Changing the tune: mixtures of network models that vary in time

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Abstract

Many of the complex systems we study in their representation as networks are growing objects, evolving by the addition of nodes and links over time. The rules governing this growth are attributed to mechanisms such as preferential attachment and triangle closure. We demonstrate a method for estimating the relative roles of these mechanisms, and further, investigating how they change as the network evolves. We show that a rich class of network evolution models can be built from a weighted mixture of these model mechanisms. Using a likelihood based formulation we show how to calculate the optimal mixture for a given set of observations of network data, and show that this framework can be used to distinguish competing models that are indistinguishable by their summary statistics. Using real data from Facebook user interactions, we show that we can improve the ability of a model to reproduce network statistics using tuned model mixtures. We further investigate the idea that the underlying model of a network can change in time, for example, a technology based network might respond to changes in the underlying technology or a financial network might respond to economic shocks. Using artificial data we show that we can recapture the time at which a known change occurred. We use the Enron email dataset to show that we can estimate how mixtures of models change over time.

1 Introduction

Networks are often a natural representation of a number of systems, for example, citation networks \cite{1, 2}, email networks \cite{3} or networks of online user interactions \cite{4, 5}. Naturally, there has been an interest in modelling network structure, aiming to generate networks which capture properties observed in real-world examples. One focus has been on which nodes attract new links. Models such as the Barabási-Albert \cite{6, 7} and Fortunato at al \cite{8} models provide a possible explanation for observations such as the power-law degree distribution of networks. As more network models are proposed it becomes increasingly important to say which model is the better fit to an observed network. The standard approach is to compare statistics from model-generated networks to some target network. In \cite{9} the authors propose FETA (Framework for Evolving Topology Analysis), a framework and software implementation for testing dynamic network models. FETA allows a rigorous likelihood to be calculated for a given model and network, provided that the order in which the links arrived to the network is known. That paper also shows how a weighted mixture of models can be used and how model likelihood can be used to find optimum weights to explain a given dataset.

In this paper we extend this framework in a number of ways. Firstly, given a number of proposed model components and network observations we can quickly calculate the weights for the model components that give highest likelihood. Our results show that tuned mixtures of models can better reproduce observed network behaviour than single model components. Secondly, we introduce a measure for similarity between the action of two different models on a particular network. Thirdly, we introduce models that change in time. This allows us to model networks that change in their evolutionary behaviour. Using artificial data where the model changes in time, we demonstrate that FETA can accurately identify the time at which this change occurs. We demonstrate that FETA can use models that vary in time to analyse real datasets, using the Enron email network \cite{10}. This framework is the only approach the authors know of that allows analysis and comparison of network models that vary in time.

Related work

The class of models that this work addresses is probabilistic models, which generate networks by the iterative
addition of nodes and links, as opposed to generating a single network snapshot such as in [10]. The value of these models is in relating the microscale rules at the level of individual edge formation events to the resulting wider network structural features. One of the features commonly modelled is a heavy-tailed degree distribution, with one possible mechanism responsible for this being a preference of new nodes to attach to existing nodes of high degree [6, 7], and another being a preference to attach to the oldest nodes [8]. Another feature is a high clustering coefficient, which measures the tendency for connected triads of vertices to form triangles, often observed in the context of social network, as a tendency for individuals with a mutual friend to be themselves friends [11]. One model suggests an explanation of this feature by the mechanism of each new node making a random connection, and with some probability connecting to a neighbour of this new connection [12], mimicking the scenario of being introduced to new connections through a mutual friend. Another proposes the mechanism of a new node copying a proportion of neighbours of an existing node [13], a process mirrored in the WWW of a new webpage offering similar hyperlinks to that of a similar existing page [14], an author picking citations from the references of a chosen paper, or the duplication/divergence process observed in protein interaction networks [15]. Other mechanisms which are often domain-specific include notions of node fitness [16], through which nodes are given a fitness parameter independent of the network structure, which combined with their degree determines their ability to attract new links, or indeed node ageing [17] in which the age of a node inhibits its ability to attract links, reflecting, for example, the decay in relevance of a scientific paper over time.

This paper aims to elucidate the role of these different mechanisms in the growth and evolution of networks, and further, investigate how these roles may change over time. Much existing research has shown that this is a difficult task. Firstly, summary statistics of a network are rarely enough to distinguish between models which may have generated it, for example, in the spirit of Anscombe’s Quartet [18], pictured in [19] are five networks which may have generated it, for example, in the spirit of Anscombe’s Quartet [18], pictured in [19] are five networks with identical degree distributions, but generated from different models and with dramatically different structural properties. Similarly, [20] showed five networks all with global inter-group assortativity 0 but very different mixing properties when focusing on more local network regions. In these cases, the networks are small enough that these differences can be identified visually, which is not always possible in much larger networks. Secondly, some of the growth mechanisms mentioned may easily be entangled — as an example, it has been shown that some local triangle closing mechanisms give rise indirectly to preferential attachment [13] and similarly that triangle closure may amplify the effects of homophily [21]. We show that our framework may be used to distinguish between two models that both generate scale-free networks with similar summary statistics, contributing to the discussion on the cautionary use of summary statistics.

The authors of [22] use a multinomial logit formulation to model connection forming as a discrete choice. This framework, like FETA, allows for mixtures of models and utilises the information of link arrival times to calculate a likelihood, but does not allow for models which change in time.

2 Theory

Model Structure

In this framework, we will consider dynamic graphs \( G(t) = (V(t), E(t)) \) where \( V(t) \) is the set of nodes and \( E(t) \) is the set of edges at time \( t \), with number of nodes \( N(t) = |V(t)| \). For simplicity we describe undirected, unweighted simple networks and consider only networks where nodes and edges arrive but not depart. These simplifications are purely for clarity of explanation and are not a fundamental constraint on the framework.

We follow the description in [9] and separate the network growth model into two parts. The operation model specifies the type of graph transformation, for example, appearance of a link between two existing nodes or a new node arriving and connecting to a number of existing nodes, occurring at each step of a network’s growth. The object model specifies which exact nodes the operation model acts on (e.g. the operation model specifies to connect two existing nodes and the object model chooses nodes 23 and 45 to be connected). The focus of this paper, as is commonly the case in the literature, is the object model.

For simplicity in this work we choose networks which grow by addition of stars eg a citation network (a star is a new paper citing many other papers) or an email network (a star is a new email from one sender to one or more recipients). Define an external star as the arrival of a new node connecting to a number of existing and/or new nodes, and an internal star as an existing node connecting to a number of existing and/or new nodes (see figure 1). Since a single link can be considered a star with one leaf, more complex operations can also be considered, as a sequence of one-leaf stars. The object model assigns a probability \( p_i(t) \) to each node \( i \) in the network reflecting its chance of attracting new links at time \( t \) with \( \sum_{i=1}^{N} p_i(t) = 1 \). Then, according to the operation model, the correct number of nodes are selected by sampling, with or without replacement depending on whether duplicate links are allowed, from the distribution provided by the object model. This part of the model is aimed at understanding the factors governing which nodes are likelier to attract new links in a growing network. Some example object models (omitting the \( t \) for simplicity) are:
Model similarity

Some models (for example rank-preference and BA) may give quite similar probabilities for most nodes. We pose the question: how do we quantify how similar two models are? Our approach is to consider the similarity of two models over a given network $G$ as an analogue to a (normalised) inner product over a vector space. Let $G$ be a graph and $M_1, M_2$ be two different object models with node $i \in G$ being assigned probability $p^1_i, p^2_i$ by $M_1, M_2$ respectively. Then we define their similarity over $G$ as:

$$\sigma_G(M_1, M_2) = K^{-1} \sum_{i=1}^{N} p^1_i p^2_i$$

where $K = \sqrt{\left(\sum_{i=1}^{N} (p^1_i)^2\right) \left(\sum_{i=1}^{N} (p^2_i)^2\right)}$ is a normalisation ensuring that (by the Cauchy-Schwarz inequality) this measure lies between 0 and 1 and is equal to 1 iff $p^1_i = p^2_i$ for all $i$. The numerator of this quantity is the probability that $M_1$ and $M_2$ would pick the same node of $G$. Drawing on the analogue of inner products, we define $M_1$ and $M_2$ as orthogonal over $G$ if $\sigma_G(M_1, M_2) = 0$. An important feature is that this quantity depends highly on the structure of $G$. For example, if $M_1, M_2$ are based strictly on node degree, then their similarity will be 1 if $G$ is a regular graph. An example highlighting this for $M_{\text{rand}}$ and $M_{\text{BA}}$ assigning probabilities $p^1_{\text{rand}} = 1/N$ and $p^2_{\text{BA}} = k_i/\sum_{j=1}^{N} k_j$ respectively. Their (squared) similarity is given by:

$$\sigma_G(M_{\text{rand}}, M_{\text{BA}})^2 = \langle k \rangle^2 / \langle k^2 \rangle$$

which is equal to 1 iff $G$ is regular. On the other hand, if $G$ is scale-free with $\langle k^2 \rangle \rightarrow \infty$ then they are orthogonal.

Likelihood calculation

Let $g_0$ be our first observation of a graph $G$, and $G_1, G_2, \ldots$ be random variables representing the subsequent states of $G$, with corresponding observations $g_1, g_2, \ldots$. We assume that our observations are high resolution enough that for any time $t$, the subgraph induced by $g_{t+1} \setminus g_t$ is a star $S_{t+1}$ as in the description of the operation model. (In practice, we relax this to allow for network growth events that may have been observed as simultaneous, e.g. two papers being published the same day in a citation network.) In this sense, the graph at observation $t$ can be described as

$$G_t = g_0 \cup_{i=1}^{t} S_i$$

where $S_i$ is a random variable with corresponding observation $s_i$.

This allows us to calculate a likelihood of a model $M$
given observations $g = g_0, g_1, \ldots, g_t$ of $G$ as
\[
l(M | G = g) = \prod_{i=1}^{t} \mathbb{P}(S_i = s_i | G_{i-1} = g_{i-1}, M) \tag{5}
\]
where the term in the product refers to the probability of the chosen leaf nodes of $S_i$ occurring, determined by object model $M$. For a more in-depth discussion of the likelihood calculation, we refer the reader to reference [9].

3 Model Mixtures

Distinguishing similar models

Our first set of tests relates to our ability to distinguish different link-forming mechanisms at play in a network’s growth, focusing on models which generate scale-free, or power-law, networks. Whilst it is disputed how prevalent this feature is in real datasets [26], many network models aim to generate scale-free networks characterised by a heavy-tailed degree distribution
\[
P(k) \sim k^{-\gamma} \tag{6}
\]
with $2 < \gamma < 3$, $P(k)$ being the proportion of vertices of degree $k$. Two such models are the Barabási-Albert [6] model which generates scale-free networks via preferential attachment to node degree, whilst work by Fortunato et al [8] demonstrated the possibility of doing this via preference to highly ranked nodes. Both models give rise to degree distributions following Equation 6, with $\gamma = 3$ in the Barabási-Albert model, and $\gamma = 1 + \frac{\alpha}{\beta}$ in the (static) rank-preference model [24], asymptotically equal if $\alpha = \frac{1}{2}$. Indeed, table 1 shows artificially generated networks from each model, which are barely distinguishable using the majority of summary statistics. To test whether we can distinguish between these mechanisms, we combine them in an object model where at time $t$, a node with label $i$ is assigned a probability
\[
p_i(t) = (1 - \beta) \frac{i^{-\alpha}}{\sum_j j^{-\alpha}} + \beta \frac{k_i}{\sum_j k_j} \tag{7}
\]
We generate networks of 1000 and 10000 nodes, with operation model comprising, at each iteration, one new node attaching to 3 existing nodes, node $i \in V(G)$ chosen with probability $p_i$ in Equation 7. We then calculate maximum likelihood estimators $\hat{\beta}$ by performing a (parallel) search through the space $\beta = 0, 0.01, 0.02, \ldots, 1$. The mean of these maximum likelihood estimators (over 10 realisations for each parameter) and SD error areas is displayed in Figure 2, showing a remarkable accuracy in detangling these two very similar mechanisms. We find intuitively that the error is smaller for the larger network, since the likelihood is calculated from ten times more datapoints in this case. The error is smaller at the extremes $\beta = 0$ and $\beta = 1$; this is because only values $\beta \in [0, 1]$ are possible, so $\hat{\beta}$ cannot overshoot at either end.

Figure 2: The upper plot shows the average model similarity values $\sigma_G(M_{BA}, M(\beta))$ and $\sigma_G(M_{RP}, M(\beta))$ where $G$ is the network of 1000 nodes generated from model $M(\beta)$. Performance of the maximum likelihood estimator $\hat{\beta}$ in recovering the true mixture parameter $\beta$ of the model combining the BA and rank-preference mechanisms.

Real Data

We test how well the mixed models capture the behaviour of different real world networks using two real datasets. We first consider the Facebook wall posts dataset [5], in which nodes represent Facebook users and an undirected timestamped edge between $A$ and $B$ at time $T$ if $A$ posts on $B$’s wall (or vice versa) at time $T$. In the Supporting Information, we consider the Stack Exchange Math Overflow dataset [4] and arXiv Hep-Ph citation dataset [1].

For each dataset, a mixed model of the form in Equation 1 was fitted to the last 10% of the links, with three components and no time dependence $\hat{\beta_i}(t) = \hat{\beta}_i$. From this best fitting object model and each of the components, networks are generated using the first 90% of the real network edgelist as a starting network seed and operation model extracted from the rest of the edgelist. We compare in Figure 3 a number of statistics from generated networks and the real networks. In the case of the Facebook wall posts dataset, the best fitting mixed model was found to be a mixture comprising 0.61 degree power (exponent 0.8), 0.33 random attachment and 0.06 triangle closure. We find that this model produces networks with maximum degree, mean squared degree, degree assortativity and singleton nodes closer to the that of the real network than any of the model components alone. The clustering coefficient is poorly captured by any of the models, and the number of triangles is only well captured by the triangle model. This is likely because the
consider an object model assigning probabilities to nodes in network growth by considering changes in the underlying model. In real networks, it is unlikely that a network’s growth behaviour is constant in time. For example, social networks may change their protocols or introduce new features to encourage different kinds of interaction, large external events may cause shocks to the underlying network, or new scientific fields may emerge which may be more responsible for the clustering observed. We find similar results for the extra datasets used (SI Figures S1 and S2).

4 Time-varying models

In real networks, it is unlikely that a network’s growth behaviour is constant in time. For example, social networks may change their protocols or introduce new features to encourage different kinds of interaction, large external events may cause shocks to the underlying network, or new scientific fields may emerge which may be observed in the corresponding citation networks. In this section, we explore how possible it is to identify changes in network growth by considering changes in the underlying model.

Recovering change-points in artificial data

As with the previous section we test our ability to recover model parameters. For the following experiments, we consider an object model assigning probabilities to nodes as

\[
p_i(t) \propto \begin{cases} 
  k_i^\alpha & t \leq T \\
  k_i^\beta & t > T 
\end{cases}
\]

for some changepoint time \( T \), i.e. the nonlinear preferential attachment model in [24] where the preferential attachment exponent changes from \( \alpha \) to \( \beta \) at time \( T \). We test our ability to estimate \( T \) from networks generated with known\( T \). Figure 4 shows results on different network sizes of 1000 and 10000 nodes, with true changepoint times \( T \) in the range 100, 200, \ldots, 900 and 9000,9100, ...,9900 respectively. A first observation is that more accurate estimates are obtained in the larger network, because observing three nodes being selected from a few thousand nodes carries more information than from a few hundred nodes. The second is that the changepoint resulting from the smaller parameter change (1.0 to 0.9) is more difficult to estimate than the larger parameter change (1.2 to 1.0), which is in line with the intuition that smaller changes should be harder to detect.

We investigate this second observation further by probing the space of parameters pre- and post-changepoint. For each pre/post parameter pair, we generate 10 networks of size 6000 with fixed changepoint time \( T = 5000 \), calculate MLEs \( \hat{T} \) from a range \( T = 4500,4510,\ldots,5490,5500 \) and plot the root mean

Table 1: Table of summary statistics of networks of size 1000 generated by the BA and RP models, showing the largest degree \( k_{max} \), average clustering coefficient \( C \), mean squared degree \( \langle k^2 \rangle \), degree assortativity \( \rho \) and power law exponent \( \gamma \) when the degree distribution is fitted to equation 6 using the methodology in [27]. Values given are means (standard deviations) over 10 realisations.

| Model            | \( k_{max} \)  | \( C \)   | \( \langle k^2 \rangle \) | \( \rho \)   | \( \gamma \)   |
|------------------|----------------|----------|--------------------------|-------------|---------------|
| Barabási-Albert  | 98.5 (13.7)    | 0.035 (0.005) | 89.5 (5.43)     | -0.077 (0.009) | 3.13 (0.059) |
| Rank-Preference  | 95.4 (8.5)     | 0.024 (0.003) | 75.3 (2.09)     | -0.012 (0.009) | 3.34 (0.134) |
squared error in Figure 5. As with Figure 4, we note that smaller parameter changes are more difficult to detect, corresponding to the band of higher error around the diagonal $\alpha = \beta$. We also notice the band increasing in width toward the extremes of the diagonal. In the lower extreme case ($\alpha$ small), this is likely because the degree distribution pre-changepoint is more homogeneous and thus many nodes have a similar likelihood of being chosen. In the higher extreme ($\alpha$ large) we suspect that this is due to an amplification of the rich-get-richer effect: after a certain point in time, the highest degree is so extreme that changing the power in the model slightly will not alter its near-certain probability of the corresponding node being selected. In Figure S3 (Supporting Info) is a heatmap showing the model similarity calculated for pairs of models $M_{DP}(\alpha)$ and $M_{DP}(\beta)$, over the networks generated from the model in Equation 8 at the time of their changepoint. We find asymmetry in the $\alpha - \beta$ phase plane, with $\sigma(G(M_{DP}(\alpha) , M_{DP}(\beta)))$ being large for large $\alpha$ values regardless of the value of $\beta$.

Figure 4: Maximum likelihood estimators for time at which degree power exponent changes value, with mean and standard deviation error areas calculated over 10 experiments. In both (a) and (b) a network is generated with each new node joining connecting to 3 existing nodes, node $i$ chosen with probability $p_i$ as in Equation 5.

Fitting time-varying models to real data

To demonstrate the relevance of this methodology, we investigate its use on a real world network dataset. The Enron email corpus, handed over to the investigation following the company’s collapse and later made public [3], comprises the inbox contents of all Enron employees during the 2001–2003 period. From this a timestamped network edgelist can be derived, where nodes are email users (not necessarily Enron employees since external emails to employees are included) and a timestamped edge represents an email sent between two users. After cleaning the dataset by removing self loops and multilinks, we fit a mixed object model with components comprising the BA model, triangle closure and random attachment:

$$p_i(t) = \beta_{BA}(t)p_i^{BA} + \beta_{tri}(t)p_i^{tri} + \beta_{rand}(t)p_i^{rand}$$

(9)

to the network evolution between August 2001 and June 2002, a period during which the most newsworthy events of the company collapse occurred. Each $\beta_{M}(t)$ is assumed to be piecewise constant, fitted over 10 evenly spaced time intervals during the whole period. Figure 6 shows the model proportions for each of these. The first observation we can make is that the BA model component has by far the largest weight for the whole evolution, with the triangle closure and random attachment components far behind, highlighting the explanatory power of the preferential attachment model. Second, there are
some trends in the changes of model weight which correspond to events in Enron’s timeline. The first is the period building up to Enron’s bankruptcy on the 2nd December 2001; we observe an increase in the BA’s weight at this time, and decrease in the random model. Indeed, in the three days following the bankruptcy announcement, the highest degree node of degree 1664 at the start of this time increases its degree by 26. Our results complement analysis on the same dataset in other works \cite{28,29,30} that have focused on the evolving community structure, spectral properties and interaction frequency, respectively.

![Figure 6: Best fitting (maximum likelihood) mixture of the BA, Triangle Closure and Random model over 10 time intervals. Annotated also are some key events in the company collapse.](image)

Figure 6: Best fitting (maximum likelihood) mixture of the BA, Triangle Closure and Random model over 10 time intervals. Annotated also are some key events in the company collapse.

5 Discussion and conclusion

In this paper we presented a methodology for modelling network growth in which different models can be compared at their ability to describe a network’s evolution. We can summarise our main contributions as follows: (i) We have contributed to the wider discussion that summary statistics are not enough to address hypotheses on network formation, by an example of two different mechanisms which generate scale-free networks with similar summary statistics. Furthermore, we show that when the network’s history is known, even these very similar network models can be distinguished. (ii) We presented a measure for similarity between two models over a given graph, which provides insight into the relationship with the model and the underlying graph. (iii) We found that a model comprised of a mixture of mechanisms can more faithfully model real networks than individual mechanisms in isolation. (iv) We show that this modelling framework is capable of recovering known changepoints in artificial networks. (v) We presented a case study of the Enron email network whose best fitting model changed over time, in response to the unfolding events in the company collapse.

The work presented adds to a toolkit of techniques for understanding time-varying networks, in a time where timestamped network data is becoming more available. Furthermore, it formally generalises and combines ideas from a wealth of literature on network growth models, into a framework that is amenable to statistical inference. The main modelling focus in this work was on which nodes attract new links in growing networks, in isolation from other considerations such as the frequency of new link/node arrivals. Our approach could be extended to addressing this aspect, investigating the interplay of these two aspects of network growth.

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Supplementary information for “Changing the tune: mixtures of network models that vary in time”

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1 Extra Datasets Used

1.1 Citation network dataset

The citation network is derived from a dataset of arXiv high energy physics phenomenology citations between 34,546 papers with submission dates spanning 1993-2003. From this we create a timestamped network edgelist, with an edge between node $A$ and $B$ if $A$ cites $B$. The network is treated as undirected, and any self-loops/duplicate edges were removed.

1.2 Stack Exchange MathOverflow dataset

The Stack Exchange MathOverflow dataset comprises UNIX-timestamped interactions (answering a user’s question, commenting on a user’s answer, commenting on a user’s question) between MathOverflow forum users. From this we construct a network by constructing an edge between two users if an interaction occurs between them, removing any self-loops/duplicate edges.

2 Use of FETA software

A Java-based implementation of the Framework for Evolving Topology Analysis used to obtain the results from this paper is available at [github.com/narnolddd/FETA3](https://github.com/narnolddd/FETA3).

Figure S1: Comparison on various network statistics between best fitting mixture model and its components for the arXiv citation dataset. The statistics shown are the maximum degree $k_{\text{max}}$, average clustering coefficient, mean squared degree $\langle k^2 \rangle$, degree assortativity, number of singleton and doubleton nodes (nodes of degree 1 and 2 respectively), and number of triangles. Each statistic is averaged over 10 realisations (apart from those of the real network) and s.d. error bars are shown.
Figure S2: Comparison on various network statistics between best fitting mixture model and its components for the Stack Exchange Math Overflow dataset. The statistics shown are the maximum degree $k_{\text{max}}$, average clustering coefficient, mean squared degree $\langle k^2 \rangle$, degree assortativity, number of singleton and doubleton nodes (nodes of degree 1 and 2 respectively), and number of triangles. Each statistic is averaged over 10 realisations (apart from those of the real network) and s.d. error bars are shown.
Figure S3: Similarity between degree power models $M_{DP}(\alpha)$ and $M_{DP}(\beta)$ over graph $G$ generated from $M_{DP}(\alpha)$.