Quantum speed limit time in relativistic frame

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We investigate the roles of relativistic effect on the speed of evolution of a quantum system coupled with amplitude damping channels. We find that the relativistic effect speed-up the quantum evolution to a uniform evolution speed of an open quantum systems for the damping parameter \(p_\tau \lesssim p_{\tau,0}\). Moreover, we point out a non-monotonic behavior of the quantum speed limit time (QSLT) with acceleration in the damping limit \(p_{\tau,0} \lesssim p_\tau \lesssim p_{\tau,1}\), where the relativistic effect first speed-up and then slow down the quantum evolution process of the damped system. For the damping strength \(p_{\tau,1} \lesssim p_\tau\), we observe a monotonic increasing behavior of QSLT, leads to slow down the quantum evolution of the damped system. In addition, we examine the roles of the relativistic effect on the speed limit time for a system coupled with the phase damping channels.

I. INTRODUCTION

The field of quantum information under relativistic constraints leads to the emergence of a new field of high research intensity, known as Relativistic Quantum Information (RQI). The most spectacular research in this field have been devoted to the study of: entanglement between quantum field modes in the accelerated frames 

\[ \text{Unruh entanglement degradation between quantum field modes in the accelerated frames} \]

The tool of quantum information theory plays a prominent role in the understanding of entanglement witness of polarization-entangled photon pairs. It has been experimentally tested that the photonic quantum entanglement persist in the accelerated frames under single mode approximation. Moreover, the observer-dependent property of entanglement has been successfully examined beyond the single-mode approximations.

The minimal time required for the evolution of a quantum system is known as “Quantum Speed Limit Time” (QSLT). There exists a considerable amount of work dedicated to estimate the minimal evolution time of a quantum system. For instance, QSLT was successfully investigated for the damped Jaynes-Cummings and the Ohmic-like dephasing model. Moreover, it was found that the relativistic effect slow down the quantum evolution of the qubit in the damped Jaynes-Cummings model. There have also been studies involving the speed of quantum evolution of a single free spin-1/2 particle coupled with phase damping channels in the relativistic framework. In addition, the nature of QSLT in Schwarzschild space-time for the damped Jaynes-Cummings and Ohmic-like dephasing models have been examined. Their results show that the QSLT decreased and increased by increasing relative distance of quantum system to event horizon for damped Jaynes-Cummings and Ohmic-like dephasing model, respectively.

Recently, the relativistic effects on the speed of quantum evolution have been reported for a free Dirac field in non-inertial frames. It is pointed out that the relativistic effects speed-up the evolution of the quantum system coupled with the amplitude damping channels. However, no relativistic effects have been encountered for the speed of quantum evolution of the phase damped-system in a non-inertial frame.

The aim of this article is to explore the role of relativistic effects on the speed of evolution of a quantum system for a free scalar field which manifests itself in the quantum noise. We point out that the QSLT initially reduces to a minimum with increasing acceleration and then trapped to a uniform fixed value for damping parameter \(p_\tau \lesssim p_{\tau,0}\). This phenomenon leads to a speed-up of the quantum evolution initially, and then reaches to a uniform evolution speed of an open quantum system. In the region \(p_{\tau,0} \lesssim p_\tau \lesssim p_{\tau,1}\), the QSLT first decreases to a minimum value, and then gradually increases to a maximum uniform value as depicted in Fig. 1.

This shows that the relativistic effect speed-up the quantum evolution in the beginning and then slow down the speed of evolution of the system. However, the quantum evolution of the system exhibits a slow down behavior with increasing acceleration for \(p_{\tau,1} \lesssim p_\tau\), leads to a larger QSLT in non-inertial frame. For each case, we notice a uniform speed of evolution of the system in the large acceleration limit, where the QSLT trapped to a
fixed value. In addition, for the phase damped-system, we obtain an acceleration independent speed limit time in relativistic frame.

![Damping parameter](p\_\tau)

![Acceleration parameter](r)

**uniform evolution**

**slow-down evolution**

**speed-up evolution**

Figure 1. (Color online) The roles of relativistic effect on quantum speed limit time of an open system coupled with the amplitude damping channels.

The structure of the paper is as follows. Sec. II is devoted to the theoretical background of the scalar field as observed by uniformly accelerated observer. In particular, we review the mathematical transformations between Minkowski and Rindler modes under single mode approximation. In Sec. III we present the physical scenario and the mathematical procedure for calculating the QSLT of a quantum system coupled with the amplitude damping channels in non-inertial frames. Moreover, we analyze the relativistic effects on QSLT for the amplitude damped open quantum system, when one observer move with a uniform acceleration. In the last section, we sum up our conclusions.

II. SCALAR FIELD

A real scalar field \( \phi \) in two dimensional Minkowski space time can be described by the massless Klein-Gordon equation, \( \Box \phi = 0 \). This field can be expressed in terms of the positive and negative energy solutions of the Klein-Gordon equation, given by \(^{13,20}\)

\[
\Phi_{\omega,M} = \int_0^\infty (a_{\omega,M} \varphi_{\omega,M} + a_{\omega,M}^\dagger \varphi_{\omega,M}^*) d\omega
\]

where \( a_{\omega,M} \) and \( a_{\omega,M}^\dagger \) are the Minkowski annihilation and creation operators, obeying the bosonic commutation relations. The positive-energy mode solution \( \varphi_{\omega,M} \) with respect to the timelike Killing vector field \( \partial_x \), for an inertial observer in Minkowski coordinates \( (t,x) \), with positive Minkowski frequency \( \omega \) is given by

\[
\varphi_{\omega,M}(t,x) = \frac{1}{\sqrt{4\pi\omega}} \exp[-i\omega(t-\varepsilon x)],
\]

where \( \varepsilon \) can takes the value \( 1 \) and \( -1 \) for modes with positive (right movers) and negative (left movers) momentum, respectively. The mode solutions satisfy the following relations

\[
\begin{align*}
(\varphi_{\omega,M}, \varphi_{\omega,M}') &= -(\varphi_{\omega,M}', \varphi_{\omega,M}) = \delta_{\omega\omega}', \\
(\varphi_{\omega,M}^*, \varphi_{\omega,M}') &= 0.
\end{align*}
\]

The Klein-Gordon equation for a uniformly accelerated observer can be more appropriately described by Rindler space-time. The Rindler coordinates and Minkowski coordinates are related by \(^{31,30}\)

\[
\eta = \arctan \left( \frac{t}{x} \right), \quad \chi = \sqrt{x^2 - t^2},
\]

where \( \chi = a^{-1} \), is the position and \( \eta/a \) is the proper time of the accelerated observer in region I. Here \( a \) is a positive constant, referred as the acceleration of the uniformly accelerated observer. The Rindler coordinates in region II can simply be obtained by replacing \( \eta = -\eta \). The Rindler coordinates have ranges \( 0 < \chi < \infty \) and \( -\infty < \eta < \infty \).

The field can be expanded in terms of the energy solutions of the Klein-Gordon equation in region I and II in the Rindler coordinates \(^{31,30}\)

\[
\Phi_{\Omega,R} = \int_0^\infty \left( a_{\Omega,1} \varphi_{\Omega,1} + a_{\Omega,1}^\dagger \varphi_{\Omega,1}^* + a_{\Omega,II} \varphi_{\Omega,II} + a_{\Omega,II}^\dagger \varphi_{\Omega,II}^* \right) d\Omega,
\]

where \( a_{\Omega,\sigma} \) and \( a_{\Omega,\sigma}^\dagger \) are the Rindler annihilation and creation operators for \( \sigma \in \{I, II\} \), respectively. It obeys the bosonic commutation relations. The \( \varphi_{\Omega,\sigma}(t,x) \) is the positive frequency mode functions with respect to the timelike Killing vector field \( \pm \partial_\eta \), for the accelerated observer in region \( \sigma \), as given by

\[
\begin{align*}
\varphi_{\Omega,1}(t,x) &= \frac{1}{\sqrt{4\pi\Omega}} \left( \frac{x-\varepsilon t}{l_\Omega} \right)^{it\Omega} e^{i\Omega t}, \\
\varphi_{\Omega,II}(t,x) &= \frac{1}{\sqrt{4\pi\Omega}} \left( \frac{\varepsilon t-x}{l_\Omega} \right)^{-it\Omega} e^{i\Omega t},
\end{align*}
\]

where \( \Omega \) is a dimensionless Rindler frequency, \( l_\Omega \) is a positive constant of dimension length.

The field solution can also be expressed in the Unruh bases \(^{31,30}\)

\[
\Phi_{\Omega,U} = \int_0^\infty \left( a_{\Omega,R} \varphi_{\Omega,R} + a_{\Omega,R}^\dagger \varphi_{\Omega,R}^* + a_{\Omega,L} \varphi_{\Omega,L} + a_{\Omega,L}^\dagger \varphi_{\Omega,L}^* \right) d\Omega,
\]

where \( a_{\Omega,\nu} \) and \( a_{\Omega,\nu}^\dagger \) are the Unruh annihilation and creation operators for \( \nu \in \{R, L\} \), respectively, obey the bosonic commutation relations. The solution of the field \( \varphi_{\Omega,\nu}(t,x) \) in the Unruh bases are related by

\[
\begin{align*}
\varphi_{\Omega,R} &= \cosh r_\Omega \varphi_{\Omega,1} + \sinh r_\Omega \varphi_{\Omega,II}, \\
\varphi_{\Omega,L} &= \cosh r_\Omega \varphi_{\Omega,II} + \sinh r_\Omega \varphi_{\Omega,1},
\end{align*}
\]
The expression Eq. [8] shows the transformation between the Unruh and the Rindler bases. Similarly, the transformation between the Minkowski and the Unruh modes are related by [39,33]

\[
\varphi_{\omega,M}(t,x) = \int_{0}^{\infty} \left( \alpha_{R \Omega}^{R} \varphi_{\Omega,R} + \alpha_{L \Omega}^{L} \varphi_{\Omega,L} \right) d\Omega, \quad (9)
\]

where

\[
\varphi_{\Omega,\nu} = \int_{0}^{\infty} \left( \alpha_{\nu \Omega}^{\nu} \right)^{*} \varphi_{\omega,M} d\omega, \quad \nu \in \{ R, L \}, \quad (10)
\]

\[
\alpha_{R \Omega}^{R} = \frac{1}{\sqrt{2\pi\omega}} (\omega l)^{r R} \Omega, \quad \alpha_{L \Omega}^{L} = (\alpha_{R \Omega}^{R})^{*}, \quad (11)
\]

where \( l \) is constant of dimension length.

A similar transformation between the Minkowski and the Unruh operators may yield the following expression [33,39]

\[
a_{\omega,M} = \int_{0}^{\infty} \left( \left( \alpha_{R \Omega}^{R} \right)^{*} a_{\omega,R} + \left( \alpha_{L \Omega}^{L} \right)^{*} a_{\omega,L} \right) d\Omega, \quad (12)
\]

where

\[
a_{\Omega,\nu} = \int_{0}^{\infty} \alpha_{\nu \Omega}^{\nu} a_{\omega,M} d\omega, \quad \nu \in \{ R, L \}, \quad (13)
\]

One can obtain a relationship between the Minkowski and the Unruh operators

\[
a_{\Omega,L} = \cosh r_{\Omega} a_{\Omega,R} + \sinh r_{\Omega} a_{\Omega,L}^{L},
\]

\[
a_{\Omega,H} = \cosh r_{\Omega} a_{\Omega,R} + \sinh r_{\Omega} a_{\Omega,L}^{L}. \quad (14)
\]

Now we are in the position to relate the vacua and excited states of the Minkowski, Rindler and Unruh modes. It is important to mention that the Minkowski and the Unruh modes have common vacuum state, i.e., \(|0_{\Omega,M}\rangle = |0_{\Omega,U}\rangle\). The Minkowski vacuum state for a free scalar field can be expressed as a two-mode squeezed state of the Rindler vacuum

\[
|0_{\Omega,M}\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh n r |n_{\Omega}\rangle_{I} |n_{\Omega}\rangle_{II}, \quad (15)
\]

where \(|n_{\Omega}\rangle_{1}\) and \(|n_{\Omega}\rangle_{II}\) indicate the Rindler particle mode in region I and II, respectively. Using single mode approximations, the Minkowski excited state can be expressed as

\[
|1_{\Omega}\rangle_{M} = \frac{1}{\cosh^{2} r} \sum_{n=0}^{\infty} \sqrt{n+1} \tanh n r |n+1_{\Omega}\rangle_{I} |n_{\Omega}\rangle_{II}, \quad (16)
\]

where \( r \) is the acceleration parameter, defined as \(\tanh r = \exp(-\pi i \Omega)\), with \( \Omega = |k| c/\alpha \), such that \( 0 \leq r \leq \infty \) for \( 0 \leq \alpha \leq \infty \). Here, \( k \) is a wave vector denotes the modes of the scalar field.

### III. RELATIVISTIC EFFECTS ON THE QSLT

In the following, we discuss the notion of a QSLT, which sets the ultimate maximal speed of evolution of an open quantum system. It basically determines the minimal time (lower bound) of evolution from a mixed state \( \rho_{0} \) to its final mixed state \( \rho_{F} \). A general expression for the QSLT of an open systems can be written as [29,31]

\[
\tau_{QSL} = \frac{\| \rho_{0} - \rho_{F} \|_{hs}}{\| \dot{\rho}_{F} \|_{hs}}, \quad (17)
\]

with

\[
\| \dot{\rho}_{F} \|_{hs} = \frac{1}{r} \int_{0}^{\tau} dt \| \dot{\rho}_{F} \|_{hs}, \quad (18)
\]

where \( \| \rho \|_{hs} \) is the Hilbert-Schmidt norm of density operator \( \rho \). The \( \| \rho \|_{hs} = \sqrt{\sum_{i} e_{i}^{2}} \), with \( e_{i} \) being the singular values of \( \rho \). The term in the numerator coined as the Euclidean distance \( D(\rho_{0}, \rho_{F}) = \| \rho_{0} - \rho_{F} \|_{hs} \). It is important to mention that the QSLT turns out to be the actual evolution time in the limit \( D(\rho_{0}, \rho_{F}) = \| \dot{\rho}_{F} \|_{hs} \).

We assume that the initial maximally entangled state between Alice \( A \) and Bob \( B \) of single Minkowski mode \( k \) reads;

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0_{k}\rangle_{A} |0_{k}\rangle_{B} + |1_{k}\rangle_{A} |1_{k}\rangle_{B} ), \quad (19)
\]

Given that the observer \( B \) undergoes a uniform acceleration with respect to the inertial observer \( A \), the scalar particle vacuum and excited state of \( B \) in Minkowski space are transformed into two causally disconnected Rindler regions I and II for particles and anti-particles, respectively. Using single mode approximation, the state (Eq. [19]) can be expressed in terms of Rindler states of the free scalar field, as given by

\[
|\Psi\rangle = \frac{1}{\sqrt{2} \cosh r} \sum_{n=0}^{\infty} \tanh n r (|0_{n}\rangle + \sqrt{n+1} \cosh r |1_{n+1}\rangle, \quad (20)
\]

where \(|xyz\rangle = |x_{k}\rangle_{A} |y_{k}\rangle_{B_{1}} |z_{-k}\rangle_{B_{2}} |, for x \in (0, 1) \) and \( y, z \in (n, n+1) \).

In what follows, we study the evolution of the amplitude-damped quantum system in the relativistic framework (especially Rindler basis in region I). Taking the trace over the inaccessible state of region II, we obtain a reduced density operator \( \rho_{AB_{I}} \) of Alice and physically accessible region of Bob, as given by

\[
\rho_{AB_{I}} = |0\rangle \langle 0| \otimes M_{nn} + |1\rangle \langle 1| \otimes M_{n+1n+1} + |1\rangle \langle 1| \otimes M_{n+1n} + |0\rangle \langle 0| \otimes M_{nn+1}, \quad (21)
\]
where
\[ M_{nn} = \frac{1}{2} \cosh^2 r \sum_{n=0}^{\infty} \tanh^{2n} r |n\rangle \langle n|, \]
\[ M_{n+1n} = \frac{1}{2} \cosh^3 r \sum_{n=0}^{\infty} \sqrt{n + 1} \tanh^{2n} r |n + 1\rangle \langle n|, \]
\[ M_{n+1n+1} = \frac{1}{2} \cosh^4 r \sum_{n=0}^{\infty} (n + 1) \tanh^{2n} r |n + 1\rangle \langle n + 1|, \]
\[ M_{nn+1} = M^*_{n+1n}. \tag{22} \]

Let consider the inertial observer \( A \) of the system is under the action of an amplitude damping channel. In this case, nothing happens, if the system is in the ground state. However, it change the dynamic of the excited state of the system. The Krauss operators \( K_i \) for \( i \in \{0, 1\} \) of this model is represented by the following positive quantum map \[ K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{p_t} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{1 - p_t} \\ 0 & 0 \end{bmatrix}, \tag{23} \]
where \( p_t = \exp (-\Gamma t) \), is a damping parameter, an exponential decay of the excited population with decay rate \( \Gamma \). The system-environment interaction can be represented by the Krauss decomposition of the quantum channel, \( \mathcal{E}(\rho_0) = \sum_i K_i^\dagger \rho_0 K_i \), satisfying \( \sum_i K_i^\dagger K_i = I \). Thus, the reduced density operator Eq. \[(21) \] after inertial observer interacting with amplitude damping channel is given by
\[ \rho_{AB_1}(p_t) = |0\rangle \langle 0| \otimes M_{nn} + \sqrt{p_t} (|1\rangle \langle 1| \otimes M_{n+1n} + h.c)
+ (p_t |1\rangle \langle 1| + (1 - p_t) |0\rangle \langle 0|) \otimes M_{n+1n+1}. \tag{24} \]

In order to characterize the speed of evolution for the amplitude damped quantum systems, we first calculate the Euclidean distance \( D(p_t, r) \), between the noise-free initial state \( \rho_{AB_1} \) to its amplitude decoherence state \( \rho_{AB_1}(p_t) \) in the accelerated frame. Calculating the singular values of the reduced system \( \rho_{AB_1} - \rho_{AB_1}(p_t) \) and taking the square root of the sum of square of them, one finds that the Euclidean distance,
\[ D(p_t, r) = \frac{1 - \sqrt{p_t}}{\sqrt{2}} \sqrt{(1 + \sqrt{p_t})^2 + a^2(r)}, \tag{25} \]
where \( a(r) \) is the acceleration dependent parameter, given by
\[ a(r) = \frac{1}{\cosh^3 r} \sum_{n=0}^{\infty} \sqrt{n + 1} \tanh^{2n} r, \]
\[ = \frac{1}{\cosh r \sinh^2 r} \text{Li}_{-\frac{1}{2}} (\tanh^2 r). \tag{26} \]

The trace norm of the \( \dot{\rho}_{AB_1}(p_t) \) can be obtained as
\[ \| \dot{\rho}_{AB_1}(p_t) \|_{1\text{h}} = -\frac{1}{2 \sqrt{2} r} \int_1^{p_t} dp_t \sqrt{\frac{4p_t + a^2(r)}{p_t}}, \tag{27} \]

Figure 2. (Color online) The \( \Delta \tau(p_t, r) \) as a function of acceleration parameter \( r \) of the accelerated bosonic observer (scalar field) for different values of damping strength \( p_t \) for \( p_t \geq 0.1 \) & \( p_t \leq 1.5 \times 10^{-3} \) of the amplitude damping channels.

Figure 3. (Color online) The \( \Delta \tau(r, p_t) \) as a function of acceleration parameter \( r \) of the accelerated bosonic observer (scalar field) for different values of damping strength \( p_t \) for \( 1.5 \times 10^{-2} \geq p_t \geq 1.5 \times 10^{-3} \) of the amplitude damping channels.

The expression for quantum speed limit time \( \tau(p_t, r) \), of the amplitude-damped state in the relativistic frame has the form
\[ \tau(p_t, r) = \frac{2 \tau(1 - \sqrt{p_t}) \sqrt{(1 + \sqrt{p_t})^2 + a^2(r)}}{\int_1^{p_t} dp_t \sqrt{\frac{4p_t + a^2(r)}{p_t}}}, \tag{28} \]
The integral in the denominator is given by \[ 31 \]. It is straightforward to show that the QSLT \( \tau(p_t, r) \), reduce
Table I. Numerical values of the critical damping parameter, where the QSLT shows different behavior with respect to acceleration.

| Critical damping parameters | Numerical values  |
|-----------------------------|-------------------|
| $p_{\tau_{c0}}$            | $1.5 \times 10^{-3}$ |
| $p_{\tau_{c1}}$            | $1.5 \times 10^{-2}$ |

The anomalous behavior of QSLT under Unruh decoherence as a function of acceleration parameter for various damping strengths, on the interval $p_\tau \in (p_{\tau_{c1}}, p_{\tau_{c0}})$, are depicted in Fig. 5. One can clearly see that the QSLT decreases to a minimum value in the beginning, then increases to a fixed value with increasing acceleration. This shows that the relativistic effect first speed-up the evolution process, then exhibit a gradual deceleration process to a uniform evolution of the system. Furthermore, we notice an acceleration independent behavior of QSLT for $p_\tau \approx 5 \times 10^{-3}$, in the large acceleration limit. The anomalous —monotonic and non-monotonic— behavior of QSLT is due to the competition between accelerated parameter $r$ and damping parameter $p_\tau$. It is easy to see an enhancement of the QSLT with acceleration for $p_\tau \geq 0.1$, trapped to a fixed value for the large acceleration limit. However, this behavior ceased out for the highly damped limit $p_\tau \sim 1$, where the QSLT turns out to be acceleration independent. In this limit, we may say that the system is completely evolved. For a weakly coupled system ($p_\tau \ll 5 \times 10^{-3}$), the acceleration parameter dominates, leads to the degradation of QSLT in non-inertial frames. In particular, the QSLT of the system in the absence of noise turns out:

$$
\tau(p_\tau = 0, r) = \frac{2\sqrt{a^2(r) + 1}}{a^2(r) + 4 + \frac{1}{2}a^2(r) \sinh^{-1}\frac{2}{a(r)}}. \quad (29)
$$

In addition, on the interval $p_\tau \in (p_{\tau_{c1}}, p_{\tau_{c0}})$, first the acceleration dominates the damping parameter, then damping parameter dominates the acceleration, results in decreasing and increasing of QSLT of the open quantum system.

Furthermore, the influence of the relativistic effects on the QSLT for the quantum systems coupled with the phase damping channels in non-inertial frames has also been examined. The relativistic QSLT can be calculated as

$$
\tau(q_\tau) = \frac{1 - \sqrt{1 - q_\tau}}{1 - \sqrt{q_\tau}}. \quad (30)
$$

where $q_\tau$ is a damping parameter of the phase damping channels. Eq. (30) suggest that the QSLT is independent of acceleration parameter for the quantum systems coupled with the phase damping channels in non-inertial frames. This acceleration-independent behavior of QSLT is in consistent with the result obtained for the phased-damped system of the fermionic field in non-inertial frames.

IV. CONCLUSIONS

We have investigated the speed of evolution of the amplitude damped quantum system under Unruh decoherence from the perspective of QSLT. For the scalar field, we have observed a speed-up of quantum evolution of the system due to Unruh decoherence for the damping parameter $p_\tau \lesssim p_{\tau_{c0}}$. The QSLT in this region turned out to be a decreasing function of the acceleration parameter for a given damping. Moreover, on the interval $p_\tau \in (p_{\tau_{c1}}, p_{\tau_{c0}})$, the relativistic effect first speeded-up, and then decelerated the quantum evolution process of the system. On the other hand, we have noticed a deceleration quantum evolution process for $p_{\tau_{c1}} \lesssim p_\tau$ in the relativistic frame. It is important to mention that the QSLT reduced/raised to a fixed value in the large acceleration limit. Furthermore, we have studied the influence of relativistic effects on the QSLT for the phase damped open quantum systems in non-inertial frames. Our results demonstrated no relativistic effects on the speed of quantum evolution of the system in accelerated frames.

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The integral has the simplified form:

\[ \int_{-1}^{1} dp_1 \sqrt{\frac{4p_1 + a^2(r)}{p_1}} = \sqrt{4 + a^2(r)} - \frac{1}{2} a^2(r) \left( \sinh^{-1} \frac{\sqrt{2}}{a(r)} - \sinh^{-1} \frac{2}{a(r)} \right). \]