PARTICLE KNOTS IN TORIC MODULAR SPACE

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Abstract
The goal of this contribution is to relate quarks to knots or loops in a 6-space $\mathbb{CP}^3$ that then collapses into a torus in real 3-space $\mathbb{CP}^3$ instantaneously after the Big Bang, and massive inflation, when 3 quarks unite to form nucleons.

Introduction
Kedia et. al. in recent paper [10] investigate knotted structures in hydrodynamic fields such as current-guiding magnetic field lines in a plasma, or vortex lines of classical or quantum fields which arise naturally as excitations that carry helicity that is a measure of the knottedness of the field. In particular their Fig. 2g is a trefoil which is our Fig. 3 without the quadrupole (that will be seen in Section 2 to collapse into a point) and the color-coding.

Fig. 1

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The torus shown in Fig.2 is due to Marcelis [13] whose calculations in a projective space with 24 vertices appear to be unpublished, but are supported to some extent by Westy [18] (from the same school) who provides a color-coded complex map of the Riemann surface z that incorporates the phases of $\omega=120$ degrees.
Murasagi [14] Ch.7 shows that Fig.2 is a trefoil (3,2) on a torus when we choose 3 points on the ends of a cylinder that can be joined to form the torus. This brings us to the goal of this contribution which is to relate the elementary particles to knots or loops in a 6-space. Here we will be guided by the work of Coxeter [5,6] who specifically labels the vertices of the torus appearing in Fig.1 by $0, \pm 1, \omega, \omega^2$ where $\omega = \exp(2i\pi/3)$ so that a knot crosses the longitude of a torus at $\omega=120$ degrees. Essentially this is a Galois Field GF(4) with permutations of $\omega$ raised to the powers $0,1,2,3$ that will be considered in more detail in the next Section where we will show how 3 quarks in 6-space unite to become a nucleon in the projective space $\mathbb{CP}^3$ which collapses to $\mathbb{P}^3$ immediately after the Big Bang. Section 3 will employ the color-coding of Fig.1 as a model for Quantum Chromodynamics or QCD. Finally according to Rovelli [15] knots or loops in the 6-space described by $\mathbb{E}_6$ employed by Coxeter may also describe Loop Quantum Gravity although details are beyond the scope of this contribution. Also Arvin [2] has considered knots on a torus as a model for elementary particles but excluding quarks. Again, Sundance O Billson Thompson, Smolin et.al. [17] also use knots as a model for Quantum Gravity and the Standard Model but utilize trinions instead of trefoils.

**Coxeter Algebra**

Fig.1 is a torus taken from Coxeter [5] which is an alternative to the graph $\mathfrak{su}(3)_{\text{c}} \times \mathfrak{su}(3)_{\text{isospin}} \times \mathfrak{su}(3)_{\text{spin}}$ which is a triality sub-algebra of $\mathbb{E}_6$. This graph is orbifolded with 27 vertices that according to Slansky [16] may be labeled by particles in the Standard Model or SM. However the actual labeling of the tritangent planes on a cubic surface (discussed by Hunt [10] Ch. 4) is new, but in line with Coxeter’s labels. For example the up-quark $u$ in Fig.1 is labeled by $(012)$ indicated by $0, \omega, \omega^2$. Then $(023)$ on the same tritangent is simply a rotation through $\omega=120$ degrees and so on. In this way we find an equilateral triangle labeled by the 3 quarks $uud$ comprising a proton and another $dud$ on a torus which is precisely the model adopted by Green, Schwarz and Witten [8] Section 9.5.2.

The quarks at the vertices of Fig.1 are trefoils illustrated by Fig.2, but the torus in Fig.1 only becomes a trefoil after the collapse of the inner ring just after the Big Bang when quarks in the 6-space $\mathbb{CP}^3$ unite to build nucleons in the projective space $\mathbb{P}^3$.

This is supported by Barth and Nieto [3], where only the 12 outer vertices and the center of Fig.1 are in $\mathbb{P}^3$. Specifically these authors find 15 synthemes, where a symtheme has 6 ‘fix-lines’ that are the edges of an invariant tetrahedron, such as $uud$ 0 representing a proton. However because there are only 3 vertices on the face of a symtheme the outer ring of Fig.1 carries the 4 stable particles proton, neutron and the anti-particles. Also since the tritangents are invariant under rotations there are actually 3x4=12 possible synthemes on the outer ring. Specifically each symtheme consists of 3 commuting operators. Thus 3 synthemes can be chosen for spin rotations about the 3 axes of 3-space. Thus introducing triality which is a characteristic of Toric-Calabi-Yau modular spaces that carry the Hessian Polyhedra in $\mathbb{E}_6$ as discussed by Lie-Yang [12] and analysed by Coxeter [5,6].

Specifically each rotation in 3-space is also accompanied by a corresponding rotation in a parity 4-space when we permute 1,2,3. Charge conjugation belongs to a second set of 3 synthemes with the same rotations in a 3-space but a parity 5-space in another charge space [7].

In this way the 12 unstable particles $sss, \tilde{u}, v, \mu, \tau, \nu$ do not appear in the compactification of $\mathbb{CP}^3$ to $\mathbb{P}^3$. This may be visualised as a collapse of the inner vertices to a point which carries the remaining 3 synthemes $e^{\pm}, \nu$, labeled by $\{011,022,033\},\{110,220,330\}$ and $\{101,202,303\}$ for the muon. In this process the masses $m_\tau, m_\mu$ of the $\tau, \mu$ reappear as stable deuterium 3 according to the relationship

$$m_\tau + m_\mu = m_p + m_n + m_e$$

(1)
There is no heavy-ion decay and the same relation holds for the anti-particles. This equation is accurate if we assume that \( m_\tau = 1777 \text{ MeV} \) and \( m_\mu = 101.4 \text{ MeV} \) instead of the Fermi decomposition of muon decay in the weak interaction yielding 106 MeV. However in a recent publication Benjamin Brau et al.\[4\] find a value of approximately 100 MeV for the mass of cosmic-ray muons so there is as yet some experimental uncertainty.

**Quantum Chromodynamics, QCD**

Returning again to Fig.1, when the inner vertices are contracted to a point at the origin the red, green and blue lines could serve as gluons on a new torus where a red upper path passes through the center before emerging at the circumference and giving way to a green gluon that in turn passes under the torus and then over to connect with a down quark and so on. The 3 color complex dimensions vanish when \( \text{CP}^3 \to \text{P}^3 \) but a torus knot remains in 3-space.

However Marcelis [13] calculates the dual set of 3 paths for the anti-gluons overline (r,g,b) which appear in Fig.3 (without the quadrupole) so the gluon, antigluon linked trefoil give us the \( SU(3)_c \) color symmetry underlining QCD as described by Griffiths [9], Section 9.1. For example when another quark is added after a rotation \( \omega \) a red gluon may unite with an anti-blue to find \( r \) anti-b, then a following rotation would bring \( r \) to anti-\( r \), and so on before blow down to \( P^3 \). In this way we can find 9 gluon pairs \( r \) anti-\( r \), \( r \) anti-\( b \), \( r \) anti-\( g \), \( b \) anti-\( r \), \( b \) anti-\( b \), \( b \) anti-\( g \), \( g \) anti-\( b \), \( g \) anti-\( g \) that are a basis for \( SU(3)_c \) symmetry.

Finally Adams [1] p 273 also envisages the 3 colors \( r, b, g \) as three extra dimensions in a 6-space.

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Fig.1 The Coxeter Polytope
Fig.2 Cayley Surface in Elliptic Space
Fig.3 Interior of Cayley Surface

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