Exact relativistic models of thin disks around static black holes in a magnetic field

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The exact superposition of a central static black hole with surrounding thin disk in presence of a magnetic field is investigated. We consider two models of disk, one of infinite extension based on a Kuzmin-Chazy-Curzon metric and other finite based on the first Morgan-Morgan disk. We also analyze a simple model of active galactic nuclei consisting of black hole, a Kuzmin-Chazy-Curzon disk and two rods representing jets, in presence of magnetic field. To explain the stability of the disks we consider the matter of the disk made of two pressureless streams of counterrotating charged particles (counterrotating model) moving along electrogodesics. Using the Rayleigh criterion we derivate for circular orbits the stability conditions of the particles of the streams. The influence of the magnetic field on the matter properties of the disk and on its stability are also analyzed.

PACS numbers: 04.20.-q, 04.20.Jb, 04.40.Nr
Keywords: General relativity; Einstein-Maxwell equations; exact solutions; thin disks

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I. INTRODUCTION

There is a strong observational evidence that active galactic nuclei (AGN), X-ray transients and gamma-ray bursts (GRBs) are associated with black holes that accrete matter via a surrounding accretion disk. The exact mechanism by which these phenomena are produced involves the interaction between a rotating black hole, the accretion disk and the electromagnetic field \[1-3\]. However, magnetic fields play a key role in understanding of these processes. The enormous observed energy in AGN is related with the presence of magnetic field in these nuclei \[4-6\]. The existence of radio jets is also attributed to the presence of strong magnetic fields in centers of AGN and quasars \[7-12\].

A general exact relativistic model that describes such astrophysical objects require an exact solution of coupled Einstein-Maxwell field equations that represent the superposition of a Kerr black hole with a stationary disk and electromagnetic fields. As a first approximation one could consider a static system composite by a Schwarzschild black hole and a thin disk immersed in a magnetic field. Exact solutions of the Einstein equations representing the field of a static thin disks without radial pressure were first studied by Bonnor and Sackfield \[13\], and Morgan and Morgan \[14\], and with radial pressure by Morgan and Morgan \[15\]. Several classes of exact solutions of the Einstein field equations corresponding to static thin disks with or without radial pressure have been obtained by different authors \[16-22\]. Rotating thin disks that can be considered as a source of a Kerr metric were presented by Bićák and Ledvinka \[23\], while rotating disks with heat flow were studied by González and Letelier \[24\].

In this work we consider the exact static superposition of a Schwarzschild black hole and a thin disk in presence of a magnetic field. The method used to include the magnetic field is the well-known complex potential formalism proposed by Ernst \[39, 40\], using as seed solutions simple vacuum spacetimes representing the field of a thin disk and a black hole.

The paper is organized as follows. In Sec. II we discuss the Einstein-Maxwell equations in the case of magnetostatic fields and we present a summary of the procedure to obtain models of thin disks with a purely azimuthal pressure and current. In order to have a stable configuration in absence of radial pressure, the matter in the disks also interpreted as made of two pressureless (dust) streams of counterrotating charged particles (counterrotating model) moving along electrogeodesic. Using the Rayleigh criterion we derive for circular orbits the stability conditions of the particles of both streams.

In Sec. III the formalism for superposing the field of a disk and a static black hole in the vacuum \[26\] is extended to the case of magnetostatic fields. In Sec. IV we consider two models of disks, one of infinite extension based on a Kuzmin-Chazy-Curzon metric and other finite based on the first Morgan-Morgan disk. Also a simple model of active galactic nuclei is studied based on the superposition of a black hole, a Kuzmin-Chazy-Curzon disk and two rods representing jets \[27\], in presence of magnetic field. Finally, in Sec. V we summarize and discuss the results obtained.

II. EINSTEIN-MAXWELL EQUATIONS AND DISKS

The line element for a static axially symmetric spacetime in Weyl’s canonical coordinates \((t, \varphi, \rho, z)\) is given by \[41\]

\[
\text{ds}^2 = -e^{2\psi}dt^2 + e^{-2\psi}[\rho^2d\varphi^2 + e^{2\Lambda}(d\rho^2 + dz^2)],
\]

where \(\psi\) and \(\Lambda\) are functions of the coordinates \(\rho\) and \(z\) only. The vacuum Einstein-Maxwell equations, in geometrized units such that \(G = c = 1\), are given by

\[
R_{ab} = 8\pi T_{ab}, \tag{2a}
\]

\[
\nabla_a F^{ab} = 0, \tag{2b}
\]

where

\[
T_{ab} = \frac{1}{4\pi} \left[ F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right] \tag{3}
\]

is the electromagnetic energy-momentum tensor,

\[
F_{ab} = A_{b,a} - A_{a,b} \tag{4}
\]
the electromagnetic field tensor, and $A_a$ the electromagnetic four potential. For magnetostatic axially symmetric fields $A_a = \delta^z_a A$, where $A$ is the magnetic potential which also is function of $r$ and $z$ only. The other symbols have the usual meaning, i.e., $(\cdot)_\alpha = \partial/\partial x^\alpha$, $\nabla_b$ covariant derivate, etc.

For the metric (1), magnetostatic Einstein-Maxwell equations give

$$
\nabla \cdot [\rho^2 f \nabla A] = 0, \\
\rho \nabla^2 f = \nabla f \cdot \nabla f + 2 \rho^2 f^3 \nabla A \cdot \nabla A,
$$

(5a)

$$
\nabla_\rho = \rho (\psi^2 \rho - \psi^2 z) + \frac{1}{\rho} \left( A^2 \rho - A^2 z \right) f,
$$

(5b)

$$
\nabla_z = 2 \rho \psi \rho \psi_z + \frac{2}{\rho} A_{\rho} A_{\rho} f,
$$

(5c)

$$
\text{where } \nabla \text{ is the standard differential operator in cylindrical coordinates and } f = e^{2\psi}. \text{ In vacuum, the solutions of the above system of equations correspond to the well-known Weyl solutions } (\psi, \Lambda) \text{ and the equation (5d) is the Laplace's equation.}
$$

Solutions of the Einstein-Maxwell equations (2a) - (2b) representing the field of a thin disk at $z = 0$ with electric current can be constructed assuming the components of the metric tensor and the electromagnetic potential continuous across the disk, and its first derivatives discontinuous in the direction normal to the disk. This can be written as

$$
b_{ab} = [g_{ab, z}] = g_{ab, z} |_{z=0^+} - g_{ab, z} |_{z=0^-} = 2 \ g_{ab, z} |_{z=0^+},
$$

(6a)

$$
a_b = [A_{b, z}] = A_{b, z} |_{z=0^+} - A_{b, z} |_{z=0^-} = 2 \ A_{b, z} |_{z=0^+}.
$$

(6b)

The application of the formalism of distributions in curved spacetimes to the Einstein-Maxwell equations (42) - (46) give us

$$
R_{ab} = 8 \pi T_{ab},
$$

(7a)

$$
T_{ab} = T_{ab}^{\text{elm}} + T_{ab}^{\text{mat}} = T_{ab}^{\text{elm}} + Q_{ab} \delta(z),
$$

(7b)

$$
\nabla_b F^{ab} = 4 \pi j^a,
$$

(7c)

$$
j^a = \frac{1}{4 \pi} [F^{ab}] \delta^a _b,
$$

(7d)

where $\delta(z)$ is the usual Dirac function with support on the disk, $T_{ab}^{\text{elm}}$ is the electromagnetic tensor (3),

$$
Q^a _b = \frac{1}{16 \pi} \{ b^{zz} \delta^a _b - b^{zz} \delta^a _b + g^{aa} b^b _b - g^{zz} b^b _b + b^c (g^{zz} \delta^a _b - g^{aa} \delta^a _b) \}
$$

(8)

is the energy-momentum tensor on plane $z = 0$, and

$$
j^a = \frac{1}{4 \pi} [F^{ab}] \delta^a _b,
$$

(9)

is the electric current density on the disk. $[F^{ab}]$ means the jump of Maxwell tensor across the disk. The “true” surface energy-momentum tensor of the disk $S_{ab}$ and the “true” surface current density $j_a$ are given by

$$
S_{ab} = \int T_{ab}^{\text{mat}} \ ds_n = \sqrt{g_{zz}} Q_{ab},
$$

(10a)

$$
j_a = \int j_a \ ds_n = \sqrt{g_{zz}} j_a,
$$

(10b)

where $ds_n = \sqrt{g_{zz}} \ dz$ is the “physical measure” of length in the direction normal to the disk. For the metric (1), the nonzero components of $S^b _a$ are

$$
S^t _t = \frac{1}{4 \pi} e^{\psi - \Lambda} \left\{ \Lambda, z - 2 \psi, z \right\},
$$

(11a)

$$
S^\varphi _\varphi = \frac{1}{4 \pi} e^{\psi - \Lambda} \Lambda, z,
$$

(11b)

and in the magnetostatic case the only nonzero component of the current density is

$$
j_c = -\frac{1}{2 \pi} e^{\psi - \Lambda} A, z,
$$

(12)
where all the quantities are evaluated at $z = 0^+$.

In terms of the orthonormal tetrad $e_{(a)}^b = \{V^b, W^b, X^b, Y^b\}$, where

$$V^a = e^{-\psi} (1, 0, 0, 0), \quad W^a = \frac{e^\psi}{\rho} (0, 1, 0, 0),$$

$$X^a = e^{-\Lambda}(0, 0, 1, 0), \quad Y^a = e^{-\Lambda}(0, 0, 0, 1),$$

the surface energy density $\epsilon$ and the azimuthal pressure $p_\varphi$ on the disk are given by

$$\epsilon = -S^t_t, \quad p_\varphi = S^\varphi_\varphi,$$

and the azimuthal current density $j$ by

$$j = W^\varphi j_\varphi.$$

Thus we have a disk only with pressure and electric current in azimuthal direction. Because there is no radial pressure or tension to support the gravitational attraction, the matter distribution is unstable. In addition, since the spacetime is static we have no rotation. In order to have a stable configuration in absence of radial pressure, we need assume the counterrotating hypothesis, that is the matter in the disk is considered made of two pressureless streams of counterrotating charged particles, i.e., that circulate in opposite directions. Even though this interpretation can be seen as merely theoretical, there are observational evidence of counterrotating matter components in certain types of galaxies [47, 51]. We assume

$$S^{ab} = S^{ab}_+ + S^{ab}_-,$$

$$j^a = j^a_+ + j^a_-,$$

where

$$S^{ab}_\pm = \epsilon_\pm u^a_\pm u^b_\pm,$$

$$j^a_\pm = \sigma_\pm u^a_\pm,$$

$\epsilon_\pm$ are the matter densities of each stream, $\sigma_\pm$ the electric charge densities, and $u^a_\pm$ the normalized four-velocities ($u^a_\pm u_{a\pm} = -1$), which for circular orbit are $(u^a_\pm) = (u^1_\pm, u^1_\pm, 0, 0) = u^0_\pm (1, \omega_\pm, 0, 0)$, where $\omega_\pm = u^1_\pm / u^0_\pm$ are the angular velocities of each stream. Now, using the continuity equation (the Bianchi identity) $T^{ab} = 0$ and Eq. (7b),

$$T^{ab} = T^{elm} + T^{mat},$$

we obtain

$$T^{ab} = -T^{ab}_{elm},$$

But from Maxwell equations $T^{ab}_{elm} = -F^a_{\varphi b} J^\varphi$, then $T^{ab}_{mat} = F^a_{\varphi b} J^\varphi$. In the disk this gives $S^{ab}_{\varphi b} = F^a_{\varphi b} J^\varphi$ and using (16a) and (16b) we have $S^{ab}_{\varphi b} = F^a_{\varphi b} j_\varphi$. Thus using (17a) and (17b) we obtain

$$\epsilon_\pm u^b_\pm u^a_\pm = \sigma_\pm F^a_\varphi j^b_\varphi,$$

i.e., each stream follow a electrogeodesic motion. For circular orbits reads

$$\frac{1}{2} \epsilon_\pm g_{ab,\varphi} u^a_\pm u^b_\pm = -\sigma_\pm F_{\rho\varphi} u^a_\pm,$$

and in the case magnetostatic

$$\frac{1}{2} \epsilon_\pm u^a_\pm \left( g_{\varphi\rho,\varphi} u^2_\pm + g_{tt,\rho} \right) = -\sigma_\pm A_{\rho} \omega_\pm,$$

where $u^0_\pm$ obtains normalizing $u^a_\pm$, that is

$$\left( u^0_\pm \right)^2 = \frac{1}{g_{\varphi\varphi} \omega^2_\pm + g_{tt}}.$$

Using (16a) - (17b) and (20) one finds [52, 53]

$$\epsilon_\pm \left( u^0_\pm \right)^2 = \frac{-\omega_\pm S_{tt}^\varphi}{\omega_\pm - \omega^\varphi_+},$$

$$\sigma_\pm u^0_\pm = \frac{j^\varphi}{\omega_\pm - \omega^\varphi_+},$$

$$\omega^2_\pm = \frac{-g_{tt,\rho}}{g_{\varphi\rho,\varphi} + 2A_{\rho} J^\varphi / S^\varphi_\varphi}.$$
With respect to the orthonormal tetrad \( \{ e_\alpha \} \), the 3-velocity has components
\[
v^{(i)}_\pm = \frac{e^{(i)}_a u^a_\pm}{e^{(0)}_b u^b_\pm},
\]
and for equatorial circular orbits the only nonvanishing velocity components is given by
\[
(v_{\pm}^{(\rho)})^2 = v^2 = -\frac{g_{\varphi\varphi}}{g_{tt}} \omega^2_\pm = \rho^2 e^{-4\psi} \omega^2_\pm,
\]
which represents the circular speed of the particles as seen by an observer at infinity. In vacuum the speed \( v \) of counterrotation (rotation curves or rotation profile) of the particles in the disk is given by
\[
v^2 = \frac{\tilde{e} \epsilon}{\epsilon}.
\]

To analyze the stability of the particles of the two streams in the case of circular orbits in the equatorial plane we use an extension of Rayleigh criteria of stability of a fluid at rest in a gravitational field. The method works as follows. Any small element of the matter distribution analyzed (in our case a test particle of the streams) is displaced slightly from its path. As a result of this displacement, forces appear which act on the displaced matter element. If the matter distribution is stable, these forces must tend to return the element to its original position.

The relativistic Lagrangian for a test particle of the streams in presence of a gravitational and magnetic field is given by
\[
\mathcal{L}_\pm = \frac{1}{2} g_{ab} u^a_\pm u^b_\pm + \tilde{e}_\pm A u^\varphi_\pm,
\]
where \( \tilde{e}_\pm \) is the specific electric charge. For magnetostatic axially symmetric fields there are two constants of motion
\[
\begin{align*}
E_\pm &= -g_{tt} u^0_\pm, \\
L_\pm &= g_{\varphi\varphi} \omega_\pm u^0_\pm + \tilde{e}_\pm A,
\end{align*}
\]
where \( E_\pm \) is the relativistic specific energy and \( L_\pm \) the specific angular momentum. The motion equation (20) can be cast as a balance equation
\[
\frac{g^{\rho\rho} g_{tt} E^2_\pm}{2g^2_{tt}} + e g^{\rho\rho} A_{,\rho} \omega_\pm u^0_\pm = -\frac{g^{\rho\rho} g_{\varphi\varphi} (L_\pm - \tilde{e}_\pm A)^2}{2g^2_{\varphi\varphi}}
\]
where the term first on the left-hand side represents the gravitational force \( F_g \), the term second the the Lorentz force \( F_L \), and the term on the right-hand side the centrifugal force \( F_c(\rho) = F_c(\rho, L_\pm(\rho)) \) acting on the test particle. So we have a balance between the total force \( F(\rho) = F_g + F_L \) and the centrifugal force. We now consider the particle to be initially in a circular orbit with radius \( \rho = \rho_0 \) and we slightly displace it to a higher orbit \( \rho > \rho_0 \). The angular momentum of particle remains equal to its initial value \( L_{\pm(0)} = L_{\pm}(\rho_0) \) which implies that the centrifugal force in its new position is \( F_c(\rho, L_{\pm(0)}) \). In order that the particle returns to it initial position must be met that \( F(\rho) > F_c(\rho, L_{\pm(0)}) \), but according to the balance equation (29) \( F(\rho) = F_c(\rho, L_{\pm}) \) so that \( F_c(\rho, L_{\pm}) > F_c(\rho, L_{\pm(0)}) \), and hence \( (L_\pm - \tilde{e}_\pm A)^2 > (L_{\pm(0)} - \tilde{e}_\pm A)^2 \). Using the expression for \( L_\pm \) (28) and defining the function \( h_\pm = g_{\varphi\varphi} \omega_\pm u^0_\pm \), follows that \( h_\pm(\rho)^2 > h_\pm(\rho_0)^2 \). The quantity \( h_\pm \) can be written as
\[
h_\pm = \frac{\rho e^{-4\psi} v}{\sqrt{1 - \tilde{e}_\pm^2}},
\]
and has the same form that the specific angular momentum in the vacuum, being the true specific angular momentum corresponding to expression (28). By doing a Taylor expansion of \( h^2_\pm(\rho) \) around \( \rho = \rho_0 \) one finds that the condition of stability for equatorial circular orbits is
\[
h_\pm h_{\pm,\rho} > 0,
\]
or, in other words, \( h^2_\pm h_{\rho} > 0 \). Thus when the counterrotating hypothesis is assumed, the stability of the disks is equivalent to the stability of the particles of the two streams.
III. SUPERPOSITION OF A BLACK HOLE AND A THIN DISK IN A MAGNETIC FIELD

We consider the superposition of a Schwarzschild black hole \((\hat{\psi}_S, \hat{\Lambda}_S)\) with a thin disk \((\hat{\psi}_D, \hat{\Lambda}_D)\) in presence of a magnetic field. The Schwarzschild black hole metric functions with mass \(m\) are given by

\[
\hat{\psi}_S = \frac{1}{2} \ln \left( \frac{x - 1}{x + 1} \right),
\]

\[
\hat{\Lambda}_S = \frac{1}{2} \ln \left( \frac{x^2 - 1}{x^2 - y^2} \right),
\]

where \((x, y)\) are prolate spheroidal coordinates which are related to Weyl coordinates \((\rho, z)\) by

\[
\rho^2 = k^2(x^2 - 1)(1 - y^2), \quad z = kxy,
\]

\[
2kx = r_+ + r_-, \quad 2ky = r_+ - r_-, \quad r_+^2 = \rho^2 + (z \pm k)^2,
\]

with \(x \geq 1\) and \(-1 < y < 1\). For the black hole \(k = m\). Since the metric function \(\hat{\psi}\) satisfies the Laplace’s equation and it is linear, then the superposition \(\hat{\psi} = \hat{\psi}_S + \hat{\psi}_D\) is also solution. The other metric function \(\hat{\Lambda}\) is nonlinear but holds the relation \[26\]

\[
\hat{\Lambda} = \hat{\Lambda}_S + \hat{\Lambda}_D + \hat{\Lambda}_{SD},
\]

where

\[
\hat{\Lambda}_{SD} = 2 \int \rho \{ (\hat{\psi}_{S,\rho} \hat{\psi}_{D,\rho} - \hat{\psi}_{S,\rho} \hat{\psi}_{D,\rho}) d\rho + (\hat{\psi}_{S,\rho} \hat{\psi}_{D,\rho} + \hat{\psi}_{S,\rho} \hat{\psi}_{D,\rho}) dz \}. \tag{35}
\]

The magnetic field can be included using the Ernst’s method. The application of such procedure to the system \[32\] \[5a]-\[5d] yield \[32\]

\[
\begin{align*}
\beta + 1 & = \frac{1}{(\beta + 1) e^{-\hat{\psi}} - (\beta - 1) e^\hat{\psi}}, \\
\Lambda & = \hat{\Lambda} = \int \rho \{ (\hat{\psi}^2_{\rho} - \hat{\psi}^2_{\rho}) d\rho + 2 \hat{\psi}_{\rho} \hat{\psi}_{\rho} dz \}, \\
A & = b \int \rho \{ - \hat{\psi}_{\rho} d\rho + \hat{\psi}_{\rho} dz \},
\end{align*}
\]

where \(\beta = \sqrt{1 + b^2}\), being \(b\) the parameter that controls the magnetic field. In absence of magnetic field \((b = 0)\), these solutions reduce to the Weyl vacuum solutions and accordingly these fields can be called magnetized Weyl fields. Thus, calling \(A_S\) the magnetic potential associated to the black hole and \(A_D\) the magnetic potential associated to the disk field, according to \[36c\] the magnetic potential for the composite system is given by

\[
A = A_S + A_D. \tag{37}
\]

The metric potential \(\Lambda\) is the same that the seed potential and the other metric function \(\psi\) obtains using \[36a\].

To obtain the components of energy-momentum tensor of the combined system we shall compute the nonzero components of \(b_{ab}\). From \[40\] we have

\[
\begin{align*}
b_{tt} & = [g_{tt,z}] = -2e^{2\hat{\psi}}[\hat{\psi}_z], \\
b_{\varphi\varphi} & = [g_{\varphi\varphi,z}] = -2\rho^2 e^{-2\hat{\psi}}[\hat{\psi}_z], \\
b_{\rho\rho} & = b_{zz} = [g_{zz,z}] = 2e^{2(\Lambda - \psi)}([\Lambda, z] - [\hat{\psi}_z]).
\end{align*}
\]

From \[36a\] \(\psi_z = F\hat{\psi}_z\) where

\[
F = \frac{(\beta + 1)e^{-\hat{\psi}} + (\beta - 1)e^{\hat{\psi}}}{(\beta + 1)e^{-\hat{\psi}} - (\beta - 1)e^{\hat{\psi}}}. \tag{39}
\]

Furthermore, for the black hole we have \([\hat{\psi}_{S,z}] = [\Lambda_{S,z}] = 0\) whereas for the disk \([\hat{\psi}_{D,z}] = 2\hat{\psi}_{D,z}\), and \([\Lambda_{D,z}] = 2\Lambda_{D,z}\). Thus

\[
[\psi_z] = [F\hat{\psi}_z] = F[\hat{\psi}_z] = F([\hat{\psi}_{S,z}] + [\hat{\psi}_{D,z}]) = 2F\hat{\psi}_{D,z}. \tag{40}
\]
Using (78) and (36b) we have
\[ [\Lambda, z] = [\Lambda_S, z] + [\Lambda_D, z] + [\Lambda_{SD}, z] \]
\[ = 2\Lambda_D, z + [\Lambda_{SD}, z] \]
\[ = 4\rho \hat{\psi}_D, \rho \hat{\psi}_D, z + [\Lambda_{SD}, z]. \]  
(41)

But from (35) \( \Lambda_{SD}, z = 2\rho (\hat{\psi}_S, \rho \hat{\psi}_D, z + \hat{\psi}_S, z \hat{\psi}_D, \rho) \), then
\[ [\Lambda_{SD}, z] = 2\rho \hat{\psi}_S, \rho [\hat{\psi}_D, z] = 4\rho \hat{\psi}_S, \rho \hat{\psi}_D, z, \]  
(42)

and hence
\[ [\Lambda, z] = 4\rho (\hat{\psi}_S + \hat{\psi}_D), \rho \hat{\psi}_D, z. \]  
(43)

Using (40) and (43) we have
\[ b_{\text{tt}} = -4e^{2\psi} F \hat{\psi}_D, z, \]  
(44a)
\[ b_{\phi \phi} = -4\rho^2 e^{-2\psi} F \hat{\psi}_D, z, \]  
(44b)
\[ b_{\rho \rho} = 4e^{2(\Lambda - \psi)} (2\rho (\hat{\psi}_S + \hat{\psi}_D), \rho - F) \hat{\psi}_D, z. \]  
(44c)

From (8), (10a), (14) and (44a)-(44c), we obtain
\[ \epsilon = -S_t^t = \frac{1}{2\pi} e^{\psi - \Lambda} (F - \rho (\hat{\psi}_S + \hat{\psi}_D), \rho) \hat{\psi}_D, z, \]  
(45)
\[ p_{\varphi} = S_{\varphi}^\varphi = \frac{1}{2\pi} e^{\psi - \Lambda} \rho (\hat{\psi}_S + \hat{\psi}_D), \rho \hat{\psi}_D, z, \]  
(46)
where all the quantities are evaluated at \( z = 0^+ \).

Similarly, for the magnetic potential we have \([A, z] = [A_S, z] + [A_D, z]\), but from (36c) \([A_S, z] = [b\rho \psi_S, \rho] = 0\), so that \([A, z] = [A_D, z]\). Thus, from (9), (10b) and (15) one finds that if \(A_D, z\) is assumed discontinuous across the disk the current density is given by
\[ j = -\frac{1}{2\pi \rho} e^{2\psi - \Lambda} A_{D, z}. \]  
(47)

The speed \( v \) of counterrotating of the particles in the disk is given by
\[ v^2 = \frac{F \rho (\hat{\psi}_S + \hat{\psi}_D), \rho}{1 - F \rho (\hat{\psi}_S + \hat{\psi}_D), \rho + \frac{1}{8\pi} b^2 f}. \]  
(48)

IV. SOME SIMPLE EXAMPLES

A. Black hole surrounded by Kuzmin-Chazy-Curzon disks in a magnetic field

Exact solutions which represent the field of a disk can be obtained using the well known “displace, cut and reflect” method that was first used by Kuzmin \[57\] and Toomre \[58\] to constructed Newtonian models of disks, and later extended to general relativity \[20, 21, 23, 24\]. Given a solution of the Einstein-Maxwell equation, this procedure is mathematically equivalent to apply the transformation \( z \rightarrow |z| + z_0 \), with \( z_0 \) constant. The resulting disks are essentially of infinite extension. This method applied to the Chazy-Curzon solution \[59, 60\] produces a Kuzmin-Chazy-Curzon disk of mass \( M \) with metric functions
\[ \hat{\psi}_D = \frac{M}{\sqrt{\rho^2 + (|z| + z_0)^2}}, \]  
(49a)
\[ \hat{\Lambda}_D = -\frac{M^2 \rho^2}{2[\rho^2 + (|z| + z_0)^2]^2}. \]  
(49b)

This solution is the relativistic generalization of the newtonian Kuzmin disk.
Now we consider the superposition of a black hole and this disk in a magnetic field. For this system the magnetic potential \(37\) is
\[
A = bm(y + 1) + bM \left( \frac{|z| + z_0}{\sqrt{\rho^2 + ((|z| + z_0)^2)} + 1} \right),
\]
where the first term on right-hand side is \(A_S\) and the second \(A_D\). The interaction term \(\Lambda_{SD}\) between the disk and the black hole can be calculated in prolate coordinates. In this coordinates the expression for \(\Lambda\) takes the form
\[
\Lambda_x = \left(1 - \frac{y^2}{x^2 - y^2}\right) \left[ x(x^2 - 1)\dot{\psi}_x^2 - x(1 - y^2)\dot{\psi}_y^2 - 2y(x^2 - 1)\dot{\psi}_x\dot{\psi}_y \right],
\]
\[
\Lambda_y = \left(\frac{x^2 - 1}{x^2 - y^2}\right) \left[ y(x^2 - 1)\dot{\psi}_x^2 - y(1 - y^2)\dot{\psi}_y^2 + 2x(1 - y^2)\dot{\psi}_x\dot{\psi}_y \right],
\]
and thus from (51b) we obtain
\[
\Lambda_{SD} = 2 \int_{-1}^{y} \left( \frac{x^2 - 1}{x^2 - y^2} \right) \left\{[y(x^2 - 1)\dot{\psi}_{D,x} + x(1 - y^2)\dot{\psi}_{D,y}]\dot{\psi}_{S,x} + (1 - y^2)(x\dot{\psi}_{D,x} - y\dot{\psi}_{D,y})\dot{\psi}_{S,y} \right\} dy,
\]
where the integral limits are chosen by requiring that the function \(\Lambda\) to be regular on the axis of symmetry. But \(\psi_{S,y} = 0\) and \(\dot{\psi}_{S,x} = 1/(x^2 - 1)\), then
\[
\Lambda_{SD} = 2 \int_{-1}^{y} \frac{1}{(x^2 - y^2)} \left[y(x^2 - 1)\dot{\psi}_{D,x} + x(1 - y^2)\dot{\psi}_{D,y} \right] dy.
\]
\[
\text{For } z > 0 \text{ this integral gives}
\[
\Lambda_{SD} = \frac{2Mm}{(z_0^2 - m^2)\sqrt{\rho^2 + (z + z_0)^2}} \left[ mx + z_0y - \sqrt{\rho^2 + (z + z_0)^2} \right],
\]
or in Weyl coordinates
\[
\Lambda_{SD} = \frac{M}{(z_0^2 - m^2)\sqrt{\rho^2 + (z + z_0)^2}} \left\{ [z_0 + m]\sqrt{\rho^2 + (z + m)^2} - [z_0 - m]\sqrt{\rho^2 + (z - m)^2} - 2m\sqrt{\rho^2 + (z + z_0)^2} \right\}.
\]

From (45)–(48), the main physical quantities associated with the system are
\[
\ddot{\epsilon} = \frac{4Me^{\psi-\Lambda}}{(\rho^2 + z_0^2)^{3/2}} \left[ \frac{m}{\sqrt{\rho^2 + m^2}} - \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right],
\]
\[
\ddot{p}_r = \frac{4Me^{\psi-\Lambda}}{(\rho^2 + z_0^2)^{3/2}} \left[ \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right],
\]
\[
j = \frac{Mb}{2\pi e^{\psi-\Lambda}} \frac{\rho}{(\rho^2 + z_0^2)^{3/2}},
\]
\[
v^2 = \frac{F \left[ \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right]}{1 - F \left[ \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right]} + \frac{1}{8\pi h^2} f,
\]
with \(\ddot{\epsilon} = 8\pi \dot{\epsilon}\) and \(\ddot{p}_r = 8\pi \dot{p}_r\).

Figs. 1 and 2 show curves of the energy density \(\ddot{\epsilon}\), the azimuthal pressure \(\ddot{p}_r\), the azimuthal electric current density \(j\), the speed \(v^2\) of counterrotation and the specific angular momentum \(h^2\) as function of \(\rho\) for \(z_0 = 2\), \(m = 0.1\), \(M = 0.2\), and different values of magnetic field parameter \(b = 0, 0.5, 1,\) and \(1.5\). We see that energy density is a positive quantity in concordance with the weak energy condition, as well as the stress in azimuthal direction (pressure). Eq. (57) shows that we have pressure for all values of parameters. Since \(\ddot{\epsilon} + p_r > 0\), the strong energy condition is also satisfied. These properties characterize a distribution of matter with the usual gravitational attractive property. However, the dominant energy condition \((\nu \leq 1)\) is not satisfied in the central region located between the black hole and the photon radius, and the magnetic field enhances the zone of superluminal speed. Since the function \(F\) tends to one and \(f\) to zero as \(\rho\) tends a zero, the expression (59) shows that these systems present the same behavior for all values of the parameters. We also observe that these structures present strong instabilities inside the photonic orbit, and that the magnetic field increases the region of instability.
B. BLACK HOLES SURROUNDED BY FINITE DISKS IN AN MAGNETIC FIELD

Solutions representing the field of a finite thin disk can be obtained resolving the Laplace equation in oblate spheroidal coordinates \((u,v)\), which are related to prolate coordinates by \(x = -iu\), \(y = v\), and \(k = ia\), where \(a\) is the radius of the disk, and with the Weyl coordinates \((\rho,z)\) by

\[
\rho^2 = a^2(u^2 + 1)(1 - v^2),
\]
\[
z = auv,
\]
with \(u \geq 0\) and \(-1 < v < 1\). These solutions are known in the literature as the Morgan-Morgan solutions [14] and the metric function \(\hat{\psi}_D\) is given by

\[
\hat{\psi}_D = -\sum_{n=0}^{\infty} c_{2n} q_{2n}(u) P_{2n}(v),
\]
where \(c_{2n}\) are constants, \(P_{2n}\) are the Legendre polynomials of order \(2n\) and

\[
q_n(u) = i^{n+1} Q_n(iu),
\]
being \(Q_n(iu)\) the Legendre functions of the second kind. For \(n = 0\) we have the zeroth order Morgan-Morgan disk and for the terms \(n = 0\) and \(n = 1\) the first Morgan-Morgan disk [56]. In this case, the metric functions are

\[
\hat{\psi}_D = -\left(\frac{M}{a}\right) \cot^{-1}(u) - \frac{1}{4} \left(\frac{M}{a}\right) \left(3u^2 - 1\right) \left(3u^2 + 1\right) \cot^{-1}(u) - 3u, \tag{63a}
\]
\[
\hat{\Lambda}_D = -\frac{9M^2}{16a^2} \rho^2 (9u^2v^2 + v^2 - u^2 - 1) + \frac{9M^2}{8a^2} u(1 - v^2)(9u^2v^2 + 7v^2 - u^2 + 1) \cot^{-1}(u)
- \frac{9M^2}{16a^2} (1 - v^2)(9u^2v^2 + 4v^2 - u^2 + 4), \tag{63b}
\]
where \(M\) is the mass of the disk.

We consider the system composed of a black hole and a disk in a magnetic field. The magnetic potential is given by

\[
A = bm(y + 1) + ab(M/a)(v + 1) + \frac{1}{2} ab(M/a)v(1 - v^2) \left[3u \left(u^2 + 1\right) \cot^{-1}(u) - 3u^2 - 2\right], \tag{64}
\]
and, in oblate coordinates, the interaction term [53] takes the form

\[
\Lambda_{SD} = \frac{2k}{a} \int_{-1}^{v} \left(\frac{v(u^2 + 1)\hat{\psi}_D,u + u(1 - v^2)\hat{\psi}_D,v}{u^2 + v^2}\right) dv. \tag{65}
\]
For \(z > 0\) this integral gives

\[
\Lambda_{SD} = \frac{3mM}{a^2} (1 - v^2) \left(u \cot^{-1}(u) - 1\right). \tag{66}
\]

In oblate coordinates the disk is located in \(u = 0\), \(-1 < v < 1\), and when the disk is crossed the coordinate \(v\) changes of sign but not its value absolute, whereas \(u\) is continuous. Hence the gravitational field is continuous across the disk whereas its normal derivate is discontinuous. The same does not occur with the magnetic potential which is discontinuous across the disk but its first derivate is continuous in the direction normal to the disk. This implies that the source of the magnetic field is non planar but of a different origin such as a remnants or fossil magnetic field [61], or can come from external sources, such as the presence of a nearby magnetars or neutron stars. These models can be interpreted as the superposition of a black hole and a finite disk immersed in a magnetic field (see Appendix).
From (45)-(47) we obtain

$$\dot{\epsilon} = \frac{12M}{a^3} e^{\psi - \Lambda} \sqrt{1 - \frac{\rho^2}{a^2}} \left[ F - \frac{3\pi M \rho^2}{4 a^2} - \frac{m}{\sqrt{m^2 + \rho^2}} \right],$$  

(67)

$$\tilde{p}_\varphi = \frac{12M}{a^3} e^{\psi - \Lambda} \sqrt{1 - \frac{\rho^2}{a^2}} \left[ \frac{3\pi M \rho^2}{4 a^2} + \frac{m}{\sqrt{m^2 + \rho^2}} \right],$$  

(68)

$$v^2 = \frac{F [ (3\pi/4) M \rho^2/a^2 + m/\sqrt{m^2 + \rho^2}] - \tilde{p}_\varphi}{1 - F [ (3\pi/4) M \rho^2/a^2 + m/\sqrt{m^2 + \rho^2}]}.$$  

(69)

Now, in order to construct a planar source for the magnetic field it is necessary to include the electric field. The electric potential is given by

$$\phi = c \left[ \frac{e^{-\psi} + e^{\psi}}{(\beta + 1) e^{-\psi} - (\beta - 1) e^{\psi}} \right],$$  

(70)

where $\beta^2 = 1 + b^2 + c^2$, being $c$ the parameter that controls the electric field. The energy density and the pressure have the same form that the expressions (67) and (68) and the only non-zero component of the current density $j_a$ on the plane $z = 0$ is

$$j_t = -\frac{1}{2\pi} e^{\psi - \Lambda} \phi_z.$$  

(71)

In terms of the tetrad $13a$ - $13b$ the electric charge density $\sigma$ is given by

$$\sigma = -V^0 j_t.$$  

(72)

However, since there is no electric current even generating the magnetic field, we need assume the counterrotating hypothesis. Using $16a$ - $17b$ and $19$ we calculate the physical quantities associated to the two streams

$$(u_0^0)^2 \epsilon_\pm = \frac{g^{00} \omega_\pm \epsilon}{(\omega_\pm - \omega_\mp)},$$  

(73a)

$$u_\pm^0 \sigma_\pm = -\frac{V^0 \omega_\pm \sigma}{(\omega_\pm - \omega_\mp)},$$  

(73b)

$$\omega_\pm = \frac{T_2 \pm \sqrt{T_2^2 - T_1 T_3}}{T_1},$$  

(73c)

where

$$T_1 = g_{\varphi \varphi, \rho},$$  

(74a)

$$T_2 = -\frac{j_0 A_{\rho}}{\epsilon},$$  

(74b)

$$T_3 = g_{tt, \rho} - 2j_0 \phi_{\rho},$$  

(74c)

and $u_0^0$ is given by (21). The counterrotating speed $v$ can be obtained from (24).

In Fig. 3 we show the energy density $\dot{\epsilon}$, the azimuthal pressure $\tilde{p}_\varphi$, the speed $v^2$ of counterrotation, and the specific angular momentum $h^2$ for a system consisting of a black hole and the first Morgan-Morgan finite disk in presence of a magnetic field with radius $a = 1$, $m = M = 0.1$, and different values of magnetic field parameter $b = 0$ (dashed curves), 0.5, 1, and 2 (dash-dotted curves), as functions of $\rho$. We find that the energy density is a positive quantity in agreement with the weak energy condition, and we have pressure for all values of the parameters. However, the dominant energy condition ($v \leq 1$) is not satisfied and the presence magnetic field enhances the zones of superluminal speed. Such behavior is also observed when the electric field is included. We also see that the magnetic field increases the regions of instability.
C. A model of AGN

We now consider a simple model of active galactic nuclei consisting of black hole, a Kuzmin-Chazy-Curzon disk and two rods of linear mass density \( \lambda \) located along \([-z_2, -z_1]\) and \([z_1, z_2]\) on the z axis, in presence of magnetic field. The metric potentials of a finite rod located on the z axis between \( z = z_1 \) and \( z = z_2 \) (\( z_2 > z_1 \)) are \([27]\)

\[
\hat{\psi}_R = \lambda \ln \left( \frac{\mu_1}{\mu_2} \right),
\]

\[
\hat{\Lambda}_R = 2\lambda^2 \ln \left[ \frac{(\rho^2 + \mu_1 \mu_2)^2}{(\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2)} \right],
\]

with \( \mu_1 = z_1 - z + \sqrt{\rho^2 + (z_1 - z)^2} \) and \( \mu_2 = z_2 - z + \sqrt{\rho^2 + (z_2 - z)^2} \), and for the combined system the metric potential \( \hat{\psi} \) is

\[
\hat{\psi} = \hat{\psi}_s + \lambda \ln \left( \frac{\mu_1}{\mu_2} \right) + \lambda \ln \left( \frac{\mu_3}{\mu_4} \right) + \hat{\psi}_D,
\]

with \( \mu_3 = -z_2 - z + \sqrt{\rho^2 + (z_2 + z)^2} \) and \( \mu_4 = -z_1 - z + \sqrt{\rho^2 + (z_1 + z)^2} \). In the case when the rods just touch the horizon of the black hole, that is \( z_1 = m \), we have

\[
\hat{\psi} = (1 - 2\lambda) \hat{\psi}_s + \lambda \ln \left( \frac{\mu_3}{\mu_2} \right) + \lambda \ln \left( \frac{\mu_4}{\mu_2} \right) + \hat{\psi}_D.
\]

The other metric function \( \Lambda = \hat{\Lambda} \) is given by

\[
\Lambda = \Lambda_s + \Lambda_{R1} + \Lambda_{R2} + \Lambda_D + \Lambda_{int}.
\]

where \( \Lambda_{R1} \) and \( \Lambda_{R2} \) are the potential associated with each rod, and by analogy with \([53]\)

\[
\Lambda_{int} = 2 \int_{-1}^1 \frac{1}{x^2 - y^2} \left[ \frac{y(x^2 - 1)}{x^2 - y^2} \frac{\hat{\psi}_{R1} + \hat{\psi}_{R2} + \hat{\psi}_D}{x} + x(1 - y^2) \frac{\hat{\psi}_{R1} + \hat{\psi}_{R2} + \hat{\psi}_D}{y} \right] dy.
\]

For \( x > 0 \) this integral gives

\[
\Lambda_{int} = 2\lambda \ln \left[ \frac{\rho m + \rho z_2 x - \sqrt{\rho^2 + (z_2 - z)^2}}{\rho m + \rho z_2 x + \sqrt{\rho^2 + (z_2 + z)^2}} \right] - 2\lambda \ln \left[ \frac{x - 1}{x + 1} \right] - 2\lambda \ln \left( \frac{x^2 - y^2}{x^2 - y^2} + \Lambda_{SD} \right),
\]

being \( \Lambda_{SD} \) the interaction term \([54]\).

From \([45],[48]\), the main physical quantities associated with the system are

\[
\tilde{\varepsilon} = \frac{4Mz_0 e^{-\Lambda}}{(\rho^2 + z_0^2)^{3/2}} \left[ F - (1 - 2\lambda) \frac{m}{\sqrt{\rho^2 + m^2}} - \frac{2\lambda z_2}{\sqrt{\rho^2 + z_2^2}} - \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right],
\]

\[
\tilde{p}_\varphi = \frac{4Mz_0 e^{-\Lambda}}{(\rho^2 + z_0^2)^{3/2}} \left[ (1 - 2\lambda) \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{2\lambda z_2}{\sqrt{\rho^2 + z_2^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right],
\]

\[
\tilde{j} = \frac{Mb}{2\pi} 2^{\varphi - \Lambda} \frac{\rho}{(\rho^2 + z_0^2)^{3/2}},
\]

\[
v^2 = \frac{F \left( 1 - 2\lambda \right) \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{2\lambda z_2}{\sqrt{\rho^2 + z_2^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}}}{1 - F \left[ (1 - 2\lambda) \frac{m}{\sqrt{\rho^2 + m^2}} + \frac{2\lambda z_2}{\sqrt{\rho^2 + z_2^2}} + \frac{M\rho^2}{(\rho^2 + z_0^2)^{3/2}} \right] + \frac{1}{\pi} \tilde{\psi}^2 f}.
\]

In Figs. 4 and 5 we plot the energy density \( \tilde{\varepsilon} \), the azimuthal pressure \( \tilde{p}_\varphi \), the azimuthal electric current density \( j \), the speed \( v^2 \) of counterrotation and the specific angular momentum \( h^2 \) as function of \( \rho \) for \( z_0 = 2, m = 0.1, M = 0.2, z_2 = 2, \lambda = 0.4 \), and different values of magnetic field parameter \( b = 0, 0.5, 1, \) and 1.5. We see that energy density is lowered with increasing magnetic field near the center of disk and then is increased, while the pressure is lowered in all regions of the disk. We also observe that energy density is a positive quantity in concordance with the weak energy condition, and Eq. \([82]\) shows that for \( \lambda \leq 1/2 \) we have pressure for all values of parameters. The dominant energy condition \( (v \leq 1) \) is also not satisfied in the central region located between the black hole and the photon radius, and increasing magnetic field enhances the zone of superluminal speed. We also observe that these structures present strong instabilities inside the photonic orbit, and that the magnetic field increases the region of instability.
V. DISCUSSION

The formalism for superposing the field of a disk and a static black hole in the vacuum was extended to the case of magnetized Weyl fields. Two relativistic models of thin disk around static black hole in presence of a magnetic field were presented. The first model is based in the Kuzmin-Chazy-Curzon infinite disk and the other in the first Morgan-Morgan finite disk. We also considered a simple model of active galactic nuclei based on the superpositions of black hole, a Kuzmin-Chazy-Curzon disk and two rods, representing the matter of jets, in presence of magnetic field. In the first and three model the source of the magnetic field is the surface electric current density presents on disk whereas in second the source is not planar. These solutions were interpreted as the exterior gravitational field of a black hole and a finite disk immersed in a magnetic field. We concluded in this case that in order to construct a planar source for the magnetic field is necessary to include the electric field.

In all cases we found values of parameters for which energy density is a positive quantity in concordance with the weak energy condition, as well as the stress in azimuthal direction (pressure). However, the dominant energy condition \( v \leq 1 \) is not satisfied in all regions of the disks and the presence magnetic field enhances the zones of superluminal speed. Such behavior is also observed in the second models when the electric field is included. We also see that the magnetic field increases the regions of instability.

Finally, in order to construct a such system that satisfies all the energy conditions, a model of a black hole surrounded by a disk with an inner edge in presence of magnetic field is being investigated.

ACKNOWLEDGMENTS

One of us (A.C.G-P.) wants to thank COLCIENCIAS, TWAS and Conacyt for financial support.

Appendix

The components of the energy-momentum tensor of the disk \( Q^b_a \) can be obtained by integration of the field equations [73] writing \( T_{ab} \) as [75] \[ 17 \]

\[
\int_{z=0^-}^{z=0^+} R_{ab}dz = 8\pi \left( \int_{z=0^-}^{z=0^+} T^\text{elm}_{ab}dz + \int_{z=0^-}^{z=0^+} Q_{ab} \delta(z)dz \right). \tag{A.1}
\]

For metric \( \Pi \), the nonzero components of \( T^\text{elm}_{ab} \) are

\[
T^\text{elm}_{00} = \frac{e^{2(3\psi-\Lambda)}}{8\pi \rho^2}(A^2_{\rho} + A^2_z), \tag{A.2a}
\]

\[
T^\text{elm}_{11} = \frac{1}{8\pi} e^{2(\psi-\Lambda)}(A^2_{\rho} + A^2_z), \tag{A.2b}
\]

\[
T^\text{elm}_{22} = -T^\text{elm}_{33} = \frac{e^{2\psi}}{8\pi \rho^2}(A^2_{\rho} - A^2_z), \tag{A.2c}
\]

\[
T^\text{elm}_{23} = \frac{e^{2\psi}}{4\pi \rho^2} A_{\rho} A_z, \tag{A.2d}
\]

and for magnetized Weyl fields (Eq. \( \text{36d} \) ) give us

\[
T^\text{elm}_{00} = \frac{b^2}{8\pi} e^{2(3\psi-\Lambda)}(\psi^2_{\rho} + \psi^2_z), \tag{A.3a}
\]

\[
T^\text{elm}_{11} = \frac{b^2}{8\pi} \rho^2 e^{2(\psi-\Lambda)}(\psi^2_{\rho} + \psi^2_z), \tag{A.3b}
\]

\[
T^\text{elm}_{22} = -T^\text{elm}_{33} = \frac{b^2}{8\pi} e^{2\psi}(\psi^2_z - \psi^2_{\rho}), \tag{A.3c}
\]

\[
T^\text{elm}_{23} = -\frac{b^2}{4\pi} e^{2\psi} \psi_{\rho} \psi_z. \tag{A.3d}
\]

Now the continuity of the metric functions across the disk implies that they are even functions of \( z \) and the discontinuity of its first derivatives in the direction normal to the disk means that these are odd functions of \( z \).
Thus $\dot{\psi}_z$ is odd and its square is even and hence continuous across the disk. It also follows that $\dot{\psi}_\rho$ is continuous. Accordingly the terms

$$\int_{z=0}^{z=0^+} T_{00}^{\text{elm}} dz = \int_{z=0}^{z=0^+} T_{11}^{\text{elm}} dz = \int_{z=0}^{z=0^+} T_{22}^{\text{elm}} dz = \int_{z=0}^{z=0^+} T_{33}^{\text{elm}} dz = 0 \quad (A.4)$$

are zero due to the continuity of the metric, $\dot{\psi}_z^2$ and $\dot{\psi}_\rho^2$ across the disk. For the other component we have

$$\int_{z=0}^{z=0^+} T_{23}^{\text{elm}} dz = -\frac{b^2}{4\pi} e^{2\psi} \dot{\psi}_\rho \int_{z=0}^{z=0^+} \dot{\psi}_z dz = -\frac{b^2}{4\pi} e^{2\psi} \dot{\psi}_\rho [\dot{\psi}] = 0, \quad (A.5)$$

which also vanishes because continuity of the metric and $\dot{\psi}_\rho$ across the disk. Therefore, the first term on right hand side of (A.1) is zero and in consequence $Q_0^\rho$ has the same form that the vacuum [5].

On the other hand, on the disk the Maxwell equations $\partial_a F^{ab} = 4\pi J^a$, where ‘bar’ denotes multiplication by $\sqrt{-g}$, are given by

$$-4\pi \bar{j}_\rho \delta(z) = \partial_z (\bar{g}^{zz} A_z) + \partial_\rho (\bar{g}^{\rho\rho} A_\rho), \quad (A.6)$$

but again using (36c) we have

$$-4\pi \bar{j}_\rho \delta(z) = b \rho \partial_z (\bar{g}^{zz} \dot{\psi}_\rho) - b \partial_\rho (\bar{g}^{\rho\rho} \dot{\psi}_z). \quad (A.7)$$

Integrating through the disk one finds that the electric current density is zero on the disk. In fact

$$-4\pi \bar{j}_\rho = b \rho \int_{z=0}^{z=0^+} \partial_z (\bar{g}^{zz} \dot{\psi}_\rho) dz - b \int_{z=0}^{z=0^+} \partial_\rho (\bar{g}^{\rho\rho} \dot{\psi}_z) dz$$

$$= b \rho (\bar{g}^{zz} \dot{\psi}_\rho)|_{z=0^+} - b \partial_\rho (\bar{g}^{\rho\rho} \dot{\psi}_z)|_{z=0^+}$$

$$= b \rho [\bar{g}^{zz} \dot{\psi}_\rho] - b \partial_\rho (\bar{g}^{\rho\rho} [\dot{\psi}])$$

$$= 0, \quad (A.8)$$

where the terms on the right hand side vanish due to the continuity of the metric and $\dot{\psi}_\rho$ across the disk. Thus, if the gravitational field is continuous across plane $z = 0$ but its normal derivative is discontinuous, the magnetostatic solutions discussed in IV (B) can be interpreted as the superposition of a black hole and a finite disk immersed in a magnetic field.

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FIG. 1. (a) The energy density $\tilde{\epsilon}$ and (b) the azimuthal pressure $\tilde{p}_\phi$ for the system black hole and a Kuzmin-Chazy-Curzon infinite disk in a magnetic field with $z_0 = 2$, $m = 0.1$, $M = 0.2$, and for values of magnetic field parameter $b = 0$ (dashed curves), 0.5, 1, and 1.5 (dash-dotted curves), as functions of $\rho$. (c) The azimuthal electric current density $j$ for $b = 0$ (axis $\rho$), 0.5, 1, and 1.5 (top curve) and the same value of other parameters.

FIG. 2. (a) The speed $v^2$ of counterrotation and (d) the specific angular momentum $h^2$ for the system black hole and a Kuzmin-Chazy-Curzon infinite disk in a magnetic field for $b = 0$ (dash curves), 0.5, 1, and 1.5 (dash-dotted curves) and the same value of other parameters.
FIG. 3. (a) The energy density $\tilde{\epsilon}$, (b) the azimuthal pressure $\tilde{p}_\phi$, (c) the speed $v^2$ of counterrotation, and (d) the specific angular momentum $h^2$ for the system black hole and the first Morgan-Morgan finite disk in a magnetic field with radius $a = 1$, $m = M = 0.1$, and for values of magnetic field parameter $\delta = 0$ (dashed curves), 0.5, 1, and 2 (dash-dotted curves), as functions of $\rho$. 
FIG. 4. (a) The energy density $\tilde{\epsilon}$ and (b) the azimuthal pressure $\tilde{p}_\varphi$ for the system composed of a black hole, a Kuzmin-Chazy-Curzon disk and two rods in a magnetic field with parameters $z_0 = 2$, $m = 0.1$, $M = 0.2$, $z_2 = 2$, $\lambda = 0.4$, and a magnetic field $b = 0$ (dashed curves), 0.5, 1, and 1.5 (dash-dotted curves), as functions of $\rho$. (c) The azimuthal electric current density $j$ for $b = 0$ (axis $\rho$), 0.5, 1, and 1.5 (top curve) and the same value of other parameters.

FIG. 5. (a) The speed $v^2$ of counterrotation and (d) the specific angular momentum $h^2$ for the above AGN model with $b = 0$ (dashed curves), 0.5, 1, and 1.5 (dash-dotted curves) and the same value of other parameters.