A Dynamical Study of the Friedmann Equations

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Abstract. Cosmology is an attracting subject for students but usually difficult to deal with if general relativity is not known. In this article, we first recall the Newtonian derivation of the Friedmann equations which govern the dynamics of our universe and discuss the validity of such a derivation. We then study the equations of evolution of the universe in terms of a dynamical system. This sums up the different behaviors of our universe and enables to address some cosmological problems.

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1. Introduction

In this article, we want to present a pedagogical approach to the equations governing the evolution of the universe, namely the Friedmann equations. Indeed, the derivation of this equations is intrinsically relativistic. Although in Newtonian theory, the universe must be static, Milne [1] and McCrea and Milne [2] showed that, surprisingly, the Friedmann equations can be derived from the simpler Newtonian theory. In section 2.1, we recall their derivation for a universe filled with pressureless matter and then discuss the introduction of a cosmological constant (§ 2.2). Indeed, it is puzzling that the Newtonian theory and general relativity give the same results; we briefly discuss this issue in § 2.3.

Once we have interpreted the Friedmann equations, we study them as a dynamical system. The first authors to consider such an approach were Stabell and Refsdal [3] who investigated the Friedmann–Lemaître model with a pressureless fluid. This was then generalised to a fluid with any equation of state [4, 5]. Then, this technique was intensively used to study the isotropisation of homogeneous models (see e.g. [6] and references therein). For a general description of the use of dynamical systems in cosmology, we refer to the book by Wainwright and Ellis [7] where most of the techniques are detailed. Our purpose here, is to present such an analysis for a fluid with any equation of state and including a cosmological constant while staying as pedagogical as possible. In section § 3, we rewrite the Friedmann equations under a form easier to handle with and we extract the dynamical system to study. We then determine the
fixed points of this system and discuss their stability. We illustrate this analytic study by a numerical integration of this set of equations (§ 4) and finish by a discussion about the initial conditions explaining the current observed state of our universe (§ 5).

2. A Newtonian derivation of the Friedmann equation

We follow the approach by Milne [1] and McCrea and Milne [2] and the reader is referred to [3] for further details.

2.1. General derivation

We consider a sphere of radius \( R \) filled with a pressureless fluid \((P = 0)\) of uniform (mass) density \( \rho \) free-falling under its own gravitational field in an otherwise empty Euclidean space. We decompose the coordinate \( x \) of any particle of the fluid as

\[
x = a(t)r
\]

where \( r \) is a constant vector referred to as the comoving coordinate, \( t \) is the time coordinate and \( a \) the scale factor. We choose \( a \) to have the dimension of a length and \( r \) to be dimensionless. It implies that the sphere undergoes a self similar expansion or contraction and that no particle can cross another one. Indeed the edge of the sphere is also moving as

\[
R(t) = a(t)R_0.
\]

Assume that while sitting on a particle labelled \( i \) we are observing a particle labelled \( j \); we see it drift with the relative velocity

\[
v_{ij} = \dot{a}(r_j - r_i) = Hx_{ij}
\]

where a dot refers to a time derivative, \( H \equiv \dot{a}/a \) and \( x_{ij} \equiv (r_j - r_i) \). As a consequence, any particle \( i \) sees any other particle \( j \) with a radial velocity proportional to its distance and the expansion is isotropic with respect to any point of the sphere, whatever the function \( a(t) \). But, note that this does not imply that all particles are equivalent (as will be discussed later).

To determine the equation of motion of any particle of this expanding sphere, we first write the equation of matter conservation stating that the mass within any comoving volume is constant (i.e. \( \rho x^3 \propto r^3 \)) implying that

\[
\rho(t) \propto a^{-3}(t),
\]

which can also be written under the form

\[
\dot{\rho} + 3H\rho = 0.
\]

Note that Eq. (5) can also be deduced from the more general conservation equation

\[
\partial_t \rho + \nabla_x j = 0 \quad \text{with} \quad j = \rho \nu, \quad \nu = Hx \quad \text{and} \quad \nabla_x x = 3.
\]
To determine the equation of evolution of the scale factor $a$, we first compute the gravitational potential energy $E_G$ of a particle of mass $m$ by applying the Gauss law

$$E_G = -\frac{GM(<x)m}{x}$$

(6)

where $G$ is the Newton constant and $M(<x)$ the mass within the sphere of radius $x$ given by

$$M(<x) = \frac{4\pi}{3}\rho x^3.$$  

(7)

We then need to evaluate its kinetic energy $E_K$ which takes the simple form

$$E_K = \frac{1}{2}m\dot{x}^2.$$  

(8)

The conservation of the total energy $E = E_G + E_K$ implies, after the use of the decomposition (1) and a simplification by $r$, that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$

(9)

where $K$ is a dimensionless constant (which can depend on $r$) given by $K = -2E/(mc^2r^2)$.

### 2.2. Introducing a cosmological constant

In the former derivation, the gravitational potential on any particle inside the sphere is proportional to the distance $x^2$. Any other force deriving from a potential proportional to $x^2$ will mimic a gravitational effect. A force deriving from the potential energy $E_\Lambda$ defined by

$$E_\Lambda = -m\frac{\Lambda c^2}{6}x^2$$

(10)

where $\Lambda$ is a constant was introduced by Einstein in 1917. As in the previous section, writing that the total energy $E = E_K + E_G + E_\Lambda$ is constant leads to the equation of motion

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}.$$  

(11)

From (10), we deduce that $\Lambda$ has the dimension of an inverse squared length. The total force on a particle is

$$F = m\left(-\frac{4\pi G}{3}\rho + \frac{\Lambda c^2}{3}\right)x$$

(12)

from which it can be concluded that (i) it opposes gravity if $\Lambda$ is positive and that (ii) it can be tuned so that $F = 0$ leading to $\dot{a} = 0$ and $\rho =$constant if

$$\Lambda = \frac{4\pi G}{c^2}\rho.$$  

(13)

‡ This scaling of $K$ with $r$ is imposed by the requirement that the expansion is self–similar (Eq. 1) and that no shell of labeled $r$ can cross a shell of label $r' > r$. 

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This enables to recover a static autogravitating sphere hence leading to a model for a static universe. The force deriving from $E_\Lambda$ is analogous to the one exerted by a spring of negative constant.

To finish, we recall on table 1 the dimension of all the quantities used in the former sections, mainly to compare with standard textbooks in which the choice $c = 1$ is usually made.

2.3. Discussion

From this Newtonian approach, the equation of evolution of the universe identified with this gravitating sphere are thus given by equation (5) and (11). These are two differential equations for the two variables $a(t)$ and $\rho(t)$ which can be solved once the two parameters $K$ and $\Lambda$ have been chosen.

In the context of general relativity, one can deduce the law of evolution for the scale factor of the universe $a$ which is given by the Friedmann equations

$$H^2 = \frac{\kappa}{3} \rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

(14)

$$\frac{\dot{a}}{a} = -\frac{\kappa}{6} (\rho + 3 \frac{P}{c^2}) + \frac{\Lambda c^2}{3}$$

(15)

with $\kappa \equiv 8\pi G$ and the conservation equation

$$\dot{\rho} + 3H(\rho + \frac{P}{c^2}) = 0.$$  

(16)

Eq. (14) reduces to (11) and, Eq. (16) to (5) when $P = 0$. The equation (15) is redundant and can be deduced from the two others. Note that now Eq. (16) is also a conservation equation but with the mass flux $j = (\rho + P/c^2)v$. This can be interpreted by remembering that the first law of thermodynamics for an adiabatic system takes the form

$$\dot{E} + P \dot{V} = 0$$

(17)

where $E = \rho V c^2$ is the energy contained in the physical volume $V$ (scaling as $a^3$).

The first thing to stress is that equations (5) and (11) do not depend on the radius $R_0$ of the sphere. It thus seems that we can let it go to infinity without changing the conclusions and hence explaining why we recover the Friedmann equations. This was the viewpoint adopted by Milne [1] and McCrea and Milne [2]. This approach leads to some problems. First, it has to be checked that the Gauss theorem still applies after taking the limit toward infinity (i.e. one has to check that the integrals and the limit
commute). This imposes that $\rho$ decreases fast enough with $r$ and thus that there is indeed a center. Equivalently, as pointed out by Layzer [10], the force on any particle of an infinite homogeneous distribution is undetermined (the integral over the angles is zero while the integral over the radial coordinate is infinite). The convergence of the force requires either the mass distribution to be finite (in which case it can be homogeneous) or to be inhomogeneous if it is infinite. The issue of the finiteness of the universe has been widely discussed and a clear presentation of the evolution of ideas in that respect are presented in [16]. Second, for distances of cosmological interests, i.e. of some hundred of Megaparsec, the recession speed of the particles of the sphere are of order of some fraction of the speed of light. One will thus require a (special) relativistic treatment of the expanding sphere. Third, the gravitational potential grows with the square of the radius of the sphere but it can not become too large otherwise, due to the virial theorem, the velocities would exceed the speed of light.

It was then proposed [12] that such an expanding sphere may describe a region the size of which is small compared with the size of the observable universe (i.e. of the Hubble size). Since all regions of a uniform and isotropic universe expand the same way, the study of a small region gives information about the whole universe (but this does not solve the problem of the computation of the gravitational force).

The center seems to be a privileged points since it is the only point to be at rest with respect to the absolute frame. But, one can show that the spacetime background of Newtonian mechanics is invariant under a larger group than the traditionally described Galilean group. As shown by Milne [1], McCrea and Milne [4] and Bonnor [13] (see also Carter and Gaffet [14] for a modern description) it includes the set of all time-dependent space translations

$$x^i \rightarrow x^i + z^i(t)$$

where $z^i(t)$ are arbitrarily differentiable functions depending only on the time coordinate $t$. This group of transformation is intermediate between the Galilean group and the group of all diffeomorphisms under which the Einstein theory is invariant. Thanks to this invariance group, each point can be chosen as a center around which there is local isotropy and homogeneity but the isotropy is broken by the existence of the boundary of the sphere (i.e. all observer can believe living at the center as long as he/she does not observe the boundary of the expanding sphere).

There are also conceptual differences between the Newtonian cosmology and the relativist cosmology. In the former we have a sphere of particle moving in a static and absolute Euclidean space and the time of evolution of the sphere is disconnected from the absolute time $t$. For instance in a recollapsing sphere, the time will go on flowing even after the crunch of the sphere. In general relativity, space is expanding and the particles are comoving. We thus identify an expanding sphere in a fixed background and an expanding spacetime with fixed particles. As long as we are dealing with a pressureless fluid, this is possible since there is no pressure gradient and each point of the sphere can be identify with one point of space (in fact, with an absolute time we are working in a synchronous reference frame and we want it to be also comoving, which is
Table 2. Comparison of the nature of the Newtonian trajectory and of the structure of space according to the value of the constant $K$ in Eq. (11).

| $E$   | $> 0$ | $0$  | $< 0$ |
|-------|-------|------|-------|
| Trajectory | hyperbolic | parabolic | elliptic |
|        | unbounded  | unbounded | bounded  |
| $K$   | $< 0$  | $0$  | $> 0$  |
| Spatial section | infinite    | infinite | finite  |

Table 2. Comparison of the nature of the Newtonian trajectory and of the structure of space according to the value of the constant $K$ in Eq. (11).

possible only if $P = 0$ [15]. Moreover, the pressure term in the Friedmann equations cannot trivially be recovered from the Newtonian argument. As shown, one gets the correct Friedmann equations if one starts from the conservation law including pressure (and derived from the first law of thermodynamics) and the conservation of energy. But if one were starting from the Newton law relating force (12) and acceleration ($m\ddot{a}$), the term containing the pressure in (17) would not have been recovered; one should have added an extra pressure contribution $F_P = -4\pi GmP/\gamma^2$ which cannot be guessed. This is a consequence that in general relativity any type of energy has a gravitational effect. In a way it is a “miracle” that the equation (14) does not depend on $P$, which makes it possible to derive from the Newtonian conservation of energy. Beside it has also to be stressed that the Newtonian derivation of the Friedmann equations by Milne came after Friedmann and Lemaître demonstrated the validity of the Friedmann equations for an unbounded homogeneous distribution of matter (using general relativity). It has to be pointed out that these Newtonian models can not explain all the observational relations since, contrary to general relativity, they do not incorporate a theory of light propagation. As outlined by Lazer [10] one can sometime legitimately treat a part of the (dust) expanding universe as an isolated system in which case the Newtonian treatment is correct, which makes McCrea [11] conclude that this is an indication that Einstein’s law of gravity must admit the same interpretation as that of Newton’s in the case of a spherically symmetric mass distribution. Note that the structural similarity of Einstein and Newton gravity were put forward by Cartan [17] who showed that these two theories are much closer that one naively thought and, in that framework (which goes far beyond our purpose) one can work out a correct derivation of the Friedmann equations (see e.g. [18]).

The most important outcome of the Newtonian derivation of the Friedmann equations is that it allows to interpret equation (14) in terms of the conservation of energy; the term in $H^2$ represents the kinetic energy, the term in $\kappa\rho/3$ the gravitational potential energy, the term in $\Lambda/3$ the energy associated with the cosmological constant and the term in $K$ the total energy of the system. The properties of the spatial sections (i.e. of the three dimensional spaces of constant time) are related to the sign of $K$ and can be compared with the property of the trajectories of the point of the sphere which are related to the sign of the total energy $E$; we sum up all these properties on table 2.
3. The Friedmann equations as a dynamical system

The Friedmann equations (14–15) and the conservation equation (16) form a set of two independent equations for three variables ($a$, $P$ and $\rho$). The usual approach is to solve this system by specifying the matter content of the universe mainly by assuming an equation of state of the form

$$P = (\gamma - 1)\rho c^2$$

where $\gamma$ may depend on $\rho$ and thus on time. For a pressureless fluid (modelling for instance a fluid of galaxies) $\gamma = 1$ and for a fluid of radiation (such as photon, neutrino,...) $\gamma = 4/3$. We assume that $\gamma \neq 0$ since such a type of matter is described by the cosmological constant and singled out from “ordinary” matter and that $\gamma \neq 2/3$ since such a type of matter mimics the curvature term and is thus incorporated with it.

One can then first integrate (16) rewritten as $d\rho/\rho = -3da/a$ to get the function $\rho(a)$ which, in the case where $\gamma$ is constant, yields

$$\rho(a) = Ca^{-3\gamma}$$

where $C$ is a positive constant of integration, and then insert the solution for $\rho(a)$ in Eq. (14) to get a closed equation for the scale factor $a$ (see e.g. [19] for such an approach and [20] for an alternative and pedagogical derivation).

In this section, we want to present another approach in which the Friedmann equations are considered as a dynamical system and to determine its phase space.

3.1. Derivation of the system

The first step is to rewrite the set of dynamical equations with the three new variables $\Omega$, $\Omega_\Lambda$ and $\Omega_K$ defined as

$$\Omega \equiv \frac{\kappa \rho}{3H^2},$$

$$\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H^2},$$

$$\Omega_K \equiv -\frac{Kc^2}{a^2H^2}.$$  

They respectively represent the relative amount of energy density present in the matter distribution, cosmological constant and curvature. $\Omega$ has to be positive and there is no constraint on the sign of both $\Omega_\Lambda$ and $\Omega_K$. With these definitions, it is straightforward to deduce from (14) that

$$\Omega + \Omega_\Lambda + \Omega_K = 1.$$  

Using that $\dot{H} = \ddot{a}/a - H^2$, expressing $\ddot{a}/a$ from Eq. (13) and $H^2$ from Eq. (14), we deduce that

$$\frac{\dot{H}}{H^2} = -(1 + q)$$
where the deceleration parameter $q$ is defined by
\[ q \equiv \frac{3\gamma - 2}{2}(1 - \Omega_K) - \frac{3\gamma}{2}\Omega. \] (25)

It is useful to rewrite the full set of equations by introducing the new dimensionless time variable $\eta \equiv \ln(a/a_0)$, $a_0$ being for instance the value of $a$ today. The derivative of any quantity $X$ with respect to $\eta$, $X'$, is then related to its derivative with respect to $t$ by $X' = \dot{X}/H$. The equation of evolution of the Hubble parameter (24) takes the form
\[ H' = -(1 + q)H. \] (26)

Now, differentiating $\Omega$, $\Omega_\Lambda$ and $\Omega_K$ with respect to $\eta$, using Eq. (26) to express $H'$, $a' = a$ and Eq. (16) to express $\rho' = -3\gamma\rho$, we obtain the system
\[ \Omega' = (2q + 2 - 3\gamma)\Omega \] (27)
\[ \Omega'_\Lambda = 2(1 + q)\Omega_\Lambda \] (28)
\[ \Omega'_K = 2q\Omega_K \] (29)
and it is trivial to check that $\Omega' + \Omega'_\Lambda + \Omega'_K = 0$ as expected form (23).

Indeed, it is useless to study the full system (26–29) (i) since $H$ does not enter the set of equations (27–29) and is solely determined by Eq. (26) once this system has been solved and (ii) since $\Omega$ can be deduced algebraically from (23). As a consequence, we retain the closed system
\[
\begin{cases}
\Omega'_\Lambda = 2(1 + q)\Omega_\Lambda \\
\Omega'_K = 2q\Omega_K
\end{cases}
\] (30)
with $q$ being a function of $\Omega_\Lambda$ and $\Omega_K$ only and defined in (24).

The system (30) is autonomous [21], which implies that there is a unique integral curve passing through a given point, except where the tangent vector is not defined (fixed points). Note that at every point on the curve the system (30) assigns a unique tangent vector to the curve at that point. It immediately follows that two trajectories cannot cross; otherwise the tangent vector at the crossing point would not be unique [21].

3.2. Determination of the fixed points

To study the system (30) as a dynamical system, we first need to determine the set of fixed points, i.e. the set of solutions such that $\Omega'_\Lambda = 0$ and $\Omega'_K = 0$. These solutions represent equilibrium positions which indeed can be either stable or unstable. The fixed points are thus solutions of
\[ (1 + q)\Omega_\Lambda = 0, \quad q\Omega_K = 0. \] (31)

We obtain the three solutions
\[ (\Omega_K, \Omega_\Lambda) \in \{(0, 0), (0, 1), (1, 0)\}. \] (32)

Each of these solutions represent a universe with different physical characteristics:
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(i) \((\Omega_K, \Omega_\Lambda) = (0, 0)\): the Einstein de Sitter space (EdS).
It is a universe with flat spatial sections, i.e. the three dimensional hypersurfaces of constant time are Euclidean and it has no cosmological constant. We deduce from (23) and (25) that
\[
\Omega = 1, \quad q = \frac{3}{2} \gamma - 1
\]
and integrating Eq. (14) gives
\[
a(t) = \left( \sqrt{\frac{\kappa C}{3}} t \right)^{\frac{2}{3\gamma}}
\]
for the solution vanishing at \(t = 0\).

(ii) \((\Omega_K, \Omega_\Lambda) = (0, 1)\): the de Sitter space (dS).
It is an empty space filled with a positive cosmological constant and with flat spatial sections. We deduce from (23) and (24) that
\[
\Omega = 0, \quad q = -1
\]
and integrating Eq. (14) gives
\[
a(t) = a_0 e^{\sqrt{\frac{2}{3}} t}.
\]
This universe is accelerating in an eternal exponential expansion.

(iii) \((\Omega_K, \Omega_\Lambda) = (1, 0)\): the Milne universe (M).
It is an empty space with no cosmological constant and with hyperbolic spatial section \((K < 0)\). We deduce from (23) and (24) that
\[
\Omega = 0, \quad q = 0
\]
and integrating Eq. (14) gives
\[
a(t) = a_0 t
\]
for the solution vanishing at \(t = 0\) and in units where \(K = a_0^2\).

It is also interesting to study the properties of the three following invariant lines which separate the phase space in disconnected regions:

(i) \(\Omega_K = 0\): The system (24) reduces to the equation of evolution for \(\Omega_\Lambda\)
\[
\Omega'_\Lambda = 3\gamma(1 - \Omega_\Lambda)\Omega_\Lambda.
\]
Thus, if initially \(\Omega_K = 0\), we stay on this line during the whole evolution and converge toward either \(\Omega_\Lambda = 1\) (i.e. the fixed point dS) or toward \(\Omega_\Lambda = 0\) (i.e. the fixed point EdS). It also follows that no integral flow lines of the system (24) can cross the line \(\Omega_K = 0\). It separates the universes with \(\Omega_K > 0\) which are compact (i.e. having a finite spatial extension) and the universes with \(\Omega_K < 0\) which are infinite (if one assumes trivial topology (23)). Crossing the line \(\Omega_K = 0\) would thus imply a change of topology. Note that if \(\gamma = 0\), the fluid behaves like a cosmological constant and thus \(\Omega'_\Lambda = 0\) since \(\Omega_K\) remains zero.
(ii) $\Omega = 0$: The system (30) reduces to the equation of evolution for $\Omega_K$

$$\Omega_K' = (3\gamma - 2)(1 - \Omega_K)\Omega_K.$$  \hfill (40)

As in the previous case, we stay on this line during the whole evolution and converge toward either $\Omega_K = 1$ (i.e. the fixed point M) or toward $\Omega_\Lambda = 0$ (i.e. the fixed point EdS). It also follows that no integral flow lines of the system (30) can cross the line $\Omega_\Lambda = 0$. Note that if $\gamma = 2/3$, the fluid behaves like a curvature term and thus $\Omega_K' = 0$ since $\Omega_\Lambda$ remains zero.

(iii) $\Omega = 0$: It is a boundary of the phase space since $\Omega$ is non negative. We now have $q = -\Omega_\Lambda$ and the system (30) reduces to

$$\Omega_\Lambda' = 2(1 - \Omega_\Lambda)\Omega_\Lambda.$$  \hfill (41)

The universe converges either toward (dS) or (M).

3.3. Stability analysis

The second step is to determine whether these fixed points are stable (i.e. attractors: A), unstable (i.e. repulsor: R) or saddle (S) points (i.e. attractor in a direction and repulsor in another). This property can be obtained by studying at the evolution of a small deviation from the equilibrium configuration. We thus decompose $\Omega_K$ and $\Omega_\Lambda$ as

$$\Omega_K \equiv \overline{\Omega}_K + \omega_K,$$
$$\Omega_\Lambda \equiv \overline{\Omega}_\Lambda + \omega_\Lambda,$$

where ($\overline{\Omega}_\Lambda, \overline{\Omega}_K$) represents the coordinates of one of the fixed points determined in the previous section and where ($\omega_\Lambda, \omega_K$) is a small deviation around this point.

Writing the system of evolution (30) as

$$\left( \begin{array}{c} \Omega_K \\ \Omega_\Lambda \end{array} \right)' = \left( \begin{array}{c} F_K(\Omega_\Lambda, \Omega_K) \\ F_\Lambda(\Omega_\Lambda, \Omega_K) \end{array} \right),$$

where $F_K$ and $F_\Lambda$ are two functions determined from (30), it can be expanded to linear order around ($\overline{\Omega}_K, \overline{\Omega}_\Lambda$) (for which $F_K$ and $F_\Lambda$ vanish) to give the equation of evolution of ($\omega_\Lambda, \omega_K$)

$$\left( \begin{array}{c} \omega_K \\ \omega_\Lambda \end{array} \right)' = \left( \begin{array}{cc} \frac{\partial F_K}{\partial \Omega_K} & \frac{\partial F_K}{\partial \Omega_\Lambda} \\ \frac{\partial F_\Lambda}{\partial \Omega_K} & \frac{\partial F_\Lambda}{\partial \Omega_\Lambda} \end{array} \right)_{(\overline{\Omega}_\Lambda, \overline{\Omega}_K)} \left( \begin{array}{c} \omega_K \\ \omega_\Lambda \end{array} \right) \equiv P_{(\overline{\Omega}_\Lambda, \overline{\Omega}_K)} \left( \begin{array}{c} \omega_K \\ \omega_\Lambda \end{array} \right).$$  \hfill (45)

The stability of a given fixed point depends on the sign of the two eigenvalues ($\lambda_{1,2}$) of the matrix $P_{(\overline{\Omega}_\Lambda, \overline{\Omega}_K)}$. If both eigenvalues are positive (resp. negative) then the fixed point is a repulsor (resp. an attractor) since ($\omega_K, \omega_\Lambda$) will respectively goes to infinity (resp. zero). In the case where the two eigenvalues have different signs, the fixed point is an attractor along the direction of the eigenvector associated with the negative eigenvalue and a repulsor along the direction of the eigenvector associated with the positive eigenvalue. We also introduce $u_{\lambda_{1,2}}$ the eigenvectors associated to the two eigenvalues which give the (eigen)–directions of attraction or repulsion.
We have to perform this stability analysis for each of the three fixed points (reminding that $\gamma \neq 0, 2/3$):

(i) **EdS fixed point**: In that case, the matrix $P$ is given by

$$P_{\text{EdS}} = \begin{pmatrix} 3\gamma - 2 & 0 \\ 0 & 3\gamma \end{pmatrix},$$

the eigenvalues of which are trivially given by $3\gamma - 2$ and $2\gamma$. We thus conclude that if $\gamma \in ]-\infty, 0]$ then EdS is an attractor, that it is a saddle point when $\gamma \in ]0, 2/3]$ and a repulsor when $\gamma \in ]2/3, +\infty]$. The matrix $P_{\text{EdS}}$ being diagonal the two eigenvectors are trivially given by

$$u_{(3\gamma)} = (0, 1), \quad u_{(3\gamma-2)} = (1, 0)$$

corresponding respectively to two invariant boundaries $\Omega_\Lambda = 0$ and $\Omega_K = 0$.

(ii) **dS fixed point**: The matrix $P$ is now given by

$$P_{\text{dS}} = \begin{pmatrix} -2 & 0 \\ 2 - 3\gamma & -3\gamma \end{pmatrix}.$$  

The eigenvalues of $P_{\text{dS}}$ are $-2$ and $-3\gamma$. It follows that the fixed point dS is never a repulsor. If $\gamma \in ]-\infty, 0]$ then dS is a saddle point and, when $\gamma \in ]0, +\infty]$, it is an attractor. The two eigenvectors are now given by

$$u_{(-3\gamma)} = (0, 1), \quad u_{(-2)} = (1, -1)$$

corresponding respectively to the two boundaries $\Omega = 0$ and $\Omega_\Lambda = 0$.

(iii) **M fixed point**: The matrix $P$ is now given by

$$P_M = \begin{pmatrix} 2 - 3\gamma & -3\gamma \\ 0 & 2 \end{pmatrix}.$$  

The eigenvalues of $P_M$ are $2$ and $2 - 3\gamma$. It follows that M is never an attractor since one of his eigenvalues is always positive. If $\gamma \in ]-\infty, 2/3]$ then M is a repulsor point and, when $\gamma \in ]2/3, +\infty]$, it is a saddle point. The two eigenvectors are now given by

$$u_{(2-3\gamma)} = (1, 0), \quad u_{(2)} = (1, -1)$$

corresponding respectively to the two boundaries $\Omega_K = 0$ and $\Omega = 0$.

Before we sum up all theses results, let us concentrate about the cases where $\gamma = 0$ or $\gamma = 2/3$ in which the matter behaves respectively either as a cosmological constant or as a curvature term. As a consequence $\Omega$ can be absorbed in a redefinition of either $\Omega_\Lambda$ or $\Omega_K$ and we can set $\Omega = 0$ from which it follows that (23) implies $\Omega_\Lambda + \Omega_K = 1$. In both cases, we deduce from (25) that $q = \Omega_K - 1 = -\Omega_\Lambda$ so that

$$\Omega_K' = 2(\Omega_K - 1)\Omega_K, \quad \Omega_\Lambda' = 2(1 - \Omega_\Lambda)\Omega_\Lambda$$

which are not independent equations due to the constraint $\Omega_\Lambda + \Omega_K = 1$. Thus, for $\gamma = 0$ or $\gamma = 2/3$, the two fixed points are either (M) or (dS) which are respectively a repulsor and an attractor.
Table 3. Stability properties of the three fixed points (EdS, dS and M) as a function of the polytropic index $\gamma$. (A: attractor, R: repulsor and S: saddle point)

| $\gamma$ | $(-\infty, 0]$ | $[0, 2/3]$ | $[2/3, +\infty]$ |
|-----------|-----------------|------------|-------------------|
| EdS       | A               | S          | N.A.              |
| dS        | S               | A          | A                 |
| M         | S               | R          | R                 |

Figure 1. The fixed points and their stability depending of the value of the index $\gamma$: (a) $\gamma < 0$, (b) $0 \leq \gamma < 2/3$ and (c) $\gamma \geq 2/3$.

As a conclusion of this study, we sum up the properties of the three spacetimes as a function of the polytropic index of the cosmic fluid in table 3 and in figure 1, we depict the fixed points, their directions of stability and instability as well as the invariant boundary in the plane $(\Omega, \Omega)$. Indeed, the attractor solution can be guessed directly from Eq. (14) and the behavior (19) of the density with the scale factor since if $\gamma < 0$ the matter energy density scales as $a^{-3\gamma}$ and comes to dominate over the cosmological constant (scaling as $a^0$) and the curvature (scaling as $a^{-2}$). On the other hand the cosmological constant always finishes by dominating if $\gamma > 0$. The curvature can never dominates in the long run since it will be caught up by either the matter or the cosmological constant.

4. Numerical examples

The full phase space picture can be obtained only through a numerical integration of the system (31) by using an implicit fourth order Runge–Kutta method [22].

Ordinary matter such as a pressureless fluid or a radiation fluid has $\gamma > 1$ and we first consider this case on figure 2 where we depict the phase space both in the $(\Omega, \Omega)$
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where the analytic study of the fixed points was performed but also in the plane \((\Omega, \Omega)\) for complementarity. On figure 2, we consider the case where \(0 < \gamma < 2/3\) which can corresponds to a scalar field slowly rolling down its potential or a tangle of domain strings (for which \(\gamma = 1/3\)) and we finish by the more theoretical case where \(\gamma < 0\) on figure 4 for which we know no simple physical example (see however [30]).
5. Discussion and conclusions

To discuss the naturalness of the initial conditions leading to our observed universe, we have to add the actual observational measures in the plane \((\Omega, \Omega_\Lambda)\) and trace them back to estimate the domain in which our universe has started. This required (i) to know what are the constraints on the cosmological constant and the curvature of the universe and (ii) determine the age of the universe, i.e. the time during which we must integrate back.

It is not the purpose of this article to detail the observational methods used in cosmology and a description can be found e.g. [19]; we now just sum up what is thought to be the current status of these observations. The current observational data such as the cosmic microwave background measurements [24], the Type Ia supernovae data [25], large scale velocity fields [26], gravitational lensing [27] and the measure of the mass to light ratio [28] tend to show that

\[ \Omega_0 \sim 0.3, \quad \Omega_\Lambda \sim 0.7. \] (53)

We refer the reader to the review by Bahcall et al. [29] for a combined study of these data and a description of all the observation methods. Let us just keep in mind that we are close to the line \(\Omega_K = 0\) and let us consider the safe area of parameter such that

\[ D_0 : \{ \Omega_0 \in [0.1, 0.5], \quad \Omega_\Lambda \in [0.5, 0.9] \}. \] (54)

and let us determinate the initial conditions allowed by these observations.

For that purpose, we need to integrate the system (30) back in time during a time equal to the age of the universe. Today, the matter content of the universe is dominated by a pressureless fluid, the energy density of which is obtained once \(\Omega_0\) has been chosen.
and is
\[ \rho_{\text{mat}} = \frac{3H_0^2}{\kappa} \Omega_0 \left( \frac{a}{a_0} \right)^{-3} = 1.80 \times 10^{-29} \Omega_0 h^2 \left( \frac{a}{a_0} \right)^{-3} \text{g.cm}^{-3} \] (55)
where \( H_0 = 100h \text{ km/s/Mpc} \) is the Hubble constant today. The energy density of the radiation is obtained by computing the energy contained in the cosmic microwave background which is the dominant contribution to the radiation in the universe. Since it is a black body with temperature \( \Theta_0 = 2.726 \text{ K} \), we deduce, from the Stephan-Boltzmann law, that
\[ \rho_{\text{rad}} = 4.47(1 + f_\nu) \times 10^{-34} \left( \frac{a}{a_0} \right)^{-4} \text{g.cm}^{-3} \] (56)
where \( f_\nu = 0.68 \) is a factor to take into account the contribution of three families of neutrinos [19]. The radiation was thus dominating over the matter for scale factors smaller than \( a_{\text{eq}} \) at which \( \rho_{\text{mat}} = \rho_{\text{rad}} \) and thus given by
\[ \frac{a_{\text{eq}}}{a_0} \simeq \frac{4.5 \times 10^{-5}}{\Omega_0 h^2}. \] (57)
We can integrate back until the Planck era for which \( a_{\text{Pl}}/a_0 \sim 10^{-30} \) and can thus approximate \( \gamma \) by
\[ \gamma = \begin{cases} 1 & a_{\text{eq}} \leq a \leq a_0 \\ 4/3 & a_{\text{Pl}} \leq a \leq a_{\text{eq}} \end{cases} \] (58)
which is a good approximation for \( \gamma = 1 + 1/3(1 + a/a_{\text{eq}}) \). In figure 5, we depict the domain \( D_0 \) of current observational values and its inverse image by the system (30) at the beginning of the matter era and at the end Planck era.

To illustrate this fine tuning problem analytically, let us just consider the simplest case where \( \Omega_\Lambda = 0 \) for which the evolution of \( \Omega \) is simply given by
\[ \Omega' = (3\gamma - 2)\Omega(\Omega - 1) \] (59)
the solution of which is
\[ \Omega = \frac{1}{1 + \frac{\Omega_{K_0} \frac{a}{a_0}}{\Omega_0 h^2} \left( \frac{a}{a_0} \right)^{3\gamma - 2}} \] (60)
and thus,
\[ \Omega = \begin{cases} \frac{1}{1 + \frac{\Omega_{K_0} \frac{a}{a_0}}{\Omega_0 h^2} \left( \frac{a}{a_0} \right)^{3\gamma - 2}} & a_{\text{eq}} \leq a \leq a_0 \\ \frac{1}{1 + \frac{\Omega_{K_0} \frac{a}{a_0}}{\Omega_0 h^2} \left( \frac{a}{a_0} \right)^{3\gamma - 2}} & a_{\text{Pl}} \leq a \leq a_{\text{eq}} \end{cases} \] (61)
From the observational data we get that \( \Omega_{K_0} \sim O(10^{-1}) \) and thus \( \Omega_0 \sim O(1) \) from which we deduce that
\[ \Omega_K|_{a=a_{\text{eq}}} \sim O(10^{-4}), \quad \Omega_K|_{a=a_{\text{Pl}}} \sim O(10^{-58}). \] (62)
This illustrate that a almost flat universe requires a fine tuning of the curvature, which is a consequence that \( \Omega_K = 0 \) is a repulsor in both the radiation and the matter era.
A solution to solve this fine tuning problem would be to add a phase prior to the radiation era in which \( \gamma \leq 0 \) so that \( \Omega_K = 0 \) becomes an attractor and to tune the duration of this era such as to have the correct initial conditions. Then, during the standard evolution EdS becomes repulsor and we evolve toward dS staying close to the line \( \Omega_K = 0 \) hence explaining current observations. Inflation is a realisation of such a scenario. What inflation really does is to change the stability property of the fixed points and invariant boundaries of the system (30). Hence during this period where a fluid with negative pressure is dominating we are attracted close to the line \( \Omega_K = 0 \) and the closer the longer this phase lasts. We then switch to a phase with normal matter and start to drift away due to repulsive property of EdS. Nevertheless, inflation does more than just explaining where we stand in this phase space, it also gives an explanation for the observed structures (galaxies, clusters...) of our universe, but this is beyond the scope of this article.

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[1] E. Milne, “A Newtonian expanding universe”, Quarterly J. of Math. 5 (1934) 64.
[2] W.H. McCrea and E. Milne, “Newtonian universes and the curvature of space”, Quarterly J. of Math. 5 (1934) 73.
A Dynamical Study of the Friedmann Equations

[3] R. Stabell and S. Refsdal, “Classification of general relativistic world models”, Mon. Not. R. Astron. Soc. 132 (1966) 379.
[4] M.S. Madsen and G.F.R. Ellis, Mon. Not. R. Astron. Soc. 234 (1988) 67.
[5] M.S. Madsen, J.-P. Minoso, J.A. Butcher, and G.F.R. Ellis, Phys. Rev. D46 (1992) 1399.
[6] M. Goliath and G.F.R. Ellis, “Homogeneous cosmology with a cosmological constant”, Phys. Rev. D60 (1999) 023502.
[7] J. Wainwright and G.F.R. Ellis, Dynamical systems in Cosmology, (Cambridge University Press, Cambridge, 1997).
[8] E.P. Harrison, Cosmology: the science of the universe (Cambridge University Press, Cambridge, 2000).
[9] R.C. Tolman, Relativity, thermodynamics and cosmology, (Clarendon Press, Oxford, 1934).
[10] D. Layzer, “On the significance of Newtonian cosmology”, Astron. J. 59 (1954) 268.
[11] W.H. McCrea, “On the significance of Newtonian cosmology”, Astron. J. 60 (1955) 271.
[12] C. Callan, R.H. Dicke, and P.J.E. Peebles, “Cosmology and Newtonian Dynamics”, Am. J. Phys. 33 (1965) 105.
[13] W.B. Bonnor, “Jean’s formula for gravitational instability”, Mon. Not. R. Astron. Soc. 117 (1957) 104.
[14] B. Carter and B. Gaffet, “Standard covariant formulation for perfect fluid dynamics”, J. Fluid Mech. 186 (1987) 1.
[15] L. Landau and E. Lifchitz, Théorie des champs, (Mir, Moscow, 1989).
[16] E.P. Harrison, “Newton and the infinite universe”, Physics Today 39 (1986) 24.
[17] E. Cartan, Ann. Sci. de l’Ecole Normale Supérieure 40 (1923) 325; ibid, 41 (1924) 1.
[18] C. Rüede and N. Straumann, “On Newton–Cartan Cosmology”, [gr-qc/9604054].
[19] P.J.E. Peebles, Principles of Physical Cosmology (Princeton Series in Physics, Princeton, New Jersey, 1993).
[20] V. Faraoni, “Solving for the dynamics of the universe”, Am. J. Phys. 67 (1999) 732.
[21] C. Bender and S. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill International Editions
[22] W. Press, S. Teukolsky, W. Vetterling and B. Flannery, Numerical Recipes 2nd edition, Cambridge University Press.
[23] M. Lachièze–Rey and J.–P. Luminet, “Cosmic Topology”, Phys. Rep. 254 (1995) 135; J-P. Uzan, “What do we know and what can we learn about the topology of the universe?”, Int. Journal of Theor. Physics, 36 (1997) 2439.
[24] P. de Bernardis et al., Nature 404 (2000) 955; A.E. Lange et al., astro-ph/0005004; A.H. Jaffe et al., astro-ph/0007333.
[25] S. Perlmutter et al., Bull. Am. Astron. Soc. 29 (1997) 1351; A.G. Riess et al., Astron. J. 116 (1998) 1009; ibid, Astron. J. 117 (1999) 107; S. Perlmutter et al., Astrophys. J. 483 (1997) 565.
[26] M. Strauss and J. Willick, Phys. Rep. 261 (1995) 271; R. Juszkiewicz et al., Science 287 (2000) 109.
[27] Y. Mellier, Ann. Rev. Astron. Astrophys. 37 (1999) 127.
[28] J-P. Ostriker, P.J.E. Peebles, and A. Yahil, Astrophys. J. 193 (1974) L1; N.A. Bahcall, L.M. Lubin, and V. Dorman, Astrophys. J. 447 (1995) L81.
[29] N. Bahcall, J.P. Ostriker, S. Perlmutter, and P.J. Steinhardt, Science 284 (1999) 1481.
[30] A. Riazuelo and J.-P. Uzan, “Quintessence and gravitational waves”, Phys. Rev. D62 (2000) 083506, astro-ph/0004155.