Zero-Temperature Dynamics of $\pm J$ Spin Glasses and Related Models

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**Abstract**

We study zero-temperature, stochastic Ising models $\sigma_t$ on $\mathbb{Z}^d$ with (disordered) nearest-neighbor couplings independently chosen from a distribution $\mu$ on $\mathbb{R}$ and an initial spin configuration chosen uniformly at random. Given $d$, call $\mu$ type $I$ (resp., type $F$) if, for every $x$ in $\mathbb{Z}^d$, $\sigma_t^x$ flips infinitely (resp., only finitely) many times as $t \to \infty$ (with probability one) — or else mixed type $M$. Models of type $I$ and $M$ exhibit a zero-temperature version of “local non-equilibration”. For $d = 1$, all types occur and the type of any $\mu$ is easy to determine. The main result of this paper is a proof that for $d = 2$, $\pm J$ models (where $\mu = \alpha \delta_J + (1 - \alpha) \delta_{-J}$) are type $M$, unlike homogeneous models (type $I$) or continuous (finite mean) $\mu$’s (type $F$). We also prove that all other noncontinuous disordered systems are type $M$ for any $d \geq 2$. The $\pm J$ proof is noteworthy in that it is much less “local” than the other (simpler) proof. Homogeneous and $\pm J$ models for $d \geq 3$ remain an open problem.

**KEY WORDS:** spin glass; nonequilibrium dynamics; deep quench; mixed type.

**I. INTRODUCTION AND RESULTS**

In this paper, we study a specific class of continuous time Markov processes $\sigma_t(t) = \sigma^t(\omega)$ with random environments. These correspond to the zero-temperature stochastic dynamics of disordered nearest-neighbor Ising models $[1-3]$. (Zero-temperature dynamics with a different sort of disorder are studied in $[5]$.) The state space is $\mathcal{S} = \{-1, +1\}^\mathbb{Z}^d$ and the initial state $\sigma^0$ is a realization of i.i.d. symmetric Bernoulli variables. The only transitions are single spin flips, where $\sigma^t_{x^+} = -\sigma^t_{x^-}$, and the transition rates depend on a realization $\mathcal{J}$ of i.i.d. random couplings $J_{x,y}$, indexed by nearest-neighbor pairs (with Euclidean distance $||x - y|| = 1$) of sites in $\mathbb{Z}^d$, with common distribution $\mu$ on $\mathbb{R}$. For a given $\mathcal{J}$, the rate for a flip at $x$ from state $\sigma^t = \sigma$ is 1 or 1/2 or 0 according to whether
\[ \Delta H_t(\sigma) \equiv 2 \sum_{y: ||y-x||=1} J_{x,y} \sigma_x \sigma_y \]  

(corresponding to the change in energy) is negative or zero or positive. The joint distribution of \( \mathcal{F} \), \( \sigma^0 \), and \( \omega \) will be denoted \( P \).

Zero-temperature dynamics without disorder have been much studied in the physics literature as a model of “coarsening” [3] and more recently because of the interesting phenomenon of persistence [4–11]. A natural question in both the disordered and non-disordered models is whether \( \sigma^t \) has a limit (with \( P \)-probability one) as \( t \to \infty \) or equivalently whether for every \( x, \sigma^t_x \) flips only finitely many times. More generally, one may call such an \( x \) an \( \mathcal{F} \)-site (\( \mathcal{F} \) for finite) and otherwise an \( \mathcal{I} \)-site (\( \mathcal{I} \) for infinite). The nonexistence of a limit corresponds to the type of “recurrence” studied in a general context and applied to various interacting particle systems in [12].

The issue of whether \( \sigma^t \) has no limit is the zero-temperature version of whether there is “local non-equilibration” at positive temperature [13]. At positive temperature, local non-equilibration concerns not the recurrence of the spin configuration \( \sigma^t \) but rather of a dynamical probability measure \( \nu_{t,\tau(t)} \) corresponding to averaging over the dynamics for times between \( t - \tau(t) \) and \( t \) (for fixed \( \mathcal{J} \), \( \sigma^0 \) and dynamics realization \( \omega \) up to time \( t - \tau(t) \)). For \( \tau(t) \) growing slowly with \( t \), \( \nu_{t,\tau(t)} \) should be (approximately) a pure Gibbs state at the given temperature (on a lengthscale growing with \( t \)). Local non-equilibration would mean that the system does not converge to a single limiting pure state as \( t \to \infty \) (depending on \( \mathcal{J}, \sigma^0, \omega \)). Although this type of non-equilibration has been proved to occur for the \( d = 2 \) homogeneous Ising model [13], it is an open problem whether it occurs at positive temperature for spin glasses (for any \( d \geq 2 \)). The focus of this paper is the study of the analogous problem at zero temperature for certain classes of spin glasses and related disordered systems.

By translation-ergodicity, the collection of \( \mathcal{F} \)-sites (resp., \( \mathcal{I} \)-sites) has (with \( P \)-probability one) a well-defined non-random spatial density \( \rho_{\mathcal{F}} \) (resp., \( \rho_{\mathcal{I}} \)). The densities \( \rho_{\mathcal{F}} \) and \( \rho_{\mathcal{I}} \) depend only on \( d \) and \( \mu \) and of course satisfy \( \rho_{\mathcal{F}} + \rho_{\mathcal{I}} = 1 \). For each \( d \), one may then characterize \( \mu \) (or more accurately, one should characterize the pair \( (d, \mu) \)) as being type \( \mathcal{F} \) or \( \mathcal{I} \) or \( \mathcal{M} \) (for mixed) according to whether \( \rho_{\mathcal{F}}(d, \mu) = 1 \) or \( \rho_{\mathcal{I}}(d, \mu) = 1 \) or \( 0 < \rho_{\mathcal{F}}, \rho_{\mathcal{I}} < 1 \).

Before reviewing previous characterization results and presenting new ones, we briefly discuss some important special cases of \( \mu \). Ferromagnetic models are those where \( \mu \) is supported on \([0, \infty)\) (so that each \( J_{x,y} \geq 0 \)) and homogeneous ones are those without disorder (i.e., where \( \mu = \delta_{\mu R} \)). In the homogeneous ferromagnet, sites flip at rate 1 or 1/2 or 0 according to whether they disagree with a strict majority or exactly one half or a strict minority of their nearest neighbors. Antiferromagnetic models are those with \( J_{x,y} \leq 0 \); on the lattice \( \mathbb{Z}^d \), these are equivalent to ferromagnetic models under the relabelling (or “gauge”) transformation in which \( \sigma_x \to -\sigma_x \) for each \( x \) on the odd sublattice while \( J_{x,y} \to -J_{x,y} \) for every \( \{x, y\} \) (leaving \( \mathcal{I} \) unchanged). Spin glasses (of the Edwards-Anderson type [14]) may be defined as those models where \( \mu \) is symmetric (under \( J_{x,y} \to -J_{x,y} \) — the most popular examples being mean zero Gaussian distributions and the \( \pm J \) spin glasses, where \( \mu = (1/2)\delta_J + (1/2)\delta_{-J} \) with \( J > 0 \)) (a standard review is [15]). As we shall see, the family of measures \( \mu = \alpha \delta_J + (1-\alpha)\delta_{-J} \) with \( \alpha \in [0, 1] \), including homogeneous ferromagnets and antiferromagnets and \( \pm J \) spin glasses, is the most difficult to characterize. Henceforth, we use the term \( \pm J \) model to refer to any \( \mu \) of the form \( \alpha \delta_J + (1-\alpha)\delta_{-J} \) with \( J > 0 \) and \( 0 < \alpha < 1 \).

Our review of known results begins with a proposition classifying all \( \mu \)'s for \( d = 1 \). The type \( \mathcal{I} \) nature of one-dimensional homogeneous ferromagnets was stated in [14], but is equivalent to a
result in \([16]\) (see also \([17]\)) because for \(d = 1\), the dynamics is the same as that for annihilating random walks or the usual voter model. It is possible that other parts of the proposition may also not be new.

**Proposition 1.** Set \(d = 1\). Then \(\mu\) is type \(I\) if it is \(\alpha \delta_J + (1 - \alpha) \delta_{-J}\) with \(J \geq 0\) and \(\alpha \in [0, 1]\); \(\mu\) is type \(F\) if it is either continuous or one of the form \(\alpha \delta_J + \beta \delta_{-J} + \nu\) with \(J > 0\), \(0 < \alpha + \beta < 1\) and a continuous \(\nu\) supported on \([-J, J]\); all other \(\mu\)'s are type \(M\).

**Proof.** Since \(d = 1\), any \(\mathcal{J} = (J_{x,x+1} : x \in \mathbb{Z})\) is equivalent (by an appropriate gauge transformation) to a ferromagnetic model with \(J_{x,x+1}\) replaced by \(|J_{x,x+1}|\). Hence, for the remainder of this proof, we can and will assume that \(\mu\) is replaced by \(\pi\), the common distribution of the \(|J_{x,x+1}|\)'s, and thus that each \(J_{x,x+1} > 0\).

If \(\pi = \delta_J\), then it is trivially type \(I\) for \(J = 0\), while for \(J > 0\), we have a homogeneous ferromagnet, for which a proof that it is type \(I\) may be found in \([1]\). For any other \(\pi\), one looks for sites \(z\) such that

\[
J_{z,z+1} > J_{z-1,z}, J_{z+1,z+2}.
\]

(2)

Since \(\pi \neq \delta_J\), this has a strictly positive probability, and hence, by translation-ergodicity, there will be (with \(P\)-probability one) a doubly infinite sequence of such sites \(z_n\) (with positive density). The conditions \((2)\) imply that \(\Delta H_z(\sigma^t)\) and \(\Delta H_{z+1}(\sigma^t)\) (see \([1]\)) are both negative or both positive according to whether \(\sigma^t_x \sigma^t_{x+1} = -1\) or \(+1\). It follows that if \(\sigma^0_x \sigma^0_{x+1} = +1\), then \(\sigma^t_x\) and \(\sigma^t_{x+1}\) will never flip, while \(\sigma^0_x \sigma^0_{x+1} = -1\) implies that (with probability one) one of them will flip exactly once and there will be no other flips of either. This already shows that \(\rho_F > 0\).

If \(\pi\) is continuous, we may rely on the proof in \([1]\) or argue as follows. Restricting attention to an interval \(\{z, z + 1, \ldots, z'\}\) with \(z = z_{n-1} + 1\) and \(z' = z_n\) (where \(z_{n-1}\) and \(z_n\) are successive sites from the special sequence defined above) and times after \(\sigma^t_x\) and \(\sigma^t_{x+1}\) have ceased flipping, we have a Markov process with a finite state space — the configurations of \((\sigma_x : z + 1 \leq x \leq z' - 1)\).

Because of the continuity of \(\pi\), each flip in this interval will strictly lower the energy,

\[
- \sum_{x = z}^{z'-1} J_{x,x+1} \sigma_x \sigma_{x+1}.
\]

(3)

Since this energy is (for a fixed \(\mathcal{J}\)) bounded below, the process in the interval must eventually stop flipping and reach an absorbing state. Applying this argument to every such interval, we conclude that a continuous \(\pi\) is type \(F\).

If \(\pi = \alpha \delta_J + \pi\) with \(J > 0\), \(0 < \alpha < 1\) and \(\pi\) a continuous measure on \([0, J]\), then we modify the above argument as follows. Instead of looking for sites \(z\) satisfying \((2)\), we look for runs of the value \(J\), i.e., for sites \(z < w\) where

\[
J_{z-1,z} < J, J_{z,z+1} = J, J_{z+1,z+2} = J, \ldots, J_{w-1,w} = J, J_{w,w+1} < J.
\]

(4)

Now let \(\{z_n, z_n + 1, \ldots, w_n\}\), as \(n\) varies over \(\mathbb{Z}\), be the doubly infinite sequence of run intervals. Focusing on the configurations in one of these intervals, and noting that the transition rates in that interval do not depend on the values of \(\sigma_{z_n-1}\) or \(\sigma_{w_n+1}\), we observe that the two constant configurations are absorbing and accessible from any other configuration, so that the process eventually reaches one of these two absorbing configurations. To conclude that this \(\pi\) is type \(F\), we need to show for each \(n\), that (after \(\sigma^t_{w_n-1}\) and \(\sigma^t_{z_n}\) have ceased flipping) the configuration in the interval
{w_{n-1}, w_{n-1} + 1, \ldots, z_n} will also reach an absorbing state. But this follows exactly as in the argument above for continuous $\overline{\mu}$, with the continuity of $\overline{\mu}$ replacing that of $\overline{\nu}$.

To complete the proof of the proposition, it remains to show that if $\overline{\nu}$ is neither $\delta_J$ nor continuous nor of the form $\alpha \delta_J + \sigma$ as above, then $\rho_{\overline{\nu}} > 0$ — i.e., some spins flip infinitely often. But for any $\overline{\nu}$ now under consideration, there will exist some $J' > 0$ and sites $z'$ and $z'' = z' + 3$ such that $z'$ and $z''$ each satisfy (3).

$$J_{z'+1,z'+2} = J_{z'+2,z'+3} (\equiv J_{z''-1,z''}) = J', \quad (5)$$

and

$$\sigma_{z'}^0 = \sigma_{z'+1}^0 = +1, \quad \sigma_{z''}^0 = \sigma_{z''+1}^0 = -1. \quad (6)$$

Under these circumstances, $\sigma_{z'+1}^t$ and $\sigma_{z'+3}^t (\equiv \sigma_{z''}^t)$ will never flip, but $\sigma_{z'+2}^t$ will flip infinitely many times because its flip rate will always be $1/2$.

Among the main results of [1] are extensions of the conclusions of Proposition 1 to $d = 2$ for the homogeneous ferromagnet (or antiferromagnet) and to $d \geq 2$ for continuous $\mu$ (satisfying some conditions). In particular, it is proved there that a continuous $\mu$ with finite mean (i.e., with $E(|J_{x,y}|) < \infty$) is type $\mathcal{F}$ for any $d$. (Certain continuous $\mu$’s with infinite means are also shown in [1] to be type $\mathcal{F}$ by the very different percolation-theoretic methods of [13]s.) The continuous finite mean $\mu$ result is actually a corollary of the following more general theorem about flips that strictly decrease the energy, which we will apply to $\pm J$ models.

**Theorem 2.** [1] For any $d$ and any $\mu$ with finite mean, (with $P$-probability one) at each site $x$ in $\mathbb{Z}^d$, there are only finitely many flips with $\Delta H_x(\sigma) < 0$.

The cases left open by the results of [1] were: (i) the homogeneous ferromagnet or antiferromagnet for $d \geq 3$, (ii) $\pm J$ models for $d \geq 2$, (iii) other noncontinuous $\mu$’s for $d \geq 2$ and finally (iv) general continuous $\mu$’s with infinite means for $d \geq 2$. The main results of this paper are the following two theorems that resolve (ii) for $d = 2$ and (iii) for $d \geq 2$.

We remark that part of the proof of Theorem 4 can be easily applied to show that $\rho_{\mathcal{F}} > 0$ for any continuous $\mu$; thus the $\mu$’s of (iv) must either be type $\mathcal{F}$ or type $\mathcal{M}$. Our guess is that (iv) is type $\mathcal{F}$ for any $d \geq 2$. As for (i), there is some numerical evidence [7] that homogeneous models remain type $\mathcal{I}$ for $d = 3$ but perhaps not for $d > 4$. For more discussion of physical background and open problems, see [2][13][19].

**Theorem 3.** $\pm J$ models are type $\mathcal{M}$ for $d = 2$.

**Theorem 4.** For any $d \geq 2$, if $\mu$ is neither continuous nor of the form $\alpha \delta_J + (1 - \alpha)\delta_{-J}$ for some $J \geq 0$ and $0 \leq \alpha \leq 1$, then $\mu$ is type $\mathcal{M}$.

The proof of Theorem 4, presented in Section 2 of the paper, is quite easy. In Sections 3 and 4, we give the proof of Theorem 3; the demonstration that $\rho_{\mathcal{I}} > 0$ (Section 3) is fairly easy but the proof that $\rho_{\mathcal{F}} > 0$ (Section 4) is not. The arguments used for the latter may be of general interest. In the proofs of both parts of Theorem 3, an important role is played by the frustration/contour representation of the $\pm J$ model for $d = 2$ (see, e.g., [20][24]); there are natural extensions of this representation for $d \geq 3$ that could be useful in determining the type in these higher dimensions.

As we shall see, there is an interesting conceptual difference between the proofs of these two theorems. The proof of Theorem 4 is essentially local in that we demonstrate that certain sites
are type \( I \) and certain are type \( F \) from knowledge of the couplings and spins (at time zero) in finite regions containing those sites. The proof of Theorem 3, on the other hand, is not local, in that using the local knowledge, we only manage to deduce that some site among a finite number must be type \( I \) (or \( F \)). This is because we are unable to find local configurations of couplings and spins that completely insulate a local region from the surroundings. The (unknown in advance) influence from the outside prevents a determination of the type of individual sites. Although we have not proved it, we suspect that \( \pm J \) models are intrinsically nonlocal in the strong sense that the type of any site cannot be ascertained from strictly local knowledge.

II. OTHER THAN \( \pm J \) MODELS: PROOF OF THEOREM 4

**Proof that** \( \rho_F > 0 \). Let \( C \) denote some cube in \( \mathbb{Z}^d \), such as the unit cube consisting of vertices \( x = (x_1, \ldots, x_d) \) with each \( x_i = 0 \) or 1, and let \( \mathcal{E}_i(C) \) (resp., \( \mathcal{E}_o(C) \)) denote the set of nearest-neighbor edges \( \{x, y\} \) such that both \( x \) and \( y \) (resp., exactly one of \( x \) and \( y \)) belong to \( C \). It suffices to show that with positive probability, the \( J_{x,y} \)'s for \( \{x, y\} \in \mathcal{E}_i(C) \cup \mathcal{E}_o(C) \) and the \( \sigma_0^x \)'s for \( x \in C \) are such that \( \sigma_t^x \) never flips for \( x \in C \).

To do this, we note that for each \( \mu \) of Theorem 4, \( |J_{x,y}| \) is nonconstant. Hence there exists some \( J' > 0 \) such that \( P(A_{J'}^+ \cup A_{-J'}^-) > 0 \), where \( A_{J'}^+ \) is the event that \( |J_{x,y}| > J' \) and \( \text{sgn}(J_{x,y}) \) is the constant value \( \pm 1 \) for every \( \{x, y\} \in \mathcal{E}_i(C) \), while for every \( \{x, y\} \) in \( \mathcal{E}_o(C) \), \( |J_{x,y}| \leq J' \). Let \( B^+(C) \) (resp., \( B^-(C) \)) denote the event that \( \sigma^0_x : x \in C \) is constant (resp., is one of the two checkerboard patterns). Then either \( A_{J'}^+(C) \cap B^+(C) \) or \( A_{-J'}^-(C) \cap B^-(C) \) (or both) have positive probability. But the occurrence of either one implies that \( \sigma_t^x \) never flips for \( x \) in \( C \) and so \( \rho_F > 0 \).

To show that \( \rho_I > 0 \), we use a slightly more complicated geometric construction involving a site \( w \) (e.g., the origin) and \( 2d \) disjoint cubes \( C_1, \ldots, C_{2d} \) that are neighbors of \( w \) in the sense that for each \( j \), \( w \notin C_j \) but \( C_j \) contains exactly one nearest neighbor \( z_j \) of \( w \) (see Fig. 1). Since \( \mu \) is not continuous, it has an atom at some value \( \tilde{J} \). We again construct events involving the couplings and spins near \( w \), but now the construction depends on which of two cases \( \mu \) falls into.
FIG. 1. Geometric construction demonstrating that a positive fraction of spins flip infinitely often for all \( d \geq 2 \), in noncontinuous disordered systems other than \( \pm J \) models. In this \( d = 2 \) figure, filled circles and solid lines denote respectively sites and edges of the original \( \mathbb{Z}^2 \) lattice. Here, the spin at site \( w \) flips infinitely often, given the events discussed in the text in Sec. 2.

**Proof that \( \rho_X > 0; \) Case 1.** Suppose \( \mu = \alpha \delta_J + \beta \delta_{-J} + \nu \) with \( J > 0 \), \( 0 < \alpha + \beta < 1 \) and a continuous \( \nu \) supported on \([-J, J]\). Then either \( J \) or \(-J\) (or both if \( \alpha, \beta > 0 \)) will work for \( \bar{J} \). We now define \( D_{\bar{j}, j} \) to be the event that \( x, y \in \bar{J} \) for every \( \{x, y\} \) in \( E_i(C_j) \) and for \( \{x, y\} = \{z_j, w\} \), but for every other \( \{x, y\} \) in \( E_o(C_j) \), \(|J_{x,y}| < |\bar{J}|\). The event \( D_{\bar{j}} \equiv \cap_{j=1}^{2d} D_{\bar{j}, j} \) has positive probability. Now, for either value of \( \text{sgn}(\bar{J}) \), consider the events \( B^+ \) and \( B^- \), defined as

\[
B^\pm = \cap_{j=1}^{2d} (B^\pm(C_j) \cap \{\sigma_{z_j}^w = (-1)^j\})
\]

and note that \( P(D_{\bar{j}} \cap B^\pm(\text{sgn}(\bar{J}))) > 0 \). We claim that if \( D_{\bar{j}} \cap B^\pm(\text{sgn}(\bar{J})) \) occurs, then \( \sigma^t_w \) flips infinitely many times and thus \( \rho_X > 0 \). To see this, note that, very much as in the proof above that \( \rho_X > 0 \), if \( D_{\bar{j}, j} \cap B^\pm(\text{sgn}(\bar{J}))(C_j) \) occurs (and here we use the fact that \( d \neq 1 \)), then no site in \( C_j \) ever flips. If in addition \( \sigma^0_{z_j} = (-1)^j \) for each \( j \), then \( w \) has at all times exactly \( d \) neighbors with \( \sigma_x = +1 \) and \( d \) with \( \sigma_x = -1 \), so its rate for flipping is always \( 1/2 \) and it will flip infinitely many times.

**Proof that \( \rho_X > 0; \) Case 2.** For any \( \mu \) satisfying the hypotheses of Theorem 4 that is not in Case 1, \( \bar{J} \) may be chosen so that \(|J_{x,y}| > |\bar{J}|\) with positive probability. We now define \( D^+_{\bar{j}, j} \) and \( D^-_{\bar{j}, j} \) as

\[
D^\pm_{\bar{j}, j} = A^\pm_{\bar{j}}(C_j) \cap \{J_{z_j, w} = \bar{J}\}
\]

and \( D^\pm_{\bar{j}} = \cap_{j=1}^{2d} D^\pm_{\bar{j}, j} \), and note that \( P(D^+_{\bar{j}} \cup D^-_{\bar{j}}) > 0 \). With \( B^\pm \) defined in (7), we have that either \( D^+_{\bar{j}} \cap B^+ \) or \( D^-_{\bar{j}} \cap B^- \) (or both) have positive probability. But if either occurs, then, as in Case 1, \( \sigma^t_z = (-1)^j \) for all \( t \) and \( \sigma^t_w \) will flip infinitely many times, which completes the proof.
III. TWO-DIMENSIONAL $\pm J$ MODELS: $\rho_{I} > 0$.

We begin this section by introducing the frustration/contour representation for the $\pm J$ model that we will use throughout this section and the next for the proof of Theorem 3. We then give the proof that $\rho_{I} > 0$, which concludes with a general lemma about recurrence that will also be used (many times) in the next section for the proof that $\rho_{F} > 0$.

The frustration/contour representation (see, e.g., [20–24]) uses variables $(\Phi, \Gamma)$ associated with the dual lattice $Z^{2*} \equiv Z^2 + (1/2, 1/2)$, that are determined by $(J, \sigma)$. A (dual) site in $Z^{2*}$ may be identified with the plaquette $p$ in $Z^2$ of which it is the center, and is called frustrated for a given $J$ if an odd number of the four couplings $J_{x,y}$ making up the edges of that $p$ are antiferromagnetic; $\Phi$ is then the set of frustrated (dual) sites. Thus $\Phi$ is determined completely by $J$, and it is not hard to see that every subset $\Phi$ of $Z^{2*}$ arises from some $J$. The edge $\{x, y\}$ in $Z^{2*}$, dual to (i.e., the perpendicular bisector of) the edge $\{x, y\}$ of $Z^2$, is said to be unsatisfied for a given $J$ and $\sigma$, if $\text{sgn}(J_{x,y} \sigma_x \sigma_y) = -1$ (and satisfied otherwise); $\Gamma$ is then the set of unsatisfied (dual) edges. We say that $(J, \sigma)$ gives rise to $(\Phi, \Gamma)$, and that $\Gamma$ is compatible with $J$ or with $\Phi$ if there exists some $\sigma$ such that $(J, \sigma)$ gives rise to $(\Phi, \Gamma)$. We define $\partial \Gamma$, the boundary of $\Gamma$, as the set of (dual) sites that touch an odd number of (dual) edges of $\Gamma$; then $\Gamma$ is compatible with $\Phi$ if and only if $\Phi = \partial \Gamma$.

A (site self-avoiding) path in $Z^{2*}$ consisting of edges from $\Gamma$ will be called a domain wall. For a given $\Phi$, domain walls can terminate (i.e., with no possibility of continuation) only on frustrated sites; this is because any termination site touches exactly one edge of $\Gamma$ and thus belongs to $\partial \Gamma$ ($= \Phi$).

For a given $J$ or $\Phi$, the Markov process $\sigma^t$ determines a process $\Gamma^t$, that is easily seen to also be Markovian. The transition associated with a spin flip at $x \in Z^2$ is a local “deformation” of the contour $\Gamma^t$ at the (dual) plaquette $x^*$ in $Z^{2*}$ that contains $x$; this deformation interchanges the satisfied and unsatisfied edges of $x^*$ and leaves the boundary $\Phi$ of $\Gamma^t$ unchanged. The only transitions with nonzero rates are those where the number of unsatisfied edges starts at $k = 4$ or $3$ or $2$ and ends at $0$ or $1$ or $2$, respectively; transitions with $k = 4$ or $3$ (resp., $k = 2$) correspond to energy-lowering (resp., zero-energy) flips and have rate $1$ (resp., $1/2$). We will continue to use the terms flip, energy-lowering, etc. for the transitions of $\Gamma^t$.

Proof that $\rho_{I} > 0$. This is by far the easier part of the proof of Theorem 3 and uses a strategy that is only a slight extension of the type of argument used in the previous section to prove $\rho_{I} > 0$ in Theorem 4. As in that proof, we will consider an event of positive probability, here denoted $D$, involving the frustration configuration in a finite region (and thus the values of only finitely many couplings). Unlike that proof, we will not then intersect $D$ with some event involving $\sigma^0$ to insure that for all $t$, some site $x$ has a positive flip rate. Instead, we will show that given $D$, and any spin (or contour) configuration in a certain fixed square $C$ of $Z^2$, there must be at least one site in $C$ with a positive flip rate. This will insure that, conditional on $D$, at least one site in $C$ will flip infinitely many times.

The region $C$ is a $6 \times 6$ square of $Z^2$ and $D$ is defined in terms of $\Phi$ restricted to the $5 \times 5$ square $\Lambda^*$ of sites of $Z^{2*}$ contained within $C$. We choose $D$ as the event that the frustrated sites of $\Lambda^*$ are exactly the nine sites (out of 25) indicated in Figure 2. These nine sites consist of a center site $w_c$ and four adjacent pairs of sites to the Southeast, Northeast, Northwest and Southwest of $w_c$. 

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FIG. 2. Geometric construction demonstrating that a positive fraction of spins flip infinitely often in $\pm J$ models for $d = 2$. In this figure the filled circles are sites in $\mathbb{Z}^2$, the empty circles are unfrustrated sites in the (dual) $\mathbb{Z}^{2*}$ lattice, and each empty circle covered by an $\times$ is a frustrated site in $\mathbb{Z}^{2*}$. Dashed lines correspond to edges in $\mathbb{Z}^{2*}$. The significance of the $\mathbb{Z}^{2*}$-sites $w_c$, $w_1$, and $w_2$ is discussed in Sec. 3.

We have to show that for any $\Gamma$ compatible with $D$, there is at least one site in $C$ (or equivalently, one (dual) plaquette touching $\Lambda^*$) with a positive flip rate, i.e., with at least two unsatisfied edges. In fact, if $D$ occurs, there must be a domain wall $\gamma_c$ starting from $w_c$; this is because $w_c$ is frustrated and so either one or three of the edges touching it belong to $\Gamma$. Either $\gamma_c$ has a “bend” within $\Lambda^*$ and thus the (dual) plaquette just inside the bend has a positive flip rate (since it has two or more unsatisfied edges) or else $\gamma_c$ runs straight out of $\Lambda^*$. In the latter case, by the invariance of $D$ with respect to rotations by $\pi/2$, we may assume (without loss of generality) that $\gamma_c$ runs from $w_c$ to the East and passes just above the (dual) edge joining the two Southeastern sites (that we will denote $w_1, w_2$). But then there must be another domain wall $\gamma_1$ starting from $w_1$. Either $\gamma_c$ and $\gamma_1$ together determine a positive flip rate site or else $\gamma_1$ runs from $w_1$ straight out of $\Lambda^*$ to the South. But then there must be another domain wall $\gamma_2$ starting from $w_2$, that (together with $\gamma_c$ and $\gamma_1$) will determine a positive flip rate site, no matter what direction it runs off to.

Let $A$ denote the set of $\Gamma$ configurations such that there is a site in $C$ with a positive flip rate and let $B$ denote the event that there is a spin flip in $C$ at some time $t \in [0, 1]$. It is easy to see that for some $\alpha > 0$,

$$\Gamma \in A \implies P(B|\Gamma^0 = \Gamma) \geq \alpha.$$  \hfill (9)

It follows from Lemma 5 below that conditional on $D$, there will (with conditional probability one) be infinitely many spin flips in $C$ and hence some site in $C$ will flip infinitely many times. Since a positive density of the translates of the event $D$ must occur (with probability one), we conclude that $\rho_D > 0$ as desired.

**Lemma 5.** Let $Z_t$ be a continuous-time Markov process with state space $Z$ and time-homogeneous transition probabilities, and let $Z_{\tau+\tau}^{(\tau)}$ denote the time-shifted process $Z_{\tau+\tau}$. For $A$ a (measurable) subset of $Z$, say $A$ recures if $\{\tau > 0 : Z_\tau \in A\}$ is unbounded. For $B$ an event measurable with respect to $\{Z_t : 0 \leq t \leq 1\}$, say $B$ recures if $\{\tau > 0 : Z^{(\tau)} \in B\}$ is unbounded. If
inf \( P(B|Z_0 = z) \geq \alpha > 0 \) \( (10) \)

and \( A \) recurs with probability one (resp., with positive probability), then so does \( B \).

**Proof.** If \( A \) occurs, then define \( T_j \) inductively by \( T_0 = 0 \) and \( T_{j+1} \) is the smallest \( \tau \geq T_j + 1 \) such that \( Z_\tau \in A \). For \( j \geq 1 \), let \( \eta_j \) denote the indicator of the event that \( Z(T_j) \in B \). It follows from (10), by conditioning on the values of the \( Z_{T_j} \)'s, that \( (\eta_1, \eta_2, \ldots) \) stochastically dominates \( (\eta'_1, \eta'_2, \ldots) \), a sequence of i. i. d. zero-or-one valued random variables with \( P(\eta'_j = 1) = \alpha \). Thus, with probability one (conditional on \( A \)), \( \sum \eta_j = \infty \) and \( B \) recurs.

**IV. TWO-DIMENSIONAL \( \pm J \) MODELS: \( \rho_F > 0 \).**

The general strategy in this section is somewhat similar to that of the last section, but the analysis is considerably more involved. We will again consider an event, now denoted \( D' \), involving the frustration configuration in a finite region \( \Lambda' \) of \( Z^{2*} \), and the spin configuration in a fixed square \( C' \) of \( Z^2 \). Our object will be to show that at least one of the sites in \( C' \) will eventually have flip rate zero and hence will flip only finitely many times, thus proving \( \rho_F > 0 \); this will be done by proving that the domain wall geometry in \( \Lambda' \) must eventually satisfy various constraints. The key technique of the proof will be to combine Theorem 2 and Lemma 5 to show that certain contour events \( A \) are *eventually absent* (e-absent), i.e., that \( A \) recurs with probability zero, since otherwise there would be infinitely many energy-lowering flips in \( \Lambda' \) with positive probability.

The region \( \Lambda' \) is an \( 8 \times 8 \) square in \( Z^{2*} \) and the event \( D' \) is that out of the 64 sites in \( \Lambda' \), the frustrated ones are exactly the 20 sites indicated in Figure 3. These are all within the “border” of \( \Lambda' \) and are those sites in the border that are at most distance two from one of the four corner sites. The region \( C' \) is the \( 7 \times 7 \) square \( C(\Lambda') \) of sites of \( Z^2 \) contained within \( \Lambda' \); these sites correspond to the 49 dual plaquettes formed by the edges of \( \Lambda' \). As indicated in Figure 3, let \( u_N, u_E, u_W \) and \( u_S \) denote the sites in the exact middles of the North, East, South and West sides of the border of \( C' \). What our proof will show is that eventually either \( u_N \) (and \( u_S \)) or \( u_E \) (and \( u_W \)) will have flip rate zero. The bulk of the proof is a lengthy series of lemmas, most of which show that certain types of contour configurations (in \( \Lambda' \)) are e-absent.
FIG. 3. Geometric construction demonstrating that a positive fraction of spins flip only finitely many times in $\pm J$ models for $d = 2$, as explained in Sec. 4. The conventions used in this figure are the same as in Fig. 2.

Here is a sketch of how the lemmas will lead to the desired conclusion. A contour configuration $\Gamma$ will be called of horizontal (resp., vertical) type if it contains a horizontal (resp., vertical) domain wall, i.e., one connecting the West and East (resp., South and North) sides of the border of $\Lambda'$; a $\Gamma$ that is of neither of these two types will be said to be of non-crossing type. It turns out (see Lemmas 12 and 13 below) that (conditional on $D'$) eventually $\Gamma_t$ is exclusively one of these three types — i.e., it will not simultaneously contain both a horizontal and a vertical domain wall, and there will be no transition in which the type changes. It also turns out (as a consequence of other lemmas and again conditional on $D'$) that for $u_N$ (resp., $u_E$) to flip, $\Gamma_t$ just before or just after the flip must either be vertical (resp., horizontal) or else must be e-absent. It follows that eventually at most one of $u_N$ and $u_E$ has positive flip rate, completing the proof. Now to the lemmas.

In the lemmas, we will consider various rectangles, denoted $\Lambda^*$ (or sometimes $\Sigma^*$) of sites in $\mathbb{Z}^2\ast$, the associated rectangles $\mathcal{C}(\Lambda^*)$ of $\mathbb{Z}^2$-sites within $\Lambda^*$, contour configurations $\Gamma(\Lambda^*)$ (and frustration configurations $\Phi(\Lambda^*)$) restricted to $\Lambda^*$, and internal (or more specifically, $\Lambda^*$-internal) transitions or flips of these restricted contour configurations, i.e., those corresponding to (energy decreasing or zero-energy) flips of sites in $\mathcal{C}(\Lambda^*)$ (these do not include external flips, i.e., of sites not in $\mathcal{C}(\Lambda^*)$ that are nearest neighbors of sites in $\mathcal{C}(\Lambda^*)$). We will call $\Gamma(\Lambda^*)$ unstable if it is the starting configuration of an energy decreasing internal transition; i.e., if $\Gamma(\Lambda^*)$ contains 3 or 4 edges of some (dual) plaquette completely within $\Lambda^*$.

**Lemma 6.** Any unstable $\Gamma(\Lambda^*)$ is e-absent.

**Proof.** This is an easy consequence of Theorem 2 and Lemma 5. Here $B$ is the event that an energy decreasing internal flip takes place in a unit time interval and $\alpha$ may be bounded below by the probability that such a flip takes place before any other (internal or external) flip that could change $\Gamma(\Lambda^*)$. We leave further details to the reader.

**Lemma 7.** Given compatible $\Phi(\Lambda^*)$ and $\Gamma(\Lambda^*)$, if there exists a rectangle $\Sigma^* \supseteq \Lambda^*$ such that for every $\Gamma(\Sigma^*)$ that coincides with $\Gamma(\Lambda^*)$ in $\Lambda^*$ and is compatible with $\Phi(\Lambda^*)$, there is a finite sequence of $\Sigma^*$-internal transitions, $\Gamma_1(\Sigma^*) = \Gamma(\Sigma^*) \rightarrow \Gamma_2(\Sigma^*) \rightarrow \ldots \rightarrow \Gamma_n(\Sigma^*)$, (possibly with
n = 1) such that \( \Gamma_n(\Sigma^*) \) is e-absent, then (conditional on \( \Phi(\Lambda^*) \)) \( \Gamma(\Lambda^*) \) is also e-absent.

**Proof.** For each \( \Gamma(\Sigma^*) \) and each \( \Sigma^*-\)internal transition from that configuration, let \( c_1(\Delta) > 0 \) denote (a lower bound for) the probability that that transition is the first (\( \Sigma^*-\)internal or \( \Sigma^*-\)external) flip to be attempted during a time interval of length \( \Delta > 0 \) and that flip is successful. Inductively, we see that with probability at least \( c(\Gamma(\Sigma^*)) = c_1(\frac{1}{n-1}) \cdots c_n(\frac{1}{m-1}) \) (or \( c(\Gamma(\Sigma^*)) = 1 \) when \( n = 1 \)) \( \Gamma(\Sigma^*) \) will transform into \( \Gamma_n(\Sigma^*) \) sometime during a time interval of unit length. We can now apply Lemma 5 with \( B \) being the event that one of these (finitely many) \( \Gamma_n(\Sigma^*) \)'s occurs during the unit time interval and with \( \alpha \) being the minimum of the \( c(\Gamma(\Sigma^*)) \)'s.

A path or domain wall in \( \mathbb{Z}^{2\ast} \) with endpoints \( z \) and \( w \) is called **monotonic** if, for one of the two directed versions of the path, either every step moves to the East or to the North or else every step moves to the East or to the South. For such a monotonic path \( \gamma \), we denote by \( R(\gamma) = R(z, w) \) the (smallest) rectangle in \( \mathbb{Z}^{2\ast} \) with \( z \) and \( w \) as two of its corners. For a non-monotonic \( \gamma \), \( R(\gamma) \) denotes the smallest rectangle containing the sites of \( \gamma \).

**Lemma 8.** \( \Gamma(\Lambda^*) \) is e-absent if it contains a non-monotonic domain wall.

**Proof.** Any non-monotonic domain wall contains as a sub-path a non-monotonic domain wall \( \gamma \), with \( R(\gamma) \) a \( 2 \times (m + 1) \) rectangle and \( \gamma \) going around one long and two short sides of the border of the rectangle. Let \( x_1, \ldots, x_m \) denote the \( \mathbb{Z}^2 \)-sites at the centers of the \( m \) (dual) plaquettes of \( R(\gamma) \) (listed in either of the two natural orders). Consider the sequence of flips of the first \( m - 1 \) of these sites (in the same order). If \( \Gamma(\Lambda^*) \) contains no other edges of \( R(\gamma) \) than those of \( \gamma \), then that sequence of flips corresponds to a sequence of transitions as in Lemma 7 (with \( \Sigma^* = \Lambda^* \)) whose final configuration is unstable; if there are other edges, then an unstable configuration may be reached earlier. In either case, we conclude from Lemmas 6 and 7 that \( \Gamma(\Lambda^*) \) is e-absent.

**Lemma 9.** Let \( \Gamma \) contain a monotonic domain wall \( \gamma \) (with endpoints \( z \) and \( w \)) but no other edge inside the rectangle \( R(\gamma) \). If \( \gamma' \) is any other monotonic path between \( z \) and \( w \), then there is a finite sequence of \( R(\gamma) \)-internal flips (i.e., flips of \( \mathbb{Z}^2 \)-sites within \( R(\gamma) \)) that transforms \( \Gamma \) into a configuration \( \Gamma' \) whose restriction to \( R(\gamma) \) consists exactly of the edges of \( \gamma' \).

**Proof.** We sketch a proof, but the reader is invited to provide her own for this elementary result. Suppose (without loss of generality) that \( z \) is the Southwest and \( w \) the Notheast corner of \( R(\gamma) \). Let \( \gamma'' \) denote the path between those corners that runs along the South and East sides of the rectangle. It suffices to show that any \( \gamma \) (and hence also \( \gamma' \)) can be transformed into \( \gamma'' \) (and vice-versa, by inversion). But this can be done (inductively) by noting that for any \( \gamma \neq \gamma'' \), there is some site within \( R(\gamma) \) whose flip will strictly reduce the area of the region between \( \gamma \) and \( \gamma'' \).

In an \( m \times n \) rectangle \( \Lambda^* \) of \( \mathbb{Z}^{2\ast} \), the border consists of those sites in \( \Lambda^* \) that are nearest neighbors of sites outside \( \Lambda^* \). The border has four (distinct, unless \( m = n = 1 \), but not disjoint) sides: North, East, West and South. There are four corners (distinct, if \( m, n > 1 \), denoted NE, NW, SW and SE, each of which is the single site at the intersection of two adjacent sides of the border. We define the *interior* of \( \Lambda^* \) (denoted \( \text{int}(\Lambda^*) \)) as those sites in \( \Lambda^* \), that are not in its border and we define the interior of any side of the border as those sites in that side that are not corners.

**Lemma 10.** Given an \( m \times n \) rectangle \( \Lambda^* \) with \( m, n > 1 \) and conditional on a frustration configuration \( \Phi(\Lambda^*) \), a contour configuration \( \Gamma(\Lambda^*) \) is e-absent if it contains a monotonic domain wall \( \gamma \) between some \( z \) and \( w \) and any one of the following four situations holds for the rectangle \( R = R(\gamma) = R(z, w) \):
(i) $\Gamma(\Lambda^*)$ contains an edge $e^*$ in $R$ that is not in $\gamma$.

(ii) There is a frustrated site in $\text{int}(R)$.

(iii) There are two frustrated sites in the interior of a single side of the border of $R$.

(iv) A corner of $R$ other than $z$, $w$ is frustrated and so is at least one site in the interior of each of the two sides of the border of $R$ touching that corner.

**Proof.** (i) Let $\gamma'$ be any monotonic path between $z$ and $w$ that contains $e^*$. Consider the sequence of flips provided by Lemma 9 that would (if there were no edges of $\Gamma(\Lambda^*)$ in $R$ other than those of $\gamma$) transform $\gamma$ into $\gamma'$. Because $e^*$ is already in $\Gamma(\Lambda^*)$ (and so may be other edges of $R$ that are not in $\gamma$), at some stage along this sequence of flips (before $e^*$ is absorbed into the evolving domain wall) $\Gamma(\Lambda^*)$ will have been transformed into an unstable configuration. The desired conclusion then follows from Lemmas 6 and 7.

(ii) Since a frustrated site must have an odd number of unsatisfied edges touching it, such a site in the interior of $R$ has an unsatisfied edge $e^*$ not in $\gamma$ (but in $R$) touching it. The result now follows from part (i).

(iii) Without loss of generality, we assume that $z$ and $w$ are the SW and NE corners of $R$ and the two frustrated sites are on the south side of the border of $R$. Since these are not endpoints of $\gamma$, but they are frustrated, they must each have at least one unsatisfied edge not from $\gamma$ touching them. By part (i), we may assume that those edges go out from $R$ to the South. Also, by part (i) we may assume that $\Gamma(\Lambda^*)$ has no edges other than those of $\gamma$ in $R$. Then by the sequence of flips provided by Lemma 9, $\Gamma$ (whose restriction to $\Lambda^*$ is $\Gamma(\Lambda^*)$) can be transformed into $\Gamma''$ where $\gamma$ is replaced by $\gamma''$, a domain wall between $z$ and $w$ lying along the South and East sides of $R$. But $\Gamma''$ also contains the two edges going South from the two frustrated sites. Thus it contains a non-monotonic domain wall in the slightly larger region $\Sigma^*$, that adds to $\Lambda^*$ its neighboring sites. The desired conclusion now follows from Lemmas 7 and 8.

(iv) We may assume that $z$ and $w$ are the SW and NE corners of $R$, that there is a frustrated site in the interior of each of the South and East sides of the border and that the SE corner is also frustrated. By the same reasoning as in part (iii), there must be an unsatisfied edge going out from $R$ starting from each of these three frustrated sites. The ones from the interior sites on the sides go to the South and the East, while the one from the corner can go in either of those two directions. Thus there are either two unsatisfied edges going South from the South side or else two going East from the East side. In either case, the proof is completed as in part (iii).

We now focus on the $8 \times 8$ square $\Lambda'$ and the frustration configuration $\Phi'(\Lambda')$ (or the event $D'$ that the frustration configuration in $\Lambda'$ is exactly $\Phi'(\Lambda')$) indicated in Figure 3. Since there are no frustrated sites in $\text{int}(\Lambda')$, domain walls of any $\Gamma(\Lambda')$ compatible with $\Phi'(\Lambda')$ must be extendable so that the endpoints $z$ and $w$ are both on the border of $\Lambda'$. By Lemma 8, if $z$ and $w$ are on a single side of the border and $\Gamma(\Lambda')$ is not e-absent, then the domain wall can only be the straight line path between $z$ and $w$. The following two lemmas cover the situations where the endpoints are on adjacent or opposite sides and give restrictions on the possible $\Gamma(\Lambda')$’s that are not e-absent. In the first of the two lemmas we write $|z - z'|$ to denote the Euclidean distance between sites in $\mathbb{Z}^{2*}$.

**Lemma 11.** Condition on $D'$. Every $\Gamma(\Lambda')$ that contains a domain wall between sites $z$ and $w$ that are on adjacent sides (but not on any single side) of the border of $\Lambda'$, is e-absent if $z$, $w$ and the common corner $c(z, w)$ of the two sides do not satisfy the following condition:

$$|z - c(z, w)| + |w - c(z, w)| \leq 3. \quad (11)$$
(If $z$ and $w$ are opposite corners of $\Lambda'$, then $c(z, w)$ can be taken as either of the two remaining corners, the condition is not satisfied and $\Gamma(\Lambda')$ is e-absent.) Every $\Gamma(\Lambda')$ that contains a domain wall $\gamma$ between $z$ and $w$ on a single side of the border of $\Lambda'$ with $z$ a corner, is e-absent unless

$$|z - w| \leq 2. \quad (12)$$

**Proof.** We may assume by Lemma 8 that the domain wall is monotonic. If $z$ and $w$ are not on a single side and (11) does not hold, then one of the following two cases occurs.

(I) One of $|z - c(z, w)|$ or $|w - c(z, w)|$ is at least 3. In this case, since we condition on $D'$, one of the sides of $R(z, w)$ contains at least two frustrated sites in its interior and part (iii) of Lemma 10 applies.

(II) $|z - c(z, w)| = 2 = |w - c(z, w)|$. In this case, since we condition on $D'$, a corner (other than $z$ or $w$) of $R(z, w)$ is frustrated and so is one site in the interior of each of the adjacent sides of the border of $R(z, w)$. Thus, part (iv) of Lemma 10 applies.

If $z$ and $w$ are on a single side with $z$ a corner, we may assume, without loss of generality, that $z$ is the NW corner and $w$ is on the North side. If (12) does not hold, then there are (at least) two frustrated sites on $\gamma$ between $z$ and $w$, and there must be unsatisfied edges not in $\gamma$ touching these two sites. One of those edges must go to the South (into $\Lambda'$), or else there would be a non-monotonic domain wall (using some edges of $\gamma$ and two edges going North just outside of $\Lambda'$). Since there is no frustration in int($\Lambda'$), that South-going edge must be extendable to a domain wall reaching some site $w'$ on the border of $\Lambda'$. Combining that extension with part of $\gamma$ yields a domain wall $\gamma'$ between $z$ and $w'$. If $w'$ is on the West or North sides, $\gamma'$ would be non-monotonic. If $w'$ is on the South or East sides, then (11) with $w$ replaced by $w'$ would not hold and $\Gamma(\Lambda')$ would be e-absent by the part of this lemma that has already been proved.

Before stating the next lemma, we recall our definition of a horizontal (resp., vertical) domain wall in $\Lambda'$ as one whose endpoints are in the West and East (resp., South and North) sides of the border.

**Lemma 12.** Conditional on $D'$, every $\Gamma(\Lambda')$ that contains both a vertical and horizontal domain wall is e-absent.

**Proof.** Let us denote the endpoints of the horizontal (resp., vertical) domain wall by $z_W$ and $z_E$ (resp., $z_S$ and $z_N$) so that the subscript indicates the side that the endpoint is located on. (Note though that the endpoints may be corners.) Since the vertical and horizontal domain walls must have at least one site of $\Lambda'$ in common, it follows that $\Gamma(\Lambda')$ has a domain wall with any pair of the points $\{z_N, z_E, z_W, z_S\}$ as endpoints.

It also follows that both the horizontal and vertical crossings must be straight lines or else there would be a non-monotonic domain wall. Hence $z_S, z_N$ are at distance $\geq 4$ from one of the East or West sides (which we take to be the East side, without loss of generality) and similarly (without loss of generality) $z_W, z_E$ may be assumed to be at distance $\geq 4$ from the South side. But then the domain wall with endpoints $z = z_S$ and $w = z_E$ violates (11) and $\Gamma(\Lambda')$ is e-absent by Lemma 11.

We recall that $\Gamma(\Lambda')$ is said to be of horizontal or vertical or non-crossing type according to whether it contains a horizontal or a vertical domain wall or neither. We will also say that $\Gamma^t$ is eventually of type A if for some (random) finite $T$, $\Gamma^t$ is of type A for all $t \geq T$, A standing for one of the above three types.
Lemma 13. Suppose $\Gamma(\Lambda')$ and $\Gamma'(\Lambda')$ are related by an internal or external flip (either $\Gamma(\Lambda') \rightarrow \Gamma'(\Lambda')$ or $\Gamma'(\Lambda') \rightarrow \Gamma(\Lambda')$), and suppose further that $\Gamma(\Lambda')$ has a vertical (or horizontal) domain wall but $\Gamma'(\Lambda')$ does not. Conditional on $D'$, any such $\Gamma'(\Lambda')$ is e-absent and, with probability one, $\Gamma'$ is eventually of one of the three types — vertical, horizontal or non-crossing.

Proof. Let $\gamma$ be a vertical domain wall in $\Gamma(\Lambda')$ between $z$ on the South side and $w$ on the North side. The flip changes the edges of a single (dual) plaquette in or next to $\Lambda'$; whether the flip is internal or external, zero-energy or energy lowering, there will remain in $\Gamma'(\Lambda')$ a portion of $\gamma$ from $z$ to some $z' \in \Lambda'$ on that plaquette and a portion from $w$ to some $w' \in \Lambda'$ on that plaquette. Without loss of generality, we may assume that the distance from $z'$ to the South side is at least 3, and we then denote by $\gamma'$ the domain wall portion from $z$ to $z'$.

If $z'$ is a border site, it cannot be on the North side since then $\gamma'$ would violate the assumption that $\Gamma'(\Lambda')$ is not of vertical type. For either the East or West side as a location for $z'$, it would follow that $\gamma'$ is either non-monotonic or else $\Gamma'(\Lambda')$ is e-absent because of violating (11) or (12) with $w$ replaced by $z'$.

If $z'$ is not a border site, then it is unfrustrated and $\gamma'$ must be extendable to a $\gamma''$ between $z$ and some border site $z''$. The e-absence of $\Gamma'(\Lambda')$ now follows by the same arguments as above but with $z'$ and $\gamma'$ replaced by $z''$ and $\gamma''$. Of course, analogous arguments work when $\Gamma(\Lambda')$ is of horizontal rather than vertical type.

The final claim of the lemma now follows by choosing $T$ to be the finite (with probability one) time beyond which no e-absent configurations in $\Lambda'$ are taken on by $\Gamma'$. By Lemma 12 and the part of this lemma already proved, no changes of type occur after that time.

Proof that $\rho_F > 0$. Since $D'$ occurs with strictly positive probability (and hence a positive density of translates of $D'$ occur with probability one), it suffices to show that conditional on $D'$, with probability one, for times beyond some finite $T$, some $Z^2$-site in the $7 \times 7$ square $C(\Lambda')$ will not flip. We take the same $T$ as in the proof of the previous lemma, namely the time beyond which no e-absent $\Gamma(\Lambda')$’s are seen. Past this time, $\Gamma'$ remains of one particular type, and we will locate a non-flipping site depending on the type.

Let $u_N$ denote the site in the middle of the North side of $C(\Lambda')$ (and $u_E$, $u_W$, $u_S$ the sites in the middle of the other sides), as indicated in Figure 3. A flip of $u_N$ corresponds to a change in $\Gamma'$ involving the edges of the (dual) plaquette inside $\Lambda'$ and just below the middle of its North side. Since e-absent configurations are no longer seen, it must be a zero-energy flip in which both before and after the flip there are exactly two unsatisfied edges from that plaquette, but with the unsatisfied and satisfied edges exchanged by the flip. Thus either before or after the flip, $\Gamma'$ must contain an edge going South from one of the two central sites (that we will denote $z$) on the North side of $\Lambda'$. That South-going edge must be extendable to a domain wall $\gamma$ between $z$ and some other border site $w$. We claim that $\gamma$ must be vertical because otherwise $\Gamma'$ would be e-absent. This is so because if $\gamma$ were not vertical, then $w$ would either be on the North side and so $\gamma$ would be non-monotonic and e-absence would follow from Lemma 8; or else $w$ would be on the West or East sides and e-absence would follow from Lemma 11. This shows that after time $T$, $u_N$ (and by symmetry $u_S$) cannot flip unless $\Gamma'$ is of vertical type. Similarly $u_E$ and $u_W$ cannot flip after $T$ unless $\Gamma'$ is of horizontal type. By Lemma 13, conditional on $D'$, after (the almost surely finite) time $T$ some site (e.g., either $u_N$ or $u_E$) does not flip. This completes the proof.

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