Achievable ranks of intersections of finitely generated free groups

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Abstract

We answer a question due to A. Myasnikov by proving that all expected ranks occur as the ranks of intersections of finitely generated subgroups of free groups.

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Let $F$ be a free group. Let $H$ and $K$ be nontrivial finitely generated subgroups of $F$. It is a theorem of Howson [1] that $H \cap K$ has finite rank. H. Neumann proved in [2] that $\text{rank}(H \cap K) - 1 \leq 2(\text{rank}(H) - 1)(\text{rank}(K) - 1)$ and asked whether or not $\text{rank}(H \cap K) - 1 \leq (\text{rank}(H) - 1)(\text{rank}(K) - 1)$.

A. Miasnikov has asked which values between 1 and $(m - 1)(n - 1)$ can be achieved as $\text{rank}(H \cap K) - 1$ for subgroups $H$ and $K$ of ranks $m$ and $n$—this is problem AUX1 of [4]. We prove that all such numbers occur by proving the following theorem.

Let $F(a, b)$ be a free group of rank two. Let

$$H_{k, \ell}^m = \langle a, bab^{-1}, \ldots, b^k a^{-1} b^{-k}, b^{k+1} a^{-\ell} b^{-(k+1)}, b^{k+2} a^{-\ell} b^{-(k+2)}, b^{k+3} a^{-\ell} b^{-(k+3)}, \ldots, b^{m-1} a^{-\ell} b^{-(m-1)} \rangle$$

and let $K = \langle b, aba^{-1}, \ldots, a^{n-1} b^{-1} a^{-1} b^{-1} \rangle$, where $0 \leq k \leq m - 2$ and $0 \leq \ell \leq n - 1$. Then the rank of $H_{k, \ell}^m \cap K$ is $k(n - 1) + \ell$.

Corollary. Let $F$ be a free group and let $m, n \geq 2$ be natural numbers. Let $N$ be a natural number such that $1 \leq N - 1 \leq (m - 1)(n - 1)$. Then there exist subgroups $H, K \leq F$, of ranks $m$ and $n$, such that the rank of $H \cap K$ is $N$.

Proof of the corollary. The theorem produces the desired subgroups for all $N$ with $N - 1 \leq (m - 1)(n - 1) - 1$ after passing to a rank two subgroup of $F$. For $N - 1 = (m - 1)(n - 1)$, simply let $H = \langle a, bab^{-1}, \ldots, b^{m-2} a^{n-2} b^{-n}, b^{m-1} \rangle$ and let $K = \langle b, aba^{-1}, \ldots, a^{n-2} b^{2-n}, a^{n-1} \rangle$.

Proof of the theorem. Let $X$ be a wedge of two circles and base $\pi_1(X)$ at the wedge point. We identify $\pi_1(X)$ with $F = F(a, b)$ by calling the homotopy class of one oriented circle $a$ and the other $b$. Given a finitely generated subgroup of $F$,
there is a covering space $\tilde{X}$ corresponding to this subgroup. Moreover, there is a compact subgraph of $\tilde{X}$ that carries the given subgroup. Given two subgroups and their associated finite graphs, one may construct the graph associated to their intersection. These procedures are laid out carefully in [3] and we assume that the reader is familiar with that paper.

In the figures, the graph associated to $H$ appears at the top, that of $K$ to the right, and that of $H \cap K$ in the center. Edges labelled with two arrowheads represent $a$, those with one arrowhead represent $b$. Our basepoint in the graph associated to $H \cap K$ is always the vertex in the upper left-hand corner.

For the moment, fix $k = m - 2$. In Figure 1, $\ell = n - 1$ and the rank of $H_{m-2,n-1}^m \cap K$ is visibly $(m-1)(n-1)$. Decreasing $\ell$ by one alters the intersection graph as depicted in Figure 2 and the rank of $H_{m-2,n-2}^m \cap K$ is $(m-1)(n-1) - 1$. Figure 3 shows the case when $\ell = n - 3$ and the rank of the intersection is $(m-1)(n-1) - 2$. When $\ell = n - j$, the rank of $H_{m-2,n-j}^m \cap K$ is $(m-1)(n-1) - (j-1)$.

Figure 4 depicts the case $\ell = 0$. Note that the graph associated to $H_{m-2,0}^m \cap K$ is the graph associated to $H_{m-3,n-1}^m \cap K$ to which a collection of trees have been attached at their roots, the graph associated to $H_{m-3,n-2}^m \cap K$ is the graph
Figure 3: $H$, $K$, and $H \cap K$ when $k = m - 2$, $\ell = n - 3$

Figure 4: $H$, $K$, and $H \cap K$ when $k = m - 2$, $\ell = 0$

associated to $H_{m-3,n-2}^{m-1} \cap K$ to which trees have been so attached, and so on. Since attaching trees in this way leaves the rank intact, we arrive at the theorem by induction on $m$.

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References

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