We establish exact, dimension-dependent, spatio-temporal, uniform and local moduli of continuity for (1) the fourth order L-Kuramoto-Sivashinsky (L-KS) SPDEs and for (2) the time-fractional stochastic partial integro-differential equations (SPIDEs), driven by space-time white noise in one-to-three dimensional space. Both classes were introduced—with Brownian-time-type kernel formulations—by Allouba in a series of articles starting in 2006, where he presented class (2) in its rigorous stochastic integral equations form. He proved existence, uniqueness, and sharp spatio-temporal Hölder regularity for the above two classes of equations in $d \in \{1, 2, 3\}$. We show that both classes are $(1/2)^{-\epsilon}$ Hölder continuously differentiable in space when $d = 1$, and we give the exact uniform and local moduli of continuity for the gradient in both cases. This is unprecedented for SPDEs driven by space-time white noise. Our results on exact moduli show that the half-derivative SPIDE is a critical case. It signals the onset of rougher modulus regularity in space than both time-fractional SPIDEs with time-derivatives of order $< 1/2$ and L-KS SPDEs. This is despite the fact that they all have identical spatial Hölder regularity, as shown earlier by Allouba. Moreover, we show that the temporal laws governing (1) and (2) are fundamentally different. We relate L-KS SPDEs to the Houdré-Villa bifractional Brownian motion, yielding a Chung-type law of the iterated logarithm for these SPDEs. We use the underlying explicit kernels and spectral/harmonic analysis to prove our results. On one hand, this work builds on the recent works on delicate sample path properties of Gaussian random fields. On the other hand, it builds on and complements Allouba’s earlier works on (1) and (2). Similar regularity results hold for Allen-Cahn nonlinear members of (1) and (2) on compacts via change of measure.
1. Introduction, statement of results, and preliminaries

1.1. Two sides of the Brownian-time coin. We delve into delicate regularity properties of paths of fourth order pattern formation stochastic PDEs (SPDEs) and time-fractional slow diffusion stochastic partial integro-differential equations (SPIDEs). The fundamental kernels associated with the deterministic versions of these two different classes are both built on the Brownian-time processes in \([12, 7, 6]\) and extensions thereof. We thus think of these two classes of equations

\[\text{References}\]

\[\text{Appendix A. Glossary of frequently used acronyms and notations}\]

\[\text{1. Introduction, statement of results, and preliminaries}\]

\[\text{1.1. Two sides of the Brownian-time coin.}\]
as “two sides of the Brownian-time coin”. It is therefore often useful and efficient to study both simultaneously and compare and contrast their various properties. In this article, we unveil a rather detailed set of results giving the exact dimension-dependent uniform and local modulus of continuity, in time and space, for two important classes of stochastic equations:

(1) the fourth order L-Kuramoto-Sivashinsky (L-KS) SPDEs connected to pattern formation phenomena accompanying the appearance of turbulence (see [1, 5, 6] for the L-KS class and for its connection to many classical and new examples of pattern formation deterministic and stochastic PDEs, and see [18, 42] for classical examples of stochastic and deterministic pattern formation PDEs; and

(2) time-fractional SPIDEs connected to slow diffusion or diffusion in material with memory (see [12, 7, 17, 19, 26, 27, 31, 32] for connected PDEs in the deterministic setting and see [2, 3, 5] for the associated stochastic integral equations (SIEs), followed later by the articles [14, 16, 35], in the stochastic setting).

We also characterize the temporal laws for these two classes of equations. More specifically, we prove our results on the exact uniform and local moduli of continuity for the canonical equations

\begin{align}
\begin{cases}
\frac{\partial U}{\partial t} = -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 U + \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
U(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
\end{align}

and

\begin{align}
\begin{cases}
C^{\partial^\beta_t} U_\beta = \frac{1}{2} \Delta U_\beta + I^\beta_t \left[ \frac{\partial^{d+1} W}{\partial t \partial x} \right], & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
U_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
\end{align}

where $\mathbb{R}_+ = (0, \infty)$; $(\varepsilon, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}$ is a pair of parameters; $\beta \in (0, 1/2]$; the noise term $\partial^{d+1} W/\partial t \partial x$ is the space-time white noise corresponding to the real-valued Brownian sheet $W$ on $\mathbb{R}_+ \times \mathbb{R}^d$, $d = 1, 2, 3$; the time fractional derivative of order $\beta$, $C^{\partial^\beta_t}$, is the Caputo fractional operator

\begin{align}
C^{\partial^\beta_t} f(t) := \begin{cases}
\frac{1}{\Gamma(1 - \beta)} \int_0^t f'(\tau) (t - \tau)^{\beta - 1} d\tau, & \text{if } 0 < \beta < 1; \\
\frac{d}{dt} f(t), & \text{if } \beta = 1,
\end{cases}
\end{align}

and the time fractional integral of order $\alpha$, $I^{\alpha}_t$, is the Riemann-Liouville fractional integral of order $\alpha$:

\begin{align}
I^{\alpha}_t f := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t - s)^{1-\alpha}} ds, \text{ for } t > 0 \text{ and } \alpha > 0,
\end{align}

In addition to equations (1.1) and (1.2), the general L-KS SPDEs and time-fractional SPIDEs classes include many nonlinear equations (both well known as well as new). We refer the reader to Theorem 5.1 and Sections 1.2 and 1.5 below for more on that. The constants in (1.1) and (1.2) can easily be changed by scaling. We will alternate freely between the notations $\partial_u f(u, v)$ and $\partial f(u, v)/\partial u$.

As in Walsh [47], we treat space-time white noise as a continuous orthogonal martingale measure, and we denote it by $\mathcal{W}$.\footnote{As in Walsh [47], we treat space-time white noise as a continuous orthogonal martingale measure, and we denote it by $\mathcal{W}$.}
and \( I_0^p = \text{Id} \), the identity operator. The initial data \( u_0 \) here is assumed Borel measurable, deterministic, and suitably regular. For convenience and for the sake of comparing (1.1) to (1.2), when \( \beta \in \{1/2^k; k \in \mathbb{N}\} \), we assume throughout this article that there is a \( 0 < \gamma \leq 1 \) such that

\[
\begin{cases}
  (a) \ u_0 \in C^{2,\gamma}_b(\mathbb{R}^d; \mathbb{R}), & \text{for (1.1)}; \\
  (b) \ u_0 \in C^{2+\gamma}_b(\mathbb{R}^d; \mathbb{R}), & \text{for (1.2), } 2^{k-1} < \beta^{-1} \leq 2^k, k \in \mathbb{N}.
\end{cases}
\]

Of course, equations (1.1) and (1.2) are formal (and nonrigorous) equations. Their rigorous formulations, which we work with in this article, are given in mild form as kernel stochastic integral equations (SIEs). These SIEs were first introduced and treated by Allouba [5, 3, 2, 1], with their genesis in [12, 7, 6]. We give them below in Section 1.5, along with the relevant details.

The results here build on the following works: (1) Allouba [5, 3, 2, 1] who established the existence/uniqueness as well as sharp dimension-dependent \( L^p \) and H"older regularity of the linear and nonlinear noise versions of (1.1) and (1.2) (he presented and treated the later in its stochastic integral equation form); and (2) Xiao [49, 50]; Meerschaert, Wang, and Xiao [34]; Wu and Xiao [45]; Xiao and Xue [51] who established several delicate analytic and geometric path properties of Gaussian processes and random fields (see also the related works in [43, 44, 24, 25]).

### 1.2. Five questions.

In a series of articles [5, 3, 2], Allouba introduced and investigated the regularity of the rigorous kernel stochastic integral form\(^5\) of the formal time-fractional SPIDEs in (1.2) with diffusion coefficient \( a \):

\[
\begin{cases}
  C \partial_t^\beta U_\beta = \frac{1}{2} \Delta U_\beta + I_1^{1-\beta} a(U_\beta) \frac{\partial^{d+1}W}{\partial t \partial x}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
  U_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d.
\end{cases}
\]

He called these stochastic integral equations time-fractional and Brownian-time Brownian motion (\( \beta = 1/2 \)) SIEs. Starting with the 2006 article [5], he proved the existence of a pathwise unique, continuous, and \( L^p \) bounded random field solution on \( (\mathbb{R}_+ \times \mathbb{R}^d), d = 1, 2, 3, \) to the stochastic integral equation formulation of (1.2) when \( \beta = 1/2 \) (the Brownian-time process or Brownian-time Brownian motion (BTBM) SIE). He proved in [5] that, in the case \( a = 1 \), the solution \( U \) satisfies the \( L^p \) bound

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}[U(t, x)]^{2p} \leq C \left[ 1 + t^{\frac{4-d}{2}} \right]; \quad t > 0, d = 1, 2, 3, \text{ and } p \geq 1.
\]

He further proved in [3, 2] that, under a Lipschitz assumption on the nonlinear \( a \), there is a pathwise Hölder continuous solution \( U_\beta \) to the SIE formulation of (1.6) such that, for any arbitrary \( T > 0 \) and \( T = [0, T] \),

\[
U_\beta \in H^{\frac{2d-1}{4d}\gamma^{-d}}(T \times \mathbb{R}^d; \mathbb{R})\wedge \frac{1}{2} \wedge \Gamma\{(0, \gamma^d) \text{ and } (0, \gamma^d)\},
\]

for every \( d = 1, 2, 3 \), \( \beta^{-1} \in \{2^k; k \in \mathbb{N}\} \); where \( H^{\gamma_1, \gamma_2}(T \times \mathbb{R}^d; \mathbb{R}) \) is the space of real-valued locally Hölder functions on \( T \times \mathbb{R}^d \) whose time and space Hölder exponents are in \( (0, \gamma_1) \) and \( (0, \gamma_2) \), respectively. He also proved in [3, 2], under just

\[\text{This is for convenience and we may relax these conditions à la those in [11].}\]

\[\text{See Section 1.5 below for the rigorous kernel or mild stochastic integral equation formulation for both L-KS SPDEs and time-fractional SPIDEs.}\]
continuity (no Lipschitz assumption) and linear growth conditions on the nonlinear $a$, the existence of lattice limits solutions to the SIE corresponding to (1.10), with the same Hölder regularity as in (1.8). In [5, 1], motivated by [6], he introduced and gave the explicit kernel stochastic integral equation formulation for a large class of stochastic equations he called L-KS SPDEs. This class includes stochastic versions of prominent nonlinear equations like the Swift-Hohenberg PDE, variants of the Kuramoto-Sivashinsky PDE, as well as many new ones (see [1]). He established in [1], among other things, the existence of a pathwise unique solution $U$ to the nonlinear L-KS SPDE (1.1) with Lipschitz diffusion coefficient $a$:

\begin{equation}
\begin{aligned}
&\frac{\partial U}{\partial t} = -\frac{\gamma}{2} (\Delta + 2\partial)^2 U + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
&U(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{aligned}
\end{equation}

with the same Hölder regularity for $U$ as the $\beta = 1/2$ case in (1.6) (BTBM SPIDE or SIE$^6$) obtained by plugging $\beta = 1/2$ in (1.8). In addition, the articles [3, 2, 1] are the first to obtain solutions to space-time white noise driven equations that are smoother in time or space—twice as smooth in space in $d = 1, 2$, as is clear from (1.8)—than the Brownian sheet $W$ corresponding to the driving white noise. Moreover, the kernels in these time-fractional SIEs, when $\beta \in \{1/2^k; k \in \mathbb{N}\}$, are fundamental solutions to higher order PDEs (where $\beta^{-1}$ is the order of the Laplacian as detailed in [2]). Thus, the regularity results in [2], given in (1.8), mean that the maximum integer number of dimensions for the existence of random field solutions for space-time white noise driven equations is 3, no matter how high the Laplacian order is. They also mean that the solutions for such equations are spatially $\gamma_s$-Hölder for all $\gamma_s \in (0, 1)$ (nearly locally Lipschitz) in dimensions $d = 1, 2$ and $\gamma_s \in (0, 1/2)$ (nearly locally Hölder 1/2) in $d = 3$. As observed in [2], when $\beta \in \{1/2^k; k \in \mathbb{N}\}$, letting $\beta \searrow 0$ (the order of the Laplacian $\beta^{-1} \not\to \infty$) does not increase the spatial Hölder regularity and the extra Hölder regularizing force is manifested entirely temporally.

These results in [1]–[3] naturally lead to the following list of motivating questions:

(Q1) Consider the L-KS SPDE and time-fractional SPIDEs in spatial dimension $d = 1$.

(a) Are the solutions to (1.9) and (1.6) actually spatially locally Lipschitz (not just nearly locally Lipschitz as in $d = 2$)? This would be unprecedented in SPDEs driven by space-time white noise, and is suggested by the sharp $L^2$ upper bounds on the kernels spatial differences in Lemma 2.4 in [3, 2] and Lemma 3.3 in [1] and is alluded to in Remark 1.2 in [3].

(b) Even more, are the solutions to (1.9) and (1.6) spatially continuously differentiable? and is the one dimensional spatial exponent $3/2$ in

\begin{equation}
\begin{aligned}
&\partial_t u + \frac{\Delta u_0}{\sqrt{4\pi t}} + \frac{1}{4} \Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = u_0(x); & x \in \mathbb{R}^d.
\end{aligned}
\end{equation}

first obtained in [12, 7]. Of course, these sharp Hölder exponents for (1.9) and (1.6) play a crucial role in our exact moduli of continuity for both the L-KS SPDE and the $\beta$ time fractional SPIDEs, as is clear from Theorem 1.1–Theorem 1.6 below.

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$^6$ We remind the reader that when $\beta = 1/2$ the scaled BTBM kernel, which is the fundamental solution to the $a \equiv 0$ version of (1.6), is also the fundamental solution to the fourth order memoryful PDE

\begin{equation}
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\Delta u_0}{\sqrt{4\pi t}} + \frac{1}{4} \Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = u_0(x); & x \in \mathbb{R}^d.
\end{aligned}
\end{equation}
the solutions Hölder exponent \((3/2 \land 1)^-\) in (1.8) indicating that the gradient of these solutions is nearly locally Hölder 1/2 in space? Also, what is the temporal Hölder regularity of the gradient?

(c) If the answer to the first two parts of (b) is yes, what are the moduli of continuity of the gradient of the solutions to (1.9) and (1.6), in the space and time variables, respectively?

(Q2) In [2], it was established that, for each \(d \in \{1, 2, 3\}\), the time-fractional SIEs (time fractional SPIDEs in (1.6)) all have the same spatial Hölder regularity—\((\frac{4-d}{2} \land 1)^-\)—for all \(\beta \in \{1/2^k; k \in \mathbb{N}\}\), as is clear from (1.8). Is the spatial modulus of continuity a more discriminating measure of regularity that depends on \(\beta \in \{1/2^k; k \in \mathbb{N}\}\), and is \(\beta = 1/2\) critical on \((0, 1/2]\)?

(Q3) It was established in [3, 2, 1] that the L-KS SPDE (1.9) and the \(\beta = 1/2\) time-fractional SIEs (time fractional SPIDEs in (1.6)) have identical spatio-temporal Hölder regularity. Does the continuity modulus capture the rougher regularity for the case \(\beta = 1/2\) time-fractional SPIDEs (1.6) (since, by footnote 6, (1.6) is also associated with the rougher positive bi-Laplacian PDE (1.10))?

(Q4) What are the exact spatio-temporal moduli of continuity for (1.1) and (1.2) in \(d = 1, 2, 3\).

(Q5) What are the temporal probability laws associated with L-KS SPDEs and time-fractional SPIDEs?

1.3. Main results: answering the questions. We answer all of the above questions at length in the \(a \equiv 1\) Gaussian case for our two classes of equations in our main results\(^7\), which we now present. First, we deal with the L-KS SPDE.

1.3.1. Exact moduli of continuity of L-KS SPDEs and their gradient, and the bifractional Brownian motion link. We start with the temporal regularity and probability law for L-KS SPDE (1.1) in spatial dimensions \(d = 1, 2, 3\). Recall that, given constants \(H \in (0, 1)\) and \(K \in (0, 1]\), the bifractional Brownian motion \((B^{H,K}_t)_{t \in [0,T]}\), introduced by Houdré and Villa in [21], is a centered Gaussian process with covariance

\[
R^{H,K}(t,s) := R(t,s) = \frac{1}{2K} \left( (t^{2H} + s^{2H})^K - |t-s|^{2HK} \right), \quad s, t \in [0,T].
\]

We refer to [22, 39, 43] for various properties of this process.

\[\text{Theorem 1.1} \quad (\text{Temporal moduli of continuity and bi-fBM connection for the L-KS SPDE in } d = 1, 2, 3). \text{ Fix } (\varepsilon, \theta) \in \mathbb{R}_+ \times \mathbb{R} \text{ and } x \in \mathbb{R}^d, \text{ and assume } d \in \{1, 2, 3\}. \text{ Assume that } (U, \mathcal{F}) \text{ is the unique solution to (1.1) on } (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}), \text{ with } u_0 \text{ satisfying (1.5) (a).}
\]

(i) There are dimension-dependent constants \(k_1^{(d)} > 0\) and \(k_2^{(d)} > 0\), independent of \(x\), such that

\(^7\)In \(d = 2\), we obtain a sharp upper bound on the uniform and local spatial moduli of continuity for the two classes of equations.
\[\left(\begin{array}{l}
\text{(a) (Uniform temporal modulus)} \text{ for any compact interval } I_{\text{time}} \subset \mathbb{R}^+ \\
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{s, t \in I_{\text{time}}} \left| \frac{U(t, x) - U(s, x)}{|t - s|^{\frac{d}{8}} \sqrt{\log |t - s|}} \right] = k_1^{(d)} \right] = 1,
\end{array}\right)\]

\[\left(\begin{array}{l}
\text{(b) (Local temporal modulus)} \text{ and for any fixed } t \geq 0 \\
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{s \in I_{\text{time}}} \left| \frac{U(t, x) - U(s, x)}{|t - s|^{\frac{d}{8}} \sqrt{\log |t - s|}} \right] = k_2^{(d)} \right] = 1.
\end{array}\right)\]

\[\left(\begin{array}{l}
\text{(ii) (B}^{H,K}\text{) link)} \text{ Assume } \vartheta = 0 \text{ in (1.1), then, } U(\cdot, x) \not\equiv c_d B^{\left(\frac{d}{8}, \frac{d}{8}\right)},
\end{array}\right)\]

where

\[c_d = (2\pi)^{-d/2} \left(\frac{\sqrt{2}}{\sqrt{d}}\right)^{d/8} \frac{2^{d-4}}{\sqrt{2^{d/2} - 1}} \int_{\mathbb{R}^d} e^{-|\xi|^4} d\xi; \quad d = 1, 2, 3.\]

In particular, we have the following Chung’s law of the iterated logarithm for \(U(\cdot, x):\)

\[\left(\begin{array}{l}
\text{(1.15)} \quad \lim_{r \searrow 0} \frac{\max_{t \in [0, r]} |U(t, x)|}{r^{(4-d)/8}} \log \log (r^{(4-d)/8}) = k_3^{(d)}
\end{array}\right)\]

for every \(x \in \mathbb{R}^d\) and for some positive finite \(d\)-dependent constant \(k_3^{(d)}\).

Remark 1.1. Theorem 1.1 establishes the temporal modulus of continuity part of Q4 and answers Q5 (when \(\vartheta = 0\)) for L-KS SPDEs. We observe that since \(u_0\) is assumed sufficiently smooth and deterministic, the deterministic part of (1.1) is \(C^{1,4}(\mathbb{R}_+ \times \mathbb{R})\) smooth (see [6, 1]) and the modulus is controled by the random parts of the SPDEs (1.1) (or their associated SIEs (1.47) below, with \(a \equiv 1\)). In addition to giving the precise dimension-dependent temporal modulus of continuity, Theorem 1.1 says that, up to a constant, the simple \((\vartheta = 0)\) L-KS SPDE solution process \(\{U(t, x), t \geq 0\}\) has the same law as a bifractional Brownian motion with indices \(H = \frac{1}{2}\) and \(K = 1 - \frac{4}{d}\). Thus, \(U\) shares all the temporal sample path properties with a \(B^{(1/2, (1-d)/4)}\), in spatial dimensions \(d = 1, 2, 3\), which can be found in [22, 39, 43].

We next state our spatial modulus result for the L-KS SPDE (1.1). Theorem 1.2, along with Theorems 1.5 and 1.6 below, give the first instance of space-time white noise driven SPDEs that have a Hölder continuous gradient.\(^8\)

\[\text{Theorem 1.2 (Spatial moduli of continuity for the L-KS SPDE in } d = 1, 2, 3).\]

Fix \((\varepsilon, \vartheta) \in \mathbb{R} \times \mathbb{R}\) and fix \(t \in \mathbb{R}_+\). Assume that \((U, \mathscr{W})\) is the unique solution to (1.1) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), with \(u_0\) satisfying (1.5) (a). In the following, \(k_i^{(d)} > 0\) \((i = 4, 5)\) are positive and finite constants depending on \(d, \varepsilon, \vartheta\) and \(t\).

\(^8\)See Appendix A for the definition of \(C^{k, \gamma}(\mathbb{R}, \mathbb{R})\) and other notations.
(i) If $d = 1$, then $U(t, \cdot) \in C^{1, \gamma}(\mathbb{R}; \mathbb{R})$, almost surely, with the Hölder exponent $\gamma \in (0, 1/2)$. Moreover,

(a) (Uniform spatial modulus) for any compact rectangle $I_{\text{space}}^{(1)} \subset \mathbb{R}$

$$
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{x,y \in I_{\text{space}}^{(1)}} \frac{|\partial_x U(t, x) - \partial_y U(t, y)|}{|x - y|^{1/2} \sqrt{\log |1/|x - y||}} = k_{d=1}^{(1)} \right] = 1,
$$

(b) (Local spatial modulus) and for any fixed $x \in \mathbb{R}$

$$
\mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x - y| < \delta} \frac{|\partial_x U(t, x) - \partial_y U(t, y)|}{\delta^{1/2} \sqrt{\log \log |1/\delta|}} = k_{d=1}^{(1)} \right] = 1.
$$

(ii) If $d = 3$, then

(a) (Uniform spatial modulus) for any compact rectangle $I_{\text{space}}^{(3)} \subset \mathbb{R}^3$

$$
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{x,y \in I_{\text{space}}^{(3)}} \frac{|U(t, x) - U(t, y)|}{|x - y|^{1/2} \sqrt{\log |1/|x - y||}} = k_{d=3}^{(3)} \right] = 1,
$$

(b) (Local spatial modulus) and for any fixed $x \in \mathbb{R}^3$

$$
\mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x - y| < \delta} \frac{|U(t, x) - U(t, y)|}{\delta^{1/2} \sqrt{\log \log |1/\delta|}} = k_{d=3}^{(3)} \right] = 1.
$$

(iii) If $d = 2$, then

(a) (Uniform spatial modulus) for any compact rectangle $I_{\text{space}}^{(2)} \subset \mathbb{R}^2$

$$
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{x,y \in I_{\text{space}}^{(2)}} \frac{|U(t, x) - U(t, y)|}{|x - y| \log |1/|x - y||} = k_{d=2}^{(2)} \right] = 1,
$$

(b) (Local spatial modulus) and for any fixed $x \in \mathbb{R}^2$

$$
\mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x - y| < \delta} \frac{|U(t, x) - U(t, y)|}{\delta \sqrt{\log |1/\delta| \log \log |1/\delta|}} = k_{d=2}^{(2)} \right] = 1.
$$

**Remark 1.2.** Theorem 1.2 answers the spatial modulus of continuity part of Q4 and answers Q1 for L-KS SPDEs. When $d = 1$, Theorem 1.2 significantly refines the Hölder conclusion of Theorem 1.1 in [1] from $U$ being nearly locally Lipschitz in space to continuously differentiable (and hence locally Lipschitz) in space. Moreover, it also says that, for $d = 1$ and for any fixed time $t$, the spatial derivative of the solution to the L-KS SPDE (1.1), $\partial_x U(t, x)$, is nearly locally Hölder $1/2$ (has Hölder exponent $\gamma \in (0, 1/2)$) in space. In addition, spatially, Theorem 1.2 gives the exact uniform and local moduli of continuity for the gradient $\partial_U U$ in $d = 1$; the exact uniform and local moduli of continuity of $U$ in $d = 3$; and sharp upper bounds on these moduli of continuity of $U$ in $d = 2$. We note that moduli of continuity of $U$ in the $d = 2$ case are different from those for $d = 3$ and the sample functions are
nearly locally Lipschitz. However, we believe that, unlike in the case of $d = 1$, the sample function $x \mapsto U(t, x)$ is nowhere differentiable in the case of $d = 2$.

For the case of $d = 2$, proving the nondifferentiability and the exact spatial moduli of continuity will need substantial extra work because, as a main technical tool for studying these problems, the property of strong local nondeterminism has only been proved in [49, 50, 51] for Gaussian random fields with (directional) Hölder exponents smaller than 1. See Remarks below for further information. We will study these and some related problems in a subsequent paper.

The comparative question Q3 will be answered completely after stating the corresponding results for time-fractional SPIDEs (Theorems 1.5 and 1.6 below).

The last main result for L-KS SPIDEs gives the sharp temporal Hölder and the exact temporal continuity modulus regularity for the spatial gradient of the L-KS SPDE. Let $H^\gamma$ be the space of locally Hölder continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ whose Hölder exponent $\gamma \in (0, \gamma_*)$.

\begin{theorem}[Sharp temporal Hölder and exact continuity moduli for the L-KS SPDE gradient] \label{thm:1.3}
Assume $d = 1$ and fix $\varepsilon, \vartheta \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in \mathbb{R}$. Assume that $(U, \mathcal{F})$ is the unique solution to (1.1) on $(\Omega; \mathcal{F}, \{\mathcal{F}_t\}, P)$, with $u_0 = \mathbb{R}$ satisfying (1.5) (a). Then, $\partial_t U(\cdot, x) \in H^{1/8 -} (\mathbb{R}_+; \mathbb{R})$, almost surely. Moreover, there exist constants $k_i \in (0, \infty)$ ($i = 0, 1, \ldots, 7$) such that

\begin{align}
&\text{(i) (Uniform temporal modulus) for any compact interval } I_{\text{time}} \subset \mathbb{R}_+ \\
&\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{(t, s) \in I_{\text{time}}} \frac{|\partial_t U(t, x) - \partial_t U(s, x)|}{|t - s|^{1/8} \sqrt{\log [1/(t - s)]}} = k_6 \right] = 1, \\
&\text{(ii) (Local temporal modulus) and for any fixed } t \in \mathbb{R}_+ \\
&\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{|t - s| < \delta} \frac{|\partial_t U(t, x) - \partial_t U(s, x)|}{\delta^{1/8} \sqrt{\log \log [1/\delta]}} = k_7 \right] = 1.
\end{align}

\end{theorem}

Remark 1.3. Theorems 1.2 and 1.3 not only tell us that, when $d = 1$, the L-KS SPDE (1.1) gradient $\partial_x U$ exists and is continuous, but they also give us thorough spatio-temporal regularity results for $\partial_x U$, in both the Hölder and modulus senses. This contrasts starkly with the standard second order heat SPDE whose solution is only spatially Hölder continuous with exponent $\gamma \in (0, 1/2)$. The spatial gradient spatio-temporal Hölder regularity in Theorems 1.2 and 1.3 tell us that the gradient of L-KS SPDEs, $\partial_x U$, is rougher ($\gamma$-Hölder with $\gamma \in (0, 1/2)$) in space than the continuously differentiable (and hence Lipschitz) solution $U$. More surprisingly, $\partial_x U$ is also rougher in time than $U$ ($\gamma$-Hölder with $\gamma \in (0, 1/8)$ vs. $\gamma \in (0, 3/8)$ as in Allouba [1]). Compared to the second order heat SPDE, the L-KS gradient $\partial_x U$ has the same spatial Hölder regularity as that of the solution to the heat SPDE; and $\partial_x U$ is twice as rough (half as smooth) as the heat SPDE solution in time, with Hölder exponent $\gamma \in (0, 1/8)$ vs. the well-known $\gamma \in (0, 1/4)$ for the heat SPDE. Similar comments apply with respect to the moduli of continuity of $\partial_x U$ as compared to those of $U$ and to the heat SPDE (see Meerschaert, Wang, and Xiao [34] for the heat SPDE moduli of continuity).
1.3.2. Exact moduli of continuity for the time-fractional SPIDEs and their gradient, and their temporal fractional laws. We now turn to the regularity of the \( \beta \)-time-fractional SPIDEs (1.2) (and their corresponding time-fractional SIEs) and to their temporal fractional law. The uniform and local temporal continuity modul for \( U_\beta \), as well as the law governing the behavior of the solution process \( U_\beta(\cdot, x) = \{U_\beta(t, x); t \geq 0\} \) of (1.2), are given by the next result.

To fully state the next theorem, we need to recall the definition of the generalized hypergeometric (or simply the hypergeometric) function \( _pF_q \):

\[
_2F_1(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad a_i, b_i, \text{ and } z \in \mathbb{R},
\]

whenever the series in the right hand side of (1.24) converges, where

\[
(u)_0 = 1 \text{ and } (u)_n = u(u + 1) \cdots (u + n - 1); u \in \mathbb{R}, \text{ and } n \geq 1.
\]

We are now ready for our result.

\[\textbf{Theorem 1.4} \text{ (Temporal moduli of continuity and laws of the } \beta \text{-time-fractional SPIDEs in } d = 1, 2, 3). \text{ Let } \beta \in (0, 1/2] \text{ and let } (U_\beta, \mathscr{W}) \text{ be the unique solution to (1.2) on } (\Omega, \mathscr{F}, \{\mathcal{F}_t\}, \mathbb{P}) \text{ with } u_0 \text{ satisfying (1.5) (b). Fix } x \in \mathbb{R}^d (d = 1, 2, 3) \text{ and let } H = \frac{2-\beta d}{4}.\]

(i) There exist constant \( k_i^{(\beta,d)} > 0 \) (\( i = 8, 9 \)), depending on \( \beta \) and \( d \) but independent of \( x \), such that

\[
\begin{align*}
\mathbb{P} \left[ \lim_{\delta \downarrow 0} \sup_{s,t \in I_{time}} \frac{|U_\beta(t, x) - U_\beta(s, x)|}{|t - s|^H \sqrt{\log \log |t-s|}} = k_8^{(\beta,d)} \right] &= 1, \\
\mathbb{P} \left[ \lim_{\delta \downarrow 0} \sup_{s,t \in I_{time}} \frac{|U_\beta(t, x) - U_\beta(s, x)|}{\delta^H \sqrt{\log \log |1/\delta|}} = k_9^{(\beta,d)} \right] &= 1.
\end{align*}
\]

(ii) (Law of the \( \beta \)-time-fractional SPIDE) The \( \beta \)-time-fractional SPIDE solution process \( \{U_\beta(t, x), t \geq 0\} \) is a mean-zero Gaussian process with covariance \( \mathbb{E}[U_\beta(t, x)U_\beta(s, x)] \) given by

\[
(2\pi)^{-d} \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \frac{t^{\beta_j} \Gamma(\beta(k-j)+1) \Gamma(\beta j+1)}{[\beta(k-j)+1] \Gamma(1+\beta j) \Gamma(1+\beta(k-j))} \right) \frac{(-1)^k |\xi|^{2k}}{2^k} d\xi.
\]

In particular, \( \{U_\beta(t, x), t \geq 0\} \) is self-similar with index \( H \), but it is not a bifractional Brownian motion. When \( \beta = 1/2 \), the BTBM SPIDE has a fundamentally different law from that of the L-KS SPIDE.

\[\textbf{Remark 1.4.} \text{ Theorem 1.4 answers the temporal modulus of continuity part of Q4 and answers Q5 for time-fractional SPIDEs. In addition to the precise temporal}\]
continuity moduli of L-KS SPDEs \((1.1)\), Theorem 1.4 gives the first contrasting behaviors of \((1.1)\) and the half-derivative or Brownian-time Brownian motion SPIDE \((1.2)\) with \(\beta = 1/2\).

The fundamental difference between the Gaussian laws of the time-fractional SPIDEs and the L-KS SPDE, even at \(\beta = 1/2\), is most easily seen in the fourth order Brownian-time Brownian motion \((\beta = 1/2)\) PDE, obtained first in [12, 7] \(^9\):

\[
\begin{align*}
\partial_t u &= \frac{\Delta u_0}{\sqrt{8\pi t}} + \frac{1}{8} \Delta^2 u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d; \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

The memory term \(\frac{\Delta u_0}{\sqrt{8\pi t}}\) in the deterministic BTBM PDE \((1.10)\), which is not shared with the deterministic version of \((1.9)\) \((a \equiv 0)\), and the opposite sign of the bi-Laplacian in \((1.28)\) vs. that in the L-KS PDE are manifestations of the fundamental reason why the L-KS SPDE and BTBM SPIDE have different laws.

We next state our spatial modulus result for the \(\beta\)-time-fractional SPIDEs \((1.2)\). We will distinguish the cases \(0 < \beta < 1/2\) and \(\beta = 1/2\), where subtle differences arise. It is interesting to notice that, for \(0 < \beta < 1/2\), the spatial moduli of SPIDEs \((1.2)\) are identical, modulo constants, to those of the L-KS SPDEs \((1.1)\).

**Theorem 1.5** (Spatial moduli of continuity for the \(\beta\)-time-fractional SPIDEs for \(0 < \beta < 1/2\) and \(d = 1, 2, 3\)). Assume that \((U_\beta, \mathcal{F})\) is the unique solution to \((1.2)\) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), with \(u_0\) satisfying \((1.5)\) \((b)\). We assume \(t \in \mathbb{R}_+\) is fixed and \(0 < \beta < 1/2\). In the following, \(k^{(\beta, d)}_i > 0\) \((i = 10, 11)\) are constants depending on \(d, t\) and \(\beta\).

(i) If \(d = 1\), then \(U_\beta(t, \cdot) \in C^{1, \gamma}(\mathbb{R}; \mathbb{R})\), almost surely, with the Hölder exponent \(\gamma \in (0, 1/2)\). Moreover,

(a) (Uniform spatial modulus) for any compact rectangle \(I^{(1)}_{\text{space}} \subset \mathbb{R}\)

\[
P \left[ \lim_{\delta \to 0} \sup_{|x-y| < \delta} \frac{\partial_x U_\beta(t, x) - \partial_y U_\beta(t, y)}{|x-y|^{1/2} \sqrt{\log [1/|x-y|]}} \right] = k^{(\beta, 1)}_{10} = 1,
\]

(b) (Local spatial modulus) and for any fixed \(x \in \mathbb{R}\)

\[
P \left[ \lim_{\delta \to 0} \sup_{|x-y| < \delta} \frac{\partial_x U_\beta(t, x) - \partial_y U_\beta(t, y)}{\delta^{1/2} \sqrt{\log [1/\delta]}} \right] = k^{(\beta, 1)}_{11} = 1.
\]

(ii) If \(d = 3\), then

(a) (Uniform spatial modulus) for any compact rectangle \(I^{(3)}_{\text{space}} \subset \mathbb{R}^3\)

\[
P \left[ \lim_{\delta \to 0} \sup_{|x-y| < \delta} \frac{|U_\beta(t, x) - U_\beta(t, y)|}{|x-y|^{1/2} \sqrt{\log [1/|x-y|]}} \right] = k^{(\beta, 3)}_{10} = 1,
\]

---

\(^9\)We alternate freely between the notations \(\partial^a f(x_1, \ldots, x_N)\) and \(\partial^a f/\partial x_i^a\), \(i = 1, \ldots, N\).
When \( \beta = 1/2 \), the next result shows that the (BTBM) SPIDE is critical, signaling the onset of rougher spatial sample paths, in \( d = 1, 2, 3 \), than both the time-fractional SPIDEs with \( \beta < 1/2 \) and the L-KS SPDE\(^{10}\). This is despite the fact that they all have the same spatial Hölder regularity as established first in Allouba [3, 2, 1].

\[ \textbf{Theorem 1.6} \text{ (Spatial continuity modulus for the critical half-derivative BTBM SPIDEs, } \beta = 1/2 \text{ and } d = 1, 2, 3). \text{ Assume that } (U_{1/2}, \mathcal{F}) \text{ is the unique solution to (1.2) on } (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}), \text{ with } \beta = 1/2 \text{ and } u_0 \text{ satisfying (1.5) (b).} \]

In the following, \( k_{i}^{(1/2,d)} > 0 \) \( (i = 10, 11) \) are constants depending on \( d \) and \( t \).

(i) If \( d = 1 \), then \( U_{1/2}(t, \cdot) \in C^{1, \gamma}(\mathbb{R}; \mathbb{R}) \), almost surely, with the Hölder exponent \( \gamma \in (0, 1/2) \). Moreover,

(a) \( \text{(Uniform spatial modulus) for any compact rectangle } I_{\text{space}}^{(1)} \subset \mathbb{R} \)

\[ \mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x-y| < \delta} \left| U_{1/2}(t,x) - U_{1/2}(t,y) \right| \leq k_{10}^{(1/2,1)} \right] = 1. \]

(b) \( \text{(Local spatial modulus) and for any fixed } x \in \mathbb{R} \)

\[ \mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x-y| < \delta} \left| \frac{\partial_x U_{1/2}(t,x) - \partial_y U_{1/2}(t,y)}{|x-y|^{1/2} \log |1/|x-y||} \right| \leq k_{11}^{(1/2,1)} \right] = 1. \]

(ii) If \( d = 3 \), then

\[ \mathbb{P} \left[ \limsup_{\delta \searrow 0} \sup_{|x-y| < \delta} \left| \frac{\partial_x U_{1/2}(t,x) - \partial_y U_{1/2}(t,y)}{\delta^{1/2} \sqrt{\log(1/|y|) \log |1/\delta|}} \right| \leq k_{10}^{(1/2,1)} \right] = 1. \]

\(^{10}\)Carefully examining Theorem 1.6, we see the extra \( 1/\sqrt{\log |1/|x-y||} \) (or \( 1/\sqrt{\log(1/\delta)} \)) term in each modulus expression as compared to the corresponding expressions in both Theorem 1.5 and Theorem 1.2 above.
(a) (Uniform spatial modulus) for any compact rectangle $I_{\text{space}}^{(3)} \subset \mathbb{R}^3$

\[
P \left[ \lim_{\delta \to 0} \sup_{|x - y| < \delta} \frac{|U_{1/2}(t, x) - U_{1/2}(t, y)|}{|x - y|^{1/2} \log \left[ \frac{1}{|x - y|} \right]} = k_{10}^{(1/2, 3)} \right] = 1,
\]

(b) (Local spatial modulus) and for any fixed $x \in \mathbb{R}^3$

\[
P \left[ \lim_{\delta \to 0} \sup_{|x - y| < \delta} \frac{|U(t, x) - U(t, y)|}{\delta^{1/2} \sqrt{\log[1/\delta] \log \log [1/\delta]}} = k_{11}^{(1/2, 3)} \right] = 1.
\]

(iii) If $d = 2$, then

(a) (Uniform spatial modulus) for any compact rectangle $I_{\text{space}}^{(2)} \subset \mathbb{R}^2$

\[
P \left[ \lim_{\delta \to 0} \sup_{|x - y| < \delta} \frac{|U(t, x) - U(t, y)|}{|x - y| \log \left( \frac{1}{|x - y|} \right)^{3/2}} \leq k_{10}^{(1/2, 2)} \right] = 1,
\]

(b) (Local spatial modulus) and for any fixed $x \in \mathbb{R}^2$

\[
P \left[ \lim_{\delta \to 0} \sup_{|x - y| < \delta} \frac{|U(t, x) - U(t, y)|}{\delta \log[1/\delta] \sqrt{\log \log [1/\delta]}} \leq k_{11}^{(1/2, 2)} \right] = 1.
\]

The last main result for time-fractional SPIDEs gives the sharp temporal Hölder and the exact temporal continuity modulus regularity for the spatial gradient of these time-fractional SPIDEs.

Theorem 1.7 (Sharp temporal Hölder and exact continuity moduli for the time-fractional SPIDEs gradient). Assume $d = 1$, $x \in \mathbb{R}$, and let $\beta \in (0, 1/2]$. Assume that $(U_0, \mathcal{W})$ is the unique solution to (1.2) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, with $u_0$ satisfying (1.5) (b). Then, $\partial_x U_\beta(t, x) \in H^{(2-3\beta)/4-}(\mathbb{R}_+; \mathbb{R})$, almost surely. Moreover, there exist constants $k_i \in (0, \infty)$ ($i = 12, 13$) such that

(i) (Uniform temporal modulus) for any compact interval $I_{\text{time}} \subset \mathbb{R}_+$

\[
P \left[ \lim_{\delta \to 0} \sup_{t, s \in I_{\text{time}}} \frac{|\partial_x U_\beta(t, x) - \partial_x U_\beta(s, x)|}{|t - s|^{2-3\beta/4} \sqrt{\log \left[ \frac{1}{|t - s|} \right]}} = k_{12} \right] = 1,
\]

(ii) (Local temporal modulus) and for any fixed $t \in \mathbb{R}_+$

\[
P \left[ \lim_{\delta \to 0} \sup_{|t - s| < \delta} \frac{|\partial_x U_\beta(t, x) - \partial_x U_\beta(s, x)|}{\delta^{2-3\beta/4} \sqrt{\log \log [1/\delta]}} = k_{13} \right] = 1.
\]

Remark 1.5. For $d = 1$, Theorems 1.5, 1.6, and 1.7 give us the existence, as well as thorough results on the spatio-temporal moduli of continuity for the gradient $\partial_x U_\beta$ of the $\beta$-time-fractional SPIDEs (1.2). Spatially, Theorems 1.5 and 1.6 say that,
even though the Hölder exponent of the gradient \( x \mapsto \partial_x U_\beta(t,x) \) is the same for all \( \beta \in (0, 1/2] \), the exact uniform and local moduli of continuity for \( \beta \in (0, 1/2) \) and \( \beta = 1/2 \). The SPIDEs gradient \( \partial_x U_\beta \) is spatially rougher in the modulus sense at \( \beta = 1/2 \) than it is for \( \beta < 1/2 \); and \( \partial_x U_\beta \) has the same spatial modulus of continuity as that of the L-KS SPDE gradient for \( \beta < 1/2 \). Theorem \( 1.7 \) shows that the time-fractional SPIDE gradient \( \partial_x U_\beta \), for \( \beta = 1/2 \), has the same Hölder and modulus regularity in the time variable as the L-KS SPDE gradient. Moreover, the temporal Hölder exponent \( \gamma \mapsto 1/2 \) [i.e., the temporal Hölder regularity increases] as \( \beta \nearrow 0 \). This is consistent with the similar phenomenon for the time-fractional solutions observed by Allouba in [2].

Theorems \( 1.1–1.7 \) together answer all the questions Q1–Q5 above except that, in the case of \( d = 2 \), extra work will be needed for completely establishing the exact spatial moduli of continuity.

1.4. The strong local nondeterminism property and modulus of continuity. The proofs of Theorems \( 1.1–1.5 \) depend on the results and methods in Meerschaert, Wang, and Xiao [34], Xue and Xiao [51] which, in turn, are based on general Gaussian methods (cf. e.g., [29]) and the properties of strong local nondeterminism in [49, 50]. More specifically we obtain an expression for the spectral measure/density associated with the solution and use it to prove the exact uniform and local moduli of continuity.

To determine many sample path properties of our Gaussian solution \( U \) to our SIEs separately and jointly in time and space, the following second moments of spatial and temporal differences are crucial\(^{11}\):

\[
\sigma_{\text{time}}^{(d)}(s,t;x) = E[U(t,x) - U(s,x)]^2; \quad s, t \in \mathbb{R}^+, x \in \mathbb{R}^d,
\]

\[
\sigma_{\text{space}}^{(d)}(t;x,y) = E[U(t,x) - U(t,y)]^2; \quad t \in \mathbb{R}^+, x, y \in \mathbb{R}^d,
\]

\[
\sigma_{\text{joint}}^{(d)}(s,t;x,y) = E[U(t,x) - U(s,y)]^2; \quad s, t \in \mathbb{R}^+, x, y \in \mathbb{R}^d.
\]

Xiao [50] gave some general conditions for effectively studying several analytic and geometric properties of Gaussian random fields. For convenience of readers, we restate these conditions below, adapting the notation slightly to our setting. Let \( I_{\text{time}} = [a,b] \) and \( I_{\text{space}}^{(d)} = \bigotimes_{k=1}^d [a_k, b_k] \) be one and \( d \)-dimensional closed intervals in \( \mathbb{R}^+ \) and \( \mathbb{R}^d \), respectively\(^{12}\). Let \( \gamma = (\gamma_1, \ldots, \gamma_{d+1}) \in (0,1]^{d+1} \) be a fixed vector, and denote by \( \rho \) the metric on \( \mathbb{R}^+ \times \mathbb{R}^d \) given by

\[
\rho(s,t;x,y) = |t-s|^{\gamma_1} + \sum_{j=2}^{d} |x_j - y_j|^{\gamma_j}; \quad s, t \in \mathbb{R}^+, x, y \in \mathbb{R}^d.
\]

(C1) (Spatio-temporal bounds) There exist positive and finite constants \( c_{2,1} \) and \( c_{2,2} \) such that

\[
c_{2,1} \rho^2(s,t;x,y) \leq \sigma_{\text{joint}}^{(d)}(s,t;x,y) \leq c_{2,2} \rho^2(s,t;x,y)
\]

for all \( s, t \in I_{\text{time}} \) and \( x, y \in I_{\text{space}}^{(d)} \).

\(^{11}\)In the case of \( \beta \) time-fractional SIEs, these quantities depend also on \( \beta \).

\(^{12}\)In this paper, unless otherwise stated we take \( 0 \leq a, a_k < 1 \) and \( b = b_k = 1 \), for all \( k \).
(C2) (SLND) There exists a constant $c_{2,3} > 0$ such that for all integers $n \geq 1$ and all $p, p^{(1)}, \ldots, p^{(n)} \in I_{\text{time}} \times I_{\text{space}}$

$$\text{Var} \left( U(p) | U(p^{(1)}), \ldots, U(p^{(n)}) \right) \geq c_{2,3} \sum_{j=1}^{d+1} \min_{0 \leq k \leq n} \left| p_j - p^{(k)}_j \right|^{2\gamma_j}.$$ 

Remark 1.6. In this article, $\gamma_1$ is the least upper bound for the temporal Hölder exponents for our SPDEs/SPIDEs, and $\gamma_2 = \cdots = \gamma_{d+1}$ are the least upper bound for the spatial Hölder exponents for our SPDEs/SPIDEs. By Theorem 1.1 in Allouba [1, 3], $\gamma_1 = (4 - d)/8$ for the L-KS SPDE and for the SPIDE (1.2) when $\beta = 1/2$; and, by Theorem 1.2 in Allouba [2], $\gamma_1 = (2\beta^{-1} - d)/4\beta^{-1}$ for the SPIDE (1.2) for $\beta = 1/2^k, k \in \mathbb{N}$. Also, $\gamma_j = [(4 - d)/2] \wedge 1, j = 2, \ldots, d+1$, and $d = 1, 2, 3$ for all SPDEs/SPIDEs in this article by Theorem 1.1 and Theorem 1.2 [3, 2] and by Theorem 1.1 in [1].

Remark 1.7. The main results of this article, Theorems 1.1–1.7, establish exact uniform and local moduli of continuity for the solutions of the L-KS SPDE and the SPIDE in the time variable $t$ and space variable $x$, separately. Also, Theorem 1.1 gives a Chung’s law of iterated logarithm for simple L-KS SPDEs. For proving these theorems, we will only use Conditions (C1) and (C2) for two special cases: either $x = y$ or $s = t$, respectively. Hence, the spectral conditions in Xiao [49] can be applied to verify these conditions. Moreover, Theorems 2.1 and 2.5 in [49] allow us to prove more general properties by replacing the power functions $|t - s|^{\gamma_1}$ and $|x_j - y_j|^{\gamma_j}$ in (1.44) by regularly varying functions of $|s - t|$ or $|x - y|$ with regularity exponents smaller than 1. Such an extension does not affect the proofs in Meerschaert, Wang, and Xiao [34], hence the theorems in Sections 4 and 5 of [34] are still applicable.

Remark 1.8. It would be interesting to study analytic and geometric properties of the solutions of L-KS SPDE and the SPIDE in both time and space variables $t$ and $x$ simultaneously. For this purpose, the full strength of Conditions (C1) and (C2) will be needed. The problems are more complicated and some new techniques will be required. We will pursue this line of research in a separate article.

1.5. Rigorous kernel stochastic integral equations formulations. For the L-KS SPDE (1.9), as done in [1, 5], we use the linearized Kuramoto-Sivashinsky kernel introduced in [6, 5, 1] to define their rigorous mild SIE formulation. This L-KS kernel is the fundamental solution to the deterministic version of (1.9) ($a \equiv 0$), as shown in [6, 5, 1], and is given by:

$$K_{t, x, y}^{\text{LKS}} = \int_{-\infty}^{0} \frac{e^{i\theta s} e^{-|x-y|^2/2is}}{(2\pi is)^{d/2}} R^{\text{0M}}_{x, t, s} ds + \int_{0}^{\infty} \frac{e^{i\theta s} e^{-|x-y|^2/2is}}{(2\pi is)^{d/2}} R^{\text{0M}}_{x, t, s} ds,$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\theta}{2} |\xi|^2} e^{i\xi (x-y)} d\xi;$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\theta}{2} |\xi|^2} \cos (\xi (x - y)) d\xi; \quad \varepsilon > 0, \quad \theta \in \mathbb{R}.$$
Let \( b: \mathbb{R} \to \mathbb{R} \) be Borel measurable. The rigorous L-KS kernel SIE (mild) formulation of the nonlinear drift-diffusion L-KS SPDE

\[
\begin{cases}
\frac{\partial U}{\partial t} = -\frac{1}{2} (\Delta + 2\partial^2) U + b(U) + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
U(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
\]

(1.46)
is the stochastic integral equation

\[
U(t, x) = \int_{\mathbb{R}^d} \mathbb{K}^{LKS}_{t; x, y} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}^{LKS}_{t-s; x, y} [b(U(s, y))dsdy + a(U(s, y))W(ds \times dy)]
\]

(1.47)

(see [5, p. 530] and [1, Definition 1.1, Eq. (1.11)]). Of course, the mild formulation of (1.1) is then obtained by setting \( a \equiv 1 \) and \( b \equiv 0 \) in (1.47).

For the time fractional SPIDE (1.6), we first explain heuristically the role of the extra time fractional integral \( I^{1-\beta} \) in the formal formulation. Succinctly, it compensates for the \( \beta \) time fractional derivative \( C^{\beta} \partial_t^\beta \) so as to end up with a standard stochastic integral in time term with respect to space-time white noise, and wind up with a simpler (and smoother) SIE formulation. To see this quickly before we give the formal computation, we first observe heuristically that to get a formulation with \( U_\beta \) on the left hand side, we only need to get rid of the \( \beta \) fractional time derivative \( C^{\beta} \partial_t^\beta \) by applying a \( \beta \) fractional integral \( I^{\beta}_t \) to it. This means we have to apply \( I^{\beta}_t \) to the right side of the SPIDE (1.6) too. So, if we want the time integral of the noise term to be of order 1 (nonfractional), we need to have started already with a fractional integral \( I^{1-\beta}_t \) of the noise so that \( I^{\beta}_t \circ I^{1-\beta}_t = I^1_t \). To put this heuristic on a firm ground and to get the SIE formulation of time fractional SPIDEs, we use Umarov’s fractional Duhamel principle (see Theorem 3.6 in [46]), which we now proceed to describe.

If we replace the bracketed terms in the nonlinear drift-diffusion time-fractional SPIDE

\[
\begin{cases}
C^{\beta} \partial_t^\beta U_\beta = \frac{1}{2} \Delta U_\beta + I^{1-\beta}_t \left[ b(U_\beta) + a(U_\beta) \frac{\partial^{d+1} W}{\partial t \partial x} \right], & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\
U_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
\]

(1.48)

by a nice forcing term \( f(t, x) \); then, using Theorem 3.6 in [46], we obtain

\[
U_\beta(t, x) = \int_{\mathbb{R}^d} \mathbb{K}^{(\beta, d)}_{t; x, y} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}^{(\beta, d)}_{t-s; x, y} \left( R^{\beta-1} \partial_t^{1-\beta} I^{1-\beta}_t f(s, y) \right) dsdy
\]

(1.49)

where \( R^{\beta}_t \) is the Riemann-Liouville fractional derivatives of order \( \alpha \):

\[
R^{\beta}_t := \begin{cases}
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau & \text{if } 0 < \alpha < 1, \\
\frac{d}{dt} f(t) & \text{if } \alpha = 1,
\end{cases}
\]

(1.50)
where we used the fact that \( R \partial_t^\beta P_t = \text{Id} \), and where \( K_{t,x}^{(\beta,d)} \) is the solution to the time-fractional PDE:

\[
\begin{align*}
\partial_t^\beta U_\beta &= \frac{1}{2} \Delta U_\beta, \\
U_\beta(0, x) &= \delta(x),
\end{align*}
\]

(1.51)

where \( \delta(x) \) is the usual Dirac delta function. These fundamental solutions \( K_{t,x}^{(\beta,d)} \) are the densities of an inverse stable Lévy time Brownian motion \( B^x(\Lambda_\beta(t)) \), at time \( t \), in which the inverse stable Lévy motion \( \Lambda_\beta \) of index \( \beta \) acts as the time clock for an independent \( d \)-dimensional Brownian motion case. Thus,

\[
K_{t,x}^{(\beta,d)} = \int_0^\infty K_{s,x}^{BM,d} K_s^{\Lambda_\beta} ds; \quad 0 < \beta < 1.
\]

(1.52)

In the case \( \beta = 1/2 \), the kernel \( K_{t,x}^{(1/2,d)} \) is the density of the Brownian-time Brownian motion as in [12, 7, 5]; and when \( \beta \in \{1/2^k; k \in \mathbb{N}\} \), the kernel \( K_{t,x}^{(\beta,d)} \) is the density of \( k \)-iterated BTBM as detailed in [2]. Namely, denote by

\[
\mathbb{B}_i \bigcap B_i(t) := B^x \left( \bigl\{ B_k(\cdots B_2(\{B_1(t)\})\cdots) \bigr\} \right)
\]

a \( k \)-iterated Brownian-time Brownian motion at time \( t \); where \( \{B_i\}_{i=1}^k \) are independent copies of a one-dimensional scaled Brownian motion starting at zero, with density \( \frac{\sqrt{2\pi}}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) = (1/\sqrt{2}) K_{t,0,x}^{BM} \), and independent from the standard \( d \)-dimensional Brownian motion \( B^x \), which starts at \( x \in \mathbb{R}^d \). When \( \beta^{-1} \in \{2^k; k \in \mathbb{N}\} \), the density \( K_{t,x}^{(\beta,d)} \) of \( \mathbb{B}_i \bigcap B_i(t) \) is given by

\[
K_{t,x}^{(\beta,d)} = 2^k \int_{\mathbb{R}^d} K_{s_1,x}^{BM,d} K_{s_1,0}^{BM} \prod_{i=0}^{k-2} K_{s_{k-i},0}^{BM} ds_1 \cdots ds_k.
\]

(1.53)

Now, denoting the white noise formally by \( \dot{W} \) and replacing the nice forcing term \( f \) by the bracketed terms in (1.48), we see that (1.49) becomes

\[
U_\beta(t, x) = \int_{\mathbb{R}^d} K_{t,x,y}^{(\beta,d)} u_0(y) dy \\
+ \int_{\mathbb{R}^d} \int_0^t K_{t-s,x,y}^{(\beta,d)} \left[ b(U_\beta(s,y)) ds dy + a(U_\beta(s,y)) \dot{W}(s,y) ds dy \right],
\]

(1.54)

which is rigorously written as

\[
U_\beta(t, x) = \int_{\mathbb{R}^d} K_{t,x,y}^{(\beta,d)} u_0(y) dy \\
+ \int_{\mathbb{R}^d} \int_0^t K_{t-s,x,y}^{(\beta,d)} \left[ b(U_\beta(s,y)) ds dy + a(U_\beta(s,y)) \dot{W}(ds \times dy) \right].
\]

(1.55)

Equation (1.55)—with \( a \equiv 1 \) and \( b \equiv 0 \)—is what we rigorously mean by the SPIDE (1.2), and it is the equation we work with. Here, we call the stochastic integral equation in (1.55) \( \beta \)-time-fractional SIE. We stress here that, in the case \( \beta = 1/2 \),

\[\text{We are using the convention } \prod_{i=0}^{k-1} c_i = 1 \text{ for any } c_i \text{ and the convention } \int_{\mathbb{R}^d} f(s) ds = f(s), \text{ for every } f. \]

Also, we use the convention that the case \( k = 0 \) (\( \beta = 1 \)) is also the standard \( d \)-dimensional Brownian motion case.
equation (1.55) is exactly equation (3.3) in [5] (when \(a \equiv 1\) and \(b \equiv 0\)), the equation in [5, top of p. 530], and equation (1.3) in [3]; and, for \(0 < \beta < 1\), equation (1.55) is exactly equation (1.14) in [2]. In the important case of \(\beta = 1/2\), we call (1.55) the BTBM SIE since in this case \(\hat{K}_{t,x}^{(\beta,d)}\) is the density of a Brownian-time Brownian motion.

**Notation 1.1.** Unless explicitly otherwise stated, \(c\) and \(C\) will denote constants whose value may change from a statement to another. We refer the reader to the convenient end-of-paper list of notations.

### 2. Kernels Fourier transforms

The following lemma gives the spatial Fourier transform of the \(\beta\)-time-fractional (including the \(\beta = 1/2\) BTBM case), and the \((\varepsilon, \vartheta)\) L-KS kernels.

**Lemma 2.1 (Spatial Fourier transforms).** Let \(K_{t,x}^{LKS, \varepsilon, \vartheta}\) and \(K_{t,x}^{(\beta,d)}\) be the \((\varepsilon, \vartheta)\) LKS kernel and the \(\beta\)-time-fractional kernel, respectively.

(i) The spatial Fourier transform of the \((\varepsilon, \vartheta)\) LKS kernel in (1.45) is given by

\[
\hat{K}_{t,\xi}^{LKS, \varepsilon, \vartheta} = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4} \left(-2\varepsilon + \vartheta|\xi|^2\right)^2}; \quad \varepsilon > 0, \quad \vartheta \in \mathbb{R}.
\]

(ii) Let \(0 < \beta < 1\). The spatial Fourier transform of the \(\beta\)-time-fractional kernel is given by

\[
\hat{K}_{t,\xi}^{(\beta,d)} = (2\pi)^{-\frac{d}{2}} E_{\beta} \left(-\frac{|\xi|^2}{2} t^{\beta}\right),
\]

where

\[
E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \beta k)}
\]

is the well known Mittag-Leffler function. In particular, the Fourier transform of the BTBM density (the case \(\beta = 1/2\)) is given by

\[
\hat{K}_{t,\xi}^{(1/2,d)} = (2\pi)^{-\frac{d}{2}} e^{\frac{1}{2} |\xi|^2} \left[\frac{2}{\sqrt{\pi}} \int_{\xi |\xi|^2}^{\infty} e^{-r^2} dr\right].
\]

**Proof.** The proof in the BTBM (the case \(\beta = 1/2\) or \(k = 1\)) and the \((\varepsilon, \vartheta)\) LKS kernels cases is given in [1, Lemma 2.1]. We now prove the general \(\beta\) case. The kernel \(K_{t,x}^{(\beta,d)}\) is given by (1.52). Since the Laplace transform of \(K_{t,s}^{(\beta,d)}\) in time is particularly simple and is given by \(\hat{K}_{t,s}^{\beta} = \theta^{\beta-1} e^{-\theta^{\beta}}\), we easily get the Fourier transforms as stated above.

---

15 In space, we are using the symmetric form of the Fourier transform \(\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx\).

16 See Haubold, Mathai, and Saxena [20] and [30] for the necessary background.

17 Strictly speaking, the \(\beta = 1/2\) BTBM Fourier transform in Lemma 2.1 is that of a BTBM in which the inner BM is time scaled. The Fourier transform of a standard BTBM is

\[
\hat{K}_{t,\xi}^{BTBM} = (2\pi)^{-\frac{d}{2}} e^{\frac{1}{2} |\xi|^2} \left[\frac{2}{\sqrt{\pi}} \int_{\xi |\xi|^2}^{\infty} e^{-r^2} dr\right].
\]
Lemma 2.2 (Mittag-Leffler Fourier transform). Assume that \( x > 0 \).

(i) If \( \beta > 0 \), then for any \( \sigma > 0 \)

\[
\mathcal{F} \left[ \mathbb{1}_{\{t > 0\}} E_\beta \left(-xt^\beta\right) e^{-t\sigma} \right] (\tau) = \frac{(\sigma + \imath \tau)^{\beta - 1}}{(\sigma + \imath \tau)^{\beta} + x}.
\]

(ii) If \( \beta \in \{1/2^k; k \in \mathbb{N}\} \), then (2.9) hold for any \( \sigma \geq 0 \). In particular,

\[
\mathcal{F} \left[ \mathbb{1}_{\{t > 0\}} E_\beta \left(-xt^\beta\right) \right] (\tau) = \frac{(\imath \tau)^{\beta - 1}}{(\imath \tau)^{\beta} + x}.
\]

Remark 2.1. The case \( \beta \in \{1/2^k; k \in \mathbb{N}\} \) is important and useful since it captures the behavior of our SPIDEs for all \( 0 < \beta \leq 1/2 \) while also representing the case where the kernels \( K_{t;x}^{(\beta,d)} \) is the fundamental solution to higher order PDEs with memory (see e.g., [2, 12, 7] and the references therein for details).
Proof. Let $\theta = \sigma + i\tau$, and suppose $x > 0$. Then,
\[
\frac{(\sigma + i\tau)^{\beta-1}}{(\sigma + i\tau)^{\beta} + x} = \mathcal{L} \left[ E_{\beta} \left( -xt^{\beta} \right) \right] (\theta)
\]
\[
= \int_{0}^{\infty} E_{\beta} \left( -xt^{\beta} \right) e^{-t\theta} dt
\]
\[
= \int_{-\infty}^{\infty} \left[ \mathbb{1}_{\{t>0\}} E_{\beta} \left( -xt^{\beta} \right) e^{-t\sigma} \right] e^{-it\tau} dt
\]
\[
= \mathcal{F} \left[ \mathbb{1}_{\{t>0\}} E_{\beta} \left( -xt^{\beta} \right) e^{-t\sigma} \right] (\tau)
\]
If $\beta \in \left\{ 1/2k; k \in \mathbb{N} \right\}$, there are no poles if we set $\sigma = 0$ in the ratio
\[
\frac{(\sigma + i\tau)^{\beta-1}}{(\sigma + i\tau)^{\beta} + x}
\]
In this case, the radius of convergence of the Laplace transform in (2.11) is $\Re(\theta) \geq 0$. Setting $\sigma = 0$ in (2.11), we thus obtain the Mittag-Leffler Fourier transform in (2.10). The proof is complete. \qed

3. The L-KS SPDEs: Proofs of Theorems 1.1–1.3

Let $U$ be the solution to the L-KS SPDE (1.1). In [1], Allouba obtained the temporal and spatial H"older exponent $\gamma_t \in (0, (4 - d)/8)$ and $\gamma_x \in (0, ((4 - d)/2) \wedge 1)$, respectively, by establishing—in [1, Lemma 3.4]—the following sharp dimension-dependent upper bounds
\[
\begin{align*}
\text{E} \left[ U(t,x) - U(s,x) \right]^{2q} &\leq C_d \left| t - s \right|^{(4-d)\alpha_d}/4, \\
\text{E} \left[ U(t,x) - U(t,y) \right]^{2q} &\leq C_d \left| x - y \right|^{2q\alpha_d}; \quad \alpha_d \in J_d,
\end{align*}
\]
for the more general nonlinear L-KS SPDE (1.9), with Lipschitz condition on $a$, for all $x, y \in \mathbb{R}^d$, for all $t, s \in [0, T]$, for $q \geq 1$, for $1 \leq d \leq 3$, and for
\[
J_d = \begin{cases} (0, 1); & d = 1, \\
(0, 1); & d = 2, \\
(0, \frac{1}{2}); & d = 3.
\end{cases}
\]
These H"older exponents determine the temporal and spatial differences exponents in the temporal and spatial moduli expressions in Theorem 1.1 and Theorem 1.2, respectively. They are also useful for getting a sharp upper bound for the uniform spatio-temporal moduli of continuity for our L-KS SPDE (1.1). Rather than complementing the upper bounds in (3.1) with corresponding lower bounds, we take a harmonic/spectral analytic route combined with a useful decomposition of our solution $U$ to get the exact uniform and local moduli of continuity in Theorem 1.1. This approach, which we also use for our time-fractional SPIDEs, builds on the results of Xiao in [49, 50] and Meerschaert, Wang, and Xiao in [34].

Assume without loss of generality that $u_0 = 0$, then the L-KS SPDE solution is given by
\[
U(t,x) = \int_{\mathbb{R}^d} \int_{0}^{t} K_{t-r}^{\text{LKS}, \theta} (dr \times dy), \quad t \geq 0, x \in \mathbb{R}^d.
\]
3.1. Temporal modulus. Throughout this subsection, let \( x \in \mathbb{R}^d \) be fixed but arbitrary. We first introduce the following auxiliary Gaussian process \( \{X(t, x), t \in \mathbb{R}_+\} \):

\[
X(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left( \mathbb{K}_{(t-r)+;x,y}^\text{LKS,\phi} - \mathbb{K}_{(s-r)+;x,y}^\text{LKS,\phi} \right) \mathcal{W}(dr \times dy),
\]

where \( a_+ = \max\{a, 0\} \) for all \( a \in \mathbb{R} \). Then the L-KS SPDE solution \( U \) may be decomposed as \( U(t, x) = X(t, x) - V(t, x) \), where

\[
V(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left( \mathbb{K}_{(t-r)+;x,y}^\text{LKS,\phi} - \mathbb{K}_{(s-r)+;x,y}^\text{LKS,\phi} \right) \mathcal{W}(dr \times dy).
\]

This idea of decomposition originated in Mueller and Tribe [36] in the second order SPDEs setting; and it has been applied in Wu and Xiao [45] and in Tudor and Xiao [44], also in the second order heat SPDE setting. See also Mueller and Wu [37] for related results on stochastic heat equation.

We first prove our results on the moduli of continuity for the auxiliary process \( X \), then using the aforementioned decomposition of \( U \), in terms of \( X \) and a smooth process \( V \), we transfer them to our L-KS SPDE solution \( U \). The following result is pivotal.

**Theorem 3.1.** Assume the spatial dimension \( d \in \{1, 2, 3\} \). Let \( X \) be as defined in (3.4) and \( x \in \mathbb{R} \) be fixed.

(i) The Gaussian process \( \{X(t, x); t \geq 0\} \) has stationary temporal increments. Moreover, we have

\[
\mathbb{E}[X(t, x) - X(s, x)]^2 = 2 \int_{\mathbb{R}} [1 - \cos((t-s)\tau)] \Delta(\tau) d\tau,
\]

where the spectral density \( \Delta \) is given by

\[
\Delta(\tau) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{\tau^2 + \frac{\tau^2}{\theta^2} (-2\theta + \xi^2)}.
\]

(ii) For each \( k \geq 1 \), there exists a modification of \( \{V(t, x), t \in \mathbb{R}_+\} \) such that its (temporal) sample function is almost surely \( k \)-times differentiable on \((0, \infty)\).

(iii) Let \( \gamma_1 = \frac{k+d}{k} \). There is a finite constant \( C \) such that

\[
\lim_{\varepsilon \to 0} \sup_{s,t \in [0, \varepsilon]} \frac{|V(t, x) - V(s, x)|}{|t-s|^{\gamma_1} \sqrt{\log \log(1/|t-s|)}} \leq C \quad a.s.
\]

**Proof.** To verify (i), we apply Parseval’s identity to the integral in \( y \) to get that for any \( 0 < s < t \):

\[
\mathbb{E}[X(t, x) - X(s, x)]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left( \mathbb{K}_{(t-r)+;x,y}^\text{LKS,\phi} - \mathbb{K}_{(s-r)+;x,y}^\text{LKS,\phi} \right)^2 \mathcal{W}(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \mathbb{K}_{(t-r)+;x,\xi}^\text{LKS,\phi} - \mathbb{K}_{(s-r)+;x,\xi}^\text{LKS,\phi} \right)^2 \mathcal{W}(d\xi) dr.
\]

Since

\[
\mathbb{K}_{(t-r)+;x,\xi}^\text{LKS,\phi} = (2\pi)^{-d/2} \cdot e^{-i(x, \xi) - \frac{\tau(x, \xi)}{\theta^2 + |\xi|^2}} \mathbb{1}_{\{t>r\}},
\]

(3.7) becomes

\[
\mathbb{E}[X(t, x) - X(s, x)]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \mathbb{K}_{(t-r)+;x,\xi}^\text{LKS,\phi} - \mathbb{K}_{(s-r)+;x,\xi}^\text{LKS,\phi} \right)^2 d\xi dr.
\]

(3.8)
equation (3.7) becomes
\[ \mathbb{E}[X(t, x) - X(s, x)]^2 \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left| e^{-\frac{(t-s-x)^2}{8}}(-2\theta + |\xi|^2)^2 \mathbb{I}_{\{t > r\}} - e^{-\frac{(s-x)^2}{8}}(-2\theta + |\xi|^2)^2 \mathbb{I}_{\{s > r\}} \right|^2 \frac{drd\xi}{(2\pi)^d}. \]
Now, we apply Parseval’s identity to the inner integral in \( r \). To this end, let
\[ \phi(r, \xi) = e^{-\frac{(t-s-x)^2}{8}}(-2\theta + |\xi|^2)^2 \mathbb{I}_{\{t > r\}} - e^{-\frac{(s-x)^2}{8}}(-2\theta + |\xi|^2)^2 \mathbb{I}_{\{s > r\}}. \]
Its Fourier transform in \( r \) is
\[ \hat{\phi}(r, \xi) = (e^{ix} - e^{is}) \frac{1}{1 + x \xi (-2\theta + |\xi|^2)^2}. \]
Hence, by Parseval’s identity, we get
\[ \mathbb{E}[X(t, x) - X(s, x)]^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\hat{\phi}(r, \xi)|^2 drd\xi \]
\[ = 2(2\pi)^{-d} \int_{\mathbb{R}} (1 - \cos((t - s)r)) \int_{\mathbb{R}^d} \frac{d\xi}{\tau^2 + \frac{\xi^4}{64}(-2\theta + |\xi|^2)^2} dr. \]
The proof of (i) is complete.

The proof of part (ii) is similar to [51, Theorem 4.8], but is more complicated in our higher order case and its corresponding kernel. For completeness, we give the main steps of the proof.

We start with the case \( k = 1 \). The mean square derivative of \( V \) at \( t \in (0, \infty) \) is given by
\[ \partial_t V(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_t \mathbb{K}_{s-r,x,y}^{LKS_d,\varphi} \mathcal{W}(dr \times dy). \]
This can be verified by checking the covariance function. For every \( s, t \in (0, \infty) \) with \( s \leq t \) we have
\[ \mathbb{E}[\partial_t V(t, x) - \partial_s V(s, x)]^2 \]
\[ = \mathbb{E} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \partial_t \mathbb{K}_{s-r,x,y}^{LKS_d,\varphi} - \partial_s \mathbb{K}_{s-r,x,y}^{LKS_d,\varphi} \right) \mathcal{W}(dr \times dy) \right]^2 \]
\[ = \int_0^t \int_{\mathbb{R}^d} \left| \partial_t \mathbb{K}_{s-r,x,y}^{LKS_d,\varphi} - \partial_s \mathbb{K}_{s-r,x,y}^{LKS_d,\varphi} \right|^2 drd\xi \]
\[ = C \int_{\mathbb{R}^d} (-2\theta + |\xi|^2)^4 \int_0^t \left| e^{-\frac{1}{8}((t-s)(-2\theta + |\xi|^2)^2)} - e^{-\frac{1}{8}((s-t)(-2\theta + |\xi|^2)^2)} \right|^2 drd\xi, \]
where we have used Parseval’s identity to the integral in \( y \) and the fact that the Fourier transform of the function \( y \mapsto \partial_t \mathbb{K}_{s-t+r,y}^{LKS_d,\varphi} \) is
\[ \hat{\partial_t \mathbb{K}_{s-t+r,y}^{LKS_d,\varphi}} = -\frac{\varepsilon(-2\theta + |\xi|^2)^2}{8(2\pi)^{d/2}} e^{-\frac{1}{8}((t-s)(-2\theta + |\xi|^2)^2)}. \]
Thus, for any \(0 < a < b < \infty\), we see that for each \(s, t \in [a, b]\) with \(s < t\) equation (3.11) becomes

\[
\mathbb{E} |\partial_t V(t, x) - \partial_s V(s, x)|^2 \\
= C \int_{\mathbb{R}^d} (-2\theta + |\xi|^2)^4 |e^{-\frac{\tau}{4}(-2\theta + |\xi|^2)} - e^{-\frac{s}{4}(-2\theta + |\xi|^2)}|^2 d\tau d\xi \\
\times \int_{\mathbb{R}^d} \frac{1}{\tau^2 + \frac{\tau^2}{64}(-2\theta + |\xi|^2)^4} d\tau d\xi \\
\leq C |t - s|^2 \int_{\mathbb{R}^d} (-2\theta + |\xi|^2)^6 e^{-\frac{\tau}{4}(-2\theta + |\xi|^2)} d\xi \\
\leq C |t - s|^2.
\]

Now, by using Kolmogorov’s continuity theorem, we can find a modification of \(V\) such that \(V(t, x)\) is continuously differentiable on in \(t\) on \([a, b]\) (see e.g. [51]). This proves (ii) for \(k = 1\). For \(k = 2\), we apply the above argument to the Gaussian process \(\{\partial_t^2 V(t, x), t \geq 0\}\), where, for each \(t > 0\), \(\partial_t^2 V\) is the second order mean-square derivative, and we find a modification of \(V\) whose temporal sample paths are twice continuously differentiable on \([a, b]\). Iterating this procedure finishes the proof of (ii).

To prove (iii), we will apply the metric-entropy method (cf. e.g., [29]). It can be verified that \(\mathbb{E}\left[V^2(t, x)\right] \asymp t^{2\gamma_1}\) for \(t \in [0, 1]\) and that \(\mathbb{E}\left[V^2(t, x)\right] \sim Ct^{2\gamma_1}\) as \(|t| \to 0\). Recall that \(\gamma_1 = (4 - d)/8\). For any \(0 < s < t\), we proceed similarly to part (ii) above to get

\[
\mathbb{E} |V(t, x) - V(s, x)|^2 \\
= C \int_{\mathbb{R}^d} \left|e^{-\frac{\tau}{4}(-2\theta + |\xi|^2)} - e^{-\frac{s}{4}(-2\theta + |\xi|^2)}\right|^2 d\tau d\xi \\
\times \int_{\mathbb{R}^d} \frac{1}{\tau^2 + \frac{\tau^2}{64}(-2\theta + |\xi|^2)^4} d\tau d\xi \\
= C \int_{\mathbb{R}^d} e^{-\frac{\tau}{4}(-2\theta + |\xi|^2)} \left|1 - e^{-\frac{s}{4}(-2\theta + |\xi|^2)}\right|^2 \frac{d\xi}{(-2\theta + |\xi|^2)^2} \\
\leq C |t - s|^2 \int_{\mathbb{R}^d} (-2\theta + |\xi|^2)^6 e^{-\frac{\tau}{4}(-2\theta + |\xi|^2)} d\xi \\
\leq Cs^{2\gamma_1 - 2} |t - s|^2.
\]

Thus, the canonical metric of \(V\) is given by

\[
d_V(s, t) = \sqrt{\mathbb{E}|V(t, x) - V(s, x)|^2} \leq C \left\{ \begin{array}{ll} 
t^{\gamma_1} & \text{if } 0 = s < t, \\
s^{\gamma_1 - 1}|t - s| & \text{if } 0 < s < t.\end{array} \right. 
\]
It follows from the Gaussian isoperimetric inequality (cf. Lemma 2.1 in [41]) that there is a constant $C \geq 1$ such that for any constant $\varepsilon > 0$, and $u > 0$,

$$\mathbb{P}\left( \max_{0 \leq t, s \leq \varepsilon} |V(t, x) - V(s, x)| \geq u \right) \leq C \exp\left( - \frac{u^2}{C \varepsilon^{2\gamma_1}} \right).$$

A standard Borel-Cantelli argument yields that for some positive and finite constant $C$,

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t, s \leq \varepsilon} \frac{|V(t, x) - V(s, x)|}{|t - s|^{\gamma_1} \sqrt{\log \log(1/|t - s|)}} \leq C, \quad \text{a.s.}$$

This proves (iii). \(\Box\)

**Corollary 3.1.** Assume the spatial dimension $d \in \{1, 2, 3\}$. The spectral density $\Delta$ is asymptotically given by

$$\Delta(\tau) \sim \frac{(2\pi)^{-d/2}}{\tau^{2-d/4}} \int_{\mathbb{R}^d} \frac{d\xi}{1 + \frac{\tau^2}{\varepsilon^2} |\xi|^8}, \quad \text{as } \tau \to \infty. \tag{3.14}$$

By combining the asymptotic behavior of the spectral density $\Delta$ in Corollary 3.1 and Theorem 2.1 in Xiao [49], we obtain the following strong local nondeterminism and double-sided bounds for $\mathbb{E}[X(t, x) - X(s, x)]^2$.

**Corollary 3.2** (Temporal SLND and double-sided bounds for $X$). Assume the spatial dimension $d \in \{1, 2, 3\}$. For any $T > 0$, there is a positive constant $c$ such that for all $t \in (0, T]$ and all $0 < r \leq 1 \wedge |t|$

$$\text{Var}(X(t, x)|X(s, x); s \in [0, T], |t - s| \geq r) \geq cr^{4-d} \tag{3.15}$$

Also,

$$\mathbb{E}[X(t, x) - X(s, x)]^2 \asymp |t - s|^{\frac{4-d}{2}}; \quad \forall s, t \in [0, T]. \tag{3.16}$$

Here and on the sequel, the notation $f \asymp g$ on $S$ means $c_L g(x) \leq f(x) \leq c_U g(x)$ for all $x \in S$ for some constants $c_L, c_U$.

**Proof.** Corollary 3.1 and Theorem 2.1 in [49] imply (4.42). Corollary 3.1 and Theorem 2.5 in [49] imply (4.43). \(\Box\)

From Corollary 3.2, the Gaussian process $\{X(t, x), t \geq 0\}$ satisfies conditions (C1) and (C2) above ((A1) and (A2) in [34]). Hence we can apply the results in [34] on the uniform and local moduli of continuity to get the following theorem on the time regularity of $X$.

**Theorem 3.2.** Let $x \in \mathbb{R}^d$, $d = 1, 2, 3$, be fixed. Let $\{X(t, x), t \geq 0\}$ be defined as above and let $\gamma_1 = \frac{4-d}{8}$. Then,

(i) (Uniform Modulus of Continuity) for every compact interval $I_{\text{time}} \subseteq \mathbb{R}_+$

$$\lim_{\delta \to 0} \sup_{s, t \in I_{\text{time}}} \left| \frac{X(t, x) - X(s, x)}{|s - t|^{\gamma_1} \sqrt{\log \log(1/|s - t|)}} \right| = k_1^{(d)}; \quad \text{a.s.} \tag{3.17}$$
(ii) *(Local Modulus of Continuity)* and for every fixed \( t \geq 0 \)

\[
\lim_{\delta \to 0} \sup_{|s-t|<\delta} \frac{|X(s, x) - X(t, x)|}{\delta^{\gamma_1} \sqrt{\log \log \frac{1}{\delta}}} = k_2^{(d)}; \ a.s.
\]

(3.18)

In the above, \( 0 < k_i^{(d)} < \infty \ (i = 1, 2) \) are \( d \)-dependent constants, independent of \( x \in \mathbb{R}^d \).

**Proof.** The uniform modulus of continuity of \( X \) in (3.17) follows from Theorem 4.1 in [34]. The local modulus of continuity of \( X \) in (3.18) follows upon applying [34, Theorem 5.1]. The constants \( k_i^{(d)} \ (i = 1, 2) \) in Theorem 3.2 do not depend on \( x \in \mathbb{R}^d \) since the distribution of the process \( \{X(t, x), t \geq 0\} \) does not depend on \( x \), see (3.14).

We believe that \( k_1^{(d)} = k_2^{(d)} \) because the large deviation behavior of the tail probabilities of the maxima \( \sup_{s,t \in [0,b],|s-t| \leq \varepsilon} |X(t, x) - X(s, x)| \) and, for fixed \( t \), \( \sup_{s \leq t} |X(s, x) - X(t, x)| \) are the same. However, the method in [34] is not enough for proving \( k_1^{(d)} = k_2^{(d)} \), a different argument may be needed.

We are now ready to use the decomposition \( U(t, x) = X(t, x) - V(t, x) \) \( (t \geq 0) \), Theorems 3.1 and 3.2 to prove part (i) of Theorem 1.1.

**Proof of Theorem 1.1 (i).** In order to derive the temporal uniform modulus of continuity for our L-KS SPDE solution process \( U \), we use part (iii) of Theorem 3.1 to see that, almost surely, there exists \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\sup_{s,t \in [0, \varepsilon]} \frac{|V(t, x) - V(s, x)|}{|t-s|^{\gamma_1} \sqrt{\log \log (1/|t-s|)}} \leq C.
\]

(3.19)

By splitting the interval \( I_{time} = [0, b] \subset \mathbb{R}_+ \) into \( [0, \varepsilon_0] \cup [\varepsilon_0, b] \) and applying (3.19) and part (ii) of Theorem 3.1 respectively, one can see that the solution process \( U \) and \( \{X(t, x), t \geq 0\} \) have the same exact uniform modulus of continuity on \( I_{time} \). Hence, it follows from Theorem 3.2 (i) that Theorem 1.1 (i) (a) holds almost surely.

To prove Theorem 1.1 (i) (b), we see that, for any \( t > 0 \), (1.13) follows from part (ii) of Theorem 3.1 and part (ii) of Theorem 3.2. When \( t = 0 \), Theorem 3.2 does not imply (1.13) because the local oscillation \( V(t, x) \) at the origin may be of the same order. We can prove (1.13) for \( t = 0 \) by using the comparison result in Lemma 7.1.10 and Remark 7.1.11 in [29]. Since this is very similar to the proof of Proposition 2 in [44], we omit the details. This finishes the proof of Theorem 1.1 part (i).
3.2. The bifractional Brownian motion link: the case $\vartheta = 0$. We now turn to proof of the L-KS SPDE bifractional Brownian motion link.

**Proof of part (ii) of Theorem 1.1.** Using Parseval’s identity to compute the covariance function of $U$, we get

$$
E[U(t,x)U(s,x)] = \int_0^s \int_{\mathbb{R}^d} K_{1-r;x,y}^{LKS,\vartheta} K_{s-r;x,y}^{LKS,\vartheta} d\xi dr \\
= (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} e^{-s(t-r)/s}(-2\theta + |\xi|^2)^2 e^{-s(t-r)/s}(-2\vartheta + |\xi|^2)^2 d\xi dr \\
= (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} e^{-s(t+r-2\vartheta)/s}(-2\theta + |\xi|^2)^2 d\xi dr.
$$

(3.20)

When $\vartheta = 0$, the above becomes:

$$
E[U(t,x)U(s,x)] = (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} e^{-(t+s-2\vartheta)/s}|\xi|^4 d\xi dr \\
= \left[ (2\pi)^{-d} \left( \frac{8}{\pi} \right)^{d/4} \frac{1}{2 - d/2} \int_{\mathbb{R}^d} e^{-|\xi|^4} d\xi \right] \left[ (t+s)^{1-\frac{d}{4}} - (t-s)^{1-\frac{d}{4}} \right].
$$

(3.21)

Hence, up to a constant, the mean zero Gaussian process $\{U(t,x), t \geq 0\}$ ($x \in \mathbb{R}^d$ fixed) is a bifractional Brownian motion with indices $H = \frac{1}{2}$ and $K = 1 - \frac{d}{4}$.

More precisely, $U(\cdot, x) \overset{d}{=} c_d B(\frac{1}{2}, \frac{1-d}{2})$, where

$$
c_d = (2\pi)^{-d/2} \left( \frac{8}{\pi} \right)^{d/8} \frac{2(d-4)/s}{\sqrt{2 - d/2}} \sqrt{\int_{\mathbb{R}^d} e^{-|\xi|^4} d\xi}.
$$

(3.22)

Hence many sample path properties of $\{U(t,x), t \geq 0\}$, including Chung’s law of the iterated logarithm in (1.15), can be derived from Tudor and Xiao [43] directly. \qed

3.3. Spatial modulus. Recall our standing assumption that $u_0 = 0$, and the solution is given by (3.3). Let $t > 0$ be fixed, we consider the L-KS Gaussian random field $\{U(t,x), x \in \mathbb{R}^d\}$. Our results are based on the following lemma.

**Lemma 3.1** (L-KS SPDE spatial spectral density). Assume the spatial dimension $d \in \{1,2,3\}$. The centered Gaussian random field $\{U(t,x), x \in \mathbb{R}^d\}$ is stationary with spectral density

$$S(\xi) = \frac{4}{\xi(2\pi)^d} \frac{1 - e^{-4\pi^2(-2\vartheta + |\xi|^2)^2}}{(-2\vartheta + |\xi|^2)^2}, \quad \forall \xi \in \mathbb{R}^d.
$$

\footnote{When $\vartheta \in \mathbb{R}\setminus\{0\}$, it is not as simple to obtain an explicit expression in terms of $s$ and $t$.}
Assume, without loss of generality, that the gradient. By applying Lemma 3.3.1.

As an immediate consequence, we get

Thus, the conclusions of the Lemma follows. □

As an immediate consequence, we get

Corollary 3.3. Assume the spatial dimension \( d \in \{1, 2, 3\} \). The spectral density \( S \) satisfies \( 0 < S(0) < \infty \) and has the asymptotic behavior

\[
S(\xi) \sim \frac{C_{r,\theta}}{|\xi|^{d+2\gamma_2}},
\]

as \( |\xi| \to \infty \), where \( \gamma_2 = 2 - \frac{d}{2} \).

3.3.1. The case \( d = 1 \): gradient spatial Hölder and modulus of continuity. We now complete the proof of the one dimensional case in Theorem 1.2 (i).

Proof of Theorem 1.2 (i). Fix \( t > 0 \). We start with the Hölder assertion for the gradient. By applying Lemma 3.1, we can show that the mean square gradient \( \partial_x U(t, x) \) exists and

\[
\mathbb{E} \left[ |\partial_x U(t, x) - \partial_x U(t, y)|^2 \right] = \int_{\mathbb{R}} \xi^2 |e^{i\xi x} - e^{i\xi y}|^2 S(\xi) d\xi
\]

\[
= \frac{4}{\varepsilon \pi} \int_{\mathbb{R}} \xi^2 [1 - \cos(\xi (x - y))] \frac{1 - e^{-\frac{4}{\varepsilon t}(-2\theta + |\xi|^2)^2}}{(-2\theta + |\xi|^2)^2} d\xi.
\]

Assume, without loss of generality, that \( \varepsilon = \theta = 1 \). We also assume that \( |x - y| \leq 1/2 \). Proceeding as in the proof of [1, Lemma 3.3], we split the last integral over three sets \( B_1 := \{ \xi \in \mathbb{R} : |\xi| < 2 \} \), \( B_2 := \{ \xi \in \mathbb{R} : 2 \leq |\xi| < \frac{1}{|x-y|} \} \) and \( B_3 := \{ \xi \in \mathbb{R} : |\xi| \geq \frac{1}{|x-y|} \} \). We will make use of the following elementary inequalities:

\[
(a) \quad 1 - \cos z \leq 2 \wedge z^2,
\]

\[
(b) \quad \frac{1 - e^{-\frac{t}{4}(-2 + |\xi|^2)^2}}{(-2 + |\xi|^2)^2} \leq \left\{ \begin{array}{ll}
\frac{t}{|\xi|^2}, & \text{on } B_1, \\
\frac{1}{|\xi|^2}, & \text{on } B_2 \cup B_3.
\end{array} \right.
\]
It follows from (3.24) and (3.25) that
\[\mathbb{E}\left|\partial_x U(t,x) - \partial_y U(t,y)\right|^2 \leq C \left[ \int_{B_1} \xi^4 |x-y|^2 d\xi + \int_{B_2} |x-y|^2 \frac{1}{|\xi|^2} d\xi + \int_{B_3} \frac{1}{\xi^2} d\xi \right] \leq C|x-y|.
\]

Thus, Kolmogorov’s continuity theorem gives us the spatial local \(\gamma\)-Hölder continuity for the L-KS gradient, \(\partial_x U\), for \(\gamma \in (0,1/2)\).

Turning now to the exact uniform and local spatial continuity moduli for the L-KS gradient, \(\partial_x U\), in Theorem 1.2 (i) (a) and (b). We first compute the gradient covariance as follows
\[
\mathbb{E}\left[\partial_x U(t,x)\partial_y U(t,y)\right] = \int_{\mathbb{R}} \xi^2 \nu^{LKS}_{\epsilon,\vartheta} (\xi) \nu^{LKS}_{\epsilon,\vartheta} (\xi) d\xi dr
\]
\[
= (2\pi)^{-d} \int_{\mathbb{R}} \xi^2 e^{i(\xi,\vartheta)} \cdot e^{-\frac{\epsilon^2 + \vartheta^2}{2} - \frac{\epsilon^2 + \vartheta^2}{2} |\xi|^2} d\xi dr
\]
\[
= \frac{4}{\epsilon} (2\pi)^{-d} \int_{\mathbb{R}} e^{i(\xi,\vartheta)} \xi^2 \left[ 1 - e^{-\frac{\epsilon^2 + \vartheta^2}{2} |\xi|^2} \right] \left( -\epsilon^2 + \vartheta^2 |\xi|^2 \right) d\xi.
\]

This means that the spatial spectral density of \(\partial_x U\), denoted by \(f\), and its asymptotic behavior are given by
\[
f(\xi) = \frac{4}{\epsilon} (2\pi)^{-d} \xi^2 \left[ 1 - e^{-\frac{\epsilon^2 + \vartheta^2}{2} |\xi|^2} \right] \left( -\epsilon^2 + \vartheta^2 |\xi|^2 \right) \sim \frac{C}{|\xi|^2}, \quad \text{as } |\xi| \to \infty.
\]

Equation (3.28) and Theorem 2.1 in [49] imply that, for every fixed \(t > 0\), the gradient of the L-KS SPDE solution \(\{\partial_x U(t,x), x \in \mathbb{R}\}\) is spatially strongly locally nondeterministic. More precisely, for every \(M > 0\), there exists a finite constant \(c > 0\) (depending on \(t\) and \(M\)) such that for every \(n \geq 1\) and for every \(x, y_1, \ldots, y_n \in [-M, M]\),
\[
\text{Var} \left[ \partial_x U(t,x)| \partial_x U(t,y_1), \ldots, \partial_x U(t,y_n) \right] \geq c \min_{0 \leq j \leq n} \{|x-y_j|\},
\]
where \(y_0 = 0\). Also, (3.28) and Theorem 2.5 in [49] imply the double sided second moment bounds
\[
\mathbb{E}[|\partial_x U(t,x) - \partial_y U(t,y)|^2] \geq |x-y|; \quad \forall x, y \in [-M, M].
\]

Thus, the uniform modulus of continuity of \(\partial_x U\) in Theorem 1.2 (i) (a) follows from Theorem 4.1 in [34]. The local modulus of continuity of \(\partial_x U\) in Theorem 1.2 (i) (b) follows upon applying [34, Theorem 5.1].

3.3.2. The fractal cases \(d = 2, 3\). We now turn to the rougher two and three dimensional cases. Starting with the \(d = 3\) case, we first obtain the strong local nondeterminism property and double-sided second moment bounds in space for the L-KS SPDE solution \(\{U(t,x); x \in \mathbb{R}^3\}\).

**Lemma 3.2** (Spatial SLND and double-sided bounds for L-KS SPDEs). For every fixed \(t > 0\), the L-KS SPDE solution \(\{U(t,x), x \in \mathbb{R}^3\}\) is spatially strongly locally nondeterministic. Namely, for every \(M > 0\), there exists a finite constant \(c > 0\)
(depending on $t$ and $M$) such that for every $n \geq 1$ and for every $x, y_1, ..., y_n \in [-M, M]$, 
\begin{equation} \label{eq:3.31} \text{Var}[U(t, x)|U(t, y_1), \ldots, U(t, y_n)] \geq c \min_{0 \leq j \leq n} \{|x - y_j|\}, \end{equation}
where $y_0 = 0$. Also,
\begin{equation} \label{eq:3.32} \mathbb{E}|U(t, x) - U(t, y)|^2 \asymp |x - y|; \quad \forall x, y \in [-M, M]. \end{equation}

**Proof.** When $d = 3$, Corollary 3.3 implies that the condition (2.17) in [49] is satisfied with $\alpha = \alpha = \gamma = \frac{1}{2}$. Hence, the conclusions in (3.31) and (3.32) follow from Theorem 2.5 in [49] with $\phi(r) = r$. \hfill $\square$

Now we can obtain the exact spatial uniform and local continuity moduli in Theorem 1.2 (ii) for the three dimensional case.

**Proof of Theorem 1.2 (ii).** With Lemma 3.2 in hand, the uniform modulus of continuity of $U$ in Theorem 1.2 (ii) (a) follows from Theorem 4.1 in [34]; and the local modulus of continuity of $U$ in Theorem 1.2 (ii) (b) follows upon applying [34, Theorem 5.1]. \hfill $\square$

Finally we turn to the proof of the upper bounds on the uniform and local continuity moduli in the critical two dimensional case in Theorem 1.2 (iii).

**Proof of Theorem 1.2 (iii).** Similarly to (3.24), we apply Lemma 3.1 to derive that for $d = 2$
\begin{equation} \label{eq:3.33} \mathbb{E}|U(t, x) - U(t, y)|^2 = \int_{\mathbb{R}^2} |e^{i(y, \xi)} - e^{i(y, \xi)}|^2 S(\xi) d\xi \end{equation}
\begin{equation} = \frac{2}{\varepsilon \pi^2} \int_{\mathbb{R}^2} \left[1 - \cos(\xi, x - y)\right] \frac{1 - e^{-\frac{\varepsilon}{(-2\theta + |\xi|^2)^2}}}\end{equation}
As in the proof of Theorem 1.2 (i), we assume $\varepsilon = \theta = 1$ and $|x - y| \leq 1/2$. Let $B_1 := \{\xi \in \mathbb{R}^2 : |\xi| < 2\}$, $B_2 := \{\xi \in \mathbb{R}^2 : 2 \leq |\xi| < \frac{1}{|x - y|}\}$ and $B_3 := \{\xi \in \mathbb{R}^2 : |\xi| \geq \frac{1}{|x - y|}\}$. By splitting the last integral in (3.33) over three sets $B_1, B_2, B_3$ and by using the inequalities in (3.25), one can derive
\begin{equation} \mathbb{E}|U(t, x) - U(t, y)|^2 \leq C|x - y|^2 \int_{B_2} \frac{d\xi}{|\xi|^2} \leq C|x - y|^2 \log \left(\frac{1}{|x - y|}\right). \end{equation}
The desired upper bounds for the uniform and local continuity moduli for the sample function $x \mapsto U(t, x)$ in $d = 2$ follow from the Gaussian isoperimetric inequality and a Borel-Cantelli argument. Since this is the same as that in the proof of part (iii) of Theorem 3.1, we omit the details. \hfill $\square$

It is natural to expect that (1.20) and (1.21) hold with “$\leq$” replaced by “$=$”, which would give the exact uniform and local continuity moduli for $x \mapsto U(t, x)$ in $d = 2$. However, substantial extra work is needed for proving these statements.
In particular, in order to apply the method in [34], one will have to establish the property of strong nondeterminism for $U(t, \cdot)$. Unfortunately, the method in [49] does not seem useful anymore and some new ideas may be needed.

### 3.4. The L-KS gradient temporal H"older and modulus of continuity.

We prove the temporal regularity of the spatial gradient $\partial_x U$ in Theorem 1.3.

**Proof of Theorem 1.3.** Let $d = 1$. We start with the H"older assertion for the gradient. Recall that $U(t, x) = X(t, x) - V(t, x)$ and that the temporal regularity of $U$ is totally determined by the rougher process $X$. Similarly, the temporal regularity of the gradient $\partial_x U$ is entirely determined by the gradient of the rougher auxiliary process $X (\partial_x X)^{19}$. Here, $\partial_x X$ plays the role of the auxiliary process for $\partial_x U$. We start with Parseval’s identity to the integral in $y$ to get:

$$
\mathbb{E} [\partial_x X(t, x) - \partial_x X(s, x)]^2 = \int \int \left| \partial_x K_{LKS,x_0}^{d, \theta}(t-r, x, y) - \partial_x K_{LKS,x_0}^{d, \theta}(s-r, x, y) \right|^2 dr dy
$$

(3.35)

$$
= \int \int \xi^2 \left| K_{LKS,x_0}^{d, \theta}(t-r, x, \xi) - K_{LKS,x_0}^{d, \theta}(s-r, x, \xi) \right|^2 d\xi dr.
$$

Since

$$
K_{LKS,x_0}^{d, \theta}(t-r, x, \xi) = (2\pi)^{-d/2} e^{-\frac{s(x, \xi)}{s} (-2\theta + |\xi|^2)^2} \mathbb{1}_{\{t \geq r\}},
$$

equation (3.35) becomes

$$
\mathbb{E} [\partial_x X(t, x) - \partial_x X(s, x)]^2
$$

(3.37)

$$
= \int \int \xi^2 \frac{e^{-\frac{\xi(\xi, s)}{s} (-2\theta + |\xi|^2)^2} \mathbb{1}_{\{t \geq r\}} - e^{-\frac{\xi(\xi, s)}{s} (-2\theta + |\xi|^2)^2} \mathbb{1}_{\{s \geq r\}}}{(2\pi)^d} d\xi dr.
$$

Now, we apply Parseval’s identity to the inner integral in $r$. To this end, let

$$
\phi(r, \xi) = e^{-\frac{\xi(\xi, s)}{s} (-2\theta + |\xi|^2)^2} \mathbb{1}_{\{t \geq r\}} - e^{-\frac{\xi(\xi, s)}{s} (-2\theta + |\xi|^2)^2} \mathbb{1}_{\{s \geq r\}}.
$$

Its Fourier transform in $r$ is

$$
\hat{\phi}(\tau, \xi) = (e^{i\tau t} - e^{i\tau s}) \frac{1}{\tau + \frac{s}{4} (-2\theta + |\xi|^2)^2}.
$$

Hence, by Parseval’s identity, inequality (3.25) (a), its related inequality

$$
1 - \cos (z \cdot \tau) \leq 2 \left( 1 \wedge |z|^{2\alpha} \right) \left[ 1 - \cos(\tau) \right]; \quad 0 < \alpha \leq 1,
$$

and the asymptotic

$$
\Delta(\tau) := (2\pi)^{-1} \int \int \frac{\xi^2}{\tau^2 + \frac{s^2}{4} (-2\theta + |\xi|^2)^4} d\xi \sim \frac{C}{|\tau|^{5/4}}, \quad \text{as } |\tau| \to \infty,
$$

---

19 It can be shown that the smoothness assertions in Theorem 3.1 (ii) and (iii) (with $\gamma_1 = 1/8$) hold for $\partial_x V$. Since the proof follows the same steps as the one for Theorem 3.1 (ii) and (iii) with straightforward modifications, we leave it to the interested reader.
we get, for a large enough $N$, that

$$
\mathbb{E} \left[ \partial_x X(t, x) - \partial_x X(s, x) \right]^2 = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 \left| \hat{\phi}(\tau, \xi) \right|^2 d\tau d\xi
$$

$$
= \frac{1}{\pi} \int_{\mathbb{R}} (1 - \cos((t - s)\tau)) \int_{\mathbb{R}} \frac{\xi^2 d\xi}{\tau^2 + \frac{\xi^2}{64}(-2\theta + |\xi|^2)^4} d\tau
$$

$$
\leq C|t - s|^{2\alpha} \left[ \int_{0}^{N} (1 - \cos(\tau))\Delta(\tau) d\tau + \int_{N}^{\infty} |\tau|^{2\alpha - \frac{\alpha}{2}} d\tau \right]
$$

$$
\leq C|t - s|^{2\alpha}, \quad 0 < \alpha < 1/8.
$$

(3.40)

It follows that $\partial_x X(\cdot, x)$ is $\gamma$-Hölder continuous in time, with $\gamma \in (0, 1/8)$. This, together with the gradient decomposition

$$
\partial_x U(t, x) = \partial_x X(t, x) - \partial_x V(t, x),
$$

and the fact that $\partial_x V$ is temporally smooth (see footnote 19) establish the Hölder regularity assertion for $\partial_x U$ in Theorem 1.3.

Turning now to the uniform and local spatial continuity moduli results for the L-KS gradient, $\partial_x U$, in Theorem 1.3. Equation (3.40) means that $\partial_x X$ has stationary increments and the spatial spectral density of $\partial_x X$ and its asymptotic behavior are given by (3.39).

Equation (3.39) and Theorem 2.1 in [49] imply that, for every fixed $x \in \mathbb{R}$, the gradient \{ $\partial_x X(t, x), t \geq 0$ \} is temporally strongly locally nondeterministic. Namely, for any $T > 0$, there is a positive constant $c$ such that for all $t \in (0, T]$ and all $0 < r \leq 1$ and $|t - s| r$,

$$
\text{Var} \left( \partial_x X(t, x) \right| \partial_x X(s, x); s \in [0, T], |t - s| \geq r \right) \geq cr^{\frac{1}{2}}
$$

(3.42)

Also, (3.28) and Theorem 2.5 in [49] imply the double sided second moment bounds

$$
\mathbb{E} \left[ \partial_x X(t, x) - \partial_x X(s, x) \right]^2 \sim |t - s|^{\frac{1}{2}}; \quad \forall s, t \in [0, T].
$$

Thus, the uniform modulus of continuity of $\partial_x X$

$$
P \left[ \lim_{\delta \searrow 0} \sup_{t, s \in I_{time}} \frac{\left| \partial_x X(t, x) - \partial_x X(s, x) \right|}{|t - s|^{1/8} \sqrt{\log \left( \frac{1}{|t - s|} \right)}} = k \right] = 1,
$$

(3.44)

for every compact interval $I_{time} \subset \mathbb{R}^+$ and for some constant $k > 0$, follows from Theorem 4.1 in [34]. The local modulus of continuity of $\partial_x X$

$$
P \left[ \lim_{\delta \searrow 0} \sup_{|t - s| < \delta} \frac{\left| \partial_x X(t, x) - \partial_x X(s, x) \right|}{\delta^{1/8} \sqrt{\log \log \left( \frac{1}{\delta} \right)}} = k \right] = 1,
$$

(3.45)

follows upon applying [34, Theorem 5.1]. The corresponding continuity moduli assertions for the gradient $\partial_x U$ in Theorem 1.3 follow from those of the auxiliary process $\partial_x X$ ((3.44) and (3.45)), the decomposition (3.41), and the smoothness of $\partial_x V$ (see footnote 19).
4. The time-fractional SPIDEs: Proofs of Theorems 1.4–1.7

As with the L-KS SPDE case, Allouba obtained in [3, 2], the time and space Hölder exponents $\gamma_\ell \in (0, (2\beta - 1 - d)/4\beta - 1)$ and $\gamma_\delta \in (0, ((4 - d)/2)/\beta)$, respectively, after establishing the sharp dimension-and-$\beta$-dependent upper bounds

\begin{equation}
(4.1) \quad \begin{cases}
\mathbb{E} [U_\beta(t, x) - U_\beta(s, x)]^2 \leq C_{d, \beta} |t - s|^\frac{(2\beta - 1 - d)}{2\beta - 1}, \\
\mathbb{E} [U_\beta(t, x) - U_\beta(t, y)]^2 \leq C_d |x - y|^{2\alpha_d}; \quad \alpha_d \in J_d,
\end{cases}
\end{equation}

for the more general nonlinear time-fractional SPIDE \((1.6)\), with Lipschitz condition on \(a\), for all \(x \in \mathbb{R}^d\), \(t, s \in [0, T]\), \(q \geq 1\), \(1 \leq d \leq 3\), \(\beta \in \{1/2^k; k \in \mathbb{N}\}\), and for the intervals \(J_d\) as in \((3.2)\). Now, we take the spectral/harmonic analysis and solution decomposition route we took in Section 3.1—with the time-fractional kernel \(K_{t,x}^{\beta,d}\) replacing the L-KS one—to get the exact dimension-dependent temporal and spatial uniform and local moduli of continuity in Theorem 1.4 and in Theorem 1.5.

Assume without loss of generality that \(u_0 = 0\), then the \(\beta\) time-fractional SPIDE solution is given by

\begin{equation}
(4.2) \quad U_\beta(t, x) = \int_{\mathbb{R}^d} \int_0^t \int_{-\xi}^{\xi} \mathbb{K}_{t,x,y}^{(\beta,d)} \mathcal{W}(ds \times dy), \quad t \geq 0, \ x \in \mathbb{R}^d.
\end{equation}

4.1. Temporal modulus. Throughout this subsection, let \(x \in \mathbb{R}^d\) be fixed but arbitrary. Let \(U_\beta\) be the solution to the time-fractional SPIDE \((1.2)\), given by \((4.2)\). Following the template used in the L-KS proofs, we first introduce the following auxiliary Gaussian process \(\{X_\beta(t, x), t \in \mathbb{R}_+\}\):

\begin{equation}
(4.3) \quad X_\beta(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \mathbb{K}_{t-r_s,x,y}^{(\beta,d)} - \mathbb{K}_{t-r_s,x,y}^{(\beta,d)} \right) \mathcal{W}(dr \times dy),
\end{equation}

where \(x \in \mathbb{R}^d\) is arbitrary but fixed. Then the solution \(U_\beta\) may be decomposed as \(U_\beta(t, x) = X_\beta(t, x) - V_\beta(t, x)\), where

\begin{equation}
(4.4) \quad V_\beta(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \mathbb{K}_{t-r_s,x,y}^{(\beta,d)} - \mathbb{K}_{t-r_s,x,y}^{(\beta,d)} \right) \mathcal{W}(dr \times dy).
\end{equation}

We start by proving the following crucial result for the auxiliary process \(X_\beta\) and the smoothness of \(V_\beta\).

**Theorem 4.1.** Assume \(d \in \{1, 2, 3\}\) and \(0 < \beta \leq 1/2\). Let \(X_\beta\) be as defined in \((4.3)\).

(i) The Gaussian process \(\{X_\beta(t, x); t \geq 0\}\) has stationary temporal increments. Moreover we have

\[\mathbb{E} [X_\beta(t, x) - X_\beta(s, x)]^2 = 2 \int_{\mathbb{R}} \left| 1 - \cos((t - s)\tau) \right| \Delta_\beta(\tau) d\tau,\]

where the spectral density \(\Delta_\beta\) is given by

\begin{equation}
(4.5) \quad \Delta_\beta(\tau) = (2\pi)^{-d} \frac{1}{|\tau|^{2-(2\beta d)/2}} \int_{\mathbb{R}^d} \frac{d\xi}{1 + |\xi|^{2} \cos \left( \frac{\tau \xi}{2} \right) + \frac{1}{4} |\xi|^4}.
\end{equation}

(ii) For each \(k \geq 1\), there exists a modification of \(\{V_\beta(t, x), t \in \mathbb{R}_+\}\) such that its (temporal) sample function is almost surely continuously \(k\)-times differentiable on \((0, \infty)\).
(iii) Let $H = \frac{2-\beta d}{4}$. There is a finite constant $C$ such that

\begin{equation}
\lim_{\varepsilon \to 0} \sup_{s,t \in [0, \varepsilon]} |V_\beta(t, x) - V_\beta(s, x)| \leq C \quad \text{a.s.}
\end{equation}

Proof. To verify (i), we apply Parseval’s identity to the integral in $y$ to get:

\begin{equation}
\mathbb{E} [X_\beta(t, x) - X_\beta(s, x)]^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K_\beta(x-y) - K_\beta(x-y)|^2 \, dr \, dy
\end{equation}

\begin{equation}
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| K_\beta(x-r) - K_\beta(s-r) \right|^2 \, d\xi \, dr.
\end{equation}

Since

\begin{equation}
\hat{K}_\beta(x-r) = (2\pi)^{-d/2} \cdot e^{-i(x, r)} E_\beta \left( - \frac{1}{2} (t-r)^\beta \right) \mathbb{1}_{\{t > r\}},
\end{equation}

equation (4.7) becomes

\begin{equation}
\mathbb{E} [X_\beta(t, x) - X_\beta(s, x)]^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\phi(r, \xi)|^2 \, dr \, d\xi,
\end{equation}

where

\begin{equation}
\phi(r, \xi) = E_\beta \left( - \frac{1}{2} (t-r)^\beta \right) \mathbb{1}_{\{t > r\}} - E_\beta \left( - \frac{1}{2} (s-r)^\beta \right) \mathbb{1}_{\{s > r\}}.
\end{equation}

Now, we apply Parseval’s identity to the inner integral in $r$. To this end, assume for simplicity and without loss of generality that $\beta \in \{1/2k; k \in \mathbb{N}\}$. In this case, using Lemma 2.2 above, the Fourier transform of $\phi$ in $r$ is

\begin{equation}
\hat{\phi}(r, \xi) = (e^{ir} - e^{is}) \frac{i^{\beta-1} \tau^{\beta-1}}{i^{\beta} \tau^{\beta} + \frac{1}{2} |\xi|^2}.
\end{equation}

Hence, by Parseval’s identity, we get

\begin{equation}
\mathbb{E} [X_\beta(t, x) - X_\beta(s, x)]^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{\phi}(r, \xi)|^2 \, dr \, d\xi
\end{equation}

\begin{equation}
= 2(2\pi)^{-d} \int_{\mathbb{R}^d} \left( 1 - \cos((t-s)\tau) \right) \int_{\mathbb{R}^d} \frac{\tau^{2(\beta-1)}}{\tau^{2\beta} + |\xi|^2 \tau^\beta \mathbb{R}(1^\beta) + \frac{1}{4} |\xi|^4} \, d\tau \, d\xi
\end{equation}

\begin{equation}
= 2(2\pi)^{-d} \int_{\mathbb{R}^d} \left( 1 - \cos((t-s)\tau) \right) \frac{d\tau}{|\tau|^{2-(\beta d)/2}} \int_{\mathbb{R}^d} \frac{d\xi}{1 + |\xi|^2 \cos \left( \frac{\pi \beta}{2} \right) + \frac{1}{4} |\xi|^4}.
\end{equation}

The proof of (i) is complete. The proof of parts (ii) and (iii) is very similar to the proof of Theorem 3.1 (ii) and (iii), with now obvious modifications. We leave the details to the interested reader. □

Using the asymptotic behavior of the spectral density $\Delta_\beta$ in (4.5) Theorem 4.1 (i), we proceed as in Section 3.1 to obtain the following SLND and two-sided bounds for $X_\beta$. 
Corollary 4.1 (Temporal SLND and double-sided bounds for $X_\beta$). Let $d \in \{1, 2, 3\}$ and let $0 < \beta \leq 1/2$. For any $T > 0$, there is a positive constant $c$ such that for all $t \in (0, T]$ and all $0 < r \leq 1 \wedge |t|$ such that

$$
(4.12) \quad \text{Var}(X_\beta(t,x)|X_\beta(s,x); s \in [0,T], |t-s| \geq r) \geq cr^{2-\beta d},
$$

and

$$
(4.13) \quad \mathbb{E}[X_\beta(t,x) - X_\beta(s,x)]^2 \asymp |t-s|^{2-\beta d}; \quad \forall s, t \in [0,T].
$$

Moreover, the function $\sigma_\beta^2(h) = \mathbb{E}[X_\beta(t+h,x) - X_\beta(t,x)]^2$ is regularly varying at $h = 0$ of order $(2 - \beta d)/2$.

Proof. The property of the spectral density $\Delta_\beta$ in Theorem 4.1 (i) and Theorem 2.1 in [49] imply (4.12). Similarly, Theorem 4.1 (i) and Theorem 2.5 in [49] imply (4.13). Finally, since the spectral density $\Delta_\beta$ is regularly varying of order $-(2-\beta d)$ at $\infty$, the last conclusion of the corollary follows from Theorem 1 in [38].

From Corollary 4.1, conditions (C1) and (C2) above ((A1) and (A2) in [34]) hold. Now, applying the results in [34] on the uniform and local continuity moduli for the auxiliary Gaussian processes to $\{X_\beta(t,x), t \geq 0\}$, we get the following theorem on the time regularity of $X_\beta$. Recall that $H = \frac{2-\beta d}{4}$.

Theorem 4.2. Let $x \in \mathbb{R}^d$, $d = 1, 2, 3$, be fixed; let $0 < \beta \leq 1/2$; and let $\{X_\beta(t,x), t \geq 0\}$ be defined as in (4.3) above. Then,

(i) \textbf{(Uniform Modulus of Continuity)} for every compact interval $I_{\text{time}} \subseteq \mathbb{R}_+$

$$
(4.14) \quad \lim_{\delta \searrow 0} \sup_{s,t \in I} \frac{|X_\beta(t,x) - X_\beta(s,x)|}{|s-t|^{H} \log \frac{1}{|s-t|}} = k_6^{(\beta,d)}; \quad \text{a.s.}
$$

(ii) \textbf{(Local Modulus of Continuity)} and for every fixed $t \geq 0$

$$
(4.15) \quad \lim_{\delta \searrow 0} \sup_{|s-t| < \delta} \frac{|X_\beta(s,x) - X_\beta(t,x)|}{\delta^H \log \log \frac{1}{\delta}} = k_7^{(\beta,d)}; \quad \text{a.s.,}
$$

where $k_i^{(\beta,d)}$ (i = 6, 7) are positive and finite constant that depend on $d$ and $\beta$, but are independent of $x$.

Proof. The uniform modulus of continuity of $X_\beta$ in (4.14) follows from Theorem 4.1 in [34]. The local modulus of continuity of $X_\beta$ in (4.15) follows upon applying [34, Theorem 5.1]. The constant $k_i^{(\beta,d)}$ (i = 6, 7) do not depend on $x \in \mathbb{R}^d$ since the distribution of the process $\{X_\beta(t,x), t \geq 0\}$ does not depend on $x$, see (3.14).

Now we use the decomposition $U_\beta(t,x) = X_\beta(t,x) - V_\beta(t,x)$ to prove Theorem 1.4 (i).

Proof of Theorem 1.4 (i). As in the proof of of Theorem 1.1 (i), we see that
(1.25) and (1.26) follow from the aforementioned decomposition and Theorems 4.1 and 4.2.

4.2. Time-fractional SPIDEs are not bifractional Brownian motions. Let $U_\beta$ be the solution to the time-fractional SPIDE (1.2), given in (4.2). We now characterize the law of $\{U_\beta(t, x); t \geq 0\}$—which we call the $\beta$ time-fractional SPIDE law—and we show that, unlike the L-KS SPDE, it’s fundamentally different from the bifractional Brownian motion law.

**Proof of Theorem 1.4 (ii).** For any $0 < s < t$, we use Parseval’s identity to get

$$
\mathbb{E}[U_\beta(t, x)U_\beta(s, x)] = \int_{\mathbb{R}^d} \int_0^s \mathbb{K}^{(\beta, d)}_{t-r, x, y} \mathbb{K}^{(\beta, d)}_{s-r, x, y} dr dy
$$

$$
= \int_0^s \int_{\mathbb{R}^d} \mathbb{K}^{(\beta, d)}_{t-r, x, y} \mathbb{K}^{(\beta, d)}_{s-r, x, y} d\xi dr
$$

$$
= (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} \mathbb{K}^{(\beta, d)}_{t-r, x, y} \mathbb{K}^{(\beta, d)}_{s-r, x, y} d\xi dr
$$

$$
= (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} E_\beta \left( -\frac{|\xi|^2}{2} (t-r)^\beta \right) E_\beta \left( -\frac{|\xi|^2}{2} (s-r)^\beta \right) d\xi dr
$$

$$
= (2\pi)^{-d} \int_0^s \int_{\mathbb{R}^d} \sum_{k=0}^\infty \sum_{j=0}^k \frac{(t-r)^{\beta j} (s-r)^{\beta(k-j)}}{\Gamma(1 + \beta j) \Gamma(1 + \beta(k-j))} \left( -\frac{1}{2} \right)^k \frac{|\xi|^{2k}}{2^k} d\xi, dr
$$

which proves the covariance assertion of Theorem 1.4 (ii). Moreover, we see from the above that for any constant $c > 0$,

$$
\mathbb{E}[U_\beta(ct, x)U_\beta(cs, x)] = c^{2-\beta d/4} \mathbb{E}[U_\beta(t, x)U_\beta(s, x)].
$$

Hence the Gaussian process $U_\beta = \{U_\beta(t, x), t \geq 0\}$ is self-similar with index $(2 - \beta d)/4$.

To show that $U_\beta$ does not have the same law as any bifractional Brownian motion, and to give an alternative form of the covariance function $\mathbb{E}[U_\beta(t, x)U_\beta(s, x)]$, we exploit the form of the kernels $\mathbb{K}^{(\beta, d)}_{t,x}$ directly rather than using their Fourier
transforms. Computing the covariance of \( U_\beta \), directly we obtain

\[
\mathbb{E}[U_\beta(t, x)U_\beta(s, x)] = \int_{\mathbb{R}^d} \int_0^s \mathbb{K}_{r; x, y}^{(\beta, d)} \mathbb{K}_{r; x, y}^{(\beta, d)} dr dy
\]

\[
= 2^k \int_0^s \left\{ \int_0^\infty \left( \int_0^\infty K_{s; x, y}^{(\beta, d)} dy \right) \right\} dr
\]

\[
\times \left( \prod_{k=0}^{k-2} K_{s-r; 0}^{(\beta, d)} ds \cdots ds \right)
\]

\[
\times \left( \prod_{i=0}^{k-2} K_{s-r; 0}^{(\beta, d)} ds \cdots ds \right)
\]

\[
\times \left( \prod_{i=0}^{k-2} K_{s-r; 0}^{(\beta, d)} ds \cdots ds \right)
\]

\[
(4.16)
\]

Gathering the two inside integrals and transforming to polar coordinates \((s_i, u_i) \mapsto (\rho_i, \theta_i), i = 1, \ldots, k\), letting \( \rho = (\rho_1, \ldots, \rho_k) \) and \( \theta = (\theta_1, \ldots, \theta_k) \), letting \( I_{\pi} = (0, \pi/2) \), and noticing that all \( \rho_i \) for \( i = 2, 3, \ldots, k \) cancel when \( k \geq 2 \); equation (4.16) gives us the covariance \( \mathbb{E}[U_\beta(t, x)U_\beta(s, x)] \) as

\[
(4.17)
\]

\[
C_{\beta, d} \int_0^s \int_{\mathbb{R}^+} \frac{e^{-\rho^2 \left[ \cos^2(\theta_1) + \sin^2(\theta_1) \right]}}{\sqrt{(t-r)(s-r)}} \prod_{i=0}^{k-2} e^{-\rho_{i+1}^2 \left[ \cos^2(\theta_{i+1}) + \sin^2(\theta_{i+1}) \right]} \frac{\rho_i^{d-1} \sin(\theta_1) + \cos(\theta_1)}{\sqrt{\sin(\theta_{i+1}) \cos(\theta_{i+1})}} \rho \, d\rho \, d\theta.
\]

To simplify our computations, it is enough for our purposes to assume that \( k = 1 \) or \( \beta = 1/2 \) (the Brownian-time Brownian motion case) and take \( d = 2 \). The integrals with respect to \( \rho \) and then \( r \) in equation (4.17) then give

\[
(4.18)
\]

\[
\int_0^s \frac{e^{-\rho^2 \left[ \frac{t-r}{(t-r)(s-r)} \cos^2(\theta) \right]}}{\sqrt{(t-r)(s-r)}} \, d\rho \, dr = \int_0^s \frac{\sqrt{\pi}}{\sqrt{t-r - (t-s) \cos^2(\theta)}} \, dr
\]

\[
= 2\sqrt{\pi} \left[ \sqrt{t \sin^2(\theta) + s \cos^2(\theta) - \sqrt{t-s} |\sin(\theta)|} \right].
\]
Finally, in the BTBM $\beta = 1/2$ and $d = 2$ case, the covariance (4.17) becomes

\begin{equation}
  2\sqrt{\pi}C_{\beta,d} \int_0^{\pi/2} \frac{\sqrt{t \sin^2(\theta) + s \cos^2(\theta)} - \sqrt{t - s \sin(\theta)}}{\sin(\theta) + \cos(\theta)} d\theta
\end{equation}

\begin{equation}
  = 2\sqrt{\pi}C_{\beta,d} \left[ \int_0^{\pi/2} \frac{\sqrt{(t - s) \sin^2(\theta) + s}}{\sin(\theta) + \cos(\theta)} d\theta - \frac{\pi}{4} \sqrt{t - s} \right]
\end{equation}

\begin{equation}
  = \frac{1}{8} \left( -2 \tanh^{-1} \left( \frac{\sqrt{2s} - \sqrt{2t - 2s}}{\sqrt{2t(s + t)}} \right) \right) \sqrt{t + s}
\end{equation}

\begin{equation}
  - 2\sqrt{t + s} \left\{ \tanh^{-1} \left( \frac{\sqrt{2s} - \sqrt{2t + 2s}}{\sqrt{2t(s + t)}} \right) - \Re \left( \tanh^{-1} \left( \frac{2s + t}{2\sqrt{s(s + t)}} \right) \right) \right\}
\end{equation}

\begin{equation}
  + \sqrt{t - s} \left\{ -2 \sin^{-1} \left( \frac{2s - t}{t} \right) - 4 \ln \left( \sqrt{t + \sqrt{t - s}} \right) + 2 \ln (s + \pi) \right\}
\end{equation}

\begin{equation}
  - \frac{\pi}{4} \sqrt{t - s}
\end{equation}

It can now be easily verified that the bracketed term is not equal to

\begin{equation}
  (4.20) \quad C \left[ \sqrt{t + s} - \sqrt{t - s} \right]
\end{equation}

for any constant $C$. Thus the law of the BTBM SPIDE is not a bifractional Brownian motion in $d = 2$. The cases $d = 1, 3$ and $\beta < 1/2$ are similar and we omit them.

4.3. **Spatial modulus.** Without loss of generality, we again assume that $u_0 = 0$, and the random field solution $U_\beta$ is given by (4.2). Fix an arbitrary $t > 0$ throughout this subsection. Our spatial results for this case crucially depend on the following Lemma.

**Lemma 4.1** (Time-fractional SPIDEs spatial spectral density). Let $d = 1, 2, 3$ and $0 < \beta \leq 1/2$. The centered Gaussian random field $\{U_\beta(t, x), x \in \mathbb{R}^d\}$ is stationary with spectral density

\begin{equation}
  S_\beta(\xi) = (2\pi)^{-d} \int_0^t E_\beta^2 \left( -\frac{|\xi|^2}{2} (t - r)^\beta \right) dr
\end{equation}

\begin{equation}
  = (2\pi)^{-d} \sum_{k=0}^{\infty} (-1)^k a_k |\xi|^{2k} (t^\beta k + 1), \quad \forall \xi \in \mathbb{R}^d,
\end{equation}

where

\begin{equation}
  a_k = \sum_{j=0}^{k} \frac{1}{\Gamma(1 + \beta j) \Gamma(1 + \beta(k - j))}.
\end{equation}
Proof. Computing the covariance of $U_\beta$, we use (4.2) and Parseval’s identity to get
\[
E[U_\beta(t,x)U_\beta(t,y)] = \int_0^t \int_{\mathbb{R}^d} K_{t-r,x,z}^{(\beta,d)} K_{t-r,y,z}^{(\beta,d)} dz dr
\]
\[
= \int_0^t \int_{\mathbb{R}^d} \hat{K}_{t-r,x,\xi}^{(\beta,d)} \hat{K}_{t-r,y,\xi}^{(\beta,d)} d\xi dr
\]
\[
= (2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} e^{i\xi(x-y)} E_\beta^2 \left( -\frac{\|\xi\|^2}{2} (t-r)^\beta \right) d\xi dr
\]
\[
= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi(x-y)} \int_0^t E_\beta^2 \left( -\frac{\|\xi\|^2}{2} (t-r)^\beta \right) dr d\xi
\]
\[
= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi(x-y)} \sum_{k=0}^{\infty} \frac{(-1)^k a_k \|\xi\|^{2k} t^{\beta k+1}}{2^k (\beta k + 1)} d\xi.
\]
(4.21)
Thus, the conclusions of the Lemma follows. \[\square\]

4.3.1. Spectral asymptotics for $0 < \beta < 1/2$. We need the asymptotic behavior of $S_\beta$ at $\infty$, which is captured in the next lemma for the case $0 < \beta < 1/2$.20

Lemma 4.2. Fix an arbitrary $t > 0$ and $d = 1, 2, 3$. If and $0 < \beta < 1/2$, then the spectral density $S_\beta$ satisfies $0 < S_\beta(0) < \infty$ and has the asymptotic behavior
\[
S_\beta(\xi) \sim C_{t,\beta,d} \frac{\|\xi\|^{d+2\gamma}}{\|\xi\|^{d+2\gamma}};
\]
(4.22)
for some finite constant $C_{t,\beta,d}$ where $\gamma = 2 - \frac{d}{2}$. Moreover, $S_\beta(\xi) \leq C_{t,\beta,d} \|\xi\|^{-(d+2\gamma)}$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Proof. Let $d \in \{1, 2, 3\}$, $0 < \beta < 1/2$, and $\gamma = 2 - \frac{d}{2}$. Clearly, $0 < S_\beta(0) < \infty$ follows from Lemma 4.1. Moreover, by using the asymptotic property of the Mittag-Leffler function in (2.8), we get that as $\|\xi\| \to \infty$,
\[
S_\beta(\xi) = \frac{\int_0^t E_\beta^2 \left( -\frac{\|\xi\|^2}{2} (t-r)^\beta \right) dr}{(2\pi)^d}
\]
\[
\sim \frac{\int_0^t \left( \frac{\|\xi\|^2}{2} (t-r)^\beta \right) \Gamma^2(1-\beta) dr}{(2\pi)^d}
\]
\[
= \frac{4\Gamma^2(1-\beta) t^{1-2\beta}}{(2\pi)^d (1-2\beta) \|\xi\|^{d+2\gamma}}.
\]
(4.23)
and (4.22) follows with
\[
C_{\beta,d,t} = \frac{4\Gamma^2(1-\beta) t^{1-2\beta}}{(2\pi)^d (1-2\beta)}.
\]
Finally, the upper bound for $S_\beta(\xi)$ follows from the first equation in (4.23) and the upper bound for $M_\beta(-x)$ in (2.7). The proof is complete. \[\square\]

20Another approach is used—and a different result is obtained—for the case $\beta = 1/2$, which we provide next.
4.3.2. Spectral asymptotics for the critical fraction $\beta = 1/2$. Since the second integral in (4.23) diverges at $\beta = 1/2$, the proof of the case $0 < \beta < 1/2$ above does not work for the case $\beta = 1/2$. The reason, as is clear from Theorem 1.5 and Theorem 1.6, is that the case $\beta = 1/2$ has a rougher modulus than that of $\beta < 1/2$. This is captured in the following lemma.

**Lemma 4.3** (Spectral asymptotic behavior at $\beta = 1/2$). Fix an arbitrary $t > 0$, and let $\beta = 1/2$. As $|\xi| \to \infty$, the spectral density has the asymptotic behavior

\[ S_{1/2}(\xi) \sim \frac{C_{t,d}}{|\xi|^{d+2\gamma}} \log |\xi|, \]  

for some finite constant $C_{t,d}$ where $\gamma = 2 - \frac{d}{2}, d = 1, 2, 3$. Moreover, $S_{1/2}(\xi) \leq C_{t,d}|\xi|^{-(d+2\gamma)} \log |\xi|$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

**Proof.** By Lemma 4.1, Lemma 2.1, and footnote 17, we have

\[ S_{1/2}(\xi) = (2\pi)^{-d} \int_0^t \left( e^{\frac{t|\xi|^4}{2}} \left( \frac{2}{\sqrt{\pi}} \int_{\frac{t|\xi|^2}{2}}^{\infty} e^{-\tau^2} d\tau \right) \right)^2 d\rho \]

\[ = \frac{(2\pi)^{-d}}{|\xi|^4} \int_0^t e^{\frac{t|\xi|^4}{2}} \left( \frac{2}{\sqrt{\pi}} \int_{\frac{t|\xi|^2}{2}}^{\infty} e^{-\tau^2} d\tau \right)^2 d\rho \]

\[ \sim \frac{(2\pi)^{-d}}{|\xi|^4} \int_1^t e^{\frac{t|\xi|^4}{2}} \left( \frac{2}{\sqrt{\pi}} \int_{\frac{T}{2}}^{\infty} e^{-\tau^2} d\tau \right)^2 d\rho, \]  

where we have used the change of variable $\rho = r|\xi|^4$. Now, using the standard asymptotic for Mills’ ratio for the standard normal random variable, $m(x) = \frac{\int_{-\infty}^x e^{-u^2/2} du}{e^{-x^2/2}} \sim 1/x$, we get

\[ S_{1/2}(\xi) \sim \frac{C_d}{|\xi|^4} \int_1^t e^{\frac{t|\xi|^4}{2}} \left( \frac{2e^{-t/\sqrt{\rho}}}{\sqrt{\rho}} \right)^2 d\rho \]

\[ = \frac{C_d}{|\xi|^4} \int_1^t d\rho \sim \frac{C_{t,d} \log |\xi|}{|\xi|^4}, \]  

as $|\xi| \to \infty$.

This proves (4.24). The last conclusion follows from the above proof by using the upper bound in Mills’ ratio to the inner integral $d\tau$ in (4.25). The lemma is now proved. 

4.3.3. Finishing the proofs of Theorems 1.5 and 1.6. We are now ready to finish the proof of Theorem 1.5 and Theorem 1.6. We start with the case $d = 1$.

**Proof of part (i) of Theorem 1.5 and Theorem 1.6.** Fix $t > 0$ and assume $0 <$
\( \beta \leq 1/2 \). We first find the spectral density of the gradient as follows: we use (4.2) and Parseval’s identity to get

\[
E[\partial_x U_\beta(t,x)\partial_y U_\beta(t,y)] = \int_0^t \int_\mathbb{R} \xi^2 e^{i\xi(x-y)} \xi^2 \hat{\mathcal{S}}_{\beta}(\xi) \, d\xi \, dr
\]

\[
= C \int_\mathbb{R} \xi^2 |\xi|^2 \int_0^t E_\beta^2 \left( -\frac{|\xi|^2}{2} (t-r)^\beta \right) \, dr \, d\xi.
\]

This means that the spatial spectral density of \( \partial_x U_\beta \) is \( \tilde{S}_\beta(\xi) = \xi^2 S_\beta(\xi) \), where \( S_\beta(\xi) \) is given in Lemma 4.1. As \( |\xi| \to \infty \), the asymptotic behavior of \( \tilde{S}_\beta(\xi) \) is—upon using Lemma 4.2, Lemma 4.3, and (4.27)—given by

\[
\tilde{S}_\beta(\xi) \sim \begin{cases} 
\frac{C}{|\xi|^2}, & \text{if } 0 < \beta < \frac{1}{2}; \\
\frac{C \log |\xi|}{|\xi|^2}, & \text{if } \beta = \frac{1}{2}.
\end{cases}
\]

We start with the Hölder assertion for the gradient in Theorem 1.5 (i). When \( 0 < \beta < 1/2 \), we apply Lemma 4.1, inequality (3.25) (a), and the Mittag-Leffler upper bound in (2.7) to obtain

\[
E \left| \partial_x U_\beta(t,x) - \partial_y U_\beta(t,y) \right|^2 = \int_\mathbb{R} \xi^2 |e^{i\xi x} - e^{i\xi y}|^2 S_\beta(\xi) \, d\xi
\]

\[
= C \int_\mathbb{R} \xi^2 \left[ 1 - \cos(\xi(x-y)) \right] \int_0^t E_\beta^2 \left( -\frac{|\xi|^2}{2} (t-r)^\beta \right) \, dr \, d\xi
\]

\[
\leq C \int_\mathbb{R} \left[ 1 - \cos(\xi(x-y)) \right] \frac{d\xi}{\xi^2} \int_0^t (t-r)^{-2\beta} \, dr
\]

\[
= Ct^{1-2\beta}|x-y|,
\]

where the last equality follows from a change of variable in the integral \( d\xi \) (or the well-known formula for the variance of fractional Brownian motion). Kolmogorov’s continuity theorem gives us the spatial local \( \gamma \)-Hölder continuity for the \( \beta \)-time-fractional SPIDEs gradient, \( \partial_x U_\beta \), for \( \gamma \in (0,1/2) \) and \( 0 < \beta < 1/2 \). For the critical \( \beta = 1/2 \) case in Theorem 1.6 (i), we use the last statement in Lemma 4.3 together with the second equality in (4.29) and inequality (3.25) (a) to obtain

\[
E \left| \partial_x U_{1/2}(t,x) - \partial_y U_{1/2}(t,y) \right|^2 \leq C \int_\mathbb{R} \left[ 1 - \cos(\xi(x-y)) \right] \frac{\log |\xi|}{\xi^2} \, d\xi
\]

\[
\leq C|x-y| \log \frac{1}{|x-y|}
\]

for all \( x, y \in \mathbb{R} \) with \( |x-y| \leq 1/2 \), where the last inequality follows from a change of variable. Hence the same Hölder assertion holds for the case of \( \beta = 1/2 \).

Turning now to the exact uniform and local spatial continuity moduli of the \( \beta \)-time-fractional SPIDEs \( \partial_x U_\beta \), in Theorem 1.5 and Theorem 1.6 (i) (a) and (b),

Combining the property of the spectral density \( \tilde{S}_\beta \) in (4.28) and Theorems 2.1 and 2.5 in [49], we can verify that the following hold: Given any constant \( M > 0 \), there exists a finite constant \( c > 0 \) (depending on \( t \) and \( M \)) such that for every
\[ n \geq 1 \text{ and for every } x, y_1, \ldots, y_n \in [-M, M], \]
\[ \text{Var} [\partial_x U_\beta(t, x)|\partial_x U_\beta(t, y_1), \ldots, \partial_x U_\beta(t, y_n)] \geq c \min_{0 \leq j \leq n} \varphi_\beta(|x - y_j|), \]
where \( y_0 = 0 \), and \( \varphi_\beta \) is defined on \((0, \infty)\) by
\[ \varphi_\beta(r) = \begin{cases} r & \text{if } 0 < \beta < 1/2, \\ r \log r & \text{if } \beta = 1/2. \end{cases} \]

Also,
\[ \mathbb{E}[\partial_x U_\beta(t, x) - \partial_x U_\beta(t, y)]^2 \asymp \varphi_\beta(|x - y|); \quad \forall x, y \in [-M, M]. \]

Hence, \( \{\partial_x U_\beta(t, x), x \in \mathbb{R}\} \) satisfies Condition (C1) and (C2) (or slight variants when \( \beta = 1/2 \)). Consequently, the desired uniform continuity in (i) (a) of Theorems 1.5 and 1.6 follow from Theorem 4.1 in [34]. The local modulus of continuity of \( \partial_x U_\beta \) Theorems 1.5 and 1.6 (i) (b) follow upon applying [34, Theorem 5.1], completing the proof of Theorems 1.5 and 1.6 part (i).

We now turn to the rougher spatial regularity in two and three dimensional fractal cases for the SPIDEs (1.2). First, we start with the \( d = 3 \) case in Theorems 1.5 and 1.6 (ii), for the cases \( 0 < \beta < 1/2 \) and \( \beta = 1/2 \), respectively.

The following lemma provides the strong local nondeterminism property and double-sided second moment bounds in space for the \( \beta \)-time-fractional SIPDE solution \( \{U_\beta(t, x); x \in \mathbb{R}^3\} \).

**Lemma 4.4** (Spatial SLND and double-sided bounds for time-fractional SPIDEs). Suppose \( 0 < \beta \leq 1/2 \) and \( d = 3 \). For every fixed \( t > 0 \), the time-fractional SIPDE solution \( \{U_\beta(t, x); x \in \mathbb{R}^3\} \) is spatially strongly locally nondeterministic. Namely, for every \( M > 0 \), there exists a finite constant \( c > 0 \) (depending on \( t \) and \( M \)) such that for every \( n \geq 1 \) and for every \( x, y_1, \ldots, y_n \in [-M, M]^3 \),
\[ \text{Var} [U_\beta(t, x)|U_\beta(t, y_1), \ldots, U_\beta(t, y_n)] \geq c \min_{0 \leq j \leq n} \varphi_\beta(|x - y_j|), \]
where \( y_0 = 0 \) and the function \( \varphi_\beta \) is defined in (4.32). Also,
\[ \mathbb{E}[U_\beta(t, x) - U_\beta(t, y)]^2 \asymp \varphi_\beta(|x - y|); \quad \forall x, y \in [-M, M]^3. \]
Moreover, as \( |x - y| \to 0, \) “\( \asymp \)” in (4.35) can be replaced by \( \sim \) [up to a constant factor].

**Proof.** The conclusions in (4.34) and (4.35) follow from Lemmas 4.2 and 4.3 together with Theorems 2.1 and 2.5 in [49]. Finally, the last statement of the lemma follows from (4.22), (4.24) and Theorem 1 of Pitman [38].

Next, we prove the results on spatial uniform and local continuity moduli in Theorem 1.5 and Theorem 1.6 (ii)-(iii).

**Proof of Theorem 1.5 and Theorem 1.6 part (ii)-(iii).** With Lemma 4.4 in hand, the uniform modulus of continuity of \( U_\beta \) in Theorem 1.5 and Theorem 1.6 (ii) (a)
follow from Theorem 4.1 in [34]; while the local modulus of continuity of $U_\beta$ in both Theorem 1.5 and Theorem 1.6 (ii) (b) follow upon applying [34, Theorem 5.1].

To prove part (iii) of Theorem 1.5 and Theorem 1.6, we start by deriving sharp upper bounds for $\mathbb{E}[U_\beta(t,x) - U_\beta(t,y) ]^2$. This is similar to the proof of Theorem 1.2 (iii), which can be obtained by using the upper bounds for the spectral density function $S_\beta$ in Lemma 4.2 and 4.3, respectively. More precisely, we can verify that

\begin{equation}
\mathbb{E}[U_\beta(t,x) - U_\beta(t,y) ]^2 \leq c \begin{cases} |x - y|^2; & \text{if } 0 < \beta < 1/2, \\ |x - y|^2 \log |x - y|^2, & \text{if } \beta = 1/2, \end{cases}
\end{equation}

for all $x, y \in \mathbb{R}$ with $|x - y| \leq 1/2$. In the above, $c \in (0, \infty)$ is a constant. The rest of the proof is similar to that of part (iii) of Theorem 1.2 and is omitted.  

\[\Box\]

\[\text{4.4. The time-fractional SPIDE gradient temporal H"older and modulus of continuity.}\] We prove the temporal regularity of the spatial gradient $\partial_s U_\beta$ in Theorem 1.7.

\textit{Proof of Theorem 1.7.} Let $d = 1$. We start with the H"older assertion for the gradient. Recall $U_\beta(t,x) = X_\beta(t,x) - V_\beta(t,x)$, where $X_\beta$ is the rougher process. Proceeding as in the proof of Theorem 1.3 for L-KS SPDEs, Parseval’s identity applied to the integral in $y$ gives

\begin{equation}
\mathbb{E} [\partial_s X_\beta(t,x) - \partial_s X_\beta(s,x)]^2 = I \int \int \left| \partial_s \Phi^{(\beta,d)}(t-r,x,y) - \partial_s \Phi^{(\beta,d)}(s-r,x,y) \right|^2 drdy
\end{equation}

\begin{equation}
= \int \int \xi^2 \left| \tilde{\Phi}^{(\beta,d)}(t-r,x,\xi) - \tilde{\Phi}^{(\beta,d)}(s-r,x,\xi) \right|^2 d\xi dr
\end{equation}

By (4.8), equation (4.37) becomes

\begin{equation}
\mathbb{E} [\partial_s X_\beta(t,x) - \partial_s X_\beta(s,x)]^2
= \int I \int \xi^2 \left[ E_\beta \left( -\frac{\xi^2}{2} (t-r)^\beta \right) \mathbb{1}_{\{t>r\}} - E_\beta \left( -\frac{\xi^2}{2} (s-r)^\beta \right) \mathbb{1}_{\{s>r\}} \right]^2 drd\xi.
\end{equation}

Taking Fourier transform in $r$, assuming without loss of generality $\beta \in \{1/2^k; k \in \mathbb{N}\}$, using Lemma 2.2 above, proceeding as in (4.10) and immediately after, and using the inequalities (3.25) (a) and (3.38), and the asymptotic

\begin{equation}
\Delta_\beta(r) := (2\pi)^{-1} \int \frac{\xi^2}{|r|^{2\beta}} \frac{2\beta}{|r|^\beta \cos \left( \frac{\pi}{2} \right) + \frac{1}{4} |\xi|^2} d\xi
\end{equation}

\begin{equation}
\sim \frac{C}{|r|^{(4\beta-1-\beta)/2\beta-1}}, \text{ as } |r| \nearrow \infty,
\end{equation}
we get, for a large enough $N$, that

$$
\mathbb{E} [\partial_x X_\beta(t, x) - \partial_x X_\beta(s, x)]^2 = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 \left| \hat{\phi}(\tau, \xi) \right|^2 d\tau d\xi \\
= 2(2\pi)^{-1} \int_{\mathbb{R}} \left(1 - \cos((t - s)\tau)\right) \frac{\left| \tau \right|^{2(\beta - 1)} \xi^2 d\xi}{\left| \tau \right|^{2\beta} + \left| \xi \right|^2 \cos \left( \frac{\pi \beta}{2} \right) + \frac{1}{4} \left| \xi \right|^4} d\tau \\
\leq C |t - s|^{2\alpha} \left[ \int_0^N (1 - \cos(\tau)) \Delta_\beta(\tau) d\tau + \int_\infty^\infty \left| \tau \right|^{2\alpha - \frac{2(\beta - 1) - 3}{2\beta - 1}} d\tau \right] \\
\leq C |t - s|^{2\alpha}, \quad 0 < \alpha < \frac{(2\beta - 1) - 3}{4\beta - 1}.
$$

(4.40)

It follows that $\partial_x X_\beta(t, x)$ is $\gamma$-Hölder continuous in time, with $\gamma \in \left(0, \frac{(2\beta - 1) - 3}{4\beta - 1}\right)$. This, together with the gradient decomposition

$$
\partial_t U_\beta(t, x) = \partial_x X_\beta(t, x) - \partial_x V_\beta(t, x),
$$

and the fact that $\partial_x V_\beta$ is temporally smooth, establishes the Hölder regularity assertion for $\partial_t U_\beta$ in Theorem 1.3.

Turning now to the uniform and local spatial continuity moduli results for the time-fractional SPIDE gradient, $\partial_t U_\beta$, in Theorem 1.7. Equation (4.40) means that $\partial_x X_\beta$ has stationary increments and the spatial spectral density of $\partial_x X_\beta$ and its asymptotic behavior are given by (4.39).

Equation (4.39) and Theorem 2.1 in [49] imply that, for every fixed $x \in \mathbb{R}$, the gradient $\{\partial_x X_\beta(t, x), t \geq 0\}$ is temporally strongly locally nondeterministic. Namely, for any $T > 0$, there is a positive constant $c$ such that for all $t \in (0, T]$ and all $0 < r < 1 \wedge |t|$ holds

$$
\text{Var} \left( X_\beta(t, x) | X_\beta(s, x); s \in [0, T], |t - s| \geq r \right) \geq c r^{\frac{2^\beta - 1 - 3}{4^\beta - 1}}
$$

(4.42)

Also, (3.28) and Theorem 2.5 in [49] imply the double sided second moment bounds

$$
\mathbb{E} [X_\beta(t, x) - X_\beta(s, x)]^2 \leq |t - s|^{\frac{2^\beta - 1 - 3}{4^\beta - 1}}, \quad \forall s, t \in [0, T].
$$

(4.43)

Thus, the uniform modulus of continuity of $\partial_x X_\beta$

$$
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{t, s \in I_{\text{time}}} \frac{\left| \partial_x X_\beta(t, x) - \partial_x X_\beta(s, x) \right|}{|t - s|^{\frac{2^\beta - 1 - 3}{4^\beta - 1}} \log \log \left( 1/|t - s| \right)} = k \right] = 1,
$$

(4.44)

for every compact interval $I_{\text{time}} \subset \mathbb{R}_+$ and for some constant $k > 0$, follows from Theorem 4.1 in [34]. The local modulus of continuity of $\partial_x X$

$$
\mathbb{P} \left[ \lim_{\delta \searrow 0} \sup_{t, s \in I_{\text{time}}} \frac{\left| \partial_x X_\beta(t, x) - \partial_x X_\beta(s, x) \right|}{\delta^{\frac{2^\beta - 1 - 3}{4^\beta - 1}} \log \log \left( 1/\delta \right)} = k \right] = 1,
$$

(4.45)

follows upon applying [34, Theorem 5.1]. The corresponding continuity moduli assertions for the gradient $\partial_t U_\beta$ in Theorem 1.7 follow from those of the auxiliary process $\partial_x X_\beta$ ((4.44) and (4.45)), the decomposition (4.41), and the smoothness of $\partial_x V_\beta$ (see footnote 21).
5. From linear to time-fractional Allen-Cahn and Swift-Hohenberg equations via measure change

We quickly remark in this section that, at their core, the space-time change of measure theorems in [10, 9, 8] are “noise” results that are independent of both the type and order of the SPDE under consideration. This makes them conveniently adaptable to different SPDEs settings. As was done in [1] for L-KS SPDEs, we can extend the results in [9, 8] to our β-time-fractional SPIDEs (1.2). The almost sure $L^2$ condition in [9, 8, 1], which is much weaker than the usual Novikov condition typically found in change-of-measure results, allows us to state an equivalence in law—and thus in all almost sure regularity results—between both (1.1) and (1.2) and their nonlinear versions the Swift-Hohenberg SPDEs

\[
\begin{align*}
(5.1) & \quad \begin{cases}
\frac{\partial U}{\partial t} = -\frac{2}{\epsilon} (\Delta + 2\partial)^2 U + b(U) + \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in (0, T) \times \mathbb{S}; \\
U(0, x) = u_0(x), & x \in \mathbb{S},
\end{cases}
\end{align*}
\]

and the β-time-fractional Allen-Cahn SPIDE

\[
(5.2) \quad \begin{cases}
\mathcal{C} \partial^\beta U_\beta = \frac{1}{2} \Delta U_\beta + I_{1-\beta} b(U) + \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in (0, T) \times \mathbb{S}; \\
U_\beta(0, x) = u_0(x), & x \in \mathbb{S},
\end{cases}
\]

respectively, where $T > 0$ is fixed and arbitrary, and where

\[
(5.3) \quad b(u) = \sum_{k=0}^{2p-1} c_k u^k, \quad \mathbb{S} = \prod_{i=1}^{d}[0, L_i], \quad \text{and with } p \in \mathbb{N}, c_{2p-1} < 0, \text{ and } d = 1, 2, 3.
\]

Let $T := [0, T]$. We supplement (5.1) and (5.3) with suitable boundary conditions\footnote{E.g., boundary conditions of Neumann type $\partial U / \partial n = \partial \Delta U / \partial n = 0$ or Dirichlet type conditions $U = \Delta U = 0$ on $\partial \mathbb{S}$ and $d = 1, 2, 3$.}, the nature of which is irrelevant to our next change of measure result. For concreteness, we assume Dirichlet boundary conditions throughout this section. We also modify the kernels $K_{t,x}^{\text{L-KS},a}$ and $K_{t,x}^{(\beta,d)}$ in the mild kernel formulations (1.47) and (1.55) (with $a \equiv 1$) to account for the boundary conditions\footnote{E.g., in the Neumann (Dirichlet) case, the propagator $e^{-|x-y|^2/2is}/(2\pi is)^d/2$ in the definition of the $(\epsilon, \partial)$ L-KS kernel $K_{t,x}^{\text{L-KS},a}$ (1.45) is replaced with the propagator with reflection (absorption) at $\partial \mathbb{S}$, respectively. Similar comments apply to the outside $d$-dimensional BM density $K_{t,x}^{\text{BM},d}$ in the definition of $K_{t,x}^{(\beta,d)}$ (1.52).}, and we replace $\mathbb{R}^d$ with $\mathbb{S}$. The linear-nonlinear equivalence result is now stated. For completeness, we restate the Swift-Hohenberg conclusions from [1].

\begin{theorem}[Swift-Hohenberg and time-fractional Allen-Cahn law equivalence to their linear counterparts] Fix $T > 0$. Let $\mathbb{S} = \prod_{i=1}^{d}[0, L_i]$, $d = 1, 2, 3$, and assume that $u_0$ satisfies (1.5) with $\mathbb{R}_+ \times \mathbb{R}^d$ replaced by $T \times \mathbb{S}$. The generalized Swift-Hohenberg SPDE, (5.1) and (5.3), admits uniqueness in law and is law equivalent to the $b \equiv 0$ version of (5.1) on $\mathcal{B}(C(T \times \mathbb{S}; \mathbb{R}))$; consequently, it has the same Hölder continuity and modulus of continuity regularity.
\end{theorem}
as the linear L-KS SPDE on $\mathbb{T} \times \mathbb{S}$. The same uniqueness assertion holds for the $\beta$-time-fractional Allen-Cahn SPIDE (5.2) and (5.3). Also, the law—and hence the Hölder continuity and the continuity modulus regularity—equivalence hold between the $\beta$-time-fractional Allen-Cahn SPIDE, (5.2) and (5.3), and its zero-drift ($b \equiv 0$) version. The uniqueness in law and the law equivalence (hence regularity equivalence) conclusions above all hold if $b(u) = \sum_{k=0}^{l} c_k u^k$, for $l \in \mathbb{N} \cup \{0\}$ and $c_k \in \mathbb{R}$.

The proof of the uniqueness and law equivalence assertions for L-KS SPDEs (in both the Allen-Cahn nonlinearity $b$ (5.3) and the general polynomial $b$ cases) was given in [1, Theorem 1.3 and Corollary 1.1]. The proof is exactly the same for the linear-to-nonlinear time-fractional SPIDEs case, and we omit it.

6. Concluding remarks

6.1. Time-fractional SPIDEs are different from time fractional SPDEs and their equivalent high-order memoryful SPDEs. Here, we make a brief but important distinction that was emphasized in [2, 3], and that the astute reader will note. For completeness, we incorporate the SPIDE coinage of our present paper here in making our point, which we now state and discuss. The time-fractional SPIDEs (1.6) are not equivalent to (their rigorous SIEs formulation (1.55) are not the mild form of) their rougher and fundamentally different relatives: (1) the time-fractional SPDEs

\[
\begin{aligned}
\partial_{\beta}^{\gamma} U_{\beta} &= \frac{1}{2} \Delta U_{\beta} + a(U_{\beta}) \frac{\partial^{d+1}W}{\partial t \partial x} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
U_{\beta}(0, x) &= u_0(x),
\end{aligned}
\]

and (2) the $2\beta^{-1} = 2\nu$ order, $\nu \in \{2^k; k \in \mathbb{N}\}$, memoryful SPDEs

\[
\begin{aligned}
\partial_{t} U_{\beta} &= \sum_{\kappa=1}^{\nu-1} C_{\beta,\kappa} \Delta^{\kappa} u_0(x) + \frac{\Delta^{\nu} U_{\beta}}{2^\nu} + a(U_{\beta}) \frac{\partial^{d+1}W}{\partial t \partial x} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\
U_{\beta}(0, x) &= u_0(x),
\end{aligned}
\]

where $C_{\beta,\kappa} = \frac{\Xi(\Lambda_{\beta}(1))^{\kappa}}{\Xi(\Lambda_{\beta}(1))}$; the process $\Lambda_{\beta}$ is the $\beta$-inverse-stable-Lévy motion, as in Section 1.5 above, which arises in the work of Meerschaert et al. [31, 33] as scaling limits of continuous time random walks and which is reviewed, along with its link to $k$-iterated Brownian-time Brownian motion, in [2].

In [3], Allouba showed that in the case $\beta = 1/2$ (the Brownian-time Brownian motion case), the $\beta$-time-fractional SIE in (1.55) is not the mild formulation of the $\beta = 1/2$ of (6.2); but rather it is an integral formulation of what he called parametrized BTBM SPDE, evaluated at the diagonals (see [3] pp. 428–431, Lemma 1.2, and footnote 3 p. 416 for the details). Similarly, as stressed in [2], for general $\beta \in \{1/2^k; k \in \mathbb{N}\}$, the $\beta$-time-fractional SIE in (1.55) is not the mild formulation of (6.2). This is contrary to what was erroneously stated in [35]. Thus, unfortunately,

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24The pathwise uniqueness, and hence uniqueness in law, of solutions trivially follows in the linear $b \equiv 0$ case from the mild formulation.
Theorem 4 of [35] claiming the equivalence between the time-fractional SPIDE\textsuperscript{25} (1.6)—corresponding to the rigorous SIE form (1.55)—and the memoryful high order SPDEs (6.2) is incorrect. In fact, in [2] it was repeatedly emphasized that the β-time-fractional SIE in (1.55) (and hence its corresponding SPIDE (1.6)) is a different and smoother stochastic version of the deterministic PDEs—obtained by setting \( a \equiv 0 \) in either (6.1) or (6.2)—than the rougher equivalent SPDEs (6.1) and (6.2) (see the discussions in [2] right before equations (1.4) and (1.8) and the discussion following equation (1.15), including footnote 15).

The formal SPDEs in (6.1) and (6.2) require rigorous formulations quite different from (1.55). This is handled, and the equivalence between (6.1) and (6.2) for suitably regular initial data \( u_0 \), is shown in an upcoming separate article. Of course, even formally, it is obvious that the time-fractional SPIDEs (1.6) are not equivalent to the time-fractional SPDEs (6.1), which lack the fractional integral and its smoothing effect.

6.2. Other remarks. Further properties on the local times and fractal behavior of the solution process for both L-KS SPDEs and time-fractional SPIDEs \( \{U(t,x), x \in \mathbb{R}^d\} \), when \( t > 0 \) is fixed, can now be derived from [48, 49, 50]. It is also possible to investigate sample path properties of the Gaussian random field \( \{U(t,x), t \geq 0, x \in \mathbb{R}^d\} \) in both time and space variables. We will carry out this in subsequent work.

APPENDIX A.GLOSSARY OF FREQUENTLY USED ACRONYMS AND NOTATIONS

I. Acronyms

- BM: Brownian motion.
- bifBM: bifractional BM.
- BTBM: Brownian-time Brownian motion.
- SIE: Stochastic integral equation.
- SLND: Strong local nondeterminism.
- SPIDE: Stochastic partial integro-differential equation.
- KS: Kuramoto-Sivashinsky.

II. Notations

- \( B^{(H,K)} \): bifractional BM with indices \( H \) and \( K \).
- \( N \): The usual set of natural numbers \( \{1, 2, 3, \ldots\} \).
- \( T \): The time interval \([0,T]\) for some arbitrary fixed \( T > 0 \).
- \( K_{BM}^{t,x} \): The density of a 1-dimensional BM, starting at 0.
- \( K_{\text{LKS}}^{t,x,\varepsilon,\vartheta} \): The generalized \((\varepsilon, \vartheta)\) L-KS kernel.
- \( K_{\text{BTBM}}^{t,x} \): The kernel or density of a \( d \)-dimensional Brownian-time Brownian motion.
- \( E \): The expectation operator.
- \( E_{\beta} \): The Mittag-Leffler function.

\textsuperscript{25}In [35] time-fractional SPIDEs are called time-fractional SPDEs, which is less precise since the name ignores the crucial smoothing effect of the time-fractional integral \( I_{1-\beta} \) in the formal formulation (1.6). We reserve the name time-fractional SPDEs for (6.1).
• $C^{k,\gamma}(\mathbb{R}, \mathbb{R})$: The set of $k$-continuously differentiable functions on $\mathbb{R}$ whose $k$-th derivative is locally Hölder continuous, with Hölder exponent $\gamma$.
• $H^{\gamma}(\mathbb{R}_+; \mathbb{R})$: The space of locally Hölder continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ whose Hölder exponent $\gamma \in (0, \gamma_0)$.
• $\partial^p_{x_i} f(x_1, \ldots, x_N) = \partial^p f / \partial x_i^n$, $i = 1, \ldots, N$ and $n \in \mathbb{N}$.
• $f(x) \preceq g(x)$ on $\mathbb{S}$ means $c_l g(x) \leq f(x) \leq c_u g(x)$ for some constants $c_l, c_u$ for every $x \in \mathbb{S}$.

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