Coding in the Finite-Blocklength Regime: Bounds based on Laplace Integrals and their Asymptotic Approximations

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Abstract—In this paper we provide new compact integral expressions and associated simple asymptotic approximations for converse and achievability bounds in the finite blocklength regime. The chosen converse and random coding union bounds were taken from the recent work of Polyanskyi-Poor-Verdú, and are investigated under parallel AWGN channels, the AWGN channels, the BI-AWGN channel, and the BSC. The technique we use, which is a generalization of some recent results available from the literature, is to map the probabilities of interest into a Laplace integral, and then solve (or approximate) the integral by use of a steepest descent technique. The proposed results are particularly useful for short packet lengths, where the normal approximation may provide unreliable results.

Index Terms—Channel capacity, Coding for noisy channels, Converse, Finite blocklength regime, Shannon theory.

I. INTRODUCTION

Coding bounds in the finite blocklength regime have recently become quite popular for their ability to capture a compact (and meaningful) description of the physical layer to be used, e.g., for upper layers optimization. These bounds date back to the work of Shannon, Gallager, and Berlekamp [1], and have received new interest now that powerful coding and decoding techniques that reach the limits of reliable communications are commonly used.

Among the many results available in the recent literature, a widely used practice is to identify the limits of communication for a coding block of size $n$ via the so called normal approximation

$$R_{\text{NA}} = C - \sqrt{\frac{V}{n}} \log_2(e) Q^{-1}(P_e) + \frac{\log_2(n)}{2n}, \quad (1)$$

where $R$ is the rate, $C$ is the channel capacity, $V$ is the channel dispersion coefficient, and $P_e$ is the average error probability. The normal approximation (1) has been proved to be a valid $O(1/n)$ asymptotic approximation for both achievability and converse bounds [2–5], where an achievability bound is intended as a performance that can be achieved by a suitable encoding/decoding couple, while a converse bound is intended as a performance that outperforms any choice of the encoding/decoding couple. Hence (1) is a good estimate of the limits of information in the finite blocklength regime.

Incidentally, Shannon already noticed in 1959 that the normal approximation, but without the $\log_2(n)/2n$ term, applies to the additive white Gaussian noise (AWGN) channel case [6 §X], although he established the inverse equation expressing an error probability bound as a function of the rate. But bound (1) has also been investigated in [5], [7–9] for AWGN and parallel AWGN channels, the binary symmetric channel (BSC) channel, the discrete memoryless channel (DMC) channel, and the erasure channel. Upper layer optimization techniques that use (1) are available in [10–12].

Although (1) provides a compact and simple description of the limits of information for sufficiently large blocklength $n$, a $O(n^{-1})$ approximation might be unreliable for small $n$, which today corresponds to the rapidly growing scenarios of low-latency machine-to-machine communications. Therefore exact bounds are still of interest for clearly assessing the region of applicability of (1), as well as more refined (but simple) approximations are needed to cover the regions where (1) fails, and to get a better theoretical understanding of the limits of communication. The further request of new methodologies, to approach the calculation of achievability and converse bounds in scenarios that were not previously practicable, fully sets the focus of the paper. The recent work of Moulin [13] goes in the same direction, using alternative techniques based upon large deviation analysis in order to provide fourth order refinements to (1).

In this paper we investigate some achievability and converse bounds that were recently proposed by Polyansky, Poor, and Verdú in [9], for which we are able to provide simple integral expressions, for numerical evaluation purposes, and reliable asymptotic approximations that outperform (1) for small $n$. The significant bounds we investigate are the Polyanskyi-Poor-Verdú (PPV) meta-converse bound [9 Theorem 28], and the random coding union (RCU) achievability bound [9 Theorem 16]. Unlike the approach used by Shannon [6], the chosen bounds are built on statistical properties (as opposed to the geometric construction associated with the sphere/cone packing problems), and can therefore be applied to any channel model. They are also known to be consistent with the normal approximation limit, and therefore are expected to be very tight. The results we are proposing are a generalization of the findings available for the AWGN channel in [14]. More specifically, while the derivation in [14] relied on a asymptotic uniform series expansion available from the work of Temme [15], in the present work we apply the (general) method used by Temme to derive new asymptotic expansions for a number
of cases of interest not previously discussed in the literature, which, in turn, will provide new expressions for converse and achievability bounds associated with the finite-blocklength regime. In its essence, the leading idea is to give a Laplace integral expression to the probabilities of interest, which can then be solved, or approximated, by using the steepest descent method [16, §7]. Some of the tools we are exploiting are in very close relation to those used in the work of Martinez, i Fabregas, et al [17], [18] as well as in the work of Tan and Tomamichel [5] for evaluating the RCU achievability bound, thus certifying the usefulness of the Laplace transform approach in this kind of problems.

The specific contribution given in this paper covers parallel AWGN channels, the standard AWGN channel, the so called BI-AWGN channel (i.e., binary transmission in an AWGN context), and the BSC. In parallel AWGN channels, for which the bound available from the literature is the normal approximation [5], [8], [19], we are able to provide an integral expression and a $O(n^{-3})$ asymptotic approximation to the PPV meta-converse bound. For the AWGN sub-case, a $O(n^{-2})$ approximation is identified for the RCU bound. The result improves the ones available in [17], [18] since, although the Laplace transform approach we are using is similar, we avoid relaxing the RCU bound through Markov’s inequality and the Chernoff bound. In such a way we are also able to identify an expression which is consistent with (i.e., which simplifies into) the normal approximation [5]. For the BI-AWGN channel we are able to provide an integral expression and a $O(n^{-3})$ asymptotic approximation to the PPV meta-converse bound which significantly outperform the state-of-the-art bounds available from Shannon [11], Valenbois and Fossorier [20], and Wiechman and Sason [21]. Although the technique used in this paper is not able to deal with the RCU achievable bound, use of the weaker $\kappa \beta$ achievability bound of [9, Theorem 25] clearly reveals the 1 dB gap existing between what we could achieve and the performance of a standard message passing decoder, the gap being valid over a very large blocklength range (i.e., for short as well as for long codes). It is anyway worth mentioning that for very short packet sizes, one can construct codes (and decoders) that approach the achievability bound, and in some regions (high error rate) even beat it [22]. For the BSC we are finally providing reliable approximations to both the PPV meta-converse and the RCU achievable bounds, thus improving over the results available in [9] and based upon a series expression. Overall, all the above results show that the RCU and the PPV meta-converse bounds are both consistent with the normal approximation (hence asymptotically optimum), that the PPV meta-converse bound is in general neater to calculate and that it can be therefore taken as (a very good approximation to) the performance limit, at least for $n \geq 200$ where upper and lower limits are sufficiently close. With proper modifications, which are not discussed in this paper and which are left for future investigation, our techniques can also be applied to other channels of practical interest, namely the DMC channel, or the AWGN channel under a specific modulation constellation choice.

The rest of the paper is organized as follows. The PPV meta-converse, RCU, and $\kappa \beta$ bounds used in this paper are defined in Section II, and a brief description of the Laplace method used is available in Section III. The bounds are evaluated in the four scenarios of interest in successive sections: Section IV deals with the parallel AWGN case, Section V deals with the simple AWGN case, Section VI deals with the BI-AWGN channel, and Section VII deals with the BSC. To keep the flow of discussion, all theorems proofs (or proof sketches) are made available in the Appendix. Fort the sake of readability, the most significant results are summarized, at the end of each section, in a compact procedural form.

II. THE Bounds

In the following we will consider two fundamental results, namely, the PPV meta-converse bound of [9, Theorem 28] which is a converse bound setting a limit to the best obtainable performance, and the RCU bound [9, Theorem 16] which is an achievable bound setting a performance that can be obtained by a suitable encoding/decoding technique. The considered performance measure is the average error probability $P_e$, but we warn the reader that some of the following results (i.e., the PPV meta-converse and the related $\kappa \beta$ bounds) are also applicable in the maximum error probability sense.

By using the notation of [14], the converse bound of interest to our investigation can be defined as follows.

Theorem 1 (PPV meta-converse bound): Assume a codebook $C$ with $M$ codewords of length $n$, with associated rate $R = (\log_2 M)/n$, and further let the codewords belong to set $K$. Assume that codewords are transmitted over a channel described by the transition probability density function (PDF) $p_{y|x}(b|a)$, and denote with $P_e$ the average error probability (in decoding) when symbols are equally likely. Choose a (arbitrary) output PDF $p_y(b)$, and define the log-likelihood function

$$\Lambda(a, b) = \frac{1}{n} \ln \frac{p_{y|x}(b|a)}{p_y(b)}.$$ (2)

Denote the Neyman-Pearson missed detection (MD) and false alarm (FA) probabilities respectively as

$$P_{FA}(a, \lambda) = P[\Lambda(a, y) \geq \lambda], \quad y \sim p_y \quad (3)$$

$$P_{MD}(a, \lambda) = P[\Lambda(a, y) < \lambda], \quad y \sim p_{y|x}, x = a.$$ (4)

If probabilities (3) are independent of $a$ for $a \in K$, then for a fixed error probability $P_e$ the rate $R$ is upper bounded by

$$R \leq \overline{R} = -\frac{1}{n} \log_2 (P_{FA}(\lambda)),$$ (5)

where $\lambda$ is set by the constraint $P_{MD}(\lambda) = P_e$.

Proof: See [9, Theorem 28] and [14].

Reading the result the other way round, for a fixed rate $R$ the error probability $P_e$ is lower bounded by

$$P_e \geq P_e \geq P_{MD}(\lambda),$$ (6)

where $\lambda$ is set by the constraint $P_{FA}(\lambda) = 2^{-n R}$. We also note that, although the theorem is defined for the average error probability, its bounds are applicable also in the maximum error probability case. This is intuitively explained by the fact that the maximum error probability is by definition greater than the average error probability, hence an upper bound established
for a given average error probability $P_e$ corresponds to a bound that holds for a maximum error probability which is greater than $P_e$ and a similar rationale applies to (4). We finally observe that the natural choice for $p_y(b)$ is the capacity achieving expression which, as we will see, fully satisfies the theorem request on Neyman-Pearson probabilities (i.e., they are independent of $a$) in all the case of practical interest to this paper, and it further sets the bound to capacity for large values of $n$. In general, however, other densities $p_y(b)$ might be used for obtaining a meaningful result.

For the achievability bound we have the following result.

**Theorem 2 (RCU achievability bound):** By using the notation of Theorem 1, the best achievable average error probability is upper bounded by expression

$$P_e \leq \mathcal{P}_e = \mathbb{E} \left[ \min(1, 2^n R g(x, y)) \right]$$

where $g(x, y) = P[\Lambda(z, y) \geq \Lambda(x, y)]$, with $y \sim p_y|x, x \sim \mathcal{U}_K,$

and where $\mathcal{U}_K$ denotes a uniform distribution over set $K$. □

*Proof: See [9, Theorem 16].

Although, in general, any distribution can be chosen for both $z$ and $x$, and not necessarily the uniform distribution $\mathcal{U}_K$, the results of this paper all rely on a uniform distribution bound. Observe also that, similarly to the PPV meta-converse case, from (6) we can obtain, for any given error probability $P_e$, a lower bound on the best achievable rate, $R \geq R_e$. Also, the natural choice for $p_y(b)$ will again be the capacity achieving expression, which, in the scenarios of interest to this paper, will guarantee an independence of the function in (6) of the specific choice of $x$.

Since the derivation of the RCU bound may be challenging in many occasions, we also introduce the (weaker) $\kappa\beta$ bound proposed by [9], which relies on the MD and FA probabilities.

**Theorem 3 ($\kappa\beta$ bound):** By using the notation and the assumptions of Theorem 1, the best achievable rate is lower bounded by expression

$$R \geq \mathcal{R}_e = \max_{P_{MD}(\lambda) = P_e - \tau} \frac{1}{\log_2(\kappa(\tau))} - \frac{1}{n} \log_2(P_{FA}(\lambda))$$

where $\lambda$ is set by the constraint $P_{MD}(\lambda) = P_e - \tau$, and where

$$\kappa(\tau) = \min_{\mathcal{R}} \int_{\mathcal{R}} p_y(b) \, db$$

$$\text{s.t.} \int_{\mathcal{R}^c} p_y|x| p_x(a) \, db \leq 1 - \tau, \forall a \in \mathcal{K}$$

for some choice of region $\mathcal{R} \subset \mathbb{R}^n$, and with $\mathcal{R}^c$ the complement region.

*Proof: See [9, Theorem 25].

Although the $\kappa\beta$ bound is naturally a bound on the maximum error probability, it is also applicable in an average error probability sense. As for the PPV meta-converse bound case, this is due to the fact that the maximum error probability is by definition greater than the average error probability, hence a lower bound established for a given maximum error probability $P_{e,\text{max}}$ corresponds to a bound that holds for an average error probability that satisfies $P_e \leq P_{e,\text{max}}$. This is also the reason why, in the average error probability sense, the $\kappa\beta$ bound is weaker than the RCU bound.

We finally note that the PPV meta-converse and the $\kappa\beta$ bounds of Theorem 1 and Theorem 3 are correct under the assumption that $y$ is a continuous distribution, and require some modifications in the case of discrete variables. Modifications to the reference results will be discussed where needed (i.e., in the BSC case of Section VII).

### III. The Method

Before delving into the derivation of the bounds, we briefly review the general method which will be used throughout the paper, with appropriate modifications depending on the specific problem considered from time to time. The method is taken from [15], and suitably adapted to all the considered scenarios. The leading idea is that a probability of the form

$$P = \Pr \left[ \sum_{i=1}^{n} u_i \leq n \lambda \right]$$

where $u_i$’s are independent identically distributed (i.i.d.) continuous random variables with PDF $f_u(a)$, can be efficiently written (and numerically evaluated) by using standard Laplace transform properties (e.g., see [23]). As a matter of fact, summing i.i.d. random variables corresponds to convolving their PDFs, that is multiplying the corresponding Laplace transforms. Hence, (10) can be written in the equivalent (inverse Laplace transform) form

$$P = \frac{1}{2\pi i} \int_{\mathcal{L}} \left( \frac{F_u(s)}{s} \right)^n e^{-n s \lambda} \, ds$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} e^{-\alpha(s)} \, ds, \quad \alpha(s) = 2 \ln(F_u(s)) + 2 \lambda s$$

where $F_u(s)$ is the bilateral Laplace transform of $f_u(a)$, namely

$$F_u(s) = \int_{-\infty}^{+\infty} f_u(a) e^{-s a} \, da, \quad s \in \mathcal{R}$$

with $\mathcal{R} \subset \mathbb{C}$ the region of convergence of the integral.

The standard integration path $\mathcal{L}$ in (11) is any line of the form $\mathcal{L} = \{s \mid \Re(s) = \gamma \in \mathbb{C} \}$, but can be modified into a more convenient integration path. A good choice is to select a path where the Laplace transform does not oscillate, in such a way to simplify (and strengthen) the numerical evaluation of the integral. This corresponds to using a path $\mathcal{D}$ which transits through one saddle point $s_\alpha$ of the function $\alpha(s)$ (see (11)), i.e., through one zero of $\alpha'(s)$, and such that the imaginary part of $\alpha(s)$ is constant in $\mathcal{D}$. Since $\alpha(s)$ is analytic by construction, hence the Cauchy–Riemann equations apply, then the resulting path is ensured to be a steepest descent integration path, i.e., one where $\alpha(s)$ decreases (e.g., see [16, §7]). This results in

$$P = \frac{1}{2\pi i} \int_{\mathcal{D}} e^{-\alpha(s)} \, ds$$

where $\mathcal{D}$ has the form $\mathcal{D} = \{s \mid \Im(s) = \Im(s_\alpha), \alpha'(s_\alpha) = 0 \}$ or, more generally, is a connection of a number of descent paths. We warn the reader that an appropriate use of the Theorem of
Residues must be considered in (13) in case the selected path \( D \) crosses some of the poles of \( \alpha(s) \).

Although (13) is good enough for numerical integration purposes (mainly because of the non-oscillating nature of \( \alpha(s) \) on \( D \), some further properties can be exploited to obtain an asymptotic expansion as in [13]. This can be done by relying on the descending nature of \( \alpha(s) \) on \( D \), in such a way to parametrize the path \( D \) in the form \( D = \{ p_\alpha(u), u \in [a, b] \} \) where the complex map \( p_\alpha(u) \in D \) such that

\[
\alpha(p_\alpha(u)) = \alpha(s_n) - u^2. \tag{14}
\]

With the newly introduced notation we can write (13) in the form

\[
P = \frac{1}{i2\pi} e^{2\alpha(s_n)} \int_a^b p'_\alpha(u) p_\alpha(u) e^{-\frac{1}{2}u^2} du, \tag{15}
\]

where the path derivative is (from [4])

\[
p'_\alpha(u) = \frac{2u}{-\alpha'(p_\alpha(u))}. \tag{16}
\]

Appropriate \( O(n^{-k}) \) asymptotic expansions can then be obtained by approximating the function \( p'_\alpha(u)/p_\alpha(u) \) with its truncated Taylor expansion at \( u = 0 \), and by recalling that the integral \( \int_a^b u e^{-\frac{1}{2}u^2} du \) is solvable in the closed form by using the (lower) incomplete Gamma function. The coefficients of the Taylor expansion will be evaluated, using standard methods, either in the closed form or in numerical form, depending on the availability of a closed form expression for \( \alpha(s) \).

IV. PARALLEL GAUSSIAN CHANNELS

A. Scenario and notation

In the parallel Gaussian channel scenario the codeword \( x \in C \), of length \( n \), is assumed to be partitioned in \( K \) parallel channels, that is \( K \) blocks, \( x_k, k = 1, \ldots, K \), of equal length \( n/k \). Hence, in the following \( n \) is assumed to be an integer multiple of \( K \). In the chosen scenario, codewords are assumed to have constant-power in each block, that is \( x \in K \) with set \( K \) defined as

\[
K = \{ x = [x_1, \ldots, x_K] ||x_k||^2 = \frac{n}{K}P_k \} \tag{17}
\]

where \( P_k \) is the average power associated with the \( k \)th block. Note that we are therefore considering a situation where the bounds are evaluated for a specific power vector \( \{P_1, \ldots, P_K\} \), which simplifies equations, and which will not prevent us to later apply power availability constraints of the form

\[
\sum_{k=1}^K P_k = P_{\text{tot}}, \tag{18}
\]

by simple use of standard constrained optimization techniques (e.g., see the water-filling application in Sect. IV-G). Our choice is dictated by the fact that a closed-form identification of the optimum power assignment under constraint (18) is not available in general, except made for the normal approximation discussed in [8 §4] where the standard water-filling solution can be used.

On the transmission side each block sees a different AWGN channel. The AWGN channel experienced by the \( k \)th block has the form \( y_k = x_k + w_k \) with \( w_k \sim N(0, \sigma_k^2) \) a zero-mean Gaussian noise vector with independent entries. The corresponding signal to noise ratio (SNR) is \( \Omega_k = P_k/\sigma_k^2 \).

Channel transition PDFs are therefore of the form

\[
p_{y|x}(b|a) = \prod_{k=1}^K \frac{1}{(2\pi\sigma_k^2)^{1/2}} \exp\left(-\frac{\|b_k - a_k\|^2}{2\sigma_k^2}\right). \tag{19}
\]

Receive PDFs \( p_y(b) \) are also of interest. By assuming capacity achieving Gaussian inputs \( x_k \in N(0, I \Omega_k) \), we have

\[
p_y(b) = \prod_{k=1}^K \frac{1}{(2\pi\sigma_k^2(1 + \Omega_k))^{1/2}} \exp\left(-\frac{\|b\|^2}{2\sigma_k^2(1 + \Omega_k)}\right), \tag{20}
\]

which denotes an auxiliary output distribution that depends weakly on the input codeword through the presence of the SNR contribution \( \Omega_k \). In this context, being the powers \( \{P_k\} \) fixed, capacity is simply expressed by

\[
C = \frac{1}{K} \sum_{k=1}^K \frac{1}{2} \log_2(1 + \Omega_k). \tag{21}
\]

As a concluding comment, we observe that in (17) we are considering a scenario where the power is fixed to a given level. Generalizations to maximum power and average power settings can be obtained by using the results of [8 §4].

B. PPV meta-converse converse bound

In order to be able to calculate the PPV meta-converse bound of Theorem [11] and to derive suitable asymptotic approximations, we can mimick the asymptotic expansion approach of Temme [15] and the results of [14], both valid in a standard Gaussian (non parallel) channel case. The leading idea is to develop a simple integral form in the Laplace domain, which can then be used to define an asymptotic expansion by means of the method of steepest descent (e.g., see [16 §7]), as we explain in the following.

We start our derivation by showing that, in a parallel AWGN channels scenario, the FA and MD probabilities (3) are independent of the specific value of \( \alpha \), and are given by

\[
P_{FA}(\lambda) = P \left[ \sum_{k=1}^K u_k \Omega_k \leq n\lambda \right] \tag{22}
\]

\[
P_{MD}(\lambda) = P \left[ \sum_{k=1}^K \frac{v_k \Omega_k}{1 + \Omega_k} > n\lambda \right]
\]

where

\[
u_k \sim \chi\left(n \frac{1 + \Omega_k}{K \Omega_k}\right), \quad v_k \sim \chi\left(n \frac{1}{K \Omega_k}\right) \tag{23}
\]
and where $\chi(n,s)$ denotes a non-central chi-squared random variable of order $n$ and parameter $s$. □

**Proof:** See the Appendix.

Starting from this result, a compact integral expression for the probabilities of interest can be derived by use of standard Laplace transform properties.

**Theorem 5:** In a parallel AWGN channels scenario the FA and MD probabilities (3) can be expressed in the form

$$
P_{FA}(\lambda) = \frac{1}{i2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{\alpha(s)}{s}} ds$$

$$P_{MD}(\lambda) = \frac{1}{i2\pi} \int_{\mu-i\infty}^{\mu+i\infty} e^{\frac{\beta(s)}{-s}} ds$$  \hspace{1cm} (24)

where

$$\alpha(s) = 2\lambda s - \frac{1}{K} \sum_{k=1}^{K} \left( 2s(1 + \Omega_k) + \ln(1 + 2s\Omega_k) \right)$$

$$\beta(s) = \alpha(s + \frac{1}{2}) + 2 \ln(2) \cdot C + 1 - \lambda$$

and where $\gamma > 0$ and $0 > 2\mu > 1 - 1 / \max_k \Omega_k$. □

**Proof:** See the Appendix.

Observe that, for $K = 1$, the result of Theorem 5 is equivalent to the findings of [15] (and [14]), although with a different notation. This is also true for the results presented later on in the text and referring to the case $K = 1$.

**C. Applying the method of steepest descent**

Efficient solution of the integrals in (24) can be achieved by appropriately changing the integration path in such a way that it corresponds to a steepest descent path [16, [7]. This requires identifying suitable saddle points, that is, points $s_\alpha$ and $s_\beta$ such that $\alpha'(s_\alpha) = 0$ and $\beta'(s_\beta) = 0$. The function symmetry further ensures that saddle points are real valued. For the sake of easier tractability, we also require saddle points to belong to the Laplace regions of convergence which generated the exponential contributions in (24), that is to sets (see also the proof of Theorem 5 for details)

$$S_\alpha = \left\{ s : \Re(s) > -\frac{1}{2s_\alpha}, k = 1, \ldots, K \right\}$$

$$S_\beta = S_\alpha - \frac{1}{2}$$  \hspace{1cm} (26)

in such a way that poles of $e^{\frac{\alpha(s)}{s}}$ and $e^{\frac{\beta(s)}{-s}}$ are not crossed by the new integration path. This is a viable option, and because of convexity of $\alpha(s)$ in $S_\alpha \cap \mathbb{R}$ we also have

$$s_\alpha = \arg\min_{s \in S_\alpha, s \in \mathbb{R}} \alpha(s)$$  \hspace{1cm} (27)

which can be efficiently solved via standard convex optimization methods. Alternatively one can solve

$$\frac{1}{K} \sum_{k=1}^{K} \frac{1 + 2\Omega_k + 2s_\alpha\Omega_k^2}{(1 + 2s_\alpha\Omega_k)^2} = \lambda$$  \hspace{1cm} (28)

From the second of (25) it is then evident that the saddle point $s_\beta$ simply satisfies

$$s_\beta = s_\alpha - \frac{1}{2}.$$  \hspace{1cm} (29)

We also note that a closed form result for (27) is available only in the AWGN case, $K = 1$, providing

$$s_\alpha = -\frac{1}{2\Omega} + \frac{1}{4\lambda} \left( 1 + \sqrt{1 + 4\lambda(1+\Omega)} \right).$$  \hspace{1cm} (30)

The steepest descent path $L_\alpha$ is then identified by the equivalence $\Re(\alpha(s)) = \Re(\alpha(s_\alpha)) = 0$, with the additional request to include the saddle point $s_\alpha$ when crossing the real axis, which ensures uniqueness of the path. The path $L_\beta$ is instead simply given by

$$L_\beta = L_\alpha - \frac{1}{2},$$  \hspace{1cm} (31)

which is again a straightforward consequence of the definition of $\beta$ in (25). We also observe that, because of the symmetry of function $\alpha(s)$, steepest descent paths are symmetric with respect to the real axis, which, incidentally, correctly ensures the integrals (24) be imaginary valued, and probabilities real valued. Hence, the steepest descent path $L_\alpha$ can be parametrized by means of a variable $u \in \mathbb{R}$ such that $\alpha(s) = \alpha(s_\alpha) - u^2$ and $\sign(u) = \sign(\Re(s))$, that is

$$u = f(s) = \sign(\Re(s)) \sqrt{\alpha(s_\alpha) - \alpha(s)}, \ s \in L_\alpha.$$  \hspace{1cm} (32)

Operatively, this defines the path in the form

$$L_\alpha = \{ p_\alpha(u), u \in \mathbb{R} \},$$  \hspace{1cm} (33)

with map $u \rightarrow p_\alpha(u) = s$ identified for $u \geq 0$ by constraints

$$\Re(s) \geq 0, \ \ \alpha(s) = \alpha(s_\alpha) - u^2$$

$$\Im(s) = 0$$

while for $u < 0$ it simply is $p_\alpha(u) = p_\alpha^*(u)$. It also naturally is $p_\alpha(0) = s_\alpha$. Alternatively, the constraint $\Re(\alpha(s)) = 0$ in (34) can be expressed in the (equivalent) explicit form

$$\lambda = \frac{1}{K} \sum_{k=1}^{K} \left( \frac{1 + \Omega_k}{1 + 2s_\alpha\Omega_k} \right) + \frac{2 \ln(2)}{\Re(2s_\alpha) \cot^{-1}\left( \Re(2s_\alpha) + \frac{1}{\Omega_k} \right)}.$$  \hspace{1cm} (35)

which, with some effort, reveals that $\Re(p_\alpha(u)) \leq s_\alpha$, and that the edge points are $p_\alpha(\pm\infty) = -\infty \pm i \pi/(2\lambda)$. Unfortunately, a closed form expression is not available for path $p_\alpha(u)$, and we must resort to numerical methods. Nevertheless, this will not prevent us in identifying an asymptotic expansion in the closed form.

By a change of the integration variable in (24), the above ensures validity of the following result.

**Theorem 6:** In a parallel AWGN channels scenario the FA and MD probabilities (3) can be expressed in the form

$$P_{FA}(\lambda) = 1(\neg s_\alpha) + \frac{1}{i2\pi} e^{\frac{\alpha(s_\alpha)}{s_\alpha}} \int_{-\infty}^{\infty} c_\alpha(u) e^{-\frac{u^2}{2}} du$$

$$P_{MD}(\lambda) = 1(s_\beta) - \frac{1}{i2\pi} e^{\frac{\beta(s_\beta)}{-s_\beta}} \int_{-\infty}^{\infty} c_\beta(u) e^{-\frac{u^2}{2}} du$$  \hspace{1cm} (36)

where $1(\cdot)$ is the unit step function, where

$$c_\alpha(u) = \frac{p_\alpha(u)}{p_\alpha(u)} - \alpha'(p_\alpha(u)) p_\alpha(u) = -c_\alpha^*(u),$$  \hspace{1cm} (37)

and where $x$ stands for either $\alpha$ or $\beta$. The path in $\alpha$ is derived, for $u \geq 0$, according to (34) where $s = p_\alpha(u)$, and for $u < 0$
it is extended by symmetry \( p_\alpha(u) = p_\alpha^*(u) \). The path in \( \beta \) is \( p_\beta(u) = p_\alpha(u) - \frac{1}{2} \).

**Proof:** See the considerations above. □

The anti-Hermitian symmetry of \( c_\alpha(u) \) stated in (37) can be further exploited in order to identify a real valued integral in the form

\[
P_{FA}(\lambda) = 1(-s_\alpha) + \frac{1}{\pi} e^{\frac{\pi \lambda}{2 \sin(\phi)}} \int_0^\infty \Im[c_\alpha(u)] e^{-\frac{\pi}{2}u^2} \, du = 1(s_\beta) - \frac{1}{\pi} e^{\frac{\pi \lambda}{2 \sin(\phi)}} \int_0^\infty \Im[c_\beta(u)] e^{-\frac{\pi}{2}u^2} \, du. \tag{38}
\]

Incidentally, note that the unit step functions in (36) and (38) simply take into account on whether the integration paths are crossing the pole at 0, in which case the Theorem of Residues was used to correctly identify the result. The expression in (37) is instead a straightforward consequence of the derivative of the inverse in (32). A summary of the overall procedure is available in Fig. 1.

We finally observe that, with a little effort, in the Gaussian case \( K = 1 \) we are able to identify from (35) the steepest descent path \( \mathcal{L}_\alpha \) in the parametric form

\[
p_\alpha(\phi) = \frac{e^{j\phi}}{4\lambda \sin(\phi)} \left( 1 + \sqrt{1 + 4\lambda \frac{1+\Omega}{2} \sin^2(\phi)} \right) - \frac{1}{2\lambda}, \tag{39}
\]

where \( \sin(\phi) = \sin(\phi)/\phi \), and \( \phi \in (-\pi, \pi) \).

**D. Asymptotic expansion**

When the map \( p_\alpha(u) \) is known either numerically or in the closed form, then the integral form of Theorem 6 can be readily used for numerical evaluation since the derivative of \( \alpha(s) \) is known from (25). As an alternative, for sufficiently large values of \( n \) we can resort to a (tight) asymptotic expansion.

The asymptotic expansion is found by using a Taylor series expansion at \( u = 0 \) for the functions \( c_\alpha \) and \( c_\beta \), and by exploiting [24] eq. 2.3.18.2. As we anticipated, the Taylor series coefficients in (40) can be evaluated in the closed form even in the absence of a closed form expression for the steepest descent paths, since the derivatives of \( p_\alpha(s) \) can be inferred from (inversion of) (32). The result can be formulated as follows.

**Procedure 7:** In a parallel AWGN channels scenario the FA and MD probabilities (3) allow the asymptotic expansion

\[
P_{FA}(\lambda) = 1(-s_\alpha) + \frac{e^{\frac{\pi \lambda}{2 \sin(\phi)}}}{\sqrt{2\pi n^2}} \sum_{k=0}^\infty \frac{(2k)!}{k! (2n)^k} \frac{c_{\alpha,2k}}{i} \tag{40}
\]

\[
P_{MD}(\lambda) = 1(s_\beta) - \frac{e^{\frac{\pi \lambda}{2 \sin(\phi)}}}{\sqrt{2\pi n^2}} \sum_{k=0}^\infty \frac{(2k)!}{k! (2n)^k} \frac{c_{\beta,2k}}{i}
\]

where \( c_{\alpha,k} \) and \( c_{\beta,k} \) are the Taylor expansion coefficients of the functions \( c_\alpha \) and \( c_\beta \), respectively, as given in (37). Operatively, for the FA probabilities the following iterative procedure can be used to identify the coefficients \( c_{\alpha,2k} \):

1) Evaluate the real valued Taylor series coefficients of the function \( \alpha(s) \) in \( s = s_\alpha \) according to rule: \( a_0 = \alpha(s_\alpha), \)

\[
a_1 = 0, \quad a_m = \frac{1}{K} \sum_{k=1}^K \left( \frac{1+\Omega_k}{1+2s_\alpha \Omega_k} + \frac{1}{m} \right) \left( -2\Omega_k \right)^m \tag{41}
\]

for \( m \geq 2 \). Note that derivatives follow an alternating sign rule, and \( a_2 \geq 0 \).

2) Evaluate the imaginary valued Taylor series coefficients in \( s = s_\alpha \) of the function \( f(s) = p_\alpha^{-1}(s) \) (defined in (32)) according to rule: \( f_0 = 0, f_1 = -i\sqrt{2}, \) and

\[
f_m = \frac{1}{2f_1} \left( a_{m+1} + \sum_{\ell=2}^{m-1} f_{\ell} f_{m+1-\ell} \right). \tag{42}
\]

3) Evaluate the Taylor series coefficients in \( u = 0 \) of the function \( p_\alpha(u) \) according to rule: \( p_0 = s_\alpha, p_1 = 1/f_1, \) and

\[
p_m = \frac{1}{f_1} \sum_{k=1}^{m-1} p_k p_{m-k}. \tag{43}
\]

with coefficients \( p_{\ell,m} \) defined by \( p_{m,m} = f_1^m \) and

\[
\ell,m = \frac{m-\ell}{k=1} \left( \frac{k! m+k+\ell}{(m-\ell)} \right) f_{k+1}^m p_{\ell,m-k} \tag{44}
\]

The even coefficients \( p_{2m} \) are real valued, and the odd coefficients \( p_{2m+1} \) are imaginary valued.

4) Evaluate the Taylor series coefficients in \( u = 0 \) of the function \( c_\alpha(u) \) (defined in (37)) according to rule:

\[
c_{\alpha,m} = \frac{1}{s_\alpha} \left( (m+1) p_{m+1} - \sum_{\ell=1}^{m-1} p_\ell c_{\alpha,m-\ell} \right) \tag{45}
\]

The even coefficients \( c_{\alpha,2m} \) are imaginary valued, and the odd coefficients \( c_{\alpha,2m+1} \) are real valued.

The coefficients \( c_{\beta,2k} \) for the MD probabilities can be found similarly, that is:

1) Evaluate the real valued Taylor series coefficients of the function \( \beta(s) \) in \( s = s_\beta \) according to rule: \( b_0 = \beta(s_\beta), b_1 = 0, \) and \( b_m = a_m \) as given by (41) for \( m \geq 2 \).

2-4) Use the method defined for the FA probabilities by replacing \( \alpha \rightarrow \beta \) and \( a_m \rightarrow b_m \).

**Proof:** See the Appendix. □

**E. Reliable approximations**

Procedure 7 is general, in that it provides a method to identify the Taylor coefficients of \( c_\alpha(u) \) of any order. However, the fact that the function \( c_\alpha(u) \) in (38) is weighted by \( e^{-\frac{\pi}{2}u^2} \) ensures that only the first coefficients are needed in practice to obtain (for sufficiently large \( n \)) a very reliable result. In this context, it is of interest investigating both the first-term and second-term approximations. We specify them by assuming that

\[
s_\beta < 0 < s_\alpha \tag{46}
\]

holds, which implies the absence of the contribution of the residues \( 1(-s_\alpha) \) and \( 1(s_\beta) \). In turn, this corresponds to \( P_{FA} < \frac{1}{2} \) and \( P_{MD} < \frac{1}{2} \), i.e., to neglecting the cases of very limited interest where \( P_e > \frac{1}{2} \) (too large error probability) and \( R < \frac{1}{n} \).
The condition is also sufficient to ensure and the series to the first terms we obtain the following result. 

**Corollary 8:** In a parallel AWGN channels scenario the FA and MD probabilities (3) allow the approximations

\[
\frac{1}{n} \ln P_{FA}^{(1)} = \frac{1}{n} \ln P_{FA}^{(1)} + \frac{1}{n} \ln \left(1 + g(s_\alpha)\right),
\]

and the \(O(n^{-3})\) asymptotic approximations

\[
\frac{1}{n} \ln P_{FA}^{(2)} = \frac{1}{n} \ln P_{FA}^{(1)} + \frac{1}{n} \ln \left(1 + g(s_\alpha)\right),
\]

\[
\frac{1}{n} \ln P_{MD}^{(2)} = \frac{1}{n} \ln P_{MD}^{(1)} + \frac{1}{n} \ln \left(1 + g(s_\beta)\right),
\]

where

\[g(s) = \frac{12a_2a_4 - 15a_3^2}{8a_2^2} - \frac{3a_3}{2a_2} - \frac{1}{a_2s^2}.
\]

and where the functions \(\alpha(s)\) and \(\beta(s)\) were defined in (25), \(s_\alpha\) was defined in (27), \(s_\beta = s_\alpha - \frac{1}{2}\), and \(a_2, a_3, a_4\) were defined in (41). The approximations hold provided that (46) is satisfied.

**Proof:** This is a straightforward application of Procedure [7].

We underline that, when (46) does not hold, then the residues in [40] are active, hence equations (47) and (48) still apply, but the FA and MD probabilities must be replaced by their complement probability counterparts.

As we already discussed, both approximation (47) and (48) are expected to be tight for a wide parameter range, but the asymptotic expansion lacks of a measure of tightness. Top this aim, we could numerically evaluate the integral via Theorem [6] or (39), which is an option. Alternatively, the approach proposed in [14] Theorem [7] can be used to identify on wether the chosen approximation is an upper or a lower bound to the true probability value, which in turn provides a measure of the approximation error (given by the difference between the two bounds). The rationale can be enunciated by the following result.

**Theorem 9:** Under (46), a sufficient condition for the validity of bounds

\[P_{FA}^{(2)} \leq P_{FA} \leq P_{FA}^{(1)},\]

\[P_{MD}^{(2)} \leq P_{MD} \leq P_{MD}^{(1)},\]

is that inequalities

\[1 - g(s_\alpha)u^2 \leq s_{\alpha} \sqrt{a_2} \Theta(c_\alpha(u)) \leq 1,\]

hold for \(u \in [0, \infty)\), and for \(x\) taking both values \(\alpha\) and \(\beta\). The condition is also sufficient to ensure that

\[\overline{R}^{(1)} \leq \overline{R} \leq \overline{R}^{(2)},\]

\[\overline{P}_c^{(2)} \leq \overline{P}_c \leq \overline{P}_c^{(1)},\]

where the superscript \((K)\) identifies a bound derived by use of either the \(K = 1\) or the \(K = 2\) term approximation.

**Proof:** See [14] Theorem [7].

The bounds of Theorem [9] have been found to apply to all the cases of practical interest.

**F. Relation with the normal approximation**

Approximation (47) is useful to derive a significant property of the PPV meta-converse bound, namely that it approaches capacity as \(n\) grows to infinity, and, more importantly, that it is consistent with the normal approximation (1). This provides a very neat relation with the results on parallel AWGN channels available from the literature, and characterizes the PPV meta-converse bound as an asymptotically optimum bound. Although this is already known from [38], approximation (47) provides a very simple way to assess the result.

**Theorem 10:** In a parallel AWGN channels scenario, the PPV meta-converse bound is consistent with the normal approximation (1) that uses the channel dispersion coefficient

\[V = \frac{1}{K} \sum_{k=1}^{K} \frac{\Omega_k(2 + \Omega_k)}{2(1 + \Omega_k)^2},\]

in the sense that \(\overline{R} = R_{FA} + O(1/n)\) for \(n \to \infty\).

**Proof:** See the Appendix.

In any case, observe that the insights provided by the true bound can be much more relevant than those given by the normal approximation, especially for low values of \(n\) where the two quantities may be significantly different.

A summary of all the above results is available in Fig. [1].

**G. Water-filling application example**

In Fig. 2 we illustrate the PPV meta-converse bound in a multiple-carrier transmission scenario with 128 complex carriers, i.e., with \(K = 256\) (recall that each symbol is a QAM symbol in an OFDM context, hence there are two real valued symbols per each channel use). The channel attenuation is the one shown in Fig. 2(a), the noise level is assumed \(N_0 = 10^{-12}\) W/Hz, and an available power of \(P_{tot} = 0.5\) W is considered. The PPV meta-converse bound is derived as the outcome of the optimization problem

\[
\max - \frac{1}{n} \log_2 P_{FA}^{(k)}(\lambda; s_\alpha, \{\Omega_k\})
\]

w.r.t. \(\lambda, s_\alpha, \Omega_k = P_k/\sigma_k^2\)

s.t. \(\alpha'(s_\alpha; \lambda, \{\Omega_k\}) = \frac{1}{n} \ln \frac{P_{MD}^{(k)}(\lambda; s_\beta = s_\alpha - \frac{1}{2}, \{\Omega_k\})}{P_c}
\]

\[
\sum_k \Omega_k \sigma_k^2 = P_{tot},
\]

which is solved by using standard routines in MatLab.

The resulting PPV meta-converse bound on rate is illustrated in Fig. 2(c), together with the normal approximation (1). Performance is illustrated as a function of the code length, and the code length is expressed in number of OFDM symbols, \(n/K\). The PPV meta-converse bound was derived by application of Corollary [8]. The very small gap observed between approximations (47) and (48), and the applicability of the sufficient condition of Theorem [9] ensure that the bound illustrated in figure is precise. Note the closeness between the PPV meta-converse and the normal approximation values, except for small block lengths.

An illustration of the optimal power allocation is given in Fig. 2(d) for the case where the block length corresponds to

\(\text{single symbol codebook, } M = 1\). Hence, by limiting the series to the first terms we obtain the following result.
PPV meta-converse bound – Generalities

The bounds in rate and error probability can be respectively expressed in the form (see Theorem 1 and 5)
\[ R = -\frac{1}{n} \log_2(P_{FA}(\lambda)) \], with \( \lambda \) the solution to \( P_{MD}(\lambda) = P_e \)
\[ P_e = P_{MD}(\lambda) \], with \( \lambda \) the solution to \( P_{FA}(\lambda) = 2^{-nR} \),
where functions \( P_{FA}(\lambda) \) and \( P_{MD}(\lambda) \) are defined in the box below.

PPV meta-converse bound – Integral representation of FA and MD probabilities

a) Identify the saddle point \( s_\alpha \) by solving the convex problem \( s_\alpha = \arg\min_{s \in \mathbb{C}, Re s \geq 0} \alpha(s) \) where function \( \alpha(\cdot) \) is defined in (25).

b) Identify the steepest descent path \( p_\alpha(u) \). For a given \( u \geq 0 \) the map \( p_\alpha : u \to s \) can be built by looking for the complex value \( s \) satisfying \( \Re(s) \geq 0 \) and guaranteeing \( \alpha(s) = \alpha(s_\alpha) - u^2 \) and \( \Im(\alpha(s)) = 0 \). Incidentally, it also is \( \lim_{u \to \infty} p_\alpha(u) = -\infty + i\pi/(2\lambda) \).

c) Evaluate FA and MD probabilities using the expressions (see Theorem 4 and (39))
\[ P_{FA}(\lambda) = 1(-s_\alpha) + \frac{1}{\pi} e^{-\alpha(s_\alpha)/2} \int_0^\infty \Re \left[ \frac{2u}{-\alpha'(p_\alpha(u)) \cdot p_\alpha(u)} \right] e^{-\frac{1}{2}\alpha^2(u)} du \]
\[ P_{MD}(\lambda) = 1(s_\alpha - \frac{1}{2}) - \frac{1}{\pi} e^{-\alpha{(s_\alpha)+2\ln(2)}C+1/4} \int_0^\infty \Re \left[ \frac{2u}{-\alpha'(p_\alpha(u)) \cdot (p_\alpha(u) - \frac{1}{2})} \right] e^{-\frac{1}{2}\alpha^2(u)} du , \]
where \( C \) is defined in the lower box, and the derivative \( \alpha'(\cdot) \) can be derived in the closed form from the first of (25).

PPV meta-converse bound – Asymptotic expansion for FA and MD probabilities

A \( O(n^{-k}) \) asymptotic expansion of FA and MD probabilities can be derived by using Procedure 7. When \( 0 < s_\alpha < \frac{1}{2} \), specific \( O(n^{-2}) \) and \( O(n^{-3}) \) expansions are available as from Corollary 8.

PPV meta-converse bound – Normal \( O(n^{-1}) \) approximation

The normal approximation is (see (1) and Theorem 10)
\[ \bar{R} \simeq C - \sqrt{\frac{V}{n}} \log_2(e) Q^{-1}(P_e) + \frac{\log_2(n)}{2n} , \quad \bar{P_e} \simeq Q \left( \sqrt{\frac{n}{V}} \left( \frac{C - R}{\log_2(e)} + \frac{\ln(n)}{2n} \right) \right) \]
where
\[ C = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{2} \log_2(1 + \Omega_k) , \quad V = \frac{1}{K} \sum_{k=1}^{K} \frac{\Omega_k(2 + \Omega_k)}{2(1 + \Omega_k)^2} . \]

Fig. 1. Summary of the most significant results for a parallel AWGN channels scenario. The scenario is defined in Section IV-A.

one OFDM, that is, \( n = K \). Interestingly, in the comparison with the common water-filling result, an excess of power is loaded in "good" carriers, while less power (or even no power) is loaded in "bad" carriers. The gap with the common water-filling approach reduces with longer lengths \( n \). Incidentally, a perfectly equivalent effect is appreciated by using the normal approximation (e.g., see (19)), although the resulting power allocation may be slightly different.

V. THE AWGN CHANNEL

A. Scenario and notation

The AWGN channel is a special case of the parallel AWGN channels scenario, where \( K = 1 \). Because of the presence of a unique component, the index \( k \) will be dropped in the following, so that \( \Omega \) is the reference SNR, \( P \) is the (average) power associated to transmission, \( n = n_1 \) is the packet length, and \( \sigma^2 \) is the noise variance.

Although with a different notation, the PPV meta-converse bound that we obtain from Section V correspond to the outcomes of (14), to which we refer the interested reader for further insights and bound properties. In the following, we instead discuss the RCU achievability bound, for which a reliable approximation can be derived by exploiting some of the methods already used in Section IV.

B. RCU achievability bound

As a first step towards our final aim, we write the result of Theorem 2 in a more usable and compact form, by revealing that the number of variables involved in the definition of the RCU bound is limited. We have:
**Theorem 11:** In a AWGN channel scenario, the RCU bound can be expressed in the form

$$P_e = E \left[ \min(1, 2^{nR} g(q)) \right], \quad \text{with } q \sim \mathcal{N}(\Omega e_n, \frac{1}{n} \Omega I_n),$$

where $e_n$ is a vector of length $n$ with entry one in first position and the rest set to zero, and $I_n$ is the identity matrix of order $n$, and where

$$g(q) = P \left[ \|q\| \geq q_1 \right], \quad \eta = \frac{\tau}{\sqrt{n-1+\tau^2}},$$

with $\tau$ a $t$-distributed random variable of order $n - 1$.\[55\]

**Proof:** See the Appendix.

Note that the RCU can be also made independent of $q$ and solely dependent on variable $\rho = q_1 / \|q\|$, to obtain a very compact formulation which is much more suitable for numerical evaluation since it requires a one dimensional integration. The result can be expressed as follows.

**Theorem 12:** In an AWGN channels scenario, the RCU bound can be formalized in the form

$$P_e = \int_{-1}^{\lambda} f_\rho(a) da + \int_{\lambda}^{1} g(a) f_\rho(a) da$$

where $f_\rho$ is the PDF of $\rho = q_1 / \|q\|$ with random vector $q$ defined in [55], and where $g(a) = P \left[ \eta > a \right]$ with random variable $\eta$ defined in [56]. The value $\lambda$ is identified by the constraint $-\frac{1}{n} \log_2 g(\lambda) = R$.\[58\]

**Proof:** The proof is trivial and it is left to the reader.\[57\]

Incidentally, we observe that we can safely consider $\lambda > 0$. As a matter of fact, since $g(0) = \frac{1}{2}$ by the symmetry of the distribution of $\eta$, a negative $\lambda$ implies $R < \frac{1}{2}$, that is $M = 2^{nR} < 2$ which corresponds to the presence of a unique symbol for transmission, i.e., to the absence of communication.

The statistical description of $\rho$ and $\eta$ can be given in the closed form, in such a way that meaningful and compact approximations to the functions in (57) can be identified. For the statistical description of $\rho$ we can exploit approximations available from the literature for non-central $t$-distributed random variables (e.g., see [23]). We have:

**Theorem 13:** The PDF of random variable $\rho$ is given by

$$f_\rho(a) = \frac{(1-a^2)^{\frac{n-3}{2} \rho}}{2^{\frac{n-1}{2}} \Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2}) / \Gamma(n)} U(n - \frac{1}{2}, -\sqrt{n} a)$$

with $a \in (-1, 1)$, and where $U(\alpha; x)$ is Weber’s form for the parabolic cylinder function and $\Gamma(\alpha)$ is the gamma function. The PDF can also be written in the form $f_\rho(a) = e^{u_n(a)}$, where $u_n(a)$ can be approximated asymptotically by the expression

$$u_n(a) = u^{(0)}(a) + \frac{\ln(n)}{2n} - \frac{u^{(1)}(a)}{2n} + O(n^{-2}),$$

\[(59)\]
where
\[ u^{(0)}(a) = \frac{1}{2} \ln(1 - a^2) - 2a^2 + (aa)^2 + aa\sqrt{1 + (aa)^2} + \ln(aa + \sqrt{1 + (aa)^2}) \]
and where we used \( \alpha = \sqrt{n/4} \).

**Proof:** See the Appendix.

We observe that even stronger asymptotic approximations can be derived from [25], but we verified that, in the cases of interest, the \( O(n^{-2}) \) approximation given by Theorem 15 is sufficient.

### C. Laplace transform expressions

Concerning \( \eta \), that is function \( g(a) \), we can instead exploit the Laplace integration method of Section IV. To this aim, we rewrite \( g(a) \) via a Laplace dual expression, to obtain a result which is suitable for being approximated by use of the steepest descent method.

**Theorem 14:** Function \( g(a) = P[\eta \geq a] \) can be written via the Laplace integral
\[ g(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mp\gamma(s)} e^{-as} ds \]
where \( \mu < 0 \) and \( \gamma(s) = G_{\nu}(s) + as \), and where the function \( G_{\nu} \) is defined as
\[ G_{\nu}(s) = \frac{1}{\nu} \ln \left( \frac{\Gamma(\nu) (\frac{1}{2} \nu s)^{1-\nu} I_{\nu}(\nu s)}{\Gamma(\nu+1)} \right) \]
with \( \Gamma \) the confluent hypergeometric limit function, and \( I_{\nu} \) the modified Bessel function of the first kind.

**Proof:** See the Appendix.

The integral can be further simplified by exploiting a steepest descent path. The idea and the procedure are perfectly identical to the MD probability expression derived in Section IV and provide (compare to (60))
\[ g(a) = 1(s_\gamma) - \frac{1}{\sqrt{2\pi}} e^{\mp\gamma(s_\gamma)} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} c_\gamma(u) du \]
where \( s_\gamma \) is the steepest descent path which corresponds to choice \( \Im[\gamma(s)] = 0 \), and which transits through the real valued saddle point
\[ s_\gamma = \text{argmin}_{s \in \mathbb{R}} \gamma(s) \, . \]

**Theorem 15:** Function \( g(a) = e^{-ns_\gamma(a)} \) can be asymptotically approximated via the expression
\[ n(a) = e^{\frac{1}{2} \ln(1 - a^2) + \frac{1}{2n} \ln(n) + O(n^{-1})} \, . \]

**Proof:** The proof is left to the reader.

To be usable, a \( O(n^{-2}) \) approximation is needed for \( G_{\nu} \) and its derivatives in Theorem 15. This can be obtained from standard asymptotic expansions for the modified Bessel functions of the first kind (e.g., see [25]), and provides the following result.

**Theorem 16:** Function \( G_{\nu} \) in (62) can be asymptotically approximated by expression
\[ G_{\nu}(s) = \sqrt{1 + s^2} - 1 - \frac{1}{\nu} \ln(\sqrt{1 + s^2} \, 2) + O(1/\nu^2) \, . \]

**Proof:** See the Appendix.

Not that (67) reveals that
\[ G_{\nu}(s) = \sqrt{1 + s^2} - 1 - \ln \left( \frac{1 + \sqrt{1 + s^2}}{2} \right) \, . \]

The rapid convergence to the limit value (68) is illustrated in Fig. 3 displaying the function \( G_{\nu}(x) + \rho x \) for \( \rho = 0.5 \), which is the core function used in the definition of \( \gamma \). The behavior is equivalent for any \( 0 < \rho < 1 \).

![Figure 3: Function \( G_{\nu}(x) + \rho x \) for \( \rho = 0.5 \) and \( \nu = \frac{1}{2}, 1, 2, 5, 10, 20, \infty \).](image)
D. Reliable approximation

With the notation assessed above the RCU bound (57) assumes the form
\[
P_e = \int_{-1}^{\lambda} e^{n u_n(\lambda)} \, d\lambda + \int_{\lambda}^{1} e^{n[u_n(\lambda)-v_n(\lambda)+R \ln(2)]} \, d\lambda \quad (70)
\]
where \( \lambda \) is defined by the relation \( v_n(\lambda) = R \ln(2) \). We also observe from (69) that
\[
\lambda = \sqrt{1 - 2^{-2R} e^{\ln(n)/n} + O(n^{-1})} \quad (71)
\]
holds for \( \lambda \).

Integral (70) can be approached numerically, which is an option, but a very compact and analytical result is obtained by reliably approximating it by use of an approach similar to the Laplace’s method [16, \S5] (which exploits a quadratic approximation of the exponent at its maximum value). In particular, since in the considered context the maximum value lies outside of the integration region, a linear approximation of the exponent will be exploited. Hence, we obtain the following result.

**Theorem 17:** In a AWGN channel scenario, for sufficiently large \( n \) the RCU bound can be approximated in the form
\[
\frac{1}{n} \ln P_e = w(0)(\lambda) - \frac{\ln(2\pi n)}{2n} - \frac{1}{n} \ln(w(0)(\lambda)) + O(n^{-2}) \quad (72)
\]
where
\[
w(0)(a) = \sqrt{\left(1 - a^2\right) a^2} \cdot \left(1 + \alpha a w(1)(a)\right) \\
\cdot \left(2a(1-a^2)w(1)(a) - a\right) \\
\cdot \left(2a - 2a(1-a^2)w(1)(a)\right)
\]
\[
w(1)(a) = \alpha a + \sqrt{1 + (\alpha a^2)} , \quad \alpha = \sqrt{\frac{\bar{n}}{2\Omega}} ,
\]
and where \( \lambda \) is defined by the equivalence \( v_n(\lambda) = R \ln(2) \) with \( v_n \) expressed by the \( O(n^{-2}) \) approximation (65) using (67). For a correct applicability of the theorem, \( w(0)(\lambda) \) must be a positive value.

**Proof:** See the Appendix.

As discussed in the proof, the request \( w(0)(\lambda) > 0 \) practically corresponds to requiring small error probabilities, \( P_e < \frac{1}{n} \), and does not limit the applicability of the result. Incidentally, \( O(n^{-3}) \) results can also be derived with some additional effort, by deriving more refined asymptotic expressions for \( v_n \) and \( v_n \), and by subsequently applying the rationale of (26).

The asymptotic expression given by Theorem 17 further allows to easily verify the validity of the normal approximation for the RCU bound, thus providing an alternative derivation to the result of [5].

**Theorem 18:** In a AWGN channel scenario, the RCU achievability bound is consistent with the normal approximation (1) that uses the channel dispersion coefficient
\[
V = \frac{\Omega(2 + \bar{n})}{2(1 + \bar{n})^2} ,
\]
in the sense that \( R = R_{\text{NA}} + O(1/n) \) for \( n \to \infty \).

**Proof:** See the Appendix.

A summary of all the above results, valid for the AWGN channel scenario, is given in Fig. [4] where we also explicitly recall that the RCU bound can be interpreted both as a bound on error probability, as well as a bound on rate (see the paragraph after Theorem 7).

E. Numerical examples

A few examples of application of the PPV meta-converse bound that uses Corollary 8 with \( K = 1 \) and of the RCU bound that uses Theorem [7] are shown in Fig. [5]. In Fig. [5] (a) and (b) we illustrate the scenarios depicted in [9, Fig. 6-7, 12-13], with the major difference that the normal approximation in [9] is neglecting the \( \log_2(n)/(2\bar{n}) \) term in (1), hence it is less accurate. For reasons of space/readability, the Shannon bounds [6] are not shown. These are, however, very close to the PPV meta-converse and the RCU bounds, the RCU bound slightly improving the achievability limit.

The closeness between the upper PPV meta-converse and the lower RCU bounds can be further appreciated in Fig. 5(c), which provides a wider look onto the spectral efficiency \( \rho = 2R \) versus the \( E_b/N_0 \) measure for quite a large range of blocklengths \( n \). Plots are given for those regions where the discussed \( O(n^{-2}) \) approximations provide a reliable result.

Note in figure how the normal approximation (1) provides a very good fit. However, as illustrated in Fig. 5(d) under an error probability perspective, the normal approximation might be loose for low block lengths, where the RCU bound occurs at a (non negligible) 0.1-0.2 dB distance in SNR.

Overall, the indications we get from Fig. 5 is that the normal approximation (for both achievability and converse bounds) is meaningful and simple with moderate to large blocklengths, while for short blocklengths the PPV meta-converse and the RCU bounds provide a reliable yet sufficiently simple alternative.

VI. BINARY-INPUT AWGN CHANNEL

A. Scenario and notation

Binary coding under an AWGN channel, also known as the BI-AWGN channel, is a scenario that corresponds to a soft-decoding receiver implementation, and which sets the limits for the (classical) performance of a binary code. In this case the codewords set of interest is
\[
\mathcal{K} = \{1, -1\}^n ,
\]
and the channel PDF is a standard AWGN expression which we write in the form
\[
p_{y|x}(b|a) = \left( \frac{\Omega}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{1}{2}\Omega \|b-a\|^2} ,
\]
where \( \Omega = P/\sigma^2 \) is the reference SNR value. Capacity is achieved with equally likely input symbols, \( p_x(a) = 2^{-n} \), and guarantees an output PDF
\[
p_y(b) = \prod_{i=1}^{n} \sqrt{\frac{\Omega}{2\pi} e^{-\frac{1}{2}\Omega(b_i-1)^2} + e^{-\frac{1}{2}\Omega(b_i+1)^2}} ,
\]
which provides a capacity expression of the form
\[
C = 1 - \frac{\Omega}{2\pi} \int e^{-\frac{1}{2}\Omega(b-1)^2} \log_2(1 + e^{-2\Omega b}) \, db .
\]
RCU achievability bound – Generalities

The bounds in rate and error probability can be respectively expressed in the form (see Theorem 12)

\[ R = -\frac{1}{n} \log_2(g(\lambda)) , \] with \( \lambda \) the solution to \( h(\lambda) = P_e \)

\[ T^c_e = h(\lambda) , \] with \( \lambda \) the solution to \( g(\lambda) = 2^{-nR} , \)

where functions \( h(\lambda) \) and \( g(\lambda) \) are defined in the box below.

---

RCU achievability bound – Integral representation of functions

Function \( h(\lambda) \) is defined as (see Eq. 57)

\[ h(\lambda) = \int_{-1}^{\lambda} f_{\rho}(a)da + \int_{\lambda}^{1} \frac{g(a)}{g(\lambda)} f_{\rho}(a)da \]

where \( f_{\rho}(a) \) is given in Eq. 58. Function \( g(a) \) can be identified via the following procedure:

a) Identify the saddle point \( s_\gamma \) by solving \( s_\gamma = \arg \min_{s} \gamma(s) \), where \( \gamma(s) = G_{\frac{1}{2}}(s) + a \), and where \( G_{\nu}(\cdot) \) is defined in Eq. 62.

For large values of \( n \), the asymptotic approximation (67) can be used.

b) Identify the steepest descent path \( p_\gamma(u) \). For a given \( u \geq 0 \) the map \( p_\gamma : u \rightarrow s \) can be built by looking for the complex value \( s \) satisfying \( \Im(s) \geq 0 \) and guaranteeing \( \gamma(s) = \gamma(s_\gamma) - u^2 \) and \( \Im(\gamma(s)) = 0 \).

c) Evaluate \( g(a) \) using the expression (see Eq. 63)

\[ g(a) = 1(s_\gamma) - \frac{1}{\pi} e^{\pi \gamma(s_\gamma)} \int_{0}^{\infty} \Im \left[ \frac{2u}{\pi \gamma(p_\gamma(u))p_\gamma(a)} \right] e^{-\pi u^2} du , \]

where the derivative of \( \gamma \) is \( \gamma'(s) = G'_{\frac{1}{2}}(s) + a \) with \( G'_{\nu}(s) = I_{\nu}(\nu s)/I_{\nu-1}(\nu s) \) (this derives from standard properties of the modified Bessel functions).

---

RCU achievability bound – Asymptotic expansion of functions

A \( O(n^{-2}) \) expansion of functions \( h \) and \( g \) is given by (see Theorem 15 and Theorem 17)

\[ \frac{1}{n} \ln(h(\lambda)) = u^{(0)}(\lambda) - \frac{\ln(2\pi n)}{2n} - \frac{1}{n} \ln(w^{(0)}(\lambda)) + O(n^{-2}) \]

\[ -\frac{1}{n} \log_2(g(\lambda)) = -\frac{1}{2 \ln(2)} \left( G_{\frac{1}{2}}(s_\gamma) + \lambda s_\gamma \right) + \frac{1}{2n} \log_2 \left( \pi n s_\gamma^2 G''_{\frac{1}{2}}(s_\gamma) \right) + O(n^{-2}) \]

where function \( w^{(0)} \) is defined in Eq. 60, function \( w^{(0)} \) is defined in Eq. 73, the asymptotic approximation (67) is used for \( G_{\frac{1}{2}}(\cdot) \) and its derivatives, and \( s_\gamma \) is the solution to \( G'_{\frac{1}{2}}(s_\gamma) + \lambda = 0 \). For the expansion to hold we require \( s_\gamma < 0 \) and \( w^{(0)}(\lambda) > 0 \).

---

RCU achievability bound – Normal \( O(n^{-1}) \) approximation

By Theorem 18 the normal approximation corresponds to that of the PPV meta-converse bound available in Fig. 1 (set \( K = 1 \)).

---

B. PPV meta-converse bound

The PPV meta-converse bound can be derived by using the same techniques employed in the parallel AWGN channel case, that is through a representation via a Lagrange integral and by its asymptotic approximation. As a starting point, we observe that the compact expression for the FA and MD probabilities is independent of the specific codeword choice, ant that Theorem 1 can be used in the following form.

Theorem 19: In a AWGN channel scenario with binary codewords the FA and MD probabilities (3) can be expressed by

\[ P_{FA}(\lambda) = P \left[ \sum_{i=1}^{n} h(u_i) \leq n\lambda \right] \]

\[ P_{MD}(\lambda) = P \left[ \sum_{i=1}^{n} h(v_i) > n\lambda \right] \]

where \( u_i \sim \mathcal{N}(d_i \Omega, \Omega) \), \( p_{d_i}(1) = p_{d_i}(-1) = \frac{1}{2} \)

\[ v_i \sim \mathcal{N}(\Omega, \Omega) \]

and \( h(x) = \ln(1 + e^{-2x}) \).

\[ \square \]
Fig. 5. The AWGN channel: (a) Converse and achievability bounds for $P_e = 10^{-3}$ and $\Omega = 0$ dB; (b) Converse and achievability bounds for $P_e = 10^{-6}$ and $\Omega = 20$ dB; (c) Converse and achievability bounds on spectral efficiency $\rho = 2R$ versus $E_b/N_0 = \Omega/\rho$ power efficiency measure, for some values of $n$, and for $P_e = 10^{-5}$; (d) Packet error rate bounds at rate $R = \frac{1}{2}$ as a function of SNR $E_b/N_0$, for some values of $n$. 
Proof: See the Appendix.

The Laplace transform method of Theorem 5 can then be exploited to write the result in an equivalent form. The proof is left to the reader, since it is a simple re-application of the concepts that led to Theorem 5.

Theorem 20: In a AWGN channel scenario with binary codewords the FA and MD probabilities (3) can be expressed as in (24) where

\[
\alpha(s) = \beta(s - 1) - 2 \ln(2) + 2\lambda \\
\beta(s) = 2\left(\lambda s + \ln(H(s))\right)
\]

(81)

and where

\[
H(s) = \frac{1}{\sqrt{2\pi\Omega}} \int_{-\infty}^{\infty} e^{-\frac{1}{\Omega}(x-\Omega)^2} e^{-s\cdot h(x)} dx
\]

(82)

is a Laplace transform which converges for any \( s \in \mathbb{C} \). □

Proof: The proof is left to the reader.

Incidentally, by exploiting the above notation we can denote capacity in the form

\[
C = 1 + \frac{H'(0)}{\ln(2)},
\]

(83)

where \( H' \) is the derivative of (82) with respect to \( s \).

The main difficulty involved with Theorem 20 is to identify a usable analytic expression for (82), which is hardly obtainable. In any case, numerical evaluation can lead to the identification of saddle points, and then steepest descent paths for numerical integration in the form of Theorem 6. A further option of interest is to exploit an asymptotic approximation in the form discussed in Procedure 7 which leads to the compact expressions (47) and (48). The result is in this case a mixture of analytical expressions and numerical integrations.

Procedure 21: In a AWGN channel scenario with binary codewords the FA and MD probabilities (3) allow the asymptotic expansion (40) whose coefficients can be evaluated according to Procedure 7 with the following substitutions:

1) the functions \( \alpha \) and \( \beta \) must be defined as in (81),
2) \( s_\alpha \) and \( s_\beta \) must be derived, respectively, from equivalences \( \alpha'(s_\alpha) = 0 \) and \( \beta'(s_\beta) = 0 \), to satisfy \( s_\alpha = s_\beta + 1 \);
3) the coefficients \( a_k \) must be defined as \( a_k = \beta^{(k)}(s_\beta)/k! \), with \( \beta^{(k)} \) the \( k \)th derivative of the function \( \beta \) in (81).

In this context, the \( O(n^{-2}) \) and \( O(n^{-3}) \) approximations given, respectively, by (47) and (48) are valid under the assumption that \( s_\alpha > 0 \), and \( s_\beta < 0 \). □

Proof: The proof is identical to the one of Procedure 7 and it is left to the reader.

We observe that, by using notation

\[
H^{(\ell)}(s) = \frac{1}{\sqrt{2\pi\Omega}} \int_{-\infty}^{\infty} e^{-\frac{1}{\Omega}(x-\Omega)^2} e^{-s\cdot h(x)} (-h(x))^\ell dx,
\]

(84)

for the derivatives of \( H \), the first few derivatives of \( \beta \) are given by

\[
\beta'(s) = 2\left(\lambda + \frac{H'(1)(s)}{H(s)}\right)
\]

\[
\beta''(s) = 2\left(\frac{H''(2)(s)}{H(s)} - \left(\frac{H'(1)(s)}{H(s)}\right)^2\right)
\]

\[
\beta'''(s) = 2\left(\frac{H'''(3)(s)}{H(s)} - 3\frac{H''(2)(s)H'(1)(s)}{H^2(s)} + 2\left(\frac{H'(1)(s)}{H(s)}\right)^3\right)
\]

\[
\beta^{(iv)}(s) = 2\left(\frac{H^{(iv)}(4)(s)}{H(s)} - 4\frac{H'''(3)(s)H'(1)(s)}{H^2(s)} - 3\left(\frac{H'(1)(s)}{H(s)}\right)^2\right.
\]

\[
+ 12\frac{H''(2)(s)H'(1)(s)^2}{H^3(s)} - 6\left(\frac{H'(1)(s)}{H(s)}\right)^4\bigg).
\]

(85)

A certificate on the approximation error (e.g., in the form of Theorem 9) is difficult to obtain in the binary codeword scenario. Nevertheless, we can always resort to Theorem 6 in order to identify a robust numerical integration method. A somewhat weaker (but much simpler) guarantee is obtained by comparing \( O(n^{-2}) \) and \( O(n^{-3}) \) approximations of Procedure 21 and by validating the result only in case of agreement. Validity of the normal approximation is instead ensured in the following form.

Theorem 22: In a AWGN channel scenario with binary codewords, the PPV meta-converse bound (4) of Theorem 1 is consistent with the normal approximation (1) that uses the channel dispersion coefficient

\[
V = \frac{1}{2} \beta''(0),
\]

(86)

that is, \( \overline{R} = R_{\text{NA}} + O(1/n) \) for \( n \to \infty \). □

Proofs: See the Appendix.

A summary of all the above results on the BI-AWGN channel scenario is given in compact procedural form in Fig. 6.

C. \( \kappa \beta \) achievability bound

In the AWGN channel scenario with binary codewords, the RCU bound can be compactly expressed in the following form.

Theorem 23: In a AWGN channel scenario with binary codewords, the RCU bound can be expressed as

\[
\overline{P}_e = E \left[ \min(1, 2^{nR(g(w))}) \right], \text{ with } w \sim \mathcal{N}(\Omega, I\Omega)
\]

(87)

and where

\[
g(w) = P \left[ \sum_{i=1}^{n} d_i w_i \leq 0 \right], \text{ with } p_{d_i}(0) = p_{d_i}(1) = \frac{1}{2}.
\]

(88)

Proof: See the Appendix.

Although it has a very simple expression, the function \( g(w) \) can hardly be mapped into a mono-dimensional integral, and the approach used for (56) (which easily mapped into (57)) is not applicable. As a consequence, the integration region of interest assumes a composite multidimensional form, which is hardly usable to obtain a satisfactory result. We therefore
a) Identify the saddle point $s_\beta$ by solving the convex problem $s_\beta = \arg\min_{s \in \mathbb{R}} \beta(s)$ where function $\beta(\cdot)$ is defined in (81)-(82).

b) Identify the steepest descent path $p_\beta(u)$. For a given $u \geq 0$ the map $p_\beta : u \to s$ can be built by looking for the complex value $s$ satisfying $\Im(s) \geq 0$ and guaranteeing $\beta(s) = \beta(s_\beta) - u^2$ and $\Im(\beta(s)) = 0$.

c) Evaluate FA and MD probabilities using the expressions (see Theorem 4 and (39))

$$P_{\text{FA}}(\lambda) = 1(-s_\beta - 1) + \frac{1}{\pi} e^{\lambda(s_\beta) - 2\ln(2) + 2\lambda} \int_0^\infty 2u \left[ -\beta'(p_\beta(u)) \cdot (p_\beta(u) + 1) \right] e^{-\frac{u}{2}u^2} du$$

$$P_{\text{MD}}(\lambda) = 1(s_\beta) - \frac{1}{\pi} e^{\frac{s_\beta}{2}} \int_0^\infty 3\left[ -\beta'(p_\beta(u)) \cdot p_\beta(u) \right] e^{-\frac{u}{2}u^2} du$$

where $\beta(\cdot)$ and the derivative $\beta'(\cdot)$ can be derived numerically using (82) and (84) where $h(x) = \ln(1 + e^{-2x})$.

### PPV meta-converse bound – Asymptotic expansion for FA and MD probabilities

An overview of the PPV meta-converse and achievability bounds, and of their relation with the counterparts to the plots of Fig. 5.(c) and (d), in such a way to provide a compact view on the characteristics of the converse and achievability bounds, and of their relation with the normal approximation. The kind of considerations which can be drawn are equivalent to the ones already discussed in (49). The function $g(\cdot)$ is defined in (85). The function $g(\cdot)$ is defined in (85).

### PPV meta-converse bound – Normal $O(n^{-1})$ approximation

The normal approximation available in Fig. 1 holds with

$$C = 1 + \frac{H'(0)}{\ln(2)}, \quad V = H''(0) - (H'(0))^2,$$

where the derivatives of $H(\cdot)$ can be numerically evaluated from (84) where $h(x) = \ln(1 + e^{-2x})$.

Fig. 6. Summary of the most significant results for a BI-AWGN channel scenario. The scenario is defined in Section VI.A.

D. Numerical examples

An overview of the PPV meta-converse and achievability bounds for the AWGN channel with binary coding is given in Fig. 7. The plots of Fig. 7(a) and (b) are, respectively, the counterparts to the plots of Fig. 5(c) and (d), in such a way to provide a compact view on the characteristics of the converse and achievability bounds, and of their relation with the normal approximation. The kind of considerations which can be drawn are equivalent to the ones already discussed in the AWGN case, to which we refer, with the addition of the
Fig. 7. The AWGN channel with binary coding: (a) Converse and achievability bounds on spectral efficiency $\rho = 2R$ versus $E_b/N_0 = \Omega/\rho$ power efficiency measure, for some values of $n$, and for $P_e = 10^{-5}$; (b) Packet error rate bounds at rate $R = \frac{1}{2}$ as a function of SNR $E_b/N_0$, for some values of $n$; (c) Converse bound improvement with respect to the ISP bound of [21]; (d) Gap between the converse and achievability bounds and the practical performance of belief propagation on binary LDPC codes.
fact that the $\kappa\beta$ bound is evidently not a strikingly tight bound. See also [14] for the comparison with the $\kappa\beta$ bound in the AWGN case.

Some further aspects are covered in Fig. 7(c) and (d).

In Fig. 7(c) we show the significant improvement given by the PPV meta-converse bound with respect to the improved sphere-packing (ISP) bound of [21]. The ISP bound is the state-of-the-art for converse bounds in an AWGN channel with binary coding, and it is an improvement over the bounds by Valembois and Fossorier [20], as well as over Shannon, Gallager, and Berlekamp’s 1967 bound [1]. Incidentally, the codes shown in Fig. 7(c) are those of [21], Fig. 2,3, and 4, to which the reader is referred. The $\kappa\beta$ bound, which is not shown in Fig. 7(c), provides a performance roughly equivalent to those of the achievable results illustrated in [21].

Finally Fig. 7(d) shows the existing 1dB gap between the PPV meta-converse (or $\kappa\beta$) bound and the practical performance of low density parity check (LDPC) decoding via belief propagation (i.e., message passing). The codes used in figure are both rate $R = \frac{1}{2}$, the shorter one being the $(k,n) = (1320,2640)$ Margulis code [27], the longer one being taken from [28]. It is in any case worth recalling that, for very short packet sizes one can construct codes (and decoders) that approach the achievability bound, and in some regions (high error rate) even beat it [22].

VII. THE BINARY SYMMETRIC CHANNEL

A. Scenario and notation

The BSC is a numerical scenario where input and output channel symbols are binary, $x, y \in \{\pm 1\}^n$, and the transition probabilities are fully described by the crossover probability $P_{\text{bit}}$, where we assume with no loss in generality that $0 < P_{\text{bit}} < \frac{1}{2}$. It covers the hard-decoder implementation under binary transmission and AWGN channel, provided that the incorrect transition probability is defined as

$$P_{\text{bit}} = Q(\sqrt{R})$$

(92)

where $Q(\cdot)$ is the Gaussian complementary cumulative distribution function (CCDF). The codewords set of interest is, equivalently to the soft-decision counterpart (75), $K = \{1, -1\}^n$, and the channel transition probabilities take the form

$$p_{y|x}(b|a) = (1 - P_{\text{bit}})^n \left( \frac{P_{\text{bit}}}{1 - P_{\text{bit}}} \right)^{\|b - a\|_H}$$

(93)

with $\|\cdot\|_H$ denoting the Hamming weight. Capacity is achieved for equally likely input symbols, $p_x(a) = 2^{-n}$, which guarantees that also the output symbols are equally likely, that is, $p_y(b) = 2^{-n}$. The closed-form capacity expression is

$$C = 1 - h(P_{\text{bit}})$$

(94)

where

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$$

(95)

is the binary random variable entropy function.

B. PPV meta-converse bound

For a BSC, the PPV meta-converse bound is expressed by the following result which takes into account the appropriate modifications to Theorem 1 needed for discrete random variables.

Theorem 25: In a BSC channel scenario the PPV meta-converse bound can be expressed in the form

$$R \leq R = -\frac{1}{n} \log_2 \left( P_{\text{FA}}(d) - \zeta q_n(d; \frac{1}{2}) \right),$$

(96)

where the integer constant $0 \leq d \leq n$ and the real valued constant $0 \leq \zeta < 1$ are defined by

$$P_{e} = P_{\text{MD}}(d) + \zeta q_n(d; P_{\text{bit}}),$$

(97)

and where we used

$$q_n(d; \epsilon) = \binom{n}{d} \left( 1 - \epsilon \right)^{d-n} \epsilon^d.$$  

(98)

In the above context, the FA and MD probabilities (3) take the form

$$P_{\text{FA}}(d) = \sum_{i=1}^{n} u_i \leq d \right]$$

(99)

$$P_{\text{MD}}(d) = \sum_{i=1}^{n} v_i > d \right],$$

(99)

where

$$p_u(i) = p_v(1) = 1 - P_{\text{bit}}, \quad p_v(1) = P_{\text{bit}}.$$

Operatively, the value of $d$ corresponds to the minimum $d$ which satisfies $P_{\text{MD}}(d) \leq P_{e}$.  

Proof: See the Appendix.

Observe that the FA probability in (99) is a binomial cumulative distribution function (CDF) of order $n$ and parameter $\frac{1}{2}$, and the MD probability is a binomial CCDF of parameter $P_{\text{bit}}$. The result of Theorem 25 is also identical to [9, Theorem 35], where binomial CDF and CCDF are implicitly given in terms of a recurrence relation. A lower bound $\underline{P}$ can also be derived from Theorem 25 by setting $P_{e} = P_{\text{MD}}(d) + \zeta q_n(d; P_{\text{bit}})$ and using constraint $P_{\text{FA}}(d) - \zeta q_n(d; \frac{1}{2}) = 2^{-nR}$. In this case the value of $d$ corresponds to the minimum $d$ which satisfies $P_{\text{FA}}(d) \geq 2^{-nR}$.

The result of Theorem 25 calls for an asymptotic expression of the binomial CDF for large values of $n$, since the standard series expansion (e.g., the above cited recurrence relation) may be hardly applicable for large values of $n$. Other results available from the literature, e.g., the asymptotic expansion given in [29]., are not applicable in the present context. Use of Hoeffding’s theorem [30, Theorem 1] is also a widely used option for approximating binomial CDFs but, although able to capture the limit behavior of the bound, it is very weak and in general unusable. We therefore follow the Laplace domain approach, and look for an asymptotic expansion which holds uniformly for large $n$. The starting point is provided by the following result.

Theorem 26: In a BSC channel scenario the FA and MD probabilities (99) can be expressed, for $d$ not an integer, as in
where
\[\alpha(s) = 2\lambda s + 2\ln(1 + e^{-s}) - 2\ln(2)\]
\[\beta(s) = \alpha(s + \delta_0) - 2(\lambda - P_{\text{bit}})\delta_0 + 2\ln(2) \cdot C,\]
with
\[\lambda = \frac{d}{n}, \quad \delta_0 = \ln((1 - P_{\text{bit}})/P_{\text{bit}}),\]
and \(\delta_0 > 0\). Values for integer \(d\) can be derived from left and right limits.

Proof: See the Appendix.

Note that in Theorem 26 we assume \(d\) not an integer in order to avoid the points of discontinuity where the inverse Laplace transform equals the average of the left and right limits. We also observe that, being variables integer valued, a perfectly equivalent result could be obtained by use of the z-transform. However, we keep the Laplace notation for coherence with the results developed so far. The final result, as well as the intermediate steps, will be the same.

C. On the correct identification of the steepest descent path

Although Theorem 26 carries the same structure of Theorem 5 integration via the method of steepest descent is more involved since, because of the discrete nature of the output, the integration path is now a collection of integration paths. As a matter of fact, the functions \(\alpha\) and \(\beta\) carry a periodic behavior. Besides the real valued saddle points
\[s_\alpha = \ln \left(1 - \frac{\lambda}{\alpha}\right), \quad s_\beta = s_\alpha - \delta_0,\]
any point of the form \(s_\alpha + \imath 2k\pi\) and \(s_\beta + \imath 2k\pi\), for any choice of \(k \in \mathbb{Z}\), is a saddle point of, respectively, \(\alpha\) or \(\beta\). The steepest descent paths associated with (102) must then be connected as graphically illustrated in Fig. 8.

![Integration paths](image)

The identification of the arc passing through saddle point \(s_\alpha\) can be performed by investigating the equivalence \(\Im(\alpha(s)) = 0\) for \(\Im(s) = \phi \in (-\pi, \pi)\), ad it provides the expression
\[\mathcal{A}_\alpha = \{p_\alpha(\phi) = \ln(v(\phi)) + i\phi, \phi \in (-\pi, \pi)\},\]
where
\[v(\phi) = \frac{\sin((1 - \lambda)\phi)}{\sin(\lambda\phi)} .\]
The arc \(\mathcal{A}_\alpha\) therefore links the points \(-i\pi\) and \(i\pi\), and transits through \(s_\alpha\), which corresponds to the choice \(\phi = 0\). The integration paths then assume the form
\[\mathcal{L}_\alpha = \bigcup_{k = -\infty}^{+\infty} \{\mathcal{A}_\alpha + i2k\pi\},\]
and \(\mathcal{L}_\beta = \mathcal{L}_\alpha - \delta_0\), which, with some effort, provides the following result.

**Theorem 27:** In a BSC channel scenario the FA and MD probabilities can be expressed, for \(d\) an integer, in the form
\[P_{\text{FA}}(d) = 1(-s_\alpha) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\frac{1}{2}(p_{\alpha}(\phi))}}{1 - e^{-p_{\alpha}(\phi)}} p_{\alpha}(\phi) \, d\phi\]
\[P_{\text{MD}}(d) = 1(s_\beta) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\frac{1}{2}(p_{\beta}(\phi))}}{1 - e^{-p_{\beta}(\phi)}} p_{\beta}(\phi) \, d\phi\]
where constants and functions are defined in (100)-(104), and \(p_{\beta}(\phi) = p_{\alpha}(\phi) - \delta_0\).

Proof: See the Appendix.

Observe that, in (105), it is
\[e\alpha(p_{\alpha}(\phi)) = 2\ln(2) \cdot h(\lambda) - 2\ln(2) - 2\ln(2)\]
\[\beta(p_{\beta}(\phi)) = 2\ln(2) \cdot h(\lambda) + 2\ln(2)\]
\[\lambda = 2\ln(2) \cdot h(\lambda) + (1 - \lambda) \ln(1 - P_{\text{bit}})\]
\[= \beta(s_\beta) - 2d(\phi)\]
where, by exploiting (with a little effort) standard trigonometric identities, we can derive that
\[d(\phi) = \lambda \ln \sin(\lambda\phi) + (1 - \lambda) \ln \sin((1 - \lambda)\phi)\]
\[- \ln \sin(\phi)\]
with \(\sin(x) = \sin(\phi)/x\). In \(\phi \in (-\pi, \pi)\) the function \(d(\phi)\) is a real valued, positive, convex function with minimum value \(d(0) = 0\), and with \(d(\pm\pi) = \infty\). A summary of the resulting integral expression for the FA and MD probabilities is available in compact explicit form in Fig. 9.

The above also allows obtaining a form equivalent to Theorem 6 by introducing variable \(u\) in the relation \(\alpha(p_{\alpha}(\phi)) = \alpha(s_\alpha) - u^2\), that is, \(u^2 = 2d(\phi)\). We obtain the following result.

**Theorem 28:** In a BSC channel scenario the FA and MD probabilities can be expressed, for \(d\) an integer, in the form (106) of Theorem 6 where
\[c_\alpha(u) = \frac{p_{\alpha}'(u)}{1 - e^{-p_{\alpha}(u)}} = -\frac{2u}{-\alpha'(p_{\alpha}(u))(1 - e^{-p_{\alpha}(u)})}\]
\[p_{\alpha}(u) = p_{\alpha}(\phi(u))\]
\[\phi(u) = f^{-1}(u), \quad f(\phi) = \text{sign}(\phi) \sqrt{2d(\phi)},\]
where \(p_{\beta}(\phi) = p_{\alpha}(\phi) - \delta_0\), and where the remaining constants and functions are defined as in (100)-(104).

Proof: The proof is left to the reader.
D. Asymptotic expansion

The result obtained implies that the asymptotic expansion of Procedure 7 holds, but a different strategy must be used for evaluating the coefficients by standard algebraic operations on Taylor series. The compact result is the following. Details are tedious and are therefore skipped, but the method has been already discussed in the proof of Procedure 7.

Procedure 29: In a BSC channel scenario the FA and MD probabilities \( c_{0,k} \) and \( c_{3,k} \) are the Taylor expansion coefficients of functions \( c_0 \) and \( c_3 \), respectively, as given in Theorem 28. Operatively, the following iterative procedure can be used to identify the coefficients, provided that, at step 4), \( q \) is set to \( q = 0 \) for the FA probabilities, and to \( q = 0 \) for the MD probabilities.

1) Evaluate the real valued Taylor series coefficients in \( \phi = 0 \) of the function \( d(\phi) \) in (108) according to rule:
   
   \[
   d_{2k+1} = 0, 
   d_0 = 0, 
   \]

   \[
   d_{2k} = \frac{(-1)^{k+1} 2^{k-1} B_{2k}}{(2k)!} \left[ 1 - \lambda^{2k+1} - (1 - \lambda)^{2k+1} \right] \quad (110)
   \]
   
   where \( B_{2k} \) is the Bernoulli number [25, \S 23]. All the coefficients are positive.

2) Evaluate the real valued Taylor series coefficients in \( \phi = 0 \) of the function \( f(\phi) \) according to rule:
   
   \[
   f_{2k} = 0, 
   f_1 = \sqrt{2d_2}, 
   f_3 = d_4 / f_1, 
   \]

   \[
   f_{2k+1} = \frac{1}{2} \left( \frac{d_{2k+2}}{d_2} f_1 - \frac{1}{k} \sum_{m=1}^{k-1} \frac{d_{2(k-m)+2}}{d_2} f_{2m+1} \right) \quad (111)
   \]

3) Evaluate the real valued Taylor series coefficients in \( u = 0 \) of the function \( \phi(u) \) according to rule:
   
   \[
   \phi_{2k+1} = -\sum_{m=0}^{k-1} \phi_{2m+1} f_{2(m-k)} F_{k-m,2m+1} \quad (112)
   \]

   where \( F_{0,n} = 1 \), and

   \[
   F_{m,n} = \sum_{k=1}^{m} \frac{(kn - m + k)}{m} F_{2k+1} f_{1} f_{m-k,n} \quad (113)
   \]

4) Evaluate the complex valued Taylor series coefficients in \( \phi = 0 \) of the function \( \phi g(\phi) \) with
   
   \[
   g(\phi) = \frac{1}{\alpha'(p_\alpha(\phi))} \frac{1}{1 - e^{-\alpha(\phi)}} \quad (114)
   \]

   \[
   = \frac{e^{i2\lambda\phi} - 1}{e^{2i\lambda\phi} - \lambda e^{2i\phi} - (1 - \lambda)} \quad (e^{\eta+1} e^{i2\lambda\phi} - e^{i\phi} - e^{-\phi}),
   \]
   
   according to rule:

   \[
   g_0 = \xi_2 / \nu_5, \quad (115)
   \]

   \[
   g_k = \frac{1}{\nu_5} \left( \xi_{k+2} - \sum_{m=1}^{k} \nu_{m+3} g_{k-m} \right)
   \]

   \[
   \xi_k = \frac{(2i)^k}{k!} \left[ (\lambda + 1)^k - \lambda^k - 2^k + 1 \right]
   \]

   \[
   \nu_k = \frac{(2i)^k}{k!} \left[ (2^k - 2)(\lambda + (1 + e^{i\eta})\lambda) - (1 + (1 + e^{i\eta})\lambda)((\lambda + 1)^k - 1 - \lambda^k) \right].
   \]

   The even coefficients are imaginary valued, and the odd coefficients are real valued. \( \square \)

Proof: The proof is left to the reader. \( \square \)

E. Asymptotic approximations

With a little effort, the first coefficients can be evaluated in the closed form, to obtain \( c_0 = i g_0(\lambda, \lambda) \) and \( c_2 = -c_0 g_2(\lambda, \lambda) \), where we used

\[
\xi_2 = 2^{k-1} \lambda^n \left( \frac{1}{\lambda} + 1 + (1 + e^{i\eta}) \right) \frac{1}{(1 - \lambda)(1 - \lambda^{2k+1})^2}.
\]

Under the further assumption that \( s_\beta < 0 < s_\alpha \), which corresponds to the request \( P_{\text{hit}} < \lambda < \frac{1}{2} \), the asymptotic approximation derived by truncating the series to the first contributions provides a result equivalent to (47) and (48).

We have

Corollary 30: In a BSC channel scenario where \( \lambda \in (P_{\text{hit}}, \frac{1}{2}) \), the FA and MD probabilities (99) allow the \( O(n^{-2}) \) asymptotic approximations

\[
\frac{1}{n} \ln P_{\text{FA}}^{(1)} = -\ln(2) - \frac{1}{2} \ln \left( \frac{2 \pi n}{g_0^2(\lambda, 0)} \right)
\]

\[
\frac{1}{n} \ln P_{\text{MD}}^{(1)} = -\ln(2) + \lambda \ln(P_{\text{hit}}) + (1 - \lambda) \ln(1 - P_{\text{hit}})
\]

\[
- \frac{1}{2} \ln \left( \frac{2 \pi n}{g_0^2(\lambda, 0)} \right),
\]

(120)
where \( \lambda = \frac{2}{n} \). Furthermore, the \( O(n^{-3}) \) approximations

\[
\begin{align*}
\frac{1}{n} \ln P_{FA}^{(2)} &= \frac{1}{n} \ln P_{FA}^{(1)} + \frac{1}{n} \ln \left( 1 - \frac{1}{n} g_2(\lambda, 0) \right) \\
\frac{1}{n} \ln P_{MD}^{(2)} &= \frac{1}{n} \ln P_{MD}^{(1)} + \frac{1}{n} \ln \left( 1 - \frac{1}{n} g_2(\lambda, \delta_0) \right),
\end{align*}
\]

where we used \((119)\), also apply.

Proof: The result is an application of Procedure \(29\).

One important characteristic of asymptotic approximations \((120)\) and \((121)\) is that they provide lower and upper bounds to the true FA and MD probabilities, so that a control on the approximation error is available. This can be proved by mimicking Theorem \(9\) but the result is much more strong and \((120)\) is that they provide lower and upper bounds \(\star\) where we used \((119)\), also apply.

Theorem 31: In a BSC channel scenario where \( \lambda \in (P_{bit}, \frac{1}{2}) \) the asymptotic approximations \((120)\) and \((121)\) of the FA and MD probabilities satisfy relation \((51)\) which ensures the validity of

\[
\begin{align*}
\mathcal{R}^{(1)} &\leq \mathcal{R} \leq \mathcal{R}^{(2)} \\
\mathcal{P}^{(1)} &\leq \mathcal{P} \leq \mathcal{P}^{(1)},
\end{align*}
\]

where the superscript \( (K) \) identifies a bound derived by use of either the \( K = 1 \) or the \( K = 2 \) term approximation. \(\square\)

Proof: See the Appendix.

The asymptotic expressions of Theorem \(30\) also allows capturing the normal approximation, which provides a proof alternative to that of [9] Theorem \(52\).

Theorem 32: In a BSC channel scenario, the PPV meta-converse bound is consistent with the normal approximation \((1)\) that uses the channel dispersion coefficient \((123)\), identifies a bound derived by use of either the \( K = 1 \) or the \( K = 2 \) term approximation.

Proof: See the Appendix.

The RCU achievability bound

To conclude our investigation, we finally discuss the RCU bound which, in the BSC scenario, can be approximated by using the tools introduced so far, which some modifications in order to be able to deal with a discrete case.

The bound is written in the following form, which is equivalent to [9] Theorem \(33\).

Theorem 33: In a BSC channel scenario, the RCU converse bound can be expressed as

\[
\begin{align*}
\mathcal{R}_c &= \sum_{d=0}^{d_0} 2^n R P_{FA}(d) q_n(d; P_{bit}) + P_{MD}(d_0)
\end{align*}
\]

where we used the FA and MD probabilities \((59)\), and where \(d_0\) is the minimum \( d \) satisfying \(-\frac{1}{n} \log_2(P_{FA}(d)) \leq R\). \(\square\)

Proof: See the Appendix.

An asymptotic approximation to \((124)\) can then be obtained by exploiting Corollary \(30\) in conjunction with the linear approximation method used in Theorem \(17\) but suitably adapted to a discrete case. The resulting approximation is available in the following form.

**Theorem 34:** In a BSC channel scenario, for sufficiently large \( n \) the RCU bound can be approximated as

\[
\begin{align*}
\frac{1}{n} \ln \mathcal{R}_c &= w_n^{(0)}(\lambda) - \frac{1}{n} \ln(w_n^{(1)}(\lambda)) + O(n^{-2})
\end{align*}
\]

where \( \lambda \) is defined through the equivalence \( w_n^{(2)}(\lambda) = R \ln(2) \), and where

\[
\begin{align*}
w_n^{(0)}(\lambda) &= h(\lambda) \ln(2) + \ln(1 - P_{bit}) - \lambda \ln(1 - P_{bit}) - \lambda \ln(1 - P_{bit}) + (1 - P_{bit}) - \lambda \ln(1 - P_{bit}) + (1 - P_{bit}) \sqrt{2} \\
w_n^{(1)}(\lambda) &= \frac{\lambda P_{bit} + (1 - P_{bit})}{P_{bit} + (1 - P_{bit})} (1 - 2) \sqrt{\lambda} \\
w_n^{(2)}(\lambda) &= [1 - h(\lambda)] \ln(2) + \frac{1}{2n} \ln \left( \frac{2 \pi \lambda (1 - 2 \lambda)^2}{1 - \lambda} \right)
\end{align*}
\]

For a correct applicability of the theorem, \( w_n^{(1)} \) in \((126)\) must be positive. \(\square\)

Proof: See the Appendix.

The approximation of Theorem \(34\) finally provides a simple means to identify the consistency with the normal approximation. The following result is therefore a proof alternative to the one available in [9] Theorem \(52\).

Theorem 35: In a BSC channel scenario, the RCU achievability bound is consistent with the normal approximation \((1)\) that uses the channel dispersion coefficient \((123)\), that is, \( \mathcal{R} = R_{NA} + O(1/n) \) for \( n \to \infty \).

Proof: See the Appendix.

A summary of all the above results on the BSC channel scenario is given in compact procedural form in Fig. \(9\). Both interpretations (rate as well as error probability bound) of the RCU bound are provided.

**G. Numerical examples**

The PPV meta-converse and the RCU bounds associated with the BSC channel are shown, together with the normal approximation, in Fig. \(10\). A hard decoding perspective is considered where \( P_{bit} = Q(\sqrt{\lambda}) \). The results of Corollary \(30\) were used for the PPV meta-converse bound, and, in the considered cases, Theorem \(31\) ensures validity of the plotted result. The approximation of Theorem \(34\) is instead used for the RCU bound. Note from Fig. \(10\) (a) that, similarly to the AWGN channel case of Fig. \(5\) (c), the PPV meta-converse and the RCU bounds are very close over a wide range of both \( n \) and SNR. A significant gap is appreciated with the normal approximation for small spectral efficiencies, and for small \( n \) (\( n = 200 \) in figure) the normal approximation is seen to be less reliable than in the AWGN case.

The packet error rate perspective is given in Fig. \(10\) (b) for rate \( R = \frac{1}{2} \), which confirms the close adherence of the RCU and the PPV meta-converse bounds over a large packet error rate range seen in the AWGN case of Fig. \(5\) (d). Observe that the difference between the RCU bound and the normal approximation is not strikingly significant at rate \( \frac{1}{2} \) (it would be much more significant at lower rates, as evident from Fig. \(10\) (a)), but Fig. \(10\) (b) completes the overview on rate \( R = \frac{1}{2} \) codes previously initiated in the AWGN case of Fig. \(5\) (d), and in the BI-AWGN case of Fig. \(7\) (b).
PPV meta-converse bound – Generalities

The bounds in rate and error probability can be respectively expressed in the form (see Theorem 25)

\[ R = -\frac{1}{n} \log_2 \left( P_{PA}(d) - \frac{P_e - P_{MD}(d)}{2^{nR} P_{bit} (1 - P_{bit})^{n-d}} \right), \]

with \( d \) the smallest integer satisfying \( P_{MD}(d) \leq P_e \)

\[ P_e = P_{MD}(d) + 2^n P_{bit} (1 - P_{bit})^{n-d} (P_{FA}(d) - 2^{-nR}), \]

with \( d \) the smallest integer satisfying \( P_{FA}(d) \geq 2^{-nR} \),

where functions \( P_{FA}(d) \) and \( P_{MD}(d) \) are defined in the box below.

PPV meta-converse bound – Integral representation of FA and MD probabilities

FA and MD probabilities can be expressed in the form (see Theorem 27 and (107)-(108))

\[ P_{FA}(d) = 1(-s_{\alpha}) + \frac{1}{\pi} e^{-n h_{\alpha}(\lambda) - \ln(2)} \int_0^\pi \Im \frac{e^{-nd(\phi)}}{1 - e^{-p_{\alpha}(\phi) P_{\alpha}'(\phi)}} d\phi \]

\[ P_{MD}(d) = 1(s_{\alpha} - \delta_0) - \frac{1}{\pi} e^{-n h_{\alpha}(\lambda) - h_{\alpha}(P_{bit}) - (\lambda - P_{bit}) \delta_0} \int_0^\pi \Im \frac{e^{-nd(\phi)}}{1 - e^{-\delta_0 - p_{\alpha}(\phi) P_{\alpha}'(\phi)}} d\phi \]

where \( s_{\alpha} = \ln((1 - \lambda)/\lambda) \), \( \lambda = \frac{d}{n} \), \( h_{\alpha}(x) = -x \log_2(x) - (1 - x) \log_2(1 - x) \), \( \delta_0 = \ln((1 - P_{bit})/P_{bit}) \), \( \phi \) is defined in (108), \( p_{\alpha}(\phi) = \ln(v(\phi)) + i\phi \) with \( v(\cdot) \) defined in (104), and the corresponding derivative is \( p_{\alpha}'(\phi) = i + v'(\phi)/v(\phi) \).

PPV meta-converse bound – Asymptotic expansion for FA and MD probabilities

A \( O(n^{-k}) \) asymptotic expansion of FA and MD probabilities can be derived by using Procedure 29.

When \( P_{bit} < \frac{d}{n} < \frac{1}{2} \), specific \( O(n^{-2}) \) and \( O(n^{-3}) \) expansions are available as from Corollary 30.

PPV meta-converse bound – Normal \( O(n^{-1}) \) approximation

The normal approximation available in Fig. 1 holds with (see Theorem 32)

\[ C = 1 - h(P_{bit}) \]

\[ V = P_{bit} (1 - P_{bit}) \ln^2 \left( \frac{1 - P_{bit}}{P_{bit}} \right), \]

where \( h(x) = -x \log_2 x - (1 - x) \log_2(1 - x) \) is the binary random variable entropy function.

RCU achievability bound – Asymptotic expansion of functions

A \( O(n^{-2}) \) expansion of the RCU bound in rate and error probability is given by (see Theorem 34)

\[ R \simeq \frac{w_n^{(2)}(\lambda)}{\ln(2)}, \]

with \( \lambda \) defined by \( \frac{1}{n} \ln P_e = w_n^{(0)}(\lambda) - \frac{1}{n} \ln(w_n^{(1)}(\lambda)) \)

\[ \overline{P}_e \simeq e^{nw_n^{(0)}(\lambda) - \ln(w_n^{(1)}(\lambda))}, \]

with \( \lambda \) defined by \( w_n^{(2)}(\lambda) = R \ln(2) \)

where functions \( w^{(i)}(\cdot) \) are defined in (126).

RCU achievability bound – Normal \( O(n^{-1}) \) approximation

By Theorem 35 the normal approximation corresponds to the one of the PPV meta-converse bound available two boxes above.

VIII. CONCLUSIONS

In this paper we discussed the application of the asymptotic uniform expansion approach of Temme [15] for the evaluation of converse and achievability bounds in the finite blocklength regime. The preliminary results available in [14] for the AWGN channel were generalized to a number of channels of practical interest, namely the parallel AWGN case, the BI-AWGN channel, and the BSC channel, but can be in principle adapted to any memoryless channel formulation, either continuos, discrete, or mixed. The method we are using is particularly suited for evaluating the PPV meta-converse bound, for which a neat integral expression as well as a simple asymptotic series expansion is always available. Calculation of the RCU achievability bound is also possible, as we show in the AWGN and BSC case, although the final result is in general weaker.
efficiency measure, for some values of \( n \), and for \( P_s = 10^{-5} \); (b) Packet error rate bounds at rate \( R = \frac{1}{2} \) as a function of SNR \( E_b/N_0 \), for some values of \( n \).

**APPENDIX**

**Proof of Theorem 4** By exploiting the fact that, in the considered context, it is \( ||a_k||^2 = \frac{n}{K} \Omega_k \sigma_n^2 \), we have

\[
\Lambda(a, b) = \frac{1}{n} \ln(1 + \Omega_k) - \frac{1}{2n} \Lambda'(a, b) \quad (127)
\]

where

\[
\Lambda'(a, b) = \sum_{k=1}^{K} \frac{\Omega_k}{1 + \Omega_k} \| b_k - \frac{(1 + \Omega_k) a_k}{\Omega_k} \| \quad (128)
\]

Hence, we can directly work on \( \Lambda' \) since it is in a linear relation with \( \Lambda \). We are therefore interested in a counterpart to (3) where \( \Lambda \) is replaced by \( \Lambda' \). With a little effort, and by exploiting the properties of a non-central chi-squared random variable [31] §2.2.3, and the same method used in [14], we obtain (22). Note that in (22) the threshold level \( \lambda' \) to be used in connection with \( \Lambda' \) has been replaced by \( \lambda \) since, operatively, they have the same meaning.

**Proof of Theorem 5** Denote \( u = \sum_{k=1}^{K} u_k \Omega_k \) and \( v = \sum_{k=1}^{K} v_k \Omega_k / (1 + \Omega_k) \) with associated PDFs \( f(u, a) \) and \( f(v, a) \), respectively. From (25) eq. 29.3.8.1 the corresponding Laplace transforms are \( F_u(s) = e^{2/\bar{a}(s) - n\lambda s} \) and \( F_v(s) = e^{2/\bar{b}(s) - n\lambda s} \), with associated region of convergence given in (26). The link with the FA and MD probabilities in Theorem 4 is \( P_{\text{FA}}(\lambda) = f_u * 1(n\lambda) \) and \( P_{\text{MD}}(\lambda) = f_v * 1_n(n\lambda) \), where * denotes convolution, \( 1(t) \) is the unit step function and \( 1_n(t) = 1(t) \).

By using standard Laplace transform properties [32] we can then be written via an inverse Laplace transform as in (24).

**Proof of Procedure 7** The proof uses the method exploited in [14] Theorem 6, to which we refer for details. The important point to observe is that \( p_x^{-1}(s) \) is defined by (32). Then, the proof is obtained by nested application of Taylor series expansion composition properties (e.g., see [33]). Taylor expansions for \( \alpha(s) \) and \( \beta(s) \) are standard. Note that \( a_2 \geq 0 \) is a consequence of (27), i.e., we are looking for a minimum. The Taylor series expansion of \( f(s) \) is an application of the squared-root map. The \( - \) sign in \( f_1 \) is chosen in such a way to guarantee the correct sign of our final coefficients. The fact that the coefficients \( f_m \) are imaginary valued is a consequence of (27). The inversion map in 3) is derived by exploiting the method used in [34]. The fact that the even coefficients \( p_{2m} \) are real and the odd coefficients \( p_{2m+1} \) are imaginary is a consequence of the fact that \( f_m \) are imaginary valued, and can be proved by induction. The result in 4) is derived from a dividing series combination rule. The fact that the even coefficients \( c_{2m} \) are imaginary and the odd coefficients \( c_{2m+1} \) are real is a consequence of the properties of \( p_m \), and can be easily proved by induction.

**Proof of Theorem 10** We first investigate the behavior at \( n = \infty \), in which case the \( O(n^{-2}) \) approximation [47] guarantees that \( \overline{R} = -\frac{1}{2} \alpha(s_n) \log_2(e) \) where \( \lambda \) is set by request \( \beta(s_\beta) = 0 \). The solution is simply \( \lambda = 1 \) and \( s_\beta = 0 \), and in fact \( \beta(0) = 0 \) for any \( \lambda \), and \( \beta'(0) = 0 \) for \( \lambda = 1 \).
These choices guarantee that $s_a = \frac{1}{2}$, and, in turn, that $\bar{R} = C$.

We then investigate the bound for large $n$. We preliminarily observe that the $Q$ function can be written in the form $Q(x) = e^{-x^2/2}I(x)$ for $x > 0$, where $\ln Q(x) = -\ln(\sqrt{2\pi x}) + O(1/x^2) [25$, eq. 26.2.12$]$. Hence, from (17), the MD probability can be approximated in the form

$$\ln P_{MD} = \frac{1}{2}\beta(s_\beta) + \ln\left(\sqrt{n}a_s s_\beta^2\right) + O(1/n) \quad (129)$$

We then Taylor expand the functions in (129) around $\lambda = 1$, which is the limit value for $n \to \infty$. To achieve our aim it is advisable to write $\beta(s; \lambda)$ in the form $2s\lambda - 2h(s)$ for a suitable choice of the function $h(s)$. With this notation, we have $s_\beta(\lambda) = [h']^{-1}(\lambda)$ and $a_2(\lambda) = -h''(s_\beta(\lambda))$, which imply the approximations

$$s_\beta(\lambda) = s_\beta^* - \frac{1}{a_2^2}(\lambda - 1) + O((\lambda - 1)^2) \quad (130)$$

$$a_2(\lambda) = a_2^* + O(\lambda - 1),$$

where the asterisk * denotes a quantity evaluated at $\lambda = 1$. Observe that $s_\beta^* = 0$ and $a_2^* = \frac{1}{4}$, hence from (131) it is $a_2 = 4V$ with $V$ the channel dispersion coefficient (53). As a consequence we also have

$$a_2(\lambda)s_\beta^2(\lambda) = \frac{1}{4V}(\lambda - 1)^2 + O((\lambda - 1)^3)$$

$$\beta(s_\beta(\lambda); \lambda) = 2\left[\lambda s_\beta(\lambda) - h(s_\beta(\lambda))\right]$$

$$= -\frac{1}{2}\lambda\left(\lambda - 1\right)^2 + O((\lambda - 1)^3),$$

which allows writing (129) in the form

$$\ln P_{MD} = \ln Q\left(x\sqrt{1 + O(\lambda - 1)}\right) + O(1/n) \quad (132)$$

where $x = (\lambda - 1)/(4\sqrt{V})$. Note that, for $n \to \infty$ and $\lambda \to 1$ the request $P_{MD} = P_e$ implies from (132) that $x \to Q^{-1}(P_e)$, and therefore $O(\lambda - 1)$ is equivalent to $O(1/\sqrt{n})$. By solving for $P_{MD} = P_e$ we then obtain

$$\lambda = 1 + \sqrt{4V/n}Q^{-1}(P_e) + O(1/n) \quad (133)$$

where the plus sign is consistent with the request $s_\beta(\lambda) < 0$.

We then inspect the FA probability which, from (47), can be approximated in the form

$$-\frac{1}{n} \ln P_{FA} = -\frac{1}{2}a(s_a) + \frac{\ln(n)}{2n} + O(1/n) \quad (134)$$

where, from (25), we have

$$-\frac{1}{2}a(s_a(\lambda); \lambda) = -\frac{1}{2}\beta(s_\beta(\lambda); \lambda) + \ln(2) C - \frac{1}{4}(\lambda - 1).$$

The approximation (11) is finally obtained by evaluating (134) with the use of (133), the second of (131), and (133).

**Proof of Theorem 12** By exploiting (127)-(128) in definition (7), and by recalling that $K = 1$, we have

$$g(x, y) = P\left[\frac{y^T (z - x)}{\sigma^2} \geq 0\right], \text{where } z \sim U_K. \quad (136)$$

Then, by the circular symmetry of both $z$ and $y$ we can further derive that the bound on $P_e$ given by (5) is independent of the choice of the PDF of $x$, provided that $x \in K$. In the following, with no loss in generality, we therefore assume that $x$ is a vector with the first entry set to $\sqrt{n}\bar{P}$ and the rest to 0. In (136) we then exploit the fact that $z$ can be written in the form

$$z = \frac{g}{\|y\|\sqrt{n}\bar{P}}, \quad g \sim N(0, I_n), \quad (137)$$

and that, because of the circular symmetry of $z$, the substitution

$$\frac{y^T z}{\sigma^2} \rightarrow \frac{\|y\|^2}{\sigma^2} \eta \sqrt{n}\Omega, \quad \eta = \frac{g_1}{\|g\|}, \quad (138)$$

provides an equivalent result. With this notation, (56) is valid by setting

$$\tau = g_1 \sqrt{n - 1}, \quad q = \frac{y}{\sigma \sqrt{n}/\Omega}, \quad (139)$$

where $\tau$ is by definition a t-distributed variable, and where $q$ is Gaussian as in (55). This proves the theorem.

**Proof of Theorem 13** We note that $\rho$ can be written in the form

$$\rho = \frac{d}{\sqrt{n - 1 + d^2}}, \quad d = q_1 \sqrt{n - 1}, \quad (140)$$

where $d$ is a non-central t-distributed random variable of order $n - 1$ an non-centrality parameter $\sqrt{n}b$. Hence, with some effort, we also find that

$$P[\rho < a] = P \left[ d \leq \frac{a\sqrt{n - 1}}{\sqrt{1 - a^2}} \right]$$

Then, by exploring the PDF of a t distributed random variable of order $\nu$ and non-centrality parameter $\mu$, namely (see [35, p. 177] and [25, eq. 19.5.3], and be advised that expression [25, eq. 26.7.9] is not correct)

$$f_t(a) = \frac{2\Gamma(\nu + 1)}{\sqrt{\nu\pi}(\frac{a}{2})^\nu} U\left(\nu + \frac{1}{2}; -\frac{\mu a}{\sqrt{\nu + a^2}}\right) \left(\nu/2 + a^2\right)^ {\nu + \frac{1}{2}} \cdot e^{-\frac{1}{2}a^2(1 + \frac{a}{\nu + a^2})} \quad (140)$$

where $U(a; x)$ is Weber’s form for the parabolic cylinder function (decreasing to 0 for large $x$), we obtain (58). With a little effort, from (25, eq. 19.10 and 6.1.40) we have

$$u_n(a) = \frac{1}{2} \left[ (1 - \frac{a}{\nu}) \ln(1 - a^2) + (aa)^2 - 2a^2 - \frac{1}{2} \ln \left( 1 - \frac{a}{\nu} \right) + a^2 \sqrt{1 + (aa)^2 - \frac{1}{2\nu}} \right. \left. + (1 - \frac{1}{2\nu}) \sinh^{-1} (aa/\sqrt{1 - \frac{1}{2\nu}}) + \frac{1}{2\nu} \ln(n) \right. \left. - \frac{1}{2\nu} \ln (2\pi e \sqrt{1 + (aa)^2}) + O(n^{-2}) \right. \quad (141)$$

Equation (59) is then obtained as a rearrangement of (141), by neglecting the $O(n^{-2})$ contributions.

**Proof of Theorem 14** The statistical properties of $\eta$ can be identified in the closed form. From (56) the CDF of $\eta$ can be written in the form

$$P[\eta \leq a] = P \left[ \tau \leq \frac{a\sqrt{n - 1}}{\sqrt{1 - a^2}} \right] \quad (142)$$
for $0 \leq a < 1$, while for $-1 < a < 0$ it simply is $P[\eta \leq a] = 1 - P[\eta \leq -a]$ because of the symmetry of $\eta$. Hence the PDF of $\eta$ is (see [25, eq. 26.7.1])

$$f_\eta(a) = \frac{\Gamma\left(\frac{\eta}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)\Gamma\left(\frac{\eta-1}{2}\right)(1-a^2)^{-\frac{\eta}{2}}} ,$$

(143)

with $a \in (-1, 1)$, and the corresponding Laplace transform is (see [25, eq. 9.6.18 and 9.6.47]):

$$F_\eta(s) = a F_1\left(\frac{s}{2}, 1, \frac{s^2}{4}\right) = \exp\left(\frac{2G_s^2(2s/n)}{G_2}\right) .$$

(144)

The result of (61) is then a counterpart to the derivation of the MD probability in Theorem 5.

Proof of Theorem 15] From [25, eq. 9.6.26, 9.6.47, 9.7.7, and 9.7.9] we have that the function $G_\nu$ can be written in the form

$$G_\nu(s) = \sqrt{1+s^2} + \frac{1}{\nu} \ln \left(\frac{\Gamma(\nu)}{\nu^{-\nu} \sqrt{2\pi}}\right) - \frac{1}{\nu} \ln(\sqrt{1+s^2})$$

$$- \left(1 - \frac{1}{\nu}\right) \ln \left(\frac{1+\sqrt{1+s^2}}{2}\right) + \frac{1}{\nu} \ln \left(1 + \sum_{k=1}^{\infty} g_k(t) \nu^{-k}\right)$$

where $t = 1/\sqrt{1+s^2}$, and where $g_k(t) = (v_k(t) + t u_k(t))/(1+t)$ with $v_k(t)$ and $u_k(t)$ the polynomials defined in [25, eq. 9.3-9.14]. By construction, $g_k(t)$ is a polynomial of order $k$ in $t$, and for the first orders we have

$$g_1(t) = -\frac{55t^3 + 12t^2 - 9t}{24}$$

$$g_2(t) = \frac{385t^6 - 840t^5 + 378t^4 + 216t^3 - 135t^2}{1152} .$$

(145)

By summing the series to the term $k = 0$, and by exploiting [25, eq. 6.1.40], we obtain the asymptotic expression (62).

Proof of Theorem 17] Recall that $\lambda$ is defined by $v_\nu(\lambda) = R \ln(2)$. The theorem is proved by assuming that the function $u_\nu(a)$ is increasing for $a \in [-1, \lambda]$, and that the function $u_\nu(a) - v_\nu(a)$ is decreasing for $a \in [1, \lambda]$, which (because of the shape of the involved functions) correspond to assuming that the function maxima lie outside the integration interval, which also implies a small error probability $P_e < \frac{1}{2}$. Incidentally note that, from the $(O(n^{-1}))$ approximations that can be derived from (59) and (69), the function maxima can be approximated in the form

$$\arg\max_a u_\nu(a) = \sqrt{\frac{\Omega}{1 + \Omega}} + O(n^{-1})$$

$$\arg\max_a u_\nu(a) - v_\nu(a) = \sqrt{\frac{4 + 12\lambda^2 - 2\Omega}{\Omega}} + O(n^{-1}) .$$

(146)

Since the Shannon bound guarantees $R < \frac{1}{2} \ln(1 + \Omega)$, from (71) we have

$$\lambda < \sqrt{\frac{\Omega}{1 + \Omega}} + O(n^{-1})$$

(147)

and the assumption is practically guaranteed for the first contribution in (70). The assumption $v'_\nu(\lambda) > u'_\nu(\lambda)$ further guarantees that it is satisfied also for the second contribution in (70).

Given the above, for the first contribution in (70) we substitute variable $a \in [-1, \lambda]$ with variable $x \in (-\infty, 0]$ such that $u_n(a) = u_\nu(\lambda) - x^2$, that is $x = f(a) = -\sqrt{u_\nu(\lambda) - u_n(a)}$. This provides the equivalent integral expression

$$\int_{-1}^{\lambda} e^{nu_n(a)} da = \int_{-\infty}^{0} e^{nu_\nu(\lambda) - nx^2} \frac{-2x}{u'_n(f^{-1}(x))} dx$$

(148)

By Taylor expansion of the function $h(x)$ in $x = 0$, and by use of [25, eq. 7.4.4-5], we obtain

$$\int_{-1}^{\lambda} e^{nu_n(a)} da = e^{nu_\nu(\lambda)} \sum_{k=0}^{\infty} h_k (-1)^k \frac{\Gamma(k+1)}{2^{k+1} k!} .$$

(149)

By then observing that $h_0 = 0$, $h_1 = -2/u'_\nu(\lambda)$, and $h_2 = 0$, the approximation

$$\int_{-1}^{\lambda} e^{nu_n(a)} da = e^{nu_\nu(\lambda)} \left(\frac{1}{u'_\nu(\lambda)} + O(n^{-2})\right)$$

(150)

holds. We can proceed equivalently for the second integral in (70), where we define $g_{\nu}(a) = u_\nu(a) - v_\nu(a) + R \ln(2)$ for compactness. In this case the integral variable is $x \in [0, \infty)$ in the relation $f_\eta(\lambda) = g_{\nu}(a)$ with $a$. This ensures

$$\int_{\lambda}^{1} e^{nu_n(a)} da = \int_{0}^{\infty} e^{nu_\nu(\lambda) - nx^2} \frac{-2x}{g'_\nu(f^{-1}(x))} dx$$

(151)

where the first Taylor coefficients are $h_0 = 0$, $h_1 = -2/g'_\nu(\lambda)$, and $h_2 = 0$. By use of [25, eq. 7.4.4-5] we then obtain

$$\int_{\lambda}^{1} e^{nu_n(a)} da = e^{nu_\nu(\lambda)} \left(\frac{1}{n(-g'_\nu(\lambda))} + O(n^{-2})\right)$$

(152)

where $g_\nu(\lambda) = u_\nu(\lambda)$ because of the definition of $\lambda$, and where $-g'_\nu(\lambda) = v'_\nu(\lambda) - u'_\nu(\lambda) > 0$ by assumption. Then

$$\frac{1}{n} \ln \frac{1}{P_e} = \ln u_\nu(\lambda) + \frac{\ln(n)}{n}$$

$$+ \ln \left(\frac{v'_\nu(\lambda)}{u'_\nu(\lambda)}\right) + O(n^{-2})$$

(153)

follows from the sum of (151) and (153). The $O(n^{-2})$ approximation (59) should be used for $u_n$, while for derivatives the $O(n^{-1})$ approximations (derived from (59) and (68))

$$v'_\nu(a) = \frac{a}{1 - a^2} + O(n^{-1})$$

$$u'_\nu(a) = \frac{-a}{1 - a^2} + 2a \left(aa + \sqrt{1 + (aa)^2}\right) + O(n^{-1})$$

(154)

can be used. Equation (72) is a compact rewrite of (154) using (155) and (59-60). The assumption $v'_\nu(\lambda) > u'_\nu(\lambda)$ corresponds to $w^{(0)}(\lambda) > 0$.

Proof of Theorem 18] The proof mimics the one of Theorem 10. We first investigate the behavior at $n = \infty$, in which case (69) and (72) guarantee that $\frac{\Delta}{\Omega} = \frac{1}{n} \log_2(1 - \lambda^2)$ where $\lambda$ is set by request $u^{(0)}(\lambda) = 0$. It can be verified
that the result is \( \lambda^* = \sqrt{\Omega/(1+\Omega)} \) and \( R = C \). We then investigate the Taylor series expansion of (72) at \( \lambda = \lambda^* \). By standard methods we obtain

\[
u^{(0)}(\lambda) = -\frac{1}{W}(\lambda - \lambda^*)^2 + O((\lambda - \lambda^*)^3)
\]

where \( W = (2 + \Omega)/(2(1 + \Omega)^3) = V/(\Omega(1 + \Omega)) \). By squaring the approximation on \( \nu^{(0)}(\lambda) \), the result we obtain is equivalent to (130). Hence, by exploiting the same method that was used in Theorem 10 we can say that (compare with \( \beta \) from which (79) straightforwardly follows by inspecting the statistical properties of products \( \beta \).

\[
u_n(\lambda) = C\ln(2) + \sqrt{\Omega/(1 + \Omega)}(\lambda - \lambda^*) + \frac{\ln(n)}{2n} + O((\lambda - \lambda^*)^2) + O(n^{-1})
\]

and by substitution of (156).

Proof of Theorem 19. We preliminarily observe that, by substitution of (75) and (77) in (3), we obtain

\[
\Lambda(a, b) = \ln 2 - \frac{1}{n} \sum_{i=1}^{n} h(\Omega a_i b_i)
\]

where \( h \) is defined in the enunciation of the theorem. Similarly to the proof of Theorem 3 we can therefore linearly modify \( \Lambda \), and introduce an alternative threshold \( \lambda \) defined through the linear relation \( \lambda_R = \ln 2 - \lambda \). By use in (3), we obtain

\[
P_{FA}(a) = P \left[ \sum_{i=1}^{n} h(\Omega a_i y_i) \leq n \lambda \right], \text{ where } y \sim p_y
\]

\[
P_{MD}(a) = P \left[ \sum_{i=1}^{n} h(\Omega a_i y_i) > n \lambda \right], \text{ where } y \sim p_{y|x}, x = a,
\]

from which (79) straightforwardly follows by inspecting the statistical properties of products \( \Omega a_i y_i \).

Proof of Theorem 22. We preliminarily prove that the PPV meta-converse bound provides capacity at \( n = \infty \). To this end, we observe that the following equivalences \( H(0) = 1 \) and \( H'(0) = (C - 1)/\log_2(e) \) hold. Therefore, from Theorem 1 and 47 with \( n = \infty \), we have that \( \lambda \) and \( s_\beta \) are set by

\[
\beta(s_\beta) = 0 \quad \text{and} \quad \beta'(s_\beta) = 0,
\]

which provides \( s_\beta = 0 \) and \( \lambda = (1 - C)/\log_2(e) \). Hence, it also is \( s_n = 1 \). Therefore the bound becomes \( R \) equals \( C \), which provides \( R = C \), and the result is proved.

The rest of the proof mimics the one of Theorem 10 by using the notation valid in the binary codewords case, and by observing that the function \( h \) used in the proof of Theorem 10 corresponds to the function \( -\ln(H(s)) \). Equation (129) is still valid because of Theorem 20 and so is (130) but we should consider that the limit value for \( \lambda \) is \( \lambda^* = (1 - C)/\log_2(e) \).

Hence (130) rewrites as

\[
s_\beta(\lambda) = s_\beta^* = \frac{1}{a^2} (\lambda - \lambda^*) + O((\lambda - \lambda^*)^2)
\]

(160)

where the asterisk * denotes a quantity evaluated at \( \lambda^* \), e.g., \( s_\beta^* = 0 \), \( s_n = 1 \). Note also, that, differently from the proof of Theorem 10 it is \( a_2^* = \frac{1}{2} \beta''(0) = V \), so that expressions hold provided that the following mappings are applied, namely: \( 4V \rightarrow V \), and \( \lambda - 1 \rightarrow \lambda - \lambda^* \). Hence, (131) turns into

\[
a_2(\lambda) s_\beta(\lambda) = \frac{1}{V} (\lambda - \lambda^*)^2 + O((\lambda - \lambda^*)^3)
\]

(161)

\[
\beta(s_\beta(\lambda); \lambda) = -\frac{1}{V} (\lambda - \lambda^*)^2 + O((\lambda - \lambda^*)^3)
\]

and the counterpart to (133) becomes

\[
\lambda = \lambda^* + \sqrt{\frac{V}{n} Q^{-1}(P_e)} + O(1/n).
\]

In the binary codewords context (135) is then reinterpreted in the form

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} h(\Omega x_i y_i) - \lambda \right)
\]

so that by substitution in (134) we obtain (1).

Proof of Theorem 23. From (158) we have

\[
g(x, y) = P \left[ \sum_{i=1}^{n} h(\Omega x_i y_i) \geq \sum_{i=1}^{n} h(\Omega z_i y_i) \right]
\]

(164)

where \( z \sim U_K \). We then consider that \( y_i \) can be written in the form \( y_i = x_i w_i / \Omega \) where \( w_i \sim U(N(\Omega, 2^\Omega)) \), which provides

\[
g(w) = P \left[ \sum_{i=1}^{n} h(w_i) \geq \sum_{i=1}^{n} h(z_i w_i) \right], \text{ where } z \sim U_K
\]

(165)

since \( x_i = 1 \) and since \( x_i z_i \) has the same statistical description of \( z_i \). By further exploiting \( h(x) - h(-x) = 2x \), we obtain (87) and (88).

Proof of Theorem 24. Denote \( g(a) = \int_R P_{y|x}(b|a)db \), and observe that

\[
\int_R p_y(b)db = \sum_{a \in K} \frac{1}{n} \int_R p_{y|x}(b|a)db = 1 - \sum_{a \in K} \frac{g(a)}{2^n}.
\]

(166)

Since \( g(a) \leq 1 - \tau \), it also is \( \kappa(\tau) \geq \tau \). The lower bound can be reached by choosing a circularly symmetric region such that \( g(a) = 1 - \tau \), for all \( a \in K \). Then (89) follows by exploiting the equality \( \kappa(\tau) = \tau \) in (83).

Proof of Theorem 25. We preliminarily observe that equations (96)-(98) provide the correct interpretation of the Neyman-Pearson criterion when dealing with discrete random variables. We then explicit the log-likelihood function (2) under (93), to have

\[
\Lambda(a, b) = \ln(2(1 - P_{\text{bit}})) - \frac{1}{n} \|b - a\|_2 \ln \left( \frac{1 - P_{\text{bit}}}{P_{\text{bit}}} \right)
\]

(167)

where the latter logarithm is guaranteed to be positive since \( P_{\text{bit}} < 1/2 \). Hence, the FA and MD probabilities (3) can be
written in the form
\[ P_{FA}(a, \lambda) = P[\|y - a\|_H \leq n\lambda], \text{ where } y \sim p_y \]
\[ P_{MD}(a, \lambda) = P[\|y - a\|_H > n\lambda], \text{ where } y \sim p_{y|x}, x = a. \]  
(168)

We then observe that \( \|y - a\|_H = \sum_i |y_i - a_i|_H \) holds in (168), with statistically independent \( y_i \)'s. By making explicit the statistical description of the binary variables \( \|y_i - a_i\|_H \) in (168), and by observing that these are independent of the value of \( a \), we then have (99) where we set \( \lambda = \frac{d}{n} \).

Proof of Theorem 26: The proof can be carried out as already seen in Theorem 5 by noting that the Laplace transforms of PDFs of \( u_i \)'s and \( v_i \)'s are, respectively, of the form \( \frac{1}{2} + \frac{1}{2} e^{-s} \), and \( 1 - P_{bit} + P_{bit} e^{-s} \). This provides \( \alpha \) as in (100), and \( \beta(s) = 2\lambda s + 2\ln(1 + e^{-(s+\delta_0)}) + 2\ln(1 - P_{bit}) \), which is equivalent to the second of (100). Note that application of the Laplace transform method provides probabilities (99) only for non integer values of \( d \). In fact, at points where the function is discontinuous, i.e., for integer values of \( d \), the inverse Laplace transform equals the average of the left and right limits.

Proof of Theorem 27: In order to identify the FA probability in (99) we need to investigate the right limit \( d^+ \) for integer \( d \), which corresponds to setting \( \lambda = \frac{d}{n} + \epsilon \), with \( \epsilon > 0 \) as small as desired, in Theorem 26. By now exploiting (105), and the equivalence \( \alpha(s + i2\pi k) = \alpha(s + 2\lambda - i2\pi k) \), then the FA probability can be written in the form
\[ P_{FA}(d) = 1(-s_\alpha) + \frac{1}{i2\pi} \sum_{k=\infty}^{+\infty} -\int_{A_{\alpha}} e^{\alpha(s)} e^{\pi s + i2\pi k} ds \]
\[ = 1(-s_\alpha) + \frac{1}{i2\pi} \int_{A_{\alpha}} e^{\alpha(s)} (\sum_{k=\infty}^{+\infty} e^{\pi s + i2\pi k}) ds \]  
(169)
where the second equivalence was obtained by exchanging integration and sum. The function in brackets in (169) can be evaluated in the closed form by means of standard Fourier transform/series properties. The rationale is the following. Start from the Fourier transform pairs
\[ 1(t) e^{qt} \rightarrow \frac{1}{i2\pi f - q} \text{ \( \Re(q) < 0 \) } \]
\[ -1(-t) e^{qt} \rightarrow \frac{1}{i2\pi f - q} \text{ \( \Re(q) > 0 \) } \]  
(170)
Recall that a sampling in frequency with sampling period \( F = 1 \) corresponds to a periodic repetition in time with period \( T_p = 1/F = 1 \), with corresponding inverse Fourier transform relations
\[ \sum_{k=\infty}^{+\infty} \frac{e^{i2\pi kt}}{i2\pi k - q} = \begin{cases} \sum_{k=\infty}^{+\infty} 1(t + k)e^{q(t + k)} & \text{\( \Re(q) < 0 \) } \\
- \sum_{k=\infty}^{+\infty} 1(-t + k)e^{q(-t + k)} & \text{\( \Re(q) > 0 \) }
\end{cases} \]  
(171)
In both cases the result is
\[ \sum_{k=\infty}^{+\infty} \frac{e^{i2\pi kt}}{i2\pi k - q} = \begin{cases} \frac{1 + e^{q}}{2(1 - e^{q})} & \text{\( t \) an integer} \\
\frac{e^{q(t^t - t^q)}}{1 - e^{q}} & \text{\( t \) not an integer} \]  
(172)
for \( \Re(q) \neq 0 \), but the result is valid also for \( \Re(q) = 0 \) by virtue of continuity. We observe that, for a sufficiently small \( \epsilon \), the application of (172) to (169) provides a contribution of the form \( 1/(1 - e^{q}) \) since \( t = n\lambda = d + n\epsilon \) and \( t - [t] = n\epsilon \simeq 0 \). We therefore obtain
\[ P_{FA}(d) = 1(-s_\alpha) + \frac{1}{i2\pi} \int_{A_{\alpha}} e^{\alpha(s)} ds \]
\[ P_{MD}(d) = 1(s_\beta) - \frac{1}{i2\pi} \int_{A_{\beta}} e^{\beta(s)} ds \]  
(173)
the result for the MD probability being derived in a perfectly equivalent way. Then, (106) is obtained by explicitly using the path expressions.

Proof of Theorem 37: The theorem proof mimics the one of Theorem 9 where inequality (51) implies (50), and, in turn, (50) implies (52) in the BSC context, the equivalent to inequality (51) is
\[ 1 - g_2(\lambda, q) u^2(\phi) \leq \frac{1}{g_0(\lambda, q)} \mathbb{Z}[c_\varepsilon(u(\phi))] \leq 1, \]  
(174)
where \( u^2(\phi) = 2d(\phi) \), and implies (50). In turn, it can be easily verified that (50) implies (52) also under adoption of (99-99). Our aim is therefore to prove (174).

We separate the main function in (174) in the form
\[ g_2(u(\phi))/g_0(\lambda, q) = f_1(\phi) f_2(\phi) \]
\[ f_1(\phi) = \frac{2u(\phi)}{\sqrt{1 + D(\phi)}} \sqrt{\lambda(1 - \lambda)} \]
\[ f_2(\phi) = \mathbb{Z} \left[ \frac{s'(\phi) + i}{1 - e^{q(s(\phi) - s\phi)}} \right] \frac{(1 - \lambda e^q + 1)}{(1 - \lambda)} \]  
(175)
and \( s(\phi) = \ln(u(\phi)) \). With a little effort we can write
\[ f_1(\phi) = \sqrt{\frac{1 + D(\phi)}{1 + C(\phi)}} \]  
(176)
with
\[ b_2 = \frac{\lambda e^q(1 + e^q)}{(1 - \lambda(1 + e^q))^2} \]  
(177)
and
\[ A(\phi) = \frac{1 + e^q(1 - \lambda)^2}{\text{sinc}^2((1 - \lambda)\phi)} - 1 \]
\[ B(\phi) = \frac{\text{sinc}^2(\lambda\phi)}{\text{sinc}^2(\phi)} - 1 \]
\[ C(\phi) = \frac{d'(\phi)}{2d_2\phi} - 1 \]
\[ D(\phi) = \frac{d(\phi)}{2d_2\phi^2} - 1 \]
\[ = \sum_{k=1}^{\infty} \frac{d_{2k}}{d_2} (k + 1)\phi^{2k} \]
(178)
Note that all constants and functions in (177) and (178) are positive, and in particular it is \( 0 \leq A(\phi) \leq B(\phi) \) because of sinc properties, and \( 2D(\phi) \leq C(\phi) \) since all the Taylor coefficients \( d_{2k} \) are positive. Incidentally, this latter property ensures \( f_1(\phi) \leq 1 \), while positivity of coefficients
and functions ensures \( f_1(\phi) \geq 0 \) and \( f_2(\phi) \leq 1 \). This proves the upper bound in (174), since \( f_1(\phi)f_2(\phi) \leq f_1(\phi) \leq 1 \).

For the lower bound we employ a different strategy for the FA and MD probabilities. Under MD, that is with \( x = 0 \) and (107), and imply the equivalences

\[
f_2(\phi) \geq 1 - 2b_2d(\phi) .
\]

(179)

To do so, we exploit inequalities \( \alpha \geq 0 \) and \( \beta \geq 1 \), which provide \( \alpha A(\phi) + \beta B(\phi) \geq B(\phi) \geq A(\phi) \). We then have

\[
2d(\phi) = (1 - \lambda) \ln(1 + A(\phi)) + \lambda \ln(1 + B(\phi))
\]

\[
\geq (1 - \lambda) \frac{A(\phi)}{A(\phi) + 1} + \lambda \left( \frac{B(\phi)}{1 + B(\phi)} \right)
\]

\[
\geq \frac{(1 - \lambda)A(\phi) + \lambda B(\phi)}{1 + \alpha A(\phi) + \beta B(\phi)}
\]

(180)

where we used \( \ln(1 + x) \geq x/(1 + x) \) in the first inequality. By substitution of (180) in the second of (176) we obtain (179).

We then observe that

\[
f_1(\phi) \geq 1 - 2a_2d(\phi), \quad a_2 = \frac{3d_4}{4d_5} = \frac{(1 - \lambda + \lambda^2)}{12\lambda(1 - \lambda)},
\]

(181)

an inequality which we verified numerically over \( \phi \in (0, \pi) \) and \( \lambda \in (0, \frac{1}{2}) \). The lower bound in (174) is then a consequence of the validity (179) and (181), of the property \( 0 \leq f_1(\phi) \leq 1 \), and of the equivalence \( a_2 + b_2 = g_2(\lambda, q) \). Under FA, where \( e^q = 1 \) and \( 0 < \lambda < \frac{1}{2} \), we numerically verified that \( f_1(\phi)f_2(\phi) \geq 1 - 2g_2(\lambda, q) \) holds for \( \phi \in (0, \pi) \) and \( \lambda \in (0, \frac{1}{2}) \). This completes the proof.

**Proof of Theorem 3.3** We mimic Theorem 10 but the proof here is complicated by the fact that we are dealing with a discrete distribution. Let \( f(x) \) be a strictly decreasing function which for integer valued \( x \) assumes the values \( \frac{1}{n} \ln(P_{\text{MD}}(x)) \), \( x \in \mathbb{N} \). Then, the value \( d \) can be obtained from the value \( d_0 \) satisfying \( f(d_0) = \frac{1}{n} \ln(P_{\text{FA}}(d)) \), in such a way that \( d = [d_0] = d_0 + O(1) \) and \( \lambda = \frac{d_0}{n} + O(1/n) \). Another important result that will be useful is that (96) ensures

\[
\mathcal{F} = -\frac{1}{n} \ln q(d) + O(n^{-1}).
\]

(182)

Now, for \( n = \infty \) we have \( f(d) = \beta(s_3) \) and the constraint is \( f(d_0) = 0 \) which provides \( \lambda_0 = \frac{d_0}{n} = P_{\text{bit}} \) and also \( \lambda = \lambda_0 \). Then, from (120) and (182), and by inspection of (107), we have

\[
\mathcal{F} = -\frac{1}{n} \ln g(d) = 1 - h(P_{\text{bit}}) = C
\]

where we also exploited the fact that \( n \to \infty \) to remove the \( O(n^{-1}) \) contributions. We then inspect the case \( n < \infty \). In the BSC case the equivalent to (129) is

\[
nf(d_0) = \frac{1}{n} \beta(s_3) \ln q \left( \sqrt{n\lambda_0^2(\lambda_0, \delta_0)} \right) + O(1/n),
\]

(183)

as can be derived by comparing (47) and (120). The values of \( g_0(\lambda_0, \delta_0) \) and \( \beta(s_3) \) are available from, respectively, (119) and (107), and imply the equivalences

\[
g_0^{-2}(\lambda_0, \delta_0) = \frac{1}{W(\lambda_0 - P_{\text{bit}})^2} + O((\lambda_0 - P_{\text{bit}})^3)
\]

\[
\beta(s_3(\lambda_0); \lambda_0) = -\frac{1}{W(\lambda_0 - P_{\text{bit}})^2} + O((\lambda_0 - P_{\text{bit}})^3)
\]

(184)

with \( W = P_{\text{bit}}(1 - P_{\text{bit}}) \). By the same arguments that lead from (131) to (132) and (133), we obtain

\[
\lambda_0 = P_{\text{bit}} + \frac{W}{n} Q^{-1}(P_{\text{bit}}) + O(1/n),
\]

(185)

and therefore \( \lambda = \lambda_0 + O(1/n) \), so that the two have the same \( O(1/n) \) approximation. We then observe that the logarithmic version of the FA probability is equivalently expressed in the form (135), and that (132) holds. Hence, the theorem is proved by observing (from (107)) that

\[
-\frac{1}{2} \ln \psi(\lambda_0; \lambda) = -\frac{1}{2} \beta(s_3(\lambda_0); \lambda) + \ln(2) + \ln(P_{\text{bit}}) + (1 - \lambda) \ln(1 - P_{\text{bit}}),
\]

(186)

and by final substitution of the second of (184) and (185) in (186).

**Proof of Theorem 3.3** By investigation of (167) we have

\[
g(x, y) = P[\|x - y\|_H \geq \|z - y\|_H], \quad z \sim U_{\mathbb{K}}
\]

\[
= P[\|z\|_H \leq \|x - y\|_H], \quad z \sim U_{\mathbb{K}}
\]

\[
g(\|x - y\|_H),
\]

(187)

where we exploited the fact that \( \|z\|_H \) and \( \|x - y\|_H \) have the same statistical description. Then, also the RCU bound is independent of \( x \), and we can therefore choose \( x = 0 \), to have

\[
\mathcal{F} = E[\min(1, 2^{nR} g(\|y\|_H))], \quad y \sim P_{y|x}, x = 0
\]

\[
= \sum_{d=0}^n \min(1, 2^{nR} g(d)) q_n(d; P_{\text{bit}})
\]

(188)

where we used (95). Then the result is obtained by exploiting the equivalences \( g(d) = P_{\text{FA}}(d) \), and \( P_{\text{MD}}(d) = \sum_{i=d+1}^n q_n(i; P_{\text{bit}}) \).

**Proof of Theorem 3.4** Note that the results of Corollary 30 provide simple approximations to estimate the value \( d_0 \) and the contribution \( P_{\text{MD}}(d_0) \). We then want to identify an \( O(n^{-1}) \) approximation to the summation in (124). To this aim we preliminarily rewrite the bound in the form

\[
\alpha = \sum_{d=0}^d e^{n[u_n(d) - d + R \ln(2)]}
\]

(189)

with \( u_n(d) = -\frac{1}{n} \ln P_{\text{FA}}(d) \), for which an \( O(n^{-2}) \) approximation is available from (120), and with

\[
u_n(d) = \frac{1}{n} \ln q_n(d; P_{\text{bit}})
\]

\[
= \frac{1}{n} \ln \left( \frac{\Gamma(n + 1)}{\Gamma(n - d + 1)\Gamma(d + 1)} \right) + \ln(1 - P_{\text{bit}}) - \frac{d}{n} \delta_0
\]

\[
= h(\lambda) \ln(2) + \ln(1 - P_{\text{bit}}) - \lambda \delta_0 - \frac{\ln(2\pi n)}{2n} - \frac{(\lambda - 1)}{2n} + O(n^{-2})
\]

(190)

where \( \lambda = \frac{d}{n} \), and whose approximation was derived from the asymptotic expression [25, eq. 6.1.41]. Use of the above \( O(n^{-2}) \) approximations provides the desired \( O(n^{-1}) \) approximation for \( \alpha \), but they do not solve the closed form evaluation
of the summation in \((199)\). To this end, we observe that, for a fixed value of \(n\), the functions \(v_n(d)\) and \(u_n(d)\) can be expressed as functions of \(\lambda = \frac{d}{n}\). Their derivative with respect to \(\lambda\) can also be identified (at least approximately) by exploiting Corollary 50 and \((190)\), to have
\[
\begin{align*}
v_n'(\lambda) &= -\ln \left(\frac{1 - \lambda}{\lambda}\right) + \frac{1}{n} \ln \left(1 - \frac{3\lambda + \lambda^2}{\lambda^2(1 - \lambda)}\right) + O(\frac{n}{n^2}) \\
u_n'(\lambda) &= \ln \left(\frac{1 - \lambda}{\lambda}\right) - \frac{1}{2\lambda} - \frac{1}{2n\lambda(1 - \lambda)} + O(\frac{n}{n^2}) .
\end{align*}
\]
(191)
The idea is then to approximate \(\alpha\) by exploiting the (truncated) Taylor expansion of the function \(g_n(\lambda) = u_n(\lambda) - v_n(\lambda)\), namely
\[
g_n(\lambda) = g_n(\lambda^*) + g_n'(\lambda^*)(\lambda - \lambda^*) + O((\lambda - \lambda^*)^2) ,
\]
(192)
for some \(\lambda^*\). A sensible choice is to choose \(\lambda^*\) as the solution to \(v_n(\lambda^*) = R \ln(2)\). With some effort it can also be verified that the \(O((\lambda - \lambda^*)^2)\) contribution at the exponent corresponds to an \(O(n^{-1})\) contribution to \(\ln(\alpha)\), which is ensured by the fact that the polylogarithmic series \(\sum_{d=0}^{\infty} e^{-d^{1/2}} d^2\) is \(O(1)\) for \(d_0 = [\sqrt{\ln n} \lambda^*] = O(n)\). Therefore, by exploiting the truncated Taylor expansion \((192)\) and the \(O(n^{-2})\) approximations for \(g_n(\lambda)\) and its derivatives, we obtain
\[
-\frac{1}{n} \ln(\alpha) = u_n(\lambda^*) + \frac{1}{n} \ln \left(1 - \frac{e^{-(d_0 + 1)g}}{1 - e^{-g}}\right) + O(n^{-2}) .
\]
(193)
where \(g = g_n'(\lambda^*)\). Note that the contribution \(e^{-(d_0 + 1)g}\) can be included in the \(O(n^{-2})\) factor since \(d_0\) is \(O(n)\), and that a \(O(n^{-1})\) approximation of \(g\) can be used, which, from inspection of \((191)\), provides
\[
-\frac{1}{n} \ln(1 - e^{-g}) = \frac{1}{n} \ln \left(1 - \frac{1 - P_{MD}(\lambda^*)^2}{P_{MD}(1 - \lambda^*)^2}\right) + O(n^{-2}) .
\]
(194)
From the second of \((120)\) and by considering that \(P_{MD}\) is a decreasing function of \(\lambda\) we further obtain
\[
-\frac{1}{n} \ln(\beta) \leq u_n(\lambda^*) + \frac{1}{n} \ln \left(1 - \frac{1 - \lambda^*}{\lambda^* - P_{MD}}\right) + O(n^{-2}) .
\]
(195)
Equation \((125)\) is finally obtained by putting altogether the above results.

Proof of Theorem 13. We are interested in the bound on rate, in which case \((125)\) must be interpreted as a constraint on \(P_c\). For \(n = \infty\), the first of \((126)\) reveals that \(\lambda = P_{bit}\). Therefore, for limited \(n\) we are interested in the Taylor expansion at \(P_{bit}\). For the first two expressions in \((126)\) we find
\[
\begin{align*}
w_n^{(0)}(\lambda) &= -\frac{\ln(2\pi n)}{2n} + \frac{(\lambda - P_{bit})^2}{2W} + O((\lambda - P_{bit})^3) \\
w_n^{(1)}(\lambda) &= \frac{(\lambda - P_{bit})^2}{W} + O((\lambda - P_{bit})^3)
\end{align*}
\]
(196)
with \(W = P_{bit}(1 - P_{bit})\). By following the rationale of Theorem 10 leading from \((129)\)–\((132)\) to \((133)\), we obtain
\[
\lambda = P_{bit} + \sqrt{\frac{W}{n} Q^{-1}(P_c) + O(1/n)} .
\]
(197)
The normal approximation is finally derived from \(R = w_n^{(2)}(\lambda) \log_2(e)\) by using the last of \((126)\) and, specifically by using \((197)\) in the \(O(n^{-1})\) approximation
\[
w_n^{(2)}(\lambda) = \left[1 - h(\lambda)\right] \ln(2) + \frac{\ln(2\pi n)}{2n} + O(1/n) .
\]
(198)

REFERENCES
[1] C. Shannon, R. Gallager, and E. Berlekamp, “Lower bounds to error probability for coding on discrete memoryless channels,” I. Information and Control, vol. 10, no. 1, pp. 65 – 103, 1967.
[2] Y. Altug and A. B. Wagner, “Moderate deviation analysis of channel coding: Discrete memoryless case,” in 2010 IEEE International Symposium on Information Theory Proceedings (ISIT). IEEE, 2010, pp. 265–269.
[3] ——, “Moderate deviations in channel coding,” IEEE Transactions on Information Theory, vol. 60, no. 8, pp. 4417–4426, 2014.
[4] Y. Polyanskiy and S. Verdú, “Channel dispersion and moderate deviations limits for memoryless channels,” in Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on, 2010, pp. 1334–1339.
[5] Y. Polyanskiy and M. Tomamichel, “The third-order term in the normal approximation for the AWGN channel,” IEEE Transactions on Information Theory, vol. 61, no. 5, pp. 2430–2438, May 2015.
[6] C. E. Shannon, “Probability of error for optimal codes in a Gaussian channel,” Bell System Technical Journal, vol. 38, no. 3, pp. 611–656, 1959.
[7] Y. Polyanskiy, H. V. Poor, and S. Verdú, “Dispersion of Gaussian channels,” in IEEE International Symposium on Information Theory (ISIT), 2009, pp. 2204–2208.
[8] Y. Polyanskiy, “Channel coding: non-asymptotic fundamental limits,” Ph.D. dissertation, Princeton University, 2010.
[9] Y. Polyanskiy, H. V. Poor, and S. Verdú, “Channel coding rate in the finite blocklength regime,” IEEE Transactions on Information Theory, vol. 56, no. 5, pp. 2307–2359, 2010.
[10] S. H. Kim, D. K. Sung, and T. Le-Ngoc, “Performance analysis of incremental redundancy type hybrid ARQ for finite-length packets in AWGN channel,” in Global Communications Conference (GLOBECOM), 2013 IEEE, 2013, pp. 2063–2068.
[11] B. Makki, T. Svensson, and M. Zorzi, “Green communication via Type-I ARQ: Finite block-length analysis,” in Global Communications Conference (GLOBECOM), 2014 IEEE, 2014, pp. 2673–2677.
[12] M. Centenaro, G. Ministeri, and L. Vangelista, “On Energy-Efficient HARQ schemes for M2M communication,” in 15th IEEE International Conference on Ubiquitous Wireless Broadband 2015: Workshop on Next Generation of Green ICT and 5G Networking (GreeNets) (IEEE ICIW 2015 WOS 07), Montreal, Canada, 2015.
[13] P. Moulin, “The log-volume of optimal codes for memoryless channels, asymptotically within a few bits,” arXiv preprint arXiv:1311.0181, 2013.
[14] T. Erseghe, “On the evaluation of the Polyanskiy-Poor-Verdu converse bound for finite blocklength coding in AWGN,” IEEE Trans. on Information Theory, vol. 61, no. 12, pp. 6578–6590, 2015.
[15] N. Terence, “Asymptotic and numerical aspects of the noncentral chi-square distribution,” Elsevier Computers & Mathematics with Applications, vol. 25, no. 5, pp. 55 – 63, 1993.
[16] N. Bleistein and R. A. Handelsman, Asymptotic expansions of integrals. Courier Corporation, 1975.
[17] A. Martínez and A. G. i Fabregas, “Saddlepoint approximation of random-coding bounds,” in Proc. Inf. Theory Applicat. Workshop (ITA), San Diego, CA, U.S.A., Feb. 2011, pp. 257–262.
[18] A. Martínez, J. Scarlett, M. Dalai, and A. G. i Fabregas, “A complex-integration approach to the saddlepoint approximation for random-coding bounds.” in Proc. IEEE Int. Symp. Wirel. Comm. Syst. (ISWCS), Barcelona, Spain, Aug. 2014, pp. 618–621.
[19] J.-H. Park and D.-J. Park, “A new power allocation method for parallel AWGN channels in the finite block length regime,” IEEE Communications Letters, vol. 16, no. 9, pp. 1392–1395, 2012.
[20] A. Valenbois and M. P. Fossorier, “Sphere-packing bounds revisited for moderate block lengths,” IEEE Transactions on Information Theory, vol. 50, no. 12, pp. 2998–3014, 2004.
[21] G. Wiechman and I. Sason, “An improved sphere-packing bound for finite-length codes over symmetric memoryless channels,” IEEE Transactions on Information Theory, vol. 54, no. 5, pp. 1962–1990, May 2008.
[22] G. Liva, E. Paolini, B. Matuz, S. Scalise, and M. Chiani, “Short turbo codes over high order fields,” Communications, IEEE Transactions on, vol. 61, no. 6, pp. 2201–2211, 2013.

[23] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, Signals and systems. Pearson, 2014.

[24] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, Integrals and series Vol. 1: Elementary functions (English translation). Gordon & Breach Science, 1986.

[25] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover, 1968, 3rd Edition.

[26] L. Tierney and J. B. Kadane, “Accurate approximations for posterior moments and marginal densities,” Journal of the American statistical association, vol. 81, no. 393, pp. 82–86, 1986.

[27] G. Margulis, “Explicit constructions of graphs without short cycles and low density codes,” Combinatorica, vol. 2, no. 1, pp. 71–78, 1982. [Online]. Available: http://dx.doi.org/10.1007/BF02579283

[28] S. J. Johnson, Iterative error correction: Turbo, low-density parity-check and repeat-accumulate codes. Cambridge University Press, 2009.

[29] N. Temme, “Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function,” Mathematics of Computation, vol. 29, no. 132, pp. 1109–1114, 10 1975.

[30] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” Journal of the American Statistical Association, vol. 58, no. 301, pp. pp. 13–30, 1963.

[31] S. Kay, Fundamentals of statistical signal processing: Vol II - Detection theory. Prentice Hall, 1993.

[32] A. Oppenheim and A. Willsky, Signals and systems. Pearson Education, 2013.

[33] I. S. Gradshteyn and I. Ryzhik, Table of integrals, series, and products. Academic Press, 1980.

[34] M. Itskov, R. Dargazany, and K. Hörnes, “Taylor expansion of the inverse function with application to the Langevin function,” Mathematics and Mechanics of Solids, vol. 17, no. 7, pp. 693–701, 2012.

[35] L. L. Scharf, Statistical signal processing. Addison-Wesley Reading, MA, 1991, vol. 98.

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