Non-asymptotically AdS/dS Solutions and Their Higher Dimensional Origins

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We look for and analyze in some details some exact solutions of Einstein-Maxwell-dilaton gravity with one or two Liouville-type dilaton potential(s) in an arbitrary dimension. Such a theory could be obtained by dimensionally reducing Einstein-Maxwell theory with a cosmological constant to a lower dimension. These (neutral/magnetic/electric charged) solutions can have a (two) black hole horizon(s), cosmological horizon, or a naked singularity. Black hole horizon or cosmological horizon of these solutions can be a hypersurface of positive, zero or negative constant curvature. These exact solutions are neither asymptotically flat, nor asymptotically AdS/dS. But some of them can be uplifted to a higher dimension, and those higher dimensional solutions are either asymptotically flat, or asymptotically AdS/dS with/without a compact constant curvature space. This observation is useful to better understand holographic properties of these non-asymptotically AdS/dS solutions.

I. INTRODUCTION

Looking for exact solutions of Einstein’s field equations with/without matter source is a subject of long standing interest. Recent years have seen a lot of activity to find and study asymptotically anti de Sitter (AdS) or de Sitter (dS) solutions. For the asymptotically AdS spaces, there are at least two reasons responsible for this. The first is related to the so-called AdS-CFT (conformal field theory) correspondence [1], which states that string/M theory on an AdS space times a compact manifold is dual to a strong coupling conformal field theory residing on the boundary of the AdS space. The radial spatial coordinate in asymptotically AdS spaces can be viewed as the energy scale in renormalization group flow of dual field theory. It has been argued by Witten [2] that the thermodynamics of black hole in AdS space can be identified with that of a certain dual CFT in the high temperature limit. For instance, the Hawking-Page phase transition [3] of black holes in AdS space is identified with the confinement/unconfinement phase transition of CFT’s. Therefore, one can gain some insights into the thermodynamic properties and phase structures of some strong coupling CFT’s by investigating the thermodynamics of AdS black holes.

The other is related to the topology structure of black hole in AdS space. Before the discovery of the AdS-CFT correspondence, it was already recognized that in an asymptotically AdS space, except for the black hole with event horizon being a positive constant curvature sphere, it is also possible to have black holes whose event horizon could be a negative constant or zero curvature hypersurface. These black hole solutions are often referred to as topological black holes in the literature [4]. Various properties associated with these topological black holes have been investigated in recent years. For example, their higher dimensional [5] and charged [6] generalizations and formation via gravitational collapse [7] have been studied. Topological black hole solutions in Lovelock gravity and associated thermodynamics have also been discussed [8]. Even some perturbative solutions in higher derivative gravity have been found and investigated in [9]. In particular, topological black holes in gauged supergravities with nontrivial scalar fields have been also discovered in [10].

In a $D$-dimensional Hilbert-Einstein action with a negative cosmological constant, $\Lambda = -(D-1)(D-2)/2l^2$,

$$ S = \frac{1}{16\pi G_D} \int d^Dx \sqrt{-g} \left( R + \frac{(D-1)(D-2)}{l^2} \right), \tag{1.1} $$

where $G_D$ is the Newton gravitational constant in $D$ dimensions, one has a neutral AdS black hole solution with the metric

$$ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Sigma_{D-2,k}^2, \tag{1.2} $$

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where
\[ f(r) = k - \frac{16\pi G_D M}{(D - 2)Vol(S)} + \frac{r^2}{l^2}, \]  
(1.3)

\( M \) is the mass of the black hole, and \( d\Sigma_{D-2,k}^2 \) represents the line element of a \((D - 2)\)-dimensional hypersurface with constant curvature \((D - 2)(D - 3)k\) and volume \(Vol(S)\). Without loss of generality, the characteristic curvature constant \( k \) can be taken \( \pm 1 \) or 0. When \( k = 1 \), the hypersurface \( \Sigma \) could be a \((D - 2)\)-dimensional round sphere \( S^{D-2} \), while \( k = 0 \), \( \Sigma \) could be a \((D - 2)\)-dimensional torus \( T^{D-2} \). When \( k = -1 \), the hypersurface \( \Sigma \) is a negative constant curvature space. In this case, one can obtain a closed hyperbolic hypersurface \( H^{D-2} \) with high genus through appropriate identification. In the AdS/CFT correspondence, the thermodynamics of black hole (1.1) can be viewed so-called mass bound conjecture in dS space [13] has been investigated in [14].

In a simpler form, the action could be written down as (1.1). On the other hand, let us note that various actions of gauge and supergravities under consistent truncation, include these gravity solutions in the spirit of holography, as the non-conformal generalizations of AdS-CFT correspondence, those (1.2) which are asymptotically AdS or dS. We have studied some aspects [14, 22, 23] of dual field theory to (1.4) have been found in some special cases. These solution are asymptotically neither AdS nor dS, different from (1.2) which are asymptotically AdS or dS. We have studied some aspects [14, 22, 23] of dual field theory to these gravity solutions in the spirit of holography, as the non-conformal generalizations of AdS-CFT correspondence and dS-CFT correspondence. Note that an action like (1.4) can also be obtained by dimensionally reducing Einstein-Maxwell theory with a cosmological constant to a lower dimension. It is certainly worth discussing the relationship between those non-asymptotically AdS/dS solutions of the action (1.4) and asymptotically AdS/dS solutions (1.2), in order to better understand the holographic properties of those non-asymptotically AdS/dS solutions. This is one of the main aims of the present paper [14]. On the other hand, we will look for and analyze in some details, exact solutions in an action like (1.4) with one or two Liouville-type potentials, without any assumption on the relations among the parameters in the theory, which generalizes existing solutions in the literature in various directions.

The organization of the present paper is as follows. In the next section, we first discuss the case where resulting action comes from a higher dimensional Einstein-Maxwell theory with a cosmological constant and the reduced subspace is Ricci flat. In this case, the resulting action is of the form (1.4). In Sec. III, we investigate the case where the reduced subspace is a nonzero constant curvature space, where resulting action has two Liouville-type potentials. The conclusions and some discussions are presented in Sec. IV.

II. DIMENSIONAL REDUCTION: RICCI FLAT CASE

A. Neutral solution

In this subsection, we first discuss the case without the Maxwell field. Let us start with a \((D + d)\)-dimensional Hilbert-Einstein action with a cosmological constant \( \Lambda_0 \),
\[ S = \frac{1}{16\pi G} \int d^{D+d}x \sqrt{-g} (R - 2\Lambda_0), \]  
(2.1)
where $G$ is the Newton gravitational constant in $D + d$ dimensions, $\tilde{g}$ and $\tilde{R}$ denote the $(D + d)$-dimensional metric determinant and curvature scalar, respectively. Consider the $(D + d)$-dimensional metric line element having the following form
\begin{equation}
\tag{2.2}
ds_{D+d}^2 = e^{2\alpha(x)} ds_D^2 + e^{2\beta(x)} ds_d^2,
\end{equation}
where $ds_d^2 = g_{ij}dy^idy^j$ denotes a $d$-dimensional constant curvature space with scalar curvature $d(d-1)k_d$. In this section, we consider the case $k_d = 0$. Namely, $g_{ij} = \delta_{ij}$, and then $ds_d^2$ is a $d$-dimensional Euclidean flat space. In addition, $\alpha$ and $\beta$ are two functions of coordinate $x$ in the subspace described by $ds_D^2$. Making dimensional reduction along $y$, and taking $\beta = (2 - D)\alpha/d$, we obtain an action from (2.1)
\begin{equation}
S = \frac{V_g}{16\pi G} \int d^Dx \sqrt{-g} \left( R - \frac{(D-2)(D+d-2)}{d} (\partial\alpha)^2 - 2\Lambda_0 e^{2\alpha} \right),
\end{equation}
where $V_g$ is the volume of the subspace described by $ds_D^2$ and $R$ is the curvature scalar of line element $ds_D^2$. Thus $G/V_g$ stands for the effective Newton gravitational constant in $D$ dimensions here. Defining
\begin{equation}
\alpha = \sqrt{\frac{d}{2(D-2)(D+d-2)}} \phi,
\end{equation}
the kinetic term of the scalar field is changed to have a canonical form
\begin{equation}
S = \frac{V_g}{16\pi G} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} (\partial\phi)^2 - 2\Lambda_0 e^{\alpha\phi} \right),
\end{equation}
where
\begin{equation}
\alpha = \sqrt{\frac{2d}{(D-2)(D+d-2)}}.
\end{equation}
The action (2.5) is just the one (2.3) with $F_2 = 0$. Varying the action (2.5) yields equations of motion
\begin{equation}
R_{\mu\nu} = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{2}{D-2} \Lambda_0 e^{\alpha\phi} g_{\mu\nu},
\end{equation}
\begin{equation}
\nabla^2 \phi - 2a\Lambda_0 e^{\alpha\phi} = 0.
\end{equation}
Now we solve the equations of motion (2.7). In the process of solving these equations, for the sake of generality, we only assume that $a$ is a positive constant (this assumption always holds because it can be made positive via $\phi \rightarrow -\phi$ if it was negative) without imposing the condition (2.6). Suppose that the metric is of the form
\begin{equation}
ds_D^2 = A(r)dt^2 + A(r)^{-1}dr^2 + R(r)^2 h_{mn}dx^m dx^n,
\end{equation}
where $h_{mn}dx^m dx^n$ denotes the line element of a $(D-2)$-dimensional hypersurface with constant curvature $(D-2)(D-3)k$ and volume $V_h$. Corresponding to the metric (2.8), we can write the equations of motion as
\begin{equation}
R'_t = -\frac{A''}{2} - (D-2)\frac{A'R'}{2R} = \frac{2}{D-2} \Lambda_0 e^{\alpha\phi},
\end{equation}
\begin{equation}
R'_r = -\frac{A''}{2} + (D-2)\frac{A'R'}{2R} - (D-2)\frac{AR''}{R} = \frac{1}{2} A\phi'^2 + \frac{2}{D-2} \Lambda_0 e^{\alpha\phi},
\end{equation}
\begin{equation}
R''_m = \delta^m_n \left\{ \frac{D-3}{R^2} k - \frac{1}{(D-2)R^{D-2}} [A(R^{D-2})']' \right\} = \frac{2}{D-2} \Lambda_0 e^{\alpha\phi} \delta^m_n,
\end{equation}
\begin{equation}
\frac{1}{R^{D-2}} (R^{D-2} A\phi')' - 2a\Lambda_0 e^{\alpha\phi} = 0,
\end{equation}
where a prime stands for the derivative with respect to $r$. From (2.9) and (2.10), one has
\begin{equation}
(D-2) \frac{R''}{R} = -\frac{1}{2} \phi'^2.
\end{equation}
Setting \( R = r^N \), where \( N \) is a constant to be determined shortly, one then has
\[
\phi'^2 = -2N(N - 1)(D - 2)/r^2.
\] (2.14)

Integrating this and substituting into Eqs. (2.9) - (2.12), we find solutions to the equations (2.9) - (2.12), depending on the characteristic curvature constant \( k \).

(1) When \( k = 0 \), we have
\[
A(r) = \frac{M}{r^{(D-2)N-1}} - \frac{2\Lambda_0 e^{\phi_0} r^{2N}}{N(ND - 1)(D - 2)},
\] \( \phi = \phi_0 - \sqrt{2N(1 - N)(D - 2)\ln r}, \quad R = r^N, \)
\[ N = \frac{2}{2 + a^2(D - 2)}. \] (2.15) (2.16)

where \( M \) and \( \phi_0 \) are two integration constants. The solution (2.15) has a singularity at \( r = 0 \) only. From (2.14) and (2.16), one can see that \( 0 < N < 1 \). When \( a = 0 \), one has \( N = 1 \) and the \( \phi \) is then a constant. In this case, the action (2.14) reduces to (1.1) and the solution (2.15) becomes (1.2) with \( k = 0 \).

We are of course interested in the case of \( a \neq 0 \) and \( \Lambda_0 < 0 \). In this case, the dilaton potential in (2.5) could be viewed as a negative effective cosmological constant. (i) When \( 1/D < N < 1 \), the second term in (2.15) is positive and dominant as \( r \to \infty \). The solution therefore describes a black hole, provided \( M > 0 \), with a conformal Ricci flat horizon \( r_+ \) determined by \( A(r)|_{r = r_+} = 0 \). Applying the quasilocal mass formulism of gravitational configurations developed in (2.16) to our case and considering the solution with \( M = 0 \) as the reference background, the quasilocal mass of the black hole is found to be (see also [22])
\[
M = \frac{(D - 2)NV_q V_h M}{16\pi G}.
\] (2.17)

Note that here the effective Newton gravitational constant is \( G/V_q \). Obviously, if \( M < 0 \), the singularity at \( r = 0 \) becomes naked. (ii) When \( 0 < N < 1/D \), the second term is negative and the first term in (2.15) is dominant as \( r \to \infty \). In this case, the solution has a black hole horizon \( r_+ \) provided \( M < 0 \) (which implies that the gravitational mass of the black hole is negative). The horizon is also determined by \( A(r)|_{r = r_+} = 0 \) and is a conformal Ricci flat hypersurface. When \( M > 0 \), the singularity at \( r = 0 \) becomes naked, again.

For completeness, we also analyze the case of \( \Lambda_0 > 0 \) here. In this case, the dilaton potential in (2.5) can be regarded as a positive effective cosmological constant. (i) When \( 1/D < N < 1 \), the second term in (2.15) is always negative and dominant as \( r \to \infty \), therefore there is a cosmological horizon provided \( M < 0 \). The solution describes a cosmological solution with a conformal Ricci flat cosmological horizon \( r_c \) determined by \( A(r)|_{r = r_c} = 0 \), and the singularity at \( r = 0 \) is a cosmological singularity. The gravitational mass of the solution is found to be (see also [14])
\[
M = \frac{- (D - 2)NV_q V_h M}{16\pi G},
\] (2.18)
in the sense of [13]. When \( M > 0 \), the solution describes a naked singularity spacetime. This solution was first found in [14] and the implication of the solution in the sense of dS/CFT correspondence was discussed there. (ii) When \( 0 < N < 1/D \), the second term is always positive and the first term is dominant as \( r \to \infty \). The solution has a cosmological horizon if \( M > 0 \), otherwise describes a naked singularity. The property of the solution is summarized in Tab. I.

TABLE I: The property of the solution (2.4) for different parameters. Here “BH” means that the outmost horizon of the solution is a black hole horizon, “CH” a cosmological horizon, and “NK” stands for that the singularity at \( r = 0 \) is naked. The same notation will be used below.

| \( \Lambda_0 \) | \( < 0 \) | \( < 0 \) | \( < 0 \) | \( < 0 \) | \( > 0 \) | \( > 0 \) | \( > 0 \) | \( > 0 \) |
|---|---|---|---|---|---|---|---|---|
| \( N \) | \( 0 < N < 1/D \) | \( 0 < N < 1/D \) | \( 1/D < N < 1 \) | \( 1/D < N < 1 \) | \( 1/D < N < 1 \) | \( 1/D < N < 1 \) | \( 0 < N < 1/D \) | \( 0 < N < 1/D \) |
| \( M \) | \( < 0 \) | \( > 0 \) | \( > 0 \) | \( < 0 \) | \( < 0 \) | \( > 0 \) | \( > 0 \) | \( < 0 \) |
| Solution | BH | NK | BH | NK | CH | NK | CH | NK |

Note that regardless of the values of parameters \( M \), \( \Lambda_0 \) and \( N \), the spacetime (2.4) with (2.15) and (2.16) is neither asymptotically dS, nor AdS. Now let us observe an interesting consequence if the parameter \( a \) in (2.16) has a relation...
In this case, one has \( N = (D + d - 2)/(D + 2d - 2) \), which obeys \( 1/2 < N < 1 \). We can uplift the \( D \)-dimensional, non-asymptotically AdS/dS solution (2.20) to the one (2.21) in \((D + d)\) dimensions and obtain
\[
ds^2_{D+d} = e^{\alpha\phi}r^{-2N(D + d - 2)} \left( -\mathcal{A} dt^2 + A^{-1} dr^2 + r^2 (\delta_{mn} dx^m dx^n + \delta_{ij} dy^i dy^j) \right) + e^{(D - 2)\alpha\phi / d} \delta_{ij} dy^i dy^j.
\]

Redefining \( r^{(D - 2)/(D + 2d - 2)} \to r \) and rescaling coordinates and the integration constant \( \mathcal{M} \), we find that the solution (2.20) can be rewritten as
\[
ds^2_{D+d} = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (\delta_{mn} dx^m dx^n + \delta_{ij} dy^i dy^j),
\]
with
\[
f(r) = -\frac{\mathcal{M}}{r^{D - 2N}} - \frac{2\Lambda_0 r^2}{(D + d - 1)(D + d - 2)}.
\]

This is nothing but the AdS black hole solution (1.2) with \( k = 0 \) in \((D + d)\) dimensions if \( \Lambda_0 < 0 \) and \( \mathcal{M} > 0 \). On the other hand, if \( \Lambda_0 > 0 \) and \( \mathcal{M} < 0 \), the solution is a topological dS solution with \( k = 0 \) in (2.19). Clearly the solution (2.21) is asymptotically AdS/dS. Therefore some of non-asymptotically AdS/dS solutions (2.15) can be understood as the dimensional reduction of a higher dimensional, asymptotically AdS/dS solution (2.20). Certainly this observation is useful to understand the holography of the non-asymptotically AdS/dS solutions (2.15).

When \( k \neq 0 \), the solution of (2.21) is found to be
\[
A(r) = -\frac{\mathcal{M}}{r^{(D - 2)N - 1}} + \frac{(D - 3)kr^{2 - 2N}}{(2N - 1)(N(D - 4) + 1)},
\]
\[
\phi = \phi_0 - \sqrt{2N(1 - N)(D - 2) \ln r}, \quad R = r^N,
\]
\[
N = \frac{a^2(D - 2)}{2 + a^2(D - 2)},
\]
\[
2\Lambda_0 = -\frac{(D - 3)(D - 2)(1 - N)ke^{-\alpha\phi}}{2N - 1}.
\]

It might be worth noticing that this set of solution with \( k \neq 0 \) does not reduce to the one (2.15) and (2.16) when \( k \to 0 \). This is caused that in the process of solving the equations (2.9), we use of \( R = r^N \) and \( \phi \) given by (2.12), we can obtain a general solution of \( A \) through (2.11). In order the equation (2.4) to hold, we have to have two constraints if \( k \neq 0 \), which result in \( N \) and \( \Lambda_0 \) in (2.15). If \( k = 0 \), there is only one constraint, giving us \( N \) in (2.10). Therefore within the ansatz of \( R = r^N \), the solutions of \( k = 0 \) and \( k \neq 0 \) belong to different branches. In addition, note that due to \( 0 < N < 1 \), the second term in (2.21) therefore is always dominant as \( r \to \infty \).

When \( k = 1 \), one can see from (2.22) that the cosmological constant \( \Lambda_0 \) is negative as \( 1/2 < N < 1 \). In this case, the solution (2.21) describes a black hole with mass (2.14) provided \( \mathcal{M} > 0 \), whose horizon is determined by \( A(r)|_{r=r_+} = 0 \), and the horizon is a \((D - 2)\)-dimensional, positive constant curvature hypersurface. If \( M < 0 \), it is a naked singularity solution. When \( 0 < N < 1/2 \), the cosmological constant has to be positive. In that case, the solution has a cosmological horizon determined by \( A(r)|_{r=r_+} = 0 \) provided \( \mathcal{M} > 0 \). Otherwise, it is a naked singularity solution. The gravitational mass is given by (2.13). This solution generalizes the topological dS solutions in (1.1). The properties of the solution are summarized in Tab. II, where the cases of naked singularity are not included.

| \( k \) | \( N \) | \( \Lambda_0 \) | Solution |
|---|---|---|---|
| \( 1 \) | \( 0 < N < 1/2 \) | \( > 0 \) | CH (\( \mathcal{M} \geq 0 \)) |
| \( 1 \) | \( 1/2 < N < 1 \) | \( < 0 \) | BH (\( \mathcal{M} > 0 \)) |
| \( -1 \) | \( 0 < N < 1/2 \) | \( < 0 \) | BH (\( \mathcal{M} > 0 \)) |
| \( -1 \) | \( 1/2 < N < 1 \) | \( > 0 \) | CH (\( \mathcal{M} < 0 \)) |

Where \( a \) takes the value of (2.4), we can uplift the solution (2.21)-(2.22) to the one in \((D + d)\)-dimensions. In that case, one has \( N = d/(D + 2d - 2) < 1/2 \). Namely we have to take a positive cosmological constant \( \Lambda_0 \) in (2.22). Upon lifting, we obtain
\[
ds^2_{D+d} = e^{\alpha\phi}r^{-2N} \left( -A(r) dt^2 + A^{-1}(r) dr^2 + e^{-(D - 2)\alpha\phi / d} r^{2N(D - 2)/d} \delta_{ij} dy^i dy^j + e^{\alpha\phi} h_{mn} dx^m dx^n \right).
\]
Redefining $r^{N(D-2)/N} \to r$ and rescaling the coordinates and the integration constant $\mathcal{M}$, the solution can be rewritten as

$$ds^2_{D+d} = -f(r)dt^2 + \frac{2\Lambda_0 r^2}{(d+1)(D+d-2)} dr^2 + r^2 \delta_{ij} dy^i dy^j + B^2 h_{mn} dx^m dx^n,$$

(2.24)

where

$$f(r) = \frac{\mathcal{M}}{r^{d-1}} - \frac{2\Lambda_0 r^2}{(d+1)(D+d-2)}, \quad B^2 = \frac{(D-3)(D+d-2)}{2\Lambda_0},$$

and the scalar curvature of the $(D-2)$-dimensional positive constant curvature space $h_{mn} dx^m dx^n$ has been normalized to $(D-2)(D-3)$. If $\mathcal{M} < 0$, this solution is nothing but a $(d+2)$-dimensional topological $dS$ solution with a conformal Ricci flat cosmological horizon times a $(D-2)$-dimensional compact space with a positive constant curvature $2(D-2)\Lambda_0/(D+d-2)$. It is easy to check that the solution (2.24) satisfies the $(D+d)$-dimensional Einstein’s field equations with a positive cosmological constant $\Lambda_0$.

(3). When $k = -1$, one can see from (2.24) that the cosmological constant is positive if $1/2 < N < 1$. In this case, the second term is negative and the solution (2.24) has a cosmological horizon which is a negative constant curvature hypersurface provided $\mathcal{M} < 0$. The gravitational mass is given by (2.18). The solution generalizes the one in [14] to the case $k = -1$. If $\mathcal{M} > 0$, the solution describes a naked singularity. On the other hand, when $0 < N < 1/2$, the cosmological constant has to be negative. In this case, the second term in (2.24) is positive and the solution describes a black hole with a negative constant curvature horizon provided $\mathcal{M} > 0$. The black hole mass is given by (2.17). If $\mathcal{M} < 0$, the singularity at $r = 0$ becomes naked. The causal structure of the solution is summarized in Tab. II.

When $a$ takes the value of (2.20), we have $N = d/(D+2d-2) < 1/2$. Namely, in this case, one has to have a negative cosmological constant. One can uplift the solution to $(D+d)$ dimensions. As the case of $k = 1$, we find that the $(D+d)$-dimensional solution can be rewritten as

$$ds^2_{D+d} = -f(r)dt^2 + \frac{2\Lambda_0 r^2}{(d+1)(D+d-2)} dr^2 + r^2 \delta_{ij} dy^i dy^j + B^2 h_{mn} dx^m dx^n,$$

(2.25)

where

$$f(r) = \frac{\mathcal{M}}{r^{d-1}} + \frac{2|\Lambda_0| r^2}{(d+1)(D+d-2)}, \quad B^2 = \frac{(D-3)(D+d-2)}{2|\Lambda_0|}.$$

Note that here $h_{mn} dx^m dx^n$ denotes a $(D-2)$-dimensional negative constant curvature space with scalar curvature $-(D-2)(D-3)$. It is easy to see that the solution (2.25) describes a $(d+2)$-dimensional AdS black hole with a conformal Ricci flat horizon times a $(D-2)$-dimensional negative constant curvature space. As the case of (2.24), the solution (2.25) obeys the $(D+d)$-dimensional Einstein’s field equations with a negative cosmological constant $\Lambda_0$.

### B. Charged solution

Next let us add a term of Maxwell field to the action (2.4),

$$S = \frac{1}{16\pi G} \int d^{D+d}x \sqrt{-g} \left( R - 2\Lambda_0 - \frac{1}{4} F_2^2 \right).$$

(2.26)

We consider magnetic charged solutions of the action (2.26). Then as the case without the Maxwell field, one can make a dimensional reduction according to (2.24) and obtain

$$S = \frac{V_q}{16\pi G} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - 2\Lambda_0 e^{a \phi} - \frac{1}{4} e^{-b \phi} F_2^2 \right),$$

(2.27)

which has the same form as (1.3). Here $\phi$ has the same relation to $\alpha$ as the case without the Maxwell field (2.1), and

$$b = a = \sqrt{\frac{2d}{(D-2)(D+d-2)}}.$$

(2.28)

Varying the action (2.27) yields the equations of motion

$$R_{\mu \nu} = \frac{1}{2} (\partial_\mu \phi)(\partial_\nu \phi) + \frac{2}{D-2} \Lambda_0 e^{a \phi} g_{\mu \nu} + \frac{1}{2} e^{-b \phi} \left( F_{\mu \lambda} F_{\nu}{}^\lambda - \frac{1}{2(D-2)} F_2^2 g_{\mu \nu} \right),$$

$$\partial_\mu (\sqrt{-g} e^{-b \phi} F^{\mu \nu}) = 0, \quad F_{\mu \nu, \lambda} + F_{\nu \lambda, \mu} + F_{\lambda \mu, \nu} = 0,$$

$$\nabla^2 \phi - 2a\Lambda_0 e^{a \phi} + \frac{b}{4} e^{-b \phi} F_2^2 = 0.$$

(2.29)
Now we find magnetic charged solution of action (2.27). In the process of solving these equations, we just keep $a$ and $b$ as two constants, and do not assume that they have a relation of (2.28). In addition, for completeness we will also find electric charged solutions of action (2.27).

1. Magnetic charged solution

Here it should be mentioned that for magnetic charged solutions in (2.26), the dimensional reduction from (2.26) to (2.27) should go to $D = 4$. With the ansatz (2.8), we find several sets of solutions, which depend on the characteristic curvature $k$.

(a) When $k = 0$, we find two sets of solutions to (2.24).

- **Solution 1**

  \[
  A(r) = -\frac{\mathcal{M}}{r^{2N-1}} - \frac{\Lambda_0 e^{a\phi_0}}{N(4N-1)} + \frac{Q^2 e^{-b\phi_0}}{4N r^{2N}},
  \]

  \[
  \phi = \phi_0 - \sqrt{4N(1-N)} \ln r, \quad R = r^N, \quad N = \frac{1}{1+a^2}, \quad b = -a,
  \]

  \[
  F_{mn} = Q \epsilon_{mn},
  \]

  where $\epsilon_{mn}$ is the volume element of $h_{mn} dx^m dx^n$ and $Q$ is an integration constant related to the magnetic charge $Q$ via $Q = Q V_0 / 4\pi$. (i) When $0 < N < 1/4$, the first term in (2.30) is dominant as $r \to \infty$. Therefore there is a cosmological horizon if $\mathcal{M} > 0$ regardless of the sign of $\Lambda_0$ provided $Q \neq 0$ (here and after when a magnetic/electric charge is present, $Q \neq 0$ is always assumed), while a black hole appears if $\mathcal{M} < 0$ and $\Lambda_0 < 0$. In other cases, the singularity at $r = 0$ is naked. (ii) when $1/4 < N < 1$, the second term in (2.30) is always dominant as $r \to \infty$. Black hole horizon will appear if $\mathcal{M} > 0$ and $\Lambda_0 < 0$. In this case, the solution may have two or one black hole horizon or a naked singularity depending on the parameters $\mathcal{M}$, $Q$ and $\Lambda_0$. For instance, if $N = 1/2$, we have two black hole horizons

  \[
  r_\pm = \frac{e^{-\alpha \phi_0}}{2|\Lambda_0|} \left( \mathcal{M} \pm \sqrt{\mathcal{M}^2 - 2|\Lambda_0| Q^2 e^{2\alpha \phi_0}} \right)
  \]

  provided $\mathcal{M}^2 - 2|\Lambda_0| Q^2 e^{2\alpha \phi_0} > 0$. If $\Lambda_0 > 0$, a cosmological horizon occurs again in spite of the sign of $\mathcal{M}$. The causal structure of the solution (2.30) is summarized in Tab. III. Due to the relation $b = -a$ for this solution, which does not satisfy $b = a$ of (2.28). Therefore this solution cannot be uplifted to the case of $D + d$ dimensions according to (2.2) unless $Q = 0$. In the case of $Q = 0$, it results in the solution (2.20).

- **Solution 2**

  \[
  A = -\frac{\mathcal{M}}{r^{2N-1}} + \frac{Q^2 e^{-b\phi_0}}{2(1-2N)},
  \]

  \[
  \phi = \phi_0 - \sqrt{4N(1-N)} \ln r, \quad R = r^N,
  \]

  \[
  N = \frac{a^2}{1+a^2}, \quad b = a,
  \]

  \[
  Q^2 e^{-b\phi_0} = 4(2N-1)\Lambda_0 e^{a\phi_0},
  \]

  The second term in (2.32) is always dominant as $r \to \infty$. The solution describes a black hole when $0 < N < 1/2$ and $\mathcal{M} > 0$. When $1/2 < N < 1$ and $\mathcal{M} < 0$, the solution has a cosmological horizon. In other cases, the singularity at $r = 0$ is naked. This solution is a special case of the solution 3 given by (2.28) below.

**TABLE III:** The property of the solution (2.30) for different parameters.

| $N$   | $0 < N < 1/4$ | $0 < N < 1/4$ | $1/4 < N < 1$ | $1/4 < N < 1$ | $\Lambda_0$ |
|-------|---------------|---------------|---------------|---------------|-----------|
|       | $< 0$         | $< 0$         | $> 0$         | $> 0$         |           |
| $\Lambda_0$ | CH ($\mathcal{M} > 0$) | BH ($\mathcal{M} < 0$) | BH ($\mathcal{M} > 0$) | CH |

- **Solution 2**
(2). When \( k \neq 0 \), we find three sets of solutions to the equations of motion \(2.29\).

- Solution 1.

\[
A = -\frac{\mathcal{M}}{r^{2N-1}} + \frac{k}{2N-1}r^{2-2N} + \frac{\mathcal{Q}^2 e^{-b\phi_0}}{4N r^{2N}},
\]

\[
\phi = \phi_0 - \sqrt{4N(1-N)} \ln r, \quad R = r^N,
\]

\[
N = \frac{a^2}{1+a^2}, \quad b = -\frac{1}{a},
\]

\[
\Lambda_0 = -\frac{(1-N)ke^{-\phi_0}}{2N-1},
\]

\[
F_{mn} = Q \epsilon_{mn}.
\]

The second term in \(2.34\) is always dominant as \( r \to \infty \). The behavior of the solution depends on the characteristic curvature \( k \). The properties of the solution are summarized in Tab. IV.

When \( k = 1 \), the causal structure is as follows. (i) When \( 0 < N < 1/2 \), the cosmological constant is positive; the second term is negative. Therefore there will be a cosmological horizon regardless of the sign of the mass parameter \( \mathcal{M} \). (ii) When \( 1/2 < N < 1 \), the cosmological constant has to be negative. In that case, black hole horizon appears when \( \mathcal{M} > 0 \). The mass of black hole mass is given by \(2.17\). If \( \mathcal{M} < 0 \), the singularity at \( r = 0 \) becomes naked. In this case, the black hole (cosmological) horizon is a positive constant curvature hypersurface.

When \( k = -1 \), the solution describes a black hole if \( 0 < N < 1/2 \) and \( \mathcal{M} > 0 \). On the other hand, in the case of \( 1/2 < N < 1 \), the cosmological constant is positive and the second term is always negative. Therefore there is a cosmological horizon. Here the black hole (cosmological) horizon is a negative constant curvature hypersurface.

This solution cannot be uplifted to \((D + d)\) dimensions according to \(2.2\) unless \( \mathcal{Q} = 0 \) or \( \Lambda_0 = 0 \). The case of \( \mathcal{Q} = 0 \) results in the higher dimensional solution \(2.24\) or \(2.26\) with \( D = 4 \), while the higher dimensional solution is included in \(2.11\) as a special case when \( \Lambda_0 = 0 \).

| \( k \) | \( N \) | \( \Lambda_0 \) | Solution |
|---|---|---|---|
| 1 | \( 0 < N < 1/2 \) | \( > 0 \) | CH |
| 1 | \( 1/2 < N < 1 \) | \( < 0 \) | BH (\( \mathcal{M} > 0 \)) |
| -1 | \( 0 < N < 1/2 \) | \( < 0 \) | BH (\( \mathcal{M} > 0 \)) |
| -1 | \( 1/2 < N < 1 \) | \( > 0 \) | CH |

- Solution 2.

\[
A = -\frac{\mathcal{M}}{r^{2N-1}} - \frac{\Lambda_0 e^{a\phi_0}}{N(4N-1)} r^{2N} + \frac{k}{r^{2N-2}},
\]

\[
\phi = \phi_0 - \sqrt{4N(1-N)} \ln r, \quad R = r^N,
\]

\[
N = \frac{1}{1+a^2}, \quad b = -\frac{Na}{1-N} = \frac{1}{a},
\]

\[
\mathcal{Q}^2 = 4(1-N)ke^{b\phi_0},
\]

\[
F_{mn} = Q \epsilon_{mn}.
\]

This solution is interesting. It follows from \(2.30\) that \( k > 0 \) since \( \mathcal{Q}^2 > 0 \). From the expression \(2.30\), we can see that the third term in \(2.36\) is dominant as \( 0 < N < 1/2 \), while the domination term is the second term if \( 1/2 < N < 1 \) as \( r \to \infty \). (i) When \( 0 < N < 1/4 \), the solution always describes a black hole in spite of the sign of the parameter \( \mathcal{M} \) if \( \Lambda_0 < 0 \); if \( \Lambda_0 > 0 \), one has to have \( \mathcal{M} > 0 \) in order to have a black hole horizon. In other cases the solution describes a naked singularity. (ii) When \( 1/4 < N < 1/2 \), the solution also describes a black hole in spite of the sign of \( \mathcal{M} \) if \( \Lambda_0 > 0 \). However, if \( \Lambda_0 < 0 \), the mass parameter has to be positive, \( \mathcal{M} > 0 \). (iii) When \( 1/2 < N < 1 \), the solution has a cosmological horizon if \( \Lambda_0 > 0 \) in spite of the sign of \( \mathcal{M} \); if \( \Lambda_0 < 0 \), it is a black hole solution provided \( \mathcal{M} > 0 \). In other cases, the singularity at \( r = 0 \) is naked. The result is summarized in Tab. V. The solution \(2.30\) cannot be uplifted to a higher dimension according to \(2.2\) unless \( \mathcal{Q} = 0 \) or \( \Lambda_0 = 0 \). For the former, the uplifting solution is just the one \(2.24\) with \( D = 4 \). On the other hand, if \( \Lambda_0 = 0 \), the resulting higher dimensional solution is included in \(2.41\) below.
TABLE V: The property of the solution for different parameters.

| $N$ | $0 < N < 1/4$ | $0 < N < 1/4$ | $1/4 < N < 1/2$ | $1/4 < N < 1/2$ | $1/2 < N < 1$ | $1/2 < N < 1$ |
|-----|---------------|---------------|-----------------|-----------------|----------------|----------------|
| $\Lambda_0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ |

Solution BH ($M > 0$) BH BH BH ($M > 0$) CH BH ($M > 0$)

- Solution 3.

\[
A = -\frac{M}{r^{2N-1}} - \frac{\Lambda_0 e^a \phi_0}{1 - N} r^{2-2N} + \frac{Q^2 e^{-b \phi_0}}{4(1 - N)} r^{-2N},
\]

\[
\phi = \phi_0 - \sqrt{4N(1 - N)} \ln r, \quad R = r^N,
\]

\[
N = \frac{a^2}{1 + a^2}, \quad b = a,
\]

\[
Q^2 e^{-b \phi_0} = 4(1 - N)k + 4\Lambda_0(2N - 1)e^{a \phi_0},
\]

\[
F_{mn} = Q\epsilon_{mn}
\]

(2.38)

The solution (2.38) can be reexpressed as

\[
A = -\frac{M}{r^{2N-1}} + \frac{1}{2N - 1} \left( k - \frac{Q^2 e^{-b \phi_0}}{2} \right) r^{-2N}.
\]

(2.40)

The second term in (2.40) is always dominant as $r \rightarrow \infty$. Therefore when the coefficient in front of $r^{2-2N}$ is positive, the solution describes a black hole if $M > 0$, otherwise a naked singularity. On the other hand, when the coefficient in front of $r^{2-2N}$ is negative, the solution has a cosmological horizon if $M < 0$, otherwise it is a naked singularity solution. When $k = 0$ this solution reduces to the solution 2 for the case of $k = 0$.

Note that this set of solutions has $b = a$, satisfying the relation of (2.28). Therefore we can uplift this solution to the case in $(D + d)$ dimensions according to (2.2). Redefining $r^{1/(d+1)} \rightarrow r$ and rescaling coordinates and parameters $M$ and $Q$, the resulting solution can be expressed as

\[
dS^2_{4+d} = -fdt^2 + f^{-1}dr^2 + r^2 \delta_{ij}dy^i dy^j + B^2 h_{mn} dx^m dx^n,
\]

\[
F_{mn} = Q\epsilon_{mn},
\]

(2.41)

where

\[
f = -\frac{M}{r^{d-1}} - \frac{2\Lambda_{\text{eff}}}{d(d + 1)} r^2,
\]

\[
B^2 = \frac{1}{4\Lambda_0} \left( (d + 2)k \pm \sqrt{(d + 2)^2k^2 - 4\Lambda_0(d + 1)Q^2} \right),
\]

\[
\Lambda_{\text{eff}} = \frac{d}{d + 2} \Lambda_0 - \frac{d}{4(d + 2)} \frac{Q^2}{B^4},
\]

(2.42)

and $h_{mn} dx^m dx^n$ denotes a two-dimensional space with constant curvature $2k$. The solution is valid only when $B^2 > 0$. To our best knowledge, this solution is new. The solution (2.41) is nothing but a $(d + 2)$-dimensional asymptotically AdS black hole (dS) solution with conformal Ricci flat black hole (cosmological) horizon, times a two dimensional space with constant curvature $2k$ with $M > 0$. One can see from (2.42) that the effective cosmological constant depends on the original cosmological constant, magnetic charge $Q$ and the curvature $k$. When $M = 0$ and $\Lambda_{\text{eff}} < 0$, this is a spacetime $AdS_{d+2} \times S^2$, known as a Freund-Rubin type solution.

2. Electric charged solution

For completeness, here we present and analyze the electric charged solution to equations (2.24). For electric charged solution we need not restrict to $D = 4$. Of course, the solutions presented below cannot be uplifted to $(D + d)$ dimensions according to (2.2).

When $k = 0$, we find two sets of solutions.
• Solution 1.

\[ A = - \frac{M}{r^{(D-2)N-1}} - \frac{2\Lambda_0 e^{a\phi_0}r^{2N}}{N(D-2)(DN - 1)} + \frac{Q^2 e^{b\phi_0}}{2N(D-2)((D-4)N + 1)r^{2(D-3)N}}, \]

\[ \phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r, \quad R = r^N, \]

\[ N = \frac{2}{2 + (D-2)a^2}, \quad b = a, \]

\[ F_{tr} = \frac{Q e^{b\phi_0}}{r^{(D-4)N+2}}, \]

(2.43)

where \( Q \) is an integration constant related to the electric charge \( Q \) of the solution via \( Q = QV_h/4\pi \). The metric function \( A \) is dominant by the first term if \( 0 < N < 1/D \), while by the second term if \( 1/D < N < 1 \) as \( r \to \infty \).

(1) When \( 0 < N < 1/D \), the case of \( \Lambda_0 < 0 \) describes a black hole spacetime if \( M > 0 \), and a naked singularity for \( M < 0 \). The case of \( \Lambda_0 > 0 \) has a cosmological horizon if \( M > 0 \), and the singularity becomes naked if \( M < 0 \). (ii) When \( 1/D < N < 1 \), if the cosmological constant is positive, the solution has a cosmological horizon regardless of the sign of \( M \). On the other hand, if the cosmological constant is negative, the solution describes a black hole as \( M > 0 \), otherwise a naked singularity, once again. The result is summarized in Tab. VI. The cases of naked singularity are not included there. In addition, we mention here that due to the electric charge, it is possible to have two black hole horizons.

**TABLE VI: The property of the solution (2.43) for different parameters.**

| \( N \) | \( 0 < N < 1/D \) | \( 0 < N < 1/D \) | \( 1/D < N < 1 \) | \( 1/D < N < 1 \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| \( \Lambda_0 \) | > 0 | < 0 | > 0 | < 0 |
| Solution | CH (\( M > 0 \)) | BH (\( M < 0 \)) | CH | BH (\( M > 0 \)) |

• Solution 2.

\[ A = - \frac{M}{r^{(D-2)N-1}} + \frac{Q^2 e^{b\phi_0}r^{2-2N}}{2(1 - 2N)((D-4)N + 1)}, \]

\[ \phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r, \quad R = r^N, \]

\[ N = \frac{(D-2)a^2}{2 + (D-2)a^2}, \quad b = -(D-3)a, \]

\[ 2\Lambda_0 e^{a\phi_0} = \frac{N(D-4) + 1}{2(2N-1)} Q^2 e^{b\phi_0}, \]

\[ F_{tr} = \frac{Q e^{b\phi_0}}{r^{N(4-D)}}. \]

(2.46)

The metric function \( A \) is always dominant by the second term as \( r \to \infty \). Therefore when \( 0 < N < 1/2 \), the cosmological constant must be negative, and the solution has a black hole horizon if \( M > 0 \), otherwise it is a naked singularity. On the other hand, when \( 1/2 < N < 1 \), the cosmological constant should be positive. In this case, the solution has a cosmological horizon if \( M < 0 \), otherwise, the singularity at \( r = 0 \) becomes naked, again. The property of the solution is summarized in Tab. VII.

**TABLE VII: The property of the solution (2.45) for different parameters.**

| \( N \) | \( 0 < N < 1/2 \) | \( 0 < N < 1/2 \) | \( 1/2 < N < 1 \) | \( 1/2 < N < 1 \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| \( \Lambda_0 \) | > 0 | < 0 | > 0 | < 0 |
| \( M \) | > 0 | < 0 | > 0 | < 0 |
| Solution | BH | NK | NK | CH |

When \( k \neq 0 \), we find three sets of solutions. They are
• Solution 1.

\[ A = -\frac{\mathcal{M}}{r^{(D-2)N-1}} + \frac{(D-3)kr^2-2N}{2(N-1)((D-4)N+1)} + \frac{Q^2e^{b\phi_0}r^{-2(D-3)N}}{2N(D-2)((D-4)N+1)} \]  

(2.47)

\[ \phi = \phi_0 - \sqrt{2N(1-N)(D-2)\ln r}, \quad R = r^N, \]

\[ N = \frac{(D-2)a^2}{2 + (D-2)a^2}, \quad b = \frac{2}{(D-2)a}, \]

\[ 2\Lambda_0e^{a\phi_0} = \frac{(D-2)(D-3)(1-N)k}{(2N-1)}, \]

\[ F_{tr} = \frac{Qe^{b\phi_0}}{r^{(D-4)N+2}}. \]  

(2.48)

For this solution, one can see that the second term in (2.47) always dominates over other two terms as \( r \to \infty \).

(i) When \( k = 1 \), the cosmological constant has to be positive for \( 0 < N < 1/2 \), the solution always has a cosmological horizon regardless of the sign of \( \mathcal{M} \); for the case \( 1/2 < N < 1 \), the cosmological constant is negative. In that case black hole horizon appears if \( \mathcal{M} > 0 \), otherwise it is a naked singularity solution.

(ii) When \( k = -1 \), the solution always has a cosmological horizon if \( 1/2 < N < 1 \), while it describes a black hole in the case of \( 0 < N < 1/2 \) with \( \mathcal{M} > 0 \). The result is summarized in Tab. VIII.

| \( k \) | 1 | -1 |
|---|---|---|
| \( N \) | 0 < \( N \) < 1/2 | 1/2 < \( N \) < 1 | 0 < \( N \) < 1/2 | 1/2 < \( N \) < 1 |
| \( \Lambda_0 \) | > 0 | < 0 | < 0 | > 0 |

Solution: CH | BH (\( \mathcal{M} > 0 \)) | BH (\( \mathcal{M} > 0 \)) | CH |

• Solution 2.

\[ A = -\frac{\mathcal{M}}{r^{(D-2)N-1}} - 2\Lambda_0e^{a\phi_0}r^{2N} + \frac{(D-3)Q^2e^{b\phi_0}}{2(1-N)(D-2)((D-4)N+1)r^{2N-2}}, \]  

(2.49)

\[ \phi = \phi_0 - \sqrt{2N(1-N)(D-2)\ln r}, \quad R = r^N, \]

\[ N = \frac{2}{2 + (D-2)a^2}, \quad b = \frac{2(D-3)}{2(D-2)a}, \]

\[ Q^2e^{b\phi_0} = \frac{2(D-3)(D-2)(1-N)k}{(D-4)N+1}, \]

\[ F_{tr} = \frac{Qe^{b\phi_0}}{r^{N(4-D)}}. \]  

(2.50)

For nonvanishing electric charge, the solution holds only for the case of \( k = 1 \). When \( 0 < N < 1/2 \), the third term in (2.49) is always dominant over other terms as \( r \to \infty \). But the second term will changes its sign around \( N = 1/D \). So we have three cases.

(i) When \( 0 < N < 1/D \), a black hole horizon always appears if \( \Lambda_0 < 0 \), otherwise in order to have a black hole horizon, the mass parameter must be positive.

(ii) When \( 1/D < N < 1/2 \), a black hole horizon always appears if \( \Lambda_0 > 0 \). Otherwise the mass parameter has to be positive.

(iii) When \( 1/2 < N < 1 \), the second term in (2.49) becomes dominant. In this case, when \( \Lambda_0 > 0 \), the solution always has a cosmological horizon; when \( \Lambda_0 < 0 \) a black hole horizon appears provided \( \mathcal{M} > 0 \). The property of the solution (2.49) is summarized in Tab. IX.
TABLE IX: The property of the solution $S_{BH}$ for different parameters.

| $N$        | $0 < N < 1/D$ | $0 < N < 1/D$ | $1/D < N < 1/2$ | $1/D < N < 1/2$ | $1/2 < N < 1$ | $1/2 < N < 1$ |
|------------|---------------|---------------|-----------------|-----------------|---------------|---------------|
| $\Lambda_0$| $> 0$         | $< 0$         | $> 0$           | $< 0$           | $> 0$         | $< 0$         |
| Solution   | BH ($M > 0$)  | BH            | BH              | BH ($M > 0$)    | CH            | BH ($M > 0$)  |

• Solution 3.

$$A = -\frac{\mathcal{M}}{r^{(D-2)N-1}} - \frac{2\Lambda_0e^{a\phi_0} - (D-3)Q^2e^{b\phi_0}/2)}{(1-N)(D-2)((D-4)N+1)} r^{2-2N},$$

$$\phi = \phi_0 - \sqrt{2N(1-N)(D-2)\ln r}, \quad R = r^N,$n

$$N = \frac{a^2}{2 + (D-2)a^2}, \quad b = -(D-3)a,$n

$$2\Lambda_0e^{a\phi_0} = -\frac{(D-2)(1-N)}{2N-1} \left( (D-3)k - \frac{(D-4)N+1}{2(D-2)(1-N)} Q^2e^{b\phi_0} \right),$$

$$F_{tr} = \frac{Qe^{b\phi_0}}{r^{N(4-D)}}.$$

The metric function $A$ is always dominant by the second term in (2.51) as $r \to \infty$. When $M > 0$, the solution has a black hole horizon if $\Lambda_0 > 0$, $\phi_0 e^{a\phi_0} - (D-3)Q^2e^{b\phi_0}/2 < 0$, otherwise, it is a naked singularity solution. On the other hand, when $M < 0$ the solution has a cosmological horizon if $\Lambda > 0$, while the singularity becomes naked again if $\Lambda < 0$.

III. DIMENSIONAL REDUCTION: CURVED CASE

In the dimensional reduction from (2.1) to (2.3), we assumed $ds_d^2$ is a $d$-dimensional Ricci flat Euclidean space. Now we relax this condition and suppose it is a nonzero constant curvature space with curvature scalar

$$R_d = d(d-1)k_d.$$ (3.1)

Without loss of generality, one may take $k_d = \pm 1$. If $k_d = 0$, the reduction goes back to the case discussed in the previous section. Doing the reduction along $y^i$ in (2.2), we obtain

$$S = \frac{V_q}{16\pi G} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - 2\Lambda_0e^{a\phi} - 2\Lambda_1e^{c\phi} \right),$$ (3.2)

where the relation between $\phi$ and $\alpha$ is the same as (2.5), $a$ is still given by (2.0) and

$$c = -\frac{D + d - 2}{d} a, \quad 2\Lambda_1 = -d(d-1)k_d.$$ (3.3)

The action (3.2) describes an Einstein-dilaton gravity theory with two Liouville-type dilaton potentials provided $a \neq c$. Otherwise, the two terms can be reduced to one. Note that there is a symmetry,

$$a \leftrightarrow c \quad \text{and} \quad \Lambda_0 \leftrightarrow \Lambda_1,$$ (3.4)

in the action (3.2). Varying the action (3.2), we have the equations of motion

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{2}{D-2} \Lambda_0e^{a\phi} g_{\mu\nu} + \frac{2}{D-2} \Lambda_1e^{c\phi} g_{\mu\nu},$$

$$\nabla^2 \phi - 2a \Lambda_0e^{a\phi} - 2c \Lambda_1e^{c\phi} = 0.$$ (3.5)

As the above, we now solve the equations of motion with the ansatz (2.8), and assuming that $\Lambda_0, \Lambda_1, a$ and $c$ are arbitrary constants. Once again, the solutions depend on the curvature $k$. 
(1) When \( k = 0 \), we find that the equations (3.5) have a consistent solution only as \( a = c \). The solution is

\[
A(r) = -\frac{\mathcal{M}}{r^{(D-2)N-1}} + \frac{2(\Lambda_0 + \Lambda_1)e^{a\phi_0}r^{2N}}{N(ND - 1)(D - 2)},
\]

\[
\phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r,
\]

\[
R = r^N, \quad c = a,
\]

\[
N = \frac{2}{2 + c^2(D - 2)}. \tag{3.6}
\]

Therefore the property of this solution is the same as the one (2.16). Only difference is that \( \Lambda_0 \) in (2.16) is replaced by \( \Lambda_0 + \Lambda_1 \) here. Because the relation between \( a \) and \( c \) in (3.6) does not satisfy (3.3), the solution therefore cannot be uplifted to \((D + d)\) dimensions unless \( \Lambda_1 = 0 \) or \( \Lambda_0 = 0 \). When \( \Lambda_1 = 0 \), the resulting solution is just the one in (2.20). Here we discuss the case of \( \Lambda_0 = 0 \). Note that in this case, one does not need the condition \( c = a \) in (3.6).

Uplifting the solution, we find that the metric can be expressed as, upon some rescalings of the coordinates and the parameters \( \mathcal{M} \),

\[
ds_{D+d}^2 = -\left( k_d - \frac{\mathcal{M}}{r^{d-1}} \right) dt^2 + \left( k_d - \frac{\mathcal{M}}{r^{d-1}} \right)^{-1} dr^2 + r^2 h_{ij} dy^i dy^j + \delta_{mn} dx^n dx^m. \tag{3.7}
\]

This is nothing but a \((d + 2)\)-dimensional Schwarzschild metric times a \((D - 2)\)-dimensional Ricci flat space. Here \( h_{ij} \) denotes the line element of a \( d \)-dimensional space with curvature scalar \( d(d-1)k_d \).

(2) When \( k \neq 0 \), we find three sets of solutions to the equations (3.5) of motion.

- **Solution 1.**

\[
A = -\frac{\mathcal{M}}{r^{(D-2)N-1}} + \frac{(D - 3)k r^{2-2N}}{2(N-1)(N(D - 4) + 1)} - \frac{2\Lambda_1 e^{a\phi_0}r^{2N}}{N(D - 2)(ND - 1)} \tag{3.8}
\]

\[
\phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r, \quad R = r^N,
\]

\[
N = \frac{a^2(D - 2)}{2 + a^2(D - 2)}, \quad c = \frac{2}{(D - 2)a},
\]

\[
2\Lambda_0 = -\frac{(D - 3)(D - 2)(1 - N)ke^{-a\phi_0}}{2N - 1}. \tag{3.9}
\]

Let us first consider the case of \( k = 1 \). As \( r \to \infty \), the second term in (3.8) is dominant if \( 0 < N < 1/2 \), while the third term does if \( 1/2 < N < 1 \). (i) In the range \( 0 < N < 1/D \), the cosmological constant \( \Lambda_0 \) must be positive. In this case, the solution must have a cosmological horizon if \( \Lambda_1 > 0 \), regardless of the sign of the mass parameter; on the other hand, in order to have a cosmological horizon, the mass parameter must be negative if \( \Lambda_1 < 0 \). (ii) In the range \( 1/D < N < 1/2 \), if \( \Lambda_1 < 0 \), the solution has a cosmological horizon regardless of the sign of \( \mathcal{M} \); to have a cosmological horizon, \( \mathcal{M} \) must be negative if \( \Lambda_1 > 0 \). (iii) If \( 1/2 < N < 1 \), the cosmological constant \( \Lambda_0 \) is negative. When \( \Lambda_1 > 0 \) the solution has a cosmological horizon regardless of the sign of \( \mathcal{M} \), while black hole horizon appears provided \( \mathcal{M} > 0 \) when \( \Lambda_1 < 0 \). In other cases, the singularity at \( r = 0 \) is naked. The result is summarized in Tab. X. This solution cannot be uplifted to \((D + d)\) dimensions according to (2.22) unless \( \Lambda_0 = 0 \) or \( \Lambda_1 = 0 \). For the latter, the resulting solution is (2.24). For the case of \( \Lambda_0 = 0 \), the higher dimensional solution turns out to be (3.7).

**TABLE X:** The property of the solution 1 in (3.8) for different parameters in the case of \( k = 1 \).

| \( N \) | \( 0 < N < 1/D \) | \( 0 < N < 1/D \) | \( 1/D < N < 1/2 \) | \( 1/D < N < 1/2 \) | \( 1/2 < N < 1 \) | \( 1/2 < N < 1 \) |
|---|---|---|---|---|---|---|
| \( \Lambda_0 \) | > 0 | > 0 | > 0 | > 0 | < 0 | < 0 |
| \( \Lambda_1 \) | > 0 | < 0 | > 0 | < 0 | > 0 | < 0 |
| Solution | CH | CH (\( \mathcal{M} < 0 \)) | CH | CH (\( \mathcal{M} < 0 \)) | CH | BH (\( \mathcal{M} > 0 \)) |

When \( k = -1 \), the cosmological constant \( \Lambda_0 \) is negative in the range \( 0 < N < 1/2 \), while it is positive in the range \( 1/2 < N < 1 \). (i) In the range \( 0 < N < 1/D \), a black hole horizon always appears if \( \Lambda_1 < 0 \) regardless of the sign of \( \mathcal{M} \). If \( \Lambda_1 > 0 \), one has to have \( \mathcal{M} > 0 \) in order to have a black hole horizon. (ii) In the range
of $1/D < N < 1/2$, the solution always has a black hole horizon if $\Lambda_1 > 0$, otherwise the mass parameter must be positive if $\Lambda_1 < 0$ in order to have a black hole horizon. (iii) In the range of $1/2 < N < 1$, a cosmological horizon will appear if $\Lambda_1 > 0$ and $\mathcal{M} < 0$; if $\Lambda_1 < 0$, the solution always describes a black hole spacetime. The property of the solution (3.8) is summarized in Tab. XI in the case of $k = -1$. Once again, because the relation between $a$ and $c$ in the solution (3.8) does not satisfy the one in (3.3), one cannot uplift the solution to higher dimensional case according to (2.22) unless $\Lambda_0 = 0$ or $\Lambda_1 = 0$. In the latter case, the uplifting solution is (2.25); while the solution turns out to be (3.7), once again, for the case of $\Lambda_0 = 0$.

### Table XI: The property of the solution 1 in (3.8) for different parameters in the case of $k = -1$.

| $N$ | $0 < N < 1/D$ | $0 < N < 1/D$ | $1/D < N < 1/2$ | $1/D < N < 1/2$ | $1/2 < N < 1$ | $1/2 < N < 1$ |
|-----|---------------|---------------|----------------|----------------|----------------|----------------|
| $\Lambda_0$ | $< 0$ | $< 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| $\Lambda_1$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ |
| Solution | BH($\mathcal{M} > 0$) | BH | BH | BH($\mathcal{M} > 0$) | CH($\mathcal{M} < 0$) | BH |

- **Solution 2.**
  
  \[
  A = \frac{\mathcal{M}}{r^{(D-2)N-1}} + \frac{(D-3)k r^{2-2N}}{(2N-1)(N(D-4)+1)} - \frac{2\Lambda_0 e^{a\phi} r^{2N}}{N(D-2)(ND-1)},
  \]
  \[\phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r, \quad R = r^N,\]
  \[N = \frac{c^2(D-2)}{2 + c^2(D-2)}, \quad a = \frac{2}{(D-2)c},\]
  \[2\Lambda_1 = -\frac{(D-3)(D-2)(1-N)ke^{-c\phi_0}}{2N-1}.\]

  This is symmetric to solution 1 according to (3.4). We do not therefore present the analysis of the solution here.

- **Solution 3.**
  
  \[
  A(r) = -\frac{\mathcal{M}}{r^{(D-2)N-1}} + \frac{(D-3)k r^{2-2N}}{(2N-1)(N(D-4)+1)},
  \]
  \[\phi = \phi_0 - \sqrt{2N(1-N)(D-2)} \ln r, \quad R = r^N,\]
  \[N = \frac{c^2(D-2)}{2 + c^2(D-2)}, \quad c = a,\]
  \[2\Lambda_0 + 2\Lambda_1 = -\frac{(D-3)(D-2)(1-N)ke^{-c\phi_0}}{2N-1}.\]

  The solution is the same as the one (2.21) for the case with one Liouville-type potential. The only difference is $\Lambda_0$ there is replaced by $\Lambda_0 + \Lambda_1$ here. The causal structure of the solution is of course the same as the one for the solution (2.21). In addition, due to $c = a$, we also cannot uplift the solution to $D + d$ dimensions according to (2.22) unless $\Lambda_1 = 0$ or $\Lambda_0 = 0$. For the former, resulting solution is given by (2.24) or (2.25) depending on the sign of $\Lambda_0$. For the latter, however, we find that the $(D + d)$-dimensional solution can be expressed as

  \[
  ds_{D+d}^2 = -\left(k - \frac{\mathcal{M}}{r^{D+d-3}}\right) dt^2 + \left(k - \frac{\mathcal{M}}{r^{D+d-3}}\right)^{-1} + r^2(q_{ij}dy^i dy^j + h_{mn} dx^m dx^n),
  \]

  The same as the above, some rescalings of coordinates and the parameter $\mathcal{M}$ have been made here, and the subspace described by $(q_{ij}dy^i dy^j + h_{mn} dx^m dx^n)$ has been normalized to have constant curvature $(D + d - 2)(D + d - 3)k$. Note that the uplifting holds only for the case where $k_0$ and $k$ have same signs since in this case we have $N = (D + d - 2)/(D + 2d - 2) > 1/2$. It can be immediately seen that the solution satisfies the vacuum Einstein equations in $(D + d)$ dimensions and is asymptotically flat.

  When the Maxwell field is present, the resulting action will have an additional term

  \[
  S = \frac{V_q}{16\pi G} \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(\partial \phi)^2 - 2\Lambda_0 e^{a\phi} - 2\Lambda_1 e^{c\phi} - \frac{1}{4} e^{-b\phi} F_{ij}^2\right),
  \]

  (3.15)
where \(a\) and \(b\) are given by (2.6) and (2.26), respectively, and \(c\) and \(\Lambda_1\) are given by (3.3). Varying the action, we have the equations of motion

\[
R_{\mu\nu} = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{2}{D-2} \Lambda_0 e^{a \phi} g_{\mu\nu} + \frac{2}{D-2} \Lambda_1 e^{c \phi} g_{\mu\nu} + \frac{1}{2} e^{-b \phi} \left( F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{2(D-2)} F_2^{\mu\nu} \right) ,
\]

\[
\partial_{\mu} \left( \sqrt{-g} e^{-b \phi} F^{\mu\nu} \right) = 0 , \quad F_{\mu\nu,\lambda} + F_{\nu,\lambda,\mu} + F_{\lambda,\mu,\nu} = 0 ,
\]

\[
\nabla^2 \phi - 2a \Lambda_0 e^{a \phi} - 2c \Lambda_1 e^{c \phi} + \frac{b}{4} e^{-b \phi} F_2^2 = 0 .
\]

As the case of one Liouville-type potential, it is straightforward to find and analyze the magnetic/electric charged solutions to equations (3.16) of motion. But the expressions of solutions and analysis are rather complicated and we shall not present them here.

### IV. CONCLUSIONS AND DISCUSSION

By dimensionally reducing an Einstein-Maxwell theory with a cosmological constant to a lower dimension, we can obtain an Einstein-Maxwell-dilaton theory with one or two dilaton potentials of Liouville-type. We have looked for and analyzed in some details exact (neutral/magnetic/electric charged) solutions of these theories. These found solutions have a rich structure depending on the parameters in the theory. These solutions could have black hole horizon(s), cosmological horizon, or a naked singularity. These horizons can be a positive, zero or negative constant curvature hypersurface. In particular, we noted that some black hole solutions have negative gravitational mass. In addition these solutions are neither asymptotically flat nor asymptotically AdS/dS. Our study generalized existing results concerning non-asymptotically AdS/dS solutions in the literature in some directions. For example, (1) in our discussions, spacetime dimensions \(D\) and \(d\) are almost arbitrary; (2) black hole horizon(s) or cosmological horizon can be a hypersurface with positive, zero or negative constant curvature; (3) the scalar potential could be one or two Liouville-type term(s). Here it would be worth mentioning some of existing literatures about non-asymptotically AdS/dS solutions: the authors of [15] looked for solutions of Einstein-Maxwell-dilaton theory with dilaton potential, but it is restricted to the case \(k = 1\); Chan in [16] discussed the modification of three dimensional BTZ black hole by a scalar potential; the authors of [17, 18] studied four dimensional topological dilaton black hole solutions with \(k = 0\) or \(-1\). Klemm in [19] found supersymmetric solution in the four dimensional gauged \(N = 4, SU(2) \times SU(2)\) supergravity; Refs. [14, 20, 22, 23] investigated domain wall solutions with \(k = 0\) in Einstein-dilaton theory with a Liouville-type potential; Cvetic et al in [21] found a domain wall solution with \(k = 0\) in an Einstein-Maxwell-dilaton theory with special coupling parameters \(a\) and \(b = \sqrt{2p}\) in (2.27); for more recent discussions on the non-asymptotically AdS/dS solutions with \(k = 1\) in four dimensional Einstein-Maxwell-dilaton theory without dilaton potential see [24, 25]; other related discussions see [26].

Since the action that we worked with could come from the dimensional reduction from a higher dimensional Einstein-Maxwell theory with a cosmological constant, some of the found solutions can be uplifted to the case of higher dimensions. We found that some of these neither asymptotically flat nor AdS/dS solutions have a higher dimensional origin; and their higher dimensional origins have well-behaved asymptotics: they are either asymptotically AdS/dS [see, for example, (2.22), (2.24), (2.25) and (2.41)], or asymptotically flat (see, for example, (3.7) and (3.14)] with/without a compact constant curvature space. From the point of view of holography, this observation is useful to better understand the holographic properties of these non-asymptotically AdS/dS solutions.

The black hole horizon and cosmological horizon of these solutions have usual thermodynamical relations. For example, the Hawking temperature of horizon is given by

\[
T = \frac{1}{4\pi} |A'(r)|_{r = r_+ \text{ or } r_+} ,
\]

where \(r_+\) and \(r_c\) denote black hole horizon and cosmological horizon, respectively. The horizon entropy is still by the so-called area formula

\[
S = \frac{V_h V_b}{4G} A ,
\]

where \(A = r^{(D-2)} N\) since we have worked in the Einstein frame, where the area formula of black hole entropy holds. With the black hole mass given by (2.17) or the gravitational mass (2.18) for the cosmological horizon, it is easy to show that the first law of thermodynamics holds for both black hole horizon and cosmological horizon

\[
dM = TdS + \Phi dQ ,
\]
where $Q$ is the electric charge or magnetic charge of the solutions, and $\Phi$ is the corresponding chemical potential. As an example, let us consider the electric charged solution given by (2.43) and (2.44). In terms of black hole horizon radius $r_+$ determined by the equation $A(r)|_{r=r_+} = 0$, the black hole mass is given by

$$M = \frac{V_q V_h r_+^{(D-2)N-1}}{16\pi G} \left( -\frac{2\Lambda_0 e^{\alpha\phi_0} r_+^{2N}}{ND-1} + \frac{Q^2 e^{b\phi_0}}{2((D-4)N+1)r_+^{2(D-3)N}} \right), \quad (4.4)$$

The Hawking temperature $T$ and the entropy are

$$T = \frac{1}{4\pi N(D-2)r_+} \left( -\frac{2\Lambda_0 e^{\alpha\phi_0} r_+^{2N}}{ND-1} - \frac{Q^2 e^{b\phi_0}}{2r_+^{2(D-3)N}} \right),$$

$$S = \frac{V_q V_h}{4G} r_+^{(D-2)N}, \quad (4.5)$$

respectively. Note that in order to have a black hole horizon, the constant $\Lambda_0$ has to be negative here so that the temperature is always positive definiteness. When $T = 0$, it indicates that the black hole is an extremal black hole with vanishing Hawking temperature. The electric charge associated with the black hole is

$$Q = \frac{V_h}{4\pi}. \quad (4.6)$$

And the conjugate chemical potential $\Phi$

$$\Phi = \frac{V_q e^{b\phi_0}}{4G((D-4)N+1)r_+^{(D-4)N+1}}. \quad (4.7)$$

These thermodynamic quantities satisfy the first law of black hole thermodynamics \[4.4\]. Indeed, as was shown by Wald \[24\], the first law of black hole mechanics always holds for any stationary black hole spacetime in any gravitational theory.

Finally we point out that the solutions we found are not well-defined for some special values of $N$, for example, $N = 1/D$ for the solution (2.15), (2.43), (2.49), and (3.6); $N = 1/2$ for (2.21), (2.32), (2.40), (2.45), (2.47), (2.51), (3.8), (3.10), and (3.12); and $N = 1/4$ for (2.30) and (2.36). In fact for these special values, we can also find solutions, but a term concerning the logarithmic function of $r$ will appear in the metric function $A$. Here we do not present those solutions, although the concerned discussions are straightforward.

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While this paper was writing up, an interesting paper [24] appears (see also [25]). The authors find that a four dimensional non-asymptotically flat black hole solution derived by Chan et al. [17] of Einstein-Maxwell-dilaton theory [16] without the dilaton potential term can be reobtained by taking a near horizon limit of asymptotically flat black hole solution, and that when $b = \sqrt{6}$ (in the notation of [24], $\alpha = \sqrt{3}$), the four dimensional non-asymptotically flat, magnetic charged black hole can be understood as a dimensional reduction of a five dimensional, asymptotically flat Schwarzschild black hole along one of the azimuthal angles.