Group-Theoretical Revision of the Unruh Effect

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Abstract. We revise the Unruh effect (vacuum radiation in uniformly relativistic accelerated frames) in a group-theoretical setting by constructing a conformal SO(4,2)-invariant quantum field theory and its spontaneous breakdown when selecting Poincaré invariant degenerated vacua (namely, coherent states of conformal zero modes). Special conformal transformations (accelerations) destabilize the Poincaré vacuum and make it to radiate.

1. Introduction

The Fulling-Davies-Unruh effect [1, 2, 3] has to do with vacuum radiation in a non-inertial reference frame and shares some features with the (black-hole) Hawking [4] effect. In simple words, whereas the Poincaré invariant vacuum |0⟩ in QFT looks the same to any inertial observer (i.e., it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

\[ T = \frac{\hbar a}{2\pi v k_B} \]  

(1)
in passing to a uniformly accelerated frame (a denotes the acceleration, v the speed of light and \( k_B \) the Boltzmann constant).

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved (viz, existence of horizons). This is the case of the natural quantization in Rindler coordinates [2, 5], which leads to a quantization inequivalent to the normal Minkowski quantization (see next Section), or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [1].

In the reference [6], it was showed that the reason for the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations (that is, the Unruh effect) is more profound and related to the spontaneous breakdown of the conformal symmetry in quantum field theory. From this point of view, a Poincaré invariant vacuum will be regarded as a coherent state of conformal zero modes, which are undetectable (“dark”) by inertial observers but unstable under special conformal transformations

\[ x^\mu \rightarrow x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx^2 + c^2x^2}, \]  

(2)
which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration \( a = 2c \) (see e.g. Ref. [7, 8, 9] and later on Eq. (14)). In the reference [6] a
quite involved “second quantization formalism on a group $G$” was developed, which was applied to the conformal group in $(1+1)$ dimensions, $SO(2,2) \simeq SO(2,1) \times SO(2,1)$, which consists of two copies of the pseudo-orthogonal group $SO(2,1)$ (left- and right-moving modes, respectively). Here we shall use more conventional methods of quantization and we shall work in realistic $(3+1)$ dimensions, using the (more involved) conformal group $SO(4,2) \simeq SU(2,2)/Z_4$.

The point of view exposed in this paper is consistent with the idea that quantum vacua are not really empty to every observer. Actually, the quantum vacuum is filled with zero-point fluctuations of quantum fields. The situation is similar to quantum many-body condensed matter systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. Moreover, we know that zero-point energy, like other non-zero vacuum expectation values, leads to observable consequences as, for instance, the Casimir effect, and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the universe (see e.g. [10] for a nice review). Indeed, dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelerating rate.

The organization of the paper is as follows. In Section 2 we briefly review the standard explanation for the Unruh effect, which has to do with space-time mutilation and Bogolyubov transformations. In Section 3 we construct a conformal-invariant quantum theory in $3+1$ dimensions and in Section 4 we discuss its (spontaneous) breakdown to a Poincaré-invariant quantum theory by selecting Poincaré-invariant pseudo-vacua which are coherent states of conformal zero modes. We compute the mean energy of the accelerated Poincaré-invariant pseudo-vacua. This is part of a work in preparation [11], where the reader will find more details and additional results.

2. Vacuum radiation as a consequence of space-time mutilation

The existence of event horizons in passing to accelerated frames of reference leads to unitarily inequivalent representations of the quantum field canonical commutation relations and to a (ill-)definition of particles depending on the state of motion of the observer.

2.1. Field decompositions and vacua

To use an explicit example, let us consider a real scalar massless field $\phi(x)$, satisfying the Klein-Gordon equation

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) = 0.$$  \hspace{1cm} (3)

Let us denote by $a_k, a_k^*$ the Fourier coefficients of the decomposition of $\phi$ into positive and negative frequency modes:

$$\phi(x) = \int dk (a_k f_k(x) + a_k^* f_k^*(x)).$$  \hspace{1cm} (4)

The Fourier coefficients $a_k, a_k^*$ are promoted to annihilation and creation operators $\hat{a}_k, \hat{a}_k^*$ of particles in the quantum field theory. The Minkowski vacuum $\ket{0}_M$ is defined as the state nullified by all annihilation operators

$$\hat{a}_k \ket{0}_M = 0, \forall k.$$  \hspace{1cm} (5)

2.2. Rindler coordinate transformations

Let us consider now the Rindler coordinate transformation (see e.g. [5]):

$$t = a^{-1} e^{az'} \sinh(at'), \quad z = a^{-1} e^{az'} \cosh(at').$$  \hspace{1cm} (6)
The worldline $z' = 0$ has constant acceleration $a$ (in natural unities). This transformation entails a mutilation of Minkowski spacetime into patches or charts with event horizons.

The new coordinate system provides a new decomposition of $\phi$ into Rindler positive and negative frequency modes:

$$\phi(x') = \int dq(a_q^+ f_q^+(x') + a_q^- f_q^-(x')).$$

(7)

The Rindler vacuum $|0\rangle_R$ is defined as the state nullified by all Rindler annihilation operators:

$$\hat{a}_q^+|0\rangle_R = 0 \forall q.$$  

(8)

Let us see that the Minkowski vacuum $|0\rangle_M$ and the Rindler vacuum $|0\rangle_R$ are not identical. In fact, the Minkowski vacuum $|0\rangle_M$ has a nontrivial content of Rindler particles.

2.3. Bogolyubov transformations

The Fourier components $a_q', a_q^+$ of the field $\phi$ in the new (accelerated) reference frame are expressed in terms of both $a_k, a_k^+$ through a Bogolyubov transformation:

$$a_q' = \int dk \left( \alpha_{qk} a_k + \beta_{qk} a_k^+ \right),$$
$$\alpha_{qk} = \langle f_q'| f_k \rangle, \quad \beta_{qk} = \langle f_q^+| f_k \rangle.$$  

(9)

The vacuum states $|0\rangle_M$ and $|0\rangle_R$, defined by the conditions (5) and (8), are not identical if the coefficients $\beta_{qk}$ in (9) are not zero. In this case the Minkowski vacuum has a non-zero average number of Rindler particles given by:

$$N_R = \langle 0|\hat{N}_R|0\rangle_M = \langle 0| \int dq \hat{a}_q^+ \hat{a}_q|0\rangle_M = \int dk dq |\beta_{qk}|^2.$$  

(10)

That is, in the second quantized theory, the vacuum states $|0\rangle_M$ and $|0\rangle_R$ are not identical if the coefficients $\beta_{qk}$ are not zero. Both quantizations are inequivalent.

3. Vacuum radiation as a spontaneous breakdown of de conformal symmetry

In this section we shall offer an alternative explanation for the Unruh effect based on symmetry grounds. Actually, in Quantum Field Theory, the vacuum state is expected to be stable under some underlying group of symmetry transformations $G$ (namely, the Poincaré group). Then the action of some spontaneously broken symmetry transformations can destabilize/excite the vacuum and make it to radiate. We shall see that this is precisely the case of the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations. Here, the Poincaré invariant vacuum looks the same to every inertial observer but converts itself into a thermal bath of radiation with temperature (1) in passing to a uniformly accelerated frame. In fact, in the reference [6], it was shown that the reason for this radiation is related to the spontaneous breakdown of the conformal symmetry in quantum field theory.

3.1. The conformal group and the compactified Minkowski space

The conformal group in 3+1 dimensions, $SO(4,2)$, is composed by Poincaré (spacetime translations $b^\mu \in \mathbb{R}^4$ and Lorentz $L^\mu_\nu(\in SO(3,1))$ transformations augmented by dilations ($c^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, $c^\mu \in \mathbb{R}^4$) which, in Minkowski spacetime, have the following realization:

$$x'^\mu = x^\mu + b^\mu, \quad x'^\mu = L^{\mu}_{\nu}(\omega)x^\nu,$$
$$x'^\mu = c^\tau x^\mu, \quad x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2c x^2 + c^2 x^2}.$$  

(11)

(12)
respectively. The interpretation of special conformal transformations
gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:
\[ z' = \frac{1}{a}(\sqrt{1 + a^2t'^2} - 1) \]
with \( a = 2\alpha \).

The infinitesimal generators (vector fields) of the transformations (12) are easily deduced:
\[
P_{\mu} = \frac{\partial}{\partial x^{\mu}}, \quad M_{\mu\nu} = x^{\mu} \frac{\partial}{\partial x^{\nu}} - x^{\nu} \frac{\partial}{\partial x^{\mu}},
\]
and they close into the conformal Lie algebra
\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho},
\]
\[
[P_{\mu}, M_{\rho\sigma}] = \eta_{\rho\mu}P_{\sigma} - \eta_{\sigma\mu}P_{\rho}, \quad [P_{\mu}, P_{\nu}] = 0,
\]
\[
[K_{\mu}, M_{\rho\sigma}] = \eta_{\rho\mu}K_{\sigma} - \eta_{\sigma\mu}K_{\rho}, \quad [K_{\mu}, K_{\nu}] = 0,
\]
\[
[D, P_{\mu}] = -P_{\mu}, \quad [D, K_{\mu}] = K_{\mu}, \quad [D, M_{\mu\nu}] = 0,
\]
\[
[K_{\mu}, P_{\nu}] = 2(\eta_{\mu\nu}D + M_{\mu\nu}).
\]
The quadratic Casimir operator has the following expression:
\[
C_2 = -2D^2 + M_{\mu\nu}M^{\mu\nu} - P_{\mu}K^{\mu} - K_{\mu}P^{\mu} = -2D^2 + M_{\mu\nu}M^{\mu\nu} - 2P_{\mu}K^{\mu} - 8D.
\]
Any group element \( g \in SO(4, 2) \) (near the identity element 1) could be written as the exponential map
\[
g = \exp(u), \quad u = \tau D + b^{\mu}P_{\mu} + a^{\mu}K_{\mu} + \omega^{\mu\nu}M_{\mu\nu},
\]
of the Lie-algebra element \( u \).

One would be tempted to blame the singular character of the special conformal transformation (13,14) to be responsible for the radiation effect, in much the same way the (singular) Rindler transformations (6) are supposedly responsible for the Unruh effect (i.e., existence of horizons). However, one could always work with the compactified Minkowski space \( \mathbb{M} = S^3 \times \mathbb{Z}_2 \times \mathbb{S}^1 \simeq U(2) \), which can be obtained as a coset \( \mathbb{M} = SO(4, 2)/\mathbb{W} \), where \( \mathbb{W} \) denotes the Weyl subgroup generated by \( K_{\mu}, M_{\mu\nu} \) and \( D \) (i.e., a Poincaré subgroup \( \mathbb{P} = SO(3, 1) \oplus \mathbb{R}^4 \) augmented by the dilations \( \mathbb{R}^+ \)). The Weyl group \( \mathbb{W} \) is the stability subgroup (the little group in physical usage) of \( x^\mu = 0 \). Now the conformal group acts transitively on \( \mathbb{M} \) and free from singularities. So, where would the radiation come from now? Let us see that the reason for vacuum radiation in relativistic uniformly accelerated frames is more profound an related to the spontaneous breakdown of the conformal symmetry.
3.2. $SU(2, 2)$ as the four-cover of $SO(4, 2)$

Instead of $SO(4, 2)$, we shall work by convenience with its four cover:

$$SU(2, 2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{4 \times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}, \quad (19)$$

where $\Gamma$ denotes a hermitian form of signature $(++, --)$.

The conformal Lie algebra (16) can also be realized in terms of gamma matrices in, for instance, the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \quad (20)$$

where $\tilde{\sigma}^\mu \equiv \sigma_\mu$ (we are using the convention $\eta = \text{diag}(1, -1, -1, -1)$) and $\sigma^\mu$ are the standard Pauli matrices

$$\sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (21)$$

Indeed, the choice

$$D = \frac{\gamma^5}{2}, \quad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4} \begin{pmatrix} \sigma^{\mu}\tilde{\sigma}^{\nu} - \sigma^{\nu}\tilde{\sigma}^{\mu} & 0 \\ 0 & \tilde{\sigma}^{\mu}\sigma^{\nu} - \tilde{\sigma}^{\nu}\sigma^{\mu} \end{pmatrix},$$

$$P^\mu = \gamma^\mu\frac{1 + \gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \quad K^\mu = \gamma^\mu\frac{1 - \gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \quad (22)$$

fulfils the commutation relations (16). These are the Lie algebra generators of the fundamental representation of $SU(2, 2)$.

The group $SU(2, 2)$ acts transitively on the compactified Minkowski space $\mathbb{M}_4 = U(2)$, with (matrix) coordinates $X$, as

$$X \rightarrow X' = (AX + B)(CX + D)^{-1}. \quad (23)$$

Setting $X = x_\mu \sigma^\mu$ (with $\sigma^\mu$ Pauli matrices) the transformations (12) can be recovered from (23) as follows:

i) Standard Lorentz transformations, $x'^\mu = \Lambda_\mu^\nu(\omega)x^\nu$, correspond to $B = C = 0$ and $A = D^{-1} = \text{SL}(2, \mathbb{C})$.

ii) Dilations correspond to $B = C = 0$ and $A = D^{-1} = a^{1/2}\sigma^0$.

iii) Spacetime translations equal $A = D = \sigma^0$, $C = 0$ and $B = b_\mu \sigma^\mu$.

iv) Special conformal transformations correspond to $A = D = \sigma^0$ and $C = c_\mu \sigma^\mu, B = 0$ by noting that $\det(CX + I) = 1 + 2cx + c^2x^2$:

$$X' = X(CX + I)^{-1} \leftrightarrow x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2x^2}.$$

3.3. Unirreps of the conformal group: discrete series

We shall consider the complex extension of $\mathbb{M}_4 = U(2)$ to the 8-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2, 2)/U(2)^2 = \{ Z \in \text{Mat}_{2 \times 2}(\mathbb{C}) : I - ZZ^\dagger > 0 \}$$
and the Unirrep
\[
[U_\lambda(g)\phi](Z) = |CZ + D|^{-\lambda}\phi(Z'), \quad Z' = (AZ + B)(CZ + D)^{-1}
\] (24)
on the space \(\mathcal{H}_\lambda(\mathbb{D}_4)\) of square-integrable holomorphic functions \(\phi\) with invariant integration measure
\[
d\mu_\lambda(Z, Z^\dagger) = \pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3)\det(I - ZZ^\dagger)^{\lambda - 4}|dZ|,
\]
where the label \(\lambda \in \mathbb{Z}, \lambda \geq 4\) is the conformal, scale or mass dimension (\(|dZ|\) denotes the Lebesgue measure in \(\mathbb{C}^4\)).

3.4. The Hilbert space of our conformal particle
It has been proved in [14] that the infinite set of homogeneous polynomials
\[
\varphi_{q_1, q_2}^{j, m}(Z) = \sqrt{\frac{2j + 1}{\lambda - 1}} \left(\frac{m + \lambda - 2}{\lambda - 2}\right)^{j}\det(Z)^m D_{q_1, q_2}^j(Z),
\] (25)
with
\[
D_{q_1, q_2}^j(Z) = \sqrt{\frac{(j + q_1)!(j - q_2)}{(j + q_2)!(j - q_1)!}} \sum_{p = \max(0, q_1 - q_2)}^{\min(q_1, q_2)} \binom{j + q_2}{p} \binom{j - q_2}{p - q_1 - q_2} z_1^{12} z_2^{j + q_1 - p} z_2^{p - q_1 - q_2}
\] (26)
the standard Wigner’s \(D\)-matrices \((j \in \mathbb{N}/2)\), verifies the following closure relation (the reproducing Bergman kernel):
\[
\sum_{j=N/2}^{\infty} \sum_{m=0}^{\infty} \sum_{q_1, q_2 = -j}^{j} \varphi_{q_1, q_2}^{j, m}(Z) \varphi_{q_1, q_2}^{j, m}(Z') = \frac{1}{\det(I - ZZ^\dagger)^{\lambda}}
\] (27)
and constitutes an orthonormal basis of \(\mathcal{H}_\lambda(\mathbb{D}_4)\), (the sum on \(j\) accounts for all non-negative half-integer numbers)

3.5. Hamiltonian and energy spectrum
In [15] we have argued that the dilation operator \(D\) plays the role of the Hamiltonian of our quantum theory. Actually, the replacement of time translations by dilations as dynamical equations of motion has already been considered in [16] and in [17], when quantizing field theories on space-like Lorentz-invariant hypersurfaces \(x^2 = x^\mu x_\mu = \tau_2^2 = \text{constant}\). In other words, if one wishes to proceed from one surface at \(x^2 = \tau_2^2\) to another at \(x^2 = \tau_2^2\), this is done by scale transformations; that is, \(D\) is the evolution operator in a proper time \(\tau\).

From the general expression (24), we can compute the finite left-action of dilations \(g = e^{\tau D}\) \((B = 0 = C\) and \(A = e^{-\tau^2/2}g^0 = D^{-1}\)\) on wave functions,
\[
[U_\lambda(g)\phi](Z) = e^{\lambda\tau}\phi(e^{\tau}Z).
\] (28)
The infinitesimal generator of this transformation is the Hamiltonian operator:
\[
H = \lambda + \sum_{i,j=1}^{2} Z_{ij} \frac{\partial}{\partial Z_{ij}} = \lambda + z_\mu \frac{\partial}{\partial z_\mu}.
\] (29)
The set of functions (25) constitutes a basis of Hamiltonian eigenfunctions (homogeneous polynomials) with energy eigenvalues \(E_n^\lambda\) (the homogeneity degree) given by:
\[
H \varphi_{q_1, q_2}^{j, m} = E_n^\lambda \varphi_{q_1, q_2}^{j, m}, \quad E_n^\lambda = \lambda + n, \quad n = 2j + 2m.
\] (30)
Actually, each energy level $E_n^\lambda$ is $(n + 1)(n + 2)(n + 3)/6$ times degenerated. This degeneracy coincides with the number of linearly independent polynomials $\prod_{i,j=1}^2 Z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^2 n_{ij}$. This proves that the set of polynomials (25) is a basis for analytic functions $\phi \in \mathcal{H}_M(M)$. The spectrum is equi-spaced and bounded from below, with ground state $\varphi_{0,0} = 1$ and zero-point energy $E_0^\lambda = \lambda$.

3.6. The ground state is Poincaré-stable and polarized by accelerations

Under a general conformal transformation, the excited ground state is:

$$\varphi_{0,0}(Z) = [U_\lambda(g)\varphi_{0,0}^{|0|}](Z) = \det(CZ + D)^{-\lambda}.$$  \hfill (31)

Therefore, for Poincaré transformations we have $C = 0$ and $\det(D) = 1$, which means that the ground state $\varphi_{0,0}$ looks the same to every inertial observer (it is stable for zero acceleration $C = 0$, i.e., it is Poincaré invariant). We shall restrict from now on to pure accelerations: $D = A = I, B = 0$ and $C = c_\mu \sigma^\mu$. For arbitrary accelerations, $C \neq 0$, we can decompose the accelerated ground state $\varphi_{0,0}$ using the Bergman kernel expansion (27) as:

$$\varphi_{0,0}(Z) = \sum_{j \in \mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1, q_2 = -j}^j \varphi_{q_1, q_2}^{j|m}(-C)\varphi_{q_1, q_2}^{j|m}(Z).$$ \hfill (32)

From here, we interpret the coefficient $\varphi_{q_1, q_2}^{j|m}(-C)$ as the probability amplitude of finding the accelerated ground state in the excited level $\varphi_{q_1, q_2}^{j|m}$ of energy $E_n^\lambda = \lambda + 2j + 2m = \lambda + n$.

3.7. Mean Energy of the accelerated ground state

The energy expectation value in the accelerated ground state (32) can be calculated as:

$$\mathcal{E}(C) = \frac{\sum_{j=0}^\infty \sum_{n=0}^\infty \sum_{q_1, q_2 = -j}^j |\varphi_{q_1, q_2}^{j|m}(C)|^2(\lambda + 2j + 2n)}{\sum_{j=0}^\infty \sum_{n=0}^\infty \sum_{q_1, q_2 = -j}^j |\varphi_{q_1, q_2}^{j|m}(C)|^2}. $$ \hfill (33)

Using (27) and its derivatives in (30), with the Hamiltonian operator given by (29), we obtain:

$$\sum_{j=0}^\infty \sum_{n=0}^\infty \sum_{q_1, q_2 = -j}^j |\varphi_{q_1, q_2}^{j|m}(C)|^2 = \frac{1}{\det(I - C\lambda)},$$

$$\lambda(\lambda + 2) + 2n = \lambda^2 - \det(C^\dagger C) \lambda + 1,$$ \hfill (34)

so that the mean energy is:

$$\mathcal{E}(C) = \lambda \frac{1 - \det(C^\dagger C)}{\det(I - C^\dagger C)}.$$ \hfill (35)

For the particular case of an acceleration along the “z” axis, $C = c_\mu \sigma^\mu = \sigma^3$, we have:

$$\mathcal{E}(c) = \frac{\lambda}{1 - c^2} = \lambda \frac{1}{\lambda/2} = \lambda + 2\lambda c^2 = \lambda + 2\lambda \frac{c^2}{1 - c^2}. $$ \hfill (36)

The mean energy (36) is of Planckian type for

$$c^2(T) \equiv e^{-\frac{h
u}{k T}}, $$ \hfill (37)
where we have introduced dimensions, with $\hbar \nu$ the quantum of energy for our harmonic oscillator. The last (*ad hoc*) assignment (37) can be in fact obtained from first thermodynamical principles (see [11] for the actual computation of the partition function, entropy and temperature). Here we shall just point out that the formula (37) gives 

$$T = -T_E / \ln(\mathcal{E}), \ T_E = \hbar \nu / k_B,$$

which differs from the Unruh formula (1). It also implies the existence of a maximal acceleration $a_{\text{max}}$, so that the dimension-full acceleration is $a = a_{\text{max}} c$ (see [18] for a discussion on the physical consequences of a maximal acceleration and [15] and [11] for more details).

4. Second-Quantized Theory, Conformal Zero Modes and $\theta$-Vacua
We have discussed the effect of relativistic accelerations in first (one particle) quantization. However, the proper setting to analyze radiation effects is in the second quantized theory. Let us denote (for space-saving notation) by $n = \{j, m, q_1, q_2\}$ the multi-index of the basis functions (25). The Fourier coefficients $a_n$ (and $a_n^*$) of the expansion in energy modes of a state

$$\phi = \sum_n a_n \varphi_n,$$

are promoted to annihilation $\hat{a}_n$ (and creation $\hat{a}_n^\dagger$) operators in second quantization. The fact that the ground state of first quantization, $\varphi_0$, is invariant under Poincaré transformations implies that $\hat{a}_0$ commutes with all Poincaré generators. It also commutes with all annihilation operators and creation operators 

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0, \ n > 0 \ (40)$$

of particles with non-zero energy. Therefore, by Schur’s Lemma, $\hat{a}_0$ must behave as a multiple of the identity in the broken theory, which means that Poincaré “$\theta$-vacua” fulfil

$$\hat{a}_0 |\theta\rangle = \theta |\theta\rangle \Rightarrow |\theta\rangle = e^{\theta \hat{a}_0 - \theta \hat{a}_0^\dagger} |0\rangle. \quad (41)$$

That is, Poincaré “$\theta$-vacua” are coherent states of conformal zero modes (see [19] and [20] for a thorough exposition on coherent states).

The second-quantized version of (32) for an acceleration $C = c \sigma^3$ along the third axis is:

$$\hat{a}_n' = \sum_{n=0}^{\infty} \varphi_n(c) \hat{a}_n, \quad (42)$$

so that accelerated Poincaré $\theta$-vacua become:

$$|\theta\rangle' = e^{\theta \hat{a}_0' - \theta \hat{a}_0'^\dagger} |0\rangle. \quad (43)$$

The average number of particles with energy $E_n$ in the accelerated vacuum (43) is then given by

$$N(c) = \langle \theta' | \hat{a}_n^\dagger \hat{a}_n | \theta' \rangle = |\theta|^2 |\varphi_n(c)|^2, \quad (44)$$

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$, and $|\varphi_n(c)|^2$ is the probability of finding a particle in the energy state $E_n$ of the accelerated vacuum $|\theta'\rangle$. One can also compute the mean energy per mode and see that it reproduces the Black-Body spectrum for (37). See more details in the coming paper [11].
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