Quantum speed limit helps interpret geometric measure of entanglement

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Using the approach offered by quantum speed limit, we show that geometric measure of multipartite entanglement for pure states [Phys. Rev. A 68, 042307(2003)] can be interpreted as the minimal time necessary to unitarily evolve a given quantum state to a separable one.

I. INTRODUCTION

More than a decade ago, rigorous quantification of entanglement was in the core of quantum information research [1]. Later on, a similar effort has been directed towards a more general theory of quantum resources [2], such as, for example, quantum coherence [3].

A general methodology behind quantification of resources like entanglement and coherence is well established. One needs to define a set of resourceless states, as well as an accompanying set of free (resourceless) operations (such which cannot generate a given resource). Then, one is in position to consider measures of the resource in question. Most importantly, such non-negative measures must be:

- faithful, which means they vanish only for resourceless states,
- monotonic, which means they must not increase under resourceless operations.

In addition, accompanying requirements such as additivity, convexity, continuity, etc. can be imposed on a candidate measure [4].

Resource-theoretic approach is a handy way of working with quantum phenomena, as it uses mathematical rigor to help decide about optimal scenarios and protocols. However, from a physical perspective, there is yet another piece of information which, a priori, is not offered by the resource theories. Given a valid measure of a resource, say an entanglement measure, we may ask about an operational interpretation of a value it can assume. If a given measure outputs the values 0.8 and 0.6, while evaluated on two concrete states respectively, we know that the first state is a more resourceful one. But what is the physical content of these exemplary values? Can they be expressed in terms of quantities which possess a clear experimental meaning? Answers to these questions, as not being by default provided by resource theories, if possible, need to be supplemented by additional considerations. Note that we are not concerned here with the problem, whether an entanglement measure can be read out from outcomes of a tailored experiment.

Within entanglement theory, there are a few known measures which are equipped with some operational interpretation [1]. Among them, we could distinguish:

- Distillable entanglement [1] which, using a different wording than is done usually, represents the success rate while transforming many copies of a given entangled state into maximally entangled states. The exemplary value 0.8 means that asymptotically (very many copies of the state are used), one can get 8 maximally entangled states by using each 10 given input states;
- Entanglement cost [4] which has a dual interpretation with respect to distillable entanglement;
- Robustness of entanglement [5] which tells us “how much” of a separable state we need to add to our state (taking a convex combination of both), in order to make the final state separable;
- Various distance measures, which tell us how far away the state is from the set of separable states.

All the above examples enjoy an interpretation which is fully satisfactory from both information-theoretic and probabilistic point of view. However, in terms of a common-sense meaning of words, certain interpretations become elusive. In particular, the distance between an entangled quantum state and the set of separable states is a mathematical distance which has little to do with distances usually measured in experiments. We speak here about a distance in a matrix space, not in our physical spacetime. Obviously, the value of some distance between two states, equal to 0.8 (in which units?), does not correspond to any physical distance measured in meters.

This conceptual issue becomes even more visible if we look at the interpretation of probably the most important entanglement measure [1], namely, the entanglement of formation [6]. We can say that this measure quantifies “minimal possible average entanglement over all pure state decompositions” [1] of the given state. It is quite hard to imagine a device which in some experimental procedure (even a thought experiment) at the end outputs a number (in physical units) which will give a direct meaning to the value 0.8 (or any other) of this measure.

We can see that even though giving an interpretation to entanglement measures (and other resource-theoretic

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measures) is usually possible to some extent, it is quite hard to provide an interpretation with an appealing physical background. Therefore, the aim of this paper is to show that a particular distance measure — geometric measure of entanglement [17] — does have such an interpretation. It turns out that for the case of entangled multipartite pure states this distance can directly be linked with the \textit{minimal time} necessary to make the state separable by means of a unitary evolution. Knowing an optimal Hamiltonian responsible for the evolution, which is always of a qubit type [3], this minimal time becomes a simple function of the entanglement measure, with time scale given by the energy gap $\delta E$ of the optimal Hamiltonian. For example, if $\delta E/\hbar = 2\text{GHz}$, which lays within a typical regime relevant for platforms suitable for quantum computation [9], the value 0.8 assumed by the geometric measure of entanglement means that the given state will, during the fastest possible evolution, reach some separable state in 1.11 ns. An another state, described by the geometric measure of entanglement equal to 0.6 will be faster, reaching the set of separable states in 0.89 ns.

The above operational meaning can be considered as physically appealing, as it converts a distance in an abstract mathematical space, to time, measured by sufficiently accurate clocks. This conceptual transformation is possible, due to links with, so called, quantum speed limit [10,12].

The paper is organized as follows. In Sec. II we briefly introduce geometric measure of entanglement and quantum speed limit for pure states. In Sec. III we connect both notions expressing, the former one by the latter one. Crucially, even though quantum speed limit only offers a bound on the minimal time, it is always possible to saturate the inequality [13] giving an exact relation between both quantities involved. In Sec. IV we discuss the case of mixed states, as well as point towards more general considerations.

**II. GEOMETRIC MEASURE OF ENTANGLEMENT AND QUANTUM SPEED LIMIT**

We conduct our discussion restricting ourselves to pure states. Usually, such a choice is dictated by simplicity, however, in this particular problem it has a more fundamental reason. We go back to this point in Sec. 4.

Given an arbitrary composite system (in general multipartite) with $K \geq 2$ subsystems, we consider entangled states $|\psi\rangle$ belonging to the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_K$. We neither need to specify dimensions of each subsystem, nor assume that all subsystems are identical.

The family of geometric measures of entanglement for pure states is defined as [17]

$$E_m |\psi\rangle = 1 - \left( \max_{\phi \in S_m} |\langle \psi | \phi \rangle|^2 \right)^2, \quad m = 2, \ldots, K. \quad (1)$$

By $S_m$ we denote the set of $m$-separable pure states [14]. Therefore standard, also called completely, separable states are in $S_K$, while $S_1 = \mathcal{H}$ is the entire Hilbert space. From obvious reasons $E_1 |\psi\rangle \equiv 0$, so we omit this trivial case starting the hierarchy at $m = 2$. Any $m$-separable state is a tensor product of $m$ states. When $m = K$ each state in the product belongs to a particular Hilbert subspace, while for $m < K$ some states constituting $|\psi\rangle$ must belong to more than one subspace. For example, if $K = 3$, the state $|\phi_{\text{sep}}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$ is completely separable, while the state $|\phi_{\text{bisep}}\rangle = |\phi_{12}\rangle \otimes |\phi_3\rangle$, with $|\phi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, is “just” bi-separable. The property that the hierarchy $E_m$ covers the whole entanglement landscape of a multipartite scenario is one of its major advantages, while computational difficulties relevant for more complex systems are a disadvantage shared with virtually all other measures. Note that certain paths have been followed in order to allow for experimental usefulness of the idea behind the geometric measure of entanglement [15,17].

The discovery of quantum speed limit [18] dates back much before the theory of quantum entanglement has been developed. Regardless of that fact, there is a lot of ongoing research in this field, for example [19–23]. There is a plethora of scenarios in which quantum speed limit can be considered, rendering better or worse performance, depending on the context [24,25]. We go back to this issue in Sec. 4. For our purpose we resort here to the most standard variant of quantum speed limit.

We know, that for two pure states $|\psi\rangle$ and $|\phi\rangle$, and a time evolution governed by a time-independent Hamiltonian $H$, the time $\tau$ necessary to unitarily evolve $|\psi\rangle$ into $|\phi\rangle$ is bounded

$$\tau \geq \hbar \frac{\arccos (|\langle \psi | \phi \rangle|)}{\Delta H}. \quad (2)$$

The usual standard deviation

$$\Delta H = \sqrt{\langle \psi | H^2 | \psi \rangle - (\langle \psi | H | \psi \rangle)^2}, \quad (3)$$

does only depend on the initial state $|\psi\rangle$, and is obviously time independent. The bound in (2) can always be saturated with the appropriate (|\phi)-dependent) choice of $H$. To simplify the discussion, without loss of generality, we assume that $\langle \psi | \phi \rangle$ is real and non-negative. This is not a restriction since we work in a complex projective space, so that all states $e^{i\theta} |\phi\rangle$ are equivalent. The optimal Hamiltonian, denoted by $H_{\text{opt}}$, is known [8] to be proportional to the $\sigma_y$ Pauli matrix$^1$ in the two dimensional subspace spanned by $\{ |\psi\rangle, |\phi\rangle \}$. Note that both states do not need to be orthogonal, so one introduces

$$|\tilde{\psi}\rangle = \frac{|\phi\rangle - \langle \psi | \phi \rangle |\psi\rangle}{\sqrt{1 - |\langle \psi | \phi \rangle|^2}}. \quad (4)$$

$^1$ Note that on page 18 in [8], Eq. 2.11 suggests that $H_{\text{opt}}$ is proportional to $\sigma_z$, which is just a minor mistake [26].
so that $\langle \psi | \tilde{\psi} \rangle = 0$. The Hamiltonian reads
\[ H_{\text{opt}} = -i\hbar \omega (\langle \psi | \langle \tilde{\psi} | - | \tilde{\psi} \rangle \langle \psi | ) , \tag{5} \]
where the frequency $\omega$ gives an energy scale. This energy scale is not subject to further optimization, as its role is just to set a time scale. By a simple calculation we can confirm that the state $e^{-iH_{\text{opt}}t/\hbar} | \psi \rangle$ equals $| \tilde{\phi} \rangle$ for $t = \tau [\psi, \phi]$, where
\[ \tau [\psi, \phi] = \frac{\arccos (|\langle \psi | \phi \rangle|)}{\omega} . \tag{6} \]
Additionally, $\langle \psi | H^2_{\text{opt}} | \psi \rangle = \hbar^2 \omega^2$ and $\langle \psi | H_{\text{opt}} | \psi \rangle = 0$, so that $\Delta H_{\text{opt}} = \hbar \omega$, and consequently the inequality [3] is saturated.

III. MAIN RESULT

As before let us consider a composite system $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_K$, and an entangled state $| \psi \rangle \in \mathcal{H}$. We ask the question: how fast can the state $| \psi \rangle$ become $m$-separable with the help of the unitary evolution given by a global Hamiltonian? Obviously, if $| \psi \rangle$ already is $m$-separable, the time necessary to achieve that task will trivially vanish. However, if the state is not $m$-separable, the time, which we denote as $\tau_m [\psi]$, will be positive.

From the previous section we know that for a given $| \phi \rangle$, saturation of quantum speed limit [4] occurs for $H = H_{\text{opt}}$. Therefore
\[ \tau_m [\psi] = \min_{\phi \in S_m} \tau [\psi, \phi] \equiv \frac{1}{\omega} \min_{\phi \in S_m} \arccos (|\langle \psi | \phi \rangle|) . \tag{7} \]

Moreover, since the arccos function is decreasing we observe that
\[ \min_{\phi \in S_m} \arccos (|\langle \psi | \phi \rangle|) = \arccos \left( \max_{\phi \in S_m} |\langle \psi | \phi \rangle| \right) = \arccos \left( \sqrt{1 - E_m [\psi]} \right) = \arcsin \left( \sqrt{E_m [\psi]} \right) . \tag{8} \]

Note that the minimum in [4] has been turned into the maximum present in [5]. We therefore obtain
\[ \tau_m [\psi] = \frac{\arcsin \left( \sqrt{E_m [\psi]} \right)}{\omega} , \tag{9} \]
which is the final result of this derivation. As expected, we can see that if the state $| \psi \rangle$ is $m$-separable, the minimal time is 0, otherwise it is strictly positive. We also transfer the ordering among classes of entangled states (provided by the hierarchy $E_m$) to the ordering among the “speeds” relevant for the quantum speed limit.

For completeness, after a simple rearrangement we can express the geometric measures of entanglement in terms of the minimal time
\[ E_m [\psi] = \sin^2 (\omega \tau_m [\psi]) . \tag{10} \]

IV. DISCUSSION

We derived the relation [10] which links the geometric measure of entanglement with the minimal time of unitary evolution. As a by-product of this analysis, maximization with respect to $m$-separable states has been replaced by minimization with respect to global Hamiltonians responsible for the time evolution. This replacement is formal, since to know the optimal Hamiltonian we need to know the optimal $m$-separable state and vice versa. This implies that if one tries to compute the minimal time $\tau_m [\psi]$, the effort to be taken is the same as while computing the sole $E_m [\psi]$. Still, the gain is on the interpretation side. While the value $E_m [\psi]$ can only be understood in information-theoretic context, $\tau_m [\psi]$ has a crystal clear physical meaning. Using qubit type Hamiltonians with energy gap
\[ \delta E = 2\hbar \omega , \tag{11} \]
the multipartite entangled state $| \psi \rangle$ cannot be made $m$-separable faster, than is specified by Eq. [9] Of course the time needs to be given in relation to the energy scale $\delta E$, since by increasing the energy we always increase the speed of time evolution.

In Fig. [1] we plot the minimal time in Eq. [9] for a few distinct values of $\omega$. We observe that the result does not explicitly depend on the index “$m$”. However, we know that $E_m [\psi] \geq E_n [\psi]$ if $m \geq n$, so the minimal time $\tau_m [\psi]$ is an increasing function of $m$.

We are now in position to address the issue of mixed states. Known entanglement measures do sometimes cover mixed states in a natural fashion. In majority of cases, however, there are issues with monotonicity, and one needs to resort to the convex roof construction. Formally speaking, this construction works always. One needs to represent a given mixed state as a convex com-
combination of pure states

\[ \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|, \]  

(12)

compute the measure individually for each \(|\psi_k\rangle\), average over the probability distribution \(\{p_k\}\) and, at the end, optimize with respect to all convex decompositions of the state in question. Consequently, using an example of the geometric measure of entanglement, one gets

\[ E_m[\rho] = \min_{\{p_k, \psi_k\}} \sum_k p_k E_m[\psi_k]. \]  

(13)

This construction, even for very simple systems, proves itself to be intractable, when it comes to concrete calculations. However, as already mentioned, on the formal level the measure for mixed states can be defined.

Our result based on the quantum speed limit does not share this feature. Before we open this part of discussion, we shall observe that even the convex roof construction, which refers to pure states, is not compatible with our approach, as the function \(\sin^2 x\) for \(0 \leq x \leq \pi/2\) is neither convex nor concave. Therefore, we abandon the convex roof construction and ask an another question, namely, whether a result similar to (10) could be derived given a mixed state \(\rho\), and some entanglement measure based on the minimal distance between this state and the set of \(m\)-separable mixed states. Even in this generalized setting we immediately meet an obstacle, because in order to unitarily evolve one state into the other one, both states need to have the same spectrum. Most likely, the optimal \(m\)-separable mixed state would not meet this criterion, therefore, it would be impossible to unitarily evolve \(\rho\) into it. One could of course consider other models for time evolution, Markovian or non-Markovian, however, a priori there is no guarantee that a fixed model of time evolution will do the job. Especially, because all known quantum speed limits involving mixed states are generally not tight [24, 25]. Because of that, one would eventually end up with a bound on a given entanglement measure. Due to lack of optimality, the minimal predicted time will be overestimated, therefore we would likely have an upper bound. While bounds (especially lower bounds) are useful from a practical perspective, they weaken the interpretation of the measure in question. In conclusion, the result reported in this paper from fundamental reasons does not extend to the case of mixed states. Regardless of that fact, we believe the interpretation valid for pure states stands on its own. As a matter of fact, an interesting future question is related with purification of mixed states and the interplay between the quantum speed limit and the way in which given quantum correlations can be immersed in a larger Hilbert space.

Since the research devoted to quantum speed limit shifts from pure theory towards applicable quantum information processing [27], it is not surprising that other quantum resources are also in the focus. For example, so called resource speed limit has recently been discussed [28]. On the other hand, formal relatives of the geometric measure of entanglement are also noticed as potentially handy quantifiers of other resources, such as non-classicality in quantum optics [29].

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