The Logarithm of the Modulus of an Entire Function as a Minorant for a Subharmonic Function outside a Small Exceptional Set

B.N. Khabibullin

Abstract. Let \( u \neq -\infty \) be a subharmonic function on the complex plane \( \mathbb{C} \). In 2016, we obtained a result on the existence of an entire function \( f \neq 0 \) satisfying the estimate \( \log |f| \leq B_u \) on \( \mathbb{C} \), where the functions \( B_u \) are integral averages of \( u \) for rapidly shrinking disks as it approaches infinity. We give another equivalent version of this result with \( \log |f| \leq u \) outside a very small exceptional set if \( u \) is of finite order.

Key Words and Phrases: subharmonic function, entire function, exceptional set, Riesz measure, integral average, covering of sets, type and order of function.

2010 Mathematics Subject Classifications: 30D20, 31A05, 26A12

1. Introduction

1.1. Definitions and notations. Preliminary result

We consider the set \( \mathbb{R} \) of real numbers mainly as the real axis in the complex plane \( \mathbb{C} \), and \( \mathbb{R}^+ := \{ x \in \mathbb{R} : x \geq 0 \} \) is the positive semiaxis in \( \mathbb{C} \). Besides, \( \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \} \) is the extended real line with the natural order \( -\infty \leq x \leq +\infty \) for every \( x \in \overline{\mathbb{R}} \), \( \mathbb{R}^+ := \mathbb{R}^+ \cup \{ +\infty \} \), \( x^+ := \sup\{ x, 0 \} \) for each \( x \in \mathbb{R} \). For an extended real function \( f : S \to \overline{\mathbb{R}} \), its positive part is the function \( f^+ : s \mapsto (f(s))^+ \).

For \( z \in \mathbb{C} \) and \( r \in \mathbb{R}^+ \), we denote by \( D(z, r) := \{ z' \in \mathbb{C} : |z' - z| < r \} \) the open disk centered at \( z \) and of radius \( r \), where \( D(z, 0) \) is the empty set \( \emptyset \), \( D(r) := D(0, r) \), \( \overline{D}(z, r) := \{ z' \in \mathbb{C} : |z' - z| \leq r \} \) is a closed disk centered at \( z \) and of radius \( r \), \( \partial D(z, r) := \overline{D}(0, r) \), and the circle \( \partial D(z, r) := \overline{D}(z, r) \setminus D(z, r) \).
centered at \( z\) and of radius \( r\), \( \partial D(r) := \partial D(0, r)\). For a function \( v: D(z, r) \to \mathbb{R} \), we define the integral averages on circles and disks as

\[
C_v(z, r) := \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) \, d\theta, \quad C^{rad}_v(r) := C_v(0, r),
\]

(1C)

\[
B_v(z, r) := \frac{2}{r^2} \int_0^r C_v(z, t) \, dt, \quad B^{rad}_v(r) := B_v(0, r),
\]

(1B)

\[
M_v(z, r) := \sup_{z' \in \partial D(z, r)} v(z'), \quad M^{rad}_v(r) := M_v(0, r),
\]

(1M)

where \( M_v(z, r) = \sup_{z' \in D(z, r)} v(z') \) if \( v \) is subharmonic on \( \mathbb{C} \) [1, Definition 2.6.7], [2].

The following result [3, Corollary 2] of 2016 found several useful applications [4, Lemma 5.1], [5], [6, Proposition 2], [7], [8, Lemma 6.3], [9], [10, 7.1] for entire functions on the complex plane:

**Theorem 1** ([3, Corollary 2], see also [4, Lemma 5.1]). Let \( u \not\equiv -\infty \) be a subharmonic function on \( \mathbb{C} \), and \( q \in \mathbb{R}^+ \) be a number with the corresponding function

\[
Q: z \mapsto \frac{1}{(1 + |z|)^q} \leq 1.
\]

(2)

Then there is an entire function \( f_q \not\equiv 0 \) on \( \mathbb{C} \) such that

\[
\log|f_q(z)| \leq B_u(z, Q(z)) \leq C_u(z, Q(z)) \leq M_u(z, Q(z)) \quad \text{for each} \ z \in \mathbb{C}.
\]

(3)

In this article, we obtain another equivalent version of Theorem 1 for subharmonic functions of finite order. This version may be useful in situations that we are not discussing here.

1.2. Main result for minorants outside an exceptional set

For an extended real function \( m: \mathbb{R}^+ \to \mathbb{R} \), we define [11], [12], [8, 2.1, (2.1t)]

\[
\text{ord}[m] := \limsup_{r \to +\infty} \frac{\log(1 + m^+(r))}{\log r} \in \mathbb{R}^+,
\]

(4)

the order of growth of \( m\); for \( p \in \mathbb{R}^+\),

\[
\text{type}_p[m] := \limsup_{r \to +\infty} \frac{m^+(r)}{r^p} \in \mathbb{R}^+,
\]

(5)

the type of growth of \( m\) at the order \( p\). Thus, it is easy to see that

\[
\text{ord}[m] = \inf\{ p \in \mathbb{R}^+: \text{type}_p[m] < +\infty\}, \quad \inf \emptyset := +\infty.
\]

(6)
If $u$ is a subharmonic function on $\mathbb{C}$, then

$$\text{ord}[u] \overset{(1M)}{=} \text{ord}[M_u^{\text{rad}}], \quad \text{type}_p[u] \overset{(1M)}{=} \text{type}_p[M_u^{\text{rad}}],$$  

and, under the condition $\text{type}_p[u] < +\infty$, the following $2\pi$-periodic function

$$\text{ind}_p[u](s) := \limsup_{r \to +\infty} \frac{u(re^{is})}{r^p} \in \mathbb{R}, \quad s \in \mathbb{R},$$  

is called the indicator of the growth of $u$ at the order $p$ [12, 3.2].

For a ray or a circle on $\mathbb{C}$, we denote by $\text{mes}$ the linear Lebesgue measure on this ray or the measure of length on this circle.

**Theorem 2.** Let $u \not\equiv -\infty$ be a subharmonic function on the complex plane, and $\text{ord}[B_u^{\text{rad}}] < +\infty$. Then the conclusion (3) of Theorem 1 with arbitrary positive numbers $q \in \mathbb{R}^+$ is equivalent to the following statement:

For any positive $q \in \mathbb{R}^+$, there are an entire function $f_q \not\equiv 0$ and a no-more-than countable set of disks $D(z_k, t_k)$, $k = 1, 2, \ldots$, such that

$$\log |f_q(z)| \leq u(z) \quad \text{for each } z \in \mathbb{C} \setminus E_q,$n

$$E_q := \bigcup_k D(z_k, r_k), \quad \sup_k t_k \leq 1, \quad \sum_{|z_k| \geq R} t_k = O\left(\frac{1}{R^q}\right) \quad \text{as } R \to +\infty.$$  

If $\text{ord}[u] < +\infty$, then the statements of Theorem 1 or the statements (9) of this Theorem 2 can be supplemented by the following restrictions:

$$\text{ord}[\log |f_q|] \overset{(4),(6),(7)}{=} \text{ord}[u],$$  

$$\text{type}_p[\log |f_q|] \overset{(5),(7)}{=} \text{type}_p[u] \quad \text{for each } p \in \mathbb{R}^+, \quad \text{ind}_p[\log |f_q|] \overset{(8)}{=} \text{ind}_p[u] \quad \text{for each } q \in \mathbb{R}^+.$$  

Besides, for any ray $L \subset \mathbb{C}$, we have

$$\text{mes}\left(L \setminus (E_q \cup D(R))\right) = O\left(\frac{1}{R^q}\right) \quad \text{as } R \to +\infty,$$  

and also

$$\text{mes}\left(E_q \cap \partial D(R)\right) = O\left(\frac{1}{R^q}\right) \quad \text{as } R \to +\infty.$$  

Theorem 2 will be proved in Sec. 3 after some preparations.
2. Preparatory results

2.1. On exceptional sets

For a Borel measure $\mu$ on $\mathbb{C}$, we set
\[ \mu(z, t) := \mu(D(z, t)), \quad z \in \mathbb{C}, \ t \in \mathbb{R}^+. \]  
(13)

For a function $d: \mathbb{C} \to \mathbb{R}^+$, $S \subset \mathbb{C}$ and $r: \mathbb{C} \to \mathbb{R}$, we define
\[ S^{\text{id}} := \bigcup_{z \in S} D(z, d(z)) \subset \mathbb{C}, \]
\[ r^{\vee d} : z \mapsto \sup_{z' \in \mathbb{C}} \left\{ r(z') : z' \in D(z, d(z)) \right\} \in \mathbb{R}, \]
and denote the indicator function of the set $S$ by
\[ 1_S : z \mapsto \begin{cases} 1 & \text{if } z \in S, \\ 0 & \text{if } z \notin S. \end{cases} \]

**Lemma 1** (cf. [13, Normal Points Lemma], [14, § 4. Normal points, Lemma]).
Let $r: \mathbb{C} \to \mathbb{R}^+$ be a Borel function such that
\[ d := 2 \sup\{r(z) : z \in \mathbb{C}\} < +\infty, \]  
(14)
and $\mu$ be a Borel positive measure on $\mathbb{C}$ with
\[ E_{\mu, r} := \left\{ z \in \mathbb{C} : \int_0^{r(z)} \frac{\mu(z, t)}{t} \, dt > 1 \right\} \subset \mathbb{C}. \]  
(15)

Then there exists a no-more-than countable set of disks $D(z_k, t_k)$, $k = 1, 2, \ldots$, such that
\[ z_k \in E_{\mu, r}, \quad t_k \leq r(z_k), \quad E_{\mu, r} \subset \bigcup_k D(z_k, t_k), \]
\[ \sup_{z \in \mathbb{C}} \#\{k : z \in D(z_k, t_k)\} \leq 2020, \]  
(16)
i.e., the multiplicity of this covering $\{D(z_k, t_k)\}_{k=1,2,\ldots}$ of set $E_{\mu, r}$ is not greater than 2020, and, for every $\mu$-measurable subset $S \subset \bigcup_k D(z_k, t_k)$,
\[ \frac{1}{2020} \sum_{S \cap D(z_k, t_k) \neq \emptyset} t_k \leq \int_{S^{\text{id}}} r^{\vee r} \, d\mu \leq \int_{S^{\text{id}}} r^{\vee d} \, d\mu. \]  
(17)
Proof. By definition (15), there is a number
\[ t_z \in (0, r(z)) \quad \text{such that} \quad 0 < t_z < r(z) \mu(z, t_z) \quad \text{for each} \quad z \in E_{\mu, r}. \quad (18) \]
Thus, the system \( D = \{ D(z, t_z) \}_{z \in E} \) of these disks has a property
\[ E_{\mu, r} \subset \bigcup_{z \in E} D(z, t_z), \quad 0 < t_z \leq r(z) \leq R. \quad (14) \]
By the Besicovitch Covering Theorem [15, 2.8.14], [16], [17], [18, I.1, Remarks], [19], [20] in the Landkof version [21, Lemma 3.2], we can select some no-more-than countable subsystem in \( D \) of disks \( D(z_k, t_k) \in D, k = 1, 2, \ldots, t_k := t_{z_k}, \) such that properties (16) are fulfilled. Consider a \( \mu \)-measurable subset \( S \subset \bigcup_k D(z_k, t_k). \)
In view of (18) it is easy to see that
\[ \bigcup\{ D(z_k, t_k): S \cap D(z_k, t_k) \neq \emptyset \} \subset \bigcup_{z \in S} D(z, d) = S^{\mu d}. \quad (19) \]
Hence, in view of (18) and (16), we obtain
\[
\sum_{S \cap D(z_k, t_k) \neq \emptyset} t_k := \sum_{S \cap D(z_k, t_k) \neq \emptyset} t_{z_k} \leq \sum_{S \cap D(z_k, t_k) \neq \emptyset} r(z_k) \mu(z, t_k) \leq \sum_{S \cap D(z_k, t_k) \neq \emptyset} \int_{D(z_k, t_k)} r \, d\mu \leq \sum_{S \cap D(z_k, t_k) \neq \emptyset} 1_{D(z_k, t_k)} r \, d\mu \leq 2020 \int_{S^{\mu d}} r \, d\mu \leq 2020 \int_{S^{\mu d}} r \, d\mu.
\]
Thus, we obtain (17). This completes the proof of Lemma 1. ▶

Lemma 2. Let \( \{ D(z_j, t_j) \}_{j \in J} \) be a system of disks in \( \mathbb{C} \), \( d := 2 \sup_{j \in J} t_j < +\infty. \) Then, for each \( z \in \mathbb{C} \), there is a positive number \( r \leq d \) such that
\[ \bigcup_{j \in J} D(z_j, t_j) \cap \partial \overline{D}(z, r) = \emptyset. \quad (20) \]
Proof. Consider a disk \( \mathcal{D}(z, d) \), where, without loss of generality, we can assume that \( z = 0 \). Then, by condition \( d := 2 \sup_{j \in J} t_j < +\infty \), the union
\[
\bigcup_{j \in J} (D(z_j, t_j) e^{-i \arg z_j}) \cap [0, d]
\] (21)
of radial projections \( (D(z_j, t_j) e^{-i \arg z_j}) \cap [0, d] \) of \( D(z_j, t_j) \) onto the radius \([0, d]\) is not empty, i.e. there is a point \( r \in [0, d] \) outside (21), which gives (20) for \( z = 0 \).

Lemma 2 is proved. ◀

Lemma 2 has the following consequence:

Lemma 3. Let \( \{ D(z_k, t_k) \}_{k=1,2,\ldots} \) be a system of disks satisfying (9E) with a strictly positive number \( q \in \mathbb{R}^+ \setminus \{0\} \), and \( q' < q \) be a positive number. Then there exists a number \( R_q \in \mathbb{R}^+ \) such that for any \( z \in \mathbb{C} \) with \( |z| > R_q \) there is a positive number \( r \leq (1 + |z|)^{-q'} \) such that (20) holds for \( J = \{1, 2, \ldots\} \).

Proof. By condition (9E), there is a constant \( C \in \mathbb{R}^+ \) such that
\[
\sum_{D(z_k, t_k) \cap D(|z| - 2) \neq \emptyset} t_k \leq \frac{C}{(1 + |z|)^q} \quad \text{for each } z \in \mathbb{C} \text{ with } |z| \geq 3,
\] (22)
and, for \(|z| \geq 3\),
\[
\text{if } D(z_k, t_k) \cap D(z, 2) \neq \emptyset, \text{ then } D(z_k, t_k) \cap D(|z| - 2) \neq \emptyset.
\] (23)

For \( 0 \leq q' < q \), we choose \( R_q \geq 3 \) so that
\[
C(1 + |z|)^{q' - q} \leq \frac{1}{2} \quad \text{for all } |z| \geq R_q \geq 3.
\] (24)

It follows from (22)–(24) that
\[
\sum_{D(z_k, t_k) \cap D(|z| - 2) \neq \emptyset} t_k \leq \frac{C}{(1 + |z|)^q} \leq \frac{1}{2} \frac{1}{(1 + |z|)^q'} \quad \text{for each } z \in \mathbb{C} \text{ with } |z| \geq R_q,
\] and
\[
\sup_{D(z_k, t_k) \cap D(|z| - 2) \neq \emptyset} t_k \leq \frac{1}{2} \frac{1}{(1 + |z|)^q'} \quad \text{for each } z \in \mathbb{C} \text{ with } |z| \geq R_q.
\] (25)

For an arbitrary fixed point \( z \in \mathbb{C} \) with \(|z| \geq R_q\), we consider
\[
J := \{ k : D(z_k, t_k) \cap D(z, 2) \neq \emptyset \}, \quad D := \{ D(z_k, t_k) \}_{k \in J}.
\]
By Lemma 2, with these $J$ and $D$ there is a circle $\partial D(z, r)$ such that

$$0 \leq r \leq (1 + |z|)^{q'}, 0 \leq \bigcup_{k \in J} D(z_k, t_k) \cap \partial D(z, r) = \emptyset.$$  

But, in view of (23), if $k \notin J$, then, as before, $D(z_k, t_k) \cap \partial D(z, r) = \emptyset$.

Lemma 3 is proved. ▽

2.2. The order and the upper density for measures on $\mathbb{C}$

For a Borel positive measure $\mu$ on $\mathbb{C}$, a function

$$\mu^{\text{rad}}: r \mapsto \mu(0, r)$$  

is called the radial counting function of $\mu$, the quantity

$$\text{ord}[\mu] := \text{ord}[\mu^{\text{rad}}]$$

is called the order of measure $\mu$, and, for $p \in \mathbb{R}^+$, the quantity

$$\text{type}_p[\mu] := \text{type}_p[\mu^{\text{rad}}]$$

is called the upper density of measure $\mu$ at the order $p$.

If $u \neq -\infty$ is a subharmonic function on $\mathbb{C}$ with the Riesz measure

$$\Delta u = \frac{1}{2\pi} \triangle u,$$

where the Laplace operator $\triangle$ acts in the sense of the theory of distributions or generalized functions [1], [2], then, by the Poisson–Jensen formula [1, 4.5], [2]

$$u(z) = C_u(z, r) - \int_0^r \frac{\Delta u(z, t)}{t} \, dt, \quad z \in \mathbb{C},$$

in a disk $D(z, r)$ in the form [4, 3, (3.3)]

$$C_u(r) - C_u(1) = \int_1^r \frac{\Delta u(t)}{t} \, dt,$$

and by (1B) together with

**Lemma 4** ([22], [23, Theorem 3]). If $u$ is a subharmonic function on $\mathbb{C}$, then $B(z, t) \leq C(z, t) \leq B(z, \sqrt{t})$ for each $z \in \mathbb{C}$ and for each $t \in \mathbb{R}^+$. 

we can easily obtain

**Lemma 5.** Let \( u \not\equiv -\infty \) be a subharmonic function on \( \mathbb{C} \) with Riesz measure (28). Then, for each \( r \geq 1 \),
\[
B_u(r) - C_u(1) \leq C_u(r) - C_u(1) \leq \int_1^r \frac{\Delta_u(t)}{t} \, dt \leq C_u(\sqrt{r}). \tag{30}
\]

In particular, we have the equalities
\[
\text{ord}[\Delta_u] = \text{ord}[C_u] = \text{ord}[B_u],
\]
and the equivalences
\[
\text{type}_p[\Delta_u] < +\infty \iff \text{type}_p[C_u] < +\infty \iff \text{type}_p[B_u] < +\infty
\]
for each strictly positive \( p \in \mathbb{R}^+ \setminus \{0\} \).

### 3. The proof of Theorem 2

#### 3.1. From Theorem 1 to (9)

Let \( q' \in \mathbb{R}^+ \). By Lemma 5, we have
\[
a_u := \text{ord}[\Delta_u] \overset{(30)}{=} \text{ord}[C_u] < +\infty. \tag{31}
\]
We choose
\[
q := a_u + q' + 3 \geq 3 \tag{32}
\]
and an entire function \( f_q \) from Theorem 1 with properties (2)–(3). Then, for entire function \( e^{-1}f_q \not\equiv 0 \), we obtain
\[
\log|e^{-1}f_q(z)| \leq C_u(z, Q(z)) - 1 \overset{(29)}{=} u(z) + \int_0^{Q(z)} \frac{\Delta_u(z, t)}{t} \, dt - 1 \text{ for each } z \in \mathbb{C} \setminus (-\infty)_u, \tag{33}
\]
where \((-\infty)_u := \{z \in \mathbb{C} : u(z) = -\infty\}\) is a minus-infinity \( G_\delta \) polar set [1, 3.5], and 1-dimensional Hausdorff measure of \((-\infty)_u\) is zero [2, 5.4]. Therefore, this set \((-\infty)_u\) can be covered by a system of disks as in (9E) with \( q' \) instead of \( q \). By Lemma 1 with
\[
r := Q, \quad d \leq 2, \quad \mu := \Delta_u, \quad E_q := \bigcup_k D(z_k, t_k) \overset{(15), (9E)}{\supset} E_{\mu, r}, \tag{34}
\]
we have, in view of (33),
\[
\log |e^{-1} f_q(z)| \leq u(z), \quad \text{for each } z \in \mathbb{C} \setminus (E_q \cup (-\infty)_u).
\]
(35)

If \(S := E_q \setminus D(R)\) and \(R \geq 4\), then, by (17),
\[
\frac{1}{2020} \sum_{|z_k| \geq R} t_k \leq \int_{S \setminus d} \frac{1}{|z|} d\Delta_u \leq \int_{|z| \geq R-2} \frac{1}{(1 + (|z| - 2))^q} d\Delta_u(z)
\]
\[
= \int_{R-2}^{+\infty} \frac{1}{(t-1)^q} d\Delta_u^{\text{rad}}(t) \leq \int_{R-2}^{+\infty} \frac{\Delta_u^{\text{rad}}(t)}{(t-1)^{q-1}} dt
\]
\[
\leq \text{const} \int_{R-2}^{+\infty} \frac{t^{a_u+1}}{(t-1)^{q-1}} dt \leq O(R^{a_u+3-q}) \quad \text{as } R \to +\infty,
\]
where const \(\in \mathbb{R}^+\) is independent of \(R\), and \(R^{a_u+3-q} \leq R^{-q'}\). The latter together with (35) gives the statements (9) of Theorem 2.

3.2. From (9) to Theorem 1

Let \(q^* \in \mathbb{R}^+\). Suppose that the statements (9) of Theorem 2 are fulfilled with \(q > q^* \geq 0\). By Lemma 3 there exists a number \(R_q \in \mathbb{R}^+\) such that for any \(z \in \mathbb{C}\) with \(|z| > R_q\) there is a positive number \(r_z \leq (1 + |z|)^{-q'}\) such that \(E_q \cap \partial D(z, r_z) = \emptyset\). Hence, by (9I), we obtain
\[
\log |f_q(z + r_z e^{i\theta})| \leq C \log |f_q(r_z)| \leq C u(r_z) \leq C \left( \frac{1}{1 + |z|^{q'}} \right) \quad \text{if } |z| \geq R_q.
\]
(36)
and for any \(z \in \mathbb{C}\) with \(|z| \geq R_q\). Therefore,
\[
\log |f_q(z)| \leq C \log |f_q(r_z)| \leq C u(r_z) \leq C \left( \frac{1}{1 + |z|^{q'}} \right) \quad \text{if } |z| \geq R_q.
\]
Hence there exist a sufficiently small number \(a > 0\) and a sufficiently large number \(R_{q^*} \geq R_q\) such that
\[
\log |af_q(z)| \leq C \left( \frac{1}{\sqrt{e}(1 + |z|)^{q'}} \right) \quad \text{if } |z| \geq R_{q^*}.
\]
The function \(\log |af_q|\) is bounded from above on \(D(R_{q^*})\), and the function
\[
C \left( \frac{1}{\sqrt{e}(1 + R_{q^*})^{q'}} \right) : z \mapsto C \left( \frac{1}{\sqrt{e}(1 + R_{q^*})^{q'}} \right)
\]
is continuous [24, Theorem 1.14]. Therefore, there exists a sufficiently small number \( b > 0 \) such that
\[
\log |abf_q(z)| \leq C(z, \frac{1}{\sqrt{e(1 + |z|^q)}})
\]
for all \( z \in \mathbb{C} \).

Hence, for \( f_{q^*} := abf_q \neq 0 \), by Lemma 4, we obtain (3) with \( q^* \in \mathbb{R}^+ \) instead of \( q \) in (2). Further, equalities (10o) and (10t) for orders and types are obvious consequences of (3) even for \( q = 0 \). Similarly, we obtain equality (10i), since indicators (8) of the growth of \( \log |f_q| \) and \( u \) are continuous. Relations (11)–(12) are obvious particular cases of (9E).

Acknowledgement

The work performed under the development program of Volga Region Mathematical Center (agreement no. 075-02-2021-1393).

References

[1] Th. Ransford, *Potential Theory in the Complex Plane*. Cambridge University Press, Cambridge, 1995.

[2] W. K. Hayman, P. B. Kennedy, *Subharmonic functions, 1*, Acad. Press, London etc., 1976.

[3] B. N. Khabibullin, T. Yu. Baiguskarov, *The Logarithm of the Modulus of a Holomorphic Function as a Minorant for a Subharmonic Function*, Mat. Zametki, 99(4), 2016, 588–602; Math. Notes, 99(4), 2016, 576–589.

[4] T. Yu. Baiguskarov, B. N. Khabibullin, A. V. Khasanova, *The Logarithm of the Modulus of a Holomorphic Function as a Minorant for a Subharmonic Function. II. The Complex Plane*, Mat. Zametki, 101(4), 2017, 483–502; Math. Notes, 101:4 (2017), 590–607.

[5] T. Yu. Baiguskarov, B. N. Khabibullin, *Holomorphic Minorants of Plurisubharmonic Functions*, Funct. Anal. Appl., 50(1), 2016, 62–65.

[6] B. N. Khabibullin, F. B. Khabibullin, *On non-uniqueness sets for spaces of holomorphic functions*, Vestnik Volgogradskogo gosudarstvennogo universiteta. Seriya 1. Mathematica. Physica, 4(35), 2016, 108–115.

[7] R..A. Baladai, B. N. Khabibullin, *From the integral estimates of functions to uniform and locally averaged*, Russian Math. (Izv. VUZ. Matematika), 61(10), 2017, 11–20.
[8] B. N. Khabibullin, A. V. Shmelyova, *Balayage of measures and subharmonic functions on a system of rays. I. Classic case*, Algebra i Analiz, 31(1), 2019, 156–210; St. Petersburg Math. J., 31(1), 2020, 117–156.

[9] B. N. Khabibullin, F. B. Khabibullin, *On the Distribution of Zero Sets of Holomorphic Functions. III. Inversion Theorems*, Funktsional. Anal. i Prilozhen., 53(2), 2019, 42–58; Funct. Anal. Appl. 53(2), 2019, 110–123.

[10] B. N. Khabibullin, A. P. Rozit, E. B. Khabibullina, *Order versions of the Hahn–Banach theorem and envelopes. II. Applications to the function theory*, Complex Analysis. Mathematical Physics, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz., 162, VINITI, Moscow, 2019, 93–135.

[11] Ch. O. Kiselman, *Order and type as measures of growth for convex or entire functions*, Proc. London Math. Soc., 66(3), 1993, 152–86.

[12] V. Azarin, *Growth Theory of Subharmonic Functions*, Birkhäuser, Advanced Texts, Basel–Boston–Berlin, 2009.

[13] B. N. Khabibullin, *Comparison of subharmonic functions by means of their associated measures*, Mat. Sb. (N.S.), 125(167)(4(12)), 1984, 522–538; Math. USSR-Sb., 53(2), 1986, 523–539.

[14] E. G. Kudasheva, B. N. Khabibullin, *Variation of subharmonic function under transformation of its Riesz measure*, Zh. Mat. Fiz. Anal. Geom., 3(1), 2007, 61–94.

[15] F. Federer, *Geometric measure theory*, Springer-Verlag, New York–Berlin, 1969.

[16] Z. Füredi, P. A. Loeb, *On the Best Constant for the Besicovich Covering Theorem*, Proc. Amer. Math. Soc., 121(4), 1994, 1063–1073.

[17] A. F. Grishin, O. F. Krizhanovskii, *An extremal problem for matrices and the Besicovich covering theorem*, Mat. Pros., Ser. 3, 14, MCCME, Moscow, 2010, 196–203 (in Russian).

[18] Miguel de Guzmán, *Differentiation of integrals in $\mathbb{R}^n$*, Lecture Notes in Math., 481, Springer-Verlag, Berlin–New York, 1975.

[19] J. M. Sullivan, *Sphere packings give an explicit bound for the Besicovich Covering Theorem*, J. Geom. Anal. 2(2), 1994, 219–230.

[20] S. G. Krantz, *The Besicovich covering lemma and maximal functions*, Rocky Mountain Jour. of Math., 49(2), 2019, 539–555.
[21] N. S. Landkof, *Foundations of modern potential theory*, 180, Springer-Verlag, Berlin – Heidelberg – New York, 1972.

[22] A. F. Beardon, *Integral means of subharmonic functions*, Proc. Camb. Philos. Soc., 69, 1971, 151–152.

[23] P. Freitas, J. P. Matos, *On the characterization of harmonic and subharmonic functions via mean-value properties*, Potential Anal., 32(2), 2010, 189–200.

[24] L. L. Helms, *Introduction to potential theory*, Wiley Interscience, New York – London – Sydney – Toronto, 1969.

Bulat N. Khabibullin
*Bashkir State University, Ufa, Russian Federation*
E-mail: khabib-bulat@mail.ru

Received 08 March 2020
Accepted 08 April 2020