Research Article

Solutions of a Class of Degenerate Kinetic Equations Using Steepest Descent in Wasserstein Space

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1. Introduction

The general model describing the kinetic equations is about an evolution equation of unknown function \( f \), representing a time-depending density of probability distribution of a material in a given domain of the space. In the present work, \( f \) may measure the density distribution of a system of identical particles of a bulk material. The density \( f \) depends on the time \( t \) and the position \( x \) and the velocity \( v \) of some particles at \( t \). Roughly speaking, the equation is considered as the evolution of the density function in the phase space \( \Omega \times \mathbb{R}^d \), with \( \Omega \) as an open bounded domain with periodic boundary. As a probability density, \( f \) remains positive in the court of time and satisfies the mass conservation principle:

\[
\int_{\Omega \times \mathbb{R}^d} f(t, x, v) \, dx \, dv = 1, \text{ for all } t \geq 0,
\]

where the initial datum \( f_0(x, v) \) is a probability density on \( \Omega \times \mathbb{R}^d \). Here, \( \Omega \) is an open, bounded, convex, and smooth domain of \( \mathbb{R}^d \), \( (d \geq 1) \) with \( \Omega \) periodic, \( c^* \) is the Legendre transform of a cost function \( c \), and \( G : [0, \infty) \to \mathbb{R} \) is a convex function.

Equation (1) can be viewed as a balance result of a streaming phenomenon with a general nonlinear interaction phenomenon between the particles described, respectively, as

\[
\begin{align*}
\frac{\partial f(t, x, v)}{\partial t} + v \cdot \nabla_x f(t, x, v) &= \text{div}_v \left[ f(t, x, v) \nabla c^* \left[ \nabla_v (G' (f(t, x, v))) \right] \right], \quad \text{in } [0, \infty) \times \Omega \times \mathbb{R}^d, \\
f(0, x, v) &= f_0(x, v), \quad \text{in } \Omega \times \mathbb{R}^d,
\end{align*}
\]

The transport equation (2) can somewhat be interpreted as a relaxation of (1) at the absence of the
interaction phenomena, whereas it reduced to (3) in absence of streaming.

One of the interests in considering (1) under a general nonlinearity is that it covers a very broad range of problems which occurred in physics and it is a purely mathematical challenge. Of course, it has been motivated by some previous works in the literature, namely, the works in [1–7], where (1) is investigated in some particular cases. Indeed, in [3], the authors dealt with the heat equation:

$$\frac{\partial \rho (t, v)}{\partial t} = \Delta \rho (t, v), \quad \text{in } [0, \infty) \times \mathbb{R}^d.$$  \hfill (4)

By fixing an probability density $\rho_0$ with $\int_{\mathbb{R}^d} |v|^2 \rho_0 (v) \, dv$ finite and a time step $h > 0$, they define the mass density $\rho^h_k$ as a discrete solution of (4) at time $t_k = kh$, which minimizes the functional

$$I (\rho) = \int_{\mathbb{R}^d} \rho \ln \rho \, dv + \frac{1}{h} W_2^2 (\rho, \rho^h_{k-1}),$$  \hfill (5)

on $P_2 (\mathbb{R}^d)$, where $P_2 (\mathbb{R}^d)$ is the set of all probability density on $\mathbb{R}^d$ having finite second moments and $W_2$ is the 2-Wasserstein metric defined as

$$W_2: \quad P_2 (\mathbb{R}^d) \times P_2 (\mathbb{R}^d) \longrightarrow \mathbb{R},$$

$$(\rho_1, \rho_2) \longrightarrow \inf_{\tau : \rho_1 \rightarrow \rho_2} \left( \int_{\mathbb{R}^d} |v - T (v)|^2 \rho_1 (v) \, dv \right)^{1/2}.$$

By defining $\rho^h_k$ as follows: $|\rho^h_k (t, v) = \rho^h_k (v)$, if $t \in [kh, h (k + 1)], \rho^h_k (0, v) = \rho_0 (v)$, they tend $h$ to 0 and then show that the sequence $(\rho^h_k)_k$ converges to a nonnegative function $\rho$, which solves (4) in a weak sense.

In [1], the existence of solutions for the spatially homogeneous equations associated with (1), that is, the equation for fixed $x$

$$\frac{\partial f}{\partial t} = \text{div}_x \{ f \mathcal{C} (\nabla_x (G' (f))) \}, \quad \text{in } [0, \infty) \times \Omega', \quad f = f (t, v),$$

has been proved by M. Agueh, using a similar variational scheme as in [6]. Here, $\Omega'$ is an bounded and convex domain (see [6] for more details).

A particular case of (1), namely, the kinetic equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f (t, x, v) = \Delta_x f (t, x, v),$$  \hfill (8)

obtained by choosing $\mathcal{C} (x) = |x|^2 / 2$ and $G (x) = x \ln x$, has been studied in [5] by using a discretization scheme basing on the “splitting method.” This enables the authors to decompose a discrete solution $f_k$ of the kinetic equation (8) in the form $f_k = f_{\text{kin}}^h f_{\text{kin}}^h$, where $f_{\text{kin}}^h$ stands for a discrete solution of the free transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f (t, x, v) = 0,$$  \hfill (9)

when $v$ is fixed and $f_{\text{kin}}^h$ a discrete solution of the diffusion equation in (8) when $x$ is fixed.

Defining $f^h$ as

$$f^h (t, x, v) = f_k ((x - (t - t_k) v), v), \quad \text{if } t \in [kh, (k + 1) h),$$

$$f^h (0, x, v) = f_0 (x, v), \quad \text{with } t_k = kh,$$

(10)

they show that $(f^h)_k$ converges to a nonnegative function $f$ which solves the kinetic equation (8) in a weak sense.

Such a decomposition is not suitable in the case of problem (1) because of its nonlinear structure. To deal with the more general class of kinetic equation (1), we combine some ideas from the splitting method in [5] along with some techniques developed in [1] for the spatially homogeneous equations:

$$\frac{\partial \rho (t, v)}{\partial t} = \text{div}_v \{ \rho \mathcal{C} (\nabla_v (G' (\rho))) \}, \quad \text{in } [0, \infty) \times \mathbb{R}^d.$$  \hfill (11)

For the best of our knowledge, our technique is new and is stated in a more general setting. It is worth mentioning that the class of the kinetic equation (1) also includes the Vlasov–Poisson equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f (t, x, v) = \text{div}_x (\omega f (t, x, v)),$$  \hfill (12)

obtained when

$$\mathcal{C} (x) = \sum_{i=1}^d x_i, \quad \omega = (1, 1, \ldots, 1) \in \mathbb{R}^d,$$

and the parabolic $p$–Laplacian equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f (t, x, v) = \text{div}_x \{ |\nabla f|^{p-2} \nabla f \},$$  \hfill (14)

in the case $\mathcal{C} (x) = |x|^p / p$ and $G (x) = x^\gamma / \gamma (\gamma - 1)$ with $\gamma = (2p - 3) / p - 1$.

In order to facilitate the reading of the paper, we summarize below the main steps and technical schemes according to which ours results will be carried out:

1. First of all, we fix a time step $h > 0$ and define $f_k$ as a discrete solution of the kinetic equation (1) at time $t_k = kh$, for $k \in \mathbb{N}$ (see Section 2.1).

2. Next, we prove that the solution $T_k$ of the Monge problem

$$(M): \quad \inf_{S_{\Omega} (\mathcal{F})^2 \mathcal{G}^{\mathbb{R}}_{\mathbb{R}}}, \quad \left\{ \int_{\mathbb{R}^d} \mathcal{C} \left( \frac{\nabla (S (v))}{h} \right) F^{\mathbb{R}}_k \, dv \right\},$$

is defined by

$$T_k (v) = v + h \mathcal{C} (\nabla_x (G_k (\rho_k F^{\mathbb{R}}_k))),$$

(16)

where $F^{\mathbb{R}}_k$ and $G^{\mathbb{R}}_k$ are as in Section 2.1. We use (16) to show that the sequence $(f_k)_k$ satisfies the time-discretization equation of the kinetic equation (1).
weakly, for $k \in \mathbb{N}$, where $A^h_k$ tends to 0 and when $h$ tends to 0.

(3) Then, we define an approximate solution $f^h$ of the kinetic equation (1) (see (118)), and we prove that the sequence $(f^h)$ converges to a nonnegative function $f$ which solves the kinetic equation (1) in a weak sense when $h$ tends to 0.

The convergence result has been achieved as follows:

(a) The weak convergence of $(f^h)$ to $f$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ for $0 < T < \infty$ follows from the displacement convexity of the functional $H(F) = \int_{\mathbb{R}^d} |F|^p \, dv$ on the set of all probability density (see Proposition 3), and its strong convergence in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ is obtained, thanks to a diagonal method combined with a result obtained in [1].

(b) The convexity of $c^*$ and the boundedness of $(\nabla_v (G^* (f^h)))_h$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ help to prove a weak convergence of the nonlinear term $(\text{div}_v (f^h c^* (\nabla_v G^* (f^h))))_h$ to $\text{div}_v (f \nabla c^* (\nabla_v G^* (f)))$ in $[C^\infty_c ([0, T] \times \Omega \times \mathbb{R}^d)]'$.

(c) Finally, the strong convergence of $(f^h)$ to $f$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and the weak convergence of the nonlinear term $(\text{div}_v (f^h c^* (\nabla_v G^* (f^h))))_h$ to $\text{div}_v (f \nabla c^* (\nabla_v G^* (f)))$ in $[C^\infty_c ([0, T] \times \Omega \times \mathbb{R}^d)]'$ enable us to establish that $f$ is a weak solution of the kinetic equation (1).

The paper is structured as follows. In Section 2, we state the required hypotheses and set some tools relevant for our problem. In Section 3, we set the variational formulation of the discrete problem related to our problem and construct the discrete solution. Section 4 concludes our main result by proving the convergence of the discrete problem to the considered problem. Section 5 ends the paper by giving an illustration example followed by an appendix on some regularity results.

2. Preliminaries

Throughout this work, we will assume the following:

$(H_G):$ $G : [0, \infty) \to \mathbb{R}$ is a convex function of class $C^3([0, \infty))$ such that $t \to t^4 G(t^{-d})$ is convex.

$(H_c):$ $c \in C^1(\mathbb{R}^d)$ is even and convex function such that $c(0) = 0$ and $A_1|x|^q \leq c(x) \leq A_2(|x|^q + 1)$, for all $x \in \mathbb{R}^d$, with $1 < q < \infty$.

$(H_f):$ $f_0$ is a probability density on $\Omega \times \mathbb{R}^d$ such that

\[
\begin{align*}
\int_{\Omega \times \mathbb{R}^d} |\nabla_v f_0(x, v)| \, dx \, dv < \infty, \\
\int_{\Omega \times \mathbb{R}^d} G(f_0(x, v)) \, dx \, dv < \infty.
\end{align*}
\]

Remark 1. Typical examples satisfying assumption $(H_G)$ are the functions $G(s) = s \ln s$, $s > 0$ and $G(s) = s^r$, $r > 1$.

Proposition 1. Assume that $c$ satisfies $(H_c)$. Let $H : [0, \infty) \to \mathbb{R}$ be a convex function such that $x \mapsto x^d H(x^{-d})$ is convex and decreasing.

Then, the functional

\[
E_0 : \mathbb{R}^d \to \mathbb{R}, \\
E_0(x) = \int_{\mathbb{R}^d} H(F(x)) \, dv,
\]

is displacement convex.

Proof. Let $F_0, F_1 \in P_q(\mathbb{R}^d)$ and $T \# F_1 = F_0$ be a $c$-optimal map that pushes $F_1$ forward to $F_0$. We define $T_t = (1 - t)F_1 + tF_1$, where $t \in [0, 1]$.

Then, we have

\[
E(F_t) = \int_{\mathbb{R}^d} H(F_t) \, dv = \int_{\mathbb{R}^d} F_t \, dv = \int_{\mathbb{R}^d} H(F_t) \, dv.
\]

From [1] and Proposition 3, we have that $F_1 = \text{det}(\nabla T) F_t \nabla T$.

Replacing (21) in (20), we obtain

\[
E(F_t) = \int_{\mathbb{R}^d} \frac{H(F_1)}{F_1(\text{det} \nabla T)^t} \, dv.
\]

Recalling again [1] and Proposition 3, we get that $\nabla T$ is diagonalizable with positive eigenvalues. So, using the fact that the map $A \mapsto (\text{det} A)^{1/d}$ is concave on the set of $d \times d$ diagonal matrices with positive eigenvalues, we get

\[
(\text{det} \nabla T)^{1/d} \geq (1 - t) + t (\text{det} \nabla T)^{1/d}.
\]

Since $x \mapsto x^d H(x^{-d})$ is decreasing, then

\[
\frac{H(F_1)}{\text{det} \nabla T} \leq \left( (1 - t)F_1^{-1/d} + tF_1^{-1/d} (\text{det} \nabla T)^{-1/d} \right)^d.
\]

From (23) and (24) and the fact that $x \mapsto x^d H(x^{-d})$ is convex, we obtain

\[
E(F_t) \leq (1 - t)E(F_1) + t \int_{\mathbb{R}^d} \frac{H(F_1)}{\text{det} \nabla T} \, dv
\]

\[
= (1 - t)E(F_1) + t \int_{\mathbb{R}^d} H(F_0) \, dv
\]

\[
= (1 - t)E(F_1) + t E(F_0).
\]
Hence, we conclude that the functional $E$ is displacement convex.

**Corollary 1.** Since the functional $E$ is displacement convex, we have that

$$E(F_0) - E(F_1) \geq \frac{dE(F_1)}{dt} \bigg|_{t=0} = \int_{\mathbb{R}^d} \left< \nabla v, Hr(F_1), T(v) - v \right> F_1 dv.$$

(26)

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} \left[ \partial_t \psi + v \cdot \nabla \psi - \nabla \psi \cdot \nabla \left( \frac{\nabla G^*}{G^*} f \right) \right] f(t, x, v) \, dt \, dx \, dv = -\int_{\Omega \times \mathbb{R}^d} \int_0^T f_0(x, v) \psi(0, x, v) \, dx \, dv.
\]

2.1. The Flow and Descend Algorithm. Assume that the probability density $f_0(x, v)$ satisfies ($H_{f_0}$) and fix $h > 0$ a time step, then we define the following:

1. $g_0^h(x, v) = f_0^0(x - h v, v)$
2. $\rho_1^h(x) = \int_{\mathbb{R}^d} g_0^h(x, v) \, dv$
3. $G^h(x) = g_1^h(x, v) / \rho_1^h(x)$ for $x \in \Omega$ fixed such that $\rho_1^h(x) > 0$
4. $J_1^h(x, v) = \rho_1^h(x) F_1^h(v)$

For each $x$ fixed, $F_1^h(x)$ denotes the unique minimizer of the variational problem:

$$\{P_1^h\} : \inf_{F \in P_q(\mathbb{R}^d)} \left\{ I(F) = E(F) + hW^h_c(F, G_1^h) \right\}. \tag{28}$$

$P_q(\mathbb{R}^d)$ is the set of all probability density on $\mathbb{R}^d$ having a finite $q$-moment, and $W_c^h(F, G_1^h)$ stands for the Kantorovich work defined as

$$W_c^h(F, G_1^h) = \inf_{\gamma \in \Pi(F, G_1^h)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left( \frac{y - z}{h} \right) \, dy \, dz \right\},$$

$$E(F) = \frac{1}{\rho_1^h(x)} \int_{\mathbb{R}^d} G\left( \rho_1^h(x) F(v) \right) \, dv. \tag{29}$$

We obtain the terms $f_k^h$, for $k \geq 2$, by induction as follows:

(i) By fixing $v$, we define $g_k^h(x, v) = f_{k-1}^h(x - h v, v)$.
(ii) By fixing $x$, we define

$$f_k^h(x, v) = \rho_k^h(x) F_k^h(v), \tag{30}$$

where

$$\rho_k^h(x) = \int_{\mathbb{R}^d} g_k^h(x, v) \, dv, \tag{31}$$

and $F_k^h$ is the unique minimizer of the variational problem.

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} \left[ \partial_t \psi + v \cdot \nabla \psi - \nabla \psi \cdot \nabla \left( \frac{\nabla G^*}{G^*} f \right) \right] f(t, x, v) \, dt \, dx \, dv = -\int_{\Omega \times \mathbb{R}^d} f_0(x, v) \psi(0, x, v) \, dx \, dv.
\]

2.2. $c$-Wasserstein Metric. In this section, we define a Wasserstein metric corresponding to a cost function $c$, and we study its topology.

**Definition 2.** Assume that $c : \mathbb{R}^d \to [0, \infty)$ satisfies $H_c$. Let $\rho_1, \rho_2 \in P(\mathbb{R}^d)$ two probability measures on $\mathbb{R}^d$. We define the $c$-Wasserstein metric between $\rho_1$ and $\rho_2$ by

$$W_c(\rho_1, \rho_2) = \inf \left\{ \lambda > 0, \inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left( \frac{y - x}{\lambda} \right) \, dy \, dx \leq 1 \right\}. \tag{34}$$

**Theorem 1.** Assume that $c : \mathbb{R}^d \to [0, \infty)$ satisfies $H_c$. Then, $W_c$ is a distance on the probability space $P(\mathbb{R}^d)$. Furthermore, if $(\rho_n)_n$ is a sequence in $P(\mathbb{R}^d)$ and $\rho \in P(\mathbb{R}^d)$, then $(\rho_n)_n$ converges to $\rho$ in the metric space $(P(\mathbb{R}^d), W_c)$ if and only if $(\rho_n)_n$ converges narrowly to $\rho$ in $P(\mathbb{R}^d)$.

**Proof.** Let $\rho_1, \rho_2 \in P(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$ such that $W_c(\rho_1, \rho_2) = 0$. Then, there exists a sequence $(\lambda_n)_n$ in $(0, \infty)$ which converges to $0$ such that

$$\inf_{\gamma \in \Pi(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left( \frac{x - y}{\lambda_n} \right) \, dy \leq 1, \quad \text{for all } n \in \mathbb{N}. \tag{35}$$

Denote by $\gamma_n$ the solution of Kantorovich problem:
\((K_n)\): \[
\inf_{y \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_n}\right) \, dy \leq 1,
\]
for all \(n \in \mathbb{N}\). \(\lambda_n\) converges to 0, then \(|x - y/\lambda_n|\) tends to \(\infty\) and when \(n\) goes to \(\infty\), for all \(x, y \in \mathbb{R}^d\) such that \(x \neq y\). Then, using the fact that \(c\) is coercive, we deduce that there exists \(N_0 \in \mathbb{N}\) such that
\[
c\left(\frac{x-y}{\lambda_n}\right) \geq \frac{|x-y|}{\lambda_n},\quad \text{for all } n \geq N_0.
\]
This with (37) implies that
\[
\inf_{y \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy \leq \lambda_n, \quad \text{for all } n \geq N_0.
\]
Let \(y_0\) be the solution of the Kantorovich problem:
\[
(K_0)\): \[
\inf_{y \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy = 0.
\]
Then, using (40), we obtain
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy_0 = 0.
\]
We deduce that \(x = y\) a.e. So for all \(\phi \in C_c (\mathbb{R}^d)\), we have
\[
\int_{\mathbb{R}^d} \phi(x)\rho_1 \, dx = \int_{\mathbb{R}^d} \phi(x)\rho_2 \, dx = \int_{\mathbb{R}^d} \phi(y)\rho_0 \, dy = \int_{\mathbb{R}^d} \phi(y)\rho_2 \, dy.
\]
Consequently, \(\rho_1 = \rho_2\).

Let us fix two probability measures \(\rho_1\) and \(\rho_2\) on \(\mathbb{R}^d\). Since \(c\) is even, then
\[
\inf_{\rho \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) \, dy = \inf_{\Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) \, dy,
\]
for all \(\lambda > 0\). We deduce from (44) that
\[
W_c (\rho_1, \rho_2) = \inf\left\{ \lambda > 0, \inf_{\Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) \, dy \leq 1 \right\}
\]
\[
= \inf\left\{ \lambda > 0, \inf_{\Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) \, dy \leq 1 \right\}
\]
\[
= W_c (\rho_2, \rho_1),
\]
Let \(\rho_1, \rho_2, \rho_3\) be three probability measures on \(\mathbb{R}^d\). Define \(\lambda_1 = W_c (\rho_1, \rho_2)\) and \(\lambda_2 = W_c (\rho_2, \rho_3)\). Denote by \(\gamma_1 \in \Pi (\rho_1, \rho_2)\) the solution of Kantorovich problem and denote by \(\gamma_2 \in \Pi (\rho_2, \rho_3)\) the solution of the Kantorovich problem:
\[
(K_2)\): \[
\inf_{y \in \Pi (\rho_2, \rho_3)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda_2}\right) \, dy.
\]
Using the Gluing lemma [8], there exists a probability measure \(\sigma\) on \(\mathbb{R}^d \times \mathbb{R}^d\) defined by
\[
\sigma (A \times B \times \mathbb{R}^d) = \gamma_1 (A \times B),
\]
\[
\sigma (\mathbb{R}^d \times A \times B) = \gamma_2 (A \times B),
\]
for some Borel subsets \(A\) and \(B\) of \(\mathbb{R}^d\). Let \(\gamma_3\) be a probability measure on \(\mathbb{R}^d \times \mathbb{R}^d\) defined by \(\gamma_3 (A \times B) = \sigma (A \times \mathbb{R}^d \times B)\). Then, \(\gamma_3 \in \Pi (\rho_1, \rho_3)\), and we use the convexity of \(c\) to get that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-z}{\lambda_1 + \lambda_2}\right) \, dy_3 \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-z}{\lambda_1}\right) \, dy_1 + \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{y-z}{\lambda_2}\right) \, dy_2.
\]

So, \(\inf_{\rho \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-z}{\lambda_1 + \lambda_2}\right) \, dy \leq 1\), and we conclude that
\[
W_c (\rho_1, \rho_2) \leq \lambda_1 + \lambda_2 = W_c (\rho_1, \rho_2) + W_c (\rho_2, \rho_3).
\]
Hence, \(W_c\) is a distance on \(P(\mathbb{R}^d)\).

Let us now study the topology of \(W_c\). Let \((\rho_n)\) be a sequence on \(P(\mathbb{R}^d)\) and \(\rho \in P(\mathbb{R}^d)\) such that \(W_c (\rho_n, \rho)\) converges to 0 when \(n\) tends to \(\infty\). Define \(\lambda_n = W_c (\rho_n, \rho)\), since \(\lambda_n\) converges to 0, then we use the fact that \(c\) is coercive to have
\[
\inf_{\rho \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy \leq \lambda_n,
\]
when \(n \to \infty\). We deduce that
\[
\lim_{n \to \infty} \inf_{\rho \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy = 0.
\]
Note that the 1-Wasserstein metric between \(\rho_n\) and \(\rho\) is
\[
W_1 (\rho_n, \rho) = \inf_{\rho \in \Pi (\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, dy.
\]
We deduce that \(W_1 (\rho_n, \rho)\) converges to 0 when \(n\) tends to \(\infty\). Since the 1-Wasserstein metric \(W_1\) induces the narrow topology of \(P(\mathbb{R}^d)\), we conclude that the sequence \((\rho_n)\) converges narrowly to \(\rho\) in \(P(\mathbb{R}^d)\).
Assume now that the sequence \((\rho_n)\) converges narrowly to \(\rho\) in \(P(\mathbb{R}^d)\). Fix \(\lambda > 0\) and denote by \(\gamma^\lambda_n\) the solution of Kantorovich problem:

\[
(K): \inf_{y \in \Pi(\rho, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) dy.
\]  

(54)

Since \(\rho_n\) converges narrowly to \(\rho\), then \(\gamma^\lambda_n\) converges narrowly to some \(\gamma^\lambda \in \Pi(\rho, \rho)\) and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) dy_n = \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) dy = 0.
\]  

(55)

So,

\[
\lim_{n \to \infty} \inf_{y \in \Pi(\rho, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) dy = 0,
\]  

(56)

for all \(\lambda > 0\). Hence, for all \(\lambda > 0\), there exists \(N_\lambda > 0\) such that

\[
\inf_{y \in \Pi(\rho, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{\lambda}\right) dy \leq 1, \quad \text{for all } n \geq N_\lambda.
\]  

(57)

So, \(W_c(\rho_n, \rho) \leq \lambda\) for all \(n \geq N_\lambda\). Then,

\[
\lim_{n \to \infty} W_c(\rho_n, \rho) = 0.
\]  

(58)

Consequently, \((\rho_n)\) converges narrowly to \(\rho\) in the metric space \((P(\Omega), W_c)\).

We establish now the existence of solution for the variational problem \((P)\) defined by

\[
(P): \inf_{\rho \in P(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} G(\rho) dx + hW_c(\rho, \rho) \right\},
\]  

(59)

in the metric space \((P(\mathbb{R}^d), W_c)\), where

\[
W_c(\rho, \rho) = \inf_{y \in \Pi(\rho, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\left(\frac{x-y}{h}\right) dy.
\]  

(60)

\(P_q(\mathbb{R}^d)\) is the set of all probability measures on \(\mathbb{R}^d\) having \(q\)-finite moment, that is,

\[
P_q(\mathbb{R}^d) = \left\{ \rho \in P(\mathbb{R}^d), \int_{\mathbb{R}^d} x^m d\rho < \infty \right\},
\]  

(61)

and \(h > 0\) being a time step.

\(\Box\)

Lemma 1. Assume that \(\rho_0\), \(G\), and \(c\) satisfy, respectively, \(H_{f_0}\), \(H_G\), and \(H_c\). Then, the following is obtained:

- (i) The map \(\rho \mapsto W_c(\rho, \rho)\) is lower semicontinuous in \((P(\mathbb{R}^d), W_c)\).
- (ii) The functional \(\rho \mapsto \int_{\mathbb{R}^d} G(\rho) dx\) is lower semicontinuous in \(L^1(\mathbb{R}^d)\).
- (iii) The set \(\{ \rho \in P(M, \mathbb{R}^d), \int_{\mathbb{R}^d} G(\rho) dx \leq \int_{\mathbb{R}^d} G(\rho_0) dx \}\) is a closed subset of \(L^1(\mathbb{R}^d)\), where \(P(M, \mathbb{R}^d) = \{ \rho \in P(\mathbb{R}^d), \rho \leq M \}\).

Proof

(i) Let \(\rho \in P(\mathbb{R}^d)\). Let \((\rho_n)\) be a sequence in \(P(\mathbb{R}^d)\) such that \((\rho_n)\) converges to \(\rho\) in metric space \((P(\mathbb{R}^d), W_c)\). Denote by \(\gamma _n^h\) the solution of the Kantorovich problem:

\[
(K_n): \inf_{y \in \Pi(\rho_n, \rho)} \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) dy.
\]  

(63)

We have

\[
\inf_{y \in \Pi(\rho_n, \rho)} \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) dy = \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) dy_n^h.
\]  

(64)

Since \((\rho_n)\) converges to \(\rho\) narrowly, then \((\gamma _n^h)\) converges to \(\gamma^h \in \Pi(\rho_0, \rho)\) narrowly and

\[
\lim_{n \to \infty} \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) dy_n^h \geq \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) dy \geq W_c(\rho_0, \rho).
\]  

(65)

This implies that

\[
\lim_{n \to \infty} W_c(\rho_n, \rho) \geq W_c(\rho_0, \rho).
\]  

(66)

Thus, we obtain the proof of (i).

(ii) Since \(G\) is convex and \(G \in C^1((0, \infty))\),

\[
\int_{\mathbb{R}^d} G(\rho) dx \geq \int_{\mathbb{R}^d} G(\rho) dx + \int_{\mathbb{R}^d} G'(\rho)(\rho_n - \rho) dx.
\]  

(67)

Hence, if \((\rho_n)\) converges to \(\rho\) weakly in \(L^1(\mathbb{R}^d)\) with \(\rho \in L^\infty(\mathbb{R}^d)\), we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} G'(\rho)(\rho_n - \rho) dx = 0,
\]  

(68)

and then

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} G(\rho_n) dx \geq \int_{\mathbb{R}^d} G(\rho) dx,
\]  

(69)

which complete the proof of (ii). The proof of (iii) is a consequence of (ii).

\(\Box\)

2.3. Existence Results for the Discrete Problem \((P_n^h)\). First of all, we introduce here for unbounded domains analogous of the maximum principle stated for bounded domains in [1]. This maximum principle plays a central role in the searching of solution for the discrete problem \((P_n^h)\). It is also used to further establish the convergence of our algorithm towards a weak solution of the kinetic equation (1).

Proposition 2 (maximum principle). Assume that the initial datum \(f_0\) satisfies \(N \leq f_0 \leq M\) with \(0 < N\), \(c\) satisfies \((H_c)\), and \(G\) satisfies \((H_G)\).

Then, any solution \(F_t^h\) of the variational problem
Taking into account some ideas from [1], we shall prove
\begin{align}
(P_{xh}^h): \inf_{P_{xh}^h} \left\{ E(F) + hW_h^h(F, G_{xh}^h) \right\},
\end{align}
satisfies $a(x) \leq P_{xh}^h \leq b(x)$, with $a(x) = (N/p_1^h(x))$ and
$b(x) = (M/p_1^h(x))$.

**Proof.** We define $E_x = \{ y \in \mathbb{R}^d, F_{xh}^h > b(x) \}$. Assume by
contradiction that $E_x$ has a positive Lebesgue measure. Then,
y_{x}(E_x^c \times E_x) > 0$, where $y_{x}$ is the minimizer of
\begin{align}
(K): \inf_{y \in \Pi \left( G_{xh}^h, P_{xh}^h \right)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c \left( \frac{y-z}{h} \right) dy, z \right\}.
\end{align}

Otherwise,
\begin{align}
b(x) | E_x | \leq \int_{E_x} P_{xh}^h dy = y_{x}(E_x^c \times E_x) = y_{x}(E_x^c \times E_x) \\
\leq y_{x}(E_x^c \times \mathbb{R}^d) = \int_{E_x^c} G_{xh}^h dy \leq |E_x| b(x).
\end{align}

This yields a contradiction.

Define $y_{x} = 1_{E_x \times E_x}, y_{x}$, i.e., $y_{x}(F) = y_{x}(F \cap E_x^c \times E_x)$, for all $F \subset \mathbb{R}^d$, and denote by $y_{0,x}$ and $y_{1,x}$ the marginals of $y_{x}$. We have $y_{x} \ll y_{x}$ with $y_{x} \in \Pi \left( G_{xh}^h, P_{xh}^h \right)$ and then
\begin{align}
y_{0,x} \ll G_{xh}^h \text{ and } y_{1,x} \ll F_{xh}^h.
\end{align}

Let $y_{0,x}$ and $y_{1,x}$ denote, respectively, the density functions of $y_{0,x}$ and $y_{1,x}$. Because
\begin{align}
\int_{E_x} y_{0,x} dy = y_{x}(E_x^c \times \mathbb{R}^d) = y_{x}(E_x^c \times E_x) = 0, \\
\int_{E_x} y_{1,x} dy = y_{x}(E_x^c \times \mathbb{R}^d) = y_{x}(E_x^c \times E_x) = 0,
\end{align}
we have $y_{0,x} = 0$ on $E_x$ and $y_{1,x} = 0$ on $E_x$.

For $\epsilon > 0$ is small enough, define $F_{xh}^{\epsilon,xh} = F_{xh}^h + \epsilon (y_{0,x} - y_{1,x})$ and
\begin{align}
\epsilon \int_{E_x^c} H(x,y) dy_x = \epsilon \int_{E_x^c} H(x,y) dy_x,
\end{align}
for all $H \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$. $F_{xh}^{\epsilon,xh} \in P_{\epsilon} (\mathbb{R}^d)$, $y_{\epsilon} \in \Pi \left( G_{xh}^{\epsilon,xh}, P_{xh}^{\epsilon,xh} \right)$, and we get
\begin{align}
I(F_{xh}^{\epsilon,xh}) - I(F_{xh}^h) = \epsilon \int_{E_x^c} [G(h(x)F_{xh}^{\epsilon,xh}) - G(h(x)F_{xh}^h)]
\end{align}
\begin{align}
- W_h^h(F_{xh}^h, G_{xh}^h) + \frac{1}{p_1^h(x)} \int_{E_x^c} (G(h(x)F_{xh}^{\epsilon,xh}) - G(h(x)F_{xh}^h))
\end{align}
\begin{align}
= \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x.
\end{align}

Since $G'$ is continuous, then
\begin{align}
W_h^h(F_{xh}^{\epsilon,xh}, G_{xh}^h) - W_h^h(F_{xh}^h, G_{xh}^h)
\end{align}
\begin{align}
= \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x - \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x
\end{align}
\begin{align}
\leq \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x - \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x
\end{align}
\begin{align}
= - \epsilon \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x.
\end{align}

Since $(y,z) \mapsto c(y-z/h)$ is nonnegative on $E_x^c \times E_x$ and $y_{x}(E_x^c \times E_x) > 0$, we have
\begin{align}
W_h^h(F_{xh}^{\epsilon,xh}, G_{xh}^h) - W_h^h(F_{xh}^h, G_{xh}^h) = - \epsilon \int_{E_x^c} \left[ \frac{y-z}{h} \right] dy_x < 0.
\end{align}

Also, we have
\begin{align}
\frac{1}{p_1^h(x)} \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - G'(h(x)F_{xh}^h) \right] dy_x
\end{align}
\begin{align}
= \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) + \epsilon \frac{h}{h} (x) (v_{0,x} - v_{1,x}) \right] - G'(h(x)F_{xh}^h) \right] dy_x
\end{align}
\begin{align}
= \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) (v_{0,x} - v_{1,x}) - G'(h(x)F_{xh}^h) \right] dy_x
\end{align}
\begin{align}
+ \frac{1}{p_1^h(x)} \int_{E_x^c} \left[ G'(h(x)F_{xh}^h) - \epsilon \frac{h}{h} (x) v_{0,x} - G'(h(x)F_{xh}^h) \right] dy_x.
\end{align}

$G$ is convex and of class $C^1$, then
\begin{align}
G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) (v_{0,x} - v_{1,x}) \geq G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{0,x} \geq G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{1,x}.
\end{align}

on $E_x$ and
\begin{align}
G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{0,x} \geq G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{0,x} \geq G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{1,x}.
\end{align}

on $E_x^c$. Hence, we have
\begin{align}
\frac{1}{p_1^h(x)} \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - G'(h(x)F_{xh}^h) \right] dy_x
\end{align}
\begin{align}
\leq \epsilon \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{0,x} \right] dy_x
\end{align}
\begin{align}
- \epsilon \int_{E_x^c} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - \epsilon \frac{h}{h} (x) v_{1,x} \right] dy_x
\end{align}
\begin{align}
= \epsilon \int_{E_x^c \times E_x} \left[ G'(h(x)F_{xh}^{\epsilon,xh}) - G'(h(x)F_{xh}^h) \right] dy_x.
\end{align}
\[
\lim_{\epsilon \to 0} \left[ G\left( \rho^{h, x}_{1, y} + \epsilon \rho^{h, v}_{0, x} \right) - G\left( \rho^{h, v}_{1, x} \right) \right] = 0.
\] (82)

Consequently, there exists \( \delta > 0 \) such that, for \( 0 < \varepsilon < \delta \), then
\[
0 \leq G\left( \rho^{h, x}_{1, y} + \epsilon \rho^{h, v}_{0, x} \right) - G\left( \rho^{h, v}_{1, x} \right) \leq \frac{1}{2\gamma_{x}(E_{x} \times E_{x})} \int_{E_{x} \times E_{x}} c\left( \frac{y - z}{\delta} \right) dz.
\] (83)

So, we fix \( \varepsilon > 0 \) small, such that \( 0 < \varepsilon < \delta \), and we use (81) and (83) to obtain
\[
\frac{1}{\rho^{h}_{1}(x)} \int_{\Omega} \left[ G(\rho^{h}_{1}(x)F^{xh}(x)) - G(\rho^{h}_{1}(x))F (x) \right] d\nu
\leq \frac{\varepsilon}{2} \int_{E_{x} \times E_{x}} c\left( \frac{y - z}{\varepsilon} \right) dz.
\] (84)

Now, fixing \( \varepsilon < \delta \) and combining (77) and (84) yield
\[
I(F^{xh}) - I(F)^{xh}
\leq -\varepsilon \int_{E_{x} \times E_{x}} c\left( \frac{y - z}{\varepsilon} \right) dz + \frac{\varepsilon}{2} \int_{E_{x} \times E_{x}} c\left( \frac{y - z}{\varepsilon} \right) dz
\]
\[
= -\varepsilon \int_{E_{x} \times E_{x}} c\left( \frac{y - z}{\varepsilon} \right) dz < 0.
\] (85)

It is a contradiction since \( F^{xh} \) is a minimizer of \( I \) on \( P_{q}(\Omega \times \mathbb{R}^{d}) \). Consequently, \( E_{x} \) is negligible and then \( F^{xh} \leq b(x) \).

The proof of \( F^{xh} \geq a(x) \) is analogous to that of \( F^{xh} \leq b(x) \). Thus, we conclude that \( a(x) \leq F^{xh} \leq b(x) \). \( \square \)

**Lemma 2.** Let \( f_{0} \in P_{q}(\Omega \times \mathbb{R}^{d}) \) be a probability density on \( \Omega \times \mathbb{R}^{d} \) such that \( 0 < \tilde{N} \leq f_{0} \leq M, \int_{\Omega \times \mathbb{R}^{d}} G(f_{0}) dxdv < \infty \) and \( \int_{\Omega \times \mathbb{R}^{d}} |\nabla f_{0}(x, v)| dxdv < \infty \).

The variational problem \( (P^{xh}) \) defined in (32) admits a unique minimizer \( F^{xh} \) in \( P_{q}(\mathbb{R}^{d}) \) and \( N(\rho^{h}_{1}(x)) \leq F^{xh} \leq (M(\rho^{h}_{1}(x)). \)

**Proof.** Since \( 0 < N \leq f_{0} \leq M \) and \( G^{xh}(v) = f_{0}(x, v)/\rho^{h}_{1}(x) \), then \( N(\rho^{h}_{1}(x)) \leq G^{xh}(v) \leq (M(\rho^{h}_{1}(x)). \) Then, we use Proposition 2, and we obtain that any minimizer \( F^{xh} \) of the variational problem \( F^{xh} \) satisfies \( a(x) \leq F^{xh} \leq b(x) \), with \( a(x) = N(\rho^{h}_{1}(x)) \) and \( b(x) = M(\rho^{h}_{1}(x)). \)

By using Proposition 2 and the fact that \( G \) is convex, we obtain that
\[
E(F) \geq 1/\rho^{h}_{1}(x) \int_{\mathbb{R}^{d}} G(\rho^{h}_{1}(x)G^{xh}(v)) dv + G'(N) - G'(M),
\] (86)

for all probability density \( F \in P_{q}(\mathbb{R}^{d}) \) such that \( N \leq F \leq M \).

We use now Lemma 1 to get that the functional \( F \mapsto \int_{\mathbb{R}^{d}} G(\rho^{h}_{1}(x)F(v)) dv \) is lower semicontinuous on the Wasserstein space \( (P_{q}(\mathbb{R}^{d}), W_{c}). \) We conclude then that the problem \( (P^{xh}) \) admits a solution \( F^{xh}. \) The strict convexity of \( G \) and \( c \) implies the strict convexity of the map \( F \mapsto W^{xh}_{c}(G^{xh}, F) \) and that of the maps \( F \mapsto \int_{\mathbb{R}^{d}} G(\rho^{h}_{1}(x)F(v)) dv \) and accordingly the uniqueness of the minimizer \( F^{xh} \) of \( (P^{xh}). \) \( \square \)

**3. Euler–Lagrange Equation for the Problem (F^{xh})**

In this section, we prove that the sequence \( (f^{xh})_{k} \) is a time discretization of the kinetic equation (1). In order to achieve it, we need the following lemma.

**Lemma 3 (explicit expression for optimal maps).** Assume that \( G \) satisfies \( H_{G} \) and \( c \) satisfies \( H_{c} \). Then, the Monge problem
\[
(M): \inf_{T_{x}} \int_{\mathbb{R}^{d}} G\left( \frac{v - T(x)}{\varepsilon} \right) F^{xh} dv,
\] (87)

admits a unique solution \( T_{x} \) such that
\[
T_{x}(v) = v + hN c^{*}(\nabla, G(\rho^{h}_{1} F^{xh}))
\] (88)

where \( F^{xh} \) is the unique minimizer of the variational problem.

\[
(F^{xh}): \inf_{F \in C^{\infty}(\mathbb{R}^{d})} \left\{ I(F) = \frac{1}{\varepsilon} \int_{\mathbb{R}^{d}} G(\rho^{h}_{1} F) + hW_{c}(G^{xh}, F) \right\}. \] (89)

**Proof.** Let \( \psi \in C^{\infty}(\mathbb{R}^{d}, \mathbb{R}) \) be a test function and consider the diffeomorphism map \( (T_{x})_{x \in \mathbb{R}^{d}} \) in \( C^{\infty}(\mathbb{R}^{d}, \mathbb{R}^{d}) \) defined by
\[
\frac{\partial T_{x}}{\partial x} = \psi \cdot T_{x},
\] (90)

Define the probability density \( G^{xh} = T_{x} \rho^{xh} \) on \( \mathbb{R}^{d}. \) Since \( G \) satisfies \( (H_{G}) \) and \( T_{x} \) is a diffeomorphism pushing \( F^{xh} \) forward to \( G^{xh}, \) we obtain the following Monge–Kantorovich-type energy inequality:
\[
\frac{1}{\rho^{h}_{1}(x)} \int_{\mathbb{R}^{d}} G(\rho^{h}_{1} G^{xh}) dv - \int_{\mathbb{R}^{d}} G(\rho^{h}_{1} F^{xh}) dv \
\geq \int_{\mathbb{R}^{d}} \left( T_{x}(v) - v, v \right)(G(\rho^{h}_{1} F^{xh})) F^{xh} dv.
\] (91)

Recalling the definition of \( T_{x}, \) we have
\[
\left| T_{x}(v) - v \right| \frac{\varepsilon}{\| \psi \|_{\infty}} \leq 0, \quad \text{for all } v \in \mathbb{R}^{d} \text{ and } \varepsilon > 0.
\] (92)

Dividing (91) by \( \varepsilon > 0 \) and using (92) and the dominated convergence theorem, we have.
\[
\lim_{\varepsilon \to 0} \frac{1}{\rho_k^0(x)} \left[ \int_{\mathbb{R}^d} G'(\rho_k^h G_k^\varepsilon) \, dv - \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv \right] \\
\geq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \left( \frac{T_k(v) - v}{\varepsilon} \cdot \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right) F_k^{\varepsilon h} \, dv \\
\geq \int_{\mathbb{R}^d} \left\langle \psi, \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right\rangle F_k^{\varepsilon h} \, dv.
\]

Furthermore, since \( T_k \# G_k^{\varepsilon h} = G_k^{\varepsilon h} \), then the Monge–Kantorovich-type energy inequality gives
\[
\lim_{\varepsilon \to 0} \frac{1}{\rho_k^0(x)} \left[ \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv - \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv \right] \\
\leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \left( \frac{T_k^{-1}(v) - v}{\varepsilon} \cdot \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right) G_k^{\varepsilon h} \, dv \\
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \left\langle \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right\rangle F_k^{\varepsilon h} \, dv.
\]

This implies that
\[
\lim_{\varepsilon \to 0} \frac{1}{\rho_k^0(x)} \left[ \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv - \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv \right] \\
= \int_{\mathbb{R}^d} \left\langle \psi, \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right\rangle F_k^{\varepsilon h} \, dv.
\]

We combine now (93) and (95) to derive
\[
\lim_{\varepsilon \to 0} \frac{1}{\rho_k^0(x)} \left[ \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv - \int_{\mathbb{R}^d} G'(\rho_k^h G_k^{\varepsilon h}) \, dv \right] \\
= \int_{\mathbb{R}^d} \left\langle \psi, \nabla_v \left( G'(\rho_k^h G_k^{\varepsilon h}) \right) \right\rangle F_k^{\varepsilon h} \, dv.
\]

Since \( T_k \# G_k^{\varepsilon h} = G_k^{\varepsilon h} \) and \( T_k \# F_k^{\varepsilon h} = G_k^{\varepsilon h} \), then
\[
W_k^h(G_k^{\varepsilon h}, G_k^* ) = \inf_{G_k \in G_k^{\varepsilon h} = G_k^* } \int_{\mathbb{R}^d} \left( \frac{v - S(v)}{h} \right) G_k^{\varepsilon h} \, dv \\
\leq \int_{\mathbb{R}^d} c \left( \frac{v - T_k^{-1}(v)}{h} \right) G_k^{\varepsilon h} \, dv.
\]

Thus, for \( \varepsilon > 0 \), we have
\[
\frac{1}{\varepsilon} \left[ c \left( \frac{v - T_k^{-1}(v)}{h} \right) - c \left( \frac{v - T_k(v)}{h} \right) \right] G_k^{\varepsilon h} \, dv.
\]

\[
\frac{h W_k^h(G_k^{\varepsilon h}, G_k^* ) - h W_k^h(G_k^{\varepsilon h}, F_k^{\varepsilon h})}{\varepsilon} \\
\leq - \int_{\mathbb{R}^d} \left\langle \nabla_v \left( \frac{T_k(v) - v}{h} \right), T_k(v) - v \right\rangle F_k^{\varepsilon h} \, dv.
\]

Now, from (92) and the dominated convergence theorem, we obtained
\[
\lim_{\varepsilon \to 0} \frac{h W_k^h(G_k^{\varepsilon h}, G_k^* ) - h W_k^h(G_k^{\varepsilon h}, F_k^{\varepsilon h})}{\varepsilon} \\
\leq - \int_{\mathbb{R}^d} \left\langle \nabla_v \left( \frac{T_k(v) - v}{h} \right), \psi \right\rangle F_k^{\varepsilon h} \, dv.
\]

Since \( F_k^{\varepsilon h} \) minimizes the functional
\[
I(F) = \frac{1}{\rho_k^0(x)} \int_{\mathbb{R}^d} G'(\rho_k^h F) \, dv + h W_k^h(G_k^{\varepsilon h}, F),
\]
on the probability space, the Euler–Lagrange equation yields
\[
\lim_{\varepsilon \to 0} \frac{I(G_k^* ) - I(F_k^{\varepsilon h})}{\varepsilon} = 0.
\]

Consequently, by using (96) and (100), we have
\[
\int_{\mathbb{R}^d} \left\langle \psi, \nabla_v \left( G'(\rho_k^h F_k^{\varepsilon h}) \right) F_k^{\varepsilon h} \, dv \\
- \int_{\mathbb{R}^d} \left\langle \nabla_v \left( \frac{T_k(v) - v}{h} \right), \psi \right\rangle F_k^{\varepsilon h} \, dv \geq 0.
\]

Next, by replacing \( \psi \) by \( -\psi \) in (96), we obtain the desired equation:
\[
\int_{\mathbb{R}^d} \left\langle \psi, \nabla_v \left( G'(\rho_k^h F_k^{\varepsilon h}) \right) F_k^{\varepsilon h} \, dv - \int_{\mathbb{R}^d} \left\langle \nabla_v \left( \frac{T_k(v) - v}{h} \right), \psi \right\rangle F_k^{\varepsilon h} \, dv = 0.
\]

Thus, we conclude that
\[
\nabla_v \left( \frac{T_k(v) - v}{h} \right) = \nabla_v \left( G'(\rho_k^h F_k^{\varepsilon h}) \right) a.e. - F_k^{\varepsilon h}.
\]

Note that \( \nabla c \) is inverse and \( (\nabla c)^{-1} = \nabla c^* \). Then, we obtain the explicit expression of the optimal map \( T\varepsilon \):
\[
T_k(v) = v + h \nabla c^* \left( \nabla_v \left( G'(\rho_k^h F_k^{\varepsilon h}) \right) \right) - a.e. F_k^{\varepsilon h}.
\]

The proof of this lemma is complete.

We are now ready to show that the sequence \( (f_k^h) \) satisfies the time discretization (17) of the kinetic equation (1).

Let \( \psi \in C_c^\infty(\Omega \times \mathbb{R}^d) \), then
\[
\int_{\Omega \times \mathbb{R}^d} \left( f_k^h(x, v) - f_k^{-1}(x, v) \right) \psi(x, v) \, dx dv \\
= \int_{\Omega \times \mathbb{R}^d} \left( g_k^h(x, v) - g_k^{-1}(x, v) \right) \psi(x, v) \, dx dv \\
+ \int_{\Omega \times \mathbb{R}^d} \left( g_k^h(x, v) - f_k^{-1}(x, v) \right) \psi(x, v) \, dx dv.
\]
By using $T_k F^{xh}_k = G^{xh}_k$, $f^h_k = \rho^h_k F^{xh}_k$, and $g^h_k = \rho^h_k G^{xh}_k$, we have

\[
\begin{align*}
\int_{\Omega \times \mathbb{R}^d} (f^h_k(x,v) - g^h_k(x,v)) \psi(x,v) \, dx \, dv &= \int_{\Omega \times \mathbb{R}^d} (\psi(x,v) - \psi(x,T_k(x))) \, f^h_k(x,v) \, dv, \quad (108) \\
\int_{\Omega \times \mathbb{R}^d} (g^h_k(x,v) - f^h_{k-1}(x,v)) \psi(x,v) \, dx \, dv &= \int_{\Omega \times \mathbb{R}^d} (\psi(x+v,hv) - \psi(x,v)) \, f^h_{k-1}(x,v) \, dv.
\end{align*}
\]

Next, we use the Taylor formula and the expression of the optimal $T_k$ to obtain

\[
\begin{align*}
&\int_{\Omega \times \mathbb{R}^d} (f^h_k(x,v) - f^h_{k-1}(x,v)) \psi(x,v) \, dx \, dv \\
&= h \int_{\Omega \times \mathbb{R}^d} \langle \psi, \nabla \psi \rangle f^h_{k-1}(x,v) \, dx \, dv \\
&\quad - h \int_{\Omega \times \mathbb{R}^d} \nabla \psi \cdot \nabla (\psi(G'(f^h_k))) f^h_{k-1} \, dx \, dv \\
&\quad + h^2 \int_{\Omega \times \mathbb{R}^d} \langle \nabla \psi, \psi + t h \theta \rangle f^h_{k-1} \, dx \, dv \\
&\quad + h^2 \int_{\Omega \times \mathbb{R}^d} \langle \sigma_k \nabla \psi, \psi + x + ah \rangle f^h_{k-1} \, dx \, dv.
\end{align*}
\]

Here, $\sigma_k = \nabla^* (\nabla \psi (G'(f^h_k)))$ and $\theta, \alpha \in [0,1]$. Now define

\[
A^h_k[\psi] = \frac{h^2}{2} \int_{\Omega \times \mathbb{R}^d} \langle \sigma_k, \nabla \psi \rangle f^h_{k-1}(x,v) \, dx \, dv
\]

\[
+ h^2 \int_{\Omega \times \mathbb{R}^d} \langle \psi, \nabla^2 \psi (x + ah, v) \rangle f^h_{k-1} \, dx \, dv.
\]

(110)

If we show that $A^h_k[\psi]$ tends to 0 as $h$ goes to 0, then we are done.

Indeed, from the maximum principle, $N \leq f^h_k \leq M$, and then

\[
\begin{align*}
|A^h_k[\psi]| &\leq \frac{h^2}{2} \|\nabla^2 \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \int_{\Omega \times \mathbb{R}^d} |\nabla \psi (G'(f^h_k))|^2 f^h_{k-1} \, dx \, dv \\
&\quad + h^2 \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)} \|f\|_{L^1(\Omega \times \mathbb{R}^d)} K_\psi \|\nabla^2 \psi\|_{L^1(\Omega \times \mathbb{R}^d)}^2 f^h_{k-1} \, dx \, dv
\end{align*}
\]

(111)

where $K_\psi$ is a compact subset of $\Omega \times \mathbb{R}^d$ such that $\text{supp} \psi \subset K_\psi$.

Since $F^{xh}_k$ minimizes the general functional energy $I(F) = 1/p^h_k \int_{\mathbb{R}^d} G(\rho^h_k F) \, dv + h W^{xh}_k (F, G^{xh}_k)$ on the metric space $(P_\gamma(\mathbb{R}^d), W_\gamma)$, then

\[
\frac{1}{p^h_k} \int_{\mathbb{R}^d} G(\rho^h_k F^{xh}_k) \, dv + h W^{xh}_k (F^{xh}_k, G^{xh}_k) \leq \frac{1}{p^h_k} \int_{\mathbb{R}^d} G(\rho^h_k G^{xh}_k) \, dv.
\]

(112)

We use in (112) the expression of the optimal maps $T_k$ and the definition of the $f^h_k$, and then we have

\[
\begin{align*}
\int_{\Omega \times \mathbb{R}^d} G(f^h_k(x,v)) \, dx \, dv &\geq h \int_{\Omega \times \mathbb{R}^d} \nabla^* (\nabla \psi (G'(f^h_k))) f^h_{k-1} \, dx \, dv.
\end{align*}
\]

(113)

Because $G \in C^1((0,\infty))$ is convex and $N \leq f^h_k \leq M$, we have

\[
\int_{\Omega \times \mathbb{R}^d} G(f^h_k(x,v)) \, dx \, dv - \int_{\Omega \times \mathbb{R}^d} G(f^h_{k-1}(x,v)) \, dx \, dv \leq G'(M) - G'(N).
\]

(114)

By using (114) and the fact that $c(x) \geq A_1 |x|^q$, then (113) becomes

\[
h \int_{\Omega \times \mathbb{R}^d} \nabla^* (\nabla \psi (G'(f^h_k))) f^h_{k-1} \, dx \, dv \leq \frac{G'(M) - G'(N)}{A_1}.
\]

(115)

Recalling (115) and (111), we obtain

\[
|A^h_k(\psi)| \leq h \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)}^2 M
\]

\[
+ h^{q+1} \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)}^2 G'(M) - G'(N) + h^{q+1}, \quad \text{if, } q \geq 2,
\]

\[
|A^h_k(\psi)| \leq h \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)}^2 M + h^{q+1} \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{R}^d)}^2 G'(M) - G'(N) + h^{q+1}, \quad \text{if, } q < 2.
\]

(116)

(117)

We combine (116) and (117) to conclude that $A^h_k(\psi)$ tends to 0, when $h$ goes to 0. Accordingly, we conclude that the sequence $(f^h_k)_k$ results from a time discretization of the kinetic equation (1).

Recalling Section 2.1, we define an approximate solution $f^h$ over $[0,\infty) \times \Omega \times \mathbb{R}^d$ of the kinetic equation (1) as follows:

\[
\begin{align*}
&f^h(t,x,v) = f^h_k(x(t - t_k),v), \quad \text{if } t \in [t_k,t_{k+1}),
\end{align*}
\]

\[
\begin{align*}
&f^h(0,x,v) = f_0(x,v).
\end{align*}
\]

(118)

In the next section, we establish the convergence of the sequence $(f^h)_h$ to a weak solution $f$ of (1). \hfill \square

4. Convergence Results

4.1. Weak Convergence of $(f^h)_h$. Let us consider the sequence $(f^h)_h$ as defined in (118).
Lemma 4. Assume that \( f_0 \) satisfies \((H_{f_0})\). Then, for all \( k \geq 1 \), we have
\[
\int_{\Omega \times \mathbb{R}^d} |f_k^h(x, v)|^p \ dx \ dv \leq \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p \ dx \ dv. \tag{119}
\]

Proof. Taking \( H(t) = t^p \), \( p > 1 \), in Corollary 1, we obtain
\[
\int_{\mathbb{R}^d} |G_k^{\text{vh}}|^p \ dv \leq \int_{\mathbb{R}^d} |F_k^{\text{vh}}|^p \ dv \geq \int_{\mathbb{R}^d} \left( \langle T_k(v) - \nabla H(F_k^{\text{vh}}) \rangle \right)^p F_k^{\text{vh}} \ dv,
\]
where \( T_k \) is \( c \)-optimal map that pushes \( F_k^{\text{vh}} \) forward to \( G_k^{\text{vh}} \).

We use expression of \( T_k \) and we obtain
\[
\int_{\mathbb{R}^d} |G_k^{\text{vh}}|^p \ dv - \int_{\mathbb{R}^d} |F_k^{\text{vh}}|^p \ dv \geq \int_{\mathbb{R}^d} \left( \langle T_k(v) - \nabla H(F_k^{\text{vh}}) \rangle \right)^p F_k^{\text{vh}} \ dv \tag{120}
\]
\[
= p(p-1)h \int_{\mathbb{R}^d} \langle \nabla c^*(\nabla G(f_k^{\text{vh}})), \nabla (F_k^{\text{vh}}) \rangle (F_k^{\text{vh}})^{p-1} \ dv. \tag{121}
\]

Since \( c \geq 0 \) and \( c(0) = 0 \), then \( c^* \geq 0 \); hence,
\[
\int_{\mathbb{R}^d} |G_k^{\text{vh}}|^p \ dv - \int_{\mathbb{R}^d} |F_k^{\text{vh}}|^p \ dv \geq 0. \tag{125}
\]

Multiplying (125) by \( |x|^p |\rho_k^h(x)|^p \), we obtain after integration
\[
\int_{\Omega \times \mathbb{R}^d} |f_k^{\text{vh}}|^p \ dx \ dv \leq \int_{\Omega \times \mathbb{R}^d} |g_h|^p \ dx \ dv = \int_{\Omega \times \mathbb{R}^d} |f_{k-1}^h|^p \ dx \ dv, \tag{126}
\]
and then by an iteration process on \( k \), we get the proof of Lemma 4. \( \square \)

4.2. Weak Convergence of the Linear Term. Here, we study the weak convergence of the linear term.

Proposition 3. Assume that \( f_0 \) satisfy \((H_{f_0})\). Then, there exists a function \( f \) in \( L^p([0, T] \times \Omega \times \mathbb{R}^d) \cap L^\infty ([0, T] \times \Omega \times \mathbb{R}^d) \) until a subsequence \( (f_k^{\text{vh}}) \) converges to \( f \) weakly in \( L^p([0, T] \times \Omega \times \mathbb{R}^d) \) and weakly-* in \( L^\infty ([0, T] \times \Omega \times \mathbb{R}^d) \).

Since \( \nabla G^* (f_k^h) = G^* (f_k^h) \nabla f_k^h \) and \( f_k^h = \rho_k^h F_k^{\text{vh}} \), (121) becomes
\[
\int_{\mathbb{R}^d} |G_k^{\text{vh}}|^p \ dv - \int_{\mathbb{R}^d} |F_k^{\text{vh}}|^p \ dv \geq \frac{h p (p-1)}{\rho_k^h(x)} \int_{\mathbb{R}^d} (F_k^{\text{vh}})^{p-1} \ dv \tag{122}
\]
\[
\geq \int_{\mathbb{R}^d} (F_k^{\text{vh}})^{p-1} \ dv \tag{123}
\]

From the convexity of \( c \) and \( c^* \), we have
\[
a b = c(a) + c^* (b), \quad \text{if } a, b \in \mathbb{R}^d, \quad \text{with } a = \nabla c^* (b). \tag{124}
\]

Thus, using (123) with \( a = \nabla c^* (\nabla G(f_k^{\text{vh}})) \) and \( b = \nabla G^* (f_k^h) \), (122) becomes
\[
\int_{\mathbb{R}^d} |G_k^{\text{vh}}|^p \ dv - \int_{\mathbb{R}^d} |F_k^{\text{vh}}|^p \ dv \geq \frac{h p (p-1)}{\rho_k^h(x)} \int_{\mathbb{R}^d} (F_k^{\text{vh}})^{p-1} \ dv \tag{125}
\]

Lemma 5. Assume that \( f_0 \in L^p(\Omega \times \mathbb{R}^d) \cap L^\infty (\Omega \times \mathbb{R}^d) \)

with \( \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p \ dx \ dv < \infty \) and \( 0 < N \leq f_0 \leq M \). For \( 0 < T < \infty \), we have
\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f_k^h(t, x, v)|^p \ dx \ dv \leq T \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p \ dx \ dv. \tag{126}
\]

Proof. Let \( 0 < T < \infty \) and let \( h > 0 \) be a step such that \( T/h \in \mathbb{N}^* \).

We use Lemma 3 and the definition of \( f_k^h \) in (118) to obtain
\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} |f_k^h(t, x, v)|^p \ dx \ dv \leq T \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p \ dx \ dv < \infty. \tag{127}
\]

Moreover,
\[
\lim_{k \to \infty} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left| \partial_t \psi + v \nabla_x \psi \right| f_k^h \ dx \ dv \tag{128}
\]
\[
= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left| \partial_t \psi + v \nabla_x \psi \right| f \ dx \ dv, \tag{129}
\]
for every test function \( \psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d) \), with \( \sup \psi(x, x, v) \subset [-T, T] \).

**Proof.** Since \( f_h \in L^p(\Omega \times \mathbb{R}^d) \cap L^\infty(\Omega \times \mathbb{R}^d) \) (see Proposition 2 and Lemma 4), then

\[
\|f_h\|_{L^p} \leq \|f\|_{L^p(\Omega \times \mathbb{R}^d)},
\]

\[
\|f_h\|_{L^\infty} \leq \|f\|_{L^\infty(\Omega \times \mathbb{R}^d)}.
\]

Hence, the sequence \((f_h)_h\) is bounded in \(L^p([0, T] \times \Omega \times \mathbb{R}^d) \cap L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\), and then, there is a subsequence of \((f_h)_h\) that converges to a nonnegative function \(f\) weakly in \(L^p([0, T] \times \Omega \times \mathbb{R}^d)\) and weakly-* in \(L^\infty([0, T] \times \Omega \times \mathbb{R}^d)\).

Since \( \partial_t \psi + v \nabla_x \psi \in L^3([0, T] \times \Omega \times \mathbb{R}^d) \), for every \( \psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d) \) such that \( \sup \psi(x, x, v) \subset [-T, T] \), we finally obtain Proposition 3. \( \square \)

### 4.3. Weak Convergence of the Nonlinear Term

In this section, we establish the weak convergence of the nonlinear term.

First, we prove that the sequence \(\nabla \psi, G(f_h)\) is bounded in \(L^3([0, T] \times \Omega \times \mathbb{R}^d)\), and that \(\nabla^c(\nabla \psi, G(f_h))\) is bounded in \(L^5([0, T] \times \Omega \times \mathbb{R}^d)\) where \(0 < T < \infty\).

**Lemma 6.** Assume that \(f, c, \) and \(G\) satisfy, respectively, \((H_f), (H_c), \) and \((H_G)\). Then, for all \(0 < T < \infty\) fixed,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \left| \nabla^c(\nabla \psi, G(f_h)) \right|^q \, dt \, dx \, dv \\
\leq \frac{T}{A_2 N} \left[ G'(M) - G'(N) \right],
\]

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \left| \nabla \psi, G(f_h) \right|^p \, dt \, dx \, dv \\
\leq \left[ A_2 T + G'(M) - G'(N) \right] \frac{pq^{p-1} A_1^{p-1}}{N}.
\]

**Proof.** Using (115), we have

\[
\int_{\Omega \times \mathbb{R}^d} \left| \nabla^c(\nabla \psi, G(f_h)) \right|^q f_h \, dx \, dv \\
\leq \frac{G'(M) - G'(N)}{A_1}.
\]

But \(f_h \geq N\), and then

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \left| \nabla^c(\nabla \psi, G(f_h)) \right|^q \, dt \, dx \, dv \\
\leq \left[ G'(M) - G'(N) \right] \frac{T}{N A_1}.
\]

Recalling (123), with \(a = \nabla^c(\nabla \psi, G(f_h))\) and \(b = \nabla \psi, G(f_h)\), we have

\[
\nabla \psi, (\nabla \psi, G(f_h)) \cdot \nabla \psi, G(f_h) \\\n- c(\nabla^c(\nabla \psi, G(f_h))) = \nabla^c(\nabla \psi, G(f_h))
\]

with \(c \geq 0\).

Then,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla^c(\nabla \psi, G(f_h)) f_h \, dt \, dx \, dv \\
\leq \int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla^c(\nabla \psi, G(f_h)) \nabla G(f_h) \, dt \, dx \, dv
\]

\[
= \int_{\Omega \times \mathbb{R}^d} \nabla^c(\nabla \psi, G(f_h)) \nabla G(f_h) \, dx \, dv
\]

\[
\leq \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} G(f_{k+1}^h) - \int_{\Omega \times \mathbb{R}^d} G(f_k^h) \, dx \, dv.
\]

Moreover,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla^c(\nabla \psi, G(f_h)) f_h \, dt \, dx \, dv \leq G'(M) - G'(N),
\]

and then,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla^c(\nabla \psi, G(f_h)) f_h \, dt \, dx \, dv \leq G'(M) - G'(N).
\]

Because \(c(y) \leq A_2 |y|^q + A_2\) we have \(x \cdot \nabla \psi + c \nabla \psi \geq x \cdot \psi + c \psi \geq A_2 \psi - A_2\) for all \(x, y \in \mathbb{R}^d\) and then

\[
c'(x) = \sup_{y \in \mathbb{R}^d} (x \cdot y - c(y)) \geq -A_2 + \frac{|x|^p}{pq^{p-1} A_2^{p-1}}
\]

Consequently,

\[
-A_2 T + \frac{1}{pq^{p-1} A_2^{p-1}} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla \psi, G(f_h) \, dx \, dv
\]

\[
\leq G'(M) - G'(N).
\]

Using \(f_h \geq N\), we obtain

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla \psi, G(f_h) \, dx \, dv
\]

\[
\leq \left[ TA_2 + G'(M) - G'(N) \right] \frac{pq^{p-1} A_2^{p-1}}{N}.
\]

\( \square \)

### 4.4. Strong Convergence of \((f_h)_h\)

In order to prove the strong convergence result, we need to establish the following compactness results for \((f_h)_h\) in \(L^p([0, T] \times \Omega \times \mathbb{R}^d)\).
**Lemma 7** (velocity-compactness). Assume that \( f_0 \) satisfies \((H_f)\). Fix \( 0 < T < \infty \) and \( 0 \not= \eta \in \mathbb{R}^d \) such that \( |\eta| \) is small. Then,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} f_k^\theta(t, x, v + \eta) - f_k^\theta(t, x, v) \, dtdxdv \leq |\eta|^p K,
\]

(141)

where \( K = \inf \{ P|\Omega|/|G^\theta(t)|^p \} |TA_2 + G^\theta(M) - G^\theta(N)\} \) is a constant.

**Proof.** We use \( \nabla_x G^\theta(f^\theta) = G''(f^\theta) \nabla_x f^\theta, N \leq f^\theta \leq M, \) and (140) to produce that

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \| \nabla_x f^\theta \|^p \, dtdxdv \leq K.
\]

(142)

Then, \( f^\theta \) and \( \nabla_x f^\theta \) are \( L^p([0, T] \times \Omega \times \mathbb{R}^d) \) and approximating \( f^\theta(t, x, \cdot) \) by \( C_c^\infty(\mathbb{R}^d) \) functions in \( W^{1,p}(\mathbb{R}^d) \), we obtain

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} f^\theta(t, x, v + \eta) - f^\theta(t, x, v) \, dtdxdv \leq |\eta|^p \int_{[0,T] \times \Omega \times \mathbb{R}^d} \| \nabla_x f^\theta \|^p \, dtdxdv.
\]

(143)

This implies that

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} f^\theta(t, x, v + \eta) - f^\theta(t, x, v) \, dtdxdv \leq |\eta|^p K.
\]

(144)

**Lemma 8** (position-compactness). Assume that \( f_0 \) satisfies \((H_f)\). We fix \( 0 < T < \infty \) and \( 0 \not= \eta \in \mathbb{R}^d \) such that \( |\eta| \) is small. Then,

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} f_k^\theta(t, x + \eta v, v) - f_k^\theta(t, x, v) \, dtdxdv \leq |\eta|^p T K_1,
\]

(145)

where \( K_1 = 2\| f_0 \|^p_{L^p(\Omega \times \mathbb{R}^d)} \).

**Proof.** By fixing \( v \), (17) becomes

\[
f_k^\theta - f_{k-1}^\theta + \eta v \cdot \nabla_x f_k^\theta = o(h),
\]

(146)

where \( o(h) \) tends to 0 when \( h \) goes to 0. We use (146), and the fact that \( f_k^\theta \) is bounded in \( L^p(\Omega \times \mathbb{R}^d) \) to deduce that \( (h v \cdot \nabla_x f_k^\theta) \) is bounded in \( L^p(\Omega \times \mathbb{R}^d) \). Then, by using Lemma 4, we have

\[
\| h v \cdot \nabla_x f_k^\theta \|_{L^p(\Omega \times \mathbb{R}^d)} \leq 2 \| f_0 \|_{L^p(\Omega \times \mathbb{R}^d)}.
\]

(147)

Thus, \( f_k^\theta(t, \cdot, v) \in L^p(\Omega) \), and approximating \( f_k^\theta(t, \cdot, v) \) by \( C_c^\infty(\mathbb{R}^d) \) functions, we use the fact that \( (h v \cdot \nabla_x f_k^\theta) \) is bounded in \( L^p(\Omega \times \mathbb{R}^d) \) to deduce that

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} f_k^\theta(t, x + \eta v, v) - f_k^\theta(t, x, v) \, dtdxdv \leq |\eta|^p \int_{[0,T] \times \Omega \times \mathbb{R}^d} \| h v \cdot \nabla_x f_k^\theta \|^p \, dtdxdv \]

(148)

\[ \leq |\eta|^p \int_{[0,T] \times \Omega \times \mathbb{R}^d} \| h v \cdot \nabla_x f_k^\theta \|^p \, dtdxdv \]

\[ \leq |\eta|^p \| f_0 \|_{L^p(\Omega \times \mathbb{R}^d)}^p T \int_{\Omega \times \mathbb{R}^d} |f_0(x, v)|^p \, dxdv. \]

(148)

**Lemma 9.** Assume that \((H_{f_{\epsilon}})\) holds. Let \( \tau > 0 \) be quite small such that \( \{t, t + \tau \} \subset [0, T], \forall t \in [0, T], \) where \( 0 < T < \infty \) is fixed. We have

\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \left[ G'(f^\theta(t + r) - G'(f^\theta(t))) \right] [f^\theta(t + r) - f^\theta(t)] \, dtdxdv \leq \tau K_2(\Omega, T, N, M, f_0, G', \eta, p).
\]

(149)

where \( K_2(\Omega, T, N, M, f_0, G', \eta, p) \) is a positive constant depending on \( \Omega, T, N, M, f_0, G, \eta, p \).

**Proof.** Choose \( h > 0 \) such that \( T/h \in \mathbb{N}^+ \) and \( \tau/h = N \in \mathbb{N}^+ \). Define

\[
A = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ G'(f_{k-1}^\theta(x, v)) - G'(f_{k-1}^\theta(x, v)) \right] \int_{\Omega \times \mathbb{R}^d} |f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v)| \, dxdv.
\]

(150)

Using the definition of \( f^\theta \) in (118), we obtain

\[
A = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ G'(f_{k-1}^\theta(x, v)) - G'(f_{k-1}^\theta(x, v)) \right].
\]

(151)

Note that

\[
f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v) = \sum_{i=0}^{N-1} \left[ f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v) \right].
\]

(152)

Using (152), we have

\[
A = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v) \right] \, dxdv = C + D,
\]

(153)

where

\[
C = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v) \right] \, dxdv,
\]

(154)

\[
D = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-1}^\theta(x, v) - f_{k-1}^\theta(x, v) \right] \, dxdv.
\]

(155)

Using \( S_{\gamma}^h \) as a c-optimal map that pushes \( F_{k-1}^h \) forward to \( G_{k-1}^h \), we have

\[
C = \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-1}^\theta(x, \gamma^h) - f_{k-1}^\theta(x, \gamma^h) \right] \, dxdv.
\]

(156)

On the contrary, \( \nabla_x G'(f_{1-1}^\theta(x, \cdot)) \in L^p(\Omega \times \mathbb{R}^d) \) (see Lemma 6), and from assumptions on \( G, G'(s)/s \) is continuous on \( \mathbb{R}^+ \), and then there exists a constant \( K \) such that \( G'(s) \leq K \cdot s \) for \( 0 < N \leq s \leq M \). Since \( 0 < N \leq f_{k-1}^\theta \leq M \) (see Lemma 2) and \( f_{k-1}^\theta \in L^p(\Omega \times \mathbb{R}^d) \), then \( G'(f_{k-1}^\theta) \in L^p(\Omega \times \mathbb{R}^d) \). So \( G'(f_{k-1}^\theta) \in W^{1,p}(\mathbb{R}^d) \).
Approximating $G' \left( f_{k-1+N-i}^{h} (x,.) \right)$ by $C_{c}^{\infty} (\mathbb{R}^{d})$ functions and using the dominated convergence theorem, we have

\[
C = \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^{d}} f_{k-1+N-i}^{h} (x, v) \left( v - s_{k,N}^{h} (x,v), \nabla \psi_{h,k} (x, v + \theta (v - s_{k,N}^{h} (x,v))) \right) dx dv, \quad \text{for } \theta \in ]0, 1[. \tag{156}
\]

Note that $v - s_{k,N}^{h} = -h \nabla c^{*} (\nabla, G' \left( f_{k-1+N-i}^{h} \right) ) \in L^{2} (\Omega \times \mathbb{R}^{d})$ and using the H"{o}lder's inequality, we have

\[
\int_{\Omega \times \mathbb{R}^{d}} |\nabla \psi_{h,k}|^{p} dx dv \leq C_{1} N^{-1} \int_{\Omega \times \mathbb{R}^{d}} |\nabla \psi_{h,k}|^{p} dx dv \leq C_{1} N^{-1} \int_{\Omega \times \mathbb{R}^{d}} |\nabla \psi_{h,k}|^{p} dx dv. \tag{157}
\]

Using the algorithm defined in Section 2.1 and Lemma 6, we have

\[
\int_{\Omega} \rho_{k-1+N-i}^{h} (x) W_{c}^{h} \left( f_{k-1+N-i}^{h} G_{k-1+N-i}^{xh} \right) dx \leq h^{q-1} \left[ \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-2+N-i}^{h} (x,v) \right) dx dv - \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-1+N-i}^{h} (x,v) \right) dx dv \right]. \tag{158}
\]

Thus,

\[
C \leq \sum_{i=1}^{N-1} \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^{d}} \left[ \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-2+N-i}^{h} (x,v) \right) dx dv - \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-1+N-i}^{h} (x,v) \right) dx dv \right]^{1/q} \times
\]

\[h^{(q-1)/q} \left( \int_{\Omega \times \mathbb{R}^{d}} |\nabla \psi_{h,k}|^{p} dx dv \right)^{1/p}. \tag{159}
\]

Applying the H"{o}lder's inequality in the previous relation yields

\[
C \leq \frac{1}{A_{1}} h^{1-(1/p)} h^{(q-1)/q} \left\{ \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-2+N-i}^{h} (x,v) \right) dx dv - \int_{\Omega \times \mathbb{R}^{d}} G \left( f_{k-1+N-i}^{h} (x,v) \right) dx dv \right\}^{1/q} \times
\]

\[\sum_{i=1}^{N-1} \left( \sum_{k=1}^{T/h} h \int_{\Omega \times \mathbb{R}^{d}} |\nabla \psi_{h,k}|^{p} dx dv \right)^{1/p}. \tag{160}
\]

We notice that
According to Lemma 8, we have

\[
\sum_{k=1}^{T/h} \left[ \int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N}^h(x, \nu)) \, d\nu \right]
\]

Next, we use Lemma 6, and we have

\[
D = \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^i \int_{\Omega \times \mathbb{R}^d} \nu \cdot \nabla_x \psi_{h,k}(x, \nu) \, d\nu
\]

We combine (160), (161), and (162) to get

\[
C \lesssim \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left( \frac{h}{(1/p)} \int_{\Omega \times \mathbb{R}^d} |\nabla \psi_{h,k}(x, \nu)| \, d\nu \right) \frac{1}{h^i} \int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N}^h(x, \nu)) \, d\nu
\]

Writing $Nh = \tau$, we then have

\[
C \lesssim \left( \frac{1}{h^i} \int_{\Omega \times \mathbb{R}^d} G(f_{k-2+N}^h(x, \nu)) \, d\nu \right)^{1/p} (M_0(T, \Omega))^{1/p}. \tag{164}
\]

On the contrary,

\[
D = \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^i \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-2+N}^h(x, \nu) - g_{k-1+N}^h(x, \nu) \right] \, d\nu
\]

Since $G'(f_{k-1+N}^h(x, \nu)) \in W^{1,p}(\Omega)$ and then $\psi_{h,k}(x, \nu) \in W^{1,p}(\Omega)$, we approximate $\psi_{h,k}(x, \nu)$ by $C^{\infty}_c(\Omega)$ functions and we get

\[
D = \sum_{k=1}^{T/h} \sum_{i=0}^{N-1} h^i \int_{\Omega \times \mathbb{R}^d} \left[ f_{k-2+N}^h(x, \nu) - f_{k-2+N}^h(x, \nu - h \nu) \right] \, d\nu.
\]

Note that $\nu \cdot \nabla_x G'(f_{k}^h(x, \nu)) = G''(f_{k}^h(x, \nu)) \nu \cdot \nabla_x f_{k}^h(x, \nu)$ and according to Lemma 8, $\nu \cdot \nabla_x f_{k}^h(x, \nu) \in L^p(\Omega \times \mathbb{R}^d)$. So, since $G'$ is continuous and $N \leq f_{k}^h \leq M$, we have

\[
(\nu \cdot \nabla_x G'(f_{k}^h))_{h,k} \in L^p(\Omega \times \mathbb{R}^d).
\]

Consequently,

\[
\left( \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} |\nabla \psi_{h,k}(x, \nu)| \right) \frac{1}{h^i} \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N}^h(x, \nu)) \, d\nu \leq (M_0(T, \Omega))^{1/p}.
\]

We combine (160), (161), and (162) to get

\[
C \lesssim \sum_{k=1}^{T/h} \int_{\Omega \times \mathbb{R}^d} \left( \frac{h}{(1/p)} \int_{\Omega \times \mathbb{R}^d} |\nabla \psi_{h,k}(x, \nu)| \, d\nu \right) \frac{1}{h^i} \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N}^h(x, \nu)) \, d\nu
\]

Writing $Nh = \tau$, we then have

\[
C \lesssim \left( \frac{1}{h^i} \int_{\Omega \times \mathbb{R}^d} G(f_{k-1+N}^h(x, \nu)) \, d\nu \right)^{1/p} (M_0(T, \Omega))^{1/p}. \tag{164}
\]
\[
\int_{[0,T] \times \mathbb{R}^d} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv
\]
\[\leq C(A_1, \Omega, T, N, M, f_0, G', q, p) \tau,\]

where \(C(A_1, \Omega, T, N, M, f_0, G', q, p)\) is positive constant.

**Proof.** Since \(G \in C^2\), we use Taylor's formula to have that
\[
G'(f^h(t + \tau, x, v) - f^h(t, x, v)) = G'(f^h(t, x, v)) + \int_0^\tau \frac{d}{d\tau} G'(f^h(t + \theta \tau, x, v)) \, d\theta,
\]
where \(\theta \in [0, 1]\) and \(f^h_0 = \theta f^h(t) + (1 - \theta)f^h(t + \tau)\).

Since \(0 < N \leq f_0 \leq M\), we have \(0 < N \leq f^h \leq M\) (see Lemma 2). Thus, using (171) and the fact that \(G^r\) is continuous and nonnegative, we have
\[
\int_{[0,T] \times \mathbb{R}^d} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv \\
\geq \inf_{\tau \in [N,M]} G'(t) \int_{[0,T] \times \mathbb{R}^d} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv.
\]

(172)

Define \(\beta = \inf_{\tau \in [N,M]} G'(t) > 0\). We use Lemma 9 and obtain
\[
\int_{[0,T] \times \mathbb{R}^d} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv
\]
\[\leq \frac{1}{\beta^p} C(A_1, \Omega, T, N, M, f_0, G', q, p) \tau.\]

Using \(N \leq f^h \leq M\), we have
\[
\int_{[0,T] \times \mathbb{R}^d} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv
\]
\[\leq \frac{1}{\beta} C(A_1, \Omega, T, N, M, f_0, G', q, p) \tau,
\]

(174)

where \(C(A_1, \Omega, T, N, M, f_0, G', q, p)\) is positive constant.

We will use Lemmas 7, 8, and 10, to prove that the sequence \((f^h_n)\) is precompact in \(L^p([0,T] \times \Omega \times \mathbb{R}^d)\).

**Lemma 11.** Assume that \(f_0, c, \) and \(G\) satisfy, respectively, \((H_f)\), \((H_c)\), and \((H_G)\). Then, \((f^h_n)\) is precompact in \(L^p([0,T] \times \Omega \times \mathbb{R}^d)\), where \(B_R\) is the open ball centered at origin of radius \(R\).

**Proof.** Define \(K = [0,T] \times \Omega \times B_R\). Then, \(K \subset \cup_{i=1}^N K_i^{\delta_i}\), with \(K_i^{\delta_i} = \{ y \in K, \text{dist}(y, K_i^{\delta_i}) \geq \delta_i \}\). \(K_i^{\delta_i} \subset K_i^{\delta_i} \subset K\), and \(K_i^{\delta_i}\) is compact and \(\text{dist}(K_i^{\delta_i}, K_i^{\delta_i}) \geq \delta_i/2\). So, there exists \(\delta_1, \delta_2, \ldots, \delta_i \geq 0\) such that \(K_i^{\delta_i} \subset \cup_{i=1}^N K_i^{\delta_i}\).

Define \(\delta_0 = \min_{i=1,2,\ldots} \delta_i\) and we have \(K_0 = K\) and \(\text{dist}(K_0, K_0) \geq \delta_0/2\).

Let \(\tau > 0, \eta_1, \eta_2 \in \mathbb{R}^d\), such that \(0 < \tau < \delta_0\), \(|\eta_1| < \delta_0\), and \(|\eta_2| < \delta_0/2\), where \(\delta_0 = \min\{\delta_0/2, \epsilon\}\).

We use Lemmas 7, 8, and 10 to have
\[
\int_{K_0} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv < c_1 \epsilon,
\]
\[
\int_{K_0} \left| f^h(t + \tau, x, v) - f^h(t, x, v) \right|^p \, dt \, dx \, dv < c_2 \epsilon^2,
\]
\[
\int_{K_0} \left| f^h(t, x, v + \eta_2) - f^h(t, x, v) \right|^p \, dt \, dx \, dv < c_3 \epsilon^3.
\]

(175)

(176)

(177)

(178)

\(c_1, c_2\), and \(c_3\) are defined in Lemmas 7, 8, and 10. So, by using the Riesz–Fréchet–Kolmogorov's theorem, we deduce that the sequence \((f^h_n)\) is precompact in \(L^p(K_i^{\delta_i})\). Therefore, \(\lim_{n \to \infty} \| K_i^{\delta_i} \| = 0\), and then for all \(\epsilon > 0\), there exists \(\delta_i > 0\) such that \(\| K_i^{\delta_i} \| < \epsilon\| f_0 \|_{L^\infty} \), and since \((f^h_n)\) is bounded in \(L^\infty([0,T] \times \Omega \times B_R)\), we have \(\| f^h_n \|_{L^p(K_i^{\delta_i})} \leq \| f_0 \|_{L^\infty} \| K_i^{\delta_i} \| < \epsilon\).

We use corollary IV.26 of [9], to conclude that the sequence \((f^h_n)\) is precompact in \(L^p([0,T] \times \Omega \times B_R)\), which implies that \((f^h_n)\) converges strongly to some function \(f_n\) in \(L^p([0,T] \times \Omega \times B_R)\).

Now, let us prove that the sequence \((f^h_n)\) is precompact in \(L^p([0,T] \times \Omega \times \mathbb{R}^d)\).

**Lemma 12.** Assume that \(f_0, c, \) and \(G\) satisfy, respectively, \((H_f)\), \((H_c)\), and \((H_G)\). Then, the sequence \((f^h_n)\) is precompact in \(L^p([0,T] \times \Omega \times \mathbb{R}^d)\), up to a subsequence.

**Proof.** Since \(f_0 \in L^0([0,T] \times \Omega \times \mathbb{R}^d)\), Lemma 3 implies that \((f^h_n)\) is bounded in \(L^0([0,T] \times \Omega \times \mathbb{R}^d)\). Then, \((f^h_n)\) converges weakly to some function \(f\) in \(L^0([0,T] \times \Omega \times \mathbb{R}^d)\) (up to a subsequence).

Let \(B_R\) be an open ball of radius \(r\) centered at origin, with \(n \in \mathbb{N}^*\). Then, \(B_{2n} \subset B_{2n+1} \cup \cup_{n \in \mathbb{N}^*} B_n = \mathbb{R}^d\).

Using previous results, the sequence \((f^h_n)\) is precompact in \(L^0([0,T] \times \Omega \times B_R)\). A subsequence \((f^h_n)\) converges strongly to some function \(f_n\) in \(L^p([0,T] \times \Omega \times B_R)\). Let us denote by \((f^h_n)\) the subsequence of \((f^h_n)\) which converges strongly to \(f_n\) in \(L^p([0,T] \times \Omega \times B_R)\). Fix \(\phi \in L^q([0,T] \times \Omega \times \mathbb{R}^d)\). We have
\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^h_n - f_n) \phi \, dt \, dx \, dv = \int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^h_n - f_n) \phi \, dt \, dx \, dv
\]
+ \[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^h_n - f_n) \phi \, dt \, dx \, dv\]
\[
\leq \| \phi \|_{L^q([0,T] \times \Omega \times \mathbb{R}^d)} \cdot 2M \| B_n \|^{1/p}.\]

(176)

(177)

(178)

\(\| B_n \|_{L^1}\) tends to zero, when \(n\) goes to \(\infty\); for \(\epsilon > 0\), there exists \(N_\epsilon > 0\) such that
\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} (f^h_n - f_n) \phi \, dt \, dx \, dv \leq \frac{\epsilon}{2},
\]

for all \(n \geq N_\epsilon\). Since \((f^h_n)\) converges strongly to \(f_n\) in \(L^p([0,T] \times \Omega \times \mathbb{R}^d)\), we deduce that
when $h_n$ goes to $0$ and $n \geq N_e$. We conclude that $(f_{h_n})$ converges weakly to $f_{w, n}$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, for $n \geq N_e$. So $f_{w, n} = f$ for $n \geq N_e$.

Now, let prove that $\|f_{h, n} - f\|_{L^p([0, T] \times \Omega \times \mathbb{R}^d)}$ tends to $0$ as $h_n \to 0$ and $n \geq N_e$.

Note that

$$\|f_{h, n} - f\|_{L^p([0, T] \times \Omega \times \mathbb{R}^d)}^p = \int_{[0, T] \times \Omega \times \mathbb{R}^d} |f_{h, n} - f|^p \, dx \, dt \, dv$$

$$\leq \int_{[0, T] \times \Omega \times \mathbb{R}^d} |f_{h, n} - f|^p \, dx \, dt \, dv + 2p |f_{0}|_p^p \|\Omega\| |B_{e_n}|.$$  

(180)

Let us recall that

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times B_{e_n}} |f_{h, n} - f|^p \, dx \, dt \, dv = 0, \quad \forall n \geq N_e.$$

(181)

Since $|B_{e_n}| < (e/2)$, for $n \geq N_e$, as

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times B_{e_n}} |f_{h, n} - f|^p \, dx \, dt \, dv = 0,$$

(182)

we can find $n_e \in \mathbb{N}$ such that for every $n \geq N_e$, we have

$$\int_{[0, T] \times \Omega \times B_{e_n}} |f_{h, n} - f|^p \, dx \, dt \, dv < \frac{\varepsilon}{2}.$$  

(183)

Then, for all $n \geq N_e$,

$$\int_{[0, T] \times \Omega \times \mathbb{R}^d} |f_{h, n} - f|^p \, dx \, dt \, dv < \varepsilon.$$  

(184)

Consequently, $(f_{h_n})$ converges strongly to $f$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

Using the previous lemma, $(f_{h_n})$ converges strongly to $f$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$ (up to a subsequence). $G'$ is being continuous, then $(G'(f_{h_n}))$ converges strongly to $G'(f)$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$. We conclude that the sequence $(\nabla \cdot (G'(f_{h_n})))$ converges to $(\nabla \cdot (G'(f)))$ in $[C_c^\infty(\mathbb{R} \times \mathbb{Omega} \times \mathbb{R}^d)]'$. Let us recall that $(\nabla \cdot (G'(f_{h_n})))$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, so $(\nabla \cdot (G'(f_{h_n})))$ converges weakly to $(\nabla \cdot (G'(f)))$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

On the contrary, $(\nabla \cdot (\nabla \cdot (G'(f_{h_n}))))$ is bounded in $L^1([0, T] \times \Omega \times \mathbb{R}^d)$, so the sequence $(\nabla \cdot (\nabla \cdot (G'(f_{h_n}))))$ converges weakly to $(\nabla \cdot (\nabla \cdot (G'(f))))$ in $[C_c^\infty(\mathbb{R} \times \mathbb{Omega} \times \mathbb{R}^d)]'$.

We derive from the proof of Theorem 3.10 in [1] that the sequence $\nabla \cdot (\nabla \cdot (G'(f_{h_n})))$ converges weakly to $(\nabla \cdot (\nabla \cdot (G'(f))))$ in $[C_c^\infty(\mathbb{R} \times \mathbb{Omega} \times \mathbb{R}^d)]'$.

Theorem 2. Assume that $f_{w, n}$, $c$, and $G$ satisfy, respectively, $(\mathcal{H}_f)$, $(\mathcal{H}_c)$, and $(\mathcal{H}_G)$. Let $0 < T < \infty$ and $u \in C_c^\infty(\mathbb{R})$ be such that $u \geq 0$ and $\text{supp} u \subset [-T, T]$. Then,

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f_{h, n} \nabla \cdot (\nabla \cdot (G'(f_{h, n}))), \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv = \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv.$$  

(185)

Moreover, $(\nabla \cdot (\nabla \cdot (G'(f))))$ converges weakly to $(\nabla \cdot (\nabla \cdot (G'(f))))$ in $[C_c^\infty(\mathbb{R} \times \mathbb{Omega} \times \mathbb{R}^d)]'$ and $\nabla \cdot (\nabla \cdot (G'(f)))$ converges weakly to $(\nabla \cdot (\nabla \cdot (G'(f))))$ in the weak sense.

Proof. Let us recall the following:

(i) $(f_{h})$ converges strongly to $f$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

(ii) $(\nabla \cdot (G'(f)))$ converges weakly to $(\nabla \cdot (G'(f)))$ in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$.

(iii) $(\sigma \nabla \cdot (\nabla \cdot (G'(f))))$ converges weakly to $\sigma$ in $L^1([0, T] \times \Omega \times \mathbb{R}^d)$.

Then, we have

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f_{h, n} \nabla \cdot (\nabla \cdot (G'(f))), \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv = \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv.$$  

(186)

Since $\nabla \cdot (\nabla \cdot (G'(f))) \in L^p([0, T] \times \Omega \times \mathbb{R}^d)$ and $(\nabla \cdot (\nabla \cdot (G'(f))))$ is bounded in $L^1([0, T] \times \Omega \times \mathbb{R}^d)$, we use (i) and (iii) to have

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f_{h, n} \sigma, \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv = \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f \sigma, \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv.$$  

(187)

Also, we have

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f_{h, n} \nabla \cdot (\nabla \cdot (G'(f))), \nabla \cdot (\nabla \cdot (G'(f))) - \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv = \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f \sigma \nabla \cdot (\nabla \cdot (G'(f))), \nabla \cdot (\nabla \cdot (G'(f)) - \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv.$$  

(188)

Using (i), (ii), and the fact that $\nabla \cdot (\nabla \cdot (G'(f)))$ is in $L^1([0, T] \times \Omega \times \mathbb{R}^d)$ and that $(\nabla \cdot (\nabla \cdot (G'(f))))$ is bounded in $L^p([0, T] \times \Omega \times \mathbb{R}^d)$, we get

$$\lim_{h_n \to 0} \int_{[0, T] \times \Omega \times \mathbb{R}^d} \langle f_{h, n} \nabla \cdot (\nabla \cdot (G'(f))), \nabla \cdot (\nabla \cdot (G'(f)) - \nabla \cdot (\nabla \cdot (G'(f))) \rangle u(t) \, dt \, dx \, dv = 0.$$  

(189)

To conclude with Theorem 2, we need to establish the following limit, whose proof is derived from the three following lemmas:
\[
\lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f^h)), \nabla, G^* (f^h)\right) u(t) \, dt \, dx \, dv
\]
\[
= \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^0 \sigma, \nabla, G^* (f)\right) u(t) \, dt \, dx \, dv.
\]
\] (190) \]

Lemma 13

\[
\lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f^h)), \nabla, G^* (f^h)\right) u(t) \, dt \, dx \, dv
\]
\[
\geq \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^0 \sigma, \nabla, G^* (f)\right) u(t) \, dt \, dx \, dv.
\]
\] (191)

Proof. Since \( c^* \) is convex and \( f^h \geq 0, u \geq 0 \), then
\[
f^h \left(\nabla c^* (\nabla, G^* (f^h)) - \nabla c^* (\nabla, G^* (f)), \nabla, G^* (f^h) - \nabla, G^* (f)\right) \geq 0.
\]
\] (192)

So,
\[
\int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f^h)), \nabla, G^* (f^h)\right) u(t) \, dt \, dx \, dv
\]
\[
\geq \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^0 \sigma, \nabla, G^* (f)\right) u(t) \, dt \, dx \, dv
\]
\[
+ \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f)), \nabla, G^* (f^h) - \nabla, G^* (f)\right) \, dt \, dx \, dv.
\]
\] (193)

Passing to the limit, we obtain
\[
\lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f^h)), \nabla, G^* (f^h)\right) u(t) \, dt \, dx \, dv
\]
\[
\geq \lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^0 \sigma, \nabla, G^* (f)\right) u(t) \, dt \, dx \, dv
\]
\[
+ \lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f)), \nabla, G^* (f^h) - \nabla, G^* (f)\right) \, dt \, dx \, dv.
\]
\] (194)

Taking into account (187) and (189) in (194) enables us to conclude the proof of Lemma 13.
\] \qed

Lemma 14

\[
\lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} \left(f^h \nabla c^* (\nabla, G^* (f^h)), \nabla, G^* (f^h)\right) u(t) \, dt \, dx \, dv
\]
\[
\leq \int_{\mathbb{R}^d} \left| f^0 G^* (f^0) - G^* (f^0) \right| u(0) \, dx \, dv
\]
\[
+ \int_{\{0, \infty\} \times \mathbb{R}^d} \left| f^h G^* (f) - G^* (f^h) \right| \, dt \, dx \, dv.
\]
\] (195)

Proof. Using the proof of Lemma 6, we have
\[
\int_{\{0, \infty\} \times \mathbb{R}^d} G(f^h k) \, dx \, dv - \int_{\{0, \infty\} \times \mathbb{R}^d} G(f^h k) \, dx \, dv
\]
\[
\geq \int_{\{0, \infty\} \times \mathbb{R}^d} \left(T_k(v) - \nabla, G^* (f^h k)\right) f^h k \, dx \, dv,
\]
where
\[
T_k(v) = v + h \nabla c^* (\nabla, G^* (f^h k)),
\]
\] (197)

is a c-optimal map that pushes \( F^h k \) forward \( G^h k \). Thus,
\[
\int_{\{0, \infty\} \times \mathbb{R}^d} \frac{G(f^h k)}{h} - \frac{G(f^h k)}{h} \, dx \, dv
\]
\[
\leq - \int_{\{0, \infty\} \times \mathbb{R}^d} \left(\nabla c^* (\nabla, G^* (f^h k)), \nabla, G^* (f^h k)\right) f^h k \, dx \, dv.
\]
\] (198)

Since \( u \geq 0 \), then
\[
\sum_{k=1}^{T/h} \int_{0}^{T/h} \int_{\{0, \infty\} \times \mathbb{R}^d} \frac{G(f^h k)}{h} - \frac{G(f^h k)}{h} \, dx \, dv
\]
\[
\leq - \int_{\{0, \infty\} \times \mathbb{R}^d} \left(\nabla c^* (\nabla, G^* (f^h k)), \nabla, G^* (f^h k)\right) f^h k \, dx \, dv.
\]
\] (199)

Also,
\[
\sum_{k=1}^{T/h} \int_{0}^{T/h} \int_{\{0, \infty\} \times \mathbb{R}^d} \frac{G(f^h k) - G(f^h k)}{h} \, dx \, dv
\]
\[
= \int_{\{0, \infty\} \times \mathbb{R}^d} G(f^h 0) u(t) \, dx \, dv
\]
\[
- \frac{1}{h} \int_{0}^{T/h} \int_{\{0, \infty\} \times \mathbb{R}^d} G(f^0) u(t) \, dx \, dv,
\]
where \( \partial u(t) = (u(t + h) - u(t))/h \). Let us prove that
\[
\lim_{h \to 0} \int_{\{0, \infty\} \times \mathbb{R}^d} G(f^h 0) \partial^h u(t) \, dx \, dv
\]
\[
= \int_{\{0, \infty\} \times \mathbb{R}^d} G(f) u(t) \, dx \, dv.
\]
\] (201)

Indeed,
\[
\left| \int_{\{0, \infty\} \times \mathbb{R}^d} G(f^h 0) \partial^h u(t) - G(f) u(t) \, dx \, dv \right|
\]
\[
\leq \int_{\{0, \infty\} \times \mathbb{R}^d} \left| G(f^h 0) - G(f) \right| u(t) \, dx \, dv
\]
\[
+ \int_{\{0, \infty\} \times \mathbb{R}^d} \left| \partial^h u(t) - u(t) \right| \left| G(f^h 0) \right| \, dx \, dv.
\]
\] (202)

Using (i), the continuity of \( G \), the boundedness of \( f^h k \) in \( L^\infty([0, \infty) \times \{0, \infty\} \times \mathbb{R}^d) \), and the fact that \( \partial^h u(t) \) converges uniformly to \( u \) on the compact \([0, T]\), we obtain...
\[
\lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f^h) \cdot \partial_t u(t) \varphi(t) \, dt \, dx \, dv = \int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f) \cdot \partial_t u(t) \varphi(t) \, dt \, dx \, dv.
\]

(203)

Next, we combine (199), (200), and (203) to get
\[
\int_{[0,T] \times \Omega \times \mathbb{R}^d} G(f) u(t) \varphi(t) \, dt \, dx \, dv + \int_{\Omega \times \mathbb{R}^d} G(f_0) u(0) \varphi(t) \, dt \, dx \, dv \geq \lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \left( \nabla c \cdot \nabla G \left( f^h \right) \right) \, dt \, dx \, dv.
\]

(204)

Since \( G \in C^1 \) is strictly convex, \( G^* \left( G^\ast (s) \right) = s G \left( s \right) - G(s) \), \( \forall s, \psi > 0 \). Consequently, replacing \( G(f) = f G^\ast (f) - G^* \left( G^\ast (f) \right) \) and \( G(f_0) = f_0 G^\ast (f_0) - G^* \left( G^\ast (f_0) \right) \) in (5), we obtain Lemma 14.

**Lemma 15**

\[
\int_{\Omega \times \mathbb{R}^d} \left[ f_0 G^\ast (f_0) - G^\ast \left( G^\ast (f_0) \right) \right] \varphi(t) \, dt \, dx \, dv + \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left[ f G(f) - G^\ast \left( G^\ast (f) \right) \right] u(t) \varphi(t) \, dt \, dx \, dv \leq \lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left\{ f^h \left( \nabla c \cdot \nabla G \left( f^h \right) \right) \varphi(t, x, v) \right\} \, dt \, dx \, dv.
\]

(205)

**Proof.** Let us write \( \psi(t, x, v) = G^\ast \left( f \right) u(t) \) for \( (t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^d \). We have \( \psi(t, x, v) \in W^{1, p} \left( \Omega \times \mathbb{R}^d \right) \).

Approximating \( \psi(t, x, v) \) by \( C_0^\infty \left( \Omega \times \mathbb{R}^d \right) \) functions and using (17), we have
\[
A_k^h \left[ \psi \right] = \int_{\Omega \times \mathbb{R}^d} f^h - g^h \varphi(t, x, v) \, dt \, dx \, dv + \int_{\Omega \times \mathbb{R}^d} \int_{0}^{T} f^h \left( \nabla c \cdot \nabla G \left( f^h \right) \right) \varphi(t, x, v) \, dt \, dx \, dv.
\]

(206)

where \( A_k^h \left[ \psi \right] \) is defined in (17) and tends to 0 as \( h \to 0 \). By integrating (206) over \( [0, T] \), one obtains
\[
\forall i = 1, \ldots, T \int_{t_i}^{t_{i+1}} \left\{ f^h \left( \nabla c \cdot \nabla G \left( f^h \right) \right) \varphi(t, x, v) \right\} \, dt \, dx \, dv + \frac{1}{h} \left[ \psi \right] \left[ \Omega \times \mathbb{R}^d \right]
\]

(207)

Next, from the definition of \( f^h \), we derive that
\[
\sum_{i=1}^{T} \int_{t_i \Delta t_{i-1}} f^h \left( \psi \right) \left[ \Omega \times \mathbb{R}^d \right] \leq \frac{h}{2} \left\| \nabla \psi \right\|^{2}_{L^\infty} + \frac{h}{2} \left\| \nabla \psi \right\|^{2}_{L^2} \left( \Omega \times \mathbb{R}^d \right) \left( \Omega \times \mathbb{R}^d \right)
\]

(212)

Recalling the fact that \( c(x) \geq A_1 |x|^{\alpha} \), we have
\[
\int_{\Omega \times \mathbb{R}^d} \left\| \nabla \left( G^\ast (f^h) \right) \right\|^{2} f^h \, dx \, dv \geq \frac{h}{A_1} \int_{\Omega \times \mathbb{R}^d} c(x) \left( F_k^h \right) dx.
\]

(213)
From the statements in Section 2.1, we get

\[
\int_\Omega \tilde{b}^h(x)W_\delta^h(F^h_{\delta}, G^h_{\delta})dx \leq \frac{1}{A_1} \left[ \int_{\Omega \times R^d} G(f^h_{\delta-1}) - \int_{\Omega \times R^d} G(f^h_{\delta}) \right] dx dv,
\]

and accordingly,

\[
|A^h_k[\psi]| \leq \frac{h}{2A_1} \|v^2\|_\infty \sum_{i=1}^{T_h} \left[ \int_{\Omega \times R^d} G(f^h_{\delta-1}) - \int_{\Omega \times R^d} G(f^h_{\delta}) + hT \right] \|f_0\|_{L^\infty(\Omega \times R^d)} \int_{\Omega \times K_{\eta}} |v|^2 dx dv.
\]

(215)

So

\[
|B^h[\psi]| \leq \frac{1}{2A_1} \|v^2\|_\infty \sum_{i=1}^{T_h} \left[ \int_{\Omega \times R^d} G(f^h_{\delta-1}) - \int_{\Omega \times R^d} G(f^h_{\delta}) + hT \right] \|v\|_{L^\infty(\Omega \times R^d)} \int_{\Omega \times K_{\eta}} |v|^2 dx dv,
\]

and then,

\[
|B^h[\psi]| \leq \frac{1}{2} \|v^2\|_\infty \sum_{i=1}^{T_h} \left[ \int_{\Omega \times R^d} G(f_0) - \int_{\Omega \times R^d} G(f^h_{\delta}) + hT \right] \|f_0\|_{L^\infty(\Omega \times R^d)} \int_{\Omega \times K_{\eta}} |v|^2 dx dv.
\]

(216)

From the proof of Lemma 6, we have

\[
\int_{\Omega \times R^d} G(f_0) dx dv - \int_{\Omega \times R^d} G(f^h_{T/h}) dx dv \leq G'(M) - G'(N).
\]

(217)

Then, we obtain

\[
|A^h_k[\psi]| \leq \frac{1}{2} \|v^2\|_\infty \sum_{i=1}^{T_h} \left[ \int_{\Omega \times R^d} G(f^h_{\delta-1}) - \int_{\Omega \times R^d} G(f^h_{\delta}) + hT \right] \|f_0\|_{L^\infty(\Omega \times R^d)} \int_{\Omega \times K_{\eta}} |v|^2 dx dv.
\]

(218)

Thus, \( B^h[\psi] \) tends to 0 as \( h \to 0 \).

(219)

By the change of variable \( t = t + h \), we have

\[
\int_{[0,T] \times \Omega \times B_\delta} \frac{G'(f(t+h))u(t+h) - G'(f(t))u(t)}{h} \] dr dx dv = \( X_1^h + X_2^h + X_3^h \),

(224)
\[ X_1^h = \int_{[0,T] \times \Omega \times B_h} f^h(t-h) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv, \]

\[ X_2^h = \int_{[-h,0] \times \Omega \times B_h} f^h(t-h) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv, \]

\[ X_3^h = -\int_{[T-h,T] \times \Omega \times B_h} f^h(t-h) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv. \]  

Recalling the definition of \( f^h, \) we get

\[ X_2^h = \int_{[-h,0] \times \Omega \times B_h} f_0 G'(f_0) \left( \frac{u(t) - u(t-h)}{h} \right) \, dr \, dv. \]  

Thus,

\[ \lim_{h \to 0} X_2^h = 0. \]  

Besides,

\[ |X_3^h| \leq M |G'(M)| \left\{ \frac{1}{h} \int_{[0,h] \times \Omega \times B_h} u(t + T - h) \, dr \, dv \right\} \]

and since \( \sup p(u) \subset [-T,T], \)

\[ \lim_{h \to 0} |X_3^h| \leq M |G'(M)| [u(T) + u(T)] = 0. \]  

We also have

\[ X_1^h = \int_{[0,T] \times \Omega \times B_h} \left( f^h(t-h) - f \right) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv \]

\[ + \int_{[0,T] \times \Omega \times B_h} f \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv. \]  

But

\[ \left| \int_{[0,T] \times \Omega \times B_h} \left( f^h(t-h) - f \right) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv \right| \leq \]

\[ \| f^h(t-h) - f \|_{L^p} \int_{[0,T] \times \Omega \times B_h} \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \, dr \, dv, \]

\[ \lim_{h \to 0} \| f^h(t-h) - f \|_{L^p} = 0, \]  

and then by the dominated convergence theorem, we have

\[ \lim_{h \to 0} \int_{[0,T] \times \Omega \times B_h} \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \, dr \, dv \]

\[ = \int_{[0,T] \times \Omega \times B_h} \left| \partial_j(G'(f(t))u(t)) \right|^p < \infty, \]  

(233)

Next, the equality

\[ \lim_{h \to 0} \int_{[0,T] \times \Omega \times B_h} \left( f^h(t-h) - f \right) \left( \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right) \, dr \, dv = 0. \]  

(234)
\[
\lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} \frac{f^h}{h} \left[ \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] \, dx \, dv
\]

Follows by recalling (229), (227), and (234), and accordingly,

\[
\lim_{h \to 0} \int_0^T \int_{\Omega \times \mathbb{R}^d} f \left[ \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right] \, dx \, dv
\]

\[
\geq \int_0^T \int_{\Omega \times \mathbb{R}^d} \left\{ f \sigma, \nabla_G \right\} u(t) \, dx \, dv.
\]

We notice that

\[
f \left[ \frac{G'(f(t))u(t) - G'(f(t-h))u(t-h)}{h} \right]
\]

\[
= fG'(f) \tilde{\partial}_t^h u(t) + \frac{1}{h} f u(t-h) \left[ G'(f) - G'(f(t-h)) \right],
\]

where \( \tilde{\partial}_t^h u(t) = u(t) - u(t-h)/h \), and since \( G \in C^1 \) is convex, then

\[
G'(G(f)) - G'(G'(f(t-h))) \leq f(G'(f) - G'(f(t-h))).
\]

Accordingly,

\[
f \left[ \frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right]
\]

\[
\geq fG'(f) \tilde{\partial}_t^h u(t) + \frac{1}{h} f u(t-h)
\]

\[
\left\{ G'(G(f)) - G'(G'(f(t-h))) \right\} ,
\]

and after integration, we get

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} f \left[ \frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] \, dx \, dv
\]

\[
\geq \int_0^T \int_{\Omega \times \mathbb{R}^d} fG'(f) \tilde{\partial}_t^h u(t) \, dx \, dv
\]

\[
+ \frac{1}{h} \int_0^T \int_{\Omega \times \mathbb{R}^d} u(t-h) \left\{ G'(G'(f)) - G'(G'(f(t-h))) \right\} \, dx \, dv.
\]

Since \( \text{supp}(u) \subset [-T,T], u(t) = 0 \) for \( t-h \in [T-h,T] \), and then, we obtain

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} f \left[ \frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] \, dx \, dv \geq
\]

\[
\int_0^T \int_{\Omega \times \mathbb{R}^d} \left[ fG'(f) - G'(G'(f(t-h))) \right] \tilde{\partial}_t^h u(t) \, dx \, dv
\]

\[
- \frac{1}{h} \int_0^T \int_{\Omega \times \mathbb{R}^d} G'(f(t-h))u(t-h) \, dx \, dv.
\]

Tending \( h \) to 0 in the relation above enables us to have

\[
\lim_{h \to 0} \int_0^T \int_{\Omega \times \mathbb{R}^d} f \left[ \frac{G'(f)u(t) - G'(f(t-h))u(t-h)}{h} \right] \, dx \, dv
\]

\[
\geq \int_0^T \int_{\Omega \times \mathbb{R}^d} \left\{ fG'(f) - G'(G'(f)) \right\} u'(t) \, dx \, dv
\]

\[
- \int_{\Omega \times \mathbb{R}^d} G'(G'(f(t)))u(0) \, dx \, dv.
\]

Moreover, expressions (236) and (242) give

\[
\int_{\Omega \times \mathbb{R}^d} \left\{ fG'(f) - G'(G'(f)) \right\} u(0) \, dx \, dv
\]

\[
+ \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left\{ fG'(f) - G'(G'(f)) \right\} u'(t) \, dx \, dv
\]

\[
\leq \int_{[0,T] \times \Omega \times \mathbb{R}^d} \left\{ fG'(f) - G'(G'(f)) \right\} u(0) \, dx \, dv.
\]

So, tending \( R \) to \( \infty \) in (243), we obtain Lemma 15. Now, we derive (185) by combining Lemmas 13–15. So, let us show that the sequence (\( \text{div}_v(f^h \sigma) \)) converges weakly to \( \text{div}_v(f \sigma) \) in \( C_{0}^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d) \) and \( \text{div}_v(f \sigma, \psi), \text{div}_v(f \nabla_G \psi) \) in the weak sense. So, fix \( T > 0 \) and let \( \psi \in C_{0}^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d) \) such that \( \text{supp}(\psi, x, v) \subset [-T,T] \).

We have

\[
\lim_{h \to 0} \int_0^T \int_{\Omega \times \mathbb{R}^d} \left\{ f \nabla_G \left( \nabla_G \psi \right) \right\} - f \sigma \psi \, dx \, dv = 0.
\]
Let us show that $\text{div}_v(f\sigma) = \text{div}_v(f\nabla c^*(\nabla G'(f^h)))$ in the weak sense. So, let $\varepsilon > 0$ and $\psi \in C_c^\infty(\Omega \times \mathbb{R}^d)$ be a test function and $u \in C^0_c(\mathbb{R})$, $u \geq 0$, with $\text{supp}(u) \subset [-T, T]$. Define $\omega_j(t, x, v) = G'(f - \varepsilon \psi).

Since $c^*$ is convex and $f^h \geq 0$, we have

$$
\begin{align*}
&\int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla G'(f^h)) - \nabla c^*(\nabla \omega_j), \nabla G'(f^h) - \nabla \omega_j \rangle u(t) \\
&= \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla G'(f^h)) - \nabla c^*(\nabla \omega_j), \nabla G'(f^h) - \nabla \omega_j \rangle u(t) \\
&+ \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla G'(f^h)) - \nabla c^*(\nabla \omega_j), \nabla G'(f) - \nabla \omega_j \rangle u(t),
\end{align*}
$$

we derive from (185) that

$$
\begin{align*}
\lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla G'(f^h)) - \nabla c^*(\nabla \omega_j), \nabla G'(f^h) - \nabla \omega_j \rangle u(t) = 0,
\end{align*}
$$

Consequently,

$$
\begin{align*}
\lim_{h \to 0} \int_{[0,T] \times \Omega \times \mathbb{R}^d} f^h \langle \nabla c^*(\nabla G'(f^h)) - \nabla c^*(\nabla \omega_j), \nabla G'(f) - \nabla \omega_j \rangle u(t) \\
= \int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma - f \nabla c^*(\nabla G'(f)) , \nabla G'(f) - \nabla \omega_j \rangle u(t) \geq 0.
\end{align*}
$$

We divide (249) by $\varepsilon$, and we tend $\varepsilon$ to 0 and have

$$
\begin{align*}
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma - f \nabla c^*(\nabla G'(f)) , \nabla \omega_j \rangle u(t) \leq 0,
\end{align*}
$$

Replacing $\psi$ by $-\psi$ in the previous relationship, we obtain

$$
\begin{align*}
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle f \sigma - f \nabla c^*(\nabla G'(f)) , \nabla \omega_j \rangle u(t) = 0.
\end{align*}
$$

We conclude that $(\text{div}_v(f^h\nabla c^*(\nabla G'(f^h))))$ converges weakly to $\text{div}_v(f\nabla c^*(\nabla G'(f)))$ in $[C_c^\infty(\Omega \times \mathbb{R}^d)]'$, and then, the proof of Theorem 2 is complete. □

### 4.5. Existence of Solutions

Now, we combine the weak convergence of the sequence $(f^h)_h$ and that of the nonlinear term $\text{div}_v(f^h\nabla c^*(\nabla G'(f^h)))$ to prove that the kinetic equation (1) admits a weak solution $f$ in the sense of Definition 1.

**Theorem 3.** Let $f_0$ be a density of probability on $\Omega \times \mathbb{R}^d$ such that

$$
\int_{\Omega \times \mathbb{R}^d} |\nabla| f_0 \, dx \, dv < \infty,
$$

and $\int_{\Omega \times \mathbb{R}^d} G(f_0) \, dx \, dv < \infty$, with $G$ and $c$ satisfying, respectively, $H_G$ and $H_c$.

Then, for any test function $\psi \in C_c^\infty(\mathbb{R} \times \Omega \times \mathbb{R}^d)$ such that $\text{supp}(\psi, x, v) \subset [-T, T]$, we have

$$
\begin{align*}
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle \nabla \omega_j \rangle \langle \nabla c^*(\nabla G'(f)) - \nabla c^*(\nabla \omega_j), \nabla G'(f) - \nabla \omega_j \rangle u(t) \geq 0.
\end{align*}
$$

Splitting (246) as follows,

$$
\begin{align*}
\int_{[0,T] \times \Omega \times \mathbb{R}^d} \langle \nabla \omega_j \rangle \langle \nabla c^*(\nabla G'(f)) - \nabla c^*(\nabla \omega_j), \nabla G'(f) - \nabla \omega_j \rangle u(t)
\end{align*}
$$

where $B^h[\psi]$ tends to 0 as $h \to 0$.

We observe that
We combine (254), (255), and (256) to obtain
\[ T/h \sum_{k=1}^{T/h} f_k \cdot \nabla \psi(x + (t - t_{k-1})v, v) dt dv \]
\[ = \int_{[0,T] \times \Omega \times \mathbb{R}^d} f_0(x - tv, v) \psi(t - h, x, v) dt dv \]

Hence,
\[ \sum_{k=1}^{T/h} f_k \cdot \nabla \psi(x + (t - t_{k-1})v, v) dt dv \]
\[ = \int_{[0,T] \times \Omega \times \mathbb{R}^d} f_0(x, v) \psi(t, x, v) dt dv \]

Finally, we use the weak convergence of \((f^h)_h\) to \(f\), the weak convergence of \(f^h \nabla \psi^* (\nabla G^* (f^h))\) to \(f \nabla \psi^* (\nabla G^* (f))\), and the uniform convergence of \(\partial_t^h \psi, \psi, v)\) to \(\partial_t \psi, \psi, v)\) on the compact set \([0, T]\) when \(h\) tends to 0, to derive the following equality:
\[ \int_{[0,T] \times \Omega \times \mathbb{R}^d} f_0 \psi(0) dx dv \]
\[ = -\int_{[0,T] \times \Omega \times \mathbb{R}^d} \nabla \psi \cdot \nabla \psi^* (\nabla G^* (f)) \]
where \( i = 1, 3, 5, \ldots; 2j + 1 \), with \( j \in \mathbb{N}^+ \) and \( u^i(t_k, x, v) = f^h_k(x, v), u^{i+1}(t_k, x, v) = f^{h+1}_k(x, v); f^h_k \) denotes the approximate solution of (1) at \( t = t_k \) defined in Section 2.1.

**Remark 2.** By fixing \( u^{i-1} \), we obtain
\[
\begin{align*}
&u^i(t, x, v) = f^h_k(x - (t - t_k)v, v) \\
&\quad + \int_{t_k}^t \text{div}_v \left\{ P^{-1}(s, x - (t - t_k)v, v) \nabla c^* \left( \nabla_s G^* \left( u^{i-1} - s, x - (t - t_k)v, v \right) \right) \right\} ds.
\end{align*}
\] (262)

By fixing \( u^i \), we obtain \( u^{i+1} \) as follows:
\[
\begin{align*}
&u^{i+1}(t, x, v) = \chi^{i+1}(t, x)U^{x,i+1}(t, v),
\end{align*}
\] (263)

where
\[
\chi^{i+1}(t, x) = \int_{\mathbb{R}^d} u^i(t - h, x - hv, v) dv,
\]
\[
U^{x,i+1} \in \text{Argmin} \left\{ \frac{1}{\chi^{i+1}(t, x)} \int_{\mathbb{R}^d} G \left( \chi^{i+1}(t, x)U \right) dv + hW^h_k(U, V^{x,i+1}) \right\},
\] (264)

with
\[
V^{x,i+1}(t, x) = \frac{u^i(t - h, x - hv, v)}{\chi^{i+1}(t, x)}
\] (265)

Note that, for \( t = t_k \) and for \( i = 1, 3, 5, \ldots; 2j + 1 \), \( j \in \mathbb{N}^+ \), we obtain \( u^i(t_k, x, v) = F^h_k(x, v), u^{i+1}(t_k, x, v) = F^{h+1}_k(x, v), \chi^{i+1}(t_k, x) = \rho^h_k(x), V^{x,i+1}(t_k, v) = G^h_k(v), \) and \( U^{x,i+1}(t_k, v) = F^{h+1}_k(v) \), where \( \rho^h_k, G^h_k, F^{h+1}_k, \) and \( F^h_k \) are defined as in the descent algorithm (see Section 2.1).

### 5.1. Error Analysis
An estimate of the rate of convergence of \( u^i \) to the exact solution of (1) is given as follows.

**Theorem 4.** Assume that \( f_0, c, \) and \( G \) satisfy, respectively, \( (H_f), (H_c), \) and \( (H_G) \), and \( \text{supp } f_0(x, \cdot) \subset B_R \) for all \( x \in \Omega; \nabla c^*(0) = 0 \). Then, we obtain the following error estimation:
\[
\left\| e^i(t) \right\|_{\text{co, L}^q(\Omega \times B_R)} \leq K \left\| e^i \right\|_{\text{co, L}^q(\Omega \times B_R)}
\] (266)

where \( e^i(t, x, v) = f(t, x, v) - u^i(t, x, v), j \in \mathbb{N}^+ \) is the error between the exact solution \( f \) and the approximate solution \( u^i \) and \( q = p/(p - 1) \leq p \) and \( e^i = f - f^{i+1} \), and \( K \) is a constant depending on \( N, M, R, \) and \( G^* \).

Proof. We use the error function \( e^i = f - u^i \) and the linearized equation of (1) via iterative splitting methods. Then, we have for \( i = 1, 3, 5, \ldots; 2j + 1 \):
\[
\begin{align*}
&\frac{\partial e^i}{\partial t} + (\nabla \cdot v) e^i = \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \right) e^{i-1} \\
&\quad + \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \nabla D^2 G^* \left( u^{i-1} \right) \right) e^{i-1} + o\left( \left\| e^{i-1} \right\| \right).
\end{align*}
\] (267)

Since \( \text{supp } f_0(x, \cdot) \subset B_R \) and \( N \leq f_0 \leq M \), then Lemma 16 gives that \( \text{supp } u^i(t, x, \cdot) \subset B_R \) and \( D^2 c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) D^2 G^* \left( u^{i-1} \right) \in L^\infty \left( [t_k, t_{k+1}] \times \Omega \times B_R \right) \) and
\[
\left\| D^2 c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) D^2 G^* \left( u^{i-1} \right) \right\|_{L^\infty \left( [t_k, t_{k+1}] \times \Omega \times B_R \right)} \leq 1 + \frac{M}{N}
\] (268)

We conclude that \( \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \right) \in L^q(\Omega \times B_R) \) and
\[
\text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \nabla D^2 G^* \left( u^{i-1} \right) \right) \in L^q(\Omega \times B_R).
\]

Thus,
\[
\left\| \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \right) \right\|_{L^q(\Omega \times B_R)} \leq K_1,
\] (269)
\[
\left\| \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \nabla D^2 G^* \left( u^{i-1} \right) \right) \right\|_{L^q(\Omega \times B_R)} \leq K_2,
\] (270)
\[
\left\| \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \nabla D^2 G^* \left( u^{i-1} \right) \right) \right\|_{L^q(\Omega \times B_R)} \leq K_3,
\] (271)

where \( K_1, K_2, \) and \( K_3 \) are constants depending on \( N, M, R, \) and \( G^* \). By using (267), we have
\[
\begin{align*}
&\frac{\partial e^i}{\partial t} + (\nabla \cdot v) e^i = \int_{t_k}^t \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \right) e^{i-1} ds \\
&\quad + \int_{t_k}^t \text{div}_v \left( u^{i-1} \nabla c^* \left( \nabla_s G^* \left( u^{i-1} \right) \right) \nabla D^2 G^* \left( u^{i-1} \right) \right) e^{i-1} ds + \int_{t_k}^t o\left( \left\| e^{i-1} \right\| \right) ds.
\end{align*}
\] (272)

We now use (270) and (271) to obtain
\[
\left\| e^i \right\|_{L^q(\Omega \times B_R)} \leq K_4 \left\| e^{i-1} \right\|_{L^q(\Omega \times B_R)},
\] (273)

with \( K_4 = K_2 + K_3 + 1 \). As in (272), we have
\[
\begin{align*}
&e^{i+1}(t, x, v) = - \int_{t_k}^t \left( \nabla \cdot v \right) e^i ds + \int_{t_k}^t \text{div}_v \left( u^{i+1} \nabla c^* \left( \nabla_s G^* \left( u^{i+1} \right) \right) \right) e^{i+1} ds \\
&\quad + \int_{t_k}^t \text{div}_v \left( u^{i+1} \nabla c^* \left( \nabla_s G^* \left( u^{i+1} \right) \right) \nabla D^2 G^* \left( u^{i+1} \right) \right) e^{i+1} ds + \int_{t_k}^t o\left( \left\| e^{i+1} \right\| \right) ds.
\end{align*}
\] (274)
\[ \|e^{i+1}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))} \leq h^2 K_4 \|e^{-1}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))} + K_4 h \|e^{i+1}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))}. \]  
\quad (275)

Accordingly,
\[ \|e^{i+1}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))} \leq K h \|e^{-1}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))}, \]
\quad (276)

where \( K \) depends on \( K_4 \) and \( K_1 \). Finally, we use (273) and (276) to conclude that
\[ \|e\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))} \leq (K h)^{-1} \|e^{i}\|_{L^\infty(T; L^2(\Omega \times \mathbb{R}^d))}. \]
\quad (277)

for \( j \in \mathbb{N}^* \)

5.2. Numerical Example. In this section, we solve (1), when \( d = 1 \), \( G(t) = t \ln t \), and \( c^t(t) = t^2/2 \), i.e., the following Boltzmann equation:
\[ \frac{\partial f}{\partial t} + v \cdot \overrightarrow{\partial_x} f = \overrightarrow{\partial^2_v} f, \quad f = f(t, x, v), \quad (t, x, v) \in [0, \infty) \times \Omega \times (0, \infty). \]
\quad (278)

We suppose \( \Omega = (0, 1)/\mathbb{Z} = \{x = x + n, x \in (0, 1), n \text{ browsing } \mathbb{Z} \} \).

The initial datum \( f(0, x, v) = f_0(x, v) \) is a probability density on \( \Omega \times (0, \infty) \).

Let us show that the function \( f \) defined by
\[
\begin{cases}
  f(t, x, v) = x + \frac{v^3}{6} + \frac{11}{12 \sqrt{\pi}(4t + 1)} e^{-(v^2/4t + 1)}, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\
  f(t, x, v) = \frac{11}{12 \sqrt{\pi}(4t + 1)} e^{-(v^2/4t + 1)}, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty),
\end{cases}
\]
\quad (279)

is one solution of the kinetic equation (278). Indeed,
\[ \frac{\partial f(t, x, v)}{\partial t} + v \cdot \overrightarrow{\partial_x} f = \overrightarrow{\partial^2_v} f, \quad \text{in } [0, +\infty) \times \Omega \times (0, +\infty), \]
\quad (280)

with the initial datum
\[
\begin{cases}
  f_0(x, v) = x + \frac{v^3}{6} + \frac{11}{12 \sqrt{\pi}} e^{-v^2}, & \text{if } (x, v) \in (0, 1) \times (0, 1), \\
  f_0(x, v) = \frac{11}{12 \sqrt{\pi}} e^{-v^2}, & \text{if } (x, v) \in (0, 1) \times [1, +\infty).
\end{cases}
\]
\quad (281)

Moreover,
\[ \frac{\partial f}{\partial t} = \left[ \frac{11v^2}{3(4t + 1)^2 \sqrt{\pi}(4t + 1)} - \frac{11}{6(4t + 1) \sqrt{\pi}(4t + 1)} \right] e^{-(v^2/4t + 1)}, \]
\quad (282)

\[ v \cdot \overrightarrow{\partial_x} f = \begin{cases} v, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\
0, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty),
\end{cases} \]
\[ \overrightarrow{\partial^2_v} f = \begin{cases} v + \left[ \frac{11v^2}{3(4t + 1)^2 \sqrt{\pi}(4t + 1)} - \frac{11}{6(4t + 1) \sqrt{\pi}(4t + 1)} \right] e^{-(v^2/4t + 1)}, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times (0, 1), \\
\left[ \frac{11v^2}{3(4t + 1)^2 \sqrt{\pi}(4t + 1)} - \frac{11}{6(4t + 1) \sqrt{\pi}(4t + 1)} \right] e^{-(v^2/4t + 1)}, & \text{if } (t, x, v) \in [0, +\infty) \times (0, 1) \times [1, +\infty). \end{cases} \]
Hence, $f$ satisfies the kinetic equation (280). Suppose $t \in [0, +\infty]$ fixed, one has

$$
\int_{\Omega \times (0, \infty)} f(t, x, v) dx dv = \int_{(0,1) \times (0,1)}
\left[ \frac{x}{6} + \frac{v^3}{12 \sqrt{\pi (4t + 1)}} e^{-\left(\frac{v^2}{4t+1}\right)} \right] dx dv
+ \left[ \frac{11}{12 \sqrt{\pi (4t + 1)}} e^{-\left(\frac{v^2}{4t+1}\right)} \right] dx dv
= \int_{(0,1)} x dx + \int_{(0,1)} \frac{v^3}{6} dv + \int_{(0,\infty)} \frac{11}{12 \sqrt{\pi (4t + 1)}} e^{-\left(\frac{v^2}{4t+1}\right)} dv,
$$

(283)

with

$$
\int_{(0,1)} x dx = \frac{1}{2},
$$

(284)

$$
\int_{(0,1)} \frac{v^3}{6} dv = \frac{1}{24},
$$

Consequently, $\int_{\Omega \times (0, \infty)} f(t, x, v) dx dv = 1$ for all $t \geq 0$.

5.3. The Figures. Here, we perform the first-order approximation $f_1$ of the solution of (278) using the algorithm in (261). We limit ourself to the first-order approximation just for the sake of simplicity and to avoid very complex and cumbersome calculations. Numerical computation carried out for $h \in \{1/10, 1/50, 1/100\}$ and the graphical representations of both the analytical solution $f$ and the approximate solution $f_1$ with Scilab software shows a coherence between the approximate solution and the analytical solution.

Moreover, an error tables obtained for some values of the time $t$ in the interval $[h, 2h]$ for different values of the step $h$ are presented below.

By a simple computation, the expression of $f_1^h$ appears as follows:

$$
\begin{align*}
&f_1^h(x, v) = \frac{x}{6} + \frac{11hv^2}{3\sqrt{\pi}} e^{-\frac{v^2}{h}} + \frac{11(1-2h)}{12\sqrt{\pi}} e^{-\frac{v^2}{h}}, \quad (x, v) \in (0,1) \times (0,1), \\
&f_1^h(x, v) = \frac{11hv^2}{3\sqrt{\pi}} e^{-\frac{v^2}{h}} + \frac{11(1-2h)}{12\sqrt{\pi}} e^{-\frac{v^2}{h}}, \quad (x, v) \in (0,1) \times [1, +\infty).
\end{align*}
$$

(285)

For some values of $h$, we draw the figures of both the analytical solution and the numerical solution (approximate solution). We notice that Figure 1 represents an approximation of Figure 2 when $h = (1/10)$, Figure 3 represents an approximation of Figure 4 when $h = (1/50)$, and Figure 5 is an approximation of Figure 6 when $h = (1/100)$.

Errors progression is in terms of the $L^\infty$ and the $L^2$ norms between the numerical solution $f_1^h$ and the analytical
Analytical solution for $t = 3/20$

Figure 2: Analytical solution $f(3/20, x, v)$ for $h = (1/10)$ and $t = 3/20 \in [1/10, 2/10)$.

Numerical solution $f_{1}$ for $h = 1/50$

Figure 3: Numerical solution $f^{1/50}(x, v)$ for $h = 1/50$.

Analytical solution for $t = 3/100$

Figure 4: Analytical solution $f(3/100, x, v)$ for $h = 1/50$ and $t = 3/100 \in [1/50, 1/25)$. 
solution $f$. One can actually notice the convergence of our method in the $L^2$ and in $L^\infty$ norms when $h$ is decreasing to 0, which is explained as follows (Table 1).

### Appendix

Here, we give the regularity result necessary in the proof of Theorem 4.

**Lemma 16.** Assume that $f_0$, $c$, and $G$ satisfy, respectively, $(H_f)$, $(H_c)$, and $(H_G)$, $\text{supp} f_0(x, .) \subset B_R$, for all $x \in \Omega$ and $\nabla c^*(0) = 0$. Then, $\text{supp} f^k_h(x, .) \subset B_R$, for all $x \in \Omega$ and for all $k \in \mathbb{N}^*$.

Moreover, $D^2 c^* (\nabla_c G^* (f^k_h)) \ast D^2_c G^* (f^k_h) \in L^\infty (\Omega \times B_R)$ and

$$\left\| h D^2 c^* (\nabla_c G^* (f^k_h)) \ast D^2_c G^* (f^k_h) \right\|_{L^\infty (\Omega \times B_R)} \leq 1 + \frac{M}{N}. \quad (A.1)$$

**Proof.** Since $\text{supp} f_0(x, .) \subset B_R$, then $f_0(x, B_R^c) = 0$, where $B_R^c$ is the complement of $B_R$. In Section 2.1, we have $f^1_t(x, v) = \rho^1_h(x) F^1_v(v)$, where

$$\rho^1_h(x) = \int_{\mathbb{R}^d} f_0(x-hv, v) dv, \quad (A.2)$$
and \( F^*_x \) is the unique solution of the variational problem.

\[
(P^*_x) : \inf_{F \in P_r (\mathbb{R}^2)} \left\{ \frac{1}{\rho^*_1(x)} \int_{\Omega} G(F \rho^*_1(x)) \, dv + h W^0_c (F, G^*_x) \right\},
\]

with

\[
G^*_1(v) = \frac{f_0(x - hv, v)}{\rho^*_1(x)},
\]

\[
W^0_c (F, G^*_x) = \inf_{T \in G^*_x} \int_{\Omega} c \left( \frac{v - T(v)}{h} \right) F \, dv.
\]

Since \( f_0(x, B^c_R) = 0 \), then \( G^*_1(B^c_R) = 0 \).

The \( c \)-optimal map \( T_1 \) that pushes \( F^*_x \) forward to \( G^*_x \) satisfies then

\[
F^*_x \left( T_1^{-1} (B^c_R) \right) = G^*_1(B^c_R) = 0.
\]

We now use the explicit expression of \( T_1 \) (see Lemma 3):

\[
T_1(v) = v + h v c' \left( \nabla_x G' \left( f^*_1 \right) \right),
\]

along with (A.5) and \( Vc' \cdot 0 \) to get

\[
T_1 \left( T_1^{-1} (B^c_R) \right) = T_1^{-1} (B^c_R).
\]

This implies that \( T_1^{-1} (B^c_R) = B^c_R \), and then, \( F^*_x (B^c_R) = 0 \).

Thus, we deduce that \( \text{supp} \, F^*_x \subset B^c_R \).

Since \( f^*_1 (x, v) = \rho^*_1(x) c^*_1(v) \), we obtain \( \text{supp} \, f^*_1 (x, v) \subset B^c_R \), for all \( x \in \Omega \). Finally, we obtain by induction \( \text{supp} \, f^*_k (x, v) \subset B^c_R \), for all \( x \in \Omega \).

Since \( F^*_k \) and \( G^*_k \) have compact support and that \( c \) is strictly convex, then the \( c \)-optimal maps \( T_k \) that pushes \( F^*_k \) forward to \( G^*_k \) is differentiable, and we have

\[
\nabla_v T_k(v) = i d + h D^2 c' \left( \nabla_x G' \left( f^*_k \right) \right) \circ D^2 v G' \left( f^*_k \right).
\]

Hence, \( \nabla_v T_k(v) \) satisfies the Jacobian equation:

\[
p^k_{h \lambda} = \det \left( \nabla_v T_k(v) \right) p^h_{\lambda} G^h \phi.
\]

Since \( N \leq f_0 \leq M \), the maximum principle gives \( N \leq f^h_k \leq M \). Then, we deduce that

\[
\frac{N}{M} \leq \det \left( \nabla_v T_k \left( v \right) \right) \leq \frac{M}{N}.
\]

Note that \( \nabla_v T_k \) is diagonalizable and has positive eigenvalues (see [1]). Then, we deduce that \( \nabla_v T_k \left( v \right) \in L^\infty (\Omega \times B^c_R) \) and \( \left\| \nabla_v T_k \left( v \right) \right\|_{L^\infty (\Omega \times B^c_R)} \leq (M/N) \). And by using (A.8), we obtain

\[
\left\| h D^2 c' \left( \nabla_x G' \left( f^*_k \right) \right) \circ D^2 v G' \left( f^*_k \right) \right\|_{L^\infty (\Omega \times B^c_R)} \leq 1 + \frac{M}{N}.
\]

\[
\square
\]

Data Availability

There are no data underlying the findings in this paper to be shared.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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