Double q-Analytic q-Hermite Binomial Formula and q-Traveling Waves

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February 3, 2016

Abstract
Motivated by derivation of the Dirac type $\delta$-function for quantum states in Fock-Bargmann representation, we find $q$-binomial expansion in terms of $q$-Hermite polynomials, analytic in two complex arguments. Based on this representation, we introduce a new class of complex functions of two complex arguments, which we call the double $q$-analytic functions. The real version of these functions describe the $q$-analogue of traveling waves, which is not preserving the shape during evolution as the usual traveling wave. For corresponding $q$-wave equation we solve IVP in the $q$-D’Alembert form.

1 Introduction

In complex analysis, a complex function $f(z)$ of one complex variable $z$ is analytic in some domain $D$ if in $D$ it satisfies

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(z) = 0. \quad (1)$$

If we consider a complex function $f(z, w)$ of two complex variables $z, w$, analytic in both variables

$$\frac{\partial}{\partial z} f(z, w) = \frac{\partial}{\partial \bar{w}} f(z, w) = 0, \quad (2)$$

it is called a double analytic if

$$\frac{1}{2} \left( \frac{\partial}{\partial z} + i \frac{\partial}{\partial w} \right) f(z, w) = 0. \quad (3)$$

As an example, $f(z, w) = (z + iw)^2$ is double analytic, while $(z - iw)^2$ is double anti-analytic.
In general, double analytic function can be written as power series in complex binomials
\[ f(z, w) = \sum_{n=0}^{\infty} a_n (z + iw)^n. \]

For such double analytic binomials here we derive the following Hermite binomial formula
\[ (z + iw)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z) H_k(w). \]  (4)

Derivation of this formula is motivated by description of the Dirac type δ-function for quantum states in the Fock-Bargmann representation. As is well known, states of a quantum system in the Fock-Bargmann representation are described by complex analytic function \( f(z) \) and visa versa [1]. In this representation, due to formula
\[ \int d\mu(z) e^{\xi \bar{z}} f(z) = f(\xi), \]  (5)

where measure \( d\mu(z) = dzd\bar{z}e^{-z\bar{z}} \), the exponential function plays the role of the Dirac type δ-function [2]. Proof of this formula is based on following identity
\[ \int d\mu(z) e^{\xi \bar{z}} z^n = \xi^n. \]  (6)

By generalization of this identity to two complex variables we find expansion (4).

In the present paper, we are going not only proof the identity (4), but also derive the \( q \)-analogue of this identity.

Recently, [5] we have introduced a complex function \( f(z; q) \) of one complex variable \( z \) according to equation
\[ D_{\bar{z}} D_q f(z; q) = \frac{1}{2} \left( D_q^z + i D_q^y \right) f(z; q) = 0 \]  (7)

and we called it as the \( q \)-analytic function. We have described a wide class of these functions, which are not analytic in the usual sense [1].

**Example:** Complex \( q \)-binomial
\[ (x + iy)^2_q = (x + iy)(x + iqy) = \frac{1}{2} \left( (1 + q)z^2 + (1 - q)\bar{z}z \right) \]
is not analytic \( \frac{\partial}{\partial z} (x + iy)_q^2 \neq 0 \), but \( q \)-analytic \( D_q^z(x + iy)_q^2 = 0 \).

A complex function \( f(z, w) \) of two complex variables \( z \) and \( w \), analytic in both variables and satisfying equation
\[ D_{z,w} f(z, w) = \frac{1}{2} \left( D_q^z + i D_q^w \right) f(z, w) = 0 \]  (8)
we call the double $q$-analytic function.

**Example:** Complex $q$-binomial

\[
(z + iw)^2_q = z^2 + [2]_q i wz - qw^2
\]

is analytic in $z$ and $w$, since $\frac{\partial}{\partial z}(z + iw)^2_q = \frac{\partial}{\partial w}(z + iw)^2_q = 0$ and double $q$-analytic due to $\bar{D}_{x,w}(z + iw)^2_q = 0$.

We will show that complex $q$-binomial $(z + iw)^n_q$, for $n$-positive integer, is double $q$-analytic. This is why any convergent power series

\[
f(z, w) = \sum_{n=0}^{\infty} a_n (z + iw)^n_q
\]

represents a double $q$-analytic function. As a central result of the paper, we find an expansion of the $q$-binomial in terms of $q$-Hermite polynomials

\[
(z + iw)^n_q = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \binom{n}{k} q^{k(k-1)/2} H_{n-k}(z; q) H_k(qw, \frac{1}{q}),
\]

In the limit $q \to 1$, this formula reduces to the Hermite binomial formula (4).

As an application of our results, we consider $q$-traveling waves in the form of power series of real $q$-binomial $(x \pm ct)^n_q$. For these traveling waves we find $q$-Hermite binomial expansion in terms of $x$ and $t$ variables. We notice that in contrast to usual traveling waves, the $q$-traveling waves are not preserving the shape during evolution. The $q$-traveling waves are subject to $q$-wave equation

\[
\left( (D_x^q)^2 - c^2 (D_t^q)^2 \right) u(x, t) = 0,
\]

for which we solve IVP in the $q$-D’Alembert form with the Jackson integral representation.

The paper organized as follows. In Section 2 we derive the Hermite binomial formula (4). In Section 3 we introduce $q$-Hermite polynomials and derive the $q$-Hermite binomial formula. The $q$-binomials are the double $q$-analytic functions, as we demonstrate in Section 4. In Section 5 we discuss relation between the double $q$-analytic binomial and the $q$-analytic one. In Section 6 we introduce $q$-traveling waves in terms of real $q$-binomials and solve IVP for $q$-wave equation in D’Alembert form. We illustrate our results for different initial values by particular examples in Section 7. Finally, in Section 8, we derive $q$-Hermite polynomial expansion of $q$-traveling waves.

### 2 Hermite Binomial Formula

In this section, we derive the Hermite binomial formula (4). We start with the following Lemma:
Lemma 2.0.1  For arbitrary complex numbers $\xi$ and $\eta$:
\[
\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4-\eta^2/4} = \left(\frac{\xi + \eta}{2}\right)^n e^{\xi^2/4-\eta^2/4} \tag{10}
\]

Proof 2.0.2
\[
\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4-\eta^2/4} = \left(\frac{d}{d\xi} - \frac{d}{d\eta}\right)^n e^{\xi^2/4-\eta^2/4} \tag{11}
\]

By changing variables $\xi$ and $\eta$ to $\lambda$ and $\mu$ according to
\[
\lambda + \mu = \xi, \quad \lambda - \mu = \eta,
\]
we have
\[
\lambda \mu = \frac{\xi^2 - \eta^2}{4} = \left(\frac{\xi - \eta}{2}\right)\left(\frac{\xi + \eta}{2}\right)
\]
and
\[
\left(\frac{d}{d\xi} - \frac{d}{d\eta}\right)^n e^{\lambda \mu} = \left(\frac{d}{d\mu}\right)^n e^{\lambda \mu} = (\frac{\xi + \eta}{2})^n e^{\xi^2/4-\eta^2/4}. \quad \Diamond \tag{12}
\]

Corollary 2.0.3
\[
\sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4}\right) \left(\frac{d^k}{d\eta^k} e^{-\xi^2/4}\right) = \xi^n \tag{13}
\]

Proof 2.0.4  By taking the limit of (11) as $\eta \to \xi \Rightarrow \mu \to 0$ and $\lambda \to \xi$,
\[
\lim_{\eta \to \xi} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4-\eta^2/4} = \lim_{\mu \to 0, \lambda \to \xi} \lambda^n e^{\lambda \mu} = \xi^n. \tag{14}
\]

Lemma 2.0.5  For an arbitrary complex number $\xi$ and $n = 1, 2, ...$,
\[
\int dz d\bar{z} e^{-\bar{z}z} e^{\xi \bar{z}z} = \xi^n. \tag{15}
\]

Proof 2.0.6  By changing complex coordinates to the Cartesian ones, the integral is expressed in terms of the sum
\[
\int dz d\bar{z} e^{-\bar{z}z} e^{\xi \bar{z}z} = \frac{1}{\pi} \int dx dy e^{\xi(x-iy)} e^{-(x^2+y^2)} (x + iy)^n
\]
\[
= \frac{1}{\pi} \sum_{k=0}^n \binom{n}{k} k \int dx \ x^{n-k} e^{-x^2} e^{\xi x} \int dy \ y^k e^{-y^2} e^{-i\xi y}
\]
\[
= \frac{1}{\pi} \sum_{k=0}^n \binom{n}{k} k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d(-i\xi)^k} \int dy \ e^{-y^2-i\xi y}
\]
\[
= \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\xi^k} e^{\xi^2/4} d\xi. \tag{16}
\]
where we have used the Gaussian integrals
\[ \int_{-\infty}^{\infty} e^{-x^2+ax} \, dx = \sqrt{\pi} \, e^{a^2/4} \]
and
\[ \int_{-\infty}^{\infty} e^{-x^2+ibx} \, dx = \sqrt{\pi} \, e^{-b^2/4} . \]

By using Corollary 2.0.3, we find desired result (15).

We can generalize this result for an arbitrary analytic function given by power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), so that,
\[ \int d\mu(z) e^{\xi z} f(z) = f(\xi) . \]

The above proof implies some interesting binomial identity for Hermite polynomials. We start from the Rodrigues formula for Hermite polynomials:
\[ H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \]
and by replacing \( z \to iz \)
\[ H_n(iz) = i^n e^{-z^2} \frac{d^n}{dz^n} e^{z^2} . \]

Then we have the following

**Identity 2.0.7**
\[ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-i)^{n-k} H_{n-k}(\frac{i}{2} \xi) H_k(\frac{\xi}{2}) = \xi^n . \]

**Proof 2.0.8 According to previous proof**
\[ \xi^n = \int d\omega \, \overline{z} \, e^{-z^2} e^{\xi \overline{z} z^n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4} \frac{d^k}{d\xi^k} e^{-\xi^2/4} . \]

By inserting \( 1 = e^{\xi^2/4} e^{-\xi^2/4} \) and using Rodrigues formulas we have
\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{\xi^2/4} \left( e^{-\xi^2/4} \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4} \right) \frac{d^k}{d\xi^k} e^{-\xi^2/4} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( \frac{1}{2i} \right)^{n-k} H_{n-k}(\frac{i}{2} \xi) \frac{d^k}{d\xi^k} e^{-\xi^2/4} \]
\[ = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( \frac{1}{2i} \right)^{n-k} H_{n-k}(\frac{i}{2} \xi) \left( \frac{1}{2i} \right)^k H_k(\frac{\xi}{2}) \]
\[ = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} i^{n-k} H_{n-k}(\frac{i}{2} \xi) H_k(\frac{\xi}{2}) \]
\[ = \xi^n . \]
Particular forms of this identity are:

\[
\frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z) H_k(-iz) = z^n, \quad (\xi \to -2iz) \tag{22}
\]

and (by reductions \(\xi = x\) and \(\xi = iy\)),

\[
\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^{n-k} H_{n-k} \left( \frac{ix}{2} \right) H_k \left( \frac{x}{2} \right) = x^n, \tag{23}
\]

\[
\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^{n-k} H_{n-k} \left( \frac{-iy}{2} \right) H_k \left( \frac{iy}{2} \right) = i^n y^n. \tag{24}
\]

This result can be generalized to the Hermite binomial formula with two complex variables \(z\) and \(w\)

**Identity 2.0.9**

\[(z + iw)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z)H_k(w) \tag{25}\]

**Proof 2.0.10** From generating function for Hermite polynomials

\[g(z, t) = e^{-t^2 + 2tz} = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} \tag{26}\]

and

\[g(w, \tau) = e^{-\tau^2 + 2\tau w} = \sum_{k=0}^{\infty} H_k(w) \frac{\tau^k}{k!} \tag{27}\]

by changing variable \(\tau = it\) in the second one and multiplying \[26\] and \[27\] we have

\[g(z, t) g(w, it) = e^{2t(z + iw)} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} H_l(z) H_k(w) \frac{t^l}{l!} \frac{\tau^k}{k!} \tag{28}\]

By changing the order of double sum with \(l + k = n\), and expanding the left hand side in \(t\) we get

\[g(z, t) g(w, it) = \sum_{n=0}^{\infty} \frac{2^n t^n (z + iw)}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(z)H_k(w) i^k. \tag{29}\]

By equating terms of the same power \(t^n\) we obtain the desired result \[25\]:

\[(z + iw)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z)H_k(w). \quad \diamond \]
Here we can give also another proof of this relation by using the holomorphic Laplace equation.

**Proof 2.0.11** \( \zeta^n \equiv (z + iw)^n \) is double analytic function of two complex variables \( z \) and \( w \). Therefore, due to (3) it satisfies the holomorphic Laplace equation \( \Delta \zeta^n = 0 \), where \( \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial w^2} = (\frac{\partial}{\partial z} - \frac{\partial}{\partial w})(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}) \), which implies \( \Delta^k \zeta^n = 0 \) for every \( k = 1, 2... \)

Expanding \( e^{-\frac{1}{4} \Delta} (z + iw)^n = \left( 1 - \frac{1}{4} \Delta + \frac{(\frac{1}{4} \Delta)^2}{2!} + ... + \frac{(-\frac{1}{4} \Delta)^n}{n!} + ... \right) (z + iw)^n = (z + iw)^n \)

we have

\[
e^{-\Delta}(z + iw)^n = e^{-\frac{1}{4} \Delta} \left( 1 - \frac{1}{4} \Delta + \frac{(\frac{1}{4} \Delta)^2}{2!} + ... + \frac{(-\frac{1}{4} \Delta)^n}{n!} + ... \right) (z + iw)^n = (z + iw)^n
\]

Then due to relation:

\[
H_n(z) = 2^n e^{-\frac{1}{4} \Delta} z^n
\]

we get the Hermit binomial formula

\[
(z + iw)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(z) H_k(w) i^k.
\]

3 **q-Hermite Binomial Formula**

Here we are going to generalize the above formula to the q-binomial case. For this, first we need to introduce the q-Hermite polynomials.

3.1 **q-Hermite Polynomials**

In paper [4], studying the q-heat and q-Burgers equations, we have defined the q-Hermite polynomials according to generating function

\[
e_q(-t^2)e_q([2]_qtx) = \sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{[n]_q!},
\]

where

\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{x^n}{[n]_q!}
\]

are Jackson’s q-exponential functions and q-numbers and q-factorials are defined as follows:

\[
[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = [1]_q [2]_q ... [n]_q.
\]
By q-differentiating the generating function (33) according to $x$ and $t$ we have the recurrence relations, correspondingly

$$D_x H_n(x; q) = [2]_q [n]_q H_{n-1}(x; q),$$  \hspace{1cm} (34)

$$H_{n+1}(x; q) = [2]_q x H_n(x; q) - [n]_q H_{n-1}(qx; q) - [n]_q q^{n+1} H_{n-1}(\sqrt{q} x; q).$$  \hspace{1cm} (35)

We get also the special values

$$H_{2n}(0; q) = (-1)^n \frac{[2n]_q q!}{[n]_q q!},$$  \hspace{1cm} (36)

$$H_{2n+1}(0; q) = 0,$$  \hspace{1cm} (37)

and the parity relations

$$H_n(-x; q) = (-1)^n H_n(x; q).$$  \hspace{1cm} (38)

For more details we refer to paper [4]. First few polynomials are

$$H_0(x; q) = 1, \quad H_1(x; q) = [2]_q x,$$

$$H_2(x; q) = [2]_q^2 x^2 - [2]_q x, \quad H_3(x; q) = [2]_q^3 x^3 - [2]_q [3]_q [2]_q x,$$

$$H_4(x; q) = [2]_q^4 x^4 - [2]_q^2 [3]_q [4]_q x^2 + [2]_q [3]_q [2]_q^2 x^2,$$

and when $q \to 1$ these polynomials reduce to the standard Hermite polynomials.

The generating function (33) for $t = 1$ gives expansion of $q$-exponential function in terms of $q$-Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{H_n(x; q)}{[n]_q q!} = e_q([2]_q x) = e_q(1).$$  \hspace{1cm} (39)

In the limiting case $q \to 1$ it gives expansion of exponential

$$e^x = e \sum_{n=0}^{\infty} \frac{H_n \left( \frac{x}{q} \right)}{n!}$$

and for $x = 1$ we find for Euler’s number $e$ as:

$$e = \sum_{n=0}^{\infty} \frac{H_n(1)}{n!}.$$  \hspace{1cm} (39)

For the Jackson q-exponential function we have the q-analog of this expansion:

$$e_q(x) = e_q \sum_{n=0}^{\infty} \frac{H_n \left( \frac{x}{[2]_q} ; q \right)}{[n]_q q!}.$$  \hspace{1cm} (39)
and as follows
\[ e_q = e_q(1) = \sum_{n=0}^{\infty} \frac{H_n \left( \frac{1}{[n]q} \right)}{[n]q!}, \quad (40) \]

where
\[ e_q = \sum_{n=0}^{\infty} \frac{1}{[n]q!}. \]
is the q-analog of Euler’s number \( e \). Relation (40) should be compared with the next one
\[ e_q(1)e_q(-1) = e_q^2 \left( \frac{1-q}{1+q} \right) = \prod_{k=1}^{\infty} \frac{1}{1 - (1-q^2)q^k} \]
which is coming for \( x = 1 \) from the following identity.

**Identity 3.1.1**
\[ e_q(x)e_q(-x) = e_q^2 \left( \frac{1-q}{1+q}x^2 \right). \]

**Proof 3.1.2** We expand and change order of the double sum
\[ e_q(x)e_q(-x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]q!} \sum_{l=0}^{\infty} \frac{(-1)^l x^l}{[l]q!} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{x^{k+l}}{[k]q! [l]q!} = \]
\[ \sum_{n=0}^{\infty} \frac{x^n}{[n]q!} \sum_{k=0}^{n} (-1)^k \frac{[n]q!}{[k]q! [n-k]q!} = \sum_{n=0}^{\infty} \frac{x^n}{[n]q!} \sum_{k=0}^{n} \frac{n}{k} (-1)^k. \quad (42) \]

Splitting the first sum to the even parts \( n = 2m \) and to the odd parts \( n = 2m + 1 \) we have
\[ \sum_{m=0}^{\infty} \frac{x^{2m}}{[2m]q!} \sum_{k=0}^{2m} \frac{2m}{k} \left[ (1-q)(1-q^3)\ldots(1-q^{2m-3})(1-q^{2m-1}) \right] (-1)^k. \quad (43) \]

Due to known identities [3]
\[ \sum_{k=0}^{2m+1} \left[ \frac{2m+1}{k} \right]_q (-1)^k = 0, \]
\[ \sum_{k=0}^{2m} \left[ \frac{2m}{k} \right]_q (-1)^k = (1-q)(1-q^3)\ldots(1-q^{2m-3})(1-q^{2m-1}), \]
the second sum vanishes and for the first sum we get
\[ \sum_{m=0}^{\infty} \frac{x^{2m}}{[2m]q!} (1-q)(1-q^3)\ldots(1-q^{2m-3}). \]

Since \([2m]q = [2]q [m]q^2\), we can rewrite this sum as

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Finally, 
\[ e_q(x) e_q(-x) = e_q^2 \left( \frac{1 - q}{1 + q} x^2 \right). \]

### 3.2 \( q \)-Hermite Binomials

Here we formulate our main result as the \( q \)-Hermite binomial identity.

**Identity 3.2.1** The \( q \)-analogue of identity (25), giving \( q \)-binomial expansion in terms of \( q \)-Hermite polynomials is

\[
(z + iw)^n_q = \frac{1}{[2]^n_q} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{t^k q^{k(k-1)}}{[k]_q!} H_k(Z; q) H_k(qw; \frac{1}{q})^i.
\] (44)

**Proof 3.2.2** By using the generating function for \( q \)-Hermite polynomials (33) and replacing \( x \to Z \) we obtain

\[
e_q(-t^2) e_q([2]_q Z t) = \sum_{n=0}^{\infty} H_n(Z; q) \frac{t^n}{[n]_q!}.
\] (45)

In this formula we replace \( t \to it, Z \to W, q \to 1/q \) so that

\[
e_q^{-1}(t^2) e_q^{-1}([2]_{1/q} i W t) = \sum_{n=0}^{\infty} H_n(W; \frac{1}{q}) i^n \frac{t^n}{[n]_q!}.
\] (46)

Multiplying (45) with (46) and using factorization of \( q \)-exponential functions

\[
e_q(x) e_q(y) = \sum_{n=0}^{\infty} \left( \frac{x + y}{[n]_q!} \right)_q = e_q(x + y)_q
\] (47)

leading to

\[ e_q(-t^2) e_q(t^2) = e_q(0) = 1, \]

we find

\[
e_q(t([2]_q Z + [2]_{1/q} i W)) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_l(Z; q) H_k(W; \frac{1}{q})}{[l]!_q [k]_q!} t^{l+k}.
\] (48)

By changing order of the double sum and expanding the left hand side in \( t \), we get

\[
\sum_{n=0}^{\infty} \frac{t^n ([2]_q Z + [2]_{1/q} i W)_q^n}{[n]_q!} = \sum_{n=0}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{t^k q^{k(k-1)}}{[k]_q!} H_k(Z; q) H_k(W; \frac{1}{q})^k.
\] (49)
Then, at power $t^n$ we have identity

$$
\left( Z + i \frac{W}{q} \right)^n_q = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{\frac{k(k-1)}{2}} H_{n-k}(Z; q) H_k(W; \frac{1}{q}) i^k,
$$

(50)

where

$$
[k]_q = \frac{1}{q^{k-1}} [k], \quad [k]_q! = \frac{1}{q^{k(k-1)/2}} [k]_q!
$$

By replacing $Z = z$ and $\frac{W}{q} = w$ the desired result is obtained

$$(z + iw)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{\frac{k(k-1)}{2}} H_{n-k}(z; q) H_k(qw; \frac{1}{q}) i^k.
$$

\diamond

4 Double q-Analytic Function

Here we consider a class of complex valued functions of two complex variables, $z$ and $w$, (or four real variables), analytic in these variables

$$
\frac{\partial}{\partial \bar{z}} f = \frac{\partial}{\partial \bar{w}} f = 0.
$$

Definition 4.0.3 A complex-valued function $f(z, w)$ of four real variables is called the double analytic in a region if the following identity holds in the region:

$$
\bar{\partial}_{z,w} f \equiv \frac{1}{2} \left( \partial_z + i \partial_w \right) f = 0,
$$

(51)

where

$$
\partial_z f = \frac{1}{2} \left( \partial_x - i \partial_y \right) f, \quad \partial_w f = \frac{1}{2} \left( \partial_u - i \partial_v \right) f
$$

and $z = x + iy, \quad w = u + iv$.

Definition 4.0.4 A complex-valued function $f(z, w)$ of four real variables is called the double $q$-analytic in a region if the following identity holds in the region:

$$
\bar{D}_{z,w} f \equiv \frac{1}{2} \left( D^z_q + i D^w_q \right) f = 0,
$$

(52)

where

$$
D^z_q f(z, w) = \frac{f(qz, w) - f(z, w)}{(q - 1)z}, \quad D^w_q f(z, w) = \frac{f(z, qw) - f(z, w)}{(q - 1)w}
$$

and $z = x + iy, \quad w = u + iv$.

Here we should notice that

$$
D^z_q \neq \frac{1}{2} \left( D^z_q - i D^w_q \right), \quad D^w_q \neq \frac{1}{2} \left( D^w_q - i D^z_q \right).
$$

The simplest set of double $q$-analytic functions is given by complex $q$-binomials

$$(z + iw)_q^n \equiv (z + iw)(z + iqw)(z + iq^2 w) ... (z + iq^{n-1} w) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{\frac{k(k-1)}{2}} k^k z^{n-k} w^k$$
\[ \frac{1}{2} (D^z_q + iD^w_q) (z + iw)^n = 0, \]

and

\[ \frac{1}{2} (D^z_q - iD^w_q) (z + iw)^n = [n]_q (z + iw)^{n-1}. \]

**Proof 4.0.5**

\[ \frac{1}{2} (D^z_q - iD^w_q) (z + iw)^n_q \]

\[ = \frac{1}{2} \left( \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} \left( D^z_q 2^z z^{n-k} \right) w^k - i \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} 2^k z^{n-k} \left( \frac{D^w_q}{2} w^k \right) \right) \]

\[ = \frac{1}{2} \left( \sum_{k=0}^{n-1} \binom{n}{k}_q q^{k(k-1)/2} (n-k) q z^{n-k-1} k w^k - i \sum_{k'=1}^{n} \binom{n}{k'}_q q^{k'(k'-1)/2} k' z^{n-k'} w^{k'-1} \frac{q^{k'-1} [k']_q}{q^{k'-(k'-1)}} \right) \]

\[ = \frac{1}{2} \left( \sum_{k=0}^{n-1} \binom{n}{k}_q q^{k(k-1)/2} (n-k) q z^{n-k-1} k w^k - i \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{k(k+1)/2} k+1 z^{n-k-1} w^k \frac{q^{k+1} [k+1]_q}{q^{k-(k+1)}} \right) \]

\[ = \frac{1}{2} \sum_{k=0}^{n-1} \left( \binom{n}{k}_q q^{k(k-1)/2} (n-k) q z^{n-k-1} k w^k - \binom{n}{k+1}_q q^{k(k+1)/2} 2 k+1 z^{n-k-1} w^k \right) \]

\[ = \frac{1}{2} \sum_{k=0}^{n-1} \left( \frac{[n]_q}{[n-k-1]_q [k]_q} q^{k(k-1)/2} z^{n-k-1} k w^k \right) \]

\[ = [n] \sum_{k=0}^{n-1} \binom{n-1}{k}_q q^{k(k-1)/2} z^{n-k-1} k w^k \]

\[ = [n] (z + iw)^{n-1}. \]

From above result follows that any convergent power series

\[ f(z + iw)_q = \sum_{n=0}^{\infty} a_n (z + iw)_q^n \]

determines a double \( q \)-analytic function. Since our relation (44) shows expansion of double \( q \)-analytic \( q \)-binomials in terms of \( q \)-Hermite polynomials, it also gives expansion of any double \( q \)-analytic function in terms of the analytic polynomials.

**Examples:** For \( n = 1 \):

\[ (z + iw)_q^1 = z + iw = \frac{1}{[2]_q} \sum_{k=0}^{1} \binom{1}{k}_q q^{k(k-1)/2} H_{1-k}(z; q) H_{k}(qw; \frac{1}{q}) i^k \]

\[ = \frac{1}{[2]_q} \left( \binom{1}{0}_q H_1(z; q) H_0(qw; \frac{1}{q}) + \binom{1}{1}_q H_0(z; q) H_1(qw; \frac{1}{q}) \right) \]
For $n = 2$:
\[
(z + iw)^2_q = (z + iw)(z + iw) = z^2 + i[2]_qzw - qw^2
\]
\[
= \frac{1}{[2]_q} \sum_{k=0}^{2} \begin{bmatrix} 2 \\ k \\ q \end{bmatrix} H_{2-k}(z; q) H_k(qw; \frac{1}{q})i^k
\]
\[
= \frac{1}{[2]_q} \left( \begin{bmatrix} 2 \\ 0 \\ q \\ 1 \\ q \end{bmatrix} H_2(z; q) H_0(qw; \frac{1}{q}) + \begin{bmatrix} 2 \\ 1 \\ q \end{bmatrix} H_1(z; q) H_1(qw; \frac{1}{q})i + \begin{bmatrix} 2 \\ 2 \\ q \end{bmatrix} qH_0(z; q) H_2(qw; \frac{1}{q})i^2 \right)
\]

4.1 $q$-Holomorphic Laplacian

Another proof of identity (4.1) can be done by noticing that $q$-binomial $(z + iw)_q^n$ is double $q$-analytic function. Then we can use the following identity and complex $q$-Laplace equation.

Identity 4.1.1
\[
e_q\left(-\frac{1}{[2]_q} \Delta_q\right)_q (z + iw)_q^n = (z + iw)_q^n \tag{54}
\]

Proof 4.1.2 Since $(z + iw)_q^n$ is $q$-analytic function, it satisfies the $q$-Laplace equation
\[
\Delta_q(z + iw)_q^n = 0
\]
and
\[
e_q\left(-\frac{1}{[2]_q} \Delta_q\right)_q (z + iw)_q^n = e_q\left(-\frac{1}{[2]_q} \left((D_q^z)^2 + (D_q^w)^2\right)\right)_q (z + iw)_q^n
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \left(-\frac{1}{[2]_q} \right)_q^n \left((D_q^z)^2 + (D_q^w)^2\right)_q (z + iw)_q^n
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \left(-\frac{1}{[2]_q} \right)_q^n \left(\Delta_q\right)_q^n (z + iw)_q^n, \tag{55}
\]

where $(\Delta_q)_q^n = \Delta_q \cdot \Delta_q^{(1)} \cdot \Delta_q^{(2)} \cdots \Delta_q^{(n-1)}$ and
\[
\Delta_q = (D_q^z)^2 + (D_q^w)^2, \quad \Delta_q^{(1)} = (D_q^z)^2 + q(D_q^w)^2, \quad \ldots, \quad \Delta_q^{(n-1)} = (D_q^z)^2 + q^{n-1}(D_q^w)^2.
\]

Using the fact that $(\Delta_q)_q^n (z + iw)_q^n = 0, \forall m = 1, 2, \ldots, \text{only the first term in expansion survives, then we get desired result.}$

Due to (47) we can factorize $q$-exponential operator function as
\[
e_q\left(-\frac{1}{[2]_q} \Delta_q\right)_q (z + iw)_q^n
\]
\[
= e_q\left(-\frac{1}{[2]_q} (D_q^z)^2\right)_q e_1\left(-\frac{1}{[2]_q} (D_q^w)^2\right)_q (z + iw)_q^n
\]
\[
= \sum_{k=0}^{n} [n]_q \frac{q^{(k+1)}}{[k]_q} q^k e_q\left(-\frac{1}{[2]_q} (D_q^z)^2\right)_q z^{n-k} e_1\left(-\frac{1}{[2]_q} (D_q^w)^2\right)_q w^k. \tag{56}
\]
By using the generating function of q-Hermite Polynomials we have the following identity:

\[ H_n(x; q) = [2]_q^n e_q \left( -\frac{1}{[2]_q^2} (D_q^x)^2 \right) x^n, \]  

(57)

which gives

\[ e_q \left( -\frac{1}{[2]_q^2} (D_q^z)^2 \right) z^{n-k} = \frac{1}{[2]_q^{n-k}} H_{n-k}(z; q), \]  

(58)

and

\[ e_q^k \left( -\frac{1}{[2]_q^2} (D_q^w)^2 \right) w^k = \frac{1}{[2]_q^k q^k} H_k(qw; \frac{1}{q}), \]  

(59)

where \( D_q^{\frac{1}{2}} = \frac{1}{q} D_q^{\frac{1}{2}} \). Substituting into (56), we get

\[ e_q \left( -\frac{1}{[2]_q^2} \Delta_q \right) (z + iw) \sum_{k=0}^n \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} \frac{1}{[2]_q^{n-k}} H_{n-k}(z; q) \frac{1}{[2]_q^k q^k} H_k(qw; \frac{1}{q}). \]

Then, according to identity (55) we obtain desired result

\[ (z + iw)^n = \sum_{k=0}^n \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} H_{n-k}(z; q) H_k(qw; \frac{1}{q}) (z+1i)^k. \]  

(60)

As a particular case of our binomial formula, we can find q-Hermite binomial expansion for the q-analytic binomial \((x + iy)^n\) as well. If in (44) we replace \( z \to x \) and \( w \to y \), then we get

\[ (x + iy)^n = \sum_{k=0}^n \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} H_{n-k}(x; q) H_k(qy; \frac{1}{q}) (x+iy)^k. \]  

(61)

Since a q-analytic function is determined by power series in q-binomials, this formula allows us to get expansion of an arbitrary q-analytic function in terms of real q-Hermite polynomials.

## 5 q-Traveling Waves

As an application of q-binomials here we consider the q-analogue of traveling waves as a solution of q-wave equation.

### 5.1 Traveling Waves:

Real functions of two real variables \( F(x, t) = F(x \pm ct) \) called the traveling waves, satisfy the following first order equations

\[ \left( \frac{\partial}{\partial t} \mp c \frac{\partial}{\partial x} \right) F(x \pm ct) = 0. \]
It describes waves with fixed shape, prorogating with constant speed $c$ in the left and in the right direction correspondingly. The general solution of the wave equation
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]
then can be written as an arbitrary superposition of these traveling waves
\[
u(x, t) = F(x + ct) + G(x - ct).
\]

### 5.2 q-Traveling Waves

Direct extension of traveling waves to q-traveling waves is not possible. This happens due to the absence in q-calculus of the chain rule and as follows, impossibility to use moving frame as an argument of the wave function. Moreover, if we try in the Fourier harmonics $f(x, t) = e^{i(kx - \omega t)}$, replace exponential function by Jackson’s q-exponential function $f(x, t) = e_q(i(kx - \omega t))$, then we find that it doesn’t work due to the absence of factorization for q-exponential function $e_q(i(kx - \omega t)) \neq e_q(ikx) e_q(i\omega t)$.

This is why, here we propose another way. First we observe that q-binomials
\[
(x \pm ct)^n = (x \pm ct)(x \pm qct)\ldots(x \pm q^{n-1}ct)
\]
for $n = 0, \pm 1, \pm 2, \ldots$, satisfy the first order one-directional q-wave equations
\[
\left(D_{\pm q}^1 \mp cD_{\pm q}^x\right)(x \pm ct)^n_q = 0. \quad (63)
\]
Then, the Laurent series expansion in terms of these q-binomials determines the q-analog of traveling waves
\[
f(x \pm ct)_q = \sum_{n=-\infty}^{\infty} a_n(x \pm ct)_q^n.
\]
Due to (63) the q-binomials (62) satisfy the q-wave equation
\[
\left((D_{\pm q}^1)^2 - c^2(D_{\pm q}^x)^2\right) u(x, t) = 0 \quad (64)
\]
and the general solution of this equation is expressed in the form of q-traveling waves
\[
u(x, t) = F(x + ct)_q + G(x - ct)_q
\]
where
\[
F(x + ct)_q = \sum_{n=-\infty}^{\infty} a_n(x + ct)_q^n
\]
and
\[
G(x - ct)_q = \sum_{n=-\infty}^{\infty} b_n(x - ct)_q^n.
\]
This allows us to solve IVP for the q-wave equation

\[
\left[ \left( D_{\frac{t}{q}} \right)^2 - c^2 (D_{\frac{q}{x}})^2 \right] u(x, t) = 0, \quad (66)
\]

\[
u(x, 0) = f(x), \quad (67)
\]

\[
D_{\frac{t}{q}} u(x, 0) = g(x), \quad (68)
\]

where \(-\infty < x < \infty\), in the D’Alembert form:

\[
u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{(x-ct)}^{(x+ct)} g(x') d_q x', \quad (69)
\]

where the Jackson integral is

\[
\int_{(x-ct)}^{(x+ct)} g(x') d_q x' = (1 - q)(x + ct) \sum_{j=0}^{\infty} q^j g(q^j(x+ct))_q - (1 - q)(x - ct) \sum_{j=0}^{\infty} q^j g(q^j(x - ct))_q. \quad (70)
\]

If the initial velocity is zero, \(g(x) = 0\), the formula reduces to

\[
u(x, t) = \frac{1}{2} \left( f(x + ct) + f(x - ct) \right). \quad (71)
\]

It should be noted here that q-traveling wave is not traveling wave in the standard sense and it is not preserving shape during evolution. It can be seen from simple observation. The traveling wave polynomial \((x - ct)_q^n = (x - ct)(x - qct)(x - q^2ct)...(x - q^{n-1}ct)\) includes the set of moving frames (as zeros of this polynomial) with re-scaled set of speeds \((c, qc, q^2c, ..., q^{n-1}c)\). It means that zeros of this polynomial are moving with different speeds and therefore the shape of polynomial wave is not preserving. Only in the linear case and in the case \(q = 1\), when speeds of all frames coincide, we are getting standard traveling wave.

### 5.3 EXAMPLES

In this section we are going to illustrate our results by several explicit solutions.

**Example 1:** We consider I.V.P. for the q-wave equation (66) with initial functions

\[
u(x, 0) = x^2, \quad (67)
\]

\[
D_{\frac{t}{q}} u(x, 0) = 0. \quad (68)
\]

Then the solution of the given I.V.P. for q-wave equation in D’Alembert form is found as

\[
u(x, t) = x^2 + qct^2. \quad (73)
\]
When $q = 1$, it reduces to well-known one as superposition of two traveling wave parabolas $(x \pm ct)^2$ moving to the right and to the left with speed $c$. Geometrically, the meaning of $q$ is the acceleration of our parabolas in vertical direction.

**Example 2:** The $q$-traveling wave

\[ u(x, t) = (x - ct)^2_q = (x - ct)(x - qct) \]

\[ = \left( x - \frac{2}{2} ct \right)^2 - \frac{(q - 1)^2}{4} c^2 t^2 \]  

(74)
gives solution of I.V.P. for the $q$-wave equation (66) with initial functions

\[ u(x, 0) = x^2 \]

\[ D^q_t u(x, 0) = -[2]_q cx. \]  

(75)

If $q = 1$ in this solution we have two degenerate zeros moving with the same speed $c$. In the case $q \neq 1$, two zeros are moving with different speeds $c$ and $qc$. It means that, the distance between zeros is growing linearly with time as $(q - 1)ct$. The solution is the parabola, moving in vertical direction with acceleration $\frac{(q - 1)^2}{4} c^2$, and in horizontal direction with constant speed $\frac{2}{q} c$. The area under the curve between moving zeros $x = ct$ and $x = qct$

\[ \int_{ct}^{qct} (x - ct)^2 dx = -\frac{(q - 1)^3 c^3 t^3}{6} \]

is changing according to time as $t^3$.

For more general initial function $f(x) = x^n$, $n = 2, 3, ...$ we get $q$-traveling wave

\[ u(x, t) = (x - ct)^n_q = (x - ct)(x - qct)...(x - q^{n-1}ct) \]

with $n$-zeros moving with speeds $c, qc, ..., q^{n-1}c$. The distance between two zeros is growing as $(q^m - q^n)ct$, and the shape of wave is changing. In parabolic case with $n = 2$, the shape of curve is not changing, but moving in horizontal direction with constant speed, and in vertical direction with constant acceleration. In contrast to this, for $n > 2$, the motion of zeros with different speeds changes the shape of the wave, and it can not be reduced to simple translation and acceleration.

**Example 3:** Given I.V.P. for the $q$-wave equation (66) with initial functions as $q$-trigonometric functions

\[ u(x, 0) = \cos_q x, \]

\[ D^q_t u(x, 0) = \sin_q x. \]  

(76)

By using the D’Alembert form (69), after $q$-integration , we get

\[ u(x, t) = \frac{1}{2} [\cos_q (x + ct)_q + \cos_q (x - ct)_q] + \frac{1}{2c} \int_{(x-ct)_q}^{(x+ct)_q} \sin_q (x') d_q x' \]

\[ = \frac{1}{2} \left[ \left( 1 + \frac{1}{c} \right) \cos_q (x - ct)_q + \left( 1 - \frac{1}{c} \right) \cos_q (x + ct)_q \right]. \]  

(77)
Example 4: \textbf{q-Gaussian Traveling Wave} For initial function in Gaussian form: \(u(x, 0) = e^{-x^2}\) in the standard case \(q = 1\) we have the Gaussian traveling wave \(u(x, t) = e^{-(x-ct)^2}\). For the \(q\)-traveling wave with Gaussian initial condition \(u(x, 0) = e^{-x^2}\), we have the \(q\)-traveling wave

\[
u(x, t) = \left(e^{-(x-ct)^2}\right)_q = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x - ct)^{2n}.
\]

5.4 \textbf{q-Traveling Waves in terms of q-Hermite Polynomials}

Identity \((25)\) allows us to rewrite the traveling wave binomial in terms of Hermite polynomials as

\[
(x + ct)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(x) H_k(-ict).
\]

Its \(q\)-analogue for \(q\)-traveling wave binomial follows from \((44)\)

\[
(x + ct)_q^n = \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k H_{n-k}(x; q) H_k(-iqct, \frac{1}{q}).
\]

Then, the general solution of \(q\)-wave equation \((64)\) can be expressed in the form of \(q\)-Hermite polynomials

\[
u(x, t) = F(x + ct)_q + G(x - ct)_q,
\]

where

\[
F(x + ct)_q = \sum_{n=-\infty}^{\infty} a_n (x + ct)_q^n = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k H_{n-k}(x; q) H_k(-iqct, \frac{1}{q}),
\]

\[
G(x - ct)_q = \sum_{n=-\infty}^{\infty} a_n (x - ct)_q^n = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k H_{n-k}(x; q) H_k(iqct, \frac{1}{q}).
\]

It is instructive to prove the \(q\)-traveling wave solution

\[
\left(D_q^+ - cD_q^x\right)_q (x + ct)_q^n = 0
\]

by using \(q\)-Hermite binomial. We have

\[
(D_q^+ - cD_q^x)_q (x + ct)_q^n = \left(D_q^+ - cD_q^x\right)_q \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k \frac{k(k-1)}{2} H_{n-k}(x; q) H_k(-iqct, \frac{1}{q})
\]

\[
= \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k \frac{k(k-1)}{2} H_{n-k}(x; q) H_k(-iqct, \frac{1}{q})
\]

\[
-c \frac{1}{2^n q^n} \sum_{k=0}^{n} \binom{n}{k} q^k \frac{k(k-1)}{2} D_q^x H_{n-k}(x; q) H_k(-iqct, \frac{1}{q})
\]

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By recursion formula for $q$-Hermite polynomials

$$D_q^n H_n(x; q) = [2]_q [n]_q H_{n-1}(x; q)$$

we get

$$(D_q^n - cD_q^{n-1})(x + ct)^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right)_q \left( \begin{array}{c} k+1 \\ n \end{array} \right)_q i^k q^{\frac{k(k+1)}{2}} H_{n-k-1}(x; q) [2]_q [k+1]_q H_k(-iqc, 1_q)$$

$$-c \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right)_q i^k q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q H_{n-k-1}(x; q) H_k(-iqc, 1_q)$$

$$= \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right)_q i^{k+1} q^{\frac{k(k+1)}{2}} H_{n-k-1}(x; q) [2]_q [k+1]_q H_k(-iqc, 1_q)$$

$$-c \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right)_q i^k q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q H_{n-k-1}(x; q) H_k(-iqc, 1_q)$$

$$= ct^n \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right)_q i^{k+1} q^{\frac{k(k+1)}{2}} [2]_q [k+1]_q q - \left( \begin{array}{c} n \\ k \end{array} \right)_q i^k q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q H_{n-k-1}(x; q) H_k(-iqc, 1_q)$$

$$= 0.$$

The expression in parenthesis is zero due to $q$-combinatorial formula and $[n]_q = \frac{[n]_q}{q^{n-1}}$.

**Acknowledgments**

This work was supported by Izmir Institute of Technology. One of the authors (S. Nalci) was partially supported by TUBITAK scholarship for graduate students.

**References**

[1] A. Perelomov, Generalized Coherent states and their applications, Springer-Verlag, 1986.

[2] E. G. Floratos, The many-body problem for $q$-oscillators J. Phys. A: Math. Gen. 24, 1991.

[3] V. Kac and P. Cheung, Quantum Calculus, Springer, New York, 2002.

[4] S. Nalci and O.K. Pashaev, $q$-analog of shock soliton solution, J. Phys. A: Math. Theor. 43, 445205, 2010.

[5] O.K. Pashaev and S. Nalci, $q$-analytic functions, fractals and generalized analytic functions, J. Phys. A: Math. Theor. 47, 045204, 2014.
[6] G. Dattoli et al., Theory of generalized Hermite polynomials, Computer Math. Applic. Vol:28, N0:4, pp. 71-83, 1994.