TANGENT BUNDLES WITH SASAKI METRIC AND ALMOST
HYPERCOMPLEX PSEUDO-HERMITIAN STRUCTURE

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Dedicated to the 60th anniversary of Prof. Kouei SEKIGAWA

ABSTRACT. The tangent bundle as a 4n-manifold is equipped with an almost hypercomplex pseudo-Hermitian structure and it is characterized with respect to the relevant classifications. A number of 8-dimensional examples of the considered type of manifold are received from the known explicit examples in that manner.

INTRODUCTION

The geometry of the almost hypercomplex manifolds with Hermitian metric is known (e. g. [1]). A parallel direction including indefinite metrics is the developing of the geometry of the almost hypercomplex manifolds with pseudo-Hermitian metric structure. It has a natural origination from the geometry of the n-dimensional quaternionic Euclidean space.

The beginning was put by our joint works with K. Gribachev and S. Dimiev in [4] and [5]. More precisely we have combined the ordinary Hermitian metrics with the so-called by us skew-Hermitian metrics with respect to the almost complex structures of a hypercomplex structure.

First, we have ascertained four types of bilinear forms compatible with a hypercomplex structure: three metrics and one Kähler form which are skew-Hermitian or Hermitian with respect to the different almost complex structures, i. e. we have constructed a pseudo-Hermitian structure.

Second, we have developed the notion of an almost hypercomplex pseudo-Hermitian manifold.

Third, we have considered mainly the (integrable) hypercomplex pseudo-Hermitian manifolds and its special subclass the class of pseudo-hyper-Kähler manifolds.

After that in [8] we have continued the investigations as we have constructed and characterized to a certain extent several 4-dimensional examples of the considered manifolds.

The aim of the present work is to generate the considered structure on a tangent bundle, to characterize it in terms of the corresponding manifolds and to construct 8-dimensional examples relevant to the known 4-dimensional examples.

In [1] we recall the notions of the almost complex manifolds with Hermitian metric or skew-Hermitian metrics and the almost hypercomplex manifolds with pseudo-Hermitian metric structure.

In [2] we remind the definitions and properties of the prolongations of vector fields to tangent bundles known as their vertical and horizontal lifts.

The main part of this paper is given in [3] where we consider an almost skew-Hermitian manifold as a base manifold and we generate its tangent bundle with a metric, which is a prolongation of the base metric, known as the Sasaki metric. In that way we get the tangent bundle with almost hypercomplex pseudo-Hermitian structure. We characterize differential-geometrically this manifold having in mind also the relevant classifications of the received manifolds.

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The results of [3] and the known examples from [4] and [5] are used in the last [4] to construct corresponding 8-dimensional examples in the discussed manner.

1. Some Kinds of Differentiable Manifolds with Almost Complex Structures

1.1. Almost Complex Manifolds with Hermitian Metric or Skew-Hermitian Metric. The notion of the almost complex manifold \((M^{2n}, J)\) is well-known. There exists a possibility it to be equipped with two different kinds of metrics. When \(J\) acts as an isometry on each tangent space then the manifold is an almost Hermitian manifold. But, in the case when \(J\) acts as an anti-isometry on each tangent space, the notion of the so-called almost skew-Hermitian manifold is available. Let us consider more precisely the last one.

Every \(n\)-dimensional complex Riemannian manifold induces a real \(2n\)-dimensional manifold \((M^{2n}, J, g, \tilde{g})\) with a complex structure \(J\), a metric \(g\) and an associated metric \(\tilde{g} = g(\cdot, J\cdot)\). Both metrics are indefinite of signature \((n, n)\). This manifold is called an almost skew-Hermitian manifold. Such metric is known also as a B-metric or a Norden metric because it is introduced by A. P. Norden in [9]. An almost skew-Hermitian manifold is a skew-Kähler manifold if \(J\) is parallel with respect to the Levi-Civita connection \(\nabla\) of the metric \(g\). The class of these manifolds is contained in every other class of the almost skew-Hermitian manifolds. A classification with respect to \(\nabla J\) consisting of three basic classes is given in [2]. The class \(W_1\) is the main class, i.e. the class where \(\nabla J\) has an explicit expression in terms of the metric tensors and the Lie form.

1.2. Almost Hypercomplex Manifolds with Pseudo-Hermitian Structure.

Let us recall the notion of the almost hypercomplex structure \(H\) on the manifold \(M^{4n}\). It is the triple \(H = (J_\alpha) (\alpha = 1, 2, 3)\) of anticommuting almost complex structures satisfying the property \(J_3 = J_1 \circ J_2\) ([1], [13]).

The introduction and the beginning of the study of the pseudo-Hermitian structures on an almost hypercomplex manifold is given in [3]. A pseudo-Riemannian metric \(g\) of signature \((2n, 2n)\) on \((M^{4n}, H)\) is introduced as follows

\[
g(\cdot, \cdot) = g(J_1\cdot, J_1\cdot) = -g(J_2\cdot, J_2\cdot) = -g(J_3\cdot, J_3\cdot).
\]

We called such metric a pseudo-Hermitian metric. It generates a Kähler 2-form \(\Phi\) and two pseudo-Hermitian metrics \(g_2\) and \(g_3\) by the following way

\[
\Phi := g(J_1\cdot, \cdot), \quad g_2 := g(J_2\cdot, \cdot), \quad g_3 := g(J_3\cdot, \cdot).
\]

Let us note that \(g\) (\(g_2, g_3\), respectively) has a Hermitian compatibility with respect to \(J_1\) (\(J_3, J_2\), respectively) and a skew-Hermitian compatibility with respect to \(J_2\) and \(J_3\) (\(J_1\) and \(J_2\), \(J_1\) and \(J_3\), respectively).

On the other hand, a quaternionic inner product \(<\cdot, \cdot>\) in \(\mathbb{H}\) generates in a natural way the bilinear forms \(g, \Phi, g_2\) and \(g_3\) by the following decomposition: \(<\cdot, \cdot> = -g + i\Phi + jg_2 + kg_3\).

We called the structure \((H, G) := (J_1, J_2, J_3; g, \Phi, g_2, g_3)\) a hypercomplex pseudo-Hermitian structure on \(M^{4n}\) or shortly a \((H, G)\)-structure on \(M^{4n}\). We called the manifold \((M, H, G)\) an almost hypercomplex pseudo-Hermitian manifold or shortly an almost \((H, G)\)-manifold.

It is well known, that the almost hypercomplex structure \(H = (J_\alpha)\) is a hypercomplex structure if the Nijenhuis tensors \(N_\alpha(\cdot, \cdot) = [\cdot, \cdot] + J_\alpha [\cdot, J_\alpha\cdot] + J_\alpha [J_\alpha\cdot, \cdot] - [J_\alpha\cdot, J_\alpha\cdot]\) vanish for each \(\alpha = 1, 2, 3\). Moreover, one \(H\) is hypercomplex iff two of \(N_\alpha\) vanish.

We introduced structural (0, 3)-tensors of the almost \((H, G)\)-manifold by

\[
F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = \langle \nabla_x g_\alpha \rangle (y, z) \quad (\alpha = 1, 2, 3),
\]

where \(\nabla\) is the Levi-Civita connection generated by \(g\) and \(x, y, z \in T_p M\) at any \(p \in M\).

There are valid relations between \(F_\alpha\)’s, e.g. \(F_1(\cdot, \cdot, \cdot) = F_2(\cdot, J_3\cdot, \cdot) + F_3(\cdot, \cdot, J_2\cdot)\).
Since $g$ is a Hermitian metric with respect to $J_1$, according to Gray-Hervella $[3]$ the basic subclass $\mathcal{W}_4$ of the Hermitian manifolds is determined by

$$F_1(x, y, z) = \frac{1}{2(2n-1)} [g(x, y)\theta_1(z) - g(x, z)\theta_1(y) - g(x, J_1 y)\theta_1(J_1 z) + g(x, J_1 z)\theta_1(J_1 y)],$$

where $\theta_1(\cdot) = g^{ij} F_1(e_i, e_j, \cdot) = \delta \Phi(\cdot)$ is the Lie form for any basis $\{e_i\}_{i=1}^{4n}$, and $\delta$ – the coderivative.

On other side, the metric $g$ is a skew-Hermitian one with respect to $J_2$ and $J_3$. According to the classification of the almost complex manifolds with skew-Hermitian metric (Norden metric or B-metric) given in $[2]$ the basic classes are defined as follows:

$$\mathcal{W}_1 : F_\alpha(x, y, z) = \frac{1}{4n} [g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y) + g(x, J_\alpha y)\theta_\alpha(J_\alpha z) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y)],$$

$$\mathcal{W}_2 : \sigma_{x,y,z} F_\alpha(x, y, J_\alpha z) = 0, \quad \mathcal{W}_3 : \sigma_{x,y,z} F_\alpha(x, y, z) = 0,$$

where $\theta_\alpha(\cdot) = g^{ij} F_\alpha(e_i, e_j, \cdot), \alpha = 2, 3$, are the corresponding Lie forms for an arbitrary basis $\{e_i\}_{i=1}^{4n}$.

Here, we denote the main subclasses of the respective complex manifolds by $\mathcal{W}(J_\alpha)$, where $\mathcal{W}(J_1) := \mathcal{W}_4(J_1)$ in $[3]$, and $\mathcal{W}(J_\alpha) := \mathcal{W}_1(J_\alpha)$ for $\alpha = 2, 3$ in $[2]$.

We obtained a sufficient condition an almost $(H, G)$-manifold to be an integrable one:

**Theorem 1.1** ($[4]$). Let $(M, H, G)$ belongs to $\mathcal{W}(J_\alpha) \cap \mathcal{W}(J_\beta)$. Then $(M, H, G)$ is of class $\mathcal{W}(J_\gamma)$ for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$.

We say that a pseudo-Hermitian manifold is a pseudo-hyper-Kähler manifold and denote $(M, H, G) \in \mathcal{K}$, if $F_\alpha = 0$ for every $\alpha = 1, 2, 3$, i.e., the manifold is of Kählerian type with respect to each $J_\alpha$.

**Theorem 1.2** ($[4]$). If $(M, H, G) \in \mathcal{K}(J_\alpha) \cap \mathcal{W}(J_\beta)$ ($\alpha \neq \beta \in \{1, 2, 3\}$) then $(M, H, G) \in \mathcal{K}$.

We gave a geometric characteristic of the pseudo-hyper-Kähler manifolds according to the curvature tensor $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ induced by the Levi-Civita connection.

**Theorem 1.3** ($[5]$). Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold with signature $(2n, 2n)$.

2. The Tangent Bundle

Let us consider a $2n$-dimensional differentiable manifold $M$ and then $T_pM$ is the tangent space at a point $p$ of $M$. It is known that the set

$$TM = \bigcup_{p \in M} T_pM$$

is called the **tangent bundle** over the manifold $M$ ($[14]$).

For any point $u$ of $TM$ such that $u \in T_pM$ the correspondence $u \to p$ defines the bundle projection $\pi : TM \to M$ by $\pi(u) = p$. The set $\pi^{-1}(p)$, i.e., $T_pM$, is called the fiber of $TM$ over $p \in M$ and $M$ – the **base manifold** of $TM$. The base space $M$ is a submanifold differentially imbedded in $TM$.

Let $(x^i)$ are the local coordinates of a point $p$ in a neighborhood $U \subset M$. Then a point $u \in T_pM$ with coordinates $(y^i)$ in $T_pM$ with respect to the natural basis $\{\frac{\partial}{\partial x^i}\}$ is represented by the ordered pair $(p, u)$. Therefore an arbitrary point $\bar{p}$ in $TM$ has local coordinates of the kind $(x^i, y^i)$ in the open set $\pi^{-1}(U) \subset TM$. These coordinates is called the **induced coordinates** in $\pi^{-1}(U)$ from $(x^i)$.

Obviously, the tangent bundle of a base manifold $M$ with almost complex structure is of dimension $4n$.

Let us denote by $T^r_s(M)$ the set of all differentiable tensor fields of type $(r, s)$ in $M$ and let $T(M) = \sum_{r=0}^{\infty} T^r_s(M)$ be the set of all tensor fields in $M$. Similarly, let us denote by $T^r_s(TM)$ and $T(TM)$ respectively the corresponding sets of tensor fields in the tangent bundle $TM$. 
2.1. Vertical Lifts. The vertical lift $f^V$ of a function $f$ in $M$ is called the composition of $\pi : TM \to M$ and $f : M \to \mathbb{R}$, i.e., $f^V = f \circ \pi$, where $f \in T_0^0(M)$ and $f^V \in T_0^0(TM)$.

Since a correspondence $TM \to \mathbb{R}$ is possible to be considered as the action of a 1-form in $M$ or a function in $TM$, then if $\omega$ is a 1-form in $M$, it is regarded, in a natural way, as a function in $TM$, denoted by $\tilde{\omega}$.

Let $\tilde{X} \in T_0^1(TM)$ be such that $\tilde{X} f^V = 0$ for all $f \in T_0^0(M)$. Then $\tilde{X}$ is called a vertical vector field in $TM$. The vector field $X^V$ in $TM$ defined by $X^V(\omega) = (\omega(X))^V$ for an arbitrary 1-form $\omega$ in $M$, is called the vertical lift of vector field $X$ in $M$ to $TM$. If $X$ has components $X^i$ in $M$, then $X^V$ has components

\begin{equation}
X^V : \left( \begin{array}{c}
0 \\
X^i 
\end{array} \right)
\end{equation}

with respect to the induced coordinates in $TM$. Consequently, it is clear, that $\left( \frac{\partial}{\partial y^i} \right)^V = \frac{\partial}{\partial y^i}$. There are valid the following formulas for any $X, Y \in \mathfrak{T}_0^0(M)$ and $f \in T_0^0(M)$.

\[ X^V f^V = 0, \quad (X + Y)^V = X^V + Y^V, \quad (f X)^V = f^V X^V, \quad [X^V, Y^V] = 0. \]

2.2. Horizontal Lifts. Let $\nabla$ be the Levi-Civita connection in $M$ induced by the pseudo-Riemannian metric $g$. Horizontal lift $f^H$ of a function $f$ in $M$ to $TM$ is defined by $f^H = \iota(df) - \gamma(\nabla f)$, where $\gamma(\nabla f) = y^k \nabla_k f$. There follows immediately that $f^H = 0$ for any $f \in T_0^0(M)$.

Suppose that $X$ and $\nabla$ have local components $X^i$ and $\Gamma_{ij}^k$, respectively, in $M$. Horizontal lift $X^H$ of a vector field $X$ in $M$ to $TM$ is called the vector field in $TM$ with components

\begin{equation}
X^H : \left( \begin{array}{c}
X^k \\
y^i \Gamma_{ij}^k X^j 
\end{array} \right)
\end{equation}

with respect to the induced coordinates in $TM$.

For the action of $X^H$ we have that $X^H f^V = (X f)^V$, $\omega^V(X^H) = \{\omega(X)\}^V$, $J^V X^H = J f X^V$. 2.3. Adapted Frames. In each coordinate neighborhood $\{U, x^k\}$ of $M$, dim $M = 2n$, we denote $e_{(i)} = \frac{\partial}{\partial x^i}$. Then $4n$ local vector fields $e^H_{(i)}$ and $e^V_{(i)}$ form a basis of the tangent space $T_p(TM)$ at each point $\tilde{p} \in \pi^{-1}(p)$ and their components are given by

\[ e^H_{(i)} : \left( \begin{array}{c}
\delta^k_i \\
-y^j \Gamma_{ij}^k 
\end{array} \right), \quad e^V_{(i)} : \left( \begin{array}{c}
0 \\
\delta^k_i 
\end{array} \right) \]

with respect to the induced coordinates $(x^k, y^k)$ in $TM$. The set $\{e^H_{(i)}, e^V_{(i)}\}$ is called the frame adapted to the connection $\nabla$ in $\pi^{-1}(U)$. On putting $\tilde{e}_{(i)} = e^H_{(i)}, \tilde{e}^V_{(i)} = e^V_{(i)}$ we write the adapted frame as $\{\tilde{e}_{(i)}, \tilde{e}^V_{(i)}\}$.

Then the horizontal and vertical lifts of $X$ with local components $X^k$ are components

\[ X^H : \left( \begin{array}{c}
X^k \\
0 
\end{array} \right), \quad X^V : \left( \begin{array}{c}
0 \\
X^k 
\end{array} \right) \]

with respect to the adapted frame $\{\tilde{e}_{(i)}\}$ in $TM$.

3. The Tangent Bundle with an Almost Hypercomplex Pseudo-Hermitian Structure

Our purpose is a determination of an almost hypercomplex pseudo-Hermitian structure $(H, G)$ on $TM$ when the base manifold $M$ has an almost skew-Hermitian structure $(J, g, \tilde{g})$.

We will use the horizontal and vertical lifts of the vector fields on $M$ to get the corresponding components of the considered tensor fields on $TM$. These components are sufficient to describe the characteristic tensor fields on $TM$ in general.
3.1. **An Almost Hypercomplex Structure on the Tangent Bundle.** As it is known [15], for any affine connection in $M$, the induced horizontal and vertical distributions in $TM$ are mutually complementary. Then we define tensor field $J_1$, $J_2$ and $J_3$ in $TM$ by their action over the horizontal and vertical lifts of an arbitrary vector field in $M$ as follows:

\[(3.1)\quad J_1 : X^H \to X^V, \quad X^V \to -X^H; \quad J_2 : X^H \to (JX)^V, \quad X^V \to (JX)^H; \quad J_3 := J_1 \circ J_2,\]

where $J$ is the given almost complex structure in $M$.

By direct computations we get the following

**Proposition 3.1.** There exists an almost hypercomplex structure $H = (J_1, J_2, J_3)$, defined by \[(3.1)\] in $TM$ over an almost complex manifold $(M, J)$ with an affine connection $\nabla$. The received 4-dimensional manifold is an almost hypercomplex manifold $(TM, H)$.

Let $N_\alpha$ denotes the Nijenhuis tensor regarding $J_\alpha$ for each $\alpha = 1, 2, 3$ and $\tilde{X}, \tilde{Y} \in T^0_0(M)$, i.e.

\[(3.2)\quad N_\alpha(\tilde{X}, \tilde{Y}) = [\tilde{X}, \tilde{Y}] + J_\alpha[J_\alpha \tilde{X}, \tilde{Y}] + J_\alpha[\tilde{X}, J_\alpha \tilde{Y}] - J_\alpha[\tilde{X}, \tilde{Y}].\]

Having in mind \[(2.1)\] and \[(2.2)\] we receive by direct computations the following

**Lemma 3.2.** Let $\nabla$ be a torsion-free affine connection in $M$ and $R$ be its curvature tensor. Then for any $X, Y \in T^0_0(M)$ at $u \in T_p M$ we have

\[
[X^H, Y^H] = [X, Y]^H - \{R(X, Y)u\}^V, \quad [X^H, Y^V] = (\nabla_X Y)^V, \\
[X^V, Y^V] = 0, \quad [X^V, Y^H] = -(\nabla_Y X)^V.
\]

Using the definitions \[(3.1), (3.2)\] and Lemma \[3.2\] we get

**Proposition 3.3.** Let $(M, J)$ be an almost complex manifold. Then the Nijenhuis tensors of the structure $H$ in $TM$ for the horizontal and vertical lifts have the form

\[(3.3)\quad -N_1(X^H, Y^H) = N_1(X^V, Y^V) = \{R(X, Y)u\}^V, \quad N_1(X^H, Y^V) = N_1(X^V, Y^H) = -\{R(X, Y)u\}^H;\]

\[(3.4)\quad N_2(X^H, Y^H) = \{J(\nabla_X J)(Y) - J(\nabla_Y J)(X)\}^H - \{R(X, Y)u\}^V, \\
N_2(X^V, Y^V) = -\{(\nabla_J X J)(Y) + (\nabla_J Y J)(X)\}^H + \{R(JX, JY)u\}^V, \\
N_2(X^H, Y^V) = \{J(\nabla_X J)(Y) + (\nabla_Y J)(X)\}^V - \{JR(X, JY)u\}^H, \\
N_2(X^V, Y^H) = -\{(\nabla_J X J)(Y) + (\nabla_Y X J)(X)\}^V - \{JR(JX, JY)u\}^H;\]

\[(3.5)\quad N_3(X^H, Y^H) = -R(X, Y)u + R(JX, JY)u + JR(JX, JY)u + JR(X, JY)u, \\
N_3(X^H, Y^V) = \{(\nabla_J X J)(Y) - (\nabla_X J)(JY)\}^V, \\
N_3(X^V, Y^H) = -\{(\nabla_Y J)(JX) - (\nabla_J J)(X)\}^V, \quad N_3(X^V, Y^V) = 0.
\]

for any $X, Y \in T^0_0(M)$ at $u \in T_p M$.

The last equalities for $N_\alpha$ imply the following necessary and sufficient conditions for integrability of $J_\alpha$ and $H$.

**Theorem 3.4.** Let $TM$ be the tangent bundle manifold with an almost hypercomplex structure $H = (J_1, J_2, J_3)$ defined as in \[(3.1)\] and $M$ be its base manifold with an almost complex structure $J$. Then the following interconnections hold:

1. $(TM, J_1)$ is complex iff $M$ is flat; (see also \[10\])
Theorem 3.9. The tangent bundle \((TM, J_3)\) for \(\beta = 2\) or 3 is complex iff \(M\) is flat and \(J\) is parallel;
(3) \((TM, H)\) is hypercomplex iff \(M\) is flat and \(J\) is parallel.

Corollary 3.5.
(1) \((TM, J_2)\) is complex iff \((TM, J_3)\) is complex.
(2) If \((TM, J_2)\) or \((TM, J_3)\) is complex then \((TM, H)\) is hypercomplex.

3.2. The Sasaki Metric on the Tangent Bundle. Let us introduce the Sasaki metric \(\hat{g}\) on \(TM\) as in \([12]\) (i. e. \(\hat{g}\) is the so-called diagonal lift of the base metric \(g\)) defined by

\[
\hat{g}(X^H, Y^H) = \hat{g}(X^V, Y^V) = g(X, Y), \quad \hat{g}(X^H, Y^V) = 0.
\]

Having in mind the definition \((3.1)\) of the structure \(H\), we verify immediately that the Sasaki metric satisfies the properties \((1.1)\) and therefore it is valid the following

Theorem 3.6. The tangent bundle \(TM\) equipped with the almost hypercomplex structure \(H\) and the Sasaki metric \(\hat{g}\), defined by \((3.1)\) and \((1.1)\), respectively, is an almost hypercomplex pseudo-Hermitian manifold denoting by \((TM, H, \hat{g})\).

Since \(\hat{\nabla}\) is the Levi-Civita connection of \(\hat{g}\) on \(TM\) as \(\nabla\) of \(g\) on \(M\), then using the property of type

\[
2\hat{g}(\nabla_X Y, Z) = X\hat{g}(Y, Z) + Y\hat{g}(X, Z) - Z\hat{g}(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)
\]

we obtain the covariant derivatives of the horizontal and vertical lifts of vector fields on \(TM\) as follows

Lemma 3.7. For any \(X, Y \in T^0_1(M)\) at \(u \in T_p M\)

\[
\hat{\nabla}_X u Y^H = (\nabla_X Y)^H - \frac{1}{2}\{R(X, Y)u\}^V, \quad \hat{\nabla}_X u Y^V = \frac{1}{2}\{R(u, Y)X\}^H + (\nabla_X Y)^V, \\
\hat{\nabla}_X u Y^V = \frac{1}{2}\{R(u, Y)X\}^H, \quad \hat{\nabla}_X u Y^H = 0.
\]

After that we calculate the components of the curvature tensor \(\hat{R}\) of \(\hat{\nabla}\) with respect to the horizontal and vertical lifts of the vector fields on \(M\) and we receive

Proposition 3.8. The following interconnections between the curvature tensors \(R\) and \(\hat{R}\) corresponding to the metrics \(g\) and \(\hat{g}\) are valid

\[
\hat{R}(X^H, Y^H, Z^H, W^H) = R(X, Y, Z, W) + \frac{1}{4}\{g(R(W, X)u, R(Y, Z)u) - g(R(W, Y)u, R(X, Z)u)\} \\
- \frac{1}{2}g(R(X, Y)u, R(Z, W)u), \\
\hat{R}(X^H, Y^H, Z^H, W^V) = -\frac{1}{2}g((\nabla_X R)(Y, Z)u - (\nabla_Y R)(X, Z)u, W), \\
\hat{R}(X^H, Y^H, Z^V, W^V) = R(X, Y, Z, W) - \frac{1}{4}\{g(R(u, W)X, R(u, Z)Y) - g(R(u, W)Y, R(u, Z)X)\}, \\
\hat{R}(X^H, Y^V, Z^H, W^H) = \frac{1}{2}g((\nabla_X R)(u, Y)Z, W), \\
\hat{R}(X^H, Y^V, Z^H, W^V) = R(X, Y, Z, W) - \frac{1}{4}\{g(R(u, Y)Z, R(u, W)X) - g(R(u, X)Z, R(u, W)Y)\}, \\
\hat{R}(X^V, Y^V, Z^H, W^V) = R(X, Y, Z, W) - \frac{1}{4}\{g(R(u, Y)Z, R(u, W)X) - g(R(u, X)Z, R(u, W)Y)\};
\]

for the lifts \((\cdot)^H, (\cdot)^V \in T^0_1(TM)\) of any \(X, Y, Z, W \in T^0_1(M)\) at \(u \in T_p M\).

Hence we get

Theorem 3.9. The tangent bundle \((TM, \hat{g})\) is flat if and only if its base manifold \((M, g)\) is flat.
Remark. The results of the last three statements are confirmed also by [7] (see also [6]), where \( g \) is a Riemannian metric.

According to [7], the tangent bundle with Sasaki metric of a Riemannian manifold is never locally symmetric unless the base manifold is locally Euclidean. Having in mind the last reasonings, the proof of the corresponding theorem in [7] and that \( g \) is a pseudo-Riemannian metric in our case, we obtain the following

Conclusion. If the tangent bundle \((TM, \hat{g})\) is locally symmetric then the curvature tensor \( R \) of its base pseudo-Riemannian manifold \((M, g)\) is zero or isotropic, i.e. \( R = 0 \) or \( g(R, R) = 0, R \neq 0 \).

3.3. The Tangent Bundles with Almost Hermitian, Almost Skew-Hermitian and Almost Hypercomplex Pseudo-Hermitian Structures. Suppose that \((M, J)\) is an almost complex manifold with skew-Hermitian metrics \( g, \hat{g} \) and that \((TM, \hat{H})\) is its almost hypercomplex tangent bundle with the pseudo-Hermitian metric structure \( \hat{G} = (\hat{g}, \hat{\Phi}, \hat{g}_2, \hat{g}_3) \) derived (as in [12]) from the Sasaki metric \( \hat{g} \) on \( TM \) - the diagonal lift of \( g \). The generated \( 4n \)-dimensional manifold we will denote by \((TM, H, \hat{G})\).

To characterize the structural tensors \( F_\alpha(\cdot, \cdot, \cdot, \cdot) = \hat{g} \left( (\nabla J_\alpha)(\cdot, \cdot, \cdot, \cdot) \right) \) at each \( u \in T_p M \) on \((TM, H, \hat{G})\) we use Lemma 3.7 and the definitions of \( J_\alpha \), whence we obtain the following

**Proposition 3.10.** The nonzero components of \( F_\alpha \) with respect to the lifts of the vector fields depend on structural tensor \( F \) and the curvature tensor \( R \) on \((M, J, \hat{g}, \hat{g})\) by the following way:

\[
\begin{align*}
-F_1(X^H, Y^H, Z^H) & = F_1(X^H, Y^V, Z^V) = F_1(X^V, Y^H, Z^V) = F_1(X^V, Y^V, Z^H) \\
& = \frac{1}{2} R(Y, Z, X, u);
\end{align*}
\]

\[
\begin{align*}
F_2(X^H, Y^H, Z^H) & = -\frac{1}{2} R(X, Y, JZ, u) + \frac{1}{2} R(Z, X, JY, u), \\
F_2(X^H, Y^V, Z^V) & = \frac{1}{2} R(X, JY, Z, u) - \frac{1}{2} R(JZ, X, Y, u), \\
F_2(X^H, Y^V, Z^V) & = F_2(X^V, Y^H, Z^H) = F(X, Y, Z), \\
F_2(X^V, Y^H, Z^V) & = \frac{1}{2} R(Y, JZ, X, u), \\
F_2(X^V, Y^V, Z^H) & = -\frac{1}{2} R(JY, Y, Z, X, u);
\end{align*}
\]

\[
\begin{align*}
F_3(X^H, Y^H, Z^H) & = -F_3(X^H, Y^V, Z^V) = -F(X, Y, Z), \\
F_3(X^H, Y^V, Z^V) & = -\frac{1}{2} R(X, JY, Z, u) - \frac{1}{2} R(Y, JZ, X, u), \\
F_3(X^H, Y^V, Z^H) & = \frac{1}{2} R(Z, X, JY, u) + \frac{1}{2} R(JZ, X, Y, u), \\
F_3(X^V, Y^H, Z^H) & = \frac{1}{2} R(JY, Y, Z, X, u) - \frac{1}{2} R(Y, JZ, X, u).
\end{align*}
\]

where \((\cdot)^H, (\cdot)^V \in T^0_p(TM)\) are the lifts of any \( X, Y, Z \in T^0_p(M) \) at \( u \in T_p M \).

Hence we compute the corresponding Lie forms with respect to the adapted frame.

Let \( \{e_i^H, e_i^V, e_i^Y, e_i^Z\} \) be the adapted frame of type \((+,-,-,+,-,\ldots,+,-,\ldots,-)\) at each point of \( TM \) derived by the orthonormal basis \( \{e_i, \overline{e}_i\} \) of signature \((n, n)\) at each point of \( M \) denoting \( e_i = J e_i \). The indices \( i \) and \( \overline{i} \) run over the ranges \( \{1, \ldots, n\} \) and \( \{n + 1, \ldots, 2n\} \), respectively. For example

\[
\theta_3(Z^H) = \sum_{i=1}^n \left\{ F_3(e_i^H, e_i^H, Z^H) - F_3(e_i^H, e_i^V, Z^H) - F_3(e_i^V, e_i^V, Z^H) - F_3(e_i^V, e_i^V, Z^H) \right\} \\
= \sum_{i=1}^n \left\{ -F(e_i, e_i, Z) + F(e_i, e_i, Z) \right\} = -\theta(Z),
\]

\[
\theta_3(Z^V) = \sum_{i=1}^n \left\{ F_3(e_i^H, e_i^H, Z^V) - F_3(e_i^H, e_i^V, Z^V) - F_3(e_i^V, e_i^V, Z^V) - F_3(e_i^V, e_i^V, Z^V) \right\} \\
= \sum_{i=1}^n \left\{ -\frac{1}{2} R(e_i, e_i, Z, u) - \frac{1}{2} R(e_i, e_i, Z, u) \right\} = 0.
\]

Then we have
Proposition 3.11.  

1. \((TM, J_1, \hat{g})\) has a zero Lie form \(\theta_1\);  
2. \((TM, J_2, \hat{g})\) has a zero Lie form \(\theta_2\) iff \((M, J, g, \hat{g})\) has a zero Lie form \(\theta\) and a zero associated Ricci tensor \(\hat{\rho}\);  
3. \((TM, J_3, \hat{g})\) has a zero Lie form \(\theta_3\) iff \((M, J, g, \hat{g})\) has a zero Lie form \(\theta\).

According to [3.7] the following property is valid.

Theorem 3.12. \((TM, J_1, \hat{g})\) is an almost Kähler manifold and it is a Kähler manifold if and only if \((M, J, g, \hat{g})\) is flat.

Remark. By comparison with the Riemannian case, in [10] it is shown that \((TM, J_1, \hat{g})\) is almost Kählerian (i.e. symplectic) for any Riemannian metric \(g\) on the base manifold when the connection used to define the horizontal lifts is the Levi-Civita connection.

Having in mind [3.7]–[3.9] and Theorem 3.4 it is easy to conclude the following sequel of properties:

Proposition 3.13.  

1. \((TM, J_1, \hat{g})\) is Kählerian iff \(M\) is flat.  
2. \((TM, J_2, \hat{g})\) for \(\beta = 2\) or \(3\) is skew-Kählerian iff \((M, J, g, \hat{g})\) is flat and skew-Kählerian.  
3. \((TM, H, G)\) is a pseudo-hyper-Kähler manifold iff \((M, J, g, \hat{g})\) is flat and skew-Kählerian.

Corollary 3.14.  

1. \((TM, J_2, \hat{g})\) is skew-Kählerian iff \((TM, J_3, \hat{g})\) is skew-Kählerian.  
2. If \((TM, J_2, \hat{g})\) or \((TM, J_3, \hat{g})\) is skew-Kählerian then \((TM, H, \hat{G})\) is pseudo-hyper-Kählerian.

Corollary 3.15.  

1. The only complex manifolds \((TM, J_\alpha, \hat{g})\) for some \(\alpha = 1, 2, 3\) are the Kähler manifolds with respect to that \(J_\alpha\).  
2. The only hypercomplex manifolds \((TM, H, \hat{G})\) are the pseudo-hyper-Kählerian manifolds.

Let us recall that the class \(\{W_2 \oplus W_3\}\) of the almost skew-Hermitian manifolds is determined by the condition for vanishing of the Lie form, the class \(W_3\) by \(\sigma F = 0\) and the class \(W_0\) by \(F = 0\) (i.e. the last is the class of the skew-Kählerian manifolds). Then we obtain the following

Theorem 3.16. Let \(M\) be \((M, J, g, \hat{g})\) and \(TM\) be \((TM, H, \hat{G})\).  

1. \(TM \in \{W_2 \oplus W_3\} (J_2)\) iff \(M \in \{W_2 \oplus W_3\} (J)\) and \(\rho = \hat{\rho} = 0\);  
2. \(TM \in W_3(J_2)\) iff \(M \in W_0(J)\) and \(\rho = \hat{\rho} = 0\);  
3. \(TM \in \{W_2 \oplus W_3\} (J_3)\) iff \(M \in \{W_2 \oplus W_3\} (J)\);  
4. \(TM \in W_3(J_3)\) iff \(M \in W_0(J)\),

where \(\rho\) and \(\hat{\rho}\) denote the Ricci tensor and the associated Ricci tensor on \(M\), respectively.

If we set some additional conditions for \((M, J, g, \hat{g})\) then we receive the corresponding specialization of \((TM, H, \hat{G})\) given in the next properties:

Proposition 3.17. Let \(M\) be \((M, J, g, \hat{g})\) and \(TM\) be \((TM, H, \hat{G})\).  

1. If \(M \in \{W_2 \oplus W_3\} (J)\) then \(TM \in AK(J_1) \cap \{W_1 \oplus W_2 \oplus W_3\} (J_2) \cap \{W_2 \oplus W_3\} (J_3)\).  
2. If \(M \in \{W_2 \oplus W_3\} (J)\) and \(\rho = \hat{\rho} = 0\) then \(TM \in AK(J_1) \cap \{W_2 \oplus W_3\} (J_2) \cap \{W_2 \oplus W_3\} (J_3)\).  
3. If \(M \in \{W_2 \oplus W_3\} (J)\) and \(R = 0\) then \(TM \in K(J_1) \cap \{W_2 \oplus W_3\} (J_2) \cap \{W_2 \oplus W_3\} (J_3)\).

Proposition 3.18. Let \(M\) be \((M, J, g, \hat{g})\) and \(TM\) be \((TM, H, \hat{G})\).
(1) If $M \in \mathcal{W}_0(J)$ then $TM \in \mathcal{AK}(J_1) \cap \{\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3\} (J_2) \cap \mathcal{W}_3(J_3)$.

(2) If $M \in \mathcal{W}_0(J)$ and $\rho = \tilde{\rho} = 0$ then $TM \in \mathcal{AK}(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3)$.

(3) If $M \in \mathcal{W}_0(J)$ and $R = 0$ then $TM \in \mathcal{K}(J_1) \cap \mathcal{W}_0(J_2) \cap \mathcal{W}_0(J_3)$.

In above properties $\mathcal{AK}(J_1)$ and $\mathcal{K}(J_1)$ denote the classes regarding $J_1$ of the almost Kähler manifolds and the Kähler manifolds determined by the conditions $\theta_1 = 0$ and $F_1 = 0$, respectively.

4. Eight-Dimensional Derived Examples

Let us recall the developed examples of 4-dimensional almost hypercomplex pseudo-Hermitian manifolds. In [4] and [8] we have constructed a lot of explicit examples of the investigated manifolds.

At first in [4], it is considered a pseudo-Riemannian spherical manifold $S^2_3$ in pseudo-Euclidean vector space $\mathbb{R}^2_3$ of signature $(2, 3)$. It admits a hypercomplex pseudo-Hermitian structure, with respect to which it is of the class $\mathcal{W} = \bigcap_{\alpha = 1}^3 \mathcal{W}(J_\alpha)$.

Secondly, in the same paper it is considered the 4-dimensional compact homogenous space $L |_{\Gamma}$, where $L = H \times S^1$ is a connected Lie group, $H$ is the Heizenberg group, $S^1$ is the circle and $\Gamma$ is the discrete subgroup of $L$ consisting of all matrices of integer entries. The manifold $M$ generated by $L$ we equipped with an $(H, G)$-structure. Then it is a $\mathcal{W}(J_1)$-manifold but it does not belong to $\mathcal{W}$.

In [8] the sequel of examples begins at Examples 1 and 2 about two Engel manifolds with almost $(H, G)$-structure. The first one is equipped with double isotropic hyper-Kählerian structures which are neither hypercomplex nor symplectic. The second one – with double isotopic hyper-Kählerian structures which are non-integrable but symplectic.

The Example 3 is constructed as a real semi-space with almost $(H, G)$-structure of the class $\mathcal{W}$.

A real quarter-space with almost $(H, G)$-structure is shown in Example 4, which is a $\mathcal{K}(J_1)$-manifold and an isotropic hyper-Kählerian manifold.

In Example 5, an almost $(H, G)$-structure is introduced on a real pseudo-hyper-cylinder in a pseudo-Euclidean real space $\mathbb{R}^3_2$. The received manifold is not integrable and the Lie forms are non-zero, regarding any $J_\alpha$.

The following three examples (Examples 6-8) concern several surfaces $S^2_3$ in a 3-dimensional complex Euclidean space $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$. There is used that the natural decomplexification of an $n$-dimensional complex Euclidean space is the $2n$-dimensional real space with a complex skew-Hermitian structure. Namely, Example 6 shows a flat pseudo-hyper-Kähler manifold as a complex cylinder. Example 7 is a complex cone with almost $(H, G)$-structure and it is a flat hypercomplex manifold which is Kählerian with respect to $J_1$ but it does not belong to $\mathcal{W}(J_2)$ or $\mathcal{W}(J_3)$ and the Lie forms $\theta_2$ and $\theta_3$ are non-zero. In Example 8, a complex sphere with almost $(H, G)$-structure is given. There is shown that it is a $\mathcal{K}(J_2)$-manifold of pointwise constant totally real sectional curvatures, but with respect to $J_1$ and $J_3$, the corresponding Nijenhuis tensors and Lie forms are non-zero.

The last two examples in [8] are inspired from an example of a locally flat almost Hermitian surface constructed in [11] about a connected Lie subgroup of $GL(4, \mathbb{R})$. There are introduced the appropriate metric in two variants and then the received $(H, G)$-manifold in Example 9 is complex with respect to $J_2$ but non-hypercomplex and the Lie forms do not vanish; and the constructed $(H, G)$-manifold in Example 10 is flat and it is Kählerian with respect to $J_1$ but with respect to $J_2$ and $J_3$ it is not complex.

We can use the given examples in [4] and [8] to construct 8-dimensional tangent bundle manifolds with almost hypercomplex pseudo-Hermitian structure which is almost Kählerian with respect to $J_1$. For this purpose we use $J_2$ or $J_3$ like a base almost complex structure $J$ of $(M, J, g, \tilde{g})$. Then applying the above mentioned results to the case $n = 2$, we receive the following
Corollary 4.1. Let $M$ and $TM$ denote for short the base 4-dimensional manifold $(M, J, g, \tilde{g})$ with almost complex skew-Hermitian structure and its tangent bundle $(TM, H, \hat{G})$ with almost hypercomplex pseudo-Hermitian structure constructed as in §3 respectively.

1. If $M$ with $J := J_2$ (or $J_3$) is the complex cylinder introduced in Example 6 of [8], then $TM$ is a flat pseudo-hyper-Kählerian.

2. If $M$ with $J := J_2$ (or $J_3$) is the complex cone introduced in Example 7 of [8] or the Lie group introduced in Example 10 of [8], then $TM$ is a flat almost hypercomplex pseudo-Hermitian and an $\mathcal{AK}(J_1)$-manifold.

3. If $M$ with $J := J_2$ is the complex sphere introduced in Example 8 of [8], then $TM$ is non-flat and its $\theta_3$ is also zero as $\theta_1$.

4. If $M$ with $J := J_2$ (or $J_3$), is some of the manifolds introduced in the other 8 known examples of [4, 8], then $TM$ is a non-flat manifold which is almost Kählerian with respect to $J_1$, but with respect to $J_2$ and $J_3$ it is not complex and the Lie forms are not zero.

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REFERENCES

[1] D. V. Alekseevsky and S. Marchiafava, Quaternionic Structures on a Manifold and Subordinated Structures, Ann. Mat. Pura Appl. (IV), CLXXI (1996), 205–273.

[2] G. Ganchev and A. Borisov, Note on the Almost Complex Manifolds with a Norden Metric, Compt. rend. Acad. bulg. Sci., 39 (1986), no. 5, 31–34.

[3] A. Gray and L. M. Hervella, The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants, Ann. Mat. Pura Appl. (IV), CXXIII (1980), 35–58.

[4] K. Gribachev, M. Manev and S. Dimiev, Almost Hypercomplex Pseudo-Hermitian Manifolds, Boll. Unione Mat. Ital. Sez. A, (to appear)

[5] K. Gribachev, M. Manev and S. Dimiev, On Almost Hypercomplex Pseudo-Hermitian Manifolds, Trends in Compl. Anal., Diff. Geom. & Math. Phys., World Sci. Publ., 2005, 51–62.

[6] S. Gudmundsson, E. Kappos, On the Geometry of Tangent Bundles, Expo. Math., 20 (2002), 1–41.

[7] O. Kowalski, Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold, J. Reine Angew. Math., 250 (1971), 124–129.

[8] M. Manev, Some Four-Dimensional Almost Hypercomplex Pseudo-Hermitian Manifolds, Contemporary Aspects in Compl. Anal., Diff. Geom. & Math. Phys., World Sci. Publ., 2005. (to appear)

[9] A. P. Norden, On the Structure of the Connections on the Manifolds of the Lines in an Euclidean Space, Russian Math. (Izv. VUZ), 17 (1960), 145–157. (in Russian)

[10] Ph. Tondeur, Structure Presque Kählerienn Naturelle sur le Fibre des Vecteurs Covariants d’une Variete Riemannienne, C. R. Acad. Sci. Paris, 254 (1962), 407–408. (in French)

[11] F. Tricerri and L. Vanhecke, Flat almost Hermitian manifolds which are not Kähler manifolds, Tensor (N.S.), 31 (1977), no. 3, 249–254.

[12] S. Sasaki, On the Differential Geometry of Tangent Bundles of Riemannian Manifolds, Tôhoku Math. J., 10 (1958), 238–354.

[13] A. Sommese, Quaternionic Manifolds, Math. Ann., 212 (1975), 191–214.

[14] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, N. J., 1951.

[15] K. Yano and S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.

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