A ROLLE TYPE THEOREM FOR CYCLICITY OF ZEROS OF
FAMILIES OF ANALYTIC FUNCTIONS

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Abstract. Let \( \{f_{\lambda_j}\}_{\lambda \in V; 1 \leq j \leq k} \) be families of holomorphic functions in the open unit
disk \( \mathbb{D} \subset \mathbb{C} \) depending holomorphically on a parameter \( \lambda \in V \subset \mathbb{C}^n \). We establish a Rolle
type theorem for the generalized multiplicity (called cyclicity) of zero of the family of
univariate holomorphic functions \( \left\{ \sum_{j=1}^{k} f_{\lambda_j} \right\}_{\lambda \in V} \) at \( 0 \in \mathbb{D} \). As a corollary, we estimate
the cyclicity of the family of generalized exponential polynomials, that is, the family of
entire functions of the form \( \sum_{k=1}^{m} P_k(z)e^{Q_k(z)}, z \in \mathbb{C}, \) where \( P_k \) and \( Q_k \) are holomorphic polynomials of
degrees \( p \) and \( q \), respectively, parameterized by vectors of coefficients of
\( P_k \) and \( Q_k \).

1. Introduction

1.1. Basic Definitions and Results. Let \( B_0(\lambda, r) \subset \mathbb{C}^n \) be the open Euclidean ball
centered at \( \lambda \) of radius \( r \). We set \( \mathbb{D}_r := B_1(0, r) \subset \mathbb{C} \) and \( \mathbb{D} := \mathbb{D}_1 \). By \( K_\varepsilon := \bigcup_{\lambda \in K} B_0(\lambda, \varepsilon) \)
we denote the \( \varepsilon \)-neighbourhood of \( K \subset \mathbb{C}^n \). Let \( V \subset \mathbb{C}^n \) be a domain and \( \mathcal{O}(V) \) be the
ring of holomorphic functions on \( V \). Consider the Maclaurin series expansion of the family
\( \mathcal{F} = \{f_\lambda\}_{\lambda \in V} \) of holomorphic functions in the disk \( \mathbb{D}_\rho \) depending holomorphically on \( \lambda \in V \),
\begin{equation}
  f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda)z^k, \quad \lambda \in V, \quad z \in \mathbb{D}_\rho, \quad a_k \in \mathcal{O}(V).
\end{equation}

Let \( U \subset V \) be a subdomain. By \( \mathcal{I}(\mathcal{F}; U) \subset \mathcal{O}(U) \) we denote the ideal generated by all
the restrictions \( a_k|_U \). The central set of \( \mathcal{F} \) in \( U \) is defined by the formula
\begin{equation}
  C(\mathcal{F}; U) := \{\lambda \in U : f_\lambda \equiv 0\}.
\end{equation}

Definition 1.1. The family \( \mathcal{F} = \{f_\lambda\}_{\lambda \in V} \) has cyclicity \( k \geq 0 \) in a compact set \( K \subset V \) if the following holds:

(a) There exist \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that for every \( \lambda \) in \( K_{\varepsilon_0} \cap (V \setminus C(\mathcal{F}; V)) \) the
function \( f_\lambda \) has at most \( k \) zeros in \( \mathbb{D}_{\delta_0} \) (counting multiplicities).

(b) For arbitrary positive numbers \( \varepsilon < \varepsilon_0 \) and \( \delta < \delta_0 \) there exists a parameter \( \lambda \in K_\varepsilon \cap (V \setminus C(\mathcal{F}; V)) \) such that \( f_\lambda \) has exactly \( k \) zeros in \( \mathbb{D}_\delta \).

The notion goes back to the famous paper [B1] (see also [B2]) of Bautin who found a
connection between cyclicity and algebraic properties of \( \mathcal{I}(\mathcal{F}; V) \) and in this way proved that a perturbation of a center in a family of quadratic vector fields can generate no more
than 3 small amplitude limit cycles.

The cyclicity of the family \( \mathcal{F} \) in \( K \) is denoted by \( c(\mathcal{F}; K) \). For a one point compact set
\( K := \{\mu\} \) we use the notation \( c(\mathcal{F}; \mu) \). For \( \mu \notin C(\mathcal{F}; V) \) this number equals the multiplicity
of the zero of \( f_\mu \) at \( 0 \in \mathbb{D} \). Therefore the term “generalized multiplicity” seems to be quite
natural for \( c(\mathcal{F}; K) \).

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The problem of bounding cyclicity for specific families $\mathcal{F}$ arises in several diverse areas of analysis, including dynamical systems (related to the, so-called, second part of Hilbert’s 16 problem), transcendental number theory (related to Hilbert’s 7 problem) and approximation theory (Bernstein-Markov-Remez’ type inequalities); for recent developments and corresponding references, see, e.g., [Y] and the Featured Review [MR 98h:34009].

The basic properties of cyclicity are obtained in [Br1]. We formulate some of them.

Let $\mathcal{F} = \{f_\lambda\}$ and $\mathcal{G} = \{g_\lambda\}$ be families of holomorphic functions in $\mathbb{D}_\rho$ depending holomorphically on $\lambda \in V$. We set

$$\mathcal{F}' := \left\{ \frac{df_\lambda}{dz} \right\}, \quad \mathcal{F} + \mathcal{G} := \{f_\lambda + g_\lambda\}, \quad \mathcal{F} \cdot \mathcal{G} := \{f_\lambda \cdot g_\lambda\} \quad \text{and} \quad e^{\mathcal{F}} := \{e^{f_\lambda}\}.$$  

The following properties are presented in Propositions 2.1, 2.2, 2.5, Theorems 1.5, 1.7 and Remark 1.6 of [Br1] (see also this paper for additional references).

(a) For a compact set $K \subset V$,

$$c(\mathcal{F}; K) = \max_{\mu \in K} c(\mathcal{F}; \mu).$$

(b) There exist a constant $A$ and an open neighbourhood $U$ of $K$ such that (see (1.1))

$$|a_k(\lambda)| \leq A \cdot \sup_{U} |a_k| \cdot \max_{0 \leq i \leq c(\mathcal{F}; K)} |a_i(\lambda)| \quad \text{for all} \quad \lambda \in U;$$

here $c(\mathcal{F}; K)$ cannot be replaced by a smaller index.

If $K = \{\mu\}$, the latter property is equivalent to the following one (see [HRT]).

Let $\mathcal{I}(\mathcal{F}; \mu)$ be the ideal in the local ring $\mathcal{O}_\mu$ of germs of holomorphic functions at $\mu \in V$ generated by germs at $\mu$ of the Taylor coefficients $a_j$ of $\mathcal{F}$, see (1.1), and $\mathcal{I}_d(\mathcal{F}; \mu) \subset \mathcal{I}(\mathcal{F}; \mu)$ be the ideal generated by germs at $\mu$ of $a_0, \ldots, a_d$. We define $d(\mathcal{F}; \mu)$ to be the minimal integer $d$ such that the integral closures of $\mathcal{I}_d(\mathcal{F}; \mu)$ and $\mathcal{I}(\mathcal{F}; \mu)$ coincide.

(b')

$$c(\mathcal{F}; \mu) = d(\mathcal{F}; \mu).$$

(c) For a compact set $K \subset V$ there exists an open neighbourhood $O_K$ of $K$ such that the central set $\mathcal{C}(\mathcal{F}; O_K)$ coincides with the set of common zeros of $a_0, \ldots, a_{c(\mathcal{F}; K)}$.

(d)

$$c(\mathcal{F}; K) \leq c(\mathcal{F}'; K) + 1.$$  

(e)

$$c(e^{\mathcal{F}}; K) = 0.$$  

(f)

$$c(\mathcal{F} \cdot \mathcal{G}; K) \leq c(\mathcal{F}; K) + c(\mathcal{G}; K).$$  

(g)

$$c(\mathcal{F} + \mathcal{G}; K) \leq \max\{c(\mathcal{F}; K), c(\mathcal{G}; K), c(W(\mathcal{F}, \mathcal{G}); K) + 1\};$$

here $W(\mathcal{F}, \mathcal{G}) := \mathcal{F}' \cdot \mathcal{G} - \mathcal{G}' \cdot \mathcal{F}$ is the family of Wronskian determinants of $\{\mathcal{F}, \mathcal{G}\}$.

(h) For every $\varepsilon > 0$ there exist an open neighbourhood $O_\varepsilon$ of $K$ and a positive number $R_\varepsilon$ such that for all $\lambda \in O_\varepsilon \setminus \mathcal{C}(\mathcal{F}; V)$

$$\sup_{\mathbb{D}_{R_\varepsilon}} |f_\lambda| - \sup_{\mathbb{D}_{R_\varepsilon/\varepsilon}} \ln |f_\lambda| \leq c(\mathcal{F}; K) + \varepsilon$$

and each $f_\lambda$ has at most $c(\mathcal{F}; K)$ zeros in $\mathbb{D}_{R_\varepsilon}$ (counting multiplicities); here $c(\mathcal{F}; K)$ cannot be replaced by a smaller number.

Property (h) generates the following Cartan-type estimates and Remez-type inequalities for the family $\mathcal{F}$, see, e.g., [Br2, Th. 3.1 and Sect. 2] and references therein.
(h') Fix \( H \in (0, 1) \), \( d > 0 \) and set \( A := e^{(\frac{d}{H})^2} \). For each \( \lambda \in O_e \setminus C(F; V) \) and \( R \in (0, R_e) \) there exists a family of open disks \( \{ D_{j;\lambda} \}_{1 \leq j \leq k} \), \( k \leq c(F; K) \), with \( \sum r_{j;\lambda}^d \leq \frac{(2H R_e)^d}{d} \), where \( r_{j;\lambda} \) is the radius of \( D_{j;\lambda} \), such that

\[
(1.3) \quad |f_\lambda(z)| \geq \sup_{D_R} |f_\lambda| \cdot A^{-e} \cdot \left( \frac{H}{A} \right)^{c(F;K)} \text{ for all } z \in \mathbb{D}_{R/e} \setminus \bigcup_j D_{j;\lambda}.
\]

(h'') Consider the function \( \Phi(t) := t + \sqrt{t^2 - 1}, t \geq 1 \). There exists an absolute constant \( \hat{c} \geq 1 \) such that for each \( \lambda \in O_1 \), an interval \( I \subset \mathbb{D}_{R_1/e} \) and a measurable subset \( \omega \subset I \) the following inequality holds:

\[
(1.4) \quad \sup_I |f_\lambda| \leq \Phi \left( \frac{2 m_1(I)}{m_1(\omega)} - 1 \right) \hat{c} \max \{ c(F;K), 1 \} \sup_\omega |f_\lambda|;
\]

here \( m_1 \) is the linear measure on \( \mathbb{C} \).

1.2. Formulation of the Main Result. In this note we generalize property (g) and establish a Rolle type theorem for the cyclicity of the family \( F := \sum_{j=1}^k F_j \), where \( F_j := \{ f_{\lambda;j} \}, 1 \leq j \leq k \), are families of holomorphic functions in \( \mathbb{D}_\mu \subset \mathbb{C} \) holomorphic in \( \lambda \in V \subset \mathbb{C}^n \). As a corollary, in the next section we estimate the cyclicity of the family of generalized exponential polynomials.

In our main result \( W(F_{i_1}, \ldots, F_{i_l}) := \{ W(f_{\lambda;i_1}, \ldots, f_{\lambda;i_l}) \}_{\lambda \in V}, 1 \leq i_1 < \cdots < i_l \leq k \), stands for the family of Wronskian determinants

\[
W(f_{\lambda;i_1}, \ldots, f_{\lambda;i_l}) := \begin{vmatrix} f_{\lambda;i_1} & f_{\lambda;i_2} & \cdots & f_{\lambda;i_l} \\ f'_{\lambda;i_1} & f'_{\lambda;i_2} & \cdots & f'_{\lambda;i_l} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\lambda;i_1}^{(l-1)} & f_{\lambda;i_2}^{(l-1)} & \cdots & f_{\lambda;i_l}^{(l-1)} \end{vmatrix}
\]

Let \( K \) be the family of all distinct subsets of \( \{ 1, \ldots, k \} \). By \( |I| \) we denote the cardinality of \( I \in K \). For \( I = \{ i_1, \ldots, i_l \} \in K \) and a compact subset \( K \subset V \) we set

\[
c(F;K) := c(W(F_{i_1}, \ldots, F_{i_l}); K).
\]

**Theorem 1.2.**

\[
c(F; K) \leq \max_{I \in K} \{ c(F; K) + |I| - 1 \}.
\]

For \( K = \{ \mu \} \subset V \) with \( \mu \notin C(F; V) \) (i.e., in the case of the multiplicity of zero at \( 0 \in \mathbb{D} \) of \( f_\mu \equiv 0 \)) this result was proved in [VP, Th. 1].

The proof of Theorem 1.2 is presented in Section 3.

2. Cyclicity of the Family of Generalized Exponential Polynomials

Consider the family \( F = \{ f_\lambda \} \) of entire functions of the form

\[
\sum_{k=1}^{m} P_k(z) e^{Q_k(z)}, \quad z \in \mathbb{C},
\]

where \( P_k(z) = \sum_{i=0}^{p} c_i z^i \) and \( Q_k(z) = \sum_{j=1}^{q} d_{kj} z^j \) are holomorphic polynomials of degrees \( p \) and \( q \), respectively. We parameterize elements of this family by strings \( \lambda = ((c_{ki})_{0 \leq i \leq p}, (d_{kj})_{1 \leq j \leq q})_{1 \leq k \leq m} \) of the coefficients of polynomials \( P_k \) and \( Q_k \) involved in their definition, i.e., by points of \( \mathbb{C}^{N_{p,q,m}} \), where \( N_{p,q,m} := m(p + q + 1) \). Then

\[
f_\lambda(z) = \sum_{n=0}^{\infty} a_n(\lambda) z^n, \quad z \in \mathbb{C},
\]
where \(a_n\) is a holomorphic polynomial in \(\lambda\) of degree \(n + 1\) with nonnegative rational coefficients. Using Taylor series expansion for the exponent together with the multinomial theorem we obtain

\[
a_n(\lambda) = \sum_{k=1}^{m} \left( \sum_{j=0}^{p} c_{kj} \left( \sum_{k_1+2k_2+\ldots+qk_q=n-j} \frac{d_{k_1}^{k_1}d_{k_2}^{k_2}\ldots d_{k_q}^{k_q}}{k_1!k_2!\ldots k_q!} \right) \right). \tag{2.6}
\]

(Here the inner summation is taken over all sequences \(k_1, \ldots, k_q \in \mathbb{Z}_+\) satisfying the required condition.)

For instance,

\[
a_0(\lambda) = \sum_{k=1}^{m} c_{k0}, \quad a_1(\lambda) = \sum_{k=1}^{m} c_{k1} + c_{k0}d_{k1}, \quad a_2(\lambda) = \sum_{k=1}^{m} c_{k2} + c_{k1}d_{k1} + c_{k0} \left( d_{k2} + \frac{d_{k1}^2}{2} \right),
\]

etc.

The center set \( \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) \) consists of points \( \lambda \in \mathbb{C}^{N_{p,q,m}} \) for which the corresponding generalized exponential polynomials \( f_{\lambda} \) are identically zero. Using a simple induction argument on \( m \), based on the comparison of exponential and polynomial growth of entire functions, one obtains that \( \sum_{k=1}^{m} P_k e^{Q_k} \equiv 0 \) if and only if there exists a subset \( I \subset \{1, \ldots, m\} \) such that

(a) \( Q_i = Q_j \) for all \( i, j \in I \), and \( Q_i \neq Q_j \) for all distinct \( i, j \notin I \) or if \( i \in I \) and \( j \notin I \);

(b) \( \sum_{k \in I} P_k = 0 \) and \( P_k = 0 \) for all \( k \notin I \).

Thus, \( \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) = \bigcup_I V_I \), where \( I \) runs over all distinct subsets of \( \{1, \ldots, m\} \) and \( V_I \subset \mathbb{C}^{N_{p,q,m}} \) is the subspace of complex dimension \((|I| - 1)(p + 1) + (m - |I| + 1)q\) defined by conditions (a) and (b); here \(|I|\) is the cardinality of \( I \).

**Remark 2.1.** All subspaces \( V_I \) with \(|I| = 1\) coincide, while all other \( V_I \) are pairwise distinct; hence, \( \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) \) is the union of \( 2^m - m \) pairwise distinct subspaces of \( \mathbb{C}^{N_{p,q,m}} \).

Let us present other methods related to the subject of the paper describing the center set \( \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) \).

If \( \mathcal{F}_k = \{f_{\lambda,k}\} \), where \( f_{\lambda,k} := P_k e^{Q_k}, \ 1 \leq k \leq m \), so that \( \sum_{k=1}^{m} f_{\lambda,k} = f_{\lambda} \), then \( \lambda \in \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) \) if and only if the functions \( f_{\lambda,1}, \ldots, f_{\lambda,k} \) are linearly dependent over \( \mathbb{C} \), that is, iff the Wronskian determinant \( W(f_{\lambda,1}, \ldots, f_{\lambda,k}) \) is identically zero. According to [VP, Lm. 4, Ex. 1], the family \( W(\mathcal{F}_1, \ldots, \mathcal{F}_m) \) has a form \( \{S_{\lambda}e^{T_{\lambda}}\}_{\lambda \in \mathbb{C}^{N_{p,q,m}}} \), where \( S_{\lambda} \) and \( T_{\lambda} \) are polynomials of degrees at most \( mp + \frac{1}{2}m(m - 1)(q - 1) \) and \( q \), respectively. Thus, due to properties (d), (e) and (f) of cyclicity,

\[
c(W(\mathcal{F}_1, \ldots, \mathcal{F}_m); \mu) \leq mp + \frac{1}{2}m(m - 1)(q - 1) \quad \text{for all} \quad \mu \in \mathbb{C}^{N_{p,q,m}}. \tag{2.7}
\]

Moreover,

\[
\mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) = \{\lambda \in \mathbb{C}^{N_{p,q,m}} : R_{\lambda} \equiv 0\},
\]

that is, the center set is defined as the zero set of at most \( mp + \frac{1}{2}m(m - 1)(q - 1) \) polynomials in \( \lambda \).

Next, (2.7) and Theorem 1.2 imply that the cyclicity of the family \( \mathcal{F} \) of generalized exponential polynomials satisfies the inequality

\[
c(\mathcal{F}; \mu) \leq m - 1 + mp + \frac{1}{2}m(m - 1)(q - 1) =: c_{p,q,m} \tag{2.8}
\]

For \( \mu \notin \mathcal{C}(\mathcal{F}; \mathbb{C}^{N_{p,q,m}}) \) (i.e., in the case of the multiplicity of zero at \( 0 \in \mathbb{D} \) of \( f_\mu \neq 0 \)) this result was proved in [VP].
Inequality (2.8) and property (c) of cyclicity give an alternative description of the center set of $F$ (cf. (2.5), (2.6)):

$$C(F; \mathbb{C}^{N_{p,q,m}}) = \{ \lambda \in \mathbb{C}^{N_{p,q,m}} : a_0(\lambda) = \cdots = a_{c_{p,q,m}}(\lambda) = 0 \}.$$ 

Also, for any relatively compact subset $K \subset \mathbb{C}^{N_{p,q,m}}$, there exists a positive number $r_K$ such that each function $f_\lambda, \lambda \in K$, has at most $c_{p,q,m}$ zeros in $\mathbb{D}_{r_K}$ (counting multiplicities).

Let us show that property (b') of cyclicity can be strengthened in our case.

**Proposition 2.2.** Integral closures in the ring of holomorphic polynomials on $\mathbb{C}^{N_{p,q,m}}$ of polynomial ideals $I_{c_{p,q,m}}(F)$ generated by $a_0, \ldots, a_{c_{p,q,m}}$ and $I(F)$ generated by all the $a_k$ coincide.

**Proof.** Let $\mathbb{D}^{N_{p,q,m}}$ be the open unit polydisk in $\mathbb{C}^{N_{p,q,m}}$. According to property (b) of cyclicity, there exists a constant $A > 0$ such that for all $k > c_{p,q,m}$ and $\lambda \in \mathbb{D}^{N_{p,q,m}}$,

$$|a_k(\lambda)| \leq A \cdot \sup_{\mathbb{D}^{N_{p,q,m}}} |a_k| \cdot \max_{0 \leq i \leq c_{p,q,m}} |a_i(\lambda)|. \tag{2.9}$$

Consider a polynomial map $\psi : \mathbb{C}^{N_{p,q,m}} \to \mathbb{C}^{N_{p,q,m}}$ which sends coordinates $c_{ki}$ to $c^{i+1}_{ki}$, $0 \leq i \leq p$, and coordinates $d_{kj}$ to $d^{kj}_{kj}$, $1 \leq j \leq q$, $1 \leq k \leq m$. Note that $\psi$ maps $\mathbb{D}^{N_{p,q,m}}$ onto $\mathbb{D}^{N_{p,q,m}}$ and, according to (2.6),

$$a_k(\psi(z \cdot \lambda)) = z^{k+1} \cdot a_k(\psi(\lambda)) \quad \text{for all} \quad k \in \mathbb{Z}_+, \quad z \in \mathbb{C}.$$

These and (2.9) imply, for all $\lambda \in \mathbb{C}^{N_{p,q,m}}$,

$$|a_k(\psi(\lambda))| \leq A \cdot \|\lambda\|^{k+1}_{\infty} \cdot \sup_{\mathbb{D}^{N_{p,q,m}}} |a_k| \cdot \max_{0 \leq i \leq c_{p,q,m}} \left\{ \frac{|a_i(\psi(\lambda))|}{\|\lambda\|^{i+1}_{\infty}} \right\}$$

$$\leq A \cdot \left( \max\{1, \|\lambda\|^{i}_{\infty}\} \right)^k \cdot \sup_{\mathbb{D}^{N_{p,q,m}}} |a_k| \cdot \max_{0 \leq i \leq c_{p,q,m}} |a_i(\psi(\lambda))|;$$

here $\| \cdot \|_{\infty}$ is $\ell_\infty$ norm on $\mathbb{C}^{N_{p,q,m}}$.

Finally, by the definition of the map $\psi$ we have for all $\lambda \in \mathbb{C}^{N_{p,q,m}}$ with $\|\lambda\|_{\infty} \geq 1$,

$$\|\lambda\|_{\infty} = \|\psi^{-1}(\psi(\lambda))\|_{\infty} \leq \|\psi(\lambda)\|_{\infty}.$$

Since $\psi$ is surjective, from the previous two formulas we get, for all $\lambda \in \mathbb{C}^{N_{p,q,m}}$,

$$|a_k(\lambda)| \leq A \cdot \left( \max\{1, \|\lambda\|_{\infty}\} \right)^k \cdot \sup_{\mathbb{D}^{N_{p,q,m}}} |a_k| \cdot \max_{0 \leq i \leq c_{p,q,m}} |a_i(\lambda)|. \tag{2.10}$$

These inequalities together with [HRT] Prop. 1.1 (c) imply that each polynomial $a_k$ with $k > c_{p,q,m}$ is integral over the ideal $I_{c_{p,q,m}}(F)$. This gives the required statement. \hfill $\square$

**Remark 2.3.** Proposition 2.2 and the Briançon–Skoda-type theorem, see, e.g., [HIS] Cor. 13.3.4], imply that all coefficients of the Maclaurin series expansion of the family $F_{c_{p,q,m}} = \{ f^{N_{p,q,m}}_\lambda \}$ belong to $I_{c_{p,q,m}}(F)$. An interesting question is whether the ideal $I_{c_{p,q,m}}(F)$ is integrally closed?

Next, we present the Cartan-type estimates for the family $F$.

For a nonpolynomial $f_\lambda = \sum_{k=1}^m P_k e^{Q_k}$ and $w \in \mathbb{C}$ by $R_{\lambda,w}$ we denote a (unique) positive number such that

$$\max_{1 \leq k \leq m, z \in \mathbb{D}_{R_{\lambda,w}(w)}} |Q_k| = 1;$$

here $\mathbb{D}_R(w) := \{ z \in \mathbb{C} : |z - w| < R \}$.\hfill $\square$
Proposition 2.4. There is a number $R_F > 0$ such that for each nonpolynomial $f_\lambda \in \mathcal{F}$ and $R \in (0, R_\lambda; w R_F)$, and fixed $H \in (0, 1]$ and $d > 0$ there exists a family of open disks \( \{D_{j;\lambda}\}_{1 \leq j \leq k} \), $k \in \mathbb{C}_{p,q,m}$, with $\sum_{j;\lambda}^{d} \leq \frac{(2H)^d}{d}$, where $r_{j;\lambda}$ is the radius of $D_{j;\lambda}$, such that

\[
(2.11) \quad |f_\lambda(z)| \geq \sup_{\mathbb{D}/e(w)} |f_\lambda| \cdot \left( \frac{H}{A} \right)^{c_{p,q,m} + 1} \quad \text{for all} \quad z \in \mathbb{D}/e(w) \setminus \bigcup_j D_{j;\lambda}.
\]

(Here $A = e^{(\frac{1}{n+1})^2}.$)

Proof. Recall that $\lambda = \{(c_{k_i})_{0 \leq i \leq p}, (d_{kj})_{1 \leq j \leq q} \}_{1 \leq k \leq m}$, where coordinates $c_{k_i}$ and $d_{kj}$ are coefficients of polynomials $P_{k}$ and $Q_{k}$, respectively. Consider the compact set

\[
K_1 := \{ \lambda \in \mathbb{C}^{N_{p,q,m}} : \sum_{k,i} |c_{k_i}| = 1, \quad \max_{1 \leq k \leq m} \sup_{\mathbb{D}} |Q_k| = 1 \} \subseteq \mathbb{C}^{N_{p,q,m}}.
\]

We apply property (h') of cyclicity (see (1.3)) to $K_1$ and $e = 1$, and set $R_F := R_1$ (here $R_1$ is as in property (h)). Then from the corresponding inequality (1.3) using that $A^{-1} \left( \frac{H}{A} \right)^{c(F,K)} \geq \left( \frac{H}{A} \right)^{c_{p,q,m} + 1}$ we get inequality (2.11) for $w = 0$, $R_{\lambda;w} = 1$. The general case is easily reduced to the previous one by the substitution $z \mapsto R_{\lambda;w}(z + w)$, $z \in \mathbb{C}$. We leave the details to the readers. \hfill $\square$

If $f_\lambda$ is a polynomial, then its degree is at most $p$ and it satisfies the analog of inequality (2.11) with $c_{p,q,m} + 1$ replaced by $p$ for all $R > 0$.

Similarly, property (h'') of cyclicity (see (1.4)) and the arguments of the proof of the previous proposition yield Remez-type inequalities for the family of generalized exponential polynomials.

Proposition 2.5. There exists an absolute constant $\hat{c} \geq 1$ such that for each nonpolynomial $f_\lambda \in \mathcal{F}$, an interval $I \subset \mathbb{D}/(R_{\lambda;w} R_F)/e(w)$ and a measurable subset $\omega \subset I$ the following inequality holds:

\[
(2.12) \quad \sup_I |f_\lambda| \leq \Phi \left( \frac{2m_1(I)}{m_1(\omega)} - 1 \right)^{\hat{c} \max\{1, c_{p,q,m}\}} \sup_\omega |f_\lambda|.
\]

If $f_\lambda$ is a polynomial, then instead of (2.12) we have the classical Remez inequality with the factor on the right-hand side replaced by $T_p \left( \frac{2m_1(I)}{m_1(\omega)} - 1 \right)$, where $T_p$ is the Chebyshev polynomial of degree $p$, valid for all measurable $\omega \subset I \subset \mathbb{C}$.

Remark 2.6. Using inequality (2.12) one obtains various local distributional inequalities for the family of multidimensional generalized exponential polynomials of the form $\sum_{k=1}^{m} P_k e^{Q_k}$, where $P_k$ and $Q_k$ are holomorphic polynomials on $\mathbb{C}^N$ of degrees $p$ and $q$, respectively (for the corresponding references and results see the Introduction and Section 2 in [Br2]).

Theorem 1.2 allows to obtain effective estimates for the cyclicity of more complicated families of functions (for instance, the family $\sum_{k=1}^{m} F_{k_1} e^{F_{k_2}}$, where all $F_{k_i}$ are families of generalized exponential polynomials). Using then the properties of cyclicity one obtains for such families results similar to those described in the present section.

3. Proof of Theorem 1.2

3.1. Equivalent Definition of Cyclicity. For a domain $O \subset \mathbb{V} \subset \mathbb{C}^n$ by $\mathcal{O}_c(\mathbb{D}, O)$ we denote the set of holomorphic maps $\varphi : \mathbb{D} \to O$ (i.e., each $\varphi$ is holomorphic in a suitable open neighbourhood of the closure $\mathbb{D}$ of $\mathbb{D}$.) Let $\mathcal{F} = \{ f_\lambda \}_{\lambda \in \mathbb{V}}$ be the family of holomorphic functions in the disk $\mathbb{D}_p$ depending holomorphically on $\lambda \in \mathbb{V}$ and $\varphi \in \mathcal{O}_c(\mathbb{D}, O)$ be such
that \( \varphi(\mathbb{D}) \not\subset \mathcal{C}(\mathcal{F}; O) \). We will assume that \( \mathcal{F} \neq 0 \), i.e., it contains nonidentically zero functions. Then the family \( \mathcal{F}_\varphi = \{ f_{\varphi(w)} \}_{w \in \mathbb{D} \setminus \varphi^{-1}(\mathcal{C}(\mathcal{F}; O))} \) consists of nonidentically zero holomorphic functions on \( \mathbb{D} \), and its center set \( \mathcal{C}(\mathcal{F}_\varphi; \mathbb{D}) = \varphi^{-1}(\mathcal{C}(\mathcal{F}; O)) \) consists of finitely many points. Let us consider the Maclaurin series expansion of \( \mathcal{F}_\varphi \),

\[
f_{\varphi(w)}(z) = \sum_{k=0}^{\infty} c_k(w) z^k, \quad z \in \mathbb{D}, \quad c_k \in \mathcal{O}(\mathbb{D}).
\]

Let \( b_\varphi(\mathcal{F}) \in \mathbb{Z}_+ \) be the minimal number such that the ideal \( \mathcal{I}(\mathcal{F}_\varphi; \mathbb{D}) \subset \mathcal{O}(\mathbb{D}) \) generated by all the \( c_k \) coincides with the ideal \( \mathcal{I}_{b_\varphi(\mathcal{F})}(\mathcal{F}_\varphi; \mathbb{D}) \subset \mathcal{O}(\mathbb{D}) \) generated by \( c_0, \ldots, c_{b_\varphi(\mathcal{F})} \). Then Theorem 1.3 of [Br1] states that for a compact subset \( K \subset V \)

\[
c(\mathcal{F}; K) = \lim_{\mathcal{O} \rightarrow K} \sup_{\varphi \in \mathcal{O}(\mathbb{D}, O)} b_\varphi(\mathcal{F}),
\]

where the limit is taken over the filter of open neighbourhoods of \( K \).

### 3.2. Proof of Theorem 1.2

According to property (a) of cyclicity, it suffices to prove the result for \( K = \{ \mu \} \), a point in \( V \). Without loss of generality we assume that \( \mathcal{F} \neq 0 \). Thus, due to (3.13), there exists \( \varphi \in \overline{\mathcal{O}}(\mathbb{D}, O), \varphi(\mathbb{D}) \not\subset \mathcal{C}(\mathcal{F}; O) \), where \( O \) is an open neighbourhood of \( \mu \) such that

\[
c(\mathcal{F}; \mu) = b_\varphi(\mathcal{F}).
\]

Recall that \( \mathcal{F} := \sum_{j=1}^{b_\varphi} \mathcal{F}_j \), where \( \mathcal{F}_j := \{ f_{\lambda,j} \}_{\lambda \in V}, 1 \leq j \leq k \). By \( S_\lambda \subset \mathcal{O}(\mathbb{D}_\rho) \) we denote the vector space generated by \( f_{\lambda,1}, \ldots, f_{\lambda,k}, \lambda \in V \), and by \( d_\lambda \) its complex dimension, so that \( 0 \leq d_\lambda \leq k \). We set

\[
d_\varphi := \sup_{w \in \mathbb{D}} d_{\varphi(w)}.
\]

By the definition, \( d_\varphi \geq 1 \) and there exist \( w_0 \in \mathbb{D} \) and \( 1 \leq j_1 < \cdots < j_{d_\varphi} \leq k \) such that \( S_{\varphi(w_0)} \subset \mathcal{O}(\mathbb{D}_\rho) \) coincides with the space generated by \( f_{\varphi(w_0); j_s}, 1 \leq s \leq d_\varphi \), and \( d_{\varphi(w_0)} = d_\varphi \).

We set

\[
g_{w;s} := f_{\varphi(w); j_s}, \quad W_{w,p} := W(g_{w;1}, \ldots, g_{w;s}), \quad 1 \leq s \leq d_\varphi, \quad \text{and} \quad W_{w;0} := 1.
\]

Then the center set of the family \( \{ W_{w; d_\varphi} \}_{w \in \mathbb{D}} \) consists of finitely many points and outside these points each \( g_w := \sum_{j=1}^{d_\varphi} f_{\varphi(w); j} \) satisfies the ordinary differential equation with meromorphic coefficients (the Frobenius formula, see [F]):

\[
\frac{W_{w; d_\varphi}}{W_{w; d_\varphi - 1}} \cdot \frac{d}{dz} \cdot \frac{W_{w; d_\varphi - 2}}{W_{w; d_\varphi - 2} \cdot W_{w; d_\varphi - 3}} \cdot \frac{d}{dz} \cdots \cdot \frac{W_{w; 1}}{w_{w; 0}} \cdot \frac{d}{dz} \cdot \frac{W_{w; 0}}{w_{w; 1}} \cdot g_w = 0.
\]

Next, consider the Taylor series expansions of families \( \{ g_w \} \) and \( \{ W_{w; s} \}, 1 \leq s \leq d_\varphi \),

\[
g_w(z) = \sum_{k=0}^{\infty} c_k(w) z^k, \quad z \in \mathbb{D}, \quad c_k \in \mathcal{O}(\mathbb{D});
\]

\[
W_{w; s}(z) = \sum_{k=0}^{\infty} b_{k,s}(w) z^k, \quad z \in \mathbb{D}, \quad b_{k,s} \in \mathcal{O}(\mathbb{D}), \quad 1 \leq s \leq d_\varphi.
\]

By definition, the center sets of these families are subsets of the center set, say \( C \), of the family \( \{ W_{w; d_\varphi} \}_{w \in \mathbb{D}} \). In particular, they are finite subsets of \( \mathbb{D} \). Thus, there exist univariate holomorphic polynomials \( P \) and \( P_s \) with zeros (if any) in \( C \), functions \( \tilde{c}_k \) and \( \tilde{b}_{k,s} \in \mathcal{O}(\mathbb{D}) \) and numbers \( t_s \in \mathbb{Z}_+, 1 \leq s \leq d_\varphi \), such that \( c_k = P \cdot \tilde{c}_k \) and \( b_{k,s} = P_s \cdot \tilde{b}_{k,s} \), for all \( k \) and \( s \), and each of the families \( \{ \tilde{c}_0, \ldots, \tilde{c}_{b_\varphi(\mathcal{F})} \} \) and \( \{ \tilde{b}_{0,s}, \ldots, \tilde{b}_{t_s,s} \}, 1 \leq s \leq d_\varphi \), has no common
zeros on \( \mathbb{D} \). (Therefore by the corona theorem, see, e.g., [GR], the ideals generated by these families in \( \mathcal{O}(\mathbb{D}) \) coincide with \( \mathcal{O}(\mathbb{D}) \).

We apply the operator on the left-hand side (3.15) with index \( w \).

(3.15) \[
\frac{\tilde{W}_{w;\phi}}{W_{w;\phi}} \frac{d}{dz} \frac{\tilde{W}_{w;\phi-1}}{W_{w;\phi-2}} = \frac{d}{dz} \frac{\tilde{W}_{w;\phi}}{W_{w;\phi-2}} \frac{d}{dz} \frac{\tilde{W}_{w;\phi-1}}{W_{w;\phi-1}} \frac{d}{dz} \frac{W_{w;\phi-2}}{W_{w;\phi-1}} \frac{d}{dz} \frac{W_{w;\phi}}{W_{w;\phi-1}} \frac{\tilde{g}_w}{\phi} = 0.
\]

Further, by the definition of \( b_\phi(F) \), there exists \( w_\ast \in \mathbb{D} \) such that \( \tilde{c}_w(F)(w_\ast) \neq 0 \) but \( \tilde{c}_w(w_\ast) = 0 \) for all \( k < b_\phi(F) (= c(F; \mu)) \). In what follows, by \( m_\ast \) we denote the multiplicity of zero of \( \tilde{W}_{w_\ast;\phi} \neq 0 \) at \( 0 \in \mathbb{D}, 1 \leq s \leq d_\phi \). By the definition of cyclicity (cf. [Br1], Th. 1.3) we have

(3.16) \[
c(F; \mu) > \max_{I \in K} \{c(F_I; \mu) + |I| - 1\}.
\]

We apply the operator on the left-hand side (3.15) with index \( w_\ast \) to \( \tilde{g}_{w_\ast} \) and compute the resulting multiplicity of zero at \( 0 \in \mathbb{D} \). By our hypothesis (3.16), the multiplicity of zero at \( 0 \in \mathbb{D} \) of

(3.17) \[
\frac{\tilde{W}_{w;\phi}}{W_{w;\phi-2}} \frac{d}{dz} \frac{\tilde{W}_{w;\phi-1}}{W_{w;\phi-1}} = \frac{d}{dz} \frac{\tilde{W}_{w;\phi}}{W_{w;\phi-1}} \frac{d}{dz} \frac{W_{w;\phi-2}}{W_{w;\phi-1}} \frac{d}{dz} \frac{W_{w;\phi}}{W_{w;\phi-1}} \frac{\tilde{g}_w}{\phi} = 0.
\]

The proof of the theorem is complete.

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