Chen–Ruan cohomology and moduli spaces of parabolic bundles over a Riemann surface

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Abstract. Let \((X, D)\) be an \(m\)-pointed compact Riemann surface of genus at least 2. For each \(x \in D\), fix full flag and concentrated weight system \(\alpha\). Let \(\mathcal{P}M_\xi\) denote the moduli space of semi-stable parabolic vector bundles of rank \(r\) and determinant \(\xi\) over \(X\) with weight system \(\alpha\), where \(r\) is a prime number and \(\xi\) is a holomorphic line bundle over \(X\) of degree \(d\) which is not a multiple of \(r\). We compute the Chen–Ruan cohomology of the orbifold for the action on \(\mathcal{P}M_\xi\) of the group of \(r\)-torsion points in \(\text{Pic}^0(X)\).

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1. Introduction

Chen and Ruan introduced a new cohomology theory in [8] for orbifolds which is named after them. It is the degree zero part of the small quantum cohomology ring, constructed in [9], of the orbifold. It contains the usual cohomology ring of the orbifold as a sub-ring.

The Chen–Ruan cohomology for the orbifolds arising from the moduli space of stable vector bundles of rank 2 and degree 1 over a compact Riemann surface was computed in [4], which was subsequently generalized for arbitrary prime rank in [5].

Let \(X\) be a compact Riemann surface of genus \(g \geq 2\), and let \(D = \{x_1, \ldots, x_m\}\) be a finite subset of \(X\); the points of \(D\) will be called parabolic points. Fix a holomorphic line bundle \(\xi\) on \(X\) of degree one. Let \(\mathcal{P}M_\xi(2)\) denote the moduli space of full flag stable parabolic vector bundles \(E_*\) over \(X\) of rank 2 and fixed determinant \(\bigwedge^2 E \cong \xi\); the parabolic weights are assumed to be generic. The moduli space \(\mathcal{P}M_\xi(2)\) is a smooth projective variety of dimension \(3g - 3 + m\). Let \(\Gamma_2 \subset \text{Pic}^0(X)\) be the subgroup defined by the points of order two, meaning the holomorphic line bundles \(L\) with \(L^\otimes 2 = \mathcal{O}_X\). Then \(\Gamma_2\) acts on \(\mathcal{P}M_\xi(2)\); the action of \(L \in \Gamma_2\) sends any \(E_* \in \mathcal{P}M_\xi(2)\) to \(E_* \otimes L\). We

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get the smooth orbifold $PM_2(2)/\Gamma_2$. The Chen–Ruan cohomology of it was computed in [3].

Here we consider the moduli space of parabolic bundles of prime rank $r \geq 3$. Fix a holomorphic line bundle $\xi$ on $X$ whose degree is not a multiple of $r$. Let $PM_\xi$ denote the moduli space of stable parabolic bundles on $X$ of rank $r$ and determinant $\xi$ with a parabolic structure over $D$. We assume the parabolic weights to be concentrated [1] (its definition is recalled in Section 2). Let $\Gamma_r$ denote the group of holomorphic line bundles $L$ on $X$ such that $L^{\otimes r} = O_X$. This group $\Gamma_r$ acts on $PM_\xi$; the action of any $L \in \Gamma_r$ sends any parabolic vector bundle $E_*$ to $E_* \otimes L$. Our aim is to determine Chen–Ruan cohomology of the corresponding orbifold.

2. Parabolic bundles

We recall basics on parabolic vector bundles and describe the moduli spaces of stable parabolic bundles.

Let $X$ be a compact Riemann surface, of genus $g \geq 2$, and $D = \{x_1, \ldots, x_m\} \subset X$. Let $E$ be a holomorphic vector bundle on $X$. A quasi-parabolic structure on $E$ is a strictly decreasing flag of linear subspaces in the fiber $E_x$,

$$E_x = E_x^1 \supset E_x^2 \supset \cdots \supset E_x^k \supset E_x^{k+1} = 0$$

over each $x \in D$. A parabolic structure on $E$ is a quasi-parabolic structure on $E$ together with a sequence of real numbers $0 \leq \alpha_1^x < \cdots < \alpha_k^x < 1$, which are called the weights. We set

$$m_j^x = \dim \mathbb{C} E_x^j - \dim \mathbb{C} E_x^{j+1}.$$

The integer $k$ is called the length of the flag and the string of integers $(m_1^x, \ldots, m_k^x)$ is called the type of the flag. We say that the flag is a full flag if $m_j^x = 1$ for all $1 \leq j \leq k = \text{rank}(E)$.

A parabolic vector bundle with parabolic structure on $D$ is a holomorphic vector bundle $E$ together with a parabolic structure on $E$; it will be denoted by $E_*$. For a parabolic bundle $E_*$ the parabolic degree is defined to be

$$p \deg(E_*) = \deg(E) + \sum_{x \in D} \sum_{j=1}^k m_j^x \alpha_j^x \in \mathbb{R},$$

where $\deg(E)$ denotes the degree of $E$, and we put

$$p\mu(E_*) = \frac{p \deg(E_*)}{\text{rank}(E)} ,$$

which is called the parabolic slope of $E_*$. A parabolic subbundle of $E_*$ is a subbundle $F$ of $E$ together with the parabolic structure induced from $E_*$. A parabolic bundle $E_*$ is called parabolic semistable if for every non-zero proper parabolic subbundle $F_*$,

$$p\mu(F_*) \leq p\mu(E_*),$$

and it is called parabolic stable if $p\mu(F_*) < p\mu(E_*)$.

The moduli space $PM^\alpha(r, d)$ of semistable parabolic bundles of rank $r$ and degree $d$ and parabolic type $\alpha$, which was constructed in [10], is a normal projective variety. The moduli space $PM$ of stable parabolic bundles is an open and smooth subset of $PM^\alpha(r, d)$. 

For a fixed rank $r$, a full flag systems of weights $\alpha = \{\alpha^x_1, \ldots, \alpha^x_r\}_{x \in D}$ is said to be concentrated if

$$\alpha^x_r - \alpha^x_1 < \frac{4}{mr^2}$$

for all $x \in D$.

In what follows, $r$ will denote a prime number. Let $\xi$ be a fixed line bundle of degree $d$ such that $(d, r) = 1$. Let $\mathcal{P}M_\xi$ denote the moduli space of semistable parabolic vector bundles $E_*$ over $X$ of rank $r$ with full flag concentrated weight system $\alpha$ together with an isomorphism $\bigwedge^r E \cong \xi$. Then all $E_* \in \mathcal{P}M_\xi$ is stable, and $\mathcal{P}M_\xi$ is a smooth complex projective variety.

We fix the parabolic weight system $\alpha$ throughout. The rank $r$ is also fixed throughout.

Let $\Gamma$ be the subgroup of $\text{Pic}^0(X)$ consisting of all holomorphic line bundles $L$ on $X$ such that $L^{\otimes r} = O_X$. Then $\Gamma$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z}) \oplus 2g$. Note that for every $L \in \Gamma$, we have $\bigwedge^r (E \otimes L) = \bigwedge^r E \otimes L^{\otimes r} = \xi$.

For each $x \in D$, we have a filtration

$$(E \otimes L)_x = E^1_x \otimes L_x \supset E^2 \otimes L_x \supset \cdots \supset E^r \otimes L_x$$

given by the parabolic structure of $E_x$ at $x$. The resulting parabolic bundle with $E \otimes L$ as the underlying vector bundle will be denoted by $E_x \otimes L$. It is to be noted that $E_x \otimes L$ is the parabolic tensor product of $E_x$ with the line bundle $L$ equipped with the trivial parabolic structure (see [2] for the parabolic tensor product). Let

$$\tilde{\phi}_L : \mathcal{P}M_\xi \to \mathcal{P}M_\xi, \quad E_* \mapsto E_* \otimes L$$

be an automorphism. This gives an action $\tilde{\phi}$ of the group $\Gamma$ on $\mathcal{P}M_\xi$,

$$\tilde{\phi}(L, E_*) = \tilde{\phi}_L(E_*) .$$

The quotient space

$$Y = \mathcal{P}M_\xi / \Gamma$$

is a smooth orbifold. Our aim is to compute the Chen–Ruan cohomology of the orbifold $Y$.

### 3. Fixed point sets

We continue with the notation of the previous section. For any $L \in \Gamma$, set

$$PS(L) := (\mathcal{P}M_\xi)^{\tilde{\phi}_L} = \{ E_* \in \mathcal{P}M_\xi \mid \tilde{\phi}_L(E_*) = E_* \} \subset \mathcal{P}M_\xi .$$

Then $PS(L)$ is a compact complex manifold, but it need not be connected.

Let $\mathcal{M}_\xi$ denote the moduli space of stable vector bundles $E$ on $X$ of rank $r$ with $\text{det}(E) = \xi$. It is a smooth projective variety. From [1, Proposition 2.6], we have a well-defined forgetful morphism

$$\gamma : \mathcal{P}M_\xi \to \mathcal{M}_\xi$$

(3.2)

that sends a parabolic vector bundle $E_*$ to its underlying vector bundle $E$. This map is surjective, and the fiber of $\gamma$ over any $E$ is the product of full flags of $E_x$ with $x \in D$ [1, Proposition 2.6]. Further, we have an action $\phi$ of $\Gamma$ on $\mathcal{M}_\xi$ given by

$$\phi(L, V) = V \otimes L$$
for $L \in \Gamma$ and $V \in \mathcal{M}_\xi$. This action gives an automorphism

$$\phi_L : \mathcal{M}_\xi \rightarrow \mathcal{M}_\xi, \ V \mapsto V \otimes L.$$  \hfill (3.3)

The morphism $\gamma$ in (3.2) is evidently $\Gamma$-equivariant. Consider

$$S(L) := (\mathcal{M}_\xi)^{\phi_L} = \{E \in \mathcal{M}_\xi \mid \phi(L, E) = E\} \subset \mathcal{M}_\xi,$$

which is a smooth compact complex manifold. As $\gamma$ is $\Gamma$-equivariant, we have $\gamma(PS(L)) \subset S(L)$.

For a description of $S(L)$, see [5, p. 499, Lemma 2.1]. Let

$$\Psi_L : PS(L) \rightarrow S(L)$$  \hfill (3.4)

be the restriction of $\gamma$.

Take a nontrivial line bundle $L \in \Gamma$. Fix a nonzero holomorphic section

$$s : X \rightarrow L^\otimes r.$$

Define

$$Y_L := \{z \in L \mid z^\otimes r \in \text{Im}(s)\} \subset L.$$

Let

$$\pi_L : Y_L \rightarrow X$$

be the restriction of the natural projection $L \rightarrow X$. Since order $r$ of $L$ is a prime number, $Y_L$ is an irreducible curve and $\pi_L$ is an unramified covering of degree $r$. More precisely, $Y_L$ is a $\mu_r$-bundle over $X$, where $\mu_r := \{a \in \mathbb{C} \mid a^r = 1\}$.

**Lemma 3.1.** Let $L \in \Gamma$ be a nontrivial line bundle on $X$.

1. The map $\Psi_L$ in (3.4) is surjective.
2. The map $\Psi_L$ is an isomorphism on each connected component of $PS(L)$. The number of connected components of $PS(L)$ is $(r!)^{m}$.

**Proof.** Let

$$\text{Prym}_\xi \subset \text{Pic}^1(Y_L)$$

be the locus of all line bundles $\eta \rightarrow Y_L$ such that $\bigwedge^r \pi_L^* \eta = \xi$. The Galois group $\text{Gal}(\pi_L) = \mathbb{Z}/r\mathbb{Z}$ acts on $\text{Prym}_\xi$; the action of any

$$\sigma \in \text{Gal}(\pi_L)$$  \hfill (3.5)

sends any $\eta \in \text{Prym}_\xi$ to $\sigma^* \eta$. We have

$$S(L) = \text{Prym}_\xi / \text{Gal}(\pi_L)$$  \hfill (3.6)

(see [5, Lemma 2.1]).

For any $\eta \in \text{Prym}_\xi$, we have

$$\pi_L^* \pi_L^* \eta = \bigoplus_{\sigma \in \text{Gal}(\pi_L)} \sigma^* \eta.$$
The fiber of $\pi_{L^*}\eta$ over $x_i \in D$ has the following decomposition:

$$
(\pi_{L^*}\eta)_{x_i} = \bigoplus_{z \in \pi_{L}^{-1}(x_i)} \eta_z = \eta_{y_i} \oplus \eta_{\sigma(y_i)} \oplus \eta_{\sigma^2(y_i)} \oplus \cdots \oplus \eta_{\sigma^{r-1}(y_i)},
$$

(3.7)

where $y_i \in \pi_{L}^{-1}(x_i)$ is a fixed point and $\sigma$ is a nontrivial automorphism as in (3.5). The filtration

$$
\eta_{y_i} \subset \eta_{y_i} \oplus \eta_{\sigma(y_i)} \subset \cdots \subset \eta_{y_i} \oplus \eta_{\sigma(y_i)} \oplus \eta_{\sigma^2(y_i)} \oplus \cdots \oplus \eta_{\sigma^{i-1}(y_i)} \subset \cdots \subset (\pi_{L^*}\eta)_{x_i}
$$

(3.8)

defines a parabolic structure on the vector bundle $\pi_{L^*}\eta$ over $x_i$. Note that we have choices for the above filtration; the direct summands in (3.7) can be permuted, so there are exactly $r!$ possible filtrations.

Thus, for any $\eta \in S(L)$ making the above choice a parabolic bundle

$$
E^* \longrightarrow X
$$
is constructed; so $E = \pi_{L^*}\eta$.

We will show that $E^* \in PS(L)$, that is, $\pi_{L^*}\eta$ is canonically isomorphic to $(\pi_{L^*}\eta) \otimes L$. The Riemann surface $Y_L$ lies in $L \setminus \{0_X\}$, where $0_X \subset L$ is the image of the zero section of $L$. Therefore $\pi_{L^*}^*L$ has a canonical trivialization. Let

$$
u : Y_L \longrightarrow \pi_{L^*}^*L
$$

be the tautological nonzero section giving the trivialization of $\pi_{L^*}^*L$. Then we have an isomorphism

$$
\eta \otimes \nu \longrightarrow \eta \otimes \pi_{L^*}^*L
$$
defined by the tensor product with $\nu$. From the projection formula, we get an isomorphism

$$
\rho : \pi_{L^*}\eta \longrightarrow \pi_{L^*}(\eta \otimes \pi_{L^*}^*L) = (\pi_{L^*}\eta) \otimes L.
$$

(3.9)

This isomorphism $\rho$ evidently preserves the decompositions of $(\pi_{L^*}\eta)_{x_i}$ and $((\pi_{L^*}\eta) \otimes L)_{x_i}$ (see (3.7)). This proves the first part of the lemma.

For the isomorphism $\rho$ in (3.9),

$$
\rho_{x_i}(V) = V \otimes L_{x_i}
$$

for some subspace $V \subset (\pi_{L^*}\eta)_{x_i}$ if and only if $V$ is the direct sum of some direct summands in (3.7). As mentioned before, for the parabolic structures on $\pi_{L^*}\eta$ over each $x_i \in D$, we have $r!$ choices for the parabolic filtration. Note that each choice gives a copy of $S(L)$. Therefore, $PS(L)$ is the disjoint union of copies of $S(L)$, and the copies are parametrized by the finite set

$$
\prod_{i=1}^{m} \text{Sym}(\pi_{L}^{-1}(x_i))
$$

which has cardinality $(r!)^m$. This completes the proof. □
COROLLARY 3.2

Let \( L \in \Gamma \) be a nontrivial line bundle. Then the cohomology algebra \( H^*(PS(L), \mathbb{Q}) \) is isomorphic to \( H^*(S(L), \mathbb{Q})^{(r)}\).

4. Action on the tangent bundle

The full length (same as complete) flag variety of a finite dimensional complex vector space \( W \) will be denoted by \( \mathcal{F}(W) \). The inverse image of any \( E \in M_\xi \) for the map \( \gamma \) defined in (3.2) is

\[
\gamma^{-1}(E) = \prod_{x_i \in D} \mathcal{F}(E_{x_i}).
\]

Therefore, elements of \( PM_\xi \) are of the form \( (E; f_1, \ldots, f_m) \) where \( E \in M_\xi \) and \( (f_1, \ldots, f_m) \in \prod_{x_i \in D} \mathcal{F}(E_{x_i}) \). Note that any \( f_i \in \mathcal{F}(E_{x_i}) \) corresponds to a filtration of the following type:

\[
f_i := \{ E_{x_i} = E_{x_i}^1 \supset E_{x_i}^2 \supset \cdots \supset E_{x_i}^r \supset E_{x_i}^{r+1} = 0 \}.
\]

Recall that the tangent space of the flag variety \( \mathcal{F}(E_{x_i}) \) is \( \text{End}_C(E_{x_i})/\text{End}_C^0(E_{x_i}) \), where \( \text{End}_C^0(E_{x_i}) \) is the space of flag preserving \( C \)-linear endomorphisms of \( E_{x_i} \).

If \( E_* \in PS(L) \), then using (3.7), the tangent space at any point of the flag variety \( \mathcal{F}(E_{x_i}) \) canonically decomposes as

\[
\text{End}_C(E_{x_i})/\text{End}_C^0(E_{x_i}) = \bigoplus_{0 \leq j < k \leq r - 1} \text{Hom}(\eta_{\sigma j}(y_i), \eta_{\sigma k}(y_i)),
\]

where \( E_* = \pi_{L_*}(\eta) \), and \( \eta \in \text{Prym}_\xi \) (see the first part of Lemma 3.1).

For \( i \in \{1, 2, \ldots, m\} \), let

\[
V_i \longrightarrow PM_\xi
\]

be the rank \( \frac{1}{2}r(r - 1) \) vector bundle whose fiber over any \( (E; f_1, \ldots, f_m) \in PM_\xi \) is the vector space

\[
V_i(E; f_1, \ldots, f_m) = \text{End}_C(E_{x_i})/\text{End}_C^0(E_{x_i}).
\]

Let \( TPM_\xi \) (respectively, \( TM_\xi \)) denote the holomorphic tangent bundle of \( PM_\xi \) (respectively, \( M_\xi \)). They fit in the following short exact sequence of vector bundles:

\[
0 \longrightarrow \bigoplus_{i=1}^m V_i \longrightarrow TPM_\xi \stackrel{d\gamma}{\longrightarrow} \gamma^*TM_\xi \longrightarrow 0,
\]

where \( d\gamma \) is the differential of the map \( \gamma \).

Take any \( L \in \Gamma \setminus \{O_X\} \). The automorphism \( \tilde{\phi}_L \) in (2.2) induces an automorphism

\[
d\tilde{\phi}_L : TPM_\xi \longrightarrow TPM_\xi
\]

of tangent bundles over the map \( \tilde{\phi}_L \). Next, \( d\tilde{\phi}_L \) induces an automorphism over \( \tilde{\phi}_L \),

\[
A^i : V_i \longrightarrow V_i,
\]

(4.4)
for every $i = 1, \ldots, m$, such that the following diagram of homomorphisms

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{m} V_i & \longrightarrow & TP \mathcal{M}_\xi & \xrightarrow{d\gamma} & \gamma^* TM_\xi & \longrightarrow & 0 \\
& & \downarrow \oplus A^i & & \downarrow d\phi_L & & \downarrow d\phi_L & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{m} V_i & \longrightarrow & TP \mathcal{M}_\xi & \xrightarrow{d\gamma} & \gamma^* TM_\xi & \longrightarrow & 0
\end{array}
$$

(4.5)

commutes.

Take any parabolic vector bundle $E_\sigma \in PS(L)$. Let $T_{E_\sigma}(P \mathcal{M}_\xi)$ denote the tangent space at $E_\sigma$, and let

$$
d\tilde{\phi}_L(E_\sigma) : T_{E_\sigma}(P \mathcal{M}_\xi) \longrightarrow T_{E_\sigma}(P \mathcal{M}_\xi)
$$

be the differential of the map $\tilde{\phi}_L$ at $E_\sigma$. The diagram in (4.5) gives the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{m} V_i(E_\sigma) & \longrightarrow & T_{E_\sigma}(P \mathcal{M}_\xi) & \xrightarrow{d\gamma(E_\sigma)} & T_{E_\sigma}(\mathcal{M}_\xi) & \longrightarrow & 0 \\
& & \downarrow \oplus A^i_{E_\sigma} & & \downarrow d\tilde{\phi}_L(E_\sigma) & & \downarrow d\phi_L(E) & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{m} V_i(E_\sigma) & \longrightarrow & T_{E_\sigma}(P \mathcal{M}_\xi) & \xrightarrow{d\gamma(E_\sigma)} & T_{E_\sigma}(\mathcal{M}_\xi) & \longrightarrow & 0
\end{array}
$$

(4.6)

Since $r$-fold composition of $\tilde{\phi}_L$ yields

$$
\tilde{\phi}_L \circ \tilde{\phi}_L \circ \cdots \circ \tilde{\phi}_L = \tilde{\phi}_{L^r} = \text{Id}_{P \mathcal{M}_\xi},
$$

d$\tilde{\phi}_L(E_\sigma)$ is a nontrivial automorphism of order $r$. Therefore, the set of eigenvalues of $d\tilde{\phi}_L(E_\sigma)$ is

$$
\mu_r := \{1, t, t^2, \ldots, t^{(r-1)}\},
$$

the group of $r$-th roots of unity. Our aim is to compute the multiplicity of each eigenvalue in $\mu_r$ of the linear operator $d\tilde{\phi}_L(E_\sigma)$.

Note that $d\phi_L(E)$ and $A^i_{E_\sigma}, i = 1, \ldots, m$ (see (4.6)) are also nontrivial automorphisms of order $r$. For any eigenvalue $v \in \mu_r$, the multiplicity of $v$ for the operator $d\tilde{\phi}_L(E_\sigma)$ is evidently the sum of the multiplicities of $v$ for the operators $\bigoplus_{i=1}^{m} A^i_{E_\sigma}$ and $d\phi_L(E)$.

The multiplicity of every $v \in \mu_r$ for the operator $d\phi_L(E)$ has been computed in [5, Proposition 3.1]. It is $r(g - 1)$ if $v \in \mu_r \setminus \{1\}$, and the multiplicity of $1 \in \mu_r$ is $(r - 1)(g - 1)$. So we need to determine the multiplicities of the eigenvalues for the operator $A^i_{E_\sigma}$ for every $i = 1, \ldots, m$.

Recall from Lemma 3.1 that $PS(L)$ is $(r!)^m$ copies of $S(L)$, and they arise from the choices of filtration in (3.8), given the decomposition in (3.7); for each point $x_i \in D$, there are $r!$ possible filtrations and there are $m$ points in $D$. Fix $y_i \in \pi^{-1}_L(x_i)$; trivialize the fiber $L_{x_i}$ using $y_i \in L_{x_i}$ (recall that $Y_L \subset L$). Let

$$
\lambda \in \mu_r \setminus \{1\}
$$

(4.7)

be such that $\sigma(y_i) = \lambda \cdot y_i \in L_{x_i}$. Note that $\lambda$ depends on $\sigma \in \text{Gal}(\pi_L)$, but it does not depend on $y_i$. Then for any $E \in S(L)$, the isomorphism

$$
E \longrightarrow E \otimes L
$$

acts on the direct summand $\eta_{\sigma, j}(y_i)$ in (3.7) as multiplication by $\lambda^j$, where $\lambda$ is the element in (4.7). Using this, it is straightforward to deduce the following.
Lemma 4.1. Let $\mathcal{C} \subset PS(L)$ be one of the $(r!)^m$ components of $PS(L)$ (recall that each component is a copy of $S(L)$). The set of eigenvalues of $\bigoplus_{i=1}^m A_{E_*}^i$ (see (4.6)), for any $E_* \in \mathcal{C}$, is $\mu_r \setminus \{1\}$. For all $1 \leq i \leq m$, there is an element

\[ s_i \in \mu_r \setminus \{1\} \]

(that depends on $\mathcal{C}$) such that for every $1 \leq c \leq r - 1$, the multiplicity of the eigenvalue $s_i^c$ of $A_{E_*}^i$ is $r - c$.

Proof. It is a straightforward computation; the details are omitted.

PROPOSITION 4.2

Let $\mathcal{C} \subset PS(L)$ be one of the $(r!)^m$ components of $PS(L)$. The set of eigenvalues of $d\tilde{\phi}_L(E_*)$, for any $E_* \in \mathcal{C}$, is $\mu_r$. For all $1 \leq i \leq m$, there is an element

\[ s_i \in \mu_r \setminus \{1\} \]

such that the multiplicity of the eigenvalue $s \in \mu_r \setminus \{1\}$ of $d\tilde{\phi}_L(E_*)$ is $r(g-1) + \sum_{i=1}^m (r - c_i)$, where $s = (s_i)^{c_i}$; the multiplicity of the eigenvalue 1 of $d\tilde{\phi}_L(E_*)$ is $(r - 1)(g - 1)$.

Proof. Considering the diagram in (4.6) it follows immediately that the collection of eigenvalues of $d\tilde{\phi}_L(E_*)$ (with multiplicities) is the union of the eigenvalues of $\bigoplus_{i=1}^m A_{E_*}^i$ and $d\phi_L(E)$. It was noted above that the multiplicity of the eigenvalue $\nu \in \mu_r \setminus \{1\}$ for $d\phi_L(E)$ is $r(g - 1)$, and the multiplicity of the eigenvalue 1 for $d\phi_L(E)$ is $(r - 1)(g - 1)$. Therefore, the proposition follows from Lemma 4.1.

Remark 4.3. Fix any $i$ with $1 \leq i \leq m$. For each component $\mathcal{C} \subset PS(L)$, consider $s_i \in \mu_r \setminus \{1\}$ in Proposition 4.2. We note that every element $s \in \mu_r \setminus \{1\}$ repeats $\frac{(r!)^m}{r-1}$ in this collection.

For any $L \in \Gamma \setminus \{O_X\}$ and $E_* \in PS(L)$, the degree shift at $E_*$ for $L$ is defined by

\[ \pi(L, E_*) := \sum_j m_j b_j, \]

where $\exp(2\pi i b_j), 0 \leq b_j < 1$, are the eigenvalues of $d\tilde{\phi}_L(E_*)$, and $m_j$ is the multiplicity of $\exp(2\pi i b_j)$.

For an integer $b$, let $0 \leq [b]_r \leq r - 1$ be such that $b = kr + [b]_r$ with $k \in \mathbb{Z}$.

As a corollary of Proposition 4.2, we get the following.

COROLLARY 4.4

With the notations used in Proposition 4.2, assume $s_i = t^{l_i}, 1 \leq l_i \leq r - 1$ and $1 \leq i \leq m$, where $t = \exp(\frac{2\pi \sqrt{-1}}{r})$. Then for any $L \in \Gamma \setminus \{O_X\}$, the degree shift at $E_*$ for $L$ is

\[ \pi(L) := \pi(L, E_*) = \frac{1}{r} \sum_{i=1}^m \left[ \sum_{k=1}^{r-1} (r - [kc_i]_r)(k l_i c_i)_r \right] + \frac{r(r-1)(g-1)}{2}. \]
Proof. As in Proposition 4.2, \( s = (s_i)_c t_c^i \) has multiplicity \( \sum_{i=1}^m (r - c_i) + r(g - 1) \) for \( d\phi_L(E_\ast) \) and the corresponding \( b_j \) will be \( \frac{k_i c_i}{r} \).

Similarly, \( s^k = (s_i)^{k} t_c^i \) has multiplicity \( \sum_{i=1}^m (r - [k c_i]_r) + r(g - 1) \) for \( d\tilde{\phi}_L(E_\ast) \) and the corresponding \( b_j \) will be \( \frac{k_i c_i}{r} \).

Remark 4.5. For \( r = 3 \) and \( m = 1 \), the degree shift at \( E_\ast \) for \( L \) is either \( \frac{4}{3} + 3(g - 1) \) or \( \frac{5}{3} + 3(g - 1) \) depending on the component in which \( E_\ast \) lies.

5. Chen–Ruan cohomology of the moduli space

Let \( S \) and \( T \) be two topological spaces, and \( \delta \in H^b(S \times T, \mathbb{Q}) \). Then the induced linear map

\[
\sigma(\delta) : H_c(S, \mathbb{Q}) \longrightarrow H^{b-c}(T, \mathbb{Q})
\]

is known as the slant product.

The moduli space \( \mathcal{P}\mathcal{M}_\xi \) is a fine moduli space, and there exists a universal parabolic bundle \( \mathcal{U} \longrightarrow X \times \mathcal{P}\mathcal{M}_\xi \) [7, Proposition 3.2]. Any two such universal bundles differ by tensoring with a line bundle pulled back from \( \mathcal{P}\mathcal{M}_\xi \). For \( 2 \leq k \leq r \), let \( a_k(\mathbb{P}(U)) \in H^{2k} (\mathcal{U}, \mathbb{Q}) \) be the characteristic classes of the projective bundle \( \mathbb{P}(U) \), where \( U \) is the vector bundle underlying the parabolic bundle \( \mathcal{U} \). Since any two universal parabolic bundles differ by tensoring with a line bundle, it follows that \( a_k(\mathbb{P}(U)) \) is independent of the choice of the universal bundle (see [6, Remark 2.1]). So \( a_k(\mathbb{P}(U)) \) induces linear maps

\[
\sigma_j(a_k(\mathbb{P}(U))) : H_j(X, \mathbb{Q}) \longrightarrow H^{2k-j} (\mathcal{P}\mathcal{M}_\xi, \mathbb{Q})
\]

for \( j = 0, 1, 2 \). Then by [6, Theorem 1.5], the cohomology algebra \( H^* (\mathcal{P}\mathcal{M}_\xi, \mathbb{Q}) \) is generated by the Chern classes \( c_j(\text{Hom}(U^k_\xi, U^{k-1}_\xi)) (x \in D) \) and the images of \( \sigma(c_1(U)) \) and \( \sigma_i(a_k(\mathbb{P}(U))) \) (\( 2 \leq k \leq r \), \( 0 \leq i \leq 2 \)). Here \( U^k_\xi \)'s are obtained from the parabolic structure of \( U \) at the point \( x \in D \). The above generators are independent of the universal parabolic bundle \( \mathcal{U} \) chosen.

Consider the action of \( \Gamma \) on \( \mathcal{P}\mathcal{M}_\xi \) induced by \( \tilde{\phi} \), and let

\[
\chi : \mathcal{P}\mathcal{M}_\xi \longrightarrow \mathcal{P}\mathcal{M}_\xi / \Gamma
\]

be the quotient map. The pullback map on the cohomologies for \( \chi \) will be denoted by \( \chi^* \). Using the same technique as in [3, Proposition 4.1], we get the following.

PROPOSITION 5.1

The homomorphism

\[
\chi^* : H^* (\mathcal{P}\mathcal{M}_\xi / \Gamma, \mathbb{Q}) \longrightarrow H^* (\mathcal{P}\mathcal{M}_\xi, \mathbb{Q})
\]

is an isomorphism.

We now describe the Chen–Ruan cohomology algebra of \( \mathcal{P}\mathcal{M}_\xi / \Gamma \). The Chen–Ruan cohomology group of \( \mathcal{P}\mathcal{M}_\xi / \Gamma \), by definition, is

\[
H^j_{CR}(\mathcal{P}\mathcal{M}_\xi / \Gamma, \mathbb{Q}) := \bigoplus_{L \in \Gamma} H^{j-2\pi (L)}(PS(L) / \Gamma, \mathbb{Q}), \quad j \geq 0,
\]
where the degree shift $\pi(L)$ is given by Corollary 4.4. The degree shift for the trivial line bundle $\mathcal{O}_X$ is zero. Using Corollary 3.2 and Proposition 5.1, we get that

$$H^*_CR(P\mathcal{M}_\xi/\Gamma, \mathbb{Q}) = H^*(P\mathcal{M}_\xi, \mathbb{Q}) \oplus \left( \bigoplus_{L \in \Gamma \setminus \{\mathcal{O}_X\}} H^{*-2\pi(L)}(S(L)/\Gamma, \mathbb{Q})^{\oplus |r|^m} \right).$$

(5.1)

The additive structure on $H^*_CR(P\mathcal{M}_\xi/\Gamma, \mathbb{Q})$ is the unique operation on it which gives the isomorphism of the groups in (5.1). We will now give the product structure ‘$\cdot$’ on it.

Let $k = (r!)^m$. From Section 2, we have that the fixed point locus $PS(L)$ is a smooth compact submanifold of $P\mathcal{M}_\xi$ of real dimension $2(r - 1)(g - 1)$ (see [4, p. 519]) having $k$ connected components which are copies of $S(L) = \text{Prym}_\xi/\text{Gal}(\pi_L)$.

Let $\tilde{\omega}$ denote the $\Gamma$-invariant differential form on $PS(L)$ which is the pullback of the differential form $\omega$ on $PS(L)/\Gamma$. Then

$$\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_k),$$

where $\tilde{\omega}_j$ is a differential form on the $j$-th copy of $S(L)$ in $PS(L)$.

Recall that the orbifold integration of a $2(r - 1)(g - 1)$-form $\omega$ on $PS(L)/\Gamma$ is defined as

$$\int_{PS(L)/\Gamma} \omega := \frac{1}{|\Gamma|} \int_{PS(L)} \tilde{\omega} = \frac{1}{|\Gamma|} \sum_{j=1}^k \int_{S(L)} \tilde{\omega}_j,$$

where $|\Gamma| = r^{2g}$ is the order of the group $\Gamma$.

The real dimension of $P\mathcal{M}_\xi/\Gamma$ is $2d = 2(r^2 - 1)(g - 1) + m(r^2 - r)$. Let $L \in \Gamma$. For any $\delta_L = (\delta_1, \ldots, \delta_k) \in H^{n-2\pi(L)}(S(L)/\Gamma, \mathbb{Q})^{\oplus k}$ and $\beta_L = (\beta_1, \ldots, \beta_k) \in H^{2d-n-2\pi(L)}(S(L)/\Gamma, \mathbb{Q})^{\oplus k}$, we define

$$\langle \delta_L, \beta_L \rangle^L = \sum_{i=1}^k \int_{S(L)/\Gamma} \delta_i \wedge \beta_i.$$

**DEFINITION 5.2**

For any integer $0 \leq j \leq 2d$, the Chen–Ruan Poincaré pairing

$$\langle -, - \rangle_{CR} : H^j_{CR}(P\mathcal{M}_\xi/\Gamma, \mathbb{Q}) \times H^{2d-j}_{CR}(P\mathcal{M}_\xi/\Gamma, \mathbb{Q}) \rightarrow \mathbb{Q}$$

is defined by

$$\langle \delta, \beta \rangle_{CR} = \sum_{L \in \Gamma} \langle \delta_L, \beta_L \rangle^L$$

for all $\delta = (\delta_L)_{L \in \Gamma} \in H^j_{CR}(P\mathcal{M}_\xi/\Gamma, \mathbb{Q}) = \bigoplus_{L \in \Gamma} H^{j-2\pi(L)}(PS(L)/\Gamma, \mathbb{Q})$ and $\beta = (\beta_L)_{L \in \Gamma} \in H^{2d-j}_{CR}(P\mathcal{M}_\xi/\Gamma, \mathbb{Q}) = \bigoplus_{L \in \Gamma} H^{2d-j-2\pi(L)}(PS(L)/\Gamma, \mathbb{Q})$.

Let $L_1, L_2 \in \Gamma$, and define $L_3 = (L_1 \otimes L_2)^\vee$. Also let $\tilde{T} = \cap_{i=1}^3 PS(L_i)$ and

$$\tilde{e}_i = \tilde{T}/\Gamma \rightarrow PS(L_i)/\Gamma, \ i = 1, 2, 3.$$
be the canonical injections.

Let \( \tilde{F}_{L_1, L_2} \) be the orbifold obstruction bundle on \( \tilde{T} / \Gamma \) (see [4, Section 4]) and \( c_{\text{top}}(\tilde{F}_{L_1, L_2}) \) its top Chern class. Note that

\[
\text{rank}(\tilde{F}_{L_1, L_2}) = \dim_{\mathbb{R}} \tilde{T} - \dim_{\mathbb{R}} P.M_{\xi} + \sum_{j=1}^{3} \pi(L_j).
\]

Then, for any \( \delta \in H_{CR}^p(PS(L_1)/\Gamma, \mathbb{Q}) \) and \( \beta \in H_{CR}^q(PS(L_2)/\Gamma, \mathbb{Q}) \), we define \( \delta \cup \beta \in H_{CR}^{p+q}(PS(L_3)/\Gamma, \mathbb{Q}) \) by the relation

\[
\langle \delta \cup \beta, \psi \rangle_{CR} = \int_{\tilde{T} / \Gamma} \tilde{e}_1^*(\delta) \wedge \tilde{e}_2^*(\beta) \wedge \tilde{e}_3^*(\psi) \wedge c_{\text{top}}(\tilde{F}_{L_1, L_2})
\]

for all \( \psi = (\psi_1, \ldots, \psi_k) \in H_{CR}^{2d-p-q}(PS(L_3)/\Gamma, \mathbb{Q}) \). Extending the product ‘\( \cup \)’ by \( \mathbb{Q} \)-linearity we get the product structure on \( H^*(P.M_{\xi}, \mathbb{Q}) \) turning it into a ring.

Let \( T = \bigcap_{i=1}^{3} S(L_i) \). This space is described in [4]. The space \( \tilde{T} \) is \( k \) copies of \( T \). The \( \Gamma \)-equivariant morphism \( \gamma : P.M_{\xi} \longrightarrow M_{\xi} \) maps \( \tilde{T} \) to \( T \). Let \( \tilde{F}_{L_1, L_2} \) be the obstruction bundle on \( \tilde{T} / \Gamma \). Thus we have

\[
\tilde{F}_{L_1, L_2}|_{T / \Gamma} \cong F_{L_1, L_2},
\]

where we have identified any connected component of \( \tilde{T} / \Gamma \) with \( T / \Gamma \) using the isomorphism between them given by \( \gamma \). It follows that

\[
\langle \delta \cup \beta, \psi \rangle_{CR} = \sum_{i=1}^{k} \langle \delta_i \cup \beta_i, \psi_i \rangle_{CR},
\]

where the pairing on the right-hand side of the equation is the non-degenerate bilinear Poincaré pairing for the Chen–Ruan cohomology on \( M_{\xi} / \Gamma \) (see [5, 6.20]). The Chen–Ruan product \( \delta_i \cup \beta_i \) for the orbifold bundle \( M_{\xi} / \Gamma \) is computed in [5]. We also have

\[
\delta \cup \beta = (\delta_1 \cup \beta_1, \ldots, \delta_k \cup \beta_k).
\]

Moreover, if \( L_1 = L_2 = O_X \), then the Chen–Ruan product \( \cup \) is the ordinary cup product on \( H^*(P.M_{\xi}, \Gamma, \mathbb{Q}) \).

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