Isoperimetric Problems of the Calculus of Variations on Time Scales

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ABSTRACT. We prove a necessary optimality condition for isoperimetric problems on time scales in the space of delta-differentiable functions with rd-continuous derivatives. The results are then applied to Sturm-Liouville eigenvalue problems on time scales.

1. Introduction

The theory of time scales (see Section 2 for basic definitions and results) is a relatively new area, that unify and generalize difference and differential equations [8]. It was initiated by Stefan Hilger in the nineties of the XX century [12, 13], and is now subject of strong current research in many different fields in which dynamic processes can be described with discrete or continuous models [1].

The study of the calculus of variations on time scales has began in 2004 with the paper [6] of Bohner, where the necessary optimality conditions of Euler-Lagrange and Legendre, as well as a sufficient Jacobi-type condition, are proved for the basic problem of the calculus of variations with fixed endpoints. Since the pioneer paper [6], the following classical results of the calculus of variations on continuous-time (\( T = \mathbb{R} \)) and discrete-time (\( T = \mathbb{Z} \)) have been unified and generalized to a time scale \( T \): the Noether’s theorem [5]; the Euler-Lagrange equations for problems of the calculus of variations with double integrals [7] and for problems with higher-order derivatives [10]; transversality conditions [14]. The more general theory of the calculus of variations on time scales seems to be useful in applications to Economics [4]. Much remains to be done [11], and here we give a step further. Our main aim is to obtain a necessary optimality condition for isoperimetric problems on time scales. Corollaries include the classical case (\( T = \mathbb{R} \)), which is extensively studied in the literature (see, e.g., [15]); and discrete-time versions [3].

The plan of the paper is as follows. Section 2 gives a short introduction to time scales, providing the definitions and results needed in the sequel. In Section 3 we prove a necessary optimality condition for the isoperimetric problem on time scales.

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Let us consider $t$ points. We denote the set of all rd-continuous functions by $C^r$, it is continuous in right-dense points and if its left-sided limit exists in left-dense points.

It is useful to provide an example to the reader with the concepts introduced so far. Consider $f$ a maximum of all delta differentiable functions with rd-continuous derivative by $C^r$, where we abbreviate $\mu(t) = \sigma(t) - t$, provided this limit exists, and

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \quad t \in [0, 1],$$

provided $f$ is continuous at $t = 1$. Let

$$f(t) = \begin{cases} t & \text{if } t \in [0, 1); \\ 2 & \text{if } t = 1. \end{cases}$$

We observe that at $t = 1$ $f$ is rd-continuous (since $\lim_{t \to 1} f(t) = 1$) but not continuous (since $f(1) = 2$).

2. The calculus on time scales and preliminaries

We begin by recalling the main definitions and properties of time scales (cf. [1, 8, 12, 13] and references therein).

A nonempty closed subset of $\mathbb{R}$ is called a Time Scale and is denoted by $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$, for all $t \in \mathbb{T}$, with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum $M$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum $m$). A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. Throughout the text we let $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ with $a, b \in \mathbb{T}$. We define $\mathbb{T}^r = \mathbb{T} \setminus (\rho(b), b)$ and $\mathbb{T}^l = (\mathbb{T}^r)^r$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, for all $t \in \mathbb{T}$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^r$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of $f$ at $t$. For delta differentiable $f$ and $g$, the next formulas hold:

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t),$$

$$(f\circ g)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t),$$

where we abbreviate $f \circ \sigma$ by $f^\sigma$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous in right-dense points and if its left-sided limit exists in left-dense points. We denote the set of all rd-continuous functions by $C^r_d$ or $C^r_d[\mathbb{T}]$ and the set of all delta differentiable functions with rd-continuous derivative by $C^1_d$ or $C^1_d[\mathbb{T}]$. It is useful to provide an example to the reader with the concepts introduced so far. Consider $\mathbb{T} = \bigcup_{k=0}^\infty [2k, 2k+1]$. For this time scale,

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^\infty [2k, 2k+1); \\ 1 & \text{if } t \in \bigcup_{k=0}^\infty \{2k+1\}. \end{cases}$$

Let us consider $t \in [0, 1] \cap \mathbb{T}$. Then, we have (see [8 Theorem 1.16])

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \quad t \in [0, 1),$$

provided this limit exists, and

$$f^\Delta(1) = \frac{f(2) - f(1)}{2 - 1},$$

provided $f$ is continuous at $t = 1$. Let

$$f(t) = \begin{cases} t & \text{if } t \in [0, 1); \\ 2 & \text{if } t = 1. \end{cases}$$

We observe that at $t = 1$ $f$ is rd-continuous (since $\lim_{t \to 1} f(t) = 1$) but not continuous (since $f(1) = 2$).
It is known that rd-continuous functions possess an antiderivative, i.e., there exists a function $F$ with $F^\Delta = f$, and in this case an integral is defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$. It satisfies

$$(2.2) \quad \int_t^{\sigma(t)} f(\tau)\Delta \tau = \mu(t)f(t).$$

Lemma 2.1 gives the integration by parts formulas of the delta integral:

**Lemma 2.1** ([8]). If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}^1$, then

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t,$$

$$\int_a^b f(t)g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$$

The following time scale DuBois-Reymond lemma will be useful for our purposes:

**Lemma 2.2** ([9]). Let $g \in C_{rd}, g : [a, b]^\ast \to \mathbb{R}^n$. Then,

$$\int_a^b g^\tau(t)\eta^\Delta(t)\Delta t = 0, \text{ for all } \eta \in C_{rd}^1 \{ \eta(a) = \eta(b) = 0$$

holds if and only if $g(t) = c,$ on $[a, b]^\ast$ for some $c \in \mathbb{R}^n$.

Finally, we prove a simple but useful technical lemma.

**Lemma 2.3.** Suppose that a continuous function $f : \mathbb{T} \to \mathbb{R}$ is such that $f^\sigma(t) = 0$ for all $t \in \mathbb{T}^c$. Then, $f(t) = 0$ for all $t \in \mathbb{T}\setminus\{a\}$ if $a$ is right-scattered.

**Proof.** First note that, since $f^\sigma(t) = 0$, then $f^\sigma(t)$ is delta differentiable, hence continuous for all $t \in \mathbb{T}^c$. Now, if $t$ is right-dense, the result is obvious. Suppose that $t$ is right-scattered. We will analyze two cases: (i) if $t$ is left-scattered, then $t \neq a$ and by hypothesis $0 = f^\sigma(\rho(t)) = f(t)$; (ii) if $t$ is left-dense, then, by the continuity of $f^\sigma$ and $f$ at $t$, we can write

$$\forall \varepsilon > 0 \exists \delta_1 > 0 : \forall s_1 \in (t - \delta_1, t], \text{ we have } |f^\sigma(s_1) - f^\sigma(t)| < \varepsilon,$$

$$\forall \varepsilon > 0 \exists \delta_2 > 0 : \forall s_2 \in (t - \delta_2, t], \text{ we have } |f(s_2) - f(t)| < \varepsilon,$$

respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and take $s_1 \in (t - \delta, t)$. As $\sigma(s_1) \in (t - \delta, t)$, take $s_2 = \sigma(s_1)$. By (2.5) and (2.6), we have:

$$|f^\sigma(t) + f(t)| = |f^\sigma(s_1) - f^\sigma(t) + f(t) - f(s_2)| \leq |f^\sigma(s_1) - f^\sigma(t)| + |f(s_2) - f(t)| < 2\varepsilon.$$

Since $\varepsilon$ is arbitrary, $|f^\sigma(t) + f(t)| = 0$, which is equivalent to $f(t) = f^\sigma(t)$. \hfill $\square$

### 3. Main results

We start in [3.1] by defining the isoperimetric problem on time scales and proving a corresponding first-order necessary optimality condition (Theorem 3.4). Then, in [3.2] we show that certain eigenvalue problems can be recast as an isoperimetric problem (Theorem 3.7).
3.1. Isoperimetric problems. Let $J : C^1_{rd} \to \mathbb{R}$ be a functional defined on the function space $(C^1_{rd}, \| \cdot \|)$ and let $S \subseteq C^1_{rd}$.

**Definition 3.1.** The functional $J$ is said to have a *local minimum* in $S$ at $y_* \in S$ if there exists a $\delta > 0$ such that $J(y_*) \leq J(y)$ for all $y \in S$ satisfying $\|y - y_*\| < \delta$.

Now, let $J : C^1_{rd} \to \mathbb{R}$ be a functional of the form

$$J(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t,$$

where $L(t, x, v) : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives $L_x(t, x, v)$ and $L_v(t, x, v)$, respectively, with respect to the second and third variables, for all $t \in [a, b]$. Such that $L(t, y^\sigma(t), y^\Delta(t))$ and $L_v(t, y^\sigma(t), y^\Delta(t))$ are rd-continuous in $t$ for all $y \in C^1_{rd}$. The *isoperimetric problem* consists of finding functions $y$ satisfying given boundary conditions

$$y(a) = y_a, \quad y(b) = y_b,$$

and a constraint of the form

$$I(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t)) \Delta t = l,$$

where $g(t, x, v) : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives with respect to the second and third variables for all $t \in [a, b]$. $g(t, y^\sigma(t), y^\Delta(t))$, $g_x(t, y^\sigma(t), y^\Delta(t))$, and $g_v(t, y^\sigma(t), y^\Delta(t))$ are rd-continuous in $t$ for all $y \in C^1_{rd}$, and $l$ is a specified real constant, that takes (3.1) to a minimum.

**Definition 3.2.** We say that a function $y \in C^1_{rd}$ is *admissible* for the isoperimetric problem if it satisfies (3.2) and (3.3).

**Definition 3.3.** An admissible function $y_*$ is said to be an *extremal* for $I$ if it satisfies the following equation (cf. (6)):

$$g_v(t, y^\sigma(t), y^\Delta(t)) - \int_a^t g_x(\tau, y^\sigma(\tau), y^\Delta(\tau)) \Delta \tau = c,$$

for all $t \in [a, b]$ and some constant $c$.

**Theorem 3.4.** Suppose that $J$ has a local minimum at $y_* \in C^1_{rd}$ subject to the boundary conditions (3.2) and the isoperimetric constraint (3.3), and that $y_*$ is not an extremal for the functional $I$. Then, there exists a Lagrange multiplier constant $\lambda$ such that $y_*$ satisfies the following equation:

$$F^\Delta_v(t, y^\sigma(t), y^\Delta(t)) - \lambda g_v(t, y^\sigma(t), y^\Delta(t)) = 0, \text{ for all } t \in [a, b],$$

where $F = L - \lambda g$ and $F^\Delta_v$ denotes the delta derivative of a composition.

**Proof.** Let $y_*$ be a local minimum for $J$ and consider neighboring functions of the form

$$\hat{y} = y_* + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2,$$

where for each $i \in \{1, 2\}$, $\varepsilon_i$ is a sufficiently small parameter ($\varepsilon_1$ and $\varepsilon_2$ must be such that $|\hat{y} - y_*| < \delta$, for some $\delta > 0$ — see Definition 3.1), $\eta_i(x) \in C^1_{rd}$ and $\eta_i(a) = \eta_i(b) = 0$. Here, $\eta_1$ is an arbitrary fixed function and $\eta_2$ is a fixed function that we will choose later.
First we show that \([3.5]\) has a subset of admissible functions for the isoperimetrical problem. Consider the quantity

\[
I(\hat{y}) = \int_a^b g(t, y^*_\epsilon(t)) + \epsilon_1 \eta_1^\Delta(t) + \epsilon_2 \eta_2^\Delta(t), y^*_\epsilon(t) + \epsilon_1 \eta_1^\Delta(t) + \epsilon_2 \eta_2^\Delta(t) \Delta t.
\]

Then we can regard \(I(\hat{y})\) as a function of \(\epsilon_1\) and \(\epsilon_2\), say \(I(\hat{y}) = \hat{Q}(\epsilon_1, \epsilon_2)\). Since \(y_*\) is a local minimum for \(I\) subject to the boundary conditions and the isoperimetric constraint, putting \(Q(\epsilon_1, \epsilon_2) = \hat{Q}(\epsilon_1, \epsilon_2) - l\) we have that

\[
(3.6) 
Q(0, 0) = 0.
\]

By the conditions imposed on \(g\), we have

\[
\frac{\partial Q}{\partial \epsilon_2}(0, 0) = \int_a^b \left[ g_x(t, y^*_\epsilon(t), y^*_\Delta(t)) \eta^\Delta(t) + g_v(t, y^*_\epsilon(t), y^*_\Delta(t)) \eta_1^\Delta(t) \right] \Delta t
\]

\[
= \int_a^b \left[ g_v(t, y^*_\epsilon(t), y^*_\Delta(t)) - \int_a^t g_x(\tau, y^*_\epsilon(\tau), y^*_\Delta(\tau)) \Delta \tau \right] \eta^\Delta_2(t) \Delta t,
\]

where \((3.7)\) follows from \(\text{(3.6)}\) and the fact that \(\eta_2(a) = \eta_2(b) = 0\). Now, the function

\[
E(t) = g_v(t, y^*_\epsilon(t), y^*_\Delta(t)) - \int_a^t g_x(\tau, y^*_\epsilon(\tau), y^*_\Delta(\tau)) \Delta \tau
\]

is rd-continuous on \([a, b]^\alpha\). Hence, we can apply Lemma 2.2 to show that there exists a function \(\eta_2 \in C^1_{rd}\) such that

\[
\int_a^b \left[ g_v(t, y^*_\epsilon(t), y^*_\Delta(t)) - \int_a^t g_x(\tau, y^*_\epsilon(\tau), y^*_\Delta(\tau)) \Delta \tau \right] \eta^\Delta_2(t) \Delta t \neq 0,
\]

provided \(y_*\) is not an extremal for \(I\), which is indeed the case. We have just proved that

\[
(3.8) 
\frac{\partial Q}{\partial \epsilon_2}(0, 0) \neq 0.
\]

Using \((3.6)\) and \((3.8)\), the implicit function theorem asserts that there exist neighborhoods \(N_1\) and \(N_2\) of 0, \(N_1 \subseteq \{\epsilon_1\ \text{from} \ (3.5)\} \cap \mathbb{R}\) and \(N_2 \subseteq \{\epsilon_2\ \text{from} \ (3.5)\} \cap \mathbb{R}\), and a function \(\epsilon_2 : N_1 \rightarrow \mathbb{R}\) such that for all \(\epsilon_1 \in N_1\) we have

\[
Q(\epsilon_1, \epsilon_2(\epsilon_1)) = 0,
\]

which is equivalent to \(\hat{Q}(\epsilon_1, \epsilon_2(\epsilon_1)) = l\). Now we derive the necessary condition \((3.4)\). Consider the quantity \(J(\hat{y}) = K(\epsilon_1, \epsilon_2)\). By hypothesis, \(K\) is minimum at \((0, 0)\) subject to the constraint \(Q(0, 0) = 0\), and we have proved that \(\nabla Q(0, 0) \neq 0\). We can appeal to the Lagrange multiplier rule (see, e.g., \(15\) Theorem 4.1.1) to assert that there exists a number \(\lambda\) such that

\[
(3.9) 
\nabla (K(0, 0) - \lambda Q(0, 0)) = 0.
\]

Having in mind that \(\eta_1(a) = \eta_1(b) = 0\), we can write:

\[
\frac{\partial K}{\partial \epsilon_1}(0, 0) = \int_a^b \left[ L_x(t, y^*_\epsilon(t), y^*_\Delta(t)) \eta^\Delta_1(t) + L_v(t, y^*_\epsilon(t), y^*_\Delta(t)) \eta_1^\Delta(t) \right] \Delta t
\]

\[
= \int_a^b \left[ L_v(t, y^*_\epsilon(t), y^*_\Delta(t)) - \int_a^t L_x(\tau, y^*_\epsilon(\tau), y^*_\Delta(\tau)) \Delta \tau \right] \eta^\Delta_1(t) \Delta t.
\]
Similarly, we have that
\[
(3.11) \quad \frac{\partial Q}{\partial \varepsilon_1}(0,0) = \int_a^b \left[ g_v(t, y^\varepsilon(t), y^{\varepsilon}_1(t)) - \int_a^t g_x(\tau, y^{\varepsilon}_x(\tau), y^{\varepsilon}_x(\tau)) \Delta \tau \right] \eta_1^{\varepsilon}(t) \Delta t.
\]
Combining (3.9), (3.10) and (3.11), we obtain
\[
\int_a^b \left\{ L_v(\cdot) - \int_a^t L_x(\cdot) \Delta \tau - \lambda \left( g_v(\cdot) - \int_a^t g_x(\cdot) \Delta \tau \right) \right\} \eta_1^{\varepsilon}(t) \Delta t = 0,
\]
where \((\cdot) = (t, y^\varepsilon(t), y^{\varepsilon}_1(t))\) and \((\cdot) = (\tau, y^\varepsilon_x(\tau), y^{\varepsilon}_x(\tau))\). Since \(\eta_1\) is arbitrary, Lemma 2.2 implies that there exists a constant \(d\) such that
\[
(3.12) \quad F_v(\cdot) - \int_a^t F_x(\cdot) \Delta \tau = d,
\]
with \(F = L - \lambda g\). Since the integral and the constant in (3.12) are delta differentiable, we obtain the desired necessary optimality condition (3.4).

**Remark 3.5.** Theorem 3.4 remains valid when \(y_*\) is assumed to be a local maximizer of the isoperimetric problem (3.1)-(3.3).

**Example 3.6.** Suppose that we want to find functions defined on \([-a, a] \cap T\) that take
\[
J(y) = \int_{-a}^a y^\sigma(t) \Delta t
\]
to its largest value (see Remark 3.5) and that satisfy the conditions
\[
y(-a) = y(a) = 0, \quad I(y) = \int_{-a}^a \sqrt{1 + (y^\Delta(t))^2} \Delta t = l > 2a.
\]
Note that if \(y\) is an extremal for \(I\), then \(y\) is a line segment [6], and therefore \(y(t) = 0\) for all \(t \in [-a, a]\). This implies that \(I(y) = 2a > 2a\), which is a contradiction. Hence, \(I\) has no extremals satisfying the boundary conditions and the isoperimetric constraint. Using Theorem 3.4 let
\[
F = L - \lambda g = y^\sigma - \lambda \sqrt{1 + (y^\Delta)^2}.
\]
Because
\[
F_x = 1, \quad F_v = \lambda \frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}},
\]
a necessary optimality condition is given by the following delta-differential equation:
\[
\lambda \left( \frac{y^\Delta}{\sqrt{1 + (y^\Delta)^2}} \right)^\Delta - 1 = 0, \quad t \in [-a, a]^{\mathbb{N}}.
\]
The reader interested in the study of delta-differential equations on time scales is referred to [9] and references therein.
If we restrict ourselves to times scales $T$ with $a(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$ ($a_0 = 0$ and $a_1 = 1$ for $T = \mathbb{R}$; $a_0 = a_1 = 1$ for $T = \mathbb{Z}$), it follows from the results in [10] that the same proof of Theorem 3.4 can be used, mutatis mutandis, to obtain a necessary optimality condition for the higher-order isoperimetric problem (i.e., when $L$ and $g$ contain higher order delta derivatives). In this case, the necessary optimality condition (3.3) is generalized to

$$\sum_{i=0}^{r} (-1)^i \left( \frac{1}{a_1} \right)^{(i-1)/2} F_{a_i} \left( t, y_\sigma(t), y_\sigma^{(r-1)}\Delta(t), \ldots, y_\sigma^{(r-r-1)}\Delta(t), y_\sigma\Delta^r(t) \right) = 0,$$

where $F = L - \lambda g$, and functions $(t, u_0, u_1, \ldots, u_r) \rightarrow L(t, u_0, u_1, \ldots, u_r)$ and $(t, u_0, u_1, \ldots, u_r) \rightarrow g(t, u_0, u_1, \ldots, u_r)$ are assumed to have (standard) partial derivatives with respect to $u_0, \ldots, u_r$, $r \geq 1$, and partial delta derivative with respect to $t$ of order $r+1$.

### 3.2. Sturm-Liouville eigenvalue problems.

Eigenvalue problems on time scales have been studied in [2]. Consider the following Sturm-Liouville eigenvalue problem: find nontrivial solutions to the delta-differential equation

$$y_\Delta^2(t) + q(t)y_\sigma(t) + \lambda y_\sigma(t) = 0, \quad t \in [a, b]^\mathbb{R},$$

with boundary values

$$y(a) = y(b) = 0.$$

Generically, the only solution to equation (3.13) is the trivial solution, $y(t) = 0$ for all $t \in [a, b]$. There are, however, certain values of $\lambda$ that lead to nontrivial solutions. These are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions. These eigenvalues may be arranged as $-\infty < \lambda_1 < \lambda_2 < \ldots$ (see Theorem 1 of [2]) and $\lambda_1$ is called the first eigenvalue.

Consider the functional defined by

$$J(y) = \int_a^b \left( (y_\Delta^2(t)) - q(t)(y_\sigma(t))^2 \right) \Delta t,$$

and suppose that $y_\ast \in C^2_{rd}$ (functions that are twice delta differentiable with rd-continuous second delta derivative) is a local minimum for $J$ subject to the boundary conditions (3.14) and the isoperimetric constraint (3.16).

$$I(y) = \int_a^b (y_\sigma(t))^2 \Delta t = 1.$$

If $y_\ast$ is an extremal for $J$, then we would have $-2y_\sigma(t) = 0$, $t \in [a, b]^\mathbb{R}$. Noting that $y(a) = 0$, using Lemma [23] we would conclude that $y(t) = 0$ for all $t \in [a, b]$. No extremals for $I$ can therefore satisfy the isoperimetric condition (3.16). Hence, by Theorem [3.3] there is a constant $\lambda$ such that $y_\ast$ satisfies

$$F_{y_\sigma}(t, y_\sigma(t), y_\Delta(t)) - F_{y_\sigma}(t, y_\sigma(t), y_\Delta(t)) = 0,$$

with $F = (y_\Delta^2 - q(y_\sigma)^2 - \lambda (y_\sigma)^2)$. It is easily seen that (3.17) is equivalent to (3.13). The isoperimetric problem thus corresponds to the Sturm-Liouville problem augmented by the normalizing condition (3.16), which simply scales the eigenfunctions. Here, the Lagrange multiplier plays the role of the eigenvalue.
Theorem 3.7. Let $\lambda_1$ be the first eigenvalue for the Sturm-Liouville problem (3.13) with boundary conditions (3.14), and let $y_1$ be the corresponding eigenfunction normalized to satisfy the isoperimetric constraint (3.16). Then, among functions in $C^2_{\alpha}$ that satisfy the boundary conditions (3.14) and the isoperimetric condition (3.16), the functional $J$ defined by (3.15) has a minimum at $y_1$. Moreover, $J(y_1) = \lambda_1$.

Proof. Suppose that $J$ has a minimum at $y$ satisfying conditions (3.14) and (3.16). Then $y$ satisfies equation (3.13) and multiplying this equation by $y_\sigma$ and integrating from $a$ to $b$, we obtain

\[ \int_a^b y_\sigma(t)y_\Delta^2(t) \Delta t + \int_a^b q(t)(y_\sigma^2(t)) \Delta t + \lambda \int_a^b (y_\sigma^2(t)) \Delta t = 0. \]  

(3.18)

Since $y(a) = y(b) = 0$,

\[ \int_a^b y_\sigma(t)y_\Delta^2(t) \Delta t = [y(t)y_\Delta(t)]_{t=a}^{t=b} - \int_a^b (y_\Delta^2) \Delta t = - \int_a^b (y_\Delta^2) \Delta t, \]

and by (3.16), (3.18) reduces to

\[ \int_a^b [(y_\Delta^2 - q(t)(y_\sigma^2(t))] \Delta t = \lambda, \]

that is, $J(y) = \lambda$. Due to the isoperimetric condition, $y$ must be a nontrivial solution to (3.13) and therefore $\lambda$ must be an eigenvalue. Since there exists a least element within the eigenvalues, $\lambda_1$, and a corresponding eigenfunction $y_1$ normalized to meet the isoperimetric condition, the minimum value for $J$ is $\lambda_1$ and $J(y_1) = \lambda_1$. \qed

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