LOCALLY COMPACT QUANTUM GROUPOIDS

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ABSTRACT. The theory of measured quantum groupoids, as defined in [L] and [E4], was made to generalize the theory of quantum groups ([KV1], [KV2]), but was only defined in a von Neumann algebra setting; Th. Timmermann constructed locally compact quantum groupoids, which is a C∗-version of quantum groupoids [Ti4]. Here, we associate to such a locally compact quantum groupoid a measured quantum groupoid in which it is weakly dense; we then associate to a measured quantum groupoid a locally compact quantum groupoid which is weakly dense in the measured quantum groupoid, but such a locally compact quantum groupoid may be not unique; we construct a duality of locally compact quantum groupoids. We give then examples of locally compact quantum groupoids.

CONTENTS

1. Introduction 2
2. Measured quantum groupoids 4
3. Locally compact quantum groupoids 17
4. From a locally compact quantum groupoid to a measured quantum groupoid 26
5. From a measured quantum groupoid to a canonical locally compact sub-quantum groupoid 31
6. Duality of locally compact groupoids 33
7. Examples 38
References 47
1. Introduction

1.1. Locally compact quantum groups. The theory of locally compact quantum groups, developed by J. Kustermans and S. Vaes ([KV1], [KV2]), provides a comprehensive framework for the study of quantum groups in the setting of $C^*$-algebras and von Neumann algebras. It includes a far reaching generalization of the classical Pontryagin duality of locally compact abelian groups, that covers all locally compact groups. Namely, if $G$ is a locally compact group, its von Neumann algebra is $L^\infty(G, \mu)$ (where $\mu$ is the left Haar measure on $G$), and its dual von Neumann algebra is $L(G)$ generated by the left regular representation $\lambda_G$ of $G$ on $L^2(\mu)$, equipped with a coproduct $\Gamma_G$ from $L(G)$ on $L(G) \otimes L(G)$ defined, for all $s \in G$, by $\Gamma_G(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s)$, and with a normal semi-finite faithful weight, called the Plancherel weight $\varphi_G$, associated via the Tomita-Takesaki construction, to the left Hilbert algebra defined by the algebra $\mathcal{K}(G)$ of continuous functions with compact support (with convolution as product), this weight $\varphi_G$ being left- and right-invariant with respect to $\Gamma_G$ ([T], VII, 3).

This theory builds on many preceding works, by G. Kac, G. Kac and L. Vainerman, J.-M. Schwartz and the author ([ES1], [ES2]), S. Baaj and G. Skandalis ([BS]), A. Van Daele ([VD1]), S. Woronowicz ([W1], [W5], [W6]) and many others. See the monography written by T. Timmermann for a survey of that theory ([Ti1]), and the introduction of [ES2] for a sketch of the historical background. It seems to have reached now a stable situation, because it fits the needs of operator algebraists for many reasons:

First, the axioms of this theory are very simple and elegant: they can be given in both $C^*$-algebras and von Neumann algebras, and these two points of view are equivalent, as A. Weil had shown it was the fact for groups (namely any measurable group equipped with a left-invariant positive non zero measure bears a topology which makes it locally compact, and this measure is then the Haar measure ([W], Appendice I)). In a von Neumann setting, a locally compact quantum group is just a von Neumann algebra, equipped with a co-associative coproduct, and two normal faithful semi-finite weights, one left-invariant with respect to that coproduct, and one right-invariant. Then, many other data are constructed, in particular a multiplicative unitary (as defined in [BS]) which is manageable (as defined in [W6]).

Second, all preceding attempts ([ES2], [W5]) appear as particular cases of locally compact quantum groups; and many interesting examples were constructed ([W2], [W3], [VV]).

Third, many constructions of harmonic analysis, or concerning group actions on $C^*$-algebras and von Neumann algebras, were generalized to locally compact quantum groups ([V2]).

Finally, many constructions made by algebraists at the level of Hopf $*$-algebras, or multipliers Hopf $*$-algebras, can be generalized for locally compact quantum groups. This is the case, for instance, for Drinfel’d double of a quantum group ([D]), and for Yetter-Drinfel’d algebras which were well-known in an algebraic approach in [M].

1.2. Measured Quantum Groupoids. In two articles ([Val1], [Val2]), J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf bimodule), in order to generalize, to the groupoid case, the classical notions of multiplicative unitary ([BS]) and of a co-associative coproduct on a von Neumann algebra. Then, F. Lesieur ([L]), starting from a Hopf bimodule, when there exist a left-invariant operator-valued weight and a right-invariant operator-valued weight, mimicking in that wider setting what was done in ([KV1], [KV2]), obtained a pseudo-multiplicative unitary, and called “measured quantum groupoids” these objects. A new set of axioms had been given in an appendix of [ES3]. In
and [E4], most of the results given in [V2] were generalized to measured quantum groupoids. Some trivial examples were given in [L]. A more interesting example was constructed in [ET]: here, a quantum transformation groupoid is defined, from a right action of a locally compact quantum group $G$ on a von Neumann algebra $N$, which generalizes the transformation groupoid given by a locally compact group $G$ having a right action $a$ on a locally compact space $X$.

This theory, up to now, had one important defect: it was only a theory in a von Neumann algebra setting.

1.3. **Locally compact quantum groupoids.** Th. Timmermann had made many attempts in order to provide a $C^*$-algebra version of it (see [Ti1] for a survey); these attempts were fruitful, but not sufficient to complete a theory equivalent to the von Neumann one.

This is the subject of this article, in order to get, for quantum groupoids, as it is for quantum groups, axioms in both $C^*$-algebras and von Neumann algebras. In a $C^*$-algebra setting, we first recall Timmermann’s theory of *locally compact quantum groupoids*; we then prove that to such an object $G$, it is possible to associate a measured quantum groupoid $\mathcal{G}$ such that the $C^*$-algebra of $G$ is a dense sub-$C^*$ algebra of the underlying von Neumann algebra of the measured quantum groupoid.

We must quote older articles about quantum groupoids, first a purely algebraic construction ([Sc]), and second a $C^*$ construction in a particular situation ([KVD1] and [KVD2]).

The article is organized as follows:

In chapter 2 are recalled the definition of measured quantum groupoids; first the definitions of the relative tensor of Hilbert space (2.1), the fiber product of von Neumann algebra (3.5) and then the definitions of Hopf-bimodules (2.4), and of measured quantum groupoids (2.5).

In chapter 3, we give a definition of locally compact quantum groupoids. First (3.1), we recall definitions of $C^*$-relative products of Hilbert spaces and $C^*$-fiber product of $C^*$-algebras ([Ti2]); after, we recall (3.2) basic results about weights on $C^*$-algebras, mostly due to F. Combes ([C1], [C2]). We then recall Kusterman’s definition of $C^*$-valued weights ([K2]), and define a class of $C^*$-valued weights which are restrictions of operator-valued weights. Using all these notions, we then give a new definition of *fiber products of $C^*$-algebras* (3.5).

In chapter 4, we prove that, to any locally compact quantum groupoid $G$, we can associate a canonical measured quantum groupoid $\mathcal{G}$ such that $G$ is a dense sub-$C^*$-algebra of $\mathcal{G}$. Then, we say that $G$ is a *locally compact sub-quantum groupoid* of $\mathcal{G}$ (3.6.3).

In chapter 5, starting from a measured quantum groupoid, we construct a canonical locally compact sub-quantum groupoid; such a sub $C^*$-algebra, which is a locally compact quantum groupoid, may be not unique (5.8).

In chapter 6, from any locally compact quantum groupoid, we construct a dual one, and prove that the bidual is isomorphic to the initial locally compact quantum groupoid (equal if we identify the canonical Hilbert spaces, on which the $C^*$-algebras are constructed).

In chapter 7, we recall several examples of locally compact quantum groupoid.

We are indebted to Thomas Timmermann who had found several mistakes in a preliminary version of this article.
2. Measured quantum groupoids

In this chapter, we recall the definition of the relative tensor product of Hilbert spaces, and of the fiber product of von Neumann algebra (2.1). Then, we recall the definition of a Hopf bimodule (2.4) and a co-inverse. We then give the definition of a measured quantum groupoids (2.9), and all the data constructed then, including duality of measured quantum groupoids (2.0).

2.1. Relative tensor products of Hilbert spaces ([C], [S], [T], [EVal]). Let \( N \) be a von Neumann algebra, \( \nu \) a normal semi-finite faithful weight on \( N \); we shall denote by \( H_\nu, R_\nu, \ldots \) the canonical objects of the Tomita-Takesaki theory associated to the weight \( \nu \).

Let \( \alpha \) be a non-degenerate faithful representation of \( N \) on a Hilbert space \( \mathcal{H} \). The set of \( \nu \)-bounded elements of the left module \( _\alpha \mathcal{H} \) is

\[
D(\alpha \mathcal{H}, \nu) = \{ \xi \in \mathcal{H} : \exists C < \infty, \| \alpha(y)\xi \| \leq C\| \Lambda_\nu(y)\|, \forall y \in \mathcal{N}_\nu \}.
\]

For any \( \xi \) in \( D(\alpha \mathcal{H}, \nu) \), there exists a bounded operator \( R^{\alpha, \nu}(\xi) \) from \( H_\nu \) to \( \mathcal{H} \) such that

\[
R^{\alpha, \nu}(\xi)\Lambda_\nu(y) = \alpha(y)\xi \quad \text{for all } y \in \mathcal{N}_\nu,
\]

and this operator exchanges the representations of \( N \). If \( \xi \) and \( \eta \) are bounded vectors, we define the operator product

\[
\langle \xi | \eta \rangle_{\alpha, \nu} = R^{\alpha, \nu}(\eta)^* R^{\alpha, \nu}(\xi),
\]

which belongs to \( \pi_\nu(N)' \). Using Tomita-Takesaki theory, this last algebra will be identified with the opposite von Neumann algebra \( N^\alpha \). We shall use also the operator

\[
\theta^{\alpha, \nu}(\xi, \eta) = R^{\alpha, \nu}(\xi)R^{\alpha, \nu}(\eta)^*
\]

which belongs to \( \alpha(N)' \). If now \( \beta \) is a non-degenerate faithful anti-representation of \( N \) on a Hilbert space \( \mathcal{K} \), the relative tensor product \( \mathcal{K} \otimes _\nu D(\alpha \mathcal{H}, \nu) \) by the scalar product defined by

\[
(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = (\beta(|\eta_1|\eta_2)_{\alpha, \nu})\xi_1|\xi_2|
\]

for all \( \xi_1, \xi_2 \in \mathcal{K} \) and \( \eta_1, \eta_2 \in D(\alpha \mathcal{H}, \nu) \). If \( \xi \in \mathcal{K} \) and \( \eta \in D(\alpha \mathcal{H}, \nu) \), we denote by \( \xi \otimes _\nu \eta \) the image of \( \xi \otimes \eta \) into \( \mathcal{K} \otimes _\nu \mathcal{H} \). Writing \( \rho^{\nu, \alpha}_\eta(\xi) = \xi \otimes _\nu \eta \), we get a bounded linear operator from \( \mathcal{H} \) into \( \mathcal{K} \otimes _\nu \mathcal{H} \), which is equal to \( 1_\mathcal{K} \otimes _\nu R^{\alpha, \nu}(\eta) \).

If \( x \in D(\sigma^\nu_{i/2}) \), then \( \alpha(x)D(\alpha \mathcal{H}, \nu) \subset D(\alpha \mathcal{H}, \nu) \), and we have ([S], 2.2 b):

\[
\xi \otimes _\nu \alpha(x)\eta = \beta(\sigma^\nu_{i/2}(x))\xi \otimes _\nu \eta
\]

Changing the weight \( \nu \) will give an isomorphic Hilbert space, but the isomorphism will not exchange elementary tensors!

We shall denote by \( \sigma_\nu \) the relative flip, which is a unitary sending \( \mathcal{K} \otimes _\nu \mathcal{H} \) onto \( \mathcal{H} \otimes _\nu \mathcal{K} \), defined by

\[
\sigma_\nu(\xi \otimes _\nu \eta) = \eta \otimes _\nu \xi
\]

for all \( \xi \in D(\mathcal{K}_\beta, \nu') \) and \( \eta \in D(\alpha \mathcal{H}, \nu) \).
In ([DC1], chap. 11), De Commer had shown that, if $N$ is finite-dimensional, the Hilbert space $\mathcal{K} \otimes_{\alpha} \mathcal{H}$ can be isometrically imbedded into the usual Hilbert tensor product $\mathcal{K} \otimes \mathcal{H}$.

2.1.1. Definition. There exists ([C], prop.3) a family $(e_i)_{i \in I}$ of $\nu$-bounded elements of $\alpha \mathcal{H}$, such that
\[ \sum_i \theta^{\alpha,\nu}(e_i, e_i) = 1 \]
Such a family will be called an $(\alpha, \nu)$-basis of $\mathcal{H}$. Then, for any $i \in I$, the image of $\theta^{\alpha,\nu}(e_i, e_i)$ is included in the closure of the subspace $\{\alpha(n)e_i, n \in N\}$.

In that situation, let us consider, for all $n \in N$ and finite $J \subset I$ with $|J| = n$, the $(1,n)$ matrix $R_I = (R^{\alpha,\nu}(e_i))_{i \in J}$. As $R_I R_I^* \leq 1$, we get that $\|R_I\| \leq 1$, and that the $(n,n)$ matrix $(<e_i, e_j>_{\alpha,\nu})_{i,j \in J}$ is less than the unit matrix.

It is possible ([EN] 2.2) to construct an $(\alpha, \nu)$-basis of $\mathcal{H}$, $(e_i)_{i \in I}$, such that the operators $R^{\alpha,\nu}(e_i)$ are partial isometries with final supports $\theta^{\alpha,\nu}(e_i, e_i)$ by 2 orthogonal, and such that, if $i \neq j$, then $<e_i, e_j>_{\alpha,\nu} = 0$. Such a family will be called an $(\alpha, \nu)$-orthogonal basis of $\mathcal{H}$.

We have, then:
\[ R^{\alpha,\nu} = \sum_i \theta^{\alpha,\nu}(e_i, e_i) R^{\alpha,\nu} \]
\[ <\xi, \eta>_{\alpha,\psi} = \sum_i <\eta, e_i>_{\alpha,\psi} <\xi, e_i>_{\alpha,\nu} \]
\[ \xi = \sum_i R^{\alpha,\nu}(e_i) J_\psi \Lambda_\psi(\xi, e_i) >_{\alpha,\nu} \]
the sums being weakly convergent.

Moreover, we get that, for all $n \in N$, $\theta^{\alpha,\nu}(e_i, e_i)\alpha(n)e_i = \alpha(n)e_i$, and $\theta^{\alpha,\nu}(e_i, e_i)$ is the orthogonal projection on the closure of the subspace $\{\alpha(n)e_i, n \in N\}$.

2.1.2. Basic construction. ([EN], 3.1) Let $M_0 \subset M_1$ be an inclusion of von Neumann algebras, $\psi_1$ a normal semi-finite faithful weight on $M_1$; then $M_2 = J_{\psi_1} M_0 J_{\psi_1}$ is a von Neumann algebra called the basic construction from the inclusion $M_0 \subset M_1$; let $\psi_0$ be a normal semi-finite faithful weight on $M_0$; if $\xi, \eta$ belong to $D(H_{\psi_1}, \psi_0^*)$, then $\theta^{\psi_0}(\xi, \eta)$ belongs to $M_2$, and the linear span of these operators is a dense ideal in $M_2$.

2.2. Operator-valued weights. Let $M_0 \subset M_1$ be an inclusion of von Neumann algebras (for simplicity, these algebras will be supposed to be $\sigma$-finite), equipped with a normal faithful semi-finite operator-valued weight $T_1$ from $M_1$ to $M_0$ (to be more precise, from $M_1^+$ to the extended positive elements of $M_0$ (cf. [IT] II.4.12)). Let $\psi_0$ be a normal faithful semi-finite weight on $M_0$, and $\psi_1 = \psi_0 \circ T_1$; for $i = 0, 1$, let $H_i = H_{\psi_i}$, $J_i = J_{\psi_i}$, $\Delta_i = \Delta_{\psi_i}$ be the usual objects constructed by the Tomita-Takesaki theory associated to these weights.

Following ([EN] 10.6), for $x$ in $\mathfrak{M}_{T_1}$, we shall define $\Lambda_{T_1}(x)$ by the following formula, for all $z$ in $\mathfrak{N}_{\psi_0}$:
\[ \Lambda_{T_1}(x) \Lambda_{\psi_0}(z) = \Lambda_{\psi_1}(xz) \]
This operator belongs to $\text{Hom}_{M_0}(H_0, H_1)$; if $x, y$ belong to $\mathcal{N}_{T_1}$, then $\Lambda_{T_1}(x) \Lambda_{T_1}(y)$ belongs to the von Neumann algebra $M_2 = J_{\psi_1} M_0 J_{\psi_1}$, which is called the basic construction made from the inclusion $M_0 \subset M_1$, and $\Lambda_{T_1}(x)^* \Lambda_{T_1}(y) = T_1(x^*y) \in M_0$.

By Tomita-Takesaki theory, the Hilbert space $H_1$ bears a natural structure of $M_1 - M_0$-bimodule, and, therefore, by restriction, of $M_0 - M_0^*$-bimodule. Let us write $r$ for the
canonical representation of $M_0$ on $H_1$, and $s$ for the canonical antirepresentation given, for all $x$ in $M_0$, by $s(x) = J_1 r(x)^* J_1$. Let us have now a closer look to the subspaces $D(H_{1s}, \psi_0^s)$ and $D(H_{1r}, \psi_0)$. If $x$ belongs to $\mathfrak{N}_{i_1} \cap \mathfrak{N}_{o_1}$, we easily get that $J_1 \Lambda_{\psi_1}(x)$ belongs to $D_{i_1} H_{1s}, \psi_0^s)$, with :

$$R_{s,\psi_0^s}(J_1 \Lambda_{\psi_1}(x)) = J_1 \Lambda_{T_1}(x) J_0$$

and $\Lambda_{\psi_1}(x)$ belongs to $D(H_{1s}, \psi_0^s)$, with :

$$R_{s,\psi_0^s}(\Lambda_{\psi_1}(x)) = \Lambda_{T_1}(x)$$

The subspace $D(H_{1s}, \psi_0^s) \cap D(H_{1r}, \psi_0)$ is dense in $H_1$; more precisely, let $\mathcal{T}_{\psi_1, T_1}$ be the algebra made of elements $x$ in $\mathfrak{N}_{o_1} \cap \mathfrak{N}_{i_1} \cap \mathfrak{N}_{o_1} \cap \mathfrak{N}_{o_1}^*$, analytical with respect to $\psi_1$, and such that, for all $z \in \mathbb{C}$, $\sigma_z^{\psi_1}(x)$ belongs to $\mathfrak{N}_{o_1} \cap \mathfrak{N}_{i_1} \cap \mathfrak{N}_{o_1} \cap \mathfrak{N}_{o_1}^*$. Then (EN2), 2.2.1:

(i) the algebra $\mathcal{T}_{\psi_1, T_1}$ is weakly dense in $M_1$; it will be called Tomita's algebra with respect to $\psi_1$ and $T_1$;

(ii) for any $x$ in $\mathcal{T}_{\psi_1, T_1}$, $\Lambda_{\psi_1}(x)$ belongs to $D(H_{1s}, \psi_0) \cap D(H_{1r}, \psi_0)$;

(iii) for any $\xi$ in $D(H_{1s}, \psi_0^s)$, there exists a sequence $x_n$ in $\mathcal{T}_{\psi_1, T_1}$ such that $\Lambda_{T_1}(x_n) = R_{s,\psi_0^s}(\Lambda_{\psi_1}(x_n))$ is weakly converging to $R_{s,\psi_0^s}(\xi)$ and $\Lambda_{\psi_1}(x_n)$ is converging to $\xi$.

More precisely, in (EN2), 2.3 was constructed an increasing sequence of projections $p_n$ in $M_1$, converging to 1, and elements $x_n$ in $\mathcal{T}_{\psi_1, T_1}$ such that $\Lambda_{\psi_1}(x_n) = p_n \xi$. We then get that :

$$T_1(x_n^* x_n) = < R_{s,\psi_0^s}(\Lambda_{\psi_1}(x_n)), R_{s,\psi_0^s}(\Lambda_{\psi_1}(x_n)) >_{s,\psi_0^s}$$

$$= < p_n \xi, p_n \xi >_{s,\psi_0^s}$$

$$= R_{s,\psi_0^s}(\xi)^* p_n R_{s,\psi_0^s}(\xi)$$

which is increasing and weakly converging to $< \xi, \xi >_{s,\psi_0^s}$.

2.2.1. Theorem. (EN, 10.3, 10.7, 10.11, EN3, 2.10) Let $M_1$ be a von Neumann algebra, $M_0$ a von Neumann subalgebra of $M_1$, $\nu$ a faithful semi-finite normal weight on $M_0$ and $T$ a normal faithful semi-finite operator-valued weight from $M_0$ onto $M_0$; let $r$ be the inclusion of $M_0$ into $B(H_{o0T})$, and $s$ the anti-$*$-homomorphism from $M_0$ into $B(H_{r0T})$ defined by $(x \in M_0, x \mapsto J_{o0T} x^* J_{r0T}$; let us define, for $x \in \mathfrak{N}_{T}$, $\Lambda_T(x) \in B(H_{\nu}, H_{r0T})$ by $(z \in \mathfrak{N}_{\nu})$ :

$$\Lambda_T(x) \Lambda_o(z) = \Lambda_{o0T}(xz)$$

(i) let $M_2$ be the basic construction made from the inclusion $M_0 \subset M_1$; then, for any $x, y$ in $\mathfrak{N}_{T}$, then $\Lambda_T(x) \Lambda_T(y)^*$ belongs to $M_2$, and the von Neumann algebra $M_2$ is generated by these operators; moreover, there exists a normal faithful semi-finite operator-valued weight $T_2$ from $M_2$ to $M_1$ such that, $T_2(\Lambda_T(x) \Lambda_T(y)^*) = xy^*$. Let $X$ belong to $\text{Hom}_{M_0^*}(H_{\nu}, H_{r0T})$ such that $XX^*$ belongs to $\mathfrak{M}_{T_2}$; then, there exists a unique element $\Phi_1(X)$ in $M_1$ such that $T_2(\Lambda_T(a)^*) = \Phi_1(X)a^*$, for all $a \in \mathfrak{N}_{T}$; this application $\Phi_1$ is an injective application of $(M_1, M_0)$-bimodule, and we have $\Phi_1(\Lambda_T(x)) = x$, for all $x \in \mathfrak{N}_{T}$.

(ii) there exists a family $(e_i)_{i \in I}$ in $\mathfrak{N}_{T} \cap \mathfrak{M}_{T_2} \cap \mathfrak{N}_{r0T} \cap \mathfrak{N}_{o0T}$ such that the operators $\Lambda_T(e_i)$ are partial isometries, with $T(e_j^* e_i) = 0$ if $j \neq i$, and with their final supports $\Lambda_T(e_i) \Lambda_T(e_i)^*$ two by two orthogonal projections of sum 1; moreover, for all $i \in I$, we have $e_i = e_i T(e_i^* e_i)$, and, for all $x \in \mathfrak{N}_{T}$, we have :

$$\Lambda_T(x) = \sum_i \Lambda_T(e_i) T(e_i^* x)$$

$$x = \sum_i e_i T(e_i^* x)$$
these two sums being weakly convergent. Such a family \((e_i)_{i \in I}\) will be called a basis for \((T, ν)\). Moreover, the family \(J_{νT}Λ_{νT}(e_i)\) is a basis for \((α, ν)\) \([\text{2.1.1}]\), where \(α\) is the inclusion \(M_0 \subset M_1\).

Moreover, the vectors \((Λ_{νT}(e_i))_{i \in I}\) are a \((s, ν^o)\)-orthogonal basis of \(H_{νT}\), and the vectors \((J_{νT}Λ_{νT}(e_i))_{i \in I}\) are a \((r, ν)\)-orthogonal basis of \(H_{νT}\).

(iii) for any \(ξ ∈ D((H_{νT})_s, ν^o)\), there exists a sequence \(x_n \in \mathfrak{s}_T \cap \mathfrak{m}_{νT}\) such that \(Λ_{νT}(x_n)\) is converging to \(ξ\), and \(Λ_T(x_n) = R^{s, ν^o}(Λ_{νT}(x_n))\) is weakly converging to \(R^{s, ν^o}(ξ)\); equivalently, for any \(η \in D(H_{νT}, ν)\), there exists a sequence \(y_n \in \mathfrak{s}_T \cap \mathfrak{m}_{νT}\) such that \(Λ_{νT}(y_n)\) is converging to \(η\) and \(J_{νT}Λ_T(y_n)J_r = R^{s, ν^o}(Λ_{νT}(y_n))\) is weakly converging to \(R^{s, ν^o}(η)\).

**Proof.** Result (i) is just \([\text{4N}], 10.3, 10.7\) and \(10.11\). Let us simplify and clarify the proof given in \([\text{4E}], 2.10\) for (ii) : let us first remark that \(Λ_T(x)^* Λ_T(x) = T(x^* x)\).

By density of \(Λ_{νT}(\mathfrak{s}_T \cap \mathfrak{m}_{νT})\) into \(H_{νT}\), we can choose a family \((x_α)_{α ∈ A}\) in \(\mathfrak{s}_T \cap \mathfrak{m}_{νT} \cap \mathfrak{m}_{ν^oT}\) such that the closed subspaces \(J_{νT}M_0J_{νT}Λ_{νT}(x_α)\) are two by two orthogonal and that \(H_{νT}\) is the sum of these subspaces. Then, for any \(α\), we can remark that, for any \(f ∈ L^∞(R^+),\) we have:

\[
f(Λ_T(x_α)Λ_T(x_α^*))Λ_T(x_α) = Λ_T(x_α)f(T(x_α^* x_α)) = Λ_T(x_α f(T(x_α^* x_α)))
\]

This formula is clear for any polynomial function, then by norm continuity, for any continuous function on \([0, ∥T(x_α^* x_α)∥]\), and by weak continuity, we get the result. So, taking the function \(f_{α,n}(t) = χ\)\(|T(x_α^* x_α)\|^n |T(x_α^* x_α)|^{n+1}\)\(t^{-1/2}\), and defining \(I\) as the subset of \(A \times N\) such that \(x_α f_{α,n}(T(x_α^* x_α)) \neq 0\) and \(e_i = x_α f_{α,n}(T(x_α^* x_α))\), we obtain the appropriate family \((e_i)_{i ∈ I}\).

Result (iii) is taken from \([\text{4E}], 2.3(i)\), or can be deduced from (i). \(\square\)

2.2.2. **Lemma.** Let \(N\) be a von Neumann algebra, \(ν\) be a faithful semi-finite normal weight on \(N\). Let \(α\) be a faithful non degenerate representation of \(N\) into a von Neumann algebra \(M_1\), and \(T_1\) be a normal faithful semi-finite operator-valued weight from \(M_1\) onto \(α(N)\). Let \(β\) be a faithful non degenerate anti-representation of \(N\) into a von Neumann algebra \(M_2\), and \(T_2\) be a normal faithful semi-finite operator-valued weight from \(M_2\) onto \(β(N)\). For \(x ∈ \mathfrak{s}_{T_1}\), let us define \(Λ_{T_1}(x) ∈ B(H_ν, H_{ναβ})\) by \((z ∈ \mathfrak{m}_ν) : \)

\[
Λ_{T_1}(x)Λ_ν(z) = Λ_{ναβ}(xα(z))
\]

and, for \(y ∈ \mathfrak{s}_{T_2}\), let us define as well \(Λ_{T_2}(y) ∈ B(H_ν, H_{ναβ})\) by :

\[
Λ_{T_2}(y)J_νΛ_ν(z) = Λ_{ναβ}(yβ(z^*))
\]

Then :

(i) for any \(x ∈ \mathfrak{s}_{T_1} \cap \mathfrak{m}_{ναβ}, J_{ναβ}Λ_{ναβ}(x)\) belongs to \(D(H_ν, H_{ναβ})\) \([\text{2.1.1}]\), \(J_{ναβ}Λ_{ναβ}(x)\) belongs to \(D(α_H_{ναβ})\) and \(R^{α, ν^o}(J_{ναβ}Λ_{ναβ}(x)) = J_{ναβ}R^{α, ν^o}(ξ)\).

Moreover, if \(x_1\) belongs to \(\mathfrak{s}_{T_1} \cap \mathfrak{m}_{ναβ}\), we have \(Λ_{T_1}(x_1)\) belongs to \(\mathfrak{s}_{T_1} \cap \mathfrak{m}_{ναβ}\), \(\lambda_ν(α^{-1}T_1(x^* x_2)) = \lambda_ν(α^{-1}T_1(x^* x_2))\).

(ii) for any \(y ∈ \mathfrak{s}_{T_2} \cap \mathfrak{m}_{ναβ}\), \(J_{ναβ}Λ_{ναβ}(y)^* J_{ναβ}Λ_{ναβ}(y)\) belongs to \(D(β_H_{ναβ})\) and \(R^{β, ν^o}(J_{ναβ}Λ_{ναβ}(y)^* J_{ναβ}Λ_{ναβ}(y)) = J_{ναβ}R^{β, ν^o}(y^* y)\).

Moreover, if \(y_1, y_2\) belong to \(\mathfrak{s}_{T_2} \cap \mathfrak{m}_{ναβ}\), we have \(Λ_{T_2}(y_1)^* Λ_{T_2}(y_2) = J_{ναβ}(T_2(y_1^* y_2))\) \([\text{2.1.1}]\), \(J_{ναβ}(T_2(y_1^* y_2)) = J_{ναβ}(β^{-1}T_2(y_1^* y_2))\).

(iii) for any \(x_1, x_2\) in \(\mathfrak{s}_{T_1} \cap \mathfrak{m}_{ναβ}\), \(y_1, y_2\) in \(\mathfrak{s}_{T_2} \cap \mathfrak{m}_{ναβ}\), the scalar product :

\[
(Λ_{ναβ}(y_1)^* Λ_{ναβ}(y_2)) = Λ_{ναβ}(α^{-1}T_1(x^* x_2))
\]

is equal to \((J_{ναβ}(β^{-1}T_2(y_1^* y_2))\).
Proof. It is straightforward to get that:

\[ J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda T_1(x) J_{\nu} \Lambda_{\nu}(z) = \alpha(z) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x) \]

The formula about \( \Lambda_T(x)^* \) is an easy calculation, and then we get the formula about \( x'_1, x'_2 \) if \( x'_1 \in \mathcal{H}_{T_1} \cap \mathcal{H}_{\nu \circ \alpha_1 \otimes \alpha_2} \), and then for all \( x'_1, x'_2 \) by continuity, which finishes the proof of (i).

The formula about \( \Lambda_{\nu \circ \beta^{-1} \circ \sigma T_2}(y) \) is just the definition of \( \Lambda_{T_2} \); the other results of (ii) are proved the same way as (i).

Using (i), we get that the scalar product is equal to:

\[ (\alpha \langle \Lambda_{\nu \circ \beta^{-1} \circ \sigma T_2}(y_1), \Lambda_{\nu \circ \beta^{-1} \circ \sigma T_2}(y_2) \rangle_{\beta, \nu}) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_1) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_2) \]

which, using (ii), is equal to

\[ (\alpha \circ \beta^{-1}(T_2(y_1^* y_2))) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_1) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_2) = \]

\[ (J_{\nu} \Lambda_{\nu}(\beta^{-1}T_2(y_1^* y_2))) J_{\nu \circ \alpha_1 \otimes \alpha_2} \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_1) \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_2) \]

and, using (i) again, is equal to:

\[ (\Lambda_T(x_2) J_{\nu} \Lambda_{\nu}(\beta^{-1}T_2(y_1^* y_2))) \Lambda_{\nu \circ \alpha_1 \otimes \alpha_2}(x_2) = \]

\[ (J_{\nu} \Lambda_{\nu}(\beta^{-1}T_2(y_1^* y_2))) \Lambda_{\nu}(\alpha^{-1}T_1(x_2^* x_1)) \]

This finishes the proof. \( \square \)

2.3. Fiber product of von Neumann algebras [Eva]. If \( x \in \beta(N)' \) and \( y \in \alpha(N)' \), it is possible to define an operator \( x \beta \otimes_{\alpha} y \) on \( \mathcal{K}_{\beta \otimes_{\alpha} \mathcal{H}} \), with natural values on the elementary tensors. As this operator does not depend upon the weight \( \nu \), it will be denoted by \( x \beta \otimes_{\alpha} y \).

If \( P \) is a von Neumann algebra on \( \mathcal{H} \) with \( \alpha(N) \subset P \), and \( Q \) a von Neumann algebra on \( \mathcal{K} \) with \( \beta(N) \subset Q \), then we define the fiber product \( Q \beta \ast_{\alpha} P \) as \( \{ x \beta \otimes_{\alpha} y : x \in Q', y \in P' \}' \).

This von Neumann algebra can be defined independently of the Hilbert spaces on which \( P \) and \( Q \) are represented. If for \( i = 1, 2, \alpha_i \) is a faithful non-degenerate homomorphism from \( N \) into \( P_i \), and \( \beta_i \) is a faithful non-degenerate anti-homomorphism from \( N \) into \( Q_i \), and \( \Phi \) (resp. \( \Psi \)) a homomorphism from \( P_1 \) to \( P_2 \) (resp. from \( Q_1 \) to \( Q_2 \)) such that \( \Phi \circ \alpha_1 = \alpha_2 \) (resp. \( \Psi \circ \beta_1 = \beta_2 \)), then, it is possible to define a homomorphism \( \Psi_{\beta_1 \ast_{\alpha_1} \Phi} \) from \( Q_1 \beta_1 \ast_{\alpha_1} P_1 \) into \( Q_2 \beta_2 \ast_{\alpha_2} P_2 \).

We define a relative flip \( \zeta_N \) from \( B(\mathcal{K}) \beta \ast_{\alpha} B(\mathcal{H}) \) onto \( B(\mathcal{H}) \alpha \ast_{\beta} B(\mathcal{K}) \) by \( \zeta_N(X) = \sigma_N X (\sigma_N)^* \) for any \( X \in B(\mathcal{K}) \beta \ast_{\alpha} B(\mathcal{H}) \) and any normal semi-finite faithful weight \( \psi \) on \( N \).

Let now \( U \) be an isometry from a Hilbert space \( \mathcal{K}_1 \) in a Hilbert space \( \mathcal{K}_2 \), which intertwines two anti-representations \( \beta_1 \) and \( \beta_2 \) of \( N \), and let \( V \) be an isometry from a Hilbert space \( \mathcal{H}_1 \) in a Hilbert space \( \mathcal{H}_2 \), which intertwines two representations \( \alpha_1 \) and \( \alpha_2 \) of \( N \). Then, it is possible to define, on linear combinations of elementary tensors, an isometry \( U \beta_1 \otimes_{\alpha_1} V \) which can be extended to the whole Hilbert space \( \mathcal{K}_1 \beta_1 \otimes_{\alpha_1} \mathcal{H}_1 \) with values in \( \mathcal{K}_2 \beta_2 \otimes_{\alpha_2} \mathcal{H}_2 \). One can show that this isometry does not depend upon the weight \( \nu \). It will be denoted by \( U \beta_1 \otimes_{\alpha_1} V \). If \( U \) and \( V \) are unitaries, then \( U \beta_1 \otimes_{\alpha_1} V \) is an unitary and \((U \beta_1 \otimes_{\alpha_1} V)^* = U^* \beta_2 \otimes_{\alpha_2} V^* \).
If $x \in D(\sigma_{\nu}^{-1/2})$, then it is possible to construct on elementary tensors an operator $\beta(x)_{\nu} \otimes_{N} 1 = 1_{\nu} \otimes_{N} \alpha(\sigma_{\nu}^{-1/2}(x))$ ([S], 2.2b)

2.4. Definition of a Hopf-bimodule. A quintuple $(N, M, \alpha, \beta, \Gamma)$ will be called a Hopf-bimodule, following ([Val2], 6.5), if $N$, $M$ are von Neumann algebras, $\alpha$ is a faithful non-degenerate representation of $N$ into $M$, $\beta$ is a faithful non-degenerate anti-representation of $N$ into $M$, with commuting ranges, and $\Gamma$ is an injective $\ast$-homomorphism from $M$ into $M_{\beta\ast\alpha}$. The von Neumann algebra $N$ will be called the basis of $(N, M, \alpha, \beta, \Gamma)$.

In ([DCl], chap. 11), De Commer has shown that, if $N$ is finite-dimensional, the Hilbert space $L^{2}(M)_{\beta\ast\alpha} L^{2}(M)$ can be isometrically imbedded into the usual Hilbert tensor product $L^{2}(M) \otimes L^{2}(M)$ and the projection $p$ on this closed subspace belongs to $M \otimes M$. Moreover, the fiber product $M_{\beta\ast\alpha} M$ can be then identified with the reduced von Neumann algebra $p(M \otimes M)p$ and we can consider $\Gamma$ as an usual coproduct $M \mapsto M \otimes M$, but with the condition $\Gamma(1) = p$.

A co-inverse $R$ for a Hopf bimodule $(N, M, \alpha, \beta, \Gamma)$ is an involutive ($R^{2} = \text{id}$) anti-$\ast$-isomorphism of $M$ satisfying $R \circ \alpha = \beta$ (and therefore $R \circ \beta = \alpha$) and $\Gamma \circ R = \zeta_{N} \circ (R \beta \ast\alpha R) \circ \Gamma$, where $\zeta_{N}$ is the flip from $M_{\beta\ast\alpha}$ into $M_{\beta\ast\alpha}$. A Hopf bimodule is called co-commutative if $N$ is abelian, $\beta = \alpha$, and $\Gamma = \zeta \circ \Gamma$.

For an example, suppose that $G$ is a measured groupoid, with $G^{(0)}$ as its set of units. We denote by $r$ and $s$ the range and source maps from $G$ to $G^{(0)}$, given by $xs^{-1} = r(x)$ and $x^{-1}x = s(x)$, and by $G^{(2)}$ the set of composable elements, i.e.

$$G^{(2)} = \{(x, y) \in G^{2} : s(x) = r(y)\}$$

Let $(\lambda^{u})_{u \in G^{(0)}}$ be a Haar system on $G$ and $\nu$ a measure on $G^{(0)}$. Let us denote by $\mu$ the measure on $G$ given by integrating $\lambda^{u}$ by $\nu$,

$$\mu = \int_{G^{(0)}} \lambda^{u}d\nu$$

By definition, $\nu$ is called quasi-invariant if $\mu$ is equivalent to its image under the inversion $x \mapsto x^{-1}$ of $G$ (see [R], [C2] II.5, [Z] and [AR] for more details, precise definitions and examples of groupoids).

In [Y1], [Y2], [Y3] and [Val2] was associated to a measured groupoid $G$, equipped with a Haar system $(\lambda^{u})_{u \in G^{(0)}}$ and a quasi-invariant measure $\nu$ on $G^{(0)}$, a Hopf bimodule with an abelian underlying von Neumann algebra $(L^{\infty}(G^{(0)}, \nu), L^{\infty}(G, \mu), r_{G}, s_{G}, \Gamma_{G})$, where $r_{G}(g) = g \circ r$ and $s_{G}(g) = g \circ s$ for all $g$ in $L^{\infty}(G^{(0)})$ and where $\Gamma_{G}(f)$, for $f$ in $L^{\infty}(G)$, is the function defined on $G^{(2)}$ by $(s, t) \mapsto f(st)$. Thus, $\Gamma_{G}$ is an involutive homomorphism from $L^{\infty}(G)$ into $L^{\infty}(G^{(2)})$, which can be identified with $L^{\infty}(G)_{s\ast\nu} L^{\infty}(G)$. 


It is straightforward to get that the inversion of the groupoid gives a co-inverse for this
Hopf bimodule structure.

2.5. Definition of measured quantum groupoids (II, [E4]). A measured quantum
groupoid is an octuple \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) such that ([E4], 3.8):

(i) \((N, M, \alpha, \beta, \Gamma)\) is a Hopf bimodule,

(ii) \(T\) is a left-invariant normal, semi-finite, faithful operator-valued weight from \(\alpha(N)\) (to be more precise, from \(M^+\) to the extended positive elements of \(\alpha(N)\) (cf. [I] IX.4.12)), which means that, for any \(x \in \mathfrak{M}_T\), we have \((\text{id}_N \otimes \alpha) T^\nu(x) = T(x) \beta \otimes \alpha 1\).

(iii) \(T'\) is a right-invariant normal, semi-finite, faithful operator-valued weight from \(M\) to \(\beta(N)\), which means that, for any \(x \in \mathfrak{M}_{T'}\), we have \((\text{id}_N \otimes \alpha) T'^\nu(x) = 1 \beta \otimes \alpha T'(x)\).

(iv) \(\nu\) is normal semi-finite faithful weight on \(N\), which is relatively invariant with respect to \(T\) and \(T'\), which means that the modular automorphisms groups of the weights \(\varphi = \nu \circ \alpha^{-1} \circ T\) and \(\psi = \nu \circ \beta^{-1} \circ T'\) commute. The weight \(\varphi\) will be called left-invariant, and \(\psi\) right-invariant.

For example, let \(\mathcal{G}\) be a measured groupoid equipped with a left Haar system \((\lambda^\nu)_{u \in \mathcal{G}(0)}\) and a quasi-invariant measure \(\nu\) on \(\mathcal{G}(0)\). Let us use the notations introduced in [E4]. If \(f \in L^\infty(\mathcal{G}, \mu)^+,\) consider the function on \(\mathcal{G}(0), u \mapsto \int_\mathcal{G} f d\lambda^u\), which belongs to \(L^\infty(\mathcal{G}(0), \nu)\). The image of this function by the homomorphism \(r_3\) is the function on \(\mathcal{G}, \gamma \mapsto \int_\mathcal{G} f d\lambda^\gamma\), and the application which sends \(f\) to this function can be considered as an operator-valued weight from \(L^\infty(\mathcal{G}, \mu)\) to \(r_3(L^\infty(\mathcal{G}(0), \nu))\) which is normal, semi-finite and faithful. By definition of the Haar system \((\lambda^u)_{u \in \mathcal{G}(0)},\) it is left-invariant in the sense of (ii). We shall denote this operator-valued weight from \(L^\infty(\mathcal{G}, \mu)\) to \(r_3(L^\infty(\mathcal{G}(0), \nu))\) by \(T_3\). If we write \(\lambda_u\) for the image of \(\lambda^u\) under the inversion \(x \mapsto x^{-1}\) of the groupoid \(\mathcal{G}\), starting from the application which sends \(f\) to the function on \(\mathcal{G}(0)\) defined by \(u \mapsto \int_\mathcal{G} f d\lambda_u\), we define a normal semifinite faithful operator-valued weight from \(L^\infty(\mathcal{G}, \mu)\) to \(s_3(L^\infty(\mathcal{G}(0), \nu))\), which is right-invariant in the sense of (ii), and which we shall denote by \(T_3^{(-1)}\).

We then get that:

\[
(L^\infty(\mathcal{G}(0), \nu), L^\infty(\mathcal{G}, \mu), r_3, s_3, \Gamma_3, T_3, T_3^{(-1)}, \nu)
\]

is a measured quantum groupoid, which we shall denote again \(\mathcal{G}\).

It can be proved ([E5]) that any measured quantum groupoid, whose underlying von
Neumann algebra is abelian, is of that type.

Let \(\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)\) be a measured quantum groupoid, then we denote by \(\mathcal{G}^\circ\) the octuplet \((N^\circ, M, \beta, \alpha, \varsigma_N \Gamma, T', T, \nu^\circ)\) (where \(\sigma_N\) is the flip from \(M\) to \(M\)) and it is another measured quantum groupoid, called the opposite measured quantum groupoid of \(\mathcal{G}\).

If \(T\) is bounded, \(\mathcal{G}\) is called "of compact type".

2.6. Pseudo-multiplicative unitary. Let \(\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)\) be an octuple
satisfying the axioms (i), (ii) (iii) of [2.3] With \(\varphi = \nu \circ \alpha^{-1} \circ T\), we shall write \(H = H_\varphi, J = J_\varphi\) and \(\tilde{\beta}(n) = J_\alpha(n^*) J\) for all \(n \in N\).

Then ([I], 3.7.3 and 3.7.4), \(\mathcal{G}\) can be equipped with a pseudo-multiplicative unitary \(W\) which is a unitary from \(H_\beta \otimes \alpha H\) onto \(H_\alpha \otimes \beta H\) ([E4], 3.6) that intertwines \(\alpha, \tilde{\beta}, \beta\) in
the following way: for all \( X \in N \), we have :
\[
W(\alpha(X)_{\beta} \otimes_{\alpha} 1) = (1_{\alpha} \otimes_{\beta} \alpha(X))W,
\]
\[
W(1_{\beta} \otimes_{\alpha} \beta(X)) = (1_{\alpha} \otimes_{\beta} \beta(X))W,
\]
\[
W(\hat{\beta}(X)_{\beta} \otimes_{\alpha} 1) = (\hat{\beta}(X)_{\alpha} \otimes_{\beta} 1)W,
\]
\[
W(1_{\beta} \otimes_{\alpha} \hat{\beta}(X)) = (\beta(X)_{\alpha} \otimes_{\beta} 1)W.
\]
Moreover, the operator \( W \) satisfies the **pentagonal relation** :
\[
(1_{\alpha} \otimes_{\beta} W)(W_{\beta} \otimes_{\alpha} 1_{H}) = (W_{\alpha} \otimes_{\beta} 1_{N})(\sigma^{23}_{\alpha,\beta})(W_{\beta} \otimes_{\alpha} 1)(1_{\beta} \otimes_{\alpha} W)
\]
where \( \sigma^{23}_{\alpha,\beta} \) goes from \((H_{\alpha} \otimes_{\beta} H)_{\beta} \otimes_{\alpha} H\) to \((H_{\beta} \otimes_{\alpha} H)_{\alpha} \otimes_{\beta} H\), and \( 1_{\beta} \otimes_{\alpha} \sigma_{\nu} \) goes from \( H_{\beta} \otimes_{\alpha} H_{\alpha} \) to \( H_{\beta} \otimes_{\alpha} H_{\beta} \otimes_{\alpha} H \). The operators in this formula are well-defined because of the intertwining relations listed above.

The operator \( W \) is defined by the following formula, for any \( a \in \mathfrak{N}_{T} \cap \mathfrak{N}_{\varphi} \), \( \nu \in D((H_{\varphi})_{\beta}, \nu') \), where \( (\xi_{i})_{i \in I} \) is a \((\beta, \nu')\) basis of \( H_{\varphi} \) (in the sense of \([2.1.1] \) (\text{4}), 3.2.10) :
\[
W^{*}(v_{\alpha} \otimes_{\beta} \Lambda_{\varphi}(a)) = \sum_{i \in I} \xi_{i} \otimes_{\varphi} \Lambda_{\varphi}((\omega_{v, \xi_{i}} \otimes_{\nu} \sigma_{\nu}(1_{\alpha} \otimes_{\beta} a))\Gamma(a))
\]
The operator \( W \) does not depend of the choice of the \((\beta, \nu')\) basis. Moreover, \( W, M \) and \( \Gamma \) are related by the following results:

(i) \( M \) is the weakly closed linear space generated by all operators \((\text{id} \ast \omega_{\xi, \eta})(W)\), where \( \xi \in D(H_{\alpha}, \nu) \) and \( \eta \in D(H_{\beta}, \nu') \) (see \([\text{4}], \) 3.8(vii)).

(ii) \( \Gamma(x) = W^{*}(1_{\alpha} \otimes_{\beta} x)W \) for all \( x \in M \) (\([\text{4}], \) 3.6).

(iii) For any \( x, y_{1}, y_{2} \in \mathfrak{N}_{T} \cap \mathfrak{N}_{\varphi} \), we have (\([\text{4}], \) 3.6):
\[
(\text{id} \ast \omega_{J_{\Lambda_{\varphi}(y_{1})} \Lambda_{\varphi}(y_{2})})(W) = (\text{id} \ast \omega_{J_{\Lambda_{\varphi}(y_{2})} \Lambda_{\varphi}(y_{1})})(W^{*})
\]

(iv) for any \( a \in \mathfrak{N}_{T} \cap \mathfrak{N}_{\varphi} \), \( \nu \in D(H_{\alpha}, \nu) \cap D(H_{\beta}, \nu') \), \( w \in D(H_{\beta}, \nu') \), we have (\([\text{4}], \) 3.3.3):
\[
(\omega_{v, w} \ast \text{id})(W^{*})\Lambda_{\varphi}(a) = \Lambda_{\varphi}((\omega_{v, w} \beta \ast_{\nu} \sigma\text{id})\Gamma(a))
\]
If \( N \) is finite-dimensional, using the fact that the relative tensor products can be identified with closed subspaces of the usual Hilbert tensor product (\([\text{2.1}] \)), we get that \( W \) can be considered as a partial isometry on the usual Hilbert tensor product, which is multiplicative in the usual sense (i.e. such that \( W_{23}W_{12} = W_{12}W_{13}W_{23} \)).

2.7. **Lemma.** Let \( W \) be an \((\alpha, \hat{\beta}, \beta)\)-pseudo-multiplicative unitary, \( \xi_{1} \) in \( D(\hat{\beta}_{2}, \nu') \), \( \xi_{2} \) in \( D(\alpha_{2}, \nu) \), \( \eta \) in \( \hat{\beta}_{2} \), let \( \xi_{1} \) in \( D(\hat{\beta}_{3}, \nu) \) and \( \xi_{1}' \) in \( \hat{\beta}_{3} \) such that \( W^{*}(\xi_{2} \otimes_{\nu} \eta) = \sum_{i} \xi_{1} \otimes_{\beta} \xi_{1}' \); then we have :
\[
\sum_{i} \alpha(<\xi_{1}, \xi_{1}'>) \xi_{1}' = (\omega_{\xi_{1} \xi_{2}} \ast \text{id})(W^{*})\eta
\]
Proof. Let \( \theta \) in \( \mathcal{S} \); we have:

\[
((\omega_{\xi,\xi_2} \ast \text{id})(W)^*|\theta) = (W^* (\xi_2 \otimes_{\beta} \eta_1) |\xi_1 \otimes_{\alpha} \theta)
\]

\[
= \left( \sum_{i} \xi_i \otimes_{\alpha} \xi_i' |\xi_1 \otimes_{\alpha} \theta \right)
\]

\[
= \left( \sum_{i} \alpha \langle \zeta_i, \zeta_i >_{\beta, \nu^o} |\zeta_i' |\theta \right)
\]

from which we get the result. \( \square \)

2.8.Lemma. Let \( W \) be an \( (\alpha, \hat{\beta}, \beta) \)-pseudo-multiplicative unitary, \( \xi_1, \zeta_1 \) in \( D(\mathcal{S}_\beta, \nu^o) \), \( \zeta, \eta_1, \eta_2 \) in \( \mathcal{S} \). Let us consider the flip \( \sigma_{1,2}^{\alpha,\beta} \) from \( H_{\beta \otimes \alpha}(H_{\alpha \otimes \beta} H) \) onto \( H_{\alpha \otimes \beta}(H_{\beta \otimes \alpha} H) \). Then, we have:

\[
(\sigma_{1,2}^{\alpha,\beta}(1_{\beta \otimes_{\alpha} N}) W(\xi_1 \otimes_{\alpha} \eta_1 \otimes_{\alpha} \xi) |\eta_2 \otimes_{\beta} (\zeta \otimes_{\alpha} \zeta_2)) = (W(\eta_1 \otimes_{\alpha} \xi) |\eta_2 \otimes_{\beta} (\zeta \otimes_{\alpha} \zeta_2))
\]

Proof. The scalar product

\[
(\sigma_{1,2}^{\alpha,\beta}(1_{\beta \otimes_{\alpha} N}) W(\xi_1 \otimes_{\alpha} \eta_1 \otimes_{\alpha} \xi) |\eta_2 \otimes_{\beta} (\zeta \otimes_{\alpha} \zeta_2))
\]

is equal to:

\[
(\xi_1 \otimes_{\alpha} W(\eta_1 \otimes_{\alpha} \xi) |\zeta \otimes_{\alpha} \eta_2)
\]

from which we get the result. \( \square \)

2.8.1. Proposition. \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) is a measured quantum groupoid in the sense of 2.6. Let \( W \) be its pseudo-multiplicative unitary, \( \xi \) in \( D(\text{a}, \mathcal{S}, \nu) \), \( \eta \) in \( D(\text{a}, \mathcal{S}, \nu^o) \). Let \( \xi_1, \eta_1 \) in \( D(\mathcal{S}_\beta, \nu^o), \xi_2, \eta_2 \) in \( D(\text{a}, \mathcal{S}, \nu) \); then, we have:

\[
(\Gamma((id \ast \omega_{\xi,\xi})(W)) |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2) = ((\omega_{\xi_1,\xi_2} \ast \text{id})(W) |\omega_{\eta_1,\eta_2} \ast \text{id})(W)|\xi_1 |\eta_1)
\]

Proof. Using the 2.6(ii) we get that:

\[
(\Gamma((id \ast \omega_{\xi,\xi})(W)) |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2) = ((1_{\alpha \otimes \beta} |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2))
\]

which is equal to:

\[
((1_{\alpha \otimes \beta} W |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2))
\]

which, using the pentagonal relation 2.6, is equal to:

\[
(\sigma_{2,3}^{\alpha,\beta}(1_{\beta \otimes_{\alpha} W} |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2))
\]

or, to:

\[
((W |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2))
\]

which is equal to:

\[
((\sigma_{1,2}^{\alpha,\beta}(1_{\beta \otimes_{\alpha} W} |\xi_1 \otimes_{\alpha} \eta_1 |\xi_2 \otimes_{\alpha} \eta_2)) (W^*(\xi_2 \otimes_{\alpha} \eta_2)))
\]
Defining now $\zeta_i, \zeta'_i$ as in \ref{2.7}, we get, using \ref{2.8} that it is equal to:

$$(W(\eta_1 \beta \otimes \alpha \xi)|\eta_2 \alpha \otimes \beta \sum_i \alpha(\zeta_i, \xi) \xi_i)$$

which, thanks to \ref{2.7}, is equal to:

$$(W(\eta_1 \beta \otimes \alpha \xi)|\eta_2 \alpha \otimes \beta (\omega_{\xi_1, \xi_2} \ast id)(W)^* \eta)$$

and, therefore, to:

$$(\omega_{\eta_1, \eta_2} \ast id)(W)\xi|(\omega_{\xi_1, \xi_2} \ast id)(W)^* \eta)$$

which finishes the proof. $\square$

2.9. Other data associated to a measured quantum groupoid \((\mathcal{G}, \mathcal{H})\). Suppose that \(\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)\) is a measured quantum groupoid in the sense of \ref{2.6}.

Let us write \(\varphi = \nu \circ \alpha^{-1} \circ T\), which is a normal semi-finite faithful left-invariant weight on \(M\). Then:

(i) There exists an anti-\(*\)-automorphism \(R\) on \(M\) such that:

\[
R^2 = id, \quad R(\alpha(n)) = \beta(n) \text{ for all } n \in N, \quad \Gamma \circ R = \varsigma_N \circ R(\beta \circ \alpha R) \Gamma
\]

and:

\[
R((id \ast \omega_{\xi, \eta})(W)) = (id \ast \omega_{\eta, \eta})(W) \text{ for all } \xi \in D(\alpha H, \nu), \eta \in D(H \beta, \nu^\alpha).
\]

This map \(R\) will be called the co-inverse.

(ii) There exists a one-parameter group \(\tau_t\) of automorphisms of \(M\) such that:

\[
R \circ \tau_t = \tau_t \circ R, \quad \tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n)), \quad \tau_t(\beta(n)) = \beta(\sigma_t^\nu(n)), \quad \Gamma \circ \sigma_t^\nu = (\tau_t \circ \alpha \circ \sigma_t^\nu) \Gamma
\]

for all \(t \in \mathbb{R}\) and \(n \in N\). This one-parameter group will be called the scaling group.

(iii) The weight \(\nu\) is relatively invariant with respect to \(T\) and \(R\)\(TR\). Moreover, \(R\) and \(\tau_t\) are still the co-inverse and the scaling group of this new measured quantum groupoid, which we shall denote by:

\[
\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, RTR, \nu),
\]

and for simplification we shall assume now that \(T' = RTR\) and \(\psi = \varphi \circ R\).

(iv) There exists a one-parameter group \(\gamma_t\) of automorphisms of \(N\) such that:

\[
\sigma_t^\beta(\beta(n)) = \beta(\gamma_t(n))
\]

for all \(t \in \mathbb{R}\) and \(n \in N\). Moreover, we get that \(\nu \circ \gamma_t = \nu\).

(v) There exist a positive non-singular operator \(\lambda\) affiliated to \(Z(M)\) and a positive non-singular operator \(\delta\) affiliated with \(M\) such that:

\[
(D\varphi \circ R : D\varphi) = \lambda^{u^2/2} \delta^u,
\]

and therefore

\[
(D\varphi \circ \sigma_s^\nu R : D\varphi) = \lambda^{s^u}.
\]

The operator \(\lambda\) will be called the scaling operator, and there exists a positive non-singular operator \(q\) affiliated to \(N\) such that \(\lambda = \alpha(q) = \beta(q)\). We have \(R(\lambda) = \lambda\).

The operator \(\delta\) will be called the modulus. We have \(R(\delta) = \delta^{-1}\) and \(\tau_t(\delta) = \delta\) for all \(t \in \mathbb{R}\), and we can define a one-parameter group of unitaries \(\delta^u \beta \otimes \alpha \delta^u\) which acts naturally on elementary tensor products and satisfies for all \(t \in \mathbb{R}\):

\[
\Gamma(\delta^u) = \delta^u \beta \otimes \alpha \delta^u.
\]
We shall say that the pseudo-multiplicative unitary $W$ for all $G$. Moreover, for all $\tau$, $\alpha$ and $\beta$, we have $\sigma_\alpha \circ \sigma_\beta = \sigma_{\alpha \circ \beta}$ for all $s, \ t$ in $R$ and allows to define a one-parameter group of unitaries by:

$$P^{it} \Lambda_\varphi(x) = \lambda^{it/2} \Lambda_\varphi(\tau_t(x)) \quad \text{for all } x \in M_\varphi.$$ 

Moreover, for any $y \in M$, we get that:

$$\tau_t(y) = P^{it} y P^{-it},$$

and it is possible to define one parameter groups of unitaries $P^{it} \beta \otimes_\alpha P^{it}$ and $P^{it} \alpha \otimes_\beta P^{it}$ such that:

$$W(P^{it} \beta \otimes_\alpha P^{it}) = (P^{it} \alpha \otimes_\beta P^{it}) W$$

Moreover, for all $v \in D(P^{-1/2})$, $w \in D(P^{1/2})$, $p$, $q$ in $D(\alpha H_\Phi, \nu) \cap D((H_\Phi)_{\beta, \nu})$, we have

$$(W^*(v \beta \otimes_\alpha p) \nu') w \beta \otimes_\alpha p = (W(P^{-1/2} v \beta \otimes_\alpha J_\Phi p) \nu) (P^{1/2} w \alpha \otimes_\beta J_\Phi q)$$

We shall say that the pseudo-multiplicative unitary $W$ is manageable, with managing operator $P$, which implies it is weakly regular in the sense of [E3], 4.1. As $\tau_\alpha \circ \sigma_\tau = \sigma_\tau \circ \tau$, we get that $J_\Phi \circ P = P$.

(vi) We have $(D\Phi \circ \tau_t : D\Phi)_t = \lambda^{-ist}$, which proves that $\tau_t \circ \sigma_s^\varphi = \sigma_s^\varphi \circ \tau_t$ for all $s, \ t$ in $R$ and allows to define a one-parameter group of unitaries by:

$$P^{it} \Lambda_\varphi(x) = \lambda^{it/2} \Lambda_\varphi(\tau_t(x)) \quad \text{for all } x \in M_\varphi.$$ 

We can prove that $\sigma_t^\varphi \circ \alpha = \alpha \circ \sigma_t^\varphi$ for all $t \in R$, which gives the existence of an operator-valued weight $T$, which appears then to be left-invariant.

As the formula $y \mapsto J y^* J$ ($y \in \hat{M}$) gives a co-inverse for the coproduct $\hat{\gamma}$, we get also a right-invariant operator-valued weight. Moreover, the pseudo-multiplicative unitary $\hat{W}$ associated to $\hat{\Theta}$ is $\hat{W} = \sigma_\nu \hat{W}^* \sigma_\nu$, its managing operator $\hat{P}$ is equal to $P$, its scaling group is given by $\hat{\gamma}_t(y) = P^{it} y P^{-it}$, its scaling operator $\hat{\lambda}$ is equal to $\lambda^{-1}$, and its one-parameter group of automorphisms $\hat{\gamma}_t$ of $N$ is equal to $\gamma_{-t}$.

We write $\hat{\varphi}$ for $\nu \circ \alpha^{-1} \circ \hat{\gamma}$, identify $H_\varphi$ with $H$, and write $\hat{J} = J_\varphi$. Then $R(x) = \hat{J} x^* \hat{J}$ for all $x \in M$ and $W^* = (\hat{J} \alpha \otimes_\beta J) W (\hat{J} \alpha \otimes_\beta J)_{N^\alpha}$.

Moreover, we have $\hat{\Theta} = \hat{\Theta}$.

For example, let $\mathcal{G}$ be a measured groupoid as in [23]. The dual $\hat{\mathcal{G}}$ of the measured quantum groupoid constructed in [23] (and denoted again by $\mathcal{G}$) is:

$$\hat{\mathcal{G}} = (L^\infty(\mathcal{G}(0), \nu), \mathcal{L}(\mathcal{G}), r_g, r_g, \hat{\gamma}_g, \hat{T}_g, \hat{T}_g)$$
where $\mathcal{L}(\mathcal{G})$ is the von Neumann algebra generated by the convolution algebra associated to the groupoid $\mathcal{G}$, the coproduct $\Gamma_\delta$ had been defined in (Val1, 3.3.2), and the operator-valued weight $\hat{T}_\delta$ had been defined in (Val1, 3.3.4). The underlying Hopf-bimodule is co-commutative.

The pseudo-multiplicative unitary $W^\circ$ associated to the opposite measured quantum groupoid $\mathcal{G}^\circ$ is:

$$W^\circ = (\hat{J}_{\alpha\beta} J)W(\hat{J}_{\alpha\beta} J)$$

2.9.1. Lemma. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid, as defined in 2.7. $W$ the pseudo-multiplicative unitary associated by 2.6; (i) we have, using the notations of 2.6:

$$(W^*_{\beta\alpha} 1)(1_{\alpha\beta} W^*) (W_{\alpha\beta} 1) = (1_{\beta\alpha} W^*) (1_{\beta\alpha} \sigma_\nu^\circ) (W^*_{\beta\alpha} 1) (\sigma_\alpha^\circ)^*$$

(ii) let $a, b$ in $\mathcal{H}_T \cap \mathcal{H}_T' \cap \mathcal{H}_\nu \cap \mathcal{H}_\nu'$; we have:

$$\Gamma((id * \omega_{\lambda \nu}(b), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W)) = \Gamma((\rho_{\beta\alpha}((\lambda \nu))) (1_{\beta\alpha} W^*) (1_{\beta\alpha} \sigma_\nu^\circ) (W^*_{\beta\alpha} 1) (\sigma_\alpha^\circ)^*)$$

Proof. Result (i) is easily obtained from 2.6 Then, by 2.6(ii), we get (ii). \qed

2.9.2. Proposition. (F1, 9.4.3). Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid, as defined in 2.7. $W$ the pseudo-multiplicative unitary associated by 2.6. $R$ the co-inverse associated by 2.7. let us define $A_n(W)$ as the norm closure of the linear span generated by all operators of the form $(id * \omega_{\xi \eta})$ for all $\xi \in D(\alpha H, \nu), \eta \in D(H^\beta, \nu^\circ)$. Then $A_n(W)$ is an algebra, $A_n(W) \cap A_n(W)^*$ is a non degenerate sub-$C^*$-algebra of $M$, weakly dense in $M$, invariant by $R, \sigma_\nu^\circ, \sigma_\nu^\circ R, \tau_t$. \qed

2.9.3. Theorem. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid; let's use all notations introduced in 2.9. Then, for any $\xi, \eta$ in $D(\alpha H, \nu)$, for all $t$ in $\mathbb{R}$, we have:

(i) $R((\xi * \omega_{\lambda \nu} J_{\xi \eta})(W))(\xi * \omega_{\lambda \nu} J_{\xi \eta})(W)$; therefore $R(A_n(W)) = A_n(W)$.

(ii) $\tau_t((\xi * \omega_{\lambda \nu} J_{\xi \eta})(W)) = (\xi * \omega_{\lambda \nu} J_{\xi \eta})(W)$

(iii) $\sigma_\nu^\circ((\xi * \omega_{\lambda \nu} J_{\xi \eta})(W)) = (\xi * \omega_{\lambda \nu} J_{\xi \eta})(W)$

Proof. Results (i) and (ii) are (F1 4.6).

Let us take $\xi = J_{\nu \lambda \nu}(y_1, y_2)$, and $\eta = J_{\nu \lambda \nu}(x)$, with $x, y_1, y_2$ in $\mathcal{H}_T \cap \mathcal{H}_\nu$; then, using 2.6 and 2.9 we get:

$$\sigma_\nu^\circ((\xi * \omega_{\lambda \nu} J_{\nu \lambda \nu}(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W)) = (id_{\beta\alpha} \omega_{\nu \lambda \nu}(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W)$$

which is equal to:

$$(id_{\beta\alpha} \omega_{\nu \lambda \nu}(\lambda \nu, \sigma_\nu^\circ R(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W)) = (id_{\beta\alpha} \omega_{\nu \lambda \nu}(\lambda \nu, \sigma_\nu^\circ R(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W))$$

which, using again 2.6 and 2.9, is equal to:

$$(i * \omega_{\nu \lambda \nu}(\lambda \nu, \sigma_\nu^\circ R(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W)) = (i * \omega_{\nu \lambda \nu}(\lambda \nu, \sigma_\nu^\circ R(y_1, y_2), \nu, \lambda \nu, \alpha, \beta, \gamma, \Gamma)(W))$$
which gives the first result of (iii), using 2.6.

By similar calculations, we obtain:

\[ \sigma_t^{\circ_0 R}((i * \alpha_j \omega_j \Lambda_j(y_1^* y_2), \Lambda_j(x)) (W)) = (id \beta^* N \omega_j \Lambda_j(y_2), J_j \Lambda_j(y_1) \circ \tau_i) \Gamma(\sigma_t^{\circ_0 R}(x^*)) \]

which is equal to:

\[ (id \beta^* N \omega_j \Lambda_j(\lambda^{1/2} \tau_1(y_2), J_j \Lambda_j(\lambda^{1/2} \tau_1(y_1))) \Gamma(\sigma_t^{\circ_0 R}(x^*)) = (i * \omega_j \Lambda_j(\lambda^{1/2} \tau_1(y_1^*)), \Lambda_j(\sigma^{\circ_0 R}(x))(W) \]

from which we obtain the second result of (iii).

\[ \square \]

2.9.4. **Lemma.** Let \( \Theta = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) be a measured quantum groupoid, as defined in 2.3. Let \( W \) the pseudo-multiplicative unitary associated by 2.6. Let \( R \) the co-inverse associated by 2.6, let \( x \) and \( y \) in \( \mathfrak{M}_\varphi \cap \mathfrak{M}_T \); then, we have:

(i) \( (id * \omega_j \Lambda_j(y), \Lambda_j(x))(W)^* = (id * \omega_j \Lambda_j(y^*), \Lambda_j(x^*))(W) \)

(ii) \( R[(id * \omega_j \Lambda_j(y), \Lambda_j(x))(W)] = (id * \omega_j \Lambda_j(y), \Lambda_j(y))(W) \)

**Proof.** Using 2.6(iii), we have, for any \( x_1, x_2, y_1, y_2 \) in \( \mathfrak{M}_\varphi \cap \mathfrak{M}_T \):

\[ (id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W) = (id \beta^* N \omega_j \Lambda_j(y_1), J_j \Lambda_j(y_2))(x_1 x_2) \]

from which we get:

\[ (id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W)^* = (id \beta^* N \omega_j \Lambda_j(y_2), J_j \Lambda_j(y_1))(x_2^* x_1) \]

and, using 2.6(iii) again, we get (i).

Using \( W^* = (\widehat{\beta} \otimes J_N) W(J_N \otimes \widehat{\beta} J) \) 2.9, we get that:

\[ R[(id * \omega_j \Lambda_j(y), \Lambda_j(x))(W)] = \widehat{\beta}(id * \omega_j \Lambda_j(y), \Lambda_j(x))(W)^* \widehat{\beta} \]

\[ = \widehat{\beta}(id * \omega_j \Lambda_j(x), \Lambda_j(y))(W^*) \widehat{\beta} \]

\[ = \widehat{\beta}(id * \omega_j \Lambda_j(x), \Lambda_j(y))[J_N \otimes \widehat{\beta} J] W(J_N \otimes \widehat{\beta} J) \widehat{\beta} \]

\[ = (id * \omega_j \Lambda_j(x), \Lambda_j(y))(W) \]

which is (ii).

\[ \square \]

2.9.5. **Proposition.** (155, 4.6). Let \( \Theta = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) be a measured quantum groupoid, as defined in 2.3. \( W \) the pseudo-multiplicative unitary associated by 2.6. \( R \) the co-inverse associated by 2.6, let \( x_1, x_2, y_1, y_2 \) in \( \mathfrak{M}_\varphi \cap \mathfrak{M}_T \); then:

(i) the operator \( (id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W) \) belongs to \( \mathfrak{M}_T \cap \mathfrak{M}_\varphi \) if \( y_1 \) belongs to \( \mathfrak{M}_{RTT} \), and we have:

\[ T((id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W) = \alpha(<RTR(y_2, 1), J_j \Lambda_j(x_2), J_j \Lambda_j(x_1)>) \]

\[ = \alpha(\Lambda_T(x_1))^* J_j RTR(y_2, 1), J_j \Lambda_T(x_2)) \]

(ii) the operator \( (id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W) \) belongs to \( \mathfrak{M}_{RTT} \cap \mathfrak{M}_{\varphi R} \) if \( x_2 x_1 \) belongs to \( \mathfrak{M}_{RTT} \), and we have then:

\[ RTR((id * \omega_j \Lambda_j(y_2, 1), \Lambda_j(x_2 x_1))(W) = \beta(<RTR(x_2 x_1), J_j \Lambda_j(y_2), J_j \Lambda_j(y_1)>) \]

\[ = \beta(\Lambda_T(y_1))^* J_j RTR(x_2 x_1), J_j \Lambda_T(y_2)) \]
Proof. Let \(x, y\) in \(\mathcal{R}_T \cap \mathcal{R}_\varphi\); Using (2.6 iii), we have
\[
(id * \omega_{\beta N} \Lambda_\varphi^{(y^x)})(W) = (id \beta \omega_{\Lambda_\varphi^{(y^x)}}(y^x))(W)
\]
(which is positive); applying the right-invariant operator valued weight \(RTR\), we get :
\[
RTR((id * \omega_{\beta N} \Lambda_\varphi^{(y^x)})(W)) = (id \beta \omega_{\Lambda_\varphi^{(y^x)}}(y^x))(1 \beta \omega_{\Lambda_\varphi^{(y^x)}}(y^x))
\]
which is a bounded positive operator if \(x\) belongs to \(\mathcal{R}_{RTR}\); it is then equal to :
\[
\beta(< RTR(x^x), J_\varphi \Lambda_\varphi^{(y^x)}(y) J_\varphi \Lambda_\varphi^{(y^x)}(y) >_{\alpha, \nu^N}) = \beta(\Lambda_T(y^x) RTR(x^x), J_\varphi \Lambda_T(y^x))
\]
by (2.2.2 i); then, we get (ii) by polarization and (i), by applying 2.9 □

3. Locally compact quantum groupoids

In this chapter, we first recall (3.1) the definition and basic properties of the C*-relative tensor product \(\tilde{\mathcal{L}}\) and \(\mathcal{L}\), and the definition of the C*-fiber product of two C*-algebras, as defined by T. Timmermann ([T12]). We then recall (3.2) the definition of a weight on a C*-algebra and the main properties : lower semi-continuous weights \(\beta\), and KMS weights \(\beta\). In (3.3) we recall the definition of C*-valued weights and lower semi-continuous C*-valued weights, as introduced by J. Kustermans ([K2]). We introduce (3.3.2) the notion of a KMS pair \((\nu, T)\), where \(\nu\) is a lower semi-continuous weight on a C*-algebra \(B\), and \(T\) a lower semi-continuous C*-valued weight from a C*-algebra \(A\) to \(M(B)\). We then give another definition of a fiber product of two C*-algebras \((\tilde{\mathcal{L}})\) and of a locally compact quantum groupoid \((\tilde{\mathcal{L}})\).

3.1. C*-relative product of Hilbert spaces and C*-fiber product of C*-algebras.

3.1.1. Definition. A C*-base \(b\) is a triple \(b = (\mathfrak{H}, B, B^t)\) where \(\mathfrak{H}\) is a Hilbert space, and \(B, B^t\) are commuting non degenerate sub-C*-algebras of \(B(\mathfrak{H})\). We denote \(b^\dagger = (\mathfrak{H}, B, B^t)\).

3.1.2. Definition. Be given a C*-base \(b = (\mathfrak{H}, B, B^t)\), a C*-b-module is a pair \((H, L)\), where \(H\) is a Hilbert space, and \(L\) is a normal closed subspace of \(B(\mathfrak{H}, H)\), such that \([L\mathfrak{H}] = H, [LB] = L, [L^*L] = B\), where \([X]\) means the closed linear space generated by \(X\) (for \(X \subset H\), or \(X \subset B(\mathfrak{H}, H)\), or \(X \subset B(\mathfrak{H})\)).

Then \((\mathfrak{H}, 2.5)\), there exists a non-degenerate normal representation \(\rho_L\) of \(B^t\) on \(H\), such that \(\rho_L(x)L_1 = L_1x\), for all \(x \in B^t, L_1 \in L\). It is easy to check that, if \(B\) is non degenerate on \(H\), then \(\rho_L\) is faithful.

Be given a C*-base \(b = (\mathfrak{H}, B, B^t)\) and two C*-b-modules \((H, L)\) and \((\tilde{H}, \tilde{L})\), a morphism from \((H, L)\) to \((\tilde{H}, \tilde{L})\), is an operator \(S \in B(H, \tilde{H})\), such that \(SL \subset \tilde{L}\) and \(S^*L \subset \tilde{L}\). Then, for all \(b^\dagger \in B^t\), we have \(S\rho_L(b^\dagger) = \rho_L(b^\dagger)S\).

3.1.3. Definition. Be given a C*-base \(b = (\mathfrak{H}, B, B^t)\), a C*-b-module \((H, L)\) and a C*-b-module \((K, P)\), the C*-relative tensor product \(H \otimes_P K\) is the Hilbert space generated by elements \(P_1 \otimes \xi \otimes L_1\), where \(P_1 \in P, \xi \in \mathfrak{H}, L_1 \in L\), with the inner product :
\[
(L_1 \otimes \xi \otimes P_1 | L_2 \otimes \xi_2 \otimes P_2) = (\xi_1 | (P_1^* P_2)(L_1^* L_2)\xi_2)
\]
In this formula, note that, by (3.1.2) \(P_1^* P_2\) belongs to \(B^\dagger\), \(L_1^* L_2\) belongs to \(B\), and therefore, by (3.1.1) commute.

The image of \(L \otimes \xi \otimes P\) in \(H \otimes_P K\) will be denoted by \(L \otimes \xi \otimes P\).
If $S$ is a morphism from the $C^*$-$b$-module $(H, L)$ to the $C^*$-$b$-module $(\tilde{H}, \tilde{L})$, and $T$ is a morphism from the $C^*$-$b^\dagger$-module $(K, P)$ to the $C^*$-$b^\dagger$-module $(\tilde{K}, \tilde{P})$, then the formula
\[(S_L \otimes_P T)(L_1 \otimes \xi \otimes P_1) = SL_1 \otimes \xi \otimes TP_1\]
defines $S_L \otimes_P T \in B(H \otimes_P K, \tilde{H} \otimes_{\tilde{P}} \tilde{K})$.

It is clear that we can define $\sigma_b : H \otimes_P K \to K_K \otimes L H$ by $\sigma_b(L \otimes \xi \otimes P) = P \otimes \xi \otimes L$, and its adjoint $\sigma_b^\dagger$. For $X \in \tilde{B}(H \otimes_P K)$, we shall write $\xi X = \sigma_b X \sigma_b^\dagger$, which belongs to $\tilde{B}(K_K \otimes L H)$.

3.1.4. **Theorem.** Be given a $C^*$-base $b = (\delta, B, B^\dagger)$, a $C^*$-$b$-module $(H, L)$, a $C^*$-$b^\dagger$-module $(K, P)$, and the relative tensor product $K \otimes_P \tilde{H}$, we have:

(i) For any $L_1 \in L$, there exists an element $\lambda_{L_1} \in B(H, H \otimes_P K)$ such that, for any $P_1 \in P$ and $\xi \in \delta$, we have $\lambda_{L_1}(P_1 \xi) = L_1 \otimes \xi \otimes P_1$.

(ii) For any $P_1 \in P$, there exists an element $\rho_{P_1} \in B(K, H \otimes_P K)$ such that, for any $L_1 \in L$ and $\xi \in \delta$, we have $\rho_{P_1}(L_1 \xi) = L_1 \otimes \xi \otimes P_1$.

*Proof.* We have $\|L_1 \otimes \xi \otimes P_1\| = \|L_1\|\|P_1\|\|\xi\| \leq \|L_1\|\|\xi\|\|P_1\|\|\xi\| = \|L_1\|\|\xi\|\|P_1\|$ from which we get (i), by density ([3.1.2]). Result (ii) is obtained the very same way. \(\square\)

3.1.5. **Definition.** Be given a $C^*$-base $b = (\delta, B, B^\dagger)$ and a $C^*$-$b$-module $(H, L)$, a $C^*$-$b$-algebra on $(H, L)$ is a non degenerate sub-$C^*$-algebra $A$ of $B(H)$, such that $\rho_L(B^\dagger) \subset M(A)$, where $\rho_L$ had been defined in [3.1.2].

If $(\tilde{H}, \tilde{L})$ is another $C^*$-$b$-module, and $\tilde{A}$ a $C^*$-algebra on $(\tilde{H}, \tilde{L})$, a $b$-morphism $\Phi$ from $A$ to $\tilde{A}$ is a strict morphism of $C^*$-algebras from $A$ to $M(\tilde{A})$, such that, for all $b^\dagger \in B^\dagger$ and $a \in A$, we have $\rho_{L}(b^\dagger) \Phi(a) = \Phi(\rho_L(b^\dagger)a)$.

3.1.6. **Definition** ([3.3]). Be given a $C^*$-base $b = (\delta, B, B^\dagger)$, a $C^*$-$b$-module $(H, L)$, a $C^*$-$b^\dagger$-module $(K, P)$, and the relative tensor product $H \otimes_P K$, let $A_1$ be a a $C^*$-$b$-algebra on $(H, L)$, and $A_2$ a $C^*$-$b^\dagger$-algebra on $(K, P)$; then the set of all elements $X \in B(H \otimes_P K)$ such that, for all $L_1 \in L$, all operators $X \lambda_{L_1}$ and $X^* \lambda_{L_1}$ in $B(H, H \otimes_P K)$ belong to the norm closure of the linear set generated by $\bigcup_{L_2 \in L} \lambda_{L_2} A_2$, and such that, for all $P_1 \in P$, all operators $X \rho_{P_1}$ and $X^* \rho_{P_1}$ in $B(K, H \otimes_P K)$ belong to the norm closure of the linear set generated by $\bigcup_{P_2 \in P} \rho_{P_2} A_1$, is a $C^*$-algebra, which will be denoted $A_1 (L^* P^* A_2) = A_{2} (L^* P^* A_1)$.

If $X$ belongs to $A_1 (L^* P^* A_2)$, then, for any $L_1, L_2 \in L$, $\lambda_{L_2}^* X \lambda_{L_1}$ belongs to $A_2$, and, for any $P_1$, $P_2$ in $P$, $\rho_{P_1}^* X \rho_{P_2}$ belongs to $A_1 ([3.1.2], 3.16(i))$.

If $(\tilde{H}, \tilde{L})$ is another $C^*$-$b$-module, and $\tilde{A}_1$ a $C^*$-$b$-algebra on $(\tilde{H}, \tilde{L})$, and $\Phi$ is a $b$-morphism from $A_1$ to $M(\tilde{A}_1)$, and if $(\tilde{K}, \tilde{P})$ is another $C^*$-$b^\dagger$-module, and $\tilde{A}_2$ a $C^*$-$b^\dagger$-algebra on $(\tilde{K}, \tilde{P})$, and $\Psi$ a $b^\dagger$-morphism from $A_2$ to $M(\tilde{A}_2)$, then, there exists a $*$-homomorphism $\Psi \rho_{L^* P} \Phi$ from $A_1 (L^* P^* A_2)$ to $M(\tilde{A}_1) (L^* P^* M(\tilde{A}_2) \subset M(\tilde{A}_1 L^* P^* A_2))$. ([3.1.2], 3.20) This homomorphism may be degenerate.
Let us remark that, if $L_1 \in L$, and $X \in B(H_{L \otimes P} K)$, then $X \lambda_{L_1}$ (and $X^* \lambda_{L_1}$) belongs to $B(H, H_{L \otimes P}^b K)$, and, therefore, to the norm closure of the linear set generated by $\bigcup_{L_2 \in L} \lambda_{L_2} B(H)$; we get also that $X \rho_{P_1}$ and $X^* \rho_{P_1}$ belong to the norm closure of the linear set generated by $\bigcup_{R_2 \in R} \lambda_{R_2} B(K)$; therefore $B(H_{L \otimes P}^b K) = B(H)_{L^*_{P}} B(K)$.

Let us suppose now that all operators $X \lambda_{L_1}$ and $X^* \lambda_{L_1}$ in $B(H, H_{L \otimes P} K)$ belong to the norm closure of the linear set generated by $\bigcup_{L_2 \in L} \lambda_{L_2} A_2$; then, as $X \rho_{P_1}$ and $X^* \rho_{P_1}$ belongs to the linear set generated by $\bigcup_{R_2 \in R} \lambda_{R_2} B(H)$, we get that $X \in B(H)_{L^*_{P}} A_2$.

3.1.7. Example. $t = (C, C, C)$ is a $C^*$-base. Then, if $H, K$ are two Hilbert spaces, it is clear that the $C^*$-relative tensor product $H \otimes K$ is just the Hilbert tensor product $H \otimes K^{[T2], 2.13}$; moreover, if $A$ is a sub-$C^*$-algebra of $B(H)$, and $K$ is a sub-$C^*$-algebra of $B(K)$, then the $C^*$-fiber product $A \rtimes B$ contains the $C^*$-algebra $M(A \otimes B) = \{X \in M(A \otimes B), \text{such that } X(1 \otimes b) \in A \otimes B, \text{for all } b \in B, \text{and } X(a \otimes 1) \in A \otimes B, \text{for all } a \in A\}^{[T2], 3.20}$.

3.2. Weights on $C^*$-algebras ($[C1]$, $[C2]$, $[K1]$).

3.2.1. Notations. Let $M$ be a von Neumann algebra, and $\alpha$ an action from a locally compact group $G$ on $M$, i.e. a homomorphism from $G$ into $Aut M$, such that, for all $x \in M$, the function $g \mapsto \alpha_g(x)$ is $\sigma$-weakly continuous. Let us denote by $C^*(\alpha)$ the set of elements $x \in M$, such that this function $t \mapsto \alpha_t(x)$ is norm continuous. It is ($[Pa], 7.5.1$) a sub-$C^*$-algebra of $M$, invariant under the $\alpha_g$, generated by the elements $(x \in N, f \in L^1(G))$:

$$\alpha_f(x) = \int_{\mathbb{R}} f(s) \alpha_s(x) ds$$

More precisely, we get that, for any $x \in M$, $\alpha_f(x)$ is $\sigma$-weakly converging to $x$ when $f$ goes in an approximate unit of $L^1(G)$, which proves that $C^*(\alpha)$ is $\sigma$-weakly dense in $M$, and that $x \in M$ belongs to $C^*(\alpha)$ if and only if this file is norm converging.

If $\alpha_t$ and $\gamma_s$ are two one-parameter automorphism groups of $M$, such that, for all $s, t$ in $\mathbb{R}$, we have $\alpha_t \circ \gamma_s = \gamma_s \circ \alpha_t$, by considering the action of $\mathbb{R}^2$ given by $(s, t) \mapsto \gamma_s \circ \alpha_t$, we obtain a dense sub-$C^*$-algebra of $M$, on which both $\alpha$ and $\gamma$ are norm continuous, we shall denote $C^*(\alpha, \gamma)$.

If $\varphi$ is a normal semi-finite faithful weight on $M$, we shall write $C^*(\varphi)$ for the norm closure of $\mathcal{M}_\varphi \cap C^*(\sigma^\varphi)$.

3.2.2. Definition. Let $A$ be a $C^*$-algebra; a weight $\nu$ on $A$ is function $A^+ \mapsto [0, +\infty]$ such that, for any $x, y \in A^+$, and $\lambda \in \mathbb{R}^+$, we have $\nu(x + y) = \nu(x) + \nu(y)$ and $\nu(\lambda x) = \lambda \nu(x)$.

We note $\mathcal{M}_\nu^+ = \{x \in A^+, \nu(x) < \infty\}$, $\mathcal{N}_\nu = \{x \in A, \nu(x^* x) < \infty\}$ and $\mathcal{M}_\nu$ for the linear space generated by $\mathcal{M}_\nu^+$ (or by all products $x^* y$, where $x, y$ are in $\mathcal{M}_\nu$).

As $\nu$ is an increasing function, we get that $\mathcal{M}_\nu^+$ is an hereditary cone, $\mathcal{N}_\nu$ a left ideal (in $M(A)$), and that $\mathcal{M}_\nu$ is a sub-$*$-algebra of $A$, and that $\mathcal{M}_\nu^+ = \mathcal{M}_\nu \cap A^+$ (which justify the notation). Moreover, $\nu$ can be extended to a linear map on $\mathcal{M}_\nu$, we shall denote again $\nu$. We denote $N_\nu$ the set of all $x \in A$, such that $\nu(x^* x) = 0$; it is clear that $N_\nu$ is a left-ideal of $A$. 
We shall say that \( \nu \) is densely defined if \( \mathfrak{M}_\nu^+ \) is dense in \( A^+ \) (or if \( \mathfrak{M}_\nu \) (resp. \( \mathfrak{N}_\nu \)) is dense in \( A \)), and that \( \nu \) is faithful if, for \( x \in A^+ \), \( \nu(x) = 0 \) implies that \( x = 0 \) (and then \( N_\nu = \{0\} \)).

3.2.3. Definition. Be given a \( C^* \)-algebra \( A \) and a weight \( \nu \) on \( A \), a GNS construction for \( \nu \) is a triple \( (H_\nu, \pi_\nu, \Lambda_\nu) \) such that :

(i) \( H_\nu \) is a Hilbert space,

(ii) \( \Lambda_\nu \) is a linear map from \( \mathfrak{N}_\nu \) in \( H_\nu \), such that \( \Lambda_\nu(\mathfrak{N}_\nu) \) is dense in \( H_\nu \), and, for any \( x, y \) in \( \mathfrak{N}_\nu \), we have \( \Lambda_\nu(x)|\Lambda_\nu(y)) = \nu(y^*x) \),

(iii) \( \pi_\nu \) is a representation of \( A \) on \( H_\nu \), such that, for any \( a \in A \), \( \Lambda_\nu(ax) = \pi_\nu(a)\Lambda_\nu(x) \).

For a construction, we refer to ([C1], 2).

3.2.4. Definition. We shall say that a weight \( \nu \) on a \( C^* \)-algebra \( A \) is lower semi-continuous \( \text{(l.s.c.)} \) if, for all \( \lambda \in \mathbb{R}^+ \), the set \( \{x \in A^+, \nu(x) \leq \lambda \} \) is closed.

Then, it is proved ([C1], 1.7) that, for any \( x \in A^+, \nu(x) = \sup \{\omega(x), \omega \in A^*_+, \omega \leq \nu \} \), and ([K1], 2.3) that \( \Lambda_\nu \) is closed. Moreover, \( \nu \) has then a natural extension to \( M(A) \).

3.2.5. Definition. Be given a \( C^* \)-algebra \( A \), a densely defined lower semi-continuous faithful weight \( \nu \) on \( A \), and a norm continuous one parameter group of automorphism \( \sigma \) on \( A \); we shall say that \( \nu \) is a \textit{KMS weight} on \( A \) (with respect to \( \sigma \)) if :

(i) for all \( t \in \mathbb{R} \), \( \nu \circ \sigma_t = \nu \);

(ii) for any \( x, y \) in \( \mathfrak{N}_\nu \cap \mathfrak{N}_\nu^* \), there exists a bounded function \( f \) in the set \( \{0 \leq \text{Im} z \leq 1\} \subset \mathbb{C} \), analytic in \( \{0 < \text{Im} z < 1\} \), such that, for all \( t \in \mathbb{R} \), \( f(t) = \nu(\sigma_t(x)y) \) and \( f(t + i) = \nu(y\sigma_t(x)) \) (the so-called KMS conditions).

Then ([EVal],2.2.3; the proof is due to F. Combes), \( \nu \) extends to a normal semi-finite faithful weight on the von Neumann algebra \( \pi_\nu(A)^\sigma \); we shall denote by \( \nu \). Then \( \sigma \) is unique and is the restriction to \( A \) of the modular group \( \sigma_\nu \).

With the notations of ([C2]), we get that, if \( \varphi \) is a normal semi-finite faithful weight on a von Neumann algebra \( M \), then \( \varphi|_{C^*(\varphi)} \) is a KMS weight on \( C^*(\varphi) \).

3.2.6. Definition. Be given a \( C^* \)-algebra \( A \), and a densely defined, lower semi-continuous, faithful weight \( \nu \), KMS with respect to a one parameter group \( \sigma \) of automorphisms, we shall say that \( x \) is \textit{analytic} if the function \( t \mapsto \sigma_t(x) \) extends to an analytic function.

If \( x \in \mathfrak{N}_\nu \), then \( x_n = \frac{1}{\pi} \int e^{-\pi z^2} \sigma_t(x) dt \) is analytic, belongs to \( \mathfrak{N}_\nu \), and \( x_n \) is norm converging to \( x \) and \( \Lambda_\nu(x_n) \) is norm converging to \( \Lambda_\nu(x) \). (K1 4.3)

3.3. \textit{C}*-valued weights ([K2]).

3.3.1. Definition. Be given two \( C^* \)-algebras \( A \) and \( B \), with \( B \subset M(A) \), and a hereditary cone \( P \) in \( A^+ \); let us write \( \mathfrak{N} = \{a \in A, a^*a \in P\} \) and \( \mathfrak{M} = \text{span} P = \mathfrak{N}^* \mathfrak{N} \); then a \textit{\( C^* \)-valued weight} from \( A \) into \( M(B) \) is a linear map \( T \) from \( \mathfrak{M} \) into \( M(B) \), such that, for any \( b \in B \) and \( x \in P \), \( b^*xb \) belongs to \( P \) and \( T(b^*xb) = b^*T(x)b \).

Then, \( \mathfrak{M} \) will be denoted \( \mathfrak{M}_T \), \( \mathfrak{N} \) will be denoted \( \mathfrak{N}_T \), and \( P \) will be denoted \( \mathfrak{P}_T^+ \). We shall say that \( T \) is densely defined if one of these sets is dense in \( A \) (or \( A^+ \)). \( T \) is faithful if \( (x \in \mathfrak{M}_T^+) \), \( Tx = 0 \) implies that \( x = 0 \). If \( T \) is faithful and densely defined, then it is easy to get that \( T(\mathfrak{M}_T) \) is a dense ideal in \( M(B) \).

A definition of lower semi-continuity is given in ([K2], 3). More precisely, Kustermans constructs a uprising set \( \mathcal{S}_T \) of bounded completely positive maps from \( A \) to \( M(B) \), and \( T \) is then said lower semi-continuous if \( \mathfrak{M}_T^* \) is the set of \( x \in A^+ \) such that \( (\rho(x))_{\rho \in \mathcal{S}_T} \) is strictly convergent in \( M(B) \), and if this limit is then equal to \( T(x) \). So, when \( \nu \) is a lower semi-continuous weight on \( B \), which then extends to \( M(B) \) ([3.2.4]), then \( \nu \circ T \) is a
lower semi-continuous weight on $A$. More precisely, if $\nu$ and $T$ are densely defined, lower semi-continuous and faithful, so is $\nu \circ T$.

3.3.2. **Definition.** Be given two $C^*$-algebras $A$ and $B$, with $B \subset M(A)$, a faithful densely defined lower semi-continuous $C^*$-weight $T$ from $A$ to $M(B)$, and a faithful densely defined lower semi-continuous weight $\nu$ on $B$, which then extends to $M(B)$. We shall say that the pair $(\nu, T)$ is KMS if:

(i) there exists a one parameter automorphism group $\sigma^\nu$ on $B$, such that $\nu$ is KMS with respect to $\sigma^\nu$;

(ii) there exists a one parameter automorphism group $\sigma^{\nu \circ T}$ on $A$, such that $\nu \circ T$ is KMS with respect to $\sigma^{\nu \circ T}$;

(iii) for all $t \in \mathbb{R}$, the restriction of $\sigma^{\nu \circ T}$ to $B$ is equal to $\sigma^\nu_t$.

In that situation, considering the GNS constructions of $\nu$ and $\nu \circ T$, we define, for any $a \in \mathfrak{M}_T$, the linear map $\Lambda_T(a) \in B(H_\nu, H_{\nu \circ T})$ by $(b \in \mathfrak{N}_\nu)$:

$$\Lambda_T(a)\Lambda_\nu(b) = \Lambda_{\nu \circ T}(ab)$$

Let us remark that $\|\Lambda_T(a)\|^2 = \|T(a^*a)\|$.

3.3.3. **Theorem.** Be given two $C^*$-algebras $A$ and $B$, with $B \subset M(A)$, a densely defined, faithful, lower semi-continuous weight $\nu$ on $B$, a densely defined, faithful, lower semi-continuous $C^*$-valued weight $T$ from $A$ into $M(B)$, such that $(\nu, T)$ is KMS, in the sense of (C3). Then:

(i) there exists a unique injective $*$-homomorphism $\Phi$ from $\pi_\nu(B)''$ into $B(H_{\nu \circ T})$ such that $\Phi \circ \pi_\nu = \pi_{\nu \circ T}$, which allows to consider $\pi_\nu(B)''$ as a sub-algebra of $\pi_{\nu \circ T}(A)'$.

(ii) moreover, there exists a normal faithful semi-finite operator-valued weight $\mathbf{T}$ from $\pi_{\nu \circ T}(A)'$ onto $\pi_\nu(B)'$ such that, for any $a$ in $\mathfrak{M}_T$, we have $\mathbf{T}(\pi_{\nu \circ T}(a)) = \pi_\nu(T(a))$.

**Proof.** For any $b \in B$ and $t \in \mathbb{R}$, we have:

$$\sigma_t^{\nu \circ T}\Phi\pi_\nu(b) = \Phi\pi_{\nu \circ T}(\sigma_t^\nu(b)) = \Phi\pi_\nu\sigma_t^\nu(b)$$

and, by continuity, we have $\sigma_t^{\nu \circ T}\Phi = \Phi\sigma_t^\nu$. By identifying $\pi_\nu(B)'$ with $\Phi(\pi_\nu(B)'')$, we may consider $\pi_\nu(B)'$ as a sub-algebra of $\pi_{\nu \circ T}(A)'$; then the restriction of $\sigma_t^{\nu \circ T}$ to this sub-algebra is equal to $\sigma_t^\nu$. This gives the existence of a normal semi-finite faithful operator valued weight $\mathbf{T}$ from $\pi_{\nu \circ T}(A)'$ onto $\pi_\nu(B)'$. The fact that the restriction of $\mathbf{T}$ to $\pi_{\nu \circ T}(A)$ is equal to $T$ is straightforward.

Note that the existence of $\Phi$ can also be deduced from (C3), 1.7 and 1.8). Thanks to that result, we get, by restriction of $\mathbf{T}$, that for all $t \in \mathbb{R}$, $\sigma_t^{\nu \circ T} = T \circ \sigma_t^{\nu \circ T}$, from which we get that $\sigma_t^{\nu \circ T}(\mathfrak{M}_T) \subset \mathfrak{M}_T$; if $x \in \mathfrak{M}_T$, then $x_n = \frac{1}{\sqrt{\pi}} \int e^{-nt^2} \sigma_t^{\nu \circ T}(x)dt$ is analytic with respect to $\sigma^{\nu \circ T}$ and belongs to $\mathfrak{M}_T$; moreover, if $x_1$, $x_2$ are in $\mathfrak{M}_T$ and analytic with respect to $\sigma^{\nu \circ T}$, then $T(x_2^*x_1)$ is analytic with respect to $\sigma^\nu$.

Moreover, we have clearly $\nu \circ T = \nu \circ T$.

3.3.4. **Theorem.** Let $M$ be a von Neumann algebra, $N$ a sub-von Neumann algebra of $M$, and $\mathbf{T}$ a normal faithful semi-finite operator-valued weight from $M$ to $N$, $\nu$ a normal semi-finite faithful weight on $N$, and $\varphi = \nu \circ T$; let us define $C^*(T, \nu)$ as the norm closure of $\mathfrak{M}_T \cap C^*(\varphi)$; then:

(i) $C^*(\sigma^\nu)$ is included into $C^*(\sigma^\varphi)$; if $x \in \mathfrak{M}_T \cap C^*(\sigma^\varphi)$, then $T(x)$ belongs to $C^*(\sigma^\nu)$;

(ii) $C^*(T, \nu)$ is equal to $C^*(\varphi)$;

(iii) the restriction of $T$ to $C^*(T, \nu)$ is a lower semi-continuous densely defined $C^*$-valued weight from $C^*(T, \nu)$ to $C^*(\nu)$ and $(\nu_{|C^*(T, \nu)}, T_{|C^*(T, \nu)})$ is KMS.
Proof. As \(\sigma^T_N = \sigma^T\), it is clear that \(C^*(\sigma^T)\) is included into \(C^*(\sigma)^{\text{r}}\); let now \(x \in \mathcal{M}_T^+ \cap C^*(\sigma^T)\). Using \(3.2.1\) we get that \(x\) is the norm limit of \(\int_R f_i(t)\sigma^T(x)dt\), when \(f_i\) is a positive continuous approximate unit of \(L^1(R)\).

Let \(\omega \in N^+\), such that \(\omega|C^*(\sigma) = 0\); as \(\omega \circ T\) is norm lower semi-continuous, we get that \(\omega \circ T\) is equal to \(\sup\{\omega' \in M^+, \omega' \leq \omega \circ T\}\); then, using the norm continuity of \(\omega'\), we have:

\[
\omega'(x) = \lim_i \int_R f_i(t)\omega'(\sigma^T(x))dt \leq \omega \circ T(\int_R f_i(t)\sigma^T(x)dt) = \lim_i \omega \circ \int_R f_i(t)\sigma^T(T(x))dt = 0
\]

because, by \(3.2.1\) again, all elements of the form \(\int_R f(t)\sigma^T(y)dt\), for any \(y \in N\) and \(f \in L^1(R)\), belong to \(C^*(\sigma^T)\); therefore, we get that \(\omega \circ T(x) = 0\), and, then, that \(T(x)\) belongs to \(C^*(\sigma^T)\).

(ii) If \(x \in \mathcal{M}_T^+ \cap C^*(\sigma^T)\) and \(y \in \mathfrak{N}_\varphi \cap C^*(\sigma^T)\), using (i), we get that \(xy\) belongs to \(\mathcal{M}_T^+ \cap \mathfrak{N}_\varphi \cap C^*(\sigma^T)\) and that such elements are dense in both \(C^*(T, \nu)\) and \(C^*(\varphi)\), which is (ii).

(iii) as \(\mathcal{M}_T^+\) is invariant under \(\sigma^T\), we get that \(\varphi|C^*(\varphi)\) is KMS; as \(\nu|C^*(\varphi)\) is also KMS, we get easily the result. \(\square\)

3.4. Basic construction for inclusion of \(C^*\)-algebras with KMS \(C^*\)-valued weights. Let \(B, A\) be two \(C^*\)-algebras, with \(B \subset M(A)\). Let \(\nu\) be a KMS weight on \(B\), and \(T\) a densely defined lower semi-continuous \(C^*\)-valued weight from \(A\) to \(M(B)\), such that the pair \((\nu, T)\) is KMS. We define \(\varphi = \nu \circ T\) which is a KMS weight on \(A\). We shall denote \(M_0 = \pi_\nu(B)^\sigma\), \(M_1 = \pi_\varphi(A)^\sigma\), \(\nu\) the canonical extension of \(\nu\) to \(M_0\) (which, by \(3.2.3\) is a normal semi-finite faithful weight on \(M_0\)), \(\varphi\) the canonical extension of \(\varphi\) to \(M_1\) (which, by \(3.2.3\) again, is a normal semi-finite faithful weight on \(M_1\)). Using again \(3.3.3\) we get that \(M_0\) can be considered as a sub-von Neumann algebra of \(M_1\), and that there exists \(T\) a canonical extension of \(T\) to \(M_1\) (which, by \(3.3.3\) is a normal semi-finite faithful operator-valued weight from \(M_1\) to \(M_0\)). Let \(M_2\) be the basic construction made from the inclusion \(M_0 \subset M_1\), and let \(T^2\) be the normal faithful semi-finite operator-valued weight constructed in \(2.2.4\) (i) from \(T^2\) to \(M_1\).

3.4.1. Lemma. Let \(a, b \in A \cap \mathfrak{N}_T \cap \mathfrak{N}_\varphi\); then:

(i) \(T(b^*a)\) belongs to \(M(B)\).

(ii) The norm closure of the linear space generated by all elements of the form \(\Lambda_T(a)\Lambda_T(b)^*\) is a \(C^*\)-algebra \(A\) and \(A \subset M(A_1)\).

(iii) There exists elements \(a_i, b_i \in A \cap \mathfrak{N}_T \cap \mathfrak{N}_\varphi \cap \mathfrak{N}_T^+ \cap \mathfrak{N}_\varphi^+\) such that, for any \(y \in A\), the sum \(y\Sigma_i\Lambda_T(a_i)\Lambda_T(b_i)^*\) is norm converging to \(y\).

Proof. The proof of (i) is trivial. For any \(a_1, a_2, b_1, b_2 \in \mathfrak{N}_T \cap \mathfrak{N}_\varphi \cap \mathfrak{N}_T^+ \cap \mathfrak{N}_\varphi^+\), we have:

\[
\Lambda_T(a_1)\Lambda_T(b_1)^*\Lambda_T(a_2)\Lambda_T(b_2)^* = \Lambda_T(a_1(T(b_1^*a_2)))\Lambda_T(b_2)^*
\]

By (i) \(T(b_1^*a_2)\) belongs to \(M(B) \subset M(A)\), so \(a_1(T(b_1^*a_2))\) belongs to \(A\). Moreover, it is easy to get that it belongs also to \(\mathfrak{N}_T \cap \mathfrak{N}_\varphi\), which gives that the linear space generated by all elements of the form \(\Lambda_T(a)\Lambda_T(b)^*\) is an algebra, and the first part of (ii). The fact that \(A \subset M(A_1)\) is trivial; which finishes the proof of (ii). Now, let us choose an approximate unit of \(A\) of the form \(\Sigma_i\Lambda_T(a_i)\Lambda_T(b_i)^*\), and we get (iii). \(\square\)
3.4.2. Lemma. Let \( x \in \mathcal{H}_T \cap B', \) \( x \) analytic with respect to \( \sigma'_T \); then \( \sigma''_{-i/2}(x^*) \) belongs to \( \mathcal{H}_T \).

Proof. Let \( b \) in \( \mathcal{H}_\nu \), analytic with respect to \( \sigma'_\nu \), such that \( \sigma_{-i/2}(b^*) \) belongs to \( \mathcal{H}_\nu \); we have :

\[
(T(x^*)J_\nu \lambda_\nu(b) | J_\nu \lambda_\nu(b)) = \nu(\sigma''_{-i/2}(b^*) T(x^*)\sigma''_{-i/2}(b^*)) = \varphi(\sigma''_{-i/2}(b^*) x \sigma''_{-i/2}(b^*)) = \\
\|J_\varphi \lambda_\nu(\sigma_{-i/2}(b^*)x)\|^2 = \|J_\varphi \lambda_\nu(\sigma_{-i/2}(b^*))\|^2 = \varphi(b^* \sigma''_{-i/2}(x^*) \sigma''_{-i/2}(x^*)b)
\]

We get then that \( \sigma''_{-i/2}(x^*) \) belongs to \( \mathcal{H}_T \). \( \square \)

3.5. Fiber product of \( C^* \)-algebras. Here, using weights and \( C^* \)-valued weights on \( C^* \)-algebras, we give another definition of the fiber product of two \( C^* \)-algebras; this definition is completely inspired by Timmermann’s work, and, in that special situation, is equivalent to ([Ti2], 3.3).

3.5.1. Definition. Be given three \( C^* \)-algebras \( A_1 \), \( A_2 \) and \( B \), \( \alpha \) an injective strict \(*\)-homomorphism from \( B \) into \( M(A_1) \), \( \beta \) an injective strict \(*\)-antihomomorphism from \( B \) into \( M(A_2) \), \( T_1 \) a densely defined faithful lower semi-continuous \( C^* \)-valued weight from \( A_1 \) to \( M(B) \), \( T_2 \) a densely defined faithful lower semi-continuous \( C^* \)-valued weight from \( A_2 \) to \( M(B) \), \( \nu \) a KMS weight on \( B \) such that both pairs \( (\nu, T_1) \) and \( (\nu, T_2) \) are KMS. Let \( N = \pi_\nu(B)^\nu \), \( \varphi_1 = \nu \circ \alpha^{-1} \circ T_1 \), \( \varphi_2 = \nu \circ \beta^{-1} \circ T_1 \) \( M_1 = \pi_{\varphi_1}(A_1)^\nu \), \( M_2 = \pi_{\varphi_2}(A_2)^\nu \); let \( (\alpha, \beta) \) be the embedding of \( N \) into \( M_1 \) (resp. \( M_2 \)) given by (3.3). Then \( \alpha \) (resp. \( \beta \)) is an injective \(*\)-homomorphism (resp. anti-\(*\)-homomorphism) of \( N \) into \( M_1 \) (resp. \( M_2 \)) and let \( M_2 \beta_\ast^\alpha M_1 \) their fiber product in the sense of (3.5). Let \( \varphi_1 \) (resp. \( \varphi_2 \)) be the extension of \( \varphi_1 \) (resp. \( \varphi_2 \)) to \( M_2^+ \) (resp. \( M_2^+ \)). Then, we shall denote \( A_2 \beta_\ast^\alpha A_1 \) the set of all elements \( X \in M_2 \beta_\ast^\alpha M_1 \) such that :

- for all \( y \in A_1 \cap \mathcal{H}_T \cap \mathcal{H}_\varphi \), \( X \rho_{x_1 \lambda_{\varphi_1}(y)} \) and \( X^* \rho_{x_1 \lambda_{\varphi_1}(y)}^\beta \) belong to the norm closure of the linear set generated by :

\[
\bigcup_{\varepsilon \in A_1 \cap \mathcal{H}_T \cap \mathcal{H}_\varphi} \rho_{x_1 \lambda_{\varphi_1}(\varepsilon)} \beta_\ast A_2
\]

- for all \( y' \in A_2 \cap \mathcal{H}_T \cap \mathcal{H}_\varphi \), \( X \lambda_{x_2 \lambda_{\varphi_2}(y')} \) and \( X^* \lambda_{x_2 \lambda_{\varphi_2}(y')}^\beta \) belong to the norm closure of the linear set generated by :

\[
\bigcup_{\varepsilon' \in A_2 \cap \mathcal{H}_T \cap \mathcal{H}_\varphi} \lambda_{x_2 \lambda_{\varphi_2}(\varepsilon')} \beta_\ast A_1
\]

It is clear that \( A_2 \beta_\ast^\alpha A_1 \) is a sub-\( C^* \)-algebra of \( M_2 \beta_\ast^\alpha M_1 \), but it may be degenerate ([Ti2], 3.20).

3.5.2. Theorem. Let’s use the notations of (3.5.1) let \( X \in A_2 \beta_\ast^\alpha A_1 \), \( x, x' \) in \( \mathcal{H}_T \cap \mathcal{H}_{\varphi_1} \), \( x_2, x'_2 \) in \( \mathcal{H}_{\varphi_2} \); then :

(i) \((\omega_{x_2 \lambda_{\varphi_2}(x_2)}, \lambda_{x_2 \lambda_{\varphi_2}(x_2)} \beta_\ast \mu) \text{id}(X) \) belongs to \( A_1 \);

(ii) \((\text{id} \beta_\ast \mu \omega_{x_2 \lambda_{\varphi_2}(x_2)} \lambda_{x_2 \lambda_{\varphi_2}(x_2)} \mu) \text{id}(X) \) belongs to \( A_2 \);

Proof. By (3.5.1) we get that the operator \( X \lambda_{x_2 \lambda_{\varphi_2}(x_2)} \) belongs to the norm closure of \( \bigcup_{x''} \lambda_{x_2 \lambda_{\varphi_2}(x'' \lambda_{\varphi_2}(x_2))} \), for all \( x'' \) in \( \mathcal{H}_{\varphi_2} \), analytic with respect to \( \varphi_2 \). Then, we get that the operator \((\omega_{x_2 \lambda_{\varphi_2}(x_2)}, \lambda_{x_2 \lambda_{\varphi_2}(x_2)} \beta_\ast \mu) \text{id}(X) \) belongs to the norm closure of all operators of the
Theorem. Which we get (i) by density. Result (ii) is proved the same way.

3.6. Locally compact quantum groupoids.

3.6.1. Theorem. Be given three $C^*$-algebras $A_1$, $A_2$ and $B$, $\alpha$ an injective strict $\ast$-homomorphism from $B$ into $M(A_1)$, $\beta$ an injective strict $\ast$-antihomomorphism from $B$ into $M(A_2)$, $T_1$ a densely defined faithful lower semi-continuous $C^*$-valued weight from $A_1$ to $M(B)$, $T_2$ a densely defined faithful lower semi-continuous $C^*$-valued weight from $A_2$ to $M(B)$, $\nu$ a KMS weight on $B$ such that both pairs $(\nu,T_1)$ and $(\nu,T_2)$ are KMS. Then, $(H_{\nu\circ\beta^{-1}\circ T_2},L_{T_1})$ is a $C^*$-$b_\nu$-module, $(H_{\nu\circ\beta^{-1}\circ T_2},L_{T_2})$ is a $C^*$-$b_\nu$-module, and the application from $H_{\nu\circ\beta^{-1}\circ T_2} \otimes_{\nu,T_2} H_{\nu\circ\beta^{-1}\circ T_2} \otimes_{T_1} H_{\nu\circ\beta^{-1}\circ T_1}$ which sends

(for any $x \in \mathcal{R}_1 \cap \mathcal{R}_2$, $y \in \mathcal{R}_2 \cap \mathcal{R}_2$, $z \in \mathcal{R}_2$) : 

$$\Lambda_{\nu\circ\beta^{-1}\circ T_2}(\beta(z^*)y) \otimes_{\nu,T_2} \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x) = \Lambda_{\nu\circ\beta^{-1}\circ T_2}(y) \otimes_{\nu,T_2} \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x \circ z)$$

on $\Lambda_{T_2}(y) \otimes J_\nu \Lambda_\nu(z) \otimes \Lambda_{T_1}(x)$ is an isomorphism of Hilbert spaces.

Proof. Note that the equality given up here is an easy corollary of [2.3]. The second term can be written also as $\Lambda_{\nu\circ\beta^{-1}\circ T_2}(y) \otimes_{\nu,T_2} \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x \circ z)$.

Let $x_1$, $x_2$ in $\mathcal{R}_1 \cap \mathcal{R}_2$, $y_1$, $y_2$ in $\mathcal{R}_2 \cap \mathcal{R}_2$, $z_1$, $z_2$ in $\mathcal{R}_2$. Using [2.2.2] we get that the scalar product :

$$(\Lambda_{\nu\circ\beta^{-1}\circ T_2}(y_1) \otimes_{\nu,T_2} \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_1) \Lambda_\nu(z_1) | \Lambda_{\nu\circ\beta^{-1}\circ T_2}(y_2) \otimes_{\nu,T_2} \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_2) \Lambda_\nu(z_2))$$

is equal to :

$$(\alpha(< \Lambda_{\nu\circ\beta^{-1}\circ T_2}(y_1), \Lambda_{\nu\circ\beta^{-1}\circ T_2}(y_2) >_{\beta,\nu,T_1}, \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_1) \Lambda_\nu(z_1) | \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_2) \Lambda_\nu(z_2)))$$

which can be written as :

$$(\alpha \circ \beta^{-1} T_2(y_2^*y_1) \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_1) \Lambda_\nu(z_1) | \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_2) \Lambda_\nu(z_2))$$

which is equal to :

$$(J_{\nu\circ\beta^{-1}\circ T_1} \otimes_{\nu,T_2} \alpha \circ \beta^{-1} T_2(y_1^*y_2) J_{\nu\circ\beta^{-1}\circ T_1} \Lambda_{\nu\circ\beta^{-1}\circ T_2}(x_2) \Lambda_\nu(z_2) | \Lambda_{\nu\circ\beta^{-1}\circ T_1}(x_1) \Lambda_\nu(z_1))$$

which, by definition [3.1.3], is equal to the scalar product :

$$(\Lambda_{T_2}(y_1) \otimes J_\nu \Lambda_\nu(z_1) \otimes \Lambda_{T_1}(x_1) | \Lambda_{T_2}(y_2) \otimes J_\nu \Lambda_\nu(z_2) \otimes \Lambda_{T_1}(x_2))$$

which proves the result.
3.6.2. Theorem. Be given three C*-algebras $A_1$, $A_2$ and $B$, $\alpha$ an injective strict *-homomorphism from $B$ into $M(A_1)$, $\beta$ an injective strict *-antihomomorphism from $B$ into $M(A_2)$, $T_1$ a densely defined faithful lower semi-continuous C*-valued weight from $A_1$ to $M(B)$, $T_2$ a densely defined faithful lower semi-continuous C*-valued weight from $A_2$ to $M(B)$, $\nu$ a KMS weight on $B$ such that both pairs $(\nu, T_1)$ and $(\nu, T_2)$ are KMS. Let $A_2L_{T_2}\ast_{L_{T_1}}A_1$ be the C*-fiber product in the sense of 3.1.3; let $N = \pi_\nu(B)^\prime$, $M_1 = \pi_{\nu\circ\alpha^{-1}T_1}(A_1)^\prime\prime$, $M_2 = \pi_{\nu\circ\beta^{-1}T_2}(A_2)^\prime\prime$; let $\alpha$ (resp. $\beta$) be the imbedding of $N$ into $M_1$ (resp. $M_2$) given by 3.3.3. Then $\alpha$ (resp. $\beta$) is an injective *-homomorphism (resp. anti-*-homomorphism) of $N$ into $M_1$ (resp. $M_2$) and let $M_2 \beta\ast_{\nu\circ\beta^{-1}N} M_1$ their fiber product in the sense of 3.3. Using 3.6.1 we shall consider that $A_2L_{T_2}\ast_{L_{T_1}}A_1$ is a sub-C*-algebra of $B(H_{\nu\circ\beta^{-1}T_2} \hat{\otimes}_{\nu\circ\alpha^{-1}N} H_{\nu\circ\alpha^{-1}T_1})$. Let $X \in A_2L_{T_2}\ast_{L_{T_1}}A_1$, $x_1, x_1' \in \mathfrak{N}_{T_1} \cap \mathfrak{N}_{\nu\circ\alpha^{-1}T_1}$, $x_2$, $x_2'$ in $\mathfrak{N}_{T_2} \cap \mathfrak{N}_{\nu\circ\beta^{-1}T_2}$; then :

(i) $(\omega_{\Lambda^{\nu\circ\beta^{-1}T_2}(x_2)}^{\nu\circ\beta^{-1}T_1}(x_1), \beta \ast_{\nu\circ\alpha^{-1}N} \alpha(id)) \in A_1$;

(ii) $(id_{\nu\circ\beta^{-1}T_2} \beta\otimes_{\nu\circ\alpha^{-1}N} \alpha(id)) \in A_2$;

(iii) The C*-fiber product $A_2L_{T_2}\ast_{L_{T_1}}A_1$ is a sub-C*-algebra of $M_2 \beta\ast_{\nu\circ\beta^{-1}N} M_1$.

Proof. Let us suppose first that $x''$ is analytic. By 3.1.6 we get that the operator $X\Lambda^{\nu\circ\beta^{-1}T_2}(x_2)$ belongs to the norm closure of $\cup x''\Lambda^{\nu\circ\beta^{-1}T_2}(x'')(A_1)$, for all $x''$ in $\mathfrak{N}_{T_2} \cap \mathfrak{N}_{\nu\circ\beta^{-1}T_2}$, analytic with respect to $\nu\circ\beta^{-1}T_2$. Using now 3.6.1 we get that the operator $(\omega_{\Lambda^{\nu\circ\beta^{-1}T_2}(x_2)}^{\nu\circ\beta^{-1}T_1}(x_1), \beta \ast_{\nu\circ\alpha^{-1}N} \alpha(id))$ belongs to the norm closure of all operators of the form $1_{\beta\otimes_{\nu\circ\alpha^{-1}N}} \alpha \circ \sigma_{\nu\circ\alpha^{-1}N}^{-1} \beta \circ \beta^{-1} \circ T_2(x_2'x'' \otimes \lambda)A_1$, which is included in $\alpha(M(B))A_1 \subset A_1$. From which we get (i) by density. Result (ii) is proved the same way. Then, it is straightforward to get that $X$ commutes with $1_{\beta\otimes_{\nu\circ\alpha^{-1}N}} y$, for all $y \in \pi_{\nu\circ\alpha^{-1}T_1}(A_1)'$, and with $x_{\nu\circ\alpha^{-1}N} 1$, for all $x \in \pi_{\nu\circ\beta^{-1}T_2}(A_2)'$; so $X$ commutes with $\beta\otimes_{\nu\circ\alpha^{-1}N} M_1$, which gives (iii). $\square$

3.6.3. Notations. Using 3.6.2 we define $(id_{L_{T_2}\ast_{L_{T_1}}T_1})$ as the restriction of $(id_{\beta\ast_{\nu\circ\alpha^{-1}N} T_1})$ to $A_2L_{T_2}\ast_{L_{T_1}}A_1$; it is a faithful lower semi-continuous C*-valued weight from $A_2L_{T_2}\ast_{L_{T_1}}A_1$ to $A_2 \cap \beta(B)'L_{T_2}\otimes_{L_{T_1}}1$. The C*-valued weight $(T_2L_{T_2}\ast_{L_{T_1}}id)$ is defined the same way.

Let us write $\varphi = \nu \circ \alpha^{-1}T_1$, which is a KMS weight on $A_1$, and $\varphi$ its extension to $M_1$. Using again 3.6.2 we define also $(id_{L_{T_2}\ast_{L_{T_1}}\varphi})$ as the restriction of $(id_{\beta\ast_{\nu\circ\alpha^{-1}N} \varphi})$ to $A_2L_{T_2}\ast_{L_{T_1}}A_1$. As $(id_{\beta\ast_{\nu\circ\alpha^{-1}N} \varphi})$ is just the inclusion of $M_2 \cap \beta(N)' = M_2 \beta\ast_{\nu\circ\alpha^{-1}N} (N)$ into $M_2$ (12, 2.5.1), we get that, for any positive $X \in A_2L_{T_2}\ast_{L_{T_1}}A_1$, we have :

$$(id_{L_{T_2}\ast_{L_{T_1}}\varphi})(x) = (id_{L_{T_2}\ast_{L_{T_1}}T_1})(x)\beta\otimes_{\nu\circ\alpha^{-1}N} 1$$

3.6.4. Definition. An octuple $G = (B, A, \alpha, \beta, \nu, T, T', \Gamma)$ will be called a locally compact quantum groupoid if :

25
(i) $A$ and $B$ are $C^*$-algebras, $\alpha$ a strict $*$-homomorphism from $B$ to $M(A)$, $\beta$ a strict anti-$*$-homomorphism from $B$ to $M(A)$;
(ii) $\nu$ is a densely defined faithful, lower semi-continuous KMS weight on $B$, in the sense of $3.2.5$.
(iii) $T$ is a densely defined faithful lower semi-continuous $C^*$-valued weight from $A$ to $\alpha(M(B))$, $T'$ is a densely defined faithful lower semi-continuous $C^*$-valued weight from $A$ to $\beta(M(B))$, such that both pairs $(T, \nu \circ \alpha^{-1})$ and $(T', \nu^\rho \circ \beta^{-1})$ are KMS, in the sense of $3.3.2$.
(iv) $\Gamma$ is a strict $*$-homomorphism from $A$ to $A_{L_T} \ast_{L_T} A$ such that, for all $b \in B$, $\Gamma(\alpha(b)) = 1_{L_T} \otimes_{L_T} \alpha(b)$, $\Gamma(\beta(b)) = \beta(b) 1_{L_T} \otimes_{L_T} L_T$ 1, and :
$$\Gamma_{L_T} \ast_{L_T} id_{L_T} \Gamma = (id_{L_T} \ast_{L_T} \Gamma) \Gamma$$
where $\Gamma_{L_T} \ast_{L_T} id_{L_T}$ and $id_{L_T} \ast_{L_T} \Gamma$ are defined using $3.1.6$. Such an application is called a coproduct.

Then, by definition of $A_{L_T} \ast_{L_T} A$, for any $x \in A$, $\Lambda_1, \Lambda_2$ in $L_T$, $\Lambda_3$ in $L_T'$, $\rho_A, \Gamma(x) \rho_A$ and $\rho_{A_2}, \Gamma(x) \rho_{A_2}$ belong to $A$. We suppose, moreover, that the two linear subsets generated by these elements are dense in $A$. Then, $\Gamma$ is called a non degenerate coproduct.

(v) the modular automorphism groups of the KMS weights $\nu \circ \alpha^{-1} \circ T$ and $\nu^\rho \circ \beta^{-1} \circ T'$ on $A$ commute;
(vi) $T$ is left-invariant, which means that, for any $x \in \mathcal{M}_T^+$, we have :
$$(id_{L_T} \ast_{L_T} T) \Gamma(x) = T(x) 1_{L_T} \otimes_{L_T} L_T 1$$
where $(id_{L_T} \ast_{L_T} T)$ had been defined in $4.11$.
(vii) $T'$ is right-invariant, which means that, for any $x' \in \mathcal{M}_{T'}^+$, we have :
$$T' \Gamma_{L_T} \ast_{L_T} id_{L_T} \Gamma = 1_{L_T} \ast_{L_T} T'(x')$$
where $(T' \ast_{L_T} id)$ had been defined in $4.11$.

3.6.5. Example. Let us suppose that $b_v = t$; then $T$ and $T'$ are KMS weights on $A$, $C^*$-tensor product over $b_v$ is the usual tensor product of Hilbert spaces, then, thanks to $3.1.7$, any reduced $C^*$-algebraic quantum group $(A, \Gamma, T, T')$ in the sense of ([KV1], 4.1) is a locally compact quantum groupoid $(C, A, id, id, id, T, T', \Gamma)$.

4. FROM A LOCALLY COMPACT QUANTUM GROUPOID TO A MEASURED QUANTUM GROUPOID

In this chapter, to any locally compact quantum groupoid $G = (B, A, \alpha, \beta, \nu, T, T', \Gamma)$, we associate a measured quantum groupoid $G = (N, M, \alpha, \beta, \Gamma, T, T')$ such that $B$ is weakly dense in $N$, $A$ is weakly dense in $M$, $\alpha$ (resp. $\beta, \Gamma, T, T'$, $\nu$) is the restriction of $\alpha$ (resp. $\beta, \Gamma, T, T'$, $\nu$). The only problem is to extend the coproduct; this is done by using $[L]$, in which Lesieur associated a pseudo-multiplicative unitary to any quantum measured groupoid; this construction was inspired by Kustermann-Vaes ([KV1]), who associated a multiplicative unitary to the $C^*$-version of any locally compact group; so, it is not a surprise that Lesieur’s construction can be extended to a $C^*$-context. Then, we shall say that $G$ is a locally compact sub-quantum groupoid of $G$ (4.12).
4.1. Notations. Let $G = (B, A, \alpha, \beta, \nu, T, T^\prime, \Gamma)$ be a locally compact quantum groupoid, as defined in \[\text{[3.6.3]},\] let us denote $\nu$ the canonical normal faithful semi-finite weight constructed on $N = \pi_\nu(B)^\prime$ in \[\text{[3.2.3]}\] and $\varphi = \nu \circ \alpha^{-1} \circ T$ the canonical faithful semi-finite weight constructed on $M = \pi_{\varphi_{\alpha^{-1} \circ T}}(A)^\prime$. Let $\alpha$ (resp. $\beta$) be the canonical injective $*$-homomorphism (resp. anti-$*$-homomorphism) from $N$ into $M$ constructed in \[\text{[3.3.3]},\] Let $T$ (resp. $T^\prime$) be the normal faithful semi-finite operator-valued weight from $M$ to $\alpha(N)$ (resp. $\beta(N)$) constructed in \[\text{[3.3.3]}\]. Finally, let us define, for all $x \in N$, $\hat{\beta}(x) = J_{e^\prime}xJ_{e}$.

4.2. Theorem. Let $G = (B, A, \alpha, \beta, \nu, T, T^\prime, \Gamma)$ be a locally compact quantum groupoid, as defined in \[\text{[3.6.3]},\] let’s use the notations of \[\text{[4.1]}\] then,

(i) for any $a \in \mathcal{H} \cap \mathcal{J}_\varphi$, and any $\nu, \xi$, in $D((H_\varphi)^{\nu})$, $(\omega_{\varphi, \xi} \beta^* \alpha \text{id}) \Gamma(a)$ belongs to $\mathcal{H} \cap \mathcal{J}_\varphi$;

(ii) there exists an isometry $U$ from $H_{\varphi} \otimes_{\nu} H_{\varphi}$ to $H_{\varphi} \otimes_{\nu} H_{\varphi}$, such that, for any orthogonal $(\beta, \nu)$-basis $(\xi_i)_{i \in I}$ of $H_{\varphi}$, any $a \in \mathcal{H} \cap \mathcal{J}_\varphi$ and any $\nu \in D((H_\varphi)^{\nu})$, we have:

$$U(v \otimes \beta \Lambda_\varphi(a)) = \sum_{i \in I} \xi_i \otimes \alpha \Lambda_\varphi((\omega_{\varphi, \xi} \beta^* \alpha \text{id}) \Gamma(a))$$

Proof. The proof of (i) is identical to (\ref{g1}, 3.2.7), and the proof of (ii) is identical to (\ref{g1}, 3.2.9 and 3.2.10).

4.3. Proposition. Let’s use the notations of \[\text{[4.1]}\] and \[\text{[4.2]}\] then, we have:

(i) for any $x \in \mathcal{H}, \ e \in \mathcal{J}_\varphi, \ \xi \in H$, we have:

$$(1 \otimes \beta \alpha N \theta_j \epsilon_j \eta_j \epsilon_j) U(\xi \otimes \beta \Lambda_\varphi(x)) = \Gamma(x)(\xi \otimes \alpha \theta_j \alpha \epsilon_j \theta_j \epsilon_j \epsilon_j)$$

(ii) for any $a \in A$, we have $\Gamma(a) U = \Gamma(1 \otimes \beta \alpha N \theta_j \epsilon_j \eta_j)$

Proof. The proof of (i) is identical to (\ref{g1}, 3.3.1); then (ii) is a direct corollary of (i), as in (\ref{g1}, 3.4.5).

4.4. Lemma. Let’s use the notations of \[\text{[4.1]}\] and \[\text{[4.2]}\] then, we have, for any $a \in \mathcal{J}_\varphi$:

(i) for any $\nu \in D((A \otimes H_\varphi)^{\nu})$, we have:

$$(\omega_{\varphi, \nu} \beta^* \alpha \text{id})(U) \Lambda_\varphi(a) = \Lambda_\varphi((\omega_{\varphi, \nu} \beta^* \alpha \text{id}) \Gamma(a))$$

(ii) for any $e \in \mathcal{J}_\varphi$, $\eta \in D((A \otimes H_\varphi)^{\eta})$, we have:

$$(\text{id} \beta^* \alpha N \theta_j \alpha \theta_j \alpha \epsilon_j \eta_j \epsilon_j) \Gamma(a) = (\text{id} \ast \omega_{\varphi, \nu}(a), J_{e^*} J_{\epsilon_j} J_{\eta_j} \eta_j)(U)$$

Proof. The proof of (i) is identical to (\ref{g1}, 3.3.3) and the proof of (ii) is identical to (\ref{g1}, 3.3.4).

4.5. Theorem. Let’s use the notations of \[\text{[4.1]}\] and \[\text{[4.2]}\] the isometry $U$ is a unitary

Proof. This is the difficult part of the result; moreover, the proof is identical to (\ref{g1}, 3.5).
4.6. **Theorem.** Let’s use the notations of 4.4 and 4.5 for any \( x \in M \), let’s define \( \Gamma(x) = U(1_{\beta} \otimes x)U^* \); then, \( \Gamma(x) \) belongs to \( M_{\beta,\alpha} M \), and \((N, M, \alpha, \beta, \Gamma)\) is a Hopf-bimodule in the sense of 2.4.

**Proof.** For any \( a \in A \), we have, thanks to 4.3(ii) and 4.6, \( \Gamma(a) = \Gamma(a) \), so, by continuity and 3.6.2 we get that \( \Gamma(x) \) belongs to \( M_{\beta,\alpha} M \). We get also that \( \Gamma \) is a coproduct by continuity. \( \square \)

4.7. **Theorem.** Let \( G = (B, A, \alpha, \beta, \nu, T, T') \) be a locally compact quantum groupoid; let’s use the notations of 4.1 and 4.6. Then, \( \mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) is a measured quantum groupoid, we shall denote \( \mathfrak{G}(G) \), and we have \( \alpha = \alpha|_B \), \( \beta = \beta|_B \), \( \Gamma = \Gamma|_A \), \( T = T|_A \), \( T' = T'|_A \), \( \nu = \nu|_B \).

**Proof.** By normality of \( T \), we easily get that \( T \) is left-invariant with respect to \( \Gamma \); the commutation property of \( \sigma^{-1} \alpha \sigma^{-1} \) and \( \sigma^{-1} \beta \sigma^{-1} \) are clear, by continuity and 3.6.4(v). \( \square \)

Using again (4.3), 3.6.2 we get easily that the pseudo-multiplicative unitary \( W \) associated to \( \mathfrak{G} \) is equal to \( U^* \).

4.8. **Theorem.** Let’s use the notations of 4.7. Then :

(i) let \( a, b, e \in \mathfrak{N}_2 \cap \mathfrak{N}_T \); we have :

\[
(id * \omega_{J_\alpha,\lambda e(e)b,\lambda e(a)})(W) = (id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(b),J_\lambda,\lambda e(c)})(\Gamma(a*))
\]

(ii) the linear set generated by all elements of the form \((id * \omega_{J_\alpha,\lambda e(b),\lambda e(a)}(W), a, b \in \mathfrak{N}_2 \cap \mathfrak{N}_T \) is norm dense in \( A \). Moreover, \( A \) is a sub-C*-algebra of the C*-algebra \( A_n(W) \cap A_n(W)^* \) introduced in 2.9.2.

(iii) the co-inverse \( \mathfrak{R} \) of \( \mathfrak{G} \) is such that \( \mathfrak{R}(A) = A \).

(iv) the modulus \( \delta \) and the scaling operator \( \lambda \) of the measured quantum groupoid \( \mathfrak{G} \) are affiliated to \( A \), in the sense of (13, [W3]).

(v) The scaling group \( \tau_t \) of \( \mathfrak{G} \) satisfies, for all \( t \in \mathbb{R} \), \( \tau_t(A) = A \), and \( \tau_{tA} \) is a norm continuous one parameter group of automorphisms of \( A \).

**Proof.** Formula (i) is just an application of 4.3(ii) and 3.6.4(iv). Then, using the notations of 2.9.2 we have \( A \subset A_n(W) \), from which we get (ii).

We have, using (E3), 3.10(v) and 3.11:

\[
R((id * \omega_{J_\alpha,\lambda e(b),\lambda e(a)}(W)) = J_{(2)}(id * \omega_{J_\lambda,\lambda e(b),\lambda e(a)}(W)) J_{(2)} = (id * \omega_{J_\alpha,\lambda e(b),\lambda e(a)}(W)) \text{ from which, using (ii), we get (v).}
\]

Using (E4), 3.8 (ii), and (i), we get :

\[
\tau_t([id * \omega_{J_\alpha,\lambda e(e+b),\lambda e(a)}(W)]) = \tau_t([id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(b),J_\lambda,\lambda e(c)})(\Gamma(a*))]
\]

\[
= (id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(b),\lambda e(c)}(\sigma^{-1}_t,\sigma^{-1}_t)\Gamma(\sigma^{-1}_t(a*))
\]

\[
= (id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(\sigma^{-1}_t(b)),J_\lambda,\lambda e(\sigma^{-1}_t(c))})(\sigma^{-1}_t(a*))
\]

\[
= (id * \omega_{J_\alpha,\lambda e(\sigma^{-1}_t(e+b)),\lambda e(\sigma^{-1}_t(a))})(W)
\]

\[
\tau_t([id * \omega_{J_\alpha,\lambda e(e+b),\lambda e(a)}(W)]) = \tau_t([id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(b),J_\lambda,\lambda e(c)})(\Gamma(a*))]
\]

\[
= (id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(b),\lambda e(c)}(\sigma^{-1}_t,\sigma^{-1}_t)\Gamma(\sigma^{-1}_t(a*))
\]

\[
= (id * \omega_{\beta,\alpha} \omega_{J_\lambda,\lambda e(\sigma^{-1}_t(b)),J_\lambda,\lambda e(\sigma^{-1}_t(c))})(\sigma^{-1}_t(a*))
\]

\[
= (id * \omega_{J_\alpha,\lambda e(\sigma^{-1}_t(e+b)),\lambda e(\sigma^{-1}_t(a))})(W)
\]
from which we get that \( \tau_t[(id * \omega_{J_{d_2}A}(b),A_{d_2}(a))](W) = (id * \omega_{J_{d_2}A}(\sigma^{-1}_t(b)),A_{d_2}(\sigma^{-1}_t(a)))(W) \), which gives that \( \tau_t(A) \subset A \), for any \( t \in \mathbb{R} \), and, therefore \( \tau_t(A) = A \). Moreover, as
\[
\|R^T_{A}(J_{d_2}A_\sigma(\sigma^{-1}_t(a)))\|^2 = \|\sigma^{-1}_t(A^*a)\|
\]
we get (iv), because \( (\sigma^{-1}_t)|_{M(B)} \) is norm continuous.
\(\square\)

4.9. **Lemma.** Let \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, R, \nu) \) be a measured quantum groupoid. Let \( A \) be a C*-algebra, weakly dense in \( M \), invariant by \( R \), and \( B \) a C*-algebra weakly dense in \( M \), such that \( T(NR_T \cap A) \subset M(B) \) and let us suppose that \( (\nu_B, T|_A) \) is KMS. Moreover, let us suppose that, for any \( a \in A, b, c \in A \cap N_\rho \cap N_T \), \((i \beta^* \alpha \omega_{J_{d_1}A}(b)_\sigma A_{d_1}(c))\Gamma(a)\) belongs to \( A \), and that \( A \) is equal to the closed linear set generated by these elements. Then, for any \( c_1, c_2 \in A \cap N_\rho \cap N_T \) and \( c_2 \in A \cap N_T \cap N_\rho \), \((id * \omega_{A_{d_1}}(c_1),A_{d_1})(W^*)\) belongs to \( A \), and \( A \) is the closed linear set generated by these elements. Moreover, if \( d_1 \), \( d_2 \) belong to \( A \cap N_\rho \cap N_T \), \((\omega_{J_{d_1}A}(d_1),\omega_{J_{d_2}A}(d_2))(\beta \ast \alpha \nu_B)\Gamma(a)\) belongs to \( A \); it belongs to \( A \cap N_T \cap N_\rho \cap N_\rho \) if \( a \) belongs to \( A \cap N_T \cap N_\rho \cap N_\rho \); using now 2.9.4 we get the first result.

The second result of is given by the hypothesis \( R(A) = A \).

\(\square\)

4.10. **Definition.** Let \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, R, \nu) \) a measured quantum groupoid. Using \( 3.2.4 \) we can define \( C^*(\sigma^\nu, \sigma^\nu R, \tau) \) as the weakly dense sub-C*-algebra of \( M \) on which the one-parameter groups \( \sigma^\nu \), \( \sigma^\nu R \), \( \tau_t \) are norm continuous, and \( C^*(\sigma^\nu, \gamma) \) a the weakly dense sub-C*-algebra of \( N \) on which the one parameter groups \( \sigma^\nu \) and \( \gamma_t \) are norm continuous. Then \( \alpha(C^*(\sigma^\nu, \gamma)) \) and \( \beta(C^*(\sigma^\nu, \gamma)) \) are included into \( C^*(\sigma^\nu, \sigma^\nu R, \tau) \). With the notations of 3.7 we get that \( A \subset C^*(\sigma^\nu, \sigma^\nu R, \tau) \).

4.11. **Notations.** Using 3.6.2 we define \((id_{T_{d_1}T_{d_1}})_{\nu} \) as the restriction of \((id_{T_{d_1}T_{d_1}})_{\nu} \) to \( A \cap \beta(B)' \cap T_{d_1} \cap T_{d_1} \). It is a faithful lower semi-continuous C*-valued weight from \( A \cap B \cap T_{d_1} \) to \( A \cap \beta(B)' \cap T_{d_1} \cap T_{d_1} \). The C*-valued weight \((id_{T_{d_1}T_{d_1}})_{\nu} \) is defined the same way.

Using again 3.6.2 we define also \((id_{T_{d_1}T_{d_1}})_{\nu} \varphi \) as the restriction of \((id_{T_{d_1}T_{d_1}})_{\nu} \varphi \) to \( A \cap \beta(B)' \cap T_{d_1} \cap T_{d_1} \). As \((id_{T_{d_1}T_{d_1}})_{\nu} \varphi \) is the inclusion of \( M \cap \beta(B)' \cap T_{d_1} \cap T_{d_1} \) into \( M \cap \beta(B)' \cap T_{d_1} \cap T_{d_1} \) (1), 2.5.1, we get that, for any positive \( X \) in \( A \cap \beta(B)' \cap T_{d_1} \), we have
\[
(id_{T_{d_1}T_{d_1}})_{\nu} \varphi(X) = (id_{T_{d_1}T_{d_1}})_{\nu} \varphi(X)
\]

4.12. **Definition.** Let \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, R, \nu) \) a measured quantum groupoid, with \( \varphi = \nu \circ T \); let \( B \) be a weakly dense sub-C*-algebra of \( N \), \( A \) a sub-C*-algebra of \( M \) such that \( \alpha(B) \subset M(A) \), \( R(A) = A \) (and, therefore, \( \beta(B) \subset M(A) \)), \( A \cap NR_T \) is dense in \( A \), and \( T(A \cap NR_T) \subset \alpha(M(B)) \). Let us suppose that \( T|_A \) is a KMS weight on \( B, T|_A \) is a densely defined faithful lower semi-continuous C*-valued weight from \( A \) to \( \alpha(M(B)) \), such that the pair \((T|_A, \varphi|_A)\) is KMS in the sense of 3.3.2. Let us suppose that \( \Gamma(A) \subset A_{RTR} \) \( A \) (which implies that \( A_{RTR} \) \( A \) is non degenerate). Then, we shall say that \((B, A, \alpha|_B, \beta|_B, \nu|_B, T|_A, RTR|_A, \Gamma|_A)\) is a locally compact quantum groupoid, or, more precisely, a locally compact sub-quantum groupoid of \( \mathcal{G} \).
4.12.1. Example. Let us suppose that $B = C$; then $T$ and $T'$ are KMS weights on $A$, then any reduced $C^*$-algebraic quantum group $(A, \Gamma, T, T')$ in the sense of ([KV1], 4.1) is a locally compact quantum groupoid $(C, A, \alpha, \beta, \delta, \Gamma, T, T', \Gamma)$. 

4.12.2. Example. Let $(C, A, \alpha, \beta, \delta, \Gamma, T, T', \Gamma)$ a locally compact quantum groupoid described in (4.12.1). Then, using (4.12.1) we obtain that $(C, \pi_T(A)''', \alpha, \beta, \delta, \Gamma, T, T', \Gamma)$ is a measured quantum groupoid. More precisely, $(\pi_T(A)''', \Gamma, T, T')$ is (the von Neumann version) of a locally compact quantum group. Let now $(A_1, \Gamma_1, T_1, T_1', \Gamma_1)$ be the the $C^*$ version of this locally compact quantum group. Using ([LS1], i) and ([KV2], 1.6), we get that $A = A_1$, and, therefore, that $\Gamma(A) \subseteq M(A \otimes A)$ and we get that that $(A, \Gamma, T, T')$ is (the $C^*$-version of) a locally compact quantum group. So, we get the converse of (4.12.1).

4.12.3. Example. Let $A$ be a $C^*$-algebra. Let $\nu$ be a densely defined lower semi-continuous faithful weight on $A$, which is KMS with respect to a norm continuous one parameter group of automorphisms $\sigma_t$ (3.2.5). Let us denote $M = \pi_{\nu}(A)''$, and $\nu$ the normal semi-finite faithful weight on $M$ which extends $\nu$. Lesieur had introduced in ([L], 14) a quantum space quantum groupoid associated to any von Neumann algebra $M$, acting on the Hilbert space $L^2(M)$, we identify $M' \ast M$ with $M' \otimes M$. Let $tr$ a normal semi-finite trace on $Z(M)$, such that $tr|_{Z(M(A))}$ is a densely defined faithful lower semi-continuous KMS trace on $Z(M(A))$, and $T$ the normal semi-finite operator-valued weight from $M$ to $Z(M)$ such that $\nu = tr \circ T$. Let $j$ be the anti-isomorphism of $\mathcal{L}(H_\nu)$ given by $j(x) = J_\nu x^* J_\nu$, for any $x \in \mathcal{L}(H_\nu)$, which will identify $M'$ with $M^\o$. Let $\alpha$ (resp. $\beta$) be the representation (resp. anti-representation) of $M$ into $M^\o \otimes M, Z(M)$, such that $\alpha(m) = 1 \otimes m$ (resp. $\beta(m) = j(m) \otimes 1$). Let $I$ be the isomorphism from $[L^2(M) \otimes L^2(M)]_\beta \otimes_\alpha [L^2(M) \otimes L^2(M)]$ onto $L^2(M) \otimes L^2(M) \otimes L^2(M)$ defined by $I[\Lambda_\nu(y) \otimes \eta j] \otimes_\alpha \Xi = \alpha(y) \Xi \otimes 1$. This isomorphism will allow us to identify the fiber product $(M^\o \otimes M)_{\beta^*} (M^\o \otimes M)$ with $M^\o \otimes M$; then we define $\Gamma$ on $M^\o \otimes M$ by $(n, m) \in M)$ : 

$$\Gamma(n^\o \otimes m) = [1 \otimes m]_{\beta^*} [n^\o \otimes 1]$$

Using the previous isomorphism, $\Gamma$ is just the identity.

The co-inverse R is then given by $R(n^\o \otimes m) = m^\o \otimes j(n)$. Then Lesieur proved that $(M, M^\o \otimes M, (M', \alpha, \beta, \Gamma, \nu, R(id \ast T) R(id \ast T))$ is a measured quantum groupoid he called a "quantum space quantum groupoid".

Let us consider now $(A, A^* \otimes A, \alpha|_A, \beta|_A, tr|_A, R(id \ast T) R(id \ast T))$, where $b$ is the $C^*$-base constructed using $tr|_{Z(A)}$. It is easy to verify that it is a locally compact sub-quantum groupoid of Lesieur’s quantum space quantum groupoid.

4.12.4. Example. Let use again the notations of ([L], 15) a measured quantum groupoid, called a "pair quantum groupoid" associated to any von Neumann algebra $M' \otimes M$ ($M'$ being the commutant of $M$ in $L^2(M)$; let $\nu$ be the canonical weight on $M'$ constructed from $\nu$.

The fiber product $(M' \otimes M)_{\beta^*} (M' \otimes M)$ can be identified with $M' \otimes Z(M) \otimes M$; then, with the normal $*$-homomorphism $\Gamma$ defined by $\Gamma(n' \otimes m) = (1 \otimes m) \beta \otimes \Gamma(n' \otimes 1)$, we
obtain a measured quantum groupoid \((M, M', \otimes, M, \alpha, \beta, \Gamma, \nu, (\nu' \otimes id), (id \otimes \nu))\). Then, it is clear that:

\[
(A, J_\Lambda A J_\Lambda \otimes_{Z(M)} A, \alpha|_A, \beta|_A, \Gamma_{\gamma_{|\Lambda}, J_\Lambda A J_\Lambda \otimes_{Z(M)} A}, \nu, (\nu' \otimes id), (id \otimes \nu))
\]

is a locally compact sub-quantum groupoid of Lesieur’s pairs quantum groupoid.

5. FROM A MEASURED QUANTUM GROUPOID TO A CANONICAL LOCALLY COMPACT SUB-QUANTUM GROUPOID

5.1. NOTATIONS. Let \(\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)\) be a measured quantum groupoid; we shall use all notations of chapter 2 using \[3.2.1\] we can define \(C^*(\sigma^T, \sigma^{\nu*R}, \tau)\) as the weakly dense sub-\(C^*\)-algebra of \(M\) on which the one-parameter groups \(\sigma_t^\nu, \sigma_t^{\nu*R}, \tau_t\) are norm continuous, and \(C^*(\alpha^\nu, \gamma)\) a the weakly dense sub-\(C^*\)-algebra of \(N\) on which the one parameter groups \(\gamma_t\) are norm continuous. Then \(\alpha(C^*(\sigma^\nu, \gamma))\) and \(\beta(C^*(\sigma^\nu, \gamma))\) are included into \(C^*(\sigma^\nu, \sigma^{\nu*R}, \tau)\). Using \[3.3.4\] let us define the \(C^*\)-algebra \(C^*(\mathcal{G})\) as \(C^*(T, \nu) \cap \sigma^T(\nu)^* \cap C^*(\sigma^\nu, \sigma^{\nu*R}, \tau)\); using \[3.3.4\] we see that \(C^*(\mathcal{G})\) contains \(\mathcal{M}_\nu \cap \mathcal{M}_{\nu*R} \cap C^*(\sigma^\nu, \sigma^{\nu*R}, \tau)\). As the modular groups \(\sigma^\nu\) and \(\sigma^{\nu*R}\) commute, we know that \(\mathcal{M}_\nu \cap \mathcal{M}_{\nu*R}\) is weakly dense in \(\mathcal{M}\); using now \[3.2.1\] we see that \(C^*(\mathcal{G})\) is weakly dense in \(\mathcal{M}\). Moreover, it is clear that \(R(C^*(\mathcal{G})) = C^*(\mathcal{G})\) and that \(T(\mathcal{M}_\nu \cap C^*(\mathcal{G})) \subset \alpha(C^*(\nu, \gamma))\) and \(RTR(\mathcal{M}_{\nu*R} \cap C^*(\mathcal{G})) \subset \beta(C^*(\nu, \gamma))\), where \(C^*(\nu, \gamma) = C^*(\nu) \cap C^*(\gamma)\).

5.2. Lemma. Let \(\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)\) be a measured quantum groupoid; we shall use all notations of chapter 2 for any \(x, y\) in \(\mathcal{M}_\nu \cap \mathcal{M}_T\), we have:

\[
\Gamma(x) \rho_{J_\nu A_\nu(x)}^{\beta, \alpha} = (1 \otimes_{\mathcal{N}} J_\nu y J_\nu) W^* \rho_{A_\nu(x)}^{\alpha, \gamma}
\]

\[\text{Proof.}\] Let \(z \in \mathcal{M}_\nu \cap \mathcal{M}_T\); using \[2.6\] iii), we have:

\[
(\rho_{J_\nu A_\nu(z)}^{\beta, \alpha}) \Gamma(x) \rho_{J_\nu A_\nu(y)}^{\beta, \alpha} = (id \otimes_{\mathcal{N}} \omega_{J_\nu A_\nu(y)} J_\nu A_\nu(z)) \Gamma(x)
\]

\[
= (id \otimes_{\mathcal{N}} \omega_{A_\nu(x)} J_\nu A_\nu(y)) \Gamma(x) \rho_{J_\nu A_\nu(z)}^{\beta, \alpha}
\]

\[
= (\rho_{J_\nu A_\nu(z)}^{\beta, \alpha}) \Gamma(x) (1 \otimes_{\mathcal{N}} J_\nu y J_\nu) W^*.
\]

Taking now a basis \((e_i)_{i \in I}\) for \(T\) \[2.2.1\], we have:

\[
\Gamma(x) \rho_{J_\nu A_\nu(y)}^{\beta, \alpha} = \sum_i \rho_{J_\nu A_\nu(e_i)}^{\beta, \alpha} (\rho_{J_\nu A_\nu(e_i)}^{\beta, \alpha})^* \Gamma(x) \rho_{J_\nu A_\nu(y)}^{\beta, \alpha}
\]

\[
= \sum_i \rho_{J_\nu A_\nu(e_i)}^{\beta, \alpha} (\rho_{J_\nu A_\nu(e_i)}^{\beta, \alpha})^* (1 \otimes_{\mathcal{N}} J_\nu y J_\nu) W^* \rho_{A_\nu(x)}^{\alpha, \gamma}
\]

\[
= (1 \otimes_{\mathcal{N}} J_\nu y J_\nu) W^* \rho_{A_\nu(x)}^{\alpha, \gamma}.
\]

5.3. Lemma. Let’s use the notations of \[5.4\].

(i) Let \(a, b\) in \(\mathcal{M}_T \cap C^*(\mathcal{G})\). The norm closure of the linear space generated by all elements of the form \(\Lambda_T(a)\Lambda_T(b)^*\) is a \(C^*\)-algebra \(A\) and \(C^*(\mathcal{G}) \subset M(A)\).

(ii) we have \(A \subset J_\nu \alpha(N)J_\nu\).
Proof. For any $a_1, a_2, b_1, b_2$ in $A \cap \mathfrak{M}_T$, we have:

$$\Lambda_T(a_1)\Lambda_T(b_1)\Lambda_T(b_2) = \Lambda_T(a_1)\Lambda_T(T(b^*a_2)))\Lambda_T(b_2)^*$$

By hypothesis, $T(b^*a_2)$ belongs to $M(C^*(\mathfrak{G}))$, so $a_1T(b^*a_2)$ belongs also to $C^*(\mathfrak{G}) \cap \mathfrak{M}_T$. Then, it is easy to get that the linear space generated by all elements of the form $\Lambda_T(a)\Lambda_T(b)^*$ is an algebra. The fact that $C^*(\mathfrak{G}) \subseteq M(A)$ is trivial; which finishes the proof of (i). Let now $n \in \mathfrak{N}_\nu$, analytic with respect to $\sigma''$, such that $\sigma''_\nu(n^*)$ belongs to $\mathfrak{N}_\nu$, and $x \in \mathfrak{R}_\nu$; we have $\Lambda_T(b)^*J_\varphi\alpha(n)J_\varphi\Lambda_\varphi(x) = T(b^*x)J_\varphi\Lambda_\varphi(n)$ and, therefore,

$$\Lambda_T(a)\Lambda_T(b)^*J_\varphi\alpha(n)J_\varphi\Lambda_\varphi(x) = \Lambda_\varphi(aT(b^*x)\alpha(\sigma''_\nu(n^*))) = J_\varphi\alpha(n)J_\varphi\Lambda_\varphi(a)\Lambda_T(b)^*\Lambda_\varphi(x)$$

from which we get, by continuity, that, for any $n \in N$

$$\Lambda_T(a)\Lambda_T(b)^*J_\varphi\alpha(n)J_\varphi = J_\varphi\alpha(n)J_\varphi\Lambda_\varphi(a)\Lambda_T(b)^*$$

which gives (ii). \hfill \Box

5.4. Proposition. Let’s use the notations of 5.1. Let $x, y$ in $C^*(\mathfrak{G}) \cap \mathfrak{M}_T \cap \mathfrak{N}_\varphi$; then the operator $(id \ast \omega_{J_\varphi\Lambda_\varphi(y),J_\varphi\Lambda_\varphi(x)})(W)$ belongs to $C^*(\mathfrak{G})$.

Proof. Using 3.4(i), we have $<J_\varphi\Lambda_\varphi((y),J_\varphi\Lambda_\varphi(y)>_{\varphi,\nu} = \alpha^{-1}T(y^*y) \in C^*(\nu,\gamma)$ and
$$<\Lambda_\varphi(x),\Lambda_\varphi(x)>_{\varphi,\nu} = \alpha^{-1}T(x^*x) \in C^*(\nu,\gamma);$$

therefore, using 3.3, 4.5 (ii), (iii) and (iv) we get the result. \hfill \Box

5.5. Proposition. Let’s use the notations of 5.1 and 3.4. Let $x, x_1, x_2, y_1, y_2$ in $C^*(\mathfrak{G}) \cap \mathfrak{M}_T \cap \mathfrak{N}_\varphi$; then, using again 3.4(iv), $\Lambda_T(x_1)\Lambda_T(x_2)^*y$ belongs to $A$; let now $\sum N\Lambda_T(a_n)\Lambda_T(b_n)^*$ be an approximate unit of $A$; then $\sum N\Lambda_T(x_1)\Lambda_T(x_2)^*y\Lambda_T(a_n)\Lambda_T(b_n)^*$ is norm converging to $\Lambda_T(x_1)\Lambda_T(x_2)^*y$. Then:

(i) $(1_{N}\otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*\Gamma(x))\rho_{J_\varphi\Lambda_\varphi(y)}^{\beta,\alpha}$ is the norm limit of:

$$\sum_N \rho_{J_\varphi\Lambda_\varphi(x_1T(x_2^*y)_n)}^{\beta,\alpha}(i \ast \omega_{J_\varphi\Lambda_\varphi(y),J_\varphi\Lambda_\varphi(x)})(W)^*$$

(ii) Let now $z \in C^*(\mathfrak{G}) \cap \mathfrak{M}_T \cap \mathfrak{M}_\varphi$ and $X, Y$ in $A$; then $(1_{N}\otimes_\alpha X)\Gamma(x)(1_{N}\otimes_\alpha Y)$ belongs to $C^*(\mathfrak{G})$._\nu \text{ RT}^*_{\nu} B(H)$._\nu \text{ RT}^*_{\nu} B(H)$.

(iii) $\Gamma(C^*(\mathfrak{G}))$ is included into $C^*(\mathfrak{G})$._\nu \text{ RT}^*_{\nu} B(H)$._\nu \text{ RT}^*_{\nu} B(H)$.

(iv) $\Gamma(C^*(\mathfrak{G}))$ is included into $C^*(\mathfrak{G})$._\nu \text{ RT}^*_{\nu} C^*(\mathfrak{G})$._\nu \text{ RT}^*_{\nu} C^*(\mathfrak{G})$

Proof. Using 5.2 and 3.4(iii), we get that $(1_{N}\otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*\Gamma(x))\rho_{J_\varphi\Lambda_\varphi(y)}^{\beta,\alpha}$ is the norm limit of:

$$\sum_N (1_{N}\otimes_\alpha J_\varphi(1_{N}\otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*)y\Lambda_T(a_n)\Lambda_T(b_n)^*)J_\varphi W^*\rho_{\Lambda_\varphi(x)}^{\beta,\alpha} =$$

$$\sum_N (1_{N}\otimes_\alpha J_\varphi \Lambda_T(x_1T(x_2^*y)_n))\Lambda_T(b_n)^*J_\varphi W^*\rho_{\Lambda_\varphi(x)}^{\beta,\alpha} =$$

$$\sum_N \rho_{\Lambda_\varphi(x_1T(x_2^*y)_n)}^{\beta,\alpha}(id \ast \omega_{\Lambda_\varphi(x),\Lambda_\varphi(b)_n})(W)^*$$

which is (i).
Let now $x_3, x_4$ in $C^*(\mathfrak{G}) \cap M_T$; we have:

\[
(1 \otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*)\Gamma(z)(1 \otimes_\alpha \Lambda_T(x_3)\Lambda_T(x_4)^*)\rho_{\beta,\alpha}^{\beta,\alpha}(y) = (1 \otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*)\Gamma(z)\rho_{\beta,\alpha}^{\beta,\alpha}(x_3T(x_2)y))
\]

So, applying (i) and 3.1.6, we get that $(1 \otimes_\alpha \Lambda_T(x_1)\Lambda_T(x_2)^*)\Gamma(z)(1 \otimes_\alpha \Lambda_T(x_3)\Lambda_T(x_4)^*)$ belongs to $C^*(\mathfrak{G}) L_{RTR}^*LT B(H)$. Then, by norm continuity, we get (ii).

Let $X$ be in $A$, invertible in $\alpha(N)$; then $1 \otimes_\alpha X^{-1}$ belongs to $M(C^*(\mathfrak{G}) L_{RTR}^*LT B(H))$; and $\Gamma(z) = (1 \otimes_\alpha X^{-1})(1 \otimes_\alpha X)\Gamma(z)(1 \otimes_\alpha X)(1 \otimes_\alpha X^{-1})$ belongs to $C^*(\mathfrak{G}) L_{RTR}^*LT B(H)$.

Then, by norm continuity, we get (iii).

Applying this result to $\mathfrak{G}$, we obtain that $\Gamma(C^*(\mathfrak{G}^o))$ is included into $B(H)L_{RTR}^*LT C^*(\mathfrak{G}^o)$; as $C^*(\mathfrak{G}^o) = C^*(\mathfrak{G})$, we get (iv).

5.6. **Theorem.** Let’s use the notations of [8.7]. Then, the octuple

\[
(C^*(\nu, \gamma), C^*(\mathfrak{G}), \alpha|_{C^*(\nu, \gamma)}, \beta|_{C^*(\nu, \gamma)}, \nu_{C^*(\nu, \gamma)}, \Gamma_{C^*(\mathfrak{G})}, RTR|_{C^*(\mathfrak{G})}, \Gamma_{|C^*(\mathfrak{G})|})
\]

is a locally compact quantum groupoid.

**Proof.** We have $\alpha(C^*(\nu, \gamma)) \subset C^*(\mathfrak{G})$, and $\beta(C^*(\nu, \gamma)) \subset C^*(\mathfrak{G})$; by construction, $\nu_{C^*(\nu, \gamma)}$ is a KMS weight on $C^*(\nu, \gamma)$, and $(T, \nu \circ \alpha^{-1})$ (resp. $(RTR, \nu \circ \beta^{-1})$) are KMS, in the sense of [3.3, 2]. And we have obtain in [3.3, iv] that $\Gamma_{|C^*(\mathfrak{G})|}$ is a coproduct in the sense of [3.iv].

It is then clear also that the restriction of $T$ to $C^*(\mathfrak{G})$ is left-invariant, and the restriction of $RTR$ to $C^*(\mathfrak{G})$ is right-invariant, and that the modular groups of $\nu \circ \alpha^{-1} \circ T_{|C^*(\mathfrak{G})|}$ and $\nu \circ \beta^{-1} \circ RTR_{|C^*(\mathfrak{G})|}$ commute, which finishes the proof.

5.7. **Proposition.** Let’s use the notations of [8.7]. The scaling operator $\lambda$ of $\mathfrak{G}$ (which is a non singular positive operator affiliated to $Z(M)$, of the form $\lambda = \alpha(q) = \beta(q)$, where $q$ is a non singular positive operator affiliated to $Z(N)$ ([12], 3.8 (vi)), is affiliated to $Z(C^*(\mathfrak{G}))$ (and $q$ is affiliated to $Z(C^*(\nu, \gamma))$) in the sense of [10], [14].

**Proof.** It is clear, using [1.3, iv] applied to $C^*(\mathfrak{G})$.

5.8. **Remark.** The examples 1.12.3 and 1.12.4 show that, in general, a locally compact sub-quantum groupoid of a given measured quantum groupoid is not unique.

6. **Duality of locally compact groupoids**

6.1. **Notations.** Let $\mathfrak{G} = (B, A, \alpha, \beta, \nu, T, T', \Gamma)$ be a locally compact quantum groupoid, as defined in 3.6.3, let $\tilde{\mathfrak{G}} = (N, M, \tilde{\alpha}, \tilde{\beta}, \tilde{T}, \tilde{T}', \tilde{\Gamma})$ be the measured quantum groupoid constructed in [1.7]. As $B \subset N$, $A \subset M$, $\alpha = \tilde{\alpha}|_B$, $\beta = \tilde{\beta}|_B$, $\nu = \tilde{\nu}|_B$, $\Gamma = \tilde{\Gamma}|_A$, $T = \tilde{T}|_A$, $T' = \	ilde{T}'|_A$, we shall use, for simplification of notations, $\alpha$, $\beta$, $\nu$, $\Gamma$, $T$, $T'$, instead of $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\nu}$, $\tilde{T}$, $\tilde{T}'$. Moreover, as indicated in [2.9, (iii)], we shall consider the measured groupoid $\tilde{\mathfrak{G}}$, and, therefore, assume that $T' = RTR$, where $R$ is the co-inverse of $\mathfrak{G}$.

As recalled in [2.9, (viii)], we consider now the dual measured quantum groupoid $\hat{\mathfrak{G}} = (N, \hat{M}, \hat{\alpha}, \hat{\beta}, \hat{T}, \hat{T}', \hat{\nu})$, and we shall construct a dual locally compact quantum groupoid $\hat{\mathfrak{G}} = (B, \hat{\alpha}|_B, \hat{\beta}|_B, \hat{T}|_A, \hat{T}'|_A, \hat{\nu}|_A)$, where $\hat{\alpha}$ is a sub-$C^*$-algebra of $\hat{M}$.
6.2. **Lemma.**  Let \( x \in \mathcal{N}_T \cap \mathfrak{N}_\varphi \); then \( \hat{J}J\lambda\varphi(x) \) belongs to \( D(H_\beta, \nu^\varphi) \).

**Proof.** Let \( n \in \mathfrak{N}_\nu \); we have:
\[
\beta(n^*) \hat{J}J\lambda\varphi(x) = \hat{J}\alpha(n)J\lambda\varphi(x) = \hat{J}J\lambda_T(x)J_\nu\lambda\nu(n)
\]
which gives the result. \( \square \)

6.3. **Proposition.**  (i) let \( x, y \), in \( A \cap \mathfrak{N}_T \cap \mathfrak{N}_\varphi \), and let \( z = (\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W) \); then \( z \) belongs to \( \mathfrak{N}_T \cap \mathfrak{N}_\varphi \), and we have \( \Lambda\varphi(z) = \hat{J}J\rho(y)\lambda\varphi(x) \), and \( \Lambda\varphi(z) = \hat{J}J\rho(y)\lambda_T(x) \).

(ii) for \( i = 1, 2 \), let \( x_i, y_i \), in \( A \cap \mathfrak{N}_T \cap \mathfrak{N}_\rho \), and let us write \( z_i = (\omega \hat{J}J\lambda\varphi(x_i), J\lambda\varphi(y_i)) * id(W) \); then \( z_i \) belongs to \( \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi \), and \( \hat{T}(z_i^2z_1) \) belongs to \( \alpha(M(B)) \).

**Proof.** Using (LEMMA, 3.10 (v)); we get that \( z \) belongs to \( \mathfrak{N}_\varphi \), and:
\[
\Lambda\varphi(z) = a\varphi(\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) = Jy*\hat{J}J\lambda\varphi(x) = \hat{J}J\rho(y)\lambda\varphi(x)
\]
Moreover, for any \( n \in B \cap \mathfrak{N}_\nu \), analytic with respect to \( \sigma^\nu \), and \( x \) analytic with respect to \( \sigma^\varphi \), we get:
\[
z\alpha(n) = (\omega_{\beta(\sigma^\nu(n^*/2))}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)
\]
\[= (\omega_{J\alpha(\sigma^\nu(-1/2))}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)
\]
\[= (\omega_{J\alpha(\sigma^\nu(-1/2))}\lambda\varphi(\sigma^\nu(-1/2)\alpha(n), J\lambda\varphi(y)) * id(W)
\]
which remains true for any \( x \in A \cap \mathfrak{N}_T \cap \mathfrak{N}_\varphi \), and any \( n \in B \cap \mathfrak{N}_\nu \), from which we get that \( z\alpha(n) \) belongs to \( \mathfrak{N}_\varphi \), and \( \Lambda\varphi(z\alpha(n)) = \hat{J}J\rho(y)\lambda\varphi(x\alpha(n)) = \hat{J}J\rho(y)\lambda_T(x\alpha(n)) \), which finishes the proof of (i). Using (i), we get that \( \hat{T}(z_i^2z_1) = T(x_2R(y_2R(y_1))x_1) \) which belongs to \( \alpha(M(B)) \), which finishes the proof. \( \square \)

6.4. **Proposition.**  Let \( x, y \) in \( A \cap \mathfrak{N}_T \cap \mathfrak{N}_\varphi \); we have:
\[
\hat{R}(\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)) = (\omega \hat{J}J\lambda\varphi(y), J\lambda\varphi(x)) * id(W)
\]

(ii) For all \( t \in \mathbb{R} \), \( \delta^u \hat{J}J\lambda\varphi(x) \) belongs to \( D(H_\beta, \nu^\rho) \), and:
\[
\delta^u((\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W))\delta^{-u} = (\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)
\]

(iii)
\[
J\delta^uJ((\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W))J\delta^{-u}J = (\omega \hat{J}J\lambda\varphi(x), \delta^uJ\lambda\varphi(y)) * id(W)
\]

(iv)
\[
P^u((\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W))P^{-u} = (\omega \hat{J}JJ^u\lambda\varphi(x), J\lambda\varphi(y)) * id(W)
\]

(v)
\[
\sigma^\varphi_t((\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)) = (\omega \hat{J}JJ^u\lambda\varphi(x), \sigma^\varphi_tJ\lambda\varphi(y)) * id(W)
\]

**Proof.** We have:
\[
\hat{R}(\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W)) = J(\omega \hat{J}J\lambda\varphi(x), J\lambda\varphi(y)) * id(W) * J = (\omega \hat{J}J\lambda\varphi(y), J\lambda\varphi(x)) * id(W)
\]

34
which gives (i).

For all \( n \in \mathbb{N} \), we have:
\[
\beta(n)\delta^u \hat{J} J A_{\varphi}(x) = \delta^u \sigma^{\varphi_0 R}_t(\beta(n)) \hat{J} J A_{\varphi}(x) = \delta^u \beta(\sigma^\varphi_t \gamma_{-1}(n)) \hat{J} J A_{\varphi}(x) = \delta^u \hat{J} \alpha(\sigma^\varphi_t \gamma_{-1}(n)) J A_{\varphi}(x) = \delta^u \hat{J} J A_{\varphi}(x) J \nu A^u H^{-u} A \nu(n)
\]

where \( H^u \) is the canonical implementation of \( \gamma_t \) on \( H_{\varphi} \). Which gives that \( \delta^u \hat{J} J A_{\varphi}(x) \) belongs to \( D(H_{\beta}, \nu^0) \). Moreover, using now that \( \Gamma(\delta^u) = \delta^u \otimes_{\alpha} \delta^u (\hat{J} J A_{\varphi}(x)) \), we finish the proof of (ii). Then (iii) is obtained easily from (ii) and (i). Using then \( W(P^u \otimes_{\alpha} P^u) = (P^u \otimes_{\gamma} P^u)(W) \) (\textbf{E3}, 3.8 (vii)), we get (iv). Then (v) is obtained using (iii), (iv) and \textbf{E3}, 3.10 (vii). \( \square \)

6.5. Proposition. Let’s use the notations of \textbf{5.4}. Let \( A_0 \) be the set of elements \( x \) in \( A \cap M_T \cap M_{RTR} \cap M_{\varphi} \cap M_{\varphi R} \), which are analytic with respect to \( \sigma^\varphi, \sigma^{\varphi R} \), and such that, for any \( z, z', z'' \) in \( C \), \( \sigma^\varphi_z \circ \sigma_z^{\varphi R} \circ \tau^\varphi(x) \) belongs to \( A \cap M_T \cap M_{RTR} \cap M_{\varphi} \cap M_{\varphi R} \). Then, \( A_0 \) is a \(*\)-subalgebra of \( A \), invariant by \( R \), norm dense in \( A \). Moreover, \( \Lambda_{\varphi}(A_0) \) is dense in \( H_{\varphi} \). Moreover, if \( A_0^0 \) denotes the linear span generated \( b \) products \( x_0 \), where \( x, y \) belong to \( A_0 \), then \( A_0^0 \) is a \(*\)-algebra of \( A_0 \), invariant by \( R \), norm dense in \( A \), and such that \( \Lambda_{\varphi}(A_0^0) \) is dense in \( H_{\varphi} \). Moreover, if \( b \in B \) is analytic with respect to \( \sigma^\varphi_t \) and \( \gamma_t \), we get that \( \alpha(b) A_0 \subset A_0 \) and \( A_0 \beta(b) \subset A_0 \).

Proof. Let’s have a closer look at (\textbf{I}, 6.0.10) and its proof. For any \( x \in A \cap M_T \cap M_{RTR} \cap M_{\varphi} \cap M_{\varphi R} \), we obtain a sequence \( x_n \) in \( A_0 \), which is norm converging to \( x \), by similar arguments as in \textbf{3.2.1}. Then, we can prove that \( A \cap M_T \cap M_{RTR} \cap M_{\varphi} \cap M_{\varphi R} \) is norm dense in \( A \), using \textbf{3.6.4} (iii) and (vi). \( \square \)

6.6. Lemma. Let’s use the notations of \textbf{5.4} and \textbf{6.2}. The linear set generated by all elements of the form \( (\omega_{\hat{J} J A_{\varphi}(x), J A_{\varphi}(y)} \ast id)(W) \) with \( x, y \) in \( A_0 \) is invariant by \( * \).

Proof. We have, using \textbf{E3} \( \Lambda_{\varphi}(\omega_{\hat{J} J A_{\varphi}(x), J A_{\varphi}(y)} \ast id)(W)) = \hat{J} J R(y) \Lambda_{\varphi}(x) \). Therefore, \( (\omega_{\hat{J} J A_{\varphi}(x), J A_{\varphi}(y)} \ast id)(W) \) belongs to \( M^*_{\varphi} \) if and only if \( J R(y) \Lambda_{\varphi}(x) \) belongs to \( D(\hat{\Delta}^{-1/2}) \).

But, if \( x, y \) belong to \( A_0 \), we get, using successively \textbf{3.10}(vii), \textbf{V1}, \textbf{E3} 3.8(vii), \textbf{E3} 3.10(v) that:
\[
\hat{\Delta}^{-1/2} J R(y) \Lambda_{\varphi}(x) = P^{-1/2} J \delta^{1/2} R(y) \Lambda_{\varphi}(x) = P^{-1/2} \lambda^{-1/4} \Lambda_{\varphi}(\sigma^{-\varphi}_{-1/2}(x^* R(y^*)) \delta^{1/2}) = \Lambda_{\varphi}(\tau_{1/2} \sigma^{-\varphi}_{-1/2}(x^* R(y^*)) \delta^{1/2}) = \hat{J} \Lambda_{\varphi}(R(\tau_{1/2} \sigma^{-\varphi}_{-1/2}(x^* R(y^*))^*)) = \hat{J} \Lambda_{\varphi}(\tau_{-1/2} \sigma^{\varphi R}(R(y) x)) = \hat{J} J \Lambda_{\varphi}(\sigma_{-1/2} \tau_{1/2} \sigma^{\varphi R}(y^* R(x^*)))
\]

which is equal to \( \Lambda_{\varphi}(\omega_{\hat{J} J A_{\varphi}(x'), J A_{\varphi}(y')} \ast id)(W)) \), with \( x' = \sigma_{-1/2} \tau_{1/2} \sigma^{\varphi R}(R(x^*)) \) and \( y' = \sigma^{\varphi R}_{-1/2} \tau_{1/2} \sigma^{\varphi R}_{-1/2}(y^*), \) which both belong to \( A_0 \). Therefore, we get that:
\[
(\omega_{\hat{J} J A_{\varphi}(x), J A_{\varphi}(y)} \ast id)(W)^* = (\omega_{\hat{J} J A_{\varphi}(x'), J A_{\varphi}(y')} \ast id)(W)^*
\]
which finishes the proof. □

6.7. Lemma. Let $X \in \mathfrak{M}_{\hat{\rho}} \cap \mathfrak{N}_{\hat{\varphi}}$ such that there exists $Y \in A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi}$ with $\Lambda_{\hat{\varphi}}(X) = \hat{J}J\Lambda_T(Y)$ and $\Lambda_{\hat{\rho}}(X) = \hat{J}J\Lambda_{\varphi}(Y)$ (for instance, using 6.3, any linear combination of operators of the form $(\omega_{\hat{J}J\Lambda_{\varphi}(x_1)\Lambda_{\varphi}(y_1)} * \text{id})(W)$ with $x_1, y_1$, in $A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi}$). Then, for any $x_2, y_2$ in $A \cap \mathfrak{M}_T \cap \mathfrak{N}_{RTR} \cap \mathfrak{N}_{\varphi} \cap \mathfrak{N}_{\varphi_0R}$, $X(\omega_{\hat{J}J\Lambda_{\varphi}(x_2)\Lambda_{\varphi}(y_2)} * \text{id})(W)$ satisfies the same property.

Proof. Let us suppose first that $x_2, y_2$ are analytics with respect to $\tau$ and that $\tau_{-\mathcal{J}/2}(x_2)$ and $R(\tau_{-\mathcal{J}/2}(y_2))$ belong to $\mathfrak{N}_{\varphi}$. Using (6.3.1.4) and 3.10.5, 5.1, we get that:

\[ \hat{J}J\Lambda_{\varphi}(R(\tau_{-\mathcal{J}/2}(y_2))) = J\lambda^{1/4}\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2)) = \delta^{1/2}\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2)) = \lambda^{-1/4}\delta^{1/2}JP^{1/2}\Lambda_{\varphi}(y_2) \]

and, using $\hat{J}J\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(x_2)) = \lambda^{1/2}\hat{J}JP^{1/2}\Lambda_{\varphi}(x_2)$

Using then 6.4 (v), we get:

\[ \sigma_{-\mathcal{J}/2}(\omega_{\hat{J}J\Lambda_{\varphi}(x_2)\Lambda_{\varphi}(y_2)} * \text{id})(W) = \lambda^{-1/4}(\omega_{\hat{J}J\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(x_2))\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2))) * \text{id})(W) \]

and, using 3.8 (ii):

\[ \Lambda_{\hat{\varphi}}(X(\omega_{\hat{J}J\Lambda_{\varphi}(x_2)\Lambda_{\varphi}(y_2)} * \text{id})(W)) = \hat{J}\sigma_{-\mathcal{J}/2}(\omega_{\hat{J}J\Lambda_{\varphi}(x_2)\Lambda_{\varphi}(y_2)} * \text{id})(W) \hat{J}\Lambda_{\varphi}(X) \]

\[ = \hat{J}\lambda^{-1/4}(\omega_{\hat{J}J\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(x_2))\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2))) * \text{id})(W) \Lambda_{\varphi}(Y) = \hat{J}\lambda^{-1/4}\Lambda_{\varphi}(\omega_{\hat{J}J\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2)))\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(x_2))) * \text{id})(W) \Lambda_{\varphi}(Y) \]

\[ = \hat{J}\lambda^{-1/4}\Lambda_{\varphi}(\omega_{\hat{J}J\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(y_2)))\Lambda_{\varphi}(\tau_{-\mathcal{J}/2}(x_2))) * \text{id})(W) \Lambda_{\varphi}(Y) \]

and, thanks to 3.6.4 (v) and 3.8 (iv), we get the result. □

6.8. Notations. Let us denote by $\hat{A}$ the subalgebra of $\hat{M}$ generated by elements of the form $(\omega_{\hat{J}J\Lambda_{\varphi}(x_1)\Lambda_{\varphi}(y_1)} * \text{id})(W)$ with $x_1, y_1$, in $A_0$. Then, using 6.6, we get that $\hat{A}$ is invariant by $\ast$. Using now 2.9.4 (ii), we get that $\hat{A}$ is invariant by $\hat{R}$. Let us now denote by $\hat{A}$ the norm closure of $\hat{A}$. It is clear that $\hat{A}$ is a sub-$\mathcal{C}^*$-algebra of $\hat{M}$, weakly dense in $\hat{M}$, invariant by $\hat{R}$.

6.9. Theorem. (i) We have $\alpha(B) \subset M(\hat{A})$ and $\beta(B) \subset M(\hat{A})$. Moreover, we have $\hat{A} \subset \mathcal{C}^*(\hat{\varphi}, \hat{\varphi} \circ \hat{R}, \hat{\tau})$.

(ii) For any $X \in \hat{A} \cap \mathfrak{M}_{\hat{\rho}}$, $\hat{T}(X)$ belongs to $\alpha(M(B))$; for any $Y \in \hat{A} \cap \mathfrak{M}_{RTR}$, $\hat{T}\hat{R}(Y)$ belongs to $\beta(M(B))$.

(iii) For any $x \in \hat{A} \cap \mathfrak{M}_{\hat{\varphi}}$, there exists $x_n$ in $\hat{A}$ (as defined in 6.8), such that, for any $b \in B$, $\Lambda_{\hat{\varphi}}(x_n)b$ is norm converging to $\Lambda_{\hat{\varphi}}(x)b$.

Proof. We have got in 6.3 that, for any $x, y$ in $A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi}$, and $n \in B \cap \mathfrak{N}_{\varphi}$, we have:

\[ (\omega_{\hat{J}J\Lambda_{\varphi}(x)\Lambda_{\varphi}(y)} * \text{id})(W)\alpha(n) = (\omega_{\hat{J}J\Lambda_{\varphi}(x\alpha(n))\Lambda_{\varphi}(y)} * \text{id})(W) \]

If now $x \in A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi} \cap \mathfrak{M}_{RTR} \cap \mathfrak{N}_{\varphi_0R}$, we get that $R(x\alpha(n)) = \beta(n)R(x)$ belongs to $A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi}$, and, therefore, that $x\alpha(n)$ belongs to $A \cap \mathfrak{M}_T \cap \mathfrak{N}_{\varphi} \cap \mathfrak{M}_{RTR} \cap \mathfrak{N}_{\varphi_0R}$. Moreover, if $x$ belongs to $\mathcal{A}$ (as defined in 6.3), and if $n$ is analytic with respect to $\sigma_{\mathcal{A}}$ and $\gamma$, then $x\alpha(n)$ belongs to $\mathcal{A}$; therefore, if $X$ belongs to $\mathcal{A}$, then $X\alpha(n)$ belongs to $\mathcal{A}$.

Using now the invariance of $\hat{A}$ by $\ast$ and $\hat{R}$, we get that $\alpha(n)X$, $\gamma(n)X$, $X\gamma(n)$ belong
to \( \hat{A} \). Then, by continuity, we get the first results of (i). Using now \ref{6.4}(v), we get that \( \hat{A} \subset C^* (\hat{\omega}, \hat{\omega} \circ \hat{R}, \hat{\tau}) \), which, by density, finishes the proof of (i).

Using \ref{6.7}, we get that, for any \( X \in \hat{A}, X \) belongs to \( \mathfrak{M}_T \cap \mathfrak{M}_\hat{\varphi} \), and \( \hat{T} (X^* X) \in \alpha (M (B)) \). Moreover, as \( \hat{A}^+ \) is norm dense in \( \hat{A} \), for any \( x \in A \cap \mathfrak{M}_+^T \), it is possible to construct a sequence \( x_n \) in \( \hat{A}^+ \), such that \( x_n \leq x \) and \( x_n \) is norm converging to \( x \). Then, using \((K2, 3.5)\) to the restriction of \( \hat{T} \) to \( C^* (\hat{\omega}, \hat{\omega} \circ \hat{R}, \hat{\tau}) \), we get that \( \hat{T} (x_n) \) is strictly converging to \( \hat{T} (x) \); therefore, we get that \( \hat{T} (x) \) belongs to \( \alpha (M (B)) \). Using the invariance of \( \hat{A} \) by \( \hat{R} \), we finish the proof of (ii). The proof of (iii) is obtained by similar arguments. \( \square \)

6.10. Theorem. For any \( x \in \hat{A} \), \( \hat{T} (x) \) belongs to \( \hat{A} L_{\hat{R} \hat{T} R} * L_{\hat{T}} \hat{A} \).

Proof. Let \( x, y \in \hat{A} \). Using \ref{6.5}(i) applied to \( \hat{G} \), and using the elements \( a_n \) and \( b_n \) defined such that \( \Sigma_N \Lambda_\hat{\varphi} (a_n) \Lambda_\hat{\varphi} (b_n) \) is an approximate unit of \( \hat{A} \), we get that:

\[
(1_{\hat{\rho}} \otimes N) \Lambda_{\hat{T} (x_1)} \Lambda_{\hat{T} (x_2)} \hat{T}^\alpha (y) = \sum_N \hat{T}^\alpha_{\hat{\Lambda}_{\hat{T} (x_1) \hat{T} (x_2) \gamma_{a_n}}} (\omega_{\Lambda_{\hat{T} (x_1)} \hat{\Lambda}_{\hat{T} (x_2) \gamma_{a_n}}} * id) (W)
\]

As \( x \) and \( b_n \) belong to \( \hat{A} \), we get that \( \Lambda_\varphi (x) \) belongs to \( \hat{J} J \Lambda_\varphi (A) \) and that \( \hat{J} \Lambda_\varphi (b_n) \) belongs to \( J \Lambda_\varphi (A) \); therefore, \( (\omega_{\Lambda_{\hat{T} (x_1)} \hat{\Lambda}_{\hat{T} (x_2) \gamma_{a_n}}} * id) (W) \) belongs to \( \hat{A} \), and we get, by continuity, that \( (1_{\hat{\rho}} \otimes N) \Lambda_{\hat{T} (x_1)} \Lambda_{\hat{T} (x_2)} \hat{T} (x) \) belongs to \( \hat{A} L_{\hat{R} \hat{T} R} * L_{\hat{T}} B (H) \). Which, by continuity, remains true for any \( x \in \hat{A} \). Using now similar arguments as in \ref{6.5}, we obtain the result. \( \square \)

6.11. Theorem. The octuple \( \hat{G} = (B, \hat{A}, \alpha \mid B, \gamma \mid B, \nu \mid B, \hat{T} \mid \hat{A}, \hat{R} \hat{T} \hat{R} \mid \hat{A}, \hat{\Gamma} \mid \hat{A}) \) is a locally compact quantum groupoid, we shall call the dual of \( G \).

Proof. This is clear, using \ref{6.9} and \ref{6.10} \( \square \)

6.12. Proposition. The octuple \( \hat{G} \), defined as the bidual of \( G \), is equal to \( G \).

Proof. Using \ref{6.8} we see that \( \hat{A} \) is the norm closure of the algebra \( \hat{A} \) generated by all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} J \Lambda_\varphi (x)) (W)\), where \( x, y \) belong to \( \hat{A}_0 \), where \( \hat{A}_0 \) is, by \ref{6.5}, the algebra of elements in \( \hat{A} \cap \mathfrak{M}_R \cap \mathfrak{M}_R \hat{T} \hat{R} \hat{T} \hat{R} \cap \mathfrak{M}_\varphi \cap \mathfrak{M}_\varphi \hat{R} \hat{R} \) which are analytic with respect to \( \sigma_t \), \( \sigma_t \hat{R} \) and \( \hat{\tau} \). As \( \hat{A}_0 \) contains \( \hat{A} \), the linear space \( \Lambda_\varphi (\hat{A}_0) \) contains \( \Lambda_\varphi (\hat{A}) \), which is equal, using \ref{6.3}(i), to \( \hat{J} J \Lambda_\varphi (A_0) \). Therefore, \( \hat{A} \) contains all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} A_\varphi (x)) (W)\), where \( x \) and \( y \) belong to \( A_0 \). By continuity, we get that \( \hat{A} \) contains all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} A_\varphi (x)) (W)\), where \( x \) and \( y \) belong to \( A_0 \), from which we get that \( A \subset \hat{A} \). But, using \ref{6.9}(iii), we get that \( \hat{A} \) is the norm closure of the set of all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} A_\varphi (x)) (W)\), where \( z_1, z_2 \) belong to \( \hat{A} \) and \( n_1, n_2 \) belong to \( B \) and are analytic with respect to \( \sigma_t \) and \( \gamma_\tau \). Therefore, it is the norm closure of the set of all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} J \Lambda_\varphi (z_1)) (W)\), where \( z_1, z_2 \) belong to \( \hat{A} \), which is, as we have seen, the norm closure of the set of all elements of the form \((id * \omega_{J \Lambda_\varphi (y)} A_\varphi (x)) (W)\), where \( x, y \) belong to \( A_0 \), which is \( A \). \( \square \)
7. Examples

7.1. Locally compact sub-quantum groupoid with finite dimensional basis. When the basis is finite dimensional, the construction of locally compact quantum groupoids had already been studied by De Commer (\cite{DC1}, chapter 11) and Baaj and Crespo (\cite{BC} 2.2, \cite{Gr}).

7.2. Continuous field of locally compact quantum groups \cite{B2}. In this chapter, we define a notion of continuous field of locally compact quantum groups \cite{7.2.2}, which was underlying in \cite{B2}. We show that these are exactly locally compact quantum groupoids with central basis, and that the dual object is of the same kind \cite{7.2.6}. We finish by recalling concrete examples \cite{7.2.9 7.2.10 7.2.11} given by Blanchard, which are examples of locally compact quantum groupoids.

7.2.1. Definition. A 6-uple \((X, \alpha, A, \Gamma^x, \varphi^x, \psi^x)\) will be called a \textit{continuous field of locally compact quantum groups} if:

(i) \((X, \alpha, A, \varphi^x)\) and \((X, \alpha, A, \psi^x)\) are measurable continuous fields of \(C^\ast\)-algebras;

(ii) for any \(x \in X\), there exists a simplifiable coproduct \(\Gamma^x\) from \(A^x\) to \(M(A^x \otimes_m A^x)\) such that \((\Gamma^x, \Gamma^x(x), \varphi^x, \psi^x)\) is a locally compact quantum group, in the sense of [KV1]. Simplifiable means that the closed linear set generated by \(\Gamma^x(A^x)(A^x \otimes_m 1)\) is equal to \(A^x \otimes_m A^x\) (resp. \(\Gamma^x(A^x)(1 \otimes_m A^x)\)).

7.2.2. Definition. (\cite{L}, 17.1.3) Let \((X, \nu)\) be a \(\sigma\)-finite standard measure space; let us take \(\{M^x, x \in X\}\) a measurable field of von Neumann algebras over \((X, \nu)\) and \(\{\varphi^x, x \in X\}\) (resp. \(\{\psi^x\}\)) a measurable field of normal semi-finite faithful weights on \(\{M^x\}\) (\cite{L}, 4.4). Moreover, let us suppose that:

(i) there exits a measurable field of injective \(*\)-homomorphisms \(\Gamma^x\) from \(M^x\) into \(M^x \otimes M^x\) (which is also a measurable field of von Neumann algebras, on the measurable field of Hilbert spaces \(H_{\varphi^x} \otimes H_{\varphi^x}\)).

(ii) for almost all \(x \in X\), \(G^x = (M^x, \Gamma^x, \varphi^x, \psi^x)\) is a locally compact quantum group (in the von Neumann sense (\cite{KV2}).

In that situation, we shall say that \((M^x, \Gamma^x, \varphi^x, \psi^x, x \in X)\) is a \textit{measurable field of locally compact quantum groups over \((X, \nu)\)}.

7.2.3. Theorem. (\cite{L}, 17.1.3, \cite{E5}, 8.2) Let \(G^x = (M^x, \Gamma^x, \varphi^x, \psi^x, x \in X)\) be a measurable field of locally compact quantum groups over \((X, \nu)\). Let us define:

(i) \(M\) as the von Neumann algebra made of decomposable operators \(\int_X M^x d\nu(x)\), and \(\alpha\) the \(*\)-isomorphism which sends \(L^\infty(X, \nu)\) into the algebra of diagonalizable operators, which is included in \(Z(M)\).

(ii) \(\Phi\) (resp. \(\Psi\)) as the direct integral \(\int_X \varphi^x d\nu(x)\) (resp. \(\int_X \psi^x d\nu(x)\)). Then, the Hilbert space \(H_\Phi\) is equal to the direct integral \(\int_X H_{\varphi^x} d\nu(x)\), the relative tensor product \(H_{\alpha \otimes \nu} H_\Phi\) is equal to the direct integral \(\int_X (H_{\varphi^x} \otimes H_{\varphi^x}) d\nu(x)\), and the product \(M_\alpha \ast N\) is equal to the direct integral \(\int_X (M^x \otimes M^x) d\nu(x)\).

(iii) \(\Gamma\) as the decomposable \(*\)-homomorphism \(\int_X \Gamma^x d\nu(x)\), which sends \(M\) into \(M_\alpha \ast N\).

(iv) \(T\) (resp. \(T'\)) as an operator-valued weight from \(M\) into \(\alpha(L^\infty(X, \nu))\) defined the following way: \(a \in M^\ast\) represented by the field \(\{a^x\}\) belongs to \(M_T\) if, for almost all \(x \in X\), \(a^x\) belongs to \(M_{\varphi^x}\) (resp. \(M_{\psi^x}\)) and the function \(x \mapsto \varphi^x(a^x)\) (resp. \(x \mapsto \psi^x(a^x)\)) is essentially bounded; then \(T(a)\) (resp. \(T'(a)\)) is defined as the image under \(\alpha\) of this function.
Then, \((L^\infty(X, \nu), M, \alpha, \alpha, \Gamma, T, T', \nu)\) is a measured quantum groupoid, we shall denote by \(\int_X G^x d\nu(x)\).

7.2.4. Proposition. ([55], 8.3) Let \((X, \nu)\) be a \(\sigma\)-finite standard measure space, and \(\{G^x, x \in X\}\) a measurable field of locally compact quantum groups, as defined in 7.2.2; let \(\int_X G^x d\nu(x)\) be the measured quantum groupoid constructed in 7.2.3; then:
(i) we have \(\alpha = \beta = \bar{\beta}\);
(ii) the pseudo-multiplicative unitary of the measured quantum groupoid is a unitary on \(H\); the multiplicative unitary associated to the locally compact quantum group.

7.2.5. Proposition. ([55], 8.4) Let \(\mathfrak{G} = (N, M, \alpha, \nu, \Gamma, T, T', \nu)\) be a measured quantum groupoid and \(\hat{\mathfrak{G}} = (N, M, \alpha, \nu, \hat{\Gamma}, \hat{T}, \hat{T}' \hat{R}, \nu)\) its dual measured quantum groupoid. Then, are equivalent:
(i) \(\alpha(N) \subset Z(M) \cap Z(\hat{M})\).
(ii) \(\alpha = \beta = \bar{\beta}\).

7.2.6. Theorem. ([55], 8.5) Let \(\mathfrak{G} = (N, M, \alpha, \nu, \Gamma, T, T', \nu)\) be a measured quantum groupoid and \(\hat{\mathfrak{G}} = (N, M, \alpha, \nu, \hat{\Gamma}, \hat{T}, \hat{T}' \hat{R}, \nu)\) its dual measured quantum groupoid; let \(W\) and \(\hat{W}\) be the pseudo-multiplicative unitaries associated, and \(\Phi = \nu \circ \alpha^{-1} \circ T\) (resp. \(\hat{\Phi} = \nu \circ \alpha^{-1} \circ \hat{T}\)); let us suppose that \(\alpha(N)\) is central in both \(M\) and \(\hat{M}\); let \(X\) be the spectrum of \(C^*\)-algebra \(C^*(\nu)\), we shall therefore identify with \(C_0(X)\); for any \(x \in X\), let \(C_x(X)\) be the sub-\(C^*\)-algebra of \(C_0(X)\) made of functions which vanish at \(x\); let \(A_n(W)\) be the sub-\(C^*\)-algebra of \(M\) introduced in 7.2.4, which is, thanks to \(\alpha|_{C_0(X)}\), a continuous field over \(X\) of \(C^*\)-algebras ([T], 4.11); let \(\varphi^x\) be the desintegration of \(\Phi|_{A_n(W)}\) over \(X\); \(\varphi^x\) is a lower semi-continuous weight on \(A_n(W)\), faithful when considered on \(A_n(W)/\alpha(C_x(X))A_n(W)\), and the representation \(\pi_{\varphi^x}\) form a continuous field of faithful representation of \(A_n(W)\).

Then:
(i) the Hilbert space \(H_{\Phi|_{\alpha \otimes \alpha}} H_{\Phi}\) is equal to \(\int_X H_{\varphi^x} \otimes H_{\varphi^x} d\nu(x)\).
(ii) the von Neumann algebra \(M = \bigoplus_{x \in X} M\) is equal to:
\[\int_X \varphi^x \left(\frac{A_n(W)}{\alpha(C_x(X))A_n(W)}\right)^* \otimes \varphi^x \left(\frac{A_n(W)}{\alpha(C_x(X))A_n(W)}\right)^* d\nu(x)\]
(iii) the coproduct \(\Gamma|_{A_n(W)}\) can be desintegrated in \(\Gamma|_{A_n(W)} = \int_X \Gamma^x d\nu(x)\), where \(\Gamma^x\) is a continuous field of coassociative coproducts on \(A_n(W)/\alpha(C_x(X))A_n(W)\).
(iv) \(R^x\) is an anti-\(\ast\)-automorphism of \(A_n(W)/\alpha(C_x(X))A_n(W)\), and, for all \(x \in X\), \((\alpha(C_x(X))A_n(W), \Gamma^x, \varphi^x \circ R^x)\) is a locally compact quantum group (in the \(C^*\)-sense), we shall denote \(G^x\). We shall denote also \(G^x\) its von Neumann version.
(iv) we have, with the notations of 7.2.3, \(\mathfrak{G} = \int_X G^x d\nu(x)\).

7.2.7. Theorem. ([55], 8.5) Let \((X, \nu)\) be a \(\sigma\)-finite standard measure space, \(G^x\) be a measurable field of locally compact quantum groups over \((X, \nu)\), ad defined in 7.2.2, and \(\int_X G^x d\nu(x)\) be the measured quantum groupoid constructed in 7.2.3. Then:
(i) there exists a locally compact set \(X\), and a positive Radon measure \(\nu\) on \(X\), such that
$L^\infty(X, \nu)$ and $L^\infty(\tilde{X}, \tilde{\nu})$ are isomorphic, and such that this isomorphism sends $\nu$ on $\tilde{\nu}$.

(ii) there exists a continuous field $(A^x)_{x \in X}$ of C*-algebras, and a continuous field of coassociative coproducts $\tilde{\Gamma}^x : A^x \to A^x \otimes^m A^x$;

(iii) there exists left-invariant (resp. right-invariant) weights $\varphi^x$ (resp. $\tilde{\varphi}^x$), such that $(A^x, \tilde{\Gamma}^x, \varphi^x, \tilde{\varphi}^x)$ is a locally compact quantum group $\tilde{G}^x$ (in the C* sense).

(iv) we have: $\int_X \tilde{G}^x d\tilde{\nu}(x) = \int_X \tilde{G}^x d\tilde{\nu}(x)$.

7.2.8. Theorem. With the notations of 7.2.7, let us define $A = \int_X A^x d\tilde{\nu}(x)$. Then, $(C_0(\tilde{X}), A, \alpha|_{C_0(\tilde{X})}, \alpha|_{C_0(\tilde{X})}, \tilde{\nu}, \Gamma|_A)$ is a locally compact quantum groupoid.

Proof. Let $a = \int_X a^x d\tilde{\nu}(x)$ in $A$, and $b = \int_X b^x d\tilde{\nu}(x)$ in $A$, with $b^x \in \mathcal{M}^x \cap \mathcal{M}^{x^*}$; then, we get that $\rho_{\tilde{\nu}, \Lambda|_A}(a) = \int_X \rho_{\tilde{\nu}, \Lambda|_A}(a^x) \Gamma_x(a^x d\tilde{\nu}(x)).$ \hfill $\Box$

7.2.9. Example. As in ([B1], 7.1), let us consider the C*-algebra $A$ whose generators $\alpha$, $\gamma$ and $f$ verify:

(i) $f$ commutes with $\alpha$ and $\gamma$;

(ii) the spectrum of $f$ is $[0, 1]$;

(iii) the matrix $\begin{pmatrix} \alpha & -f \gamma \\ \gamma & \alpha^* \end{pmatrix}$ is unitary in $M_2(A)$. Then, using the sub C*-algebra generated by $f$, $A$ is a $C([0, 1])$-algebra; let us consider now $A$ as a $C_0([0, 1])$-algebra. Then, Blanchard had proved ([B2] 7.1) that $A$ is a continuous field over $[0, 1]$ of C*-algebras, and that, for all $q \in [0, 1]$, we have $A^q = SU_q(2)$, where the $SU_q(2)$ are the compact quantum groups constructed by Woronowicz and $A^1 = C(SU(2))$. Moreover, using the coproducts $\Gamma^q$ defined by Woronowicz as

$$\Gamma^q(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma$$

$$\Gamma^q(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

and the (left and right-invariant) Haar state $\varphi^q$, which verifies:

$\varphi^q(\alpha^k \gamma^m \gamma^n) = 0$, for all $k \geq 0$, and $m \neq n$,

$\varphi^q(\alpha^k \gamma^m \gamma^n) = 0$, for all $k < 0$, and $m \neq n$,

$\varphi^q((\gamma \gamma^*)^m) = \frac{1-q^2}{1-q^2m^2}$,

we obtain this way a continuous field of compact quantum groups (see [B2], 6.6 for a definition); this leads to put on $A$ a structure of locally compact quantum groupoid (of compact type, in the sense of [B2], because $1 \in \mathcal{A}$).

This structure is given by a coproduct $\Gamma$ which is $C_0([0, 1])$-linear from $A$ to $\otimes^m C_0([0, 1])$, and given by:

$$\Gamma(\alpha) = \alpha \otimes^m_{C_0([0, 1])} \alpha - f \gamma^* \otimes^m_{C_0([0, 1])} \gamma$$

$$\Gamma(\gamma) = \gamma \otimes^m_{C_0([0, 1])} \alpha + \alpha^* \otimes^m_{C_0([0, 1])} \gamma$$

and by a conditional expectation $E$ from $A$ on $M(C_0([0, 1]))$ given by:

$E(\alpha^k \gamma^m \gamma^n) = 0$, for all $k \geq 0$, and $m \neq n$,

$E(\alpha^k \gamma^m \gamma^n) = 0$, for all $k < 0$, and $m \neq n$,

$E((\gamma \gamma^*)^m)$ is the bounded function $x \mapsto \frac{1-q^2}{1-q^2m^2}$.

Then $E$ is both left and right-invariant with respect to $\Gamma$. This example gives results at the level of C*-algebras, which are more precise than theorem 7.2.3.
7.2.10. **Example.** One can find in [B2] another example of a continuous field of locally compact quantum group. Namely, in (B2, 7.2), Blanchard constructs a C*-algebra $A$ which is a continuous field of C*-algebras over $X$, where $X$ is a compact included in $[0, 1]$, with $1 \in X$. For any $q \in X$, $q \neq 1$, we have $A^q = SU_q(2)$, and $A^1 = C^*_r(G)$, where $G$ is the "ax + b" group. (B2, 7.6).

Moreover, he constructs a coproduct (denoted $\delta$) (B2, 7.7(c)), and "the system of Haar weights" $\Phi$ (B2 7.2.3), which bear left-invariant-like properties (end of remark after B2 7.2.3).

Finally, he constructs a unitary $U$ in $\mathcal{L}(\mathcal{E}_\Phi)$ (B2 7.10), with which it is possible to construct a co-inverse $R$ of $(A, \delta)$, which leads to the fact that $\Phi \circ R$ is right-invariant. Clearly, the fact that we are here dealing with non-compact locally compact quantum groups made the results more problematic at the level of C*-algebra; at the level of von Neumann algebra, (7.2.3) allow us to construct an example of measured quantum groupoid from these data.

7.2.11. **Example.** Let us finish by quoting a last example given by Blanchard in (B2, 7.4): for $X$ compact in $[1, \infty[$, with $1 \in X$, he constructs a C*-algebra which is a continuous field over $X$ of C* algebras, whose fibers, for $\mu \in X$, are $A^\mu = E_\mu(2)$, with a coproduct $\delta$ and a continuous field of weights $\Phi$, which is left-invariant. As in 7.2.10 he then constructs a unitary $U$ on $\mathcal{L}(\mathcal{E}_\Phi)$, which will lead to a co-inverse, and, therefore, to a right-invariant C*-weight.

7.2.12. **Example.** (L, 17.1) Let us return to (7.2.10) let $I$ be a (discrete) set, and, for all $i$ in $I$, let $G_i = (M_i, \Gamma_i, \varphi_i, \psi_i)$ be a locally compact quantum group; then the product $\Pi_i G_i$ is a measured field of locally compact quantum groups, and can be given a natural structure of measured quantum groupoid, described in (L, 17.1).

7.2.13. **Example.** (DC2 3.17)(E5, 9) Let $(C^2, M, \alpha, \beta, \Gamma, T, RTR, \nu)$ be a measured quantum groupoid; let $(e_1, e_2)$ be the canonical basis of $C^2$ et let us suppose that $\nu(e_1) = \nu(e_2) = 1/2$; let us define $M_{i,j} = M\alpha(e_i)\beta(e_j)$; then, the fiber product $M_{\alpha, \beta} M$ can be identified with the reduced von Neumann algebra $(M \otimes M)_{\beta(e_1)\alpha(e_2)+\beta(e_2)\alpha(e_1)}$, and $\Gamma$ can be identified with an injective *-homomorph from $M$ to $M \otimes M$, such that $(\Gamma \otimes id)\Gamma = (id \otimes \Gamma)\Gamma$ and:

$$\Gamma(1) = \beta(e_1) \otimes \alpha(e_1) + \beta(e_2) \otimes \alpha(e_2)$$

$$\Gamma(\alpha(e_i)\beta(e_j)) = \alpha(e_i)\beta(e_1) \otimes \alpha(e_i)\beta(e_j) + \alpha(e_i)\beta(e_2) \otimes \alpha(e_2)\beta(e_j)$$

Let $M_{i,j} = M\alpha(e_i)\beta(e_j)$, we have $M_{1,1} \neq \{0\}, M_{2,2} \neq \{0\}$, and:

$$\Gamma(M_{i,j}) \subset (M_{i,1} \otimes M_{1,j}) \oplus (M_{i,2} \otimes M_{2,j})$$

There exist normal semi-finite faithful weights $\varphi_{i,j}$ on $M_{i,j}$ such that, for any $X \in \mathcal{M}_+$, $X = \sum_{i,j} x_{i,j}$, with $x_{i,j} \in M_{i,j}^+$, we get that $T(X)$ is the image under $\alpha$ of:

$$(\varphi_{1,1}(x_{1,1} + \varphi_{1,2}(x_{1,2}))e_1 + \varphi_{2,1}(x_{2,1}) + \varphi_{2,2}(x_{2,2}))e_2$$

For $x_{i,j} \in M_{i,j}$, and $k = 1, 2$, let us define $\Gamma_{i,j}^k(x_{i,j}) = \Gamma(x_{i,j})(\alpha((e_i)\beta(e_k) \otimes \alpha(e_k)\beta(e_j))$ Then, $G^1 = (M_{1,1}, \Gamma^1_{1,1}, \varphi_{1,1}, \psi_{1,1})$ and $G^2 = (M_{2,2}, \Gamma^2_{2,2}, \varphi_{2,2}, \psi_{2,2})$ are two locally compact quantum groups; if $\alpha(C^2) \subset Z(M)$, then $\beta = \alpha$ and $G = G^1 \oplus G^2$ (i.e. $M_{1,2} = M_{2,1} = \{0\}$).

If $\alpha(e_1) \notin Z(M)$, then $M_{1,2} \neq \{0\}$, $M_{2,1} = R(M_{1,2}) \neq \{0\}$, $\Gamma^2_{1,2} : M_{1,2} \to M_{1,2} \otimes M_{2,2}$ is a right action of $G^2$ on $M_{1,2}$, $\Gamma^1_{1,2} : M_{1,2} \to M_{1,1} \otimes M_{1,2}$ is a left action of $G^1$ on $M_{1,2}$,
7.3. Abelian measured quantum groupoid. We consider now the case of an "abelian" measured quantum groupoid (i.e. a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T')$ where the underlying von Neumann algebra $M$ is abelian); then we prove that it is possible to put on the spectrum of the $C^*$-algebra $A_{\nu}(W)$ a structure of a locally compact groupoid, whose basis is the spectrum of $C^*(\nu)$ ([E5], 7.1). Starting from a measured groupoid equipped with a left-invariant Haar system, we recover Ramsay’s theorem which says that this groupoid is measure-equivalent to a locally compact one (7.3.3).

7.3.1. Theorem. Let $G = (B, A, \alpha, \beta, \nu, T, T', \Gamma)$ be a locally compact quantum groupoid, such that the $C^*$-algebra $A$ (and, therefore, the $C^*$-algebra $B$) is abelian. Let $\mathcal{G}$ be the spectrum of $A$, and $\mathcal{G}^{(0)}$ be the spectrum of $B$. Then:

(i) there are two continuous open applications $r$, $s$ from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ such that, for all $f \in C_0(\mathcal{G}^{(0)})$, we have $\alpha(f) = f \circ r$ and $\beta(f) = f \circ s$.

(ii) there is a partially defined multiplication on $\mathcal{G}$, which gives to $\mathcal{G}$ a structure of locally compact groupoid, with $\mathcal{G}^{(0)}$ as set of units. Then, for any $f \in C_0(\mathcal{G})$, $\Gamma(f) \in C_b(\mathcal{G}^{(2)})$.

(iii) for all $f \in \mathcal{X}(\mathcal{G})$ and $u \in \mathcal{G}^{(0)}$, the application $f \rightarrow \alpha^{-1}(T(f))(u)$ defines a positive Radon measure $\lambda^u$ on $\mathcal{G}$, whose support is $\mathcal{G}^u = \{x \in \mathcal{G}, r(x) = u\}$. The measures $\langle \lambda^u \rangle_{u \in \mathcal{G}^{(0)}}$ are a Haar system on $\mathcal{G}$.

(iv) the trace $\nu$ on $C_0(\mathcal{G}^{(0)})$ is a measure on $\mathcal{G}^{(0)}$, which is quasi-invariant with respect to this Haar system.

(v) we have $G = (C_0(\mathcal{G}^{(0)}), C_0(\mathcal{G}), r_3, s_3, \Gamma_3, (\lambda_u)_{u \in \mathcal{G}^{(0)}}, (\lambda_u)_{u \in \mathcal{G}^{(0)}}, \nu)$, where $r_3(f) = f \circ r$, $s_3 = f \circ s$, $\Gamma_3(f)(x_1, x_2) = f(x_1x_2)$, for any $f$ in $C_0(\mathcal{G})$, $(x_1, x_2) \in \mathcal{G}^{(2)}$, and $\lambda_u$ is the image of $\lambda^u$ by $x \rightarrow x^{-1}$.

Proof. Completely similar to ([E5], 7.1). □

7.3.2. Example. Let $\mathcal{G}$ be a locally compact groupoid, equipped with a left Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$ and a quasi-invariant Radon measure $\nu$ on $\mathcal{G}^{(0)}$. Then, using the same notations as in [2,5], we get that $(C_0(\mathcal{G}^{(0)}), C_0(\mathcal{G}), r_3, s_3, \nu, T_3, T_3^{-1}, \Gamma_3)$ is a locally compact quantum groupoid, and the measured quantum groupoid constructed by [4,7] is then $(L^\infty(\mathcal{G}^{(0)}), \nu), L^\infty(\mathcal{G}, \mu_1), r_3, s_3, \Gamma_3, T_3, T_3^{-1}, \nu)$ where $\mu_1 = \int_{\mathcal{G}^{(0)}} \lambda^u d\nu$ and $\mu_r = \int_{\mathcal{G}^{(0)}} \lambda^u d\nu$.

The canonical locally compact quantum groupoid constructed by [5,6] is then:

$$(C^*(\nu), C^*(\mu_1), r_3, s_3, \nu, T_3, T_3^{-1}, \Gamma_3),$$

where $C^*(\nu)$ (resp. $C^*(\mu_1)$) is the norm closure of $L^1(\mathcal{G}^{(0)}, \nu) \cap L^\infty(\mathcal{G}^{(0)}, \nu)$ (resp. $L^1(\mathcal{G}, \mu_1) \cap L^1(\mathcal{G}, \mu_2) \cap L^\infty(\mathcal{G}, \mu_1)$) in $L^\infty(\mathcal{G}, \nu)$ (resp. in $L^\infty(\mathcal{G}, \mu_1)$), which is not, in general, the initial one.

Moreover, this example shows that there is no hope for a unicity theorem for a locally compact sub-quantum groupoid of a measured quantum groupoid. In particular, if $X$ is a locally compact space, and $\mu$ a Radon measure on $X$ with full support, then $X$ (with $X^{(0)} = X$ and then $X^{(2)} = X$) can be considered as a locally compact groupoid (called space groupoid), which has $id$ as left Haar system, and $\mu$ as quasi-invariant measure. Therefore, $(C_0(X), C_0(X), id, id, \mu, id, id, id, id)$ is a locally compact quantum groupoid. The measured quantum groupoid associated by [4,7] is then $(L^\infty(X, \mu), L^\infty(X, \mu), id, id, id, id, id, id, id, \mu)$. But then, any dense sub-$C^*$-algebra $A$ of $L^\infty(X, \mu)$ such that $A \cap L^1(\mu)$ is dense in $A$ gives another locally compact quantum groupoid.
7.3.3. Ramsay’s theorem. [Ra] Let $\mathcal{G}$ be a measured groupoid, with $\mathcal{G}^{(0)}$ as space of units, and $r$ and $s$ the range and source functions from $\mathcal{G}$ to $\mathcal{G}^{(0)}$, with a Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$ and a quasi-invariant measure $\nu$ on $\mathcal{G}^{(0)}$. Let us write $\mu = \int_{\mathcal{G}^{(0)}} \lambda^ud\nu$. Let $\Gamma_{\mathcal{G}}$, $r_{\mathcal{G}}$, $s_{\mathcal{G}}$ be the morphisms associated in [7.3.3]. Then, there exists a locally compact groupoid $\hat{\mathcal{G}}$, with set of units $\hat{\mathcal{G}}^{(0)}$, with a Haar system $(\hat{\lambda}^u)_{u \in \hat{\mathcal{G}}^{(0)}}$, and a quasi-invariant measure $\hat{\nu}$ on $\hat{\mathcal{G}}^{(0)}$, such that, if $\hat{\mu} = \int_{\hat{\mathcal{G}}^{(0)}} \hat{\lambda}^ud\hat{\nu}$, we get that the abelian measured quantum groupoids $\mathcal{G}(\mathcal{G})$ and $\mathcal{G}(\hat{\mathcal{G}})$ are isomorphic.

7.4. Locally compact transformation groupoid. In [ET] had been defined the ”measured quantum transformation groupoids”, which are quantum groupoids constructed on a crossed product of a von Neumann algebra $N$ by a specific action of a quantum group $G$; more precisely, $N$ must be a braided commutative Yetter-Drinfeld algebra. Frank Taine ([Ta]) had proved that, if the quantum group is constructed from an algebraic quantum group, as defined by Van Daele in [VD], acting on a $*$-algebra, with appropriate axioms, then the crossed product of the action of the quantum group on the norm closure of the algebra is a locally compact quantum transformation groupoid. Here, mimicking this thesis, we obtain the same result for appropriate action of a locally compact quantum group on a $C^*$-algebra. I must thank Frank Taine who gave me new ideas on the subject.

7.4.1. Definition. [ET] (2.4) Let $G = (M, \Gamma, \varphi \circ R)$ be (the von Neumann version of) a locally compact quantum group, and $\hat{G} = (\hat{M}, \hat{\Gamma}, \hat{\varphi} \circ \hat{R})$ its dual. A $G$-Yetter-Drinfeld algebra is a von Neumann algebra $N$, with a left action $\alpha$ of $G$ on $N$, and a left action $\hat{\alpha}$ of $\hat{G}$ on $N$, such that, for all $x \in N$:

$$(id \otimes \alpha)\hat{\alpha}(x) = Ad(\sigma W \otimes 1)(id \otimes \hat{\alpha})\alpha(x)$$

7.4.2. Definition. [ET] (2.5.3) For any action $\alpha$ of a von Neumann algebra $N$, we define the action $\alpha^c$ of $G^c$ on $N^o$ and the action $\alpha^o$ of $G^o$ on $N^o$, where $\alpha^c = (j \otimes \sigma^c)\alpha$, and $\alpha^o = (R \otimes \sigma^o)\alpha$, where, for any $x \in M$, $j(x) = Jx^*J$.

If $(N, \alpha, \hat{\alpha})$ is a $G$-Yetter-Drinfeld’s algebra, then, are equivalent:

(i) $\alpha^c(N^o)$ and $\hat{\alpha}^c(N^o)$ commute;
(ii) $\alpha^o(N^o)$ and $\hat{\alpha}^o(N^o)$ commute.

In that case, we shall say that $(N, \alpha, \hat{\alpha})$ is braided-commutative. Let $\nu$ be a normal faithful semi-finite weight on $N$, and let $U^c_{\nu} = J_P(J \otimes J_P)$ be the standard implementation of $\alpha$ on $H \otimes H_{\nu}$ ([ET]2.2). Let us define an injective anti-$*$-homomorphism $\beta$ by $\beta(x) = U^c_{\nu}\hat{\alpha}^c(x^o)(U^c_{\nu})^*$; then $(N, \alpha, \hat{\alpha})$ is braided-commutative if and only if $\beta(N)$ is included in the crossed product $G \ltimes_{\alpha} N$ of $N$ by $\alpha$.

7.4.3. Definition. [ET] If $(N, \alpha, \hat{\alpha})$ is a braided commutative $G$-Yetter-Drinfel’d algebra, we can construct on the crossed product $G \ltimes_{\alpha} N$ a structure of Hopf bimodule ([ET], 4) by the following way:

On the crossed-product $G \ltimes_{\alpha} N$, let $\hat{\alpha}$ be the dual action of $\hat{G}^o$ given, for $X \in G \ltimes_{\alpha} N$, by

$$\hat{\alpha}(X) = (\hat{W}^o \otimes 1)(1 \otimes X)(\hat{W}^o \otimes 1)$$

For any $\eta \in H$ and $p \in \mathfrak{m}_{\nu}$, the vector $U^c_{\nu}(\eta \otimes J_{\nu} \Lambda_{\nu}(p))$ belongs to $D_{\nu}(H \otimes H_{\nu})$ ([ET]4.3(i)) and we can define a unitary $V_1$ from $(H \otimes H_{\nu}) \otimes_{\alpha} (H \otimes H_{\nu})$ onto $H \otimes H \otimes H_{\nu}$ by (for all $\Xi$ in $H \otimes H_{\nu}$):

$$V_1(\Xi \otimes_{\alpha} U^c_{\nu}(\eta \otimes J_{\nu} \Lambda_{\nu}(p))) = \eta \otimes \beta(p^*) \Xi$$
By (ET, 2.2), we have that $U'_\alpha = J_\nu(\hat{J} \otimes J_\nu)$ and, for $x \in M_\nu$ and $y \in M_{\hat{\varphi}}$, $(y \otimes 1)\alpha(x)$ belongs to $M_\nu$, and $J_\nu \Lambda_\varphi((y \otimes 1)\alpha(x)) = U'_\alpha(\hat{J} \Lambda_{\varphi}(y) \otimes J_\nu \Lambda_\varphi(x))$.

For any $X \in G \ltimes_a N$, let us define $\hat{\Gamma}(X) = V_1^* \alpha(X) V_1$; then $(N, G \ltimes_a N, \alpha, \beta, \hat{\Gamma})$ is a Hopf-bimodule (ET, 4.4). This Hopf-bimodule can be equipped with a co-inverse $\hat{R}$, a left-invariant operator-valued weight $T_\alpha$ and a right-invariant operator-valued weight $\hat{R} \circ T_\beta \circ \hat{R}$ (ET, 5.4); if the automorphism groups $\sigma_\nu^\hat{R}$ and $\sigma_\nu^{\hat{R}^{-1}}$ commute, for all $s, t$ in $R$, then $(N, G \ltimes_a N, \alpha, \beta, \hat{\Gamma}, T_\alpha, \hat{R} \circ T_\alpha \circ \hat{R}, \nu)$ is a measured quantum groupoid (ET, 5.9). If $z \in \hat{M}$ and $x \in N$, we have

$$\hat{\Gamma}((z \otimes 1)\alpha(x)) = V_1^* (\hat{\Gamma}^\alpha(z) \otimes 1) V_1 (\alpha(x) \otimes_\alpha 1)$$

7.4.4. **Definition.** Let $\alpha$ an action of a (von Neumann version of a) locally compact quantum group $G$ on a von Neumann algebra $N$; let $G = (A, \Gamma|_A, \varphi|_A, \varphi \circ R|_A)$ be the C* version of $G$, and $\hat{G} = (\hat{A}, \hat{\Gamma}|_A, \hat{\varphi}|_A, \hat{\varphi} \circ \hat{R}|_A)$ its dual; let $B$ a sub-C*-algebra of $N$, weakly dense in $N$; then $\alpha|_N$ is an action of $G$ on $B$ if $\alpha(B) \subset M(A \otimes B)$.

Let us suppose that the weight $\nu_B$ is semifinite; then $T_{\alpha|_G \ltimes_a B}$ is a semi-finite operator-valued weight from $G \ltimes_a B$ to $\alpha(M(B))$.

7.4.5. **Definition.** Let $G = (M, \Gamma, \varphi, \varphi \circ R)$ be (the von Neumann version of) a locally compact quantum group, $(A, \Gamma|_A, \varphi|_A, \varphi \circ R|_A)$ its C*-version, and $\hat{G} = (\hat{M}, \hat{\Gamma}, \hat{\varphi} \circ \hat{R})$ its dual (von Neumann version) or $(\hat{A}, \hat{\Gamma}|_A, \hat{\varphi}|_A, \hat{\varphi} \circ \hat{R}|_A, \hat{\Gamma}|_A)$ (C* version). Let $(N, \alpha, \hat{\alpha})$ be a $G$-Yetter-Drinfeld algebra as defined in [7.4.1]. Let $B$ be sub-C*-algebra of $N$, such that $\alpha(B) \subset M(A \otimes B)$, and $\hat{\alpha}(B) \subset M(\hat{A} \otimes B)$; then, we say that $(B, \alpha|_B, \hat{\alpha}|_B)$ is a C*-G-Yetter-Drinfel’d algebra. If $(N, \alpha, \hat{\alpha})$ is braided-commutative (as defined in [7.4.2]), we shall say that $(B, \alpha|_B, \hat{\alpha}|_B)$ is braided-commutative if $\beta(B) \subset G \ltimes_a B$ (as defined in [7.4.3]). Then $R(G \ltimes_a B) = G \ltimes_a B$.

7.4.6. **Lemma.** Let $x \in A$, and $y \in A \cap M_{\varphi}$; then, there exist $n_x \in A \cap M_{\varphi}$ and $y_n \in A$ such that, for all $\xi \in H$, $\Gamma(x)(\xi \otimes \Lambda_{\varphi}(y))$ is equal to the norm limit of $\sum_n (y_n \otimes x_n)$.

**Proof.** Let us write $y = y_1 y_2$, with $y_1$ and $y_2$ in $A \cap M_{\varphi}$. We have then

$$\Gamma(x)(\xi \otimes \Lambda_{\varphi}(y)) = \Gamma(x)(1 \otimes y_1) [\xi \otimes \Lambda_{\varphi}(y_2)]$$

By [KV1] 4.4, we know that $\Gamma(x)(1 \otimes y_1)$ belongs to $A \otimes A$, and, therefore, is a norm limit of a sum $\sum_n (y_n \otimes x_n)$, with $x_n \in A$ and $y_n \in A$; then, we get that $\Gamma(x)(\xi \otimes \Lambda_{\varphi}(y))$ is the norm limit of $\sum_n (y_n \otimes \Lambda_{\varphi}(x_1 y_2))$; writing $x_n = x_n y_2$, we get the result. \hfill $\square$

7.4.7. **Proposition.** Let $X \in G \ltimes_a B$, $a \in M_{\varphi}$, $y \in M_{\varphi}$. Then $\hat{\Gamma}(X) \rho_{y_1}^{\alpha}(y \otimes 1)\alpha(a)$ is equal to the norm limit of elements of the form $\rho_{y_1}^{\alpha}(y_1 \otimes 1)\alpha(a)$, $y_1 \in \hat{A} \cap M_{\varphi}$, $a \in B$, $y_1 \in B \cap M_{\varphi}$.

**Proof.** Using [7.4.3], we get that $\hat{\Gamma}((z \otimes 1)\alpha(x)$) $\otimes_\alpha 1) \Lambda_{\varphi}(y \otimes 1)\alpha(a)$) is equal to

$$V_1^* (\hat{\Gamma}^\alpha(z) \otimes 1) V_1 [\alpha(x) \otimes_\alpha U_{\nu}^\alpha(\hat{J} \Lambda_{\varphi}(y) \otimes J_\nu \Lambda_\varphi(a))]$$

and to:

$$V_1^* (\hat{\Gamma}^\alpha(z) \otimes 1)(\hat{J} \Lambda_{\varphi}(y) \otimes (a^*) \alpha(x) \otimes_\alpha 1) = V_1^* (\hat{\Gamma}^\alpha(z) \otimes 1)(\Lambda_{\varphi}(\sigma_\nu^{\hat{R}}(y^*)) \otimes (a^*) \alpha(x) \otimes_\alpha 1)$$
Using \[7.4.9\] applied to the \(C^*\)-version of \(\hat{G}\) and \[7.4.10\] there exist \(z_p \in \hat{A} \cap \mathfrak{M}_\varphi\) and \(y_p \in \hat{A}\), such that it is equal to the norm limit of:

\[
\Sigma_p V_1^* \Lambda_\varphi(z_p) \otimes (y_p \otimes 1) \beta(a^*) \alpha(x) \Xi
\]

Using \[7.4.3\] we get that \((y_p \otimes 1) \beta(a^*) = \tilde{R}[\alpha(a^*) (\tilde{R}(y_p) \otimes 1)]\) is the norm limit of a sum \(\Sigma_n \tilde{R}[(y_{n,p} \otimes 1) \alpha(a_n)] = \Sigma_n \beta(a_n) (\tilde{R}(y_{n,p}) \otimes 1)\), with \(y_{n,p} \in \hat{A}\) and \(a_n \in A \cap \mathfrak{M}_\nu\) such that \(a_n\) is analytic and \(\sigma_{i/2}(a_n^*) \in \mathfrak{M}_\nu\) and, therefore, we get that:

\[
\tilde{R}((z \otimes 1) \alpha(x)) \Xi \otimes_{\nu} \rho_{x^\nu \Lambda_\varphi((y \otimes 1) \alpha(a))}
\]

is equal to the norm limit of:

\[
\Sigma_p \Sigma_n V_1^* \Lambda_\varphi(z_p) \otimes \beta(a_n) (\tilde{R}(y_{n,p}) \otimes 1) \alpha(x) \Xi = \Sigma_p \Sigma_n (\tilde{R}(y_{n,p}) \otimes 1) \alpha(x) \Xi \otimes_{\nu} U_{\nu}^a (\Lambda_\varphi(z_p) \otimes \Lambda_\nu(a_n))
\]

Therefore, we get that \(\tilde{R}((z \otimes 1) \alpha(x)) \rho_{x^\nu \Lambda_\varphi((y \otimes 1) \alpha(a))}\) is the norm limit of:

\[
\Sigma_p \Sigma_n \rho_{x^\nu \Lambda_\varphi((y \otimes 1) \alpha(a))} (\tilde{R}(y_{n,p}) \otimes 1) \alpha(x)
\]

By continuity of \(\tilde{R}\), we obtain the result. \(\square\)

**7.4.8. Lemma.** Let \(y_n \in \hat{A} \cap \mathfrak{M}_\varphi\), such that \((\Lambda_\varphi(y_n))_n\) is an orthonormal basis of \(H\). Let \(X \in G \otimes_{\alpha} B \cap \mathfrak{M}_{\mathcal{T}_\alpha} \cap \mathfrak{M}_\nu\) (resp. \(X' \in \hat{A} \cap \mathfrak{M}_{\mathcal{T}_\alpha} \cap \mathfrak{M}_\nu\)) and \(\epsilon > 0\).

(i) There exists \(N\) in \(\mathbb{N}\), and, for any \(n \in \mathbb{N}\), such that \(1 \leq n \leq N\), there exists \(a_{n,N} \in B\) such that:

\[||X - \Sigma_{n=1}^N (y \otimes 1) \alpha(a_{n,N})|| < \epsilon\]

(ii) There exists \(N\) in \(\mathbb{N}\), and, for any \(n \in \mathbb{N}\), such that \(1 \leq n \leq N\), there exist \(a_n \in B \cap \mathfrak{M}_\nu\) such that:

\[||\Lambda_\varphi(X) - \Sigma_{n=1}^N \Lambda_\varphi(y_n) \otimes \Lambda_\nu(a_n)||^2 < \epsilon\]

(iii) \(\Lambda_{\mathcal{T}_\alpha}(X)\) is the weak limit of \(\Sigma_n \Lambda_\varphi(y_n) \otimes a_n\) and \(\Lambda_{\mathcal{T}_\alpha}(X^*X)\) is the weak limit of \(\alpha(\Sigma_n a_n^* a_n)\).

(iv) Let us suppose that \(\mathfrak{T}_{\varphi}(X^*X)\) belongs to \(\alpha(B)\); then \(\mathfrak{T}_{\varphi}(X^*X)\) is the norm limit of \(\alpha(a_n^* a_n)\) and \(\Lambda_{\mathcal{T}_\alpha}(X)\) is the norm limit of \(\Sigma_n \Lambda_\varphi(y_n) \otimes a_n\).

**Proof.** Using \[7.4.3\] we get that, for all \(\epsilon > 0\), there exist \(z_p \in \hat{A} \cap \mathfrak{M}_\varphi\) and \(b_p \in B \cap \mathfrak{M}_\nu\) such that:

\[\|X - \Sigma_p (z \otimes 1) \alpha(b_p)\| < \epsilon\]

We have \(\Lambda_{\varphi}(\Sigma_p (z \otimes 1) \alpha(b_p)) = \Sigma_p \Lambda_{\varphi}(z) \otimes \Lambda_\nu(b_p)\). Using now the basis \((\Lambda_\varphi(y_n))_n\) we get that there exists \(N\) and \(\lambda_{p,n} \in C\) such that, for all \(1 \leq p \leq P\), we have \(\Lambda_{\varphi}(z_p) = \Sigma_{n=1}^N \lambda_{p,n} \Lambda_\varphi(y_n)\), and, therefore, \(\Lambda_{\varphi}(\Sigma_p (z \otimes 1) \alpha(b_p)) = \Sigma_{p=1}^N \Lambda_\varphi(y_n) \otimes \Lambda_\nu(\Sigma_{p=1}^P \lambda_{p,n} b_p)\). From which we get that \(\Sigma_p (z \otimes 1) \alpha(b_p) = \Sigma_{n=1}^N (y \otimes 1) \alpha(\Sigma_p \lambda_{p,n} b_p)\).

Writing \(a_{n,N} = \Sigma_p \lambda_{p,n} b_p\), we get (i).

There exists a family \(\xi_n\) in \(H\) such that \(\Lambda_{\varphi}(X) = \Sigma_n \Lambda_\varphi(y_n) \otimes \xi_n\). We have \(\tilde{\nu}(X^*X) = \Sigma_n ||\xi_n||^2\). Therefore, there exists \(N\) such that \(\Sigma_n > N ||\xi_n||^2 < \epsilon/2\). And, for all \(n \leq N\), there exist \(a_n \in B \cap \mathfrak{M}_\nu\) such that \(||\xi_n - \Lambda_\nu(a_n)||^2 < \epsilon/2N\). From which we get that:

\[||\Lambda_{\varphi}(X) - \Sigma_{n=1}^N \Lambda_\varphi(y_n) \otimes \Lambda_\nu(a_n)||^2 = ||\Sigma_{n=1}^N \Lambda_\varphi(y_n) \otimes (\xi_n - \Lambda_\nu(a_n))||^2 + ||\Sigma_{n>N} \Lambda_\varphi(y_n) \otimes \xi_n||^2 < \epsilon\]

which is (ii).
Let now $x \in \mathfrak{N}_\nu$; we have, using (ii):

$$\Lambda_{\tau_\nu}(X)\Lambda_\nu(x) = \Lambda_\nu(X\alpha(x)) = \Sigma_n \Lambda_\nu((y_n \otimes 1)\alpha(a_n x)) = \Sigma_n \Lambda_\nu(y_n) \otimes \Lambda_\nu(a_n x) = \Sigma_n \Lambda_\nu(y_n) \otimes a_n \Lambda_\nu(x)$$

from which we get (iii). Let us suppose now that $T_\beta(X^*X)$ belongs to $\alpha(B)$; there exists $b_n \in B \cap \mathfrak{N}_\nu$ such that $T_\beta(X^*X)$ is the norm limit of $\Sigma_n b_n^* b_n$. Let us define $Y = \Sigma_n \Lambda_\nu(y_n) \otimes b_n \in B(H_\nu, H_\nu)$. We get that, for all $\xi \in H_\nu$, we have $||\Lambda_{\tau_\nu}(X)\xi|| = ||Y\xi||$; therefore, there exists an isometry $U \in B(H_\nu)$ such that $UY = \Lambda_{\tau_\nu}(X)$; so, we get that $a_n = (\omega_{\lambda_\nu}(y_n) \otimes id)(U)b_n$; then, we have $\Sigma_{n \in N} a_n^* a_n \leq \Sigma_{n \in N} b_n^* b_n$, and, therefore $\Sigma_n a_n^* a_n$ is norm converging to $T_\beta(X^*X)$, and, then, we deduce that $\Lambda_{\tau_\nu}(X)$ is the norm limit of $\Sigma_n \Lambda_\nu(y_n) \otimes a_n$, which is (iv).

7.4.9. Proposition. Let $X \in \mathcal{G}_\beta \alpha B$, and $Y \in \mathcal{G}_\beta \alpha B \cap \mathfrak{N}_{\tau_\nu} \cap \mathfrak{N}_\nu$; then:

(i) Let us suppose that $T_\beta(Y^*Y) \in \alpha(A)$; then there exist $y_p \in \widehat{A}$, $z_p \in \widehat{A} \cap \mathfrak{N}_\nu$, $a_p \in B$, $x_p \in B \cap \mathfrak{N}_\nu$ such that $\hat{\Gamma}(X)\rho_{\beta, \alpha}(\nu(y_p))$ is the norm limit of $\rho_{\beta, \alpha}(\nu(z_p \otimes \nu(x_p)))(y_p \otimes 1)\alpha(a_p)$.

(ii) Let $a \in \mathfrak{N}_\nu$. Then there exist $y_p \in \widehat{A}$, $z_p \in \widehat{A} \cap \mathfrak{N}_\nu$, $a_p \in B$, $x_p \in B \cap \mathfrak{N}_\nu$ such that $\hat{\Gamma}(X)\rho_{\beta, \alpha}(\nu(y_p))$ is the norm limit of $\rho_{\beta, \alpha}(\nu(z_p \otimes \nu(x_p)))(y_p \otimes 1)\alpha(a_p)$.

(iii) Let $a \in \mathfrak{N}_\nu$, analytic with respect to $\sigma_\nu$. There exist $y_p \in \widehat{A}$, $z_p \in \widehat{A} \cap \mathfrak{N}_\nu$, $a_p \in B$, $x_p \in B \cap \mathfrak{N}_\nu$ such that $\hat{\Gamma}(X)(\beta(a^* \beta \otimes 1)\rho_{\beta, \alpha}(\nu(y_p))$ is the norm limit of $\rho_{\beta, \alpha}(\nu(z_p \otimes \nu(x_p)))(y_p \otimes 1)\alpha(a_p)$.

Proof. Using [4.4.8] (iv) applied to $Y$ and [4.4.7], we get (i). Then, as

$$T((\alpha(a))^*T(Y^* Y)\alpha(a)$$

belongs to $\alpha(A)$, using (i) applied to $Y\alpha(a)$, we get (ii). As we have:

$$\hat{\Gamma}(X)(\beta(a^* \beta \otimes 1)\rho_{\beta, \alpha}(\nu(y_p)) = \hat{\Gamma}(X)(1_{\beta \otimes 1} \alpha(\sigma_{\nu/2}(a^*))(\nu(z_p \otimes (x_p))$$

we get that (ii) implies (iii). Let $\Xi \in H_\nu$; then there exist $a_n$ in $B$, analytic with respect to $\sigma_\nu$, such that $\Xi$ is the norm limit of $\beta(a_n^* \Xi$; then $\hat{\Gamma}(X)\Xi = \beta \otimes \sigma_\nu J_\nu a_{\beta, \alpha}(\nu(y_p))$ is the norm limit of

$$\hat{\Gamma}(X)\beta(a_n^* \Xi) = \beta \otimes \sigma_\nu J_\nu a_{\beta, \alpha}(\nu(y_p))$$

and, using (iii), there exist $y_p^n$, $z_p^n$, $a_p^n$ and $x_p^n$ such that it is the norm limit of elements of the form $(y_p^n \otimes 1)\alpha(a_p^n)\Xi = \beta \otimes \sigma_\nu J_\nu a_{\beta, \alpha}(\nu(z_p^n \otimes 1)\alpha(x_p^n))$. So, we get (iv).

7.4.10. Theorem. Let’s use the notations of [4.4.5]; then:

$$(B, \mathcal{G}_\beta \alpha B, \alpha_B, \beta_B, T_\beta\mathcal{G}_\beta \alpha B, (\hat{\Gamma}T \hat{\Gamma})\mathcal{G}_\beta \alpha B, \Gamma_{\beta \sigma_\nu} B)$$

is a locally compact quantum groupoid.

Proof. Using [4.4.9] (iv), we get that $\hat{\Gamma}(\mathcal{G}_\beta \alpha B) \subseteq \mathcal{G}_\beta \alpha B \otimes \mathcal{G}_\beta \alpha B$, as $\hat{\Gamma}(\mathcal{G}_\beta \alpha B) = \mathcal{G}_\beta \alpha B \otimes \mathcal{G}_\beta \alpha B$.
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