WAVE EXCITATION IN DISKS AROUND ROTATING MAGNETIC STARS

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ABSTRACT

The accretion disk around a rotating magnetic star (neutron star, white dwarf, or T Tauri star) is subjected to periodic vertical magnetic forces from the star, with the forcing frequency equal to the stellar spin frequency or twice the spin frequency. This gives rise to bending waves in the disk that may influence the variabilities of the system. We study the excitation, propagation, and dissipation of these waves using a purely hydrodynamical model coupled with a generic model description of the magnetic forces. The $m = 1$ bending waves are excited at the Lindblad/vertical resonance and propagate either to larger radii or inward toward the corotation resonance where dissipation takes place. While the resonant torque is negligible compared to the accretion torque, the wave nevertheless may reach an appreciable amplitude and can cause or modulate flux variabilities from the system. We discuss the possible relevance of our result to the observed quasi-periodic oscillations from various systems, in particular neutron star low-mass X-ray binaries.

Subject headings: accretion, accretion disks — binaries: general — hydrodynamics — stars: magnetic fields — stars: neutron — waves

1. INTRODUCTION

Disk accretion onto magnetic central objects occurs in a variety of astrophysical systems, ranging from classical T Tauri stars (e.g., Bouvier et al. 2007a) and cataclysmic variables (intermediate polars; e.g., Warner 2004) to accretion-powered X-ray pulsars (e.g., Lewin & van der Klis 2006). The basic picture of the disk-magnetosphere interaction is well known. The stellar magnetic field disrupts the accretion flow at the magnetospheric boundary and funnels the plasma onto the polar caps of the star or ejects it to infinity. The magnetosphere boundary is located where the magnetic and plasma stresses balance,

$$r_m = \xi \left[ \mu_0 \left( \frac{G M^2}{T} \right) \right]^{1/7},$$

(1)

where $M$ and $\mu$ is the mass and magnetic moment of the central object, $M$ is the mass accretion rate, and $\xi$ is a dimensionless constant of order 0.5–1. The funnel flow occurs when $r_m$ is less than the corotation radius $r_c$ (where the disk rotates at the same rate as the star). For $r_m > r_c$, centrifugal forces may lead to ejection of the accreting matter ("propeller" effect). Over the last several decades, numerous theoretical studies have been devoted to understanding the interaction between accretion disks and magnetized stars (e.g., Ghosh & Lamb 1979; Aly 1980; Lipunov & Shakura 1980; Anzer & Börner 1983; Arons 1993; Aly & Kuijpers 1990; Spruit & Taam 1993; Shu et al. 1994, 2000; Wang 1995; Lovelace et al. 1995, 1999; Li et al. 1996; Campbell 1997; Lai 1998, 1999; Terquem & Papaloizou 2000; Shirakawa & Lai 2002a, 2002b; Pfeiffer & Lai 2004; Uzdensky 2004; Matt & Pudritz 2005). There have also been many numerical simulations on accretion onto magnetic stars, with increasing sophistication and realism (e.g., Stone & Norman 1994; Hayashi et al. 1996; Goodson et al. 1997; Miller & Stone 1997; Fendt 2003; Romanova et al. 2003, 2006; Ustyugova et al. 2006; Long et al. 2007).

In this paper we carry out an analytical study of wave excitation in disks around rotating magnetic stars. The idea is very simple: a rotating dipole, inclined relative to the stellar spin axis, induces a time-dependent magnetic force on the disk. The forcing frequency normally equals the spin frequency $\omega_s$, but we show that under certain conditions the magnetic force could also have a $2\omega_s$ component (see § 2). The magnetic forces excite $m = 1$ density/bending waves in the disk, most prominently at the Lindblad/vertical resonance (L/VR). This magnetically driven resonance was noted before (Lai 1999) but not studied. Since details of the magnetic field–disk interaction are complex (e.g., the interaction depends on magnetic field dissipation/reconnection), we model the magnetic force using simple models that, we conjecture, capture the essential dynamical feature of the system. Petri (2005) examined several aspects of the forced oscillations in magnetized disks, but did not study the resonantly excited bending waves and their evolution that are of interest in this paper (see also related work by Agapitou et al. 1997; Terquem & Papaloizou 2000).

Wave excitations in magnetic disks may be relevant to understanding the flux variabilities that have been observed in many accreting systems, from pre-main-sequence stars (e.g., O'Sullivan et al. 2005; Bouvier et al. 2007b and references therein) to accreting magnetic white dwarfs and neutron stars. It has been suggested that the phenomenology of the quasi-periodic brightness modulations in cataclysmic variables might be explained by magnetically excited traveling waves in the disk (Warner 2004). Of great interest are the high-frequency (kHz) quasi-periodic oscillations (QPOs) observed in the X-ray fluxes of more than 20 neutron stars in low-mass X-ray binaries (LMXBs; see van der Klis 2006 for a review). These kHz QPOs often occur in pairs, with intriguing correlation between the upper frequency $\nu_{\text{up}}$, the lower frequency $\nu_l$, and the spin frequency $\nu_s$. In particular, in the accreting millisecond pulsar SAX J1808.4-3658, $\nu_0 = 694 \pm 4$ Hz, $\nu_l = 499 \pm 4$ Hz, and $\nu_{\text{up}} - \nu_l$ equals $\nu_0/2 = 401/2$ Hz to within a few Hz (Wijnands et al. 2003). In another accreting millisecond pulsar XTE J1807.4-294, $\nu_0 - \nu_l$ remains constant with an average value of $205 \pm 6$ Hz, approximately equal to $\nu_0 = 191$ Hz, even when both $\nu_l$ and $\nu_{\text{up}}$ vary over a range of more than 200 Hz (Linares et al. 2005). In some systems, however, $\nu_{\text{up}} - \nu_l$ are found to vary significantly (e.g., Circinus X-1; Boutloukos et al.

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2. Periodic Magnetic Forces on the Disk

The interaction between the dipole magnetic field of a star and its accretion disk is complex, and many papers have been published on it (§ 1). To focus on the dynamics of waves driven by magnetic forces, we consider two simple analytical models, representing two limiting behaviors of the disk. The basic geometric setup is as follows. The disk lies in the $x$-$y$ plane with its normal vector along the $z$-axis; the spin ($\omega_s$) of the star is inclined with respect to the $z$-axis by an angle $\beta$ and lies in the $y$-$z$ plane. Because of the possible size of large-scale disk warping driven by the (static) stellar magnetic field, we allow for $\beta \neq 0$ (Lai 1999; Pfeiffer & Lai 2004; see below). The stellar dipole $\mu$ rotates around $\omega_s$, and the angle between $\mu$ and $\omega_s$ is $\theta$, such that $\mu$ varies in time according to $\mu = \mu(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. First consider the limiting case where the disk is a perfect conductor and has no large-scale magnetic field of its own. The inner radius of the disk is located at $r = r_m$. The magnetic field produced by the stellar dipole cannot penetrate the disk, and a diamagnetic surface current is induced. Aly (1980) has found the exact analytic solution for the magnetic field outside the thin disk. At a point $(r, \phi, z = 0)$ (cylindrical coordinates) on the disk surface ($z = 0, r > r_m$), the field is given by

$$B_r = \frac{2\mu}{r^3} \sin \chi \cos (\varphi - \varphi_\mu) + \frac{4\mu}{r^3 D} \cos \chi,$$  

$$B_\varphi = \frac{\mu}{r^3} \sin \chi \sin (\varphi - \varphi_\mu),$$  

and $B_z = 0$, where $\chi = \chi(t)$ is the angle between $\mu$ and the $z$-axis. In equation (2), the top (bottom) sign applies to the upper (lower) disk surface, and $D = \max [(r^2/r_m^2 - 1)^{1/2}, (2h/r_m)^{1/2}]$, where $h \ll r_m$ is the half-thickness of the disk. The vertical magnetic force per unit area on the disk is

$$F_z = \frac{2\mu^2}{\pi^2 r^6 D} \sin 2\chi \cos (\varphi - \varphi_\mu).$$  

Using the relations between various angles, we have

$$F_z = \text{Re} \sum_{\omega_f} F_{\omega_f}(r) \exp (i\omega_f t),$$

where $\omega_f = 0, \pm \omega_s, \pm 2\omega_s$, with

$$F_0(r) = -iF_D(2 - 3 \sin^2 \theta) \sin 2\beta,$$

$$F_\omega(r) = F_D \sin 2\beta(\cos \beta + \cos 2\beta),$$

$$F_{-\omega}(r) = F_D \sin 2\beta(\cos \beta - \cos 2\beta),$$

$$F_{2\omega}(r) = -iF_D \sin^2 \theta \sin (\cos \beta + 1),$$

$$F_{-2\omega}(r) = -iF_D \sin^2 \theta \sin (\cos \beta - 1),$$

where

$$F_D = \frac{\mu^2}{\pi^2 r^6 D}.$$

The disk is unlikely to be completely diamagnetic: various instabilities and dissipation mechanisms will allow some of the stellar magnetic field to partially penetrate the disk. As a simple model, we decompose the vertical stellar dipole field on the disk into a static component and a time-varying component,

$$B_{z0} = -\frac{\mu}{r^3 \cos \chi} = \frac{\mu}{r^3} (\sin \beta \sin \theta \sin \omega_s t - \cos \beta \cos \theta).$$

We assume that the static field penetrates the disk, while the variable field is shielded out of the disk by a screening current. Because of the shear between the disk and the plasma outside the disk, the threaded vertical field is wound to produce an azimuthal field that has different signs on the upper and lower surfaces of the disk. We thus adopt the following Ansatz for the magnetic field in the disk (see Lai 1999),

$$B_r = \frac{2\mu}{r^3} \sin \chi \cos (\varphi - \varphi_\mu) + \frac{4\mu}{r^3 D} \sin \beta \sin \theta \sin \omega_s t,$$

$$B_\varphi = \frac{\mu}{r^3} \sin \chi \sin (\varphi - \varphi_\mu) + \frac{\zeta}{r^3} \cos \beta \cos \theta,$$

$$B_z = -\frac{\mu}{r^3} \cos \beta \cos \theta,$$

where $\zeta$ is a positive constant of order unity. The vertical magnetic force (per unit area) on the disk is given by

$$F_z = -\frac{4\mu^2}{\pi^2 r^6 D} \sin \beta \sin \theta \sin \omega_s t \sin \chi \cos (\varphi - \varphi_\mu)$$

$$-\frac{\zeta \mu^2}{2\pi r^6} \cos \beta \cos \theta \sin \chi \sin (\varphi - \varphi_\mu).$$

This force can be written in the form of equation (5), with

$$F_0(r) = iF_D \sin^2 \theta \sin 2\beta + F_T \cos^2 \theta \sin 2\beta,$$

$$F_\omega(r) = F_D \sin 2\theta(\cos \beta + \cos 2\beta),$$

$$F_{-\omega}(r) = -F_D \sin 2\theta(\cos \beta - \cos 2\beta),$$

$$F_{2\omega}(r) = -iF_D \sin^2 \theta \sin (\cos \beta + 1),$$

$$F_{-2\omega}(r) = iF_D \sin^2 \theta \sin (\cos \beta - 1),$$

where

$$F_T = \frac{\zeta \mu^2}{4\pi r^6}.$$
2.2. General Property of Magnetic Forces

The magnetic forces discussed above share some general properties. The zero-frequency force components \( F_0 \) have been shown to lead to secular warping and precession of the disk (Lai 1999; Shirakawa & Lai 2002a, 2002b; Pfeiffer & Lai 2004). The force resulting from threaded field \((F_s, F_r)\) induces a warping instability, while the force resulting from the dielectric response of the disk \((F_d, F_t)\) gives rise to precession. The negative-frequency components \( (F_{-\omega}, F_{-2\omega}) \) average to zero on the timescale of spin period.

The positive-frequency forces, \( F_+, F_{2\omega}, \) are of interest for this paper. They are generally present for nonzero magnetic field inclination angle \( \theta \). Note that the \( 2\omega \) component arises because the dipole field varies as \( \sin \omega t \) or \( \cos \omega t \), and the induced screening current also varies as \( \sin \omega t \) or \( \cos \omega t \), and \( F_{2\omega} \) is present only when \( \sin \beta \neq 0 \). We note that periodic magnetic forces in the radial direction are also present in some disk models, but we focus on the dynamics of the disk under the periodic vertical force in this paper.

3. BASIC EQUATIONS: DISK DYNAMICS UNDER A PERIODIC NONPOTENTIAL FORCING

The key approximation underlying our study is that we treat the waves in the disk using pure hydrodynamics, with the magnetic effect entering only as external forces. This simplification allows us to focus on the dynamics of wave excitation, propagation, and dissipation. As we will see, the wave dynamics is sufficiently complex and subtle even without magnetic fields, and we believe that such a hydrodynamical treatment can be considered a useful first step. Future study using MHD will be of great interest (and obviously much more difficult). We conjecture that in a real system, the wave dynamics will not differ qualitatively from that described in this paper, since the waves of interest here are basically sound waves modified by disk rotation, and in the presence of magnetic fields, the sound waves simply become fast magnetosonic waves (with the sound speed replaced by fast wave speed).

In this section we develop the general hydrodynamical equations for three-dimensional disks under generic nonpotential forces (cf. Zhang & Lai 2006). We consider a geometrically thin gas disk and adopt cylindrical coordinates \((r, \varphi, z)\). The unperturbed disk has velocity \( v_0 = (0, r\Omega, 0) \), where the angular velocity \( \Omega = \Omega(r) \) is taken to be a function of \( r \) alone. The disk is assumed to be isothermal in the vertical direction and non-self-gravitating.

Thus, the vertical density profile is given by

\[
\rho_0(r, z) = \frac{\sigma}{\sqrt{2\pi\hbar}} \exp\left(-\frac{Z^2}{2}\right), \quad \text{with } Z = z/h,
\]

where \( h = h(r) = c/\Omega_z \) is the disk scale height, \( c = c(r) = (\rho_0/\rho_0)^{1/2} \) is the isothermal sound speed, \( \sigma = \sigma(r) = \int \rho \text{ d}z \) is the surface density, and \( \Omega_z \) is the vertical oscillation frequency of the disk.

We now consider perturbation of the disk driven by an external force (per unit mass) \( \mathbf{f} \). For simplicity, we shall assume that the perturbation is isothermal, so that the \((\text{Eulerian})\) density and pressure perturbations are related by \( \delta P = c^2 \delta \rho \).

The linear perturbation equations read

\[
\frac{\partial \mathbf{u} \cdot \nabla}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v}_0 = -\nabla \eta + \mathbf{f},
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u} + \mathbf{v}_0 \delta \rho) = 0,
\]

where \( \delta \rho \) is the \((\text{Eulerian})\) density perturbation and \( \eta \equiv \delta \rho/\rho_0 \) is the enthalpy perturbation. Without loss of generality, each perturbation variable and the external force are assumed to depend on \( t \) and \( \varphi \) as \( \exp(i m \varphi - i \omega t) \), where \( m > 0 \) is an integer and \( \omega \) is allowed to be either positive or negative, corresponding to the prograde or retrograde wave, respectively.

Equations (16) and (17) then reduce to

\[
-i\tilde{\omega} u_r - 2\Omega u_{\varphi} = -\frac{\partial}{\partial r} \eta + f_r,
\]

\[
-i\tilde{\omega} u_\varphi + \frac{\kappa^2}{2\Omega} u_r = -\frac{im}{r} \eta + f_\varphi,
\]

\[
-i\tilde{\omega} u_z = -\frac{\partial}{\partial z} \eta + f_z,
\]

\[
-i\tilde{\omega} \rho_0 \frac{1}{c^2} \eta + \frac{1}{r} \frac{\partial}{\partial r} (\rho_0 u_r) + \frac{im}{r} \rho_0 u_\varphi + \frac{\partial}{\partial z} (\rho_0 u_z) = 0,
\]

where \( \tilde{\omega} \) is the ‘‘Doppler-shifted’’ frequency

\[
\tilde{\omega} = \omega - m\Omega,
\]

and \( \kappa \) is the epicyclic frequency, defined by

\[
\kappa^2 = \frac{2\Omega}{r} \frac{d}{dr} (r^2 \Omega).
\]

In this paper we consider cold, Keplerian (Newtonian) disks, for which the three characteristic frequencies, \( \Omega, \Omega_z, \) and \( \kappa \), are identical and equal to the Keplerian frequency \( \Omega_K = (GM/r^3)^{1/2} \). However, we continue to use different notations \((\Omega, \Omega_z, \kappa)\) for them in our treatment below when possible so that the physical origins of various terms are clear.

To separate out the \( z \)-dependence, we expand the perturbations with Hermite polynomials \( H_n \) (cf. Okazaki et al. 1987; Kato 2001),

\[
\begin{bmatrix}
 f_r(r, z) \\
 f_\varphi(r, z) \\
 \eta(r, z) \\
 u_r(r, z) \\
 u_\varphi(r, z) \\
 u_z(r, z)
\end{bmatrix} = \sum_n \begin{bmatrix}
 f_{rn}(r) \\
 f_{\varphi n}(r) \\
 \eta_{rn}(r) \\
 u_{rn}(r) \\
 u_{\varphi n}(r) \\
 u_{zn}(r)
\end{bmatrix} H_n(Z),
\]

\[
\begin{bmatrix}
 f_r(r, z) \\
 f_\varphi(r, z) \\
 \eta(r, z) \\
 u_r(r, z) \\
 u_\varphi(r, z) \\
 u_z(r, z)
\end{bmatrix} = \sum_n \begin{bmatrix}
 f_{rn}(r) \\
 f_{\varphi n}(r) \\
 \eta_{zn}(r) \\
 u_{rn}(r) \\
 u_{\varphi n}(r) \\
 u_{zn}(r)
\end{bmatrix} H_n'(Z),
\]

where \( H'_n = dH_n/dZ \), and the Hermite polynomial is defined by \( H_n(Z) = (-1)^n e^{-Z^2/2} d^n(e^{-Z^2/2})/dZ^n \). Note that since \( H_1 = Z \), \( H_2 = Z^2 - 1 \), the \( n = 1 \) mode coincides with the bending mode studied by Papaloizou & Lin (1995; who considered disks with no resonance), and the \( n = 2 \) mode is similar to the mode studied by Lubow (1981). With the expansion in equation (24), the fluid equations (21) become

\[
-i\tilde{\omega} u_{rn} - 2\Omega u_{\varphi n} = -\frac{d}{dr} \eta_{zn} + \frac{n\mu}{r} \eta_{zn} + \frac{(n+1)(n+2)\mu}{r} \eta_{zn+2} + f_{rn},
\]

\[
-i\tilde{\omega} u_{zn} + \frac{\kappa^2}{2\Omega} u_{rn} = -\frac{im}{r} \eta_{zn} + f_{zn},
\]

\[
-i\tilde{\omega} u_{zn} = -\frac{\eta_{zn}}{h} + f_{zn},
\]

\[
-i\tilde{\omega} \eta_{zn} + \left( \frac{d}{dr} \ln r + \frac{n\mu}{r} \right) u_{zn} + \frac{\mu}{r} u_{r, n-2} + \frac{d}{dr} u_{zn} + \frac{im}{r} u_{zn} - \frac{n}{h} u_{zn} = 0,
\]
where $\mu \equiv d \ln h / d \ln r$. Eliminating $u_{zn}$ and $u_{zn}$ from equations (25)--(28), we have

$$\frac{d\eta_1}{dr} = \frac{2m\Omega}{r\omega} \eta_1 - \frac{D}{\omega} u_{rn} + \frac{\mu}{r} [m_{n1} + (n+1)(n+2)\eta_{n+2}]$$

$$+ f_{rn} + i\frac{2\Omega}{\omega} f_{zn},$$

(29)

$$\frac{du_{rn}}{dr} = -\left[ \frac{d\ln(\sigma \omega)}{dr} + \frac{m^2}{2r^2 \sigma \omega} \right] u_{rn} + \frac{1}{r^2} \left( \frac{m^2}{r^2} + \frac{n}{h} \right) \eta_1$$

$$+ \frac{i\omega}{e^2} \eta_1 - \frac{\mu}{r} (mu_{rn} + u_{r,n-2}) + \frac{m}{\omega r} f_{zn} + \frac{in}{\omega h} f_{zn},$$

(30)

where we have defined

$$D \equiv \kappa^2 - \omega^2 = \kappa^2 - (\omega - m\Omega)^2.$$  

(31)

We can check that for an external potential force, $f = -\nabla \phi$, these equations reduce to those of Zhang & Lai (2006).

Consider a local free wave solution of the form $\eta_1 \propto \exp [i \int k(z) ds]$. For $|k| \gg 1$, from equations (29)--(30), in the absence of the external force, we find (see Okazaki et al. 1987; Kato 2001)

$$(\bar{\omega}^2 - \kappa^2) (\bar{\omega}^2 - n\Omega^2) / \bar{\omega}^2 = k^2 c^2,$$  

(32)

where we have used $h = c/\Omega_\perp \ll r$ (thin disk) and $m, n \ll r/h$. The dispersion relation is useful for identifying wave propagation zones in the disk (see Zhang & Lai 2006).

4. WAVES AT LINDBLAD/VERTICAL RESONANCE AND COROTATION RESONANCE

We now consider the magnetic force on the disk as given in §2, i.e., the force per unit area is $F_z = F_z(r) \exp (i\phi - i\omega t)$, with $\omega = \omega_e$ or $2\omega_e$. This corresponds to $m = n = 1$, with

$$f_{\Sigma}(r) = \frac{1}{\sigma} F_{\Sigma}(r), \quad f_e = f_{\phi} = 0 \quad (\omega = \omega_e, 2\omega_e).$$

(33)

The wave propagation diagram (based on the dispersion relation [32] with $m = n = 1$) is shown in Figure 1.

Note that for $\omega \ll \Omega = \kappa = \Omega_\perp$, the $m = n = 1$ mode dispersion relation (32) reduces to $\omega = \pm c \kappa/2$, i.e., the bending wave propagates nondispersively at speed $c/2$. This is the regime explored in a number of previous works (e.g., Papaloizou & Lin 1995; Ogilvie 2006). However, here we are particularly interested in the disk region where $\Omega$ is of the same order as $\omega$.

Consider equations (29)--(30) for $n = 1$. The coupling terms can be neglected if $|\eta_1| \gtrsim |\eta_2|$, which is justified since there is no $n = 3$ driving force. Keeping only the potentially singular terms ($\propto D^{-1}, \bar{\omega}^{-2}$), we have

$$\left[ \frac{d^2}{dr^2} + \left( \frac{d}{dr} \ln \frac{\sigma \Omega}{D} \right) \frac{d}{dr} - \frac{D}{h^2 \bar{\omega}^2} \right] \eta_1 = \frac{D}{\bar{\omega}^2 h} f_{\bar{\Sigma}}.$$  

(34)

This equation can be simplified further by changing variables via

$$\eta_1(r) = (-D/\sigma \omega)^{1/2} y(r).$$

(35)

Fig. 1.— Sketch of the function $G = (\kappa^2 - \omega^2)(1 - \Omega^2 / \omega^2)$ as a function of $r$ for $m = n = 1$. The WKB dispersion relation is $G = -k^2 c^2$, and thus, waves propagate in the region $G < 0$. Note that the Lindblad/vertical resonance (L/VR) is the turning point $k \to 0$ and the corotation resonance (CR) is a singularity ($k \to \infty$).

Keeping the leading-order terms, we obtain

$$\left[ \frac{d^2}{dr^2} + \left( \frac{\bar{\omega}^2 - \Omega^2_\perp}{h^2 \bar{\omega}^2 \Omega^2_\perp} \right) - \frac{3(\bar{\omega}^2 d\Omega/dr + \kappa d\kappa/dr)^2}{D^2} \right] y = \frac{(-D \sigma \omega)^{1/2}}{h \bar{\omega}^2} f_{\bar{\Sigma}}.$$  

(36)

The terms dropped inside the square brackets in the above equation are considered to be smaller than the retained terms for the following reasons. First, away from the $D = 0$ region, they are negligible compared to the term $\propto h^{-2}$ owing to the assumed smallness of $h$, and second, they are insignificant with respect to the term $\propto D^{-2}$ in the vicinity of $D = 0$.

Equation (36) reveals two special points: one is at $\bar{\omega} = 0$ (CR), and the other is at $\bar{\omega} = \kappa$ (Lindblad resonance). The CR is a singularity of the wave equation, and the singularity implies that a steady emission or absorption of waves may occur there. The Lindblad resonances are turning points at which waves are generated or reflected. Another interesting point is at $\bar{\omega}^2 = \Omega^2_\perp$. It is a transition point where the vertical resonance occurs. For a Keplerian disk, $\kappa = \Omega_\perp = \Omega$, the vertical resonance coincides with the Lindblad resonance, and we call it the Lindblad/vertical resonance (L/VR). Equation (36) then reduces to

$$\left[ \frac{d^2}{dr^2} + \left( \frac{\bar{\omega}^2 - \Omega^2_\perp}{h^2 \bar{\omega}^2 \Omega^2_\perp} \right) - \frac{3(\bar{\omega}^2 d\Omega/dr)^2}{(\bar{\omega}^2 - \kappa^2)^2} \right] y = \frac{\sigma \omega^{1/2} \kappa^{1/2}}{h \bar{\omega}^2} f_{\bar{\Sigma}}.$$  

(37)

This is our basic working equation.

4.1. Lindblad/Vertical Resonance

We now study wave excitations near a L/VR, where $D = 0$, or $\bar{\omega} = \kappa = \Omega, \omega = 2\Omega$, and the (outer) L/VR radius is denoted by $r_\perp$. Changing $r$ to the new variable $x \equiv (r - r_\perp)/r_\perp$ and keeping
the leading-order terms, we find that, for \( |x| \ll 1 \), equation (37)
reduces to
\[
d^2 y + \left( b^2 x^2 - \frac{3}{4x^2} \right) y = S_r r L^2, \tag{38}
\]
where (the subscript “L” implies that the quantity should be eval-
uated at \( r = r_L \))
\[
b^2 = \frac{64}{\omega^2} \left( \frac{d \Omega / dr}{h^2} \right)^2 \left| \frac{1}{L} \right| = 36 \left( \frac{r}{h} \right)^2. \tag{39}
\]
The two independent solutions of the homogeneous version of equation (38) are
\[
y_\pm \propto (b x)^{-1/2} e^{\pm i b x^{1/2}}. \tag{40}
\]
The general solution of the inhomogeneous equation (38) is then
\[
y = -y_+ \int x y- \frac{S_r r L^2}{W} dx + y_- \int x y_+ \frac{S_r r L^2}{W} dx + C_+ y_+ + C_- y_-,
\tag{41}
\]
where \( W = y_- \frac{d y_-}{dx} - y_+ \frac{d y_+}{dx} = -2i \) is the Wronskian.
The constants \( C_\pm \) can be fixed by requiring waves propagating away
from the resonance. Using equation (35), we obtain
\[
\eta_i = i \left( \frac{r L}{24} \right)^{1/2} f(z) \left[ e^{i \zeta} \int_0^\infty e^{-i \zeta^2} d\zeta \right.

\left. + e^{-i \zeta^2} \int_\infty^0 e^{i \zeta^2} d\zeta \right], \tag{42}
\]
where \( f(z) \) is evaluated at \( r = r_L \), and we have defined \( \zeta = b^{1/2} x \).
Note that although our analysis here is limited to \( |x| \ll 1 \), we have
extended the integration limit to \( \zeta \approx \pm \infty \) in equation (42).
This is valid because \( b = \frac{6 \pi r_0}{h} \gg 1 \) for a thin disk, and the inte-
grands in the integrals are highly oscillatory for \( |\zeta| \gg 1 \) (so that the
contribution to the integrals from the \( |\zeta| \gg 1 \) region is negligible).

To calculate the angular momentum transfer through the res-
sonance, we note that for \( \zeta \rightarrow +\infty \) (but still \( x \ll 1 \)), equation (42)
becomes
\[
\eta_i = i \left( \frac{r L}{24} \right)^{1/2} f(z) (-2 \pi i)^{1/2} e^{-i \zeta^2} (\eta \rightarrow \infty). \tag{43}
\]
The angular momentum flux to the \( r > r_L \) region is
\[
F(r > r_L) = \frac{\pi \sigma \eta}{D} \left| \eta_i \frac{d \eta_i}{dr} \right| = \frac{\pi^2}{2} \left( \sigma \frac{dr}{dr} \right)^2 \left| f(z) \right|^2 \left( \frac{r^2 \sigma | f(z) |^2}{L} \right)_L. \tag{44}
\]
Similarly, for \( \zeta \rightarrow -\infty \), equation (42) gives
\[
\eta_i = i \left( \frac{r L}{24} \right)^{1/2} f(z) (2 \pi i)^{1/2} e^{-i \zeta^2} (\eta \rightarrow -\infty). \tag{45}
\]
The angular momentum flux to the \( r < r_L \) region is identical to
\( F(r > r_L) \). The total torque on the disk acted through the outer
L/VR is then
\[
T_L = 2 F(r > r_L) = \frac{2 \pi^2}{3 \omega^2} \left( r^2 \sigma | f(z) |^2 \right)_L. \tag{46}
\]
To estimate the torque \( T_L \), we consider \( \omega = \omega_c \) and use \( f(z) = F / \sigma \) with \( F \sim \mu^2 / (4 \pi 6) \) (see 2). For a Keplerian disk, \( r_L = 2 \xi r_c \), and we assume \( r / r_m = \lambda \) (for accretion to occur, we require
\( \lambda > 1 \). Using equation (1), we then obtain
\[
T_L \sim 4 \times 10^{-4} \frac{\dot{M}^2}{(\xi \lambda)^{1/2} \sigma(r_L)}. \tag{47}
\]
The canonical accretion torque is \( T_d \equiv M (GM \xi) \lambda^3 \). Using \( M =
2 \pi (r \sigma | u_r |) L \), where \( u_r \) is the radial velocity in the unperturbed
accretion flow, we find
\[
\frac{T_L}{T_d} \sim 3 \times 10^{-3} \xi^{-7} \lambda^{-6} \left( \frac{| u_r |}{\gamma r \delta \xi} \right)_L. \tag{48}
\]
Thus, the resonant torque is much smaller than the accretion
torque.

4.2. Corotation Resonance

The corotation resonance (CR) is located where \( \bar{\omega} = 0 \) or
\( \omega = m \Omega(r_c) \). The WKB dispersion relation shows that for \( n = 0 \),
waves are evanescent in the region around the corotation radius
\( r_c \), while for \( n > 0 \), wave propagation is possible around \( r_c \).

We again focus on the \( n = m = 1 \) waves. In the vicinity of
corotation, the terms \( \propto h^{-2} \) in equations (34) or (37) are
dominant, and we only need to keep these terms and the second-order
differential term. Equation (34) then reduces to
\[
\frac{d^2}{dx^2} \eta_i = \frac{D(\xi^2 - \Omega^2)}{h^2 \Omega^2} \eta_i = \frac{D}{h^2 \Omega} f(z). \tag{49}
\]
Defining \( x = (r - r_c) / r_c \) and expanding equation (49) around \( x = 0 \),
we have
\[
\left[ \frac{d}{dx^2} + \frac{C}{(x + i \epsilon)^2} \right] \eta_i = \frac{h f(z)}{(x + i \epsilon)^2}, \tag{50}
\]
where \( f(z) \) is evaluated at \( r = r_c \), and (where the subscript “c”
means that the quantity is evaluated at \( r = r_c \))
\[
C = \frac{\Omega}{h d \Omega / dr} \left( \frac{r_c}{3 h} \right) \gg 1. \tag{51}
\]
In equation (50), we have inserted a small imaginary part \( i \epsilon \) (with
\( \epsilon > 0 \)) in \( 1/x^2 \), because we consider the response of the disk to a
slowly increasing perturbation.

The general solution to equation (50) is
\[
\eta_i = h f(z) + M e^{i z} + N e^{-i z}, \tag{52}
\]
where \( M = [C (1/4)]^{1/2} \gg 1 \), \( x = x + i \epsilon \) (with \( \epsilon > 0 \)), and \( M \)
and \( N \) are constants. The first term, the nonwave part, is a particular
solution, while the other two terms are solutions to the homogeneous equation, depicting the waves. The $z^{1/2}z''$ term has a local wave number $\nu = d(\nu \ln z)/dr = \nu/(r \omega)$, with the group velocity $v_g = d\nu/dk = -\nu/k = -3r \omega^2(2\nu) < 0$; thus, it represents waves propagating toward small $r$. Similarly, the $z^{1/2}z''$ term has $v_g > 0$ and represents waves propagating toward large $r$.

As shown in Zhang & Lai (2006), waves with $n = 1$ can propagate into the corotation region and get absorbed there. Consider an incident wave propagating from $x > 0$ region toward $x = 0$,

$$\eta_1(x > 0) = A_+ x^{1/2} e^{i \theta \ln x},$$  

(53)

where $A_+$ is constant specifying the wave amplitude. The transmitted wave is given by

$$\eta_1(x < 0) = i A_+ e^{-\nu \theta}(-1)^{1/2} e^{i \theta \ln x}.$$

(54)

Since $\nu \gg 1$, the wave amplitude is vastly decreased by a factor $e^{-\nu \theta}$ after propagating through the corotation. The net angular momentum flux absorbed at corotation is

$$\Delta F_c = \pi i \left(\frac{\sigma}{k^2}\right) \nu |A_+|^2 (1 + e^{-2\nu \theta}) \approx \left(\frac{2\pi \rho \sigma}{\hbar \omega^2}\right) |A_+|^2,$$

(55)

where in the last equality we have used $\nu \simeq \sqrt{C} \gg 1$. Thus, a wave propagating from $r > r_c$ into the CR deposits almost all of its positive angular momentum $\theta = r \omega$ at $r = r_c$. Similarly, a wave propagating from $r < r_c$ into the CR deposits its negative angular momentum at $r = r_c$.

5. GLOBAL DISK RESPONSE

Having studied the behavior of waves near the L/VR and CR, we can now construct a global solution for the waves excited by the external force. Away from the L/VR, the two linearly independent WKB solutions to the homogeneous wave equation (37) are

$$y_\pm = c^{-1/2} \exp \left(\pm i \int_{r}^{r_c} k_r \, dr\right),$$

(56)

where

$$k_r \equiv \frac{\omega^2 - \Omega^2}{\hbar \Omega \omega} = \frac{\omega(\omega - 2\Omega)}{\hbar \Omega(\omega - \Omega)}.$$  

(57)

It is easy to check that equation (56) reduces to equation (40) for $|r - r_c| \ll r_c$; thus, it is valid even near the L/VR. From equation (35), we have

$$\eta_\pm = \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right]^{1/2} \exp \left(\pm i \int_{r}^{r_c} k_r \, dr\right).$$

(58)

Near the CR, this solution reduces to

$$\eta_\pm \propto x^{1/2} e^{i \theta \ln x}, \quad \text{for} \quad |x| = |(r - r_c)/r_c| \ll 1.$$  

(59)

Thus equation (58) represents the two independent solutions of the homogeneous wave equation for all radii.

As discussed above, waves are mainly excited at the L/VR, where the waveform is given by equation (42). Matching equation (43) or equation (45) with the general solution, $\eta_1 = C_+ \eta_1^+ + C_- \eta_-$ (where $C_\pm$ are constants), we find that the waves away from the L/VR are

$$\eta_1 = \left(\frac{i \pi r^2 \sigma}{3 \omega^2}\right)^{1/2} f_1(r_c) \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right] \exp \left[i \int_{r_c}^{r} k_r \, dr\right] (r > r_c),$$

(60)

$$\eta_1 = \left(-\frac{i \pi r^2 \sigma}{3 \omega^2}\right)^{1/2} f_1(r_c) \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right] \exp \left[-i \int_{r_c}^{r} k_r \, dr\right] (r_c < r < r_r).$$

(61)

Note that although for $r_c < r < r_r$ the wave (61) has positive phase velocity (since $k_c < 0$ for $r_c < r < r_r$), the group velocity is negative, i.e., the wave propagates toward small radii. As this wave approaches $r_c$, the waveform reduces to (see eq. [53])

$$\eta_1 = A_+ x^{1/2} e^{i \theta \ln x} \quad \left(0 < x = \frac{r - r_c}{r_c} << 1\right),$$

(62)

where $\nu = 2r_c/(3h_c)$ and

$$A_+ = \left(-\frac{i \pi r^2 \sigma}{3 \omega^2}\right)^{1/2} f_1(r_c) \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right] \exp \left[-i \int_{r_c}^{r} k_r \, dr - i \nu \ln x\right],$$

(63)

where $0 < x_+ \equiv (r_r - r_c)/r_c < 1$. After passing through the CR, the wave amplitude is significantly reduced,

$$\eta_1 = \left(\frac{i \pi r^2 \sigma}{3 \omega^2}\right)^{1/2} f_1(r_c) \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right] \exp \left[-i \int_{r}^{r_c} k_r \, dr - i \nu \ln x\right] \quad \left(x = \frac{r - r_c}{r_c} < 0, \quad |x| \ll 1\right).$$

(64)

This then joins onto the inward-going wave solution

$$\eta_1 = e^{-\nu \theta} \left(\frac{i \pi r^2 \sigma}{3 \omega^2}\right)^{1/2} f_1(r_c) \left[\frac{\hbar \Omega(\omega - \Omega)}{r \sigma}\right] \exp \left[-i \int_{r}^{r_c} k_r \, dr - i \nu \ln x\right] \quad (r < r_c),$$

(65)

with $\varphi = \int_{r}^{r_c} k_r \, dr + \nu \ln (-x_+/x), \quad \text{where} \quad x_+ = (r_r - r_c)/r_c < 0$ and $|x_+| << 1$. Using equation (55), we find that the angular momentum flux deposited at $r_c$ is

$$\Delta F_c = \frac{\pi^2}{3 \omega^2} \left(r^2 \sigma f_1^2 \right)_c,$$

(66)

in agreement with the angular momentum flux emitted from $r_L$ toward corotation (see eq. [46]). Figure 2 depicts the global solution describing waves excited at the L/VR, which either propagate outward or inward toward the CR at which the angular momentum deposition occurs.

6. APPLICATION TO QPOs

The main result of the paper is illustrated in Figure 2. It shows that under a periodic magnetic vertical forcing, the disk responds by launching $m = n = 1$ waves from the Lindblad/vertical resonance (where $\omega = 2\Omega$, with the forcing frequency $\omega = \omega_r$ or
The wave either propagates to large radii or propagates inward where it gets absorbed at the corotation resonance (where $\omega = \Omega$).

We have already shown that the torque carried by the excited wave is small compared to the canonical accretion torque (see §4.1). Nevertheless, the wave may manifest itself by inducing or modulating variabilities. Clearly, the wave is most visible at the Lindblad-vertical resonance, since the wavelength is the largest there. Variations caused by the wave are most likely due to fluid elements around $r_L$. From equations (60)–(61), we find that the amplitude of enthalpy perturbation at $r_L$ is given by $|\eta| = (\pi r/12 h)^{1/2} (F h/\sigma)$ (all quantities are evaluated at $r = r_L$). With $F_\omega \sim \mu^2/(4\pi c^6)$ (see §2), $r_m = r_c/\lambda$ (where $\lambda > 1$ is a constant), $r_L = 2^{2/3} r_c$, and using equation (1) and $M = 2\pi r^2|\eta|$, we find the dimensionless amplitude

$$|\eta|/c^2 \sim 0.05(\lambda^2)^{-7/2} \left( \frac{F c^2}{h \omega^2} \right)^{1/2}$$

where $\omega_c$ is the radial (inflow) velocity. For an $\alpha$-disk, with $|\eta|/c^2 \sim (3/2r)\alpha h$ not too close to the disk inner edge, we have

$$|\eta|/c^2 \sim 0.1\alpha(\lambda^2)^{-7/2} \left( \frac{h}{r_L} \right)^{1/2}.$$  \hspace{1cm} (68)

The dimensionless vertical velocity perturbation, $|\eta_z|/c$, is of the same order of magnitude at $r \sim r_L$ as $|\eta|/c^2$. Since $\lambda^2 \sim 1$, the dimensionless perturbation amplitude may reach a few percent, while definitely remaining in the linear regime. Note that since the vertical velocity perturbation amplitude increases toward the corotation radius, nonlinear saturation will occur there. But since the wavelength approaches zero as $r \to r_c$, variations caused by fluid elements around the corotation resonance may not be visible.

It seems inevitable that the bending waves studied in this paper will be excited in disks around magnetic stars. We may suggest various ways that these resonantly excited waves may induce variabilities and QPOs. In the context of LMXBs (van der Klis...
2006), one may imagine that the oscillating fluid perturbation at the Lindblad resonance can produce a beat phenomenon by modulating the “seed” radiation from the inner region of the disk (e.g., Lamb & Miller 2003). Suppose the “seed” radiation has a quasi-periodic variability with frequency $v_h$ (e.g., due to orbital motion of blobs at the disk inner edge or some other mechanisms). The fluid element at the Lindblad/vertical resonance has orbital frequency $v_l = \Omega_l / 2$ (since $\Omega_l / 2 = \Omega_l$ for $m = 1$, and $\Omega_l \approx \Omega$). The reprocessed radiation would then show an additional QPO at frequency $v_l = v_h/2$. In this regard, it is particularly interesting that in our model, the driving frequency $v_d = \Omega / (2\Omega_s)$ can be either the spin frequency $\Omega_s$ or twice that, depending on the disk geometry with respect to the stellar spin and dipole axes (see § 2). As noted in § 1, both $v_h - v_l = \nu_s$ and $\nu_l = \nu_s/2$ have been observed in LMXBs. This feature has not been explained by current models (see van der Klis 2006). For many systems, the beat is not perfect. This might be due to the fact that the excited wave still has significant strength away from the Lindblad/vertical resonance (see Fig. 2).

Finally we note that our treatment of wave excitation by magnetic forces is performed under simplifying assumptions, as detailed in § 3. We hope that the novel features of our model in connection with QPOs will motivate future, more rigorous studies of the dynamics of waves in disks around magnetic stars.

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REFERENCES

Agapitou, V., Papaloizou, J. C. B., & Terquem, C. 1997, MNRAS, 292, 631
Altamirano, D., et al. 2008, ApJ, 674, L45
Aly, J. J. 1980, A&A, 86, 192
Aly, J. J., & Kuippers, J. 1990, A&A, 227, 473
Anzer, U., & Börner, G. 1983, A&A, 122, 73
Arons, J. 1993, ApJ, 408, 160
Boutloukos, S., et al. 2006, ApJ, 653, 1435
Bouvier, J., et al. 2007a, in Protostars and Planets V, ed. B. Reipurth, D. Jewitt, & K. Keil (Tucson: Univ. Arizona Press), 479
———. 2007b, A&A, 463, 1017
Campbell, C. G. 1997, Magnetohydrodynamics in Binary Stars (Dordrecht: Kluwer)
Casella, P., Altamirano, D., Wijnands, R., & van der Klis, M. 2008, ApJ, 674, L41
Fendt, C. 2003, Ap&SS, 287, 59
Ghosh, P., & Lamb, F. K. 1979, ApJ, 232, 259
Goodson, A. P., Winglee, R. M., & Böhm, K.-H. 1997, ApJ, 489, 199
Hayashi, M. R., Shibata, K., & Matsumoto, R. 1996, ApJ, 468, L37
Kato, S. 2001, PASJ, 53, 1
Lai, D. 1998, ApJ, 502, 721
———. 1999, ApJ, 524, 1030
Lamb, F. K., & Miller, M. C. 2003, preprint (astro-ph/0308179)
Lewin, W., & van der Klis, M. 2006, Compact Stellar X-Ray Sources (Cambridge: Cambridge Univ. Press)
Li, J., Wickramasinghe, D. T., & Rüdiger, G. 1996, ApJ, 469, 765
Linares, M., et al. 2005, ApJ, 634, 1250
Lubow, S. H., 1981, ApJ, 245, 274
Lubow, S. H., et al. 2007, ApJ, 674, L45
Lloyd, C. C., & Pudritz, R. E. 2005, MNRAS, 356, 167
Mendez, M., & Belloni, T. 2007, MNRAS, 381, 790
Miller, K. A., & Stone, J. M. 1997, ApJ, 489, 890
Ogilvie, G. I. 2006, MNRAS, 365, 977
Ogilvie, G. I., & Lin, D. N. C. 1999, PASJ, 51, 3
O'Sullivan, M., et al. 2005, MNRAS, 358, 632
Papaloizou, J. C. B., & Lin, D. N. C. 1995, ApJ, 438, 841
Petri, J. 2005, A&A, 439, 443
Pfeiffer, H. P., & Lai, D. 2004, ApJ, 604, 766
Romanova, M. M., Ustyugova, G. V., Koldoba, A. V., Wick, J. V., & Lovelace, R. V. E. 2003, ApJ, 595, 1099
Romanova, M. M., et al. 2006, Adv. Space Res., 38, 2887
Shirakawa, A., & Lai, D. 2002a, ApJ, 564, 361
———. 2002b, ApJ, 565, 1134
Shu, F. H., Najita, J. R., Shang, H., & Li, Z.-Y. 2000, in Protostars and Planets IV, ed. V. Mannings, A. P. Boss, & S. S. Russell (Tucson: Univ. Arizona Press), 789
Shu, F. H., et al. 1994, ApJ, 429, 781
Spruit, H. C., & Taam, R. E. 1993, ApJ, 402, 593
Stone, J. M., & Norman, M. L. 1994, ApJ, 433, 746
Terquem, C., & Papaloizou, J. C. B. 2000, A&A, 360, 1031
Ustyugova, G. V., Koldoba, A. V., Romanova, M. M., & Lovelace, R. V. E. 2006, ApJ, 646, 304
Uzdensky, D. A. 2004, Ap&SS, 292, 573
van der Klis, M. 2006, in Compact Stellar X-Ray Sources, ed. W. Lewin & M. van der Klis (Cambridge: Cambridge Univ. Press), 39
Wang, Y.-M. 1995, ApJ, 449, L153
Warner, B. 2004, PASP, 116, 115
Wijnands, R., et al. 2003, Nature, 424, 44
Yin, H. X., et al. 2007, A&A, 471, 381
Zhang, H., & Lai, D. 2006, MNRAS, 368, 917

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