and \(\gamma_0\) are odd. Consequently, \(\gamma_0\) is zero at the boundary. This action of \(\Omega\) on the basic BRST multiplet can be uniquely extended to the whole field content of the theory, compatibly with its symmetries.

Observables in the closed sector of the theory are given by the descendants \(\sigma_n\) of the puncture operator \(P\):

\[
P = c\bar{c} \cdot \delta(\gamma)\delta(\bar{\gamma}), \quad \sigma_n = \gamma_0^n \cdot P.
\] (4.2) (4.3)

The descendants, \(\sigma_n\), are are even or odd under \(\Omega\), depending on \(n\) being even or odd. In the open sector, we can define the boundary puncture operator

\[
\tilde{P} = c \cdot \delta(\gamma),
\] (4.4)

but by virtue of the boundary conditions satisfied by \(\gamma_0\), all composites of the form \(\gamma_0^n\tilde{P}\) for \(n \neq 0\) are zero! It thus seems that the boundary puncture operator has no descendants.
There is a natural mathematical support to this conjecture. Analogously as in the sigma model case, the moduli spaces of equivariant surfaces can be viewed upon as real parts of the moduli spaces of Riemann surfaces [44]. The Chern classes of the line bundles $\mathcal{L}_{(i)}$, which Witten has used in the topological expression for the correlation functions, can be pulled back to the moduli spaces of equivariant surfaces. It is thus natural to expect that

$$\langle \sigma_1 \ldots \sigma_n \cdot \bar{P} \ldots \bar{P} \rangle = \int_{R\mathcal{M}} \iota^* c_1(\mathcal{L}_{(1)})^{n_1} \wedge \ldots \wedge \iota^* c_1(\mathcal{L}_{(s)})^{n_s} \quad (4.5)$$

might represent the (formal) topological expression for the correlation functions on surfaces with boundaries and crosscaps, with possible insertions of the boundary puncture operator at $\partial \Sigma$. (Indeed, the integral is taken over the appropriate moduli space of equivariant surfaces with punctures.) In this topological framework, boundary puncture descendants can be expected to enter the topological expression for correlation functions by inserting characteristic classes of a real line bundle on $R\mathcal{M}$ on the right hand side of (4.5). It is however a well known fact that non-trivial characteristic classes of real line bundles take values in $\mathbb{Z}_2$-cohomologies [45], and their real versions are zero. This seems to offer some further support to the conjecture that there are no descendants of the boundary puncture operator in equivariant topological gravity. Note that problems with observables living on the world-sheet boundary have also been observed by Myers in [13].

At the quantum level, the situation with equivariant topological gravity is even worse than it might have appeared from our classical considerations. Indeed, the correlation functions of $\omega_n$ do not respect the $\mathbb{Z}_2$ symmetry that we would like to use in the orbifold procedure. As a consequence of this $\mathbb{Z}_2$ anomaly, we cannot insist on the decoupling of $\Omega$-odd descendants from the correlation functions, and the standard mechanism of constructing an equivariant theory out of a non-equivariant one cannot be used.

One possibility of curing this problem may be connected with the doubling phenomenon noticed in matrix models [3, 46]. It is well known by now that topological gravity and Hermitian matrix models of even potential are not naively equivalent; rather, the partition function of topological gravity corresponds to the square root

\[1\] For the cognoscenti: the cohomology of the classifying space $B\mathbb{Z}_2 \equiv RP^\infty$ of real line bundles is given by $H^0(B\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}$, $H^i(B\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$ for $i \geq 1$, and zero otherwise. After tensoring with the reals, the only survivor is the generator of the zeroth cohomology, in which we recognize the boundary puncture operator $\bar{P}$. 


of the partition function of the matrix model. This doubling of degrees of freedom might help in constructing an equivariant version of topological gravity. Alternatively, we might hope to extract some information from the conjectured topological definition of the correlation functions on surfaces with boundaries and crosscaps, equation (4.5). Clearly, the structure of topological open string theory in the gravitational sector remains still unclear and deserves further investigation.

Note Added (September 1993)

This paper was originally published as Prague Institute of Physics preprint PRAHEP-90/18 (December 1990); the present version contains just minor changes and misprint corrections as compared to the original.

Since 1990, many exciting new ideas and results have indeed been obtained in the subject of topological field theories in general, and in topological sigma models in particular. It is clearly impossible to review all of them in this short note, and I will just mention those that seem to be most directly connected to the present paper.

While the present paper discusses what is now called the A model, it is possible to formulate its mirror-related counterpart, the B model (for conformal targets), and study the still enigmatic mirror symmetry between them [47]. Also, for targets of complex dimension three, the sigma model defines a legitimate (classical) string theory by itself [18], without any necessity of coupling to topological gravity on the worldsheet. For these relatively simple models, one can develop the corresponding string field theory [18], and observe that in the open string sector the result coincides with the Chern-Simons gauge theory on the real part of the target [18].

Some important recent observations indicate [19] that the structure of topological sigma models is actually slightly more sophisticated than believed thus far. When considered carefully, the path integral of the topological theory contains a holomorphic anomaly, which makes the A model background-dependent. The topological sigma models considered thus far, including those of the present paper, then represent a specific choice of the background, defined by sending the corresponding coupling constants to infinity (for details, see [19, 50]). The existence of holomorphic anomaly seems to lead to some new insight into the puzzles of background
independence in string theory, at least in its toy-model topological incarnation [50].

The paper has been restricted to simply connected targets; recently, some progress has been achieved in some simple cases with targets whose fundamental group is non-trivial, such as the torus [51]. The structure of observables is much more complicated than for the simply-connected targets. In the specific case of the target torus, the infinite fundamental group generates an infinite number of physical states and an infinite-dimensional spacetime symmetry algebra, leading to topological $\mathcal{w}_\infty$ supersymmetry, odd-symplectic geometry [51] and Batalin-Vilkovisky geometry [51, 52] in the target. These results fit nicely into a broad picture [53] that indicates the existence of topological symmetry in the spacetime of topological string theory.
Appendix:
Axiomatics of Equivariant TQFT

Let $G$ be a discrete group, allowed to act effectively by orientation-preserving diffeomorphisms on our “spacetimes,” making $G$-manifolds out of them. To the same extend as Atiyah’s axiomatics is related to usual cobordisms of manifolds, the equivariant axiomatics will be related to equivariant cobordisms of manifolds. To define a $G$-equivariant topological quantum field theory, we first associate with every oriented $(D-1)$-dimensional $G$-manifold $\Sigma$ a vector space $H_\Sigma$ (of physical states).

To any $D$-dimensional $G$-manifold $Y$ we assign an element $\Psi_Y$ of $H_{\partial Y}$, with the induced $G$-action on $\partial Y$ implicitly understood, and assume that these data satisfy the following system of axioms:

1. **Topological invariance.** An equivariant isomorphism $f : \Sigma \to \Sigma'$ induces an equivariant isomorphism $H(f) : H_\Sigma \to H_{\Sigma'}$ compatible with the $G$-action, and these induced isomorphisms compose in the obvious way.

Bearing in mind that the vector space associated to a given $(D-1)$-dimensional manifold $\Sigma$ is the space of physical states on $\Sigma$, it should behave appropriately under disjoint union of two Hamiltonian slices:

2. **Multiplicativity.** If $\Sigma_1 \cup \Sigma_2$ is the disjoint union, then $H_{\Sigma_1 \cup \Sigma_2} = H_{\Sigma_1} \otimes H_{\Sigma_2}$, with the obvious $G$-action.

The fact that transition amplitudes can be defined is ensured by

3. **Duality.** If $\Sigma^*$ is $\Sigma$ with the opposite orientation, then $H_{\Sigma^*} = H_{\Sigma}^*$ is the dual space.

The central axiom of topological quantum field theory is the requirement of associativity, or the factorization property of physical transition amplitudes, which represents the possibility to sum over the full set of intermediate states in any channel:
4. **Factorization of amplitudes.** If $\Sigma$ is a component of $Y$, and $\Sigma^*$ is a component of $Y'$, then
$$
\Psi_W = (\Psi_{Y'}, \Psi_Y)_{\Sigma},
$$
where we have denoted by $W$ the result of sewing $Y$ and $Y'$ along their common boundary component $\Sigma$, and $(\ , \ )_{\Sigma}$ denotes the canonical pairing of $H_{\Sigma}$ with $H_{\Sigma^*}$, *i.e.* the contraction in the corresponding indices. An analogous identity should be valid for non-separating cuts as well.

5. **Completeness.** The states assigned to manifolds $Y$ with boundary $\partial Y = \Sigma$ span the whole vector space $H_{\Sigma}$.

If $H_{\Sigma}$ have canonical identifications with their duals, as it is the case when $H_{\Sigma}$ are Hilbert spaces, it is natural to require:

6. **Conjugation.** For any oriented $D$-dimensional manifold $Y$,
$$
\Psi_Y = \Psi_{Y^*}.
$$

We have assumed implicitly in these axioms that all of the physical states are bosonic. The axioms could be obviously generalized to allow for fermions as well.

One particular way of constructing equivariant topological field theory is to mod out a topological field theory by the action of a discrete subgroup of its symmetry group, which is a generalization of the orbifold construction known from critical string theory. For this reason, $G$ is sometimes referred to in the paper as the “orbifold group.”
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**Figure Captions**

1. Stable open string geometries at the tree level. Inserting physical states in the punctures of the discs, the functional integral will give $\hat{n}_{\alpha\beta}$ and $\hat{c}_{-\alpha\beta}$, respectively.

2. The disc geometry that gives the three point function $d_{-\alpha\beta}$ is reducible by factorization to two stable discs with punctures, thereby leading to equation (2.10).

3. The operator product expansion of one closed and one open string observable gives another open string observable, which has been “evaluated” in the picture, using figure (2).

4. Insertion of the boundary or crosscap state on the world-sheet corresponds to integrating out a boundary or a crosscap.

5. The surface that gives via the functional integral the $\Upsilon$ operator of open string – closed string transition.

6. A typical open string configuration on an equivariant topological target manifold. Both ends of the string are confined to the real part $\mathbb{R}M$ of the target.