SPECULATIVE BEHAVIOR AND CHAOTIC ASSET PRICE DYNAMICS: ON THE EMERGENCE OF A BANDCOUNT ACCRETION BIFURCATION STRUCTURE

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Abstract. We study a simple financial market model with interacting chartists and fundamentalists that may give rise to multiband chaotic attractors. In particular, asset prices fluctuate erratically around their fundamental values, displaying a significant bull and bear market behavior. An in-depth analytical and numerical study of our model furthermore reveals the emergence of a new bifurcation structure, a phenomenon that we call a bandcount accretion bifurcation structure. The latter consists of regions associated with chaotic dynamics only, the boundaries of which are not defined by homoclinic bifurcations, but mainly by contact bifurcations of particular type where two distinct critical points of certain ranks coincide.

1. Introduction. Financial markets are excessively volatile and regularly display severe bubbles and crashes. Prominent economists such as Galbraith, Kindleberger and Shiller [13, 24, 30] illustrate that the turbulences of financial markets may be quite harmful for the real economy. Fortunately, models with heterogeneous interacting agents have improved our understanding of the functioning of financial markets in recent years. In turn, this may help policymakers to implement trading environments that yield more stable asset price dynamics. See [38, 4, 7, 8, 9, 6, 25, 12] for pioneering models, [1, 5, 10, 16, 27, 37] for policy applications and [18, 11, 36] for comprehensive surveys.

The seminal contribution by Day and Huang [8], for instance, explains the intricate bull and bear market behavior of actual financial markets via the interplay of three different types of market participants: chartists, fundamentalists and market makers. Chartists follow a linear trading rule and buy (sell) assets in bull (bear) markets. Fundamentalists rely on a nonlinear trading rule and buy (sell) assets in undervalued (overvalued) markets. An important assumption of this model is that fundamentalists’ trading intensity increases with mispricing of the asset. Finally, a market maker increases (decreases) the price of the asset if speculators’ buying

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orders exceed (fall short of) their selling orders. Formally, the model by Day and Huang [8] corresponds to a one-dimensional nonlinear map and its basic functioning may be summarized as follows. As long as mispricing of the asset is relatively low, destabilizing orders placed by chartists will dominate the financial market and may set a bubble process in motion. As mispricing of the asset grows, however, fundamentalists become increasingly aggressive, eventually causing the bubble to collapse. Since fundamentalists reduce their trading intensity as the asset price approaches its fundamental value again, chartists once again rule the financial market and start the next bubble. To gain deeper analytical insights, Huang and Day [19] transformed this model into a one-dimensional piecewise linear model. Further variations of this fascinating theme are introduced and discussed in [21, 20, 34, 22, 23] and, with an even closer link to our work, in [33, 26].

Inspired by this line of research, in this paper we study a simple financial market model with chartists, fundamentalists and a market maker that may produce chaotic asset price dynamics. As in the original model by Day and Huang [8], chartists believe in the persistence of bull and bear markets while fundamentalists bet on mean reversion. However, fundamentalists only become active when mispricing of the asset exceeds a critical threshold. The market maker adjusts the asset price with respect to speculators’ orders in the usual way. Technically, our model corresponds to a one-dimensional discontinuous piecewise-linear map with three branches. Expressing the model in deviations from the fundamental value reveals that the inner branch crosses the origin with a slope larger than one. The two outer branches have the same slope as the inner branch, but the left (right) branch has a positive (negative) offset. Since fundamentalists’ market entry level in the bear market deviates from their market entry level in the bull market, the map is asymmetric with respect to the origin.

Against this backdrop, the economic explanation for the appearance of endogenous (chaotic) asset-price dynamics within our model may be outlined as follows. Near the asset’s fundamental value, chartists’ orders push the price of the asset away from its fundamental value. Further away from the asset’s fundamental value, however, fundamentalists enter the financial market and their orders create a temporal mean reversion. As the asset price is driven closer to its fundamental value, fundamentalists exit the market and, consequently, chartists create either the next boom or bust episode, depending on the condition of the financial market. This pattern repeats itself, albeit in an intricate manner, leading to (multiband) chaotic attractors. Assuming that fundamentalists enter bear markets earlier than they enter bull markets tends to generate bull markets that are more pronounced than bear markets, as documented in [28] for actual financial markets.

Moreover, we use our model to describe a new bifurcation structure associated with multiband chaotic attractors. In general, for piecewise smooth maps, bifurcation structures located in the chaotic domain, such as bandcount adding or bandcount incrementing structures, are known to be related to homoclinic bifurcations of certain repelling cycles. At the bifurcation moment, a point of the corresponding cycle coincides with a critical point of a definite rank; this point is then periodic of period, say, $n$. In other words, a critical point of rank $i$ coincides with the same critical point of rank $i + n$. Due to such a bifurcation, the number of bands of the chaotic attractor changes as the cycle switches between being homoclinic and non-homoclinic. In the latter case, points of the non-homoclinic repelling cycle are located in between the bands of the attractor. For a detailed description of known
bifurcation structures related to chaotic attractors, we refer the reader to [3] and references therein.

Contrariwise, in the bifurcation structure described in this work, which we refer to as a \textit{bandcount accretion bifurcation structure}, boundaries of the related chaoticity regions are mainly associated with contact bifurcations for critical points, at which two distinct critical points of certain ranks coincide, involving no cycles. The number of bands of the attractor changes only due to the noninvertibility of the map and the appearance/disappearance of intervals that have no preimages inside the absorbing set (and hence, cannot belong to the attractor). The main formation principles of the bandcount accretion bifurcation structure can be described as follows. Its first tier consists of regions related to chaotic attractors, where the number of bands is \((n + 1), \ n \geq 1\). There is a single band on the left-hand side of the origin and \(n\) bands on the right. The number of regions of the first tier is limited from above and depends on the slope, which is identical for all three branches of the map. The expression for this limit value is obtained explicitly. Between the two successive regions of the first tier there are an infinite number of regions belonging to the second tier associated with \((n + k)\)-band attractors, which have \(k, \ k \geq 2\) bands on the left-hand side of the origin and \(n\) bands on the right. The regions of the second tier accumulate to a certain point, the coordinates of which can be obtained analytically.

The rest of our paper is organized as follows. Section 2 presents a simple financial market model and a number of preliminary remarks. Section 3 discusses the emergence of a bandcount accretion bifurcation structure in a rather simple scenario; Section 4 extends our analysis to a more general case. Finally, Section 5 concludes the paper.

2. Economic and mathematical background.

2.1. A simple asset-pricing model. The basic structure of the asset-pricing model by Panchuk, Sushko and Westerhoff, developed in [26] and resting on [8, 19, 33, 34], may be summarized as follows. They consider a financial market that is populated by a market maker and four different types of speculators. The market maker mediates transactions on the financial market and sets the asset price with respect to speculators’ aggregate order flow. There are two types of chartists and two types of fundamentalists. Type 1 chartists believe in the persistence of bull and bear markets. Accordingly, they optimistically buy assets in bull markets and pessimistically sell assets in bear markets. In contrast, type 1 fundamentalists bet on mean reversion, i.e. they buy assets if the financial market is undervalued and sell assets if it is overvalued. While type 1 chartists and type 1 fundamentalists are always present in the financial market, type 2 chartists and type 2 fundamentalists are more reluctant and only enter the financial market if the distance between the asset price and its fundamental value exceeds a critical threshold, i.e. if they receive a clear and significant price signal.

Let us turn to the model’s mathematical representation. The market maker increases (decreases) the asset price if speculators’ buying orders exceed (fall short of) their selling orders. Since the market maker uses a linear price adjustment rule, the asset price in period \(t + 1\) is given by

\[
P_{t+1} = P_t + a \left( D^{C,1}_t + D^{F,1}_t + D^{C,2}_t + D^{F,2}_t \right),
\]

(1)
where the four terms in the bracket on the right-hand side of (1) capture the orders placed by type 1 chartists, type 1 fundamentalists, type 2 chartists and type 2 fundamentalists, respectively. Without loss of generality, parameter \( a > 0 \), regulating the market maker’s price adjustment behavior, is set to \( a = 1 \).

Chartists regard a market as a bull (bear) market if the price of the asset is above (below) its fundamental value, represented by \( F \). Orders placed by type 1 chartists are written as

\[
D^C_{t,1} = c_1 (P_t - F).
\]

Note that type 1 chartists receive a buy (sell) signal if the financial market is in a bull (bear) state, where parameter \( c_1 > 0 \) indicates how aggressively type 1 chartists react to their price signals.

Fundamentalists believe that the price of the asset will always return towards its fundamental value in the long run. Orders placed by type 1 fundamentalists are formalized as

\[
D^F_{t,1} = f_1 (F - P_t).
\]

Parameter \( f_1 \) is positive and reflects the trading intensity of type 1 fundamentalists with respect to mispricing of the asset. Since type 1 fundamentalists sell when the financial market is overvalued and buy when the financial market is undervalued, they are a mirror image of type 1 chartists.

Type 2 chartists are only active if the asset price is sufficiently distant from its fundamental value. For type 2 chartists, this critical distance is given with \( z^- > 0 \) in the bear market and with \( z^+ > 0 \) in the bull market. Orders placed by type 2 chartists are captured by

\[
D^C_{t,2} = \begin{cases} 
  c_2 (P_t - F) + c_3 & \text{for } P_t - F \geq z^+, \\
  0 & \text{for } -z^- < P_t - F < z^+, \\
  c_2 (P_t - F) - c_3 & \text{for } P_t - F \leq -z^-.
\end{cases}
\]

Since parameter \( c_3 \) is positive, the trading aggressiveness of type 2 chartists increases with mispricing of the asset. Their orders also depend on parameter \( c_3 \), with \( c_3 \geq \max\{-c_2 z^+, -c_2 z^-\} \). The latter restriction ensures that type 2 chartists’ transactions are non-negative in the bull market and non-positive in the bear market.

Type 2 fundamentalists rely on the same market entry levels as type 2 chartists do. Consequently, their orders are expressed as

\[
D^F_{t,2} = \begin{cases} 
  f_2 (F - P_t) - f_3 & \text{for } P_t - F \geq z^+, \\
  0 & \text{for } -z^- < P_t - F < z^+, \\
  f_2 (F - P_t) + f_3 & \text{for } P_t - F \leq -z^-.
\end{cases}
\]

where, of course, the restrictions \( f_2 > 0 \) and \( f_3 \geq \max\{-f_2 z^+, -f_2 z^-\} \) hold. Needless to say, type 2 fundamentalists increase their orders with mispricing of the asset.

It then follows from (1) to (5) that

\[
P_{t+1} = \begin{cases} 
  P_t + (c_1 + c_2 - f_1 - f_2) (P_t - F) + c_3 - f_3 & \text{for } P_t - F \geq z^+, \\
  P_t + (c_1 - f_1) (P_t - F) & \text{for } -z^- < P_t - F < z^+, \\
  P_t + (c_1 + c_2 - f_1 - f_2) (P_t - F) - c_3 + f_3 & \text{for } P_t - F \leq -z^-.
\end{cases}
\]

i.e. the asset price in period \( t+1 \) depends on its value in period \( t \), on its current mispricing and on speculators’ eight behavioral parameters.
2.2. Preliminaries and notations. After introducing the following notations (as in [26])
\[
\begin{align*}
{a}_M &= 1 + c_1 - f_1, \quad {a}_\mathcal{L} = a_R = a_M + b, \quad b = c_2 - f_2, \\
\mu_\mathcal{L} &= -\mu, \quad \mu_R = \mu, \quad \mu = c_3 - f_3,
\end{align*}
\]
map (6) acquires the form
\[
f : x \to f(x) =
\begin{cases}
  f_R(x) = a_R x + \mu_R = (a_M + b) x + \mu & \text{for } x \geq z^+, \\
  f_M(x) = a_M x & \text{for } -z^- < x < z^+, \\
  f_L(x) = a_L x + \mu_L = (a_M + b) x - \mu & \text{for } x \leq -z^-,
\end{cases}
\]
with \(x := x_t = P_t - F\). Parameters \(z^-\) and \(z^+\) are positive and \(a_M, b, \mu \in \mathbb{R}\), in general. Following [19], however, we assume that \(c_1 > f_1\) and hence \(a_M > 1\).

In general, map \(f\) has two discontinuities at \(x = -z^-\) and \(x = z^+\), unless additional conditions \(f_L(-z^-) = f_M(-z^-)\) and/or \(f_R(z^+) = f_M(z^+)\) are imposed. Intervals \(I_\mathcal{L} = (-\infty, -z^-)\), \(I_M = (-z^-, z^+)\) and \(I_R = (z^+, \infty)\) are called partitions. An arbitrary point \(x \in I_s, s \in \{\mathcal{L}, M, R\}\) is coupled with the relevant symbol \(s\). Consequently, any orbit \(\{f^i(x)\}_{i=0}^\infty = \{x_{s_{i+1}}\}_{i=0}^\infty\) can be associated with a particular symbolic sequence \(\sigma = \{s_i\}_{i=1}^\infty =: \sigma(x)\) consisting of symbols \(\mathcal{L}, M\) and \(R\). Sequence \(\sigma(x)\) is also referred to as the itinerary of a point \(x\), and the latter is sometimes denoted as \(x_\sigma\). If the orbit is periodic, the assigned symbolic sequence is finite, otherwise it is infinite. Fixed points of \(f\) clearly have the shortest symbolic sequences consisting of only one symbol, namely, \(\sigma(x^*_\mathcal{L}) = \mathcal{L}, \quad \sigma(x^*_M) = M, \quad \sigma(x^*_R) = R\), where
\[
\begin{align*}
x^*_\mathcal{L} &= -\frac{\mu}{1 - a_M - b} \in I_\mathcal{L}, \quad x^*_M = 0 \in I_M, \\
x^*_R &= -\frac{\mu}{1 - a_M - b} \in I_R.
\end{align*}
\]
Likewise, for an arbitrary \(n\)-cycle \(O = \{x_i\}_{i=1}^n = \{s_{i_1} s_{i_2} \ldots s_{i_n}\}\), the related symbolic sequence is \(\sigma = \sigma(O) = s_1 s_2 \ldots s_n\) and the cycle is denoted as \(O_\sigma\). In a similar way (for the sake of brevity), an arbitrary composition of functions \(f_L, f_M\) and \(f_R\) can be associated with the respective symbolic sequence, that is, \(f_\sigma := f_{s_1} \circ f_{s_2} \circ \ldots \circ f_{s_k}\) with \(\sigma = s_1 s_2 \ldots s_k\).

Recall that the region in the parameter space of a map related to the stable cycle \(O_\sigma\) is called the periodicity region and its boundaries can be defined by (see, e.g. [31])

- a border collision bifurcation, occurring when a point of the cycle collides with a discontinuity point;
- a degenerate flip bifurcation associated with the cycle’s eigenvalue crossing \(-1\);
- a degenerate bifurcation associated with eigenvalue \(+1\).

Note that it is often also necessary to distinguish between intervals \(I_{M_-} = (-z^-, 0)\) and \(I_{M_+} = (0, z^+)\), for which the corresponding symbols are \(M_-\) and \(M_+\). The symbolic sequence of an orbit (a cycle) or a function composition is adjusted accordingly.

\[1\] Clearly, any cyclic shift \(\sigma_i\) of \(\sigma\), where \(\sigma_i = s_1 \ldots s_{n-1} s_{n}\), is associated with the same \(n\)-cycle as well, that is, \(O_{\sigma_i} = O_\sigma\).
Another important object that is often involved in bifurcations of piecewise smooth maps is a critical point, which, for discontinuous maps, represents the limiting value of the function at the discontinuity from the left or from the right. Indeed, [3] shows that the overall bifurcation structure of the parameter space of such a map does not depend on its definition at the border point, but on its left and right limits at this point. Since map (8) has two discontinuity points, it has four critical points, namely,

\[\ell = f_L(-z^-), \quad m^- = f_M(-z^-),\]
\[m^+ = f_M(z^+), \quad r = f_R(z^+).\]  

The image \(c_j = f^j(c), j \geq 1\) of a critical point \(c \in \{\ell, m^-, m^+, r\}\) is referred to as the critical point of rank \(j\). If map \(f\) has an absorbing interval, its boundaries are given by two different critical points or by one critical point and its image. Clearly, this absorbing interval can include either one or both discontinuity points. For simplicity, we denote the absorbing interval as \(J^-\) if it contains only point \(x = -z^-\), as \(J^+\) if it contains only \(x = z^+\) and as \(J^\pm\) if it contains both. Clearly, for the appropriate parameter values, absorbing intervals \(J^-\) and \(J^+\) may coexist.

Besides the absorbing intervals, critical points of different ranks can also represent boundaries of chaotic attractors consisting of \(n\) pieces, called bands by convention, while the intervals between bands are sometimes referred to as gaps. An \(n\)-band chaotic attractor (which clearly has \(n-1\) gaps) is denoted by \(Q_n\); the related region \(C_n\) in the parameter space is referred to as the chaoticity region. When a parameter is varied, the number of intervals of a chaotic attractor may change due to one of the following bifurcations:

- A merging bifurcation, at which the pieces of a chaotic attractor merge pairwise, associated with a homoclinic bifurcation of a repelling cycle with a negative multiplier;
- An expansion bifurcation, characterized by an abrupt increase in size of the attractor, associated with a homoclinic bifurcation of a repelling cycle with a positive multiplier;
- One may also observe a change in the number of pieces and/or their width as a result of a fold BCB, leading to the appearance of two repelling cycles, at least one of which is non-homoclinic.

In addition, a chaotic attractor may become a chaotic repeller due to a final bifurcation (also called a boundary crisis), associated with a homoclinic bifurcation of a repelling cycle located at the immediate basin boundary of the attractor. Possible transformations of chaotic attractors and the related bifurcations are described in detail, e.g. in [2, 3].

Finally, a particular bifurcation may occur for map \(f\). This is referred to as a contact bifurcation for critical points, at which for two different critical points \(c, d \in \{\ell, m^-, m^+, r\}\) there exist ranks \(i, j \geq 0\) such that \(c_i = d_j\). For the sake of brevity, we use below the notation \(\nu_{c_i,d_j}\) for the related curve in the parameter space. For piecewise monotone maps with more than one discontinuity, this occurrence may lead to the merging of chaotic attractors or to a change in the number of bands of a single chaotic attractor. In contrast, for maps with one border point, such a contact of critical points cannot be responsible for a bifurcation (see [32]).

3. Bandcount accretion bifurcation structure: a simple case. For \(z^- = z^+\), map (8) is symmetric with respect to the origin. Bifurcation structures observed for such a map have been described in [34] and [26] for regular and chaotic dynamics,
respectively. However, the aim of our work is to study a novel bifurcation structure that is related to the contact of different critical points and, in general, not associated with homoclinic bifurcations. This structure can occur only when the symmetry of map (8) is broken, by assuming that (i) $z^+ \neq z^-$, (ii) $\mu_L \neq -\mu_R$ or (iii) $a_L \neq a_R$.

The first inequality implies that market entry thresholds $z^+$ and $z^-$ for type 2 speculators are not equal. In fact, [28] reports that bull markets are usually more pronounced than bear markets, suggesting that $z^- = z^+$ and, with a view to the market entry of additional stabilizing fundamentalists, that $z^- < z^+$. The other two inequalities may also be interpreted economically. Obviously, if speculators trade more/less aggressively in bull markets than in bear markets, the two offsets $\mu_L$ and $\mu_R$ and/or the two slope parameters $a_L$ and $a_R$ are different. Of course, conditions (i), (ii) and (iii) may hold simultaneously, rendering map (8) even more asymmetric. To be able to study the underlying mechanism of bandcount accretion bifurcation structures in detail, however, we consider case (i). Following the same arguments as in [33], and to tie in more closely with the results established in [26], we put $z^- = 1$, $z^+ = 1 + \varepsilon$ and $\varepsilon > 0$.

For the sake of simplicity, we furthermore assume that $b = 0$, implying that $c_2$ and $f_2$ are equal, and that $\mu < 0$, implying that $c_3$ is smaller than $f_3$. By setting $a_L = a_R = a_M := a$, map (8) becomes

$$f : x \rightarrow f(x) = \begin{cases} f_R(x) = ax + \mu & \text{for } x \geq 1 + \varepsilon, \\ f_M(x) = ax & \text{for } -1 < x < 1 + \varepsilon, \\ f_L(x) = ax - \mu & \text{for } x \leq -1. \end{cases} \tag{11}$$

To sum up, map (11) has three parameters, restricted to $a > 1$, $\varepsilon > 0$ and $\mu < 0$. This parameter constellation excludes stable cycles, but chaotic attractors may exist.

Before we continue, a comment is in order. To retain a link between our work and that conducted in [33] and [26], we derive our results using a specification of their financial market model. Importantly, however, map (11) may also be obtained more easily and interpreted as a simple stand-alone financial market model, resting on key elements of the original models by Day and Huang [8, 19]. In a stylized way, map (11) may also describe the behavior of other real-life dynamical systems (e.g. national income fluctuations or complex population dynamics). See Appendix A for details.

### 3.1. Main principles of the bifurcation structure formation.

To illustrate a number of properties of map (11), we fix parameter $a$ and vary parameters $\mu$ and $\varepsilon$ in Figure 1. As it was already mentioned, the bandcount accretion bifurcation structure cannot occur for a map with only one discontinuity; hence, we require that the absorbing interval contains both border points. With the current model setup (with all slopes being equal and greater than one, and with $z^+ > z^-$), we must require that $r < 0$, in which case $J^\pm = [m^-, m^+]$ or $J^\pm = [r, \ell]$. Otherwise, either there is $J^- = [m^-, \ell]$ and/or $J^+ = [r, m^+]$ or a typical orbit diverges. In Figure 1(a) we plot a schematic representation of the $(\varepsilon, \mu)$ parameter plane with several bifurcation curves related to the existence/configuration of the absorbing interval(s). Above the final bifurcation line $\chi_{L}^{\mu_2} = \{(\varepsilon, \mu) : \mu = a(a-1)\}$ associated
with the condition \( x_* = m^- \), map \( f \) has no absorbing interval\(^2\) (Fig. 1(b)). Interval \( J^- \) exists between \( \chi^{m^-}_L \) and the line \( \chi^f_M = \{(\varepsilon, \mu) : \mu = a\} \), corresponding to the final bifurcation \( \ell = 0 \), (Figs. 1(c) and (d)). Interval \( J^+ \) exists between the two final bifurcation lines, \( \chi^{m^+}_R = \{(\varepsilon, \mu) : \mu = a(1 / 2 + 1)\} \) (related to the condition \( r = \ell \)) and \( \chi^r_M = \{(\varepsilon, \mu) : \mu = a(1 + \varepsilon)\} \) (related to \( r = 0 \), (Figs. 1(d) and (e)). Note that the line \( \psi_{\ell, r} = \{(\varepsilon, \mu) : \mu = a(1 + \varepsilon)\} \) (shown by the dashed line in Fig. 1(a)), at which a contact of the two critical points \( \ell = r \) occurs, does not correspond to any bifurcation for the current map configuration. Indeed, even when \( \ell \) becomes greater than \( r \), the absorbing interval still remains \( J^+ = [r, m^+] \), including only the right border point \( x = z^+ \) (provided that \( r > 0 \)). Below \( \chi^r_M \), the absorbing interval contains both border points; this parameter region is of interest to us (Figs. 1(f) and (g)). More precisely, the target is the region between bifurcation lines \( \chi^r_M \) and \( \psi^{m^- \cdot r} = \psi^{m^+ \cdot \ell} = \{(\varepsilon, \mu) : \mu = a(1 + \varepsilon)\} \), at which there is \( m^- = r \) and \( m^+ = \ell \).

Figure 2 shows the 2D bifurcation diagram related to chaotic multiband attractors (different colors correspond to a different number of bands). Above \( \chi^r_M \), there exists an absorbing interval \( J^+ \) that includes only the right border point \( x = z^+ \). Between lines \( \chi^f_M \) and \( \psi^{r \cdot m^-} \), one can see a new bifurcation structure whose main tier consists of chaoticity regions \( C_{i+1} \), \( i = 1, 2, 3 \) (the aim of choosing such a particular index notation will become clearer later). Between these regions, other substructures are observed, which are related to chaotic attractors with a larger number of bands, namely, \( C_{2+j} \) and \( C_{3+j} \), \( j = 2, 3, \ldots \) (see also Fig. 2(b)).

\(^2\)There is also the other region shaded gray below \( \chi^f_M \) and above \( \chi^{m^+}_R \), in which the absorbing interval does not exist either, since \( J^- \) has already been destroyed and \( J^+ \) has not yet appeared.
Let us examine which bifurcations are associated with the boundaries of the aforementioned chaoticity regions. To this end, Figure 3 shows the 1D bifurcation diagram along the path marked with the blue vertical arrow in Figure 2(a). Here solid lines of four different colors (blue, green, orange and red) correspond to the critical points $\ell_i^-, m_i^-, m_i^+$ and $r_i$ of different ranks are shown by blue, green, orange and red lines, respectively.
following, when referring to the existing chaotic attractor in general (without indicating how many pieces it has, e.g. when it is not known or not important), we denote it simply as $Q$. If it is essential that the attractor has $n$ pieces, we write $Q_n$ (or $Q_{n,j}$ to distinguish between attractors with the same number of bands but different configurations, as explained below).

As one can see in Figure 3, for $\mu$ being greater than
\begin{equation}
\mu_{m_i^1} = -a^2(1 + \varepsilon),
\end{equation}

and $\ell$, where namely, different configurations, as explained below)

corresponding to the expansion bifurcation $\zeta_{m_i^1}$, at which the critical point $m_i^1$ collides with the repelling fixed point $x_{m}^*$, the attractor is 1-piece $Q_1 = J^k = [m^{-}, m^{+}]$. At $\zeta_{m_i^1}$, the fixed point $x_{m}^*$ becomes non-homoclinic and the attractor becomes 2-piece $Q_{1+1}$ (namely, having exactly one “right” band and one “left” band). Let us denote the right interval (band) of $Q_{1+1}$ as $B^R = [m_i^1, m_i^+]$ and the left one as $B^L = [m_i^-, m_i^-]$. It is easy to see that, for $\mu < \mu_{m_i^1}$, for the set $S = B^L \cup B^R$ there is $f(S) \subset S$, as long as $\ell \in B^R$, $r \in B^L$ and $x_{m}^*$ is non-homoclinic (clearly, $f_{m}^{-1}(x_{m}^*) > m^+$ and $f_{m}^{-1}(x_{m}^*) < m^-$. It then follows that, for smaller $\mu$, the attractor $Q$ (no matter how many bands it has) belongs to $S$ and the interval $(m_i^+, m_i^-) \not\subset Q$ represents one of its gaps.

For a certain value of
\begin{equation}
\mu = \mu_{m_i^1} = -a(a + 1),
\end{equation}

$\ell$, associated with the contact bifurcation of critical points $v_{\ell, m_i^-}$, that is, $\ell = m_i^-$, the attractor $Q_{1+1}$ transforms to $Q_{3+1}$. This is because two new gaps occur in the right interval $B^R$, namely $G_1 = (\ell, m_i^-)$ and $G_2 = f(G_1) = (\ell_1, m_i^-)$ (so now the attractor has three “right” bands and one “left” band). First of all, we remark that the structural change observed in the attractor cannot be associated with any repelling cycle switching from homoclinic to non-homoclinic. Indeed, the gaps are $G_1 \subset I_{m_i+}$ and $G_2 \subset I_{m_i-}$, while the image $f(G_2) \subset I_{m_i-}$. If a related cycle existed, it would have had at least one point in $I_{m_i-}$. Hence, together with $G_1$ and $G_2$, at least one gap belonging to $I_{m_i-}$ should have appeared, which is not revealed at this stage. This implies that the bifurcation is related solely to the contact of critical points $\ell = m_i^-$. To explain the underlying mechanism, we plot in Figure 4 map $f$ for $\mu = \mu_1 = -2.9$, which is slightly below $\mu_{m_i^-}$ (see the respective vertical pink dashed line in Fig. 3). The blue, green and orange lines show several successive images of critical points $\ell$, $m_i^-$ and $m_i^+$, respectively. The dot-dashed lines mark the (alternative) preimages by $f_{m_i^1}$, $f_{m_i^1}^{-1}$ or $f_{m_i^1}^{-1}$.

It is easy to see that, for any point $x > m^+$ ($x < m^-$), the set of its successive preimages $\{x^{-i}\}_{i=1}^{\infty}$ is such that $x^{-i} > m^+$ ($x^{-i} < m^-$) and $\lim_{i \to \infty} x^{-i} = x_{m_i^*}^\ell$ ($\lim_{i \to \infty} x^{-i} = x_{m_i^*}^-$. Note also that $m_i^- < \ell < m_i^+$, which is one of the key conditions for the appearance of the bandcount accretion bifurcation structure\footnote{Indeed, in case $m_i^- < m_i^- < \ell$, interval $(m_i^-, \ell)$ has a preimage in $B^L$ by $f_{m_i^1}$, and no additional gaps appear.}. Let us consider an arbitrary point $x \in G_1 = (\ell, m_i^-)$. This point has two preimages, namely, $f_{m_i^-}^{-1}(x) =: x^{-1, m_i^-} \in (\ell, m_i^-) \subset (m_i^-, m_i^+)$ and $f_{m_i^+}^{-1}(x) =: x^{-1, m_i^+} > m^+$, where $\ell_{m_i^1} := f_{m_i^1}(\ell) < m_i^-$ is another preimage of $\ell$ belonging to $I_{m_i-}$. None of these preimages belongs to set $S$, and hence, point $x$ cannot belong to the current attractor $Q$. Consequently, interval $G_1 \not\subset Q$. The same can be stated about image $G_2 = (\ell_1, m_i^-) = f(G_1) \subset I_{m_i-}$ in contrast, the second image $\tilde{G}_2 = f^2(G_1) = (\ell_2, m_i^-) \subset \tilde{G}_2$. Indeed, in case $m_i^- < m_i^- < \ell$, interval $(m_i^-, \ell)$ has a preimage in $B^L$ by $f_{m_i^1}$, and no additional gaps appear.
$I_L \cup I_{M-}$ has a preimage belonging to $B^L \cap I_{M-}$, provided that $m_i^- < m_2^+$ (cf. in Fig. 4c the respective values $\ell_1^- = f_{M}^{-1}(\ell_2) < m_3^{-M} = f_{M}^{-1}(m_4^-) < m_1^+$. Moreover, any point $x \in (m_2^+, m_4^+)$ in $B^L$ necessarily has one preimage $x^{-1,M} \in B^L \cap I_{M-}$, while for interval $(m_3^-, m_4^+)$ its preimage under $f_R^{-1}$ is clearly interval $(m_3^-, m_4^+)$. This means that $B^L \subset Q$, proving that the attractor is indeed a 4-piece $Q = Q_{3+1}$ consisting of intervals $[m_3^-, m_4^+)$, $[m_4^-, m_1^+]$, $[m_1^-, \ell_1]$, $[\ell_2, \ell_1]$ and $[m_3^-, m_4^+]$.

Note that the contact of critical points $\ell = m_2^-$ can also occur before the fixed point $x^*_M$ becomes non-homoclinic. Then, due to the expansion bifurcation $\mu_{m_2}^+$, one observes a transition from the 1-piece chaotic attractor to a chaotic attractor with more than two pieces. For instance, Figure 2 shows for certain larger values of $\varepsilon$ (when the line related to $\nu_{\mu_1} = \ell_2 < m_2^+ < m_4^-$, preimage $m_1^+ \in G_2$). Similarly, the critical point $m_i^- = f_{M}^{-1}(m_i^-)$ has two other preimages, $m_1^+ \in G_1$. By the same argument as above, this new gap cannot be related to any unstable cycle switching from homoclinic to non-homoclinic.

![Figure 4](image-url)  

**Figure 4.** Plot of function $f$ and formation of gaps in the upper part $B^R$ of the chaotic attractor $Q$.

We return to the 1D bifurcation diagram versus $\mu$ shown in Figure 3. The next bifurcation occurring at

$$\mu = \mu_{m_i^-} = \frac{-a^3(a^2 + \varepsilon + 1)}{a^3 + a - 1}$$

is $\nu_{m_i^-} \mu_{m_2}^+$ associated with the contact of critical points $m_i^- = m_2^+$. Similarly to the mechanism described above, which caused the gaps in $B^R$ to appear, this contact of critical points entails the occurrence of an additional gap $G_3 = (m_2^+, m_4^-)$ in the left part $B^L$ of attractor $Q$. In Figure 5, we plot the related function for $\mu = \mu_2 = -2.92 < \mu_{m_i^-} \mu_{m_2}^+$. Critical point $m_2^+ = f_{M}^{-1}(m_2^+)$ has two other preimages, $m_1^+ = f_{M}^{-1}(m_2^+) \in I_R$ and $m_1^+ = f_{L}^{-1}(m_2^+) \in I_L$. Preimage $m_1^+ \in G_1$. Since when $\mu < \mu_{m_i^-} \mu_{m_2}^+$ there is $\ell_2 < m_2^+ < m_4^-$, preimage $m_1^+ \in G_2$. Similarly, the critical point $m_4^- = f_{R}^{-1}(m_3^-)$ has two other
preimages, \( m_3^{−L} = f_3^{-1}(m_4) < m^− \) and \( m_3^{−M} = f_3^{-1}(m_4) \in (m^+, 0) \), none of which belongs to \( S \) (and hence not to \( \mathcal{Q} \) either). Then any point \( x \in G_3 \) has three preimages, \( x^{−1} < m_3^{−L} < m^− \) and \( x^{−1} \in (m^+, 0) \) and \( x^{−1} \in G_2 \), none of which can belong to attractor \( \mathcal{Q} \). In such a way, due to \( \nu_m^{−3} \) we observe a transition \( \mathcal{Q}_{3+1} \rightarrow \mathcal{Q}_{3+2} \), with the latter consisting of intervals \( [m^−, m^+] \), \( [m_1^−, \ell] \), \( [m_1^−, \ell] \) and \( [m_1^−, m^+] \) (that is, three “right” bands and two “left” bands). In the 2D bifurcation diagram shown in Figure 2, one can see the related chaoticity region \( C_{3+2} \) of the respective finer substructure located between \( C_{3+1} \) and \( C_{2+1} \).

Finally, at the contact bifurcation \( \nu_m^{−3} \) (associated with \( m_3^{−3} = m^+ \)), the bands \( [m_4^−, m^+] \) and \( [m_1^−, m^+] \), belonging to \( B^L \) and \( B^R \), respectively, vanish and the attractor becomes a 3-piece \( \mathcal{Q}_{2+1} \) consisting of intervals \( [m^−, f(q)] \), \( [m_1^−, \ell] \), \( [m_1^−, q] \), where \( q = \ell_1 \) or \( q = m^+ \) (that is, two “right” bands and one “left” band).

When \( \mu \) decreases further, an analogous bifurcation sequence, i.e. \( \nu_m^{m_3^{−3}} \) and \( M^m \), leads to transformation \( \mathcal{Q}_{2+1} \rightarrow \mathcal{Q}_{2+2} \rightarrow \mathcal{Q}_{1+1} \).

The bifurcation sequence (and the associated bifurcation structure) described above is located in the parameter region for which the fixed point \( x^*_M \) is non-homoclinic, that is, below the bifurcation line \( \zeta_{m^+}^{m_3^{−3}} \). However, Figure 2(a) shows that region \( C_{2+1} \) related to a three-band chaotic attractor has a triangular shaped “jut”, which is located above \( \zeta_{m^+}^{m_3^{−3}} \). Figure 6 shows the respective part of the \((\varepsilon, \mu)\) parameter plane. As one can see, the aforementioned “jut” is confined by \( \nu_m^{m_3^{−3}} \) (from above) and by \( \zeta_{m^+_1}^{m^{−3}} \) (from below). One also observes small regions related to multiband chaotic attractors with a larger number of pieces denoted as \( C_{2+1+j} \), \( j = 11, 12, \ldots \).

To shed light on how this additional part of region \( C_{2+1} \) is formed, we first plot the 1D bifurcation diagram versus \( \mu \) for \( \varepsilon = 1.37 \) in Figure 7 (see the corresponding vertical blue arrow in Fig. 6(b) marked by “1”). As one can see, transition \( \mathcal{Q}_1 \rightarrow \mathcal{Q}_{2+1} \) occurs at \( \nu_m^{m_3^{−3}} \). As shown below, after such a contact (for \( \mu < \nu_m^{m_3^{−3}} \)) the fixed point \( x_3^{m_3^{−3}} \) is only one-side homoclinic (until the next homoclinic bifurcation \( \zeta_{m^+}^{m_3^{−3}} \).
occurs). We start with the value of $\mu = -3.44$ and plot function $f$ in Figure 8. As before, blue, green, and orange lines mark forward iterations of critical points $\ell$, $m^-$, and $m^+$, respectively. As long as $m^+ > 0$ (that is, above $\zeta_{m^+}$), the $f_R$-preimage of the zero fixed point is $y_{-1} := f_R^{-1}(0) < m^+$, and hence, there exists a sequence of preimages $y_{-1} = f_M^{-i+1}(y_{-1}) \in I_{M+}$, $i \geq 2$ (whose itinerary is shown by the pink line in Fig. 8) such that $\lim_{i \to \infty} y_{-i} = 0^+$. Since $m^- < m^+$ (above $\nu_{m^-}$), interval $I := [m_1^-, m_1^+] \subset I_{M-}$ has a sequence of preimages $I_{-i} := f_M^{-i} \circ f_E^{-2} \circ f_{\epsilon_2}^{-1} \circ f_M^{-1}(I)$, $i \geq 1$, such that all $I_{-i}$ belong to $I_{M-}$ and approach the point $x = 0$ from the left as $i$ increases. Although no $y_{-i}$ falls into interval $[m_1^-, \ell_2]$ (as $\ell_2 < 0 < m_1^-$), some $k > 3$ may exist such that $y_{-k} \in I$ (in the considered case, $k = 12$), and hence, there also exists a sequence of preimages $\{u_{-i}\}_{i=1}^\infty$ (whose itinerary is shown by a dark-red line) of $x_{M}^*$ such that $u_{-1} := f_R^{-1}(y_{-k})$ and $\lim_{i \to \infty} u_{-i} = 0^-$. This implies that $x_M^*$ is double-side homoclinic and the attractor is 1-piece $Q_1$. Indeed, let us consider the preimages of the border point $z^+$ under $f_M$. Clearly, $z^+_+ \in I$ (in general, $z^+_{k+2} \in I$). Then the respective image of interval $\hat{I} := [m^-_1, z^+_+ \in I].$ When $k = 10$, this means that there are points located in $I_{M-}$ arbitrarily close to zero, orbits of which, after a finite number of iterations, enter the neighborhood of zero in $I_{M+}$, and vice versa.

Returning to the 1D bifurcation diagram in Figure 7, we notice that, for a certain value of $\mu = \mu_{M-}$, the fixed point $x_M^*$ undergoes a homoclinic bifurcation (the corresponding bifurcation boundary is denoted by $\zeta_{M-}$). For $\mu < \mu_{M-}$, the preimage of zero $y_{-12}$ is located outside interval $I$ and the aforementioned sequence $\{u_{-i}\}_{i=1}^\infty$ disappears. However, the attractor is still 1-piece. In Figure 9, we plot function $f$ for $\mu = \tilde{\mu} = -3.454$, corresponding to the vertical pink dashed line in Figure 7. Here the dark green line marks the iterates of $m^-$ for $i \leq 16$, while light
Figure 7. 1D bifurcation diagram corresponding to the blue arrow marked “1” in Figure 6(b), with $\varepsilon = 1.37$ and $a = 1.25$. The critical points $\ell_i, m_i^-, m_i^+$ and $r_i$ of different ranks are shown by blue, green, orange and red lines, respectively.

Figure 8. Plot of function $f$, with $a = 1.25$, $\varepsilon = 1.37$ and $\mu = \tilde{\mu} = -3.44$. Pink and dark-red lines show the related sequences of preimages of zero. Green marks the iterates for $17 \leq i \leq 35$. The meaning of the other colors is as in Figure 8. As one can see, not interval $I$ but the interval $f^{12}(I) = [m_{16}, m_{13}^+]$ now contains several preimages of $x_M$, and, therefore, there again exists another sequence\(^5\) $\{w^{-i}\}_{i=1}^{\infty}$ of its preimages such that $\lim_{i \to \infty} w^{-i} = 0$—(as an example, the preimages of $y^{-16}$ are shown by a dark-red line). As with the previous case,

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\(^5\)To be precise, there exist several such sequences, each generated by some $y^{-i} \in [m_{16}, m_{13}^+]$.
considering interval $I := [m^{-19}, z^{-14}] \ni y^{-16}$, taking its images under $f$ and using the same argument as before, we conclude that the attractor is 1-piece.

To sum up, until there exists interval $I = [m^{-14}, m^{+1}] \subset I_{M^+}$ and provided that no other homoclinic bifurcation occurs, the fixed point $x^*_{M^+}$ is double-side homoclinic and the attractor is $Q_1$. At the contact bifurcation $\nu_{M^+}$, interval $I$ vanishes and point $x^*_{M^+}$ becomes one-side homoclinic, having only the sequence of preimages $\{y^{-i}\}^{\infty}_{i=1}$ that approach it from the right. It can easily be shown that set $S = [m^{-}, \ell_2] \cup [m^{-1}, \ell] \cup [m^{-2}, \ell_1]$ (no longer containing the border point $z^+$) is $f$-invariant. This means that the attractor becomes a 3-piece $Q_3$. Note that the sequence of $y^{-i}$ is completely located in the complement set $\bar{S} = (\ell_2, m^{-1}) \cup (\ell, m^{-2}) \cup (\ell_1, m^{-1})$.

Let us now consider two different bifurcation scenarios related to the small regions, adjacent to $C_{2+1}$, associated with chaotic attractors that have a larger number of bands. In Figures 10 and 11, we plot 1D bifurcation diagrams corresponding to two blue vertical arrows marked in Figure 6(b) by “2” and “3”, respectively.

In the first scenario (Fig. 10), an expansion bifurcation $\zeta_{M^+}$ occurs as $\mu$ decreases. As in the case of $\zeta_{M^+}$ described above, due to this bifurcation the preimage of zero $y^{-12}$ exits interval $I = [m^{-1}, m^{+1}] \subset I_{M^+}$. However, the difference is now that $f^{12}(I) \subset I_{M^+}$ and the alternative sequence of preimages $\{w^{-i}\}^{\infty}_{i=1}$ (mentioned above) does not occur, implying that $x^*_{M^+}$ is only one-side homoclinic. As a result, set $S^o = [m^{-}, \ell_2] \cup [m^{-1}, \ell] \cup [m^{-2}, \ell_1] \cup (\cup_{i=0}^{12} f^i([m^{-1}, m^{+1}]))$ is $f$-invariant. Since intervals $[m^{-1}, \ell]$ and $[m^{-14}, m^{+1}]$ overlap, as well as intervals $[m^{-1}, \ell_1]$ and $[m^{-14}, m^{+1}]$, the resulting chaotic attractor consists of 14 pieces (see the respective region $C_{2+1+11}$).

In the second scenario (Fig. 11), the homoclinic bifurcation, due to which the number of bands of the attractor changes from 1 to 14, is not associated with the zero fixed point, but with a repelling 13-cycle $O_{R, M^+}$, namely, this structural change is related to an expansion bifurcation $\zeta_{R, M^+}$. 

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**Figure 9.** Plot of function $f$, with $a = 1.25$, $\varepsilon = 1.37$ and $\mu = -3.454$. Pink and dark-red lines show the related sequences of preimages of zero.
3.2. Economic insight. From an economic perspective, the functioning of our model may be explained as follows. Figure 12 shows the dynamics of Figure 9 in the time domain. Apparently, our model is able to produce irregularly alternating bull and bear market dynamics, with more pronounced bull markets than bear markets. Let us suppose that the financial market has just entered a bull market, i.e. the...
The financial market is slightly overvalued. Then the buying orders placed by optimistic chartists initiate an upward movement of the asset price with increasing momentum. Eventually, however, fundamentalists enter the financial market and their orders force the asset price back towards its fundamental value. Depending on the extent of the crash, we may observe the emergence of the next bull market (which occurs when the asset price remains above its fundamental value, keeping chartists optimistic) or the appearance of a bear market (which occurs when the asset price is pushed below its fundamental value, making chartists pessimistic). In both cases, the orders placed by chartists push the asset price away from its fundamental value—until fundamentalists re-enter the financial market, bringing the asset price back to more moderate values. As can be seen, the interaction between chartists and fundamentalists creates sustained bull and bear market dynamics. For a given parameter setting, these (chaotic) dynamics are located within a union of certain intervals (bands). If one of these parameters changes, characteristics of the bands (e.g. their number and width) may vary too. Since even a tiny parameter shift may have a significant impact on the market’s dynamics, it is easily imaginable that sporadic parameter changes, triggered, for instance, by exogenous political or economic events, may drastically increase the dynamics’ complexity.

Figure 12. The functioning of the financial market model. The panel depicts the dynamics of Figure 9 in the time domain. The parameters are $\varepsilon = 1.37$, $a = 1.25$, and $\mu = -3.454$.

4. Bandcount accretion bifurcation structure: a general case. The bandcount accretion structure described in the previous section represents one of the simplest cases of such bifurcation structures that are primarily unrelated to homoclinic orbits. In this example, we observe only three chaoticity regions of the main tier, namely, $C_{3+1}$, $C_{2+1}$ and $C_{1+1}$, related to attractors, with the left part $B^L$ does not have any gaps. In general, there may be more such regions. In Figure 13, for example, we plot the bifurcation structure of the $(\varepsilon, \mu)$ parameter plane for $a = 1.1$. As one can clearly see, there are more chaoticity regions of the first tier related to attractors $Q_{n+1}$ (that is, with $n$ bands in $B^R$). As with the case $a = 1.25$, the
region related to the highest $n = 7$ issues from the intersection point of $v^{\ell,m^-}$ and $\zeta_M^m$. The following question arises: What does the number $n$ depend on? Let us consider in general the moment of bifurcation $v^{\ell,m^-}$, at which it clearly also holds $\ell_i = m_{n+2}^i$, $i = 0, 1, \ldots$. Moreover, for $0 \leq i < n - 2$ it is $\ell_i \in I_{M^+}$ and $\ell_{n-2} \in I_R$ with a certain $n = n(a, \varepsilon)$. The successive image is $\ell_{n-1} < 0$. For the values of $\mu$ immediately after the bifurcation, using the same argument as above (for the case $a = 1.25$ with $n = 3$), we conclude that interval $G_1 = (\ell, m_{n+1}^-) \not\subset Q$. Its successive images $G_i+1 = (\ell_i, m_{n+2}^-) = f^i(G_1) \subset I_{M^+} \cup I_R$, $i < n - 1$ do not belong to attractor $Q$ either, the next image $G_n = f^{n-1}(G_1) = (\ell_{n-1}, m_{n+1}^-) \subset I_L \cup I_{M^-}$ has a preimage belonging to $B^L$, provided that $m_{n+1}^- < m_+^L$. This implies that $B^R$ contains $n$ pieces of $Q$, which are denoted here as $B^R_i = [m_{n-i}, \ell_{i-1}]$, $i = 1, n - 1$ and $B^R_n = [m_{n-i}, m_+]$, while intervals $G_i = (\ell_{i-1}, m_{i+1}^-)$, $i = 1, n - 1$ represent the respective gaps. The left interval is completely $B^L \subset Q$.

![Figure 13](image.png)

Now we estimate how many chaoticity regions of the first tier $C_{n+1}$ are revealed in the bandcount accretion bifurcation structure, depending on the value of $a$. For this, at the moment of bifurcation $v^{\ell,m^-}$ we compute the number $n = n(a, \varepsilon)$ such that $\ell_{n-2} > 0$ (or alternatively $\ell_{n-3} \in I_{M^+}$). For $\mu = \mu^{\ell,m^-}$, given in (13), the critical point $\ell_i > 0$, $i = 0, n - 2$, is obtained as

$$\ell_i = f_i^M(\ell) = a\ell = -a(a + \mu^{\ell,m^-}) = a^{i+2}. \quad (14)$$

The condition for $\ell_{n-3} \in I_{M^+}$ is

$$\ell_{n-3} = a^n < z_+ = 1 + \varepsilon. \quad (15)$$

It is clear from (15) that the larger $\varepsilon$ is (with fixed $a$), the higher the related number $n$ is. To obtain the final expression for the maximum possible value $\bar{n} = \bar{n}(a)$ of the number of pieces of $Q \cap B^R$, we have to eliminate $\varepsilon$ from (15). As mentioned above,
Moreover, for a larger $\bar{\nu}$ (Fig. 13) and $C_m$ that immediately get bifurcation value of $Q \cap \Lambda$. Thus, the number of pieces of $Q$ belonging to $\bar{\nu}$ be larger. More precisely, it depends on the number of bands in $B$ which the left part intersect. Hence, the maximum $\bar{n}$ is achieved for the attractor whose chaoticity region issues from point $A = A(\bar{\epsilon}, \mu^{\ell, m\bar{\nu}})$, with

$$\bar{\epsilon} = \frac{1}{a},$$

(16)

Substituting (16) into (15), we get

$$\bar{n} = \left[\frac{\ln(a + 1)}{\ln a}\right],$$

(17)

where $[\cdot]$ denotes the integer part. Setting $a = 1.1$ and $a = 1.25$ in (17), we immediately get $\bar{n} = 7$ and $\bar{n} = 3$, respectively, which correspond exactly to $C_{7+1}$ (Fig. 13) and $C_{3+1}$ (Fig. 2). From (17) it also follows that the closer $\bar{\epsilon}$ is to unity, the larger $\bar{n}$ is and the more chaoticity regions in the bifurcation structure are revealed. Moreover, for $a > (\sqrt{5} + 1)/2$ the value of $\bar{n}$ equals 1 and the bandcount accretion bifurcation structure is not observed.

Using similar analysis, we can estimate the number $k = k(a, \epsilon)$ of pieces into which the left part $B^L$ splits due to another contact bifurcation of critical points. In Figure 3, the related bifurcation is $\nu^{m_{\bar{\nu}}, m\bar{\nu}}$, but in general the rank of $m$ may be larger. More precisely, it depends on the number of bands in $Q \cap B_R$, which is $n = n(a, \epsilon) \leq \bar{n}(a)$ such that $\ell_{n-2} \in I_R$, and consequently, $m^-_n \in I_R$. Then the successive image is $m^-_{n+1} < 0$ and if it becomes $m^-_{n+1} > m^+_2$, then the part of the attractor $Q \cap B^L$ also splits into at least two pieces (recall the argumentation above and Figures 3 and 5). Thus, the related bifurcation condition is, in general, the contact $m^-_{n+1} = m^+_2$. At the moment of this contact, there is $\ell_{n-2} < m^-_n$ (since $\ell_{n-2} < m^-_n$). Using the same argument as in the simple case with $\nu^{m_{\bar{\nu}}, m\bar{\nu}}$, it can be also shown that immediately after bifurcation $\nu^{m^-_{n+1}, m\bar{\nu}}$, interval $G_n = (m^+_2, m^-_n)$ has three preimages, none of which can belong to attractor $Q$. Indeed, $f^{-1}_L(G_n) \subset (\infty, m^-)$, $f^{-1}_M(G_n) \subset (m^+_1, m^-_1)$ and $f^{-1}_R(G_n) \subset G_{n-1}$. Then the total number of gaps in $Q \cap B^L$ depends on how many images of $G_n$ do not have preimages belonging to $Q$. Let us consider an image $G_{n+1} = f^i(G_n) = (m^+_2, m^-_{n+1})$. If $m^-_{n+1} > m^+_2 > \ell_{n-1}$, from similar arguments as for $G_n$ it follows that $G_{n+1} \notin Q$. Thus, the number of pieces of $Q \cap B^L$ that occur due to $\nu^{m^-_{n+1}, m\bar{\nu}}$ equals $k$ such that $m^-_{n+k-1} = m^+_k > \ell_{n-1}$ at the bifurcation moment. Let us first compute the bifurcation value of $\mu = \mu^{m^-_{n+1}, m\bar{\nu}}$. We have

$$m^-_{n+1} = f_{\Lambda^{n+1}}(m^-) = -\alpha^{n+2} - a^n \mu + \mu,$$

$$m^+_2 = f_{\hat{R}^n}(m^+) = a\mu + a(1 + \epsilon).$$

(18)

Equating the two expressions (18), we get

$$\mu^{m^-_{n+1}, m^+_2} = \frac{-\alpha^3(a^{n-1} + \epsilon + 1)}{a^n + a - 1}.$$
Then there is
\[ \ell_{n-1}|_{\mu=\mu_{n+1}^{-}m_{2}^{+}} = f_{Mn-2}(\ell)|_{\mu=\mu_{n+1}^{-}m_{2}^{+}} = \frac{a^{3}((a^{n-1}-1)\varepsilon + a^{2n-2} - 1)}{a^{n} + a - 1} - a^{n}, \quad (20) \]
\[ m_{k}^{+}|_{\mu=\mu_{n+1}^{-}m_{2}^{+}} = f_{R,M}(m^{+})|_{\mu=\mu_{n+1}^{-}m_{2}^{+}} = \frac{a^{k+1}((a^{n} - 1)\varepsilon - 1)}{a^{n} + a - 1}. \quad (21) \]
Again, equating (20) and (21), we obtain the following estimation for the value of $k$ which means that there is no upper limit for $k$.

The latter implies
\[ k = \left[ \frac{1}{\ln a} \cdot \ln \frac{a^{3}((1 + \varepsilon) - a^{n}(a^{n+1} - an + a^{2}\varepsilon - a + 1)}{a(1 + \varepsilon - \varepsilon a^{n})} \right], \quad (23) \]
where $[\cdot]$ denotes the integer part. Setting in (23) $a = 1.25$, $\varepsilon = 0.72$ and $n = 3$, for example, we compute $k$, which corresponds to the 5-band attractor existing in the region $C_{3+2}$ (see Fig. 2). Or similarly, for $a = 1.1$, $\varepsilon = 1$, $n = 5$ it is $k = 3$, which corresponds to the 8-band attractor existing in the region $C_{3+3}$ (see Fig. 13).

Furthermore, regions $C_{n+k}$ related to the attractors with $n$ bands in $B_{L} \cap Q$ and $k$ bands in $B_{L} \cap Q$ are located between the two bifurcation lines $\nu_{m_{n+1}^{-}m_{2}^{+}}$ and $\nu_{m_{n}^{-}m_{2}^{+}}$. For the latter, the corresponding value of $\mu$ is given by
\[ \mu_{m_{n}^{-}m_{2}^{+}} = -\frac{a^{n} + \varepsilon + 1}{a^{n-2}}. \quad (24) \]

It can be shown that, for a fixed $n$, the right-hand side expression of (22) increases with increasing $\varepsilon$ (see Appendix B). This means that the maximum value of $k$ is achieved at the intersection of $\nu_{m_{n+1}^{-}m_{2}^{+}}$ and $\nu_{m_{n}^{-}m_{2}^{+}}$, that is, at the parameter point $B_{n} = B_{n}(\bar{\varepsilon}_{n}, \bar{\mu}_{n})$ with
\[ \bar{\varepsilon}_{n} = \frac{a - 1}{a^{n+1} - a^{n} - a + 1}, \quad \bar{\mu}_{n} = -\frac{a^{n+2}}{a^{n} - 1}. \quad (25) \]
On the other hand, in between lines $\nu_{m_{n+1}^{-}m_{2}^{+}}$ and $\nu_{m_{n}^{-}m_{2}^{+}}$, the regions related to chaotic attractors with the number of bands $n + k$ and $n + k + 1$ are separated by bifurcation lines $\nu_{m_{n+1}^{-}m_{2}^{+}}$, given as
\[ \mu = \mu_{m_{n+1}^{-}m_{2}^{+}} = -\frac{a^{n}(a^{k+1} - 1)}{a^{n+k-1} - a^{k-1} - a^{n-1} + 1}. \quad (26) \]
It is easy to show that
\[ \lim_{k \to \infty} \mu_{m_{n+1}^{-}m_{2}^{+}} = -\frac{a^{n+2}}{a^{n} - 1} = \bar{\mu}_{n}, \]
which means that there is no upper limit for $k$, and with decreasing $\mu$ (increasing $\varepsilon$) the number of bands in $B_{L} \cap Q$ increases and regions $C_{n+k}$ accumulate to point $B_{n}$. 


5. **Conclusions.** In this paper, we study a simple financial market model in the spirit of [7, 21, 24, 25, 26]. Due to nonlinear interactions between heterogeneous speculators, we find that our model may produce chaotic asset price dynamics. Our analysis furthermore reveals that the destabilizing trading behavior of chartists may spark significant bubbles. Eventually, however, the market entry of stabilizing fundamentalists creates some kind of mean reversion pressure. Since fundamentalists enter bull markets more reluctantly than bear markets, bull markets tend to be more pronounced than bear markets, as observed in actual financial markets.

The analysis of our model furthermore reveals the emergence of a novel bifurcation structure associated with chaotic attractors. To our knowledge, structures of such kind have not been yet described, since they cannot appear in maps with only one discontinuity, which often serve as examples for characterizing transformations of chaotic attractors. Unlike well-known bandcount adding and bandcount incrementing structures, a bandcount accretion bifurcation structure is, in general, not related to homoclinic bifurcations of repelling cycles. Contrariwise, the boundaries of chaoticity regions involved are defined by particular contact bifurcations, at which two distinct critical points of certain ranks coincide. The keystone here is that for the current model setup, there always exists a fixed point \( x_M^* = 0 \) on the middle branch of the map. The bandcount accretion bifurcation structure is located in the area of parameters for which this fixed point is non-homoclinic. This means that inside the absorbing interval there exists an interval \( G_0 \) including \( x_M^* \) that cannot be a part of a chaotic attractor \( Q \), and the latter must have at least two pieces. With changing parameter values, some critical points \( c_i \) and \( d_j \) have a contact, \( c_i = d_j \), due to which the interval \( G_1 \) confined by \( c_i \) and \( d_j \) may become a new gap of the chaotic attractor. This happens as the preimage of \( G_1 \) that previously belonged to \( Q \) either disappears or moves to \( G_0 \). Therefore, the number of bands of \( Q \) increases (together with decreasing of the band width). If additionally the image \( G_2 = f(G_1) \) does not have alternative preimages belonging to \( Q \), the interval \( G_2 \) becomes a gap as well. The total number of newly appearing gaps depends on the maximum number \( k \) such that \( f^k(G_1) \) does not have alternative preimages inside \( Q \) after the contact \( c_i = d_j \).

The bandcount accretion bifurcation structure observed for the current model consists of at least two tiers. Chaoticity regions \( C_{n+1} \) of the first tier correspond to attractors with a single band on the left-hand side of the origin and \( n \geq 1 \) bands on the right. There exists a limit value \( \bar{n} \) depending on slope \( a \), for which an explicit expression is obtained. Between the two successive regions \( C_{n+1} \) and \( C_{(n+1)+1} \) of the first tier there are an infinite number of regions \( C_{n+k} \) of the second tier associated with chaotic attractors that have \( k \), \( k \geq 2 \) bands on the left-hand side of the origin and \( n \) bands on the right. As \( k \) increases, regions \( C_{n+k} \) accumulate to a certain point, the coordinates of which are computed analytically.

In the cases of the bandcount accretion bifurcation structure described in this work, only three out of four critical points are involved in structural changes of chaotic attractors. Discovering whether the fourth critical point can also be engaged and how it would then influence the overall bifurcation structure is considered a challenging task for future research. We also seek to understand if the bandcount accretion bifurcation structure can have any more tiers (third, fourth, etc.) related to chaotic attractors that have even more bands. Since the outer branches of the model’s map may also be associated with the behavior of policymakers that seek to stabilize the market’s dynamics (as indicated in Appendix A), it might also be
interesting to explore how policymakers may manipulate them such that markets become more efficient. See [14, 15] for recent effort in that direction.

**Appendix A. Alternative derivation for map (11).** In this appendix, we provide an alternative derivation for map (11). Let us assume that chartists still regard a market as a bull (bear) market if the price of the asset is above (below) its fundamental value. Representing the asset’s fundamental value by \( F \), we may express their orders as

\[
D_t^C = c (P_t - F),
\]

with \( c > 0 \). As usual, fundamentalists believe that the price of the asset will always return towards its fundamental value, although they only enter the market if the asset price is sufficiently distant from its fundamental value. Let \( z^- \) and \( z^+ \) stand for their market entry levels in the bear and bull market, respectively. We may then formalize the orders of fundamentalists as

\[
D_t^F = \begin{cases} 
-f & \text{for } P_t - F \geq z^+, \\
0 & \text{for } -z^- < P_t - F < z^+, \\
f & \text{for } P_t - F \leq -z^-.
\end{cases}
\]

with \( f > 0 \). Since there are only two types of speculators, the market maker’s price adjustment turns into

\[
P_{t+1} = P_t + a (D_t^C + D_t^F)
\]

with \( a = 1 \), implying that the model’s law of motion takes the form

\[
P_{t+1} = \begin{cases} 
P_t + c (P_t - F) - f & \text{for } P_t - F \geq z^+, \\
P_t + c (P_t - F) & \text{for } -z^- < P_t - F < z^+, \\
P_t + c (P_t - F) + f & \text{for } P_t - F \leq -z^-.
\end{cases}
\]

Finally, setting \( x_t = P_t - F \) reveals that

\[
x_{t+1} = \begin{cases} 
a x_t + \mu & \text{for } x_t \geq 1 + \varepsilon, \\
a x_t & \text{for } -1 < x_t < 1 + \varepsilon, \\
a x_t - \mu & \text{for } x_t \leq -1,
\end{cases}
\]

where \( a = 1 + c > 1, \mu = -f < 0, z^- = 1 \) and \( z^+ = 1 + \varepsilon > 1 \), formally identical to map (11) studied in the main body of the paper.

Due to its simplicity, map (11) may also—in a stylized way—capture the behavior of other real-life dynamical systems. To see this point, recall that the (inner) slope of map (11) is larger than one, implying that the model’s steady state is repelling. However, its dynamics does not explode. When the system is sufficiently distant from its steady state, an additional stabilizing force becomes active. In our financial market model, the destabilizing trading behavior of chartists is eventually stopped by the stabilizing trading behavior of fundamentalists. In a business cycle model, the destabilizing sentiment of consumers and/or investors, as modeled in [35], may be countered by a government that alters its expenditures. In a population model, a regulatory authority may start a harvesting campaign when the population gets out of bounds and restock the population when it becomes too rare, as discussed in [17, 29].
Appendix B. On the right-hand side of expression (22). In this appendix, we show that, for a fixed (admissible) $n$, the right-hand side of (22) grows as $\varepsilon$ increases. Let us denote the corresponding expression as
\[
\psi(a, n) := a^{2n+1} - a^{2n} - 2a^{n+1} + 2a^n - a^3 + a^2 + a - 1.
\]
After a simple transformation, we have
\[
\psi(a, n) = (a-1)((a^n-1)^2 - a^2).
\]
Since $a > 1$ and $n \leq \bar{n}$, given by (17), there is $(a^n - 1)^2 \leq a^2$ and, consequently, $\psi(a, n) \leq 0$. This implies $d\phi(\varepsilon)/d\varepsilon > 0$ for any $\varepsilon$, and hence function $\psi$ is increasing in the whole domain of its definition.

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