On the solution of the Zakharov-Shabat system, which arises in the analysis of the largest real eigenvalue in the real Ginibre ensemble

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Abstract

Let $\lambda_{\text{max}}$ be a shifted maximal real eigenvalue of a random $N \times N$ matrix with independent $N(0,1)$ entries (the ‘real Ginibre matrix’) in the $N \to \infty$ limit.

It was shown by Poplavskyi, Tribe, Zaboronski [9] that the limiting distribution of the maximal real eigenvalue has $s \to -\infty$ asymptotics

$$P[\lambda_{\text{max}} < s] = e^{-\frac{1}{2}\sqrt{2\pi} \zeta\left(\frac{3}{2}\right)s} + O(1),$$

where $\zeta$ is the Riemann zeta-function.

This limiting distribution was expressed by Baik, Bothner [1] in terms of the solution $q(x)$ of a certain Zakharov-Shabat inverse scattering problem, and the asymptotics was extended to the form

$$P[\lambda_{\text{max}} < s] = e^{-\frac{1}{2}\sqrt{2\pi} \zeta\left(\frac{3}{2}\right)s c(1 + O(1))}, s \to -\infty.$$

We show that $q(x)$ is a smooth function, which behaves as $\frac{1}{x}$ as $x \to -\infty$. Second, we show that the error term in the asymptotics is subexponential, i.e. smaller than $e^{-C|x|}$ for any $C$.

Third, we identify the constant $c$ as a conserved quantity of a certain fast decaying solution $u(x,t)$ of the Korteweg-de Vries equation. This, in principle, gives a way to determine $c$ via the known long-time $t \to +\infty$ asymptotics of $u(x,t)$. We also conjecture a representation for the $c$ in terms of an integral of the Hastings-MacLeod solution of Painlevé II equation.

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1 Introduction

For $\gamma \in [0, 1]$, define

$$R(k; \gamma) = -\sqrt{\gamma}e^{-k^2/4},$$

(1)

and consider the following Riemann-Hilbert problem (RHP):\(^2\)

**Riemann-Hilbert problem 1.** To find a $2 \times 2$ matrix-valued function $\bfM = \bfM(x, t; k; \gamma)$ that satisfies the following properties:

- **analyticity:** $\bfM(x, t; k; \gamma)$ is analytic in $k \in \mathbb{C} \setminus \mathbb{R}$, and continuous up to the boundary $k \in \mathbb{R}$;
- **jumps:** $\bfM_+ = \bfM_+ \bfJ_\bfM$, where

$$\bfJ_\bfM = \begin{pmatrix} 1 & R(k; \gamma) \cdot e^{-2i\hat{\theta}(x, t; k)} \\ -R(k; \gamma) e^{2i\hat{\theta}(x, t; k)} & 1 - |R(k; \gamma)|^2 \end{pmatrix}, \quad k \in \mathbb{R},$$

where $\hat{\theta}(x, t; k) = kx + 4k^3t$;
- **asymptotics at the infinity:**

$$\bfM(k) \to \bfI \quad \text{as} \quad k \to \infty.$$  

Define the functions $q(x, t; \gamma), u(x, t; \gamma)$ by the formulas\(^3\)

$$q(x, t; \gamma) = -2i \lim_{k \to \infty} k(\bfM(x, t; k; \gamma) - \bfI)_{12} = 2i \lim_{k \to \infty} k(\bfM(x, t; k; \gamma) - \bfI)_{21} \in \mathbb{R},$$

$$u(x, t; \gamma) = q^2(x, t; \gamma) - q_x(x, t; \gamma), \quad \int_x^{+\infty} u(x, t; \gamma) = q(x, t; \gamma) + \int_x^{+\infty} q^2(z, t; \gamma)dz,$$

(2)

where the subscript $12$ means the element situated on the intersection of the first row and the second column in the matrix. For $t = 0$, we denote $q(x; \gamma) := q(x, 0; \gamma)$, $u(x; \gamma) := u(x, 0; \gamma)$. The $q(x, t; \gamma)$ satisfies the (defocusing) modified Korteweg-de Vries equation (MKdV) and $u(x, t; \gamma)$ satisfies the Korteweg-de Vries equation,

$$q_t - 6q^2q_x + q_{xxx} = 0,$$

(3a)

$$u_t - 6uu_x + u_{xxx} = 0,$$

(3b)

and $R(k; \gamma)$ is the reflection coefficient, associated with $q$ via MKdV scattering problem, and is the reflection coefficient, associated with $u$ via KdV scattering problem (see sections 7, 8 for a short explanation what does it mean). Moreover, $q(x, t; \gamma)$ for $\gamma \in [0, 1)$ is an example of a classical solution of MKdV, which is exponentially decaying as $x \to \pm \infty$ for all times $t$. The $u(x, t; \gamma)$ is such an example for KdV, but already for all $\gamma \in [0, 1]$, including the case $\gamma = 1$.

Define the function

$$F(2s; \gamma) = \exp \left[ -\frac{1}{2} \int_s^{+\infty} (z - s)q^2(z, 0; \gamma)dz \right] \sqrt{\cosh(\sigma(s; \gamma)) - \sqrt{\gamma} \sinh(\sigma(s; \gamma))},$$

where\(^4\)

$$\sigma(s; \gamma) := \int_s^{+\infty} q(x, 0; \gamma)dx.$$  

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\(^1\)The $r$ from $[1]$ equals $iR(k; \gamma)$.

\(^2\)The $\bfX$ from $[1]$ equals $e^{i\sigma/4} \bfM e^{-i\sigma s/4}$, where $\sigma_3 = \text{diag}[1, -1]$.

\(^3\)The $y(x; \gamma)$ from $[1]$ equals $y = iq|_{x=0}$.

\(^4\)The $\mu(\cdot; \gamma)$ from $[1]$ equals $\mu(2s; \gamma) = \sigma(s; \gamma)$.  

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For $\gamma = 1$ the above expression equals

$$F(2s; 1) = \exp \left[ -\frac{1}{2} \int_0^\infty \int_0^\infty (z-s)q^2(z,0;1)dz - \frac{1}{2} \int_0^\infty q(z,0;1)dz \right] = \exp \left[ \frac{1}{2} \int_0^\infty q(z;1)dz - \frac{1}{2} \int_z^\infty q^2(x,0;1)dx \right] = \exp \left[ \frac{1}{2} \int_0^\infty (z-s)u(z,0;1)dz \right].$$

(4)

It was shown in [1] that the function $F(s; 1)$ with $\gamma = 1$ plays an important role in the analysis of real eigenvalues in the real Ginibre ensemble. Namely,

**Theorem 1.** (Baik, Bothner, [1]) Let $\{z_j(X)\}_{j=1}^n$ denote the eigenvalues of a $n \times n$ matrix with independent $N(0,1)$ entries (the ‘real Ginibre matrix’). Then

$$\lim_{n \to \infty} \mathbb{P} \left( \max_{j,\gamma} |z_j(X)| = \sqrt{n} + s \right) = F(s; 1), \quad s \in \mathbb{R}. \quad (5)$$

The work [1] is based on a previous work of Rider, C. Sinclair [7]; Poplavsky, Tribe, Zaboronski [9], where the left-hand-side of (5) is identified with a certain Fredholm determinant.

It was noticed in [1] that for $\gamma \in [0,1)$ the function $q(x; \gamma) := q(x,0;\gamma)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, while for $\gamma = 1$ it does not. Our first goal here is to answer the following question: *to which class does $q(x;1)$ belong?* We show that $q(x;1)$ is infinitely smooth in $x$, decays exponentially as $x \to +\infty$, and decays as $x^{-1}$ for $x \to -\infty$ (see formulas (8), (9) below). In more details, we show

**Theorem 2.** (a) For any $\gamma \in [0,1]$, the function $q(x; \gamma) \in C^\infty(\mathbb{R})$; 
(b) For any $x \in \mathbb{R}$, the function $q(x; \gamma) \in C(\gamma \in [0,1])$; 
(c) for fixed $\gamma \in (0,1)$ and $x \to -\infty$, for any $C > 0$,

$$q(x; \gamma) = \frac{8 \kappa_2 e^{2x\kappa_2} L_{-1}(\gamma)}{4 \kappa_2^2 - e^{4x\kappa_2} L_{-1}(\gamma)^2} + O(e^{-C|x|})$$

$$\int_x^{+\infty} q^2(\tilde{x}; \gamma) d\tilde{x} = 2T_1(\gamma) - \frac{4 \kappa_2 e^{2x\kappa_2} L_{-1}(\gamma)^2}{4 \kappa_2^2 - e^{4x\kappa_2} L_{-1}(\gamma)^2} + O(e^{-C|x|}). \quad (6)$$

Here $\kappa_\gamma = \sqrt{-2 \ln \gamma} \geq 0$ (so that $\kappa_1 = 0$), and

$$T_1(\gamma) = \frac{-1}{2\pi} \int_{-\infty}^{+\infty} \ln(1 - \gamma e^{-\frac{s^2}{2}}) ds = \frac{1}{\sqrt{2\pi}} L_{i\gamma}(\gamma) > 0,$$ 

$$L_{-1}(\gamma) = \frac{1}{\kappa_\gamma} \exp \left[ \frac{1}{\pi i} \int_{-\infty}^{+\infty} \ln(1 - \gamma e^{-\frac{s^2}{2}}) ds \right] \geq 0. \quad (7)$$

(d) for $\gamma = 1$, as $x \to -\infty$, for any $C > 0$,

$$q(x;1) = \frac{2}{-2x + L_1(1)} + O(e^{-C|x|}),$$

$$\int_x^{+\infty} q^2(\tilde{x}; \gamma) d\tilde{x} = 2T_1(1) + \frac{2}{2x - L_1(1)} + O(e^{-C|x|}). \quad (8)$$
where \( T_1(1) \) is as in (4),
\[
T_1(1) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \ln(1 - e^{-\frac{s^2}{2}}) ds = \frac{1}{\sqrt{2\pi}} L_1(1) = \frac{1}{\sqrt{2\pi}} \cdot \left( \frac{3}{2} \right) \approx 1.042 186 978 869,
\]
and
\[
L_1(1) = 2 - \frac{1}{\pi} \int_{\Sigma_{1/4}} \ln \left( \left[ 1 - e^{-\frac{t^2}{2}} \right] \frac{s^2 + 1}{s^2} \right) \frac{ds}{s^2} \approx 1.165 194 315 878 021 340 410 354,
\]
with the integral over the oriented contour \( \Sigma_{1/4} = (\infty, -\frac{1}{4}) \cup (-\frac{1}{4}, -\frac{1}{4}) \cup (-\frac{1}{4}, \frac{1}{4}) \cup (\frac{1}{4}, +\infty) \).

(e)
\[
\int_{x}^{+\infty} q(\tilde{x}; 1) d\tilde{x} = \ln (-2x + L_1(1)) + \frac{1}{2} \ln 2 + O(e^{-C|x|}), \quad x \to -\infty. \tag{9}
\]

Denote,
for \( \gamma \in [0, 1] \), \( H(\gamma) = 3 \int_{-\infty}^{+\infty} u^2(x, t; \gamma) dx \), \( K(\gamma) = \int_{-\infty}^{+\infty} xu(x, t; \gamma) dx + H(\gamma)t \); \tag{10a}
for \( \gamma \in [0, 1] \), \( N(\gamma) = 3 \int_{-\infty}^{+\infty} (q^4(x, t; \gamma) + q_{x}^2(x, t; \gamma)) dx > 0 \); \tag{10b}
for \( \gamma \in (0, 1) \), \( M(\gamma) = \int_{-\infty}^{x} x q^2(x, t; \gamma) dx + N(\gamma)t - \frac{1}{2} \ln |\ln \gamma| - \ln 2 \); \tag{10c}
for \( \gamma = 1 \) and \( x < 0 \),
\[
M(1) = \int_{-\infty}^{x} \left( z q^2(z, t; \gamma) + \frac{1}{z} \right) dz + \int_{x}^{+\infty} z q^2(z, t; \gamma) dz + N(1)t + \ln |x| + \frac{3}{2} \ln 2 - 1. \tag{10d}
\]

All the functions \( K(\gamma), H(\gamma), M(\gamma), N(\gamma) \) are conserved quantities, i.e. they do not depend on time \( t \). Quantity \( M(1) \) do not depend on the choice of \( x < 0 \) (Lemma [11]).

**Corollary 3.** We have as \( s \to -\infty \), for any \( C > 0 \),
\[
F(s; 1) = e^{\frac{s}{2} T_1(1)} e^{-\frac{s}{2} K(1)} (1 + O(e^{-C|s|})),
\]
and \( K(1) = M(1) \).

**Proof.** Using asymptotics (5), (9) and conservation law (10d) at the time \( t = 0 \), we find
\[
\frac{1}{2} \int_{s}^{+\infty} z q^2(z; 1) dz = \frac{s}{L_1 - 2s} + \frac{1}{2} \ln(L_1 - 2s) - \frac{1}{2} M(1) + \frac{1}{4} \ln 2 + O(e^{-C|s|}),
\]
\[
\frac{s}{2} \int_{s}^{+\infty} q^2(z; 1) dz = s T_1(1) + \frac{s}{2s - L_1(1)} + O(e^{-C|s|}),
\]
\[
\frac{1}{2} \int_{s}^{+\infty} q(z; 1) dz = \frac{1}{2} \ln(L_1(1) - 2s) - \frac{1}{4} \ln 2 + O(e^{-C|s|}).
\]
Substitute this into the first formula of (4), then

\[ F(2s; 1) = \exp \left[ T_1(1)s - \frac{1}{2} M(1) + O(e^{-C|s|}) \right]. \]

Furthermore, using the third of formulas (4), asymptotics (5), (9), and expression (2) of \( u = q^2 - q_x \), integrating by parts, we find that

\[ F(2s; 1) = \exp \left[ T_1(1)s - \frac{1}{2} K(1) + O(e^{-C|s|}) \right]. \]

Hence, \( K(1) = M(1) \).

\[ \square \]

Remark 1. The quantity \( K(1) \) from the formula in Corollary 1 was found by non-rigorous computations by Forrester [10, (2.26), (2.30)], in the form of a slowly convergent series,

\[ K(1) = -2 \left( \ln 2 - \frac{1}{4} + \frac{1}{4\pi} \sum_{n=2}^{\infty} \frac{1}{n} \left( -\pi + \sum_{j=1}^{n-1} \frac{1}{\sqrt{j(n-j)}} \right) \right) \approx -0.1254. \]

On the other hand, Baik and Bothner [1], unnumbered formula for \( \eta_0(1) = e^{-\frac{1}{4} K(1)} \) on p.6, formula (1.16) found numerically another value of \( K(1) \),

\[ K(1) \approx 0.56798925. \]

The fact, that \( K(1) \) is a conserved quantity of the KdV, allows, in principle, to compute \( K(1) \) by using (known) large time \( t \to +\infty \) asymptotics of the \( u(x, t; 1) \). Indeed, for \( t = 0 \) we might study only the asymptotics \( x \to \pm \infty \) of \( u(x, 0; \gamma) \). When \( t \to +\infty \), we know in principle the asymptotics for \( u(x, t; \gamma) \) for all \( x \), which means that we can find an expression for integral of \( u(x, t; \gamma) \). Easier said than done, and we do not pursue this issue here. For a note, we list the known leading asymptotic as \( t \to +\infty \) terms for \( u(x, t; \gamma) \) (15, [4], Thm 5.4),

1. \( x < -\varepsilon t \) (similarity asymptotics):

\[ u(x, t) \sim \sqrt{\frac{4\nu(\xi)}{3t}} \sin \left( \frac{16t\nu_0(\xi)}{3t} + \nu(\xi) \ln \left( 192t\nu_0(\xi) + \delta(\xi) \right) \right), \]

\[ \int_{x}^{+\infty} u(\tilde{x}, t) d\tilde{x} \sim -\frac{1}{\pi} \ln \frac{1}{k_0} \int_{-k_0}^{k_0} \left( |R(z)|^2 \right) dz - \sqrt{\frac{\nu(\xi)}{3k_0t}} \cos \left( 192t\nu_0(\xi) + \nu(\xi) \ln \left( 192t\nu_0(\xi) + \delta(\xi) \right) \right), \]

where

\[ \xi = \frac{1}{t^{2/3}}, \ k_0 = k_0(\xi) = \sqrt{-\xi}, \ \nu(\xi) = \frac{1}{t} \ln \left( 1 - |R(k_0(\xi))|^2 \right), \]

\[ \delta(\xi) = \frac{2t}{3} + \arg R(k_0(\xi)) + \Gamma \left( iv(k_0(\xi)) \right) - \frac{1}{2} \int_{-k_0}^{k_0} \ln \left( \frac{1-|R(\xi)|^2}{2} \right) \frac{d\xi}{\xi + k_0}. \]

2. \(-C < \frac{1}{t^{2/3}} < C\):

\[ u(x, t) \sim \frac{1}{(3t)^{2/3}} \left( p^2(s) - p'(s) \right), \quad \text{where} \ s = \frac{x}{(3t)^{1/3}}, \]

and \( p(s) \) is the solution of the Painlevé II equation

\[ p''(s) - sp(s) - 2p^3(s) = 0, \]

fixed by its asymptotics \( p(s) \sim -R(0)Ai(s), \ s \to +\infty \). For \( R(0) > -1, \ p(s) \) is oscillating and vanishing as \( s \to -\infty \), and for \( R(0) = -1, \ p(s) \sim \sqrt{\frac{4}{3}s} \) as \( s \to -\infty \) (see Hastings, McLeod [4] for details).
3. In the case $R(0) = -1$, there is an additional region $-C_2 < \frac{\gamma}{(\alpha + 1)} < -C_1$; with an elliptic asymptotics,
\[ u(x, t) \sim \frac{-2x}{3t} \left[ A(\alpha) + B(\alpha) \text{e}^{2} (2K(\alpha) + \theta_0; \alpha) \right], \]
where the slow parameter $\alpha = \alpha(s)$ is determined by
\[ \alpha = \alpha(s) = 1 - \frac{a^2(s)}{b^2(s)} \quad \text{where } 0 \leq a(s) \leq b(s) \leq \sqrt{2} \text{ are determined by the system} \]
\[ a^2 + b^2 = 2, s = 24 \int_a^b \sqrt{(y^2 - a^2)(b^2 - y^2)} dy, 0 \leq s \leq 8^{3/2} \]
and we refer the reader to the original paper [2] for details about the other quantities in the above formula.

4. $x > \epsilon t$: \[ u(x, t) \sim 0 \] (there are no solitons in our case).

Here $\varepsilon, C, C_1, C_2$ are generic positive constants. Between the regions there are gaps, which to the best of our knowledge were not studied in the literature.

**Conjectural and non rigorous Remark 2.** Substituting the above asymptotics of $u$ into the expression \((10n)\) of $K(\gamma)$, and making some heuristic computations

(like those: since $\int_{-\infty}^{\epsilon t} u(x, t) dx = O(1)$, then $\int_{-\infty}^{\epsilon t} x u(x, t) dx = O(t), t \to +\infty$)

furthermore, $\int_{C^{1/3}}^{C^{1/3}} \frac{1}{(x_0)^{2/3}} \left( p^2 \left( \frac{\theta_0}{(x_0)^{2/3}} \right) - p \left( \frac{\theta_0}{(x_0)^{2/3}} \right) \right) dx = O(1), t \to +\infty$,

we guess that the similarity asymptotics give the contribution of the order $t^4 \int K(\gamma)$, and Painleve asymptotics give a contribution of the order $t^0$. Let us mention, that the contribution of the order $t^4$ is always non-zero, even when there are no solitons, as in our case. Furthermore, the integral $\int_{-\infty}^{\infty} s (p^2(s) - p'(s)) ds$, which might be convergent for $\gamma < 1$, but definitely divergent for $\gamma = 1$, might be regularized for $\gamma = 1$.

Indeed, function $p(s)$, corresponding to the case $\gamma = 1$, has the asymptotics as $s \to \pm \infty$:
\[ p(s) \sim \text{Ai}(s), s \to +\infty, \quad p(s) = \frac{\sqrt{-s}}{\sqrt{2}} \left( 1 + \frac{1}{8s} + \frac{73}{128s^6} + O(s^{-9}) \right), s \to -\infty, \]
which admits element-wise differentiation w.r.t. $s$, so that
\[ s(p^2(s) - p'(s)) = \frac{1}{2} s^2 - \frac{\sqrt{-s}}{2\sqrt{2}} + \frac{1}{8} s^{-1} + O(|s|^{-5/2}), s \to -\infty. \]
We have a convergent integral
\[ P = \int_{-\infty}^{s} \left[ \text{Ai}(p^2(\tilde{s})) - \frac{1}{2} s^2 + \frac{\sqrt{-s}}{2\sqrt{2}} - \frac{1}{8} s \right] d\tilde{s} + \int_{s}^{+\infty} \left( p^2(\tilde{s}) - p'(\tilde{s}) \right) d\tilde{s}, \]
which does not depend on the choice of $s < 0$.

We would expect that $K(1)$ from Corollary 3 is related to $P$, $K(1) \approx P$.

**Remark 3.** A more practical way to compute $K(\gamma)$, gamma $\in [0, 1]$, numerically is to do this at the time $t = 0$, by using the main integral equations of the inverse scattering problem (a.k.a Marchenko equations, Gelfand-Levitan-Marchenko equations) [6] formulas (3.5.18), (3.5.18'), (3.5.21), p.290], which are Volterra integral equations. For the spectral problem $-\partial_x^2 \psi + u(x, 0; \gamma) \psi = k^2 \psi$ they are
\[ K_+ (x, y) + R_+ (x + y) + \int_{x}^{+\infty} K_+ (x, z) R_+ (z + y) dz = 0, \quad y \geq x, \]
\[ K_- (x, y) + R_- (x + y) + \int_{-\infty}^{x} K_- (x, z) R_- (z + y) dz = 0, \quad y \leq x, \]
where
\[ \mathcal{R}_+(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(k)e^{ikx} dk = -\sqrt{\frac{2}{\pi}} e^{-x^2}, \quad \mathcal{R}_-(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(k)e^{-ikx} dk, \]

and \( R(k) = R(k; \gamma) \) is defined in [11], and \( L(k) = L(k; \gamma) \) is defined in [15]. The link with the function \( u(x; \gamma) = u(x, 0; \gamma) \) is given by the formulas
\[ K_+(x, x) = \frac{1}{2} \int_{-\infty}^{+\infty} u(\tilde{x}) d\tilde{x}, \quad K_-(x, x) = \frac{1}{2} \int_{-\infty}^{x} u(x) d\tilde{x}. \]

Then the integral \([11, 10]\), computed at \( t = 0 \), can be treated as follows: for any real \( x_0 \),
\[ K(\gamma) = \int_{-\infty}^{+\infty} xu(x, 0; \gamma) dx = \int_{-\infty}^{x_0} xu(x, 0; \gamma) dx + \int_{x_0}^{+\infty} xu(x, 0; \gamma) dx = \]
\[ = -\int_{-\infty}^{x_0} (x_0 - x) u(x, 0; \gamma) dx + \int_{x_0}^{+\infty} (x - x_0) u(x, 0; \gamma) dx + x_0 \int_{-\infty}^{+\infty} u(x) dx = \]
\[ = -\int_{-\infty}^{x_0} \left( \int_{-\infty}^{x} u(\tilde{x}, 0; \gamma) d\tilde{x} \right) dx + \int_{x_0}^{+\infty} \left( \int_{-\infty}^{+\infty} u(\tilde{x}, 0; \gamma) d\tilde{x} \right) dx + x_0 \int_{-\infty}^{+\infty} u(x, 0; \gamma) dx = \]
\[ = -\int_{-\infty}^{x_0} 2K_-(x, x) dx + \int_{x_0}^{+\infty} 2K_+(x, x) dx + x_0 \cdot \frac{1}{2\pi} \text{Li}_2(\gamma), \]

since \( \int_{-\infty}^{+\infty} u(x, 0; \gamma) dx = 2T_1(\gamma) = \sqrt{\frac{2}{\pi}} \text{Li}_2(\gamma) \) in view of formulas [12] and asymptotics [6, 8].

**Remark 4.** Function \( q(x; \gamma) \) seems to be positive. For \( x \to -\infty \), approximately
\[ \int_{-\infty}^{+\infty} q^2(z; \gamma) dz - 2T_1(\gamma) \approx -q(x; \gamma) \left( \frac{e^{2\kappa\gamma} - 1}{2\kappa\gamma} \right), \quad \gamma < 1, \]
\[ \int_{-\infty}^{+\infty} q^2(z; 1) dz - 2T_1(1) \approx -q(x; 1). \]

**Remark 5.** Consider rarefaction problem for KdV, \( u \to c^2, x \to -\infty, \quad u \to 0, x \to +\infty \). It has conserved quantities, (independent of \( x \) and \( t \))
\[ K = 3 \int_{-\infty}^{x} (u^2 - c^4) d\tilde{x} + 3 \int_{x}^{+\infty} u^2 d\tilde{x} + 3c^4 (x + 4c^2 t), \]
\[ H = \int_{-\infty}^{x} z (u(z, t) - c^2) dz + \int_{x}^{+\infty} zu(z, t) dz + \frac{c^2}{2} x^2 - 6c^2 t^2 + K t. \]

Conjectural and non rigorous **Remark 6.** Numerical experiment (based on section [4, 7]) allows us to conjecture that
\[ \frac{L_{-1}(\gamma)}{2\kappa_1} = 1 - L_1(1) - \frac{L_2(1)}{\kappa_1} - \frac{L_3(1)}{\kappa_1^3} + O(\kappa_1^4), \quad \gamma \to 1 - 0. \]
Then formulas (5) might be obtained from formulas (4) by taking formal limit $\kappa, \gamma \to 0$, and neglecting terms of positive order in $\kappa, \gamma$.

Indeed, we get formally that for $\gamma \to 1 - 0$,

$$q(x; \gamma) \sim \frac{2}{L_1(1) - 2x} + \frac{(2l_2 - L_1(1)^2)\kappa, \gamma}{(L_1(1) - 2x)^2} + O(\kappa, \gamma).$$

Numerics $l_2 \approx 0.678\,838\,896\,877 \approx \frac{1}{4}L_1(1)^2$ suggest us to conjecture $l_2 = \frac{1}{4}L_1(1)^2$, and then we can simplify the expression for $\kappa, \gamma^2$ term:

$$q(x; \gamma) \sim \frac{2}{L_1(1) - 2x} + \frac{6l_3 + 2x(4x^2 - 6L_1(1)x + 3L_1(1)^2)}{3(L_1(1) - 2x)^2} \kappa^2 + O(\kappa, \gamma).$$

Sweet life ends here: because of presence of $x$, we can not make this term to be equal to 0. We have

$$l_3 \approx 0.236\,014\,873 \neq \frac{1}{6}L_1(1)^3 \approx 0.263\,659\,741.$$

Conjectural and non rigorous Remark 7. It seems that $M(\gamma) \to M(1)$ as $\gamma \to 1 - 0$. Indeed, splitting the integral for $M(\gamma)$, $\gamma < 1$ into two parts $(-\infty, x)$ and $(x, +\infty)$ for $x$ sufficiently large negative, and substituting asymptotics (6), we find

$$\gamma < 1 : \int_{-\infty}^{x} \tilde{x}q^2(\tilde{x}; \gamma) d\tilde{x} \sim \frac{4x R L_{-1}(\gamma)^2 e^{4x \kappa}}{1 - L_{-1}(\gamma)^{2}\kappa, \gamma} + \ln \left( 1 - \frac{L_{-1}(\gamma)^2 e^{4x \kappa}}{4 \kappa^2} \right),$$

which in the $\kappa, \gamma \to 0$ limit gives, using (12),

$$\frac{2x}{L_1(1) - 2x} + \ln(L_1(1) - 2x) + \ln(2\kappa, \gamma) + O(\kappa, \gamma).$$

On the other hand,

$$\gamma = 1 : \int_{-\infty}^{x} \left( \tilde{x}q^2(\tilde{x}; 1) - \frac{1}{\tilde{x}} \right) d\tilde{x} \sim \frac{2x}{L_1(1) - 2x} + \ln(L_1(1) - 2x) + 1 - \ln 2 - \ln |x|.$$

Comparing, we come to a formal conclusion that $M(1 - 0) = M(1)$.

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2 Proof of (a),(b)

Lemma 4. 1. For any fixed $\gamma \in [0, 1]$, $x \in \mathbb{R}$, the Riemann-Hilbert problem (7) has the unique solution. This solution is continuous in parameters $(x; \gamma) \in \mathbb{R} \times [0, 1]$.

2. For any $\gamma \in [0, 1]$ and $x \in \mathbb{R}$, the solution of the RHP (7) is infinitely differentiable in $x$.

Proof. The proof is almost word-to-word repetition of the similar proof from [5], p. 13-17 (for the existence part also cf [1]). For the convenience of the reader we present it also here.

Existence. Let $x \in \mathbb{R}$ and $\gamma \in [0, 1]$ be fixed. We look for the solution $M(x; k; \gamma)$ of the RHP (7) in the form:

$$M(x; k; \gamma) = 1 + \frac{1}{2\pi i} \int_{\mathbb{R}} \left[ 1 + Z(x; s; \gamma) \right] \left[ 1 - J_M(x; s; \gamma) \right] ds, \quad s \in \mathbb{C} \setminus \mathbb{R}. \quad (13)$$
One can show that the Cauchy integral (13) satisfies all the properties of the RHP if and only if the matrix $Z(x; k; \gamma)$ satisfies the singular integral equation

$$Z(x; s; \gamma) - K[Z](x; s; \gamma) = F(x; s; \gamma), \quad s \in \mathbb{R}. \quad (14)$$

The singular integral operator $K$ and the right-hand side $F(x; k; \gamma)$ are as follows:

$$K[Z](x; s; \gamma) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{Z(z; s; \gamma)(1 - JM(x; z; \gamma))}{(z - s)_+} dz,$$

$$F(x; s; \gamma) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1 - JM(x; s; \gamma)}{(z - s)_+} dz.$$

We consider this integral equation in the space $L^2(\mathbb{R})$ of $2 \times 2$ matrix complex-valued functions $Z(k) := Z(x; k; \gamma)$. The operator $K$ is defined by the jump matrix $JM(x; k; \gamma)$ and the generalized function $1 - JM(x; k; \gamma)$ is bounded as a function of variable $k$. Hence, the Cauchy operator $F(x; k; \gamma)$ is also in $L^2(\mathbb{R})$. The function $1 - JM(x; k; \gamma)$ satisfies all the properties of the RHP if and only if the Cauchy operator $Z(x; s; \gamma) - K[Z](x; s; \gamma) = F(x; s; \gamma)$ is bounded in the space $L^2(\mathbb{R})$.

The matrix-valued function $1 - JM(x; k; \gamma)$ as a function of variable $k$ is in the space $L^2(\mathbb{R})$. Hence, the function $F(x; k; \gamma)$ is also in $L^2(\mathbb{R})$. The matrix-valued function $1 - JM(x; k; \gamma)$ is bounded as a function of variable $k$.

It is a classical fact that the Cauchy operator

$$C_+[f](s) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(z)}{(z - s)_+} dz$$

is bounded in the space $L^2(\mathbb{R})$. The matrix-valued function $1 - JM(x; k; \gamma)$ as a function of variable $k$ is in the space $L^2(\mathbb{R})$. Hence, the function $F(x; k; \gamma)$ is also in $L^2(\mathbb{R})$. The matrix-valued function $1 - JM(x; k; \gamma)$ is bounded as a function of variable $k$. Hence, the Cauchy operator $F(x; k; \gamma)$ is also bounded.

In our case the contour coincides with the real axis, and hence the second condition of the Schwartz reflection principle is trivial in our case.

Then Theorem 9.3 from [8] (p.984) guarantees the $L^2$ invertibility of the operator $Id - K$. Therefore, the singular integral equation (14) has a unique solution $Z(x; k; \gamma) \in L^2(\mathbb{R})$ for any fixed $x \in \mathbb{R}$, $\gamma \in [0, 1]$ and formula (13) gives the solution of the above RHP.

The operator $Id - K$ depends continuously on the parameters $(x; \gamma) \in \mathbb{R} \times [0, 1]$. Therefore the inverse operator $(Id - K)^{-1}$ also has this property. Hence, the solution $Z(x; k; \gamma)$ of the singular integral equation (14) also depends continuously on $x$, $\gamma$. From representation (13) we obtain the required statement for $M(x; k; \gamma)$.

Uniqueness. The uniqueness for the RHP (1) in the space $L^2(\mathbb{R})$ is proved in [7] (p.194-198). Smoothness. We can differentiate the singular integral equation (13) in $x$ as many times as desired. Indeed, to differentiate this equation and matrix $Z$ it is sufficient that its formal derivatives are convergent. The function $1 - JM(x; s; gamma)$ is responsible for decaying of integrands in the singular integral equation. Since on the real line $1 - JM(x; s; gamma)$ decays exponentially fast w.r.t. $s \rightarrow \pm \infty$. Singular integral equations obtained from (14) by differentiation w.r.t. $x$ are of the same form as the original one (14), only the r.h.s. of these equations vary. Indeed, writing (14) in the form

$$Z(x; s; \gamma) - C_+[Z(x; s; \gamma)(1 - JM(x; s; \gamma))] = F(x; s; \gamma),$$

for its formal derivative w.r.t. $x$ we get

$$Z'_x(x; s; \gamma) - C_+[Z'_x(x; s; \gamma)(1 - JM(x; s; \gamma))] = F'_x(x; s; \gamma) := F_x(x; s; \gamma) := F(x; s; \gamma) - C_+[Z(x; s; \gamma)JM(x; s; \gamma)],$$

for its formal derivative w.r.t. $s$ we get

$$Z'_s(x; s; \gamma) - C_+[Z'_s(x; s; \gamma)(1 - JM(x; s; \gamma))] = F'_s(x; s; \gamma) := F_s(x; s; \gamma) := F(x; s; \gamma) - C_+[Z(x; s; \gamma)JM(x; s; \gamma)].$$

for its formal derivative w.r.t. $\gamma$ we get

$$Z'_\gamma(x; s; \gamma) - C_+[Z'_\gamma(x; s; \gamma)(1 - JM(x; s; \gamma))] = F'_\gamma(x; s; \gamma) := F_\gamma(x; s; \gamma) := F(x; s; \gamma) - C_+[Z(x; s; \gamma)JM(x; s; \gamma)].$$
Finally, writing an expansion real line. The symmetries of the elements of the expansion follows from the symmetries in Lemma 6. This follows from the corresponding symmetries for the jump matrix, Proof. The proof is standard and uses the fact that the derivative $M(x; k; \gamma)$, $A = 1 \in J(x, k; \gamma)$, where all $A_j = A_j(x; \gamma)$, $B_j = B_j(x; \gamma)$ are real. Furthermore, $q(x; \gamma) = 2B_1(x; \gamma) \in \mathbb{R}$, $\partial_x(A_1(x; \gamma)) = \frac{1}{2}q^2(x; \gamma) = 2B_1(x; \gamma)^2 \in \mathbb{R}$. Proof. The possibility to expand the function $M(x; k; \gamma)$ for large $k$ follows from the representation and the fact that the $1 - J_M(x; s; \gamma)$ is exponentially small for $s$ on the infinite part of the real line. The symmetries of the elements of the expansion follows from the symmetries in Lemma 5. Finally, writing an expansion

$$M(x; k; \gamma) = 1 + \frac{m_1(x; \gamma)}{k} + \frac{m_2(x; \gamma)}{k^2} + \ldots$$
and substituting this into
\[ M_x + ik[\sigma_3, M] = QM, \]
where \([A, B] = AB - BA\) is the matrix commutator, we obtain
\[
\begin{align*}
\frac{m_{1,x}}{k} + \frac{m_{2,x}}{k^2} + \ldots + i[\sigma_3, m_1] + \frac{i[\sigma_3, m_2]}{k} + \frac{i[\sigma_3, m_3]}{k^2} + \ldots = Q + \frac{Qm_1}{k} + \frac{Qm_2}{k^2} + \ldots
\end{align*}
\]
Comparing the (off-diagonal) terms of the order \(k^0\), and diagonal terms of the order \(k^{-1}\), we find that
\[ q(x; \gamma) = -2i(m_1)_{12}, \quad \partial_x(m_1)_{11} = \frac{i}{2}q^2, \]
which finishes the proof. \(\square\)

3 Analysis for \(x \to +\infty\).

Lemma 9. Let \(A_1(x; \gamma)\) be as in Lemma 8. Then
\[
A_1(x; \gamma) = -\frac{1}{2} \int_x^{+\infty} q^2(\tilde{x}; \gamma) d\tilde{x} = -2 \int_x^{+\infty} B_1(\tilde{x}; \gamma)^2 d\tilde{x}.
\]

Proof. Let us draw two lines \(L_1 = \mathbb{R} + i, L_2 = \mathbb{R} - i\), with orientation as on the real line. Denote the domain between \(L_1\) and \(\mathbb{R}\) by \(\Omega_1\), the other domain in \(3k > 0\) by \(\Omega_3\), the domain between \(L_2\) and \(\mathbb{R}\) by \(\Omega_2\), and the remaining domain by \(\Omega_4\). Denote \(\Sigma = L_1 \cup L_2\) to be an oriented contour.

Define a function
\[
P(x; k; \gamma) = M(x; k; \gamma) \cdot \begin{cases}
1 & k \in \Omega_1, \\
-R(k; \gamma) e^{2ikx} & k \in \Omega_2, \\
0 & k \in \Omega_4.
\end{cases}
\]

The function \(P(x; k; \gamma)\) solves the following RHP.

Riemann-Hilbert problem 2. To find a \(2 \times 2\) matrix-valued function \(P(x, t; k; \gamma)\) that satisfies the following properties:

- **analyticity:** \(P(x; k; \gamma)\) is analytic in \(k \in \mathbb{C} \setminus \Sigma\),
- **jumps:** \(P_- = P_+ J_P\), where
  \[
  J_P = \begin{pmatrix}
  1 & 0 \\
  -R(k; \gamma) e^{2ikx} & 1
  \end{pmatrix}, \quad k \in L_1, \\
  \begin{pmatrix}
  1 & \overline{R(k; \gamma)} e^{-2ikx} \\
  0 & 1
  \end{pmatrix}, \quad k \in L_2;
  \]
- **asymptotics at the infinity:** \(P(k) \to 1\) as \(k \to \infty\).

For \(x \to +\infty\), the jump matrix for \(P\) is uniformly exponentially close to \(1\) everywhere on the contour \(\Sigma = L_1 \cup L_2\), and hence the matrix
\[
\begin{pmatrix}
  iA_1(x; \gamma) & iB_1(x; \gamma) \\
  -iB_1(x; \gamma) & -iA_1(x; \gamma)
\end{pmatrix} = \lim_{k \to \infty} k(M(x; k; \gamma) - I) = \lim_{k \to \infty} k(P(x; k; \gamma) - I)
\]
tends to 0 exponentially fast as \(x \to +\infty\). Then, firstly, \(\int_x^{+\infty} B_1^2(\tilde{x}; \gamma)\) exists (converges), and secondly, by Lemma 8 from
\[
\partial_x A_1(x; \gamma) = 2B_1(x; \gamma)^2
\]
we get
\[ A_1(x; \gamma) = -2 \int_x^{+\infty} B_1(\tilde{x}; \gamma)^2 d\tilde{x}. \]

\[ \square \]

4 \hspace{1em} Analysis for \( x \to -\infty \) and proof of (c), (d)

4.1 Functions \( T(k; \gamma), L(k; \gamma), \delta(k; \gamma) \).

First of all, let us introduce some auxiliary functions.
Define an entire function \( T(k; \gamma) \) by the formula

\[
T(k; \gamma) = \begin{cases}
\exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - |R(s; \gamma)|^2) \frac{ds}{s - k} \right], & \Re k > 0, \\
(1 - R(k; \gamma)\overline{R(k; \gamma)}) \cdot \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - |R(s; \gamma)|^2) \frac{ds}{s - k} \right], & \Re k < 0.
\end{cases}
\]

Furthermore, define the left reflection coefficient \( L(k; \gamma) \) by the formula

\[ L(k; \gamma) = -\frac{\overline{R(k; \gamma)} T(k; \gamma)}{\overline{T(k; \gamma)}} = -\frac{R(k; \gamma) T(k; \gamma)^2}{1 - R(k; \gamma)\overline{R(k; \gamma)}} \tag{15} \]

We collect the properties of \( T(k; \gamma), L(k; \gamma) \) in the following lemma.

Lemma 10. \hspace{1em} 1. \( T(k; \gamma), R(k; \gamma) \) are entire function in \( k \), \( L(k; \gamma) \) is a meromorphic function with a pole at \( k = i\kappa, \gamma \).

2. Zeros and poles.
   - (a) The only zero of \( T(k; \gamma) \) is a simple pole at \( k = -i\kappa, \gamma \), where
     \[ \kappa, \gamma = \sqrt{-2\ln \gamma} \geq 0. \]
     It is a simple zero in both cases \( \gamma < 1 \) and \( \gamma = 1 \).
   - (b) For \( \gamma \in (0, 1) \), the function \( L(k; \gamma) \) has a simple pole at \( k = i\kappa, \gamma \), and a simple zero at \( k = -i\kappa, \gamma \).
     For \( \gamma = 1 \), the function \( L(k; 1) \) is an entire function, and does not vanish at \( k = 0 \).

3. Scattering relations:
   - \( T(k; \gamma) \overline{T(k; \gamma)} = 1 - R(k; \gamma)\overline{R(k; \gamma)} = 1 - L(k; \gamma)\overline{L(k; \gamma)} \), \( k \in \mathbb{C} \).
   - \( T(k; \gamma) \overline{T(k; \gamma)} = 1 - L(k; \gamma)\overline{L(k; \gamma)} \), \( k \in \mathbb{C} \).
   - \( T(k; \gamma)R(k; \gamma) + L(k; \gamma)\overline{T(k; \gamma)} = 0 \).

4. Symmetries:
   - \( \overline{T(-k; \gamma)} = T(k; \gamma) \), \( \overline{L(-k; \gamma)} = L(k; \gamma) \), \( \overline{R(-k; \gamma)} = R(k; \gamma) \).

5. Large \( k \) asymptotics of \( T(k; \gamma) \) in \( \Re k \geq 0 \):
   - \( T(k; \gamma) = 1 - \frac{iT_1(\gamma)}{k} + O(k^{-2}) \), \( k \to \infty, \Re k \geq 0 \),
   - where \( T_1(\gamma) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln(1 - \gamma e^{-s})ds = \frac{1}{\sqrt{2\pi}} Li_\frac{1}{2}(\gamma) > 0. \)
6. Pole condition of $L(k; \gamma)$ at $k = \pm ik_\gamma$

$$L(k; \gamma) = -\left(\frac{i}{k - ik_\gamma} L_0(\gamma) + iL_1(\gamma)(k - ik_\gamma) + L_2(\gamma)(k - ik_\gamma)^2 + \ldots\right)$$

For $\gamma = 1$,

$$L(k; \gamma) = -\left(1 + iL_1(1)k + L_2(1)k^2 + O(k^3)\right), \quad L_2(1) = \frac{-1}{2}L_1(1)^2 - \frac{1}{4}$$

and all $L_j(\gamma)$ are real.

Furthermore,

$$\frac{L_{-1}(\gamma)}{2k_\gamma} = 1 - \frac{1}{2}L_1(1)\kappa_\gamma + \frac{1}{2}L_1(1)^2\kappa_\gamma^2 + O(\kappa_\gamma^3), \quad \gamma \to 1 - 0.$$ 

Proof. To prove that the function $T(k; \gamma)$ is indeed entire, it suffices to establish continuity across the real line. This follows by Sokhotsky-Plemelj formula. The scattering relation follows from the definition of $T(k; \gamma)$.

Regarding poles and zeros, observe first that for $\gamma \in (0, 1]$, the only zeros of the function

$$1 - R(k)\overline{R(k)} = 1 - \gamma e^{-\frac{1}{2}k^2} = 1 - e^{-\frac{1}{2}(k^2 + \kappa_\gamma^2)}$$

are $k = \pm ik_\gamma$.

This is sufficient to prove all the statements about zeros and poles for $\gamma \neq 1$.

To treat also $\gamma = 1$, and being able to make transition for $\gamma \to 1 - 0$, it is useful to introduce two auxiliary functions, $\delta(k; \gamma)$ and $\hat{\delta}(k; \gamma; a)$.

Namely, define

$$\delta(k; \gamma) = \begin{cases} T(k; \gamma), & \exists k > 0, \\ \overline{T(k; \gamma)}, & \exists k < 0. \end{cases}$$

The function $\delta(k; \gamma)$ solves the conjugation problem

$$\frac{\delta_+(k; \gamma)}{\delta_-(k; \gamma)} = 1 - |R(k; \gamma)|^2, \quad k \in \mathbb{R},$$

and $\delta(k; \gamma) \to 1$ as $k \to \infty$. Furthermore, define

$$\hat{\delta}(k; \gamma; a) = \begin{cases} \frac{k + a}{k + ik_\gamma} \delta(k; \gamma), & \exists k > 0, \\ \frac{k - a}{k - ik_\gamma} \delta(k; \gamma), & \exists k < 0. \end{cases}$$

Here $a > \kappa_\gamma$ is an arbitrary parameter; we can keep $a = 1$ for all $\gamma \in \left(\frac{1}{\sqrt{2}}, 1\right]$. The latter formula is valid also for $\gamma = 1$, when $\kappa_1 = 0$. Function $\hat{\delta}(k; \gamma)$ solves the following scalar conjugation problem:

$$\frac{\hat{\delta}_+(k; \gamma)}{\hat{\delta}_-(k; \gamma)} = \frac{k^2 + a^2}{k^2 + \kappa_\gamma^2} (1 - |R(k; \gamma)|^2) \equiv (k^2 + a^2) \frac{1 - e^{-\frac{1}{2}(k^2 + \kappa_\gamma^2)}}{k^2 + \kappa_\gamma^2}, \quad k \in \mathbb{R},$$

and $\hat{\delta}(k; \gamma) \to 1$ as $k \to \infty$.

The functions $\delta, \hat{\delta}$ possesses the symmetries

$$\delta(-k; \gamma) = \delta(k; \gamma) = \frac{1}{\overline{\delta(k; \gamma)}} = \frac{1}{\overline{\delta(-k; \gamma)}}, \quad \hat{\delta}(-k; \gamma; a) = \hat{\delta}(k; \gamma; a) = \frac{1}{\overline{\hat{\delta}(k; \gamma; a)}}, \quad \overline{\hat{\delta}(-k; \gamma; a)} = \frac{1}{\overline{\hat{\delta}(-k; \gamma; a)}.}$$

The function $\hat{\delta}(k; \gamma; a)$ can be written explicitly,

$$\hat{\delta}(k; \gamma; a) = \exp \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln \left\{ \frac{x^2 + a^2}{x^2 + k^2} \left(1 - e^{-\frac{1}{2}(x^2 + \kappa^2)}\right)\right\}}{x - k} \, dx \right].$$
Denote for further usage the coefficient of square of \( \hat{d} \) for \( k \to ik_{\gamma}, \)

\[
\hat{d}^2(k; \gamma; a) = \exp \left( c_0 + ic_1(k - ik_{\gamma}) + c_2(k - ik_{\gamma})^2 + ic_3(k - ik_{\gamma})^3 + \ldots \right) \quad k \to ik_{\gamma},
\]

\[
\hat{d}^2(k; \gamma; a) = \exp \left( -c_0 + ic_1(k + ik_{\gamma}) - c_2(k + ik_{\gamma})^2 + ic_3(k + ik_{\gamma})^3 + \ldots \right) \quad k \to -ik_{\gamma},
\]

where

\[
c_j = c_j(\gamma; a) := \frac{1}{\pi i} \int_{\Sigma_a} \ln \frac{s^2 + a^2}{s + ik_{\gamma}} \left( 1 + e^{-\frac{i}{2}(s^2 + \kappa_{\gamma}^2)} \right) ds \in \mathbb{R}, \quad j \text{ is even, } j \geq 0,
\]

\[
= -\frac{1}{\pi i} \int_{\Sigma_a} \ln \frac{s^2 + a^2}{s + ik_{\gamma}} \left( 1 + e^{-\frac{i}{2}(s^2 + \kappa_{\gamma}^2)} \right) ds \in \mathbb{R}, \quad j \text{ is odd, } j \geq 0,
\]

In the case \( \gamma = 1 \) the limits in \((19)\) should be understood in the sense \( k \to 0, \Im k > 0 \) and \( k \to 0, \Im k < 0 \), respectively.

Since the r.h.s. of \((17)\) does not vanish in the layer \( |\Im k| < \alpha \), the logarithm in the latter integral is well-defined not only on the real line, but also in the above mentioned layer.

Hence, when computing \( \hat{d}(k; \gamma; a) \) numerically at the point \( ik_{\gamma} \), for \( \gamma = 1 \) or \( \gamma \) close to 1, we can deform the contour of integration, integrating instead over the contour \( \Sigma_a = (-\infty, -a/4) \cup (-\frac{a}{4}, -\frac{a}{2}) \cup (-\frac{a}{2}, \frac{a}{2}) \cup (\frac{a}{2}, +\infty) \).

Furthermore, since the r.h.s. in \((17)\) is uniformly continuous as \( \gamma \to 1 - 0 \) and non-vanishing, the function \( \hat{d}(k; \gamma) \) is also uniformly continuous as \( \gamma \to 1 - 0 \). This means that

\[
\hat{d}(k; \gamma; a) \to \hat{d}(k; 1; a) \quad \text{as} \quad \gamma \to 1 - 0 \quad \text{uniformly w.r.t.} \quad k \in \mathbb{C}.
\]

The function \( L(k; \gamma) \) can be written with the help of function \( \hat{d}(k; \gamma; a) \) as follows:

\[
L(k; \gamma) = \frac{e^{-\frac{i}{2}(k^2 + \kappa_{\gamma}^2)}}{1 - e^{-\frac{i}{2}(k^2 + \kappa_{\gamma}^2)}} \frac{(k + ik_{\gamma})^2}{(k + ia)^2} \hat{d}^2(k; \gamma; a), \quad \Im k > 0,
\]

\[
= e^{-\frac{i}{2}(k^2 + \kappa_{\gamma}^2)} \left( 1 - e^{-\frac{i}{2}(k^2 + \kappa_{\gamma}^2)} \right) \frac{(k - ia)^2}{(k - ik_{\gamma})^2} \hat{d}^2(k; \gamma; a), \quad \Im k < 0.
\]

From this representation we see that indeed, for \( \gamma < 1 \), in \( \Im k > 0 \) there is a simple pole at \( k = ik_{\gamma} \), and in \( \Im k < 0 \) there is a simple zero at \( k = -ik_{\gamma} \). Furthermore, for \( \gamma = 1 \), \( \kappa_1 = 0 \), the \( L(k; 1) \) does not have neither zero nor pole at \( k = 0 \).

Furthermore, expanding \((20)\) into series for \( k \to ik_{\gamma}, \) for \( \gamma < 1 \) we obtain

\[
L(k; \gamma) = -\left( \frac{L_{-1}(\gamma)}{k - ik_{\gamma}} + L_0(\gamma) + L_1(\gamma)k_{\gamma}^2 + \ldots \right) := \frac{-4ie^{\gamma a}(\gamma a)}{(a + \kappa_{\gamma}^2)^2 (k - ik_{\gamma})} + \mathcal{O}(1), \quad k \to ik_{\gamma},
\]

whence

\[
L_1(\gamma) = \frac{4ie^{\gamma a}(\gamma a)}{(a + \kappa_{\gamma}^2)^2}.
\]

For \( \gamma = 1 \), \( \kappa_1 = 0 \), both expressions in \((20)\) must give the same series at \( k \to 0 \). Thus,

\[
e^{a(1; a)} = \frac{\alpha^2}{2}, \quad c_2(1; a) = -\frac{1}{4} + \frac{1}{a^2}.
\]

\[
L(k; 1) = -\left( 1 + i\left( c_1(1; a) + \frac{2}{a} \right) k - \left( \frac{1}{4} + \frac{1}{2} \left( c_1(1; a) + \frac{2}{a} \right)^2 \right) k^2 + \mathcal{O}(k^3) \right).
\]

\[
\square
\]
Remark 8. Let us mention, that for $\gamma = 1$, we have locally as $k \to 0$

$$T(k;1) = \frac{-ik}{\sqrt{2}} + O(k^2), \quad k \to 0.$$  

$$L(k; 1) = -i \left(1 + iL_{1}(1)k + O(k^2)\right), \quad k \to 0, \quad \text{and} \quad L_{1}(1) \in \mathbb{R}.$$ 

The fact that $L_{1}(1)$ is real follows from $|L(k;\gamma)|^2 < 1$ for $k \in \mathbb{R}$. Furthermore, for $\gamma < 1$, we have $R(i\kappa;\gamma) = -i$, and

$$L_{-1}(\gamma) = \frac{1}{\kappa_{\gamma}} T(i\kappa;\gamma)^{2} > 0.$$ 

Let us also mention another formula for $L_{-1}(\gamma)$, which can be derived from the previous ones,

$$\ln \frac{L_{-1}(\gamma)}{2\kappa_{\gamma}} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \ln \frac{1 - e^{-\frac{1}{4}k_{\gamma}^{2}}} {s - i} ds.$$ 

It follows from (21) and the first of the formulas (22) that

$$\lim_{\gamma \to 1^{-}} \frac{L_{-1}(\gamma)}{2\kappa_{\gamma}} = 2 \pi i \exp \left\{ \frac{1}{\pi i} \int_{\Sigma_{\delta}} \ln \left( \frac{1 - e^{-\frac{1}{4}k_{\gamma}^{2}}}{s} \right) ds \right\} = 1.$$  

4.2 Long $x \to -\infty$ analysis for $\gamma < 1$, and proof of (c)

Since we are mostly interested in $\gamma$ that are close to 1, we restrict here our attention to $\gamma \in (\frac{1}{\sqrt{6}}, 1) \approx (0.6065, 1)$ (for $\gamma < 1/\sqrt{6}$ the analysis can be done in a more simple fashion). For such $\gamma$, we have $\kappa_{\gamma} < 1$ and hence the point $i\kappa_{\gamma}$ lies in the domain $\Omega_{4}$.

Define a function

$$N(x; k; \gamma) = M(x; k; \gamma) \cdot \begin{cases} T(k;\gamma)^{-\sigma_{3}} \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right), k \in \Omega_{1}, \\ \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right)^{-\sigma_{3}}, k \in \Omega_{3}, \\ \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right)^{-\sigma_{3}}, k \in \Omega_{4}. \end{cases}$$

The function $N(x; k; \gamma)$ solves the following RHP.

Riemann-Hilbert problem 3. To find a $2 \times 2$ matrix-valued function $N(x; k; \gamma)$ that satisfies the following properties:

- **analyticity:** $N(x, t; k; \gamma)$ is meromorphic in $k \in (\mathbb{C} \setminus \Sigma)$, with simple poles at $k = \pm i\kappa_{\gamma}$, and continuous up to the boundary $k \in \Sigma = L_{1} \cup L_{2}$;
- **jumps:** $N_{-}(x; k; \gamma) = N_{+}(x; k; \gamma) J_{N}(x; k; \gamma)$, where

$$J_{N} = \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right), k \in L_{1}, \quad J_{N} = \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right), k \in L_{2};$$

- **poles at $k = \pm i\kappa_{\gamma}$ function:**

$$N(x; k; \gamma) \left( \begin{array}{cc} 1 & \frac{R(k;\gamma) T(k;\gamma)^{2}}{1-R(k;\gamma) R(k;\gamma)} e^{-2ikx} \\ 0 & 1 \end{array} \right).$$
is regular at \( k = \kappa_\gamma \), function

\[
\mathbf{N}(x; k; \gamma) \begin{pmatrix} 1 & 0 \\ \frac{1 - R(k ; \gamma)}{1 - R(k ; \gamma) R(E; \gamma)} \, e^{2ikx} & 1 \end{pmatrix}
\]

is regular at \( k = -i\kappa \);

- asymptotics at the infinity:

\[
\mathbf{N}(x; k; \gamma) \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as} \quad k \to \infty.
\]

**Remark 9.** To see that the above RHP is indeed well-posed, we can rewrite the pole conditions as jump conditions across some circles of small radius \( \varepsilon_\gamma \leq \frac{\kappa_\gamma}{r} \) around the points \( \pm i\kappa \). To this end, define

\[
\mathbf{N}_{\text{reg}}(x; k; \gamma) = \mathbf{N}(x; k; \gamma) \begin{pmatrix} 1 & 0 \\ \frac{1 - R(k ; \gamma)}{1 - R(k ; \gamma) R(E; \gamma)} \, e^{-2ikx} & 1 \end{pmatrix}, \quad |k - i\kappa_\gamma| < \varepsilon_\gamma,
\]

\[
= \mathbf{N}(x; k; \gamma) \begin{pmatrix} 1 & 0 \\ \frac{-R(k ; \gamma)}{1 - R(k ; \gamma) R(E; \gamma)} \, e^{2ikx} & 1 \end{pmatrix}, \quad |k + i\kappa_\gamma| < \varepsilon_\gamma
\]

\[
= \mathbf{N}(x; k; \gamma), \quad \text{elsewhere}.
\]

Then \( \mathbf{N}_{\text{reg}} \) solves the RHP for \( \mathbf{N} \), with the pole conditions being replaced by the jump conditions

\[
\mathbf{N}_{\text{reg},-}(x; k; \gamma) = \mathbf{N}_{\text{reg},+}(x; k; \gamma) \begin{pmatrix} 1 & 0 \\ \frac{R(E; \gamma)}{1 - R(k ; \gamma) R(E; \gamma)} \, e^{-2ikx} & 1 \end{pmatrix}, \quad k \in C_{\gamma}(i\kappa_\gamma),
\]

\[
\mathbf{N}_{\text{reg},-}(x; k; \gamma) = \mathbf{N}_{\text{reg},+}(x; k; \gamma) \begin{pmatrix} 1 & 0 \\ \frac{-R(k ; \gamma)}{1 - R(k ; \gamma) R(E; \gamma)} \, e^{2ikx} & 1 \end{pmatrix}, \quad k \in C_{\gamma}(-i\kappa_\gamma),
\]

where by \( C_{\gamma}(a) \) we denote the circle with the center \( a \) and radius \( r \), oriented counter-close-wise, so that the positive side of the contour is inside the circle.

**Model problem \( \mathbf{N}_{\text{mod}}(x; k; \gamma) \).**

We see that the jumps for the \( \mathbf{N}(x; k; \gamma) \) are exponentially close to \( I \) as \( x \to -\infty \). This suggests that the main contribution to the asymptotics of \( \mathbf{N}(x; k; \gamma) \) comes from the pole conditions at the points \( k = \pm i\kappa_\gamma \). Introduce an ansatz

\[
\mathbf{N}_{\text{mod}}(x; k; \gamma) = \begin{pmatrix} 1 + \frac{\alpha(x; \gamma)}{k + i\kappa_\gamma} & \frac{i\beta(x; \gamma)}{k + i\kappa_\gamma} \\ \frac{-i\beta(x; \gamma)}{k - i\kappa_\gamma} & 1 - \frac{\alpha(x; \gamma)}{k - i\kappa_\gamma} \end{pmatrix}
\]

with real \( \alpha(x; \gamma), \beta(x; \gamma) \) which are to be determined from the condition that \( \mathbf{N}_{\text{mod}} \) satisfies the pole conditions of the RHP. Then the error matrix \( \mathbf{N}_{\text{err}}(x; k; \gamma) = \mathbf{N}(x; k; \gamma)\mathbf{N}_{\text{mod}}(x; k; \gamma)^{-1} \) will be regular at the points \( k = \pm i\kappa_\gamma \) (the simplest way to see this is to rewrite again the pole conditions as jump conditions), and the jumps for it will be exponentially close to \( I \) (smaller than \( e^{-C|x|} \) for any \( C > 0 \), which we can achieve by moving the contours \( L_1, L_2 \) towards \( \pm i\infty \)), provided that \( A, B \) are uniformly bounded. Hence, we would see that indeed \( \mathbf{N}_{\text{mod}}(x; k; \gamma) \) is close to \( \mathbf{N}(x; k; \gamma) \) for \( x \to -\infty \).

Substituting the ansatz (24) into pole condition at \( k = i\kappa_\gamma \) of RHP (23) (just one of the condition suffices in view of symmetries), and recalling the definition (15), (16) of the left reflection coefficient...
Finally, we obtain the following conditions for $\alpha(x; \gamma)$, $\beta(x; \gamma)$:

\[
\begin{cases}
1 + \frac{\alpha(x; \gamma)}{2k_\gamma} L_{-1}(\gamma)e^{2k_\gamma x} - \beta(x; \gamma) = 0, \\
\frac{\beta(x; \gamma)}{2k_\gamma} L_{-1}(\gamma)e^{2k_\gamma x} - \alpha(x; \gamma) = 0, \\
\end{cases}
\]

whence

\[
\begin{cases}
\alpha(x; \gamma) = \frac{2k_\gamma e^{2k_\gamma \gamma} L_{-1}(\gamma)^2}{4k_\gamma^2 - e^{4k_\gamma \gamma} L_{-1}(\gamma)^2}, \\
\beta(x; \gamma) = \frac{4k_\gamma^2 e^{2k_\gamma \gamma} L_{-1}(\gamma)}{4k_\gamma^2 - e^{4k_\gamma \gamma} L_{-1}(\gamma)^2}. \\
\end{cases}
\]

We see that indeed $\alpha, \beta$ are bounded as $x \to -\infty$, and both of them are positive. One can check that for such choice of $\alpha, \beta$, we have $\det N_{mod}(x; k; \gamma) \equiv 1$. We have

\[
\lim_{k \to \infty} k(N(x; k; \gamma) - I) = \lim_{k \to \infty} k(M(x; k; \gamma)T(x; k; \gamma)^{-\sigma_3} - 1) = \begin{pmatrix}
iA_1(x; \gamma) + iT_1(\gamma) \\
iB_1(x; \gamma)
\end{pmatrix} = \begin{pmatrix}
iA_1(x; \gamma) - iT_1(\gamma)
\end{pmatrix},
\]

and hence

\[
A_1(x; \gamma) = -T_1(\gamma) + \alpha(x; \gamma) + \mathcal{O}(e^{-C|x|}), \quad B_1(x; \gamma) = \beta(x; \gamma) + \mathcal{O}(e^{-C|x|}),
\]

for any $C > 0$. This finishes proof for (c).

### 4.3 Long $x \to -\infty$ analysis for $\gamma = 1$, and proof of (d)

Here we again define function $N(x; k; \gamma = 1)$ by formula (23). The function $N(x; k; 1)$ solves the following RHP:

**Riemann-Hilbert problem 4.** To find a $2 \times 2$ matrix-valued function $N(x; k; 1)$ that satisfies the following properties:

- **analyticity:** $N(x; k; 1)$ is meromorphic in $k \in \mathbb{C} \setminus \Sigma$, with a simple pole at $k = 0$, and continuous up to the boundary $k \in \Sigma = L_1 \cup L_2$;
- **jumps:** $N_-(x; k; 1) = N_+(x; k; 1)J_N(x; k; 1)$, where

\[
J_N = \begin{pmatrix} \frac{R(k; 1)}{1 - R(k; 1)} & e^{-2ik_\gamma x} \\ 0 & 1 \end{pmatrix}, k \in L_1, \quad J_N = \begin{pmatrix} \frac{-R(k; 1)}{1 - R(k; 1)} & e^{2ik_\gamma x} \\ 1 & 1 \end{pmatrix}, k \in L_2;
\]

- **singularity at $k = 0$:** function

\[
N(x; k; 1) \begin{pmatrix} 1 - \frac{R(k; 1)}{1 - R(k; 1)} e^{-2ik_\gamma x} \\ 0 \\ 1 \end{pmatrix} k^{\sigma_3}
\]

is regular at $k = 0$, $\exists k \geq 0$,

-function

\[
N(x; k; 1) \begin{pmatrix} 1 - \frac{-R(k; 1)}{1 - R(k; 1)} e^{2ik_\gamma x} \\ 0 \\ 1 \end{pmatrix} k^{-\sigma_3}
\]

is regular at $k = 0$, $\exists k \leq 0$;

- **asymptotics at the infinity:**

\[
N(x; k; 1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ as } k \to \infty.
\]

**Remark 10.** To see that the above RHP is indeed well-posed, we can rewrite the pole conditions as jump conditions across a circle of a small radius $\varepsilon$ around the point 0, and the segment $(-\varepsilon, \varepsilon)$. 17
To this end, define

\[
N_{\text{reg}}(x; k; 1) = N(x; k; 1) \begin{pmatrix} \frac{1}{1-R(k; 1)} T(k; 1)^2 e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix} T(k; 1)^{-\sigma_3}, \quad |k| < \varepsilon, \Im k > 0,
\]

\[
= N(x; k; 1) \begin{pmatrix} 0 & 1 \\ \frac{1}{1-R(k; 1)} T(k; 1)^{-\sigma_3} & 1 \end{pmatrix} e^{2ikx} T(k; 1)^{-\sigma_3} = N(x; k; 1), \quad |k| < \varepsilon, \Im k < 0,
\]

Then \(N_{\text{reg}}\) solves the RHP for \(N\), with the singularity condition being replaced by the jump conditions

\[
N_{\text{reg},-}(x; k; \gamma) = N_{\text{reg},+}(x; k; \gamma) T(k; 1)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{1}{1-R(k; \gamma)} T(k; \gamma)^2 e^{2ikx} & 1 \end{pmatrix}, \quad k \in C^+_\varepsilon(0),
\]

\[
N_{\text{reg},-}(x; k; \gamma) = N_{\text{reg},+}(x; k; \gamma) T(k; 1)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{1}{1-R(k; \gamma)} T(k; \gamma)^2 e^{2ikx} & 1 \end{pmatrix}, \quad k \in C^-\varepsilon(0),
\]

\[
N_{\text{reg},-}(x; k; 1) = N_{\text{reg},+}(x; k; 1) \begin{pmatrix} 0 & 1 \\ -\frac{1}{1-R(k; 1)} e^{2ikx} & 1 - |R(k; 1)|^2 \end{pmatrix}, k \in (-\varepsilon, \varepsilon),
\]

where by \(C^+_\varepsilon(0)\) we denote part of the oriented counter-clock-wise circle \(C_\varepsilon(0)\), which lies in \(\Im k > 0\), and similar for \(C^-\varepsilon\).

Model problem \(N_{\text{mod}}(x; k; 1)\).

We see that the jumps for the \(N(x; k; 1)\) are exponentially close to \(I\) as \(x \to -\infty\). This suggests that the main contribution to the asymptotics of \(N(x; k; 1)\) comes from the singularity condition at the point \(k = 0\). Introduce an anzatz

\[
N_{\text{mod}}(x; k; 1) = \begin{pmatrix} 1 + \frac{i\alpha(x; 1)}{k} & \frac{i\beta(x; 1)}{k} \\ -\frac{i\beta(x; 1)}{k} & 1 - \frac{i\alpha(x; 1)}{k} \end{pmatrix}
\]

(25)

with real \(A(x; 1), B(x; 1)\) which are to be determined from the condition that \(N_{\text{mod}}\) satisfies the singularity conditions of the RHP \(\mathbb{R}\). Then the error matrix \(N_{\text{err}}(x; k; 1) = N(x; k; 1)N_{\text{mod}}(x; k; 1)^{-1}\) will be regular at the points \(k = 0\) (the simplest way to see this is to rewrite again the pole conditions as jump conditions), and the jumps for it will be exponentially close to \(I\) (smaller than \(e^{-C|x|}\) for any \(C > 0\), which we can achieve by moving the contours \(L_1, L_2\) towards \(\pm i\infty\)), provided that \(A, B\) are uniformly bounded. Hence, we would see that indeed \(N_{\text{mod}}(x; k; 1)\) is close to \(N(x; k; 1)\) for \(x \to -\infty\).

Substituting the anzatz (25) into the singularity condition at \(k = 0, \Im k > 0\) of RHP \(\mathbb{R}\) (just one of the condition suffices in view of symmetries), and recalling the definition (15), (19) of the left reflection coefficient \(L(k; 1)\), we obtain the following conditions for \(\alpha(x; 1), \beta(x; 1)\):

\[
\begin{cases}
\alpha(x; 1) = \beta(x; 1), \\
(\alpha(x; 1), L(1) - 2x) = 1
\end{cases}
\]

whence \(\alpha(x; 1) = \beta(x; 1) = \frac{1}{-2x + L(1)}\).

We see that indeed \(\alpha, \beta\) are bounded as \(x \to -\infty\), and both of them are positive. One can check that for such choice of \(\alpha, \beta\), we have \(\det N_{\text{mod}}(x; k; 1) \equiv 1\). We have

\[
\lim_{k \to i\infty} k(N(x; k; 1) - I) = \lim_{k \to i\infty} k(M(x; k; 1)T(x; k; 1)^{-\sigma_3} - I) = \begin{pmatrix} iA_1(x; 1) + iT_1(1) & iB_1(x; 1) \\
-iB_1(x; 1) & -iA_1(x; 1) - iT_1(1) \end{pmatrix},
\]

and hence

\[
A_1(x; 1) = -T_1(1) + \alpha(x; 1) + O(e^{-C|x|}), \quad B_1(x; 1) = \beta(x; 1) + O(e^{-C|x|}),
\]

for any \(C > 0\). This finishes proof for (d).
5 Expression for $\int_{x}^{+\infty} y(z)dz$, and proof of (e)

Expand the solution $M(x; k; \gamma)$ of the original RHP at $k = 0$,

$$M(x; k; \gamma) = M_0(x; \gamma) + kM_1(x; \gamma) + k^2M_2(x; \gamma) + k^3M_3(x; \gamma) + \ldots,$$

and substitute this into the differential equation for $M$:

$$M_x(x; k; \gamma) + ik[\sigma_3, M(x; k; \gamma)] = Q(x; \gamma)M(x; k; \gamma).$$

Comparing the elements of $k^0, k^1, \ldots$, we obtain

$$\partial M_0(x; \gamma) = Q(x; \gamma)M_0(x; k; \gamma), \quad \partial M_1(x; \gamma) + [\sigma_3, M_0(x; k; \gamma)] = Q(x; \gamma)M_1(x; k; \gamma), \ldots$$

Let us treat the first one. Denote, using symmetries (Lemma [5]),

$$M_0(x; \gamma) = \begin{pmatrix} r(x; \gamma) & w(x; \gamma) \\ w(x; \gamma) & r(x; \gamma) \end{pmatrix},$$

then we obtain

$$r_x(x; \gamma) = -q(x; \gamma)w(x; \gamma), \quad w_x(x; \gamma) = -q(x; \gamma)r(x; \gamma),$$

and the boundary conditions are

$$\lim_{x \to +\infty} r(x; \gamma) = 1, \quad \lim_{x \to +\infty} w(x; \gamma) = 0.$$

We obtain

$$(r + w)_x = -q(x)(r + w), \quad (r - w)_x = q(x)(r - w)$$

whence

$$r + w = \exp \left( \int_x^{+\infty} q(z; \gamma)dz \right), \quad r - w = \exp \left( -\int_x^{+\infty} q(z; \gamma)dz \right),$$

and finally

$$r(x; \gamma) = \cosh \left( \int_x^{+\infty} q(z; \gamma)dz \right), \quad w(x; \gamma) = \sinh \left( \int_x^{+\infty} q(z; \gamma)dz \right).$$

Now substitute this in the ingredients of the asymptotic analysis. We have

$$P = \begin{cases} \begin{pmatrix} M_1 - Re^{2ikx}M_2, & M_2 \\ M_1, & M_2 - 7Re^{-2ikx}M_1 \end{pmatrix}, & k \in \Omega_1, \\ \begin{pmatrix} M_1, & M_2 - 7Re^{-2ikx}M_1 \end{pmatrix}, & k \in \Omega_2, \end{cases}$$

and

$$P(k) = \begin{pmatrix} r(x; \gamma) & w(x; \gamma) \\ w(x; \gamma) & r(x; \gamma) \end{pmatrix} + O(k), \quad k \to 0.$$
Expanding the middle term, we see that the $k^{-2}$ term vanish because of $\alpha(x;1) = \beta(x;1)$, and the $k^{-1}$ term vanish because $\alpha(x;1) = \beta(x;1) = \frac{1}{2x+L_1(1)}$. Then, comparing the $k^0$ terms gives us

$$r(x) = \frac{(L_1(1) - 2x)^2 + 2L_2(1) + L_1(1)^2 + 1}{\sqrt{2}(L_1(1) - 2x)},$$

$$w(x) = -\frac{(L_1(1) - 2x)^2 + L_1(1)^2 + 2L_2(1)}{\sqrt{2}(L_1(1) - 2x)},$$

were we denoted (16)

$$T(k;1) = -\frac{ik}{\sqrt{2}} + T_2k^2 + T_3k^3 + \ldots, \quad L(k;1) = -(1 + iL_1(1)k + L_2(1)k^2 + L_3k^3 + \ldots).$$

and $r^2 - w^2 = 1$ gives us $L_1^2(1) + 2L_2(1) = -\frac{1}{2}$. Hence,

$$\exp \left( \int_{x}^{+\infty} q(z;1)dz \right) = r(x) - w(x) = \sqrt{2}(L_1(1) - 2x),$$

$$\exp \left( -\int_{x}^{+\infty} q(z;1)dz \right) = r(x) + w(x) = \frac{1}{\sqrt{2}(L_1(1) - 2x)}.$$

This proves (e).

6 Some conservative quantities.

Lemma 11. The quantities (10) do not depend on time $t$, and the quantity $M(1)$ does not depend on $x < 0$.

Proof. First of all, the integrals are convergent\(^5\). Then it is enough to differentiate w.r.t. $t$ and $x$, using (3). Let us consider for example (10c) and (10d).

(10c) The integral converges. Since $\partial_t M = (x(3q^4 - 2qq_{xx} + q_{x}^2) + 2qq_{x})^{+\infty}_{-\infty} - 3 \int_{-\infty}^{+\infty} (q^4 + q_x^2)dx + N = 0.$

(10d) The integral converges. Since $\partial_x M(1) = 0$, we conclude that it does not depend on $x < 0$.

$$(-4y^6 - 4y^3y_{xx} + 12y^2y_x^2 + 2y_{xx}y_{xxx} - (y_x)^2) = 4y^3y_t - 2y_{tt}y_{tx}.$$  

\(^5\)For $t \neq 0$ there are asymptotic formulas similar to (11). Because $\nu(\xi)$ is exponentially small for $x \to -\infty$, the $q(x,t)$ is also exponentially small, and hence integrals are convergent.
7 One-pager on Korteweg-de Vries equation

The KdV equation, \( u = u(x, t) \),
\[ u_t(x, t) - 6u(x, t) u_x(x, t) + u_{xxx}(x, t) = 0 \]
is the compatibility condition of the ordinary differential equations (Lax pair) for a function \( f = f(x, t; k) \),
\[
(26): -f_{xx} + u(x, t)f = k^2 f, \quad (26_0): f_t = (4k^2 + u(x, t)) f_x - (4k + u(x, t) + c) f,
\]
where \( c \) is an arbitrary constant. Let \( u_0(x) = u(x, 0) \) be an initial function, \( u_0(x) \to 0 \) as \( x \to \pm \infty \). Let \( f_x(k; x) \) be a solution of (26) for \( t = 0 \) with asymptotics
\[
f^+(x; k) = \alpha ikx (1 + \mathcal{O} (1)), \quad x \to +\infty, \quad f^-(x; k) = e^{-ikx} (1 + \mathcal{O} (1)), \quad x \to -\infty, \quad k \in \mathbb{R}.
\]
Define the spectral functions \( a(k), b(k), R(k) = \frac{\alpha(k)}{\alpha(1)} \) by relations
\[
f^-(x; k) = a(k) f^+(x; k) + b(k) f(x; k), \quad |a(k)|^2 - |b(k)|^2 = 1, \quad 2ika(k) = W(k) = \{ f^-(x; k), f^+(x; k) \}, \quad 2ikb(k) = \{ f^+(x; k), f^-(x; k) \},
\]
where the brackets \( \{ f, g \} = fg_x - fg \) denote the Wronskian. The \( 1 \times 2 \) vector function \( V(x; k) \),
\[
V(x; k) = \left( \begin{array}{c}
\frac{1}{a(k)} f^-(x; k)e^{ikx}, & f^+(x; k)e^{-ikx}
\end{array} \right), \quad \exists k > 0,
\]
has the jump across the real line
\[
V_-(x; k) = V_+(x; k) \left( \frac{1}{-R(k)e^{2ikx}} \frac{R(k)e^{-2ikx}}{1 - |R(k)|^2} \right), \quad k \in \mathbb{R}, \quad (27)
\]
where \( V_+(x; k) = V(x; k \pm i0) \) for real \( k \). Let \( 1 \times 2 \) vector-valued function \( V(x; t; k) \) has the jump (27) with \( 2ikx \) substituted with \( 2ikx + 8ik^3 t \) for all \( t \), and let \( V \) be normalized by the \( (1, 1) \) vector as \( k \to \infty \). Expanding \( f^\pm(x; k) = e^{\pm ikx} \left( 1 + \frac{1}{2k^2} + \frac{1}{3k^4} + \ldots \right) \), for \( k \to \infty, \exists k \geq 0 \), and substituting into (26), we find
\[
f_1^+ = \frac{i}{2} \int_{-\infty}^{+\infty} u(x)dx, \quad f_1^- = \frac{i}{2} \int_{-\infty}^{+\infty} u(x)dx, \quad f_2^\pm = \frac{i}{2}(f_1^\pm)^2 + \frac{i}{2} f_1^\pm, \quad a(k) = 1 - \frac{1}{2k^2} \int_{-\infty}^{+\infty} u + O(k^{-2}),
\]
\[
V(x; k) = (1, 1) + \frac{1}{2ik} \int_{-\infty}^{+\infty} u(x)dx \left( 1 - 1 + O(k^{-2}) \right), \quad V_{[1]} V_{[2]} = 1 + \frac{u}{2k^2} + O(k^{-3}), \quad (28)
\]
Function \( u(x, t) \) obtained from \( V(x; t; k) \) by formulas (28) is a solution of KdV for all \( t \). Furthermore, define a singular at \( k = 0 \) matrix
\[
M_{\text{sing}}(x; k) = \frac{1}{2} \frac{\alpha(1)}{\alpha(0)} \left( f^- + \frac{ik}{2k^2} f_x^- \right) e^{ikx} \left( f^+ + \frac{ik}{2k^2} f_x^+ \right) e^{-ikx}, \quad \exists k > 0,
\]
\[
= \sigma_1 M_{\text{sing}}(\overline{k}) \sigma_1 = \sigma_1 M_{\text{sing}}(-k) \sigma_1, \quad \text{where} \quad \sigma_1 := \left( \begin{array}{c}
0
1
\end{array} \right)
\]
and a regular at \( k = 0 \) matrix function
\[
M_{\text{reg}}(x; k) = \left( \frac{1 + i\alpha(x)}{\alpha(0)} \right) \frac{\alpha(1)}{\alpha(0)} \left( 1 - \frac{\alpha(0)}{\alpha(1)} \right) M_{\text{sing}}(x; k), \quad \text{where} \quad \alpha(x) = -\frac{1}{2f^+} \bigg|_{k=0}.
\]
Define
\[
q(x, t) = 2\alpha(x, t) := -\frac{\partial_t V_{[2]}(x; t; k)}{V_{[2]}(x; t; k)} \bigg|_{k=0}.
\]
Then
\[
u(x, t) = q^2(x, t) - q_x(x, t), \quad \text{and} \quad u_t - 6uu_x + u_{xxx} = \left( q_t - 6q^2 q_x + q_{xxx} \right)_x.
\]
8 One-pager on modified Korteweg-de Vries equation

The (defocusing) MKdV equation, \( q = q(x,t) \),
\[
q_t - 6q^2q_x + q_{xxx} = 0
\]
is the compatibility condition of the system, \( \Phi = \Phi(x,t;k) \),
\[
\Phi_x + ik\sigma_3\Phi = Q(x,t)\Phi, \quad \Phi_t + 4i k^2\sigma_3\Phi = Q_2(x,t;k)\Phi,
\]
where
\[
Q = Q(x,t) = \begin{pmatrix} 0 & -q(x,t) \\ -q(x,t) & 0 \end{pmatrix}, \quad Q_2(x,t;k) = 4k^2 Q - 2i (Q_x + Q^2)\sigma_3 k + (2Q^3 - Q_{xx}).
\]
Let \( q_0(x) = q(x,0) \to 0 \) as \( x \to \pm \infty \) be an initial function. Let
\[
\Phi^-(x,t;k) = \begin{pmatrix} \varphi^-(x;k) & \varphi^-(x;k) \\ \psi^-(x,t;k) & \varphi^-(x;k) \end{pmatrix}, \quad \Phi^+(x,t;k) = \begin{pmatrix} \psi^+(x;k) & \psi^+(x;k) \\ \varphi^+(x;k) & \varphi^+(x;k) \end{pmatrix}
\]
be solutions of the first of (29), normalized as \( \Phi^\pm(x;k) = e^{-ikx\sigma_3}(1 + O(1)), \quad x \to \pm \infty, k \in \mathbb{R}. \)
Define the transition matrix and spectral functions \( a(k), b(k), R(k) = \frac{b(k)}{a(k)} \),
\[
T(k) = (\Phi^+(x;k))^{-1}\Phi^-(x;k) = \begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix}, \quad |a(k)|^2 - |b(k)|^2 = 1.
\]
The \( 2 \times 2 \) matrix-valued function
\[
M(x;k) = \begin{pmatrix} \frac{1}{a(k)} \Phi^-_{[1]} e^{ikx}, & \Phi^+_{[2]} e^{-ikx} \\ \Phi^+_{[1]} e^{ikx}, & \frac{1}{a(k)} \Phi^-_{[2]} e^{-ikx} \end{pmatrix}, \quad \exists k > 0,
\]
\[
= \begin{pmatrix} \Phi^+_{[1]} e^{ikx}, & \Phi^+_{[2]} e^{-ikx} \\ \frac{1}{a(k)} \Phi^-_{[1]} e^{ikx}, & \frac{1}{a(k)} \Phi^-_{[2]} e^{-ikx} \end{pmatrix}, \quad \exists k < 0,
\]
has the jump
\[
M_-(x;k) = M_+(x;k) \begin{pmatrix} 1 & -\frac{R(k) e^{-2ikx}}{1 - |R(k)|^2} \\ -R(k) e^{2ikx} & 1 \end{pmatrix}, \quad k \in \mathbb{R},
\]
and symmetries \( \sigma_3 M(k) \sigma_3 = \overline{M(k)} = M(-k) \). Function \( M(x,t;k) \) which for all \( t \) has the jump \( \Phi_\pm \) with \( 2ikx \) changed with \( 2ikx + 8ik^3 t \), and normalized to \( 1 \) as \( k \to \infty \), generates a solution \( q(x,t) \) of MKdV by formulas
\[
\lim_{k \to 0, \exists k > 0} M(x,t;k) = 1 + \frac{1}{2ik} \begin{pmatrix} f^{+\infty}_x q^2 & -q \frac{f^{+\infty}_x}{k} \\ -q & f^{+\infty}_x q \end{pmatrix} + O(k^{-2}), \quad k \to \infty,
\]
\[
M(x,t;k) \left( \begin{pmatrix} 1 & 0 \\ -R(k) e^{2ikx + 8ik^3 t} & 1 \end{pmatrix} \right) = \begin{pmatrix} \cosh \left[ f^{+\infty}_x q \right] & \sinh \left[ f^{+\infty}_x q \right] \\ \sinh \left[ f^{+\infty}_x q \right] & \cosh \left[ f^{+\infty}_x q \right] \end{pmatrix}.
\]
If \( q(x,t) \) is a solution to MKdV, then also \( \tilde{q}(x,t) = -q(x,t) \) is a solution. The corresponding
quantities
\[
\tilde{\Phi}_\pm = \sigma_3 \Phi^\pm \sigma_3, \quad \tilde{a}(k) = a(k), \quad \tilde{b}(k) = b(k), \quad \tilde{R}(k) = -R(k).
\]
Functions \( u(x,t) = q^2 - q_x, \tilde{u} = q^2 + q_x \) are solutions to KdV, and the associated with \( u \) spectral functions \( a, b, R \) are the same as the ones associated with \( q \). The associated with \( \tilde{u} \) spectral functions are the same as the ones associated with \( \tilde{q} \), i.e the reflection coefficient is the opposite. If \( R(0) = -1 \), then \( u \) is a fast decaying solution of KdV and \( \tilde{u} \) is a slowly decaying solution of KdV, and if \( R(0) = 1 \), then vice versa.

The \( a_{MKdV}(k) \neq 0 \) for \( 3k \geq 0 \), while \( a_{KdV} \) might have simple zeros \( a_{i\kappa} = 0 \), for some \( \kappa > 0 \). In the latter case the corresponding solution \( q \) of MKdV will have poles for real \( x \).
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