Asymptotic Spectral Efficiency of the Uplink in Spatially Distributed Wireless Networks With Multi-Antenna Base Stations

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Abstract

The spectral efficiency of a representative uplink (with appropriate normalization) in interference-limited, spatially-distributed wireless networks with hexagonal cells and linear Minimum-Mean-Square-Error estimation is found to converge to an asymptotic limit as the numbers of base-station antennas N and wireless nodes go to infinity. A simple approximation for the mean spectral efficiency is also found for systems with both hexagonal and random cells when transmit power budgets are large. It is found that for large N in the interference-limited regime, the mean spectral efficiency is primarily a function of the ratio of the product of N and the ratio of base-station to wireless node density, indicating that it is possible to scale such networks by linearly increasing the product of the number of base-station antennas and base-station density with wireless node density. This work is useful for designers of wireless systems with high inter-cell interference because it provides simple expressions for spectral efficiency as a function of tangible system parameters like base-station and wireless node densities, and number of antennas. These results were derived combining infinite random matrix theory and stochastic geometry.

Index Terms

Cellular Networks, MIMO, Wireless Networks, Antenna Arrays, Stochastic Geometry, Hexagonal Cells.

I. INTRODUCTION

It is increasingly common for multiple wireless networks to be within interfering distance of each other in urban environments today due to proliferation of systems such as city-wide wireless internet access, pico-cells for mobile telephony, and wireless local-area networks. Antenna arrays at base stations can significantly increase data rates in such systems. It is thus important to study the spectral efficiencies (b/s/Hz) of wireless links with multiple antennas in environments that have high base-station and wireless node densities. In such systems the densities of nodes (both in-and out-of-cell) and their distribution in space are important factors as they influence inter-node distances and hence signal and interference strength, which directly impact the Signal-to-Interference-Plus-Noise-Ratio (SINR), spectral efficiency and ultimately data rates.

Most works on wireless networks with multi-antenna base-stations do not explicitly consider out-of-cell interference as resulting from spatially distributed in-band interferers. For example, Dai and Poor [2] used random matrix techniques similar to those used here to obtain asymptotic expressions for the spectral efficiency in multi-cellular environments where multiple base-stations cooperate to jointly decode signals. Recently, [3] analyzed the capacity region of multi-user MIMO channels with correlated channels in the asymptotic regime when the number of nodes is constant but the number of transmitter and receiver antennas go to infinity. However, neither of these works models path-loss as a function of distance and thus do not capture the distribution of interference resulting from spatially distributed nodes. Monte-carlo simulations were used in [4] and [5] to analyze small, spatially-distributed multi-antenna cellular networks.

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Cellular networks with single-antenna base-stations and spatially distributed nodes have been analyzed in works such as [6], [7], [8], [9], [10], [11] using stochastic geometry to model the spatial distribution of nodes. An extensive set of mathematical techniques to analyze such networks can be found in [12] and [13].

Stochastic geometry has also been used to study ad-hoc wireless networks with both multi and single antenna nodes in works such as [14]. Ad-hoc wireless networks with spatially distributed multi-antenna nodes have been analyzed in [15] which derived an asymptotic expression for the spectral efficiency of multi-antenna links in ad-hoc wireless networks as a function of path-loss exponent, link length and the ratio of the number of receiver antennas to node-density, and [16] which found that it is possible to linearly scale the network spectral efficiency density by linearly increasing the density of transmitting users with the the number of receiver antennas using a partial-zero-forcing receiver. More recently, [17] and [18] have found exact expressions for the CDF of the SINR of ad-hoc wireless networks in Rayleigh fading with MMSE receivers. The key difference between these works and this paper is that this paper explicitly models link lengths resulting from a cellular architecture and uses power control that is based on the distance of nodes from their respective base-stations rather than assuming constant link lengths and transmit powers. Please see [19] for an comprehensive survey of works utilizing stochastic geometry in both cellular and ad-hoc wireless networks.

II. SYSTEM MODELS

In this section, we describe the two system models for the main results of this paper that follow in Sections III and IV. We first present the system model for a network with wireless nodes transmitting at random, independent and identically distributed (IID) power levels, followed by a model for a wireless network with base-stations which we shall also call tethered nodes.

A. Purely Wireless Network

Consider a planar wireless network with \( n \) wireless transmitters located at random IID points in a circle of radius \( R \) such that

\[
n = \rho_w \pi R^2. \tag{1}\]
We shall consider the spectral efficiency of a representative link between a receiver placed at the center of the circular network and an additional transmitting node (node-1) located at a distance \( r_1 \) as illustrated in Figure 2. Note that the base-stations at hexagonal lattice sites represented by the solid circles should be ignored for this part. The remaining \( n \) transmitting nodes numbered 2, 3, \ldots, \( n+1 \) are interferers to this link.

Suppose that each transmitting node uses a random, IID power level with PDF \( f_P(p) \) and all nodes transmit simultaneously in the same frequency band. Let \( P_i \) equal the transmit power of node-\( i \) and the received power due to node-\( i \) at a distance \( r_i \) from the representative receiver at the origin \( p_i = G_i P_i r_i^{-\alpha} \), where \( \alpha > 2 \) is the path-loss exponent which we assume is a rational number for technical reasons. We allow \( P_i \) to be dependent on \( r_i \) as long as the following holds as \( R, n, N \to \infty \) such that \( n/N = c > 0 \) and (1) always hold.

\[
\Pr \left\{ \frac{r_i}{\sqrt{N}} \geq \left( \frac{P_i}{x} \right)^{\frac{2}{\alpha}} ; P_i \right\} \to \left[ \left( 1 - \frac{\pi \rho w}{c} \left( \frac{P_i}{x} \right)^{\frac{2}{\alpha}} \right) I_p \left( \frac{\pi \rho w}{c} \right)^{\alpha/2} \right] f_P(P_i).
\]  

(2) requires the transmit power of a node randomly distributed with uniform probability in the circular network be asymptotically independent of its normalized distance from the origin as the quantity in the brackets in (2) equals the probability that \( r_i/\sqrt{N} \) exceeds \( (P_i/x)^{\frac{2}{\alpha}} \). Both the hexagonal-cell and Poisson cell models have this property as shown in Appendix B.

We further assume that the representative receiver has an array of \( N \) antenna elements and each wireless node has an isotropic antenna. We assume frequency-flat fading with circularly symmetric complex Gaussian channel coefficients between all pairs of antennas. Let \( y \) be an \( N \)-element vector of sampled received signals at the \( N \) antennas of the representative receiver. Let the \((n+1) \times 1\) vector \( x \) contain the transmitted signals from node-1 and the \( n \) interferers with entries normalized to unit power, and the \( N \times 1 \) vector \( w \) contain zero-mean, IID complex Gaussian noise terms of variance \( \sigma^2 \) denoted by \( C \mathcal{N}(0, \sigma^2) \). This system can be represented by the following equation:

\[
y = Hx + w
\]

where the \( N \times (n+1) \) matrix \( H \) is the channel matrix whose \( ij \)-th entry is the channel coefficient between transmitting node \( j \) and antenna element \( i \) of the receiver. Let the \( i \)-th column of \( H \) equal \( \sqrt{p_i} g_i \) where \( g_i \) has IID \( C \mathcal{N}(0, 1) \) entries. Thus, \( g_i \) captures the Rayleigh fading and \( p_i \) captures the combined transmit power and path loss associated with node-\( i \). We assume that the base stations use linear Minimum-Mean-Square-Error (MMSE) estimators, and the transmitting nodes use Gaussian codebooks resulting in Gaussian interference. Note that the linear MMSE receiver is the optimal linear receiver for Gaussian signals as it maximizes the Signal-to-Interference-plus-Noise-Ratio (SINR) (e.g. see [21]) which maximizes the spectral efficiency.

For technical reasons, we assume a noise power \( \bar{\sigma}^2 \) that is a function of \( N \) as follows:

\[
\bar{\sigma}^2 = \sigma^2 \left( N^{1-\frac{2}{\alpha}} \right)
\]

where \( \sigma^2 \) is a constant. This assumption enables the asymptotic analysis of the SINR as \( N \to \infty \). Without this assumption, as \( N \to \infty \), the MMSE receiver will suppress interference to levels comparable to the thermal noise which results in a system that is not interference limited. Defining the thermal noise power as in (4) makes the thermal noise power increase at the correct rate such that the system remains interference-limited. Note that while this assumption is necessary to make precise statements about the SINR, it does not impact the approximations to the SINR and spectral efficiency in interference-limited regime which is the focus of this work.

### B. Base-station Architecture

Consider a plane divided into cells with base stations at arbitrary locations with effective area density \( \rho_t \). Each cell is associated with one base station and is the region of the plane that is closer to that base station than any other. Thus, each cell is the Voronoi cell associated with its base station. We assume that all cells are bounded. Suppose the network model from the previous section is overlaid on this network of base-stations such that one of the base stations is the representative receiver. Figure 2 shows one such case where base stations are at hexagonal lattice sites separated by distance \( d \). Assume that the \( i \)-th wireless node transmits data to its nearest base station located at a distance \( r_{ti} \) away with power \( P_i \) where

\[
P_i = \min \left( \frac{p_i}{G_i} r_{ti}^{-\alpha}, P_M \right).
\]

(5)
Thus, the \( i \)-th wireless node tries to achieve a target received power (relative to path-loss) of \( p_t \) at its nearest base station, subject to a maximum power constraint \( P_M \).

For a set of base stations locations, the link lengths and hence transmit powers are independent random variables as they depend solely on the locations of the wireless nodes, which are independent by assumption. Hence, results derived using the assumptions of the previous section can be applied to this network model with the transmit-power PDF \( f_P(p) \) arising from the base-station locations as long as (2) holds.

### III. Main Results For Receivers At Arbitrary Locations

In this section, we present results for the spectral efficiency of the representative link between node-1 and the representative receiver at the origin of the network described in Section [II-A]. To avoid degenerate expressions in the derivation, we define a normalized SINR of the representative link at the output of the MMSE receiver: \( \beta_N = N^{-\alpha/2}\text{SINR} \). The SINR can then be found by multiplying \( \beta_N \) by \( N^{\alpha/2} \). Using this definition, we introduce the following theorem proved in the Appendix [A]

**Theorem 1:** Consider the network model from Section [II-A]. As the number of interferers \( n \to \infty \), the number of antennas \( N \to \infty \), and the outer radius of the network \( R \to \infty \) such that \( c = n/N > 0 \) and \( \rho_w = \frac{n}{\pi R^2} \) are constants, then \( \beta_N \to \beta \) with probability 1 where \( \beta \) is the unique, non-negative real solution to the following equation:

\[
E[P^{2/\alpha}]^{2/\alpha} \frac{\pi}{\alpha} \csc \left( \frac{2\pi}{\alpha} \right) = -2\pi \rho_w \beta r_1^{-\alpha - 2} \int_0^\infty \frac{\tau^{\frac{2}{\alpha} - 1}}{1 + \tau \beta} \int_{\tau/b}^{\infty} f_P(x) x^{\frac{2}{\alpha}} dx d\tau + \frac{\beta r_1^{-\alpha - 2} \sigma^2}{2G_t \rho_w \pi P_1^{1-\frac{2}{\alpha}}} = \frac{P_1^{\frac{2}{\alpha}}}{2 \rho_w \pi r_1^2} \tag{6}
\]

where \( b = \left( \frac{2\rho_w}{\pi} \right)^{\frac{2}{\alpha}} \), \( E[P^{\frac{2}{\alpha}}] \) is the expected value of the transmit power of the wireless nodes raised to \( \frac{2}{\alpha} \), and \( P_1 \) is the power of the representative transmitter.

Since all nodes use Gaussian codebooks the residual interference at the output of the MMSE receiver is Gaussian, so we use the Shannon formula to estimate the per-link spectral efficiency of the system:

\[
C = \log_2(1 + \text{SINR}) = \log_2(1 + N^{\alpha/2} \beta). \]

Since the log function is continuous, as \( n, N \to \infty \) in the manner described in Theorem 1, the following expression holds with probability 1 (e.g. see [22]):

\[
C - \log_2(N^{\alpha/2}) \to \log_2(\beta). \tag{7}
\]

Hence, with appropriate normalization, the spectral efficiency converges to an asymptote with probability 1 as \( N \to \infty \). Following the steps of the derivation in Appendix E of [23] which is essentially an application of the dominated convergence theorem (e.g. see [22]), the deviation of the mean spectral efficiency from is asymptotic value approaches zero, i.e.

\[
\left| E[C] - \log_2(1 + N^{\alpha/2} \beta) \right| \to 0. \tag{8}
\]

Hence, \( \log_2(1 + N^{\alpha/2} \beta) \) is a good approximation for the mean spectral efficiency for large \( N \).

From Theorem [II], the value of the limiting normalized SINR \( \beta \) is unclear as it is given implicitly by the solution to (6). To obtain a more meaningful expression for \( \beta \) and the spectral efficiency, we can simplify Equation (6) using Lemma 3 of [23] which indicates that as \( c \to \infty, b \to 0 \) and the second term on the LHS of (6) vanishes. Thus, when the ratio of the number of antennas at the representative receiver to the number of interferers is high (i.e. large \( c \)), (6) can be written as [1]:

\[
\frac{\pi E[P^{\frac{2}{\alpha}}]^{\frac{2}{\alpha}}}{\alpha} \csc \left( \frac{2\pi}{\alpha} \right) + \frac{\beta r_1^{-\alpha - 2} \sigma^2}{2G_t \rho_w \pi P_1^{1-\frac{2}{\alpha}}} \approx \frac{P_1^{\frac{2}{\alpha}}}{2 \rho_w \pi r_1^2}. \tag{9}
\]

[1] This approximation requires the solution of (6) to be a continuous function of \( \beta \). If \( \alpha \) is rational, (6) can be raised to a sufficiently high power resulting in a polynomial equation in \( \beta \) with real coefficients that are known to have continuous roots.
Additionally, in the interference-limited regime, we assume that \( \sigma^2 \) is sufficiently small that the second term on the LHS of (9) is dominated by the first. Writing \( G_\alpha = \left( \frac{\alpha}{P^2} \sin\left( \frac{2\pi}{\alpha} \right) \right)^{\alpha/2} \) for convenience, neglecting the second term on the LHS of (9), substituting the definition of \( \beta_N \) and rearranging terms yields the following approximation for the SINR when \( N \) is large:

\[
\text{SINR} \approx P_1 G_\alpha \left( \frac{N}{E[P_1] \pi \rho_w r_1^2} \right)^{\alpha/2}.
\]

We have made several approximations in deriving (10). The validity of these approximations for reasonable values of \( N \), \( n \), and \( \sigma^2 \) are verified in simulations presented in Section IV-C. Results of more extensive simulations can be found in [24].

With Gaussian codebooks, the mean spectral efficiency can be approximated as:

\[
E[C(r_1, P_1)] \approx \log \left( 1 + P_1 G_\alpha \left( \frac{N}{E[P_1] \pi \rho_w r_1^2} \right)^{\alpha/2} \right).
\]

Suppose that the maximum distance between any transmitting node and its desired receiver \( r_M \leq \left( \frac{G_\alpha P_o}{P_t} \right)^{\frac{1}{\alpha}} \). We call this the sufficient-power case since every wireless node can satisfy the target received power \( P_t \) at its desired receiver. For the cellular model described in the next section, this corresponds to the base-station separation being sufficiently small that the target received power is attained by each wireless node. Substituting (5) into (11),

\[
E[C] \approx \log_2 \left( 1 + \frac{P_t}{G_\alpha r_1^2} G_\alpha \left( \frac{N}{E \left[ \frac{p_t}{G_\alpha r_1^2} \right] \pi \rho_w r_1^2} \right)^{\alpha/2} \right) = \log_2 \left( 1 + G_\alpha \left( \frac{N}{E[r_1^2] \pi \rho_w} \right)^{\alpha/2} \right)
\]

which is a function of the second moment of the link-lengths.

IV. HEXAGONAL CELLS

Suppose that the base stations are located at hexagonal lattice sites on the plane separated by distance \( d \) which results in hexagonal cells as illustrated in Figure I. The following lemma statistically characterizes the link-lengths for this model.

Lemma 1: The PDF \( f_X(x) \), Cumulative-Distribution-Function (CDF) \( F_X(x) \), and \( k \)-th moment of the link length \( x \) between a randomly located wireless node and its closest base station in a hexagonal-cellular system with minimum base station separation \( d \) are the following:

\[
f_x(x) = \begin{cases} 
\frac{4\pi}{\sqrt{3}d^2} x, & \text{if } 0 < x < \frac{d}{2} \\
\frac{4\pi}{\sqrt{3}d^2} x - \frac{2\sqrt{3}x}{d^2} \cos^{-1} \left( \frac{d}{2x} \right), & \text{if } \frac{d}{2} < x < \frac{\sqrt{3}d}{3} \\
0, & \text{otherwise.}
\end{cases}
\]

\[
F_X(x) = \begin{cases} 
0, & \text{if } x < 0, \\
\frac{2\sqrt{3}x^2}{3d^2} - \frac{4\sqrt{3}x^2}{d^2} \cos^{-1} \left( \frac{d}{2x} \right) + 2\sqrt{3} \left( \frac{d^2}{4x^2} - \frac{1}{2} \right)^{\frac{3}{2}}, & \text{if } \frac{d}{2} \leq x < \frac{\sqrt{3}d}{3} \\
1, & \text{if } x \geq \frac{\sqrt{3}d}{3}.
\end{cases}
\]

\[
E(x^k) = \frac{2\sqrt{3}}{k + 2} \left( \frac{d}{2} \right)^k \int_0^{\frac{\pi}{2}} \frac{1}{(\cos \tau)^{k+2}} d\tau.
\]

Proof: Consider Fig. I which illustrates a portion of a wireless network with hexagonal cells. Each wireless node in the network falls on some random point in an equilateral triangle formed by the three base stations closest
to it, and forms a link with the base station at the closest vertex of that triangle as illustrated in Fig. 1. Thus, the link-lengths are statistically equivalent to the distance between a randomly selected point in an equilateral triangle to the closest vertex of that triangle. The CDF, PDF and k-th moments of the distance between a random point in an equilateral triangle to the closest vertex are known [25], and are precisely the formulae in Lemma 1. Note that the PDF of link-lengths associated with a single hexagonal cell which equals (13), has been given without proof before in [26].

\[ E[C] \approx \log_2 \left( 1 + G \alpha \left( \frac{N}{0.14d^2 \pi \rho_w} \right)^{\frac{3}{2}} \right). \]  

(16)

A. Sufficient Transmit Power

If \( d \leq \frac{3}{\sqrt{3}} \left( \frac{G \rho_w}{p_t} \right)^{\frac{1}{3}} \), all wireless nodes have sufficient transmit power to meet the target received power \( p_t \) at their base-stations. From (12), the spectral efficiency depends on the second moment of link-lengths given by (15) with \( k = 2 \):

\[ E[t^2] = \frac{5}{36} d^2 \approx 0.14d^2. \]

Substituting into (12) yields the following expression for the mean uplink spectral efficiency of interference-limited, hexagonal-cell systems with a large number of base station antennas:

\[ E[C] \approx \log_2 \left( 1 + G \alpha \left( \frac{N}{0.14d^2 \pi \rho_w} \right)^{\frac{3}{2}} \right). \]  

(16)

B. Insufficient Transmit Power

If \( d > \frac{3}{\sqrt{3}} \left( \frac{G \rho_w}{p_t} \right)^{\frac{1}{3}} \), the transmit power budget is insufficient for all nodes to meet the target received power at their base stations which results in some wireless nodes transmitting at full power. In this case, \( E[P^{2/\alpha}] \) (which is required to find the mean spectral efficiency using (11)) is given by the following lemma which can be proved by direct computation using Lemma 1.

Lemma 2: If \( P_M < \frac{p_t}{G_t} \left( \frac{d}{2} \right)^{\alpha}, \)

\[ E[P^{2/\alpha}] = P^{2/\alpha}_M - \frac{\sqrt{3\pi}}{3d^2} \left( \frac{G_t}{p_t} \right)^{\frac{2}{3}} P^{4/\alpha}_M. \]  

(17)

If \( \frac{p_t}{G_t} \left( \frac{d}{2} \right)^{\alpha} \leq P_M < \frac{p_t}{G_t} \left( \frac{\sqrt{3\pi}}{d} \right)^{\alpha} \),

\[ E[P^{2/\alpha}] = P^{2/\alpha}_M - \frac{\sqrt{3\pi}}{3d^2} \left( \frac{p_t}{G_t} \right)^{\frac{2}{3}} P^{4/\alpha}_M + 2 \frac{\sqrt{3}}{d^2} \left( \frac{p_t}{G_t} \right)^{\frac{2}{3}} P^{4/\alpha}_M \cos^{-1} \left( \frac{1}{2} \left( \frac{p_t}{G_t P_M} \right)^{\frac{2}{3}} \right) \]

\[ + \left( 1 - \frac{p_t}{G_t P_M} \right)^{1/3} \left( \frac{p_t}{G_t} \right)^{\frac{2}{3}} \sqrt{\frac{3}{d^2}} \left( \frac{p_t}{G_t P_M} \right)^{2/3} \right) - d^2. \]  

(18)

Lemma 2 substituted into (11) yields the mean spectral efficiency for a link of length \( r_1 \). Averaged over the PDF of link-lengths arising from hexagonal cells, the mean spectral efficiency is:

\[ E[C] = \int_0^{2\pi} \log_2 \left( 1 + G \alpha \left( \frac{N}{E[P^{2/\alpha}_M \pi \rho_w x^2]} \right)^{\frac{2}{3}} \right) f_x(x) dx \]

\[ + \int_0^{\frac{3\pi}{2\sqrt{3}}} \log_2 \left( 1 + G \alpha P_M \left( \frac{N}{E[P^{2/\alpha}_M \pi \rho_w x^2]} \right)^{\frac{2}{3}} \right) f_x(x) dx \]

\[ = F_x \left( \left( \frac{p_t}{G_t P_M} \right)^{-1/\alpha} \right) \log_2 \left( 1 + G \alpha P_M \left( \frac{N}{E[P^{2/\alpha}_M \pi \rho_w x^2]} \right)^{\frac{2}{3}} \right) f_x(x) dx \]

\[ + \int_0^{\frac{3\pi}{2\sqrt{3}}} \log_2 \left( 1 + G \alpha P_M \left( \frac{N}{E[P^{2/\alpha}_M \pi \rho_w x^2]} \right)^{\frac{2}{3}} \right) f_x(x) dx \]  

(19)
where \( F_x(x) \) and \( f_x(x) \) are from Lemma 1, and \( E[P^{2/\alpha}] \) is from Lemma 2. We were not able to integrate the second term on the RHS of (19) and thus use numerical integration to compute \( E[C] \) for this case.

### C. Monte Carlo Simulations

To verify the asymptotic results of the previous section, we simulated network topologies with base stations at hexagonal lattice sites, and wireless nodes distributed randomly on a large circular network on the plane. We simulated each configuration 5000 times. To reduce edge effects, we evaluate uplink spectral efficiencies in the center-most cell using the Shannon formula.

For each trial, we placed 4000 wireless nodes randomly in circular networks with radii selected to meet target wireless node densities of \( 10^{-2}, 10^{-3}, \) and \( 10^{-4} \) nodes \( m^{-2} \). The circular network was overlaid on a hexagonal grid of base stations which extends beyond the edge of the circular network of wireless nodes. The base stations were spaced such that their densities were 20%, 10%, 5% and 2.5% of the wireless node density.

The thermal noise power was fixed at \( 10^{-15} \text{W} \), equivalent to an antenna temperature of \( \sim 300 \text{ K} \) for 200kHz bandwidth. The target received power at the base station \( p_t \), was such that the received SNR \( = 30 \text{dB} \). We used a high value for \( p_t \) to ensure that the system is interference-limited for low densities of wireless nodes. We simulated systems with both unlimited transmit powers (to simulate the sufficient-power case) and powers limited to \( P_M = 200mW \).

The channel coefficient between the antenna of wireless node \( i \) and antenna \( j \) of the representative base station was modeled as \( \sqrt{G r_i^{-\alpha} g_{ij}} \), where \( \alpha = 4, G_i = 10^{-5} \text{m}^4 \), \( P_i \) is the transmit power of the wireless node (a function of the distance between the wireless node and its nearest base station) and \( g_{ij} \) are IID \( CN(0, 1) \) random variables which represents the narrow-band Rayleigh fading channel.

1) **Sufficient Transmit Power:** Figure 3 illustrates the mean uplink spectral efficiency for wireless node densities of \( \rho_w = 10^{-3} \) and \( \rho_w = 10^{-2} \) nodes \( m^{-2} \), and unlimited transmit powers per node versus the number of antennas at the representative base station. The square and asterisk markers represent wireless node densities of \( 10^{-2} \), and \( 10^{-3} \) nodes \( m^{-2} \), respectively and the solid lines represent the asymptotic mean spectral efficiency from (16).

Note that the asterisk and square markers coincide indicating that the absolute density of wireless nodes does not affect the mean spectral efficiency, and it is the relative density of wireless to base stations that matters. Furthermore, it is clear that the asymptotic approximation (19) holds when \( N \) is sufficiently large. For instance, when the base station density is 20% of the wireless node density, the asymptotic and simulated mean spectral efficiency differ by less than 10% when \( N \geq 10 \). For lower densities of base stations, the convergence is slower, e.g. when the base station density is 5% of the wireless node density, the difference between the simulated and asymptotic mean spectral efficiency drops below 10% only when \( N > 37 \).

We analyzed the outage spectral efficiencies from the simulated data, where spectral efficiency with outage probability \( P_o \) means that a fraction \( 1 - P_o \) of the links in our simulations achieved that spectral efficiency or greater. Figure 4 illustrates the outage spectral efficiencies vs. number of receive antennas at the representative base station for \( \rho_w = 10^{-2} \) nodes \( m^{-2} \) with 5%, 25% and 50% outage probabilities.

Note that the intersection of the line with the circular markers and the 1 \( \text{bs}^{-1} \text{Hz}^{-1} \) mark in Figure 4 occurs approximately at \( N = 14 \) indicating that it is possible for 95% of links to achieve 1 \( \text{bs}^{-1} \text{Hz}^{-1} \) with \( N \geq 14 \) antennas at the base stations when the base station density is 10% of the density of transmitting wireless nodes. In real systems, the number of nodes transmitting at any time is far smaller than the total number of nodes in the network. Suppose that at any one time, 10% of nodes are actively transmitting in the network. Figure 4 indicates that with a base station density equaling 1% of total wireless node density (including inactive ones), it is possible for 95% of links to achieve 1 \( \text{bs}^{-1} \text{Hz}^{-1} \) with 14 antennas at each base station.

2) **Insufficient Transmit Power:** Figures 5 and 6 illustrate the mean spectral efficiency vs. number of receive antennas for \( \rho_w = 10^{-4} \) and \( \rho_w = 10^{-2} \) respectively, with \( P_M = 200mW \). The markers represent the simulated mean spectral efficiencies for different relative densities of base stations to wireless nodes. The solid lines are the predicted asymptotic mean spectral efficiencies obtained by numerically evaluating equation (19).

It is clear from Figures 5 and 6 that the asymptotic approximation (19) holds when \( N \) is sufficiently large. In Figure 5 the simulated and asymptotic mean spectral efficiencies agree to within 5% for \( N \geq 2 \) for all the base-station densities considered. In Figure 6 however, for base-station densities that are 5% of the wireless node density of \( 10^{-2} \) nodes \( m^{-2} \), the simulated and asymptotic spectral efficiencies differ by less than 13% only when
there are 13 or more antenna elements at the receiver. For base station densities that are 20% of the wireless node density, the simulated and asymptotic spectral efficiencies agree to within 13% when \( N \geq 3 \).

At low wireless node densities, the simulated spectral efficiencies converge more rapidly (compared to high densities) to the asymptotic values because a large fraction of nodes transmit at the 200 mW power limit. The empirical distribution function (e.d.f.) of interference powers at the representative receiver thus converges more rapidly to its asymptotic value. The rate of convergence of the e.d.f. of interference powers controls the rate of convergence of the eigenvalues of the spatial interference covariance matrix \( \mathbf{GPG} \) (see Appendix A and Section 3 of [27]) which affects the convergence rates of the normalized SINR and spectral efficiency.

Figure 7 shows the outage and mean spectral efficiencies for \( \rho_w = 10^{-4} \) (solid lines) and \( \rho_w = 10^{-3} \) (dashed lines) nodes \( m^{-2} \), 10% relative density of base-stations to wireless nodes and \( P_M = 200 mW \). Note that with 10 antennas at the receiver, approximately 0.2 and 0.3 b/s/Hz are achievable for \( \rho_w = 10^{-4} \) and \( \rho_w = 10^{-3} \) respectively. The discrepancy in the spectral efficiency is a result of the maximum transmit power. For \( \rho_w = 10^{-4} \), a larger fraction of nodes transmit at \( P_M \) compared to \( \rho_w = 10^{-3} \), resulting in higher Signal-to-Interference-Ratios (SIR) for \( \rho_w = 10^{-4} \). The higher total interference power for \( \rho_w = 10^{-3} \) is offset by increased signal powers due to shorter links since the relative base-station to wireless node density is fixed.

V. RANDOM CELLS

A. Mean Spectral Efficiency

Suppose that instead of at hexagonal lattice sites, the base stations were located at random points on the plane according to a homogenous PPP with intensity \( \rho_t \) nodes \( m^{-2} \). The random cells generated constitute a homogenous Poisson-Voronoi tessellation (PVT) of the plane where the Voronoi cell associated with each base station is the set of points on the plane that are closer to that base station than any other. We shall henceforth omit the term homogenous, i.e. PPP and PVT shall refer to the homogenous PPP and homogenous PVT respectively. Figure 8 illustrates a portion of such a network where the base stations are the circles and the cell boundaries are the solid lines. The square is a representative wireless node connected to its closest base station.

The distances between wireless nodes and their closest base station (and their transmit powers) are correlated as they are related by the random locations of the base stations. This correlation prevents direct application of Theorem 1 which requires independent transmit powers. However, conditioned on a realization of the PPP, the transmit powers are independent as they are simply functions of the node locations which are independent by assumption. Thus, we first find the mean spectral efficiency of the representative link conditioned on a realization of the base station PPP \( \Pi_t \), and then average over all \( \Pi_t \) to find the unconditional mean spectral efficiency. First, place the representative transmitter at an arbitrary point on the plane. Next, consider a realization of \( \Pi_t \) and let the closest base station to the representative transmitter be the representative receiver. We assume that \( \Pi_t \) does not result in any Voronoi cell of infinite area. Excluding such realizations of PVTs does not change the mean spectral efficiency since they are zero-probability events (e.g., see [28]). Additionally, we shift the coordinates of the system such that the representative receiver is at the origin. \( n \) IID interferers are then placed in a radius-\( R \) disk centered at the origin. We shall consider the limit as \( R, N \) and \( n \to \infty \) as in the previous section.

Conditioned on \( \Pi_t \) and \( r_1 \), the mean spectral efficiency is approximated by:

\[
E[C|\Pi_t, r_1] \approx \log_2 \left( 1 + G_\alpha P_1 \left( \frac{N}{E[P_\alpha^z | \Pi_t] \pi \rho r_1^2} \right)^{\frac{1}{z}} \right) \quad (20)
\]

By the ergodicity of the PPP, \( E[P_\alpha^z | \Pi_t] = E[P_\alpha^z] \) with probability 1 (see e.g. [28] for a discussion on ergodicity). That is to say, \( E[P_\alpha^z | \Pi_t] \) which is averaged over all the cells in the PVT, is independent of the specific realization of the PVT with probability 1. Hence,

\[
E[C|\Pi_t, r_1] \approx \log_2 \left( 1 + G_\alpha P_1 \left( \frac{N}{E[P_\alpha^z] \pi \rho r_1^2} \right)^{\frac{1}{z}} \right) \quad (21)
\]
The PDF of the distance \( r \) between any wireless node and its closest base station \( f_r(r) = 2\pi \rho_t r e^{-\pi \rho_t r^2} \), for \( r > 0 \) (e.g. see [25]). Using this expression and the power control of (5),

\[
E[P^{2/a}] = \int_0^\infty \left( \min \left( \frac{p_t}{G_t} r_1^\alpha, P_M \right) \right)^{\frac{a}{2}} 2\pi \rho_t r e^{-\pi \rho_t r^2} dr = \int_0^\infty \left( \frac{a_1}{p_t G_t} \right)^{\frac{a}{2}} \left( \frac{p_t}{G_t} \right)^{\frac{a}{2}} 2\pi \rho_t r^3 e^{-\pi \rho_t r^2} dr \\
+ \int_0^\infty \left( \frac{a_1}{p_t G_t} \right)^{\frac{a}{2}} P_M^{\frac{a}{2}} 2\pi \rho_t r e^{-\pi \rho_t r^2} dr = \left( \frac{p_t}{G_t} \right)^{\frac{a}{2}} \frac{1 - e^{-\pi \rho_t \left( \frac{a_1}{p_t G_t} \right)^{\frac{a}{2}}} \pi \rho_t}{
\int_0^\infty \left( \frac{a_1}{p_t G_t} \right)^{\frac{a}{2}} P_M^{\frac{a}{2}} 2\pi \rho_t r e^{-\pi \rho_t r^2} dr
\right) \frac{1}{2\pi \rho_t} e^{-\pi \rho_t r_1^2} dr_1.
\] (22)

Substituting (5) into (21) and averaging with respect to \( r_1 \),

\[
E[C] \approx \int_0^\infty \log_2 \left( 1 + \min \left( \frac{p_t}{G_t} r_1^\alpha, P_M \right) r_1^{-\alpha} G_\alpha \left( \frac{N}{E[P^{2/a}] \pi \rho_w} \right)^{\frac{a}{2}} \right) 2\pi \rho_t r_1 e^{-\pi \rho_t r_1^2} dr_1
= \left( 1 - e^{-\pi \rho_t \left( \frac{a_1}{p_t G_t} \right)^{\frac{a}{2}}} \right) \log_2 \left( 1 + \frac{p_t}{G_t} G_\alpha \left( \frac{N}{E[P^{2/a}] \pi \rho_w} \right)^{\frac{a}{2}} \right)
+ \int_0^\infty \log_2 \left( 1 + P_M r_1^{-\alpha} G_\alpha \left( \frac{N}{E[P^{2/a}] \pi \rho_w} \right)^{\frac{a}{2}} \right) 2\pi \rho_t r_1 e^{-\pi \rho_t r_1^2} dr_1.
\] (23)

We were unable to find a closed form expression for the second term on the RHS of (23) and thus use numerical integration to evaluate it. However, if the transmit power budget of each wireless node is large (or the density of base stations is high), (22) is approximated by

\[
E[P_t^{2/a}] \approx \frac{p_t^{2/a}}{G_t^{2/a} \pi \rho_t}
\] (24)

and (23) is approximated by

\[
E[C] \approx \log_2 \left( 1 + p_t G_\alpha \left( \frac{N}{E[P_t^{2/a}] \pi \rho_w} \right)^{\frac{a}{2}} \right).
\] (25)

Substituting (24) into (25),

\[
E[C] \approx \log_2 \left( 1 + G_\alpha \left( \frac{N p_t}{\rho_w} \right)^{\frac{a}{2}} \right),
\] (26)

i.e., the mean spectral efficiency primarily depends on the ratio of \( p_t \) and \( \rho_w \) implying scale invariance in dense networks where the transmit power limit \( P_M \) is not significant.

Note that while (26) does not depend on the choice of \( p_t \) the original equation used to derive (26) was based on the assumption that the system is interference limited which means (26) is valid only when \( p_t \) and \( \rho_w \) are sufficiently high that the system is interference limited.

The scale invariance implied by (26) indicates that with hexagonal cells, constant mean spectral efficiency can be maintained by fixing the relative density of base stations to wireless nodes.

**B. Monte Carlo Simulations**

We verified (23) and (26) by Monte Carlo simulations of the network topology. We placed base stations in a circular network of radius \( 8R \). The numbers of base stations were selected to achieve relative densities of base stations to wireless nodes of 20%, 10% and 5%. The network of base stations was then re-centered such that a base-station exists at the origin. 4000 wireless nodes were then placed in a circular network of radius \( R \), centered on the network of base stations with \( R \) selected to achieve a wireless node density of \( 1 \times 10^{-3} \) nodes \( m^{-2} \). This experiment was repeated 1000 times. For each trial, the spectral efficiency of a randomly selected link in the center-most cell (to reduce edge-effects) was collected and averaged. The transmit power of each wireless node
was set according to (5) with $P_M = \infty$ (to simulate the sufficient power case) or $P_M = 200\ mW$. $G_t = 10^{-5}m^\alpha$, thermal noise power of $10^{-14}$ W, and $\alpha = 4$, were assumed.

Figure 9 shows results of Monte Carlo simulations and the asymptotic expression given by (26) for systems with unlimited transmit powers per node. Note that the simulations match the asymptotic results to within 10% when $N \geq 9$ for a relative base station to wireless node density of 20%. For lower relative densities, the convergence is slower. For 10% relative density, the simulations match the asymptotic expression to within 10% only when $N \geq 19$ and only when $N \geq 38$ for 5% relative density. The rate of convergence for random cells is slower than that for hexagonal cells because the range of transmit powers is much larger for random cells compared to hexagonal cells which results in slower convergence, as explained in Section IV-C.

Figure 10 shows simulations of systems with a 200 mW transmit power limit. The target received power $p_t$ was set such that the target SNR, $p_t/\sigma^2 = 30dB$. For relative base station to wireless node densities of 20%, 10%, and 5%, the simulated mean spectral efficiencies are within 10% of the asymptotic prediction when $N \geq 6$, $N \geq 7$ and $N \geq 9$ respectively. The convergence of the simulated mean spectral efficiencies to the asymptotic values is faster for systems with limited transmit power as the range of transmit powers in the network is smaller when there is a bound on the transmit power.

These simulations indicate that the asymptotic expressions are useful for reasonable numbers of base-station antennas.

C. The Cost of Random Cells

For systems with limited transmit powers, we numerically evaluated and plotted equations for the spectral efficiency corresponding to random and hexagonal cells in Figure 11 where the solid and dashed lines represent hexagonal and random cells respectively. The transmit power budget was 200 mW and wireless node density was $10^{-3}$ with different relative density of base stations to wireless nodes as shown in the plot. Note that the difference in mean spectral efficiencies diminishes with the number of antennas. However, for high base station densities the mean spectral efficiency for random cells is significantly lower. For instance, with 10 antennas at the base stations and 20% relative density of base stations to wireless nodes, the mean spectral efficiency with hexagonal cells is twice that of random cells.

When base-station density and/or transmit power budgets are high, the mean spectral efficiency given by (16) can be rewritten in terms of the effective base-station density $\rho_h$ as:

$$E[C] \approx \log_2 \left( 1 + G_\alpha \left( \frac{1.95N\rho_h}{\rho_w} \right)^{\frac{2}{\alpha}} \right).$$  (27)

When compared to the mean spectral efficiency with random cells given by (26), (27) indicates that several-fold (but not orders of magnitude) gains in mean spectral efficiency can be achieved by regularly distributing base stations in planar networks compared to randomly distributing them, and furthermore, the difference diminishes with the number of base-station antennas.

VI. SUMMARY AND CONCLUSIONS

We have derived an asymptotic expression for the mean spectral efficiency of the uplink in wireless networks with multi-antenna base-stations (tethered-nodes) in networks with hexagonal cells. We assumed a power control algorithm for which wireless nodes try to achieve a target received power at the base stations to which they are connected. This power control algorithm which has also been used in [14] and related works ensures that uplink spectral efficiencies are close to the mean value with high probability when the number of antennas per link is large and the wireless nodes have high power budgets.

If the spacing between base stations is small enough that all wireless nodes are able to achieve the target received signal power at their base stations (which we call the sufficient-power case), the mean spectral efficiency takes a simple form given by (16). Note that for a fixed ratio of base station to wireless node densities $p_t/\rho_w$, as $\rho_w$ increases, the system eventually moves to the sufficient-power case so this is an effective way of scaling the density of such networks.

From (16), note that with 7 antenna elements per base station and $p_t/\rho_w \approx 0.1$, the mean spectral efficiency is approximately $1\ bs^{-1}Hz^{-1}$. If we assume that 10% of all wireless nodes are actively transmitting at any one time,
the ratio of base station to total wireless node density has to be just 1% to achieve a mean spectral efficiency of 1 $bs^{-1}Hz^{-1}$, as given by (16). For systems with insufficient power, i.e., the base stations are far enough apart that some fraction of the wireless nodes will not achieve the target received power, the expression for the mean spectral efficiency is more complicated and has to be evaluated by numerical integration.

We verified the accuracy of the derived expressions by Monte Carlo simulations. We also used the simulations to study the outage spectral efficiency, i.e., the spectral efficiency achievable with a given probability. We found that in the sufficient power case, with 14 antennas per base station (a reasonable number for base-stations) and single antennas at each wireless node, and with 10% of wireless nodes transmitting simultaneously at any one time, over 1 $bs^{-1}Hz^{-1}$ is achievable by 95% of wireless nodes when the ratio of base station to wireless node densities is 1%.

We also found an expression for the mean spectral efficiency of links with base stations at random locations with area density $\rho_t$. We find that the penalty of random cells compared to hexagonal cells diminishes with increasing $N$. At modest $N$ we found that hexagonal cells can increase the mean spectral efficiency over random cells, several-fold as illustrated in Figure [11]

The findings of this work are useful for designers of cellular wireless systems such as pico-cells and city-wide wi-fi access as they provide compact expressions for the spectral efficiency and hence data rates as a function of tangible system parameters such as user and base-station densities, number of base-station antennas and random versus regular distribution of base-stations.

**APPENDIX**

**A. Derivation of Asymptotic Normalized SINR for General Link Lengths (Proof of Theorem 1)**

Let the representative transmitter be node-1, at a distance $r_1$ from the representative receiver at the origin. The remaining transmitting nodes numbered 2, 3, $\cdots$ $n$ + 1 are interferers. To find the normalized SINR, $\beta_N = N^{-1/2}\bar{I}$SINR, we scale the interference and noise powers by $N^{-1/2}$ and find the SINR of this new system. Let $\tilde{p}_i = N^{-1/2}p_i = N^{-1/2}p_i G_{\beta} r_i^{-\alpha}$ where $p_i$ is the transmit power of node-$i$, and $r_i$ its distance from the origin. Let the matrix $P = \text{diag}(\tilde{p}_2, \tilde{p}_3, \cdots, \tilde{p}_{n+1}).$ Since the noise power of the original system is $\sigma^2$ from (4), the noise power of the new system is $N^{\alpha/2}\sigma^2 = N\sigma^2$. By the well-known formula for SINR of MMSE estimators,

$$\beta_N = \frac{1}{N} \mathbf{h}^\dagger \left( \frac{1}{N} \mathbf{GG}^\dagger + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}$$

where $\mathbf{h}$ is an $N \times 1$ vector of channel coefficients between the representative transmitter and the antennas at the representative receiver, and $\mathbf{G}$ is an $n \times N$ matrix of IID $\mathcal{CN}(0, 1)$ Rayleigh fading coefficients between the antennas of the interferers and the representative receiver.

By Theorem 7.1 of [20] which is a strengthening of Theorem 3.1 of [21], if the e.d.f of the received interference powers (i.e. $\tilde{p}_2, \tilde{p}_3, \cdots, \tilde{p}_{n+1}$) converges with probability 1 to a limiting function $H(x)$ as $N$ and $n \rightarrow \infty$ with $n/N = c > 0$, the SINR converges with probability one to the value of $m(z)$ that satisfies (29) with $z = -\sigma^2$.

$$zm(z) + 1 = m(z) c \int_0^\infty \frac{\tau dH(\tau)}{1 + \tau m(z)}.$$  

(29)

To show convergence of the interference powers, recall that $n$ interferers are distributed in a disk of radius $R$ centered at the origin. Setting $G_t = 1$ for notational convenience (it will be reintroduced in the final expressions), the CDF of the received power from wireless node $i$ is

$$\Pr\{\tilde{p}_i \leq x\} = \Pr\{p_i N^{-1/2} r_i^{-\alpha} \leq x\} = \int \Pr\left\{ \frac{r_i}{N} \geq \left( \frac{P_i}{x} \right)^{1/\alpha} \right\} f_P(P_i) dP_i.$$  

(30)

$$\rightarrow \int f_P(P) \left( 1 - \frac{\pi \rho \omega}{c} \left( \frac{P}{x} \right)^{1/\alpha} \right) I_{\{P < x\}} dP \hspace{1cm} \text{(31)}$$

$$= F_P\left( \frac{x}{b} \right) - \frac{\pi \rho \omega}{c} x^{-\alpha} E\left[ P^{2/\alpha} \right] + \frac{\pi \rho \omega}{c} x^{-\alpha} \int_0^\infty f_P(P) P^{2/\alpha} dP.$$  

(32)
Taking the derivative of the RHS of (32) yields:

\[
\frac{dH(x)}{dx} = \frac{2\pi \rho_w}{c_\alpha} E \left[P_{\hat{w}}^2\right] x^{-\frac{\hat{w}}{\alpha} - 1} - \frac{2\pi \rho_w}{c_\alpha} x^{-\frac{1}{\alpha} - 1} \int_{x/b}^{\infty} f_{P}(p) \frac{p - \hat{w}}{\alpha} dp. \tag{33}
\]

Substituting (33) and applying Lemma 1 of [15], the RHS of (29) becomes

\[
zm(z) + 1 = \frac{2\pi \rho_w}{\alpha} E \left[P_{\hat{w}}^2\right] m(z)\frac{2}{\pi} \csc \left(\frac{2\pi}{\alpha}\right) - \frac{2\pi \rho_w m(z)}{\alpha} \int_{0}^{\infty} \frac{\tau^{-\frac{2}{\alpha}}}{1 + m(z)\tau} d\tau \int_{\tau/b}^{\infty} f_{P}(p) x^\frac{2}{\alpha} dx. \tag{34}
\]

Re-arranging terms in the RHS of the last equation yields (2).

B. Mild Dependence of Transmit Power on Position for Cellular Networks

We show that \( r_i / \sqrt{N} \) and \( P_i \) are asymptotically independent for both hexagonal and Poisson Voronoi cells. Assume that node \( i \) is distributed with uniform probability in the radius \( R \) circular network and recall that \( r_i \), \( r_{\tilde{t}i} \), and \( P_i \) are its distance to the origin, distance to its closest base station and transmit power respectively. Let \( B_\varepsilon \) be a disk of radius \( x \) centered at the origin and \( \Xi_v \) be the union of the set of disks of radius \( v \) centered at the base stations. Conditioned on \( r_i \leq y \sqrt{N} \), node-i is uniformly distributed in \( B_{y\sqrt{N}} \). Hence,

\[
\Pr \left\{ r_{\tilde{t}i} \leq v \bigg| \frac{r_i}{\sqrt{N}} \leq y \right\} = \frac{\Pr \left\{ r_{\tilde{t}i} \leq v \big| r_i \leq y \sqrt{N} \right\} \text{Area} \left( \Xi_v \cap B_{y\sqrt{N}} \right)}{\pi y^2 N} \tag{37}
\]

For hexagonal cells, as \( N \) and \( R \to \infty \), the RHS approaches \( F_x(v) \) from (14) as the edge effects diminish. For Poisson cells, the set \( \Xi_v \) forms a Boolean model and as \( N \to \infty \), the RHS of (37) converges to \( 1 - e^{-\rho_i x v^2} \) with probability 1 (see Chapter 3 of [28]). Hence, \( r_{\tilde{t}i} \) and \( r_i / \sqrt{N} \) are asymptotically independent and since \( P_i \) is a function of \( r_{\tilde{t}i} \), \( P_i \) is asymptotically independent of \( r_i / \sqrt{N} \). Thus, as \( N \to \infty \), with probability approaching unity,

\[
\Pr \left\{ \frac{r_i}{\sqrt{N}} \geq \left( \frac{P_i}{x} \right)^\frac{1}{\alpha} \bigg| P_i \right\} \to \Pr \left\{ \frac{r_i}{\sqrt{N}} \geq \left( \frac{P_i}{x} \right)^\frac{1}{\alpha} \right\} = \frac{R^2 - N \left( \frac{P_i}{x} \right)^\frac{1}{\alpha}}{R^2} I \left\{ 0 < \left( \frac{P_i}{x} \right)^\frac{1}{\alpha} \sqrt{N < R} \right\}.
\]

Note that the RHS is simply the CDF of \( r_i \) evaluated at \( \sqrt{N} \left( \frac{P_i}{x} \right)^\frac{1}{\alpha} \). Substituting \( c = n/N \), Bayes’ rule, and rearranging terms in the RHS of the last equation yields (2).
Fig. 1. Illustration of base stations at hexagonal lattice sites.

Fig. 2. Illustration of wireless network with representative link and base-stations at hexagonal lattice sites.
Fig. 3. Mean spectral efficiency vs. number of receive antennas for $\rho_w = 10^{-3}$ and $\rho_w = 10^{-2}$ nodes $m^{-2}$ with unlimited transmit powers and hexagonal cells.

Fig. 4. Outage spectral efficiency vs. number of receive antennas for $\rho_w = 10^{-2}$ nodes $m^{-2}$ with unlimited transmit powers and base station density equaling 10% of wireless node density, with hexagonal cells.

Fig. 5. Mean spectral efficiency for $\rho_w = 10^{-4}$ nodes $m^{-2}$ with different relative density of base station to wireless nodes and hexagonal cells.
Fig. 6. Mean spectral efficiency vs. number of receive antennas for $\rho_w = 10^{-2}$ nodes $m^{-2}$ with different relative density of base stations to wireless nodes and hexagonal cells.

Fig. 7. Outage spectral efficiency vs. number of receive antennas for $\rho_w = 10^{-4}$ and $10^{-3}$ nodes $m^{-2}$ and base-station density equal to 10% of wireless node density, hexagonal cells and 200mW transmit power budget. The solid lines represent $\rho_w = 10^{-3}$ and dashed lines represent $\rho_w = 10^{-4}$. The markers represent the different outage probabilities shown in the legend.
Fig. 8. Illustration of network with base-stations distributed according to a Poisson point process.

Fig. 9. Mean spectral efficiency of uplink communications with random cells and unlimited transmit powers.

Fig. 10. Mean spectral efficiency of uplink communications with random cells and 200mW transmit power limit per node.
Fig. 11. Mean spectral efficiency of the uplink with random cells and hexagonal cells and transmit power limited to 200 mW. Solid and dashed lines represent hexagonal and random cells respectively.