(2-)PSEUDOCOMPACT PARATOPOLOGICAL GROUPS THAT ARE TOPOLOGICAL

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Abstract. We obtain sufficient conditions when (2-)pseudocompact paratopological group is topological. (2-)pseudocompact paratopological groups that are not topological are constructed. Our particular attention is devoted to periodic and topologically periodic groups. In addition, we prove that the product of a nonempty family of pseudocompact paratopological groups is pseudocompact.

1. Introduction

Under a paratopological group we understand a pair \((G, \tau)\) consisting of a group \(G\) and a topology \(\tau\) on \(G\) making the group operation \(\cdot : G \times G \to G\) of \(G\) continuous (such a topology \(\tau\) will be called a semigroup topology on \(G\)). If, in addition, the operation \((\cdot)^{-1} : G \to G\) of taking the inverse is continuous with respect to the topology \(\tau\), then \((G, \tau)\) is a topological group. A standard example of a paratopological group failing to be a topological group is the Sorgenfrey line, that is the real line endowed with the Sorgenfrey topology (generated by the base consisting of half-intervals \([a, b), a < b\)).

In this paper we search conditions when a paratopological group is topological. An interested reader can find known results on this subject in [AlaSan, Introduction], in the survey [Rav3, Section 5.1], and in the sequel.

2. Definitions

In this paper the word ”space” means ”topological space”.

We recall the following definitions. A space \(X\) is called

- sequentially compact if each sequence of \(X\) contains a convergent subsequence,
- \(\omega\)-bounded if each countable subset of \(X\) has the compact closure,
- totally countably compact if each sequence of \(X\) it contains a subsequence with the compact closure,
- countably compact at a subset \(A\) of \(X\) if each infinite subset \(B\) of \(A\) has an accumulation point \(x\) in the space \(X\) (the latter means that each neighborhood of \(x\) contains infinitely many points of the set \(B\)),
- countably compact if \(X\) is countably compact at itself,
- countably pracompact if \(X\) is countably compact at a dense subset of \(X\),
- pseudocompact if each locally finite family of nonempty open subsets of the space \(X\) is finite,
- finally compact if each open cover of \(X\) has a countable subcover.

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The following inclusions hold.
- Each compact space is ω-bounded.
- Each ω-bounded space is totally countably compact.
- Each totally countably compact space is countably compact.
- Each sequentially compact space is countably compact.
- Each countably compact space is countably pracompact.
- Each countably pracompact space is pseudocompact.

In these terms, a space X is compact if and only if X is countably compact and finally compact. A Tychonov space X is pseudocompact if and only if each continuous real-valued function on X is bounded.

A space X is quasiregular if each nonempty open subset A of X contains the closure of some nonempty open subset B of X.

A paratopological group G is precompact if for each neighborhood U of the unit of G there exists a finite subset F of G such that FU = G if and only if for each neighborhood U of the unit of G there exists a finite subset F of G such that UF = G. A sequence \{U_n : n \in \omega\} of subsets of a space X is non-increasing if \(U_n \supseteq U_{n+1}\) for each \(n \in \omega\). A (not necessarily \(T_0\)) paratopological group G is 2-pseudocompact if \(\bigcap U_n^{-1} \neq \emptyset\) for each non-increasing sequence \(\{U_n : n \in \omega\}\) of nonempty open subsets of G. Clearly, each countably compact paratopological group is 2-pseudocompact. A paratopological group G is left ω-precompact if for each neighborhood U of the unit of G there exists a countable subset F of G such that FU = G.

A paratopological group G is saturated if for each nonempty open subset U of G there exists a nonempty open subset V of G such that \(V^{-1} \subset U\).

Denote by \(\mathcal{B}^G\) the open base at the unit of a paratopological group G.

Suppose A is a subset of a group G. Denote by \(\langle A \rangle \subset G\) the subgroup generated by the set A.

3. (2-)pseudocompact paratopological groups that are topological

**Proposition 1.** Each totally countably compact paratopological group is a topological group.

**Proof.** Let G be such a group. Put \(B = \bigcap \mathcal{B}^G\). From Pontrjagin conditions \cite[Rav 1.1]{Rav} it follows that B is a semigroup. Put \(B' = \{e\} = \{x \in G : (\forall U \in \mathcal{B}^G)(xU \ni e)\} = \{x \in G : xB \ni e\} = B^{-1}\). Hence \(B'\) is a closed subsemigroup of the group G. Since \(B'\) is a compact semigroup, there is a minimal closed right ideal \(H \subset B'\). Let \(x\) be an arbitrary element of \(H\). Then \(x^2H = H \ni x\). Hence \(x^{-1} \in H\) and \(e \in H\). Consequently \(H = B'\) and \(xB' = B'\) for each element \(x \in B'\). Therefore \(B'\) is a group and \(B' = B\). Since \(B = \bigcap \mathcal{B}^G\), we see that \(g^{-1}Bg \subset B\) for each \(g \in G\). Hence \(B\) is a normal subgroup of the group G. Since B has the antidiscrete topology, B is a topological group.

Since the set \(B\) is closed, we see that the quotient group G/B is a T\(_1\)-space. Now let \(\{x_n : n \in \omega\}\) be an arbitrary sequence of the group G/B and let \(\pi : G \rightarrow G/B\) be the quotient map. Choose a sequence \(\{x'_n : n \in \omega\}\) of the group G such that \(\pi(x'_n) = x_n\) for each \(n \in \omega\). Since G is totally countable compact, we see that there is a subsequence \(A\) of \(\{x'_n : n \in \omega\}\) such that the closure \(\overline{A}\) is compact. Since the set \(\overline{A}\) is closed, \(\overline{A} = \overline{AB}\). Hence the closed compact set \(\pi(\overline{A})\) contains a subsequence of the sequence \(\{x_n : n \in \omega\}\). Therefore G/B is a T\(_1\) totally countably compact paratopological group. Consequently, from \cite[Cor 2.3]{AlaSan} it follows that G/B is a topological group. Hence both B and
$G/B$ are topological groups. Then by [Rav4, 1.3] or [Rav3, Pr. 5.3], $G$ is a topological group too.

Proposition 1 generalizes Corollary 2.3 in [AlaSan] and Lemma 5.4 in [Rav3]. To prove the next proposition, we need the following six lemmas.

Lemma 1. [Rav2, Pr. 1.7] Each saturated paratopological group is quasiregular.

Given a topological space $(X, \tau)$ Stone [Sto] and Katetov [Kat] consider the topology $\tau_r$ on $X$ generated by the base consisting of all canonically open sets of the space $(X, \tau)$. This topology is called the regularization of the topology $\tau$. If $(X, \tau)$ is a paratopological group then $(X, \tau_r)$ is a $T_3$ paratopological group [Rav, Ex. 1.9], [Rav3, p. 31], and [Rav3, p. 28]. We remind that a paratopological group $G$ is topologically periodic if for each $x \in G$ and a neighborhood $U \subset G$ of the unit there is a number $n \geq 1$ such that $x^n \in U$, see [BokGur].

Lemma 2. [RavRez] Suppose $(G, \tau)$ is a quasiregular paratopological group such that $(G, \tau_r)$ is a topological group; then $(G, \tau)$ is a topological group.

Lemma 3. Each topologically periodic Baire paratopological group is saturated.

Proof. In [AlaSan], Alas and Sanchis proved that each $T_0$ topologically periodic Baire paratopological group is saturated. This result answers my Question 2 from [Rav5]. Moreover, from their proof follows that each topologically periodic Baire paratopological group is saturated.

Lemma 4. Each 2-pseudocompact paratopological group is a Baire space.

Proof. In [AlaSan, Th. 2.2], Alas and Sanchis proved that each $T_0$ 2-pseudocompact paratopological group is a Baire space. Nevertheless their proof also proves this lemma.

Lemma 5. (also, see [AlaSan, Th. 2.5]) Each Baire left $\omega$-precompact paratopological group is saturated.

Lemma 6. Each paratopological group that is a dense $G_\delta$-subset of a $T_3$ pseudocompact space is a topological group.

Proof. In [ArkRez, Th. 1.7], Arkhangel’skii and Reznichenko proved that for each paratopological group $G$ such that $G$ is a dense $G_\delta$-subset of a regular pseudocompact space, it follows that $G$ is a topological group. Their proof with respective changes also proves this lemma.

Proposition 2. Suppose $G$ is a paratopological group; then the following conditions are equivalent:

1. The group $G$ is pseudocompact quasiregular;
2. The group $G$ is pseudocompact saturated;
3. The group $G$ is pseudocompact topologically periodic Baire;
4. The group $G$ is pseudocompact left $\omega$-precompact Baire;
5. The group $G$ is 2-pseudocompact saturated;
6. The group $G$ is 2-pseudocompact topologically periodic;
7. The group $G$ is 2-pseudocompact left $\omega$-precompact;
8. The group $G$ is a pseudocompact topological group.
Proof. It is easy to prove that Condition 8 implies all other conditions.

(1 ⇒ 8) The regularization \((G, \tau_r)\) of the group \((G, \tau)\) is a \(T_3\) pseudocompact paratopological group. By Lemma 6, \((G, \tau_r)\) is a topological group. Then by Lemma 2, \((G, \tau)\), is a topological group too.

(2 ⇒ 1) By Lemma 1, the group \(G\) is quasiregular.

(3 ⇒ 1) By Lemma 3, the group \(G\) is saturated. Then by Lemma 1, the group \(G\) is quasiregular.

(4 ⇒ 2) By Lemma 5, the group \(G\) is saturated.

(5 ⇒ 1) By Lemma 1, the group \(G\) is quasiregular. Now let \(\{U_n : n \in \omega\}\) be an infinite sequence of nonempty open subsets of \(G\). Choose a sequence \(\{V_n : n \in \omega\}\) of nonempty open subsets of the group \(G\) such that \(V_n^{-1} \subset U_n\) for each \(n \in \omega\). Since \(G\) is 2-pseudocompact, there is a point \(x \in \bigcap_{n \in \omega} \bigcup_{n \geq i} V_n^{-1}\). Each neighborhood of the point \(x\) intersects infinitely many members of the sequence \(\{U_n\}\). Thus the group \(G\) is pseudocompact.

(6 ⇒ 5) By Lemma 4, the group \(G\) is Baire. Then by Lemma 3, \(G\) is saturated.

(7 ⇒ 5) By Lemma 4, the group \(G\) is Baire space. Then by Lemma 5, \(G\) is saturated. \(\Box\)

Implication (1 ⇒ 8) generalizes Proposition 2 from [RavRez]. Implication (7 ⇒ 8) generalizes Proposition 2.6 from [AlaSan]. Implication (6 ⇒ 8) answers Question C from [AlaSan] and generalizes Theorem 3 from [BokGur].

A group has two basic operations: the multiplication and the inversion. The first operation is continuous on a paratopological group, but the second may be not continuous. But there are continuous operations on a paratopological group besides the multiplication. For instance, a power. Now suppose that there exists an open subset \(U\) of a paratopological group \(G\) such that the inversion on the set \(U\) coincides with a continuous operation. Then, since the inversion on the set \(U\) is continuous, it follows that the group \(G\) is topological. Let us consider a trivial application of this idea. Let \(G\) be a paratopological group of finite exponent. There is a number \(n\) such that the inversion on the group \(G\) coincides with the \(n\)-th power. Thus the group \(G\) is topological. Another applications are formulated in two next propositions.

Proposition 3. Each Baire periodic paratopological group is a topological group.

Proof. Let \(G\) be such a group. Put \(B' = \bigcap\{U^{-1} : U \in B^G\}\). Then \(B'\) is closed subset of \(G\). Since \(B'\) is a periodic semigroup, we see that \(B'\) is a group. For each \(n > 1\) we put \(G_n = \{x \in G : x^n \in B'\}\). Since the group \(G\) is Baire, there is a number \(n > 1\) such that the set \(G_n\) has the nonempty interior.

Therefore there are a point \(x \in G_n\) and a neighborhood \(U\) of the unit of \(G\) such that \((xu)^n \in B'\) for each \(u \in U\). Then \(V^{-1} \subset B'(xV)^{n-1} x\) for each subset \(V\) of \(U\). Let \(U_1\) be an arbitrary neighborhood of the unit of \(G\). Then \(x^n \in B'U_1\). By the continuity of the multiplication there is a neighborhood \(W \subset U\) of the unit such that \((xW)^{n-1} x \subset B'U_1\). Since \(B' \subset U_1\), we see that \(W^{-1} \subset B'U_1 \subset U_1^2\). \(\Box\)

Proposition 4. Each Hausdorff pseudocompact periodic paratopological group is a topological group.

Proof. Let \((G, \tau)\) be such a group and let \((G, \tau_r)\) be the regularization of the group \((G, \tau)\). Then \((G, \tau_r)\) is a regular pseudocompact periodic topological group. Let \(G\) be the Rajkov completion of the group \((G, \tau_r)\).
We claim that the group \( \hat{G} \) is periodic. Assume the converse. Suppose there exists a non-periodic element \( x \in \hat{G} \). Then for each positive integer \( n \) there exists a neighborhood \( V_n \ni x \) such that \( V_n^n \not\ni e \). Since the group \( G \) is \( G_\delta \)-dense in the group \( \hat{G} \), we see that there exists a point \( y \in G \cap \bigcap V_n \). Then there is a positive integer \( n \) such that \( y^n = e \). This contradiction proves the periodicity of the group \( \hat{G} \).

For each \( n > 1 \) we put \( \hat{G}_n = \{ x \in \hat{G} : x^n = e \} \). Since the group \( \hat{G} \) is Baire, there is a number \( n > 1 \) such that the set \( \hat{G}_n \) has the nonempty interior. Since the group \( G \) is dense in \( \hat{G} \), we see that there is a point \( x \in G \) and a neighborhood \( U \) of the unit of \( G \) such that \( (xu)^n = e \) for each \( u \in U \).

Let \( y \) be an arbitrary point of \( G \) and \( V \) be an arbitrary neighborhood of the point \( y^{-1} \). Then \( x^n y^{-1} = y^{-1} \in V \). By the continuity of the multiplication there is a neighborhood \( W \in \tau \) of the unit such that \( W \subset U \) and \( (xW)^{n-1} y^{-1} \in V \). Let \( z \in yW \) be an arbitrary point. Then \( (xy^{-1}z)^n = e \). Therefore \( z^{-1} = (xy^{-1}z)^{n-1}y^{-1} \in (xW)^{n-1}y^{-1} \subset V \). Thus the inversion on the group \( G \) is continuous.

T. Banakh build the next example. This result answers some my questions and shows that in general, Proposition 4 cannot be generalized for \( T_1 \) groups (even for countably pracompact and periodic groups). Moreover, in [AlaSan], Alas and Sanchis proved that each topologically periodic \( T_0 \) paratopological group is \( T_1 \). Example 1 shows that there is a \( T_1 \) periodic paratopological group \( G \) such that \( G \) is not Hausdorff.

**Example 1.** There exists a \( T_1 \) periodic paratopological group \( G \) such that each power of \( G \) is countably pracompact but \( G \) is not a topological group.

**Proof.** For each positive integer \( n \) let \( C_n \) be the set \( \{0, \ldots, n - 1\} \) endowed with the discrete topology and the binary operation \( ^n + ^n \) such that \( x + ^ny \equiv x + y (\text{mod} \ n) \) for each \( x, y \in C_n \). Let \( G = \bigoplus_{n=1}^{\infty} C_n \) be the direct sum. Let \( F \) be the family of all non-decreasing unbounded functions from \( \omega \setminus \{0\} \) to \( \omega \). For each \( f \in F \) put

\[
O_f = \{0\} \cup \{(x_n) \in G : (\exists m \geq 1)((\forall n > m)(x_n = 0) \& (0 < x_m < f(m)))\}.
\]

It is easy to check that the family \( \{O_f : f \in F\} \) is a base at the zero of a \( T_1 \) semigroup topology on \( G \).

For each positive integer \( m \geq 2 \) we put \( a^m = (a^m_n) \in G \), where \( a^m_n = 1 \) if \( n = m \); \( a^m_n = 0 \) in the opposite case. Let \( f, g \in F \) be arbitrary functions and let \( x \in G \) be an arbitrary element. There exists a number \( m \) such that \( f(m) \geq 2 \), \( g(m) \geq 2 \), and \( x_n = 0 \) for each \( n \geq m \). Then \( a^m \in O_g \) and \( x + a^m \in O_f \). Therefore \( x \in \overline{O_f} \). Hence \( \overline{O_f} = G \) for each \( f \in F \). Thus \( G \) is not Hausdorff. Since each two nonempty open subsets of \( G \) intersects, we see that each two nonempty open subsets of each power of \( G \) in the box topology intersects too. Therefore each power of \( G \) in the box topology is pseudocompact. Put \( A = \{0\} \cup \{a^m : m \geq 2\} \). Then \( A \) is compact and dense in \( G \). Let \( \kappa \) be an arbitrary cardinal. Then \( A^\kappa \) is compact and dense in \( G^\kappa \). Therefore the space \( G^\kappa \) is countably compact at a dense subset of \( G^\kappa \). Thus the space \( G^\kappa \) is countably pracompact. \( \square \)

Denote by TT the following axiomatic assumption: there is an infinite torsion-free abelian countably compact topological group without non-trivial convergent sequences. The first example of such a group constructed by M. Tkachenko under the Continuum Hypothesis [Tka]. Later, the Continuum Hypothesis weakened to the Martin Axiom for \( \sigma \)-centered posets by Tomita in [Tom2], for countable posets in [KosTomWat], and finally to the existence continuum many incomparable selective ultrafilters in [MadTom]. Yet,
the problem of the existence of a countably compact group without convergent sequences in ZFC seems to be open, see [DikSha].

The proof of [BanDimGut, Lemma 6.4] implies the following

Lemma 7. (TT) Let \( G \) be a free abelian group generated by the set \( c \). There exists a Hausdorff group topology on \( G \) such that for each countable subset \( M \) of the group \( G \) there exists an element \( \alpha \in \overline{M} \cap c \) such that \( M \subseteq \langle \alpha \rangle \).

Example 2. (TT) There exists a functionally Hausdorff countably compact free abelian paratopological group \((G, \sigma)\) such that \((G, \sigma)\) is not a topological group.

Proof. Let \( G \) be a free abelian group generated by the set \( c \). Lemma 7 implies that there exists a Hausdorff group topology \( \tau \) on the group \( G \) such that for each countably subset \( M \) of the group \( G \) there exists an element \( \alpha \in \overline{M} \cap c \) such that \( M \subseteq \langle \alpha \rangle \). Each element \( x \in G \) has a unique representation \( x = \sum_{\alpha \in A_x} n_\alpha \alpha \) such that the integer number \( n_\alpha \) is non-zero for each \( \alpha \in A_x \). Put \( S_0 = \{ x \in G : n_{\sup A_x} < 0 \} \) and \( S = S_0 \cup \{ 0 \} \). Clearly, \( S \) is a subsemigroup of the group \( G \). It is easy to check that the family \( U \cap S : U \in \tau, U \ni 0 \) is a base at the zero of a semigroup topology on \( G \). Denote this semigroup topology by \( \sigma \).

Let \( M \) be an arbitrary countable subset of the group \( G \). There exists an element \( \alpha \in \overline{M} \cap c \) such that \( M \subseteq \langle \alpha \rangle \). Since \( M \subseteq \alpha + S_0 \), we see that \( \alpha \in \overline{M} \). Hence \((G, \sigma)\) is a countably compact Hausdorff paratopological group.

Suppose \((G, \sigma)\) is a topological group; then there exists a neighborhood \( U \in \tau \) of the zero such that \( U \cap S \subseteq S \cap (\neg S) = \{ 0 \} \). Fix any element \( \alpha \in c \). Then \( -\alpha \in S \). Since the topological group \((G, \tau)\) is countably compact, there exists a number \( n < 0 \) such that \( n\alpha \in U \). Then \( 0 \neq n\alpha \in U \cap S \). This contradiction proves that \((G, \sigma)\) is not a topological group.

Example 2 negatively answers Problem 1 from [Gur] under TT. Since each countably compact left \( \omega \)-precompact paratopological group is a topological group [Rav3, p. 82], we see that Example 2 implies the negative answers to Question A from [AlaSan] and to Problem 2 from [Gur] under TT.

The following proposition answers Questions B from [AlaSan].

Proposition 5. Each 2-pseudocompact paratopological group of countable pseudocharacter is a topological group.

Proof. Let \( G \) be such a group. By induction we can build a sequence \( \{ V_i : i \in \omega \} \) of open neighborhoods of the unit \( e \) of \( G \) such that \( V_i^2 \subseteq V_i \) for each \( i \in \omega \) and \( \bigcap \{ V_i : i \in \omega \} = \{ e \} \). Suppose \( G \) is not a topological group; then there exists a neighborhood \( U \in B^G \) such that \( V_i \not\subseteq U^{-1} \subset (U^{-1})^2 \) for each \( i \in \omega \). Since the group \( G \) is 2-pseudocompact, there is a point \( x \in G \) such that \( xW^{-1} \cap (V_i \setminus U^{-1}) \neq \emptyset \) for each \( W \in B^G \) and each \( i > 0 \). Then \( xW^{-1} \cap (V_i \setminus U^{-1}) \neq \emptyset \) and \( Wx^{-1} \cap (V_i \setminus U^{-1}) \neq \emptyset \). Therefore \( x^{-1} \in V_i^{-1} \subseteq V_i^{-1} \). Hence \( x = e \). But \( e \in U^{-1} \) and \( U^{-1} \cap (V_i \setminus U^{-1}) = \emptyset \). This contradiction proves that \( G \) is a topological group.

Example 3 shows that counterparts of Proposition 5 and Implication \((7 \Rightarrow 1)\) of Proposition 2 do not hold for countably pracompact paratopological groups.

Example 3. There exists a functionally Hausdorff left \( \omega \)-precompact first countable paratopological group \( G \) such that each power of \( G \) is countably pracompact but \( G \) is not a topological group.
The following conditions are equivalent:

**Proposition 6.** Let $G$ be a countably compact paratopological group such that the closure of each cyclic subgroup of $G$ is a group. Then $G$ is a topological group.

**Proof.** Let $x$ be an arbitrary point of $G$. Put $H = \langle x \rangle$. Then $H$ is a separable countably compact paratopological group. Since each countably compact left $\omega$-precompact paratopological group is a topological group [Rav3, p. 82], we see that $H$ is a topological group. Therefore $H$ is a topologically periodic group. Hence $G$ is a topologically periodic group. By [BokGur], $H$ is a topological group.

4. Cone topologies

In this section $G$ is an abelian group and $S \ni 0$ is a subsemigroup of $G$. In this section, we consider an interplay between the algebraic properties of the semigroup $S$ and properties of two semigroup topologies generated by $S$ on the group $G$. The main aim of this section is Example 5 of two 2-pseudocompact paratopological groups which are not topological groups.

We need the following technical lemmas.

**Lemma 8.** If there is an element $a \in S$ such that $S \subset a - S$, then $S$ is a group.

**Lemma 9.** The following conditions are equivalent:

1. For each countable subset $C$ of $S$ there is an element $a \in S$ such that $C \subset a - S$.
2. For each countable subset $C$ of $S$ there is an element $a \in S$ such that $C \subset a - S$.
3. For each countable infinite subset \( C \) of \( S \) there are an element \( a \in S \) and an infinite subset \( B \) of \( C \) such that \( B \subset a - S \).

Proof. (\( 3 \Rightarrow 2 \)). Let \( C \) be an arbitrary countable subset of \( S \). If \( C \) is finite, then \( S \subset (\sum_{c \in C} c) - S \). Now suppose the set \( C \) is infinite. Let \( C = \{c_n : n \in \omega \} \). Put \( D = \{d_n : n \in \omega \} \), where \( d_n = \sum_{i=0}^{n} c_i \) for each \( n \in \omega \). If the set \( D \) if finite, then \( S \subset (\sum_{d \in D} d) - S \). Now suppose the set \( D \) is infinite. There are an element \( a \in S \) and an infinite subset \( B \) of \( D \) such that \( B \subset a - S \). Therefore for each number \( i \in \omega \) there exists a number \( n \in \omega \) such that \( i \leq n \) and \( d_n \in B \). Then \( a_i \in d_n - S \in a - S - S \in a - S \).

\( (2 \Rightarrow 1) \). Let \( \{c_n : n \in \omega \} \) be a countable subset of the group \( \overline{S} \). Fix sequences \( \{a_n : n \in \omega \} \) and \( \{b_n : n \in \omega \} \) of \( S \) such that \( c_n = b_n - a_n \) for each \( n \). There exist an element \( a \in S \) such that \( b_n \in a - S \) for each \( n \). Then \( c_n \in a - S \) for each \( n \).

4.1. Cone topology. The one-element family \( \{S\} \) satisfies Pontrjagin conditions (see \[Rav\] Pr. 1]). Therefore there is a semigroup topology on \( G \) with the base \( \{S\} \) at the unit. This topology is called the cone topology generated by the semigroup \( S \) on \( G \). Denote by \( G_S \), the group \( G \) endowed with this topology.

It is easy to check that the closure \( \overline{S} \) of the semigroup \( S \) is equal to \( S - S \) and \( \overline{S} = \langle S \rangle \).

Moreover, \( \overline{S} \) is clopen.

Proposition 7. The group \( G_S \) is \( T_0 \) if and only if \( S \cap (-S) = \{0\} \).

Proposition 8. The group \( G_S \) is \( T_1 \) if and only if \( S = \{0\} \).

Lemma 10. A subset \( K \) of \( G_S \) is compact if and only if there is a finite subset \( F \) of \( K \) such that \( F + S \supseteq K \).

Lemma 11. Let \( A \) be a subset of \( G_S \). Then \( \overline{A} = A - S \).

Proposition 9. The following conditions are equivalent:

1. The group \( G_S \) is compact
2. The group \( G_S \) is \( \omega \)-bounded.
3. The group \( G_S \) is totally countably compact.
4. The group \( G_S \) is precompact.
5. The semigroup \( S \) is a subgroup of \( G \) of finite index.

Proof. Implications (\( 1 \Rightarrow 4 \)), (\( 1 \Rightarrow 2 \)) and (\( 2 \Rightarrow 3 \)) are trivial.

\( (4 \Rightarrow 5) \). Choose a finite subset \( F \) of \( G \) such that \( F + S = G \). Since \( f + S \subset f + \overline{S} \) for every \( f \in F \), \( \overline{S} \subset (F \cap \overline{S}) + S \). Choose a number \( n \) and points \( x_1, \ldots, x_n, y_1, \ldots, y_n \in S \) such that \( F \cap \overline{S} = \{x_1 - y_1, \ldots, x_n - y_n\} \). Then \( \overline{S} = \bigcup x_i - y_i + S \subset \bigcup -y_i + S \). Put \( y = \sum y_i \). Then \( \overline{S} \subset -y + S \). Therefore there is an element \( z \in S \) such that \( -2y = -y + z \). Then \( z = -y \) and hence \( -y_i \in S \) for every \( i \). Therefore \( S = S + S \supseteq \overline{S} \). Consequently \( S \) is a group. Since \( F + S = G \), we see that the index \( |G : S| \) is finite.

\( (5 \Rightarrow 1) \). Since \( S \) is a subgroup of \( G \) of finite index, we see that there exists a finite subset \( F \) of \( G \) such that \( F + S = G \). For each point \( f \in F \) find set \( U_f \in \mathcal{U} \) such that \( f \in U_f \). Let \( f \in F \) be an arbitrary point. Since the set \( U_f \) is open, \( f + S \subset \bigcup U_f \). Therefore \( \{U_f : f \in F\} \) is a finite subcover of \( \mathcal{U} \) and \( \bigcup \{U_f : f \in F\} = G \).

\( (3 \Rightarrow 5) \). The set \( -S = \{0\} \) is compact. Therefore there is a finite subset \( F \) of \( S \) such that \( -F + S \supseteq -S \). Let \( F = \{f_1, \ldots, f_n\} \). Put \( f = \sum f_i \). Then \( -f + S \supseteq F + S \supseteq -S \). Moreover, \( -f + S \supseteq -f + S + S \supseteq S - S \). Hence there is an element \( g \in S \) such that \( -2f = -f + g \). Then \( g = -f \) and \( -f + S \supseteq S \). Therefore \( S = \overline{S} \). Suppose the index \( |G : S| \) is infinite. Then there is a countable set \( A \subset G \) that contains at most one point
of each coset of the group \( S \) in \( G \). Let \( B \) be an arbitrary infinite subset of \( A \). The set \( \mathcal{B} = B - S \) intersects an infinite number of the cosets. Therefore \( \mathcal{B} \neq F + S \) for each finite subset \( F \) of \( G \). This contradiction proves that the index \( |G : S| \) is finite. \( \square \)

**Proposition 10.** The following conditions are equivalent:

1. The group \( G_S \) is sequentially compact.
2. The group \( G_S \) is countably compact.
3. The group \( G_S \) is 2-pseudocompact.
4. The index \( |G : S| \) is finite and for each countable subset \( C \) of \( S \) there is an element \( a \in S \) such that \( C \subset a - S \).

Moreover, in this case each power of the space \( G_S \) is countably compact.

**Proof.** Implications (1 \( \Rightarrow \) 2) and (2 \( \Rightarrow \) 3) are trivial.

(3 \( \Rightarrow \) 4). Let \( C \) be an arbitrary countable infinite subset of \( S \). Let \( \{x_n : n \in \omega\} \) be an enumeration of elements of the set \( C \). For each \( n \in \omega \) put \( U_n = \bigcup_{i \geq n} x_i + S \). Then \( \{U_n : n \in \omega\} \) is a non-increasing sequence of nonempty open subsets of \( G_S \). Since the group \( G_S \) is 2-pseudocompact then there is a point \( b \in G \) such that \( (b + S) \cap -U_n \neq \emptyset \) for each \( n \in \omega \). Then for each \( n \in \omega \) there is a number \( i \geq n \) such that \( (b + S) \cap -(x_i - S) \neq \emptyset \) and hence \( -x_i \in b + S \). Let \( B \) be the set of all such elements \( x_i \). Then \( B \) is an infinite subset of \( -b - S \). Since \( B \subset S \), we see that \( -b \in B + S \subset S \). Now Lemma 9 implies Condition 4.

Suppose the index \( |G : S| \) is infinite. Then there is a countable infinite set \( C \subset G \) that contains at most one point of each coset of the group \( \mathcal{S}_G \) in \( G \). Let \( \{x_n : n \in \omega\} \) be an enumeration of the elements of the set \( C \). For each \( n \in \omega \) put \( U_n = \bigcup_{i \geq n} x_i + S \). Then \( \{U_n\} \) is a non-increasing sequence of nonempty open subsets of \( G_S \). But for each point \( x \in G \) an open set \( x + \mathcal{S}_G \) intersects only finitely many sets of the family \( \{-U_n : n \in \omega\} \). This contradiction proves that the index \( |G : S| \) is finite.

(4 \( \Rightarrow \) 1). Let \( \{c_n : n \in \omega\} \) be a sequence of elements of the group \( G_S \). There is a point \( x \in G \) such that the set \( I = \{n \in \omega : x_n \in x + \mathcal{S}_G\} \) is infinite. By Lemma 9 there is a point \( a \in S \) such that \( x - c_n \in a - S \) for each \( n \in I \). Thus the sequence \( \{c_n : n \in I\} \) converges to the point \( x - a \).

Now let \( \kappa \) be an arbitrary cardinal. Condition 1 and Lemma 9 imply that for each countable subset \( C \) of \( \mathcal{S}_G^\kappa \) there is a point \( a \in S^\kappa \) such that \( C \subset S^\kappa - a \). Consequently, the space \( \mathcal{S}_G^\kappa \) is sequentially compact. Since \( \mathcal{S}_G \) is a clopen subgroup of \( G_S \), we see that the space \( G_S \) is homeomorphic to \( \mathcal{S}_G \times D \), where \( D \) is a finite discrete space. Then a space \( G_S^\kappa \) is homeomorphic to the countably compact space \( \mathcal{S}_G^\kappa \times D^\kappa \). \( \square \)

The proof of the following lemma is straightforward.

**Lemma 12.** If \( X \) is a countably pracompact space and \( Y \) is a compact space then the space \( X \times Y \) is countably pracompact. \( \square \)

**Proposition 11.** The following conditions are equivalent:

1. The group \( G_S \) is countably pracompact.
2. The group \( G_S \) is pseudocompact.
3. The index \( |G : S| \) is finite.

Moreover, in this case each power of the space \( G_S \) is countably pracompact.

**Proof.** Implication (1 \( \Rightarrow \) 2) is trivial.

(2 \( \Rightarrow \) 3). Suppose the index \( |G : S| \) is infinite. Then there is a countable infinite set \( C \subset G \) that contains at most one point of each coset of the group \( \mathcal{S}_G \) in \( G \). Let \( \{x_n : n \in \omega\} \)
be an enumeration of the elements of the set \( C \). For each \( n \in \omega \) put \( U_n = \bigcup_{i \geq n} x_i + S \).
Then \( \{U_n\} \) is a non-increasing sequence of nonempty open subsets of \( G_S \). But for each point \( x \in G \) an open set \( x + S \) intersects only finitely many sets of the family \( \{U_n : n \in \omega\} \).
This contradiction proves that the index \( |G : S| \) is finite.

(3 \( \Rightarrow \) 1). Choose a finite set \( A_0 \subset G \) such that the intersection of \( A_0 \) with each coset of the group \( S \) in \( G \) is a singleton. Put \( A = A_0 + S \). Let \( x \in G \) be an arbitrary point. Then there is an element \( a \in A_0 \) such that \( x \in a + S \). Hence \( (x + S) \cap (a + S) \neq \emptyset \). Therefore \( (x + S) \cap A \neq \emptyset \). Thus \( A \) is dense in \( G_S \).

Let \( C \) be an arbitrary countable infinite subset of \( A \). There is a point \( a \in A_0 \) such that the set \( B = C \cap (a + S) \) is finite. But since \( C \subset A_0 + S \), we see that \( B \subset a + S \). Therefore \( a \) is an accumulation point of the set \( C \).

Now let \( \kappa \) be an arbitrary cardinal. Since the set \( S \) is dense in \( \overline{S} \), we see that the set \( S^\kappa \) is dense in \( \overline{S}^\kappa \). Since each neighborhood of the zero of the group \( \overline{S}^\kappa \) contains the set \( S^\kappa \), the space \( \overline{S}^\kappa \) is countably compact at \( S^\kappa \). Hence the space \( \overline{S}^\kappa \) is countably pracoompact. Since \( \overline{S} \) is a clopen subgroup of \( G_S \), the space \( G_S^\kappa \) is homeomorphic to \( \overline{S}^\kappa \times D^\kappa \), where \( D \) is a finite discrete space. Then a space \( G_S^\kappa \) is homeomorphic to \( \overline{S}^\kappa \times D^\kappa \) by Lemma \( \text{[12]} \) the space \( \overline{S}^\kappa \times D^\kappa \) is countably pracoompact.

\[ \text{Proposition 12.} \quad \text{The group } G_S \text{ is finally compact if and only if } G_S \text{ is left } \omega\text{-precompact.} \]

\[ \text{4.2. Cone* topology.} \quad \text{The family } B_S = \{\{0\} \cup (x + S) : x \in S\} \text{ satisfies Pontryagin conditions (see [Rav], Pr. 1]). Therefore there is a semigroup topology on } G \text{ with the base } B_S \text{ at the unit. This topology is called the a cone* topology generated by the semigroup } S \text{ and on } G. \text{ Denote by } G_S^* \text{, the group } G \text{ endowed with this topology.}

\text{It is easy to check that the closure } \overline{S} \text{ of the semigroup } S \text{ is equal to } S - S \text{ and } \overline{S} = \langle S \rangle. \text{ Moreover, } \overline{S} \text{ is clopen.}

\[ \text{Proposition 13.} \quad \text{The group } G_S^* \text{ is Hausdorff if and only if } S = \{0\}. \]

\[ \text{Proposition 14.} \quad \text{The following conditions are equivalent:}

1. The group \( G_S^* \) is \( T_1 \).
2. The group \( G_S^* \) is \( T_0 \).
3. \( S = \{0\} \) or \( S \) is not a group.
4. \( S = \{0\} \) or \( \bigcap \{x + S : x \in S\} = \emptyset \).

\text{Proof. Implications } (1 \Rightarrow 4), (4 \Rightarrow 2), \text{ and } (2 \Rightarrow 3) \text{ are trivial. We show that } (3 \Rightarrow 1). \text{ Suppose there is a point } y \in \bigcap \{x + S : x \in S\}. \text{ Then } y \in 2y + S. \text{ Hence there is an element } z \in S \text{ such that } y = 2y + z. \text{ Therefore } -y = z \in S. \text{ Now let } x \text{ be an arbitrary element of } S. \text{ There is an element } s \in S \text{ such that } y = x + s. \text{ Then } -x = -(x + s) + s \in S. \text{ Therefore } S \text{ is a group. Thus } y = 0. \]

\[ \text{Proposition 15.} \quad \text{The following conditions are equivalent:}

1. The group \( G_S^* \) is compact.
2. The group \( G_S^* \) is \( \omega \)-bounded.
3. The group \( G_S^* \) is totally countably compact.
4. The group \( G_S^* \) is sequentially compact.
5. The group \( G_S^* \) is countably compact.
6. The group \( G_S^* \) is precompact.
7. The semigroup \( S \) is a subgroup of \( G \) of finite index. \]
Proof. Implications (1 \Rightarrow 2), (2 \Rightarrow 3), (3 \Rightarrow 5), (5 \Rightarrow 7), (4 \Rightarrow 5) and (1 \Rightarrow 6) are trivial. From (7) follows that \( G^*_S = G_S \). Therefore by Proposition 9, (7 \Rightarrow 1) and by Proposition 10 (7 \Rightarrow 4). From (6) follows that the group \( G_S \) is precompact. Therefore by Proposition 11 (6 \Rightarrow 7).

Finally we prove Implication (5 \Rightarrow 7). Suppose (5). Let \( x \in S \) be an arbitrary element. We claim that there is \( n \geq 1 \) such that \( -nx \in S \). Assume the converse. Then the set \( X = \{ -nx : n \geq 1 \} \) is infinite. There is a point \( b \in G \) such that the set \( (b + s + S) \cap X \) is infinite for each \( s \in S \). Since \( X \subset S \), we see that \( b \in S \). Therefore there are elements \( y, z \in S \) such that \( b = y - z \). But since \( S \supset (y - z) + z + S \), the set \( S \cap X \) is infinite too. This contradiction proves that there is \( n \geq 1 \) such that \( -nx \in S \). Therefore \( S \) is a group. By Proposition 10 the index \( |G : S| \) is finite. \( \square \)

**Proposition 16.** The group \( G^*_S \) is countably pracompact if and only if the index \( |G : S| \) is finite and there is a countable subset \( C \) of \( S \) such that \( S \subset C - S \).

Moreover, in this case each power of the space \( G^*_S \) is countably pracompact.

**Proof.** The necessity. Since the group \( G_S \) is countably pracompact, by Proposition 11 the index \( |G : S| \) is finite. Let \( A \) be a dense subset of \( S \) such that the space \( S \) is countably compact at \( A \). Let \( B \) be an arbitrary countable infinite subset of \( A \). There are elements \( x, y \in S \) such that the set \( (x - y + s + S) \cap B \) is infinite for each \( s \in S \). Since \( s + S \supset (x - y) + y + s + S \), we see that the set \( (s + S) \cap B \) is infinite for each \( s \in S \). This condition holds for each countable infinite subset \( B \) of \( A \). Therefore the set \( A\setminus(s + S) \) is finite for each \( s \in S \).

There is a countable subset \( C \) of \( S \) such that \( a + s \in S \) for each \( a \in A \). \( \square \)

Choose a finite set \( A_0 \subset G \) such that the intersection of \( A_0 \) with each coset of the group \( S \) in \( G \) is a singleton. Put \( A = A_0 + C' \). Let \( x \in G \) be an arbitrary point. Then there is an element \( a \in A_0 \) such that \( x \in a + S \). Therefore there are elements \( s, s' \in S \) such that \( x = a = s - s' \). There is an element \( c \in C' \) such that \( s \in c - S \). Then \( a + c = x + s' - s + c \in x + s' + s - c + c \subset x + S \). Hence \( A \) is dense in \( G^*_S \).

Let \( A' \) be an arbitrary countable infinite subset of \( A \). There is a point \( a \in A_0 \) such that the set \( B' = A' \cap (a + S) \) is infinite. Let \( s \in S \) be an arbitrary element. Since \( B' - a \) is an infinite subset of \( C' \), the set \( (s + S) \cap (B' - a) \) is infinite. Then the set \( (a + s + S) \cap B' \) is infinite too. Therefore \( a \) is an accumulation point of the set \( B' \subset A' \).

Let \( \kappa \) be an arbitrary cardinal. Put \( A = \{0\} \cup C' \). Then \( A \) is compact and dense in \( S \). Therefore \( A^\kappa \) is compact and dense in \( S^\kappa \). Hence the space \( S^\kappa \) is countably compact at its dense subset. Consequently \( S^\kappa \) is countably pracompact. Since \( S \) is a clopen subgroup of \( G^*_S \), we see that the space \( G^*_S \) is homeomorphic to \( S^\kappa \times D^\kappa \), where \( D \) is a finite discrete space. Then a space \( G^*_S \) is homeomorphic to \( S^\kappa \times D^\kappa \). by Lemma 12 the space \( S^\kappa \times D^\kappa \) is countably pracompact. \( \square \)
Corollary 1. The group $G^*_S$ is compact if and only if the group $G^*_S$ is countably pracompact and 2-pseudocompact.

Proof. The proof is implied from Proposition 17, Proposition 16 and Lemma 8.

Proposition 18. The group $G^*_S$ is pseudocompact if and only if the index $|G : S|$ is finite.

Proof. The necessity. If the group $G^*_S$ is pseudocompact then the group $G^*_S$ is pseudocompact too. Thus Proposition 11 implies the necessity.

The sufficiency. Let $\{U_n : n \in \omega\}$ be a family of nonempty open subsets of $G$. For each $n$ choose a point $z_n \in U_n$ such that $z_n + S \subset U_n$. Since the index $|G : S|$ is finite, there are a finite subset $I$ of $\omega$ and an element $z \in G$ such that $z_n \in z + S$ for each $n \in I$. For each $n \in I$ fix elements $x_n, y_n \in S$ such that $z_n - z = x_n - y_n$. There is an element $a \in S$ such that $x_n \in a - S$ for each $n \in I$. Let $n \in I$ be an arbitrary number. Then $z_n - z = x_n - y_n \in a - S$. Therefore $U_n \ni z_n + S \ni a + z$. Since the family $\{U_n\}$ is non-increasing, we see that $a + z \in \bigcap\{U_n : n \in \omega\}$.

Now let $\kappa$ be an arbitrary cardinal and $G = S^\kappa$. Let $\{U_n : n \in \omega\}$ be an arbitrary non-increasing sequence of nonempty open subsets of $G$. For each $n$ choose a point $z_n \in U_n$ such that $z_n + S^\kappa \subset U_n$ and fix elements $x_n, y_n \in S^\kappa$ such that $z_n = x_n - y_n$. There is an element $a \in S^\kappa$ such that $x_n \in a - S^\kappa$ for each $n \in \omega$. Thus $U_n \ni z_n + S^\kappa \ni a$. □

Corollary 2. The countably pracompact group $G^*_S$ is Baire if and only if $S$ is a group.
Corollary 3. The countable group $G_S^*$ is 2-pseudocompact if and only if the semigroup $S$ is a subgroup of $G$ of finite index.

Proof. If the semigroup $S$ is a subgroup of $G$ of finite index then from Proposition 15 it follows that $G_S^*$ is compact. If the group $G_S^*$ is 2-pseudocompact then from Lemma 1 it follows that $G_S^*$ is Baire. By Proposition 19, $S$ is a group. Proposition 17 implies that the index $|G : S| = |\overline{G} : \overline{S}|$ is finite. □

Example 5. There are an abelian group $G$ and a subsemigroup $S$ of the group $G$ such that the paratopological group $G_S$ is $T_0$ sequentially compact, not totally countably compact, not precompact, and not a topological group and the paratopological group $G_S^*$ is $T_1$ 2-pseudocompact, pseudocompact, not countably pracompact, not precompact and not a topological group.

Proof. Let $G = \bigoplus_{\alpha \in \omega_1} \mathbb{Z}$ be the direct sum of the groups $\mathbb{Z}$. Let $$S = \{0\} \cup \{(x_\alpha) \in G : (\exists \beta \in \omega_1)((\forall \alpha > \beta)(x_\alpha = 0) \& (x_\beta > 0))\}.$$ Since $S \cap (-S) = \{0\}$ then by Proposition 7 the group $G_S$ is $T_0$ and by Proposition 14 the group $G_S^*$ is $T_1$. Since $G = S - S$ and for each countable infinite subset $C$ of $S$ there is an element $a \in S$ such that $C \subset a - S$, we see that by Proposition 10 the group $G_S$ is sequentially compact and by Proposition 17 the group $G_S^*$ is 2-pseudocompact. By Proposition 18 the group $G_S^*$ is pseudocompact. By Proposition 16 the group $G_S^*$ is not countably pracompact. By Proposition 9 the group $G_S$ is not totally countably compact and not precompact. Therefore the group $G_S^*$ is not precompact too. □

From Proposition 3 in [RavRez] it follows that each $T_1$ paratopological group $G$ such that $G \times G$ is countably compact is a topological group. Example 1 shows that in general this proposition cannot be generalized for $T_1$ countably pracompact groups. Example 5 shows that the proposition cannot be generalized for $T_0$ sequentially compact groups.

5. Products of pseudocompact paratopological groups

The product of an arbitrary nonempty family of compact spaces is compact by Tychonov Theorem. But there are Tychonov countably compact spaces $X$ and $Y$ such that $X \times Y$ is not pseudocompact [Eng 3.10.19]. The product of an arbitrary nonempty family of Tychonov pseudocompact topological groups is pseudocompact [ComRos, Th. 1.4]. Under Martin Axiom van Douwen obtained the first example of countably compact topological groups $G_1$ and $G_2$ such that the product $G_1 \times G_2$ is not countably compact. Later, under $MA_{countable}$ Hart and van Mill proved that there is a countably compact topological group $G$ such that $G \times G$ is not countably compact. As far as the author knows, there are no ZFC-example of a family $\{G_\alpha : \alpha \in A\}$ of countably compact Tychonoff topological groups such that the product $\prod \{G_\alpha : \alpha \in A\}$ is countably compact, see the surveys [Com] and [ComHofRem].

The next proposition generalizes the result of Comfort and Ross [ComRos, Th. 1.4] for paratopological groups.

Proposition 20. The product of an arbitrary nonempty family of pseudocompact paratopological groups is pseudocompact.

To prove Proposition 20 we need the following lemmas. The first of them is quite easy and probably is known.
Lemma 13. Let \((X, \tau)\) be a topological space. Then \((X, \tau)\) is pseudocompact if and only if the regularization \((X, \tau_r)\) is pseudocompact. \(\square\)

The proof of the following technical lemma is straightforward.

Lemma 14. Let \((X, \tau)\) be the product of a family \(\{(X_\alpha, \tau_\alpha) : \alpha \in A\}\) of topological spaces. Then \((X, \tau_r) = \prod_{\alpha \in A} (X_\alpha, \tau_{\alpha r})\). \(\square\)

Let \((G, \tau)\) be a paratopological group and \(H \subset G\) be a normal subgroup of \(G\). Then the quotient group \(G/H\) endowed with the quotient topology is a paratopological group, see [Rav]. If \(H = \bigcap\{U \cap U^{-1} : U \in \mathcal{B}^G\}\) then we denote the quotient group \(G/H\) by \(T_0G\). It is easy to check that the group \(T_0G\) is a \(T_0\) space for each paratopological group \(G\).

Lemma 15. A paratopological group \(G\) is pseudocompact if and only if the paratopological group \(T_0G\) is pseudocompact.

Proof. If the group \(G\) is pseudocompact then the group \(T_0G\) is a continuous image of a pseudocompact space. Thus \(T_0G\) is pseudocompact. Now suppose the group \(T_0G\) is pseudocompact. Let \(U\) be a locally finite family of open subsets of the space \(G\). Put \(H = \bigcap\{U \cap U^{-1} : U \in \mathcal{B}^G\}\). Then \(UH = H\) for each open subset \(U\) of the space \(G\). Let \(\pi : G \to T_0G\) be the quotient homomorphism. Therefore the family \(\{\pi(U) : U \in U\}\) is locally finite too. Since the space \(T_0\) is pseudocompact, the family \(\{\pi(U) : U \in U\}\) is finite. Thus the family \(U\) is finite too. \(\square\)

Proof. (Of Proposition 20). Let \((G, \tau)\) be the product of a nonempty family \(\{(G_\alpha, \tau_\alpha) : \alpha \in A\}\) of pseudocompact paratopological groups. Let \(\mathcal{B}_r\) be an open base of the topology \(\tau_r\) at the unit \(e\) of the paratopological group \((G, \tau_r)\). Put \(H = \bigcap\{U \cap U^{-1} : U \in \mathcal{B}_r\}\). Lemma 14 implies that \((G, \tau_r) = \prod_{\alpha \in A} (G_\alpha, \tau_{\alpha r})\). Let \(\pi : G \to G/H\) be the quotient map.

Let \(\alpha \in A\) be an arbitrary element, \(\mathcal{B}_{\alpha r}\) an open base of the topology \(\tau_{\alpha r}\) at the unit \(e_\alpha\) of the paratopological group \((G_\alpha, \tau_{\alpha r})\), \(H_\alpha = \bigcap\{U \cap U^{-1} : U \in \mathcal{B}_{\alpha r}\}\), \(\pi_\alpha : (G_\alpha, \tau_{\alpha r}) \to (G_\alpha, \tau_{\alpha r})/H_\alpha\) the quotient map, and \(p_\alpha : (G, \tau_r) \to (G_\alpha, \tau_{\alpha r})\) the projection.

Let \(i : \prod_{\alpha \in A} (G_\alpha, \tau_{\alpha r}) \to \prod_{\alpha \in A} (G_\alpha, \tau_{\alpha r})/H_\alpha\) be the product of the family \(\{\pi_\alpha : \alpha \in A\}\). Proposition 2.329 from [Eng] implies that the map \(i\) is open. Since \(H = \{e\}^{\tau_r} \cap (\{e\}^{\tau_r})^{-1} \text{ and } H_\alpha = \{e_\alpha\}^{\tau_{\alpha r}} \cap (\{e\}^{\tau_{\alpha r}})^{-1}\) for each \(\alpha \in A\), we see that \(H = \prod_{\alpha \in A} H_\alpha = \bigcap_{\alpha \in A} p_\alpha^{-1}(H_\alpha) = \ker i\). Let \(i' : (G, \tau_r)/H \to \prod_{\alpha \in A} (G_\alpha, \tau_{\alpha r})/H_\alpha\) be a map such that \(i'(xH) = i(x)\) for each \(x \in G\). Theorem on continuous epimorphism [Rav, p. 42] implies that the map \(i'\) is well-defined and a topological isomorphism.

Let \(\alpha \in A\) be an arbitrary element. Then \((G_\alpha, \tau_{\alpha r})\) is a \(T_3\) pseudocompact paratopological group (see the remark after Lemma 14). So \((G_\alpha, \tau_{\alpha r})\) is a pseudocompact topological group by Lemma 6. Then the group \((G_\alpha, \tau_{\alpha r})/H_\alpha\) is a \(T_0\) pseudocompact topological group. Therefore by Theorem 1.4 from [ComRos], the product \(\prod_{\alpha \in A} (G_\alpha, \tau_{\alpha r})/H_\alpha\) is pseudocompact. Hence the group \((G, \tau_r)/H = T_0(G, \tau_r)\) is pseudocompact. By Lemma 15 the group \((G, \tau_r)\) is pseudocompact. Thus by Lemma 13 the group \((G, \tau)\) is pseudocompact. \(\square\)

6. Open Problems

This section is preliminary and it will be edited in a submitted version of this paper.

The next two problems are related to conditions when a paratopological group is topological.
Problem 1. Is there a ZFC-example of a paratopological group $G$ such that $G$ is not a topological group but $G$ satisfies one of the following conditions:

- $G$ is Hausdorff countably compact,
- $G$ is Hausdorff countably pracompact Baire,
- $G$ is Hausdorff pseudocompact Baire,
- $G$ is Hausdorff 2-pseudocompact,
- $G$ is $T_1$ countably compact.

If $G$ is such a group then $G$ is not topologically periodic, not saturated, not pseudocompact quasiregular, and not left $\omega$-precompact. Moreover, $G \times G$ is not countably compact. By Theorem 2.2 from [AlaSan], a group $(G, \sigma)$ from Example 2 is Baire. Therefore under TT there is a group $G$ such that $G$ satisfies all conditions listed in Problem 1.

Problem 2. Suppose a paratopological group $G$ satisfies one of the following conditions:

- $G$ is Hausdorff countably pracompact topologically periodic left $\omega$-precompact,
- $G$ is Hausdorff countably pracompact topologically periodic,
- $G$ is Hausdorff pseudocompact topologically periodic left $\omega$-precompact,
- $G$ is Hausdorff pseudocompact topologically periodic,
- $G$ is regular 2-pseudocompact.

Is $G$ a topological group?

If a paratopological group $G$ satisfies one of Problem 2 Conditions 1-4 and the group $G$ is quasiregular or saturated or Baire then $G$ is a topological group. If $G$ is a regular 2-pseudocompact paratopological group $G$ and the group $G$ saturated or topologically periodic or left $\omega$-precompact or has a countable pseudocharacter then $G$ is a topological group.

Each countably compact paratopological group and each pseudocompact topological group are both 2-pseudocompact and pseudocompact. Answers to the next two problems would help to understand the relations between 2-pseudocompact and pseudocompact paratopological groups.

Problem 3. Is each (Hausdorff) 2-pseudocompact paratopological group pseudocompact?

The answer to Problem 3 is positive provided the group is saturated or topologically periodic or left $\omega$-precompact.

Since each 2-pseudocompact paratopological group is Baire and there are pseudocompact not Baire paratopological groups, we see that an opposite inclusion does not hold.

Problem 4. Is each pseudocompact (countably pracompact) Baire paratopological group 2-pseudocompact?

The answer to Problem 4 is positive provided the group is saturated or topologically periodic or left $\omega$-precompact.

Problem 5. Is the product of two 2-pseudocompact paratopological groups 2-pseudocompact?

Of a countable nonempty family? Of an arbitrary nonempty family?

Is the product of two countably pracompact paratopological groups countably pracompact?

Of a countable nonempty family? Of an arbitrary nonempty family?

The product of a nonempty family of 2-pseudocompact topological groups is 2-pseudocompact. Are there other nontrivial conditions sufficient for 2-pseudocompactness of the product? Is
the product of two countably compact paratopological groups 2-pseudocompact? Of a countable nonempty family? Of an arbitrary nonempty family? In particular, by [RavRez, Pr. 3], the square \((G, \sigma) \times (G, \sigma)\) of a group \((G, \sigma)\) from Example 2 is not countably compact. By Proposition 20, \((G, \sigma) \times (G, \sigma)\) is pseudocompact. Is \((G, \sigma) \times (G, \sigma)\) 2-pseudocompact? Countably pracompact?

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