The Structure Fault-Tolerance of Enhanced Hypercube Networks

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Abstract. An important indicator of a network’s robustness is the connectivity of the network which is directly related to its reliability and fault tolerability. Let H be a connected subgraph of graph G, the H-structure-connectivity of graph G, denoted by \( \kappa(G, H) \), is the cardinality of a minimal set of subgraphs \( F=\{ J_1, J_2, \ldots, J_m \} \), such that every element of F is isomorphic to H, and \( G-F \) is disconnected. The H-substructure-connectivity of graph G, denoted by \( \kappa^s(G,H) \), is the cardinality of a minimal set of subgraphs \( F=\{ J_1, J_2, \ldots, J_m \} \), such that every is a connected subgraph of H, and \( G-F \) is disconnected. In this paper, the H-structure-connectivity \( \kappa(Q_{n,k}, H) \) and \( \kappa^s(Q_{n,k}, H) \) are considered in enhanced hypercube \( Q_{n,k} \) when \( H\in\{ P_1, P_2 \} \).

Introduction

In parallel and distributed systems, interconnection networks play an important role. An interconnection network topology structure can be modeled by graph \( G=(V, E) \), where \( V \) corresponds the set of processors and, \( E \) corresponds the set of communication links.

The hypercube and its variants, such as folded hypercube ([1]), enhanced hypercube ([2]–[6]), generalized hypercube ([7]), are important interconnection network architectures developed for multiprocessor system and large computation in industrial. This paper focus on the enhanced hypercube.

The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of nodes such that \( G-\kappa(G) \) is disconnected. The connectivity is one of the most important measures of the reliability and fault-tolerance of networks. To more accurately measure the fault-tolerance of an interconnection network, [8] proposed the restricted \( h \)-connectivity. [9] introduced the g-extra connectivity. The g-extra connectivity of a graph \( G \), denoted by \( \kappa_g(G) \), is the minimum number of nodes so that \( G-\kappa_g(G) \) is disconnected and each component has more than \( g \) nodes. [10] suggested the structure connectivity and substructure connectivity recently. Let \( H \) be a connected subgraph of \( G \), \( F \) be a set of subgraph of \( G \) such that each element of \( F \) is isomorphic to \( H \). Then \( F \) is called a H-structure–cut if \( G-H \) is disconnected. The H-structure connectivity, \( \kappa(G, H)=\min\{|F|: F \text{ is H-structure–cut}\} \). Similarly, Let \( F \) be a set of subgraph of \( G \) such that every element of \( F \) is isomorphic to a connected subgraph of \( H \). Then \( F \) is called a H-substructure–cut if \( G-H \) is disconnected. The H-substructure connectivity , \( \kappa^s(G,H)=\min\{|F|: F \text{ is H-substructure–cut}\} \). The definition implies \( \kappa^s(G,H) \leq \kappa(G,H) \). Certainly, \( K_1 \)-structure connectivity and \( K_1 \)-substructure connectivity are just the connectivity.

Definitions and Preliminaries

A network is usually modeled by a connected graph \( G=(V; E) \), where \( V \) denotes the set of processors and \( E \) denotes the set of communication links between processors. Two vertices \( x, y \in V \) are adjacent if they are incident with a common edge. The set of vertices \( N_G(v) = \{ u : uv \in E \} \) is called the neighbor set of vertex \( v \) in \( G \). \( d(v) = |N_G(v)| \) is called the degree of vertex \( v \) in \( G \) when no loop occurs. Let \( A \subseteq V \), \( N_G(A) \) denotes the vertex set \( \cup_{v \in A} N_G(v) \setminus A \) and \( N_G[A] = N_G(A) \cup A \). Let \( P_k = v_1v_2 \ldots v_k \) and \( C_k = v_1v_2 \ldots v_k v_1 \) be a path and cycle of \( k \) nodes, respectively. The length of a path \( P \) is defined as the number of edges contained in \( P \). The distance \( d_G(x; y) \) between any two nodes \( x \) and \( y \) is the length of a shortest
path of joining $x$ and $y$. The length of a shortest cycle is defined as the girth of graph $G$, denoted by $g(G)$. A graph $G$ is bipartite if the vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$, such that every edge in $G$ joins a vertex in $V_1$ with a vertex in $V_2$. A graph $G$ is bipartite if and only if $G$ contains no odd cycle. Two graphs $G_1$ and $G_2$ are isomorphic, denoted as $G_1 \cong G_2$, if there is a one to one mapping $f$ from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$.

An $n$-dimensional hypercube, denoted by $Q_n$, has $2^n$ vertices represented by the set $V(Q_n) = \{x_1 x_2 \cdots x_n : x_i = 0 \text{ or } 1; 1 \leq i \leq n\}$, two vertices $x_1 x_2 \cdots x_n$ and $y_1 y_2 \cdots y_n$ are adjacent if and only if $\Sigma_{i=1}^n |x_i - y_i| = 1$. Let $x, y \in Q_n$, the Hamming distance between $x$ and $y$, denoted by $h(x; y) = \Sigma_{i=1}^n |x_i - y_i|$, is the number of different bits between the corresponding strings of $x$ and $y$. Obviously, $d_{Q_n}(x; y) = h(x; y)$. The weight of a vertex $x$ is defined as $w(x) = \Sigma_{i=1}^n |x_i|$ (or the number of 1’s in $x$).

**Definition.** Enhanced hypercube $Q_{n,k} = (V, E)$ is an undirected simple graph. $V(Q_{n,k}) = \{x_1, x_2, \ldots x_n : x_i \in \{0, 1\} \text{ for } 1 \leq i \leq n\}$. Two vertices $x = (x_1, x_2 \cdots x_n)$ and $y$ are connected by an edge of $E$ if and only if $y$ satisfies one of the following two conditions:

(i) $y = x' = x_1 x_2 \ldots x_{i-1} \bar{x}_i x_{i+1} \ldots x_n, 1 \leq i \leq n$; or
(ii) $y = \bar{x} = x_1 x_2 \ldots x_{i-1} \bar{x}_i x_{i+1} \ldots x_n$.

Hypercube $Q_n$ is a subgraph of enhanced hypercube $Q_{n,k}$, obtained by removing all edges of $\bar{x}x$ called complimentary edges of $Q_{n,k}$. The edges of type $x'x$ are called the $i$-dimensional edges of $Q_{n,k}$ or $Q_{n,k}$. Let $E_i = \{x' : x \in V(FQ_n)\} (i=1,2,\ldots,n)$, and $E_c = \{\bar{x} : x \in V(FQ_n)\}$.

**Main Results**

**Lemma 1** ([11]). $Q_{n,k}$ is a bipartite graph if and only if $n, k$ have the same parity. When $n, k$ have different parity, $Q_{n,k}$ contains odd cycle, and the smallest odd cycle contains exactly one complementary edge and the length of $n - k + 2$.

Lemma 1 leads to lemma 2.

**Lemma 2.** The girth of $Q_{n,k}$ is $g(Q_{n,k}) = 4$ for $n \geq 3, 2 \leq k \leq n - 2$, and $g(Q_{n,n-1}) = 3$.

**Lemma 3.** Let $x, y$ be any two vertices in $Q_{n,k}$ for $n \geq 4$, then one of the following holds.

(i) $x, y \in V(Q_{n,k})$ for $2 \leq k \leq n - 4$, then $x$ and $y$ have exactly two common neighbors if they have.

(ii) $x, y \in V(Q_{n,n-3})$

If $\bar{x} \in N_{Q_{n,n-3}}(x) \cap N_{Q_{n,n-3}}(y)$, then $x, y$ have exactly two common neighbors.

If $x \notin N_{Q_{n,n-3}}(x) \cap N_{Q_{n,n-3}}(y)$, then $x$ and $y$ have exactly two common neighbors if they have.

(iii) $x, y \in V(Q_{n,n-2})$

\[
x = x_1 x_2 \ldots x_{i-1} x_i y_{i+1} \ldots x_n
\]

$y \notin x_1 x_2 \ldots x_{i-1} \bar{y}_i y_{i+1} \ldots x_{n}$

where $\{i,j\} \subseteq \{n-2,n-1\}$, then $x$ and $y$ have exactly two common neighbors if they have.

(iv) $x, y \in V(Q_{n,n-2})$

\[
x = x_1 x_2 \ldots x_{i-1} \bar{y}_i y_{i+1} \ldots x_n
\]

$y = x_1 x_2 \ldots x_{i-1} \bar{x}_i x_{i+1} \ldots y_{n}$

where $\{j,i\} \subseteq \{n-2,n-1\}$, then $x$ and $y$ have exactly two common neighbors if they have.

(v) $x, y \in V(Q_{n,n-1})$

If $y \notin \{x^{n-1}, x^n\}$, then $x, y$ have exactly two common neighbors if they have.

If $y \in \{x^{n-1}, x^n\}$, then $x, y$ have exactly two common neighbors $\{\bar{x}, y\}$.

Proof: Let $x = x_1 x_2 \ldots x_{i-1} x_i x_{i+1} \ldots x_n, y = x_1 x_2 \ldots x_{i-1} \bar{y}_i y_{i+1} \ldots x_{j-1} y_j y_{j+1} \ldots x_n$, then $x, y$ have some common neighbours.

(i) $x, y$ have exactly two common neighbours $x' = x_1 x_2 \ldots x_{i-1} \bar{y}_i x_{i+1} \ldots x_n, y' = x_1 x_2 \ldots x_{j-1} \bar{x}_j y_j x_{j+1} \ldots x_n$.

(ii) $x, y \in V(Q_{n,n-3})$

If $\bar{x} \in N_{Q_{n,n-3}}(x) \cap N_{Q_{n,n-3}}(y)$, then $x, y$ have exactly two common neighbours $\{\bar{x}, y\} = N_{Q_{n,n-3}}(x) \cap N_{Q_{n,n-3}}(y)$.
If \( x \notin N_{Q_{n,k}}(x) \cap N_{Q_{n,k}}(y) \), and \( x, y \) have some common neighbours, then similar to case (i), \( x, y \) have exact two common neighbours.

(iii) The proof is similar to (i).

(iv) \( x, y \in V(Q_{n,n-2}) \), \( x = x_1x_2...x_{j-1}x_{j+1}...x_{n}\), \( y = x_1x_2...x_{j-1}x_{j+1}...x_{n}\), then \( \bar{x} = x_1x_2...x_{j-2}x_{j+2}...x_{n} \), \( \bar{y} = x_1x_2...x_{j-2}x_{j+2}...x_{n} \) and \( x, y \) have exact four common neighbors \( \{x^{n-2}, x^{n-1}, x^n, \bar{x}\} \).

(v) \( x, y \in V(Q_{n,n-1}) \), if \( y \notin \{x^{n-1}, x^n\} \), the proof is similar to (i) and \( x, y \) have exact two common neighbors. If \( y = x^{n-1}, x^n \), then \( x, y \) have exact two common neighbors \( \{ \bar{x}, \bar{y} \} \) because when \( y = x^{n-1} = x_1x_2...x_{n-2} \bar{x}_n...x_n \) (or \( x^n = x_1x_2...x_{n-2} \bar{x}_n...x_n \)), then \( \bar{y} = x^\ell \) (or \( x^n \)). The proof is finished.

The following lemma is benefit for our results.

**Lemma 4 ([12]).** If \( n \geq 4 \), then

\[
\kappa_s(Q_n) = \begin{cases} 
(g+1)n - 2g - \left( \frac{n}{2} \right) & \text{for } 0 \leq g \leq n-4, \\
\frac{n(n-1)}{2} & \text{for } n - 3 \leq g \leq n.
\end{cases}
\]

By definition, \( \kappa(Q_{n,k};P_1) = \kappa'(Q_{n,k};P_1) = \kappa(Q_{n,k}) = n + 1 \) for \( n \geq 4 \).

**Lemma 5.** \( \kappa(Q_{n,k};P_2) \leq n \) and \( \kappa'(Q_{n,k};P_2) \leq n \) for \( n \geq 4 \).

**Proof:** Set \( x = 00...0 \) and \( y = x^\ell \) being two adjacent nodes in \( Q_{n,k} \), and let \( S = \{ (x', y') | 2 \leq i \leq n \} \cup \{ x, y \} \). Now \( S \) forms a \( P_2 \)-structure-cut of and \( |S| = n \). Thus, \( \kappa(Q_{n,k};P_2) \leq n \) and certainly \( \kappa'(Q_{n,k};P_2) \leq n \).

**Lemma 6.** \( \kappa(Q_{n,k};P_2) \geq n \) and \( \kappa'(Q_{n,k};P_2) \geq n \) for \( n \geq 4 \).

**Proof:** Let \( F = \{ P_1, \ldots, P_i, P_{i+1}, \ldots, P_j \} \) and \( |F| = l + t \leq n - 1 \) for \( l, k \geq 0 \). By contradiction, suppose that \( F \) is a cut of \( Q_{n,k} \), \( Q_{n,k} - F \) is disconnected, \( Q_{n,k} - F \) has at least two components, say, \( W \) is a smallest component of \( Q_{n,k} - F \). We consider the several cases as bollow.

Case 1. \( |V(W)| = 1 \), that is \( W \) is a isolated point, say \( V(W) = \{ u \} \), then the definition means \( |N_{Q_{n,k}}(u)| = n + 1 \). This implies that one has to delete at least \( n + 1 \) nodes to isolate the vertex \( u \), but it is impossible because of \( |F| < n \).

Case 2. \( |V(W)| \geq 2 \). With lemma 4 and \( Q_n \subseteq Q_{n,k} \), we have \( \kappa(Q_{n,k}) \geq \kappa(Q_n) = 2n - 2 \). This means that we have to delete more than \( n - 2 \) nodes to isolate the component \( W \) in \( Q_{n,k} \). But \( |F| < n \). It is a contradiction. Therefore \( \kappa'(Q_{n,k};P_2) \geq n \) and \( \kappa(Q_{n,k};P_2) \geq n \).

Lemma 5 and lemma 6 lead to theorem 7 hold.

**Theorem 7.** \( \kappa(Q_{n,k};P_2) = n \) and \( \kappa'(Q_{n,k};P_2) = n \) for \( n \geq 4 \).

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