Heat equation with a nonlinear boundary condition and uniformly local $L^r$ spaces

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Abstract

We establish the local existence and the uniqueness of solutions of the heat equation with a nonlinear boundary condition for the initial data in uniformly local $L^r$ spaces. Furthermore, we study the sharp lower estimates of the blow-up time of the solutions with the initial data $\lambda \psi$ as $\lambda \to 0$ or $\lambda \to \infty$ and the lower blow-up estimates of the solutions.

1 Introduction

This paper is concerned with the heat equation with a nonlinear boundary condition,

\[
\begin{aligned}
\partial_t u &= \Delta u, \quad x \in \Omega, \ t > 0, \\
\nabla u \cdot \nu(x) &= |u|^{p-1}u, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= \varphi(x), \quad x \in \Omega,
\end{aligned}
\]

(1.1)

where $N \geq 1$, $p > 1$, $\Omega$ is a smooth domain in $\mathbb{R}^N$, $\partial _t = \partial / \partial t$ and $\nu = \nu(x)$ is the outer unit normal vector to $\partial \Omega$. For any $\varphi \in \text{BUC}(\Omega)$, problem (1.1) has a unique solution

\[u \in \text{C}^{2,1}(\Omega \times (0, T)) \cap \text{C}^{1,0}(\overline{\Omega} \times (0, T)) \cap \text{BUC}(\overline{\Omega} \times [0, T])\]

for some $T > 0$ and the maximal existence time $T(\varphi)$ of the solution can be defined. If $T(\varphi) < \infty$, then

\[\limsup_{t \to T(\varphi)} \|u(t)\|_{L^\infty(\Omega)} = \infty\]

and we call $T(\varphi)$ the blow-up time of the solution $u$.

Problem (1.1) has been studied in many papers from various points of view (see e.g. [4]–[8], [10]–[14], [16]–[21], [23], [24], [30] and references therein) while there are few results related to the dependence of the blow-up time on the initial function even in the case $\Omega = \mathbb{R}^N$. We remark that the blow-up time for problem (1.1) cannot be chosen uniform for all initial functions lying in a bounded set of $L^r(\mathbb{R}^N_+)$ with $1 \leq r \leq N(p - 1)$. Indeed, similarly to [29] Remark 15.4 (i), for any solution $u$ blowing up at $t = T < \infty$ and $\mu > 0$,

\[u_{\mu}(x, t) := \mu^{1/(p-1)}u(\mu x, \mu^2 t)\]

(1.2)
is a solution of (1.1) blowing up at $t = \mu^{-2}T$ while

$$
\|u_\mu(0)\|_{L^r(\mathbb{R}^N)} = \mu^{\frac{1}{p-1} - \frac{N}{r}} \|\varphi\|_{L^r(\mathbb{R}^N)} \leq \|\varphi\|_{L^r(\mathbb{R}^N)}
$$

for any $\mu \geq 1$.

For $1 \leq r < \infty$ and $\rho > 0$, let $L^r_{uloc,\rho}(\Omega)$ be the uniformly local $L^r$ space in $\Omega$ equipped with the norm

$$
\|f\|_{r,\rho} := \sup_{x \in \Omega} \left( \int_{\Omega \cap B(x,\rho)} |f(y)|^r \, dy \right)^{1/r}.
$$

We denote by $L^\infty_{uloc,\rho}(\Omega)$ the completion of bounded uniformly continuous functions in $\Omega$ with respect to the norm $\| \cdot \|_{r,\rho}$, that is,

$$
L^\infty_{uloc,\rho}(\Omega) := \overline{BUC(\Omega)}_{\| \cdot \|_{r,\rho}}.
$$

We set $L^\infty_{uloc,\rho}(\Omega) = L^\infty(\Omega)$ and $L^\infty_{uloc,\rho}(\Omega) = BUC(\Omega)$.

In this paper we prove the local existence and the uniqueness of the solutions of problem (1.1) with initial functions in $L^r_{uloc,\rho}(\Omega)$, and study the dependence of the blow-up time on the initial functions. As an application of the main results of this paper, we study the asymptotic behavior of the blow-up time $T(\varphi)$ with $\varphi = \lambda \psi$ as $\lambda \to 0$ or $\lambda \to \infty$ and show the validity of our arguments. Furthermore, we obtain a lower estimate of the blow-up rate of the solutions (see Section 5).

Throughout this paper, following [29, Section 1], we assume that $\Omega \subset \mathbb{R}^N$ is a uniformly regular domain of class $C^1$. For any $x \in \mathbb{R}^N$ and $\rho > 0$, define

$$
B(x, \rho) := \{ y \in \mathbb{R}^N : |x - y| < \rho \}, \quad \Omega(x, \rho) := \Omega \cap B(x, \rho), \quad \partial \Omega(x, \rho) := \partial \Omega \cap B(x, \rho).
$$

By the trace inequality for $W^{1,1}(\Omega)$-functions and the Gagliardo-Nirenberg inequality we can find $\rho_* \in (0, \infty]$ with the following properties (see Lemma 2.2).

- There exists a positive constant $c_1$ such that

$$
\int_{\partial \Omega(x, \rho)} |v| \, d\sigma \leq c_1 \int_{\Omega(x, \rho)} |\nabla v| \, dy \tag{1.3}
$$

for all $v \in C^1_0(B(x, \rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

- Let $1 \leq \alpha, \beta \leq \infty$ and $\sigma \in [0, 1]$ be such that

$$
\frac{1}{\alpha} = \sigma \left( \frac{1}{2} - \frac{1}{N} \right) + (1 - \sigma) \frac{1}{\beta}. \tag{1.4}
$$

Assume, if $N \geq 2$, that $\alpha \neq \infty$ or $N \neq 2$. Then there exists a constant $c_2$ such that

$$
\|v\|_{L^\infty(\Omega(x, \rho))} \leq c_2 \|v\|^{1-\sigma}_{L^\alpha(\Omega(x, \rho))} \|\nabla v\|^{\sigma}_{L^2(\Omega(x, \rho))} \tag{1.5}
$$

for all $v \in C^1_0(B(x, \rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$. 

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We remark that, in the case
\[ \Omega = \{ (x', x_N) \in \mathbb{R}^N : x_N > \Phi(x') \}, \]
where \( N \geq 2 \) and \( \Phi \in C^1(\mathbb{R}^{N-1}) \) with \( \| \nabla \Phi \|_{L^\infty(\mathbb{R}^{N-1})} < \infty \), (1.3) and (1.5) hold with \( \rho_* = \infty \) (see Lemma 2.2). Inequalities (1.3) and (1.5) are used to treat the nonlinear boundary condition.

Next we state the definition of the solution of (1.1).

**Definition 1.1** Let \( 0 < T \leq \infty \) and \( 1 \leq r < \infty \). Let \( u \) be a continuous function in \( \Omega \times (0, T] \). We say that \( u \) is an \( L^r_{uloc}(\Omega) \)-solution of (1.1) in \( \Omega \times [0, T) \) if

- \( u \in L^\infty(\tau, T : L^\infty(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0, R))) \) for any \( \tau \in (0, T) \) and \( R > 0 \),
- \( u \in C([0, T] : L^r_{uloc,\rho}(\Omega)) \) with \( \lim_{t \to 0} \| u(t) - \varphi \|_{r,\rho} = 0 \) for some \( \rho > 0 \),
- \( u \) satisfies
  \[ \int_0^T \int_{\Omega} \{ -u \partial_t \phi + \nabla u \cdot \nabla \phi \} \, dy \, ds = \int_0^T \int_{\partial \Omega} |u|^{p-1} u \phi \, d\sigma \, ds \quad (1.6) \]
  for all \( \phi \in C_0^\infty(\mathbb{R}^N \times (0, T)) \).

Here \( d\sigma \) is the surface measure on \( \partial \Omega \). Furthermore, for any continuous function \( u \) in \( \Omega \times (0, T) \), we say that \( u \) is a \( L^r_{uloc}(\Omega) \)-solution of (1.1) in \( \Omega \times [0, T) \) if \( u \) is a \( L^r_{uloc,\varrho}(\Omega) \)-solution of (1.1) in \( \Omega \times [0, \eta] \) for any \( \eta \in (0, T) \).

We remark the following for any \( \rho, \rho' \in (0, \infty) \):
- \( f \in L^r_{uloc,\rho}(\Omega) \) is equivalent to \( f \in L^r_{uloc,\rho'}(\Omega) \);
- \( u \in C([0, T] : L^r_{uloc,\rho}(\Omega)) \) is equivalent to \( u \in C([0, T] : L^r_{uloc,\rho'}(\Omega)) \).

These follow from property (i) in Section 2.

Now we are ready to state the main results of this paper. Let \( p_* = 1 + 1/N \).

**Theorem 1.1** Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a uniformly regular domain of class \( C^1 \). Let \( p_* \) satisfy (1.3) and (1.5). Then, for any \( 1 \leq r < \infty \) with

\[
\begin{align*}
  r & \geq N(p - 1) \quad \text{if} \quad p > p_*, \\
  r & > 1 \quad \text{if} \quad p = p_*, \\
  r & \geq 1 \quad \text{if} \quad 1 < p < p_*,
\end{align*}
\]

there exists a positive constant \( \gamma_1 \) such that, for any \( \varphi \in L^r_{uloc,\rho}(\Omega) \) with

\[ \rho^{\frac{1}{p-1} - \frac{N}{r}} \| \varphi \|_{r,\rho} \leq \gamma_1 \quad (1.8) \]
for some $\rho \in (0, \rho_*/2)$, problem (1.1) possesses a $L^r_{uloc}(\Omega)$-solution $u$ of (1.1) in $\Omega \times [0, \mu \rho^2]$ satisfying

$$\sup_{0 < t < \mu \rho^2} \|u(t)\|_{r, \rho} \leq C \|\varphi\|_{r, \rho},$$

(1.9)

$$\sup_{0 < t < \mu \rho^2} \int_0^t \|u(t)\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{r, \rho}.$$  

(1.10)

Here $C$ and $\mu$ are constants depending only on $N$, $\Omega$, $p$ and $r$.

**Theorem 1.2** Assume the same conditions as in Theorem 1.1. Let $v$ and $w$ be $L^r_{uloc}(\Omega)$-solutions in $\Omega \times [0, T)$ such that $v(x, 0) \leq w(x, 0)$ for almost all $x \in \Omega$, where $T > 0$ and $r$ is as in (1.7). Assume, if $r = 1$, that

$$\limsup_{t \to +0} t^{\frac{1}{p-1} - \frac{N}{r}} \left[ \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \right] < \infty.$$  

(1.11)

Then there exists a positive constant $\gamma_2$ such that, if

$$\rho^{\frac{1}{p-1} - \frac{N}{r}} \left[ \|v(0)\|_{r, \rho} + \|w(0)\|_{r, \rho} \right] \leq \gamma_2$$  

(1.12)

for some $\rho \in (0, \rho_*/2)$, then

$$v(x, t) \leq w(x, t) \quad \text{in} \quad \Omega \times (0, T).$$

We give some comments related to Theorems 1.1 and 1.2.

(i) Let $u$ be a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, T)$. It follows from Definition 1.1 that $u \in L^\infty(\tau, \sigma : L^\infty(\Omega))$ for any $0 < \tau < \sigma < T$. This together with Theorem 6.2 of [8] implies that $u(t) \in BUC(\Omega)$ for any $t \in (0, T)$. This means that $u(0) \in L^r_{uloc}(\Omega)$ for any $\rho > 0$.

(ii) Consider the case $\Omega = \mathbb{R}^N_+$. Let $u$ be a $L^r_{uloc}(\Omega)$-solution of (1.1) blowing up at $t = T < \infty$, where $r$ is as in (1.7). Then, for any $\mu > 0$, $u_\mu$ defined by (1.2) satisfies

$$\mu^{\frac{1}{p-1} - \frac{N}{r}} \|u_\mu(0)\|_{r, \mu^{-1}} = \|u(0)\|_{r, 1}$$

and it blows up at $t = \mu^{-2}T$. This means that Theorem 1.1 holds with $\rho = 1$ if and only if Theorem 1.1 holds for any $\rho > 0$.

(iii) Let $1 \leq r < \infty$. If, either

(a) $f \in L^r_{uloc, 1}(\Omega)$, $r > N(p - 1)$ or
(b) $f \in L^r(\Omega)$, $r \geq N(p - 1)$,

then, for any $\gamma > 0$, we can find a constant $\rho > 0$ such that $\rho^{\frac{1}{p-1} - \frac{N}{r}} \|f\|_{r, \rho} \leq \gamma$.

As a corollary of Theorem 1.1 we have:

**Corollary 1.1** Assume the same conditions as in Theorem 1.1 and $p > p_*$. 


(i) For any $\varphi \in L^{N(p-1)}(\Omega)$, problem (1.1) has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution in $\Omega \times [0, T]$ for some $T > 0$.

(ii) Assume $\rho_*=\infty$. Then there exists a constant $\gamma$ such that, if

$$\| \varphi \|_{L^{N(p-1)}(\Omega)} \leq \gamma,$$

then problem (1.1) has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution $u$ such that

$$\sup_{0 < t < \infty} \| u(t) \|_{L^{N(p-1)}(\Omega)} + \sup_{0 < t < \infty} t^{\frac{1}{2(p-1)}} \| u(t) \|_{L^\infty(\Omega)} < \infty.$$

For further applications of our theorems, see Section 5.

**Remark 1.1** Let $\Omega = \mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N : x_N > 0\}$. If $1 < p \leq p_*$, then problem (1.1) possesses no positive global-in-time solutions. See [7] and [14]. For the case $p > p_*$, it is proved in [24] (see also [23]) that, if $\varphi \geq 0$, $\varphi \not\equiv 0$ in $\Omega$ and

$$\| \varphi \|_{L^1(\mathbb{R}^N_+)} \| \varphi \|_{L^{N(p-1)-1}_\infty(\mathbb{R}^N_+)}$$

is sufficiently small,

then there exists a positive global-in-time solution of (1.1). This also immediately follows from assertion (ii) of Corollary 1.1 and the comparison principle.

We explain the idea of the proof of Theorem 1.1. Under the assumptions of Theorem 1.1 there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset BUC(\Omega)$ such that

$$\lim_{n \to \infty} \| \varphi - \varphi_n \|_{r, \rho} = 0, \quad \sup_n \| \varphi_n \|_{r, \rho} \leq 2 \| \varphi \|_{r, \rho}. \quad (1.14)$$

For any $n = 1, 2, \ldots$, let $u_n$ satisfy in the classical sense

$$\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, T_n), \\
\nabla u \cdot \nu(x) = |u|^{p-1}u & \text{on } \partial \Omega \times (0, T_n), \\
u(x,0) = \varphi_n(x) & \text{in } \Omega,
\end{cases} \quad (1.15)$$

where $T_n$ is the blow-up time of the solution $u_n$. By regularity theorems for parabolic equations (see e.g. [8] and [25] Chapters III and IV) we see that

$$u_n \in BUC(\overline{\Omega} \times [0, T]), \quad \nabla u_n \in L^\infty(\Omega \times (\tau, T)),$$

for any $0 < \tau < T < T_n$, which imply that $u_n$ is a $L^r_{uloc}(\Omega)$-solution in $\Omega \times (0, T_n)$ for any $1 \leq r < \infty$. Set

$$\Psi_{r,\rho}[u_n](t) := \sup \sup_{0 \leq \tau \leq t} \int_{\Omega(x,\rho)} |u_n(y, \tau)|^r \, dy, \quad 0 \leq t < T_n.$$

It follows from (1.8) and (1.14) that

$$\Psi_{r,\rho}[u_n](0)^{\frac{1}{r}} = \| \varphi_n \|_{r, \rho} \leq 2 \| \varphi \|_{r, \rho} \leq 2 \gamma_1 \rho^{-\frac{1}{p-1} + \frac{N}{2p-1}}. \quad (1.17)$$
Define
\[ T_n^* := \sup \{ \sigma \in (0, T_n) : \Psi_{r,\rho}[u_n](t) \leq 6M\Psi_{r,\rho}[u_n](0) \text{ in } [0, \sigma] \}, \]
\[ T_{n}^{**} := \sup \{ \sigma \in (0, T_n) : \rho^{-\frac{1}{2}} \| u_n(t) \|_{L^\infty(\Omega)}^{\frac{n}{q} - 1} \leq 2t^{-\frac{1}{2}} \text{ in } (0, \sigma) \}, \]
\( \text{for some } \mu > 0. \) This enables us to prove Theorem 1.1. Theorem 1.2 follows from a similar argument as in Theorem 1.1.

The rest of this paper is organized as follows. In Section 2 we give some preliminary lemmas related to \( \rho^* \). In Sections 3 and 4 we prove Theorems 1.1 and 1.2. In Section 5, as applications of Theorem 1.1 we give some results on the blow-up time and the blow-up rate of the solutions.

2 Preliminaries

In this section we recall some properties of uniformly local \( L^r \) spaces and prove some lemmas related to \( \rho^* \). Furthermore, we give some inequalities used in Sections 3 and 4. In what follows, the letter \( C \) denotes a generic constant independent of \( x \in \overline{\Omega} \), \( n \) and \( \rho \).

Let \( 1 \leq r < \infty \). We first recall the following properties of \( L^r_{uloc,\rho}(\Omega) \):

(i) if \( f \in L^r_{uloc,\rho}(\Omega) \) for some \( \rho > 0 \), then, for any \( \rho' > 0 \), \( f \in L^r_{uloc,\rho'}(\Omega) \) and
\[ \| f \|_{r,\rho'} \leq C_1 \| f \|_{r,\rho} \]
for some constant \( C_1 \) depending only on \( N \), \( \rho \) and \( \rho' \);

(ii) there exists a constant \( C_2 \) depending only on \( N \) such that
\[ \| f \|_{r,\rho} \leq C_2 \rho^{\frac{N}{q} - 1} \| f \|_{r,\rho}, \quad f \in L^q_{uloc,\rho}(\Omega), \]
for any \( 1 \leq r \leq q < \infty \), \( \rho > 0 \), \( q \) and \( \rho' \);

(iii) if \( f \in L^r(\Omega) \), then \( f \in L^r_{uloc,\rho}(\Omega) \) for any \( \rho > 0 \) and
\[ \lim_{\rho \to +0} \| f \|_{r,\rho} = 0. \]

Properties (ii) and (iii) are proved by the Hölder inequality and the absolute continuity of \( |f|^r \, dy \) with respect to \( dy \). Property (i) follows from the following lemma.

**Lemma 2.1** Let \( N \geq 1 \) and \( \Omega \) be a domain in \( \mathbb{R}^N \). Then there exists \( M \in \{1, 2, \ldots \} \) depending only on \( N \) such that, for any \( x \in \overline{\Omega} \) and \( \rho > 0 \),
\[ \Omega(x, 2\rho) \subset \bigcup_{k=1}^{n} \Omega(x_k, \rho) \]
for some \( \{ x_k \}_{k=1}^{n} \subset \overline{\Omega} \) with \( n \leq M \).

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Proof. There exist \( M \in \{1, 2, \ldots \} \) and \( \{y_k\}_{k=1}^M \subset B(0, 2) \) such that

\[
B(0, 2) \subset \bigcup_{k=1}^M B(y_k, 1/2).
\]

Then, for any \( x \in \overline{\Omega} \) and \( \rho > 0 \), we can find \( \{y_i\}_{i=1}^n \subset \{y_k\}_{k=1}^M \) such that

\[
\Omega(x + \rho y_i, \rho/2) \neq \emptyset \quad \text{and} \quad \Omega(x, 2\rho) \subset \bigcup_{i=1}^n \Omega(x + \rho y_i, \rho/2).
\]

(2.4)

Furthermore, for any \( i \in \{1, \ldots, n\} \), there exists \( x_i \in \overline{\Omega} \) such that

\[
x_i \in \Omega(x + \rho y_i, \rho/2) \quad \text{and} \quad \Omega(x + \rho y_i, \rho/2) \subset \Omega(x_i, \rho).
\]

This together with (2.4) implies (2.3), and Lemma 2.1 follows. \( \square \)

We state a lemma on the existence of \( \rho_* \) satisfying (1.3) and (1.5).

Lemma 2.2 Let \( N \geq 1 \) and \( \Omega \) be a uniformly regular domain of class \( C^1 \). Then there exists \( \rho_* > 0 \) such that (1.3) and (1.5) hold. In particular, if

\[
\Omega = \{(x', x_N) \in \mathbb{R}^N : x_N > \Phi(x')\},
\]

(2.5)

where \( N \geq 2 \) and \( \Phi \in C^1(\mathbb{R}^{N-1}) \) with \( \|
abla \Phi\|_{L^\infty(\mathbb{R}^{N-1})} < \infty \), then (1.3) and (1.5) hold with \( \rho_* = \infty \).

Proof. By the definition of uniformly regular domain, it suffices to consider the case (2.5). Let \( f \in C^1_0(B(x_*, \rho)) \), where \( x_* \in \overline{\Omega} \) and \( \rho > 0 \). Set \( f = 0 \) outside \( B(x_*, \rho) \). We first consider the case of \( \partial \Omega(x_*, \rho) \neq \emptyset \). Then there exists \( y_* \in \partial \Omega \) such that \( B(x_*, \rho) \subset B(y_*, 2\rho) \). Set

\[
g(x', x_N) := \begin{cases} f(x' - y'_N, x_N + \Phi(x')) & \text{for } x_N \geq 0, \\ f(x' - y'_N, -x_N + \Phi(x')) & \text{for } x_N < 0, \end{cases} \quad \tilde{g}(z) := g(2\rho' z),
\]

where

\[
\rho' = \rho \left(1 + \|
abla \Phi\|_{L^\infty(\mathbb{R}^{N-1})}^2\right)^{1/2}.
\]

Then \( \tilde{g} \in C^1_0(B(0, 1)) \). Applying the Gagliardo-Nirenberg inequality (see e.g. [15]) and the trace imbedding theorem (see e.g. [2, Theorem 5.22]), we obtain

\[
\|\tilde{g}\|_{L^\beta(B(0, 1))} \leq C\|\tilde{g}\|_{L^\beta(B(0, 1))}^{1-\sigma}\|\nabla \tilde{g}\|_{L^\alpha(B(0, 1))}^{\sigma},
\]

\[
\int_{B(0, 1) \cap \partial \mathbb{R}^N_+} |\tilde{g}| \, d\sigma \leq C\|
abla \tilde{g}\|_{W^{1, 1}(B(0, 1) \cap \mathbb{R}^N_+)} \leq C\|
abla \tilde{g}\|_{L^1(B(0, 1) \cap \mathbb{R}^N_+)},
\]

where \( \alpha, \beta \) and \( \sigma \) are as in (1.4) and \( \alpha \neq \infty \) if \( N = 2 \). These imply that

\[
\|g\|_{L^\beta(B(0, 2\rho'))} \leq C\|g\|_{L^\beta(B(0, 2\rho'))}^{1-\sigma}\|\nabla g\|_{L^\alpha(B(0, 2\rho'))}^{\sigma},
\]

\[
\int_{B(0, 2\rho') \cap \partial \mathbb{R}^N_+} |g| \, d\sigma \leq C\|
abla g\|_{L^1(B(0, 2\rho') \cap \mathbb{R}^N_+)},
\]

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for some constants \( C \) independent of \( \rho \). Then we have

\[
\| f \|_{L^2(\Omega(x, \rho))} = \| f \|_{L^2(\Omega(y_0, 2\rho))} \leq C \| g \|_{L^2(B(0, 2\rho'))} \\
\leq C \| g \|_{L^2(B(0, 2\rho'))} \| \nabla g \|_{L^2(B(0, 2\rho'))} \\
\leq C \| f \|_{L^2(\Omega(x, \rho))} \| \nabla f \|_{L^2(\Omega(x, \rho))},
\]

(2.6)

\[
\int_{\partial \Omega(x, \rho)} |f| d\sigma \leq C \int_{B(0, 2\rho') \cap \Omega} |g| d\sigma \leq C \| \nabla g \|_{L^1(B(0, 2\rho') \cap \Omega)}.
\]

(2.7)

Therefore we obtain (1.3) and (1.5) for any \( \rho > 0 \) in the case of \( \partial \Omega(x, \rho) \neq \emptyset \). Similarly, we get (1.3) and (1.5) for all \( \rho > 0 \) in the case of \( \partial \Omega(x, \rho) = \emptyset \). Thus (1.3) and (1.5) hold with \( \rho_* = \infty \) in the case (2.5), and the proof is complete. \( \Box \)

We obtain the following two lemmas by using (1.3) and (1.5).

**Lemma 2.3** Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a uniformly regular domain of class \( C^1 \). Let \( \rho_* \) satisfy (1.3) and (1.5). Then there exists a constant \( C_1 \) such that

\[
\int_{\partial \Omega(x, \rho)} \phi^2 d\sigma \leq \epsilon \int_{\Omega(x, \rho)} |\nabla \phi|^2 d\sigma + \frac{C_1}{\epsilon} \int_{\Omega(x, \rho)} \phi^2 dy
\]

(2.9)

for all \( \phi \in C^1_0(B(x, \rho)) \), \( \epsilon > 0 \), \( x \in \overline{\Omega} \) and \( \rho \in (0, \rho_*) \). Furthermore, for any \( p > 1 \) and \( r > 0 \), there exists a constant \( C_2 \) such that

\[
\int_{\Omega(x, \rho)} f^{2p+r-2} dy \leq C_2 \left( \int_{\Omega(x, \rho)} f^{N(p-1)} dy \right)^{\frac{2}{N}} \int_{\Omega(x, \rho)} |\nabla f^2|^2 dy
\]

(2.10)

for all nonnegative functions \( f \) satisfying \( f^{r/2} \in C^1(\Omega(x, \rho)) \) with \( f = 0 \) near \( \Omega \cap \partial B(x, \rho) \), \( \rho \in (0, \rho_*) \) and \( x \in \overline{\Omega} \).

**Proof.** It follows from (1.5) that

\[
\int_{\partial \Omega(x, \rho)} \phi^2 d\sigma \leq C \int_{\Omega(x, \rho)} |\nabla \phi|^2 dy \leq 2C \int_{\Omega(x, \rho)} |\phi| |\nabla \phi| dy
\]

\[
\leq \epsilon \int_{\Omega(x, \rho)} |\nabla \phi|^2 dy + \frac{C^2}{\epsilon} \int_{\Omega(x, \rho)} \phi^2 dy
\]

for all \( \phi \in W^{1,2}_0(B(x, \rho)) \), \( \epsilon > 0 \), \( x \in \overline{\Omega} \) and \( \rho \in (0, \rho_*) \). This implies (2.9).

Let \( r > 0 \) and \( 0 < \rho < \rho_* \). If \( 2N(p-1) \geq r \), then, by (1.5) we have

\[
\int_{\Omega(x, \rho)} g^{\frac{2}{r}(p-1)+2} dy \leq C \left( \int_{\Omega(x, \rho)} g^{2N(p-1)} dy \right)^{\frac{2}{r}} \int_{\Omega(x, \rho)} |\nabla g|^2 dy
\]

(2.11)

for all \( g \in C^1_0(B(x, \rho)) \) and \( x \in \overline{\Omega} \). Furthermore, we obtain (2.11) by the Hölder inequality and (1.5) even for the case \( 2N(p-1) < r \) (see e.g. \[28\] Lemma 3). Then, setting \( g = f^{r/2} \), we obtain (2.10), and the proof is complete. \( \Box \)
Lemma 2.4 Assume the same conditions as in Theorem 1.1. Let \( r \geq 1, T > 0 \) and \( f \) be a nonnegative function such that

\[
f \in C([0, T] : L_{uloc}^r(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0, R)))
\]

for any \( \rho \in (0, \rho_*/2), \tau \in (0, T) \) and \( R > 0 \). Let \( x \in \tilde{\Omega} \) and \( \zeta \) be a smooth function in \( \mathbb{R}^N \) such that

\[
0 \leq \zeta \leq 1 \quad \text{and} \quad |\nabla \zeta| \leq 2\rho^{-1} \quad \text{in} \quad \mathbb{R}^N,
\]

\[
\zeta = 1 \quad \text{on} \quad B(x, \rho), \quad \zeta = 0 \quad \text{outside} \quad B(x, 2\rho).
\]

Set \( f_\epsilon = f + \epsilon \) for \( \epsilon > 0 \). Then, for any sufficiently large \( k \geq 2 \), there exists a constant \( C \) such that

\[
\sup_{x \in \tilde{\Omega}} \int_{T}^{t} \int_{\partial\Omega(x, 2\rho)} f_\epsilon^{p+r-1}\zeta^k d\sigma ds
\]

\[
\leq C \left[ \rho^{-N} \Psi_{r,\rho}[f_\epsilon](t) \right]^{p-1} \left[ \sup_{x \in \Omega} \int_{\Omega(x, \rho)} |\nabla f_\epsilon^p|^2 dy ds + \rho^{-2}(t - \tau)\Psi_{r,\rho}[f_\epsilon](t) \right]^{p-2} \quad (2.12)
\]

for all \( 0 < \tau < t \leq T, \rho \in (0, \rho_*/2) \) and \( \epsilon > 0 \).

**Proof.** Let \( \rho \in (0, \rho_*/2) \). It suffices to consider the case where \( \partial\Omega(x, \rho) \neq \emptyset \). Let \( k \geq 2 \) be such that

\[
k = \frac{2p + r - 2}{2} \geq 1. \quad (2.13)
\]

By (1.3) and Lemma 2.1, for any \( \delta > 0 \), we have

\[
\int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{p+r-1}\zeta^k d\sigma ds \leq C \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} |\nabla (f_\epsilon^{p+r-1}\zeta^k)| dy ds
\]

\[
\leq C \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{p+r-1} |\nabla f_\epsilon^p|^k \zeta^k dy ds + C \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{p+r-1} |\nabla \zeta|^k \zeta^{-1} dy ds
\]

\[
\leq C\delta \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{2p+r-2} \zeta^k dy ds + C\delta^{-1} \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{p} |\nabla f_\epsilon^p|^2 \zeta^{k-2} \zeta^2 dy ds \quad (2.14)
\]

\[
\leq C\delta \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^{2p+r-2} \zeta^k dy ds + C\delta^{-1} \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} f_\epsilon^p |\nabla \zeta|^2 \zeta^2 dy ds
\]

\[
+ C\delta^{-1} \sup_{x \in \Omega} \int_{\tau}^{t} \int_{\Omega(x, \rho)} |\nabla f_\epsilon^p|^2 dy ds + C\delta^{-1} \rho^{-2}(t - \tau)\Psi_{r,\rho}[f_\epsilon](t)
\]

for \( 0 < \tau < t \leq T \), where \( C \) is a constant independent of \( \epsilon \) and \( \delta \). Set \( g_\epsilon := f_\epsilon^{k/(2p+r-2)} \). It follows from (2.13) that \( f_\epsilon^{r/2} = 0 \) near \( \partial B(x, 2\rho) \cap \Omega \). Then, by Lemmas 2.1 and 2.3 we
have
\[
\int_t^\tau \int_{\Omega(x,2\rho)} f_{\epsilon}(y,\tau)^{2p+r-2} \zeta^k \, dy \, ds = \int_t^\tau \int_{\Omega(x,2\rho)} g_{\epsilon}(y,\tau)^{2p+r-2} \, dy \, ds
\]
\[
\leq C \sup_{0<\rho<s<\tau} \left( \int_{\Omega(x,2\rho)} g_{\epsilon}(y,s)^{N(p-1)} \, dy \right)^{\frac{1}{p-1}} \int_t^\tau \int_{\Omega(x,2\rho)} |\nabla g_{\epsilon}|^2 \, dy \, ds
\]
\[
\leq C \sup_{0<\rho<s<\tau} \left( \int_{\Omega(x,2\rho)} f_{\epsilon}(y,s) \, dy \right)^{\frac{1}{p-1}} \times \left[ \int_t^\tau \int_{\Omega(x,2\rho)} |\nabla f_{\epsilon}|^2 \, dy \, ds + \rho^{-2} \int_t^\tau \int_{\Omega(x,2\rho)} f_{\epsilon}^r \, dy \, ds \right]
\]
\[
\leq C \left[ \rho^{\frac{r}{2p-r-N}} \Psi_{r,\rho}[f_{\epsilon}](t) \right]^{\frac{2(p-1)}{2(r-1)}} \times \left[ \sup_{x\in\Omega} \int_t^\tau \int_{\Omega(x,\rho)} |\nabla f_{\epsilon}|^2 \, dy \, ds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[f_{\epsilon}](t) \right]
\]
for \( 0 < \tau < t \leq T \). Therefore, taking \( \delta = [\rho^{\frac{r}{2p-r-N}} \Psi_{r,\rho}[f_{\epsilon}](t)]^{-(p-1)/r} \), by (2.14) and (2.15) we obtain (2.12), and the proof is complete. \( \square \)

### 3 Proof of Theorems 1.1 and 1.2 in the case \( r > 1 \).

Let \( v \) and \( w \) be \( L^r_{uloc}(\Omega) \)-solutions of (1.1) in \( \Omega \times [0,T] \), where \( 0 < T < \infty \) and \( r \) is as in (1.7). Set \( z := v - w \) and \( z_{\epsilon} := \max\{z,0\} + \epsilon \) for \( \epsilon \geq 0 \). Then \( z_{\epsilon} \) satisfies
\[
\partial_t z_{\epsilon} \leq \Delta z_{\epsilon} \quad \text{in} \quad \Omega \times (0,T], \\
\nabla z_{\epsilon} \cdot \nu(x) \leq a(x,t)z_{\epsilon} \quad \text{on} \quad \partial\Omega \times (0,T],
\]
in the weak sense (see e.g. [9, Chapter II]). Here
\[
a(x,t) := \begin{cases} 
|v(x,t)|^{p-1}v(x,t) - |w(x,t)|^{p-1}w(x,t) & \text{if} \ v(x,t) \neq w(x,t), \\
|v(x,t)|^{p-1} & \text{if} \ v(x,t) = w(x,t),
\end{cases}
\]
which satisfies
\[
0 \leq a(x,t) \leq C(|v|^{p-1} + |w|^{p-1}) \quad \text{in} \quad \Omega \times (0,T].
\]
In this section we give some estimates of \( z \), and prove Theorems 1.1 and 1.2 in the case \( r > 1 \).

We first give an \( L^\infty_{uloc} \) estimate of \( z_0 \) by using the Moser iteration method with the aid of (1.18). For related results, see [13].

**Lemma 3.1** Assume the same conditions as in Theorem 1.1. Let \( v \) and \( w \) be \( L^r_{uloc}(\Omega) \)-solutions of (1.1) in \( \Omega \times [0,T] \), where \( 0 < T < \infty \) and \( r \geq 1 \). Set \( z_0 := \max\{v - w,0\} \).
and $a = a(x,t)$ as in \((3.2)\). Then there exists a constant $C$ such that

$$
\|z_0(t)\|_{L^\infty(\Omega(x,R_1) \times (t_1,t))} \leq CD^{N/2} \left( \int_{t_2}^{t} \int_{\Omega(x,R_2)} z_0^r \, dyds \right)^{1/r}, \tag{3.4}
$$

$$
\int_{t_1}^{t} \int_{\Omega(x,R_1)} |\nabla z_0|^2 \, dyds \leq CD \int_{t_2}^{t} \int_{\Omega(x,R_2)} z_0^2 \, dyds, \tag{3.5}
$$

for all $x \in \Omega$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \leq T$, where

$$
D := \|a\|_{L^\infty(\Omega(x,R_2) \times (t_2,t))}^2 + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.
$$

**Proof.** Let $x \in \Omega$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \leq T$. For $j = 0, 1, 2, \ldots$, set

$$
r_j := R_1 + (R_2 - R_1)2^{-j}, \quad \tau_j := t_1 - (t_1 - t_2)2^{-j}, \quad Q_j := \Omega(x,r_j) \times (\tau_j,t).
$$

Let $\zeta_j$ be a piecewise smooth function in $Q_j$ such that

$$
0 \leq \zeta_j \leq 1 \quad \text{in} \quad \mathbb{R}^N, \quad \zeta_j = 1 \quad \text{on} \quad Q_{j+1},
$$

$$
\zeta_j = 0 \quad \text{near} \quad \partial \Omega(x,r_j) \times [\tau_j,t] \cup \Omega(x,r_j) \times \{\tau_j\},
$$

$$
|\nabla \zeta_j| \leq \frac{2^{j+1}}{R_2 - R_1} \quad \text{and} \quad 0 \leq \partial_t \zeta_j \leq \frac{2^{j+1}}{t_1 - t_2} \quad \text{in} \quad Q_j.
$$

For any $\alpha \geq \alpha_0$, multiplying \((3.3)\) by $z_\epsilon^{\alpha - 1} \zeta_j^2$ and integrating it on $Q_j$, we obtain

$$
\frac{1}{\alpha} \sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \frac{\alpha - 1}{2} \int_{Q_j} z_\epsilon^{\alpha - 2} |\nabla z_\epsilon| \zeta_j^2 \, dyds
\leq 4 \int_{Q_j} z_\epsilon^\alpha \zeta_j |\partial_t \zeta_j| \, dyds + \frac{4}{\alpha - 1} \int_{Q_j} z_\epsilon^\alpha |
\nabla \zeta_j|^2 \, dyds + 2 \int_{\tau_j}^{t} \int_{\partial \Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds. \tag{3.7}
$$

This calculation is somewhat formal, however it is justified by the same argument as in \cite{25} Chapter III \(\text{(see also \cite{9})}\). Then it follows that

$$
\sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \int_{Q_j} |\nabla [z_\epsilon^\frac{\alpha}{2} \zeta_j]|^2 \, dyds \leq C \int_{Q_j} z_\epsilon^\alpha \zeta_j |\partial_t \zeta_j| \, dyds
$$

$$
+ C \int_{Q_j} z_\epsilon^\alpha |
\nabla \zeta_j|^2 \, dyds + C \alpha \int_{\tau_j}^{t} \int_{\partial \Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds \tag{3.8}
$$

for all $j = 0, 1, 2, \ldots$ and $\alpha \geq \alpha_0$. On the other hand, by Lemma \cite{23} we have

$$
C \alpha \int_{\tau_j}^{t} \int_{\partial \Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds \leq C \alpha \|a\|_{L^\infty(Q_0)} \int_{\tau_j}^{t} \int_{\partial \Omega_j} z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds
$$

$$
\leq \frac{1}{2} \int_{Q_j} |\nabla [z_\epsilon^\frac{\alpha}{2} \zeta_j]|^2 \, dyds + C \alpha^2 \|a\|_{L^\infty(Q_0)}^2 \int_{Q_j} z_\epsilon^\alpha \zeta_j^2 \, dyds. \tag{3.9}
$$

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We deduce from (3.6), (3.8) and (3.9) that

\[
\sup_{\tau_j < s < t} \int_{\Omega(x,r)} z^\alpha_{c\ell} \zeta_j^2 \, dy + \int_{Q_j} |\nabla [z^\ell \zeta_j]|^2 \, dyds \\
\leq C \left[ \alpha^2 \|a\|^2_{L^\infty(Q_0)} + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right] \int_{Q_j} z^\alpha \, dyds
\]

(3.10)

for all \( j = 0, 1, 2, \ldots \) and \( \alpha \geq \alpha_0 \). This together with (1.5) implies that

\[
\left( \int_{Q_{j+1}} z^\alpha \, dyds \right)^{1/\kappa} \leq C \left[ \alpha^2 \|a\|^2_{L^\infty(Q_0)} + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right] \int_{Q_j} z^\alpha \, dyds
\]

(3.11)

for all \( j = 0, 1, 2, \ldots \) and \( \alpha \geq \alpha_0 \), where \( \kappa := 1 + 2/N \). Furthermore, by (3.10) with \( \alpha = 2 \) we have (3.5).

We prove (3.4) in the case \( r \geq 2 \). Setting

\[
I_j := \|z^\ell\|_{L^\alpha_j(Q_j)}, \quad \alpha_j := r\kappa^j,
\]

by (3.11) we have

\[
I_{j+1} \leq C^{\frac{1}{\alpha_j}} \left[ \alpha_j^2 \|a\|^2_{L^\infty(Q_0)} + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right]^{\frac{1}{\alpha_j}} I_j \leq C^{\frac{1}{\alpha_j}} (CD)^{\frac{1}{\alpha_j}} I_j
\]

(3.12)

for all \( j = 0, 1, 2, \ldots \), where \( D := \|a\|^2_{L^\infty(Q_0)} + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1} \). Since

\[
\sum_{j=0}^{\infty} \frac{1}{\alpha_j} = \frac{1}{r} \sum_{j=0}^{\infty} \kappa^{-j} = \frac{1}{r(1 - \kappa^{-1})} = \frac{N + 2}{2r}, \quad \sum_{j=0}^{\infty} \frac{j}{\alpha_j} < \infty,
\]

we deduce from (3.12) that

\[
\|z^\ell\|_{L^\infty(Q_\infty)} = \lim_{j \to \infty} I_j \leq C^{\sum_{j=0}^{\infty} \frac{1}{\alpha_j}} (CD)^{\sum_{j=0}^{\infty} \frac{1}{\alpha_j}} I_0 \leq CD^{(N+2)/2r} \|z^\ell\|_{L^r(Q_0)},
\]

which implies

\[
\|z^\ell\|_{L^\infty(\Omega(x,R_1) \times (t_1,t))} \leq CD^{\frac{N+2}{2r}} \left( \int_{t_2}^{t} \int_{\Omega(x,R_2)} z^\ell_r \, dyds \right)^{1/r},
\]

(3.13)

where \( r \geq 2 \). Then, passing the limit as \( \epsilon \to 0 \), we obtain (3.5).

On the other hand, for the case \( 1 \leq r < 2 \), applying (3.13) with \( r = 2 \) to the cylinders \( Q_j \) and \( Q_{j+1} \), we have

\[
\|z^\ell\|_{L^\infty(Q_{j+1})} \leq C \left( 2^{2j} D \right)^{\frac{N+2}{2r}} \left( \int_{Q_j} z^\ell \, dyds \right)^{1/2} \leq Cb^j \|z^\ell\|_{L^\infty(Q_j)} \left( D^{(N+2)/2} \int_{Q_j} z^\ell \, dyds \right)^{1/2},
\]

where \( b := (2^{2j} D)^{1/2} \leq 2^{j/2} \).
where \( b = 2^{(N+2)/2} \). Then, for any \( \nu > 0 \), we have

\[
\| z_\epsilon \|_{L^\infty(Q_{j+1})} \leq \nu \| z_\epsilon \|_{L^\infty(Q_j)} + C \nu^{j+1} \left( \int_{Q_j} \frac{z_\epsilon}{r^{2j}} \, dyds \right)^{1/r} 
\]

for \( j = 1, 2, \ldots \). Taking a sufficiently small \( \nu \) if necessary, we see that

\[
\| z_\epsilon \|_{L^\infty(Q_{j+1})} \leq \nu^{j+1} \| z_\epsilon \|_{L^\infty(Q_j)} + C \left( \int_{Q_j} z_\epsilon \, dyds \right)^{1/r} 
\]

for \( j = 1, 2, \ldots \). Passing to the limit as \( j \to \infty \) and \( \epsilon \to 0 \), we obtain

\[
\| z_0 \|_{L^\infty(Q_{\infty})} \leq C \left( \int_{Q_0} z_0 \, dyds \right)^{1/r},
\]

which implies (3.5) in the case \( 1 \leq r < 2 \). Thus Lemma 3.1 follows. \( \square \)

**Lemma 3.2** Assume the same conditions as in Theorem 1.1. Let \( r \) satisfy (1.7) and \( r > 1 \). Let \( v \) be a \( L^r_{uloc}(\Omega) \)-solution of (1.1) in \( \Omega \times [0,T] \), where \( T > 0 \). Then there exists a positive constant \( \Lambda \) such that, if

\[
\rho^{\frac{r}{r-1}} N \Psi_{r,\rho}[v](T) \leq \Lambda
\]

for some \( \rho \in (0, \rho_*/2) \), then

\[
\Psi_{r,\rho}[v](t) \leq 5M \Psi_{r,\rho}[v](\tau), \tag{3.15}
\]

\[
\sup_{x \in T} \int_{\tau}^{t} \int_{\partial \Omega(x, \rho)} |v|^p \, d\sigma ds \leq C \Lambda^{\frac{p-1}{r}} \Psi_{r,\rho}[v](\tau), \tag{3.16}
\]

for all \( 0 \leq \tau \leq t \leq T \) with \( t - \tau \leq \mu \rho^2 \), where \( C \) and \( \mu \) are positive constants depending only on \( N, \Omega, p \) and \( r \).

**Proof.** Let \( x \in \overline{\Omega} \) and let \( \zeta \) and \( k \) be as in Lemma 2.4. By (3.14) we can take a sufficiently small \( \epsilon > 0 \) so that

\[
\rho^{\frac{r}{r-1}} N \Psi_{r,\rho}[v_\epsilon](T) \leq 2\Lambda, \tag{3.17}
\]

where \( v_\epsilon := \max\{ \pm v, 0 \} + \epsilon \). Similarly to (3.8), for any \( 0 < \tau < t \leq T \), multiplying (1.1) by \( v_\epsilon^{r-1} \zeta^k \) and integrating it in \( \Omega \times (\tau, t) \), we obtain

\[
\int_{\Omega(x, 2\rho)} v_\epsilon(y, s)^r \zeta^k \, dy \bigg|_{s=\tau}^{s=t} + \int_{\tau}^{t} \int_{\Omega(x, \rho)} |\nabla v_\epsilon^r|^2 \, dyds
\]

\[
\leq C \rho^{-2} \int_{\tau}^{t} \int_{\Omega(x, 2\rho)} v_\epsilon^r \, dyds + C \int_{\tau}^{t} \int_{\partial \Omega(x, 2\rho)} v_\epsilon^{p+r-1} \zeta^k \, d\sigma ds. \tag{3.18}
\]
This together with \( v \in C(\Omega \times [\tau, T]) \cap L^\infty(\tau, T : L^\infty(\Omega)) \) (see Definition 1.1) implies that

\[
\sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds < \infty. \tag{3.19}
\]

Furthermore, by Lemma 2.4, (3.17) and (3.18) we have

\[
\int_{\Omega(x, 2\rho)} v_r(y, s)^r \, dy \bigg|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds \leq C \rho^{-2} \int_\tau^t \int_{\Omega(x, 2\rho)} v_r \, dyds
\]

\[+ C(2\Lambda)^{^\frac{p-1}{p-r}} \left[ \sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_{\epsilon}](t) \right] \tag{3.20}
\]

for \( 0 < \tau < t \leq T \). Therefore, by Lemma 2.1 (1.18) and (3.20) we obtain

\[
\sup_{x \in \Omega} \int_{\Omega(x, 2\rho)} v_r(y, t)^r \, dy + \sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds \leq M \sup_{x \in \Omega} \int_{\Omega(x, \rho)} v_r(y, \tau)^r \, dy + C \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_{\epsilon}](t)
\]

\[+ C(2\Lambda)^{^\frac{p-1}{p-r}} \left[ \sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_{\epsilon}](t) \right] \tag{3.21}
\]

for \( 0 < \tau < t \leq T \). Taking a sufficiently small \( \Lambda \) if necessary, we deduce from (3.19) and (3.21) that

\[
\sup_{x \in \Omega} \int_{\Omega(x, \rho)} v_r(y, t)^r \, dy + \frac{1}{2} \sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds
\]

\[\leq M \sup_{x \in \Omega} \int_{\Omega(x, \rho)} v_r(y, \tau)^r \, dy + C \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_{\epsilon}](t)
\]

for \( 0 < \tau < t \leq T \) with \( t-\tau \leq \mu \rho^2 \). This implies that

\[
\Psi_{r,\rho}[v_{\epsilon}](t) + \frac{1}{2} \sup_{x \in \Omega} \int_\tau^t \int_{\Omega(x, \rho)} |\nabla \tilde{v}_x|^2 \, dyds \leq 2M\Psi_{r,\rho}[v_{\epsilon}](\tau) + \frac{1}{2}\Psi_{r,\rho}[v_{\epsilon}](t) \tag{3.22}
\]

for \( 0 < \tau < t \leq T \) with \( t-\tau \leq \mu \rho^2 \). Furthermore, by Lemma 2.4 (3.22) and (3.23) we have

\[
\int_\tau^t \int_{\partial \Omega(x, \rho)} \max\{\pm\epsilon, 0\} \, d\sigma ds \leq \int_\tau^t \int_{\partial \Omega(x, \rho)} v_{\epsilon}^{p+r-1} \, d\sigma ds
\]

\[\leq CA^{^\frac{p+1}{p-1}} \Psi_{r,\rho}[v_{\epsilon}](t) \leq CA^{^\frac{p-1}{p-r}} \Psi_{r,\rho}[v](t) + C\epsilon^r \rho^N. \tag{3.24}
\]

Since \( \tau \) and \( \epsilon \) is arbitrary, by (3.23) and (3.24) we obtain (3.15) and (3.16). Thus Lemma 3.2 follows. \( \Box \)
Lemma 3.3  Assume the same conditions as in Lemma 3.1. Let \( r \) satisfy (1.7) and \( r > 1 \). Then there exists a positive constant \( \Lambda \) such that, if
\[
\rho \frac{t - r^N}{t} (\Psi_{r,p}[v](T) + \Psi_{r,p}[w](T)) \leq \Lambda
\]
for some \( \rho \in (0, \rho_* / 2) \), then
\[
\Psi_{r,p}[\zeta_0](t) \leq C \Psi_{r,p}[\zeta_0](\tau)
\]
for \( 0 \leq \tau < t \leq T \) with \( t - \tau \leq \mu \rho^2 \), where \( C \) and \( \mu \) are positive constants depending only on \( N \), \( \Omega \), \( p \) and \( r \).

**Proof.** Let \( x \in \overline{\Omega} \) and \( \zeta \) be as in Lemma 2.4. Let \( k \) be as in Lemma 2.4 and \( \epsilon > 0 \). Similarly to (3.18), we have
\[
\int_{\Omega(x,2\rho)} \zeta_{\epsilon}(y,s)^r \zeta_{\epsilon}^k \, dy \bigg|_{s=\tau}^{s=t} + \int_{\tau}^{t} \int_{\Omega(x,2\rho)} |\nabla \zeta_{\epsilon}^r z_{\epsilon}^k| \, dy \, ds
\]
\[
\leq C \rho^{-2} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} \zeta_{\epsilon}^r \, dy \, ds + C \int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} a(y,s) \zeta_{\epsilon}^r \zeta_{\epsilon}^k \, \sigma \, ds
\]
for all \( 0 < \tau < t \leq T \). This together with \( \zeta_{\epsilon} \), \( a \in C(\overline{\Omega} \times [\tau,T]) \cap L^\infty(\Omega \times (\tau,T)) \) implies that
\[
\sup_{x \in \Omega} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} |\nabla \zeta_{\epsilon}^r z_{\epsilon}^k| \, dy \, ds < \infty
\]
for \( 0 < \tau < t \leq T \). On the other hand, by the Hölder inequality and (3.3) we have
\[
\int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} a(y,s) \zeta_{\epsilon}^r \zeta_{\epsilon}^k \, \sigma \, ds \leq C \left( \int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, \sigma \, ds \right)^{\frac{1}{p+r}} \times \left( \int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} z_{\epsilon}^{p+r-1} \zeta_{\epsilon}^k \, \sigma \, ds \right)^{\frac{1}{p+r}}
\]
Let \( \Lambda \) and \( \mu \) be sufficiently small positive constants. Then, by Lemma 2.4, (3.10) and (3.25) we see that
\[
\int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, \sigma \, ds
\]
\[
\leq M \sup_{x \in \Omega} \int_{\tau}^{t} \int_{\partial\Omega(x,\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, \sigma \, ds
\]
\[
\leq C \Lambda \frac{\rho^{-1}}{r} \left\{ \Psi_{r,p}[v](\tau) + \Psi_{r,p}[w](\tau) \right\} \leq C \Lambda \frac{\rho^{-1}}{r} \frac{\rho^{-1}}{r} + N
\]
for all \( 0 < \tau < t \leq T \) with \( t - \tau \leq \mu \rho^2 \). Similarly, by Lemma 2.4 we obtain
\[
\int_{\tau}^{t} \int_{\partial\Omega(x,2\rho)} z_{\epsilon}^{p+r-1} \zeta_{\epsilon}^k \, \sigma \, ds \leq C \left( \rho^{\frac{r}{r-1} - N} \Psi_{r,p}[\zeta_{\epsilon}](t) \right)^{\frac{1}{r}} \times \left[ \sup_{x \in \Omega} \int_{\tau}^{t} \int_{\Omega(x,\rho)} |\nabla \zeta_{\epsilon}|^{\frac{r}{r-1}} \, dy \, ds + \rho^{-2}(t - \tau) \Psi_{r,p}[\zeta_{\epsilon}](\tau) \right]
\]
for all $0 < \tau < t \leq T$ with $t - \tau \leq \mu \rho^2$. Then we deduce from (3.29)–(3.31) that

$$\int_{\tau}^{t} \int_{\partial \Omega(x, 2\rho)} a(y, t) z_{\epsilon}^r \zeta^k \, d\sigma ds \leq C \Lambda^{\frac{r-1}{p}} (\Psi_{r, \rho}[z_{\epsilon}](t))^{\frac{r-1}{p}}$$

$$\times \left[ \sup_{x \in \Omega} \int_{\tau}^{t} \int_{\Omega(x, \rho)} |\nabla (z_{\epsilon})^2 \, dyds + \rho^{-2}(t - \tau) \Psi_{r, \rho}[z_{\epsilon}](t) \right]^{\frac{r}{p+r-1}}$$

(3.32)

for all $0 < \tau < t \leq T$ with $t - \tau \leq \mu \rho^2$. Then, taking sufficiently small constants $\Lambda$ and $\mu$ if necessary, we obtain

$$\Psi_{r, \rho}[z_{\epsilon}](t) \leq 4 M \Psi_{r, \rho}[z_{\epsilon}](\tau)$$

for all $0 < \tau < t \leq T$ with $t - \tau \leq \mu \rho^2$. This implies (3.26), and the proof is complete.

Now we are ready to complete the proof of Theorems 1.1 and 1.2 in the case $r > 1$.

**Proof of Theorem 1.1 in the case $r > 1$.** Let $\gamma_1$ be a sufficiently small positive constant and assume (1.8). Let $\{\varphi_n\}$ satisfy (1.14) and define $T_n^*$ and $T_n^{**}$ as in (1.18). Then it follows from (1.17) that

$$\rho^{\frac{r}{r-1}-N} \Psi_{r, \rho}[u_n](t) \leq 6 M \rho^{\frac{r}{r-1}-N} \Psi_{r, \rho}[u_n](0) \leq 6 M (2\gamma_1)^r$$

(3.33)

for all $0 \leq t \leq T_n^*$. Taking a sufficiently small $\gamma_1$ if necessary, by Lemma 3.2 (1.17) and (3.33), we can find a constant $\mu > 0$ such that

$$\Psi_{r, \rho}[u_n](t) \leq 5 M \Psi_{r, \rho}[u_n](0) < 6 M \Psi_{r, \rho}[u_n](0) \leq C \varphi^{\mu} \Rightarrow_{r, \rho}$$

(3.34)

for $0 \leq t \leq \min\{T_n^*, \mu \rho^2\}$. On the other hand, we apply Lemma 3.1 with $R_1 = \rho^2 / 2$, $R_2 = \rho$, $t_1 = t / 2$ and $t_2 = t / 4$ to obtain

$$\|u_n(t)\|_{L^{\infty}(\Omega(x, \rho^2))} \leq CD^{\frac{n+2}{2r}} \left( \int_{t/4}^{t} \int_{\Omega(x, \rho)} |u_n|^r \, dyds \right)^{1/r}$$

(3.35)

$$\int_{t/4}^{t} \int_{\Omega(x, \rho)} |\nabla u_n|^2 \, dyds \leq CD \int_{t/4}^{t} \int_{\Omega(x, \rho)} |u_n|^2 \, dyds,$$

(3.36)
for all \( x \in \Omega \) and \( t \in (0, T_n) \), where \( D = \| u_{n_l} \|_{L^\infty(\Omega \times (t/4, t))}^2 + \rho^{-2} + t^{-1} \). By (1.18), (3.34) and (3.35) we have
\[
\| u_n(t) \|_{L^\infty(\Omega)} \leq C t^{-\frac{N}{2p}} \| \varphi \|_{r, \rho} \leq C \gamma_1 t^{-\frac{1}{2(p-1)}} (\rho^{-2} t)^{-\frac{N}{2p} + \frac{1}{2(p-1)}},
\]
(3.37)
\[
\sup_{x \in \Omega} \int_{t/2}^t \int_{\Omega(x, \rho)} \| \nabla u_n \|^2 \, dyds \leq C \rho^N \| u_n \|_{L^\infty(\Omega \times (t/4, t))}^2 \leq C \rho^N t^{-\frac{N}{2}} \| \varphi \|_{r, \rho}^2,
\]
(3.38)
for all \( 0 < t \leq \min\{\rho^2, T_n^*, T_n^{**}\} \). Since \( r \geq N(p-1) \), taking sufficiently small \( \gamma_1 > 0 \) and \( \mu > 0 \) if necessary, by (3.37) we have
\[
(\rho^{-2} t)^{\frac{1}{2}} + t^{\frac{1}{2}} \| u_n(t) \|_{L^\infty(\Omega)}^{p-1} \leq \mu^{\frac{1}{2}} + (C \gamma_1)^{p-1} \mu^{-\frac{N(p-1)}{2} + \frac{1}{2}} \leq 1
\]
for \( 0 < t \leq \min\{\rho^2, T_n^*, T_n^{**}\} \). This implies that \( T_n > T_n^{**} > \min\{T_n^*, \rho^2\} \) for \( n = 1, 2, \ldots \). Then, by (3.34) we see that \( T_n^* > \rho^2 \) for \( n = 1, 2, \ldots \). Therefore, by (3.34), (3.37) and (3.38) we obtain
\[
\| u_n(t) \|_{L^\infty(\Omega)} \leq C t^{-\frac{N}{2p}} \| \varphi \|_{r, \rho},
\]
(3.39)
\[
\sup_{x \in \Omega} \int_{t/2}^t \int_{\Omega(x, \rho)} | \nabla u_n |^2 \, dyds \leq C \rho^N t^{-\frac{N}{2}} \| \varphi \|_{r, \rho}^2,
\]
(3.40)
\[
\sup_{0 < t < \rho^2} \| u_n(t) \|_{r, \rho} \leq C \| \varphi \|_{r, \rho},
\]
(3.41)
for \( 0 < t \leq \rho^2 \) and \( n = 1, 2, \ldots \).

Applying [8, Theorem 6.2] with the aid of (3.39), we see that \( u_n \) (\( n = 1, 2, \ldots \)) are uniformly bounded and equicontinuous on \( K \times [\tau, \rho^2] \) for any compact set \( K \subset \Omega \) and \( \tau \in (0, \rho^2) \). Then, by the Ascoli-Arzelà theorem and the diagonal argument we can find a subsequence \( \{u_{n_l}\} \) and a continuous function \( u \) in \( \Omega \times (0, \rho^2) \) such that
\[
\lim_{n_l \to \infty} \| u_{n_l} - u \|_{L^\infty(K \times [\tau, \rho^2])} = 0
\]
for any compact set \( K \subset \Omega \) and \( \tau \in (0, \rho^2) \). This together with (3.39) and (3.41) implies (1.9) and (1.10). Furthermore, by (3.40), taking a subsequence if necessary, we see that
\[
\lim_{n_l \to \infty} u_{n_l} = u \quad \text{weakly in } L^2([\tau, \rho^2] : W^{1,2}(\Omega \cap B(0, R)))
\]
for any \( R > 0 \) and \( 0 < \tau < \rho^2 \). This implies that \( u \) satisfies (1.6).

On the other hand, since \( u_n \) is a \( L^*_{uloc}(\Omega) \)-solution of (1.11) (see (1.16)), we see that
\[
u_n \in C([0, \rho^2] : L^*_{uloc,\rho}(\Omega)).
\]
Furthermore, by Lemma [3.3] and (3.33), taking a sufficiently small \( \gamma_1 \) if necessary, we have
\[
\sup_{0 < \tau < \rho^2} \| u_m(\tau) - u_n(\tau) \|_{r, \rho} \leq C \| u_m(0) - u_n(0) \|_{r, \rho}, \quad m, n = 1, 2, \ldots
\]
This means that \( \{u_n\} \) is a Cauchy sequence in \( C([0, \rho^2] : L^*_{uloc,\rho}(\Omega)) \), which implies
\[
u \in C([0, \rho^2] : L^*_{uloc,\rho}(\Omega)).
\]
Therefore we see that $u$ is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, \mu\rho^2]$ satisfying (1.9) and (1.10), and the proof of Theorem 1.1 for the case $r > 1$ is complete. □

**Proof of Theorem 1.2 in the case $r > 1$.** Let $v$ and $w$ be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T)$, where $T > 0$. Let $\gamma_2$ be a sufficiently small constant and assume (1.12). We can assume, without loss of generality, that $\rho \in (0, \rho_*/2)$. Since $v, w \in C([0, T] : L^r_{uloc}(\Omega))$, we can find a constant $T' \in (0, T)$ such that

$$\rho^{-1} \sup_{0 < t < T'} \sup_{\frac{1}{2} \leq \tau < T} \langle \tau \rangle^{\frac{1}{N}} \leq 2 \gamma_2.$$

(3.43)

Furthermore, for any $T'' \in (T', T)$, since $v, w \in L^\infty(\Omega \times (T', T''))$, we see that

$$\rho^{-1} \sup_{T' < t < T''} \sup_{\frac{1}{2} \leq \tau < T_1} \langle \tau \rangle^{\frac{1}{N}} \leq \gamma_2,$$

(3.44)

for some $\tilde{\rho} \in (0, \rho)$. Since $v(x, 0) \leq w(x, 0)$ for almost all $x \in \Omega$, by (3.43) and (3.44) we apply Lemma 3.3 to obtain

$$\sup_{0 < \tau < \min\{\mu \tilde{\rho}^2, T''\}} \|v(t) - w(t)\|_{0, \rho} \leq C\|v(0) - w(0)\|_{0, \rho} = 0$$

for some constant $\mu > 0$. This implies that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, \min\{\mu \tilde{\rho}^2, T''\})$. Repeating this argument, we see that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, T''')$. Finally, since $T'''$ is arbitrary, we see that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, T)$, and the proof is complete. □

### 4 Proof of Theorems 1.1 and 1.2 in the case $r = 1$

In this section we consider the case $1 < p < 1 + 1/N$ and $r = 1$, and complete the proof of Theorems 1.1 and 1.2. Furthermore, we prove Corollary 1.1. We use the same notation as in Section 3.

**Lemma 4.1** Assume the same conditions as in Theorem 1.1. Let $v$ and $w$ be $L^1_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T)$, where $0 < T < \infty$, such that

$$\|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \leq C_1 t^{-\frac{1}{2(p-1)}}, \quad 0 < t \leq T,$$

(4.1)

for some $C_1 > 0$. Then there exists a constant $C_2$ such that

$$\|v(t)\|_{L^\infty(\Omega)} \leq C_2 t^{-\frac{N}{2}} \Psi_{1, \rho}[v](t),$$

(4.2)

$$\|z_0(t)\|_{L^\infty(\Omega)} \leq C_2 t^{-\frac{N}{2}} \Psi_{1, \rho}[z_0](t),$$

(4.3)

for all $0 < t \leq \min\{T, \rho^2\}$ and $0 < \rho < \rho_*$. 

**Proof.** Similarly to (3.32), by Lemma 3.1 and (1.1) we have

$$\|z_0(t)\|_{L^\infty(\Omega \times \{\rho^2/2\})} \leq C \left[ \|v(t)\|_{L^\infty(\Omega \times \{t/4\})}^{2(p-1)} + \rho^{-2} + t^{-1} \right]^{\frac{N+2}{N-2}} \int_{t/4}^t \int_{\Omega(x, \rho)} |z_0(y, s)| dy ds$$

$$\leq C(1 + C_1^2 t)^{\frac{N+2}{2}} t^{-\frac{N}{2}} \Psi_{1, \rho}[z_0](t)$$
for all \( x \in \overline{\Omega} \) and \( 0 < t \leq \min\{T, \rho^2\} \). This implies (4.3). Furthermore, (4.2) follows from (4.3), and the proof is complete. \( \square \)

**Lemma 4.2** Assume the same conditions as in Theorem 1.1 and \( 1 < p < 1 + 1/N \). Let \( v \) and \( w \) be \( L^1_{uloc}(\Omega) \)-solutions of (1.1) in \( \Omega \times [0,T] \), where \( 0 < T < \infty \), and assume (4.1) for some constant \( C_1 > 0 \). Let \( 0 < \rho < \rho_* \) and \( \Lambda \) be such that

\[
\rho^{\frac{1}{1-N}} [\Psi_{\rho,v}(T) + \Psi_{\rho,w}(T)] \leq \Lambda. \tag{4.4}
\]

Then, for any \( \sigma \in (0,1) \) and \( \delta \in (0,1) \) with \( \sigma > \delta N/2 \), there exists a positive constant \( C_2 \) such that

\[
\limsup_{\epsilon \to 0} \sup_{x \in \overline{\Omega}} \int_0^t \int_{\Omega(x,\rho)} (\rho^{-2}s)^{\sigma} \frac{\left| \nabla z_e \right|^2}{z_e^{1-\delta}} dyds \leq C_2 \mu^{\sigma-\frac{\delta N}{2}} \rho^{-\delta N} \Psi_{\rho,v}(t)^{1+\delta} \tag{4.5}
\]

for \( 0 < t \leq \min\{T, \mu \rho^2\} \) and \( 0 < \mu \leq 1 \).

**Proof.** Let \( \sigma \in (0,1) \) and \( \delta \in (0,1) \) be such that \( \sigma > \delta N/2 \). Let \( x \in \overline{\Omega} \) and \( \zeta \) be as in Lemma 2.4. Similarly to (3.8), multiplying (3.1) by \((\rho^{-2})^\sigma z_e(x_t) \delta^\zeta(x)\) and integrating it on \( \Omega(x,2\rho) \times (\tau, t) \), we obtain

\[
\begin{align*}
\frac{\delta}{2} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} \frac{\left| \nabla z_e \right|^2}{z_e^{1-\delta}} dyds & \leq (\rho^{-2}\tau)^{\sigma} \int_\Omega z_e(y, \tau)^{1+\delta} dy \\
& + \frac{\sigma}{1+\delta} \rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma-1} z_e^{1+\delta} \delta^\zeta dyds \\
& + C \rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} z_e^{1+\delta} dyds + \int_\tau^t \int_{\partial\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} a(y,s) z_e^{1+\delta} \delta^\zeta ds
\end{align*}
\tag{4.6}
\]

for \( 0 < \tau < t \leq T \). On the other hand, it follows from Lemma 4.1, (3.3), (4.1) and (4.4) that

\[
\|a(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N(p-1)}{2}} \left[ \Psi_{\rho,v}(t)^{p-1} + \Psi_{\rho,w}(t)^{p-1} \right] \leq CA^{-1}(\rho^{-2}t)^{-\frac{N(p-1)}{2}} \tag{4.7}
\]

for all \( 0 < t \leq \min\{T, \rho^2\} \). Furthermore, by Lemma 2.3 we have

\[
\int_{\partial\Omega(x,2\rho)} z_e^{1+\delta} \delta^\zeta ds \leq \nu \int_{\Omega(x,2\rho)} \left| \nabla z_e \right|^2 dy + \frac{C}{\nu} \int_{\Omega(x,2\rho)} \zeta \delta^\zeta dy \\
\leq 2\nu \left( 1 + \frac{\delta}{2} \right)^2 \frac{\left| \nabla z_e \right|^2}{z_e^{1-\delta}} dy + 2\nu \int_{\Omega(x,2\rho)} z_e^{1+\delta} \delta^\zeta dy \\
+ \frac{C}{\nu} \int_{\Omega(x,2\rho)} z_e^{1+\delta} \delta^\zeta dy
\tag{4.8}
\]

for all \( 0 < t \leq T \) and \( \nu > 0 \). By (4.7) and (4.8) we obtain

\[
\begin{align*}
\int_\tau^t \int_{\partial\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} a(y,s) z_e^{1+\delta} \delta^\zeta ds & \leq \frac{\delta}{4} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} \frac{\left| \nabla z_e \right|^2}{z_e^{1-\delta}} dyds + C \rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma-1} z_e^{1+\delta} dyds \\
& + C \rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{-\frac{N(p-1)}{2}} z_e^{1+\delta} dyds
\end{align*}
\tag{4.9}
\]
for all $0 < \tau < t \leq \min\{T, \rho^2\}$. We deduce from (4.6)–(4.9) that
\[
\frac{\delta}{4} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} \frac{\nabla z_{\epsilon}}{\delta_{x-\delta}} \nabla z_{\epsilon}^2 dyds \leq \frac{(\rho^{-2})^{\sigma}}{1 + \delta} \int_{\Omega(x,2\rho)} z_{\epsilon}(y,\tau)^{1+\delta} dy + C \rho^{-2} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} [(\rho^{-2}s)^{\sigma-1} + (\rho^{-2}s)^{\sigma} + (\rho^{-2}s)^{\sigma-N(p-1)}] z_{\epsilon}^{1+\delta} dyds
\]
for all $0 < \tau < t \leq \min\{T, \rho^2\}$. Furthermore, by Lemmas 2.1 and 4.1 we have
\[
\sup_{x \in \Omega} \int_{\Omega(x,2\rho)} z_{\epsilon}(y,s)^{1+\delta} dy \leq 2M \sup_{x \in \Omega} \int_{\Omega(x,\rho)} z_{0}(y,s)^{1+\delta} dy + C \epsilon^{1+\delta} \rho^N
\]
\[
\leq 2M \|z_{0}(s)\|_{L^{\infty}(\Omega)}^{\delta} \Psi_{1,\rho}[z_{0}](t) + C \epsilon^{1+\delta} \rho^N
\]
\[
\leq C(\rho^{-2}s)^{\frac{-2N}{2}} \rho^{-\delta N} \Psi_{1,\rho}[z_{0}](t)^{1+\delta} + C \epsilon^{1+\delta} \rho^N
\]
for all $0 < s < t \leq \min\{T, \rho^2\}$. It follows from $N(p-1) < 1$ and $\sigma > \delta N/2$ that
\[
\sigma - N(p-1) - \frac{\delta N}{2} > \sigma - 1 - \frac{\delta N}{2} > -1.
\]
Then, by (4.10) and (4.11), passing to the limit as $\tau \to 0$ and $\epsilon \to 0$, we have
\[
\limsup_{\epsilon \to 0} \sup_{x \in \Omega} \int_{\tau}^{t} \int_{\Omega(x,\rho)} (\rho^{-2}s)^{\sigma} \frac{\nabla z_{\epsilon}}{\delta_{x-\delta}} \nabla z_{\epsilon}^2 dyds
\]
\[
\leq C \rho^{-2-\delta N} \Psi_{1,\rho}[z_{0}](t)^{1+\delta} \int_{0}^{t} (\rho^{-2}s)^{-\frac{2N}{2}} [(\rho^{-2}s)^{\sigma-1} + (\rho^{-2}s)^{\sigma} + (\rho^{-2}s)^{\sigma-N(p-1)}] ds
\]
\[
\leq C \rho^{-\delta N} \sigma^{-\frac{2N}{2}} \Psi_{1,\rho}[z_{0}](t)^{1+\delta}
\]
for all $0 < t \leq \min\{T, \mu \rho^2\}$ and $0 < \mu \leq 1$. This implies (4.15), and Lemma 4.2 follows.

Lemma 4.3 Assume the same conditions as in Lemma 4.2 with $\rho \in (0, \rho_*/2)$. Then there exists a constant $\mu \in (0,1)$ such that
\[
\Psi_{1,\rho}[z_{0}](t) \leq 2M \Psi_{1,\rho}[z_{0}](0), \quad 0 < t \leq \min\{T, \mu \rho^2\}.
\]

Proof. Let $x \in \overline{\Omega}$ and $\zeta$ be as in Lemma 2.4. Let $\sigma \in (0,1)$ and $\delta \in (0,1)$ be such that
\[
\frac{\delta N}{2} < \sigma < 1 - N(p-1) \quad \text{and} \quad p - 1 > \delta.
\]
By (3.1) we have
\[
\int_{\Omega(x,2\rho)} z_{0} \zeta^{2} dy \bigg|_{s=t}^{s=\tau} \leq 2 \int_{\tau}^{t} \int_{\Omega(x,2\rho)} |\nabla z_{0}| |\nabla \zeta| \zeta dyds + \int_{\tau}^{t} \int_{\partial \Omega(x,2\rho)} a(y,s) z_{0} \zeta^{2} d\sigma ds
\]
for $0 < \tau < t \leq T$. Furthermore, we have
\[
2 \int_{\tau}^{t} \int_{\Omega(x,2\rho)} |\nabla z_{0}| |\nabla \zeta| \zeta dyds \leq \nu \limsup_{\epsilon \to 0} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} \frac{|\nabla z_{\epsilon}|^{2}}{|\delta_{x-\delta}|^{\delta}} \zeta dyds
\]
\[
+ C \nu^{-1} \rho^{-2} \int_{\tau}^{t} \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{-\sigma} z_{0}^{1-\delta} dyds
\]
(4.15)
for \( \nu > 0 \). On the other hand, by (4.3) and (4.7) we obtain

\[
\int_{\tau}^{t} \int_{\partial \Omega(x,2\rho)} a(y,s) z_0^2 \, d\sigma ds \leq \int_{\tau}^{t} \int_{\partial \Omega(x,2\rho)} a(y,s) z_0^2 \, d\sigma ds \\
\leq CA^{p-1} \rho^{-1} \int_{\tau}^{t} \int_{\partial \Omega(x,2\rho)} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} z_0^2 \, d\sigma ds \\
\leq C\rho^{-1} \int_{\tau}^{t} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} \int_{\Omega(x,2\rho)} ||\nabla z_0||^2 + 2z_0 \zeta \, dyds \\
\leq C\nu \int_{\tau}^{t} \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} \frac{|\nabla z_0|}{z_0^{1-\delta}} \zeta^2 \, dyds \\
+ C\rho^{-2} \nu^{-1} \int_{\tau}^{t} (\rho^{-2}s)^{-\sigma-N(p-1)} \int_{\Omega(x,2\rho)} z_0^{1-\delta} \, dyds \\
+ C\rho^{-2} \int_{\tau}^{t} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} \int_{\Omega(x,2\rho)} z_0 \, dyds
\]

(4.16)

for \( 0 < t \leq \min\{T, \rho^2\} \), \( \epsilon > 0 \) and \( \nu > 0 \). Then it follows from Lemma 2.1 and (4.14)–(4.16) that

\[
\sup_{x \in \Omega} \int_{\Omega(x,\rho)} z_0(y,t) \, dy \leq M \sup_{x \in \Omega} \int_{\Omega(x,\rho)} z_0(y,0) \, dy \\
+ C\nu \limsup_{\epsilon \to 0} \sup_{x \in \Omega} \int_{0}^{t} \int_{\Omega(x,2\rho)} (\rho^{-2}s)^{\sigma} \frac{|\nabla z_0|}{z_0^{1-\delta}} \zeta^2 \, dyds \\
+ C\nu^{-1} \rho^{-2} \sup_{x \in \Omega} \int_{0}^{t} \int_{\Omega(x,\rho)} (\rho^{-2}s)^{-\sigma-N(p-1)} \int_{\Omega(x,\rho)} z_0^{1-\delta} \, dyds \\
+ C\rho^{-2} \nu^{-1} \sup_{x \in \Omega} \int_{0}^{t} (\rho^{-2}s)^{-\sigma-N(p-1)} \int_{\Omega(x,\rho)} z_0 \, dyds \\
+ C\rho^{-2} \sup_{x \in \Omega} \int_{0}^{t} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} \int_{\Omega(x,\rho)} z_0 \, dyds
\]

(4.17)

for \( 0 < t \leq \min\{T, \rho^2\} \) and \( \nu > 0 \). Furthermore, by the H"older inequality we have

\[
\sup_{x \in \Omega} \int_{\Omega(x,\rho)} z_0(y,t)^{1-\delta} \, dy \leq C\rho^{\delta N} \Psi_{1,\rho}[z_0](t)^{1-\delta}, \quad t > 0.
\]

(4.18)

Then we deduce from (4.5), (4.17) and (4.18) that

\[
\Psi_{1,\rho}[z_0](t) \leq M\Psi_{1,\rho}[z_0](0) + C\rho^{\delta \rho^{-2}} \rho^{-\delta N} \Psi_{1,\rho}[z_0](t)^{1+\delta} \\
+ C\nu^{-1} \rho^{\delta N} \Psi_{1,\rho}[z_0](t)^{1-\delta} \rho^{-2} \int_{0}^{t} [(\rho^{-2}s)^{-\sigma} + (\rho^{-2}s)^{-\sigma-N(p-1)}] \, ds \\
+ C\rho^{-2} \Psi_{1,\rho}[z_0](t) \int_{0}^{t} (\rho^{-2}s)^{-\frac{N(n-1)}{2}} \, ds
\]

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Theorem 1.1 in the case \( r \leq 1 \). Then, by the same argument as in the proof for the case \( r > 1 \), we can find a positive constant \( \mu \in (0, 1) \) such that

\[
\Psi_{1,\rho}[z_0](t) \leq M\Psi_{1,\rho}[z_0](0) + C(\mu^{\sigma - \frac{\delta N}{2}} + \mu^{1 - \sigma - N(p-1)} + \mu^{1 - \frac{N(p-1)}{2}})\Psi_{1,\rho}[z_0](t)
\]

\[
\leq M\Psi_{1,\rho}[z_0](0) + \frac{1}{2}\Psi_{1,\rho}[z_0](t)
\]

for \( 0 < t \leq \min\{T, \mu \rho^2\} \). This implies (4.12), and Lemma 4.3 follows. \( \square \)

Now we are ready to prove Theorem 1.1 in the case \( r = 1 \).

**Proof of Theorem 1.1 in the case** \( r = 1 \). It suffices to consider the case \( 1 < p < 1 + 1/N \). Let \( \gamma_1 \) be a sufficiently small positive constant and assume (1.3). Let \( \{\varphi_n\} \) satisfy (1.14) and define \( T_1^* \) and \( T_1^{**} \) as in (1.18). Then it follows from (1.17) that

\[
2T_1^{**}N\gamma_1 < \rho^\frac{1}{p-1} - N\Psi_{1,\rho}[u_n](t) \leq 6M\rho^\frac{1}{p-1} - N\Psi_{1,\rho}[u_n](0) \leq 12M\gamma_1
\]

(4.19)

for all \( 0 \leq t \leq T_1^* \). By Lemma 4.3 we have

\[
\|u_n(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2}}\Psi_{1,\rho}[u_n](t)
\]

(4.20)

for \( 0 < t \leq \min\{T_1^*, \mu \rho^2\} < T_1^* \) and \( n = 1, 2, \ldots \). Then, taking a sufficiently small \( \gamma_1 \) and applying Lemma 4.3 with \( v = u_n \) and \( w = 0 \), we can find a constant \( \mu \in (0, 1) \) such that

\[
\Psi_{1,\rho}[u_n](t) \leq 2M\Psi_{1,\rho}[u_n](0)
\]

(4.21)

for \( 0 < t \leq \min\{T_1^*, T_1^{**}, \mu \rho^2\} \) and \( n = 1, 2, \ldots \). This implies that \( \min\{T_1^{**}, \mu \rho^2\} < T_1^* \) for \( n = 1, 2, \ldots \). Furthermore, by (4.19)–(4.21), taking a sufficiently small \( \mu \) if necessary, we obtain

\[
(\rho^{-2}t)\frac{1}{2} + t^{\frac{1}{2}}\|u_n(t)\|_{L^\infty(\Omega)}^{p-1} \leq \mu^{\frac{1}{2}} + C(\rho^{-2}t)^{-\frac{N(p-1)}{2}} + \frac{1}{2}\gamma_1^{p-1}
\]

\[
\leq \mu^{\frac{1}{2}} + C\mu^{-\frac{N(p-1)}{2}} + \frac{1}{2}\gamma_1^{p-1} \leq 1
\]

for \( 0 < t \leq \min\{\mu \rho^2, T_1^{**}\} \). This yields \( T_1^{**} > \mu \rho^2 \) for \( n = 1, 2, \ldots \). Therefore, by (1.17), (4.20), and (4.21) we obtain

\[
\sup_{0 < \tau \leq t} \|u_n(\tau)\|_{1,\rho} = \Psi_{1,\rho}[u_n](t) \leq C\|\varphi\|_{1,\rho}, \quad \|u_n(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2}}\|\varphi\|_{1,\rho},
\]

(4.22)

for all \( 0 < t \leq \mu \rho^2 \) and \( n = 1, 2, \ldots \). Furthermore, applying Lemma 4.3 with \( v = u_m \) and \( w = u_n \) and taking a sufficiently small \( \mu \) if necessary, we see that

\[
\sup_{0 < \tau < \mu \rho^2} \|u_m - u_n\|_{1,\rho} \leq 2M\|u_m(0) - u_n(0)\|_{1,\rho}
\]

Then, by the same argument as in the proof for the case \( r > 1 \) we see that there exists a \( L^1_{loc}(\Omega) \)-solution \( u \) of (1.1) in \( \Omega \times [0, \mu \rho^2] \) satisfying (1.9) and (1.10). Thus the proof of Theorem 1.1 in the case \( r = 1 \) is complete. \( \square \)
Proof of Theorem 1.2 in the case \( r = 1 \). Let \( v \) and \( w \) be \( L^1_{\text{loc}}(\Omega) \)-solutions of (1.1) in \( \Omega \times [0, T) \), where \( 0 < T \leq \infty \). Assume (1.11). Then, for any \( 0 < T' < T \), we have

\[
\|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{2(p-1)}}, \quad 0 < t \leq T'.
\]

By Lemma 4.3 we can find a positive constant \( \mu \in (0, 1) \) such that

\[
\|(v(t) - w(t))_+\|_{1,\rho} \leq 2M\|(v(0) - w(0))_+\|_{1,\rho} = 0
\]

for all \( 0 < t \leq \min\{T', \mu\rho\} \). Repeating this argument, we see that

\[
\|(v(t) - w(t))_+\|_{1,\rho} \leq 0
\]

for all \( 0 < t \leq T' \). Since \( T' \) is arbitrary, we deduce that \( v(x, t) \leq w(x, t) \) in \( \Omega \times (0, T) \). Thus Theorem 1.2 in the case \( r = 1 \) follows. \( \square \)

Proof of Corollary 1.1. Let \( p > 1 + 1/N \) and \( \varphi \in L^{N(p-1)}(\Omega) \). By (2.2) we can find \( \rho \in (0, \rho_*) \) such that

\[
\|\varphi\|_{N(p-1),\rho} \leq \gamma_1,
\]

where \( \gamma_1 \) is the constant given in Theorem 1.1. Then assertion (i) follows from Theorem 1.1. Furthermore, if \( \rho_* = \infty \) and \( \varphi \) satisfies (1.13), then assertion (i) of Theorem 1.1 holds for any \( \rho > 0 \). This implies assertion (ii), and Corollary 1.1 follows. \( \square \)

5 Applications

In this section, as an application of Theorem 1.1 we give lower estimates of the blow-up time and the blow-up rate for problem (1.1).

5.1 Blow-up time

Let \( T(\lambda \psi) \) be the blow-up time of the solution of (1.1) with the initial function \( \varphi = \lambda \psi \). In this subsection we study the behavior of \( T(\lambda \psi) \) as \( \lambda \to \infty \) or \( \lambda \to 0 \).

Theorem 5.1 Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a uniformly regular domain of class \( C^1 \). Let \( r \) satisfy

\[
N(p-1) < r \leq \infty \quad \text{if} \quad p \geq p_* \quad \text{and} \quad 1 \leq r \leq \infty \quad \text{if} \quad 1 < p < p_*.
\]

Then, for any \( \psi \in L^r_{\text{loc},\rho}(\Omega) \) with \( \rho > 0 \), there exists a positive constant \( C \) such that

\[
T(\lambda \psi) \geq \begin{cases} 
C(\lambda \|\psi\|_{r,\rho})^{-\frac{2r(p-1)}{r-N(p-1)}} & \text{if} \quad r < \infty, \\
C(\lambda \|\psi\|_{L^\infty(\Omega)})^{-2(p-1)} & \text{if} \quad r = \infty,
\end{cases}
\]

for all sufficiently large \( \lambda \).
Proof. Let $\gamma_1$ and $\mu$ be constants given in Theorem 1.1. If $r < \infty$, by Theorem 1.1 we see that

$$T(\lambda \psi) \geq \mu \left( \frac{\gamma_1}{\lambda \|\psi\|_{r, \rho}} \right)^{2(p-1)} \geq C(\lambda \|\psi\|_{r, \rho})^{-\frac{2r(p-1)}{r-N(p-1)}}$$

for all sufficiently large $\lambda$. If $r = \infty$, then

$$\|\lambda \psi\|_{N(p-1), \rho} \leq C \lambda \|\psi\|_{L^\infty(\Omega)} \rho^{\frac{1}{p-1}}.$$  

It follows from Theorem 1.1 that

$$T(\lambda \psi) \geq \mu \left( \frac{\gamma_1}{C \lambda \|\psi\|_{L^\infty(\Omega)}} \right)^{2(p-1)} \geq C(\lambda \|\psi\|_{L^\infty(\Omega)})^{-2(p-1)}$$

for all sufficiently large $\lambda$. Thus Theorem 5.1 follows. $\square$

Theorem 5.2 Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a uniformly regular domain of class $C^1$. Assume

$$\sup_{x \in \Omega} \|x|^\beta \psi(x)\| < \infty,$$  

where $0 \leq \beta < N$ if $1 < p < p_*$ and $0 \leq \beta < 1/(p-1)$ if $p \geq p_*$. Then there exists a positive constant $C_1$ such that

$$T(\lambda \psi) \geq C_1 \lambda^{-\frac{2(p-1)}{1-\beta(p-1)}}$$

for all sufficiently large $\lambda$. Furthermore, if $\Omega = \mathbb{R}^N_+$ and

$$\inf_{x \in \Omega(0, \delta)} \|x|^\beta \psi(x)\| > 0,$$  

for some $\delta > 0$, then there exists a positive constant $C_2$ such that

$$T(\lambda \psi) \leq C_2 \lambda^{-\frac{2(p-1)}{1-\beta(p-1)}}$$

for all sufficiently large $\lambda$.

Proof. In the case $1 < p < p_*$, let $r > 1$, $r > N(p-1)$ and $\beta < N/r$. In the case $1 < p < p_*$, let $r = 1$. It follows from (5.1) that $\rho^{\frac{1}{r-1}} \|\psi\|_{r, \rho} \leq C \rho^{\frac{1}{p-1}}$ for all sufficiently small $\rho > 0$. This together with Theorem 1.1 implies (5.2).

Assume (5.3). Let $v$ be a solution of

$$\begin{cases}
\partial_t v = \Delta v & \text{in } \mathbb{R}^N_+ \times (0, \infty), \\
\nabla v \cdot \nu(x) = 0 & \text{in } \partial \mathbb{R}^N_+ \times (0, \infty), \\
v(x, 0) = A|x|^{-\beta} \chi_{B(0, \delta)} & \text{in } \mathbb{R}^N_+,
\end{cases}$$

where $A$ is a positive constant to be chosen as $\psi(x) \geq v(x, 0)$ in $\mathbb{R}^N_+$. By [7, Lemma 2.1.2] we can find a constant $c_p$ depending only on $p$ such that

$$\lambda \|v(\cdot, 0, t)\|_{L^\infty(\mathbb{R}^{N-1})} \leq c_p t^{-\frac{1}{2(p-1)}}, \quad 0 < t < T(\lambda v(0)).$$  

(5.5)
On the other hand, since $T(\lambda \psi) \leq T(\lambda v(0))$ and
\[
\|v(\cdot, 0, t)\|_{L^\infty(\mathbb{R}^{N-1})} \geq C t^{-\frac{\beta}{2}}, \quad 0 < t \leq 1,
\]
we have
\[
\lambda T(\lambda \psi)^{\frac{1}{2(p-1)} - \frac{\beta}{2}} \leq \lambda T(\lambda v(0))^{\frac{1}{2(p-1)} - \frac{\beta}{2}} \leq C c_p,
\]
which implies (5.4). Thus Theorem 5.2 follows. \(\Box\)

**Remark 5.1** For the case $\Omega = (0, \infty)$, Fernández Bonder and Rossi [10] proved
\[
\lim_{\lambda \to \infty} \lambda^{2(p-1)} T(\lambda \psi) = T(\psi(0))
\]
provided that $\psi$ is bounded continuous and positive on $[0, \infty)$.

Motivated by [26], we consider the case $\Omega = \mathbb{R}^N$ and study the behavior of the blow-up time $T(\lambda \psi)$ as $\lambda \to 0$.

**Theorem 5.3** Let $\Omega = \mathbb{R}^N$ and assume
\[
\sup_{x \in \mathbb{R}^N} (1 + |x|^\beta) |\psi(x)| < \infty
\]
for some $\beta \geq 0$. Let $\lambda > 0$ and consider problem (1.1) with $\varphi = \lambda \psi$. Then there exists a positive constant $C_1$ such that
\[
T(\lambda \psi) \geq C_1 f(\lambda)
\]
(5.7)
for all sufficiently small $\lambda > 0$, where
\[
f(\lambda) := \begin{cases}
\lambda^{-\frac{2(p-1)}{1-\beta}} & \text{if } p \geq p_*, 0 \leq \beta < \frac{1}{p-1}, \\
\lambda^{-\frac{2(p-1)}{1-\beta}} & \text{if } 1 < p < p_*, 0 \leq \beta < N,
\end{cases}
\]
\[
(\lambda \log \lambda)^{-\frac{2(p-1)}{1-N(p-1)}} & \text{if } 1 < p < p_*, \quad \beta = N,
\]
\[
\lambda^{-\frac{2(p-1)}{1-N(p-1)}} & \text{if } 1 < p < p_*, \quad \beta > N.
\]
Furthermore, if
\[
\inf_{x \in \mathbb{R}^N} (1 + |x|^\beta) \psi(x) > 0,
\]
then there exists a positive constant $C_2$ such that
\[
T(\lambda \psi) \leq C_2 f(\lambda)
\]
(5.8)
for all sufficiently small $\lambda > 0$.

**Proof.** Consider the case $p \geq p_*$. Let $0 \leq \beta < 1/(p - 1), r > N(p - 1)$ and $\beta < N/r$. By (5.6) we have
\[
\rho^{-\frac{1}{p-1} - \frac{N}{r}} \|\lambda \psi\|_{r, \rho} \leq C \lambda \rho^{-\beta + \frac{1}{p-1}}.
\]

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for all sufficiently large $\rho$. Similarly, in the case $p < p_*$, it follows from (5.6) that

$$
\rho^{\frac{1}{p-1}-N} \|\psi\|_{1,\rho} \leq \begin{cases} 
C\lambda \rho^{\frac{-\beta + 1}{p-1}} & \text{if } 0 \leq \beta < N, \\
C\lambda \rho^{\frac{1}{p-1}-N} \log \rho & \text{if } \beta = N, \\
C\lambda \rho^{\frac{1}{p-1}-N} & \text{if } \beta > N,
\end{cases}
$$

for all sufficiently large $\rho$. Therefore, by Theorem 1.1 we obtain in the case $p \geq p_*$

$$
T(\psi) \geq C\lambda^{\frac{2}{p-1}} = C\lambda^{-\frac{2(p-1)}{1-\beta(p-1)}}
$$

and in the case $1 < p < p_*$

$$
T(\psi) \geq \begin{cases} 
C\lambda^{\frac{-2(p-1)}{1-\beta(p-1)}} & \text{if } 0 \leq \beta < N, \\
C(\lambda |\log \lambda|)^{-\frac{2(p-1)}{1-N(p-1)}} & \text{if } \beta = N, \\
C\lambda^{-\frac{2(p-1)}{1-N(p-1)}} & \text{if } \beta > N,
\end{cases}
$$

for all sufficiently small $\lambda > 0$. These imply (5.7).

Let $v$ be a solution of

$$
\begin{cases}
\partial_t v = \Delta v & \text{in } \mathbb{R}_+^N \times (0, \infty), \\
\nabla v \cdot \nu(x) = 0 & \text{in } \partial \mathbb{R}_+^N \times (0, \infty), \\
v(x, 0) = A(1 + |x|)^{-\beta} & \text{in } \mathbb{R}_+^N,
\end{cases}
$$

where $A$ is a positive constant to be chosen as $\psi(x) \geq v(x, 0)$ in $\mathbb{R}_+^N$. Since $T(\psi) \leq T(\psi(0))$ and

$$
\|v(\cdot, 0, t)\|_{L^\infty(\mathbb{R}_+^{N-1})} \geq \begin{cases} 
Ct^{\frac{\beta}{p}} & \text{if } 0 \leq \beta < N, \\
Ct^{-\frac{N}{2}} \log t & \text{if } \beta = N, \\
Ct^{-\frac{N}{2}} & \text{if } \beta > N,
\end{cases}
$$

for all sufficiently large $t$, by a similar argument as in the proof of (5.4) we obtain (5.8). Thus Theorem 5.3 follows.

### 5.2 Blow-up rate

Let $u$ be a solution of (1.1) in $\Omega \times [0, T)$, where $0 < T < \infty$, such that $u$ blows up at $t = T$. In this subsection, as a corollary of Theorem 1.1 we state a result on lower estimates of the blow-up rate of the solution $u$. Blow-up rate of positive solutions for problem (1.1) was first obtained by Fila and Quittner [12], where it was shown that

$$
\limsup_{t \to T} (T - t)^{\frac{1}{2(p-1)}} \|u(t)\|_{L^\infty(\Omega)} < \infty
$$

holds in the case where $\Omega$ is a ball, the initial function $\varphi$ is radially symmetric and satisfies some monotonicity assumptions. Subsequently, it was proved that (5.9) holds for positive solutions in the following cases:
• Ω is a bounded smooth domain, \((N-2)p < N\) and \(\partial_t u \geq 0\) in \(\Omega \times (0,T)\) (see [16], [18] and [21]);

• Ω is a bounded smooth domain and \(p \leq 1 + 1/N\) (see [20]);

• \(\Omega = \mathbb{R}^N_+\) and \((N-2)p < N\) (see [5]).

See [30] for sign changing solutions. On the other hand, for positive solutions, it was shown in [21] that

\[
\liminf_{t \to T} (T - t)^\frac{1}{2(p-1)} \|u(t)\|_{L^\infty(\Omega)} > 0
\]  

(5.10)

holds if \(\Omega\) is a bounded smooth domain (see also [16] and [18]).

We state a result on lower estimates of the blow-up rate of the solutions. Theorem 5.4 is a generalization of (5.10) and it holds without the boundedness of the domain \(\Omega\) and the positivity of the solutions.

**Theorem 5.4** Let \(N \geq 1\) and \(\Omega \subset \mathbb{R}^N\) be a uniformly regular domain of class \(C^1\). Let \(u\) be a solution of (1.1) blowing up at \(t = T < \infty\). Then

\[
\liminf_{t \to T} (T - t)^\frac{1}{2(p-1)} \|u(t)\|_{L^r(\Omega)} > 0,
\]  

(5.11)

where

\[
\begin{cases}
N(p-1) \leq r \leq \infty & \text{if } p > 1 + 1/N, \\
1 < r \leq \infty & \text{if } p = 1 + 1/N, \\
1 \leq r \leq \infty & \text{if } 1 < p < 1 + 1/N.
\end{cases}
\]

(5.12)

**Proof.** Let \(1 \leq r < \infty\) satisfy (5.12). By Theorem 1.1 we can find positive constants \(\gamma_1\) and \(\mu\) such that, if

\[
||u(T - t)||_{r,\rho} \leq \gamma_1 \rho^\frac{N}{r-1}
\]

for some \(\rho \in (0, \rho_*/2)\), then the solution \(u\) exists in \(\Omega \times (0, T - t + \mu \rho^2]\). Since the solution \(u\) blows up at \(t = T\), we can find a constant \(\delta > 0\) such that

\[
||u(T - t)||_{r,\rho(t)} > \gamma_1 \rho(t)^\frac{N}{r-1} \quad \text{for } \quad t \in (T - \delta, T),
\]  

(5.13)

where

\[
\rho(t) := \left(\frac{T - t}{\mu}\right)^\frac{1}{2}.
\]

This implies (5.11) in the case \(r < \infty\). Furthermore, by (5.13), for any \(t \in (T - \delta, T)\), there exist \(x(t) \in \Omega\) and \(y(t) \in \Omega(x(t), \rho(t))\) such that

\[
C \rho(t)^N u(y(t), t)^r \geq \int_{\Omega(x(t), \rho(t))} u(y, t)^r \, dy \geq \frac{\gamma_1}{2} \rho(t)^{N - \frac{r}{p-1}}.
\]

This yields (5.11) in the case \(r = \infty\), and Theorem 5.4 follows. □

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