Matrix commuting differential operators of rank 2

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Abstract

In this paper we propose a very effective method for constructing matrix commuting differential operators of rank 2 and vector rank (2,2). We find new matrix commuting differential operators $L, M$ of orders 2 and $2g$ respectively.

Introduction

Let us consider two differential operators

$$L_n = \sum_{i=0}^{n} u_i(x) \partial_x^i, \quad L_m = \sum_{i=0}^{m} v_i(x) \partial_x^i,$$

where coefficients $u_i(x)$ and $v_i(x)$ are scalar or matrix valued functions. The commutativity condition $L_n L_m = L_m L_n$ is equivalent to a very complicated system of nonlinear differential equations. The theory of commuting ordinary differential operators was first developed in the beginning of the XX century in the works of Wallenberg [1], Schur [2].

If two differential operators with scalar or matrix valued coefficients commute, then there exists a nonzero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$ (see [3], [4]). The curve $\Gamma$ defined by $R(z, w) = 0$ is called the spectral curve. If

$$L_n \psi = z\psi, \quad L_m \psi = w\psi,$$

then $(z, w) \in \Gamma$.

If coefficients are scalar functions, then for almost all $(z, w) \in \Gamma$, the dimension of the space of common eigenfunctions $\psi$ is the same. The dimension of the space of common eigenfunctions of two commuting scalar differential operators is called the rank of this pair. The rank is a common divisor of $m$ and $n$. The genus of the spectral curve of a pair of commuting operators is called the genus of this pair.

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If the rank of two commuting scalar differential operators equals 1, then there are explicit formulas for coefficients of commutative operators in terms of Riemann theta-functions (see [3]).

The case when rank of scalar commuting operators is greater than 1 is much more difficult. The first examples of commuting scalar differential operators of the nontrivial rank 2 and the nontrivial genus $g = 1$ were constructed by Dixmier [6] for the nonsingular elliptic spectral curve.

A general classification of commuting scalar differential operators was obtained by Krichever [7]. The general form of commuting scalar operators of rank 2 for an arbitrary elliptic spectral curve was found by Krichever and Novikov [8]. The general form of scalar commuting operators of rank 3 with arbitrary elliptic spectral curve was found by Mokhov [9], [10]. In [11] Mironov developed theory of self-adjoint scalar operators of rank 2 and found examples of commuting scalar operators of rank 2 and arbitrary genus. Using Mironov’s method many examples of scalar commuting operators of rank 2 and arbitrary genus were found (see [12], [13], [14], [15], [16]). Moreover, examples of commuting scalar differential operators of arbitrary genus and arbitrary rank with polynomial coefficients were constructed by Mokhov in [17], [18].

Theory of commuting differential operators helps to find solutions of nonlinear partial differential equations from mathematical physics (see [19], [20], [21], [22]). Also there are deep connections between theory of commuting scalar differential operators and Schottky problem (see [23], [24]). The theory of commuting differential operators with polynomial coefficients has connections with the Dixmier conjecture and Jacobian conjecture (see [25], [26]).

A general classification of commuting matrix differential operators was obtained by Grinevich [4]. Grinevich considered two differential operators

\[ L = \sum_{i=0}^{m} U_i \partial_x^i, \quad M = \sum_{i=0}^{n} V_i \partial_x^i, \]

where $U_i$ and $V_i$ are smooth and complex-valued $s \times s$ matrices. Let us suppose the following conditions

1) \( \det(U_m) \neq 0. \)
2) Eigenvalues $\lambda_1(x), ..., \lambda_s(x)$ of $U_m$ are distinct.
3) Matrix $V_n$ is diagonalizable. Let $\mu_1(x), ..., \mu_s(x)$ be eigenvalues of matrix $V_n$. Suppose that functions $\frac{\mu_i}{\lambda_i}$ are distinct constants for all $i = 1, ..., s$.

Easy to see that if $L$ and $M$ commute, then $U_m$ and $V_n$ commute. If operators
$L$ and $M$ commute, then $FLF^{-1}$ and $FMF^{-1}$ commute, where $F$ is matrix. We also can change variable. So, without loss of generality we can suppose that

$$(U_m)_{ij} = \delta_{ij} \lambda_i, \quad (V_n)_{ij} = \delta_{ij} \mu_i, \quad tr(U_{m-1}) = 0$$

Let $\Gamma$ be the spectral curve of commuting matrix operators $L, M$. Spectral curve of matrix commuting operators can be reducible. Let $\Gamma_i$ be an irreducible component of the spectral curve. The dimension of the space of common eigenfunctions

$$L\psi = z\psi, \quad M\psi = w\psi, \quad (z, w) \in \Gamma_i$$

is called the rank of commuting pair on $\Gamma_i$. Grinevich discovered that the spectral curve $\Gamma$ has $s$ points at infinity. So, $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$, where $k \leq s$. Let $l_i$ be the rank of operators on $\Gamma_i$. Operators $L, M$ are called commuting operators of vector rank $(l_1, ..., l_k)$, where $k \leq s$. Numbers $l_i$ are common divisors of $m$ and $n$. For more details see [4]. Also see [27], [28], [29].

If the rank of commuting matrix differential operators equals 1, then there exists explicit formulas for coefficients in terms of Riemann theta-functions [30].

In this paper we propose a very effective method for constructing matrix commuting operators of rank 2 and vector rank $(2,2)$. We find new commuting operators $L, M$ of orders 2 and $2g$ respectively.

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Explicit examples of commuting matrix differential operators of rank 2

Let us consider the operator

$$L = E(x)\partial_x^2 + R(x)\partial + Q(x), \quad (1)$$

where

$$E = \begin{pmatrix} \lambda_1(x) & \lambda_3(x) \\ 0 & \lambda_2(x) \end{pmatrix}, \quad R = \begin{pmatrix} r_1(x) & r_3(x) \\ r_2(x) & -r_1(x) \end{pmatrix}, \quad Q = \begin{pmatrix} q_1(x) & q_3(x) \\ q_2(x) & q_4(x) \end{pmatrix}$$
We want to find an operator $M$ of order $2g$ such that $[L, M] = 0$. Let us consider the operator

$$M = B_0(x) L^g + (A_1(x) \partial_x + B_1(x)) L^{g-1} + (A_2(x) \partial_x + B_2(x)) L^{g-2} + \ldots + (A_{g-1}(x) \partial_x + B_{g-1}(x)) L + A_g(x) \partial_x + B_g(x),$$

where

$$A_{g-k} = \begin{pmatrix} a^{g-k}_1(x) & a^{g-k}_3(x) \\ a^{g-k}_2(x) & a^{g-k}_4(x) \end{pmatrix}, \quad B_{g-k} = \begin{pmatrix} b^{g-k}_1(x) & b^{g-k}_3(x) \\ b^{g-k}_2(x) & b^{g-k}_4(x) \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

We note that in the formulas above the number $g - k$ is index not degree. The index $g - k$ says us that functions $a_i^{g-k}$ and $b_i^{g-k}$ are elements of matrices $A_{g-k}$ and $B_{g-k}$ respectively.

Let us note that if $[L, N] = 0$, then $[L, MN] = LMN - MNL = LMN - MLN = [L, M]N$, where $L, M, N$ are matrix differential operators. So, we see that

$$[L, M] = \sum_{k=0}^{g} [L, (A_{g-k} \partial_x + B_{g-k}) L^k] = \sum_{k=0}^{g} [L, (A_{g-k} \partial_x + B_{g-k})] L^k = \sum_{k=0}^{g} [L, (A_{g-k} \partial_x + B_{g-k})] L^k.$$ 

Direct calculations show that

$$[L, A_{g-k} \partial_x + B_{g-k}] = [E \partial_x^2 + R \partial + Q, A_{g-k} \partial_x + B_{g-k}] =$$

$$(EA_{g-k} - A_{g-k} E) \partial_x^3 + (2EA'_{g-k} + EB_{g-k} + RA_{g-k} - A_{g-k} E' - A_{g-k} R - B_{g-k} E) \partial_x^2 +$$

$$(EA''_{g-k} + 2EB'_{g-k} + R_{g-k} A' + RB_{g-k} + QA_{g-k} - A_{g-k} R' - A_{g-k} Q - B_{g-k} R) \partial_x +$$

$$(EB''_{g-k} + RB'_{g-k} + QB_{g-k} - A_{g-k} Q' - B_{g-k} Q) =$$

$$= K_{g-k} \partial_x^3 + P_{g-k} \partial_x^2 + T_{g-k} \partial_x + F_{g-k},$$

where

$$K_{g-k} = EA_{g-k} - A_{g-k} E,$$

$$P_{g-k} = 2EA'_{g-k} + EB_{g-k} + RA_{g-k} - A_{g-k} E' - A_{g-k} R - B_{g-k} E,$$

$$T_{g-k} = EA''_{g-k} + 2EB'_{g-k} + R_{g-k} A' + RB_{g-k} + QA_{g-k} - A_{g-k} R' - A_{g-k} Q - B_{g-k} R,$$

$$F_{g-k} = EB''_{g-k} + RB'_{g-k} + QB_{g-k} - A_{g-k} Q' - B_{g-k} Q.$$
Using the fact that $\partial_x L = E \partial_x^3 + (E' + R) \partial_x^2 + (R' + Q) \partial_x + Q'$ we get

$$
K_{g-k} \partial_x^3 + P_{g-k} \partial_x^2 + T_{g-k} \partial_x + F_{g-k} =
$$

$$
K_{g-k} E^{-1} \partial_x L + (P_{g-k} - K_{g-k} E^{-1}(E' + R)) \partial_x^2 +
(T_{g-k} - K_{g-k} E^{-1}(R' + Q)) \partial_x + (F_{g-k} - K_{g-k} E^{-1}Q') =
$$

$$
K_{g-k} E^{-1} \partial_x L + (P_{g-k} - K_{g-k} E^{-1}(E' + R)) E^{-1}L +
(T_{g-k} - K_{g-k} E^{-1}(R' + Q) - (P_{g-k} - K_{g-k} E^{-1}(E' + R)) E^{-1}R) \partial_x + +
(F_{g-k} - K_{g-k} E^{-1}Q' - (P_{g-k} - K_{g-k} E^{-1}(E' + R)) E^{-1}Q) =
$$

$$
\tilde{K}_{g-k} \partial_x L + \tilde{P}_{g-k} L + \tilde{T}_{g-k} \partial_x + \tilde{F}_{g-k},
$$

where

$$
\tilde{K}_{g-k} = K_{g-k} E^{-1}
$$

$$
\tilde{P}_{g-k} = (P_{g-k} - K_{g-k} E^{-1}(E' + R)) E^{-1}
$$

$$
\tilde{T}_{g-k} = (T_{g-k} - K_{g-k} E^{-1}(R' + Q) - K_{g-k} E^{-1}(E' + R) E^{-1}R)
$$

$$
\tilde{F}_{g-k} = (F_{g-k} - K_{g-k} E^{-1}Q' - K_{g-k} E^{-1}(E' + R) E^{-1}Q).
$$

Finally we obtain

$$
[L, M] = \sum_{k=0}^{g} [L, (A_{g-k} \partial_x + B_{g-k})] L^k =
$$

$$
= \left(\tilde{K}_0 \partial_x + \tilde{P}_0\right) L^{g+1} + \left((\tilde{T}_0 + \tilde{K}_1) \partial_x + (\tilde{F}_0 + \tilde{P}_1)\right) L^{g-1} +
+ \left((\tilde{T}_1 + \tilde{K}_2) \partial_x + (\tilde{F}_1 + \tilde{P}_2)\right) L^{g-1} + ... +
+ \left(\tilde{T}_{g-1} + \tilde{K}_g\right) \partial_x + (\tilde{F}_{g-1} + \tilde{P}_g)\right) L + T_g \partial_x + F_g.
$$

So, if

$$
\tilde{K}_0 = 0, \tilde{P}_0 = 0, \tilde{K}_1 = -\tilde{T}_0, \tilde{P}_1 = -\tilde{F}_0, ...,
$$

$$
\tilde{K}_m = -\tilde{T}_{m-1}, \tilde{P}_m = -\tilde{F}_{m-1}, ...,
$$

$$
\tilde{K}_g = -\tilde{T}_{g-1}, \tilde{P}_g = -\tilde{F}_{g-1},
$$

$$
\tilde{T}_g = 0, \tilde{F}_g = 0,
$$

(4)
then \([L, M] = 0\).

Let us calculate \(\tilde{K}_{g-k}, \tilde{P}_{g-k}, \tilde{T}_{g-k}, \tilde{F}_{g-k}\) from (3). This formulas are too hard for analyzing but in the sequel only special cases are considered. We are going to show that in some cases formulas (4) give a very effective methods for finding commuting operators. Let us describe the main idea. We know that \(A_0 = 0\) because \(M\) is operator of order \(2g\). Let us take \(B_0\) such that \(\tilde{K}_0 = 0\) and \(\tilde{P}_0 = 0\). Then we can calculate \(\tilde{T}_1, \tilde{F}_1\). Using (4) we can find \(\tilde{K}_1, \tilde{P}_1\), then \(a_1^1, a_3^1, b_1^1, b_2^1, b_3^1, b_4^1\). And using (4) we can find \(\tilde{T}_1\) and \(\tilde{F}_1\). So, we get recurrence relations

\[
\begin{align*}
\begin{cases}
 a_i^{m+1} = g_i(a_{i-1}^{m}, a_{i-2}^{m}, a_{i-3}^{m}, a_{i-4}^m, b_{i-1}^m, b_{i-2}^m, b_{i-3}^m, b_{i-4}^m, r_1, r_2, r_3, q_1, q_2, q_3, q_4), & i = 1, 2, 3, 4 \\
 b_i^{m+1} = h_i(a_{i-1}^{m}, a_{i-2}^{m}, a_{i-3}^{m}, a_{i-4}^m, b_{i-1}^m, b_{i-2}^m, b_{i-3}^m, b_{i-4}^m, r_1, r_2, r_3, q_1, q_2, q_3, q_4) & i = 1, 2, 3, 4.
\end{cases}
\end{align*}
\]

We will see that if there exists \(g\) such that

\[
\begin{cases}
 a_i^{g+1} = 0 & i = 1, ..., 4 \\
 b_i^{g+1} = 0 
\end{cases}
\]

then the operator \(L\) commutes with operator \(M\).

Let us suppose that \(\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0, r_2(x) = r_3(x) = 0, q_3 = q_2\) and \(q_4 = -q_1\). Then we have

\[
\begin{align*}
\tilde{K}_{g-k} &= \begin{pmatrix} 0 & -2a_2^{g-k} \\ -2a_2^{g-k} & 0 \end{pmatrix}, \quad \tilde{P}_{g-k} = \begin{pmatrix} 2(a_1^{g-k})' & -2b_2^{g-k} - 2(a_2^{g-k})' \\ -2b_2^{g-k} - 2(a_2^{g-k})' & 2(a_4^{g-k})' \end{pmatrix}, \\
\tilde{T}_{g-k} &= \begin{pmatrix} \tilde{T}_{g-k,1}^{1} & \tilde{T}_{g-k,2}^{1} \\ \tilde{T}_{g-k,1}^{2} & \tilde{T}_{g-k,2}^{2} \end{pmatrix}, \quad \tilde{F}_{g-k} = \begin{pmatrix} \tilde{F}_{g-k,1}^{1} & \tilde{F}_{g-k,2}^{1} \\ \tilde{F}_{g-k,1}^{2} & \tilde{F}_{g-k,2}^{2} \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\tilde{T}_{g-k,1}^{1} &= a_2^{g-k}q_2 + a_3^{g-k}q_2 - r_1(a_1^{g-k})' + 2(b_1^{g-k})' - a_1^{g-k}r_1' + (a_1^{g-k})', \\
\tilde{T}_{g-k,2}^{1} &= -a_2^{g-k}q_2 + a_4^{g-k}q_2 - r_1(a_3^{g-k})' + 2(b_3^{g-k})' - a_3^{g-k}r_1' + (a_3^{g-k})', \\
\tilde{T}_{g-k,1}^{2} &= a_1^{g-k}q_2 - a_4^{g-k}q_2 + r_1(a_2^{g-k})' - 2(b_2^{g-k})' + a_2^{g-k}r_1' - (a_2^{g-k})', \\
\tilde{T}_{g-k,2}^{2} &= a_2^{g-k}q_2 + a_3^{g-k}q_2 + r_1(a_4^{g-k})' - 2(b_4^{g-k})' + a_4^{g-k}r_1' - (a_4^{g-k})', \\
\tilde{F}_{g-k,1}^{1} &= b_2^{g-k}q_2 + b_3^{g-k}q_2 - 2q_1(a_1^{g-k})' + 2q_2(a_3^{g-k})' + r_1(b_1^{g-k})' - q_1'a_1^{g-k} + q_2'a_3^{g-k} + (b_1^{g-k})', \\
\tilde{F}_{g-k,2}^{1} &= b_4^{g-k}q_2 - b_2^{g-k}q_2 - 2q_1(a_1^{g-k})' - 2q_2(a_3^{g-k})' + r_1(b_3^{g-k})' - q_1'a_1^{g-k} - q_2'a_3^{g-k} + (b_3^{g-k})', \\
\tilde{F}_{g-k,1}^{2} &= b_1^{g-k}q_2 - b_2^{g-k}q_2 + 2q_1(a_2^{g-k})' - 2q_2(a_4^{g-k})' - r_1(b_2^{g-k})' + q_1'a_2^{g-k} - q_2'a_4^{g-k} - (b_2^{g-k})', \\
\tilde{F}_{g-k,2}^{2} &= b_3^{g-k}q_2 + b_4^{g-k}q_2 + 2q_1(a_2^{g-k})' + 2q_2(a_4^{g-k})' - r_1(b_4^{g-k})' + q_1'a_2^{g-k} + q_2'a_4^{g-k} - (b_4^{g-k})'.
\end{align*}
\]
From (4) we get
\[ \tilde{K}_{g-k+1} = -\tilde{T}_{g-k}, \quad \tilde{P}_{g-k+1} = -\tilde{F}_{g-k}. \]

So, we obtain recurrence relations

\begin{align*}
    a^0_i(x) & \equiv 0, \quad i = 1, \ldots, 4, \\
    b^0_2(x) & = b^0_3 \equiv 0 \\
    \tilde{T}^1_{g-k,1} & = 0 \iff \int \left( a^g_{g-k} q_2 + a^g_{g-k} q_2 - (a^{g-k}_1)' r_1 - a^g_{g-k} r_1' + (a^{g-k}_1)'' \right) dx + C^{g-k}_1, \\
    \tilde{T}^2_{g-k,2} & = 0 \iff \int \left( a^g_{g-k} q_2 + a^g_{g-k} q_2 + (a^{g-k}_1)' r_1 + a^g_{g-k} r_1' - (a^{g-k}_1)'' \right) dx + C^{g-k}_2, \\
    a^{g-k+1}_3 & = \frac{1}{2} \left( -a^g_{g-k} q_2 + a^g_{g-k} q_2 - (a^{g-k}_1)' r_1 + 2(b^g_{g-k})' - a^g_{g-k} r_1' + (a^{g-k}_1)'' \right), \\
    a^{g-k+1}_2 & = \frac{1}{2} \left( a^g_{g-k} q_2 - a^g_{g-k} q_2 + (a^{g-k}_2)' r_1 - 2(b^g_{g-k})' + a^g_{g-k} r_1' - (a^{g-k}_2)'' \right), \\
    2(a^{g-k+1}_1)' & = -\tilde{F}^1_{g-k,1} \iff \int \left( b^g_{g-k} q_2 + b^g_{g-k} q_2 - 2(a^{g-k}_1)' q_1 + 2q_2 (a^{g-k}_3)' + (b^{g-k}_1)' r_1 - a^g_{g-k} q_1 + a^g_{g-k} q_2 + (b^{g-k}_1)'' \right) dx + C^{g-k+1}_3, \\
    2(a^{g-k+1}_4)' & = -\tilde{F}^2_{g-k,2} \iff \int \left( b^g_{g-k} q_2 + b^g_{g-k} q_2 + 2(a^{g-k}_1)' q_2 + 2(a^{g-k}_1)' q_1 - (b^{g-k}_4)' r_1 + a^g_{g-k} q_1 + a^g_{g-k} q_2 - (b^{g-k}_4)'' \right) dx + C^{g-k+1}_4, \\
    a^{g-k+1}_4 & = -\frac{1}{2} \int \left( b^g_{g-k} q_2 + b^g_{g-k} q_2 + 2(a^{g-k}_1)' q_2 + 2(a^{g-k}_1)' q_1 - (b^{g-k}_4)' r_1 + a^g_{g-k} q_1 + a^g_{g-k} q_2 - (b^{g-k}_4)'' \right) dx + C^{g-k+1}_4,
\end{align*}
We obtain the following theorem

\[
-2b_3^{g-k+1} - (a_3^{g-k+1})' = -\tilde{F}_{g-k,2}^1 \
\]

\[
b_3^{g-k+1} = \frac{1}{2} \left( -b_1^{g-k}q_2 + b_4^{g-k}q_2 - 2(a_1^{g-k})'q_2 - 2(a_3^{g-k})'q_1 + (b_3^{g-k})'r_1 - a_3^{g-k}q_1' - a_1^{g-k}q_2' + (b_3^{g-k})'' \right) - (a_3^{g-k+1})',
\]

\[
-2b_2^{g-k+1} - (a_2^{g-k+1})' = -\tilde{F}_{g-k,1}^2 \
\]

\[
b_2^{g-k+1} = \frac{1}{2} \left( b_1^{g-k}q_2 - b_4^{g-k}q_2 + 2(a_2^{g-k})'q_1 - 2(a_4^{g-k})'q_2 - (b_2^{g-k})'r_1 + a_2^{g-k}q_1' - a_4^{g-k}q_2' - (b_2^{g-k})'' \right) - (a_2^{g-k+1})'.
\]

We see that

\[
\begin{cases}
\tilde{T}_g = 0 \\
\tilde{F}_g = 0
\end{cases} \iff
\begin{cases}
\tilde{K}_{g+1} = 0 \\
\tilde{P}_{g+1} = 0
\end{cases}
\]

where

\[
\tilde{K}_{g+1} = \begin{pmatrix} 0 & -2a_3^{g+1} \\ 2a_2^{g+1} & 0 \end{pmatrix}, \quad \tilde{P}_{g+1} = \begin{pmatrix} 2(a_1^{g+1})' & -2b_3^{g+1} - 2(a_3^{g+1})' \\ -2b_2^{g+1} - 2(a_2^{g+1})' & 2(a_4^{g+1})' \end{pmatrix}
\]

where \( C_i^j \) are arbitrary constants. Let us note that if \( a_1^0(x) = 0, b_2^0(x) = 0, b_3^0(x) = 0 \) for all \( i = 1, \ldots, 4 \), then from (6) and (7) we get that \( b_1^0(x) = \text{const} \) and \( b_4^0(x) = \text{const} \).

We obtain the following theorem

**Theorem 1.** If there exists number \( g \) and constants of integration \( C_i^m \) such that \( a_i^{g+1} = 0, b_2^{g+1} = 0, b_3^{g+1} = 0 \) for all \( i = 1, \ldots, 4 \), then the operator

\[
L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} r_1(x) & 0 \\ 0 & -r_1(x) \end{pmatrix} \partial_x + \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{pmatrix}
\]

commutes with operator

\[
M = B_0L^g + (A_1\partial_x + B_1)L^{g-1} + \ldots + A_0\partial_x + B_0,
\]

\[
B_0 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} C_1^1 + \frac{C_3^1r_1(x)}{2} & -\frac{(\mu_1 - \mu_2)q_2}{2} \\ \frac{(\mu_1 - \mu_2)q_2}{2} & C_4^1 + \frac{C_3^4r_1(x)}{2} \end{pmatrix}.
\]
where $\mu_1, \mu_2$ are arbitrary constants and $C^j_i$ are some constants. We see that if $\mu_1 \neq \mu_2$ and $q_2 \neq \text{const}$, then $B_1$ is not constant matrix hence $M$ is not polynomial in $L$.

**Theorem 2.** The operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} \alpha_2x^2 + \alpha_0 & 0 \\ 0 & -\alpha_2x^2 - \alpha_0 \end{pmatrix} \partial_x + \begin{pmatrix} \beta x^2 + \alpha_2x & \gamma x \\ \gamma x & -\beta x^2 - \alpha_2x \end{pmatrix},$$

where

$$\gamma^2 = -n^2 \alpha_2^2, \quad n \in \mathbb{N}$$

and $\alpha_2, \alpha_0, \beta$ are arbitrary constants, commutes with differential operator (14) of order $4n$, where $g = 2n$. The order of operator $M$ equals $4n$.

**Remark.** Calculations show that if $n \leq 3$, then the spectral curve of operators $L, M$ from Theorem 2 is nonsingular for almost all $\alpha_0, \alpha_2, \beta$ and is hyperelliptic. Hence $L$ and $M$ are operators of rank 2. In some cases spectral curve is reducible and we get commuting operators of rank $(2, 2)$. Note that operators from Theorem 2 can’t be operators of rank 1. Also note that from Theorem 1 we see that the matrix operator $M$ from Theorem 2 is operator with polynomial coefficients.

**Example 1.** If $n = 1$ and $\mu_1 = 1, \mu_2 = -1$, then the operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} \alpha_2x^2 + \alpha_0 & 0 \\ 0 & -\alpha_2x^2 - \alpha_0 \end{pmatrix} \partial_x + \begin{pmatrix} \beta x^2 + \alpha_2x & i\alpha_2x \\ i\alpha_2x & -\beta x^2 - \alpha_2x \end{pmatrix}$$

commutes with operator $M = B_0L^2 + A_1\partial_xL + B_1L + A_2\partial_x + B_2$. Calculations show that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^4 + \begin{pmatrix} 2(\alpha_2x^2 + \alpha_0) & 0 \\ 0 & -2(\alpha_2x^2 + \alpha_0) \end{pmatrix} \partial_x^3 + \begin{pmatrix} \alpha_2x^4 + 2(\alpha_0\alpha_2 + \beta)x^2 + 6\alpha_2x + \alpha_0^2 & i\alpha_2x \\ i\alpha_2x & -\alpha_2x^4 - 2(\alpha_0\alpha_2 + \beta)x^2 - 6\alpha_2x - \alpha_0^2 \end{pmatrix} \partial_x^2 + \begin{pmatrix} m_1 & m_2 \\ m_2 & -m_1 \end{pmatrix} \partial_x + \begin{pmatrix} h_1 & h_2 \\ h_2 & -h_1 + 2\beta - \alpha_0\alpha_2 \end{pmatrix} + C_1L + \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix},$$

$$m_1 = 2\alpha_2\beta x^4 + 4\alpha_2^2x^3 + 2\alpha_0\beta x^2 + 4(\alpha_0\alpha_2 + \beta)x + 4\alpha_2,$$

$$m_2 = i\alpha_2x^3 + i\alpha_0\alpha_2x + i\alpha_2,$$

$$h_1 = \beta_2x^4 + 4\alpha_2\beta x^3 + \frac{3\alpha_2^2}{2}x^2 + 2\alpha_0\beta x + 4\beta,$$
\[ h_2 = i\alpha_2\beta x^3 + \frac{3}{2}i\alpha_2^2 x^2 + \frac{i\alpha_2\alpha_0}{2}, \]

where \( C_1 \) and \( C_0 \) are arbitrary constants. The spectral curve of operators \( L, M \) has the form

\[
\left( w - C_1 z - \left( C_0 - \frac{\alpha_2\alpha_0 - 2\beta}{2} \right) \right)^2 = z^4 - (\alpha_0\alpha_2 - 2\beta)z^2 - \alpha_2\alpha_0\beta + \beta^2.
\]

If we take \( C_0 = \frac{\alpha_2\alpha_0 - 2\beta}{2}, \) \( C_1 = 0, \) then we get

\[ w^2 = z^4 - (\alpha_0\alpha_2 - 2\beta)z^2 - \alpha_2\alpha_0\beta + \beta^2. \]

This spectral curve is nonsingular if \( \alpha_2\alpha_0\beta(\alpha_2\alpha_0 - \beta) \neq 0. \) So in nonsingular case we get that operators \( L, M \) are operators of rank 2. If \( \alpha_0 = 0, \) then the spectral curve has the form

\[ w^2 = (z^2 + \beta)^2 \iff (w - z^2 - \beta)(w + z^2 + \beta) = 0 \]

We see that if \( \alpha_0 = 0, \) then the spectral curve is reducible. Note that \( M \neq L^2 + \beta \) and \( M \neq -L^2 - \beta \) but \( (M - L^2 - \beta)(M + L^2 + \beta) = 0 \) and we have operators of vector rank (2,2).

**Example 2.** If \( n = 1 \) and \( \mu_1 = 1, \mu_2 = 2, \) then the operator

\[ L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} \alpha_2 x^2 + \alpha_0 & 0 \\ 0 & -\alpha_2 x^2 - \alpha_0 \end{pmatrix} \partial_x + \begin{pmatrix} \beta x^2 + \alpha_2 x & i\alpha_2 x \\ i\alpha_2 x & -\beta x^2 - \alpha_2 x \end{pmatrix} \]

commutes with operator \( M = B_0 L^2 + A_1 \partial_x L + B_1 L + A_2 \partial_x + B_2. \) Direct calculations show that

\[
M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \partial_x^4 + \begin{pmatrix} 2(\alpha_2 x^2 + \alpha_0) & 0 \\ 0 & 4(\alpha_2 x^2 + \alpha_0) \end{pmatrix} \partial_x^3 + \begin{pmatrix} \alpha_2^2 x^4 + 2(\alpha_0\alpha_2 + \beta)x^2 + 6\alpha_2 x + \alpha_0^2 & -\frac{i\alpha_2 x}{2} \\ -\frac{i\alpha_2 x}{2} & 2\alpha_2^2 x^4 + 4(\alpha_0\alpha_2 + \beta)x^2 + 12\alpha_2 x + 2\alpha_0^2 \end{pmatrix} \partial_x^2 + \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \partial_x + \begin{pmatrix} h_1 & h_3 \\ h_2 & h_4 \end{pmatrix} + C_1 L + \begin{pmatrix} C_0 & 0 \\ 0 & C_0 \end{pmatrix},
\]

\[ m_1 = 2\alpha_2\beta x^4 + 4\alpha_2^2 x^3 + 2\alpha_0\beta x^2 + 4(\alpha_0\alpha_2 + \beta)x + 4\alpha_2, \]

\[ m_2 = -\frac{i\alpha_2^2 x^3}{2} - \frac{i\alpha_0\alpha_2 x}{2} - \frac{7i\alpha_2}{2}, \]

\[ m_3 = -\frac{i\alpha_2^2 x^3}{2} - \frac{i\alpha_0\alpha_2 x}{2} + \frac{5i\alpha_2}{2}, \]

\[ m_4 = \frac{i\alpha_2^2 x^3}{2} + \frac{i\alpha_0\alpha_2 x}{2} + \frac{3i\alpha_2}{2}. \]
\[ m_4 = 4\alpha_2 \beta x^4 + 8\alpha_2^2 x^3 + 4\alpha_0 \beta x^2 + 8(\alpha_0 \alpha_2 + \beta)x + 8\alpha_2, \]

\[ h_1 = \beta^2 x^4 + 4\alpha_2 \beta x^3 + \frac{3\alpha_2^2}{4} x^2 + 2\alpha_0 \beta x + \beta + \frac{3\alpha_2 \alpha_0}{2}, \]

\[ h_2 = -\frac{i\alpha_2 \beta x^3}{2} - \frac{9i\alpha_2^2 x^2}{4} - \frac{7i\alpha_2 \alpha_0}{4}, \]

\[ h_3 = -\frac{i\alpha_2 \beta x^3}{2} + \frac{3i\alpha_2^2 x^2}{4} + \frac{5i\alpha_2 \alpha_0}{4}, \]

\[ h_4 = 2\beta^2 x^4 + 8\alpha_2 \beta x^3 + \frac{9\alpha_2^2}{4} x^2 + 4\alpha_0 \beta x + 4\beta + 2\alpha_2 \alpha_0, \]

where \( C_1 \) and \( C_0 \) are arbitrary constants. If we take \( C_1 = 0 \) and \( C_0 = 0 \), then the spectral curve of operators \( L, M \) has the form

\[ 16w^2 - 8w(\alpha_2 \alpha_0 - 2\beta + 6z^2) + 32z^4 + 16(\alpha_2 \alpha_0 - 2\beta)z^2 + \alpha_2^2 \alpha_0^2 = 0 \]

We see that the spectral curve is nonsingular for almost all \( \alpha_2, \alpha_0, \beta \) and \( L, M \) are operators of rank 2. If \( \alpha_0 = 0 \), then the spectral curve has the form

\[ (w - 2z^2)(w - z^2 + \beta) = 0 \]

and \( L, M \) are operators of vector rank \((2, 2)\).

**Theorem 3.** Let \( \wp(x) \) be the Weierstrass elliptic function satisfying the equation \((\wp'(x))^2 = 4\wp^3(x) + g_2 \wp(x)\). The operator

\[ L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} 0 & \alpha \wp(x) \\ \alpha \wp(x) & 0 \end{pmatrix}, \]

\[ \alpha^2 = 64n^4 - 4n^2, \quad n \in \mathbb{N} \]

commutes with a differential operator \((14)\), of order \(4n\), where \( g = 2n \). The order of operator \( M \) equals \( 4n \).

**Proofs of Theorem 2 and Theorem 3**

We prove Theorem 2 and Theorem 3 using Theorem 1. Let us suppose that \( C_2^k = C_3^k = C_4^k = 0 \) and \( C_1^{2k+1} = 0 \) for all \( k \). We know that

\[ A_0 \equiv 0, \quad B_0 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}. \]
Direct calculations using (6) – (13) show that

\[ a_1 = a_2 = a_3 = a_4 = 0, \]
\[ b_2^1 = \frac{\mu_1 - \mu_2}{2} q_2, \quad b_3^1 = -\frac{\mu_1 - \mu_2}{2} q_2 = -b_2^1. \]  

(16)

Then

\[ a_1^2 = a_4^2 = 0, \quad a_2^2 = a_3^2 = -\frac{\mu_1 - \mu_2}{2} q_2' = -(b_2^1)', \]
\[ b_2^2 = b_3^2 = \frac{r_1 a_2^2 - (a_2^2)'}{2} \]
\[ a_1^3 = a_3^3 = a_4^3 = 0, \]
\[ b_3^3 = -b_3^2. \]  

(17)

Lemma 1. If \( k = 2m + 1 \), then

\[ a_1^k = a_2^k = a_3^k = a_4^k = 0, \quad b_2^k = -b_3^k \]  

(18)

If \( k = 2m \), then

\[ a_1^k = a_4^k = 0, \quad a_2^k = a_3^k = -(b_2^{k-1})', \quad b_3^k = b_3^k = \frac{r_1 a_2^k - (a_2^k)'}{2} \]  

(19)

Proof

We see that relations (18) and (19) is true when \( k = 1 \) and \( k = 2 \). Let us suppose that (18) and (19) is true for some \( k \).

If \( k = 2m + 1 \), then using (6) – (13) we get

\[ a_1^{2m+2} = a_4^{2m+2} = 0, \quad a_2^{2m+2} = a_3^{2m+2} = -(b_2^{2m+1})', \quad b_3^{2m+2} = b_2^{2m+2} = \frac{r_1 a_2^{2m+2} - (a_2^{2m+2})'}{2}. \]

If \( k = 2m \), then again using (6) – (13) we have

\[ a_1^{2m+1} = a_2^{2m+1} = a_3^{2m+1} = a_4^{2m+1} = 0, \quad b_3^{2m+1} = -b_2^{2m+1}. \]

The Lemma is proved.

Proof of Theorem 2

From (16) and (17) we get

\[ a_1^1 = a_2^1 = a_3^1 = a_4^1 = 0, \]
\[ b_2^1 = \frac{\mu_1 - \mu_2}{2} q_2 = \frac{\mu_1 - \mu_2}{2} \gamma x, \quad b_3^1 = -\frac{\mu_1 - \mu_2}{2} q_2 = -\frac{\mu_1 - \mu_2}{2} \gamma x. \]
Then
\[ a_1^2 = a_4^2 = 0, \quad a_2^2 = a_3^2 = \frac{\mu_1 - \mu_2}{2} \gamma = -(b_2^1)', \]
\[ b_2^3 = b_3^2 = -\frac{\mu_1 - \mu_2}{4} \alpha_0 \gamma - \frac{\mu_1 - \mu_2}{4} \alpha_2 \gamma x^2, \]
\[ a_1^3 = a_2^3 = a_3^2 = a_4^2 = 0, \]
\[ b_2^3 = \frac{2C_1^2}{4} + (\mu_1 - \mu_2)(\alpha_0 \alpha_2 - 2\beta) \gamma x + \frac{\mu_1 - \mu_2}{4} \gamma (\gamma^2 + \alpha_2^2) x^3. \]

We want to prove that \( L \) commutes with differential operator (14), where \( g = 2n \).
From Theorem 1 and Lemma 1 we know that we must prove that there exists constants \( C_1^{2k} \) such that \( b_2^{2n+1} \equiv 0 \). Let us note that recurrence relations (6) – (13) are linear in \( a_i^{k+1} \) and \( b_i^{k+1} \). Assume that \( b_i^{2m-1} = x^{2m-1} \). Then we have
\[ b_2^{2m} = (2m - 1)(m - 1)x^{2m-3} - \frac{\alpha_0 (2m - 1)x^{2m-2}}{2} - \frac{\alpha_2 (2m - 1)x^{2m}}{2} = -b_3^2, \]
\[ a_2^{2m} = -(b_2^{2m-1})' = -(2m - 1)x^{2m-2} = a_3^{2m}. \]

Again using (6) – (13) we obtain
\[ a_1^{2m+1} = a_2^{2m+1} = a_3^{2m+1} = a_4^{2m+1} = 0, \]
\[ b_2^{2m+1} = \frac{(2m - 1)(\alpha_0^2 m^2 + \gamma^2)}{2m} x^{2m+1} + \frac{(\alpha_2 \alpha_0 - 2\beta)(2m - 1)^2}{4} x^{2m-1} + \frac{\alpha_2^2 (2m - 2)(2m - 1)}{4} x^{2m-3} - \frac{(2m - 1)(2m - 2)(2m - 3)(2m - 4)}{4} x^{2m-5} + \frac{C_1^{2m} \beta}{2} x. \]

From (20) we see that \( b_2^3 = K_1^3 x + K_3^3 x^3 \), where \( K_1^3 = K_1^3 (C_1^2) \) is constant and depends on \( C_1^2, K_3^3 = \frac{\mu_1 - \mu_2}{4} \gamma (\gamma^2 + \alpha_2^2) \). Let us suppose that for some \( m \)
\[ b_2^{2m-1} = K_1^{2m-1} x + K_3^{2m-1} x^3 + \ldots + K_{2m-1}^{2m-1} x^{2m-1}, \]
where
\[ K_{2m-1}^{2m-1} = \frac{(\mu_1 - \mu_2) \prod_{j=1}^{m-1} ((2j - 1)(\alpha_2 j^2 + \gamma^2))}{2^m (m - 1)!}, \]
\( K_{2m-3}^{2m-1} \) is constant and depends on \( C_1^2, K_{2m-5}^{2m-1} \) is constant and depends on \( C_1^2, C_1^4, K_{2m-7}^{2m-1} \) depends on \( C_1^2, \ldots, C_1^{2j} \) and \( K_{2m-1}^{2m-1} \) depends on \( C_1^2, C_1^4, \ldots, C_1^{2m-2} \). We see
that it is true when \( m = 2 \). Using (21) we get

\[
\begin{align*}
\frac{b}{2}^{2m+1} &= K_1^{2m+1}x + K_3^{2m+1}x^3 + \ldots + K_{2m-1}^{2m+1}x^{2m-1} + K_{2m+1}^{2m+1}x^{2m+1}, \\
K_1^{2m+1} &= \frac{C_2^{2m+1}}{2} - 30K_3^{2m+1} + \frac{3a_0^2}{2}K_3^{2m+1} + \frac{\alpha_2 \alpha_0 - 2\beta}{2}K_1^{2m+1} \\
K_3^{2m+1} &= \frac{\alpha_2^2 + \gamma^2}{2}K_1^{2m+1} + \frac{9(\alpha_2 \alpha_0 - 2\beta)}{2}K_3^{2m+1} + 5\alpha_0^2K_5^{2m+1} - 210K_7^{2m+1} \\
\ldots \\
K_2^{2m+1} &= \frac{(2m - 3)(\alpha_2^2(m - 1)^2 + \gamma^2)}{2m - 2}K_2^{2m+1} + \frac{\alpha_2 \alpha_0 - 2\beta}{2}(2m - 1)^2K_2^{2m+1} \\
K_2^{2m+1} &= \frac{(2m - 1)(\alpha_2^2m^2 + \gamma^2)}{2m}K_2^{2m+1}.
\end{align*}
\]

Easy to see that \( K_2^{2m+1} \) depends on constant of integration \( C_i^2 \) because \( K_2^{2m+1} \) depends on \( C_1^2 \) and \( K_2^{2m+1} \) depends on \( C_2^2, C_3^2 \). The last coefficient \( K_1^{2n+1} \) depends on constants of integrations \( C_1^2, \ldots, C_1^{2n} \).

Now let us consider

\[
\begin{align*}
\frac{b}{2}^{2n+1} &= K_1^{2n+1}x + K_3^{2n+1}x^3 + \ldots + K_{2n-1}^{2n+1}x^{2n-1} + K_{2n+1}^{2n+1}x^{2n+1} \\
K_1^{2n+1} &= \frac{C_2^{2n+1}}{2} - 30K_3^{2n+1} + \frac{3a_0^2}{2}K_3^{2n+1} + \frac{\alpha_2 \alpha_0 - 2\beta}{2}K_1^{2n+1} \\
K_3^{2n+1} &= \frac{\alpha_2^2 + \gamma^2}{2}K_1^{2n+1} + \frac{9(\alpha_2 \alpha_0 - 2\beta)}{2}K_3^{2n+1} + 5\alpha_0^2K_5^{2n+1} - 210K_7^{2n+1} \\
\ldots \\
K_2^{2n+1} &= \frac{(2n - 3)(\alpha_2^2(m - 1)^2 + \gamma^2)}{2n - 2}K_2^{2n+1} + \frac{\alpha_2 \alpha_0 - 2\beta}{2}(2n - 1)^2K_2^{2n+1} \\
K_2^{2n+1} &= \frac{(2n - 1)(\alpha_2^2m^2 + \gamma^2)}{2n}K_2^{2n+1}.
\end{align*}
\]

We know that \( \gamma^2 + n^2 \alpha_2 = 0 \) and hence \( K_2^{2n+1} = 0 \). To prove Theorem 2 we must find constants \( C_1^2, \ldots, C_1^{2n} \) such that \( K_1^{2n+1} = K_3^{2n+1} = \ldots = K_{2n}^{2n+1} = 0 \). It is always possible because \( K_2^{2n+1} \) depends on \( C_i^2 \) and \( K_2^{2n+1} \) depends on \( C_1^2, C_3^2 \) etc. The last coefficient \( K_1^{2n+1} \) depends on constants of integration \( C_1^2, \ldots, C_1^{2n} \).

**Theorem 2 is proved.**

**Proof of Theorem 3**

The proof of Theorem 3 coincides with the proof of Theorem 2. Let us prove that principle parts of functions \( a_i^{2n+1} \) and \( b_i^{2n+1} \) equals zero. We see from (6) – (13) that \( a_i^2 \) and \( b_i^2 \) are elliptic functions for any \( i, j \). Hence if principle parts of functions \( a_i^{2n+1} \) and \( b_i^{2n+1} \) equals zero, then \( a_i^{2n+1} \) and \( b_i^{2n+1} \) havn’t poles and hence are constants. In our case these constants are zeroes.

From (16) and (17) we get

\[
\begin{align*}
\frac{a}{2}^i &= a^o^i = a^i = a^i = 0, \\
\frac{b}{2}^i &= \frac{\mu_1 - \mu_2}{2}q_2 = \frac{\mu_1 - \mu_2}{2x^2} \alpha + O(x^2), \\
\frac{b}{3}^i &= -\frac{\mu_1 - \mu_2}{2x^2} \alpha + O(x^2).
\end{align*}
\]
Then
\[ a_1^2 = a_4^2 = 0, \quad a_2^2 = a_3^2 = \frac{\mu_1 - \mu_2}{x^3} \alpha + O(x) = -(b_2^1)', \]
\[ b_2^2 = b_3^2 = \frac{3(\mu_1 - \mu_2)}{2x^4} \alpha + \frac{(\mu_1 - \mu_2)g_2\alpha}{40} + O(x^4), \]
\[ a_1^3 = a_2^3 = a_3^3 = a_4^3 = 0, \]
\[ b_2^3 = -b_3^3 = \frac{(\mu_1 - \mu_2)\alpha(\alpha^2 - 60)}{4x^6} + \frac{\alpha(80C_1^2 + 6(\mu_1 - \mu_2)g_2\alpha^2)}{160x^2} + O(x^2) \]

We mentioned before that recurrence relations (6) – (13) are linear in \( a_i^{k+1} \) and \( b_i^{k+1} \).
Assume that \( b_3^{2m-1} = -b_2^{2m-1} = \frac{1}{x^{4m-2}} \) and \( a_1^{2m-1} = a_2^{2m-1} = a_3^{2m-1} = a_4^{2m-1} = 0. \)
Then we have
\[ b_2^{2m} = b_3^{2m} = \frac{8m^2 - 6m + 1}{x^{4m}}, \]
\[ a_2^{2m} = -(b_2^{2m-1})' = \frac{4m - 2}{x^{4m-1}}. \]

Again using (6) – (13) we obtain
\[ a_1^{2m+1} = a_2^{2m+1} = a_3^{2m+1} = a_4^{2m+1} = 0, \]
\[ b_2^{2m+1} = \frac{(2m - 1)(\alpha^2 + 4m^2 - 64m^4)}{2m} + \frac{\text{const}}{x^{4m-2}} + \frac{\text{const}}{x^{4m-6}} + ... + \frac{C_1^{2m} \alpha}{2x^2} + O(x^2) \]
\[ (23) \]

From (22) we see that \( b_2^2 = \frac{K_6^3}{x^6} + \frac{K_2^3}{x^2} + O(x^2) \), where \( K_2^3 = K_2^3(C_1^2) \) is constant and depends on \( C_1^2, K_6^3 = \frac{(\mu_1 - \mu_2)\alpha(\alpha^2 - 60)}{4} \).
Let us suppose that for some \( m \)
\[ b_2^{2m-1} = \frac{K_4^{2m-1}}{x^{4m-2}} + \frac{K_4^{2m-1}}{x^{4m-6}} + ... + \frac{K_2^{2m-1}}{x^2} + O(x^2), \]
where
\[ K_{4m-2}^{2m-1} = -\frac{(\mu_1 - \mu_2) \prod_{j=1}^{m-1} ((2j - 1)(\alpha^2 + 4j^2 - 64j^4))}{2^m(m - 1)!}, \]

\( K_{4m-6}^{2m-1} \) is constant and depends on \( C_1^2 \), \( K_{4m-10}^{2m-1} \) is constant and depends on \( C_1^2, C_4^4 \), \( K_{4m-4j-2}^{2m-1} \) depends on \( C_1^2, ..., C_1^{2j} \) and \( K_2^{2m-1} \) depends on \( C_1^2, C_4^4, ..., C_1^{2m-2} \). We see that it is true when \( m = 2 \). Using (23) we get
\[ b_2^{2m+1} = K_1^{2m+1} x + K_3^{2m+1} x^3 + ... + K_{2m-1}^{2m+1} x^{2m-1} + K_{2m+1}^{2m+1} x^{2m+1}. \]
Easy to see that \( K^{2m+1}_{4m-2} \) depends on constant of integration \( C_1^2 \). And \( K^{2m+1}_{4m-6} \) depends on \( C_1^2, C_3^2 \). The last coefficient \( K^{2m+1}_2 \) depends on constants of integrations \( C_1^2, ..., C_{2m}^2 \).

Now let us consider

\[
K_{2n+1}^{2n+1} = K_{1}^{2n+1}x + K_{3}^{2n+1}x^3 + ... + K_{2n-1}^{2n+1}x^{2n-1} + K_{2n+1}^{2n+1}x^{2n+1}
\]

We know that \( \alpha^2 + 4n^2 - 64n^4 = 0 \) hence \( K_{4n+2}^{2n+1} = 0 \). To prove Theorem 3 we must find constants \( C_1^2, ..., C_{2n}^2 \) such that

\[
K_{2n+1}^{2n+1} = K_6^{2n+1} = ... = K_{4n-2}^{2n+1} = 0.
\]

It is always possible because \( K_{4n-2}^{2n+1} \) depends on \( C_1^2 \) and \( K_{2n+1}^{2n+1} \) depends on \( C_1^2, C_3^2 \) etc. The last coefficient \( K_{2n+1}^{2n+1} \) depends on constants of integration \( C_1^2, ..., C_{2n}^2 \).

**Theorem 3 is proved.**

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