SPECIAL SYMPLECTIC SIX-MANIFOLDS

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Abstract. We classify nilmanifolds with an invariant symplectic half-flat structure. We solve the half-flat evolution equations in one example, writing down the resulting Ricci-flat metric. We study the geometry of the orbit space of 6-manifolds with an SU(3)-structure preserved by a U(1) action, giving characterizations in the symplectic half-flat and integrable case.

1. Introduction

A half-flat manifold is a six-dimensional manifold endowed with an SU(3)-structure whose intrinsic torsion is symmetric. An SU(3)-structure defines a non-degenerate two-form $\omega$, an almost-complex structure $J$, and a complex volume form $\Psi$; the half-flat condition is equivalent to requiring $\omega \wedge \omega$ and the real part of $\Psi$ to be closed [5].

Hypersurfaces in seven-dimensional manifolds with holonomy $G_2$ have a natural half-flat structure, given by the restriction of the holonomy group representation; the intrinsic torsion can then be identified with the second fundamental form. The converse is not obvious. In [13], Hitchin proved that, starting with a half-flat manifold $(M, \omega, \Psi)$, if certain evolution equations admit a solution coinciding with $(\omega, \Psi)$ at time zero, then $(M, \omega, \Psi)$ can be embedded isometrically as a hypersurface in a manifold with holonomy contained in $G_2$.

Given a six-manifold $M$ with an SU(3)-structure, one can consider the product $G_2$-structure on $M \times S^1$. Half-flat manifolds satisfying special conditions on this product $G_2$-structure have been studied: in [6] and [4], the six-dimensional nilmanifolds carrying invariant SU(3)-structures of these types have been classified.

More generally, the problem of classifying nilmanifolds admitting invariant half-flat structures is open.

In this paper we focus on the symplectic case; that is to say, we take into consideration half-flat structures for which $\omega$ is closed. In this context, one can introduce special Lagrangian submanifolds, namely the three-dimensional submanifolds $\iota: L \to M$ such that both $\iota^*\omega$ and $\iota^*\text{Im} \Psi$ vanish. Like in the integrable case, special Lagrangian submanifolds are exactly the submanifolds calibrated by $\text{Re} \Psi$ [12].

Symplectic half-flat manifolds can also be viewed as symplectic manifolds $(M, \omega)$ endowed with two extra objects, namely an $\omega$-calibrated almost-complex structure and a $(3,0)$-form with closed real part which is parallel with respect to the Chern connection [8], [9].

Our main result is the classification of nilmanifolds carrying an invariant symplectic half-flat structure; the special case $b_1 \geq 4$ was carried out in [2].

The contents of this paper are organized as follows. Section 2 consists of the classification. Since six-dimensional nilmanifolds can be realized as circle bundles

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over nilmanifolds of dimension five, this classification problem can be reduced to a problem in five dimensions: indeed, the symplectic half-flat structure induces an SU(2)-structure on the base of the circle bundle, satisfying certain conditions involving the curvature (see Lemma 2.2). In Lemma 2.3 we classify the five-dimensional nilmanifolds with this type of induced structure. We then use this lemma to show that every six-dimensional nilmanifold admitting a symplectic half-flat structure is modelled on one of a list of three Lie algebras (Theorem 2.4). The corresponding nilmanifolds are the torus, a torus bundle over the four-dimensional torus (see [9] and [11]) and a torus bundle over a three-dimensional torus (see [2] and [11]). In the last case, the fibres are actually special Lagrangian submanifolds.

In Section 3 we fix a symplectic half-flat structure on the irreducible nilmanifold appearing in Theorem 2.4 and solve the evolution equations. We write down the resulting metric, computing the curvature and proving that the holonomy coincides with $G_2$. We also observe that if one starts with the reducible example, where the half-flat manifold is the product of a five-manifold with a circle, the resulting seven-dimensional manifold is also reducible.

In Section 4 we generalize this situation and the construction of Lemma 2.2 to the non-invariant case. More precisely, we consider circle bundles with a $U(1)$-invariant SU(3)-structure, and we define an induced SU(2)-structure on the base; the manifolds are assumed to be compact. The metric underlying the SU(2)-structure is not the same as the quotient metric, but it is obtained from it by rescaling along certain directions, as in [11].

The compactness assumption is dropped in Section 5 where we compute the intrinsic torsion of the SU(2)-structure in terms of the intrinsic torsion of the SU(3)-structure and curvature of the bundle. As far as we know, this is the first detailed application of intrinsic torsion for SU(2)-structures on five-manifolds. We then characterize the $U(1)$-invariant symplectic half-flat manifolds in terms of the quotient structure (Proposition 5.2). In the case that the $U(1)$-invariant SU(3)-structure is integrable, we obtain a stronger result: indeed, the intrinsic torsion of the quotient structure and the curvature are determined by the length of the fibres (Theorem 5.3).

2. Invariant structures on nilmanifolds

In this section we introduce half-flat structures on 6-manifolds and classify invariant symplectic half-flat structures on 6-dimensional nilmanifolds.

Let $M$ be a 6-dimensional manifold. An SU(3)-structure on $M$ is a pair $(\omega, \Psi)$, where $\omega$ is a non-degenerate two-form and $\Psi$ is a decomposable complex three-form, such that the following compatibility conditions hold:

$$\begin{align*}
\Psi \wedge \omega &= 0 \\
\Psi \wedge \overline{\Psi} &= \frac{4}{3} i \omega^3
\end{align*}$$

Indeed, a decomposable complex three-form $\Psi = \theta^1 \wedge \theta^2 \wedge \theta^3$ determines an almost-complex structure $J$ for which $\theta^1$, $\theta^2$ and $\theta^3$ span the space of forms of type $(1,0)$. The second compatibility condition implies in particular that $J$ is $\omega$-tamed; the first condition asserts that $\omega$ is of type $(1,1)$, so $J$ is actually calibrated by $\omega$.

At each point $x$ of $M$, the three-form $\Psi_x$ has stabilizer conjugate to $SL(3, \mathbb{C})$ for the natural action of $GL^+(T_x M)$; on the other hand, the stabilizer of $\omega_x$ is conjugate
to the symplectic group $\text{Sp}(3, \mathbb{R})$ for the natural action of $\text{GL}(T_x M)$. The compatibility conditions ensure that the intersection of the two stabilizers is conjugate to $\text{SU}(3)$, so that an $\text{SU}(3)$-structure is defined.

We shall denote by $\psi^+$ and $\psi^-$ the real and imaginary part of $\Psi$, respectively. It was shown by Hitchin \[13\] that having fixed the orientation, the real three-form $\psi^+$ is sufficient to determine the almost-complex structure $J$, and therefore $\psi^-$. Thus, an $\text{SU}(3)$-structure $(\omega, \Psi)$ is really determined by the pair $(\omega, \psi^+)$. We now introduce a special class of $\text{SU}(3)$-structures on 6-manifolds, related to 7-dimensional Riemannian manifolds with holonomy contained in $G_2$ \[4\]:

**Definition 2.1.** An $\text{SU}(3)$-structure $(\omega, \psi^+)$ on a 6-manifold is *half-flat* if $\omega \wedge \omega$ and $\psi^+$ are closed.

The 2-form $\omega$ appearing in the characterization of $\text{SU}(3)$-structures is required to be non-degenerate; if it is also closed, it defines a symplectic structure. In this case, we say that the $\text{SU}(3)$-structure is symplectic.

Consider a nilmanifold $M$, i.e. a compact manifold of the form $\Gamma \backslash G$, where $G$ is a 6-dimensional nilpotent group, and $\Gamma$ a discrete subgroup of $G$. Recall that in six dimensions, every nilpotent Lie algebra $\mathfrak{g}$ gives rise to such a nilmanifold. We say that a structure on the nilmanifold $M$ is *invariant* if it pulls back to a left-invariant structure on $G$. Invariant structures can be viewed as structures on the Lie algebra $\mathfrak{g}$; using the above characterization of $\text{SU}(3)$-structures in terms of differential forms, we shall mainly work with the dual $\mathfrak{g}^*$.

We start by reducing the problem to a problem in five dimensions; the idea is to realize $M$ as a circle bundle over a 5-dimensional manifold, in such a way that the $\text{SU}(3)$-structure on $M$ is invariant under the circle action. The geometry of this construction will be studied in Section \[4\] here, we shall use the following algebraic result:

**Lemma 2.2.** Let $(\omega, \psi^+)$ be a symplectic half-flat structure on a nilpotent Lie algebra $\mathfrak{g}$; we have an orthogonal decomposition

$$\mathfrak{g}^* = (\eta) \oplus V^5,$$

where $\eta$ is a unit form, and $d(\mathfrak{g}^*) \subseteq \Lambda^2 V^5$.

Define forms $\alpha$, $\omega_1$, $\omega_2$, and $\omega_3$ on $\ker \eta$ by

$$\begin{cases} 
\omega = \omega_3 + \eta \wedge \alpha \\
\Psi = (\omega_1 + i \omega_2) \wedge (\eta + i \alpha) 
\end{cases}$$

Setting $\phi = d\eta$, the following hold:

$$\begin{cases} 
0 = d\alpha \\
0 = d\omega_1 \\
-\phi \wedge \alpha, \quad d(\omega_2 \wedge \alpha) = \omega_1 \wedge \phi .
\end{cases}$$

**Proof.** By Engel’s theorem, some non-zero $\xi$ in $\mathfrak{g}$ satisfies $\text{ad}(\xi) = 0$.

Choosing for $\eta$ a suitable multiple of $\xi^\perp$, and setting $V^5 = \eta^\perp$, the first part of the Lemma is satisfied. By definition,

$$0 = d\omega = d\omega_3 + \phi \wedge \alpha - \eta \wedge d\alpha ;$$
isolating the component in $\eta \wedge \Lambda^2 V^5$, we deduce that $\alpha$ is closed and $d\omega_3$ satisfies the required equation.

Similarly, the rest of (3) follow from:

$$0 = d\phi^+ = d\omega_1 \wedge \eta + \omega_1 \wedge \phi - d(\omega_2 \wedge \alpha).$$

**Remark.** The forms $(\alpha, \omega_i)$ introduced in Lemma 2.2 define an SU(2)-structure on $\mathfrak{g}$. More generally, we recall that differential forms $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-manifold define an SU(2)-structure if and only if at each point (and hence locally) there exists a coframe $e^1, \ldots, e^5$ such that

$$\begin{aligned}
\alpha &= e^5 \\
\omega_1 &= e^{12} + e^{34} \\
\omega_2 &= e^{13} + e^{42} \\
\omega_3 &= e^{14} + e^{23}
\end{aligned}$$

(4)

Here and in the sequel, $e^{12}$ is short for $e^1 \wedge e^2$, and so on. If one fixes an orientation, the condition above is equivalent to the existence of a triplet $(\omega_1, \psi_2, \psi_3)$ with

$$\omega_1 = e^{12} + e^{34}, \quad \psi_2 = e^{135} + e^{425}, \quad \psi_3 = e^{145} + e^{235}.$$ 

By construction, $V^5$ is itself the dual of a nilpotent Lie algebra; we shall proceed by listing the 5-dimensional Lie algebras that arise this way. To describe Lie algebras, we shall use symbolic expressions such as

$$\mathfrak{g} = (0, 0, 0, 0, 12),$$

meaning that $\mathfrak{g}^*$ has a basis $\eta^1, \ldots, \eta^6$ such that $d\eta^6 = \eta^1 \wedge \eta^2$ and $\eta^i$ is closed for $i = 1, \ldots, 5$.

**Lemma 2.3.** In the hypotheses of Lemma 2.2, $V^5$ is one of

$$(0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12), \quad (0, 0, 12, 13).$$

**Proof.** By the same argument we used in the proof of Lemma 2.2, we can construct a filtration

$$V^0 \subset \cdots \subset V^5, \quad \dim V_i = i, \quad dV^{i+1} \subset \Lambda^2 V^i.$$ 

Moreover, by Lemma 2.2, we can assume that $\alpha$ lies in $V^1 \subset V^4$; therefore, using the fact that the 4-dimensional representation of SU(2) is transitive, we can fix a basis $e^1, \ldots, e^5$ of $V^5$ satisfying $\mathfrak{g}$, with $e^4$ in $(V^4)^\perp$.

We have to show that the first Betti number satisfies $b_1 \geq 3$; by the classification of 5-dimensional nilpotent Lie algebras, it will then suffice to show that for some choice of the $V^i$ as above, $de^4$ lies in $\Lambda^2 V^3$ (in particular, this implies that the step of $V^5$ is less than two).

By Lemma 2.2,

$$0 = de^{12} + de^{34} = de^3 \wedge e^4 \mod \Lambda^2 V^4,$$

implying that $e^3$ is closed. Thus, we can assume

$$V^2 = \langle e^3, e^5 \rangle; \quad V^4 = \langle e^1, e^2, e^3, e^5 \rangle.$$

Define a real constant $h$, a 2-form $\gamma \in \Lambda^2 \langle e^1, e^2, e^3 \rangle$ and 1-forms $\phi_4, \phi_5$ in $\langle e^1, e^2, e^3 \rangle$, such that

$$\phi = \phi_5 \wedge e^5 + \phi_4 \wedge e^4 + he^{45} + \gamma.$$ 

Since $\phi$ is closed,

$$d\phi_5 \wedge e^5 - \phi_4 \wedge de^4 + hde^4 \wedge e^5 + d\gamma = -d\phi_4 \wedge e^4;$$

$$\Box$$
the left-hand side lies in $\Lambda^3 V^4$, i.e. it has no component containing $e^4$, so both sides are zero and $d\phi_4 = 0$.

By Lemma 2.2 $d\omega_3 = -\phi \wedge \alpha$, giving
\begin{equation}
  de^{14} + de^2 \wedge e^3 = -\phi_4 \wedge e^{45} - \gamma \wedge e^5 ;
\end{equation}
comparing the components in $e^4 \wedge \Lambda^2 V^4$, we obtain
\begin{equation}
  de^1 = \phi_4 \wedge e^5 .
\end{equation}

Again by Lemma 2.2 $d\psi_2 = \omega_1 \wedge \phi$; on the other hand, $d\psi_2 = d\omega_2 \wedge \alpha$, so we can drop the component of $\phi$ not containing $e^5$ and write
\begin{equation}
  (de^{13} + de^{42}) \wedge e^5 = (e^{12} + e^{34}) \wedge (\phi_5 \wedge e^5 + he^{45}) .
\end{equation}
The components containing $e^4$ give
\begin{equation}
  de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5 - he^{125} ,
\end{equation}
and wedging with $e^3$,
\begin{equation}
  de^2 \wedge e^{35} = -he^{1235} .
\end{equation}
Since $de^2$ is in $\Lambda^2 V^3$, the left hand side is in $\Lambda^4 V^3$, so it is zero. We conclude that $h = 0$ and
\begin{equation}
  de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5 ;
\end{equation}
now, either $V^3 = \langle e^1, e^3, e^5 \rangle$, or some linear combination $\lambda e^1 + e^2$ lies in $V^3$, and consequently $0 = (\lambda de^1 + de^2) \wedge e^5 = de^2 \wedge e^5$. Either way,
\begin{equation}
  \phi_5 \in \langle e^1, e^3 \rangle .
\end{equation}

By Lemma 2.2 $\omega_1$ is closed, giving
\begin{equation}
  \phi_4 \wedge e^{52} - e^1 \wedge de^2 = e^3 \wedge de^4 ;
\end{equation}
to proceed further, we must distinguish three cases.

i) Suppose that $\phi_4$ is not a multiple of $e^3$; then
\begin{equation}
  V^3 = \langle e^3, e^5, \phi_4 \rangle ,
\end{equation}
and $d$ is zero on $V^3$. Moreover $e^1$ is not closed, so $V^4 = V^3 \oplus \langle e^1 \rangle$. Since $e^2$ is in $V^4$, we have $de^2 = ke^3 \wedge \phi_4$ for some (possibly zero) constant $k$. So (9) becomes
\begin{equation}
  \phi_4 \wedge e^{52} - ke^{15} \wedge \phi_4 = e^3 \wedge de^4 ,
\end{equation}
implying that
\begin{equation}
  de^4 \wedge e^3 \wedge \phi_4 = 0 = de^4 \wedge e^{35} .
\end{equation}
Since the space of closed two-forms in $\Lambda^2 V^4$ is
\begin{equation}
  \Lambda^2 V^3 \oplus \langle e^{15}, e^1 \wedge \gamma_4 \rangle ,
\end{equation}
we can conclude that $de^4$ lies in $\Lambda^2 V^3$; we already know that $b_1 \geq 3$, so there is nothing left to prove in this case.

In general, the component of (6) in $\Lambda^3 V^4$ gives
\begin{equation}
  -e^1 \wedge de^4 + de^2 \wedge e^3 = -\gamma \wedge e^5 ;
\end{equation}
in particular, $de^4 \wedge e^{15} = 0$. Moreover, we can rewrite (10) as
\begin{equation}
  -\phi_4 \wedge de^4 + d\gamma = 0 .
\end{equation}
ii) Suppose now that \( \phi_4 = 0 \); then \( e^1 \) is closed and we can assume that
\( V^3 = \langle e^1, e^3, e^5 \rangle \). By (9),
\[-e^1 \wedge de^2 = e^3 \wedge de^4,
\]
so clearly \( de^4 \wedge e^{13} = 0 \); moreover \( de^4 \wedge e^{35} = 0 \), since \( de^2 \wedge e^{15} \) is zero. It follows that \( de^4 \) lies in \( \Lambda^2 V^3 \), completing the proof in this case.

iii) The remaining case is the one where \( \phi_4 = ae^3 \) for some non-zero \( a \). By (11) and (9), this condition implies
\[d\gamma = ae^3 \wedge de^4 = a^2 e^{352} - ae^1 \wedge de^2.
\]
Equations (8) and (9) show that
\[de^2 \wedge e^{15} = 0 = de^2 \wedge e^{13} = de^2 \wedge e^{35},
\]
so \( de^4 \) lies in \( \Lambda^2 \langle e^1, e^3, e^5 \rangle \). Hence the space of closed forms in \( \Lambda^2 V^4 \) is contained in
\[\Lambda^2 \langle e^1, e^3, e^5 \rangle \oplus e^2 \wedge \langle e^3, e^5 \rangle;
\]
wedging the closed two-form \( \gamma - ae^{12} \) with \( e^{35} \) we deduce
\[\gamma \wedge e^{35} = ae^{1235}.
\]
Comparing with (10), we find \( e^{13} \wedge de^4 = -ae^{1235} \), which together with (12) gives the contradiction
\[-a^2 e^{1235} = a^2 e^{1352}.
\]
\[\square
\]
All three possibilities listed in Lemma 2.3 can occur:

- On \( V^5 = (0, 0, 0, 0, 0) \), set
\( \omega_1 = \eta^{12} + \eta^{34}, \quad \psi_2 = \eta^{135} + \eta^{425}, \quad \psi_3 = \eta^{145} + \eta^{235}, \quad \phi = 0 \).

- On \( V^5 = (0, 0, 0, 0, 12) \), set
\( \omega_1 = \eta^{34} + \eta^{15}, \quad \psi_2 = \eta^{312} + \eta^{542}, \quad \psi_3 = \eta^{352} + \eta^{412}, \quad \phi = -\eta^{13} \).

- On \( V^5 = (0, 0, 0, 12, 13) \), set
\( \omega_1 = \eta^{24} + \eta^{35}, \quad \psi_2 = \eta^{123} + \eta^{154}, \quad \psi_3 = \eta^{125} + \eta^{143}, \quad \phi = -2\eta^{23}; \) or
\( \omega_1 = \eta^{24} - \eta^{35}, \quad \psi_2 = -\eta^{123} + \eta^{154}, \quad \psi_3 = \eta^{125} - \eta^{143}, \quad \phi = 0 \).

It is easy to verify that Equations (8) are satisfied in these cases. The construction can then be inverted: define
\[g^* = \langle \eta \rangle \oplus V^5,\]
declaring that \( d\eta = \phi \); clearly, \( g^* \) is the dual of a nilpotent Lie algebra \( g \). A straightforward calculation shows that the SU(3)-structure on \( g \) defined by (2) is half-flat and symplectic.

So, there are three non-isomorphic nilpotent Lie algebras that admit a symplectic half-flat structure. It only remains to show that this list is complete.

**Theorem 2.4.** The 6-dimensional nilpotent Lie algebras whose corresponding nilmanifold carries an invariant symplectic half-flat structure are
\[(0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 12, 13, 23).\]
Proof. We retain the notation from the proof of Lemma 2.3. We first show that \( \phi_4 \) is zero; in other words, case i) of Lemma 2.3 cannot occur, like case iii), which we have already ruled out. Indeed, suppose that \( \phi_4 \) is independent of \( e^3 \) and \( e^5 \). Hence \( V^5 = (0, 0, 0, 12, 13) \); indeed, \( de^1 \) and \( de^4 \) must be independent, since otherwise a combination of \( e^1 \) and \( e^4 \) would lie in \( \ker d \), which is orthogonal to \( e^4 \) by construction.

Observe that \( \langle e^1, e^3, \phi^4 \rangle \) has dimension three, because \( \phi_4 \) is closed but \( e^1 \) is not, and we are assuming that \( \phi_4 \) is not a multiple of \( e^3 \). Therefore,

\[
\langle e^1, e^3, \phi^4 \rangle = \langle e^1, e^2, e^3 \rangle ,
\]

and we can write \( \gamma = a e^{13} + b e^4 \wedge \phi_4 + c e^3 \wedge \phi_4 \); Equation 10 then yields

\[
-e^1 \wedge de^4 + ke^5 \wedge \phi_4 \wedge e^3 = -ae^{135} + be^{15} \wedge \phi_4 + ce^{35} \wedge \phi_4 ,
\]

so in particular

\[
de^4 = ae^{35} - be^5 \wedge \phi_4 ;
\]

substituting in 11, it follows that

\[
-a \phi_4 \wedge e^{35} + a \phi_4 \wedge e^{53} = 0 ,
\]

i.e. \( a = 0 \); but then \( de^4 = b e^5 \wedge \phi_4 \) is a multiple of \( de^1 \), which is absurd.

We have proved that \( \phi_4 \) is necessarily zero; now assume that \( e^2 \) is closed. Then 9 and 10 give

\[
e^3 \wedge de^4 = 0 , \quad e^1 \wedge de^4 = \gamma \wedge e^5 .
\]

It follows that \( de^4 = \lambda e^{35} + \mu e^{13} \) and \( \gamma = \lambda e^{13} \) for some constants \( \lambda \) and \( \mu \). The components of 7 not containing \( e^4 \) give

\[
\mu e^{1325} = -e^{125} \wedge \phi_5 ;
\]

Since as a consequence of 5, \( \phi_5 \) is a multiple of \( e^3 \), it follows that \( \phi_5 = -\mu e^3 \).

Summing up,

\[
\phi = -\mu e^{35} + \lambda e^{13} ,
\]

so that \( \phi \) and \( de^4 \) are either linearly independent or both zero. The resulting 6-dimensional Lie algebras are

\[
(0, 0, 0, 0, 12, 13) , \quad (0, 0, 0, 0, 0, 0) .
\]

If \( e^2 \) is not closed, \( V^5 \) is \( (0, 0, 0, 12, 13) \). As both \( de^4 \) and \( de^2 \) are in \( \Lambda^2 V^3 \), 10 implies that \( \gamma \) is a multiple of \( e^{13} \). Therefore \( \phi \) lies in \( \Lambda^2 V^3 \) as well, forcing \( g \) to be either \( (0, 0, 0, 12, 13, 23) \) or \( (0, 0, 0, 0, 12, 13) \).

\[\square\]

3. Associated Ricci-flat metrics

In this section we show that the symplectic half-flat manifolds of Theorem 2.4 can be realized as hypersurfaces in Ricci-flat manifolds; in one example, we compute explicitly the metric and its curvature, proving that the holonomy is \( G_2 \).

Recall that a \( G_2 \)-structure on a 7-manifold \( N \) is defined by a three-form \( \varphi \) which at each point \( x \) lies in the \( \text{GL}(T_x N) \) orbit of

\[
e^{147} + e^{257} + e^{367} + e^{123} - e^{156} - e^{426} - e^{453} ,
\]

where \( e^1, \ldots, e^7 \) is any coframe at \( x \). Since \( G_2 \) is contained in \( \text{SO}(7) \), \( \varphi \) determines a metric and an orientation on \( N \).

The \( G_2 \)-structure defined by \( \varphi \) is integrable if and only if \( \varphi \) is closed and co-closed 10. Then the corresponding Riemannian metric has holonomy contained in \( G_2 \).
and is therefore Ricci-flat.

Now let $N$ be a manifold with an integrable $G_2$-structure and let $\iota: M \to N$ be a hypersurface. Then there exists a unique SU(3)-structure $(\omega, \psi^+)$ on $M$ such that

$$\psi^+ = \iota^* \varphi, \quad \omega^2 = 2\iota^* \varphi;$$

from the integrability of the $G_2$-structure it clearly follows that this induced structure is half-flat. Conversely, it is known that if $(\omega(t), \psi^+(t))$ is a one-parameter family of half-flat structures on $M$, for $t$ ranging in $(a,b)$, then

$$(\psi^+ + \iota^* \phi, \omega^2) = -\frac{1}{2} \iota^* \omega^2;$$

defines a $G_2$-structure on $M \times (a,b)$, which is integrable if and only if $(\omega(t), \psi^+(t))$ satisfies the half-flat evolution equations:

$$(d\omega = \frac{\partial}{\partial t} \psi^+, \quad d\psi^+ = -\frac{1}{2} \frac{\partial}{\partial t} \omega^2).$$

In this construction, $(\omega(t), \psi^+(t))$ must satisfy the compatibility conditions (1) for all $t$. However, it turns out that $\psi^+$ is still defined for small deformations of an SU(3)-structure, and if (15) are satisfied, then (1) are preserved in time. Indeed, we have the following [14]:

**Theorem 3.1** (Hitchin). Let $M$ be a compact 6-manifold, and let $(\omega(t), \psi^+(t))$ be a one-parameter family of sections of $\Lambda^2(M) \oplus \Lambda^3(M)$ satisfying the evolution equations (15). If $(\omega(0), \psi^+(0))$ is a half-flat structure, and $\omega(t)^3$ is nowhere zero for $t$ in $(a,b)$, then $(\omega(t), \psi^+(t))$ defines a half-flat structure for all $t \in (a,b)$. In particular, $M \times (a,b)$ has a Riemannian metric with holonomy contained in $G_2$.

We now solve the evolution equations (15) for the nilmanifold with Lie algebra $(0, 0, 0, 12, 13, 23)$. Our solution is different from the one given in [4], since we choose symplectic initial data. Consider the one-parameter family of SU(3)-structures given by

$$(\omega, \psi^+) = \begin{cases} \omega = \frac{1}{u} \eta^{16} - \frac{1}{u} \eta^{25} - \frac{3u^2 - 1}{u} \eta^{34} \\ \psi^+ = \frac{(3u^2 - 1)^2}{4u^6} \eta^{123} - 2 \eta^{154} + 2 \eta^{624} + \eta^{653} \end{cases}$$

Clearly, $(\omega, \psi^+)$ is half-flat for all values of $u$, and symplectic for $u = \pm 1$. Setting

$$t = -12 + \frac{1}{2u^3} - \frac{1}{10u^6},$$

Equations (16) give a solution of (15). An orthonormal basis of 1-forms is given by

$$E^1 = \sqrt{\frac{3u^2 - 1}{2u^4}} \eta^1, \quad E^2 = \sqrt{\frac{3u^2 - 1}{2u^4}} \eta^2, \quad E^3 = \frac{3u^2 - 1}{2u^2} \eta^3, \quad E^4 = \sqrt{\frac{2u^2}{3u^2 - 1}} \eta^6, \quad E^5 = -\sqrt{\frac{2u^2}{3u^2 - 1}} \eta^8, \quad E^6 = -2u \eta^4,$$

where indices and signs have been adjusted for compatibility with (13). Let $N$ be the 7-dimensional Riemannian manifold obtained by the above construction; set $E^7 = dt$, and consider the inclusion $\Lambda^2(N) \subset \text{End}(TN)$, where the two-form $E^7$ is...
identified with the skew-symmetric endomorphism mapping $E^i$ to $E^j$. As a section of $S^2(\Lambda^2(N))$, the curvature is given by

$$-\frac{4u^10}{(3u^2-1)^4} \left( 3 \left( E^{17} + E^{35} \right)^2 + 3 \left( E^{34} - E^{27} \right)^2 + (E^{14} - E^{25})^2 - (E^{12} + E^{45}) \right) +$$

$$-\frac{12u^10(2u^2-1)}{(3u^2-1)^3} \left( (E^{16} + E^{27})^2 + (E^{17} - E^{26})^2 - 2 (E^{12} - E^{67})^2 \right) +$$

$$+ \frac{12u^10(u^2-1)}{(3u^2-1)^4} \left( (E^{24} + E^{37})^2 + (E^{15} - E^{37})^2 \right) +$$

$$- \frac{12u^10(u^2-2)}{(3u^2-1)^4} \left( (E^{13} + E^{57})^2 + (E^{23} - E^{47})^2 \right) +$$

$$- \frac{4u^10}{(3u^2-1)^2} \left( (E^{23} + E^{56})^2 + (E^{13} + E^{46})^2 - (E^{14} - E^{36})^2 \right).$$

This shows that the metric is not reducible and the holonomy equals $G_2$.

Similar computations for a symplectic half-flat structure on the Lie algebra $\mathfrak{g} = (0, 0, 0, 12, 13)$ were carried out in [7]. In this case though, as one can see by computing the curvature, the resulting 7-manifold is the product of a 6-manifold with holonomy SU(3) and a circle. In fact, the nilmanifold is a trivial circle bundle with connection form $\eta = \eta^4$. The symplectic half-flat structure induces an SU(2)-structure $(\alpha, \omega)$ on the five-dimensional base by (2); this structure is hypo in the sense of [7] (see also Section 4), and can therefore be evolved to give a 6-manifold with holonomy SU(3). The reducible 7-manifold is nothing but the product of this 6-manifold with a circle.

**Remark.** Whilst by Hitchin’s theorem the half-flat conditions are automatically preserved in time, the symplectic condition is not, as shown in the above example. This is a general fact: the evolution flow is transverse to the space of symplectic half-flat structures, except where it vanishes (namely, at points defining integrable structures).

4. **SU(3)-structures on circle bundles**

In this section we pursue an idea introduced in Section 2, namely that of reducing a 6-dimensional manifold to a 5-dimensional manifold by means of a quotient, and establishing a relation between the two geometries in terms of $G$-structures. Here we work in a more general context, without requiring invariance under a transitive action; however, for the construction to make sense we still need invariance along one direction, i.e. a Killing field on the 6-manifold. More precisely, we shall establish a one-to-one correspondence between a class of 6-manifolds with an SU(3)-structure and a regular vector field preserving the structure, and a class of 5-manifolds with an SU(2)-structure plus some additional data; since this correspondence only holds “up to isomorphism”, it will be natural to state it in terms of categories. For the moment, we impose no integrability conditions on the SU(3)-structure.

We define a category $\mathcal{K}$ whose objects are 4-tuples $(M, \omega, \psi^+, X)$, where $M$ is a compact 6-dimensional manifold, $(\omega, \psi^+)$ is an SU(3)-structure on $M$, and $X$ is a regular vector field on $M$ which preserves the SU(3)-structure, i.e.

$$\mathcal{L}_X \omega = 0 = \mathcal{L}_X \psi^+.$$
For brevity, we shall often write $M$ for $(M, \omega, \psi^+, X)$; so, when $M$ is referred to as an object of $\mathcal{K}$, it will be understood that $\omega$, $\psi^+$ and $X$ are also fixed on $M$. Sometimes we will need to consider two distinct objects $M$, $\tilde{M}$, and it will be understood that $\tilde{M}$ stands for $(\tilde{M}, \tilde{\omega}, \tilde{\psi}^+, \tilde{X})$.

A morphism $f \in \text{Hom}(M, \tilde{M})$ is a smooth map $f : M \to \tilde{M}$ such that
\[
f^*\tilde{\omega} = \omega, \quad f^*\tilde{\psi}^+ = \psi^+, \quad X \text{ is } f\text{-related to } \tilde{X}.
\]
In particular, morphisms are orientation-preserving isometries, and therefore covering maps.

We are going to relate $\mathcal{K}$ to a category $\mathcal{C}$, whose objects are 6-tuples
\[(N, \alpha, \omega_1, \omega_2, \omega_3, \phi, t),\]
where $N$ is a compact 5-manifold, $(\alpha, \omega_i)$ is an SU(2)-structure on $N$, $t$ is a function on $N$, and $\phi$ is a closed two-form on $N$ such that
\[
\frac{1}{2\pi} \phi \in H^2(N, \mathbb{Z}) .
\]
Again, we shall write $N$ for an object $(N, \alpha, \omega_1, \omega_2, \omega_3, \phi, t)$ of $\mathcal{C}$.

A morphism $f \in \text{Hom}(N, \tilde{N})$ is a smooth map $f : N \to \tilde{N}$ such that
\[
f^*\tilde{\alpha} = \alpha, \quad f^*\tilde{\omega}_i = \omega_i, \quad i = 1, 2, 3, \quad f^*\tilde{\phi} = \phi, \quad \tilde{i} \circ f = t .
\]

We shall construct a functor $F : \mathcal{K} \to \mathcal{C}$ which realizes each object $M$ of $\mathcal{K}$ as a circle bundle over $F(M)$; the 2-form $\phi$ represents the curvature, and the function $t$ the length of the fibres.

Let us first recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is faithful (resp. full) if
\[
\text{Hom}(A, B) \to \text{Hom}(F(A), F(B))
\]
is one-to-one (resp. onto) for all objects $A, B$ of $\mathcal{C}$; it is representative if every object in $\mathcal{D}$ is isomorphic to $F(A)$ for some object $A$ of $\mathcal{C}$. A full, representative, faithful functor is called an equivalence.

The functor we consider does not quite establish an equivalence, but we shall show that it induces an equivalence between categories derived from $\mathcal{K}$ and $\mathcal{C}$. The starting point is the following observation: if $M$ is an object of $\mathcal{K}$, the maximal integral curves of $X$ are closed subsets of the compact manifold $M$, and therefore diffeomorphic to circles. More precisely, we have the following [3, p. 27]:

**Lemma 4.1.** For every object $M$ of $\mathcal{K}$, the maximal integral curves of $X$ viewed as maps $\phi_x : \mathbb{R} \to M$ are periodic, with period not depending on the point $x$. In particular $M$ is the total space of a circle bundle, and $X$ is a fundamental vector field.

We can now prove the following:

**Proposition 4.2.** There is a representative functor $F : \mathcal{K} \to \mathcal{C}$.

**Proof.** We define the covariant functor $F : \mathcal{K} \to \mathcal{C}$ as follows: let $M$ be an object of $\mathcal{K}$. The space of integral lines of $X$ is a compact 5-manifold $N$, and by Lemma 4.1 $M$ is the total space of a circle bundle over $N$. Since $X$ is a Killing vector field, the norm of $X$ is constant on the fibres, and so defines a smooth function $t$ on $N$. By Lemma 4.1, the maximal integral curves of $X$ have constant period; we can rescale $X$ so that this period is $2\pi$. Let $\eta$ be the connection 1-form determined
by \( X \), i.e. the dual form to \( X \) rescaled so that \( \eta(X) = 1 \); set \( \phi = d\eta \). Then the cohomology class

\[
c_1 = \left[ \frac{1}{2\pi} \phi \right]
\]

is the Chern class of the \( U(1) \)-bundle \( M \to N \); as such, it is integral. Define forms \((\alpha, \omega_i)\) on \( M \) by

\[
\begin{align*}
\alpha &= X \cdot \omega, \\
\omega_3 &= t X \cdot (\omega \wedge \eta), \\
\omega_1 &= X \cdot \psi^+, \\
\omega_2 &= X \cdot \psi^-.
\end{align*}
\]

By construction, all the objects appearing on the right-hand side of (17) are invariant under the action of \( U(1) \); therefore, each form \( \alpha, \omega_1, \omega_2, \omega_3 \) is the pullback of a form on \( N \), which we denote by the same symbol.

Now choose a local orthonormal basis of 1-forms

\[
\frac{1}{\sqrt{t}} e_1, \ldots, \frac{1}{\sqrt{t}} e_4, e_5, e_6
\]

on \( M \), with \( \eta = t^{-1} e_6 \), such that

\[
\omega = \frac{1}{t}(e^{14} + e^{23} + e^{65}), \quad \Psi = \frac{1}{t}(e^{1} + i e^{4}) \wedge (e^{2} + i e^{3}) \wedge (e^{6} + \frac{1}{t} i e^{5}).
\]

Then (18) is satisfied, so \((\alpha, \omega_i)\) defines an \( SU(2) \)-structure on \( N \). In particular, \( e_1, \ldots, e_5 \) is an orthonormal basis of 1-forms on \( N \); so, we are not using the quotient metric on the 5-manifold, but a deformation of it.

If \( M \) and \( \tilde{M} \) are objects of \( K \) and \( f \in \text{Hom}(M, \tilde{M}) \), then \( f \) maps integral curves of \( X \) to integral curves of \( \tilde{X} \). Therefore, \( f \) induces a smooth map \( F(f) : N \to \tilde{N} \), where \( N = F(M), \tilde{N} = F(\tilde{M}) \); we must show that \( F(f) \) is a morphism. From the fact that \( X \) is \( f \)-related to \( \tilde{X} \) and \( f \) is a local isometry, it follows that \( \tilde{t} \circ F(f) = t \).

Now consider the diagram of maps

\[
\begin{array}{ccc}
M & \xrightarrow{f} & \tilde{M} \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
N & \xrightarrow{F(f)} & \tilde{N}
\end{array}
\]

By the commutativity of the diagram and (17),

\[
\pi^* (F(f)^* \tilde{\alpha}) = f^* (\tilde{\pi}^* \tilde{\alpha}) = \alpha;
\]

therefore, \( F(f)^* \tilde{\alpha} = \alpha \) on \( N \). By the same argument \( F(f)^* \tilde{\omega}_i = \omega_i \) and \( F(f)^* \tilde{\phi} = \phi \).

Next we show that \( F \) is representative; more precisely, that every object \( N \) of \( C \) can be written as \( F(M) \) for some object \( M \) of \( K \). Indeed, let \( M \) be a circle bundle over \( N \) with Chern class \( [\phi/(2\pi)] \). Let \( A \) be the standard generator of the Lie algebra \( \mathfrak{u}(1) \), and let \( X = A^* \) be the associated fundamental vector field. Choose a connection form \( \eta \) such that \( d\eta = \phi \) and define

\[
\begin{align*}
\omega &= t^{-1} \omega_3 + \eta \wedge \alpha \\
\Psi &= (\omega_1 + i \omega_2) \wedge (\eta + it^{-2} \alpha)
\end{align*}
\]

It is clear that this defines an \( SU(3) \)-structure preserved by \( X \), which has norm \( t\eta(X) = t \), and Equations (17) give back the original \( SU(2) \)-structure on \( N \). □
Remark. One might wonder at the advantage of referring explicitly to categories, rather than just defining $F$ and studying its properties. With the latter approach, a problem arises when one tries to construct an inverse to $F$: indeed, “the” circle bundle with a given Chern class is not a well-defined manifold, but something only defined up to isomorphism.

Remark. The reduction we have chosen behaves well with respect to evolution theory. Indeed, let $N$ be a 5-manifold with an SU(2)-structure; recall that $N$ is called hypo if the forms $\omega_1, \omega_2 \wedge \alpha,$ and $\omega_3 \wedge \alpha$ are closed. It is easy to verify that the SU(3)-structure induced on $N \times S^1$ by the above construction (corresponding to taking $\phi = 0$ and $t = 1$) is half-flat if and only if the structure on $N$ is hypo. Moreover, hypo geometry also has evolution equations similar to (15), and it turns out that a one-parameter solution of the hypo evolution equations lifts to a solution of the half-flat evolution equations. An example of this situation is the reducible half-flat nilmanifold mentioned in Section 3.

The functor $F$ fails to be an equivalence in two respects: it is not full, and it is not faithful. We start by addressing the first issue. Let $C'$ be the subcategory of $C$ consisting of objects $N$ such that $N$ is simply-connected as a manifold; let $K'$ be the subcategory of $K$ of objects $M$ such that $F(M)$ is simply-connected.

Lemma 4.3. The functor $F: K' \to C'$ is full.

Proof. Consider two objects $M, \tilde{M}$ in $K'$; let $F(M) = N, F(\tilde{M}) = \tilde{N},$ and let $h \in \text{Hom}(N, \tilde{N})$. Fix a point $u$ in $M$; every path $\sigma$ in $N$ based at $\pi(u)$ has a horizontal lift $\gamma$ with $\gamma(0) = u$. Fix also a point $\tilde{u}$ in $\tilde{M}$, lying over $\pi(u)$; then $\sigma$ lifts to $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{u}$. Now define $f(\gamma(t)) = \tilde{\gamma}(t)$; we have to show that the definition does not depend on the path $\sigma$.

Indeed, let $I = [0, 1]$, and let $\Sigma: I \times I \to N$ be a smooth homotopy with fixed endpoints between two paths $\Sigma_0$ and $\Sigma_1$ starting at $\pi(u)$, where $\Sigma_t = \Sigma(t, \cdot)$. The pullback bundle $\Sigma^*M$ is trivial; we can therefore choose a section

$$\Gamma: I \times I \to M,$$

where $\Gamma_0$ and $\Gamma_1$ are horizontal lifts of $\Sigma_0, \Sigma_1$ respectively, and $\Gamma(\cdot, 0) = u$.

The “vertical distance” from $\Gamma_0(1)$ to $\Gamma_1(1)$ is measured by

$$\int_{\Gamma(\cdot, 1)} \eta = \int_{I \times I} \Gamma^*d\eta = \int_{I \times I} \Sigma^*\phi,$$

where we have used Stokes’ theorem. Quite similarly if $\tilde{\Sigma} = h \circ \Sigma$ and $\tilde{\Gamma}$ is constructed as above, imposing this time $\tilde{\Gamma}(\cdot, 0) = \tilde{u}$, we obtain

$$\int_{\tilde{\Gamma}(\cdot, 1)} \tilde{\eta} = \int_{I \times I} \tilde{\Gamma}^*d\tilde{\eta} = \int_{I \times I} \tilde{\Sigma}^*\tilde{\phi} = \int_{I \times I} \Sigma^*\phi,$$

because $h^*\tilde{\phi} = \phi$.

Hence, we can lift $h$ to an equivariant map $f: M \to \tilde{M}$ satisfying $f^*\tilde{\eta} = \eta$. Thus,

$$F: \text{Hom}(M, \tilde{M}) \to \text{Hom}(N, \tilde{N})$$

is onto as required. □

Remark. The definition of the lift $f$ in the proof of Lemma 4.3 is also a characterization, because morphisms preserve the connection form, and therefore map horizontal paths to horizontal paths. So, in the non-simply-connected case one
cannot expect to be able to produce a lift, i.e. $F$ is not full as a functor from $\mathcal{K}$ to $\mathcal{C}$.

We now come to faithfulness. Fix an object $M$ in $\mathcal{K}'$ and consider the periodic integral lines $\phi_x$; the map

$M \ni x \xrightarrow{f_{\theta}} \phi_x(\theta) \in M$

is an isomorphism for all constants $\theta$. Clearly, $F(f_{\theta})$ is the identity; so,

$F : \text{Hom}(M, M) \to \text{Hom}(F(M), F(M))$

is not one-to-one and $F$ is not faithful.

However, this is the only amount to which $F$ fails to be an equivalence. Indeed, for each $M, \tilde{M}$ in $\mathcal{K}'$, define an equivalence relation in $\text{Hom}(M, \tilde{M})$ by

$f_1 \sim f_2 \iff f_1 = f_2 \circ f_{\theta}, \theta \in \mathbb{R}$.

We can then consider the quotient category $\mathcal{K}''$, whose objects are the objects of $\mathcal{K}'$, and whose morphisms are defined by

$\text{Hom}_{\mathcal{K}''}(M, \tilde{M}) = \text{Hom}_{\mathcal{K}'}(M, \tilde{M}) / \sim$.

**Proposition 4.4.** $\mathcal{K}''$ is a category, and the induced functor $F'' : \mathcal{K}'' \to \mathcal{C}'$ is an equivalence.

**Proof.** By construction, elements of $\text{Hom}(M, \tilde{M})$ are $U(1)$-equivariant. Therefore,

$$(f_1 \circ f_{\theta_1}) \circ (f_2 \circ f_{\theta_2}) = f_1 \circ f_2 \circ f_{\theta_1 + \theta_2}.$$ 

It follows that if $g_1 \sim f_1$ and $g_2 \sim f_2$ then $g_1 \circ g_2 \sim f_1 \circ f_2$. This is sufficient to conclude that $\mathcal{K}''$ is a category.

Observe that the induced functor $F''$ is well defined, because clearly $f_1 \sim f_2$ implies $F''(f_1) = F''(f_2)$.

Now, recall from the proof of Proposition 4.2 that the connection form is defined only by the metric and the Killing field; since a morphism $f \in \text{Hom}(M, \tilde{M})$ is an isometry and $X$ is $f$-related to $\tilde{X}$, we have $f^*\tilde{\eta} = \eta$. Therefore, $f$ is uniquely determined by its value at a point, or in other words,

$F'' : \text{Hom}_{\mathcal{K}''}(M, M) \to \text{Hom}(F''(M), F''(M))$

is one-to-one, as required. \qed

5. **Intrinsic torsion of the quotient structure**

In this section we drop the assumptions of compactness and global regularity, and study the local behaviour of the construction of Section 4 in terms of intrinsic torsion. In particular, we characterize the intrinsic torsion of the SU(2)-structures obtained by taking the quotient of a symplectic half-flat structure, generalizing Lemma 2.2. Then, in the assumption that the starting SU(3)-structure is integrable, we write down a differential equation that the function $t$ must satisfy, and prove that the intrinsic torsion of the quotient SU(2)-structure depends only on $t$. More precisely, we give necessary and sufficient conditions on $(N, \alpha, \omega_i, t)$ for it to arise, locally, as the quotient of a 6-manifold with an integrable SU(3)-structure. Observe that in the integrable case the 6-manifold cannot be compact, unless it is reducible.

We shall work in a neighbourhood of a point where the Killing field is non-zero; thus, we assume that $M$ is a 6-manifold with an SU(3)-structure preserved by some regular Killing field $X$. Recall from the proof of Proposition 4.2 that the quotient
is a 5-manifold on which an SU(2)-structure \((\alpha, \omega)\) is induced (see (17)), as well as a function \(t\), the norm of \(X\), and a two-form \(\phi\), which in the case of a circle bundle is the curvature form.

Recall from [5] that the intrinsic torsion of an SU(3)-structure takes values in a 42-dimensional space, and its components can be represented as follows:

\[
\begin{array}{c|c|c}
W_1^+ & W_1^- & \\
W_2^+ & W_2^- & \\
W_3 & \quad & \\
W_4 & \quad & \\
W_5 & \quad & \\
\end{array}
\in \begin{array}{c|c|c}
\mathbb{R} & \mathbb{R} & \\
[\Lambda_{0,1}^1] & [\Lambda_{0,1}^1] & \\
[\Lambda_{0,0}^2] & [\Lambda_{0,0}^2] & \\
[\Lambda_{1,0}^1] & [\Lambda_{1,0}^1] & \\
[\Lambda_{1,0}^2] & [\Lambda_{1,0}^2] & \\
\end{array}
\]

meaning that the component \(W_1^+\) takes values in \(\mathbb{R}\), and so on. Explicitly, we can write

\[
\begin{aligned}
    d\psi^+ &= \psi^+ \wedge W_5^+ + W_2^+ \wedge \omega + W_1^+ \omega^2 \\
    d\psi^- &= \psi^- \wedge W_5^- + W_2^- \wedge \omega + W_1^- \omega^2 \\
    d\omega &= -\frac{3}{2}W_1^- \psi^+ + \frac{3}{2}W_1^+ \psi^- + W_3 + W_4 \wedge \omega
\end{aligned}
\]

We can do the same for SU(2)-structures on 5-manifolds [7]; the intrinsic torsion now takes values in a 35-dimensional space, and we can arrange its components in the following table:

\[
\begin{array}{c|c|c|c}
\lambda & f_1 & f_2 & f_3 \\
g_1^1 & g_1^2 & g_1^3 \\
\beta & \quad & \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\omega^- & \quad & \\
\sigma_1^- & \sigma_2^- & \sigma_3^- \\
\end{array}
\in \begin{array}{c|c|c|c}
\mathbb{R} & \mathbb{R} & \mathbb{R} & \\
\mathbb{R} & \mathbb{R} & \mathbb{R} & \\
\Lambda^1 & \Lambda^1 & \Lambda^1 & \\
\Lambda^2 & \Lambda^2 & \Lambda^2 & \\
\end{array}
\]

In the above table, \(\Lambda^1\) is the 4-dimensional representation of SU(2) such that the tangent space at a point is

\[
T = \Lambda^1 \oplus \mathbb{R} ,
\]

whereas \(\Lambda^2\) is the 3-dimensional representation of SU(2) consisting of anti-self-dual two-forms on \(\Lambda^1\). We shall write, say, \((\omega)_{\Lambda^2}\) for the \(\Lambda^2\) component of a two-form \(\omega\).

Setting \(g^i_j = -g^j_i\), the components of the intrinsic torsion are given by

\[
\begin{aligned}
    d\alpha &= \alpha \wedge \beta + \frac{3}{2} \sum_{j=1}^3 f^j \omega_j + \omega^- \\
    d\omega_i &= \gamma_i \wedge \omega_i + \lambda \alpha \wedge \omega_i + \sum_{j \neq i} g^j_i \alpha \wedge \omega_j + \alpha \wedge \sigma_i^-
\end{aligned}
\]

We can now prove the following:

**Proposition 5.1.** Define the intrinsic torsion of \(M\) as above, and write

\[
W_i = \eta \wedge \Xi_i + \Delta_i, \quad \text{where} \quad \Xi_i = X \lrcorner W_i ;
\]
then the intrinsic torsion of the quotient is given by

\[
\begin{array}{|c|c|c|}
\hline
\frac{3}{2}W_- & -\langle \Delta_5, \alpha \rangle & -t^{-1}\Xi_4 - \frac{1}{2}\langle \Xi_3, \omega_3 \rangle \\
-t^{-2}\Xi_5 & -2t^{-1}W_1^+ - t^{-1}\langle \Xi^+_3, \alpha \rangle & -2t^{-1}W_1^- - t^{-1}\langle \Xi^-_3, \alpha \rangle \\
-(\Delta_4)_{A^1} & -t^{-1}\Xi^+_3 \omega_2 & -t^{-1}\Xi^-_3 \omega_1 \\
\hline
-\Delta_5^1 & -\Delta_5^2 & t(\alpha \Delta_3 - \phi)_{\Lambda^2} \\
\hline
\end{array}
\]

**Proof.** Taking the interior product of (20) with X, then substituting (17) in the right-hand side, one easily computes:

\[
d\alpha = \frac{3}{2}W_1^- \omega_1 - \frac{3}{2}W_1^+ \omega_2 - \Xi_3 - t^{-1}\Xi_4 \omega_3 + \Delta_4 \wedge \alpha \\
d\omega_1 = -\omega_1 \wedge \Delta_5 - t^{-2}\Xi_5 \omega_2 \wedge \alpha - t^{-1}\Xi^+_3 \wedge \omega_3 - \Delta^+_3 \wedge \alpha - 2t^{-1}W_1^+ \omega_3 \wedge \alpha \\
d\omega_2 = -\omega_2 \wedge \Delta_5 + t^{-2}\Xi_5 \omega_1 \wedge \alpha - t^{-1}\Xi^-_3 \wedge \omega_3 - \Delta^-_3 \wedge \alpha - 2t^{-1}W_1^- \omega_1 \wedge \alpha
\]

On the other hand, \(d\omega_3 = d \log t \wedge \omega_3 - tX_3 d(\omega \wedge \eta)\) by (17). Using (20) and then (15), we obtain

\[
d(\omega \wedge \eta) = \frac{3}{2}t^{-2}W_1^- \omega_2 \wedge \alpha \wedge \eta + \frac{3}{2}t^{-2}W_1^+ \omega_1 \wedge \alpha \wedge \eta + \Delta_3 \wedge \eta + \\
\quad + t^{-1}\Delta_4 \wedge \omega_3 \wedge \eta + t^{-1}\omega_3 \wedge \phi + \eta \wedge \alpha \wedge \phi;
\]

therefore

\[
d\omega_3 = d \log t \wedge \omega_3 + \frac{3}{2}t^{-1}W_1^- \omega_2 \wedge \alpha + \frac{3}{2}t^{-1}W_1^+ \omega_1 \wedge \alpha + t\Delta_3 + \Delta_4 \wedge \omega_3 - t\alpha \wedge \phi.
\]

The decompositions (22) of the components \(W_3\) and \(W_2^\pm\) correspond to projections

\[
[\Lambda_0^{2,1}] \to \omega^\perp \oplus (\alpha \wedge \omega_1)^\perp, \quad [\Lambda_0^{1,1}] \to T \oplus \Lambda^2, 
\]

respectively; the statement is now a straightforward consequence of (21). \(\square\)

**Remark.** Of the decompositions (23), the first is not surjective; in other words, the components \(\Xi_3\) and \(\Delta_3\) are not independent.

It is clear that one can write down a converse to Proposition 5.1 because the quotient determines the intrinsic torsion of \(M\); one can then characterize the \(M\) with special intrinsic torsion in terms of the quotient. For example, in the symplectic half-flat case one obtains this generalization of Lemma 2.2.

**Proposition 5.2.** \(M\) is symplectic half-flat if and only if the quotient satisfies

\[
\begin{align*}
d\alpha &= 0, \\
d\omega_1 &= 0, \\
d\omega_2 \wedge \alpha &= t^2\omega_1 \wedge \phi + 2d \log t \wedge \omega_2 \wedge \alpha, \\
d\omega_3 &= d \log t \wedge \omega_3 - t\alpha \wedge \phi. 
\end{align*}
\]

**Proof.** Follows immediately from (15). \(\square\)

We now consider the case where \(M\) is integrable. In order to state our theorem, we need to introduce two differential operators on 5-manifolds with an
SU(2)-structure. The first one is $\partial$, which maps a function $f$ to $\langle \alpha, df \rangle$. Secondly, consider the endomorphism $J_3$ of $T^*N$ characterized by

$$J_3\alpha = 0, \quad \omega_1 \wedge \beta = \omega_2 \wedge J_3\beta \text{ for } \beta \in \alpha \perp ;$$

we can then define an operator $d^c$ which maps a function $f$ to $d^c f = J_3 df$.

**Theorem 5.3.** If the SU(3)-structure on $M$ is integrable, the function $t$ is a solution of

$$\partial^2_\alpha \log t - (\partial_\alpha \log t)^2 - 2t^{-1} \| (d \log t) \|_\Lambda^1 = 0 .$$

The intrinsic torsion is determined by $t$ as follows: $\alpha$, $\omega_1$ and $\omega_2$ are closed, and $\omega_3$ satisfies

$$d\omega_3 = (d \log t) \wedge \omega_3 + \frac{1}{\partial_\alpha t} \alpha \wedge (2d \log t \wedge d^c \log t - dd^c \log t) \Lambda_2 ;$$

moreover the “curvature form” is

$$\phi = t^{-1} \partial_\alpha t \log t - \frac{1}{t^2 \partial_\alpha \log t} (2d \log t \wedge d^c \log t - dd^c \log t) \Lambda_2 - 2t^{-2} \alpha \wedge d^c \log t .$$

Conversely, let $N$ be a 5-manifold with an SU(2)-structure $(\alpha, \omega_i)$ and a function $t$, where $\alpha$, $\omega_1$ and $\omega_2$ are closed, and $(25)$, $(26)$ are satisfied. Then the two-form $\phi$ defined by $(27)$ is closed; if the cohomology class $[\phi / 2\pi]$ is an element of $H^2(N, \mathbb{Z})$, there is a circle bundle over $N$ on which an integrable SU(3)-structure is defined by $(18)$, where $\eta$ is a connection form such that $d\eta = \phi$.

Locally, Theorem 5.3 is a characterization. Indeed, in the second part one can restrict $N$ to a contractible open subset $N'$, so that

$$0 = [\phi / 2\pi] \in H^2(N', \mathbb{Z}) .$$

**Proof.** It is clear from $(18)$ that $\alpha$, $\omega_1$ and $\omega_2$ are closed. Hence, the only non-vanishing components of the intrinsic torsion are $\gamma_3$ and $\sigma_3^-$, determined by

$$d\omega_3 = \gamma_3 \wedge \omega_3 + \alpha \wedge \sigma_3^- .$$

Using $(18)$, we find

$$t^{-1} \omega_3 \wedge (\gamma_3 - d \log t) + \alpha \wedge (t^{-1} \sigma_3^- + \phi) = 0 .$$

Hence $\gamma_3 = (d \log t) \Lambda_1$, and the component of $\phi$ in $\Lambda^2(\alpha \perp)$ is determined by

$$\langle \phi, \omega_1 \rangle = 0 = \langle \phi, \omega_2 \rangle , \quad \langle \phi, \omega_3 \rangle = 2t^{-1} \partial_\alpha t \log t , \quad \langle \phi \rangle \Lambda_2 = -t^{-1} \sigma_3^- .$$

It also follows from $(18)$ that

$$\omega_1 \wedge \phi + 2t^{-3} dt \wedge \omega_2 \wedge \alpha = 0$$

$$\omega_2 \wedge \phi - 2t^{-3} dt \wedge \omega_1 \wedge \alpha = 0$$

Therefore

$$\alpha \wedge \phi = -2t^{-2} d^c \log t .$$

For brevity, we set $s = \partial_\alpha t$. By construction $\phi$ is closed, so

$$0 = t^{-1} (-s d \log t \wedge \omega_3 + ds \wedge \omega_3 + s (d \log t - s \alpha) \wedge \omega_3 + \alpha \wedge \sigma_3^- +$$

$$+ dd^c \log t \wedge \sigma_3^- - d\sigma_3^-) + 2t^{-2} \alpha \wedge (-2d \log t \wedge d^c \log t + dd^c \log t) .$$
We can split (28) into two equations by taking the wedge and the interior product with $\alpha$. One of these is satisfied automatically: indeed, taking $d$ of $d\omega_3$ we find

$$0 = \alpha \wedge (d \log t \wedge \sigma_3^- - d\sigma_3^- + ds \wedge \omega_3),$$

so the right-hand side of (28) vanishes on wedging with $\alpha$. Taking the interior product gives

$$(\partial_\alpha s - s^2)\omega_3 + 2s\sigma_3^- + 2t^{-1}(-2d \log t \wedge d^c \log t + dd^c \log t)_{\Lambda^2(\alpha^+)}.$$ 

It is now clear that $\sigma_3^-$ can be expressed in terms of $t$, giving (26). Using the general formula

$$\langle \beta \wedge J_3 \beta, \omega_3 \rangle = \parallel \beta \parallel^2 - \langle \beta, \alpha \rangle^2,$$

we also deduce that $t$ satisfies (25).

Conversely, suppose (26) and (25) are satisfied, and define $\phi$ by (27). The above calculations show that $\phi$ is closed and the construction of Proposition 4.2 defines an integrable SU(3)-structure. □

Remark. The condition of Theorem 5.3 implies in particular that 28 out of the 35 components of the intrinsic torsion of $(N,\alpha,\omega_i)$ vanish. A similar construction was described in [1], starting with a 7-manifold with holonomy $G_2$, and defining an SU(3)-structure on the quotient. In that case, the vanishing components of the intrinsic torsion of the quotient are also 28, though out of 42.

In general (26) and (25) are not independent, because the norm on one-forms depends on $\omega_3$. Motivated by this observation, we consider the special case

$$d \log t_{\Lambda^1} = 0;$$

in order to apply Theorem 5.3 we have to assume that the SU(2)-structure is integrable. Let $x$ be a coordinate in the direction of $\alpha$, so that $\alpha = dx$. Set $t = (1 - x)^{-1}$; then (26) is satisfied. Suppose that one has a circle bundle over $N$ with Chern class $[\frac{1}{2}\pi \omega_3]$; then the hypotheses of Theorem 5.3 hold. Define a connection form $\eta$ such that $d\eta = \omega_3$; then

$$\omega = (1 - x)\omega_3 + \eta \wedge \alpha, \quad \Psi = (\omega_1 + i\omega_2) \wedge (\eta + i(1 - x)^2\alpha),$$

defines an integrable SU(3)-structure. One can actually prove that if the original 5-manifold has holonomy SU(2), then the Calabi-Yau 6-manifold has holonomy SU(3).

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