Unique continuation principle for the
one-dimensional time-fractional diffusion
equation

Zhiyuan Li†, Masahiro Yamamoto‡

Abstract This paper deals with the unique continuation of solutions for a one-dimensional anomalous diffusion equation with Caputo derivative of order \( \alpha \in (0, 1) \). Firstly, the uniqueness of solutions to a lateral Cauchy problem for the anomalous diffusion equation is given via the Theta function method, from which we further verify the unique continuation principle.

1 Introduction and main result

The anomalous diffusion processes whose mean square displacement behaves like \( \langle \Delta x^2 \rangle \sim C_\alpha t^\alpha \) as \( t \to \infty \) were found in many problems in the fields of science and engineering. For the qualitative analysis of these anomalous diffusion, a macro-model based on the continuous time random walk, which is called a time-fractional diffusion equation, is derived:

\[
\partial_t^\alpha u(x, t) - \partial_x^2 u(x, t) = 0, \quad (x, t) \in (0, 1) \times (0, T),
\]

with the Caputo derivative \( \partial_t^\alpha \) \( (0 < \alpha < 1) \) which is usually defined by

\[
\partial_t^\alpha \phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \phi'(\tau) (t-\tau)^{\alpha-1} d\tau, \quad t > 0,
\]

where \( \Gamma(\cdot) \) is a usual Gamma function. For various properties of the Caputo derivative, we refer to Kilbas, Srivastava and Trujillo [6], Podlubny [15] and the references therein.

The fractional diffusion models have received great attention in applied disciplines, e.g., in describing some anomalous phenomena including the non-Fickian growth rates, skewness and long-tailed profile which are poorly characterized by the classical diffusion equations (see e.g., Benson, Wheatcraft and Meerschaert [1], Levy and Berkowitz [13] and the references therein). In contrast to the success in the practice, theoretical researches related to the fractional diffusion equation are still under development. The Caputo derivative is inherently nonlocal in time with a history dependence, and there are crucial differences between fractional models and classical models (i.e., \( \alpha = 1 \)), for example, concerning long-time asymptotic behavior (see, e.g., Li, Luchko and Yamamoto [8] and Li, Liu and Yamamoto [9]). There are also some publications on some important properties. For example, a maximum principle in the usual setting still holds similarly to the parabolic equation (see, e.g., Luchko [10]). As is known, the unique continuation property (UCP) is one of remarkable properties of parabolic equations, which asserts that if a solution to a homogeneous equation vanishes in an open subset, then the solution is identically

†School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255049, China. E-mail: zyli@sdut.edu.cn
‡Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: myama@ms.u-tokyo.ac.jp.
zero in the whole domain (see, e.g., Saut and Scheurer [17]). For the time-fractional diffusion equation, there are not affirmative answers except for some special cases. For the special half-order fractional diffusion equation (i.e., $\alpha = \frac{1}{2}$), under the assumption that the initial-value vanishes, in Xu, Cheng and Yamamoto [21] for the one-dimensional case, and Cheng, Lin and Nakamura [2] for the two-dimensional case, the uniqueness results are proved: if a solution $u$ to a homogeneous fractional diffusion equation satisfies $u = 0$ in $\omega \times (0, T)$ and $u(\cdot, 0) = 0$ in $\Omega$ with some subdomain $\omega \subset \Omega$, then $u = 0$ in $\Omega \times (0, T)$. Here $\Omega \subset \mathbb{R}^d$ is a spatial domain where we are considering the fractional diffusion equation. The proof is done via Carleman estimates for the operator $\partial_t - \Delta^2$. For a general fractional order $\alpha \in (0, 1)$, a recent work Lin and Nakamura [11] obtained a similar uniqueness result with the zero initial condition by using a newly established Carleman estimate based on calculus of pseudo-differential operators. Sakamoto and Yamamoto [19] showed that for a solution of the time-fractional diffusion equations with the homogeneous Dirichlet boundary condition on the whole boundary, if the Neumann data vanish on arbitrary subboundary, then it vanishes identically. The paper Jiang, Li, Liu and Yamamoto [5] generalized the result in [19] to the multi-term case. Recently, for the multi-term case with the first order time-derivative, the UCP was established by Li, Huang and Yamamoto [7] via a Carleman type estimate for the parabolic equation. All these results should be considered as a weak type of uniqueness because the homogeneous differential operators. This theorem is exactly corresponding to the unique continuation in the case of $\alpha = 1$, assuming the zero initial condition like [2], [11], [21], but we do not know if the same unique continuation holds for general dimensions or equations with variable coefficients even in the one dimension.

2 Proof of Theorem 1.1

In this paper, we will show that the classical unique continuation property for solutions of (1) is valid. More precisely, we have the following main theorem.

**Theorem 1.1.** Let $T > 0$ be fixed constant and $u \in L^\infty(0, T; H^2(0, 1))$ be a solution to the fractional diffusion equation (1). Then we have

$$u(x, t) = 0, \quad (x, t) \in [0, 1] \times [0, T]$$

provided that $u \equiv 0$ in $I \times [0, T]$, where $I$ is a non-empty open subinterval of $(0, 1)$.

In the theorem, we consider a class of solutions satisfying $u \in L^\infty(0, T; H^2(0, 1))$ and $\partial^\alpha_t u \in L^\infty(0, T; L^2(0, 1))$. In the case where $u(0, t) = u(1, t) = 0$ for $t > 0$ and $u(\cdot, 0) \in H^2_0(0, 1)$, we can prove that $u \in L^\infty(0, T; H^2(0, 1))$ (e.g., [19]).

This theorem is exactly corresponding to the unique continuation in the case of $\alpha = 1$, not assuming the zero initial condition like [2], [11], [21], but we do not know if the same unique continuation holds for general dimensions or equations with variable coefficients even in the one dimension.
which has the form from Luchko and Zuo [14] and Rundell, Xu and Zuo [16]

\[ K_\alpha(x,t) = \frac{1}{2} t^{-\frac{\alpha}{2}} M_\alpha(|x| t^{-\frac{1}{2}}), \quad x \in \mathbb{R}, \ t > 0, \]

where \( M_\alpha \) is a special member of the family of the Wright functions (see, e.g., Mainardi, Luchko and Pagnini [12] and [14]) which is defined by

\[ M_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k! \Gamma(-\alpha k + (1 - \alpha))} \]

and its Laplace transform with respect to the time \( t \) has the form

\[ \mathcal{L}\{K_\alpha(x,t); s\} = \frac{1}{2} s^{\frac{\alpha}{2} - 1} e^{-\frac{|x|}{s^{\frac{1}{2} - \alpha}}}, \quad s > 0, \ x \in \mathbb{R}. \] (3)

Moreover, based on the fundamental solution \( K_\alpha \), we introduce an important special function named Theta function \( \theta_\alpha(x,t) \), \( \alpha > 0 \) which plays an important role in representing the solution to (1) with non-homogeneous boundary conditions. We consider the Theta function \( \theta_\alpha \) by the following form of series:

\[ \theta_\alpha(x,t) := \sum_{m=-\infty}^{\infty} K_\alpha(x+2m, t), \quad t > 0. \]

We now list some properties of the function \( \theta_\alpha \) which can be found in Eidelman and Kochubei [4], Luchko and Zuo [14] and Rundell, Xu and Zuo [16].

**Lemma 2.1.** The functions \( K_\alpha(x,t) \) and \( \theta_\alpha(x,t) \) are even with respect to \( x \), and \( \theta_\alpha(1,t) = \theta_\alpha(-1,t) \) is \( C^\infty \) on \( [0, \infty) \) and

\[ \frac{d^m \theta_\alpha(1,0)}{dt^m} = 0, \quad m = 0, 1, \ldots. \]

Moreover, \( \theta_\alpha \) satisfies the following estimates.

(a) If \( |x|^2 \geq t^\alpha > 0 \), then there exist constants \( C > 0 \) and \( \sigma > 0 \) depending on \( \alpha \) such that

\[ |K_\alpha(x,t)| \leq C t^{-\frac{\alpha}{2}} e^{-\sigma t^{\frac{1}{2}} |x|^{\frac{2}{1-\alpha}}}, \] (4)

and

\[ |D_t^{1-\alpha} K_\alpha(x,t)| \leq C t^{\frac{\alpha}{2} - 1} e^{-\sigma t^{\frac{1}{2}} |x|^{\frac{2}{1-\alpha}}}. \] (5)

(b) If \( 0 < |x|^2 \leq t^\alpha \), then there exists a constant \( C > 0 \) depending on \( \alpha \) such that

\[ |K_\alpha(x,t)| \leq C t^{-\frac{\alpha}{2}}, \] (6)

and

\[ |D_t^{1-\alpha} K_\alpha(x,t)| \leq C t^{\frac{\alpha}{2} - 1}. \] (7)

Now let us turn to considering the following lateral Cauchy problem

\[ \begin{align*}
\partial_t^\alpha u - \partial_x^2 u &= 0 \quad \text{in } (0,1) \times (0,T], \\
u(0, \cdot) = u_x(0, \cdot) &= 0 \quad \text{in } [0,T].
\end{align*} \] (8)

Assuming that \( u \in L^{\infty}(0,T; H^2(0,1)) \) satisfies (8), we will now focus on the representation of the solution to (8), as this will be essential to our approach. For this, we first set \( u_0(x) := u(x,0) \) and \( g(t) := u_x(1,t) \). We extend the function \( g \) to the interval \([0, \infty)\) by letting \( g = 0 \) outside of
(0, T + 1) and letting \( g(t) = g(T)(T + 1 - t) \) if \( t \in (T, T + 1) \), and by \( \tilde{g} \) we denote the extension, and by \( \tilde{u} \) we denote the solution to the following auxiliary system

\[
\begin{aligned}
\partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} &= 0 \quad \text{in } (0, 1) \times (0, \infty), \\
\tilde{u}(1, 0) &= u_0, \quad \text{in } (0, 1), \\
\tilde{u}_x(0, \cdot) &= 0, \quad \tilde{u}_x(1, \cdot) = \tilde{g} \quad \text{in } (0, \infty).
\end{aligned}
\]  

(9)

On the basis of the properties of the fundamental solution \( K_\alpha \) and the Theta function \( \theta_\alpha \) in Lemma 2.1, we see that the Theta function \( \theta_\alpha \), finishing the proof of the lemma, it remains to show the estimates for \( w \) and \( v \). For this, from Lemma 2.1 we see that the Theta function \( \theta_\alpha \) is even with respect to \( x \). Thus

\[
\tilde{u}(0, t) = u(x, t) + v(x, t), \quad (x, t) \in (0, 1) \times (0, \infty),
\]

(10)

where

\[
w(x, t) = \int_0^1 (\theta_\alpha(x - \xi, t) + \theta_\alpha(x + \xi, t))u_0(\xi)d\xi,
\]

(11)

\[
v(x, t) = 2 \int_0^t (D_t^{1-\alpha}\theta_\alpha)(x - 1, t - \tau)\tilde{g}(\tau)d\tau, \quad (x, t) \in (0, 1) \times (0, \infty).
\]

(12)

Moreover, the following estimate

\[
|\tilde{u}(0, t)| \leq C \left( t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}} + t^{\frac{\alpha}{\alpha - 2}} + t^{\frac{\alpha}{\alpha - 2}} \right), \quad t > 0.
\]

holds true for \( t > 0 \).

**Proof.** The representation formula (10) is directly derived from Lemma 3.1 in [16]. In order to finishing the proof of the lemma, it remains to show the estimates for \( w \) and \( v \). For this, from Lemma 2.1 we see that the Theta function \( \theta_\alpha \) is even with respect to \( x \). Thus

\[
w(0, t) = 2 \int_0^1 \theta_\alpha(\xi, t)u_0(\xi)d\xi.
\]

We need to evaluate \( \theta_\alpha(x, t) \) for \((x, t) \in (0, 1) \times (0, \infty)\). In the case of \(|x + 2m|^2 \geq t^\alpha\), from the estimate (4), it follows that

\[
|K_\alpha(x + 2m, t)| \leq C t^{-\frac{\alpha}{2}} e^{-\sigma t^{-\frac{\alpha}{2 \alpha - 2}} |x + 2m|^{\frac{2 \alpha}{\alpha - 2}}},
\]

Moreover, if \( m = 0 \), then we have

\[
|K_\alpha(x, t)| \leq \begin{cases}
C t^{-\frac{\alpha}{2}} e^{-\sigma t^{-\frac{\alpha}{2 \alpha - 2}}} & x^2 \geq t^\alpha, \\
C t^{-\frac{\alpha}{2}} & x^2 \leq t^\alpha.
\end{cases}
\]

For \( m \neq 0 \), we have \(|x + 2m| \geq 2|m| - 1 \geq 1 \) by \( 0 < x < 1 \), which further implies that \(|x + 2m|^{\frac{2 \alpha}{\alpha - 2}} \geq 2|m|^{\alpha - 1} \), because of \( \frac{2}{x - \alpha} > 1 \). Consequently

\[
|K_\alpha(x + 2m, t)| \leq C t^{-\frac{\alpha}{2}} e^{-\sigma t^{-\frac{\alpha}{2 \alpha - 2}} (2|m|^{-1})}, \quad |x + 2m|^2 \geq t^\alpha, \quad m \neq 0.
\]

On the other hand, if \(|x + 2m|^2 \leq t^\alpha\), then the estimate (6) implies

\[
|K_\alpha(x + 2m, t)| \leq C t^{-\frac{\alpha}{2}}, \quad |x + 2m|^2 \leq t^\alpha.
\]
Collecting all the above estimates, we obtain

\[ |\theta_\alpha(x, t)| \leq |K_\alpha(x, t)| + \left( \sum_{|x+2m|^2 \geq t^\alpha, m \neq 0} + \sum_{|x+2m|^2 < t^\alpha, m \neq 0} \right) |K_\alpha(x + 2m, t)| \]

\[ \leq Ct^{-\frac{\alpha}{2}} + \sum_{|x+2m|^2 \geq t^\alpha, m \neq 0} Ct^{-\frac{\alpha}{2}} e^{-\sigma t^{-\frac{\alpha}{2}}(2|m|-1)} + \sum_{|x+2m|^2 < t^\alpha} Ct^{-\frac{\alpha}{2}}, \]

where \((x, t) \in (0, 1) \times (0, \infty)\). Again by noting that \(|x+2m| \geq 2|m|-1\), we see that \(|x+2m|^2 < t^\alpha\) implies \(2|m| \leq 1 + t^{\frac{\alpha}{2}}\), so that

\[ \sum_{|x+2m|^2 < t^\alpha} Ct^{-\frac{\alpha}{2}} = Ct^{-\frac{\alpha}{2}} \sum_{|x+2m|^2 < t^\alpha} \leq Ct^{-\frac{\alpha}{2}}(2 + t^{\frac{\alpha}{2}}) = C\left(t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}}\right). \]

By direct calculations, we find

\[ \sum_{|x+2m|^2 \geq t^\alpha, m \neq 0} e^{-\sigma t^{-\frac{\alpha}{2}}(2|m|-1)} \leq 2 \sum_{m=1}^{\infty} e^{-\sigma t^{-\frac{\alpha}{2}}(2m-1)} = \frac{2e^{\sigma t^{-\frac{\alpha}{2}}}}{e^{2\sigma t^{-\frac{\alpha}{2}}} - 1}. \]

Moreover we can directly verify that

\[ \frac{2e^{\sigma t^{-\frac{\alpha}{2}}}}{e^{2\sigma t^{-\frac{\alpha}{2}}} - 1} \leq Ct^{\frac{\alpha}{2}}, \quad t > 0, \]

which implies that

\[ \sum_{|x+2m|^2 \geq t^\alpha, m \neq 0} Ct^{-\frac{\alpha}{2}} e^{-\sigma t^{-\frac{\alpha}{2}}(2|m|-1)} \leq Ct^{\frac{\alpha^2}{2}}, \quad t > 0. \]

Finally we obtain

\[ |\theta_\alpha(x, t)| \leq C\left(t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}} + t^{\frac{\alpha^2}{2}}\right), \quad t > 0. \]

Similarly

\[ |D_\tau^{1-\alpha} \theta_\alpha(x, t)| \leq C\left(t^{\frac{\alpha^2}{2}} + t^{\frac{\alpha}{2}-1} + t^{\frac{\alpha^2}{2}}\right), \quad t > 0. \]

Therefore

\[ |w(0, t)| \leq 2C\left(t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}} + t^{\frac{\alpha^2}{2}}\right) \int_{0}^{1} |u_0(\xi)| d\xi \leq 2C\|u_0\|_{L^{1}(0,1)} \left(t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}} + t^{\frac{\alpha^2}{2}}\right). \]

In view of the definition of \(v\), by direct calculations and (13), we arrive at the estimate for \(v(0, t)\):

\[ |v(0, t)| \leq 2\|g\|_{L^{\infty}(0,\infty)} \int_{0}^{t} |D_\tau^{1-\alpha} \theta_\alpha(-1, \tau)| d\tau \]

\[ \leq 2\|g\|_{L^{\infty}(0,\tau)} \left(\frac{2}{\alpha}t^{\frac{\alpha}{2}} + \frac{2}{3\alpha}t^{\frac{2\alpha}{2}} + \frac{2}{2\alpha}t^{\frac{2\alpha-2}{2}}\right), \quad t > 0. \]

Collecting all the above estimates, we finally find that

\[ |\bar{u}(0, t)| \leq C\left(t^{-\frac{\alpha}{2}} + t^{\frac{\alpha}{2}} + t^{\frac{\alpha^2}{2}} + t^{\frac{\alpha^2}{2}}\right), \quad t > 0. \]
We also need a classical result from the complex analysis:

**Lemma 2.3** (Phragmén-Lindelöf principle). Let $F(z)$ be a holomorphic function in a sector $S = \{ z \in \mathbb{C}; \theta_1 < \arg z < \theta_2 \}$ of angle $\pi/\beta = \theta_1 - \theta_2$, and continuous on the closure $\overline{S}$. If

\[
|F(z)| \leq 1
\]

for $z \in \partial S$: the boundary of $S$, and

\[
|F(z)| \leq Ce^{C|z|^{\gamma}}
\]

for all $z \in S$, where $0 \leq \gamma < \beta$ and $C > 0$, then (15) holds also for all $z$ in $S$.

The proof of the above lemma can be found in Stein and Shakarchi [18]. Furthermore, Phragmén-Lindelöf principle yields the following useful corollary:

**Corollary 2.1.** Let $f$ be a real-valued continuous function on the interval $[0, 1]$ and satisfy the following estimate

\[
\left| \int_0^1 f(t)e^{st}dt \right| \leq Ce^{as}, \quad s \geq 0,
\]

where $0 < a < 1$ and the constant $C$ is independent of $s$. Then $f$ is identically zero on $[a, 1]$.

**Proof.** By splitting $0 \leq t \leq 1$ into the two parts $[0, a]$ and $[a, 1]$, we see that

\[
\left| \int_a^1 f(t)e^{st}dt \right| = \left| \int_0^1 f(t)e^{st}dt - \int_0^a f(t)e^{st}dt \right| \leq (C + a\|f\|_{C[0,1)})e^{as}, \quad s \geq 0.
\]

After the change $\eta = t - a$ of variables, we arrive at

\[
\left| \int_0^{1-a} f(t+a)e^{st}e^{as}dt \right| \leq (C + a\|f\|_{C[0,1)})e^{as}, \quad s \geq 0,
\]

hence that

\[
\left| \int_0^{1-a} f(t+a)e^{st}dt \right| \leq C_1, \quad s \geq 0,
\]

where $C_1 := C + a\|f\|_{C[0,1]}$. Therefore our statement in this corollary is equivalent to the following:

If $G(z) = \int_0^b g(t)e^{tz}dt$ is bounded for $z = s \geq 0$, then $g \equiv 0$ in $[0, b]$.

Here $b := 1 - a$ and $g(t) := f(t + a)$.

From the definition of the function $G$ and the above estimation, we see that $G$ is bounded on the imaginary axis and as well on $\{ z \in \mathbb{C}; \arg z = 0 \}$. Setting $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{\beta}$ in Lemma 2.3, so that $\beta = 2$, and noting the estimate

\[
|G(z)| \leq \int_0^b |g(t)|e^{t|z|}dt \leq b\|g\|_{L^\infty(0,b)}e^{b|z|}, \quad 0 < \arg z < \frac{\pi}{2},
\]

we conclude from Lemma 2.3 that $G$ must be bounded on the whole sector $\{ z \in \mathbb{C}; 0 < \arg z < \frac{\pi}{2} \}$. Similarly, $G$ must be bounded on the sector $\{ z \in \mathbb{C}; -\frac{\pi}{2} < \arg z < 0 \}$. Thus $G$ is bounded on the right half plane. Moreover we can directly see that $G(z)$ is bounded on the left half plane. Thus, since $G$ is holomorphic on $\mathbb{C}$, iy follows that $G$ is a constant function, and the constant is zero, because

\[
\lim_{z \to -\infty} \int_0^b g(t)e^{tz}dt = 0,
\]

which implies our desired conclusion: $f \equiv 0$ in $[a, 1]$.
Now we are ready to prove the uniqueness of solutions to the lateral Cauchy problem \( \mathcal{S} \). We have

**Lemma 2.4.** Let \( T > 0 \) be a fixed constant and \( u \in L^\infty(0,T; H^2(0,1)) \) be a solution to the lateral Cauchy problem \( \mathcal{S} \). Then we have

\[
u(x, t) = 0, \quad (x, t) \in [0,1] \times [0,T].
\]

Theorem 1.1 directly follows from Lemma 2.4. Indeed, setting \( I = (a,b) \) with \( 0 < a < b < 1 \), by \( u|_{I \times (0,T)} = 0 \), we have \( u(a, \cdot) = u_x(a, \cdot) = 0 \) and \( u(b, \cdot) = u_x(b, \cdot) = 0 \) in \( (0,T) \). Changing independent variables \( x \to a - x \) and \( x \to x - b \) in the intervals \( (0,a) \) and \( (b,1) \) respectively, and applying Lemma 2.4, we obtain \( u = 0 \) in \( (0,a) \times (0,T) \) and \( (b,1) \times (0,T) \).

Thus the rest of the paper is devoted to the proof of Lemma 2.4.

**Proof.** From the above calculation and settings, and noting Lemma 2.2 we see that \( \tilde{u} \) is an extension of \( u \) which solves the Cauchy problem \( \mathcal{S} \), that is, \( \tilde{u} = u \in [0,1] \times [0,T] \). Using the assumption that \( u(0, t) = 0 \) for \( t \in [0,T] \), we find

\[
2 \int_0^1 \theta_\alpha(x,t) u_0(x) dx + 2 \int_0^1 D_\alpha \theta_\alpha(1,t-s) \tilde{g}(s) ds = \begin{cases} 0, & t \in (0,T), \\ \tilde{u}(0,t), & t \in [T,\infty). \end{cases}
\]

Taking the Laplace transforms on both sides of the above equation, we have

\[
2 \int_0^1 \mathcal{L}\{\theta_\alpha(x,t); s\} u_0(x) dx + 2 \mathcal{L}\{D_\alpha \theta_\alpha(1,t); s\} \mathcal{L}\{\tilde{g}(t); s\} = \int_T^{\infty} \tilde{u}(0,t)e^{-st} dt.
\]

We will show several useful estimates which mainly describe the rate of the convergence of the terms in \( \mathcal{S} \) as \( s \to \infty \).

First, from the definition of the Theta function and the formula \( \mathcal{B} \), it follows that

\[
2 \mathcal{L}\{\theta_\alpha(x,t); s\} = s^{\alpha-1} \sum_{m=-\infty}^{\infty} e^{-\alpha |x+2ms|s^{\alpha}}, \quad s > 0.
\]

For \( x \in [0,1] \), we further treat the above identity as follows

\[
2 \mathcal{L}\{\theta_\alpha(x,t); s\} = s^{\alpha-1} \left( e^{xs^{\alpha}} \sum_{m=-\infty}^{-1} e^{2ms^{\alpha}} + e^{-xs^{\alpha}} \sum_{m=0}^{\infty} e^{-2ms^{\alpha}} \right)
\]

\[
= s^{\alpha-1} \left( e^{xs^{\alpha}} \sum_{m=1}^{\infty} e^{-2ms^{\alpha}} + e^{-xs^{\alpha}} \sum_{m=0}^{\infty} e^{-2ms^{\alpha}} \right)
\]

\[
= s^{\alpha-1} \frac{e^{xs^{\alpha}}}{e^{2s^{\alpha}} - 1} + s^{\alpha-1} \frac{e^{-xs^{\alpha}}}{e^{-2s^{\alpha}} - 1}, \quad s > 0.
\]

From Lemma 2.1 we have \( \theta_\alpha(1,0) = 0 \). Then using the formula

\[
\mathcal{L}\{D_\alpha \theta_\alpha(1,t); s\} = s^{1-\alpha} \mathcal{L}\{\theta_\alpha(1,t); s\},
\]

we are led to

\[
\mathcal{L}\{D_\alpha \theta_\alpha(1,t); s\} = s^{\alpha} \frac{e^{xs^{\alpha}}}{e^{2s^{\alpha}} - 1}, \quad s > 0.
\]
Solving (17) with respect to \( L\{\tilde{g}(t); s\} \) and substituting the above representations of \( L\{\theta_{\alpha}(\xi, t); s\} \) and \( L\{D_{t}^{-\alpha}\theta_{\alpha}(1, t); s\} \), we have

\[
L\{\tilde{g}(t); s\} = \frac{1}{2} s^{\frac{s}{2}} \left( e^{s\frac{\pi}{2}} - e^{-s\frac{\pi}{2}} \right) \int_{T}^{\infty} \tilde{u}(0, t)e^{-st}dt - \frac{1}{2} s^{\alpha-1} \int_{0}^{1} e^{(\xi-1)s\frac{\pi}{2}} u_{0}(\xi)d\xi
\]

\[\quad - \frac{1}{2} s^{\alpha-1} \int_{0}^{1} e^{(1-\xi)s\frac{\pi}{2}} u_{0}(\xi)d\xi
= : I_{1}(s) + I_{2}(s) - I_{3}(s), \quad s > 0,
\]

that is,

\[
I_{3}(t) = \frac{1}{2} s^{\alpha-1} \int_{0}^{1} e^{(1-\xi)s\frac{\pi}{2}} u_{0}(\xi)d\xi
\]

\[= \frac{1}{2} s^{\frac{s}{2}} \left( e^{s\frac{\pi}{2}} - e^{-s\frac{\pi}{2}} \right) \int_{T}^{\infty} \tilde{u}(0, t)e^{-\xi t}dt
\]

\[\quad - \frac{1}{2} s^{\alpha-1} \int_{0}^{1} e^{(\xi-1)s\frac{\pi}{2}} u_{0}(\xi)d\xi - L\{\tilde{g}(t); s\}
= : I_{1}(s) + I_{2}(s) - L\{\tilde{g}(t); s\}.
\]

From the choice of the extension \( \tilde{g} \) of the function \( g \), we conclude that the left-hand side of the above equation can be rephrased as follows

\[
L\{\tilde{g}(t); s\} = \int_{0}^{T} g(t)e^{-st}dt + g(T) \int_{T}^{T+1} (T + 1 - t)e^{-st}dt,
\]

which implies that

\[
|L\{\tilde{g}(t); s\}| \leq \|g\|_{L^{\infty}(0,T)} \int_{0}^{T+1} e^{-st}dt = \frac{\|g\|_{L^{\infty}(0,T)}(1 - e^{-s(T+1)})}{s} \leq \|g\|_{L^{\infty}(0,T)} s^{-1}, \quad s > 0.
\]

Here we used that \( g = u_{x}(1, \cdot) \in L^{\infty}(0,T) \) by \( u_{x} \in L^{\infty}(0,T; H^{1}(0,1)) \) and \( H^{1}(0,1) \subset C[0,1] \). Now by letting \( s \) sufficiently large, we conclude that

\[
L\{\tilde{g}(t); s\} \to 0, \quad as \quad s \to \infty.
\]

The final conclusion of Lemma 2.2 yields \( |\tilde{u}(0, t)| < Ce^{MT} \) for \( t \in [T, \infty) \), where \( C, M > 0 \) are constants only depend on \( \alpha, T, u_{0} \) and \( g \), which implies that

\[
\left| \int_{T}^{\infty} \tilde{u}(0, t)e^{-st}dt \right| \leq \int_{T}^{\infty} Ce^{(M-s)t}dt = \frac{Ce^{MT}}{s-M}e^{-sT}, \quad s > 2M.
\]

By \( \alpha \in (0, 1) \), we can choose \( C_{1} > 0 \) such that

\[
|I_{1}(s)| \leq \frac{Ce^{MT}}{M} s^{\frac{s}{2}} e^{s\frac{\pi}{2}} - sT \leq Ce^{C_{1}s}, \quad s > 2M.
\]

For \( I_{2} \), since \( u_{0} := u(\cdot, 0) \in C[0, 1] \), we have

\[
|I_{2}(s)| \leq \frac{1}{2} s^{\alpha-1} \|u_{0}\|_{C[0,1]} \int_{0}^{1} e^{(\xi-1)s\frac{\pi}{2}} d\xi
\]

\[= \frac{1}{2} \|u_{0}\|_{C[0,1]} s^{\alpha-1}(1 - e^{-s\frac{\pi}{2}}) \leq \frac{1}{2} \|u_{0}\|_{C[0,1]} s^{\frac{s}{2}} - s^{-1}, \quad s > 0.
\]
Finally, noting the equality \(11\), from \(14\) and the estimates for \(I_j, j = 1,2\), we obtain an estimate for \(I_3(s)\):
\[
|I_3(s)| \leq Ce^{-C_1s} + \|g\|_{L^\infty(0,T)}s^{-1} + \frac{1}{2}\|u_0\|_{C[0,1]}s^{\frac{2}{N}-1}, \quad s > 2M,
\]
where \(C_2 := \sup_{s \geq 0} |I_1(s)| < \infty\), which further implies
\[
\left| \int_0^1 e^{(1-\xi)s^{\frac{2}{N}}} u_0(\xi) d\xi \right| \leq C_2(s^{-\frac{2}{N}} + s^{-\alpha}), \quad s > 2M.
\]
The change of variables implies
\[
\int_0^1 e^{n^2(1-\eta)} u_0(1-\eta) d\eta = \int_0^1 e^{(1-\xi)s^{\frac{2}{N}}} u_0(\xi) d\xi.
\]
Therefore, after the change of variable \(z := s^{\frac{2}{N}}\), we find
\[
\left| \int_0^1 e^{n^2u_0(1-\eta)} d\eta \right| \leq C_2(z^{1} + z^{-2}), \quad z > (2M)^{\frac{2}{N}}.
\]
For \(0 < z \leq (2M)^{\frac{2}{N}}\), we have
\[
\left| \int_0^1 e^{n^2u_0(1-\eta)} d\eta \right| \leq \|u_0\|_{C[0,1]}e^{z} \leq \|u_0\|_{C[0,1]}e^{(2M)^{\frac{2}{N}}}.
\]
Therefore we can choose constants \(C_3 > 0\) and \(a \in (0,1)\) such that
\[
\left| \int_0^1 e^{n^2u_0(1-\eta)} d\eta \right| \leq C_3e^{az}, \quad z > 0.
\]
Hence Corollary \(24\) yields that \(u_0 \equiv 0\) in \([0,1]\). From \(16\), it follows that
\[
\int_0^t D_1^{1-\alpha}\theta_\alpha(1,\tau)g(t-\tau)d\tau = 0, \quad t \in (0,T).
\]

Therefore, the Titchmarsh convolution theorem (see Doss \(3\) and Titchmarsh \(20\)) implies the existence of \(T_1, T_2 \geq 0\) satisfying \(T_1 + T_2 \geq T\) such that \(D_1^{1-\alpha}\theta_\alpha(1,t) = 0\) for almost all \(t \in (0,T_1)\) and \(g(t) = 0\) for all \(t \in (0,T_2)\). However, recalling the definition of Wright function, we see that the Theta function \(\theta_\alpha(t)\) is analytic in \(t \in (0,T)\), hence \(D_1^{1-\alpha}\theta_\alpha(t)\) is \(t\)-analytic. Thus \(T_1\) must be zero, that is, \(g \equiv 0\) in \((0,T)\). Finally we can prove the uniqueness of the solution of the initial-boundary value problem \(24\) similarly to \(19\). Although in \(19\), the Dirichlet boundary condition is considered but the case of the Neumann boundary condition is treated in the same way. Thus \(u \equiv 0\) in \([0,1] \times (0,T)\). This completes the proof of the lemma. 

3 Concluding remarks

In this paper, we first investigated the lateral Cauchy problem for the 1-D time-fractional diffusion equation. On the basis of the Theta function method, we gave a representation formula of the solution and showed the uniqueness of the solution to the Cauchy problem by the use of the Laplace transform argument. As a direct conclusion of the uniqueness of the Cauchy problem, we proved that the classical unique continuation is valid. Let us mention that the proof of the unique continuation principle heavily relies on the Theta function method which enable one to derive an explicit representation formula of the solution. It would be interesting to investigate what happens about the unique continuation property of the solution in the general dimensional case.
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