A note on periods of Calabi–Yau fractional complete intersections

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Abstract
We prove that the GKZ $D$-module $M^β_A$ arising from Calabi–Yau fractional complete intersections in toric varieties is complete, i.e., all the solutions to $M^β_A$ are period integrals. This particularly implies that $M^β_A$ is equivalent to the Picard–Fuchs system. As an application, we give explicit formulae of the period integrals of Calabi–Yau threefolds coming from double covers of $\mathbb{P}^3$ branched over eight hyperplanes in general position.

Keywords  GKZ systems · Calabi–Yau fractional complete intersections · Period integrals

Mathematics Subject Classification  14J10 · 14J32

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1 Introduction

Mirror symmetry from physics has successfully made numerous predictions in enumerative geometry and led to many deep conjectures in various branches of mathematics. Roughly, mirror symmetry predicts that for a Calabi–Yau manifold $Y$ there is a Calabi–Yau manifold $Y^\vee$ such that the genus zero Gromov–Witten theory of $Y$ (resp. $Y^\vee$) can be computed via the complex deformation of $Y^\vee$ (resp. $Y$). Sometimes, the genus zero Gromov–Witten theory

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is called the A model while the complex deformation theory is called the B model in the literature.

One can study the B model via period integrals. It is an old strategy to study periods by first constructing a PDE system that governs them, called a Picard–Fuchs system, then computing the solutions to the Picard–Fuchs system, and ultimately trying to recover the periods from the solutions. However, it is difficult to write down all the equations which govern the periods, or even a set of generators. Nevertheless, in some examples of Calabi–Yau hypersurfaces or complete intersections in toric varieties, Batyrev [1] observed that their period integrals satisfy a certain type of GKZ systems, introduced by Gel’fand, Graev, Kapranov, and Zelevinsky [5, 8]: the input of GKZ systems consists of a matrix $A \in \text{Mat}_{d \times m}(\mathbb{Z})$ and an exponent vector $\beta \in \mathbb{C}^d$ and the output is a PDE system $\mathcal{M}_A^\beta$ on $\mathbb{C}^m$ (cf. Definition 3.1). Consequently, in the present case, the GKZ system is a subsystem of the Picard–Fuchs system but it is far away from being full in general (we say that the GKZ system is incomplete). To see this, note that the GKZ system $\mathcal{M}_A^\beta$ here is a regular holonomic $\mathcal{D}$-module whose holonomic rank is equal to the normalized volume of a polytope determined by $A$ and the said volume is greater than the expected dimension in most cases. In fact, a result by Huang et al. implies that for Calabi–Yau hypersurfaces in Fano toric manifolds the set of solutions to $\mathcal{M}_A^\beta$ at any point can be identified with certain relative homology group [13, Corollary 3.6] and later the author and Zhang generalized this result to arbitrary complete intersections [15]. Finally, there are several recipes to recover period integrals. Inspired by mirror symmetry, Hosono et al. proposed a mechanism, also known as the hyperplane conjecture, to characterize the honest period integrals among the solutions to $\mathcal{M}_A^\beta$ [12]. Another way is to enlarge the incomplete PDE system by adding additional operators from the symmetries of the ambient variety. In this direction, Hosono et al. introduced the extended GKZ system to tackle the case of Calabi–Yau hypersurfaces in toric varieties [12] and later Lian et al. introduced tautological systems by bringing this idea to general manifolds with a Lie group action [17]. Using tautological systems, Lian et al. were able to prove the hyperplane conjecture when the ambient toric variety is $\mathbb{P}^n$ [18].

The recent work of Hosono et al. shed light on mirror symmetry for singular Calabi–Yau varieties [10, 11]: they investigated singular $K3$ surfaces arising from configurations of six lines in general position on $\mathbb{P}^2$ and found their lattice-theoretical mirror. Based on their results, in [9], we constructed pairs $(Y, Y^\vee)$ of singular Calabi–Yau varieties from double covers of toric manifolds branched over a divisor with strictly normal crossings and showed that $(Y, Y^\vee)$ are topological mirror pairs when the dimension is less than 5. In threefold cases, we also studied explicit examples and verified that $A(Y) \cong B(Y^\vee)$. In the present case, $A(Y)$ is the untwisted part of the genus zero orbifold Gromov–Witten theory and $B(Y^\vee)$ is the complex deformation of $Y^\vee$ coming from particular deformations of the branch locus. Given these, it is natural to study the family of such a Calabi–Yau double cover.

It is known that the period integrals of the family of Calabi–Yau double covers coming from a nef-partition also satisfy a GKZ system with $A$ being the matrix associated to the dual nef-partition but the exponent $\beta$ being fractional only. Owing to the combinatorial nature of $A$, one can extend classical techniques for Calabi–Yau complete intersections in toric varieties to this new class of singular Calabi–Yau varieties. Because of the striking similarity, such a Calabi–Yau is called a fractional complete intersection in [14].

In this paper, we investigate the B model of Calabi–Yau fractional complete intersections. We prove that the GKZ system associated with a family of Calabi–Yau fractional complete intersections is equivalent to the Picard–Fuchs system.
Theorem A (=Theorem 3.4) Let $\mathcal{M}_A^\beta$ be the GKZ system associated with the family of Calabi–Yau double covers coming from a nef-partition. Then $\mathcal{M}_A^\beta$ is complete, i.e., the (classical) solutions to $\mathcal{M}_A^\beta$ are precisely the period integrals.

See Sect. 2.3 for the explicit construction of the Calabi–Yau double covers from a nef-partition and Sect. 3 for the definition of $A$ and $\beta$ in the present situation.

Theorem 1 says there is no need to enlarge the GKZ system although the base toric variety might have non-torus automorphisms. This is the case since we only consider particular complex deformations; the branch locus of our Calabi–Yau double cover is a divisor whose support is a union of toric divisors and some numerically effective divisors and we deform those numerically effective divisors only. The infinitesimal automorphisms fixing the union of toric divisor are all from the maximal torus. This gives a conceptual explanation of Theorem 1.

Combined with the results in [14], we obtain all period integrals via the generalized Frobenius method (cf. Corollary 4.8). As an application, we give an explicit formula for the period integrals of the family of Calabi–Yau threefolds coming from double covers of $\mathbb{P}^3$ branch over eight hyperplanes in general position (cf. Example 4.9).

Our proof of Theorem 1 heavily relies on a classical result of Gel’fand, Kapranov, and Zelevinsky. Indeed, in the case of Calabi–Yau double covers coming from a nef-partition, the associated GKZ system $\mathcal{M}_A^\beta$ is rather special: the exponent $\beta$ is always non-resonant with respect to $A$ (cf. Definition 3.2) and Theorem 1 follows from the main result in [6]. The proof of the non-resonance of $\beta$ relies on the combinatorial structure of $A$ which boils down to the convexity of the support function of numerically effective toric divisors.

We remark that for general cyclic covers of toric varieties, not necessarily coming from a nef-partition, the exponent $\beta$ might be resonant. We will deal with the case when $\beta$ is semi-nonresonant in a forthcoming paper [16].

2 Calabi–Yau fractional complete intersections

2.1 Toric varieties

Let $N = \mathbb{Z}^n$ be a rank $n$ lattice and $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ be the dual lattice. Denote by $\langle -, - \rangle$ the canonical dual pairing between $M$ and $N$. Recall that a fan $\Sigma$ in $N_\mathbb{R} := N \otimes \mathbb{R}$ is a collection of strongly convex rational polyhedral cones in $N_\mathbb{R}$ such that

- if $\sigma \in \Sigma$, then all the faces of $\sigma$ belong to $\Sigma$;
- the intersection of two cones is a face of each.

The support $|\Sigma|$ of a fan $\Sigma$ is the union of all the cones belonging to $\Sigma$. A fan $\Sigma$ is called proper if $|\Sigma| = N_\mathbb{R}$. A fan $\Sigma$ is called simplicial if every cone in $\Sigma$ is simplicial, i.e., generated by a $\mathbb{Q}$-linearly independent subset. The toric variety defined by $\Sigma$ is denoted by $X_\Sigma$.

Recall that a polyhedron is a finite intersection of closed half-spaces in an Euclidean space. A polyhedron is called a polytope if it is bounded. A lattice polytope is a polytope whose vertices are integral. Let $\Delta$ be a polytope in $\mathbb{R}^n$. The dimension of $\Delta$ is the smallest integer $d$ such that $\Delta \subset \mathbb{R}^d$. $\Delta$ is called full-dimensional if $d = n$. If $\Delta$ is a full-dimensional lattice polytope in $M_\mathbb{R} := M \otimes \mathbb{R}$ and 0 is its interior point, we can define the dual polytope or the polar polytope

$$\Delta^\vee := \{ u \in N_\mathbb{R} \mid \langle m, u \rangle \geq -1 \text{ for all } m \in \Delta \}.$$
A lattice polytope $\Delta$ containing $0$ in its interior is called a reflexive polytope if $\Delta^\vee$ is again a lattice polytope. One can check that $(\Delta^\vee)^\vee = \Delta$ for $\Delta$ being reflexive.

Given a lattice polytope $\Delta \subset M_\mathbb{R}$, one can define the normal fan $\mathcal{N}(\Delta)$ of $\Delta$. If $0 \in \text{int}(\Delta)$, one can define the face fan $\mathcal{F}(\Delta)$ of $\Delta$. If $\Delta$ is a reflexive polytope, one can easily check

$$\mathcal{N}(\Delta) = \mathcal{F}(\Delta^\vee)$$

and

$$\mathcal{N}(\Delta^\vee) = \mathcal{F}(\Delta).$$

If $\Delta \subset M_\mathbb{R}$ a lattice polytope, we denote by $P_\Delta$ the toric variety defined by $\mathcal{N}(\Delta)$. According to our notation, we have $P_\Delta = X_{\mathcal{N}(\Delta)}$.

Let $\Sigma$ be a fan in $N_\mathbb{R}$. Denote by $\Sigma(k)$ the set of $k$-dimensional cones in $\Sigma$. Each $\rho \in \Sigma(1)$ determines a torus invariant divisor $D_\rho$ on $X$. By abuse of notation, the same notation $\rho$ also stands for the primitive generator of the 1-cone $\rho$ and $\Sigma(1)$ is also regarded as the set of primitive generators of 1-cones in $\Sigma$. Thus the notation $\rho \in \Sigma(1)$ has two meanings but we will not explicitly spell them out when the context is clear. For a torus invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, we define the divisor polyhedron

$$\Delta_D := \{ m \in M_\mathbb{R} | \langle m, \rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1) \}.$$

Note that $\Delta_D$ is a polytope when $\Sigma$ is proper. In which case $\Delta_D$ is called the divisor polytope and the set of integral points in $\Delta_D$ gives a canonical basis of $H^0(X, \mathcal{O}_X(D))$. If moreover $D$ is Cartier, for $\sigma \in \Sigma(n)$, one can find uniquely an $m_\sigma \in M$ such that

$$\langle m_\sigma, \rho \rangle = -a_\rho \text{ if } \rho \in \Sigma(1) \text{ and } \rho \subset \sigma.$$

Suppose that every maximal cone in $\Sigma$ is $n$-dimensional. Then a Cartier divisor $D$ determines a collection $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ which is called the Cartier data of $D$. One can define the support function of $D$

$$\varphi_D : |\Sigma| \to \mathbb{R} \text{ via } \varphi_D|_\sigma(u) := \langle m_\sigma, u \rangle.$$  

Now assume further that $|\Sigma|$ is convex. It is known that (cf. [4, Theorem 6.1.7 and Theorem 6.3.12])

$$D \text{ is basepoint free } \iff D \text{ is numerically effective } \iff \varphi_D \text{ is convex.}$$

### 2.2 Batyrev–Borisov’s duality construction

Let $\Delta \subset M_\mathbb{R}$ be a reflexive polytope and $\mathcal{N}(\Delta)$ be the normal fan of $\Delta$.

**Definition 2.1** A partition $I_1 \sqcup \cdots \sqcup I_r = \mathcal{N}(\Delta)(1)$ is called a nef-partition if for each $1 \leq j \leq r$ the divisor

$$E_j := \sum_{\rho \in I_j} D_\rho$$

is Cartier and numerically effective on $P_\Delta$. Note that $E_1 + \cdots + E_r = -K_{P_\Delta}$. A nef-partition also induces a decomposition

$$\Delta = \Delta_1 + \cdots + \Delta_r$$

where $\Delta_j := \Delta_{E_j}$.

By abuse of the terminology, both $E_1 + \cdots + E_r = -K_{P_\Delta}$ and $\Delta = \Delta_1 + \cdots + \Delta_r$ are also called nef-partitions.
A nef-partition \( I_1 \sqcup \cdots \sqcup I_r = \mathcal{N}(\Delta)(1) \) gives rise to polytopes
\[
\nabla_j := \text{Conv}(I_j \cup \{0\}).
\]
One can check \( \Delta^\vee = \text{Conv}(\nabla_1, \ldots, \nabla_r) \). The direction \("\subseteq\"\) is clear. To prove the opposite inclusion, we pick \( \mu \in \Delta^\vee \). Then \( \mu \) belongs to a cone \( \sigma \in \mathcal{N}(\Delta) \) of maximal dimension. There are 1-cones \( \rho_1, \ldots, \rho_s \in \sigma(1) \) and non-negative scalars \( c_1, \ldots, c_s \) such that \( \mu = \sum_{i=1}^s c_i \rho_i \). Since \( \sigma \) is a maximal cone in the normal fan \( \mathcal{N}(\Delta) \), there exists an \( m \in M_{\mathbb{R}} \) (indeed a vertex of \( \Delta \)) such that \( \langle m, \rho_i \rangle = -1 \) for all \( i = 1, \ldots, s \) and \( \langle m, \mu \rangle > -1 \) for \( \mu \in \mathcal{N}(\Delta)(1) \setminus \{\rho_1, \ldots, \rho_s\} \). We then have
\[
1 \geq -\langle m, \mu \rangle = -\sum_{i=1}^s c_i \langle m, \rho_i \rangle = \sum_{i=1}^s c_i
\]
and hence \( \mu \in \text{Conv}(\nabla_1, \ldots, \nabla_r) \). A fundamental result in [3] states that their Minkowski sum
\[
\nabla = \nabla_1 + \cdots + \nabla_r \tag{2.1}
\]
is again a reflexive polytope and (2.1) induces a nef-partition. This is called the dual nef-partition in [2].

### 2.3 Calabi–Yau fractional complete intersections

Let \( \Delta \subset M_{\mathbb{R}} \) be a reflexive polytope. There exists a maximal projective crepant partial desingularization (MPCP desingularization for short hereafter) of \( \mathbf{P}_\Delta \). Recall that a MPCP desingularization of \( \mathbf{P}_\Delta \) is a projective toric variety \( \mathbf{X} / \Sigma_1 \) such that
\[
\begin{align*}
\Sigma & \text{ is a refinement of } \mathcal{N}(\Delta); \\
\Sigma & \text{ is simplicial}; \\
\Sigma(1) & = \Delta^\vee \cap N \setminus \{0\}.
\end{align*}
\]
Note that MPCP desingularizations exist and may not be unique.

Given a nef-partition \( E_1 + \cdots + E_r = -K_{\mathbf{P}_\Delta} \) and a MPCP desingularization \( \mathbf{X} / \Sigma \to \mathbf{P}_\Delta \), the pullback of the nef-partition is again a nef-partition on \( \mathbf{X} / \Sigma \). By abuse of notation, it is also denoted by \( E_1 + \cdots + E_r = -K_{\mathbf{X}} \).

Let \( S_1 \sqcup \cdots \sqcup S_r = \mathcal{N}(\nabla)(1) \) be a nef-partition representing the dual nef-partition \( \nabla = \nabla_1 + \cdots + \nabla_r \). Then the divisor polytope of \( F_j = \sum_{\rho \in S_j} D_\rho \) is \( \nabla_j \). By the duality construction, we have
\[
\Delta_i = \text{Conv}(S_i \cup \{0\}) \text{ and } \nabla^\vee = \text{Conv}(\Delta_1, \ldots, \Delta_r).
\]

We recall the construction of the Calabi–Yau double covers introduced in [9]. We make the following assumption.

**Assumption** Both \( \mathbf{P}_\Delta \) and \( \mathbf{P}_\nabla \) admit a smooth MPCP desingularization.

Let \( X \to \mathbf{P}_\Delta \) and \( X^\vee \to \mathbf{P}_\nabla \) be smooth MPCP desingularizations. Let \( E_1 + \cdots + E_r = -K_X \) and \( F_1 + \cdots + F_r = -K_{X^\vee} \) be nef-partitions on \( X \) and \( X^\vee \) respectively.

Let \( s_{j,1} \in H^0(X^\vee, \mathcal{O}(F_j)) \) be the global section corresponding to \( 0 \in \nabla_j \) and \( s_{j,2} \in H^0(X^\vee, \mathcal{O}(F_j)) \) be a general section such that their product
\[
s := \prod_{j=1}^r s_{j,1} s_{j,2} \in H^0 \left( X^\vee, \omega_{X^\vee}^{-2} \right)
\]

\( \Delta_0 \)
is a divisor with strictly normal crossings on \( X \). Then \( s \) determines a double cover \( Y^\vee \) of \( X \) branch over \( \{ s = 0 \} \). Moreover, \( Y^\vee \) is Calabi–Yau, i.e., its canonical bundle is trivial and \( H^i (Y^\vee, \mathcal{O}_{Y^\vee}) = 0 \) for \( 1 \leq i \leq n - 1 \).

**Definition 2.2**  The double cover \( Y^\vee \rightarrow X^\vee \) is called the Calabi–Yau double cover from a nef-partition, or a Calabi–Yau fractional complete intersection.

Deforming \( s_{j,2} \) yields a family of singular Calabi–Yau double covers over \( X \)
\[
\mathcal{Y}^\vee \rightarrow V \subset H^0(X^\vee, \mathcal{O}(F_1)) \times \cdots \times H^0(X^\vee, \mathcal{O}(F_r)).
\]
We can apply the above construction to \( X \) which gives rise to another family of singular Calabi–Yau double covers over \( X \)
\[
\mathcal{Y} \rightarrow U \subset H^0(X, \mathcal{O}(E_1)) \times \cdots \times H^0(X, \mathcal{O}(E_r)).
\]
It is conjectured in [9] that

**Conjecture**  \( Y \) and \( Y^\vee \) are mirror.

In this note, we will focus on their \( B \) model, i.e., the families \( Y \rightarrow U \) and \( Y^\vee \rightarrow V \).

### 3 GKZ A-hypergeometric systems

We can study the \( B \) model of the family \( Y^\vee \rightarrow V \) via period integrals and it is known that those period integrals are governed by a certain type of GKZ systems.

To begin, let us recall the definition of GKZ systems and the notion of non-resonance.

**Definition 3.1**  Let \( A = (a_{ij}) \in \text{Mat}_{d \times m}(\mathbb{Z}) \) be an integral matrix and \( \beta = (\beta_i) \in \mathbb{C}^d \). The GKZ system associated with \( A \) and \( \beta \) is the cyclic \( \mathcal{D} \)-module
\[
\mathcal{D} / \mathcal{I}
\]
where \( \mathcal{D} = \mathbb{C}[x_1, \ldots, x_m, \partial_1, \ldots, \partial_m] \) with \( \partial_j \equiv \partial / \partial x_j \) is the Weyl algebra on an affine space \( \mathbb{C}^m \) with coordinates \( x_1, \ldots, x_m \) and \( \mathcal{I} \) is the left ideal generated by

- \( \partial^{v_+} - \partial^{v_-} \), where \( v_\pm \in \mathbb{Z}^m_{\geq 0} \) such that \( A v_+ = A v_- \);
- \( \sum_{j=1}^{m} a_{ij} x_j \partial_j - \beta_i \) for \( i = 1, \ldots, d \).

Let \( A \in \text{Mat}_{d \times m}(\mathbb{Z}) \) be homogeneous, i.e., the column vectors of \( A \) are contained in an affine hyperplane in \( \mathbb{R}^d \), and \( \mathbb{R}_+ A \) denote the cone generated by columns of \( A \) in \( \mathbb{R}^d \). Assume that the columns of \( A \) generate \( \mathbb{Z}^d \) as an abelian group.

**Definition 3.2**  A parameter \( \beta \in \mathbb{C}^d \) is called non-resonant with respect to \( A \) if
\[
\beta \notin \bigcup_{F \subset \mathbb{R}_+ A} \left( \mathbb{C} F + \mathbb{Z}^d \right)
\]
where the union runs through all proper faces of \( \mathbb{R}_+ A \). It is also clear that it suffices to take the union over all facets of \( \mathbb{R}_+ A \).

We will be only interested in the case when \( A \) and \( \beta \) are of special types. To this end, let us fix the following notation for the rest of the note.

\[ \text{Springer} \]
Notation Suppose we are given the data in Sect. 2.3 and let notation be the same as there.

(1) Let \( X = X_\Sigma \) be a smooth MPCP desingularization of \( P_\Delta \) and \( E_1 + \cdots + E_r = -K_X \) be the pullback of the nef-partition \( I_1 \sqcup \cdots \sqcup I_r = N(\Delta)(1) \). We denote by \( J_1 \sqcup \cdots \sqcup J_r = \Sigma(1) \) the corresponding nef-partition on \( X \). Let \( p = |\Sigma(1)| \). We write \( J_k = \{ \rho_{k,1}, \ldots, \rho_{k,m_k} \} \) and put \( \rho_{k,0} := 0 \in N \) for each \( 1 \leq k \leq r \). Then \( p = m_1 + \cdots + m_r \).

(2) Regard \( x_{k,j}, 0 \leq j \leq m_k \), as coordinate functions on the affine space \( W_k^\gamma := \mathbb{C}^{m_k+1} \). Denote by \( (t_1, \ldots, t_n) \) the coordinate on \( T \subset X^\gamma \), the maximal torus of \( X^\gamma \). Consider

\[
    s_{k,2} = \sum_{j=0}^{m_k} x_{k,j}^t k^j .
\]

For \( x = (s_{1,2}, \ldots, s_{r,2}) \in W_1^\gamma \times \cdots \times W_r^\gamma \), we denote by \( U_x \) the region \( T \setminus \bigcup_{k=1}^r \{ s_{k,2} = 0 \} \). Let \( \mathcal{E}_x \) be the local system on \( U_x \) whose monodromy exponent around \( \{ s_{k,2} = 0 \} \) is 1/2 for each \( k \).

We will justify the notation later, i.e., \( s_{k,2} \) can be regard as an element in \( H^0(X^\gamma, \mathcal{O}(F_k)) \) (cf. Corollary 3.7).

(3) Let \( \{ e_1, \ldots, e_r \} \) be the standard basis of \( \mathbb{R}^r \). Set \( \mu_{i,j} := (e_i, \rho_{i,j}) \in \mathbb{Z}^r \times N \). Define matrices

\[
    A_i := \left[ \begin{array}{cccc} 1 & \mu_{i,0} & \cdots & \mu_{i,m_i} \\ \vdots & \ddots & \ddots & \ddots \\ \mu_{i,m_i} & \cdots & 1 \end{array} \right] \in \text{Mat}_{(r+n) \times (m_i+1)}(\mathbb{Z})
\]

and

\[
    A := \left[ A_1 \quad \cdots \quad A_r \right] \in \text{Mat}_{(r+n) \times (r+p)}(\mathbb{Z}).
\]

For convenience, we will label the columns of \( A \) by a double index \((i, j)\), i.e., the \((i, j)\)th column of \( A \) is exactly the vector \( \mu_{i,j} \). Let

\[
    \beta = \left[ \begin{array}{c} -1/2 \\ \vdots \\ -1/2 \\ 0 \end{array} \right] \in \mathbb{Q}^{r+n}.
\]

We remark that by the smoothness of \( X \), the columns of \( A \) generate \( \mathbb{Z}^{r+n} \) as an abelian group.

Throughout this note, we will be only interested in the matrix \( A \) and the parameter \( \beta \) defined in (3) above (\( d = r + n \) and \( m = r + p \) according to our notation). In this case, since \( A \) is homogeneous and contains \((1, \ldots, 1)\) in its row span, \( M_\beta^A \) is regular holonomic.

From now on, let \( A \) and \( \beta \) be the matrix and the parameter defined in Sect. 3 (3).

Definition 3.3 We define affine period integrals to be

\[
    \Pi_\gamma(x) := \int_{\gamma} \frac{1}{s_{1,2}^{1/2} \cdots s_{r,2}^{1/2}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}, \quad (3.2)
\]

where \( \gamma \in H_n(U_x, \mathcal{E}_x) \) and \( s_{k,2} = \sum_{j=0}^{m_k} x_{k,j}^t k^j \in W_i^\gamma \).
It is easy to check that the affine period integrals (3.2) are solutions to $\mathcal{M}^\beta_A$. We have the chain-integral map

$$H_n(U_x, \mathcal{E}_x) \to \text{Sol}^0(\mathcal{M}^\beta_A)_x, \gamma \mapsto \Pi_\gamma(x)$$

(3.3)

Here $\text{Sol}^0(\mathcal{M}^\beta_A) := R^0\text{Hom}_{\mathcal{O}_X}(\mathcal{M}^\beta_A, \mathcal{O}_X)$ is the classical solution functor.

Now we can state our main result in this note.

**Theorem 3.4** The morphism (3.3) is an isomorphism.

Recall the following result in [6].

**Theorem 3.5** [6, §2.10] Let $A$ and $\beta$ be as in Sect. 3 (3). If $\beta$ is non-resonant, then the map (3.3) is an isomorphism for every $x$.

In the rest of this section, we explain how $s_{k,2}$ defined in Sect. 3 (2) is related to an element in $H^0(X^\vee, \mathcal{O}(F_k))$ and therefore justify the notation $s_{k,2}$.

**Lemma 3.6** There exists an $1 \leq i \leq r$ such that $V(F_v) \subseteq I_i$.

**Proof** Since $I_1 \sqcup \cdots \sqcup I_r = \mathcal{N}(\Delta)(1)$ is a nef-partition representing the Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$, the Weil divisor

$$\sum_{\rho \in I_i} D_\rho$$

is Cartier on $X_{\mathcal{N}(\Delta)}$ and it follows that for any $\tau \in \mathcal{N}(\Delta)$ there exists an $m_\tau \in M$ such that

$$\begin{cases} 
\langle m_\tau, \rho \rangle = -1, & \text{if } \rho \in \tau(1) \cap I_i \\
\langle m_\tau, \rho \rangle = 0, & \text{if } \rho \in \tau(1) \text{ but } \rho \notin I_i.
\end{cases}$$

Note that $F_v$ is contained in a facet of $\Delta^\vee$. We may assume that there exists an $m \in M$ such that

$$F_v \subseteq \{ n \in N^\vee \mid \langle m, n \rangle = -1 \}.$$

On one hand, if $v = \sum_{v \in V(F_v)} c_v v$ with $c_v > 0$, we have

$$-1 = \langle m, v \rangle = \sum_{v \in V(F_v)} c_v \langle m, v \rangle = - \sum_{v \in V(F_v)} c_v.$$

On the other hand, suppose $V(F_v) \not\subseteq I_i$ for all $i$. Let $\tau$ be the cone over $F_v$. By the discussion above, for each $i$, there exists an $m^i_\tau \in M$ such that

$$\begin{cases} 
\langle m^i_\tau, v \rangle = -1, & \text{if } v \in \tau(1) \cap I_i \\
\langle m^i_\tau, v \rangle = 0, & \text{if } v \in \tau(1) \text{ but } v \notin I_i.
\end{cases}$$

By hypothesis, we can always find distinct $i, j \in \{1, \ldots, r\}$ so that $V(F_v) \cap I_i \neq \emptyset$ and $V(F_v) \cap I_j \neq \emptyset$. Then we have

$$0 > \langle m^i_\tau, v \rangle = \sum_{v \in V(F_v)} c_v \langle m^i_\tau, v \rangle > \sum_{v \in V(F_v)} c_v = -1.$$
Here the first inequality holds since $V(F_r) \cap I_i \neq \emptyset$, and the second one holds since $V(F_r) \cap I_j \neq \emptyset$. This contradicts to the fact that $\langle m_r, v \rangle \in \mathbb{Z}$. 

Let $J_1 \sqcup \cdots \sqcup J_r = \Sigma(1)$ be the nef-partition on $X$ induced by the pullback of $I_1 \sqcup \cdots \sqcup I_r = \mathcal{N}(\Delta(1))$. We immediately have the following corollary.

**Corollary 3.7** We have $J_i = \nabla_i \cap N \setminus \{0\}$. Therefore,

$$H^0(X^\vee, \mathcal{O}(F_k)) \cong \bigoplus_{0 \leq j \leq m_k} \mathbb{C}r^{p_k,j}$$

and $W_i^\vee$ in Sect. 3 (2) is identified with $H^0(X^\vee, \mathcal{O}(F_k))$. In particular,

$$N \cap \Delta^\vee = \bigcup_{i=1}^r (\nabla_i \cap N).$$

In other words, any integral point in $\Delta^\vee$ belongs to $\nabla_i$ for some $1 \leq i \leq r$.

Now we explain how the cycle we integrate over in the period integrals is related to a cycle in $H_n(Y^\vee, \mathbb{C})$. Recall that $X^\vee \to \mathbb{P}^r_Y$ is a smooth $\mathbb{C}$ desingularization. Pick $x = (s_1, 2, \ldots, s_r, 2) \in W_1^\vee \times \cdots \times W_r^\vee$ and let $\pi: Y^\vee \to X^\vee$ be the double cover representing $x$. Denote by $R_x$ the branch divisor of $\pi$. Then $U_x = X^\vee \setminus R_x$. Put $V_x = Y^\vee \setminus R_x$ and let $i: R_x \to Y^\vee$ and $j: V_x \to Y^\vee$. We have a distinguished triangle (indeed a short exact sequence)

$$j_i j^{-1} C_{Y^\vee} \to C_{Y^\vee} \to i_* i^{-1} C_{Y^\vee} \to .$$

Applying the functor $R \mathcal{F} = R \mathcal{F}$ to (3.4) and the base change theorem from the commutative diagram

$$\begin{array}{ccc}
V_x & \xrightarrow{j} & Y^\vee \\
\downarrow \pi & & \downarrow \pi \\
U_x & \xrightarrow{j} & X^\vee
\end{array}$$

we arrive at a triangle

$$j_! R \mathcal{F}_{\mathcal{C}_{V_x}} \to R \mathcal{F}_{\mathcal{C}_{Y^\vee}} \to i_* \mathcal{C}_{R_x} \to .$$

$R \mathcal{F}_{\mathcal{C}_{V_x}}$ (resp. $R \mathcal{F}_{\mathcal{C}_{Y^\vee}}$) is decomposed into eigensubsheaves $\mathcal{C}_{U_x} \oplus \mathcal{E}_x$ (resp. $\mathcal{C}_{X^\vee} \oplus \mathcal{F}$) according to the $\mathbb{Z}_2$-action coming from the double cover $Y^\vee \to X^\vee$ structure. Here $\mathcal{E}_x$ (resp. $\mathcal{F}$) is the eigensubsheaf of $R \mathcal{F}_{\mathcal{C}_{V_x}}$ (resp. $R \mathcal{F}_{\mathcal{C}_{Y^\vee}}$) associated to the non-trivial character $\mathbb{Z}_2 \to \mathbb{C}^\times$. Moreover, $\mathcal{F} \cong j_! \mathcal{E}_x$. Taking $R \Gamma_2(X, -)$, we see that

$$H^n_0(Y^\vee, \mathbb{C}) \cong H^n_c(X^\vee, \mathbb{C}) \oplus H^n_0(U_x, \mathcal{E}_x).$$

Combined with Poincaré duality, since the cycles in the periods (3.2) belongs to $H^n_0(U_x, \mathcal{E}_x) \subset H^n_0(Y^\vee, \mathbb{C}) \cong H_n(Y^\vee, \mathbb{C})$, we conclude that the affine period integrals defined in Definition 3.3 are exactly the restriction of all possibly non-trivial period integrals of $Y^\vee \to V$ to $T$. Note that when $n$ is odd, we have

$$H_n(Y^\vee, \mathbb{C}) \cong H^n_c(Y^\vee, \mathbb{C}) \cong H^n_0(U_x, \mathcal{E}_x) \cong H_n(U_x, \mathcal{E}_x).$$

**Corollary 3.8** The affine period integrals are exactly the restriction of all possibly non-trivial period integrals of $Y^\vee \to V$ to $T$. 
4 $\beta$ is non-resonant

We continue to assume that $A$ and $\beta$ are the same as in Sect. 3 (3) in this section. In this section, we show that in the present case the exponent $\beta$ is non-resonant with respect to $A$. Combined with Theorem 3.5, we obtain Theorem 3.4.

Let us briefly summarize the idea. In our situation, the columns of the matrix $A$ correspond to (the primitive generator of) 1-cones in the fan $\Theta$ defining the total space $\text{Tot}(\mathcal{O}_X(-E_1) \oplus \cdots \oplus \mathcal{O}_X(-E_r))$ of the vector bundle. Since each $E_i$ is numerically effective, the convexity of the support function guarantees that the cone $\text{Cone}(A)$ generated by the columns of $A$ is equal to the support of $\Theta$. This enables us to characterize the facets of $\text{Cone}(A)$ and hence to verify that $\beta$ is nonresonant. The same observation also allows to compute the volume of $A$, which turns out to be the Euler characteristic of $X$.

The defining fan $\Sigma$ of $X$ is a MPCP desingularization of $\mathcal{P}_\Delta$, the toric variety defined by $F(\Delta^\vee)$, the face fan of the Cayley polytope $\text{Conv}(\nabla_1, \ldots, \nabla_r) = \Delta^\vee$ (or equivalently a MPCP desingularization of $N(\Delta)$, the normal fan of $\Delta$). Recall that by Batyrev–Borisov’s duality construction, if $I_1 \cup \cdots \cup I_r = N(\Delta)(1)$ is the nef-partition on $\mathcal{P}_\Delta$ representing the Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$, then $\nabla_i = \text{Conv}(I_i \cup \{0\})$.

For each $\sigma \in \Sigma(n)$, let

$$\hat{\sigma} := \text{Cone}(\{\hat{\rho} \mid \rho \in \sigma(1) \cup \{e_1 \times 0, \ldots, e_r \times 0\})$$

and

$$\text{Poly}(\hat{\sigma}) := \text{Conv}(\{\hat{\rho} \mid \rho \in \sigma(1) \cup \{e_1 \times 0, \ldots, e_r \times 0\})$$

where $\hat{\rho} = (e_i, \rho)$ if $\rho \in J_i$. Under our notation in Sect. 3, we have $\hat{\rho}_{i,j} = \mu_{i,j}$. Then $\text{Poly}(\hat{\sigma})$ is a convex polytope in $\mathbb{R}^r \times N_\mathbb{R}$. Moreover, owing to our hypothesis on $X$, $\text{Poly}(\hat{\sigma})$ is indeed a simplex (its normalized volume is 1). Recall that

$$E_i := \sum_{\rho \in I_i} D_\rho.$$

is a numerically effective divisor on $X$; the support function $\varphi_i$ of $E_i$ is convex. We have the following observation.

**Lemma 4.1** Let $\sigma \in \Sigma(n)$ and $\rho_1, \ldots, \rho_n$ be the primitive generator of the 1-cones in $\sigma$. For non-negative scalars $u_1, \ldots, u_n$, we have

$$\sum_{k=1}^{n} u_k \hat{\rho}_k = \begin{bmatrix} -\varphi_1 \left( \sum_{k=1}^{n} u_k \rho_k \right) \\ \vdots \\ -\varphi_r \left( \sum_{k=1}^{n} u_k \rho_k \right) \\ \sum_{k=1}^{n} u_k \rho_k \end{bmatrix} \in \mathbb{R}^r \times N_\mathbb{R}. \quad (4.1)$$

**Proof** Let $m^i_\sigma \in M$ be the element defining the support function $\varphi_i$ on $\sigma$, i.e., $\varphi_i(v) = \langle v, m^i_\sigma \rangle$ for $v \in \sigma$. Then

$$\varphi_i(\rho_j) = \langle m^i_\sigma, \rho_j \rangle = \begin{cases} -1 & \text{if } \rho_j \in I_i \\ 0 & \text{if } \rho_j \notin I_i \end{cases}. \quad (4.2)$$

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Said differently, we have

$$\hat{\rho}_j = \begin{bmatrix} -\varphi_1(\rho_j) \\ \vdots \\ -\varphi_r(\rho_j) \\ \rho_j \end{bmatrix} \in \mathbb{Z}^r \times N.$$  \hfill (4.3)

Since $\varphi_i$ is linear on $\sigma$ and $u_k \geq 0$ for all $k$, the result follows.

**Proposition 4.2** We have

$$\text{Conv}(A \cup \{0\}) = \bigcup_{\sigma \in \Sigma(n)} \text{Poly}(\hat{\sigma}).$$  \hfill (4.4)

**Proof** The inclusion “$\supseteq$” is clear. To establish the reverse inclusion, it suffices to show that the right hand side of (4.4), which is denoted by $B$ in the sequel, is convex. Pick $\nu_0$ and $\nu_1$ from $B$. We shall prove that the line segment connecting $\nu_0$ and $\nu_1$ is also contained in $B$.

We may assume $\nu_j \in \text{Poly}(\hat{\sigma}_j)$ for some $\sigma_j \in \Sigma(n)$. Denote by $\rho^{1}_{j}, \ldots, \rho^{r}_{j}$ the primitive generator of the 1-cones contained in $\sigma_j$. Write

$$\nu_j = \sum_{k=1}^{n} c_{j,k} \hat{\rho}_{k}^{j} + \sum_{l=1}^{r} d_{j,l} (e_{l} \times 0)$$

with $c_{j,k}, d_{j,l} \geq 0$ and $\sum_{k=1}^{n} c_{j,k} + \sum_{l=1}^{r} d_{j,l} \leq 1$.

Our goal is to prove that $\nu_t := t\nu_1 + (1-t)\nu_0 \in B$ for all $t \in [0, 1]$. First we observe that $B$ projects onto the Cayley polytope $\text{Conv}(\nabla_1, \ldots, \nabla_r) = \Delta^\vee$ under the canonical projection $\pi: \mathbb{R}^r \times N_{\mathbb{R}} \to N_{\mathbb{R}}$; the line segment $t\nu_1 + (1-t)\nu_0$ projects onto a line segment in $\Delta^\vee$. Fix $t \in [0, 1]$ and put $\nu \equiv \nu_t$. Let $\sigma \in \Sigma(n)$ be a maximal cone containing $\pi(\nu)$ and $\rho^{1}_{1}, \ldots, \rho^{r}_{1}$ be the primitive generator of the 1-cones in $\sigma$. We shall now prove that $\nu \in \text{Poly}(\hat{\sigma})$, i.e., we have to prove that

(1) There are non-negative scalars $c_1, \ldots, c_n$ and $d_1, \ldots, d_r$ such that

$$\nu = \sum_{k=1}^{n} c_k \hat{\rho}_k + \sum_{l=1}^{r} d_l (e_l \times 0);$$

(2) $\sum_{k=1}^{n} c_k + \sum_{l=1}^{r} d_l \leq 1$.

Let us prove (1). Since $\pi(\nu) \in \Delta^\vee \cap \sigma$, there are non-negative scalars $c_1, \ldots, c_n$ such that

$$\pi(\nu) = \sum_{k=1}^{n} c_k \rho_k.$$  

By Lemma 4.1,

$$\nu - \sum_{k=1}^{n} c_k \hat{\rho}_k = \nu - \begin{bmatrix} -\varphi_1 \left( \sum_{k=1}^{n} c_k \rho_k \right) \\ \vdots \\ -\varphi_r \left( \sum_{k=1}^{n} c_k \rho_k \right) \\ \sum_{k=1}^{n} c_k \rho_k \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_r \\ 0 \end{bmatrix}$$
with \( d_i = -t\varphi_i(\sum_{k=1}^{n} c_{1,k}\rho_k^1) - (1-t)\varphi_i(\sum_{k=1}^{n} c_{0,k}\rho_k^0) + \varphi_i(\sum_{k=1}^{n} c_k\rho_k) \). Moreover, \( d_i \geq 0 \) since \( \varpi(v) = tv_1 + (1-t)v_0 \) and \( \varphi_i \) is convex.

Now we prove (2). Since \( J_1 \cup \cdots \cup J_r = \Sigma(1) \) is a nef-partition, for each \( 1 \leq j \leq n \) there exists exactly one \( i \in \{1, \ldots, r\} \) satisfying \( \varphi_i(\rho_j) = -1 \). Therefore

\[
\sum_{k=1}^{n} c_k + \sum_{l=1}^{r} d_l = \sum_{l=1}^{r} \left( -\varphi_i \left( \sum_{k=1}^{n} c_k\rho_k \right) + d_l \right) = \langle n, v \rangle,
\]

where

\[
n = \begin{bmatrix}
1 \\
\vdots \\
1 \\
0
\end{bmatrix} \in \mathbb{Z}^r \times M.
\]

Now \( \langle n, v_j \rangle = \sum_{k=1}^{n} c_{j,k} + \sum_{l=1}^{r} d_{j,l} \leq 1 \) for \( j = 0, 1 \). It follows from the linearity that \( \langle n, v \rangle = \langle n, tv_1 + (1-t)v_0 \rangle \leq 1 \). This establishes (2). \( \square \)

**Corollary 4.3** The normalized volume of \( \text{Conv}(A \cup \{0\}) \) is equal to \( |\Sigma(n)| \), the number of maximal cones in \( \Sigma \). Therefore, the holonomic rank of \( M_A^B \) is \( |\Sigma(n)| \).

**Proof** Since \( X \) is smooth by our hypothesis, each cone \( \sigma \) has normalized volume 1 and so does \( \hat{\sigma} \). \( \square \)

Next we examine the facets of \( \mathbb{R}_+A \).

**Proposition 4.4**

\[
\mathbb{R}_+A = \bigcup_{\sigma \in \Sigma(n)} \hat{\sigma}.
\]

**Proof** The direction “\( \supseteq \)” is clear. Now we prove the converse. Let

\[
v := \sum_{\rho \in \Sigma(1)} c_{\rho}\hat{\rho} + \sum_{l=1}^{r} d_l(e_l \times \mathbf{0}) \quad \text{with } c_{\rho}, d_l \geq 0.
\]

Here by abuse of notation we denote the primitive generator of \( \rho \in \Sigma(1) \) by \( \rho \).

Since \( \Sigma \) is proper, the vector \( \sum_{\rho \in \Sigma(1)} c_{\rho}\rho \) is contained in some maximal cone \( \sigma \in \Sigma(n) \). Let \( \rho_1, \ldots, \rho_n \) be the primitive generator of the 1-cones in \( \sigma \) as before. There are non-negative scalars \( c_1, \ldots, c_n \) such that

\[
\sum_{\rho \in \Sigma(1)} c_{\rho}\rho = \sum_{k=1}^{n} c_k\rho_k.
\]

Now we compare \( \sum_{\rho \in \Sigma(1)} c_{\rho}\hat{\rho} \) with \( \sum_{k=1}^{n} c_k\hat{\rho}_k \). Notice that by the convexity of \( \varphi_i \)

\[
\sum_{\rho \in \Sigma(1)} -c_{\rho}\varphi_i(\rho) \geq -\varphi_i \left( \sum_{\rho \in \Sigma(1)} c_{\rho}\rho \right) = -\varphi_i \left( \sum_{k=1}^{n} c_k\rho_k \right).
\]

\( \square \) Springer
It follows that
\[ \sum_{\rho \in \Sigma(n)} c_\rho \hat{\rho} - \sum_{k=1}^n c_k \hat{\rho}_k \in \text{Cone}([e_1 \times 0, \ldots, e_r \times 0]). \]
Consequently, \( v \in \hat{\sigma} \) as desired. \( \square \)

Let \( F \) be a facet of \( \mathbb{R}_+ A \). By Proposition 4.4, \( F \) must be a facet of \( \hat{\sigma} \) for some \( \sigma \in \Sigma(n) \). Let again \( \rho_1, \ldots, \rho_n \) be the primitive generator of the 1-cones in \( \sigma \). Since \( \hat{\sigma} \) is simplicial, \( F \) must contain all but one vectors in
\[ \{ \hat{\rho}_1, \ldots, \hat{\rho}_n, e_1 \times 0, \ldots, e_r \times 0 \}. \]
We claim that \( F \) can not contain all \( e_i \times 0 \). Otherwise, if \( F \) is defined by \( (a, m) \in \mathbb{R}^r \times M_{\mathbb{R}} \) (i.e., \( F = \mathbb{R}_+ A \cap \{(b, n) \in \mathbb{R}^r \times N_{\mathbb{R}} \mid (a, m), (b, n) = 0\} \), we must have \( a = 0 \). Because \( \Sigma \) is proper, \( \mathbb{R}_+ A \) can not fall into the closed half-space defined by \( (0, m) \) and \( F \) can not be a face. This is a contradiction.

Assume that \( e_j \times 0 \) is omitted. It follows that if \( F \) is defined by \( (a, m) \in \mathbb{R}^r \times M_{\mathbb{R}} \) as above, we have \( a_i = 0 \) for \( i \neq j \). Moreover, we may assume \( a_j = 1 \) (and hence \( a = e_j \)). Then \( m \) satisfies the equations \( \langle \hat{\rho}_k, (e_j, m) \rangle = 0 \), i.e.,
\[ \begin{cases} 
\langle m, \rho_k \rangle = 0 & \text{if } \rho_k \notin I_j \\
\langle m, \rho_k \rangle = -1 & \text{if } \rho_k \in I_j.
\end{cases} \]
We see that \( m = m_j^\sigma \), the Cartier data of \( E_j \) on \( \sigma \). In particular, \( m \in M \).

**Theorem 4.5** Let \( A \) and \( \beta \) be as in Sect. 3 (3). Then \( \beta \) is non-resonant, i.e.,
\[ \beta \notin \bigcup_{F \subset \mathbb{R}_+ A}(CF + \mathbb{Z}^{r+n}) \]
where the union runs through all the proper faces of \( \mathbb{R}_+ A \).

**Proof** Suppose \( \beta \in CF + \mathbb{Z}^{r+n} \) for some facet \( F \). Write \( \beta = f + z \) with \( f \in CF \) and \( z \in \mathbb{Z}^{r+n} \). According to our discussion above, \( F \) is defined by an element of the form \( (e_j, m) \) and \( m \in M \). But then
\[ \frac{1}{2} = \langle (e_j, m), \beta \rangle = \langle (e_j, m), f + z \rangle = \langle (e_j, m), z \rangle \in \mathbb{Z} \]
which is absurd. \( \square \)

**Proof of Theorem 3.4** By Theorem 4.5, \( \beta \) is non-resonant with respect to \( A \). Theorem 3.4 is now a direct consequence of Theorem 3.5. \( \square \)

**Corollary 4.6** \( \mathcal{M}_A^\beta \) admits a rank one point, i.e., there is a \( x \in W_1^\vee \times \cdots \times W_r^\vee \) such that \( \text{Sol}^0(\mathcal{M}_A^\beta) \) is 1-dimensional.

**Proof** It suffices to pick \( x = (1, \ldots, 1) \in W_1^\vee \times \cdots \times W_r^\vee \). In which case, \( U_x = T \) and \( \mathcal{E}_x = \mathbb{C} \) is the constant sheaf. Therefore, \( H_n(U_x, \mathcal{E}_x) \cong \mathbb{C} \). \( \square \)

**Remark 4.7** It is clear from the proof that the assumption made in Sect. 2.3 can be weakened. We only need to assume that \( P_A \) admits a MPCP desingularization \( X_\Sigma \) such that \( \Sigma(1) \) generates \( \mathbb{Z}^n \) as an abelian group; this assumption allows us to apply the result in [6]. Under this weakened assumption, all the results in Sect. 3 (except for Corollary 4.3) are still valid and the chain-integral map (3.3) is an isomorphism.
Using the generalized Frobenius method in [14], we can write down explicitly all the power series solutions to \( \mathcal{M}_A^\beta \).

Put \( D_{i,j} = D_{\rho_{i,j}} \) for \( 1 \leq i \leq r \) and \( 1 \leq j \leq m_i \) and \( D_{i,0} := -\sum_{j=1}^{m_i} D_{i,j} \). We can define a cohomology-valued series

\[
B^\alpha_X(x) := \left( \sum_{\ell \in \mathbb{N}(X)^r \cap \ker(A)} \mathcal{O}_\ell^\alpha x^{\ell + \alpha} \right) \exp \left( \sum_{i=1}^{r} \sum_{j=0}^{m_i} (\log x_{i,j}) D_{i,j} \right).
\]

Here we think of \( A \) as a linear map \( \mathbb{Z}^{r+p} \to \mathbb{Z}^{r+p} \) and identify the Mori cone \( \mathbb{N}(X) \) of \( X \) with a cone in \( \mathbb{R}^{r+p} \). Similar to Sect. 3 (3), the components of elements \( \ell \in \ker(A) \) are labeled by \( (i, j) \). Finally

\[
\mathcal{O}_\ell^\alpha := \frac{\prod_{i=1}^{r}(1)_{\ell_{i,0}} \Gamma(-D_{i,0} - \ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^{r} \Gamma(-\alpha_{i,0}) \prod_{i=1}^{r} \prod_{j=1}^{m_i} \Gamma(D_{i,j} + \ell_{i,j} + \alpha_{i,j} + 1)}.
\]

and \( \alpha = (\alpha_{i,j}) \in \mathbb{Q}^{r+p} \) with \( \alpha_{i,0} = -1/2 \) and \( \alpha_{i,j} = 0 \) for \( 1 \leq j \leq m_i \). We refer the reader to [14, §2] for detailed explanations.

The element \( B^\alpha_X(x) \) is understood as a \( \mathcal{H}^*(X, \mathbb{C}) \)-valued function. Note that \( \dim \mathcal{H}^*(X, \mathbb{C}) = |\Sigma(n)| \) is equal to the normalized volume of \( \text{Conv}(A \cup \{0\}) \) by Corollary 4.3 and it is shown that the coefficients of \( B^\alpha_X(x) \) form a basis of the set of solutions to \( \mathcal{M}_A^\beta \) (cf. [14, Corollary 3.4]). Combined with Theorem 1, we obtain

**Corollary 4.8** The coefficients of the vector-valued function \( B^\alpha_X(x) \) form a basis of the set of period integrals for \( \mathcal{Y}^\vee \to \mathcal{V} \).

**Example 4.9** We can apply our results to cyclic covers of \( X^\vee = \mathbb{P}^3 \) branch over eight hyperplanes in general position. It is known that such a Calabi–Yau double cover \( Y^\vee \) admits a crepant resolution \( \tilde{Y}^\vee \to Y^\vee \) and its middle cohomology has Hodge numbers \( (1, 9, 9, 1) \). Moreover, it is proven that the moduli space of the complex deformation of \( \tilde{Y}^\vee \) can not be embedded as a Zariski open subset of a locally hermitian symmetric domain by the period map [7].

In this case, we have

\[
\begin{align*}
\nabla_1 &= \text{Conv}(\{0, (1, 0, 0), (0, 1, 0), (0, 0, 1)\}), \\
\nabla_2 &= \text{Conv}(\{0, (-1, 0, 0), (-1, 1, 0), (-1, 0, 1)\}), \\
\nabla_3 &= \text{Conv}(\{0, (0, -1, 0), (1, -1, 0), (0, -1, 1)\}), \\
\nabla_4 &= \text{Conv}(\{0, (0, 0, -1), (1, 0, -1), (0, 1, -1)\})
\end{align*}
\]

\( \nabla = \nabla_1 + \cdots + \nabla_4 \). One can check that \( \mathbb{P}_\nabla = \mathbb{P}^3 = X^\vee \). Now it is straightforward to check that \( \mathbb{P}_\Delta \) with \( \Delta = \text{Conv}(\nabla_1, \ldots, \nabla_4) \) admits a smooth MPCP desingularization.
Applying our results, the solutions to $\mathcal{M}_A^\beta$ with

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1/2 \\
-1/2 \\
-1/2 \\
0 \\
0 \\
0
\end{bmatrix},$$

$\beta = \begin{bmatrix}
-1/2 \\
-1/2 \\
-1/2 \\
0 \\
0 \\
0
\end{bmatrix}$

are precisely the period integrals of the family $Y^\vee \to V$ of Calabi–Yau fractional complete intersections and $B^a_X(x)$ gives the power series expansion of the period integrals.

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