Integral theorems for monogenic functions in commutative algebras

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Let \( A_{m}^{n} \) be an arbitrary \( n \)-dimensional commutative associative algebra over the field of complex numbers with \( m \) idempotents. Let \( e_1 = 1, e_2, \ldots, e_k \) with \( 2 \leq k \leq 2n \) be elements of \( A_{m}^{n} \) which are linearly independent over the field of real numbers. We consider monogenic (i.e. continuous and differentiable in the sense of Gateaux) functions of the variable \( \sum_{j=1}^{k} x_j e_j \), where \( x_1, x_2, \ldots, x_k \) are real, and we prove curvilinear analogues of the Cauchy integral theorem, the Morera theorem and the Cauchy integral formula in \( k \)-dimensional \( (2 \leq k \leq 2n) \) real subset of the algebra \( A_{m}^{n} \). The present article is generalized of the author’s paper [1], where mentioned results are obtained for \( k = 3 \).

1 Introduction

The Cauchy integral theorem and Cauchy integral formula for the holomorphic function of the complex variable are a fundamental result of the classical complex analysis. Analogues of these results are also an important tool in commutative algebras of dimensional more that 2.

In the paper of E. R. Lorch [2] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the curvilinear integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. E. K. Blum [3] withdrew a convexity condition of a domain in the mentioned results from [2].

Let us note that a priori the differentiability of a function in the sense of Gateaux is a restriction weaker than the differentiability of this function in the sense of Lorch.

Therefore, we consider a monogenic functions defined as a continuous and differentiable in the sense of Gateaux. Also we assume that a monogenic function is given in a domain of three-dimensional subspace of an arbitrary commutative associative algebra with unit over the field of complex numbers. In this situation the results established in the papers [2, 3] is not applicable for a mentioned monogenic function, because it deals with an integration along a curve on which the function is not given, generally speaking.

In the papers [4, 5, 6] for monogenic function the curvilinear analogues of the Cauchy integral theorem, the Cauchy integral formula and the Morera theorem are obtained in special finite-dimensional commutative associative algebras. The results of the papers [4, 5, 6] are generalized in the article [1] to an arbitrary commutative associative algebra. At the same time in [1] monogenic functions were defined in a domain of real three-dimensional subspace of an algebra.
In this paper we generalize results of the papers [1] assuming that monogenic functions are defined in a domain of real \( k \)-dimensional subspace of an algebra.

Let us note that some analogues of the curvilinear Cauchy’s integral theorem and the Cauchy’s integral formula for another classes of functions in special commutative algebras are established in the papers [7, 8, 9, 10].

2 The algebra \( \mathbb{A}_n^{m} \)

Let \( \mathbb{N} \) be the set of natural numbers. We fix the numbers \( m, n \in \mathbb{N} \) such that \( m \leq n \). Let \( \mathbb{A}_n^{m} \) be an arbitrary commutative associative algebra with unit over the field of complex number \( \mathbb{C} \). E. Cartan [11, pp. 33 – 34] proved that in the algebra \( \mathbb{A}_n^{m} \) there exist a basis \( \{I_k\}_{k=1}^{n} \) satisfies the following multiplication rules:

1. \( \forall \ r, s \in [1, m] \cap \mathbb{N} : \quad I_r I_s = \begin{cases} 0 & \text{if } r \neq s, \\ I_r & \text{if } r = s; \end{cases} \)

2. \( \forall \ r, s \in [m + 1, n] \cap \mathbb{N} : \quad I_r I_s = \sum_{k=\max\{r,s\}+1}^{n} \Upsilon_{r,k}^s I_k ; \)

3. \( \forall \ s \in [m + 1, n] \cap \mathbb{N} \exists! u_s \in [1, m] \cap \mathbb{N} \forall \ r \in [1, m] \cap \mathbb{N} : \)

\[
I_r I_s = \begin{cases} 0 & \text{if } r \neq u_s, \\ I_s & \text{if } r = u_s. \end{cases}
\]

Furthermore, the structure constants \( \Upsilon_{r,k}^s \in \mathbb{C} \) satisfy the associativity conditions:

(A 1). \( (I_r I_s) I_p = I_r (I_s I_p) \quad \forall \ r, s, p \in [m + 1, n] \cap \mathbb{N}; \)

(A 2). \( (I_u I_s) I_p = I_u (I_s I_p) \quad \forall \ u \in [1, m] \cap \mathbb{N} \forall \ s, p \in [m + 1, n] \cap \mathbb{N}. \)

Obviously, the first \( m \) basis vectors \( \{I_u\}_{u=1}^{m} \) are the idempotents and, respectively, form the semi-simple subalgebra. Also the vectors \( \{I_r\}_{r=m+1}^{n} \) form the nilpotent subalgebra of algebra \( \mathbb{A}_n^{m} \). The unit of \( \mathbb{A}_n^{m} \) is the element \( 1 = \sum_{u=1}^{m} I_u \). Therefore, we will write that the algebra \( \mathbb{A}_n^{m} \) is a semi-direct sum of the \( m \)-dimensional semi-simple subalgebra \( S \) and \( (n - m) \)-dimensional nilpotent subalgebra \( N \), i. e.

\[
\mathbb{A}_n^{m} = S \oplus_s N.
\]

Let us note that nilpotent algebras are fully described for the dimensions 1, 2, 3 in the paper [12], and some four-dimensional nilpotent algebras can be found in the papers [13, 14].

The algebra \( \mathbb{A}_n^{m} \) contains \( m \) maximal ideals

\[
\mathcal{I}_u := \left\{ \sum_{k=1, k \neq u}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}, \quad u = 1, 2, \ldots, m,
\]

the intersection of which is the radical

\[
\mathcal{R} := \left\{ \sum_{k=m+1}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}.
\]

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We define $m$ linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ by put
$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall \omega \in \mathcal{I}_u, \quad u = 1, 2, \ldots, m.$$ Since the kernels of functionals $f_u$ are, respectively, the maximal ideals $\mathcal{I}_u$, then these functionals are also continuous and multiplicative (see [16, p. 147]).

## 3 Monogenic functions in $E_k$

Let us consider the vectors $e_1 = 1, e_2, \ldots, e_k$ in $\mathbb{A}_n^m$, where $2 \leq k \leq 2n$, and these vectors are linearly independent over the field of real numbers $\mathbb{R}$ (see [6]). It means that the equality
$$\sum_{j=1}^k \alpha_j e_j = 0, \quad \alpha_j \in \mathbb{R},$$ holds if and only if $\alpha_j = 0$ for all $j = 1, 2, \ldots, k$.

Let the vectors $e_1 = 1, e_2, \ldots, e_k$ have the following decompositions with respect to the basis $\{I_r\}_{r=1}^n$:
$$e_1 = \sum_{r=1}^m I_r, \quad e_j = \sum_{r=1}^n a_{jr} I_r, \quad a_{jr} \in \mathbb{C}, \quad j = 2, 3, \ldots, k. \quad (2)$$

Let $\zeta := \sum_{j=1}^k x_j e_j$, where $x_j \in \mathbb{R}$. It is obvious that
$$\xi_u := f_u(\zeta) = x_1 + \sum_{j=2}^k x_j a_{ju}, \quad u = 1, 2, \ldots, m.$$ Let $E_k := \{\zeta = \sum_{j=1}^k x_j e_j : x_j \in \mathbb{R}\}$ be the linear span of vectors $e_1 = 1, e_2, \ldots, e_k$ over the field $\mathbb{R}$. We note that in the further investigations, it is essential assumption: $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Obviously, it holds if and only if for every fixed $u = 1, 2, \ldots, m$ at least one of the numbers $a_{2u}, a_{3u}, \ldots, a_{ku}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

With a set $Q_\mathbb{R} \subset \mathbb{R}^k$ we associate the set $Q := \{\zeta = \sum_{j=1}^k x_j e_j : (x_1, \ldots, x_k) \in Q_\mathbb{R}\}$ in $E_k$. We also note that the topological properties of a set $Q$ in $E_k$ understood as a corresponding topological properties of a set $Q_\mathbb{R}$ in $\mathbb{R}^k$. For example, a homotopicity of a curve $\gamma \subset E_k$ to the zero means a homotopicity of $\gamma_\mathbb{R} \subset \mathbb{R}^k$ to the zero; a rectifiability of a curve $\gamma \subset E_k$ we understand as a rectifiability of the curve $\gamma_\mathbb{R} \subset \mathbb{R}^k$, etc.

Let $\Omega$ be a domain in $E_k$. With a domain $\Omega \subset E_k$ we associate the domain
$$\Omega_\mathbb{R} := \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : \zeta = \sum_{j=1}^k x_j e_j \in \Omega\}$$ in $\mathbb{R}^k$. 
We say that a continuous function \( \Phi : \Omega \to \mathbb{A}_n^m \) is monogenic in \( \Omega \) if \( \Phi \) is differentiable in the sense of Gateaux in every point of \( \Omega \), i.e. if for every \( \zeta \in \Omega \) there exists an element \( \Phi'(\zeta) \in \mathbb{A}_n^m \) such that
\[
\lim_{\varepsilon \to 0^+} (\Phi(\zeta + \varepsilon h) - \Phi(\zeta)) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_k. \tag{3}
\]

\( \Phi'(\zeta) \) is the Gateaux derivative of the function \( \Phi \) in the point \( \zeta \).

Consider the decomposition of a function \( \Phi : \Omega \to \mathbb{A}_n^m \) with respect to the basis \( \{I_r\}_{r=1}^n \):
\[
\Phi(\zeta) = \sum_{r=1}^n U_r(x_1, x_2, \ldots, x_k) I_r. \tag{4}
\]

In the case where the functions \( U_r : \Omega_\mathbb{R} \to \mathbb{C} \) are \( \mathbb{R} \)-differentiable in \( \Omega_\mathbb{R} \), i.e. for every \((x_1, x_2, \ldots, x_k) \in \Omega_\mathbb{R}\),
\[
U_r(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) - U_r(x_1, x_2, \ldots, x_k) =
\sum_{j=1}^k \frac{\partial U_r}{\partial x_j} \Delta x_j + o\left(\sqrt{\sum_{j=1}^k (\Delta x_j)^2}\right), \quad \sum_{j=1}^k (\Delta x_j)^2 \to 0,
\]

the function \( \Phi \) is monogenic in the domain \( \Omega \) if and only if the following Cauchy – Riemann conditions are satisfied in \( \Omega \):
\[
\frac{\partial \Phi}{\partial x_j} = \frac{\partial \Phi}{\partial x_1} e_j \quad \text{for all} \quad j = 2, 3, \ldots, k. \tag{5}
\]

An expansion of the resolvent is of the form (see [17]):
\[
(te_1 - \zeta)^{-1} = \sum_{u=1}^m \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^n \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_{us})} I_s \tag{6}
\]
\[
\forall t \in \mathbb{C} : t \neq \xi_u, \quad u = 1, 2, \ldots, m,
\]

where the coefficients \( Q_{r,s} \) are determined by the following recurrence relations:
\[
Q_{2,s} = T_s, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q} B_{q,s}, \quad r = 3, 4, \ldots, s - m + 1, \tag{7}
\]
with
\[
T_s := \sum_{j=2}^k x_j a_{js}, \quad B_{q,s} := \sum_{p=m+1}^{s-1} T_p \gamma_{q,s}^p, \quad p = m + 2, m + 3, \ldots, n, \tag{8}
\]
and the natural numbers \( u_s \) are defined in the rule 3 of the multiplication table of algebra \( \mathbb{A}_n^m \).

In the paper [15] an expansion of the resolvent is obtained for \( k = 3 \).
It follows from the relation (6) that the points \((x_1, x_2, \ldots, x_k) \in \mathbb{R}^k\) corresponding to the noninvertible elements \(\zeta = \sum_{j=1}^{k} x_j e_j\) form the set

\[ M^R_u := \left\{ \begin{array}{l} x_1 + \sum_{j=2}^{k} x_j \Re a_{ju} = 0, \\ \sum_{j=2}^{k} x_j \Im a_{ju} = 0, \end{array} \right. \]

in the \(k\)-dimensional space \(\mathbb{R}^k\). Also we consider the set \(M_u := \{ \zeta \in E_k : f_u(\zeta) = 0 \}\) for \(u = 1, 2, \ldots, m\). It is obvious that the set \(M^R_u \subset \mathbb{R}^k\) is congruent with the set \(M_u \subset E_k\).

Denote by \(D_u \subset \mathbb{C}\) the image of \(\Omega\) under the mapping \(f_u, \ u = 1, 2, \ldots, m\). We say that a domain \(\Omega \subset E_k\) is convex with respect to the set of directions \(M_u\) if \(\Omega\) contains the segment \(\{ \zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0, 1] \}\) for all \(\zeta_1, \zeta_2 \in \Omega\) such that \(\zeta_2 - \zeta_1 \in M_u\). A constructive description of all monogenic functions in the algebra \(A_n^m\) by means of holomorphic functions of the complex variable are obtained in the paper [17]. Namely, it is proved the theorem:

Let a domain \(\Omega \subset E_k\) be convex with respect to the set of directions \(M_u\) and \(f_u(E_k) = \mathbb{C}\) for all \(u = 1, 2, \ldots, m\). Then every monogenic function \(\Phi : \Omega \rightarrow A_n^m\) can be expressed in the form

\[ \Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int_{\Gamma_u} G_s(t)(te_1 - \zeta)^{-1} dt, \]  

(9)

where \(F_u\) and \(G_s\) are certain holomorphic functions in the domains \(D_u\) and \(D_{us}\), respectively, and \(\Gamma_q\) is a closed Jordan rectifiable curve in \(D_q\) which surrounds the point \(\xi_q\) and contains no points \(\xi_{q'}, q = 1, 2, \ldots, m, q' \neq q\).

We note that in the paper [13] the previous result is proved for \(k = 3\).

4 Cauchy integral theorem for a curvilinear integral

Let \(\gamma\) be a Jordan rectifiable curve in \(E_k\). For a continuous function \(\Psi : \gamma \rightarrow A_n^m\) of the form

\[ \Psi(\zeta) = \sum_{r=1}^{n} U_r(x_1, x_2, \ldots, x_k) I_r + i \sum_{r=1}^{n} V_r(x_1, x_2, \ldots, x_k) I_r, \]  

(10)

where \((x_1, x_2, \ldots, x_k) \in \gamma_\mathbb{R}\) and \(U_r : \gamma_\mathbb{R} \rightarrow \mathbb{R}\), \(V_r : \gamma_\mathbb{R} \rightarrow \mathbb{R}\), we define an integral along a Jordan rectifiable curve \(\gamma\) by the equality:

\[
\int_\gamma \Psi(\zeta) d\zeta := \sum_{j=1}^{k} e_j \sum_{r=1}^{n} I_r \int_{\gamma_\mathbb{R}} U_r(x_1, x_2, \ldots, x_k) dx_j + i \sum_{j=1}^{k} e_j \sum_{r=1}^{n} I_r \int_{\gamma_\mathbb{R}} V_r(x_1, x_2, \ldots, x_k) dx_j,
\]

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where \( d\zeta := e_1 dx_1 + e_2 dx_2 + \ldots + e_k dx_k \).

Also we define a surface integral. Let \( \Sigma \) be a piece-smooth hypersurface in \( E_k \). For a continuous function \( \Psi : \Sigma \rightarrow \mathbb{A}_m^m \) of the form (10), where \((x_1, x_2, \ldots, x_k) \in \Sigma \) and \( U_r : \Sigma, V_r : \Sigma \rightarrow \mathbb{R} \), we define a surface integral on \( \Sigma \) with the differential form \( dx_p \land dx_q \), by the equality

\[
\int_{\Sigma} \Psi(\zeta) dx_p \land dx_q := \sum_{r=1}^{n} I_r \int_{\Sigma} U_r(x_1, x_2, \ldots, x_k) dx_p \land dx_q + \ni \sum_{r=1}^{n} I_r \int_{\Sigma} V_r(x_1, x_2, \ldots, x_k) dx_p \land dx_q.
\]

If a function \( \Phi : \Omega \rightarrow \mathbb{A}_n^m \) is continuous together with partial derivatives of the first order in a domain \( \Omega \), and \( \Sigma \) is a piece-smooth hypersurface in \( \Omega \), and the edge \( \gamma \) of surface \( \Sigma \) is a rectifiable Jordan curve, then the following analogue of the Stokes formula is true:

\[
\int_{\gamma} \Phi(\zeta) d\zeta = \int_{\Sigma} \left( \frac{\partial \Psi}{\partial x_1} e_2 - \frac{\partial \Psi}{\partial x_2} e_1 \right) dx_1 \land dx_2 + \left( \frac{\partial \Psi}{\partial x_2} e_3 - \frac{\partial \Psi}{\partial x_3} e_2 \right) dx_2 \land dx_3 + \ldots + \left( \frac{\partial \Psi}{\partial x_k} e_1 - \frac{\partial \Psi}{\partial x_1} e_k \right) dx_k \land dx_1. \tag{11}
\]

Now, the next theorem is a result of the formula (11) and the equalities (11).

**Theorem 1.** Suppose that \( \Phi : \Omega \rightarrow \mathbb{A}_n^m \) is a monogenic function in a domain \( \Omega \), and \( \Sigma \) is a piece-smooth surface in \( \Omega \), and the edge \( \gamma \) of surface \( \Sigma \) is a rectifiable Jordan curve. Then

\[
\int_{\gamma} \Phi(\zeta) d\zeta = 0. \tag{12}
\]

In the case where a domain \( \Omega \) is convex, then by the usual way (see, e.g., [10]) the equality (12) can be prove for an arbitrary closed Jordan rectifiable curve \( \gamma \).

In the case where a domain \( \Omega \) is an arbitrary, then similarly to the proof of Theorem 3.2 [3] we can prove the following

**Theorem 2.** Let \( \Phi : \Omega \rightarrow \mathbb{A}_n^m \) be a monogenic function in a domain \( \Omega \). Then for every closed Jordan rectifiable curve \( \gamma \) homotopic to a point in \( \Omega \), the equality (12) is true.

5 **The Morera theorem**

To prove the analogue of Morera theorem in the algebra \( \mathbb{A}_n^m \), we introduce auxiliary notions and prove some auxiliary statements.

Let us consider the algebra \( \mathbb{A}_n^m(\mathbb{R}) \) with the basis \( \{I_k, iI_k\}_{k=1}^{n} \) over the field \( \mathbb{R} \) which is isomorphic to the algebra \( \mathbb{A}_n^m \) over the field \( \mathbb{C} \). In the algebra \( \mathbb{A}_n^m(\mathbb{R}) \) there exist another basis \( \{e_r\}_{r=1}^{2n} \), where the vectors \( e_1, e_2, \ldots, e_k \) are the same as in the Section 3.
For the element \( a := \sum_{r=1}^{2n} a_r e_r, \ a_r \in \mathbb{R} \) we define the Euclidian norm
\[
\|a\| := \sqrt{\sum_{r=1}^{2n} a_r^2}.
\]
Accordingly, \( \|\zeta\| = \sqrt{\sum_{j=1}^{k} x_j^2} \) and \( \|e_j\| = 1 \) for all \( j = 1, 2, \ldots, k \).

Using the Theorem on equivalents of norms, for the element \( b := \sum_{r=1}^{n} (b_{1r} + ib_{2r}) I_r \), \( b_{1r}, b_{2r} \in \mathbb{R} \) we have the following inequalities
\[
|b_{1r} + ib_{2r}| \leq \sqrt{\sum_{r=1}^{2n} (b_{1r}^2 + b_{2r}^2)} \leq c\|b\|,
\]
where \( c \) is a positive constant does not depend on \( b \).

**Lemma 1.** If \( \gamma \) is a closed Jordan rectifiable curve in \( E_k \) and function \( \Psi : \gamma \rightarrow \mathbb{A}_n^m \) is continuous, then
\[
\left\| \int_{\gamma} \Psi(\zeta) \, d\zeta \right\| \leq c \int_{\gamma} \|\Psi(\zeta)\| \|d\zeta\|,
\]
where \( c \) is a positive absolutely constant.

**Proof.** Using the representation of function \( \Psi \) in the form (10) for \((x_1, x_2, \ldots, x_k) \in \gamma \), we obtain
\[
\left\| \int_{\gamma} \Psi(\zeta) \, d\zeta \right\| \leq \sum_{r=1}^{n} \|e_1 I_r\| \int_{\gamma} \left| U_r(x_1, x_2, \ldots, x_k) + i V_r(x_1, x_2, \ldots, x_k) \right| \, dx_1 + \\
\ldots + \sum_{r=1}^{n} \|e_k I_r\| \int_{\gamma} \left| U_r(x_1, x_2, \ldots, x_k) + i V_r(x_1, x_2, \ldots, x_k) \right| \, dx_k.
\]
Now, taking into account the inequality (13) for \( b = \Psi(\zeta) \) and the inequalities \( \|e_j I_r\| \leq c_j, \ j = 1, 2, \ldots, k \), where \( c_j \) are positive absolutely constants, we obtain the relation (14). The lemma is proved.

We understand a triangle \( \Delta \) as a plane figure bounded by three line segments connecting three its vertices. Denote by \( \partial \Delta \) the boundary of triangle \( \Delta \) in relative topology of its plane.

Using Lemma 1, for functions taking values in the algebra \( \mathbb{A}_n^m \), the following Morera theorem can be established in the usual way.

**Theorem 3.** If a function \( \Phi : \Omega \rightarrow \mathbb{A}_n^m \) is continuous in a domain \( \Omega \) and satisfies the equality
\[
\int_{\partial \Delta} \Phi(\zeta) \, d\zeta = 0
\]
(15)
for every triangle $\triangle$ such that closure $\overline{\triangle} \subset \Omega$, then the function $\Phi$ is monogenic in the domain $\Omega$.

6 Cauchy integral formula for a curvilinear integral

Let $\zeta_0 := \sum_{j=1}^{k} x_{j}^{(0)} e_{j}$ be a point in a domain $\Omega \subset E_k$. Let us take any 2-dimensional plan containing the point $\zeta_0$ and in this plane we take a circle $C(\zeta_0, \varepsilon)$ of radius $\varepsilon$ with the center at the point $\zeta_0$, such that this circle completely contained in $\Omega$. By $C_{u}(\zeta_{u}^{(0)}, \varepsilon) \subset \mathbb{C}$ we denote the image of $C(\zeta_0, \varepsilon)$ under the mapping $f_{u}$, $u = 1, 2, \ldots, m$. We assume that the circle $C(\zeta_0, \varepsilon)$ embraces the set $\{\zeta - \zeta_0 : \zeta \in \bigcup_{u=1}^{m} M_{u}\}$. It means that the curve $C_{u}(\zeta_{u}^{(0)}, \varepsilon)$ bounds some domain $D'_{u}$ and $f_{u}(\zeta_0) = \zeta_{u}^{(0)} \in D'_{u}$, $u = 1, 2, \ldots, m$.

We say that the curve $\gamma \subset \Omega$ embraces once the set $\{\zeta - \zeta_0 : \zeta \in \bigcup_{u=1}^{m} M_{u}\}$, if there exists a circle $C(\zeta_0, \varepsilon)$ which embraces the mentioned set and is homotopic to $\gamma$ in the domain $\Omega \setminus \{\zeta - \zeta_0 : \zeta \in \bigcup_{u=1}^{m} M_{u}\}$.

Since the function $\zeta^{-1}$ is continuous on the curve $C(0, \varepsilon)$, then there exists the integral

$$\lambda := \int_{C(0, \varepsilon)} \zeta^{-1} d\zeta. \quad (16)$$

The following theorem is an analogue of Cauchy integral theorem for monogenic function $\Phi : \Omega \to \mathbb{A}_n^m$.

**Theorem 4.** Suppose that a domain $\Omega \subset E_k$ is convex with respect to the set of directions $M_u$ and $f_{u}(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Suppose also that $\Phi : \Omega \to \mathbb{A}_n^m$ is a monogenic function in $\Omega$. Then for every point $\zeta_0 \in \Omega$ the following equality is true:

$$\lambda \Phi(\zeta_0) = \int_{\gamma} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta, \quad (17)$$

where $\gamma$ is an arbitrary closed Jordan rectifiable curve in $\Omega$, that embraces once the set $\{\zeta - \zeta_0 : \zeta \in \bigcup_{u=1}^{m} M_{u}\}$.

The proof is similar to the proof of Theorem 4 of [1].

7 A constant $\lambda$

In some special algebras (see [1, 5, 6]) the Cauchy integral formula (17) has the form

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int_{\gamma} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta, \quad (18)$$
In this Section we indicate a set of algebras $A^{m}_n$ for which (19) holds. In this way we first consider some auxiliary statements.

As a consequence of the expansion (6), we obtain the following equality:

$$\zeta^{-1} = \sum_{r=1}^{n} \tilde{A}_r I_r$$

(20)

with the coefficients $\tilde{A}_r$ determined by the following relations:

$$\tilde{A}_u = \frac{1}{\xi_u}, \quad u = 1, 2, \ldots, m,$$

$$\tilde{A}_s = \sum_{r=2}^{s-m+1} \frac{\tilde{Q}_{r,s}}{\xi_{u_s}}, \quad s = m + 1, m + 2, \ldots, n,$$

(21)

where $\tilde{Q}_{r,s}$ are determined by the following recurrence relations:

$$\tilde{Q}_{2,s} = -T_s, \quad \tilde{Q}_{r,s} = -\sum_{q=r+m-2}^{s-1} \tilde{Q}_{r-1,q} B_{q,s}, \quad r = 3, 4, \ldots, s - m + 1,$$

(22)

where $T_s$ and $B_{q,s}$ are the same as in the equalities (8), and natural numbers $u_s$ are defined in the rule 3 of the multiplication table of the algebra $A^{m}_n$.

Taking into account the equality (20) and the relation

$$d\zeta = \sum_{j=1}^{k} dx_j e_j = \sum_{u=1}^{m} (dx_1 + \sum_{j=2}^{k} dx_j a_{ju}) I_u +$$

$$+ \sum_{r=m+1}^{n} \sum_{j=2}^{k} dx_j a_{js} I_r = \sum_{u=1}^{m} d\xi_u I_u + \sum_{r=m+1}^{n} dT_r I_r,$$

we have the following equality

$$\zeta^{-1} d\zeta = \sum_{u=1}^{m} \tilde{A}_u d\xi_u I_u + \sum_{r=m+1}^{n} \tilde{A}_u dT_r I_r +$$

$$+ \sum_{s=m+1}^{n} \tilde{A}_s d\xi_{u_s} I_s + \sum_{s=m+1}^{n} \sum_{r=m+1}^{n} \tilde{A}_s dT_r I_s I_r =: \sum_{r=1}^{n} \sigma_r I_r.$$

(23)

Now, taking into account the denotation (23) and the equality (21), we calculate:

$$\int_{C(0, R)} \sum_{u=1}^{m} \sigma_u I_u = \sum_{u=1}^{m} I_u \int_{C_u(\zeta_u, R)} \frac{d\xi_u}{\xi_u} = 2\pi i \sum_{u=1}^{m} I_u = 2\pi i.$$

Thus,

$$\lambda = 2\pi i + \sum_{r=m+1}^{n} I_r \int_{C(0, R)} \sigma_r.$$

(24)
Therefore, the equality (19) holds if and only if
\[
\int_{C(0,R)} \sigma_r = 0 \quad \forall \ r = m + 1, \ldots, n.
\] (25)

But, for satisfying the equality (25) the differential form \( \sigma_r \) must be a total differential of some function. We note that the property of being a total differential is invariant under admissible transformations of coordinates [19, p. 328, Theorem 2]. In our situation, if we show that \( \sigma_r \) is a total differential of some function depend of the variables \( \xi_{m+1}, \ldots, \xi_r \), then it means that \( \sigma_r \) is a total differential of some function depending on \( x_1, x_2, \ldots, x_k \).

7.1 7.1

In this subsection we indicate a set of algebras in which the vectors (2) chosen arbitrarily and the equality (19) holds. We remind that an arbitrary commutative associative algebra, \( \mathbb{A}_n^m \), with unit over the field of complex number \( \mathbb{C} \) can be represented as \( \mathbb{A}_n^m = S \oplus_s N \), where \( S \) is \( m \)-dimensional semi-simple subalgebra and \( N \) is \((n - m)\)-dimensional nilpotent subalgebra (see Section 2).

**Theorem 5.** Let \( \mathbb{A}_n^m = S \oplus_s N \). Then the equality (19) holds if at least one of the following conditions is satisfied:

1. \( \mathbb{A}_n^m \equiv S \);
2. \( N \) is a zero nilpotent subalgebra;
3. \( \dim_{\mathbb{C}} N \leq 3 \);
4. \( \dim_{\mathbb{C}} N = 4 \) and
   \[\gamma_{m+1}^{m+1} \gamma_{m+2}^{m+2} \gamma_{m+3}^{m+3} \gamma_{m+4}^{m+4} = \gamma_{m+1}^{m+1} \gamma_{m+2}^{m+2} \gamma_{m+3}^{m+3} \gamma_{m+4}^{m+4} = \gamma_{m+1}^{m+1} \gamma_{m+2}^{m+2} \gamma_{m+3}^{m+3} \gamma_{m+4}^{m+4} = \gamma_{m+1}^{m+1} \gamma_{m+2}^{m+2} \gamma_{m+3}^{m+3} \gamma_{m+4}^{m+4} = \gamma_{m+1}^{m+1} \gamma_{m+2}^{m+2} \gamma_{m+3}^{m+3} \gamma_{m+4}^{m+4} = 0.\] (26)

The proof is analogous to the proofs of Theorems 5—8 of [1].

Further we consider some examples of algebras, which satisfy the relations (26).

**Examples.**

- Consider the algebra with the basis \( \{ I_1 := 1, I_2, I_3, I_4, I_5 \} \) and multiplication rules:
  \[
  I_2^2 = I_3 \quad I_2 I_4 = I_5
  \]
  and other products are zeros (for nilpotent subalgebra see [14], Table 21, algebra \( \mathcal{J}_{69} \) and [13], page 590, algebra \( A_{1,4} \)).
• Consider the algebra with the basis \( \{ I_1 := 1, I_2, I_3, I_4, I_5 \} \) and multiplication rules:
\[
I_2^2 = I_3
\]
and other products are zeros (for nilpotent subalgebra see [13], page 590, algebra \( A_{1,2} \oplus A_{2,0}^2 \)).
• The algebra with the basis \( \{ I_1 := 1, I_2, I_3, I_4, I_5 \} \) and multiplication rules:
\[
I_2^2 = I_3, \quad I_4^2 = I_5
\]
and other products are zeros (for nilpotent subalgebra see [13], page 590, algebra \( A_{1,2} \oplus A_{1,2} \)).
• The algebra with the basis \( \{ I_1 := 1, I_2, I_3, I_4, I_5 \} \) and multiplication rules:
\[
I_2^2 = I_3, \quad I_2 I_3 = I_4
\]
and other products are zeros (for nilpotent subalgebra see [14], Table 21, algebra \( J_{71} \)).

In the paper [1] is considered an example of algebra, which does not satisfy the relations (26). Moreover, in [1] is selected the vectors \( e_1, e_2, e_3 \) of the form (2) such that the equality (19) is not true.

7.2 7.2

In this subsection we indicate sufficient conditions on a choose of the vectors (2) for which the equality (19) is true. Let the algebra \( A_{m,n} = S \oplus s N \) be represented as \( S \oplus s N \). Let us note that the condition \( \zeta \in E_k \subset S \) means that in the decomposition (2) \( a_{jr} = 0 \) for all \( j = 2, 3, \ldots, k \) and \( r = m + 1, \ldots, n \).

**Theorem 6.** If \( A_{m,n} = S \oplus s N \) and \( \zeta \in E_k \subset S \), then the equality (19) holds.

**Proof.** Since \( \zeta \in S \), then \( T_s = 0 \) for \( s = m + 1, \ldots, n \) (see denotation (8)). We note that from the relation (23) follows the equalities
\[
\begin{align*}
\sigma_{m+1} &= \frac{dT_{m+1}}{\zeta_{u_{m+1}}} + \tilde{A}_{m+1} d\xi_{u_{m+1}}, \\
\sigma_r &= \frac{dT_r}{\zeta_{u_r}} + \tilde{A}_r d\xi_{u_r} + \sum_{q,s=m+1}^{r-1} \tilde{A}_q dT_s \gamma_{q,r}^s, \quad r = m + 2, \ldots, n.
\end{align*}
\] (27)

Now, from (22), (24) follows that \( \tilde{A}_s = 0 \), and then from the (27) follows that \( \sigma_r = 0 \) for \( r = m + 1, \ldots, n \). The equality (19) is a consequence of the equality \( \sigma_r = 0 \) and the relation (24). The theorem is proved.

Let us note that by essentially the Theorem 7 generalizes the Theorem 3 of the paper [21] and generalizes the Theorem 9 of [1].

Now we consider a case where \( \zeta \notin S \). If \( A_{m,n} = S \oplus s N \) and \( \dim \mathbb{C} N \leq 3 \), then by Theorem 7.2 the equality (19) holds for any \( \zeta \in E_k \).

**Theorem 7.** Let \( A_{m,n} = S \oplus s N \) and \( \dim \mathbb{C} N = 4 \). Then the equality (19) holds if the following two conditions satisfied:
1. \( a_{j,m+1} = 0 \) for all \( j = 2, 3, \ldots, k \);
2. at least one of the relations \( a_{j,m+2} = 0 \) or \( a_{j,m+3} = 0 \) is true for all \( j = 2, 3, \ldots, k \).

The proof is similar to the proof of Theorem 10 of [1].

References

1. Shpakivskyi V. S. Curvilinear integral theorems for monogenic functions in commutative associative algebras // submitted to Adv. Appl. Clifford Alg., http://arxiv.org/pdf/1503.03464v1.pdf
2. Lorch E. R. The theory of analytic function in normed abelin vector rings // Trans. Amer. Math. Soc., 54 (1943), 414 – 425.
3. Blum E. K. A theory of analytic functions in banach algebras // Trans. Amer. Math. Soc., 78 (1955), 343 – 370.
4. Shpakivskyi V. S., Plaksa S. A. Integral theorems and a Cauchy formula in a commutative three-dimensional harmonic algebra // Bulletin Soc. Sci. Lettr. Lódz, 60 (2010), 47 – 54.
5. Plaksa S. A., Shpakivskyi V. S. Monogenic functions in a finite-dimensional algebra with unit and radical of maximal dimensionality // J. Algerian Math. Soc., 1 (2014), 1 – 13.
6. Plaksa S. A., Pukhtaievych R. P. Constructive description of monogenic functions in \( n \)-dimensional semi-simple algebra // An. Şt. Univ. Ovidius Constanţa, 22 (2014), no. 1, 221 – 235.
7. Ketchum P. W. Analytic functions of hypercomplex variables // Trans. Amer. Math. Soc., 30 (1928), no. 4, 641 – 667.
8. Ketchum P. W. A complete solution of Laplace’s equation by an infinite hypervariable // Amer. J. Math., 51 (1929), 179 – 188.
9. Roşculeţ M. N. O teorie a funcţiilor de o variabilă hipercomplexă în spaţiul cu trei dimensiuni // Studii şi Cercetări Matematice, 5, nr. 3–4 (1954), 361 – 401.
10. Roşculeţ M. N. Algebre liniare asociative și comutative și funcții monogene atașate lor // Studii și Cercetări Matematice, 6, nr. 1–2 (1955), 135 – 173.
11. Cartan E. Les groupes bilinéares et les systèmes de nombres complexes // Annales de la faculté des sciences de Toulouse, 12 (1898), no. 1, 1 – 64.
12. Burde D., de Graaf W. Classification of Novicov algebras // Applicable Algebra in Engineering, Communication and Computing, 24(2013), no. 1, 1 – 15.
13. Burde D., Fialowski A. Jacobi–Jordan algebras // Linear Algebra Appl., 459 (2014), 586 – 594.
14. Martin M. E. Four-dimensional Jordan algebras // Int. J. Math. Game Theory Algebra 20 (4) (2013) 41 – 59.
15. Shpakivskyi V. S. Constructive description of monogenic functions in a finite-dimensional commutative associative algebra // submitted to J. Math. Anal. Appl., http://arxiv.org/pdf/1411.4643v1.pdf
16. Hille E., Phillips R. S. Functional analysis and semi-groups [Russian translation], Inostr. Lit., Moscow (1962).

17. Shpakivskyi V. S. Monogenic functions in finite-dimensional commutative associative algebras // accepted to Zb. Pr. Inst. Mat. NAN Ukr.

18. Privalov I. I. Introduction to the Theory of Functions of a Complex Variable, GITTL, Moscow, 1977. (Russian)

19. Shabat B. V. Introduction to complex analysis, Part 2, Nauka, Moskow (1976). (Russian)

20. Plaksa S. A., Pukhtaevich R. P. Constructive description of monogenic functions in a three-dimensional harmonic algebra with one-dimensional radical // Ukr. Math. J., 65 (2013), no. 5, 740 – 751.

21. Shpakivskyi V. S., Kuzmenko T. S. Integral theorems for the quaternionic G-monogenic mappings // accepted to An. Şt. Univ. Ovidius Constanţa.