Quantum simulation of PT-arbitrary-phase–symmetric systems

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Abstract – Parity-time-reversal (PT) symmetric quantum mechanics promotes the increasing research interest of non-Hermitian (NH) systems for the theoretical value, novel properties, and links to open and dissipative systems in various areas. Recently, anti-PT-symmetric systems and their featured properties have started to be investigated. In this work, we develop the PT and anti-PT symmetry to PT-arbitrary-phase symmetry (or PT-ϕ symmetry) for the first time, being analogous to bosons, fermions and anyons. It can also be seen as a complex extension of the PT symmetry, unifying the PT and anti-PT symmetries and having intermediate properties between them. Many of the established concepts and mathematics in the PT-symmetric system are still compatible. We mainly investigate quantum simulation of this novel NH system in two dimensions in detail and discuss for higher-dimensional cases in general using the linear combinations of unitaries in the scheme of duality quantum computing, enabling implementations and experimental investigations of novel properties on both small quantum devices and near-term quantum computers.

Introduction. – From Feynman’s idea of investigating nature by itself[1] to nowadays available technologies [2,3], quantum simulation has become a strong tool of scientific research in practice. The time evolution of a quantum system can be simulated as long as the effective Hamiltonian is constructed or the relevant quantum circuit is designed. Plenty of phenomena have been investigated by quantum simulation especially for Hermitian systems [4–13]. Besides, it has provided an effective and efficient way to investigate non-Hermitian (NH) systems [14–21].

The non-Hermitian quantum system attracts more and more research interest because it develops the conventional quantum theory, links to open and dissipative systems, and has a lot of novel properties and applications. Among them is a class of systems with PT symmetry [22–24], exhibiting entirely real spectra of the Hamiltonians. As the significances both in theory and potential applications, it is developed fast [25–32] and investigated thoroughly from different aspects in a variety of systems [33–39]. In recent years, quantum simulations of PT-symmetric systems have been carried out, e.g., fast and slow evolutions in the quantum brachistochrone problem [14,26,27], a superluminal information transmission [29], a general PT two-level system [15–18], etc.

As the anti-symmetric counterpart of PT symmetry, anti-PT–symmetric systems attract research interest for their appealing features [19,20,40–50], such as designing balanced positive and negative index of optical materials [40], constructing constant refraction [44] in optical systems, simulating time evolutions and information flows [19,20], etc. In fact, PT– and anti-PT–symmetric Hamiltonians can be seen as the real and imaginary counterparts of each other linked by i. It is an interesting question whether they can be unified or developed to their complex extensions. If possible, what the fundamental properties and potential applications are and how to simulate this novel NH systems on Hermitian quantum devices are the subsequent questions.

In this work, we investigate a novel non-Hermitian quantum system of PT-arbitrary-phase symmetry (or PT-ϕ symmetry) for the first time, which unifies PT and anti-PT symmetries. Especially, we give the general explicit form of the Hamiltonian in the two-dimensional case, and discuss its basic properties, such as the eigenvalues, conditions whether the PT symmetry is spontaneously broken or not, etc. Then we propose how to simulate the general PT-arbitrary-phase–symmetric two-level system

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using linear combinations of unitaries (LCU) [51–55] in qubit-qudit–hybrid and pure-qubit Hermitian systems, designing the quantum circuit and calculate the successful probabilities. At last, we analyze the experimental implementations and the practicabilities.

**Basic theory and two-level systems.** – PT– and anti-PT– (APT-) symmetric Hamiltonians, \( H_{PT} \) and \( H_{APT} \), are those that satisfy the commutation and anti-commutation relations \([PT, H_{PT}] = 0 \) and \([PT, H_{APT}] = 0 \), respectively, where \([A, B] = AB - BA\). \((A, B) = AB + BA\), the parity \( P \) inverts the positions and the time reversal \( T \) has the effect as the complex conjugation changing \( i \rightarrow -i \). \( H_{PT} \) and \( H_{APT} \) can be seen as the real and imaginary counterparts of each other since the former times \( i \) will become an anti-PT–symmetric Hamiltonian, and vice versa. Consider a non-Hermitian Hamiltonian \( H \), which satisfies

\[
H^{PT} \overset{\text{def}}{=} (PT) H (PT)^{-1} = e^{i\varphi} H. \quad (1)
\]

Clearly, \( H \) can be obtained by a phase factor \( e^{-i\frac{\pi}{2}} \) timing an \( H_{PT} = e^{i\frac{\pi}{2}} H \), which is a complex generalization of the PT-symmetric Hamiltonian

\[
H = e^{-i\frac{\pi}{2}} H_{PT} = \cos \frac{\varphi}{2} H_{PT} - \sin \frac{\varphi}{2} (i H_{PT}). \quad (2)
\]

If we set \( H_{APT} = i H_{PT} \) that satisfies anti-PT symmetry, then

\[
H = \cos \frac{\varphi}{2} H_{PT} - \sin \frac{\varphi}{2} H_{APT}. \quad (3)
\]

Thus, \( H \) can also be seen as a hybrid combination of an \( H_{PT} \) and an \( H_{APT} \). Therefore, we say that \( H \) is of PT-arbitrary-phase symmetry or PT-\( \varphi \) symmetry. The relations of PT, anti-PT and PT-\( \varphi \) symmetry can be analogous to that of the boson, fermion and anyon. Thus, \( H \) should have intermediate properties between the former two. In fact, the relation in eq. (1) unifies the PT and anti-PT symmetries for \( \varphi = 2k\pi \) and \( \varphi = (2k + 1)\pi \) (\( k \) is integral), respectively. Notice that eqs. (1), (2) and (3) are valid for not only two-dimensional cases but also general cases.

In the two-dimensional case, the most general form is

\[
H = e^{-i\frac{\pi}{2}} \left( \begin{array}{cc} r e^{i\theta} & s + i w \\ s - i w & r e^{-i\theta} \end{array} \right), \quad (4)
\]

where \( \varphi, r, s, w \) and \( \theta \) are real parameters, the parity \( P = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) and \( \varphi \) is fixed linking to the symmetry. In fact, it can be obtained by a general PT-symmetric Hamiltonian [23] times \( e^{-i\frac{\pi}{2}} \).

The eigenvalues of \( H \) are \( e_{+} = e^{-i\frac{\pi}{2}} (r \cos \theta \pm \frac{\sqrt{3}}{2} w) \) with respect to the two eigenvectors \( |e_{\pm}\rangle \), respectively, where \( \Delta_0 = e^{i\frac{\pi}{2}} (e_{+} - e_{-}) = 2\sqrt{3} s^2 + w^2 - r^2 \sin^2 \theta \) is the energy difference of \( H_0 \). \( H_0 \) is of exact PT symmetry (or PT symmetry of \( H \) is unbroken) when \( s^2 + w^2 \geq r^2 \sin^2 \theta \), in which case the two eigenstates of \( H_0 \) are those of the PT operation. Otherwise, the PT symmetry of \( H_0 \) is spontaneously broken. The exceptional points (EPs) of \( H \) in the parametric space [56] are those leading \( \Delta_0 \) to be zero, which form the boundary of PT symmetry broken and unbroken. In the next section, our quantum simulation method is applicable to all of the parametric space, including the neighborhoods of EPs. While the PT inner product introduced by Bender et al. [23] is well defined in this system, we still use the Hilbert-Schmidt inner product of the convention quantum mechanics in the next section because this system will be simulated by a Hermitian system.

**Duality quantum simulation.** – Investigations of novel systems and phenomena using a controllable system is one of the main tasks of quantum simulation, which can be realized by quantum devices being available nowadays. For a Hermitian system, the time evolution can be simulated directly by a conventional quantum system in a Hilbert space of same dimensions because of its unitarity.

The time evolution operator of the PT-arbitrary-phase–symmetric system is governed by

\[
e^{-i\frac{\pi}{2}H}, \quad (5)
\]

where \( H \) is that in eq. (4). However, \( e^{-i\frac{\pi}{2}H} \) is not unitary. Thus, it cannot be simulated directly in a Hermitian quantum system using the conventional method. We will construct a general PT-\( \varphi \)-symmetric subsystem in a larger Hilbert space and simulate the time evolution in the scheme of duality quantum computing using the LCU method. At the first step, the unitary expansion (UE) techniques [57] will be applied to the non-unitary time evolution operator.

**UE of the time evolution.** – We calculate details of the unitary expansions of \( e^{-i\frac{\pi}{2}H} \). In general cases, it can be expanded by four or three UE terms as follows:

\[
e^{-i\frac{\pi}{2}H} = f_0 |0\rangle\langle 0| + f_1 |1\rangle\langle 1| + f_2 (|2\rangle\langle 2| + f_3 |3\rangle\langle 3|), \quad (6)
\]

where \( f_k = |f_k| e^{i\theta_k} \) \((k = 0, 1, 2, 3)\) are the UE parameters, and \( \sigma_0 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \sigma_1 = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \sigma_2 = \left[ \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right] \) and \( \sigma_3 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \) are the Pauli matrices. Equation (6) can be further combined to three terms as

\[
e^{-i\frac{\pi}{2}H} = e^{i\theta_0} |0\rangle\langle 0| + e^{i\theta_3} |g_1\rangle\langle g_1| + e^{i\theta_3} |g_2\rangle\langle g_2|, \quad (7)
\]

where the UE parameters \( g_1 \) and \( g_2 \) are complex functions of \( f_k \)'s \((k = 0, 1, 2, 3)\) in eq. (6). All the explicit forms of the parameters, \( U_1 \) and \( U_2 \) are presented in the appendix. Notice that the UE parameters are time-dependent complex functions depending on \( H \) in eq. (4), and have no limits on the norms.

If \( f_k \)'s \((k = 0, 1, 2, 3)\) in eq. (6) meet one of the phase matching conditions, which can be referred to in the supplementary information of ref. [57], then the time evolution operator in eq. (5) can be expressed by two UE terms as

\[
e^{-i\frac{\pi}{2}H} = h_0 V_0 + h_1 V_1, \quad (8)
\]

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where \( h_k \)'s and the elements of \( V_k \in SU(2) \) \((k = 0, 1)\) are complex functions of time \( t \). The explicit forms of \( h_0, h_1, V_0 \) and \( V_1 \) vary in different phase matching conditions, which can be referred to in ref. \[57\]. We can apply our UE technics to quantum simulation of the non-unitary evolution in eq. (5) using the method of linear combinations of unitaries (LCU) in the scheme of duality quantum computing.

### Duality quantum computing and LCU

Duality quantum computing was proposed in 2002 \[51\] at first, pointing out that not only the products but also the linear combinations of unitary operators can be used for constructing new quantum algorithms. Then LCU and duality quantum computing were developed fast \[51–55,58–60\] and became one of the strongest tools in designing quantum algorithms \[61\]. Recently, we developed LCU to simulate NH systems \[14–21\], other novel systems \[12,13\], and generalized time-dependent non-unitary operators \[57\].

In the following subsections, we first show how to simulate the time evolution in eq. (5) in a qubit-qudit system based on eq. (6) to illustrate the principle of LCU and duality quantum computing, and then give theories to perform quantum simulation in qubits systems based on eq. (7), being of higher efficiency and practicability realized by nowadays available small quantum devices.

### Qubit-qudit–hybrid simulation

The hybrid system consists of a work qubit \( e \) and a four-dimensional ancillary qudit \( a \), constructing an eight-dimensional Hilbert space. A qudit is a basic building block of high-dimensional quantum computing. A four-dimensional qudit has four orthogonal logical bases \(|0\rangle, |1\rangle, |2\rangle, |3\rangle\), which can be realized by a four-state quantum system such as an ultracold atom with four non-degenerate energy levels, a nuclear spin with four split levels, a spin-(3/2) particle, etc. In some quantum algorithms, qudits take advantages over qubits. For example, it reaches a higher accuracy to solve the eigenvalue problem using the quantum phase estimation algorithm by qudits \[65\] rather than by qubits \[62\].

Although this is not an efficient nor practical protocol, it is clear to show the principle of duality quantum simulation. The quantum circuit is shown in fig. 1. At the beginning, the system is initialized to a pure state \(|0\rangle_a|\psi\rangle_e\), where \(|\psi\rangle_e\) is an arbitrary state. Subsequently, we will construct the PT-arbitrary-phase–symmetric subsystem. A single-qudit operator \( U_P = (u_{jk}) \in SO(4) \) is applied on the ancillary qudit at first, where \( u_{jk} \)'s are the matrix elements and \( k, j = 1, 2, 3, 4 \), assigning the UE parameters \( f_k \)'s to the probabilistic amplitudes of the ancillary qudit. While the explicit form of \( U \) is not unique, it is required that if \( U_1U_F = I_4 \) and the matrix element in the first column satisfies \( u_{j1} = f - 1/j \) \((j = 1, 2, 3, 4)\), where

\[
    f = \frac{3}{k=0} \sqrt{|f_k|^2} \quad (9)
\]

is a normalizing factor. In the second step, four controlled gates are applied on the whole system, whose work qubit is controlled by the ancillary qudit. They are 0-controlled \( \sigma_0 \), 1-controlled \( \sigma_1 \), 2-controlled \( \sigma_2 \) and 3-controlled \( \sigma_3 \), where the effects of the controlled operators are to generate the four UE terms in eq. (6) and to entangle them with the qudit. In fact, the first controlled gate is a trivial matrix of \( I_4 \). In the third step, a Hadamard \( H_2 \otimes H_2 \in SU(4) \) is applied on the ancillary qudit (where \( H_2 \) is the Hadamard in \( SU(2) \)), and the whole system evolves to a superposition state

\[
    \frac{1}{2f} \left( |0\rangle_a e^{-i\frac{H}{\hbar}} |\psi\rangle_e + f \sum_{k=1}^3 |k\rangle_a |s_k\rangle_e \right) . \quad (10)
\]

The explicit forms of \(|s_k\rangle_e\)'s are neglected because they will be discarded if \(|k\rangle_a\) is output \((k = 1, 2, 3)\).

Now, if a quantum measurement is performed on the ancillary qudit and the result indicates an output of \(|0\rangle_a\), the work qubit will evolve as the time evolution operator in eq. (5) neglecting a normalizing factor. Therefore, \( e^{-i\frac{H}{\hbar}} \) is simulated in an indeterministic way with a successful probability of

\[
    \frac{1}{4f^2} e^{i\langle \psi | e^{-i\frac{H}{\hbar}} |0\rangle_a |\psi\rangle_e} . \quad (11)
\]
Notice that though the normalizing factor $f$ affects the successful probability, it does not change the evolved state governed by $e^{-i\frac{t}{\hbar}H}$ after quantum measurement.

In other cases, the work qubit will evolve to some work qubit $\ket{e}$, which is extended by $C_{00-\sigma_0}$ can be removed in practice, the other two jointly controlled gates are essential. They can be expressed as

\begin{equation}
C_{01-U_1} = \begin{bmatrix}
\sigma_0 & 0 & 0 & 0 \\
0 & U_1 & 0 & 0 \\
0 & 0 & \sigma_0 & 0 \\
0 & 0 & 0 & \sigma_0 \\
\end{bmatrix}
\end{equation}

and

\begin{equation}
C_{10-U_2} = \begin{bmatrix}
\sigma_0 & 0 & 0 & 0 \\
0 & \sigma_0 & 0 & 0 \\
0 & 0 & U_2 & 0 \\
0 & 0 & 0 & \sigma_0 \\
\end{bmatrix},
\end{equation}

respectively, referring to eq. (7) and the appendix for $U_1$ and $U_2$. It is achieved that generating and entangling the three UE terms with the three bases of the ancillary subspace.

Now the three UE terms will be superposed by swapping the two ancillary qubits three times together with $H_2$ and $R_3$ in between as shown in fig. 2, where

\begin{equation}
R_3 = \frac{1}{\sqrt{3}} \begin{bmatrix}
\sqrt{2} & 1 & -\sqrt{2} \\
\end{bmatrix}
\end{equation}

The whole system evolves to a superposition state

\begin{equation}
\frac{1}{\sqrt{3}f} \left[ |00\rangle_a e^{-i\frac{t}{\hbar}H} |\psi\rangle_e + f \sum_{k=0,1,10} |k\rangle_a |s_{k}\rangle_e \right],
\end{equation}

where the three UE terms are superposed in the first term as the time evolution. The rest terms in eq. (17) are not shown explicitly because they will be discarded after measurements.

Finally, quantum measurements are performed on the ancillary qubits. If the ancillary subsystem outputs a state $|00\rangle_a$, the work qubit will evolve to $e^{-i\frac{t}{\hbar}H} |\psi\rangle_e$, governed by the NH Hamiltonian in eq. (4). If any one of the other
two results of $|01\rangle_a$ or $|01\rangle_a$ is obtained, the simulation will be terminated and started over until $|00\rangle_a$ is observed. Therefore, it is an indeterministic protocol to simulate the time evolution of the PT-arbitrary-phase–symmetric two-level system. The successful probability is

$$\frac{1}{3f^2}e^{\langle \psi |e^{i\frac{\hat{H}t}{\hbar}}e^{-i\frac{\hat{H}t}{\hbar}}|\psi \rangle},$$

(18)

indicating that the successful probability can be increased by saving the two-dimensional subspace than using the full eight dimensions.

The number of the qubits can be reduced to two if the time evolution operator in eq. (5) can be expressed by two UE terms as that in eq. (8). In this case, the quantum circuit is shown in fig. 3.

At the beginning, the two-qubit system is initialized to $|0\rangle_a |\psi\rangle_e$ as needed. A single-qubit unitary

$$V = \frac{1}{f} \begin{bmatrix} h_0 & -h_1^\dagger \\ h_1 & h_0^\dagger \end{bmatrix},$$

(19)

is applied on the ancillary qubit. Although the explicit forms of $h_0$ and $h_1$ vary in the parameters of $H$ and the phase matching conditions (refer to ref. [57] for details), they always satisfy that

$$f = \sqrt{|h_0|^2 + |h_1|^2}.$$

(20)

Two controlled gates follow, which are

$$C_{0-V_0} = \begin{bmatrix} V_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}$$

(21)

and

$$C_{1-V_1} = \begin{bmatrix} \sigma_0 & 0 \\ 0 & V_1 \end{bmatrix},$$

(22)

where the explicit forms of $V_0$ and $V_1$ vary in phase matching conditions [57]. Then a Hadamard operation $H_2$ is performed on the ancillary qubit, and the two-qubit system evolves to

$$\frac{1}{\sqrt{2f}} \left[ |0\rangle_a e^{-i\frac{\hat{H}t}{\hbar}}|\psi\rangle_e + |1\rangle_a (h_0 V_0 - h_1 V_1)|\psi\rangle_e \right].$$

(23)

Similarly, the first term is relevant to the time evolution, while the second term will be discarded.

Finally, a measurement is performed on the ancillary qubit. If $|0\rangle_a$ is observed, the work qubit $e$ will evolve as $e^{-i\frac{\hat{H}t}{\hbar}}|\psi\rangle_e$ with a successful probability of

$$\frac{1}{2f^2}e^{\langle \psi |e^{i\frac{\hat{H}t}{\hbar}}e^{-i\frac{\hat{H}t}{\hbar}}|\psi \rangle},$$

(24)

decided by both the evolution operator and the initial state. If the ancillary qubit is measured in state $|1\rangle_a$, the result will be discarded and the process will be started over until outputting $|0\rangle_a$. From eqs. (18) and (24), the successful probability using two qubits is higher than that using three qubits. Therefore, it is valuable to reduce the UE terms before quantum simulation to save qubit and increase efficiency, decreasing the complexities and difficulties for experiments.

To simulate a general PT-arbitrary-phase$(\varphi)$–symmetric system $H$ of $d_\varphi$ dimensions, we first expand the non-unitary time evolution operator into $d_\varphi$ UE terms. Thus, a $d_\varphi$-dimensional ancillary qudit is needed to assist the $d_\varphi$ dimensional work qudit to evolve as $e^{-i\frac{\hat{H}t}{\hbar}}$. The quantum circuit can refer to fig. 1 by substituting the work qubit by a $d_\varphi$-dimensional qudit and applying $d_\varphi$ relevant controlled gates in the middle part. The quantum simulation process can also be achieved by a number of log$_2(d_\varphi \cdot d_e)$ qubits at least.

**Experimental proposals.** We propose experimental implementations in qubit quantum devices to achieve the simulation of time evolutions of PT-arbitrary-phase–symmetric two-level systems. Take the nuclear magnetic resonance (NMR) quantum simulator as an example, three nuclei of spin-1/2 take the roles of one work qubit and two ancillary qubits. The spatial-averaging method [66] can be adopted to initialize the pseudo-pure state $|00\rangle_a |0\rangle_e$ at the beginning, and a unitary can be applied on the work qubit to prepare the system into $|00\rangle_a |\psi\rangle_e$. Then, a series of magnetic pulse sequences can realize the relevant quantum gates. Specifically, a single qubit rotation can be realized by hard pulses, whereas a controlled two-qubit
gate can be achieved by the free evolutions of the two nuclei of spin-$\frac{1}{2}$ in a period [14].

Besides, quantum optics can also be chosen to realize the simulation task experimentally. Two orthogonal polarized directions of a photon can be a qubit, and a single-qubit gate can be realized by a series of half-wave plates and quarter-wave plates [67, 68]. It is possible to realize a two-polarization-qubit gate using measurement-induced nonlinearity [69]. However, the efficiency is too low in practice. The efficiency may be improved with the assistance of location degree of freedom [70].

Other qubit systems can be implementation candidates, such as superconductor quantum systems, two energy levels of ultracold atoms, ion-trap systems, etc. The operations in the quantum circuit can be realized by the relevant controlling techniques.

Conclusions. – We extend the PT– and anti-PT–symmetric systems to PT-arbitrary-phase($\varphi$)-symmetric systems for the first time, which can be seen as a complex extension of the pure real (PT) and imaginary (anti-PT) cases. When the phase angle is fixed to $2k\pi$ or $(2k+1)\pi$ ($k$ is the integral), the system will become a PT– or anti-PT–symmetric system. Many concepts of the PT-symmetric quantum theory are still well defined, and novel properties combining the PT and anti-PT symmetries can be expected, e.g., quantum optics, photonics, etc. We then theoretically investigate quantum simulation of the time evolution of a PT-$\varphi$-symmetric two-level system by a conventional Hermitian system. We adopt the unitary expansion technics and LCU method to design the quantum circuits for qubit-qudit hybrid and pure qubit devices. Both proposals are in indeterministic ways, consisting of a work qubit and an ancillary subsystem. The former proposal shows our simulation method clearly, while the latter has more practicality and higher successful probability. In general, three qubits are necessary to achieve the simulation, though only a six-dimensional Hilbert subspace is used. In special cases when the UE parameters meet phase matching conditions, two qubits are enough to achieve the simulation with a higher successful probability. At last, we analyze experimental realizations especially for NMR and quantum optics systems, expecting implementations on small quantum devices.

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Appendix

i) Explicit forms of $f_k$’s ($k = 0, 1, 2, 3$):

\[ f_0 = m [a_0a_0 + i(-a_1b_1 - a_2b_2 + a_3b_3)], \]
\[ f_1 = m [a_0b_1 + i(-a_1b_0 - a_2b_3 + a_3b_2)], \]
\[ f_2 = m [(a_2b_0 + a_3b_1 - a_1b_3) + ia_0b_2], \]
\[ f_3 = m [(a_3b_0 - a_1b_2 + a_2b_1) + ia_0b_3], \]

where
\[ m = e^{-i\frac{\varphi}{2}e^{(\varphi/2)}\cos \theta}, \]
\[ a_0 = \cos \alpha = \cos (\Delta_{nt}/2\hbar), \]
\[ a_1 = 2s \cos (\varphi/2) \sin \alpha/\Delta_n, \]
\[ a_2 = 2w \cos (\varphi/2) \sin \alpha/\Delta_n, \]
\[ a_3 = 2r \cos (\varphi/2) \sin \theta \sin \alpha/\Delta_n, \]
\[ \Delta_n = 2 |\cos (\varphi/2)| \sqrt{s^2 + w^2 - r^2 \sin^2 \theta}; \]

and
\[ b_0 = \cos \beta = \cos (\Delta_{nt}/2\hbar), \]
\[ b_1 = -2s \sin (\varphi/2) \sin \beta/\Delta_n, \]
\[ b_2 = -2w \sin (\varphi/2) \sin \beta/\Delta_n, \]
\[ b_3 = -2r \sin (\varphi/2) \sin \theta \sin \beta/\Delta_n, \]
\[ \Delta_\beta = 2 |\sin (\varphi/2)| \sqrt{r^2 \sin^2 \theta - s^2 + w^2}. \]

ii) Explicit forms of $g_1$ and $g_2$: As in the main text under eq. (6), where $f_k = |f_k|e^{i\theta_k}$ ($k = 0, 1, 2, 3$), $g_1$ and $g_2$ are

\[ g_1 = e^{i\theta_1} \sqrt{|f_1|^2 + |f_2|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}; \]
\[ g_2 = e^{i\theta_2} \sqrt{|f_1|^2 \sin^2 \varphi_1 + |f_2|^2 \cos^2 \varphi_2}, \]

where
\[ \varphi_1 = \theta_1 - \theta_3 \quad \text{and} \quad \varphi_2 = \theta_2 - \theta_3. \]

iii) Explicit forms of $U_1$ and $U_2$:

\[ \begin{bmatrix} \cos \zeta_1 & e^{i\phi_1} \sin \zeta_1 \\ e^{-i\phi_1} \sin \zeta_1 & -\cos \zeta_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & e^{i\phi_2} \\ -e^{-i\phi_2} & 0 \end{bmatrix}, \]

where $\phi_1$ is decided by

\[ \cos \phi_1 = \frac{|f_1| \cos \varphi_1}{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}} \quad \text{and} \quad \sin \phi_1 = \frac{|f_2| \sin \varphi_2}{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}}; \]

and $\phi_2$ is decided by

\[ \cos \phi_2 = \frac{|f_2| \cos \varphi_2}{|g_2|} \quad \text{and} \quad \sin \phi_2 = \frac{|f_1| \sin \varphi_1}{|g_2|}; \]

$\zeta_1$ is decided by

\[ \cos \zeta_1 = \frac{|f_3|}{|g_1|} \quad \text{and} \quad \sin \zeta_1 = \frac{\sqrt{|f_1|^2 \cos^2 \varphi_1 + |f_2|^2 \sin^2 \varphi_2}}{|g_1|}. \]
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