Vacuum high-harmonic generation in the shock regime and photon-photon scattering dynamics

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Abstract. The presence of charged quantum virtual states permits a nonlinear self-interaction of the electromagnetic field in vacuum. This interaction can be described as real photon-photon scattering. This effect has been calculated for the case of colliding plane waves, when the centre-of-mass energy is much less than the electron rest energy. The quantum effect is included in the classical electromagnetic field equations of motion by a standard effective approach based upon a weak-field expansion of the Heisenberg-Euler Lagrangian. Solving for the resultant electromagnetic field indicates a signal for real photon-photon scattering when the plane waves overlap, which can be significantly larger than the usually-considered asymptotic values that reach detectors. By considering arbitrary numbers of four- and six-photon scattering, the process of vacuum higher harmonic generation has been studied both analytically and numerically. A route to prolific harmonic generation is identified that does not depend on field strengths being necessarily close to the Schwinger limit. The resulting vacuum electromagnetic shock wave has been studied and a nonlinear shock parameter identified.

1. Introduction
Real photons are predicted to be able to scatter off one another in vacuum due to the presence of charged virtual states. The study of this process has a long history, with Weisskopf [1] and Heisenberg and Euler [2] producing the first comprehensive calculations, which were later reformulated by Schwinger [3]. For laser-based experiments, a useful approximation is to assume the centre-of-mass energy is much less than the rest electron energy. This permits using an approach based on the Heisenberg-Euler Lagrangian to include the quantum process of photon-photon scattering as an effective correction to the classical field equations of motion. A second useful approximation is that the peak electromagnetic field strength \( E \) is much lower than the critical, so-called “Schwinger” field strength \( E_{\text{cr}} = m^2c^3/\hbar\epsilon = 1.3 \times 10^{16} \text{ Vcm}^{-1} \). This permits performing a weak-field approximation of the Heisenberg-Euler Lagrangian, the leading-order of which describes four-photon scattering [4], which can be interpreted in the classical paradigm as “vacuum four-wave mixing”. Higher orders in this expansion describe vacuum 2\( n \)-wave mixing.

In vacuum four-wave mixing, the final photons can have the same energies as the initial photons, but with an altered wavevector [5, 6, 7, 8] and altered polarisation [9, 10, 11, 12]. Alternatively, final photons can be produced with an energy different to the probe, in either case equal to the sum or difference of the incoming photons [13, 14, 15]. Ongoing experiments
measure real photon-photon scattering through the corresponding phase shift in a laser beam as it passes through a magnetised cavity [16, 17]. Reviews of calculations of real photon-photon scattering using intense lasers can be be found in [18, 19, 20].

One aim of the project was to include real photon-photon scattering in numerical simulation. Of particular interest was the dynamics of the electromagnetic field at finite time. The focus of this proceedings will be on the overlap scattered field, so-called as it is a source of scattered photons that is only present when the electromagnetic invariants are non-vanishing, such as in the overlap of counter-propagating plane waves. The generation of the zeroth, fundamental and second harmonics were simulated and showed excellent agreement with analytical calculations.

2. Low-frequency, weak-field photon-photon scattering

We consider an oscillating probe plane-wave pulse of electric field $E_p$, counterpropagating with a slowly-varying background of electric field $E_b$. The probe is right-moving $E_p(x^-)$ whereas the background is left-moving $E_b(x^+)$ where $x^\pm = t \mp z$ and we concentrate on the scattering of the probe wave. The electric (and magnetic fields $B$) will be expressed in units of $E_b$. The wave equation for the total electric field $E = E_p + E_b$, which includes the quantum vacuum interaction can be written:

$$[\partial_t^2 - \partial_x^2] E = T[E, B]$$

where $B$ is the total magnetic field and$^1$

$$T[E, B] = -4\pi \left[ \nabla \wedge \partial_t M + \partial_t^2 P - \nabla (\nabla \cdot P) \right],$$

for dimensionless vacuum magnetisation $M = (\alpha/m^4)\partial L_{HE}/\partial B$ and polarisation $P = (\alpha/m^4)\partial L_{HE}/\partial E$, where the Heisenberg-Euler Lagrangian $L_{HE}$ is given by [3]:

$$L_{HE} = -\frac{m^4}{8\pi^2} \int_0^\infty ds \frac{e^{-s}}{s^3} \left[ s^2 ab \cot as \coth bs - 1 + \frac{s^2}{3} (a^2 - b^2) \right],$$

and the two electromagnetic and two secular invariants are defined thus:

$$F = -F^2/4E_b^2, \quad G = -FF^*/4E_b^2 = E \cdot B,$$

$$a = \left[ \sqrt{F^2 + G^2 + F^*} \right]^{1/2}, \quad b = \left[ \sqrt{F^2 + G^2 - F} \right]^{1/2}. $$

Here $F$ and $F^*$ are the Faraday tensor and its dual and we note that all fields are normalised to $E_b$. When $E \ll 1$, one can perform a weak-field expansion of Eq. (3) to give:

$$L_{HE} = \frac{m^4}{\alpha} \sum_{i=1}^\infty L_i,$$

$$L_1 = \frac{\mu_1}{4\pi} \left[ (E^2 - B^2)^2 + 7(E \cdot B)^2 \right],$$

$$L_2 = \frac{\mu_2}{4\pi} \left[ 2 (E^2 - B^2)^2 + 13 (E \cdot B)^2 \right],$$

$$L_3 = \frac{\mu_3}{4\pi} \left[ 3 (E^2 - B^2)^4 + 22 (E^2 - B^2)^2 (E \cdot B)^2 + 19 (E \cdot B)^4 \right],$$

$^1$ This equation corrects the missing minus signs in [21, 22].
where $\mu_1 = \alpha/90\pi$, $\mu_2 = \alpha/315\pi$, $\mu_3 = 4\alpha/945\pi$. Then $L_j$ describes photon-photon scattering involving $2 + 2j$ photons. For counterpropagating plane waves, $\nabla(\nabla \cdot P) = 0$, and the solution to Eq. (1) can be written:

$$E^{(1)} = E^{(0)} + \Delta E^{(0)},$$

where $E^{(0)} = E^{(0)}_p + E^{(0)}_s$ is the trivial vacuum solution, for which $(\partial_t^2 - \partial_z^2) E^{(0)} = 0$ and the scattered field for a single photon-photon scattering event is:

$$\Delta E^{(0)} = \Delta E^{(0)}_p + \Delta E^{(0)}_s,$$

which correspond to scattering in the forward and backwards directions respectively\(^2\) and:

$$\Delta E^{(0)}_p(t, z) = -\int_{-\infty}^{z} \frac{dz'}{2} J^{(0)}(t' = x' + z', z'),$$

$$\Delta E^{(0)}_s(t, z) = -\int_{z}^{\infty} \frac{dz'}{2} J^{(0)}(t' = x' - z', z'),$$

where the vacuum current $J^{(0)}$, fulfilling $\partial_t J^{(0)} = -T[E^{(0)}_x B^{(0)}_y - E^{(0)}_y B^{(0)}_x]$, is the standard current term occurring in the modified Maxwell equations. Since all fields are plane waves, and the solution is also assumed to be of the form of a plane wave, the electromagnetic field can be expressed in terms of just the probe and background electric fields. Since each term of the weak-field expansion of the Heisenberg-Euler Lagrangian is expressed in terms of the electromagnetic invariants, and since these vanish identically for plane waves, only the probe-background cross terms survive so $L_n \sim (E_s E_p)^{n+1}$. Since the vacuum current that occurs in the field equation of motions is acquired by differentiating $L_n$ by the field, the current at each order is given by $J_n \sim (E_s E_p)^n E$.

This simple power-counting argument shows how the vacuum current can be further written:

$$J_n = J_{n,s} + J_{n,p},$$

where:

$$J_{n,s} = \varepsilon_{n,s} \mu_n E_{n+1}^p(x^+) \partial_{-} E_n^p(x^-)$$

$$J_{n,p} = \varepsilon_{n,p} \mu_n E_{n+1}^p(x^-) \partial_{+} E_n^p(x^+),$$

and the polarisation vectors $\varepsilon_{n,s}, \varepsilon_{n,p}$ are complicated functions of the wavevectors $k_p, k_s$ and polarisation vectors $\varepsilon_p, \varepsilon_s$ of the probe and background plane-wave solutions to the classical vacuum wave equation [21]. It is clearer to write quantities in terms of lightcone co-ordinates $x^\pm$ rather than $t$ and $z$. Of most interest to us was the forward-scattered field, so let us specify the discussion to $\Delta \hat{E}^{(0)}$. Then since we can write:

$$\Delta \hat{E}(x^+, x^-) = -\int_{-\infty}^{x^+} dy \frac{dy}{4} J(x^+ = y, x^- = x^-),$$

the different parts of the current give two sources of real photon-photon scattering:

$$\Delta \hat{E}_n(x^+, x^-) = -\varepsilon_{n,p} \frac{\mu_n}{4} \partial_{-} E_n^p(x^-) \int_{-\infty}^{x^+} dy E_1^{1+n}(y) - \varepsilon_{n,s} \frac{\mu_n}{4} E_1^{1+n}(x^-) \int_{-\infty}^{x^+} dy \partial_{y} E_1^{n}(y).$$

\(^2\) The current has been redefined with a minus sign, correcting [21, 22]
We identify the first term with the asymptotic forward-scattered field as it is the term that remains when the two plane waves have passed through each other and are again well-separated:

$$\lim_{x^+ \to \infty} \Delta \mathbf{E}_n(x^+, x^-) = -\varepsilon_{n,p} \frac{\mu_n}{4} \partial_y E_p^{1+n}(x) \int_{-\infty}^{\infty} dy \ E_s^{1+n}(y).$$

(19)

The remaining term is referred to as the overlap forward-scattered field and can be written as a surface term:

$$\Delta \mathbf{E}_n^o(x^+, x^-) = -\varepsilon_{n,s} \frac{\mu_n}{4} E_p^{1+n}(x) \int_{-\infty}^{x^+} dy \ \partial_y E_s^n(y)$$

$$= -\varepsilon_{n,s} \frac{\mu_n}{4} E_p^{1+n}(x^-) \left[ E_s^n(x^+) - \lim_{y \to -\infty} E_s(y) \right],$$

(20)

and is most significant when the probe and background most overlap, viz. when $E_p(x^-)E_s(x^+)$ is a maximum.

2.1. Overlap forward-scattered field

The overlap field $\Delta \mathbf{E}_n^o(x^+, x^-)$ is only present if there is some inhomogeneity in the background field. For this reason, it does not occur in traditional treatments of photon-photon scattering that are used to define a vacuum refractive index. However, if a vacuum modified refractive index is employed at finite time to describe the outcome of photon-photon scattering experiments using high-intensity lasers, it would seem to be consistent to also include this overlap field. The modification of the vacuum refractive index due to an inhomogeneity in the background was recently derived in [23] and was found to be equivalent to the overlap field term studied here. Moreover, a similar surface term appeared also in the same year in calculations of the polarisation flip in four photon-photon scattering [11]. However, the Heisenberg-Euler Lagrangian is derived for an eternally constant electromagnetic field. Therefore, it is perhaps questionable to replace this constant background with a field of a finite duration and to make conclusions about effects at finite times. Much more satisfactory would be to derive an effective theory for a slowly-varying background, which would have some notion of what “finite time” meant. Extensions of the Heisenberg-Euler Lagrangian that contain field derivatives already exist [24, 25, 26], although a naive application of these extended Lagrangians appear to be at a higher derivative order.

It is interesting to continue with a naive interpretation of the overlap field. First, it would imply a difference in the number of photons scattered in the two cases: i) an ever-present constant background; ii) an adiabatically evolved quasi-constant background. Second, the generation of the second probe harmonic as it counterpropagates in a more slowly-varying background is possible in four-photon scattering although it vanishes identically in the standard asymptotic calculation [27]. Third, second-harmonic generation occurs at a lower order of the weak-field expansion in the overlap signal (in four-photon scattering) than in the asymptotic signal (in six-photon scattering). If the duration over which the strong background varies is of the order $\tau_s$ and the probe of frequency $\omega_p$, then when the fields overlap, the second harmonic is dominated by the overlap signal if $E_s^2 \omega_p \tau_s \ll 1$.

3. Numerical Method

The numerical treatment is based on the solution of the corresponding system of Maxwell equations following from (6) using the “pseudocharacteristic method of lines” (PCMOL) [28] matrix inversion. This allows one to convert the equations of motion into a system of ordinary
differential equations (ODEs) and use the ODE-Solver CVODE [29]. In the linear case, Maxwell’s equations in the plane wave setup considered here, can be written in matrix form:

\[ \mathbf{I}_4 \partial_t \mathbf{f} + \mathbf{Q} \partial_z \mathbf{f} = 0 \]  

(21)

where \( \mathbf{f} = (E_x, E_y, B_x, B_y)^T \), \( \mathbf{I}_4 \) is the identity matrix in four dimensions and \( \mathbf{Q} = \text{diag}(1, -1, -1, 1) \) is an anti-diagonal matrix. Since the system is hyperbolic (i.e. it can be diagonalised with real eigenvalues), we can apply the PCMOL to discretise the system in space. As a first step, the system is transformed to the new diagonal basis \( \mathbf{u} := \mathbf{S} \mathbf{f} \), in which one obtains four decoupled advection equations:

\[ \partial_t \mathbf{u} + \mathbf{A} \partial_z \mathbf{u} = 0 \]  

(22)

with \( \mathbf{A} = \mathbf{S} \mathbf{Q} \mathbf{S}^{-1} = \text{diag}(-1, -1, 1, 1) \) being a diagonal matrix and

\[ \mathbf{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \]

\[ \mathbf{u} := \mathbf{S} \mathbf{f} = \frac{1}{\sqrt{2}} \begin{pmatrix} B_x - E_x \\ E_y + B_y \\ E_x + B_y \\ B_x - E_y \end{pmatrix}. \]  

(23)

The eigenvalues \( \lambda_i \) of \( \mathbf{A} \) are called the “characteristic speeds” of the system. A positive (negative) sign of \( \lambda_i \) thereby corresponds to the component \( \mathbf{u}_i \) travelling in the positive (negative) \( z \)-direction. Then the components \( \mathbf{u}_i \) are mapped onto a 4\( N \)-dimensional vector representing \( \mathbf{u} \) on a co-located grid with \( N \) points such that the grid point with index \( 0 < n < N \) is assigned the field values \( (\mathbf{u}_1^{(n)}, \mathbf{u}_2^{(n)}, \mathbf{u}_3^{(n)}, \mathbf{u}_4^{(n)}) \). Now the spatial derivatives are approximated using biased differences of fourth order accuracy. For \( \lambda_i > 0 \), \( \partial_z \mathbf{u}_i \) is calculated using backward differences and for \( \lambda_i < 0 \) using forward differences. At the boundary, the derivatives are calculated using values inside the box. The initial conditions are chosen such that the field invariants \( \mathcal{F} \) and \( \mathcal{G} \) and the field values at the boundary are essentially zero at the beginning. The flow-character of the advection equations implements open boundary conditions. The resulting set of ordinary differential equations is then integrated using the ODE-Solver CVODE. As a final step, the fields are transformed back to the original basis \( \mathbf{f} \) for output.

For the nonlinear case, the resulting Maxwell equations can be written as

\[ (\mathbf{I}_4 + \mathbf{X}) \partial_t \mathbf{f} + (\mathbf{Q} + \mathbf{Y}) \partial_z \mathbf{f} = 0 \]  

(24)

where \( \mathbf{X} \) and \( \mathbf{Y} \) are the nonlinear corrections resulting from (6). Since the nonlinear corrections \( \mathbf{X}, \mathbf{Y} \) depend only on local field values at each point, their discretised version is of block diagonal form: \( \mathbf{X} = \oplus_{n=1}^N \mathbf{X}^{(n)} \). The upper index indicates the matrix \( \mathbf{X} \) to be evaluated at point \( n \) and one has a similar expression for \( \mathbf{Y} \). The main difference to the linear case is the appearance of the matrix \( \mathbf{X} \) in front of the temporal derivatives. To convert the system to an ODE-form, we invert the matrices \( \mathbf{I}_4 + \mathbf{X}^{(n)} \) at each grid point separately. The biased differencing is the same as in the linear case. The resulting system is then again integrated using CVODE with the same parameters as in the linear case. The signals are analysed using a discrete Fourier Transform in Wolfram Mathematica [30] under the assumption \( \omega = |\mathbf{k}| \). For further details, the reader is referred to [22].

4. Single four and six photon scattering

The probe and background fields considered were of the form:

\[ \mathbf{E}_p(x^-) = \varepsilon_p \mathbf{E}_p e^{-\left(\frac{x^-}{\tau_p}\right)^2} \cos \omega_p x^-; \quad \mathbf{E}_s(x^+) = \varepsilon_s \mathbf{E}_s e^{-\left(\frac{x^+}{\tau_s}\right)^2}, \]  

(25)

where \( \varepsilon_p \cdot \varepsilon_p = \varepsilon_s \cdot \varepsilon_s = 1 \) and we consider \( \omega_p \tau_p \gg 1 \) and the momentum three-vectors are \( \mathbf{k}_p = \omega_p (0, 0, 1), \mathbf{k}_s = -\omega_s (0, 0, 1) \). Both four- and six-photon scattering were included in the vacuum interaction. In what follows, we concentrate on the scattered probe field, \( \Delta \mathbf{E}_p \).
4.1. Fundamental Harmonic
The signal of the fundamental harmonic is dominated by the usual, asymptotic signal, but acquires a small term due to the overlap field:

\[ \Delta E_p = \mu_1 \varepsilon_{1,s} E_p \varepsilon_s^2 e^{-\left(\frac{x^-}{\tau_p}\right)^2} \left[ \sqrt{\frac{\pi}{2}} \frac{1 + \text{erf} \left(\frac{\sqrt{2} x^+}{\tau_s}\right)}{\tau_s} \sin \omega_p x^- - e^{-2 \left(\frac{x^-}{\tau_p}\right)^2} \cos \omega_p x^- \right], \] (26)

where \( \varepsilon_{1,s} = c_{1,1} \varepsilon_s + c_{1,2} \hat{k}_s \wedge \varepsilon_s \) with coefficients \( c_{1,1} = 4 \varepsilon_s \cdot \varepsilon_p (1 - \hat{k}_s \cdot \hat{k}_p) \) and \( c_{1,2} = 7 (\varepsilon_s \cdot \hat{k}_p \wedge \varepsilon_p + \varepsilon_p \cdot \hat{k}_s \wedge \varepsilon_s) \). When \( \Delta E_p \ll E_p \), one can express the scattered field as an effective refractive index that leads to \( \Delta n \). Suppose we take the limit \( \tau_p \to \infty \) and consider a monochromatic probe and study the case of parallel probe and background polarisations. Then for a phase change \( \delta \varphi_p \),

\[ E_p (\omega_p x^- + \delta \varphi_p) = E_p (\omega_p x^-) + \delta \varphi_p E_p (\omega_p x^-) + O(\delta \varphi_p^2), \] (27)

then we see if \( \delta \varphi_p = -\omega_p \delta z \), then since:

\[ \delta z = \frac{1}{2} \int_{-\infty}^{x^+} \delta n\parallel (x^+) dx^+, \] (28)

we can infer:

\[ \delta n\parallel = 16 \mu_1 \left[ E_s^2 + \frac{E_p (\omega_p x^-)}{E_p (\omega_p x^-)} \frac{1}{\omega_p} \partial_z E_s^2 \right] = \delta n\parallel_{\infty} + \delta n\parallel_0, \] (29)

where we label the first term the asymptotic modification to the vacuum refractive index \( \delta n\parallel_{\infty} \) compared to the overlap modification of \( \delta n\parallel_0 \) that includes derivative terms. This agrees with Eq. (29) in [23] for a monochromatic probe (\( \tau_p \to \infty \)) when one takes \( \omega_0 = \omega_s \), \( \omega = \omega_p \), \( \xi(\varphi) = E_p (\varphi_p) \), \( \omega_s \Gamma_2 = [\partial_+ E_s (x^+)]/E_s (x^+). \) We note the similarity with the ponderomotive force on a charge, which is proportional to the gradient of the field squared [31]. Whether the change of vacuum refractive index due to the overlap field is a true refractive index effect would have to be investigated by including higher orders of four-photon scattering.

4.2. Second Harmonic
We find two sources of the second harmonic in the overlap field, originating from single four and six photon-photon scattering. The first of these can dominate the asymptotic signal from single six photon scattering, as demonstrated in Fig. 1:

\[ \Delta E_p = \mu_2 \varepsilon_{2,s} E_p^2 E_s^3 e^{-2 \left(\frac{x^-}{\tau_p}\right)^2} \left[ \sqrt{\frac{\pi}{3}} \frac{1 + \text{erf} \left(\frac{\sqrt{3} x^+}{\tau_s}\right)}{\tau_s} \sin 2 \omega_p x^- - \frac{1}{2} e^{-3 \left(\frac{x^-}{\tau_p}\right)^2} \cos 2 \omega_p x^- \right] \]

\[ -\frac{1}{2} \mu_1 \varepsilon_{1,p} E_p^2 E_s e^{-\left(\frac{x^-}{\tau_p}\right)^2} \cos 2 \omega_p x^- \] (30)

where the polarisation vectors are now \( \varepsilon_{1,p} = c_{1,1} \varepsilon_p + c_{1,2} \hat{k}_p \wedge \varepsilon_p \) and \( \varepsilon_{2,s} = c_{2,1} \varepsilon_s + c_{2,2} \hat{k}_s \wedge \varepsilon_s \). The overlap field can dominate the second-harmonic signal when \( E_s^2 \omega_p \tau_s \ll 1 \). Such a situation is shown in Fig. 1 for parameters \( E_s = 0.02 \), \( E_p = 0.005 \), \( \omega_p = 0.6 \text{ eV} \), \( \tau_s = 6.4 \lambda_p \) and \( \tau_p = 5 \lambda_p \), where the field is in units of the probe amplitude, \( E_p \). The agreement between theory and numerical simulation is displayed in Fig. 2.

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Figure 1. The thinner panels plot snapshots of the total electric field at various stages of
the corresponding scattered overlap (greenish-blue solid) and asymptotic (black dashed) second
harmonic field at times $t_4 > t_3 > t_2 > t_1$. Electric fields are normalised by the probe field
amplitude, $E_p$.

Figure 2. Agreement between numerical simulation (points) and analytical calculation (solid
line) for generation of the second harmonic with $E_s = 0.02$, $E_p = 0.005$, $\omega_p = 0.6$ eV, $\tau_s = 6.4\lambda_p$
and $\tau_p = 5\lambda_p$ (the field is in units of $E_p$). The leading-order overlap signal from four-
photon scattering (thin dashes) can dominate the leading-order asymptotic field from six-photon
scattering (thick dashes).

5. Conclusion
In this conference proceeding, we have expounded upon the results in [21]. The process of four
and six photon-photon scattering in counterpropagating plane waves has been studied using
numerical simulation, which was shown to agree with analytical predictions. The main result
was the prediction of an increased photon-photon scattering signal that occurs at finite time when
probe and background pulses overlap. This signal occurs when one naively solves the classical
wave equation in the presence of an effective vacuum current, derived from the Heisenberg-Euler
Lagrangian. The standard vacuum refractive index [32] is derived for a constant electromagnetic
background. If it is to be used to describe photon-photon scattering at finite time in a slowly-
varying background, then it would seem to be inconsistent to neglect the interaction studied
here, namely the part of the vacuum current proportional to the derivative of a power of the background. For the fundamental harmonic, the change in the field is proportional to the gradient of the background squared, and we note the similarity to the ponderomotive force on a charge in the field. If the duration over which the strong background varies is of the order $\tau_s$ and the probe of frequency $\omega_p$, then the second harmonic is dominated by the overlap signal if $E_2^2 \omega_p \tau_s \ll 1$.

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