On the classification of 3-dimensional complex hom-Lie algebras

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Abstract

We classify hom-Lie structures with nilpotent twisting map on 3-dimensional complex Lie algebras, up to isomorphism, and classify all degenerations in such family. The ideas and techniques presented here can be easily extrapolated to study similar problems in other algebraic structures and provide different perspectives from where one can tackle classical open problems of interest in rigid Lie algebras.

1 Introduction

The concept of hom-Lie algebra has its origin in Quantum Calculus. In the discrete context, when one replace the usual way of differentiating functions by a $\sigma$-derivation, it is expected that similar properties, identities and formulas to the ones in differential calculus appear (e.g. Twisted Leibniz rule). In the case of the notion of hom-Lie algebra, such definition was introduced in the Daniel Larsson’s doctoral thesis, [34], (see also publications resulting from this thesis, as [25]) and it is motivated by some examples of quantum deformations of algebras of vector fields which satisfied a twisted Jacobi identity with respect to an appropriate (Lie) bracket operation. Potential applications of such notion were considered in the mentioned thesis and subsequently in [35]. Over the last fifteen years, the hom-Lie algebras have become a research topic in algebra and many papers on such algebraic structures have been dedicated to obtain results about (regular) hom-Lie algebras generalizing notions and ideas well-known from Lie algebra theory (in some cases, mutatis mutandis).

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Our motivation to study hom-Lie algebras comes from classical open problems in Lie theory, such as a conjecture due to Michèle Vergne, which states that there are no rigid complex nilpotent Lie algebras in the algebraic set of all \( n \)-dimensional complex Lie algebras ([65, p. 83]), or more generally the Grunewald-O’Halloran Conjecture: every complex nilpotent Lie algebra is the degeneration of another nonisomorphic Lie algebra. Both of these problems are relevant for the understanding of the geometry and algebraic properties of Nilpotent Lie groups. For reasons that we will explain in Remark 2.8, regular hom-Lie algebra structures on a Lie algebra can be used to deform it linearly. It is also of interest to study how much we can say of a Lie algebra about its different hom-Lie algebra structures; which correspond to the solution set of a linear equation system depending on the Lie algebra. A precursor result about this kind of ideas is the well known Jacobson’s theorem concerning Lie algebras admitting non-singular derivations and its very nice generalization due to Wolfgang Moens (see [44]), or more generally, the work of Alice Fialowski, Abror Khudoyberdiyev and Bakhrom Omirov for the Leibniz algebra case in [14].

In this paper, we tackle the problem of classifying hom-Lie structures on 3-dimension complex Lie algebras and their degenerations. We give more importance to the techniques and ideas for doing so, we then focus our attention to present only the case when the twisting map is a nilpotent operator, although the general case can be obtained in the same way. Of the different ways to obtain the classification, we have preferred a more practical approach, while avoiding symbolic computations or any no easy verifiable result. As far as degenerations of the mentioned structures are concerned, we use the classification of orbit closures of 3-dimensional complex Lie algebras to reduce our exposition and do a more substantial presentation of the results and techniques, which can be implemented in many other problems of a similar nature ([12, 57]); in fact, we are guided by the forgotten ancient proverb that says

“Algebras are like persons; to known how an algebra is, we need to observe what your algebra does, likes ... and how your algebra behaves towards all things around it”.

Clearly results such as Élie Cartan’s criterion for semisimplicity (for solvability), the Jacobson-Moens theorem mentioned earlier, the work of Dietrich Burde and Manuel Ceballos in [11], the results obtained by Valerii Filippov in [15, 19] or by George Leger and Eugene Luks in [36] can be considered as illustrative examples of this saying. Here, we can also mention the Jiří Hrivnáč’s doctoral thesis [30] (and its publications [29, 51]) which introduced and studied some Lie algebra invariants defined by slightly modifying the linear equations corresponding to the notions of derivation of a Lie algebra and cocycle of the adjoint representation, and which have been used effectively to re-examine the classification of Lie algebras and their degenerations in low dimensions.

Originally, the study of degenerations of algebraic structures, especially in the context of Lie algebras or associative algebras, was largely driven by applications in Mathematical physics ([46]), Geometry and Algebra: examples include the solution by Peter Gabriel to a question of Maurice Auslander concerning associative algebras of finite representation type ([17]), results by Johannes Grassberger, Alastair King and Paulo Tirao on the homology of 2-step nilpotent Lie algebras in [21, Section 4], and in Geometry, we might mention Ernst Heintze’s
result \[27\] Theorem 1] and the Theorem 2.5 in the Milnor’s seminal paper \[43\]; both of these results use (implicitly) degenerations of Lie algebras to prove existence the Riemannian metrics on Lie groups with certain curvature properties. More recently, Yuri Nikolayevsky and Yurii Nikonorov have given results on the curvature of solvable Lie groups by using explicitly degenerations (see \[47\]).

At present, there is an overwhelmed proliferation of papers dealing with degenerations of different kinds of (linear) algebraic structures (including some families of Lie algebras), some of these ones are motivated by abstruse incentives and purposes. We expect our techniques would motivate more developments and different strategies to attempt solving the above-mentioned problems in Lie theory or new potential applications of the notion of hom-Lie algebra in Lie theory.

2 Preliminaries

2.1 Classification of 3-dimensional complex Lie algebras and their automorphisms

The classification of low dimensional complex Lie algebras was started earlier by Sophus Lie himself (see \[38\] Abtheilung VI, Kap. 28, §136). Three-dimensional real Lie algebras have been investigated and completely classified by Luigi Bianchi in \[4, §198-199\] and have distinguished role not only in algebra and geometry, but also in cosmology; which was first noted by the Kurt Gödel in 1949 (see \[31\]). Since then, the classification of the Lie algebras mentioned above have been revised in several different ways by using methods from representation theory, Lie theory or algebraic geometry (see \[55\], which is an arXiv extended version of \[54\]), or can be studied as an (guided) exercise in a usual Lie algebra course (see for instance \[33\] Chapter I, Section 15, Problems 28-35).

Theorem 2.1 (\[9\] Lemma 2]). Every complex Lie algebra of dimension 3 is isomorphic to one and only one of the Lie algebras in the following list:

- \(\mathfrak{L}_0 := \mathfrak{a}_3(\mathbb{C})\), the 3-dimensional abelian Lie algebra.
- \(\mathfrak{L}_1 := \mathfrak{n}_3(\mathbb{C})\), the 3-dimensional Heisenberg Lie algebra: \([e_1, e_2] = e_3\).
- \(\mathfrak{L}_2 := \mathfrak{r}_3(\mathbb{C})\), \([e_1, e_2] = e_2, \ [e_1, e_3] = e_2 + e_3\).
- \(\mathfrak{L}_3 := \mathfrak{r}_{3,1}(\mathbb{C})\), \([e_1, e_2] = e_2, \ [e_1, e_3] = e_3\).
- \(\mathfrak{L}_4 := \mathfrak{r}_{3,-1}(\mathbb{C})\), the Poincaré algebra \(\mathfrak{p}(1,1)\): \([e_1, e_2] = e_2, \ [e_1, e_3] = -e_3\).
- \(\mathfrak{L}_5(\mathfrak{z}) := \mathfrak{r}_{3,\mathfrak{z}}(\mathbb{C})\) with \(0 < |\mathfrak{z}| < 1\ or, |\mathfrak{z}| = 1\ and \text{Im}(\mathfrak{z}) > 0): \ [e_1, e_2] = e_2, \ [e_1, e_3] = ze_3\).
- \(\mathfrak{L}_6 := \mathfrak{r}_{3,0}(\mathbb{C}) = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}: \ [e_1, e_2] = e_2\).
- \(\mathfrak{L}_7 := \mathfrak{s}\mathfrak{o}_3(\mathbb{C}) \cong \mathfrak{g}\mathfrak{o}_3(\mathbb{C})\): \([e_1, e_2] = e_3, \ [e_2, e_3] = e_1, \ [e_3, e_1] = e_2\).

Remark 2.2. It is important to note that \(\mathfrak{r}_{3,\mathfrak{z}} \cong \mathfrak{r}_{3,\bar{\mathfrak{z}}} if and only if (z - \bar{z})(z\bar{z} - 1) = 0\).

Definition 2.3. An almost Abelian Lie algebra is a Lie algebra with a codimension one Abelian ideal.
It is shown by Dietrich Burde and Manuel Ceballos in [11, Proposition 3.1] that a Lie algebra \( g \) is almost Abelian if and only if \( g \) has a codimension one Abelian subalgebra. Any \( n + 1 \)-dimensional almost Abelian Lie algebra is in particular a solvable Lie algebra and can be identified with a square matrix of size \( n \). In fact, if \( g \) is an almost Abelian Lie algebra, \( h \) is a codimension 1 Abelian ideal of \( g \) and \( \text{span}_C(v_0) \) is a linear complement of \( h \), then \( g \) is determined by the linear endomorphism \( A = \text{ad}(v_0)|_h \) (\( g \cong \text{Span}_C(v_0) \times h \)). Conversely, given a linear endomorphism \( A \) of \( C^n \), it is defined the almost Abelian Lie algebra \( r_A \) by the Lie algebra structure on the vector space \( CA \oplus C^n \) satisfying \([A, v] = Av\) for any \( v \in C^n \) and \([v, w] = 0\) for any \( v, w \) in \( C^n \).

The following result is a rewrite of Theorem 2.1 and will be used throughout the paper.

**Proposition 2.4.** Let \( g \) be a 3-dimensional complex Lie algebra. Then \( g \) is a simple Lie algebra or \( g \) is a solvable Lie algebra. Moreover, if \( g \) is simple then \( g \) is isomorphic to \( \mathfrak{so}(3, C) \) (\( \cong \mathfrak{sl}(2, C) \)), and if \( g \) is solvable then \( g \) is isomorphic to an almost Abelian Lie algebra \( r_A \), which is isomorphic to

- the 3-dimensional Abelian Lie algebra if and only if \( A = 0 \),
- the Heisenberg Lie algebra if and only if \( A \neq 0 \), \( \det(A) = 0 \) and \( \text{Trace}(A) = 0 \),
- \( r_{3,1}(C) \) if and only if \( A = t \text{Id} \) for some \( t \in C^* \).
- \( r_3(C) \) if and only if \( A \neq t\text{Id} \) for all \( t \in C \), \( \text{Trace}(A)^2 = 4\det(A) \) and \( \text{Trace}(A) \neq 0 \),
- \( r_{3, -1}(C) \) if and only if \( \text{Trace}(A) = 0 \) and \( \det(A) \neq 0 \),
- \( r_2(C) \times C \) if and only if \( \text{Trace}(A) \neq 0 \) and \( \det(A) = 0 \),
- \( r_{3, z}(C) \) for some \( z \in C \) such that \( z(z^2 - 1) \neq 0 \) if and only if \( \text{Trace}(A)^2 \neq 4\det(A) \), \( \text{Trace}(A) \neq 0 \) and \( \det(A) \neq 0 \).

The automorphisms of all three-dimensional real Lie algebras, as far as we know, were first investigated by Alex Harvey in [26], where it is given the identity component of each group.

**Proposition 2.5.** The automorphism group of each Lie algebra given in Theorem 2.1 is given in the following list, by identifying an automorphism with its matrix representation in the ordered basis \( \{e_1, e_2, e_3\} \):

| Lie algebra | Automorphism | Lie algebra | Automorphism |
|-------------|-------------|-------------|-------------|
| \( \mathfrak{a}_3(C) \) | any \( g \in \text{GL}(3, C) \) | \( \mathfrak{so}_3(C) \) | any \( g \in \text{SO}(3, C) \) |
| \( r_3(C) \) | \[
\begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  x & y & ad - bc \\
\end{pmatrix}
\] | \( r_{3,1}(C) \) | \[
\begin{pmatrix}
  1 & 0 & 0 \\
  x & a & b \\
  y & c & d \\
\end{pmatrix}
\] |
| \( r_{3, -1}(C) \) | \[
\begin{pmatrix}
  1 & 0 & 0 \\
  x & a & w \\
  y & 0 & a \\
\end{pmatrix}
\] | \( r_{3, z}(C) \) \( (z^2 - 1) \neq 0 \) | \[
\begin{pmatrix}
  1 & 0 & 0 \\
  x & a & 0 \\
  y & 0 & b \\
\end{pmatrix}
\] |

In all cases the complex matrix must be non-singular.
2.2 hom-Lie algebras

**Definition 2.6** ([25, Definition 14], [39, Definition 1.3] and [61, Definition 2.1]). A complex hom-Lie algebra is a triple \((V, \cdot, A)\) where \(V\) is a complex vector space, \((V, \cdot)\) is a skew-symmetric algebra (which is the same as an anti-commutative algebra), i.e.

\[x \cdot y = -y \cdot x, \forall x, y \in V\]

and \(A : V \to V\) is a linear transformation, called the twisting map, satisfying the hom-Jacobi identity:

\[\text{Jac}_{(\cdot, A)}(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma)Ax_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot x_{\sigma(3)}) = 0\]

for any \(x_1, x_2\) and \(x_3\) in \(V\). Here, \(S_3\) is the permutation group of degree 3 and \text{sign}(\sigma) is the signum of a permutation \(\sigma\). If \(A\) is also an algebra homomorphism of \((V, \cdot)\), i.e.

\[A(x \cdot y) = Ax \cdot Ay, \forall x, y \in V,\]

then the hom-Lie algebra \((V, \cdot, A)\) is called multiplicative. A regular hom-Lie algebra is one for which the twisting map is an algebra automorphism.

**Remark 2.7.** Note that any Lie algebra defines a hom-Lie algebra by taking the twisting map to be the identity map. More generally, as it is noted in [68, Corollary 2.6], it is easy to check that if \((V, \cdot)\) is a skew-symmetric algebra and \(A\) is an algebra automorphism of \((V, \cdot)\), then \((V, \cdot, A)\) is a hom-Lie algebra if and only if \((V, [-, -])\) is a Lie algebra where \([- , -]\) is defined by

\[[x, y] := A^{-1} (x \cdot y), \forall x, y \in V. \quad (2)\]

Note that any regular hom-Lie algebra \((V, \cdot, A)\) and its associated Lie algebra \((V, [-, -])\), as it is defined in [2], share so many algebraic properties. For instance, the most obvious observations are that \(A\) is an algebra automorphism of both algebras \((V, \cdot)\) and \((V, [-, -])\) and also, such algebras have the same lower central series and the derived series.

**Remark 2.8.** Our primary interest in hom-Lie structures on Lie algebras is mainly motivated by the apparent relation between the hom-Jacobi identity and deformations of Lie algebras. In fact, given a Lie algebra \((V, [-, -])\), we can naturally consider the vector space

\[Z := \left\{ A \in \mathfrak{g}l(V) : \sum_{\sigma \in S_3} \text{sign}(\sigma)[x_{\sigma(1)}, A[x_{\sigma(2)}, x_{\sigma(3)}]] = 0 \right\}. \quad (3)\]

It is easy to check that any \(A \in Z\) defines a deformation of the Lie algebra \((V, [-, -])\) given by \(\varphi := A[-, -]\). Even better, the skew-symmetric bilinear map \(\varphi\) is a trivial solution to the Maurer-Cartan (deformation) equation of the Lie algebra \((V, [-, -]) =: \mu\)

\[\delta_{\mu} \varphi + \frac{1}{2}[[\varphi, \varphi]] = 0, \quad (4)\]
since \( \varphi \) is a Lie bracket on \( V \) and is a 2-cocycle for the adjoint representation of the Lie algebra \((V, \mu)\); and so \( \mu + t \varphi \) is a Lie bracket on \( V \) for all \( t \in \mathbb{C} \).

If \( A \) is an algebra automorphism of \((V, [-, -] = \mu)\), then \( A \in Z \) if and only if \((V, [-, -], A^{-1})\) is a regular hom-Lie algebra.

**Definition 2.9.** A derivation of a hom-Lie algebra \((V, \cdot, A)\) is a linear transformation \( D : V \to V \) such that \( D \) is a usual derivation of the algebra \((V, \cdot)\), i.e.

\[
D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy), \quad \forall x, y \in V, \tag{5}
\]

and also, \( D \) commutes with the twisting map \( A \). We denote by \( \text{Der}(V, \cdot, A) \) the Lie algebra of all derivations of the hom-Lie algebra \((V, \cdot, A)\).

**Remark 2.10.** In [41, Definition 1.7], it is introduced a well-founded notion of a derivation of a hom-Lie algebra \((V, \cdot, A)\); among others substantiated definitions so that these notions can be analogous concepts from Lie algebra theory. Thus, in [41, Section 5], the authors ask whether the Jacobson’s result about invertible derivations is true for hom-Lie algebras. In fact, it is easy to check that if a regular hom-Lie algebra \((V, \cdot, A)\) admits an invertible \( A \)-derivation \( D \), then \((V, \cdot, A)\) is nilpotent (in the sense described in [41, Definition 3.1]), since \( A^{-1}D \) is an invertible derivation of the Lie algebra \((V, [-, -])\) with \([-, -]\) as it is defined in Equation (2). But it is not true in general: consider the Lie algebra \( \mathbb{R}^{2} \times \mathbb{C} \) with multiplicative hom-Lie algebra structure given by the twisting map \( A = \text{diag}(1, 0, 1) \) and \( A \)-derivation defined by \( D = \text{diag}(1, 1, 1) \); here \( A \) and \( D \) are written with respect to the ordered basis \( \{e_1, e_2, e_3\} \) given in Theorem 2.1.

In our case, the definition given in 2.9 is different from that of [41, Definition 1.7] and is related to what could be the automorphism group of a hom-Lie algebra.

**Definition 2.11** ([25, Pag. 331],[61, Section 2]). If \((V, \cdot, A)\) and \((U, \ast, B)\) are hom-Lie algebras, a homomorphism of hom-Lie algebras from \((V, \cdot, A)\) to \((U, \ast, B)\) is a linear transformation \( g : V \to U \) such that \( g \) is an algebra homomorphism between \((V, \cdot)\) and \((U, \ast)\), i.e.

\[
g(x \cdot y) = (gx) \ast (gy), \quad \forall x, y \in V,
\]

and also

\[
g \circ A = B \circ g.
\]

An invertible hom-Lie algebra homomorphism is called a hom-Lie algebra isomorphism.

From now on, we identify our \( n \)-dimensional complex vector space \( V \) with \( \mathbb{C}^n \) and we denote by \( \{e_1, e_2, \ldots, e_n\} \) the canonical basis of \( \mathbb{C}^n \).

**Definition 2.12.** Let \( W \) be the vector space \( \mathbb{C}^2 \times \mathbb{C}^1 \) where \( C^k \) (with \( k \geq 1 \)) is the vector space

\[
\{ \varphi : \mathbb{C}^n \times \ldots \times \mathbb{C}^n \to \mathbb{C}^n : \varphi \text{ is an alternating multilinear map of } k \text{ variables} \},
\]

i.e. \( C^k \cong \text{Hom}(\Lambda^k \mathbb{C}^n, \mathbb{C}^n) \cong \Lambda^k \mathbb{C}^n \otimes \mathbb{C}^n \).

We denote by \( \text{hom-} \mathcal{L}_n(\mathbb{C}) \) the subset of \( W \) of all hom-Lie algebra structures on \( \mathbb{C}^n \).
Note that $\text{hom-}L_n(\mathbb{C})$ is an affine algebraic subset of $W$. If we consider structure constants of a element $(\mu, A)$ in $W$, $\mu(e_i, e_j) = \sum c_{i,j}^ke_k$ and $Ae_k = \sum a_{h,i}c_{k,h}$, we have $(\mu, A) \in \text{hom-}L_n(\mathbb{C})$ if and only if $\{c_{i,j}^k\} \cup \{a_{h,i}\}$ satisfy the polynomial relations determined by the hom-Jacobi identity
\[
\sum_{p,q=1}^n (a_{p,i}c_{j,k}^q + a_{p,j}c_{k,i}^q + a_{p,k}c_{i,j}^q)e_{p,q} = 0.
\]
for all $1 \leq i, j, k, l \leq n$. In practice, it is very easy to determine whether a given pair $(\mu, A) \in W$ is a hom-Lie algebra structure on $\mathbb{C}^n$, since the map $\text{Jac}_{\mu,A}$ is an alternating tensor, and so we only need to check that $\text{Jac}_{\mu,A}(e_i, e_j, e_k) = 0$ for any $i, j, k$ with $1 \leq i < j < k \leq n$ (one straightforward verification is enough in dimension 3).

Recall that the complex general linear group $\text{GL}(n, \mathbb{C})$ acts on $O^k$ ($k \geq 1$) via change of basis, i.e., given $\varphi \in \mathbb{C}^k$
\[
g \circ \varphi(x_1, x_2, \ldots, x_k) := g \varphi(g^{-1}x_1, g^{-1}x_2, \ldots, g^{-1}x_k)
\]
for $g \in \text{GL}(n, \mathbb{C})$ and $x_1, x_2, \ldots, x_k \in \mathbb{C}^n$. And so, $\text{GL}(n, \mathbb{C})$ acts on the vector space $W$ in a natural way and $\text{hom-}L_n(\mathbb{C})$ is a $\text{GL}(n, \mathbb{C})$-invariant subset of $W$. The orbit of a point $(\mu, A)$ in $W$, denoted by $O(\mu, A)$ (or by $\text{GL}(n, \mathbb{C}) \circ (\mu, A)$) is the set of images of $(\mu, A)$ under the action by elements of $\text{GL}(n, \mathbb{C})$:
\[
O(\mu, A) := \{ g \circ (\mu, A) : g \in \text{GL}(n, \mathbb{C}) \}.
\]
Given a hom-Lie algebra structure $(\mu, A)$ in $\text{hom-}L_n(\mathbb{C})$, it follows from Definition 2.11 that the orbit $O(\mu, A)$ corresponds to the isomorphism class of the hom-Lie algebra $(\mathbb{C}^n, \mu, A)$.

For each $(\mu, A) \in \text{hom-}L_n(\mathbb{C})$, the isotropy group of $(\mu, A)$ is the set of elements of $\text{GL}(n, \mathbb{C})$ that fix $(\mu, A)$, i.e., $g \in \text{GL}(n, \mathbb{C})$ such that $g \circ \mu = \mu$ and $g \circ A = A$ (or equivalently, $gAg^{-1} = A$). We can think of such group as the automorphism group of the hom-Lie algebra $(\mathbb{C}^n, \mu, A)$, $\text{Aut}(\mathbb{C}^n, \mu, A)$; in this way $\text{Der}(\mathbb{C}^n, \mu, A)$ (see Definition 2.9) is the Lie algebra of $\text{Aut}(\mathbb{C}^n, \mu, A)$.

The Zariski closure of an arbitrary subset $Y \subseteq W$, denoted by $\overline{Y}$, is the smallest Zariski closed set in $W$ containing $Y$; which is $V(\mathbb{I}(Y))$. In the case $Y = O(\mu, A)$ with $(\mu, A)$ a hom-Lie algebra structure on $\mathbb{C}^n$, since $\text{hom-}L_n(\mathbb{C})$ is a Zariski closed set of $W$, $\overline{O(\mu, A)} \subseteq \text{hom-}L_n(\mathbb{C})$. Even though an orbit $O(v)$ in a $G$-variety is not necessarily a Zariski closed set, the dimension of an orbit is by definition the dimension of its Zariski closure and it is well-known that the isotropy group of $v, G_v$, is an algebraic subgroup of $G$ and
\[
\dim(O(v)) = \dim(G) - \dim(G_v);
\]
see for instance [6], Sections 21.4, 23.2].

**Definition 2.13 (Degeneration).** Let $(\mu, A)$ and $(\lambda, B)$ be two hom-Lie structures on $\mathbb{C}^n$. It is said that $(\mu, A)$ degenerates to $(\lambda, B)$ if $(\lambda, B) \in \overline{O(\mu, A)}^\#$ and this is denoted by $(\mu, A) \xrightarrow{\text{deg}} (\lambda, B)$. The degeneration $(\mu, A) \xrightarrow{\text{deg}} (\lambda, B)$ is called proper if $(\lambda, B)$ is in the boundary of the orbit $O(\mu, A)$, or equivalently, if $(\mathbb{C}^n, \mu, A)$ and $(\mathbb{C}^n, \lambda, B)$ are not isomorphic hom-Lie algebras.
The following proposition is due to Armand Borel (see [6, Proposition 15.4]) and is very important to study degenerations of linear (algebraic) structures:

**Proposition 2.14** ([7, Closed orbit lemma], [64, Proposition 21.4.5]). Let \( G \) be an algebraic group acting morphically on a non-empty variety \( Z \). Then each orbit is a smooth variety which is open in its closure in \( Z \). Its boundary is a union of orbits of strictly lower dimension. In particular, the orbits of minimal dimension are closed.

An immediate consequence of the proposition above is that the dimension of the algebra of derivations of a hom-Lie algebra is an obstruction to study its degenerations.

**Corollary 2.15.** If \( (\mu, A) \xrightarrow{deg} (\lambda, B) \) is a proper degeneration then

\[
\dim(\text{Der})(\mathbb{C}^n, \mu, A) < \dim(\text{Der})(\mathbb{C}^n, \lambda, B).
\]

Although it is generally quite difficult to work with the Zariski topology, the following observation that has originally been proved by Jean-Pierre Serre in [60, Proposition 5.] (see also [8, Section 2.2], [66, Page 165] or the exposition given by David Mumford of *Complex varieties* in [15, §10]) shows how to study Zariski closure of orbits by using tools from elementary calculus.

**Proposition 2.16.** Let \( Z \) be a complex algebraic variety and Let \( U \) be a Zariski open and Zariski dense subset of \( Z \). Then \( U \) is dense in \( Z \) with respect to Euclidean subspace topology, and, in consequence, the closures of \( U \) with respect to both topologies are the same.

Other basic concept to consider is related with the notion of *rigidity*

**Definition 2.17** (Rigid hom-Lie algebra). A hom-Lie algebra \( (\mathbb{C}^n, \mu, A) \) is called rigid (in hom-\( L_n(\mathbb{C}) \)) if the \( \text{GL}(n, \mathbb{C}) \)-orbit of \( (\mu, A) \) is a Zariski open set of hom-\( L_n(\mathbb{C}) \).

A consequence of this definition is that a rigid hom-Lie algebra is not a proper degeneration of other hom-Lie algebra. In fact, a rigid hom-Lie algebra is responsible for one of the irreducibles components of hom-\( L_n(\mathbb{C}) \); i.e., the Zariski closure of a rigid hom-Lie algebra is an irreducible component in the mentioned algebraic set. Therefore, there exists only a finite number of rigid hom-Lie algebra in each dimension.

**Remark 2.18.** A hom-Lie algebra \( (\mathbb{C}^n, \mu, A) \) with \( (\mathbb{C}^n, \mu) \) a Lie algebra cannot be rigid in hom-\( L_n(\mathbb{C}) \) because \( (\mathbb{C}^n, \mu, A + t \text{Id}) \) (with \( t \) sufficiently close to zero) is a one-parameter family of hom-Lie algebras which are not isomorphic to \( (\mathbb{C}^n, \mu, A) \). If \( (\mathbb{C}^n, \mu, A) \) is a hom-Lie algebra with \( A \) a non-nilpotent transformation, then it can be shown quickly that \( (\mathbb{C}^n, \mu, A) \) is not rigid because \( (\mathbb{C}^n, \mu, (1 + t)A) \) is a one-parameter family of hom-Lie algebras which are not isomorphic to \( (\mathbb{C}^n, \mu, A) \).

Other interesting consequence of the Proposition 2.14 is worth noting that the notion of degeneration defines a partial order on the orbit space hom-\( L_n(\mathbb{C})/\text{GL}(n, \mathbb{C}) \) given by \( \text{GL}(n, \mathbb{C}) \cdot (\lambda, B) \leq \text{GL}(n, \mathbb{C}) \cdot (\mu, A) \) if \( (\mu, A) \) degenerates to \( (\lambda, B) \). And so, once we have an “enumeration” of \( n \)-dimensional hom-Lie algebras (with certain properties), the next natural step would be to keep “order” in
such set. This kind of ideas originally go back to works of Albert Nijenhuis and Roger Wolcott Richardson in [18, 19], Murray Gerstenhaber (see [20]), and Peter Gabriel [17]. Obtain a classification of orbit closures in a variety of algebraic structures of certain type is occasionally called a geometric classification (term introduced by Guerino Mazzola in [42] inspired by [17]).

### 2.2.1 Other varieties of hom-Lie algebras

One can consider other algebraic sets related with hom-Lie algebras in the same way as we have introduced the set hom-$\mathcal{A}_n(\mathbb{C})$ and can also study the notions of rigidity and degenerations; for instance, the algebraic set of $n$-dimensional complex multiplicative hom-Lie algebras, hom$^m$-$\mathcal{A}_n(\mathbb{C})$ and if the twisting map is prescribed, say $A$, it induces the algebraic subset of $C^2$, hom$_A$-$\mathcal{A}_n(\mathbb{C})$, given by the skew-symmetric bilinear maps $\mu \in C^2$ such that $(\mathbb{C}^n, \mu, A)$ is a hom-Lie algebra (or hom$^m$$_A$-$\mathcal{A}_n(\mathbb{C})$ for the multiplicative case). Whereas GL$(n, \mathbb{C})$ naturally acts on hom-$\mathcal{A}_n(\mathbb{C})$, we only have the action of GL$(n, \mathbb{C})_A$, the set of invertible matrices that commute with $A$, in the set hom$_A$-$\mathcal{A}_n(\mathbb{C})$.

By following [49] Section 24], we have sufficient conditions for a hom-Lie algebra to be rigid in each algebraic set. In fact, let $(\mathbb{C}^n, \mu, A)$ be a (multiplicative) hom-Lie algebra and let us denote by gl$(n, \mathbb{C}) \bullet (\mu, A)$ and gl$_A(n, \mathbb{C}) \bullet \mu$ the first approximations to the “Zariski tangent” to GL$(n, \mathbb{C})_A (\mu, A)$ and GL$_A(n, \mathbb{C}) \bullet \mu$ at $(\mu, A)$ and $\mu$ respectively. More precisely, this means

\[
\text{gl}(n, \mathbb{C}) \bullet (\mu, A) := \left\{ (\lambda, B) \in C^2 \times C^1 : \lambda = \delta_\mu(X) \text{ and } B = AX -XA, \right. \\
& \left. \text{ with } X \in \text{gl}(n, \mathbb{C}) \right\}
\]

\[
\text{gl}_A(n, \mathbb{C}) \bullet \mu := \left\{ \lambda \in C^2 : \lambda = \delta_\mu(X) \text{ with } X \in \text{gl}_A(n, \mathbb{C}) \right\}
\]

where

\[
\delta_\mu(X)(y, z) := X\mu(y, z) - \mu(Xy, z) - \mu(y, Xz), \forall y, z \in \mathbb{C}^n
\]

and gl$(n, \mathbb{C})_A$ is the set of complex matrices $n \times n$ that commute with $A$.

Similarly, let $T_1(\mu, A)$, $T_2(\mu, A)$, $T_3(\mu)$ and $T_4(\mu)$ be denote the first approximations to the Zariski tangent to hom-$\mathcal{A}_n(\mathbb{C})$, hom$^m$-$\mathcal{A}_n(\mathbb{C})$, hom$_A$-$\mathcal{A}_n(\mathbb{C})$ and hom$^m$$_A$-$\mathcal{A}_n(\mathbb{C})$ at $(\mu, A)$ and $\mu$ respectively: for $(\mu, A)$ we have

\[
T_1(\mu, A) := \left\{ (\lambda, B) \in C^2 \times C^1 : (d\text{Jac})_{(\mu, A)}(\lambda, B) = 0 \right\}
\]

\[
T_2(\mu, A) := \left\{ (\lambda, B) \in C^2 \times C^1 : (d\text{Jac})_{(\mu, A)}(\lambda, B) = 0 \text{ and } (d\mu)_{(\mu, A)}(\lambda, B) = 0 \right\},
\]

where

\[
(d\text{Jac})_{(\mu, A)}(\lambda, B)(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma) \mu(Ax_{\sigma(1)}, \lambda(x_{\sigma(2)}, x_{\sigma(3)})) + \sum_{\sigma \in S_3} \text{sign}(\sigma) \lambda(Ax_{\sigma(1)}, \mu(x_{\sigma(2)}, x_{\sigma(3)}))
\]

\[
(7)
\]

and

\[
(d\mu)_{(\mu, A)}(\lambda, B)(y, z) = (A\lambda(y, z) - \lambda(Ay, Az)) + (B\mu(y, z) - \mu(Ay, Bz) - \mu(By, Az)), \forall y, z \in \mathbb{C}^n.
\]

(8)

With respect to the sets hom$_A$-$\mathcal{A}_n(\mathbb{C})$ and hom$^m$$_A$-$\mathcal{A}_n(\mathbb{C})$ we have

\[
T_3(\mu) := \left\{ \lambda \in C^2 : (d\text{Jac})_{\mu}\lambda = 0 \right\}
\]

\[
T_4(\mu) := \left\{ \lambda \in C^2 : (d\text{Jac})_{A}\lambda = 0 \text{ and } (d\mu)_A\lambda = 0 \right\},
\]
where
\[
(d \text{Jac}_A)_\mu(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sign}(\sigma) \mu(Ax_{\sigma(1)}, \lambda(x_{\sigma(2)}, x_{\sigma(3)})) + \sum_{\sigma \in S_3} \text{sign}(\sigma) \lambda(Ax_{\sigma(1)}, \mu(x_{\sigma(2)}, x_{\sigma(3)}))
\] (9)
and
\[
(d m)_A \lambda(y, z) = A\lambda(y, z) - \lambda(Ay, Az), \forall y, z \in \mathbb{C}^n.
\] (10)

If \( gl(n, \mathbb{C}) \bullet (\mu, A) \) coincides with \( T_1(\mu, A) \) or \( gl\mathcal{L}_n(\mathbb{C}) \bullet \mu \) or \( gl\mathcal{L}_n(\mathbb{C}) \bullet \mu \) with \( T_3(\mu) \), then \( (\mu, A) \) is rigid in hom-\( \mathcal{L}_n(\mathbb{C}) \), respectively \( \mu \) is rigid in hom\( \mathcal{L}_n(\mathbb{C}) \) (and analogously in the multiplicative case).

It is to be expected that the above conditions are not necessary for the rigidity of a hom-Lie algebra in such algebraic sets; since a rigid hom-Lie algebra can satisfy additional (intrinsic) identities which are “independent” of the hom-Jacobi condition. For instance, it is easy to give a linear transformation \( A \) of \( \mathbb{C}^3 \) such that hom\( \mathcal{L}_3(\mathbb{C}) \) has only two \( GL(A, 3, \mathbb{C}) \)-orbits, one of which corresponds to a rigid hom-Lie algebra not satisfying the above sufficient condition (consider \( A \) defined by \( A e_1 = 0, A e_2 = 2e_2 + e_3 \) and \( A e_3 = e_3 \)).

3 Classification

Our purpose in this section is to give the classification of the hom-Lie algebra structures on 3-dimensional complex Lie algebras with nilpotent twisting map, up to isomorphism of hom-Lie algebras. Let \((V, \cdot)\) be a skew-symmetric algebra and let us consider the vector space of hom-Lie structures on \((V, \cdot)\):
\[
\text{hom-Lie}(V, \cdot) := \{ A \in \text{End}(V) : \text{Jac}_{\cdot, A} = 0 \}.
\]

It follows from Definition 2.11 that two hom-Lie algebras \((V, \cdot, A)\) and \((V, \cdot, B)\) are isomorphic if and only if \( A \) and \( B \) are conjugate with respect to an automorphism of the algebra \((V, \cdot)\). Therefore, for each Lie algebra \( g \) in Theorem 2.1, we need to study \( \text{Aut}(g) \)-conjugacy classes in hom-Lie\( (g) \). And so, for instance, by the cyclic decomposition of a nilpotent operator, the 3-dimensional complex abelian Lie algebra has only three hom-Lie algebra structures with nilpotent twisting map, up to isomorphism:

**Proposition 3.1.** The hom-Lie algebra structures with nilpotent twisting map on the 3-dimensional complex abelian Lie algebra are given by (up to isomorphism):

1. \( \mathcal{L}_0^0 \): \( (\mathfrak{a}_3(\mathbb{C}), A_0) \) with \( A_0 \) the zero map,
2. \( \mathcal{L}_1^0 \): \( (\mathfrak{a}_3(\mathbb{C}), A_1) \) with \( A_1 e_1 = 0, A_1 e_2 = 0 \) and \( A_1 e_3 = e_2 \),
3. \( \mathcal{L}_2^0 \): \( (\mathfrak{a}_3(\mathbb{C}), A_2) \) with \( A_2 e_1 = 0, A_2 e_2 = e_1 \) and \( A_2 e_3 = e_2 \).

3.1 hom-Lie structures on \( \mathfrak{so}_3(\mathbb{C}) \)

As we mentioned above, it is important to determine the vector space hom-Lie\( (\mathfrak{so}_3(\mathbb{C})) \). It is straightforward to show that:
Lemma 3.2. An endomorphism $A$ of $\mathbb{C}^3$ defines a hom-Lie algebra structure on $\mathfrak{so}_3(\mathbb{C})$ if and only if $A$ is a symmetric operator for the Killing form of $\mathfrak{so}_3(\mathbb{C})$, or equivalently the matrix of $A$ with respect to the ordered basis $\{e_1, e_2, e_3\}$ is a symmetric complex matrix.

Therefore, since $\text{Aut}(\mathfrak{so}_3(\mathbb{C})) = \text{SO}(3, \mathbb{C})$, we need to get canonical forms for complex symmetric matrices under complex orthogonal similarity. Nigel Scott presents a solution to this problem in [59] (or see also the references given there), which can be used to classify, up to isomorphism, the hom-Lie algebra structures on $\mathfrak{so}(3, \mathbb{C})$ (not just the ones corresponding to nilpotent twisting maps). In our case, we have:

**Proposition 3.3.** Any hom-Lie algebra $(\mathfrak{so}(3, \mathbb{C}), A)$ with $A$ nilpotent operator is isomorphic to some hom-Lie algebra $L_i := (\mathfrak{so}(3, \mathbb{C}), A_i)$ where $A_0$ is the Zero map and the matrix of $A_i$ with respect to the ordered basis $\{e_1, e_2, e_3\}$ is:

$$
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & \sqrt{-1} \\
0 & \sqrt{-1} & -1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & \sqrt{-1} \\
1 & 0 & 0 \\
\sqrt{-1} & 0 & 0
\end{pmatrix}
$$

Two hom-Lie algebras $(\mathfrak{so}(3, \mathbb{C}), A_i)$ and $(\mathfrak{so}(3, \mathbb{C}), A_j)$ are isomorphic if and only if $A_i = A_j$.

3.2 hom-Lie structures on $\mathfrak{r}_{3,z}(\mathbb{C})$

It is easy to see that a linear map $A$ defines a hom-Lie algebra structure on $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$ if and only if $A e_3 \in \text{Span}_\mathbb{C}\{e_2, e_3\}$. With respect to $\mathfrak{r}_{3,z}(\mathbb{C}) (z \neq 0)$, its hom-Lie algebras structures are exactly those linear maps that leave its derived algebra invariant. On the other hand, the Lie algebras $\mathfrak{r}_{3,z}(\mathbb{C})$, for all $z \in \mathbb{C}$ have almost the same automorphism group; $\mathfrak{r}_{3,-1}$ has one more automorphism and $\mathfrak{r}_{3,1}$ has others more.

So, we first study the action by conjugation of the group

$$
G := \left\{ \begin{pmatrix}
1 & 0 & 0 \\
x & a & 0 \\
y & 0 & b
\end{pmatrix} \in M(3, \mathbb{C}) : a, b \in \mathbb{C}^* \right\}
$$

on the set

$$
\mathcal{N}_1 := \left\{ A \in M(3, \mathbb{C}) : A \text{ is a nilpotent matrix and } A e_3 \in \text{Span}_\mathbb{C}\{e_2, e_3\} \right\}
$$

which contains the $G$-invariant set

$$
\mathcal{N}_2 := \left\{ A \in M(3, \mathbb{C}) : A \text{ is a nilpotent matrix and } A \text{Span}_\mathbb{C}\{e_2, e_3\} \subseteq \text{Span}_\mathbb{C}\{e_2, e_3\} \right\}
$$

3.2.1 $G$-conjugacy classes in $\mathcal{N}_2$

Let us denote by $\mathfrak{k}$ the vector space $\text{span}_\mathbb{C}\{e_2, e_3\}$. Note that $\mathfrak{k}$ is a codimension-1 invariant subspace for any $A$ in $\mathcal{N}_2$, and so $\text{Im} A \subseteq \mathfrak{k}$. Also, $A|_\mathfrak{k}$ is obviously a nilpotent operator.
First consider the case in which \( A|_{\mathfrak{t}} \) is the zero map. If \( A \) is non-zero then the degree of nilpotency of \( A \) is 2. Since \( A e_1 \in \mathfrak{t} \) is easy to see that \( A \) is \( G \)-similar to some of the following nilpotent matrices of \( \mathcal{N}_2 \) given by

- \( A_1 e_1 = e_2, A_1 e_2 = 0, A_1 e_3 = 0 \),
- \( A_2 e_1 = e_3, A_2 e_2 = 0, A_2 e_3 = 0 \),
- \( A_3 e_1 = e_2 + e_3, A_3 e_2 = 0, A_3 e_3 = 0 \).

For example, if \( A e_1 = ae_2 + be_3 \) with \( a, b \neq 0 \), let \( g \) be the matrix of \( G \) defined by \( g e_1 = e_1, g e_2 = ae_2 \) and \( g e_3 = be_3 \). Since \( g(e_2 + e_3) = A e_1 \), we have \( g^{-1} A g = A_3 \).

The case in which \( A|_{\mathfrak{t}} \) has nilpotency degree equal to 2, according to how \( \text{Ker} \ A|_{\mathfrak{t}} \) is positioned in \( \mathfrak{t} \), we have three types of possible \( \tilde{G} \)-conjugacy classes for \( A|_{\mathfrak{t}} \):

- \( B_1 e_2 = 0, B_1 e_3 = e_2 \),
- \( B_2 e_2 = e_3, B_2 e_3 = e_0 \),
- \( B_3 e_2 = B_1 e_3 = \lambda (e_2 - e_3) \) with \( \lambda \in \mathbb{C}^* \).

Here, \( \tilde{G} \) is the group \( \{ g : \mathfrak{k} \rightarrow \mathfrak{k} : g e_2 = ae_2, g e_3 = be_3 \text{ with } a, b \in \mathbb{C}^* \} \). For example, if both \( e_2 \) and \( e_3 \) do not belong to the \( \text{Ker} \ A|_{\mathfrak{t}} \), we have \( A|_{\mathfrak{t}} \) sends \( e_2 \) to a nonzero vector \( v_0 \), and \( e_3 \) to \( t v_0 \) with \( t \neq 0 \) (\( \dim(\text{Im}(A|_{\mathfrak{t}})) = 1 \)). Let \( g \) be the linear transformation of \( \tilde{G} \) given by \( g e_2 = e_2 \) and \( g e_3 = \frac{1}{t} e_3 \), we have \( g^{-1} A|_{\mathfrak{g}} \) is of the form described in the third type.

If \( A \in \mathcal{N}_2 \) is a nilpotent matrix of degree 2 with \( A|_{\mathfrak{t}} \) nonzero, we have a vector \( v_0 \) in \( \text{Ker}(A) \) of the form \( e_1 + w_0 \) with \( w_0 \in \mathfrak{t} \), and so \( A \) is \( G \)-similar to a matrix \( \tilde{A} \) in \( \mathcal{N}_2 \) with \( A e_1 = 0 \); consider \( g e_1 = v_0 \), \( g e_2 = e_2 \) and \( g e_3 = e_3 \), we see that \( g \in G \) and \( g^{-1} A g e_1 = 0 \). In turn \( \tilde{A} \) is \( G \)-similar to a matrix \( A_i \) (\( 4 \leq i \leq 6 \)) such that \( A_i e_1 = 0 \) and \( A_i|_{\mathfrak{t}} \) coincides with \( B_{i-3} \); as we observed above.

If \( A \in \mathcal{N}_2 \) is a nilpotent matrix of degree 3, then \( A|_{\mathfrak{t}} \) is nonzero and \( A \) is \( G \)-similar to a matrix \( \tilde{A} \) in \( \mathcal{N}_2 \) with \( A|_{\mathfrak{t}} \) equal to some \( B_i \). When \( i = 2 \) or 3, since \( \text{Im}(\tilde{A}) = \mathfrak{t} \), there exists a vector \( v_1 \) of the form \( e_1 + w_1 \), with \( w_1 \in \mathfrak{t} \) such that \( A v_1 = t v_2 \), \( t \neq 0 \). By considering the linear map \( g \in G \) given by \( g e_1 = v_1 \), \( g e_2 = t e_2 \) and \( g e_3 = t e_3 \), we have \( \tilde{A} \) is \( G \)-similar to \( g^{-1} \tilde{A} g \):

- \( A e_1 = e_2, A e_2 = e_3, A e_3 = 0 \),
- \( A e_1 = e_2, A e_2 = A e_3 = \lambda (e_2 - e_3) \).

In the case when \( i = 1 \), by a similar argument as above (with \( e_3 \) playing the role of \( e_2 \)), we have \( \tilde{A} \) is \( G \)-similar to a matrix of the form

- \( A e_1 = e_3, A e_2 = 0, A e_3 = e_2 \).

The above discussion can be rephrased in terms of hom-Lie algebra structures on \( \mathfrak{t} \) (\( z(z^2 - 1) \neq 0 \)) with nilpotent twisting map:

**Proposition 3.4.** Any hom-Lie algebra \( (\mathfrak{t}, A) \) with \( z(z^2 - 1) \neq 0 \) and \( A \) nilpotent operator is isomorphic to some hom-Lie algebra \( \Sigma(z) := (\mathfrak{t}, A_i) \) where \( A_0 \) is the zero map and the matrix of \( A_i \) with respect to the ordered basis \( \{ e_1, e_2, e_3 \} \) is:
Two hom-Lie algebras \((\mathfrak{r}_{3, z}, A_i)\) and \((\mathfrak{r}_{3, z}, A_j)\) are isomorphic if and only if \(A_i = A_j\).

The last assertion follows by using the invariants given in the Tables 3 and 6, or alternatively, by taking advantage of the fact that we have all elements of \(\text{Aut}(\mathfrak{r}_{3, z})\) expressed explicitly and by showing the affirmation with straightforward computations.

Now, by considering the group \(\text{Aut}(\mathfrak{r}_{3, -1})\), which contains the group \(G\) and has (essentially) one element more: \(ge_1 = -e_1, ge_2 = e_3\) and \(ge_3 = e_2\), it is easy to check that the linear maps \(A_1\) and \(A_2\) given in the above discussion are \(\text{Aut}(\mathfrak{r}_{3, -1})\)-similar, and the same is true for the pairs of maps \(A_4\) and \(A_5\), \(A_7\) and \(A_8\).

**Proposition 3.5.** Any hom-Lie algebra \((\mathfrak{r}_{3, -1}, A)\) with nilpotent twisting map is isomorphic to some hom-Lie algebra \(L_i := (\mathfrak{r}_{3, -1}, A_i)\) where \(A_0\) is the Zero map and the matrix of \(A_i\) with respect to the ordered basis \(\{e_1, e_2, e_3\}\) is:

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}\]

\[A_1 \quad A_2 \quad A_3\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[A_4 \quad A_5 \quad A_6(\lambda)\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[A_7 \quad A_8 \quad A_9(\lambda)\]

\[A_5 \quad A_6(\lambda) \quad A_7 \quad A_8 \quad A_9(\lambda)\]

\[\lambda \in \mathbb{R}_{>0} \text{ or } \text{Im}(\lambda) > 0\]

Two hom-Lie algebras \((\mathfrak{r}_{3, -1}, A_i)\) and \((\mathfrak{r}_{3, -1}, A_j)\) are isomorphic if and only if \(A_i = A_j\).

With respect to the hom-Lie algebra structures on \(\mathfrak{r}_{3, 1}\) with nilpotent twisting map, by using that \(\text{Aut}(\mathfrak{r}_{3, 1})\) contains to the group \(G\) and studying the \(\text{Aut}(\mathfrak{r}_{3, 1})\)-conjugacy classes of the matrices obtained at the beginning of 3.2.1, we deduce that:
Proposition 3.6. Any hom-Lie algebra \((\mathfrak{t}_{3,1}, A)\) with nilpotent twisting map is isomorphic to some hom-Lie algebra \(\mathfrak{g}_A^1 := (\mathfrak{t}_{3,1}, A_1)\) where \(A_0\) is the Zero map and the matrix of \(A_i\) with respect to the ordered basis \(\{e_1, e_2, e_3\}\) is

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Two hom-Lie algebras \((\mathfrak{t}_{3,1}, A_i)\) and \((\mathfrak{t}_{3,1}, A_j)\) are isomorphic if and only if \(A_i = A_j\).

Remark 3.7. An alternative proof can be given by noting that any matrix \(A\) in \(N_2\) is \(G\)-similar to a matrix \(\hat{A}\) in \(N_2\) such that \(\hat{A}\) is a nilpotent Jordan block, where \(G\) is the subgroup of Aut(\(\mathfrak{t}_{3,1}\)) consisting of linear maps of Aut(\(\mathfrak{t}_{3,1}\)) that fix to the vector \(e_1\).

3.2.2 \(G\)-conjugacy classes in \(N_1 \setminus N_2\)

Let \(A \in N_1 \setminus N_2\). For simplicity, let us suppose that \(Ae_2 = e_1 + w\) with \(w \in \mathfrak{t}\); since it is easy to see that any matrix in such set is \(G\)-similar to a matrix like \(A\).

If \(A\) is a nilpotent matrix of degree 2, we have \(Ae_2\) spans \(\text{Im}(A)\). It follows that \(e_3 \in \text{Ker}(A)\), since \(Ae_3 \in \mathfrak{t} \cap \text{Span}_{\mathbb{C}}\{Ae_2\}\). Let \(g\) be the linear map defined by \(ge_1 = Ae_2\), \(ge_2 = e_2\) and \(ge_3 = e_3\). So, \(g \in G\) and \(\hat{A} = g^{-1}Ag\) is the matrix given by

- \(\tilde{A}e_1 = 0, \tilde{A}e_2 = g^{-1}Ae_2 = e_1, \tilde{A}e_3 = 0\).

If the nilpotency degree of \(A\) is 3 and \(e_3 \in \text{Ker}(A)\), we have \(e_2\) is a cyclic vector for \(A\), since \(Ae_2 \notin \text{span}_{\mathbb{C}}\{e_3\} = \text{Ker}(A)\). As \(A^2e_2 \in \text{Ker}(A)\) we have \(A^2e_2 = te_3\) with \(t \neq 0\). We consider the linear map \(g\) defined by \(ge_1 = Ae_2\), \(ge_2 = e_2\) and \(ge_3 = te_3 = A^2e_2\). It is clear that \(g \in G\) and we have \(A = g^{-1}Ag\) has the form:

- \(\tilde{A}e_1 = g^{-1}A^2e_2 = e_3, \tilde{A}e_2 = g^{-1}Ae_2 = e_1, \tilde{A}e_3 = 0\).

The case in which \(A\) is nilpotent matrix of degree 3 and \(e_3 \notin \text{Ker}(A)\), we have \(\{Ae_2, Ae_3\}\) spans \(\text{Im}(A)\) and \(e_3 \notin \text{Ker}(A^2)\). Since \(\text{Im}(A)\) is an invariant subspace of \(A\), we \(A|_{\text{Im}(A)}\) is a nilpotent operator of degree 2, and so there are two cases: whether \(Ae_2\) is a vector in \(\text{Ker}\ A|_{\text{Im}(A)}\) or not. If \(Ae_2 \in \text{Ker}\ A|_{\text{Im}(A)}\), equivalently \(e_2 \in \text{Ker}(A^2)\), since \(\text{Im}(A) = \text{Ker}(A^2)\), we have \(Ae_3 = te_2\) with \(t \neq 0\) and so the cyclic basis of \(\mathbb{C}^3\) is given by \(e_3\) is \(\{e_3, Ae_3 = te_2, A^2e_3 = tAe_2\}\). Let \(g\) be the linear map defined by \(ge_1 = Ae_2\), \(ge_2 = e_2\) and \(ge_3 = \frac{1}{t}e_3\). Then \(g \in G\) and \(\hat{A} = g^{-1}Ag\) is define by:

- \(\tilde{A}e_1 = g^{-1}A^2e_2 = 0, \tilde{A}e_2 = g^{-1}Ae_2 = e_1, \tilde{A}e_3 = g^{-1}e_2 = e_2\).

The remaining case, when \(e_2, e_3 \notin \text{Ker}(A^2)\), we require a bit more work to find a “canonical form” for \(A\) in \(N_1 \setminus N_2\) with respect to the action of the group \(G\). Let \(v_2\) be a vector such that \(v_2\) spans \(\text{Ker}(A)\). Then

\[v_2 = aAe_2 - bAe_3\]
with \( a, b \neq 0 \). We may assume that \( a = 1 \), and so \( v_2 = e_1 + w' \) with \( w' \in \mathfrak{t} \). Since \( A^2 e_3 \in \text{Ker}(A) \), we have
\[
A^2 e_3 = \frac{\lambda}{b} v_2, \lambda \neq 0.
\]
If we write \( A e_3 = x e_2 + y e_3 \), we obtain
\[
v_2 = \frac{bx}{\lambda} A e_2 + \frac{by}{\lambda} A e_3
\]
by applying \( A \) to \( A e_3 \), and so \( bx = \lambda \), \( y = -\lambda \). Let \( g \) be the linear map defined by \( ge_1 = v_2, ge_2 = e_2 \) and \( ge_3 = be_3 \). We have \( g \in G \) and \( \tilde{A} = g^{-1} A g \) is of the form:

- \( \tilde{A} e_1 = 0, \tilde{A} e_2 = e_1 + \lambda(e_2 - e_3), \tilde{A} e_3 = \lambda(e_2 - e_3) \)

since \( \tilde{A} e_2 - \tilde{A} e_3 = e_1 \).

It follows from subsections 3.2.1 and the above analysis that:

**Proposition 3.8.** Any hom-Lie algebra \( (\mathfrak{t}_{3,0} = \mathfrak{t}_2 \times \mathbb{C}, A) \) with \( A \) nilpotent operator is isomorphic to some hom-Lie algebra \( \mathcal{L}_{i} := (\mathfrak{t}_2 \times \mathbb{C}, A_i) \) where \( A_0 \) is the zero map and the matrix of \( A_i \) with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
A_1 & A_2 & A_3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_4 & A_5 & A_6(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda & -\lambda \\
\end{bmatrix}
\begin{bmatrix}
A_7 & A_8 & A_9(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{10} & A_{11} & A_{12}(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{13} & A_{14} & A_{15}(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Two hom-Lie algebras \( (\mathfrak{t}_2 \times \mathbb{C}, A_i) \) and \( (\mathfrak{t}_2 \times \mathbb{C}, A_j) \) are isomorphic if and only if \( A_i = A_j \).

### 3.3 hom-Lie structures on \( \mathfrak{t}_3(\mathbb{C}) \)

It is easy to verify that hom-Lie(\( \mathfrak{t}_3(\mathbb{C}) \)) represented by matrices with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is the set \( \mathcal{N}_2 \); as in [3.2.1] So, we have to study
conjugacy classes on $N_2$ with respect to the natural action of
\[
\text{Aut}(\mathfrak{r}_3(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & a & z \\ y & 0 & a \end{pmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \right\}.
\]

Recall all elements of $N_2$ leave $\mathfrak{k} = \text{Span}_\mathbb{C}\{e_2, e_3\}$ invariant. An analysis similar to the one in 3.2.1 shows that any $A \in N_2$ is Aut$(\mathfrak{r}_3(\mathbb{C}))$-similar to a matrix $\tilde{A} \in N_2$ such that $A|_{\mathfrak{k}}$ is the zero map or

\begin{itemize}
  \item $\tilde{A}e_2 = 0$, $\tilde{A}e_3 = \lambda e_2$ with $\lambda \in \mathbb{C}^*$
  \item $\tilde{\tilde{A}}e_2 = \lambda e_3$, $\tilde{\tilde{A}}e_3 = 0$ with $\lambda \in \mathbb{C}^*$,
\end{itemize}

depending on whether $e_2$ spans Ker $A|_{\mathfrak{k}}$ or not. For instance, if $e_2 \notin \text{Ker} A|_{\mathfrak{k}}$, then there exists a vector $v_1 \in \mathfrak{k}$ of the form $ze_2 + ae_3$, $a \neq 0$, that spans Ker $A|_{\mathfrak{k}}$. Since $Ae_2 \in \text{Ker} A|_{\mathfrak{k}}$, we have $Ae_2 = 0$. Let $g$ be the linear map defined by $ge_1 = e_1$, $ge_2 = ae_2$ and $ge_3 = v_1$. Clearly $g \in \text{Aut}(\mathfrak{r}_3(\mathbb{C}))$. $A = g^{-1}Ag$ satisfies $Ae_2 = g^{-1}(e_2) = \lambda e_2$ and $Ae_3 = 0$.

And so, if we repeat the same reasoning as in 3.2.1 by considering the nilpotency degree of $A$, it is easy to verify that $A$ is Aut$(\mathfrak{r}_3(\mathbb{C}))$-similar to a matrix $\hat{A}$ such that $\hat{A}e_1 \in \{0, e_2, e_3\}$ and $\hat{A}|_{\mathfrak{k}} = \hat{A}|_{\mathfrak{k}}$. For example, if $A$ is a nilpotent matrix of degree 3 and $\hat{A}$ is as given in the first type: $\hat{A}e_2 = 0$ and $\hat{A}e_3 = \lambda e_2$. Since $\text{Im} \hat{A} = \mathfrak{k}$, there must exist a vector $v_2 = e_1 + w$ with $w \in \mathfrak{k}$ such that $\hat{A}v_2 = te_3$ with $t \neq 0$. Let $g$ be the linear map defined by $ge_1 = v_2$, $ge_2 = te_2$, $ge_3 = te_3$, then $A = g^{-1}Ag$ satisfies $\hat{A}e_1 = e_3$, $\hat{A}e_2 = 0$ and $\hat{A}e_3 = \lambda e_2$.

**Proposition 3.9.** Any hom-Lie algebra $(\mathfrak{r}_3, A)$ with a nilpotent operator is isomorphic to some hom-Lie algebra $\mathfrak{L}_2 := (\mathfrak{r}_3, A_i)$ where $A_0$ is the zero map and the matrix of $A_i$ with respect to the ordered basis $\{e_1, e_2, e_3\}$ is:

\[
\begin{pmatrix}
  A_1 \\
  A_2 \\
  A_3(\lambda) \\
  A_4(\lambda) \\
  A_5(\lambda) \\
  A_6(\lambda)
\end{pmatrix}
\]

Two hom-Lie algebras $(\mathfrak{r}_3, A_i)$ and $(\mathfrak{r}_3, A_j)$ are isomorphic if and only if $A_i = A_j$.

### 3.4 hom-Lie structures on $\mathfrak{n}_3(\mathbb{C})$

Note that any linear map on a 2-step nilpotent Lie algebra gives a hom-Lie algebra structure. In this case, we need to study Aut$(\mathfrak{n}_3(\mathbb{C}))$-conjugacy classes of nilpotent matrices in $M(3, \mathbb{C})$. Since span$_\mathbb{C}\{e_3\}$ is an invariant subspace for
any automorphism in Aut($\mathfrak{h}_3(\mathbb{C})$), we will consider several cases for a hom-Lie algebra structure $A$ on $\mathfrak{h}_3(\mathbb{C})$, with $A$ nilpotent matrix, depending on how $e_3$ is related with $\text{Ker}(A)$ or $\text{Im}(A)$ and the nilpotency degree of $A$.

First, we suppose that $A^2 = 0$ and $A \neq 0$.

**Case 1:** $e_3 \in \text{Im}(A) \subset \text{Ker}(A)$. And so, let $v_1 \in \mathbb{C}^3$ such that $Av_1 = e_3$ and let $v_2 \in \text{Ker}(A)$ be a linear complement of $e_3$ in $\text{Ker}(A)$. Define $g$ the linear map given by $ge_1 = \frac{1}{\lambda}v_1$, $ge_2 = v_1$ and $ge_3 = e_3$, where $\lambda = e^1 \wedge e^2(v_1, v_2)$. We have $g \in \text{Aut}(\mathfrak{h}_3(\mathbb{C}))$ and $\tilde{A} = g^{-1}Ag$ satisfies

- $\tilde{A}e_1 = 0$, $\tilde{A}e_2 = e_3$ and $\tilde{A}e_3 = 0$

**Case 2:** $e_3 \in \text{Ker}(A)$ and $e_3 \notin \text{Im}(A)$. Let $v_2 \in \text{Im}(A)$ and $v_1 \in \mathbb{C}^3$ such that $Av_1 = v_2$. We have $\{v_1, v_2, e_3\}$ is a basis of $\mathbb{C}^3$ and the linear transformation $g$ given by $ge_1 = v_2$, $ge_2 = v_1$ and $ge_3 = \lambda e_3$ with $\lambda = e^1 \wedge e^2(v_1, v_2)$ is in $g \in \text{Aut}(\mathfrak{h}_3(\mathbb{C}))$. Let $\tilde{A} = g^{-1}Ag$, we have

- $\tilde{A}e_1 = 0$, $\tilde{A}e_2 = e_1$ and $\tilde{A}e_3 = 0$

**Case 3:** $e_3 \notin \text{Ker}(A)$. Let $v_2 = Ae_3$. Since $v_2 \in \text{Ker}(A)$, let $v_1$ be a linear complement of $v_2$ in $\text{Ker}(A)$. Consider the linear map defined by $ge_1 = \frac{1}{\lambda}v_1$, $ge_2 = v_2$ and $ge_3 = e_3$ with $\lambda = e^1 \wedge e^2(v_1, v_2)$. Then $g \in \text{Aut}(\mathfrak{h}_3(\mathbb{C}))$ and $\tilde{A} = g^{-1}Ag$ is such that:

- $\tilde{A}e_1 = 0$, $\tilde{A}e_2 = 0$ and $\tilde{A}e_3 = e_2$

Now, if the degree of nilpotency of $A$ is 3, we have $\dim \text{Ker}(A) = 1$ and $\text{Ker}(A) \subset \text{Im}(A)$.

**Case 4:** $e_3 \in \text{Ker}(A) \subset \text{Im}(A)$. Let $v_2$ be a linear complement of $e_3$ in $\text{Im}(A)$ and let $v_1$ be a vector such that $Av_1 = v_2$. Since $Av_2 = A^2v_1 \in \text{Ker}(A)$, $Av_2 = \lambda e_3$ with $\lambda \neq 0$. We have $\{v_1, v_2, e_3\}$ is a cyclic basis for $\mathbb{C}^3$ from which it follows that the linear map $g$ defined by $ge_1 = \alpha v_1$, $ge_2 = \alpha e_2$ and $ge_3 = \alpha^2 e_1 \wedge e^2(v_1, v_2)$ with $\alpha = \frac{\lambda}{e^1 \wedge e^2(v_1, v_2)}$ is an automorphism of $\mathfrak{h}_3(\mathbb{C})$ and $\tilde{A} = g^{-1}Ag$ satisfies

- $\tilde{A}e_1 = e_2$, $\tilde{A}e_2 = e_3$ and $\tilde{A}e_3 = 0$

**Case 5:** $e_3 \in \text{Im}(A)$ and $e_3 \notin \text{Ker}(A)$. Let $v_1$ a vector in $\mathbb{C}^3$ such that $Av_1 = e_3$ and let $v_2 = Ae_3$. We have $\{\lambda v_1, \lambda e_3, \lambda v_2\}$ is a cyclic basis for $\mathbb{C}^3$ and the linear map $g$ given by $ge_1 = \lambda v_1$, $ge_2 = \lambda v_2$ and $ge_3 = \lambda e_3$ with $\lambda = \frac{1}{e^1 \wedge e^2(v_1, v_2)}$ is in $\text{Aut}(\mathfrak{h}_3(\mathbb{C}))$. The linear map $\tilde{A} = g^{-1}Ag$ is such that:

- $\tilde{A}e_1 = e_3$, $\tilde{A}e_2 = 0$ and $\tilde{A}e_3 = e_2$

**Case 6:** $e_3 \notin \text{Im}(A)$. And so, $e_3 \notin \text{Ker}(A^2)$ and $\{e_3, v_1 := Ae_3, v_2 := A^2e_3\}$ is a cyclic basis for $\mathbb{C}^3$ and the linear map $g$ defined by $ge_1 = \lambda v_2$, $ge_2 = \lambda v_1$ and $ge_3 = \lambda e_3$ with $\lambda = \frac{1}{e^1 \wedge e^2(v_1, v_2)}$ is such that

- $\tilde{A}e_1 = 0$, $\tilde{A}e_2 = e_1$ and $\tilde{A}e_3 = e_2$
The following proposition summarizes the computations above.

**Proposition 3.10.** Any hom-Lie algebra \((\mathfrak{h}_3(\mathbb{C}), A)\) with a nilpotent operator is isomorphic to some hom-Lie algebra \(L_i := (\mathfrak{h}_3(\mathbb{C}), A_i)\) where \(A_0\) is the zero map and the matrix of \(A_i\) with respect to the ordered basis \(\{e_1, e_2, e_3\}\) is:

\[
\begin{align*}
A_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & A_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
A_4 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & A_5 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & A_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\end{align*}
\]

Two hom-Lie algebras \((\mathfrak{h}_3(\mathbb{C}), A_i)\) and \((\mathfrak{h}_3(\mathbb{C}), A_j)\) are isomorphic if and only if \(A_i = A_j\).

### 4 Classification of Orbit Closures

The aim of this section is to study the partial order given by the degeneration relation on the family of hom-Lie algebras obtained above. We want to provide useful invariants of hom-Lie algebras which are preserved under the process of degeneration and can be easily computed/verified. In addition to the above requirements, we also want to take advantage of the well-known classification of orbit closures of 3-dimensional complex Lie algebra (\([9]\)).

We begin by studying degenerations between hom-Lie algebras given in section 3 with isomorphic underlying Lie algebra. By a case-by-case analysis, we have seen that a hom-Lie algebra \(L_i := (\mathfrak{h}_3(\mathbb{C}), A_i)\) degenerates to a hom-Lie algebra \(L_k := (\mathfrak{h}_3(\mathbb{C}), A_k)\) if and only if \(A_k\) in the Euclidean closure of \(\text{Aut}(L_j) \cdot A_i\).

#### 4.1 Invariants

As we observed earlier in the Corollary 2.15, the dimension of the algebra of Derivations of a hom-Lie algebra is an important invariant to study degenerations. By elementary computations, such invariant for each hom-Lie algebra \(L_i\) is given in Table 3.

#### 4.1.1 Lie algebra realizations by hom-Lie algebras

Let \(\Omega\) be a finite subset of \(\mathbb{Z}^3_{\geq 0}\), and let \((\alpha_\lambda)_{\lambda \in \Omega}\) be a family of complex constants, indexed by \(\Omega\). We consider the (continuous) map \(\varphi_{(\alpha_\lambda)_{\lambda \in \Omega}} : C^2 \times C^1 \to L^2\), given by

\[
\varphi_{(\alpha_\lambda)_{\lambda \in \Omega}}(\mu, A) = \sum_{\lambda=(i,j,k) \in \Omega} \alpha_\lambda A^i \mu(A^j-, A^k-). \tag{11}
\]

Here, \(L^2\) is the vector space of bilinear maps from \(\mathbb{C}^3 \times \mathbb{C}^3\) to \(\mathbb{C}^3\). Note that for certain constants, we can restrict the codomain of \(\varphi_{(\alpha_\lambda)_{\lambda \in \Omega}}\) to \(C^2\). Let \(\psi_{\alpha,\beta}, \phi_{\beta}, \rho\) be functions from \(C^2 \times C^1 \to C^2\) defined by
\[ \psi_{\alpha,\beta}(\mu, A) := \mu(\cdot, \cdot) + \alpha A\mu(\cdot, \cdot) + \beta \mu(A\cdot, \cdot) + \beta \mu(\cdot, A\cdot) \]
\[ \phi_{\beta}(\mu, A) := A\mu(\cdot, \cdot) + \beta \mu(A\cdot, \cdot) + \beta \mu(\cdot, A\cdot) \]
\[ \rho(\mu, A) := \mu(A\cdot, \cdot) + \mu(\cdot, A\cdot). \]

Note that if \((\mu, A) \xrightarrow{\deg} (\lambda, B)\) then \(\varphi((\alpha,\lambda)_{\Lambda_{\mathbb{E}}} (\mu, A) \xrightarrow{\deg} \varphi((\alpha,\lambda)_{\Lambda_{\mathbb{E}}} (\lambda, B),\)

analogously for \(\phi_{\beta}\) and \(\rho\). If \((C^3, \mu)\) is an almost Abelian Lie algebra and \(A\) leaves an ideal codimension 1 of \((C^3, \mu)\) invariant, then \(\psi_{\alpha,\beta}, \phi_{\beta}\) and \(\rho\) sends to \((\mu, A)\) to an almost Abelian Lie algebra structure. The purpose of this part is to illustrate how evaluate the functions in (12) at each hom-Lie algebra given in section 3. The next example show how we obtained the results of the Tables 4, 5, 6 and 7.

**Example 4.1.** We consider the hom-Lie algebras \(\mathfrak{L}_5^2(z, \lambda)\). As we mentioned above, we have \(\varphi((\alpha,\lambda)_{\Lambda_{\mathbb{E}}} (\mathfrak{L}_5^2(z, \lambda))\) is an Almost abelian Lie algebra. We want to determine the isomorphism class of \(\psi_{\alpha,\beta}(\mathfrak{L}_5^2(z, \lambda))\):

\[ \psi_{\alpha,\beta}(\mathfrak{L}_5^2(z, \lambda)) = \begin{cases} [e_1, e_2] = (1 + \alpha \lambda + \beta \lambda) e_2 - \lambda (\alpha + \beta \lambda) e_1, \\ [e_1, e_3] = \lambda (\beta + \alpha z) e_2 + z (1 - \alpha \lambda - \beta \lambda) e_3. \end{cases} \]

From Proposition 2.4, we just have to study \(\text{Det}(B), \text{Tr}(B)\) and the discriminant of \(B\) where \(B\) is

\[ B = \begin{bmatrix} 1 + \alpha \lambda + \beta \lambda & \lambda (\beta + \alpha z) \\ -\lambda (\alpha + \beta \lambda) & z (1 - \alpha \lambda - \beta \lambda) \end{bmatrix} \]

Note that \(\psi_{\alpha,\beta}(\mathfrak{L}_5^2(z, \lambda))\) is never the abelian Lie algebra, and so, \(\psi_{\alpha,\beta}(\mathfrak{L}_5^2(z, \lambda))\) is isomorphic to \(\mathfrak{n}_1(C)\) if and only if \(\text{Det}(B) = 0\) and \(\text{Tr}(B) = 0\); i.e.

\[ \alpha \beta = -\frac{z}{(z - 1)^2 \lambda^2} \]
\[ \alpha + \beta = \frac{1 + z}{(z - 1)^2 \lambda}. \]

or equivalently,

\[ \alpha = \frac{z + 1 \pm \sqrt{z^2 + 6z + 1}}{2(z - 1) \lambda} \]
\[ \beta = \frac{z + 1 \mp \sqrt{z^2 + 6z + 1}}{2(z - 1) \lambda}. \]

Similarly \(B\) is never equal to a multiple of the identity matrix, therefore \(\psi_{\alpha,\beta}(\mathfrak{L}_5^2(z, \lambda))\) is isomorphic to \(r_2(C)\) if and only if \(\text{Trace}(B) \neq 0\) and \(\text{Trace}(B)^2 - 4 \text{Det}(B) = 0\), equivalently

\[ \begin{cases} (1 - z) (\alpha + \beta) + 1 + z s - 1 = 0 \\ (z - 1) (\alpha - \beta)^2 \lambda^2 - 2 (1 + z) (\alpha + \beta) + (z - 1) = 0 \end{cases} \]

for some \(s \in \mathbb{C}^*\), which is the same as

\[ \alpha = \frac{s + sz - 1 \pm \sqrt{s (s - 2 + 6sz - 2z + z^2s)}}{2s \lambda (z - 1)} \]
\[ \beta = \frac{s + sz - 1 \mp \sqrt{s (s - 2 + 6sz - 2z + z^2s)}}{2s \lambda (z - 1)} . \]
In other case, $\psi_{\alpha,\beta}(\mathcal{L}_\beta^0(z,\lambda))$ is isomorphic to $r_2(\mathbb{C}) \times \mathbb{C}$ if and only if $\text{Trace}(B)s - 1 = 0$, for some $s \in \mathbb{C}^*$ and $\text{Det}(B) = 0$; this is

$$\alpha = \frac{sz + s - 1 \pm \sqrt{s}z^2 + 6sz + s^2 - 2sz - 2s + 1}{2\lambda s (z - 1)},$$

$$\beta = \frac{sz + s - 1 \mp \sqrt{s}z^2 + 6sz + s^2 - 2sz - 2s + 1}{2\lambda s (z - 1)},$$

and similarly, $\psi_{\alpha,\beta}(\mathcal{L}_\beta^0(z,\lambda))$ is isomorphic to $\mathfrak{r}_{3,-1}$ if and only if $\text{Det}(B)s - 1 = 0$, for some $s \in \mathbb{C}^*$ and $\text{Trace}(B) = 0$:

$$\alpha = \frac{sz + s \pm \sqrt{s}(sz^2 + 6sz + s^2 + 1)}{2(z - 1)\lambda s},$$

$$\beta = \frac{sz + s \mp \sqrt{s}(sz^2 + 6sz + s^2 + 1)}{2(z - 1)\lambda s}.$$  

In the remaining case, we have $\psi_{\alpha,\beta}(\mathcal{L}_\beta^0(z,\lambda))$ is isomorphic to $\mathfrak{r}_{3,2}$ for some $z \in \mathbb{C}$, with $z(z^2 - 1) \neq 0$, if and only if $\text{Trace}(B)s_1 - 1 = 0$, $\text{Det}(B)s_2 - 1 = 0$ and $\text{Trace}(B)^2 - 4\text{Det}(B)s_3 - 1 = 0$, for some $s_1$, $s_2$ and $s_3$ in $\mathbb{C}^*$. By solving the three equations for $\alpha$ and $\beta$ we have

$$\alpha = \frac{s_2(sz_1 + s_1 - 1) \pm \sqrt{f(s_1, s_2)}}{2s_1s_2(z - 1)\lambda},$$

$$\beta = \frac{s_2(sz_1 + s_1 - 1) \mp \sqrt{f(s_1, s_2)}}{2s_1s_2(z - 1)\lambda},$$

$$f(s_1, s_2) = (1 + (z^2 + 1 + 6z)s_1^2 - 2(z + 1)s_1)s_2^2 - 4s_2s_1^2$$

with $4s_1^2 - s_2 \neq 0$.

In the case of the hom-Lie algebras $\mathcal{L}_\beta^1$, a straightforward verification shows that $\psi_{\alpha,\beta}(\mathcal{L}_\beta^1)$ is a Lie algebra with nondegenerate Killing form, and so we have such Lie algebra is isomorphic to $\mathfrak{so}(3,\mathbb{C})$. With respect to $\mathcal{L}_\beta^{12}$ and $\mathcal{L}_\beta^{13}(\lambda)$, the function $\psi_{\alpha,\beta}$ does not necessarily send them to a Lie algebra.

### 4.1.2 Others derivations

**Given a finite subset of $\mathbb{Z}^3_0$, say $\Omega$, a family of complex constants $\{\alpha_I, \beta_I, \gamma_I : I = (i_1, i_2, i_3) \in \Omega\}$ and a hom-Lie algebra $(\mathbb{C}^4, \mu, A)$, we can obtain a new algebra as in the equation (11). It is evident that the (dimension of) extended derivations of this new algebra is an invariant of $(\mathbb{C}^3, \mu, A)$; we mean by such derivations to the kernel of $T(\mu, A) : (C^1)^{[r]} \times (C^1)^{[r]} \times (C^1)^{[r]} \to L^2$ defined by $T(\mu, A)(D_1, \ldots, F_1, \ldots, G_1)$ equal to**

$$\sum_{I \in \Omega} \alpha_I D_I A^I(\mu(A^I - A^I)) + \beta_I A^I(\mu(F_I A^I - A^I)) + \gamma_I A^I(\mu(A^I - G_I A^I)).$$

**In addition, note that we can impose extra conditions of linear dependence of the matrices $D_I$, $F_I$ and $G_I$ and commuting relations or others relations with the matrices $A^I$.**
It follows easily by the upper semi-continuity of the nullity function on linear operators that the dimension of such subspace of derivations of a hom-Lie algebra \((\mathbb{C}^n, \mu, A)\) is less or equal that in the orbit closure of \((\mu, A)\).

For our purposes, it suffices to consider the following subspaces of extended derivations: let \((\mathbb{C}^3, \mu, A)\) be a hom-Lie algebra and let \(l\) be a complex constant

\[
\begin{align*}
D_l\mu(\cdot, \cdot) + \mu(D_l\cdot, \cdot) + \mu(\cdot, D_l\cdot) &= 0 \\
D_l + lD_1 &= 0 \\
D_lA - AD_l &= 0, \\
D_l(\cdot, \cdot, \cdot) &= 0 \\
D_1A = AD_1 &= 0, \\
D_3A - AD_3 &= 0.
\end{align*}
\]

We denote by \(\text{der}^1_l(\mathbb{C}^3, \mu, A)\) the dimension of such subspace. Similarly, \(\text{der}^2_l(\mathbb{C}^3, \mu, A)\) stands for the dimension of the vector space

\[
\begin{align*}
D\mu(\cdot, \cdot, \cdot) &= 0 \\
DA - AD &= 0.
\end{align*}
\]

**Example 4.2.** We consider the hom-Lie algebra \(\mathcal{L}^5_3(z) = (r_{3,2}, A_3)\). If \(D_l(w')s\), \(D_2(z's)\) and \(D_3(z's)\) are commuting matrices with \(A_5\) and \(D_1 = -tD_3\), then \(\Lambda(\cdot, \cdot) := D_1[\cdot, \cdot] + [D_2, \cdot] + [\cdot, D_3] \) is

\[
e_1 \cdot e_1 = (x_{3,1} + y_{3,1})ze_3, \quad e_1 \cdot e_3 = (x_{1,1} - y_{3,3} + y_{3,3})ze_3,
\]

\[
e_1 \cdot e_2 = -y_{1,2}e_1 + (x_{1,1} - y_{3,3} + y_{3,3})e_2 + (-y_{3,3} + y_{3,3}z)e_3,
\]

\[
e_2 \cdot e_1 = y_{1,2}e_1 + (-y_{1,1} + y_{3,3} - y_{3,3})e_2 + (y_{3,2} - y_{3,2}z)e_3,
\]

\[
e_2 \cdot e_2 = (x_{1,2} - y_{1,2})e_2, \quad e_2 \cdot e_3 = x_{1,2}ze_3, \quad e_3 \cdot e_2 = -y_{1,2}e_3,
\]

\[
e_3 \cdot e_1 = (-y_{1,1} + y_{3,3} - y_{3,3})ze_3.
\]

Since \(z \neq 0\), \(x_{1,2} = 0\), \(y_{1,2} = 0\), \(x_{3,1} = y_{3,1}\), \(x_{3,2} = y_{3,2}\), \(x_{1,1} = y_{3,3}(t - 1)\), \(x_{3,3} = y_{3,3} - y_{1,1}\) and \((t - z)y_{3,2} = 0\). It follows that if \(t = z\) then \(\text{der}^1_2 = 4\), and \(\text{der}^1_1 = 3\) in other case.

| hom-Lie | \(\text{der}^1_1\) | hom-Lie | \(\text{der}^1_2\) |
|---------|-----------------|---------|-----------------|
| \(\mathcal{L}^5_3(z)\) | \(t = z\) | \(\mathcal{L}^5_3(z)\) | \(t = 1/z\) |
| \(t \neq z\) | \(3\) | \(t \neq 1/z\) | \(3\) |
| \(\mathcal{L}^5_2(z)\) | \(t = 1\) | \(\mathcal{L}^5_3(z)\) | \(t = 1\) |
| \(t = z\) | \(4\) | \(t = 1/z\) | \(4\) |
| \(t \neq 1, z\) | \(3\) | \(t \neq 1, 1/z\) | \(3\) |

**Table 1:** The \(\text{der}^1_1\)-function on \(\mathcal{L}^5_i(z), i = 1, ..., 5\)

### 4.2 Degenerations in the family \(\mathcal{L}^j_i\) with \(j\) fixed

Before proceeding further with the degenerations of all hom-Lie algebras obtained before, we focus our attention for a while on the special case of degenerations of ones with isomorphic underlying Lie algebra. Given \((\mathbb{C}^3, \mu, A)\) and \((\mathbb{C}^3, \mu, B)\) two hom-Lie algebras with \((\mathbb{C}^3, \mu)\) a Lie algebra, our strategy to show that \((\mathbb{C}^3, \mu, A)\) degenerates in \((\mathbb{C}^n, \mu, B)\) is to verify that \(B\) is in the Euclidean closure of the \(\text{Aut}(\mathbb{C}^3, \mu)\)-orbit of \(A\). We recall that this type of problems are well-known in the literature and go back at least as far as the contributions by
Wim Hesselink and Murray Gerstenhaber to the study the adjoint action of an algebraic group on the nilpotent elements of its Lie algebra (see the nice review of the problem given by Joyce O’Halloran in [52]).

**Theorem 4.3** ([10, Proposition 1.6], [23, Theorem 3.10], [52, Section 2]). Let $A$ and $B$ be $m \times m$ complex nilpotent matrices and let $GL(m, \mathbb{C}) \cdot A$ denote the orbit of $A$ given by its similarity class. $B$ is in the (Zariski) closure of $GL(m, \mathbb{C}) \cdot A$ if and only if $\text{rank } A^k \geq \text{rank } B^k$, $k = 1, \ldots, m - 1$.

In our case, we are working with $3 \times 3$ complex nilpotent matrices, and so the above theorem can be showed by an easy verification.

Also, it is important mention the result about the classification of the degenerations of 3-dimensional complex Lie algebras, which plays an important role in our approach to obtain the analogous result in the family of hom-Lie algebras studied here.

**Proposition 4.4** ([10, Proposition 3], [3, Section 4]). All degenerations of the 3-dimensional complex Lie algebras are:

1. $\text{C}^3 \xrightarrow{\text{deg}} \text{C}^3$,
2. $\mathfrak{n}_3(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{n}_3(\mathbb{C}), \text{C}^3$,
3. $\mathfrak{r}_3(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{r}_3(\mathbb{C}), \mathfrak{r}_3(\mathbb{C}), \mathfrak{n}_3(\mathbb{C}), \text{C}^3$,
4. $\mathfrak{r}_{3,1}(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{r}_{3,1}(\mathbb{C}), \text{C}^3$,
5. $\mathfrak{r}_{3,1}(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{r}_{3,1}(\mathbb{C}), \mathfrak{n}_3(\mathbb{C}), \text{C}^3$,
6. If $z \neq \pm 1$, $\mathfrak{r}_{3,z}(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{r}_{3,z}(\mathbb{C}), \mathfrak{n}_3(\mathbb{C}), \text{C}^3$,
7. $\mathfrak{r}_2 \times \mathbb{C} \xrightarrow{\text{deg}} \mathfrak{r}_2 \times \mathbb{C}, \mathfrak{n}_3(\mathbb{C}), \text{C}^3$,
8. $\mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\text{deg}} \mathfrak{sl}_2(\mathbb{C}), \mathfrak{r}_{3,1}(\mathbb{C}), \mathfrak{n}_3(\mathbb{C}), \text{C}^3$.

The Hasse diagram is given by:

```
  sl_2(C)
   ↓
  r_{3,-1}(C)
  ↓
  r_{3,z}(C)
  ↓
  r_2(C) \times \mathbb{C}
  ↓
  r_3(C)
  ↓
  n_3(C)
  ↓
  r_{3,1}(C)
  ↓
  C^3
```

To illustrate, we study the degenerations in the family of hom-Lie algebras $\mathfrak{L}_0^i$; we can now proceed analogously to analyze the degenerations in the remaining families (the invariant $\text{der}_1^i$ given in Table 1 can be used to study degenerations in the family $\mathfrak{L}_0^i(z)$).

We use the Table 3 like a starting point for our analysis because Corollary 2.15. The table 2 shows the information about the degenerations in the
family $\mathcal{L}_0^1$. To be more precise, the checkmark in the intersection of row $\mathcal{L}_0^1$ with column $\mathcal{L}_0^1$ denotes that there is a degeneration of the hom-Lie algebra $\mathcal{L}_0^1$ in $\mathcal{L}_0^1$. For example, $\mathcal{L}_0^{13}(\lambda)$ degenerates to $\mathcal{L}_0^{13}(\kappa)$ if and only if they are isomorphic, equivalently $\lambda = \kappa$; because of the invariant $\psi$. Other example is that $\mathcal{L}_0^{13}(\kappa)$ is a degeneration of $\mathcal{L}_0^{13}(\lambda)$ if and only if $\lambda = \kappa$. To show that $\mathcal{L}_0^{13}(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_0^{13}(\kappa)$, we explicitly find a family $g_t$ in $\text{Aut}(t_2(\mathbb{C}) \times \mathbb{C})$ such that $A_0 = \lim_{t \to \infty} g_t \cdot A_{13}$ (see example 4.5). Conversely, if $\mathcal{L}_0^{13}(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_0^{13}(\kappa)$ then $\psi_{\alpha,\beta}(\mathcal{L}_0^{13}(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha,\beta}(\mathcal{L}_0^{13}(\kappa))$, and so, by letting $\alpha = 0$ and $\beta = -1/\lambda$ we get $n_0 \xrightarrow{\text{deg}} \psi_{\alpha,\beta}(\mathcal{L}_0^{13}(\kappa))$, which follows from Proposition 4.4 and Table 5 that $\psi_{0,-1/\lambda}(\mathcal{L}_0^{13}(\kappa))$ is isomorphic to $n_1$, and consequently $\beta = -1/\kappa = -1/\lambda$.

The symbol Der + invariant in the intersection of row $\mathcal{L}_0^1$ with column $\mathcal{L}_0^1$ stands that the dimensions of the algebras of derivations of $\mathcal{L}_0^1$ and $\mathcal{L}_0^1$ are equal but $\mathcal{L}_0^1$ and $\mathcal{L}_0^1$ are not isomorphic because of such invariant; hence there is no degeneration from $\mathcal{L}_0^1$ to $\mathcal{L}_0^1$. Analogously, the symbol Der denotes that $\mathcal{L}_0^1$ cannot degenerate to $\mathcal{L}_0^1$ since the dimension of algebra of derivations of $\mathcal{L}_0^1$ is greater than to the dimension of algebra de derivations of $\mathcal{L}_0^1$, and the symbol $\psi$ represents that there is no degeneration from $\mathcal{L}_0^1$ to $\mathcal{L}_0^1$ because it is easy give values of $\alpha$ and $\beta$ such that $\psi_{\alpha,\beta}(\mathcal{L}_0^1)$ does not degenerate to $\psi_{\alpha,\beta}(\mathcal{L}_0^1)$ (here it is important to be aware of Proposition 4.4 as we showed above). Similarly $\phi$ and $\rho$ are used. Where we have used a clover symbol, the invariants mentioned above do not give reasons why such a degeneration is impossible, and so we give other reasons; which are explained below.

\textbf{\textbullet} 1 $\mathcal{L}_0^1$ and $\mathcal{L}_0^1$ are not isomorphic: $\mathcal{L}_0^1$ is a multiplicative hom-Lie algebra and $\mathcal{L}_0^1$ is not.

\textbf{\textbullet} 2 $\mathcal{L}_0^1$ satisfies the identity $[A_5, \cdot] \equiv 0$ but $\mathcal{L}_0^1$ does not.

\textbf{\textbullet} 3 An easy computation shows that $\text{der}^2(\mathcal{L}_0^1) = 4$ and $\text{der}^2(\mathcal{L}_0^1) = 3$; therefore $\mathcal{L}_0^1$ does not degenerate to $\mathcal{L}_0^1$ (see the definition of the invariant $\text{der}^2$ in equation (14)).

\textbf{\textbullet} 4 $\mathcal{L}_0^1$ and $\mathcal{L}_0^1$ are not isomorphic: $\mathcal{L}_0^1$ satisfies the identity $[A_2, \cdot] \equiv 0$ but $\mathcal{L}_0^1$ does not.

**Example 4.5.** To show that $(\mathcal{L}_6, A_{13}(\lambda))$ degenerates to $(\mathcal{L}_6, A_9(\lambda))$, given that the underlying algebra is the same, we will find a family $g_t$ in $\text{GL}(3, \mathbb{C})$ such that $g_t \cdot \mathcal{L}_6 = \mathcal{L}_6$; i.e. $g_t \in \text{Aut}(\mathcal{L}_6)$, and $g_t A_{13}(\lambda) g_t^{-1} \xrightarrow{\text{deg}} A_9(\lambda)$ as $t$ tends to infinity. Here, we take advantage of the Proposition 4.3 in the following way. Let $g$ be an arbitrary automorphism of $\mathcal{L}_6$, then we know that $g$ has the following expression:

$$
 g = \begin{pmatrix}
 1 & 0 & 0 \\
 x & a & 0 \\
 y & 0 & b
 \end{pmatrix}, \ a, b \in \mathbb{C}^*.
$$

Let $E := g A_{13}(\lambda) g^{-1} - A_9(\lambda)$

$$
 E = \begin{bmatrix}
 -x & \frac{1}{a} & 0 \\
 \frac{-x^2 b + x b a \lambda + \lambda y a^2 + a b}{a} & \frac{\lambda (a-b)}{b} & \frac{\lambda}{a} \\
 \frac{-x b + x b a \lambda + \lambda y a}{a} & \frac{y - b a + a \lambda}{a} & 0
 \end{bmatrix}
$$

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The basic idea is to try that $E$ is as close as possible to being the zero map. To this end, we can start letting some entries of $E$ to be zero; for instance, by solving the equations $E[2,1], E[3,1], E[3,2]$ for $a$, $b$ and $y$. Here, we can introduce a new variable $z$ such that $z^2 = 1 + 4x\lambda$ and so

$$a = \frac{(z + 1)^2(z - 1)}{8\lambda^2}$$

$$b = a$$

$$y = \frac{(z + 1)(z - 1)}{4\lambda}$$

and

$$E = \begin{bmatrix}
\frac{-2\lambda}{z+1} & \frac{8\lambda^2}{(z-1)(z+1)^2} & 0 \\
0 & \frac{2\lambda}{z+1} & \frac{2\lambda}{z-1} \\
0 & 0 & 0
\end{bmatrix}.$$  

If we let $z = 1 + \exp(t)$, then we obtain a parametric family of automorphisms of $L_6$, $g_t$, and it satisfies $g_t \cdot A_{13}(\lambda) \xrightarrow{t \to \infty} A_9(\lambda)$

The respective Hasse diagram of the family $L^j_6$, $j = 0, \ldots, 13$ with respect to the partial order defined by the degeneration relation is:
| deg | $\mathcal{L}_0^{13}(\kappa)$ | $\mathcal{L}_0^{12}$ | $\mathcal{L}_0^{11}$ | $\mathcal{L}_0^{10}$ | $\mathcal{L}_0^9(\kappa)$ | $\mathcal{L}_0^8$ | $\mathcal{L}_0^7$ | $\mathcal{L}_0^6(\kappa)$ | $\mathcal{L}_0^5$ | $\mathcal{L}_0^4$ | $\mathcal{L}_0^3$ | $\mathcal{L}_0^2$ | $\mathcal{L}_0^1$ | $\mathcal{L}_0^0$ |
|-----|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\mathcal{L}_0^{13}(\lambda)$ | $\lambda \equiv \kappa, \psi$ | Der + $\rho$ | Der + $\rho$ | $\psi$ | $\lambda \equiv \kappa, \psi$ | $\psi$ | $\lambda = \kappa, \psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ |
| $\mathcal{L}_0^{12}$ | Der + $\rho$ | $\checkmark$ | Der + $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ |
| $\mathcal{L}_0^{11}$ | Der + $\rho$ | Der + $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ | $\checkmark$ | $\rho$ |
| $\mathcal{L}_0^{10}$ | Der | Der | Der | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | $\lambda \equiv \kappa, \psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ |
| $\mathcal{L}_0^{9}(\lambda)$ | Der | Der | Der | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | $\lambda \equiv \kappa, \psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ |
| $\mathcal{L}_0^{8}$ | Der | Der | Der | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | $\lambda \equiv \kappa, \psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ |
| $\mathcal{L}_0^{7}$ | Der | Der | Der + $\phi$ | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | Der + $\rho$ | $\lambda \equiv \kappa, \psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ | $\psi$ |

Table 2: Degenerations in the family $\mathcal{L}_0^i$
By using similar arguments we obtain the Hasse diagrams for the closure ordering in the remaining families:

0. Degenerations in the family $\mathcal{L}_0^i$

1. Degenerations in the family $\mathcal{L}_1^i$

2. Degenerations in the family $\mathcal{L}_2^i$

3. Degenerations in the family $\mathcal{L}_3^i$

4. Degenerations in the family $\mathcal{L}_4^i$

5. Degenerations in the family $\mathcal{L}_5^i(z)$
4.3 Degenerations 3-dimensional hom-Lie algebras structures with nilpotent twisting map on 3-dimensional Lie algebras

The remainder of this paper will be devoted to explain how to obtain the classification of the orbits closure for the hom-Lie algebras obtained above. Again, we use, as much as possible, Theorem 4.4 and Proposition 2.16. In fact, if a hom-Lie algebra \((\mathbb{C}^n, \mu, A)\) degenerates to \((\mathbb{C}^n, \lambda, B)\), then the algebra \((\mathbb{C}^n, \mu)\) degenerates to \((\mathbb{C}^n, \lambda)\) (because of Proposition 2.16). In a similar way as in subsection 4.2, to show that \((\mathbb{C}^n, \mu, A)\) degenerates to \((\mathbb{C}^n, \lambda, A)\), we can try to find a family \(g_t\) in \(GL(3, \mathbb{C})\) the stabilizer of \(A\) under conjugation, such that \(g_t \cdot \mu\) tends to \(\lambda\) as \(t\) tends to infinity. Based on a case-by-case analysis, all the degenerations in such situation can be obtained by executing such idea.

Example 4.6. We will show that \(L^9_6(\lambda) = (r_2(\mathbb{C}) \times \mathbb{C}, A(\lambda))\) degenerates to \(L^5_1 = (n_3(C), B)\). Since the twisting maps of each such hom-Lie algebras are conjugate, we will find a one-parameter family of transformations in the left coset \(gGL(3, \mathbb{C})A(\lambda)\) where \(gA(\lambda)g^{-1} = B\), to realize the degeneration of \(L^6_0 = r_2(\mathbb{C}) \times \mathbb{C}\) in \(L^1_1 = h_3(\mathbb{C})\).

Note that any such \(g\) has the form

\[
g := \begin{bmatrix} a & 0 & 0 \\ x & a & a \\ y & x & \frac{\lambda x - a}{\lambda} \end{bmatrix}, \quad a \in \mathbb{C}^* \]

Now, by the change of basis given by \(g\) we have the Lie algebra \(r_2(\mathbb{C}) \times \mathbb{C}\) in this new basis has the form

\[
[e_1, e_2] = \frac{(a - \lambda x)}{a^2} e_2 + \frac{x(a - \lambda x)}{a^3} e_3, \quad [e_1, e_3] = \frac{\lambda}{a} e_2 + \frac{x \lambda}{a^2} e_3.
\] (15)

We want to try that the above Lie algebra is as close as possible to being the Heisenberg Lie algebra. To this end, we can start letting the structure constant \(e_3^3([e_1, e_2]) = 1\) by solving the equation \(\frac{x(a - \lambda x)}{a^3} = 1\) for \(x\). Here, we can introduce a new variable \(z\) such that \(z^2 = 1 - 4a \lambda\) and so \(x = \frac{(-1 + z^2)(-1 + \lambda z)}{4a^2}\) and (15) has the form

\[
[e_1, e_2] = \frac{2 \lambda}{1 - z} e_2 + e_3, \quad [e_1, e_3] = \frac{4 \lambda^2}{1 - z^2} e_2 + \frac{2 \lambda}{z + 1} e_3.
\]
If we let $z = 1 + \exp(t)$, then we obtain a parametric family of transformations, $g_t$, such that $g_t A(\lambda) g_t^{-1} = B$ and $g_t \bullet \tau_2(\mathbb{C}) \times \mathbb{C} \xrightarrow{t \to \infty} n_3(\mathbb{C})$.

To obtain the classification of orbits closure in our family of hom-Lie algebras, we proceed in the same way as in the examples of Subsection 4.2; we use the invariants $\text{Der}$, $\psi_{\alpha,\beta}$, $\psi_{\beta}$ and $\rho$ to determine impossible degenerations and the ideas illustrated in Examples 4.5 and 4.6. Here, we must mention there are five impossible degenerations that cannot be explained by the above-mentioned invariants; we need to give a different argument.

**Proposition 4.7 (♣-5).** The hom-Lie algebras $L_3^4$, $L_5^5$, $L_6^4$ and $L_6^5$ cannot degenerate to $L_2^1$.

**Proof.** We consider the continuous map $\varpi : C^2 \times C^1 \to L^2 \times C^1$ given by

$$(\mu, A) \mapsto (\lambda := \mu(A_{--}), A).$$

The map $\varpi$ sends $L_3^4$, $L_5^5$, $L_6^4$ and $L_6^5$ to the same pair $(\lambda_1, B_1)$

$$\lambda_1 := \{ e_3 \cdot e_1 = -e_2 \\ B_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

while $\varpi$ sends $L_2^1$ to $(\lambda_0, B_0)$

$$\lambda_0 := \{ e_2 \cdot e_2 = e_3 \\ B_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

If $L_3^4$, $L_5^5$, $L_6^4$ degenerate to $L_2^1$, then $(\lambda_0, B_0)$ is in the closure of $\text{GL}(3, \mathbb{C})$-orbit of $(\lambda_1, B_1)$, which cannot happen. In fact, we consider the $(\text{GL}(3, \mathbb{C})$-equivariant) continuous map

$$T : L^2 \times C^1 \to \text{End}(C^1, L^2 \times C^1)$$

where $T(\lambda, B) := T_{(\lambda, B)}$ is the linear map from $C^1$ to $L^2 \times C^1$ given by

$$T_{(\lambda, B)}(X) = (X\lambda(-,-), XB - BX).$$

As in Equation (14), we consider the invariant $\text{Ker} \ T_{(\lambda, B)}$. We have $\text{Ker} \ T_{(\lambda_1, B_1)}$ is equal to 4 and so $\text{Ker} \ T_{(\lambda_1, B_1)}$ is 4 for any $g \in \text{GL}(3, \mathbb{C})$ and on the other hand $\text{Ker} \ T_{(\lambda_0, B_0)}$ is equal to 3. It follows that $(\lambda_1, B_1)$ does not degenerate to $(\lambda_0, B_0)$ since $T_{g(\lambda_1, B_1)}$ cannot be near $T_{(\lambda_0, B_0)}$ for any $g \in \text{GL}(3, \mathbb{C})$ (by the upper semi-continuity of the nullity function on linear operators).

We can now proceed analogously to the previous case to prove that $L_3^4$ and $L_5^5$ do not degenerate to $L_2^1$ by considering the same invariant functions. ■

### Others classifications

We must mention that there have been some attempts to classify complex hom-Lie algebras in dimension 3. We have found several redundancies and mistakes in such works.
In [53, Theorem 4.3], by using symbolic calculations in Wolfram Mathematica®, it is given five non-isomorphic families of Hom-Lie algebras to parameterize 3-dimensional hom-Lie algebras with nilpotent twisting map. As the authors already mentioned, the provided families in [53, Theorem 4.3] present redundancies. For example, we could take the subfamily of $H^3_7$ given by

$$
\mu(a,b,c) := \begin{cases} 
[e_1, e_2] = -a^2 ce_2 + (a^4 + ab) e_3, \\
[e_2, e_3] = e_1 + ce_2 - a^2 e_3.
\end{cases}
$$

It is easy to see that $(\mu(a_0, b_0, c_0), \alpha_3)$ is isomorphic to $(\mu(a_1, b_1, c_1), \alpha_3)$ if and only if $(a_1, b_1, c_1) = (a_0, b_0, c_0)$ or $(a_1, b_1, c_1) = (c_0, b_0 + (a_0 - c_0)(a_0^2 + c_0^2), a_0)$. The algebra $(\mathbb{C}^3, \mu(a,b,c))$ is a Lie algebra if and only if $a = 0$ and $b = 0$, or $c = 0$ and $b = -a^3$. In these cases the hom-Lie algebra $(\mathbb{C}^3, \mu(0,0,c), \alpha_3)$ is isomorphic to $(\mathbb{C}^3, \mu(c,-c^3,0), \alpha_3)$ and also, it is isomorphic to our hom-Lie algebra $L^{13}_6$. We recall the $\alpha_3$ is the linear map defined by $\alpha_3 e_1 = e_2$, $\alpha_3 e_2 = e_3$ and $\alpha_3 e_3 = 0$.

With respect to [18], there are also visible mistakes and redundancies in the classification obtained there. By way of example, in [18, Proposition 4.3], it is easily seen that the given twisting maps are far from being canonical forms for the different hom-Lie structures on $\mathfrak{sl}(2, \mathbb{C})$. For instance, for any $x \in \mathbb{C}^*$, the hom-Lie algebra $(\mathfrak{sl}(2, \mathbb{C}), A(a,x))$ with $A(a,x)$ equal to

$$
\begin{pmatrix}
 a & 0 & 0 \\
 0 & a^{x^2} & x \\
 0 & 0 & a
\end{pmatrix}
$$

is isomorphic to $(\mathfrak{sl}(2, \mathbb{C}), A(a,1))$ via the $g \in \text{Aut}(\mathfrak{sl}(2, \mathbb{C}))$ defined by $g H = H$, $g E = \frac{1}{2} E$ and $g F = x F$. Similarly, in the second family, the hom-Lie algebra $(\mathfrak{sl}(2, \mathbb{C}), B(a,x))$ is isomorphic to $(\mathfrak{sl}(2, \mathbb{C}), B(a,1))$ via the same $g \in \text{Aut}(\mathfrak{sl}(2, \mathbb{C}))$ given above, where $B(a,x)$ is

$$
\begin{pmatrix}
 a & 0 & x \\
 2x & a & 0 \\
 0 & 0 & a
\end{pmatrix}.
$$

In the third case, a true canonical form could be

$$
\begin{pmatrix}
 a & x & 1 \\
 -2 & a & 0 \\
 2x & 0 & a
\end{pmatrix}, \text{ with } a, x \in \mathbb{C}, x \neq 0,
$$

and in the last one, a canonical form could be $C(a,x,y)$ equal to

$$
\begin{pmatrix}
 a & x & y \\
 2y & a & 1 \\
 2x & 0 & a
\end{pmatrix}, \text{ with } a, x, y \in \mathbb{C}, xy \neq 0,
$$

and $x \in \mathbb{R}_{>0}$ or $\text{Im}(x) > 0$. 

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| Dim Der | $\mathfrak{L}_0$ | $\mathfrak{L}_1$ | $\mathfrak{L}_2$ | $\mathfrak{L}_3$ | $\mathfrak{L}_4$ | $\mathfrak{L}_5(z)$ | $\mathfrak{L}_6$ | $\mathfrak{L}_7$ |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0       | $\mathfrak{L}_0^1$ | $\mathfrak{L}_0^5(\lambda)$ | $\mathfrak{L}_0^7(\lambda)$ | $\mathfrak{L}_0^7(z)$ | $\mathfrak{L}_0^8(z)$ | $\mathfrak{L}_0^{10}(\lambda)$, $\mathfrak{L}_0^{12}$, $\mathfrak{L}_0^{11}$ | $\mathfrak{L}_0^7$ | $\mathfrak{L}_0^7$ |
| 1       | $\mathfrak{L}_1^1$, $\mathfrak{L}_1^7$ | $\mathfrak{L}_1^2(\lambda)$, $\mathfrak{L}_1^3(\lambda)$, $\mathfrak{L}_1^4$ | $\mathfrak{L}_1^4(z)$ | $\mathfrak{L}_1^4(z)$ | $\mathfrak{L}_1^4(z)$, $\mathfrak{L}_1^4(z)$ | $\mathfrak{L}_1^4(z)$, $\mathfrak{L}_1^4(z)$ | $\mathfrak{L}_1^4(z)$ | $\mathfrak{L}_1^4(z)$ |
| 2       | $\mathfrak{L}_2^1$, $\mathfrak{L}_2^7$ | $\mathfrak{L}_2^2(\lambda)$, $\mathfrak{L}_2^3(\lambda)$, $\mathfrak{L}_2^4$ | $\mathfrak{L}_2^4(z)$ | $\mathfrak{L}_2^4(z)$ | $\mathfrak{L}_2^4(z)$, $\mathfrak{L}_2^4(z)$ | $\mathfrak{L}_2^4(z)$, $\mathfrak{L}_2^4(z)$ | $\mathfrak{L}_2^4(z)$ | $\mathfrak{L}_2^4(z)$ |
| 3       | $\mathfrak{L}_3^1$, $\mathfrak{L}_3^7$ | $\mathfrak{L}_3^2(\lambda)$, $\mathfrak{L}_3^3(\lambda)$, $\mathfrak{L}_3^4$ | $\mathfrak{L}_3^4(z)$ | $\mathfrak{L}_3^4(z)$ | $\mathfrak{L}_3^4(z)$, $\mathfrak{L}_3^4(z)$ | $\mathfrak{L}_3^4(z)$, $\mathfrak{L}_3^4(z)$ | $\mathfrak{L}_3^4(z)$ | $\mathfrak{L}_3^4(z)$ |
| 4       | $\mathfrak{L}_4^1$, $\mathfrak{L}_4^7$ | $\mathfrak{L}_4^2(\lambda)$, $\mathfrak{L}_4^3(\lambda)$, $\mathfrak{L}_4^4$ | $\mathfrak{L}_4^4(z)$ | $\mathfrak{L}_4^4(z)$ | $\mathfrak{L}_4^4(z)$, $\mathfrak{L}_4^4(z)$ | $\mathfrak{L}_4^4(z)$, $\mathfrak{L}_4^4(z)$ | $\mathfrak{L}_4^4(z)$ | $\mathfrak{L}_4^4(z)$ |
| 5       | $\mathfrak{L}_5^1$, $\mathfrak{L}_5^7$ | $\mathfrak{L}_5^2(\lambda)$, $\mathfrak{L}_5^3(\lambda)$, $\mathfrak{L}_5^4$ | $\mathfrak{L}_5^4(z)$ | $\mathfrak{L}_5^4(z)$ | $\mathfrak{L}_5^4(z)$, $\mathfrak{L}_5^4(z)$ | $\mathfrak{L}_5^4(z)$, $\mathfrak{L}_5^4(z)$ | $\mathfrak{L}_5^4(z)$ | $\mathfrak{L}_5^4(z)$ |
| 6       | $\mathfrak{L}_6^1$ | $\mathfrak{L}_6^7$ | $\mathfrak{L}_6^7$ | $\mathfrak{L}_6^7(z)$ | $\mathfrak{L}_6^7(z)$ | $\mathfrak{L}_6^7(z)$, $\mathfrak{L}_6^7(z)$ | $\mathfrak{L}_6^7(z)$ | $\mathfrak{L}_6^7(z)$ |
| 7       | $\mathfrak{L}_7^1$ | $\mathfrak{L}_7^7$ | $\mathfrak{L}_7^7$ | $\mathfrak{L}_7^7(z)$ | $\mathfrak{L}_7^7(z)$ | $\mathfrak{L}_7^7(z)$, $\mathfrak{L}_7^7(z)$ | $\mathfrak{L}_7^7(z)$ | $\mathfrak{L}_7^7(z)$ |

Table 3: Dimension of the algebra of derivations of the hom-Lie algebras in Section 3.
\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$\varphi_{1}^{1}$, $\varphi_{1}^{2}$, $\varphi_{1}^{3}$ & $\alpha$ & $n_{3}$ \\
\hline
$\varphi_{1}^{1}$, $\varphi_{1}^{2}$, $\varphi_{1}^{4}$ & $\beta$ & $r_{1}$ \\
\hline
$\alpha = \beta$ & $n_{3}$ & \\
$\alpha = 0$ & & \\
$\alpha \neq 0$ & & \\
$\alpha \neq \beta$ & $r_{1}$ & \\
$\alpha = -\beta$ & & \\
$\alpha < 0$ & & \\
$\alpha \beta (\alpha^2 - \beta^2) \neq 0$ & $r_{1, z}$ & \\
\hline
$\varphi_{2}^{1}$ & $\alpha$ & $r_{1, 1}$ \\
\hline
$\varphi_{2}^{1}$ & $\beta$ & $r_{1, 1}$ \\
\hline
$\varphi_{2}^{1}$ & $\alpha = -\beta$ & $r_{1, 1}$ \\
\hline
$\varphi_{2}^{1}$ & $\alpha \neq -\beta$ & $r_{1, 1}$ \\
\hline
$\varphi_{2}^{1}$, $\varphi_{2}^{2}$, $\varphi_{2}^{3}$, $\varphi_{2}^{4}$ & $\alpha$ & $r_{3, 1}$ \\
$\varphi_{2}^{1}$, $\varphi_{2}^{2}$, $\varphi_{2}^{3}$, $\varphi_{2}^{4}$ & $\beta$ & $r_{3, 1}$ \\
\hline
$\varphi_{3}^{1}$, $\varphi_{3}^{2}$, $\varphi_{3}^{3}$ & $\alpha$ & $r_{3, 1}$ \\
\hline
$\varphi_{3}^{1}$, $\varphi_{3}^{2}$, $\varphi_{3}^{3}$ & $\beta$ & $r_{3, 1}$ \\
\hline
$\varphi_{3}^{1}$, $\varphi_{3}^{2}$, $\varphi_{3}^{3}$ & $\alpha = -\beta$ & $r_{3, 1}$ \\
$\varphi_{3}^{1}$, $\varphi_{3}^{2}$, $\varphi_{3}^{3}$ & $\alpha \neq -\beta$ & $r_{3, 1}$ \\
\hline
$\varphi_{3}^{1}$, $\varphi_{3}^{2}$, $\varphi_{3}^{3}$, $\varphi_{3}^{4}$, $\varphi_{3}^{5}$, $\varphi_{3}^{6}$ & $\alpha$, $\beta$ & $r_{3, 1}$ \\
\hline
\end{tabular}
\caption{$\psi_{\alpha, \beta}$-invariant of the hom-Lie algebras in Section 3}
\end{table}
| $\mathfrak{L}_n^0(z, \lambda)$, $\mathfrak{L}_n^0(z, \lambda)$ | $\mathfrak{L}_n^1(z, \lambda)$, $\mathfrak{L}_n^1(z, \lambda)$ | $\mathfrak{L}_n^2(z, \lambda)$, $\mathfrak{L}_n^2(z, \lambda)$ | $\mathfrak{L}_n^0(z, \lambda)$, $\mathfrak{L}_n^0(z, \lambda)$ | $\mathfrak{L}_n^1(z, \lambda)$, $\mathfrak{L}_n^1(z, \lambda)$ | $\mathfrak{L}_n^0(z, \lambda)$, $\mathfrak{L}_n^0(z, \lambda)$ |
|---|---|---|---|---|---|
| $\psi_{\alpha, \beta}$ | $\psi_{\alpha, \beta}$ | $\psi_{\alpha, \beta}$ | $\psi_{\alpha, \beta}$ | $\psi_{\alpha, \beta}$ | $\psi_{\alpha, \beta}$ |
| $\alpha$, $\beta$ | $\alpha$, $\beta$ | $\alpha$, $\beta$ | $\alpha$, $\beta$ | $\alpha$, $\beta$ | $\alpha$, $\beta$ |
| $\alpha = (z + 1 \mp \sqrt{z^2 + 6 z + 1})/2 (1 + z) \lambda$ | $\alpha = (z + 1 \pm \sqrt{z^2 + 6 z + 1})/2 (1 + z) \lambda$ | $\alpha = s^2 + s^{-1} z/(s - 2 + 6 z + 2 s z^2) / 2 \lambda (1 + z)$ | $\alpha = z^2 + z^{-1} \sqrt{s(2s^2 + 4s + 1)} / 2 \lambda (1 + z)$ | $\alpha = (s \pm \sqrt{s(4 - s)}) / 2 \lambda s$ | $\alpha = s^2 + s^{-1} \sqrt{s(2s^2 + 4s + 1)} / 2 \lambda (1 + z)$ |
| $\beta = (z + 1 \mp \sqrt{z^2 + 6 z + 1})/2 (1 + z) \lambda$ | $\beta = (z + 1 \pm \sqrt{z^2 + 6 z + 1})/2 (1 + z) \lambda$ | $\beta = s^2 + s^{-1} z/(s - 2 + 6 z + 2 s z^2) / 2 \lambda (1 + z)$ | $\beta = z^2 + z^{-1} \sqrt{s(2s^2 + 4s + 1)} / 2 \lambda (1 + z)$ | $\beta = (s \pm \sqrt{s(4 - s)}) / 2 \lambda s$ | $\beta = s^2 + s^{-1} \sqrt{s(2s^2 + 4s + 1)} / 2 \lambda (1 + z)$ |
| $\psi_{\alpha, \beta}$ invariant of the hom-Lie algebras in Section 5

Table 5: $\psi_{\alpha, \beta}$-invariant of the hom-Lie algebras in Section 5
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₁ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₂ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₃ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₄ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₅ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₆ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₇ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₈ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₉ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₁₀ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₁₁ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₁₂ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |
| L₀, L₁, L₂, L₅, L₀², L₁², L₂², L₅² | φ₁₃ | a₀, a₁, a₂, a₃, a₄, a₅, a₆, a₇, a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃. |

Table 6: φ₉-invariant of the hom-Lie algebras in Section 3.
| $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$, $\mathcal{L}_6$, $\mathcal{L}_7$, $\mathcal{L}_8$, $\mathcal{L}_9$, $\mathcal{L}_{10}$, $\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{13}$ | $\rho$ |
|---|---|
| $(a_3, a_3, a_5, a_7, a_9, a_10, a_11, a_{13})$ | $a_3$ |
| $2 \times \mathbb{C}$ | $a_3 \times \mathbb{C}$ |
| $(n_3, n_3)$ | $n_3$ |
| $(r_3, r_3)$ | $r_3$ |

Table 7: $\rho$-invariant of the hom-Lie algebras in Section 3
Figure 1: Degenerations of hom-Lie algebras with underlying algebra isomorphic to \(t_3(C), t_{3,1}(C), h_3(C)\) or \(a_3\), and nilpotent twisting map.
Figure 2: Degenerations of hom-Lie algebras with underlying algebra isomorphic to $\mathfrak{r}_{3,z}(\mathbb{C})$, $\mathfrak{h}_{3}(\mathbb{C})$ or $\mathfrak{a}_{3}$, and nilpotent twisting map.
Figure 3: Degenerations of hom-Lie algebras with underlying algebra isomorphic to \(\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}, \mathfrak{h}_3(\mathbb{C})\) or \(\mathfrak{a}_3\), and nilpotent twisting map.
Figure 4: Degenerations of hom-Lie algebras with underlying algebra isomorphic to a unimodular Lie algebra and nilpotent twisting map.
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1 Invariants

1.1 Invariants

Let \( V_2 := \{ \mu : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n : \mu \) is a bilinear map \} \). Given \( \alpha_0, \ldots, \alpha_{26} \in \mathbb{C} \), let 
\[ \varphi_{\alpha_0, \ldots, \alpha_{26}} : V_2 \times M(n, \mathbb{C}) \to V_2 \]
be defined by
\[
\begin{align*}
\alpha_0 \mu(\cdot, \cdot) + \\
\alpha_1 \mu(\cdot, \cdot) + \alpha_2 \mu(A, \cdot) + \alpha_3 \mu(\cdot, A) + \\
\alpha_4 \mu(A, \cdot) + \alpha_5 \mu(\cdot, A) + \alpha_6 \mu(A, A) + \\
\alpha_7 \mu(A, A) + \\
\alpha_8 \mu(\cdot, \cdot) + \alpha_9 \mu(A^2, \cdot) + \alpha_{10} \mu(\cdot, A^2) + \\
\alpha_{11} \mu(\cdot, A^2) + \alpha_{12} \mu(A, A^2) + \alpha_{13} \mu(A^2, A) + \\
\alpha_{14} \mu(A^2, \cdot) + \alpha_{15} \mu(\cdot, A^2) + \alpha_{16} \mu(A, A^2) + \\
\alpha_{17} \mu(A^2, \cdot) + \alpha_{18} \mu(\cdot, A^2) + \alpha_{19} \mu(A^2, A) + \\
\alpha_{20} \mu(A^2, A) + \alpha_{21} \mu(A^2, A) + \alpha_{22} \mu(A, A^2) + \\
\alpha_{23} \mu(\cdot, A^2) + \alpha_{24} \mu(A, A^2) + \alpha_{25} \mu(A^2, A) + \\
\alpha_{26} \mu(A^2, A^2).
\end{align*}
\]

and let \( \theta_{\alpha_0, \ldots, \alpha_{26}}(\mu, A) := (\varphi_{\alpha_0, \ldots, \alpha_{26}}(\mu, A), A) \). Set
\[
\begin{align*}
\psi_{\alpha, \beta} & := \varphi_{1, \alpha, \beta, 0, \ldots, 0} \\
\theta_{\beta} & := \varphi_{0, 1, \beta, 0, \ldots, 0} \\
\rho & := \varphi_{0, 0, 1, 1, 0, \ldots, 0}
\end{align*}
\]

1.2 Some invariant vector spaces

Given \( \theta_1, \ldots, \theta_6 \) and \( \beta_1, \ldots, \beta_6 \) complex numbers, let \( W_{\theta_1, \ldots, \theta_6} \) denote the vector space
\[
W_{\theta_1, \ldots, \theta_6} := \left\{ (D_1, D_2, D_3) \in M(n, \mathbb{C}) \times M(n, \mathbb{C}) \times M(n, \mathbb{C}) : \begin{array}{c}
\theta_1 D_1 + \theta_2 D_2 + \theta_3 D_3 = 0 \\
\theta_4 D_1 + \theta_5 D_2 + \theta_6 D_3 = 0
\end{array} \right\}
\]
and we consider the function
\[
T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_6} : V \times M(n, \mathbb{C}) \to \text{End}(W_{\theta_1, \ldots, \theta_6}; M(n, \mathbb{C}) \times M(n, \mathbb{C}) \times M(n, \mathbb{C}) \times V),
\]
which is given by
\[
T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_6} : \begin{array}{c}
W_{\theta_1, \ldots, \theta_6} \\
(D_1, D_2, D_3)
\end{array} \to (M_1, M_2, M_3, \nu)
\]
where
\[
\begin{align*}
M_1 & = \beta_1 A D_1 + \beta_2 D_1 A + \beta_3 A D_1 A \\
M_2 & = \beta_4 A D_2 + \beta_5 D_2 A + \beta_6 A D_2 A \\
M_3 & = \beta_7 A D_3 + \beta_8 D_3 A + \beta_9 A D_3 A \\
\nu & = D_1 \mu(\cdot, \cdot) + \mu(D_2, \cdot) + \mu(\cdot, D_3)
\end{align*}
\]

Note that if a hom-Lie algebra \((\mu, A)\) degenerates to a hom-Lie algebra \((\nu, B)\), then
\[
\dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_6})) \leq \dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6, \beta_1, \ldots, \beta_6}(\mu, A))). \tag{1}
\]
Furthermore, when \( \theta_1 = 1, \theta_2 = 1, \theta_3 = 0, \theta_4 = 1, \theta_5 = 0, \theta_6 = 1 \) and \( \beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0 \) we have \( \text{Ker}(T_{1,1,0,1,1,1,-1,0,-1,0,1,-1,0}(\mu, A)) \) is the
Lie algebra of derivations of the hom-Lie algebra \((\mu,A)\) and if \((\mu,A)\) degenerates properly to a hom-Lie algebra \((\nu,B)\) then

\[
\dim \ker T_{1,1,0,1,0,1,1,-1,0,1,-1,0,1,-1,0,1,0}(\mu,A) < \dim \ker T_{1,1,0,1,0,1,1,-1,0,1,-1,0,1,-1,0,1,0}(\nu,B).
\]

**Theorem 1.1** All the degenerations for 3-dimensional complex Lie algebra are:

1. \(C^3 \xrightarrow{\text{deg}} C^3\).
2. \(n_3(C) \xrightarrow{\text{deg}} n_3(C), C^3\).
3. \(r_3(C) \xrightarrow{\text{deg}} r_3(C), r_{3,1}(C), n_3(C), C^3\).
4. \(r_{3,1}(C) \xrightarrow{\text{deg}} r_{3,1}(C), C^3\).
5. \(r_{3,-1}(C) \xrightarrow{\text{deg}} r_{3,-1}(C), n_3(C), C^3\).
6. If \(z \neq \pm 1\), \(r_{3,z}(C) \xrightarrow{\text{deg}} r_{3,z}(C), n_3(C), C^3\),
7. \(r_2 \times C \xrightarrow{\text{deg}} r_2 \times C, n_3(C), C^3\).
8. \(sl_2(C) \xrightarrow{\text{deg}} sl_2(C), r_{3,-1}(C), n_3(C), C^3\).
2 \( \mathfrak{L}_0 \): 3-dimensional Abelian Lie algebra

\( \text{Aut}(\mathfrak{L}_0) = GL(3, \mathbb{C}) \).
\( \text{Der}(\mathfrak{L}_0) = M(3, \mathbb{C}) \)

2.1 \( \mathfrak{L}_0^0 \)

Let \( A \) be the Zero map and let us denote by \( \mathfrak{L}_0^0 \) the hom-Lie algebra \((\mathfrak{L}_0, A)\). We have:

\( \text{Aut}(\mathfrak{L}_0^0) = GL(3, \mathbb{C}) \).
\( \text{Der}(\mathfrak{L}_0^0) = M(3, \mathbb{C}) \)

2.2 \( \mathfrak{L}_0^1 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and let us denote by \( \mathfrak{L}_0^1 \) the hom-Lie algebra \((\mathfrak{L}_0, A)\). We have:

\( \text{Aut}(\mathfrak{L}_0^1) = \left\{ \begin{pmatrix} a & x & z \\ y & b & x \\ 0 & 0 & b \end{pmatrix} \in M(3, \mathbb{C}) : a, b \in \mathbb{C}^*, \quad x, y, z \in \mathbb{C} \right\} \).
\( \text{Der}(\mathfrak{L}_0^1) = \left\{ \begin{pmatrix} t_4 & t_2 & t_3 \\ t_1 & 0 & t_5 \\ 0 & 0 & t_2 \end{pmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\} \cong (\mathbb{C} \ltimes n_3(\mathbb{C})) \times \mathbb{C} \).

2.3 \( \mathfrak{L}_0^2 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

and let us denote by \( \mathfrak{L}_0^2 \) the hom-Lie algebra \((\mathfrak{L}_0, A)\). We have:

\( \text{Aut}(\mathfrak{L}_0^2) = \left\{ \begin{pmatrix} a & x & y \\ 0 & a & x \\ 0 & 0 & a \end{pmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^*, \quad x, y \in \mathbb{C} \right\} \).
\( \text{Der}(\mathfrak{L}_0^2) = \left\{ \begin{pmatrix} t_4 & t_2 & t_3 \\ t_1 & 0 & t_3 \\ 0 & 0 & t_1 \end{pmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{L}_0 \).

2.4 Degenerations between hom-Lie algebras \((\mathfrak{L}_i^j)\)

If \( \mathfrak{L}_i^j \xrightarrow{\text{deg}} \mathfrak{L}_k^k \) then \( \text{Der}(\mathfrak{L}_i^j) \leq \text{Der}(\mathfrak{L}_k^k) \). Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|----------------|
| 3        | \( \mathfrak{L}_0^0 \) |
| 5        | \( \mathfrak{L}_0^1 \) |
| 9        | \( \mathfrak{L}_0^2 \) |
2.4.1 Degenerations

1. $\mathfrak{L}_0^2 \xrightarrow{\text{deg}} \mathfrak{L}_0^1$
   In fact, set
   \[
   g(t) = \begin{pmatrix}
   e^{-t} & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]
   We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^2$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
   \[
   A(t) = \begin{pmatrix}
   0 & e^{-t} & 0 \\
   0 & 0 & 1 \\
   0 & 0 & 0
   \end{pmatrix}.
   \]
   It is easy to check that $(\mathfrak{L}_0, A(t)) \rightarrow \mathfrak{L}_0^0$ as $t$ tends to infinity.

2. $\mathfrak{L}_0^1 \xrightarrow{\text{deg}} \mathfrak{L}_0^0$
   In fact, set
   \[
   g(t) = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & e^{-t} & 0 \\
   0 & 0 & 1
   \end{pmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]
   We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^1$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
   \[
   A(t) = \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & e^{-t} \\
   0 & 0 & 0
   \end{pmatrix}.
   \]
   It is easy to check that $(\mathfrak{L}_0, A(t)) \rightarrow \mathfrak{L}_0^0$ as $t$ tends to infinity.
3 \ \mathfrak{L}_1: 3\text{-dimensional Heisenberg Lie algebra, } n_3(\mathbb{C})

\mathfrak{L}_1 := \{[e_1, e_2] = e_3 \}

\text{Aut}(\mathfrak{L}_1) = \left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ x & y & a d - c b \end{bmatrix} \in M(3, \mathbb{C}) : a, b, c, d, x, y \in \mathbb{C}, \quad a d - c b \neq 0 \right\}.

\mathrm{Der}(\mathfrak{L}_1) = \left\{ \begin{bmatrix} t_4 & t_2 & 0 \\ t_5 & t_6 & t_1 + t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{C} \right\} \cong \mathfrak{sl}(2, \mathbb{C}) \ltimes \mathfrak{v}_{3,1}(\mathbb{C}).

3.1 \ \mathfrak{L}_0^0

Let \( A \) be the Zero map and let us denote by \( \mathfrak{L}_0^0 \) the hom-Lie algebra \((\mathfrak{L}_1, A)\). We have:
\text{Aut}(\mathfrak{L}_0^0) = \text{Aut}(\mathfrak{L}_1),
\text{Der}(\mathfrak{L}_0^0) = \text{Der}(\mathfrak{L}_1).

3.2 \ \mathfrak{L}_1^1

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
and let us denote by \( \mathfrak{L}_1^1 \) the hom-Lie algebra \((\mathfrak{L}_1, A)\). We have:
\text{Aut}(\mathfrak{L}_1^1) = \left\{ \begin{bmatrix} 1 & b & 0 \\ 0 & a & 0 \\ x & y & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^*, b, x, y \in \mathbb{C} \right\},
\text{Der}(\mathfrak{L}_1^1) = \left\{ \begin{bmatrix} 0 & t_1 & 0 \\ t_3 & t_4 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong \mathbb{C} \ltimes n_3(\mathbb{C}).

3.2.1 \ \psi_{\alpha,\beta}(\mathfrak{L}_1^1)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_1^1) = t_{\alpha,\beta} \):
\[
t_{\alpha,\beta} = \{[e_1, e_2] = e_3, \text{ which is the Heisenberg Lie algebra.}

3.2.2 \ \phi_{\beta}(\mathfrak{L}_1^1)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_1^1) = \tilde{t}_{\beta} \) is the 3-dimensional abelian Lie algebra.

3.2.3 \ \rho(\mathfrak{L}_1^1)

In this case, we have \( \rho(\mathfrak{L}_1^1) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

3.2.4 \ Identities
\begin{itemize}
\item \( A[\cdot, \cdot] = 0 \)
\item \([A\cdot, \cdot] = 0 \)
\end{itemize}
In particular, \( \mathfrak{L}_1^1 \) is a multiplicative hom-Lie algebra.
3.3 \( \mathcal{L}_1^2 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
and let us denote by \( \mathcal{L}_1^2 \) the hom-Lie algebra \((\mathcal{L}_1, A)\). We have:
\[
\text{Aut}(\mathcal{L}_1^2) = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & x & a^2 \end{bmatrix} \in M(3, \mathbb{C}) : \begin{array}{c} a \in \mathbb{C}^* \\ b, x \in \mathbb{C} \end{array} \right\}.
\]
\[
\text{Der}(\mathcal{L}_1^2) = \left\{ \begin{bmatrix} t_1 & t_2 & 0 \\ 0 & t_1 & 0 \\ 0 & t_3 & 2t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{t}_2 \times \mathbb{C}.
\]

3.3.1 \( \psi_{\alpha, \beta}(\mathcal{L}_1^2) \)

In this case, we have \( \psi_{\alpha, \beta}(\mathcal{L}_1^2) = t_{\alpha, \beta} \):
\[
t_{\alpha, \beta} = [[e_1, e_2] = e_3.
\]
which is the Heisenberg Lie algebra.

3.3.2 \( \phi_{\beta}(\mathcal{L}_1^2) \)

In this case, we have \( \phi_{\beta}(\mathcal{L}_1^2) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.

3.3.3 \( \rho(\mathcal{L}_1^2) \)

In this case, we have \( \rho(\mathcal{L}_1^2) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

3.3.4 Identities

- \( A[., .] = 0 \)
- \( [A., .] + [., A.] = 0 \)
  \[- [A., A.] = 0 \]

In particular, \( \mathcal{L}_1^2 \) is a multiplicative hom-Lie algebra.

3.4 \( \mathcal{L}_1^3 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
and let us denote by $\mathfrak{L}_3^1$ the hom-Lie algebra $(\mathfrak{L}_1, A)$. We have:

$$\text{Aut}(\mathfrak{L}_3^1) = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \\ b \in \mathbb{C} \end{cases}$$

$$\text{Der}(\mathfrak{L}_3^1) = \begin{cases} \begin{bmatrix} t_2 & t_1 & 0 \\ 0 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \end{cases} \cong \mathfrak{t}_2(\mathbb{C}).$$

### 3.4.1 $\psi_{\alpha, \beta}(\mathfrak{L}_3^1)$

In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}_3^1) = \tilde{t}_{\alpha, \beta}$:

$$\tilde{t}_{\alpha, \beta} = \begin{bmatrix} [e_1, e_2] = \alpha e_2 + e_3, [e_1, e_3] = \beta e_3. \\$$

We note that $\tilde{t}_{\alpha, \beta}$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_{\alpha, \beta}$; we can identify $\tilde{t}_{\alpha, \beta}$ the matrix

$$B = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}.$$ 

The isomorphism class of $\tilde{t}_{\alpha, \beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $\alpha = \beta$

   (a) $0$ is an repeated eigenvalue of $B$ if and only if $\alpha = \beta = 0$

   i. $t_{\alpha, \beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$

   ii. $t_{\alpha, \beta}$ is isomorphic to $t_3(\mathbb{C})$ if and only if $B \neq 0$ if and only if $\alpha = \beta = 0$.

   (b) $B$ has a non-zero repeated eigenvalue if and only if $\alpha = \beta \neq 0$

   i. $t_{\alpha, \beta}$ is $t_{\lambda, 1}$ if and only if $B = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $1 = 0$

   ii. $t_{\alpha, \beta}$ is isomorphic to $t_3$ if and only if $B \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $\alpha = \beta \neq 0$

2. $B$ has distinct eigenvalues eigenvalues if and only if $\alpha \neq \beta$

   (a) $t_{\alpha, \beta}$ is isomorphic to $t_3(\mathbb{C}) \times \mathbb{C}$ if and only if $0$ is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\alpha = 0$ or $\beta = 0$, and $\alpha \neq \beta$.

   (b) $t_{\alpha, \beta}$ is isomorphic to $t_{\lambda, -1}$ if and only if $B$ has distinct eigenvalues eigenvalues and $\text{Trace}(B) = 0$ if and only if $\beta = -\alpha \neq 0$.

   (c) $t_{\alpha, \beta}$ is isomorphic to $t_{\beta, z}$ for some $z \in \mathbb{C}$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\alpha \neq \beta$, $\alpha, \beta \neq 0$ and $\beta \neq -\alpha$.

### 3.4.2 $\phi_{\beta}(\mathfrak{L}_3^1)$

In this case, we have $\phi_{\beta}(\mathfrak{L}_3^1) = \widetilde{t}_{\beta}$:

$$\widetilde{t}_{\beta} = \begin{bmatrix} [e_1, e_2] = e_2, [e_1, e_3] = \beta e_3. \\$$

We note that $\widetilde{t}_{\beta}$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\widetilde{t}_{\beta}$; we can identify $\widetilde{t}_{\beta}$ the matrix

$$\widetilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}.$$ 

The isomorphism class of $\widetilde{t}_{\beta}$ is determined by the eigenvalues of $\widetilde{B}$:
1. \( \tilde{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)
   
   (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if 1 = 0
   
   (b) \( B \) has a non-zero repeated eigenvalue if and only if \( \beta = 1 \)

   i. \( \tilde{t}_\beta \) is isomorphic to \( r_3,1 \) if and only if \( \tilde{B} = \lambda \text{Id} \) for some \( \lambda \neq 0 \) if and only if \( \beta = 1 \)

   ii. \( \tilde{t}_\beta \) is isomorphic to \( r_3,1 \) if and only if for all \( \lambda \in \mathbb{C} \) if and only if 1 = 0

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( \beta \neq 1 \)

   (a) \( \tilde{t}_\beta \) is isomorphic to \( r_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \beta = 0 \)

   (b) \( \tilde{t}_\beta \) is isomorphic to \( r_{3,-1} \) if and only if \( \text{Trace}(\tilde{B}) = 0 \) if and only if \( \beta = -1 \).

   (c) \( \tilde{t}_\beta \) is isomorphic to \( r_{3,z} \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( \beta \neq 1, \beta \neq 0 \) and \( \beta \neq -1 \).

3.4.3 \( \rho(\mathcal{L}_1^3) \)

In this case, we have \( \rho(\mathcal{L}_1^3) = \hat{t} \):

\[
\hat{t} = \{ [e1, e3] = e3 \}.
\]
which is isomorphic to \( r_2 \times \mathbb{C} \)

3.4.4 Identities

- \( [A_\cdot A] = 0 \)

The hom-Lie algebra \( \mathcal{L}_1^3 \) is a non-multiplicative hom-Lie algebra.

3.5 \( \mathcal{L}_1^4 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( \mathcal{L}_1^4 \) the hom-Lie algebra \((\mathcal{L}_1^4, A)\). We have:

\[
\text{Aut}(\mathcal{L}_1^4) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.
\]

\[
\text{Der}(\mathcal{L}_1^4) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.
\]

3.5.1 \( \psi_{\alpha,\beta}(\mathcal{L}_1^4) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathcal{L}_1^4) = t_{\alpha,\beta} \):

\[
t_{\alpha,\beta} = \{ [e1, e2] = \alpha e2 + e3, [e1, e3] = \beta e3 \}.
\]

We note that \( t_{\alpha,\beta} \) is an almost abelian Lie algebra where \( \text{span}_{\mathbb{C}} \{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha,\beta} \); we can identify \( t_{\alpha,\beta} \) the matrix

\[
B = \begin{bmatrix}
\alpha & 0 \\
1 & \beta
\end{bmatrix}.
\]
The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $\alpha = \beta$
   
   (a) 0 is a repeated eigenvalue of $B$ if and only if $\alpha = \beta = 0$
   
   i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$
   
   ii. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{t}_4(\mathbb{C})$ if and only if $\alpha = \beta = 0$

   (b) $B$ has a non-zero repeated eigenvalue if and only if $\alpha = \beta \neq 0$
   
   i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $1 = 0$
   
   ii. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{t}_3$ if and only if $B \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $\alpha = \beta \neq 0$

2. $B$ has distinct eigenvalues eigenvalues if and only if $\alpha \neq \beta$

   (a) $\tilde{t}_\beta$ is isomorphic to $t_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\alpha = 0$ or $\beta = 0$, and $\alpha \neq \beta$.

   (b) $t_{\alpha,\beta}$ is isomorphic to $t_{3,1}$ if and only if $B$ has distinct eigenvalues eigenvalues and $\text{Trace}(B) = 0$ if and only if $\beta = -\alpha \neq 0$.

   (c) $t_{\alpha,\beta}$ is isomorphic to $t_{3,z}$ for some $z \in \mathbb{C}$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\alpha \neq \beta$, $\alpha, \beta \neq 0$ and $\beta \neq -\alpha$.

### 3.5.2 $\phi_\beta(\mathfrak{L}_4^1)$

In this case, we have $\phi_\beta(\mathfrak{L}_4^1) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e 1, e 2] = e 2, [e 1, e 3] = \beta e 3\}.$$  

We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_{\mathbb{C}}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\mathfrak{t}_\beta$; we can identify $\tilde{t}_\beta$ the matrix

$$\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}.$$  

The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $\beta = 1$
   
   (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $1 = 0$

   (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $\beta = 1$
   
   i. $\tilde{t}_\beta$ is $t_{3,1}$ if and only if $\tilde{B} = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $\beta = 1$

   ii. $\tilde{t}_\beta$ is isomorphic to $\mathfrak{t}_3$ if and only if $\tilde{B} \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues eigenvalues if and only if $\beta \neq 1$

   (a) $\tilde{t}_\beta$ is isomorphic to $t_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\beta = 0$

   (b) $\tilde{t}_\beta$ is isomorphic to $t_{3,1}$ if and only if $\text{Trace}(\tilde{B}) = 0$ if and only if $\beta = -1$.

   (c) $\tilde{t}_\beta$ is isomorphic to $t_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\beta \neq 1$, $\beta \neq 0$ and $\beta \neq -1$.

### 3.5.3 $\rho(\mathfrak{L}_4^1)$

In this case, we have $\rho(\mathfrak{L}_4^1) = \hat{t}$:

$$\hat{t} = \{[e 1, e 3] = e 3\}.$$ 

which is isomorphic to $t_2 \times \mathbb{C}$.
3.5.4 Identities

- \([A^2, \cdot] + [\cdot, A] + [\cdot, A^2]\)
- \([A^2, A^2]\)

The hom-Lie algebra \(\mathfrak{L}_4^1\) is a non-multiplicative hom-Lie algebra.

3.6 \(\mathcal{L}_1^5\)

Let \(A\) be the endomorphism of \(\mathbb{C}^3\) whose matrix representation with respect to the ordered basis \(\{e_1, e_2, e_3\}\) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and let us denote by \(\mathfrak{L}_1^5\) the hom-Lie algebra \((\mathfrak{L}_1, A)\). We have:

**Aut(\(\mathfrak{L}_1^5\))**

\[
\begin{cases}
1 & 0 & 0 \\
y & 1 & 0 \\
x & y & 1 \\
x & y & 1
\end{cases} \in M(3, \mathbb{C}) : x, y \in \mathbb{C}
\]

**Der(\(\mathfrak{L}_1^5\))**

\[
\begin{cases}
t_2 & 0 & 0 \\
t_1 & t_2 & 0 \\
0 & 0 & 0
\end{cases} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \cong \mathbb{C}^2.
\]

3.6.1 Identities

- \(A[\cdot, \cdot] = 0\)
- \([A\cdot, \cdot] + [\cdot, A\cdot] = 0\)
- \([A\cdot, A\cdot] = 0\)

The hom-Lie algebra \(\mathfrak{L}_1^5\) is a multiplicative hom-Lie algebra.

3.6.2 \(\psi_{\alpha,\beta}(\mathfrak{L}_1^5)\)

In this case, we have \(\psi_{\alpha,\beta}(\mathfrak{L}_1^5) = t_{\alpha,\beta}\):

\(t_{\alpha,\beta} = \{[e_1, e_2] = e_3\}

which is the Heisenberg Lie algebra.

3.6.3 \(\phi_\beta(\mathfrak{L}_1^5)\)

In this case, we have \(\phi_\beta(\mathfrak{L}_1^5) = \tilde{t}_\beta\) is the 3-dimensional abelian Lie algebra.

3.6.4 \(\rho(\mathfrak{L}_1^5)\)

In this case, we have \(\rho(\mathfrak{L}_1^5) = \hat{t}\) is the 3-dimensional abelian Lie algebra.
Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis \{e_1, e_2, e_3\} is
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
and let us denote by $\mathfrak{L}_1^6$ the hom-Lie algebra $(\mathfrak{L}_1, A)$. We have:
\[
\text{Aut}(\mathfrak{L}) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.
\]
\[
\text{Der}(\mathfrak{L}) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.
\]

### 3.7.1 $\psi_{\alpha,\beta}(\mathfrak{L}_1^6)$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_1^6) = t_{\alpha,\beta}$:
\[
t_{\alpha,\beta} = [[e1, e2] = \alpha e2 + e3, [e1, e3] = \beta e3.
\]
We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where span$_{\mathbb{C}}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix
\[
B = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}.
\]
The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $\alpha = \beta$
   - (a) If $0$ is a repeated eigenvalue of $B$ if and only if $\alpha = \beta = 0$
     i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$
     ii. $t_{\alpha,\beta}$ is isomorphic to $n_3(\mathbb{C})$ if and only if $B \neq 0$ if and only if $\alpha = \beta = 0$
   - (b) $B$ has a non-zero repeated eigenvalue if and only if $\alpha = \beta \neq 0$
     i. $t_{\alpha,\beta}$ is isomorphic to $r_3, 1$ if and only if $B = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $1 = 0$
     ii. $t_{\alpha,\beta}$ is isomorphic to $t_3$ if and only if $B \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $\alpha = \beta \neq 0$

2. $B$ has distinct eigenvalues eigenvalues if and only if $\alpha \neq \beta$
   - (a) $t_{\alpha,\beta}$ is isomorphic to $r_2(\mathbb{C}) \times \mathbb{C}$ if and only if $0$ is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\alpha = 0$ or $\beta = 0$, and $\alpha \neq \beta$.
   - (b) $t_{\alpha,\beta}$ is isomorphic to $r_{3, -1}$ if and only if $B$ has distinct eigenvalues eigenvalues and $\text{Trace}(B) = 0$ if and only if $\beta = -\alpha \neq 0$.
   - (c) $t_{\alpha,\beta}$ is isomorphic to $r_{3,z}$ for some $z \in \mathbb{C}$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\alpha \neq \beta$, $\alpha, \beta \neq 0$ and $\beta \neq -\alpha$. 

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3.7.2 $\phi_\beta(L_6^1)$
In this case, we have $\phi_\beta(L_6^1) = \tilde{t}_\beta$:
\[
\tilde{t}_\beta = \{[e1, e2] = e2, [e1, e3] = \beta e3\}.
\]
We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_\beta$; we can identify $\tilde{t}_\beta$ the matrix
\[
\tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}.
\]
The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $\beta = 1$
   - (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $1 = 0$
   - (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $\beta = 1$
     i. $\tilde{t}_\beta$ is $\mathfrak{r}_3$ if and only if $\tilde{B} = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $\beta = 1$
     ii. $\tilde{t}_\beta$ is isomorphic to $\mathfrak{r}_3$ if and only if $\tilde{B} \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues eigenvalues if and only if $\beta \neq 1$
   - (a) $\tilde{t}_\beta$ is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\beta = 0$
   - (b) $\tilde{t}_\beta$ is isomorphic to $\mathfrak{r}_{3,-1}$ if and only if Trace($\tilde{B}$) = 0 if and only if $\beta = -1$.
   - (c) $\tilde{t}_\beta$ is isomorphic to $\mathfrak{r}_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\beta \neq 1, \beta \neq 0$ and $\beta \neq -1$.

3.7.3 $\rho(L_6^1)$
In this case, we have $\rho(L_6^1) = \hat{t}$:
\[
\hat{t} = \{[e1, e3] = e3\}.
\]
which is isomorphic to $\mathfrak{r}_2 \times \mathbb{C}$.

3.7.4 Identities
- $[A, A] = 0$
- $A^2[., .] = 0$
- $[A^2, .] + [., A^2] = 0$

The hom-Lie algebra $\mathfrak{L}_0^1$ is a non-multiplicative hom-Lie algebra.

3.8 Degenerations between hom-Lie algebras ($\mathfrak{L}_1^i$)
If $\mathfrak{L}_i^j \xrightarrow{\text{deg}} \mathfrak{L}_k^i$ then $\text{Der}(\mathfrak{L}_i^j) \leq \text{Der}(\mathfrak{L}_k^i)$. Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 0        | $\mathfrak{L}_0^i$                                  |
| 1        | $\mathfrak{L}_1^i$                                  |
| 2        | $\mathfrak{L}_1^i, \mathfrak{L}_1^j$               |
| 3        | $\mathfrak{L}_1^i$                                  |
| 4        | $\mathfrak{L}_1^i$                                  |
| 6        | $\mathfrak{L}_1^i$                                  |
3.8.1 Degenerations

1. \( \mathfrak{L}_4 \xrightarrow{\text{deg}} \mathfrak{L}_6 \)

In fact, set
\[
g(t) = \begin{bmatrix} 0 & e^t & 0 \\ e^{2t} & -e^{3t} & 0 \\ 0 & e^{2t} & -e^{3t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_4 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix} e^{-t} & 0 & -e^{-2t} \\ 0 & 0 & 1 \\ 1 & 0 & -e^{-t} \end{bmatrix}.
\]

It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_6 \) as \( t \) tends to infinity.

2. \( \mathfrak{L}_6 \xrightarrow{\text{deg}} \mathfrak{L}_3 \)

In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_6 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ e^{-t} & 0 & 0 \end{bmatrix}.
\]

It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_3 \) as \( t \) tends to infinity.

3. \( \mathfrak{L}_6 \xrightarrow{\text{deg}} \mathfrak{L}_5 \)

In fact, set
\[
g(t) = \begin{bmatrix} 0 & -1 & 0 \\ -e^{3t} & e^t & 0 \\ 0 & -e^{2t} & -e^{3t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_6 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix}
-e^{-t} & 0 & e^{-3t} \\
1 & 0 & -e^{-2t} \\
0 & 1 & e^{-t}
\end{bmatrix}.
\]
It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_1^5 \) as \( t \) tends to infinity.

4. \( \mathfrak{L}_1^3 \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)
   In fact, set
\[
g(t) = \begin{bmatrix}
-e^{3t} & e^{2t} & 0 \\
0 & 1 & 0 \\
0 & e^t & -e^{3t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_3 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix}
0 & 1 & -e^{-t} \\
0 & e^{-2t} & -e^{-3t} \\
0 & e^{-t} & -e^{-2t}
\end{bmatrix}.
\]
It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_1^2 \) as \( t \) tends to infinity.

5. \( \mathfrak{L}_1^5 \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)
   In fact, set
\[
g(t) = \begin{bmatrix}
0 & e^{-t} & 0 \\
e^{-t} & e^{-2t} & 0 \\
0 & 0 & -e^{-2t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_5 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix}
-e^{-t} & 1 & 0 \\
-e^{-2t} & e^{-t} & 0 \\
-e^{-t} & 0 & 0
\end{bmatrix}.
\]
It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_1^2 \) as \( t \) tends to infinity.

6. \( \mathfrak{L}_1^2 \xrightarrow{\text{deg}} \mathfrak{L}_1^1 \)
   In fact, set
\[
g(t) = \begin{bmatrix}
e^t & -e^{3t} & 0 \\
1 & 0 & 0 \\
e^{2t} & 0 & e^{3t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_1) \) and \( g(t) \cdot \mathfrak{L}_2 \) is the hom-Lie algebra \((\mathfrak{L}_1, A(t))\) with
\[
A(t) = \begin{bmatrix}
-e^{-2t} & e^{-t} & 0 \\
-e^{-3t} & e^{-2t} & 0 \\
-e^{-t} & 1 & 0
\end{bmatrix}.
\]
It is easy to check that \((\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_1^1 \) as \( t \) tends to infinity.
7. $\mathfrak{L}_1^1 \xrightarrow{\text{deg}} \mathfrak{L}_1^0$

In fact, set

$$g(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(\mathfrak{L}_1)$ and $g(t) \cdot \mathfrak{L}_1^1$ is the hom-Lie algebra $(\mathfrak{L}_1, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-t} & 0 \end{bmatrix}.$$

It is easy to check that $(\mathfrak{L}_1, A(t)) \rightarrow \mathfrak{L}_1^0$ as $t$ tends to infinity.
4 \mathfrak{L}_2: r_3(\mathbb{C})

\mathfrak{L}_1 := \{[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3\}

\text{Aut}(\mathfrak{L}_2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & a & x \\ z & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : \begin{array}{c} a \in \mathbb{C}^* \\ x, y, z \in \mathbb{C} \end{array} \right\}

\text{Der}(\mathfrak{L}_2) = \left\{ \begin{bmatrix} t_3 & t_1 & t_2 \\ t_4 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong \mathbb{C} \ltimes \mathfrak{n}_3(\mathbb{C}).

4.1 \mathfrak{L}_2^0

Let \( A \) be the Zero map and let us denote by \( \mathfrak{L}_2^0 \) the hom-Lie algebra \((\mathfrak{L}_2, A)\). We have:

\text{Aut}(\mathfrak{L}_2^0) = \text{Aut}(\mathfrak{L}_2),

\text{Der}(\mathfrak{L}_2^0) = \text{Der}(\mathfrak{L}_2).

4.1.1 \psi_{\alpha,\beta}(\mathfrak{L}_2^0)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_2^0) = t_{\alpha,\beta} \) is the Lie algebra \( r_3 \).

4.1.2 \phi_{\beta}(\mathfrak{L}_2^0)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_2^0) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.

4.1.3 \rho(\mathfrak{L}_2^0)

In this case, we have \( \rho(\mathfrak{L}_2^0) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

4.2 \mathfrak{L}_2^1

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_2^1 \) the hom-Lie algebra \((\mathfrak{L}_2, A)\). We have:

\text{Aut}(\mathfrak{L}_2^1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & x \\ z & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x, y, z \in \mathbb{C} \right\}

\text{Der}(\mathfrak{L}_2^1) = \left\{ \begin{bmatrix} t_2 & 0 & t_3 \\ t_1 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{n}_3(\mathbb{C}).

4.2.1 \psi_{\alpha,\beta}(\mathfrak{L}_2^1)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_2^1) = t_{\alpha,\beta} \) is the Lie algebra \( r_3 \).

4.2.2 \phi_{\beta}(\mathfrak{L}_2^1)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_2^1) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.
4.2.3 \( \rho(L_1^2) \)
In this case, we have \( \rho(L_1^2) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

4.2.4 Identities

- \( A[\cdot, \cdot] = 0 \)
- \( [A, \cdot] + [\cdot, A] = 0 \)
  - \( [A, A] = 0 \)

In particular, the hom-Lie algebra \( L_1^2 \) is a multiplicative hom-Lie algebra.

4.3 \( L_2^2 \)
Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
and let us denote by \( L_2^2 \) the hom-Lie algebra \( (L_2^2, A) \). We have:
\[
\text{Aut}(L_2^2) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
y & 0 & 1
\end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\}.
\]
\[
\text{Der}(L_2^2) = \left\{ \begin{bmatrix}
t_1 & 0 & 0 \\
t_2 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2.
\]

4.3.1 \( \psi_{\alpha, \beta}(L_2^2) \)
In this case, we have \( \psi_{\alpha, \beta}(L_2^2) = t_{\alpha, \beta} \) is the Lie algebra \( t_3 \).

4.3.2 \( \phi_{\beta}(L_2^2) \)
In this case, we have \( \phi_{\beta}(L_2^2) = \tilde{t}_{\beta} \) is the 3-dimensional abelian Lie algebra.

4.3.3 \( \rho(L_2^2) \)
In this case, we have \( \rho(L_2^2) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

4.3.4 Identities

- \( A[\cdot, \cdot] = 0 \)
- \( [A, \cdot] + [\cdot, A] = 0 \)
  - \( [A, A] = 0 \)

In particular, the hom-Lie algebra \( L_2^2 \) is a multiplicative hom-Lie algebra.
4.4 \( \mathfrak{L}_3^3(\lambda) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \lambda \\
0 & 0 & 0 \\
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_3^3(\lambda) \) the hom-Lie algebra \((\mathfrak{L}_2, A)\), with \( \lambda \neq 0 \). We have:

\[
\text{Aut}(\mathfrak{L}_3^3(\lambda)) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
y & a & x \\
0 & 0 & a \\
\end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \\ x, y \in \mathbb{C} \right\}.
\]

\[
\text{Der}(\mathfrak{L}_3^3(\lambda)) = \left\{ \begin{bmatrix}
t_3 & t_1 & t_2 \\
0 & 0 & 0 \\
0 & 0 & t_1 \\
\end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}.
\]

4.4.1 \( \psi_{\alpha, \beta}(\mathfrak{L}_3^3(\lambda)) \)

In this case, we have \( \psi_{\alpha, \beta}(\mathfrak{L}_3^3(\lambda)) = t_{\alpha, \beta} : \)

\[
t_{\alpha, \beta} = \{ [e_1, e_2] = e_2, [e_1, e_3], (1 + \beta \lambda + \alpha \lambda) e_2 + e_3 \}.
\]

We note that \( t_{\alpha, \beta} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha, \beta} \); we can identify \( t_{\alpha, \beta} \) the matrix

\[
B = \begin{bmatrix}
1 & 1 + \beta \lambda + \alpha \lambda \\
0 & 1 \\
\end{bmatrix}.
\]

The isomorphism class of \( t_{\alpha, \beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 = 1 \)
   (a) \( 0 \) is a repeated eigenvalue of \( B \) if and only if \( 1 = 0 \)
   (b) \( B \) has a non-zero repeated eigenvalue if and only if \( 1 = 1 \)
      i. \( t_{\alpha, \beta} \) is \( \mathfrak{r}_3 \) if and only if \( B = \lambda \text{Id} \) for some \( \lambda \neq 0 \) if and only if \( 1 + \beta \lambda + \alpha \lambda = 0 \)
      ii. \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{r}_3 \) if and only if \( B \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( 1 + \beta \lambda + \alpha \lambda \neq 0 \)

2. \( B \) has distinct eigenvalues if and only if \( 1 = 0 \)

4.4.2 \( \phi_\beta(\mathfrak{L}_3^3) \)

In this case, we have \( \phi_\beta(\mathfrak{L}_3^3) = \tilde{t}_\beta : \)

\[
\tilde{t}_\beta = \{ [e_1, e_2] = (\beta \lambda + \lambda) e_2 \}.
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
0 & \beta \lambda + \lambda \\
0 & 0 \\
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):
1. \( \widetilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   (a) 0 is a repeated eigenvalue of \( \widetilde{B} \) if and only if \( 0 = 0 \)
   i. \( \widetilde{t}_β \) is the 3-dimensional abelian Lie algebra if and only if \( \widetilde{B} = 0 \) if and only if \( βλ + λ = 0 \)
   ii. \( \widetilde{t}_β \) is isomorphic to \( \mathfrak{n}_3(\mathbb{C}) \) if and only if \( \widetilde{B} \neq 0 \) if and only if \( βλ + λ \neq 0 \)
   (b) \( B \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \widetilde{B} \) has distinct eigenvalues if and only if \( 1 = 0 \)

4.4.3 \( ρ(\mathfrak{L}^2_3) \)
In this case, we have \( ρ(\mathfrak{L}^2_3) = \hat{t} \):
\[
\hat{t} = \{ [e1, e3] = λ e2 \}
\]
which is isomorphic to \( \mathfrak{n}_3 \)

4.4.4 Identities
- \( A[·, ·] = [A · , ·] + [·, A ·] \) (Derivation)
- \( [A · , A ·] = 0 \)
  - \( A[·, ·] = 0 \)
The hom-Lie algebra \( \mathfrak{L}^2_3(λ) \) is a non-multiplicative hom-Lie algebra.

4.5 \( \mathfrak{L}^2_4(λ) \)
Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & λ & 0
\end{bmatrix}
\]
with \( λ \in \mathbb{C}, λ \neq 0 \) and let us denote by \( \mathfrak{L}^2_4(λ) \) the hom-Lie algebra \( (\mathfrak{L}_2, A) \). We have:
\[
\text{Aut}(\mathfrak{L}^2_4(λ)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\
0 & a & 0 \\
x & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^*, x \in \mathbb{C} \right\}
\]
\[
\text{Der}(\mathfrak{L}^2_4(λ)) = \left\{ \begin{bmatrix} 0 & t_1 & 0 \\
t_2 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C})
\]

4.5.1 \( ψ_{α, β}(\mathfrak{L}^4_2(λ)) \)
In this case, we have \( ψ_{α, β}(\mathfrak{L}^4_2(λ)) = t_{α, β} \):
\[
t_{α, β} = \{ [e1, e2] = (1 + β λ) e2 + (α λ + β λ) e3, [e1, e3] = e2 + (1 + α λ) e3 \}
\]
We note that \( t_{α, β} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C} \{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{α, β} \); we can identify \( t_{α, β} \) the matrix
\[
B = \begin{bmatrix}
1 + β λ & 1 \\
α λ + β λ & 1 + α λ
\end{bmatrix}
\]
The isomorphism class of \( t_{α, β} \) is determined by the eigenvalues of \( B \):
1. \( B \) has repeated eigenvalues if and only if \((\alpha - \beta)^2 \lambda + 4 (\alpha + \beta) = 0 \)

(a) 0 is a repeated eigenvalue of \( B \) if and only if \( \alpha = \frac{-1+\sqrt{2}}{\lambda}, \beta = \frac{-1+\sqrt{2}}{\lambda} \).

i. \( t_{\alpha,\beta} \) is the 3-dimensional abelian Lie algebra if and only if \( B = 0 \) if and only if \( t = 0 \)

ii. \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3(\mathbb{C}) \) if and only if \( B \neq 0 \) if and only if \( \alpha = \frac{-1+\sqrt{2}}{\lambda}, \beta = \frac{-1+\sqrt{2}}{\lambda} \).

(b) \( B \) has a non-zero repeated eigenvalue if and only if \( \{ \alpha = \frac{-s - 1 \pm \sqrt{2 s^2 - 2}}{2}, \beta = \frac{-s + 1 \pm \sqrt{2 s^2 - 2}}{2} \} \) for any \( s \in \mathbb{C} \setminus \{0\} \).

i. \( t_{\alpha,\beta} \) is \( t_{3,1} \) if and only if \( B = t \text{Id} \) for some \( t \neq 0 \) if and only if \( \beta = 0 \)

ii. \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3 \) if and only if \( B \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( \{ \alpha = \frac{-s - 1 \pm \sqrt{2 s^2 - 2}}{2}, \beta = \frac{-s + 1 \pm \sqrt{2 s^2 - 2}}{2} \} \) for any \( s \in \mathbb{C} \setminus \{0\} \).

2. \( B \) has distinct eigenvalues if and only if \((\alpha - \beta)^2 \lambda + 4 (\alpha + \beta) \neq 0 \)

(a) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \{ \alpha = \frac{-2 s - 1 \pm \sqrt{4 s^2 - 8 s + 1}}{2 \lambda}, \beta = \frac{-2 s - 1 \pm \sqrt{4 s^2 - 8 s + 1}}{2 \lambda} \} \) for any \( s \in \mathbb{C} \setminus \{0\} \).

(b) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_{3,1} \) if and only if \( \text{Trace}(B) = 0 \) if and only if \( \{ \alpha = \frac{-2 s \pm \sqrt{(8 s + 1)}}{2 \lambda}, \beta = \frac{-2 s \pm \sqrt{(8 s + 1)}}{2 \lambda} \} \) for any \( s \in \mathbb{C} \setminus \{0\} \).

(c) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_{3,2} \) for some \( z \) such that \( z(z^2 - 1) \neq 0 \) if and only if \( \{ \alpha = \frac{-2 s_1 s_2 - s_1 \pm \sqrt{s_1 s_2 (8 s_1 s_2 - 4 s_1 + s_2)}}{2 s_1 s_2 \lambda}, \beta = \frac{-2 s_1 s_2 - s_1 \pm \sqrt{s_1 s_2 (8 s_1 s_2 - 4 s_1 + s_2)}}{2 s_1 s_2 \lambda} \} \) for any \( s_1, s_2 \in \mathbb{C} \setminus \{0\} \) such that \( s_1 - s_2 \neq 0 \).

4.5.2 \( \phi_\beta(\mathfrak{L}_2^1(\lambda)) \)

In this case, we have \( \phi_\beta(\mathfrak{L}_2^1(\lambda)) = \widehat{\mathfrak{t}}_\beta: \)

\[ \widehat{\mathfrak{t}}_\beta = \{ [e1, e2] = \beta e2 + (\lambda + \beta \lambda) e3, [e1, e3], \lambda e3 \}. \]

We note that \( \widehat{\mathfrak{t}}_\beta \) is an almost abelian Lie algebra where \( \text{span}_{\mathbb{C}}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \widehat{\mathfrak{t}}_\beta \); we can identify \( \widehat{\mathfrak{t}}_\beta \) the matrix

\[ \widehat{B} = \begin{bmatrix} \beta \lambda & 0 \\ \lambda + \beta \lambda & \lambda \end{bmatrix}. \]

The isomorphism class of \( \widehat{\mathfrak{t}}_\beta \) is determined by the eigenvalues of \( \widehat{B} \):

1. \( \widehat{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)

(a) 0 is a repeated eigenvalue of \( \widehat{B} \) if and only if \( \lambda = 0 \) (false)

(b) \( B \) has a non-zero repeated eigenvalue if and only if \( \beta = 1 \)

i. \( t_\beta \) is \( t_{3,1} \) if and only if \( \widehat{B} = t \text{Id} \) for some \( t \neq 0 \) if and only if \( \lambda = 0 \) (false)

ii. \( t_\beta \) is isomorphic to \( t_3 \) if and only if \( \widehat{B} \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( \beta = 1 \)

2. \( \widehat{B} \) has distinct eigenvalues if and only if \( \beta \neq 1 \)

(a) \( \widehat{t}_\beta \) is isomorphic to \( \mathfrak{t}_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \beta = 0 \)

(b) \( \widehat{t}_\beta \) is isomorphic to \( \mathfrak{t}_{3,-1} \) if and only if \( \text{Trace}(\widehat{B}) = 0 \) if and only if \( \beta = 1 \)

(c) \( \widehat{t}_\beta \) is isomorphic to \( \mathfrak{t}_{3,2} \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( \beta \neq 1 \) and \( \beta \neq 0 \) and \( \beta \neq -1 \)
4.5.3 \( \rho(L^4_2) \)

In this case, we have \( \rho(L^4_2) = \hat{t} : \)

\[ \hat{t} = \{ [e_1, e_2] = \lambda e_2 + \lambda e_3. \]  

which is isomorphic to \( r_2 \times \mathbb{C} \).

4.5.4 Identities

- \( [A, A] = 0 \)

The hom-Lie algebra \( L^4_2(\lambda) \) is a non-multiplicative hom-Lie algebra.

4.6 \( L^5_2(\lambda) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{ e_1, e_2, e_3 \} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \lambda \\
1 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( L^5_2(\lambda) \) the hom-Lie algebra \( (L_2, A) \). We have:

\[ \text{Aut}(L^5_2(\lambda)) = \left\{ \begin{bmatrix}
y & 1 & x \lambda \\
x & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\} , \]

\[ \text{Der}(L^5_2(\lambda)) = \left\{ \begin{bmatrix}
t_2 & 0 & \lambda t_1 \\
t_1 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2. \]

4.6.1 \( \psi_{\alpha,\beta}(L^5_2(\lambda)) \)

In this case, we have \( \psi_{\alpha,\beta}(L^5_2(\lambda)) = t_{\alpha,\beta} : \)

\[ t_{\alpha,\beta} = \{ [e_1, e_2] = e_2, [e_1, e_3], (1 + \beta \lambda + \alpha \lambda) e_2 + e_3 \}. \]

We note that \( t_{\alpha,\beta} \) is an almost abelian Lie algebra where span_{\mathbb{C}}\{e_2, e_3\} is a codimension 1 abelian ideal of \( t_{\alpha,\beta} \); we can identify \( t_{\alpha,\beta} \) the matrix

\[
B = \begin{bmatrix}
1 & 1 + \beta \lambda + \alpha \lambda \\
0 & 1
\end{bmatrix}.
\]

The isomorphism class of \( t_{\alpha,\beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 = 1 \)
   (a) \( 0 \) is a repeated eigenvalue of \( B \) if and only if \( 1 = 0 \)
   (b) \( B \) has a non-zero repeated eigenvalue if and only if \( 1 = 1 \)
      i. \( t_{\alpha,\beta} \) is \( t_{3,1} \) if and only if \( B = \lambda \text{Id} \) for some \( \lambda \neq 0 \) if and only if \( 1 + \beta \lambda + \alpha \lambda = 0 \)
      ii. \( t_{\alpha,\beta} \) is isomorphic to \( t_3 \) if and only if \( B \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( 1 + \beta \lambda + \alpha \lambda \neq 0 \)
2. \( B \) has distinct eigenvalues eigenvalues if and only if \( 1 = 0 \)
4.6.2 $\phi_\beta(L^2_5)$

In this case, we have $\phi_\beta(L^2_5) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e_1, e_3] = (\beta \lambda + \lambda) e_2 \}.$$

We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_\beta$; we can identify $\tilde{t}_\beta$ the matrix

$$\tilde{B} = \begin{bmatrix} 0 & \beta \lambda + \lambda \\ 0 & 0 \end{bmatrix}.$$

The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $0 = 0$
   
   (a) $0$ is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$
   
   i. $\tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\beta \lambda + \lambda = 0$
   
   ii. $\tilde{t}_\beta$ is isomorphic to $n_3(\mathbb{C})$ if and only if $\tilde{B} \neq 0$ if and only if $\beta \lambda + \lambda \neq 0$

(b) $B$ has a non-zero repeated eigenvalue if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues if and only if $1 = 0$

4.6.3 $\rho(L^2_6)$

In this case, we have $\rho(L^2_6) = \hat{\tilde{t}}$:

$$\hat{\tilde{t}} = \{[e_1, e_3] = \lambda e_2 \}.$$

which is isomorphic to $n_3$

4.6.4 Identities

- $A[\cdot, \cdot] = [A\cdot, \cdot] + [\cdot, A\cdot]$ (Derivation)
- $[A\cdot, A\cdot] = 0$
- $A^2[\cdot, \cdot] = 0$

The hom-Lie algebra $L^2_6(\lambda)$ is a non-multiplicative hom-Lie algebra.

4.7 $L^6_6(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix}$$

and let us denote by $L^6_6(\lambda)$ the hom-Lie algebra $(L_2, A)$. We have:

$$\text{Aut}(L^6_6(\lambda)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.$$

$$\text{Der}(L^6_6(\lambda)) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_1 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.$$
4.7.1 $\psi_{\alpha,\beta}(L^6_2(\lambda))$

In this case, we have $\psi_{\alpha,\beta}(L^6_2(\lambda)) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e_1, e_2] = (1 + \beta \lambda) e_2 + (\alpha \lambda + \beta \lambda) e_3, [e_1, e_3] = e_2 + (1 + \alpha \lambda) e_3\}.$$ 

We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix

$$B = \begin{bmatrix}
1 + \beta \lambda & 1 \\
\alpha \lambda + \beta \lambda & 1 + \alpha \lambda
\end{bmatrix}.$$ 

The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $(\alpha - \beta)^2 \lambda + 4 (\alpha + \beta) = 0$.
   (a) 0 is a repeated eigenvalue of $B$ if and only if $\alpha = -\frac{1 + \sqrt{2}}{\lambda}$, $\beta = \sqrt{2} - 1$ or $\alpha = -\frac{1 - \sqrt{2}}{\lambda}$, $\beta = -\sqrt{2} - 1$.
   i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$
   ii. $t_{\alpha,\beta}$ is isomorphic to $t_3(C)$ if and only if $B \neq 0$ if and only if $\alpha = -\frac{1 + \sqrt{2}}{\lambda}$, $\beta = -\sqrt{2} - 1$.

(b) $B$ has a non-zero repeated eigenvalue if and only if
   $$\{\alpha = -s - 1 + \sqrt{2} - z, \beta = -s^2 + 1 + \sqrt{2} - z\} \text{ for any } s \in C \setminus \{0\}.$$
   i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = t \text{Id}$ for some $t \neq 0$ if and only if $1 = 0$
   ii. $t_{\alpha,\beta}$ is isomorphic to $t_3$ if and only if $B \neq t \text{Id}$ for all $t \in C$ if and only if
      $$\{\alpha = -s - 1 + \sqrt{2} - z, \beta = -s^2 + 1 + \sqrt{2} - z\} \text{ for any } s \in C \setminus \{0\}.$$

2. $B$ has distinct eigenvalues if and only if $(\alpha - \beta)^2 \lambda + 4 (\alpha + \beta) \neq 0$.
   (a) $t_{\alpha,\beta}$ is isomorphic to $t_2(C) \times C$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if
      $$\{\alpha = -\frac{2s + 1 + \sqrt{2} - 4s + 8s + 1}{2s}, \beta = -\frac{2s + 1 + \sqrt{2} - 4s + 8s + 1}{2s}\} \text{ for any } s \in C \setminus \{0\}.$$
   (b) $t_{\alpha,\beta}$ is isomorphic to $t_{3, -1}$ if and only if $\text{Trace}(B) = 0$ if and only if
      $$\{\alpha = -\frac{2s + \sqrt{8s + 1}}{2s}, \beta = -\frac{2s + \sqrt{8s + 1}}{2s}\} \text{ for any } s \in C \setminus \{0\}.$$
   (c) $t_{\alpha,\beta}$ is isomorphic to $t_{3, z}$ for some $z$ such that $z(z^2 - 1) \neq 0$ if and only if
      $$\{\alpha = -\frac{2s_1 s_2 - s_1 s_3 \sqrt{s_1 s_2 (s_1 s_2 - 4 s_1 + s_2)}}{2s_1 s_2}, \beta = -\frac{-2s_1 s_2 s_3 + \sqrt{s_1 s_2 (s_1 s_2 - 4 s_1 + s_2)}}{2s_1 s_2}\} \text{ for any } s_1, s_2 \in C \setminus \{0\} \text{ such that } s_1 - s^2 \neq 0.$$

4.7.2 $\phi_{\beta}(L^6_2(\lambda))$

In this case, we have $\phi_{\beta}(L^6_2(\lambda)) = \tilde{t}_{\beta}$:

$$\tilde{t}_{\beta} = \{[e_1, e_2] = \beta \lambda e_2 + (\lambda + \beta \lambda) e_3, [e_1, e_3] = e_2 + (1 + \alpha \lambda) e_3\}.$$ 

We note that $\tilde{t}_{\beta}$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_{\beta}$; we can identify $\tilde{t}_{\beta}$ the matrix

$$\tilde{B} = \begin{bmatrix}
\beta \lambda & 0 \\
\lambda + \beta \lambda & \lambda
\end{bmatrix}.$$ 

The isomorphism class of $\tilde{t}_{\beta}$ is determined by the eigenvalues of $\tilde{B}$:
1. $\tilde{B}$ has repeated eigenvalues if and only if $\beta = 1$
   (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $\lambda = 0$ (false)
   (b) $B$ has a non-zero repeated eigenvalue if and only if $\beta = 1$
      i. $\tilde{1}_\beta$ is $r_{3,1}$ if and only if $\tilde{B} = t \text{Id}$ for some $t \neq 0$ if and only if $\lambda = 0$ (false)
      ii. $\tilde{1}_\beta$ is isomorphic to $r_3$ if and only if $\tilde{B} \neq t \text{Id}$ for all $t \in \mathbb{C}$ if and only if $\beta = 1$

2. $\tilde{B}$ has distinct eigenvalues if and only if $\beta \neq 1$
   (a) $\tilde{1}_\beta$ is isomorphic to $r_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\beta = 0$
   (b) $\tilde{1}_\beta$ is isomorphic to $r_{3,-1}$ if and only if $\text{Trace}(\tilde{B}) = 0$ if and only if $\beta = -1$.
   (c) $\tilde{1}_\beta$ is isomorphic to $r_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\beta \neq 1$ and $\beta \neq 0$ and $\beta \neq -1$

4.7.3 $\rho(\mathfrak{L}_2^6)$
In this case, we have $\rho(\mathfrak{L}_2^6) = \tilde{1}$:

$$\tilde{1}_\beta = \{e_1, e_2\} = \lambda e_2 + \lambda e_3,$$

which is isomorphic to $r_2 \times \mathbb{C}$

4.7.4 Identities
- $[A, A] = 0$
- $A^2[\cdot, \cdot] = 0$
- $[A^2, \cdot] + [\cdot, A^2] = 0$

The hom-Lie algebra $\mathfrak{L}_2^6(\lambda)$ is a non-multiplicative hom-Lie algebra.

4.8 Degenerations between hom-Lie algebras ($\mathfrak{L}_2^i$)
If $\mathfrak{L}_i^j \xrightarrow{\text{deg}} \mathfrak{L}_i^k$ then $\text{Der}(\mathfrak{L}_i^j) \leq \text{Der}(\mathfrak{L}_i^k)$. Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|----------------|
| 1        | $\mathfrak{L}_2^6(\lambda)$ |
| 2        | $\mathfrak{L}_2^4(\lambda), \mathfrak{L}_2^5(\lambda), \mathfrak{L}_2^6$ |
| 3        | $\mathfrak{L}_2^4(\lambda), \mathfrak{L}_2^5, \mathfrak{L}_2^6$ |
| 4        | $\mathfrak{L}_2^5, \mathfrak{L}_2^6$ |

4.8.1 Degenerations

$$\begin{array}{c}
\mathfrak{L}_2^6 \\
\mathfrak{L}_2^5 \\
\mathfrak{L}_2^4 \\
\mathfrak{L}_2^3 \\
\mathfrak{L}_2^2 \\
\mathfrak{L}_2^1 \\
\mathfrak{L}_2^0
\end{array}$$
1. $L_6^2(\lambda) \xrightarrow{\text{deg}} L_4^2(\lambda)$
In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(L_2)$ and $g(t) \cdot L_6^2(\lambda)$ is the hom-Lie algebra $(L_2, A(\lambda, t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix}$$

It is easy to check that $(L_2, A(\lambda, t)) \rightarrow L_3^2(\lambda)$ as $t$ tends to infinity.

2. $L_5^2(\lambda) \xrightarrow{\text{deg}} L_3^2(\lambda)$
In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(L_2)$ and $g(t) \cdot L_5^2(\lambda)$ is the hom-Lie algebra $(L_2, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ e^{-t} & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_2, A(t)) \rightarrow L_3^2(\lambda)$ as $t$ tends to infinity.

3. $L_2^2 \xrightarrow{\text{deg}} L_1^2$
In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 1 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(L_2)$ and $g(t) \cdot L_2^2(\lambda)$ is the hom-Lie algebra $(L_2, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ e^{-t} & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_2, A(t)) \rightarrow L_1^2(\lambda)$ as $t$ tends to infinity.

4. $L_1^2 \xrightarrow{\text{deg}} L_0^2$
In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(L_2)$ and $g(t) \cdot L_1^2(\lambda)$ is the hom-Lie algebra $(L_2, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
It is easy to check that $(L_2, A(t)) \to L_0^2$ as $t$ tends to infinity.

5. $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_4^2(\kappa)$ with $(\lambda \neq \kappa)$.
   Suppose, contrary to our claim, that $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_4^2(\kappa)$ with $(\lambda \neq \kappa)$. In such case, we have
   \[
   \psi_{\alpha, \beta}(L_6^2(\lambda)) \overset{\text{deg}}{\longrightarrow} \psi_{\alpha, \beta}(L_4^2(\kappa)).
   \]
   By taking $\alpha = -\frac{1 + \sqrt{2}}{\kappa}$, $\beta = \frac{\sqrt{2} - 1}{\kappa}$, we have from 4.7.1 $\psi_{\alpha, \beta}(L_6^2(\lambda))$ is a Lie algebra isomorphic to $n_3(\mathbb{C})$, therefore, $\psi_{\alpha, \beta}(L_4^2(\kappa))$ is a Lie algebra which is isomorphic to one of the following: $n_2(\mathbb{C})$ or $a_3(\mathbb{C})$ (by Theorem 1.1). We must have $\psi_{\alpha, \beta}(L_4^2(\kappa))$ is isomorphic to $n_3(\mathbb{C})$ (from 4.5.1). But, we now apply again 4.5.1, to obtain $\{\alpha = -\frac{1 + \sqrt{2}}{\kappa}, \beta = \frac{\sqrt{2} - 1}{\kappa}\}$ or $\{\alpha = -\frac{1 + \sqrt{2}}{\kappa}, \beta = \frac{\sqrt{2} - 1}{\kappa}\}$; this is a contradiction.

6. $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_3^2(\kappa)$ with $(\kappa \in \mathbb{C}^*)$.
   Suppose, contrary to our claim, that $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_3^2(\kappa)$ with $(\kappa \in \mathbb{C}^*)$. In such case, we have
   \[
   \psi_{\alpha, \beta}(L_6^2(\lambda)) \overset{\text{deg}}{\longrightarrow} \psi_{\alpha, \beta}(L_3^2(\kappa)).
   \]
   By taking $\alpha = (-1 + \sqrt{2})/\lambda$ and $\beta = (-1 - \sqrt{2})/\lambda$, we have from 4.7.1 that $\psi_{\alpha, \beta}(L_6^2(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L_3^2(\kappa))$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). But, from 4.4.1, $\psi_{\alpha, \beta}(L_3^2(\kappa))$ is isomorphic to $\tau_3, 1$ or $\tau_3$; this is a contradiction.

7. $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_2^0$ with $(\lambda \in \mathbb{C}^*)$.
   Suppose, contrary to our claim, that $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_2^0$ with $(\kappa \in \mathbb{C}^*)$. In such case, we have
   \[
   \psi_{\alpha, \beta}(L_6^2(\lambda)) \overset{\text{deg}}{\longrightarrow} \psi_{\alpha, \beta}(L_2^0).
   \]
   By taking $\alpha = (-1 + \sqrt{2})/\lambda$ and $\beta = (-1 - \sqrt{2})/\lambda$, we have from 4.7.1 that $\psi_{\alpha, \beta}(L_6^2(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L_2^0)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). But, from 4.4.1, $\psi_{\alpha, \beta}(L_2^0)$ is isomorphic to $\tau_3$; this is a contradiction.

8. $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_3^2(\kappa)$ with $(\lambda \neq \kappa)$.
   Suppose, contrary to our claim, that $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_3^2(\kappa)$ with $(\lambda \neq \kappa)$. In such case, we have
   \[
   \psi_{\alpha, \beta}(L_6^2(\lambda)) \overset{\text{deg}}{\longrightarrow} \psi_{\alpha, \beta}(L_3^2(\kappa)).
   \]
   By taking $\alpha = -\frac{1}{\kappa}$ and $\beta = 0$, we have from 4.6.1 that $\psi_{\alpha, \beta}(L_6^2(\lambda))$ is a Lie algebra isomorphic to $\tau_{3,1}$, therefore, $\psi_{\alpha, \beta}(L_3^2(\kappa))$ is a Lie algebra which is isomorphic to one of the following: $\tau_{3,1}$ or $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha, \beta}(L_3^2(\kappa))$ is isomorphic to $\tau_{3,1}$ (from 4.4.1). But, we now apply again 4.4.1, to obtain $1 + \beta \kappa + \alpha \kappa = 0$ and therefore $\lambda = \kappa$; this is a contradiction.

9. $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_2^0$.
   Suppose, contrary to our claim, that $L_6^2(\lambda) \overset{\text{deg}}{\longrightarrow} L_2^0$. In such case, we have
   \[
   \psi_{\alpha, \beta}(L_6^2(\lambda)) \overset{\text{deg}}{\longrightarrow} \psi_{\alpha, \beta}(L_2^0).
   \]
   By taking $\alpha = -\frac{1}{\kappa}$ and $\beta = 0$, we have from 4.6.1 that $\psi_{\alpha, \beta}(L_6^2(\lambda))$ is a Lie algebra isomorphic to $\tau_{3,1}$, therefore, $\psi_{\alpha, \beta}(L_2^0)$ is a Lie algebra which is isomorphic to one of the following: $\tau_{3,1}$ or $a_3$ (by Theorem 1.1). But, we now apply again 4.4.1 and we have $\psi_{\alpha, \beta}(L_2^0)$ is isomorphic to $\tau_3$; this is a contradiction.
10. $\mathfrak{L}_2^2 \xrightarrow{\deg} \mathfrak{L}_3^2(\lambda)$ with $\lambda \neq 0$.

Suppose, contrary to our claim, that $\mathfrak{L}_2^2 \xrightarrow{\deg} \mathfrak{L}_3^2(\lambda)$ with $\lambda \neq 0$. In such case, we have

$$\phi_\beta(\mathfrak{L}_2^2) \xrightarrow{\deg} \psi_\beta(\mathfrak{L}_3^2(\lambda)).$$

By taking $\beta \neq -1$, we have from 4.3.2 that $\phi_\beta(\mathfrak{L}_2^2)$ is a Lie algebra isomorphic to $\mathfrak{a}_3$, therefore, $\phi_\beta(\mathfrak{L}_3^2(\lambda))$ is a Lie algebra which is isomorphic to $\mathfrak{a}_3$. But, from 4.4.2, we have $\phi_\beta(\mathfrak{L}_3^2(\lambda))$ is isomorphic to $\mathfrak{n}_3$; this is a contradiction.
5 $\mathfrak{L}_3$: Bianchi type V Lie algebra, $\mathfrak{r}_{3,1}$

$$\mathfrak{L}_3 := \{[e_1, e_2] = e_2, [e_1, e_3] = e_3\}.$$ 

$$\begin{align*}
\text{Aut}(\mathfrak{L}_3) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & y & h \end{pmatrix} \in M(3, \mathbb{C}) : \ h \in GL(2, \mathbb{C}), \\ x, y \in \mathbb{C} \right\}. \\
\text{Der}(\mathfrak{L}_3) &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ t_1 & t_2 & B \end{pmatrix} \in M(3, \mathbb{C}) : \ B \in M(2, \mathbb{C}), \\ t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{sl}(2, \mathbb{C}) \ltimes \mathfrak{r}_{3,1}. 
\end{align*}$$

5.1 $\mathfrak{L}_{3}^0$

Let $A$ be the Zero map and let us denote by $\mathfrak{L}_{3}^0$ the hom-Lie algebra $(\mathfrak{L}_3, A)$. We have:

$$\text{Aut}(\mathfrak{L}_{3}^0) = \text{Aut}(\mathfrak{L}_3).$$

$$\text{Der}(\mathfrak{L}_{3}^0) = \text{Der}(\mathfrak{L}_3).$$

5.1.1 $\psi_{\alpha,\beta}(\mathfrak{L}_{3}^0)$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_{3}^0) = t_{\alpha,\beta}$ is the Lie algebra $\mathfrak{r}_{3,1}$.

5.1.2 $\phi_{\beta}(\mathfrak{L}_{3}^0)$

In this case, we have $\phi_{\beta}(\mathfrak{L}_{3}^0) = \tilde{t}_{\beta}$ is the 3-dimensional abelian Lie algebra.

5.1.3 $\rho(\mathfrak{L}_{3}^0)$

In this case, we have $\rho(\mathfrak{L}_{3}^0) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

5.2 $\mathfrak{L}_{3}^1$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

and let us denote by $\mathfrak{L}_{3}^1$ the hom-Lie algebra $(\mathfrak{L}_3, A)$. We have:

$$\begin{align*}
\text{Aut}(\mathfrak{L}_{3}^1) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & x \\ z & 0 & a \end{pmatrix} \in M(3, \mathbb{C}) : \ a \in \mathbb{C}^* \\ x, y, z \in \mathbb{C} \right\}. \\
\text{Der}(\mathfrak{L}_{3}^1) &= \left\{ \begin{pmatrix} t_1 & 0 & t_3 \\ 0 & t_2 & t_4 \end{pmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong \mathbb{C} \ltimes n_3(\mathbb{C}). 
\end{align*}$$

5.2.1 $\psi_{\alpha,\beta}(\mathfrak{L}_{3}^1)$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_{3}^1) = t_{\alpha,\beta}$ is the Lie algebra $\mathfrak{r}_{3,1}$.

5.2.2 $\phi_{\beta}(\mathfrak{L}_{3}^1)$

In this case, we have $\phi_{\beta}(\mathfrak{L}_{3}^1) = \tilde{t}_{\beta}$ is the 3-dimensional abelian Lie algebra.
5.2.3 \( \rho(L_1^3) \)

In this case, we have \( \rho(L_1^3) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

5.2.4 Identities

- \( A[\cdot, \cdot] = 0 \)
- \( [A\cdot, \cdot] + [\cdot, A\cdot] = 0 \)
- \( [A\cdot, A\cdot] = 0 \)

In particular, the hom-Lie algebra \( L_1^3 \) is a multiplicative hom-Lie algebra.

5.3 \( L_2^3 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( L_2^3 \) the hom-Lie algebra \( (L_3, A) \). We have:

\[
\text{Aut}(L_2^3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & a & x \\ 0 & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \right\}
\]

\[
\text{Der}(L_2^3) = \left\{ \begin{bmatrix} t_3 & t_1 & t_2 \\ 0 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}.
\]

5.3.1 \( \psi_{\alpha, \beta}(L_2^3) \)

In this case, we have \( \psi_{\alpha, \beta}(L_2^3) = t_{\alpha, \beta} \):

\[
t_{\alpha, \beta} = \{ [e_1, e_2] = e_2, [e_1, e_3] = (\beta + \alpha) e_2 + e_3 \}.
\]

We note that \( t_{\alpha, \beta} \) is an almost abelian Lie algebra where \( \text{span}_{\mathbb{C}} \{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha, \beta} \); we can identify \( t_{\alpha, \beta} \) the matrix

\[
B = \begin{bmatrix} 1 & \beta + \alpha \\ 0 & 1 \end{bmatrix}
\]

The isomorphism class of \( t_{\alpha, \beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 = 1 \)
   (a) 0 is a repeated eigenvalue of \( B \) if and only if \( 1 = 0 \)
   (b) \( B \) has a non-zero repeated eigenvalue if and only if \( 1 = 1 \)
      i. \( t_{\alpha, \beta} \) is \( r_{3,1} \) if and only if \( B = \lambda \text{Id} \) for some \( \lambda \neq 0 \) if and only if \( \beta = -\alpha \)
      ii. \( t_{\alpha, \beta} \) is isomorphic to \( r_3 \) if and only if \( B \neq \lambda \text{Id} \) for all \( \lambda \in \mathbb{C} \) if and only if \( \beta \neq -\alpha \)
2. \( B \) has distinct eigenvalues eigenvalues if and only if \( 1 = 0 \)
5.3.2 \( \phi_\beta(\mathfrak{L}_3^2) \)

In this case, we have \( \phi_\beta(\mathfrak{L}_3^2) = \tilde{t}_\beta \):

\[
\tilde{t}_\beta = \{ [e_1, e_3] = (\beta + 1) e_2 \}.
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_\beta \); we can identify \( t_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix} 0 & \beta + 1 \\ 0 & 0 \end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   - (a) \( 0 \) is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
     - i. \( t_\beta \) is the 3-dimensional abelian Lie algebra if and only if \( \tilde{B} = 0 \) if and only if \( \beta = -1 \)
     - ii. \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{n}_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \beta 

2. \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

5.3.3 \( \rho(\mathfrak{L}_3^2) \)

In this case, we have \( \rho(\mathfrak{L}_3^2) = \hat{t} \):

\[
\hat{t} = \{ [e_1, e_3] = e_2 \}.
\]

which is isomorphic to \( \mathfrak{n}_3 \)

5.3.4 **Identities**

- \( A[\cdot, \cdot] = [A \cdot, \cdot] + [\cdot, A \cdot] \)
- \( A[A \cdot, \cdot] - [A \cdot, A \cdot] = 0 \)

The hom-Lie algebra \( \mathfrak{L}_3^3 \) is a non-multiplicative hom-Lie algebra.

5.4 \( \mathfrak{L}_3^3 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_3^3 \) the hom-Lie algebra \( (\mathfrak{L}_3, A) \). We have:

\[
\text{Aut}(\mathfrak{L}_3^3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & x & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\}.
\]

\[
\text{Der}(\mathfrak{L}_3^3) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ t_2 & t_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \simeq \mathbb{C}^2.
\]
5.4.1 $\psi_{\alpha,\beta}(L^3_3)$

In this case, we have $\psi_{\alpha,\beta}(L^3_3) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e_1, e_2] = e_2 + (\alpha + \beta) e_3, [e_1, e_3] = e_3\}$$

We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where span$_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix

$$B = \begin{bmatrix} 1 & 0 \\ \alpha + \beta & 1 \end{bmatrix}.$$ 

The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $1 = 1$
   
   (a) 0 is a repeated eigenvalue of $B$ if and only if $1 = 0$
   
   (b) $B$ has a non-zero repeated eigenvalue if and only if $1 = 1$
      
      i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = \lambda \text{Id}$ for some $\lambda \neq 0$ if and only if $\beta = -\alpha$
      
      ii. $t_{\alpha,\beta}$ is isomorphic to $t_3$ if and only if $B \neq \lambda \text{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $\beta \neq -\alpha$

2. $B$ has distinct eigenvalues eigenvalues if and only if $1 = 0$

5.4.2 $\phi_\beta(L^3_3)$

In this case, we have $\phi_\beta(L^3_3) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e_1, e_2] = (1 + \beta) e_3\}.$$ 

We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where span$_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_\beta$; we can identify $\tilde{t}_\beta$ the matrix

$$\tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 + \beta & 0 \end{bmatrix}.$$ 

The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $0 = 0$
   
   (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$
   
   i. $\tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\beta = -1$
   
   ii. $\tilde{t}_\beta$ is isomorphic to $n_3(\mathbb{C})$ if and only if $\tilde{B} \neq 0$ if and only if $\beta \neq -1$
   
   (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues eigenvalues if and only if $1 = 0$

5.4.3 $\rho(L^3_3)$

In this case, we have $\rho(L^3_3) = \hat{t}$:

$$\hat{t}_3 = \{[e_1, e_2] = e_3\}.$$ 

which is isomorphic to $n_3$

5.4.4 Identities

- $A[\cdot, \cdot] = [A\cdot, \cdot] + [\cdot, A\cdot] = 0$
- $A^2[\cdot, \cdot] = 0$
- $[A\cdot, A\cdot] = 0$

The hom-Lie algebra $L^3_3$ is a non-multiplicative hom-Lie algebra.
5.5 Degenerations between hom-Lie algebras ($\mathfrak{L}_3^i$)

If $\mathfrak{L}_j^i \rightarrow \deg \rightarrow \mathfrak{L}_k^h$ then $\text{Der}(\mathfrak{L}_j^i) \leq \text{Der}(\mathfrak{L}_k^h)$. Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 2        | $\mathfrak{L}_3^3$ |
| 3        | $\mathfrak{L}_3^2$ |
| 4        | $\mathfrak{L}_3^1$ |
| 6        | $\mathfrak{L}_3^0$ |

5.5.1 Degenerations

1. $\mathfrak{L}_3^3 \rightarrow \deg \rightarrow \mathfrak{L}_3^2$

   In fact, set

   $g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-t} \\ 0 & e^{-t} & e^{-2t} \end{bmatrix}$, with $t \in \mathbb{R}$.

   We have $g(t) \in \text{Aut}(\mathfrak{L}_3)$ and $g(t) \cdot \mathfrak{L}_3^3$ is the hom-Lie algebra ($\mathfrak{L}_3, A(t)$) with

   $$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -e^{-t} & 1 \\ e^{-t} & -e^{-2t} & e^{-t} \end{bmatrix}.$$  

   It is easy to check that ($\mathfrak{L}_3, A(t)$) $\rightarrow \mathfrak{L}_3^2$ as $t$ tends to infinity.

2. $\mathfrak{L}_3^2 \rightarrow \deg \rightarrow \mathfrak{L}_3^1$

   In fact, set

   $g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{t} & 0 & e^{t} \end{bmatrix}$, with $t \in \mathbb{R}$.

   We have $g(t) \in \text{Aut}(\mathfrak{L}_3)$ and $g(t) \cdot \mathfrak{L}_3^2$ is the hom-Lie algebra ($\mathfrak{L}_3, A(t)$) with

   $$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & e^{-t} \\ 0 & 0 & 0 \end{bmatrix}.$$  

   It is easy to check that ($\mathfrak{L}_3, A(t)$) $\rightarrow \mathfrak{L}_3^1$ as $t$ tends to infinity.
3. $\mathfrak{L}_3^1 \xrightarrow{\text{deg}} \mathfrak{L}_3^0$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(\mathfrak{L}_3)$ and $g(t) \cdot \mathfrak{L}_3^1$ is the hom-Lie algebra $(\mathfrak{L}_3, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(\mathfrak{L}_3, A(t)) \rightarrow \mathfrak{L}_3^0$ as $t$ tends to infinity.
6  \( \mathfrak{L}_4 \): The Lie algebra of the isometry group of 2-dimensional Minkowski spacetime, \( \mathfrak{r}_{3,-1} \)

\[ \mathfrak{L}_1 := \{ [e_1, e_2] = e_2, [e_1, e_3] = -e_3 \} \]

\[
\text{Aut}(\mathfrak{L}_4) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ y & 0 & b \end{bmatrix} \in M(3, \mathbb{C}) : \begin{bmatrix} -1 & 0 & 0 \\ x & 0 & a \\ y & b & 0 \end{bmatrix} \in M(3, \mathbb{C}) \right\} \bigcup \left\{ \begin{bmatrix} t_3 & t_1 & 0 \\ t_4 & 0 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathfrak{r}_2(\mathbb{C}). \]

6.1  \( \mathfrak{L}_0^0 \)

Let \( A \) be the zero map and let us denote by \( \mathfrak{L}_0^0 \) the hom-Lie algebra \( (\mathfrak{L}_4, A) \). We have:
\[
\text{Aut}(\mathfrak{L}_0^0) = \text{Aut}(\mathfrak{L}_4).
\]
\[
\text{Der}(\mathfrak{L}_0^0) = \text{Der}(\mathfrak{L}_4).
\]

6.1.1  \( \psi_{\alpha,\beta}(\mathfrak{L}_0^0) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_0^0) = t_{\alpha,\beta} \) is the Lie algebra \( \mathfrak{r}_{3,-1} \)

6.1.2  \( \phi_{\beta}(\mathfrak{L}_0^0) \)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_0^0) = \tilde{t}_{\beta} \) is the 3-dimensional abelian Lie algebra.

6.1.3  \( \rho(\mathfrak{L}_0^0) \)

In this case, we have \( \rho(\mathfrak{L}_0^0) = \tilde{t} \) is the 3-dimensional abelian Lie algebra.

6.2  \( \mathfrak{L}_1^1 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{ e_1, e_2, e_3 \} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
and let us denote by \( \mathfrak{L}_1^1 \) the hom-Lie algebra \( (\mathfrak{L}_4, A) \). We have:
\[
\text{Aut}(\mathfrak{L}_1^1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \right\}.
\]
\[
\text{Der}(\mathfrak{L}_1^1) = \left\{ \begin{bmatrix} t_2 & 0 & 0 \\ t_3 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}.
\]

6.2.1  \( \psi_{\alpha,\beta}(\mathfrak{L}_1^1) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_1^1) = t_{\alpha,\beta} \) is the Lie algebra \( \mathfrak{r}_{3,-1} \)
6.2.2 $\phi_{\beta}(\mathfrak{L}^1_4)$
In this case, we have $\phi_{\beta}(\mathfrak{L}^1_4) = \tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra.

6.2.3 $\rho(\mathfrak{L}^1_4)$
In this case, we have $\rho(\mathfrak{L}^1_4) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

6.2.4 Identities
- $A[\cdot, \cdot] = 0$
- $[A\cdot, \cdot] + [\cdot, A\cdot] = 0$
  \[ - [A\cdot, A\cdot] = 0 \]
In particular, the hom-Lie algebra $\mathfrak{L}^1_4$ is a multiplicative hom-Lie algebra.

6.3 $\mathfrak{L}^2_4$
Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is
\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]
and let us denote by $\mathfrak{L}^2_4$ the hom-Lie algebra $(\mathfrak{L}_4, A)$. We have:
\[
\text{Aut}(\mathfrak{L}^2_4) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\} \bigcup \left\{ \begin{bmatrix} -1 & 0 & 0 \\ x & 0 & -1 \\ y & -1 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\}.
\]
\[
\text{Der}(\mathfrak{L}^2_4) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ t_2 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2.
\]

6.3.1 $\psi_{\alpha, \beta}(\mathfrak{L}^2_4)$
In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}^2_4) = t_{\alpha, \beta}$ is the Lie algebra $\mathfrak{r}_{3,-1}$

6.3.2 $\phi_{\beta}(\mathfrak{L}^2_4)$
In this case, we have $\phi_{\beta}(\mathfrak{L}^2_4) = \tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra.

6.3.3 $\rho(\mathfrak{L}^2_4)$
In this case, we have $\rho(\mathfrak{L}^2_4) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

6.3.4 Identities
- $A[\cdot, \cdot] = 0$
- $[A\cdot, \cdot] + [\cdot, A\cdot] = 0$
  \[ - [A\cdot, A\cdot] = 0 \]
In particular, the hom-Lie algebra $\mathfrak{L}^2_4$ is a multiplicative hom-Lie algebra.
6.4 \( \mathfrak{L}_4^3 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_4^3 \) the hom-Lie algebra \((\mathfrak{L}_4, A)\). We have:

\[
\text{Aut}(\mathfrak{L}_4^3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ 0 & 0 & a \end{bmatrix} : a \in \mathbb{C}^*, x \in \mathbb{C} \right\},
\]

\[
\text{Der}(\mathfrak{L}_4^3) = \left\{ \begin{bmatrix} t_2 & t_1 & 0 \\ 0 & 0 & t_1 \end{bmatrix} : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2.
\]

6.4.1 \( \psi_{\alpha,\beta}(\mathfrak{L}_4^3) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathfrak{L}_4^3) = t_{\alpha,\beta} : \)

\[
t_{\alpha,\beta} = \lbrace [e_1, e_2] = e_2, [e_1, e_3] = (\beta - \alpha) e_2 - e_3.\rbrace
\]

We note that \( t_{\alpha,\beta} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha,\beta} \); we can identify \( t_{\alpha,\beta} \) the matrix

\[
B = \begin{bmatrix}
1 & \beta - \alpha \\
0 & -1
\end{bmatrix}
\]

The isomorphism class of \( t_{\alpha,\beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 = 0 \)
2. \( B \) has distinct eigenvalues eigenvalues if and only if \( 1 = 1 \)
   (a) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_2(\mathbb{C}) \times \mathbb{C} \) if and only if \( 0 \) is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( 0 = 1 \)
   (b) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_{3,-1} \) if and only if \( \text{Trace}(B) = 0 \) if and only if \( 1 = 1 \).
   (c) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_{3,z} \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( 1 = 0 \).

6.4.2 \( \phi_{\beta}(\mathfrak{L}_4^3) \)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_4^3) = \tilde{t}_{\beta} : \)

\[
\tilde{t}_{\beta} = \lbrace [e_1, e_3] = (\beta - 1) e_2.\rbrace
\]

We note that \( \tilde{t}_{\beta} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_{\beta} \); we can identify \( \tilde{t}_{\beta} \) the matrix

\[
\tilde{B} = \begin{bmatrix}
0 & \beta - 1 \\
0 & 0
\end{bmatrix}
\]

The isomorphism class of \( \tilde{t}_{\beta} \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
(a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$

i. $\tilde{t}_β$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\beta = 1$

ii. $\tilde{t}_β$ is isomorphic to $t_3 (\mathbb{C})$ if and only if $\tilde{B} \neq 0$ if and only if $\beta \neq 1$

(b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues eigenvalues if and only if $1 = 0$

$6.4.3 \ \rho(L^3_4)$

In this case, we have $\rho(L^3_4) = \hat{t}$:

$$\hat{t} = \{[e1, e3] = e2. \}

which is isomorphic to $t_3$

6.4.4 Identities

- $A[., ] + [A[., ] + [., A]] = 0$
- $A[A[. . ] = 0$
- $[A[., A] = 0$

The hom-Lie algebra $L^3_4$ is a non-multiplicative hom-Lie algebra.

6.5 $L^4_4(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}
$$

and let us denote by $L^4_4(\lambda)$ the hom-Lie algebra $(L_4, A)$. We have:

$\text{Aut}(L^4_4(\lambda)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\
-x & a & 0 \\
x & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : \begin{array}{c} a \in \mathbb{C}^* \\ x \in \mathbb{C} \end{array} \right\}$.

$\text{Der}(L^4_4(\lambda)) = \left\{ \begin{bmatrix} -t_2 & t_1 & 0 \\
t_2 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{v}_2(\mathbb{C})$.

We note that the hom-Lie algebra $L^4_4(\lambda)$ is isomorphic to $L^4_4(-\lambda)$. In fact, $L^4_4(\lambda)$ is similar to $L^4_4(-\lambda)$ via the invertible matrix $g \in \text{Aut}(L_4)$:

$$g = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}$$
6.5.1 \( \psi_{\alpha,\beta}(L^4_1(\lambda)) \)

In this case, we have \( \psi_{\alpha,\beta}(L^4_1(\lambda)) = t_{\alpha,\beta} \):

\[
t_{\alpha,\beta} = \{ [e_1, e_2] = (1 + \alpha \lambda + \beta \lambda) e_2 + (-\alpha \lambda + \beta \lambda) e_3, [e_1, e_3] = (-\alpha \lambda + \beta \lambda) e_2 + (-1 + \alpha \lambda + \beta \lambda) e_3 \}.
\]

We note that \( t_{\alpha,\beta} \) is an almost abelian Lie algebra where span\(_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha,\beta} \); we can identify \( t_{\alpha,\beta} \) the matrix

\[
B = \begin{bmatrix}
1 + \alpha \lambda + \beta \lambda & -\alpha \lambda + \beta \lambda \\
-\alpha \lambda + \beta \lambda & -1 + \alpha \lambda + \beta \lambda
\end{bmatrix}.
\]

The isomorphism class of \( t_{\alpha,\beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 + (\alpha - \beta)^2 \lambda^2 = 0 \)
   (a) \( 0 \) is a repeated eigenvalue of \( B \) if and only if \( \alpha = -\beta = \pm \sqrt[2]{-1} \)
   i. \( t_{\alpha,\beta} \) is the 3-dimensional abelian Lie algebra if and only if \( B = 0 \) if and only if \( 1 = 0 \)
   ii. \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3(\mathbb{C}) \) if and only if \( 0 \) if and only if \( \alpha = -\beta = \pm \sqrt[2]{-1} \)

   (b) \( B \) has a non-zero repeated eigenvalue if and only if
   \[
   \{ \alpha = -\frac{1+\sqrt{1-4\lambda}}{2\lambda}, \beta = \frac{1+\sqrt{1-4\lambda}}{2\lambda} \} \text{ for any } s \in \mathbb{C} \setminus \{0\}.
   \]

   i. \( t_{\alpha,\beta} \) is \( \mathfrak{t}_3(1) \) if and only if \( B = t \text{ Id} \) for some \( t \neq 0 \) if and only if \( 1 = 0 \)
   ii. \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3 \) if and only if \( B \neq t \text{ Id} \) for all \( t \in \mathbb{C} \) if and only if
   \[
   \{ \alpha = -\frac{1+\sqrt{1-4\lambda}}{2\lambda}, \beta = \frac{1+\sqrt{1-4\lambda}}{2\lambda} \} \text{ for any } s \in \mathbb{C} \setminus \{0\}.
   \]

2. \( B \) has distinct eigenvalues if and only if \( 1 + (\alpha - \beta)^2 \lambda^2 \neq 0 \)
   (a) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_2(\mathbb{C}) \times \mathbb{C} \) if and only if \( 0 \) is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if
   \[
   \{ \alpha = -\frac{1+\sqrt{1-4\lambda}}{4\lambda}, \beta = \frac{1+\sqrt{1-4\lambda}}{4\lambda} \} \text{ for any } s \in \mathbb{C} \setminus \{0\}.
   \]

   (b) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3(-1) \) if and only if Trace\( (B) \neq 0 \) if and only if
   \[
   \{ \alpha = \frac{\sqrt{-1}}{2\lambda}, \beta = \pm \frac{\sqrt{-1}}{2\lambda} \} \text{ for any } s \in \mathbb{C} \setminus \{0\}.
   \]

   (c) \( t_{\alpha,\beta} \) is isomorphic to \( \mathfrak{t}_3(z) \) for some \( z \) such that \( z(z^2 - 1) \neq 0 \) if and only if
   \[
   \{ \alpha = \frac{s_1\sqrt{s_1s_2^2(1-s_1)}}{2s_1s_2}, \beta = \frac{s_1\sqrt{s_1s_2^2(1-s_1)}}{2s_1s_2} \} \text{ for any } s_1, s_2 \in \mathbb{C} \setminus \{0\} \text{ such that } s_1 - s_2^2 \neq 0.
   \]

6.5.2 \( \phi_\beta(L^4_1(\lambda)) \)

In this case, we have \( \phi_\beta(L^4_1(\lambda)) = \tilde{t}_\beta \):

\[
\tilde{t}_\beta = \{ [e_1, e_2] = (\lambda + \beta \lambda) e_2 + (-\lambda + \beta \lambda) e_3, [e_1, e_3] = (-\lambda + \beta \lambda) e_2 + (\lambda + \beta \lambda) e_3 \}.
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where span\(_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
\lambda + \beta \lambda & -\lambda + \beta \lambda \\
-\lambda + \beta \lambda & \lambda + \beta \lambda
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)
(a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $\lambda = 0$ (False)

(b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $\beta = 1$
   i. $\tilde{t}_\beta$ is $t_{3,1}$ if and only if $\tilde{B} = t \operatorname{Id}$ for some $\lambda \neq 0$ if and only if $\lambda = \lambda$
   ii. $\tilde{t}_\beta$ is isomorphic to $t_3$ if and only if $\tilde{B} \neq t \operatorname{Id}$ for all $\lambda \in \mathbb{C}$ if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues if and only if $\beta \neq 1$

   (a) $\tilde{t}_\beta$ is isomorphic to $t_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\beta = 0$

   (b) $\tilde{t}_\beta$ is isomorphic to $t_{3,-1}$ if and only if $\operatorname{Trace}(\tilde{B}) = 0$ if and only if $\beta = -1$.

   (c) $\tilde{t}_\beta$ is isomorphic to $t_{3,2}$ for some $z$ such that $z(z^2 - 1) \neq 0$ if and only if $\beta \neq -1$ and $\beta \neq 0$.

6.5.3 $\rho(\mathfrak{L}_4^4(\lambda))$

In this case, we have $\rho(\mathfrak{L}_4^4(\lambda)) = \tilde{t}$:

$$\tilde{t}_\beta = \{[e1, e2] = \lambda e2 + \lambda e3, [e1, e3] = \lambda e2 + \lambda e3\}.$$

which is isomorphic to $t_2 \times \mathbb{C}$

6.5.4 Identities

- $[A, A] = 0$

The hom-Lie algebra $\mathfrak{L}_4^4(\lambda)$ is a non-multiplicative hom-Lie algebra.

6.6 $\mathfrak{L}_4^5$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

and let us denote by $\mathfrak{L}_4^5$ the hom-Lie algebra $(\mathfrak{L}_4^4, A)$. We have:

$$\text{Aut}(\mathfrak{L}_4^5) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x & 0 & 1
\end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.$$

$$\text{Der}(\mathfrak{L}_4^5) = \left\{ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
t_1 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.$$

6.6.1 $\psi_{\alpha, \beta}(\mathfrak{L}_4^5)$

In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}_4^5) = t_{\alpha, \beta}$:

$$t_{\alpha, \beta} = [e1, e2] = e2 + (\alpha - \beta) e3, [e1, e3] = -e3.$$

We note that $t_{\alpha, \beta}$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha, \beta}$; we can identify $t_{\alpha, \beta}$ the matrix

$$B = \begin{bmatrix}
1 & 0 \\
\alpha - \beta & -1
\end{bmatrix}.$$

The isomorphism class of $t_{\alpha, \beta}$ is determined by the eigenvalues of $B$:
1. \( B \) has repeated eigenvalues if and only if \( 1 = 0 \)

2. \( B \) has distinct eigenvalues if and only if \( 1 = 1 \)
   
   (a) \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{t}_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( 0 = 1 \)
   
   (b) \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{t}_3, -1 \) if and only if \( \text{Trace}(B) = 0 \) if and only if \( 1 = 1 \).
   
   (c) \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{t}_3, z \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( 1 = 0 \).

### 6.6.2 \( \phi_\beta(\mathfrak{L}^4_4) \)

In this case, we have \( \phi_\beta(\mathfrak{L}^5_4) = \tilde{t}_\beta \):

\[
\tilde{t}_\beta = \{ [e_1, e_2] = (1 - \beta) e_3 \}.
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 - \beta & 0 \end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   
   (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
   
   i. \( \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra if and only if \( \tilde{B} = 0 \) if and only if \( \beta = 1 \)
   
   ii. \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{n}_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \beta \neq 1 \)
   
   (b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( 1 = 0 \)

### 6.6.3 \( \rho(\mathfrak{L}^5_4) \)

In this case, we have \( \rho(\mathfrak{L}^5_4) = \hat{t} \):

\[
\hat{t} = \{ [e_1, e_2] = -e_3 \}.
\]

which is isomorphic to \( \mathfrak{n}_3 \)

### 6.6.4 Identities

- \( A[, , ] + [A, , ] + [, A] = 0 \)
- \( A^2[, , ] = 0 \)
- \( [A, A] = 0 \)

The hom-Lie algebra \( \mathfrak{L}^3_4 \) is a non-multiplicative hom-Lie algebra.
6.7 $\mathfrak{L}_4^6(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis \{e_1, e_2, e_3\} is

$$
\begin{bmatrix}
0 & 0 & 0 \\
1 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}
$$

and let us denote by $\mathfrak{L}_4^6(\lambda)$ the hom-Lie algebra $(\mathfrak{L}_4, A)$. We have:

$$\text{Aut}(\mathfrak{L}_4^6(\lambda)) = \begin{cases} 
\begin{bmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
-x & 0 & 1
\end{bmatrix} & \in M(3, \mathbb{C}) : x \in \mathbb{C} 
\end{cases}.$$

$$\text{Der}(\mathfrak{L}_4^6(\lambda)) = \begin{cases} 
\begin{bmatrix}
0 & 0 & 0 \\
t_1 & 0 & 0 \\
t_1 & 0 & 0
\end{bmatrix} & \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} 
\end{cases} \cong \mathbb{C}.
$$

We note that the hom-Lie algebra $\mathfrak{L}_4^6(\lambda)$ is isomorphic to $\mathfrak{L}_4^6(-\lambda)$. In fact, $\begin{bmatrix}
0 & 0 & 0 \\
1 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}$ is similar to $\begin{bmatrix}
0 & 0 & 0 \\
1 & -\lambda & -\lambda \\
0 & \lambda & \lambda
\end{bmatrix}$ via the invertible matrix $g \in \text{Aut}(\mathfrak{L}_4)$:

$$
g = \begin{bmatrix}
-1 & 0 & 0 \\
- \frac{1}{2\lambda} & 0 & -1 \\
- \frac{1}{2\lambda} & -1 & 0
\end{bmatrix}.
$$

6.7.1 $\psi_{\alpha,\beta}(\mathfrak{L}_4^6(\lambda))$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_4^6(\lambda)) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e_1, e_2] = (1 + \alpha \lambda + \beta \lambda) e_2 + (-\alpha \lambda + \beta \lambda) e_3, [e_1, e_3] = (-\alpha \lambda + \beta \lambda) e_2 + (-1 + \alpha \lambda + \beta \lambda) e_3\}.$$

We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix

$$B = \begin{bmatrix}
1 + \alpha \lambda + \beta \lambda & -\alpha \lambda + \beta \lambda & -\alpha \lambda + \beta \lambda \\
-\alpha \lambda + \beta \lambda & -1 + \alpha \lambda + \beta \lambda & -\alpha \lambda + \beta \lambda
\end{bmatrix}.$$

The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $1 + (\alpha - \beta)^2 \lambda^2 = 0$
   
   (a) 0 is a repeated eigenvalue of $B$ if and only if $\alpha = -\beta = \pm \frac{\sqrt{\lambda}}{2\lambda}$
   
   i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$
   
   ii. $t_{\alpha,\beta}$ is isomorphic to $n_3(\mathbb{C})$ if and only if $B \neq 0$ if and only if $\alpha = -\beta = \pm \frac{\sqrt{\lambda}}{2\lambda}$

   (b) $B$ has a non-zero repeated eigenvalue if and only if $\{\alpha = -\frac{1 + \sqrt{7\lambda}}{2\lambda}, \beta = \frac{1 + \sqrt{7\lambda}}{2\lambda}\}$ for any $s \in \mathbb{C} \setminus \{0\}$
   
   i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = t \text{Id}$ for some $t \neq 0$ if and only if $1 = 0$
ii. \( t_{\alpha, \beta} \) is isomorphic to \( r_3 \) if and only if \( B \neq t \text{Id} \) for all \( t \in C \) if and only if 
\[
\alpha = \frac{-1 + \sqrt{1 - 4s}}{2s}, \beta = \frac{1 + \sqrt{1 - 4s}}{2s}
\] for any \( s \in C \setminus \{0\} \).

2. \( B \) has distinct eigenvalues if and only if \( 1 + (\alpha - \beta)^2 \lambda^2 \neq 0 \)

   (a) \( t_{\alpha, \beta} \) is isomorphic to \( r_2(C) \times C \) if and only if \( 0 \) is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if 
   \[
   \alpha = \frac{-1 + \sqrt{1 - 4s}}{2s}, \beta = \frac{1 + \sqrt{1 - 4s}}{2s}
   \] for any \( s \in C \setminus \{0\} \).

   (b) \( t_{\alpha, \beta} \) is isomorphic to \( r_{3,-1} \) if and only if \( \text{Trace}(B) = 0 \) if and only if 
   \[
   \alpha = \frac{\sqrt{s - \lambda}}{2s\lambda}, \beta = \frac{-\sqrt{s - \lambda}}{2s\lambda}
   \] for any \( s \in C \setminus \{0\} \).

   (c) \( t_{\alpha, \beta} \) is isomorphic to \( r_{3,2} \) for some \( z \) such that \( z(z^2 - 1) \neq 0 \) if and only if
   \[
   \alpha = \frac{s_1 \pm \sqrt{s_1 s_2 (1 - s_1)}}{2s_1 s_2 \lambda}, \beta = \frac{s_1 \pm \sqrt{s_1 s_2 (1 - s_1)}}{2s_1 s_2 \lambda}
   \] for any \( s_1, s_2 \in C \setminus \{0\} \) such that \( s_1 - s_2^2 \neq 0 \).

### 6.7.2 \( \phi_\beta(\mathfrak{L}^0_4(\lambda)) \)

In this case, we have \( \phi_\beta(\mathfrak{L}^0_4(\lambda)) = \hat{t}_3 \):

\[
\hat{t}_3 = \{[e1, e2] = (\lambda + \beta \lambda) e2 + (-\lambda + \beta \lambda) e3, [e1, e3] = (-\lambda + \beta \lambda) e2 + (\lambda + \beta \lambda) e3 \}.
\]

We note that \( \hat{t}_3 \) is an almost abelian Lie algebra where \( \text{span}_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \hat{t}_3 \); we can identify \( \hat{t}_3 \) the matrix

\[
\hat{B} = \begin{bmatrix}
\lambda + \beta \lambda & -\lambda + \beta \lambda \\
-\lambda + \beta \lambda & \lambda + \beta \lambda
\end{bmatrix}.
\]

The isomorphism class of \( \hat{t}_3 \) is determined by the eigenvalues of \( \hat{B} \):

1. \( \hat{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)
   
   (a) \( 0 \) is a repeated eigenvalue of \( \hat{B} \) if and only if \( \lambda = 0 \) (False)
   
   (b) \( \hat{B} \) has a non-zero repeated eigenvalue if and only if \( \beta = 1 \)

   i. \( \hat{t}_3 \) is \( t_{3,1} \) if and only if \( \hat{B} = t \text{Id} \) for some \( t \neq 0 \) if and only if \( \lambda = \lambda \)

   ii. \( \hat{t}_3 \) is isomorphic to \( t_3 \) if and only if \( \hat{B} \neq t \text{Id} \) for all \( \lambda \in C \) if and only if \( 1 = 0 \)

2. \( \hat{B} \) has distinct eigenvalues if and only if \( \beta \neq 1 \)

   (a) \( \hat{t}_3 \) is isomorphic to \( t_3(C) \times C \) if and only if \( 0 \) is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \beta = 0 \)

   (b) \( \hat{t}_3 \) is isomorphic to \( t_{3,-1} \) if and only if \( \text{Trace}(\hat{B}) = 0 \) if and only if \( \beta = -1 \).

   (c) \( \hat{t}_3 \) is isomorphic to \( t_{3,2} \) for some \( z \) such that \( z(z^2 - 1) \neq 0 \) if and only if \( \beta \neq 1 \) and \( \beta \neq -1 \) and \( \beta \neq 0 \).

### 6.7.3 \( \rho(\mathfrak{L}^0_4) \)

In this case, we have \( \rho(\mathfrak{L}^0_4) = \hat{t} \):

\[
\hat{t} = \{[e1, e2] = \lambda e2 + \lambda e3, [e1, e3] = \lambda e2 + \lambda e3 \}.
\]

which is isomorphic to \( t_2 \times C \).
6.7.4 Identities

- \([A, A] = 0\)
- \(A^2[.,.] = 0\)
- \([A^2, .] + [., A^2] = 0\)

The hom-Lie algebra \(\mathfrak{L}_6^4(\lambda)\) is a non-multiplicative hom-Lie algebra.

6.8 Degenerations between hom-Lie algebras \(\mathfrak{L}_4^i\)

If \(\mathfrak{L}_i^j \xrightarrow{\text{deg}} \mathfrak{L}_k^h\) then \(\text{Der}(\mathfrak{L}_i^j) \leq \text{Der}(\mathfrak{L}_k^h)\). Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 1        | \(\mathfrak{L}_4^1(\lambda)\), \(\mathfrak{L}_4^2\) |
| 2        | \(\mathfrak{L}_4^3(\lambda)\), \(\mathfrak{L}_4^4\) |
| 3        | \(\mathfrak{L}_4^5\) |
| 4        | \(\mathfrak{L}_4^6\) |

6.8.1 Degenerations

1. \(\mathfrak{L}_4^6(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_4^1(\pm \lambda)\)

In fact, set

\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \(g(t) \in \text{Aut}(\mathfrak{L}_4)\) and \(g(t) \cdot \mathfrak{L}_4^6(\lambda)\) is the hom-Lie algebra \((\mathfrak{L}_4, A(t))\) with

\[
A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & \lambda & \lambda \\ 0 & -\lambda & -\lambda \end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_4, A(t)) \rightarrow \mathfrak{L}_4^1(\lambda)\) as \(t\) tends to infinity.

Since, \(\mathfrak{L}_4^3(\lambda) \equiv \mathfrak{L}_4^2(-\lambda)\), then \(\mathfrak{L}_4^5(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_4^2(-\lambda)\).

2. \(\mathfrak{L}_4^5 \xrightarrow{\text{deg}} \mathfrak{L}_4^3\)

In fact, set

\[
g(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ e^{-t} & 0 & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathcal{L}_4) \) and \( g(t) \cdot \mathcal{L}_4^5 \) is the hom-Lie algebra \((\mathcal{L}_4, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
e^{-t} & 0 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathcal{L}_4, A(t)) \to \mathcal{L}_4^3\) as \( t \) tends to infinity.

3. \( \mathcal{L}_4^5 \xrightarrow{\text{deg}} \mathcal{L}_4^2 \)
   In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
-e^t & 1 & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathcal{L}_4) \) and \( g(t) \cdot \mathcal{L}_4^5 \) is the hom-Lie algebra \((\mathcal{L}_4, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & e^{-t} & 0
\end{bmatrix}
\]

It is easy to check that \((\mathcal{L}_4, A(t)) \to \mathcal{L}_4^2\) as \( t \) tends to infinity.

4. \( \mathcal{L}_4^3 \xrightarrow{\text{deg}} \mathcal{L}_4^1 \)
   In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-e^t & 0 & e^t
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathcal{L}_4) \) and \( g(t) \cdot \mathcal{L}_4^3 \) is the hom-Lie algebra \((\mathcal{L}_4, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & e^{-t} \\
0 & 0 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathcal{L}_4, A(t)) \to \mathcal{L}_4^1\) as \( t \) tends to infinity.

5. \( \mathcal{L}_4^2 \xrightarrow{\text{deg}} \mathcal{L}_4^1 \)
   In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathcal{L}_4) \) and \( g(t) \cdot \mathcal{L}_4^2 \) is the hom-Lie algebra \((\mathcal{L}_4, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
e^{-t} & 0 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathcal{L}_4, A(t)) \to \mathcal{L}_4^1\) as \( t \) tends to infinity.
6. $L^1_4 \xrightarrow{\deg} L^0_4$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(L_4)$ and $g(t) \cdot L^1_4$ is the hom-Lie algebra $(L_4, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

It is easy to check that $(L_4, A(t)) \rightarrow L^0_4$ as $t$ tends to infinity.

7. $L^6_4(\lambda) \xrightarrow{\deg} L^4_4(\kappa)$ with $(\kappa \neq \pm \lambda)$.

Suppose, contrary to our claim, that $L^6_4(\lambda) \xrightarrow{\deg} L^4_4(\kappa)$ with $(\kappa \neq \pm \lambda)$. In such case, we have

$$\psi_{\alpha,\beta}(L^6_4(\lambda)) \xrightarrow{\deg} \psi_{\alpha,\beta}(L^4_4(\kappa)).$$

By taking $\alpha = -\beta$ and $\alpha = \pm \frac{\sqrt{-1}}{2\kappa}$, we have from 6.7.1 that $\psi_{\alpha,\beta}(L^6_4(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha,\beta}(L^4_4(\kappa))$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha,\beta}(L^4_4(\kappa))$ is isomorphic to $n_3$ (from 6.5.1). But, we now apply again 6.5.1, to obtain $\alpha = -\beta$ and $\alpha = \pm \frac{\sqrt{-1}}{2\kappa}$ and therefore $\lambda = \pm \kappa$; this is a contradiction.

8. $L^0_4(\lambda) \xrightarrow{\deg} L^0_4$ with $(\lambda \neq 0)$.

Suppose, contrary to our claim, that $L^0_4(\lambda) \xrightarrow{\deg} L^0_4$ with $(\lambda \neq 0)$. In such case, we have

$$\psi_{\alpha,\beta}(L^0_4(\lambda)) \xrightarrow{\deg} \psi_{\alpha,\beta}(L^0_4).$$

By taking $\alpha = -\beta$ and $\alpha = \pm \frac{\sqrt{-1}}{2\kappa}$, we have from 6.7.1 that $\psi_{\alpha,\beta}(L^0_4(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha,\beta}(L^0_4)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). But, from 6.1.1, we have $\psi_{\alpha,\beta}(L^0_4)$ is isomorphic to $r_3,1$; this is a contradiction.

9. $L^5_4 \xrightarrow{\deg} L^5_4(\lambda)$ with $(\lambda \neq 0)$.

Suppose, contrary to our claim, that $L^5_4 \xrightarrow{\deg} L^5_4(\lambda)$ with $(\lambda \neq 0)$. In such case, we have

$$\phi_{\beta}(L^5_4) \xrightarrow{\deg} \phi_{\beta}(L^5_4(\lambda)).$$

By taking $\beta = 1$, we have from 6.6.2 that $\phi_{\beta}(L^5_4)$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_{\beta}(L^5_4(\lambda))$ is the 3-dimensional abelian Lie algebra. But, from 6.5.2, we have $\phi_{\beta}(L^5_4(\lambda))$ is isomorphic to $r_{3,1}$; this is a contradiction.
7 \( L_5(z) \) with \( z(z^2-1) \neq 0 \): Bianchi type VI Lie algebra, 
\( r_{3,z} \) with \( z(z^2-1) \neq 0 \)

\[ L_1 := \{ [e_1, e_2] = e_2, [e_1, e_3] = ze_3 \} \]

\[ \text{Aut}(L_5(z)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ y & 0 & b \end{bmatrix} \in M(3, \mathbb{C}) : a, b \in \mathbb{C}^* \right\} \]

\[ \text{Der}(L_5(z)) = \left\{ \begin{bmatrix} t_3 & t_1 & 0 \\ t_4 & 0 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathfrak{r}_2(\mathbb{C}). \]

7.1 \( L_0^5(z) \)

Let \( A \) be the Zero map and let us denote by \( L_0^5(z) \) the hom-Lie algebra \( (L_5(z), A) \). We have:
\[ \text{Aut}(L_0^5(z)) = \text{Aut}(L_5(z)). \]
\[ \text{Der}(L_0^5(z)) = \text{Der}(L_5(z)). \]

7.1.1 \( \psi_{\alpha,\beta}(L_0^5(z)) \)

In this case, we have \( \psi_{\alpha,\beta}(L_0^5(z)) = t_{\alpha,\beta} \) is the Lie algebra \( r_{3,z} \).

7.1.2 \( \phi_{\beta}(L_0^5(z)) \)

In this case, we have \( \phi_{\beta}(L_0^5(z)) = \tilde{t}_{\beta} \) is the 3-dimensional abelian Lie algebra.

7.1.3 \( \rho(L_0^5) \)

In this case, we have \( \rho(L_0^5) = \hat{\mathfrak{i}} \) is the 3-dimensional abelian Lie algebra.

7.2 \( L_1^5(z) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and let us denote by \( L_1^5(z) \) the hom-Lie algebra \( (L_5(z), A) \). We have:
\[ \text{gl}(n, \mathbb{C}) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ t_3 & t_1 & t_5 \\ t_4 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, \ldots, t_5 \in \mathbb{C} \right\} \]
\[ \text{Aut}(L_1^5(z)) = \left\{ \begin{bmatrix} x & 1 & 0 \\ y & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \right\}. \]
\[ \text{Der}(L_1^5(z)) = \left\{ \begin{bmatrix} t_2 & 0 & 0 \\ t_3 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}. \]
Given $D_1(w')$, $D_2(x')$ and $D_3(y')$ in $\mathfrak{gl}(n, \mathbb{C})$, we have

$$\lambda := D_1[\cdot, \cdot] + [D_2\cdot, \cdot] + \ldots + [\cdot, A\cdot] - [A\cdot, A\cdot] = 0$$

In particular, the hom-Lie algebra $L_1^5(z)$ is a multiplicative hom-Lie algebra.

By taking $\{\theta_1 = 1, \theta_2 = 0, \theta_3 = t, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0\}$ and $\{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0\}$ and then, by replacing $D_1$ by $-tD_3$, we have $\text{Ker}(T_{\theta_1, \ldots, \theta_6, \beta_1, \ldots, \beta_9}(\mathfrak{L}_5^1))$ is determined by the equations

$$\begin{aligned}
-x_{2,1} + y_{2,1} &= 0, \\
x_{3,2} - ty_{2,2} + y_{2,2} &= 0, \\
y_{2,3} - ty_{2,3} &= 0, \\
x_{2,2} - ty_{3,2} + y_{3,2} &= 0, \\
y_{2,3} + ty_{3,3} - x_{3,3} &= 0, \\
x_{2,3} - ty_{3,3} &= 0, \\
y_{2,2} + ty_{3,2} - x_{3,2} &= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
x_{2,1} &= y_{2,1}, \\
x_{2,2} &= -y_{2,2} + ty_{2,2}, \\
x_{2,3} &= ty_{3,2}, \\
x_{3,1} &= y_{3,1}, \\
x_{3,3} &= -y_{2,2} + ty_{3,3},
\end{aligned}$$

and so:

$$\begin{aligned}
y_{2,3}(-1 + tz) &= 0, \\
y_{2,2} - y_{3,3}(t - 1) &= 0.
\end{aligned}$$

Since $z \neq 0, \pm 1$, we have

$$\dim T_{\theta_1, \ldots, \theta_6, \beta_1, \ldots, \beta_9}(\mathfrak{L}_5^1) = \begin{cases} 4, & \text{if } t = 1 \text{ or } t = 1/z; \\ 3, & \text{in other case.} \end{cases}$$

In this case, we have $\psi_0(z) = t_{0,0}$ is the Lie algebra $t_{0,0}$.

In this case, we have $\phi_0(z) = t_{0,0}$ is the 3-dimensional abelian Lie algebra.

In this case, we have $\rho(z) = \tilde{t}$ is the 3-dimensional abelian Lie algebra.

**Identities**

- $A[\cdot, \cdot]$
- $[A\cdot, \cdot] + [\cdot, A\cdot] - [A\cdot, A\cdot] = 0$

In particular, the hom-Lie algebra $\mathfrak{L}_5^1(z)$ is a multiplicative hom-Lie algebra.
7.3 \( \mathfrak{L}_2^2(z) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_2^2(z) \) the hom-Lie algebra \((\mathfrak{L}_5(z), A)\). We have:

\[
\begin{align*}
\text{gl}(n, \mathbb{C})_A &= \left\{ \begin{bmatrix} t_1 & 0 & 0 \\
t_3 & t_2 & 0 \\
t_5 & t_4 & t_1 \end{bmatrix} : t_1, t_2, t_5 \in \mathbb{C} \right\} \\
\text{Aut}(\mathfrak{L}_2^2(z)) &= \left\{ \begin{bmatrix} x & a & 0 \\
y & 0 & 1 \\
0 & 0 & 0 \end{bmatrix} : a \in \mathbb{C}^*, x, y \in \mathbb{C} \right\} \\
\text{Der}(\mathfrak{L}_2^2(z)) &= \left\{ \begin{bmatrix} t_2 & t_1 & 0 \\
t_3 & 0 & 0 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}.
\end{align*}
\]

7.3.1 \( T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_5}(\mathfrak{L}_2^2) \)

Given \( D_1(w's), D_2(x's) \) and \( D_3(y's) \) in \( \text{gl}(n, \mathbb{C})_A \), we have \( \lambda := D_1[\cdot, \cdot] + [D_2, \cdot] + [\cdot, D_3] \)

\[
\begin{align*}
\lambda &= \left\{ \begin{align*}
e 1 \cdot e 2 &= (-x_{2,1} + y_1) e 2 + (-x_{3,1} + y_{3,1}) e 3, \\
e 1 \cdot e 2 &= (x_{3,3} + w_{2,2} + y_{2,1}) e 2 + (w_{3,2} + y_{3,2}) e 3, \\
e 1 \cdot e 3 &= (y_{3,3} + x y_{3,3} + y_{3,3}) e 3, \\
e 2 \cdot e 1 &= (-y_{3,3} - w_{2,2} - x_{2,1}) e 2 + (-w_{3,2} - x_{3,2}) e 3, \\
e 3 \cdot e 1 &= (-y_{3,3} - x y_{3,3} - x_{3,3}) e 3
\end{align*} \right\}
\end{align*}
\]

By taking \( \{\theta_1 = 1, \theta_2 = 0, \theta_3 = t, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0\} \) and \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0\} \) and then, by replacing \( D_1 \) by \(-tD_3\), we have \( \text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_5}(\mathfrak{L}_2^2)) \) is determined by the equations

\[
\begin{align*}
x_{2,1} + y_{2,1} &= 0, \\
y_{2,2} + y_{3,2} &= 0, \\
y_{3,1} - x_{3,1} &= 0, \\
x_{3,3} + y_{2,2} + y_{2,2} &= 0, \\
y_{3,3} + y_{2,2} + y_{2,2} &= 0, \\
x_{3,1} + y_{3,1} + y_{3,1} &= 0, \\
x_{3,3} + y_{3,3} + y_{3,3} &= 0, \\
x_{3,3} + y_{3,3} + y_{3,3} &= 0.
\end{align*}
\]

Therefore, since \( z \neq 0 \), \( x_{2,1} = y_{2,1}, x_{2,2} = -y_{3,3} + y_{2,2}, x_{3,1} = y_{3,1}, x_{3,2} = t \frac{y_{3,2}}{z} \) and \( x_{3,3} = t y_{2,2} - y_{2,2} \), and so:

\[
\begin{align*}
-y_{3,2} (t - z) &= 0, \\
(-y_{3,3} + y_{2,2}) (t - 1) &= 0
\end{align*}
\]

Since

\[
\text{Dim } T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_5}(\mathfrak{L}_2^2) = \left\{ \begin{array}{ll} 
4, & \text{if } t = 1 \text{ or } t = z; \\
3, & \text{in other case.}
\end{array} \right.
\]

7.3.2 \( \psi_{\alpha, \beta}(\mathfrak{L}_2^2(z)) \)

In this case, we have \( \psi_{\alpha, \beta}(\mathfrak{L}_2^2(z)) = t_{\alpha, \beta} \) is the Lie algebra \( \mathfrak{r}_{3,z} \).
7.3.3 \( \phi_\beta(L^3_5(z)) \)
In this case, we have \( \phi_\beta(L^3_5(z)) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.

7.3.4 \( \rho(L^3_5(z)) \)
In this case, we have \( \rho(L^3_5(z)) = \tilde{t} \) is the 3-dimensional abelian Lie algebra.

7.3.5 Identities
- \( A[,\cdot] = 0 \)
- \([A[,\cdot] + [,A] = 0\)
- \([-A, A] = 0\)

In particular, the hom-Lie algebra \( L^3_5(z) \) is a multiplicative hom-Lie algebra.

7.4 \( L^3_5(z) \)
Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
and let us denote by \( L^3_5(z) \) the hom-Lie algebra \( (L^3_5(z), A) \). We have:
\[
\text{Aut}(L^3_5(z)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x, y \right\}
\]
\[
\text{Der}(L^3_5(z)) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \\ t_2 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2.
\]

7.4.1 \( T_{\theta_1, \theta_2, \theta_3, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9} \)
Given \( D_1(u^s), D_2(x^s) \) and \( D_3(y^s) \) in \( \text{gl}(n, \mathbb{C})_A \), we have \( \lambda := D_4[,\cdot] + [D_2^*,[,\cdot] + [,D_3^*] \]
\[
\lambda := \begin{cases}
eq 1 \cdot e_1 = (-x_{2,1} + y_{2,1}) e_2 + (-x_{3,1} z + y_{3,1} z) e_3, \\
eq 1 \cdot e_2 = (x_{3,2} + x_{3,3} + w_{3,2} + w_{3,3} - w_{2,3} + y_{3,2} + y_{3,3} - y_{2,3}) e_2 + (w_{3,2} - y_{3,2} z) e_3, \\
eq 1 \cdot e_3 = (y_{2,3} + w_{2,3} z) e_2 + ((x_{2,2} + x_{2,3}) z + w_{2,3} z + y_{3,3} z) e_3, \\
eq 2 \cdot e_1 = (-y_{3,2} - y_{3,3} - w_{3,2} - w_{3,3} + w_{2,3} - x_{3,2} + x_{2,3}) e_2 + (-w_{3,2} - x_{3,2} z) e_3, \\
eq 2 \cdot e_2 = (y_{2,3} + w_{2,3} z) e_2 + ((x_{2,2} + x_{2,3}) z + w_{2,3} z + y_{3,3} z) e_3, \\
eq 2 \cdot e_3 = (y_{2,3} + w_{2,3} z) e_2 + ((x_{2,2} + x_{2,3}) z + w_{2,3} z + y_{3,3} z) e_3
\end{cases}
\]
By taking \( \{\theta_1 = 1, \theta_2 = 0, \theta_3 = t, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0\} \) and \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0\} \) and then, by replacing \( D_1 \) by \(-tD_3\), we have
$\text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(L_3^2))$ is determined by the equations

$$\begin{align*}
-x_{2,1} + y_{2,1} &= 0, \\
-x_{2,3} + t y_{2,3}z &= 0, \\
y_{2,3} - t y_{2,3}z &= 0, \\
y_{3,2} - y_{3,2}z &= 0, \\
t y_{3,2} - x_{3,2}z &= 0, \\
x_{3,2}z + x_{3,3}z - ty_{3,3}z + y_{3,3}z &= 0, \\
y_{3,2}z - y_{3,3}z + t y_{3,3}z - x_{3,3}z &= 0, \\
x_{3,2} + x_{3,3} - t y_{3,3}z + t y_{2,3} + y_{3,2} + y_{3,3} - y_{2,3} &= 0, \\
y_{3,2} - y_{3,3} + t y_{3,3} - t y_{2,3} - x_{3,2} - x_{3,3} + x_{2,3} &= 0.
\end{align*}$$

Therefore $x_{2,1} = y_{2,1}$, $x_{2,3} = t y_{2,3}z$, $x_{3,1} = y_{3,1}$, $x_{3,2} = \frac{t y_{3,2}z}{z}$ and $x_{3,3} = -y_{3,2} - y_{3,3} + t y_{3,3}$, and so:

$$\begin{align*}
(t (z - 1) (y_{2,3}z + y_{3,3}) &= 0, \\
-y_{2,3} (1 + tz) &= 0, \\
-y_{3,3} (t - z) &= 0.
\end{align*}$$

Since $z \neq \pm 1, 0$, we have $y_{2,3} = 0$ and $y_{3,2} = 0$, and

$$\text{Dim } T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(L_3^2) = 3, \forall t \in \mathbb{C}.$$
\[ \text{Aut}(\mathfrak{L}_4^4(z)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ 0 & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : \begin{array}{c} a \in \mathbb{C}^* \\ x \in \mathbb{C} \end{array} \right\} \]

\[ \text{Der}(\mathfrak{L}_4^4(z)) = \left\{ \begin{bmatrix} t_2 & t_1 & 0 \\ 0 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{t}_3(\mathbb{C}). \]

### 7.5.1 \( T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\mathfrak{L}_4^4) \)

Given \( D_1(w's), D_2(x's) \) and \( D_3(y's) \) in \( \mathfrak{gl}(n, \mathbb{C}) \), we have \( \lambda := D_1[\cdot, \cdot] + [D_2, \cdot] + [\cdot, D_3] \)

\[ \lambda := \begin{cases} 
  e_1, e_1 = (-x_{2,1} + y_{2,1}) e_2, \\
  e_1, e_2 = (x_{1,1} + w_{3,3} + y_{3,3}) e_2, \\
  e_1, e_3 = z w_{1,3} e_1 + (y_{2,3} + z w_{2,3}) e_2 + (x_{1,1} z + z w_{3,3} + y_{3,3} z) e_3, \\
  e_2, e_1 = (-y_{1,1} - w_{3,3} - x_{3,3}) e_2, \\
  e_2, e_3 = -y_{1,3} e_2, \\
  e_3, e_1 = -z w_{1,3} e_1 + (-x_{2,3} - z w_{2,3}) e_2 + (-y_{1,1} z - w_{3,3} - x_{3,3}) e_3, \\
  e_3, e_2 = x_{1,3} e_2, \\
  e_3, e_3 = (x_{1,3} z - y_{1,3} z) e_3 
\end{cases} \]

By taking \( \{ \theta_1 = 1, \theta_2 = 0, \theta_3 = t, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0 \} \) and \( \{ \beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0 \} \) and then, by replacing \( D_1 \) by \(-tD_3\), we have \( \text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\mathfrak{L}_4^4)) \) is determined by the equations

\[ \begin{cases} 
  x_{1,3} = 0, \\
  -y_{1,3} = 0, \\
  -x_{2,1} + y_{2,1} = 0, \\
  -x_{2,3} + ty_{2,3} z = 0, \\
  y_{2,3} - ty_{2,3} z = 0, \\
  -x_{1,3} z - y_{1,3} z = 0, \\
  x_{1,1} - ty_{3,3} + y_{3,3} = 0, \\
  -y_{1,1} + ty_{3,3} - x_{3,3} = 0. 
\end{cases} \]

Therefore, since \( z \neq 0, x_{1,1} = ty_{3,3} - y_{3,3}, x_{1,3} = 0, x_{2,1} = y_{2,1}, x_{2,3} = ty_{2,3} z, x_{3,3} = -y_{1,1} + ty_{3,3} \) and \( y_{1,3} = 0 \), and so:

\[ \{ y_{2,3} -t = 0 \] and consequently so

\[ \dim T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\mathfrak{L}_4^4) = \begin{cases} 
  4, & \text{if } t = 1/z; \\
  3, & \text{in other case}. 
\end{cases} \]

### 7.5.2 \( \psi_{\alpha, \beta}(\mathfrak{L}_4^4(z)) \)

In this case, we have \( \psi_{\alpha, \beta}(\mathfrak{L}_4^4(z)) = t_{\alpha, \beta} \)

\[ t_{\alpha, \beta} = \{ [e_1, e_2] = e_2, [e_1, e_3] = (\alpha + z) e_2 + z e_3 \}. \]

We note that \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{t}_{3,3} \).

### 7.5.3 \( \phi_{\beta}(\mathfrak{L}_4^4(z)) \)

In this case, we have \( \phi_{\beta}(\mathfrak{L}_4^4(z)) = \tilde{t}_{\beta} \)

\[ \tilde{t}_{\beta} = \{ [e_1, e_3] = (\beta + z) e_2 \} \]
We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where span_{\mathbb{C}} \{e_2, e_3\} is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
0 & \beta + z \\
0 & 0
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
   i. \( \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra if and only if \( \tilde{B} = 0 \) if and only if \( \beta = -z \)
   ii. \( \tilde{t}_\beta \) is isomorphic to \( n_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \beta \neq -z \)
   (b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( 0 = 1 \)

7.5.4 \( \rho(\mathcal{L}_3^4(z)) \)

In this case, we have \( \rho(\mathcal{L}_3^4(z)) = \tilde{t} \):

\[
\tilde{t} = \{[eI, e\beta] = e2.\}
\]

which is isomorphic to \( n_3 \).

7.5.5 Identities

- \( A[\cdot, \cdot] - z[A, \cdot] - z[\cdot, A] = 0 \) (Generalized derivation)
- \( [A, A] = 0 \)

The hom-Lie algebra \( \mathcal{L}_3^4(z) \) is a non-multiplicative hom-Lie algebra.

7.6 \( \mathcal{L}_3^5(z) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and let us denote by \( \mathcal{L}_3^5(z) \) the hom-Lie algebra \( (\mathcal{L}_3^5(z), A) \). We have:

\[
\text{gl}(n, \mathbb{C})_A = \left\{ \begin{bmatrix} t_1 & t_5 & 0 \\ 0 & t_2 & 0 \\ t_3 & t_4 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, \ldots, t_5 \in \mathbb{C} \right\}.
\]

\[
\text{Aut}(\mathcal{L}_3^5(z)) = \left\{ \begin{bmatrix} 0 & a & 0 \\ x & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^*, x \in \mathbb{C} \right\}.
\]

\[
\text{Der}(\mathcal{L}_3^5(z)) = \left\{ \begin{bmatrix} 0 & t_1 & 0 \\ t_2 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{v}_2(\mathbb{C}).
\]
7.6.1 \( T_{\theta_1,\ldots,\theta_6,\beta_1,\ldots,\beta_9}(L_5^0) \)

Given \( D_1(w's), D_2(x's) \) and \( D_3(y's) \) in \( \mathfrak{gl}(n, \mathbb{C})A \), we have \( \lambda := D_1[\cdot,\cdot] + [D_2\cdot,\cdot] + \cdots \).

\[ \begin{align*}
\lambda := & \begin{cases}
    e1 \cdot e1 = (-x_{1,1} + y_{1,1}) e\beta, \\
    e1 \cdot e2 = w_{1,1} e1 + (x_{1,1} + w_{3,3} + y_{3,3}) e2 + (w_{3,2} + y_{3,3}) e\beta, \\
    e1 \cdot e3 = (x_{1,1} + y_{3,3}) e\beta, \\
    e2 \cdot e1 = -w_{1,2} e1 + (-y_{1,1} - w_{3,3} - x_{3,3}) e2 + (-w_{3,2} - x_{3,3}) e\beta, \\
    e2 \cdot e2 = (x_{1,2} - y_{1,2}) e2, [e2, e\beta], x_{1,2} e\beta, \\
    e2 \cdot e3 = (-y_{1,1} z - w_{3,3} z - x_{3,3} z) e\beta, \\
    e3 \cdot e1 = (-y_{1,1} z - w_{3,3} z - x_{3,3} z) e\beta, \\
    e3 \cdot e2 = -y_{1,2} z e\beta.
\end{cases}
\end{align*} \]

By taking \( \{\theta_1 = 1, \theta_2 = 0, \theta_3 = t, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0\} \) and \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0\} \) and then, by replacing \( D_1 \) by \( -tD_1 \), we have \( \text{Ker}(T_{\theta_1,\ldots,\theta_6,\beta_1,\ldots,\beta_9}(L_5^0)) \) is determined by the equations

\[
\begin{aligned}
x_{1,2} &= 0, \\
y_{1,2} &= 0, \\
x_{1,2} - y_{1,2} &= 0, \\
ty_{3,2} + y_{3,2} z &= 0, \\
ty_{3,2} - x_{3,2} z &= 0, \\
x_{3,1} + y_{3,1} &= 0, \\
x_{3,1} - ty_{3,3} + y_{3,3} &= 0, \\
y_{1,1} + ty_{3,3} - x_{3,3} &= 0.
\end{aligned}
\]

Therefore \( x_{1,1} = ty_{3,3} - y_{3,3}, x_{1,2} = 0, x_{3,1} = y_{3,1}, x_{3,2} = \frac{ty_{3,2}}{z}, x_{3,3} = -y_{1,1} + ty_{3,3} \) and \( y_{1,2} = 0 \), and so:

\[
\left\{ \begin{array}{l}
y_{3,2} (t - z) = 0 \\
\end{array} \right. \]

and consequently so

\[
\text{Dim } T_{\theta_1,\ldots,\theta_6,\beta_1,\ldots,\beta_9}(L_5^0) = \left\{ \begin{array}{ll}
4, & \text{if } t = z; \\
3, & \text{in other case}.
\end{array} \right.
\]

7.6.2 \( \psi_{\alpha,\beta}(L_5^0(z)) \)

In this case, we have \( \psi_{\alpha,\beta}(L_5^0(z)) = t_{\alpha,\beta} \):

\[
t_{\alpha,\beta} = \{[e1, e2] = e2 + (\alpha + \beta z) e\beta, [e1, e\beta] = z e\beta.\}
\]

We note that \( t_{\alpha,\beta} \) is isomorphic to \( t_{3,2} \).

7.6.3 \( \phi_{\beta}(L_5^0(z)) \)

In this case, we have \( \phi_{\beta}(L_5^0(z)) = \tilde{t}_\beta \):

\[
\tilde{t}_\beta = \{[e1, e2] = (1 + \beta z) e\beta.\}
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
0 & 0 \\
1 + \beta z & 0
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   
   (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
\[ \tilde{t}_\beta \text{ is the 3-dimensional abelian Lie algebra if and only if } \tilde{B} = 0 \text{ if and only if } \{ \beta = -\frac{1}{z} \} \]

ii. \( t_\beta \) is isomorphic to \( \mathfrak{n}_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \{ \beta \neq -\frac{1}{z} \} \)

(b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( 0 = 1 \)

7.6.4 \( \rho(\Sigma_5^6(z)) \)

In this case, we have \( \rho(\Sigma_5^6(z)) = \tilde{t} \):

\[ \tilde{t} = \{ [e1, e2] = ze3 \} \]

which is isomorphic to \( \mathfrak{n}_3 \).

7.6.5 Identities

- \( A[\cdot, \cdot] - \frac{1}{2}[A\cdot, \cdot] - \frac{1}{2}[, A\cdot] = 0 \) (Generalized derivation)
- \( [A\cdot, A\cdot] \)

The hom-Lie algebra \( \Sigma_6^6(z) \) is a non-multiplicative hom-Lie algebra.

7.7 \( \Sigma_6^6(z, \lambda) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{ e_1, e_2, e_3 \} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}
\]

and let us denote by \( \Sigma_6^6(z, \lambda) \) the hom-Lie algebra \( (\Sigma_6^6(z), A) \). We have:

\[ \text{Aut}(\Sigma_6^6(z, \lambda)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ -x & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C}, a \in \mathbb{C}^* \right\} \]

\[ \text{Der}(\Sigma_6^6(z, \lambda)) = \left\{ \begin{bmatrix} t_2 & t_1 & 0 \\ -t_2 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong t_2(\mathbb{C}) \times \mathbb{C}. \]

7.7.1 \( \psi_{\alpha, \beta}(\Sigma_6^6(z)) \)

In this case, we have \( \psi_{\alpha, \beta}(\Sigma_6^6(z)) = t_{\alpha, \beta} \):

\[ t_{\alpha, \beta} = \{ [e1, e2] = (1 + \alpha \lambda + \beta \lambda) e2 + (-\alpha \lambda - \beta \lambda z) e3, [e1, e3] = (\beta \lambda + \alpha z \lambda) e2 + (z - \alpha z \lambda - \beta \lambda z) e3 \}. \]

We note that \( t_{\alpha, \beta} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha, \beta} \); we can identify \( t_{\alpha, \beta} \) the matrix

\[ B = \begin{bmatrix}
1 + \alpha \lambda + \beta \lambda & \beta \lambda + \alpha z \lambda \\
-\alpha \lambda - \beta \lambda z & \lambda - \alpha z \lambda - \beta \lambda z
\end{bmatrix} \]

The isomorphism class of \( t_{\alpha, \beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \((-1 + z) (\alpha - \beta)^2 \lambda^2 - 2 (z + 1) (\beta + \alpha) \lambda - 1 + z = 0 \)
(a) 0 is a repeated eigenvalue of $B$ if and only if \[ \alpha = -\frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda}, \beta = \frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda} \].

i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $z = 0$ (False)

ii. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{n}_3(\mathbb{C})$ if and only if $B \neq 0$ if and only if \[ \{ \alpha = -\frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda}, \beta = \frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda} \} \] for some $t \in \mathbb{C} \setminus \{0\}$.

b) $B$ has a non-zero repeated eigenvalue if and only if \[ \alpha = -\frac{t_2-t+1+\sqrt{t_2-1+(1+3z)(1+z)^2}}{2(1+z)\lambda}, \beta = \frac{t_1+t+\sqrt{t_2-1+(1+3z)(1+z)^2}}{2(1+z)\lambda} \] for some $t \in \mathbb{C} \setminus \{0\}$.

i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = s \text{Id}$ for some $s \neq 0$ if and only if $t = 1$

ii. $t_{\alpha,\beta}$ is isomorphic to $t_3$ if and only if $B \neq s \text{Id}$ for all $s \in \mathbb{C}$ if and only if \[ \{ \alpha = -\frac{t_2-t+1+\sqrt{t_2-1+(1+3z)(1+z)^2}}{2(1+z)\lambda}, \beta = \frac{t_1+t+\sqrt{t_2-1+(1+3z)(1+z)^2}}{2(1+z)\lambda} \} \] for some $t \in \mathbb{C} \setminus \{0\}$.

2. $B$ has distinct eigenvalues if and only if $(−1 + z)(\alpha - \beta)^2 \lambda^2 - 2(z + 1)(\beta + \alpha)\lambda - 1 + z \neq 0$

(a) $t_{\alpha,\beta}$ is isomorphic to $t_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\alpha = -\frac{z}{(1+z)^2\lambda^2}$ and $\beta \neq \frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda} \neq 0$

(b) $t_{\alpha,\beta}$ is isomorphic to $t_{3,-1}$ if and only if Trace$(B) = 0$ if and only if $\alpha = -\frac{1-\beta-\lambda z}{\lambda(1+z)}$ and $\beta \neq \frac{z-1+\sqrt{z^2+6z+1}}{2(1+z)\lambda}$

(c) $t_{\alpha,\beta}$ is isomorphic to $t_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $(−1 + z)(\alpha - \beta)^2 \lambda^2 - 2(z + 1)(\beta + \alpha)\lambda - 1 + z \neq 0$ and $1 + \alpha + \beta + z - \alpha \lambda - \beta \lambda z \neq 0$ and $z + \beta \lambda^2 \alpha - 2 \alpha \lambda^2 \beta + z \neq 0$

7.7.2 $\phi_\beta(\mathcal{L}_3^\beta(z))$

In this case, we have $\phi_\beta(\mathcal{L}_3^\beta(z)) = \widetilde{t}_\beta$:

$\widetilde{t}_\beta = \{[e_1, e_2] = (\lambda + \beta \lambda) e_2 + (−\lambda - \beta \lambda z) e_3, [e_1, e_3] = (\beta \lambda + z\lambda) e_2 + (−z\lambda - \beta \lambda z) e_3\}$.

We note that $\widetilde{t}_\beta$ is an almost abelian Lie algebra where span$_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\widetilde{t}_\beta$; we can identify $\widetilde{t}_\beta$ the matrix

$$\widetilde{B} = \begin{bmatrix} \lambda + \beta \lambda & \beta \lambda + z\lambda \\ -\lambda - \beta \lambda z & -z\lambda - \beta \lambda z \end{bmatrix}.$$ 

The isomorphism class of $\widetilde{t}_\beta$ is determined by the eigenvalues of $\widetilde{B}$:

1. $\widetilde{B}$ has repeated eigenvalues if and only if $\beta = 1$

   (a) 0 is a repeated eigenvalue of $\widetilde{B}$ if and only if $\lambda = 0$ (False)

   (b) $\widetilde{B}$ has a non-zero repeated eigenvalue if and only if $\beta = 1$

      i. $\widetilde{t}_\beta$ is $t_{3,1}$ if and only if $\widetilde{B} = t \text{Id}$ for some $t \neq 0$ if and only if $z = -1$ (False)

      ii. $\widetilde{t}_\beta$ is isomorphic to $t_3$ if and only if $\widetilde{B} \neq t \text{Id}$ for all $t \in \mathbb{C}$ if and only if $\beta = 1$

2. $\widetilde{B}$ has distinct eigenvalues if and only if $\beta \neq 1$

   (a) $\widetilde{t}_\beta$ is isomorphic to $t_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $\widetilde{B}$ with algebraic multiplicity 1 if and only if $\beta = 0$

   (b) $\widetilde{t}_\beta$ is isomorphic to $t_{3,-1}$ if and only if Trace$(\widetilde{B}) = 0$ if and only if $\beta = -1$.

   (c) $\widetilde{t}_\beta$ is isomorphic to $t_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\beta \neq 1$ and $\beta \neq 0$ and $\beta \neq -1$.  

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7.7.3 $\rho(L^6_5(z, \lambda))$

In this case, we have $\rho(L^6_5(z, \lambda)) = \hat{t}$:

$$\hat{t} = \{[e_1, e_2] = \lambda e_2 - \lambda ze_3, [e_1, e_3] = \lambda e_2 - \lambda ze_3\}.$$ 

which is isomorphic to $r_2 \times \mathbb{C}$.

7.7.4 Identities

- $[A, A] = 0$

The hom-Lie algebra $L^6_5(z, \lambda)$ is a non-multiplicative hom-Lie algebra.

7.8 $L^7_5(z)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
$$

and let us denote by $L^7_5(z)$ the hom-Lie algebra $(L^5_5(z), A)$. We have:

$$\text{Aut}(L^7_5(z)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.$$

$$\text{Der}(L^7_5(z)) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_1 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\}.$$

7.8.1 $\psi_{\alpha, \beta}(L^7_5(z))$

In this case, we have $\psi_{\alpha, \beta}(L^7_5(z)) = t_{\alpha, \beta}$:

$$t_{\alpha, \beta} = \{[e_1, e_2] = e_2 + (\alpha + \beta z) e_3, [e_1, e_3] = ze_3\}.$$

We note that $t_{\alpha, \beta}$ is isomorphic to $r_3 \times \mathbb{C}$.

7.8.2 $\phi_{\beta}(L^7_5(z))$

In this case, we have $\phi_{\beta}(L^7_5(z)) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e_1, e_2] = (1 + \beta z) e_3\}.$$

We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_{\mathbb{C}}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_\beta$; we can identify $\tilde{t}_\beta$ the matrix

$$\tilde{B} = \begin{bmatrix}
0 & 0 \\
1 + \beta z & 0 \\
\end{bmatrix}.$$

The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $0 = 0$

   (a) $0$ is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$
i. $\tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\{\beta = -\frac{1}{z}\}$

ii. $t_\beta$ is isomorphic to $n_3(\mathbb{C})$ if and only if $\tilde{B} \neq 0$ if and only if $\{\beta \neq -\frac{1}{z}\}$

(b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues if and only if $0 = 1$

7.8.3 $\rho(\mathcal{L}_7^2(z))$

In this case, we have $\rho(\mathcal{L}_7^2(z)) = \tilde{t}$:

$$\tilde{t} = \{[e_1, e_2] = ze_3\}.$$

which is isomorphic to $n_3$

7.8.4 Identities

• $A[\cdot, \cdot] - \frac{1}{z}[A, \cdot] - \frac{1}{z}[A, \cdot] = 0$ (Generalized derivation)

• $A^2[\cdot, \cdot] = 0$

• $[A, A] = 0$

The hom-Lie algebra $\mathcal{L}_7^2(z)$ is a non-multiplicative hom-Lie algebra.

7.9 $\mathcal{L}_8^5(z)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}$$

and let us denote by $\mathcal{L}_8^5(z)$ the hom-Lie algebra $(\mathcal{L}(z), A)$. We have:

$$\text{Aut}(\mathcal{L}_8^5(z)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\},$$

$$\text{Der}(\mathcal{L}_8^5(z)) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.$$

7.9.1 $\psi_{\alpha,\beta}(\mathcal{L}_8^5(z))$

In this case, we have $\psi_{\alpha,\beta}(\mathcal{L}_8^5(z)) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e_1, e_2] = e_2, [e_1, e_3] = (\beta + \alpha z) e_2 + ze_3\}.$$

We note that $t_{\alpha,\beta}$ is isomorphic to $t_{3,z}$. 58
7.9.2 \( \phi_\beta(L^5_8(z)) \)

In this case, we have \( \phi_\beta(L^5_8(z)) = \tilde{t}_\beta \):
\[
\tilde{t}_\beta = \{[e1, e3] = (\beta + z)e2. \}
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix
\[
\tilde{B} = \begin{bmatrix}
0 & \beta + z \\
0 & 0
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   - (a) \( 0 \) is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
     - i. \( \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra if and only if \( \tilde{B} = 0 \) if and only if \( \beta = -z \)
     - ii. \( \tilde{t}_\beta \) is isomorphic to \( n_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \beta \neq -z \)
   - (b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( 0 = 1 \)

7.9.3 \( \rho(L^5_8(z)) \)

In this case, we have \( \rho(L^5_8(z)) = \hat{t} \):
\[
\hat{t} = \{[e1, e3] = e2. \}
\]

which is isomorphic to \( n_3 \).

7.9.4 Identities

- \( A[\cdot, \cdot] - z[A\cdot, \cdot] - \cdot[A, \cdot] = 0 \)
- \( [A\cdot, A\cdot] = 0 \)
- \( A^2[\cdot, \cdot] = 0 \)
  \[ \rightarrow [A^2, \cdot] + [\cdot, A^2] \]

The hom-Lie algebra \( L^5_8(z) \) is a non-multiplicative hom-Lie algebra.

7.10 \( L^9_5(z, \lambda) \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is
\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}
\]

and let us denote by \( L^9_5(z, \lambda) \) the hom-Lie algebra \( (L_5(z), A) \). We have:

\[
\text{Aut}(L^9_5(z, \lambda)) = \left\{ \begin{bmatrix}
x & 1 & 0 \\
x & 0 & 1 \\
-x & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.
\]

\[
\text{Der}(L^9_5(z, \lambda)) = \left\{ \begin{bmatrix}
t_1 & 0 & 0 \\
-t_1 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.
\]
7.10.1 $\psi_{\alpha,\beta}(L^0_5(z))$

In this case, we have $\psi_{\alpha,\beta}(L^0_5(z)) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{ [e_1, e_2] = (1 + \alpha \lambda + \beta \lambda) e_2 + (-\alpha \lambda - \beta \lambda z) e_3, [e_1, e_3] = (\beta \lambda + \alpha z \lambda) e_2 + (z - \alpha z \lambda - \beta \lambda z) e_3 \}.$$

We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ as the matrix

$$B = \begin{bmatrix}
1 + \alpha \lambda + \beta \lambda & \lambda(\beta + \alpha z) \\
-\lambda(\alpha + z \lambda) & z(1 - \alpha \lambda - \beta \lambda)
\end{bmatrix}.$$

The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $(-1 + z)(\alpha - \beta)^2 \lambda^2 - 2(z + 1)(\beta + \alpha)\lambda - 1 + z = 0$
   (a) 0 is a repeated eigenvalue of $B$ if and only if $\alpha = -\frac{1 + \sqrt{7 + 6z + 2z^2}}{2(-1 + z)\lambda}, \beta = \frac{1 + \sqrt{7 + 6z + 2z^2}}{2(-1 + z)\lambda}$
   i. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{n}_3(C)$ if and only if $B = 0$ if and only if $z = 0$
      (False)
   ii. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{t}_3(1)$ if and only if $B \neq 0$ if and only if
   (b) $B$ has a non-zero repeated eigenvalue if and only if $\alpha = -\frac{1 + \sqrt{7 + 6z + 2z^2}}{2(-1 + z)\lambda}, \beta = \frac{1 + \sqrt{7 + 6z + 2z^2}}{2(-1 + z)\lambda}$ for any $s \in C^*$.
   i. $t_{\alpha,\beta}$ is $\mathfrak{r}_3$ if and only if $B = s \text{Id}$ for some $s \neq 0$ if and only if $1 = 0$
   ii. $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{r}_2$ if and only if $B \neq t \text{Id}$ for some $t$$ \in C$ if and only if
   (c) $\alpha = -\frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}, \beta = \frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}$ for any $s \in C^*$.

2. $B$ has distinct eigenvalues eigenvalues if and only if $(-1 + z)(\alpha - \beta)^2 \lambda^2 - 2(z + 1)(\beta + \alpha)\lambda - 1 + z \neq 0$
   (a) $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{r}_2(C) \times C$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if
   $$\alpha = -\frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}, \beta = \frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}$$ for any $s \in C^*$.
   (b) $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{r}_{3,-1}$ if and only if $\text{Trace}(B) = 0$ if and only if
   $$\alpha = -\frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}, \beta = \frac{1 + \sqrt{7 + 6z + 2z^2}}{2s \lambda(-1 + z)}$$
   (c) $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{r}_{3,z}$ for some $z$ such that $z(z^2 - 1) \neq 0$ if and only if
   $$\alpha = \frac{2s_1s_2 - s_1 + s_1s_2 \sqrt{s_1s_2(s_1^2s_2^2 - 2s_1s_2s_2 - 2s_1s_2 + s_2^2)}}{2s_1s_2 \lambda(-1 + z)}$$ and
   $$\beta = \frac{2s_1s_2 - s_1 + s_1s_2 + s_2 \sqrt{s_1s_2(s_1^2s_2^2 - 2s_1s_2s_2 - 2s_1s_2 + s_2^2)}}{2s_1s_2 \lambda(-1 + z)}$$ for any $s_1, s_2 \in C^*$ with $s_1 - s_2^2 \neq 0$.

7.10.2 $\phi_{\beta}(L^0_5(z))$

In this case, we have $\phi_{\beta}(L^0_5(z)) = \tilde{t}_{\beta}$:

$$\tilde{t}_{\beta} = \{ [e_1, e_2] = (\lambda + \beta \lambda) e_2 + (-\lambda - \beta \lambda z) e_3, [e_1, e_3] = (\beta \lambda + \alpha z \lambda) e_2 + (z - \alpha z \lambda - \beta \lambda z) e_3 \}.$$
We note that \( \tilde{\mathfrak{t}}_\beta \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C}\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{\mathfrak{t}}_\beta \); we can identify \( \tilde{\mathfrak{t}}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
\lambda + \beta \lambda & \beta \lambda + z \lambda \\
-\lambda - \beta \lambda z & -z \lambda - \beta \lambda z
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{\mathfrak{t}}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)
   
   (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if \( \lambda = 0 \) (False)
   
   (b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( \beta = 1 \)
   
   i. \( \tilde{t}_\beta \) is \( \mathfrak{r}_3, 1 \) if and only if \( \tilde{B} = t \text{Id} \) for some \( t \neq 0 \) if and only if \( z = -1 \) (False)
   
   ii. \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{r}_3 \) if and only if \( \tilde{B} \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( \beta = 1 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( \beta \neq 1 \)
   
   (a) \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \beta = 0 \)
   
   (b) \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{r}_3, -1 \) if and only if \( \text{Trace}(\tilde{B}) = 0 \) if and only if \( \beta = -1 \).
   
   (c) \( \tilde{t}_\beta \) is isomorphic to \( \mathfrak{r}_3, z \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( \beta \neq 1 \) and \( \beta \neq 0 \) and \( \beta \neq -1 \).

### 7.10.3 \( \rho(\mathcal{L}_5^0(z, \lambda)) \)

In this case, we have \( \rho(\mathcal{L}_5^0(z, \lambda)) = \tilde{t} \):

\[
\tilde{t} = \{\{e_1, e_2\} = \lambda e_2 - \lambda ze_3, \{e_1, e_3\} = \lambda e_2 - \lambda ze_3\}.
\]

which is isomorphic to \( \mathfrak{r}_2 \times \mathbb{C} \).

### 7.10.4 Identities

- \([A, A] = 0\)
- \(A^2[,] = 0\)
- \([A^2, ,] + [, A^2] = 0\)

The hom-Lie algebra \( \mathcal{L}_5^0(z, \lambda) \) is a non-multiplicative hom-Lie algebra.

### 7.11 Degenerations between hom-Lie algebras (\( \mathcal{L}_5^0 \))

If \( \mathcal{L}_i \xrightarrow{\text{deg}} \mathcal{L}_k^0 \) then \( \text{Der}(\mathcal{L}_i) \leq \text{Der}(\mathcal{L}_k^0) \). Therefore, we can organize the hom-Lie algebras in the following way:

\[
\begin{array}{c|c}
\text{Dim}(\text{Der}) & \text{hom-Lie algebra} \\
\hline
1 & \mathcal{L}_5^0(\lambda), \mathcal{L}_3^0, \mathcal{L}_4^0 \\
2 & \mathcal{L}_5^0(\lambda), \mathcal{L}_3^0, \mathcal{L}_4^0, \mathcal{L}_2^0 \\
3 & \mathcal{L}_5^0, \mathcal{L}_3^0, \mathcal{L}_4^0, \mathcal{L}_2^0, \mathcal{L}_1^0 \\
4 & \mathcal{L}_5^0, \mathcal{L}_3^0, \mathcal{L}_4^0, \mathcal{L}_2^0, \mathcal{L}_1^0
\end{array}
\]
7.11.1 Degenerations

1. $\mathfrak{L}_5^0(z, \lambda) \xrightarrow{\text{deg}} \mathfrak{L}_6^0(z, \lambda)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(\mathfrak{L}_5(z))$ and $g(t) \cdot \mathfrak{L}_5^0(z)$ is the hom-Lie algebra $(\mathfrak{L}_5(z), A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ -e^{-t}(-1 + \lambda) & \lambda & \lambda \\ e^{-t}\lambda & -\lambda & -\lambda \end{bmatrix}$$

It is easy to check that $(\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_6^0(z, \lambda)$ as $t$ tends to infinity.

2. $\mathfrak{L}_5^8(z) \xrightarrow{\text{deg}} \mathfrak{L}_4^8(z)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(\mathfrak{L}_5(z))$ and $g(t) \cdot \mathfrak{L}_5^8(z)$ is the hom-Lie algebra $(\mathfrak{L}_5(z), A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ -e^{-t} & 0 & 1 \\ e^{-t} & 0 & 0 \end{bmatrix}$$

It is easy to check that $(\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_4^8(z)$ as $t$ tends to infinity.

3. $\mathfrak{L}_5^8(z) \xrightarrow{\text{deg}} \mathfrak{L}_3^8(z)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -e^{-t} & 0 \\ e^t & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \in \text{Aut}(\mathfrak{L}_5(z))$ and $g(t) \cdot \mathfrak{L}_5^8(z)$ is the hom-Lie algebra $(\mathfrak{L}_5(z), A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -e^{-t} \\ 1 & 0 & 0 \end{bmatrix}$$

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It is easy to check that \((\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_3^5(z)\) as \(t\) tends to infinity.

4. \(\mathfrak{L}_7^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^5(z)\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & e^{-t} & 0 \\
    0 & 0 & e^{-t}
  \end{bmatrix}, \text{ with } t \in \mathbb{R}.
  \]

   We have \(g(t) \in \text{Aut}(\mathfrak{L}_5(z))\) and \(g(t) \cdot \mathfrak{L}_7^5(z)\) is the hom-Lie algebra \((\mathfrak{L}_5, A(t))\) with
   
   \[
   A(t) = \begin{bmatrix}
    0 & 0 & 0 \\
    e^{-t} & 0 & 0 \\
    0 & 1 & 0
  \end{bmatrix}.
  \]

   It is easy to check that \((\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_3^5(z)\) as \(t\) tends to infinity.

5. \(\mathfrak{L}_5^7(z) \xrightarrow{\text{deg}} \mathfrak{L}_7^5(z)\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
    1 & 0 & 0 \\
    -e^t & 1 & 0 \\
    0 & 0 & e^{-t}
  \end{bmatrix}, \text{ with } t \in \mathbb{R}.
  \]

   We have \(g(t) \in \text{Aut}(\mathfrak{L}_5(z))\) and \(g(t) \cdot \mathfrak{L}_5^7(z)\) is the hom-Lie algebra \((\mathfrak{L}_5, A(t))\) with
   
   \[
   A(t) = \begin{bmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    1 & e^{-t} & 0
  \end{bmatrix}.
  \]

   It is easy to check that \((\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_3^7(z)\) as \(t\) tends to infinity.

6. \(\mathfrak{L}_4^7(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^7(z)\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
    1 & 0 & 0 \\
    e^t & -e^t & 0 \\
    0 & 0 & 1
  \end{bmatrix}, \text{ with } t \in \mathbb{R}.
  \]

   We have \(g(t) \in \text{Aut}(\mathfrak{L}_5(z))\) and \(g(t) \cdot \mathfrak{L}_4^7(z)\) is the hom-Lie algebra \((\mathfrak{L}_5, A(t))\) with
   
   \[
   A(t) = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    1 & -e^{-t} & 0
  \end{bmatrix}.
  \]

   It is easy to check that \((\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_2^4(z)\) as \(t\) tends to infinity.

7. \(\mathfrak{L}_5^4(z) \xrightarrow{\text{deg}} \mathfrak{L}_1^4(z)\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    e^t & 0 & -e^t
  \end{bmatrix}, \text{ with } t \in \mathbb{R}.
  \]

   It is easy to check that \((\mathfrak{L}_5(z), A(t)) \rightarrow \mathfrak{L}_2^4(z)\) as \(t\) tends to infinity.
We have $g(t) \in \text{Aut}(L_5)$ and $g(t) \cdot L_3^4(z)$ is the hom-Lie algebra $(L_5, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -e^{-t} \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_5, A(t)) \to L_1^5(z)$ as $t$ tends to infinity.

8. $L_3^4(z) \xrightarrow{\text{deg}} L_2^5(z)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \in \text{Aut}(L_5)$ and $g(t) \cdot L_3^4(z)$ is the hom-Lie algebra $(L_5, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_5, A(t)) \to L_2^5(z)$ as $t$ tends to infinity.

9. $L_3^4(z) \xrightarrow{\text{deg}} L_1^5(z)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \in \text{Aut}(L_5)$ and $g(t) \cdot L_3^4(z)$ is the hom-Lie algebra $(L_5, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ e^{-t} & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_5, A(t)) \to L_1^5(z)$ as $t$ tends to infinity.

10. $L_3^4(z) \xrightarrow{\text{deg}} L_0^5(z)$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \in \text{Aut}(L_5)$ and $g(t) \cdot L_3^4(z)$ is the hom-Lie algebra $(L_5, A(t))$ with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-t} & 0 & 0 \end{bmatrix}$$

It is easy to check that $(L_5, A(t)) \to L_0^5(z)$ as $t$ tends to infinity.
11. $L_1^0(z) \xrightarrow{\deg} L_0^0(z)$

In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have $g(t) \in \text{Aut}(L_5)$ and $g(t) \cdot L_1^0(z)$ is the hom-Lie algebra $(L_5, A(t))$ with
\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
e^{-t} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

It is easy to check that $(L_5, A(t)) \to L_0^0(z)$ as $t$ tends to infinity.

12. $L_1^0(z, \lambda) \xrightarrow{\deg} L_0^0(z, \kappa)$ with $(\lambda \neq \kappa)$.

Suppose, contrary to our claim, that $L_1^0(z, \lambda) \xrightarrow{\deg} L_0^0(z, \kappa)$ with $(\lambda \neq \kappa)$. In such case, we have
\[
\psi_{\alpha, \beta}(L_1^0(z, \lambda)) \xrightarrow{\deg} \psi_{\alpha, \beta}(L_0^0(z, \kappa)).
\]

By taking $\alpha = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \lambda$ and $\beta = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \lambda$, we have from 7.10.1 that $\psi_{\alpha, \beta}(L_1^0(z, \lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L_1^0(z, \lambda))$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha, \beta}(L_0^0(z, \kappa))$ is isomorphic to $n_3$ (from 7.7.1). But, we now apply again 7.7.1, to obtain $\alpha = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \kappa$ and $\beta = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \kappa$, and therefore $\lambda = \kappa$; this is a contradiction.

13. $L_1^0(z, \lambda) \xrightarrow{\deg} L_0^0(z)$. 

Suppose, contrary to our claim, that $L_1^0(z, \lambda) \xrightarrow{\deg} L_0^0(z)$. In such case, we have
\[
\psi_{\alpha, \beta}(L_1^0(z, \lambda)) \xrightarrow{\deg} \psi_{\alpha, \beta}(L_0^0(z)),
\]

By taking $\alpha = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \lambda$ and $\beta = (z + 1 + \sqrt{z^2 + 6 z + 1})/2 (-1 + z) \lambda$, we have from 7.10.1 that $\psi_{\alpha, \beta}(L_1^0(z, \lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L_1^0(z, \lambda))$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). But, from 7.1.1, we have $\psi_{\alpha, \beta}(L_1^0(z, \lambda))$ is isomorphic to $t_4, z$; this is contradiction.

14. $L_1^0(z) \xrightarrow{\deg} L_0^0(z, \lambda)$ with $(\lambda \neq 0)$.

Suppose, contrary to our claim, that $L_1^0(z) \xrightarrow{\deg} L_0^0(z, \lambda)$ with $(\lambda \neq 0)$. In such case, we have
\[
\phi_{\beta}(L_1^0(z)) \xrightarrow{\deg} \phi_{\beta}(L_0^0(z, \lambda)),
\]

By taking $\beta = -z$, we have from 7.9.2 that $\phi_{\beta}(L_1^0(z))$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_{\beta}(L_1^0(z, \lambda))$ is a Lie algebra which is isomorphic to $a_3$ (by Theorem 1.1), but this contradicts to 7.7.2.

15. $L_1^0(z) \xrightarrow{\deg} L_0^0(z)$.

Suppose, contrary to our claim, that $L_1^0(z) \xrightarrow{\deg} L_0^0(z)$. In such case, we have
\[
\phi_{\beta}(L_1^0(z)) \xrightarrow{\deg} \phi_{\beta}(L_0^0(z)),
\]

By taking $\beta = -z$, we have from 7.9.2 that $\phi_{\beta}(L_1^0(z))$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_{\beta}(L_0^0(z))$ is a Lie algebra which is isomorphic to $a_3$ (by Theorem 1.1). From 7.6.3, we have $\beta = -1/z$ and so $z^2 = 1$; this is a contradiction.
16. $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^6(z, \lambda)$ with $\lambda \neq 0$.
   Suppose, contrary to our claim, that $\mathfrak{L}_5^7(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^6(z, \lambda)$ with $\lambda \neq 0$). In such case, we have
   $$\phi_\beta(\mathfrak{L}_5^5(z)) \xrightarrow{\text{deg}} \phi_\beta(\mathfrak{L}_5^6(z, \lambda)).$$
   By taking $\beta = -1/z$, we have from 7.8.2 that $\phi_\beta(\mathfrak{L}_5^5(z))$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_\beta(\mathfrak{L}_5^6(z, \lambda))$ is a Lie algebra which is isomorphic to $\mathfrak{a}_3$ (by Theorem 1.1), but this contradicts to 7.7.2.

17. $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^2(z)$.
   Suppose, contrary to our claim, that $\mathfrak{L}_5^7(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^4(z)$. In such case, we have
   $$\phi_\beta(\mathfrak{L}_5^5(z)) \xrightarrow{\text{deg}} \phi_\beta(\mathfrak{L}_5^4(z)).$$
   By taking $\beta = -1/z$, we have from 7.9.2 that $\phi_\beta(\mathfrak{L}_5^5(z))$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_\beta(\mathfrak{L}_5^4(z))$ is a Lie algebra which is isomorphic to $\mathfrak{a}_3$ (by Theorem 1.1). From 7.5.3, we have $\beta = -z$ and so $z^2 = 1$; this is a contradiction.

18. $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^1(z)$.
   Suppose, contrary to our claim, that $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^2(z)$. In such case, for $t = z$, we have
   $$\dim \ker(T_{1,0,0,0,0,0,1,-1,0,1,-1,0,1,-1,0,1}) = 4 \leq \dim \ker(T_{1,0,0,0,0,0,1,-1,0,1,-1,0,1,-1,0,1})$$
   From 7.2.1, we have $\dim \ker(T_{1,0,0,0,0,0,1,-1,0,1,-1,0,1,-1,0,1}) = 4$, and therefore, $z = 1$ or $z = 1/z$. Since $z \neq \pm 1$; we have a contradiction.

19. $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^2(z)$.
   Suppose, contrary to our claim, that $\mathfrak{L}_5^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_5^2(z)$. In such case, for $t = 1/z$, we have
   $$\dim(\ker(T_{1,0,1,z,0,0,0,1,-1,0,1,-1,0,1,-1,0,1})) = 4 \leq \dim(\ker(T_{1,0,1/z,0,0,0,1,-1,0,1,-1,0,1,-1,0,1}))$$
   From 7.3.1, we have $\dim \ker(T_{1,0,1/z,0,0,0,1,-1,0,1,-1,0,1,-1,0,1}) = 4$, and therefore, $z = 1$ or $z = 1/z$. Since $z \neq \pm 1$; we have a contradiction.
8 \( \mathcal{L}_6 \): \( r_2(\mathbb{C}) \times \mathbb{C} \)

\[ \mathcal{L}_6 := \{ [e_1, e_2] = e_2 \} \]

\( \text{Aut}(\mathcal{L}_6) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & a & 0 \\ y & 0 & b \end{bmatrix} \in M(3, \mathbb{C}) : a, b \in \mathbb{C}^* \right\} \).

\( \text{Der}(\mathcal{L}_6) = \left\{ \begin{bmatrix} t_3 & t_1 & 0 \\ t_4 & 0 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3, t_4 \in \mathbb{C} \right\} \cong r_2(\mathbb{C}) \times r_2(\mathbb{C}) \).

8.1 \( \mathcal{L}_6^0 \)

Let \( A \) be the zero map and let us denote by \( \mathcal{L}_6^0 \) the hom-Lie algebra \((\mathcal{L}_6, A)\). We have:

\( \text{Aut}(\mathcal{L}_6^0) = \text{Aut}(\mathcal{L}_6) \).

\( \text{Der}(\mathcal{L}_6^0) = \text{Der}(\mathcal{L}_6) \).

8.1.1 \( \psi_{\alpha,\beta}(\mathcal{L}_6^0) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathcal{L}_6^0) = t_{\alpha,\beta} \) is the Lie algebra \( r_2(\mathbb{C}) \times \mathbb{C} \).

8.1.2 \( \phi_{\beta}(\mathcal{L}_6^0) \)

In this case, we have \( \phi_{\beta}(\mathcal{L}_6^0) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.

8.1.3 \( \rho(\mathcal{L}_6^0) \)

In this case, we have \( \rho(\mathcal{L}_6^0) = \hat{t} \) is the 3-dimensional abelian Lie algebra.

8.2 \( \mathcal{L}_6^1 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{ e_1, e_2, e_3 \} \) is

\[ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and let us denote by \( \mathcal{L}_6^1 \) the hom-Lie algebra \((\mathcal{L}_6, A)\). We have:

\( \mathfrak{gl}(n, \mathbb{C})_A = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ t_3 & t_1 & t_5 \\ t_4 & 0 & t_2 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_5 \in \mathbb{C} \right\} \).

\( \text{Aut}(\mathcal{L}_6^1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\} \).

\( \text{Der}(\mathcal{L}_6^1) = \left\{ \begin{bmatrix} t_2 & 0 & 0 \\ t_3 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong r_2(\mathbb{C}) \times \mathbb{C} \).

8.2.1 \( \psi_{\alpha,\beta}(\mathcal{L}_6^1) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathcal{L}_6^1) = t_{\alpha,\beta} \) is the Lie algebra \( r_2(\mathbb{C}) \times \mathbb{C} \).
8.2.2 \( \phi_\beta(L_1^1) \)
In this case, we have \( \phi_\beta(L_1^1) = \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra.

8.2.3 \( \rho(L_1^1) \)
In this case, we have \( \rho(L_1^1) = \tilde{t} \) is the 3-dimensional abelian Lie algebra.

8.2.4 Identities
- \( A[\cdot, \cdot] = 0 \)
- \( [A, \cdot] + [\cdot, A] = 0 \)
- \( -[A, A] = 0 \)

In particular, the hom-Lie algebra \( L \) is a multiplicative hom-Lie algebra.

8.3 \( L_6^2 \)
Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

and let us denote by \( L_6^2 \) the hom-Lie algebra \( (L_6, A) \). We have:

\[
\text{gl}(n, \mathbb{C})_A = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\
t_3 & t_2 & 0 \\
t_5 & t_4 & t_1 \end{pmatrix} \in M(3, \mathbb{C}) : t_1, \ldots, t_5 \in \mathbb{C} \right\}.
\]

\[
\text{Aut}(L_6^2) = \left\{ \begin{pmatrix} x & a & 0 \\
y & 0 & 1 \\
0 & 0 & 0 \end{pmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\}.
\]

\[
\text{Der}(L_6^2) = \left\{ \begin{pmatrix} t_2 & t_1 & 0 \\
t_3 & 0 & 0 \end{pmatrix} \in M(3, \mathbb{C}) : t_1, t_2, t_3 \in \mathbb{C} \right\} \cong \mathfrak{sl}_2(\mathbb{C}) \times \mathbb{C}.
\]

8.3.1 \( T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(L_6^2) \)
Given \( D_1(u's) \), \( D_2(x's) \) and \( D_3(y's) \) in \( \text{gl}(n, \mathbb{C})_A \), we have \( \lambda := D_1[\cdot, \cdot] + [D_2, \cdot] + [\cdot, D_3] \)

\[
\lambda := \begin{cases}
      e_1 \cdot e_1 = (x_{2,1} + y_{1,2}) e_2, \\
      e_1 \cdot e_2 = (x_{3,3} + w_{2,2} + y_{2,2}) e_2 + w_{3,2} e_3, \\
      e_2 \cdot e_1 = (-y_{3,3} - w_{2,2} - x_{2,2}) e_2 - w_{3,2} e_3.
   \end{cases}
\]

By taking \( \{\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 1\} \) and \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0, \} \) and then, by replacing \( D_2 \) and \( D_3 \) by 0, we have \( \text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(L_6^2)) \) is determined by the equations

\[
\begin{cases}
w_{2,2} = 0, & w_{3,2} = 0
\end{cases}
\]

Therefore

\[
\dim T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(L_6^2) = 3.
\]
In this case, we have ψα,β(L2\text{6}) = tα,β is the Lie algebra r2(C) × C.

In this case, we have φβ(L2\text{6}) = \tilde{t}_β is the 3-dimensional abelian Lie algebra.

In this case, we have ρ(L2\text{6}) = \hat{t} is the 3-dimensional abelian Lie algebra.

8.3.5 Identities

- A[·, ·] = 0
- [A·, ·] = 0

In particular, the hom-Lie algebra L2\text{6} is a multiplicative hom-Lie algebra.

8.4 L3\text{6}

Let A be the endomorphism of C^3 whose matrix representation with respect to the ordered basis \{e_1, e_2, e_3\} is

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

and let us denote by L3\text{6} the hom-Lie algebra (L_3, A). We have:

\[\text{Aut}(L_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x, y \in \mathbb{C} \right\}.\]

\[\text{Der}(L_3) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathbb{C}^2.\]

8.4.1 ψα,β(L3\text{6})

In this case, we have ψα,β(L3\text{6}) = tα,β is the Lie algebra r2(C) × C.

8.4.2 φβ(L3\text{6})

In this case, we have φβ(L3\text{6}) = \tilde{t}_β is the 3-dimensional abelian Lie algebra.

8.4.3 ρ(L3\text{6})

In this case, we have ρ(L3\text{6}) = \hat{t} is the 3-dimensional abelian Lie algebra.

8.4.4 Identities

- A[·, ·] = 0
- [A·, ·] + [·, A·] = 0
  - [A·, A·] = 0

In particular, the hom-Lie algebra L3\text{6} is a multiplicative hom-Lie algebra.
8.5 \( \mathfrak{L}_6^4 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \( \mathfrak{L}_6^4 \) the hom-Lie algebra \( (\mathfrak{L}_6, A) \). We have:

\[
\text{gl}(n, \mathbb{C})_A = \left\{ \begin{bmatrix}
t_1 & 0 & t_3 \\
t_5 & t_2 & t_4 \\
0 & 0 & t_2
\end{bmatrix} \in M(3, \mathbb{C}) : t_1, \ldots, t_5 \in \mathbb{C} \right\}.
\]

\[
\text{Aut}(\mathfrak{L}_6^4) = \left\{ \begin{bmatrix}
x & a & 0 \\
0 & 0 & a \\
0 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : \begin{array}{l}
a \in \mathbb{C}^* \\
x \in \mathbb{C}
\end{array} \right\}.
\]

\[
\text{Der}(\mathfrak{L}_6^4) = \left\{ \begin{bmatrix}
t_2 & t_1 & 0 \\
0 & 0 & t_1
\end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C}).
\]

8.5.1 \( T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9} (\mathfrak{L}_6^4) \)

Given \( D_1(y')s \), \( D_2(x')s \) and \( D_3(y')s \) in \( \text{gl}(n, \mathbb{C})_A \), we have \( \lambda : = D_1[\cdot, \cdot] + [D_2, \cdot] + [\cdot, D_3] \)

\[
\lambda := \begin{cases}
    e_1 \cdot e_1 = (-x_{2,1} + y_{2,1}) e_2, \\
    e_1 \cdot e_2 = (x_{1,1} + w_{3,3} + y_{3,3}) e_2, \\
    e_1 \cdot e_3 = y_{2,3} e_2, \\
    e_2 \cdot e_1 = (-y_{1,1} - w_{3,3} - x_{3,3}) e_2, \\
    e_2 \cdot e_2 = -y_{1,3} e_2, \\
    e_2 \cdot e_3 = x_{1,3} e_2
\end{cases}
\]

By taking \( \{\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 1\} \) and \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0\} \) and then, by replacing \( D_2 \) and \( D_3 \) by 0, we have \( \text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9} (\mathfrak{L}_6^4)) \) is determined by the equations

\[
\left\{ \begin{array}{l}
w_{3,3} = 0
\end{array} \right\}
\]

Therefore

\[
\dim T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9} (\mathfrak{L}_6^4) = 4.
\]

8.5.2 \( \psi_{\alpha, \beta}(\mathfrak{L}_6^4) \)

In this case, we have \( \psi_{\alpha, \beta}(\mathfrak{L}_6^4) = t_{\alpha, \beta} \):

\[
t_{\alpha, \beta} = \langle [e_1, e_2] = e_2, [e_1, e_3] = \beta e_2 \rangle.
\]

We note that \( t_{\alpha, \beta} \) is an almost abelian Lie algebra where \( \text{span}_\mathbb{C} \{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha, \beta} \); we can identify \( t_{\alpha, \beta} \) the matrix

\[
B = \begin{bmatrix}
1 & \beta \\
0 & 0
\end{bmatrix}.
\]

Therefore, the Lie algebra \( t_{\alpha, \beta} \) is isomorphic to \( \mathfrak{r}_2(\mathbb{C}) \times \mathbb{C} \).
8.5.3 \( \phi_\beta(L^4_6) \)

In this case, we have \( \phi_\beta(L^4_6) = \tilde{t}_\beta \):

\[
\tilde{t}_\beta = \{ [e_1, e_3] = \beta e_2 \}.
\]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[
\tilde{B} = \begin{bmatrix}
0 & \beta \\
0 & 0
\end{bmatrix}.
\]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( 0 = 0 \)
   (a) \( 0 \) is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 0 = 0 \)
      i. \( \tilde{t}_\beta \) is the 3-dimensional abelian Lie algebra if and only if \( \tilde{B} = 0 \) if and only if \( \beta = 0 \)
      ii. \( \tilde{t}_\beta \) is isomorphic to \( n_3(\mathbb{C}) \) if and only if \( \tilde{B} \neq 0 \) if and only if \( \beta \neq 0 \)
   (b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( 1 = 0 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( 1 = 0 \)

8.5.4 \( \rho(L^4_6) \)

In this case, we have \( \rho(L^4_6) = \hat{t} \):

\[
\hat{t} = \{ [e_1, e_3] = e_2 \}.
\]

which is isomorphic to \( n_3 \)

8.5.5 Identities

- \( A[\cdot, \cdot] = 0 \)
  - \( [A, A] = 0 \)

In particular, the hom-Lie algebra \( L^4_6 \) is a multiplicative hom-Lie algebra.

8.6 \( L^5_6 \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and let us denote by \( L^5_6 \) the hom-Lie algebra \( (L^5_6, A) \). We have:

\[
\text{gl}(n, \mathbb{C}) = \left\{ \begin{bmatrix}
t_1 & t_5 & 0 \\
0 & t_2 & 0 \\
t_3 & t_4 & t_2
\end{bmatrix} \mid t_1, \ldots, t_5 \in \mathbb{C} \right\}.
\]

\[
\text{Aut}(L^5_6) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
0 & a & 0 \\
x & 0 & a
\end{bmatrix} \mid a \in \mathbb{C}^*, x \in \mathbb{C} \right\}.
\]

\[
\text{Der}(L^5_6) = \left\{ \begin{bmatrix}
0 & t_1 & 0 \\
t_2 & 0 & t_1
\end{bmatrix} \mid t_1, t_2 \in \mathbb{C} \right\} \cong r_2(\mathbb{C}).
\]
8.6.1 $\psi_{\alpha,\beta}(\mathfrak{L}_6^5)$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_6^5) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e1, e2] = e2 + \alpha e3.$$  

We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix

$$B = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix}.$$  

Therefore, the Lie algebra $t_{\alpha,\beta}$ is isomorphic to $\mathfrak{r}_2(C) \times C$.

8.6.2 $\phi_\beta(\mathfrak{L}_6^5)$

In this case, we have $\phi_\beta(\mathfrak{L}_6^5) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e1, e2] = e2.$$  

is the Heisenberg Lie algebra, $\mathfrak{n}_3(C)$.

8.6.3 $\rho(\mathfrak{L}_6^5)$

In this case, we have $\rho(\mathfrak{L}_6^5) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

8.6.4 Identities

- $[A, -] = 0$

The hom-Lie algebra $\mathfrak{L}_6^5$ is a non-multiplicative hom-Lie algebra.

8.7 $\mathfrak{L}_6^6(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & \lambda \\ 0 & -\lambda & -\lambda \end{bmatrix}$$

and let us denote by $\mathfrak{L}_6^6(\lambda)$ the hom-Lie algebra $(\mathfrak{L}_6, A)$. We have:

$$\text{Aut}(\mathfrak{L}_6^6(\lambda)) = \begin{Bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -x & a & 0 \\ x & 0 & a \end{bmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^* \\ x \in \mathbb{C} \end{Bmatrix}.$$  

$$\text{Der}(\mathfrak{L}_6^6(\lambda)) = \begin{Bmatrix} \begin{bmatrix} -t_2 & t_1 & 0 \\ 0 & 0 & 0 \\ t_2 & 0 & t_1 \end{bmatrix} \in M(3, \mathbb{C}) : t_1, t_2 \in \mathbb{C} \end{Bmatrix} \cong \mathfrak{r}_2(\mathbb{C}).$$

8.7.1 $\psi_{\alpha,\beta}(\mathfrak{L}_6^6(\lambda))$

In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}_6^6(\lambda)) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e1, e2], (1 + \alpha \lambda + \beta \lambda) e2 - \alpha \lambda e3, [e1, e3], \beta \lambda e2.$$
We note that $t_{\alpha,\beta}$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha,\beta}$; we can identify $t_{\alpha,\beta}$ the matrix

$$B = \begin{bmatrix} 1 + \alpha \lambda + \beta \lambda & \beta \lambda \\ -\alpha \lambda & 0 \end{bmatrix}.$$  

The isomorphism class of $t_{\alpha,\beta}$ is determined by the eigenvalues of $B$:

1. $B$ has repeated eigenvalues if and only if $1 + (\alpha - \beta)^2 \lambda^2 + 2 (\alpha + \beta) \lambda = 0$
   (a) 0 is a repeated eigenvalue of $B$ if and only if $\{ \alpha = -\frac{1}{\lambda}, \beta = 0 \}$ or $\{ \alpha = 0, \beta = -\frac{1}{\lambda} \}$.
   i. $t_{\alpha,\beta}$ is the 3-dimensional abelian Lie algebra if and only if $B = 0$ if and only if $1 = 0$
   ii. $t_{\alpha,\beta}$ is isomorphic to $n_3(C)$ if and only if $B \neq 0$ if and only if $\{ \alpha = -\frac{1}{\lambda}, \beta = 0 \}$ or $\{ \alpha = 0, \beta = -\frac{1}{\lambda} \}$.
   (b) $B$ has a non-zero repeated eigenvalue if and only if
      $$\{ \alpha = -\frac{s^2 + \sqrt{s^2 - 4}}{2\lambda s}, \beta = \frac{2s^2 + \sqrt{s^2 - 4}}{2\lambda s} \}$$ for all $s \in C^*$.
      i. $t_{\alpha,\beta}$ is $t_{3,1}$ if and only if $B = t \text{Id}$ for some $t \neq 0$ if and only if $1 = 0$
      ii. $t_{\alpha,\beta}$ is isomorphic to $t_3$ if and only if $B \neq t \text{Id}$ for all $t \in C$ if and only if
          $$\{ \alpha = -\frac{s^2 + \sqrt{s^2 - 4}}{2\lambda s}, \beta = \frac{2s^2 + \sqrt{s^2 - 4}}{2\lambda s} \}$$ for any $s \in C \setminus \{0\}$.

2. $B$ has distinct eigenvalues if and only if $1 + (\alpha - \beta)^2 \lambda^2 + 2 (\alpha + \beta) \lambda \neq 0$
   (a) $t_{\alpha,\beta}$ is isomorphic to $t_2(C) \times C$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\{ \alpha = -\frac{1}{\lambda}, \beta = 0 \}$ or $\{ \alpha = 0, \beta = -\frac{1}{\lambda} \}$, for any $s \in C^*$.
   (b) $t_{\alpha,\beta}$ is isomorphic to $t_{3,2}$ if and only if Trace$(B) = 0$ if and only if
       $$\{ \alpha = \frac{2s^2 + \sqrt{s^2 - 4}}{2\lambda s}, \beta = \frac{s^2 + \sqrt{s^2 - 4}}{2\lambda s} \}$$ for any $s \in C^*$.
   (c) $t_{\alpha,\beta}$ is isomorphic to $t_{3,2}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if
       $$\{ \alpha = \frac{-s_1s_2 + s_1 \pm \sqrt{s_1s_2(s_1s_2 - 2s_1 + s_2)}}{2s_1s_2\lambda}, \beta = \frac{-s_1s_2 + s_1 \mp \sqrt{s_1s_2(s_1s_2 - 2s_1 + s_2)}}{2s_1s_2\lambda} \}$$ for any $s_1, s_2 \in C^*$ such that $s_1 - s_2^2 \neq 0$.

8.7.2 $\phi_\beta(\mathfrak{g}_0^\ast(\lambda))$
In this case, we have $\phi_\beta(\mathfrak{g}_0^\ast(\lambda)) = \tilde{t}_3$:

$$\tilde{t}_3 = \{(e_1, e_2) = (\lambda + \beta \lambda) e_2 - \lambda e_3, [e_1, e_2] = \beta \lambda e_2 \}.$$  

We note that $\tilde{t}_3$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_3$; we can identify $\tilde{t}_3$ the matrix

$$\tilde{B} = \begin{bmatrix} \lambda + \beta \lambda & \beta \lambda \\ -\lambda & 0 \end{bmatrix}.$$  

The isomorphism class of $\tilde{t}_3$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $\beta = 1$
   (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $1 = 0$
   (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $\beta = 1$
      i. $\tilde{t}_3$ is $t_{3,1}$ if and only if $\tilde{B} = t \text{Id}$ for some $\lambda \neq 0$ if and only if $\lambda = 0$ (False)
      ii. $\tilde{t}_3$ is isomorphic to $t_3$ if and only if $\tilde{B} \neq t \text{Id}$ for all $\lambda \in C$ if and only if $\beta = 1$.
2. $\tilde{B}$ has distinct eigenvalues if and only if $\beta \neq 1$

(a) $\tilde{t}_\beta$ is isomorphic to $r_2(\mathbb{C}) \times \mathbb{C}$ if and only if 0 is an eigenvalue of $B$ with algebraic multiplicity 1 if and only if $\beta = 0$

(b) $\tilde{t}_\beta$ is isomorphic to $r_{3,-1}$ if and only if $\text{Trace}(\tilde{B}) = 0$ if and only if $\beta = -1$.

(c) $\tilde{t}_\beta$ is isomorphic to $r_{3,z}$ for some $z$ such that $z^2 \neq 1$ and $z \neq 0$ if and only if $\beta \neq 1$ and $\beta \neq -1$ and $\beta \neq 0$.

8.7.3 $\rho(\mathfrak{L}'^6(\lambda))$

In this case, we have $\rho(\mathfrak{L}'^6(\lambda)) = \hat{t}$:

$$\hat{t} = \{ [e1, e2] = \lambda e2, [e1, e3] = \lambda e2 \}.$$

which is isomorphic to $r_2 \times \mathbb{C}$.

8.7.4 Identities

• $[A, A] = 0$

The hom-Lie algebra $\mathfrak{L}'^7_6(\lambda)$ is a non-multiplicative hom-Lie algebra.

8.8 $\mathfrak{L}'^7_6$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}$$

and let us denote by $\mathfrak{L}'^7_6$ the hom-Lie algebra $(\mathfrak{L}'^7_6, A)$. We have:

$$\text{Aut}(\mathfrak{L}'^7_6) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x & 0 & 1
\end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.$$  

$$\text{Der}(\mathfrak{L}'^7_6) = \left\{ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
t_1 & 0 & 0
\end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.$$

8.8.1 $\psi_{\alpha, \beta}(\mathfrak{L}'^7_6)$

In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}'^7_6) = t_{\alpha, \beta}$:

$$t_{\alpha, \beta} = \{ [e1, e2] = e2 + \alpha e3 \}.$$  

We note that $t_{\alpha, \beta}$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha, \beta}$; we can identify $t_{\alpha, \beta}$ the matrix

$$B = \begin{bmatrix}
1 & 0 \\
\alpha & 0
\end{bmatrix}.$$  

Therefore, the Lie algebra $t_{\alpha, \beta}$ is isomorphic to $r_2(\mathbb{C}) \times \mathbb{C}$.
8.8.2 $\phi_\beta(L^7_6)$

In this case, we have $\phi_\beta(L^7_6) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e_1, e_2] = e_3\}.$$

is the Heisenberg Lie algebra, $\mathfrak{n}_3(\mathbb{C})$.

8.8.3 $\rho(L^7_6)$

In this case, we have $\rho(L^7_6) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

8.8.4 Identities

- $[A, \cdot] + [\cdot, A] = 0$
- $[A, A] = 0$
- $A^2[\cdot, \cdot] = 0$

The hom-Lie algebra $L^7_6$ is a non-multiplicative hom-Lie algebra.

8.9 $L^8_6$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
$$

and let us denote by $L^8_6$ the hom-Lie algebra $(L_6, A)$. We have:

$$
\begin{align*}
\text{Aut}(L^8_6) &= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\} \\
\text{Der}(L^8_6) &= \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.
\end{align*}
$$

8.9.1 $\psi_{\alpha, \beta}(L^8_6)$

In this case, we have $\psi_{\alpha, \beta}(L^8_6) = t_{\alpha, \beta}$:

$$t_{\alpha, \beta} = \{[e_1, e_2] = e_2, [e_1, e_3] = \beta e_2\}.$$

We note that $t_{\alpha, \beta}$ is an almost abelian Lie algebra where $\text{span}_{\mathbb{C}}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_{\alpha, \beta}$; we can identify $t_{\alpha, \beta}$ the matrix

$$B = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}.$$

Therefore, the Lie algebra $t_{\alpha, \beta}$ is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$. 

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8.9.2 $\phi_\beta(\mathfrak{L}_6)$

In this case, we have $\phi_\beta(\mathfrak{L}_6) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e_1, e_3] = \beta e_2\}.$$ 

We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_\mathbb{C}\{e_2, e_3\}$ is a codimension 1 abelian ideal of $t_\beta$; we can identify $t_\beta$ the matrix

$$\tilde{B} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}.$$ 

The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:

1. $\tilde{B}$ has repeated eigenvalues if and only if $0 = 0$
   (a) $0$ is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$
   i. $t_\beta$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\beta = 0$
   ii. $\tilde{t}_\beta$ is isomorphic to $\mathfrak{n}_3(\mathbb{C})$ if and only if $\tilde{B} \neq 0$ if and only if $\beta \neq 0$
   (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $1 = 0$

2. $\tilde{B}$ has distinct eigenvalues if and only if $1 = 0$

8.9.3 $\rho(\mathfrak{L}_6)$

In this case, we have $\rho(\mathfrak{L}_6) = \hat{t}$:

$$\hat{t} = \{[e_1, e_3] = e_2\}.$$ 

which is isomorphic to $\mathfrak{n}_3$

8.9.4 Identities

- $A[.,.] = 0$
- $[A., A.] = 0$
- $[A_2.,.] + [., A_2.] = 0$

In particular, the hom-Lie algebra $\mathfrak{L}_6^\beta$ is a multiplicative hom-Lie algebra.

8.10 $\mathfrak{L}_6^\lambda(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & \lambda & \lambda \\ 0 & -\lambda & -\lambda \end{bmatrix}$$

and let us denote by $\mathfrak{L}_6^\lambda(\lambda)$ the hom-Lie algebra $(\mathfrak{L}_6, A)$. We have:

$$\text{Aut}(\mathfrak{L}_6^\lambda(\lambda)) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -x & 0 & 1 \end{bmatrix} \in M(3, \mathbb{C}) : x \in \mathbb{C} \right\}.$$ 

$$\text{Der}(\mathfrak{L}_6^\lambda(\lambda)) = \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \\ -t_1 & 0 & 0 \end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}.$$
8.10.1 \( \psi_{\alpha,\beta}(L_9^\lambda) \)

In this case, we have \( \psi_{\alpha,\beta}(L_9^\lambda) = t_{\alpha,\beta} \):

\[ t_{\alpha,\beta} = \{ [e_1, e_2], (1 + \alpha \lambda + \beta \lambda) e_2 - \alpha \lambda e_3, [e_1, e_3], \beta \lambda e_2 \}. \]

We note that \( t_{\alpha,\beta} \) is an almost abelian Lie algebra where \( \text{span}_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( t_{\alpha,\beta} \); we can identify \( t_{\alpha,\beta} \) the matrix

\[ B = \begin{bmatrix} 1 + \alpha \lambda + \beta \lambda & \beta \lambda \\ -\alpha \lambda & 0 \end{bmatrix}. \]

The isomorphism class of \( t_{\alpha,\beta} \) is determined by the eigenvalues of \( B \):

1. \( B \) has repeated eigenvalues if and only if \( 1 + (\alpha - \beta)^2 \lambda^2 + 2 (\alpha + \beta) \lambda = 0 \)
   - (a) 0 is a repeated eigenvalue of \( B \) if and only if \( \alpha = -\frac{1}{\lambda}, \beta = 0 \) or \( \alpha = 0, \beta = -\frac{1}{\lambda} \).
     - i. \( t_{\alpha,\beta} \) is the 3-dimensional abelian Lie algebra if and only if \( B = 0 \) if and only if \( 1 = 0 \)
     - ii. \( t_{\alpha,\beta} \) is isomorphic to \( n_3(C) \) if and only if \( B \neq 0 \) if and only if \( \alpha = -\frac{1}{\lambda}, \beta = 0 \) or \( \alpha = 0, \beta = -\frac{1}{\lambda} \).
   - (b) \( B \) has a non-zero repeated eigenvalue if and only if
     \[ \left\{ \alpha = -\frac{s - 2 + \sqrt{s^2 - 4s}}{2 \lambda s}, \beta = -\frac{s + 2 + \sqrt{s^2 - 4s}}{2 \lambda s} \right\} \text{ for all } s \in C^*. \]
     - i. \( t_{\alpha,\beta} \) is \( t_{3,1} \) if and only if \( B = t \text{Id} \) for some \( t \neq 0 \) if and only if \( 1 = 0 \)
     - ii. \( t_{\alpha,\beta} \) is isomorphic to \( n_3 \) if and only if \( B \neq t \text{Id} \) for all \( t \in C \) if and only if
     \[ \left\{ \alpha = -\frac{s - 2 + \sqrt{s^2 - 4s}}{2 \lambda s}, \beta = -\frac{s + 2 + \sqrt{s^2 - 4s}}{2 \lambda s} \right\} \text{ for all } s \in C \setminus \{0\}. \]

2. \( B \) has distinct eigenvalues if and only if \( 1 + (\alpha - \beta)^2 \lambda^2 + 2 (\alpha + \beta) \lambda 
eq 0 \)
   - (a) \( t_{\alpha,\beta} \) is isomorphic to \( \tau_2(C) \times \mathbb{C} \) if and only if \( 0 \) is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \alpha = -\frac{1}{\lambda^2}, \beta = 0 \) or \( \alpha = 0, \beta = -\frac{1}{\lambda} \), for any \( s \in C^* \).
   - (b) \( t_{\alpha,\beta} \) is isomorphic to \( \tau_{3,-1} \) if and only if \( \text{Trace}(B) = 0 \) if and only if
     \[ \left\{ \alpha = -\frac{s - 2 + \sqrt{s^2 - 4s}}{2 \lambda s}, \beta = -\frac{s + 2 + \sqrt{s^2 - 4s}}{2 \lambda s} \right\} \text{ for any } s \in C^*. \]
   - (c) \( t_{\alpha,\beta} \) is isomorphic to \( \tau_{3,z} \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if
     \[ \left\{ \alpha = \frac{s_1 s_2 + s_1 z + s_2 (s_1 s_2 - 2 s_1 z + s_2)}{2 s_1 s_2 \lambda}, \beta = \frac{s_1 s_2 + s_1 z s_2 (s_1 s_2 - 2 s_1 + s_2)}{2 s_1 s_2 \lambda} \right\} \text{ for any } s_1, s_2 \in C^* \text{ such that } s_1 - s_2^2 \neq 0. \]

8.10.2 \( \phi_\beta(L_9^\lambda) \)

In this case, we have \( \phi_\beta(L_9^\lambda) = \tilde{t}_\beta \):

\[ \tilde{t}_\beta = \{ [e_1, e_2] = (\lambda + \beta \lambda) e_2 - \lambda e_3, [e_1, e_3] = \beta \lambda e_2 \}. \]

We note that \( \tilde{t}_\beta \) is an almost abelian Lie algebra where \( \text{span}_C\{e_2, e_3\} \) is a codimension 1 abelian ideal of \( \tilde{t}_\beta \); we can identify \( \tilde{t}_\beta \) the matrix

\[ \tilde{B} = \begin{bmatrix} \lambda + \beta \lambda & \beta \lambda \\ -\lambda & 0 \end{bmatrix}. \]

The isomorphism class of \( \tilde{t}_\beta \) is determined by the eigenvalues of \( \tilde{B} \):

1. \( \tilde{B} \) has repeated eigenvalues if and only if \( \beta = 1 \)
   - (a) 0 is a repeated eigenvalue of \( \tilde{B} \) if and only if \( 1 = 0 \)

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(b) \( \tilde{B} \) has a non-zero repeated eigenvalue if and only if \( \beta = 1 \)

i. \( \tilde{t}_\beta \) is \( r_{3,1} \) if and only if \( \tilde{B} = t \text{Id} \) for some \( \lambda \neq 0 \) if and only if \( \lambda = 0 \) (False)

ii. \( \tilde{t}_\beta \) is isomorphic to \( r_3 \) if and only if \( \tilde{B} \neq t \text{Id} \) for all \( t \in \mathbb{C} \) if and only if \( \beta = 1 \)

2. \( \tilde{B} \) has distinct eigenvalues if and only if \( \beta \neq 1 \)

(a) \( \tilde{t}_\beta \) is isomorphic to \( r_2(\mathbb{C}) \times \mathbb{C} \) if and only if 0 is an eigenvalue of \( B \) with algebraic multiplicity 1 if and only if \( \beta = 0 \)

(b) \( \tilde{t}_\beta \) is isomorphic to \( r_{3,-1} \) if and only if \( \text{Trace}(\tilde{B}) = 0 \) if and only if \( \beta = -1 \).

(c) \( \tilde{t}_\beta \) is isomorphic to \( r_{3,z} \) for some \( z \) such that \( z^2 \neq 1 \) and \( z \neq 0 \) if and only if \( \beta \neq 1 \) and \( \beta \neq -1 \) and \( \beta \neq 0 \).

8.10.3 \( \rho(\mathcal{L}_6^0(\lambda)) \)

In this case, we have \( \rho(\mathcal{L}_6^0(\lambda)) = \hat{t} \):

\[
\hat{t} = \{ [e1, e2] = \lambda e2, [e1, e3] = \lambda e2 \}.
\]

which is isomorphic to \( r_2 \times \mathbb{C} \).

8.10.4 Identities

- \([A, A] = 0 \)
- \( A^2[\cdot, \cdot] = 0 \)
- \([A^2, \cdot] + [\cdot, A^2] = 0 \)

The hom-Lie algebra \( \mathcal{L}_6^0(\lambda) \) is a non-multiplicative hom-Lie algebra.

8.11 \( \mathcal{L}_6^{10} \)

Let \( A \) be the endomorphism of \( \mathbb{C}^3 \) whose matrix representation with respect to the ordered basis \( \{e_1, e_2, e_3\} \) is

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and let us denote by \( \mathcal{L}_6^{10} \) the hom-Lie algebra \( (\mathcal{L}_6, A) \). We have:

\[
\text{Aut}(\mathcal{L}_6^{10}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \in M(3, \mathbb{C}) : a \in \mathbb{C}^\ast \right\}.
\]

\[
\text{Der}(\mathcal{L}_6^{10}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t_1 \end{pmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\}.
\]

8.11.1 \( \psi_{\alpha,\beta}(\mathcal{L}_6^{10}) \)

In this case, we have \( \psi_{\alpha,\beta}(\mathcal{L}_6^{10}) = t_{\alpha,\beta} \):

\[
t_{\alpha,\beta} = \{ [e1, e2] = \alpha e1 + e2 \}.
\]

which is isomorphic to \( r_2(\mathbb{C}) \times \mathbb{C} \).
8.11.2 $\phi_\beta(\mathfrak{L}^{10}_6)$
In this case, we have $\phi_\beta(\mathfrak{L}^{10}_6) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e1, e2] = e1.$$

which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$.

8.11.3 $\rho(\mathfrak{L}^{10}_6)$
In this case, we have $\rho(\mathfrak{L}^{10}_6) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

8.11.4 Identities
- $[A, ] + [\cdot, A] = 0$
- $[A, A] = 0$

The hom-Lie algebra $\mathfrak{L}^{10}_6$ is a non-multiplicative hom-Lie algebra.

8.12 $\mathfrak{L}^{11}_6$
Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

and let us denote by $\mathfrak{L}^{11}_6$ the hom-Lie algebra $(\mathfrak{L}_6, A)$. We have:

$\text{Aut}(\mathfrak{L}^{11}_6) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

$\text{Der}(\mathfrak{L}^{11}_6) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

8.12.1 $\psi_{\alpha,\beta}(\mathfrak{L}^{11}_6)$
In this case, we have $\psi_{\alpha,\beta}(\mathfrak{L}^{11}_6) = t_{\alpha,\beta}$:

$$t_{\alpha,\beta} = \{[e1, e2] = \alpha e1 + e2.$$

which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$.

8.12.2 $\phi_\beta(\mathfrak{L}^{11}_6)$
In this case, we have $\phi_\beta(\mathfrak{L}^{11}_6) = \tilde{t}_\beta$:

$$\tilde{t}_\beta = \{[e1, e2] = e1.$$

which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$.

8.12.3 $\rho(\mathfrak{L}^{11}_6)$
In this case, we have $\rho(\mathfrak{L}^{11}_6) = \hat{t}$ is the 3-dimensional abelian Lie algebra.
8.12.4 Identities

- \([A_-, ] + [, A] = 0\)
- \([A_-, A] = 0\)
- \([A^2_, -] = 0\)

The hom-Lie algebra \(L_1\) is a non-multiplicative hom-Lie algebra.

8.13 \(\mathcal{L}^{12}_6\)

Let \(A\) be the endomorphism of \(\mathbb{C}^3\) whose matrix representation with respect to the ordered basis \(\{e_1, e_2, e_3\}\) is

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

and let us denote by \(\mathcal{L}^{12}_6\) the hom-Lie algebra \((\mathcal{L}_6, A)\). We have:

\[
\text{Aut}(\mathcal{L}^{12}_6) = \left\{ \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \right\}
\]

\[
\text{Der}(\mathcal{L}^{12}_6) = \left\{ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \right\}
\]

8.13.1 \(\psi_{\alpha, \beta}(\mathcal{L}^{12}_6)\)

In this case, we have \(\psi_{\alpha, \beta}(\mathcal{L}^{12}_6) = t_{\alpha, \beta}\):

\(t_{\alpha, \beta} = \{[eI, e2] = \alpha eI + e2, [eI, e3] = \beta e2, \}

We have that \(t_{\alpha, \beta}\) is a Lie algebra if and only if \(\beta = 0\) or \(\alpha = 0\); in such case \(t_{\alpha, \beta}\) is isomorphic to \(\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}\).

8.13.2 \(\phi_{\beta}(\mathcal{L}^{12}_6)\)

In this case, we have \(\phi_{\beta}(\mathcal{L}^{12}_6) = \tilde{t}_{\beta}\):

\(\tilde{t}_{\beta} = \{[eI, e2] = eI, [eI, e3] = \beta e2, \}

We have that \(\tilde{t}_{\beta}\) is a Lie algebra if and only if \(\beta = 0\); in such case \(\tilde{t}_{\beta}\) is isomorphic to \(\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}\).

8.13.3 \(\rho(\mathcal{L}^{12}_6)\)

In this case, we have \(\rho(\mathcal{L}^{12}_6) = \hat{t}\):

\(\hat{t} = \{[eI, e3] = e2, \}

which is isomorphic to \(\mathfrak{n}_3\).

8.13.4 Identities

- \([A^2, -] = 0\)
- \([A^2, A^2] = 0\)
- \([A_-, A] + [A^2_, -] + [, A^2_] = 0\)

The hom-Lie algebra \(\mathcal{L}^{12}_6\) is a non-multiplicative hom-Lie algebra.
8.14 $\Sigma_6^{13}(\lambda)$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{pmatrix}
$$

and let us denote by $\Sigma_6^{13}(\lambda)$ the hom-Lie algebra $(\mathfrak{L}_6, A)$. We have:

$$
\text{Aut}(\Sigma_6^{13}(\lambda)) = \begin{cases} 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} & , \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} & .
\end{cases}
$$

$$
\text{Der}(\Sigma_6^{13}(\lambda)) = \begin{cases} 
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} & .
\end{cases}
$$

8.14.1 $\psi_{\alpha,\beta}(\Sigma_6^{13})$

In this case, we have $\psi_{\alpha,\beta}(\Sigma_6^{13}) = t_{\alpha,\beta}$:

$$
t_{\alpha,\beta} = \{[e_1, e_2] = \alpha e_1 + (1 + \alpha \lambda + \beta \lambda) e_2 - \alpha \lambda e_3, [e_1, e_3] = \beta \lambda e_2. $

We have that $t_{\alpha,\beta}$ is a Lie algebra if and only if $\beta = 0$ or $\alpha = 0$. If $\alpha = 0$, then $t_{0,\beta}$ is the Lie algebra

$$
t_{0,\beta} = \{[e_1, e_2] = (1 + \beta \lambda) e_2, [e_1, e_3] = \beta \lambda e_2. $$

which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$ if and only if $\beta \neq -\frac{1}{\lambda}$; in other case, $t_{0,\beta} = -\frac{1}{\lambda}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{n}_3$. If $\beta = 0$, then $t_{\alpha,0}$ is the Lie algebra

$$
t_{\alpha,0} = \{[e_1, e_2] = \alpha e_1 + (1 + \alpha \lambda) e_2 - \alpha \lambda e_3. $$

which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$. In fact, if $\alpha \neq -\frac{1}{\lambda}$ we can consider the change of basis

$$
g = \begin{pmatrix}
1 + \alpha \lambda & -\alpha & 0 \\
0 & 1 & 0 \\
\lambda & 0 & 1
\end{pmatrix}
$$

and, if $\alpha = -\frac{1}{\lambda}$ we can consider the change of basis given by

$$
\tilde{g} = \begin{pmatrix}
0 & \lambda^{-1} & 0 \\
1 & 0 & 0 \\
\lambda & 0 & 1
\end{pmatrix}
$$

8.14.2 $\phi_\beta(\Sigma_6^{13})$

In this case, we have $\phi_\beta(\Sigma_6^{13}) = \tilde{t}_\beta$:

$$
\tilde{t}_\beta = \{[e_1, e_2] = e_1 + (\lambda + \beta \lambda) e_2 - \lambda e_3, [e_1, e_3] = \beta \lambda e_2. $$

We have that $\tilde{t}_\beta$ is a Lie algebra if and only if $\beta = 0$; in such case is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \times \mathbb{C}$. In fact, we can consider the change of basis given by

$$
g = \begin{pmatrix}
\lambda & -1 & 0 \\
0 & 1 & 0 \\
\lambda & 0 & 1
\end{pmatrix}
$$
8.14.3 $\rho(\mathfrak{L}_{6}^{13}(\lambda))$

In this case, we have $\rho(\mathfrak{L}_{6}^{13}(\lambda)) = \hat{t}$:

$$\hat{t} = \{[e_1, e_2] = \lambda e_2, [e_1, e_3] = \lambda e_2\}$$

which is isomorphic to $\mathfrak{r}_2 \times \mathbb{C}$.

8.14.4 Identities

- $[A \cdot, A ] + [A^2 \cdot, ] + [\cdot, A^2 \cdot] = 0$
- $[A^2 \cdot, A^2 \cdot] = 0$

The hom-Lie algebra $\mathfrak{L}_{6}^{13}(\lambda)$ is a non-multiplicative hom-Lie algebra.

8.15 Degenerations between hom-Lie algebras ($\mathfrak{L}_{6}^{i}$)

If $\mathfrak{L}_{i}^{j} \xrightarrow{\text{deg}} \mathfrak{L}_{k}^{h}$ then $\text{Der}(\mathfrak{L}_{j}^{i}) \leq \text{Der}(\mathfrak{L}_{k}^{h})$. Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 0        | $\mathfrak{L}_{6}^{0}(\lambda)$, $\mathfrak{L}_{6}^{2}$, $\mathfrak{L}_{6}^{4}$ |
| 1        | $\mathfrak{L}_{6}^{0}(\lambda)$, $\mathfrak{L}_{6}^{1}$, $\mathfrak{L}_{6}^{3}$ |
| 2        | $\mathfrak{L}_{6}^{0}(\lambda)$, $\mathfrak{L}_{6}^{4}$, $\mathfrak{L}_{6}^{5}$ |
| 3        | $\mathfrak{L}_{6}^{0}$, $\mathfrak{L}_{6}^{2}$, $\mathfrak{L}_{6}^{4}$ |
| 4        | $\mathfrak{L}_{6}^{0}$ |

8.15.1 Degenerations

$$\mathfrak{L}_{6}^{0}(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_{6}^{2}(\lambda)$$

In fact, set

$$g(t) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{e^{t(1+e^t)}}{\lambda} & -\frac{e^{t(1+e^t)}^2}{\lambda^2} & 0 \\ -\frac{(1+e^t)^2}{\lambda} & 0 & -\frac{e^{t(1+e^t)}^2}{\lambda^2} \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$
We have \( g(t) \in \text{Aut}(\mathfrak{L}_0) \) and \( g(t) \cdot \mathfrak{L}_6^{13} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with
\[
A(t) = \begin{bmatrix}
\frac{\lambda}{1+e^t} & -\frac{\lambda^2}{e(t-1+e^t)^2} & 0 \\
1 & \frac{\lambda(-2+e^t)}{1+e^t} & \lambda \\
0 & \frac{1}{e^t} - \lambda & -\lambda
\end{bmatrix}
\]
It is easy to check that \((\mathfrak{L}_6, A(t)) \to \mathfrak{L}_0^9(\lambda)\) as \(t\) tends to infinity.

2. \( \mathfrak{L}_6^{12} \xrightarrow{\text{deg}} \mathfrak{L}_6^{10} \)
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^t
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_0) \) and \( g(t) \cdot \mathfrak{L}_6^{12} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with
\[
A(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & e^{-t} \\
0 & 0 & 0
\end{bmatrix}
\]
It is easy to check that \((\mathfrak{L}_6, A(t)) \to \mathfrak{L}_6^{10}\) as \(t\) tends to infinity.

3. \( \mathfrak{L}_6^{12} \xrightarrow{\text{deg}} \mathfrak{L}_6^{8} \)
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^{2t} & -e^{3t} & 0 \\
e^t & 0 & -e^{3t}
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_0) \) and \( g(t) \cdot \mathfrak{L}_6^{12} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with
\[
A(t) = \begin{bmatrix}
e^{-t} & -e^{-3t} & 0 \\
0 & -e^{-t} & 1 \\
1 & -e^{-2t} & 0
\end{bmatrix}
\]
It is easy to check that \((\mathfrak{L}_6, A(t)) \to \mathfrak{L}_6^{8}\) as \(t\) tends to infinity.

4. \( \mathfrak{L}_6^{12} \xrightarrow{\text{deg}} \mathfrak{L}_6^{7} \)
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^t & 0 \\
e^t & 0 & -e^{-2t}
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_0) \) and \( g(t) \cdot \mathfrak{L}_6^{12} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with
\[
A(t) = \begin{bmatrix}
0 & e^{-t} & 0 \\
1 & 0 & -e^{-t} \\
0 & 1 & 0
\end{bmatrix}
\]
It is easy to check that \((\mathfrak{L}_6, A(t)) \to \mathfrak{L}_6^{7}\) as \(t\) tends to infinity.

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5. \( \mathfrak{L}_6^{11} \xrightarrow{\text{deg}} \mathfrak{L}_6^{10} \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^{-t} & 1 & 0 \\
e^{-t} & 0 & e^{-2t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_6^{11} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
-e^{-t} & 1 & 0 \\
-e^{-2t} & e^{-t} & 0 \\
0 & e^{-t} & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^{10}\) as \( t \) tends to infinity.

6. \( \mathfrak{L}_6^{11} \xrightarrow{\text{deg}} \mathfrak{L}_6^{7} \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^t & -e^{2t} & 0 \\
e^{2t} & 0 & e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_6^{11} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
e^{-t} & -e^{-2t} & 0 \\
1 & -e^{-t} & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^{7}\) as \( t \) tends to infinity.

7. \( \mathfrak{L}_6^{10} \xrightarrow{\text{deg}} \mathfrak{L}_6^{5} \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^t & 0 \\
e^t & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_6^{10} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & e^{-t} & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^{5}\) as \( t \) tends to infinity.

8. \( \mathfrak{L}_6^{10} \xrightarrow{\text{deg}} \mathfrak{L}_6^{3} \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^t & -e^{2t} & 0 \\
e^t & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_6^{10} \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
e^{-t} & -e^{-2t} & 0 \\
1 & -e^{-t} & 0 \\
1 & -e^{-t} & 0
\end{bmatrix}
\]
It is easy to check that \((L_6, A(t)) \to L_3\) as \(t\) tends to infinity.

9. \(L_9^\lambda \xrightarrow{\text{deg}} L_6^\lambda\)

In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \(g(t) \in \text{Aut}(L_6)\) and \(g(t) \cdot L_9^\lambda\) is the hom-Lie algebra \((L_6, A(t))\) with
\[
A(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{-t} & \lambda & \lambda \\ 0 & -\lambda & -\lambda \end{bmatrix}
\]

It is easy to check that \((L_6, A(t)) \to L_6^\lambda\) as \(t\) tends to infinity.

10. \(L_8^\lambda \xrightarrow{\text{deg}} L_4^\lambda\)

In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \(g(t) \in \text{Aut}(L_6)\) and \(g(t) \cdot L_8^\lambda\) is the hom-Lie algebra \((L_6, A(t))\) with
\[
A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ e^{-t} & 0 & 0 \end{bmatrix}
\]

It is easy to check that \((L_6, A(t)) \to L_4^\lambda\) as \(t\) tends to infinity.

11. \(L_8^\lambda \xrightarrow{\text{deg}} L_6^3\)

In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ -e^t & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \(g(t) \in \text{Aut}(L_6)\) and \(g(t) \cdot L_8^\lambda\) is the hom-Lie algebra \((L_6, A(t))\) with
\[
A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & e^{-t} \\ 1 & 0 & 0 \end{bmatrix}
\]

It is easy to check that \((L_6, A(t)) \to L_6^3\) as \(t\) tends to infinity.

12. \(L_7^\lambda \xrightarrow{\text{deg}} L_6^5\)

In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_7^6 \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
e^{-t} & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^5\) as \( t \) tends to infinity.

13. \( \mathfrak{L}_6^7 \xrightarrow{\text{deg}} \mathfrak{L}_6^3 \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^t & 1 & 0 \\
0 & 0 & -(e^t)^{-1}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_7^6 \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & -e^{-t} & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^3\) as \( t \) tends to infinity.

14. \( \mathfrak{L}_6^5 \xrightarrow{\text{deg}} \mathfrak{L}_6^2 \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
e^t & -e^t & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_7^6 \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & -e^{-t} & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^2\) as \( t \) tends to infinity.

15. \( \mathfrak{L}_6^4 \xrightarrow{\text{deg}} \mathfrak{L}_6^1 \)

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
e^t & 0 & -e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \in \text{Aut}(\mathfrak{L}_6) \) and \( g(t) \cdot \mathfrak{L}_7^6 \) is the hom-Lie algebra \((\mathfrak{L}_6, A(t))\) with

\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -e^{-t} \\
0 & 0 & 0
\end{bmatrix}
\]

It is easy to check that \((\mathfrak{L}_6, A(t)) \rightarrow \mathfrak{L}_6^1\) as \( t \) tends to infinity.
16. $\mathfrak{L}_0^3 \xrightarrow{\text{deg}} \mathfrak{L}_0^2$
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^3$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
e^{-t} & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
It is easy to check that $(\mathfrak{L}_0, A(t)) \rightarrow \mathfrak{L}_0^2$ as $t$ tends to infinity.

17. $\mathfrak{L}_0^3 \xrightarrow{\text{deg}} \mathfrak{L}_0^1$
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^3$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
e^{-t} & 0 & 0
\end{bmatrix}
\]
It is easy to check that $(\mathfrak{L}_0, A(t)) \rightarrow \mathfrak{L}_0^1$ as $t$ tends to infinity.

18. $\mathfrak{L}_0^2 \xrightarrow{\text{deg}} \mathfrak{L}_0^0$
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^2$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
e^{-t} & 0 & 0
\end{bmatrix}
\]
It is easy to check that $(\mathfrak{L}_0, A(t)) \rightarrow \mathfrak{L}_0^0$ as $t$ tends to infinity.

19. $\mathfrak{L}_0^1 \xrightarrow{\text{deg}} \mathfrak{L}_0^0$
In fact, set
\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \in \text{Aut}(\mathfrak{L}_0)$ and $g(t) \cdot \mathfrak{L}_0^1$ is the hom-Lie algebra $(\mathfrak{L}_0, A(t))$ with
\[
A(t) = \begin{bmatrix}
0 & 0 & 0 \\
e^{-t} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
It is easy to check that \((L_6, A(t)) \xrightarrow{t \to \infty} L_0\).

20. Suppose, contrary to our claim, that \(L_6^{13}(\lambda) \xrightarrow{\deg} L_6^0(\kappa)\) with \((\lambda \neq \kappa)\). In such case, we have
\[
\psi_{\alpha,\beta}(L_6^{13}(\lambda)) \xrightarrow{\deg} \psi_{\alpha,\beta}(L_6^0(\kappa)).
\]
By taking \(\alpha = 0\) and \(\beta = -1/\lambda\), we have from 8.14.1 that \(\psi_{\alpha,\beta}(L_6^{13}(\lambda))\) is a Lie algebra isomorphic to \(n_3\), therefore, \(\psi_{\alpha,\beta}(L_6^0(\kappa))\) is a Lie algebra which is isomorphic to one of the following: \(n_3\) or \(a_3\) (by Theorem 1.1). We must have \(\psi_{\alpha,\beta}(L_6^0(\kappa))\) is isomorphic to \(n_3\) (from 8.7.1). But, we now apply again 8.7.1, to obtain \(\alpha = 0\) or \(\beta = -1/\kappa\), and therefore \(\lambda = \kappa\); this is a contradiction.

21. Suppose, contrary to our claim, that \(L_6^{13}(\lambda) \xrightarrow{\deg} L_6^0\). In such case, we have
\[
\psi_{\alpha,\beta}(L_6^{13}(\lambda)) \xrightarrow{\deg} \psi_{\alpha,\beta}(L_6^0(\lambda)).
\]
By taking \(\alpha = 0\) and \(\beta = -1/\lambda\), we have from 8.14.1 that \(\psi_{\alpha,\beta}(L_6^{13}(\lambda))\) is a Lie algebra isomorphic to \(n_3\), therefore, \(\psi_{\alpha,\beta}(L_6^0(\lambda))\) is a Lie algebra which is isomorphic to one of the following: \(n_3\) or \(a_3\) (by Theorem 1.1). But, from 8.1.1, \(\psi_{\alpha,\beta}(L_6^0(\lambda))\) is isomorphic to \(r_2 \times \mathbb{C}\); this is a contradiction.

22. Suppose, contrary to our claim, that \(L_6^{12} \xrightarrow{\deg} L_6^0(\lambda)\) with \((\lambda \neq 0)\).

We have from 8.13.3 that \(\rho(L_6^{12})\) is a Lie algebra isomorphic to \(n_3\), therefore, \(\rho(L_6^0(\lambda))\) is a Lie algebra which is isomorphic to one of the following: \(n_3\) or \(a_3\) (by Theorem 1.1). But, from 8.7.3, we have \(\rho(L_6^0(\lambda))\) is isomorphic to \(r_2 \times \mathbb{C}\); this is a contradiction.

23. Suppose, contrary to our claim, that \(L_6^{11} \xrightarrow{\deg} L_6^0(\lambda)\) with \((\lambda \neq 0)\).

By taking \(\beta = -1\), we have from 8.12.2 that \(\phi_{\beta}(L_6^{11})\) is a Lie algebra isomorphic to \(r_2 \times \mathbb{C}\), therefore, \(\phi_{\beta}(L_6^0(\lambda))\) is a Lie algebra which is isomorphic to one of the following: \(r_2 \times \mathbb{C}\), \(n_3\) or \(a_3\) (by Theorem 1.1). We must have \(\phi_{\beta}(L_6^0(\lambda))\) is isomorphic to \(r_2 \times \mathbb{C}\) (from 8.7.2). But, we now apply again 8.7.2, to obtain \(\beta = 0\); this is a contradiction.

24. Suppose, contrary to our claim, that \(L_6^{11} \xrightarrow{\deg} L_6^4\). In such case, we have
\[
\rho(L_6^{11}) \xrightarrow{\deg} \rho(L_6^4).
\]
We have from 8.12.3 that \(\rho(L_6^{11})\) is a Lie algebra isomorphic to \(a_3\), therefore, \(\rho(L_6^4)\) is the 3-dimensional abelian Lie algebra. But, from 8.5.4, \(\rho(L_6^4)\) is isomorphic to \(n_3\); this is a contradiction.

25. Suppose, contrary to our claim, that \(L_6^8 \xrightarrow{\deg} L_6^5\). In such case, we have
\[
\phi_{\beta}(L_6^8) \xrightarrow{\deg} \phi_{\beta}(L_6^5).
\]
By taking $\beta = 0$, we have from 8.9.2 that $\phi_\beta(\mathfrak{L}^8_6)$ is the 3-dimensional Lie algebra algebra, therefore, $\phi_\beta(\mathfrak{L}^5_6)$ is the abelian Lie algebra. But, from 8.6.2, we have $\phi_\beta(\mathfrak{L}^5_6)$ is isomorphic to $\mathfrak{n}_3$; this is a contradiction.

26. $\mathfrak{L}^5_6 \xrightarrow{\text{deg}} \mathfrak{L}^1_6$.

For $\mathfrak{L}^5_6$ satisfies the identity $[A_5, \cdot , \cdot] \equiv 0$ but $\mathfrak{L}^1_6$ does not.

27. $\mathfrak{L}^4_6 \xrightarrow{\text{deg}} \mathfrak{L}^2_6$.

Suppose, contrary to our claim, that $\mathfrak{L}^4_6 \xrightarrow{\text{deg}} \mathfrak{L}^2_6$. We have

$$\dim(\ker T_{0,1,0,0,0,1,1,-1,0,1,-1,0,1,-1,0,1,-1,0}(\mathfrak{L}^4_6)) \leq 4 \leq \dim(\ker T_{0,1,0,0,0,1,1,-1,0,1,-1,0,1,-1,0}(\mathfrak{L}^2_6))$$

From 8.3.1, $\dim(\ker T_{0,1,0,0,0,1,1,-1,0,1,-1,0,1,-1,0}(\mathfrak{L}^2_6)) = 3$; we have a contradiction.
9 $\mathfrak{L}_7$: $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$

$\mathfrak{L}_7 := \{[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2\}$

$\text{Aut}(\mathfrak{L}_7) = \left\{ g \in M(3, \mathbb{C}) : \begin{array}{c} g^Tg = \text{Id}, \\
\det(g) = 1 \end{array} \right\} = \text{SO}(3, \mathbb{C})$.

$\text{Der}(\mathfrak{L}_7) = \{ X \in M(3, \mathbb{C}) : X^T + X = 0 \} = \mathfrak{so}(3, \mathbb{C})$.

9.1 $\mathfrak{L}^0_7$

Let $A$ be the Zero map and let us denote by $\mathfrak{L}^0_7$ the hom-Lie algebra $(\mathfrak{L}_0, A)$. We have:

$\text{Aut}(\mathfrak{L}^0_7) = \mathfrak{L}_7$.

$\text{Der}(\mathfrak{L}^0_7) = \mathfrak{L}_7$

9.1.1 $\psi_{\alpha, \beta}(\mathfrak{L}^0_7)$

In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}^0_7) = t_{\alpha, \beta}$ is the Lie algebra $\mathfrak{so}(3, \mathbb{C})$.

9.1.2 $\phi_\beta(\mathfrak{L}^0_7)$

In this case, we have $\phi_\beta(\mathfrak{L}^0_7) = \tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra.

9.1.3 $\rho(\mathfrak{L}^0_7)$

In this case, we have $\rho(\mathfrak{L}^0_7) = \hat{t}$ is the 3-dimensional abelian Lie algebra.

9.2 $\mathfrak{L}^1_7$

Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & \sqrt{-1} \\
0 & \sqrt{-1} & -1
\end{bmatrix}
$$

and let us denote by $\mathfrak{L}^1_7$ the hom-Lie algebra $(\mathfrak{L}_7, A)$. We have:

$\text{Aut}(\mathfrak{L}^1_7) = \left\{ \begin{bmatrix} 1 & \sqrt{-1}t & -t \\
-\sqrt{-1}t & \frac{1}{2}t^2 + 1 & \frac{1}{2} \sqrt{-1}t^2 \\
t & \frac{1}{2} \sqrt{-1}t^2 & 1 - \frac{1}{2} t^2 \\
\end{bmatrix} \in M(3, \mathbb{C}) : t \in \mathbb{C} \right\}$

$\bigcup \left\{ \begin{bmatrix} 1 & \sqrt{-1}t & -t \\
-\sqrt{-1}t & -\frac{1}{2}t^2 - 1 & -\frac{1}{2} \sqrt{-1}t^2 \\
t & -\frac{1}{2} \sqrt{-1}t^2 & -1 + \frac{1}{2} t^2 \\
\end{bmatrix} \in M(3, \mathbb{C}) : t \in \mathbb{C} \right\}$

$\text{Der}(\mathfrak{L}^1_7) = \left\{ \begin{bmatrix} -\sqrt{-1}t_1 & 0 & 0 \\
t_1 & 0 & 0 \\
\end{bmatrix} \in M(3, \mathbb{C}) : t_1 \in \mathbb{C} \right\} \cong \mathbb{C}$.

9.2.1 $\psi_{\alpha, \beta}(\mathfrak{L}^1_7)$

In this case, we have $\psi_{\alpha, \beta}(\mathfrak{L}^1_7) = t_{\alpha, \beta}$:

$t_{\alpha, \beta} = \{[e_1, e_2] = (i\alpha - i\beta) e_2 + (1 + \beta - \alpha) e_3, [e_1, e_3] = (-1 - \alpha + \beta) e_2 + (-i\alpha + i\beta) e_3, [e_2, e_3] = e_1 \}$. 

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is a Lie algebra isomorphic to so(3, C). In fact, by considering the matrix of the Killing form of $t_{\alpha, \beta}$ in the basis $\{e_1, e_2, e_3\}$
\[
\begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 - 2\beta + 2\alpha & 2i\alpha - 2i\beta \\
0 & 2i\alpha - 2i\beta & -2 - 2\alpha + 2\beta
\end{bmatrix}
\]
we note that it is not degenerate and it follows the conclusion.

9.2.2 $\phi_\beta(L^1)$
In this case, we have $\phi_\beta(L^1) = \tilde{t}_\beta$:
\[
\tilde{t}_\beta = \{[e1, e2] = (i - i\beta) e2 + (\beta - 1) e3, [e1, e3] = (\beta - 1) e2 + (-i + i\beta) e3\}.
\]
We note that $\tilde{t}_\beta$ is an almost abelian Lie algebra where $\text{span}_C\{e_2, e_3\}$ is a codimension 1 abelian ideal of $\tilde{t}_\beta$; we can identify $\tilde{t}_\beta$ the matrix
\[
\tilde{B} = \begin{bmatrix}
i - i\beta & \beta - 1 \\
\beta - 1 & -i + i\beta
\end{bmatrix}.
\]
The isomorphism class of $\tilde{t}_\beta$ is determined by the eigenvalues of $\tilde{B}$:
1. $\tilde{B}$ has repeated eigenvalues if and only if $0 = 0$
   (a) 0 is a repeated eigenvalue of $\tilde{B}$ if and only if $0 = 0$
   i. $\tilde{t}_\beta$ is the 3-dimensional abelian Lie algebra if and only if $\tilde{B} = 0$ if and only if $\beta = 1$
   ii. $\tilde{t}_\beta$ is isomorphic to $n_3(C)$ if and only if $\tilde{B} \neq 0$ if and only if $\beta \neq 1$
   (b) $\tilde{B}$ has a non-zero repeated eigenvalue if and only if $1 = 0$
2. $\tilde{B}$ has distinct eigenvalues if and only if $1 = 0$

9.2.3 $\rho(L^1)$
In this case, we have $\rho(L^1) = \hat{t}$:
\[
\hat{t} = \{[e1, e2] = -ie2 + e3, [e1, e3] = e2 + ie3\}.
\]
which is isomorphic to $n_3$

9.2.4 Identities
- $A[\cdot, \cdot] + [A\cdot, \cdot] + [\cdot, A\cdot] = 0$ (Generalized Derivation)
- $[A\cdot, A\cdot] = 0$

The hom-Lie algebra $L^1$ is a non-multiplicative hom-Lie algebra.

9.3 $L^2$
Let $A$ be the endomorphism of $\mathbb{C}^3$ whose matrix representation with respect to the ordered basis $\{e_1, e_2, e_3\}$ is
\[
\begin{bmatrix}
0 & 1 & \sqrt{-1} \\
1 & 0 & 0 \\
\sqrt{-1} & 0 & 0
\end{bmatrix}
\]
and let us denote by \( L^2_7 \) the hom-Lie algebra \((L^7, A)\). We have:

\[
\text{Aut}(L^2_7) = \begin{cases} 
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{cases}, \\
\text{Der}(L^2_7) = \begin{cases} 
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{cases}.
\]

\[9.3.1\quad \psi_{\alpha,\beta}(L^2_7)\]

In this case, we have \( \psi_{\alpha,\beta}(L^2_7) = t_{\alpha,\beta} \):

\[
t_{\alpha,\beta} = \{ [e1, e2] = (i\alpha - i\beta) e1 + e3, [e1, e3] = (-\alpha + \beta) e1 - e2, [e2, e3] = e1 + (\alpha - \beta) e2 + (i\alpha - i\beta) e3. \]

is a Lie algebra isomorphic to \( \text{sl}(2, \mathbb{C}) \). In fact, by considering the matrix of the Killing form of \( t_{\alpha,\beta} \) in the basis \( \{e_1, e_2, e_3\} \):

\[
\begin{pmatrix} 
-2 & 2\alpha - 2\beta & 2i\alpha - 2i\beta \\
2\alpha - 2\beta & -(\alpha - \beta)^2 - 2 & -2i(\alpha - \beta)^2 \\
2i\alpha - 2i\beta & -2i(\alpha - \beta)^2 & 2(\alpha - \beta)^2 - 2 
\end{pmatrix}
\]

we note that it is not degenerate and it follows the conclusion.

\[9.3.2\quad \phi_{\beta}(L^2_7)\]

In this case, we have \( \phi_{\beta}(L^2_7) = \tilde{t}_{\beta} \):

\[
\tilde{t}_{\beta} = \{ [e1, e2] = (i - i\beta) e1, [e1, e3] = (-1 + \beta) e1, [e2, e3] = (1 - \beta) e2 + (i - i\beta) e3. \]

is a unimodular Lie algebra.

We note that \( t_{\beta} \) is 3-dimensional abelian Lie algebra if and only if \( \beta = 1 \). In other case, \( \tilde{t}_{\beta} \) is isomorphic to \( \text{r}_{3,-1} \) In fact, we can consider the change of basis given by

\[
g = \begin{pmatrix} 
0 & -i(-1 + \beta) & -1 + \beta \\
0 & 1 & 0 \\
1 & 0 & 0 
\end{pmatrix}
\]

we have \( g \cdot \tilde{t}_{\beta} = \text{r}_{3,-1} \).

\[9.3.3\quad \rho(L^2_7)\]

In this case, we have \( \rho(L^2_7) = \hat{t} \):

\[
\hat{t} = \{ [e1, e2] = -i e1, [e1, e3] = e1, [e2, e3] = -e2 - ie3. \}

which is isomorphic to \( \text{r}_{3,-1} \). In fact, by considering the change of basis given by

\[
g := \begin{pmatrix} 
0 & -i & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 
\end{pmatrix}
\]

we have \( g \cdot \hat{t} = \text{r}_{3,-1} \).
9.3.4 Identities

- $[A, \cdot] + [A \cdot, \cdot] + [\cdot, A \cdot] = 0$ (Generalized Derivation)
- $A^2[\cdot, \cdot] = [A, A \cdot]$  

The hom-Lie algebra $\mathfrak{L}_7^2$ is a non-multiplicative hom-Lie algebra.

9.4 Degenerations between hom-Lie algebras ($\mathfrak{L}_7^i$)

If $\mathfrak{L}_7^j \xrightarrow{\text{deg}} \mathfrak{L}_7^k$ then $\text{Der}(\mathfrak{L}_7^j) \leq \text{Der}(\mathfrak{L}_7^k)$. Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|---------|-----------------|
| 0       | $\mathfrak{L}_7^2$ |
| 1       | $\mathfrak{L}_7^1$ |
| 3       | $\mathfrak{L}_7^0$ |

9.4.1 Degenerations

1. $\mathfrak{L}_7^2 \xrightarrow{\text{deg}} \mathfrak{L}_7^1$

   In fact, set
   \[
g(t) = \frac{1}{8t^3} \begin{bmatrix} 4t & -8t^3 & 4\sqrt{-1}t \\ 0 & 4t^4 + 4t^2 - 1 & 4t^2 - \sqrt{-1} (4t^4 - 4t^2 + 1) \\ \sqrt{-1} (-1 + 4t^4 - 4t^2) & 4\sqrt{-1}t^2 & 1 + 4t^2 + 4t^4 \end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]
   whose matrix inverse is $g(t)^T$. We have $g(t) \in \text{Aut}(\mathfrak{L}_7)$ and $g(t) \cdot \mathfrak{L}_7^2$ is the hom-Lie algebra $(\mathfrak{L}_7, A(t))$ with
   \[
   A(t) = \begin{bmatrix} 0 & -t & -\sqrt{-1}t \\ -t & 1 & \sqrt{-1} \\ -\sqrt{-1}t & \sqrt{-1} & -1 \end{bmatrix}
   \]
   It is easy to check that $(\mathfrak{L}_7, A(t)) \to \mathfrak{L}_7^1$ as $t$ tends to zero.

2. $\mathfrak{L}_7^1 \xrightarrow{\text{deg}} \mathfrak{L}_7^0$

   In fact, set
   \[
g(t) = \frac{1}{2t} \begin{bmatrix} 2t & 0 & 0 \\ 0 & -1 - t^2 & \sqrt{-1} (-1 + t^2) \\ 0 & -\sqrt{-1} (-1 + t^2) & -1 - t^2 \end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]
   We have $g(t) \in \text{Aut}(\mathfrak{L}_7)$ and $g(t) \cdot \mathfrak{L}_7^2$ is the hom-Lie algebra $(\mathfrak{L}_7, A(t))$ with
   \[
   A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t^2 & \sqrt{-1}t^2 \\ 0 & \sqrt{-1}t^2 & -t^2 \end{bmatrix}
   \]
It is easy to check that \((\mathcal{L}_7, A(t)) \rightarrow \mathcal{L}_7^0\) as \(t\) tends to zero.
9.5 Degenerations between hom-Lie algebras \((\mathfrak{L}_j^i)\) with \(\mathfrak{L}_j^i\) being an unimodular Lie algebra

If \(\mathfrak{L}_j^i \xrightarrow{\deg} \mathfrak{L}_h^k\) then \(\mathfrak{L}_i^j \xrightarrow{\deg} \mathfrak{L}_h^k\) and \(\text{Der}(\mathfrak{L}_j^i) \leq \text{Der}(\mathfrak{L}_h^k)\).

Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|----------------|
| 0        | \(\mathfrak{L}_j^i\) |
| 1        | \(\mathfrak{L}_j^i\), \(\mathfrak{L}_j^*(\lambda), \mathfrak{L}_j^*\) |
| 2        | \(\mathfrak{L}_j^*(\lambda), \mathfrak{L}_j^2, \mathfrak{L}_j^i, \mathfrak{L}_j^i\) |
| 3        | \(\mathfrak{L}_j^3\), \(\mathfrak{L}_j^4\), \(\mathfrak{L}_j^5\) |
| 4        | \(\mathfrak{L}_j^4\), \(\mathfrak{L}_j^4\) |
| 5        | \(\mathfrak{L}_j^5\), \(\mathfrak{L}_j^5\) |
| 6        | \(\mathfrak{L}_j^6\), \(\mathfrak{L}_j^6\) |
| 9        | \(\mathfrak{L}_j^9\) |

1. \(\mathfrak{L}_j^2 \xrightarrow{\deg} \mathfrak{L}_j^5\)
   In fact, set
   \[
g(t) = \begin{bmatrix}
0 & e^t & \sqrt{-1}e^t \\
0 & e^t & \frac{1}{2} \sqrt{-1}e^{2t} & -\frac{1}{2} e^{2t} \\
\frac{1}{2} \sqrt{-1}e^{2t} & \frac{1}{8} e^t (4 + e^{2t}) & \frac{1}{8} \sqrt{-1}e^t (-4 + e^{2t})
\end{bmatrix}, \text{with } t \in \mathbb{R}.
   \]
   We have \(g(t) \cdot \mathfrak{L}_j^2\) is the hom-Lie algebra \((\mu(t), A)\) with
   \[
   [e_1, e_2] = -\sqrt{-1}e^{-t}e_t + e_2, [e_1, e_3] = \sqrt{-1}e^{-t}e_2 - e_3, [e_2, e_3] = -\sqrt{-1}e^{-t}e_3,
   \]
   \[
   A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
   \]
   It is easy to check that \((\mu(t), A) \to \mathfrak{L}_j^5\) as \(t\) tends to infinity.

2. \(\mathfrak{L}_j^3 \xrightarrow{\deg} \mathfrak{L}_j^3\)
   In fact, set
   \[
g(t) = \begin{bmatrix}
-\sqrt{-1} & 0 & 0 \\
0 & -\frac{1}{2} \sqrt{-1}e^t & -\frac{1}{2} e^t \\
0 & -\sqrt{-1}e^t & e^t
\end{bmatrix}, \text{with } t \in \mathbb{R}.
   \]
   We have \(g(t) \cdot \mathfrak{L}_j^3\) is the hom-Lie algebra \((\mu(t), A)\) with
   \[
   [e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e^{-2t}e_t
   \]
   \[
   A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
   \]
   It is easy to check that \((\mu(t), A) \to \mathfrak{L}_j^3\) as \(t\) tends to infinity.

3. \(\mathfrak{L}_j^4 \xrightarrow{\deg} \mathfrak{L}_j^2\)
   In fact, set
   \[
g(t) = \begin{bmatrix}
0 & e^t & \sqrt{-1}e^t \\
e^t & 0 & -\sqrt{-1}e^t \\
2 \sqrt{-1}e^{2t} + e^t & 2e^{3t} - 2 \sqrt{-1}e^{2t} & 2 \sqrt{-1}e^{3t} + 2e^{2t} - \sqrt{-1}e^t
\end{bmatrix}, \text{with } t \in \mathbb{R}.
   \]
We have $g(t) \cdot \mathfrak{L}_1^2$ is the hom-Lie algebra $(\mu(t), A)$ with

\[
\begin{cases}
[e_1, e_2] = \frac{1}{2} e^{-2t} e_1 + e_2, [e_1, e_3] = -\frac{1}{2} e^{-2t} e_1 - e_3, [e_2, e_3] = \frac{1}{2} e^{-2t} e_2 + \frac{1}{2} e^{-2t} e_3 \\
A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\end{cases}
\]

It is easy to check that $(\mu(t), A) \rightarrow \mathfrak{L}_1^2$ as $t$ tends to infinity.

4. $\mathfrak{L}_4^0 \xrightarrow{\text{deg}} \mathfrak{L}_1^0$

In fact, set

\[
g(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have $g(t) \cdot \mathfrak{L}_1^0$ is the hom-Lie algebra $(\mu(t), A)$ with

\[
\begin{cases}
[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e^{-2t} e_1 \\
A = 0
\end{cases}
\]

It is easy to check that $(\mu(t), A) \rightarrow \mathfrak{L}_1^0$ as $t$ tends to infinity.

5. $\mathfrak{L}_4^0(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_1^5$

In fact, set

\[
g(t) = \begin{bmatrix}
-e^{2t} & 0 & 0 \\
-\frac{1}{2} \frac{e^{3t} \sqrt{\lambda}}{\sqrt{\lambda}} - e^{2t} & -e^{2t} & -e^{2t} \\
0 & -\frac{1}{2} \frac{e^{3t} \sqrt{\lambda}}{\sqrt{\lambda}} - e^{2t} & -\frac{1}{2} \sqrt{\lambda} e^{3t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have $g(t) \cdot \mathfrak{L}_1^5$ is the hom-Lie algebra $(\mu(t), A)$ with

\[
\begin{cases}
[e_1, e_2] = \left( \frac{\sqrt{\lambda}}{e^t} + e^{-2t} \right) e_2 + \left( 1 + \frac{\sqrt{\lambda}}{e^{2t}} \right) e_3, [e_1, e_3] = -2 e^{-2t} \lambda e_2 - \left( \frac{\sqrt{\lambda}}{e^t} + e^{-2t} \right) e_3 \\
A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\end{cases}
\]

It is easy to check that $(\mu(t), A) \rightarrow \mathfrak{L}_1^5$ as $t$ tends to infinity.

6. $\mathfrak{L}_4^5 \xrightarrow{\text{deg}} \mathfrak{L}_1^5$

In fact, set

\[
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
\frac{1}{2} e^{2t} & e^t & 0 \\
0 & \frac{1}{2} e^{2t} & e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have $g(t) \cdot \mathfrak{L}_1^5$ is the hom-Lie algebra $(\mu(t), A)$ with

\[
\begin{cases}
[e_1, e_2] = e^{-t} e_2 + e_3, [e_1, e_3] = -e^{-t} e_3 \\
A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\end{cases}
\]

It is easy to check that $(\mu(t), A) \rightarrow \mathfrak{L}_1^5$ as $t$ tends to infinity.
7. \( \mathfrak{L}_4^1(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)

In fact, set

\[
g(t) = \begin{bmatrix}
0 & \frac{1}{2} \lambda^{-1} & -\frac{1}{2} \lambda^{-1} \\
0 & 1 & 1 \\
2 \lambda e^{2t} & e^t & e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \cdot \mathfrak{L}_4^1(\lambda) \) is the hom-Lie algebra \((\mu(t), A)\) with

\[
\begin{aligned}
[e_1, e_2] &= e^{-t} e_2 + e_3, \\
[e_1, e_3] &= -e^{-2t} e_2 - e^{-t} e_3, \\
[e_2, e_3] &= -\frac{e^{-2t}}{4\lambda^2} e_1
\end{aligned}
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

It is easy to check that \((\mu(t), A) \to \mathfrak{L}_1^2\) as \(t\) tends to infinity.

8. \( \mathfrak{L}_4^2 \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)

In fact, set

\[
g(t) = \begin{bmatrix}
0 & e^t & e^t \\
2 e^t & 0 & 0 \\
0 & -2 e^{2t} & 2 e^{2t}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \cdot \mathfrak{L}_4^2 \) is the hom-Lie algebra \((\mu(t), A)\) with

\[
\begin{aligned}
[e_1, e_2] &= e_3, \\
[e_1, e_3] &= -\frac{\lambda}{4} e^{-2t} e_1
\end{aligned}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \((\mu(t), A) \to \mathfrak{L}_1^2\) as \(t\) tends to infinity.

9. \( \mathfrak{L}_4^1 \xrightarrow{\text{deg}} \mathfrak{L}_1^1 \)

In fact, set

\[
g(t) = \begin{bmatrix}
0 & 0 & 1 \\
e^t & 0 & 0 \\
0 & e^t & \frac{1}{2} e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \cdot \mathfrak{L}_4^1 \) is the hom-Lie algebra \((\mu(t), A)\) with

\[
\begin{aligned}
[e_1, e_2] &= e^{-t} e_1 + e_3, \\
[e_2, e_3] &= e^{-t} e_3
\end{aligned}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \((\mu(t), A) \to \mathfrak{L}_1^1\) as \(t\) tends to infinity.

10. \( \mathfrak{L}_4^0 \xrightarrow{\text{deg}} \mathfrak{L}_1^0 \)

In fact, set

\[
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} e^t & 1
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_0^4 \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
\{ [e_1, e_2] &= e^{-t}e_2 + e_3, [e_1, e_3] = -e^{-t}e_3 \\
A &= 0.
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to \mathfrak{L}_1^0\) as \( t \) tends to infinity.

11. \( \mathfrak{L}_1^0 \xrightarrow{\text{deg}} \mathfrak{L}_0^2 \)
In fact, set
\[
g(t) = \begin{bmatrix} 0 & 0 & e^t \\ 0 & e^t & 0 \\ e^t & 0 & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_1^0 \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_2, e_3] &= -e^{-t}e_1 \\
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to \mathfrak{L}_0^2\) as \( t \) tends to infinity.

12. \( \mathfrak{L}_1^1 \xrightarrow{\text{deg}} \mathfrak{L}_0^1 \)
In fact, set
\[
g(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_1^1 \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= e^{-t}e_2 \\
A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to \mathfrak{L}_0^1\) as \( t \) tends to infinity.

13. \( \mathfrak{L}_0^1 \xrightarrow{\text{deg}} \mathfrak{L}_0^0 \)
In fact, set
\[
g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_0^1 \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= e^{-t}e_3 \\
A &= 0
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to \mathfrak{L}_0^0\) as \( t \) tends to infinity.

14. \( \mathfrak{L}_2^4 \xrightarrow{\text{deg}} \mathfrak{L}_4^2(\lambda) \) with \( \lambda \neq 0 \).
Suppose, contrary to our claim, that \( \mathfrak{L}_2^4 \xrightarrow{\text{deg}} \mathfrak{L}_4^2(\lambda) \) with \( \lambda \neq 0 \). In such case, we have
\[
\phi_{\beta}(\mathfrak{L}_2^4) \xrightarrow{\text{deg}} \phi_{\beta}(\mathfrak{L}_4^2(\lambda)).
\]
By taking $\beta = 1$, we have from 9.3.2 that $\phi_\beta(L^2_7)$ is the 3-dimensional Lie abelian Lie algebra therefore, $\phi_\beta(L^2_1(\lambda))$ is the Lie algebra $a_3$. But, from 6.5.2, $\phi_\beta(L^2_1(\lambda))$ is isomorphic to $\tau_{3,1}$; this is a contradiction.

15. $L^2_7 \xrightarrow{\text{deg}} L^1_3$.

Suppose, contrary to our claim, that $L^2_7 \xrightarrow{\text{deg}} L^3_\lambda$. In such case, we have

$$\phi_\beta(L^2_7) \xrightarrow{\text{deg}} \phi_\beta(L^3_1).$$

By taking $\beta = 1$, we have from 9.3.2 that $\phi_\beta(L^2_7)$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_\beta(L^3_1)$ is the Lie algebra $a_3$. But, from 3.4.2, $\phi_\beta(L^3_1)$ is isomorphic to $\tau_{3,1}$; this is a contradiction.

16. $L^1_4 \xrightarrow{\text{deg}} L^5_1$.

Suppose, contrary to our claim, that $L^1_4 \xrightarrow{\text{deg}} L^5_1$. In such case, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{-1} \\ 0 & \sqrt{-1} & -1 \end{pmatrix} \xrightarrow{\text{deg}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

17. $L^0_7 \xrightarrow{\text{deg}} L^1_0$.

Suppose, contrary to our claim, that $L^0_7 \xrightarrow{\text{deg}} L^1_0$. In such case, we have

$$0 \xrightarrow{\text{deg}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 2, we have a contradiction.

18. $L^0_4(\lambda) \xrightarrow{\text{deg}} L^3_1$ with $(\lambda \neq 0)$.

Suppose, contrary to our claim, that $L^0_4(\lambda) \xrightarrow{\text{deg}} L^3_1$ with $(\lambda \neq 0)$. In such case, we have

$$\psi_{\alpha, \beta}^{(0)}(L^0_4(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha, \beta}(L^3_1).$$

By taking $\alpha = -\beta$ and $\alpha = \pm \frac{\sqrt{-1}}{\lambda}$, we have from 6.7.1 that $\psi_{\alpha, \beta}(L^0_4(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L^3_1)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha, \beta}(L^3_1)$ is isomorphic to $n_3$ (from 3.4.1). But, we now apply again 3.4.1, to obtain $\alpha = \beta = 0$; this is a contradiction.

19. $L^5_1 \xrightarrow{\text{deg}} L^3_1$.

Suppose, contrary to our claim, that $L^5_1 \xrightarrow{\text{deg}} L^3_1$. In such case, we have

$$\phi_\beta(L^5_1) \xrightarrow{\text{deg}} \phi_\beta(L^3_1).$$

By taking $\beta = 1$, we have from 6.6.2 that $\phi_\beta(L^5_1)$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_\beta(L^3_1)$ is the Lie algebra $a_3$. But, from 3.4.2, $\phi_\beta(L^3_1)$ is isomorphic to $\tau_{3,1}$; this is a contradiction.
20. $L_4^4(\lambda) \xrightarrow{\text{deg}} L_0^2$.
Suppose, contrary to our claim, that $L_4^4(\lambda) \xrightarrow{\text{deg}} L_0^2$. In such case, we have
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

21. $L_3^4 \xrightarrow{\text{deg}} L_1^2$.
Suppose, contrary to our claim, that $L_3^4 \xrightarrow{\text{deg}} L_1^2$. In such case, we have
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
If $\{\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \ldots, \alpha_{26} = 0\}$ we have $\vartheta_{\alpha_0, \ldots, \alpha_{26}}(L_3^4)$ is the pair
\[
(\nu_1, A_1) = \begin{cases}
\nu_1 = \{e_3 \cdot e_1 = -e_2, 0, 0, 0\} \\
A_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\end{cases}
\]
and $\vartheta_{\alpha_0, \ldots, \alpha_{26}}(L_1^2)$ is the pair
\[
(\nu_2, A_2) = \begin{cases}
\nu_2 = \{e_2 \cdot e_3 = -e_3, 0, 0, 0\} \\
A_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{cases}
\]
Therefore
\[
\dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9 (\nu_1, A_1)})) \leq \dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9 (\nu_2, A_2)})).
\tag{2}
\]
Now, we consider $\{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0, \beta_1 = 1, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 1\}$.
The set of 3-by-3 matrices that commute with $A_1$ is
\[
\text{gl}(n, \mathbb{C})_{A_1} = \left\{ \begin{bmatrix}
t_1 & 0 & t_4 \\
t_3 & t_2 & t_5 \\
0 & 0 & t_2
\end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]
Given $D_1(u^s)$, $D_2(x^s)$ and $D_3(y^s)$ in $\text{gl}(n, \mathbb{C})_{A_1}$, with $D_2$ and $D_3$ equal to zero, we have $\lambda_1 := D_1 \nu(\cdot, \cdot) + \nu(D_2 \cdot, \cdot) + \nu(\cdot, D_3 \cdot)$,
\[
\lambda_1 := \{ e_3 \cdot e_1 = (-w_{3,3}) e_2, \}
\]
and so $\dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9 (\nu_1, A_1)})) = 4.$
On the other hand, the set of 3-by-3 matrices that commute with $A_2$ is

$$\mathfrak{gl}(n, \mathbb{C})_{A_2} = \left\{ \begin{bmatrix} t_1 & t_3 & t_4 \\ 0 & t_1 & 0 \\ 0 & t_5 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.$$ 

and given $D_1(w's)$, $D_2(x's)$ and $D_3(y's)$ in $\mathfrak{gl}(n, \mathbb{C})_{A_2}$, with $D_2$ and $D_3$ equal to zero, we have $\lambda_2 := D_1 \nu(v, \cdot) + \nu(D_2', \cdot) + \nu(\cdot, D_3')$,

and so $\dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6}; \beta_1, \ldots, \beta_9(A_2))) = 3$; this is a contradiction.

22. $\mathfrak{L}_4^3 \xrightarrow{\deg} \mathfrak{L}_0^2$.
Suppose, contrary to our claim, that $\mathfrak{L}_4^3 \xrightarrow{\deg} \mathfrak{L}_0^2$. In such case, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\deg} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

23. $\mathfrak{L}_4^2 \xrightarrow{\deg} \mathfrak{L}_0^2$.
Suppose, contrary to our claim, that $\mathfrak{L}_4^2 \xrightarrow{\deg} \mathfrak{L}_0^2$. In such case, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\deg} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

24. $\mathfrak{L}_4^0 \xrightarrow{\deg} \mathfrak{L}_0^1$.
Suppose, contrary to our claim, that $\mathfrak{L}_4^0 \xrightarrow{\deg} \mathfrak{L}_0^1$. In such case, we have

$$0 \xrightarrow{\deg} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 1 and the second one has nilpotent degree equal to 2, we have a contradiction.

25. $\mathfrak{L}_4^1 \xrightarrow{\deg} \mathfrak{L}_0^2$.
Suppose, contrary to our claim, that $\mathfrak{L}_4^1 \xrightarrow{\deg} \mathfrak{L}_0^2$. In such case, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\deg} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.
9.6 Degenerations between hom-Lie algebras \( (\mathcal{L}_j^i) \) with \( j = 6, 1, 0 \)

If \( \mathcal{L}_i^j \xrightarrow{\text{deg}} \mathcal{L}_h^k \) then \( \mathcal{L}_i \xrightarrow{\text{deg}} \mathcal{L}_h \) and \( \text{Der}(\mathcal{L}_j^i) \leq \text{Der}(\mathcal{L}_h^k) \).

Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 0        | \( \mathcal{L}_i^6(\lambda), \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 1        | \( \mathcal{L}_i^6(\lambda), \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 2        | \( \mathcal{L}_i^6(\lambda), \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 3        | \( \mathcal{L}_i^6, \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 4        | \( \mathcal{L}_i^6, \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 5        | \( \mathcal{L}_i^6, \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 6        | \( \mathcal{L}_i^6, \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |
| 9        | \( \mathcal{L}_i^6, \mathcal{L}_i^4, \mathcal{L}_i^2 \) \( \mathcal{L}_{i1}^4 \) |

9.6.1 Degenerations

1. \( \mathcal{L}_0^6(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_1^5 \)
   In fact, set
   \[
g(t) = \begin{bmatrix}
   e^{2t} & 0 & 0 \\
   \frac{e^{2t}(\sqrt{-1}e^{\lambda}+\sqrt{\lambda})}{\lambda^{3/2}} & e^{2t} & e^{2t} \\
   0 & \frac{e^{2t}(\sqrt{-1}e^{\lambda}+\sqrt{\lambda})}{\lambda^{3/2}} & \frac{e^{2t}}{\sqrt{\lambda}}
   \end{bmatrix}, \text{ with } t \in \mathbb{R}.
   \]
   We have \( g(t) : \mathcal{L}_0^6(\lambda) \) is the hom-Lie algebra \((\mu(t), A)\) with
   \[
   [e_1, e_2] = -\sqrt{-1}\sqrt{\lambda}e_2 + \left(1 - \frac{e^{2t}}{e^{\sqrt{\lambda}}}\right) e_3
   
   [e_1, e_3] = e^{2t}\lambda e_3 + \left(\sqrt{-1}\sqrt{\lambda} + e^{-2t}\right) e_3
   
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0
   \end{bmatrix}
   \]
   It is easy to check that \((\mu(t), A) \rightarrow \mathcal{L}_1^5\) as \( t \) tends to infinity.

2. \( \mathcal{L}_0^8 \xrightarrow{\text{deg}} \mathcal{L}_1^5 \)
   In fact, set
   \[
g(t) = \begin{bmatrix}
   e^t & 0 & 0 \\
   -e^{2t} & 0 & e^t \\
   0 & e^t & -e^{2t}
   \end{bmatrix}, \text{ with } t \in \mathbb{R}.
   \]
   We have \( g(t) : \mathcal{L}_0^8 \) is the hom-Lie algebra \((\mu(t), A)\) with
   \[
   [e_1, e_2] = e_3, [e_1, e_3] = e^{-t} e_3
   
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0
   \end{bmatrix}
   \]
   It is easy to check that \((\mu(t), A) \rightarrow \mathcal{L}_1^5\) as \( t \) tends to infinity.
3. $L^7_0 \to L^5_1$

In fact, set

$$g(t) = \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} & e^t & 0 \\ 0 & e^{2t} & e^t \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \cdot L^7_0$ is the hom-Lie algebra $(\mu(t), A)$ with

$$[e_1, e_2] = e^{-t}e_3 + e_3, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(\mu(t), A) \to L^5_1$ as $t$ tends to infinity.

4. $L^6_0(\lambda) \to L^2_1$

In fact, set

$$g(t) = \begin{bmatrix} 0 & e^{-t} & 0 \\ 0 & 0 & \lambda e^{-t} \lambda e^{-t} \\ e^{2t} & 1 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \cdot L^6_0(\lambda)$ is the hom-Lie algebra $(\mu(t), A)$ with

$$[e_1, e_2] = e^{-t}e_3 + \lambda e^{-t}e_2 + e_3, [e_1, e_3] = -\lambda e^{-2t}e_1 - \lambda e^{-2t}e_2 - \lambda e^{-t}e_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(\mu(t), A) \to L^2_1$ as $t$ tends to infinity.

5. $L^3_0 \to L^2_1$

In fact, set

$$g(t) = \begin{bmatrix} 0 & 0 & e^t \\ e^t & 0 & 0 \\ 0 & -(e^t)^2 & (e^t)^2 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$ 

We have $g(t) \cdot L^3_0$ is the hom-Lie algebra $(\mu(t), A)$ with

$$[e_1, e_2] = e_3, [e_2, e_3] = e^{-t}e_3$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $(\mu(t), A) \to L^2_1$ as $t$ tends to infinity.

6. $L^2_0 \to L^1_1$

In fact, set

$$g(t) = \begin{bmatrix} 0 & 1 & 0 \\ e^t & 0 & 0 \\ 0 & -e^t & e^t \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$
We have $g(t) \cdot L_2^6$ is the hom-Lie algebra $(\mu(t), A)$ with

$$
\begin{align*}
[e_1, e_2] &= -e^{-t} e_1 + e_3 \\
A &= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\end{align*}
$$

It is easy to check that $(\mu(t), A) \to L_1^1$ as $t$ tends to infinity.

7. $L_0^1 \xrightarrow{\deg} L_1^1$

In fact, set

$$
g(t) = \begin{bmatrix}
0 & 0 & 1 \\
e^t & 0 & 0 \\
0 & e^t & 1
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
$$

We have $g(t) \cdot L_1^1$ is the hom-Lie algebra $(\mu(t), A)$ with

$$
\begin{align*}
[e_1, e_2] &= e_3, [e_2, e_3] = e^{-t} e_3 \\
A &= \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\end{align*}
$$

It is easy to check that $(\mu(t), A) \to L_1^1$ as $t$ tends to infinity.

8. $L_0^0 \xrightarrow{\deg} L_1^0$

In fact, set

$$
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
0 & 1 & 0 \\
0 & e^t & 1
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
$$

We have $g(t) \cdot L_1^0$ is the hom-Lie algebra $(\mu(t), A)$ with

$$
\begin{align*}
[e_1, e_2] &= -e^{-t} e_2 + e_3 \\
A &= 0
\end{align*}
$$

It is easy to check that $(\mu(t), A) \to L_1^0$ as $t$ tends to infinity.

9. $L_0^1(\lambda) \xrightarrow{\deg} L_1^1$ with $(\lambda \neq 0)$.

Suppose, contrary to our claim, that $L_0^1(\lambda) \xrightarrow{\deg} L_1^1$ with $(\lambda \neq 0)$. In such case, we have

$$
\psi_{\alpha,\beta}(L_0^1(\lambda)) \xrightarrow{\deg} \psi_{\alpha,\beta}(L_1^1).
$$

By taking $\alpha = 0$ and $\beta = -1/\lambda$, we have from 8.14.1 that $\psi_{\alpha,\beta}(L_0^1(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha,\beta}(L_1^1)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha,\beta}(L_1^1)$ is isomorphic to $n_3$ (from 3.4.1). But, we now apply again 3.4.1, to obtain $\alpha = \beta = 0$; this is a contradiction.

10. $L_0^1 \xrightarrow{\deg} L_1^3$

Suppose, contrary to our claim, that $L_0^1 \xrightarrow{\deg} L_1^3$. In such case, we have

$$
\rho(L_0^1) \xrightarrow{\deg} \rho(L_1^3).
$$

We have from 8.13.3 that $\rho(L_0^1)$ is a Lie algebra isomorphic to $n_3$, therefore, $\rho(L_1^3)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$(by Theorem 1.1). But, from 3.4.3, $\rho(L_1^3)$ is isomorphic to $\mathfrak{u}_2 \times \mathbb{C}$; this is a contradiction.
11. \( \mathfrak{L}_{6}^{11} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{3} \)

Suppose, contrary to our claim, that \( \mathfrak{L}_{6}^{11} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{3} \). In such case, we have

\[ \phi_{\beta}(\mathfrak{L}_{6}^{11}) \xrightarrow{\text{deg}} \phi_{\beta}(\mathfrak{L}_{1}^{3}). \]

By taking \( \beta = -1 \), we have from 8.12.2 that \( \phi_{\beta}(\mathfrak{L}_{6}^{11}) \) is a Lie algebra isomorphic to \( \mathfrak{r}_{2} \times \mathbb{C} \), therefore, \( \phi_{\beta}(\mathfrak{L}_{1}^{3}) \) is a Lie algebra which is isomorphic to one of the following: \( \mathfrak{r}_{2} \times \mathbb{C}, \mathfrak{n}_{3} \) or \( \mathfrak{a}_{3} \) (by Theorem 1.1). But, from 3.4.2, \( \phi_{\beta}(\mathfrak{L}_{1}^{3}) \) is isomorphic to \( \mathfrak{r}_{3} \), this is a contradiction.

12. \( \mathfrak{L}_{6}^{10} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{5} \)

Suppose, contrary to our claim, that \( \mathfrak{L}_{6}^{10} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{5} \). In such case, we have

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

13. \( \mathfrak{L}_{6}^{6}(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_{0}^{2} \)

Suppose, contrary to our claim, that \( \mathfrak{L}_{6}^{6}(\lambda) \xrightarrow{\text{deg}} \mathfrak{L}_{0}^{2} \). In such case, we have

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

14. \( \mathfrak{L}_{6}^{5} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{2} \)

Suppose, contrary to our claim, that \( \mathfrak{L}_{6}^{5} \xrightarrow{\text{deg}} \mathfrak{L}_{1}^{2} \). In such case, we have

\[ \varphi_{\alpha_{0},\ldots,\alpha_{26}}(\mathfrak{L}_{6}^{5}) \xrightarrow{\text{deg}} \varphi_{\alpha_{0},\ldots,\alpha_{26}}(\mathfrak{L}_{1}^{2}). \]

If \( \{\alpha_{0} = 0, \alpha_{1} = 0, \alpha_{2} = 1, \alpha_{3} = 0, \ldots, \alpha_{26} = 0\} \) we have \( \varphi_{\alpha_{0},\ldots,\alpha_{26}}(\mathfrak{L}_{6}^{5}) \) is the Abelian Lie algebra and \( \varphi_{\alpha_{0},\ldots,\alpha_{26}}(\mathfrak{L}_{1}^{2}) \) is the non Lie algebra

\[
\left\{ e_{2} \cdot e_{2} = e_{3} \right\}
\]

this is a contradiction.

15. \( \mathfrak{L}_{6}^{5} \xrightarrow{\text{deg}} \mathfrak{L}_{0}^{2} \)

Suppose, contrary to our claim, that \( \mathfrak{L}_{6}^{5} \xrightarrow{\text{deg}} \mathfrak{L}_{0}^{2} \). In such case, we have

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.
16. \( L_0^4 \xrightarrow{\deg} L_1^2 \).

Suppose, contrary to our claim, that \( L_0^4 \xrightarrow{\deg} L_1^2 \). In such case, we have

\[
\vartheta_{\alpha_0,\ldots,\alpha_{26}}(L_0^4) \xrightarrow{\deg} \vartheta_{\alpha_0,\ldots,\alpha_{26}}(L_1^2).
\]

If \( \{\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \ldots, \alpha_{26} = 0\} \) we have \( \vartheta_{\alpha_0,\ldots,\alpha_{26}}(L_0^4) \) is the pair

\[
(\nu_1, A_1) = \begin{cases}
\nu_1 = \{[e_2, e_1] = -e_2 \\
0 \quad 0 \quad 1 \\
0 \quad 0 \quad 0
\end{cases}
A_1 =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and \( \vartheta_{\alpha_0,\ldots,\alpha_{26}}(L_1^2) \) is the pair

\[
(\nu_2, A_2) = \begin{cases}
\nu_2 = \{[e_2, e_3] = e_3 \\
0 \quad 0 \quad 1 \\
0 \quad 0 \quad 0
\end{cases}
A_2 =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Therefore

\[
\dim(\ker(T_{\theta_1,\ldots,\theta_6;\beta_1,\ldots,\beta_9}(\nu_1, A_1))) \leq \dim(\ker(T_{\theta_1,\ldots,\theta_6;\beta_1,\ldots,\beta_9}(\nu_2, A_2))).
\] (3)

Now, we consider \( \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0.\} \) and \( \{\theta_1 = 0, \theta_2 = 0, \theta_3 = 1, \theta_4 = 0, \theta_5 = 0, \theta_6 = 0.\} \).

The set of 3-by-3 matrices that commute with \( A_1 \) is

\[
\mathfrak{gl}(n, \mathbb{C})_{A_1} = \left\{ \begin{bmatrix} t_1 & 0 & t_4 \\ t_3 & t_2 & t_5 \\ 0 & 0 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]

Given \( D_1(w's) \), \( D_2(x's) \) and \( D_3(y's) \) in \( \mathfrak{gl}(n, \mathbb{C})_{A_1} \), with \( D_2 \) and \( D_3 \) equal to zero, we have \( \lambda_1 := \nu(D_1, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3) \),

\[
\lambda_1 := \{ [e_3, e_1] = (-w_{3,3}) e_2,
\]

and so \( \dim(\ker(T_{\theta_1,\ldots,\theta_6;\beta_1,\ldots,\beta_9}(\nu_1, A_1))) = 4. \)

On the other hand, the set of 3-by-3 matrices that commute with \( A_2 \) is

\[
\mathfrak{gl}(n, \mathbb{C})_{A_2} = \left\{ \begin{bmatrix} t_1 & t_3 & t_4 \\ 0 & t_1 & 0 \\ 0 & t_5 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]

and given \( D_1(w's) \), \( D_2(x's) \) and \( D_3(y's) \) in \( \mathfrak{gl}(n, \mathbb{C})_{A_2} \), with \( D_2 \) and \( D_3 \) equal to zero, we have \( \lambda_2 := \nu(D_1, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3) \),

\[
\lambda_2 := \{ [e_2, e_2], w_{1,3} e_1 + w_{3,3} e_3
\]

and so \( \dim(\ker(T_{\theta_1,\ldots,\theta_6;\beta_1,\ldots,\beta_9}(\nu_2, A_2))) = 3; \) this is a contradiction.
17. \( L_6^4 \xrightarrow{\text{deg}} L_0^2 \).

Suppose, contrary to our claim, that \( L_6^4 \xrightarrow{\text{deg}} L_0^2 \). In such case, we have

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \xrightarrow{\text{deg}}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

18. \( L_6^3 \xrightarrow{\text{deg}} L_0^2 \).

Suppose, contrary to our claim, that \( L_6^3 \xrightarrow{\text{deg}} L_0^2 \). In such case, we have

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} \xrightarrow{\text{deg}}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

19. \( L_6^0 \xrightarrow{\text{deg}} L_0^1 \).

Suppose, contrary to our claim, that \( L_6^0 \xrightarrow{\text{deg}} L_0^1 \). In such case, we have

\[
0 \xrightarrow{\text{deg}}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Since the first matrix has nilpotent degree equal to 1 and the second one has nilpotent degree equal to 2, we have a contradiction.
9.7 Degenerations between hom-Lie algebras \( (\mathcal{L}_j') \) with \( j = 5, 1, 0 \)

If \( \mathcal{L}_i' \xrightarrow{\text{deg}} \mathcal{L}_k' \) then \( \mathcal{L}_i \xrightarrow{\text{deg}} \mathcal{L}_h \) and \( \text{Der}(\mathcal{L}_i') \leq \text{Der}(\mathcal{L}_k') \).

Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|----------|-----------------|
| 0        | \( \mathcal{L}_1' \) |
| 1        | \( \mathcal{L}_2'(z, \lambda), \mathcal{L}_3'(z), \mathcal{L}_4'(z) \) |
| 2        | \( \mathcal{L}_2'(z, \lambda), \mathcal{L}_3'(z), \mathcal{L}_4'(z), \mathcal{L}_5'(z) \) |
| 3        | \( \mathcal{L}_3'(z) \) |
| 4        | \( \mathcal{L}_3'(z) \) |
| 5        | \( \mathcal{L}_4'(z) \) |
| 6        | \( \mathcal{L}_4'(z) \) |
| 7        | \( \mathcal{L}_4'(z) \) |
| 8        | \( \mathcal{L}_4'(z) \) |
| 9        | \( \mathcal{L}_4'(z) \) |

9.7.1 Degenerations

1. \( \mathcal{L}_3'(z, \lambda) \xrightarrow{\text{deg}} \mathcal{L}_5' \) with \( z^2 \neq 1, 0 \)

In fact, set

\[
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
\frac{e^{2t}}{\sqrt{\lambda(z-1)}+\lambda z-\lambda} & e^t & e^{2t} \\
0 & \frac{e^{2t}}{\sqrt{\lambda(z-1)}+\lambda z-\lambda} & \frac{e^{2t}}{\sqrt{\lambda(z-1)}+\lambda z-\lambda}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \cdot \mathcal{L}_3'(z, \lambda) \) is the hom-Lie algebra \( (\mu(t), A) \) with

\[
[\epsilon_1, \epsilon_2] = e^{-t} \left( \sqrt{\lambda(z-1)+\lambda z} - \lambda \right) \epsilon_2 + \left( \frac{\sqrt{\lambda(z-1)+\lambda z}}{\sqrt{\lambda(z-1)}} + 1 - \frac{1}{\sqrt{\lambda(z-1)}} \right) \epsilon_3,
\]

\[
[\epsilon_1, \epsilon_3] = -\lambda \left( z - 1 \right) e^{-2t} \epsilon_2 + \frac{e^{-t} \left( -\lambda z + e^{-2t} \sqrt{\lambda(z-1)+\lambda} \right)}{\sqrt{\lambda(z-1)}} \epsilon_3
\]

\[
A = \begin{bmatrix}0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \( (\mu(t), A) \rightarrow \mathcal{L}_1' \) as \( t \) tends to infinity.

2. \( \mathcal{L}_5'(z) \xrightarrow{\text{deg}} \mathcal{L}_1' \)

In fact, set

\[
g(t) = \begin{bmatrix}e^t & 0 & 0 \\
\frac{e^{2t}}{z^{-t}} & 0 & e^t \\
0 & e^t & \frac{e^{2t}}{z^{-t}}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]

We have \( g(t) \cdot \mathcal{L}_5'(z) \) is the hom-Lie algebra \( (\mu(t), A) \) with

\[
[\epsilon_1, \epsilon_2] = e^{-t} \epsilon_2 + \epsilon_3, \quad [\epsilon_1, \epsilon_3] = e^{-t} \epsilon_3
\]

\[
A = \begin{bmatrix}0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

It is easy to check that \( (\mu(t), A) \rightarrow \mathcal{L}_1' \) as \( t \) tends to infinity.
3. \( \mathfrak{L}_7^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_1^5 \)
In fact, set
\[
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
-\frac{e^{2t}}{z-1} & e^t & 0 \\
0 & -\frac{e^{2t}}{z-1} & e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_7^5(z) \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= e^{-t}e_2 + e_3, \\
[e_2, e_3] &= e^{-t}ze_3
\end{align*}
\]
It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{L}_1^5\) as \( t \) tends to infinity.

4. \( \mathfrak{L}_6^5(z, \lambda) \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)
In fact, set
\[
g(t) = \begin{bmatrix}
0 & 1 & z \\
0 & -\lambda (z-1) & -\lambda (z-1) \\
(z-1)e^{2t} & \sqrt{-1}e^t (z-1) \sqrt{\lambda} & \sqrt{-1}e^t (z-1) \sqrt{\lambda}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_6^5(z, \lambda) \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= -\sqrt{\frac{(z+1)e^{-t}}{z-1}} e_1 + \sqrt{-1} \sqrt{\lambda} e_2 + e_3, \\
[e_1, e_3] &= -\frac{(z+1)e^{-2t}}{z^2-1} e_1 + e^{-2t} \lambda e_2 - \sqrt{-1} \sqrt{\lambda} e_3, \\
[e_2, e_3] &= \frac{(z^2-2z+1)\lambda}{z^2-1} e_1
\end{align*}
\]
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{L}_1^2\) as \( t \) tends to infinity.

5. \( \mathfrak{L}_3^5(z) \xrightarrow{\text{deg}} \mathfrak{L}_1^2 \)
In fact, set
\[
g(t) = \begin{bmatrix}
0 & e^t & 0 \\
e^t & 0 & 0 \\
0 & \frac{(e^t)^2}{z-1} & \frac{(e^t)^2}{z-1}
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot \mathfrak{L}_3^5(z) \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= -e^{-t}e_1 + e_3, \\
[e_2, e_3] &= ze^{-t}e_3
\end{align*}
\]
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{L}_1^2\) as \( t \) tends to infinity.
6. \( L^3_\delta(z) \xrightarrow{\text{deg}} L^3_1 \)  
In fact, set 
\[
g(t) = \begin{pmatrix} 0 & 1 & 0 \\ e^t & 0 & 0 \\ 0 & e^t/z & e^t \end{pmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot L^3_\delta(z) \) is the hom-Lie algebra \((\mu(t), A)\) with 
\[
\begin{align*}
[e_1, e_2] &= -e^{-t}e_1 + e_3, [e_2, e_3] = e^{-t}e_3 \\
A &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to L^3_1\) as \(t\) tends to infinity.

7. \( L^3_1(z) \xrightarrow{\text{deg}} L^3_1 \)  
In fact, set 
\[
g(t) = \begin{pmatrix} 0 & 0 & 1 \\ e^t & 0 & 0 \\ 0 & e^t & -e^t/z-1 \end{pmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot L^3_1(z) \) is the hom-Lie algebra \((\mu(t), A)\) with 
\[
\begin{align*}
[e_1, e_2] &= -e^{-t}ze_1 + e_3, [e_2, e_3] = e^{-t}e_3 \\
A &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to L^3_1\) as \(t\) tends to infinity.

8. \( L^0_\delta(z) \xrightarrow{\text{deg}} L^0_1 \)  
In fact, set 
\[
g(t) = \begin{pmatrix} -e^t(z - 1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^t & 1 \end{pmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have \( g(t) \cdot L^0_\delta(z) \) is the hom-Lie algebra \((\mu(t), A)\) with 
\[
\begin{align*}
[e_1, e_2] &= -e^{-1}z/e_2 + e_3, [e_2, e_3] = -e^{-1}z/e_3 \\
A &= 0
\end{align*}
\]
It is easy to check that \((\mu(t), A) \to L^0_1\) as \(t\) tends to infinity.

9. \( L^3_\delta(z, \lambda) \xrightarrow{\text{deg}} L^3_1 \) with \((z(z^2 - 1) \neq 0)\).  
Suppose, contrary to our claim, that \( L^3_\delta(z, \lambda) \xrightarrow{\text{deg}} L^3_1 \) with \((z(z^2 - 1) \neq 0)\). In such case, we have 
\[
\psi_{\alpha, \beta}(L^3_\delta(z, \lambda)) \xrightarrow{\text{deg}} \psi_{\alpha, \beta}(L^3_1).
\]
By taking \( \alpha = (z + 1 = \sqrt{z^2 + 6z + 1})/2 (1 + z) \lambda \) and \( \beta = (z + 1 = \sqrt{z^2 + 6z + 1})/2 (1 + z) \lambda \), we have from 7.10.1 that \( \psi_{\alpha, \beta}(L^3_\delta(z, \lambda)) \) is a Lie algebra isomorphic to \( n_3 \), therefore, \( \psi_{\alpha, \beta}(L^3_1) \) is a Lie algebra which is isomorphic to one of the following: \( n_3 \) or \( n_3 \) (by Theorem 1.1). We must have \( \psi_{\alpha, \beta}(L^3_1) \) is isomorphic to \( n_3 \) (from 3.4.1). But, we now apply again 3.4.1, to obtain \( \alpha = \beta = 0 \); this is a contradiction.
10. \( L^8_{5} \xrightarrow{\text{deg}} L^3_{1} \).

Suppose, contrary to our claim, that \( L^8_{5} \xrightarrow{\text{deg}} L^3_{1} \). In such case, we have

\[
\phi_\beta(L^8_{5}) \xrightarrow{\text{deg}} \phi_\beta(L^3_{1}).
\]

By taking \( \beta = -z \), we have from 7.9.2 that \( \phi_\beta(L^8_{5}) \) is the 3-dimensional abelian Lie algebra, therefore, \( \phi_\beta(L^3_{1}) \) is \( a_3 \); this contradicts to 3.4.2.

11. \( L^7_{5} \xrightarrow{\text{deg}} L^3_{1} \).

Suppose, contrary to our claim, that \( L^7_{5} \xrightarrow{\text{deg}} L^3_{1} \). In such case, we have

\[
\phi_\beta(L^7_{5}) \xrightarrow{\text{deg}} \phi_\beta(L^3_{1}).
\]

By taking \( \beta = -1/z \), we have from 7.8.2 that \( \phi_\beta(L^7_{5}) \) is the 3-dimensional abelian Lie algebra, therefore, \( \phi_\beta(L^3_{1}) \) is \( a_3 \); this contradicts to 3.4.2.

12. \( L^6_{5}(z, \lambda) \xrightarrow{\text{deg}} L^2_{0} \).

Suppose, contrary to our claim, that \( L^6_{5}(z, \lambda) \xrightarrow{\text{deg}} L^2_{0} \). In such case, we have

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & \lambda & \lambda \\
0 & -\lambda & -\lambda
\end{bmatrix}
\xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

13. \( L^5_{5}(z) \xrightarrow{\text{deg}} L^2_{2} \).

Suppose, contrary to our claim, that \( L^5_{5}(z) \xrightarrow{\text{deg}} L^2_{2} \). In such case, we have

\[
\vartheta_{\alpha_0, \ldots, \alpha_{26}}(L^5_{5}(z)) \xrightarrow{\text{deg}} \vartheta_{\alpha_0, \ldots, \alpha_{26}}(L^2_{2}).
\]

If \( \left\{ \alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \ldots, \alpha_{26} = 0 \right\} \) we have \( \vartheta_{\alpha_0, \ldots, \alpha_{26}}(L) \) is the pair

\[
(\nu_1, A_1) = \begin{cases}
\nu_1 = \{ e_2 \cdot e_1 = -z e_3 \\
A_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\end{cases}
\]

and \( \vartheta_{\alpha_0, \ldots, \alpha_{26}}(L) \) is the pair

\[
(\nu_2, A_2) = \begin{cases}
\nu_2 = \{ e_2 \cdot e_2 = e_3 \\
A_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{cases}
\]

Therefore

\[
\dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6}; \beta_1, \beta_9, \beta_2, \beta_8, \beta_3, \beta_7, \beta_6, \beta_4, \beta_5, \beta_0)) \leq \dim(\text{Ker}(T_{\theta_1, \ldots, \theta_6}; \beta_1, \beta_9, \beta_2, \beta_8, \beta_3, \beta_7, \beta_6, \beta_4, \beta_5, \beta_0)). \tag{4}
\]

Now, we consider \( \{ \beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0, \} \) and \( \{ \theta_1 = 0, \theta_2 = 1, \theta_3 = 0, \theta_4 = 0, \theta_5 = 0, \theta_6 = 1 \} \).
The set of 3-by-3 matrices that commute with $A_1$ is
\[
\text{gl}(n, \mathbb{C})_{A_1} = \left\{ \begin{bmatrix} t_1 & t_3 & 0 \\
0 & t_2 & 0 \\
t_4 & t_5 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]

Given $D_1(w')$, $D_2(x')$ and $D_3(y')$ in $\text{gl}(n, \mathbb{C})_{A_1}$, with $D_1$ and $D_2$ equal to zero, we have
\[
\lambda_1 := D_1 \nu(\cdot, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3),
\]
\[
\lambda_1 = \{ e_2 \cdot e_1 = -w_{3,3} e_3,
\]
and so $\dim(\ker(T_{\theta_1, \ldots, \theta_6}(\nu_1, A_1))) = 4$.

On the other hand, the set of 3-by-3 matrices that commute with $A_2$ is
\[
\text{gl}(n, \mathbb{C})_{A_2} = \left\{ \begin{bmatrix} t_1 & t_3 & t_4 \\
0 & t_1 & 0 \\
0 & t_5 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\},
\]
and given $D_1(w')$, $D_2(x')$ and $D_3(y')$ in $\text{gl}(n, \mathbb{C})_{A_2}$, with $D_1$ and $D_2$ equal to zero, we have
\[
\lambda_2 := D_1 \nu(\cdot, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3),
\]
\[
\lambda_2 = \{ e_2 \cdot e_2 = w_{1,3} e_l + w_{3,3} e_3,
\]
and so $\dim(\ker(T_{\theta_1, \ldots, \theta_6}(\nu_2, A_2))) = 3$; this is a contradiction.

14. $\mathfrak{L}^5_3(z) \xrightarrow{\text{deg}} \mathfrak{L}^2_6$.

Suppose, contrary to our claim, that $\mathfrak{L}^5_3(z) \xrightarrow{\text{deg}} \mathfrak{L}^2_6$. In such case, we have
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \xrightarrow{\text{deg}}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

15. $\mathfrak{L}^4_5(z) \xrightarrow{\text{deg}} \mathfrak{L}^2_1$.

Suppose, contrary to our claim, that $\mathfrak{L}^4_5(z) \xrightarrow{\text{deg}} \mathfrak{L}^2_1$. In such case, we have
\[
\vartheta_{\alpha_0, \ldots, \alpha_{26}}(\mathfrak{L}^4_5(z)) \xrightarrow{\text{deg}} \vartheta_{\alpha_0, \ldots, \alpha_{26}}(\mathfrak{L}^2_1).
\]
If $\{\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \ldots, \alpha_{26} = 0\}$ we have $\vartheta_{\alpha_0, \ldots, \alpha_{26}}(\mathfrak{L}^4_5(z))$ is the pair
\[
(\nu_1, A_1) = \left\{ \begin{array}{c}
\nu_1 = \{ e_3 \cdot e_1 = -e_2 \\
A_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\end{array} \right\}
\]
and $\vartheta_{\alpha_0, \ldots, \alpha_{26}}(\mathfrak{L}^2_1)$ is the pair
\[
(\nu_2, A_2) = \left\{ \begin{array}{c}
\nu_2 = \{ e_2 \cdot e_2 = e_3 \\
A_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{array} \right\}
\]

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Therefore

\[ \dim(Ker(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\nu_1, A_1))) \leq \dim(Ker(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\nu_2, A_2))). \tag{5} \]

Now, we consider \{\beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = 1, \beta_5 = -1, \beta_6 = 0, \beta_7 = 1, \beta_8 = -1, \beta_9 = 0, A_1) \} \} and \{\theta_1 = 0, \theta_2 = 1, \theta_3 = 0, \theta_4 = 0, \theta_5 = 0, \theta_6 = 1\}. \}

The set of 3-by-3 matrices that commute with \( A_1 \) is

\[
\mathfrak{gl}(n, \mathbb{C})_{A_1} = \left\{ \begin{bmatrix} t_1 & 0 & t_4 \\ t_3 & t_2 & t_5 \\ 0 & 0 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]

Given \( D_1(w'), D_2(x') \) and \( D_3(y') \) in \( \mathfrak{gl}(n, \mathbb{C})_{A_1} \), with \( D_2 \) and \( D_3 \) equal to 0, we have

\[
\lambda_1 := D_1 \nu(\cdot, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3),
\]

and so \( \dim(Ker(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\nu_1, A_1))) = 4 \).

On the other hand, the set of 3-by-3 matrices that commute with \( A_2 \) is

\[
\mathfrak{gl}(n, \mathbb{C})_{A_2} = \left\{ \begin{bmatrix} t_1 & t_3 & t_4 \\ 0 & t_1 & 0 \\ 0 & t_5 & t_2 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5 \in \mathbb{C} \right\}.
\]

and given \( D_1(w'), D_2(x') \) and \( D_3(y') \) in \( \mathfrak{gl}(n, \mathbb{C})_{A_2} \), with \( D_2 \) and \( D_3 \) equal to 0, we have

\[
\lambda_2 := D_1 \nu(\cdot, \cdot) + \nu(D_2, \cdot) + \nu(\cdot, D_3),
\]

and so \( \dim(Ker(T_{\theta_1, \ldots, \theta_6; \beta_1, \ldots, \beta_9}(\nu_2, A_2))) = 3 \); this is a contradiction.

16. \( \mathfrak{L}_3^4(z) \xrightarrow{\deg} \mathfrak{L}_6^2 \).

Suppose, contrary to our claim, that \( \mathfrak{L}_3^4(z) \xrightarrow{\deg} \mathfrak{L}_6^2 \). In such case, we have

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \xrightarrow{\deg} 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

17. \( \mathfrak{L}_3^3(z) \xrightarrow{\deg} \mathfrak{L}_6^2 \).

Suppose, contrary to our claim, that \( \mathfrak{L}_3^3(z) \xrightarrow{\deg} \mathfrak{L}_6^2 \). In such case, we have

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} \xrightarrow{\deg} 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.
Suppose, contrary to our claim, that $\mathbf{L}_0^5(z) \xrightarrow{\text{deg}} \mathbf{L}_0^1$. In such case, we have

$$0 \xrightarrow{\text{deg}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 1 and the second one has nilpotent degree equal to 2, we have a contradiction.
9.8 Degenerations between hom-Lie algebras \((\mathcal{L}_j^i)\) with \(j = 2, 3, 1, 0\)

If \(\mathcal{L}_i^j \xrightarrow{\text{deg}} \mathcal{L}_h^k\) then \(\mathcal{L}_i^j \xrightarrow{\text{deg}} \mathcal{L}_h^k\) and \(\text{Der}(\mathcal{L}_i^j) \leq \text{Der}(\mathcal{L}_h^k)\).

Therefore, we can organize the hom-Lie algebras in the following way:

| Dim(Der) | hom-Lie algebra |
|---------|------------------|
| 0       | \(\mathcal{L}_4^4\) |
| 1       | \(\mathcal{L}_2^2(\lambda)\) |
| 2       | \(\mathcal{L}_2^2(\lambda), \mathcal{L}_2^3, \mathcal{L}_3^2\) |
| 3       | \(\mathcal{L}_2^2(\lambda), \mathcal{L}_2^3, \mathcal{L}_3^2, \mathcal{L}_2^1\) |
| 4       | \(\mathcal{L}_2^1\) |
| 5       | \(\mathcal{L}_3^3\) |
| 6       | \(\mathcal{L}_3^2\) |
| 9       | \(\mathcal{L}_0^0\) |

9.8.1 Degenerations

1. \(\mathcal{L}_2^2(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_1^5\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
-e^{2t}\lambda & 0 & 0 \\
-e^{3t}\lambda & -e^{2t}\lambda & 0 \\
0 & -e^{3t}\lambda & -e^{2t}
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]

   We have \(g(t) \cdot \mathcal{L}_2^2(\lambda)\) is the hom-Lie algebra \((\mu(t), A)\) with

   \[
   [e_1, e_2] = \frac{e^{-t}\left(-\lambda e^{-t}\right)}{1} e_2 + e_3, [e_1, e_3] = -e^{-2t} e_2 - \frac{e^{-t}(e^{-t} + \lambda)}{\lambda} e_2
   \]

   \[
   A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
   \]

   It is easy to check that \((\mu(t), A) \rightarrow \mathcal{L}_1^5\) as \(t\) tends to infinity.

2. \(\mathcal{L}_2^2(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_0^2\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
0 & e^t & 0 \\
0 & 0 & e^t\lambda \\
e^t\lambda & 0 & 0
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]

   We have \(g(t) \cdot \mathcal{L}_2^2(\lambda)\) is the hom-Lie algebra \((\mu(t), A)\) with

   \[
   [e_1, e_2] = \frac{-e^{-t}}{\lambda} e_1, [e_2, e_3] = -\frac{e^{-t}}{\lambda} e_1 - \frac{e^{-t}}{\lambda} e_2
   \]

   \[
   A = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
   \]

   It is easy to check that \((\mu(t), A) \rightarrow \mathcal{L}_0^2\) as \(t\) tends to infinity.

3. \(\mathcal{L}_2^2(\lambda) \xrightarrow{\text{deg}} \mathcal{L}_1^2\)
   
   In fact, set
   
   \[
g(t) = \begin{bmatrix}
0 & 1 & 1 \\
0 & \lambda & 0 \\
0 & e^{2t} & 0
\end{bmatrix}, \quad \text{with } t \in \mathbb{R}.
   \]

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We have $g(t) \cdot L_4^2(\lambda)$ is the hom-Lie algebra $(\mu(t), A)$ with
\[
\begin{align*}
[e_1, e_2] &= 2 e^{-t} e_1 + e^{-t} \lambda e_2 + e_3, [e_1, e_3] = -2 \lambda e^{-2t} e_1 - \lambda^2 e^{-2t} e_2 - e^{-t} \lambda e_3, [e_2, e_3] = e^{-2t} e_1 \\
A &= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]
It is easy to check that $(\mu(t), A) \to L_1^2$ as $t$ tends to infinity.

4. $L_3^2 \xrightarrow{\deg} L_1^1$
In fact, set
\[
g(t) = \begin{bmatrix}
0 & 0 & 1 \\
0 & \lambda^{-1} & 0 \\
e^t & 0 & -e^t
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \cdot L_3^2$ is the hom-Lie algebra $(\mu(t), A)$ with
\[
\begin{align*}
[e_1, e_2] &= e_2, [e_1, e_3] = -e^{-t} e_1 - \frac{e^{-t} e_2}{\lambda} + e_3, [e_2, e_3] = -e^{-t} e_2 \\
A &= \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]
It is easy to check that $(\mu(t), A) \to L_1^1$ as $t$ tends to infinity.

5. $L_2^3(\lambda) \xrightarrow{\deg} L_0^1$
In fact, set
\[
g(t) = \begin{bmatrix}
e^t & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \cdot L_2^3$ is the hom-Lie algebra $(\mu(t), A)$ with
\[
\begin{align*}
[e_1, e_2] &= e^{-t} e_2, [e_1, e_3] = \frac{e^{-t} e_2}{\lambda} + e_3 \\
A &= \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]
It is easy to check that $(\mu(t), A) \to L_0^1$ as $t$ tends to infinity.

6. $L_2^2 \xrightarrow{\deg} L_1^2$
In fact, set
\[
g(t) = \begin{bmatrix}
0 & 0 & e^t \\
e^t & 0 & 0 \\
0 & -(e^t)^2 & 0
\end{bmatrix}, \text{ with } t \in \mathbb{R}.
\]
We have $g(t) \cdot L_2^2$ is the hom-Lie algebra $(\mu(t), A)$ with
\[
\begin{align*}
[e_1, e_2] &= -e^{-t} e_1 + e_3, [e_2, e_3] = e^{-t} e_3 \\
A &= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]
It is easy to check that $(\mu(t), A) \to L_1^2$ as $t$ tends to infinity.
In fact, set
\[ g(t) = \begin{bmatrix} 0 & 0 & -1 \\ e^t & 0 & 0 \\ 0 & e^t & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}. \]

We have \( g(t) \cdot \mathfrak{l}_1^{\text{deg}} \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= -e^{-t}e_1 + e_3, [e_2, e_3] = e^{-t}e_3
\end{align*}
\]

It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{l}_1^{\text{deg}}\) as \( t \) tends to infinity.

In fact, set
\[ g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}, \text{ with } t \in \mathbb{R}. \]

We have \( g(t) \cdot \mathfrak{l}_1^{\text{deg}} \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= e_2, [e_1, e_3] = e^{-t}e_2 + e_3 \\
A &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{l}_1^{\text{deg}}\) as \( t \) tends to infinity.

In fact, set
\[ g(t) = \begin{bmatrix} -e^{2t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^t & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}. \]

We have \( g(t) \cdot \mathfrak{l}_2^{\text{deg}} \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= (e^{-t} - e^{-2t}) e_2 + e_3, [e_1, e_3] = -e^{-2t}e_2 + (-e^{-2t} - e^{-t}) e_3 \\
A &= 0
\end{align*}
\]

It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{l}_2^{\text{deg}}\) as \( t \) tends to infinity.

In fact, set
\[ g(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}, \text{ with } t \in \mathbb{R}. \]

We have \( g(t) \cdot \mathfrak{l}_{02}^{\text{deg}} \) is the hom-Lie algebra \((\mu(t), A)\) with
\[
\begin{align*}
[e_1, e_2] &= e_2, [e_1, e_3] = e^{-t}e_2 + e_3 \\
A &= 0
\end{align*}
\]

It is easy to check that \((\mu(t), A) \rightarrow \mathfrak{l}_{02}^{\text{deg}}\) as \( t \) tends to infinity.
11. $L_3^3 \xrightarrow{\text{deg}} L_2^2$

In fact, set

$$g(t) = \begin{bmatrix} 0 & 0 & e^t \\ 0 & e^t & 0 \\ e^t & 0 & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \cdot L_3^3$ is the hom-Lie algebra $(\mu(t), A)$ with

$$\begin{cases} [e_1, e_3] = -e^{-t}e_1, [e_2, e_3] = -e^{-t}e_2 \\ A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{cases}$$

It is easy to check that $(\mu(t), A) \rightarrow L_2^0$ as $t$ tends to infinity.

12. $L_3^1 \xrightarrow{\text{deg}} L_1^1$

In fact, set

$$g(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & e^t & 0 \\ e^t & 0 & 0 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \cdot L_1^1$ is the hom-Lie algebra $(\mu(t), A)$ with

$$\begin{cases} [e_1, e_3] = -e^{-t}e_1, [e_2, e_3] = -e^{-t}e_2 \\ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{cases}$$

It is easy to check that $(\mu(t), A) \rightarrow L_1^0$ as $t$ tends to infinity.

13. $L_3^0 \xrightarrow{\text{deg}} L_0^0$

In fact, set

$$g(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } t \in \mathbb{R}.$$  

We have $g(t) \cdot L_0^0$ is the hom-Lie algebra $(\mu(t), A)$ with

$$\begin{cases} [e_1, e_2] = e^{-t}e_2, [e_1, e_3] = e^{-t}e_3 \\ A = 0 \end{cases}$$

It is easy to check that $(\mu(t), A) \rightarrow L_0^0$ as $t$ tends to infinity.

14. $L_2^2(\lambda) \xrightarrow{\text{deg}} L_1^3$ with $\lambda \neq 0$.

Suppose, contrary to our claim, that $L_2^3(\lambda) \xrightarrow{\text{deg}} L_1^3$ with $\lambda \neq 0$. In such case, we have

$$\psi_{\alpha, \beta}(L_2^3(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha, \beta}(L_1^3).$$

By taking $\alpha = (-1 \mp \sqrt{2})/\lambda$ and $\beta = (-1 \pm \sqrt{2})/\lambda$, we have from 4.7.1 that $\psi_{\alpha, \beta}(L_2^3(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore, $\psi_{\alpha, \beta}(L_1^3)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $n_3$ (by Theorem 1.1). We must have $\psi_{\alpha, \beta}(L_1^3)$ is isomorphic to $n_3$ (from 3.4.1). But, we now apply again 3.4.1, to obtain $\alpha = \beta = 0$; this is a contradiction.
15. $L^6_2(\lambda) \xrightarrow{\text{deg}} L^0_3$ with $(\lambda \neq 0)$. Suppose, contrary to our claim, that $L^6_2(\lambda) \xrightarrow{\text{deg}} L^0_3$ with $(\lambda \neq 0)$. In such case, we have

$$\psi_{\alpha,\beta}(L^6_2(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha,\beta}(L^0_3).$$

By taking $\alpha = (-1 + \sqrt{2})/\lambda$ and $\beta = (-1 + \sqrt{2})/\lambda$, we have from 4.7.1 that $\psi_{\alpha,\beta}(L^6_2(\lambda))$ is a Lie algebra isomorphic to $n_3$, therefore $\psi_{\alpha,\beta}(L^0_3)$ is a Lie algebra which is isomorphic to one of the following: $n_3$ or $a_3$ (by Theorem 1.1). But, from 5.1.1, we have $\psi_{\alpha,\beta}(L^0_3)$ is isomorphic to $t_{3,1}$; this is a contradiction.

16. $L^5_2(\lambda) \xrightarrow{\text{deg}} L^3_2$. Suppose, contrary to our claim, that $L^5_2(\lambda) \xrightarrow{\text{deg}} L^3_2$. In such case, we have

$$\psi_{\alpha,\beta}(L^5_2(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha,\beta}(L^3_2).$$

From 4.6.1, we have $\psi_{\alpha=-\frac{1}{\lambda},\beta=-\frac{1}{\lambda}}(L^5_2(\lambda))$ is a Lie algebra isomorphic to $t_{3,1}$, therefore, $\psi_{\alpha=-\frac{1}{\lambda},\beta=-\frac{1}{\lambda}}(L^3_2)$ is a Lie algebra which is isomorphic to one of the following: $t_{3,1}$, $a_3$ (by Theorem 1.1). We must have $\psi_{\alpha=-\frac{1}{\lambda},\beta=-\frac{1}{\lambda}}(L^3_2)$ is isomorphic to $t_{3,1}$ (from 5.3.1). But, we now apply again 5.3.1, to obtain $\alpha = -\beta$ but $\alpha = \beta$, and therefore $\alpha = 0 = -\frac{1}{2\lambda}$; this is a contradiction.

17. $L^5_2(\lambda) \xrightarrow{\text{deg}} L^0_1$ with $(\lambda \neq 0)$. Suppose, contrary to our claim, that $L^5_2(\lambda) \xrightarrow{\text{deg}} L^0_1$ with $(\lambda \neq 0)$. In such case, we have

$$\psi_{\alpha \neq 0,\beta = \frac{1}{\lambda}}(L^5_2(\lambda)) \xrightarrow{\text{deg}} \psi_{\alpha \neq 0,\beta = \frac{1}{\lambda}}(L^0_1).$$

From 4.6.1, we have $\psi_{\alpha \neq 0,\beta = \frac{1}{\lambda}}(L^5_2(\lambda))$ is a Lie algebra isomorphic to $t_{3,1}$, but $\psi_{\alpha \neq 0,\beta = \frac{1}{\lambda}}(L^0_1)$ is isomorphic to $n_3(\mathbb{C})$; this contradicts theorem 1.1.

18. $L^4_2(\lambda) \xrightarrow{\text{deg}} L^0_2$. Suppose, contrary to our claim, that $L^4_2(\lambda) \xrightarrow{\text{deg}} L^0_2$. In such case, we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{deg}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

19. $L^2_2 \xrightarrow{\text{deg}} L^3_2$. Suppose, contrary to our claim, that $L^2_2 \xrightarrow{\text{deg}} L^3_2$. In such case, we have

$$\phi_{\beta}(L^2_2) \xrightarrow{\text{deg}} \phi_{\beta}(L^3_2).$$

By taking $\beta \neq -1$, we have from 4.3.2 that $\phi_{\beta}(L^2_2)$ is the 3-dimensional abelian Lie algebra, therefore, $\phi_{\beta}(L^3_2)$ is the Lie algebra $a_3$. But, from 5.3.2, we have $\phi_{\beta}(L^2_2)$ is isomorphic to $n_3$; this is a contradiction.

20. $L^2_2 \xrightarrow{\text{deg}} L^0_2$. Suppose, contrary to our claim, that $L^2_2 \xrightarrow{\text{deg}} L^0_2$. In such case, we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{deg}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Since the first matrix has nilpotent degree equal to 2 and the second one has nilpotent degree equal to 3, we have a contradiction.

21. \( L^0_2 \xrightarrow{\text{deg}} L^1_0 \).

Suppose, contrary to our claim, that \( L^0_2 \xrightarrow{\text{deg}} L^1_0 \). In such case, we have

\[
0 \xrightarrow{\text{deg}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Since the first matrix has nilpotent degree equal to 1 and the second one has nilpotent degree equal to 2, we have a contradiction.