Searching for polarization in signed graphs: a local spectral approach

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ABSTRACT
Signed graphs have been used to model interactions in social networks, which can be either positive (friendly) or negative (antagonistic). The model has been used to study polarization and other related phenomena in social networks, which can be harmful to the process of democratic deliberation in our society. An interesting and challenging task in this application domain is to detect polarized communities in signed graphs. A number of different methods have been proposed for this task. However, existing approaches aim at finding globally optimal solutions. Instead, in this paper we are interested in finding polarized communities that are related to a small set of seed nodes provided as input. Seed nodes may consist of two sets, which constitute the two sides of a polarized structure.

In this paper we formulate the problem of finding local polarized communities in signed graphs as a locally-biased eigen-problem. By viewing the eigenvector associated with the smallest eigenvalue of the Laplacian matrix as the solution of a constrained optimization problem, we are able to incorporate the local information as an additional constraint. In addition, we show that the locally-biased vector can be used to find communities with approximation guarantee with respect to a local analogue of the Cheeger constant on signed graphs. By exploiting the sparsity in the input graph, an indicator-vector for the polarized communities can be found in time linear to the graph size.

Our experiments on real-world networks validate the proposed algorithm and demonstrate its usefulness in finding local structures in this semi-supervised manner.

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1 INTRODUCTION
The issue of polarization in social media is becoming of increasing interest, due to its impact on the health of public discourse and the integrity of democratic processes. Detecting polarized structures in social networks is therefore a well-motivated problem. One way to accomplish this task is to employ graph clustering [27] or community-detection methods [11]. These techniques help us locate dense structures in networks, which might correspond to so-called echo chambers [13].

Spectral methods are commonly used for graph clustering. By inspecting the eigenvectors of the graph Laplacian it is possible to find partitions with quality guarantees [9], often formulated in terms of conductance—a well-known measure of graph-partition quality. A shortcoming of these techniques is that they are inherently global, and therefore not useful for finding local solutions, which could be relevant to the interests of a user. To overcome this limitation, Mahoney et al. propose a locally-biased community-extraction formulation [24]. The idea is to find a solution that is correlated with a given input vector. The approach enables us to detect dense communities that are close to a subgraph of interest. Previous approaches for similar tasks employ PageRank vectors [1, 4, 14, 17].

To encode the attitude or sentiment of interactions between individuals in a social network, we can annotate every edge in the corresponding graph with a positive or negative sign, so as to signify whether the interaction is friendly or hostile. A graph featuring such edge annotations is known as a signed graph. The analysis of signed graphs, in particular for clustering and community detection, has received considerable attention in recent years [8, 19, 21, 30]. A number of different methods have been proposed for finding polarized communities in signed graphs [8, 23], however, existing methods aim at finding globally optimal solutions. To our knowledge, this is the first paper that addresses the problem of finding local polarized communities in signed graphs.

Motivation. Finding polarized groups related to a set of seed nodes has several important applications. As an example, consider a scenario in which a polarized debate emerges around a topic discussed in an online social-media platform (e.g., Facebook). Assume further that a small number of high-profile users are known to be engaged in hostile interactions. Our algorithm could then determine the remaining members of the two rival groups, using the set of known users as a seed. Finding the two polarized communities can be useful for computational social scientists who are interested in understanding the structure of such communities and the mechanics of their discussions [31]. In addition, finding the polarized communities can be of interest to the social-media platform, if they wish to reduce the degree of polarization by recommending a thought-provoking article to the users involved in the argument.

Contributions. In this paper we propose a local spectral method to find polarized communities in signed graphs. In particular, given a set of seed nodes, our method finds two antagonistic groups close
results and proofs are required. Second, we rely on recent results on signed graphs (i.e., Proposition 1 in Section 3), not present in the literature on unsigned graphs to the best of our knowledge. Third, efficient rounding of fractional solutions is more complicated on signed graphs than on unsigned graphs, and new techniques are necessary.

**Spectral graph theory on signed graphs.** Spectral methods have been studied extensively also for signed graphs. Kunegis et al. [19] study normalized cuts, while Chiang et al. [7] study $k$-way partitioning. Mercado et al. [25] propose the use of the geometric mean of the Laplacians of the positive and negative parts of the graph to correctly cluster graphs under the stochastic block model. Note that all these methods are global, while our method is local.

The spectrum of the signed Laplacian reveals interesting properties about the corresponding graph. The smallest eigenvalue bounds the frustration number (resp. index) [3, 22], which is the minimum number of nodes (resp. edges) to remove to make graph balanced. Notably, Atay et al. [2] shows that the smallest eigenvalue of the normalized Laplacian is closely related to signed bipartiteness ratio, which can be seen as a signed analogue of conductance.

**Antagonistic communities.** Finding antagonistic communities in signed graphs [5, 8, 12, 23] has attracted attention in recent years. Existing works differ in the definition of an antagonistic community. Most works focus on enumerating antagonistic communities at the global level, and implicitly reduce information redundancy by avoiding overlapping communities. In contrast, we focus on locating antagonistic communities given seed nodes. It should be noted that our setting is one approach to deal with overlapping structures, e.g., as depicted in Figure 1. Furthermore, our approach yields approximation guarantees in terms of our definition of antagonism, while previous methods employ heuristics without quality guarantees.

### 2 Problem Formulation

We represent column vectors by boldface lower-case letters, e.g., $x, y$, and matrices by upper-case letters, e.g., $A, D$. We write $x_i$ to denote the $i$-th entry in vector $x$ and $A_{i,j}$ to denote the $j$-th entry in the $i$-th row of matrix $A$. Our notation is shown in Table 1.

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**Figure 1:** A motivating example of local polarization search on the New Guinea Highland Tribes graph [26]: individual tribes are shown as vertices, with friendly relations shown as solid line and enemy relations as dashed line. Given a pair of node sets $(S_1, S_2)$ as query, we are interested in finding two subgraphs $C_1$ and $C_2$ such that: (i) $C_1$ and $C_2$ are antagonistic to each other and (ii) $C_1$ and $C_2$ are friendly inside themselves and (iii) nodes in $C_1$ and $C_2$ are related to $S_1$ and $S_2$. In the figure, we show two querying scenarios. (a) querying $S_1 = \{a\}$ and $S_2 = \{b\}$ produces $C_1$ and $C_2$ as the red and blue nodes respectively. (b) querying $S_1 = \{a\}$ and $S_2 = \{c\}$ produces $C_1$ and $C_2$ as the red and green nodes. Note that the two pairs of subgraphs are overlapping in red-colored nodes.

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**Table 1: Notation**

| Symbol | Meaning |
|--------|---------|
| $E^+, E^-$ | positive/negative edges |
| $A^+, A^-, A$ | positive/negative/signed adjacency matrix |
| $D$ | degree matrix |
| $\text{vol}(C)$ | volume of node set $C$ |
| $\lambda_1, \nu_1$ | smallest eigenvalue of $L$ and its eigenvector |
| $C_1, C_2$ | two bands of a polarized community |
| $S_1, S_2$ | seed sets corresponding to the two bands |
| $\beta(C_1, C_2)$ | signed bipartiteness ratio of $C_1, C_2$ |
| $k$ | volume parameter in $\text{LocalPolar-Discrete}$ |
| $s$ | seed vector in $\text{LocalPolar}$ |
| $\kappa$ | correlation parameter in $\text{LocalPolar}$ |
| $h(S_1, S_2, k)$ | local signed Cheeger’s constant w.r.t $S_1, S_2$ and $k$ |
| $\lambda(s, \kappa)$ | locally-biased smallest eigenvalue w.r.t. $s$ and $\kappa$ |

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**1.1 Related work**

**Local community detection.** The problem of finding local communities with respect to seed nodes has been studied extensively. In general, given a set of seed nodes $S$ the task is to find a subgraph that contains $S$ and optimizes some quality measure. Various measures have been proposed, such as conductance [1], $k$-core [29], $k$-truss [16], and quasi-cliques [10]. One line of effort utilizes PageRank methods [1, 4, 14, 17], which enjoy provable performance guarantee [1] and are effective in practice [18]. Mahoney et al. [24] combine seed information with spectral partitioning and obtain guarantees with respect to conductance.

Our approach is inspired by the work of Mahoney et al. [24], and our paper extends their results to signed graphs. Yet, to make the extension possible, several technical results are introduced. First, given the spectral properties of signed graphs, we work with $v_1$, the eigenvector of the smallest eigenvalue, while Mahoney’s work use $v_2$, the eigenvector of the second smallest eigenvalue. This key difference leads to a different problem, as the solution is no longer orthogonal to the first vector. In addition, different intermediate

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Signed graphs. A signed graph is defined as an undirected graph $G = (V, E^+ \cup E^-)$, where $V = \{1, \ldots, n\}$ is a set of nodes, $E^+$ a set of positive edges, and $E^-$ a set of negative edges. The set of all edges is $E = E^+ \cup E^-$. The adjacency matrix of the positive edges of the graph is denoted by $A^+$. The entry $A^+_{i,j}$ is defined to be 1 (or another positive value if the graph is weighted) if $(i,j) \in E^+$, and 0 otherwise. The adjacency matrix of the negative edges of the graph is denoted by $A^-$. \( A^- \) is defined a similar way. We combine $A^+$ and $A^-$ into a single term, the signed adjacency matrix $A = A^+ - A^-$. We focus on undirected graphs, so $A^+$, $A^-$, and $A$ are all symmetric.

Spectral graph theory on signed graphs. Given a signed graph $G = (V, E^+ \cup E^-)$ we define the following matrices: the degree matrix $D$ is a diagonal matrix with $D_{i,i} = \deg(i) = \sum_{j=1}^{n} |A_{i,j}|$; the signed Laplacian matrix is defined as $L = D - A$. Normalizing $L$ gives the signed normalized Laplacian $L = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. \(^1\)

Given a vector $x \in \mathbb{R}^n$ and associated vector $y = D^\frac{1}{2}x$, we define the Rayleigh quotient $\mathcal{R}_L(y)$:

$$\mathcal{R}_L(y) = \frac{x^T Ly}{y^T y} = \frac{x^T Dx}{y^T y}. \quad (1)$$

Minimizing $\mathcal{R}_L(y)$ yields the smallest eigenvalue $\lambda_1$ of $L$ and its associated eigenvector $v_1$. If $G$ is connected, a small value of $\lambda_1$ indicates a more “balanced” graph, where a graph $G$ is perfectly balanced if its nodes can be partitioned into two sets $C_1$ and $C_2$, such that all edges within $C_1$ and $C_2$ are positive and all edges across are negative. If $G$ is perfectly balanced, $\lambda_1$ is zero. Otherwise, we say it is unbalanced. The eigenvector $v_1$ gives hints on a node partitioning to minimize $\mathcal{R}_L$.

Measuring polarization. We define a polarized community $(C_1, C_2)$ as two disjoint sets of nodes $C_1, C_2 \subseteq V$, such that (i) there are relatively few (resp. many) negative (resp. positive) edges within $C_1$ and within $C_2$; (ii) there are relatively few (resp. many) positive (resp. negative) edges across $C_1$ and $C_2$; and (iii) there are relatively few edges (of either sign) from $C_1$ and $C_2$ to the rest of the graph. We refer to $C_1$ and $C_2$ as the two bands of the community.

Before formalizing the degree of polarization of a community, we introduce some additional notation. Given a set of edges $F$ and two node sets $X, Y \subseteq V$, we consider the edges of $F(X,Y)$ that have one endpoint in $X$ and one in $Y$, i.e., $F(X,Y) = \{(u,v) \in F \mid u \in X, v \in Y\}$. We define $F(X) = F(X,X)$. For example, $E^+ (X)$ is the set of positive edges having both endpoints in $X$. The volume of a node set $S$ is defined as $\text{vol}(S) = \sum_{i \in S} \deg(i)$.

We also represent a community $(C_1, C_2)$ in vector form. Given bands $C_1$ and $C_2$, we consider a vector $x \in \{-1,0,1\}^n$ such that $x_i = 1$ if $i \in C_1$, $x_i = -1$ if $i \in C_2$, and $x_i = 0$ otherwise.

We now proceed to define our polarization measure: given a community with two bands $(C_1, C_2)$ we measure the degree of polarization in the community by

$$\beta(C_1, C_2) = \frac{2|E^+(C_1, C_2)| + |E^-(C_1)| + |E^-(C_2)|}{\text{vol}(C_1 \cup C_2)} + |E(C_1 \cup C_2, V \setminus (C_1 \cup C_2))| \quad (2)$$

We refer to the quantity $\beta$ as signed bipartiveness ratio \(^2\). In particular, the more polarized the community $(C_1, C_2)$ the smaller the value of $\beta(C_1, C_2)$. Using $x$ (defined over $C_1, C_2$) and $y = D^\frac{1}{2}x$, the following relation holds:

$$\mathcal{R}_L(y) = \frac{x^T Ly}{x^T y} = \frac{4|E^+(C_1, C_2)| + 4|E^-(C_1)| + 4|E^-(C_2)|}{\text{vol}(C_1 \cup C_2)} + |E(C_1 \cup C_2, V \setminus (C_1 \cup C_2))| \quad (3)$$

Note that $\mathcal{R}_L(y)$ and $\beta(C_1, C_2)$ bound each other by a constant factor:

$$\beta(C_1, C_2) \leq \mathcal{R}_L(y) \leq 4\beta(C_1, C_2). \quad (4)$$

Intuitively, minimizing $\beta(C_1, C_2)$ is closely related to minimizing the corresponding $\mathcal{R}_L(y)$.

As in the work of Atay and Liu \(^2\), we refer to

$$h(G) = \min_{C_1, C_2} \beta(C_1, C_2)$$

as the signed Cheeger constant of the graph $G$.

Encoding information about seed nodes. In this paper we consider a setting where a set of seed nodes is given as input. We sometimes refer to seeds as query nodes. The seed nodes are specified as two disjoint sets $(S_1, S_2)$, i.e., $S_1, S_2 \subseteq V$ and $S_1 \cap S_2 = \emptyset$. The sets $S_1$ and $S_2$ are meant to provide example nodes for the two bands of the polarized community we are interested in discovering. Typically, $|S_1|$ and $|S_2|$ are small.

Intuitively, we are seeking a community $(C_1, C_2)$ such that each band contains one of the seed sets, e.g., $S_1 \subseteq C_1$ and $S_2 \subseteq C_2$ and at the same time minimizes $\beta(C_1, C_2)$. In addition, we are interested in solutions whose size is bounded with respect to the size of the seed nodes and are preferably well connected to these. We express this as a constraint of the volume of the set $C_1 \cup C_2$ normalized by the volume of the set $S_1 \cup S_2$. In particular, we require that the volume of the solution $\text{vol}(C_1 \cup C_2)$ is at most $k$ times the volume of the seed nodes $\text{vol}(S_1 \cup S_2)$, where $k$ is a user-defined parameter.

We formalize this problem as follows:

**Problem (LocalPolar-Discrete).** Given a signed graph $G$, a pair of seed sets $(S_1, S_2)$ and a positive number $k \in \mathbb{R}^+$ our goal is to find a polarized community $(C_1, C_2)$ so that

1. $(C_1, C_2)$ minimizes the signed bipartiveness ratio $\beta(C_1, C_2)$;
2. $S_1 \subseteq C_1$ and $S_2 \subseteq C_2$;
3. \( \frac{\text{vol}(C_1 \cup C_2)}{\text{vol}(S_1 \cup S_2)} \leq k \);

Adapting the notion of signed Cheeger’s constant $h(G)$, we define a local variant of $h(G)$. Given a signed graph $G$, two disjoint node sets $S_1$ and $S_2$, and a positive number $k$, local signed Cheeger’s constant is defined as:

$$h(S_1, S_2, k) = \min_{\substack{C_1, C_2 | V \setminus \{S_1 \cup S_2\} \subseteq C_1 \cup C_2 \subseteq V \setminus \{S_1 \cup S_2\} \subseteq C_1 \cup C_2 \subseteq V \setminus \{S_1 \cup S_2\}}} \beta(C_1, C_2) \quad (5)$$

Next, we consider a continuous variant of the LocalPolar-Discrete problem. The theoretical properties of the continuous problem will be studied soon. First, let $x$ be continuous, i.e., $x \in \mathbb{R}^n$.

Second, we encode the query information via a seed vector $s \in \mathbb{R}^n$. Seed nodes take non-zero values in $s$ and signs indicate membership in $S_1$, or $S_2$: by convention, $s_i > 0$ if $i \in S_1$, $s_i < 0$ if $i \in S_2$, and...
Given a signed graph $G$, a seed vector $s \in \mathbb{R}^n$, and a correlation parameter $\kappa$, our goal is to find a vector $x \in \mathbb{R}^n$ such that

$$\min_x \ x^T L x$$

$$\text{such that} \quad x^T D x = 1 \quad (6)$$

The parameter $\kappa$ controls the correlation between $x$ and $s$. Notice that LocalPolar is not convex due to the constraint $x^T D x = 1$.

It can be shown that LocalPolar is indeed a continuous relaxation of LocalPolar-Discrete. This is shown in Claim 1 below. Using this result, we can solve an instance of the LocalPolar-Discrete problem, by first solving its continuous relaxation and then rounding the solution to a discrete one. The result is also used to prove the approximation guarantee for LocalPolar-Discrete.

Before stating the claim, we introduce some additional notation. For $S \subseteq V$, we let $1_S \in \{0,1\}^n$ be a vector that has 1 at entries in $S$ and 0 otherwise. Then, for two disjoint sets $A, B \subseteq V$, we define the vector $s_{A,B} = (1_A - 1_B)/\sqrt{\text{vol}(A \cup B)}$. It can be checked that $s_{A,B}^T D s_{A,B} = 1$.

**Claim 1.** For any $S_1, S_2 \subseteq V$, LocalPolar($G, s_{S_1, S_2}, \sqrt{1/k}$) is a relaxation of LocalPolar-Discrete($G, S_1, S_2, k$). In particular,

$$\lambda(s_{S_1, S_2}, \sqrt{1/k}) \leq 4h(S_1, S_2, k),$$

where $\lambda(s_{S_1, S_2}, \sqrt{1/k})$ is the optimal value of LocalPolar($G, s_{S_1, S_2}, \sqrt{1/k}$) and $h(S_1, S_2, k)$ is the optimal value of LocalPolar-Discrete($G, S_1, S_2, k$).

**Proof.** First the objective of LocalPolar relaxes the objective of LocalPolar-Discrete because LocalPolar optimizes over a continuous space and the two objectives bound each other up to constant factors (Equation 4).

Second, LocalPolar relaxes both constraints of LocalPolar-Discrete and translates them into the correlation constraint. Consider any feasible solution $C_1$ and $C_2$ of LocalPolar-Discrete, and let $C = C_1 \cup C_2$ and $S = S_1 \cup S_2$. Also let $x = s_{C_1, C_1}$ and $s = s_{S_1, S_2}$ to be used in LocalPolar. Expanding $x^T D s$ gives:

$$x^T D s = \frac{1}{\sqrt{\text{vol}(C) \text{vol}(S)}} (1_{C_1} - 1_{C_2})^T D (1_{S_1} - 1_{S_2})$$

$$= \frac{1}{\sqrt{\text{vol}(C) \text{vol}(S)}} (1_{C_1}^T D 1_{S_1} + 1_{C_2}^T D 1_{S_2})$$

$$= \frac{1}{\sqrt{\text{vol}(C) \text{vol}(S)}} (\text{vol}(S_1) + \text{vol}(S_2))$$

$$= \frac{1}{\sqrt{\text{vol}(C)}} \geq \sqrt{\frac{1}{k}}.$$

Line 2 follows from: $C_1 \cap C_2 = \emptyset$, $S_1 \cap S_2 = \emptyset$ and $S_1 \subseteq C_1, S_2 \subseteq C_2$.

Last, it is easy to verify $x^T L x \leq 4\beta(C_1, C_2)$. Also by definition, we have $\lambda(s_{S_1, S_2}, \sqrt{1/k}) \leq x^T L x$. So it follows immediately that $\lambda(s_{S_1, S_2}, \sqrt{1/k}) \leq 4h(S_1, S_2, k)$. $\square$

In the next section, we analyze the properties of LocalPolar and propose an approximation algorithm for LocalPolar-Discrete.

## 3 THEORETICAL ANALYSIS

We first present how to solve the continuous problem LocalPolar and obtain an optimal solution $x^*$. This is stated in Theorem 1. Then, we describe how to round the continuous optimal solution $x^*$ so as to obtain a discrete solution to LocalPolar-Discrete with approximation ratio $O(\sqrt{h(S_1, S_2, k)})$. Finally, we present an efficient algorithm PolarSeeds (Algorithm 1) that leverages these results.

**Theorem 1 (Solution characterization).** Given an unbalanced signed graph $G$ with normalized Laplacian matrix $L$, let $s \in \mathbb{R}^n$ be a seed vector, with $s^T D s = 1$ and $s^T D v_1 \neq 0$, where $v_1$ is the eigenvector of the smallest eigenvalue of $L$. Let $0 \leq \kappa < 1$ be a correlation parameter, and let $x^*$ be an optimal solution to LocalPolar($G, s, \kappa$). Then, there exists some $\alpha \in (-\infty, 1)$ and a $\beta \in [0, \infty]$ such that

$$x^* = \beta (L - \alpha D)^{-1} D s$$

where $(L - \alpha D)^{-1}$ is the pseudo inverse of $L - \alpha D$.

**Theorem 2 (Approximation guarantee).** Given an unbalanced signed graph $G$, two disjoint node sets $S_1, S_2 \subseteq V$, and a positive integer $k$, we can find two disjoint sets $C_1$ and $C_2$ that achieve $\beta(C_1, C_2) = O(\sqrt{h(S_1, S_2, k)})$ from the optimal solution of LocalPolar($G, s_{S_1, S_2}, \sqrt{1/k}$).

We illustrate the high-level structure of our proof in Figure 2.

### 3.1 Proof of Theorem 1

We first relax LocalPolar($G, s, \kappa$) to a convex semi-definite program $SDP_p(G, s, \kappa)$ shown in Figure 3. Then we show that the optimality gap between $SDP_p$ and LocalPolar is zero.\(^2\) By using the complementary slackness conditions of $SDP_p$ and its dual we show that the solution of $SDP_p$ has rank 1, which characterizes its form. Our proof technique is adapted from Mahoney et al. [24].

To see that $SDP_p$ is a relaxation of LocalPolar, consider a feasible solution $x$ of LocalPolar. Then $x = xx^T$ is also a feasible solution for $SDP_p$ and $x^T L = x^T L x$.

The following two claims are based on known results in convex optimization [6]:

**Claim 2.** Strong duality holds between $SDP_p(G, s, \kappa)$ and $SDP_d(G, s, \kappa)$.

**Proof.** As $SDP_p(G, s, \kappa)$ is a convex problem, Slater’s condition is sufficient to verify zero duality gap [6] (Chapter 5.5.3). By setting $X = ss^T$, we have $(D s)(D s)^T = ss^T = (s^T D s)^2 = 1 > \kappa$, i.e., strong feasibility holds. Therefore Slater’s condition is satisfied. $\square$

**Claim 3.** Strong duality holds between $SDP_{p}(G, s, \kappa)$ and $SDP_{d}(G, s, \kappa)$. Therefore, the feasibility and complementary slackness conditions suffice to establish optimality.

\(^2\) We omit problem arguments when the context is clear, e.g., we use $SDP_p$ for brevity.
SDP LocalPolar

Figure 3: SDP relaxation of Equation (9) we can rewrite Equation (11) as:

\[
\begin{align*}
\min_X & \quad L \circ X \\
\text{s.t.} & \quad D \circ X = 1, \\
& \quad (Ds)(Ds)^T \circ X \geq \kappa, \\
& \quad L - \alpha^* D - \beta^*(Ds)(Ds)^T \geq 0, \quad \beta^* \geq 0 \\
\end{align*}
\]

and complementary slackness conditions are:

\[
\begin{align*}
\beta^*((Ds)(Ds)^T \circ X - \kappa) &= 0, \quad \text{and} \\
X^* \circ (L - \alpha^* D - \beta^*(Ds)(Ds)^T) &= 0
\end{align*}
\]

Proof. These conditions directly follow from the optimality conditions of semi-definite programs [6] (Chapter 5.9.2).

A key finding is the following claim. The proof is given in the Appendix.

Claim 4. The rank of \(X^*\) is 1, i.e., \(X^* = x^*(x^*)^T\), and therefore the optimality gap between \(\text{LOCALPOLAR}(G, s, \kappa)\) and \(\text{SDP}_p(G, s, \kappa)\) is 0.

Claim 5. Given that \(X^*\) has the form \(x^*(x^*)^T\), the feasibility and complementary slackness conditions imply,

\[
\begin{align*}
x^* &= \pm \beta^* \sqrt{\kappa}(L - \alpha^* D)^\dagger (Ds).
\end{align*}
\]

Proof. Using Claim 4, we have \(X^* = x^*(x^*)^T\). Relying on equation (9) we can rewrite Equation (11) as:

\[
\begin{align*}
(L - \alpha^* D - \beta^*(Ds)(Ds)^T)x^* &= 0 \\
(L - \alpha^* D)x^* &= \pm \beta^* \sqrt{\kappa} (Ds) \\
x^* &= \pm \beta^* \sqrt{\kappa}(L - \alpha^* D)^\dagger (Ds).
\end{align*}
\]

The 2nd line follows from Equation (10), as \((Ds)^T x^* = \pm \sqrt{\kappa} \).}

3.2 Proof of Theorem 2

Our proof relies on Cheeger’s inequality for signed graphs:

PROPOSITION 1. (Lemma 4.2 by Atay et al. [2]) For any non-zero vector \(x \in \mathbb{R}^{|V|}\), there exists a \(t \in \mathbb{R}\) such that

\[
\beta(V_k(t), V_k(-t)) \leq \sqrt{\frac{x^T L x}{x^T D x}}
\]

where \(V_k(t) = \{ u \in V \mid x_u \geq t \} \) and \(V_k(-t) = \{ u \in V \mid x_u \leq -t \} \).

Proof of Theorem 2. Given a problem instance of \(\text{LOCALPOLAR}\) with parameters \(s_{s_1, s_2}\) and \(\kappa = \sqrt{1/k}\), we apply Theorem 1 and get the optimal vector \(x^*\). By Proposition 1, we can get two sets \(C_1\) and \(C_2\) that have \(\beta(C_1, C_2) \leq \sqrt{2/3}\). Finally, by Claim 1, \(\beta(C_1, C_2) = O(\sqrt{\kappa(C_1, C_2, \kappa)})\).

3.3 POLARSEEDS algorithm

We now summarize the complete algorithm, which we call POLARSEEDS. Pseudocode is presented as Algorithm 1. The POLARSEEDS algorithm consists of 2 steps: (i) approximate within small error the optimal solution \(x^*\) of the problem \(SDP_p\), using Theorem 1; and (ii) round \(x^*\) to obtain an integral solution, using Proposition 1. We discuss these two steps in more detail.

Computing the optimal solution \(x^*\). We discuss how to find the optimal solution \(x^* = \beta(L - \alpha^* D)^\dagger D s\) without knowing the value of \(\alpha^*\) beforehand.\(^3\) If we assume that \(\alpha^*\) is known, then \((L - \alpha^* D)^\dagger\) can be approximated using Conjugate Gradient Method [15] in time \(O(m v e)\), where \(v e\) is the condition number of \(L - \alpha^* D\).

Then, to approximate \(x^*\), we use complementary slackness condition Equation (10) and apply binary search on \(\alpha \in [-vol(G), 1]\). The search stops when \(x^T (D_s)(D_s)^T x\) is close enough to \(\kappa\) (controlled by \(\epsilon\)). The whole process requires time \(O(\sqrt{em} \log(m))\).

Lines 3 to 10 in Algorithm 1 describe the above process.

Rounding \(x^*\). We want to find a value \(t \in \mathbb{R}^+\) that minimizes \(\beta(V_k(t), V_k(-t))\). A naive way is to try each value in \(x\) for \(t\), compute the corresponding \(\beta(V_k(t), V_k(-t))\) from scratch (doable in time \(O(m)\) and take the value of \(t\) that minimizes \(\beta\). The total runtime cost is \(O(m n + \log(n))\).

We propose a faster procedure, which incrementally computes \(\beta(V_k(t), V_k(-t))\) for values of \(t\) in ascending order. The time cost reduces to \(O(m n \log(n))\). Moreover, our implementation leverages

\(^3\)Our statement is a special case of the original lemma: we set \(\mu\) to be node degree. Also, we use \(t\) instead of \(\sqrt{T}\). It’s easy to see they’re equivalent.

\(^3\)Note that once \(\alpha^*\) is found, the value \(\beta^*\) can be computed by normalizing \((L - \alpha^* D)^\dagger D s\) with respect to \(\kappa\).
We use the Conjugate Gradient solver implemented in Scipy 1.2.1.

Table 2: Real-world signed graphs:

| Signed graph          | |          |          | \(\lambda_1\) | Category |
|-----------------------|-----------------|----------|---------------|----------|
| Word                  | 5 K             | 47 K     | 0.20          | 0.030    | language |
| Bitcoin               | 6 K             | 21 K     | 0.15          | 0.040    | social   |
| Referendum            | 11 K            | 251 K    | 0.05          | 0.039    | political|
| Slashdot              | 82 K            | 500 K    | 0.24          | 0.017    | social   |
| Epinions              | 119 K           | 704 K    | 0.17          | 0.011    | social   |
| WikiConflict          | 115 K           | 2 000 K  | 0.63          | 0.075    | edit conflict |
| WikiConflict-IM       | 1 M             | 33 M     | 0.63          | 0.075    | artificial |

Real-world datasets. We choose publicly-available real-world signed graphs, whose statistics are summarized in Table 2. Word [28] captures the synonym-and-antonym relation among words in the English language. A synonym (resp. antonym) pair is connected by an edge with positive (resp. negative) weight, where a weight represents the strength of similarity (resp. opposition).

Algorithm 1: PolarSeeds. The algorithm first approximates \(x^*\) by performing binary search on \(\alpha\) (lines 3–10). Then, \((C_1, C_2)\) is attained by rounding \(x\) at line 11. An \(\mathcal{O}(m + n \log n)\) rounding procedure is described in Appendix A.2.

**Data:** Signed graph \(G = (V, E^+, E^-)\), seeds \((S_1, S_2)\), \(k \in \mathbb{R}^+\) and tolerance parameter \(\epsilon > 0\).

**Result:** two bands \((C_1, C_2)\)

1. Create seed vector \(s\) s.t. \(s_i = 1\) if \(i \in C_1\) and \(s_i = -1\) if \(i \in C_2\);
2. Normalize \(s\) s.t. \(s^T D s = 1\);
3. \(\kappa = 1/\sqrt{\kappa}\)
4. \(\ell \leftarrow -\text{vol}(V)\), \(u \leftarrow \lambda_1^*\);
5. \(\alpha \leftarrow (\ell + u)/2\);
6. Solve \((L - \alpha D)x = Ds\) using Conjugate Gradient method;
7. Normalize \(x\) s.t. \(x^T D x = 1\);
8. If \(\sqrt{\kappa} - x^T D x < 0\) then \(\ell \leftarrow \alpha\);
9. Else \(u \leftarrow \alpha\);
10. Until \(\epsilon \leq \sqrt{\kappa} - x^T D x \leq 0\);
11. Find \(t \in \mathbb{R}^+\) that minimizes \(\beta(V_t(t), V_{\ell}(t))\);
12. Return \((V_t(t), V_{\ell}(t))\)

basic vector and matrix operations, which further boost speed in practice. The details are described in Appendix A.2.

### 4 Experimental Evaluation

In this section we evaluate the performance of the proposed algorithm. Our experimental evaluation is structured as follows. First in Section 4.1, we experiment with synthetic graphs to better understand the behavior of our method under different scenarios. In Section 4.2, we compare with a state-of-the-art method [8] for polarized community detection on real-world graphs. In Section 4.3 we discuss two case studies, and finally, in Section 4.4 we demonstrate the scalability of our algorithm.

We implement our algorithm in Python and perform the experiment on a Linux machine with a 10-core CPU and 64 GB of memory. We use the Conjugate Gradient solver implemented in Scipy 1.2.1 with the default parameters. We set the binary search parameter \(\epsilon = 10^{-3}\) (used by Algorithm 1).

The implementation of the code will be available in the non-anonymized version of the paper.

**Baseline.** We compare our method with a state-of-the-art method on discovering polarized communities on real-world graphs. The FOCG algorithm [8] is designed to discover \(k\)-way polarized communities. Note that FOCG is a global method, and thus, not directly comparable with our approach; we select FOCG as baseline as it is the most related work in the literature. FOCG returns a set of \(k\)-way polarized communities, \(\{C_i\}_{i=1}^\ell\), where \(C_i = (C_{i1}, \ldots, C_{ik})\) and \(\ell\) is determined by the algorithm. FOCG ensures that \(C_i \cap C_j = \emptyset\) for all \(i \neq j\). We use \(k = 2\) and set the other parameters as suggested. We exclude communities that either (i) are too small, in particular, \(|C_1| \leq 5\) or (ii) the two bands intersect.

### 4.1 Evaluation on synthetic graphs

By studying the effects of various parameters including algorithm and graph generation parameters, we aim to answer the following research questions:

3 http://snap.stanford.edu/data/soc-Slashdot0902.html
4 http://snap.stanford.edu/data/soc-Epinions1.html
5 http://konect.uni-koblenz.de/networks/wikiconflict
Local polarized communities in signed graphs

Q1: what is the effect of parameter $\kappa$ on the characteristics of the solution, as well as in the accuracy of finding the ground truth?

Q2: what is the effect of noise in the behavior of PolarSeeds?

Q3: can we find a polarized community more accurately by giving more seeds to PolarSeeds?

We report the average performance for each configuration over a combination of 10 randomly generated graphs and 10 randomly selected ground-truth communities paired with random seeds in them.

**Evaluation metric.** Given a solution $(C_1, C_2)$ and ground truth $(C_1^*, C_2^*)$, we measure its average precision.\(^8\)

$$AP(C_1, C_2) = \frac{1}{2} \left( \frac{|C_1 \cap C_1^*|}{|C_1|} + \frac{|C_2 \cap C_2^*|}{|C_2|} \right).$$

We also report the ratio $\beta(C_1, C_2)/\beta(C_1^*, C_2^*)$, referred as $\beta$-ratio for brevity. The motivation is to see whether our method can find a solution that is even better than the ground truth, according to the optimized measure, which happens when $\beta$ ratio < 1. Finally, we report the solution volume $\text{vol}(C_1 \cup C_2)$.

We make the following observations:

**Effect of correlation parameter $\kappa$.** As a part of the answer to Q1, Figures 4(a) and (d) indicate that smaller $\kappa$ produces solutions with larger $\text{vol}(C_1 \cup C_2)$. This phenomenon can be explained by Claim 1. Recall that $\kappa = \sqrt{1/k}$, where $k$ upper bounds the solution volume in LOCALPOLAR-DISCRETE and $\kappa$ is the correlation parameter in LOCALPOLAR. A smaller $\kappa$ implies a larger $k$. Thus, the constraint $\text{vol}(C_1 \cup C_2) \leq k \text{vol}(S_1 \cup S_2)$ in LOCALPOLAR-DISCRETE encourages solutions with larger $\text{vol}(C_1 \cup C_2)$ if a larger $k$ is given.

With respect to the second part of the question Q1, we observe that larger values of $\kappa$ make PolarSeeds more noise-resistant. Figure 4(c) shows directly that $\kappa = 0.9$ achieves the best AP scores. Internally, large $\kappa$ values make PolarSeeds select solutions with small volume, which increases AP scores. This concludes the answer to Q1.

**Effect of noise.** In Figures 4(a), (b), and (c), we vary $\eta$ from 0.01 to 0.3 and set seed size equal to 2, i.e., $|S_1| = |S_2| = 1$. Thus, with respect to our research question Q2, we observe that as we increase $\eta$, PolarSeeds is more likely to pick some $C_1, C_2$ other than $C_1^*, C_2^*$, thus, lowering the AP scores. We explain this phenomenon next.

First, PolarSeeds misses $C_1^*, C_2^*$ when there is another $C_1, C_2$ that gives better $\beta$, i.e., $\beta(C_1, C_2) < \beta(C_1^*, C_2^*)$. This happens when $\eta$ is large enough (shown in Figure 4(b)). On the other hand, as we add more noise, $C_1^*, C_2^*$ becomes less polarized i.e., $\beta(C_1^*, C_2^*)$ increases. When there is a sufficiently large amount of noise, the polarization structure is destroyed, i.e., $\beta(C_1^*, C_2^*) \approx 1$. On the other hand, as $\beta(C_1^*, C_2^*)$ approaches 1, solutions with larger $\text{vol}(C_1, C_2)$ are preferred as it makes $\beta(C_1, C_2)$ smaller. This is shown by the increased solution volume in Figure 4(a). Combining the above observations, it follows that when the noise increases, the accuracy of PolarSeeds finds $C_1^*, C_2^*$ decreases.

**Effect of seed size.** In Figures 4(d), (e), and (f), we vary the seed size on $C_1^*$ and $C_2^*$ from 2 to 20 and we set edge noise $\eta = 0.05$.

We notice that the effects on all 3 metrics are limited when $\kappa$ is either too large (e.g., 0.9) or too small (e.g., 0.1). In contrast, when $\kappa = 0.5$, seed size starts to play a remarkable role. For example,

\[^8\text{W}t.o.g., \text{we omit the issue of label permutation in the definition of AP.}\]
We scan the communities sequentially (in random order), keep track of the nodes and Polarity, larger values are better.

Communities by design. We randomly seed PolarSeeds communities, we make the following adaptation to make them comparable. We randomly seed PolarSeeds graphs. Since FOOG does not enforce connectivity on neither $S_1$ nor $S_2$, we allow $vol(S_1 \cup S_2)$ to increase. As a consequence, this brings down the ratio $\beta(C_1, C_2)/\beta(C_1^*, C_2^*)$ (Figure 4(e)), which further decreases AP scores (Figure 4(f)).

So with respect to question Q3, we conclude that adding more seeds does not necessarily improve the AP score. The value of $\kappa$ affects how much influence the seed number can exert. The influence is limited when $\kappa$ is either too large or too small. On the other hand, when $\kappa$ lies in between, more seeds encourage PolarSeeds to produce larger-volume solutions, reducing AP scores.

### 4.2 Evaluation on real-world graphs

Next, we compare PolarSeeds with FOOG and evaluate their capability to finding high-quality polarized communities in real-world graphs. Since FOOG is “query-less” and tries to enumerate polarized communities, we make the following adaptation to make them comparable. We randomly seed PolarSeeds over multiple rounds and collect the results.\(^8\) As PolarSeeds can produce communities that overlap, for fairness, we keep only one of the overlapping communities.\(^9\) We set $\kappa = 0.9$ as we observe that this choice is effective on a wide range of real-world graphs. In addition, $\kappa = 0.9$ ensures that the seed nodes have dominant magnitude in the solution, i.e., $|x_s|$ dominates at $s \in S_1 \cup S_2$.

#### Evaluation metrics.\(^{10}\)

We use 3 metrics to measure quality:

- (a) **Signed bipartiteness ratio** $\beta$, the objective of our problem;
- (b) **HAM**, the harmonic mean of Cohesion and Opposition measures, defined by Chu et al.\(^8\). In particular,

\[
\text{Cohesion}(C_1, C_2) = \frac{1}{2}[d^+(C_1) + d^+(C_2)]
\]

\[
\text{Opposition}(C_1, C_2) = d^+(C_1, C_2)
\]

with $d^+(C) = \frac{|E^+(C)|}{|E^+(C)| + |E^-(C)|}$ and $d^+(C_1, C_2) = \frac{|E^+(C_1 \cup C_2)|}{|E^+(C_1 \cup C_2)|}$;

- (c) **Polarity**\(^5\), which counts the number of edges that agree with the polarized structure and penalizes large communities:

\[
\text{Polarity}(C_1, C_2) = \frac{|E^+(C_1) \cup E^+(C_2)| + 2|E^-(C_1, C_2)|}{|C_1 \cup C_2|}
\]

Results. The distributions of the 3 evaluation metrics for both methods are shown in Figure 5. In Figure 5(a), PolarSeeds always outperforms FOOG w.r.t. the median of $\beta$, which is expected because PolarSeeds optimizes $\beta$. In terms of HAM (Figure 5(b)), our method achieves better median value on 4 out of the 5 graphs (except WikiConflict). This result is somewhat surprising because FOOG optimizes an objective closer to HAM than our approach. Last in Figure 5(c), there is no clear winner for the Polarity metric.

### 4.3 Case studies on real-world graphs

**Overlapping communities in Word.** We demonstrate the usefulness of our method in finding overlapping communities on the Word graph. Inspired by the phenomenon of polysemy in natural language, we consider *fair* and two of its meanings, *without cheating* and *not excessive*. For each meaning, we seed our method with words of both similar and opposite meaning. The goal is to find antonyms and synonyms of *fair* of the given meaning. The result is shown in Figures 6(a) and (b).

**Unilateral distrust in Bitcoin.** We highlight the capability of our method in finding subgraph pairs that have distrust only in one direction. We seed PolarSeeds only with nodes from one group, e.g., $S_1 = \emptyset$ and $S_2 = \emptyset$. We found one community (of size 62) that has a cohesive band $C_1$, which radiates out many negative edges towards another band $C_2$, which is not cohesive at all. We visualized the adjacency matrix\(^{11}\) of this community in Figure 6(c). Note that since LocalPolar does not enforce connectivity on neither $C_1$ nor $C_2$, it is understandable that $C_2$ is not well connected within itself.

### 4.4 Running time on WikiConflict-IM.

To study the scalability of our method, we produce a large graph WikiConflict-IM by (i) adding dummy nodes into WikiConflict so

---

\(^8\) We use a simple heuristic to avoid enumerating all possible seeds. Nodes $u$ and $v$ are considered seeds for $C_1$ and $C_2$ respectively if $(u, v) \in E$ and $\deg^-(w) \geq t$ and $\deg^+(v) \geq t$, where $t$ is some pre-defined positive number.

\(^9\) We scan the communities sequentially (in random order), keep track of the nodes that are covered so far and drop any communities that intersect with the covered nodes.

\(^{10}\) For visualization, we use the original adjacency matrix here, which is asymmetric.
that |V| = 1 million and (ii) adding edges so that both the average degree and the ratio of negative edges remain the same with respect to WikiConflict. Then, we randomly select 100 seeds using the same heuristic described in Section 4.2 and report the average running time and its standard deviation. We decompose running time into: (1) approximating \(x^k\) takes 17.2 (± 8.1) seconds and (2) rounding \(x^k\) to obtain \((C_1, C_2)\) takes 1.4 (± 0.1) seconds. We conclude that our method scales nicely and can be applied to find local polarized communities in large signed graphs.

5 CONCLUSION

We propose a local spectral method to find polarized communities in signed graphs. We characterize the solutions in close-form using techniques in duality theory and linear algebra. Based on results from spectral graph theory on signed graphs, our method yields solutions with approximation guarantees with respect to a local variant of signed Cheeger’s constant. We empirically demonstrate our method’s scalability and effectiveness in finding polarized communities on synthetic and real-world graphs.

Our works opens interesting questions for future research. For example, (i) can we extend the current work to finding k-way polarized communities \((k > 2)\)? (ii) how does the signed bipartiteness ratio compare with other polarization measures, such as Polarity proposed in [5]? (iii) how to enforce graph connectivity on both bands in a polarized community?

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A APPENDIX

A.1 Proof of Claim 4

We first present a few lemmas which will be used to prove Claim 4.

Lemma 1. If \(\lambda_1 > 0\) and \(\lambda_1 > \alpha\), the rank of \(L - \alpha D\) is \(n\).

Proof. Our assumption that \(G\) is unbalanced (Theorem 1) translates into \(\lambda_1 > 0\) [19]. We rewrite \(L - \alpha D = D^{1/2}(L - \alpha I)D^{1/2}\). We slightly abuse the notation and denote \(\lambda_i(M)\) as the \(i\)th smallest eigenvalue of \(M\). Then we have \(\lambda_i(L - \alpha I) = \lambda_i(L) - \alpha\) for all \(i = 1, \ldots, n\). As \(\lambda_1 > \alpha\), it holds that \(\lambda_i(L) - \alpha > 0\) for all \(i\). Given \(n\) positive eigenvalues, the rank of \(L - \alpha I\) is \(n\). Multiplying \(L - \alpha I\) by positive definite matrix \(D^{1/2}\) does not change its rank.

\[\text{Lemma 2. Given } A, B \in \mathbb{R}^{n \times m} \text{ and } A, B \geq 0, A \circ B = 0 \text{ implies } \mathcal{R}(A) \subseteq \mathcal{N}(B), \text{ where } \mathcal{R}(A) \text{ denotes the range of } A \text{ and } \mathcal{N}(B) \text{ denotes null space of } B.\]

Proof. It is \(A \circ B = B \circ A = \text{tr}(B^T A) = \text{tr}(BA) = \sum_{i=1}^{m} \lambda_i(BA) = 0\). \(A, B \geq 0\) implies \(\lambda_i(BA) \geq 0\) for all \(i\). Combining the above, we have \(\lambda_i(BA) = 0\) for all \(i = 1, \ldots, m\).

Then \(BA\) can be diagonalized into \(SAS^{-1}\), where \(S\) has eigenvectors of \(BA\) as its columns and \(A\) is a diagonal matrix with \(\lambda_i(BA) = 0\) as the diagonal entries. In other words, for any \(y \in \mathbb{R}^m\), \(BAy = 0\) always holds, note that \(Ay\) corresponds to \(\mathcal{R}(A)\).

\[\text{Lemma 3. } -\beta^*(D)(D) = \text{rank of } L_G - \alpha^* L_K_n \text{ by 1}.\]

Proof. Denote \(B = (D)(D)\), \(A = L_G - \alpha^* L_K_n\). Both \(B\) and \(A\) are positive semi-definite, and therefore their eigenvalues are non-negative. We can consider the eigenvalue decomposition \(B = QDQ^T\). Since \(B\) has rank equal to 1, \(D\) is a diagonal matrix with a single non-zero element. The spectrum of \(Q^T AQ\) and \(A\) is the same, as this is a similarity transformation. Therefore, we can analyze the difference \(Q^T (A - \beta^* B)Q = Q^T AQ - \beta^* D\).

Now, observe that \(\sum_i \lambda_i(A) = \text{tr}(A) = \text{tr}(Q^T AQ)\), where \(\lambda_i(A)\) is the \(i\)-th eigenvalue of \(A\). Since \(Q^T AQ - \beta^* D\) results in the modification of a single element of the diagonal of \(Q^T AQ\), by increasing \(\beta^*\) we can make \(\text{tr}(Q^T AQ)\) arbitrarily close to \(-\infty\). As the eigenvalues of a matrix are continuous functions of its entries, we have that for some \(\beta^*\), at least one eigenvalue of \(Q^T AQ - \beta^* D\) must become zero.

The fact that the difference in rank is at most 1 follows trivially from the fact that only one row is modified.

\[\text{Proof of Claim 4. Multiplying } v_1^T \text{ and } v_1 \text{ on both sides of Eq. 9, we get}\]
$v_t^T L v_1 - \alpha^* v_t^T D v_1 = \beta^* v_t^T (D_s (D s)^T) v_1$

$$= \lambda_1 - \alpha^* - \beta^* (v_t^T D_s)^2 \geq 0$$

The second line follows by $(v_1, v_1)_D = 1$ and $v_t^T L v_1 = \lambda_1$. As $\beta^* (v_t^T D_s)^2 \geq 0$, it must be the case $\lambda_1 \geq \alpha^*$.

**Case 1:** $\lambda_1 = \alpha^*$. As we assume $s_t^D v_1 \neq 0$, $\beta^* = 0$. Plugging this into Eq. 11, we have

$$X^* \circ (L - \alpha^* D) = 0 \implies X^* \circ L = \alpha^* X^* \circ D = 0$$

$$\Rightarrow X^* \circ L = \alpha^* \implies X^* \circ L = \lambda_1$$

The 2nd line follows by Eq. 8. Decomposing $X^*$ as the outer product of its eigenvectors gives $X^* = \sum_i \sigma_i w_i w_i^T$. As $X$ is PSD and $\lambda_1 = \min_x x^T L x$, we conclude $X^* = v_1 v_1^T$, which has rank 1.

**Case 2:** $\lambda_1 > \alpha^*$. Rank of matrix $L - \alpha^* D$ is $n$ (by Lemma 1). From Eq. 11, we have that the range of $X^*$ lies in the null space of $L - \alpha^* D - \beta^* (D_s (D s)^T)$ (by Lemma 2). As $(D_s (D s)^T)$ is a rank-1 matrix and using Eq. 9, we have the rank of $L - \alpha^* D$ decreases by 1 after subtracting $\beta^* (D_s (D s)^T)$ (Lemma 3). Therefore, $X^*$ has rank 1. □

### A.2 Efficient Using of $x$

In Proposition 1, we set $x \in \mathbb{R}^n$ to round the solution from $x$ to $(C_1(t), C_2(t))$, where $C_1(t) = \{ u \in V \mid (u, x) \geq t \}$ and $C_2(t) = \{ u \in V \mid (x, u) \geq t \}$. Our goal is to select a value $t$ that minimizes $\beta(C_1(t), C_2(t))$. As a convention, when we say “order”, we refer to descending order by default. We describe how to find the optimal $t$ in $O(n log(n))$, where $n log(n)$ is the sorting cost.

First, it’s easy to see that we only evaluate $t$ values from the set $\{ |x_i| \mid i = 1, \ldots, n \}$ and $|unique(x)| \leq n$. Moreover, we assume $|x|$ is strictly ordered, e.g., $|x_i| > |x_{i+1}|$ for all $i = 1, \ldots, n-1$. In other words, thresholding on $t = |x_i|$ takes the top-$i$ nodes ordered by $|x|$. We denote these nodes as $C_{|x_i|}[i]$. Similarly, we define the top-$i$ nodes ordered by $x$ and $-x$ as $C_x[i]$ and $C_{-x}[i]$ respectively.

Further, we partition $C_{|x|}[i]$ into $C_{x}[j]$ and $C_{-x}[k]$ where $j = \{ u \in C_{|x|}[i] \mid x_u \geq 0 \}$ and $k = \{ u \in C_{|x|}[i] \mid x_u < 0 \}$. Then, $C_{x}[j]$ and $C_{-x}[k]$ relate to $C_1(t)$ and $C_2(t)$ respectively.

Therefore, finding $t^* = \arg \min_{t \in [0, \ldots, n]} \beta(C_1(t), C_2(t))$ is equivalent to finding a position:

$$t^* = \arg \min_{t \in [0, \ldots, n]} \frac{2 ||E^*(C_x[j], C_{-x}[k])|| + ||E^*(C_{-x}[j], C_x[k])||}{\text{vol}(C_{|x|}[i])}$$

$$+ \frac{E(C_{|x|}[i], V \setminus C_{|x|}[i])}{\text{vol}(C_{|x|}[i])}$$

In other words, $t^* = |x_{i_0}|$.

To make notation lighter, we make the following short-cut notation which depends on vector $x$ and position $i$:

$$cut_x[i] = E(C_x[i], V \setminus C_x[i])$$

$$cut_{-x}[i] = E(C_{-x}[i], V \setminus C_{-x}[i])$$

$$in_{x}[i] = E^*(C_x[i])$$

$$in_{-x}[i] = E^*(C_{-x}[i])$$

Our algorithm evaluates at all possible $i$ and pick the optimal position. However, it achieves efficiency by re-using intermediate results. Specifically, we compute the related terms at position $i$ from those at $1, \ldots, i-1$ which are already computed.

Next, we show how to achieve this in $O(n + m)$. **Computing $\text{vol}(C_{|x|}[i])$.** We define the $x$-ordered volume, $\text{vol}_x$, which is affected by node ordering induced by $x \in \mathbb{R}^n$. Denote the ordering as $\delta : [n] \rightarrow [n]$. Then define

$$\text{vol}_x[i] = \sum_{j=1}^{i} \text{deg}(\delta(j))$$

We also consider signed variants of $x$-ordered volume, e.g., $\text{vol}_x^s[i]$, which only count degrees from positive edges.

It is easy to see the following equivalence for $i = 1, \ldots, n$: $\text{vol}(C_{|x|}[i]) = \text{vol}_x[i]$.

Then, the following recursion holds for $i = 1, \ldots, n$:

$$\text{vol}_x[i] = \text{vol}_x[i-1] + \text{deg}(i)$$

where $\text{vol}_x[0] = 0$. In other words, $\text{vol}_x[i]$ can be computed incrementally in $O(m + n)$. In a similar fashion, we can compute $\text{vol}_x^s$, $\text{vol}_x^{-s}$, $\text{vol}_{-x}$ and $\text{vol}_{-x}^{-s}$ in $O(m + n)$.

**Computing $\text{cut}_x[i]$.** We will use the following matrices to compute $\text{cut}_x[i]$ and $\text{cut}_{-x}[i]$. $L^+$ and $L^+$ are the lower triangular matrices of $A^+$ and $A^+$ respectively. Given a vector $x \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n \times n}$, we define $M_x$ as the $x$-ordered matrix $M$. $M_x$ equals $M$ with both rows and columns permuted by the ordering induced by $x$. Using this notation, we have $A^+$ and $A^+$ permuted by the order in $x$, $-x$, or $|x|$.

It is easy to see the following recursions hold:

$$\text{in}_{x^+}[i] = \text{in}_{x^+}[i-1] + 2 \sum_{j=1}^{i} \text{L}_{x^+}[i,j]$$

$$\text{in}_{x^-}[i] = \text{in}_{x^-}[i-1] + 2 \sum_{j=1}^{i} \text{L}_{x^-}[i,j]$$

Both of which are computable in $O(m + n)$. Since

$$\text{cut}_{x^+}[i] = \text{vol}_{x^+}[i] - \text{in}_{x^+}[i]$$

$$\text{cut}_{x^-}[i] = \text{vol}_{x^-}[i] - \text{in}_{x^-}[i]$$

$$\text{cut}_{x}[i] = \text{cut}_{x^+}[i] + \text{cut}_{x^-}[i]$$

Therefore, $\text{cut}_x[i]$ can be computed in $O(m + n)$.

**Computing $\text{in}_{x^+}[i] + \text{in}_{x^-}[i]$ given $t$.** Recall that $t$ and $k$ are determined implicitly by $i$. In practice, for all $i = 1, \ldots, n$, the list of $|j|$ and $|k|$ can be computed in $O(n)$. Note that both $\text{in}_{x^+}(\cdot)$ and $\text{in}_{x^-}(\cdot)$ are already computed before. Therefore, this step is done in $O(n)$.

**Computing $E(C_x[i], C_{-x}[i])$.** We observe the following:

$$E(C_x[i], C_{-x}[i]) = E^*(C_x[i], V \setminus C_x[i]) \cup E^*(C_{-x}[i], V \setminus C_{-x}[i])$$

It follows immediately:

$$E^*(C_x[i], C_{-x}[i]) = \text{cut}_{x}[i] + \text{cut}_{x^-}[i] - \text{cut}_{x^+}[i]$$

which is computable in $O(m + n)$ for all $i = 1, \ldots, n$. 

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12We omit $x$ in certain notation when context is clear.

13To make notation lighter, $j$ and $k$’s values are implicitly determined by $i$ and $x$. 

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