DEFORMATIONS AND MODULI OF STRUCTURES ON MANIFOLDS: GENERAL EXISTENCE THEOREM AND APPLICATION TO THE SASAKIAN CASE

LAURENT MEERSSEMAN AND MARCEL NICOLAU

Abstract. In this paper, we prove an existence theorem of a local moduli space for geometric structures in a very general setting. Then to show the interest of this result, we apply it to the case of sasakian structures.

Contents

1. Introduction 1
2. Existence of local moduli sections and spaces 4
   2.1. Global setting 4
   2.2. Existence of a local section 5
   2.3. Minimality conditions 6
   2.4. Local moduli sections 9
   2.5. Local moduli space 10
   2.6. Rigidity 12
   2.7. The case of a differential operator 12
   2.8. Smooth versus Sobolev Kuranishi type spaces 16
3. Two classical examples 17
   3.1. Complex structures 18
   3.2. Riemannian metrics 19
4. Deforming sasakian manifolds 20
   4.1. Background 20
   4.2. Encoding the structures 23
   4.3. Deformations of the contact form of a sasakian structure 26
   4.4. General deformations 31
   4.5. Comparison of the different deformation spaces 35
References 36

1. Introduction

The aim of this paper is twofold

Date: March 30, 2015.

1991 Mathematics Subject Classification. 58H15, 32G05, 32G07, 53C25.

This work was partially supported by the grants Marie Curie 271141 DEFFOL and MTM2011-26674-C02-01 from the Ministry of Economy and Competitiveness of Spain. The first author enjoyed the warmful atmosphere of the CRM at Bellaterra during part of the preparation of this work.

1
• proving an existence theorem of a local moduli space for geometric structures in a very general setting.
• applying this theorem to the case of sasakian structures.

Both problems are very classical so we shall first recall the previous major ideas and results; then turn to the main difficulties to generalize them; and finally discuss the solutions we bring in this paper.

Let us begin with making some general comments. Solving the local moduli problem for a certain class of structures supposes attacking and solving three different but related sides of the question.

• a conceptual side: one has to define precisely what is meant by a local moduli space. It must be general enough to allow an existence theorem under reasonable hypotheses, and natural enough so that such a result is meaningful.
• a theoretical side: to prove and state an existence theorem, as well as associated results (as a rigidity result).
• a practical side: to provide technical tools to compute the local moduli space on explicit examples.

To explain the first point, let us emphasize that in moduli theory, one does not classify structures, but structures up to an equivalence relation. An ideal local moduli space should encode every class of structures close to the base one as a single point. However, recall the case of complex structures, whose classical deformation theory inspires all this work. Kuranishi’s Theorem asserts that every compact complex manifold \(X\) has a local moduli space (called here the Kuranishi space of \(X\)), which is a (germ of a) finite-dimensional analytic space. However, it is not a local moduli space in the ideal case, nor in the classical sense recalled in section 2.5. For example, if \(X\) is the second Hirzebruch surface, then the Kuranishi space of \(X\) is (the germ of) a disk \(D\) with \(0 \in D\) encoding \(X\) and all other points encoding \(\mathbb{P}^1 \times \mathbb{P}^1\). Hence this disk encodes only two different complex structures up to biholomorphism.

Indeed, the Kuranishi space encodes every class of structures close to the base one (it is complete), but the requirement of single coding is replaced by the fact that it is minimal amongst all complete analytic spaces (it is versal (also called semi-universal) deformation space, cf. 9 or 17). This notion of versal space is also adapted to the case of transversely holomorphic foliations, for which an existence theorem is proved in 7 and in 9 for the case with fixed differentiable foliated type. In both cases, associated to the classical theory of the Kodaira-Spencer map and the criteria for a deformation to be complete and versal, it provides powerful tools to compute the Kuranishi space.

However, apart from these two classes and some variants, and although versality can be defined in a very general setting, in many other situations (CR structures, smooth foliations, kählerian metrics, ...), it is not true, or not known, whether a versal space exists.

Nevertheless, Ebin proved in 5 the existence of a local moduli space for riemannian metrics, and Donaldson for ASD hermitian connections in 2. Both constructions are inspired in Kuranishi’s approach. Both are indeed global moduli spaces, benefiting from the fact that the isotropy groups are
finite (at least for some generic choices). This illustrates the fact that, although Kuranishi’s work was focused on complex structures, his arguments and strategy only involve very general tools that have nothing to do with the cohomological aspects of the classical deformation theory of Kodaira-Spencer.

The strategy of this paper is to clean up Kuranishi’s proof of the existence of a versal space for compact complex manifolds, to keep from it only the scheme that was applied in Ebin’s and Donaldson’s cases, and to give general hypotheses under which it can be applied to. This supposes to be in a coherent general setting and to have a definition of a (generalized) local moduli space. Both comes from [16], with slight modifications. Let us be more specific. Kuranishi’s Theorem is a slice Theorem and not a deformation Theorem. At a fixed base point, one looks for a local slice to the action of the diffeomorphism group onto the space of complex operators. Versality can then be interpreted as a minimality condition of the slice: the orbit of the base point must intersect the slice only at a discrete set, see [16] and section 2.3. All this can easily be transposed to the abstract setting of a topological group acting on a Hilbert manifold.

In this framework, one forgets about families and deformations, since one then tries to show the existence of a minimal slice using only the classical inverse function Theorem. We prove that there are basically two conditions to succeed. The first one is geometric and obvious: the local orbit of the base point must be closed. The second one is analytic and more surprising: the isotropy group must contain only more regular elements. To understand it, think of the case where the structures under consideration are sections of some smooth bundle. To run the program, one uses sections of Sobolev class, say \( l \), (to be in a Banach space). Then, the condition is that the isotropy group contains only elements of class \( l+1 \). Of course, this is a direct consequence of the fact that composition of diffeomorphisms of same Sobolev class is only a continuous and not a \( C^1 \) map. And it is obviously satisfied for complex structures (since automorphisms are holomorphic), riemannian metrics (since isometries are of class \( C^\infty \)), and generic ASD connections (since their isotropy group is plus or minus the identity). However, it is not clear at all if this condition just comes from the Banach formalism or really reveals some severe obstruction to the existence of any reasonable kind of local moduli space.

The main difference with the classical use of Kuranishi’s methods in deformation theory is that we do not have elliptic complexes nor cohomology groups ruling the infinitesimal deformations.

As an application, we show the existence of a Kuranishi type moduli space for sasakian structures. A sasakian manifold is an odd-dimensional riemannian manifold endowed with a very special metric (cf. section [4]) and comes equipped with many induced additional structures: a strongly pseudo-convex CR structure, a contact form, a flow by isometries, a transversely kähler foliation. All these structures are closely linked one to each other. Hence there are plenty of deformation problems one can state and study, depending on what is kept fixed or not. Literature on sasakian manifolds
is full of deformation results and arguments (see for example the reference book [1]).

However, a detailed look shows that most of these deformation arguments involve constructing special deformations and not searching for a moduli space. Indeed, the only detailed and complete treatment we know of such a moduli space is that of the transversely kähler structure (the smooth foliation being fixed), and is a direct application of the results of [6]. In particular, there is no comprehensive construction of a general local moduli space, general meaning that we allow deformations of all structures at the same time.

This is what we do in Theorem 4.26. It shows the interest of our generalization. In the same vein, we also construct in Theorem 4.3 a Kuranishi type moduli space for deformations of the contact form of a sasakiian manifold. Finally, we describe the relationship between these different Kuranishi type moduli spaces associated to a sasakiian structure.

2. Existence of local moduli sections and spaces

In this part, we prove a general theorem on the existence of local moduli sections and spaces in the setting of [16].

2.1. Global setting. We follow, with slight modifications, the setting introduced in [16].

We denote by $X$ a compact, connected, smooth manifold, by $\mathcal{E}$ a set of structures on $X$. We assume that $\mathcal{E}$ is a Banach manifold (over $\mathbb{R}$ or $\mathbb{C}$) and contains a closed subspace $\mathcal{I}$ of integrable structures (cf. Section 3 or [16] for concrete examples).

We consider a topological group $G$ with countable topology. We assume that $G$ acts continuously on $\mathcal{E}$ by smooth transformations, preserving $\mathcal{I}$.

Given $J_0 \in \mathcal{I}$, we are interested in finding a local section to the action of $G$ at $J_0$, which has moreover good properties of minimality. As explained in Section 1, the crucial point is to define minimality in this very general context.

Before proceeding, let us precise some notations and definitions.

Given two Banach manifolds, by a smooth map between them, we mean a $C^\infty$ one. In the case where they are modeled over a complex vector space, we assume that the differential of the map commutes with complex multiplication (hence the map has to be thought of as a holomorphic map). A smooth bijective map with smooth inverse is an isomorphism (hence we use the generic word isomorphism instead of diffeomorphism and biholomorphism). In a Banach vector space, a map is analytic if it is analytic in the sense of [3]. An analytic set is the zero set of a finite number of analytic functions.

We require that a smooth submanifold of a Banach manifold has closed tangent space at each point. If $K$ is a closed subset of $\mathcal{E}$, then a map from $K$ to a Banach manifold is $C^k$, respectively smooth, if it is the restriction of a $C^k$, respectively smooth map defined on some open set of $\mathcal{E}$ containing $K$. 
A smooth bijective map between closed subsets of Banach manifolds whose inverse is smooth is called an isomorphism.

We denote by \( J \cdot g \) the action\(^1\) of an element \( g \in G \) onto a structure \( J \in \mathcal{E} \).

We say that two structures \( J_1 \) and \( J_2 \) are equivalent, and write \( J_1 \sim J_2 \), if they belong to the same \( G \)-orbit.

2.2. Existence of a local section. Let \( J_0 \in \mathcal{I} \). We assume that there exists a Banach vector space \( T \) and a homeomorphism

\[
\Phi \quad 0 \in V \subset T \quad \phi \quad e \in W \subset G
\]

between a connected open neighborhood \( V \) of 0 and a connected open neighborhood \( W \) of the neutral element \( e \) of \( G \) such that the action of \( G \) onto \( \mathcal{E} \) is smooth in this chart, that is

\[
(\xi,J) \in V \times U \quad \mapsto \quad J \cdot \phi(\xi) \in \mathcal{E}
\]

is smooth for \( U \) a connected open neighborhood of \( J_0 \).

Remark 2.1. Observe that chart (2.1) depends on \( J_0 \). This may seem curious at first sight, but this is exactly what happens in the classical case of complex structures (cf. Section 3.1).

Call \( L \) the differential of (2.2) at \((0,J_0)\). Let

\[
E := \ker L|_{F \times \{0\}}
\]

seen as a subspace of \( T \).

**Hypotheses.**

- **(H1):** The vector subspace \( E \) admits a closed complement \( E^\perp \) in \( T \).
- **(H2):** The differential \( L \) has the form \( L(\xi,\omega) = \omega + P\xi \) for some linear bounded operator \( P : T \to T_{J_0}\mathcal{E} \).
- **(H3):** Set \( F := \text{Im } P \). Then \( F \) is closed in \( T_{J_0}\mathcal{E} \) and admits a closed complement \( F^\perp \).

Remark 2.2. (H1) and the second part of (H3) are automatically satisfied if \( \mathcal{E} \) is an Hilbert manifold, which is often the case in practice (cf. Section 3.1). (H2) is satisfied in many interesting cases, and, in any case, it is very easy to check. So the crucial point is the first part of (H3). Checking that \( \text{Im } P \) is closed is usually the hard point.

Remark 2.3. \( P \) will often - but not always - be a differential operator, cf. section 2.7.

Let \( \tilde{K} \) be a submanifold of \( U \) passing through \( J_0 \) and tangent to \( F^\perp \) at \( J_0 \) (if \( \exp : U' \subset T_{J_0}\mathcal{E} \to U \) is a local chart at \( J_0 \), just take \( \exp(F^\perp \cap U') \) as \( \tilde{K} \)).

We are now in position to prove the existence of a local section.

**Proposition 2.4.** Assume (H1), (H2) and (H3). Then, shrinking \( V \) and \( U \) if necessary, the map \( \Phi \) from \( E^\perp \cap V \times \tilde{K} \) to \( U \subset \mathcal{E} \) defined by

\[
(\xi,J) \mapsto \Phi(\xi,J) := J \cdot \phi(\xi)
\]

\(^1\)We make \( G \) act on the right. In the many cases where \( J \) is encoded by a 1-form with values in some bundle, this means that \( G \) acts by pullback.
is an isomorphism at \((0, J_0)\).

**Proof.** The map (2.3) is smooth with differential at \((0, J_0)\) equal to

\[(\xi, \omega) \in E^\perp \times F^\perp \mapsto \omega \oplus P\xi \in F^\perp \oplus F.\]

It is an isomorphism, hence the conclusion follows from the inverse function theorem. \(\square\)

Set now \(K = \tilde{K} \cap \mathcal{I}\). This is a closed subset of \(\mathcal{I}\). It follows from the \(G\)-invariance of \(\mathcal{I}\) that the map (2.3) is an isomorphism from \(E^\perp \cap V \times K\) to \(U \cap \mathcal{I}\).

The set \(K\) is the local section we were looking for. Notice that we have a smooth retraction map

\[(2.4) \quad J \in U \cap \mathcal{I} \mapsto \Xi(J) := (\Phi^{-1})_2(J)\]

whose fibers are included in the \(G\)-orbits. Here \((\Phi^{-1})_2\) denotes the second component of \(\Phi^{-1}\).

**Remark 2.5.** We insist on the fact that \(K\) is just a closed set and that a function on \(K\) is an isomorphism means that both this function and its inverse are restriction of a smooth map between manifolds. Of course, in many cases, \(K\) is a manifold or an analytic space and the isomorphism is ”really” an isomorphism, cf. Section 3.

**Remark 2.6.** Here retraction just means that

\[\Xi \circ \Xi \equiv \Xi.\]

**Remark 2.7.** In the Hilbert case, since \(\text{Im} \ P\) is closed, a complementary subspace is given by the kernel of the Hilbert adjoint \(P^*\). If \(P\) is a differential operator, using the appropriate norms, the kernel of the formal adjoint gives also a complementary subspace.

### 2.3. Minimality conditions

We want to prove that, under some additional assumptions, the local section has good properties of minimality. We will consider two minimality conditions.

In the classical case of complex structures (cf. Section 3.1), versality in the sense of [9] is the good minimality condition. However, it cannot be adapted to our general setting for it supposes to have an associated Kodaira-Spencer theory. To be more precise, in the complex case, Kodaira-Spencer theory tells us that versality means bijectivity of the Kodaira-Spencer map at \(J_0\). But here we do not have a well defined Kodaira-Spencer map; we do not even have a well defined notion of a deformation as a flat morphism in some sense. Indeed, the point here is that we try to define minimality in situations where there is no associated Kodaira-Spencer map.

As a substitute, versality in the complex case can also be defined as minimality of the dimension of the local section at the base point. But this is also unadapted to our setting since it supposes finite dimension of the local section, which is precisely an hypothesis we want to discard, since it is not satisfied in many examples.

In [16], we proposed a definition of a local moduli section which has to be
thought of as a substitute for the notion of versal deformation space. We prove it to be equivalent to versality for complex structures.

We deal with this notion in Section 2.4. But, before that, we will see now slightly more general minimality conditions, which appear naturally in our setting. The central idea is very simple: in a minimal local section, the repetitions (that is the subset of points encoding a fixed structure up to $G$-action) should be minimal. The case of complex structures shows that we cannot prevent repetitions, even repetitions of the base point $J_0$. It also shows that it is too much to expect these repetitions to be countable subsets for all $J \in K$, but that we should ask for this at $J_0$. In other words, one should call minimal a local section in which there is no path in $K$ starting at $J_0$ encoding a trivial deformation of $J_0$. Both the minimality conditions introduced here and that of [16] are precise statements saying that. The differences lie in defining what is a trivial deformation, and are related to the validity or not of the Fischer-Grauert property (see Section 2.4).

Let us introduce the following two minimality conditions.

\textbf{(MC1):} Let $c : [0, \epsilon) \to G$ be a continuous path starting at $e$. For $t \in [0, \epsilon)$, define $J(t) := \Xi(J_0 \cdot c(t))$. Then the continuous path $J$ in $K$ satisfies $J \equiv J_0$.

In other words, if (MC1) is fulfilled, the intersection of $K$ with the local $G$-orbit of $J_0$ does not contain any non-constant continuous path.

\textbf{(MC2):} Up to shrinking $W$, we have $g \in W, J_0 \cdot g \in K \implies J_0 \cdot g = J_0$.

In other words, if (MC2) is satisfied, the intersection of $K$ with the local $G$-orbit of $J_0$ is equal to $\{J_0\}$.

Of course, (MC2) implies (MC1); but more is true.

\textbf{Proposition 2.8.} (MC1) $\iff$ (MC2).

\textbf{Proof.} Assume that (MC2) is not satisfied. Then, since $G$ has a countable topology, we can find a sequence $(g_n)$ in $G$ converging to $e$ and such that $J_0 \cdot g_n$ is in $K \setminus \{J_0\}$ for all $n$.

For $n$ big enough, $g_n$ belongs to $W$ hence $g_n = \phi(\xi)$ for some $\xi$. Set $c(t) = \phi(t\xi)$ and $J(t) = \Xi(J_0 \cdot c(t))$.

This is a continuous path in $K$ and in the local orbit of $J_0$. Besides, it is non-constant since $J(1) = J_0 \cdot g_n$ is different from $J_0$. Hence (MC1) is not satisfied.

Because of Proposition 2.8, we will from now on refer to both (MC1) and (MC2) as the shortened (MC).

\textbf{Definition 2.9.} If a local section fulfills (MC), we say it is \textit{minimal}.

We now need to define some more assumptions.

\textbf{Hypotheses.}
(H4): The isotropy group

\[ G_{J_0} := \{ g \in G \mid J_0 \cdot g = J_0 \} \]

is a local Banach submanifold at \( J_0 \), that is there exists a smooth map

\[ E \cap V \xrightarrow{\psi} G_{J_0} \cap W. \]

(H5): The map

\[ (g, h) \mapsto \mu(g, h) = g \circ h \in W \]

for \( g \in G_{J_0} \) and \( h \in G \), is \( C^1 \) (that is \( C^1 \) in the chart \( 2.4 \)), with differential at \((e, e)\) equal to

\[ (\xi, \eta) \in E \times T \mapsto \xi + \eta \in T. \]

Remark 2.10. We emphasize that (H5) means that the composition of elements \( g \) and \( h \) of \( G \) is \( C^1 \), only when \( g \) belongs to the isotropy group of \( J_0 \). In many cases, for example when \( G \) is the group of diffeomorphisms of class \( C^k \) or of Sobolev class \( W^l \), the composition map in the group is only continuous (see the proof of Proposition 2.27). So for (H5) to be fulfilled, we need that \( G_{J_0} \) contains only diffeomorphisms of higher regularity, as it the case in both examples of Section 3.

We have now

**Proposition 2.11.** Assume (H1), (H2), (H3), (H4) and (H5). Then the local section \( K \) of Proposition 2.4 is minimal.

**Proof.** We shall prove (MC2). As permitted by assumption (H4), let \( \psi \) be a local chart of \( G_{J_0} \) at \( e \). We assume that its differential at 0 (written in the chart \( 2.4 \)) is the identity. Then, shrinking \( V \) and \( W \) if necessary, we may assume that the map

\[ (\xi, \chi) \in (E \cap V) \times (E^\perp \cap V) \mapsto \mu(\psi(\xi), \phi(\chi)) \in W \]

is an isomorphism, since it is a smooth map by (H5) and since its differential at \((0, 0)\) is the identity.

Let \( g \in W \) be such that \( J_0 \cdot g \in K \). Then \( g \) can be written as \( \psi(\xi) \circ \phi(\chi) \) and we have

\[ J_0 \cdot g = J_0 \cdot (\psi(\xi) \circ \phi(\chi)) = J_0 \cdot \phi(\chi) \]

the last equality coming from the fact that \( \psi(\xi) \) belongs to the isotropy group of \( J_0 \).

But

\[ \Xi(J_0 \cdot \phi(\chi)) = J_0 = \Xi(J_0 \cdot g) = J_0 \cdot g \]

the last equality coming from the fact that \( J_0 \cdot g \) is supposed to be in \( K \). Hence \( J_0 \cdot g = J_0 \) and (MC2) is verified.
2.4. Local moduli sections. In [16], we proposed a definition of a local moduli section. It says that $K$ is a local moduli section if there exists a smooth retraction from a neighborhood of $J_0$ in $I$ onto $K$ (condition A1) and if any smooth path $J : [0, \epsilon) \to K$ starting at $J_0$ and all of whose points are equivalent to $J_0$ is indeed constant (condition A2).

We want to compare this definition with that of a minimal local section. First, observe that condition A1 is always fulfilled, even for a general local section, because of the existence of (2.4). Then, notice that smoothness is not a problem, since replacing continuous by smooth in (MC1) yields an equivalent condition. This is due to the fact that a continuous path in a Banach space can be approximated by smooth ones.

Also, condition A2 obviously implies (MC1). However, the converse is only true if we have:

\textbf{(FG) property:} Given $J : 0 \in [0, \epsilon) \to J_0 \in K$ a smooth path of structures all equivalent to $J_0$, then there exists a smooth path $c : 0 \in [0, \epsilon) \to e \in G$ such that $J(t) = J_0 \cdot c(t)$ for all $t$.

Of course, (FG) stands for Fischer-Grauert. Define a smooth deformation of $J_0$ as a smooth map from a smooth base manifold $B$ to $K$ sending a base point 0 onto $J_0$, and a trivial deformation of $J_0$ as a smooth deformation $J : b \in B \mapsto J_0 \cdot c(b)$ for $c$ a smooth map from $B$ to $G$. Then, these definitions are consistent with those used for complex structures (cf. [14]) and (FG) property can be rephrased as: a smooth deformation of $J_0$ all of whose points are equivalent to $J_0$ is trivial. This is exactly the (smooth version) of Fischer-Grauert’s Theorem (see [13]).

To sum up, we have

\textbf{Proposition 2.12.} A minimal local section is a local moduli section if and only if (FG) property is true.

To rephrase this proposition, minimality states that there is no non-constant path in $K$ encoding a trivial deformation of $J_0$, whereas condition A2 means that there is no non-constant path all of whose points are equivalent to $J_0$.

In a setting where (FG) property is true (as in the case of complex structures), both definitions are the same. However, in a setting where (FG) property is not true, then the good definition to take is that of minimality, the philosophy being that a path all whose points encode the same structure but which is not a trivial deformation encodes important information and cannot be removed from the local section. Indeed, if (FG) property is not true, there cannot exist a local moduli section.

We may now state

\textbf{Definition 2.13.} A minimal local section at $J_0$ is also called a Kuranishi type space of $J_0$.

Proposition 3.1 should justify this terminology.

\textbf{Remark 2.14.} To be more precise, a local moduli section as defined in [16] is supposed to be an analytic space in the sense of [3]. In this section, we drop
this requirement. However, notice that the analytic structure is important to prove [16, Proposition 4.1], see Section 3 and remark 3.2.

2.5. Local moduli space. To finish with this part, we want to compare the previous definitions with the more classical one of local moduli space. Recall the

Definition 2.15. A local section $K$ is a local moduli space if there exists an open neighborhood $W$ of $e$ in $G$ such that

$$J \in K, \ g \in W, \ J \cdot g \in K \implies J \cdot g = J.$$ 

This is stronger than the notions of minimal local section and local moduli section. Indeed,

Proposition 2.16. Assume that $K$ is a local moduli space. Then $K$ is a local moduli section at any point $J_0 \in K$.

Proof. Take $J_0$ in $K$. Obviously, $K$ is minimal, since (MC1) condition is nothing else than the condition in definition 2.15 applied to $J_0$.

Moreover, let

$$J : 0 \in [0, \epsilon) \longrightarrow J_0 \in K$$

be a smooth path all of whose points are equivalent to $J_0$. For every $t$, we may choose an element $g_t$ of $G$ such that $J(t) = J_0 \cdot g_t$. Assume $J$ is non-constant. Since it is smooth, it is locally injective at each point of an open subset $I$ of $[0, \epsilon)$. Moreover, we may assume that $J$ is not constant on any interval containing 0 without any loss of generality. Hence, we have that 0 belongs to the closure of $I$. Since $I$ is uncountable whereas $G$ has a countable topology, the uncountable family $((g_t)_{t \in I})$ of $G$ must have an accumulation point. The same trick shows that there exists such an accumulation point arbitrarily close to 0. All that says that we can find $t_\infty$ arbitrarily close to 0 and $t_n \to t_\infty$ with all $t_n$ different from $t_\infty$ and with $g_{t_n}$ converging to $g_{t_\infty}$.

For $n$ big enough, $(g_{t_\infty})^{-1} \circ g_{t_n}$ belongs to $W$ and

$$J(t_\infty) \cdot ((g_{t_\infty})^{-1} \circ g_{t_n}) = J_0 \cdot g_{t_n} = J(t_n) \in K$$

so we have $J(t_n) = J(t_\infty)$, contradicting local injectivity at $t_\infty$. \hfill \Box

Let us try to give additional hypotheses under which we have a local moduli space.

Hypotheses.

(H4'): For all $J \in K$, its isotropy group $G_J$ is a local Banach submanifold with tangent space at identity $E_J$ isomorphic to $E$. Moreover, there exists a smooth map $\psi : U \times (E \cap V) \to G$ such that, for all $J \in K$,

$$\psi(J, E \cap V) = G_J \cap W$$

and

$$\psi(J, 0) \equiv e.$$
(H5'): The map
\[(J,\xi,h) \in \tilde{K} \times E \cap V \times W \mapsto \psi(J,\xi) \circ h \in G\]
is $C^1$ (that is $C^1$ in the chart \((2.1)\)), and the differential at \((0,e)\) of
\[(\xi,h) \mapsto \psi(J_0,\xi) \circ h\]
is equal to
\[(\xi,\eta) \in E \times T \mapsto \xi + \eta \in T.\]

Remark 2.17. Set
\[C = \{(J,\xi) \in K \times V \mid \xi \in E_J\}.\]
Then the natural projection map $C \to K$ satisfies
\[\xymatrix{C \ar[r]^{\psi^{-1}} \ar[d] & K \times (E \cap V) \ar[d]^{\text{1st projection}} \\
K & K}
up to shrinking $K$ and $V$. If $C$ and $K$ are Banach $C$-analytic spaces, then this is exactly saying that $C \to K$ is a smooth morphism (cf. \[3, \text{p.28}\]).

Remark 2.18. (H4'), respectively (H5'), is stronger than (H4), respectively (H5), and is meant to replace it.

Then, we have

Proposition 2.19. Assume (H1), (H2), (H3), (H4') and (H5'). Then $K$ is a local moduli space.

Proof. This is close to the proof of proposition \[2.11\]. We need a sort of uniform version of it. Let
\[(2.6) \quad (J,\xi,\chi) \in \tilde{K} \times (V \cap (E \times E^\perp)) \mapsto (J,\psi(J,\xi) \circ \phi(\chi)) \in \tilde{K} \times G.\]
By (H4') and (H5'), this is a smooth map. By (H5'), its differential at $(J_0,0,0)$ is equal to
\[(\omega,\xi,\chi) \in T_{J_0} \tilde{K} \times E \times E^\perp \mapsto (\omega,\xi \oplus \chi) \in T_{J_0} \tilde{K} \times T\]
so is an isomorphism. There is no $\omega$ term in the second component because of the condition $\psi(J,0) \equiv e$. Up to shrinking $\tilde{K}$, $V$ and $W$, we may assume that \[(2.6)\] is an isomorphism onto $\tilde{K} \times W$. Then, take $g \in W$ and $J \in K$. We may write $g$ as $\psi(J,\xi) \circ \phi(\chi)$ using \[(2.6)\]. And we have
\[J \cdot (\psi(J,\xi) \circ \phi(\chi)) = J \cdot \phi(\chi)\]
since $\psi(J,\xi)$ belongs to $G_J$.
Assume now that $J \cdot g \in K$, then, since $J \in K$ and $g \in W$, by \[(2.3)\] and \[(2.4)\], we have
\[\phi(\chi) = e \quad \text{and} \quad J \cdot g = J.\]
as claimed. $\square$

Observe that conditions (H4'), (H5') are satisfied in the following case.

Corollary 2.20. Assume that for all $J \in K$, we have $G_J \cap W$ reduced to $e$. Then $K$ is a local moduli space.
Proof. Just take \( \psi \) as the constant mapping onto \( e \). (H4') and (H5') are trivially satisfied. \( \square \)

Let us sum up all the previous result in the following Theorem.

**Theorem 2.21.** Let \( X \) be a compact smooth manifold. Consider \( E, I \) and \( G \) as in Section 2.2. Let \( J_0 \in I \). Assume (H1), (H2) and (H3).

1. There exists a local section \( K \) to the \( G \)-action. It satisfies (2.3).
2. Assume (H4) and (H5). Then \( K \) is a Kuranishi type space. And it is a local moduli section if and only if (FG) property is true.
3. Assume (H4') and (H5'). Then \( K \) is a local moduli space.

2.6. **Rigidity.** From Theorem 2.21 we may easily deduce a rigidity result in the spirit of that saying that a compact complex manifold \( X \) with first cohomology group with values in the sheaf of holomorphic vector fields \( H^1(X, \Theta) \) being zero is rigid.

As in Section 2.4, define a smooth (or holomorphic) deformation of \( J_0 \) as a smooth/holomorphic map from a smooth/holomorphic base manifold \( B \) to \( K \) sending a base point 0 onto \( J_0 \); and a trivial deformation of \( J_0 \) as a deformation \( J : b \in B \mapsto J_0 \cdot c(b) \) for \( c \) a map from \( B \) to \( G \).

Then, rigidity is defined classically as follows.

**Definition 2.22.** The structure \( J_0 \) is called **rigid** if every germ of deformation of \( J_0 \) is isomorphic to a trivial deformation.

**Corollary 2.23.** Assume (H1), (H2) and (H3). Assume that \( K \) is reduced to a point. Then \( J_0 \) is rigid.

**Remark 2.24.** Assume that \( E \) and \( G \) are local Hilbert manifolds. Assume that \( I \) is given locally as the zero set of some analytic map \( Q \). Denote by \( Q^{\text{lin}} \) the linear part of \( Q \). Then, consider the vector space

\[
\{ J \in T_{J_0}E \mid P^* J = Q^{\text{lin}} J = 0 \}.
\]

This is the "tangent space" of \( K \) at \( J_0 \), in the sense that the derivative at 0 of any smooth map \( c \) into \( K \) with \( c(0) = J_0 \) lies in it. Assume that \( K \) is a closed subset of a submanifold of \( E \) whose tangent space is (2.7) (this is obviously the case if \( K \) itself is a submanifold of \( E \) or an analytic subspace of \( E \) in any reasonable sense). We have

**Corollary 2.25.** In the setting of remark 2.24, assume (H1), (H2) and (H3). Assume also that (2.7) is reduced to a point. Then \( J_0 \) is rigid.

2.7. **The case of a differential operator.** In many geometric situations, the operator \( P \) can be interpreted as a differential operator. To be more precise, that means that

- The set \( E \) is a subset of the space of sections of a fiber bundle over \( X \), and \( T_{J_0}E \) is the space of sections of the corresponding vector bundle over \( X \).
- The group \( G \) is a subgroup of the group of diffeomorphisms of \( X \), and the vector space \( T \) is a subspace of the Lie algebra of vector fields on \( X \).
- The operator \( P \) is a differential operator from \( T \) to \( T_{J_0}E \).
Here, as usual, we are interested in $C^\infty$ sections. Hence $E$ is a Fréchet manifold, $G$ has a Fréchet chart \((2.1)\) and there exist two smooth vector bundles $B_0$ and $B_1$ over $X$ such that

\[ T = \Gamma^\infty(B_0) \quad \text{and} \quad T_{J_0}E = \Gamma^\infty(B_1) \]

where $\Gamma^\infty$ denotes the set of smooth sections of a bundle.

However, to run Theorem 2.21, we need to use a Sobolev completion and work with $W^l$ sections with $l > 1 + \dim X/2$; that is, denoting by $k$ the order of the differential operator $P$, we consider its natural extension from the Sobolev $(l + k)$-completion of $T$ into the Sobolev $l$-completion of $T_{J_0}E$. We also assume that the Fréchet structure of $E$ extends as an Hilbert structure on $E^l$, the corresponding subset of $W^l$ sections. In other words, given a Fréchet chart of $E$ at some point $J$ modeled on $\Gamma^\infty(B_1)$, we may assume that it extends as a Hilbert chart of $E^l$, modeled on the completion $\Gamma^l(B_1)$ of $\Gamma^\infty(B_1)$. In the same way, we assume that the Fréchet chart \((2.1)\) from $T$ to $G$ extends as an Hilbert map from the $(l + k)$-completion $T^{l+k}$ to the $(l + k)$-completion $G^{l+k}$.

Then, fixing such an $l$, we are in the Hilbert setting. Let us state

**Hypotheses.**

\[ \text{(H2diff): The differential } L \text{ has the form } L(\xi, \omega) = \omega + P\xi \text{ for some differential operator } P : T \to T_{J_0}E. \]

\[ \text{(H3diff): The differential operator } P \text{ is elliptic with } C^\infty \text{ coefficients.} \]

**Remark 2.26.** Here, by elliptic, we mean that $P$ has an injective symbol, not a bijective one. In this latest case, we speak of a strongly elliptic operator.

Also we add a $l$ to an hypothesis to say that it is valid for the particular Sobolev class $W^l$. For example, (H4l) means that (H4) is valid for the particular Sobolev class $W^l$, whereas (H4) means it is valid for all classes (always assuming implicitly that $l$ is big enough). We have now

**Proposition 2.27.** Assume (H2diff) and (H3diff). Then there exists a local section $K^l$ for all $l$. Moreover, assuming (H4l) for a particular choice of $l$, then the corresponding $K^l$ is a Kuranishi type space.

**Proof.** Fix some $l > 1 + \dim X/2$. Since our vector spaces are Hilbert spaces, conditions (H1) and (H3), second part, are automatically satisfied. Moreover, following [15] §3.9, since $P$ is an elliptic operator by (H3diff), its image is closed in any Sobolev class and (H3) is completely fulfilled. Hence, by Theorem 2.21 there exists a local section $K^l$.

Consider the composition map

\[ (f, g) \in G^{l+k} \times G^{l+k} \rightarrow f \circ g \in G^{l+k}. \]

Using the above assumption that $G^{l+k}$ is a subgroup of the $(l+k)$-diffeomorphism group of $X$, then we see that its (formal) differential at a point $(f, g)$ takes the form

\[ (\xi', \chi') \rightarrow \xi' \circ g + df(\chi') \]

(cf. [8] Example 4.4.5). Because of the term $df$, it does not map $W^{l+k}$-vector fields onto $W^{l+k}$-vector fields. So \((2.8)\) is not even a $C^1$ map. But
it is a smooth map when we take \( f \) in a submanifold of \( G^{l+k} \) that contains only \( C^\infty \) points.

Now, still by (H3diff), \( E \) is the kernel of an elliptic operator with smooth coefficients, hence contains only \( C^\infty \) elements [18, §3.7]. By (H4) and our assumptions on the chart (2.1), the same is true for \( G_{\eta_0} \cap W \). From this, it follows that \((H5l)\) is fulfilled. We conclude by Theorem 2.21. \( \square \)

**Remark 2.28.** It is crucial to emphasize that, in the setting we use here, the fact that \( P \) is elliptic does not imply that the local section \( K^l \) is finite-dimensional. This is because the tangent space to \( K^l \) at the base point is not given by the kernel of the laplacian associated to \( P \) as in the classical case; but by the kernel of (2.7) which may be completely different from this laplacian. Indeed, perhaps the main idea of our construction is to separate the integrability condition from the existence of a local section, so that the operator encoding integrability (that is the linear part of the integrability equation) is not supposed to be the same as the operator encoding the orbit (that is \( P \)). This allows us to gain flexibility and treat cases that cannot be treated in the classical setting.

In this framework, one is faced to a delicate analytic problem. If (H2diff), (H3diff), and (H4) are satisfied, then there exists a Kuranishi type space \( K^l \) for any \( l > 1 + \dim X/2 \). Nevertheless, the interesting geometric situation is the \( C^\infty \) class, for which we do not have a Kuranishi type space, for we cannot use Theorem 2.21 the spaces of sections not being Banach spaces.

Since our deformation problems arise mainly from geometric situations, one may expect that Theorem 2.21 is still valid in the \( C^\infty \) case, taking as Kuranishi type space the set of \( C^\infty \) points of \( K^l \) (which should be the same for all \( l \)).

However, this is not evident at all from the point of view of differential operator theory. Indeed, in the general case (that is if \( P \) is not elliptic but has closed image in each class \( W^l \)), we do not even know if the set of \( C^\infty \) points of \( K^l \) is not empty.

The only case where this problem easily disappears is the case where the tangent space of \( K^l \) given in (2.7) is the kernel of an elliptic operator with \( C^\infty \) coefficients. Then, \( K^l \) contains only \( C^\infty \) structures (see [18], §3.7), hence \( K^l \) does not depend on \( l \) and is a Kuranishi type space for \( C^\infty \)-structures. However, this forces this space to be finite-dimensional, which is not the case in many geometric problems, as in subsection 3.2 and section 4, and which is not the case in our setting, cf. remark 2.28.

If \( P \) is elliptic, we have the following weaker result.

**Proposition 2.29.** Assume (H2diff) and (H3diff). Then, \( K^l \) is equal to the \( l \)-completion of \( K^\infty \).

**Proof.** We owe this proof to J.A. Alvarez López.

Because of (2.7), it is enough to prove that the kernel of \( P^* \) applied to \( W^l \) points is the \( l \)-completion of the kernel of \( P^* \) applied to smooth points.

First assume that \( P \) is the first morphism of an elliptic complex \( (E_i, P_i) \)
(that is, \( T = E_0, T_{\partial_0} \mathcal{E} = E_1 \) and \( P = P_0 \)). Hodge decomposition Theorem implies
\[
\text{Ker } P_0^* = \text{Ker } (P_1 + P_0^*) \oplus \Gamma^0(\mathcal{E}^\infty(E_2))
\]
in \( \Gamma^\infty(E_1) \), and
\[
\text{Ker } P_0^* = \text{Ker } (P_1 + P_0^*) \oplus \Gamma^1(\mathcal{E}^\infty(E_2))
\]
in \( \Gamma^1(E_1) \). Indeed, \( P_1^*(\mathcal{E}^\infty(E_2)) \) is dense in \( P_1^*(\mathcal{E}^1+k(E_2)) \) because \( \Gamma^\infty(E_2) \) is dense in \( \Gamma^1_k(E_2) \) and \( P_1^* : \Gamma^1_k(E_2) \to \Gamma^0(E_1) \) is continuous. Besides \( \text{Ker } (P_1 + P_0^*) \) is the same in both decompositions because it contains only sections of class \( C^\infty \) and has finite dimension. So \( \text{Ker } P_0^* \) in \( \Gamma^\infty(E_1) \) is dense in \( \text{Ker } P_0^* \) in each \( \Gamma^i(E_1) \).

To deal with the general case, let \( \sigma_0(x, \xi) : (E_0)_x \to (E_1)_x \) be the injective symbol of \( P \) where \( x \in X \) and \( 0 \neq \xi \in T_x X^* \). Let
\[
E_2 = ((TX^*)^\otimes k \otimes E_1)/I^2,
\]
where \( I^2 \) is the vector bundle whose fiber at \( x \) is generated by vectors
\[
\xi^\otimes k \otimes \sigma_0(x, \xi)(v)
\]
for \( \xi \in T_x X^* \) and \( v \in E_0^\otimes x \). This \( I^2 \) is a subbundle because \( \sigma_0(x, \xi) \) is injective. Also it depends differentiably in \( (x, \xi) \). For \( x \in X \) and \( \xi \in T_x X^* \), define \( \sigma_1(x, \xi) : (E_1)_x \to (E_2)_x \) as
\[
\sigma_1(x, \xi)(v) = [(\xi^\otimes k \otimes v),
\]
where the brackets denote the class modulo \( I^2_x \). Then \( \sigma_1(x, \xi)(v) \) is linear in \( v \), differentiable in \( (x, \xi) \), and homogeneous of order \( k \) in \( \xi \). Hence it is the principal symbol of some pseudodifferential operator \( P_1 : \Gamma^\infty(E_1) \to \Gamma^\infty(E_2) \) of order \( k \), since pseudodifferential operators are locally defined by their symbols. Moreover, the sequence
\[
0 \to (E_0)_x \xrightarrow{\sigma_0(x, \xi)} (E_1)_x \xrightarrow{\sigma_1(x, \xi)} (E_2)_x
\]
is exact if \( \xi \neq 0 \).

By induction, we construct vector bundles \( E_i \) and operators
\[
P_i : \Gamma^\infty(E_i) \to \Gamma^\infty(E_{i+1})
\]
with symbol \( \sigma_i \) for all \( i \). The sequence of symbols \( \sigma_i \) is exact (although infinite). From the properties fulfilled by \( \sigma_0 \) and \( \sigma_1 \) we have that
\[
\sigma_0(x, \xi)\sigma_0(x, \xi)^* + \sigma_1(x, \xi)^*\sigma_1(x, \xi)
\]
is an isomorphism if \( \xi \neq 0 \). Hence \( T = P_0P_0^* + P_1^*P_1 \) is an elliptic selfadjoint operator, yielding a Hodge decomposition
\[
\Gamma^\infty(E_1) = \text{Ker } T \oplus \text{Im } T
\]
It can be refined as
\[
\Gamma^\infty(E_1) = (\text{Ker } P_0^* \cap \text{Ker } P_1) \oplus \text{Im } P_0 \oplus \text{Im } P_1^*
\]
from which it follows that
\[
\text{Ker } P_0^* = (\text{Ker } P_0^* \cap \text{Ker } P_1) \oplus K
\]
with \( K \cong \text{Im } P_1^* \) given by canonical projection.
Corollary 2.30. Fix $l$. Then the set of $C^\infty$ points of $K^l$ is dense in $K^l$ and does not depend on $l$.

2.8. Smooth versus Sobolev Kuranishi type spaces. More generally, the problem of smooth points appears when $E$ and $G$ satisfy all the hypotheses of section 2.7, even if the corresponding $P$ is not a differential operator. Then, to run Theorem 2.21 one needs to pass to Sobolev $W^l$ sections, say $E^l$. Of course, at the end of the journey, one would like to go back to the smooth world and show that one has a Kuranishi type space for the original $E$.

One option is to use the Nash-Moser Theorem instead of the classical inverse function theorem. But it is much more complicated to use and we do not know of such a general setting where this could be done automatically.

A variant is to use, if it exists, a riemannian structure on $E$. The following is directly inspired in [5]. Assume that

**Hypotheses.**

(H6): For all $g \in V$, the structure $J_0 \cdot \phi(g)$ is of class $C^\infty$ if and only if $g$ is of class $C^\infty$.

(H7): There exists a smooth affine connection on $\mathcal{E}^l$ which is invariant under the action of $G^{l+k}$.

(H8): Action (2.2) is given by pull-back, i.e., $J$ can be identified with some differential form with values in a vector bundle and we have $J \cdot \phi(\xi) = \phi(\xi)^*J$.

In particular, (H6) implies that $J_0$ itself is assumed to be of class $C^\infty$.

Then, we can associate to the affine connection an exponential map exp, [12], §IV.4 and §VII.7. Consider the local orbit $O$ of $G$ at $J_0$. Assuming (H1), (H2) and (H3) then, by (2.3), it is closed with tangent space equal to $\text{Im } P$. For $J$ a point of this orbit close to $J_0$, define

$$(2.9) \quad N_J := \{ \omega \in T_J \mathcal{E}^l \mid h(\omega, T_J O) = 0 \}.$$  

Define locally the normal bundle to this orbit

$$(2.10) \quad N = \{ (\xi, \omega) \mid \xi \in V, \omega \in N_{J_0, \phi(\xi)} \}.$$  

with projection map

$$(\xi, \omega) \in N \mapsto \pi(\xi, \omega) := \xi \in V.$$  

Then we have

**Lemma 2.31.** The map

$$(2.11) \quad (\xi, \omega) \in N \mapsto \exp_N(\xi, \omega) := \exp(J_0 \cdot \phi(\xi), \omega)$$  

is a local isomorphism at $(0,0)$ onto the neighborhood of $J_0$ in $\mathcal{E}^l$.

**Proof.** This is a standard fact about the exponential map that the composition map

$$\mathcal{E}^l \to T \mathcal{E}^l \to \mathcal{E}^l$$  

of the projection map with the inclusion map as zero section is the identity. From this, the differential of (2.11) at $(0,0)$ is given by

$$(\xi, \omega) \in T \times N_{J_0} \mapsto \omega + P_\xi \in T_{J_0} \mathcal{E}.$$  

and this is now a direct application of the inverse function theorem. □

Then we have

**Proposition 2.32.** Assume (H2diff) and (H3diff), or (H1l), (H2l) and (H3l). Assume also (H7l). Then we may define the local section of (2.3) as

\[ \tilde{K} := \{ \exp(J_0, \omega) \mid \omega \text{ close to } 0 \} \]

Moreover the inverse map of (2.3) is given by

\[ J \mapsto \Phi^{-1}(J) := (\pi((\exp_N)^{-1}(J)), J \cdot (\phi(\pi((\exp_N)^{-1}(J)))^{-1})). \]

**Proof.** This is a straightforward computation using lemma 2.11. We have

\[
\Phi^{-1} \circ \Phi(\xi, J) = \Phi^{-1} \circ \Phi(\xi, \exp(J_0, \omega)) = \Phi^{-1}(\exp(J_0, \omega) \cdot \phi(\xi)) = \Phi^{-1}(\exp((J_0, \omega) \cdot \phi(\xi))) = \Phi^{-1}(\exp_N(\xi, \omega \cdot \phi(\xi)))
\]

using the invariance of the exponential map. Hence,

\[ \pi((\exp_N)^{-1}(\exp_N(\xi, \omega \cdot \phi(\xi)))) = \xi \]

and, using (2.11) and writing \( J \) as \( \exp(J_0, \omega) \),

\[ \Phi^{-1} \circ \Phi(\xi, J) = (\xi, J \cdot (\phi(\xi))^{-1} \cdot \phi(\xi)) = (\xi, J). \]

□

As in Section 2.7, define \( K^\infty \) as the set of \( C^\infty \) points of \( K^l \). We have now

**Proposition 2.33.** Assume (H2diff) and (H3diff), or (H1l), (H2l) and (H3l). Assume also (H6), (H7l) and (H8). Define \( \tilde{K} \) as in (2.12). Then both maps (2.3) and (2.13) preserve the \( C^\infty \) class. In particular, the map (2.3) is an isomorphism for \( C^\infty \) structures and \( K^\infty \) is a local section, and a Kuranishi type space if \( K^l \) is.

**Proof.** By (H8), the exponential map is invariant by pull-back by diffeomorphisms, hence commutes with Lie derivatives, so is a homeomorphism from the set of smooth points of \( TE \) to the set of smooth points of \( E \) (cf. [5, Theorem 7.5]). This, together with (H6) implies that the map (2.11) is also a homeomorphism from the set of smooth points of \( N \) onto that of \( E \). Then, we deduce that both formulas (2.3) and (2.13) preserve the \( C^\infty \) class. □

### 3. Two classical examples

In this part, we run the previous definitions and propositions in two classical cases: complex structures and riemannian metrics.

---

2However, here, \( l \) is fixed and, strictly speaking, \( K^\infty \) depends on \( l \).
3.1. **Complex structures.** This is the foundational example, which inspires in all the definitions we gave. Here $\mathcal{E}$ is the set of almost complex operators of, say, class $W^l$, and $\mathcal{I}$ is the subspace of integrable ones, hence of complex structures. Then $G$ is the group of diffeomorphisms of class $W^{l+1}$. It is a classical fact that $\mathcal{E}$ is a Hilbert manifold over $\mathbb{C}$, and $\mathcal{I}$ a closed subset. It is even an analytic subspace in the sense of [3]. Let $J_0$ be a complex structure of class $C^\infty$. Then $T$ is the Hilbert space of $(1,0)$-vector fields of class $W^{l+1}$ on $X$ (for the structure $J_0$), whereas $T_{J_0}\mathcal{E}$ is the Hilbert space of $(0,1)$-forms of same class with values in $T$. The map $\phi$ is defined as the exponential of some analytic riemannian metric, see [10] for more details. Considering $G$ as a subset of the set of maps of class $W^l$ from $X$ onto the complex manifold $(X,J_0)$, then we get a $C^\infty$-analytic structure on $G$ with chart (2.1).

The space $E$ is the vector space of $J_0$-holomorphic vector fields. We also have

$$L(\xi,\omega) = \omega + \bar{\partial}\xi$$

so that $P = \bar{\partial}$. It is known to be an elliptic operator on vector fields and so (H2diff) and (H3diff) are satisfied. It can also be easily checked that hypotheses (H4) and (H5) are satisfied (just take the time 1 flow of a vector field as map $\psi$; since $G_{J_0}$ is the automorphism group of $(X,J_0)$, by definition, it contains only holomorphic, hence $C^\infty$, maps). As a consequence, there is a minimal local section $K$. Using remark 2.7 and the integrability condition given in [10], it is given by

$$(3.1)\quad K = \{\omega \in U \mid \bar{\partial}\omega + \frac{1}{2}[\omega,\omega] = \bar{\partial}^*\omega = 0\}.$$ 

It is important to notice that $K$ is not only a closed subset but also has a natural structure of analytic space with Zariski tangent space (2.7) at the base point. More precisely, it is given by the kernel of the $\bar{\partial}$-laplacian, a strongly elliptic operator. Hence it is finite dimensional and contains only $C^\infty$ solutions (and there is no dependance at all in the class $l$ which explains that we denoted it as $K$ and not $K^l$).

Moreover, (FG) property is true by Fischer-Grauert’s theorem, so $K$ is indeed a local moduli section.

Last, but not least, it follows directly from (3.1) that the germ of $K$ at $J_0$ is the Kuranishi space of $(X,J_0)$. Indeed, it is proven in [16] that versality is equivalent to being a local moduli section.

Finally, it is known ([23] or [17]), that $K$ is not in general a local moduli space, but that it is as soon as the dimension of the space of $J$-holomorphic vector fields on $X$ is constant when $J$ varies in $K$.

Let us compare with Proposition 2.19. Indeed, [17] contains the construction of a map $\psi$ satisfying (H4'). And (H5') follows easily, taking into account that all isotropy groups contain only $C^\infty$ elements. So we have

**Proposition 3.1.** Consider the case of complex structures. Then,

1. Properties (H2diff), (H3diff), (H4), (H5) as well as (FG) are always satisfied.
(2) Any Kuranishi type space $K$ is isomorphic (as a germ) to the Kuranishi space of $(X, J_0)$.

(3) The same $K$ is a Kuranishi type space for both smooth and Sobolev structures.

(4) Conditions (H4') and (H5') are satisfied if and only if the dimension of the space of $J$-holomorphic vector fields on $X$ is constant when $J$ varies in $K$.

Remark 3.2. To be precise, $K$ as an analytic space is not always reduced, hence does not always identifies with $K$ as an analytic set. Hence, there are slight differences between Proposition 3.1 and the results in the literature on deformations of complex structures. For example, Douady proved that isomorphism (2.3) is indeed an isomorphism of Banach $C$-analytic spaces, cf. [3]. Here, we just recover the isomorphism between the reductions of the involved spaces. Indeed, to avoid all the difficulties, one can read Proposition 3.1 replacing $K$ with its reduction.

However, it must be noticed that point (ii) of Proposition 3.1, namely the equivalence between Kuranishi space and Kuranishi type space, is shown to be an isomorphism of analytic spaces even in the non-reduced case in [16] by imposing that (2.4) is analytic.

Remark 3.3. Corollary 2.20 is nothing else in this context that the statement: if $H^0(X, J, \Theta_J)$ is zero for all $J \in K$, then the Kuranishi space is a local moduli space (also called universal). Indeed, due to the semi-continuity results of [9], it is enough to have $H^0(X, J_0, \Theta_{J_0})$ equal to zero.

3.2. Riemannian metrics. The case of riemannian metrics on a smooth compact manifold $X$ is due to Ebin [5]. It perfectly fits to this setting. Here $\mathcal{E} = \mathcal{I}$ is the set of $W^l$ riemannian metrics on $X$, encoded as the open positive convex cone of definite positive symmetric 2-tensors. This is an open set of the Hilbert space of symmetric contravariant 2-tensors. The group $G$ is the set of diffeomorphisms of class $W^{l+1}$ of $X$ acting by pullback on $\mathcal{E}$, so $T$ is just the vector space of $W^{l+1}$-vector fields on $X$. Let $g_0 \in \mathcal{E}$. The (riemannian) exponential map associated to $g_0$ can be used as map $\phi$.

By a direct computation,

$$(\xi, h) \in T \times T_{g_0} \mathcal{E} \mapsto L(\xi, h) = h + L_\xi g_0$$

where $L$ is the Lie derivative (cf. [5], Lemma 6.2). So $P$ is just the Lie derivative of $g_0$. It is elliptic by [5], Proposition 6.10, hence (H2diff) and (H3diff) are satisfied.

Hence, we may apply proposition 2.4 and obtain a local section. Also (H4) is satisfied as well as (H5) by defining $\psi$ as the exponential map associated to $g_0$. Therefore the local section of [5] is a Kuranishi type space.

Moreover, the Kuranishi type space $K$ of [5] enjoys the following property: if $f$ is a diffeomorphism such that $K \cdot f$ intersects $K$, then $f$ must be an isometry of $g_0$, [5], Theorem 7.1. This implies (FG) property, since if $c$ is a continuous path of $K$ all of whose points encode $g_0$, then all points are in fact equal to $g_0 \cdot \phi$, with $\phi$ an isometry of $g_0$. Hence the path $c$ is constant.

Finally, it is proven in [5] that if the isotropy group of $g_0$ is the identity, it is
still the identity for \( g \) close to \( g_0 \). So in this case, \((H4')\) and \((H5')\) are satisfied and we obtain a local moduli space (this is indeed a direct application of Corollary 2.20). Last but not least, it is proven in [5], Theorem 7.4, that the result are still valid for \( C^\infty \) metrics by taking as \( K \) the subset of \( C^\infty \) points of \( K^l \). Indeed, Ebin shows the existence of a smooth invariant riemannian metric on \( \mathcal{E} \), hence a smooth invariant affine connection. The result follows now from Proposition 2.33. Hypothesis \((H6)\) is proved in [5], Proposition 6.13. To sum up,

**Proposition 3.4.** Consider the case of riemannian metrics. Then,

1. Properties \((H2\text{diff}), (H3\text{diff}), (H4), (H5)\) are always satisfied, so given a riemannian metric \( g_0 \), for all \( l \), it has a Kuranishi type space \( K^l \) given as a neighborhood of 0 in the kernel of \( P^* \).
2. \((FG)\) property is fulfilled so \( K^l \) is a local moduli section.
3. If the isotropy group of \( g_0 \) is the identity (which is the case on an open and dense subset of \( \mathcal{E} \)), then conditions \((H4')\) and \((H5')\) are satisfied and \( K^l \) is a local moduli space.
4. Properties \((H6), (H7)\) and \((H8)\) are satisfied. Hence, defining \( K^\infty \) as the subset of \( C^\infty \) points of \( K^l \), then \(1), (2)\) and \(3)\) are still valid for \( C^\infty \) metrics.

**Remark 3.5.** In [5], the author constructs two riemannian metrics on \( \mathcal{E}^l \), the strong one and the weak one. Here, to run Proposition 2.33 we use the strong riemannian metric, which depends on a particular choice of \( l \). Ebin prefers using the weak one (weak in the sense that it induces on each tangent space to \( \mathcal{E}^l \) the \( W \) topology and not the required \( W^l \) topology), because it has the advantage of being independent of \( l \). However, with such a weak metric, the existence of the affine connection and the exponential map, as well as the fact that (2.9) is a smooth bundle is not immediate.

4. **Deforming sasakian manifolds**

4.1. **Background.** We start with some classical facts about sasakian manifolds, see [1] and [22] for more details.

**Definition 4.1.** A compact smooth riemannian manifold \((S, g)\) is called a *sasakian manifold* if the cone \( S \times \mathbb{R}^{>0} \) admits a complex structure which is kähler for the metric \( r^2g + dr \otimes dr \) (where \( r \) is the coordinate on \( \mathbb{R}^{>0} \)).

A sasakian manifold comes equipped with many structures. Identifying \( S \) with the hypersurface \( S \times \{1\} \) of its cone and denoting by \( J \) the complex operator on the cone, we have:

- The unit vector field
  \[ \xi := J \left( r \frac{\partial}{\partial r} \right) \]
  is tangent to \( S \), acts by isometries on \((S, g)\), and is called the *Reeb vector field*.
- The contact form
  \[ \eta := J \left( \frac{dr}{r} \right) \]
is tangent to $S$ and satisfies
\begin{equation}
    i_\xi \eta \equiv 1 \quad \text{and} \quad i_\xi d\eta \equiv 0.
\end{equation}

- The operator defined by
\begin{equation}
    \Phi(\xi) := 0 \quad \text{and} \quad \Phi(V) := JV \quad \text{on Ker } \eta
\end{equation}
is an endomorphism of $TS$ which induces an integrable CR operator on $D := \text{Ker } \eta$.

A sasakian manifold enjoys the following properties.

- $\xi$ acts by CR isomorphisms, i.e. its flow preserves $D$ and $J$.
- The foliation $\mathcal{F}$ induced by $\xi$ on $S$ is transversely kähler, with holomorphic normal bundle identified with $D$, and transverse kähler form $\omega := d\eta$.
- The CR structure $(D, J)$ is strictly pseudo-convex with Levi form equal to $\omega$.

We denote by $(S, g, \xi, \eta, \Phi)$ a sasakian manifold. Note that
\begin{equation}
    J \equiv \Phi + \frac{1}{r} dr \otimes \xi - \eta \otimes \frac{\partial}{\partial r}.
\end{equation}
and, for $V$ tangent to $S$,
\begin{equation}
    \Phi(V) = J(V - \eta(V)\xi).
\end{equation}

All these datas are not independent, for example fixing $\eta$ and $g$ gives a unique $\xi$ through (4.1), and a unique $\Phi$ through
\begin{equation}
    \Phi(V) = \nabla_\xi V
\end{equation}
where $\nabla$ is the Levi-Civita connection of $g$. Moreover, $g$ and $\eta$ are related one to the other through the equations
\begin{equation}
    \eta(V) = g(\xi, V) \quad \text{and} \quad g(V, W) = \frac{1}{2} d\eta(V, \Phi(W)) + \eta(V) \eta(W).
\end{equation}

However for our deformation purposes, it is important to keep track of these four structures, as well as the associated structures $\mathcal{F}$ and $(D, J)$ on $S$. Indeed, when deforming sasakian structures, one has to be very careful and precise about which structure(s) is (are) fixed, and which is (are) deformed; and one has to decide if we only consider deformations which are still sasakian or allow general deformations (in some problems there is no difference but not in all). Unfortunately, this is not always the case in the existing litterature.

We first focus on the sasakian deformations of the contact form $\eta$, keeping the transverse holomorphic structure of $\mathcal{F}$ fixed. Recall that the normal bundle to $\mathcal{F}$ is the quotient bundle
\begin{equation}
    N\mathcal{F} := TS/\mathcal{T}\mathcal{F}
\end{equation}
and that the transverse holomorphic structure of $\mathcal{F}$ is determined by a splitting of the complexified normal bundle to the foliation
\begin{equation}
    N_{\mathbb{C}}\mathcal{F} := N\mathcal{F} \otimes \mathbb{C} = N^{0,1}\mathcal{F} \oplus N^{1,0}\mathcal{F}
\end{equation}
to $(1, 0)$ and $(0, 1)$ vectors. The subbundles $N^{0,1}\mathcal{F}$ and $N^{1,0}\mathcal{F}$ are complex conjugate and involutive (for the quotient Lie bracket). Fixing the transverse holomorphic structure means keeping $N^{0,1}\mathcal{F}$ fixed.
Equivalently, looking at the natural projection map
\[(4.6)\quad \pi : T_C S \to N_C F \to N^{1,0} F\]
and setting
\[(4.7)\quad E = \text{Ker} \pi\]
the transverse holomorphic structure is given by the involutive subbundle $E$ of $T_C S$, so fixing the transverse holomorphic structure means fixing $E$ (cf. [19]). In the sasakian case, observe that
\[(4.8)\quad E = D^{0,1} \oplus C \xi\]
where $D^{0,1}$ is the subbundle of $(0,1)$-vectors of the complexification of the CR distribution $(D, J)$. In particular, given $E$, we have
\[D^{0,1} = E \cap D_C\]
so $J$ is uniquely determined on $D_C$ and thus on $D$: it acts as multiplication by $-i$ on $D^{0,1}$ and as multiplication by $+i$ on its complex conjugate.

The reason for dealing with this problem is that this is perhaps the simplest case where infinite-dimensionality occurs, so that the classical theory of deformations does not apply.

Observe that $E$ being fixed, $\xi$ is only changed by a multiplicative factor. As a variant to this deformation problem, one can deform $\eta$ keeping $E$ and $\xi$ fixed. The resulting Kuranishi type space will be essentially the same (see remark 4.17).

**Remark 4.2.** If $\Phi$ is fixed, then so is $E$, simply because
\[E = \text{Im} (Id + i\Phi)\]
However, the converse is false. Indeed, $\Phi$ determines also $D^{0,1}$ as the kernel of $\Phi + iId$ (and thus $D$), whereas $E$ does not. In other words, fixing $E$ means fixing $N^{0,1} F$, whereas fixing $\Phi$ means fixing $D^{0,1}$, which is a precise realization of $N^{0,1} F$ as a subbundle of $T_C S$. Deformations of $S$ with $\Phi$ fixed correspond to deformations of the induced polarized CR structure as defined and studied in [15].

Note that we cannot fix more structures, since we have

**Lemma 4.3.** Let $(S, g, \eta, \xi, \Phi)$ and $(S, g', \eta', \xi', \Phi')$ be two sasakian structures on the same manifold $S$.

1. If $g = g'$ and $\xi' = \xi$, then $\eta' = \eta$ and $\Phi' = \Phi$ so both structures are the same.
2. If $\xi' = \xi$ and $\Phi' = \Phi$, then $\eta' = \eta$ and $g' = g$ so both structures are the same.

**Proof.** If $g = g'$ and $\xi' = \xi$, then $\eta'$ is equal to $\eta$ because of (4.3) and $\Phi'$ is equal to $\Phi$ because of (4.4).

And if we let $\Phi$ fixed, as observed in remark 4.2, then $D$ is fixed. Since $\xi$ is also fixed, then $\eta$ is fixed, since it is zero on $D$, and 1 on $\xi$. Finally $g$ is fixed because of (4.5). \[\square\]
4.2. Encoding the structures. We first need the following characterization of sasakian manifolds.

**Proposition 4.4.** Let $(S, \eta, J)$ be a triple: compact smooth manifold, contact form, integrable CR-structure on the kernel $D$ of $\eta$. Define $\xi$ using \ref{4.1}. Assume that

1. $L_\xi J \equiv 0.$
2. $d\eta(V, JY) > 0$ for all non zero $V$ tangent to $D$.

Then, defining $\Phi$ by \ref{4.2} and $g$ by \ref{4.5}, the manifold $(S, g, \eta, \xi, \Phi)$ is sasakian.

**Proof.** We will first show that $g$ (defined through \ref{4.5}) is a riemannian metric. The integrability of the CR-structure implies that, for all vector fields $X$ and $Y$ tangent to $D$, we have

\[ [X + iJX, Y + iJY] = Z + iJZ \tag{4.9} \]

for some $Z$ tangent to $D$. It follows from \ref{4.9} that

\[
\begin{cases}
  [JX, Y] + [X, JY] \in \Gamma(D) \\
  [X, Y] - [JX, JY] \in \Gamma(D)
\end{cases}
\tag{4.10}
\]

Using the fact that $D$ is the kernel of $\eta$ and applying $\eta$ to \ref{4.10}, we obtain

\[ d\eta(X, Y) = d\eta(JX, JY) \tag{4.11} \]
and

\[ d\eta(JX, Y) + d\eta(X, JY) = 0. \tag{4.12} \]

Now, \ref{4.11} means that $d\eta$ is a $(1,1)$-form, and \ref{4.12} means that $g$ is symmetric. By (iii), this is enough to prove that $g$ is a riemannian metric.

We consider now the riemannian cone $(S \times \mathbb{R}^+, \bar{g} = r^2 g + dr \otimes dr)$. We extend $J$ to the cone by setting

\[ J_r \frac{\partial}{\partial r} = \xi \quad \text{and} \quad J_\xi = -r \frac{\partial}{\partial r} \tag{4.13} \]
and we set

\[ \omega := \bar{g}(J-,-). \tag{4.14} \]

We will now show that $J$ defines a complex structure on the cone. Indeed, the bundle of $(1,0)$-vectors, say $Q^{1,0}$, satisfies

\[ Q^{1,0} = D^{1,0} \oplus \mathbb{C}(\xi + ir \frac{\partial}{\partial r}). \]

But, for $X$ tangent to $D$, we have

\[ [X - iJX, \xi + ir \frac{\partial}{\partial r}] = -L_\xi X + iL_\xi (JX) = -L_\xi X + iJL_\xi X \]
because of condition (ii). So it belongs to $D^{1,0}$. Since this bundle is involutive, this proves the involutivity of $Q^{1,0}$.

Our last step is to prove that $d\omega$ is zero. Since $\omega$ is the $(1,1)$-form associated to the $J$-invariant riemannian metric $\bar{g}$, this shows that $\bar{g}$ is a kählerian metric, so, by definition \ref{1.1} we are done.
We first claim that

\[(4.15) \quad i_\xi d\omega = 0.\]

This can be proven as follows. Take \(X\) and \(Y\) local vector fields tangent to \(D\) and commuting with \(\xi\). Using the relations

\[L_\xi \eta = L_\xi d\eta = L_\xi X = L_\xi Y = L_\xi J = 0\]

we deduce that

\[(4.16) \quad (L_\xi g)(X,Y) = L_\xi(g(X,Y)) = L_\xi(1/2d\eta(X,JY)) = 0.\]

Similar computations replacing \((X,Y)\) with \((X,\xi)\) and then \((\xi,\xi)\) show that

\[(4.17) \quad L_\xi g = L_\xi \bar{g} = r^2 L_\xi g + L_\xi (dr \otimes dr) = 0\]

and from \((4.17)\) that

\[(4.18) \quad L_\xi \omega = 0.\]

Moreover,

\[i_\xi \omega(X) = \bar{g}(-r \frac{\partial}{\partial r}, X) = 0\]

and

\[i_\xi \omega(r \frac{\partial}{\partial r}) = g(-r \frac{\partial}{\partial r}, r \frac{\partial}{\partial r}) = r^2\]

yielding

\[(4.19) \quad i_\xi \omega = -r dr.\]

Combining \((4.18)\) and \((4.19)\) gives \((4.15)\).

We are now in position to show that \(d\omega\) is zero and thus to finish with the proof. Choose local coordinates and local vector fields \(Y_i\) tangent to \(D\) which satisfy

\[(4.20) \quad [Y_i, Y_j] = [Y_i, r \frac{\partial}{\partial r}] = [Y_i, \xi] = 0.\]

This gives the relation

\[d\omega(Y_i, Y_j, Y_k) = Y_i \cdot \omega(Y_j, Y_k) - Y_j \cdot \omega(Y_i, Y_k) + Y_k \cdot \omega(Y_i, Y_j) + r^2 Y_i \cdot (1/2d\eta(Y_j, Y_k)) - r^2 Y_j \cdot (1/2d\eta(Y_i, Y_k)) + r^2 Y_k \cdot (1/2d\eta(Y_i, Y_j))\]

hence to

\[r^2/2(d\eta([Y_i, Y_j], Y_k) - d\eta([Y_j, Y_k], Y_i) + d\eta([Y_k, Y_i], Y_j)) = 0\]

since \(D\) and \(\xi\) are \(\bar{g}\)-orthogonal and since, by \((4.20)\), all Lie brackets are zero. But this implies that

\[d\omega(r \frac{\partial}{\partial r}, Y_i, Y_j) = r^2 d\eta(JY_i, JY_j) = r^2 d\eta(Y_i, Y_j) = -r^2 \eta([Y_i, Y_j]) = 0\]

so \(d\omega\) is zero and we are done.

\[\square\]

**Corollary 4.5.** Let \(S\) be a compact smooth manifold. Then, a sasakian structure on \(S\) determines uniquely and is uniquely determined by the data of a subbundle \(E\) of the complexified tangent bundle and a contact form \(\eta\) satisfying
(1) $E + \bar{E} = T_C S$.
(2) $E \cap \bar{E} = \mathbb{C}\xi$.
(3) $[E, E] \subset E$.
(4) The form $\eta$ satisfies $\eta|_E = 0$.
(5) For all non-zero vector of $E \cap D_C$, one has $\eta(V, i\bar{V}) + \eta(\bar{V}, -iV) > 0$.

where $\xi$ is defined through (4.1) and $D$ is the kernel of $\eta$.

Remark 4.6. In the proof of corollary 4.5, the sasakian structure will be made explicit from $E$ and $\eta$.

Proof. Let $(g, \eta, \xi, \Phi)$ be a sasakian structure on $S$. Define $E$ through (4.8). Then the conditions above are easily verified.

Conversely, let $(E, \eta)$ be as above. By a result of [20], the first three conditions imply that the foliation $F$ induced by $\xi$ is transversely holomorphic. Set

$$D^{0,1} := D_C \cap E.$$  

We have

$$T_C S = D_C \oplus \mathbb{C}\xi \implies E = D^{0,1} \oplus \mathbb{C}\xi.$$  

The fourth and fifth conditions can now be rewritten as: $d\eta$ is a basic $(1,1)$-form and $d\eta(V, J\bar{V}) > 0$ for all non-zero $V$ tangent to $D$. Moreover, for $V$ and $W$ tangent to $D^{0,1}$,

$$\eta([V, W]) = d\eta(V, W) = 0$$

since $d\eta$ is $(1,1)$. So $[V, W]$ belongs to $D_C$, hence to $D^{0,1}$, which is thus involutive, proving the integrability of $(D, J)$.

Notice that $V$ tangent to $D$ implies that $[\xi, D]$ is tangent to $D$, since

$$\eta([\xi, V]) = d\eta(\xi, V) = 0.$$  

Finally, the transverse holomorphic structure is by definition invariant by holonomy, hence we have

$$[\xi, J\bar{V}] - J[\xi, V] \in \mathbb{C}\xi$$

Since (4.21) is tangent to $D$ by the previous remark, it must be zero. This can be rephrased as: $\mathcal{L}_\xi J$ is zero. Applying proposition 4.4 yields the result.  

We are now in position to give a good encoding of the sasakian deformations with the transverse holomorphic structure fixed.

Corollary 4.7. Let $(S, g, \xi, \eta, \Phi)$ be a sasakian manifold. Define $E$ as in (4.8). Then, there exists a neighborhood $U_0$ of 0 in the space of real 1-forms $\eta$ such that, for all $\alpha \in U_0$, the following conditions are equivalent

(1) The triple $(S, E, \eta + \alpha)$ is a sasakian manifold.
(2) We have $d\alpha|_E = 0$.

Remark 4.8. Of course, to make Corollary 4.7 precise, one has to fix the same regularity on the space of 1-forms and basic 1-forms: $C^\infty$ or $W^1$, etc...
Proof. Choose $U_0$ so that, for all $\alpha \in U_0$, we have that $\eta + \alpha$ is a contact form and the fifth condition of corollary 4.5 is fulfilled. This is possible since both properties are open.

Assume that $(S,E,\eta')$ is sasakian, with $\eta' - \eta$ in $U_0$. Define $\alpha := \eta' - \eta$. Let $\xi'$ be the Reeb vector field associated to $\eta'$ through (4.1). Since $E$ is kept fixed, it follows from Corollary 4.5 that $d\alpha |_E$ is zero. Conversely, let $\alpha$ belong to $U_0$ and satisfy (2). Then, for $V = \lambda \xi + W + \bar{W}$ a real vector field (here with $W$ tangent to $D^{1,0}$), we have

$$i_\xi d\eta(V) = i_\xi d\alpha(V) = d\alpha(\xi, W) + d\alpha(\xi, \bar{W}) = 0$$

because of (2) and because $E \cap E = C \xi$. Hence, $\xi'$ is a multiple of $\xi$. Applying corollary 4.5 gives the result. □

We are now in position to prove a existence of a Kuranishi type space for the deformations of $\eta$ with the transverse structure of $F$ fixed.

4.3. Deformations of the contact form of a sasakian structure. In this subsection, we construct a Kuranishi type space for the deformations of the contact form of a sasakian manifold. Let $(S,E,\eta)$ be a sasakian manifold. Fix $l > \dim S + 6$ and consider only 1-forms of class $W^l$. We let $T^*S$ be the cotangent bundle of $S$. The notation $\Gamma^l(-)$ stands for the vector space of $W^l$ sections of the corresponding bundle. Let

$$(4.22) \quad \mathcal{E} = \mathcal{I} = \{ \alpha \in \Gamma^l(T^*S) \mid d\alpha |_E = 0 \}.$$ 

Let $G = G^l$ be the connected component of the identity of the topological group of diffeomorphisms $f$ of $S$ of class $W^{l+1}$ such that $f^*E \equiv E$. Let $\chi^\infty_F$ denote the Lie subalgebra of $\Gamma^\infty(TS)$ of those vector fields that define infinitesimal automorphisms of the transversely holomorphic foliation $F$. In foliated local coordinates $(z, \bar{z}, t)$, a vector field $v \in \Gamma^\infty(TS)$ is written

$$v = a \xi + \sum b^i \frac{\partial}{\partial z^i} + \sum \bar{b}^i \frac{\partial}{\partial \bar{z}^i}$$

where $a$ is a real function and $b^i$ are complex-valued. The vector field $v$ belongs to $\chi^\infty_F$ if and only if $b^i = b^i(z)$ are holomorphic functions depending only on the transverse coordinates $z$. We denote by $\chi^\infty_F$ the $W^l$-completion of $\chi^\infty_F$. We consider the following exact sequence of complex vector bundles over $S$

$$0 \longrightarrow (Q^{1,0})^* \longrightarrow T^*_C S \longrightarrow E^* \longrightarrow 0$$

and we denote $A^\perp = E^* \otimes Q^{1,0}$. Sections of $A^\perp$ can be viewed as classes of vector-valued 1-forms $\beta$ that, in local foliated coordinates, are written

$$\beta = \sum_k [\gamma^k dt + \sum_i \delta^k_i d\bar{z}^i] \otimes \left[ \frac{\partial}{\partial z^k} \right]$$
where \( \gamma^k, \delta^k \) are complex-valued functions. For a vector field \( v \), locally given as in (4.23), we define

\[
d_Fv = \sum_k \left[ \frac{\partial b^k}{\partial t} dt + \frac{\partial b^k}{\partial \bar{z}^i} d\bar{z}^i \right] \otimes \left[ \frac{\partial}{\partial \bar{z}^k} \right].
\]

A straightforward computation shows that \( d_Fv \) is a well-defined section of the vector bundle \( A^1_F \). Therefore the map

\[
d_F: \Gamma^\infty(\mathcal{I}) \to \Gamma^\infty(A^1_F)
\]

is a differential operator of degree 1. In [4], Duchamp and Kalka extend the map \( d_F \) to vector-forms of any degree and show that the sequence

\[
\Gamma^\infty(Q^{1,0}) \overset{d_F}{\to} \Gamma^\infty(A^1_F) \overset{d_F}{\to} \Gamma^\infty(A^2_F) \to \ldots
\]

is an elliptic complex. Since the image by \( d_F \) of \( \Gamma^\infty(\mathcal{I}) \) and of \( \Gamma^\infty(Q^{1,0}) \) is the same, we deduce from the ellipticity of the above complex that the image \( d_F(\Gamma^\infty(\mathcal{I})) \) of the map

\[
d_F: \Gamma^l(\mathcal{T}) \to \Gamma^{l-1}(A^1_F)
\]

is closed in \( \Gamma^{l-1}(A^1_F) \). Notice also that the kernel of that map is \( \chi^l_F \). Thus we are in position to apply Theorem 2 and Corollary 1 in [21]. These results imply in particular that \( G^l \) is a Hilbert manifold in such a way that a neighborhood \( W \) of \( e \) in \( G^l \) is identified to a neighborhood \( V \) of 0 in \( \chi^l_F \). This identification \( \phi: V \to W \) provides the local chart used in (2.1). □

Remark 4.10. In this subsection we are assuming the stronger condition \( l > \dim S + 6 > \dim S/2 + 1 \) in order to apply Omori’s theorem. Notice however that the result of Omori that we are using here is not sufficient to assure that the subgroup \( G^\infty \) of \( C^\infty \) elements of \( G \) is a Fréchet Lie group.

We use the notations of section 2.1. In particular, \( V \) (respectively \( U \)) denotes a neighborhood of 0 in \( T = \mathcal{X} \) (respectively \( \mathcal{E} \)). Action (2.2) is given by

\[
(\chi, \alpha) \in V \times U \mapsto \alpha \cdot \phi(\chi) = \phi(\chi)^*(\eta + \alpha) - \eta \in \mathcal{E}.
\]

Looking at the differential at \((0, \eta)\),

**Lemma 4.11.** We have

\[
L(\chi, \alpha) = \alpha + P_\chi = \alpha + \mathcal{L}_\chi \eta.
\]

**Proof.** By definition, we have

\[
L(\chi, \alpha) = \frac{d}{ds} \bigg|_{s=0} (\phi(s\chi)^s(\eta + s\alpha) - \eta).
\]

Observe now that, in a local chart, for \( s \) sufficiently small, we have

\[
\phi(s\chi) = Id + s\chi + \epsilon(s)
\]

Now, (4.20) is exactly the Taylor development in \( s \) of the flow of \( \chi \) at \( s \). Hence, up to order 1, \( \phi(s\chi) \) coincide with the flow \( \phi^s \) of \( \chi \) at time \( s \). As a consequence, we immediately deduce from (4.25) that

\[
L(\chi, \alpha) = \alpha + \frac{d}{ds} \bigg|_{s=0} ((\phi^s)^* \eta)
\]
and thus
\[ L(\chi, \alpha) = \alpha + P\chi = \alpha + L\chi\eta. \]

Finally,

**Lemma 4.12.** The image of \( P \) is closed in \( \mathcal{E} \).

**Proof.** Let \( \chi \in V \). Set \( \chi = h\xi + \chi_N \).

We have
\[ L\chi\eta = d(ih\xi\eta) + hi\xi d\eta + d(i\chi_N \eta) + i\chi_N d\eta. \]

Now, the third term is zero because the kernel of \( \eta \) is generated by the vector fields \( \chi_N \). Using (4.1), it follows that
\[ L\chi\eta = dh + i\chi_N d\eta. \]

But this formula shows that the image of \( P \) is the sum of the image of the de Rham differential applied to the set of functions and of the image of a finite-dimensional vector space under a bounded linear operator. Hence it is the sum of a closed subspace and of a finite-dimensional one. So it is closed.

Hence, hypotheses (H1), (H2), (H3), (H4) and (H5) are satisfied. Observe that the isotropy group of \( \eta \) is the automorphism group of the sasakian manifold \((S, E, \eta)\). Hence it is finite-dimensional and we can use the time 1 flow as chart \( \psi \) fulfilling (H4). Define
\[ (4.27) K^l_\eta := \{ \alpha \in U_0 \mid P^*\alpha = d\alpha |_E = 0 \} \]
we now have, using Theorem 2.21

**Theorem 4.13.** The space \( K^l_\eta \) defined in (4.27) is an open neighborhood of 0 in a infinite-dimensional Hilbert space and is a Kuranishi type space for \( \eta \).

**Proof.** The equations in (4.27) are all linear and continuous, hence \( K^l_\eta \) is an open neighborhood of 0 in a Hilbert space. Besides, it contains all the basic 1-forms whose differential is \((1, 1)\), and in particular all the \( \partial\bar{\partial} f \) for \( f \) a basic function. Finally, the Reeb flow of a sasakian manifold has no dense orbit, cf. [1] or [22]. Hence (4.27) is infinite-dimensional.

**Lemma 4.14.** Hypotheses (H6), (H7) and (H8) are satisfied.

**Proof.** Assume that \( f^*\eta - \eta \) and \( \eta \) are \( C^\infty \), with \( f \) of class \( W^l \) preserving \( E \). We want to prove that \( f \) is indeed \( C^\infty \).

From \( f^*\eta \) of class \( C^\infty \) and (4.1), we deduce that \( f_*\xi \) is also \( C^\infty \). Moreover, since
\[ (4.28) f^* D^{0,1} = f^* E \cap \text{Ker } (f^* \eta) = E \cap \text{Ker } (f^* \eta) \]
we have that \( f \cdot \Phi \) is \( C^\infty \). Finally (4.23) implies that \( f^* g \) is \( C^\infty \). In particular, \( f \) sends geodesics onto geodesics, a property that classically implies that \( f \) is \( C^\infty \).

Let us focus now on (H7). We construct an invariant riemannian metric on
\( \mathcal{E} \). Each structure \( \alpha \in \mathcal{E} \) encodes a unique sasakian metric \( g_\alpha \) on \( S \) through (4.5). This induces a unique riemannian metric on the cotangent bundle of \( S \), still denoted by \( g_\alpha \). By integrating over \( S \), we obtain a scalar product \( h_\alpha \) on the space of 1-forms on \( S \). The collection \( (h_\alpha) \) is a riemannian metric on \( \mathcal{E} \). It is obviously invariant under the action of the diffeomorphism group. To show it is smooth, we proceed as follows. Given \( \alpha \in \mathcal{E} \), we define \( D_\alpha \) as the kernel of \( \eta + \alpha \), then \( D_0^{1,1} \alpha \) as the intersection of the complexification of \( D_\alpha \) with \( E \). This allows us to define \( \Phi_\alpha \) and finally \( g_\alpha \) through (4.5).

In this process, observe that

(i): To know \( g_\alpha \) in a point \( x \in M \), it is enough to know \( \alpha(x) \) and \( d\alpha(x) \).

(ii): If \( \alpha \) varies smoothly, then so does \( g_\alpha \).

In other words, the map \( \alpha \mapsto g_\alpha \) is a map from the \( W^1 \) sections of the bundle of 1-jets of differential forms of degree 1 into the \( W^1 \) sections of the bundle of symmetric 2-tensors, which comes from a smooth vector bundle map. This is enough to show that it is smooth. It is then easy to see that \( \alpha \mapsto h_\alpha \) is smooth, cf. [5, p.18-19]. This gives a weak invariant metric. To get a strong one, one simply has to play the same game to obtain a weak metric on the bundle of \( l \)-jets of sections of \( E \), see [5, p.20]. To be more precise, the weak invariant metric induces a weak invariant metric on each associated tensor bundle, and thus induces a weak riemannian metric on the bundle of \( l \)-jets of sections of \( E \). But this is equivalent to endowing the bundle of \( W^l \)-sections of \( E \) with a strong riemannian metric.

Finally Hypothesis (H8) is immediate. \( \square \)

As an application of Lemma 4.14 and Proposition 2.33, we thus have

**Corollary 4.15.** Let \( K^\infty_\eta \) be the subset of \( C^\infty \) points of \( K^l_\eta \). Then it is a Kuranishi type space for \( C^\infty \) structures.

We can give a more precise description of (4.27) by computing \( P^* \). We can rewrite \( P \) as

\[
(4.29) \quad (h, \chi) \in \Gamma^{l+1}(\mathbb{R}) \times \mathcal{X}_N \mapsto dh + i_{\chi_N} d\eta \in \mathcal{E}.
\]

As usual, we let \( g \) denote the sasakian metric of the base structure. In what follows, we extend \( g \) to the 1-forms and all the tensor fields. Then we use the \( L^2 \) product associated to \( g \) on the tensor fields. In particular, on \( \Gamma^{l+1}(\mathbb{R}) \times \mathcal{X}_N \), we use the sum of the \( L^2 \) product on the functions and that on the vector fields. We denote this sum as well as all the \( L^2 \) products by the same symbol \( \langle - , - \rangle \).

Going back to (4.29), observe that the sum in the right expression is not a direct sum. But, defining the closed vector subspace

\[
\mathcal{X}'_N := (P|_{[0] \times \mathcal{X}_N})^{-1} (P|_{[0]}(\Gamma^{l+1}(\mathbb{R}) \times \{0\}))^\perp
\]

then (4.29) becomes

\[
(4.30) \quad (h, \chi) \in \Gamma^{l}(\mathbb{R}) \times \mathcal{X}'_N \mapsto dh + i_{\chi_N} d\eta \in \mathcal{E}.
\]

Define now

\[
(4.31) \quad \beta \in \Gamma^{l}(T^* S) \mapsto \beta^l \in \Gamma^{l}(TS)
\]
by
(4.32) \[ g(\beta^t, -) = \beta. \]
Observe that, with this convention,
(4.33) \[ g(\alpha, \beta) = g(\alpha^t, \beta^t) = \alpha(\beta^t) = \beta(\alpha^t). \]
We have

Lemma 4.16. The adjoint of (4.30) is given by the formula
\[ P^*\alpha = (d^*\alpha, -(i_{\alpha^t}d\eta)^\sharp) \]
for \( \alpha \in E \) and \( d^* \) the codifferential on 1-forms.

Proof. Just compute
\[ \langle P^*\alpha, (h, \chi) \rangle = \langle d^*\alpha, h \rangle + \int_S d\eta(\alpha^t, \chi) \text{vol}_g \]
because of (4.32) and of (4.33). But this is exactly
\[ \langle \alpha, dh \rangle - \int_S i_\chi d\eta(\alpha^t, \chi) \text{vol}_g = \langle \alpha, dh \rangle - \langle \alpha, i_\chi \eta \rangle \]
finishing the proof. \( \Box \)

Remark 4.17. Let us treat rapidly the associated case where the contact form is deformed, keeping \( E \) and \( \xi \) fixed. The following statement is analogous to Corollary 4.7 and is easy to prove.

Corollary 4.18. Let \((S, g, \xi, \eta, \Phi)\) be a sasakian manifold. Define \( E \) as in (4.7). Then, there exists a neighborhood \( U_0 \) of 0 in the space of 1-forms on \( S \) such that, for all \( \alpha \in U_0 \), the following conditions are equivalent
(1) The triple \((S, E, \eta + \alpha)\) is a sasakian manifold with Reeb vector field \( \xi \).
(2) The 1-form \( \alpha \) is basic and its differential is \((1, 1)\), that is satisfies
\[ i_{\xi} \alpha = d\alpha|_E = 0. \]

Let \( G \) be the topological group of diffeomorphisms \( f \) of \( S \) of class \( W^{l+1} \) such that
\[ f_*\xi = \xi \quad \text{and} \quad f_*E = E. \]
With this new statement and this new group on mind, one obtains easily the

Corollary 4.19. The infinite-dimensional analytic set
\[ (K')_\eta^l := \{ \alpha \in U_0 \mid P^*\alpha = i_{\xi} \alpha = d\alpha|_E = 0 \} \]
is a Kuranishi-type space for \( \eta \).

Here, the operator \( P \) is the same as that appearing in Lemma 4.11, but restricted to the subspace \( X_b \) of vector fields of \( X \) whose \( \xi \)-coordinate is basic. This implies that \( P^* \) is slightly different from that of (4.27). Indeed it is the composition of this latter operator with the projection onto \( X_b \).

Also, using the same arguments as above,

Corollary 4.20. Let \((K')_\eta^{\infty} \) be the subset of \( C^\infty \) points of \((K')_\eta^l \). Then it is a Kuranishi type space for \( C^\infty \) structures.
Finally, we have

**Proposition 4.21.** Assume that the Lie algebra $\mathcal{X}_N$ is zero. Then $K^l_\eta$ and $K^{\infty}_\eta$ (respectively $(K')^l_\eta$ and $(K')^{\infty}_\eta$) are local moduli spaces.

**Remark 4.22.** By [4], this is equivalent to saying that the group of holomorphic basic infinitesimal automorphisms $H^0(S, \Theta)$ is zero.

**Proof.** The automorphism group of $(E, \eta)$ consists of diffeomorphisms fixing $E$ and $\eta$. It is a finite dimensional Lie group whose Lie algebra consists of vector fields

$$\chi = h\xi \oplus \chi_N$$

such that $[\chi, E] \subset E$, $\mathcal{L}_\chi \eta = 0$.

The $E$-preservation implies that $\chi_N$ belongs to $\mathcal{X}_N$.

If $\mathcal{X}_N$ is zero, then $\chi$ is a multiple of $\xi$ and, since $\mathcal{L}_\chi \eta = dh$, it is a constant multiple. Hence this Lie algebra is reduced to constant multiples of $\xi$ and is one-dimensional.

Consider firstly the case of $E$ and $\xi$ fixed. For any other structure $(E, \eta')$, the Lie algebra of infinitesimal automorphisms is still equal to $\mathbb{C} \cdot \xi$. Hence, defining $\psi(\eta', \lambda \xi)$ as the time 1 flow of the vector field $\lambda \xi$, we immediately have (H4') and (H5') fulfilled. The conclusion follows from Theorem 2.21.

Consider secondly the case of $E$ fixed. For any other structure $(E, \eta')$, since we keep the same $E$, the Lie algebra of infinitesimal automorphisms it is still one-dimensional and generated by the constant multiples of the corresponding Reeb vector field $\xi'$. Observe that the mapping $\eta' \mapsto \xi'$ is smooth. Hence, defining $\psi(\eta', \lambda \xi)$ as the time 1 flow of the vector field $\lambda \xi'$, it is easy to check that (H4'), and (H5') are satisfied. So, once again, we may apply Theorem 2.21. □

### 4.4. General deformations.

We now deal with the case of general deformations of sasakian manifolds. Using corollary 4.5, this means deforming both $E$ and $\eta$.

Let $S$ be a compact smooth manifold of dimension $2n + 1$. Let $\mathcal{G}$ be the grassmannian bundle of complex $(n + 1)$-planes of $T_C S$. As usual, we fix some positive $l$ and consider sections of class $W^l$ of the bundles.

Set

$$\mathcal{E} = \{(E, \eta) \in \Gamma^l(\mathcal{G}) \times \Gamma^l(T^* S) \mid E + \bar{E} = T_C S, \eta \text{ positive contact}\}$$

Here by $\eta$ positive contact, we mean that $\eta$ is a contact form satisfying

$$(4.34)\quad d\eta(V, i\bar{V}) + d\eta(\bar{V}, -iV) > 0$$

for all non-zero vector of $E \cap D_C$.

Observe that $\mathcal{E}$ is an open subset of the Hilbert space $\Gamma^l(\mathcal{G}) \times \Gamma^l(T^* S)$. Set now

$$\mathcal{I} = \{(E, \eta) \in \mathcal{E} \mid [E, E] \subset E, \quad d\eta|_E \equiv 0\}$$

By corollary 4.5, the closed set $\mathcal{I}$ of $\mathcal{E}$ is exactly the set of sasakian structures of class $W^l$ on $S$. 


Let \( (E, \eta) \in \mathcal{I} \). A local chart for \( E \) at \( (E, \eta) \) is given by
\[
(\omega, \alpha) \in \Gamma^1((E^* \otimes D^{1,0}) \oplus T^*S) \to ((\text{Id} - \omega)E, \eta + \alpha) \in \mathcal{E}
\]
Recall that
\[
T_{\mathcal{C}}S = D^{1,0} \oplus E = D^{1,0} \oplus D^{0,1} \oplus \mathbb{C}\xi.
\]
These three subbundles are involutive and correspond to foliated coordinates \((z, \bar{z}, t)\). In local foliated coordinates, we may thus decompose the de Rham differential as
\[
d = \partial + \bar{\partial} + \partial_t.
\]
One can show that the operator \( \bar{\partial} + \partial_t \) is indeed globally defined, whereas \( \bar{\partial} \) and \( \partial_t \) are not, cf. [4].

For \( \chi \) a smooth vector field, using the natural injection of \( TS \) into \( T_{\mathcal{C}}S \), we decompose it accordingly into
\[
\chi = \chi^{1,0} \oplus \chi^E = \chi^{1,0} \oplus \chi^{0,1} \oplus \chi^\xi
\]
where \( \chi^{0,1} = \overline{\chi^{1,0}} \) and \( \chi^\xi \) is real.

The bundle \( D^{1,0} \) is isomorphic to \( N^{1,0}\mathcal{F} \) through the map (4.6), hence, in local foliated coordinates \((z, \bar{z}, t)\), is locally generated by the vector fields
\[
e_i = \frac{\partial}{\partial z_i} + a_i \xi
\]
for some complex valued functions \( a_i \). Such a field belongs to \( D^{1,0} \) if it is in the kernel of \( \eta \), hence we have
\[
e_i = \frac{\partial}{\partial z_i} - \eta \left( \frac{\partial}{\partial z_i} \right) \xi.
\]
Since \( D^{1,0} \) is invariant under the flow of \( \xi \), the transition functions of the bundle \( D^{1,0} \) can be chosen as the transverse changes of charts of \( \mathcal{F} \). Hence they are holomorphic and independent of \( t \), and we may thus extend the operator \( \bar{\partial} + \partial_t \) as a global operator acting on \((1,0)\)-vector fields.

We are now in position to compute the integrability conditions and the differential of the action.

**Lemma 4.23.** The closed set \( \mathcal{I} \) is locally isomorphic to the analytic set in \( \Gamma^1((E^* \otimes D^{1,0}) \oplus T^*S) \) given by the equations
\[
\begin{align*}
(\bar{\partial} + \partial_t)\omega + \frac{1}{2} [\omega, \omega] &= 0 \\
Q(\omega, \alpha) := \left( d\alpha(I\text{d} - \omega, I\text{d} - \omega) - d\eta(\omega, I\text{d}) - d\eta(\text{Id}, \omega) \right)_{|E} &\equiv 0.
\end{align*}
\]

**Proof.** The first equation is the integrability condition of a transversely holomorphic foliation, see [4]. For the second one, by Corollary 4.5, it is given by
\[
d(\eta + \alpha)|_{(I\text{d} - \omega)E} \equiv 0.
\]
Using bilinearity, \( d\eta|_E \equiv 0 \) and \( d\eta|_{D^{1,0}} \equiv 0 \), we immediately obtain the result. \( \square \)
The group acting is just \( \text{Diff}^{d+1}(S) \) with chart \((2.1)\) given by the exponential associated to a fixed real analytic riemannian metric. Action \((2.2)\) is
\[
(\omega, \alpha) \cdot \phi(v) = (\omega \cdot \phi(v), \phi(v)^* (\eta + \alpha) - \eta)
\]
where \(\omega \cdot \phi(v)\) is characterized by
\[
(4.37) \quad \phi(v)_* \{ w - (\omega \cdot \phi(v))(w) \mid w \in E \} = \{ w - \omega(w) \mid w \in E \}
\]
We have now

**Lemma 4.24.** The differential \(L\) of \((4.37)\) at \((E, \eta)\) is
\[
L(\chi, \omega, \alpha) = (\omega + (\bar{\partial} + \partial_t)\chi^{1,0}, \alpha + \mathcal{L}_\chi \eta).
\]

**Proof.** The first component is computed in [4], and the second one in Lemma 4.11. \(\square\)

Following the notations of section \(2.2\), we define the operator \((4.39)\)
\[
\chi \in \Gamma^1(TS) \mapsto P(\chi) = ((\bar{\partial} + \partial_t)\chi^{1,0}, \mathcal{L}_\chi \eta) \in T_{(E, \eta)} E.
\]

**Lemma 4.25.** The operator \(P\) is an elliptic differential operator of order 1 from \(TS\) into \((\Omega^1(E) \otimes D^{1,0}) \oplus T^*S\).

Hence (H2diff) and (H3diff) are fulfilled.

**Proof.** From its definition \((4.39)\), \(P\) is clearly a differential operator from \(TS\) into \((\Omega^1(E) \otimes D^{1,0}) \oplus T^*S\). Let us compute its symbol \(\sigma\). Let \(x \in S\) and \(v \in T_x S \setminus \{0\}\). Choose local foliated coordinates \((z, \bar{z}, t)\), where we assume that \(\xi = \partial/\partial t\). Then a direct computation shows that
\[
(4.40) \quad \sigma_{(x,v)}(\chi) = (v^E \otimes \chi^{1,0}, i_{\chi \eta} v).
\]
Assume now that \(\sigma_{(x,v)}(\chi)\) is zero. Since \(v\) is real and not zero, \(v^E\) is not zero, so \(\chi^{1,0}\) must be zero. This implies that \(\chi^{0,1}\) is also zero, but it is not enough to conclude that \(\chi\) is zero.

However, looking at the second component of \((4.40)\), we have
\[
i_{\chi \eta} = \chi_\xi (i_{\xi \eta}) = \chi_\xi = 0.
\]
This is exactly what was missing to conclude that \(\chi\) is zero. Hence \(\sigma_{(x,v)}(\chi)\) is injective and \(P\) is elliptic. \(\square\)

Finally, the automorphism group of a sasakian manifold is known to be a finite-dimensional Lie group and we can use the time 1 flow as chart \(\psi\) in (H4). We conclude from Proposition 2.27 and lemma 4.23 that, setting
\[
(4.41) \quad K^1 := \{(\omega, \alpha) \mid P^*(\omega, \alpha) = (\bar{\partial} + \partial_t)\omega + \frac{1}{2}[\omega, \omega] = Q(\omega, \alpha) = 0\}
\]
with \(P^*\) the composition of the formal adjoint to \(P\) and of chart \((4.35)\), we have

**Theorem 4.26.** The infinite-dimensional analytic set \((4.41)\) is a Kuranishi type space for sasakian structures of class \(W^1\) at \((E, \eta)\).
Moreover, each structure \((E, \eta) \in \mathcal{E}\) encodes a unique riemannian metric 
\(g_{(E, \eta)}\) on \(S\) as follows. Look at the second formula of (4.5). Starting from 
\((E, \eta)\) sasakian, it defines a riemannian metric. However, starting from 
\((E, \eta)\) only in \(\mathcal{E}\), it does not give a symmetric expression. We claim that its 
symmetrization, that is
\[
(4.42) \quad g_{(E, \eta)}(V, W) = \frac{1}{4}(d\eta(V, \Phi(W)) + d\eta(\Phi(V), W)) + \eta(V)\eta(W)
\]
is a riemannian metric. Indeed it is definite positive on \(E\) because of (4.34) 
and then on the whole \(TS\) because of (4.1).

This induces a unique riemannian metric on the bundle \(E^* \otimes D^{1,0} \oplus T^* S\), still 
denoted by \(g_{(E, \eta)}\). By integrating over \(S\), we obtain a product scalar 
\(h_{(E, \eta)}\) on the space of global sections \(\Gamma((E^* \otimes D^{1,0} \oplus T^* S))\). The collection \((h_{\alpha})\) is 
a weak riemannian metric on \(E\), from which one deduces a strong riemannian metric. It is obviously invariant under the action of the diffeomorphism group, and it is smooth by arguing as in the proof of Lemma 4.14. Hypothesis (H6) and (H7) are thus satisfied. Hypothesis (H8) is only satisfied on the second component, but this is enough for the proof of Proposition 2.33 to be applied (see (4.37)). As an application of Proposition 2.33, we thus have

**Corollary 4.27.** Let \(K^\infty\) be the subset of \(C^\infty\) points of \(K^l\). Then it is a 
Kuranishi type space for \(C^\infty\) structures.

Observe that the equations in (4.41) are cubic, and not quadratic as in the classical case of complex structures. Let us compute more precisely the adjoint \(P^*\). This is similar to the computation of (4.16). Write
\[
(4.43) \quad (h, \chi) \in \Gamma^{l+1}(\mathbb{R} \times D^{1,0}) \mapsto P(h, \chi) = ((\bar{\partial} + \partial_t)\chi, i_{\chi + \bar{\chi}}d\eta + dh) \in \mathcal{E}
\]
and defining
\[
\Gamma_0 := \left(\left(P|_{\{0\} \times \Gamma^{l+1}(D^{1,0})}\right)^{-1}\left(P(\Gamma^{l+1}(\mathbb{R}) \times \{0\})\right)^{\perp}\right)
\]
then (4.43) becomes
\[
(4.44) \quad (h, \chi) \in \Gamma^{l+1}(\mathbb{R}) \times \Gamma_0 \mapsto P(h, \chi) = ((\bar{\partial} + \partial_t)\chi, i_{\chi + \bar{\chi}}d\eta \oplus dh) \in \mathcal{E}.
\]
and we have

**Lemma 4.28.** The adjoint of (4.44) is given by the formula
\[
(4.45) \quad P^*(\omega, \alpha) = (d^*\alpha, (\bar{\partial} + \partial_t)^*\omega - (i_{\alpha};d\eta)^*)
\]
The proof is a direct computation and is completely similar to that of Lemma 4.16. Observe that \(g\) being invariant by the flow of \(\xi\), we may use the Hodge operator associated to \(g\) to define both \(d^*\) and \((\bar{\partial} + \partial_t)^*\).

As in Proposition 4.21 we have

**Proposition 4.29.** Assume that the group of basic infinitesimal automorphisms \(H^0(S, \Theta)\) is zero. Then \(K^l\) and \(K^\infty\) is a local moduli space.
Proof. If $H^0(S, \Theta)$ is zero, then by the semi-continuity theorems of [4], it is zero also for $S'$ close to $S$. Hence, we may apply the proof of Proposition 4.21 and obtain that the automorphism group of any $S'$ in $K^j$ is equal to $\mathbb{R} \xi'$, and that (H4') and (H5') are satisfied. Theorem 2.21 gives the result. □

4.5. Comparison of the different deformation spaces. Let $S$ be a sasakian manifold. As Proposition 4.4 suggests, it depends only on two structures: the transversely holomorphic foliation encoded in the subbundle $E$, and the contact form $\eta$. We want to compare the associated three deformation spaces:

1. The Kuranishi type space $K^\infty_\eta$ of $\eta$-deformations defined in (4.27).
2. The Kuranishi type space $K^\infty$ of general deformations given in (4.41).
3. The versal space $K_E$ of the transversely holomorphic foliation $(S, E)$.

The space $K_E$ was obtained in [7]. It is finite-dimensional and contains only smooth structures (so we drop the exponent since it is not relevant here).

Since $K_E$ is versal, there is a natural holomorphic map $\pi$ from $K^\infty$ to $K_E$ fixing 0.

Observe that the set

$$K^\infty_0 := \{(0, \alpha) \in K^\infty\}$$

is exactly $K^\infty_\eta$ (up to shrinking). Hence we have a natural inclusion of $K^\infty_0$ into $K^\infty$. Of course, $K^\infty_0$ is included in $\pi^{-1}(\{0\})$, but there is no reason for this inclusion to be an equality in general. Nevertheless, we have

**Proposition 4.30.** If $K^\infty$ is a local moduli space, then, up to shrinking, the central fiber $\pi^{-1}(\{0\})$ is equal to $K^\infty_\eta$.

**Proof.** Call $I^\infty$ (respectively $I^\infty_\eta$ and $I^\infty_E$) the set of $C^\infty$ sasakian structures (respectively $C^\infty$ sasakian structures with fixed $E$ and $C^\infty$ sasakian structures with fixed $\eta$). Using the natural local encodings of these structures (see (4.22) and (4.35)), we have a natural projection map

$$\omega, \alpha \in I^\infty \mapsto \omega \in I^\infty_E$$

and isomorphisms onto their image

$$\chi, \omega, \alpha \in V \times K^\infty \mapsto (\omega, \alpha) \cdot \phi(\chi) \in W^\infty \subset I^\infty$$

and

$$\chi, 0, \alpha \in V' \times K^\infty_0 \mapsto (0, \alpha) \cdot \phi'(\chi) \in W^\infty_\eta \subset I^\infty_\eta.$$  

Be careful that $V$ is a neighborhood of 0 in the Lie algebra of smooth vector fields of $S$, whereas $V'$ is a neighborhood of 0 in $\mathcal{X}$. Also the maps $\phi$ and $\phi'$ are not the same, cf. lemma 4.9. Finally, we have an isomorphism

$$\chi, \alpha \in V'' \times K_E \mapsto \omega \cdot \phi(\chi) \in W^\infty \subset I^\infty_E$$

where $V''$ is a neighborhood of 0 in the Lie algebra of smooth vector fields of $S$. We assume, restricting the Kuranishi spaces if necessary, that the image

$$\{\phi'(\chi') \circ \phi(\chi) \mid \chi \in V'', \chi' \in V'\}$$

is included in $\phi(V)$.
Now, let \((\omega, \alpha) \in \pi^{-1}(0)\). By (4.49), that means that there exists some \(\chi \in V''\) such that
\[
\omega \cdot \phi(\chi) = 0.
\]
Set
\[
(0, \alpha_0) := (\omega, \alpha) \cdot \phi(\chi).
\]
Using (4.48), we know that there exists \(\chi' \in V'\) such that
\[
(0, \alpha_0) \cdot \phi'(\chi') \in K_0^\infty.
\]
Hence,
\[
(0, \alpha_1) := (0, \alpha_0) \cdot \phi'(\chi') = (\omega, \alpha) \cdot (\phi'(\chi') \circ \phi(\chi))
\]
so \((0, \alpha_1)\) and \((\omega, \alpha)\) both belong to \(K^\infty\) and represent the same sasakian structure. Moreover, they belong to the same local orbit of \(\text{Diff}(S)\) in \(W^\infty\). Now condition (4.50) associated to (4.47) and the local moduli space assumption shows that
\[
(0, \alpha_1) = (\omega, \alpha).
\]
As \((0, \alpha_1)\) belongs to \(K_0^\infty\), this implies
\[
\pi^{-1}(0) \subset K_0^\infty.
\]
Since we already noticed that the other inclusion is clear, we are done. □

**Remark 4.31.** The map \(\pi\) is not surjective. Indeed, consider the vector field
\[
\chi = z \frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w}
\]
in \(\mathbb{C}^2\). For \(\lambda \in \mathbb{C} \setminus (-\infty, 0]\), the flow of \(\chi\) is transverse to the unit sphere \(S^3\) and induces a transversely holomorphic flow on it. However, there does not exist an associated sasakian metric for all such \(\lambda\) (see [1]).

**References**

[1] Boyer, C.P. and Galicki, K. *Sasakian Geometry*. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

[2] Donaldson, S.K. *An application of gauge theory to four dimensional topology*. J. Differential Geometry, 18 (1983), 269–278.

[3] Douady, A. *Le problème des modules pour les sous-espaces analytiques compacts d’un espace analytique donné*. Ann. Inst. Fourier 16, (1966). 1–95.

[4] Duchamp, T. and Kalka, M. *Deformation theory for holomorphic foliations*. J. Differential Geom. 14 (1979), no. 3, 317–337.

[5] Ehricke, D. G. *The manifold of riemannian metrics*. Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 11-740, Amer. Math. Soc., Providence, R.I., 1970.

[6] El Kacimi-Alaoui, A. and Nicolau, M. *Déformations des feuilletages transversalement holomorphes à type différentiable fixe*. Publ. Mat. 33 (1989), no. 3, 485–500.

[7] Girbau, J.; Haefliger, A. and Sundararaman, D. *On deformations of transversely holomorphic foliations*. J. Reine Angew. Math. 345 (1983), 122–147.

[8] Hamilton, R. S. *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.

[9] Kodaira, K. and Spencer, D.C. *On deformations of complex analytic structures I*. Ann. of Math. 67 (1958), 328–402.

[10] Kuranishi, M. *New proof for the existence of locally complete families of complex structures*. Proc. Conf. Complex Analysis (Minneapolis, 1964) pp. 142–154, Springer, Berlin, 1965.

[11] Kuranishi, M. *A note on families of complex structures*. Global Analysis, Papers in honor of K. Kodaira pp. 309–313, Princeton University Press, Princeton NJ, 1969.
Laurent Meersseman, LAREMA, Université d’Angers, F-49045 Angers Cedex, France, laurent.meersseman@univ-angers.fr

Marcel Nicolau, Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain, nicolaue@mat.uab.cat