INVERSE FORMULA FOR THE BLASCHKE-LEVY REPRESENTATION WITH APPLICATIONS TO ZONOID S AND SECTIONS OF STAR BODIES.

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Abstract. We say that an even continuous function $H$ on the unit sphere $\Omega$ in $\mathbb{R}^n$ admits the Blaschke-Levy representation with $q > 0$ if there exists an even function $b \in L_1(\Omega)$ so that $H^q(x) = \int_{\Omega} |(x, \xi)|^q b(\xi) \, d\xi$ for every $x \in \Omega$. This representation has numerous applications in convex geometry, probability and Banach space theory. In this paper, we present a simple formula (in terms of the derivatives of $H$) for calculating $b$ out of $H$. We use this formula to give a sufficient condition for isometric embedding of a space into $L_p$ which contributes to the 1937 P.Levy’s problem and to the study of zonoids. Another application gives a Fourier transform formula for the volume of $(n-1)$-dimensional central sections of star bodies in $\mathbb{R}^n$. We apply this formula to find the minimal and maximal volume of central sections of the unit balls of the spaces $\ell_p^n$ with $0 < p < 2$.

1. Introduction

For $q > 0$, we say that an even continuous function $H$ on $\mathbb{R}^n$ admits the Blaschke-Levy representation with the exponent $q$ if there exists an even function $b$ on the unit sphere $\Omega$ in $\mathbb{R}^n$ so that $b \in L_1(\Omega)$ and, for every $x \in \mathbb{R}^n$,

(1) \[ H(x) = \int_{\Omega} |(x, \xi)|^q b(\xi) \, d\xi, \]

where $(x, \xi)$ stands for the scalar product.

It was known to Blaschke [3] that every infinitely differentiable function on the sphere admits the representation (1) with $q = 1$. On the other hand, the representation (1) is known in the probability theory under the name of P.Levy, and it was an important part of P.Levy’s theory of stable processes [19] that the function $\|x\|^q$ admits the representation (1) with a measure in place of the function $b$, where $(\mathbb{R}^n, \| \cdot \|)$ is any $n$-dimensional subspace of $L_q$. In mathematical physics the representation (1) is called the plain-wave expansion.

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The Blaschke-Levy representation has had numerous applications to convex geometry, probability and Banach space theory. One of the most popular ways to apply the Blaschke-Levy representation is based on the fact that the representation is unique for every \( q > 0 \) which is not an even integer (the uniqueness fails if \( q \) is an even integer, because only a finite number of moments of the functions must be equal). The uniqueness was first shown by Blaschke [3] in the case where \( q = 1 \) and \( n = 3 \). Aleksandrov [1] proved the uniqueness for \( q = 1 \) and arbitrary dimension, and P. Levy [19] did it for \( 0 < q < 2 \). The last two results are valid for signed measures in place of \( b \). The uniqueness for every \( q \) which is not an even integer was established by Kanter [13]. For different proves and applications of the uniqueness theorem see [11, 20, 23, 24, 14]. In Section 2 we present a Fourier transform proof which is close to that from [14].

The existence of the representation (1) with \( q = 1 \) for infinitely differentiable functions was known to Blaschke [3]. A precise proof under a weaker assumption that \( H \in C^{n+2}(\Omega) \) was given by Schneider [26] who found a spherical harmonics expansion for the function \( b \) (which turned out to be a continuous function on \( \Omega \).) Later Goodey and Weil [10] proved the existence of the representation (1) (also with \( q = 1 \)) for the functions \( H \) of the class \( C^{(n+5)/2} \) where the function \( b \) appears to belong to the space \( L_2(\Omega) \). Weil [28] found a generating distribution for the support function of any centered convex body. Richards [25] showed that the representation (1) exists for any \( q \in (0, 2) \) and any \( H \in C^{n+q+1}(\Omega) \). A generalization of this result to the case of arbitrary \( q > 0 \) which is not an even integer was given in [16].

All the results mentioned above were based on the use of spherical harmonics. A connection between the Blaschke-Levy representation and the Fourier transform was found in [14] where it was shown that the function \( b \) is the restriction to the sphere \( \Omega \) of the Fourier transform of \( H \) (we present a short version of that proof in Theorem 1 below; in fact, in [14] the Fourier transform of \( H \) was restricted to a hyperplane). This fact was used to show that every norm in \( \mathbb{R}^n \) admits the Blaschke-Levy representation with every \( q > 0 \) which is not an even integer, but we must allow \( b \) to be a distribution and the representation (1) is considered in a generalized form. Note that the Fourier transform connection was used in [14, 15] to obtain exact representations for certain norms, which, in particular, led to applications to positive definite functions and embedding of Banach spaces.

A remarkable feature of Schneider’s spherical harmonics construction is that it allows to gain control over the function \( b \) by estimating the \( L_\infty(\Omega) \)-norm of \( b \) in terms of \( H \). Namely, Schneider [26] showed that, for any \( H \in C^{n+2}(\Omega) \), the function \( b \) appearing in the Blaschke-Levy representation with \( q = 1 \) satisfies the inequality

\[
\|b\|_\infty \leq K\|H\|_{L_2(\Omega)} + L\|\Delta_\Omega^r H\|_{L_2(\Omega)},
\]

where \( \Delta_\Omega \) is the Laplace-Beltrami operator, \( r > (n+2)/2 \) and \( K \) and \( L \) are constants which are given as the sum of certain series’. Schneider [27] used this inequality...
to construct non-trivial zonoids whose polars are zonoids. In order to do that, he considered a perturbation of the Euclidean norm by means of an infinitely differentiable function \( f \) on the sphere \( \Omega \): put \( H(x) = \|x\|_2(1 + \lambda f(x/\|x\|_2)) \), \( x \in \mathbb{R}^n \), where \( \| \cdot \|_2 \) is the Euclidean norm and \( \lambda \) is a (small) real number. If the function \( b \) corresponding to \( H \) in the representation (1) is non-negative, then \( H \) is the norm of a subspace of \( L_1 \), and, therefore, it is the support function of a body whose polar is a zonoid. Since the function \( b \) corresponding to the Euclidean norm \( \|x\|_2 \) in the Blaschke-Levy representation is a constant, and the \( \ell_\infty \)-norm of the perturbing function \( b \) is controlled by \( \lambda \) because of (2), one can choose \( \lambda \) small enough so that the function \( b \) corresponding to \( H \) is non-negative. It is easy to see that making \( \lambda \) even smaller (if necessary) one can make the body \( \{ x : H(x) \leq 1 \} \) to be a zonoid too.

The inequality (2) was generalized in [16] to the case of the Blaschke-Levy representation with any \( q > 0 \) which is not an even integer. This led to a construction of common subspaces of \( L_q \)-spaces: for any \( n \in \mathbb{N} \) and any compact subset \( Q \) of \( (0, \infty) \setminus \{ \text{even integers} \} \), there exists an \( n \)-dimensional non-Hilbertian Banach space which is isometric to a subspace of \( L_q \) for every \( q \in Q \).

This paper is an attempt to gain more control over the function \( b \) by presenting an inverse formula for the representation (1) which does not involve spherical harmonics or the Fourier transform, and by giving a simpler version of the inequality (2) with computable constants. We start with the Fourier transform inverse formula showing that \( b \) is the restriction to the sphere of the Fourier transform of the function \( H \) (which is homogeneous of degree \( q \) because of (1)). However, to avoid the calculation of the Fourier transform, we first apply the Laplace operator to the function \( H \) as many times as it is necessary to make the result homogeneous of degree less or equal than \(-n + 1\). Note that action of the Laplace operator does not change the restriction of the Fourier transform to the sphere (up to a sign). The crucial point is that, by Lemmas 3 and 4, the Fourier transform of a homogeneous function of degree less or equal than \(-n + 1\) can easily be expressed in terms of the function itself.

In this way we show that, for every \( q > 0 \) which is not an integer and every even homogeneous function \( H \) of degree \( q \) on \( \mathbb{R}^n \) such that the restriction to sphere \( H|_\Omega \) belongs to the space \( C^{n+[q]}(\Omega) \), there exists the Blaschke-Levy representation with the exponent \( q \), where the corresponding function \( b \) is given by

\[
b(\xi) = (-1)^k \frac{\pi}{2(2\pi)^{n-1}C_{-n-q+2k}C_q} \int_{\Omega} |(\theta, \xi)|^{-n-q+2k} (\Delta^k H^q)(\theta) \, d\theta,
\]

for every \( \xi \in \Omega \), where \( k = (n + [q])/2 \) if \( n + [q] \) is an even integer, and \( k = (n + [q] + 1)/2 \) if \( n + [q] \) is an odd integer.

If \( q \) is an odd integer and the dimension \( n \) is an even integer the expression for
\[ b(\xi) = \frac{(-1)^{(n+q-1)/2}}{2\pi^{n-1}} C_q \int_{\Omega \cap \{(\theta,\xi)=0\}} \Delta^{(n+q-1)/2} H^q(\theta) \, d\theta. \]

If both \( q \) and \( n \) are odd integers the technique of this paper does not work for the reason that, in this case, the Laplace transform of \( H \) may contain a part supported at zero. As it was mentioned above, if \( q \) is an even integer the uniqueness fails.

In Section 4 we apply the inverse formulae to get a new criterion for the existence of an isometric embedding of a given space into \( L_q \). Finding such criteria is a matter of the 1937 P. Levy’s problem (see [19]). We calculate the functions \( b \) for certain perturbations of the Euclidean norm, and show the way to get exact constants \( \lambda \) in Schneider’s construction.

In Section 5 we use our results to get a Fourier transform formula for the volume of central \((n-1)\)-dimensional sections of centrally symmetric star bodies in \( \mathbb{R}^n \). If \( K \) is such a body then, for every \( \xi \in \Omega \),

\[ Vol_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)}(\|x\|^{-n+1})^\wedge(\xi) \]

where \( \xi^\perp = \{x \in \mathbb{R}^n : (x,\xi) = 0\} \), and \( \|x\| = \min\{a \in \mathbb{R}; ax \in K\} \). Finally, we use this formula to show that the minimal volume of central sections of the unit ball of the space \( \ell\mathbb{R}_p \), \( p \in (0,2) \) occurs if the section is perpendicular to the vector \( \xi = (1,1,..,1) \). This result proves a conjecture of Meyer and Pajor [21].

2. Connection between the Blaschke-Levy representation and the Fourier transform.

The main tool of this paper is the Fourier transform of distributions. As usual, we denote by \( \mathcal{S}(\mathbb{R}^n) \) the space of rapidly decreasing infinitely differentiable functions (test functions) in \( \mathbb{R}^n \), and \( \mathcal{S}'(\mathbb{R}^n) \) is the space of distributions over \( \mathcal{S}(\mathbb{R}^n) \). The Fourier transform of a distribution \( f \in \mathcal{S}'(\mathbb{R}^n) \) is defined by \( (\hat{f},\hat{\phi}) = (2\pi)^n (f,\phi) \) for every test function \( \phi \). A distribution is called even homogeneous of degree \( q \in \mathbb{R} \) if \( (f(x),\phi(x/\alpha)) = |\alpha|^{n+q}(f,\phi) \) for every test function \( \phi \) and every \( \alpha \in \mathbb{R} \), \( \alpha \neq 0 \). The Fourier transform of an even homogeneous distribution of degree \( q \) is an even homogeneous distribution of degree \(-n-q\).

If \( q > -1 \) and \( q \) is not an even integer, then the Fourier transform of the function \( h(z) = |z|^q \), \( z \in \mathbb{R} \) is equal to \( (|z|^q)^\wedge(t) = C_q|t|^{-1-q} \) (see [8, p. 173]), where

\[ C_q = \frac{2^{q+1}\sqrt{\pi} \Gamma((q+1)/2)}{\Gamma(-q/2)}. \]

Throughout the paper, we use the following fact which is a simple consequence of the connection between the Fourier transform and the Radon transform.
Lemma 1. Let \( q > -1 \), \( q \) is not an even integer. Then for every even test function \( \phi \) with \( 0 \notin \text{supp}(\phi) \) and every fixed vector \( \xi \in \mathbb{R}^n \), \( \xi \neq 0 \), we have

\[
(3) \quad \int_{\mathbb{R}^n} |(x, \xi)|^q \hat{\phi}(x) \, dx = (2\pi)^{n-1} C_q \int_{\mathbb{R}} |t|^{-1-q} \phi(t\xi) \, dt.
\]

Proof. By the well-known connection between the Fourier transform and the Radon transform (see [12]), the function \( t \to (2\pi)^n \phi(-t\xi) \) is the Fourier transform of the function \( z \to \int_{(x,\xi)=z} \hat{\phi}(x) \, dx \). (Recall that \( (\hat{\phi})^\wedge(x) = (2\pi)^n \phi(-x) \).) Using this fact and the Fubini theorem, for every test function \( \phi \) with \( 0 \notin \text{supp}(\phi) \), we get

\[
\int_{\mathbb{R}^n} |(x, \xi)|^q \hat{\phi}(x) \, dx = \int_{\mathbb{R}} |z|^q \left( \int_{(x, \xi)=z} \hat{\phi}(x) \, dx \right) \, dz = \left( \int_{(x, \xi)=z} \hat{\phi}(x) \, dx \right) \left( \int_{\mathbb{R}} |t|^{-1-q} (2\pi)^n \phi(-t\xi) \, dt \right) = (2\pi)^{n-1} C_q \int_{\mathbb{R}} |t|^{-1-q} \phi(t\xi) \, dt.
\]

\( \square \)

Remark 1. If \( q > -1 \) and \( \mu \) is a Borel signed measure with bounded variation on \( \Omega \), then the integral \( G(x) = \int_{\Omega} |(x, \xi)|^q \, d\mu(\xi) \) converges for almost all \( x \in \Omega \) with respect to the uniform measure on \( \Omega \). This follows from the fact that, for \( q > -1 \) and any \( \xi \in \Omega \),

\[
W_q = \int_{\Omega} |(x, \theta)|^q \, dx = 2\Gamma((q + 1)/2) \pi^{(n-1)/2} \Gamma((n + q)/2) < \infty,
\]

and, therefore, the restriction \( G|_{\Omega} \) of the function \( G \) to \( \Omega \) satisfies

\[
\|G|_{\Omega}\|_1 \leq \int_{\Omega} d|\mu|(\xi) \int_{\Omega} |(x, \theta)|^q \, dx = W_q \|\mu\|_1(\Omega).
\]

If \( \mu \) has the density \( f \in L_\infty(\Omega) \) then \( \|G|_{\Omega}\|_1 \leq W_q \|f\|_\infty \). We denote by \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \) the norms of the spaces \( L_1(\Omega) \) and \( L_\infty(\Omega) \), respectively.

Let us calculate the Fourier transform of the function \( G \) from Remark 1.

Lemma 2. Let \( q > -1 \), \( q \) is not an even integer, and let \( \mu \) be a Borel symmetric signed measure with bounded variation on \( \Omega \). Then the Fourier transform \( \hat{G} \) of the function \( G(x) = \int_{\Omega} |(x, \xi)|^q \, d\mu(\xi) \) has the property that for every even test function \( \phi \) with \( 0 \notin \text{supp}(\phi) \),

\[
(4) \quad (\hat{G}, \phi) = (2\pi)^{n-1} C_q \int_{\Omega} d\mu(\xi) \int_{\mathbb{R}} |t|^{-1-q} \phi(t\xi) \, dt.
\]

Proof. By Remark 1, \( G \) is an even homogeneous function of degree \( q \) whose restriction to the sphere belongs to the space \( L_1(\Omega) \). For every even test function \( \phi \) with \( 0 \notin \text{supp}(\phi) \), using Lemma 1 and the Fubini theorem we get

\[
(\hat{G}, \phi) = \int_{\mathbb{R}^n} G(x) \hat{\phi}(x) \, dx = \int_{\mathbb{R}^n} \left( \int_{\Omega} |(x, \xi)|^q d\mu(\xi) \right) \hat{\phi}(x) \, dx = \int_{\Omega} d\mu(\xi) \int_{\mathbb{R}^n} |(x, \xi)|^q \hat{\phi}(x) \, dx = (2\pi)^{n-1} C_q \int_{\Omega} d\mu(\xi) \int_{\mathbb{R}} |t|^{-1-q} \phi(t\xi) \, dt. \quad \square
\]
**Remark 2.** Lemma 2 was proved in [14] in a slightly different form, and it was used there to give a new Fourier transform proof of the following well-known uniqueness theorem (see introduction for the history of the problem and other applications): if \( q > 0 \), \( q \) is not an even integer, and \( \mu \) and \( \nu \) are symmetric measures with bounded variation on \( \Omega \) so that, for every \( x \in \Omega \)

\[
\int_\Omega |(x, \xi)|^q \, d\mu(\xi) = \int_\Omega |(x, \xi)|^q \, d\nu(\xi),
\]

then \( \mu = \nu \). To see that, it is enough to apply Lemma 2 to the test functions of the form \( \phi(x) = u(t)v(\xi) \), where \( x = t\xi, \ t > 0, \ \xi \in \Omega, \ u \) is any test function on \( \mathbb{R} \) with \( 0 \notin \text{supp}(u) \), and \( v \) is any even infinitely differentiable function on the sphere \( \Omega \). For such functions \( \phi \), we have \( \int_{\mathbb{R}} |t|^{-q-1}_q \phi(t\xi) \, dt = v(\xi) \int_{\mathbb{R}} |t|^{-q-1}_q u(t) \, dt \).

Since the Fourier transforms of both sides of (5) are equal and have the property of Lemma 2, we derive from (4) that \( \int_\Omega v(\xi)d\mu(\xi) = \int_\Omega v(\xi) \, d\nu(\xi) \) for any infinitely differentiable function \( v \) on \( \Omega \), which implies \( \mu = \nu \). Note that if \( q \) is an even integer the uniqueness theorem fails to be true because only a finite number of moments of the measures \( \mu \) and \( \nu \) must be equal.

Now we are ready to show the connection between the Fourier transform and the Blaschke-Levy representation.

**Theorem 1.** Let \( H \) be a continuous, non-negative, even homogeneous function of degree 1 on \( \mathbb{R}^n \). Suppose that, for some \( q > 0 \) which is not an even integer, \( (H^q)^\wedge \) is a function on \( \mathbb{R}^n \setminus \{0\} \) so that \( (H^q)^\wedge|_\Omega \) belongs to the space \( L_1(\Omega) \). Then the function \( H^q \) admits the Blaschke-Levy representation with the exponent \( q \), and the corresponding function \( b \in L_1(\Omega) \) is given by \( b(\xi) = (1/\langle 2\pi \rangle^{n-1} C_q) (H^q)^\wedge(\xi) \) for every \( \xi \in \Omega \).

**Proof.** The Fourier transform of an even homogeneous (of degree \( q \)) function \( H^q \) is an even homogeneous distribution of degree \(-n-q\). Fix any even test function \( \phi \) with \( 0 \notin \text{supp}(\phi) \). Since we know that \( (H^q)^\wedge \) is a function on \( \mathbb{R}^n \setminus \{0\} \) whose restriction to the sphere is an \( L_1 \)-function, we can write the value of the distribution \( (H^q)^\wedge \) at the test function \( \phi \) as an integral, and then pass to the spherical coordinates:

\[
(6) \quad (\langle H^q \rangle^\wedge, \phi) = \int_{\mathbb{R}^n} (H^q)^\wedge(x) \phi(x) \, dx = (1/2) \int_{\Omega} (H^q)^\wedge(\xi) \, d\xi \int_{\mathbb{R}} |t|^{-1-q}_q \phi(t\xi) \, dt.
\]

Put \( b(\xi) = (1/(2\pi)^{n-1} C_q) (H^q)^\wedge(\xi) \) for every \( \xi \in \Omega \), and let us show that this function \( b \) provides the equality (1). Since \( b \in L_1(\Omega) \), the integral \( \int_{\Omega} |(x, \xi)|^q b(\xi) \, d\xi \) is a homogeneous function (of the variable \( x \in \mathbb{R}^n \)) of degree \( q \) whose restriction to the sphere is an \( L_1 \)-function. By Lemma 2,

\[
(7) \quad (\langle \int |(x, \xi)|^q b(\xi) \, d\xi \rangle^\wedge, \phi) = (2\pi)^{n-1} C_q \int \langle b(\xi) \rangle \int |t|^{-1-q}_q \phi(t\xi) \, dt.
\]
Because of the definition of the function $b$, the right-hand sides of (6) and (7) are equal. Since $\phi$ is an arbitrary even test function supported in $\mathbb{R}^n \setminus \{0\}$, the even functions $(H^q)^\wedge$ and $(\int_{\Omega} |(x, \xi)|^q b(\xi) \, d\xi)^\wedge$ are equal distributions in $\mathbb{R}^n \setminus \{0\}$. Therefore, $H^q$ and $x \to \int_{\Omega} |(x, \xi)|^q b(\xi) \, d\xi$ are functions in $\mathbb{R}^n$ which can differ by a polynomial only (see [9, p. 119]). Since both of those functions are even homogeneous of the order $q$, and $q$ is not an even integer, we conclude that the polynomial must be equal to zero, and we have (1). The uniqueness follows from Remark 2. □

We end this section by showing that the Fourier transform of a homogeneous function of degree $p \leq -n + 1$ can be expressed in terms of the function itself. We have to treat the cases $p < -n + 1$ and $p = -n + 1$ separately.

**Lemma 3.** Let $p < -n + 1$ so that $-n - p$ is not an even integer, and let $f$ be an even homogeneous function of degree $p$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$ such that $f|\Omega \in L_1(\Omega)$. Then for every $\xi \in \mathbb{R}^n$

\begin{equation}
\hat{f}(\xi) = \frac{\pi}{C_{-n-p}} \int_{\Omega} |(\theta, \xi)|^{-n-p} f(\theta) \, d\theta,
\end{equation}

so $\hat{f}|\Omega \in L_1(\Omega)$, and $\|\hat{f}|\Omega\|_1 \leq (\pi W_{-n-p}/C_{-n-p})\|f|\Omega\|_1$. Also if $f|\Omega \in L_\infty(\Omega)$ then $\hat{f}|\Omega \in L_\infty(\Omega)$ and $\|\hat{f}|\Omega\|_\infty \leq (\pi W_{-n-p}/C_{-n-p})\|f|\Omega\|_\infty$.

**Proof.** Since $f|\Omega \in L_1(\Omega)$ and $-n - p > -1$, Remark 1 implies that the right-hand side of (8) is a homogeneous function of degree $-n - p$ whose restriction to the sphere is an $L_1$-function. Let $\phi$ be an even test function with $0 \notin \text{supp}(\hat{\phi})$. Switching to the spherical coordinates and using the fact that $f$ is even homogeneous we get

\begin{equation}
(\hat{f}, \phi) = \int_{\mathbb{R}^n} f(z) \hat{\phi}(z) \, dz = (1/2) \int_{\Omega} \int_{\mathbb{R}} f(t\theta) |t|^{n-1} \hat{\phi}(t\theta) \, dt \, d\theta =
\end{equation}

\begin{equation}
(1/2) \int_{\Omega} f(\theta) \, d\theta \int_{\mathbb{R}} |t|^{n+p-1} \hat{\phi}(t\theta) \, dt.
\end{equation}

Now we apply Lemma 1 with $q = -n - p$. Recall that $(\hat{\phi})^\wedge = (2\pi)^n \hat{\phi}$. The right-hand side of (9) is equal to

\begin{align*}
\frac{(2\pi)^n}{2(2\pi)^{n-1}C_{-n-p}} & \int_{\Omega} f(\theta) \, d\theta \int_{\mathbb{R}^n} |(\theta, \xi)|^{-n-p} \phi(\xi) \, d\xi = \\
& \frac{\pi}{C_{-n-p}} \left( \int_{\Omega} |(\theta, \xi)|^{-n-p} f(\theta) \, d\theta, \phi \right).
\end{align*}

Since $\phi$ is an arbitrary even test function with $0 \notin \text{supp}(\hat{\phi})$ we conclude (similarly to the end of the proof of Theorem 1) that the functions $\hat{f}(\xi)$ and $\xi \to (\pi/C_{-n-p}) \int_{S} |(\theta, \xi)|^{-n-p} f(\theta) \, d\theta$ are even homogeneous functions of the order $-n - p$ which are equal up to an even homogeneous polynomial, and that polynomial must be equal to zero because the number $-n - p$ is not an even integer. So we get (8), and the inequalities for the norms follow. □
Lemma 4. Let $f$ be an even homogeneous function of degree $-n + 1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$ so that $f|_\Omega \in L_1(\Omega)$. Then, for every $\xi \in \Omega$,

$$\hat{f}(\xi) = \pi \int_{\Omega \cap \{(\theta, \xi) = 0\}} f(\theta) \, d\theta.$$ 

In particular, if $f|_\Omega \in L_\infty(\Omega)$ then $\hat{f}|_\Omega \in L_\infty(\Omega)$, and

$$\|\hat{f}|_\Omega\|_\infty \leq (2\pi^{(n+1)/2}/\Gamma((n-1)/2)) \|f|_\Omega\|_\infty.$$ 

Proof. Because of the connection between the Fourier transform and the Radon transform, for every even test function $\phi$ and every $\theta \in \Omega$, the Fourier transform of the function $t \to \hat{\phi}(t\theta)$ at zero is equal to $\int_{\mathbb{R}} \hat{\phi}(t\theta) \, dt = 2\pi \int_{(\theta, \xi) = 0} \phi(\xi) \, d\xi$. Also the Fourier transform of the $\delta$-function (defined by $(\delta, \phi) = \phi(0)$) is the constant function $h(t) = 1$. Therefore, passing to the spherical coordinates we get

$$(\hat{f}, \phi) = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x) \, dx = \int_{\Omega} \int_0^\infty f(t\xi)t^{n-1}\hat{\phi}(t\xi) \, dt \, d\xi = (1/2) \int_{\mathbb{R}} f(\theta) \, d\theta \int_{\mathbb{R}} \hat{\phi}(t\theta) \, dt = \pi \int_{\Omega} f(\theta) \, d\theta \int_{(\theta, \xi) = 0} \phi(\xi) \, d\xi = \pi \int_{\mathbb{R}^n} \left( \int_{\Omega \cap \{(\theta, \xi) = 0\}} f(\theta) \, d\theta \right) \phi(\xi) \, d\xi,$$

and the result follows since $\phi$ is an arbitrary even test function. □

3. The inverse formula.

Theorem 1 gives a condition for the existence of the Blaschke-Levy representation and the inverse formula in terms of the Fourier transform of the original function. Though this criterion has a few applications (see [14]), it is often difficult to calculate the Fourier transform. However, using Lemmas 3 and 4 we can replace the Fourier transform condition by a condition in terms of the derivatives of the original function which is sometimes more convenient for applications.

Let us explain what is going to happen. Suppose we want to find the Blaschke-Levy representation for a function $H$. Theorem 1 reduces this problem to calculating the Fourier transform of $H$. Instead of doing that, let us consider the distribution $\Delta^k H$, where $\Delta$ is the Laplace operator and $k$ is an integer so that the distribution $\Delta^k H$ is homogeneous of degree less or equal than $-n + 1$. The Fourier transform of $\Delta^k H$ has (up to a sign) the same restriction to the sphere as the Fourier transform of $H$. On the other hand, by Lemma 3 (or Lemma 4) if $\Delta^k H$ is an $L_1$-function on the sphere so is its Fourier transform, and there is a simple formula expressing the Fourier transform of $\Delta^k H$ in terms of the function itself. That is why we can replace the Fourier transform condition for the existence of the Blaschke-Levy representation by a condition in terms of the function $\Delta^k H$. 
First, let us consider the case where \( q \) is not an integer.

**Theorem 2.** Let \( q > 0, \) \( q \) is not an integer, and let \( H \) be a continuous, non-negative, even homogeneous function of degree 1 on \( \mathbb{R}^n, \) \( n > 1. \) Suppose that \( \Delta^k H^q \) is a function in \( \mathbb{R}^n \ \{0\} \) so that \( (\Delta^k H^q)|_\Omega \in L_1(\Omega), \) where \( k = (n+[q])/2 \) if \( n+[q] \) is an even integer, \( k = (n+[q]+1)/2 \) if \( n+[q] \) is an odd integer, and differentiation is considered in the sense of distributions. Then the function \( H^q \) admits the Blaschke-Levy representation (1) with the exponent \( q, \) where the function \( b \in L_1(\Omega) \) can be calculated by

\[
b(\xi) = (-1)^k \frac{\pi}{2(2\pi)^{n-1}C_{n-q+2k}C_q} \int_\Omega |(\theta, \xi)|^{-n-q+2k}(\Delta^k H^q)(\theta) \, d\theta
\]

for every \( \xi \in \Omega. \) Moreover,

\[
\|b\|_1 \leq \frac{\pi W_{n-q+2k}}{2(2\pi)^{n-1}C_{n-q+2k}C_q} \|(\Delta^k H^q)|_\Omega\|_1.
\]

If the function \( (\Delta^k H^q)|_\Omega \) belongs to \( L_\infty(\Omega) \) then \( b \in L_\infty(\Omega) \) and

\[
\|b\|_\infty \leq \frac{\pi W_{n-q+2k}}{2(2\pi)^{n-1}C_{n-q+2k}C_q} \|(\Delta^k H^q)|_\Omega\|_\infty.
\]

**Proof.** Since the function \( H^q \) is even homogeneous, the distribution \( \Delta^k H^q \) is even homogeneous of the order \( q-2k < -n+1. \) Also \( -n-q+2k \) is not an even integer, so \( \Delta^k H^q \) satisfies the conditions of Lemma 3. By Lemma 3, the Fourier transform

\[
(\Delta^k H^q)^\wedge(\xi) = \frac{\pi}{C_{n-q+2k}} \int_\Omega |(\theta, \xi)|^{-n-q+2k}(\Delta^k H^q)(\theta) \, d\theta
\]

is an \( L_1 \)-function on \( \Omega. \) Because of the connection between the Fourier transform and differentiation we have

\[
(\Delta^k H^q)^\wedge(\xi) = (-1)^k (\xi_1^2 + \ldots + \xi_n^2)^k (H^q)^\wedge(\xi),
\]

and the restrictions to the sphere of \( (\Delta^k H^q)^\wedge \) and \( (-1)^k (H^q)^\wedge \) are equal. In particular, \( (H^q)^\wedge|_\Omega \in L_1(\Omega) \). This means that we can apply Theorem 1, and the result follows. \( \square \)

If \( q \) is an even integer the uniqueness in the Blaschke representation fails (as mentioned in Remark 2). Therefore, it remains to consider the case where \( q \) is an odd integer.

First, suppose that the dimension \( n \) is even. Then we apply the Laplace operator to the function \( H^q \) until it becomes a homogeneous function of degree \( -n+1, \) and then we use Lemma 4 instead of Lemma 3. The rest of the proof of Theorem 3 is similar to that of Theorem 2.
Theorem 3. Let \( n \in \mathbb{N} \) be an even integer, \( q > 0 \) be an odd integer, and \( H \) is a continuous, non-negative, even homogeneous function of degree 1 on \( \mathbb{R}^n \), \( n > 1 \). Suppose that \( \Delta^{(q+1)/2} H^q \) is a function in \( \mathbb{R}^n \setminus \{0\} \) so that \( (\Delta^{(q+1)/2} H^q)|_\Omega \in L_1(\Omega) \), where differentiation is considered in the sense of distributions. Then the function \( H^q \) admits the Blaschke-Levy representation (1) with the exponent \( q \), and the corresponding function \( b \in L_1(\Omega) \) is given by

\[
b(\xi) = \frac{(-1)^{n+1}/2\pi}{(2\pi)^{n-1}C_q} \int_{\Omega \cap \{\theta, \xi\} = 0} \Delta^{(q+1)/2} H^q(\theta) \, d\theta
\]

for every \( \xi \in \Omega \). Moreover, if \( b \in L_\infty(\Omega) \) then

\[
\|b\|_\infty \leq \frac{2\pi^{(n+1)/2}}{\Gamma((n-1)/2)(2\pi)^{n-1}C_q}}(\Delta^{(q+1)/2} H^q)|_\Omega\|_\infty.
\]

In the case where \( q \) and \( n \) are both odd integers, the technique of this paper does not work. The reason is that the polynomials, which appear at the end of the proofs of Theorem 1 and Lemma 3 (and can easily be eliminated in those cases), start playing active role when \( n + q \) is an even integer. To illustrate this, let us just note that, for the Euclidean norm \( \|x\|_2 \) in \( \mathbb{R}^n \) with \( n \) being an odd integer, the distribution \( \Delta^2 \|x\|_2 \) vanishes everywhere in \( \mathbb{R}^n \setminus \{0\} \), and, therefore, it is a linear combination of the derivatives of the \( \delta \)-function. So in the case where \( q \) and \( n \) are odd integers, the Fourier transform of \( \Delta^k H^q \) not always can be expressed in terms of the restriction of the function \( \Delta^k H^q \) to the sphere.

Let us give a scheme of how Theorem 3 works in the case where \( q = 1 \), \( n \) is an even integer, and the function \( H \) is of the form \( H(x) = P_m(x)\|x\|_2^{m+1} \) on \( \mathbb{R}^n \), where \( P_m \) is an even homogeneous polynomial of degree \( m > 0 \). First, by Euler’s formula for homogeneous functions, we have \( \sum x_i(\partial P_m/\partial x_i) = mP_m \), and, for every \( \beta \),

\[
\Delta(P_m\|x\|_2^{\beta}) = \Delta(P_m)\|x\|_2^{\beta} + \beta(n + 2m + \beta - 2)P_m\|x\|^{\beta-2}.
\]

Iterating the latter formula one can calculate \( \Delta^k H \) for every \( k \), and find the polynomial which is the restriction of \( \Delta^k H \) to the sphere. Now the problem of finding the Blaschke-Levy representation for the function \( H \) is reduced to calculating the integrals of the form

\[
\int_{\Omega \cap \{x, \xi\} = 0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, dx,
\]

where \( \alpha_i \) are even integers and \( \xi \in \Omega \). To calculate these integrals we use an argument similar to that of [17]. Namely, we start with the equality

\[
\|

\|\xi\|_2^{\alpha_1 + \cdots + \alpha_n - 1} = (1/W_{\alpha_1 + \cdots + \alpha_n - 1}) \int_{\Omega} \|x, \xi\|^{\alpha_1 + \cdots + \alpha_n - 1} \, dx.
\]

Differentiating this equality we see that the integral (10) is equal to

\[
\frac{\partial^{\alpha_1 + \cdots + \alpha_n} \|\xi\|^{(\alpha_1 + \cdots + \alpha_n - 1)/2}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}} W_{\alpha_1 + \cdots + \alpha_n - 1} \frac{(\alpha_1 + \cdots + \alpha_n - 1)!}{(\alpha_1 + \cdots + \alpha_n - 1)!},
\]

where the derivative is calculated at the point \( \xi \in \Omega \). In Section 4 we give a numerical example.
This calculation includes differentiation only. A different way of calculating the function $b$ is to find the spherical harmonics expansion of the polynomial $P_m$, and then use Rodriguez’s formula (see [22] for the properties of spherical harmonics).

If $H$ is not of a polynomial form, it is sometimes impossible to calculate $b$ precisely using our inverse formulae. However, Theorems 2 and 3 give estimates for the $L_1$ and $L_\infty$-norms of the function $b$ with computable constants. This seems to be an advantage of our approach over the one using spherical harmonics where the constants appear as the sums of certain series’.

4. A characterization of subspaces of $L_q$.

The question of how to check whether a given space is isometric to a subspace of $L_q$ is a matter of an old problem raised by P.Levy [19]. In [19] P.Levy showed that an $n$-dimensional space is isometric to a subspace of $L_q$ if and only if its norm admits the Blaschke-Levy representation with the exponent $q$ (and with a non-negative measure in place of the function $b$.) Bretagnolle, Dacunha-Castelle and Krivine [5] proved that, for $0 < q \leq 2$, a Banach space is isometric to a subspace of $L_q$ if and only if the function $\exp(-\|x\|^q)$ is positive definite, and, in particular, showed that the space $L_p$ embeds isometrically into $L_q$ if $0 < q < p \leq 2$. Another criterion involving the Fourier transform (which, in fact, is our Theorem 1 in a slightly stronger form) was given in [14], [15]: for any $q \in (0, \infty) \setminus \{\text{even integers}\}$, an $n$-dimensional space is isometric to a subspace of $L_q$ if and only if the restriction of the Fourier transform of $\|x\|^q \Gamma(-q/2)$ to the sphere $\Omega$ is a finite Borel (non-negative) measure on $\Omega$. Though the Fourier transform criteria work for certain spaces, calculating the Fourier transform of a norm precisely is not always possible. That is why a condition involving the derivatives of the norm instead of the Fourier transform could be useful. A necessary condition in terms of the derivatives of the norm was given by Zastanvy [29] who proved that a three dimensional space is not isometric to a subspace of $L_q$ with $0 < q \leq 2$ if there exists a basis $e_1, e_2, e_3$ so that the function

$$(y, z) \mapsto \|xe_1 + ye_2 + ze_3\|'_x(1, y, z)/\|e_1 + ye_2 + ze_3\|, \quad y, z \in \mathbb{R}$$

belongs to the space $L_1(\mathbb{R}^2)$.

In this section, we use the inverse formula for the Blaschke-Levy representation to give a sufficient condition for the existence of isometric embedding of a space into $L_q$ which is formulated in terms of the Laplace operator of the norm.

We start with a well-known fact which explains the connection between the Blaschke-Levy representation and isometric embedding into $L_q$.

**Lemma 5.** Let $q$ be a positive number which is not an even integer, $(X, \|\cdot\|)$ be an $n$-dimensional space, and suppose that the function $\|\cdot\|^q$ admits the Blaschke-Levy representation of the form $\|\cdot\|^q = \gamma + \sum_{m=1}^\infty a_m P_m(u)$. Then $X$ is isometric to a subspace of $L_q$ if and only if $\mathcal{L}(u) = -\sum_{m=1}^\infty a_m \frac{\partial}{\partial u_m}$ is a positive definite measure on $S^{n-1}$.
representation with a function \( b \in L_1(\Omega) \): for every \( x \in \mathbb{R}^n \),

\[
\|x\|^q = \int_\Omega |(x, \xi)|^q \, b(\xi) \, d\xi.
\]

Then \( X \) is isometric to a subspace of \( L_q \) if and only if \( b \) is a non-negative (not identically zero) function.

**Proof.** If \( b \) is a non-negative function we can assume without loss of generality that \( \int_\Omega b(\xi) \, d\xi = 1 \). Choose any measurable (with respect to Lebesgue measure) functions \( f_1, \ldots, f_n \) on \([0, 1]\) so that their joint distribution is the measure \( b(\xi) \, d\xi \) on the sphere \( \Omega \). Then, by (11), the operator \( x \mapsto \sum x_i f_i, \, x \in \mathbb{R}^n \) is an isometry from \( X \) to \( L_q([0, 1]) \).

Conversely, if \( X \) is a subspace of \( L_q([0, 1]) \) choose any functions \( f_1, \ldots, f_n \in L_q \) which form a basis in \( X \), and let \( \mu \) be the joint distribution of the functions \( f_1, \ldots, f_n \) with respect to Lebesgue measure. Then, for every \( x \in \mathbb{R}^n \),

\[
\|x\|^q = \left\| \sum_{k=1}^n x_k f_k \right\|^q = \int_0^1 \left\| \sum_{k=1}^n x_k f_k(t) \right\|^q dt =
\]

\[
\int_{\mathbb{R}^n} |(x, \xi)|^q \, d\mu(\xi) = \int_{\Omega} |(x, \xi)|^q \, d\mu_\Omega(\xi)
\]

where \( \mu_\Omega \) is the projection of \( \mu \) to the sphere. (For every Borel subset \( A \) of \( \Omega \), \( \mu_\Omega(A) = (1/2) \int_{\{\xi, \xi \in R\}} \|x\|^q d\mu(x) \). It follows from (11) and (12) that

\[
\int_{\Omega} |(x, \xi)|^q \, b(\xi) \, d\xi = \int_{\Omega} |(x, \xi)|^q \, d\mu_\Omega(\xi)
\]

for every \( x \in \mathbb{R}^n \). Since \( q \) is not an even integer, we can apply the uniqueness theorem for measures on the sphere (see Remark 2) to show that \( d\mu_\Omega(\xi) = b(\xi) \, d\xi \) which means that \( b(\xi) \, d\xi \) is a measure, and the function \( b \) is non-negative. \( \square \)

In view of Lemma 5, the inverse formulae from Section 3 lead to the following criteria of isometric embedding into \( L_q \).

First, if \( q \) is not an integer we use Lemma 5 and Theorem 2: under the assumption that \( (\Delta^k\|x\|^q)|_\Omega \in L_1(\Omega) \), an \( n \)-dimensional normed (quasi-normed) space \( (\mathbb{R}^n, \|\cdot\|) \) embeds isometrically in \( L_q \) if and only if

\[
\xi \to \frac{(-1)^k}{C_{-n-q+2k}C_q} \int_\Omega |(\theta, \xi)|^{-n-q+2k} \Delta^k\|\theta\|^q \, d\theta,
\]

is a non-negative function on \( \Omega \), where \( k \) is as in Theorem 2. If for some reason it is impossible to calculate the latter integral precisely, one can use the following sufficient condition: if the function \( ((-1)^k/(C_{-n-q+2k}C_q))\Delta^k\|x\|^q \) is non-negative on \( \Omega \) and its restriction to \( \Omega \) belongs to \( L_1(\Omega) \) then the space \( (\mathbb{R}^n, \|\cdot\|) \) embeds isometrically into \( L_q \).
If \( q \) is an odd integer and the dimension \( n \) is an even integer, similar criteria follow from Lemma 5 and Theorem 3. Under the assumption that \( (\Delta^{(n+q-1)/2}\|x\|^q)|\Omega \in L_1(\Omega) \), a space \((\mathbb{R}^n, \| \cdot \|)\) is isometric to a subspace of \( L_q \) if and only if

\[
\xi \rightarrow \frac{(-1)^{(n+q-1)/2}}{C_q} \int_{\Omega \cap \{ (\theta, \xi) = 0 \}} \Delta^{(n+q-1)/2}\|\theta\|^q \, d\theta
\]

is a non-negative function on \( \Omega \). The related sufficient condition is that the function \((-1)^{(n+q-1)/2}/C_q)\Delta^{(n+q-1)/2}\|x\|^q\) is a non-negative \( L_1 \)-function on \( \Omega \).

**Example 1.** Consider the function \( \|x\| = \|x\|_2 + \lambda x_1^2\|x\|_2^{-1} \) which is an even homogeneous function of degree 1 on \( \mathbb{R}^n \). For which values of \( \lambda \) does the space \((\mathbb{R}^d, \| \cdot \|)\) embed isometrically in \( L_1 \)? An equivalent question asks for the values of \( \lambda \) for which the polar set to \( \{ x : \|x\| \leq 1 \} \) is a zonoid (see [4] for the connection between zonoids and embedding into \( L_1 \)).

Let us apply Theorem 3 with \( q = 1 \), \( n = 4 \) to find the function \( b \) corresponding to \( H(x) = \|x\| \). Since \( (n + q - 1)/2 = 2 \) we calculate

\[
\Delta^2\|x\| = -3\|x\|_2^{-3} + \lambda(-12\|x\|_2^{-1} + 45x_1^2\|x\|_2^{-3}).
\]

Therefore, \( \Delta^2H|\Omega = -3 - 12\lambda + 45\lambda x_1^2 \). Also \( C_1 = -1 \), and, by Theorem 3, for every \( \xi \in \Omega \)

\[
b(\xi) = \frac{1}{8\pi^2} \int_{\Omega \cap \{ (\theta, \xi) = 0 \}} (3 - 12\lambda - 45\lambda x_1^2) \, dx.
\]

To calculate the integral note that \( \int_{\Omega \cap \{ (\theta, \xi) = 0 \}} x_1^2 \, dx \) is equal to the second derivative by \( \xi_1 \) of the integral \( \int_{\Omega} |(x, \xi)| \, dx = W_1\|\xi\|_2 \). Also the surface area of the 3-dimensional sphere \( \Omega \cap \{ (\theta, \xi) = 0 \} \) is equal to \( 2\pi^{3/2}/\Gamma(3/2) \).

Finally, \( b(\xi) = (1/(8\pi))(4 - 8\lambda + 24\lambda \xi_1^2) \). Clearly, \( b \) is a non-negative function if and only if \(-1/4 \leq \lambda \leq 1/2 \), and these are all the values of \( \lambda \) for which the space embeds in \( L_1 \).

**Example 2.** Let \( \|x\| = \|x\|_2 + \lambda P(x) \), where \( P \) is an even homogeneous function of degree 1 on \( \mathbb{R}^n \), \( n \) is an even integer, and \( P|\Omega \in C^{n/2}(\Omega) \). To find the values of \( \lambda \) for which \((\mathbb{R}^n, \| \cdot \|)\) embeds isometrically in \( L_1 \), we calculate

\[
\Delta^{n/2}\|x\| = (-1)^{(n-2)/2}(n - 1)!!(n - 3)!!\|x\|_2^{-n+1} + \lambda \Delta^{n/2}P.
\]

The sufficient condition formulated above shows that the space embeds in \( L_1 \) if \((n - 1)!!(n - 3)!! - (-1)^{(n-2)/2}\lambda(\Delta^{n/2}P)|\Omega \) is a non-negative function (note that \( C_1 = -1. \)) Hence, if

\[
|\lambda| \leq \frac{(n - 3)!!(n - 1)!!}{\|\Delta^{n/2}P\|_\infty},
\]

then the space embeds in \( L_1 \).
5. A Fourier transform formula for the central sections of star bodies

Let $K$ be a centrally symmetric star body in $\mathbb{R}^n$ so that the norming functional $\|x\| = \min\{a > 0 : x \in aK\}$, $x \in \mathbb{R}^n$ generated by $K$ is a continuous, non-negative, even homogeneous function of degree 1 on $\mathbb{R}^n$. It is easy to see that, for every $\xi$ in the unit sphere $\Omega$, the $(n - 1)$-dimensional volume of the section of $K$ by the hyperplane $\xi^\perp = \{(x, \xi) = 0\}$ satisfies the equality

\[
\frac{Vol_{n-1}(K \cap \xi^\perp)}{Vol_{n-1}(B_{n-1})} = \frac{\int_{\Omega \cap \xi^\perp} \|x\|^{-n+1} \, dx}{A_{n-1}},
\]

where $Vol_{n-1}(B_{n-1}) = \pi^{(n-1)/2}/\Gamma((n + 1)/2)$ is the volume of the Euclidean unit ball $B_{n-1}$ in $\mathbb{R}^{n-1}$, and $A_{n-1} = 2\pi^{(n-1)/2}/\Gamma((n - 1)/2)$ is the surface area of the Euclidean unit sphere in $\mathbb{R}^{n-1}$.

The integral in the right-hand side of (13) is equal to the integral in Lemma 4 with $f(x) = \|x\|^{-n+1}$. Therefore, Lemma 4 and (13) imply the following Fourier transform formula for the volume of central sections of $K$:

**Theorem 4.** For every $\xi \in \Omega$,

\[
Vol_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi (n - 1)^n} (\|x\|^{-n+1})^\wedge(\xi).
\]

The Fourier transforms of powers of different norms have been calculated in [18] (for the $\ell^n_\infty$-norm), [15] (for the $\ell^n_p$-norms), [6] (for the Lorentz norm). In view of Theorem 4, one can use those calculations to obtain formulae for the volume of central sections. For example, the Fourier transform of the functions of the form $f(\|x\|_\infty)$ was calculated in [18], where $\|x\|_\infty$ stands for the norm of the space $\ell^n_\infty$, and $f$ belongs to a large class of functions on $\mathbb{R}$. (Note that a multiplier $(-1)^{n-1}$ is missing in the formula in [18].) If we apply the formula from [18] to the function $f(t) = |t|^p$ with $p \in (-1, 0)$, use the formulae for the Fourier transform of the functions $|t|^p$ and $|t|^p \text{sgn}(t)$ (see [8, p.173]), and then use analytic extension by $p$, we get an expression for the Fourier transform of $\|x\|^{-n+1}_\infty$ : for every $\xi \in \mathbb{R}^n$ with non-zero coordinates, if the dimension $n$ is odd we have

\[
(\|x\|^{-n+1}_\infty)^\wedge(\xi) = \frac{(-1)^{(n-1)/2} \sqrt{\pi} \Gamma((-n + 2)/2)}{\Gamma((n - 1)/2) \prod_{k=1}^n \xi_k} \sum_{\delta} \delta_1 \ldots \delta_n \sum_{j=1}^n \delta_j \xi_j |\xi_j|^{n-1} \text{sgn}(\sum_{j=1}^n \delta_j \xi_j).
\]

If the dimension $n$ is even we have

\[
(\|x\|^{-n+1}_\infty)^\wedge(\xi) = \frac{(-1)^{(n-2)/2} \sqrt{\pi} \Gamma((-n + 3)/2)}{\Gamma(n/2) \prod_{k=1}^n \xi_k} \sum_{\delta} \delta_1 \ldots \delta_n \sum_{j=1}^n \delta_j \xi_j |\xi_j|^{n-1}.
\]
The outer sum is taken over all changes of sign \( \delta = (\delta_1, \ldots, \delta_n) \), \( \delta_j = \pm 1 \). These formulae, in conjunction with Theorem 4, imply simple formulae for the volume of central sections of the cube \([-1, 1]^n\). Previously, similar formulae were obtained using probabilistic arguments specifically designed for the cube. Ball [2] has shown that the exact lower and upper bounds for the volume of central sections of the unit ball of the space \( \ell_p^n \) are \( 2^n \) and \( 2^n \sqrt{2} \), respectively. We refer the reader to [7] for a historical survey and more information about sections.

Meyer and Pajor [21] have proved that the minimal section of the unit ball of the space \( \ell_1^n \) is the one perpendicular to the vector \((1, 1, \ldots, 1)\), and the maximal section is perpendicular to the vector \((1, 0, 0, \ldots, 0)\). They also showed that, for the unit balls of the spaces \( \ell_p^n \) with \( 1 < p < 2 \), the upper bound occurs in the same direction as for \( p = 1 \), and raised the question of whether the situation is the same for the lower bound.

We end this paper by confirming the conjecture of Meyer and Pajor. First, let us compute the Fourier transform of the functions \( \|x\|^\beta_p \), where \( \|x\|_p \) stands for the norm of the space \( \ell_p^n \). Denote by \( \gamma_p \) the Fourier transform of the function \( z \mapsto \exp(-|z|^p) \), \( z \in \mathbb{R} \). For \( 0 < p \leq 2 \), \( \gamma_p \) is (up to a constant) the density of the standard \( p \)-stable measure on \( \mathbb{R} \), so \( \gamma_p \) is a non-negative function. For every \( p > 0 \),

\[
\lim_{t \to \infty} t^{1+p} \gamma_p(t) = 2\Gamma(p+1) \sin(\pi p/2),
\]

so \( \gamma_p \) decreases at infinity as \( |t|^{-1-p} \) (see [30]). Also simple calculations show that \( \gamma_p(0) = 2\Gamma(1+1/p) \), and \( \int_0^\infty \gamma_p(t) \, dt = \pi \). The following calculation is taken from [15].

**Lemma 6.** Let \( p > 0 \), \( n \in \mathbb{N} \), \( -n < \beta < pn \), \( \beta/p \notin \mathbb{N} \cup \{0\} \), \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \), \( \xi_k \neq 0 \), \( 1 \leq k \leq n \). Then

\[
(\|x\|_p^\beta)^\wedge(\xi) = ((|x_1|^p + \cdots + |x_n|^p)^{\beta/p})^\wedge(\xi) = \frac{p}{\Gamma(-\beta/p)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_p(t\xi_k) \, dt.
\]

**Proof.** Assume that \( -1 < \beta < 0 \). By the definition of the \( \Gamma \)-function

\[
(|x_1|^p + \cdots + |x_n|^p)^{\beta/p} = \frac{p}{\Gamma(-\beta/p)} \int_0^\infty y^{-1-\beta} \exp(-y^p(|x_1|^p + \cdots + |x_n|^p)) \, dy.
\]

For every fixed \( y > 0 \), the Fourier transform of the function \( x \mapsto \exp(-y^p(|x_1|^p + \cdots + |x_n|^p)) \) at any point \( \xi \in \mathbb{R}^n \) is equal to \( y^{-n} \prod_{k=1}^n \gamma_p(\xi_k/y) \). Making the change of variables \( t = 1/y \) we get

\[
((|x_1|^p + \cdots + |x_n|^p)^{\beta/p})^\wedge(\xi) = \frac{p}{\Gamma(-\beta/p)} \int_0^\infty y^{-n-\beta-1} \prod_{k=1}^n \gamma_p(\xi_k/y) \, dy =
\]
\[ (14) \quad \frac{p}{\Gamma(-\beta/p)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_p(t\xi_k) \, dt. \]

The latter integral converges if \(-n < \beta < pn\) since the function \(t \to \prod_{k=1}^n \gamma_p(t\xi_k)\) decreases at infinity like \(t^{-n-p}\) (recall that \(\xi_k \neq 0, 1 \leq k \leq n\)).

If \(\beta\) is allowed to assume complex values then the both sides of (14) are analytic functions of \(\beta\) in the domain \(\{-n < \Re \beta < np, \beta/p \notin \mathbb{N} \cup \{0\}\}\). These two functions admit unique analytic continuation from the interval \((-1, 0)\). Thus the equality (14) remains valid for all \(\beta \in (-n, pn), \beta/p \notin \mathbb{N} \cup \{0\}\) (see [8] for details of analytic continuation in such situations). \(\square\)

Now we can use Lemma 6 with \(\beta = -n+1\) and Theorem 4 to get an expression for the volume of central sections. Note that the condition of Lemma 6 that \(\xi\) has non-zero coordinates may be removed in Corollary 1 because the volume of a section is a continuous function of \(\xi\). Denote by \(B_p\) the unit ball of the space \(\ell_p^n, p > 0, n > 1\).

**Corollary 1.** For every \(p > 0\) and \(\xi \in \Omega\),

\[ \text{Vol}_{n-1}(B_p \cap \xi^\perp) = \frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \prod_{k=1}^n \gamma_p(t\xi_k) \, dt. \]

For \(p \in (1, 2)\), the latter equality was established by Meyer and Pajor [21] using a probabilistic argument. Note that when \(p \to \infty\) the formula (14) turns into the expression used by Ball [2] for the slices of the unit cube.

The following fact is a property of the functions \(\gamma_p\) with \(p \in (0, 2)\) only.

**Lemma 7.** For every \(p \in (0, 2),\) the function \(\gamma_p(\sqrt{t})\) is log-convex on \((0, \infty)\). In other words, the function \(\gamma_p'(t)/(t\gamma_p(t))\) is increasing on \((0, \infty)\). Also, for every \(k, m \in \mathbb{N}, k < m\) and every \(t > 0\), we have \(\gamma_p^k(k^{-1/2}t)\gamma_p^{m-k}(0) \geq \gamma_p^m(m^{-1/2}t)\).

**Proof.** A well-known fact is that there exists a measure \(\mu\) on \([0, \infty)\) whose Laplace transform is equal to \(\exp(-tp^{p/2})\). This is a stable measure, and its properties and asymptotic behavior of its density (which decreases at infinity as \(|t|^{-1-p/2}\), up to a constant) are described, for example, in [30]. For every \(z \in \mathbb{R}\), we have

\[ \exp(-|z|^p) = \int_0^\infty \exp(-uz^2) \, d\mu(u). \]

Calculating the Fourier transform of both sides of the latter equality as functions of the variable \(z\), we get, for every \(t \in \mathbb{R}\),

\[ \gamma_p(t) = \sqrt{2\pi} \int_0^\infty u^{-1/2} \exp\left(\frac{-t^2}{4u}\right) \, d\mu(u). \]

where the integral converges because of the asymptotics of the density of \(\mu\) at infinity, as mentioned above. Now the fact that \(\gamma_p(\sqrt{(t^2 + t^2)/2}) \leq \gamma_p(\sqrt{t^2})\) follows from \(\left(\sqrt{t^2 + t^2}/2\right)^2 = \sqrt{t^2} \cdot (\sqrt{t^2})\).
follows from the Cauchy-Schwarz inequality applied to the functions \( \exp(-t_1/(8u)) \) and \( \exp(-t_2/(8u)) \) and the measure \( u^{-1/2} \, d\mu(u) \), where \( t_1, t_2 \) are arbitrary positive numbers. Therefore, the function \( \gamma_p(\sqrt{t}) \) is log-convex which implies the other two statements of Lemma 7. □

**Theorem 5.** For every \( p \in (0, 2) \) and every \( \xi \in \Omega \),

\[
\frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \gamma_p^n(t/\sqrt{n}) \, dt \leq \operatorname{Vol}_{n-1}(B_p \cap \xi^\perp) \leq \frac{p}{\pi(n-1)\Gamma((n-1)/p)} \gamma_p^{n-1}(0) \int_0^\infty \gamma_p(t) \, dt = 2
\]

with the left inequality turning into an equality if and only if \( |\xi_i| = 1/\sqrt{n} \) for every \( i \), and the upper bound occurs if and only if one of the coordinates of the vector \( \xi \) is equal to \( \pm 1 \) and the others are equal to zero.

**Proof.** Consider the function

\[
F(\xi_1, \ldots, \xi_n) = \int_0^\infty \gamma_p(t\xi_1) \cdots \gamma_p(t\xi_n) \, dt + \lambda(\xi_1^2 + \cdots + \xi_n^2 - 1),
\]

where \( \lambda \) is the Lagrange multiplier. It suffices to find the maximal and minimal value of the function \( F \) in the positive octant under the condition \( \xi_2 + \cdots + \xi_n^2 = 1 \). To find the critical points of the function \( F \), we have to solve the system of equations

\[
\frac{\partial F}{\partial \xi_i}(\xi) = \int_0^\infty t\gamma_p'(t\xi_i) \prod_{k \neq i} \gamma_p(t\xi_k) \, dt + 2\lambda \xi_i = 0,
\]

where \( i = 1, \ldots, n \). For each \( i \) with \( \xi_i \neq 0 \) we can write the latter equality in the following form:

(14.)

\[
\int_0^\infty t \frac{\gamma_p'(t\xi_i)}{\xi_i \gamma_p(t\xi_i)} \prod_{k=1}^n \gamma_p(t\xi_k) \, dt = -2\lambda
\]

Since (by Lemma 7) the function \( \gamma_p'(t\xi_i)/(\xi_i \gamma_p(t\xi_i)) \) is increasing and \( \gamma_p \) is non-negative, we can have (14) for different values of \( i \) simultaneously only if the corresponding coordinates of the vector \( \xi \) are equal. Therefore, the critical points of the function \( F \) are only those points \( \xi \) for which some of the coordinates are zero, and the absolute values of the rest are equal. Hence, the problem is reduced to comparing the values of \( F \) at the points \( \xi^{(k)}, \ k = 1, \ldots, n \), where the first \( k \) coordinates of \( \xi^{(k)} \) are equal to \( 1/\sqrt{k} \) and the last \( n-k \) coordinates are equal to zero. It follows from the inequality of Lemma 7 that the maximal value of \( F \) on the sphere \( \Omega \) occurs at the point \( \xi^{(1)} \), and the minimal value is at the point \( \xi^{(n)} \). Now the result of Theorem 5 follows from Corollary 1. □
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