Universal relation between the static structure factor and the condensate fraction

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(Dated: August 3, 2020)

We present an analytical method to estimate the condensate fraction \(n_0/n\) in strongly correlated systems for which the zero-temperature static structure factor \(S(p)\) is known. The advantage of the proposed method is that it allows to predict the long-range behavior of the one-body density matrix (i) in macroscopic and mesoscopic systems, (ii) in three- and two-dimensional geometry, (iii) at zero and low finite temperature, and (iv) in weakly and strongly correlated regimes. Our method is tested against exact values obtained with quantum Monte Carlo methods in a number of strongly correlated systems showing an excellent agreement. The proposed technique is also useful in numerical simulations as it allows to extrapolate the condensate fraction to the thermodynamic limit for particle numbers as small as hundreds. Our method is specially valuable for extracting the condensate fraction from the experimentally measured static structure factor \(S(p)\), thus providing a new simple alternative technique for the estimation of \(n_0/n\). We analyze available experimental data for \(S(p)\) of superfluid helium and find an excellent agreement with the experimental value of \(n_0/n\).

Strongly-interacting Bose systems represent a promising platform in modern quantum physics. Their numerous realizations, such as excitons in quantum wells [1–3], transition metal dichalcogenides (TMDs) [4, 5], ultracold gases [6] in optical lattices [7], at Feshbach resonance [8–10], Rydberg atoms [11, 12] as well as superfluid \(^4\)He [13], are now attracting a special attention.

Whereas weakly correlated systems are routinely described by mean-field theory [14], the regime of strong correlations hardly allows for an analytical description. Importantly, the Bose-Einstein condensate (BEC) density, being the main characteristic quantifying the macroscopic coherence, cannot be calculated perturbatively in the regime of strong correlations. This failure is caused by the large condensate depletion produced by interparticle correlations. For example, the condensate fraction of superfluid \(^4\)He is only 7% [15] and in two-dimensional dipolar excitons in GaAs quantum wells [16, 17] is 40% [1], 25% [18] or even smaller, 10% [19]. Large condensate depletion entails significant complications for a precise determination of the condensate density both in experiment and in theory. Unfortunately, at present there is not a simple analytical theory capable of making a quantitative prediction for the condensate fraction in the regime of strong correlations. The experimental estimation of the condensate fraction in liquid \(^4\)He was done, among others, by Glyde and collaborators [20], after intensive neutron scattering experiments. The analysis of the experiment is quite elaborated as final state effects (FSE) in the scattering process have to be taken into account. The estimation of the FSE is rather involved and several theories have been devised to account for them, normally acting as a convoluting function applied to the impulse approximation [21–31]. The experimental values obtained in this way are in close agreement with quantum Monte Carlo (QMC) simulations in a wide range of densities [13]. Still, the high-precision neutron scattering experiment is quite complicated and expensive so the community would benefit from a simpler alternative to estimate the condensate fraction.

In this Letter, we establish a universal relation between the condensate fraction \(n_0/n\) and the zero-temperature static structure factor \(S(p)\). We present a new, fully analytical method for describing the long-wavelength properties of cold bosonic systems in terms of \(S(p)\), which enters as the only input quantity. Our method is based on an empirical choice of the ultraviolet (UV) cutoff in quantum-field hydrodynamics (HD) [32–35]. We exploit a number of advanced quantum Monte Carlo (QMC) methods (variational, diffusion, path-integral) in order to verify our theory. We find a remarkable quantitative agreement in a number of mesoscopic and macroscopic systems and in different dimensionalities (2D and 3D), both at zero and low temperatures. Our method is valid, not only in the Bogoliubov perturbative regime, but in the regime of strong correlations, since the small parameter in our theory is the normal fraction \((n - n_0)/n\) rather than the noncondensate fraction \((n - n_0)/n\). Remarkably, and in contrast to standard approaches [32–35], we obtain a quantitative agreement with QMC results for a spatial range as small as few interparticle distances. This allows one to make a reliable extrapolation to the thermodynamic limit based only from simulations based on hundreds or even tens of particles. As a further check, we apply our method to probably the most famous and difficult example of strongly correlated system, namely superfluid Helium. Even if the condensate fraction is as small as 7%, our method is able to reproduce it correctly when QMC data for \(S(p)\) is taken as an input. Finally, we use experimental data [13, 36, 37] of \(S(p)\) and find that the condensate fraction obtained in this way are in agreement with the ones derived from deep-inelastic neutron scattering. This opens the door to an alternative and easier way of determining the condensate fraction of...
quantum systems in which strong correlations produce a deep depletion of the condensate.

In order to be able to predict properties in strongly correlated systems we use quantum hydrodynamic theory (for details see Supplemental Material). Based on physical grounds, we introduce an effective interaction potential and choose the ultraviolet cutoff factor $\kappa_p (r)$. This allows us to obtain explicit expressions for the one-body density matrix $g_1 (r)$ in terms of the static structure factor at zero temperature $S(p)$ for both finite-size and macroscopic systems. We obtain the following expression for the one-body density matrix (OBDM),

$$
\frac{g_1 (r)}{n} = \exp \left[ \frac{1}{V_D} \sum_{p \neq 0} \kappa_p (r) e^{\varepsilon_p / T} + \frac{1}{4n_s S(p)} e^{\varepsilon_p / T} - 1 \right] (\cos \frac{p r}{h} - 1),
$$

where the effective dispersion relation is defined as $\varepsilon_p \equiv p^2 / (2m S(p))$, $V_D = L^D$ is the quantization volume, and $D = 2$ or 3 is the system dimensionality. We choose the cutoff factor $\kappa_p (r)$ in the form

$$
\kappa_p (r) = |1 - S(p) |^{2/(2 - g^{T=0}(r)/n)},
$$

where $g^{T=0}_1 (r)$ is the OBDM at zero temperature [38]. The cutoff function (2) satisfies the conditions $\kappa \to 0 (r) = 1$ and $\kappa \to \infty (r) = 0$ at any distance $r$, as it must be in a proper HD theory [33]. As the OBDM also appears in the cutoff, the calculation of $g_1 (r)$ has to be done by solving an algebraic equation for $g^{T=0}_1 (r)$. In the continuum thermodynamic limit, the sum in Eq. (1) is substituted by an integral in the usual form, $1/V_D \sum_p = 1/(2\pi h)^D \int dp$

Equations (1)-(2) represent the main result of our work and allow to calculate $g_1 (r)$ (a non-diagonal function) relying only on the knowledge of the static density factor (a diagonal function) at zero temperature $S(p)$. In doing so, we assume that the superfluid fraction is large and neglect the normal component. In this way, our approach is applicable even if the condensate fraction is tiny thus providing predictions in parameter regions characterized by the failure of Gross-Pitaevskii, Bogoliubov and other theories which are perturbative in the non-condensed occupation. In order to verify the correctness of our theory we compare the results obtained for $g_1 (r)$, according to the prescription given above, and the exact result obtained in non-perturbative quantum Monte Carlo methods. We use the most appropriate QMC methods to calculate the OBDM, that is diffusion Monte Carlo (DMC) and path-integral ground state (PIGS) methods at zero temperature, and path-integral Monte Carlo (PIMC) at finite temperature.

In Fig. 1, we show a comparison of the OBDM for the ground state of some 2D systems. The path-integral ground state (PIGS) method is used for 2D liquid $^4$He at its equilibrium density and the DMC method for 2D quantum dipoles with all the dipolar moments oriented perpendicular to the plane, at densities $n = 1$ and $n = 290$ (in reduced dipolar units). As it is known, in the $T = 0$ limit a 2D quantum fluid has a finite condensate fraction or, in other words, it manifests off-diagonal long-range order. In the same figure, we report the results obtained for the same conditions and system using HD theory. We emphasize that our analytic approach requires only the knowledge of the static structure factor at zero temperature. To this end, we use the function $S(p)$ provided by the same QMC methods used to estimate $g_1 (r)$. As anticipated, our approach is accurate for small $p$, or equivalently for large $r$. And this is, in fact, observed in Fig. 1 where predictions of hydrodynamics are compared with the exact QMC results. Notably, we find out that even at intermediate and smaller distances both results are not so different, specially for the dipolar system. Focusing on the limit of large distances, we can estimate the condensate fraction from the long-range asymptotic limit of $g_1 (r)$. HD produces results for $n_0 / n$ which match the QMC ones within their statistical error. We obtain $n_0 / n = 0.23$ for $^4$He and $n_0 / n = 0.37$ and 0.015 for dipoles at densities $n = 1$ and 290, respectively. Remarkably, even for such small values of the condensate fraction, corresponding to strongly interacting systems, we obtain a perfect agreement. It is worth mentioning that standard perturbative theories are not applicable for such strongly-correlated systems.

It can be seen in Fig. 1 that the OBDM is correctly described by HD theory already at small distances, $r \gtrsim \ell n^{-1/D}$, starting from $\ell \approx 1.5 - 3$ interparticle separations. Hence, the HD approach should be valid in mesoscopic systems, i.e., for numbers of particles start-
FIG. 2. The OBDM at \( r = L/2 \), as a function of the particle number \( N \) and at two different interaction strengths. The points stand for the DMC results and the dashed lines correspond to HD theory. The solid lines are the results of extrapolation from the finite-size data to the thermodynamic limit using the HD theory. The data correspond to \( T = 0 \) 2D dipoles at densities \( n = 1 \) (top) and 32 (bottom).

FIG. 3. HD prediction for the OBDM at \( r = L/2 \) as a function of the particle number \( N \) in 2D \(^4\)He at the equilibrium density, \( n = 0.04347 \) Å\(^{-2}\), and \( T = 0 \) (red line). It is compared to its thermodynamic value (dashed line). All curves are based on the PIMC static structure factor calculated at 500 mK and \( N = 64 \), and considering a linear phononic behavior for small momenta.

FIG. 4. The one-body density matrix \( g_1(\mathbf{r}) \) for 2D liquid \(^4\)He as obtained from the HD method, at different temperatures, expressed in units of the BKT critical temperature \( T_{BKT} \). Dashed lines show the long-range power-law asymptotic behavior predicted by the BKT theory below \( T_{BKT} \).

We envision a practical utility of our method in predicting the OBDM for large system sizes, which are hard or even impossible to simulate directly. As an example, we can calculate the OBDM in 2D \(^4\)He at zero temperature for system sizes as large as one million particles. Fig. 3 shows the convergence of the condensate fraction to the macroscopic limit in this case. Even for systems consisting of thousands of particles, the difference between the finite-size condensate fraction and its thermodynamic value is significant. Only for system sizes as large as \( N \approx 10^6 \) the differences become negligible. It is appropriate to note that a direct simulation with any QMC method consisting of \( 10^6 \) particles, if possible, would require the use of large supercomputing facilities. On the contrary, the use of our extrapolation method yields a high-precision estimation at the thermodynamic limit already starting from \( N \approx 100 \), as shown in Fig. 2.

Interestingly, our approach is able to describe correctly the effect of a finite temperature on the spatial coherence. The advantage of using HD theory is that it properly takes into account both thermal and zero-point phase fluctuations. In Fig. 4, we show the long-range behavior of \( g_1(\mathbf{r}) \) for 2D \(^4\)He at temperatures below the BKT...
critical temperature. It is worth noticing that, to produce these results, we only need the zero-temperature static structure factor calculated for a single finite size of the simulated system, with periodic boundary conditions. As one can see, our results perfectly match the expected BKT power-law decay, \( g_1(r) \sim r^{-\alpha} \), with \( \alpha = mT/\left(2\pi\hbar n_s\right) \).

We verify that our method works also correctly in 3D quantum liquids. For that purpose, we make a comparison for bulk superfluid \(^4\)He, where we know that the condensate fraction is small due its strongly interacting nature. We show a comparison between QMC and HD in Fig. 5. The results corresponding to \( T = 0 \) are obtained with PIGS and the ones at \( T = 1 \) K with PIMC. The equilibrium density, \( n_0 = 0.02186 \, \text{Å}^{-3} \), is used in both cases. As one can see, the agreement in the plateau is excellent and quite good at short and intermediate distances. The condensate fraction obtained with the two methods are again in agreement within the error bars of the QMC estimation, \( n_0/n \approx 0.07 \).

The comparison between the hydrodynamic predictions for the condensate fraction and available experimental results [13] is the more exigent test to conclude on the usefulness of our method. This check can be made for the paradigmatic case of liquid \(^4\)He. In Fig. 6, we report results of \( n_0/n \) for superfluid \(^4\)He as a function of the density. The functions \( S(p) \), required for the one-body density matrix calculation, are obtained using the PIGS method. As we can see in the figure, an overall agreement is achieved with the experimental results [13], obtained by using deep-inelastic neutron scattering, and after an accurate inclusion of FSE in the analysis. Remarkably, the accuracy of our theory does not fall even when the freezing density is approached and the condensate fraction is significantly reduced.

One may wonder whether the HD approach can also provide an alternative way of measuring the condensate fraction instead of the present way, based on deep inelastic neutron scattering at large momentum transfer. In other words, whether the experimental measure of the static structure factor can be used to infer \( n_0/n \). We have verified this possibility by using in our theory three independent experimental measurements of \( S(p) \), two obtained by x-rays [36, 39] and the other by neutron scattering [37] at low momentum transfer. Our method predicts \( n_0/n = 0.071 \) based on the experimental \( S(p) \) from Ref. [37], and \( n_0/n = 0.086 \) for data from Ref. [39] (see Fig. 6). In the latter case, \( S(p) \) is reported only up to a rather small \( p \) value producing a larger value for the condensate fraction using HD. Data form Ref. [36] produces results at different densities that are very close to the results from Ref. [13], obtained through deep inelastic neutron scattering, thus validating the usefulness of our method for extracting the condensate fraction in an alternative way.

In conclusion, we have developed new quantum hydrodynamic theory for superfluid systems. Using our phenomenological approach for the interaction potential and the ultraviolet cutoff factor, we arrive to both finite-size and macroscopic expressions for the one-body density matrix \( g_1(r) \) that rely only on a previous knowledge of the static structure factor at zero temperature. In this way, we provide access to a non-diagonal property starting from a diagonal one. In contrast to standard perturbation theories, which rely on a large condensate fraction and fail for a small one, the hydrodynamic method used requires a large superfluid fraction (collisionless regime) and it can be applied for large condensate depletion as
well. We have verified that the values of the condensate fraction, derived from the long-range behavior of \( g_1(r) \), match closely the ones obtained from ab initio QMC simulations. Our approach provides an enhanced convergence of the condensate fraction to its thermodynamic-limit value because it is based on the static structure factor, which typically has reduced finite-size effects. Using the static structure factor \( S(p) \), obtained in a QMC calculation, we reproduce experimental results for the condensate fraction in superfluid \( ^4\text{He} \) for a wide range of densities. Moreover, using experimental measurements of \( S(p) \) we obtain predictions for the condensate fraction that are statistically indistinguishable from the ones obtained by deep inelastic neutron scattering at large momentum transfer. Moreover, our method applies even to cases in which the two-body interaction potential is not exactly known (for example, for dense excitons or Rydberg gas), thus impeding a direct simulation of the system, whereas \( S(p) \) is experimentally accessible. The present theory is addressed to both experimentalists and theoreticians and it provides an alternative procedure to the usual and difficult estimation of the condensate fraction, derived from the long-range behavior of \( g_0) = \langle \hat{c}_p \hat{c}^-_p \rangle / n \) at \( T = 0 \) is our single input parameter. Therefore, it must be reproduced self-consistently. This dictates the following choice

\[
U(p) = \frac{p^2}{4mn_s} \left( \frac{1}{(S(p))^2} - 1 \right)
\]

(9)

for the momentum-dependent potential, with the \( U(p \to 0) = g \) condition being automatically hold. Then, the empirical Hamiltonian (8) is diagonalized by the transformation

\[
\hat{p}_p = i\sqrt{n_s S(p)} (\hat{c}_p - \hat{c}^-_p), \quad \hat{\psi}_p = \frac{i \hat{p} / m}{\sqrt{4n_s S(p)}} (\hat{c}_p + \hat{c}^-_p),
\]

(10)

SUPPLEMENTAL MATERIAL

We consider a bulk homogeneous three-dimensional (3D) or two-dimensional (2D) bosonic superfluid system in absence of static currents. The effective Hamiltonian corresponding to the free-energy functional can be conveniently written in the form [33, 35]

\[
\hat{H} - \mu \hat{N} = \int \left( \frac{m}{2} \hat{\rho}_s(r)(\hat{\nu}'(r))^2 + f(\hat{\rho}'(r)) \right) dr,
\]

with \( m \) the particle mass, \( \mu \) the chemical potential, \( \hat{N} \) the particle number operator. The density operator \( \hat{\rho}_s(r) \) of the superfluid component is conveniently split into superfluid density \( n_s \) and density fluctuation operators \( \hat{\rho}'(r) \) as \( \hat{\rho}_s(r) = n_s + \hat{\rho}'(r) \). The density \( \hat{\rho}'(r) \) and velocity \( \hat{\nu}'(r) \) fluctuation operators of the superfluid component have zero Gibbs averages, \( \langle \hat{\rho}'(r) \rangle = \langle \hat{\nu}'(r) \rangle = 0 \) and zero volume averages, \( \int \rho'(r) dr = \int \hat{\nu}'(r) dr = 0 \). As usual, brackets \( \langle \cdots \rangle \) denote averaging over the equilibrium state of the system that preserves the number of particles. Finally, the function \( f(\hat{\rho}'(r)) \) in Eq. (3) describes two-, three- or many-body interactions.

The Hamiltonian (3) \( \hat{H} - \mu \hat{N} \) can be decomposed into quadratic and anharmonic terms. In order to find the correlation functions, we need only the quadratic form

\[
\hat{H}_0 = \int \left( \frac{mn_s}{2} (\hat{\nu}'(r))^2 + \frac{g}{2} (\hat{\rho}'(r))^2 \right) dr,
\]

(4)

with \( g \equiv m^2 / \chi \) a generalization of the coupling constant to the case of strong correlations, \( \chi \) being the adiabatic compressibility [35]. Let us expand the density and velocity fluctuations operators in Fourier series,

\[
\hat{\rho}'(r) = \frac{1}{\sqrt{V_D}} \sum_{p \neq 0} e^{i p r / \hbar} \hat{\rho}_p, \quad \hat{\nu}'(r) = \frac{1}{\sqrt{V_D}} \sum_{p \neq 0} e^{i p r / \hbar} \hat{\nu}_p,
\]

(5)

and decompose the velocity fluctuation operator \( \hat{\nu}_p \) into phononic \( \hat{\nu}_p^\| \) and vortex \( \hat{\nu}_p^\perp \) contributions, where \( \hat{\nu}_p^\| \parallel p \) and \( \hat{\nu}_p^\perp \perp p \), with \( \hat{\nu}_p^\| + \hat{\nu}_p^\perp = \hat{\nu}_p \), and prime over \( \hat{\nu} \) is dropped as \( p \neq 0 \). Then, the quadratic Hamiltonian (4) in reciprocal space turns to

\[
\hat{H}_0 = \hat{H}_0^\| + \sum_{p \neq 0} \frac{mn_s}{2} \hat{\nu}_p^\| \hat{\nu}^\|_p,
\]

(6)

where

\[
\hat{H}_0^\| = \sum_{p \neq 0} \left( \frac{mn_s}{2} \hat{\nu}_p^\| \hat{\nu}^\|_p + \frac{g}{2} \hat{\rho}_p \hat{\rho}^\|_p \right),
\]

(7)

is the phononic part of the Hamiltonian.

We generalize the standard hydrodynamic Hamiltonian (7), by introducing a momentum-dependent potential \( U(p) \) satisfying the limiting condition \( U(p \to 0) = g \),

\[
\hat{H}_0^\| = \sum_{p \neq 0} \left( \frac{mn_s}{2} \hat{\nu}_p^\| \hat{\nu}^\|_p + \frac{U(p)}{2} \hat{\rho}_p \hat{\rho}^\|_p \right),
\]

(8)

The zero-temperature static structure factor \( S(p) = \langle \hat{\rho}_p \hat{\rho}^\|_p \rangle / n \) at \( T = 0 \) is our single input parameter. Therefore, it must be reproduced self-consistently. This dictates the following choice

\[
U(p) = \frac{p^2}{4mn_s} \left( \frac{1}{(S(p))^2} - 1 \right).
\]

(9)
with \( p \neq 0 \). In Eqs. (10), \( \hat{c}_{p} \) is the annihilation operator of a phonon with momentum \( p \), satisfying the usual bosonic commutation relations \([34], [\hat{c}_{p}, \hat{c}_{q}] = 0, [\hat{c}_{p}, \hat{c}_{q}^\dagger] = \delta_{pq}\). We recast the empirical Hamiltonian (8) in the final form

\[
\hat{H}_0^\parallel = \sum_{p \neq 0} \hat{\varepsilon}_p \hat{c}_p^\dagger \hat{c}_p,
\]

where the energy of the quasi-particles \( \hat{\varepsilon}_p \) corresponds to the Feynman excitation energy. Therefore, \( \hat{\varepsilon}_p \) is an empirical dispersion relation, which differs from the true one, and coincides with it only in the phononic long-wavelength limit \( p \to 0 \), in which the described approach is valid.

Our main interest focuses on the estimation of the one-body density matrix. To this end, we use the Kubo cumulant expansion \([32]\),

\[
g_{1}(\mathbf{r}) \equiv \langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{0}) \rangle = \text{const} \exp \left[ -\frac{ \langle (\hat{\varphi}(\mathbf{r}) - \hat{\varphi}(\mathbf{0}))^2 \rangle}{2} \right],
\]

where \( \hat{\psi}(\mathbf{r}) \) is the operator of the Bose particle field, the constant prefactor depends on a particular choice of the ultraviolet cutoff. The phase fluctuation operator \( \hat{\varphi}(\mathbf{r}) \) can be expressed in terms of the superfluid velocity \( \hat{\mathbf{v}}(\mathbf{r}) \),

\[
\hat{\varphi}(\mathbf{r}) - \hat{\varphi}(\mathbf{0}) = \frac{m}{\hbar} \int_0^\mathbf{r} \hat{\mathbf{v}}(\mathbf{r}')d\mathbf{r}'.
\]

We assume sufficiently low temperatures so that the thermal activation of the vortex rings (3D) or vortices (2D) is exponentially suppressed, and we set the corresponding contribution to zero, \( \hat{\mathbf{v}}_p^\dagger = 0 \). In the 3D case, vortex effects are not important up to sufficiently high temperatures, where our method is no longer expected to be applicable. In the 2D case, the vortex effects must be taken into account, since they are necessary for an appropriate description of the BKT phase transition. Renormalization by free vortices (if they are generated) can be taken into account following Kosterlitz \([41]\) by introducing a \( e^{-r/\xi_c} \) factor in Eq. (12). Instead, a renormalization originating from pairs of vortices can be included in \( n_s \) \([42]\). Therefore, in both 2D and 3D cases we arrive at the result

\[
g_{1}(\mathbf{r}) = \text{const} \cdot \exp \left[ -\frac{ \langle (\hat{\varphi}_\| (\mathbf{r}) - \hat{\varphi}_\| (\mathbf{0}))^2 \rangle}{2} \right] e^{-r/\xi_c},
\]

where the phononic contribution to the phase fluctuation is

\[
\hat{\varphi}_\| (\mathbf{r}) - \hat{\varphi}_\| (\mathbf{0}) = \frac{1}{\sqrt{V_D}} \sum_{p \neq 0} \frac{m p}{ip^2} \hat{\mathbf{v}}_\| (e^{ipr/h} - 1),
\]

and \( \xi_c \) denotes the distance between free vortices \([43]\). In 2D, if free vortices are absent (below the BKT critical temperature) one should set the corresponding distance to infinity. In 3D, one should always set \( \xi_c = \infty \).

After substituting Eq. (15) into Eq. (14) with \( \xi_c = \infty \) one has

\[
g_{1}(\mathbf{r}) = \frac{1}{n} \exp \left[ \frac{1}{V_D} \sum_{p \neq 0} \kappa_p \left( \frac{m^2}{p^2} (\hat{\mathbf{v}}^\dagger \hat{\mathbf{v}}_\| - 1) \right) \right],
\]

where we inserted an ultraviolet cutoff factor \( \kappa_p \) “by hand” and take into account the fact that \( g_{1}(0) = n \).

Averages of Bose operators satisfy \( \langle \hat{c}_p \hat{c}_q \rangle = \langle \hat{c}_p \rangle \langle \hat{c}_q \rangle = 0 \) and \( \langle \hat{c}_p^\dagger \hat{c}_q \rangle = \delta_{pq} / (e^{\varepsilon_p/T} - 1) \), thus the velocity-velocity correlation function becomes \([\text{see Eq. (10)}]\)

\[
\langle \hat{\mathbf{v}}_p \hat{\mathbf{v}}_\| p \rangle = \frac{g_p}{4m^2 n_s S(\mathbf{p})} e^{\varepsilon_p/T^2} + 1.
\]

Finally, by substituting Eq. (17) to Eq. (16), we arrive to the following final expression for OBDM in a finite system

\[
g_{1}(\mathbf{r}) = \frac{1}{n} \exp \left[ \int \frac{dp}{2\pi\hbar^2} \frac{\kappa_p \left( e^{\varepsilon_p/T} + 1 \right)}{4n_s S(\mathbf{p})} e^{\varepsilon_p/T} - 1 \right].
\]

In the thermodynamic limit, the summation over momenta in Eq. (19) should be replaced by an integral. This yields the final form for OBDM in a macroscopic system

\[
g_{1}(\mathbf{r}) = \exp \left[ \int \frac{d\mathbf{p}}{(2\pi\hbar)^D} \frac{\kappa_p \left( e^{\varepsilon_p/T} + 1 \right)}{4n_s S(\mathbf{p})} e^{\varepsilon_p/T} - 1 \right].
\]

The integral is convergent in both the infrared and ultraviolet integration limits, \( p \to 0 \) and \( p \to \infty \), if we take the cutoff factor in the form of

\[
\kappa_p = \left| 1 - S(\mathbf{p}) \right|^{2/(2-g^2)} \langle \mathbf{r} \rangle / n_S,
\]

and consider that the structure factor converges sufficiently fast to unity at large momenta.

[1] A. A. High, J. R. Leonard, A. T. Hammack, M. M. Fogler, L. V. Butov, A. V. Kavokin, K. L. Campman, and A. C. Gossard, “Spontaneous coherence in a cold exciton gas,” Nature 483, 584–588 (2012).
[2] Mathieu Alloing, Mussie Beian, Maciej Lewenstein, David Fuster, Yolanda González, Luisa González, Roland Combescot, Monique Combescot, and François Dubin, “Evidence for a bose-einstein condensate of excitons,” EPL (Europhysics Letters) 107, 10012 (2014).
[3] Yotam Mazuż-Harpaz, Kobi Cohen, Michael Leveson, Ken West, Loren Pfeiffer, Maxim Khodas, and Ronen Rapaport, “Dynamical formation of a strongly correlated dark condensate of dipolar excitons,” Proceedings of the National Academy of Sciences 116, 18328–18333 (2019).
[4] A. K. Geim and I. V. Grigorieva, “Van der waals heterostructures,” Nature 499, 419–425 (2013).
S. O. Diallo, R. T. Azuah, D. L. Abernathy, R. Rota, A. Griffin, “Conserving and gapless approximations for Henry Glyde, A. Urvoy, F. Ripka, I. Lesanovsky, D. Booth, J. P. Shaf- Yurii E. Lozovik, I. L. Kurbakov, G. E. Astrakharchik, V. V. Solov’ev, I. V. Kukushkin, J. H. Klitzing, and W. Dietsche, “Kinetics of indirect electron-hole recombination in a wide single quantum well in a strong electric field,” JETP Letters 84, 222–225 (2006).

H. R. Glyde, R. T. Azuah, and W. G. Stirling, “Condensate, momentum distribution, and final-state effects in liquid 4He,” Phys. Rev. B 62, 14337–14349 (2000).

H. A. Gersch and L. J. Rodriguez, “Final-state effects on thermal-neutron scattering at high-energy transfer,” Phys. Rev. A 8, 905–912 (1973).
[40] P. C. Hohenberg, “Existence of long-range order in one and two dimensions,” Phys. Rev. 158, 383–386 (1967).
[41] J M Kosterlitz, “The critical properties of the two-dimensional xy model,” Journal of Physics C: Solid State Physics 7, 1046–1060 (1974).
[42] Petter Minnhagen and G. G. Warren, “Superfluid density of a two-dimensional fluid,” Phys. Rev. B 24, 2526–2532 (1981).
[43] Vinay Ambegaokar, B. I. Halperin, David R. Nelson, and Eric D. Siggia, “Dissipation in two-dimensional superfluids,” Phys. Rev. Lett. 40, 783–786 (1978).