INTEGER POINTS IN DOMAINS AND ADIABATIC LIMITS

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Abstract. We prove an asymptotic formula for the number of integer points in a family of bounded domains with smooth boundary in the Euclidean space, which remain unchanged along some linear subspace and expand in the directions, orthogonal to this subspace. A more precise estimate for the remainder is obtained in the case when the domains are strictly convex.

Using these results, we improved the remainder estimate in the adiabatic limit formula (due to the first author) for the eigenvalue distribution function of the Laplace operator associated with a bundle-like metric on a compact manifold equipped with a Riemannian foliation in the particular case when the foliation is a linear foliation on the torus and the metric is the standard Euclidean metric on the torus.

1. Statement of the problem and the main results

A classical problem on integer points distribution consists in the study of the asymptotic behavior of the number of points of the integer lattice \( \mathbb{Z}^n \) in a family of homothetic domains in \( \mathbb{R}^n \). This problem is originated in the Gauss problem on the number of integer points in the disk, where it is directly related with the arithmetic problem on the number of representations of an integer as a sum of two squares, and sufficiently well studied (see, for instance, books [2, 3, 4, 8] and the references therein).

In this paper we investigate much less studied problem on counting of integer points in a family of anisotropically expanding domains. More precisely, let \( F \) be a \( p \)-dimensional linear subspace of \( \mathbb{R}^n \) and \( H = F^\perp \) the \( q \)-dimensional orthogonal complement of \( F \) with respect to the standard inner product \((\cdot, \cdot)\) in \( \mathbb{R}^n \), \( p + q = n \). For any \( \varepsilon > 0 \), consider the linear transformation \( T_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) given as follows:

\[
T_\varepsilon(x) = \begin{cases} x, & \text{if } x \in F, \\ \varepsilon^{-1}x, & \text{if } x \in H. \end{cases}
\]

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For any bounded domain $S$ with smooth boundary in $\mathbb{R}^n$, we put

$$n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$  

The main goal of the paper is to study the asymptotic behavior of the function $n_\varepsilon(S)$ as $\varepsilon \to 0$. Before we state the main results of the paper, we introduce some auxiliary notions.

Let $\Gamma = \mathbb{Z}^n \cap F$. $\Gamma$ is a free abelian group. Denote by $r = \text{rank} \Gamma \leq p$ the rank of $\Gamma$. For $r \geq 1$, denote by $(\ell_1, \ell_2, ... \ell_r)$ some base in $\Gamma$. Let $V$ be the $r$-dimensional subspace of $\mathbb{R}^n$ spanned by $(\ell_1, \ell_2, ... \ell_r)$. Observe that $\Gamma$ is a lattice in $V$.

Denote by $Q \subset V$ the parallelepiped spanned by the vectors $(\ell_1, \ell_2, ... \ell_r)$ and by $|Q|$ its $r$-dimensional Euclidean volume:

$$|Q| = \text{vol}_r(\ell_1, \ell_2, ... \ell_r) = \text{vol}(V/\Gamma).$$

Let $\Gamma^*$ denote the lattice in $V$, dual to the lattice $\Gamma$:

$$\Gamma^* = \{ \gamma^* \in V : (\gamma^*, \Gamma) \subset \mathbb{Z} \}.$$  

For $r = 0$, the groups $\Gamma$ and $\Gamma^*$ are trivial, and it is natural to put $|Q| = 1$.

For any $x \in V$, we denote by $P_x$ the $(n - r)$-dimensional affine subspace of $\mathbb{R}^n$, passing through $x$ orthogonal to $V$.

**Theorem 1.1.** For any bounded open set $S$ with smooth boundary in $\mathbb{R}^n$, the formula holds:

$$n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{1-r+1-q}), \quad \varepsilon \to 0.$$  

Note that, in the general case, the leading term in the asymptotic formula for $n_\varepsilon(S)$ as $\varepsilon \to 0$ was unknown.

**Remark 1.** In the case when $F$ is trivial, we have $p = r = 0$, $q = n$. The problem is reduced to the classical problem on the asymptotics of the number of integer points in a family of homothetic domains in $\mathbb{R}^n$. The formula (1.2) is reduced to the classical formula, going back to Gauss:

$$n_\varepsilon(S) = \varepsilon^{-n} \text{vol}_n(S) + O(\varepsilon^{1-n}), \quad \varepsilon \to 0.$$  

**Remark 2.** In the case when $\Gamma$ is trivial, we have $r = 0$. The formula (1.2) takes the form

$$n_\varepsilon(S) = \varepsilon^{-q} \text{vol}_n(S) + O(\varepsilon^{1-r-q}), \quad \varepsilon \to 0.$$  

In order to obtain a more precise estimate for the remainder, we need to impose some restrictions to the boundary of the domain $S$. We give just one result of similar type.

**Theorem 1.2.** For any bounded open set $S$ with smooth boundary in $\mathbb{R}^n$ such that, for any $x \in F$, the intersection $S \cap \{x + H\}$ is strictly convex, the formula holds:

$$n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{k-q}), \quad \varepsilon \to 0.$$
where

\[ k = \begin{cases} 
q + 1 & \text{if } \frac{q-1}{2} \leq p - r \\
\frac{q}{2(p-r)} & \text{if } \frac{q-1}{2} > p - r.
\end{cases} \]

Remark 3. In the case when \( F \) is trivial, we get \( k = 2n/(n + 1) \), and the formula (1.2) is reduced to the following formula:

\[ n_\varepsilon(S) = \varepsilon^{-n}\text{vol}_n(S) + O(\varepsilon^{-n+2-\frac{2}{n+1}}), \quad \varepsilon \to 0. \]

This formula was proved by Randol [12, 13].

Example 1.3. As the simplest non-trivial example, we consider the case when \( n = 2 \) and \( p = 1 \). Thus, let \( F \) be the one-dimensional linear subspace of \( \mathbb{R}^2 \) spanned by \((1, \alpha) \in \mathbb{R}^2\). Then its orthogonal complement \( H \) is spanned by \((-\alpha, 1) \in \mathbb{R}^2\). One can distinguish two cases.

Case 1: \( \alpha \not\in \mathbb{Q} \). In this case, \( \Gamma \) is trivial, therefore, \( r = 0 \). Moreover, \( \Gamma^* \) is trivial, and, as mentioned above, it is natural to put \( |Q| = 1 \). For any bounded domain \( S \) with smooth boundary in \( \mathbb{R}^2 \), we get

\[ n_\varepsilon(S) = \varepsilon^{-1}\text{area}(S) + O(\varepsilon^{-1/2}), \quad \varepsilon \to 0. \]

Case 2: \( \alpha \in \mathbb{Q} \). Write \( \alpha = \frac{p}{q} \), where \( p \) and \( q \) are coprime numbers. Then \( \Gamma \) is generated by \( \ell_1 = (q, p) \). Therefore,

\[ |Q| = |\ell_1| = \sqrt{p^2 + q^2}. \]

The dual lattice \( \Gamma^* \) is generated by \( \frac{1}{p^2+q^2}\ell_1 \). An arbitrary element of \( \Gamma^* \) has the form

\[ \gamma^* = \frac{1}{p^2+q^2}k\ell_1, \quad k \in \mathbb{Z}. \]

The corresponding subspace \( P_{\gamma^*} \) is the line \( L_k \) on the plane \( \mathbb{R}^2 \) given by the equation \( qx + py - k = 0 \). Therefore, for any bounded domain \( S \) with smooth boundary in \( \mathbb{R}^2 \), we get

\[ n_\varepsilon(S) = \varepsilon^{-1}\frac{1}{\sqrt{p^2 + q^2}}\sum_{k \in \mathbb{Z}}|S \cap L_k| + O(1), \quad \varepsilon \to 0. \]

The problem on counting of integer points in anisotropically expanding domains in a slightly different context was studied in considerable detail in [14] [15] [10] [11]. More precisely, these papers were devoted to the estimates of the number \( N(S, \Gamma) \) of points of a lattice \( \Gamma \) in \( \mathbb{R}^n \) lying in a bounded domain \( S \) for special domains and lattices. Let us briefly describe some results of these papers and show how they can be applied to the problem under consideration.

Let \( k \) be a totally real algebraic number field of degree \( n \), \( \sigma \) the canonical embedding of \( k \) in the Euclidean space \( \mathbb{R}^n \), \( M \subset k \) an arbitrary \( \mathbb{Z} \)-module of rank \( n \) and \( \Gamma_M = \sigma(M) \) the corresponding algebraic lattice in \( \mathbb{R}^n \). Let
Let \( \Pi \subset \mathbb{R}^n \) be an \( n \)-dimensional parallelepiped centered at the origin and with edges parallel to the coordinated axes:

\[
\Pi = \prod_{j=1}^{n} (-a_j, a_j).
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), we put

\[
\Nm \lambda = \prod_{j=1}^{n} \lambda_j.
\]

Let \( \{e_1, \ldots, e_n\} \) be the standard basis of \( \mathbb{R}^n \). Any vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) defines a linear transformation of \( \mathbb{R}^n \) by

\[
\lambda \cdot e_j = \lambda_j e_j, \quad j \in \{1, \ldots, n\}.
\]

By [15, Theorem 1.1] (see also [14, 11]), for any \( \lambda \in \mathbb{R}^n \) we have the estimate

\[
(1.4) \quad \left| N(\lambda \cdot \Pi, \Gamma_M) - \Nm \lambda \left( \frac{\vol_n(P)}{d(\Gamma_M)} \right) \right| < C_1 \ln |\lambda|^{n-1}, \quad \lambda \rightarrow 0,
\]

where \( d(\Gamma_M) \) is the volume of a fundamental domain of the lattice \( \Gamma_M \) and \( C > 0 \) is a constant, independent of \( \lambda \).

In [10 11], this result was extended to the case of algebraic lattices associated with an arbitrary algebraic number field as well as to a wider class of domains. As an application, the authors obtain nontrivial remainder estimates in the asymptotic formula due to Lang [9] for the number of elements of an algebraic number field in a parallelootope determined by the canonical system of valuations.

The study carried out in [14 15 10 11] shows that the remainder estimate in the asymptotic formula for \( N(\lambda \cdot S, \Gamma) \) turns out to be very sensitive to the Diophantian properties of the lattice \( \Gamma \) and to the geometric properties of the domain \( S \).

In order to apply the results described above to the problems considered in this paper, let us fix \( p \in \{1, \ldots, n\} \) and make use of the formula (1.4) for \( \lambda = \lambda(\varepsilon) \in \mathbb{R}^n \) of the form

\[
\lambda_j = \begin{cases} 1, & \text{if } 1 \leq j \leq p, \\ \varepsilon^{-1}, & \text{if } p + 1 \leq j \leq n, \end{cases}
\]

for some \( \varepsilon > 0 \). We get

\[
\left| N(\lambda(\varepsilon) \cdot \Pi, \Gamma_M) - \varepsilon^{-q} \frac{\vol_n(P)}{d(\Gamma_M)} \right| < C_1 |\varepsilon|^{n-1}, \quad \varepsilon \rightarrow 0,
\]

where \( C_1 > 0 \) is a constant, independent of \( \varepsilon \), and \( p + q = n \).

On the other hand, the lattice \( \Gamma_M \) can be represented as \( \Gamma_M = A(\mathbb{Z}^n) \) with some linear isomorphism \( A \) of \( \mathbb{R}^n \). Therefore, we have

\[
N(\Pi, \Gamma_M) = N(\Pi, A(\mathbb{Z}^n)) = N(A^{-1}(\Pi), \mathbb{Z}^n).
\]
It is not difficult to see from here that
\[ N(\lambda(\varepsilon) \cdot \Pi, \Gamma_M) = n_\varepsilon(A^{-1}(\Pi)), \]
where the right-hand side of this identity is defined by \(1.1\) with the subspace \(F \subset \mathbb{R}^n\) spanned by \(A^{-1}(e_1), \ldots, A^{-1}(e_p)\) and the subspace \(H \subset \mathbb{R}^n\) spanned by \(A^{-1}(e_{p+1}), \ldots, A^{-1}(e_n)\). Observe that \(\mathbb{Z}^n \cap F = \{0\}\).

Thus, finally we get
\[ n_\varepsilon(A^{-1}(\Pi)) = \varepsilon^{-q} \text{vol}_n(A^{-1}(\Pi)) + O(\ln \varepsilon |n^{-1}|), \quad \varepsilon \to 0, \]
which is the formula \(1.2\) with a more precise remainder estimate in the case when \(S = A^{-1}\Pi\) is a parallelepiped centered at the origin and with edges parallel to the vectors \(A^{-1}(e_1), \ldots, A^{-1}(e_n)\). Note that, in this case, the subspaces \(F\) and \(H\) have rather special form and, in general, are not orthogonal.

Similarly, one can use the results of \([10, 11]\) and get more precise remainder estimates in \(1.2\) for some special subspaces \(F\) and \(H\) and domains \(S\).

It would be interesting to continue the study of remainder estimates in the asymptotic formula \(1.2\), depending on geometry of a domain \(S\) and properties of \(F\) and \(H\).

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2. Applications to adiabatic limits

It is well known that the Gauss problem on counting of integer points in the disk is equivalent to the problem on the asymptotic behavior of the eigenvalue distribution function of some elliptic differential operator on a compact manifold, namely, of the Laplace operator on a torus. In the case under consideration, there is also an equivalent asymptotic spectral problem, namely, the problem on the asymptotic behavior of the eigenvalue distribution function of the Laplace operator on a torus in the adiabatic limit associated with a linear foliation.

Consider the \(n\)-dimensional torus \(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\). Let \(\mathcal{F}\) be a linear foliation on \(\mathbb{T}^n\): the leaf \(L_x\) of \(\mathcal{F}\) through \(x \in \mathbb{T}^n\) has the form:
\[ L_x = x + F \mod \mathbb{Z}^n. \]
The decomposition of \(\mathbb{R}^n\) into the direct sum of subspaces \(\mathbb{R}^n = F \oplus H\) induces the corresponding decomposition \(g = g_F + g_H\) of the Euclidean metric into the sum of the tangential and transversal components. Define a one-parameter family \(g_\varepsilon\) of Euclidean metrics on \(\mathbb{R}^n\) by
\[ g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0. \]
We will also consider the metrics \(g_\varepsilon\) as Riemannian metrics on \(\mathbb{T}^n\).

Let \(A = (a_1, \ldots, a_n) \in \mathbb{R}^n\) be some point. Define a 1-form \(A\) on \(\mathbb{T}^n\) by
\[ A = \sum_{j=1}^{n} a_j dx_j. \]
Consider the operator $d - 2\pi i A$, acting from $C^\infty(T^n)$ to the space $\Omega^1(T^n)$ of smooth 1-forms on $T^n$, where $d$ is the de Rham differential, and $A$ is the multiplication operator by $A$. Let $(d - 2\pi i A)^\ast_{g_\varepsilon} : \Omega^1(T^n) \to C^\infty(T^n)$ be the adjoint of $d - 2\pi i A$ with respect to the inner products in $C^\infty(T^n)$ and $\Omega^1(T^n)$ determined by $g_\varepsilon$.

For any $\varepsilon > 0$, consider the operator $H_\varepsilon$ in $C^\infty(T^n)$ defined by

$$H_\varepsilon = (d - 2\pi i A)^\ast_{g_\varepsilon} (d - 2\pi i A).$$

In the local coordinates $(x_1, x_2, \ldots, x_n)$ of the space $\mathbb{R}^n$, the operator $H_\varepsilon$ is written in the form

$$H_\varepsilon = \sum_{j,\ell=1}^n g_{j\ell}^\varepsilon \left( \frac{\partial}{\partial x_j} - 2\pi i a_j \right) \left( \frac{\partial}{\partial x_\ell} - 2\pi i a_\ell \right),$$

where $g_{j\ell}^\varepsilon$ are the elements of the inverse matrix of $g_\varepsilon$.

The operator $H_\varepsilon$ can be considered as the magnetic Schrödinger operator on the torus $T^n$, associated with the metric $g_\varepsilon$, with the constant magnetic potential $A$. It has a complete orthogonal systems of eigenfunctions $U_k(x) = e^{2\pi i (k,x)}$, $x \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$, with the corresponding eigenvalues

$$\lambda_k = (2\pi)^2 \|k - A\|_{g_\varepsilon}^{-2} = (2\pi)^2 \sum_{j,\ell=1}^n g_{j\ell}^\varepsilon (k_j - a_j) (k_\ell - a_\ell).$$

Denote by $N_\varepsilon(\lambda)$ the eigenvalue distribution function of $H_\varepsilon$:

$$N_\varepsilon(\lambda) = \sharp \{ k \in \mathbb{Z}^n : \lambda_k < \lambda \}, \quad \lambda \in \mathbb{R}.$$ 

It is easy to see that

$$n_\varepsilon(B_{\sqrt{\lambda}}(A)) = N_\varepsilon(4\pi^2 \lambda), \quad \lambda \in \mathbb{R}.$$

Thus, the problem on the asymptotic behavior of the number $n_\varepsilon(B_{\sqrt{\lambda}}(A))$ of integer points in the ellipsoid $T_\varepsilon(B_{\sqrt{\lambda}}(A))$ as $\varepsilon \to 0$ is equivalent to the problem on the asymptotic behavior of the eigenvalue distribution function $N_\varepsilon(\lambda)$ as $\varepsilon \to 0$. The limiting procedure $\varepsilon \to 0$ will be called adiabatic limit. This notion was introduced by Witten in 1985 in the study of global anomalies in string theory. We refer the reader to a survey paper [7] for some historic remarks and references.

In [5] (see also [6]), the first author computed the leading term of the asymptotics of the eigenvalue distribution function of the Laplace operator associated with a bundle-like metric on a compact manifold equipped with a Riemannian foliation, in adiabatic limit. The linear foliation on the torus is a Riemannian foliation, and a Euclidean metric on the torus is bundle-like. As a straightforward consequence of Theorem 1.2, we obtain a more precise estimate for the remainder in the asymptotic formula of [5] for this particular case (see also [16]).
Theorem 2.1. For $\lambda > 0$, the following asymptotic formula holds as $\varepsilon \to 0$:

$$N_\varepsilon(\lambda) = \varepsilon^{-q} \frac{\omega_{n-r}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \left( \frac{\lambda}{4\pi^2} - |\gamma^* - A|^2 \right)^{(n-r)/2} + O(\varepsilon^{k-q}),$$

where $\omega_{n-r}$ is the volume of the unit ball in $\mathbb{R}^{n-r}$ and $k$ is defined as

$$k = \begin{cases} \frac{q+1}{2(p-r+1)}, & \text{if } \frac{q-1}{2} \leq p-r \vspace{1mm} \\ \frac{2q}{q+1+2(p-r)}, & \text{if } \frac{q-1}{2} > p-r. \end{cases}$$

3. Proofs of the Main Results

We will use a method based on the Poisson summation formula and the method of stationary phase (Van der Corput, Randol [12, 13], Colin de Verdière [1]). First of all, we observe that we have the inclusion

$$(3.1) \quad \mathbb{Z}^n \subset \bigcup_{\gamma^* \in \Gamma^*} P_{\gamma^*}.$$ 

Indeed, let $k \in \mathbb{Z}^n$. Denote by $\pi_V : \mathbb{R}^n \to V$ the orthogonal projection on $V$. Then, for any $\gamma \in \Gamma$, we have

$$(\pi_V(k), \gamma) = (k, \gamma) \in \mathbb{Z},$$

since $\gamma \in \mathbb{Z}^n$ and $\gamma \in F \subset V$. Hence, $\pi_V(k) \in \Gamma^*$, that immediately implies (3.1).

For any $\gamma^* \in \Gamma^*$, denote

$$\mathbb{Z}_{\gamma^*}^n = \mathbb{Z}^n \bigcap P_{\gamma^*} = \{k \in \mathbb{Z}^n : \pi_V(k) = \gamma^*\}.$$ 

We identify the affine subspace $P_{\gamma^*}$ with the linear space $V^\perp$, fixing an arbitrary point $k_{\gamma^*} \in \mathbb{Z}_{\gamma^*}^n$:

$$P_{\gamma^*} = k_{\gamma^*} + V^\perp.$$ 

It is easy to see that

$$\mathbb{Z}_{\gamma^*}^n = k_{\gamma^*} + \Gamma^\perp,$$

where

$$\Gamma^\perp = \mathbb{Z}^n \bigcap V^\perp$$

is a lattice in $V^\perp$.

Let us observe the following relation, which will be needed later:

$$\text{vol}(V^\perp / \Gamma^\perp) = |Q|.$$

For its proof, let us choose some base $(\ell_1, \ldots, \ell_r)$ in $\Gamma$. Denote by $(\ell_1^*, \ldots, \ell_r^*)$ the dual base in $\Gamma^*$: $(\ell_i, \ell_j^*) = \delta_{ij}$ for any $i, j = 1, \ldots, r$. For any $i = 1, \ldots, r$, choose some $k_i^* \in \mathbb{Z}^n$ such that $\pi_V(k_i^*) = \ell_i^*$. Let $(k_1^*, \ldots, k_{n-r}^*)$ be a base in $\Gamma^\perp$. Using (3.1), it easy to show that the system $(k_1^*, \ldots, k_r^*, k_1^\perp, \ldots, k_{n-r}^\perp)$ is a base in $\mathbb{Z}^n$. Therefore, for the volume of the parallelepiped spanned by $(k_1^*, \ldots, k_r^*, k_1^\perp, \ldots, k_{n-r}^\perp)$, one has the following formula:

$$\text{vol}_n(k_1^*, \ldots, k_r^*, k_1^\perp, \ldots, k_{n-r}^\perp) = \text{vol}(\mathbb{R}^n / \mathbb{Z}^n) = 1.$$
On the other hand, since $k_i^* - \ell_i^* \in V^\perp$ for any $i = 1, \ldots, r$, and the systems $(\ell_1^*, \ldots, \ell_r^*)$ and $(k_1^\perp, \ldots, k_{n-r}^\perp)$ are mutually orthogonal, we get

$$\text{vol}_n(k_1^*, \ldots, k_r^*, k_1^\perp, \ldots, k_{n-r}^\perp) = \text{vol}_n(\ell_1^*, \ldots, \ell_r^*, k_1^\perp, \ldots, k_{n-r}^\perp) = \text{vol}(V^*/\Gamma) \text{vol}(V^\perp/\Gamma^\perp).$$

Thus, we have

$$\text{vol}(V^\perp/\Gamma^\perp) = \frac{1}{\text{vol}(V^*/\Gamma)}.\$$

In order to complete the proof of (3.2), it remains to apply the well known relation

$$\text{vol}(V/\Gamma) \text{vol}(V^*/\Gamma) = 1.$$

Thus, we can write

$$n_\varepsilon(S) = \sum_{\gamma^* \in \Gamma^*} n_\varepsilon(S, \gamma^*),$$

where

$$n_\varepsilon(S, \gamma^*) = \#(T_\varepsilon(S) \cap \mathbb{Z}_n^\gamma).$$

Note that, since $S$ is bounded, the sum in the right hand side of (3.3) has finitely many non-vanishing terms.

Fix $\gamma^* \in \Gamma^*$. Let $\chi_{S, \gamma^*}$ be the indicator of the set $S, \gamma^* = S \cap P_{\gamma^*}$. It is easy to see that

$$n_\varepsilon(S, \gamma^*) = \sum_{k \in \mathbb{Z}_n^\gamma} \chi_{T_\varepsilon(S, \gamma^*)}(k) = \sum_{\gamma \in \Gamma^\perp} \chi_{T_\varepsilon(S, \gamma^*)}(k_{\gamma^*} + \gamma)$$

$$= \sum_{\gamma \in \Gamma^\perp} \chi_{S, \gamma^*}(T_\varepsilon^{-1}(k_{\gamma^*} + \gamma))$$

$$= \sum_{\gamma \in \Gamma^\perp} \chi_{S, \gamma^*}(k_{\gamma^*} + (T_\varepsilon^{-1}(k_{\gamma^*}) - k_{\gamma^*}) + T_\varepsilon^{-1}(\gamma))).$$

The space $V^\perp$ decomposes into the direct sum

$$V^\perp = F_V \bigoplus H,$$

where $F_V = F \cap V^\perp$. We will write the decomposition of $x \in V^\perp$, corresponding to (3.4), as follows:

$$x = x_F + x_H, \quad x_F \in F_V, x_H \in H.$$

Note that

$$T_\varepsilon(x) = x_F + \varepsilon^{-1}x_H.$$

Let $\rho \in C_0^\infty(\mathbb{R})$ be an even function such that $0 \leq \rho(x) \leq 1$ for any $x \in \mathbb{R}$ and $\text{supp} \rho \subset (-1, 1)$. For any $t_F > 0$ and $t_H > 0$, define a function
\[ \rho_{t_F,t_H} \in C_0^\infty(V^\perp) \] by
\[ \rho_{t_F,t_H}(x) = \frac{c}{t_F^p t_H^q} \rho \left( \left( t_F^{-2} x_F^2 + t_H^{-2} x_H^2 \right)^{1/2} \right), \quad x \in V^\perp, \]
where the constant \( c > 0 \) is chosen so that \( \int_{V^\perp} \rho_{1,1}(x) \, dx = 1 \). The function \( \rho_{t_F,t_H} \) is supported in the ellipsoid
\[ B(0,t_F,t_H) = \left\{ x \in V^\perp : \frac{x_F^2}{t_F^2} + \frac{x_H^2}{t_H^2} < 1 \right\}. \]
Define the function \( n_{\varepsilon,t_F,t_H}(S, \gamma^*) \) by
\[ n_{\varepsilon,t_F,t_H}(S, \gamma^*) = \sum_{k \in \mathbb{Z}_{\gamma^*}} (\chi_{T_\varepsilon(S,*)} * \rho_{t_F,t_H})(k), \]
where the function \( \chi_{T_\varepsilon(S,*)} * \rho_{t_F,t_H} \in C_0^\infty(P_{\gamma^*}) \) is defined by
\[ (\chi_{T_\varepsilon(S,*)} * \rho_{t_F,t_H})(y) = \int_{V^\perp} \chi_{T_\varepsilon(S,*)}(y-x) \rho_{t_F,t_H}(x) \, dx, \quad y \in P_{\gamma^*}. \]
For any domain \( S \subset P_{\gamma^*} \) and for any \( t_F > 0 \) and \( t_H > 0 \), denote
\[ S_{t_F,t_H} = \bigcup_{x \in S} (x+B(0,t_F,t_H)), \]
and
\[ S_{-t_F,-t_H} = P_{\gamma^*} \setminus (P_{\gamma^*} \setminus S)_{t_F,t_H}. \]
It is easy to see that, for any \( \varepsilon > 0 \), \( t_F > 0 \) and \( t_H > 0 \), one has
\[ T_\varepsilon(S_{t_F,t_H}) = (T_\varepsilon(S))_{t_F,t_H}. \]

**Lemma 3.1.** For any \( \varepsilon > 0 \), \( t_F > 0 \) and \( t_H > 0 \), the following inequalities hold:
\[ n_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{-t_F,-t_H}, \gamma^*) \leq n_{\varepsilon}(S, \gamma^*) \leq n_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{t_F,t_H}, \gamma^*). \]

**Proof.** Suppose that \( k \in \mathbb{Z}^n \cap T_\varepsilon(S_{\gamma^*}) \). For any \( x \in B(0,t_F,t_H) \), the point \( k-x \) belongs to \( (T_\varepsilon(S_{\gamma^*}))_{t_F,t_H} \). Therefore, \( \chi_{T_\varepsilon((S_{\gamma^*})_{t_F,t_H})}(k-x) = 1 \) and
\[ (\chi_{T_\varepsilon((S_{\gamma^*})_{t_F,t_H})} * \rho_{t_F,t_H})(k) = \int_{V^\perp} \rho_{t_F,t_H}(x) \, dx = 1 = \chi_{T_\varepsilon(S_{\gamma^*})}(k). \]
If \( k \not\in \mathbb{Z}^n \cap T_\varepsilon(S_{\gamma^*}) \), then
\[ \chi_{T_\varepsilon(S_{\gamma^*})}(k) = 0 \leq (\chi_{T_\varepsilon((S_{\gamma^*})_{t_F,t_H})} * \rho_{t_F,t_H})(k). \]
We get
\[ n_{\varepsilon}(S, \gamma^*) = \sum_{k \in \mathbb{Z}_{\gamma^*}} \chi_{T_\varepsilon(S_{\gamma^*})}(k) \leq \sum_{k \in \mathbb{Z}_{\gamma^*}} (\chi_{T_\varepsilon((S_{\gamma^*})_{t_F,t_H})} * \rho_{t_F,t_H})(k) = n_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{t_F,t_H}, \gamma^*). \]
The second inequality follows of the proven one applied to the domain \( B \setminus S \), where \( B \) is a sufficiently large ball, containing \( S \). \( \square \)
For any \( f \in S(V^\perp) \), define its Fourier transform \( \hat{f} \in S(V^\perp) \) by
\[
\hat{f}(\xi) = \int_{V^\perp} e^{-2\pi i (\xi, x)} f(x) \, dx.
\]

Recall the Poisson summation formula
\[
\sum_{k \in \Gamma^\perp} f(k) = \frac{1}{|Q|} \sum_{k^* \in \Gamma^\perp} \hat{f}(k^*), \quad f \in S(V^\perp),
\]
where \( \Gamma^\perp \subset V^\perp \) is the dual lattice of \( \Gamma^\perp \), and we used the relation (3.2).

Let us apply (3.5) to the function
\[
f(x) = (\chi_{T_\varepsilon((S_{\gamma^*})_{t_F}, t_H)} \ast \rho_{t_F, t_H})(k_{\gamma^*} + x), \quad x \in V^\perp.
\]
The formula (3.5) can be applied, because, for any \( N > 0 \), we have the estimate
\[
|\hat{\rho}_{t_F, t_H}(\xi)| \leq C_N \frac{1}{1 + t_F^N|\xi|_F^N + t_H^N|\xi|_H^N}, \quad \xi \in V^\perp.
\]

One has the relation
\[
\hat{\chi}_{T_\varepsilon((S_{\gamma^*})_{t_F}, t_H)}(\xi) = e^{-q} e^{2\pi i (\xi, (1-T_\varepsilon)(k_{\gamma^*})]} \hat{\chi}_{(S_{\gamma^*})_{t_F}, t_H}(T_\varepsilon(\xi)).
\]
Indeed, for any subset \( S \subset V^\perp \), we have
\[
\hat{\chi}_{T_\varepsilon(S)}(\xi) = \int_{V^\perp} e^{-2\pi i (\xi, x)} \chi_{T_\varepsilon(S)}(k_{\gamma^*} + x) \, dx
\]
\[
= \int_{V^\perp} e^{-2\pi i (\xi, x)} \chi_S(k_{\gamma^*} + (T_{\varepsilon^{-1}}(k_{\gamma^*}) - k_{\gamma^*}) + T_{\varepsilon^{-1}}(x)) \, dx
\]
\[
= e^{-q} \int_{V^\perp} e^{-2\pi i (\xi, T_{\varepsilon^{-1}}(x') + (T_{\varepsilon^{-1}}(k_{\gamma^*}) - k_{\gamma^*}))} \chi_S(k_{\gamma^*} + x') \, dx
\]
\[
= e^{-q} e^{2\pi i (\xi, (1-T_\varepsilon)(k_{\gamma^*}))} \hat{\chi}_S(T_\varepsilon(\xi)).
\]
We also have
\[
\hat{\rho}_{t_F, t_H}(\xi) = \hat{\rho}_{1,1}(t_F \xi_F + t_H \xi_H), \quad \xi \in V^\perp.
\]
By these relations, the Poisson formula (3.5) applied to the function \( f \) given by (3.6) is written as
\[
n_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{t_F}, t_H; \gamma^*) = \frac{e^{-q}}{|Q|} \sum_{k \in \Gamma^\perp} e^{2\pi i (k^*, (1-T_\varepsilon)(k_{\gamma^*}))} \hat{\chi}_{(S_{\gamma^*})_{t_F}, t_H}(T_\varepsilon(\xi)) \hat{\rho}_{1,1}(t_F k_F + t_H k_H).
\]
The series in the right hand side of (3.8) converges uniformly by the estimate (3.7).

Let us write
\[
n_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{t_F}, t_H; \gamma^*) = n_{\varepsilon,t_F,t_H}'((S_{\gamma^*})_{t_F}, t_H; \gamma^*) + n_{\varepsilon,t_F,t_H}''((S_{\gamma^*})_{t_F}, t_H; \gamma^*),
\]
where
\[
n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \sum_{k \in \Gamma_0^+, k_H = 0} \hat{\chi}(S_{\gamma^*})_{t_F, t_H} (T_\varepsilon(k)) \hat{\rho}_{1,1} (t_F k_F + t_H k_H),
\]
and
\[
n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \sum_{k \in \Gamma_0^+, k_H \neq 0} e^{2\pi i (1 - \varepsilon^{-1})(k_H, k_{\gamma^*})} \hat{\chi}(S_{\gamma^*})_{t_F, t_H} (T_\varepsilon(k)) \hat{\rho}_{1,1} (t_F k_F + t_H k_H).
\]

Let \( k \in \Gamma_0^+ \) be such that \( k_H = 0 \). Then \( k \in F_V \). Since \( \Gamma_0^+ \subset Q^n \) and \( F_V \cap Q^n = \{0\} \), we get \( k = 0 \). Thus, we have
\[
n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \hat{\chi}(S_{\gamma^*})_{t_F, t_H} (0)
= \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}((S_{\gamma^*})_{t_F, t_H})
= \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}((S_{\gamma^*})_{t_F, t_H} \setminus S_{\gamma^*}).
\]

We have the estimate
\[
\text{vol}_{n-r}((S_{\gamma^*})_{t_F, t_H} \setminus S_{\gamma^*}) \leq C(t_F + t_H \varepsilon),
\]
therefore, we obtain that
\[
(3.9) \quad n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(t_F \varepsilon^{-q} + t_H \varepsilon^{1-q}).
\]

Consider the case when \( k \in \Gamma_0^+ \) and \( k_H \neq 0 \). For any \( t \in F_V \) and for any domain \( D \subset P_{\gamma^*} \), we denote
\[
D(t) = \{ x_H \in H : k_{\gamma^*} + t + x_H \in D \} \subset H.
\]
For any function \( \phi \in S(H) \), denote by \( F_H(\phi) \in S(H) \) its Fourier transform:
\[
[F_H(\phi)](\xi_H) = \int_H \phi(x_H) e^{-2\pi i (\xi_H, x_H)} \, dx_H, \quad \xi_H \in H.
\]
It is easy to see that
\[
[F_H(\chi_D(t))](\xi_H) = \int_H \chi_D(k_{\gamma^*} + t + x_H) e^{-2\pi i (\xi_H, x_H)} \, dx_H, \quad \xi_H \in H.
\]
Therefore, we get
\[
\hat{\chi}_{(S_{\gamma^*})_{t_F, t_H}}(T_{\varepsilon}(k)) = \int_{V_{\perp}} \chi_{(S_{\gamma^*})_{t_F, t_H}}(k_{\gamma^*} + x) e^{-2\pi i(T_{\varepsilon}(k), x)} dx
\]
\[
= \int_{F_V} \int_{H_{\varepsilon}} \chi_{(S_{\gamma^*})_{t_F, t_H}}(k_{\gamma^*} + x_F + x_H) e^{-2\pi i((k_F, x_F) + \varepsilon^{-1}(k_H, x_H))} dx_F dx_H
\]
\[
= \int_{F_V} e^{-2\pi i(k_F, x_F)} F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\varepsilon^{-1}k_H) dx_F,
\]
and, as a consequence, we obtain the estimate
\[
(3.10) \quad |\hat{\chi}_{(S_{\gamma^*})_{t_F, t_H}}(T_{\varepsilon}(k))| \leq \int_{F_V} |F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\varepsilon^{-1}k_H)| dx_F.
\]
Hence, our considerations are reduced to the sufficiently well studied problem of estimating the Fourier transform of the indicator of a domain, and we can apply existing results.

**Proof of Theorem 1.1.** For any sufficiently small \( \varepsilon > 0 \), \( t_F > 0 \) and \( t_H > 0 \), the domain \( (S_{\gamma^*})_{t_F, t_H}(x_F) \) has smooth boundary. The Stokes formula allows us to write the Fourier transform \( F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)] \) as an oscillating integral over the boundary of \( (S_{\gamma^*})_{t_F, t_H}(x_F) \):

\[
(3.11) \quad F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\tau \omega) = \frac{1}{\tau} \int_{\partial[(S_{\gamma^*})_{t_F, t_H}(x_F)]} e^{-i\tau(\omega, x)} i_{\omega}(dx_1 \wedge \ldots \wedge dx_q),
\]
that implies the estimate
\[
(3.12) \quad |F_H[\chi_{(S_{\gamma^*})_{t_F, t_H}}(x_F)](\xi)| = O(|\xi|^{-1}), \quad |\xi| \to \infty.
\]
Hence, using the estimates \( (3.10), (3.12) \) and \( (3.7) \), we get
\[
(3.13) \quad |n_{\varepsilon, t_F, t_H}^{''}((S_{\gamma^*})_{t_F, t_H}, \gamma^*)| \leq C \varepsilon^{-q} \sum_{k \in \Gamma_{\perp, k_H \neq 0}} \varepsilon |k_H|^{-1} \frac{1}{1 + t_F^N |k_F|^N + t_H^N |k_H|^N}
\]
\[
\leq C \varepsilon^{-q} \int_{V_{\perp}} |x_H|^{-1} \frac{dx_F dx_H}{1 + t_F^N |x_F|^N + t_H^N |x_H|^N}
\]
\[
\leq C \varepsilon^{-q} t_F^{-(p-r)q} t_H^{1-q}.
\]
Putting \( t_H = c_H \) to be a constant, independent of \( \varepsilon > 0 \), and \( t_F = c_F \varepsilon^{\alpha} \), where
\[
\alpha = \frac{1}{p - r + 1},
\]
by \( (3.9) \) and \( (3.13) \), we obtain the statement of the theorem. \( \square \)
Proof of Theorem 1.2. By assumption, the domain $S_{\gamma^*}(x_F) = S \cap \{\gamma^* + x_F + H\}$ is strictly convex. Therefore, for any sufficiently small $\varepsilon > 0$, $t_F > 0$ and $t_H > 0$, the domain $(S_{\gamma^*})_{t_F,\varepsilon t_H}(x_F)$ is strictly convex. By the stationary phase method, we derive from the representation (3.11) the following estimate

\[(3.14) \quad |F_{H}[\chi(S_{\gamma^*})_{t_F,\varepsilon t_H}(x_F)](\xi)| = O(|\xi|^{-(q+1)/2}), \quad |\xi| \to \infty.\]

Thus, using the estimate (3.10), (3.14) and (3.7), we obtain that

\[(3.15) \quad |n''_{\varepsilon,t_F,t_H}((S_{\gamma^*})_{t_F,\varepsilon t_H}, \gamma^*)| \leq C\varepsilon^{-(q+1)/2} \sum_{k \in \Gamma^\perp, k_H \neq 0} \varepsilon^{(q+1)/2}|k_H|^{-(q+1)/2} \frac{1}{1 + t_F^N |k_F|^N + t_H^N |k_H|^N} \]

\[\leq C\varepsilon^{-(q+1)/2} \int_{\gamma^*}^{\perp} |x_H|^{-(q+1)/2} \frac{dx_F \, dx_H}{1 + t_F^N |x_F|^N + t_H^N |x_H|^N} \]

\[\leq C\varepsilon^{-(q-1)/2} t_F^{(p-r)} t_H^{-(q-1)/2}.\]

Put $t_F = \varepsilon^{\alpha_F}$, $t_H = \varepsilon^{\alpha_H}$, where $\alpha_F \geq 0$ and $\alpha_H \geq 0$ are chosen as follows. If $\frac{q}{2} \geq p - r$, then

\[\alpha_F = \frac{2q}{q + 1 + 2(p - r)}, \quad \alpha_H = \frac{q - 1 - 2(p - r)}{q + 1 + 2(p - r)}.\]

If $\frac{q}{2} \leq p - r$, then

\[\alpha_F = \frac{q + 1}{2(p - r + 1)}, \quad \alpha_H = 0.\]

Using the estimates (3.9) and (3.15), we immediately conclude the proof. \( \square \)

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