Non-radial solutions to a bi-harmonic equation with negative exponent

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Abstract
We prove the existence of smooth non-radial entire solution to
\[ \Delta^2 u + u^{-q} = 0 \quad \text{in } \mathbb{R}^3, \quad u > 0, \]
for \( q > 1 \). This answers an open question raised by McKenna and Reichel (Electron J Differ Equ 37:1–3, 2003).

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1 Introduction

We consider the following bi-harmonic equation with negative exponent
\[ \Delta^2 u + u^{-q} = 0 \quad \text{in } \mathbb{R}^3, \quad u > 0, \quad (1) \]
where \( q > 0 \).

For \( q = 7 \), every positive smooth solution to (1) corresponds to a conformal metric on \( \mathbb{R}^3 \) with constant \( Q \)-curvature. Let us recall that the Paneitz operator on a three dimension manifold \((M, g)\) is defined by (see [1,5,22])
\[ P_g = (-\Delta_g)^2 + \delta \left( \frac{5}{4} R_g g - 4 Ric_g \right) d - \frac{1}{2} Q_g, \]

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where the $Q$-curvature is given by

$$ Q_g = -2 |\text{Ric}_g|^2 + \frac{23}{32} R_g^2 - \frac{1}{4} \Delta_g R_g. $$

Here, $\Delta_g$ denotes the Laplace–Beltrami operator, $\delta$ the divergence, $d$ the differential, $R_g$ the scalar curvature, and $\text{Ric}_g$ the Ricci tensor of the metric $g$. Under a conformal change of metrics $g_u = u^{-4} g$ with $u > 0$, the Paneitz operator enjoys the following conformal covariance property

$$ P_{g_u} w = u^7 P(u w). $$

Moreover, the $Q$-curvatures are related by the equation

$$ P_{g_u} u = -\frac{1}{2} Q_{g_u} u^{-7}. $$

When $(M, g) = (\mathbb{R}^3, |dx|^2)$ ($|dx|^2$ is the Euclidean metric on $\mathbb{R}^3$), the above equation with $Q_{g_u} \equiv 2$ reduces to (1) with $q = 7$.

In the recent past, radial solutions to Eq. (1) have been studied by many authors, especially the existence and asymptotic behavior:

**Theorem A** [5,6,8,11,16,21]

(i) There is no entire solution to (1) for $0 < q \leq 1$.

(ii) If $u$ has exactly linear growth at infinity, that is,

$$ \lim_{|x| \to +\infty} \frac{u(x)}{|x|} = C > 0, $$

then $q > 3$. Moreover, for $q = 7$, $u$ is given by $u(x) = \sqrt{1/15} + |x|^2$, and is unique up to dilation and translations.

(iii) For $q > 3$ there exists a radial solution with exactly linear growth.

(iv) For $q > 1$ there exists a radial solution with exactly quadratic growth, that is,

$$ \lim_{|x| \to +\infty} \frac{u(x)}{|x|^2} = C > 0. $$

(v) For $1 < q < 3$ there exists a radial solution $u$ such that $r^{-\frac{4}{q+1}} u(r) \to C(q) > 0$ as $r \to \infty$ (the constant $C(q)$ is explicitly known).

(vi) For $q = 3$ there exists a radial solution $u$ such that $r^{-1} (\log r)^{-\frac{1}{2}} u(r) \to 2^\frac{1}{2}$ as $r \to \infty$.

It has been shown by Choi–Xu [5] that if $u$ is a solution to (1) with $q > 4$, and $u$ has exact linear growth at infinity then $u$ satisfies the integral equation

$$ u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x-y|}{u^q(y)} dy + \gamma, \quad (2) $$

for some $\gamma \in \mathbb{R}$, and $\gamma = 0$ if and only if $q = 7$. In fact, every positive solution $u$ to

$$ (-\Delta)^n u + u^{-4n-1} = 0 \quad \text{in } \mathbb{R}^{2n-1}, \quad n \geq 2, $$

with exact linear growth at infinity satisfies

$$ u(x) = c_n \int_{\mathbb{R}^{2n-1}} \frac{|x-y|}{u^{4n-1}(y)} dy, $$

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where \( c_n \) is a dimensional constant, see [7,17]. For the classification of solutions to the above integral equation we refer the reader to [12,21].

In [16] McKenna–Reichel proved the existence of non-radial solution to

\[
\Delta^2 w + w^{-q} = 0 \quad \text{in } \mathbb{R}^n, \quad w > 0
\]  

for \( n \geq 4 \). This was a simple consequence of their existence results to (3) in lower dimension. More precisely, if \( u \) is a radial solution to (3) with \( n \geq 3 \) then \( w(x) := u(x') \) is a non-radial solution to \( \Delta^2 w + w^{-q} = 0 \) in \( \mathbb{R}^{n+1} \), where \( x = (x',x'') \in \mathbb{R}^n \times \mathbb{R} \). Then they asked whether in \( \mathbb{R}^3 \) non-radial positive entire solution exist. (See [Open Questions (1), [16]].)

We answer this question affirmatively. (See Theorem 1.2 below.) In fact we prove the following theorems.

**Theorem 1.1** Let \( u \) be a solution to (1) for some \( q > 1 \). Assume that

\[
\beta := \frac{1}{8\pi} \int_{\mathbb{R}^3} u^{-q} \, dx < +\infty.
\]

Then, up to a rotation and translation, we have

\[
u(x) = (\beta + o(1))|x| + \sum_{i \in I_1} a_i x_i^2 + \sum_{i \in I_2} b_i x_i + c, \quad o(1) \xrightarrow{|x| \to \infty} 0,
\]

where

\[ I_1, I_2 \subset \{1, 2, 3\}, \quad I_1 \cap I_2 = \emptyset, \quad a_i > 0 \text{ for } i \in I_1, \quad |b_i| < \beta \text{ for } i \in I_2, \quad c > 0.\]

**Theorem 1.2** Let \( q > 1 \). Then for every \( 0 < \kappa_1 < \kappa_2 \) there exists a non-radial solution \( u \) to (1) such that

\[
\lim \inf_{|x| \to \infty} \frac{u(x)}{|x|^2} = \kappa_1 \quad \text{and} \quad \lim \sup_{|x| \to \infty} \frac{u(x)}{|x|^2} = \kappa_2.
\]

**Theorem 1.3** Let \( q > 7 \). Then for every \( \kappa > 0 \) there exists a non-radial solution \( u \) to (1) such that

\[
\lim \inf_{|x| \to \infty} \frac{u(x)}{|x|} \in (0, \infty) \quad \text{and} \quad \lim \sup_{|x| \to \infty} \frac{u(x)}{|x|^2} = \kappa.
\]

The non-radial solutions constructed in Theorem 1.2 also satisfy the following integral condition

\[
\int_{\mathbb{R}^3} u^{-q} \, dx < +\infty,
\]

for \( q > \frac{3}{2} \). Note that McKenna-Reichel’s non-radial example has infinite \( L^1 \) bound: \( \int_{\mathbb{R}^{n+1}} w^{-q} \, dx = +\infty \).

The existence of infinitely many entire non-radial solutions with different growth rates for the conformally invariant equation \( \Delta^2 u + u^{-7} = 0 \) in \( \mathbb{R}^3 \) is in striking contrast to other conformally invariant equations \(-\Delta u = u^{\frac{n+2}{n-2}} \) in \( \mathbb{R}^n \), \( n \geq 3 \) and \( (-\Delta)^m u = u^{\frac{n+2m}{n-2m}} \) in \( \mathbb{R}^n \), \( n > 2m \). In both cases all solutions are radially symmetric with respect to some point in \( \mathbb{R}^n \), see [2,4,13,19].

Our motivation in the proof of Theorems 1.2–1.3 come from a similar phenomenon exhibited in the following equation

\[
(-\Delta)^{\frac{n}{2}} u = e^{nu} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{nu} \, dx < +\infty.
\]
It is well-known that every solution to (8) with $n \geq 4$ is bounded from above, and it has the following asymptotic behavior at infinity:

$$u(x) = -c_0 + o(1)) \log |x| + P(x), \quad o(1) \xrightarrow{|x| \to \infty} 0,$$

where $c_0$ is a positive constant, $P$ is a polynomial of degree at most $n - 1$, and $P$ is bounded from above. Moreover, for $n \geq 4$, problem (8) admits non-radial entire solutions having the above asymptotic behavior, see [3,9,10,13–15,20] and the references therein.

In the remaining part of the paper we prove Theorems 1.1–1.3 respectively. We also give a new proof of (iii)–(iv) of Theorem A, see Sect. 2.1.

2 Proof of the theorems

We begin by proving Theorem 1.1.

**Proof of Theorem 1.1** Let $u$ be a solution to (1), (4). We set

$$v(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{u^q(y)} dy, \quad w := u - v. \quad (9)$$

Fixing $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ so that

$$\int_{B_R} \frac{dx}{u^q(x)} < 8\pi \varepsilon,$$

one gets

$$v(x) \geq \frac{1}{8\pi} \int_{B_R} \frac{|x| - 2|y|}{u^q(y)} dy - \frac{1}{8\pi} \int_{B_R} \frac{|x|}{u^q(y)} dy \geq (\beta - 2\varepsilon)|x| - C(R).$$

Using that $||x - y| - |y|| \leq |x|$, from (9), we obtain

$$|v(x)| \leq \beta |x| \text{ in } \mathbb{R}^3.$$

Combining these estimates we deduce that

$$\lim_{|x| \to \infty} \frac{v(x)}{|x|} = \beta.$$

It follows that $w$ satisfies

$$\Delta^2 w = 0 \text{ in } \mathbb{R}^3, \quad w(x) \geq -\beta |x|,$$

and hence, $w$ is a polynomial of degree at most 2, see for instance [14, Theorem 5]. Indeed, up to a rotation and translation, we can write

$$w(x) = \sum_{i \in \mathcal{I}_1} a_i x_i^2 + \sum_{i \in \mathcal{I}_2} b_i x_i + c_0,$$

where $\mathcal{I}_1, \mathcal{I}_2$ are two disjoint (possibly empty) subsets of $\{1, 2, 3\}$, $a_i \neq 0$ for $i \in \mathcal{I}_1$, $b_i \neq 0$ for $i \in \mathcal{I}_2$ and $c_0 \in \mathbb{R}$. Therefore, up to a rotation and translation, we have

$$u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{u^q(y)} dy + \sum_{i \in \mathcal{I}_1} a_i x_i^2 + \sum_{i \in \mathcal{I}_2} b_i x_i + c.$$

Now $u > 0$ and $|v(x)| \leq \beta |x|$ lead to $a_i > 0$ for $i \in \mathcal{I}_1$, $|b_i| \leq \beta$ for $i \in \mathcal{I}_2$ and $c = u(0) > 0$. 

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In order to prove that $|b_1| < \beta$ we assume by contradiction that $|b_{i_0}| = \beta$ for some $i_0 \in I_2$. Up to relabelling we may assume that $i_0 = 1$. Then

$$ u(x) \leq C + |b_1 x_1| + b_1 x_1 \text{ on } C := \{x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 : |\bar{x}| \leq 1\}, $$

a contradiction to (4).

We conclude the proof. \square

Now we move on to the existence results. We look for solutions to (1) of the form $u = v + P$ where $P$ is a polynomial of degree 2. Notice that $u = v + P$ satisfies (1) if and only if $v$ satisfies

$$ \Delta^2 v = -(v + P)^{-q}, \quad v + P > 0. \quad (10) $$

In particular, if $P \geq 0$, and $v$ satisfies the integral equation

$$ v(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{(P(y) + v(y))^{q}} dy, \quad (11) $$

then $v$ satisfies (10). Thus, we only need to find solutions to (11) (or a variant of it), and we shall do that by a fixed point argument. Let us first define the spaces on which we shall work:

$$ X := \{v \in C^0(\mathbb{R}^3) : \|v\|_X < \infty\}, \quad \|v\|_X := \sup_{x \in \mathbb{R}^3} \frac{|v(x)|}{1 + |x|}, $$

$$ X_{ev} := \{v \in X : v(x) = v(-x) \forall x \in \mathbb{R}^3\}, \quad \|v\|_{X_{ev}} := \|v\|_X, $$

$$ X_{rad} := \{v \in X : v \text{ is radially symmetric}\}, \quad \|v\|_{X_{rad}} := \|v\|_X. $$

The following proposition is crucial in proving Theorem 1.2.

**Proposition 2.1** Let $P$ be a positive continuous function on $\mathbb{R}^3$ such that $P(-x) = P(x)$ and for some $q > 0$

$$ \int_{\mathbb{R}^3} \frac{|x|}{(P(x))^{q}} dx < \infty. $$

Then there exists a function $v \in X_{ev}$ satisfying $\min_{\mathbb{R}^3} v = v(0) = 0,$

$$ v(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(P(y) + v(y))^{q}} dy, \quad (12) $$

and

$$ \lim_{|x| \to \infty} \frac{v(x)}{|x|} = \alpha_{P,v} := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{dy}{(P(y) + v(y))^{q}}. $$

Moreover, if $P$ is radially symmetric then there exists a solution to (12) in $X_{rad}$.

**Proof** Let us define an operator $T : X_{ev} \to X_{ev}, v \mapsto \tilde{v}$, (In case $P$ is radial we restrict the operator $T$ on $X_{rad}$.) Notice that $T(X_{rad}) \subset X_{rad}$.) where

$$ \tilde{v}(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(P(y) + |v(y)|)^{q}} dy. \quad (13) $$

We note that $\tilde{v} \in C^3_{rad}(\mathbb{R}^3)$ for every $0 \leq \alpha < 1$. We prove the proposition in few steps.

**Step 1** $T$ is compact.
Using that $||x - y| - |y|| \leq |x|$ we bound

$$\|\bar{v}\|_X \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{(P(y))^q} dy \leq C \quad \text{for every } v \in X.$$  \hfill (14)

Differentiating under the integral sign one gets

$$|\nabla \bar{v}(x)| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|y|^i}{(P(y) + |v_k(y)|)^q} dy \xrightarrow{k \to \infty} c_i.$$  

We rewrite (13) (with \(v = v_k\) and \(\bar{v} = \bar{v}_k\)) as

$$\bar{v}_k(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{(P(y) + |v_k(y)|)^q} dy + \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |x|}{(P(y) + |v_k(y)|)^q} dy =: I_{1,k}(x) + I_{2,k}(x).$$

It follows that

$$I_{1,k}(x) \to c_0|x| - c_1 \quad \text{in } X \quad \text{as } k \to \infty.$$  

Using that $||x - y| - |x|| \leq |y|$ we bound

$$|I_{2,k}(x)| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|y|}{(P(y))^q} dy \leq C.$$  

This implies that

$$\lim_{R \to \infty} \sup_k \sup_{x \in \mathbb{R}^3 \setminus B_R} \frac{I_{2,k}(x)}{1 + |x|} = 0.$$  

Since

$$\sup_k \sup_{x \in \mathbb{R}^3} |\nabla I_{2,k}(x)| < \infty,$$  

up to a subsequence,

$$I_{2,k} \to I \quad \text{in } X_{ev},$$

for some \(I \in X_{ev}\). This proves Step 1 as \(T\) is continuous.

**Step 2** \(T\) has a fixed point in \(X_{ev}\).

It follows from (14) that there exists \(M > 0\) such that \(T(X_{ev}) \subset B_M \subset X_{ev}\). In particular, \(T(\mathcal{B}_M) \subset B_M\). Hence, by Schauder fixed point theorem there exists a fixed point of \(T\) in \(B_M\).

**Step 3** For every \(v \in X_{ev}\) we have \(\lim_{|x| \to \infty} \frac{\bar{v}(x)}{|x|} = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{dy}{(P(y) + |v(y)|)^q} =: \alpha(P, v)\).

Step 3 follows from

$$|\bar{v}(x) - \alpha(P, v)|x|| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{||x - y| - |y| - |x||}{(P(y) + |v(y)|)^q} dy \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|y|}{(P(y))^q} dy \leq C.$$  

**Step 4** If \(v\) is a fixed point of \(T\) then \(v \geq 0\).
Differentiating under the integral sign, from (13) one can show that the hessian $D^2 \bar{v}$ is strictly positive definite, and hence $\bar{v}$ is strictly convex. Moreover, using that $(P + |v|)$ is an even function, one obtains $\nabla \bar{v}(0) = 0$. This leads to

$$\min_{x \in \mathbb{R}^3} \bar{v}(x) = \bar{v}(0) = 0.$$  

We conclude the proposition. \hfill $\square$

In the same spirit one can prove the following proposition.

**Proposition 2.2** Let $P$ be a continuous positive even function on $\mathbb{R}^3$ such that for some $q > 0$

$$\int_{\mathbb{R}^3} \frac{|x|}{(P(x))^q} dx < \infty.$$  

Then there exists a positive function $v \in X_{ev}$ satisfying

$$v(x) := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{(P(y) + v(y))^q} dy, \quad \min_{x \in \mathbb{R}^3} v = v(0).$$  \hspace{1cm} (15)

**Proof of Theorem 1.2** Let $q > 1$ and $0 < \kappa_1 < \kappa_2$ be fixed. For every $\varepsilon > 0$ let $v_\varepsilon \in X_{ev}$ be a solution of (12), that is,

$$v_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(P_\varepsilon(y) + v_\varepsilon(y))^q} dy,$$  \hspace{1cm} (16)

where

$$P_\varepsilon(x) := 1 + \kappa_1 x_1^2 + \kappa_2 (x_2^2 + x_3^2) + \varepsilon |x|^4, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$  

We claim that for every multi-index $\beta \in \mathbb{N}^3$ with $|\beta| = 2$

$$|D^\beta v_\varepsilon(x)| \leq C \quad \text{on } B_2 \quad \text{and} \quad |D^\beta v_\varepsilon(x)| \leq C f_q(x) \quad \text{on } B_2^c,$$  \hspace{1cm} (17)

where

$$f_q(x) := \begin{cases} |x|^{-1} & \text{if } q > 3/2 \\ |x|^{-1} \log |x| & \text{if } q = 3/2 \\ |x|^{2-2q} & \text{if } q < 3/2. \end{cases}$$

For $|\beta| = 2$, differentiating under the integral sign, from (16), we obtain

$$|D^\beta v_\varepsilon(x)| \leq C \int_{\mathbb{R}^3} \frac{1}{|x - y|} \frac{dy}{(P_\varepsilon(y) + v_\varepsilon(y))^q} \leq C \int_{\mathbb{R}^3} \frac{1}{|x - y|} \frac{dy}{(1 + \kappa_1 |y|^2)^q}$$

$$= C \sum_{i=1}^{3} I_i(x),$$

where

$$I_i(x) := \int_{A_i} \frac{dy}{|x - y|} \frac{1}{(1 + \kappa_1 |y|^2)^q}, \quad A_1 := B_{|x|}, \quad A_2 := B_{2|x|} \setminus A_1, \quad A_3 := \mathbb{R}^3 \setminus B_{2|x|}.$$
Since \( q > 1 \) we have \( |D^\beta v_\varepsilon| \leq C \) on \( B_2 \). For \( |x| \geq 2 \) we bound
\[
I_1(x) \leq \frac{2}{|x|} \left[ \int_{A_1} \frac{dy}{(1 + \kappa_1|y|^2)^q} \right] \leq Cf_q(x),
\]
\[
I_2(x) \leq \frac{C}{|x|^{2q}} \left[ \int_{A_2} \frac{dy}{|x - y|} \right] \leq \frac{C}{|x|^{2q}} \left[ \int_{|y| \leq 3|x|} \frac{dy}{|y|} \right] \leq C|x|^{-2q},
\]
\[
I_3(x) \leq 2 \left[ \int_{A_3} \frac{dy}{|y|(1 + \kappa_1|y|^2)^q} \right] \leq C|x|^{-2q}.
\]
This proves (17). Since \( v_\varepsilon(0) = |\nabla v_\varepsilon(0)| = 0 \), by (17), we have that \( (|\nabla v_\varepsilon|) \) is bounded in \( C^0_{loc}(\mathbb{R}^3) \), and that
\[
v_\varepsilon(x) \leq C \begin{cases} 
(1 + |x|) \log(2 + |x|) & \text{if } q > 3/2 \\
(1 + |x|)(1 + |x|)^2 & \text{if } q = 3/2 \\
(1 + |x|)^{4-2q} & \text{if } q < 3/2.
\end{cases}
\]

Thus \( (v_\varepsilon) \) is bounded in \( C^2_{loc}(\mathbb{R}^3) \). In fact, as \( P_\varepsilon \) is smooth, using the integral Eq. (16) one can show that \( (v_\varepsilon) \) is bounded in \( C^5_{loc}(\mathbb{R}^3) \). Therefore, for some \( \varepsilon_k \downarrow 0 \) we must have \( v_{\varepsilon_k} \to v \) in \( C^4_{loc}(\mathbb{R}^3) \) for some \( v \in C^4(\mathbb{R}^3) \), where \( v \) satisfies
\[
\Delta^2 v = -\frac{1}{(v + P_0)^q} \quad \text{in } \mathbb{R}^3, \quad v \geq 0 \quad \text{in } \mathbb{R}^3, \quad P_0(x) := 1 + \kappa_1 x_1^2 + \kappa_2 (x_2^2 + x_3^2).
\]

Hence, \( u = v + P_0 \) is a solution to (1). Moreover, as \( v \) satisfies (18), we have
\[
\liminf_{|x| \to \infty} \frac{u(x)}{|x|^2} = \liminf_{|x| \to \infty} \frac{P_0(x)}{|x|^2} = \kappa_1, \quad \limsup_{|x| \to \infty} \frac{u(x)}{|x|^2} = \limsup_{|x| \to \infty} \frac{P_0(x)}{|x|^2} = \kappa_2.
\]

This completes the proof.

\textbf{Proof of Theorem 1.3} Let \( q > 7 \) be fixed. Then for every \( \varepsilon > 0 \) there exists a positive solution \( v_\varepsilon \) to (15) with
\[
P(x) = P_\varepsilon(x) := 1 + \varepsilon x_1^2 + \kappa (x_2^2 + x_3^2).
\]
Setting \( u_\varepsilon := v_\varepsilon + P_\varepsilon \) one gets
\[
u_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u_\varepsilon^q(y)} \, dy + P_\varepsilon(x), \quad \min_{\mathbb{R}^3} u_\varepsilon = u_\varepsilon(0).
\]

Setting \( u_\varepsilon := v_\varepsilon + P_\varepsilon \) one gets
\[
u_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u_\varepsilon^q(y)} \, dy + P_\varepsilon(x), \quad \min_{\mathbb{R}^3} u_\varepsilon = u_\varepsilon(0).
\]

Since \( c_q := \frac{1}{2} - \frac{3}{q^2 q} > 0 \) for \( q > 7 \), from (22), one obtains
\[
0 = c_q \int_{\mathbb{R}^3} \frac{1}{u_\varepsilon^{q-1}(x)} \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \frac{2x \cdot \nabla P_\varepsilon(x) - P_\varepsilon(x)}{u^q(x)} \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} \frac{3 P_\varepsilon(x) + 2c_q u_\varepsilon(x) - 4}{u_\varepsilon^q(x)} \, dx,
\]
which implies that \( 2c_q u_\varepsilon(0) < 4 \), that is, \( u_\varepsilon(0) \leq C \). Therefore, by (19)
\[
\frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|y|}{u_\varepsilon^q(y)} \, dy = u_\varepsilon(0) - 1 \leq C.
\]
Hence, differentiating under the integral sign, from (19)

\[ |\nabla (u_\epsilon(x) - P_\epsilon(x))| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{dy}{u_\epsilon^q(y)} \leq C. \]

Thus, \((u_\epsilon)_{0<\epsilon\leq 1}\) is bounded in \(C^1_{loc}(\mathbb{R}^3)\). This yields

\[ u_\epsilon(x) \geq \frac{1}{8\pi} \int_{B_1} \frac{|x - y|}{u_\epsilon^q(y)} dy \geq \delta |x| \quad \text{for} \quad |x| \geq 2, \]

for some \(\delta > 0\). Using this, and recalling that \(q > 4\), we deduce

\[ \lim_{R \to \infty} \sup_{0<\epsilon\leq 1} \int_{|y| \geq R} \frac{|y|}{u_\epsilon^q(y)} dy = 0. \]

Therefore, for some \(\epsilon_k \downarrow 0\), we have \(u_{\epsilon_k} \to u\), where \(u\) satisfies

\[ u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u^q(y)} dy + 1 + \kappa(x_2^2 + x_3^2). \]

We conclude the proof. \(\square\)

2.1 A new proof of (iii)–(iv) of Theorem A

Proof of (iii) Let \(q > 3\) be fixed. Then by Proposition 2.1, for every \(\epsilon > 0\), there exists a radial function \(u_\epsilon\) satisfying

\[ u_\epsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u_\epsilon^q(y)} dy + 1 + \epsilon |x|^2, \quad \min_{\mathbb{R}^3} u_\epsilon = u_\epsilon(0) = 1. \]

Since \(u_\epsilon\) is radially symmetric, one has (see Eq. (3.3) in [5])

\[ u_\epsilon(r) \geq \delta (1 + r^4)^{\frac{1}{q+1}}, \]

for some \(\delta > 0\). Therefore, as \(q > 3\)

\[ \int_{\mathbb{R}^3} \frac{dx}{u_\epsilon^q(x)} \leq C \int_{\mathbb{R}^3} \frac{dx}{(1 + |x|^4)^{\frac{q}{q+1}}} \leq C, \]

which gives

\[ |\nabla u_\epsilon(x)| \leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{u_\epsilon^q(y)} dy + 2\epsilon |x| \leq C + 2\epsilon |x|. \]

As \(u_\epsilon(0) = 1\), one would get

\[ u_\epsilon(x) \leq 1 + C|x| + C\epsilon |x|^2. \]

Thus, the family \((u_\epsilon)_{0<\epsilon\leq 1}\) is bounded in \(C^1_{loc}(\mathbb{R}^3)\). Hence, for some \(\epsilon_k \downarrow 0\) we have \(u_{\epsilon_k} \to u\) where \(u\) satisfies

\[ u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{u^q(y)} dy + 1, \quad \min_{\mathbb{R}^3} u = u(0) = 1. \]

Finally, as before, we have

\[ \lim_{|x| \to \infty} \frac{u(x)}{|x|} = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{dy}{u^q(y)}. \]
Since \( q > 1 \) be fixed. Then by Proposition 2.1, for every \( \varepsilon > 0 \), there exists a non-negative radial function \( v_\varepsilon \) satisfying
\[
v_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(1 + |y|^2 + \varepsilon |y|^4 + v_\varepsilon(y))^q} dy.
\]
The rest of the proof is similar to that of Theorem 1.2.

\[ \square \]

Proof of (iv) Let \( q > 1 \) be fixed. Then by Proposition 2.1, for every \( \varepsilon > 0 \), there exists a non-negative radial function \( v_\varepsilon \) satisfying
\[
v_\varepsilon(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y| - |y|}{(1 + |y|^2 + \varepsilon |y|^4 + v_\varepsilon(y))^q} dy.
\]
The rest of the proof is similar to that of Theorem 1.2.

\[ \square \]

In the spirit of [5, Lemma 4.9] we prove the following Pohozaev type identity (see [18]).

Lemma 2.3 (Pohozaev identity) Let \( u \) be a positive solution to
\[
u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u^q(y)} dy + P(x), \tag{21}
\]
for some non-negative polynomial \( P \) of degree at most 2 and \( q > 4 \).

Then
\[
\left( \frac{1}{2} - \frac{3}{q - 1} \right) \int_{\mathbb{R}^3} \frac{1}{u^{q-1}(x)} dx + \frac{1}{2} \int_{\mathbb{R}^3} \frac{2x \cdot \nabla P(x) - P(x)}{u^q(x)} dx = 0 \tag{22}
\]

Proof Differentiating under the integral sign, from (21)
\[
x \cdot \nabla u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{x \cdot (x - y)}{|x - y|} \frac{1}{u^q(y)} dy + x \cdot \nabla P(x).
\]

Multiplying the above identity by \( u^{-q}(x) \) and integrating on \( B_R \)
\[
\int_{B_R} \frac{x \cdot \nabla u(x)}{u^q(x)} dx = \frac{1}{8\pi} \int_{B_R} \int_{\mathbb{R}^3} \frac{x \cdot (x - y)}{|x - y|} \frac{1}{u^q(y)u^q(y)} dy dx + \int_{B_R} \frac{x \cdot \nabla P(x)}{u^q(x)} dx.
\]

Integration by parts yields
\[
\int_{B_R} \frac{x \cdot \nabla u(x)}{u^q(x)} dx = \frac{1}{1 - q} \int_{B_R} x \cdot \nabla (u^{1-q}(x)) dx
\]
\[= -\frac{3}{1 - q} \int_{B_R} u^{1-q} dx + \frac{R}{1 - q} \int_{\partial B_R} u^{1-q} d\sigma.
\]

Since \( q > 4 \) and \( u(x) \geq \delta |x| \) for some \( \delta > 0 \) and \( |x| \) large
\[
\lim_{R \to \infty} R \int_{\partial B_R} u^{1-q} d\sigma = 0.
\]

Writing \( x = \frac{1}{2} ((x + y) + (x - y)) \), and setting
\[
F(x, y) := \frac{(x + y) \cdot (x - y)}{|x - y|} \frac{1}{u^q(x)u^q(y)}
\]
we get
\[
\frac{1}{8\pi} \int_{B_R} \int_{\mathbb{R}^3} \frac{x \cdot (x - y)}{|x - y|} \frac{1}{u^q(x)u^q(y)} dy dx
\]
\[= \frac{1}{2} \int_{B_R} \frac{1}{u^q(x)} \left( \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|x - y|}{u^q(y)} dy \right) dx + \frac{1}{16\pi} \int_{B_R} \int_{\mathbb{R}^3} F(x, y) dy dx
\]
\[= \frac{1}{2} \int_{B_R} \frac{1}{u^q(x)} (u(x) - P(x)) dx + \frac{1}{16\pi} \int_{B_R} \int_{\mathbb{R}^3} F(x, y) dy dx.
\]
Notice that \( F(x, y) = -F(y, x) \). Hence,
\[ \int_{B_R} \int_{B_R} F(x, y) dy dx = 0, \]

and

\[ \lim_{R \to \infty} \int_{B_R} \int_{B_R} F(x, y) dy dx = \lim_{R \to \infty} \int_{B_R} \int_{B_R} F(x, y) dy dx = 0, \]

where the last equality follows from \(|x| u^{-q}(x) \in L^1(\mathbb{R}^3)\). Combining these estimates and taking \(R \to \infty\) in (23) one gets (22).

\[ \square \]

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