QUASI-LOCAL ENERGY IN PRESENCE OF
GRAVITATIONAL RADIATION

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Abstract. We discuss our recent work [4] in which gravitational radiation was studied by evaluating the Wang-Yau quasi-local mass of surfaces of fixed size at the infinity of both axial and polar perturbations of the Schwarzschild spacetime, à la Chandrasekhar [1].

We compute the Wang-Yau quasi-local mass [7, 8] of “spheres of unit size” at null infinity to capture the information of gravitational radiation. The set-up, following Chandrasekhar [1], is a gravitational perturbation of the Schwarzschild solution, which is governed by the Regge-Wheeler equation (see below). We take a sphere of a fixed areal radius and push it all the way to null infinity. The limit of the geometric data is that of a standard configuration and thus the optimal embedding equation [7, 8, 2] can be solved.

Let us first consider the axial perturbations. The metric perturbation is of the form:

\[-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\phi - q_2 dr - q_3 d\theta)^2.\]

The linearized vacuum Einstein equation is solved by a separation of variable Ansatz in which \(q_2^3\) and \(q_3\) are explicitly given by the Teukolsky function and the Legendre function.

In particular,

\[q_3 = \sin(\sigma t) \frac{C_\mu(\theta)}{\sin \theta} \left( \frac{\sigma^2 r^4}{r^2 - 2mr} \right) \frac{d}{dr} (rZ(-))\]

for a solution of frequency \(\sigma\) and a separation of variable constant \(\mu\). Here \(C_\mu(\theta)\) is related to the \(\mu\)-th Legendre function \(P_\mu\) by

\[C_\mu(\theta) = \sin \theta \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{dP_\mu(\cos \theta)}{d\theta} \right).\]

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After the change of variable
\[ r_* = r + 2m \ln\left(\frac{r}{2m} - 1\right), \]
\(Z(-)\) satisfies the Regge-Wheeler equation:
\[ \left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z(-) = V(-) Z(-), \]
where
\[ V(-) = \frac{r^2 - 2mr}{r^5} \left[(\mu^2 + 2)r - 6m\right], \]
and \(\mu\) is a separation of variable constant.

On the Schwarzschild spacetime
\[-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,\]
we consider an asymptotically flat Cartesian coordinate system \((t, y_1, y_2, y_3)\) with \(y_1 = r \sin \theta \sin \phi, y_2 = r \sin \theta \cos \phi, y_3 = r \cos \theta\). Given \((d_1, d_2, d_3) \in \mathbb{R}^3\) with \(d^2 = \sum_{i=1}^{3} d_i^2\), consider the 2-surface
\[ \Sigma_{t,d} = \{(t, y_1, y_2, y_3) : \sum_{i=1}^{3} (y_i - d_i)^2 = 1\}. \]

We compute the quasi-local mass of \(\Sigma_{t,d}\) as \(d \to \infty\).

Denote
\[ A(r) = \frac{(r^2 - 2mr)}{\sigma^2 r^3} \frac{d}{dr} (rZ(-)). \]
The linearized optimal embedding equation of $\Sigma_{t,d}$ is reduced to two linear elliptic equations on the unit 2-sphere $S^2$:
\[
\Delta(\Delta + 2)\tau = [-A''(1 - Z_1^2) + 6A'Z_1 + 12A]Z_2Z_3
\]
\[
(\Delta + 2)N = (A'' - 2A'Z_1 + 4A)Z_2Z_3,
\]
where $\tau$ and $N$ are the respective time and radial components of the solution, and $Z_1, Z_2, Z_3$ are the three standard first eigenfunctions of $S^2$. $A'$ and $A''$ are derivatives with respect to $r$, and $r^2$ is substituted by $r^2 = d^2 + 2Z_1 + 1$ in the above equations.

The quasi-local mass of $\Sigma_{t,d}$ with respect to the optimal isometric embedding is then
\[
E(\Sigma_{t,d}) = C^2\{\sin^2(\sigma t)E_1 + \sigma^2 \cos^2(\sigma t)E_2\} + O(\frac{1}{d^3}),
\]
where $E_1$ and $E_2$ are two integrals on the standard unit 2-sphere, that depend on the solution $\tau$ and $N$ of the optimal isometric embedding equation. Explicitly,
\[
E_1 = \int_{S^2} \frac{1}{2} \left[ A^2Z_2^2(7Z_3^2 + 1) + 2AA'Z_1Z_3^2(3Z_2^2 - 1) - N(\Delta + 2)N \right]
\]
\[
E_2 = \int_{S^2} \left[ A^2Z_2^2Z_3^2 - \tau(\Delta + 2)\tau \right].
\]
In particular,
\[
\partial_t E(\Sigma_{t,d}) = \frac{\sigma \sin(2\sigma t)C^2(\theta)}{d^2} \{E_1 - \sigma^2 E_2\} + O(\frac{1}{d^3}).
\]
Let us compare the quasi-local mass on the small spheres $\Sigma_{t,d}$ along a certain direction to the quasi-local mass of the large coordinate spheres $S_{t,r}$. 

![Diagram](image-url)
Naively, one may expect to recover \( \partial_t E(S_t, r) \) by integrating the energy radiated away at all directions \( \partial_t E(\Sigma_{t,d}) \). However, our calculation indicates that there are nonlinear correction terms from the quasi-local energy that should be taken into account.

We can also consider the polar perturbation of the Schwarzschild space-time in which the metric coefficients \( g_{tt}, g_{rr}, g_{\theta\theta}, \) and \( g_{\phi\phi} \) are perturbed in

\[
-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]

The gravitational perturbation is governed by the Zerilli equation

\[
\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^+(\Sigma_t, r) = V^+(\Sigma_t, r),
\]

where

\[
V^+(\Sigma_t, r) = \frac{2(r^2 - 2mr)}{r^5(nr + 3m)^2} [n^2(n + 1)r^3 + 3mn^2r^2 + 9m^2nr + 9m^3],
\]

and \( n \) is the separation of variable constant. Again, we compute the quasi-local mass of spheres of unit-size at null infinity. The calculation is similar to the axial perturbation case but the result is different as the leading term is of the order \( \frac{1}{d} \) (as opposed to \( \frac{1}{d^2} \) for axial-perturbation) with nonzero coefficients. If such a linear perturbation can be realized as an actual perturbation of the Schwarzschild spacetime, the result would contradict the positivity of the quasi-local mass \([6, 7, 8]\). From this, we deduce the following conclusion: There does not exist any gravitational perturbation of the Schwarzschild spacetime that is of purely polar type in the sense of Chandrasekhar \([1]\).

For an actual gravitational perturbation of the Schwarzschild solution, the vanishing of the \( \frac{1}{d} \) gives a limiting integrand that integrates to zero on the limiting 2-sphere at null infinity. In fact, the quasi-local mass density \( \rho \) (see \([3\) equation 2.2\)) of \( \Sigma_{t,d} \) can be computed at the pointwise level. Up to an \( O(\frac{1}{d^2}) \) term

\[
\rho = (K - \frac{1}{4}|H|^2)
\]

\[
- \frac{(|H| - 2)^2}{4} + \frac{1}{d^2} \left\{\frac{1}{2} |\nabla^2 N| + ((\Delta + 2)N)^2 - \frac{1}{4} (\Delta N)^2
\right. \\
- \frac{1}{4} (\Delta \tau)^2 + \frac{1}{2} [\nabla_a \nabla_b (\tau_a \tau_b) - |\nabla \tau|^2 - \Delta |\nabla \tau|^2] \},
\]

where \( K \) is the Gauss curvature of \( \Sigma_{t,d} \). The first line, which integrates to zero, is of the order of \( \frac{1}{d} \) and is exactly the mass aspect function of the Hawking mass \([5]\). The \( \frac{1}{d} \) term of the quasi local mass \( \int_{\Sigma_{t,d}} \rho \, d\mu_{\Sigma_{t,d}} \) has contributions from the second and third lines (of the order of \( \frac{1}{d^2} \)), the \( \frac{1}{d^2} \) term of the first line, and the \( \frac{1}{d} \) term of the area element \( d\mu_{\Sigma_{t,d}} \). The
above integral formula is obtained after performing integrations by parts and applying the optimal embedding equation several times.

To each closed loop on the limiting 2-sphere at null infinity, we can thus associate a non-vanishing arc integral that is of the order of $\frac{1}{d}$, where $d$ is the distance from the source. We expect the freedom in varying the shape of the loop can increase the detectability of gravitational waves.

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