An Atiyah-Singer Theorem for gerbes.

Introduction.

This paper has been motivated by the following problem: Let \( M \) be a compact riemannian manifold, the curvatures are very useful to study the topology of \( M \). If the manifold \( M \) is spin, the study of the bundles of spinors provides results in this way. Unfortunately every compact manifold is not spin. The obstruction to the existence of a spin—structure on \( M \) is the second Stiefel-Whitney class \( w_2(M) \) of \( M \). Nevertheless the class \( w_2(M) \) is the classifying cocycle associated to an \( \mathbb{Z}/2 \) gerbe on \( M \) that we call the spin gerbe, which is according to Brylinski, and Mc Laughlin, an illuminating example of gerbe. The objects of this gerbe are naturally endowed with a riemannian metric, invariant by the automorphisms of the gerbe. It is natural to think that the study of this spin gerbe can have topological applications. For example one may expect to generalize the Lichnerowicz theorem. On this purpose, we need first to prove an Atiyah-Singer type theorem for gerbes, which is our purpose.

1. On the notion of vectorial gerbes.

The aim of this section is to develop the notion of vectorial gerbes.

Definition 1.

Let \( M \) be a manifold, a sheaf \( S \) of categories on \( M \), is a map \( U \to S(U) \), where \( U \) is an open set of \( M \), and \( S(U) \) a category which satisfies the following properties:

- To each inclusion \( U \to V \), there exists a map \( r_{U,V} : S(V) \to S(U) \) such that \( r_{U,V} \circ r_{V,W} = r_{U,W} \).
- Gluing conditions for objects,

Consider a covering family \((U_i)_{i \in I}\) of an open set \( U \) of \( B \), and for each \( i \), an object \( x_i \) of \( S(U_i) \), suppose that there exists a map \( g_{ij} : r_{U_i \cap U_j, U_i}(x_i) \to r_{U_i \cap U_j, U_j}(x_j) \) such that \( g_{ij}g_{jk} = g_{ik} \), then there exists an object \( x \) of \( C(U) \) such that \( r_{U_i,U}(x) = x_i \).

Gluing conditions for arrows,

Consider two objects \( P \) and \( Q \) of \( S(M) \), then the map \( U \to \text{Hom}(r_{U,M}(P), r_{U,M}(Q)) \) is a sheaf.

Moreover, if the following conditions are satisfied the sheaf of categories \( S \) is called a gerbe

\[G1\]
There exists a covering family \((U_i)_{i \in I}\) of \(M\) such that for each \(i\) the category \(S(U_i)\) is not empty.

\(G2\)

Let \(U\) be an open set of \(M\), for each objects \(x\) and \(y\) of \(U\), there exists a covering family \((U_i)_{i \in I}\) of \(U\) such that \(r_{U_i, U}(x)\) and \(r_{U_i, U}(y)\) are isomorphic.

\(G3\)

Every arrow of \(S(U)\) is invertible, and there exists a sheaf \(A\) in groups on \(M\), such that for each object \(x\) of \(S(U)\), \(\text{Hom}(x, x) = A(U)\), and the elements of this family of isomorphisms commute with the restriction maps.

The sheaf \(A\) is called the band of the gerbe \(S\), in the sequel, we will consider only gerbes with commutative band.

**Notation.**

For a covering family \((U_i)_{i \in I}\) of \(B\), and an object \(x_i\) of \(S(U_i)\), we denote by \(x^1_{i_1 \cdots i_n}\) the element \(r_{U_i \cap \cdots \cap U_{i_n}, U_i}(x_i)\), and by \(U_{i_1 \cdots i_n}\) the intersection \(U_{i_1} \cap \cdots \cap U_{i_n}\).

**Definition 2.**

A gerbe is a vectorial gerbe if and only if for each open set \(U\), the category \(S(U)\) is a category of vector bundles over \(U\) with typical fiber the vector space \(V\) and maps between objects are isomorphisms of vector bundles. The vector space \(V\) will be called the typical fiber of the vectorial gerbe. More precisely, there exists a covering family \((U_i)_{i \in I}\) of \(M\), a commutative subgroup \(H\) of \(\text{Gl}(V)\), such that there exist maps \(g'_{ij} : U_i \cap U_j \rightarrow \text{Gl}(V)\), which define isomorphisms

\[
g_{ij} : U_i \cap U_j \times V \rightarrow U_i \cap U_j \times V
\]

\[
(x, y) \rightarrow (x, g_{ij}(x)y)
\]

such that \(c_{ijk} = g_{ij}g_{jk}g_{ki}\) is an \(H\)-2-Cech cocycle.

**Examples.**

**The Clifford gerbe associated to a riemannian structure.**

Let \(M\) be a \(n\)-riemannian manifold, \(O(M)\) the reduction of the bundle of linear frames which defines the riemannian structure of the manifold \(M\). The bundle \(O(M)\) is a locally trivial principal bundle over \(M\) which typical fiber is \(O(n)\). There is an exact sequence \(1 \rightarrow \mathbb{Z}/2 \rightarrow \text{Spin}(n) \rightarrow O(n) \rightarrow 1\), where \(\text{Spin}(n)\) is the universal cover of \(O(n)\). We can associate to this problem a gerbe which band is \(\mathbb{Z}/2\) and such that for each open set \(U\), \(\text{Spin}(U)\) is the category of \(\text{Spin}(U)\) bundles over \(U\) such that the quotient of each of its elements by \(\mathbb{Z}/2\) is \(O(U)\), the restriction of \(O(M)\) to \(U\). The classifying cocycle of this gerbe is the second Stiefel-Whitney class.

One can associate to this gerbe a vectorial gerbe named the Clifford gerbe \(Cl(M)\). For each open set \(U\) of \(M\), \(Cl(U)\) is the category which objects are Clifford bundles associated to the objects of \(\text{Spin}(U)\). The gerbe \(Cl(M)\) is a vectorial gerbe.

Let \(g'_{ij}\) be the transition functions of the bundle \(O(M)\), for each map \(g_{ij}\), consider an element \(g_{ij}\) over \(g'_{ij}\) in \(\text{Spin}(n)\). The the element \(g_{ij}(x)\) acts on
by left multiplication, we will denote by $h_{ij}(x)$ the resulting automorphism of $Cl(\mathbb{R}^n)$. The Clifford gerbe is thus defined by $h_{ij} : U_i \cap U_j \rightarrow Spin(n)$.

The gerbe defined by the lifting problem associated to a vectorial bundle.

Consider a vector bundle $E$ over $M$ which typical fiber is the vector space $V$. One can associate to $E$, a principal $Gl(V)$ bundle. We suppose that this bundle has a reduction $E_K$ where $K$ is a subgroup of $Gl(V)$. Consider a central extension $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$. This central extension defines a gerbe $C_H$ on $M$, such that for each open set $U$ of $M$, the objects of $C_H(U)$ are $G$–principal bundles over $U$ which quotient by $H$ is the restriction of $E_K$ to $U$. We denote by $\pi$ the projection $\pi : G \rightarrow K$.

Suppose moreover defined a representation $r : G \rightarrow Gl(W)$, and a surjection $f : W \rightarrow V$ such that the following square is commutative

$$
\begin{array}{ccc}
W & \xrightarrow{r(h)} & W \\
\downarrow f & & \downarrow f \\
V & \xrightarrow{\pi(h)} & V
\end{array}
$$

then one can defined the vectorial gerbe $C_{H,W}$ on $M$ such that the object of $C(U)$ are $e_U \propto r$, where $e_U$ is an object of $C_H(U)$. Let $(U_i)_{i \in I}$ be a trivialization of $E$ by the transitions functions $g'_{ij}$, we consider a map $g_{ij} : U_i \cap U_j \rightarrow G$ over $g'_{ij}$. The gerbe $C_{H,W}$ is defined by $r(g_{ij})$.

Let $T$ be the set of elements of $W$ fixed by elements of $H$. The action of $G$ on $W$ defines an action of $K$ on $T$. This action defines a vector bundle $S_T$ on $M$ with typical fiber is $T$. Let $e_U$ be an object of the category $C_{H,W}$. then the restriction of $S_T$ to $U$ is the set of elements of $e_U$ invariant by $H$.

Definition 3.
- A riemannian metric on a vectorial gerbe $C$ is defined by the following data:
  - For each objects $e_U$ of $C$, a riemannian metric $<,>$ on the vector bundle $e_U$ which is preserved by morphisms of between objects of $C(U)$. We remark that the band need to be contained in a compact group in this case since it preserves the riemannian metric.

An example of a scalar product on a gerbe is the following: consider the Clifford gerbe $Cl(M)$, we know that the group $Spin$ is a compact group, its action on $Cl(\mathbb{R}^n)$ preserves a scalar product. This scalar product defines on each fiber of an object $e_U$ of $Cl(U)$ a scalar product which defines, a riemannian metric $<,>_e$ on $e_U$. The family of riemannian metrics $<,>_e$ is a riemannian metric defined on the gerbe $Cl(M)$.

Definition 4.
- A global section of a vectorial gerbe associated to a 1–Cech chain $(g_{ij})$ is defined by a covering space $(U_i)_{i \in I}$ of $M$, for each element $i$ of $I$, an object $e_i$ of $C(U_i)$, a section $s_i$ of $e_i$, a family of morphisms $g_{ij} : e_j \rightarrow e_i^s$ such that on $U_{ij}$ we have $s_i = g_{ij}(s_j)$.
Let \((s_i)_{i \in I}\) be an element of \(S(g_{ij})\), on \(U_{ijk}\) we have: \(s_j = g_{jk}(s_k)\), \(s_i = g_{ij}(s_j)\). This implies that \(s_i = g_{ij}g_{jk}(s_k) = g_{ik}(s_k)\). Or equivalently \(g_{ik}^{-1}g_{ij}g_{jk}(s_k) = s_k\). Remark that the restriction of the element \(s_k\) is not necessarily preserved by all the band.

Suppose that \(M\) is compact, and \(I\) is finite. We can suppose that there exists \(i_0\) such that \(T = U_{i_0} - \cup_{i \neq i_0} U_i\) is not empty. Consider a section \(s_{i_0}\) of \(e_{i_0}\) which support is contained in \(T\), then we can define a global section \((u_i)_{i \in I}\) such that \(u_{i_0} = s_{i_0}\), and if \(i \neq i_0, u_i = 0\). This ensures that \(S(g_{ij})\) is not empty.

We will denote by \(S(g_{ij})\) the family of global sections associated to \((g_{ij})_{i,j \in I}\). Remark that \(S(g_{ij})\) is a vector space.

**Proposition 5.**

Suppose that the vectorial gerbe \(C\) is the gerbe associated to the lifting problem defined by the extension \(1 \to H \to G \to K \to 1\) and the vector bundle \(E\). Let \(\rho : G \to GL(W)\) be a representation, we suppose that the condition of the previous diagram is satisfied. Then each for \(G\)–chain \(g_{ij}\), each element \((s_i)_{i \in I}\) of the vector space of global sections \(S(g_{ij})\), satisfies the following condition: there exists a section \(s\) of \(E\), such that \(s|_{U_i} = f \circ s_i\).

**Proof.**

Let \((s_i)_{i \in I}\) be a global section associated to the chain \(S(g_{ij})\), then on \(U_{ij}\), we have \(s_i = g_{ij}(s_j)\), this implies that on \(U_{ij}\), \(f(s_i) = f(s_j)\). Thus the family \((f(s_i))_{i \in I}\) of local sections of \(E\) defines a global section \(s\) of \(E\).

**Remark.**

Let \(s\) be a section of the bundle \(E\), locally we can define a family of sections \(s_i\) of \(e_i\), such that \(f(s_i) = s|_{U_i}\). We can consider the chain \(s_{ij} = s_i - g_{ij}(s_j)\). We have \(s_{jk} - s_{ik} + s_{ij} = s_{ijk}\) is 2–cocycle. Whenever there exists a chain \(g_{ij}\) a global section \(s = (s_i)_{i \in I}\) such that \(s_i = g_{ij}(s_j)\), and \(f(s_i) = s|_{U_i}\), it is not sure that such a global section exists for another chain \(h_{ij}\). This motivates the following definition:

**Definition 6.**

We will define the vector space \(S\) of formal global sections of the vector gerbe \(C\), the vector space which generators are \([s]\) where \(s\) is an element of a set of global sections \(S(g_{ij})\).

The elements of \(S\), are formal finite sum of global sections.

**The Prehilbertian structure of \(S(g_{ij})\).**

First we remark that \(S(g_{ij})\) is a vector space. Let \(s\) and \(t\) be elements of \(S(g_{ij})\), we will denote by \(s_i\) and \(t_i\) the sections of \(e_i\) which define respectively the global sections \(s\) and \(t\). We have \(s_i = g_{ij}(s_j)\) and \(t_i = g_{ij}(t_j)\) this implies that \(as_i + bt_i = g_{ij}(as_j + bt_j)\) where \(a\) and \(b\) are real numbers.

- The scalar structure of \(S(g_{ij})\).
Let \((V_k, f_k)_{k \in K}\) be a partition of unity subordinate to \((U_i)_{i \in I}\), this means that for each \(k\) there exists an \(i(k)\) such that \(V_k\) is a subset of \(U_{i(k)}\). Since the support of \(f_k\) is a compact subset of \(V_k\), we can calculate \(\int <s_{ii(k)}, t_{ii(k)}>\) where \(s_{ii(k)}\) and \(t_{ii(k)}\) are the respective restrictions of \(s_{i(k)}\) and \(t_{i(k)}\) to \(V_k\), remark that since we have supposed that \(s_i = g_{ij}(s_j)\) and \(g_{ij}\) is a riemannian isomorphism between \(\theta^j_i\) and \(\theta^j_i\), if \(V_k\) is also included in \(U_j\), then \(<s_{ii(k)}, t_{ii(k)}> = <s_{jj(k)}, t_{jj(k)}>\) on \(V_k\). We can define \(<s, t> = \sum_k \int <f_k s_{ii(k)}, f_k t_{ii(k)}>\).

We will denote by \(L^2(S(g_{ij}))\) the Hilbert completion of the Pre Hilbert structure of \((S(g_{ij}), <, >)\).

**The scalar structure on set of formal global sections \(S\).**

Let \(s\) and \(t\) be two formal global sections, we have \(s = [s_{n_1}] + .. + [s_{n_p}]\), and \(t = [t_{m_1}] + .. + [t_{m_q}]\), where \(s_{n_i}\) and \(t_{m_j}\) are global sections.

We will define a scalar product on \(S\) as follows: if \(s\) and \(t\) are elements of the same set of global sections \(S(g_{ij})\), \(<s, t> = <s, t>_{(g_{ij})}\). If \(s\) and \(t\) are not elements of the same set of global sections, then \(<s, t> = 0\).

**Proposition 7.**

An element of \(L^2(S(g_{ij}))\) is a family of \(L^2\) sections \(s_i\) of \(e_i\) such that \(s_i = g_{ij}(s_j)\).

**Proof.**

Let \((s^l)_{l \in N}\) be a Cauchy sequence of \((S(g_{ij}), <, >)\). We can suppose that the open sets \(V_k\) used to construct the riemannian metric are such that the restriction \(e^k_i\) of \(e_i\) to \(V_k\) is a trivial vector bundle. The sequence \((f_k s^l_{ii(k)})\) is a Cauchy sequence defined on the support \(T_k\) of \(f_k\). Since this support is compact, we obtain that \((f_k s^l_{ii(k)})_{l \in N}\) goes to an \(L^2\) section \(s_{ii(k)}\) of \(e^k_i\). We can define \(s_i = \sum_k \sum_{V_k \cap U_i \neq \emptyset} f_k s_{ii(k)}\). The family \((s_i)_{i \in I}\) defines the requested limit.

Suppose that morphisms between objects commute with laplacian, we can then endow \(S(g_{ij})\) with the prehilbertian structure defined by \(<u, v> = \int <\Delta^*(u), v>\), where \(\Delta^*(u)\) is the global section defined by \(\Delta^*(u)_i = \Delta^*(u_i)\). We will denote by \(H_s(S(g_{ij}))\) the Hilbert completion of this prehilbertian space.

We will define the formal \(s\)–distributional global sections \(H_s(S)\) as the vector space generated by finite sums \([s_1] + .. + [s_k]\) where \(s_i\) is an element of an Hilbert space \(H_s(S(g_{ij}))\).

**Connection on riemannian gerbes and characteristic classes.**

The notion of connection is not well-defined for general vectorial gerbe, nevertheless the existence of a riemannian structure on a vectorial gerbe, \(C\), gives rise to a riemannian connection on each object \(e_U\) of \(C(U)\), this family of riemannian connections will be the riemannian connection of the gerbe \(C\).

Let \((U_i)_{i \in I}\) be an open covering of \(M\), suppose that the objects of \(C(U_i)\) are trivial bundles. The riemannian connection of the object \(e_i\) of \(C(e_i)\) is defined by a \(1\)–form \(w_i\) on \(TU_i\), and the covariant derivative of this connection evaluated to a section \(s_i\) of \(e_i\) is \(ds_i + w_is_i\). The curvature of this connection is the \(2\)–form \(\Omega_i = dw_i + w_i \wedge w_i\).
The $2k^{eme}$ Chern class of $e_i$, $e^{i}_{2k}$ is defined by $\text{Trace}[\left(\frac{1}{2\pi i}\Omega_i\right)^k]$. Let $e'_i$ be another object of $C(U_i)$. There exist isomorphisms $\phi_i : e_i \rightarrow U_i \times V$, and $\phi'_i : e'_i \rightarrow U_i \times V.$, and $g_i : e_i \rightarrow e'_i$. The map $\phi'_i \circ g_i \circ \phi_i^{-1}$ is an automorphism of $U_i \times V$ defined as follows:

$$h_i : (x, y) \rightarrow (x, u_i(x)y)$$

The riemannian $\phi_i^{-1} <,> = \phi'_i^{-1} <,>$ is preserved by $h_i$. We have $\phi_i^{-1*}\Omega_i = u_i\phi'^{-1}_i\Omega'iu_i^{-1}$. This implies that $e^{2k}_{2k} = e^{i}_{2k}$.

There exists an isomorphism between the respective restrictions $e^i_j$ and $e^i_j$ on $U_I$. As above we can show that this implies that the $2k$--Chern classes of $e^i_j$ and $e^i_j$ coincide on $U_i \cap U_j$, and defines a global class $e_{2k}$ on $M$ which is the $2k$-- Chern class of the riemannian gerbe.

We can define the $c(C) = c_1(M) + \ldots + c_n(M)$ the total Chern form of the gerbe, and the form define locally by $ch(C)|_{U_i} = Tr(exp(i\frac{1}{2\pi x} ))$ the total Chern character.

2. Operators on riemannian gerbe.

We begin by recalling the definition of pseudo-differential operators for open sets of $\mathbb{R}^n$ and manifolds.

Let $U$, be an open set of $\mathbb{R}^n$, we denote by $S^m(U)$ the set of smooth functions $p(x, u)$ defined on $U \times \mathbb{R}^n$, such that for every compact set $K \subset U$, and every multi-indices $\alpha$ and $\beta$, we have $\|D^\alpha D^\beta p(x, u)\| < C_{\alpha,\beta,K} (1+\|u\|)^{m-|\alpha|}$.

Let $K(U)$ and $L(U)$ denote respectively the smooth functions with compact support defined on $U$ and the smooth functions on $U$. We can define the map, $P : K(U) \rightarrow L(U)$ such that

$$P(f) = \int p(x, u)\hat{f}(u)e^{i<x,u>} du$$

where $\hat{f}$ is the Fourier transform of $f$.

Definition 1.
An operator on $U$ is pseudo-differential, if it is locally of the above type.

Definition 2.
Let $P$ be a pseudo-differential operator, $(U_i)_{i \in I}$ a covering family of $U$ such that the restriction of $P$ to $U_i$ is defined by $P(f) = \int p_i(x, u)\hat{f}(u)e^{i<x,u>} du$. The operator is of degree $m$ if $\sigma(p_{U_i}) = lim_{t \rightarrow \infty} \frac{p_i(x, tu)}{t^m}$ exists. In this case $\sigma(p)$ is called the symbol of $P$.

Let $E$ be a vector bundle over the riemannian manifold $M$, endowed with a scalar metric. We denote by $K(E)$ and $L(E)$ the respectively set of smooth sections of $E$ with compact support and the set of smooth sections of $E$. An operator on the vector bundle $E$, is a map $P : K(E) \rightarrow L(E)$ such that there exists a covering family $(U_i)_{i \in I}$ which satisfies:

- The restriction of $E$ to $U_i$ is trivial
- The restriction of \( P_i \) of \( P \) to \( U_i \) is a map \( P_i : K(U_i \times V) \to L(U_i' \times V) \)
where \( V \) is the typical fiber of \( E \).

- If we consider charts \( \phi_i \) and \( \psi_i \) such that \( \phi_i(U_i \times V) = \psi_i(U_i' \times V) = U \times \mathbb{R}^n \), then the map \( P_i \) is defined by the matrix \((p_{kl})\) where \( p_{kl} \) define an operator of degree \( m \). More precisely, if \( s' \) is a section of \( E \) over \( U_i \) and \( s = (s_1, ..., s_n) = \phi_i(s') \), we can define \( t_k = \sum_{l=1}^{l=n} \int p_{lk}(x,u)\hat{s}_l(u)e^{i<x,u>}du \), and \( P_i(s') = \psi_i^{-1}(t_1, ..., t_n) \).

Consider \( SM \) the sphere bundle of the cotangent space \( T^*M \) of \( M \), and \( \pi^*E \) the pull-back of \( E \) to \( T^*M \), the symbols defined by \((p_{ij})\) define a map \( \sigma : \pi^*E \to \pi^*E \). Consider now the projection \( \pi_S : SM \to M \), then \( \sigma \) induces a map \( \sigma_S : \pi_S^*E \to \pi_S^*E \).

Let \( s \) be a positive integer we denote by \( H_s^{loc}(M, E) \), the space of distributions sections \( u \) of \( E \) such that \( D(u) \) is a \( L^{2loc} \) section, where \( D \) is any differential operator of order less than \( s \), and by \( H_s^{comp}(M, E) \) the subset of elements of \( H_s^{loc}(M, E) \) with compact support. Remark that if \( M \) is compact, then \( H_s^{loc}(M, E) = H_s^{comp}(M, E) \). We define by \( H_s^{loc}(M, E) \) to be the dual space of \( H_s^{comp}(M, E) \), and by \( H_s^{comp}(M, E) \) the dual space of \( H_s^{loc}(M, E) \).

The Sobolev space \( H_s \) is an Hilbert space endowed with the norm defined by \( (\int |f| < Du, u >)^{1/2} \).
Every operator \( P \) of order less than \( m \) can be extended to a continuous morphism \( H_s \to H_{s-m} \).

**Definition 3.**

Let \( C \) be a riemannian gerbe defined on the manifold \( M \), an operator \( D \) of degree \( m \) on \( C \), is a family of operators \( D_e \) of degree \( m \) defined on \( e \) where \( e \) is an object of the category \( C(U) \). We suppose that for each morphism \( g : e \to f \), \( D_1g^* = g^*D_e \).

**Remark.**

The last condition in the previous definition implies that \( D_e \) is invariant by the automorphisms of \( e \). The map \( g^* \) is the map which transforms a section \( s \) to \( g(s) \). The operator considered in the sequel will be assumed to be continue. It defines a map \( D_e : H_s^{comp}(U, e) \to H_{s-m}^{loc}(U, e) \).

**Proposition 4.** Let \( D \) be an operator of degree \( m \) defined on the riemannian gerbe \( C \), then \( D \) induces a map \( D_{\pi(g_{ij})} : H_s(S(g_{ij})) \to H_{s-m}(S(g_{ij})) \) and a map \( D_S : H_s(S) \to H_{s-m}(S) \).

**Proof.**

Consider a global distributional section \( s \) which is an element of \( H_s(S_{g_{ij}}) \), we have \( g_{ij}(D_{e_i}(s_i)) = D_{e_j}(s_j) \). This implies the result.

In the sequel we will consider only pseudo-differential operators that preserve \( C^\infty \) sections.

**The symbol of an operator.**

Let \( C \) be a vectorial gerbe defined on \( M \) endowed with the operator \( D \) of degree \( m \), for each object \( e \) of \( C(U) \), we can pulls back the bundle \( e \) by the
projection map $\pi_{SU}: SU \to U$ to a bundle $\pi^*_{SU}e$ over $SU$, where $SU$ is the restriction of the cosphere bundle defined by a fixed riemannian metric of $T^*M$.

The family $C_\pi(U)$ which elements are $\pi^*_{SU}e$ is a category. The maps between objects of this category are induced by maps between elements of $C(U)$. The map $U \to C_\pi(U)$ is a gerbe which has the same band than $C$.

Now on the object $e$, we can define symbol $\sigma_{De}: \pi^*_{SU}e \to \pi^*_{SU}e$.

Remark that for every automorphism $g$ of $e$, the fact that $g^* \circ De = De \circ g^*$ implies that $\sigma_{gDe} g^{-1} = \sigma_{De}$.

**Proposition 5. Rellich Lemma for the family $S(g_{ij})$.**

Let $f_n$ be a sequence of elements of $H_s(S(g_{ij}))$, we suppose that there is a constant $L$ such that $\|f_n\|_s < L$, then for every $s > t$, there exits a subsequence $f_{nk}$ which converges in $H_t$.

**Proof.**

Let $(s_n)$ be a sequence of sections which satisfy the condition of the proposition, and $(V_\alpha, f_\alpha)$ a partition of unity subordinate to $(U_i)_{i \in I}$. We suppose that the support of $f_\alpha$ is a compact space $K_\alpha$. We denote by $s^i_n$ the section of $e_i$ which defines $s_n$, and by $s^i_\alpha$ the restriction of $s^i_n$ to the restriction of $e_i$ to $V_\alpha$. The family $(f_\alpha s^i_\alpha)$ goes to the element $s_\alpha$ in $V_\alpha$ by the classical Rellich lemma. We can write then $s^i = \sum_{V_\alpha \subset U_i} f_\alpha s^i_\alpha$. this is an $H_t$ map since the family of $V_\alpha$ can be supposed to be finite, since $M$ is compact. The family $(s^i)$ define a global $H_s$ section which is the requested limit.

**Remark.**

A compact operator between Hilbert space is an operator which transforms bounded spaces to compact spaces. The previous lemma implies that if $s > t$, then the inclusion $H_s(S(g_{ij})) \to H_t(S(g_{ij}))$ is compact, since as $H_s(S(g_{ij}))$ is a separate space, a compact subspace of $H_s(S(g_{ij}))$ is a set such that we can extract a convergent sequence from every bounded sequence.

**Proposition 6.**

The space $Op(C)$ of continuous linear maps of $H_s(S(g_{ij}))$ is a Banach space.

**Proof.**

Let $D_n$ be a Cauchy sequence of elements of $Op(C)$, for each global section $s$, the sequence $D_n(s)$ is a Cauchy sequence in respect to the norm of $H_s$, we conclude that it goes to an element $D(s)$. The map $D: s \to D(s)$ is the requested limit. It is bounded since $(D_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

The previous Lemma allows us to define $O^m$, the completion of the pseudo-differential operators in $OP(C)$ of order $m$, and to extend the symbol $\sigma$ to $O^m$.

Now we will show that the kernel of the extension of the symbol to $O^m$ contains only compact operators.

**Definition 7.**

We say that an operator is elliptic if the family of symbol $\sigma_{De}$ are invertible maps.
Proposition 8.
The kernel of the f of the symbol map contains only compact operators.

Proof.
The symbol $\sigma(P)$ of the operator $P$ is zero if and only if the order $m$ of the operator is less than $-1$. This implies that the operator $P$ is compact. To see it, we can suppose our operator to be an $L^2(S(g_{ij}))$ operator, by composing it by the inclusion map $H_{2-s}(S(g_{ij})) \to L^2(S(g_{ij}))$, we conclude by using the previous Rellich lemma.

3. $K$–theory and the index.

In this part we will give the definitions of the $K$–theory groups $K_0$, and $K_1$, and show how we can use them to associate to a symbol of an operator on a gerbe, an element of $K_0(T^*M)$.

We will denote by $M_n$ the vector space of $n \times n$ complex matrix. For $n \leq m$, we consider the natural injection $M_n \to M_m$. We will call $M_\infty$, the inductive limit of the vector spaces $M_n, n \in \mathbb{N}$.

Let $R$ be a ring and $p$, and $q$ be two idempotents of $R_\infty = R \otimes M_\infty$, we will say that $p \sim q$ if and only if there exists elements $u$ and $v$ of $R_\infty$ such that $p = uv$, and $q = vu$. We denote by $[p]$ the class of $p$, and by $Idem(R_\infty)$ the set of equivalence classes.

If $[p]$ and $[q]$ are represented respectively by elements of $R \otimes M_n$ and $R \otimes M_m$, we can define an idempotent of $R \otimes M_{n+m}$ represented by the matrix $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = [p + q]$.

Definition 1.
We will denote by $K_0(R)$, the semi-group $Idem(R_\infty)$, endowed with the law $[p] + [q] = [p + q]$.

Let $M$ be a compact manifold, and $C(M)$ the set of complex valued functions on $M$. It is a well-known fact that for a complex vector bundle $V$ on $M$, there exists a bundle $W$, such that $V \oplus W$ is a trivial bundle isomorphic to $M \times L^k$. We can thus identify a vector bundle over $M$ to an idempotent of $C(X) \otimes M_k$ which is also an idempotent of $C(X)_\infty$. This enables to identify $K_0(M)$ to $K_0(C(X))$.

In fact the semi-group $K(R)$ is a group.

Let $Gl_k(R)$ be the group of invertible elements of $M_k(R)$, if $l \leq k$ we have the canonical inclusion map $Gl_l(R) \to Gl_k(R)$. We will denote by $Gl_\infty(R)$, the inductive limit of the groups $Gl_k(R)$.

Definition 2.
Let $Gl_\infty(R)_0$ be the connected component of $Gl_\infty(R)$. We will denote by $K_1(R)$ the quotient of $Gl_\infty(R)$ by $Gl_\infty(R)_0$.

For a compact manifold $M$, we define $K_1(M)$ by $K_1(C(X))$. 

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Consider now an exact sequence $0 \to R_1 \to R_2 \to R_3 \to 0$ of $C^*$ algebras, we have the following exact sequences in $K-$theory:

$$K_1(R_1) \to K_1(R_2) \to K_1(R_3) \to K_0(R_1) \to K_0(R_2) \to K_0(R_3)$$

Let $\mathcal{H}$ be an Hilbert space, we denote by $B(\mathcal{H})$ the space of continuous operators defined on $\mathcal{H}$, and $\mathcal{K}$ the subspace of compact continuous operators. We have the following exact sequence:

$$0 \to \mathcal{K} \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K} = Ca \to 0.$$ 

It is a well-known fact that $K_0(\mathcal{K}) = \mathbb{Z}$.

Let $M$ be a riemannian manifold, and $C$ a riemannian gerbe defined on $M$. Consider an elliptic operator $D$ defined on $C$, of degree $l$. The operator $D$ induces a morphism: $D : L^2(S(g_{ij})) \to H_{2-l}(S(g_{ij}))$. Consider the operator $(1 - \Delta)^{-m}$ of degree $-l$. The operator $D' = (1 - \Delta)^{-m}D$ is a morphism of $L^2(S(g_{ij}))$. The symbol of $(1 - \Delta)^{-m}D = \sigma(D)$. This implies that the image of the operator $D$ in the Calkin algebra $L^2(S(g_{ij}))$ is invertible. It thus define a class $[\sigma(D')]$ of $K_1(Ca)$. The image of $[\sigma(D')]$ in $K_0(\mathcal{K})$ is the index of $D$. We remark that the index of the operator depends only of the symbol.

For every object $e$ of $C(U)$, the symbol $\sigma(D_e)$ is an automorphism of $\pi_{SU}e$, it defines an element $[\sigma(D_e)]$ of $K_1(C(S(U)))$ (recall that $S(U)$ is the cosphere bundle over $U$ defined by the riemannian metric).

**Remark.**

Let $U$ be an open set such that the objects of $C(U)$ are trivial bundles. Consider an object $e$ of $C(U)$, and a trivialization map $\phi_e : e \to U \times V$. For every object $f$ of $C(U)$, we have $\phi_e^{-1*}(\sigma_{D_e}) = \phi_f^{-1*}(\sigma_{D_f})$.

**Proposition 3.**

Let $C$ be a vectorial gerbe defined over the compact manifold $M$, there is a trivial complex bundle $f_m = M \times \mathcal{E}^m$ such that each object of $C(U)$ where $U$ is an open set of $M$ is isomorphic to a subbundle of the restriction of $f_m$ to $U$.

**Proof.**

Let $(U_i)_{i \in I}$ be a finite covering family such that for each $i$ the objects of the category $C(U_i)$ are trivial bundles. Let $e_{U_i}$ be an object of $C(U_i)$, the restriction $e_i$ of $e$ to $U_i$ is a trivial vectorial bundle. We consider a fixed object $e_0$ of $C(U_i)$. Consider a finite partition of unity $(f_p)_{p \in 1...l}$ subordinate to the covering family $(U_i)$, and $g_i : e_0 \to V$ the composition of the trivialization and the second projection. Let $h_i : e_i \to e_0^0$, we can define the map $k : e_U \to \mathcal{L}^{dimV}$ such that $k(x) = (f_1(x)(x)_1h_1(x),...f_i(x)(x)_i h_i(x))$. $k$ induces a map $K : M \to G_m(\mathcal{E}^N)$, where $m = dimV$ and $N = ldim(V)$. $e_U$ is the pull-back of the canonical $m-$ vector bundle over $G_m(\mathcal{E}^N)$. We remark that its a subbundle of the pull-back of the trivial bundle $G_m(\mathcal{E}^N) \times \mathcal{E}^N$.

The orthogonal bundle of $e_U$ in the previous result can be chosen canonically by considering the orthogonal bundle of the canonical $\mathcal{E}^m$ bundle over $G_m(\mathcal{E}^N)$. 

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in $G_m(L^N) \times L^N$. We will suppose that this bundle is chosen canonically in the sequel.

**Proposition 4.**

Let $C$ be a riemannian gerbe defined over the compact manifold $M$. Then we can associate naturally to the symbol of the elliptic operator $D$ a class $[\sigma_D]$ in $K_1(T^*M)$.

**Proof.**

Let $(U_i)_{i \in I}$, a finite covering family of $M$ such that for each $i$, each object of $e_i$ is a trivial bundle. Consider for each $i$ a trivialization $\phi_i : e_i \to U_i \times V$. The map $\phi_i^{-1}(\sigma_{D_{e_i}})$ is an automorphism of the bundle $\phi_i^{-1}(\pi_{S^*(e)}) = S(e)(U_i)$.

This enables to extend $\phi_i^{-1}(\sigma_{D_{e_i}})$ to a morphism of the restriction of $f_m$ to $U_i$, completing by $1$ on the diagonal since the orthogonal of $e_U$ is chosen canonically. It results a morphism $\sigma'_{D}$ of $C(M) \otimes L^m$, that is as an element of $C(S^* M) \otimes \text{Gl}(V)$.

This element defines the requested element $[\sigma'_{D}]$ of $K_1(C(S^* M) \otimes M_k) \simeq K_1(C(S^* M))$.

Let $B^* M$ be the compactification of $T^* M$ with fibers isomorphic to the unit ball. We consider the bundle $B^* M/T^* M$ that we can identify to the sphere bundle $S^* M$. We have the following exact sequence

$$0 \to C_0(T^* M) \to C(B^* M) \to C(S^* M) \to 0.$$ 

This sequence gives rise to the following exact sequence in $K-$theory:

$$K_1(C(S^* M) \otimes M_k) \to K_0(C(T^* M) \otimes M_k) \to K_0(C(B^* M) \otimes M_k) \to K_0(C(S^* M) \otimes M_k) \to 0.$$ 

We can define the boundary $\delta([\sigma'_{D}])$ which is an element of $K_0(T^* M) \otimes M_k) \simeq K_0(T^* M)$.

**Proposition 5.**

The index of $D$ depends only of the class of $\delta([\sigma'_{D}])$ in $K_0(T^* M)$.

**Proof.**

We remark that the kernel of the image of a symbol by the map $K_1(S^* M \otimes M_k) \to K_0(T^* M \otimes M_k)$ is zero, if it is the restriction of a map defines on $B^* M$. This implies that this symbol is homotopic to a function which does not depend of the second variable $u$. Thus the operator it defines is homotopic to a multiplication by a constant which index is zero.

E. Meinrenken asked me the following question:

Let $M$ be a manifold which is the union of two open sets $U_1$ and $U_2$ such that there exists objects $e_1$ and $e_2$ of the respective categories $C(U_1)$ and $C(U_2)$. Given operators $D_{e_1}$ and $D_{e_2}$ on $e_1$ and $e_2$ is it possible to associate to these operators an element of the $K-$theory? We do not request any compatibility between $D_{e_1}$ and $D_{e_2}$.
The vectors bundles $e_1$ and $e_2$ are subbundles of the respective trivial bundles $U_1 \times \mathbb{C}^n$ and $U_2 \times \mathbb{C}^k$. Let $SU_1$ and $SU_2$ be the restriction of the sphere bundle of the cotangent space of $M$ to $U_1$ and $U_2$, we can canonically extends the symbols of $D_{e_1}$ and $D_{e_2}$, to respective automorphisms $\sigma_{D_1}$ and $\sigma_{D_2}$ of $F_1 = SU_1 \times \mathbb{C}^k$, and $F_2 = SU_2 \times \mathbb{C}^k$, thus elements of $K_1(C(U_1) \otimes M_k) = K_1(C(U_1))$, and $K_1(C(U_2) \otimes M_k) = K_1(C(U_2))$, where $C(U_1)$ and $C(U_2)$ are respectively the set of differentiable functions of $U_1$ and $U_2$. 

The Mayer-Vietoris sequence for algebraic $K$–theory gives rise to the sequence 

$$K_2(C(S(U_1 \cap U_2)) \to K_1(C(SM)) \to K_1(C(SU_1)) \oplus K_1(C(SU_2)) \to K_1(C((S(U_1 \cap U_2)))$$

if the image of $[\sigma_{D_{i_1}}] + [\sigma_{D_{i_2}}]$ in the previous sequence is zero, then there exists a class $[\sigma_{D}]$ of $K_1(SM)$, which image by the map of the previous exact sequence is the element $[\sigma_{D_{i_1}}] + [\sigma_{D_{i_2}}]$ of $K_1(C(SU_1) \oplus K_1(C(SU_2))$. The class $[\sigma_{D}]$ is not necessarily unique.

**The index formula for operator on gerbes.**

Now, we will deduce an Atiyah-Singer type theorem for riemannian gerbe. We know that the Chern character of the cotangent bundle induces an isomorphism:

$$K_0(T^*M) \otimes \mathbb{R} \to H^*_c\text{even}(M, \mathbb{R})$$

$$x \otimes t \to \text{tch}(x)$$

Consider $\text{Vect(Ind)}$ the subspace of $K_0(T^*M)$ generated by $\sigma_P$, where $P$ is an operator on the riemannian gerbe. It can be considered as a subspace of $H^*_c\text{even}(M, \mathbb{R})$. The map $\text{Vect}(\sigma_P) \to \mathbb{R}$ determined by $\text{ch}(\sigma_P) \to \text{ind}(P)$ can be extended to a linear map $H^*_c\text{even}(M, \mathbb{R}) \to \mathbb{R}$.

The Poincare duality implies the existence of a class $t(M)$ such that

$$\text{Ind}(P) = \int_{T^*M} \text{ch}(\sigma_P) \wedge t(M)$$

4. Applications.

We will apply now this theory to the problem that has motivated is construction.

Let $M$ be a riemannian manifold, consider the Clifford gerbe on $M$.

Let $(U_i)_{i \in I}$ be an open covering of $M$, the riemannian connection $\omega_{\text{Civ}}$ is defined by a family of $\text{so}(n)$ forms $w_i$ on $U_i$ which satisfy $w_j = ad g_{ij}^{-1}w_i + g_{ij}^{-1}dg_{ij}$. The covariant derivative of the Levi-Civita connection is $d + w_i$. We will fixes an orthogonal basis $(e_1, ..., e_n)$ of the tangent space $TU_i$ of $U_i$, and write $w_i = \sum_{k=1}^{n} w_{ik}e_k$.

We can define the spinorial covariant derivative by setting $\phi_{ij} = -\frac{1}{2}w_{ij}$. In the orthogonal basis $(e_1, ..., e_n)$ we have $\phi_{ij} = \sum_{k,l} \phi_{kl}e_k e_l$, with $\phi_{kl} = -\phi_{lk}$.
The Dirac operator.

Let \( e_U \) be an object of \( \text{Cl}(U) \), \( U \) a trivialization of \( \text{Cl}(U) \), we will define 
\[
D_{e_U} = \sum_{k=1}^{n} e_i \nabla_{\text{spin} e_i},
\]
On each object \( e_U \) of \( C(U) \), we have the Lichnerowicz-Weitzenbock formula: 
\[
D^2 = \nabla^* \nabla + \frac{1}{4} s,
\]
where \( s \) is the scalar curvature. In this formula \( \nabla^* \nabla \) is the connection laplacian.

We will say the global spinor is harmonic if \( D_{e_i}(s_i) = 0 \), for each \( s_i \).

**Proposition 1.**

Suppose that the scalar curvature \( s \) of \( M \) is strictly positive, and \( M \) is compact then every harmonic global spinor is 0.

**Proof.**

Let \( \psi \) be an harmonic global spinor, we can represent \( \psi \) by a family of spinor \( s_i \) defined on an open cover \( (U_i)_{i \in I} \) of \( M \), we have on each \( U_i \), \( D_{e_i}(s_i) = 0 \), this implies that \( D_{e_i}^2(s_i) = 0 \), we can write
\[
\int_{U_i} \nabla \nabla^* s_i + \frac{1}{4} s = 0
\]
this implies that \( \int_M s = 0 \), which contradicts the fact that the scalar curvature is strictly positive.

**Corollary 2.** Suppose that the sectional curvature of a compact riemannian manifold is strictly positive then the class \( \tau(M) \) associated to the index formula for operators on the \( \text{Cl}(M) \) gerbe is zero.

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