Designing linear lattices for round beam in electron storage rings using SLIM

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For some synchrotron light source beamline applications, a round beam is preferable to a flat one. A conventional method of obtaining round beam is to shift an electron storage ring’s tune close to a linear difference resonance. Then the linearly coupled beam dynamics is analyzed with perturbation theories, which have certain limitations. In this paper, we adopt the Solution by LInear Matrices (SLIM) analysis to calculate exact beam sizes to design round beam lattices. The SLIM analysis can deal with a generally linearly coupled accelerator lattice. The effects of various coupling sources on beam emittances and sizes can be studied within a self-consistent frame. Both the on- and off-resonance schemes to obtain round beams are explained with examples. The SLIM formalism for two widely used magnet models: combined-function bending magnets, and planar wigglers and undulators, is also derived.

I. INTRODUCTION

Round beam rather than a flat one is preferable for some beamline applications in the synchrotron light source community. Concurrently, an increased vertical beam size can significantly improve beam lifetime as well, particularly in extremely low emittance rings. Therefore, some future diffraction-limited light sources, such as ALS-U [1] and APS-U [2], are planning to operate with round beam lattices. Most of light source rings only have horizontal bending magnets, which leads to an intrinsically flat beam. Either dedicated devices, such as skew quadrupoles, or some imperfections in magnets, such as normal quadrupole roll errors, can couple the beam motion transversely. Conventionally, a geometric emittance re-distribution can be considered necessary.

In this investigation. The main reason for breaking down is that, when the linear coupling can significantly change the focusing property of the origin lattice, electron bunches will reach an a new equilibrium state.

In this paper, to design round beam lattices for light source rings, we adopt an exact and self-consistent analysis – the Solution by LInear Matrices (SLIM) technique, developed by Chao back in the 1970–1980s [13–15]. This analysis can yield fruitful results such as the trajectory of the electron distribution center and the beam sizes and shapes in phase space. Linear coupling effects among the horizontal, vertical, and longitudinal motions are included in a straightforward manner even without introducing the auxiliary Twiss functions. An alternate, and also exact approach is to calculate the coupled synchrotron-radiation integrals [16], which could also be used for this purpose, although it was not tested in this investigation.

The remainder of this paper is outlined as follows: Sect. III briefly explains the principle of the SLIM analysis. Sect. III introduces the linear couplings due to two random errors: Subsect. III A reviews the methods to take closed orbit errors into account; and the effect of normal quadrupole roll errors is investigated in Subsect. III B. Sect. IV explains two round beam schemes, i.e., on- and off-resonance optics using examples. Some discussion and a brief summary is given in Sect. V.
The formalism of two commonly used magnetic devices, combined-function bending magnets and planar wiggler and undulators, are given in the Appendix A and B respectively.

II. SLIM AND AUXILIARY TWISS FUNCTIONS

The detailed SLIM formalism can be found in the references [13–15, 17] and has been implemented in some accelerator codes, such as MAD-X [18]. Here we will briefly explain its basic principles. It deals with the motion of a charged particle in a linear electromagnetic device by purely using their transport matrices. First, symplectic one-turn linear matrices for a storage ring are used to compute the eigenvalues and eigenvectors. The eigenvalues indicate whether the linear motion is stable or not, and provides the fractional parts of the tunes when the motion is stable. The eigenvectors evolving along the ring provide information about closed orbit, beam size, etc. For electron rings, non-symplectic one-turn matrices including the radiation damping are used to compute the damping rates. Then equilibrium emittances are obtained by balancing the quantum diffusion and radiation damping in all radiating magnets around the ring. The particle distributions within a bunch along the ring can be given with 21 independent second moments. In the presence of linear couplings, no approximation is needed and therefore, the computations are exact. The ring’s global emittances and the local s-dependent beam sizes are derived within a self-consistent frame.

When there is no linear coupling, two SLIM’s second moments $\langle xx \rangle, \langle yy \rangle$ were confirmed to agree with the beam sizes obtained with Sands’s formalism [19] using the Courant-Snyder parameterization. Throughout this paper the National Synchrotron Light Source II (NSLS-II) double bend achromat (DBA) bare lattice was used for the proof of concept. Fig. 1 shows the magnet layout of its one supercell composed of two mirror symmetrical DBA cells, and the uncoupled beam sizes along s, computed using these two methods.

Linear coupling can then be introduced into the lattice by inserting two 0.2 m long skew quadrupoles inside each supercell as shown in Fig. 2. The first family of skew quadrupoles (SQ1) are located within the achromat dispersive sections, while the second family (SQ2) are located at the original dispersion-free sections. As a numerical experiment, all 30 skew quads were set with the same normalized focusing value $|K_1| = \frac{1}{B_ρ} \frac{dB_ρ}{dx}$ but given alternating polarities (+, −, + · · ·) along the ring. Here $B_ρ$ is the beam rigidity. The variation of the emittances with $|K_1|$ can then be computed with the SLIM analysis as shown in Fig. 3. The vertical emittance is observed to grow fast with the coupling strength, which leads to the sum of two emittances becoming variable and no longer constant. Eq. (1) is only valid when $K_1$ is sufficiently small.

No auxiliary Twiss functions are needed in the SLIM analysis as the particle distributions are obtained by computing 21 independent second moments projected on the corresponding subspaces. However, it is important to note that once the emittances $\epsilon_{x,y}$ and energy spread $\sigma_δ = \sqrt{\langle \delta \delta \rangle}$ are obtained with the SLIM analysis, the beam size along the ring can also be computed using the
Fig. 2, in which two dispersion functions \( \eta \) Ripken parameterization [7, 8] as shown in Eq. (2) and which leads to the breakdown of Eq. (1).

In other words, although the coupled Twiss functions are unnecessary in the SLIM frame, these auxiliary functions are still useful in interpreting the same physics meanings as the uncoupled case.

### III. COUPLING FROM RANDOM ERRORS

Linear coupling can be from not only dedicated magnets, such as skew quadrupoles and solenoids, but some random errors. In this section, the effects of two primary sources, the vertical closed orbit through sextupoles and the roll errors of normal quadrupoles, are discussed.

#### A. Closed orbit errors

In reality, small misalignments and magnet imperfections are unavoidable. Therefore, when circulating beam reaches an equilibrium distribution around a closed orbit, it always differs, at least slightly, from the design orbit. In this section, the technique for studying the effect of closed orbit on beam size is briefly explained. Closed orbit, if it exists, can be accurately obtained by performing an iterative one-turn 6-dimensional tracking till a convergence is reached. This method is widely used in many existing lattice codes, such as ELEGANT [20] and MAD [21]. Alternatively, the SLIM analysis adds a seventh component, which is always an unitary constant, to expand one-turn transport matrices to a 7 x 7 format. In this case, the closed orbit corresponds to the eigenvectors with eigenvalues of 1. The nonlinear kicks from nonlinear multipoles can also be accounted for by iteratively updating the transport matrices around the local closed orbit to reach a convergence.

Once the closed orbit is obtained, the transport matrix of each magnet needs to be updated with the reference to it. Thick-lens off-axis transport matrices can be computed in different ways. A simple kick-drift model is often used to obtain an approximation of thick elements. First, a thick element is sliced into multiple pieces, e.g., \( n \) pieces. Each piece is approximated by two components of drift with a thin-lens kick in-between. By concatenating those slices sequentially, a thick model is obtained. For example, a thick lens sextupole can be approximated as the sequential products of the following kick-drift matrices,

\[
R_{co,i} = M_d M_k M_d,
\]

with

\[
M_d = \begin{bmatrix}
1 & \frac{1}{\gamma} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

and

\[
M_k = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\lambda x_{0,i} & 1 & \lambda y_{0,i} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-\frac{1}{\gamma^2} (x_{0,i}^2 - y_{0,i}^2) & 0 & \lambda x_{0,i} y_{0,i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

Here \( R_{co,i} \) stands for the \( i \)th slice’s matrix around the local closed orbit coordinates \( (x_{0,i}, y_{0,i}) \), \( \lambda = \frac{l_s}{\gamma} \frac{\partial^2 B_y}{\partial x^2} \) is the normalized focusing integral of this slice, and \( \gamma \) is the relativistic factor. When each slice’s \( l_s \) is reasonably small, the thick-lens \( R_{co} = R_{co,n} \cdots R_{co,2} R_{co,1} \) will be sufficiently accurate. Since the kick-drift scheme is adopted, such thick-lens matrices are automatically symplectic. This 2\( nd \)-order thick-lens model has been adopted in our investigation. More complicated high order kick-drift schemes, as explained in ref. [22], are also available.

Another method to obtain a thick element’s off-axis transport matrix is to compute its higher order on-axis matrices first, then the following truncated Taylor map [23] is used,

\[
R_{co} = R + 2 T \lambda x_0 + 3 U x_0^2 + \cdots,
\]

where \( X_0 \) represents the 6-dimensional closed orbit vector at the magnet entrance and \( R, T, U \) are the \( 1^{st} - 3^{rd} \) order on-axis transport matrices which can be obtained.
with the truncated power series algorithm [24]. However, the truncated map is usually not perfectly symplectic, and can introduce an artificial damping or excitation. A symplectification process might be considered, if needed.

From Eq. (3), a vertical offset through sextupoles introduces linear coupling (non-zero $m_{23}$ and $m_{41}$). In modern high brightness light source rings, strong sextupoles are needed to correct chromaticity and enlarge dynamic aperture. Small closed orbit errors might introduce some coupling which can blow up the vertical emittance as shown in Fig. 4. The summation of transverse emittances increases gradually with the amplitude of vertical orbit. The idea of the constant emittance assumption from Eq. (1) appears valid only when the root-mean-square (RMS) closed orbit error is less than about 200µm for the NSLS-II storage ring.

Figure 4. Correlation between the emittances and RMS vertical orbit distortions $y_{co,rms}$. With gradually increasing orbit distortion, a very flat beam becomes a near-round one. When the closed orbit distortion is sufficiently large, it becomes flat (and tilted) but with larger emittances. The total emittance is observed to be blown up, albeit rather slowly.

The above computation is carried out when the machine’s fractional tune (0.22/0.26) is not near the difference resonance. When the tune is close to the difference resonance $\nu_I - \nu_{II} = n$, the vertical beam emittance can be easily increased as will be discussed in Sect. IV A.

B. Normal quadrupole roll errors

Another linear coupling source is from the random normal quadrupole roll errors. Quadrupoles can be aligned within several hundreds of microradians roll angles using the modern alignment techniques. If the linear tune is sufficiently separated from resonances, even though the total beam emittances are only slightly increased (about 1-2%), a significant part of the transverse emittance can be gradually redistributed to the vertical plane, as shown in Fig. 5. While the machine’s tune is sufficiently close to a difference resonance, even small roll errors can easily couple the transverse motion. This is the most common way to obtain an approximately round beam. Although it can be explained well with perturbation theory, this particular case will be re-analyzed exactly with the SLIM technique in Sect. IV A.

Figure 5. Emittance variation with quadrupole roll errors when the machine’s tune (0.22/0.26) is off a difference resonance.

IV. TWO ROUND BEAM SCHEMES

Thus far, we have explained how to compute exact beam emittances and sizes in the presence of various coupling sources using the SLIM analysis. To obtain round beam in an electron storage ring, its vertical emittance needs to be blown up with either dedicated coupling elements (skew quadrupoles, solenoids), or by shifting the machine’s tunes close to a difference resonance, and letting random coupling errors (such as quadrupole roll errors) to couple the emittances transversely. Below we quantitatively investigate two schemes with examples.

A. Round beam with on-resonance tune

Currently, the most common method to obtain round beam is by shifting the machine’s tune close to a difference resonance. With this on-resonance scheme, random imperfections are usually sufficient to couple the transverse motion. Because orbit displacements through sextupoles centers can be well controlled using the beam-based-alignment technique [25, 26], quadrupoles’ roll errors are regarded as the primary coupling sources, which are often at a level of several hundred microradians.

Although this on-resonance scheme can be explained with the perturbation theory, we re-investigated it with the exact SLIM analysis. Our design goal is to make the beam to be round at the short straight centers (SSC). First, the local quadrupoles $QL_{1-3}$ there were re-tuned to let the local eigenvectors absolute values to be close (or the coupled $\beta_{i(x,y),(I,II)}$ functions to be close if one prefers to use Eq. (2) instead.) Then the quadrupoles in the long straight sections $QH_{1-3}$ were tuned to shift the
fractional tune close to a difference resonance. Here we assumed the RMS roll angles for quadrupoles is 500 µrad, which can be easily achieved with the current alignment technique. The needed adjustments on quadrupoles settings as listed in Tab. I can switch the nominal lattice to an on-resonance round beam lattice. The corresponding beam emittances, fractional tune and beam sizes (at SCCs) are also listed there.

Table I. Parameters from flat beam to round beam with on-resonance tune

| Name       | original value | new value | unit |
|------------|----------------|-----------|------|
| K₁,QL₁    | -1.61785       | -1.39864  | m⁻²  |
| K₁,QL₂    | 1.76477        | 1.73601   | m⁻²  |
| K₁,QL₃    | -1.51868       | -1.58219  | m⁻²  |
| K₁,QH₁    | -0.64196       | -0.62274  | m⁻²  |
| K₁,QH₂    | 1.43673        | 1.42475   | m⁻²  |
| K₁,QH₃    | -1.75355       | -1.76958  | m⁻²  |
| RMS quad. roll | 0 | 500 | µrad |
| ϵ₁/ϵ₁I    | 2.096/1.7 × 10⁻⁴ | 1.185/0.925 | nm |
| fractional υ₁/υ₁I | 0.22/0.26 | 0.23/0.23 | |
| σₓ/σᵧ @ SSC | 62.08/0.45 | 43.1/44.2 | µm |

The eigen-emittances and beam sizes’ variation with the quadrupole roll angles are illustrated in Fig. 6 and Fig. 7, respectively. Because quadrupole roll errors are random, the corresponding beam emittances and sizes have some statistical fluctuations as represented by error bars there. These figures illustrate the needed quadrupoles roll errors to get a geometric round beam. With the quadrupole settings in Tab. I, a RMS error greater than 400 µrad will be sufficient to result in σₓ ≈ σᵧ. It is interesting to note that, the round beam size (∼ 44 µm) is significantly greater than the half of the uncoupled horizontal flat beam size (∼ 60 µm) as seen in Fig. 7. It can be explained with Eq. (2): although two modes’ emittances ϵ₁,II are approximately equal to each other, the sum of coupled β₁,(x,y) + β₁,(x,y) is greater than the uncouple β₁,(x,y) as illustrated in Fig. 8. The beam sizes for one supercell with one specific random seed is illustrated in Fig. 9 which can be used for the beam lifetime estimation.

If quadrupoles are aligned accurately with smaller roll angles, eigen-emittances might not be equally distributed by the resonance coupling, i.e., ϵ₁ > ϵ₁I. In this case, we can still tune the local quadrupoles (QL₁₋₃) to make β₁,(I,II),x < β₁,(I,II),y correspondly to get a geometric round beam.

Now we consider a more realistic situation by including the closed orbit distortion. Thus the skew quadrupolar components due to the vertical offsets through sextupoles can be taken into account. A simulation shows that the round beam profiles can still be maintained when the closed orbit errors are within a quite wide range (see Fig. 10). Once an even larger closed orbit distortion can drive the tune further away from the resonance, however, the beam profiles will gradually become flatter. Usually the closed orbit can be well controlled after the beam-based alignment technique is implemented, this error shouldn’t be a concern.

B. Round beam with off-resonance tune

The second scheme to obtain round beam would be to use some strong skew/tilted quadrupoles (or solenoids), which would allow the machine’s tune to stay off resonances. In this case, the perturbation theory is not applicable. To obtain round beam at specific locations, the linear lattice design can be summarized as an optimization problem:

1. Assign some magnet’s focusing strengths (such as normal and skew quadrupoles) K₁, and/or tilt angles φ as knobs.
Figures 8. On-resonance scheme: four coupled $\beta$ functions of one supercell for one random seed with 500$\mu$rad quadrupole roll errors. The local $\beta_{(I,II),x}$ at short straight sections are tuned close to $\beta_{(I,II),y}$ to obtain a geometric round beam there.

Figure 9. On-resonance scheme: horizontal and vertical beam sizes of one supercell for one specific random seed with 500$\mu$rad quadrupole roll errors. Such data can be used for the beam lifetime estimation.

Figure 10. On-resonance scheme: round beam profile can be maintained with gradually increased closed orbit errors when the machine’s tune stays near a difference resonance.

2. Simultaneously minimize:
   (a) $\epsilon_I$;
   (b) $\epsilon_{II}$;
   (c) $|\frac{\sigma_y}{\sigma_x} - 1|$ at specific $s$ locations.

3. subject to the following constraints:
   (a) the existence of stable linear solutions;
   (b) the fractional tune is sufficiently away from low order resonances;
   (c) $\epsilon_{I,II} < \epsilon_{\text{threshold}}$
   (d) the tilt angle $|\theta_{xy}| < \theta_{\text{threshold}}$;

Here, the stable linear solution constraint means that three eigenvalues of the one-turn symplectic matrix must stay on the complex unity circle, $\epsilon_{\text{threshold}}$ is the emittance threshold to guarantee the required brightness, $\tan 2\theta_{xy} = \frac{2\langle xy \rangle}{\langle xx \rangle - \langle yy \rangle}$ defines the tilt angle for the $x-y$ beam profile relative to the horizontal axis, $d$ represents an allowed tolerance for the beam to be a perfectly round shape. This linear lattice design eventually need to be optimized iteratively after taking the dynamic aperture and energy acceptance into account, but was not covered in this paper.

A possible linear solution for the NSLS-II ring is listed in Tab. II. Three families of quadrupoles $QL_1-3$, and two families of skew quadrupoles $[SQ_1, SQ_2]$ are used. The beam size for one supercell is illustrated in Fig. 11.

| Name   | original value | new value | unit |
|--------|----------------|-----------|------|
| $K_1,QL_1$ | -1.61785 | -1.23000 | $m^{-2}$ |
| $K_1,QL_2$ | 1.76477 | 1.70791 | $m^{-2}$ |
| $K_1,QL_3$ | -1.51868 | -1.61387 | $m^{-2}$ |
| $K_1,SQ_1$ | 0 | 0.085 | $m^{-2}$ |
| $K_1,SQ_2$ | 0 | -0.000 | $m^{-2}$ |
| $\epsilon_{I}/\epsilon_{II}$ | 2.096/1.7 $\times 10^{-4}$ | 2.095/1.460 | nmm |
| fractional $\nu_I/\nu_{II}$ | 0.22/0.26 | 0.41/0.34 | |
| $\sigma_x/\sigma_y$ SSC | 62.08/0.45 | 55.5/53.8 | $\mu m$ |

Once the imperfections of magnets and the misalignments are accounted for, beam will reach an equilibrium around a closed orbit with random quadrupole roll errors. A large quadrupole roll errors and closed orbit distortions could overstretch and tilt the round beams. For example, a simulation studying the deformation of a round beam...
due to vertical closed orbit distortions is shown in Fig. 12. For this particular lattice, since its tune is away from the difference resonance, the profile of the round beam is not noticeably sensitive to closed orbit distortions if they can be constrained to within $100^{-200} \mu m$.

Recently a hybrid flat-round beam scheme [27] is being under investigation for a steady-state microbunching [28] ring. A similar idea has been studied using a different method [29] for a diffraction-limited light source ring. The schematic diagram of this hybrid flat-round beam mode for a storage ring is illustrated in Fig. 13 in which a pair of coupling elements are used to generate a local closed section with a round beam, while maintaining a flat beam in the rest part of the ring. Obviously this scheme is also an off-resonance scheme, which can be exactly analyzed with the SLIM technique.

It is worth to note that the off-resonance scheme is more robust to random errors than the on-resonance one by comparing Fig. 10 and ?? . The reason is simply because the on-resonance scheme solely relies on the random errors amplified by the linear resonance. While the off-resonance scheme takes advantage of strong and dedicated linear coupling magnets and the tune is far away from the resonance.

V. SUMMARY

Two schemes to obtain round beam, i.e., with machine’s tune sitting on- or off-difference resonance, are studied with the SLIM analysis. This exact analysis accounts for the linear coupling by calculating the quantum diffusion and radiation damping rate around a 6-dimensional closed orbit. The on-resonance scheme takes the advantage of the random imperfections to drive the beam to be coupled when the tune stays close to a difference resonance. This scheme is easy to implement in a real machine, however, beam profiles, coupled optics functions, and dispersions etc. have quite large and uncontrollable fluctuations. The off-resonance scheme can provide a more controllable and robust round beam, but needs to integrate dedicated magnets into the lattice to generate strong coupling.

In modern light source storage rings, high-harmonic cavities are often used to stretch the bunch longitudinally. Under desired conditions, the local potential well basin is flattened. Strictly speaking, if a linear focusing is missing from any dimension, the SLIM analysis cannot be applied. In this case, the head and tail of the bunch are still focused by the nonlinear potential well, but the bunch center will coast because there is
no linear focusing at \( z = 0 \). To deal with this special case, an artificial, small linear RF focusing can be introduced. Although the bunches will have long longitudinal sizes, the transverse dynamics and energy spread computations are still correct because the longitudinal focusing is weak at most rings. However, this calculation will miss one class of effects, i.e., the nonlinear synchro-betatron coupling effects. When the longitudinal focusing comes purely from a nonlinear potential well, these nonlinear synchro-betatron effects could be significant. For cases like this, the SLIM analysis can still be used to design the linear lattice first, then detailed simulations are needed to study the nonlinear effects [27].

In real machine operations, maintaining round beam lattices can still be complicated and challenging. For example, when some users open or close their insertion device gaps, not only do the devices focusing properties change, but the equilibrium between the damping and quantum excitation changes simultaneously. Such changes are observed for all beamlines. Therefore, a feed-forward or feedback system (or both) would be needed to stabilize machine’s tune, coupling, local beam sizes etc. to account for these changes.

As previously mentioned, the nonlinear beam dynamics is not covered in this paper. Therefore, an iterative optimization between the linear and nonlinear lattice design would be needed once the SLIM analysis was deployed.

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Appendix A: Combined-function bending magnets

Two magnet models commonly used in light source rings, combined-function bending magnets and wigglers and undulators, are not included in Chao’s original literature. The appendices summarize their transport matrices, the corresponding changes when the radiation damping and closed orbit are considered. The quantum diffusion rate for planar wigglers and undulators are derived.

The Hamiltonian of combined function bending magnet is given in ref. [30]. Here a thick dipole is sliced into multiple pieces. The transport matrix around a closed orbit \( x_0, y_0 \) for one slice can be approximated as,

\[
R_{co,i} = M_d M_k M_d,
\]

with

\[
M_k = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-K_h l_s & 1 & 0 & 0 & K_h l_s x_0 + l_s h & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & K_1 l_s & 1 & 0 & -K_1 l_s x_0 \\
-K_h l_s x_0 - l_s h & 0 & K_1 l_s x_0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

here \( M_d \) is the drift’s transport matrix as already given in Eq. (9), \( K_h = K_1 + h^2 \) with \( h = \frac{B_x}{\beta_p} \) the reciprocal of bending radius. Correspondly, after including the transverse gradient contribution, the radiation damping changes the diagonal \( m_{66} \) element as,

\[
m_{66} = 1 - \frac{C_s E^3 l_s (B_x^2 + B_y^2)}{(B\rho)^2}
\]

Appendix B: Planar insertion devices

Insertion devices such as wigglers and undulators are used as the main X-ray radiators on modern light source rings. In this appendix a planar Halbach type wiggler or undulator model [31] is integrated into the SLIM framework. The \( s \)-dependent vertical magnetic field of such a devices reads as,

\[
B_y = \hat{B}_y \cosh(k_x x) \cosh(k_y y) \sin(ks),
\]

Here \( k_x^2 + k_y^2 = k^2 = (\frac{2\pi}{\lambda_l})^2 \), \( \lambda_l \) is the period length of the device, \( \hat{B}_y \) is the peak field in the vertical plane. The linear focusing of such type devices can be extracted as explained in ref. [32]. If the magnetic poles are sufficiently wide, the weak focusing in the horizontal plane is negligible, i.e., \( k_x \approx 0 \), the nonvanishing linear transport
The quantum diffusion rate for one period is

\[ m_{11} = m_{22} = m_{55} = m_{66} = 1 \]

\[ m_{12} = \lambda_l \]

\[ m_{33} = m_{44} = \cos(\sqrt{K_y} \lambda_l) \]

\[ m_{34} = \frac{\sin(\sqrt{K_y} \lambda_l)}{\sqrt{K_y}} \]

\[ m_{43} = -\sqrt{K_y} \sin(\sqrt{K_y} \lambda_l) \]

\[ m_{56} = \lambda_l / \gamma^2. \]

(B2)

Here \( K_y = \frac{B_y^2}{2(B_ρ)^2} \). Note that a complete wiggler and undulator period is an achromat system, therefore, \( m_{16} = m_{26} = 0 \). Other unlisted elements are all zeros. The radiation damping changes the diagonal element \( m_{66} \) for one period as

\[ m_{66} = 1 - \frac{C_s E^3 B_y^2 \lambda_i}{2\pi(B_ρ)^2}. \]

(B3)

The quantum diffusion rate for one period is

\[ \langle |A_{\pm k}|^2 \rangle_{\text{sideways}} = C_L \frac{\gamma^5 |E_{k(1,3)}(s)|^2}{c \alpha_k} \frac{2B_y^2 \lambda_i}{3\pi(B_ρ)^3}, \]

(B4)

here \( C_L = \frac{55}{48} \frac{r_e h}{m_c} \), \( h \) is the reduced Planck’s constant, \( r_e \) is the classical electron radius, \( m_c \) is the electron mass, \( c \) is the speed of light, \( \alpha_k \) is the radiation damping constant for mode \( k = I, II, III, E_{k5} \) is the 5th component of the normalized eigenvector for the mode \( k \) at the location of this period. A relative small contributions from sideways photons can be included as well,

\[ \langle |A_{\pm k}|^2 \rangle_{\text{sideways}} = C_L \frac{\gamma^5 |E_{k(1,3)}(s)|^2}{c \alpha_k} \frac{2B_y^2 \lambda_i}{3\pi(B_ρ)^3}, \]

(B5)

Here \( E_{k(1,3)}(s) \) is the 1st or 3rd component of the normalized \( k \)th mode’s eigenvector in the horizontal or vertical plane. Here all photon emission events are assumed to be uncorrelated. The intrinsic dispersion effect \( \overline{K} \) is ignored because it is negligible comparing with bending magnets in the third and fourth generation light sources rings.

When calculating closed orbit, nonvanishing elements for one period in the 7th column are

\[ m_{67} = -\frac{C_s E^3 B_y^2 \lambda_i}{2\pi(B_ρ)^2} \]

m_{77} = 1.

(B6)

Unlike bending magnets, \( m_{27} \) for a complete wiggler period is zero due to its alternating field polarities.

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