KNOTS AND WORDS

VLADIMIR TURAEV

Abstract. Knots and links are interpreted as homotopy classes of nanowords and nanophrases in an alphabet consisting of 4 letters. Similar results hold for curves on surfaces. We also discuss versions of the Jones link polynomial and the link quandles for nanophrases.

1. Introduction

C. F. Gauss [Ga] introduced a method allowing to encode closed planar curves by words of a certain type called now Gauss words. This method extends to planar knot diagrams and their isotopy. This gives a description of the set of isotopy types of classical knots in terms of words and their transformations.

To state our results we need to generalize both knots and words. An appropriate generalization of knots is provided by long virtual knots, see [Kau2], [GPV]. We shall use an equivalent formulation in terms of stable equivalence classes of (pointed oriented) knot diagrams on surfaces, see [KK], [CKS]. The set of these equivalence classes $\mathcal{K}$ contains the set $\mathcal{S}$ of isotopy classes of oriented knots in $S^3$.

On the combinatorial side, the Gauss words generalize to so-called nanowords, see [Tu2]. Our main result is a bijection between $\mathcal{K}$ and the set of (appropriately defined) homotopy classes of nanowords in the alphabet consisting of 4 letters. Note that the image of $\mathcal{S} \subset \mathcal{K}$ under this bijection can be described via the Rosenstiehl theorem [Ro] giving necessary and sufficient conditions for the curve corresponding to a given Gauss word to be planar. Thus, isotopy classification of knots in $S^3$ is a special instance of homotopy classification of words. Similarly, classical and virtual links can be interpreted as nanophrases. This gives a broader perspective to knot theory. A number of methods of knot theory including the Kauffman bracket polynomial, the Jones polynomial, the knot quandle etc. can be extended to the more general setting of words and phrases in arbitrary alphabets.

The theory of knots and links being very reach, it is natural to study related simpler objects. One simplification of links in cylinders over surfaces is obtained by projecting them to the surfaces, i.e., by forgetting the over/under-crossing information in link diagrams. The stable equivalence classes of curves on surfaces were studied in [Kad], [Tu1]. We describe these classes in terms of nanowords and nanophrases over an alphabet consisting of only two letters. We also suggest two different simplifications of knot theory in terms of nanophrases over a 2-letter alphabet. The resulting combinatorial objects are called pseudo-links and quasi-links. Every oriented link in a cylinder over an oriented surface projects
to a (multi-component) curve on the surface, to a pseudo-link and to a quasi-link. Their distinctive features are contained in the following facts: the Jones polynomial of a link depends only on the underlying pseudo-link; the fundamental group of the 2-fold branched covering of a classical link depends only on the underlying quasi-link.

The plan of the paper is as follows. In Sect. 2 we recall nanowords and their homotopy. In Sect. 3 we identify the stable equivalence classes of pointed curves on surfaces with homotopy classes of nanowords in a 2-letter alphabet. In Sect. 4 we identify the stable equivalence classes of pointed knot diagrams on surfaces with homotopy classes of nanowords in a 4-letter alphabet. In Sect. 5 we show how to get rid of the base points. In Sect. 6 we extend these results to links. In Sect. 7 we introduce pseudo-links and quasi-links. In Sect. 8 we discuss the bracket polynomial and the Jones polynomial. In Sect. 9 and 10 we discuss the knes of nanophrases.

Conventions. Throughout the paper, all surfaces, curves, knots, links, and knot and link diagrams are oriented, unless explicitly stated to the contrary.

2. NANOWORDS AND HOMOTOPY

2.1. Words and nanowords. An alphabet is a set and letters are its elements. A word of length \( n \geq 1 \) in an alphabet \( A \) is a mapping \( w: \hat{n} \to A \) where \( \hat{n} = \{1, 2, ..., n\} \). A word \( w: \hat{n} \to A \) is usually encoded by the sequence of letters \( w(1)w(2) \cdots w(n) \). A word \( w: \hat{n} \to A \) is a Gauss word if each element of \( A \) is the image of precisely two elements of \( \hat{n} \).

For a set \( \alpha \), an \( \alpha \)-alphabet is a set \( A \) endowed with a mapping \( A \to \alpha \) called projection. The image of \( A \in A \) under this mapping is denoted \( |A| \). A nanoword over \( \alpha \) is a pair \((\alpha, w)\) where \( \alpha \) is a Gauss word in the alphabet \( A \). For example, any Gauss word \( w \) in the alphabet \( \alpha \) yields a nanoword \((\alpha, w)\), see [Tu2] for more examples and further details. By definition, there is a unique empty nanoword \( \emptyset \) of length 0.

An isomorphism of \( \alpha \)-alphabets \( \alpha_1, \alpha_2 \) is a bijection \( f: \alpha_1 \to \alpha_2 \) such that \( |\alpha_1| = |f(\alpha_1)| \) for all \( \alpha_1 \in \alpha_1 \). Two nanowords \((\alpha_1, w_1)\) and \((\alpha_2, w_2)\) over \( \alpha \) are isomorphic if there is an isomorphism of \( \alpha \)-alphabets \( f: \alpha_1 \to \alpha_2 \) such that \( w_2 = f w_1 \).

2.2. Homotopy of nanowords. A homotopy data consists of a set \( \alpha \) with involution \( \tau: \alpha \to \alpha \) and a set \( S \subset \alpha^3 = \alpha \times \alpha \times \alpha \). The following three transformations of nanowords over \( \alpha \) are called \( S \)-homotopy moves or simply homotopy moves.

(1). The first move applies to any nanoword of the form \((\alpha, xAAy)\) where \( \alpha \in A \) and \( x, y \) are words in the alphabet \( \alpha' = \alpha - \{A\} \). It transforms \((\alpha, xAAy)\) into the nanoword \((\alpha', xy)\) where the structure of an \( \alpha \)-alphabet in \( \alpha' \) is obtained by restricting the one in \( \alpha \). Note that \( xy \) is a Gauss word in the alphabet \( \alpha' \).

The inverse move \((\alpha', xy) \mapsto (\alpha' \cup \{A\}, xAAy)\) adds a new letter \( A \) with arbitrary \( |A| \in \alpha \) and replaces the Gauss word \( xy \) in the \( \alpha \)-alphabet \( \alpha' \) with \( xAAy \).
where $A, B, C \in A$ with $|B| = \tau(|A|)$ and $x, y, z$ are words in the alphabet $A' = A - \{A, B\}$. This nanoword is transformed into $(A', xyz)$ where the structure of an $\alpha$-alphabet in $A'$ is obtained by restricting the one in $A$.

The inverse move $(A', xyz) \mapsto (A' \cup \{A, B\}, xAByBAz)$ adds two new letters $A, B$ with arbitrary $|A| \in \alpha$ and $|B| = \tau(|A|)$ and replaces the Gauss word $xyz$ in the alphabet $A'$ with $xAByBAz$.

(3) The third move applies to a nanoword of the form $(A, xAByACzBCt)$ where $A, B, C \in A$ are distinct letters such that $(|A|, |B|, |C|) \in S$ and $x, y, z, t$ are words in the alphabet $A - \{A, B, C\}$. The move transforms $(A, xAByACzBCt)$ into $(A, xBAyCAzCBt)$. The inverse move applies if $(|A|, |B|, |C|) \in S$ and transforms a nanoword $(A, xBAyCAzCBt)$ into $(A, xAByACzBCt)$.

Two nanowords over $\alpha$ are $S$-homotopic if they can be obtained from each other by a finite sequence of homotopy moves (1) – (3), the inverse moves, and isomorphisms. The relation of $S$-homotopy is denoted $\simeq_S$. The set of $S$-homotopy classes of nanowords over $\alpha$ is denoted $\mathcal{N}(\alpha, S)$. (The involution $\tau$ is omitted in this notation for shortness.)

Recall the following two lemmas from [Tu2, Sect. 3.2.]

**Lemma 2.2.1.** Let $A, B, C$ be distinct letters in an $\alpha$-alphabet $A$ and let $x, y, z, t$ be words in the alphabet $A - \{A, B, C\}$ such that $xyzt$ is a Gauss word in this alphabet. Then

(i) $(A, xAByCAzBCt) \simeq_S (A, xBAyACzCBt)$ for $(|A|, \tau(|B|), |C|) \in S$;

(ii) $(A, xBAyCAzCBt) \simeq_S (A, xAByACzBCt)$ for $(\tau(|A|), |B|, |C|) \in S$;

(iii) $(A, xAByACzCBt) \simeq_S (A, xBAyCAzCBt)$ for $(\tau(|A|), |B|, |C|) \in S$.

A homotopy data $(\alpha, S)$ is admissible if $S \cap (\alpha \times \alpha \times \alpha) \neq \emptyset$ for all $\alpha \in \alpha$. For instance, if $S$ contains the diagonal $\{(a, a, a)\}_{a \in \alpha}$ of $\alpha^3$, then $(\alpha, S)$ is admissible.

**Lemma 2.2.2.** Let $(A, xAByABz)$ be a nanoword over $\alpha$ where $A, B \in A$ with $|B| = \tau(|A|)$ and $x, y, z$ are words in the alphabet $A - \{A, B\}$. If $(\alpha, S)$ is admissible, then $(A, xAByABz) \simeq_S (A - \{A, B\}, xyz)$.

A morphism $(\alpha, S) \to (\alpha', S')$ between two homotopy data is an equivariant mapping $f : \alpha \to \alpha'$ such that $(f \times f \times f)(S) \subset S'$. Given $f$, we can transform a nanoword $(A, w)$ over $\alpha$ into a nanoword $(\alpha', w')$ over $\alpha'$ where $\alpha' = A$ as sets, $w' = w$, and the projection $\alpha' \to \alpha'$ is the composition of the projection $\alpha' = A \to \alpha$ with $f$. This transformation is compatible with homotopy and induces a monoid homomorphism $\mathcal{N}(\alpha, S) \to \mathcal{N}(\alpha', S')$.

3. Curves versus words

3.1. Curves. By a curve, we mean the image of a generic immersion of an oriented circle into an oriented surface. The word “generic” means that the curve has only a finite set of self-intersections which are all double and transversal. A curve is pointed if it is endowed with a base point (the origin) which is not a self-intersection. Two pointed curves are stably homeomorphic if there is a homeomorphism of their regular neighborhoods in the ambient surfaces mapping the
first curve onto the second one and preserving the origin of the curve and the orientations of the curve and the surface. In particular, attaching a 1-handle to the ambient surface away from a curve or removing such a handle does not change the stable homeomorphism type of the curve.

Following [KK], [CKS], we call two pointed curves \( \text{stably equivalent} \) if they can be related by a finite sequence of the following transformations: (i) replacing the curve with a stably homeomorphic one; (ii) homotopy of the curve in its ambient surface away from the origin. Note that such a homotopy may push a branch of the curve across another branch or a double point but not across the origin of the curve.

Denote \( \mathcal{C} \) the set of stable equivalence classes of pointed curves. This set is a monoid with multiplication defined by connected sum at the origin. We are far from understanding the algebraic structure of \( \mathcal{C} \). Several stable equivalence invariants of pointed curves were introduced in [Tu1], [SW] where curves are studied in terms of virtual strings.

We show now that the study of \( \mathcal{C} \) is an instance of homotopy theory of words.

3.2. Homotopy data \((\alpha_0, S_0)\). Consider the homotopy data \((\alpha_0, S_0)\) where \( \alpha_0 \) is the set \{a, b\} with involution \( \tau : \alpha_0 \to \alpha_0 \) permuting \( a, b \) and \( S_0 = \{(a, a, a)\}_{a \in \alpha_0} \) is the diagonal. This homotopy data is admissible in the sense of Sect. 2.2.

**Theorem 3.2.1.** There is a canonical bijection \( \mathcal{C} = \mathcal{N}(\alpha_0, S_0) \).

**Proof.** We associate with any pointed curve \( f \) a nanoword \( w(f) \) over \( \alpha_0 \). Let us label the double points of \( f \) by (distinct) letters \( A_1, ..., A_n \) where \( n \) is the number of double points. Starting at the origin of \( f \) and following along \( f \) in the positive direction we write down the labels of all double points until the return to the origin. Since every double point is traversed twice, this gives a Gauss word \( w(f) \) in the alphabet \( \mathcal{A} = \{A_1, ..., A_n\} \). Let \( t_1^i \) (resp. \( t_2^i \)) be the tangent vector to \( f \) at the crossing point labeled by \( A_i \) appearing at the first (resp. second) passage through this crossing. Set \( |A_i| = a \) if the pair \((t_1^i, t_2^i)\) is positively oriented and \( |A_i| = b \) otherwise. This makes \( \mathcal{A} \) into an \( \alpha_0 \)-alphabet and makes \( w \) into a nanoword over \( \alpha_0 \). This nanoword is well defined up to isomorphism.

We claim that stably equivalent pointed curves give rise to \( S_0 \)-homotopic nanowords. Stable homeomorphisms of curves preserve the nanoword up to isomorphism. We need to show that a homotopy of a curve \( f \) in its ambient surface away from the origin does not change the \( S_0 \)-homotopy class of \( w(f) \). Such a homotopy can be obtained by an ambient isotopy and a finite sequence of local deformations shown in Figure 1 and the inverse deformations. It is understood that all deformations in Figure 1 are effected away from the origin of \( f \).

An ambient isotopy does not change \( w(f) \). A local deformation of the first type changes \( w(f) \) via the first homotopy move. Depending on the orientations of the two branches of \( f \), a local deformation of the second type changes \( w(f) \) via one of the moves \( xAByBAz \to xyz \) or \( xAByABz \to xyz \) where \( |B| = \tau(|A|) \).

In both cases the \( S_0 \)-homotopy class of \( w(f) \) is preserved. Consider the local deformation of the third type. It suffices to consider the case where all three
branches are oriented upwards. (The deformations involving other orientations of the branches can be obtained as compositions of this one with ambient isotopy and local deformations of the second type.) There are 6 cases to consider depending on the order in which one traverses the three branches involved. Let \( I \) (resp. \( II, III \)) be the branch connecting the leftmost (resp. intermediate, rightmost) bottom point to the rightmost (resp. intermediate, leftmost) top point. If one traverses these branches in the order \( I, II, III \) (resp. \( III, II, I \)), then \(|A| = |B| = |C| = a\) (resp. \(|A| = |B| = |C| = b\)) and the deformation changes \( w(f) \) via the third homotopy move (resp. its inverse). If one traverses these branches in the order \( II, III, I \) (resp. \( I, III, II \)), then the deformation changes \( w(f) \) via the second homotopy from Lemma 2.2.1 (resp. its inverse). Finally, if one traverses these three branches in the order \( II, I, III \) (resp. \( III, I, II \)), then the deformation changes \( w(f) \) via the third homotopy from Lemma 2.2.1 (resp. its inverse).

We can conclude that the formula \( f \mapsto w(f) \) defines a mapping \( W : C \to \mathcal{N}(\alpha_0, S_0) \). We claim that it is bijective. It is easy to see that for every nanoword \( w \) over \( \alpha_0 \) there is a unique (up to stable homeomorphism) pointed curve \( f \) such that \( w(f) = w \). Indeed, knowing \( w \) we can uniquely recover an oriented regular neighborhood of such a curve in the ambient surface (this well known construction is described in detail in \([Tu1]\), Sect. 4.1 in terms of virtual strings. Note that the notion of an open virtual string is equivalent to the one of an isomorphism class of a nanoword over \( \alpha_0 \).) This implies that \( W \) is surjective.

The injectivity of \( W \) follows from the fact that if two nanowords are related by the \( i \)-th homotopy move with \( i = 1, 2, 3 \) then they can be represented by pointed curves related by the \( i \)-th deformation in Figure 1 (effected away from the origin). Here for \( i = 2 \) the two branches in Figure 1 are oriented in opposite directions and for \( i = 3 \) all the branches are oriented upwards and traversed in the order \( I, II, III \).

4. Knots versus words

4.1. Knot diagrams. By a knot diagram, we mean a (generic oriented) curve on an (oriented) surface such that at each crossing point of the curve one of the two branches is distinguished. The distinguished branch is the over-crossing and the second branch is the under-crossing. A knot diagram is pointed if it is endowed with a base point (the origin) distinct from the crossing points. Two pointed knot diagrams are stably homeomorphic if there is a homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first diagram onto the second.
one and preserving the origin, the over/undercrossings, and the orientations of the surface and the curve.

Following [KK], [CKS], we call two pointed knot diagrams \textit{stably equivalent} if they can be related by a finite sequence of the following transformations: (i) replacing a knot diagram with a stably homeomorphic one; (ii) the usual Reidermeister moves on a knot diagram in its ambient surface away from the origin. The latter moves may push a branch of the diagram above or below a double point or another branch but not across the origin. It should be stressed that removing a closed subset from the ambient surface away from a knot diagram or attaching a 1-handle away from the diagram does not change the stable equivalence type of the diagram.

Denote \( \mathcal{K} \) the set of stable equivalence classes of pointed knot diagrams. The elements of \( \mathcal{K} \) bijectively correspond to long virtual knots in the sense of [Kau2], [GPV]. Every (oriented) knot \( K \subset S^3 \) determines an element of \( \mathcal{K} \) obtained by presenting \( K \) by a diagram on \( S^2 \) and picking an arbitrary base point. This yields a well defined mapping from the set of isotopy classes of classical knots into \( \mathcal{K} \). This mapping is essentially injective, see [Kau2], [GPV].

Forgetting the over/under-crossing information we obtain a natural projection \( K \to C \). We now interpret \( K \) in terms of words.

4.2. \textbf{Homotopy data} \((\alpha_*, S_*)\). Consider the homotopy data \((\alpha_*, S_*)\) where \( \alpha_* = \{a_+, a_-, b_+, b_-\} \) with involution \( \tau : \alpha_* \to \alpha_* \) defined by \( \tau(a_+) = b_+, \tau(b_+) = a_+ \) and \( S_* \subset \alpha_* \times \alpha_* \times \alpha_* \) consists of the following 12 triples:

\[
(a_+, a_+, a_+) , (a_+, a_+, a_-) , (a_+, a_+ , a_+), (b_+, b_+, b_+) , (b_+, b_+, b_-), (b_+, b_-, b_+). \]

This homotopy data is admissible in the sense of Sect. 2.2.

Forgetting the signs, we obtain a projection \( \alpha_* \to \alpha_0 \). Applying it, we can transform a nanoword over \( \alpha_* \) into a nanoword over \( \alpha_0 \). This induces a monoid homomorphism \( \mathcal{N}(\alpha_*, S_*) \to \mathcal{N}(\alpha_0, S_0) \).

\textbf{Theorem 4.2.1.} There is a canonical bijection \( \mathcal{K} = \mathcal{N}(\alpha_*, S_*) \). Under this bijection, the monoid homomorphism \( \mathcal{N}(\alpha_*, S_*) \to \mathcal{N}(\alpha_0, S_0) \) corresponds to the natural projection \( K \to C \).

\textbf{Proof.} The proof reproduces the proof of Theorem 3.2.1 with a few changes. We begin by associating with any pointed knot diagram \( F \) a nanoword \( w = w(F) \) over \( \alpha_* \). As usual, each crossing of \( F \) gives rise to a sign \( \pm \). It is + if the over-going branch crosses the under-going branch from left to right and \(- \) otherwise. To define \( w \), label the double points of \( F \) by (distinct) letters \( A_1, ..., A_n \) where \( n \) is the number of double points. Starting at the origin of \( F \) and following along \( F \) we write down the labels of all double points until the return to the origin. This gives a Gauss word \( w \) in the alphabet \( A = \{A_1, ..., A_n\} \). Let \( t_1^i \) (resp. \( t_2^i \)) be the tangent vector to \( F \) at the crossing labeled \( A_i \) appearing at the first (resp. second) passage through this crossing. Let $\varepsilon(i) = \pm$ be the sign of this crossing. Set $|A_i| = a_{\varepsilon(i)}$ if the pair \((t_1^i, t_2^i)\) is positively oriented and $|A_i| = b_{\varepsilon(i)}$ otherwise. This makes \( A \) into an \( \alpha_* \)-alphabet and makes \( w = w(F) \) into a nanoword over \( \alpha_* \). This nanoword is well defined up to isomorphism.
We need to verify that stably equivalent pointed knot diagrams give rise to $S_*$-homotopic nanowords. Stable homeomorphisms preserve the nanoword up to isomorphism. We need to show that the $S_*$-homotopy class of $w(F)$ is preserved under the Reidemeister moves on $F$ away from the origin. The first Reidemeister move changes $w(F)$ via the first homotopy move. Depending on the orientations of the two branches of $F$ involved in the second Reidemeister move, the nanoword $w(F)$ changes via one of the moves $(A, xAbyBAz) \mapsto (A - \{A, B\}, xy)$ or $(A, xAbyABz) \mapsto (A - \{A, B\}, xy)$ where $A, B \in A$ with $|B| = \tau(|A|)$. In both cases the $S_*$-homotopy class of $w(F)$ is preserved. Consider the third Reidemeister move. It suffices to consider the case where all three branches are oriented in the same direction, say upwards, and the signs of all crossings are $+$. (This is the classical “braid move” $\sigma_1 \sigma_2 \sigma_1 \mapsto \sigma_2 \sigma_1 \sigma_2$; the moves involving other orientations of the branches and/or other signs of crossings can be obtained as compositions of this move with second Reidemeister moves.) There are 6 cases to consider depending on the order in which one traverses the three branches involved. Let $I$ (resp. $II, III$) be the branch connecting the leftmost (resp. intermediate, rightmost) bottom point to the rightmost (resp. intermediate, leftmost) top point. If one traverses these branches in the order $I, II, III$ (resp. $III, II, I$), then $|A| = |B| = |C| = a_+$ (resp. $|A| = |B| = |C| = b_+$) and the deformation changes $w(F)$ via the second homotopy move (resp. its inverse) where we use that $(a_+, a_+, a_+) \in S_*$ (resp. that $(b_+, b_+, b_+) \in S_*$). If one traverses these three branches in the order $II, III, I$ (resp. $I, III, II$), then the deformation changes $w(F)$ via the second homotopy move from Lemma [2.21] (resp. its inverse) where we use that $(a_-, a_-, a_+ \in S_*$ (resp. that $(b_-, b_-, b_+) \in S_*$). Finally, if one traverses these branches in the order $(II, I, III)$ (resp. $III, I, II$), then the deformation changes $w(F)$ via the third homotopy move from Lemma [2.21] (resp. its inverse) where we use that $(b_+, b_-, b_-) \in S_*$ (resp. that $(a_+, a_-, a_-) \in S_*$).

Thus the formula $F \mapsto w(F)$ defines a mapping $W: K \to N(\alpha_*, S_*)$. We claim that $W$ is a bijection. As in the case of curves, for every nanoword $w$ over $\alpha_*$ there is a unique (up to stable homeomorphism) pointed knot diagram $F$ such that $w(F) = w$. Indeed it suffices to realize the underlying nanoword over $\alpha_0$ by a pointed curve and then to choose the over/under-crossings to ensure the right signs at all crossings. This implies that $W$ is surjective.

To prove the injectivity of $W$ it suffices to observe that if two nanowords over $\alpha_*$ are related by the $i$-th homotopy move with $i = 1, 2, 3$ then they can be represented by pointed knot diagrams related by the $i$-th Reidemeister move (effected away from the origin). The cases $i = 1, 2$ are straightforward. For $i = 3$ the 12 elements of the set $S_*$ lead to all 12 possible choices of over/under-crossings in the third move on Figure 1 leading to admissible Reidemeister moves (for this argument we can assume that the branches $I, II, III$ are oriented upwards and traversed either in the order $I, II, III$ or $III, II, I$).

The last claim of the theorem follows from the definitions. \hfill $\square$

4.3. Examples. The nanoword $ABCABC$ with $|A| = |C| = a_+, |B| = b_+$ represents a pointed trefoil, see Figure 2 where the thick point is the origin of the
Figure 2. Trefoil and figure eight knot

diagram. The nanoword $ABCADCB$ with $|A| = |D| = b_+, |B| = b_-, |C| = a_-$ represents a pointed figure eight knot, see Figure 2.

5. Eliminating the origin

5.1. Shifts. Fix an involution $\nu$ in a set $\alpha$ called the shift involution. The $\nu$-shift of a nanoword $(A, w : \hat{n} \to A)$ over $\alpha$ is the nanoword $(A', w' : \hat{n} \to A')$ obtained by moving the first letter $A = w(1)$ of $w$ to the end and applying $\nu$ to $|A| \in \alpha$. More precisely, $A' = (A - \{A\}) \cup \{A_\nu\}$ where $A_\nu$ is a “new” letter not belonging to $A$. The projection $A' \to \alpha$ extends the given projection $A - \{A\} \to \alpha$ by $|A_\nu| = \nu(|A|)$. The word $w'$ in the alphabet $A'$ is defined by $w' = xA_\nu y A_\nu$ for $w = Ax Ay$.

Given a homotopy data $(\alpha, S)$ and a shift involution $\nu$ in $\alpha$, we can quotient the set of nanowords over $\alpha$ by the equivalence relation generated by $S$-homotopy and $\nu$-shifts. The resulting set is denoted $N(\alpha, S, \nu)$. There is a natural projection $N(\alpha, S) \to N(\alpha, S, \nu)$ but there is no natural multiplication in $N(\alpha, S, \nu)$.

5.2. Non-pointed knots. Stable equivalence can be defined for (non-pointed) knot diagrams through repeating the definition in the pointed case but omitting all references to base points. Denote $\hat{K}$ the set of stable equivalence classes of knot diagrams. Each knot in the cylinder over a surface represents an element in $\hat{K}$ depending only on the isotopy type of the knot.

As we know, a pointed knot diagram gives rise to a nanoword in the alphabet $\alpha_* = \{a_+, a_-, b_+, b_-\}$. This nanoword is preserved when the origin is pushed along the generic part of the diagram. When the origin jumps over a double point, the nanoword is modified by the $\nu$-shift where $\nu : \alpha_* \to \alpha_*$ is the involution sending $a_\pm$ to $b_\pm$. Theorem 4.2.1 implies that $\hat{K} = N(\alpha_*, S_*, \nu)$.

Non-oriented knots can be treated similarly, we do it in the next section in a more general setting of links.

6. Nanophrases and links

6.1. Nanophrases. A nanaphrase of length $k \geq 0$ over a set $\alpha$ is a tuple consisting of an $\alpha$-alphabet $\mathcal{A}$ and a sequence of $k$ words $w_1, ..., w_k$ in the alphabet $\mathcal{A}$ such that their concatenation $w_1 w_2 \cdots w_k$ is a Gauss word in this alphabet.
We denote this nanophrase by \((A, (w_1|w_2|\cdots|w_k))\) or shorter by \((w_1|w_2|\cdots|w_k)\). Note that some of the words \(w_1, \ldots, w_k\) may be empty.

By definition, there is a unique empty nanophrase of length 0 (the corresponding \(\alpha\)-alphabet \(A\) is void).

Any nanoword \(w\) over \(\alpha\) yields a nanophrase \((w)\) of length 1. In the sequel we make no difference between nanowords and nanophrases of length 1.

Isomorphism of two nanophrases is an isomorphism of \(\alpha\)-alphabets transforming the first sequence of words into the second one. Given a homotopy data \(\alpha, \tau, S\), we define homotopy moves on nanophrases as in Sect. 6.2 with the only difference that the 2-letter sub-words \(AA, AB, BA, AC, BC\) etc. modified by these moves may belong to different words of the phrase. Isomorphisms and homotopy moves generate an equivalence relation \(\simeq_S\) of \(S\)-homotopy on the class of nanophrases over \(\alpha\). Examples:

\[(AB|AC|BC) \simeq_S (BA|CA|CB), \quad (AB|ADDCBC) \simeq_S (BA|CACB)\]

provided \((|A|, |B|, |C|) \in S\). The length of a nanophrase is preserved under \(S\)-homotopy.

Lemmas 2.2.1 and 2.2.2 extend to nanophrases with the only change that the 2-letter sub-words \(AB, BA, CA\) etc. may belong to different words of the phrase.

6.2. Operations on nanophrases. Fix a homotopy data \((\alpha, \tau, S)\) and a shift involution \(\nu\) in \(\alpha\). We define \(\nu\)-shifts, \(\nu\)-inversions, and \(\nu\)-permutations of words in a nanophrase \(P = (A, (w_1|w_2|\cdots|w_k))\) over \(\alpha\).

We can \(\nu\)-shift the \(i\)-th word \(w_i\) in \(P\) through moving the first letter, say \(A\), of \(w_i\) to the end of \(w_i\) keeping \(|A| \in \alpha\) if \(A\) appears in \(w_i\) only once and applying \(\nu\) to \(|A|\) if \(A\) appears in \(w_i\) twice. All other words in \(P\) are preserved.

To define inversions, we need more notation. For a word \(w\) in \(A\), denote by \(\mathcal{A}_w\) the same alphabet \(A\) with new projection \(|\ldots|_w\) to \(\alpha\) defined as follows: for \(A \in \mathcal{A}\) set \(|A|_w = \tau(|A|)\) if \(A\) occurs in \(w\) once, \(|A|_w = \nu(|A|)\) if \(A\) occurs in \(w\) twice, and \(|A|_w = |A|\) otherwise. The \(\nu\)-inversion of the \(i\)-th word in \(P\) replaces \(w_i\) with the opposite word \((w_i)^-\) obtained by reading \(w_i\) from right to left and replaces the \(\alpha\)-alphabet \(A\) with \(\mathcal{A}_w\). All other words in \(P\) are preserved.

The words in \(P\) can be permuted in an arbitrary way, producing thus new nanophrases over \(\alpha\). We will need more sophisticated permutations of words depending on \(\nu\). We begin with notation. For two words \(u, v\) in the alphabet \(\mathcal{A}\), consider the mapping \(\mathcal{A} \rightarrow \alpha\) sending \(A \in \mathcal{A}\) to \(\nu(|A|) \in \alpha\) if \(A\) appears both in \(u\) and \(v\) and sending \(A\) to \(|A|\) otherwise. This mapping makes the set \(\mathcal{A}\) into an \(\alpha\)-alphabet denoted \(\mathcal{A}_{\nu\circ\alpha}\). For \(i = 1, \ldots, k - 1\), the \(\nu\)-permutation of the \(i\)-th and \((i + 1)\)-st words transforms \(P = (A, (w_1|w_2|\cdots|w_k))\) into the nanophrase

\[(\mathcal{A}_{w_i\circ w_{i+1}}, (w_1|w_2|\cdots|w_{i-1}|w_{i+1}|w_i|w_{i+2}|\cdots|w_k)).\]

This operation is involutive. The \(\nu\)-permutations define an action of the symmetric group \(S_k\) on the set of nanophrases of length \(k\).

Denote \(\mathcal{P}(\alpha, S, \nu)\) the set of nanophrases over \(\alpha\) quotiented by the equivalence relation generated by \(S\)-homotopy, \(\nu\)-permutations and \(\nu\)-shifts on words. Denote
\( \mathcal{P}_u(\alpha, S, \nu) \) the set of nanophrases over \( \alpha \) quotiented by the equivalence relation generated by the same operations and the \( \nu \)-inversions.

### 6.3. Link diagrams

*Link diagrams* on (oriented) surfaces are defined in the same way as knot diagrams with the difference that they may be formed by several (transversal generic oriented closed) curves rather than only one curve. These curves are *components* of the diagram. A link diagram is *pointed* if each component is endowed with a base point (the origin) distinct from the crossing points of the diagram. A link diagram is *ordered* if its components are numerated by \( 1, 2, \ldots, k \) where \( k \) is the number of the components. Two ordered pointed link diagrams are *stably homeomorphic* if there is an orientation preserving homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first diagram onto the second one and preserving the over/undercrossings and the order, the origins and the orientations of the components.

The *stable equivalence* of ordered pointed link diagrams is generated by the same transformations as in the case of knots. These transformations should preserve the order and the origins of the components; the Reidemeister moves are allowed only away from the origins.

Denote \( \mathcal{L} \) the set of stable equivalence classes of ordered pointed link diagrams. Recall the homotopy data \( \alpha^*, \tau, S^* \) defined in Sect. 4.2.

**Theorem 6.3.1.** There is a canonical bijection \( \mathcal{L} = \mathcal{P}(\alpha^*, S^*) \).

The proof of this theorem is analogous to the proof of Theorem 4.2.1. To write down the nanophrase associated with an ordered pointed link diagram one goes along the first component starting at its origin, then along the second component, etc.

Forgetting the order and the origins of link components, we obtain a notion of stable equivalence for (non-ordered non-pointed) link diagrams. Denote \( \hat{\mathcal{L}} \) the set of equivalence classes of link diagrams. As in the case of knots, each link in the cylinder over a surface represents an element in \( \hat{\mathcal{L}} \) depending only on the isotopy class of this link. Theorem 6.3.1 implies that \( \hat{\mathcal{L}} = \mathcal{P}(\alpha^* S^* \nu) \) where \( \nu : \alpha^* \to \alpha^* \) is the involution sending \( a^\pm \) to \( b^\pm \).

Additionally forgetting link orientations, we obtain a notion of stable equivalence for unoriented link diagrams (on oriented surfaces). Denote \( \hat{\mathcal{L}}_u \) the set of equivalence classes of unoriented link diagrams. Theorem 6.3.1 implies that \( \hat{\mathcal{L}}_u = \mathcal{P}_u(\alpha^* S^* \nu) \).

### 6.4. Remarks

1. **Theorem 6.3.1** can be extended to framed links. Consider the involution \( \nu \tau = \tau \nu : \alpha^* \to \alpha^* \) sending \( a^\pm, b^\pm \) to \( a^\mp, b^\mp \), respectively. A *framed homotopy* \( \sim \) of nanophrases over \( \alpha^* \) is defined as the \( S^* \)-homotopy with the first homotopy move replaced by the following “framed homotopy move” on a nanoword in a nanophrase: \( x A y B z \to x y z \) provided \( |A| = \nu \tau(|B|) \). Framed homotopic nanophrases are \( S^* \)-homotopic; the converse is in general not true. Lemmas 2.2.1 and 2.2.2 for \( \alpha = \alpha^* \) extend to this setting by replacing \( \simeq_S \) with \( \sim \). The proof of Lemma 2.2.1 in [102] does not use the first homotopy move. The
proof of Lemma 2.2.2 in [12] uses the first homotopy move but can be easily modified to use the framed move instead.

Let us show that a nanoword $xABB$ is framed homotopic to $xy$ provided $|A| = \tau(|B|)$. Pick letters $C, D$ not appearing in $x, y$ with $|C| = |B|, |D| = |A|$. Then

$$xABB \sim xACDABB\sim xCADBAC\sim xDBBD \sim xy.$$ 

Here we insert $CD\cdot CD$, apply homotopy (iii) of Lemma 2.2.1 delete $CA\cdot AC$, and finally delete $DBBD$. A similar argument shows that a transformation $xABB \mapsto xBA\sim y$ preserves the framed homotopy class. These observations easily imply that framed homotopy classes of nanophrases over $\alpha_s$ bijectively correspond to stable equivalence classes of framed ordered point end link diagrams.

2. The results of Sect. 4 and 5 have an obvious version for systems of transversal curves on surfaces; it suffices to replace $\alpha_s$ by $\alpha_0 = \{a, b\}$.

7. PSEUDO-LINKS AND QUASI-LINKS

7.1. Pseudo-links. Set $\alpha_1 = \{1, -1\}$ with involution $\tau$ permuting 1 and $-1$ and let $S_1 \subset \alpha_1 \times \alpha_1 \times \alpha_1$ consist of the following 6 triples:

$$(1, 1, 1), (1, 1, -1), (-1, 1, 1), (-1, -1, -1), (-1, -1, 1), (1, -1, 1).$$

This homotopy data is admissible in the sense of Sect. 2.2. As a shift involution in $\alpha_1$, we take the identity mapping $\text{id} : \alpha_1 \to \alpha_1$. The corresponding permutations and shifts of words in nanophrases over $\alpha_1$ are the ordinary permutations and cyclic shifts of words (involving no modification of the underlying $\alpha_1$-alphabets). Nanophrases over $\alpha_1$ considered up to permutations and cyclic shifts of words are called pseudo-links.

This terminology is justified by the following connections to knot theory. Consider the projection $\alpha_s \to \alpha_1$ sending $a_+, b_+$ to 1 and $a_-, b_-$ to $-1$. This projection transforms $S_s \subset (\alpha_s)^3$ into $S_1$. It commutes with $\tau$ and with the shift involution in $\alpha_s, \alpha_1$ (the shift involution in $\alpha_s$ is defined by $\nu(a_\pm) = b_\pm$).

Applying the projection $\alpha_s \to \alpha_1$, we can transform any nanophrase over $\alpha_s$ into a nanophrase over $\alpha_1$. Clearly, $S_s$-homotopic nanophrases over $\alpha_s$ yield $S_1$-homotopic nanophrases over $\alpha_1$. This induces a mapping $P(\alpha_s, S_s) \to P(\alpha_1, S_1)$. Quotienting by permutations and shifts of words we obtain a mapping from $P(\alpha_s, S_s, \nu)$ to $P(\alpha_1, S_1, \text{id})$. Further quotienting by inversions of words we obtain a mapping $P_u(\alpha_s, S_s, \nu) \to P_u(\alpha_1, S_1, \text{id})$. All these mappings are surjective.

By Sect. 5 a link diagram $D$ on a surface yields a nanophrase over $\alpha_s$. Projecting to $\alpha_1$, we obtain a nanophrase $p(D)$ over $\alpha_1$. If $D$ is pointed and ordered, then $p(D)$ is well-defined, otherwise $p(D)$ is defined only up to permutations and shifts of words. The class of $p(D)$ in $P(\alpha_1, S_1, \text{id})$ is an invariant of stable equivalence of $D$. We call $p(D)$ the underly ing pseudo-link of $D$. Further projecting to $P_u(\alpha_1, S_1, \text{id})$ we obtain an invariant independent of the orientation of $D$.

In the next section we explain that the Jones polynomial of a link depends only on the underlying pseudo-link. This shows that pseudo-links are highly non-trivial objects retaining important features of links.
7.2. **Quasi-links.** Set \( \alpha_2 = \{c, d\} \) with the identity involution \( \tau = \text{id} : \alpha_2 \to \alpha_2 \) and let \( S_2 \subset \alpha_2 \times \alpha_2 \times \alpha_2 \) consist of the following 6 triples:

\[(c, c, c), (c, c, d), (d, c, c), (d, d, d), (d, d, c), (c, d, d).\]

This homotopy data essentially differs from \((\alpha_1, S_1)\) by the choice of \( \tau \). The homotopy data \((\alpha_2, S_2)\) is admissible. As a shift involution \( \nu_2 \) in \( \alpha_2 \), we take the permutation of \( c \) and \( d \). Nanophrases over \( \alpha_2 \) considered up to \( \nu_2 \)-permutations and \( \nu_2 \)-shifts of words are called **quasi-links**.

Connections to knot theory go as follows. Consider the projection \( \alpha_* \to \alpha_2 \) sending \( a_+, b_- \) to \( c \) and \( a_-, b_+ \) to \( d \). This projection transforms \( S_* \subset (\alpha_*)^3 \) into \( S_2 \), commutes with \( \tau \) and with the shift. Applying this projection, we can transform any nanophrase over \( \alpha_* \) into a nanophrase over \( \alpha_2 \). This induces a mapping \( \mathcal{P}(\alpha_*, S_*) \to \mathcal{P}(\alpha_2, S_2) \). Quotienting by permutations and shifts of words (and eventually by inversions of words) we obtain projections \( \mathcal{P}(\alpha_*, S_*, \nu) \to \mathcal{P}(\alpha_2, S_2, \nu_2) \) and \( \mathcal{P}_u(\alpha_*, S_*, \nu) \to \mathcal{P}_u(\alpha_2, S_2, \nu_2) \).

A link diagram \( D \) yields a nanophrase over \( \alpha_* \) whose projection to \( \alpha_2 \) is a nanophrase over \( \alpha_2 \) denoted \( q(D) \). If \( D \) is pointed and ordered, then \( q(D) \) is well defined, otherwise \( q(D) \) is defined only up to \( \nu_2 \)-permutations and \( \nu_2 \)-shifts of words. The class of \( q(D) \) in \( \mathcal{P}(\alpha_2, S_2, \nu_2) \) is an invariant of stable equivalence of \( D \). We call \( q(D) \) the **underlying quasi-link** of \( D \). Further projecting to \( \mathcal{P}_u(\alpha_2, S_2, \nu_2) \) we obtain an invariant independent of the orientation of \( D \).

Quasi-links will be further discussed in Sect. 9.

7.3. **Remarks.** 1. As explained above, there are three natural projections from the set of link diagrams on surfaces to simpler objects. They map a link diagram to the underlying family of curves, the underlying pseudo-link and the underlying quasi-link. The length of the resulting nanophrases is equal to the number of link components.

2. The homotopy data \((\alpha_0, S_0)\) and \((\alpha_1, S_1)\) are closely related. Consider the bijection from \( \alpha_0 = \{a, b\} \) to \( \alpha_1 = \{1, -1\} \) sending \( a \) to \( 1 \) and \( b \) to \( -1 \). This bijection commutes with the involution \( \tau \) in \( \alpha_0, \alpha_1 \) and transforms \( S_0 \subset (\alpha_0)^3 \) into a subset of \( S_1 \subset (\alpha_1)^3 \). In this way any nanophrase over \( \alpha_0 \) determines a pseudo-link and homotopic nanophrases yield homotopic pseudo-links. Thus, the homotopy theory of pseudo-links is a quotient of the homotopy theory of curves. However, there is no way to recover the pseudo-link \( p(D) \) underlying a link diagram \( D \) from the system of curves underlying \( D \). Note also that the shift involutions in \( \alpha_0 \) and \( \alpha_1 \) do not match: the first one permutes \( a \) and \( b \) while the second one is the identity.

8. **The bracket polynomial**

The aim of this section is to construct a polynomial invariant of pseudo-links whose value on the underlying pseudo-link of a link diagram is equal to the Jones polynomial of the link. We begin by recalling Kauffman’s bracket polynomial.
8.1. Bracket polynomial of links. L. Kauffman \cite{Kau1} associated with every non-empty link diagram \( D \) on a surface a 1-variable Laurent polynomial \( \langle D \rangle \) called the bracket polynomial of \( D \). This polynomial is defined by expanding each crossing of \( D \) as a linear combination of two uncrossings with coefficients \( t \) and \( t^{-1} \), see Figure 3. This expands \( D \) as a linear combination of diagrams with no crossings. Each \( d \)-component diagram with no crossings is then replaced with \(- (t^2 + t^{-2})^d - 1\). The bracket polynomial depends neither on the orientation of \( D \), nor on an order of its components, nor on a choice of base points. The bracket polynomial is invariant under the second and the third Reidemeister moves and is multiplied by \( t^{\pm 3} \) under the first Reidemeister move.

We can translate the bracket polynomial to the language of nanophrases over \( \alpha_\ast \). A nanophrase \( P \) over \( \alpha_\ast \) gives rise to a pointed ordered link diagram on a surface. Let \( \langle P \rangle \in \mathbb{Z}[t, t^{-1}] \) be the bracket polynomial of this diagram. This polynomial is invariant under \( \nu \)-shifts, \( \nu \)-inversions, and \( \nu \)-permutations on words in \( P \) since they are translated to diagrams as change of base points, orientation reversal, and change of order of components. The polynomial \( \langle P \rangle \) is invariant under the second and third \( S_\ast \)-homotopy moves on \( P \) since they are translated to diagrams as the second and third Reidemeister moves. Under the move deleting \( AA \) from a word of \( P \), the bracket polynomial is multiplied by \(- t^{\varepsilon(A)} \) where 
\[ \varepsilon(A) = +1 \text{ if } |A| \in \{a_+, b_+\} \text{ and } \varepsilon(A) = -1 \text{ if } |A| \in \{a_-, b_-\}. \]
When an empty word \( \emptyset \) is deleted from a nanophrase \( P \) of length \( \geq 2 \) the polynomial \( \langle P \rangle \) is divided by \(- (t^2 + t^{-2}) \). Clearly, \( \langle \emptyset \rangle = 1 \).

To translate the Kauffman crossing expansion to this setting, we need the following notation. Given a phrase \( P \) in an \( \alpha_\ast \)-alphabet \( \mathcal{A} \) and a word \( w \) in the alphabet \( \mathcal{A} \), denote by \( P_w \) the same phrase \( P \) in the \( \alpha_\ast \)-alphabet \( \mathcal{A}_w \) defined in Sect. \ref{sect:nanophrases}. Denote by \( w^- \) the word in the alphabet \( \mathcal{A} \) obtained by writing the letters of \( w \) in the opposite order.

The Kauffman crossing expansion applied to a self-crossing of a link component and to a crossing of two different components implies the following two recursive relations for the bracket polynomial of nanophrases:

\[
\langle (P_1|AwAz|P_2) \rangle = t^{\varepsilon(A)} \langle (P_1|w|z|P_2) \rangle + t^{-\varepsilon(A)} \langle (P_1|w^{-}\,z|P_2) \rangle_w,
\]
\[
\langle (P_1|Aw|Az|P_2) \rangle = t^{\varepsilon(A)} \langle (P_1|w|z|P_2) \rangle + t^{-\varepsilon(A)} \langle (P_1|w^{-}\,z|P_2) \rangle_w.
\]

Here \( w \) and \( z \) are words in an \( \alpha_\ast \)-alphabet \( \mathcal{A} \), \( A \) is a letter in \( \mathcal{A} \), and \( P_1, P_2 \) are finite sequences of words in \( \mathcal{A} \) such that every letter of \( \mathcal{A} \) appears in the phrase \( (P_1|AwAz|P_2) \) twice. In the first formula \( w \) and \( z \) are parts of a word \( AwAz \) while...
in the second formula $Aw$ and $Az$ are two consecutive words. These formulas and the properties of $\langle P \rangle$ listed above allow us to compute this polynomial recursively.

8.2. Bracket polynomial of pseudo-links. In the recursive formulas above, the right hand side depends on $\varepsilon(A)$ rather than on $A$. This implies (by induction on the number of letters in a nanophrase) that the bracket of a nanophrase over $\alpha_s$ depends only on the underlying nanophrase over $\alpha_1$. Any pseudo-link $p$ determines a Laurent polynomial $\langle p \rangle \in \mathbb{Z}[t^{\pm 1}]$ by $\langle p \rangle = \langle \tilde{p} \rangle$ where $\tilde{p}$ is any nanophrase over $\alpha_1$ whose projection to $\alpha_1$ equals $p$. The polynomial $\langle p \rangle$ is invariant under shifts, inversions, and permutations of words in $p$. It is preserved under the second and third $S_1$-homotopy moves and is multiplied by $-t^{-3|A|}$ under the move deleting $AA$ from a word of $p$ (where $|A| \in \alpha_1 = \{1, -1\}$). To compute $\langle p \rangle$ one can use the recursive relations above with $\varepsilon(A)$ replaced everywhere by $|A| \in \{1, -1\}$.

As an illustration, we compute the bracket for the nanoword $ABCABC$ over $\alpha_1$ where $|A| = |B| = |C| = 1$. We have

$$\langle ABCABC \rangle = t(\langle BC|BC \rangle) + t^{-1}(\langle C_7B_7B_7C_7 \rangle)$$

$$= t(t(\langle CC \rangle) + t^{-1}(\langle C_7C_7 \rangle)) + t^{-1}(\langle t^3B_7 \rangle(\langle C_7C_7 \rangle))$$

$$= t^2(-t^3|C|) - t^3|C| + t^{-1}(\langle t^3B_7 \rangle(-t^3|C|)) = -t^5 - t^{-3} + t^{-7}$$

where $|B_7| = \tau(|B|) = -1$ and $|C_7| = \tau(|C|) = -1$. This is compatible with the usual formula for the bracket of the standard diagram of a trefoil.

8.3. The Jones polynomial of pseudo-links. For a pseudo-link $p$ we define the writhe $|p| = \sum \lambda |A| \in \mathbb{Z}$ where $A$ runs over all letters occurring in $p$. The polynomial $J(p) = (-t)^{-|p|}\langle p \rangle$ is invariant under all $S_1$-homotopy moves on $p$. For a pseudo-link $p$ arising from a link in $S^3$, the polynomial $\langle p \rangle$ is equal to the Jones polynomial of this link up to a re-parametrization.

8.4. Polynomials of phrases. An $\alpha_1$-alphabet is nothing but a bipartitioned set, that is a set decomposed as a disjoint union of two subsets (the preimages of $\pm 1 \in \alpha_1$). Any phrase $P$ in an $\alpha_1$-alphabet $\mathcal{A}$ gives rise to a polynomial $\langle P \rangle \in \mathbb{Z}[t^{\pm 1}]$ as follows. It is explained in [112] that a word $w$ in any alphabet determines in a canonical way a nanoword $w^d$ over this alphabet. The same procedure applies to phrases and derives from $P$ a nanophrase $P^d$ over $\mathcal{A}$. (Each letter $A \in \mathcal{A}$ occurring $m_A$ times in $P$ gives rise to $m_A(m_A+1)/2$ distinct letters each occurring in $P$ twice.) Composing with projection $\mathcal{A} \to \alpha_1$ we obtain from $P^d$ a pseudo-link $\langle P^d \rangle_1$. Set $\langle P \rangle = \langle (P^d)_1 \rangle$. Similarly, we define the Jones polynomial of $P$ by $J(P) = J((P^d)_1) = (-t)^{-|P|}\langle P \rangle$ with $|P| = \sum_{A \in \mathcal{A}} |A|m_A(m_A+1)/2$ where $|A| = \pm 1$ is the image of $A$ in $\alpha_1$ and $m_A$ is the number of entries of $A$ in $P$. The polynomials $\langle P \rangle, J(P)$ are interesting invariants of phrases in bipartitioned alphabets. One natural question is to characterize the polynomials that arise from phrases in this way.
9. Keis

Keis were introduced by M. Takasaki in 1942 as abstractions of symmetries, see [Kam] for a survey of keis and related objects (quandles, racks, etc.). Here we recall from [Tu2] the concept of an \( \alpha \)-kei where \( \alpha \) is a set with involution \( \tau \). This will be instrumental in the next section where we discuss keis of nanophrases.

9.1. \( \alpha \)-keis. Consider a set \( X \) and suppose that each \( a \in \alpha \) gives rise to a bijection \( x \mapsto ax : X \to X \) and to a binary operation \( (x, y) \mapsto x * a y \) on \( X \). The set \( X \) is an \( \alpha \)-kei and the mappings \( x \mapsto ax, (x, y) \mapsto x * a y \) are kei operations if the following axioms are satisfied:

(i) \( ax * a x = x \) for all \( a \in \alpha, x \in X \);
(ii) \( a(x * a y) = ax * a y \) for all \( a \in \alpha, x, y \in X \);
(iii) \( (x * a y) * a z = (x * a z) * a (y * a z) \) for all \( a \in \alpha, x, y, z \in X \);
(iv) \( a \tau(a) x = x \) for all \( x \in X, a \in \alpha \) and
(v) \( (x * a y) \tau(a) ay = x \) for all \( x, y \in X, a \in \alpha \).

A morphism of \( \alpha \)-keis \( X \to X' \) is a set-theoretic mapping commuting with the kei operations in \( X, X' \). Given an \( \alpha \)-kei \( X \), we define an \( \alpha \)-kei \( \overline{X} \) to be the same set \( X \) with new kei operations \( ax := \tau(a)x, x * a y := x * \tau(a) y \) for \( x, y \in X, a \in \alpha \).

Clearly, \( \overline{X} = X \).

In analogy with group theory, one can define presentations of \( \alpha \)-keis by generators and relations. A presentation of an \( \alpha \)-kei \( X \) by generators and relations yields a presentation of \( \overline{X} \) by generators and relations by replacing every letter \( a \in \alpha \) appearing in the relations by \( \tau(a) \).

In the simplest case where \( \alpha = \{a\} \) is a 1-element set, an \( \alpha \)-kei is a set \( X \) with involution \( x \mapsto \tilde{x} = ax \) and a binary operation \( (x, y) \mapsto x \circ y = x * a y \) such that \( \tilde{x} \circ x = x; \tilde{x} \circ \tilde{y} = \tilde{x} \circ \tilde{y}; (x \circ y) \circ z = (x \circ \tilde{z}) \circ (y \circ z) \), and \( (x \circ y) \circ \tilde{y} = x \) for all \( x, y, z \in X \). When the involution \( x \mapsto \tilde{x} \) is the identity, these axioms are equivalent to those of a kei, see [Kam].

The \( \alpha \)-keis generalize quandles: there is a canonical bijection between quandles and \( \alpha \)-keis \( X \) such that \( \alpha = \{a, b\} \) is a 2-element set with involution permuting \( a, b \) and \( ax = bx = x \) for all \( x \in X \), cf. [Tu2], Lemma 14.7.1.

9.2. Core \( \alpha \)-keis. The following construction of \( \alpha \)-keis provides a vast set of examples. By a \( \tau \)-compatible action of \( \alpha \) on a group \( G \) we mean a set of group automorphisms \( \{G \to G, g \mapsto ag\}_{a \in \alpha} \) such that \( a \tau(a) g = g \) for all \( a \in \alpha, g \in G \).

It is easy to check that such an action together with kei operations \( g \circ h = h(\tau(a)g) \circ h \) make \( G \) into an \( \alpha \)-kei. It is called the core of \( G \) and denoted \( \text{core}(G) \). For \( \alpha \) consisting of one element that acts on \( G \) as the identity, this construction is due to D. Joyce (cf. [FR], p. 349).

The construction of the core has a natural adjoint associating with an arbitrary \( \alpha \)-kei \( X \) a group \( \Gamma_X \) with generators \( \{[x]\}_{x \in X} \) and relations \( [a(x \circ y)] = [ay][a \tau(b)x]^{-1}[ay] \) for all \( a, b \in \alpha, x, y \in X \). We endow \( \Gamma_X \) with the \( \tau \)-compatible action of \( \alpha \) defined on the generators by \( a[x] = [ax] \) for \( a \in \alpha, x \in X \). Given a group \( G \) with \( \tau \)-compatible action of \( \alpha \) and a kei morphism \( f : X \to \text{core}(G) \), there is a unique group homomorphism \( \Gamma_X \to G \) whose composition with the
inclusion $X \hookrightarrow \Gamma_X, x \mapsto [x]$ is equal to $f$. This universal property characterizes $\Gamma_X$ up to isomorphism.

A presentation of $\Gamma_X$ by generators and relations can be read from an arbitrary presentation $[S : R]$ of $X$ by generators and relations (cf. [FR], Lemma 4.3). Namely, $\Gamma_X$ is generated by the symbols $\{a s\}_{a \in \alpha, s \in S}$ subject to the relations obtained from $R$ by replacing all terms of type $a(x \ast y)$ by $(ay)(a \tau(b)x)^{-1}(ay)$.

10. **Keis of nanophrases**

A well known construction due to S. Matveev and D. Joyce associates quandles with link diagrams. Since link diagrams are nanophrases over $\alpha$, one may attempt to generalize this construction to nanophrases over an arbitrary alphabet $\alpha$ with involution $\tau$. We do it here starting with certain additional data.

10.1. **Keis and homotopy.** Fix an equivalence relation $\sim$ on $\alpha$ such that $a \sim b \Rightarrow \tau(a) \sim \tau(b)$ for $a, b \in \alpha$. Let $\underline{\alpha} = \alpha / \sim$ with involution $\underline{\tau}$ induced by $\tau$. For $a \in \alpha$, denote its projection to $\underline{\alpha}$ by $\underline{a}$.

Fix a set (possibly empty) $\beta \subset \alpha$ such that $\tau(\beta) = \beta$. We associate with any nanophrase $P = (A, (w_1, \ldots, w_k))$ over $\alpha$ an $\underline{\alpha}$-kei $\kappa_\beta(P)$ as follows. Let $n_r$ be the length of the word $w_r$ for $r = 1, \ldots, k$. Each letter $a \in A$ appears in $P$ twice, say, first time at the $i_1$-th position in $w_{r_1}$ and second time at the $i_2$-th position in $w_{r_2}$ where $1 \leq i_1 \leq n_{r_1}, 1 \leq i_2 \leq n_{r_2}, r_1 \leq r_2$, and $r_1 = r_2 \Rightarrow i_1 < i_2$. The $\underline{\alpha}$-kei $\kappa_\beta(P)$ is generated by the symbols $\{x^r_s\}$ where $1 \leq r \leq k$ and $0 \leq s \leq n_r$. Each $A \in A$ gives rise to two defining relation: if $a = |A| \in \beta$, then

$$x^r_{i_1} = \underline{a} x^r_{i_1-1}, \quad x^r_{i_2} = x^r_{i_2-1} \ast \underline{a} x^r_{i_1-1},$$

and if $a = |A| \in \alpha - \beta$, then

$$x^r_{i_1} = x^r_{i_1-1} \ast \underline{a} x^r_{i_2-1}, \quad x^r_{i_2} = \underline{a} x^r_{i_2-1}.$$

These generators and relations define the $\underline{\alpha}$-kei $\kappa_\beta(P)$. It has two sets of distinguished elements $x^0_0, x^2_0, \ldots, x^k_0$ (the inputs) and $x^{r_0}_{n_1}, x^{r_2}_{n_2}, \ldots, x^{r_k}_{n_k}$ (the outputs). Adding the relations $x^r_0 = x^r_{n_r}$ for $r = 1, \ldots, k$ we obtain a quotient $\underline{\alpha}$-kei $\tilde{\kappa}_\beta(P)$.

Note the obvious $\underline{\alpha}$-kei isomorphism $\kappa_\beta(P) \approx \kappa_{\alpha - \beta}(P^-)$ where $P^-$ is $P$ read from right to left. This isomorphism transforms the $r$-th input (resp. output) into the $(n + 1 - r)$-th output (resp. input). Clearly, $\tilde{\kappa}_\beta(P^-) \approx \tilde{\kappa}_{\alpha - \beta}(P)$.

For the next definition, it is convenient to set $\beta_0 = \beta$ and $\beta_1 = \alpha - \beta$. Let $S = S(\beta, \sim) \subset \alpha^3$ consist of all triples $(a, b, c) \in \alpha^3$ such that

- $a \sim b \sim c$ and $a, b, c \in \beta_i$ for some $i \in \{0, 1\}$;
- or $a \sim b \sim \tau(c)$ and $a, b \in \beta_i, c \in \beta_{1-i}$ for some $i \in \{0, 1\}$;
- or $\tau(a) \sim b \sim c$ and $b, c \in \beta_i, a \in \beta_{1-i}$ for some $i \in \{0, 1\}$.

The set $S$ contains the diagonal of $\alpha^3$ and therefore the homotopy data $(\alpha, S)$ is admissible.

**Theorem 10.1.1.** For any nanophrase $P$ over $\alpha$, the $\underline{\alpha}$-kei $\kappa_\beta(P)$ is invariant under $S$-homotopy moves.
The proof goes by repeating the proof of Lemma 15.1.1 in [Tu2].

The next theorem shows that for an appropriate choice of the shift involution \( \nu \), the kei \( \kappa_\beta(P) \) is also preserved under \( \nu \)-permutations on \( P \) and its quotient \( \hat{\kappa}_\beta(P) \) is preserved under \( \nu \)-permutations and \( \nu \)-shifts.

**Theorem 10.1.2.** Let \( \nu : \alpha \to \alpha \) be an involution such that \( \nu(\beta) = \alpha - \beta \) and \( a \sim \nu(a) \) for all \( a \in \alpha \). For a nanophrase \( P \) over \( \alpha \), the \( \alpha \)-kei \( \kappa_\beta(P) \) is invariant under \( \nu \)-permutations on the words of \( P \). The quotient \( \alpha \)-kei \( \hat{\kappa}_\beta(P) \) is invariant under \( \nu \)-permutations and \( \nu \)-shifts on the words of \( P \).

We leave the proof to the reader as an exercise.

10.2. **Examples.** 1. Consider the alphabet \( \alpha_* = \{a_+, a_-, b_+, b_-, \} \) with involution \( \tau(a_\pm) = b_\mp \), shift involution \( \nu(a_\pm) = b_\pm \), and distinguished subset \( \beta = \{a_+, b_-\} \). Provide \( \alpha_* \) with equivalence relation \( a_+ \sim b_+, a_- \sim b_- \). This data satisfies all the conditions of Theorems 10.1.1 and 10.1.2. Clearly, \( S(\beta, \sim) = S_* \subset (\alpha_*^3) \) is the set defined in Sect. 12. This yields for any nanophrase \( P \) over \( \alpha_* \) an \( \alpha_* \)-kei \( \kappa_\beta(P) \) invariant under \( S_* \)-homotopy. The quotient \( \alpha_* \)-kei \( \hat{\kappa}_\beta(P) \) is also invariant under \( \nu \)-shifts and \( \nu \)-permutations. The set \( \alpha_* = \{+, -\} \) consists of 2 elements permuted by \( \nu \). For the nanophrase \( P \) associated with a link diagram on a surface, the \( \alpha_* \)-kei \( \hat{\kappa}_\beta(P) \) is invariant under stable equivalence and independent of the choice of the order and the base points of the link components. The quotient of \( \hat{\kappa}_\beta(P) \) by \( ax = x \) for all \( a \in \alpha_* = \{+, -\}, x \in \hat{\kappa}_\beta(P) \) with binary operation \( *_+ \) is the standard link quandle (see [FR], [Kam], [Kau2]).

2. Consider the alphabet \( \alpha_0 = \{a, b\} \) with involution \( \tau(a) = b \) and distinguished subset \( \beta_0 = \alpha \). As the equivalence relation \( \sim \) in \( \alpha_0 \) we take the equality \( = \). This data satisfies the conditions of Theorem 10.1.1 where \( S(\beta_0, \sim) = S_0 \subset (\alpha_0^3) \) is the diagonal. For any nanophrase \( P \) over \( \alpha_0 \), we obtain an \( \alpha_0 \)-kei \( \kappa_{\beta_0}(P) \) invariant under \( S_0 \)-homotopy. This example is contained in the previous one: the mapping \( \alpha_0 \to \alpha_* \) defined by \( a \mapsto a_+, b \mapsto b_- \) transforms \( P \) into a nanophrase \( P_* \) over \( \alpha_* \) and \( \kappa_{\beta_0}(P) = \kappa_{\beta_0}(P_*) \).

3. The homotopy data \( (\alpha_1, S_1) \) from Sect. 7 cannot be obtained by the methods of Sect. 10.1 and does not lead to keis. Pseudo-links have no keis.

4. Consider the alphabet \( \alpha_2 = \{c, d\} \) with trivial involution \( \tau = \text{id} \), shift involution \( \nu \) permuting \( c \) and \( d \), and distinguished subset \( \beta = \{c\} \). Provide \( \alpha_2 \) with trivial equivalence relation \( \sim \) (all elements are equivalent). This data satisfies the conditions of Theorems 10.1.1 and 10.1.2. Clearly, \( S(\beta, \sim) = S_2 \subset (\alpha_2^3) \) is the set defined in Sect. 12. This yields for any nanophrase \( P (= \text{a nanophrase over } \alpha_2) \) an \( \alpha_2 \)-kei \( \kappa_{\beta}(P) \) invariant under \( S_2 \)-homotopy. The quotient \( \alpha_2 \)-kei \( \hat{\kappa}_\beta(P) \) is also invariant under \( \nu \)-shifts and \( \nu \)-permutations. The set \( \alpha_2 \) consists here of 1 element. A study of this kei should lead to interesting homotopy invariants of quasi-links.

When \( P \) is obtained from an oriented link \( L \subset S^3 \) by taking the associated nanophrase over \( \alpha_* \) and projecting to \( \alpha_2 \), the group \( \Gamma = \Gamma_{\hat{\kappa}_\beta(P)} \) (defined in Sect. 12) is closely related to the fundamental group of the 2-fold branched cover \( M \) of \( S^3 \) with branching set \( L \). Namely, the group \( \pi_1(M) * \mathbb{Z} \) is the quotient of \( \Gamma \)
by the relations \( ag = g \) for the unique \( a \in \mathbb{Q} \) and all \( g \in \Gamma \). This is obtained by comparing the presentation of \( \hat{\kappa}_\beta(P) \) as above with the Wada presentation of \( \pi_1(M) \ast \mathbb{Z} \), both computed from a diagram of \( L \). This observation easily extends to the nanophrase derived from (a diagram of) a link \( L \subset \Sigma \times [0, 1] \) where \( \Sigma \) is a surface. Here one should use the 2-fold branched cover of \( \Sigma \times [0, 1]/\Sigma \times 1 \) with branching set \( L \), cf. the argument in [KK], Prop. 5.1 and the proof of Wada’s theorem in [Pr], p. 287.

5. Let \( \alpha_\ast, \tau, \nu, \beta, \sim \) be as in Example 1. Pick a set \( \gamma \) and consider the alphabet \( \alpha_\gamma = \alpha_\ast \times \gamma \) with involution \( \tau \times \text{id} \), shift involution \( \nu_\gamma = \nu \times \text{id} \), and distinguished subset \( \beta \times \gamma \). Provide \( \alpha_\gamma \) with equivalence relation \( \sim_\gamma \) as follows: two pairs \( (x, c), (y, d) \) with \( x, y \in \alpha_\ast, c, d \in \gamma \) are equivalent if \( x \sim y \) and \( c = d \). This data satisfies the conditions of Theorems 10.1.1 and 10.1.2. The set \( S_\gamma = S(\beta \times \gamma, \sim_\gamma) \) is the product of \( S_\ast \subset (\alpha_\ast)^3 \) and the diagonal of \( \gamma^3 \). This yields for any nanophrase \( P \) over \( \alpha_\gamma \) an \( \alpha_\gamma \)-kei \( \kappa_\beta \times \gamma(P) \) invariant under \( S_\gamma \)-homotopy where \( \alpha_\gamma = \alpha_\gamma/\sim_\gamma = \{+, -\} \times \gamma \). The quotient \( \alpha_\gamma \)-kei \( \kappa_\beta \times \gamma(P) \) is also invariant under \( \nu_\gamma \)-shifts and \( \nu_\gamma \)-permutations.

Nanophrases over \( \alpha_\gamma \) have a simple geometric interpretation. Let us call an (ordered pointed) link diagram on an (oriented) surface \( \gamma \)-colored if all its crossings are endowed with elements of \( \gamma \) (the colors). Homeomorphisms of \( \gamma \)-colored link diagrams should preserve the colors of the crossings. Stable equivalence of \( \gamma \)-colored link diagrams is defined as in the non-colored case with the following restrictions on the Reidemeister moves: the second move is allowed only when it involves two crossings of the same color, the third move is allowed only when it involves three crossings of the same color which is kept under the move. The crossings not involved in the moves keep their color. Denote \( L_\gamma \) the set of stable equivalence classes of \( \gamma \)-colored ordered pointed link diagrams. The same arguments as in Theorem 6.3.1 show that \( L_\gamma = P(\alpha_\gamma, S_\gamma) \). This equality implies a similar equality for non-ordered non-pointed link diagrams. The construction above associates with every \( \gamma \)-colored link diagram an \( \alpha_\gamma \)-kei invariant under stable equivalence.

10.3. Remark. Further invariants of nanophrases can be derived from their keis by abelianization [Tu2]. Another interesting possibility is to define cohomology of \( \alpha \)-keis and to derive homotopy invariants of nanophrases from cocycles and state sums.

References

[CKS] J. S. Carter, S. Kamada, M. Saito, Stable equivalence of knots on surfaces and virtual knot cobordisms. J. Knot Theory Ramifications 11 (2002), no. 3, 311–322.

[FR] R. Fenn, C. Rourke, Racks and links in codimension two. J. Knot Theory Ramifications 1 (1992), no. 4, 343–406.

[Ga] C. F. Gauss, Werke, Vol. VIII, Teubner, Leipzig, 1900, pp. 272, 282-286.

[GPV] M. Goussarov, M. Polyak, O. Viro, Finite-type invariants of classical and virtual knots. Topology 39 (2000), no. 5, 1045–1068.

[Kad] T. Kadokami, Detecting non-triviality of virtual links. J. Knot Theory Ramifications 12 (2003), no. 6, 781–803.
[Kam] S. Kamada, *Knot invariants derived from quandles and racks*. Invariants of knots and 3-manifolds (Kyoto, 2001), 103–117 (electronic), Geom. Topol. Monogr., 4, Geom. Topol. Publ., Coventry, 2002.

[KK] N. Kamada, S. Kamada, *Abstract link diagrams and virtual knots*. J. Knot Theory Ramifications 9 (2000), no. 1, 93–106.

[Kau1] L.H. Kauffman, *State models and the Jones polynomial*. Topology 26 (1987), no. 3, 395–407.

[Kau2] L.H. Kauffman, *Virtual knot theory*. European J. Combin. 20 (1999), no. 7, 663–690.

[Pr] J. Przytycki, *3-coloring and other elementary invariants of knots*. Knot theory (Warsaw, 1995), 275–295, Banach Center Publ., 42, Polish Acad. Sci., Warsaw, 1998.

[Ro] P. Rosenstiehl, *Caractrisation des graphes planaires par une diagonale algébrique*. C. R. Acad. Sci. Paris Sr. A-B 283 (1976), no. 7, A417–A419.

[SW] D. Silver, S. Williams, *An invariant for open virtual strings*, math.GT/0409185

[Tu1] V. Turaev, *Virtual strings*, Ann. Inst. Fourier 54 (2004), no. 7, 2455–2525.

[Tu2] V. Turaev, *Topology of words*, math.CO/0503683

IRMA, Université Louis Pasteur - C.N.R.S.,
7 rue René Descartes
F-67084 Strasbourg
France
E-mail: turaev@math.u-strasbg.fr