Z-MEASURES ON PARTITIONS RELATED TO THE INFINITE GELFAND PAIR \((S(2\infty), H(\infty))\)

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Abstract. The paper deals with the z-measures on partitions with the deformation (Jack) parameters 2 or 1/2. We provide a detailed explanation of the representation-theoretic origin of these measures, and of their role in the harmonic analysis on the infinite symmetric group.

Keywords. Infinite symmetric group, symmetric functions, Gelfand pairs, random partitions.

1. Introduction

Let \(S(\infty)\) denote the group whose elements are finite permutations of \(\{1, 2, 3, \ldots\}\). The group \(S(\infty)\) is called the infinite symmetric group, and it is a model example of a "big" group. The harmonic analysis for such groups is an active topic of modern research, with connections to different areas of mathematics from enumerative combinatorics to random growth models and to the theory of Painlevé equations. A theory of harmonic analysis on the infinite symmetric and infinite-dimensional unitary groups is developed by Kerov, Olshanski and Vershik \[15, 16\], Borodin \[4\], Borodin and Olshanski \[2, 8\]. For an introduction to harmonic analysis on the infinite symmetric group see Olshanski \[22\]. Paper by Borodin and Deift \[3\] studies differential equations arising in the context of harmonic analysis on the infinite-dimensional unitary group, and paper by Borodin and Olshanski \[7\] describes the link to problems of enumerative combinatorics, and to certain random growth models. For very recent works on the subject see, for example, Vershik and Tsilevich \[26\], Borodin and Kuan \[13\].

Set

\[ G = S(\infty) \times S(\infty), \]
\[ K = \text{diag} \, S(\infty) = \{(g, g) \in G \mid g \in S(\infty)\} \subset G. \]

Then \((G, K)\) is an infinite dimensional Gelfand pair in the sense of Olshanski \[21\]. It can be shown that the biregular spherical representation of \((G, K)\) in the space \(\ell^2(S(\infty))\) is irreducible. Thus the conventional scheme of non-commutative harmonic analysis is not applicable to the case of the infinite symmetric group.

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In 1993, Kerov, Olshanski and Vershik [13] (Kerov, Olshanski and Vershik [16] contains the details) constructed a family \( \{T_z : z \in \mathbb{C}\} \) of unitary representations of the bisymmetric infinite group \( G = S(\infty) \times S(\infty) \). Each representation \( T_z \) acts in the Hilbert space \( L^2(\mathfrak{G}, \mu_t) \), where \( \mathfrak{G} \) is a certain compact space called the space of virtual permutations, and \( \mu_t \) is a distinguished \( G \)-invariant probability measure on \( \mathfrak{G} \) (here \( t = |z|^2 \)). The representations \( T_z \) (called the generalized regular representations) are reducible. Moreover, it is possible to extend the definition of \( T_z \) to the limit values \( z = 0 \) and \( z = \infty \), and it turns out that \( T_\infty \) is equivalent to the biregular representation of \( S(\infty) \times S(\infty) \). Thus, the family \( \{T_z\} \) can be viewed as a deformation of the biregular representation. Once the representations \( T_z \) are constructed, the main problem of the harmonic analysis on the infinite symmetric group is in decomposition of the generalized regular representations \( T_z \) into irreducible ones.

One of the initial steps in this direction can be described as follows. Let \( 1 \) denote the function on \( \mathfrak{G} \) identically equal to 1. Consider this function as a vector of \( L^2(\mathfrak{G}, \mu_t) \). Then \( 1 \) is a spherical vector, and the pair \( (T_z, 1) \) is a spherical representation of the pair \( (G, K) \), see, for example, Olshanski [22], Section 2. The spherical function of \( (T_z, 1) \) is the matrix coefficient \( (T_z(g_1, g_2)1, 1) \), where \( (g_1, g_2) \in S(\infty) \times S(\infty) \). Set

\[
\chi_z(g) = (T_z(g, e)1, 1), \quad g \in S(\infty).
\]

The function \( \chi_z \) can be understood as a character of the group \( S(\infty) \) corresponding to \( T_z \). Kerov, Olshanski and Vershik [13,16] found the restriction of \( \chi_z \) to \( S(n) \) in terms of irreducible characters of \( S(n) \). Namely, let \( \mathbb{Y}_n \) be the set of Young diagrams with \( n \) boxes. For \( \lambda \in \mathbb{Y}_n \) denote by \( \chi^\lambda \) the corresponding irreducible character of the symmetric group \( S(n) \) of degree \( n \). Then for any \( n = 1, 2, \ldots \) the following formula holds true

\[
(1.1) \quad \chi_z \bigg|_{S(n)} = \sum_{\lambda \in \mathbb{Y}_n} M^{(n)}_{z, \bar{z}}(\lambda) \frac{\chi^\lambda}{\chi^\lambda(e)}.
\]

In this formula \( M^{(n)}_{z, \bar{z}} \) is a probability measure (called the \( z \)-measure) on the set of Young diagrams with \( n \) boxes, or on the set of integer partitions of \( n \). Formula (1.1) defines the \( z \)-measure \( M^{(n)}_{z, \bar{z}} \) as a weight attached to the corresponding Young diagram in the decomposition of the restriction of \( \chi_z \) to \( S(n) \) in irreducible characters of \( S(n) \). Expression (1.1) enables to reduce the problem of decomposition of \( T_z \) into irreducible components to the problem on the computation of spectral counterparts of \( M^{(n)}_{z, \bar{z}} \).

In addition to their role in the harmonic analysis on the infinite symmetric group the \( z \)-measures described above are quite interesting objects by themselves. It is possible to introduce more general objects, namely measures \( M^{(n)}_{z, z'} \) on Young diagrams with \( n \) boxes. Such measures depend on two complex parameters \( z, z' \). If \( z' = \bar{z} \), then \( M^{(n)}_{z, \bar{z}} \) coincide with the \( z \)-measures in equation (1.1). Under suitable restrictions on \( z \) and \( z' \) the weights \( M^{(n)}_{z, z'} \) are nonnegative and their sum is equal to 1. Thus \( M^{(n)}_{z, z'} \) can be understood...
as probability measures on $\mathbb{Y}_n$. For special values of parameters $z, z'$ the z-measures turn into discrete orthogonal polynomial ensembles which in turn related to interesting probabilistic models, see Borodin and Olshanski [7]. In addition, the z-measures are a particular case of the Schur measures introduced by Okounkov in [20]. The z-measures $M^{(n)}_{z,z'}$ were studied in details in the series of papers by Borodin and Olshanski [5, 7, 9, 10], in Okounkov [19], and in Borodin, Olshanski, and Strahov [11].

Moreover, as it follows from Kerov [14], Borodin and Olshanski [9] it is natural to consider a deformation $M^{(n)}_{z,z',\theta}$ of $M^{(n)}_{z,z'}$, where $\theta > 0$ is called the parameter of deformation (or the Jack parameter). Then the measures $M^{(n)}_{z,z'}$ can be thought as the z-measures with the Jack parameter $\theta = 1$. It is shown in Borodin and Olshanski [9], that $M^{(n)}_{z,z'}$ are in many ways similar to log-gas (random-matrix) models with arbitrary $\beta = 2\theta$. In particular, if $\theta = 2$ or $\theta = 1/2$ one expects that $M^{(n)}_{z,z'}$ will lead to Pfaffian point processes, similar to ensembles of Random Matrix Theory of $\beta = 4$ or $\beta = 1$ symmetry types, see Borodin and Strahov [12], Strahov [25] for the available results in this direction.

It is the purpose of the present paper to describe the origin of z-measures with the Jack parameters $\theta = 2$ and $\theta = 1/2$ in the representation theory. First we recall the notion of the z-measures with an arbitrary Jack parameter $\theta > 0$. Then we consider the symmetric group $S(2n)$ viewed as the group of permutations of the set $\{-n, \ldots, -1, 1, \ldots, n\}$, and its subgroup $H(n)$ defined as the centralizer of the product of transpositions $(-n, n), (-n+1, n-1), \ldots, (-1, 1)$. The group $H(n)$ is called the hyperoctahedral group of degree $n$. One knows that $(S(2n), H(n))$ are Gelfand pairs, and their inductive limit, $(S(2\infty), H(\infty))$, is an infinite dimensional Gelfand pair, see Olshanski [21]. We describe the construction of a family of unitary spherical representations $T_{z,\frac{1}{2}}^{\theta}$ of the infinite dimensional Gelfand pair $(S(2\infty), H(\infty))$ and show that z-measures with the Jack parameters $\theta = 1/2$ appear as coefficients in the decomposition of the spherical functions of $T_{z,\frac{1}{2}}^{\theta}$ into spherical functions of the Gelfand pair $(S(2n), H(n))$. Due to the fact that z-measures with the Jack parameters $\theta = 2$ and $\theta = 1/2$ are related to each other in a very simple way, see Proposition 2.2 the construction described above provides a representation-theoretic interpretation for z-measures with the Jack parameter $\theta = 2$ as well. Therefore, it is natural to refer to such z-measures as to the z-measures for the infinite dimensional Gelfand pair $(S(2\infty), H(\infty))$, or, more precisely, as to the z-measures of the representation $T_{z,\frac{1}{2}}^{\theta}$.

The fact that these measures play a role in the harmonic analysis was mentioned in Borodin and Olshanski [9], and in our explanation of this representation-theoretic aspect we used many ideas from Olshanski [23].

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2. The $z$-measures on partitions with the general parameter $\theta > 0$

We use Macdonald [17] as a basic reference for the notations related to integer partitions and to symmetric functions. In particular, every decomposition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) : n = \lambda_1 + \lambda_2 + \ldots + \lambda_l,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$ are positive integers, is called an integer partition. We identify integer partitions with the corresponding Young diagrams. The set of Young diagrams with $n$ boxes is denoted by $\mathbb{Y}_n$.

Following Borodin and Olshanski [9], Section 1, and Kerov [14] let $M_{z, z', \theta}^{(n)}$ be a complex measure on $\mathbb{Y}_n$ defined by

$$M_{z, z', \theta}^{(n)}(\lambda) = \frac{n!(z)_{\lambda, \theta}(z')_{\lambda, \theta}}{(t)_{n}H(\lambda, \theta)H'(\lambda, \theta)},$$

where $n = 1, 2, \ldots$, and where we use the following notation

- $z, z' \in \mathbb{C}$ and $\theta > 0$ are parameters, the parameter $t$ is defined by $t = \frac{zz'}{\theta}$.

- $(t)_n$ stands for the Pochhammer symbol,

$$\frac{\Gamma(t + n)}{\Gamma(t)} = (t)_n = t(t + 1) \ldots (t + n - 1).$$

- $(z)_{\lambda, \theta}$ is a multidimensional analogue of the Pochhammer symbol defined by

$$\prod_{(i,j) \in \lambda} (z + (j - 1) - (i - 1)\theta) = \prod_{i=1}^{l(\lambda)} (z - (i - 1)\theta)_{\lambda_i}.$$

Here $(i, j) \in \lambda$ stands for the box in the $i$th row and the $j$th column of the Young diagram $\lambda$, and we denote by $l(\lambda)$ the number of nonempty rows in the Young diagram $\lambda$.

- $H(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1)$,

$$H'(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta),$$

where $\lambda'$ denotes the transposed diagram.

**Proposition 2.1.** The following symmetry relations hold true

$$H(\lambda, \theta) = \theta^{l(\lambda)}H'(\lambda', \frac{1}{\theta}), \quad (z)_{\lambda, \theta} = \theta^{-l(\lambda)}\left(\frac{z}{\theta}\right)_{\lambda', \frac{1}{\theta}}.$$

Here $|\lambda|$ stands for the number of boxes in the diagram $\lambda$.

**Proof.** These relations follow immediately from definitions of $H(\lambda, \theta)$ and $(z)_{\lambda, \theta}$. $\square$
Proposition 2.2. We have

\[ M_{z,z',\theta}(\lambda) = M_{z/\theta,z'/\theta,1/\theta}(\lambda'). \]

Proof. Use definition of \( M_{z,z',\theta}(\lambda) \), equation (2.1), and apply Proposition 2.1. \( \square \)

Proposition 2.3. We have

\[ \sum_{\lambda \in \mathcal{Y}_n} M_{z,z',\theta}(\lambda) = 1. \]

Proof. See Kerov [14], Borodin and Olshanski [9, 6]. \( \square \)

Proposition 2.4. If parameters \( z, z' \) satisfy one of the three conditions listed below, then the measure \( M_{z,z',\theta}^{(n)} \) defined by expression (2.1) is a probability measure on \( \mathcal{Y}_n \). The conditions are as follows.

- **Principal series:** either \( z \in \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} + \mathbb{Z}_{\geq 0}\theta) \) and \( z' = \overline{z} \).
- **The complementary series:** the parameter \( \theta \) is a rational number, and both \( z, z' \) are real numbers lying in one of the intervals between two consecutive numbers from the lattice \( \mathbb{Z} + \mathbb{Z}\theta \).
- **The degenerate series:** \( z, z' \) satisfy one of the following conditions
  - (1) \( (z = m\theta, z' > (m - 1)\theta) \) or \( (z' = m\theta, z > (m - 1)\theta) \);
  - (2) \( (z = -m, z' < -m + 1) \) or \( (z' = -m, z < -m + 1) \).

Proof. See Propositions 1.2, 1.3 in Borodin and Olshanski [9]. \( \square \)

Thus, if the conditions in the Proposition above are satisfied, then \( M_{z,z',\theta}^{(n)} \) is a probability measure defined on \( \mathcal{Y}_n \), as follows from Proposition 2.3. In case when \( z, z' \) are taken either from the principal series or the complementary series we refer to \( z, z' \) as admissible parameters of the \( z \)-measure under considerations. We will refer to \( M_{z,z',\theta}^{(n)}(\lambda) \) as to the \( z \)-measure with the deformation (Jack) parameter \( \theta \).

Remark 2.5. When both \( z, z' \) go to infinity, expression (2.1) has a limit

\[ M_{\infty,\infty,\theta}^{(n)}(\lambda) = \frac{n!\theta^n}{H(\lambda, \theta)H'(\lambda, \theta)} \]

called the Plancherel measure on \( \mathcal{Y}_n \) with general \( \theta > 0 \). Statistics of the Plancherel measure with the general Jack parameter \( \theta > 0 \) is discussed in many papers, see, for example, a very recent paper by Matsumoto [18], and references therein. Matsumoto [18] compares limiting distributions of rows of random partitions with distributions of certain random variables from a traceless Gaussian \( \beta \)-ensemble.

3. The spaces \( X(n) \) and their projective limit

3.1. The homogeneous space \( X(n) = H(n) \setminus S(2n) \). Let \( S(2n) \) be the permutation group of \( 2n \) symbols realized as that of the set \( \{ -n, \ldots, -1, 1, \ldots, n \} \). Let \( \tilde{t} \in S(2n) \) be the product of the transpositions \( (-n, n), (-n + 1, n - 1), \ldots \).
1), \ldots, (-1, 1). By definition, the group $H(n)$ is the centralizer of $\tilde{t}$ in $S(2n)$. We can write

$$H(n) = \left\{ \sigma \in S(2n), \sigma \tilde{t} \sigma^{-1} = \tilde{t} \right\}.$$ 

The group $H(n)$ is called the hyperoctahedral group of degree $n$.

Set $X(n) = H(n) \setminus S(2n)$. Thus $X(n)$ is the space of right cosets of the subgroup $H(n)$ in $S(2n)$.

It is not hard to check that the set $X(n)$ can be realized as the set of all pairings of $\{-n, \ldots, -1, 1, \ldots, n\}$ into $n$ unordered pairs. Thus every element $\tilde{x}$ of $X(n)$ is representable as a collection of $n$ unordered pairs,

$$(3.1) \quad \tilde{x} \in X(n) \iff \tilde{x} = \left\{ i_1, i_2, \ldots, i_{2n-1}, i_{2n} \right\},$$

where $i_1, i_2, \ldots, i_{2n}$ are distinct elements of the set $\{-n, \ldots, -1, 1, \ldots, n\}$.

For example, if $n = 2$, then $S(4)$ is the permutation group of $\{-2, -1, 1, 2\}$, the element $\tilde{t}$ is the product of transpositions $(-2, -1)$ and $(1, 2)$, the subgroup $H(2)$ is

$$H(2) = \left\{ \begin{pmatrix} -2 & -1 & 1 & 2 \\ -2 & -1 & 1 & 2 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -1 & 1 & 2 \\ 2 & -1 & 1 & -2 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 \end{pmatrix} \right\},$$

and the set $X(2)$ is the set consisting of three elements, namely

$$\left\{ \{-2, -1\}, \{1, 2\} \right\}, \left\{ \{-2, 1\}, \{-1, 2\} \right\}, \text{ and } \left\{ \{-2, -2\}, \{-1, 1\} \right\}.$$ 

So each element of $X(2)$ is the pairing of $\{-2, -1, 1, 2\}$ into (two) unordered pairs.

We have

$$|X(n)| = \frac{|S(2n)|}{|H(n)|} = \frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdots (2n-1).$$

3.2. Canonical projections $p_{n,n+1} : X(n+1) \to X(n)$. The projective limit of the spaces $X(n)$. Given an element $\tilde{x}' \in X(n+1)$ we define its derivative element $\tilde{x} \in X(n)$ as follows. Represent $\tilde{x}'$ as $n + 1$ unordered pairs, as it is explained in the previous Section. If $n + 1$ and $-n - 1$ are in the same pair, then $\tilde{x}$ is obtained from $\tilde{x}'$ by deleting this pair. Suppose that $n + 1$ and $-n - 1$ are in different pairs. Then $\tilde{x}'$ can be written as

$$\tilde{x}' = \left\{ i_1, i_2, \ldots, i_m, -n - 1, \ldots, i_k, n + 1, \ldots, i_{2n+1}, i_{2n+2} \right\}.$$
In this case $\tilde{x}$ is obtained from $\tilde{x}'$ by removing $-n - 1$, $n + 1$ from pairs \( \{i_m, -n - 1\} \) and \( \{i_k, n + 1\} \) correspondingly, and by replacing two these pairs, \( \{i_m, -n - 1\} \) and \( \{i_k, n + 1\} \), by one pair \( \{i_m, i_k\} \). The map $\tilde{x}' \to \tilde{x}$, denoted by $p_{n,n+1}$, will be referred to as the canonical projection of $X(n+1)$ onto $X(n)$.

Consider the sequence

\[ X(1) \leftarrow \ldots \leftarrow X(n) \leftarrow X(n+1) \leftarrow \ldots \]

of canonical projections, and let

\[ X = \varprojlim X(n) \]

denote the projective limit of the sets $X(n)$. By definition, the elements of $X$ are arbitrary sequences $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots)$, such that $\tilde{x}_n \in X(n)$, and $p_{n,n+1}(\tilde{x}_{n+1}) = \tilde{x}_n$. The set $X$ is a closed subset of the compact space of all sequences $(\tilde{x}_n)$, therefore, it is a compact space itself.

In what follows we denote by $p_n$ the projection $X \to X(n)$ defined by $p_n(\tilde{x}) = \tilde{x}_n$.

### 3.3. Cycles. Representation of elements of $X(n)$ in terms of arrow configurations on circles.

Let $\tilde{x}$ be an element of $X(n)$. Then $\tilde{x}$ can be identified with arrow configurations on circles. Such arrow configurations can be constructed as follows. Once $\tilde{x}$ is written as a collection of $n$ unordered pairs, one can represent $\tilde{x}$ as a union of cycles of the form

\[
\begin{align*}
  j_1 & \to -j_2 & \to j_2 & \to -j_3 & \to j_3 & \to \ldots & \to -j_k & \to j_k & \to -j_1 & \to j_1,
\end{align*}
\]

where $j_1, j_2, \ldots, j_k$ are distinct integers from the set \(-n, \ldots, n\). For example, take

\[
\tilde{x} = \left\{ \{1, 3\}, \{-2, 5\}, \{2, -1\}, \{-3, -5\}, \{4, -6\}, \{-4, 6\} \right\}.
\]

Then $\tilde{x} \in X(3)$, and it is possible to think about $\tilde{x}$ as a union of two cycles, namely

\[ 1 \to 3 \to -3 \to -5 \to 5 \to -2 \to 2 \to -1 \to 1, \]

and

\[ 4 \to -6 \to 6 \to -4 \to 4. \]

Cycle (3.2) can be represented as a circle with attached arrows. Namely, we put on a circle points labelled by $|j_1|$, $|j_2|$, $\ldots$, $|j_k|$, and attach arrows to these points according to the following rules. The arrow attached to $|j_1|$ is directed clockwise. If the next integer in the cycle (3.2), $j_2$, has the same sign as $j_1$, then the direction of the arrow attached to $|j_2|$ is the same as the direction of the arrow attached to $|j_1|$, i.e. clockwise. Otherwise, if the sign of $j_2$ is opposite to the sign of $j_1$, the direction of the arrow attached to $|j_2|$ is opposite to the direction of the arrow attached to $|j_1|$, i.e. counterclockwise. Next, if the integer $j_3$ has the same sign as $j_2$, then the direction of the arrow attached to $|j_3|$ is the same as the direction of the arrow attached to $|j_2|$, etc. For example, the representation of of the element $\tilde{x}$ defined by (3.3) in terms of arrow configurations on circles is shown on Fig. 1.
In terms of arrow configurations on circles. The first circle (from the left) represents cycle $1 \rightarrow 3 \rightarrow -3 \rightarrow -5 \rightarrow 5 \rightarrow -2 \rightarrow 2 \rightarrow -1 \rightarrow 1$, and the second circle represents cycle $4 \rightarrow -6 \rightarrow 6 \rightarrow -4 \rightarrow 4$.

In this representation the projection $p_{n,n+1} : X(n+1) \rightarrow X(n)$ is reduced to removing the point $n + 1$ together with the attached arrow.

4. The $t$-measures on $X$

4.1. Probability measures $\mu_t^{(n)}$ on $X(n)$, and $\mu_t$ on $X$. The measures $\mu_t^{(n)}$ on the spaces $X(n)$ are natural analogues of the Ewens measures on the group $S(n)$ described in Kerov, Olshanski and Vershik [16].

Definition 4.1. For $t > 0$ we set

$$\mu_t^{(n)}(\tilde{x}) = \frac{t^{|[\tilde{x}]_n}}{t(t + 2) \ldots (t + 2n - 2)},$$

where $\tilde{x} \in X(n)$, and $[\tilde{x}]_n$ denotes the number of cycles in $\tilde{x}$, or the number of circles in the representation of $\tilde{x}$ in terms of arrow configurations, see Section 3.3.

Proposition 4.2. a) We have

$$\sum_{\tilde{x} \in X(n)} \mu_t^{(n)}(\tilde{x}) = 1.$$ 

Thus $\mu_t^{(n)}(\tilde{x})$ can be understood as a probability measure on $X(n)$.

b) Given $t > 0$, the canonical projections $p_{n,n+1}$ preserve the measures $\mu_t^{(n)}(\tilde{x})$, which means that the condition

$$\mu_t^{(n+1)}\left(\{\tilde{x}' \mid \tilde{x}' \in X(n+1), p_{n,n+1}(\tilde{x}') = \tilde{x}\}\right) = \mu_t^{(n)}(\tilde{x})$$

is satisfied for each $\tilde{x} \in X(n)$.
**Proof.** If $n = 1$, then $X(1)$ consists of only one element, namely $\{-1, 1\}$, and from Definition 4.1 we immediately see that equation (4.1) is satisfied in this case.

Let $\bar{x}$ be an arbitrary element of $X(n)$. Represent $\bar{x}$ in terms of circles with attached arrows, as it is explained in Section 3.3. Consider the set

$$\{\bar{x}' \mid \bar{x}' \in X(n + 1), p_{n,n+1}(\bar{x}') = \bar{x}\}.$$ (4.3)

It is not hard to see that this set consists of $2^n + 1$ points. Indeed, given $\bar{x} \in X(n)$ we can obtain $\bar{x}'$ from set (4.3) (i.e. $\bar{x}'$ which lies above $\bar{x}$ with respect to the canonical projection $p_{n,n+1}$) by adding an arrow to existing circle in $2^n$ ways, or by creating a new circle.

If $\bar{x}'$ is obtained from $\bar{x}$ by creating a new circle, then

$$[\bar{x}']_{n+1} = [\bar{x}]_n + 1, \quad \text{and} \quad t[\bar{x}']_{n+1} = t[\bar{x}]_{n+1}.$$ (4.4)

If $\bar{x}'$ is obtained from $\bar{x}$ by adding an arrow to an existing circle, then

$$[\bar{x}']_{n+1} = [\bar{x}]_n.$$ (4.5)

Therefore, the relation

$$\sum_{\bar{x}' \in X(n + 1)} t[\bar{x}']_{n+1} = (t + 2n) \sum_{\bar{x} \in X(n)} t[\bar{x}]_n$$

is satisfied. From the recurrent relation above we obtain

$$\sum_{\bar{x}' \in X(n + 1)} t[\bar{x}']_{n+1} = t(t + 2) \ldots (t + 2n).$$ (4.6)

This formula is equivalent to equation (4.1), and the first statement of the Proposition is proved.

Let us now prove the second statement of the Proposition. We need to show that the condition (4.2) is satisfied for each $\bar{x} \in X(n)$. We have

$$\mu_t^{(n+1)}(\{\bar{x}' \mid \bar{x}' \in X(n), p_{n,n+1}(\bar{x}') = \bar{x}\}) = \sum_{\bar{x}', \bar{x}' \in X(n+1), p_{n,n+1}(\bar{x}')=\bar{x}} \frac{t[\bar{x}']_{n+1}}{t(t + 2) \ldots (t + 2n)},$$

where we have used the definition of $\mu_t^{(n)}$, Definition 4.1. By the same argument as in the proof of the first statement of the Proposition the sum in the righthand side of equation (4.1) can be decomposed into two sums. This first sum runs over those $\bar{x}'$ that are obtained from $\bar{x}$ by adding an arrow to one of the existing circles of $\bar{x}$. This sum is equal to

$$\frac{(2n)t[\bar{x}]_n}{t(t + 2) \ldots (t + 2n)}.$$ (4.7)

The second sum runs over those $\bar{x}'$ that are obtained from $\bar{x}$ by creating a new circle. There is only one such $\bar{x}'$, and its contribution is

$$\frac{t t[\bar{x}]_n}{t(t + 2) \ldots (t + 2n)}.$$ (4.8)
Adding expressions (4.5) and (4.6) we obtain
\[
\mu_t^{(n+1)} \left( \{ x' | x' \in X(n), p_{n,n+1}(x') = x \} \right) = \frac{(2n)t[x^n_n]}{t(t+2)\ldots(t+2n)} + \frac{t t[x^n_{n+1}]}{t(t+2)\ldots(t+2n)},
\]
and the righthand side of the above equation is \( \mu_t^{(n)} \tilde{x} \).

It follows from Proposition 4.2 that for any given \( t > 0 \), the canonical projection \( p_{n-1,n} \) preserves the measures \( \mu_t^{(n)} \). Hence the measure
\[
\mu_t = \lim_{n \to \infty} \mu_t^{(n)}
\]
on \( X \) is correctly defined, and it is a probability measure. Note that as in the case considered in Kerov, Olshanski, and Vershik [16], Section 2, the probability space \( \langle X, \mu_t \rangle \) is closely related to the Chinese Restaurant Process construction, see Aldous [1], Pitman [24].

4.2. The group \( S(2\infty) \) and its action on the space \( X \). First we describe the right action of the group \( S(2n) \) on the space \( X(n) \), and then we extend it to the right action of \( S(2\infty) \) on \( X \).

Let \( \tilde{x}_n \in X(n) \). Then \( \tilde{x}_n \) can be written as a collection of \( n \) unordered pairs (equation (3.1)). Let \( g \) be a permutation from \( S(2n) \),
\[
g: \begin{pmatrix} -n & -n+1 & \ldots & n-1 & n \\ g(-n) & g(-n+1) & \ldots & g(n-1) & g(n) \end{pmatrix}.
\]
The right action of the group \( S(2n) \) on the space \( X(n) \) is defined by
\[
\tilde{x}_n \cdot g = \{ (g(i_1), g(i_2)), (g(i_3), g(i_4)), \ldots, (g(i_{2n-1}), g(i_{2n})) \}.
\]

**Proposition 4.3.** The canonical projection \( p_{n,n+1} \) is equivariant with respect to the right action of the group \( S(2n) \) on the space \( X(n) \), which means
\[
p_{n,n+1}(\tilde{x} \cdot g) = p_{n,n+1}(\tilde{x}) \cdot g,
\]
for all \( \tilde{x} \in X(n+1) \), and all \( g \in S(2n) \).

**Proof.** Let \( \tilde{x} \) be an arbitrary element of \( X(n+1) \). Represent \( \tilde{x} \in X(n+1) \) in terms of configurations of arrows on circles, as it is described in Section 3.3. In this representation the right action of an element \( g \) from \( S(2n) \) on \( x \) is reduced to permutations of numbers \( 1, 2, \ldots, n \) on the circles, and to changes in the directions of the arrows attached to these numbers. The number \( n+1 \), and the direction of the arrow attached to \( n+1 \) remains unaffected by the action of \( S(2n) \). Since \( p_{n,n+1}(\tilde{x}) \) is obtained from \( \tilde{x} \) by deleting \( n+1 \) together with the attached arrow, the statement of the Proposition follows.

Since the canonical projection \( p_{n,n+1} \) is equivariant, the right action of \( S(2n) \) on \( X(n) \) can be extended to the right action of \( S(2\infty) \) on \( X \). For \( n = 1, 2, \ldots \) we identify \( S(2n) \) with the subgroup of permutations \( g \in S(2n+2) \) preserving the elements \( -n-1 \) and \( n+1 \) of the set \( \{-n-1, -n, \ldots, -1, 1, \ldots, n, n+1\} \), i.e.
\[
S(2n) = \left\{ g | g \in S(2n+2), g(-n-1) = -n-1, \text{ and } g(n+1) = n+1 \right\}.
\]
Let $S(2) \subset S(4) \subset S(4) \ldots$ be the collection of such subgroups. Set

$$S(2\infty) = \bigcup_{n=1}^{\infty} S(2n).$$

Thus $S(2\infty)$ is the inductive limit of subgroups $S(2n)$,

$$S(2\infty) = \lim\downarrow S(2n).$$

If $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots) \in X$, and $g \in S(2\infty)$, then the right action of $S(2\infty)$ on $X = \lim\downarrow X(n)$,

$$X \times S(2\infty) \rightarrow X,$$

is defined as $\tilde{x} \cdot g = \tilde{y}$, where $\tilde{x}_n \cdot g = \tilde{y}_n$ for all $n$ so large that $g \in S(2\infty)$ lies in $S(2n)$.

**Proposition 4.4.** We have

$$p_n(\tilde{x} \cdot g) = p_n(\tilde{x}) \cdot g$$

for all $\tilde{x} \in X$, $g \in S(2\infty)$, and for all $n$ so large that $g \in S(2n)$.

**Proof.** The claim follows immediately from the very definition of the projection $p_n$, and of the right action of $S(2\infty)$ on $X$. \hfill $\square$

### 4.3. The fundamental cocycle.

Recall that $\lfloor . \rfloor_n$ denotes the number of cycles in the cycle representation of an element from $X(n)$ (see Section 3.3 where the cycle structure of the elements from $X(n)$ was introduced).

**Proposition 4.5.** For any $\tilde{x} = (\tilde{x}_n) \in X$, and $g \in S(2\infty)$, the quantity

$$c(\tilde{x}; g) = [p_n(\tilde{x} \cdot g)]_n - [p_n(\tilde{x})]_n = [p_n(\tilde{x}) \cdot g]_n - [p_n(\tilde{x})]_n,$$

does not depend on $n$ provided that $n$ is so large that $g \in S(2n)$.

**Proof.** Let $g$ be an element of $S(2n)$. To prove the Proposition it is enough to show that the condition

$$(4.7) \quad [p_{n,n+1}(\tilde{x}) \cdot g]_n - [p_{n,n+1}(\tilde{x})]_n = [\tilde{x} \cdot g]_{n+1} - [\tilde{x}]_{n+1}$$

is satisfied for any element $\tilde{x}$ of $X(n + 1)$. Since $g \in S(2n)$ can be always represented as a product of transpositions, and since $p_{n,n+1}$ is equivariant with respect to the right action of the group $S(2n)$, it is enough to prove (4.7) for the case when $g$ is a transposition. Thus we assume that $g$ is a transposition $(ij) \in S(2n)$, where $i$ and $j$ are two different elements of the set $\{-n, \ldots, -1, 1, \ldots, n\}$.

Let $\tilde{x}$ be an element of $X(n + 1)$. Write $\tilde{x}$ as a collection of cycles as it is explained in Section 3.3. Assume that both $i$ and $j$ belong to the same cycle of $\tilde{x}$. We check that the multiplication of $\tilde{x}$ by $(i,j)$ from the right either splits this cycle into two, or transforms it into a different cycle. Thus we have

$$[\tilde{x} \cdot g]_{n+1} - [\tilde{x}]_{n+1} = 1 \text{ or } 0.$$

The value of the difference $[\tilde{x} \cdot g]_{n+1} - [\tilde{x}]_{n+1}$ depends on the mutual configuration of $-i, i, -j$, and $j$ in the cycle containing $i, j$. More explicitly, if the pair with $-i$ is situated from the left to the pair with $i$, and, at the same
time, the pair with \(-j\) is situated from the left to the pair with \(j\), then the value of \([\dot{x} \cdot g]_{n+1} - [\bar{x}]_{n+1}\) is 1. In this case the cycle under considerations has the form
\[
k_1 \rightarrow \ldots \rightarrow k_m \rightarrow -i \rightarrow i \rightarrow -k_{m+1} \rightarrow \ldots \rightarrow k_p \rightarrow -j \rightarrow j \rightarrow -k_{p+1} \rightarrow \ldots \rightarrow -k_1 \rightarrow k_1,
\]
or the form
\[
k_1 \rightarrow \ldots \rightarrow k_m \rightarrow -j \rightarrow j \rightarrow -k_{m+1} \rightarrow \ldots \rightarrow k_p \rightarrow -i \rightarrow i \rightarrow -k_{p+1} \rightarrow \ldots \rightarrow -k_1 \rightarrow k_1,
\]
and the corresponding mutual configuration of \(-i, i, -j, \) and \(j\) is
\[
\{., -i\}\{., i\} \ldots \{., -j\}\{j, .\},
\]
or
\[
\{., -j\}\{j, .\} \ldots \{., i\}\{i, .\}.
\]
If in the cycle under considerations the pair with \(-i\) stands from the right to the pair with \(i\), and, at the same time, the pair with \(-j\) is situated from the right to the pair with \(j\), then the value of \([\dot{x} \cdot g]_{n+1} - [\bar{x}]_{n+1}\) is 1 as well. In this case the mutual configuration of \(-i, i, -j, \) and \(j\) is
\[
\{., i\}\{., -i\} \ldots \{., j\}\{-j, .\},
\]
or
\[
\{., j\}\{-j, .\} \ldots \{., i\}\{-i, .\}.
\]
Otherwise, if the mutual configuration of \(-i, i, -j, \) and \(j\) is different from those described above, then \([\dot{x} \cdot g]_{n+1} - [\bar{x}]_{n+1} = 0\).

On the other hand, the numbers \(i\) and \(j\) belong to the one and the same cycle of \(p_{n,n+1}(\bar{x})\) if and only if they belong to one and the same cycle of \(\bar{x}\). Moreover, if \(i\) and \(j\) belong to the same cycle of \(\bar{x}\), then the mutual configuration of \(-i, i, -j, \) and \(j\) is the same as in \(p_{n,n+1}(\bar{x})\). Thus we conclude that equation (4.7) holds true if \(i\) and \(j\) belong to the same cycle of \(\bar{x}\).

If \(i\) and \(j\) belong to different cycles then the two cycles of \(\bar{x}\) containing the elements \(i\) and \(j\) merge into a single cycle of the product \(\dot{x} \cdot (ij)\), and we clearly have
\[
[\dot{x} \cdot (ij)]_{n+1} - [\bar{x}]_{n+1} = -1.
\]
The same equation holds true if we replace \(\bar{x}\) by \(p_{n,n+1}(\bar{x})\), so equation (4.7) holds true when \(i\) and \(j\) belong to different cycles as well. \(\square\)

4.4. Quasiinvariance of \(\mu_t\).

**Proposition 4.6.** Each of measures \(\mu_t\), \(0 < t < \infty\), is quasiinvariant with respect to the action of \(S(2\infty)\) on the space \(X = \varprojlim X(n)\). More precisely,
\[
\frac{\mu_t(d\dot{x} \cdot g)}{\mu_t(d\bar{x})} = t^{c(\dot{x}; g)}; \quad \dot{x} \in X, \ g \in S(2\infty),
\]
where \(c(\dot{x}; g)\) is the fundamental cocycle of Section 4.3.

**Proof.** We need to check that
\[
\mu_t(V \cdot g) = \int_V t^{c(\dot{x}; g)} \mu_t(d\dot{x}), \ g \in S(2\infty)
\]
(4.8)
for every Borel subset \( V \subseteq X \). Choose \( m \) so large that \( g \in S(2m) \), and let \( n \geq m \). Take \( \bar{y} \in X(n) \), and set \( V_n(\bar{y}) = p_n^{-1}(\bar{y}) \subset X \). This is a cylinder set. It is enough to check equation (4.8) for \( V = V_n(\bar{y}) \). Note that \( V_n(\bar{y}) \cdot g = V_n(\bar{y} \cdot g) \). This follows from the fact that the projection \( p_n \) is equivariant with respect to the right action of the group, see Proposition 4.4.

From the definition of \( \mu_t \) we conclude that \( \mu_t(V_n(\bar{y})) = \mu_t^{(n)}(\{\bar{y}\}) \), hence

\[
\mu_t(V_n(g) \cdot g) = \mu_t^{(n)}(\{\bar{y} \cdot g\}).
\]

On the other hand,

\[
c(\bar{x}; g) = [p_n(\bar{x} \cdot g)]_n - [p_n(\bar{x})]_n = [\bar{y} \cdot g]_n - [\bar{y}]_n
\]

for all \( \bar{x} \in V_n(y) \). Therefore, equation (4.8) takes the form

\[
\mu_t^{(n)}(\{\bar{y} \cdot g\}) = t^{[\bar{y} \cdot g]_n - [\bar{y}]_n} \mu_t^{(n)}(\{\bar{y}\}).
\]

Using the very definition of \( \mu_t^{(n)} \) we check that the equation just written above holds true. Therefore, equation (4.8) holds true as well. \( \square \)

5. The representations \( T_{z, \frac{1}{2}} \)

The aim of this Section is to introduce a family \( T_{z, \frac{1}{2}} \) of unitary representations of the group \( S(2\infty) \). These representations are parameterized by points \( z \in \mathbb{C} \setminus \{0\} \), and can be viewed as the analogues of the generalized regular representations introduced in Kerov, Olshanski, and Vershik [15] [16]. As in the case of the generalized regular representations, each element of the family \( T_{z, \frac{1}{2}} \) can be approximated by the regular representation of the group \( S(2n) \). This enables us to give an explicit formula for the restriction of the spherical function of the representation \( T_{z, \frac{1}{2}} \) to \( S(2n) \), and to introduce the measures on Young diagrams associated with representations \( T_{z, \frac{1}{2}} \). Then it will be shown that these measures can be understood as the \( z \)-measures with the Jack parameter \( \theta = \frac{1}{2} \) in the notation of Section 2. Thus the \( z \)-measures with the Jack parameter \( \theta = \frac{1}{2} \) will be associated to representations \( T_{z, \frac{1}{2}} \) in a similar way as the \( z \)-measures with the Jack parameter \( \theta = 1 \) are associated with generalized regular representations in Kerov, Olshanski, and Vershik [16], Section 4.

5.1. Definition of \( T_{z, \frac{1}{2}} \). Let \( (\mathcal{X}, \Sigma, \mu) \) be a measurable space. Let \( G \) be a group which acts on \( \mathcal{X} \) from the right, and preserves the Borel structure. Assume that the measure \( \mu \) is quasiinvariant, i.e. the condition

\[
d\mu(\bar{x} \cdot g) = \delta(\bar{x}; g)d\mu(\bar{x})
\]

is satisfied for some nonnegative \( \mu \)-integrable function \( \delta(\bar{x}; g) \) on \( \mathcal{X} \), and for every \( g, \bar{g} \in G \). Set

\[
(T(g)f)(\bar{x}) = \tau(\bar{x}; g)f(\bar{x} \cdot g), \quad f \in L^2(\mathcal{X}, \mu),
\]

where \( |\tau(\bar{x}; g)|^2 = \delta(\bar{x}; g) \). If

\[
\tau(\bar{x}; g_1g_2) = \tau(\bar{x} \cdot g_1; g_2)\tau(\bar{x}; g_1), \quad \bar{x} \in \mathcal{X}, g_1, g_2 \in G,
\]

\[
\mu_n(\bar{x}) = \frac{1}{n!} \int_{\mathcal{X}^n} \delta(\bar{x}; g_1g_2) \cdots \delta(\bar{x}; g_n)d\mu(\bar{x}g_1) \cdots d\mu(\bar{x}g_n)
\]

where \( \mathcal{X}^n = \{\bar{x}_1, \ldots, \bar{x}_n\} \) for every \( \bar{x}_1, \ldots, \bar{x}_n \in \mathcal{X} \). Then \( \mu_n \) is quasiinvariant, i.e. the condition

\[
d\mu_n(\bar{x}_1, \ldots, \bar{x}_n) = \delta(\bar{x}_1; g_1, \ldots, g_n)d\mu_n(\bar{x}_1, \ldots, \bar{x}_n)
\]

is satisfied for some nonnegative \( \mu_n \)-integrable function \( \delta(\bar{x}_1; g_1, \ldots, g_n) \) on \( \mathcal{X}^n \), and for every \( g_1, \ldots, g_n \in G \). Set

\[
(T^*g_1g_2f)(\bar{x}_1, \ldots, \bar{x}_n) = \tau(\bar{x}_1; g_2)f(\bar{x}_1 \cdot g_2, \ldots, \bar{x}_n \cdot g_2), \quad f \in L^2(\mathcal{X}^n, \mu_n),
\]

where \( |\tau(\bar{x}_1; g_2)|^2 = \delta(\bar{x}_1; g_2) \). If

\[
\tau(\bar{x}_1; g_1g_2) = \tau(\bar{x}_1 \cdot g_1; g_2)\tau(\bar{x}_1; g_1), \quad \bar{x}_1 \in \mathcal{X}, g_1, g_2 \in G.
\]
then equation (5.1) defines a unitary representation $T$ of $G$ acting in the Hilbert space $L^2(X; \mu)$. The function $\tau(\bar{x}; g)$ is called a multiplicative cocycle.

Let $z \in \mathbb{C}$ be a nonzero complex number. We apply the general construction described above for the space $X = X$, the group $G = S(2\infty)$, the measure $\mu = \mu_t$ (where $t = |z|^2$), and the cocycle $\tau(\bar{x}; g) = z^{c(\bar{x}; g)}$. In this way we get a unitary representation of $S(2\infty)$, $T_{z,\frac{1}{2}}$, acting in the Hilbert space $L^2(X, \mu_t)$ according to the formula

\[
(T_{z,\frac{1}{2}}(\alpha f))(\bar{x}) = z^{c(\bar{x}; g)} f(\bar{x} \cdot g), \quad f \in L^2(X, \mu_t), \quad \bar{x} \in X, \quad g \in S(2\infty).
\]

5.2. Approximation by quasi-regular representations.

**Definition 5.1.** For $n = 1, 2, \ldots$ let $\mu_1^{(n)}$ denote the normalized Haar measure on $X(n)$. The regular representation $\text{Reg}^n$ of the group $S(2n)$ acting in the Hilbert space $L^2(X(n), \mu_1^{(n)})$ is defined by

\[
(\text{Reg}^n(g)f)(\bar{x}) = f(\bar{x} \cdot g), \quad \bar{x} \in X(n), \quad g \in S(2n), \quad f \in L^2(X(n), \mu_t).
\]

**Proposition 5.2.** The representations $\text{Reg}^n$ and $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu_1^{(n)})}$ of $S(2n)$ are equivalent.

**Proof.** Set

\[
F_z^{(n)}(\bar{x}) = \left( \frac{1 \cdot 3 \cdot \ldots \cdot (2n - 1)}{t \cdot (t + 2) \cdot \ldots \cdot (t + 2n - 2)} \right)^{1/2} z^{\|z\|^2}, \quad \bar{x} \in X(n),
\]

and denote by $f_z^{(n)}$ the operator of multiplication by $F_z^{(n)}$. Since

\[
|F_z^{(n)}(\bar{x})|^2 = \frac{\mu_t^{(n)}(\bar{x})}{\mu_1^{(n)}(\bar{x})},
\]

the operator $f_z^{(n)}$ carries $L^2(X(n), \mu_t^{(n)})$ onto $L^2(X(n), \mu_1^{(n)})$, and defines an isometry. Moreover, it is straightforward to check that $f_z^{(n)}$ intertwines for the $S(2n)$-representations $\text{Reg}^n$ and $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu_1^{(n)})}$. \qed

Next we need the notion of the inductive limits of representations. Let $G(1) \subseteq G(2) \subseteq \ldots$ be a collection of finite groups, and set $G = \bigcup_{n=1}^{\infty} G(n)$. Thus $G$ is the inductive limit of the groups $G(n)$. Assume that for each $n$ a unitary representation $T_n$ of $G(n)$ is defined. Denote by $H(T_n)$ the Hilbert space in which the representation $T_n$ acts, and denote by $H$ the Hilbert completion of the space $\bigcup_{n=1}^{\infty} H(T_n)$. We also assume that an isometric embedding $\alpha_n : H(T_n) \rightarrow H(T_{n+1})$ is given, and that this embedding is intertwining for the $G(n)$-representations $T_n$ and $T_{n+1}|_{G(n)}$.

**Definition 5.3.** A unitary representation $T$ of the group $G$ acting in the Hilbert space $H$, and uniquely defined by

\[
T(g)\xi = T_n(g)\xi, \quad \text{if } g \in G(n) \text{ and } \xi \in H(T_n)
\]

is called the inductive limit of representations $\{T_n\}$. 

Consider the following diagram

\[
\begin{array}{ccccccc}
H(T_1) & \xrightarrow{f_1} & H(T_2) & \xrightarrow{f_2} & H(T_3) & \xrightarrow{f_3} & \ldots \\
| & | & | & | & | & | \\
F_1 & \downarrow & F_2 & \downarrow & F_3 & \downarrow & \\
H(S_1) & \xrightarrow{\rho_1} & H(S_2) & \xrightarrow{\rho_2} & H(S_3) & \xrightarrow{\rho_3} & \ldots \\
\end{array}
\]

Here \( \{T_n\}_{n=1}^{\infty} \) and \( \{S_n\}_{n=1}^{\infty} \) are collections of representations of \( G(1), G(2), \ldots, \) and \( G(1) \subseteq G(2) \subseteq \ldots. \) The following fact is almost obvious, and we formulate it as a Proposition without proof.

**Proposition 5.4.** Assume that for each \( n = 1, 2, \ldots \) the following conditions are satisfied

- The linear map \( F_n \) is from \( H(T_n) \) onto \( H(S_n) \), which is intertwining for \( T_n \) and \( S_n \).
- The linear map \( f_n \) is an isometric embedding of \( H(T_n) \) into \( H(T_{n+1}) \), which is intertwining for the \( G(n) \)-representations \( T_n \) and \( T_{n+1} \).
- The map \( \rho_n \) is an isometric embedding of \( H(S_n) \) into \( H(S_{n+1}) \) such that the diagram

\[
\begin{array}{ccc}
H(T_n) & \xrightarrow{f_n} & H(T_n) \\
| & | & | \\
F_n & \xrightarrow{\rho_n} & F_{n+1} \\
H(S_n) & \xrightarrow{\rho_n} & H(S_{n+1}) \\
\end{array}
\]

is commutative, i.e., the condition \( f_n = F_n^{-1} \circ \rho_n \circ f_n \) holds true.

Then the inductive limits of \( \{T_n\}_{n=1}^{\infty} \), and of \( \{S_n\}_{n=1}^{\infty} \) are well-defined, and these inductive limits are equivalent.

**Proposition 5.5.** Define the operator \( L_z^{(n)} \),

\[
L_z^{(n)} : L^2(X(n), \mu_1^{(n)}) \longrightarrow L^2(X(n+1), \mu_1^{(n)})
\]

as follows: if \( f \in L^2(X(n), \mu_1^{(n)}) \), and \( \tilde{x} \in X(n+1) \), then

\[
(L_z^{(n)} f)(\tilde{x}) = \begin{cases} 
  z \sqrt{\frac{2n+1}{2n+\tau}} f(\tilde{x}), & \tilde{x} \in X(n) \subset X(n+1), \\
  \sqrt{\frac{2n+1}{2n+\tau}} f(p_{n,n+1}(\tilde{x})), & \tilde{x} \in X(n+1) \setminus X(n).
\end{cases}
\]

(5.3)

For any nonzero complex number \( z \) the operator \( L_z^{(n)} \) provides an isometric embedding \( L^2(X(n), \mu_1^{(n)}) \longrightarrow L^2(X(n+1), \mu_1^{(n)}) \) which intertwines for the \( S(2n) \)-representations \( \text{Reg}^n \) and \( \text{Reg}^{n+1} \). Let \( T_{z,\frac{1}{2}} \) denote the inductive limit of the representations \( \text{Reg}^n \) with respect to the embedding

\[
L^2(X(1), \mu_1^{(1)}) \xrightarrow{L_z^{(1)}} L^2(X(2), \mu_1^{(2)}) \xrightarrow{L_z^{(2)}} \ldots
\]

Then the representations \( T_{z,\frac{1}{2}} \) and \( T_{z,\frac{1}{2}} \) are equivalent.
Proof. For $f \in L^2(X(n), \mu^{(n)}_t)$, and $\tilde{x} \in X(n + 1)$ set
\[
(\alpha^{(n)} f) (\tilde{x}) = f(p_{n,n+1}(\tilde{x})).
\]
Then $\alpha^{(n)}$ is an isometric embedding of $L^2(X(n), \mu^{(n)}_t)$ into $L^2(X(n+1), \mu^{(n+1)}_t)$. Using the definition of the representation $T_{z,\frac{1}{2}}$ it is straightforward to verify that $\alpha^{(n)}$ intertwines for the $S(2n)$-representations $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu^{(n)}_t)}$ and $T_{z,\frac{1}{2}}|_{L^2(X(n+1), \mu^{(n+1)}_t)}$. This enables us to consider $T_{z,\frac{1}{2}}$ as the inductive limit of $S(2n)$-representations of $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu^{(n)}_t)}$. Now examine the following diagram
\[
\begin{array}{ccccccc}
L^2(X(1), \mu^{(1)}_t) & \xrightarrow{\alpha^{(1)}} & L^2(X(2), \mu^{(2)}_t) & \xrightarrow{\alpha^{(2)}} & L^2(X(3), \mu^{(3)}_t) & \xrightarrow{\alpha^{(3)}} & \ldots \\
| f^{(1)}_z | & | f^{(2)}_z | & | f^{(3)}_z | & & & & \\
\downarrow & \downarrow & \downarrow & & & & \\
L^2(X(1), \mu^{(1)}_t) & \xrightarrow{L^{(1)}_z} & L^2(X(2), \mu^{(2)}_t) & \xrightarrow{L^{(2)}_z} & L^2(X(3), \mu^{(3)}_t) & \xrightarrow{L^{(3)}_z} & \ldots
\end{array}
\]
where the operators $f^{(n)}_z$ are that of multiplications by $F^{(n)}_z$ introduced in the proof of Proposition \[5.2\]. Recall that $f^{(n)}_z$ intertwines for the $S(2n)$-representations $\text{Reg}^n$ and $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu^{(n)}_t)}$. We determine $L^{(n)}_z$ from the condition of commutativity of the diagram
\[
\begin{array}{ccccccc}
L^2(X(n), \mu^{(n)}_t) & \xrightarrow{\alpha^{(n)}} & L^2(X(n+1), \mu^{(n+1)}_t) \\
| f^{(n)}_z | & | f^{(n)}_z | & & & & \\
\downarrow & \downarrow & & & & \\
L^2(X(n), \mu^{(n)}_t) & \xrightarrow{L^{(n+1)}_z} & L^2(X(n+1), \mu^{(n+1)}_t)
\end{array}
\]
and obtain that $L^{(n)}_z$ is given by formula \[5.3\]. Moreover, from equation \[5.3\] we see that $L^{(n)}_z$ defines the isometric embedding of $L^2(X(n), \mu^{(n)}_t)$ into $L^2(X(n+1), \mu^{(n+1)}_t)$. Now we use Proposition \[5.3\] to conclude that the inductive limit $T'_{z,\frac{1}{2}}$ of the representations $\text{Reg}^n$ with respect to the embedding
\[
\begin{array}{ccccccc}
L^2(X(1), \mu^{(1)}_1) & \xrightarrow{L^{(1)}_1} & L^2(X(2), \mu^{(2)}_2) & \xrightarrow{L^{(2)}_2} & \ldots
\end{array}
\]
is well-defined, and it is equivalent to $T_{z,\frac{1}{2}}$. \[\square\]

5.3. A formula for the spherical function of $T_{z,\frac{1}{2}}$. Let $(G, K)$ be a Gelfand pair, and let $T$ be a unitary representation of $G$ acting in the Hilbert space $H(T)$. Assume that $\xi$ is a unit vector in $H(T)$ such that $\xi$ is $K$-invariant, and such that the span of vectors of the form $T(g)\xi$ (where $g \in G$) is dense in $H(T)$. In this case $\xi$ is called the spherical vector, and the matrix coefficient $(T(g)\xi, \xi)$ is called the spherical function of the representation $T$. Two spherical representations are equivalent if and only if their spherical functions are coincide.
Proposition 5.6. Denote by $\varphi_z$ the spherical function of $T_{z,\frac{1}{2}}$. Then we have

$$\varphi_z|_{S(2n)}(g) = (\text{Reg}^n(g)F_z^{(n)}, F_z^{(n)})_{L^2(X(n), \mu_1^{(n)})}. \quad (5.4)$$

Proof. Let $f_0 \equiv 1$ be a unit vector, and let us consider $f_0$ as an element of $L^2(X(n), \mu_1^{(n)})$. Then we find

$$\left(T_{z,\frac{1}{2}}(g)f_0\right)(\tilde{x}) = z^{c(\tilde{x}; g)}, \quad \tilde{x} \in X(n), \ g \in S(2n).$$

If $g \in H(n)$, then $c(\tilde{x}; g) = 0$. In this case we obtain that $f_0$ is invariant under the action of $H(n)$, so $f_0$ can be understood as the cyclic vector of the $S(2n)$-representation $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu_1^{(n)})}$. On the other hand, the $S(2n)$-representation $T_{z,\frac{1}{2}}|_{L^2(X(n), \mu_1^{(n)})}$ is equivalent to $\text{Reg}^n$. This representation, $\text{Reg}^n$, acts in the space $L^2(X(n), \mu_1^{(n)})$, and from the proof of Proposition 5.2 we conclude that the cyclic vector of the $S(2n)$-representation $\text{Reg}^n$ is $F_z^{(n)}$ defined by formula (5.2). This gives expression for the spherical function of $T_{z,\frac{1}{2}}$ in the statement of the Proposition. \hfill \Box

6. Definition of $z$-measures associated with the representations $T_{z,\frac{1}{2}}$

6.1. The space $C(S(2n), H(n))$. Consider the set of functions on $S(2n)$ constant on each double coset $H(n)gH(n)$ in $S(2n)$. We shall denote this set by $C(S(2n), H(n))$. Therefore,

$$C(S(2n), H(n)) = \{f | f(hgh') = f(g), \text{ where } h, h' \in H(n), \text{ and } g \in S(2n)\}.$$ 

We equip $C(S(2n), H(n))$ with the scalar product $\langle ., . \rangle|_{S(2n)}$ defined by

$$\langle f_1, f_2 \rangle_{S(2n)} = \frac{1}{|S(2n)|} \sum_{g \in S(2n)} f_1(g)\overline{f_2(g)}.$$

Proposition 6.1. The space $C(S(2n), H(n))$ is isometrically isomorphic to the space $L^2(X(n), \mu_1^{(n)})^{H(n)}$ defined as a subset of functions from $L^2(X(n), \mu_1^{(n)})$ invariant with respect to the right action of $H(n)$,

$$L^2(X(n), \mu_1^{(n)})^{H(n)} = \left\{ f | f \in L^2(X(n), \mu_1^{(n)}), f(\tilde{x}) = f(\tilde{x} \cdot h), \right\}$$

where $\tilde{x} \in X(n)$, and $h \in H(n)$.\hfill \Box

Proof. The claim of the Proposition is almost trivial. Indeed, the fact that $C(S(2n), H(n))$ is isomorphic to $L^2(X(n), \mu_1^{(n)})^{H(n)}$ is obvious from the definition of these spaces. We have

$$\langle f_1, f_2 \rangle_{S(2n)} = \frac{1}{|S(2n)|} \sum_{g \in S(2n)} f_1(g)\overline{f_2(g)}$$

$$= \frac{1}{|X(n)|} \sum_{\tilde{x} \in X(n)} f_1(\tilde{x})\overline{f_2(\tilde{x})} = \langle f_1, f_2 \rangle_{L^2(X(n), \mu_1^{(n)})^{H(n)}},$$
for any two functions \( f_1, f_2 \) from \( C(S(2n), H(n)) \). Therefore, the isomorphism between \( C(S(2n), H(n)) \) and \( L^2(X(n), \mu_1^{(n)})^H(n) \) is isometric. \( \square \)

6.2. **The spherical functions of the Gelfand pair** \((S(2n), H(n))\). It is known (see Macdonald [17], Section VII.2) that \((S(2n), H(n))\) is a Gelfand pair. In particular, this implies that there is an orthogonal basis \( \{ w^\lambda \} \) in \( C(S(2n), H(n)) \) whose elements, \( w^\lambda \), are the spherical functions of \((S(2n), H(n))\). The elements \( w^\lambda \) are parameterized by Young diagrams with \( n \) boxes, and are defined by

\[
w^\lambda(g) = \frac{1}{|H(n)|} \sum_{h \in H(n)} \chi^{2\lambda}(gh),
\]

see Macdonald [17], Sections VII.1 and VII.2. Here \( \chi^{2\lambda} \) is the character of the irreducible \( S(2n) \)-module corresponding to \( 2\lambda = (2\lambda_1, 2\lambda_2, \ldots) \). By Proposition 6.1, the spherical functions \( w^\lambda \) define an orthogonal basis in \( L^2(X(n), \mu_1^{(n)})^H(n) \). Besides, the zonal spherical functions \( w^\lambda \) satisfy the following relations

\[(6.1)\]
\[w^\lambda(e) = 1, \text{ for any } \lambda \in \mathbb{Y}_n,\]

\[(6.2)\]
\[\langle w^\lambda, w^\mu \rangle_{L^2(X(n), \mu_1^{(n)})^H(n)} = \frac{\delta_{\lambda, \mu}}{\dim 2\lambda},\]

\[(6.3)\]
\[\frac{1}{|X(n)|} \sum_{\bar{x} \in X(n)} w^\lambda(\bar{x} \cdot g) w^\mu(\bar{x}) = \delta_{\lambda, \mu} \frac{w^\lambda(g)}{\dim 2\lambda}, \quad g \in S(2n).\]

Here \( \dim 2\lambda = \chi^{2\lambda}(e) \). The relations just written above follow from general properties of spherical functions, see Macdonald [17], Section VII.1.

6.3. **The z-measures \( M_{z, \frac{1}{2}}^{(n)} \) of the representation** \( T_{z, \frac{1}{2}} \).

**Definition 6.2.** Let \( z \) be a nonzero complex number, \( \lambda \) be a Young diagram with \( n \) boxes, and let

\[\tilde{w}^\lambda = (\dim 2\lambda)^{1/2} \cdot w^\lambda\]

be the normalized zonal spherical function of the Gelfand pair \((S(2n), H(n))\) parameterized by \( \lambda \). Set

\[(6.4)\]
\[M_{z, \frac{1}{2}}^{(n)}(\lambda) = \left| \left(F_{z}^{(n)}(\tilde{w}^\lambda)_{L^2(X(n), \mu_1^{(n)})}\right)^2 \right|,
\]

where \( F_{z}^{(n)} \) is a vector from \( L^2(X(n), \mu_1^{(n)}) \) defined by equation (5.2). The function \( M_{z, \frac{1}{2}}^{(n)} \) defined on the set of Young diagrams with \( n \) boxes is called the z-measure of the representation \( T_{z, \frac{1}{2}} \).

The relation with the representation \( T_{z, \frac{1}{2}} \) is clear from the following Proposition.

**Proposition 6.3.** Denote by \( \varphi_z \) the spherical function of \( T_{z, \frac{1}{2}} \). We have

\[(6.5)\]
\[\varphi_z|_{S(2n)}(g) = \sum_{|\lambda|=n} M_{z, \frac{1}{2}}^{(n)}(\lambda) w^\lambda(g), \quad g \in S(2n).\]
Proof. The functions \{\tilde{w}^\lambda\} define an orthonormal basis in \(L^2(X(n), \mu_1^{(n)})^H(n)\). On the other hand, we can check that \(F_z^{(n)}\) is an element of \(L^2(X(n), \mu_1^{(n)})^H(n)\). Therefore, we must have

\[
    F_z^{(n)}(\tilde{x}) = \sum_{|\lambda| = n} a_z^{(n)}(\lambda) \tilde{w}^\lambda(\tilde{x}), \quad \tilde{x} \in X(n).
\]

We insert expression (6.6) into formula (5.4). This gives

\[
    \varphi_z|S(2n)(g) = \frac{1}{|X(n)|} \sum_{\tilde{x} \in X(n)} \sum_{|\lambda| = n} \sum_{|\mu| = n} a_z^{(n)}(\lambda) a_z^{(n)}(\mu) \tilde{w}^\lambda(\tilde{x} \cdot g) \tilde{w}^\mu(\tilde{x}).
\]

Using equation (6.3) we find that

\[
    \varphi_z|S(2n)(g) = \sum_{|\lambda| = n} |a_z^{(n)}(\lambda)|^2 w^\lambda(g).
\]

From equations (6.4) and (6.6) we see that

\[
    M_z^{(n)}(\lambda) = |a_z^{(n)}(\lambda)|^2,
\]

which gives the formula in the statement of the Proposition. \(\Box\)

**Corollary 6.4.** We have

\[
    \sum_{|\lambda| = n} M_z^{(n)}(\lambda) = 1,
\]

i.e. \(M_z^{(n)}(\lambda)\) can be understood as a probability measure on the set of Young diagrams with \(n\) boxes.

**Proof.** This follows from equations (6.7), (6.8), and from the fact that

\[
    \varphi_z|S(2n)(e) = w^\lambda(e) = 1.
\]

\(\Box\)

### 6.4. An explicit formula for \(M_z^{(n)}\).

**Proposition 6.5.** (Olshanski [23]) The \(z\)-measure \(M_z^{(n)}(\lambda)\) admits the following explicit formula

\[
    M_z^{(n)}(\lambda) = \frac{n!}{(\frac{z}{2})_n \cdot h(2\lambda)} \cdot \prod_{(i,j) \in \lambda} (z + 2(j-1) - (i-1))(\bar{z} + 2(j-1) - (i-1)),
\]

where \(h(2\lambda)\) denotes the product of the hook-lengths of \(2\lambda = (2\lambda_1, 2\lambda_2, \ldots)\), and \((.)_n\) stands for the Pochhammer symbol,

\[
    (a)_n = a(a+1) \ldots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.
\]

In particular, it follows that \(M_z^{(n)}(\lambda)\) is exactly the \(z\)-measure with the Jack parameter \(\theta = 1/2\) in the notation of Section 2,

\[
    M_z^{(n)}(\lambda) = M_{z,z,\theta = 1/2}^{(n)}(\lambda).
\]
Proof. We start from formula (6.4), and observe that this formula can be rewritten as
\[
M^{(n)}_{z, \frac{1}{2}}(\lambda) = \frac{1}{[(2n)!]^2} \left| \left( \hat{F}^{(n)}_z, \hat{\bar{w}}^{\lambda} \right) \right|^2,
\]
where \( F^{(n)}_z, \hat{\bar{w}}^{\lambda} \) are understood as two functions from \( C(S(2n), H(n)) \), and the scalar product \( (f_1, f_2) \) is defined by
\[
(f_1, f_2) = \sum_{g \in S(2n)} f_1(g) \overline{f_2(g)}.
\]
To compute the scalar product in equation (6.9), we use the characteristic map,
\[
C(S(2n), H(n)) \xrightarrow{ch''} \Lambda^n_C,
\]
introduced in Macdonald [17], Section VII.2. Here \( \Lambda^n \) denotes the set of the homogeneous symmetric polynomials of degree \( n \), and \( \Lambda^n_C \) is the linear span of these polynomials with complex coefficients. The characteristic map, \( ch'' \), is defined by
\[
ch''(f) = |H(n)| \sum_{|\rho| = n} z_\rho^{-1} 2^{-l(\rho)} p_\rho f(\rho).
\]
Here the symbol \( z_\rho \) is defined by
\[
z_\rho = \prod_{i \geq 1} i^{m_i} m_i!,
\]
where \( m_i = m_i(\rho) \) is the number of parts of \( \rho \) equal to \( i \). In equation (6.10) \( l(\rho) \) stands for the number of nonzero parts in \( \rho \), \( p_\rho = p_{\rho_1}p_{\rho_2} \cdots \), where \( p_k \) stands for \( k \)th power sum, and \( f(\rho) \) is the value of \( f \) at elements of the double coset parameterized by the Young diagram \( \rho \), see Macdonald, Section VII.2. The map \( ch'' \) is an isometry of \( C(S(2n), H(n)) \) onto \( \Lambda^n_C \). Therefore,
\[
M^{(n)}_{z, \frac{1}{2}}(\lambda) = \frac{1}{[(2n)!]^2} \left| (ch''(F^{(n)}_z), ch''(\hat{\bar{w}}^{\lambda})) \right|^2,
\]
where the scalar product is defined by
\[
(p_\rho, p_\sigma) = \delta_{\rho \sigma} 2^{l(\rho)} z_\rho.
\]
It remains to find \( ch''(F^{(n)}_z) \), \( ch''(\hat{\bar{w}}^{\lambda}) \), and to compute the scalar product in the righthand side of equation (6.11). We have
\[
ch''(\hat{\bar{w}}^{\lambda}) = (\dim 2\lambda)^{1/2} J^{(\alpha=2)}_\lambda,
\]
where \( J^{(\alpha)}_\lambda \) stands for the Jack polynomial with the Jack parameter \( \alpha \) parameterized by the Young diagram \( \lambda \) (in notation of Macdonald, Section VI). In order to find \( ch''(F^{(n)}_z) \) it is enough to obtain a formula for \( ch''(N^{[\lambda]}_{1,n}) \). We have
\[
ch''(N^{[\lambda]}_{1,n}) = |H(n)| \sum_{|\rho| = n} z_\rho^{-1} p_\rho \left(\frac{N}{2}\right)^{l(\rho)}.
\]
Since
\[
\left( \frac{N}{2} \right)^{l(\rho)} = p_\rho(1, \ldots, 1),
\]
we can use equation (1.4) in Section I.4 of Macdonald [17], and write
\[
\left( \frac{N}{2} \right)^{l(\rho)} = |H(n)| \left\{ \prod_{i=1}^\infty (1 - x_i)^{-\frac{N}{2}} \right\}_n.
\]
Here \(\{.\}_n\) denotes the component of degree \(n\). Now we have
\[
\prod_{i=1}^\infty (1 - x_i)^{-\frac{N}{2}} = \sum_{\lambda} \frac{1}{h(2\lambda)} J^{(2)}_\lambda(x) J^{(2)}_{\lambda}(1, \ldots, 1).
\]
The value \(J^{(2)}_{\lambda}(1, \ldots, 1)\) is known,
\[
J^{(2)}_{\lambda}(1, \ldots, 1) = \prod_{i,j \in \lambda} (N + 2(j - 1) - (i - 1)).
\]
This gives us the following formula
\[
\prod_{i=1}^\infty (1 - x_i)^{-\frac{N}{2}} = \sum_{|\lambda|=n} \frac{1}{h(2\lambda)} J^{(2)}_\lambda(x) \prod_{(i,j) \in \lambda} (N + 2(j - 1) - (i - 1)),
\]
and we obtain
(6.13)
\[
ch''(F^{(n)}_z) = \left( \frac{1 \cdot 3 \cdots (2n - 1)}{t(t + 2) \cdots (t + 2n - 2)} \right)^{1/2} \left( \frac{1 \cdot 3 \cdots (2n - 1)}{t(t + 2) \cdots (t + 2n - 2)} \right)^{1/2} |H(n)| \sum_{|\lambda|=n} \frac{1}{h(2\lambda)} J^{(2)}_\lambda(x) \prod_{(i,j) \in \lambda} (N + 2(j - 1) - (i - 1)).
\]
Finally, using the orthogonality relation
\[
(J^{(2)}_\lambda, J^{(2)}_\mu) = \delta_{\lambda\mu} h(2\lambda)
\]
we find from equations (6.11)-(6.13) that
\[
M^{(n)}_{z,2}(\lambda) = \frac{|H(n)|^2}{[2n]!^2} \left( \frac{1 \cdot 3 \cdots (2n - 1)}{t(t + 2) \cdots (t + 2n - 2)} \right) \dim 2\lambda \times \prod_{(i,j) \in \lambda} (z + 2(j - 1) - (i - 1))(\bar{z} + 2(j - 1) - (i - 1)).
\]
Noting that \(|H(n)| = 2^n n!\), and that \(\dim 2\lambda = \frac{(2n)!}{h(2\lambda)}\) we arrive to the first formula in the statement of the Proposition. The fact that \(M^{(n)}_{z,2}(\lambda)\) coincides with the \(z\)-measure with \(\theta = 1/2\) in the notation of the Section 2 can now be checked directly using formulae for \(H(\lambda, \theta)\) and \(H'(\lambda, \theta)\) stated in Section 2.
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