ON TIMELIKE SLANT HELICES IN $S_1^2$

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Abstract. In this paper, we investigate timelike slant helix in $S_1^2$ and we obtain parametric equation of timelike slant helix in $S_1^2$. Also related examples and their illustrations are given.

Key Words: Minkowski 3-space, timelike slant helix, spherical curve.

1. Introduction

Izumiya and Takeuchi, in [5], have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $E_3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that $\gamma$ is a slant helix in if and only if the geodesic curvature of the principal normal of a space curve $\gamma$ is a constant function.

In [7], L. Kula and Y. Yayli studied the spherical images under both tangent and binormal indicatrices of slant helices and obtained that the spherical images of a slant helix are spherical helix. In [8], the author characterize slant helices by certain differential equations verified for each one of obtained spherical indicatrix in Euclidean 3-space. Recently, Ali and Lopez, in [1], have studied slant helix in Minkowski 3-space. They showed that the spherical indicatrix of a slant helix in $E_1^3$ are helices. Also in [2], Ali and Turgut, studied the position vector of a timelike slant helix in $E_1^3$.

In [3] we consider the spherical slant helices in $\mathbb{R}^3$. We also present the parametric slant helices, their curvatures and torsions. Moreover, we show how could be obtained to a spherical slant helix and we give some slant helix examples in Euclidean 3-space.

In this paper, we investigate spherical timelike slant helix in Minkowski 3-space $E_1^3$ and we obtain parametric equation of spherical timelike slant helix.

2. Preliminaries

The Minkowski 3–space $E_1^3$ is the Euclidean 3-space $E_3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E_1^3$. Recall that a vector $v \in E_1^3$ is called spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector $v$ is given by

\[2000 \text{ Mathematics Subject Classification:} \quad 53C40, 53C50.\]
\[ ||v|| = \sqrt{g(v,v)} \] and two vectors \( v \) and \( w \) are said to be orthogonal if \( g(v,w) = 0 \).

An arbitrary curve \( \alpha(s) \) in \( E^3_1 \) can locally be spacelike, timelike or null (lightlike), if all its velocity vectors \( \alpha'(s) \) are spacelike, timelike or null, respectively. Spacelike or a timelike curve \( \alpha \) has unit speed, if \( g(\alpha'(s), \alpha'(s)) = \pm 1 \). A null curve \( \alpha \) is parameterized by pseudo-arc \( s \), if \( g(\alpha''(s), \alpha''(s)) = 1 \) ([9]). For a timelike space curve \( \alpha(s) \) in the space \( E^3_1 \), the following Frenet formulae are given in [1]

\[
\begin{align*}
T'(s) &= \nu(s) \kappa(s) N(s) \\
N'(s) &= -\nu(s) \kappa(s) T(s) + \nu(s) \tau(s) B(s) \\
B'(s) &= -\nu(s) \tau(s) N(s),
\end{align*}
\]

where \( g(\alpha'(s), \alpha'(s)) = \nu^2(s) \) and the Minkowski vector products of Frenet vectors are given as

\[
\begin{align*}
T(s) \times N(s) &= B(s) \\
N(s) \times B(s) &= -T(s) \\
B(s) \times T(s) &= N(s),
\end{align*}
\]

where \( g(T(s), T(s)) = \varepsilon_0 = \pm 1 \), \( g(N(s), N(s)) = \varepsilon_1 = \pm 1 \) and \( g(B(s), B(s)) = \varepsilon_2 = \pm 1 \) and two \( \varepsilon_i \)'s are equal to 1 the other \( \varepsilon_i \) is -1.

In this case, \( \kappa \) can take only two values: \( \kappa = 0 \) when \( \alpha \) is a straight null line or \( \kappa = 1 \) in all other cases.

It is well known that, the pseudo-Riemannian sphere with radius \( r = 1 \) and centered at origin is defined by

\[
S^2_1 = \{ p \in E^3_1 : g(p,p) = 1 \},
\]

the pseudohyperbolic space of radius \( r = 1 \) and centered at origin is defined by

\[
H^2_0 = \{ p \in E^3_1 : g(p,p) = -1 \}
\]

are the hyperquadrics with dimension 2 and index 1 and with dimension 2 and index 0, respectively, ([9]).

3. Spherical Timelike Slant Helix in Minkowski 3-space

In \( E^3_1 \), the definition of spherical space curve is similar with the Euclidean case but richer than Euclidean case. For example for a timelike curve, if its position vector is a spacelike then the curve lies on the pseudo-Riemannian sphere \( S^2_1 \), if its position vector is a timelike then the curve lies on pseudohyperbolic space \( H^2_0 \). In Minkowski space, for the characterizations of spherical curves, we refer the papers of Petrović-Torgašev and Šućurović, ([10, 11]) and Inoguchi and Lee ([6]).

In this section, we give relation between the curvature of timelike slant helix, the axis of timelike slant helix and timelike slant helix in Minkowski 3-space. Moreover, we investigate parametric equation of timelike slant helix. Here, by \( \varepsilon \) we consider

\[
\varepsilon = \begin{cases} 
1, & \tau^2 - \kappa^2 > 0 \\
-1, & \tau^2 - \kappa^2 < 0
\end{cases}
\]

We will give to "Lemma 3.1" as unproved, since we will use in next sections.
Lemma 3.1. Let $\alpha$ be a timelike curve in $E^3_1$. Geodesic curvature of the spherical image of spacelike principal normal indicatrix $(N)$ of $\alpha$ is

$$\sigma_1 = \frac{\kappa^2}{\nu (\tau^2 - \kappa^2)^{3/2}} \left( \frac{T}{\kappa} \right)'$$

and geodesic curvature of the spherical image of timelike principal normal indicatrix $(N)$ of $\alpha$ is

$$\sigma_2 = \frac{\kappa^2}{\nu (\tau^2 - \kappa^2)^{3/2}} \left( \frac{T}{\kappa} \right)'$$

where $\tau^2 - \kappa^2$ does not vanish.[1]

Theorem 3.1. Let $\alpha$ be a timelike curve in Minkowski space $E^3_1$ with Frenet vectors $T, N, B$ and curvatures $\kappa, \tau$. The following statements are equivalent:

1. $\alpha$ is slant helix
2. $\kappa = \frac{1}{a} \nu \theta' \sin \theta$
3. $\tau = \frac{1}{a} \nu \theta' \cos \theta$

(3)

$$\vec{U} = \frac{\tau}{a \sqrt{a^2 - 1}} T + \frac{\kappa}{a \sqrt{a^2 - 1}} B = \text{constant}$$

Proof. $(1 \Rightarrow 3, 4)$: Let $\alpha : I \rightarrow E^3_1$ be a timelike slant helix and the axis of $\alpha$ be $\vec{a}$. Since $\alpha$ is a timelike slant helix, If

(3.1) $\vec{a} = \cos \theta_1 T + \cos \theta_2 N + \cos \theta_3 B$,

then

(3.2) $\langle \vec{a}, N \rangle = \cos \theta_2 = \text{constant}$.

Differentiating the eq. (3.1), we get

(3.3) $\langle \vec{a}, T \rangle = -\frac{\tau}{\kappa} \langle \vec{a}, B \rangle$

Differentiating the eq. (3.3), we obtain

(3.4) $\frac{\nu(\tau^2 - \kappa^2)}{\kappa} \langle \vec{a}, N \rangle = (\frac{\tau}{\kappa})' \langle \vec{a}, B \rangle$.

Since $g(\vec{a}, \vec{a}) = 1$,

(3.5) $\vec{a} = \frac{a}{\sqrt{a^2 - 1}} \left( \frac{\tau}{a \sqrt{a^2 - 1}} T + \frac{\kappa}{a \sqrt{a^2 - 1}} B \right)$.

If we choose $\vec{a} = \frac{\sigma}{\sqrt{\sigma^2 - 1}} \vec{U}$, then

(3.6) $\vec{U} = \frac{\tau}{a \sqrt{a^2 - 1}} T + \frac{\kappa}{a \sqrt{a^2 - 1}} B$. 
\( \vec{a} \) is constant vector and \( \sigma \) is constant, \( \vec{U} \) is constant vector, i.e.

\[
\vec{U}' = \left( \frac{\tau}{a \sqrt{\varepsilon (\tau^2 - \kappa^2)}} \right)' + \nu \kappa \right] T + \left[ \left( \frac{\kappa}{a \sqrt{\varepsilon (\tau^2 - \kappa^2)}} \right)' + \nu \tau \right] B = 0,
\]

where

\[
\left( \frac{\tau}{\sqrt{\varepsilon (\tau^2 - \kappa^2)}} \right)' = -a \nu \kappa
\]

\( \left( \frac{\kappa}{\sqrt{\varepsilon (\tau^2 - \kappa^2)}} \right)' = -a \nu \tau. \)

(3.7)

Consequently, by simple calculation,

\[
\kappa = \frac{1}{a \nu} \theta' \sinh \theta
\]

\[
\tau = \frac{1}{a \nu} \theta' \cosh \theta.
\]

(2 \( \Rightarrow \) 1): The proof is obvious.

(3 \( \Rightarrow \) 1): Let \( \vec{U} \) be a constant vector. We can easily show that

\[
\cos \theta_2 = \frac{\sigma}{\sqrt{\sigma^2 - 1}} = \text{constant}
\]

which means that \( \alpha \) is a slant helix. These complete the proof.

\[\square\]

**Theorem 3.2.** Let \( \alpha \) be a curve in \( E^3_1 \) with equation

\[
x(s) = \frac{1}{\sqrt{-1 + a^2}} (-\sqrt{-1 + a^2} \cosh \theta \sinh \left[ \frac{-1 + a^2}{a} \theta \right] \sin \left[ \frac{\sinh \theta}{a} \right] + \cosh \left[ \frac{-1 + a^2}{a} \theta \right] \left( \cos \left[ \frac{\sinh \theta}{a} \right] + a \sin \left[ \frac{\sinh \theta}{a} \right] \sinh \theta \right)),
\]

\[
y(s) = \frac{1}{\sqrt{-1 + a^2}} (-\sqrt{-1 + a^2} \cosh \theta \cosh \left[ \frac{-1 + a^2}{a} \theta \right] \sin \left[ \frac{\sinh \theta}{a} \right] + \sinh \left[ \frac{-1 + a^2}{a} \theta \right] \left( \cos \left[ \frac{\sinh \theta}{a} \right] + a \sin \left[ \frac{\sinh \theta}{a} \right] \sinh \theta \right)),
\]

\[
z(s) = \frac{1}{\sqrt{-1 + a^2}} (-a \cos \left[ \frac{\sinh \theta}{a} \right] + \sin \left[ \frac{\sinh \theta}{a} \right] \sinh \theta).
\]

where \( a \) is constant and \( \theta = \theta(s) \). \( \alpha \) is timelike slant helix in \( S^2_1 \).

**Proof.** Let \( \alpha \) be a curve in \( E^3_1 \) with Frenet frame \( \{ T, N, B \} \), curvature \( \kappa \) and torsion \( \tau \).

In this case, we will show that \( \sigma(s) = \text{constant} \) and \( \alpha \in S^2_1 \)
By simple calculation, spherical indicatrices $T(s)$, $N(s)$, $B(s)$ of the curve $\alpha$, respectively, are

\[
T(s) = (\frac{a \cosh \theta \cos \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right)}{\sqrt{-1 + a^2}} - \sinh \theta \sinh \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right),
\]

\[
N(s) = (\frac{a \cosh \theta \sinh \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right)}{\sqrt{-1 + a^2}} - \sinh \theta \cosh \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right), -\cosh \theta),
\]

\[
B(s) = (-\frac{a \sinh \theta \cosh \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right)}{\sqrt{-1 + a^2}} + \cosh \theta \sinh \left(\sqrt{-1 + \frac{a^2}{a^2}} \theta\right), \frac{\sinh \theta}{\sqrt{-1 + a^2}}).
\]

We can easily see that $g(T, T) = -1$, $g(N, N) = 1$ and $g(B, B) = 1$, that is, $T$ is a timelike vector, $N$ is a spacelike vector and $B$ is a spacelike vector. Moreover, by the formulae of the curvature and the torsion for a general parameter, we can calculate that

\[
\kappa(s) = \sec \left[\frac{\sinh \theta}{a}\right],
\]

\[
\tau(s) = \coth \theta \sec \left[\frac{\sinh \theta}{a}\right].
\]

Therefore,

\[
\sigma(s) = a = \text{constant}
\]

which means that $\alpha$ is a timelike slant helix.

Finally,

\[
g(\alpha(s), \alpha(s)) = 1.
\]

Then, $\alpha$ is slant helix in $S^2_1$. Thus the proof of theorem is completed. \qed
Theorem 3.3. Let $\alpha$ be a curve in $E^3_1$ with equation
\begin{align*}
  x(s) &= \frac{1}{\sqrt{-1+a^2}}(-\sqrt{-1+a^2}\cosh \theta \sinh \left[\frac{\sqrt{1+a^2}}{a} \theta\right] \sin \left[\frac{\sinh \theta}{a}\right] \\
  &\quad + \cosh \left[\frac{\sqrt{-1+a^2}}{a} \theta\right] \left( \cos \left[\frac{\sinh \theta}{a}\right] + a \sin \left[\frac{\sinh \theta}{a}\right] \sinh \theta \right) ), \\
  y(s) &= \frac{1}{\sqrt{-1+a^2}}(-a \cos \left[\frac{\sinh \theta}{a}\right] + \sin \left[\frac{\sinh \theta}{a}\right] \sinh \theta ) \\
  z(s) &= \frac{1}{\sqrt{-1+a^2}}\left(-\sqrt{-1+a^2}\cosh \theta \cosh \left[\frac{\sqrt{1+a^2}}{a} \theta\right] \sin \left[\frac{\sinh \theta}{a}\right] \\
  &\quad + \sinh \left[\frac{\sqrt{-1+a^2}}{a} \theta\right] \left( \cos \left[\frac{\sinh \theta}{a}\right] + a \sin \left[\frac{\sinh \theta}{a}\right] \sinh \theta \right) ,
\end{align*}

where $a$ is constant and $\theta = \theta(s)$. $\alpha$ is timelike slant helix in $S^2_1$.

**Proof.** By using the method in the above theorem the proof of the theorem is obvious. \qed

4. Example

In this section we give an example of Spherical timelike slant helix $\alpha$ in Minkowski 3-space and draw its pictures and its tangent indicatrix, normal indicatrix, and binormal indicatrix by using Mathematica.

**Example 4.1.** We consider a timelike slant helix $\alpha$ in $S^2_1$ is defined by
\begin{align*}
  x(s) &= -\cosh \theta \sin \left[\frac{\sinh \theta}{3}\right] \sinh \left[\frac{2\sqrt{2}}{3} \theta\right] \\
  &\quad + \frac{1}{2\sqrt{2}} \cosh \left[\frac{2\sqrt{2}}{3} \theta\right] \left( \cosh \left[\frac{\sinh \theta}{3}\right] + 3 \sin \left[\frac{\sinh \theta}{3}\right] \sinh \theta \right) \\
  y(s) &= -\cosh \theta \sin \left[\frac{\sinh \theta}{3}\right] \cosh \left[\frac{2\sqrt{2}}{3} \theta\right] \\
  &\quad + \frac{1}{2\sqrt{2}} \sinh \left[\frac{2\sqrt{2}}{3} \theta\right] \left( \cosh \left[\frac{\sinh \theta}{3}\right] + 3 \sin \left[\frac{\sinh \theta}{3}\right] \sinh \theta \right) \\
  z(s) &= \frac{1}{2\sqrt{2}} \left(-3 \cos \left[\frac{\sinh \theta}{3}\right] + \sin \left[\frac{\sinh \theta}{3}\right] \sin \theta \right) .
\end{align*}

The picture of the curve $\alpha$ is rendered in Figure 1.

The parametrization of the tangent indicatrix $T = (T_1, T_2, T_3)$ of the slant helix $\alpha$ is
Figure 1. For $a = 3$, timelike slant helix $\alpha$ in $S^2_1$.

\[
T_1(s) = \frac{3}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta - \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta,
\]

\[
T_2(s) = \frac{3}{2\sqrt{2}} \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta - \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta,
\]

\[
T_3(s) = -\frac{1}{2\sqrt{2}} \cosh \theta.
\]

The picture of the tangent indicatrix is rendered in Figure 2 (a).

The parametrization of the normal indicatrix $N = (N_1, N_2, N_3)$ of the slant helix $\alpha$ is

\[
N_1(s) = -\frac{1}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right],
\]

\[
N_2(s) = -\frac{1}{2\sqrt{2}} \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right],
\]

\[
N_3(s) = \frac{3}{2\sqrt{2}}.
\]

The picture of the normal indicatrix is rendered in Figure 2 (b).

The parametrization of the binormal indicatrix $B = (B_1, B_2, B_3)$ of the slant helix $\alpha$ is
\[ B_1(s) = -\frac{3}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta + \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta, \]

\[ B_2(s) = -\frac{3}{2\sqrt{2}} \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta + \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta, \]

\[ B_3(s) = \frac{1}{2\sqrt{2}} \sinh \theta. \]

The picture of the binormal indicatrix is rendered in Figure 2 (c).

\[ \text{Figure 2. For } \alpha = 1, \text{ tangent indicatrix of the slant helix } \alpha \text{ in } H_2^3 \]
\[ \text{(a), normal indicatrix of the pseudo-spherical slant helix } \alpha \text{ in } S_3^1 \]
\[ \text{(b) and binormal indicatrix of the pseudo-spherical slant helix } \alpha \text{ in } S_1^2 \text{ (c).} \]

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