Vacua and walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$

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Abstract

We construct walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$, by using the moduli matrix formalism and the simple roots of $SO(2N)$. Penetrable walls are observed in the nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$. 

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1 Introduction

The moduli matrix formalism which was developed in [1, 2], is one of powerful tools to study Bogomol’nyi–Prasad–Sommerfield (BPS) objects. The moduli matrices contain all the moduli parameters of BPS solutions. In [1, 2], moduli matrices of walls are constructed in a mass-deformed hyper-Kähler nonlinear sigma model whose target space is the cotangent bundle over the complex Grassmann manifold $T^*G_{N_F,N_C}$, which is the strong coupling limit of supersymmetric $U(N_C)$ gauge theory with $N_F > N_C$ [3]. It is also shown that the vacua and the walls are on the Kähler manifold.

In the moduli matrix formalism, walls are generated by operators, which are positive root generators of the system. Elementary wall operators are generators of simple roots [4]. Thus elementary walls and compressed walls can be labelled by simple roots and by linear combinations of simple roots respectively. We can also figure out structures of multiwalls from the linear combinations of simple roots.

Various BPS objects are constructed in moduli matrix formalism. Domain walls and domain wall webs are discussed in [5, 6, 7]. Other BPS objects and composite objects are discussed in [8, 9, 10, 11, 12, 13, 14].

Moduli matrices, which are constrained by a single quadratic constraint $SO(2N)$ or $Sp(N)$, are constructed for walls in [6] and for kink monopoles [14]. In [6], walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2, 3$ are discussed. It is known that there are isomorphisms

$$SO(4)/U(2) \simeq CP^1, \quad SO(6)/U(3) \simeq CP^3.$$ (1.1)

Therefore the nonlinear sigma models discussed in [6] are Abelian theories. In Abelian theories, the ordering of walls is absolute. In non-Abelian theories, however, walls can pass through each other. Penetrable walls are observed in nonlinear sigma models on the Grassmann manifold [2].

The purpose of this paper is to construct walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$, which are non-Abelian theories. We use the convention and part of the formalism used in [14]. In Section 2, we review the moduli matrix method discussed in [2, 6, 14]. In Section 3, we review the results of [6] with the $O(2N)$ invariant tensor, which is used in [14]. We also identify elementary walls with the simple roots of $SO(2N)$. In Section 4, we study elementary walls of nonlinear sigma models on $SO(2N)/U(N)$ with $N = 4, 5, 6, 7$, which are labelled by the simple roots of $SO(2N)$. In Section 5, we discuss the vacuum structure connected to the maximum number of elementary walls. In Section 6, we make some observations about walls of nonlinear sigma model on $SO(12)/U(6)$, which is the simplest nontrivial case. In Section 7, we summarize our result.

\footnote{This condition is required to define the Grassmann manifold $G_{N_F,N_C} = \frac{SU(N_F)}{SU(N_C) \times SU(N_F-N_C) \times U(1)}$.}
In this section, we review the formalism of [6, 14]. The mass-deformed Kähler nonlinear sigma model on $SO(2N)/U(N)$ in (2+1) dimensions [6] is obtained by dimensional reduction [16] of $N = 1$ (3+1)-dimensional Kähler nonlinear sigma model on $SO(2N)/U(N)$ [15]. As we are interested in solitonic objects, we consider only the bosonic part of the model. The Lagrangian in (2+1) dimensions is

$$
L = -|D_\mu \phi_i^a|^2 - i\phi_i^a M_j^i - i\Sigma^b_a \phi_b i^a|^2 + |F_a^i|^2 + (D_b^a \phi_i^a - D_a^a) + \left((F_0)_{ab}^i \phi_i^b J_{ij}^{Tj} + \phi_0^a F_b^i J_{ij}^{Tj} + (\phi_0)_{ab}^i J_{ij} F_b^T_{Tj} + (h.c)\right),
$$

where the Greek letter $\mu$ denotes three-dimensional spacetime indices and the covariant derivative is defined by $(D_\mu \phi_i^a) = \partial_\mu \phi_i^a - iA_{\mu a}^b \phi_b^i$. The indices $i, j(= 1, \cdots, 2N)$ are flavour indices and the indices $a, b(= 1, \cdots, N)$ are color indices. The last term (h.c) stands for the Hermitian conjugate. The $O(2N)$ invariant tensor $J$ is defined by

$$
J = \sigma^i \otimes I_N,
$$

where $I_N$ stands for the $N \times N$ identity matrix. In this basis, the $SO(2N)$ Cartan generators are

$$
H_n = \left( \begin{array}{c} h_n \\ -\overline{h}_n \end{array} \right), \ (n = 1, \cdots, N),
$$

with $N \times N$ matrix $h_n$ which has an only component 1 in $(n, n)$-element. The mass matrix is a linear combination of the Cartan generators. By defining vectors

$$
m = (m_1, m_2, \cdots, m_N), $$
$$
H = (H_1, H_2, \cdots, H_N),
$$

the mass matrix is formulated as

$$
M = m \cdot H.
$$

In the basis of $SO(2N)$ with the invariant tensor (2.2), the mass matrix (2.5) is

$$
M = \sigma_3 \otimes \text{diag}(m_1, m_2, \cdots, m_N).
$$

The constraints of the Lagrangian (2.1) are

$$
\phi_a^i \overline{\phi}_i^a - \delta_a^b = 0,
$$
$$
\phi_a^i J_{ij} \phi_b^{Tj} = 0, \ \text{(hermitian conjugate) = 0}.
$$
Eq.(2.7) is from the D-term constraint which defines the Grassmann manifold \( G_{2N,N} \) and eqs.(2.8) are from the F-term constraints which correspond to the (anti-)holomorphic embedding of \( SO(2N) \) manifold. By eliminating auxiliary fields, the potential term of the Lagrangian becomes

\[
V = |i\phi_a^j M_j^i - i\Sigma_a^b \phi_b^i|^2 + 4|\phi_0^a \phi_b^i|^2.
\] (2.9)

Therefore the vacuum conditions are

\[
\phi_a^j M_j^i - \Sigma_a^b \phi_b^i = 0,
\] (2.10)

\[
(\phi_0)^{ab} = 0.
\] (2.11)

\( \Sigma \) can be diagonalized as

\[
\Sigma = \text{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_N),
\] (2.12)

by a \( U(N) \) gauge transformation. Then the vacua are labelled by

\[
(\Sigma_1, \Sigma_2, \cdots, \Sigma_N) = \left( \pm m_1, \pm m_2, \cdots, \pm m_N \right).
\] (2.13)

Since the tensor (2.2) is invariant under \( O(2N) \) transformation, only the half of the vacua in (2.13) belong to a nonlinear sigma model and the other half belong to the other nonlinear sigma model, which is related by parity. Therefore the number of vacua of the model is \( 2^{N-1} \). This number is the Euler characteristic of the manifold on which the nonlinear sigma model lives [17, 18, 19, 20].

The BPS equation for wall solutions is derived from the Bogomol’nyi completion of the Hamiltonian. It is assumed that fields are static and all the fields depend only on \( x_1 \equiv x \) coordinate. It is also assumed that there is Poincaré invariance on the two-dimensional worldvolume of walls to set \( A_0 = A_2 = 0 \). The energy density along the \( x \)-direction is

\[
\mathcal{E} = \left( |(D\phi)^a_e|^2 + |\phi_a^j M_j^i - \Sigma_a^b \phi_b^i|^2 + 4|\phi_0^{ab} \phi_b^i|^2 \right)
\geq \pm T.
\] (2.14)

This is also constrained by (2.7) and (2.8). The tension of the wall is

\[
T = \partial_i (\phi_a^i M_j^i \phi_j^a).
\] (2.15)

We choose the upper sign for the BPS equation and the lower sign for the anti-BPS equation. The BPS equation is

\[
(D\phi)^a_e - (\phi_a^j M_j^i - \Sigma_a^b \phi_b^i) = 0.
\] (2.16)
By introducing complex matrix functions $S^b_a(x)$ and $f^i_b(x)$, which are defined by

$$
\Sigma^b_a - i A^b_a = (S^{-1})^b_a \partial S^b_a, \quad \phi^i_a = (S^{-1})^b_a f^i_b.
$$

(2.17)

the equation (2.16) is solved by

$$
\phi^i_a = (S^{-1})^b_a H^j_0 (e^{Mx})^i_j.
$$

(2.18)

The coefficient matrix $H_0$ is called moduli matrix. $\Sigma$, $A$ and $\phi$ are invariant under transformations

$$
S'^b_a = V^c_a S^b_c, \quad H'^j_0 = V^c_a H^i_j c
$$

where $V \in GL(N, \mathbb{C})$. The $V$ defines an equivalent class of $(S, H_0)$. This is called the worldvolume symmetry [1, 2]. The constraints (2.7) and (2.8) correspond to

$$
H^i_0 (e^{2Mx})^j_i H^j_0 = (S\bar{S})^b_a \equiv \Omega^b_a,
$$

(2.20)

$$
H^i_0 J_{ij} H^T_{ij} = 0, \quad \text{(hermitian conjugate) } 0,
$$

(2.21)

respectively. The worldvolume symmetry of $H_0$ in (2.19) defines the Grassmann manifold. The constraint (2.21) is a holomorphic embedding of $SO(2N)$. Therefore moduli matrices $H_0$’s parametrize $SO(2N)/U(N)$.

There are quantities that we can consider

$$
\text{tr} \Sigma = \frac{1}{2} \partial \ln \det \Omega,
$$

$$
T = \frac{1}{2} \partial^2 \ln \det \Omega.
$$

(2.22)

In [2], walls are algebraically constructed from elementary walls. In the nonlinear sigma model on the Grassmann manifold, two nearest vacua have the same color number and differ by one flavour number. The elementary wall interpolating two nearest vacua, vacuum $\langle A \rangle$ and vacuum $\langle B \rangle$, which are in the flavour $I$ and $I + 1$ in the same color is $H_{0\langle A\leftarrow B \rangle} = H_{0\langle A \rangle} e^{E_I(r)}$ where $E_I(r) \equiv e^r E_I, \quad (r \in \mathbb{C})$. The elementary wall operator $E_I$ is defined by

$$
[cM, E_I] = c(m_I - m_{I+1}) E_I = T_{\langle I\leftarrow I+1 \rangle} E_I,
$$

(2.23)

where $c$ is a constant, $M$ is the mass matrix and $E_I$ is an $N_f \times N_f$ square matrix generating an elementary wall. $T_{\langle I\leftarrow I+1 \rangle}$ is the tension of the wall. The $E_I$ has a nonzero component only in the $(I, I + 1)$-th element. However in the nonlinear sigma model on $SO(2N)/U(N)$, the definitions of the nearest vacua and the elementary wall are not valid due to the quadratic constraint (2.21). Moduli matrices, which are constrained by a single quadratic constraint
\(SO(2N)\) or \(Sp(N)\) are studied in [6, 14]. We review the methods for walls of nonlinear sigma models on \(SO(2N)/U(N)\).

Since the mass matrix is a linear combination of the Cartan generators as it is defined in (2.6), we can generalize (2.23) as
\[
 c[M, E_i] = c(m \cdot \alpha_i)E_i = T_iE_i. \tag{2.24}
\]
\(E_i\) are the elementary wall operators which are the positive root generators of the simple roots \(\alpha_i\) of \(SO(2N)\). The elementary wall \(H_{0(A \leftarrow B)} = H_{0(A)} e^{E_i(r)}\) where \(E_i(r) \equiv e^r E_i\), \((r \in \mathbb{C})\) can be labelled by the simple root \(\alpha_i\). The wall has tension \(T_i\). We can restrict ourselves to the case where \(m_1 > m_2 > \cdots > m_N\). Then the vector \(\underline{m}\) of (2.4) is a vector in the interior of the positive Weyl chamber,
\[
 \underline{m} \cdot \underline{\alpha_i} > 0. \tag{2.25}
\]
Elementary walls can be identified with simple roots [4]. The corresponding positive root generators are elementary wall operators. The formula (2.24) is valid in nonlinear sigma models on the Grassmann manifold as well.

We should only consider positive roots which correspond to the BPS solutions. Any linear combinations of positive roots and negative roots break supersymmetries. These linear combinations are called non-BPS sectors.

The simple roots of \(SO(2N)\) are
\[
\begin{align*}
\alpha_1 &= (1, -1, 0, \cdots, 0, 0, 0), \\
\alpha_2 &= (0, 1, -1, \cdots, 0, 0, 0), \\
&\cdots \\
\alpha_{N-1} &= (0, 0, 0, \cdots, 0, 1, -1), \\
\alpha_N &= (0, 0, 0, \cdots, 0, 1, 1). \tag{2.26}
\end{align*}
\]
These simple roots describe elementary walls of nonlinear sigma models on \(SO(2N)/U(N)\).
The corresponding elementary wall operators are

\[
E_i = \begin{pmatrix}
i + 1 & i + N \\
i & 1 \\
i + N + 1 & -1
\end{pmatrix}, \quad (i = 1, \ldots, N - 1),
\]

\[
E_N = \begin{pmatrix}
N - 1 & 2N \\
N & -1 \\
2N - 1 & 2N
\end{pmatrix}.
\] (2.27)

The elementary wall, which connects two nearest vacua \( \langle A \rangle \) and \( \langle B \rangle \) is defined by

\[
H_{0(\langle A \leftarrow B \rangle)} = H_{0(\langle A \rangle)} e^{E_a(r)},
\]

\[
E_a(r) \equiv e^r E_a, \quad (a = 1, \ldots, N),
\] (2.28)

where \( E_a \) is the elementary wall operator and \( r \) is a complex parameter ranging \( -\infty < \text{Re}(r) < \infty \). Elementary walls can be compressed to a single compressed wall. A compressed wall of level \( n \), which connects \( \langle A \rangle \) and \( \langle A' \rangle \) is defined by

\[
H_{0(\langle A \leftarrow A' \rangle)} = H_{0(\langle A \rangle)} e^{E_{a_1}(r)} e^{E_{a_2}(r_2)} \cdots e^{E_{a_n}(r_n)},
\]

\[
(a_i = 1, \ldots, N).
\] (2.29)

There can exist multiwalls which connect two vacua \( \langle A \rangle \) and \( \langle B \rangle \)

\[
H_{0(\langle A \leftarrow B \rangle)} = H_{0(\langle A \rangle)} e^{E_{a_1}(r_1)} e^{E_{a_2}(r_2)} \cdots e^{E_{a_n}(r_n)},
\] (2.30)

where parameters \( r_i \) \( (i = 1, 2, \cdots) \) are complex parameters ranging \( -\infty < \text{Re}(r_i) < \infty \). Elementary walls can pass through each other if

\[
[E_{a_i}, E_{a_j}] = 0.
\] (2.31)

These walls are called penetrable walls.
3 Review on $SO(2N)/U(N)$ with $N = 2, 3$

In this section, we review the moduli matrices of vacua and walls of the nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2, 3$ [6] by using the formalism of [14] and the simple roots of $SO(2N)$ (2.26), which are summarized in Section 2. The moduli matrices of the vacua are discussed in Appendix A. We label the moduli matrices of vacua in descending order as follows:

\[
\begin{align*}
(\Sigma_1, \Sigma_2, \ldots, \Sigma_{N-1}, \Sigma_N) &= (m_1, m_2, \ldots, m_{N-1}, m_N), \\
(\Sigma_1, \Sigma_2, \ldots, \Sigma_{N-1}, \Sigma_N) &= (m_1, m_2, \ldots, -m_{N-1}, -m_N), \\
&\vdots \\
(\Sigma_1, \Sigma_2, \ldots, \Sigma_{N-1}, \Sigma_N) &= (\pm m_1, -m_2, \ldots, -m_{N-1}, -m_N),
\end{align*}
\]

(3.1)

where the sign $\pm$ is $+$ for odd $N$ and $-$ for even $N$. $H_0(A)$ and $\langle A \rangle$ denote a vacuum. $H_0(A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_p)$ and $\langle A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_p \rangle$ denote a wall which connects vacua $\langle A_1 \rangle \leftarrow \langle A_2 \rangle \leftarrow \cdots \leftarrow \langle A_p \rangle$.

The vacua of the nonlinear sigma model on $SO(4)/U(2)$ (A.4) are

\[
\begin{align*}
H_0(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, m_2), \\
H_0(2) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2).
\end{align*}
\]

(3.2)

There is only one elementary wall operator

\[
E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(3.3)

The elementary wall is

\[
H_{0(1 \leftarrow 2)} = H_{0(1)} e^{E(r)} = \begin{pmatrix} 1 & 0 \\ 0 & e^r \end{pmatrix}.
\]

(3.4)

This is the only wall of the nonlinear sigma model on $SO(4)/U(2)$. 

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The vacua of the nonlinear sigma model on $SO(6)/U(3)$ (A.8) are

$$H_0^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, m_3),$$

$$H_0^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, -m_3),$$

$$H_0^{(3)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, -m_3),$$

$$H_0^{(4)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, m_3).$$

(3.5)

The positive root generators of $SO(6)$, which are the elementary wall operators of $SO(6)/U(3)$ are

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (3.6)$$
The elementary walls are

\[ H_{0(1\leftarrow 2)} = H_{0(1)} e^{E_3(r_1)} = \begin{pmatrix} 1 & 0 & e^{r_1} \\ 1 & 0 & -e^{r_1} \\ 0 & -e^{r_1} & 1 \end{pmatrix}, \]

\[ H_{0(2\leftarrow 3)} = H_{0(2)} e^{E_1(r_1)} = \begin{pmatrix} 1 & e^{r_1} & 0 \\ 0 & 0 & e^{r_1} \\ 0 & -e^{r_1} & 1 \end{pmatrix}, \]

\[ H_{0(3\leftarrow 4)} = H_{0(3)} e^{E_2(r_1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -e^{r_1} \\ 0 & e^{r_1} & 1 \end{pmatrix}. \] (3.7)

There are two compressed walls of level one. These are generated by following operators

\[ \tilde{E}_4 = [E_3, E_1], \quad \tilde{E}_5 = [E_1, E_2]. \] (3.8)

We have used \( \sim \) to distinguish the operators from elementary wall operators. The walls are

\[ H_{0(1\leftarrow 3)} = H_{0(1)} e^{\tilde{E}_4(r_1)}, \quad H_{0(2\leftarrow 4)} = H_{0(2)} e^{\tilde{E}_5(r_1)}. \] (3.9)

There is only one compressed wall of level two, which is generated by

\[ \tilde{E}_6 = [E_3, \tilde{E}_5] = [\tilde{E}_4, E_2]. \] (3.10)

The moduli matrix for the wall is

\[ H_{0(1\leftarrow 4)} = H_{0(1)} e^{\tilde{E}_6(r_1)}. \] (3.11)

There are four double walls

\[ H_{0(1\leftarrow 2\leftarrow 3)} = H_{0(1\leftarrow 2)} e^{E_1(r_2)} = \begin{pmatrix} 1 & e^{r_2} & 0 & e^{r_1} \\ 1 & 1 & e^{r_1+r_2} & -e^{r_1} \\ 0 & e^{r_1+r_2} & 0 & e^{r_1} \\ 0 & 0 & -e^{r_2} & 1 \end{pmatrix}, \]

\[ H_{0(2\leftarrow 3\leftarrow 4)} = H_{0(2\leftarrow 3)} e^{E_2(r_2)} = \begin{pmatrix} 1 & e^{r_2} & 0 & e^{r_1} \\ 0 & 0 & e^{r_1+r_2} & -e^{r_1} \\ 1 & -e^{r_1+r_2} & 0 & e^{r_1} \\ 1 & 1 & -e^{r_1} & 0 \end{pmatrix}, \]

\[ H_{0(1\leftarrow 2\leftarrow 4)} = H_{0(1\leftarrow 2)} e^{\tilde{E}_5(r_2)} = \begin{pmatrix} 1 & e^{r_2} & 0 & e^{r_1} \\ 1 & 0 & -e^{r_1+r_2} & 0 \\ e^{r_1+r_2} & 0 & 0 & e^{r_1} \\ e^{r_1} & 1 & e^{r_1} & 0 \end{pmatrix}, \]

\[ H_{0(1\leftarrow 3\leftarrow 4)} = H_{0(1\leftarrow 3)} e^{E_2(r_2)} = \begin{pmatrix} 1 & e^{r_2} & 0 & e^{r_1} \\ 1 & 0 & -e^{r_1+r_2} & 0 \\ e^{r_1+r_2} & 0 & 0 & e^{r_1} \\ e^{r_1} & 1 & e^{r_1} & 0 \end{pmatrix}. \] (3.12)
and one triple wall

\[ H_{0(1\leftarrow 2\leftarrow 3\leftarrow 4)} = H_{0(1\leftarrow 2\leftarrow 3)} e^{E_2(r_3)} = \begin{pmatrix} 1 & e^{r_2} & 0 \\ e^{r_2} & 1 & -e^{r_1+r_3} \\ 0 & e^{r_1+r_2} & -e^{r_1} \end{pmatrix} \] \quad (3.13) \]

We investigate the walls by using positive roots. The \(SO(6)\) Cartan generators are

\[
H_1 = \text{diag}(1, 0, 0, -1, 0, 0),
H_2 = \text{diag}(0, 1, 0, 0, -1, 0),
H_3 = \text{diag}(0, 0, 1, 0, 0, -1),
\] \quad (3.14)

and the simple roots are

\[
\alpha_1 := (1, -1, 0),
\alpha_2 := (0, 1, -1),
\alpha_3 := (0, 1, 1). \] \quad (3.15)

The subscripts correspond to the subscripts of the elementary wall operators in (3.6). We indicate the root which connects two vacua \(\langle i \rangle \leftarrow \langle j \rangle\) as \(g_{(i\leftarrow j)}\). Then the elementary walls are

\[
g_{(1\leftarrow 2)} = \alpha_3,
g_{(2\leftarrow 3)} = \alpha_1,
g_{(3\leftarrow 4)} = \alpha_2, \] \quad (3.16)

and the roots of compressed walls are

\[
g_{(1\leftarrow 3)} = \alpha_3 + \alpha_1,
g_{(2\leftarrow 4)} = \alpha_1 + \alpha_2,
g_{(1\leftarrow 4)} = \alpha_1 + \alpha_2 + \alpha_3. \] \quad (3.17)

Diagrams of simple roots which connect the vacua of nonlinear sigma models on \(SO(4)/U(2)\) and \(SO(6)/U(3)\) are drawn in Fig.1.

Once we identify the moduli matrices of vacua and the simple roots which connect the vacua, we can construct all the operators and the moduli matrices of walls. From the next section we concentrate only on the moduli matrices of vacua and the simple roots.
Figure 1: (a) $SO(4)/U(2)$ (b) $SO(6)/U(3)$. The numbers indicate the subscript $i$’s of roots $\omega_i$.

4. $SO(2N)/U(N)$ with $N = 4, 5, 6, 7$

As discussed in Section 1, nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$ are non-Abelian theories. Therefore we expect that some of walls are penetrable.

For $N = 4$ there are $8(=2^3)$ vacua. The moduli matrices of vacua are defined by

\begin{align}
H_{0(1)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (m_1, m_2, m_3, m_4), \\
H_{0(2)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (m_1, m_2, -m_3, -m_4), \\
H_{0(3)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (m_1, -m_2, m_3, -m_4), \\
H_{0(4)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (m_1, -m_2, -m_3, m_4), \\
H_{0(5)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (-m_1, m_2, m_3, -m_4), \\
H_{0(6)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (-m_1, m_2, -m_3, m_4), \\
H_{0(7)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (-m_1, -m_2, m_3, m_4), \\
H_{0(8)} & : (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = (-m_1, -m_2, -m_3, -m_4).
\end{align}

(4.1)

The simple roots of $SO(8)$ are

\begin{align}
\alpha_1 & := (1, -1, 0, 0), \\
\alpha_2 & := (0, 1, -1, 0), \\
\alpha_3 & := (0, 0, 1, -1), \\
\alpha_4 & := (0, 0, 1, 1).
\end{align}

(4.2)

Then the roots of elementary walls $g_{(i \leftarrow j)}$ are

\begin{align}
g_{(3 \leftarrow 4)} & = g_{(5 \leftarrow 6)} = \alpha_1, \\
g_{(2 \leftarrow 3)} & = g_{(6 \leftarrow 7)} = \alpha_2, \\
g_{(3 \leftarrow 5)} & = g_{(4 \leftarrow 6)} = \alpha_3, \\
g_{(1 \leftarrow 2)} & = g_{(7 \leftarrow 8)} = \alpha_4.
\end{align}

(4.3)

The roots of penetrable walls are orthogonal

\begin{align}
\alpha_i \cdot \alpha_j & = 0.
\end{align}

(4.4)
The vacua and the roots are depicted in Figure 2. A pair of orthogonal simple roots in this diagram makes a parallelogram. The roots $\alpha_1$ and $\alpha_3$ are orthogonal. Therefore elementary

![Diagram](image)

Figure 2: (a) Vacua of $SO(8)/U(4)$. The numbers indicate the labels of vacua. (b) Elementary walls of $SO(8)/U(4)$. The numbers indicate the subscript $i$'s of simple roots $\alpha_i$, ($i = 1, \ldots, 4$).

walls $\langle 3 \leftrightarrow 5 \rangle$ and $\langle 5 \leftrightarrow 6 \rangle$ are penetrable. Elementary walls $\langle 3 \leftrightarrow 4 \rangle$ and $\langle 4 \leftrightarrow 6 \rangle$ are also penetrable. The observable quantities $\text{tr}\Sigma$ and $T$ (2.22) of double wall $\langle 3 \leftrightarrow 5 \leftrightarrow 6 \rangle$ are plotted in Figure 3.

Double wall $\langle 2 \leftrightarrow 3 \leftrightarrow 5 \rangle$ can be compressed. The moduli matrix is

$$
H_{0(2\leftarrow3\leftarrow5)} = H_{0(2)} e^{E_2(r_1)} e^{E_1(r_2)}
= \begin{pmatrix}
1 & e^{r_2} & 0 & 0 \\
1 & e^{r_1} & e^{r_1+r_2} & -e^{r_1} \\
0 & e^{r_1} & 1 & 1
\end{pmatrix}.
$$

Under the worldvolume symmetry transformation, $H_{0(2\leftarrow3\leftarrow5)}$ can be transformed to

$$
H_{0(2\leftarrow3\leftarrow5)} \rightarrow \begin{pmatrix}
1 & -e^{r_2} \\
1 & 1 \\
1 & e^{r_1+r_2}
\end{pmatrix} \begin{pmatrix}
1 & e^{r_2} & 0 & 0 \\
1 & e^{r_1} & e^{r_1+r_2} & -e^{r_1} \\
0 & e^{r_1} & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & -e^{r_1+r_2} \\
1 & e^{r_1} \\
0 & e^{r_1+r_2} & -e^{r_1} & 1
\end{pmatrix}.
$$

(4.6)
Figure 3: Double wall $\langle 3 \leftrightarrow 5 \leftrightarrow 6 \rangle$ in $SO(8)/U(4)$, which consists of two penetrable walls. $m_1 = 70$, $m_2 = 50$, $m_3 = 30$, $m_4 = 20$. (a)$r_1 = 80$, $r_2 = 100$ (b)$r_1 = 200$, $r_2 = 100$ (c)$r_1 = 250$, $r_2 = 100$.

In the limit where $r_1 + r_2 = r$ (finite) and $r_1 \to -\infty$, double wall $H_{0(2\leftrightarrow 3\leftrightarrow 5)}$ becomes a compressed wall of level one $H_{0(2\leftrightarrow 5)} = H_{0(2)} e^{|E_2,E_1|r}$.

In $SO(10)/U(5)$, there are $16(= 2^4)$ vacua. The five simple roots of $SO(10)$ are

$$\alpha_1 := (1, -1, 0, 0, 0),$$
$$\alpha_2 := (0, 1, -1, 0, 0),$$
$$\alpha_3 := (0, 0, 1, -1, 0),$$
$$\alpha_4 := (0, 0, 0, 1, -1),$$
$$\alpha_5 := (0, 0, 0, 1, 1).$$

(4.7)
The roots of elementary walls are
\[
g(4 \leftarrow 5) = g(7 \leftarrow 8) = g(9 \leftarrow 10) = g(12 \leftarrow 13) = \Omega_1, \\
g(3 \leftarrow 4) = g(6 \leftarrow 7) = g(10 \leftarrow 11) = g(13 \leftarrow 14) = \Omega_2, \\
g(2 \leftarrow 3) = g(7 \leftarrow 9) = g(8 \leftarrow 10) = g(14 \leftarrow 15) = \Omega_3, \\
g(3 \leftarrow 6) = g(4 \leftarrow 7) = g(5 \leftarrow 8) = g(15 \leftarrow 16) = \Omega_4, \\
g(1 \leftarrow 2) = g(9 \leftarrow 12) = g(10 \leftarrow 13) = g(11 \leftarrow 14) = \Omega_5. \\
(4.8)
\]

The roots of elementary walls are depicted in Figure 4. As before a pair of facing sides of each parallelogram are the same and a pair of adjacent sides of each parallelogram are orthogonal.

![Diagram of SO(10)/U(5) vacua and elementary walls](image)

Figure 4: (a) Vacua of $SO(10)/U(5)$. The numbers indicate the labels of vacua. (b) Elementary walls of $SO(10)/U(5)$. The numbers indicate the subscript $i$ of simple roots $\alpha_i$, ($i = 1, \cdots, 5$). Since facing sides are the same, only one sides of each parallelogram are labelled.

The roots of elementary walls of $SO(12)/U(6)$ and $SO(14)/U(7)$ are depicted in Figure 5 and Figure 6. We present the same diagrams labelled by the vacua in Appendix B to avoid cluttering up pages.

### 5 Generalization

The vacua are parametrized by $(\Sigma_1, \cdots, \Sigma_N) = (\pm m_1, \cdots, \pm m_N)$ as mentioned previously. The half of the vacua which differ by even numbers of minus signs belong to one nonlinear sigma model and the other half of the vacua belong to the other nonlinear sigma model.
which is related by parity. Therefore there are $2^{N-1}$ vacua in the nonlinear sigma models on $SO(2N)/U(N)$.

From the diagrams for $N = 2, \cdots, 7$, we make the following observations:

- $N = 2$

\begin{align}
\langle 1 \rangle & \leftarrow 2, \\
2 & \leftarrow \langle 2 \rangle.
\end{align}  \quad (5.1)
\( N = 3 \)
\[
\begin{align*}
3 & \leftarrow \langle 2 \rangle \leftarrow 1 \leftarrow \langle 3 \rangle \leftarrow 2. \\
(5.2)
\end{align*}
\]

\( N = 4 \)
\[
\begin{align*}
4 & \leftarrow 2 \leftarrow \langle 3 \rangle \leftarrow \{1, 3\} \leftarrow \cdots \leftarrow \{1, 3\} \leftarrow \langle 6 \rangle \leftarrow 2 \leftarrow 4. \\
(5.3)
\end{align*}
\]

\( N = 5 \)
\[
\begin{align*}
5 & \leftarrow \cdots \leftarrow \{2, 4\} \leftarrow \langle 6 \rangle \leftarrow \{1, 3\} \leftarrow \cdots \\
& \quad \cdots \leftarrow \{1, 3\} \leftarrow \langle 11 \rangle \leftarrow \{2, 5\} \leftarrow \cdots \leftarrow 4. \\
(5.4)
\end{align*}
\]

\( N = 6 \)
\[
\begin{align*}
6 & \leftarrow \cdots \leftarrow \{2, 4\} \leftarrow \langle 11 \rangle \leftarrow \{1, 3, 6\} \leftarrow \cdots \\
& \quad \cdots \leftarrow \{1, 3, 6\} \leftarrow \langle 22 \rangle \leftarrow \{2, 4\} \leftarrow \cdots \leftarrow 6. \\
(5.5)
\end{align*}
\]

\( N = 7 \)
\[
\begin{align*}
7 & \leftarrow \cdots \leftarrow \{2, 4, 7\} \leftarrow \langle 22 \rangle \leftarrow \{1, 3, 5\} \leftarrow \cdots \\
& \quad \cdots \leftarrow \{1, 3, 5\} \leftarrow \langle 43 \rangle \leftarrow \{2, 4, 6\} \leftarrow \cdots \leftarrow 6. \\
(5.6)
\end{align*}
\]

The vacuum structures which are connected to the maximum number of simple roots are as follows.

\( N = 4m - 2 \ (m \geq 2) \)
\[
\begin{align*}
& N(= 4m - 2) \leftarrow \cdots \\
& \cdots \leftarrow \left\{ \underbrace{2, 4, \cdots, 4m - 4}_{(2m-2)} \right\} \leftarrow \langle A \rangle \leftarrow \\
& \left\{ \underbrace{1, 3, \cdots, 4m - 5, 4m - 2}_{(2m-2)} \right\} \leftarrow \cdots \\
& \cdots \left\{ \underbrace{1, 3, \cdots, 4m - 5, 4m - 2}_{(2m-1)} \right\} \leftarrow \langle B \rangle \leftarrow \\
& \left\{ \underbrace{2, 4, \cdots, 4m - 4}_{(2m-1)} \right\} \leftarrow \cdots \\
& \cdots \leftarrow N(= 4m - 2). \\
(5.7)
\end{align*}
\]
• $N = 4m - 1$ ($m \geq 2$)

\[
N(= 4m - 1) \leftarrow \cdots \leftarrow \{2, 4, \cdots, 4m - 4, 4m - 1\} \leftarrow \langle A \rangle \leftarrow \underbrace{\{2, 4, \cdots, 4m - 4, 4m - 1\}}_{(2m-2)} \leftarrow \underbrace{\{1, 3, \cdots, 4m - 5, 4m - 3\}}_{(2m-1)} \leftarrow \cdots
\]

\[
\leftarrow \{1, 3, \cdots, 4m - 5, 4m - 3\} \leftarrow \langle B \rangle \leftarrow \underbrace{\{1, 3, \cdots, 4m - 5, 4m - 3\}}_{(2m-1)} \leftarrow \{2, 4, \cdots, 4m - 2\} \leftarrow \cdots \\
\leftarrow \underbrace{\{2, 4, \cdots, 4m - 2\}}_{(2m-1)} \leftarrow N - 1(= 4m - 2).
\] (5.8)

• $N = 4m$ ($m \geq 2$)

\[
N(= 4m) \leftarrow \cdots \leftarrow \{2, 4, \cdots, 4m - 2\} \leftarrow \langle A \rangle \leftarrow \{2, 4, \cdots, 4m - 2\} \leftarrow \underbrace{\{1, 3, \cdots, 4m - 1\}}_{(2m)} \leftarrow \cdots
\]

\[
\leftarrow \{1, 3, \cdots, 4m - 1\} \leftarrow \langle B \rangle \leftarrow \{1, 3, \cdots, 4m - 1\} \leftarrow \{2, 4, \cdots, 4m - 2\} \leftarrow \cdots \\
\leftarrow \underbrace{\{2, 4, \cdots, 4m - 2\}}_{(2m-1)} \leftarrow \cdots \leftarrow N(= 4m).
\] (5.9)
\[ N = 4m + 1 \quad (m \geq 2) \]

\[ N(= 4m + 1) \leftarrow \cdots \]
\[ \cdots \leftarrow \begin{cases} 2, 4, \ldots, 4m \\ (2m) \end{cases} \leftarrow \langle A \rangle \leftarrow \]
\[ \leftarrow \begin{cases} 1, 3, \ldots, 4m - 1 \\ (2m) \end{cases} \leftarrow \cdots \]
\[ \cdots \leftarrow \begin{cases} 1, 3, \ldots, 4m - 1 \\ (2m) \end{cases} \leftarrow \langle B \rangle \leftarrow \]
\[ \leftarrow \begin{cases} 2, 4, \ldots, 4m - 2, 4m + 1 \\ (2m-1) \end{cases} \leftarrow \cdots \]
\[ \cdots \leftarrow N - 1(= 4m) . \quad (5.10) \]

\langle A \rangle and \langle B \rangle are the vacua which are connected to the maximum number of elementary walls. \langle A \rangle denotes the vacuum near \langle 1 \rangle and \langle B \rangle denotes the vacuum near \langle 2^{N-1} \rangle. (5.7), (5.8), (5.9) and (5.10) are proved in Appendix C.

6 Walls of nonlinear sigma model on \( SO(12)/U(6) \)

We have studied the vacuum structures which are connected to the maximum number of elementary walls for general \( N \). Walls can be penetrable or compressed to a single wall. We discuss some physical consequences on \( SO(12)/U(6) \), which is the simplest nontrivial case. As it is shown previously, the vacua which are connected to the maximum number of elementary walls are \langle 11 \rangle and \langle 22 \rangle. The structure near \langle 11 \rangle is

\[ \langle 7 \rangle \leftarrow 2 \leftarrow \langle 11 \rangle \leftarrow \frac{1}{3} \leftarrow \langle 19 \rangle \]
\[ \langle 10 \rangle \leftarrow \frac{4}{3} \leftarrow \langle 11 \rangle \leftarrow \frac{3}{6} \leftarrow \langle 13 \rangle \]
\[ \leftarrow \langle 12 \rangle . \quad (6.1) \]

In (6.1), \( \alpha_4 \cdot \alpha_1 = 0 \). Therefore the elementary wall which interpolates \langle 10 \rangle and \langle 11 \rangle and the elementary wall which interpolates \langle 11 \rangle and \langle 19 \rangle are penetrable. The moduli matrix of
the double wall which connects \langle 10 \rangle, \langle 11 \rangle and \langle 19 \rangle is

\[ H_{0(10\leftarrow 11\leftarrow 19)} = H_{0(10)} e^{E_4(r_1)} e^{E_1(r_2)} = \begin{pmatrix}
1 & e^{r_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -e^{r_2} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & e^{r_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \].

(6.2)

The tension of the walls are plotted in Figure 7. Elementary wall \langle 10 \leftarrow 11 \rangle and elementary wall \langle 11 \leftarrow 19 \rangle pass through each other.

![Figure 7: Double wall ⟨10 ← 11 ← 19⟩ in SO(12)/U(6), which consists of two penetrable walls.](image)

Figure 7: Double wall \langle 10 \leftarrow 11 \leftarrow 19 \rangle in SO(12)/U(6), which consists of two penetrable walls. \( m_1 = 80, m_2 = 60, m_3 = 40, m_4 = 20, m_5 = 10, m_6 = 5 \). (a)\( r_1 = 50, r_2 = 20 \) (b)\( r_1 = 50, r_2 = 100 \) (c)\( r_1 = 50, r_2 = 150 \).

In (6.1), \( \alpha_4, \alpha_3 \neq 0 \). Therefore elementary wall \langle 10 \leftarrow 11 \rangle and elementary wall \langle 11 \leftarrow 13 \rangle can be compressed to a single wall. The moduli matrix of double wall \langle 10 \leftarrow 11 \leftarrow 13 \rangle is

\[ H_{0(10\leftarrow 11\leftarrow 13)} = H_{0(10)} e^{E_4(r_1)} e^{E_3(r_2)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & e^{r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & e^{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{r_1+r_2} & -e^{r_1} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

(6.3)
By using the worldvolume symmetry the moduli matrix can be transformed to

\[
H_{0\langle 10\leftarrow 11\leftarrow 13 \rangle} \rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -e^{r_2} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e^{r_1} & 0 & 0 & 0 & 0 & 0 \\
e^{r_1} & 0 & 0 & 0 & 0 & 0 \\
e^{r_1} & 0 & 0 & 0 & 0 & 0 \\
e^{r_1} & 0 & 0 & 0 & 0 & 0 \\
e^{r_1} & 0 & 0 & 0 & 0 & 0 \\
e^{r_1} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0 \\
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0 \\
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0 \\
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0 \\
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0 \\
e^{r_1+r_2} & -e^{r_1} & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In the limit where \( r_1 + r_2 = r \) (finite) and \( r_1 \to -\infty \), double wall \( \langle 10 \leftarrow 11 \leftarrow 13 \rangle \) becomes a compressed wall of level one. The tension of double wall \( \langle 10 \leftarrow 11 \leftarrow 13 \rangle \) is plotted in Figure 8. Two elementary walls are compressed to a single wall.

Figure 8: Double wall \( \langle 10 \leftarrow 11 \leftarrow 13 \rangle \) in \( SO(12)/U(6) \), which consists of two penetrable walls. \( m_1 = 80, m_2 = 60, m_3 = 40, m_4 = 20, m_5 = 10, m_6 = 5 \). (a) \( r_1 = 20, r_2 = 6 \) (b) \( r_1 = 10, r_2 = 16 \) (c) \( r_1 = -10, r_2 = 36 \).
7 Discussion

We have discussed the vacua and the walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N \geq 2$ by using the moduli matrix formalism and the simple roots of $SO(2N)$. We have observed that there are penetrable walls in the cases of $N > 3$, which are non-Abelian theories. We have discussed the vacuum structures which are connected to the maximum number of elementary walls and proved them by induction.

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Appendix A  Vacuum structure of nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2, 3$

The vacua of nonlinear sigma models are obtained from the vacuum condition (2.10) in Section 2.

The vacua of nonlinear sigma model on $SO(4)/U(2)$ are

$$
\Phi_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, m_2),
$$

$$
\Phi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2),
$$

$$
\Phi_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, -m_2),
$$

$$
\Phi_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, m_2). \quad (A.1)
$$

There are four solutions to the vacuum condition (2.10), but only the half of them are the vacua of a nonlinear sigma model on $SO(4)/U(2)$ and the other half are the vacua of the
other nonlinear sigma model which is related by a parity transformation. Let us define a rotation transformation $R$ and a parity transformation $P$ as follows:

$$R = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}. \tag{A.2}$$

The vacua are related by

$$\Phi_2 = \Phi_1 \cdot R, \quad \Phi_4 = \Phi_3 \cdot R,$$

$$\Phi_3 = \Phi_1 \cdot P, \quad \Phi_4 = \Phi_2 \cdot P. \tag{A.3}$$

The vacua $\Phi_1$ and $\Phi_2$ are on the same manifold. $\Phi_3$ and $\Phi_4$ are on the other manifold which are related by the parity transformation $P$. Therefore we can focus only on $\Phi_1$ and $\Phi_2$ without loss of generality. The corresponding moduli matrices of the vacua, which are related to the vacuum solution by (2.18) are

$$H_{0(1)} = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
\end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (m_1, m_2),$$

$$H_{0(2)} = \begin{pmatrix}
0 & 1 \\
0 & 1 \\
\end{pmatrix}, \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2). \tag{A.4}$$
The vacua of nonlinear sigma model on $SO(6)/U(3)$ are

\[
\Phi_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, m_3),
\]

\[
\Phi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, -m_3),
\]

\[
\Phi_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, -m_3),
\]

\[
\Phi_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, m_3),
\]

\[
\Phi_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, -m_3),
\]

\[
\Phi_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, m_3),
\]

\[
\Phi_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, m_3),
\]

\[
\Phi_8 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, -m_3).
\]

(A.5)

There are eight vacua, but only the half of them are the vacua of a nonlinear sigma model on $SO(6)/U(3)$. We can identify them by rotational transformations. We have three rotational
transformations

\[
R_1 = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix},
R_2 = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1
\end{pmatrix},
R_3 = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]  \hspace{1cm} (A.6)

The vacua are related by

\[\Phi_2 = \Phi_1 \cdot R_1, \quad \Phi_3 = \Phi_1 \cdot R_2, \quad \Phi_4 = \Phi_1 \cdot R_3.\]  \hspace{1cm} (A.7)

The solutions \(\Phi_1, \Phi_2, \Phi_3\) and \(\Phi_4\) are the vacua of a nonlinear sigma model on \(SO(6)/U(3)\) and the others are the vacua of a nonlinear sigma model on the other \(SO(6)/U(3)\) which are related by parity. The moduli matrices of the vacua are

\[
H_{0(1)} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Sigma_1, \Sigma_2, \Sigma_3 = (m_1, m_2, m_3),
\]

\[
H_{0(2)} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Sigma_1, \Sigma_2, \Sigma_3 = (m_1, -m_2, -m_3)
\]

\[
H_{0(3)} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Sigma_1, \Sigma_2, \Sigma_3 = (-m_1, m_2, -m_3)
\]

\[
H_{0(4)} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \Sigma_1, \Sigma_2, \Sigma_3 = (-m_1, -m_2, m_3).
\]  \hspace{1cm} (A.8)

We use the same method to compute moduli matrices of vacua for any \(N\). All the vacua are labelled by sets of \((\Sigma_1, \cdots, \Sigma_N) = (\pm m_1, \cdots, \pm m_N)\) of which the numbers of minus signs differ by even numbers.
Appendix B  Vacua

Figure 9: Vacua of $SO(12)/U(6)$. The numbers indicate the labels of vacua.

Figure 10: Vacua of $SO(14)/U(7)$. The numbers indicate the labels of vacua.

Appendix C  Vacuuum structures

We prove (5.7), (5.8), (5.9) and (5.10). The vacua which are connected to the maximum number of elementary walls can be found by decomposing the diagrams into two-dimensional
diagrams. The only rule is that simple roots which already have appeared in the previous diagrams should not be repeated.

In Figure 5 and Figure 9, the vacuum which is connected to the maximum number of simple roots near $\langle 1 \rangle$ is $\langle 11 \rangle$. The vacuum structure near $\langle 1 \rangle$ decomposes into two diagrams as it is shown in Figure 11. From this we can conclude that $\langle 11 \rangle$ is connected to the maximum number of elementary walls. Figure (b) is the same as Figure 1 (a). In Figure 5 and Figure 9, the vacuum which is connected to the maximum number of simple roots near $\langle 32 \rangle$ is $\langle 22 \rangle$. This can also be seen by decomposing the vacuum structure near $\langle 32 \rangle$. We get the same two-dimensional diagrams as Figure 11 by replacing $\langle 11 \rangle$ and $\langle 12 \rangle$ with $\langle 22 \rangle$ and $\langle 21 \rangle$. In this case the vacua on the left hand side are in the $x \to -\infty$ limit and the vacua on the right hand side are in the $x \to \infty$ limit. In the same manner, Figure 6 and Figure 10 decompose as it is shown in Figure 12. The vacua which are connected to the maximum number of simple roots are $\langle 22 \rangle$ and $\langle 43 \rangle$. Figure 12 (b) is the same as Figure 1 (b). $N = 8$ and $N = 9$ cases are depicted in Figure 13 and in Figure 14. The vacuum structures repeat the four diagrams in Figure 1, Figure 2 and Figure 4.

The vacuum structure near $\langle 1 \rangle$ decomposes into the diagrams in Figure 15 while the vacuum structure near $\langle 2^{N-1} \rangle$ decomposes into the diagrams in Figure 16. In both Figures, only the first two diagrams are shown. By repeating them all the cases fall into four categories. The vacuum which is connected to the maximum number of simple roots is circled in each diagram in Figure 17.

There are two vacua which are connected to the maximum number of elementary walls. $\langle A \rangle$ denotes the vacuum near $\langle 1 \rangle$ and $\langle B \rangle$ denotes the vacuum near $\langle 2^{N-1} \rangle$. The common parts of each vacuum structure is shown in Figure 18 and Figure 19. The rest of the vacuum structure can be derived from Figure 17. The vacuum structures near $\langle A \rangle$ and $\langle B \rangle$ are illustrated in Figure 20 and Figure 21 respectively.
Figure 12: $N = 7$ case. Diagrams near $\langle 1 \rangle (\langle 64 \rangle)$. Diagram (b) is the same as Figure 1 (b).

Figure 13: $N = 8$ case. Diagrams near $\langle 1 \rangle (\langle 128 \rangle)$. Diagram (b) is the same as Figure 2.
Figure 14: $N = 9$ case. Diagrams near $\langle 1 \rangle (\langle 256 \rangle )$. Diagram (b) is the same as Figure 4.

Figure 15: First two diagrams of the vacuum structure near $\langle 1 \rangle$ for $N$.

Figure 16: First two diagrams of the vacuum structure near $\langle 2^{N-1} \rangle$ for $N$. 
Figure 17: Four types of vacuum structures. The circle indicates the vacuum which is connected to the maximum number of simple roots.

Figure 18: Common part of the vacuum structure near $\langle A \rangle$.

Figure 19: Common part of the vacuum structure near $\langle B \rangle$. 
Figure 20: Rest part of the vacuum structure near $\langle A \rangle$.

Figure 21: Rest part of the vacuum structure near $\langle B \rangle$. 
Figure 18 and Figure 20 lead to the following:

- $N = 4m - 2, \ (m \geq 2)$

$$\left\{ \frac{2, 4, \cdots, 4m - 4}{2m - 2} \right\} \leftarrow \langle A \rangle \leftarrow \left\{ \frac{1, 3, \cdots, 4m - 5, 4m - 2}{2m - 2} \right\}$$  \hspace{1cm} (C.1)

- $N = 4m - 1, \ (m \geq 2)$

$$\left\{ \frac{2, 4, \cdots, 4m - 4, 4m - 1}{2m - 2} \right\} \leftarrow \langle A \rangle \leftarrow \left\{ \frac{1, 3, \cdots, 4m - 5, 4m - 3}{2m - 1} \right\}$$  \hspace{1cm} (C.2)

- $N = 4m, \ (m \geq 2)$

$$\left\{ \frac{2, 4, \cdots, 4m - 2}{2m - 1} \right\} \leftarrow \langle A \rangle \leftarrow \left\{ \frac{1, 3, \cdots, 4m - 1}{2m} \right\}$$  \hspace{1cm} (C.3)

- $N = 4m + 1, \ (m \geq 2)$

$$\left\{ \frac{2, 4, \cdots, 4m}{2m} \right\} \leftarrow \langle A \rangle \leftarrow \left\{ \frac{1, 3, \cdots, 4m - 1}{2m} \right\}$$  \hspace{1cm} (C.4)

Each case with $m = 2$ is shown in Figure 11, Figure 12, Figure 13 and Figure 14. Let us assume that (C.1), (C.2), (C.3) and (C.4) are true. Then these are true for $m' = m + 1$ since it corresponds to adding one more diagram in Figure 18. Therefore (C.1), (C.2), (C.3) and (C.4) are true.

Figure 19 and Figure 21 lead to the following:

- $N = 4m - 2, \ (m \geq 2)$

$$\left\{ \frac{1, 3, \cdots, 4m - 5, 4m - 2}{2m - 2} \right\} \leftarrow \langle B \rangle \leftarrow \left\{ \frac{2, 4, \cdots, 4m - 4}{2m - 2} \right\}$$  \hspace{1cm} (C.5)

- $N = 4m - 1, \ (m \geq 2)$

$$\left\{ \frac{1, 3, \cdots, 4m - 5, 4m - 3}{2m - 1} \right\} \leftarrow \langle B \rangle \leftarrow \left\{ \frac{2, 4, \cdots, 4m - 2}{2m - 1} \right\}$$  \hspace{1cm} (C.6)
• $N = 4m$, $(m \geq 2)$

$$\left\{ \frac{1, 3, \cdots, 4m - 1}{2m} \right\} \leftarrow \langle B \rangle \leftarrow \left\{ \frac{2, 4, \cdots, 4m - 2}{2m-1} \right\}$$

(C.7)

• $N = 4m + 1$, $(m \geq 2)$

$$\left\{ \frac{1, 3, \cdots, 4m - 1}{2m} \right\} \leftarrow \langle B \rangle \leftarrow \left\{ \frac{2, 4, \cdots, 4m - 2, 4m + 1}{2m-1} \right\}$$

(C.8)

Each case with $m = 2$ is shown in Figure 11, Figure 12, Figure 13 and Figure 14. Let us assume that (C.5), (C.6), (C.7) and (C.8) are true. Then these are true for $m' = m + 1$ since it corresponds to adding one more diagram in Figure 19. Therefore (C.5), (C.6), (C.7) and (C.8) are true.

The vacuum structure is

$$\overline{N} \leftarrow \cdots \leftarrow \langle A \rangle \leftarrow \cdots \leftarrow \langle B \rangle \leftarrow \cdots \leftarrow \overline{N}$$

(C.9)

for even $N$ and

$$\overline{N} \leftarrow \cdots \leftarrow \langle A \rangle \leftarrow \cdots \leftarrow \langle B \rangle \leftarrow \cdots \leftarrow \overline{N} - 1$$

(C.10)

for odd $N$.

Therefore (5.7), (5.8), (5.9) and (5.10) are proved.
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