Granular flows are ubiquitous in the environment and in industry but there are still no known equations for general granular systems. For flow on an inclined plane, however, progress has been made [1, 2, 3, 4], and current theories give a reasonable explanation of uniform flowing states. The simplicity of the inclined-plane geometry plays a crucial role in such descriptions because boundary effects, which can be exceedingly complicated in granular flows, are well defined. This system is also well suited to Discrete Element Method (DEM) simulations [5, 6, 7] because it can be treated periodically and steady flows often result, so good statistics can be obtained. The combination of detailed simulations and experiments has led to a solid understanding of steady flow states in this system where experiments and simulations can be well summarized [2, 6, 8, 9] by a recently proposed model [8]. Little is known, however, about the instabilities of this model or indeed of most granular flows, and the model has only been well tested in simple shear states. In contrast, the Navier-Stokes equation of fluid flow has been known for over a century, and its accuracy has been repeatedly tested by comparing the results of linear and weakly-nonlinear stability analysis to experimental systems displaying an instability from a simple state to one with a distinct pattern [10]. Such an approach, that is, investigating the growth of instabilities from their respective steady states [11, 12], will certainly be very useful in testing granular constitutive models and will provide critical tests for emerging theories of granular flow.

The steady and fully developed state of a rapid, dilute granular flow on a rough inclined plane was shown experimentally to be unstable to the formation of longitudinal vortices observed as lateral stripes [11]. In this pattern the downstream velocity and the layer height vary periodically across the flow consisting of higher-slower and lower-faster regions. The development is attributed to a mechanism analogous to the Rayleigh-Bénard instability in heated liquid layers [13]. The average packing fraction \( \eta_{av} \) of this flow was below 0.3 corresponding to a relative density \( \rho_r = \eta_{av}/\eta_s \approx 0.5 \), where \( \eta_s \) is the static packing fraction (\( \eta_s \) is in between the random-closed-packed and random-loose-packed packing fractions [14]). This low density state, however, is hard to find either numerically or for some materials experimentally. Instead, we show that when increasing the plane inclination angle, the stripe state that emerges naturally is an instability of a dense uniform flow state. This stripe state is robust and easy to find, and that the maxima of the downstream surface velocity correspond to the highest regions of the modulated height profile in qualitative agreement with the flow rule for the uniform state [13].

The flow was analyzed for a wide range of granular materials including different sized sand and glass beads and various copper samples with different particle shape. We demonstrate, using the apparatus illustrated in Fig. 1, that longitudinal stripes are robustly observed over a broad range of flow conditions with relative densities in the range \( 0.2 < \rho_r < 0.95 \) (corresponding to about \( 0.12 < \eta_{av} < 0.57 \)), separated into the dense flow state for \( \rho_r \geq 0.6 \) (i.e. \( \eta_{av} \geq 0.36 \)) and the dilute stripes for lower \( \rho_r \) as described below.

In the present study two experimental setups were used. The first apparatus, described in detail elsewhere [10], was used to characterize flow regimes over a wide range of \( \theta \). Because the system was enclosed in a cylindrical tube, precise measurements of the height pro-

---

**FIG. 1:** Schematic illustration of the experimental setup.
periodic in the ing fraction to normalize the results. The system was to rest with a packing fraction of 0.6. We use this pack-

◦

The instability was found over a range of parameter val-

dimensionalized using the particle diameters and gravity.

was allowed to form randomly. All quantities were non-

force (coefficient of friction 0.5). Particle stiffness was

damping linear spring for the normal force (coefficient of

materials tested.

DE simulations were also performed to investigate the instability. A soft particle model was used with a
damped linear spring for the normal force (coefficient of

 restitution 0.8) and Coulomb friction for the tangential

friction for the tangential force (coefficient of friction 0.5). Particle stiffness was

chosen so that the maximum overlap was less than 1%.

The time step was 1/10 of the binary collision time. The

base was made of identical particles held at fixed posi-

tions. The stresses, particle positions, particle velocities

of [17] using a Dirac delta function as the weight func-

Contact stresses were calculated according to the method

varied between $20 \leq r \leq 36$. In the fol-

lowing, we characterize the stripe structure in the dense

FIG. 2: (a) The phase diagram of the system presenting

the mean flow density $\rho_r$ as a function of normalized plane

inclination $\tan \theta/\tan \theta_r$ and the normalized hopper opening

$H = H/d$ based on data taken for sand, glass beads and vari-

cous copper samples. Various flow regimes are indicated, the

bullet in the dense stripe domain corresponds to the simula-

tion data presented. Illustration of the flow structure in (b)
dense and (c) dilute regimes, gray levels indicate local density.

regimes is characterized by increasing height of the slow

moving region as the plane inclination is increased as il-

ustrated in Fig. 3a-c or in movies taken for various ma-

terials at [18]. The density decreases with increasing $\theta$ in a similar way for all materials as shown in Fig. 2,

where $\rho_r$ is plotted as a function of the normalized hop-

per opening $H = H/d$ and normalized plane inclination

$\tan \theta/\tan \theta_r$. Generally, stripes are only observed for

$\tan \theta/\tan \theta_r > 1.25$. Stripes with the structure typical for the dense regime are observed for $0.6 < \rho_r < 0.95$
(corresponding to $0.36 < \eta_{av} < 0.57$), whereas the dil-

ute regime, when it is observed, exists in the range for

$0.2 < \rho_r < 0.7$ (i.e. $0.12 < \eta_{av} < 0.42$). In the fol-

lowing, we characterize the stripe structure in the dense

plate with dimensions $2 \times 0.4$ m inclined at an angle

$\theta = 41.3^\circ$, see Fig. 1. One layer of the $d = 0.4$ mm sand

grains was glued onto the surface of the glass plate in both setups to provide a random roughness. Measure-

ments were repeated using sandpaper with characteristic

roughness of 0.19 mm (grit 80) and produced very simi-

lar results. The surface velocity of the flow, with down-

stream and transverse components $u^s$ and $v^s$, and the

height profile $h(y)$, determined by a laser sheet, were ob-

tained simultaneously using two cameras (Fig. 1). Using

camera 3, the velocity at the bottom of the layer ($u^b$

and $v^b$) was taken at a location where the glass plate

was clean. The flow velocities were determined using

particle image velocimetry on image sequences taken at

2000 frames/s. The relative density $\rho_r$ was measured us-

ing a method described in detail elsewhere [16]. The

majority of the data presented in this paper was ob-

tained with sorted sand with $d = 0.4 \pm 0.05$ mm and

d = 0.2 \pm 0.05 mm. The stripe state was also detected for

glass beads with $d = 0.18 \pm 0.05$ mm, $d = 0.36 \pm 0.05$ mm, and for four sets of copper particles with similar size

($d = 0.16 \pm 0.03$ mm) but various shapes. The angle

of repose $\theta_r$ varied between $20.9^\circ \leq \theta_r \leq 33.8^\circ$ for the materials tested.

The instability was found over a range of parameter val-

ues: slope angle $34–39^\circ$, restitution $0.80–0.95$ and width

greater than 50. Below we present results from one typi-

cal simulation with a slope angle $37^\circ$. If the slope angle in

the simulations is reduced below $\theta_r$, then the flows come
to rest with a packing fraction of 0.6. We use this pack-
ing fraction to normalize the results. The system was

periodic in the $x$ direction (down-slope length 24.3) and the

$y$ direction (cross-slope width 120.15). The number of

particles simulated was 55 761, so the volume of the par-
inicles over the $xy$ area corresponded to a height of

10 (height at 100% packing fraction). The system was

run for several months until steady state was achieved.

Contact stresses were calculated according to the method

of [17] using a Dirac delta function as the weight func-
tion. The stresses, particle positions, particle velocities

(first and second moments), were grided with spacing

0.1 vertically and 0.45 laterally using linear interpolation.

These were calculated every program step and averaged

over 500 time units.

In the experiments the stripe state for all the mate-

materials tested has qualitatively similar characteristics, and

there is no sharp transition between the states observed

in the dense and dilute flow regimes. Nevertheless, the

flow structure of the dense flow state is quite different

from the dilute flow case reported earlier [11]. The struc-

ture of the dense stripe state consists of relatively-narrow,
dense, fast-moving regions that are also the highest. In

the dilute regime the fast moving region corresponds to a

height minimum, as schematically illustrated in Fig. 2b,c.

In both cases, however, the grains sink in the fastest mov-
ing regions. The continuous transition between the two
regime using experimental data shown in Fig. 3 obtained for sand with \(d = 0.4\) mm and \(d = 0.2\) mm and numerical results shown in Fig. 4 obtained from DEM simulations.

The flow pattern has a downstream surface velocity \(u^s(y)\) and a lateral surface velocity \(v^s(y)\). The downstream velocity has a relatively large modulation of \((u^s_{\text{max}} - u_{\text{av}}^s)/u_{\text{av}}^s \leq 0.2\), whereas the lateral velocity is very slow with maximal value \(v^s_{\text{max}} \leq 0.04 u_{\text{av}}^s\), where \(u_{\text{av}}^s\) denotes the average downstream surface velocity. Cross sections of the velocity of the fully developed state in simulations are shown in Fig. 3a,b showing very similar downstream velocity modulation, but somewhat smaller lateral velocities. In the experiments the lateral velocity measured at the surface \(v^s(y)\) and at the bottom of the layer \(v^b(y)\), is of opposite direction in accord with the flow structure obtained from the simulations (see Fig. 4b).

The experimentally observed height variation in Fig. 3b agrees nicely with the simulation results (Fig. 4d). The fast stripes correspond to a higher narrow maximum of the \(h(y)\) curve but another set of less pronounced maxima is present between them. Thus, instead of a sinusoidal \(h(y)\) profile observed for the dilute regime, a more complex \(h(y)\) is seen where a higher, global maximum corresponds to the fast flowing region and a lower, local maximum corresponds to the slower region. The double peak is also seen in the transmitted light intensity in Fig. 3, but the wavelength of other important measures of the pattern, e.g., the downstream velocity or velocity in the \(yz\) plane (see Fig. 3c and Fig. 4b) do not change during the dilute-dense transition so the emergence of a nonsinusoidal height profile does not correspond to a full frequency-doubling. The pattern is observed only in a finite range of the flow thickness with the strongest amplitude at \(10 < h < 18\). The wavelength of the pattern \(\lambda\) is related to the mean flow thickness \(h_{\text{av}}\) as \(2.8 < \lambda / h_{\text{av}} < 4.5\) as shown in Fig. 3f, so the cross section of a roll is elongated as compared to the traditional Rayleigh-Bénard rolls with nearly circular cross section.

Numerical simulations enable us to visualize spatial variations of the relative density, Fig. 4c, and of the inertial number \(I\) in Fig. 4d. The inertial number, usually defined for incompressible flows [8], can be extended for the compressible case. We define \(I = d \sqrt{\rho} D' / \sqrt{\rho}\), where

\[
D' = E \frac{\rho}{D} (D - \text{Tr}(D) / 3),
\]

\(\rho\) the density, \(p\) the pressure, and we use the norm \(|A| = \sqrt{\text{Tr}(AA^\top) / 2}\). We do not consider normal pressure differences and define \(p = -\text{Tr}(\sigma) / 3\), where \(\sigma\) is the stress tensor and \(\sigma' = \sigma + p\). This is equivalent to

\[
\tilde{u}^s = \frac{u^s}{\sqrt{\rho d}}
\]

as a function of the normalized transverse coordinate \(\tilde{y} = y / d\) for sand with \(d = 0.2\) mm and downstream distance from the outlet \(x = 1.55\) m at plane inclinations (a) \(\theta = 42.6\); (b) \(48.5\); and (c) \(52.2\), corresponding to \(\tan \theta / \tan \theta_o = 1.56, 1.92\) and \(2.19\), respectively. Data obtained at \(x = 2.13\) m for sand with \(d = 0.4\) mm at \(\theta = 41.3\); (d) height profiles \(h(y)\) taken at various hopper openings \(H\), (e) laser-line intensity (exposure time 4 ms), (f) transmitted light intensity, and (g) dimensionless wavelength \(\lambda / h_{\text{av}}\) of the pattern as a function of the normalized mean flow thickness \(h_{\text{av}} = h_{\text{av}} / d\). To adjust \(h_{\text{av}}\) the hopper opening \(H\) was varied.

\[
\tilde{v}^s = \frac{v^s}{\sqrt{\rho d}}
\]

FIG. 3: Images of the flow taken in reflected light (illumination from the right) and normalized downstream surface velocity \(\tilde{u}^s = u^s / \sqrt{\rho d}\) as a function of the normalized transverse coordinate \(\tilde{y} = y / d\) for sand with \(d = 0.2\) mm and downstream distance from the outlet \(x = 1.55\) m at plane inclinations (a) \(\theta = 42.6\); (b) \(48.5\); and (c) \(52.2\), corresponding to \(\tan \theta / \tan \theta_o = 1.56, 1.92\) and \(2.19\), respectively. Data obtained at \(x = 2.13\) m for sand with \(d = 0.4\) mm at \(\theta = 41.3\); (d) height profiles \(h(y)\) taken at various hopper openings \(H\), (e) laser-line intensity (exposure time 4 ms), (f) transmitted light intensity, and (g) dimensionless wavelength \(\lambda / h_{\text{av}}\) of the pattern as a function of the normalized mean flow thickness \(h_{\text{av}} = h_{\text{av}} / d\). To adjust \(h_{\text{av}}\) the hopper opening \(H\) was varied.

\[
\tilde{v}^s = \frac{v^s}{\sqrt{\rho d}}
\]

FIG. 4: Results of the DEM simulations: (a) cross section of the downstream velocity \(\tilde{u}\), (b) speed in the \(yz\) plane (\(\tilde{s} = \sqrt{\tilde{u}^2 + v^2}\)) with streamlines, (c) relative density \(\rho_s = \eta / 0.6\) or packing fraction \(\eta\), (d) the inertial number \(I\), (e) dependence of relative density on \(I\) and best quadratic fit, and (f) dependence of effective friction \(\mu\) on the inertial number \(I\). Solid line is best cubic fit. Dashed line is best fit to Pouliquen model (Eq. 2 in [8]).
the Pouliquen definition when the flow is incompressible ($\text{Tr}(D) = 0$). The inertial number is proportional to the shear rate and to the ratio of the collisional stress to the total stress. The flow has the highest density in the fast moving region where $I$ (and shear rate) is lowest. We identify this region as a “plug” sliding fast on top of a “boiling” region with very low relative density and high inertial number and shear rate, a configuration reminiscent of the Leidenfrost effect [19] when a droplet of liquid lifted by its vapor is hovering above a hot surface. Experimental data visualizing the level of fluidization at the surface (Fig. 3) and the profile of the transmitted light intensity (Fig. 3) fully agree with this picture.

The relative density (or packing fraction) determined from the simulation data decreases monotonically with increasing shear rate, see Fig. 3, and shows an amazing collapse over a wide range of densities and values of the inertial number $I$. This suggests that, at least for fast chute flows, a simple equation of state giving the pressure as a function of shear rate and density, is possible. To test the rheology we calculate the effective friction coefficient $\mu$ by $\mu = -\text{Tr}((\sigma' D')/p D')$. This is the $\mu$ that minimizes the residual error $\sigma' + \mu D'$. Simpler calculations of $\mu$, e.g., $\mu = -\sigma_{zz}/\sigma_{zz}$ or $\mu = -\sigma_{zz}/p$, produce poor results (no collapse would be seen in Fig. 3), due to the complicated strain field. This definition is an extension of the Pouliquen model [8] to include compressible flows. Fig. 4 shows the effective friction coefficient as a function of $I$ and demonstrates a reasonably good collapse. The data does not fit well with Pouliquen rheology and is much more strongly curved and appears to even lapse. The data does not fit well with Pouliquen rheology but at $I = 0.7$ there appears to be a turnover above which $\mu$ decreases with increasing $I$. We believe that this behavior of the system is a key feature leading to the instability. Namely, by increasing the flow thickness above a certain value, the inertial number near the plane reaches a threshold above which the effective friction starts decreasing. As a consequence the local inertial number (shear rate) grows even more, leading to stronger fluidization near the plane. At the same time the fluidization drops in the upper layer (a plug develops) and this plug slides even faster on top of the expanded fluidized region. This mechanism agrees with other results for chute flows [9], but in that case the simulations were too narrow for the lateral instability to develop. On the surface the plug absorbs material from the two sides as the surface fluidization is larger in that region, so the plug grows. In the simulations, as the instability develops the mean flow speed decreases until a new lower velocity is reached at which point the instability has saturated. Thus, the instability may play an important role on steeper slopes, where no simple, steady state is expected, by increasing the effective viscosity. Though we have framed our discussion in terms of $\mu$, it is difficult to draw too many conclusions from these results for a Pouliquen rheology. Our simulation data shows density differences, normal stress difference and that the deviatoric stress tensor is not aligned with the strain tensor. These all disagree with the assumptions of the Pouliquen rheology indicating that a considerably more complicated rheology is necessary along with an equation of state to describe these flows.

This system provides a very interesting case for studying granular rheologies because there is a complicated strain and stress field but very simple boundary conditions. Since the flow is steady, accurate measurements of all the flow variables in a simulation are possible, the only constraint being computer time. This system is therefore ideal for developing and validating granular theories in new ways. An intriguing possibility is that the lateral ridges and furrows observed in large rock avalanches [20] may be the result of the same instability.

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[1] R.A. Bagnold, Proc. Roy. Soc. London, Ser. A 225, 49 (1954); 205, 219 (1966).
[2] GDR MiDi, Eur. Phys. J. E 14, 341 (2004).
[3] C.S. Campbell, Powder Tech. 162, 208 (2006); I.S. Aranson and I.S. Tsimring, Rev. Mod. Phys. 78, 641 (2006).
[4] S.B. Savage, J. Fluid Mech. 92, 53 (1979).
[5] R. Delannay, M. Louge, P. Richard, N. Taberlet and A. Valance, Nature Materials 6, 99 (2007).
[6] N. Taberlet, P. Richard, J.T. Jenkins and R. Delannay, Eur. Phys. J. E 22, 17 (2007).
[7] L.E. Silbert, D. Ertas, G.S. Grest, T.C. Halsey, D. Levine and S.J. Plimpton, Phys. Rev. E 64, 051302 (2001); D. Ertas, and T.C. Halsey, Europhys. Lett. 60, 931 (2002); L.E. Silbert, J.W. Landry and G.S. Grest, Phys. Fluids 15, 1 (2003); D.M. Hanes and O.R. Walton, Powder Tech. 109, 133 (2000).
[8] P. Jop, Y. Forterre and O. Pouliquen, Nature 441, 727 (2006), J. Fluid Mech. 541, 167 (2005).
[9] I.S. Aranson, L.S. Tsimring, F. Malloggi and E. Clément, Phys. Rev. E. 78, 031303 (2008).
[10] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
[11] Y. Forterre and O. Pouliquen, Phys. Rev. Lett. 86, 5886 (2001); J. Fluid Mech. 467, 361 (2002).
[12] S.N. Prasad, D. Pal and J.M. Römkins, J. Fluid Mech. 413, 89 (2000); M.Y. Louge and S.C. Keast, Phys. Fluids 13, 1213 (2001); Y. Forterre and O. Pouliquen, J. Fluid Mech. 486, 21 (2003); S.L. Conway, D.J. Goldfarb, T.
Shinbrot and B.J. Glasser, Phys. Rev. Lett. 90, 074301 (2003).

[13] E.M. Sparrow and R.B. Husar, J. Fluid Mech. 37, 251-255 (1969).

[14] J.D. Bernal and J. Mason, Nature 188, 910 (1960); G.Y. Onoda and E.G. Liniger, Phys. Rev. Lett. 64, 2727 (1990).

[15] O. Pouliquen, Phys. Fluids 11, 542 (1999).

[16] T. Börzsönyi and R.E. Ecke, Phys. Rev. E 74, 061301 (2006).

[17] I. Goldhirsch and C. Goldenberg, Eur. Phys. J. E 9, 245 (2002).

[18] Movies and images of the pattern recorded in experiments for various materials can be downloaded from http://www.szfki.hu/~btamas/gran/stripes.html

[19] J.G. Leidenfrost, De Aquae Communis Nonnullis Qualitatibus Tractatus (University of Duisburg, Duisburg, Germany, 1756) [Int. J. Heat Mass Transfer 9, 1153 (1966)]; A.-L. Bianc, C. Clanet and D. Quéré, Phys. Fluids 15, 1632 (2003).

[20] K. Kelfoun, T. Druitt, B. van Wyk de Vries, M.-N. Guilbaud, Bull. Volcanol. 70, 1169 (2008).