Abstract

In 4-dimensional supergravity theories, covariant under symplectic electric-magnetic duality rotations, a significant role is played by the symplectic matrix $\mathcal{M}(\varphi)$, related to the coupling of scalars $\varphi$ to vector field-strengths. In particular, this matrix enters the twisted self-duality condition for 2-form field strengths in the symplectic formulation of generalized Maxwell equations in presence of scalar fields.

In this investigation, we compute several properties of this matrix in relation to the attractor mechanism of extremal (asymptotically flat) black holes. At the attractor points with no flat directions (as in the $\mathcal{N} = 2$ BPS case), this matrix enjoys a universal form in terms of the dyonic charge vector $\mathbf{Q}$ and the invariants of the corresponding symplectic representation $R_{\mathbf{Q}}$ of the duality group $G$, whenever the scalar manifold is a symmetric space with $G$ simple.

At attractors with flat directions, $\mathcal{M}$ still depends on flat directions, but not $\mathcal{M}\mathbf{Q}$, defining the so-called Freudenthal dual of $\mathbf{Q}$ itself. This allows for a universal expression of the symplectic vector field strengths in terms of $\mathbf{Q}$, in the near-horizon Bertotti-Robinson black hole geometry.
1 Introduction

One of the most appealing properties of extended (ungauged) four-dimensional supergravities (i.e. locally supersymmetric models with no less than 8 supercharges) is their on-shell global symmetry which is conjectured to encode the known string/M-theory dualities [1]. The corresponding global symmetry group $G$, to be also dubbed $U$-duality, is the isometry group of the scalar manifold (i.e., global symmetry of the scalar field sigma-model), whose (non-linear) action on the scalar fields is combined with a linear symplectic action on the $n$ electric field strengths $F^A_{\mu \nu}$, $A = 0, \ldots, n-1$, and their magnetic duals $G^A_{\Lambda \mu \nu}$ (electric-magnetic duality action of $G$). The latter action being defined by an embedding of $G$ in the symplectic group $Sp(2n, \mathbb{R})$, so that $F^A_{\mu \nu}$, together with $G^A_{\Lambda \mu \nu}$, transform under electric-magnetic duality in a symplectic representation $R_Q$ of $G$. This embedding, which determines the couplings of the vector fields to all the other fields in the action, is built-in the definition of a flat symplectic bundle over the scalar manifold, which is a common mathematical feature of $\mathcal{N} \geq 2$-extended supergravities [3, 4, 5].

Solutions to these theories naturally arrange themselves in orbits with respect to the action of $G$, and important physical properties are captured by $G$-invariant quantities. A notable example are the extremal, static, asymptotically-flat black holes in $D = 4$, which have deserved considerable attention in the literature during the last 20 years or so, since they provide a
valuable tool for studying string/M-theory dualities. These solutions are naturally coupled to scalar fields as a consequence of the non-minimal couplings of these to the vector fields in the supergravity action. Near the horizon, however, they exhibit an attractor mechanism [6, 7]:

the near-horizon geometry, which is described by an $AdS_2 \times S^2$ Bertotti-Robinson space-time [8], is independent of the values of the scalar fields at radial infinity, and it only depends on the quantized magnetic and electric charges $p^\Lambda, q^\Lambda$. In particular the horizon area $A_H$, which is related to the entropy $S$ of the solution through the Bekenstein- Hawking formula [9], is expressed in terms of the quartic invariant $I_4(p, q)$ of the representation $R_Q$ of $G$, only depending on $p^\Lambda, q^\Lambda$ (we set $8\pi G_N = c = \hbar = 1$):

$$ S = \frac{A_H}{4} = \pi \sqrt{|I_4(p, q)|} . $$

(1.1)

This is a consequence of the fact that the horizon represents an asymptotically stable equilibrium point for the radial evolution of those scalar fields which are effectively coupled to the solution and thus affect its geometry. In other words, such scalars flow from radial infinity to the horizon toward values which only depend on the quantized charges (fixed values). The horizon fixed point is defined by extremizing an effective potential $V_{BH}(\varphi; p, q)$ ($\varphi$ generically denoting the scalar fields) [7]:

$$ V_{BH}(\varphi, Q) := -\frac{1}{2} Q^T M(\varphi) Q , $$

(1.2)

where $Q = (p^\Lambda, q^\Lambda)$ is the vector quantized charges in the representation $R_Q$ of $G$. The value of this potential at the horizon defines its area, being equal to $\sqrt{|I_4(p, q)|}$. The scalar fields which are not fixed at the horizon are those which are not effectively coupled to the black hole charges, and they are flat directions of $V_{BH}$. They will be denoted by $\varphi_{flat}$. In the above formula, $M(\varphi)$ is a $2n \times 2n$ symmetric, symplectic, negative-definite matrix-valued function of the scalar fields. In all extended supergravities it is defined by the flat symplectic bundle over the scalar manifold. In fact, it encodes all the information about the non-minimal couplings of the scalar to the vector fields in the action through the kinetic term of the latter and the generalized theta-term. Moreover it allows to define the so called Freudenthal duality [10], a recently studied on-shell symmetry [11, 12, 13] which we shall be dealing with in the following.

An interesting question to be posed is what happens to the geometric structures associated with the scalar manifold, e.g. pertaining to its symplectic bundle, near the horizon. In the present investigation, we focus on the matrix $M(\varphi)$, because of its relevance to the geometry of the supergravity model.

At the horizon $M(\varphi)$ depends on $Q$, through the fixed scalars, and on the flat directions:

$$ M(\varphi)|_{\text{horizon}} = M^H(Q, \varphi_{flat}) . $$

(1.3)

As we shall prove in what follows, the dependence on the flat directions drops out already when we contract $M^H$ once with the charge vector. This implies the independence of the vector field-strengths at the horizon from $\varphi_{flat}$. On general grounds, using the properties of $M(\varphi)$, one can show that if we act on the solution by means of an element $g$ of $G$, which maps $\varphi$ into $\varphi'$ and $Q$ into $Q'$, the matrix $M(\varphi)$ at the horizon transforms as follows:

$$ M^H(Q', \varphi_{flat}') = g^{-T} M^H(Q, \varphi_{flat}) g^{-1} , $$

(1.4)

Here and in the following we use the short-hand notation $g^{-T} := (g^{-1})^T$. 

2
where, with an abuse of notation, we have denoted by $g$ also the symplectic $2n \times 2n$ matrix representing the corresponding $G$-element on contravariant vectors of $R_2$.

In absence of flat directions, the above equation suggests that $\mathcal{M}^H(Q)$ should be described in terms a symmetric, symplectic, negative-definite matrix defined on the $G$-orbit of $Q$, and thus constructed out of $Q$ and of $G$–invariant tensors. Restricting our analysis to the case of simple groups $G$, with the exception of the STU model, for charge vectors $Q$ with $I_4(Q) > 0$ we could construct such a matrix $M(Q)$ using a simple Ansatz, which involves only $Q$ and $G$-invariant tensors, and imposing the following properties of $\mathcal{M}^H$:

$$MCM = C;$$

$$MQ = -\frac{\epsilon}{2\sqrt{|I_4|}} \frac{\partial I_4}{\partial Q},$$

where $I_4 = |I_4|$, and $C$ is the symplectic invariant $2n \times 2n$ antisymmetric matrix. Note that the second of $(1.13)$ [12] implies

$$Q^TMQ = -2\sqrt{|I_4(Q)|};$$

however, it can be checked that this yields the same condition (namely, $(1.1)$ further below) on the real coefficients $A$, $B$ and $C$ of the Ansatz $(1.9)-(1.10)$. Starting from the same Ansatz we actually find two solutions to the above equations, denoted by $M_+(Q)$ and $M_-(Q)$. For charges with $I_4(Q) > 0$ and no flat directions, we give arguments in favor of the identification of one of these matrices ($M_+$) with $\mathcal{M}^H(Q)$. The other solution ($M_-$), on the other hand, is never negative definite and has the general form:

$$M_{-,MN} = -\frac{\partial^2 \sqrt{|I_4(Q)|}}{\partial Q^M \partial Q^N}.$$  

This Hessian has been considered in the literature, see [14] [15], though in different contexts.

As far as regular BPS solutions in $\mathcal{N} = 2$ supergravities are concerned, the two matrices $M_\pm$ enjoy an interesting interpretation as the value at the horizon of two characteristic symplectic, symmetric matrices of the theories: the matrix $M$ which is constructed out of the real and imaginary parts of the period matrix $\mathcal{N}_{\Lambda\Sigma}(\varphi)$ (defining the generalized theta-term and the kinetic term for the vector fields, respectively), and a matrix $M^{(F)}$, constructed just as $M$, but in terms of the real and imaginary parts of a different complex matrix, namely the Hessian $\mathcal{F}_{\Lambda\Sigma}$ of the holomorphic prepotential of the special Kähler manifold. In terms of the covariantly holomorphic section $V = (V^\imath)$ of the special Kähler manifold describing the vector multiplet scalars $z^\imath$, and of its covariant derivatives $U_\imath = D_\imath V = (U_\imath^\imath)$ (we use the notations of [16]), the two matrices have the following expressions:

$$\mathcal{M}(z, \bar{z}) = C \left( V V^T + \bar{V} \bar{V}^T + U_\imath g^{\imath\bar{\imath}} \bar{U}_\bar{\imath}^T + \bar{U}_\bar{\imath} g^{\bar{\imath}\imath} U_\imath^T \right) C,$$

$$\mathcal{M}^{(F)}(z, \bar{z}) = C \left( V V^T + \bar{V} \bar{V}^T - U_\imath g^{\imath\bar{\imath}} \bar{U}_\bar{\imath}^T - \bar{U}_\bar{\imath} g^{\bar{\imath}\imath} U_\imath^T \right) C.$$  

The former was given in [5] and [17], and it is the real part of the identity $(1.16)$ of [11]. On the other hand, the latter expression follows from $(1.13)$ of [12]; furthermore, $Q^T \mathcal{M}^{(F)}(z, \bar{z}) Q$ agrees with Eq. (57) of [18]. In $\mathcal{N} \geq 2$-extended supergravity, for charge orbits characterized by $I_4(Q) < 0$, the two matrices $M_\pm$, though still satisfying the second of $(1.13)$, are anti-symplectic, namely for them the following property holds:

$$M_\pm C M_\pm = -C.$$  

(1.10)
The matrix $M_+$, in particular, for all regular charge-orbits, as opposed to $M_-$, has the notable property of being an automorphism of the U-duality algebra $g$, that is $g$, in the representation $R_Q$, is invariant under the adjoint action of $M_+$ (if $I_4 < 0$, being $M_+$ anti-symplectic, it is an outer automorphism). On the other hand $M_-$ is still, in all regular orbits, identified with the Hessian $H$. Moreover both $M_\pm$ are invariant, up to a sign, under Freudenthal duality at the horizon.

For a generic regular charge-orbit we will find the following relation between $\mathcal{M}^H$ and the automorphism $M_+$:

$$\mathcal{M}^H = M_+ A,$$

(1.11)

where $A$ is an involutive automorphism of $G$ in the stabilizer of $Q$, depending in general on $Q$ and $\varphi_{\text{flat}}$. For $I_4 < 0$, both $M_+$ and $A$ are anti-symplectic outer-automorphisms of $G$, while for $I_4 > 0$, $A \in G$ and, in absence of flat directions, it is the identity matrix.

Besides the interpretation in terms of $\mathcal{M}$ at the horizon, which holds only for $M_+$ in specific orbits, the solution $M_-$ is the symplectic metric on the $G$-orbit of $Q$ [15] and thus it has a mathematical relevance per se.

We then consider $\mathcal{M}^H$ for solutions with flat directions, and prove a general factorization property:

$$\mathcal{M}^H(Q, \varphi_{\text{flat}}) = \mathcal{M}^H_1(Q) \mathcal{M}_0(\varphi_{\text{flat}}),$$

(1.12)

where the two factor-matrices are elements of $G$ and commute. The former $\mathcal{M}^H_1(Q)$ is negative definite while the latter $\mathcal{M}_0(\varphi_{\text{flat}})$ is in the stability group of $Q$. As anticipated above, from this it follows that the dependence on the flat directions drops out once we contract $\mathcal{M}^H$ with a charge vector, so that the vector field strengths at the horizon are $\varphi_{\text{flat}}$-independent.

The relation between the decompositions (1.12) and (1.11) is that $\mathcal{M}^H_1(Q)$ can be written as the product $M_+(Q) A_0(Q)$, $A_0$ being an involutive automorphism in the stabilizer of $Q$, so that $A(Q, \varphi_{\text{flat}})$ in (1.11) is given by the product $A_0(Q) \mathcal{M}_0(\varphi_{\text{flat}})$.

The plan of the paper is the following.

In Sect. 2 we recall some basic facts about extremal black hole solutions in extended supergravities, as well as their properties under the global symmetry of the models. This includes a review of the Freudenthal duality, and sets the stage for the discussion of our results.

In Sect. 3 we recall the main properties of the independent lowest-order invariant tensors, namely $C_{MN}$ (symplectic metric) and $K_{MNPQ}$ ($K$-tensor), in the symplectic black hole charge representation $R_Q$ of the $U$-duality groups of symmetric four-dimensional Maxwell-Einstein (super)gravity theories (to which we restrict our present investigation).

In Sec. 4, which focuses on the cases without flat directions, we construct, out of a general ansatz involving suitable contractions of the $K$-tensor and of the symplectic metric $C_{MN}$ with a number of charge vectors $Q$, a symmetric matrix $M$ satisfying conditions (1.5). As anticipated above, restricting our analysis to simple $U$-duality groups, treated in Subsec. 4.1 we actually find, for $I_4(Q) > 0$ two solutions $M_+$ and $M_-$. The former is identified with $\mathcal{M}^H$, while the properties of the latter are studied at the end of the same Section. The definition of the matrices $M_\pm$ is the generalized to the $I_4 < 0$ orbit, in Sec. 4.2 here general properties of $M_\pm$, in any regular charge-orbit $I_4 \neq 0$, are discussed.

In Sec. 4.3 we consider $\mathcal{N} = 2$ theories, where we show that $M_-$, in the BPS–orbit, is identified with the matrix $\mathcal{M}^{(F)}$.

The peculiar cases of the non-BPS attractors in the $T^3$-model as well as in $\mathcal{N} = 2$ minimally coupled Maxwell-Einstein theory and in $\mathcal{N} = 3$ supergravity are considered in Subsecs. 4.4 and 4.5 respectively.
Then, in Sec. 5, flat directions are taken into account, for both cases \( I_4 < 0 \) (Subsec. 5.1) and \( I_4 > 0 \) (Subsec. 5.2).

A summary of results and general properties of \( M_+ \) and \( M_- \), as well as their relation to the matrix \( \mathcal{M}^H \), is finally given in Sec. 6.

Appendices A and B contain details of the derivation of some results of Sec. 4, while Appendix C containing a discussion of anti-symplectic outer-automorphisms of the U-duality algebra, concludes the paper.

## 2 Symmetry Properties of Extremal Black Holes in Extended Supergravities

One of the basic ingredients of the symplectic formulation of electric-magnetic duality in \( \mathcal{N} \geq 2 \)-extended supergravity theories in four dimensions, whose bosonic sector reads (in absence of gauging)

\[
\mathcal{L} = \frac{-R}{2} + \frac{1}{2} g^{ij} \partial_{i} \varphi \partial_{j} \varphi + \frac{1}{4} I_{\Lambda \Sigma} (\varphi) F_{\mu \nu}^{A} F_{\mu \nu}^{A} + \frac{1}{8 \sqrt{\mathcal{G}}} R_{\Lambda \Sigma} (\varphi) \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu}^{A} F_{\rho \sigma}^{A},
\]

is the \( (2n_V + 2) \times (2n_V + 2) \) real, negative definite, symmetric matrix \( M \):

\[
M = \begin{pmatrix}
I & -R \\
0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & I^{-1}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-R & I
\end{pmatrix}
= \begin{pmatrix}
I + RI^{-1}R & -RI^{-1} \\
-I^{-1}R & I^{-1}
\end{pmatrix},
\]

where \( n_V + 1 \) denotes the number of Abelian vector fields and \( I \) denotes the \( (n_V + 1) \)-dimensional identity matrix. \( I_{\Lambda \Sigma} \) is the kinetic vector matrix, and \( R_{\Lambda \Sigma} \) enters the topological theta term in (2.1); they are usually regarded as the imaginary resp. real part of a complex kinetic matrix \( \mathcal{N}_{\Lambda \Sigma} \), such that (2.2) yields \( M = M[R, I] = M[\text{Re}(\mathcal{N}), \text{Im}(\mathcal{N})] \). Moreover, it is symplectic, namely it satisfies the first of (1.5).

The symplectic structure of the generalized special geometry of scalar fields yields that \( M \) can be equivalently rewritten as

\[
M = - (LL^T)^{-1} = -L^{-T}L^{-1},
\]

where \( L \) is an element of the \( \text{Sp}(2n_v + 2, \mathbb{R}) \)-valued symplectic bundle of generalized special geometry.

\( M \) enters the symplectic-covariant form of the Maxwell equations (twisted self-duality condition) \[19\]:

\[
H = \mathcal{C} M * H,
\]

where \( (\Lambda = 0, 1, \ldots, n_V; \text{ in } \mathcal{N} = 2 \text{ theories, the naught index is reserved for the graviphoton}) \)

\[
H := (F^A, G_{\Lambda})^T;
\]

\[
*G_{\Lambda \mu \nu} := 2 \frac{\delta \mathcal{L}}{\delta F^\Lambda_{\mu \nu}}.
\]

\[2\]The notation “\( n_V + 1 \)” is actually relevant for \( \mathcal{N} = 2 \) supergravity, in which one can distinguish between the graviphoton and the \( n_V = n - 1 \) matter multiplets’ vector fields.

\[3\]Throughout this paper we use for the symplectic invariant matrix the following form: \( \mathcal{C} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \).
is the symplectic vector of the 2-form Abelian field strengths and of their duals, and the four-dimensional Hodge dual is denoted, as usual, by \(*\). Correspondingly, the equations of motion and Bianchi identities take the form
\[
dH = 0. \tag{2.6}
\]

In the background of a static, spherically symmetric, asymptotically flat, dyonic extremal black hole (\(\tau := -1/r\))
\[
ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)} \left[ \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2} \left( d\theta^2 + \sin \theta d\psi^2 \right) \right], \tag{2.7}
\]
one can introduce the symplectic vector \(Q\) of asymptotic magnetic and electric fluxes of \(H\) as follows:
\[
Q : = \int_{\mathbb{S}^2} H = (p^\Lambda, q^\Lambda)^T; \tag{2.8}
\]
\[
p^\Lambda : = \frac{1}{4\pi} \int_{\mathbb{S}^2} F^\Lambda, \quad q^\Lambda = \frac{1}{4\pi} \int_{\mathbb{S}^2} G^\Lambda. \tag{2.9}
\]
Thus, the twisted self-duality condition (2.4) and the symplecticity of \(\mathcal{M}\) imply that
\[
\int_{\mathbb{S}^2} *H = -\mathcal{C}M \int_{\mathbb{S}^2} H = -\mathcal{C}M Q = : \mathfrak{F}(Q); \quad \updownarrow \quad \mathfrak{F} \left( \int_{\mathbb{S}^2} H \right) = \int_{\mathbb{S}^2} *H, \tag{2.10}
\]
where \(\mathfrak{F}\) denotes the “non-critical”, scalar-dependent generalization of the so-called *Freudenthal duality* [10], defined as a scalar-dependent involution on the symplectic vector \(Q\) [11]:
\[
\mathfrak{F} : Q \to \mathfrak{F}(Q) := -\mathcal{C}M (\varphi) Q; \tag{2.11}
\]
\[
\mathfrak{F}^2 = -Id. \tag{2.12}
\]
It should be stressed that the anti-involutivity (2.12) of \(\mathfrak{F}\) is a direct consequence of the symplecticity of \(\mathcal{M}\) itself. Thus, in every generalized special geometry [5], one can define a scalar-dependent almost-complex structure as follows [13]:
\[
\mathcal{S}(\varphi) : = \mathcal{C}M (\varphi); \tag{2.13}
\]
\[
\mathcal{S}^2 (\varphi) = \mathcal{C}M (\varphi) \mathcal{C}M (\varphi) = \mathbb{C}^2 = -\mathbb{I}, \tag{2.14}
\]
such that
\[
\mathfrak{F}(Q) := -\mathcal{S}(\varphi) (Q). \tag{2.15}
\]
For *U-duality* [4] groups \(G\) of type \(E_7\) [21], \(\mathcal{S}(\varphi) \in Aut(\mathcal{F}) \equiv G\), where \(\mathcal{F}\) denotes the corresponding *Freudenthal triple system* [13]; in these theories, \(\mathcal{S}\) may be regarded as the projection onto the adjoint in the symmetric tensor product of the symplectic representation \(\mathbb{R}_Q\) of \(G\), carried by \(\mathcal{F}\) itself.

\textsuperscript{4}Here *U-duality* is referred to as the “continuous” symmetries of [20]. Their discrete versions are the *U*-duality non-perturbative string theory symmetries introduced by Hull and Townsend [1].
By virtue of the definition (2.11), the twisted self-duality condition (2.14) can be recast as

\[ H = - \mathfrak{F} (\ast H) = - \ast \mathfrak{F} (H), \] (2.16)

which is nothing but the “unfluxed” version of (2.10). (2.16) expresses the compatibility of the two different almost-complex (anti-involutive) structures defined by the Hodge *-duality (in $D = 4$ spacetime, $\ast^2 = - \text{Id}$) and by the operator $\mathfrak{F}$ (in the symplectic vector $G$-space $\mathbb{R}_Q$ spanned by $H$ (2.5) or by its asymptotic flux $Q$ (2.8); recall (2.12)). An equivalent restatement of such a compatibility is given by (2.10), as well.

Furthermore, $\mathcal{M}$ is relevant to define the Abelian 2-form field strengths $H$ in the background (2.7) (cfr. e.g. [22, 23, 24])

\[ H (\varphi, U, Q) = e^{2U} C \mathcal{M} (\varphi) Q dt \wedge d\tau + Q \sin \theta d\theta \wedge d\psi; \] (2.17)

\[ = - e^{2U} \mathfrak{F} (Q) dt \wedge d\tau + Q \sin \theta d\theta \wedge d\psi; \] (2.18)

thus implying that (recall (2.4))

\[ \ast H (\varphi, U, Q) = \mathfrak{F} (H (\varphi, U, Q)) \] (2.19)

\[ = e^{2U} Q dt \wedge d\tau + \mathfrak{F} (Q) \sin \theta d\theta \wedge d\psi \] (2.20)

\[ = H (\varphi, U, \mathfrak{F} (Q)) =: \mathfrak{F} (H (\varphi, U, Q)), \] (2.21)

consistently with (2.16). Note that the dependence of $H$ on scalars is completely encoded in $C \mathcal{M} (\varphi)$, or, equivalently, in the “non-critical” Freudenthal duality $\mathfrak{F}$ (2.11).

$\mathcal{M}$ also defines the (positive definite) effective black hole potential (1.2), such that $\mathfrak{F}$ (2.11) can equivalently be defined as

\[ \mathfrak{F} : Q \rightarrow \mathfrak{F} (Q) := C \frac{\partial V_{BH}}{\partial Q}. \] (2.22)

The potential $V_{BH}$ (1.2) governs the radial evolution of the scalar fields $\varphi(\tau)$ as well as of the warp factor $U (\tau)$:

\[ \begin{align*}
\frac{dU}{d\tau} &= e^{2U} V_{BH}; \\
\frac{d^2 \varphi}{d\tau^2} &= g^{ij} e^{2U} \frac{\partial V_{BH}}{\partial \varphi^j}.
\end{align*} \] (2.23)

At the event horizon of the extremal black hole ($\tau \rightarrow - \infty$), the attractor mechanism [6, 7] yields that, regardless of the initial (asymptotic) conditions:

\[ \lim_{\tau \rightarrow - \infty} \varphi^a = \varphi_H^a (Q), \] (2.24)

up to flat directions [25]. It can be shown [11] that

\[ \lim_{\tau \rightarrow - \infty} \mathfrak{F} (Q) =: \mathfrak{F}_H (Q) = - C \mathcal{M}_H Q = C \frac{\partial S_{BH}}{\partial Q} =: \tilde{Q}, \] (2.25)

where $S_{BH}$ denotes the Bekenstein-Hawking entropy [9] of the extremal black hole (2.7) as

\[ S_{BH} = \frac{A_H}{4} = - \frac{\pi}{2} \mathcal{M}^M_{MN} Q^M Q^N; \] (2.26)

and the horizon limit of the matrix $\mathcal{M}$ has been defined as

\[ \lim_{\tau \rightarrow - \infty} \mathcal{M} (\varphi (\tau)) =: \mathcal{M}^H. \] (2.27)
Note that (2.26) implies $\mathcal{M}^H$ to be homogeneous of degree zero in the charges.
For $U$-duality groups of type $E_7$ [21], $\tilde{Q}$ can also be written as [10] [11]

$$\tilde{Q}_M = \frac{2}{\sqrt{|I_4|}} K_{MNPQ} Q^N Q^P Q^Q,$$
$$\tilde{Q}^M : = C^{MN} Q_N. \quad (2.28)$$

where $K_{MNPQ}$ is the so-called invariant $K$-tensor (see below, and cfr. Sec. 3), which allows for the definition of a quartic homogeneous polynomial $I_4$ in the charges $Q$:

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon |I_4|, \quad (2.30)$$
in turn determining the Bekenstein-Hawking entropy $S_{BH}$ (2.26) as

$$S_{BH} = \pi \sqrt{|I_4|} = - \frac{\pi}{2} \mathcal{M}_{MN}^H Q^M Q^N. \quad (2.31)$$

Therefore, at the event horizon of the extremal black hole, the symplectic field strengths vector

$$H_H := \lim_{\tau \to -\infty} H(\varphi(\tau)) \quad (2.32)$$

reads

$$H_H = e^{2U_H} C^M \mathcal{M}^H Q dt \wedge d\tau + Q \sin \theta d\theta \wedge d\psi \quad (2.33)$$

$$= -e^{2U_H} \tilde{Q} dt \wedge d\tau + Q \sin \theta d\theta \wedge d\psi = -\tilde{F}_H(\ast H_H), \quad (2.34)$$

where $U_H$ is the leading order contribution in $\tau$ of the near-horizon limit of $U(\tau)$. (2.34) implies

$$\tilde{F}_H(H_H) = \ast H_H, \quad (2.35)$$

which is nothing but the evaluation of (2.19)-(2.21), or equivalently of the twisted self-duality condition (2.16), at the horizon.

It should also be recalled that $\mathcal{M}$ occurs in the expression of the metric of the enlarged scalar manifold, obtained by dimensional reduction of the $D = 4$ Maxwell-Einstein-scalar Lagrangian density (2.1) down to $D = 3$ (and by subsequent dualization of the $D = 3$ vector fields to scalars $Z$) [19]:

$$ds^2_{D=3} = \frac{1}{4} \left[ 4dU^2 + 2g_{ij} d\varphi^i d\varphi^j + e^{-2U} (dA + Z^T C dZ)^2 - 2e^{-2U} dZ^T \mathcal{M}(\varphi) dZ \right]. \quad (2.36)$$

When (2.1) is regarded as the bosonic sector of $\mathcal{N} = 2$, $D = 4$ (un gauged) Maxwell-Einstein supergravity coupled to $n_V$ Abelian vector multiplets, such a dimensional reduction is named $c$-map [20], and the $ds^2_{D=3}$ (2.36) is quaternionic-Kähler, which can also be considered as the metric of the quaternionic scalars of the $\mathcal{N} = 2$ hypermultiplets.

In the present investigation, we spell out some additional basic properties of the matrix $\mathcal{M}$.

Along the radial evolution of the scalar flow, $\mathcal{M}$ has a complicate dependence on the scalars; for instance, in generic $d$-geometries, the expression of the real symmetric matrices $I_{\Lambda \Sigma}$ and $R_{\Lambda \Sigma}$ is given e.g. in Sec. 2 of [27].
In absence of flat directions, when all the scalars \( \varphi \)'s are stabilized to a (purely) Q-dependent value \( \varphi_H (Q) \) (2.24) at the (unique) event horizon of the extremal “large” black hole, the non-linear action of an element \( g \) of the U-duality group \( G \) on the scalars

\[
\varphi_H (g^{-1} Q) = g^* \varphi_H (Q)
\]

induces a linear transformation on \( \mathcal{M} \) (cfr. (1.4)):

\[
\mathcal{M} (g^* \varphi_H (Q)) = g^{-T} \mathcal{M} (\varphi_H (Q)) g^{-1}.
\]

The attractor mechanism [6, 7] is responsible for the stabilization of \( \varphi \) (through (2.24)), and thus of \( \mathcal{M} (\varphi_H) \), in terms of \( Q \), which transforms in the representation \( R_Q \) of \( G \). This allows for a group-theoretical characterization of the horizon expression \( \mathcal{M}^H \) (2.27) of \( \mathcal{M} \). Indeed, a U-duality transformation \( g \in G \) acting on \( Q \) yields to a symplectic transformation on \( \mathcal{M}^H \) itself:

\[
\mathcal{M} (\varphi_H (g^{-1} Q)) = g^{-T} \mathcal{M} (\varphi_H (Q)) g^{-1} =: g^{-T} \mathcal{M}^H g^{-1}.
\]

Certainly an \( Sp(2n_V + 2, \mathbb{R}) \)-covariant, symmetric matrix \( M(Q) \), only built out of \( Q \) and of \( G \)-invariant tensors in products of the representation \( R_Q \), satisfies the above transformation property. These invariant tensors include the symplectic metric \( C^{MN} \), which allows for the definition of the invariant product of two symplectic vectors \( B_1 \) and \( B_2 \):

\[
\langle B_1, B_2 \rangle := C^{MN} B_1^M B_2^N,
\]

and the rank-4 completely symmetric invariant \( K \)-tensor \( K^MNPQ \) (cfr. Sec. 3). In the next sections we address the problem of expressing \( \mathcal{M}^H \) in terms of a matrix \( M(Q) \) of this kind, restricting ourselves to \( D = 4 \) Maxwell-Einstein (super)gravity theories whose scalar manifold is a symmetric space \( G/H \) (which correspond to characterizing \( G \) as a group of type \( E_7 \) [21]). We find a simple identification for \( I_4 > 0 \) orbits in absence of flat directions.

As we will elucidate in the subsequent treatment, things change in presence of flat directions (short-hand denoted by \( \varphi_{\text{flat}} \)), namely, of scalar degrees of freedom which are not stabilized at the horizon of the extremal black hole, and thus for which the attractor mechanism does not hold, at least at Einsteinian level [25]. In this case, \( \mathcal{M} \) depends on flat directions (unlike the ADM mass and \( V_{BH} \) itself), also at the horizon (i.e., setting the scalar fields at their attractor value \( \varphi_H (Q) \) (2.24)):

\[
\frac{\partial \mathcal{M}}{\partial \varphi_{\text{flat}}} \neq 0; \quad \frac{\partial \mathcal{M}^H}{\partial \varphi_{\text{flat}}} \neq 0.
\]

On the other hand, we will prove that \( \mathcal{M}^H Q \) does not depend on \( \varphi_{\text{flat}} \):

\[
\frac{\partial}{\partial \varphi_{\text{flat}}} (\mathcal{M}^H Q) = \frac{\partial \mathcal{M}^H}{\partial \varphi_{\text{flat}}} Q = 0,
\]

even if \( \mathcal{M} Q \) does:

\[
\frac{\partial}{\partial \varphi_{\text{flat}}} (\mathcal{M} Q) = \frac{\partial \mathcal{M}}{\partial \varphi_{\text{flat}}} Q \neq 0.
\]

(2.42) is consistent with the universality of the conformally-flat Bertotti-Robinson [8] near-horizon geometry of an extremal black hole [6, 7]; indeed, by (2.33), it implies that at the event horizon the symplectic field strengths vector \( H_H \) (2.32) are independent of the flat directions:

\[
\frac{\partial H_H}{\partial \varphi_{\text{flat}}} = 0,
\]
but, from (2.17)-(2.21) and (2.43), the same generally does not hold away from the horizon:
\[
\frac{\partial H}{\partial \phi_{\text{flat}}} \neq 0. \tag{2.45}
\]

(2.42) and (2.43) also yield a crucial difference between the “non-critical” Freudenthal duality \(\mathcal{F} (2.11) \ [11]\) and its “critical”, horizon limit \(\mathcal{F}_H (2.25) \ [10]\): while the former depends on flat directions, the latter does not:
\[
\frac{\partial \mathcal{F}(\mathcal{Q})}{\partial \phi_{\text{flat}}} \neq 0; \quad \frac{\partial \mathcal{F}_H(\mathcal{Q})}{\partial \phi_{\text{flat}}} = 0. \tag{2.46-2.47}
\]

(2.46)-(2.47) can be regarded as an attractor mechanism for the Freudenthal duality: \(\mathcal{F}_H(\mathcal{Q}) =: \mathcal{Q}\) does not depend on flat directions \(\phi_{\text{flat}}\), and thus it is purely \(\mathcal{Q}\)-dependent.

3 The \(K\)-Tensor

Let us consider a \(D = 4\) U-duality group \(G\) of real dimension \(d\), with generators \(t^\alpha\) in the adjoint representation \((\alpha = 1, \ldots, d)\). The Gaillard-Zumino [2] symplectic maximal embedding
\[
G \subset Sp(2n, \mathbb{R}) \quad \mathcal{R}_\mathcal{Q} = 2n \tag{3.1}
\]
is provided by \((M, N = 1, \ldots, 2n)\)
\[
t^\alpha_{MN} := t^\alpha_M \mathcal{C}_{PN}, \tag{3.2}
\]
defining the Cartan-Killing metric \(k_{\alpha\beta}\) of \(G\) as
\[
\left( t_{\alpha|M}^N t_{\beta|N}^M \right) \equiv k_{\alpha\beta}, \tag{3.3}
\]
so that \(t_{\alpha|M}^N t^\alpha_N^M = d\). \(t^\alpha_{MN}\) is a singlet of \(G\), as expected by analyzing the product of the representations. In fact, at least all electric-magnetic duality groups consistent with (not necessarily symmetric nor homogeneous) generalized special geometry [5] share the property that
\[
(\exists!) t^\alpha_{MN} := 1 \in \text{Adj} \times_s (\mathcal{R}_\mathcal{Q} \times_s \mathcal{R}_\mathcal{Q}), \tag{3.4}
\]
where \(\text{Adj}\) denote the adjoint representation of \(G\) (e.g. 133 in \(E_7\)). A key property of \(t^\alpha_{MN}\) defined by (3.2) is the even symmetry of the symplectic indices:
\[
t^\alpha_{MN} = t^\alpha_{(MN)}. \tag{3.5}
\]

At least for groups \(G\) “of type \(E_7\)” [21], it is possible to construct the aforementioned rank-4 completely symmetric invariant tensor, dubbed \(K\)-tensor [28]:
\[
(\exists!) K_{MNPQ} := 1 \in (\mathcal{R}_\mathcal{Q} \times \mathcal{R}_\mathcal{Q} \times \mathcal{R}_\mathcal{Q} \times \mathcal{R}_\mathcal{Q})_s, \tag{3.6}
\]
which can be generally defined as follows:

\[
K_{MNPQ} \propto t_{(MNTa|PQ)}^a = \frac{1}{3} \left( t_{MN}t_{a|PQ} + t_{MP}t_{a|QN} + t_{MQ}t_{a|PN} \right)
= \frac{1}{4!} \left( 8t_{MN}t_{a|PQ} + 16t_{MP}t_{a|QN} + 16t_{MQ}t_{a|PN} \right).
\]

(3.7)

Needless to say, the prototype of groups “of type E7” is E7 itself (pertaining to \( \mathcal{N} = 8 \) and \( \mathcal{N} = 2 \) supergravity, in its real forms \( E_{7(7)} \) and \( E_{7(-25)} \), respectively), with \( R_Q = 56 \). By following the treatment of \([28]\), one can prove that

\[
K_{MNPQ} = \xi \left[ t_{MN}t_{a|PQ} - \tau C_{M(P}C_{Q)N} \right],
\]

(3.8)

where the real constants \( \xi \) and \( \tau \) have been introduced; the latter can be determined by imposing the skew-tracelessness condition \( C_{NP}K_{MNPQ} = 0 \), yielding \([28]\)

\[
\tau = \frac{d}{n(2n+1)},
\]

(3.9)

whereas, by consistency with the definitions used in literature (cfr. \([29]\), taking into account the different normalization conventions), \( \xi \) is fixed as

\[
\xi = -\frac{1}{6\tau} = -\frac{n(2n+1)}{6d}.
\]

(3.10)

Thus, the following general expression for the \( K \)-tensor is obtained:

\[
K_{MNPQ} = -\frac{n(2n+1)}{6d} \left[ t_{MN}t_{a|PQ} - \frac{d}{n(2n+1)} C_{M(P}C_{Q)N} \right],
\]

(3.11)

The formula (3.11) will be relevant to many subsequent computations (most of them reported in Appendix A). By contracting the \( K \)-tensor with four charge vectors \( Q \)'s, one obtains the quartic \( G \)-invariant homogeneous polynomial \( I_4 \) \([30] \) \((2.30)\) in \( R_Q \), which can therefore be rewritten as

\[
I_4 := K_{MNPQ}Q^M Q^NN^P Q^Q = -\frac{1}{6\tau} t_{MN}t_{a|PQ}Q^M Q^NN^P Q^Q.
\]

(3.12)

### 4 The \( M \)-Matrix and \( \mathcal{M}^H \) without Flat Directions

In the present section we focus on a class of four-dimensional Maxwell-Einstein (super)gravity theories with symmetric scalar manifolds \( G/H \), and construct a symplectic matrix \( M(Q)_{MN} \) out of \( K_{MNPQ} \), \( C_{MN} \) and \( Q \), satisfying Eqs.

\[
MC = \epsilon C ;
\]

\[
MQ = -\frac{\epsilon}{2\sqrt{|\mu|}} \frac{\partial \mu}{\partial Q} .
\]

(4.1)

For each regular orbit \( (I_4 > 0, I_4 < 0) \), we find two distinct solutions \( M_\pm \) with different properties. As mentioned in the Introduction, in the present investigation we will consider only simple \( U \)-duality groups \( G \); we leave the treatment of semi-simple cases elsewhere. In the absence of flat directions and for \( I_4 > 0 \) \((\epsilon = +1)\), we can identify one of the two matrices \( (M_+) \) with \( \mathcal{M}^H_{MN} \). Thus, even if the definition of \( M_\pm \) is general, the identification \( \mathcal{M}^H = M_+ \) turns out to hold only for (cfr. e.g. \([25]\]):
1. (1/2-)BPS attractors in all $\mathcal{N} = 2$ (symmetric) models with $G$ simple ($I_4 > 0$) and the same class of solutions in the STU model (in spite the fact that this model has a semi-simple $G = [SL(2, \mathbb{R})]^3$); 

2. (1/3-)BPS attractors in “pure” $\mathcal{N} = 3$ supergravity ($I_2 > 0$); 

3. non-BPS attractor in $N = 2$ $\mathbb{C}P^1$ (the so-called $T^2$) model ($I_2 < 0$); 

4. non-BPS $Z_{AB,H} = 0$ attractors in $\mathcal{N} = 3$ supergravity coupled to 1 vector multiplet ($I_2 < 0$). 

5. non-BPS $Z_H = 0$ attractors in $STU$ model with $I_4 > 0$. 

The $I_4 < 0$ attractor in the $\mathcal{N} = 2$ $T^3$-model, though still having no flat-directions, deserves a separate treatment which will be given in Subsec. 4.4. Indeed, in this case, the identification $\mathcal{M}_H = M_\pm$ cannot work since $M_\pm$ is antisymplectic.

Simple $U$-duality groups $G$ “of type $E_7$” [21] will be considered in Subsec.s 4.1, 4.2, 4.3 and 4.4. Here we first construct the solutions $M_\pm$ for $I_4 > 0$, discuss their geometric properties and the relation of one of them to $\mathcal{M}^H$. Then we move to the definition of $M_\pm$ in the $I_4 < 0$ case, generalizing some of their properties to all regular orbits.

The particular case of minimal coupling of Abelian vector multiplets to $\mathcal{N} = 2$ supergravity, in which $K_{MNPQ}$ is reducible (corresponding to degenerate groups “of type $E_7$” [33]), will be considered in Subsec. 4.5.

4.1 The $I_4 > 0$ Case and $\mathcal{M}^H$

We start and look for a $G$-covariant symmetric matrix $M(Q)$, solution to the equations (1.5):

\[
M_{MN}M_{PQ}C^{NP} = C_{MQ}; \quad (4.2)
\]

\[
M_{MN}Q^N = -\frac{\epsilon}{2\sqrt{|I_4|}} \frac{\partial I_4}{\partial Q^M} = -\tilde{Q}_M. \quad (4.3)
\]

We use for $M$ the following general Ansatz $(A, B, C \in \mathbb{R})$:

\[
M_{MN}(Q) = \frac{A}{|I_4|^{3/2}} K_M K_N + \frac{B}{|I_4|^{1/2}} K_{MN} + \frac{C}{|I_4|^{1/2}} K_{MB_1B_2} K_{NB_3B_4} C^{B_1B_3} C^{B_2B_4}, \quad (4.4)
\]

where:

\[
K_{MNP} := K_{MNPQ} Q^Q, \quad K_{MN} := K_{MNPQ} Q^P Q^Q, \quad K_M := K_{MNPQ} Q^N Q^P Q^Q. \quad (4.5)
\]

By recalling (2.28) [10, 11], it holds that

\[
K_M = \frac{1}{2\epsilon} |I_4|^{1/2} \tilde{Q}_M, \quad (4.6)
\]

such that (4.4) can be rewritten as

\[
M_{MN}(Q) = \frac{A}{4|I_4|^{1/2}} \tilde{Q}_M \tilde{Q}_N + \frac{B}{|I_4|^{1/2}} K_{MN} Q^P Q^Q + \frac{C}{|I_4|^{1/2}} K_{MB_1B_2} K_{NB_3B_4} C^{B_1B_3} C^{B_2B_4}. \quad (4.7)
\]
By exploiting the identity\footnote{As discussed in \cite{28} and in \cite{38}, this is a consequence of a general identity involving the quantity $K_{MNPA_i} K_{P_i Q R A_2} C_{A_1 A_2}$, given by (5.16) of \cite{28}.}

$$K_{M_A A_2 K_{PA_i A_4}} C^{A_1 A_3} C^{A_2 A_4} = -\frac{1}{6\tau} \left[ (2\tau - 1) K_{MP} + \frac{1}{12} (\tau - 1) C_{A_1 (M} C_{P) A_2} Q^{A_1} Q^{A_2} \right], \quad (4.8)$$

the Ansatz (4.4) (or, equivalently (4.7)) can be further simplified as

$$M_{MN}(Q) = \frac{A}{|I_4|^{3/2}} K_M K_N + \frac{1}{|I_4|^{3/2}} (B + \frac{(1 - 2\tau)}{6\tau} C) K_{MN} + \frac{C}{72|I_4|^{3/2}} (\tau - 1) Q_M Q_N$$

$$= \frac{A}{4|I_4|^{3/2}} \tilde{Q}_M \tilde{Q}_N + \frac{1}{|I_4|^{3/2}} (B + \frac{(1 - 2\tau)}{6\tau} C) K_{MN} + \frac{C}{72|I_4|^{3/2}} (\tau - 1) Q_M Q_N. \quad (4.9)$$

In App. A, the real coefficients $A$, $B$ and $C$ in (4.4) and (4.7) are determined by exploiting the properties of the most general $G$-covariant symmetric matrix $M(Q)$ at the horizon, namely (cfr. (1.5), (2.28); as already mentioned, (4.3) implies (1.6)).

It should be remarked that a term proportional to $Q_{(M} Q_{N)}$ cannot occur in (4.7) (or, equivalently, in (4.10)), because it is not consistent with (4.3) \cite{11}.

A consistent solution to (4.2)-(4.3) within the Ansatz (4.4) can be found only for $\epsilon = +1 \Leftrightarrow I_4 > 0$, and it reads

$$A_{\pm} = -2 \mp 6, \quad B_{\pm} = \frac{6(1 - 2\tau \mp \tau)}{(\tau - 1)}, \quad C_{\pm} = -\frac{36\tau (1 \pm 1)}{(\tau - 1)}. \quad (4.11)$$

The splitting into “+” branches generally corresponds to two independent expressions, namely $M_+$ and $M_-$, in terms of suitable contractions of the $K$-tensor itself and of the symplectic metric $C_{MN}$ with charge vectors $Q$’s; note that $M_-$ lacks the term proportional to $Q_M Q_N$, because $C_- = 0$.

This “±” degeneracy can be removed when considering the relation to the negative-definite matrix $M^H$. Indeed $M_-(Q)$ always has (at least) a positive eigenvalue and thus can never be identified with $M^H$. This result is illustrated in App. B by a direct computation in the $STU$ model (and its rank-2 ($ST^2$) and rank-1 ($T^3$) “degenerations” determine the corresponding symmetric models), and thus holds at least in all rank-3 symmetric models of which the $STU$ one is a universal sector. This check allows one to conclude that only the “+” branch is consistent with the properties required for the matrix $M$ (at the horizon).

Using (4.9)-(4.11), direct computation in some cases (recall $I_4 > 0$) suggests the following identification

$$M^H_{MN}(Q) = M_{+;MN}(Q) = -\frac{1}{\sqrt{I_4}} \left( \frac{8}{I_4} K_{MK} K_N - 6K_{MN} + Q_M Q_N \right)$$

$$= -\frac{1}{\sqrt{I_4}} \left( 2 \tilde{Q}_M \tilde{Q}_N - 6K_{MN} + Q_M Q_N \right). \quad (4.12)$$

Using general properties of $M_+$, to be discussed below, we shall give, at the end of Sect. C an alternative argument in favor of the above identification.
Let us comment on the properties of the above matrices under Freudenthal duality $\mathcal{F}$ (2.11), and in particular under its “critical”/horizon version $\mathcal{F}_H$ (2.25). Using the property
\[
\mathcal{F}_H(K_{MN}) = K_{MNPQ} \tilde{Q}^P \tilde{Q}^Q = K_{MN} - \frac{1}{6} \tilde{Q}_M \tilde{Q}_N + \frac{1}{6} \tilde{Q}_M \tilde{Q}_N,
\]
as it can be verified by exploiting the properties of groups “of type $E_7$” [21], it is straightforward to show that
\[
\mathcal{F}_H(M \pm (Q)) \equiv M \pm (\mathcal{F}_H(Q)) = M \pm (Q).
\]
Thus the identification (4.12) is consistent with the invariance of $M_{H}^{M,N}$ under $\mathcal{F}_H$, as given Eq. (1.9) of [11]:
\[
\mathcal{F}_H (M_{H}^{M,N}) := M_{H}^{M,N} (\tilde{Q}) = M_{H}^{M,N} (Q).
\]

Furthermore, the result (4.11), as discussed in App. A, is constrained by the consistency condition
\[
d = \frac{3n (2n + 1)}{n + 8},
\]
relating the dimension $d$ of the adjoint irrep. $\text{Adj}$ and the dimension $2n = 2n_V + 2$ of the black hole charge irrep. $\mathbf{R}_Q$ of $G$. As observed in [28], (4.16) actually characterizes at least all the pairs $(G, \mathbf{R}_Q)$ related to simple rank-3 Euclidean Jordan algebras [35] (such pairs are example of simple, non-degenerate groups “of type $E_7$” [33]).

The cases related to $D = 4$ Maxwell-Einstein gravity theories with local supersymmetry are reported in Table 1; within this class, the so-called $STU$ model [34] is an exception: the corresponding rank-3 Jordan algebra is semi-simple ($\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$), but however it still satisfies (4.16).

The condition (4.16) can be further elaborated, by observing that, in all the cases under consideration, it holds that
\[
n = 3q + 4,
\]
thus implying
\[
d = \frac{3(3q + 4) (2q + 3)}{q + 4}.
\]
For $J_3^{A(s)}$-related models (“magic” (super)gravities [35]), the parameter $q$ can be defined as
\[
q := \dim \mathbb{R} A(s) = 8, 4, 2, 1 \text{ for } A(s) = O(s), H(s), C(s), \mathbb{R},
\]
while $q = -2/3$ and $q = 0$ for $T^3$ and $STU$ model, respectively (and $q = 2$ for $\mathcal{N} = 5$ theory).

**Interpretation of $M_-$.** Interestingly, also
\[
M_{-J_4>0}^{M,N}(Q) = \frac{4}{(I_4)^{3/2}} K_M K_N - \frac{6}{\sqrt{I_4}} K_{MN} = \frac{1}{\sqrt{I_4}} \tilde{Q}_M \tilde{Q}_N - \frac{6}{\sqrt{I_4}} K_{MN}
\]
\[
= -\partial_M \partial_N \sqrt{I_4}
\]
can be given a meaning within the stratification of $\mathbf{R}_Q$ into $G$-orbits.

Indeed, $M_{-J_4>0}^{M,N}$ (4.21) can be regarded as the metric of the non-compact pseudo-Riemannian rigid special Kähler manifold [15]
\[
M_{J_4>0} := \mathcal{O}_{J_4>0} \times \mathbb{R}^+,
\]
Table 1: Four-dimensional $U$-duality groups $G$, black hole charge representation $\mathbf{R}_Q$, and data $d := \dim \mathbf{Adj}$ and $n := \dim \mathbf{R}_Q/2$. The corresponding scalar manifolds are the symmetric cosets $G/H$, where $H$ is the maximal compact subgroup (with symmetric embedding) of $G$. $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$ is the split form of octonions. $M_{1,2}(\mathbb{O})$ is the Jordan triple system (not upliftable to $D = 5$) generated by $2 \times 1$ Hermitian matrices over $\mathbb{O}$ [35]. Note that the STU model [34], based on $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, is reducible, but triality symmetric. All cases pertain to models with 8 supersymmetries, with exception of $M_{1,2}(\mathbb{O})$ and $J_3^{\mathbb{O}_s}$, related to 20 and 32 supersymmetries, respectively. The $D = 5$ uplift of the $T^3$ model based on $\mathbb{R}$ is the pure $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^\mathbb{R}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories share the very same bosonic sector [35, 38]. All data $d$ and $n$ satisfy the relations (4.16)-(4.18).

| $J_3$ | $G$ | $\mathbf{R}_Q$ | $(d, n)$ |
|-------|-----|----------------|----------|
| $J_3^{\mathbb{O}_s}$ | $E_7(7)$ | 56 | (133, 28) |
| $J_3^\mathbb{O}$ | $E_7(-25)$ | 56 | (133, 28) |
| $J_3^\mathbb{H}$ | $SO^*(12)$ | 32$^{(')}$ | (66, 16) |
| $J_3^\mathbb{C}$ | $SU(3, 3)$ | 20 | (35, 10) |
| $J_3^\mathbb{R}$ | $Sp(6, \mathbb{R})$ | 14$'$ | (21, 7) |
| $M_{1,2}(\mathbb{O})$ | $SU(1, 5)$ | 20 | (35, 10) |
| $\mathbb{R}$ | $SL(2, \mathbb{R})$ | 4 | (3, 2) |
| $T^3$ | | | |
| $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ | $[SL(2, \mathbb{R})]^3$ | (2, 2, 2) | (9, 4) |

with real dimension $2n_V + 2$; $\mathcal{O}_{I_4>0}$ denotes the corresponding “large” $G$-orbit defined by the $G$-invariant constraint $I_4 > 0$ on the charge representation $\mathbf{R}_Q$ of $G$; the $\mathbb{R}^+$ factor in (4.22) simply corresponds to the non-vanishing (strictly positive) values of $I_4$ itself. The signature along the $\mathbb{R}^+$-direction is negative, while the metric on $\mathcal{O}_{I_4>0}$ is the opposite of the Cartan-Killing metric.
on the coset \( G/G_0 \), \( G_0 \) being the stabilizer of \( Q \), namely its positive and negative eigenvalues correspond to the non-compact and compact generators in the coset space, respectively.

In \( \mathcal{N} = 2 \) (symmetric) theories, two \( G \)-orbits are defined by the constraint \( I_4 > 0 \); the \((\frac{1}{8})\)BPS orbit, and the non-BPS \( Z_H = 0 \) orbit \([39]\). Let us consider for instance the \( \mathcal{N} = 2 \) exceptional magic theory \([35]\) \((G = E_{7(-25)}, \mathbf{R}_Q = 56)\), for which one can define the two pseudo-Riemannian 56-dimensional rigid special Kähler manifolds:

\[
\mathbf{M}_{I_4 > 0, BPS} = \mathcal{O}_{I_4 > 0, BPS} \times \mathbb{R}^+ = \left( \frac{E_{7(-25)}}{E_{6(-14)}} \right) \times \mathbb{R}^+ , \quad \text{metric } M_{-|MN} \text{ with } (n_+, n_-) = (54, 2) ;
\]

\[
\mathbf{M}_{I_4 > 0, nBPS} = \mathcal{O}_{I_4 > 0, nBPS} \times \mathbb{R}^+ = \left( \frac{E_{7(-25)}}{E_{6(-14)}} \right) \times \mathbb{R}^+ , \quad \text{metric } M_{-|MN} \text{ with } (n_+, n_-) = (22, 34) .
\]

(4.23)

In general, the metric \( M_{-|MN} \) of \( \mathbf{M}_{I_4 > 0, BPS} \) always has signature \((n_+, n_-) = (2n_V, 2)\). This, indeed, is nothing but the signature of the symplectic matrix \( \mathbf{M}^{(F)} \) in \((1.45)\), or \((1.48)\), which will be proven in Sect. 4.3 to coincide, for the BPS orbit, with \( M_- \). In the example of the STU truncation, for instance, one of the two positive eigenvalues of \( M_- \) \((4.20)-(4.21)\) is computed in App. \([3]\) for the charge configuration \((q_0, p^1, p^2, p^3)\), the other is implied by \( M_- \) being symplectic.

On the other hand, in the maximal \( \mathcal{N} = 8 \) theory \((G = E_{7(7)}, \mathbf{R}_Q = 56)\) there is only one \( G \)-orbit defined by the constraint \( I_4 > 0 \), namely the \( \frac{1}{8} \)-BPS “large” orbit, which thus allows to define the pseudo-Riemannian 56-dimensional rigid special Kähler manifold \([15]\):

\[
\mathbf{M}_{I_4 > 0, \frac{1}{8}-BPS} := \mathcal{O}_{I_4 > 0, \frac{1}{8}-BPS} \times \mathbb{R}^+ = \left( \frac{E_{7(7)}}{E_{6(2)}} \right) \times \mathbb{R}^+ , \quad \text{metric } M_{-|MN} \text{ with } (n_+, n_-) = (30, 26) .
\]

(4.24)

### 4.2 Generalizing the Solutions \( M_\pm \) to all \( I_4 \neq 0 \) Orbits

If we extend the expressions for \( M_\pm \), given Section \([1.1]\) to \( I_4 < 0 \):

\[
M_{+, I_4 < 0 MN} = \frac{1}{(-I_4)^{\frac{1}{2}}} \left( -8 K_M K_N + 6 I_4 K_{MN} - I_4 Q_M Q_N \right),
\]

(4.25)

\[
M_{-, I_4 < 0 MN} = \frac{1}{(-I_4)^{\frac{1}{2}}} \left( 4 K_M K_N - 6 I_4 K_{MN} \right).
\]

(4.26)

we find that, in contrast to the \( I_4 > 0 \) case, these matrices, though still satisfying the second of conditions \([1.2]\), are anti-symplectic, namely exhibit the property \([1.40]\). Under the “critical”/horizon version \( \mathfrak{F}_H \) \((2.25)\) of Freudenthal duality, \( M_{\pm, I_4 < 0} \) transform as follows:

\[
\mathfrak{F}_H (M_{\pm, I_4 < 0}) = -M_{\pm, I_4 < 0}.
\]

(4.27)

This can be proved by using

\[
\mathfrak{F}_H (K_{MN}) = K_{MNPQ} \tilde{Q}^P \tilde{Q}^Q = \epsilon K_{MN} - \frac{1}{6} \tilde{Q}_M \tilde{Q}_N + \frac{\epsilon}{6} Q_M Q_N ,
\]

(4.28)

where \( I_4 = \epsilon |I_4| \), which generalizes \([1.13]\) for any sign of \( I_4 \). Note that properties \([1.14]\) and \((4.27)\) can be summarized, for any sign of \( I_4 \), as follows:

\[
\mathfrak{F}_H (M_\pm) = \epsilon M_\pm .
\]

(4.29)
As far as $M_-$ is concerned, for $I_4 < 0$, it coincides with the Hessian of $-\sqrt{-I_4}$. As a consequence of this, in all regular orbits, we can write, as a general property of $M_-$,

$$M_{-I_4 > 0} R^{MN}(Q) = -\partial_M \partial_N \sqrt{|I_4|}.$$  

(4.30)

Thus also for $I_4 < 0$, $M_-$ can be given the same interpretation as for the $I_4 > 0$ case: $M_{-I_4 < 0}$ can be regarded as the metric of the non-compact pseudo-Riemannian rigid special Kähler manifold

$$M_{I_4 < 0} := O_{I_4 < 0} \times \mathbb{R}^+.$$  

(4.31)

with real dimension $2n_V + 2$; $O_{I_4 < 0}$ denotes the unique “large” non-BPS $G$-orbit defined by the $G$-invariant constraint $I_4 < 0$ on the charge representation $R_Q$ of $G$; the $\mathbb{R}^+$ factor in (4.31) simply corresponds to the non-vanishing values of $|I_4|$ itself. For the $N = 2$ exceptional magic theory and $N = 8$ supergravity, the manifold (4.31) is respectively given by

$N = 2 : G = E_7(-25), R_Q = 56 : M_{I_4 < 0} := \frac{E_7(-25)}{E_6(-26)} \times \mathbb{R}^+$, metric $M_{-|MN}$ with $(n_+, n_-) = (28, 28)$;

$N = 8 : G = E_7(7), R_Q = 56 : M_{I_4 < 0} := \frac{E_7(7)}{E_6(6)} \times \mathbb{R}^+$, metric $M_{-|MN}$ with $(n_+, n_-) = (28, 28)$.

Interestingly, the two manifolds share the same signature.

As opposite to $M_-$, the adjoint action of $M_+$ defines, just as in the $I_4 > 0$ case, an automorphism of the $U$-duality group $G$:

$$(M_+)^{-1} \hat{R}_Q M_+ \subset \hat{R}_Q \iff M_+ \in \text{Aut}(G),$$  

(4.32)

where $\hat{R}_Q$ denotes the $2n \times 2n$ matrix representation of $G$ in $R_Q$. Since

$$M_+ \mathbb{C} M_+ = \mathbb{C},$$  

(4.33)

for $I_4 > 0$ $M_+$ thus belongs to the inner-automorphisms Inn$(G) \subset \text{Aut}(G)$, whereas for $I_4 < 0$ the anti-symplecticity of $M_-$ implies that it belongs to the outer-automorphisms Aut$(G)''/\text{Inn}(G)$

see Appendix [C] because the matrix realization $\hat{R}_Q$ of the elements of $G$ in the representation $R_Q$ is symplectic:

$$\hat{R}_Q^T \hat{R}_Q = \mathbb{C}.$$  

(4.34)

The same generally does not hold for the adjoint action of $M_-$:

$$(M_-)^{-1} \hat{R}_Q M_- \nsubseteq \hat{R}_Q \iff M_- \notin \text{Aut}(G).$$  

(4.35)

We can define the matrix $S_+ := \mathbb{C} M_+$, which is still in Aut$(G)$, since $M_+$ is. Moreover $S_+ Q = \mathbb{C} M^H Q = -\mathbb{3}_H(Q)$. We can then use (4.29) and write:

$$S_+^{-T} M_-(Q) S_+^{-1} = M_-(\mathbb{3}_H(Q)) = \epsilon M_-(Q),$$  

(4.36)

\footnote{Anticipating an argument to be given in Sect. [C] here we assume that Out$(G) = \text{Aut}(G)/\text{Inn}(G)$ is contained in $\mathbb{Z}_2$, as it seems to be common for groups of “type $E_7$” considered here. Since, as we shall prove in Appendix [C] the part of $\text{Aut}(G)$ not connected to the identity is implemented by anti-symplectic matrices, symplectic automorphisms are inner.}
from which we can easily derive the following property:

\[ M_+ \mathcal{C} M_- \mathcal{C} M_+ = -\epsilon M_- , \tag{4.37} \]

or, equivalently:

\[ M_- M_+^{-1} = M_+ M_-^{-1} . \tag{4.38} \]

Finally it can be easily shown from their definition in both \( I_4 > 0 \) and \( I_4 < 0 \) cases, that

\[ M_{\pm MN} \mathcal{Q}^N = \mathcal{M}_{MN}^{ij} \mathcal{Q}^N = -\partial M \sqrt{|I_4|} . \tag{4.39} \]

### 4.3 Interpretation of \( M_{\pm} \) in \( \mathcal{N} = 2 \) Theories

In the vector multiplet sector of an \( \mathcal{N} = 2 \) supergravity, we can define two symmetric, symplectic matrices: one is the matrix \( \mathcal{M} \) constructed out of the real and imaginary parts of \( \mathcal{N}_{\Lambda \Sigma} \), as in (2.2), the other is a matrix \( \mathcal{M}^{(F)} \) defined by having the same matrix form as in (2.2), but in terms of the real and imaginary parts of the complex \( n \times n \) matrix

\[ \mathcal{F}_{\Lambda \Sigma}(X) = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} , \tag{4.40} \]

\( F(X) \) being the holomorphic prepotential, homogeneous function of degree 2 of \( X^\Lambda(z) \) (we use the notations of [16]). We can write then:

\[ \mathcal{M}(z, \bar{z}) = \mathcal{M}[\Re \mathcal{N}, \Im \mathcal{N}] , \tag{4.41} \]

\[ \mathcal{M}^{(F)}(z, \bar{z}) = \mathcal{M}[\Re \mathcal{F}, \Im \mathcal{F}] , \tag{4.42} \]

where \( \mathcal{M}[R, I] \) is the function of the matrices \( R, I \) defined in (2.2). As anticipated in the introduction, can write the matrix \( \mathcal{M}(z, \bar{z}) \) in the manifestly symplectic-covariant form [5, 17]

\[ \mathcal{M}(z, \bar{z}) = \mathcal{C} \left( V V^T + V V^T + U_i g^{ij} \bar{U}_j^T + \bar{U}_j g^{ij} U_i^T \right) \mathcal{C} . \tag{4.43} \]

Note that the r.h.s. is the sum of two symmetric matrices:

\[ A_1 = \mathcal{C} \left( V V^T + V V^T \right) \mathcal{C} ; \quad A_2 = \mathcal{C} \left( U_i g^{ij} \bar{U}_j^T + \bar{U}_j g^{ij} U_i^T \right) \mathcal{C} , \tag{4.44} \]

which satisfy the condition \( A_1 \mathcal{C} A_2 = 0 \), which follow from the general properties: \( V^T \mathcal{C} U_i = V^T \mathcal{C} U_i = 0 \). Therefore, if \( \mathcal{M} = A_1 + A_2 \) is symmetric and symplectic, also \( A_1 - A_2 \) is. The latter is just the matrix \( \mathcal{M}^{(F)} \):

\[ \mathcal{M}^{(F)}(z, \bar{z}) = \mathcal{C} \left( V V^T + V V^T - U_i g^{ij} \bar{U}_j^T - \bar{U}_j g^{ij} U_i^T \right) \mathcal{C} , \tag{4.45} \]

The relation between the two matrices being then [7]:

\[ \mathcal{M}(z, \bar{z}) = -\mathcal{M}^{(F)}(z, \bar{z}) + 2 \mathcal{C} \left( V V^T + V V^T \right) \mathcal{C} , \tag{4.46} \]

which is consistent with the relation between the lower diagonal blocks of the two matrices given e.g. in [18]:

\[ \Im \mathcal{N}^{-1 \Lambda \Sigma} = -\Im \mathcal{F}^{-1 \Lambda \Sigma} - 4 L^{(\Lambda \Sigma)} . \tag{4.47} \]

---

7This relation is also given in (1.13) of [12], in terms of the so-called Hesse potential (defined in (1.10) therein).
In $\mathcal{N} = 2$ theories, we can express the matrix $\mathcal{M}^{(F)}$ in a form similar to Eq. (2.3) for $\mathcal{M}$, namely:

$$\mathcal{M}^{(F)} = -L^{-T} \eta L^{-1}, \quad (4.48)$$

where $L$ is an $\text{Sp}(2n, \mathbb{R})$-matrix of the form:

$$L = \sqrt{2} (\text{Re}(V), \text{Re}(U), -\text{Im}(V), \text{Im}(U)); \quad (4.49)$$

moreover, $U_I = E_I^i U_i$, $E_I^i$ being the complex Vielbein matrix of the special Kähler manifold, and $\eta$ is the diagonal matrix:

$$\eta = \text{diag}(1, -\mathbb{I}_{n-1}, 1, -\mathbb{I}_{n-1}), \quad (4.50)$$

where $\mathbb{I}_{n-1}$ denotes the $(n-1) \times (n-1)$ identity matrix.

Let us now evaluate relation (4.46) at the horizon of a regular BPS black hole (thus, with $I_4 > 0$) and show that it yields the relation between $M_\pm$, proving thus that, if $M_\pm$ coincides with the matrix $\mathcal{M}^H$, $M_\pm$ coincides with $\mathcal{M}^{(F)}$ at the horizon. To this end, we use the relations [5]:

$$2i \bar{Z} V^M \big|_{\text{horizon}} = Q^M - i C^{MN} \partial_N \sqrt{I_4} = Q^M - \frac{2i}{\sqrt{I_4}} C^{MN} K_N, \quad (4.51)$$

which hold at the horizon of the solution. Using the property that, at the horizon, $|Z|_{\text{horizon}}^2 = \sqrt{I_4}$, we end up with

$$4 V^{(M\bar{V}N)} \big|_{\text{horizon}} = \frac{1}{\sqrt{I_4}} Q^M Q^N + \frac{4}{\sqrt{I_4^3}} C^{MP} C^{NQ} K_P K_Q, \quad (4.52)$$

so that

$$\mathcal{M}^H = -\mathcal{M}^{(F)} \big|_{\text{horizon}} - \frac{1}{\sqrt{I_4}} Q_M Q_N - \frac{4}{\sqrt{I_4^3}} K_M K_N, \quad (4.53)$$

which is the same relation holding between $M_+$ and $M_-$. Indeed, from (6.6) and (6.4), it follows that

$$M_+|_{MN} = -M_-|_{MN} - \epsilon \frac{1}{\sqrt{|I_4|}} Q_M Q_N - \frac{1}{\sqrt{|I_4|}} \tilde{Q}_M \tilde{Q}_N = -M_-|_{MN} - \epsilon \frac{1}{\sqrt{|I_4|}} Q_M Q_N - \frac{4}{|I_4|^{3/2}} K_M K_N, \quad (4.54)$$

which for $I_4 > 0$ reduces to the same relation (4.53).

### 4.4 $T^3$ Model with $I_4 < 0$

Among the symmetric models, this is the unique case in which the non-BPS $I_4 < 0$ attractors do not exhibit any flat direction [39, 25], and thus it deserves a separate treatment.

In this case, however, we can write

$$\mathcal{M}^H(Q) = M_+(Q) A_0(Q), \quad (4.55)$$

where $M_+(Q)$ is the anti-symplectic matrix given by (4.25), and $A_0(Q)$ is defined as:

$$A_0(Q) := M_+^{-1}(Q) \mathcal{M}^H(Q). \quad (4.56)$$
From Eq. (4.39) we have in particular that:
\[ M_+ (Q)_{MN} Q^N = M^H (Q)_{MN} Q^N = - \partial_M \sqrt{-I_4}, \] (4.57)
which implies
\[ A_0 (Q) = Q \Rightarrow A_0 (Q) \in \text{Stab}_Q [\text{GL}(4, \mathbb{R})], \] (4.58)
where \( \text{Stab}_Q [\text{GL}(4, \mathbb{R})] \) denotes the stabilizer of \( Q \) in \( \text{GL}(4, \mathbb{R}) \).

Since, as anticipated in Sec. 4.2 (see footnote 6) and as discussed in Sect. 6, the adjoint action of \( M_+ \) defines an outer-automorphism of \( G \), and since \( M^H \) is an element of \( G \), it follows that also the adjoint action of \( A_0 (Q) \) defines an outer-automorphism of \( G \) (\( \text{Out}(G) := \text{Aut}(G) / \text{Inn}(G) \); cfr. App. [C]):
\[ A_0 (Q) \in \text{Out}(G) \cap \text{Stab}_Q [\text{GL}(4, \mathbb{R})]. \] (4.59)

### 4.5 \( \mathcal{N} = 2 \) minimally coupled and \( \mathcal{N} = 3 \)

As mentioned above, the cases of \( 1/2 \)-BPS attractors in \( \mathcal{N} = 2 \) minimally coupled supergravity, non-BPS attractors in the \( T^2 \) model, \( 1/3 \)-BPS attractors in \( \mathcal{N} = 3 \) “pure” supergravity, and non-BPS attractors in the same theory coupled to 1 vector multiplet, deserve a separate treatment.

Besides not exhibiting any flat direction, they indeed share the peculiar property that the \( K \)-tensor is non-primitive: namely, it can be expressed in terms of a rank-2 symmetric invariant tensor \( S_{MN} \) of the corresponding \( U \)-duality group.

The aforementioned cases correspond to particular cases \( (r = 1 \) and \( r = 3 \), respectively) of the class of pseudo-unitary duality group \( U(r, n) \) with symplectic black hole charge representation given by the (complex) fundamental representation \( r + n \) \([33, 40]\).

After \([33]\), the following definition and relations holds\(^8\):
\[ I_2 = \zeta S_{MN} Q^M Q^N; \] (4.60)
\[ K_{MNPQ} = \frac{1}{6} S_{MN(SPQ)} \Rightarrow I_4 = \frac{1}{2} I_2^2 > 0; \] (4.61)
\[ K_{MNP} = \frac{1}{24} (S_{MP} S_{NQ} + S_{NP} S_{MQ} + S_{MN} S_{PQ}); \] (4.62)
\[ K_{MN} = \frac{1}{12} S_{MP} S_{NQ} Q^P Q^Q + \frac{\zeta}{6} I_2 S_{MN}; \] (4.63)
\[ K_M = \frac{\zeta}{2} I_2 S_{MP} Q^P. \] (4.64)

Indeed, \( U(r, n) \) has an Hermitian invariant rank-2 tensor in its (complex) fundamental representation \( r + n \), whose real and imaginary part defines \( S_{MN} \) (symmetric) and the symplectic metric \( C_{MN} \) (antisymmetric) (cfr. e.g. (2.33) of \([33]\)). The quadratic invariant \( I_2 \) occurs in the Bekenstein-Hawking formula (2.31) for extremal black holes in the corresponding (super)gravity theories:
\[ S_{BH} = \pi |I_2| = - \frac{\pi}{2} M_{MN}^H Q^M Q^N. \] (4.65)

As discussed in Sec. 10 of \([33]\), in these cases the horizon/“critical” limit \( \mathcal{F}_H \) of the “non-critical” Freudenthal duality \( \mathcal{F} \) (2.11) is nothing but a particular anti-involutive symplectic transformation of \( U(r, n) \).

\(^8\)A different normalization with respect to \([33]\), e.g. implying \( \zeta^2 = 1/4 \), has been adopted.
By exploiting (4.60)-(4.64), the result (4.12) simplifies as follows:

\[ M_{MN}(Q) = -\frac{3}{\sqrt{2}|I_2|} \left( S_{MP}S_{NQ}Q^PQ^Q - \varrho I_2 S_{MN} + \frac{2}{3} Q_M Q_N \right) , \]

(4.66)

where \( \varrho := \zeta / |\zeta| \); indeed, let us remark that the result (4.66) holds for any sign of \( I_2 \), and also that the sign of the real normalization constant \( \zeta \) occurring in the definition (4.60) is not fixed (as discussed in [33], only \( \zeta^2 \) is fixed).

5 \textit{M}^H \textit{with Flat Directions}

In presence of flat directions \( \varphi_{flat} \) (namely, in all - symmetric - cases not treated in the analysis of Sec. 4), the matrix \( \text{M}^H \) is generally not \( G \)-covariant, \emph{i.e.} it cannot be computed only in terms of purely \( Q \)-dependent covariant quantities. Thus, the procedure outlined in Sec. 4 cannot be applied.

However, in the following treatment we will determine the most general expression of \( \text{M}^H \) also in presence of flat directions. These generically occur in the case \( I_4 < 0 \) (genuine non-BPS attractors), to be dealt with in Subsec. 5.1, and in the case \( I_4 > 0 \) for non-BPS solutions and BPS ones in theories with \( \mathcal{N} > 2 \), to be discussed in Subsec. 5.2.

5.1 \( I_4 < 0 \)

For \( d \)-geometries, in the 4D/5D special coordinate symplectic frame, one can decompose the element \( \text{L} \) of the \( \text{Sp}(2n + 2, \mathbb{R}) \)-valued symplectic bundle of generalized special geometry as follows\(^9\) (cfr. (2.2) of [37]):

\[ \text{L}(a, \phi, E) = T(a) \mathcal{D}(\phi) G(E) = L_1(a, \phi) L_0(E) ; \]

\[ L_1(a, \phi) = T(a) \mathcal{D}(\phi) ; \quad L_0(E) := G(E) . \]

(5.1)

(5.2)

\( L_0(E) \) depends only on the \( D = 5 \) real scalar fields, while the \( a^I \)'s are axionic fields (with the index \( I \) running over the number of \( D = 5 \) vector fields) and \( \phi \) denotes the KK scalar.

At least for symmetric \( d \)-geometries (or when \( U \)-orbit structure is present), by choosing a representative \( Q = (p^0, q_0) \) belonging to the genuine non-BPS orbit (corresponding to \( I_4(Q) < 0 \) in symmetric models), one obtains that

\[ \text{L}^{-1}Q = G^{-1}(E) \mathcal{D}^{-1}(\phi) T^{-1}(a) Q = G(E^{-1}) \mathcal{D}(-\phi) T(-a) Q \]

\[ = \mathcal{D}(-\phi) T(-aE^{-1}) G(E^{-1}) Q = \mathcal{D}(-\hat{\phi}) T(-\hat{a}) Q , \]

(5.3)

where \( \hat{a} := aE^{-1} \). Thus, up to suitable redefinitions of the axions \( (a \rightarrow \hat{a} := aE^{-1}) \), \( \text{L}^{-1}Q \) is independent of flat directions (spanning the scalar manifold of the theory uplifted to \( D = 5 \)), also \textit{off-shell} [24].

Therefore, by recalling (2.3) and observing that \( G^{-1}(E) = G(E^{-1}) \), one obtains the following \textit{off-shell} decomposition:

\[ \text{M} \left( a, \phi, (EE^T)^{-1} \right) = \text{M}_1(a, \phi, EE^T) \text{M}_0(EE^T) , \]

(5.4)

\(^9\)The decomposition (5.1) is related to the decomposition (35) of [24] by a field redefinition.
where

\[ \mathcal{M}_1 (a, \phi, EE^T) := -T^T (-a) D (-2\phi) T \left( -a \left( EE^T \right)^{-1} \right), \quad (5.5) \]

\[ \mathcal{M}_0 (EE^T) := \begin{pmatrix} 1 & \left( EE^T \right)^{-1} \\ EE^T \end{pmatrix}, \quad (5.6) \]

with \( E \) denoting the \( D = 5 \) kinetic vector matrix. Note that \textit{off-shell} it generally holds that

\[ [\mathcal{M}_1, \mathcal{M}_0] \neq 0. \quad (5.7) \]

At the \( I_4 < 0 \) critical points (\textit{i.e.}, \textit{on-shell} at genuinely non-BPS critical points of \( V_{BH} \)), in the \((p^0, q_0)\) charge configuration the axions can be set to zero without any loss of generality (\( a_H = 0, \hat{a}_H = 0 \)) \cite{27}, and therefore

\[ \mathcal{M}^H_{I_4 < 0} = \mathcal{M}^H_1 (a = 0, \phi_H, EE^T) \mathcal{M}_0 (EE^T) = \mathcal{M}^H_1 (\phi_H) \mathcal{M}_0 (EE^T), \quad (5.8) \]

where the result

\[ \mathcal{M}^H_1 (\phi_H) = -D(-2\phi_H), \quad (5.9) \]

is due to the fact that \( \mathcal{M}_1 \) depends on \( EE^T \) \textit{only} through \( a \)’s. Thus, in contrast with (5.7), \textit{on-shell} it generally holds that

\[ [\mathcal{M}^H_1, \mathcal{M}_0] = 0. \quad (5.10) \]

While in symmetric models the result above can be extended by duality all along the genuine non-BPS \( U \)-orbit (\( I_4 < 0 \)), it should be stressed that for \textit{non-symmetric} \( d \)-geometries, the result (5.8)-(5.9) strictly holds for the \((p^0, q_0)\) charge configuration. Indeed, for any \( d \)-geometry (see \textit{e.g.} \cite{27,37}), the general solution to the Attractor Equations \( \partial V_{BH} = 0 \) supported by the \((p^0, q_0)\) charge configuration reads (cfr. (1.12)-(1.13) of \cite{37})

\[ a^I_H = 0, \quad e^{6\phi_H} = 4 \left| \frac{q_0}{p^0} \right|. \quad (5.11) \]

In \textit{symmetric} models, all these genuine non-BPS solutions (or suitable truncations thereof) can be uplifted to non-BPS \( I_4 < 0 \) solutions in \( \mathcal{N} = 8, D = 4 \) supergravity, and a prototypical solution is given \textit{e.g.} by the Kaluza-Klein black hole, as discussed in \cite{41}. By using (5.11) and recalling the block-diagonal form of the generator \( D \) of the \( SO(1,1) \) Kaluza-Klein dilatation (\( \mathbb{I} \) here denoting the \( n_V \times n_V \) identity):

\[ D = \begin{pmatrix} -3 & -\mathbb{I} \\ -\mathbb{I} & 3 \end{pmatrix}, \quad (5.12) \]

the expression of \( \mathcal{M}^H_1 \) can be recast as follows:

\[ \mathcal{M}^H_1 = -D(-2\phi_H) = -\exp (-2\phi_H D) = \begin{pmatrix} e^{6\phi_H} & 0 \\ 0 & e^{-6\phi_H} \end{pmatrix}, \quad (5.13) \]
By recalling (1.2), the relations above entail the dependence of the black hole effective potential \( V_{BH} \), as well as of related quantities, on the various kinds of \( D = 4 \) scalar fields:

\[
V_{BH} = \frac{1}{2} Q^T T^T (\sim \hat{a}) \mathcal{D} (\sim 2\phi) T (\sim \hat{a}) \mathcal{Q} = V_{BH} (\hat{a}, \phi),
\]

where (5.3) was used; this shows that, up to suitable redefinitions of the axions \( (a \rightarrow \hat{a} := aE^{-1}) \), \( V_{BH} \) is independent of flat directions, also off-shell [24] (and the same holds for the ADM mass, which can be computed as the asymptotic (radial) limit of the first order superpotential \( \mathcal{W} \).

On the other hand, as anticipated in (2.11), the scalar-dependent, “non-critical” Freudenthal duality (2.11) does depend on flat directions off-shell:

\[
\mathfrak{F} := -C\mathcal{M} \mathcal{Q} = \frac{\partial V_{BH}}{\partial \mathcal{Q}} = CT^T (\sim a) \mathcal{D} (\sim 2\phi) T (\sim \hat{a}E^{-1}) \mathcal{Q} = \mathfrak{F} (a, \phi, E),
\]

but, as anticipated in (2.47), it does not when evaluated on-shell:

\[
\mathfrak{F}_{\mathcal{L}_\phi < 0} (\mathcal{Q}; a_H = 0, \phi_H) = C\mathcal{D} (\sim 2\phi) \mathcal{Q} = -C\mathcal{M}^H (\phi_H) \mathcal{Q} = \frac{1}{\pi} C \frac{\partial S_{BH}}{\partial \mathcal{Q}},
\]

where in the last step a result of [11] was exploited. By recalling (2.42)-(2.44), Eq. (5.16) can thus be considered as expressing the validity of the attractor mechanism for the Freudenthal duality \( \mathfrak{F} \) (2.11), or, equivalently, by recalling (2.17)-(2.21) and (2.44), for the Abelian two-form field strengths \( H \) in the near-horizon Bertotti-Robinson metric of the extremal black hole (2.7).

In order to determine the most general expression for \( \mathcal{M}^H_{\mathcal{L}_\phi < 0} \), one needs to generalize the expression \( \mathcal{M}^H \) (5.9) to a generic representative of the \( I_4 < 0 \) \( U \)-orbit, in which all components of the charge vector \( \mathcal{Q} \) are non-vanishing. This can be achieved by applying the most general set of axionic translations to \( \mathcal{M}^H \) (5.9); such translations are a “universal” sector of the \( D = 4 \) electric-magnetic symmetry, common to all \( d \)-geometries (recently studied in [37]):

\[
\mathcal{M}^H (\phi_H) \rightarrow g^{-T} \mathcal{M}^H (\phi_H) g^{-1},
\]

\[
g : = \exp (\alpha^I T_I) \exp (\alpha^J T^J),
\]

where \( T_I \) and \( T^J \) denote the generators of the contravariant (namely, axionic) and covariant translations (with \( \alpha^I \) and \( \alpha_J \) denoting the corresponding parameters; \( (T^I)^T = T_I \)). After the treatment e.g. of [13, 37], the explicit form of the finite “universal” translations \( g \) (5.17) can be computed to read

\[
g = \exp (\alpha^I T_I) \exp (\alpha^J T^J)
\]

\[
= \begin{pmatrix}
1 & 0 & -\tilde{d}/6 & \frac{d^I}{2} \\
\alpha^I & \delta_I^J + \alpha^I \alpha_J & -\frac{1}{6} \left( d \alpha^I + 3d^I \right) & \frac{1}{2} \alpha^I d^I + d^{IJ} \\
-\tilde{d}/6 & -\frac{1}{6} \left( d \alpha_I + 3d_I \right) & 1 + \alpha_I \alpha^I + d^I d_I / 4 + \tilde{d} d_I / 36 & \left( \alpha^I + d^I / 12 + d^{IJ} d_J / 2 \right) \\
d_I / 2 & \frac{1}{2} \alpha_I d_J + d_{IJ} & -\left( \alpha_I + \tilde{d} d_I / 12 + d_{IJ} d^J / 2 \right) & \delta_I^J + d^K d_K / 4 + d^{JK} d_{JK}
\end{pmatrix},
\]

where

\[
d : = d_{IJK} \alpha^I \alpha^J \alpha^K, \quad d_I : = d_{IJK} \alpha^I \alpha^K, \quad d_{IJ} := d_{IJK} \alpha^K, \quad d^{IJ} := d^{IJK} \alpha_K.
\]

\[
\tilde{d} := d^{IJK} \alpha_I \alpha_J \alpha_K, \quad d^I := d^{IJK} \alpha_J \alpha_K, \quad d^J := d^{IJK} \alpha_K.
\]
In order to determine the expression of $\mathcal{M}_1^H$ when all charges are non-vanishing (but still constrained by $I_4 < 0$), the system:

$$\begin{pmatrix} p^0 \\ P^I \\ Q_0 \\ Q_I \end{pmatrix} = \exp (\alpha^I T_I) \exp (\alpha_I T^I) \begin{pmatrix} p^0 \\ q^I_0 \\ q_0 \\ 0_I \end{pmatrix}$$ (5.21)

must be solved, constrained by\(^{10}\)

$$- (p^0 q_0)^2 = I_4 (p^0, P^I, Q_0, Q_I) < 0;$$ (5.22)

$$I_{4(Q)} = -(p^0 Q_0 + P^I Q_I)^2$$ (5.23)

$$+ 4 \left[ Q_0 \frac{d_{IJK}}{3!} P^J P^K - p^0 \frac{d_{IJK}}{3!} Q_I Q_J Q_K + \frac{1}{4} d_{IJK} d_{IJKLM} Q_J Q_K P^L P^M \right].$$

The solution of the system \((5.21)-(5.23)\) is provided by the parameters $\alpha^I$, $\alpha_I$, as well as the charges $p^0$ and $q_0$, expressed in terms of the charges $P^0, P^I, Q_0, Q_I$, and it can be computed to read

$$p^0 = \frac{P^0}{2} \pm \frac{1}{2 \sqrt{-I_4}} \left( \frac{2}{3!} d_{IJK} P^I P^J P^K - P^0 \left( p^0 Q_0 + P^I Q_I \right) \right);$$ (5.24)

$$q_0 = \frac{I_4 \left[ \frac{1}{2} d_{IJK} P^I P^J P^K - P^0 \left( P^0 Q_0 + P^I Q_I \right) + P^0 \sqrt{-I_4} \right]}{\prod I : (-P^0 Q_0 + \frac{1}{2} d_{IJK} P^I P^K)};$$ (5.25)

$$\alpha^I = \frac{P^0 Q_0 - P^I Q_I \pm \sqrt{-I_4}}{2 \left( -P^0 Q_0 + \frac{1}{2} d_{IJK} P^J P^K \right)};$$ (5.26)

$$\alpha_I = \pm \frac{\left( P^0 Q_0 - \frac{1}{2} d_{IJK} P^J P^K \right)}{\sqrt{-I_4}}.$$ (5.27)

where no sum on repeated indices is assumed in the numerator of the r.h.s. of (5.26). It is amusing to observe that, in the “wrong” assumption $I_4 > 0$, (5.26) $\alpha^I$ becomes complex and matches the attractor value of the $D = 4$ scalar fields at BPS critical points (cfr. the discussion in Subsec. 5.2).

Therefore, by plugging \((5.24)-(5.27)\) back into \((5.18)\), one obtains the general expression $\mathcal{M}_{1,\text{gen}}^H$ supported by a “generic” representative of the non-BPS $I_4 < 0$ $U$-orbit with all charges non-vanishing:

$$\mathcal{M}_{1,\text{gen}}^H (P^0, P^I, Q_0, Q_I) := [g^{-T} \mathcal{M}_1^H (\phi_H) g^{-1}] \mid_{5.24} \mid_{5.29}.$$ (5.28)

As mentioned, $\mathcal{M}_{1,\text{gen}}^H (P^0, P^I, Q_0, Q_I)$ is not covariant with respect to the $D = 4$ $U$-duality group $G_4$, but its covariance is broken down to the non-compact stabilizer of the non-BPS $U$-orbit, which, in the symmetric $\mathcal{N} = 2$ $d$-geometries, is nothing but the $D = 5$ $U$-duality group $G_5$:

$$\mathcal{O}_{I_4<0} = \frac{G}{G_5}.$$ (5.29)

\(^{10}\)In general, $I_4$ \((5.23)\) is independent of scalar fields (and the treatment itself makes sense) only for symmetric $d$-geometries.
Thus, by recalling (5.8) and (5.6), the general expression of $\mathcal{M}_{I_4 < 0}^H$ supported by a “generic” representative of the non-BPS $I_4 < 0$ $U$-orbit with all charges non-vanishing is given by:

$$
\mathcal{M}_{I_4 < 0}^H (P^0, P^I, Q_0, Q_I; EE^T) = \mathcal{M}_{1, \text{gen}}^H (P^0, P^I, Q_0, Q_I) \mathcal{M}'_0,
$$

with $\mathcal{M}_0$ and $g$ respectively defined by (5.6) and (5.17).

### 5.2 $I_4 > 0$

At non-BPS $I_4 > 0$ and BPS critical points of $V_{BH}$ in symmetric $d$-geometries (in which flat directions generally occur), the situation is even simpler (in this section we only consider BPS attractors in $\mathcal{N} > 2$ theories in which they exhibit flat directions).

Indeed, all such solutions (or suitable truncations thereof) can be uplifted to (“large”) $\frac{1}{8}$-BPS $I_4 > 0$ solutions in $\mathcal{N} = 8$, $D = 4$ supergravity, and the corresponding prototypical solution is given by the Reissner-Nordström (RN) black hole, discussed e.g. in [41], which is consistent with an attractor solution in which all scalar fields (but not necessarily the flat directions) vanish.

Let us recall that the (non-BPS and BPS) $I_4 > 0$ $U$-orbits are generically described by a non-symmetric coset of the form

$$
\mathcal{O}_{I_4 > 0} = \frac{G}{G_0},
$$

where, for the non-BPS attractors in $\mathcal{N} = 2$ theories, $G_0$ is nothing but a different non-compact real form of the $D = 5$ $U$-duality group $G_5$. By denoting the maximal compact subgroup (mcs) of $G_0$ by $H_0$, the $D = 4$ scalar fields can then be $H_0$-covariantly decomposed as

$$
\varphi = \varphi_{\text{flat}} + \tilde{\varphi},
$$

with $\varphi_{\text{flat}}$ and $\tilde{\varphi}$ respectively denoting the flat directions and the attracted scalar degrees of freedom supported by $\mathcal{O}_{I_4 > 0}$.

The RN solution, supported by a dyonic charge vector $Q$ with only two non-vanishing charges (say, $p$ and $q$) consistently fixes $\tilde{\varphi}_H = 0$, and thus correspondingly the matrix $\mathcal{M}$ at the horizon reads

$$
\mathcal{M}_{RN}^H = -\mathcal{M}_0 (\varphi_{\text{flat}}). \tag{5.33}
$$

In this case, the general expression of $\mathcal{M}_{I_4 > 0}^H$ (supported by a “generic” representative of the $I_4 > 0$ non-BPS or BPS $U$-orbit with all charges non-vanishing) is formally obtained by the same procedure described in Subsec. 5.1, with the important difference that in this case the KK $SO(1, 1)$ is replaced by a compact $U(1)$ symmetry, and different non-compact real forms do occur with respect to the previous treatment. The “translations” are now $U(1)$-charged, and indeed the corresponding parameters are complex ($\tilde{\alpha}_I = \bar{\alpha}_I^T$):

$$
\begin{align*}
\mathcal{M}_{I_4 > 0}^H &= \mathcal{M}_{1, \text{gen}}^H \mathcal{M}'_0 = -\tilde{g}^{-T} \tilde{g}^{-1} \mathcal{M}'_0 = \tilde{g}^{-T} \mathcal{M}_0 (\varphi_{\text{flat}}) \tilde{g}^{-1} \tag{5.34} \\
\tilde{g} : &= \exp (\tilde{\alpha}_I T_I) \exp (\bar{\alpha}_I T_I^T) \tag{5.35} \\
\mathcal{M}'_0 &= g \mathcal{M}_0 (\varphi_{\text{flat}}) g^{-1} \tag{5.36}
\end{align*}
$$

where $\tilde{\alpha}_I$ and $\bar{\alpha}_I$ solves the system

$$
\begin{pmatrix}
P^\Lambda \\
Q_\Lambda
\end{pmatrix}
= \exp (\tilde{\alpha}_I T_I) \exp (\bar{\alpha}_I T_I^T)
\begin{pmatrix}
p \\
0 \\
q \\
0
\end{pmatrix} \tag{5.37}
$$

25
by: We have constructed two symmetric real matrices $G$ corresponding which is never negative definite, enjoys an interpretation as symplectic metric of the corresponding system (as instead they were in (5.23)). The solution $\tilde{\alpha}^I = \alpha^I (P^A, Q_A)$ of the system (5.37)–(5.38) is formally given by (5.26) with $\alpha^I \rightarrow \tilde{\alpha}^I$, which indeed yields complex $\tilde{\alpha}^I$'s for $I_4 > 0$, as observed below (5.27) : in other words, amusingly, $\tilde{\alpha}^I (P^A, Q_A)$ as well as $\tilde{\alpha}^I (P^A, Q_A) = \tilde{\alpha}^I (P^A, Q_A)$ are nothing but the attracted values of the scalar fields (and of their complex conjugates) at $1/2$-BPS critical points.

Thus, by recalling (5.34), the general expression of $\mathcal{M}^H_{I_4>0}$ supported by a “generic” representative of the (BPS or non-BPS) $I_4 > 0$ $U$-orbit with all charges non-vanishing is given by:

$$\mathcal{M}^H_{I_4>0} = \left[g^{-T} M_0 (\varphi_{flat}) g^{-1}\right]\bigg|_{(5.26)I_4>0}.$$  

(5.39)

### 6 Summary of Results and General Properties

We have constructed two symmetric real matrices $M_\pm (Q)$ satisfying the conditions (1.5):

$$M_\pm (Q)^T C M_\pm (Q) = \epsilon C ;$$

$$Q^T M_\pm (Q) Q = -2 \sqrt{|I_4|},$$

(6.1)

(6.2)

where $I_4 =: \epsilon |I_4|$. These matrices also satisfy relations (4.39) :

$$M_{\pm MN} Q^N = M^H_{MN} Q^N = -\partial M \sqrt{|I_4|}.$$

(6.3)

The matrix

$$M_{-|MN} = \frac{4}{|I_4|^{3/2}} K_M K_N - \epsilon \frac{6}{\sqrt{|I_4|}} K_{MN} = \frac{1}{\sqrt{|I_4|}} \tilde{Q}_M \tilde{Q}_N - \epsilon \frac{6}{\sqrt{|I_4|}} K_{MN} = -\partial M \partial N \sqrt{|I_4|},$$

(6.4)

which is never negative definite, enjoys an interpretation as symplectic metric of the corresponding $G$-orbit of $Q$ (see above as well as the final part of Sec. 1.11). Moreover it does not belong to $\text{Aut}(G)$.

On the other hand, the matrix

$$M_{+|MN} = -\frac{8}{|I_4|^{3/2}} K_M K_N + \epsilon \frac{6}{\sqrt{|I_4|}} K_{MN} - \epsilon \frac{1}{\sqrt{|I_4|}} Q_M Q_N

(6.5)

= -\frac{2}{\sqrt{|I_4|}} \tilde{Q}_M \tilde{Q}_N + \epsilon \frac{6}{\sqrt{|I_4|}} K_{MN} - \epsilon \frac{1}{\sqrt{|I_4|}} Q_M Q_N

(6.6)

belongs to $\text{Aut}(G)$ (in particular, see below, $M_{+,I_4>0} \in \text{Inn}(G)$ and $M_{+,I_4<0} \in \text{Aut}(G)/\text{Inn}(G) =: \text{Out}(G)$; cfr. e.g. App. C).

Both matrices under $\tilde{S}_H$ (2.23) transform as in (1.29).

For charges in a generic regular $G$-orbit (also in presence of flat directions), one can construct the matrix:

$$\mathcal{A} (Q, \varphi_{flat}) := M_+ (Q)^{-1} \mathcal{M}^H (Q, \varphi_{flat}),$$

(6.7)

so that

$$\mathcal{M}^H (Q, \varphi_{flat}) = M_+ (Q) \mathcal{A} (Q, \varphi_{flat}).$$

(6.8)
The above decomposition is related to the one given in Sect. 5 \[\text{namely } \mathcal{M}^H = \mathcal{M}_1^H \mathcal{M}_0 \text{ as follows. Just as for } \mathcal{M}^H \text{ we define for the negative-definite G-matrix } \mathcal{M}^H(Q) \text{ a matrix } \mathcal{A}_0(Q) = M_+^{-1} \mathcal{M}^H_1, \text{ so that } \mathcal{M}^H_1 = M_+ \mathcal{A}_0. \text{ The matrix } \mathcal{A}_0 \text{ will have the same properties as } \mathcal{A}, \text{ except that it does not depend on the flat directions. Then, comparing } \mathcal{M}^H = \mathcal{M}^H_1 \mathcal{M}_0 \text{ with } (6.8), \text{ we can write } \mathcal{A}(Q, \varphi_{\text{flat}}) = \mathcal{A}_0(Q) \mathcal{M}_0(\varphi_{\text{flat}}).

Let us illustrate some properties of \( \mathcal{A} \); as it follows from from Eq. (6.3), \( \mathcal{A}(Q, \varphi_{\text{flat}}) \) is in the stabilizer of \( Q \) in GL(2n, \( \mathbb{R} \)). Moreover, since \( M_+ \in \text{Aut}(G) \) and \( \mathcal{M}^H \in G \subset \text{Aut}(G) \), both are invariant under \( H_0 \) (denoting the stabilizer of \( \varphi_{\text{flat}} \)), also \( \mathcal{A} \) is, and thus we can write (4.59):

\[
\mathcal{A}(Q, \varphi_{\text{flat}}) \in \frac{\text{Aut}(G)}{H_0} \cap \text{Stab}[\text{GL}(2n, \mathbb{R})].
\]  

An important property of \( \mathcal{A} \) is the following:

\[
\mathcal{A}^T M_+(Q) \mathcal{A} = M_+(A^{-1} Q) = M_+(Q),
\]  

which follows from (6.9), but can be alternatively be proven using Eq.s (6.7), (2.13), (6.1), and (4.29):

\[
\mathcal{A}^T M_+(Q) \mathcal{A} = \mathcal{M}^H M_+(Q)^{-1} \mathcal{M}^H = -C S^H M_+(Q)^{-1} (S^H)^T C = -C M_+(S^H Q)^{-1} C = \epsilon M_+(S^H Q) = \epsilon \mathcal{S}_H(M_+) = M_+(Q).
\]  

From this, it also follows that \( \mathcal{A} \) is \textit{involutive}:

\[
\mathcal{A}^2 = (M_+)^{-1} \mathcal{M}^H (M_+)^{-1} \mathcal{M}^H = (M_+)^{-1} M_+ = \mathbb{I}.
\]  

Note that a property analogous to (6.11) holds for \( M_- \):

\[
\mathcal{A}^T M_-(Q) \mathcal{A} = M_-, \tag{6.13}
\]

as it can be shown along the same lines as in (6.11) and using property (4.38).

If \( I_+ < 0 \), \( M_+(Q) \) is \textit{anti-symplectic}, and thus (6.7) yields that \( \mathcal{A} \) is \textit{anti-symplectic} as well. Therefore, as \( M_+(Q) \), it defines an \textit{outer}-automorphism of \( G \) (see Appendix C for a discussion on anti-symplectic outer-automorphisms of the U-duality algebra), and one can write \( (Q \in O_{I_+ < 0} \text{ (5.29)}; H_0 = H_5) \):

\[
M_+(Q) \in \text{Out}(G);
\]

\[
\mathcal{A}_{I_+ < 0}(Q, \varphi_{\text{flat}}) = \mathcal{A}_0(Q) \mathcal{M}_0(\varphi_{\text{flat}}) \in \text{Out}(G) \cap \text{Stab}[\text{GL}(2n, \mathbb{R})],
\]  

where \( \mathcal{M}_0(\varphi_{\text{flat}}) \) \( \in G_0/H_0 \) was defined in Sec. 5.1 and \( \mathcal{A}_0(Q) \) is a purely charge-dependent anti-symplectic outer-automorphism of \( G \). By recalling the \textit{on-shell} decomposition (5.8) discussed in Sec. 5.1

\[
\mathcal{M}^H_{I_+ < 0} = \mathcal{M}^H_1(\phi_H) \mathcal{M}_0(\varphi_{\text{flat}}),
\]

(6.8) and (6.15) thus imply

\[
\mathcal{M}^H_1(\phi_H) = M_+(Q) \mathcal{A}_0(Q).
\]  

For the \( T^3 \)-model (cfr. Sec. 4.4), there are no flat directions \( \varphi_{\text{flat}} \), and thus \( \mathcal{M}_0 = \mathbb{I} \), and (6.16)-(6.17) consistently yield (4.55):

\[
\mathcal{M}^H_{I_+ < 0} = M_+(Q) \mathcal{A}_0(Q).
\]  

(6.18)
Then, one can generalize to a generic $Q \in O_{I_4 < 0}$ by following the procedure outlined in Sec. 5.1.

Note that, at least in those cases in which $Out(G) \subset \mathbb{Z}_2$, which seems to be common for groups “of type $E_7$” (including for instance $E_7(7)$ itself) [21], all non-trivial outer-automorphisms are implemented by an anti-symplectic transformation.

If $I_4 > 0$, $M_+(Q)$ (cfr. (6.11)) is symplectic, and thus (6.7) yields that $A$ is symplectic as well. Therefore, as $M_+(Q)$, it defines an inner-automorphism of $G$, and one can write (with $Q$ belonging to regular $G$-orbits with $I_4 > 0$; $H_0 = mcs (G) / U(1)$ in the BPS case, while $H_0 = H_0$ in the non-BPS case (5.31)):

\[
M_+(Q) \in \text{Inn}(G);
\]

\[
A_{I_4>0}(Q, \varphi_{\text{flat}}) = A_0(Q) M_0(\varphi_{\text{flat}}) \in \frac{\text{Inn}(G)}{H_0} \cap \text{Stab}_Q[\text{Sp}(2n, \mathbb{R})],
\]

where $M_0(\varphi_{\text{flat}})$ was defined in Sec. 5.2, and here $\varphi_{\text{flat}}$ denotes the flat directions at $I_4 > 0$ (generally different from the flat directions at $I_4 < 0$, considered above; cfr. the procedure outlined in Sec. 5.2), and $A_0(Q)$ is a purely charge-dependent symplectic inner-automorphism of $G$.

In absence of flat directions $\varphi_{\text{flat}}$ (such as for $\mathcal{N} = 2$ regular BPS orbit), namely in those cases considered in Sec. 4, $G_0 = H_0$, $M_0(\varphi_{\text{flat}}) = 1$ and property (6.9) implies

\[
A_{I_4>0}(Q, \varphi_{\text{flat}}) = A_0(Q) = 1,
\]

which is consistent with the identification $M^H = M_+$ made in Sect. 4 (cfr. (4.12)).

Let us conclude with a few comments.

A special role in our discussion has been played by outer-automorphisms of the U-duality algebra which are implemented by anti-symplectic transformations. These should correspond, modulo $U$-dualities, to a discrete symmetry of ungauged extended supergravities, see Appendix C, which deserves a separate discussion [45].

Finally it would be interesting to extend our analysis to “small orbits” of $R_Q$, for which $I_4 = 0$.

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An interesting reference in which these properties of real forms of simple Lie groups are listed is http://en.wikipedia.org/wiki/List_of_simple_Lie_groups. We thank G. Dall’Agata for pointing it out to us.
A Computing the Coefficients $A$, $B$ and $C$

We will here report the derivation of result (4.11), which can actually be obtained in (at least) two equivalent ways.

A.1 With the Invariant Tensor $S^\alpha\beta_{MQ}…$

We start from the condition (4.3), which can be easily recast as

$$A + \epsilon \left( B - \frac{(2\tau - 1)}{6\tau} C \right) = -2.$$  \hspace{1cm} (A.1)

On the other hand, the implementation of the symplectic condition (4.2) requires some further manipulations. By exploiting (4.8), one can rewrite (4.2) as

$$C_{MQ} = M_{MN} M_{PQ} C^{NP}$$

where the result (obtained by explicit computation)

$$K_{N[M} K_{Q]P} C^{NP} = \frac{I_4}{72\tau} K_{[M} C_{Q]A} Q^A = K_N K_{[Q} K_{M]} C^{NP}$$  \hspace{1cm} (A.2)

was used. The skew-trace of (A.2) yields to (recall $n = n_V + 1$)

$$2n = M_{MN} M_{PQ} C^{NP} C^{MQ}$$

where the result

$$K_{MN} K_{PQ} C^{NP} C^{MQ} = -\frac{(2\tau - 1)}{6\tau} I_4$$  \hspace{1cm} (A.3)

has been taken into account.
Since the l.h.s. of Eq. (A.2) is skew-symmetric, the only way to obtain from (A.2) a further constraint (not proportional to the skew-trace condition (A.4)) on the real coefficients \(A, B\) and \(C\) is to single out the terms not proportional to the symplectic metric \(\mathcal{C}_{MQ}\) itself. Group theoretical arguments (cfr. e.g. App. C of [28]) lead to the following decomposition:

\[
K_{MN}K_{PQ}\mathcal{C}^{NP} = \frac{1}{18n} \frac{1}{12} I_4 \mathcal{C}_{MQ} - \frac{2}{9n} \frac{1}{12} K_{[M}\mathcal{C}_{Q]A} Q^A - \frac{1}{36\tau^2} t_{\alpha\alpha\beta\beta(M}^S_{MQ)} (Q^A t_{\beta\beta|A_3A_4|}) Q^{A_1} Q^{A_2} Q^{A_3} Q^{A_4},
\]

where \(S^\alpha_{MQ}\) is a \(G\)-invariant tensor, satisfying [28]

\[
S^\alpha_{MQ} = S^{(\alpha\beta)}_{[MQ]}, \quad S^{\alpha\beta}_{MQ} \mathcal{C}^{MQ} = 0,
\]

and the result

\[
f_{\alpha\beta\gamma} t^\alpha_{(MA_1} t^\beta_{A_2)} (A_3 t^\gamma_{A_4 Q}) Q^{A_1} Q^{A_2} Q^{A_3} Q^{A_4} = 0
\]

has been used.

Using the irreducible decomposition

\[
- \frac{1}{6\tau} t_{\alpha\beta[M}^S_{NPQ]} S^\alpha_{MQ} = \mathcal{A} K_{(MN} Q \mathcal{C}_{R)S}
\]

(where \(\mathcal{A}\) is a constant to be determined), one can prove that the three terms in the r.h.s. of (A.6) are not independent. In fact, the following relation holds:

\[
K_{[M} \mathcal{C}_{Q]A} Q^A = \frac{1}{4} I_4 \mathcal{C}_{MQ} + \frac{1}{4A} t_{\alpha\beta\gamma(M} Q^A t_{\beta\gamma|A_3A_4|)} Q^{A_1} Q^{A_2} Q^{A_3} Q^{A_4},
\]

thus implying (A.6) to reduce to

\[
K_{MN} K_{PQ} \mathcal{C}^{NP} = - \left( 1 + \frac{1}{2n\mathcal{A}} \right) \frac{1}{36\tau^2} t_{\alpha\alpha\beta\beta(M}^S_{MQ)} (Q^A t_{\beta\beta|A_3A_4|}) Q^{A_1} Q^{A_2} Q^{A_3} Q^{A_4}.
\]

Therefore, the finite symplecticity condition (A.2) for \(\mathcal{M}^H\) can be rewritten as follows:

\[
\mathcal{C}_{MQ} = M_{MN} M_{PQ} \mathcal{C}^{NP}
\]

\[
= - \frac{1}{24\tau} \left\{ \epsilon \tau A \left[ B - C \left( \frac{2\tau - 1}{6\tau} \right) \right] \right. + C \left( \frac{\tau - 1}{6} \right) \left[ B - C \left( \frac{2\tau - 1}{6\tau} \right) \right] \\
- \left( \frac{2}{9} \left( \frac{1}{n} + 2A \right) \left[ B - C \left( \frac{2\tau - 1}{6\tau} \right) \right]^2 \right. \\
+ \epsilon \tau A \left[ B - C \left( \frac{2\tau - 1}{6\tau} \right) \right] \\
+ C \left( \frac{\tau - 1}{6} \right) \left[ B - C \left( \frac{2\tau - 1}{6\tau} \right) \right] \\
+ \epsilon A C \left( \frac{\tau - 1}{6} \right) \right\} t_{\alpha\alpha\beta\beta(M}^S_{MQ)} (Q^A t_{\beta\beta|A_3A_4|}) Q^{A_1} Q^{A_2} Q^{A_3} Q^{A_4}.
\]

\[
(A.12)
\]

It is clear that \(t_{\alpha\alpha\beta\beta(M}^S_{MQ)} (Q^A t_{\beta\beta|A_3A_4|})\) contains \(t_{\alpha\alpha\beta\beta(M}^S_{MQ)} (Q^A t_{\beta\beta|A_3A_4|})\) which, due to (A.7), is orthogonal to (and thus independent of) the symplectic metric \(\mathcal{C}_{MQ}\). Thus, the related coefficient has
to be set to zero. This argument leads to the following independent conditions:

\[-\frac{\epsilon}{6\tau} \left\{ \epsilon \tau A \left[ B - C \frac{(2\tau - 1)}{6\tau} \right] + C \frac{(\tau - 1)}{6} \left[ B - C \frac{(2\tau - 1)}{6\tau} \right] + \epsilon AC \frac{(\tau - 1)}{6} \right\} = 4; \]

(A.13)

\[-\frac{1}{9} \epsilon \left[ B - C \frac{(2\tau - 1)}{6\tau} \right]^2 = -4. \]

(A.14)

In these relations, the real constant $A$ introduced in the decomposition (A.9) has been set to

\[A = \frac{1}{2} \left( 3\tau - \frac{1}{n} \right). \]

(A.15)

The result (A.15) can be achieved by noticing that, using (A.9), the following equation holds:

\[K_N K_{\left[ M K_Q P \right]} C^{NP} = -\frac{1}{36\tau} \left( \frac{1}{n} + 2A \right) I_4 K_{\left[ M C Q \right]} A Q^A. \]

(A.16)

$K_N K_{\left[ M K_Q P \right]} C^{NP}$ can also be elaborated through explicit computation, and the result is given by Eq. (A.3). By comparing the skew-traces of (A.16) and (A.3), (A.15) follows.

It should be stressed that Eqs. (A.13) and (A.14) are consistent with the skew-tracelessness condition (A.4) iff the relation (4.16) holds. This means that only two conditions out of the three ones given by Eqs. (A.4), (A.13) and (A.14) are independent. The third independent condition is given by (A.1).

Thus, the solutions of the resulting system of three independent conditions on the coefficients $A$, $B$ and $C$ occurring in the Ansatz (4.4) read as follows:

\[A = -2 \mp 6\sqrt{\epsilon}, \quad B = \frac{6(1 - 2\tau \mp \tau\sqrt{\epsilon})}{(\tau - 1)}, \quad C = \frac{-36\tau (1 \pm \sqrt{\epsilon})}{(\tau - 1)}. \]

(A.17)

Since $A$, $B$ and $C$ must be real, (A.17) implies that the treatment is consistent only for $I_4 > 0 \Leftrightarrow \epsilon = +1$. Then, specifying $\epsilon = +1$, (A.17) simplifies down to the final result (4.11).

We also add that the results (A.10) and (A.11) yield

\[K_{MN} K_{PQ} C^{NP} = -\frac{1}{27\tau} \left( \frac{1}{n} + 2A \right) K_{\left[ M C Q \right]} A Q^A + \frac{1}{18} \left( \frac{1}{n} + 2A \right) \frac{1}{6\tau} I_4 C_{MQ}. \]

(A.18)

Clearly, the skew-trace of the Eq. (A.18) must coincide with Eq. (4.5), thus implying the consistency condition (4.16).

A.2 ...and without $S^\alpha_{MQ}$

By inserting (A.15) into (A.18), one obtains

\[K_{MN} K_{PQ} C^{NP} = \frac{1}{9} K_{\left[ M C Q \right]} P Q^P + \frac{1}{36} I_4 C_{MQ} = -\frac{1}{9} K_{\left[ M Q N \right]} + \frac{1}{36} I_4 C_{MQ}, \]

(A.19)

which, by further contracting with $Q^Q$, yields

\[K_{MN} K_{P} C^{NP} = -K_{MP} C^{NP} K_N = \frac{1}{12} I_4 Q_M. \]

(A.20)
Results (A.19)-(A.20) actually hint for a simpler derivation of result (4.11), not involving of the use of the $G$-invariant tensor $S^{\alpha\beta}_{MQ} \ \ (A.7) \ [28]$ at all.

Indeed, starting from the Ansatz (cfr. (4.9); $a, b, c \in \mathbb{R}$)

\[ M_{MN}(Q) = a K_M K_N + b K_{MN} + c Q_M Q_N, \]  

(A.21)

and observing that\(^{12}\)

\[ -\frac{1}{2} f_{\alpha\beta\gamma} t^\alpha_M t^\beta_{NP} t^\gamma_{RS} Q^P Q^Q Q^R Q^S = \tau^2 I_4 C_{MN} + 2 \tau^2 K[M Q_N], \]  

(A.22)

after a little algebra Eqs. (A.19)-(A.20) yield (4.11):

\[
\begin{aligned}
    a &= -(2 \pm 6) / |I_4|^{3/2}; \\
    b &= \pm 6 / |I_4|^{1/2}; \\
    c &= -(1 \pm 1) / 2 |I_4|^{1/2}.
\end{aligned}
\]  

(A.23)

Thus, in order to study its definiteness, it suffices to analyze the signs of its diagonal elements. In the \(STU\) truncation under consideration, it can be explicitly computed that the first diagonal element is strictly positive (\(I_4 = q_0 p^1 p^2 p^3 > 0\)):

\[ M_{00} = q_0^2 \sqrt{q_0 p^1 p^2 p^3} > 0, \]  

(B.3)

thus implying that \(M_{-|MN}\) is not negative definite.

On the other hand, it can be calculated that \(M_+ (Q)\), given by (4.9)-(4.10) and (4.11) in the branch “+”, is diagonal, with all strictly negative elements, and thus trivially negative definite.

\(^{12}\)Note that (A.22) implies (A.8).
C Outer (Anti-symplectic) Automorphisms of $\mathfrak{g}$

In symmetric extended $D = 4$ supergravities, the U–duality algebra $\mathfrak{g}$ admit an automorphism implemented, in the representation $\mathbf{R}_Q$, by an anti-symplectic transformation. Consider the symplectic frame in which the subgroup $H \cap \text{SO}(n) \subset \text{Sp}(2n, \mathbb{R})$ has a block-diagonal representation. Such frame is obtained through a Cayley transformation of the complex basis in which the whole $H$ is block-diagonal. In this frame the conjugation by the anti-symplectic matrix:

$$O = \begin{pmatrix} I_n & 0_n \\ 0_n & -I_n \end{pmatrix},$$

defines an automorphism:

$$O^{-1} \mathbf{R}_Q[\mathfrak{g}] O = \mathbf{R}_Q[\mathfrak{g}].$$

For instance, in the maximal theory, such transformation switches the sign of the generators in the $35_c$ (parametrized by the pseudo-scalars) and $35_s$ (compact generators in $\mathfrak{su}(8) \oplus \mathfrak{so}(8)$), leaving the other generators unaltered [44].

Since all $G$ transformations in $\mathbf{R}_Q$ are implemented by symplectic matrices, $O$ is not in $G$ and defines a non-trivial outer automorphism of $\mathfrak{g}$:

$$O \in \frac{\text{Aut}(G)}{\text{Inn}(G)} = \text{Out}(G).$$

We can give an alternative representation to $O$, for those supergravities admitting a $D = 5$ uplift, in the symplectic frame originating from the $D = 5 \to D = 4$ reduction. In this frame the generators $t_\alpha$ of $\mathfrak{g}$ have a characteristic matrix form given in [37], defined by branching the $D = 4$ duality algebra with respect to $O(1,1) \times G_5$, $G_5$ being the global symmetry group of the $D = 5$ parent theory. The algebra $\mathfrak{g}$ decomposes accordingly:

$$\mathfrak{g} = [\mathfrak{o}(1,1) \oplus \mathfrak{g}_5]_0 \oplus [\mathbf{R}_{-2} + \overline{\mathbf{R}}_{+2}],$$

where the subscripts refer to $O(1,1)$-gradings, $\mathbf{R}$, $\overline{\mathbf{R}}$ are $(n-1)$-dimensional (Abelian) spaces of nilpotent generators transforming in the representations $\mathbf{R}$ and $\overline{\mathbf{R}}$ under the adjoint action of $G_5$, respectively. Generators of $\mathfrak{g}$ in each of the subspaces on the right-hand-side of (C.4), have the following matrix form in $\mathbf{R}_Q$:

$$D \in \mathfrak{o}(1,1) ; \quad D = \text{diag}(-3, -I_{n-1}, 3, I_{n-1}),$$

$$E(\lambda) \in \mathfrak{g}_5 ; \quad E(\lambda) = \text{diag}(1, E(\lambda), 1, -E(\lambda)^T),$$

$$T(a^I) \in \overline{\mathbf{R}}_{+2} ; \quad T(a^I) = a^I T_I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a^I & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^I \\ 0 & d_{IJ} & 0 & 0 \end{pmatrix},$$

$$\bar{T}(b_I) \in \mathbf{R}_{-2} ; \quad \bar{T}(b_I) = b_I (T_I)^T,$$

13Strictly speaking, to show that $O$ is an outer-automorphism, one should prove that no other element of $G$ can induce the same transformation on $\mathfrak{g}$. This is immediate if $\mathbf{R}_Q$ is irreducible since any other real matrix inducing the same transformation, must be proportional to $O$, and thus non-symplectic. Inspection of supergravities in which $\mathbf{R}_Q$ is reducible, however, leads to the same conclusion: No element of $G$ can induce the same automorphism as $O$.

14The simplest example of a real Lie group admitting a symplectic representation, in which an outer automorphism is implemented by an anti-symplectic transformation, is $\text{SL}(2, \mathbb{R})$. The fundamental representation 2 is symplectic and the anti-symplectic matrix $\sigma_3 = \text{diag}(+1, -1)$ implements an outer-automorphism.
where $\mathcal{E}(\lambda)$ are $(n-1) \times (n-1)$ matrices representing the generic element $E(\lambda)$ of $g_5$. In this basis the matrix there is the following anti-symplectic automorphism $\mathcal{O}$:

$$
\mathcal{O} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -I_{n-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & I_{n-1}
\end{pmatrix},
$$

(C.5)

whose action on the $\mathfrak{g}$-generators is:

$$
\mathcal{O}^{-1} D \mathcal{O} = D; \quad \mathcal{O}^{-1} E(\lambda) \mathcal{O} = E(\lambda); \quad \mathcal{O}^{-1} T(a^I) \mathcal{O} = -T(a^I); \quad \mathcal{O}^{-1} \bar{T}(a^I) \mathcal{O} = -\bar{T}(a^I). \quad (C.6)
$$

The anti-symplectic automorphism $\mathcal{O}$ is relevant for defining the $CP$-transformation in supergravity [15].

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