Cyclic quadrilaterals and smooth Jordan curves

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Abstract
For every smooth Jordan curve \( \gamma \) and cyclic quadrilateral \( Q \) in the Euclidean plane, we show that there exists an orientation-preserving similarity taking the vertices of \( Q \) to \( \gamma \). The proof relies on the theorem of Polterovich and Viterbo that an embedded Lagrangian torus in \( \mathbb{C}^2 \) has minimum Maslov number 2.

A quadrilateral \( Q \) inscribes in a smooth Jordan curve \( \gamma \) in the Euclidean plane if there exists an orientation-preserving similarity of the plane taking the vertices of \( Q \) to \( \gamma \); it is cyclic if it inscribes in a circle. The result of this paper is the solution of the cyclic quadrilateral peg problem [5, Conjecture 9]:

**Theorem** Every cyclic quadrilateral inscribes in every smooth Jordan curve in the Euclidean plane.

The result is best possible, by considering the case in which the smooth Jordan curve is itself a circle. Moreover, some regularity hypothesis on the Jordan curve is necessary in order for the Theorem to hold, as the only cyclic quadrilaterals that inscribe in all triangles are the isosceles trapezoids [8, § 3.6]. See [2–5] for earlier progress towards this result.

**Proof** For a fixed cyclic quadrilateral \( Q \) and smooth Jordan curve \( \gamma \), we construct a pair of Lagrangian tori \( T_1 \) and \( T_2 \) in standard symplectic \( \mathbb{C}^2 \). They intersect cleanly along \( \gamma \times \{0\} \) and in a disjoint set of points \( P \) which parametrize the inscriptions of \( Q \) in \( \gamma \). By smoothing the intersection along \( \gamma \times \{0\} \), we obtain an immersed Lagrangian torus \( T \) whose set of self-intersections is \( P \). As we show, \( T \) has minimum Maslov number 4. On the other hand, a theorem independently due to Polterovich and Viterbo...
asserts that an embedded Lagrangian torus in $\mathbb{C}^2$ has minimum Maslov number 2 \cite{9, 12}. Therefore $P$ is non-empty, so $Q$ inscribes in $\gamma$. □

The strategy of proof of the Theorem resembles that of our earlier result, which treated the case in which $Q$ is a rectangle \cite{2}. In that case, we additionally arranged that $T$ is invariant under a symplectic involution $\tau$ of $\mathbb{C}^2$. Passing to the quotient by $\tau$, we obtained an immersed Lagrangian Klein bottle $K = T/\tau$ in $\mathbb{C}^2$ whose self-intersections $P/\tau$ parametrize inscriptions of $Q$ in $\gamma$ up to rotation by $\pi$. A theorem independently due to Shevchishin and Nemirovski asserts that there is no embedded Lagrangian Klein bottle in $\mathbb{C}^2$ \cite{7, 11}, thereby ensuring that $P$ is non-empty, so $Q$ inscribes in $\gamma$. In the more general case of a cyclic quadrilateral, $T$ does not admit any apparent symmetry, which impedes reusing the same approach. Our revised approach produces a stronger result and somewhat more directly.

Cyclic quadrilaterals. We begin by characterizing the set of cyclic quadrilaterals. Let $Q$ denote a convex quadrilateral in the plane whose vertices are labeled $ABCD$ in counterclockwise order. Its diagonals $AC$ and $BD$ intersect in a point $X$. Euclid’s chord theorem asserts that $Q$ is cyclic if and only if $|AX| \cdot |CX| = |BX| \cdot |DX|$ \cite{1, Theorem III.35}.

By a cyclic permutation of the vertex labels, we may assume that $|AX| \leq |CX|$ and $|BX| \leq |DX|$. We thereby obtain real values $s = |AX|/|AC|$ and $t = |BX|/|BD|$ in $(0, 1/2]$ and an angle $\phi = \angle AXB$ in $(0, \pi)$. The triple of values $(s, t, \phi)$ uniquely determines the oriented similarity class of $Q$, unless one of $s$ and $t$ equals $1/2$, in which case $(s, t, \phi)$ and $(t, s, \pi - \phi)$ determine the same oriented similarity class.

We reformulate the preceding description for our present purposes. Identify the Euclidean plane with the complex numbers $\mathbb{C}$. Define $\mathbb{C}$-linear automorphisms of $\mathbb{C}^2$ by the matrices

$$F_r = \begin{pmatrix} 1 - r & r \\ \sqrt{r(1 - r)} & -\sqrt{r(1 - r)} \end{pmatrix} \quad \text{and} \quad R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

for values $r \in (0, 1/2]$ and $\phi \in (0, \pi)$.

**Lemma 1** Points $A, B, C, D \in \mathbb{C}$ correspond as above to vertices of a cyclic quadrilateral with parameters $(s, t, \phi)$ if and only if

$$R_\phi \circ F_s(A, C) = F_t(B, D) \quad \text{and} \quad A \neq C \ (\text{equivalently} \ B \neq D). \quad (1)$$

**Proof** Equality in the first coordinate of (1) is equivalent to the assertion that segments $AC$ and $BD$ intersect at a point $X$ so that $|AX| = s \cdot |AC|$ and $|BX| = t \cdot |BD|$. Equality in the second coordinate given the first then ensures that $\angle AXB = \phi$ and that $|AX| \cdot |CX| = s(1 - s) \cdot |AC|^2 = t(1 - t) \cdot |BD|^2 = |BX| \cdot |DX|$. Insisting that $A \neq C$ or $B \neq D$ ensures that $Q$ does not degenerate to a point. □

Two embedded Lagrangian tori. Suppose that $Q$ is a cyclic quadrilateral with parameters $(s, t, \phi)$ as above and that $\gamma$ is a smooth Jordan curve in $\mathbb{C}$. Note that $\gamma \times \gamma$

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1 Euclid proves the forward direction, which can be used to prove the reverse.
is a smoothly embedded torus in $\mathbb{C}^2$. Define tori

$$T_1 = R_\phi \circ F_s(\gamma \times \gamma) \quad \text{and} \quad T_2 = F_t(\gamma \times \gamma).$$

Note that both $R_\phi \circ F_s$ and $F_t$ map the point $(z, z)$ to $(z, 0)$ for all $z \in \mathbb{C}$. From Lemma 1 we see that the set of inscriptions of $Q$ in $\gamma$ is parametrized by the set of points

$$P = T_1 \cap T_2 - \gamma \times \{0\}.$$  

In addition, the hypothesis that $\gamma$ is a smooth Jordan curve ensures that $\gamma \times \{0\}$ is a smooth submanifold of both $T_1$ and $T_2$. Let $\omega = dz \wedge d\bar{z} + dw \wedge d\bar{w}$ denote the standard symplectic form on $\mathbb{C}^2$, up to scale.

**Lemma 2** The tori $T_1$ and $T_2$ are Lagrangian with respect to $\omega$ and intersect cleanly along $\gamma \times \{0\}$:

$$T_{(p, 0)} T_1 \cap T_{(p, 0)} T_2 = T_{(p, 0)} (\gamma \times \{0\}), \quad \text{for all } p \in \gamma.$$  

**Proof** A direct calculation shows that

$$\omega_r := F_r^* \omega = (1 - r) \cdot dz \wedge d\bar{z} + r \cdot dw \wedge d\bar{w}$$

for $r \in (0, 1/2]$. Note that $\gamma \times \gamma$ is Lagrangian with respect to $\omega_r$ and $R_\phi^* \omega = \omega$. It follows that $T_1$ and $T_2$ are Lagrangian with respect to $\omega$.

If $p \in \gamma$ is a point on the Jordan curve, then $T_{p, \gamma} \subset \mathbb{C}$ is a 1-dimensional real subspace. A direct calculation shows that

$$T_{(p, 0)} T_1 = T_{p, \gamma} \times \{0\} \oplus \{0\} \times e^{i\phi} T_{p, \gamma} \quad \text{and} \quad T_{(p, 0)} T_2 = T_{p, \gamma} \times \{0\} \oplus \{0\} \times T_{p, \gamma},$$

so

$$T_{(p, 0)} T_1 \cap T_{(p, 0)} T_2 = T_{p, \gamma} \times \{0\} = T_{(p, 0)} (\gamma \times \{0\}),$$

and the intersection along $\gamma \times \{0\}$ is clean, as required. 

**A surgered immersed Lagrangian torus.** Because $T_1$ and $T_2$ intersect cleanly along $\gamma \times \{0\}$, a version of the Weinstein neighborhood theorem due to Poźniak [10, Proposition 3.4.1] implies that we can select coordinates $(x_1, y_1, x_2, y_2)$ in a neighborhood $N \approx (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^3$ of $\gamma \times \{0\}$ such that

- $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$,
- $T_1 \cap N = \{y_1 = y_2 = 0\},$ and
- $T_2 \cap N = \{y_1 = x_2 = 0\}.$

We smooth the intersection of $T_1$ and $T_2$ in $N$ as suggested by Fig. 1 and let $T$ denote the result. The tangent plane to $T$ at a point in $N$ is spanned by $\partial/\partial x_1$ and a vector of the form $a \cdot \partial/\partial x_2 + b \cdot \partial/\partial y_2$, which are $\omega$-orthogonal. Thus, $T$ is an immersed Lagrangian torus in $(\mathbb{C}^2, \omega)$, and its set of self-intersections equals $P$, which parametrizes the set of inscriptions of $Q$ in $\gamma$. 

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Fig. 1 Cross-section of smoothing in the $x_1 = \text{constant}, \ y_1 = 0$ plane

The minimum Maslov number. Equip $\mathbb{C}^n$ with a product symplectic form $\omega_0 = \sum_{i=1}^n c_i \cdot d z_i \land d\overline{z}_i$. An immersed Lagrangian submanifold $i : L \to (\mathbb{C}^n, \omega_0)$ has a Maslov class $\mu \in H^1(L; \mathbb{Z})$, given as follows (cf. [6, pp.117-118]). The tangent planes to $i(L)$ along the image of an embedded loop $\alpha \subset L$ determine a loop $\alpha^\sharp$ in $\mathcal{L}(\omega_0)$, the Grassmannian of Lagrangian $n$-planes in $(\mathbb{C}^n, \omega_0)$. The Maslov index of $\alpha$ is the value $\mu([\alpha]) := [\alpha^\sharp] \in H_1(\mathcal{L}(\omega_0); \mathbb{Z}) \approx \mathbb{Z}$, and the minimum Maslov number of $L$ is the non-negative integer $m(L)$ such that $\mu(H_1(L; \mathbb{Z})) = m(L) \cdot \mathbb{Z}$.

Proposition The minimum Maslov number of $T$ is 4.

Proof Orienting $\gamma \subset \mathbb{C}$ counterclockwise, its Maslov index equals 2 with respect to $c \cdot dz \land d\overline{z}$. Hence $\gamma \times \{\text{pt}\}$ and $\{\text{pt}\} \times \gamma$ both have Maslov index 2 in $\gamma \times \gamma$ with respect to the product form $\omega_r$. Since their homology classes generate $H_1(\gamma \times \gamma; \mathbb{Z})$, we obtain $m(\gamma \times \gamma) = 2$. The diagonal loop $\{(z, z) : z \in \gamma\}$ is homologous to their sum, so it has Maslov index 4 in $\gamma \times \gamma$ with respect to $\omega_r$. Applying $R_\phi \circ F_s$ and $F_t$, we deduce that $\gamma \times \{0\}$ has Maslov index 4 in both $T_1$ and $T_2$ with respect to $\omega$ and that $m(T_1) = m(T_2) = 2$. Let $\delta$ denote a push-off of $\gamma \times \{0\}$ in $T_1$ away from the site of surgery. A neighborhood of $\delta$ survives the surgery, so the Maslov index of $[\delta]$ in $T$ is 4 with respect to $\omega$.

Next, select oriented loops $\lambda_1 \subset T_1$, $\lambda_2 \subset T_2$, and $\lambda \subset T$ such that $\lambda_1 \cup \lambda_2$ and $\lambda$ coincide outside the neighborhood $N$ above and meet it in a single slice $x_1 = \text{constant}, \ y_1 = 0$, as displayed in Fig. 1. The tangent planes to $T \cup T_1 \cup T_2$ along the difference 1-cycle $\lambda - \lambda_1 - \lambda_2$ describe a nullhomotopic loop in $\mathcal{L}(\omega)$. Consequently, $[\lambda^\sharp] = [\lambda_1^\sharp] + [\lambda_2^\sharp] \in H_1(\mathcal{L}(\omega); \mathbb{Z}) \approx \mathbb{Z}$. The class $[\lambda_j]$ completes to a basis of $H_1(T_j; \mathbb{Z})$ with $[\gamma \times \{0\}]$ for $j = 1, 2$. Since $m(T_j) = 2$ and $[\gamma \times \{0\}]$ has Maslov index 4 in $T_j$, $j = 1, 2$, it follows that $[\lambda_j]$ has Maslov index 2 (mod 4), $j = 1, 2$. Therefore, the Maslov index of $[\lambda]$ in $T$ is a multiple of 4.

Since $[\delta]$ and $[\lambda]$ form a basis for $H_1(T; \mathbb{Z})$, it follows that $m(T) = 4$. □

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