A gauge-invariant discrete analog of the Yang-Mills equations on a double complex

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Abstract
An intrinsically defined gauge-invariant discrete model of the Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$ is constructed. We develop several algebraic structures on the matrix-valued cochains (discrete forms) that are analogs of objects in differential geometry. We define a combinatorial Hodge star operator based on the use of a double complex construction. Difference self-dual and anti-self-dual equations will be given. In the last section we discuss the question of generalizing our constructions to the case of a 4-dimensional combinatorial sphere.

Key words and phrases: Yang-Mills equations, gauge invariance, difference equations
Math. Subj. Clas.: 81T13, 39A12

1 Introduction
In this paper we construct a gauge-invariant discrete model of the Yang-Mills equations and study combinatorial analogs of some objects in differential geometry, namely the Hodge star operator, the self-dual and anti-self-dual equations. We define these structures on a combinatorial analog of $\mathbb{R}^4$ based on the use of a double complex.

Using the approach first introduced by Dezin [7], in [19, 20] we consider gauge-invariant discrete models of the Yang-Mills equations in $\mathbb{R}^n$ and in Minkowski space. Our approach based also on some constructions from [8], where certain 2-dimensional models connected with the Yang-Mills equations are discussed. In [19] the combinatorial Hodge star operator $*$ is defined using both an inner product on discrete matrix-valued forms (cochains) and Poincaré duality but the operation $(*)^2$ is equivalent to a shift with corresponding sign. This
is one of the main distinctive features of the formalism [7] as compared to the continual case, where the operator \((\ast)^2\) is either an involution or antiinvolution.

In this paper we introduce a combinatorial object, namely a double complex, in which the discrete Hodge star operator is defined in such way that \((\ast)^2 = \pm Id\). At the same time we consider discrete forms, the product \(\cup\) on cochains (discrete analog of the exterior product) and the coboundary operator \(d^c\) (discrete analog of the exterior differential operator) similarly as in [19, 20].

There is another approach presented in Dodziuk’s paper [9]. In [9, 10] the authors using an embedding of simplicial cochains into differential forms (due to [22]) show that a combinatorial Laplacian on the cochains provide a good approximation of the smooth Laplacian on a closed Riemannian manifold. Using the techniques [9], Wilson [23] defines a combinatorial star operator on the simplicial cochains of a triangulated Riemannian manifold and proves its convergence to the smooth star operator. Other related results on the subject can be found in [3, 13, 4].

In section 4 we construct a discrete analog of the Yang-Mills equations on the cochains of the double complex. We try to be as close to continual Yang-Mills theory as possible. We’ll define the discrete Yang-Mills equations using both a geometric structure of the object and a gauge invariance of these equations.

A large number of paper in the physical literature have been devoted to discretization of gauge theories (see, for example, [2, 11, 17, 5, 16] and the references therein). Discrete version of Yang-Mills theories using lattices and graphs, as well as their applications to finite dimensional versions of gauge theories, have been studied in [15, 18, 12]. Some other interesting results on gauge invariant lattice models of Yang-Mills actions using the geometry of finite groups can be found in [6].

It is well known that in 4-dimensional non-abelian gauge theory the self-dual and anti-self-dual connections are the most important extrema of the Yang-Mills action. In section 5 we study discrete analogs of the self-dual and anti-self-dual equations on the double complex and show that some interesting relations amongst the curvature form and its self-dual and anti-self-dual parts, that hold in the continual theory, also hold in the combinatorial case. We also describe difference self-dual equations as a system of nonlinear matrix equations.

In section 6 we consider a combinatorial construction which is homeomorphic to a 4-dimensional sphere. The technique introduced, namely the combinatorial Hodge star operator on the cochains of the double complex, allows us to describe a discrete model of the Yang-Mills equations on a combinatorial 4-dimensional sphere. Using the same approach a discrete analog of the Laplacian on a combinatorial 2-dimensional sphere is considered in [21]. It is interesting to study problems like this since the question concerning the global approximations throughout the surface of a ball has not been study enough.
2 Continual setting

In this section we briefly recall some well-known definitions of smooth Yang-Mills theory (see, for example, [14]). Let $M$ be a smooth oriented Riemannian manifold. Consider the trivial bundle $P = M \times SU(2)$. Let $T^*P$ be the cotangent bundle of $P$ and let $(x, g)$, $x \in M$, $g \in SU(2)$, be local coordinates of the bundle $P$. It is known [14] that a connection can be shown to arise from a certain 1-form $\omega \in T^*P$, where $\omega$ is required to have values in the Lie algebra $su(2)$ of the Lie group $SU(2)$. This form is given by

$$\omega = g^{-1} dg + g^{-1} Ag.$$  \hfill (2.1)

The connection 1-form $A$ is defined by

$$A = \sum_{\alpha, \mu} A^\alpha_\mu(x) \lambda_\alpha dx^\mu,$$  \hfill (2.2)

where $\lambda_\alpha$ is a basis of $su(2)$ and $A^\alpha_\mu(x)$ is a smooth function for any $\mu, \alpha$. Here we take $\lambda_\alpha = \frac{\sigma_\alpha}{2}$, where $\sigma_\alpha$, $\alpha = 1, 2, 3$, are the standard Pauli matrices and $i$ is the imaginary unit.

Let the coordinates of $P$ change (locally) from $(x, g)$ to $(x', g')$. Let us only make a change of fibre coordinates, i.e. $x' = x$ and $g'$ is given by

$$g' = hg, \quad h \in SU(2).$$  \hfill (2.3)

Under the change of coordinates (2.3) the 1-form $\omega$ induces a certain transformation law for the connection form $A$. Suppose that the form (2.1) is invariant under transformations (2.3), i.e.

$$(g')^{-1} dg' + (g')^{-1} A' g' = g^{-1} dg + g^{-1} Ag.$$  \hfill (2.4)

From which we obtain

$$A' = hdh^{-1} + hAh^{-1}.$$  \hfill (2.4)

This is the transformation law of the connection form $A$. It is what is called in Yang-Mills theories the gauge transformation law.

We define the curvature 2-form $F$ in the following way

$$F = dA + A \wedge A.$$  \hfill (2.5)

Under the gauge transformation (2.3) the curvature $F$ changes as follows

$$F' = hFh^{-1}.$$  \hfill (2.6)

Define the covariant exterior differential operator $d_A$ by

$$d_A \Omega = d\Omega + A \wedge \Omega + (-1)^{r+1} \Omega \wedge A,$$  \hfill (2.7)

where $\Omega$ is a $su(2)$-valued $r$-form.
Then the Yang-Mills equations can be written as

\[ d_A F = 0, \quad (2.8) \]
\[ d_A \ast F = 0, \quad (2.9) \]

where \( \ast \) is the Hodge star operator. Equation (2.8) is known as the Bianchi identity. Let \( \Phi, \Psi \) be \( su(2) \)-valued \( r \)-forms on \( M \). The inner product is defined by

\[ (\Phi, \Psi) = -\text{tr} \int_M \Phi \wedge \ast \Psi, \quad (2.10) \]

where \( \text{tr} \) is the trace operator.

Let \( M \) be 4-dimensional. The Yang-Mills action \( S \) can be expressed in terms of the 2-forms \( F \) and \( \ast F \) as

\[ S = -\text{tr} \int_M F \wedge \ast F. \quad (2.11) \]

Equations (2.8), (2.9) are the Euler-Lagrange equations for the extrema of \( S \). In 4-dimensional Yang-Mills theories the following equations

\[ F = \ast F, \quad F = -\ast F \quad (2.12) \]

are called self-dual and anti-self-dual respectively. Solutions of (2.12) – the self-dual and anti-self-dual connections – are called also instantons and anti-instantons [11]. It is known that the self-dual and anti-self-dual connections are the most important minima of the action \( S \).

### 3 Double complex

Let the tensor product \( C(4) = C \otimes C \otimes C \otimes C \) of an 1-dimensional complex \( C \) be a combinatorial model of Euclidean space \( \mathbb{R}^4 \) (see for details [7, 20]). The 1-dimensional complex \( C \) is defined in the following way. Let \( C^0 \) denotes the real linear space of 0-dimensional chains generated by basis elements \( x_\kappa \) (points), \( \kappa \in \mathbb{Z} \). It is convenient to introduce the shift operators \( \tau, \sigma \) in the set of indices by

\[ \tau \kappa = \kappa + 1, \quad \sigma \kappa = \kappa - 1. \]

We denote the open interval \( (x_\kappa, x_{\tau \kappa}) \) by \( e_\kappa \). We’ll regards the set \( \{e_\kappa\} \) as a set of basis elements of the real linear space \( C^1 \) of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the introduced spaces \( C = C^0 \oplus C^1 \). The boundary operator \( \partial \) on the basis elements of \( C \) is given by

\[ \partial x_\kappa = 0, \quad \partial e_\kappa = x_{\tau \kappa} - x_\kappa. \quad (3.1) \]

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements \( x_\kappa, e_\kappa \) in various way we obtain basis elements of \( C(4) \). Let \( s_k^p \), where \( k = (k_1, k_2, k_3, k_4), \ k_i \in \mathbb{Z} \), be an arbitrary basis element of \( C(4) \). We suppose that the superscript \( (p) \) contains the whole requisite
information about the number and places of the 1-dimensional "components" $e_k$ in $s_k^{(p)}$. For example, 1-dimensional basis elements of $C(4)$ can be written as

$$
e_1^k = e_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{k_4}, \quad e_2^k = e_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes x_{k_4},$$

$$e_3^k = x_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes x_{k_4}, \quad e_4^k = x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes e_{k_4}$$

(3.2)

and for the 2-dimensional basis elements $ε_{ij}^k$ we have

$$ε_{12}^k = e_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4}, \quad ε_{23}^k = x_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes x_{k_4},$$

$$ε_{13}^k = e_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}, \quad ε_{24}^k = x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4},$$

$$ε_{34}^k = x_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes e_{k_4}.$$  

(3.3)

Using (3.1), we define the boundary operator $∂$ on chains of $C(4)$ in the following way: if $c_p$, $c_q$ are chains of the indicated dimension, belonging to the complexes being multiplied, then

$$∂(c_p \otimes c_q) = ∂c_p \otimes c_q + (-1)^p c_p \otimes ∂c_q.$$  

(3.4)

For example, for the basis element $ε_{24}^k$ we have

$$∂ε_{24}^k = (∂x_{k_1} \otimes e_{k_2}) \otimes x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes (∂e_{k_2} \otimes x_{k_3} \otimes e_{k_4})$$

$$= ∂x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4} + x_{k_1} \otimes ∂e_{k_2} \otimes x_{k_3} \otimes e_{k_4}$$

$$- x_{k_1} \otimes e_{k_2} \otimes ∂x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes ∂e_{k_2} \otimes x_{k_3} \otimes e_{k_4}$$

$$= x_{k_1} \otimes x_{r_2} \otimes x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes e_{k_4}$$

$$- x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{r_4} + x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{k_4}.$$

We now describe the construction of a double complex. Together with the complex $C(4)$ we consider its "double", namely the complex $C(4)$ of exactly the same structure. Define the one-to-one correspondence

$$* : C(4) \to \tilde{C}(4), \quad * : \tilde{C}(4) \to C(4)$$

(3.5)

in the following way. Let $s_k^{(p)}$ be an arbitrary $p$-dimensional basis element of $C(4)$, i.e. the product $s_k^{(p)} = s_{k_1} \otimes s_{k_2} \otimes s_{k_3} \otimes s_{k_4}$ contains exactly $p$ 1-dimensional elements $e_{k_i}$ and $4 - p$ 0-dimensional elements $x_{k_i}, \ p = 0, 1, 2, 3, 4, \ k_i \in \mathbb{Z}$. Then

$$* : s_k^{(p)} \to ± s_k^{(4-p)}, \quad * : s_k^{(4-p)} \to ± s_k^{(p)},$$

(3.6)

where

$$s_k^{(4-p)} = *s_{k_1} \otimes *s_{k_2} \otimes *s_{k_3} \otimes *s_{k_4}$$

and $*s_{k_i} = ε_{k_i}$ if $s_{k_i} = x_{k_i}$ and $*s_{k_i} = x_{k_i}$ if $s_{k_i} = e_{k_i}$. In the first of mapping (3.6) we take "+" if the permutation $( (p), (4-p) )$ of $(1, 2, 3, 4)$ is even and "−" if the permutation $( (p), (4-p) )$ is odd. Recall that in symbol $(p)$ the number of components is contained. For example, for the 2-dimensional basis element $ε_{13}^k = e_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}$ we have $*ε_{13}^k = - ε_{24}^k$ since the permutation $(1, 3, 2, 4)$ is odd. The mapping $* : s_k^{(4-p)} \to ± s_k^{(p)}$ is defined by analogy.
Proposition 3.1. Let \( c_r \in C(4) \) be an \( r \)-dimensional chain. Then we have
\[
**c_r = (-1)^{r(4-r)}c_r. \tag{3.7}
\]

Proof. For \( r = 0, 4 \) it is obviously. Let \( r = 1 \), then for the 1-dimensional basis elements \( \varepsilon \) we have
\[
\begin{align*}
*\varepsilon^1_k = \tilde{x}_k & \otimes \tilde{e}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{e}_{k_4} = \varepsilon^{234}_k, \\
*\varepsilon^2_k = -\tilde{e}_k & \otimes \tilde{x}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{e}_{k_4} = -\varepsilon^{134}_k, \\
*\varepsilon^3_k = \tilde{e}_k & \otimes \tilde{e}_{k_2} \otimes \tilde{x}_{k_3} \otimes \tilde{e}_{k_4} = \varepsilon^{124}_k, \\
*\varepsilon^4_k = -\tilde{e}_k & \otimes \tilde{e}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{x}_{k_4} = -\varepsilon^{123}_k
\end{align*}
\]
and
\[
*\varepsilon^{-13}_{12} = e^4_k, \quad *\varepsilon^{12}_{34} = -e^3_k, \quad *\varepsilon^{13}_{4} = e^2_k, \quad *\varepsilon^{24}_{3} = -e^1_k.
\]
Hence \(* \varepsilon^i_k = -e^i_k \) for any \( i = 1, 2, 3, 4 \). The case \( r = 3 \) is similar.

Let now \( \varepsilon^{ij}_k \in C(4) \) be a 2-dimensional basis element \( \varepsilon \). Then
\[
\begin{align*}
*\varepsilon^{12}_k = *\varepsilon_{12}^k = \varepsilon^{12}_k, & \quad *\varepsilon^{13}_k = -*\varepsilon_{13}^k = \varepsilon^{13}_k, \quad *\varepsilon^{14}_k = *\varepsilon_{14}^k = \varepsilon^{14}_k, \\
*\varepsilon^{23}_k = *\varepsilon_{23}^k = \varepsilon^{23}_k, & \quad *\varepsilon^{24}_k = -*\varepsilon_{24}^k = \varepsilon^{24}_k, \quad *\varepsilon^{34}_k = *\varepsilon_{34}^k = \varepsilon^{34}_k
\end{align*}
\]

To an arbitrary chain \( c_r \) the operation \(* \) extends by linearity.

Now we consider a dual object of the complex \( C(4) \). Let \( K(4) \) be a cochain complex with \( gl(2, \mathbb{C}) \)-valued coefficients, where \( gl(2, \mathbb{C}) \) is the Lie algebra of all complex \( 2 \times 2 \) matrices. We suppose that the complex \( K(4) \), which is a conjugate of \( C(4) \), has a similar structure: \( K(4) = K \otimes K \otimes K \otimes K \), where \( K \) is a dual of the 1-dimensional complex \( C \). Basis elements of \( K \) can be written as \( \{ x^a \}, \{ e^i \} \). Then an arbitrary basis element of \( K(4) \) is given by \( s^k = s^{k_1} \otimes s^{k_2} \otimes s^{k_3} \otimes s^{k_4} \), where \( s^{k_j} \) is either \( x^{k_j} \) or \( e^{k_j} \). For example, we denote the 1-, 2-dimensional basis elements of \( K(4) \) by \( \varepsilon^i_k, \varepsilon^{ij}_k \), respectively, cf. \( \varepsilon \), \( \varepsilon^{ij} \).

We define the pairing operation for arbitrary basis elements \( \varepsilon^i_k \in C(4), s^k \in K(4) \) by the rule
\[
< \varepsilon^i_k, a s^k > = \begin{cases} 0, & \varepsilon^i_k \neq s_k \\ a, & \varepsilon^i_k = s_k, \ a \in gl(2, \mathbb{C}) \end{cases} \tag{3.8}
\]
The operation \( \varepsilon \) is linearly extended to cochains. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

The operation \( \partial \) induces the dual operation \( d^c \) on \( K(4) \) in the following way:
\[
< \partial \varepsilon^i_k, a s^k > = < \varepsilon^i_k, a d^c s^k > . \tag{3.9}
\]
The coboundary operator \( d^c \) is an analog of the exterior differentiation operator.

Now we describe a cochain product on the forms of \( K(4) \). See \[7, 19, 20\] for details. We denote this product by \( \cup \). In terms of the homology theory this is the so-called Whitney product. First we introduce the \( \cup \)-product on the chains
of the 1-dimensional complex $K$. For the basis elements of $K$ the $\cup$-product is defined as follows

$$x^\kappa \cup x^\kappa = x^\kappa, \quad e^\kappa \cup x^\kappa = e^\kappa, \quad x^\kappa \cup e^\kappa = e^\kappa, \quad \kappa \in \mathbb{Z},$$

supposing the product to be zero in all other case. To arbitrary forms the $\cup$-product be extended linearly. Let us introduce an $r$-dimensional complex $K(r)$, $r = 1, 2, 3$, in an obvious notation. Let $s^k_{(p)}$ be an arbitrary $p$-dimensional basis element of $K(r)$. It is convenient to write the basis element of $K(r+1)$ in the form $s^k_{(p)} \otimes s^\kappa$, where $s^k_{(p)}$ is a basis element of $K(r)$ and $s^\kappa$ is either $e^\kappa$ or $x^\kappa$, $\kappa \in \mathbb{Z}$. Then, supposing that the $\cup$-product in $K(r)$ has been defined, we introduce it for basis elements of $K(r+1)$ by the rule

$$(s^k_{(p)} \otimes s^\kappa) \cup (s^k_{(q)} \otimes s^\mu) = Q(\kappa, q)(s^k_{(p)} \cup s^k_{(q)}) \otimes (s^\kappa \cup s^\mu), \quad (3.10)$$

where the signum function $Q(\kappa, q)$ is equal to $-1$ if the dimension of both elements $s^\kappa, s^k_{(q)}$ is odd and to $+1$ otherwise (see [7]). The extension of the $\cup$-product to arbitrary forms of $K(r+1)$ is linear. Note that the coefficients of forms multiply as matrices.

**Proposition 3.2.** Let $\varphi$ and $\psi$ be arbitrary forms of $K(4)$. Then

$$d^c(\varphi \cup \psi) = d^c \varphi \cup \psi + (-1)^p \varphi \cup d^c \psi, \quad (3.11)$$

where $p$ is the dimension of a form $\varphi$.

The proof of Proposition 3.2 is totally analogous to one in [7, p. 147] for the case of discrete forms with real coefficients.

The complex of the cochains $\tilde{K}(4)$ over the double complex $\tilde{C}(4)$, with the operator $d^c$ defined in it by (3.9), has the same structure as $K(4)$. The operation (3.9) induces the respective mapping

$$*: K(4) \rightarrow \tilde{K}(4), \quad *: \tilde{K}(4) \rightarrow K(4)$$

by the rule:

$$<\tilde{c}, *\varphi> = <*\tilde{c}, \varphi>, \quad <c, *\tilde{\psi}> = <*c, \tilde{\psi}>, \quad (3.12)$$

where $c \in C(4)$, $\tilde{c} \in \tilde{C}(4)$, $\varphi \in K(4)$, $\tilde{\psi} \in \tilde{K}(4)$. It is obviously that Proposition 3.1 is true for any $r$-dimensional cochain $c^r \in K(4)$. So we have

$$* \ast \varphi = (-1)^{r(4-r)} \varphi$$

for any discrete $r$-form $\varphi$ on $K(4)$ and note that the same relation holds in the continual case.

Let $V \subset C(4)$ be a "domain" of the complex $C(4)$. We define its as follows

$$V = \sum_k V_k, \quad k = (k_1, k_2, k_3, k_4), \quad k_i = 1, 2, ..., N_i, \quad (3.13)$$
where \( V_k = e_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes e_{k_4} \) is a 4-dimensional basis element of \( C(4) \). Let \( s_k^{(p)} \) be a \( p \)-dimensional basis element of \( C(4) \). We set
\[
V_p = \sum_k \sum_{(p)} s_k^{(p)} \otimes *s_k^{(p)},
\]
where the subscripts \( k_i, i = 1, 2, 3, 4 \), run the set of values indicated in (3.13). For example,
\[
V_1 = \sum_{k} \sum_{i=1}^{4} e_k^i \otimes *e_k^i = \sum_k (e_k^1 \otimes \tilde{e}_k^{234} - e_k^2 \otimes \tilde{e}_k^{134} + e_k^3 \otimes \tilde{e}_k^{124} - e_k^4 \otimes \tilde{e}_k^{123}).
\]

Let \( K(V) \) denotes \( K(4) \) restricted to \( V \) and let
\[
\mathbb{V} = \sum_{p=0}^{4} V_p.
\]

Consider the following discrete \( p \)-forms
\[
\varphi = \sum_k \sum_{(p)} \varphi_k^{(p)} s_k^{(p)}, \quad \varphi^* = \sum_k \sum_{(p)} (\varphi_k^{(p)})^* s_k^{(p)},
\]
where \( \varphi_k^{(p)} \in gl(2, \mathbb{C}) \) and \((\varphi_k^{(p)})^*\) denotes the conjugate transpose of the matrix \( \varphi_k^{(p)} \), i.e. \((\varphi_k^{(p)})^* = (\varphi_k^{(p)})^T\).

For any \( p \)-forms \( \varphi, \psi \in K(V) \) we define the inner product \( (\ , \ )_V \) by
\[
(\varphi, \psi)_V = tr < \mathbb{V}, \varphi \otimes *\psi^* > = tr < V_p, \varphi \otimes *\psi^* > = tr \sum_k \sum_{(p)} < s_k^{(p)}, \varphi > *s_k^{(p)}, *\psi^* >.
\]

Using (3.6), (3.8), it is easy to check that
\[
(\varphi, \psi)_V = tr \sum_k \sum_{(p)} \varphi_k^{(p)} (\psi_k^{(p)})^*,
\]
where \( \varphi_k^{(p)}, (\psi_k^{(p)})^* \in gl(2, \mathbb{C}) \) are components of \( \varphi, \psi^* \in K(V) \).

Note that for \( su(2) \)-valued \( p \)-forms on \( V \) (cf. (2.10)) Relation (3.16) can be rewritten as follows
\[
(\varphi, \psi)_V = -tr < \mathbb{V}, \varphi \otimes \psi^* > = -tr \sum_k \sum_{(p)} \varphi_k^{(p)} \psi_k^{(p)}.
\]

The inner product makes it possible to define the adjoint of \( d^c \), denoted \( \delta^c \).

**Proposition 3.3.** For any \((p-1)\)-form \( \varphi \) and \( p \)-form \( \omega \) we have
\[
(d^c \varphi, \omega)_V = tr < \partial \mathbb{V}, \varphi \otimes *\omega^* > + (\varphi, \delta^c \omega)_V,
\]
where
\[
\delta^c = (-1)^p *^{-1} d^c *
\]
and \(*^{-1} = Id\).
Proof. The proof is a computation. From the definition (3.9) it follows that
induces the similar relation for the coboundary operator $d^c$ on forms:
\[
d^c (\varphi \otimes \ast \omega) = d^c \varphi \otimes \ast \omega + (-1)^{p-1} \varphi \otimes d^c (\ast \omega).
\]
Using this, we compute
\[
(d^c \varphi, \omega)_V = \text{tr} < \nabla, d^c \varphi \otimes \ast \omega > = \text{tr} < \nabla, d^c (\varphi \otimes \ast \omega) > = \text{tr} < \nabla, \varphi \otimes (\ast (d^c \ast \omega)) >.
\]
It immediately follows (3.17).

Relation (3.17) is a discrete analog of the Green formula. It should be noted that from (3.7) we have:
\[
\ast^{-1} = (-1)^{p(4-p)} \ast.
\]

4 Discrete Yang-Mills equations

In this section we’ll construct a discrete model of the Yang-Mills equations (2.8), (2.9) using the double complex introduced above. Let $A \in K(4)$ be a discrete 1-form. We can write $A$ as
\[
A = \sum_{k} \sum_{i=1}^{4} A^i_k e^k_i,
\]
where $A^i_k \in su(2)$ and $e^k_i$ is an 1-dimensional basis element of $K(4)$. Suppose that the $su(2)$-valued 1-form (4.1) is a discrete analog of the connection form (2.2).

Let us introduce some discrete 0-dimensional form with coefficients belonging to the Lie group $SU(2)$. We put
\[
h = \sum_k h_k x^k,
\]
where $h_k \in SU(2)$ and $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3} \otimes x^{k_4}$ is a 0-dimensional basis element of $K(4)$.

The discrete analog of the transformations (2.3), (2.4) are defined to be
\[
g' = h \cup g, \quad A' = h \cup d^c h^{-1} + h \cup A \cup h^{-1},
\]
where $h, h^{-1}, g$ are 0-forms of the type (4.2) and $h^{-1}$ denotes the form with inverse coefficients (inverse matrices). We’ll call this transformation a gauge transformation for the discrete model.

Remark 4.1. The set of the 0-forms (4.2) is a group with respect to the $\cup$-product.
It is obviously, since by definition of the $\cup$-product for the 0-forms $h, g$ we have
\[ h \cup g = \left( \sum_k h_k x^k \right) \cup \left( \sum_k g_k x^k \right) = \sum_k h_k g_k x^k, \]
where $h_k, g_k$ are multiplied as matrices.

The discrete curvature form $F$ is defined by
\[ F = d^c A + A \cup A. \tag{4.4} \]

The 2-form $F \in K(4)$ we can write also as follows
\[ F = \sum_k \sum_{i<j} F^i_j \varepsilon^k_{ij}, \tag{4.5} \]
where $F^i_j \in gl(2, \mathbb{C})$, $\varepsilon^k_{ij}$ is a 2-dimensional basis elements of $K(4)$ and $1 \leq i, j \leq 4$, $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$.

Let us introduce for convenient the shifts operator $\tau_i$ and $\sigma_i$ as
\[ \tau_i k = (k_1, ..., \tau k_i, ..., k_4), \quad \sigma_i k = (k_1, ..., \sigma k_i, ..., k_4). \]
Similarly, we denote by $\tau_{ij}$ ($\sigma_{ij}$) the operator shifting to the right (to the left) two differ components of $k = (k_1, k_2, k_3, k_4)$. For example,
\[ \tau_{12} k = (\tau k_1, \tau k_2, k_3, k_4), \quad \sigma_{14} k = (\sigma k_1, k_2, k_3, \sigma k_4). \]

Combining (4.4) and (4.5) and using (3.8) – (3.10), we obtain
\[ F^i_j = \Delta_k A^i_k A^j_k - A^i_k A^j_k A_{\tau_i k} - A^j_k A^i_k A_{\tau j k}, \tag{4.6} \]
where $\Delta_k A^i_k = A^j_{i, k} - A^i_k$.

**Remark 4.2.** In the continual case the curvature form $F (2.5)$ takes values in the algebra $su(2)$ for any $su(2)$-valued connection form $A$. Unfortunately, it is not true in the discrete case because, generally speaking, the components $A^i_k A^j_k - A^j_k A^i_k$ of the form $A \cup A$ (see (4.6)) do not belong to $su(2)$.

It is easy to check that the combinatorial Bianchi identity:
\[ d^c F = A \cup F + F \cup A \tag{4.7} \]
holds for the discrete curvature form (4.4) (cf. (2.8)). The discrete analog of the exterior covariant differentiation operator (2.7) is defined by
\[ d^c A \Omega = d^c \Omega + A \cup \Omega + (-1)^{r+1} \Omega \cup A, \tag{4.8} \]
where $\Omega$ is an arbitrary $r$-form of $K(4)$.

**Theorem 4.3.** Under the gauge transformation (4.3) the curvature form (4.4) changes as
\[ F' = h \cup h^{-1}. \tag{4.9} \]
Proof. The proof is similar to that of Proposition 2, \[20\].

Let us introduce the following operation
\[
\tilde{i} : K(4) \to \tilde{K}(4), \quad \tilde{i} : \tilde{K}(4) \to K(4)
\]
by setting
\[
\tilde{s}_k^p = s_k^p, \quad \tilde{s}_k^p = s_k^p,
\]
where \(s_k^p\) and \(\tilde{s}_k^p\) are basis elements of \(K(4)\) and \(\tilde{K}(4)\). So, for a \(p\)-form \(\varphi \in K(4)\) we have \(\tilde{i}\varphi = \tilde{\varphi}\). Recall that the coefficients of \(\tilde{\varphi} \in \tilde{K}(4)\) and \(\varphi \in K(4)\) are the same.

Proposition 4.4. The following hold
\[
i^2 = Id, \quad i* = *i, \quad i d^c = d^c i,
\]
\[
i(\varphi \cup \psi) = i\varphi \cup i\psi,
\]
where \(\varphi, \psi \in K(4)\).

Proof. The proof immediately follows from definitions of the corresponding operations. \(\square\)

The discrete analog of Equation (2.9) can be written as
\[
d^c_A * i F = 0. \tag{4.12}
\]
Using (4.3), we have
\[
d^c_A * i F = d^c * i F + A \cup *i F = *i F \cup A. \tag{4.13}
\]

Lemma 4.5. Let \(h\) be a discrete 0-form. Then for an arbitrary \(p\)-form \(f \in K(4)\) we have
\[
i* (h \cup f) = h \cup i* f. \tag{4.14}
\]
Proof. Applying (4.10), the proof is analogous to the proof of Lemma 1 in \[20\]. \(\square\)

Lemma 4.6. Let \(f \in K(4)\) be a 2-form. We have
\[
i* (f \cup h) = i* f \cup h \tag{4.15}
\]
if and only if the coefficients of a 0-form \(h\) satisfy the following conditions
\[
h_{\tau_{2k}} = h_{\tau_{3k}}, \quad h_{\tau_{3k}} = h_{\tau_{4k}}, \quad h_{\tau_{4k}} = h_{\tau_{23k}} \tag{4.16}
\]
for all \(k = (k_1, k_2, k_3, k_4), k_i \in \mathbb{Z} \).
Proof. The proof is computational. See also the proof of Lemma 2 in [20]. Using (3.10) and (3.6), we compute
\[ f \cup h = \sum_{k} \sum_{i<j} f_{k}^{ij} h_{\tau_{ij},k} \varepsilon_{ij}^{k}, \]
and
\[ *f = \sum_{k} (f_{k}^{12} \varepsilon_{34}^{k} - f_{k}^{13} \varepsilon_{24}^{k} + f_{k}^{14} \varepsilon_{23}^{k} + f_{k}^{23} \varepsilon_{14}^{k} - f_{k}^{24} \varepsilon_{13}^{k} + f_{k}^{34} \varepsilon_{12}^{k}), \]
where \( \varepsilon_{ij}^{k} \) is a 2-dimensional basis element of \( K(4) \). Then, by the definition of \( \tilde{i} \), we obtain
\[ \tilde{i} * (f \cup h) = \sum_{k} (f_{k}^{12} h_{\tau_{12},k} \varepsilon_{34}^{k} - f_{k}^{13} h_{\tau_{13},k} \varepsilon_{24}^{k} + f_{k}^{14} h_{\tau_{14},k} \varepsilon_{23}^{k} + f_{k}^{23} h_{\tau_{23},k} \varepsilon_{14}^{k} - f_{k}^{24} h_{\tau_{24},k} \varepsilon_{13}^{k} + f_{k}^{34} h_{\tau_{34},k} \varepsilon_{12}^{k}). \]
On the other hand, we have
\[ \tilde{i} * f \cup h = \sum_{k} (f_{k}^{12} h_{\tau_{34},k} \varepsilon_{34}^{k} - f_{k}^{13} h_{\tau_{13},k} \varepsilon_{24}^{k} + f_{k}^{14} h_{\tau_{14},k} \varepsilon_{23}^{k} + f_{k}^{23} h_{\tau_{13},k} \varepsilon_{14}^{k} - f_{k}^{24} h_{\tau_{14},k} \varepsilon_{13}^{k} + f_{k}^{34} h_{\tau_{34},k} \varepsilon_{12}^{k}). \]
Combining the last two expressions with one another, we conclude that (4.15) implies (4.16) and vice versa.

It should be noted that the set of 0-forms \( F \), which satisfy Conditions (4.16), is a group under \( \cup \)-product (see Remark 4.1).

Theorem 4.7. Under Conditions (4.16) the discrete Yang-Mills equation (4.12) is gauge invariant.

Proof. The proof is analogous to the proof of Theorem 1 in [20]. By Theorem 4.2 and Lemmas 4.4, 4.5, we have
\[ \tilde{i} * F'' = \tilde{i} * (h \cup F \cup h^{-1}) = h \cup \tilde{i} * F \cup h^{-1}. \]
Note that \( h^{-1} \) also satisfies Conditions (4.16). Using (3.18) we compute
\[ d^c \tilde{i} * F'' = d^c h \cup \tilde{i} * F \cup h^{-1} - h \cup d^c \tilde{i} * F \cup h^{-1} + h \cup \tilde{i} * F \cup d^c h^{-1}. \]
Since \( d^c h \cup h^{-1} = -d^c h \cup h^{-1} \) and taking into account (4.13) and (4.19), we get
\[ A' \cup \tilde{i} * F' = -d^c h \cup \tilde{i} * F \cup h^{-1} + h \cup A \cup \tilde{i} * F \cup h^{-1} \]
and
\[ \tilde{i} * F'' \cup A' = h \cup \tilde{i} * F \cup d^c h^{-1} + h \cup \tilde{i} * F \cup A \cup h^{-1}. \]
Putting the last three expressions in (4.13), one obtains:
\[ d^c A' \tilde{i} * F'' = h \cup d^c \tilde{i} * F \cup h^{-1} + h \cup A \cup \tilde{i} * F \cup h^{-1} - h \cup \tilde{i} * F \cup A \cup h^{-1} = h \cup d^c A \tilde{i} * F \cup h^{-1}. \]
Thus, if \( d^c A \tilde{i} * F = 0 \), then \( d^c A \tilde{i} * F' = 0. \)
5 Difference self-dual and anti-self-dual equations

The discrete analog of Equations (2.12) is defined by

\[ F = i \ast F, \quad F = -i \ast F, \]  

where \( F \) is the discrete curvature form \([4.4]\). Using \([4.5]\), by the definitions of \( i \) and \( \ast \), the first equation (self-dual) of (5.1) can be rewritten as follows

\[ F_{k}^{12} = F_{k}^{34}, \quad F_{k}^{13} = -F_{k}^{24}, \quad F_{k}^{14} = F_{k}^{23}. \]  

We call these equations difference self-dual equations. Using \([4.6]\), we obtain

\[
\begin{align*}
\Delta k \Lambda_{k}^{2} - \Delta k_{1} \Lambda_{k}^{3} - \Delta k_{2} \Lambda_{k}^{1} - \Lambda_{k}^{1} \Lambda_{k}^{2} & = \Delta k_{1} \Lambda_{k}^{3} - \Delta k_{3} \Lambda_{k}^{1} - \Lambda_{k}^{1} \Lambda_{k}^{3} + \Lambda_{k}^{2} \Lambda_{k}^{4} - \Lambda_{k}^{3} \Lambda_{k}^{4}, \\
\Delta k \Lambda_{k}^{3} - \Delta k_{1} \Lambda_{k}^{4} + \Lambda_{k}^{1} \Lambda_{k}^{3} - \Lambda_{k}^{1} \Lambda_{k}^{4} & = \Delta k_{1} \Lambda_{k}^{4} - \Delta k_{4} \Lambda_{k}^{1} + \Lambda_{k}^{1} \Lambda_{k}^{4} + \Lambda_{k}^{2} \Lambda_{k}^{3} - \Lambda_{k}^{3} \Lambda_{k}^{4}, \\
\Delta k \Lambda_{k}^{4} - \Delta k_{1} \Lambda_{k}^{2} + \Lambda_{k}^{1} \Lambda_{k}^{4} - \Lambda_{k}^{1} \Lambda_{k}^{2} & = \Delta k_{1} \Lambda_{k}^{2} - \Delta k_{2} \Lambda_{k}^{1} + \Lambda_{k}^{1} \Lambda_{k}^{2} - \Lambda_{k}^{3} \Lambda_{k}^{2} - \Lambda_{k}^{4} \Lambda_{k}^{2}.
\end{align*}
\]

Recall that \( \Lambda_{k}^{i} \in su(2) \) is a component of the discrete connection 1-form \([4.1]\). Of course, changing the sign on the right hand side of Equations (5.2), we obtain the difference anti-self-dual equations.

As in the continual case (see, for example, \([14]\)), we can decompose our arbitrary discrete 2-form \( F \) into its self-dual and anti-self-dual parts as follows

\[ F = F^{+} + F^{-}, \]

where \( F^{+} = \frac{1}{2}(F + i \ast F) \) and \( F^{-} = \frac{1}{2}(F - i \ast F) \). The form \( F^{+} \) is self-dual, i.e. \( F^{+} = i \ast F^{+} \). Indeed, using Proposition 3.1 and \([4.11]\), we compute

\[ i \ast F^{+} = i \ast \frac{1}{2}(F + i \ast F) = \frac{1}{2}(i \ast F + i \ast i \ast F) = \frac{1}{2}(i \ast F + F) = F^{+}. \]

Similarly,

\[ i \ast F^{-} = i \ast \frac{1}{2}(F - i \ast F) = \frac{1}{2}(i \ast F - i \ast i \ast F) = \frac{1}{2}(i \ast F - F) = -F^{-}. \]

So, \( F^{-} \) is anti-self-dual.

Let \( \| \cdot \|_{\nu} \) denote the norm on \( K(V) \) generated by the inner product \([3.15]\). Then a discrete analog of the Yang-Mills action \([2.11]\) can be written as

\[ S = \| F \|^2_{\nu} = (F, F)_{\nu} = tr < V_{2}, F \otimes F^{*} >. \]

See also \([3.13]\)–\([3.16]\).

**Theorem 5.1.** For the discrete curvature form \([4.4]\) we have

\[ \| F \|^2_{\nu} = \| F^{+} \|^2_{\nu} + \| F^{-} \|^2_{\nu}. \]  

(5.3)
Proof. By definition (3.15) we have
\[
\|F\|_V^2 = \|F^+ + F^-\|_V^2
\]
\[
= tr < V_2, F^+ \otimes *(F^+)^* > + tr < V_2, F^- \otimes *(F^-)^* >
\]
\[
+ tr < V_2, F^+ \otimes *(F^-)^* > + tr < V_2, F^- \otimes *(F^+)^* >
\]
\[
= \|F^+\|_V^2 + \|F^-\|_V^2 + (F^+, F^-)_V + (F^-, F^+)_V.
\]
Denote the components of \(F^+\), \(F^-\) by \((F^+_{ij})^k\), \((F^-_{ij})^k\) respectively. For \((F^+_{ij})^k\) we have (5.2) and for \((F^-_{ij})^k\) we can write the following relations
\[
(F^+_{ij})^k = -(F^+_{ji})^k, \quad (F^-_{ij})^k = (F^-_{ji})^k, \quad (F^+_{ij})^k = -(F^-_{ji})^k.
\]
Then, using (3.16), we obtain
\[
(F^+, F^-)_V = tr \sum_k \sum_{i<j} (F^+_{ij})^k [(F^+_{ij})^-]^k = - tr \sum_k \sum_{i<j} (F^+_{ij})^k [(F^+_{ij})^-]^k
\]
\[
= -(F^+, F^-)_V.
\]
Thus, \((F^+, F^-)_V = 0\). Similarly, we have \((F^-, F^+)_V = 0\).

It should be noted that in the continual case Relation (5.3) implies that the self-dual and anti-self-dual connections (solutions of (2.12)) are always absolute minima of the action \(S\) (see [14]).

6 Combinatorial model of the 4-sphere

In this section we discuss the question of generalizing our constructions introduced above to the case of a 4-dimensional complex which is the boundary of a 5-dimensional domain. Note that constructions used in [19, 20], namely the operation \(*\), are inappropriate to this case. It is convenient to employ here a construction based on the use of the double complex. We will use a standard technique ("gluing of the double") that turns a manifold with boundary into a manifold without boundary.

Let \(V \in C(4)\) be a "domain" in the form (3.13). Together with \(V \in C(4)\) we introduce its counterpart \(\hat{V} \in C(4)\). Considering now \(V, \hat{V}\) to be two distinct domains and identifying the respective elements of the boundary, we obtain the 4-dimensional combinatorial manifold \(M = V \cup \hat{V}\) which is homeomorphic to the 4-dimensional sphere. Let \(s^{(p)}_k\) be a basis element of \(C(V)\). Denote by \(s^{(p)}_{k_1 \ldots k_4}\) the corresponding basis element of \(\hat{C}(V)\). The "gluing" conditions of \(V\) and \(\hat{V}\) are defined by
\[
\begin{align*}
\hat{s}^{(p)}_{k_1 \ldots k_4} = & s^{(p)}_{k_1 \ldots k_4}, & s^{(p)}_{k_1 \ldots k_4} & = \hat{s}^{(p)}_{k_1 \ldots k_4} = s^{(p)}_{k_1 \ldots k_4}, \\
\hat{s}^{(p)}_{k_1 \ldots k_4} = & s^{(p)}_{k_1 \ldots k_4}, & s^{(p)}_{k_1 \ldots k_4} & = \hat{s}^{(p)}_{k_1 \ldots k_4} = s^{(p)}_{k_1 \ldots k_4},
\end{align*}
\] (6.1)
where \(0 \leq k_i \leq N_i\), see (3.13). On the other hand, a new combinatorial object, namely the complex \(C(M)\), is defined by Conditions (6.1). The boundary operator \(\partial\) on \(C(M)\) is given by (3.3). We call the complex \(C(M)\) a combinatorial 4-dimensional sphere. As in section 3, we introduce the dual complex \(K(M)\). No essential modifications are needed to carry out constructions, considered in \(K(4)\), in the complex \(K(M)\). An arbitrary \(p\)-form \(\varphi \in K(M)\) can be written as

\[
\varphi = \sum_k \sum_{(p)} (\varphi_k^{(p)} s_k^{(p)} + \hat{\varphi}_k^{(p)} \hat{s}_k^{(p)}),
\]

where \(\varphi_k^{(p)}\), \(\hat{\varphi}_k^{(p)} \in gl(2, \mathbb{C})\) and \(s_k^{(p)} \in K(V)\), \(\hat{s}_k^{(p)} \in K(\hat{V})\) are corresponding basis elements, \(k = (k_1, k_2, k_3, k_4)\), \(k_i = 1, 2, ..., N_i\). Due to the definition (3.13), Conditions (6.1) imply the following conditions for the form \(\varphi\):

\[
\varphi_k^{(p)} \varphi_{k_1, \ldots, k_4} = \varphi_k^{(p)} \varphi_{k_1, \ldots, k_4}, \quad \varphi_{k_1, \ldots, k_4} = \varphi_{k_1, \ldots, k_4}.
\]

Recall that the components \(\varphi_k^{(p)}\), \(\varphi_k^{(p)}\) appear when we consider the coboundary operator \(\partial\) and its applications.

It is obvious that the double complex construction extends to the complex \(C(M)\) (or \(K(M)\)). The star operation * on \(K(M)\) is also defined by (3.13). So, for any \(p\)-forms \(\varphi, \psi \in K(M)\) the inner product can be written as

\[
(\varphi, \psi)_M = tr < V_p, \varphi \otimes * \psi > + tr < \hat{V}_p, \hat{\varphi} \otimes * \hat{\psi} >
\]

or

\[
(\varphi, \psi)_M = tr \sum_k \sum_{(p)} (\varphi_k^{(p)} (\psi_k^{(p)})^* + \hat{\varphi}_k^{(p)} (\hat{\psi}_k^{(p)})^*).
\]

The discrete connection 1-form over \(C(M)\) is defined by

\[
A = \sum_{i=1}^4 \sum_k (A_{ik}^i e_k^i + \hat{A}_{ik}^i \hat{e}_k^i),
\]

where \(A_{ik}, \hat{A}_{ik} \in su(2)\), \(e_k^i, \hat{e}_k^i\) are 1-dimensional basis elements of \(K(M)\), \(k = (k_1, k_2, k_3, k_4)\), \(k_i = 1, 2, ..., N_i\). Similarly, the discrete curvature 2-form (4.4) can be written as

\[
F = \sum_k \sum_{i<j} (F_{ik}^{ij} e_k^i \epsilon_{ij}^k + \hat{F}_{ik}^{ij} \hat{e}_k^i \hat{\epsilon}_{ij}^k),
\]

where \(F_{ik}^{ij}, \hat{F}_{ik}^{ij} \in gl(2, \mathbb{C})\), \(e_k^i, \hat{e}_k^i\) are 2-dimensional basis elements of \(K(M)\). The components \(A_{ik}, \hat{A}_{ik}, F_{ik}^{ij}, \hat{F}_{ik}^{ij}\) satisfy Conditions (6.2).

It is easy to check that all constructions from Sections 4, 5 carry out in the complex \(K(M)\). Thus, we can write the discrete Yang-Mills equations in the form (4.7), (4.12) and Theorem 4.7 holds on the combinatorial sphere \(C(M)\). In \(K(M)\) the difference self-dual and anti-self-dual equations (5.2) are completed by the same equations for the components \(\hat{F}_{ik}^{ij}\). So, under Conditions 6.2 we obtain the finite-dimensional system of matrices equations.
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