A QUANTIFIED TAUBERIAN THEOREM AND LOCAL DECAY OF $C_0$-SEMIGROUPS

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Abstract. We prove a quantified Tauberian theorem for functions under a new kind of Tauberian condition. In this condition we assume in particular that the Laplace transform of the considered function extends to a domain to the left of the imaginary axis, given in terms of an increasing function $M$ and is bounded at infinity within this domain in terms of a different increasing function $K$. Our result generalizes [4, Theorem 4.1]. We also prove that the obtained decay rates are optimal for a very large class of functions $M$ and $K$. Finally we explain in detail how our main result improves known decay rates for the local energy of waves in odd-dimensional exterior domains.

1. Introduction

In the last decade there has been much activity in the field of quantified Tauberian theorems for functions of a real variable [20, 2, 10, 3, 6, 21, 5, 9, 4]. See also [23, 24] and references therein for quantified Tauberian theorems on sequences and [14] for Dirichlet series. We refer to [15] and [1, Chapter 4] for a general overview on Tauberian theory.

Let $X$ be a Banach space and $f : \mathbb{R}_+ \to X$ be a locally integrable function. For some continuous and increasing function $M : \mathbb{R}_+ \to [2, \infty)$ let us define

$$\Omega_M = \left\{ z \in \mathbb{C}; 0 > \Re z > -\frac{1}{M(|\Im z|)} \right\}. \quad \text{(1)}$$

The above mentioned articles impose essentially the Tauberian condition that the function $f$ has a bounded derivative (in the weak sense), the Laplace transform $\hat{f}$ extends across the imaginary axis to $\Omega_M$ and it satisfies a growth condition, also expressed in terms of $M$ in $\Omega_M$ at infinity. The decay rate (the rate of convergence to zero) is then determined in terms of $M$. For example, a polynomially growing $M$ gives a polynomial decay rate and an exponentially growing $M$ gives a logarithmic decay rate. In general $\hat{f}$ could also have a finite number of singularities on the imaginary axis [21], but we are not interested in this situation in the present article.

The pioneering works [21, 2] focus on polynomial decay for orbits of $C_0$-semigroups. A generalization for functions (as formulated above) and to arbitrary decay rates was given in [3] for the first time. There the authors also improved the decay rates from [20, 2]. In [6] it was shown that the results of [3] are optimal in the case of polynomial decay. We want to emphasize at this point that the main result of [3] for the special case of a truncated orbit of a unitary group $U$ of operators (i.e. $f(t) = P_2 U(t) P_1$ for some bounded operators $P_1, P_2$) were already obtained in the

MSC2010: Primary 40E05. Secondary 47D06, 35B40.

Keywords and phrases: Tauberian theorem, quantified, rates of decay, $C_0$-semigroups, local energy decay.
earlier article \cite{22} with the same rate of decay. Actually the authors only formulated a theorem on polynomial decay but in the retrospective it is not difficult to generalize their proof to arbitrary decay rates.

A major contribution to the field of Tauberian theorems is the recent article \cite{4}. The authors extended the known Tauberian theorems to $L^p$-rates of decay. On the basis of a technique already applied in \cite{6} the authors showed the optimality of their results in the case of polynomial decay. Another important observation, made in \cite{6}, concerns the above mentioned growth condition. In \cite{3} it was assumed that the norm of $\hat{f}(z)$ is bounded by $M(|3z|)$ in $\Omega_M$. This condition was weakened in \cite{6} in case of polynomial decay, and later in \cite{4} assuming merely that $\hat{f}(z)$ can be bounded by a polynomial in $(1 + |3z|)M(|3z|)$.

The aim of the present article is to further generalize the growth condition on $\hat{f}$ in $\Omega_M$. That is, we introduce a second continuous and increasing function $K : \mathbb{R}_+ \to [2, \infty)$ and assume that the norm of $\hat{f}(z)$ is bounded by $K(|3z|)$ in $\Omega_M$. The decay rate is then given in terms of $M$ and $K$.

Let $M^{-1}$ denote the right-continuous right-inverse of $M$ given by $M^{-1}(t) = \sup\{s \geq 0; M(s) = t\}$ for all $t \geq 0$. Let

$$w_M(t) = \begin{cases} M^{-1}(t) & \text{if } t \geq M(1) \\ 1 & \text{else}. \end{cases}$$

We are now ready to state our first main result, a generalization of \cite{4}, Theorem 4.1.

**Theorem 1.1.** Let $(X, \|\cdot\|)$ be a Banach space, $m \in \mathbb{N}$, and $f : \mathbb{R}_+ \to X$ be a locally integrable function such that its $m$-th weak derivative $f^{(m)}$ is in $L^p(\mathbb{R}_+; X)$ for some $1 < p \leq \infty$. Assume that there exist continuous and increasing functions $M, K : \mathbb{R}_+ \to [2, \infty)$ satisfying

(i) $\forall s > 1 : K(s) \geq \max\{s, M(s)\}$,

(ii) $\exists \varepsilon \in (0, 1) : K(s) = O\left(e^{c(s^{1-\varepsilon})}\right)$ as $s \to \infty$.

such that the Laplace transform $\hat{f}$ of $f$ extends analytically to $\Omega_M \cup \mathbb{C}_+$ and

(1) $\left\|\hat{f}(z)\right\| \leq K(|3z|)$ for all $z \in \Omega_M$.

Then there exists a constant $c_1 > 0$ such that

(2) $(t \mapsto \|w_{MK}(c_1 t)^m f(t)\|) \in L^p(\mathbb{R}_+)$,

where $M_K(s) := M(s) \log(K(s))$.

**Remark 1.2.** Note that a function $f \in L^1_{loc}(\mathbb{R}_+; X)$ with $f^{(m)} \in L^p(\mathbb{R}_+; X)$ is polynomially bounded. In fact, $\|f(t)\| \leq C(1 + t)^{m-1/p}$ holds for all $t \geq 0$. In particular the Laplace transform of $f$ is well-defined in the interior of $\mathbb{C}_+$ as an absolutely convergent integral.

**Remark 1.3.** One can drop condition (i) on $K$ but then one has to replace $M_K$ by the function given by $M(s) \log((2 + s)M(s)K(s))$.

**Remark 1.4.** We are not able to prove the theorem for $\varepsilon = 0$ in condition (ii). In Section \cite{2,3} the reader can find a short discussion on a slightly weaker constraint on $K$. 
If we replace $K(|\mathbb{R}z|)$ in (1) by $((1 + |\mathbb{R}z|)M(|\mathbb{R}z|))^{\alpha}$ for some $\alpha > 0$ and set $m = 1$ we recover [4, Theorem 4.1]. Our theorem applies perfectly to local energy decay of waves in odd-dimensional exterior domains. Here $f$ is typically a spatially truncated orbit of a solution to the wave equation and one is often confronted with the situation that $M$ is constant and $K$ is asymptotically larger than any polynomial. In this situation no known Tauberian result applies directly. One might guess that one can apply the Phragmén-Lindelöf principle to get a better estimate on $f$ on a smaller domain to the left of the imaginary axis. Indeed this works, and as shown in [9] one can apply known Tauberian theorems after this procedure. However in Section 5 we discuss the application to local decay of waves in exterior domains in detail and show that this procedure yields a weaker estimate than a direct application of Theorem 1.

We prove Theorem 1.1 as a corollary to the following variant which is a generalization of [2, Theorem 2.1(b)]:

**Theorem 1.5.** Let $(X, \|\cdot\|)$ be a Banach space, $m \in \mathbb{N}$, and $f : \mathbb{R}_+ \to X$ be a locally integrable function such that $f^{(m)} \in L^p(\mathbb{R}_+; X)$ for some $1 < p \leq \infty$. Let $M$ and $K$ be as in Theorem 1.7. Assume that the Fourier transform $\hat{F}$ of $f$ is of class $C^\infty$ and its derivatives satisfy for all $j \in \mathbb{N}_0$

\[ \left\| F^{(j)}(s) \right\| \leq j! K(|s|) M(|s|)^j \] for all $s \in \mathbb{R}$.

Then there exists a constant $c_1 > 0$ such that

\[ (t \mapsto \|w_{MK}(c_1 t)^m f(t)\|) \in L^p(\mathbb{R}_+), \]

where $M_K(s) := M(s) \log(K(s))$.

**Remark 1.6.** Note that the Fourier transform of $f$ is well-defined in the sense of tempered distributions since $f$ is polynomially bounded (compare with Remark 1.2).

A theorem of this type (for $p = \infty$, $m = 1$ and $K = M$) was formulated for the first time in [3]. A main contribution of the authors was also to provide a new and easier to understand technique - on the basis of Ingham’s original proof of the unquantified version [17] - for proving Tauberian theorems. For example in [3] one main difficulty is to choose contours for integration in the complex plane in a clever way. In [9] the authors avoid this technicality by considering the derivatives of the Fourier transform of $f$ instead of the Laplace transform.

To prove Theorem 1.5 we adapt the proof of [8, Theorem 2.1(b)]. That is - for $m = 1$ - we decompose $f = |f - \phi_R * f| + \phi_R * f = J_1 + J_2$ into two terms with the help of some suitably chosen and scaled convolution kernel $\phi_R(t) = R\phi(Rt)$ with $\int_R \phi(t)dt = 1$. Then we estimate the $X$-norm of $J_1(t, R)$ and $J_2(t, R)$ in terms of $R$ and $t$, solely assuming $f' \in L^p$ respectively the bounds on all derivatives $F^{(j)}$. Finally we optimize the sum of these two estimates by choosing $R = w_{MK}(c_1 t)$ for a sufficiently small $c_1$.

We improve the techniques of [9] in the following way: We estimate $J_1(t, R)$ from above by a Poisson integral $R^{-1}P_{R^{-1}} * \|f'\| (t)$ which makes it possible to apply a fundamental result on Carleson measures. We note that this technique was already applied in [4]. Compared to the proof in [9] we get a better estimate on $J_2(t, R)$ by choosing a better convolution kernel $\phi$. Also the Fourier transform $\psi$ of our
convolution kernel is a $C^\infty_c$-function which simplifies the prove slightly. Our choice of $\psi$ is based on the Denjoy-Carleman theorem on quasi-analytic functions.

The paper is organized as follows. In Section 2 and 3 we prove Theorems 1.5 and 1.1, respectively. In Section 4 we prove the optimality of Theorem 1.1 for a very large class of possible choices of $M$ and $K$. This is even new in the case where $K = M$. To prove the optimality we make a similar construction as in [6]. As a side product this construction also shows that there actually exist functions $f$ satisfying the hypotheses of Theorem 1.1 for $K$ increasing faster than any polynomial in $sM(s)$, but do not satisfy (1) if one replaces $K$ by a polynomial in $sM(s)$ (see Remark 4.7). This proves that Theorem 1.1 is a proper generalization of [4, Theorem 4.1]. A short discussion on the optimal choice of $c_1$ in (2) is included in Subsection 4.1. In Subsection 5.1 we explain how to get local decay rates for $C_0$-semigroups from our results. Finally in Subsection 5.2 we apply this to local energy decay of waves in odd-dimensional exterior domains.

1.1. Notation. We denote $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}_+ = \{ z \in \mathbb{C}; \Re z \geq 0 \}$. By $\mathbb{N}_0$ we denote the natural numbers including 0. For $m \in \mathbb{N}_0$ we define $\mathbb{N}_m$ to be the natural numbers greater or equal to $m$. By $C$ we denote a strictly positive constant which may change implicitly their value from line to line. Every statement in this article which includes $C$ remains true if one replaces $C$ by a larger constant. Other strictly positive constants, having the names $C_1, C_2, \ldots$ are not allowed to change their values - except it is explicitly stated. Analogously $c, c_1, c_2, \ldots$ are strictly positive constants which might be replaced by smaller constants without invalidating any statement in our article. We say that a function $\phi: \mathbb{R} \to \mathbb{R}$ decays rapidly if for any $n \in \mathbb{N}_0$ there exists a constant $C$ such that $|\phi(t)| \leq C(1 + t)^{-n}$.

2. Proof of Theorem 1.5

Without loss of generality we may assume that $f(0) = f'(0) = \ldots = f^{(m-1)}(0) = 0$. If this was not satisfied we could replace $f$ by $f - g$ for some function $g \in C^m_c([0, t_1); X)$ with $g(0) = f(0), \ldots, g^{(m-1)}(0) = f^{(m-1)}(0)$ and $t_1 > 0$ arbitrary. This neither changes the asymptotics of $f$ at infinity nor does it change the growth of $F$ and its derivatives at infinity considerably. To see this note that the Fourier transform $G$ of $g$ satisfies

$$\left\| G^{(j)}(s) \right\| \leq t_1^{j+1} \|g\|_\infty \text{ for } j \in \mathbb{N}_0 \text{ and } s \in \mathbb{R}.$$ 

Now let us extend $f$ by zero on the negative numbers. By our additional assumptions we see that the extended function is $(m-1)$-times continuously differentiable on the whole real line and $f^{(m)} \in L^p(\mathbb{R}; X)$.

Let $\psi \in C^\infty_c(\mathbb{R})$ with supp $\psi \subseteq [-1, 1]$ and $\psi(0) = 1$ be a function to be fixed later in the proof. Let

$$\phi(t) = F^{-1}\psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} \psi(s) ds$$
be its inverse Fourier transform. Note that $\phi$ is a Schwartz function with $\int_{-\infty}^{\infty} \phi dt = \psi(0) = 1$. For $R > 0$ let $\phi_R(t) = R\phi(Rt)$ and $\psi_R(s) = \psi(s/R)$. Let us decompose

$$f(t) = (\delta - \phi_R)^* m * f(t) - [(\delta - \phi_R)^* m - \delta] * f(t)$$

$$= \left[ \sum_{j=0}^{m} \binom{m}{j} (-1)^j \phi^*_R * f \right] (t) - \left[ \sum_{j=1}^{m} \binom{m}{j} (-1)^j \phi^{*j}_R * f \right] (t)$$

$$=: J_1(t, R) + J_2(t, R).$$

Here by $\phi^{*j}$ we denote the $j$-times convolution of $\phi$ with itself. We also define $\phi^{*0} = \delta$ (delta-function). Note that $(\phi_R)^{*j} = (\phi^{*j})_R$.

2.1. Estimation of $J_1$. Let us define the Poisson kernel by

$$P_y(t) = \frac{1}{\pi} \cdot \frac{y}{t^2 + y^2}.$$

Recall that by Young’s inequality the Poisson kernel acts as a continuous operator on $L^p(\mathbb{R})$ via convolution.

**Lemma 2.1.** Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}_1$. Let $f : \mathbb{R} \to X$ be a locally integrable function such that $f^{(m)} \in L^p(\mathbb{R}; X)$. Let $\phi$ be as above. Then there exists a constant $C > 0$ (only depending on $\phi$ and $m$) such that

$$\| (\delta - \phi_R)^* m * f(t) \| \leq \frac{C}{R^m} P_y \| f^{(m)} \| (t)$$

holds for all $t \geq 0$ and $R > 0$.

**Remark 2.2.** It is clear from the proof that in the statement of the lemma one can replace $P = P_1$ by any positive and integrable kernel bounded from below by $c(1+t)^{-\alpha}$ for some $\alpha > 1$. We then define $P_y(t) = y^{-1}P(y^{-1}t)$. Unfortunately this is not consistent with the definition of $\phi_R$, but for the Carleson measure argument below it is more convenient to define $P_y$ as above.

**Proof.** Let us define two antiderivatives of $\phi$

$$\Phi_-(t) = \int_{-\infty}^{t} \phi(\tau) d\tau, \quad \Phi_+(t) = -\int_{t}^{\infty} \phi(\tau) d\tau.$$

Furthermore we define the following auxiliary function

$$\Phi(t) = \begin{cases} 
\Phi_-(t) & \text{if } t < 0 \\
\Phi_+(t) & \text{if } t \geq 0 
\end{cases}.$$

We observe that the derivative of $\Phi$ is $\phi$ plus a factor times the delta function at zero. This observation is the reason why we split the integral from the following calculation at 0.
First we consider the case \(m = 1\).

\[
[f - \phi_R * f](t) = \int_{-\infty}^{\infty} (f(t) - f(t - \tau)) \phi_R(\tau) d\tau
\]

\[
= \left[ \int_{-\infty}^{0} + \int_{0}^{\infty} \right] (f(t) - f(t - \frac{\tau}{R})) \phi(\tau) d\tau
\]

\[
= \frac{1}{R} \int_{-\infty}^{\infty} f'(t - \frac{\tau}{R}) \Phi(\tau) d\tau
\]

\[
= -\frac{1}{R} \Phi_R * f'(t).
\]

(7)

We need to explain why the partial integration executed from line two to three produces no boundary terms at \(-\infty, 0\) and \(\infty\). At zero there are no boundary terms since \((f(t) - f(t - \frac{\tau}{R}))\) vanishes at \(\tau = 0\) and the two limits \(\lim_{\tau \to 0^+} \Phi(t)\) exist. Recall that \(f\) is polynomially bounded. Moreover the function \(\Phi\) decays rapidly at infinity. Thus there are no boundary terms at plus or minus infinity. Finally the last equality together with the fact that \(\Phi\) decays rapidly implies

\[
\|f - \phi_R * f(t)\| \leq \frac{C}{R} \int_{-\infty}^{\infty} \|f'(t - \frac{\tau}{R})\| \frac{1}{\tau^2 + 1} d\tau
\]

\[
\leq \frac{C}{\pi R} \int_{-\infty}^{\infty} \|f'(t - \tau)\| \frac{R^{-1}}{\tau^2 + R^{-2}} d\tau
\]

\[
= \frac{C}{R} \mathcal{P}_1 \|f\|(t).
\]

Now we consider the case \(m \in \mathbb{N}_2\). Let us define recursively \(f_{j+1} = f_j - \phi_R * f_j, f_0 = f\) for \(j \in \{0, 1, \ldots, m - 1\}\). Clearly \(f_m = (\delta - \phi_R)^m * f\). We prove now \(f_j = (-1/R)^j \Phi^{(j)}_R * f(j)\) via induction on \(j\). Observe that for any \(j \in \mathbb{N}_1\) the function \(\Phi^{(j)}\) decays rapidly. For \(j = 1\) the inductive hypothesis is precisely (7). Assume that the hypothesis is valid for some \(j < m\). Then by (7) for \(f\) replaced by \(f_j\)

\[
f_{j+1} = f_j - \phi_R * f_j = -\frac{1}{R} \Phi_R * f_j = \left( -\frac{1}{R} \right)^{j+1} \Phi^{(j+1)}_R * f(j+1).
\]

From here we can finish the proof as in the case \(m = 1\).

Since the \(L^1\)-norm of the Poisson kernel is 1 (for any \(y > 0\)) we see from Young’s inequality that for any \(g \in L^p(\mathbb{R})\) and \(y > 0\) it holds that \(\|P_y * g\|_{L^p} \leq \|g\|_{L^p}\). If \(p = \infty\) and if we set \(R = R(t) = w_{M_k}(c_1 t)\) we deduce from Lemma 2.1 that

(8)

\[
R(t)^m \|((\delta - \phi_{R(t)})^m * f(t))\| \leq C_{c_1} < \infty
\]

holds for all \(t \geq 0\). If we compare this with (4) we see that this already yields the desired estimate on \(J_1\) in the case \(p = \infty\). If \(p < \infty\) we need a slightly more involved argument based on a property of Carleson-measures.

Therefore let \(P * g(t, y) := P_y * g(t)\) and let \(\mu\) be a Borel measure on the upper half-plane \(H = \{(t, y) \in \mathbb{R}^2; y > 0\}\). Now we ask for which measures \(\mu\) an inequality

(9)

\[
\|P * g\|_{L^p(H, d\mu)} \leq C_p \|g\|_{L^p(\mathbb{R})}
\]

holds for all \(g \in L^p(\mathbb{R})\) with a constant \(C_p\) not depending on \(g\)? Note that the inequality \(\|P_y * g\|_{L^p} \leq C_{y} \|g\|_{L^p}\) is a special case of (9) for \(C_p = 1\) with \(\mu\) being the one-dimensional Hausdorff measure of the line \(\{(t, y) \in H; t \in \mathbb{R}\} \subset H\). Actually for \(1 < p < \infty\) one can characterize the class of all measures \(\mu\) for which (9) holds
for all \( g \). These measures are called Carleson measures (see \[13\], Theorem 1.5.6). Let \( \gamma : \mathbb{R} \to (0, \infty) \) be a bounded continuous function with bounded variation. Then the one-dimensional Hausdorff measure of
\[
\Gamma = \{ t + \gamma (t); t \in \mathbb{R} \} \subset H
\]
is a Carleson measure. Now let \( \gamma \) gives a description of those sequences \( \sum_{\gamma} \). The Denjoy-Carleman theorem implies that our choice of non-zero functions \( \gamma \) at the same time. The Denjoy-Carleman \( 1 \) \( \psi \) such a \( \psi \) every \( \gamma \) to be the Carleson measure corresponding to this particular choice of \( \gamma \) then we deduce that for \( 1 < p < \infty \)
\[
(10) \quad P \star \left\| f^{(m)} \right\|_{L^p(H, d\mu_{M_K})} \leq C_p \left\| f^{(m)} \right\|_{L^p(R_+; X)} < \infty.
\]
From this together with Lemma 2.4 we deduce

**Lemma 2.3.** Let \( c_1 > \) and define \( R(t) = w_{M_K}(c_1 t) \). (i) Then for \( p = \infty \) we have
\[
\sup_{0 < t < \infty} R(t)^m \left\| (\delta - \phi_{R(t)})^{*m} \ast f(t) \right\| \leq C \left\| f^{(m)} \right\|_{L^\infty(R_+; X)}.
\]
(ii) For \( 1 < p < \infty \) we have
\[
\int_0^\infty \left\| R(t)^m (\delta - \phi_{R(t)})^{*m} \ast f(t) \right\|^p dt \leq C \left\| f^{(m)} \right\|^p_{L^p(R_+; X)}.
\]
In both cases \( C \) does not depend on \( f \).

2.2. Estimation of \( J_2 \). The following Lemma is only necessary if \( p \neq \infty \).

**Lemma 2.4.** There exists a \( \delta > 0 \) such that \( K(w_{M_K}(t)) \geq t^\delta \) for all \( t \geq M_K(1) \).

**Proof.** Let \( R = w_{M_K}(t) \). Since \( w_{M_K} \) is essentially the right-inverse of \( M_K \) we have
\[
t = M_K(R) = M(R) \log(K(R)) \geq M(R) \log(M(R)).
\]
The inverse of the function \( x \mapsto x \log(x) \) is asymptotically equal to \( y \mapsto y/\log(y) \) for large \( y \). Hence there exists a \( \delta > 0 \) such that \( M(R) \leq \delta^{-1} t/\log(t) \). Thus
\[
K(R) = \exp(\log(K(R))) = \exp \left( \frac{t}{M(R)} \right) \geq \exp(\delta \log(t)) = t^\delta.
\]

\( \square \)

At this point in the proof we fix a \( \psi \) having one additional property. We assume that the derivatives of \( \psi \) satisfy for some \( C_1 > 0 \)
\[
(11) \quad \forall j \in \mathbb{N}_0 : \sup_{s \in [-1, 1]} \left| \psi^{(j)} (s) \right| \leq C_1^{j+1} A_j \text{ with } A_j = (j \log(2 + j)^{1+\varepsilon})^j.
\]

Note that (11) can not be satisfied by any \( \psi \) if we would replace \( A_j \) by \( j! \) since then \( \psi \) would be analytic and hence can not have compact support and \( \psi(0) = 1 \) at the same time. The Denjoy-Carleman\(^1\) theorem (see e.g. \[13\], Theorem 1.3.8 or \[11\]) gives a description of those sequences \((A_j)\) which allow for compactly supported non-zero functions \( \psi \) satisfying the inequality in (11). In particular, the Denjoy-Carleman theorem implies that our choice of \( A_j \) is admissible for the existence of such a \( \psi \). Conversely it implies that there is no \( \psi \in C_c^\infty(\mathbb{R}) \setminus \{0\} \) which satisfies (11) with \( \varepsilon = 0 \).

\(^1\)A special version of the Denjoy-Carleman theorem (sufficient for our considerations) reads as follows: Let \( S \) be the set of \( C^\infty \)-functions on \( \mathbb{R} \) supported on \([-1, 1]\) such that (11) holds for a sequence \((A_j)\) such that \((\sqrt{A_j})\) is increasing. Then \( S \) contains a non-zero function if and only if \( \sum_{j} 1/\sqrt{A_j} < \infty \).
Now we proceed with the estimation of \( J_2(t, R) \). Therefore we have to estimate \( J_{2,j}(t, R) = \phi_R^{*j} \ast f(t) \) for \( j \in \{1, \ldots, m\} \). First let us consider \( J_{2,1} \). Let \( N \in \mathbb{N}_0 \).

Integration by parts \( N \)-times yields

\[
J_{2,1}(t, R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} F(s) \psi_R(s) \, ds = \frac{1}{2\pi} \left( \frac{i}{t} \right)^N \int_{-R}^{R} e^{ist} \sum_{j=0}^{N} \binom{N}{j} F^{(N-j)}(\psi(j))_R \, ds.
\]

To verify the following calculations recall \([3]\) and \([11]\). We estimate the integral very roughly from above: length of interval of integration times supremum of the integrand within this interval. We also use Stirling’s formula implying for example

\[
\text{length of interval} \cdot \sup \text{of integrand within interval} \approx (N-1)! \cdot e^{-\epsilon^2/2}.
\]

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J_{2,1}(t, R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} F(s) \psi_R(s) \, ds
\]

\[
= \frac{1}{2\pi} \left( \frac{i}{t} \right)^N \int_{-R}^{R} e^{ist} \sum_{j=0}^{N} \binom{N}{j} F^{(N-j)}(\psi(j))_R \, ds.
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\[
\text{length of interval} \cdot \sup \text{of integrand within interval} \approx (N-1)! \cdot e^{-\epsilon^2/2}.
\]

The second inequality is valid for sufficiently large \( C_2, C_3 > 0 \). Now let us set \( N = \lceil t/(C_2M(R)) \rceil \) and \( R = w_{M_k}(c_1t) \). The constant \( c_1 > 0 \) will be chosen later. Then the condition (ii) on \( K \) implies

\[
B \leq \sum_{j=0}^{N} \left( \frac{C_3 \log((c_1 C_2)^{-1} \log(K(R)))^{1+\epsilon}}{RM(R)} \right)^j \leq \sum_{j=0}^{N} \left( \frac{C_4 (RM(R))^{1-\epsilon^2}}{RM(R)} \right)^j \leq C.
\]

The constant in the last inequality does not depend on \( t \). Moreover

\[
A \leq CR^{m+1}K(R)e^{-N} \leq CR^{m+1}K(R)e^{-\log(K(R))/c_1} = CR^{m+1}K(R)^{1-1/c_1^2}.
\]

If we choose \( c_1 \) sufficiently small Lemma \([2,3]\) implies that

\[
\|w_{M_k}(c_1 t)^m J_{2,1}(t, w_{M_k}(c_1 t))\| \leq \begin{cases} C \frac{1}{(1+t)^{1/p}} & \text{if } p = \infty, \\ C \left( \frac{1}{(1+t)^{1/p}} \right)^{p} & \text{if } 1 \leq p < \infty. \end{cases}
\]

Clearly \([12]\) remains valid if one replaces \( J_{2,1} \) by \( J_{2,k} \) and \( \psi \) by its \( k \)-th power \( \psi^k \). It is not difficult to check that \( \psi^k \) also satisfies \([11]\) if one replaces \( C_{j+1}^k \) by \( C_k^k \). Therefore \([14]\) remains true after replacing \( J_{2,1} \) by \( J_2 \). This together with Lemma \([2,3]\) proves Theorem \([1,5]\).

2.3. A remark on condition (ii) for \( K \). Our proof breaks down if we allow \( \varepsilon \) to be zero in condition (ii) in Theorem \([1,4]\) and \([1,5]\). This is essentially due to the fact that by the Denjoy-Carleman theorem a function \( \psi \) satisfying \([11]\) for \( \varepsilon = 0 \) is necessarily quasi-analytic. This means that \( \psi^{(j)}(s_0) = 0 \) for a single \( s_0 \in \mathbb{R} \) but all \( j \in \mathbb{N} \) automatically implies \( \psi = 0 \). However, one can weaken (ii) slightly by
choosing for some given \( \varepsilon \in (0,1) \) and \( n \in \mathbb{N} \)

\[
A_j = j \cdot L_1(j) \cdot L_2(j) \cdots \cdot L_n(j) \cdot L_{n+1}(j)^{1+\varepsilon} \quad \text{with} \quad L_k(j) = \log \circ \cdots \circ \log(1 + k + j).
\]

This allows to replace (ii) by the condition

\[
K(s) = O\left(\exp\left(\exp\left(\frac{sM(s)}{L_1(sM(s)) \cdots \cdot L_n(sM(s)) \cdot L_{n+1}(sM(s))^{1+\varepsilon}}\right)\right)\right).
\]

Again choosing \( \varepsilon = 0 \) is forbidden for any \( n \).

3. PROOF OF THEOREM 1.1

Lemma 3.1 below implies that Theorem 1.1 and Theorem 1.5 are equivalent. To prepare the formulation of this lemma we introduce some notation. Let \( M_1, M_2, K_1, K_2 : \mathbb{R}^+ \to [2, \infty) \) be continuous and increasing functions. For \( f : \mathbb{R}^+ \to X \) measurable and polynomially bounded and extended by zero on the negative real numbers we consider two distinct conditions. The first one is

\[
\forall z \in \Omega_M : \bigg|\hat{f}(z)\bigg| \leq K_1(|z|).
\]

This condition implicitly states that the Laplace transform of \( f \) can be extended to \( \Omega_M \). Let \( F \) be the Fourier-transform of \( f \). The second condition is

\[
\forall j \in \mathbb{N}_0, s \in \mathbb{R} : \bigg|F^{(j)}(s)\bigg| \leq j!K_2(|s|M_2(|s|)^j).
\]

This condition implicitly states that the Fourier transform is a \( C^\infty \)-function.

The following lemma relates these conditions to each other under a mild condition on \( f \).

**Lemma 3.1.** Let \( f : \mathbb{R}^+ \to X \) be a measurable and polynomially bounded function with \( f^{(m)} \in L^p(\mathbb{R}^+; X) \) for some \( 1 \leq p \leq \infty \) and \( m \in \mathbb{N}_1 \). We extend \( f \) by zero on the negative real numbers and denote by \( F \) its Fourier transform. (a) If \( F \) satisfies (10) then \( f \) satisfies (13) with

\[
M_1(s) = (1 - \varepsilon)^{-1}M_2(s) \quad \text{and} \quad K_1(s) = \varepsilon^{-1}K_2(s)
\]

for any \( \varepsilon \in (0,1) \). (b) If \( f \) satisfies (13) then \( F \) satisfies (10) with

\[
M_2(s) = M_1\left(s + \frac{1}{M_1(s)}\right) \quad \text{and} \quad K_2(s) = K_1\left(s + \frac{1}{M_1(s)}\right) + C_f M_1\left(s + \frac{1}{M_1(s)}\right)^{2 - \frac{1}{p}} + C_f^2,
\]

The constant \( C_f \) depends only on \( \|f^{(m)}\|_{L^p} \), the constant \( C_f^2 \) depends only on \( \|f(0)\|, \ldots, \|f^{(m-1)}(0)\| \).

Before proving this lemma we finish the proof of Theorem 1.1. Since \( f \) satisfies (13) for \( M_1 = M \) and \( K_1 = K \), Lemma 3.1 implies that (10) is true for \( M_2 \) and \( K_2 \) given as in part (b) of the lemma. In the following we assume \( s > 0 \) large enough to satisfy \( 1/M_1(s) \leq s \). Note that condition (i) in Theorem 1.1 implies the existence of a (small) constant \( c > 0 \) such that (for large \( s \))

\[
cM_2(s) \log(K_2(s)) \leq M(2s) \log(K(2s)).
\]
This immediately yields for large $t$

\[ w_{M_k}(ct) \leq 2w_{(M_2)K_2}(t). \]

Therefore $w_{(M_2)K_2}(c_1)^m f \in L^p$ for some $c_1 > 0$ implies that $w_{M_k}(cc_1)^m f \in L^p$. The proof of Theorem 1.1 is complete.

**Proof of Lemma 3.2** Let us begin with the easier part (a). Hadamard’s formula shows that (16) implies that $\hat{f}$ is analytic in $\Omega_{M_2} \supset \Omega_{M_1}$. Let $z \in \Omega_{M_1}$ and let $s = 3z$. Then

\[
\left\| \hat{f}(z) \right\| = \left\| \sum_{j=0}^{\infty} \frac{1}{j!} \hat{f}^{(j)}(is - is)^j \right\| \leq \sum_{j=0}^{\infty} K_2(s) M_2(s)^j \left( \frac{1 - \varepsilon}{M_2(s)} \right)^j = \varepsilon^{-1} K_2(s).
\]

Let us now prove part (b). Let us fix $s \in \mathbb{R}$, let $r = 1/M_1(|s| + 1/M(|s|))$ and let $\gamma$ be the positively oriented circle of radius $r$ around $is$ in the complex plane. Note that $\gamma$ is indeed included in the closure of the union of $\Omega_{M_1}$ and $\mathbb{C}_+$. Let $\gamma_+$ and $\gamma_-$ be the intersection of $\gamma$ with $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. By Cauchy’s formula we have

\[
\hat{f}^{(j)}(is) = \frac{j!}{2\pi i} \left[ \int_{\gamma_-} + \int_{\gamma_+} \right] \frac{\hat{f}(z)}{(z - is)^{k+1}} \left( 1 + \frac{(z - is)^2}{r^2} \right) dz \cdot \varepsilon^{-1} K_2(s).
\]

Let us first estimate $I_-$:

\[
\|I_-\| \leq \frac{1}{2\pi} \cdot r^{-j-1} \sup_{z \in \gamma_-} \|\hat{f}(z)\| \cdot \pi r \cdot 2 \leq K_1 \left( |s| + \frac{1}{M_1(|s|)} \right) M_1 \left( |s| + \frac{1}{M_1(|s|)} \right)^j.
\]

Let us now estimate $I_+$:

\[
I_+ = \frac{1}{2\pi i} \int_{\gamma_+} \frac{1 + (z - is)^2}{(z - is)^{k+1}} \left( \sum_{k=0}^{m-1} z^{-j-1} f^{(k)}(0) + z^{-m} \int_0^\infty e^{-zt} f^{(m)}(t) dt \right) dz
\]

\[=: \sum_{k=0}^{m-1} I_{+,k} + I_{+,m}.
\]

It is an easy exercise to show that the integral of $e^{-rt \cos(\theta) \cos(\theta)}$ over $\theta \in (\pi/2, \pi/2)$ can be estimated from above by a constant times $(rt)^2 + 1)^{-1}$. Therefore by Hölder’s inequality we get for large $|s|

\[
\|I_{+,m}\| \leq \frac{C}{|s|^m r^j + 1} \int_0^\infty \int_{\pi}^\pi e^{-rt \cos(\theta) \cos(\theta)} d\theta \left\| f^{(m)}(t) \right\| dt
\]

\[
\leq \frac{C}{|s|^m r^j + 2 - 1/p} \left\| f^{(m)} \right\|_{L^p}
\]

\[
\leq \frac{C}{|s|^m} |s|^{2j + 2 - 1/p} M_1(|s| + 1/M_1(|s|))^{j+2-1/p}.
\]

A similar (and easier) estimate is true for the other summands $I_{+,k}$. This together with (17) yields the claim. \(\square\)
4. Optimality of Theorem 4.1

In this section we show that under the assumptions of Theorem 4.1 and for $p = \infty, m = 1$ one can - up to improvement of the constant $c_1$ - not get a faster decay rate than the one already given by the theorem. To show this we use almost the same method as in [6]. There the authors showed the optimality in the very particular case that $M(s) = C(1 + s^\sigma)$ and $K(s) = C(1 + s^\beta)$ for $\beta > \alpha/2 > 0$.

**Theorem 4.1.** Let $c_1 > 0$ and let $M, K : \mathbb{R}_+ \to [2, \infty)$ be continuous and increasing functions satisfying for some increasing function $N : \mathbb{R}_+ \to [1, \infty)$

(i) $\lim_{s \to \infty} \frac{M_K(s)}{\log(2 + s)} = \infty$ and $\exists \varepsilon > 0, s_0 > 0 \forall s \geq s_0 : K(s) \geq s^\varepsilon$,

(ii) $\exists s_0 > 0 \forall s \geq s_0, s' \geq 0 : M(s + s') \leq N(s')M(s)$. 

Then there exists a real number $\gamma \geq 0$, not depending on $c_1$ and a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$ with $f' \in L^\infty(\mathbb{R}_+)$ such that

\[
|\hat{f}(z)| \leq \frac{C}{R} M(|\Im z|) \frac{2}{s} K(|\Im z|) \frac{\beta}{\alpha} \text{ for all } z \in \Omega_M
\]

and

\[
\limsup_{t \to \infty} M_K^{-1}(c_1 t) |f(t)| \geq c > 0.
\]

If instead of (ii) we have the stronger assumption that there exists a $\gamma_0 \geq 1$ such that

(iii) $\forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s + s') \leq \gamma_0 M(s)$

and if $\gamma > \gamma_0$ then it is possible to choose $f$ in such a way that (18) holds for this choice of $\gamma$. If in addition $M$ is unbounded then it is possible to choose $f$ in such a way that (18) holds for all $\gamma > \gamma_0$.

**Remark 4.2.** Note that condition (i) is only a very mild restriction. In fact, a typical situation where (i) is violated is that $M$ is a constant and $K$ grows at most polynomially. But then Theorem 4.1 implies exponential decay for $f$. This in turn implies, that the integral which defines $\hat{f}$ is absolutely convergent in a small strip to the left of the imaginary axis. In particular $\hat{f}$ extends analytically to this strip and is bounded there. So our results are trivially optimal in that case.

Before we prove the Theorem we need a similar lemma as in [6]. Given a compactly supported measure $\mu$ on $\mathbb{C}\setminus\Omega_M \cup \mathbb{C}_+$ we use the following notation for $z \in \Omega_M \cup \mathbb{C}_+$ and $t \geq 0$

\[
\mathcal{L}_\mu(z) = \int \frac{1}{z - \zeta} d\mu(\zeta), \mathcal{L}_\mu(t) = \int e^{\xi t} d\mu(\zeta), \mathcal{L}'\mu(t) = \int \zeta e^{\xi t} d\mu(\zeta).
\]

To simplify the notation we extend $M$ and $K$ symmetrically to the negative real axis.

**Lemma 4.3.** Let $c_1, M$ and $K$ be as in Theorem 4.1. There exists a $\delta > 0$ and $\gamma > 0$, only depending on $M$ and $\delta$, such that for all $\varepsilon > 0$ and $\varepsilon_0 \in \mathbb{N}_0$ there exists
$k \in \mathbb{N}_{k_0}$ and a compactly supported Borel measure $\mu$ on $\mathbb{C}\setminus \Omega_M \cup \mathbb{C}^+$ such that

\begin{align}
|\mathcal{L}\mu(z)| &\leq \frac{C}{R} M^{\frac{1}{2}} K^{\gamma} 1_{[R-2\delta, R+2\delta]}(\Im z) + \varepsilon, \\
|\mathcal{L}'\mu(t)| &\leq C1_{[\frac{1}{2}, \frac{2}{3}]}(t) + \varepsilon, \\
|\mathcal{L}\mu(t)| &\leq \frac{C}{R} 1_{[\frac{1}{2}, \frac{2}{3}]}(t) + \frac{\varepsilon}{\max\{R, M^{-1}(c_1t)\}}, \\
\left|\mathcal{L}\mu\left(\frac{k}{\delta}\right)\right| &\geq \frac{c}{R}
\end{align}

holds for all $z \in \Omega_M$ and $t \geq 0$. Here $R$ is the largest real number such that $c_1k = \delta M_K(R)$. If instead of (ii) we have the stronger assumption that there exists a $\gamma_0 \geq 1$ such that

\begin{align}
(ii') \forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s + s') \leq \gamma_0 M(s)
\end{align}

and if $\gamma > \gamma_0$ then it is possible to choose $f$ in such a way that (22) holds for this choice of $\gamma$. If in addition $M$ is unbounded then it is possible to choose $f$ in such a way that (22) holds for all $\gamma > \gamma_0$.

Remark 4.4. For $\Im z = R$ the inequality (20) holds also in the reverse direction (for a different value of $C$). This will be indicated in the proof.

**Proof.** Let $\delta > 1/M(0)$ be a real number to be fixed later. Let $k \in \mathbb{N}_{k_0}$ to be fixed later. Let us define

$$w = iR - \delta, \quad q = e^{2\pi i/(k+1)}, \quad \delta A = kl(k)$$

where $l : \mathbb{R}_+ \to (0, \infty)$ is a strictly increasing function such that $l(t) \geq \beta \log(e + t)$ for some $\beta \geq 1$ to be fixed later. By $\delta z_0$ we denote the Dirac-measure at $z_0 \in \mathbb{C}$. Let us define

$$\mu = \frac{\tau}{R} \sum_{j=0}^{k} q^j \delta_{w+A^{-1}q^j}.$$

The constant $\tau > 0$ will be chosen later. Before we go on we state a simple lemma which will be frequently applied in the following.

**Lemma 4.5.** Let $n > 0$ be a real number. The function $s \mapsto s^ne^{-s}$ has a unique maximum on $\mathbb{R}_+$. Before this maximum the function is strictly increasing and after that maximum it is strictly decreasing.

One can prove the lemma by simply taking the derivative of the function.

**Part 1:** Estimation of $\mathcal{L}\mu$. We distinguish the two cases $t \leq A$ and $t > A$.

**Case 1:** $t \leq A$. We calculate
\[ \mathcal{L}_\mu(t) = \frac{\tau}{R} \sum_{j=0}^{k} q^j e^{t(w+A^{-1}q')} \]

\[ = \frac{\tau}{R} e^{tw} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{t}{A} \right)^m \sum_{j=0}^{k} q^{m+1}j \]

\[ = \frac{\tau}{R} \cdot e^{tw} \sum_{m=0}^{\infty} \frac{k!}{A^k k!} \cdot \left( \frac{t}{A} \right)^{n(k+1)} \]

\[ = \cdot \frac{\tau}{R} \cdot e^{tw} \cdot \left( \frac{t}{A} \right)^{(n-1)(k+1)} \]

Clearly \( II \) is bounded from below by 1 and bounded from above by a constant which does not depend on \( k \) or \( A \). Thus by Stirling’s formula we get

\[ \mathcal{L}_\mu(t) \geq c \frac{\tau}{R} \sqrt{ke^{-\delta t}} \left( \frac{e^{\delta t}}{\delta Ak} \right)^k. \]

As a function in \( t \) we can maximize the right-hand side by setting \( \delta t = k \). If we furthermore define

\[ (24) \quad \tau = \frac{1}{\sqrt{k}} (\delta A)^k \]

we see that (23) is proved. Since \( II \) is bounded from above we have

\[ (25) \quad \mathcal{L}_\mu(t) \leq C \frac{\tau}{R} \sqrt{ke^{-\delta t}} \left( \frac{e^{\delta t}}{\delta Ak} \right)^k. \]

Again we maximize the right-hand side by setting \( \delta t = k \) and plugging in (24). This leads to

\[ \mathcal{L}_\mu(t) \leq C \frac{\tau}{R} \sqrt{ke^{-k}} \left( \frac{e^{-k}}{\delta A} \right)^k \leq \frac{C}{R} \]

For \( t \in [k/2\delta, 2k/\delta] \) this is already what we want to have in (22).

Case 1.1: \( \delta t \leq k/2 \). In this case the maximum in (25) with respect to \( t \) is attained for \( \delta t = k/2 \). This yields

\[ |\mathcal{L}_\mu(t)| \leq C \frac{\tau}{R} \sqrt{ke^{-k}} \left( \frac{e^{-k}}{\delta A} \right)^k = C \frac{\tau}{R} \left( \frac{e^{-k}}{\delta A} \right)^k \leq \frac{\varepsilon}{R} \]

The last inequality holds for sufficiently large \( k \). We proved (22) for \( \delta t \leq k/2 \).

Case 1.2: \( 2k \leq \delta t \leq \delta A \). Condition (i) from Theorem 4.1 yields \( M_{-1}^{-1}(c_1 t) \leq e^{\delta t/\alpha} \) for any \( \alpha > 0 \) as long as \( t \) is large enough. Thus, if we multiply (25) by \( M_{-1}^{-1}(c_1 t) \) we get

\[ M_{-1}^{-1}(c_1 t) |\mathcal{L}_\mu(t)| \leq C \frac{\tau}{R} \sqrt{ke^{-(1-\frac{1}{\alpha})\delta t}} \left( \frac{e^{\delta t}}{\delta Ak} \right)^k \]

\[ \leq C \frac{\sqrt{k}}{R} \left( \frac{2}{\delta} \right)^k \leq \varepsilon \]

for sufficiently large \( k \). From the first to the second line we used that the maximum of the right-hand side of the first line is attained at \( \delta t = 2k \) if \( \alpha \geq 2 \). In the last estimate we used \( e^{1-\frac{1}{\alpha}} > 2 \) which is true if \( \alpha \) is large enough. We proved (22) for \( 2k \leq \delta t \leq \delta A \).
Case 2: \( t > A \). Then we have
\[
|\mathcal{L}\mu(t)| \leq \frac{\tau}{R} (k+1) e^{-(\delta-A^{-1})t} \\
\leq \frac{C}{R} \sqrt{k} (\delta A)^k e^{-\delta A} e^{-(\delta-A^{-1})(t-A)}
\]

In the following we assume that \( \delta - A^{-1} > 0 \) which is true for large \( k \).

Case 2.1: \( A < t < 2A \). In this case (using again \( M^{-1}_K(c_1 t) \leq e^{\delta t/\alpha} \) for large \( t \))
we get
\[
M^{-1}_K(2c_1 A) |\mathcal{L}\mu(t)| \leq \frac{C}{R} \sqrt{k} \left( kl(k) e^{-l(k)} \right)^k e^{2k(l(k))} \\
= \frac{C}{R} \sqrt{k} \left( kl(k) e^{-(1-\frac{1}{A})l(k)} \right)^k \leq \varepsilon
\]
if we choose \( \beta > 1 \) and let \( \alpha \) satisfy \( (1 - \frac{2}{A})^{-1} < \beta \) and if \( k \) is large enough. We
proved (22) for \( A < t < 2A \).

Case 2.2: \( t \geq 2A \). If we use \( \sqrt{k} (\delta A)^k e^{-\delta A} \leq 1 \) for large \( k \) we can calculate for
an \( \alpha > 4 \)
\[
M^{-1}_K(c_1 t) |\mathcal{L}\mu(t)| \leq \frac{C}{R} e^{-\left(1-\frac{1}{A}\right)(\delta t - \delta A)} e^{-\frac{\delta t}{\alpha}} \\
\leq \frac{C}{R} e^{\left(\frac{1}{A} - \frac{1}{\alpha}\right)\delta t} \leq \varepsilon.
\]

This finishes the proof of (22).

Part 2: Estimation of \( \mathcal{C}\mu \). First observe that as long as \( z \) is no \((k+1)\)-th root of unity we have
\[
\sum_{j=0}^{k} \frac{q^j}{z-q} = \frac{k+1}{z^{k+1} - 1}.
\]

Clearly this equation must hold for some \( k \)-th order polynomial \( p \) if one replace the
term \( k+1 \) on the right-hand side by \( p(z) \). Moreover the left-hand side is invariant
under the substitution which replaces \( z \) by \( qz \). Thus \( p(z) = p(qz) \). But this implies
that \( p \) is a constant. By plugging in \( z = 0 \) we see that \( p = k+1 \).

The observation yields for \( z \in \Omega_M \)
\[
(26) \quad \mathcal{C}\mu(z) = \frac{\tau}{R} \frac{(k+1)A}{(A(z-w))^{k+1} - 1}.
\]

Now it is not difficult to prove (20) for \( |\Im z - R| > 2\delta \). The latter condition implies
\( |z-w| > 2\delta \). Thus, using (26) we get for \( |\Im z - R| > 2\delta \) and \( k \) large:
\[
|\mathcal{C}\mu(z)| \leq C \frac{\tau}{R} k A (2\delta A)^{-k-1} \leq \frac{C \sqrt{k}}{\delta R} 2^{-k} \leq \varepsilon.
\]
If we don’t have \(|z - R| > 2\delta\) we can merely estimate \(|z - w| \geq \delta - 1/M(3z)|. This yields for \(z \in \Omega_M\) with \(|z - R| > 2\delta\) and for all \(\gamma_1 > 1\)

\[
|C\mu(z)| \leq C \frac{\tau}{R} A(\delta A(1 - \frac{1}{\delta M(3z)})^{-k-1} < 1 \leq C \frac{\sqrt{\delta R}}{\delta R} e^{\gamma_1 N(2\delta)} 
\leq C \frac{\sqrt{\delta R}}{\delta R} e^{\gamma_1 N(2\delta)} 
\leq C \frac{\sqrt{\delta R}}{\delta R} \sqrt{M_K(R) K(R)}^{\gamma_1 N(2\delta)}. 
\]

From the first to the second line we use the inequality \(1 - x \geq e^{-\gamma_1 x}\) which is valid for small \(x \geq 0\). If \(M\) is bounded we choose \(\delta\) large enough to make use of this inequality. From the second to the third line we used condition (ii) from Theorem \(1\). Choosing \(\gamma = \gamma_1 N(2\delta)\) we get \((20)\). Concerning Remark \(1\) a reverse inequality for \(3z = R\) can be proved analogously but in an even simpler way by using the inequality \(1 - x \leq e^{-x}\) which is valid for all \(x \geq 0\).

**Part 3:** Estimation of \(L'\mu\). Finally we want to estimate the derivative of \(L\mu\).

**Case 1:** \(t \geq A\). In this case we directly get for large \(k\)

\[
|L'\mu(t)| \leq \frac{\tau}{R} (k + 1)(R + A^{-1})e^{(\delta - A^{-1})t} 
\leq C \frac{\sqrt{\delta R}}{R} (\delta A)^k R e^{-\delta A} \leq \varepsilon.
\]

**Case 2:** \(t < A\). Let us first get a different representation of \(L\mu\):

\[
L'\mu(t) = \frac{\tau}{R} \sum_{j=0}^{k} q^j (w + A^{-1}q^j) e^{(w + A^{-1}q^j)t} 
= \frac{\tau}{R} \sum_{j=0}^{k} q^j \frac{1}{m!} \sum_{m=0}^{j} (wq^{(m+1)}j + A^{-1}q^{(m+2)}j) 
= \frac{w}{A^k k!} \sum_{j=0}^{k} \frac{k!}{n(k + 1)!} \frac{t^{(n-1)(k+1)}}{n(k + 1)!} \left[ 1 + \frac{n(k + 1) - 1}{wt} \right].
\]

Note that if \(t > t_0 > 0\) the sum at the end of the calculation is bounded by a constant which only depends on \(t_0\).

\[
|L'\mu(t)| \leq C\sqrt{\delta} e^{-\delta t} \left( \frac{e\delta t}{\delta A k} \right)^k \left[ 1 + \frac{k}{Rt} \right] 
\leq C e^{-\delta t} \left( \frac{e\delta t}{k} \right)^k \left[ 1 + \frac{k}{Rt} \right] 
\leq C e^{-\delta t} \left( \frac{e\delta t}{k} \right)^k \left[ 1 + \frac{k}{Rt} \right] (27)
\]

Note that \((27)\) as a function in \(t\) is increasing for \(\delta t < k - 1\) and decreasing for \(\delta t > k\). Therefore we see that \(|L'\mu(t)|\) bounded by a constant not depending on \(t\). This shows \((21)\) for \(k/2\delta \leq t \leq 2k/\delta\).

**Case 2.1:** \(\delta t \leq k/2\). The maximum in \((24)\) is then attained for \(\delta t = k/2\). This yields

\[
|L'\mu(t)| \leq C e^{-\frac{k}{2}} \left( \frac{e}{2} \right)^k \leq C \left( \frac{e}{4} \right)^{\frac{k}{2}} \leq \varepsilon.
\]
if $k$ is large enough. 

Case 2.2: $2k \leq \delta t \leq A$. The maximum in (27) is then attained for $\delta t = 2k$. This yields

$$|L' \mu(t)| \leq Ce^{-2k} (2e)^k \leq C \left( \frac{2}{e} \right)^k \leq \varepsilon$$

if $k$ is large enough. This finishes the proof of Lemma 4.3.

\[ \Box \]

Proof of Theorem 4.1. For an $\varepsilon_0 > 0$ to be chosen later we define a sequence $(\varepsilon_n)$ by $\varepsilon_n = 2^{-n} \varepsilon_0$. There exists a $\delta > 0$, an increasing sequence of natural numbers $(k_n)$ and a sequence of measures $(\mu_n)$ according to Lemma 4.3. We may assume that $\left([R_n - 2\delta, R_n + 2\delta]\right)$ and $\left([k_n/2\delta, 2k_n/\delta]\right)$ are sequences of pairwise disjoint intervals. Let us define

$$f(t) = \sum_{n=1}^{\infty} L_n^{(1)} \mu_n(t) \text{ for } t \geq 0.$$  

The sum is uniformly convergent because of (22). The function $f$ is therefore continuous and since the sequence of derivatives converges uniformly (by (21)) we see that $f$ has a bounded weak derivative given by

$$f'(t) = \sum_{n=1}^{\infty} L_n^{(1)} \mu_n(t) \text{ for } t \geq 0.$$  

By a similar argument the Laplace transform has the form

$$\hat{f}(z) = \sum_{n=1}^{\infty} \mathcal{L} \mu_n(z) \text{ for } z \in \Omega_M.$$  

Here the sum converges uniformly on compact subsets of $\Omega_M \cup \mathbb{C}_+$ (by (20)). We already know that the derivative of $f$ is bounded. The estimate (18) follows immediately from (20). It remains to prove (19). Let us set $t_n = k_n/\delta$ then we deduce from (22) and (23) that

$$|f(t_n)| \geq \frac{c}{R_n} - \varepsilon_0 \sum_{j \neq n} \frac{2^{-j}}{\max\{R_j, M^{-1}_R(c_1 t_n)\}}$$

$$\geq \frac{c}{R_n} - \varepsilon_0 \sum_{j \neq n} \frac{2^{-j}}{R_n}$$

$$\geq \frac{c}{R_n} = \frac{c}{M^{-1}_R(c_1 t_n)}.$$  

In the last line we chose $\varepsilon_0$ small enough.

Remark 4.6. By the same technique one can also prove the optimality of Theorem 1.1 for $m > 1$. To achieve this one just has to define the measure $\mu$ in Lemma 4.3 by $\mu = \tau R^{m-2} \sum_{j=0}^{k} q^j \delta_{w+A-1,q^j}$. 

Remark 4.7. With the help of remark 4.4 one easily sees that for $z = R_n$ the inequality (18) holds also in the reverse direction (for a different constant $C$).
4.1. On the optimality of the constant $c_1$ in Theorem 1.1. The literature seems not to pay much attention to the constant $c_1$ appearing in Theorem 1.1. If we are interested in polynomial decay the constant does not influence the decay rate much. However, if for example $M_K^{-1}(t) = \exp(t^{\alpha})$ for some $\alpha \in (0, 1)$ we immediately see that $c_1$ influences the decay rate in a crucial way. The aim of this subsection is to give a partial answer concerning the question of the optimality of $c_1$. Under not too restrictive conditions on $M$ and $K$ we show that Theorem 1.1 is valid for any $c_1 < 1$ and false for $c_1 > 1$. Unfortunately we have to exclude the important special case of exponential decay from our discussion.

**Theorem 4.8.** Let $p = \infty$. (a) In addition to the assumptions in Theorem 1.1 assume that $K$ increases faster than any polynomial and assume that $K(s) \geq c(1 + s)^{-m}M(2s)^2$. Then (3) holds for all $c_1 < 1$. (b) Let $M, K$ satisfy the assumptions of Theorem 1.1. Assume in addition that for some $\gamma_0 \geq 1$

\[
(28) \quad \forall s_1 > 0 \exists s_0 > 0 \forall s \geq s_0, s' \leq s_1 : M(s + s') \leq \gamma_0 M(s).
\]

Assume furthermore that $K$ increases faster than any polynomial in $sM(s)$. Let $c_1 > \gamma_0$. Then there exists a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$, satisfying the assumptions of Theorem 1.1 such that (4) does not hold for this choice of $c_1$.

**Remark 4.9.** It is not difficult to find functions $M$ which satisfy (28) for any $\gamma_0 > 1$. Take for example $M$ to be a constant, a logarithm or a polynomial. It is also possible to take $M(s) = \exp(s^\alpha)$ for $\alpha \in (0, 1)$. On the other hand the example $M(s) = \exp(s)$ does not satisfy this condition for any $\gamma > 1$.

**Remark 4.10.** We think that the condition that $K$ increases faster than a polynomial in $s$ is natural in both parts of the theorem. On the other hand we don’t know whether the growth condition on $K$ in terms of $M(s)$ or $M(2s)$ is a necessary assumption for the conclusion of Theorem 4.8 to hold. Concerning (a) this condition is only necessary in the proof since we do not know whether Lemma 3.1 is valid for $C_f = 0$. Concerning (b) we need it because of the factor $M(|z|)^{1/2}$ appearing in (13).

**Proof.** (a) The claim is proved by having a look into the proof of Theorem 1.1. It is not difficult to see that in (13) is true for any $C_2 > 1$. To get (14) one has to choose $c_1$ in such a way that $K(R)R^{-1} \geq cR^{m+1}$. Since $K$ grows super-polynomially in $s$ this means $c_1 < 1/C_2$. Now observe that in in the final step of the proof in Section 3 before the proof of Lemma 3.1 one can choose any $c < 1$. Here we use that $K(s) \geq c(1 + s)^{-m}M(2s)^2$. Since $C_2$ can be chosen arbitrary close to 1 the first assertion is proved.

(b) Let $\gamma_0 < \gamma < c_1$. First observe that the assumptions of Theorem 4.1 (including (ii')) are satisfied (concerning $m > 1$ see also Remark 4.10). Thus there exists a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$ such that the conclusion of Theorem 4.1 is satisfied. Since $K$ grows faster than any polynomial of $M(s)$ we can withdraw the factor $M(|z|)^{1/2}$ from (13) if we replace $\gamma/c_1$ by 1 in this inequality. Now the function satisfies the assumptions of Theorem 1.1 but it fails to satisfy (2) for our choice of $c_1$ by Theorem 1.1.

5. Application: Local decay rates

Our results can be applied to calculate local decay rates for $C_0$-semigroups. To fix some of our notation let $T = (T(t))_{t \geq 0}$ be a $C_0$-semigroup on a Banach space.
Let $G$ increasing functions satisfying

In Subsection 5.1 we apply the abstract setting from Subsection 5.1 to local energy decay for the wave equation in an odd-dimensional exterior domain. In Subsection 5.2 we naturally restrict our considerations to the case $p = \infty$. A discussion of $L^p$-rates for semigroups and an application to the wave equation can be found in [4, Section 6].

5.1. Local decay of $C_0$-semigroups. The following is an immediate consequence of our main result Theorem 1.1.

Corollary 5.1 (to Theorem 1.1). Let $T$ be a $C_0$-semigroup on a Banach space $(X, \| \cdot \|)$ with generator $A$ and $\omega_0(T) \geq 0$. Let $P_1$ and $P_2$ be two bounded operators on $X$, let $x \in X$ and let $1 < p \leq \infty$. Let $M, K : \mathbb{R}_+ \to [2, \infty)$ be continuous and increasing functions satisfying

(i) $\forall s > 1 : K(s) \geq \max \{ s, M(s) \}$,
(ii) $\exists \varepsilon \in (0, 1) : K(s) = O \left( e^{R(s)} \right)$ as $s \to \infty$.

Let $G(z) = P_2(z - A)^{-1}P_1x$ for $\Re z > 0$. Assume that $G$ extends analytically to the domain $\Omega_M \cup \mathbb{C}_+$ and satisfies the estimate

$$\|G(z)\| \leq K(|\Im z|) \text{ for } z \in \Omega_M.$$  

Assume furthermore that $(t \mapsto \|P_2T(t)P_1x\|) \in L^p(\mathbb{R}_+)$. Then for all $m \in \mathbb{N}_1$ and $\omega > \omega_0(T)$ we have

$$(t \mapsto w_{MK}(t)^m \|P_2T(t)(\omega - A)^{-m}P_1x\|) \in L^p(\mathbb{R}_+)$$

where $MK(s) = M(s) \log(K(s))$.

Remark 5.2. Observe that the condition $(t \mapsto \|P_2T(t)P_1x\|) \in L^p(\mathbb{R}_+)$ is trivially satisfied if $T$ is a bounded $C_0$-semigroup and $p = \infty$. If in this case we also have that $A$ is invertible then - as is clear from the proof - one can also take $\omega = 0$. In the case $P_1 = P_2 = 1$ we note that if $p \neq \infty$ and $(t \mapsto \|T(t)x\|) \in L^p(\mathbb{R}_+)$ is true for all $x \in X$ then by Datko’s theorem (see e.g. [3, Theorem 5.1.2] the semigroup is automatically exponentially stable.

Remark 5.3. In the particular case $P_1 = P_2 = 1$ one typically assumes that the resolvent extends continuously to the imaginary axis and satisfies an estimate $\|(is - A)^{-1}\| \leq M(|s|)$ for $s \in \mathbb{R}$. This then implies that the resolvent extends analytically to $\Omega_M$ and it satisfies (29) with $K$ being a multiple of $M$ in a slightly smaller domain. So in this situation our corollary does not improve known results.

However, our main interest in applying this theorem is to consider the case where $P_1$ and $P_2$ are not the identity. We think that a typical situation is that $M$ is a slowly increasing function (possibly constant) and $K$ is a (possibly much) faster increasing function. That is, we assume that the perturbed resolvent extends to a relatively large domain to the left of the imaginary axis, but only has to satisfy a mild growth condition. We illustrate this philosophy in Subsection 5.2.
Proof. Let us define $f(t) = P_2 T(t)(\omega - A)^{-m} P_1 x$. Then we have for $t > 0$ and for $z \in \Omega_M$

$$f^{(m)}(t) = P_2 T(t)[\omega(\omega - A)^{-1} - 1]^m P_1 x$$

and write

$$\hat{f}(z) = \sum_{j=0}^{m-1} (\omega - z)^{-(j+1)} P_2 (\omega - A)^{-(m-j)} P_1 x + (\omega - z)^{-m} G(z).$$

The second line immediately implies (1) up to a constant factor. The first line implies $\|f^{(m)}\| \in L^p(\mathbb{R}_+)$ since

$$\|P_2 T(t)(\omega - A)^{-1} P_1 x\| = \left\| \int_0^\infty P_2 e^{-\omega^* T(t + \tau) P_1 x} d\tau \right\| \leq \|P_2 T(t) P_1 x\|.$$

Thus the conclusion of the corollary follows from Theorem 1.1. □

5.2. Local energy decay for waves in exterior domains. We want to show that Corollary 5.1 applies naturally to local energy decay for waves in exterior domains. It improves known decay rates and even simplifies the proofs.

Let $\Omega \subseteq \mathbb{R}^d$ be a connected open set with bounded complement and non-empty $C^\infty$-boundary. The dimension $d$ is assumed to be at least 2. We consider the wave equation on this domain:

$$\begin{aligned}
&\begin{aligned}
&u_{tt}(t, x) - \Delta u(t, x) = 0 & (t \in (0, \infty), x \in \Omega), \\
u(t, x) = 0 & (t \in (0, \infty), x \in \partial \Omega), \\
u(0, x) = \nu_0(x), u_t(0, x) = u_1(x) & (x \in \Omega).
\end{aligned}
\end{aligned}$$

(30)

Let us fix a radius $r > 0$ such that the obstacle $K = \mathbb{R}^d \setminus \Omega$ is included in the open ball $B_r$ of radius $r$ and center 0. We define a state (at time $t$) of the system by $x(t) := (u, v)(t) := (u(t), u_t(t))$. We define the local energy of a state by

$$E^{loc}(x) = \int_{\Omega \cap B_r} |\nabla u|^2 + |v|^2 \, dx.$$ 

Clearly equation (31) is well defined for all $u \in C^\infty_c(\Omega)$ and $v \in L^2(\Omega)$. Therefore it is also well defined on the energy space

$$\mathcal{H} = H^1_D(\Omega) \times L^2(\Omega),$$

where $H^1_D(\Omega)$ is the completion of $C^\infty_c(\Omega)$ under the norm given by the quadratic form $u \mapsto \int_{\Omega} |\nabla u|^2$.

The wave equation (30) on the energy space $\mathcal{H}$ can be reformulated in the language of $C_0$-semigroups. Therefore we write $x(t) = (u(t), u_t(t))$ set $x_0 = (u_0, u_1)$ and write

$$\begin{aligned}
&\begin{aligned}
&\dot{x}(t) = Ax(t), \\
x(0) = x_0 \in \mathcal{H}
\end{aligned}
\end{aligned}$$

(32)

where $A = \begin{pmatrix} 0 & 1 \\ \Delta D & 0 \end{pmatrix}$ with $D(A) = D(\Delta_D) \times H^1_D(\Omega)$.

The Dirichlet-Laplace operator $\Delta_D$ has the domain $D(\Delta_D) = \{ u \in H^1_D(\Omega); \Delta u \in L^2(\Omega) \}$, where $\Delta$ denotes the Laplace operator in the sense of distributions. It can be proved that the wave operator $A$ is skew-adjoint (see e.g. [19, Theorem V.1.2]). Therefore the following theorem follows by Stone’s theorem (see e.g. [19, Appendix 1, Theorem 2]).

**Theorem 5.4.** The wave operator $A$ generates a unitary $C_0$-group on $\mathcal{H}$. 

Let \( m \in \mathbb{N}_0 \). We are interested in the uniform decay rate of the local energy with respect to sufficiently smooth initial data, compactly supported in the ball of radius \( r \):

\[
(33) \quad p_m(t) := \sup \left\{ \left( \frac{E^{loc}(x(t))}{||x_0||^{m+1}_m} \right)^{\frac{1}{2}} : x_0 \in H^{m+1}_\text{comp} \times H^m_\text{comp}(\Omega \cap B_r) \right\}.
\]

Here by \( H^{m}_\text{comp}(\Omega \cap B_r) \) we denote all square-integrable functions, supported on \( \Omega \cap B_r \) for which all weak derivatives up to order \( m \) are square-integrable too. It is well known that \( p_0 \) either does not decay to zero, or decays exponentially for \( d \) odd and as \( t^{-d} \) for \( d \) even. Moreover the decay can be characterized by boundedness of the local resolvent of \( A \) on the imaginary axis. We refer to \[25\] and references therein for these facts.

In the following we assume \( m \in \mathbb{N}_1 \). Following the philosophy of the present article we see that we have to investigate the resolvent of \( A \). In the literature on local energy decay it is common to investigate the outgoing resolvent therein for these facts. For \( \Re z > 0 \) and \( f \in L^2(\Omega) \) the outgoing resolvent is defined as a Laplace transform:

\[
R(z)f = \int_0^\infty e^{-zt}u(t)dt
\]

where \( u \) is the first component of the solution to (32) for \( x_0 = (0, f) \). It is not difficult to show that \( w = R(z)f \) then satisfies the stationary wave equation

\[
\left\{ \begin{array}{ll}
 z^2w(x) - \Delta w(x) = f(x) & (x \in \Omega), \\
w(x) = 0 & (x \in \partial \Omega).
\end{array} \right.
\]

There is an important relation between this operator and the resolvent of \( A \): For \( \Re z > 0 \) we have

\[
(35) \quad (z - A)^{-1} = \begin{pmatrix}
 zR(z) & R(z) \\
 z^2R(z) - 1 & zR(z)
\end{pmatrix}.
\]

Let us fix a cut-off function \( \chi \in C_c^\infty(\mathbb{R}^d) \) with \( 0 \leq \chi \leq 1 \) such that \( \chi = 1 \) on a neighbourhood of \( K \). We define the truncated resolvent by \( R_\chi(z) = \chi R(z) \chi \) where we consider \( \chi \) as a multiplication operator on \( L^2(\Omega) \). From the definition we see that the outgoing truncated resolvent is an analytic function in the interior of \( \mathbb{C}_+ \). The next proposition illuminates its behaviour on the other half of the complex plane.

**Proposition 5.5.** (i)\[A\] Appendix B] The truncated outgoing resolvent \( R_\chi \) extends analytically across \( i\mathbb{R} \setminus \{0\} \). Moreover, for any \( \varepsilon > 0 \) \( R_\chi : L^2(\Omega) \to L^2(\Omega) \) is bounded in a small neighbourhood of \( 0 \) intersected with a sector \( \{z \in \mathbb{C} \setminus \{0\} ; |\arg z - \pi| \geq \varepsilon\} \). (ii)\[B\] Corollary V.3.3 together with Remark V.4.3] If the dimension \( d \) is odd \( R_\chi \) extends meromorphically to \( \mathbb{C} \).

In the following we want to restrict our considerations to the odd-dimensional (i.e. \( d \) is odd) case only. By Proposition \[5.5\] together with \[6.3\] we immediately see that \( (z \mapsto \chi(z - A)^{-1} \chi) \) for \( \Re z > 0 \) extends to a meromorphic function \( G_\chi \) on \( \mathbb{C} \) which has no poles on \( i\mathbb{R} \). Here we consider \( \chi \) as an operator on \( \mathcal{H} \) acting as \( \chi(u_0, u_1) = (\chi u_0, \chi u_1) \). Since the spectrum of \( A \) is the entire imaginary axis the equality \( G_\chi(z) = \chi(z - A)^{-1} \chi \) does not hold for \( \Re z < 0 \) in general.
The following proposition is well-known in the literature on exterior wave equations. The proof is not difficult but rather lengthy. Unfortunately we could not find a proof in the literature but we refer to [8, Section] for a similar statement and the idea of the proof.

**Proposition 5.6.** Let $\delta > 0$ and let $\tilde{\chi}$ be defined as $\chi$ but with $\tilde{\chi} = 1$ on the support of $\chi$. Let $z$ with $-\delta < \Re z < 0$ be no pole of $R_{\chi}$, then

$$\|G_{\chi}(z)\| \leq C(1 + |\Im z|) \|R_{\tilde{\chi}}(z)\|_{L^2 \to L^2}$$

holds with a constant $C > 0$ independent of $z$. The reverse inequality - with a different constant and $\tilde{\chi}$ replaced by $\chi$ - is also true.

It can happen that a whole strip $\{z \in \mathbb{C}; -\delta < \Re z < 0\}$ is free of poles - see for instance [16]. In [12] the impact of the presence of such a strip on local energy decay was studied. There it was shown in a first step that such a strip implies that the norm of $G_{\chi}$ can be estimated by $C \exp(C|\Im(z)|^\alpha)$ on this strip for some $\alpha \geq 1$. Indeed $\alpha = d - 1$ in this article but it was not shown that this is optimal. In a second step the authors showed that this implies a bound of the form $(1 + |\Im z|)^\alpha$ on $G_{\chi}$ in a region of the form $\{z \in \mathbb{C}; -C(1 + |\Im z|)^{-\alpha} < \Re z < 0\}$. Finally in a third step they applied a Tauberian theorem (more precisely [22, Proposition 1.4]) to get a $(\log(t)/t)^{1/\alpha}$ decay rate.

However, given a polynomial bound on the resolvent it would be desirable to have a polynomial decay of the local energy - without the logarithmic loss! If we were not in a local situation then the results of [6] would help us to deduce our desired result without the logarithmic loss. Unfortunately it is not known whether [6, Theorem 2.4] generalizes to local decay of semigroups on Hilbert spaces. In the following we show that with the help of Corollary 5.1 we get rid of the logarithmic loss. It even simplifies the proof in the sense that the second step is not necessary anymore since our preconditions in Corollary 5.1 are fulfilled by the local resolvent on the strip.

By the preceding discussion it is reasonable to assume from now on the following conditions to be satisfied:

(i) There is a $\delta > 0$ such that $R_{\chi}$ has no poles in $S_\delta = \{z \in \mathbb{C}; -\delta < \Re z < 0\}$.

(ii) There is a continuous and increasing function $\tilde{M} : \mathbb{R}_+ \to [2, \infty)$ satisfying $\tilde{M}(s) \geq c\log(2 + s)$ for any $s \geq 0$ such that $|\Im z| \|R_{\chi}(z)\|_{L^2 \to L^2} \leq C \exp(C\tilde{M}(|\Im z|))$ holds for all $z \in S_\delta$.

Under these assumptions we can prove:

**Theorem 5.7.** Let $d$ be odd and let (i) and (ii) above be satisfied. Let $m \in \mathbb{N}_1$. Then

$$p_m(t) \leq \frac{C}{\tilde{M}^{-1}(c_1 t)^m}$$

holds for a sufficiently small constant $c_1$ and a sufficiently large constant $C$. Here $\tilde{M}^{-1}$ denotes the right-continuous right-inverse of $\tilde{M}$.

**Proof.** For $\Re z > 0$ let $G_{\chi}(z) = \chi(z - A)^{-1}\chi$. Assumptions (i) and (ii) together with Proposition 5.6 imply that $G_{\chi}$ extends analytically to $S_\delta \cup \mathbb{C}_+$ and satisfies

$$\|G_{\chi}(z)\| \leq C \exp(C\tilde{M}(|\Im z|))$$

for $z \in S_\delta$. 
Thus by Corollary 5.1 (put $M = \delta$ and $K = C \exp o(C \tilde{M})$) we get (uniformly in $x_0 \in \mathcal{H}$)

$$
\| e^{tA}(1 - A)^{-m}x_0 \| \leq \frac{C}{M_{c_1 t}^{-1}(c_1 t)^m} \| x_0 \|.
$$

The uniformity in $\mathcal{H}$ follows from the closed graph theorem. For simplicity we assume $m = 1$ in the following. The general case can be treated almost the same way.

Let $\chi_1 \in C_c^\infty (\mathbb{R}^d)$ be a function such that $0 \leq \chi_1 \leq 1$ and $\chi_1 = 1$ on $\text{supp} \chi$. Of course Propositions 5.5 and 5.6 remain valid if one replaces $\chi$ by $\chi_1$. Note that the commutator $[\chi, 1 - A]$ is a bounded operator on $\mathcal{H}$. Let $x_1 = (1 - A)^{-1}x_0 \in D(A)$. Observe

$$
\| e^{tA}\chi x_1 \| \leq \| e^{tA}(1 - A)^{-1}\chi x_0 \| + \| \chi(\chi_1 e^{tA}(1 - A)^{-1}\chi) \chi_1 (1 - A) x_1 \|
$$

$$
\leq \frac{C}{M_{c_1 t}^{-1}(c_1 t)}(\| x_0 \| + \| x_1 \|)
$$

$$
\leq \frac{C}{M_{c_1 t}^{-1}(c_1 t)} \| x_1 \|_{D(A)}.
$$

Without loss of generality we may assume that $\chi = 1$ on $B_r$. Observe that the norm of elements of $D(A)$, supported in $\overline{\Omega} \cap B_r$, is equivalent to the norm in the space $H^2 \times H^1(\Omega)$. This follows from maximal regularity of the Dirichlet-Laplace operator on the bounded and smooth domain $\Omega \cap B_r$. Thus the last inequality (restricted to those $x_1$ with support in $B_r$) implies the conclusion of the theorem. □

**Acknowledgements.** I am most grateful to Ralph Chill and Yuri Tomilov for valuable discussions on the topic of this article. I would like to thank the department of mathematics of the Nicolaus Copernicus University in Toruń for its hospitality. The idea to work on this topic came to me during a visit in december 2016.

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