1 Introduction

A census taker knocks on a door. A mother answers.

The census taker says, “I need to know the number of children you have, and their ages.” The woman responds in puzzle-ese, “I have three daughters, the product of their ages is 36, and the sum of their ages is equal to the house number next door.”

The census taker, who never wastes questions, computes for a while and then asks, “Does your oldest daughter love dogs?” The mother answers affirmatively. The census taker says, “Thank you. I now know the ages.”

What are the ages of the children?

With seemingly insufficient information, such number-theoretic puzzles belong to what is now known as the census-taker problem.

To someone encountering the problem for the first time, the information transpired during the conversation is not sufficient to determine the ages. Aside from this initial reaction, many questions arise. How can one determine the ages from the given product and the house number next door? What does the oldest daughter’s loving of dogs have to do with the problem? To solve the problem, one simply puts his feet on the shoes of the census taker. The readers do not know the house number next door, but the census taker does. Moreover, once one lists down the eight triples of positive integers whose product is 36, the answers to these questions will become clearer. This list is presented in Table 1, where each triple consists of the ages of the daughters. Looking at the table, there are two triples whose sums are equal to 13. Because the census taker never wastes a question, this common sum is the reason why he asks
another question, and this sum is also the house number next door. The affirmative answer of the mother confirms that she has an oldest daughter, which nails down the triple \( \{9, 2, 2\} \) to give the ages of the girls.

| Triple  | Sum | Triple  | Sum |
|---------|-----|---------|-----|
| \{36, 1, 1\} | 38  | \{9, 2, 2\} | 13  |
| \{18, 2, 1\} | 21  | \{6, 6, 1\} | 13  |
| \{12, 3, 1\} | 16  | \{6, 3, 2\} | 11  |
| \{9, 4, 1\} | 14  | \{4, 3, 3\} | 10  |

Table 1: Triples whose product is 36.

The last question of the census taker is important. Its answer decides which of the triples \( \{9, 2, 2\} \) and \( \{6, 6, 1\} \) gives the ages of the daughters. What if the mother’s answer to the last question is negative? Does it point to the triple \( \{6, 6, 1\} \)? Not necessarily. It depends on how the mother phrases her negative answer. If she simply says “No,” this means that she has an oldest daughter who does not love dogs, which points again to the triple \( \{9, 2, 2\} \). However, if the mother says something like “No, but my youngest daughter does,” then this answer confirms that she does have a youngest daughter, which points to the triple \( \{6, 6, 1\} \). Asking the last question then only tells us that the census taker is choosing between two triples, but the mother’s appropriate answer settles the dilemma.

According to [1], the census-taker problem originated during World War II. In literature, it has appeared in different stories in a puzzle-ese style. As a result, many number-theoretic problems with seemingly insufficient information have been proposed in some journals. See [4] and [5].

The current work revisits the results in [3], and presents the census-taker problem as a motivation to introduce the beautiful theory of numbers.

## 2 The Census-Taker Numbers

Table [1] depicts a special property of the number 36. Among the distinct unordered triples of positive integers whose product is 36, there are exactly two triples with equal sums. With such special property, the number 36 is called a census-taker number, and 36 is the smallest such positive integer.

More formally, a positive integer \( N \) is said to be a census-taker number (CTN) if there is exactly one pair \( \{\{a, b, c\}, \{d, e, f\}\} \) of different (unordered) triples of positive integers such that

\[
abc = def = N
\]  

(1)
and
\[ a + b + c = d + e + f. \]  \hfill (2)

We shall refer equation (1) as the \textit{product condition}, equation (2) as the \textit{sum condition}, the common sum as the \textit{magic sum}, and the triples \( \{a, b, c\} \) and \( \{d, e, f\} \) as the \textit{mysterious triples} of the CTN.

It can be easily checked that there are no one-digit CTNs. Moreover, Table 2 lists down all the two-digit CTNs. For the curious minds, there are twenty-nine three-digit CTNs.

| CTN | Magic Sum | \( \{a, b, c\} \) | \( \{d, e, f\} \) |
|-----|-----------|-------------------|-------------------|
| 36  | 13        | \{9, 2, 2\}       | \{6, 6, 1\}       |
| 40  | 14        | \{10, 2, 2\}      | \{8, 5, 1\}       |
| 72  | 14        | \{8, 3, 3\}       | \{6, 6, 2\}       |
| 96  | 21        | \{16, 3, 2\}      | \{12, 8, 1\}      |

Table 2: The two-digit CTNs, and their magic sums and mysterious triples.

There is an interesting geometric interpretation of the census-taker number. Every CTN produces a unique pair of “mysterious” rectangular boxes with special properties. Each of the mysterious triples constructs a corresponding mysterious rectangular box whose dimensions are the elements of the corresponding triple. The product condition implies that these boxes have the same volume (which is the given CTN), while the sum condition implies that the boxes have the same perimeter (which is four times the magic sum). The volume being a census-taker number denies the existence of a third box of integral dimensions with the same volume and perimeter. Figure 1 shows the two mysterious rectangular boxes whose dimensions correspond to the mysterious triples \( \{9, 2, 2\} \) and \( \{6, 6, 1\} \) of the census-taker number 36.

![Figure 1: The “mysterious” rectangular boxes with volume equal to 36.](image)

To show that a number is a census-taker, it is not enough to find two triples that satisfy the product and the sum conditions. For example, consider \( N = 144 \). Note that
\[ 144 = 9 \cdot 4 \cdot 4 = 8 \cdot 6 \cdot 3 \quad \text{and} \quad 9 + 4 + 4 = 8 + 6 + 3, \]
and so the triples \( \{9, 4, 4\} \) and \( \{8, 6, 3\} \) satisfy the product and the sum conditions. However, \( N = 144 \) is not a census-taker number because another pair of triples satisfies the product and the sum conditions, and these are \( \{12, 4, 3\} \) and \( \{9, 8, 2\} \). For a positive integer to be a census-taker number, there must be exactly two triples that satisfy the product and the sum conditions.

3 Properties of CTNs

Table 2 listed down the four two-digit CTNs, and it was mentioned that there are twenty-nine three-digit CTNs. For smaller integers like 36, the determination of the CTNs can be done by enumerating and then checking every possible triple. However, for larger integers, the impracticality of this procedure demands a more elegant way to determine the CTNs. Some nice properties of these numbers help us eliminate most of those integers that are not CTNs in a faster way. As concrete illustration, we apply this sieving technique to positive integers up to 100.

Let \( \{a, b, c\} \) and \( \{d, e, f\} \) be the mysterious triples associated with the census-taker number \( N \). Since the order of the triples and the order of the elements in each triple are not important, we assume from now on that
\[
a \geq b \geq c, \quad d \geq e \geq f, \quad \text{and} \quad c \geq f.
\]
This assumption will be further restricted after the following theorem.

**Theorem 1.** The mysterious triples of a CTN cannot have a common entry.

**Proof.** Let \( \{a, b, c\} \) and \( \{d, e, f\} \) be the mysterious triples of the CTN. Without loss of generality, we suppose that \( a = d \). By the sum and the product conditions, we have
\[
f - c = b - e \quad \text{and} \quad bc = ef.
\]
Using the identities \( bc = ec + c(b - e) \) and \( ef = ec + e(f - c) \), we get
\[
c(b - e) = e(b - e),
\]
which implies that \( b = e \) or \( c = e \). The former implies that \( c = f \), while the latter implies that \( b = c = e = f \). In both cases, we conclude that \( \{a, b, c\} = \{d, e, f\} \), a contradiction. Q.E.D.

With Theorem 1, we further assume that \( c > f \). More than this assumption, Theorem 1 eliminates a good number positive integers as candidates for CTNs. A prime cannot be a CTN because all triples of positive integers whose product is the given prime have the integer 1 as a common element. In fact, this quick consequence of Theorem 1 also eliminates any integer of the form \( pq \), where \( p \) and \( q \) are (not necessarily distinct) primes, from the list of CTNs. This is so because the only possible triples for these integers \( \{pq, 1, 1\} \) and \( \{p, q, 1\} \), and these triples have a common element 1.
Theorem 2. A CTN is not a power of a prime.

Proof. Suppose, on the contrary, that \( N = p^n \) is a CTN, where \( p \) is prime and \( n \) is a positive integer. Since any divisor of \( N \) is also a power of \( p \), we may assume that \( \{p^r, p^s, p^t\} \) and \( \{p^u, p^v, p^w\} \) are the mysterious triples of \( N \), where \( n = r + s + t = u + v + w, r \geq s \geq t, u \geq v \geq w, \) and \( t > w \). From the sum condition, we have

\[
p^{t-w}(p^{r-t} + p^{s-t} + 1) = p^{u-w} + p^{v-w} + 1.
\]

(\(*\))

Note that the left-hand side of equation (\(*\)) is divisible by \( p \). However, for \( p \neq 3 \), the right-hand side of (\(*\)) is not divisible by \( p \). Finally, for \( p = 3 \), the right-hand side is divisible by \( p \) if and only if \( u = v = w \), which implies that the right-hand side is exactly equal to 3, while the left-hand side is at least 9. Both scenarios lead to a contradiction. Q.E.D.

Theorem 3. Every CTN is a product of at least four (not necessary distinct) primes.

Proof. By Theorem 1, it suffices to prove that a CTN is not a product of exactly three primes.

Suppose that \( N = pqr \) is a CTN, where \( p, q, \) and \( r \) are primes with \( p \geq q \geq r \). Then there are only five possible triples for \( N \), and these are \( \{pqr, 1, 1\} \), \( \{pq, r, 1\} \), \( \{pr, q, 1\} \), \( \{qr, p, 1\} \), or \( \{p, qr, 1\} \). By Theorem 1, the only possible pair of mysterious triples is \( \{pqr, 1, 1\} \) and \( \{p, q, r\} \). However, since \( p+q+r < pqr + 2 \), this pair does not satisfy the sum condition. Q.E.D.

Table 3 shows the remaining integers from 1 to 100 after applying Theorems 1, 2, and 3, where the integers are crossed out with /, //, and —, respectively.

Table 3: Applying Theorems 1, 2, and 3 to integers from 1 to 100.

In Table 3, fourteen two-digit integers remain after the sieving techniques due to the theorems. As claimed in Table 2, ten of these integers are not CTNs. Albeit there
are advanced properties of the CTNs that are capable of eliminating some of these from the list, we do not mention them anymore in this paper, and we remain with the elementary treatment of these interesting numbers. To check whether or not the remaining fourteen integers in Table 3 are CTNs, we simply list down all the triples for each remaining integer, and verify the definition. As examples, Table 4 shows that $N = 56$ is not a CTN by simply looking at the different sums produced by the triples, and Table 5 proves that $N = 96$ is a CTN.

| Triple      | Sum | Triple      | Sum |
|-------------|-----|-------------|-----|
| \{56, 1, 1\} | 58  | \{14, 2, 2\} | 18  |
| \{28, 2, 1\} | 31  | \{8, 7, 1\}  | 16  |
| \{14, 4, 1\} | 19  | \{7, 4, 2\}  | 13  |

Table 4: Triples whose product is 56.

| Triple      | Sum | Triple      | Sum |
|-------------|-----|-------------|-----|
| \{96, 1, 1\} | 98  | \{16, 3, 2\} | 21  |
| \{48, 2, 1\} | 51  | \{12, 8, 1\} | 21  |
| \{32, 3, 1\} | 36  | \{12, 4, 2\} | 18  |
| \{24, 4, 1\} | 29  | \{8, 6, 2\}  | 16  |
| \{24, 2, 2\} | 28  | \{8, 4, 3\}  | 15  |
| \{16, 6, 1\} | 23  | \{6, 4, 4\}  | 14  |

Table 5: Triples whose product is 96.

4 How many CTNs are there?

It is not known whether or not there are infinitely many census-taker numbers. However, if we assume that there are infinitely many primes $p$ such that $2p - 1$ is also prime, we claim that there are also infinitely many CTNs.

We need the following inequalities to prove our claim about the infinitude of CTNs.

**Lemma 4.** For all real numbers $x \neq 1$, the following inequalities hold:

\[(i) \quad 4x^4 - 4x^3 + x^2 + 2 > 4x^3 - 4x^2 + 2x + 1\]

\[(ii) \quad 5x^2 - 4x + 2 > 4x^2 - 2x + 1\]
(iii) \(4x^2 - 2x + 1 > 2x^2 + 2x - 1\)

(iv) \(2x^2 + 2x - 1 > x^2 + 4x - 2\)

Proof. To present the proof technique to establish these inequalities, we completely prove (i).

Observe that
\[
4x^4 - 4x^3 + x^2 + 2 = 4x^3 - 4x^2 + 2x + 1 + (4x^2 + 1)(x - 1)^2.
\]

Since \((4x^2 + 1)(x - 1)^2 > 0\) for all \(x \neq 1\), inequality (i) follows.

For the other inequalities, we simply observe the following equations:
\[
\begin{align*}
5x^2 - 4x + 2 &= 4x^2 - 2x + 1 + (x - 1)^2 \\
4x^2 - 2x + 1 &= 2x^2 + 2x - 1 + 2(x - 1)^2 \\
2x^2 + 2x - 1 &= x^2 + 4x - 2 + (x - 1)^2.
\end{align*}
\]

Since \((x - 1)^2 > 0\) for all \(x \neq 1\), the other inequalities follow. Q.E.D.

With slight modifications, the technique used in the proof of the previous lemma can also be utilized to prove the following lemmas.

Lemma 5. For all real numbers \(x > 1\), the following inequality holds:
\[
2x^3 - x^2 + 2x > 5x^2 - 4x + 2.
\]

Lemma 6. For all real numbers \(x\) that satisfy \(x > -\frac{1}{2}\) and \(x \neq 1\), the following inequality holds:
\[
4x^3 - 4x^2 + 2x + 1 > 2x^3 - x^2 + 2x.
\]

We are now ready to prove a theorem that will lead to our claim.

Theorem 7. Let \(p\) be prime such that \(2p - 1\) is also prime. Then \(N = p^2(2p - 1)^2\) is a CTN.

Proof. Table 6 presents the eight possible triples of \(N\). Using Lemmas 4, 5, and 6, the sums can be arranged from highest to lowest in the following inequalities:
\[
\begin{align*}
4p^4 - 4p^3 + p^2 + 2 &> 4p^3 - 4p^2 + 2p + 1 \\
&> 2p^3 - p^2 + 2p \\
&> 5p^2 - 4p + 2 \\
&> 4p^2 - 2p + 1 \\
&> 2p^2 + 2p - 1 \\
&> p^2 + 4p - 2.
\end{align*}
\]

It follows that \(N = p^2(2p - 1)^2\) is a CTN. Q.E.D.
Corollary 8. If there are infinitely many primes \( p \) such that \( 2p - 1 \) is also prime, then there are also infinitely many CTNs.

As a serious remark to the infinitude of CTNs established in the above corollary, it is not yet known, however, whether or not there are indeed infinitely many such primes \( p \). Nevertheless, because there are CTNs that are not of the form \( p^2(2p - 1)^2 \), Corollary 8 is not the only direction to prove or disprove the infinitude of CTNs.

## 5 Concluding Remarks

A related problem was studied in [2], where it was shown that, for every integer \( M > 18 \), there exist triples with sum \( M \) and equal (unspecified) products. Though this problem can be thought of as a dual of the census-taker problem, it does not consider the uniqueness condition that there should be exactly two triples with equal products. In fact, this uniqueness condition is what makes the census-taker problem more interesting and difficult.

For the curious minds, we mention again that there are no one-digit, 4 two-digit, and 29 three-digit CTNs. Adding to these lists, with 9996 as the largest, there are 277 four-digit CTNs.

With further twists and turns, the census-taker problem still has a long journey to go.

Acknowledgments. The authors thank Giovanni Mazzarello for generously sharing his programming skills by providing the list of all CTNs, with magic sums and mysterious triples, up to 10000. Moreover, they also thank the anonymous referees for the constructive reviews, especially the referee who suggested to include a separate section about the number of CTNs.

## References

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