A simplified criterion for MDP convolutional codes

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Abstract

Maximum distance profile (MDP) convolutional codes have the property that their column distances are as large as possible. There exists a well-known criterion to check whether a code is MDP using the parity-check matrix of the code. However, this criterion has only been shown under the assumption that the parity-check matrix is left prime. Proving left primeness of a polynomial matrix is not an easy task and also, in almost all the previous papers which provide a construction of MDP convolutional codes, there is a gap in the application of the mentioned criterion to show the MDP property, since it is not proven that the parity-check matrix is left prime. In this paper we close this gap. In particular, we show that under the assumption that \((n - k) \mid \delta\) or \(k \mid \delta\), a polynomial matrix that fulfills the MDP criterion is actually always left prime. Moreover, when \((n - k) \nmid \delta\) and \(k \nmid \delta\), we show that the MDP criterion is not enough to ensure left primeness. In this case, with one more assumption, we can guarantee the result.

1 Introduction

In the algebraic theory of error correcting codes, one important family of codes for telecommunication is given by convolutional codes. Mathematically, a convolutional code is a submodule of \(F[z]^n\), of rank \(k\), where \(F\) is a finite field. Unfortunately, constructions of these codes, having some good designed minimum distance, are quite rare. Convolutional codes have the flexibility of grouping blocks of information in an appropriate way, according to the erasures location, and then decoding first the part of the sequence with less erasures or the part of the sequence where the distribution of erasures allows a complete correction.

For convolutional codes, there is a notion of distance that is more important than the definition of minimum distance used in classical block codes. This notion results in the \(j\)-th column distances of a convolutional code \(C\), which satisfy an upper bound given in [7]. If all the column distances meet this bound with equality, then \(C\) is called maximum distance profile (MDP), see [2]. It was shown in [8] that MDP convolutional codes have optimal recovery rate for windows of a certain length, depending on the code parameters.

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MDP convolutional codes can be regarded as the convolutional counterpart of MDS block codes. However, in contrast to the case of MDS block codes there are very few algebraic constructions of MDP codes, all based on a characterization provided in [2], which works with the assumption that a parity-check matrix $H(z)$ (or equivalently a generator matrix $G(z)$) of a convolutional code is left prime. In particular, the criterion says that a left prime matrix $H(z) \in \mathbb{F}[z]^{(n-k)\times n}$ is a parity-check (equivalently a left prime $G(z) \in \mathbb{F}[z]^{k\times n}$ is a generator) matrix of an MDP convolutional code if all the full-size minors of the truncated sliding parity-check matrix (equivalently the sliding generator matrix), taken in a particular way, are non zero. Left prime matrices have been widely investigated in module theory, in system theory and also in the theory of convolutional codes. In the literature one can find several properties equivalent to left primeness (see for instance [3]). However, proving even one of them for a polynomial matrix is in general not easy.

Several papers that provide a (concrete) construction for MDP convolutional codes, for example [8], [1], [4], are based on the characterization mentioned above. Unfortunately, in all of them there is no discussion on the left primeness of the constructed matrices. They show only the criterion on the minors of the sliding parity-check (or generator) matrix, without being able to apply it, because they miss to show left primeness of the parity-check (or generator) matrix.

In this paper, we show that if $(n-k)$ divides $\delta$ or $k$ divides $\delta$ and the criterion on the minors of the truncated sliding parity-check $H_L^c$ (resp. generator $G_L^c$) matrix of a convolutional code is satisfied then the parity-check $H(z)$ (resp. generator $G(z)$) matrix has to be left prime. This simplifies the criterion and makes all the previous constructions work. Moreover, for this result we assume that the degree of the parity-check (resp. generator) matrix is $\nu = \delta/(n-k)$ (resp. $\mu = \delta/k$). Observe that if $(n-k)$ divides $\delta$, we consider the parity-check matrix and if $k$ divides $\delta$ we consider the generator matrix. If $(n-k)$ and $k$ do not divide $\delta$, then we require some technical assumption in addition to the criterion on the minors of the sliding matrices to obtain the left primeness property.

The paper is structured as follows. In Section 2, we give some background on convolutional codes, focusing on the family of MDP convolutional codes. Moreover, we point out the importance of the left primeness property for generator or parity-check matrices for having noncatastrophic convolutional codes. In Section 3, we prove the main result of the paper, namely we show that the MDP property on the truncated sliding parity-check matrix $H_L$ of a convolutional code, implies that $H(z)$ is left prime. We conclude with some more remarks in Section 4.

2 Preliminaries

In this section we give the basic notions and results on the theory of convolutional codes. For a more detailed treatment we refer to [4]. In particular, we will focus on the property of left primeness of a parity-check (or generator) matrix of a noncatastrophic convolutional code.
Let $\mathbb{F}$ be a finite field, $\mathbb{F}[z]$ be the polynomial ring over $\mathbb{F}$ and let $k, n$ be positive integers, such that $k < n$. A rate $k/n$ convolutional code $C$ is a submodule of $\mathbb{F}[z]^n$ of rank $k$.

Since $\mathbb{F}[z]$ is a principal ideal domain, every submodule of $\mathbb{F}[z]^n$ is free. Hence, there exists a matrix $G(z) \in \mathbb{F}[z]^{k \times n}$, whose rows are a basis for $C$. We call such a matrix generator matrix of the convolutional code $C$. Note that the generator matrix of a convolutional code is not unique. Assume that $G(z) \in \mathbb{F}[z]^{k \times n}$ and $\bar{G}(z) \in \mathbb{F}[z]^{k \times n}$ are two generator matrices for the same convolutional code $C$, then, there exists a unimodular matrix $U(z) \in \mathbb{F}[z]^{k \times k}$ such that $\bar{G}(z) = U(z)G(z)$. We say that $G(z)$ and $\bar{G}(z)$ are equivalent.

Given a rate $k/n$ convolutional code $C \subseteq \mathbb{F}[z]^n$, with generator matrix $G(z) \in \mathbb{F}[z]^{k \times n}$, we define the largest degree among the entries in the $i$-th row of $G(z)$ as the $i$-th row degree and we denote it by $\delta_i$. We say that $G(z)$ is row-reduced or minimal if its row degrees attain the minimum possible value. We define the degree $\delta$ of $C$ to be the highest degree of the $k \times k$ minors of $G(z)$. We denote a rate $k/n$ convolutional code defined over $\mathbb{F}$ of degree $\delta$ as $(n, k, \delta)$. Note that if $G(z)$ is row-reduced, then $\delta = \delta_1 + \cdots + \delta_k$.

Moreover, given a generator matrix $G(z)$ of a convolutional code $C$, there always exists a row-reduced generator matrix equivalent to $G(z)$.

There is another important property of matrices that is useful in the context of convolutional codes.

**Definition 2.1.** Let $G(z) \in \mathbb{F}[z]^{k \times n}$ be a matrix. Then $G(z)$ is said to be left prime if in all the factorizations $G(z) = M(z)\bar{G}(z)$, with $M(z) \in \mathbb{F}[z]^{k \times k}$ and $\bar{G}(z) \in \mathbb{F}[z]^{k \times n}$, the left factor $M(z)$ is unimodular, i.e. $M(z) \in \text{GL}_k(\mathbb{F}[z])$.

There are several characterizations for left prime matrices. In particular, $G(z) \in \mathbb{F}[z]^{k \times n}$ is left prime if and only if it admits a right $n \times k$ polynomial inverse (see [3] for details).

Since all generator matrices of a convolutional code $C$ are equivalent up to multiplication by unimodular matrices, if $C$ admits a left prime generator matrix, then all its generator matrices are left prime. In this case, we say that $C$ is a noncatastrophic convolutional code.

Let $C$ be a noncatastrophic $(n, k, \delta)$ convolutional code and $G(z) \in \mathbb{F}[z]^{k \times n}$ be a generator matrix of $C$. Then there exists a matrix $H(z) \in \mathbb{F}[z]^{(n-k) \times n}$, such that

$$c(z) \in C \iff H(z)c(z)^\top = 0. \quad (1)$$

Such a matrix $H(z)$ is called a parity-check matrix of $C$. In [9], it has been shown that a convolutional code $C$ is noncatastrophic if and only if it admits a parity-check matrix. If $H(z) \in \mathbb{F}[z]^{(n-k) \times n}$ is a left prime and row reduced parity check matrix of an $(n, k, \delta)$ convolutional code $C$, then the sum of the row degrees of $H(z)$ is equal to $\delta$ (see [3]). Throughout this paper, we assume that $H(z)$ has so-called generic row degrees, i.e. its row degrees are equal to $\left\lfloor \frac{\delta}{n-k} \right\rfloor + 1$ with multiplicity $t := \delta - (n-k) \left\lfloor \frac{\delta}{n-k} \right\rfloor$ and $\left\lfloor \frac{\delta}{n-k} \right\rfloor$ with multiplicity $n-k-t$. This ensures that for given parameters $n, k, \delta$, we consider...
parity-check matrices with the minimal possible degree (as polynomial) \( \deg(H) = \nu \),
where \( \nu = \frac{\delta}{n-k} \) if \( (n-k) \mid \delta \) and \( \nu = \left\lceil \frac{\delta}{n-k} \right\rceil + 1 \) if \( (n-k) \nmid \delta \).

Given a codeword \( v(z) = \sum_{i=0}^{r} v_i z^i \in C \), we define the weight of \( v(z) \) as
\[
\text{wt}(v(z)) := \sum_{i=0}^{r} \text{wt}(v_i) \in \mathbb{N}_0,
\]
where \( \text{wt}(v_i) \) denotes the Hamming weight of \( v_i \in \mathbb{F}_n \), i.e. the number of its nonzero components. Finally, the free distance of a convolutional code \( C \) is defined as
\[
d_{\text{free}}(C) := \min\{\text{wt}(v(z)) \mid v(z) \in C, v(z) \neq 0\}.
\]
The generalized Singleton bound for an \((n, k, \delta)\) convolutional code \( C \), derived by Rosen-\(\text{thal} \) and Smarandache in [7], relates the parameters of a convolutional code via the following inequality:
\[
d_{\text{free}}(C) \leq (n-k) \left( \left\lceil \frac{\delta}{k} \right\rceil + 1 \right) + \delta + 1.
\]
A convolutional code whose free distance reaches the bound (2) with equality is called maximum distance separable (MDS) code.

2.1 MDP convolutional codes

In this section we briefly define what MDP convolutional codes are and why their study is important.

In the context of convolutional codes, one aims to build codes which can correct as many errors as possible within windows of different sizes. This property is described by the notion of column distances. More formally, we introduce the following definition.

Definition 2.2. Let \( v(z) = \sum_{i=0}^{r} v_i z^i \in \mathbb{F}[z]^n \). For any \( j \leq r \), let \( v_{[0,j]}(z) := \sum_{i=0}^{j} v_i z^i \). The \( j \)-th column distance \( d^j \) of the code \( C \) is defined as
\[
d^j := \min_{v(z) \in C} \{\text{wt}(v_{[0,j]}(z)) \mid v_0 \neq 0\}
\]
The column distances of an \((n, k, \delta)\) convolutional code \( C \) satisfy the following bound.

Theorem 2.3. [2] Proposition 2.3 For any integer \( j \geq 0 \),
\[
d^j \leq (n-k)(j+1) + 1. \tag{3}
\]

Obviously, \( d^j \leq d_{\text{free}}(C) \) for any \( j \). It is easy to see that the maximum \( j \) for which the bound (3) is achievable is for \( j = L \), where
\[
L := \left\lceil \frac{\delta}{k} \right\rceil + \left\lceil \frac{\delta}{n-k} \right\rceil.
\]
The \((L+1)\)-tuple of numbers \( (d^0, \ldots, d^L) \) is called the column distance profile of the code \( C \).
Definition 2.4. An \((n, k, \delta)\) convolutional code \(C\) whose column distances \(d_c^j\) meet the bound of Theorem 2.3 with equality, for all \(j = 0, \ldots, L\) is called maximum distance profile (MDP).

Recall that the encoding map of an \((n, k, \delta)\) convolutional code \(C\) is given by the action of a polynomial matrix \(G(z)\) and it can be expressed via the multiplication by the following polynomial:

\[
G(z) := G_0 + G_1z + \cdots + G_mz^m,
\]

where \(G_i \in \mathbb{F}^{k \times n}\) and \(G_m \neq 0\). In the same way, the parity check matrix is given by

\[
H(z) := H_0 + H_1z + \cdots + H_\mu z^\mu,
\]

with \(H_i \in \mathbb{F}^{(n-k) \times n}\) and \(H_\mu \neq 0\).

Let \(G(z)\) be a generator matrix of an \((n, k, \delta)\) convolutional code \(C\) and let \(H(z)\) be a parity-check matrix for \(C\). For any \(j \in \mathbb{N} \cup \{0\}\), we define the \(j\)-th truncated sliding generator matrix and the \(j\)-th truncated sliding parity-check matrix as

\[
G^c_j := \begin{pmatrix}
G_0 & G_1 & \cdots & G_j \\
G_0 & \cdots & G_{j-1} \\
\vdots & & \ddots & \vdots \\
G_0 & & & \\
\end{pmatrix} \in \mathbb{F}^{(j+1)k \times (j+1)n},
\]

\[
H^c_j := \begin{pmatrix}
H_0 & H_1 & \cdots & H_j \\
H_0 & \cdots & H_{j-1} \\
\vdots & & \ddots & \vdots \\
H_0 & & & \\
\end{pmatrix} \in \mathbb{F}^{(j+1)(n-k) \times (j+1)n},
\]

where \(G_j = 0\), whenever \(j > m\) and \(H_j = 0\) when \(j > \mu\).

These sliding matrices are relevant for the following characterization of MDP convolutional codes.

Theorem 2.5. [2, Corollary 2.3 and Theorem 2.4] Let \(G(z) = \sum_{i=0}^{m} G_i z^i\) and \(H(z) = \sum_{i=0}^{\mu} H_i z^i\) be a left prime generator matrix and a left prime parity-check matrix, respectively, of an \((n, k, \delta)\) convolutional code \(C\). The following are equivalent:

1. \(d_c^j(C) = (n-k)(j+1) + 1\),

2. every \((j+1)k \times (j+1)k\) full-size minor of \(G^c_j\) formed by the columns with indices \(1 \leq t_1 < \cdots < t_{(j+1)k}\), where \(t_{sk+1} > sn\) for \(s = 1, \ldots, j\), is nonzero,

3. every \((j+1)(n-k) \times (j+1)(n-k)\) full-size minor of \(H^c_j\) formed by the columns with indices \(1 \leq t_1 < \cdots < t_{(j+1)(n-k)}\), where \(t_{sn-k+1} \leq sn\) for \(s = 1, \ldots, j\), is nonzero.

In particular, \(C\) is MDP if and only if one of the above equivalent conditions holds for \(j = L\).
The minors considered in Theorem 2.5 are the only full-size minors of $G^c_j$ and $H^c_j$ that can possibly be non-zero. For this reason, we call these minors \textit{non trivially zero.}

In the following, we refer to the conditions of Theorem 2.5 as MDP property on the sliding generator or parity-check matrix of an MDP convolutional code.

\textbf{Remark 2.6.} Observe that for Theorem 2.5 it is necessary to assume that $G(z)$ and $H(z)$ are left prime.

\section{Simplified criterion and left primeness of polynomial matrices}

In this section, we simplify the criterion of Theorem 2.5, by getting rid of the assumption that the generator or parity-check matrix of the convolutional code has to be left prime. In particular, we show that if the MDP property on the minors of the sliding parity-check (generator, resp.) matrix is satisfied, then the parity-check (generator, resp.) matrix is left prime.

\textbf{Theorem 3.1.} Consider $H(z) \in \mathbb{F}[z]^{(n-k)\times n}$ with $\deg(H) = \nu$ and set $\delta = (n-k)\nu$, $r = \left\lfloor \frac{\delta}{k} \right\rfloor$. If the matrix

$$\check{H} := \begin{bmatrix} H_0 & \ddots & \ddots & \vdots \\ \vdots & H_\nu & H_0 & \ddots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & H_\nu \\ I_{n-k} \\ 0_{n-k} \\ \vdots \\ 0_{n-k} \end{bmatrix} \in \mathbb{F}^{(n-k)(r+\nu+1)\times n(r+1)}$$

has full (row) rank, then $H(z)$ is left prime.

\textit{Proof.} First note that since $(n-k)(r+\nu+1) = n(r+1) + \delta - k(r+1) < n(r+1)$, $\check{H}$ has more columns than rows. As $\check{H}$ has full row rank, the map $\mathbb{F}^{n(r+1)} \to \mathbb{F}^{(n-k)(r+\nu+1)}$, $v \mapsto \check{H}v$ is surjective and there exists $\check{X} = \begin{pmatrix} X_0 \\ \vdots \\ X_r \end{pmatrix} \in \mathbb{F}^{(r+1)n\times(n-k)}$ with $X_i \in \mathbb{F}^{n\times(n-k)}$ for $i = 1, \ldots, r$ such that $\check{H}\check{X} = \begin{pmatrix} I_{n-k} \\ 0_{n-k} \\ \vdots \\ 0_{n-k} \end{pmatrix}$. Defining $X(z) = \sum_{i=0}^{r} X_iz^i$, one gets $H(z)X(z) = I_{n-k}$ and hence $H(z)$ is left prime.

\textbf{Corollary 3.2.} Let $n, k, \delta \in \mathbb{N}$ with $k < n$ and $(n-k) \mid \delta$ and set $\nu = \frac{\delta}{n-k}$. If $H(z) = \sum_{i=0}^{\nu} H_iz^i \in \mathbb{F}[z]^{(n-k)\times n}$ with $H_\nu$ full rank has the property that all full-size minors of $H_\nu^L$ with $L = \left\lfloor \frac{\delta}{k} \right\rfloor + \frac{\delta}{n-k}$ that are not trivially zero are nonzero, then $H(z)$ is the parity-check matrix of an MDP convolutional code with rate $\frac{k}{n}$ and degree $\delta$.\hfill \Box
\textbf{Example 3.4.} Let \(1 \leq \delta < k\) and \(\delta < n - k\), i.e. \(L = 0\). We get that \(\deg(H) = \lceil \frac{\delta}{n-k} \rceil + 1 = 1\), so \(H(z) = H_0 + H_1z\) and \(H_L^r = H_0\). If we choose \(H_0\) such that all full-size minors are nonzero and \(H_1 = -H_0\), then \(H\) fulfills the MDP property but \(H(z) = (z^{-1})I_{n-k}^r\bar{H}\) for some \(\bar{H} \in \mathbb{F}^{(n-k)\times n}\), i.e. \(H(z)\) is not left prime and the degree of the code with this parity-check matrix is zero. Hence, this cannot be an \((n,k,\delta)\) MDP convolutional code.

However, it is possible to modify Theorem 3.1, imposing stronger assumptions to get similar results for the case \((n-k) \nmid \delta\).

\textbf{Theorem 3.5.} Let \((n-k) \nmid \delta\). Consider \(H(z) \in \mathbb{F}[z]^{(n-k)\times n}\) with \(\deg(H) = \nu = \lfloor \frac{\delta}{n-k} \rfloor + 1\). If \(r \geq \frac{n\nu-k\nu-k}{k}\) and the matrix

\[ H := \begin{bmatrix} H_0 & \cdots & H_0 \\ \vdots & \ddots & \vdots \\ H_\nu & \cdots & H_\nu \end{bmatrix} \in \mathbb{F}^{(n-k)(r+\nu+1)\times n(r+1)} \]

has full (row) rank, then \(H(z)\) is left prime.

\textit{Proof.} The result can be shown exactly in the same way as Theorem 3.1. \qed

\textbf{Corollary 3.6.} Let \(n, k, \delta \in \mathbb{N}\) be such that \(k < n\) and \((n-k) \nmid \delta\) and set \(\nu = \lfloor \frac{\delta}{n-k} \rfloor + 1\). Assume that \(H(z) = \sum_{i=0}^{\nu} H_iz^i \in \mathbb{F}[z]^{(n-k)\times n}\) is row reduced and it has the property that all the full-size minors of \(H_L^r\) with \(L = \lfloor \frac{\delta}{k} \rfloor + \nu - 1\) that are not trivially zero are

\[ \text{...} \]
nonzero. Moreover, assume that there exists \( r \geq \frac{nu - kv - k}{k} \) such that \( \bar{H} \) as defined in the preceding theorem is full row rank. Then, \( H(z) \) is the parity-check matrix of an MDP convolutional code with rate \( \frac{k}{n} \) and degree \( \delta \).

**Remark 3.7.** The inequality \( r \geq \frac{nu - kv - k}{k} > \frac{\delta}{k} - 1 \) implies \( L - \nu < r \). Hence, for \( (n - k) \nmid \delta \), the matrix \( \bar{H} \) cannot be a submatrix of \( H_L^c \) and we always need to impose this additional condition. Here, we also see why it is of advantage to choose generic row degrees to keep \( \nu \), the degree (as a polynomial) of \( H(z) \), small, because only if \( r + \nu \leq L \), criterion 3 of Theorem 2.5 implies that \( H \) is left prime.

4 Conclusion

In this paper, we gave a simplified version of the criterion provided in [2] to show that a convolutional code is MDP, providing that \( (n - k) \) divides \( \delta \) and the parity-check matrix has generic row degrees. The same criterion can be applied to the generator matrix provided that \( k \) divides \( \delta \). We were able to remove the assumption of left primeness in the characterization. Moreover, when \( n - k \) and \( k \) do not divide \( \delta \), we can add some technical assumption to still get the same result. Moreover, with this work, we show that even if in almost all the known constructions of MDP convolutional codes there is a gap, they are still all correct.

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