A linear cosmological perturbation theory of an almost homogeneous and isotropic perfect fluid Universe with dynamically evolving Newton constant $G$ and cosmological constant $\Lambda$ is presented. A gauge-invariant formalism is developed by means of the covariant approach, and the acoustic propagation equations governing the evolution of the comoving fractional spatial gradients of the matter density, $G$, and $\Lambda$ are thus obtained. Explicit solutions are discussed in cosmologies where both $G$ and $\Lambda$ vary according to renormalization group equations in the vicinity of a fixed point.
I. INTRODUCTION

The subject of cosmological perturbations continues to attract much attention because it is an essential step in understanding any theory of cosmological structure formation. The simple idea that the observed structure in the Universe has resulted from the gravitational amplification of small primordial fluctuations works remarkably well and it must be discussed in any theory of relativistic cosmology.

In inflationary cosmology the presently observed structures in the Universe are generated by quantum fluctuations during an early de Sitter phase. The subsequent evolution is classical and depends on the interplay between pressure forces, the rate of growth of the expansion factor and on the content and the nature of the, yet unknown, dark matter.

Very recently, an alternative scenario has been proposed. In [1] we discussed a cosmology of the Planck Era, valid immediately after the initial singularity, in which the Newton constant and the cosmological constant are dynamically coupled to the geometry by “improving” the Einstein equations with the renormalization group (RG) equation for Quantum Einstein Gravity [2]. It has then been shown that in this new scenario a solution to the horizon and flatness problem of standard cosmology is possible without the introduction of an ad hoc scalar field. There is also a natural mechanism generating primordial density fluctuations with spectral index $n = 1$, at least at short wavelengths.

The recent evidence for a non-trivial ultraviolet fixed point in Quantum Einstein Gravity [4,5] has opened the possibility of a non-perturbative renormalizability of the theory along the lines of the asymptotic safety conjectured by Weinberg [6]. This has triggered a number of investigations in black hole physics [10,11] and cosmology [1]. According to the findings of [2–5,7–9] Newton’s constant is an asymptotically free coupling at very high energy scales. Given the RG flow near this non-trivial fixed point it is possible to “RG improve” the
Einstein field equations by replacing Newton’s constant $G$ and the cosmological constant $\Lambda$ by their scale-dependent ("running") counterparts. The RG improved Einstein equations lead to a mathematically tractable system of evolution equations for the scale factor, the pressure, the density, $G$, and $\Lambda$. Near the fixed point explicit solutions are available [1].

A similar mechanism could in principle be operating also far from the Big Bang, in a completely different era of the Universe. In fact, in the late Universe, the possibility that another, this time infrared attractive fixed point, might be present in the RG flow for gravity and governs its long distance behavior has been discussed in [12]. In this case a solution of the “cosmic coincidence problem” [13] arises naturally without the introduction of a quintessence field. It can be shown that in the fixed point regime the vacuum energy density $\rho_\Lambda \equiv \Lambda / 8\pi G$ is automatically adjusted so as to equal the matter energy density, i.e. $\Omega_\Lambda = \Omega_M = 1/2$, and that the deceleration parameter approaches $q = -1/4$. Moreover, an analysis of the high-redshift SNe Ia data leads to the conclusion that this infrared fixed point cosmology is in good agreement with the observations [14], and that it is a promising candidate for describing the dynamics of the Universe on very large scales. It is important to note that the experimental determination of $\Omega_M$ and $\Omega_\Lambda$ is model dependent. The numbers which are usually quoted, $\Omega_M \approx 0.3$ and $\Omega_\Lambda \approx 0.7$, are obtained when one fits the data to the cosmological standard model. Our prediction $\Omega_M = \Omega_\Lambda = 0.5$ is not inconsistent with this result since we use a different model; in fact, we find that it fits the supernova data as accurately as the standard model.

Cosmologies with a time dependent $G$ have been discussed in [15], while homogeneous and isotropic cosmological models with variable $G$ and $\Lambda$ have been discussed in [16] and [17,18]. Also for this reason a formalism where the evolution of small fluctuations of the density, of $G$, and of $\Lambda$ can be consistently discussed is needed. The aim of this paper is to present such a formalism and to apply it to the case of the RG derived cosmologies.
We shall follow the so-called covariant approach, pioneered by Hawking [19], further extended by Ellis and Bruni [20], Jackson [21] and Zimdahl [22], in order to provide a gauge-invariant formulation of the problem. The gauge issue is in fact a major problem in discussing the evolution of the long wavelength fluctuations, where the presence of spurious gauge modes can be more difficult to avoid [23]. The second order acoustic propagation equations, governing the comoving fractional spatial gradients of the density and of $G$ and $\Lambda$ are then obtained for a class of background spacetimes with a time-dependent Newton constant and cosmological constant where the matter field is described by a perfect fluid with conserved energy momentum tensor. The properties of the solutions are then discussed for a class of RG derived cosmologies. It turns out that it is possible to have both growing and decaying modes for the fractional quantities describing the perturbations of the density, $G$ and $\Lambda$, depending on the equation of state.

In particular, if the late time evolution of the background Universe is governed by the infrared fixed point cosmology we shall find that all the relevant perturbations decay with a power law of the cosmic time or stay constant, i.e. the infrared fixed point cosmology is stable under small deviations from perfect homogeneity and isotropy. According to this result, the ultimate fate of the Universe at very large scales is a sort of “eternal dilution” where a homogeneous and isotropic state is reached for $t \to \infty$.

II. GAUGE-INvariant PERTURBATION THEORY

In this section we generalize the formalism of refs. [20–22] by allowing $G \equiv G(x^\mu)$ and $\Lambda \equiv \Lambda(x^\mu)$ to be scalar functions on spacetime.

A "fundamental observer" describing the cosmological fluid flow lines has 4-velocity

$$u^\mu = dx^\mu/d\tau, \quad u^\mu u_\mu = -1$$

(2.1)
where \( \tau \) is the proper time along the fluid flow lines. The projection tensor onto the tangent 3-space orthogonal to \( u^\mu \) is

\[
h_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu \tag{2.2}
\]

with \( h^\mu_\nu h^\nu_\sigma = h^\mu_\sigma \) and \( h^\mu_\nu u^\nu = 0 \). The covariant derivative of \( u^\mu \) is

\[
u \frac{\partial}{\partial x^\nu} \right) u_\nu - \frac{1}{3} \Theta h_{\mu \nu} - \dot{u}_\mu u_\nu \tag{2.3}
\]

where \( \omega_{\mu \nu} = h^{\alpha}_\mu h^{\beta}_\nu u_{[\alpha;\beta]} \) is the vorticity tensor, \( \sigma_{\mu \nu} = \frac{1}{2} h^{\alpha}_\mu h^{\beta}_\nu u_{(\alpha;\beta)} \) is the shear tensor, \( \Theta = u_{\mu ; \mu} \) is the expansion scalar and \( \ddot{u}^\mu = u^\mu ;_\nu u^\nu \) is the acceleration four-vector; square and round brackets denote anti-symmetrization and symmetrization, respectively. The Riemann tensor is defined by

\[
u \frac{\partial}{\partial x^\nu} \right) u_\nu = R^\sigma_{\mu \lambda \tau} u_\sigma \tag{2.4}
\]

where \( \Lambda = \Lambda(x^\mu) \) is the position dependent cosmological constant and \( G = G(x^\mu) \) the position dependent Newton constant. The energy-momentum tensor is assumed to be conserved. For a perfect fluid it has the form

\[
T_{\mu \nu} = \rho u^\mu u^\nu + p h_{\mu \nu} \tag{2.5}
\]

The conservation law \( T^{\mu \nu} ;_\nu = 0 \) leads to mass-energy conservation

\[
\frac{\dot{\rho}}{\rho + p} + \Theta = 0 \tag{2.6}
\]

and the equation of motion

\[
\dot{u}^\mu + \frac{h_{\mu \nu} p_\nu}{\rho + p} = 0. \tag{2.7}
\]
The Bianchi identities require the RHS of (2.4) to be covariantly conserved. This consistency condition together with the conservation laws (2.6) and (2.7) provides the equations for Λ and G,

\[ \dot{\Lambda} + 8\pi\dot{G}_\rho = 0 \quad (2.8) \]

\[ h^{\mu\nu}\Lambda_{;\nu} - 8\pi p h^{\mu\nu}G_{;\nu} = 0 \quad (2.9) \]

by projecting along \( u^\mu \) and onto the hyperplane orthogonal to \( u^\mu \). The Raychaudhuri equation is obtained with the help of the Einstein field equations and of Eq.(2.3),

\[ \dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}_{;\mu} + 4\pi G(\rho + 3p) - \Lambda = 0 \quad (2.10) \]

where \( 2\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} \) and \( 2\omega^2 \equiv \omega_{\mu\nu}\omega^{\mu\nu} \). The term \( \dot{u}_{;\mu} \) can be rewritten as [22]

\[ \dot{u}_{;\mu} = -h^{\lambda\tau}\left( h^{\nu}\frac{p_{\nu}}{\rho + p}\right)_{;\tau} + h^{\lambda\tau}\frac{p_{\lambda}}{\rho + p}\frac{p_{\tau}}{\rho + p}. \quad (2.11) \]

The scalar \( K \) is defined as

\[ K \equiv 2\sigma^2 - \frac{\Theta^2}{3} + 16\pi G\rho + 2\Lambda \quad (2.12) \]

It is possible to show that for zero vorticity, \( \omega_{\mu\nu} = 0 \), it coincides with the Ricci scalar \( ^{(3)}R \) of the 3-dimensional hyperplane everywhere orthogonal to \( u^\mu \). An auxiliary length scale \( S(t) \) is introduced as the solution of the equation

\[ \frac{\dot{S}}{S} = \frac{1}{3}\Theta. \quad (2.13) \]

Suitable quantities useful to characterize the spatial inhomogeneities of density, pressure and expansion are, respectively,

\[ D_\mu \equiv \frac{S h_\mu^\nu \rho_{;\nu}}{\rho + p}, \quad P_\mu \equiv \frac{S h_\mu^\nu p_{;\nu}}{\rho + p}, \quad t_\mu \equiv S h_\mu^\nu \Theta_{;\nu} \quad (2.14) \]
In order to characterize spatial inhomogeneities of \( G \) and \( \Lambda \) it is convenient to introduce the following dimensionless quantities:

\[
\Gamma_{\mu} \equiv \frac{1}{G} S h_{\mu}^{\nu} G_{\nu}, \quad \Delta_{\mu} \equiv \frac{1}{\Lambda} S h_{\mu}^{\nu} \Lambda_{\nu}.
\] (2.15)

We then have from (2.6) and (2.7)

\[
\dot{h}^{\alpha}_{\gamma}(S h^{\beta}_{\alpha \rho \beta} \dot{\rho}) = \frac{S}{3} \Theta h^{\alpha}_{\gamma \rho \alpha} + S \dot{h}^{\alpha}_{\gamma \rho \alpha} + Sh^{\alpha}_{\gamma \rho}(\dot{\rho} - \dot{\rho} - \dot{\rho} + \dot{\rho} + \dot{\rho})
\] (2.16)

In the first line we used the identity \((\rho \dot{\rho} + \rho \dot{\rho} + \rho \dot{\rho})\) and in the second line (2.3). Eq.(2.16) can now be rewritten as

\[
h^{\nu}_{\mu} \dot{D}_{\nu} + \frac{\dot{P}}{\rho + p} D_{\mu} + (\omega^{\nu}_{\mu} + \sigma^{\nu}_{\mu}) D_{\nu} + t_{\mu} = 0.
\] (2.17)

We can obtain the equation of motion for \( t_{\mu} \) by following similar manipulations:

\[
h^{\nu}_{\mu} \dot{t}_{\nu} = \dot{S} h^{\nu}_{\mu} \Theta_{\nu} + S h^{\nu}_{\mu} \dot{u}_{\nu} \dot{\Theta} + Sh^{\nu}_{\mu}(\Theta_{\nu})
\]

\[
= -\dot{\Theta} P_{\mu} - (\omega^{\nu}_{\mu} + \sigma^{\nu}_{\mu}) t_{\nu} - \frac{2}{3} \Theta t_{\mu} - Sh^{\nu}_{\mu}(2\sigma^{2} - 2\omega^{2})_{\nu} + Sh^{\nu}_{\mu}(u^{\tau}_{\tau})_{\nu}
\]

\[-4\pi G(\rho + p)[D_{\mu} + 3P_{\mu}] - 4\pi G \Gamma_{\mu}(\rho + p)
\] (2.18)

In the last line we have used the Raychaudhuri equation (2.10) and Eq.(2.9). We stress that Eq.(2.17) and Eq.(2.18) are completely general in the sense that no assumption on the functional form of \( G \) or \( \Lambda \) has been made. In particular, Eq.(2.18) coincides with Eq.(27) of [22] for constant \( G \) and \( \Lambda \).

Given an equation of state \( p = p(\rho) \) it is possible to express \( P_{\mu} \) in terms of \( D_{\mu} \), but (2.17) and (2.18) still do not form a closed system since an evolution equation for \( \Gamma_{\mu} \) is needed. It can be obtained in the following way: from the definition of \( \Gamma_{\mu} \) we have
\[ h_\nu^\mu (G\Gamma_\nu) = -P_\mu \dot{G} + Sh_\mu^\nu \dot{G}_\nu - G\Gamma_\nu (\omega_\nu^\mu + \sigma_\nu^\mu) \] (2.19)

\[ h_\nu^\mu (\Lambda \Delta_\nu) = -P_\mu \dot{\Lambda} + Sh_\mu^\nu \dot{\Lambda}_\nu - \Lambda \Delta_\nu (\omega_\nu^\mu + \sigma_\nu^\mu) \] (2.20)

where we have used Eq.(2.3). From Eq.(2.8) and Eq.(2.9) we get instead

\[ h_\nu^\mu (\Lambda \Delta_\nu) = 8\pi \dot{p} G\Gamma_\mu + 8\pi p h_\nu^\mu (G\Gamma_\nu) \] (2.21)

\[ h^{\mu\nu} \dot{\Lambda}_\nu + 8\pi h^{\mu\nu} \rho_\nu + 8\pi \rho h^{\mu\nu} \dot{G}_\nu = 0 \] (2.22)

Therefore by using (2.20) in (2.21) and using (2.22) in the result, (2.19) becomes

\[ h_\nu^\mu (G\Gamma_\nu) = -\frac{\dot{p}}{\rho + p} G\Gamma_\mu - \dot{G} D_\mu - \frac{G}{\rho + p} \Gamma_\nu (\omega_\nu^\mu + \sigma_\nu^\mu) \] (2.23)

This is the equation for \( \Gamma_\mu \) we were looking for.

From now on we shall assume the background Universe to be homogeneous and isotropic, i.e. \( \omega_{\mu\nu} = \sigma_{\mu\nu} = u_\mu = 0 \). In this Universe we consider small perturbations of the motion of the fluid and consequently, up to first order in the inhomogeneities, the factors multiplying the quantities \( D_\mu, P_\mu, t_\mu \) and \( \Gamma_\mu \) in (2.17) and (2.18) refer to the background. Thus the linearized equation for \( D_\mu \) becomes

\[ h_\nu^\mu \dot{D}_\nu + \frac{\dot{p}}{\rho + p} D_\mu + t_\mu = 0 \] (2.24)

and the linearized equation for \( t_\mu \) reads

\[ h_\nu^\mu \dot{t}_\nu = -\dot{\Theta} P_\mu - \frac{2}{3} \Theta t_\mu + Sh_\nu^\mu (\dot{u}_\tau^\nu)_\nu - 4\pi G(\rho + p)[D_\mu + 3P_\mu] - 4\pi G\Gamma_\mu (\rho + p) \] (2.25)

These equations can be further simplified if we specify the factors in front of the quantities \( D_\mu, P_\mu \) for the background spacetime. In particular, in a homogeneous Universe with variable \( G \) and \( \Lambda \) the relevant equations for the background evolution read
\[ K = 2\left( -\frac{1}{3} \dot{\Theta}^2 + 8\pi G\rho + \Lambda \right) = (3) R \]  

(2.26a)

\[ \dot{\Theta} + 12\pi G(\rho + p) = (3) R/2 \]  

(2.26b)

\[ \dot{\Lambda} + 8\pi \dot{G}\rho = 0 \]  

(2.26c)

These equations imply that we may set \( S(t) = a(t) \). In fact, we observe that Eq.(2.26a) and Eq.(2.26b) are the 00-component and \( ii \)-component of Einstein’s equation, respectively, provided we identify \( S(t) \) with the scale factor \( a(t) \) of the Robertson-Walker line element

\[ ds^2 = -dt^2 + a(t)^2\left[ \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \]  

(2.27)

For \( K = 0 \) the 3-spaces of constant cosmological time \( t \) are flat, and for \( K = +1 \) and \( -1 \) they are spheres and pseudo-spheres, respectively. The 3-curvature of the \( t = const \) hyperplane is

\[ (3) R = \frac{6K}{a^2}. \]  

(2.28)

As a consequence of the above results, Eq.(2.25) now reads

\[ h^\nu_{\mu} \dot{t}_\nu = -\frac{(3) R}{2} P_\mu - 2\frac{\dot{a}}{a} t_\mu - 4\pi G(\rho + p)D_\mu + Sh^\nu_{\mu}(\dot{u}^\tau_{\nu}; \tau) - 4\pi G\Gamma_\mu(\rho + p) \]  

(2.29)

Eq.(2.11), up to linear order, implies [22]:

\[ Sh^\tau_{\nu}(\dot{u}^\mu_{\nu};\tau) = -\frac{\nabla^2}{a^2} P_\nu \]  

(2.30)

where \( \nabla^2 \) is the Laplacian on the maximally symmetric 3-space. For an equation of state \( p = p(\rho) \) we may write

\[ \dot{p} = -c_s^2 \Theta (\rho + p), \quad P_\mu = c_s^2 D_\mu. \]  

(2.31)

where \( c_s = (\partial p/\partial \rho)^{1/2} \) denotes the velocity of sound in the medium. Therefore Eq.(2.29) becomes
\[ h^\nu_{\mu} \dot{t}_\nu = -2 \frac{\dot{a}}{a} t_\mu - \left[ 4\pi G (\rho + p) + c_s^2 \frac{3K}{a^2} + c_s^2 \frac{\nabla^2}{a^2} \right] D_\mu - 4\pi G \Gamma_\mu (\rho + p) \]  

(2.32)

By combining (2.32) with (2.24) and (2.31) and noticing that \( h^\nu_{\tau}(h^\nu_{\mu} \dot{D}_\nu) = h^\nu_{\tau} \dot{D}_\nu - \dot{u}_\tau \dot{u}\nu D_\nu = h^\nu_{\tau} \ddot{D}_\nu \) up to second order, we get the desired differential equation for the fractional density which is of second order in \( t \):

\[ h^\nu_{\mu} \ddot{D}_\nu + (2 - 3 c_s^2) \frac{\dot{a}}{a} h^\nu_{\mu} \dot{D}_\nu - \left[ 3(c_s^2) \frac{\dot{a}}{a} + 4\pi G(\rho - 3p) + 2\Lambda c_s^2 + 4\pi G(\rho + p) + c_s^2 \frac{\nabla^2}{a^2} \right] D_\mu - 4\pi G \Gamma_\mu (\rho + p) = 0 \]  

(2.33)

The corresponding first order evolution equation for \( \Gamma_\mu \) reads according to Eq.(2.23)

\[ h^\nu_{\mu} \dot{(G \Gamma_\nu)} = c_s^2 \Theta G \Gamma_\mu - \dot{G} D_\mu \]  

(2.34)

Once the solution for \( \Gamma_\mu \) is obtained also \( \Delta_\mu \) is known. Eq. (2.9) yields the simple relationship

\[ \Delta_\mu = \frac{8\pi p G}{\Lambda} \Gamma_\mu \]  

(2.35)

The coupled system (2.33), (2.34), (2.35) are the general gauge-invariant equations which govern the first order perturbations of cosmologies with variable \( G \) and \( \Lambda \). They are one of our main results. As an immediate consequence of (2.35) we observe that \( \Delta_\mu = 0 \) in any Universe with \( p = 0 \) so that \( \Lambda \) is always spatially constant in this case.

### III. RG DERIVED COSMOLOGIES

The idea of using the RG flow equation in gravity is borrowed from particle physics where the RG improvement is a standard device in order to add, for instance, the dominant quantum corrections to the Born approximation of a scattering cross section. However, instead of improving solutions, in [1,12] the more powerful improvement of the basic equations...
has been discussed. We shall now briefly review the main results for an homogeneous and isotropic Universe.

We assume that there is a fundamental scale dependence of Newton’s constant which is governed by an exact RG equation for a Wilsonian effective action whose precise nature needs not to be specified here. At a typical length scale $\ell$ or mass scale $k = \ell^{-1}$ those “constants” assume the values $G(k)$ and $\Lambda(k)$, respectively. In trying to “RG-improve” Einstein’s equation the crucial step is the identification of the scale $\ell$ or $k$ which is relevant in the situation under consideration [24]. In cosmology, the postulate of homogeneity and isotropy implies that $k$ can depend on the cosmological time only so that the scale dependence is turned into a time dependence:

$$G(t) \equiv G(k = k(t)), \quad \Lambda(t) \equiv \Lambda(k = k(t))$$  \hspace{1cm} (3.1)

In principle the time dependence of $k$ can be either explicit or implicit via the scale factor: $k = k(t, a(t), \dot{a}(t), \ddot{a}(t), \cdots)$. In ref. [1] we gave detailed arguments as to why the purely explicit time dependence

$$k(t) = \xi/t$$  \hspace{1cm} (3.2)

with $\xi$ being a positive constant of order unity is the correct identification. In a nutshell, the argument is that, when the age of the Universe is $t$, no (quantum) fluctuation with a frequency smaller than $1/t$ can have played any role yet. Hence the integrating-out of modes (“coarse graining”) which underlies the Wilson renormalization group should be stopped at $k \approx 1/t$. Moreover, other plausible cutoffs, such as $k \approx H(t)$, are equivalent to (3.2) under very general conditions. (See [1] and [12] for further details.)

Once the RG trajectory $k \mapsto (G(k), \Lambda(k))$ is known, the system (2.26) together with (3.1,3.2) can be solved. In the following we focus on the vicinity of a fixed point where simple analytical solutions can be found [1,12].
To be precise, we assume that within a large basin of attraction the dimensionless quantities $g(k) \equiv k^2 G(k)$ and $\lambda(k) \equiv \Lambda(k)/k^2$ get attracted either towards an ultraviolet (UV) fixed point for $k \to \infty$ or towards an infrared (IR) fixed point for $k \to 0$. While the physics of these two situations is quite different (early vs. late Universe) they are rather similar mathematically and we can discuss the two cases in parallel *.

As for the actual derivation of those fixed points from the RG equations, a non-gaussian UV fixed point is known to exist in the Einstein-Hilbert truncation of pure Quantum Einstein Gravity [2–4,7,8] and in more general truncations [5]. Recently it was also found within the 2 Killing-vector reduction of Quantum Einstein Gravity [9]. For the time being, the existence of an IR fixed point cannot be deduced yet from a (truncated) RG equation because from a technical point of view it is very difficult to follow the RG flow very far towards the IR. The reason is that a reliable description of IR physics most probably requires truncations containing nonlocal invariants. In the second paper of [7] a first progress was made in this direction. Using a simple, mathematically tractable nonlocal truncation it was shown that there is a scale invariant IR fixed point at which the size of the universe is completely unrelated to the value of the bare cosmological constant. The problem is that this truncation has an interpretation only within euclidean quantum gravity and does not directly imply the IR fixed point postulated in the present paper. Nevertheless, this investigation suggests that the late Universe, at very large scales, may be regarded as a kind of scale free “critical phenomenon”. A natural way of implementing this general physical picture within Lorentzian gravity is by means of a fixed point for $g$ and $\lambda$.

*If both of the fixed points are present one could have a very symmetric scenario with respect to the “birth” and “death” of the Universe where the cosmological evolution amounts to a cross-over from the UV to the IR fixed point. See ref. [25] for a similar cross-over in 2D gravity.
It is also conceivable that the postulated IR fixed point is of a completely classical nature [26] or that the quantum effects of matter fields play an important role [27].

In the vicinity of a fixed point \((g_*, \lambda_*)\) the evolution of the dimensionful \(G\) and \(\Lambda\) is approximately given by

\[
G(k) = \frac{g_*}{k^2}, \quad \Lambda(k) = \lambda_* k^2
\]  

(3.3)

From (3.3) with (3.2) we obtain the time dependent Newton constant and cosmological constant:

\[
G(t) = g_\xi t^2, \quad \Lambda(t) = \frac{\lambda_\xi^2 t^2}{t^2}
\]  

(3.4)

The power laws (3.4) are valid for \(t \searrow 0\) (UV case) or for \(t \to \infty\) (IR case), respectively. If we use these functions \(G(t)\) and \(\Lambda(t)\) in the coupled system (2.26), its solution gives us the scale factor \(a(t)\) and the density \(\rho(t)\) of the “RG improved cosmology”.

In the case of a spatially flat Universe \((K = 0)\) and the equation of state \(p = w\rho\), which we shall consider in this paper, the system (2.26) with (3.4) has the following one-parameter family of solutions:

\[
a(t) = \left[\frac{3}{8}\right]^2 (1 + w)^4 g_* \lambda_* \mathcal{M}^{1/(3+3w)} t^{4/(3+3w)}
\]

(3.5a)

\[
\rho(t) = \frac{8}{9\pi(1 + w)^4 g_* \lambda_*} \frac{1}{t^4}
\]

(3.5b)

\[
G(t) = \frac{3}{8} (1 + w)^2 g_* \lambda_* t^2
\]

(3.5c)

\[
\Lambda(t) = \frac{8}{3(1 + w)^2} \frac{1}{t^2}
\]

(3.5d)

Note in particular that \(a \propto t\) for \(w = 1/3\) and \(a \propto t^{4/3}\) for \(w = 0\). Apart from the constant \(w\) and the product \(g_* \lambda_*\), the solution (3.5) depends only on a single constant of integration, \(\mathcal{M}\).
whose value affects only the overall scale of $a(t)$. Numerically it equals $8\pi \rho(t)(a(t))^{3+3w} \equiv \mathcal{M}$ which, like in standard cosmology, is a conserved quantity. Introducing the critical density

$$\rho_{\text{crit}}(t) \equiv \frac{3}{8\pi G(t)} \left(\frac{\dot{a}}{a}\right)^2$$

we find for any value of $w$, $g_* \lambda_*$, and $\mathcal{M}$ that $\rho_{\text{crit}}(t) = 2\rho(t)$ and $\rho_\Lambda(t) = \rho(t)$. Hence

$$\rho = \rho_\Lambda = \frac{1}{2}\rho_{\text{crit}}$$

Thus the total energy density $\rho_{\text{tot}} \equiv \rho + \rho_\Lambda$ equals precisely the critical one: $\rho_{\text{tot}}(t) = \rho_{\text{crit}}(t)$. The exact equality, at any time, of the matter energy density $\rho$ and the vacuum energy density $\rho_\Lambda$ is a nontrivial prediction of the fixed point solution. In terms of the relative densities,

$$\Omega_M = \Omega_\Lambda = \frac{1}{2}, \quad \Omega_{\text{tot}} = 1$$

Also the Hubble parameter of the solution (3.5)

$$H \equiv \frac{\dot{a}}{a} = \frac{4}{3+3w} \frac{1}{t}$$

and its deceleration parameter

$$q \equiv -\frac{a \ddot{a}}{a^2} = \frac{3w - 1}{4}$$

are independent of $g_*$, $\lambda_*$ and $\mathcal{M}$ [28]. (It can be shown that the standard formula for $q$ in terms of the relative densities continues to be correct for all solutions of the improved system (2.6) with an arbitrary RG trajectory: $q = \frac{1}{2} (3w + 1) \Omega_M - \Omega_\Lambda$.)

IV. STABILITY OF FIXED POINT COSMOLOGIES

In Section II we derived a general system of equations governing the dynamics of small perturbations about a cosmology with variable $G$ and $\Lambda$. In the following we apply this formalism to the fixed point background cosmology of (3.5). We write
\[ D^\mu = \delta^n(t) \Psi_n^\mu \quad \Gamma^\mu = \gamma^n(t) \Psi_n^\mu \] (4.1)

where \( \Psi_n^\mu \) is an eigenfunction of the Laplacian operator \( \nabla^2 \) with (negative) eigenvalues \( \nu_n^2 \):

\[ -\nabla^2 \Psi_n^\mu = \nu_n^2 \Psi_n^\mu \] (4.2)

For an equation of state of the type \( p = w\rho \) we then obtain the following closed system for the perturbed quantities:

\[ \ddot{\delta}^n + (2 - 3w)\frac{a}{\dot{a}}\dot{\delta}^n - \left[ 4\pi G \rho (1 - w)(3w + 1) + 2w\Lambda \right]\delta^n + w \nu_n^2 \delta^n = 4\pi \rho G \gamma^n(1 + w) \] (4.3)

\[ \dot{\gamma}^n = 3w \frac{a}{\dot{a}}\gamma^n - \frac{\dot{G}}{G} \gamma^n - \frac{\dot{G}}{G} \delta^n. \] (4.4)

We consider this system in the long wavelength limit for which \( \nu_n \approx 0 \). Its solutions are simple power-laws. If we set \( \delta = At^\alpha \) and \( \gamma = Bt^\beta \), then, by direct substitution in (4.3), one finds that \( \beta = \alpha \), and \( \alpha \) is obtained by solving a cubic equation. It has three real roots:

\[ \alpha_1 = \frac{4w}{1 + w} \quad \alpha_2 = \frac{2(3w - 1)}{3(w + 1)} \quad \alpha_3 = \frac{w - 3}{w + 1} \] (4.5)

In the late Universe governed by an IR fixed point, \( p = 0 \) i.e. \( w = 0 \) should be a good approximation to the equation of state. In this case the exponents are \( \alpha = (0, -2/3, -3) \). It is not necessary to invoke the long wavelength limit in order to arrive at this result; for \( w = 0 \) the eigenvalues \( \nu_n^2 \) drops out from (4.4) so that we are left with the same equation for perturbations at any wavelength. As a consequence, there are no growing modes of the \( \rho \)- and \( G \)-perturbations. Furthermore, as we mentioned already, \( \Lambda \) is strictly spatially constant in a \( p = 0 \) Universe so that there are no inhomogeneous \( \Lambda \)-perturbations at all. We conclude that in an IR-fixed point Universe there exist no linear perturbations which would drive the evolution away from the ideal homogeneous and isotropic state. This implies that there is no Jeans-type instability giving rise to the formation of structures on cosmological scales.
This last restriction stems from the fact that the above derivation made essential use of the cutoff identification (3.2) which assumes that the relevant momentum scale $k$ is given by the inverse cosmological time. For a perfectly homogeneous and isotropic Universe this is essentially the unique choice [1,12]. Therefore the above analysis should apply to perturbations with proper wavelengths on the Megaparsec scale or beyond where the Universe starts looking homogeneous and isotropic. The dynamics of perturbations on smaller length scales is much harder to analyze. The reasons are: (i) It is not clear how to identify the running scale $k$ in terms of physical quantities since in presence of inhomogeneities or anisotropies $1/t$ is presumably not the only relevant scale. (ii) Smaller lengths correspond to higher values of $k$ for which the RG trajectory might still be far away from the IR fixed point. Thus more detailed knowledge about the trajectory $k \mapsto (G(k), \Lambda(k))$ is necessary.

In order to get a first understanding of what could happen at small length scales let us consider the following “toy-model” RG trajectory. Let us assume that for $k$ below a certain critical value, $G(k)$ and $\Lambda(k)$ run according to the fixed point law (3.4) and they are approximately constant for $k$ well above this critical value. This behavior is motivated by the fact that from laboratory to galactic scales, say, we do not see any variation of $G$ and $\Lambda$. In this model the evolution of the homogeneous and isotropic background is still described by the fixed point cosmology: $k$ is small, the RG-trajectory is close to the fixed point, and $k \propto 1/t$ is the unique cutoff identification. However, if “lumps” form due to the gravitational attraction and if they are sufficiently small then their size can correspond to a $k$-value smaller than the critical one so that $G$ and $\Lambda$ are constant across these structures. As a consequence, a density perturbation is not accompanied by a perturbation (spatial inhomogeneity) of $G$ in this case.

Small scale perturbations of this type are described by Eq.(4.3) with $\gamma^n \equiv 0$ on the RHS,

$$
\ddot{\delta}^n + (2 - 3w)\frac{\dot{a}}{a} \dot{\delta}^n - \left[4\pi G \rho (1 - w)(3w + 1) + 2w \Lambda\right] \delta^n + w \nu^2_a \delta^n = 0
$$

(4.6)
Formally this equation is the same as in standard cosmology, but now the background evolution is given by the fixed point solution (3.5). If we set \( \delta_n(t) = A t^\alpha \), we obtain

\[
\alpha_1 = \frac{5(3w-1) - \sqrt{81w^2 + 138w + 73}}{6(1+w)} \quad \alpha_2 = \frac{5(3w-1) + \sqrt{81w^2 + 138w + 73}}{6(1+w)}
\]

(4.7)

and there is always a growing mode and a decaying mode, as in the standard case. In deriving (4.7) we used the long wavelength limit \( \nu_n^2 = 0 \) in (4.6). As we are interested in small structures this is not necessarily always allowed, but for the most relevant case \( w = 0 \) the eigenvalue \( \nu_n^2 \) drops out again so that (4.7) is indeed valid for small scale perturbations. For \( w = 0 \) the density perturbations grow approximately as

\[
\delta(t) \propto t^{0.59} \propto a^{0.44}
\]

(4.8)

In [1] we developed a cosmology of the Planck era immediately after the Big Bang which was governed by an UV fixed point. In this scenario \( w = 1/3 \) is a distinguished choice for the equation of state. From Eq.(4.5) we read off that in this “radiation dominated Planck era” there exist perturbations which grow as \( \delta(t) \propto t \propto a(t) \). Interestingly, for \( w = 1/3 \), Eq.(4.6) with \( \nu_n^2 = 0 \) yields a qualitatively similar result; from (4.7) we obtain growing modes with \( \delta(t) \propto t^{\sqrt{\pi}} \propto a^{\sqrt{\pi}} \).

V. CONCLUSION

We have presented a general covariant framework to investigate the evolution of cosmological perturbations in Robertson-Walker spacetimes with variable \( G \) and \( \Lambda \). In comparison to earlier work [15–18,29] on cosmologies with a time dependent \( G, \Lambda \) and possibly fine structure constant \( \alpha \) the new feature of our model is that the time dependence of \( G \) and \( \Lambda \) is a secondary effect which derives from a more fundamental scale dependence. In a typical Brans-Dicke type theory, say, the dynamics of the Brans-Dicke field \( \Phi = 1/G \) is governed
by a standard local lagrangian with a kinetic term \( \propto (D_\mu \Phi)^2 \). In our approach there is no simple lagrangian description of the \( G \)-dynamics. It rather arises from an RG equation for \( G(k) \) and a cutoff identification \( k = k(x) \). From the point of view of the gravitational field equations, \( G(x) \) has the status of an external scalar field.

Our main interest was to analyze a special type of spacetimes where the evolution of \( G \) and \( \Lambda \) in the background Universe is governed by a RG flow near a fixed point. In this case the flow is very simple and it is possible to obtain explicit solutions in a model independent way. In fact, as for the IR fixed point governing the late-time behavior of the Universe, no further assumptions beyond the very fixed point hypothesis need to be made. In particular, the IR fixed point is not necessarily due to quantum gravity effects as is the UV fixed point, it could have a purely classical origin (for instance within a classical averaging scenario [26]).

We performed a detailed stability analysis of the fixed point solution (3.5) which was first discussed in [1,12]. With the equation of state \( p = 0 \), appropriate for the late Universe, we found that the IR fixed point cosmology is stable against the formation of large scale inhomogeneities. As a consequence, structure formation on cosmological scales stops by the time when the Universe enters the fixed point regime. We also found that perturbations on small length scales still can grow even in the fixed point epoch. As a very rough estimate, we expect the dividing line between large- and small-scale perturbations to be of the order of the Hubble length.

It also turned out that if the evolution of the Universe immediately after the initial singularity at \( t = 0 \) is described by a UV fixed point solution (3.5) with \( w > 0 \), there is always an amplification of the small disturbances generated during the quantum gravity era. In particular for \( w = 1/3 \) density perturbations grow as \( \delta_n \propto a \). These fluctuations would then emerge as “primordial” density perturbations at the end of the Planck era and follow the subsequent evolution according to the standard Friedmann-Robertson-Walker dynamics.
It is clear, though, that the classical isotropy problem of the Big Bang cosmology is not solved by the IR fixed point postulate. A satisfactory solution to this problem would involve starting from a generic inhomogeneous and anisotropic Universe and showing that it evolves into an almost homogeneous and isotropic one long before the time of decoupling. It is intriguing to speculate that RG effects play a crucial role here. In this context it is an encouraging first result that the UV fixed point cosmology is free from the particle horizon present in the radiation dominated standard model [1].

Clearly more work is needed in order to confront the (IR) fixed point cosmology with the observations. As a first step we compare in [14] our predictions to the recent high-redshift supernova data. It turns out that, at least as far as these data are concerned, the fixed point model is phenomenologically viable. In ref. [14] we also extend the fixed point model proper by assuming that the fixed point epoch is preceded by an era with a constant $G$ and $\Lambda$. In this model, because of the accelerated $t^{4/3}$-expansion after the transition, perturbations first evolve inside the horizon, then they become larger than the horizon. If we assume that the dividing line between “large” and “small” perturbations is indeed of the order $H^{-1}$, then, in the fixed point regime, perturbations first grow inside the horizon with the exponents (4.8) and at some point they reach the asymptotic constant regime with (4.5). This behavior is similar to the one found in $\Lambda$-dominated cosmologies and it is a consequence of the cosmic-no hair theorem [30]. We shall come back to the extended model elsewhere.

We also hope to implement this formalism in a more realistic framework, taking into account the presence of dark matter in a multifluid description, in a $N$-body simulation where the background solution is given by the IR fixed point cosmology. Also, it will be important to analyze the model in the context of the recent data on the cosmic microwave background radiation and the cluster density data.
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