Twisted Tensor Products of $K^n$ with $K^m$

Jack Arce$^1$ · Jorge A. Guccione$^{2,3}$ · Juan J. Guccione$^{2,4}$ · Christian Valqui$^{1,5}$

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Abstract
We find three families of twisting maps of $K^m$ with $K^n$, where $K$ is a field, and we make a detailed study of its properties. One of them is related to truncated quiver algebras, the second one consists of deformations of the first and the third one requires $m = n$ and yields algebras isomorphic to $M_n(K)$.

Keywords Twisted tensor products · Quivers

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Juan J. Guccione
jjgucci@dm.uba.ar

Jack Arce
jarcef@pucp.edu.pe

Jorge A. Guccione
vander@dm.uba.ar

Christian Valqui
cvalqui@pucp.edu.pe

1 Sección Matemáticas, PUCP, Pontificia Universidad Católica del Perú, Av. Universitaria 1801, San Miguel, Lima 32, Perú
2 Facultad de Ciencias Exactas y Naturales, Departamento de Matemática, Universidad de Buenos Aires, Buenos Aires, Argentina
3 Instituto de Investigaciones Matemáticas “Luis A. Santaló” (IMAS), CONICET-Universidad de Buenos Aires, Buenos Aires, Argentina
4 CONICET, Instituto Argentino de Matemática (IAM), Buenos Aires, Argentina
5 Instituto de Matemática y Ciencias Afines (IMCA) Calle Los Biólogos 245. Urb San César. La Molina, Lima 12, Perú
1 Introduction

Let $A$ and $C$ be unitary $K$-algebras, where $K$ is a commutative ring. By definition, a twisted tensor product of $A$ with $C$ over $K$, is an algebra structure defined on $A \otimes_K C$, with unit $1 \otimes 1$, such that the canonical maps $i_A : A \to A \otimes_K C$ and $i_C : C \to A \otimes_K C$ are algebra maps satisfying $a \otimes c = i_A(a)i_C(c)$. This structure was introduced independently in [16] and [20], and it has been formerly studied by many people with different motivations (in addition to the previous references see also [1–5, 9, 13, 17, 21]). A number of examples of classical and recently defined constructions in ring theory fits into this construction. For instance, Ore extensions, skew group algebras, smash products, etcetera (for the definitions and properties of these structures we refer to [18] and [14]). On the other hand, it has been applied to braided geometry and it arises as a natural representative for the product of noncommutative spaces, this being based on the existing duality between the categories of algebraic affine spaces and commutative algebras, under which the cartesian product of spaces corresponds to the tensor product of algebras. Twisted tensor products arise also as a tool for building algebras starting with simpler ones.

Given algebras $A$ and $C$, a basic problem is to determine all the twisted tensor products of $A$ with $C$. To our knowledge, the first paper in which this problem was attacked in a systematic way was [6], in which C. Cibils studied and solved the case $C := K \times K$, when $K$ is a field. In [12], the case $C := K^n$ is analysed and some partial classification result were achieved.

In [7] the authors analyzed in detail the case $\dim(C) = 2$. There are only three possible algebra structures on $C$, namely $C \cong K^2$, $C \cong K[x]/\langle x^2 \rangle$ or $C$ is a quadratic field extension of $K$. The case of $\dim = 2$ serves as a toy model in order to describe a physical system consisting of two parallel, but very close world sheets. In the search of a model that would describe a physical system with more than two interacting world sheets, we have to generalize the results of [7] to higher dimensions. If $C$ is a quadratic extension, then certainly Galois theory would be the canonical generalisation. On the other hand the twisted tensor product with the dual numbers $K[x]/\langle x^2 \rangle$ also was analysed in [10] (classifying completely the twisted tensor products of $K[y]$ with $K[x]/\langle x^2 \rangle$), and twisted tensor products with the higher dimensional algebras $K[x]/\langle x^n \rangle$ were studied in [11].

Thus we focus on the generalization of the remaining case, which means that we assume $C = K^n$. The interpretation in geometrical physics of this case would be to take $n$ copies of a manifold $M$ in Euclidean space that are parallel but very close, so they interact. In the case $C = K^2$ satisfactory classification results in literature (see [6],[7] and [12]) were achieved only in the case when $A = K^m$ and $K$ is a field, although the authors also analyze the product for general $A$ and have general results.

In this paper we also assume that $K$ is a field, and we consider the case $A = K^m$ and $C = K^n$. This corresponds to the case where the manifold $M$ mentioned above consists of $m$ distinct points. The noncommutative phenomena could be caused by limitations on the accuracy of measurement.

It is well known that there is a canonical bijection between the twisted tensor products of $A$ with $C$ and the so called twisting maps $\chi : C \otimes_K A \to A \otimes_K C$. So, each twisting map $\chi$ is associated with a twisted tensor product of $A$ with $C$ over $K$, which will be denoted by $A \otimes_{\chi} C$. 

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It is evident that each $K$-linear map $\chi : K^n \otimes K^m \to K^m \otimes K^n$ determines and is determined by unique scalars $\lambda_{ij}^{kl}$, such that

$$\chi(e_i \otimes f_j) = \sum_{k,l} \lambda_{ij}^{kl} f_k \otimes e_l \quad \text{for all } e_i \text{ and } f_j.$$

Given such a map $\chi$, for all $i, l \in \mathbb{N}_n^*$ and $j, k \in \mathbb{N}_m^*$, we let $A_\chi(i, l) \in M_n(K)$ and $B_\chi(j, k) \in M_m(K)$ denote the matrices defined by

$$A_\chi(i, l)_{kj} := \lambda_{ij}^{kl} =: B_\chi(j, k)_{li}.$$

In Proposition 2.3 we show that $\chi$ is a twisting map if and only if these matrices satisfy certain natural conditions. This translates the problem of finding all twisting maps into a problem of linear algebra. When one tries to find all twisting maps of $K^3$ with $K^3$ using this linear algebra approach, one encounters that nearly all cases of twisting maps have a very special form. We call these twisting maps standard and we prove that the resulting twisted tensor product algebras are isomorphic to certain square zero radical truncated quiver algebras. Moreover, there arises a second type of twisting maps, which we call quasi-standard twisting maps, which yield algebras that are formal deformations of the standard ones. We also construct a third family of twisting maps when $n = m$, and we show that the resulting algebras are isomorphic to $M_n(K)$. These three families cover nearly all twisting maps of $K^3$ with $K^3$. The remaining cases are more difficult to describe. We will determine them in a forthcoming paper.

The paper is organized as follows: in Section 2 we make a quick review of the basic theory of twisting maps and the notion of $(n - 1)$-ary cross product of vectors. The unique new result in this section is Proposition 1.3, in which we obtain a necessary and sufficient condition for a twisting map $\chi : (B \times C) \otimes A \to A \otimes (B \times C)$ to be the extension of a twisting map $\chi_B : B \otimes A \to A \otimes B$. Section 3 is divided in three subsections. In the first one we present several characterizations of the twisting maps of $K^m$ with $K^n$. The main results are Propositions 2.3, 2.5 and 2.11, and Corollary 2.6. In the second one we determine when two twisting maps are isomorphic, and in the third one we introduce several natural representations of a twisted tensor product $K^n \otimes_\chi K^m$ in the matrix algebras $M_n(K)$ and $M_m(K)$. In Section 4 we reprove the results of [6] in our language. In Section 5 we obtain some basic results on the idempotent matrices $A_\chi(i, l)$.

In Section 6 we construct a family of twisting maps $\chi$ of $K^n$ with $K^n$, such that the rank of the matrices $A_\chi(i, l)$ and $B_\chi(j, k)$ are 1, and we prove that their associated twisting tensor products are isomorphic to $M_n(K)$.

In Section 7 we introduce the concepts of standard column and standard twisting map and we prove several results about them. In particular, in Proposition 6.9 we prove that a twisting map $\chi$ is a standard twisting map if and only if the dual map $\tilde{\chi}$, introduced in Remark 2.2, is also, and in Theorem 6.11 we show that the matrices $A_\chi(i, i)$ and $B_\chi(k, k)$ completely determine the twisting map $\chi$ if it is standard.

In Section 8 we introduce the concepts of quasi-standard column and quasi-standard twisting map and we begin the study of their properties. The main results are Theorem 7.13, in which we obtain necessary and sufficient conditions for a quasi-standard column of a pre-twisting (see Definition 6.2) to satisfy all the conditions of Corollary 2.6; Proposition 7.20,
in which we prove that $\chi$ is a quasi-standard twisting map if and only if $\tilde{\chi}$ is also; and Theorem 7.21, in which we obtain necessary and sufficient conditions for a pre-twisting $A = (A(i, l))_{1 \leq i, l \leq m}$ to be a twisting map, under the hypothesis that $(A(i, l))_{1 \leq i, l \leq r}$ defines a twisting map, $(A(i, l))_{i > r, l \leq r}$ is the zero matrix and the $l$-th column of $A$ is quasi-standard for all $l > r$. In Section 8 we classify completely the case of reduced rank 1.

Section 10 is divided in three subsections. In the first one we associate a quiver $\chi Q$ to each standard twisting map $\chi$ of $K^n$ with $K^n$ and we prove that $K^n \otimes_\chi K^m$ is isomorphic to the radical square zero algebra $K\chi Q / \{\chi Q_1^2\}$. The main result of the second one is Corollary 9.15, in which we give a recursive method for construing the quasi-standard twisting maps from the standard ones. In an appendix we list all quasi-standard twisting maps. We made the necessary calculations for this using Theorem 6.11 and this corollary. Finally, in the third one we compute the Jacobson radical of a quasi-standard twisted tensor product $C := K^n \otimes_\chi K^m$ and we prove that there exists a subalgebra $A$ of $C$ such that $C = A \oplus J(C)$.

## 2 Preliminaries

Let $K$ be a field. From now on we assume implicitly that all the maps whose domain and codomain are $K$-vector spaces are $K$-linear maps, that all the algebras are associative and unitary algebras over $K$, and that all the algebra homomorphisms are unital. We set $K^\times := K \setminus \{0\}$. For each natural number $i$, we set $\mathbb{N}_i^\times := \{1, \ldots, i\}$. The tensor product over $K$ is denoted by $\otimes$, without any subscript. Given a matrix $X$, we let $X^T$ denote the transpose matrix of $X$. Moreover, we denote with a juxtaposition the multiplication of two matrices and with a bullet the multiplication in $K^n$. So,

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1b_1, \ldots, a_nb_n).$$

Note that an element $a = (a_1, \ldots, a_n)$ is invertible respect to this multiplication map if and only if $\mu_n(a) := a_1 \cdots a_n \neq 0$. In this case we let $a^{-1}$ denote the inverse $(a_1^{-1}, \ldots, a_n^{-1})$ of $a$. We let $E^{ij} \in M_n(K)$ denote the matrix with 1 in the $(i, j)$-entry and 0 otherwise. So, $\{E^{ij} : 1 \leq i, j \leq n\}$ is the canonical basis of $M_n(K)$. For the sake of simplicity we write $\mathbb{1} = \mathbb{1}_n := 1^n_{K_n}$. Finally, the symbol $\tau_{nm}$ denotes the flip $K^n \otimes K^m \longrightarrow K^n \otimes K^m$.

### 2.1 Twisting Maps

Let $A$ and $C$ be two algebras and let $\mu_A$, $\eta_A$, $\mu_C$ and $\eta_C$ be the multiplication and unit maps of $A$ and $C$, respectively. A twisted tensor product of $A$ with $C$ is an algebra $B$, with underlying vector space $A \otimes C$, such that the canonical maps $i_A : A \longrightarrow A \otimes C$ and $i_C : C \longrightarrow A \otimes C$ are algebra homomorphisms and $\mu \circ (i_A \otimes i_C) = \text{id}_{A \otimes C}$, where $\mu$ denotes the multiplication map of $B$. For a twisted tensor product of $A$ with $C$, the map

$$\chi : C \otimes A \longrightarrow A \otimes C,$$

defined by $\chi := \mu \circ (i_C \otimes i_A)$, satisfies:

1. $\chi \circ (\eta_C \otimes A) = A \otimes \eta_C$,
2. $\chi \circ (C \otimes \eta_A) = \eta_A \otimes C$,
3. $\chi \circ (\mu_C \otimes A) = (A \otimes \mu_C) \circ (\chi \otimes C) \circ (C \otimes \chi)$,
4. $\chi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \chi) \circ (\chi \otimes A)$.

A map satisfying these conditions is called a twisting map of $C$ with $A$. Conversely, if

$$\chi : C \otimes A \longrightarrow A \otimes C$$
is a twisting map, then \( A \otimes C \) becomes a twisted tensor product, denoted \( A \otimes_{\chi} C \), via
\[
\mu_{\chi} := (\mu_A \otimes \mu_C) \circ (A \otimes \chi \otimes C).
\]
Furthermore, these constructions are inverse one to each other.

**Definition 2.1** Let \( \chi : C \otimes A \to A \otimes C \) and \( \chi' : C' \otimes A' \to A' \otimes C' \) be twisting maps. A morphism \( F_{gh} : \chi \to \chi' \), from \( \chi \) to \( \chi' \), is a pair \( (g, h) \) of algebra homomorphisms \( g : C \to C' \) and \( h : A \to A' \), such that \( \chi' \circ (g \otimes h) = (h \otimes g) \circ \chi \).

**Remark 2.2** Let \( \chi \) and \( \chi' \) be as above. If \( F_{gh} : \chi \to \chi' \) is a morphism of twisting maps, then the map \( h \otimes g : A \otimes \chi C \to A' \otimes \chi' C' \) is an algebra homomorphism. Moreover this correspondence is functorial in an evident sense.

**Proposition 2.3** Let \( \chi : (B \times C) \otimes A \to A \otimes (B \times C) \) be a twisting map. Denote by \( \iota_B, \iota_C, p_B, p_C \) the evident inclusions and projections. The map \( \chi_B : B \otimes A \to A \otimes B \), defined by
\[
\chi_B := (A \otimes p_B) \circ \chi \circ (\iota_B \otimes A),
\]
is a twisting map if and only if \( (A \otimes p_B) \circ \chi \circ (\iota_C \otimes A) = 0 \). Moreover, in this case \( F_{p_B, \text{id}_A} \) is a morphism of twisting maps from \( \chi \) to \( \chi_B \). We say that \( p_B(\chi) := \chi_B \) is the twisting map of \( B \) with \( A \) induced by \( \chi \), and that \( \chi \) is an extension of \( \chi_B \).

**Proof** Since \( \chi \) is a twisting map
\[
\chi((1_B, 0) \otimes a) = \chi((1_B, 1_C) \otimes a) - \chi((0, 1_C) \otimes a) = a \otimes (1_B, 1_C) - \chi((0, 1_C) \otimes a).
\]
Consequently, if \( \chi_B \) is also a twisting map, then
\[
a \otimes 1_B = \chi_B((1_B \otimes a) = a \otimes 1_B - (A \otimes p_B) \circ \chi((0, 1_C) \otimes a),
\]
or, equivalently, \( (A \otimes p_B) \circ \chi((0, 1_C) \otimes a) = 0 \). Evaluating now the equalities
\[
\begin{align*}
\begin{array}{c}
\chi_B = \\
\iota_B \otimes A
\end{array}
\end{align*}
\]
in \( (0, 1_C) \otimes (0, c) \otimes a \) for all \( c \in C \) and \( a \in A \), we conclude that \( (A \otimes p_B) \circ \chi \circ (\iota_C \otimes A) = 0 \). We leave to the reader the task to check the other assertions. \( \square \)

### 2.2 Cross Product

We recall that the **cross product** is the \((n - 1)\)-ary operation
\[
(v_1, \ldots, v_{n-1}) \mapsto v_1 \times \cdots \times v_{n-1}
\]
on \( K^n \), determined by
\[
(v_1 \times \cdots \times v_{n-1})x^T = \det \begin{pmatrix} x \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}
\]
for all $x \in K^n$. From this definition it follows immediately that $v_1 \times \cdots \times v_{n-1}$ is orthogonal to the subspace $\langle v_1, \ldots, v_{n-1} \rangle$ generated by $v_1, \ldots, v_{n-1}$, and that $v_1 \times \cdots \times v_{n-1} = 0$ if $v_1, \ldots, v_{n-1}$ are linearly dependent. It is well known (and very easy to check) that

$$v_1 \times \cdots \times v_{n-1} = \det \begin{pmatrix} e_1 & \cdots & e_n \\ v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \cdots & v_{n-1,n} \end{pmatrix},$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $K^n$, $v_j = (v_{j1}, \ldots, v_{jn})$ and the determinant is computed by the Laplace expansion along the first row. From this it follows immediately that if $X$ is the matrix with rows $x_1, \ldots, x_n$ and columns $y_1, \ldots, y_n$, then

$$(y_1^T \times \cdots \times y_j^T \times \cdots \times y_n^T) \cdot e_j = (x_1 \times \cdots \times \hat{x}_j \times \cdots \times x_n) \cdot e_j \quad \text{for all } j,$$

(2.1)

where, as usual, the notation $\hat{a}$ means that the term $a$ is omitted.

**Proposition 2.4** If $x \in K^n$ is invertible, then

$$x \cdot (v_1 \times \cdots \times v_{n-1}) = \mu_n(x) (x' \cdot v_1) \times \cdots \times (x' \cdot v_{n-1}) \quad \text{for all } v_1, \ldots, v_{n-1} \in K^n,$$

where, as at the beginning of this section, $\mu_n(x) = x_1 \cdots x_n$.

**Proof** This assertion is an immediate consequence of the fact that

$$y(x \cdot (v_1 \times \cdots \times v_{n-1}))^T = (x \cdot y)(v_1 \times \cdots \times v_{n-1})^T$$

$$= \det((x \cdot y)^T v_1^T \cdots v_{n-1}^T)$$

$$= \mu_n(x) \det(y^T (x' \cdot v_1)^T \cdots (x' \cdot v_{n-1})^T)$$

$$= \mu_n(x) y((x' \cdot v_1) \times \cdots \times (x' \cdot v_{n-1}))^T$$

for all $y \in K^n$. \hfill $\square$

### 3 Twisted Tensor Products of $K^n$ with $K^m$

Let $\chi : K^m \otimes K^n \rightarrow K^m \otimes K^n$ be a map and let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be the canonical bases of $K^m$ and $K^n$, respectively. There exist unique scalars $\lambda_{ij}^{kl}$ such that

$$\chi(e_i \otimes f_j) = \sum_{k,l} \lambda_{ij}^{kl} f_k \otimes e_l \quad \text{for all } e_i \text{ and } f_j.$$  

(3.1)

Given such a map $\chi$, for all $i, l \in \mathbb{N}_n^*$ and $j, k \in \mathbb{N}_m^*$, we let $A(i, l) \in M_n(K)$ and $B(j, k) \in M_m(K)$ denote the matrices defined by

$$A(i, l)_{kj} := \lambda_{ij}^{kl}:= B(j, k)_{li}.$$  

(3.2)

If necessary we will specify these matrices with a subscript, writing $A_\chi(i, l)$ and $B_\chi(j, k)$. Also, we let $A = A_\chi$ denote the family $(A(i, l))_{i,l \in \mathbb{N}_n^*}$ and $B = B_\chi$ denote the family $(B(j, k))_{j,k \in \mathbb{N}_m^*}$.

**Notation 3.1** For each $i, l \in \mathbb{N}_m^*$ we set $J_i(l) := \{j \in \mathbb{N}_n^* : A(i, l)_{jj} = 1\}$. If there is no danger of confusion (as is the case, for example, when we work with the matrices $A(1, l), \ldots, A(m, l)$ of a fixed column of $A$), we write $J_i$ instead of $J_i(l)$. Similarly, for
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each $i, l \in \mathbb{N}^*_n$ we set $\tilde{J}_u(k) := \{i \in \mathbb{N}^*_m : B(u, k)_{ii} = 1\}$, and we write $\tilde{J}_u$ instead of $\tilde{J}_u(k)$ whenever there is no danger of confusion.

Remark 3.2 Let $\chi : K^m \otimes K^n \rightarrow K^n \otimes K^m$ be a map and let $\tilde{\chi} := \tau_{mn} \circ \chi \circ \tau_{nm}$. An immediate computation shows that $A_{\tilde{\chi}}(j, k)_{ll} = B_{\chi}(j, k)_{ll}$ and $B_{\tilde{\chi}}(i, l)_{lj} = A_{\chi}(i, l)_{lj}$, and that $\tilde{\chi}$ is a twisting map if and only if $\chi$ is. Moreover, in this case the map

$$\theta : K^n \otimes \chi K^m \rightarrow K^m \otimes \tilde{\chi} K^n,$$

defined by $\theta(f_j \otimes e_i) := e_i \otimes f_j$, is an algebra isomorphism. We say that $\chi$ and $\tilde{\chi}$ are dual of each other.

3.1 Characterizations of twisting maps of $K^m$ with $K^n$

In this subsection we give several characterizations of the twisting maps of $K^m$ with $K^n$, in terms of properties of the matrices introduced in Eq. 3.2.

Proposition 3.3 The map $\chi$ is a twisting map if and only if the following facts hold:

1. $\delta_{ii'} A(i, l) = A(i, l) A(i', l)$ for all $i, i'$ and $l$,
2. $\delta_{jj'} B(j, k) = B(j, k) B(j', k)$ for all $j, j'$ and $k$,
3. $A(i, l) 1 = \delta_{il} 1$ for all $i$ and $l$, \[1605\]
4. $B(j, k) 1 = \delta_{jk} 1$ for all $j$ and $k$.

Proof A direct computation shows that

$$\chi \circ (\mu K^m \otimes K^n) = (K^n \otimes \mu K^m) \circ (\chi \otimes K^m) \circ (K^m \otimes \chi),$$

if and only if

$$\delta_{ii'}^{ikl} \delta_{lj} = \sum_{u=1}^n \chi_{i'u} \chi_{ul}$$

for all $i, i', j, k, l$, which is equivalent to Condition (1), and that

$$\chi \circ (K^m \otimes \mu K^n) = (\mu K^n \otimes K^m) \circ (K^n \otimes \chi) \circ (K^m \otimes \chi)$$

if and only if

$$\delta_{jj'}^{jk} \delta_{li}^{kl} = \sum_{u=1}^m \chi_{ij} \chi_{ul}$$

for all $i, j, j', k, l$, which is equivalent to Condition (2). Finally it is easy to check that

$$\chi \circ (K^m \otimes \eta K^n) = \eta K^n \otimes K^m \quad \text{and} \quad \chi \circ (\eta K^m \otimes K^n) = K^n \otimes \eta K^m$$

if and only if Conditions (3) and (4) are fulfilled.

Remark 3.4 Statement (1) says that for each $l \in \mathbb{N}^*_n$, the matrices $A(1, l), \ldots, A(m, l)$ are a family of orthogonal idempotents, and statement (2) says that for each $k \in \mathbb{N}^*_m$, the matrices $B(1, k), \ldots, B(n, k)$ are also a family of orthogonal idempotents. Statement (1) implies that statement (3) holds if and only if $1_n \in \text{Im} A(i, i)$ for all $i$. Similarly, if statement (2) is fulfilled, then statement (4) is true if and only if $1_m \in \text{Im} B(j, j)$ for all $j$. \[\square\]
Proposition 3.5 For each map $\chi : K^m \otimes K^n \to K^n \otimes K^m$ the following assertions hold:

1. $\sum_{i=1}^{m} B(j, k)_{li} = \delta_{jk}$ if and only if $\sum_{i=1}^{m} A(i, l)_{kj} = \delta_{jk}$.
2. $\sum_{j=1}^{n} A(i, l)_{kj} = \delta_{il}$ if and only if $\sum_{j=1}^{n} B(j, k)_{li} = \delta_{il}$.
3. $\delta_{jj'} B(j, k)_{li} = B(j, k)_{li} \iff \sum_{h=1}^{m} A(i, h)_{kj} A(h, l)_{kj} = \delta_{jj'}$ for all $i$ and $l$.
4. $\delta_{ii'} A(i, l)_{kj} = A(i, l)_{kj} \iff \sum_{h=1}^{m} B(j, h)_{li} B(h, k)_{li} = \delta_{ii'}$ for all $j$ and $k$.

Proof Items (1) and (3) follows immediately from the fact that, by Equality (3.2),

$$\sum_{i=1}^{m} B(j, k)_{li} = \sum_{i=1}^{m} A(i, l)_{kj} \quad \text{and} \quad \sum_{h=1}^{m} B(j, k)_{hi} B(j', k)_{hi} = \sum_{h=1}^{m} A(i, h)_{kj} A(h, l)_{kj}.$$ 

The proof of items (2) and (4) are similar. \qed

Corollary 3.6 The map $\chi$ is a twisting map if and only if the following conditions are fulfilled:

1. $\delta_{ii'} A(i, l)_{kj} = A(i, l)_{kj} \iff \sum_{h=1}^{m} B(j, h)_{li} B(h, k)_{li} = \delta_{ii'}$ for all $i$, $j$, $j'$, $k$ and $l$.
2. $\delta_{il} A(i, l)_{kj} = \delta_{il} \iff \sum_{h=1}^{m} B(j, h)_{li} B(h, k)_{li} = \delta_{il}$ for all $i$ and all $l$.
3. $\sum_{i=1}^{m} A(i, l)_{kj} = \text{id}$ for all $l$.
4. $\sum_{h=1}^{m} A(i, h)_{kj} A(h, l)_{kj} = \delta_{jj'} A(i, l)_{kj}$ for all $i$, $j$, $j'$, $k$ and $l$.

Remark 3.7 By Remark 3.2 and the fact that $\chi$ is a twisting map if and only if $\tilde{\chi}$ is, there is a similar corollary with the matrices $A(i, l)$ replaced by the matrices $B(j, k)$.

Remark 3.8 Corollary 3.6(4) says in particular that the vector $(A(i, 1)_{kj}, \ldots, A(i, m)_{kj})$ is orthogonal to the vector $(A(1, l)_{kj}, \ldots, A(m, l)_{kj})$ for each $i$, $j$, $j'$, $k$ and $l$ with $j \neq j'$.

Remark 3.9 Let $X_1, \ldots, X_k \in M_n(K)$ be such that $\sum_{j=1}^{k} X_j = \text{id}_n$. A straightforward computation shows that if $\sum_{j=1}^{k} \text{rk}(X_j) \leq n$, then the $X_i$’s are orthogonal idempotents.

Remark 3.10 If $X_1, \ldots, X_k \in M_n(K)$ are idempotent matrices such that $\sum_{i=1}^{k} X_i = \text{id}_n$, then the $X_i$’s are orthogonal idempotents. In fact, since $\text{rk}(X_i) = \text{Tr}(X_i)$ and

$$\sum_{i} \text{Tr}(X_i) = \text{Tr}\left(\sum_{i} X_i\right) = \text{Tr}(\text{id}) = n,$$

this follows from Remark 3.9.

Proposition 3.11 A map $\chi : K^m \otimes K^n \to K^n \otimes K^m$ is a twisting map if and only if the following conditions are fulfilled:

1. $A(i, l)$ is idempotent for all $i$ and $l$,
2. $\sum_{i=1}^{m} A(i, l) = \text{id}$ for all $l$,
3. $A(i, l) \cdot \text{id} = \delta_{il} \cdot \text{id}$ for all $i$ and all $l$,
4. $\sum_{h=1}^{m} A(i, h)_{kj} A(h, l)_{kj} = A(i, l)_{kj}$ for all $i$, $j$, $k$ and $l$. 

\section*{Remark}
**Proof** By Remark 3.10, Proposition 3.5 and Corollary 3.6.

**Definition 3.12** The matrices $\Gamma_{\chi} \in M_m(K)$, of $A$-ranks, and $\tilde{\Gamma}_{\chi} \in M_n(K)$, of $B$-ranks, are defined by

$$
\Gamma_{\chi} := \begin{pmatrix}
\gamma_{11} & \cdots & \gamma_{1m} \\
\vdots & \ddots & \vdots \\
\gamma_{m1} & \cdots & \gamma_{mm}
\end{pmatrix}
$$

and

$$
\tilde{\Gamma}_{\chi} := \begin{pmatrix}
\tilde{\gamma}_{11} & \cdots & \tilde{\gamma}_{1n} \\
\vdots & \ddots & \vdots \\
\tilde{\gamma}_{n1} & \cdots & \tilde{\gamma}_{nn}
\end{pmatrix},
$$

where $\gamma_{il} := \text{rk}(A(i, l))$ and $\tilde{\gamma}_{jk} := \text{rk}(B(j, k))$.

**Corollary 3.13** If $\chi$ is a twisting map, then the rank matrices have the following properties:

1. $\delta_{il} \leq \gamma_{il} \leq n$ for all $i$ and $l$.
2. $\sum_{i=1}^{m} \gamma_{il} = n$ for all $l$.
3. $\delta_{jk} \leq \tilde{\gamma}_{jk} \leq m$ for all $j$ and $k$.
4. $\sum_{j=1}^{n} \tilde{\gamma}_{jk} = m$ for all $k$.

**Proof** Items (1) and (2) follow from items (1) and (3) of Corollary 3.6, and items (3) and (4), from the corresponding properties of the $B(j, k)$’s.

**Remark 3.14** Let $r < m$ and $n$ be natural numbers. By Proposition 2.3, a twisting map

$$
\chi : (K^r \times K^{m-r}) \otimes K^n \longrightarrow K^n \otimes (K^r \times K^{m-r})
$$

is an extension of a twisting map $\tilde{\chi}$ of $K^r$ with $K^n$ if and only if $\gamma_{il} = 0$ for all $i > r$ and $l \leq r$. Moreover, in this case $A_{\tilde{\chi}} = (A_{\chi}(i, l))_{1 \leq i, l \leq r}$.

### 3.2 Isomorphisms of twisting maps

In this subsection we obtain a necessary a sufficient condition for two twisting maps of $K^m$ with $K^n$ be isomorphic. In this case we also say that they are equivalent.

**Proposition 3.15** Two twisting maps $\chi, \chi' : K^m \otimes K^n \longrightarrow K^n \otimes K^m$ are isomorphic if and only if there exists $\sigma \in S_m$ and $\varsigma \in S_n$ such that

$$
A_{\chi'}(i, l)_{kj} = A_{\chi}(\sigma(i), \sigma(l))_{\varsigma(k)\varsigma(j)} \quad \text{(or, equivalently, } B_{\chi'}(j, k)_{li} = B_{\chi}(\varsigma(j), \varsigma(k))_{\sigma(l)\sigma(i)})
$$

**Proof** By definition $\chi$ and $\chi'$ are isomorphic if and only if there are algebra automorphisms $g : K^m \rightarrow K^m$ and $h : K^n \rightarrow K^n$ such that $\chi' = (h^{-1} \otimes g^{-1}) \circ \chi \circ (g \otimes h)$. Since the automorphisms of $K^n$ and $K^m$ are given by permutation of the entries, there exist $\varsigma \in S_n$ and $\sigma \in S_m$ such that $g(e_i) = e_{\sigma(i)}$ and $h(f_j) = f_{\varsigma(j)}$ for all $i \in \mathbb{N}_m^*$ and $j \in \mathbb{N}_n^*$, and so

$$
\chi'(e_i \otimes f_j) = (h^{-1} \otimes g^{-1})\chi(e_{\sigma(i)} \otimes f_{\varsigma(j)}) = \sum_{k,l} \lambda_{\varsigma(k)\sigma(l)}^{\varsigma(k)\sigma(l)}(h^{-1} \otimes g^{-1})(f_{\varsigma(k)} \otimes e_{\sigma(l)})
$$

Now the result follows immediately from Eqs. 3.1 and 3.2.  

\[\Box\]
3.3 Representations in Matrix Algebras

In this subsection $\chi: K^m \otimes K^n \rightarrow K^n \otimes K^m$ denotes a twisting map and $\lambda_{ij}^{kl}$, $A(i, l)$ and $B(j, k)$ are as at the beginning of Section 3.

Proposition 3.16 For each $1 \leq u \leq m$ the formulas

$$\rho_u(f_j \otimes 1) := E^{lj} \quad \text{and} \quad \rho_u(1 \otimes e_i) := A(i, u)$$

define a representation $\rho_u: K^n \otimes \chi K^m \rightarrow M_n(K)$. Similarly, for each $1 \leq v \leq n$ the formulas

$$\tilde{\rho}_v(1 \otimes e_i) := E^{ii} \quad \text{and} \quad \tilde{\rho}_v(f_j \otimes 1) := B(j, v)$$

define a representation $\tilde{\rho}_v: K^n \otimes \chi K^m \rightarrow M_m(K)$.

Proof Clearly the restriction of $\rho_u$ to $K^n \otimes K \cdot 1 K^m$ is a morphism of algebras. Moreover, by items (1) and (3) of Corollary 3.6, the restriction of $\rho_u$ to $K \cdot 1 K^n \otimes K^m$ is also a morphism of algebras. Consequently, since

$$(1 \otimes e_i) (f_j \otimes 1) = \sum_{k,l} \lambda_{ij}^{kl} (f_k \otimes 1)(1 \otimes e_l),$$

in order to prove that $\rho_u$ defines a representation, it suffices to note that, by Eq. 3.2 and Corollary 3.6(4),

$$\sum_{k,l} \lambda_{ij}^{kl} E^{kk} A(l, u) = \sum_{k,l,s} \lambda_{ij}^{kl} A(l, u)_{ks} E^{ks} = \sum_{k,l,s} A(i, l)_{kj} A(l, u)_{ks} E^{ks} = \sum_k A(i, u)_{kj} E^{kj} = A(i, u) E^{ij}.$$

The proof for $\tilde{\rho}_v$ is similar. \(\square\)

Remark 3.17 We can give a complete description of the image of $\rho_u$ and $\tilde{\rho}_v$. For this, note that if $A(i, u)_{kj} \neq 0$ for some $i$, $j$ and $k$, then $E^{kj} \in \text{Im}(\rho_u)$. In fact,

$$E^{kj} A(i, u)_{kj} = E^{kk} A(i, u) E^{ij} = \rho_u((f_k \otimes 1)(1 \otimes e_i)(f_j \otimes 1)).$$

Hence

$$E^{kj} = \rho_u \left( \frac{(f_k \otimes 1)(1 \otimes e_i)(f_j \otimes 1)}{A(i, u)_{kj}} \right).$$

So, the image of $\rho_u$ is the matrix incidence algebra of the preorder on $\{1, \ldots, n\}$, given by $k \leq j$ if and only if $k = j$ or there exists $i$ such that $A(i, u)_{kj} \neq 0$. Consequently $\rho_u$ is surjective if and only if for all $k \neq j$ there exists $i$ with $A(i, u)_{kj} \neq 0$. Similarly, the image of $\tilde{\rho}_v$ is the matrix incidence algebra of the preorder on $\{1, \ldots, m\}$, given by $l \leq i$ if and only if $l = i$ or there exists $j$ such that $B(j, v)_{li} \neq 0$.

Remark 3.18 Set $x_{ji} := f_j \otimes e_i$. A straightforward computation shows that, in $K^n \otimes \chi K^m$,

$$x_{kl} x_{jl} = \lambda_{ij}^{kl} x_{kl} = A(i, l)_{kj} x_{kl} = B(j, k)_{li} x_{kl}.$$
We also can prove that all two-sided ideals of the algebra $K^n \otimes \chi K^m$ are generated by monomials. In fact, let $I \subseteq K^n \otimes \chi K^m$ be an ideal and let $\sum_{r,s} \alpha_{rs} x_{rs} \in I$. Then

$$(f_j \otimes 1) \left( \sum_{r,s} \alpha_{rs} x_{rs} \right) (1 \otimes e_i) = \sum_{r,s} \alpha_{rs} (f_j \otimes 1)(f_r \otimes 1)(1 \otimes e_s)(1 \otimes e_i) = \alpha_{ji} x_{ji},$$

and so, if $\alpha_{ji} \neq 0$, then $x_{ji} \in I$. This shows that $I$ is linearly generated by a set of elements $x_{ji}$.

4 Twisting Maps of $K^m$ with $K^2$

The proofs given in this section could be lightly simplified using some of the results obtained in Section 5, but we prefer to use the least machinery possible in order to give a flavour of how our methods work, reproducing the beautiful result of Cibils in [6]. Therefore we restrict ourselves to use the results established in the previous sections and the following remark:

Remark 4.1 Let $A \in M_2(K)$ be such that $A^2 = A$. If $A I = I$ and $\text{rk}(A) = 1$, then there exists $a \in K$ such that

$$A = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}. \hspace{1cm} (4.1)$$

The twisting maps of $K^m$ with $K^2$ have been classified by Cibils in terms of colored quivers $Q_{f,\delta}$. In order to obtain his result the first step is to describe the quiver $Q_f$. Consider a map

$$\chi: C \otimes \frac{K[X]}{\langle X(1-X) \rangle} \rightarrow \frac{K[X]}{\langle X(1-X) \rangle} \otimes C,$$

where $C := K^m$. In [6, Section 3] it was proved that $\chi$ is a twisting map if and only if there exists an algebra morphism $f: C \rightarrow C$ and an idempotent derivation $\delta: C \rightarrow fC$ (where $fC$ is $C$ endowed with the $C$-bimodule structure given by $c \cdot c' \cdot c'' := f(c)c'c''$), satisfying

$$f = f^2 + \delta f + f \delta,$$

such that

$$\chi(e_i \otimes X) = X \otimes f(e_i) + 1 \otimes \delta(e_i) = X \otimes (f + \delta)(e_i) + (1 - X) \otimes \delta(e_i),$$

where $(e_i)_{i \in \mathbb{N}_m^*}$ is the canonical basis of $C$. With our notations, we have

$$\chi(e_i \otimes f_1) = \sum_l (\lambda^1_{i1} f_1 \otimes e_l + \lambda^2_{i1} f_2 \otimes e_l) = \sum_l (A(i,l)_{11} f_1 \otimes e_l + A(i,l)_{21} f_2 \otimes e_l),$$

where $f_1$ is the class of $X$ in $k[X]/\langle X(1-X) \rangle$ and $f_2$ is the class of $1 - X$ in $k[X]/\langle X(1-X) \rangle$. Hence

$$f(e_i) = \sum_l (A(i,l)_{11} - A(i,l)_{21}) e_l \hspace{1cm} \text{and} \hspace{1cm} \delta(e_i) = \sum_l A(i,l)_{21} e_l. \hspace{1cm} (4.1)$$

The quiver $Q_f$ in [6] is constructed in the following way. Since $f$ is an algebra map, there exists a unique map $\varphi: \mathbb{N}_m^* \rightarrow \mathbb{N}_m^*$ such that

$$f(e_i) = \sum_{\{l: \varphi(i) = l\}} e_l. \hspace{1cm} (4.2)$$

By definition, $Q_f$ has set of vertices $\mathbb{N}_m^*$ and an arrow from $i$ to $\varphi(i)$ for each $i \in \mathbb{N}_m^*$. As the following proposition shows, we can obtain this quiver directly from our $A$-rank matrix.
Proposition 4.2 Let $\chi$ be a twisting map and let $f$ be as above. The adjacency matrix of the quiver $Q_f$ is $M(\chi) := (\Gamma_\chi - \text{id})^T$, where $\Gamma_\chi$ is as in Definition 3.12.

Proof] Let $l \in \mathbb{N}_m^*$. By Corollary 3.13 we know that $\text{rk}(A(l, l)) = 2$ and $A(i, l) = 0$ for all $i \neq l$, or $\text{rk}(A(l, l)) = 1$ and there exists a unique $i \neq l$ such that $\text{rk}(A(i, l)) = 1$ and $A(j, l) = 0$ for all $j \neq \{i, l\}$. Thus, if $\text{rk}(A(l, l)) = 2$, then $A(l, l) = \text{id}$, and so $A(l, l)_{11} - A(l, l)_{21} = 1$. On the other hand if $\text{rk}(A(l, l)) = 1$, then by Proposition 3.3 and Remark 4.1 there exists $a_l \in K$ such that $A(l, l) = \left(\begin{array}{c} a_l - 1 & a_l \\ a_l & 1 - a_l \end{array}\right)$, and hence $A(l, l)_{11} - A(l, l)_{21} = 0$. Moreover, since $A(i, l) + A(l, l) = \text{id}$, we have $A(i, l) = \left(\begin{array}{c} 1 - a_l & a_l - 1 \\ -a_l & a_l \end{array}\right)$, and so $A(i, l)_{11} - A(i, l)_{21} = 1$. Finally, if $\text{rk}(A(j, l)) = 0$, then (of course) $A(j, l)_{11} - A(j, l)_{21} = 0$.

Consequently, by the first equality in Eq. (4.1) and Equality (4.2),

\[ M(\chi)_{il} = \begin{cases} 1 & \text{if } \varphi(i) = l, \\ 0 & \text{otherwise}, \end{cases} \]

which finishes the proof.

\[ \square \]

Corollary 4.3 A vertex $i$ of $Q_f$ is a loop vertex if and only if $\text{rk}(A(i, i)) = 2$.

In the rest of this section, for each $i \in \mathbb{N}_m^*$ we let $a_i$ denote $A(i, i)_{11}$. We want to determine the possible matrices $A(i, l)$ which can occur in a twisting map of $K^m$ with $K^2$:

1. If $\text{rk}(A(l, l)) = 2$, then $A(l, l) = \text{id}$ and $A(i, l) = 0$ for all $i \neq l$.
2. If $\text{rk}(A(l, l)) = 1$, then there exists $i \neq l$ such that

\[ A(l, l) = \left(\begin{array}{c} a_l - 1 & a_l \\ a_l & 1 - a_l \end{array}\right), \quad A(i, l) = \left(\begin{array}{c} 1 - a_l & a_l - 1 \\ -a_l & a_l \end{array}\right) \quad \text{and} \quad A(h, l) = 0 \forall h \neq \{i, l\}. \]

Now we have several possibilities:

- If $\text{rk}(A(i, i)) = 2$, then $A(l, i) = 0$, and so, by Eq. (3.2) and Proposition 3.3(2),

\[ a_l - a_l^2 = B(1, 1)_{ll} - (B(1, 1)^2)_{ll} = 0, \tag{4.3} \]

which implies that $a_l \in \{0, 1\}$.

- If $\text{rk}(A(i, i)) = 1$, then we have $A(i, i) = \left(\begin{array}{c} a_l - 1 & a_l \\ a_l & 1 - a_l \end{array}\right)$, and, again by Eq. (3.2) and Proposition 3.3(2),

\[ (1 - a_l)(1 - a_l - a_l) = B(1, 1)_{ii} - (B(1, 1)^2)_{ii} = 0 \] (4.4)

and

\[ a_l(a_l + a_l - 1) = B(2, 2)_{ii} - (B(2, 2)^2)_{ii} = 0. \tag{4.5} \]

Hence $a_l + a_l = 1$. If $A(l, i) \neq 0$, then we do not obtain additional conditions on $a_l$, while if $A(l, i) = 0$, then by Eq. (4.3) we have $a_l \in \{0, 1\}$, and so there are only two cases: $a_l = 0$ and $a_l = 1$ or $a_l = 1$ and $a_l = 0$.

Next we recall the definition of a coloration on $Q_f$ in [6, Definition 3.12], but we take the opposite coloration.
**Definition 4.4** A coloration of $Q_f$ is an element $c = \sum c_i e_i \in C$ such that:

1. For a connected component reduced to the round trip quiver with vertices $i$ and $j$ the coefficients $c_i$ and $c_j$ satisfy $c_i + c_j = 1$.
2. For other connected components:
   (a) In case $i$ is a non loop vertex $c_i \in \{0, 1\}$.
   (b) For each arrow having no loop vertex target, one extremity value is 0 and the other one is 1.
   (c) At a loop vertex $i$ we have $c_i = 0$.

Given a twisting map $\chi: K_m \otimes K_2 \rightarrow K_2 \otimes K_m$ consider the matrices $A(i, l) := A(\chi(i, l))$. By Proposition 4.2 and the discussion above Definition 4.4, the element $c := (c_1, \ldots, c_m) \in C$ is a coloration. Conversely, given a coloration $c = (c_1, \ldots, c_m) \in C$ on a one-valued quiver $Q_f$ with set of vertices $\mathbb{N}_m^*$, we can construct matrices $A(i, l) \in M_2(K)$ in the following way: if $l$ is a loop vertex, then $A(l, l) := id$ and $A(i, l) := 0$ for $i \neq l$. Otherwise

- we set $A(l, l) := \begin{pmatrix} a_l & 1 - a_l \\ 1 - a_l & a_l \end{pmatrix}$, where $a_l := c_l$,
- for the target $t(l)$ of the arrow starting at $l$, we set $A(t(l), l) := \begin{pmatrix} 1 - a_l & a_l - 1 \\ -a_l & a_l \end{pmatrix}$,
- for all $i \notin \{t(l), l\}$, we set $A(h, l) := 0$.

In order to verify that these matrices define a twisting map, we must check the hypothesis of Proposition 3.3, where the matrices $B(j, k)$ are defined by Eq. 3.2. Conditions (1) and (3) are satisfied by construction. Condition (2) is equivalent to

$$\sum_i A(i, l)_{kj} = \delta_{jk} \quad \text{for all } l, j \text{ and } k,$$

which holds because

$$\sum_i A(i, l)_{kj} = \begin{cases} A(l, l)_{kj} = \delta_{jk}, & \text{if } \text{rk}(A(l, l)) = 2 \\ A(l, l)_{kj} + A(t(l), l)_{kj} = \delta_{jk}, & \text{if } \text{rk}(A(l, l)) = 1. \end{cases}$$

Finally we check Condition (4), which is equivalent to

$$\delta_{jk} A(i, l)_{kj} = \sum_u A(i, u)_{kj} A(u, l)_{kj} \quad \text{for all } i, j, j', k \text{ and } l. \quad (4.6)$$

When $t(l) = l$, then $A(u, l) = \delta_{ul} id$ for all $u$, which implies that Equality (4.6) holds. Assume that $t(l) \neq l$. We consider three cases: $i = l$, $i = t(l)$ and $i \notin \{t(l), l\}$. If $i = l$, then Equality (4.6) reads

$$\delta_{jk} A(l, l)_{kj} = A(l, l)_{kj} + A(l, l)_{kj} A(t(l), l)_{kj} \quad \text{for all } j, j' \text{ and } k;$$

if $i = t(l)$, then Equality (4.6) reads

$$\delta_{jk} A(t(l), l)_{kj} = A(t(l), l)_{kj} + A(t(l), t(l))_{kj} A(t(l), l)_{kj} \quad \text{for all } j, j' \text{ and } k;$$

and finally, if $i \notin \{t(l), l\}$, then Equality (4.6) reads

$$0 = A(t(l), l)_{kj} A(t(l), l)_{kj} \quad \text{for all } j, j' \text{ and } k.$$

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All these equalities are easily verified using that, since $(c_1, \ldots, c_m)$ is a coloration,

$$A(l, l) = \begin{pmatrix} a_l & 1 - a_l \\ a_l & 1 - a_l \end{pmatrix}, \quad A(t(l), l) = \begin{pmatrix} 1 - a_l & a_l - 1 \\ -a_l & a_l \end{pmatrix},$$

and that

- if $t(t(l)) = t(l)$, then $A(t(t(l)), t(l)) = \text{id}$;
- if $t(t(l)) \neq t(l)$, then $A(t(t(l)), t(l)) = \begin{pmatrix} a_l & -a_l \\ a_l - 1 & 1 - a_l \end{pmatrix}$ and $A(u, t(l)) = 0$ for all $u \notin \{l, t(l)\}$;
- if $t(t(l)) = l$, then $A(l, t(l)) = \begin{pmatrix} a_l & -a_l \\ a_l - 1 & 1 - a_l \end{pmatrix}$ and $A(u, t(l)) = 0$ for all $u \notin \{t(l), t(t(l))\}$.

## 5 Miscellaneous Results

Throughout this section $\chi : K^m \otimes K^n \rightarrow K^n \otimes K^m$ denotes a map and $\lambda_{ij}^{kl}$, $A(i, l)$ and $B(j, k)$ are as at the beginning of Section 3. We also assume that $A(i, l)$ and $B(j, k)$ are idempotent matrices for all $i, l \in \mathbb{N}_m^*$ and $j, k \in \mathbb{N}_n^*$. The following results are useful in our quest of classifying the twisted tensor products $K^n \otimes_{\chi} K^m$.

### 5.1 General Properties

**Remark 5.1** Since the matrices $A(i, l)$ are idempotent, we know that $\text{rk}(A(i, l)) = \text{Tr}(A(i, l))$. Consequently,

$$\text{rk}(A(i, l)) = \sum_j A(i, l)_{jj} = \sum_j B(j, j)_{ii}.$$ 

Similarly, $\text{rk}(B(j, k)) = \sum_i B(j, k)_{ii} = \sum_i A(i, i)_{kj}$.

**Remark 5.2** The rank matrices $\Gamma_{\chi}$ and $\tilde{\Gamma}_{\chi}$, introduced in Definition 3.12, have the same trace. In fact,

$$\text{Tr}(\Gamma_{\chi}) = \sum_i \text{rk}(A(i, i)) = \sum_{i, j} \lambda_{ij}^{ji} = \sum_j \text{rk}(B(j, j)) = \text{Tr}(\tilde{\Gamma}_{\chi}).$$

### 5.2 Standard Idempotent 0, 1-matrices

**Definition 5.3** A 0,1-matrix $A \in M_n(K)$ is called a *standard idempotent 0,1-matrix* if there exist $r \in \mathbb{N}_n^*$ and $C \in M_{n-r \times r}(K)$ with exactly one non-zero entry in each row, such that

$$A = \begin{pmatrix} \text{id}_r & 0 \\ C & 0 \end{pmatrix}, \quad (5.1)$$

where $\text{id}_r$ is the identity of $M_r(K)$.

**Definition 5.4** Two matrices $A, A' \in M_n(K)$ are *equivalent via identical permutations in rows and columns* if there exists a permutation $\sigma \in S_n$ such that $A'_{kj} = A_{\sigma(k)\sigma(j)}$ for all $k, j$. 

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Remark 5.5 A matrix $A \in M_n(K)$ is equivalent via identical permutations in rows and columns to a standard idempotent 0,1-matrix if and only if it is a 0,1-matrix with exactly one nonzero entry in every row, and for each $j \in \mathbb{N}_n^*$ it is true that if $A_{jj} = 0$, then $A_{kj} = 0$ for all $k$.

Notation 5.6 Let $A \in M_n(K)$ be a 0,1-matrix such that $A 1 = 1$. For each $k$ such that $A_{kk} = 0$, we let $c_k = c_k(A)$ denote the unique index such that $A_{k c_k} = 1$.

Proposition 5.7 Let $A \in M_n(K)$ be a 0,1-matrix. If $A$ is idempotent and $A 1 = 1$, then $A$ is equivalent via identical permutations in rows and columns to a standard idempotent 0,1-matrix.

Proof Let $r := \text{Tr}(A)$. Clearly we have $r$ times the entry 1 and $n - r$ times the entry 0 on the diagonal of $A$. Applying an identical permutations in rows and columns we can assume that the 1’s are in the first $r$ entries. Since $A 1 = 1$, each row of this matrix has only one 1, and the other entries are zero. Thus, the first $r$ rows of $A$ are as in Eq. 5.1. Now the fact that $r = \text{rk}(A)$ implies that, again as in Eq. 5.1, the right lower block of $A$ is the zero matrix and its left lower block is a matrix $C$ that satisfies the required properties.

Remark 5.8 If $A$ is as in Proposition 5.7, then $A_{c_k c_k} = 1$ for each $k$ such that $A_{kk} = 0$.

Corollary 5.9 Assume that $\chi$ is a twisting map. If $A(l, l)$ is a 0,1-matrix, then it is equivalent via identical permutations in rows and columns to a standard idempotent 0,1-matrix.

Proposition 5.10 Assume that $\chi$ is a twisting map and let $l \in \mathbb{N}_m^*$. If for all $i \neq l$ it is true that $A(i, l) = 0$ or $A(l, i) = 0$, then $A(l, l)$ is a 0,1-matrix.

Proof By Corollary 3.6(4), the hypothesis implies that $A(l, l)_{kj} = A(l, l)_{kj}^2$ for all $k, j$.

Corollary 5.11 If $\chi$ is a twisting map and $\Gamma_\chi$ is upper or lower triangular, then each one of the matrices $A(l, l)$ is a 0,1-matrix.

Remark 5.12 Proposition 5.10 and Corollaries 5.9 and 5.11 are valid for the matrices $B(j, j)$ (in the second corollary we replace $\Gamma_\chi$ by $\tilde{\Gamma}_\chi$).

5.3 Rank 1 idempotent matrices

Remark 5.13 At the beginning of Section 4 we noted that if $A \in M_2(K)$ satisfies $A^2 = A$, $A 1 = 1$ and $\text{rk}(A) = 1$, then there exists $a \in K$ such that $A = \begin{pmatrix} a & 1 - a \\ a & 1 - a \end{pmatrix}$. More generally, it is clear that if $A \in M_n(K)$ satisfies $A^2 = A$, $A 1 = 1$ and $\text{rk}(A) = 1$, then there exist $a_1, \ldots, a_n \in K$ with $\sum a_j = 1$, such that

$$A = \begin{pmatrix} a_1 & \ldots & a_n \\ \vdots & \ddots & \vdots \\ a_1 & \ldots & a_n \end{pmatrix}.$$
6 Columns of 1's in $\Gamma_{\chi}$

In this section we associate a twisting map $\chi: K^n \otimes K^n \rightarrow K^n \otimes K^n$ to each family $v_1, \ldots, v_n$ of $n$ invertible elements of $K^n$, and we prove that the rank matrix $\Gamma_{\chi}$ is the matrix $J_n$ whose entries are all 1, and that the associated twisted tensor product $K^n \otimes_{\chi} K^n$ is isomorphic to the matrix algebra $M_n(K)$. Note that in general $\Gamma_{\chi} = J_n$ doesn’t imply that $K^n \otimes K^n \cong M_n(K)$.

**Proposition 6.1** Let $\chi: K^n \otimes K^n \rightarrow K^n \otimes K^n$ be a twisting map and let $A(i, l)$ and $B(j, k)$ be as at the beginning of Section 3. If $\text{Diag}(\Gamma_{\chi}) = (1, 1, \ldots, 1)$, then $\Gamma_{\chi} = \tilde{\Gamma}_{\chi}$ is the matrix $J_n$.

**Proof** By Remark 5.2 and Corollary 3.13(3), we know that $\text{Diag}(\tilde{\Gamma}_{\chi}) = (1, 1, \ldots, 1)$. In other words, $\text{rk}(B(j, j)) = 1$ for all $j$. Assume by contradiction that $\Gamma_{\chi} \neq J_n$. Then, by items (1) and (2) of Corollary 3.13 there exist $i, l$ such that $A(i, l) = 0$. Hence, by Remark 5.13 the $i$-th column of $B(j, j)$ is zero for all $j$. But then $\text{Diag}(A(i, i)) = (0, 0, \ldots, 0)$, which, since $A(i, i)$ is idempotent, implies that $A(i, i) = 0$, a contradiction. For $\tilde{\Gamma}_{\chi}$ proceed in a similar way. \hfill $\Box$

**Proposition 6.2** Let $\chi$, $A(i, l)$ and $B(j, k)$ be as in Proposition 6.1 and let $l \in \mathbb{N}^n$. Assume that $\Gamma_{\chi} = J_n$ and that there exists $k$ such that $A(l, k)_{kj} \neq 0$ for all $j$. Let $v = (v_1, \ldots, v_n) \in K^n \setminus \{0\}$. If $v^T \in \text{Im}(A(i, l))$ for some $i$, then $v_k \neq 0$.

**Proof** Since $\text{rk}(A(i, l)) = 1$ and $v^T \in \text{Im}(A(i, l))$, there exists $w = (w_1, \ldots, w_n) \in K^n$ such that $A(i, l)v = v^Tw$. Assume by contradiction that $v_k = 0$. Then $A(i, l)_{kj} = v_kw_j = 0$ for all $j$. By Eq. 3.2 this means that $B(j, k)_{l_1} = 0$ for all $j$, and so

$$\det\begin{pmatrix} B(1, k)_{l_1} & \ldots & B(1, k)_{l_l} & B(1, k)_{l_{n}} \\ \vdots & \ddots & \vdots & \vdots \\ B(n, k)_{l_1} & \ldots & B(n, k)_{l_l} & B(n, k)_{l_{n}} \end{pmatrix} = 0. \quad (6.1)$$

On the other hand, by Remark 3.7 and Proposition 6.1, clearly $(B(1, k), \ldots, B(n, k))$ is a complete family of orthogonal idempotent matrices of rank 1. But then, also $(B(1, k)^T, \ldots, B(n, k)^T)$ is. Since $B(j, k)_{l_1} = A(l, k)_{l_j} \neq 0$ for all $j$, implies that the vector $(B(j, k)_{l_1}, \ldots, B(j, k)_{l_n})^T$ generates $\text{Im}(B(j, k)^T)$ for all $j$, the determinant (6.1) cannot be zero, a contradiction which concludes the proof. \hfill $\Box$

**Theorem 6.3** Let $v_1, \ldots, v_n$ be $n$ invertible elements of $K^n$. If

$$\det(v_1^T \ldots v_n^T) = 1,$$

then there exists a twisting map $\chi: K^n \otimes K^n \rightarrow K^n \otimes K^n$ with

$$A_{\chi}(i, l) := (-1)^{j-1}(v_i^T \cdot v_i)^T(v_i^T \cdot (v_1 \times \ldots \times \hat{v}_i \times \ldots \times v_n)) \quad \text{for all } i \text{ and } l.$$

Moreover $\Gamma_{\chi} = J_n$ and the twisted tensor product algebra $K^n \otimes_{\chi} K^n$ is isomorphic to $M_n(K)$.

**Proof** We assert that the $A_{\chi}(i, l)$’s

1. $A_{\chi}(i, l)A_{\chi}(i', l) = \delta_{ii'}A_{\chi}(i, l)$ for all $i, i'$ and $l$,
2. $A_{\chi}(i, l)1 = \delta_{ii}1$ for all $i$ and $l$,
(3) \( \sum_{i=1}^{n} A_{\chi}(i, l) = \text{id} \) for all \( l \).

In fact, since by Proposition 2.4

\[ v_l \cdot (v_1 \times \cdots \times \widehat{v}_j \times \cdots \times v_n) = \mu_n(v_l) (v_j^* \cdot v_1) \times \cdots \times (v_j^* \cdot v_i) \times \cdots \times (v_j^* \cdot v_n), \]

we have

\[
(v_l \cdot (v_1 \times \cdots \times \widehat{v}_j \times \cdots \times v_n))^T = \mu_n(v_l) \det \begin{pmatrix} v_j^* \cdot v_j \\ v_j^* \cdot v_1 \\ \vdots \\ v_j^* \cdot v_{i-1} \\ v_j^* \cdot v_{i+1} \\ \vdots \\ v_j^* \cdot v_n \end{pmatrix}
\]

\[
= (-1)^{i-j} \mu_n(v_l) \delta_{ij} \det \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}
\]

\[
= (-1)^{i-j} \delta_{ij}
\]

This implies that \( A_{\chi}(i, l) \) is the idempotent with image \( K(v_j^* \cdot v_j)^T \) and kernel \( (v_j^* \cdot v_j)^T : j \neq i \), which implies items (1), (2) and (3) (for item (2) use that \( v_j^* \cdot v_l = 1_{K^n} \)). Note also that

\[
A_{\chi}(i, l)_{jk} = (-1)^{j+k} v_{ij}^{-1} v_{ij} v_{lk} \det
\]

\[
\begin{pmatrix}
v_{11} & \ldots & v_{1,k-1} & v_{1,k+1} & \ldots & v_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
v_{i-1,1} & \ldots & v_{i-1,k-1} & v_{i-1,k+1} & \ldots & v_{i-1,n} \\
v_{i+1,1} & \ldots & v_{i+1,k-1} & v_{i+1,k+1} & \ldots & v_{i+1,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
v_{n1} & \ldots & v_{n,k-1} & v_{n,k+1} & \ldots & v_{nn}
\end{pmatrix}
\]

where we write \( v_j = (v_{j1}, \ldots, v_{jn}) \) for each \( j \). Now we consider the vectors \( w_1, \ldots, w_n \in K^n \) determined by the equality

\[
\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} := (v_1^T \cdots v_n^T),
\]

and we define the matrices

\[
B_{\chi}(j, k) := (-1)^{j-1} (w_k^* \cdot w_j)^T (w_k \cdot (w_1 \times \cdots \times \widehat{w}_j \times \cdots \times w_n)) \quad \text{for all} \ j \text{ and } k.
\]

One checks that \( A_{\chi}(i, l)_{kj} = B_{\chi}(j, k)_{li} \). Moreover, arguing as above for the \( A_{\chi}(i, l)'s \), it can be proven that

\[
B_{\chi}(j, k) B_{\chi}(j', k) = \delta_{jj'} B_{\chi}(j, k) \quad \text{for all} \ j, j' \text{ and } k.
\]
From this it follows immediately that the matrices $A_\chi(i, l)$ satisfy Condition (4) of Corollary 3.6, which finishes the proof of the existence of $\chi$. Clearly $\Gamma_\chi = \mathbb{S}_n$. So it only remains to prove that $K^n \otimes K^n$ is isomorphic to $M_n(K)$. By Remark 3.17 for this it suffices to prove that, for any $l$ and all $k$ and $j$, there exists $i$ such that $A_\chi(i, l)_{jk} \neq 0$, since then the representation $\rho_1$ is a surjective morphism between two algebras of the same dimension, and hence it is an isomorphism. So fix $l, k$ and $j$. From the equality $\sum_i A_\chi(i, l) = \text{id}$ it follows that there exists $i$ such that $A_\chi(i, l)_{kk} \neq 0$. But then

$$A_\chi(i, l)_{jk} = \frac{v_{jk}v_{ij}}{v_{ik}v_{lj}} A_\chi(i, l)_{kk} \neq 0,$$

as desired. $\square$

Of course, the twisting map constructed in Theorem 6.3 is unique by definition. This fact can be improved as the following proposition shows.

**Proposition 6.4** If two twisting maps $\chi$ and $\tilde{\chi}$ with $\Gamma_\chi = \Gamma_{\tilde{\chi}} = \mathbb{S}_n$ satisfy $A_\chi(i, l_0) = A_{\tilde{\chi}}(i, l_0)$ for a fixed $l_0$ and all $i$, and all the entries of $A_\chi(l_0, l_0)$ are non-zero, then $\chi = \tilde{\chi}$.

**Proof** By equalities (3.2),

$$B_\chi(j, k)_{l_0i} = A_\chi(i, l_0)_{kj} = A_{\tilde{\chi}}(i, l_0)_{kj} = B_{\tilde{\chi}}(j, k)_{l_0i} \quad \text{for all } i, j \text{ and } k. \quad (6.2)$$

So, by Proposition 6.1 and Remark 5.13,

$$B_\chi(j, j)_{li} = B_\chi(j, j)_{li} = B_{\tilde{\chi}}(j, j)_{li} = B_{\tilde{\chi}}(j, j)_{li} \quad \text{for all } i, j \text{ and } l.$$

Hence, again by equalities (3.2),

$$A_\chi(i, l)_{jj} = B_\chi(j, j)_{li} = B_{\tilde{\chi}}(j, j)_{li} = A_{\tilde{\chi}}(i, l)_{jj} \quad \text{for all } i, j \text{ and } l.$$ 

Consequently, again by Remark 5.13,

$$A_\chi(l, l)_{kj} = A_\chi(l, l)_{jj} = A_{\tilde{\chi}}(l, l)_{jj} = A_{\tilde{\chi}}(l, l)_{kj} \quad \text{for all } j, k \text{ and } l.$$ 

Therefore, again by equalities (3.2),

$$B_\chi(j, k)_{lj} = A_\chi(l, l)_{kj} = A_{\tilde{\chi}}(l, l)_{kj} = B_{\tilde{\chi}}(j, k)_{lj} \quad \text{for all } j, k \text{ and } l. \quad (6.3)$$

Moreover, by Proposition 6.2 we have $A_\chi(i, l_0)_{kj} \neq 0$ for all $i$, $j$, and $k$, and so

$$B_\chi(j, k)_{l_0i} \neq 0 \quad \text{and} \quad B_{\tilde{\chi}}(j, k)_{l_0i} \neq 0 \quad \text{for all } i, j, k \text{ and } l.$$ 

Since $\text{rk}(B_\chi(j, k)) = 1 = \text{rk}(B_{\tilde{\chi}}(j, k))$, using this and Eqs. 6.2 and 6.3, we obtain that

$$B_\chi(j, k) = B_{\tilde{\chi}}(j, k) \quad \text{for all } j \text{ and } k,$$

which, again by equalities (3.2), implies that $A_\chi(i, l) = A_{\tilde{\chi}}(i, l)$ for all $i$ and $l$. $\square$

**Proposition 6.5** Let $(A_1, \ldots, A_n)$ be a complete family of orthogonal idempotent matrices of $M_n(K)$ and let $l \in \mathbb{N}_n^*$. Assume that $\text{rk}(A_i) = 1$ for all $i$, that the image of each $A_i$ is generated by an invertible element $v_i \in K^n$ and that $A_i\mathbb{I} = \mathbb{I}$. Then there exists a unique twisting map $\chi: K^n \otimes K^n \rightarrow K^n \otimes K^n$ such that $A(i, l)_{\chi} = A_i$ for all $i$.

**Proof** Clearly we can choose $\{v_1, \ldots, v_n\}$ in such a way that $v_i = \mathbb{I}$ and that $\det(v_1^T \ldots v_n^T) = 1$. Since $v_i^T (v_1 \cdots \hat{v_i} \cdots v_n)v_i = (-1)^{l-1} A_i v_i$, it is clear that

$$A_i = (-1)^{l-1} v_i^T (v_1 \cdots \hat{v_i} \cdots v_n) \mathbb{I} = (-1)^{l-1} A_i v_i, \quad \text{for all } i.$$
So, the existence of \( \chi \) follows from Theorem 6.3. Moreover, \( \chi \) is unique by Proposition 6.4.

\[ \square \]

7 Standard Columns and Standard Twisting Maps

In this section we introduce the notion of standard column, which is the most simple type of column that can appear in the matrix \( A_\chi \) of a twisting map \( \chi \) of \( K^m \) with \( K^n \), and we study its properties. We also introduce the notion of standard twisting map of \( K^m \) with \( K^n \), namely, a twisting map \( \chi \) such that all the columns of the matrix \( A_\chi \) are standard, and we also study its properties.

**Definition 7.1** The support of a matrix \( A \in M_n(K) \) is the set

\[ \text{Supp}(A) := \{(i, j) \in \mathbb{N}^*_n \times \mathbb{N}^*_n : a_{ij} \neq 0\}, \]

and the support of the \( k \)-th row of \( A \) is the set \( \text{Supp}(A_{\star k}) := \{j \in \mathbb{N} : a_{kj} \neq 0\} \).

**Definition 7.2** A family \( (A(i, l))_{i, l} \in \mathbb{N}^*_m \) of matrices \( A(i, l) \in M_n(K) \), is called a pre-twisting of \( K^m \) with \( K^n \) if it satisfies Conditions (1), (2) and (3) of Corollary 3.6.

Throughout this section \( A := (A(i, l))_{i, l} \in \mathbb{N}^*_m \) denotes a pre-twisting of \( K^m \) with \( K^n \).

**Definition 7.3** We say that the \( l_0 \)-th column of \( A \) is a standard column if

1. \( A(l_0, l_0) \) is a 0,1-matrix,
2. \( \text{Supp}(A_{\star l_0}) \subseteq \text{Supp}(A(l_0, l_0)) \cup \text{Supp} \text{(id)} \) for all \( i \).

**Remark 7.4** Assume that \( (A(i, l_0))_{i} \in \mathbb{N}^*_m \) is a standard column of \( A \) and let \( k \in \mathbb{N}^*_n \). By Corollary 5.9 the matrix \( A(l_0, l_0) \) is equivalent via identical permutations in rows and columns to a standard idempotent 0,1-matrix. So, all the matrices \( A(i, l_0) \) are equivalent, via identical permutations in rows and columns, to idempotent lower triangular matrices. Consequently,

1. For each index \( i \), we have \( A(i, l_0)_{kk} \in \{0, 1\} \).
2. \( A(i, l_0)_{kk} = 1 \) for exactly one \( i \). We let \( i(k) = i(k, l_0) \) denote this index.
3. If \( i \neq i(k) \) and \( i \neq l_0 \), then \( A(i, l_0)_{kj} = 0 \) for all \( j \).
4. \( A(i, l_0)_{kj} = -1 \) if and only if \( i = i(k) \neq l_0 \) and \( j = c_k(A(l_0, l_0)) \). Moreover, \( A(i, l_0)_{kj} = 0 \) for all \( j \neq \{k, c_k(A(l_0, l_0))\} \).
5. \( A(i, l_0)_{kj} \in \{1, 0, -1\} \) for all \( i, k, j \), and \( A(i, l_0)_{kj} = 1 \) implies \( i = l_0 \) or \( j = k \).

**Remark 7.5** From Remark 7.4 it follows that each standard column \( A(i, l_0)_{i} \in \mathbb{N}^*_m \), of a pre-twisting of \( K^m \) with \( K^n \), can be obtained in the following way:

1. Take a matrix \( A \in M_n(K) \), which is equivalent via identical permutations in rows and columns to a standard idempotent 0,1-matrix, and set \( A(l_0, l_0) := A \).
2. Set \( J_{l_0} := \{k \in \mathbb{N}^*_n : A(l_0, l_0)_{kk} = 1\} \).
(3) For all \( i \in \mathbb{N}_m^* \setminus \{l_0\} \) choose \( J_i \subseteq \mathbb{N}_n^* \setminus J_{l_0} \) such that \( J_i \cap J_{i'} = \emptyset \) if \( i \neq i' \) and \( \bigcup_{i=1}^{m} J_i = \mathbb{N}_n^* \).

(4) For \( i \neq l_0 \) define \( A(i, l_0) \in M_n(K) \) by

\[
A(i, l_0)_{kj} := \begin{cases} 
1 & \text{if } k \in J_i \text{ and } j = k, \\
-1 & \text{if } k \in J_i \text{ and } j = c_k(A(l_0, l_0)), \\
0 & \text{otherwise}.
\end{cases}
\]

Next we generalize the notation introduced in Remark 7.4(2).

**Remark 7.6** Let \( l_0 \in \mathbb{N}_m^* \) and \( k \in \mathbb{N}_n^* \). If \( A(i, l_0)_{kk} \in \{0, 1\} \) for all \( i \), then there is a unique index \( i_0 \), which is denoted \( i(k) = i(k, l_0) = i(k, l_0, \mathcal{A}) \), such that \( A(i_0, l_0)_{kk} = 1 \). So, \( A(i, l_0)_{kk} = \delta_{i_0} \).

**Definition 7.7** We say that a twisting map \( \chi : K^m \otimes K^n \rightarrow K^n \otimes K^m \) is standard if the columns of \( \mathcal{A}_\chi \) are standard columns. In this case we also say that \( \chi \) is standard if and only if \( \chi \) is standard.

**Remark 7.8** Let \( \chi \) and \( \chi' \) be two equivalent twisting maps of \( K^m \) with \( K^n \). Using Proposition 3.15 it is easy to see that \( \chi \) is standard if and only if \( \chi' \) is.

**Proposition 7.9** A map \( \chi \) is a standard twisting map if and only if the map \( \tilde{\chi} \), introduced in Remark 3.2, is.

**Proof** By Remark 3.2 we know that \( \mathcal{A}_{\tilde{\chi}} = \mathcal{B}_\chi \). Thus, since \( \tilde{\chi} \) is a twisting map, we only must check that the \( l_0 \)-th column of \( \mathcal{B}_\chi \) is a standard column for all \( l_0 \in \mathbb{N}_n^* \). Item (1) of Definition 7.3 is an immediate consequence of Remark 7.4(1). For item (2) it suffices to consider the case \( i \neq l_0 \). By Remark 7.4(4) we know that \( B_\chi(i, l_0)_{kk} \in \{0, 1\} \) for all \( j, k \), and that \( B_\chi(i, l_0)_{kj} \neq 1 \) for \( j \neq k \). Since \( \sum_{j=1}^{m} B_\chi(i, l_0)_{kj} = 0 \), this implies that if \( B_\chi(i, l_0)_{kk} = 0 \), then the \( k \)-th row vanishes. Else \( B_\chi(i, l_0)_{kk} = 1 \) and there exists exactly one index \( j' \) such that \( B_\chi(i, l_0)_{kj'} = -1 \). It remains to check that \( j' = c_k(B_\chi(l_0, l_0)) \). Using that \( B_\chi(i, l_0) \) is idempotent, we obtain that

\[
-1 = B_\chi(i, l_0)_{kj'} = \sum_{j=1}^{m} B_\chi(i, l_0)_{kj} B_\chi(i, l_0)_{jj'} = B_\chi(i, l_0)_{kj'} - B_\chi(i, l_0)_{j'j'} = -1 - B_\chi(i, l_0)_{j'j'}.
\]

Set \( i_0 := i(j', l_0, \mathcal{A}_{\tilde{\chi}}) \). Since \( B_\chi(i_0, l_0)_{j'j'} = 1 \) the above equality implies that \( i \neq i_0 \). Thus

\[
0 = \sum_{j=1}^{m} B_\chi(i, l_0)_{kj} B_\chi(i_0, l_0)_{jj'} = B_\chi(i_0, l_0)_{kj'} - B_\chi(i_0, l_0)_{j'j'} = B_\chi(i_0, l_0)_{kj'} - 1,
\]

where the first equality holds because \( B_\chi(i, l_0) B_\chi(i_0, l_0) = 0 \). Therefore \( B_\chi(i_0, l_0)_{kj'} = 1 \), and so \( i_0 = l_0 \), because \( j' \neq k \). Hence, \( j' = c_k(B_\chi(l_0, l_0)) \), as desired. □

**Remark 7.10** Let \( \chi \) be a standard twisting map and let \( i \neq l \) and \( k \neq j \). Then \( A_\chi(i, l)_{kj} = -1 \) if and only if \( B_\chi(k, k)_{ii} = 1 \) and \( A_\chi(l, l)_{kj} = 1 \). In fact, by Remark 7.4(4),

\[
A_\chi(i, l)_{kj} = -1 \Rightarrow B_\chi(k, k)_{ii} = A_\chi(i, l)_{kk} = 1.
\]

Since \( \mathcal{A}_{\tilde{\chi}} = (B_\chi(i, l))_{i, l} \in \mathbb{N}_n^* \) and, by Proposition 7.9 and Remark 3.2, the map \( \tilde{\chi} \) is a standard twisting map, we also have \( A_\chi(l, l)_{kj} = 1 \). Conversely,

\[
1 = B_\chi(k, k)_{ii} = A_\chi(i, l)_{kk} \Rightarrow \exists! j \text{ such that } A_\chi(i, l)_{kj} = -1.
\]
So \( j = c_k(A_X(l, l)) \).

**Theorem 7.11** Let \((A(i))_{i \in \mathbb{N}_m^*}\) and \((B(k))_{k \in \mathbb{N}_n^*}\) be two families of idempotent 0,1-matrices \(A(i) \in M_n(K)\) and \(B(k) \in M_m(K)\) such that, for all \(i\) and \(k\),

1. \(A(i) \mathbb{1} = \mathbb{1}\) and \(B(k) \mathbb{1} = \mathbb{1}\),
2. \(A(i)_{kk} = B(k)_{ii}\).

The family \(A_X = (A_X(i, l))_{i, l \in \mathbb{N}_m^*}\), of matrices \(A_X(i, l) \in M_n(K)\) defined by

\[
A_X(i, l)_{kj} := \begin{cases} 
A(l)_{kj} & \text{if } i = l, \\
B(k)_{li} & \text{if } k = j, \\
-1 & \text{if } i \neq l, k \neq j \text{ and } A(l)_{kj} = B(li) \neq 1, \\
0 & \text{otherwise},
\end{cases}
\]

gives the unique standard twisting map \(\chi : K^m \otimes K^n \rightarrow K^n \otimes K^m\) such that \(A_X(i, i) = A(i)\) and \(B_X(k, k) = B(k)\).

**Proof** The uniqueness holds since the definition of \(A_X\) is forced. Set \(B_X(j, k)_{li} := A_X(i, l)_{kj}\). Note that \(B_X(k, k) = B(k)\). We must check that Conditions (1)–(4) of Proposition 3.3 are fulfilled and that \(\chi\) is standard. For Condition (3) we must verify that

\[
\delta_{il} = \sum_j A_X(i, l)_{kj} \quad \text{for all } i, l \text{ and } k. \tag{7.1}
\]

When \(i = l\) this is true by assumption. When \(i \neq l\) and \(B(k)_{li} = 0\), we have \(A_X(i, l)_{kj} = 0\) for all \(j\), and thus Equality (7.1) is true. Finally, when \(i \neq l\) and \(B(k)_{li} = 1\), we have \(A_X(i, l)_{kk} = 1\), \(A_X(i, l)_{kk} = -1\) (where \(c_k := c_k(A(l))\)) and \(A_X(i, l)_{kj} = 0\) for \(j \neq \{k, c_k\}\), and again Equality (7.1) is true. The proof of Condition (4) is similar. Since \(B_X(j, k)_{li} = A_X(i, l)_{kj}\), Conditions (3) and (4) say that \(\sum_i A_X(i, l) = \id\) and \(\sum_j B_X(j, k) = \id\) for all \(l\) and for all \(k\). Hence, by Remark 3.9, in order to check Condition (1) it suffices to prove that

\[
\sum_i \text{rk}(A_X(i, l)) \leq n \quad \text{for all } l. \tag{7.2}
\]

Fix \(l \in \mathbb{N}_m^*\). Since the \(B(k)\)'s are equivalent, via identical permutations in rows and columns, to a standard idempotent 0,1-matrices, we know that for each \(k\) there exists a unique \(i\) such that \(A_X(i, l)_{kk} = B(k)_{li} = 1\). Thus \(\sum_i \#\{k : A_X(i, l)_{kk} = 1\} = n\). Consequently, to conclude that Inequality (7.2) holds, it is enough to show that

\[
\text{rk}(A_X(i, l)) \leq \#\{k : A_X(i, l)_{kk} = 1\} \quad \text{for all } i.
\]

But, for \(i = l\) we know that \(\text{rk}(A_X(l, l)) = \#\{k : A_X(l, l)_{kk} = 1\}\), because \(A(l)\) is an idempotent 0,1-matrix, while, for \(i \neq l\), from the fact that \(A_X(i, l)_{kk} \in \{0, 1\}\) and \(A_X(i, l)_{kk} = 0\) implies that \(A_X(i, l)_{kj} = 0\) for all \(j\), it follows that \(\#\{k : A_X(i, l)_{kk} = 1\}\) is the number of non zero rows of \(A_X(i, l)\), which is greater than or equal to \(\text{rk}(A_X(i, l))\). This concludes the proof of Condition (1) of Proposition 3.3. The proof of Condition (2) is similar.

**Notation 7.12** For all \(l \in \mathbb{N}_m^*\) we set \(F_0(A, l) := \{k \in \mathbb{N}_n^* : A(i)_l = \delta_{il} \delta_{kj}\},\) for all \(i\) and \(j\), and for all \(i, l \in \mathbb{N}_m^*\) we set \(F(A(i, l)) := \{j \in \mathbb{N}_n^* : A(i)_lj = 1\}\).
Remark 7.13  The set $F(A(i, l))$ was introduced in Notation 3.1, where it was denoted $J_i(l)$, but in some places we prefer to use the longer, but more precise notation, $F(A(i, l))$.

Definition 7.14  We will say that Corollary 3.6(4) is satisfied in the $l_0$-th column of $A$ if

$$
\sum_{h=1}^{m} A(i, h)_{kj} A(h, l_0)_{kj'} = \delta_{j'} A(i, l_0)_{kj} \quad \text{for all } i, j, j' \text{ and } k. \quad (7.3)
$$

Proposition 7.15  If the $l_0$-th column of $A$ is standard, then Corollary 3.6(4) is satisfied in the $l_0$-th column of $A$ if and only if $F(A(v, l_0)) \subseteq F_0(A, v)$ for all $v \in \mathbb{N}_m^n$.

Proof $\Rightarrow$ Let $v \in \mathbb{N}_m^n$ and $k \in \mathbb{N}_m^n$. If $k \in F(A(v, l_0))$, then $A(u, l_0)_{kk} = \delta_{uv}$ for all $u \in \mathbb{N}_m^n$ (see Remark 7.4). So, from Eq. 7.3 with $j = k$, we obtain that

$$
A(i, v)_{kj} = \sum_{u=1}^{m} A(i, u)_{kj} A(u, l_0)_{kk} = \delta_{jk} A(i, l_0)_{kk} = \delta_{jk} \delta_{iv}
$$

for all $i, j$, which says that $k \in F_0(A, v)$, as desired.

$\Leftarrow$ Fix $k \in \mathbb{N}_n^n$. If $i(k, l_0) = l_0$, then $k \in F(A(l_0, l_0)) \subseteq F_0(A, l_0)$, and so Condition (7.3) holds if and only if

$$
A(i, l_0)_{kj} \delta_{jj'} = \delta_{i,j'} A(i, l_0)_{kj} \quad \text{for all } i, j \text{ and } j'.
$$

But this is true for $i \neq l_0$, since then $A(i, l_0)_{kj} = 0$, and also for $i = l_0$, since $A(l_0, l_0)_{kj} = \delta_{kj}$.

If $h_0 := i(k, l_0) \neq l_0$, then Equality (7.3) holds if and only if

$$
A(i, h_0)_{kj} A(h_0, l_0)_{kj'} + A(i, l_0)_{kj} A(l_0, l_0)_{kj'} = \delta_{j'} A(i, l_0)_{kj} \quad \text{for all } i, j \text{ and } j', \quad (7.4)
$$

since, for $h \notin \{h_0, l_0\}$, we have $A(h, l_0)_{kj'} = 0$ for all $j'$. In order to prove that Eq. 7.4 is true, we consider the cases $j = k$, $j = c_k = c_k(A(l_0, l_0))$ and $j \notin \{k, c_k\}$. We will use that $A(i, h_0)_{kj} = \delta_{i,h_0} \delta_{kj}$ for all $i, j$, which is true, because $k \in F(A(h_0, l_0)) \subseteq F_0(A, h_0)$.

If $j = k$, then we must prove that

$$
A(i, h_0)_{kk} A(h_0, l_0)_{kj'} + A(i, l_0)_{kk} A(l_0, l_0)_{kj'} = \delta_{j'} A(i, l_0)_{kk} \quad \text{for all } i \text{ and all } j'.
$$

But this is true, since by the above discussion, Remark 7.4 and Proposition 5.7,

$$
A(i, h_0)_{kk} = \delta_{i,h_0}, \quad A(h_0, l_0)_{kj'} = \delta_{k,j'} - \delta_{j'c_k}, \quad A(i, l_0)_{kk} = \delta_{i,h_0} \text{ and } A(l_0, l_0)_{kj'} = \delta_{j'c_k}.
$$

Since $A(i, h_0)_{kk} = 0$ for all $i$, when $j = c_k$ we are reduced to prove that

$$
A(i, l_0)_{c_k} A(l_0, l_0)_{kj'} = \delta_{c_k,j} A(i, l_0)_{c_k} \quad \text{for all } i \text{ and all } j'.
$$

But this is true, since $A(l_0, l_0)_{kj'} = \delta_{j'c_k}$. Finally, if $j \notin \{k, c_k\}$, then both sides of Eq. 7.4 vanish. Thus, Eq. 7.3 holds in all the cases.

Corollary 7.16  Let $\chi : K^m \otimes K^n \rightarrow K^n \otimes K^m$ be a $k$-linear map such that $A_\chi$ is a pre-twisting. If each column of $A_\chi$ is standard, then $\chi$ is a twisting map if and only if $F(A(i, l)) \subseteq F_0(A, i)$ for all $i, l \in \mathbb{N}_m^n$.

Given sets $X, Y$, in the sequel we let $M_{X,Y}(K)$ denote the set of functions from $X \times Y$ to $K$. We also denote by $\text{id}_X$ the identity matrix in $M_X(K) := M_{X,X}(K)$. 

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8 Quasi-Standard Columns and Quasi-Standard Twisting Maps

In this section we introduce the notions of quasi-standard column and quasi-standard twisting map and we study their properties. As in Section 7, throughout this section \( A = (A(i,l))_{i,l \in \mathbb{N}_m^*} \) denotes a pre-twisting of \( K^m \) with \( K^n \).

**Proposition 8.1** Let \( l \in \mathbb{N}_m^* \) and let \( A(1), \ldots, A(m) \in M_n(K) \) be matrices such that \( A(l) \) is an idempotent 0,1-matrix with \( A(l)1 = 1 \). Set \( J_l := \{ k : A(l)_{kk} = 1 \} \) and \( J_l^c := \mathbb{N}_n^* \setminus J_l \). For each \( i, l \), let

\[
X_i := A(i)|J_l \times J_l, \quad Y_i := A(i)|J_l^c \times J_l^c, \quad U_i := A(i)|J_l^c \times J_l^c, \quad \text{and} \quad W_i := A(i)|J_l \times J_l^c.
\]

The matrices \( A(i)'s \) are orthogonal idempotents satisfying \( \sum_i A(i) = \text{id} \) if and only if the following facts hold:

1. \( X_i = 0 \) for all \( i \neq l \),
2. \( Y_i = 0 \) for all \( i \),
3. \( W_i W_j = \delta_{ij} W_i \) for all \( i \),
4. \( U_i = -W_i U_i \) for all \( i \neq l \),
5. \( \sum_i W_i = \text{id}_{J_l^c} \).

Moreover, if the \( A(i)'s \) satisfy the required conditions, then \( A(l)1 = \delta_{il}1 \).

**Proof** Without loss of generality we can assume that \( J_l = \mathbb{N}_r^* \), where \( r := \text{rk}(A(l)) \). Then

\[
A(l) = \begin{pmatrix} \text{id}_r & 0 \\ U_l & 0 \end{pmatrix} \quad \text{and} \quad A(i) = \begin{pmatrix} X_i & Y_i \\ U_i & W_i \end{pmatrix}.
\]

Let \( i \neq l \). A direct computation shows that \( A(l)A(i) = 0 \) if and only if \( X_i = 0 \) and \( Y_i = 0 \). Under this condition, \( A(i)A(l) = 0 \) if and only if \( U_i = -W_i U_i \). Assuming all the previous conditions for all \( i \neq l \), we have \( A(i)A(j) = \delta_{ij} A(i) \) if and only if \( W_i W_j = \delta_{ij} W_i \), and, under the same conditions, \( \sum_i A(i) = \text{id}_n \) if and only if \( \sum_i W_i = \text{id}_{J_l^c} \). The last assertion follows from the fact that \( U_i = -W_i U_i \) and \( U_i 1_{J_l^c} = 1_{J_l^c} \). \( \square \)

**Definition 8.2** Let \( l_0 \in \mathbb{N}_m^* \). For all \( i, u, v \in \mathbb{N}_m^* \), set \( D^{uv}_{(i),l_0} := D^{uv}_{(i)} := A(i,l_0)|J_u \times J_v \), where \( J_u := J_u(l_0) \) and \( J_v := J_v(l_0) \). We say that \( (A(i,l_0))_{i \in \mathbb{N}_m^*} \) is a quasi-standard column of \( A \) if

1. \( A(l_0,l_0) \) is a 0,1-matrix,
2. \( A(i,l_0)_{kk} \in \{0, 1\} \) for all \( i \) and \( k \),
3. \( D^{uv}_{(i)} = 0 \) if \( u \neq i \) and \( v \neq \{i, l_0\} \),
4. For \( u, i \in \mathbb{N}_m^* \), \( v \in \mathbb{N}_m^* \setminus \{l_0\} \) and \( k \in J_u \), we have \( \# \text{Supp}(D^{uv}_{(i)}) \leq 1 \). Moreover, if \( d \in \text{Supp}(D^{uv}_{(i)}) \), then \( c_d = c_k \), where \( c_d := c_d(A(l_0, l_0)) \) and \( c_k := c_k(A(l_0, l_0)) \).

If necessary we will write \( d^{(v)} \) or \( d^{(v)}_{k} \) instead of \( d \).

**Remark 8.3** Let \( k \in J_l \) and let \( i \neq l_0 \). By items (1) and (2) of Proposition 8.1 we know that \( A(i,l_0)_{kj} = 0 \) for all \( j \). Consequently \( D^{0v}_{(i)} = 0 \) for all \( v \in \mathbb{N}_m^* \). Note that this implies that \( F(A(l_0, l_0)) = F_0(A, l_0) \).

\footnote{Note that \( c_k \) exists since necessarily \( u \neq l_0 \) (see Remark 8.3 and the beginning of Remark 8.7).}
Remark 8.4 Since $\sum_{i \in \mathbb{N}^*} A(i, l_0) = \text{id}$, by item (2) of Definition 8.2 we have $\mathbb{N}^*_m = \bigcup_{i=1}^m J_i$ and $J_i \cap J_{i'} = \emptyset$ if $i \neq i'$. Moreover by Remark 5.1 we know that $\#(J_i) = \text{rk}(A(i, l_0))$ for all $i$.

Remark 8.5 Since $\sum_i A(i, l_0) = \text{id}$, we have $\sum_i D_{(i)}^{uu} = \text{id}$ for all $u \in \mathbb{N}^*_m$, which by Condition (3) implies that $D_{(u)}^{uu} = \text{id}$ for all $u \neq l_0$ (by Proposition 5.7, also $D_{(l_0)}^{l_0 l_0} = \text{id}$).

Remark 8.6 Since $\sum_i A(i, l_0) = \text{id}$, we have $\sum_i D_{(i)}^{uv} = 0$ for all $u \neq v$ in $\mathbb{N}^*_m$, which by Condition (3) implies that $D_{(u)}^{uv} = -D_{(v)}^{uv}$ for all $u \in \mathbb{N}^*_m$ and $v \in \mathbb{N}^*_m \setminus \{u, l_0\}$.

Remark 7.3 is valid for pre-twistings that satisfy Condition (1) of Definition 8.2, while Remarks 8.5 and 8.6 are true for pre-twistings that satisfy Conditions (1) and (3) of the same definition.

Remark 8.7 From the fact that $A(l_0, l_0)$ is a 0,1-matrix it follows immediately that $D_{(l_0)}^{uv} = 0$ for all $u \in \mathbb{N}^*_m$ and $v \in \mathbb{N}^*_m \setminus \{l_0\}$. Combining this with Remarks 8.3 and 8.6 we obtain that Conditions (3) and (4) of Definition 8.2 can be replaced by

(3') $D_{(i)}^{uv} = 0$ if $i \neq l_0$ and $u, v \notin [i, l_0]$,
(4') $\#\text{Supp}(\langle D_{(v)}^{uv}(k) \rangle) \leq 1$ for $u, v \in \mathbb{N}^*_m \setminus \{l_0\}$ and $k \in J_u$. Moreover, if $d \in \text{Supp}(\langle D_{(v)}^{uv}(k) \rangle)$, then $c_d = c_k$, where $c_d := c_d(A(l_0, l_0))$ and $c_k := c_k(A(l_0, l_0))$, respectively.

Remark 8.8 Each standard column of $A$ is quasi-standard.

Example 8.9 Assume for example that $n = 10$, $J_{l_0} = \{1, 2\}$ and $J_i = \{5, 6, 7\}$. If the $l_0$-th column of $A$ is quasi-standard, then the only entries where the matrix $A(i, l_0)$ may have nonzero values are the entries indicated by stars. In this example and in Example 8.12 below, the elements of each family $J_u$ are consecutive, but of course this need not be the case.

\[ A(i, l_0) = \begin{pmatrix}
J_{l_0} & J_i \\
J_{l_0} & J_i \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & 0 & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
\ast & \ast & 0 & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast & \ast & \ast & 0 & 0 & \ast & \ast \\
\end{pmatrix} \]

Lemma 8.10 Assume that the $l_0$-th column of $A$ satisfies Conditions (1)–(3) of Definition 8.2. Take $i, u \in \mathbb{N}^*_m \setminus \{l_0\}$ and $k \in J_u$. If $A(i, l_0)_{kc_k} \neq 0$, then there exist indices $v \in \mathbb{N}^*_m \setminus \{l_0\}$ and $d \in J_v$ such that $D_{(i)}^{uv} = D_{(v)}^{uv} \neq 0$. Moreover, if $u \neq i$, then necessarily $v = i$. 

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Proof By Remark 8.3 we know that \( A(i, l_0)_{j c_k} = 0 \) for all \( j \in J_{l_0} \). So
\[
\sum_{v \in \mathbb{N}^* \setminus \{ l_0 \}} \sum_{d \in J_v} A(i, l_0)_{kd} A(i, l_0)_{dc_k} = \sum_{d \in \mathbb{N}^*} A(i, l_0)_{kd} A(i, l_0)_{dc_k} = A(i, l_0)_{k c_k} \neq 0.
\]
Consequently, there exists \( v \in \mathbb{N}^* \setminus \{ l_0 \} \) and \( d \in J_v \) such that \( (D^u)^{(v)}_{(i)} = A(i, l_0)_{k j} \neq 0 \). The last assertion is true by item (3) of Definition 8.2.

For \( u \in \mathbb{N}^* \setminus \{ l_0 \} \) and \( k \in J_u = J_u(l_0) \), we set
\[
\mathcal{X}_k := \{ v \in \mathbb{N}^* \setminus \{ u, l_0 \} : \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) \neq \emptyset \} \quad \text{and} \quad d^{(\mathcal{X}_k)} := \{ d^{(v)} : v \in \mathcal{X}_k \}.
\]

**Lemma 8.11** Assume that the \( l_0 \)-th column of \( A \) is quasi-standard. Take \( k \in \mathbb{N}^* \setminus \{ l_0 \} \) and write \( u := i(k, l_0) \). The following assertions hold:

1. \( \text{Supp}\left( A(l_0, l_0)_{k} \right) = \{ c_k \} \).
2. If \( v \notin \{ u, l_0 \} \) and \( v \notin \mathcal{X}_k \), then \( \text{Supp}\left( A(v, l_0)_{k} \right) = \emptyset \), while if \( v \notin \{ u, l_0 \} \) and \( v \in \mathcal{X}_k \), then \( \text{Supp}\left( A(v, l_0)_{k} \right) = \{ c_k, d^{(v)} \} \) and \( A(v, l_0)_{k c_k} + A(v, l_0)_{k d^{(v)}} = 0 \).
3. If \( v = u \), then \( \text{Supp}\left( A(v, l_0)_{k} \right) \subseteq \{ k, c_k \} \cup d^{(\mathcal{X}_k)} \).

**Proof** (1) This is clear.

(2) By the very definition of quasi-standard column and Remark 8.6,
\[
\text{Supp}\left( A(v, l_0)_{k} \right) \subseteq \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) \cup \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) \quad \text{and} \quad \# \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) \leq 1.
\]
So, by Corollary 3.6(2) we are reduced to prove that \( \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) \subseteq \{ c_k \} \). Since \( D^u_{(i)} = 0 \) for \( i \notin \{ v, l_0 \} \), we have
\[
(D^u)^{(v)}_{(i)} D^u_{(i)} + D^u_{(i)} D^v_{(i)} = A(v, l_0) A(l_0, l_0) |_{J_u \times J_0} = 0.
\]
Since \( D^u_{(i)} = \text{id} \), this yields
\[
D^u_{(i)} = -D^{v^*}_{(i)} D^v_{(i)}.
\]
Thus, if \( \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) = \emptyset \), then \( \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) = \emptyset \). Else \( \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) = \{ d^{(v)} \} \), and so
\[
(D^u)^{(v)}_{(i)} = -D^{u^*}_{(i)} D^v_{(i)} d^{(v^*)}.
\]
Combining this with the fact that
\[
\text{Supp}\left( (D^u)^{(v)}_{(i)} d^{(v^*)} \right) = \text{Supp}\left( A(l_0, l_0) d^{(v^*)} \right) = \{ c_{d^{(v)}} \} = \{ c_k \},
\]
we obtain that \( \text{Supp}\left( (D^u)^{(v)}_{(i)} \right) = \{ c_k \} \), as desired.

(3) Using that \( A(u, l_0)_{k} = \delta_{k \ast} - \sum_{i \neq u} A(i, l_0)_{k} \) we obtain that
\[
\text{Supp}\left( A(u, l_0)_{k} \right) \subseteq \{ k \} \cup \bigcup_{i \neq u} \text{Supp}\left( A(i, l_0)_{k} \right),
\]
which, combined with items (1) and (2), finishes the proof.
\[\square\]
Example 8.12 The matrices

\[
A(1, 1) := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A(2, 1) := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 - \lambda_1 & 0 & 1 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 - \lambda_2 & 0 & 1 & 0 & 0 & \lambda_2 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda_3 & 0 & 0 & \lambda_3 & 0 & 0 & 0 \\
0 & -\lambda_4 & 0 & 0 & \lambda_4 & 0 & 0 & 0
\end{pmatrix}
\]

and \(A(3, 1) := \text{id} - A(1, 1) - A(2, 1)\) form a quasi-standard column of each pre-twisting of \(K^3\) with \(K^8\) that include them (for instance we can take \(A(1, 2) = A(3, 2) = A(1, 3) = A(2, 3) = 0\) and \(A(2, 2) = A(3, 3) = \text{id}\)). In this example \(J_1 = \{1, 2\}, J_2 = \{3, 4, 5\}\) and \(J_3 = \{6, 7, 8\}\).

Theorem 8.13 Assume that the \(l_0\)-th column of \(A\) is quasi-standard. Then Corollary 3.6(4) is satisfied in the \(l_0\)-th column of \(A\) (that is, Condition (7.3) is fulfilled) if and only if the following conditions hold:

1. \(J_i = F(A(i, l_0)) \subseteq F_0(A, i)\) for all \(i \in \mathbb{N}_m^*\).  
2. If \((D^u_{(u)})_{kd} \neq 0\) and \(u \neq v \neq l_0\), then
   (a) \(A(u, v)_{kj} = \delta_{kj} - \delta_{jd}\) for all \(j\),
   (b) \(A(v, v)_{kj} = \delta_{jd}\) for all \(j\),
   (c) \(A(i, v)_{kj} = 0\) for \(i \neq \{u, v\}\) and for all \(j\).

Proof \(\Rightarrow\) The arguments given in the proof of Proposition 7.15 show that Condition (1) is fulfilled. So we only must prove Condition (2). By Definition 8.2(3)

\[
A(i, l_0)_{kd} = 0 \quad \text{for} \quad i \neq \{u, v\}, \quad (8.1)
\]

which, since \(\sum_i A(i, l_0) = \text{id}\) and \(k \neq d\), implies that

\[
A(v, l_0)_{kd} = -A(u, l_0)_{kd} = -(D^u_{(v)})_{kd} \neq 0. \quad (8.2)
\]

Moreover, by Condition (1) we know that \(k \in J_u \subseteq F_0(A, u)\), and so

\[
A(i, u)_{kj} = \delta_{iu} \delta_{kj} \quad \text{for all} \quad i \quad \text{and} \quad j. \quad (8.3)
\]

By Eqs. 8.1 and 8.3, the Equality (7.3) with \(j' = d\) reads

\[
\delta_{iu} \delta_{kj} A(u, l_0)_{kd} + A(i, v)_{kj} A(v, l_0)_{kd} = \delta_{jd} A(i, l_0)_{kd} \quad \text{for all} \quad i \quad \text{and} \quad j. \quad (8.4)
\]

When \(i = u\), from Eqs. 8.2 and 8.4, we obtain that

\[
\delta_{kj} A(u, l_0)_{kd} - A(u, v)_{kj} A(u, l_0)_{kd} = \delta_{jd} A(u, l_0)_{kd} \quad \text{for all} \quad j,
\]

which gives (a), since \(A(u, l_0)_{kd} \neq 0\). On the other hand, when \(i \neq u\) Equality (8.4) reduces to

\[
A(i, v)_{kj} A(v, l_0)_{kd} = \delta_{jd} A(i, l_0)_{kd} \quad \text{for all} \quad j,
\]

which, combined with Eqs. 8.1 and 8.2, gives items (b) and (c).

Proof \(\Leftarrow\) We must prove that

\[
\sum_{h \in \mathbb{N}_m^*} A(i, h)_{kj} A(h, l_0)_{kj'} = \delta_{jj'} A(i, l_0)_{kj'} \quad \text{for all} \quad i, k, j \quad \text{and} \quad j'. \quad (8.5)
\]
Fix \( k \in \mathbb{N}_n^{*} \) and set \( u := i(k, l_0) \). If \( k \in J_0 \) then \( A(l_0, l_0) = F_0(A(l_0, l_0)) \), then \( A(h, l_0)_{k'} = \delta_{h_0} \delta_{k'} \) for all \( h \) and \( j' \), and Equality (8.5) is trivially true. Consequently we can assume that \( u \neq l_0 \). So, by \( \text{Lemma 8.11} \),

\[
(3) \supp(A(l_0, l_0)_{k*}) = \{c_k\},
\]

(4) If \( h \notin \{u, l_0\} \) and \( h \notin \mathcal{X}_k \), then \( \supp(A(h, l_0)_{k*}) = 0 \), while if \( h \notin \{u, l_0\} \) and \( h \notin \mathcal{X}_k \), then \( \supp(A(h, l_0)_{k*}) = \{c_k, d^{(h)}\} \) and \( A(h, l_0)_{k\{c_k\}} = A(h, l_0)_{k\{d^{(h)}\}} = 0 \),

(5) If \( h = u \), then \( \supp(A(h, l_0)_{k*}) \subseteq \{k, c_k\} \cup \{d^{(h)}\} \).

Thus, if \( j' \neq \{k, c_k\} \cup \{d^{(h)}\} \) both sides of Equality (8.5) are zero. On the other hand, by Condition (1) we have \( A(i, u)_{kj} = \delta_{jk} \delta_{iu} \), and so Eq. (8.5) becomes

\[
\delta_{jk} \delta_{iu} A(u, l_0)_{kj'} + \sum_{h \in \mathbb{N}_n^{*} \setminus \{u\}} A(i, h)_{kj} A(h, l_0)_{kj'} = \delta_{jj'} A(i, l_0)_{kj'} \quad \text{for all } i, j \text{ and } j'.
\]

(8.6)

This equality holds for \( j' = k \), since Conditions (3)–(5) imply \( A(h, l_0)_{k*} = 0 \) for all \( h \in \mathbb{N}_n^{*} \setminus \{u\} \). Assume now that \( j' = d^{(v)} \) with \( v \in \mathcal{X}_k \). By Conditions (3)–(5), we have \( A(h, l_0)_{k\{d^{(v)}\}} = 0 \) if \( h \notin \{v, u\} \). Moreover, by the definition of \( \mathcal{X}_k \) we know that \( v \neq u \), and so by Remark 8.6

\[
A(u, l_0)_{k\{d^{(v)}\}} = -A(v, l_0)_{k\{d^{(v)}\}}.
\]

Hence, for all \( j \) Eq. (8.6) reads

\[
(A(i, v)_{kj} - \delta_{iu} \delta_{jk}) A(v, l_0)_{k\{d^{(v)}\}} = \begin{cases} -\delta_{jd^{(v)}} A(v, l_0)_{k\{d^{(v)}\}} & \text{if } i = u, \\ \delta_{jd^{(v)}} A(v, l_0)_{k\{d^{(v)}\}} & \text{if } i = v, \\ 0 & \text{otherwise.} \end{cases}
\]

Since \( (D_{(u)}^{(v)})_{k\{d^{(v)}\}} \neq 0 \) and \( u \neq v \neq l_0 \), this equality holds by item (2). Assume now that \( j' = c_k \). By Condition (4) we know that \( h \notin \{u, l_0\} \) and \( A(h, l_0)_{kc_k} = 0 \) if and only if \( h \notin \mathcal{X}_k \). Thus, by Condition (2)(c) and the fact that \( A(l_0, l_0)_{kc_k} = 1 \), when \( i \neq u \) Eq. (8.6) reads

\[
\delta_{jk} \delta_{iu} A(u, l_0)_{kc_k} + A(i, l_0)_{kj} + \sum_{h \in [1] \setminus \{u, l_0\}} A(i, h)_{kj} A(h, l_0)_{kc_k} = \delta_{jc_k} A(i, l_0)_{kc_k}.
\]

(8.7)

while, by Condition (2)(a) and the fact that \( A(l_0, l_0)_{kc_k} = 1 \), when \( i = u \) the same equation reads

\[
\delta_{jk} A(u, l_0)_{kc_k} + A(u, l_0)_{kj} + \sum_{h \in \mathcal{X}_k} (\delta_{kj} - \delta_{jd^{(v)}}) A(h, l_0)_{kc_k} = \delta_{jc_k} A(u, l_0)_{kc_k}.
\]

(8.8)

If \( i = l_0 \), then Equality (8.7) holds because \( \delta_{l_0u} = 0 \) and \( A(l_0, l_0)_{kj} = \delta_{jc_k} \). Assume now that \( i \neq \{u, l_0\} \). Since \( \delta_{iu} = 0 \) and \( A(l_0, l_0)_{kc_k} = 1 \), Eq. (8.7) becomes

\[
A(i, i)_{kj} A(i, l_0)_{kc_k} + A(i, l_0)_{kj} = \delta_{jc_k} A(i, l_0)_{kc_k} \quad \text{for all } j.
\]

(8.9)

If \( i \notin \mathcal{X}_k \), then this equality is trivially satisfied, since \( \supp(A(i, l_0)_{k*}) = \emptyset \) by Condition (4). On the other hand, if \( i \in \mathcal{X}_k \), then \( A(i, i)_{kj} = \delta_{jd^{(v)}} \) by Condition (2)(b), and so Equality (8.9) follows easily from the fact that by Condition (4),

\[
\supp(A(i, l_0)_{k*}) = \{c_k, d^{(i)}\} \quad \text{and} \quad A(i, l_0)_{kc_k} + A(i, l_0)_{k\{d^{(i)}\}} = 0.
\]

It remains to consider the case \( i = u \). If \( j = c_k \), then Equality (8.8) is clearly true, while if \( j = k \), then it holds because

\[
\sum_{h \in \mathbb{N}_n^{*}} A(h, l_0)_{kc_k} = 0 \quad \text{and} \quad A(l_0, l_0)_{kc_k} = A(u, l_0)_{k} = 1.
\]
Assume that \( j = d(v) \) with \( v \in \mathcal{X}_k \). In this case Equality (8.8) holds, since, by Condition (4) and Remark 8.6,
\[
A(v, l_0)_{kd} = -A(u, l_0)_{kd} = A(u, l_0)_{kd}.
\]
Finally, if \( j \notin \{ k, c_k \} \cup d(\mathcal{X}_k) \), then Equality (8.8) is trivial.

**Definition 8.14** We say that the \( l_0 \)-th column of \( A \) has reduced rank \( r \) if there are exactly \( r \) indices \( i \neq l_0 \) such that \( A(i, l_0) \neq 0 \). In this case we write \( \text{rrank}_A(l_0) = r \). If \( A \) is associated with a map \( \chi \) as at the beginning of Section 3, then we use \( \text{rrank}_\chi(l_0) \) as a synonym of \( \text{rrank}_A(l_0) \).

**Remark 8.15** Let \( A = (A(i, l))_{i,l \in \mathbb{N}_m} \) be a pre-twisting of \( K^m \) with \( K^n \). If the \( l \)-th column of \( A \) has reduced rank 1 and \( A(l, l) \) is a 0,1-matrix, then the \( l \)-th column of \( A \) is standard.

**Remark 8.16** Let \( l_0, u \in \mathbb{N}_m \) and let \( k \in J_u \). Assume that \( A \) is a family of matrices associated with a twisting map of \( K^m \) with \( K^n \) and that Conditions (1) and (2) of Definition 8.2 are fulfilled for the \( l_0 \)-th column of \( A \). By Remark 7.6 we have \( A(v, l_0)_{kk} = \delta_{uv} \) for all \( v \). Consequently, from Corollary 3.6(4) with \( j' = d \), it follows that
\[
A(i, u)_{kj} = \delta_{iu} \delta_{kj} \quad \text{for all } i \text{ and } j. \tag{8.10}
\]

The following proposition is a variant of Remark 8.15.

**Proposition 8.17** Let \( l_0 \in \mathbb{N}_m^* \). Assume that \( A \) is a family of matrices associated with a twisting map of \( K^m \) with \( K^n \) (and not just with a pre-twisting) and that Conditions (1) and (2) of Definition 8.2 are fulfilled for the \( l_0 \)-th column of \( A \).

(a) If Condition (3) is also fulfilled, then the \( l_0 \)-th column of \( A \) is quasi-standard.

(b) If the reduced rank of the \( l_0 \)-th column of \( A \) is lower than or equal to 2, then the \( l_0 \)-th column of \( A \) is quasi-standard.

**Proof** As in Definition 7.2, for all \( i, u, v \in \mathbb{N}_m \) we set \( D_{uv}^{(i)} := A(i, l_0)|_{J_u \times J_v} \), where \( J_i := J_i(l) \).

(a) We must prove that if Condition (3) of Definition 8.2 is fulfilled, then Condition (4') of Remark 8.7 is also. We begin by proving that
\[
\# \text{Supp}(\{D_{uv}^{(i)}\}_{k*}) \leq 1 \quad \text{for all } u, v \in \mathbb{N}_m \setminus \{l_0\} \text{ and } k \in J_u.
\]

For \( u = v = i \) this is true by Remark 8.5. So, we can assume that \( u \neq v \). Assume on the contrary that there exist \( d_1 \neq d_2 \) in \( \text{Supp}(\{D_{uv}^{(i)}\}_{k*}) \). Since \( A(i, l_0)_{kd_1} = 0 \) for \( i \neq \{u, v\} \) and \( A(v, l_0)_{kd_1} = -A(u, l_0)_{kd_1} \neq 0 \), Corollary 3.6(4) with \( i = u, l = l_0 \) and \( j = j' = d_1 \), gives
\[
A(v, v)_{kd_1} - A(v, u)_{kd_1} = 1.
\]

A similar argument shows that Corollary 3.6(4) with \( i = u, l = l_0, j = d_1 \) and \( j' = d_2 \), gives
\[
A(v, v)_{kd_1} - A(v, u)_{kd_1} = 0,
\]
a contradiction.

It remains to check that if \( d \in \text{Supp}(\{D_{uv}^{(i)}\}_{k*}) \), then \( c_d = c_k \). When \( v = u \) this follows again from Remark 8.5. Assume that \( v \neq u \). We assert that
\[
\text{Supp}(A(v, l_0)_{k*}) = \{d, c_d\}. \tag{8.11}
\]
Since $\text{Supp}((D_{(v)}^{uv})_{k_+}) = \{d\}$ and $D_{(v)}^{ui} = 0$ for $i \notin \{v, l_0\}$, this is true if and only if

$$\text{Supp}((D_{(v)}^{ul_0})_{k_+}) = \{c_d\}.$$  

In order to check this, note that $A(v, l_0)A(l_0, l_0) = 0$ implies

$$D_{(v)}^{ul_0}D_{(l_0)}^{ul_0} + D_{(v)}^{uv}D_{(l_0)}^{vl_0} = \sum_i D_{(v)}^{ui}D_{(l_0)}^{il_0} = 0.$$  

Since $D_{(l_0)}^{l_0} = \text{id}$ and $\text{Supp}((D_{(v)}^{uv})_{k_+}) = \{d\}$, this yields $(D_{(v)}^{ul_0})_{k_+} = -(D_{(v)}^{uv})_{kd}(D_{(l_0)}^{vl_0})_{d}$. Combining this with the fact that

$$\text{Supp}((D_{(l_0)}^{vl_0})_{d}) = \text{Supp}(A(l_0, l_0)_{d_+}) = \{c_d\},$$

we obtain that $\text{Supp}((D_{(v)}^{ul_0})_{k_+}) = \{c_d\}$, as we need.

By Lemma 8.10, if $A(h, l_0)_{kck} \neq 0$, then $h \in \{u, v, l_0\}$. So Corollary 3.6(4) with $j = d$, $j' = c_k$ and $i = v$, gives

$$A(v, l_0)_{kd} + A(v, v)_{kd}A(v, l_0)_{kck} + A(v, u)_{k}A(u, l_0)_{kck} = 0,$$

where we use that $A(l_0, l_0)_{kck} = 1$. But by Eq. 8.10 we have $A(v, u)_{kd} = \delta_{vu}\delta_{kd} = 0$, and so, necessarily $A(v, l_0)_{kck} \neq 0$, which, by Equality (8.11), implies $c_k = c_d$.  

(b) If the reduced rank of the $l_0$-th column of $A$ is lower than 2, then that column is standard and the result is trivial (see Remark 8.8). So we can assume that its reduced rank is 2. By item (a) and Remark 8.7 it suffices to prove that $D_{(v)}^{ul_0} = 0$, if $i \neq l_0$ and $u, v \notin \{i, l_0\}$. Since the reduced rank of the $l_0$-th column is 2, there exist two indices $i_0, i_1 \neq l_0$ such that $A(i, l_0) \neq 0$ if and only if $i \in \{l_0, i_0, i_1\}$. So we must prove that $D_{(l_0)}^{ib_0} = 0$ for $a \in \{0, 1\}$ and $b := 1 - a$. Take $k \in J_{ib}$. We first prove that either

$$\text{Supp}(A(i_b, l_0)_{k_+}) \subseteq \{k, c_k\} \quad \text{or} \quad \exists! d \text{ such that } d \in \text{Supp}(A(i_b, l_0)_{k_+}) \setminus \{k, c_k\}. \quad (8.12)$$  

Assume by contradiction that there exist $d \neq e \in \text{Supp}(A(i_b, l_0)_{k_+}) \setminus \{k, c_k\}$. First note that, since $A(l_0, l_0) + A(i_a, l_0) + A(i_b, l_0) = \text{id}$ and $\text{Supp}(A(l_0, l_0)_{k_+}) = \{c_k\}$, if $f \notin \{k, c_k\}$, then

$$A(i_a, l_0)_{kf} = -A(i_b, l_0)_{kf}. \quad (8.13)$$

By Eq. 8.10 we know that $A(i_b, i_b)_{kd} = 0$. Moreover, since $\text{Supp}(A(l_0, l_0)_{k_+}) = \{c_k\}$,

$$A(l_0, l_0)_{kd} = A(l_0, l_0)_{ke} = 0.$$  

Consequently from Corollary 3.6(4) with $j = j' = d$ and $i = i_b$, we obtain that

$$A(i_b, i_a)_{ke}A(i_a, l_0)_{kd} = \sum_u A(i_b, u)_{kd}A(u, l_0)_{kd} = A(i_b, l_0)_{kd} \neq 0,$$

which implies $A(i_b, i_a)_{kd} \neq 0$. On the other hand from Corollary 3.6(4) with $j = d$, $j' = e$ and $i = i_b$, we obtain that

$$A(i_b, i_a)_{kd}A(i_a, l_0)_{ke} = \sum_u A(i_b, u)_{kd}A(u, l_0)_{ke} = 0,$$

and so, necessarily $A(i_a, l_0)_{ke} = 0$. But this is impossible since $A(i_a, l_0)_{ke} = -A(i_b, l_0)_{ke} \neq 0$. Hence Condition (8.12) is satisfied. We claim that if it exists, then $d \in J_{i_a}$. In fact, since $k \in J_{ib}$ we have $A(i_a, l_0)_{kk} = 0$, and thus, by Equality (8.13), if
\( d \in \text{Supp}(A(i_b, l_0)_{k^*}) \setminus \{k, c_k\}, \) then \( \text{Supp}(A(i_a, l_0)_{k^*}) = \{c_k, d\}. \) Using now that \( A(i_a, l_0) \)

is idempotent, we obtain that

\[
A(i_a, l_0)_{kd} = A(i_a, l_0)_{k^*}A(i_a, l_0)_{sd}
\]

\[
= A(i_a, l_0)_{kd}A(i_a, l_0)_{ck d} + A(i_a, l_0)_{kd}A(i_a, l_0)_{dd}
\]

\[
= A(i_a, l_0)_{kd}A(i_a, l_0)_{dd},
\]

since \( A(i_a, l_0)_{ck d} = 0 \) by Remark 8.3. But then \( A(i_a, l_0)_{dd} = 1 \), which means that \( d \in J_{ia} \), as we claim. Thus

\[
\text{Supp}(A(i_a, l_0)_{k^*}) \subseteq J_{l_0} \cup J_{ia},
\]

which implies that \( \text{Supp}((D(i_a)_{k^*})) = \text{Supp}(A(i_a, l_0)_{k^*}) \cap J_{i_b} = \emptyset \) for all \( k \in J_{i_b} \), as desired.

**Definition 8.18** We say that a twisting map \( \chi : K^n \otimes K^n \to K^n \otimes K^n \) is quasi-standard if the columns of \( A_\chi \) are quasi-standard.

**Remark 8.19** Let \( \chi \) and \( \chi' \) be two equivalent twisting maps of \( K^n \) with \( K^n \). Using Proposition 3.15 it is easy to see that \( \chi \) is quasi-standard if and only if \( \chi' \) is.

**Proposition 8.20** A twisting map \( \chi : K^n \otimes K^n \to K^n \otimes K^n \) is quasi-standard if and only if the twisting map \( \tilde{\chi} \), introduced in Remark 3.2, is.

**Proof** By Proposition 8.17, Remark 3.2 and the fact that \( \chi \) is a twisting map if and only if \( \tilde{\chi} \) is, in order to prove the proposition it suffices to check that if every column of \( A_\chi \) is quasi-standard, then each column of \( A_{\tilde{\chi}} = B_\chi \) satisfies items (1), (2) and (3) of Definition 8.2. Assume that each column of \( A_\chi \) is quasi-standard. Using Equality (3.2) it is easy to check that items (1) and (2) are satisfied by the columns of \( B_\chi \). Consequently, by Remark 8.7 we only must prove that \( \tilde{D}^{uw}_{ij} := B_\chi(j,k)|_{\tilde{J}_u \times \tilde{J}_w} \) (where \( \tilde{J}_u := \tilde{J}_u(k) \)) are null matrices for \( j \neq k \) and \( u, v \notin \{j, k\} \). So we are reduced to prove that

\[
B_\chi(j,k)_{l s} = A_\chi(s,l)_{kj} = 0 \quad \text{for all } k \notin \{j, u, v\}, j \notin \{u, v\}, l \in \tilde{J}_u \text{ and } s \in \tilde{J}_v.
\]

But, since

\[
l \in \tilde{J}_u \text{ and } s \in \tilde{J}_v \quad \text{if and only if} \quad A_\chi(l,l)_{ku} = 1 \text{ and } A_\chi(s,s)_{kv} = 1,
\]

and, in that case,

\[
 j \notin \{u,v\} \quad \text{if and only if} \quad A_\chi(l,l)_{kj} = 0 \text{ and } A_\chi(s,s)_{kj} = 0,
\]

for this it suffices to check that if \( k \notin \{j, u, v\} \) and the \( l \)-th column of \( A_\chi \) is quasi-standard, then

\[
\begin{align*}
A_\chi(l,l)_{ku} &= 1 \\
A_\chi(s,s)_{kv} &= 1 \\
A_\chi(l,l)_{kj} &= 0 \\
A_\chi(s,s)_{kj} &\neq 0
\end{align*}
\]

Clearly \( s \neq l \). Moreover \( k \in J_{w(l)} \) with \( w \neq s \), since otherwise \( A_\chi(s,l)_{kj} = 0 \) by Theorem 8.13(1). Suppose that \( k \notin J_{l(l)} \) and \( j \notin \{k, c_k(A_\chi(l,l))\} \). Then, by Lemma 8.11 we have \( j \notin J_{l(l)} \cup J_{w(l)} \). Consequently, by Definition 8.2(3) and Remark 8.6,

\[
 j \in \text{Supp}((D^{uw}_{i_s})_{k^*}) = \text{Supp}((D^{uw}_{j_s})_{k^*}) \quad \text{and} \quad w \neq s \neq l.
\]

Thus, from Theorem 8.13(2b) we obtain that \( A_\chi(s,s)_{kj} = \delta_{jj} \neq 0 \), as desired. So, in order to finish the proof, we must check that \( k \notin J_{l(l)} \) and \( j \notin \{k, c_k(A_\chi(l,l))\} \). But \( k \notin J_{l(l)} \), Springer
because $A_X(l, l)_{ku} = 1$ implies that $A_X(l, l)_{kk} = 0$; $j \neq k$, because, by Theorem 8.13(1), if $j = k$, then $A_X(s, s)_{kv} = \delta_{ss}\delta_{kv} = 0$; and $j \neq c_k(A_X(l, l))$, since $A_X(l, l)_{kj} = 0$. □

Let $\chi : K^m \otimes K^n \to K^n \otimes K^m$ be a twisting map and let $r < m$. By Remark 3.14 we know that there exists a twisting map $\tilde{\chi} : K^r \otimes K^n \to K^n \otimes K^r$ such that $A_{\tilde{\chi}} = \left(A_X(i, l)\right)_{1 \leq i, l \leq r}$ if and only if $A_X(i, l) = 0$ for all $i > r$ and $l \leq r$.

Now suppose that we have a twisting map $\tilde{\chi} : K^r \otimes K^n \to K^n \otimes K^r$. Let $A = (A(i, l))_{1 \leq i, l \leq m}$ be a pre-twisting which is a extension of the family $A_{\tilde{\chi}} = (A_X(i, l))_{1 \leq i, l \leq r}$ such that

- $A(i, l) = 0$ if $i > r$ and $l \leq r$,
- for $l > r$, the $l$-th column of $A$ is a quasi-standard column.

In the following theorem we give necessary and sufficient conditions in order that $A$ defines a twisting map.

**Theorem 8.21** Let $A$ be as above. For all $u, v, l \in \mathbb{N}_m^*$ with $l > r$, set $D_{(u, l)}^{uv} := A(i, l)J_u \times J_v$, where $J_u := J_u(l)$ and $J_v := J_v(l)$. The family $A$ defines a twisting map if and only if

1. for all $i \in \mathbb{N}_m^*$, it is true that $\cup_{l > r} F(A(i, l)) \subseteq F_0(A, i)$.
2. If $(D_{(u, l)}^{uv})_{kd} \neq 0$, with $u \neq v \neq l$,
   (a) $A(u, v)_{kj} = \delta_{kj} - \delta_{jd}$ for all $j$,
   (b) $A(v, v)_{kj} = \delta_{jd}$ for all $j$,
   (c) $A(i, v)_{kj} = 0$ for $i \notin \{u, v\}$ and for all $j$.

Moreover, there exist $u \neq v \neq l$ such that $D_{(u, l)}^{uv} \neq 0$ if and only if the $l$-th column is not a standard column.

**Proof** The last assertion follows immediately from the definition of standard column. Next we prove the main part of the theorem.

$\Leftarrow$) We only must show that Condition (4) of Corollary 3.6 is satisfied. For $l \leq r$ this is true since

$$
\sum_{h=1}^{m} A(i, h)_{kj} A(h, l)_{kj'} = \sum_{h=1}^{r} A(i, h)_{kj} A(h, l)_{kj'} = \delta_{jj'}A(i, l)_{kj},
$$

because $A(h, l) = 0$ if $h > r$ and $\tilde{\chi}$ is a twisting map; while for $l > r$ this follows from Theorem 8.13.

$\Rightarrow$) This follows immediately from Theorem 8.13. □

**Proposition 8.22** Let $A$ be a pre-twisting of $K^m$ with $K^n$. Assume that $A(i, i) = \text{id}$ for all $i \in \mathbb{N}_m^* - 1$. Then $A$ is the family $A_X$ of matrices associated with a twisting map $\chi$ if and only if $(A(l, m))_{l \in \mathbb{N}_m^*}$ is a standard column.
Proof \( \Rightarrow \) By the assumptions it is clear that the rank matrix \( \Gamma_\chi \) introduced in Definition 3.12 has the form
\[
\Gamma_\chi = \begin{pmatrix}
 id_m & * \\
0 & *
\end{pmatrix}.
\] (8.14)
Consequently, \( \Gamma_\chi \) satisfies the hypothesis of Proposition 5.10 for \( l = m \), and so \( A(m, m) \) is a 0,1-matrix. It remains to check that item (2) of Definition 7.3 is fulfilled for \( l_0 = m \), i.e., that
\[
A(k, m)_{ij} \neq 0 \Rightarrow A(m, m)_{ij} \neq 0 \quad \text{for } k < m \text{ and } i \neq j;
\]
but this follows immediately from the fact that
\[
A(m, m)_{ij} = \sum_t A(t, t)_{ij} = \sum_t B_\chi(j, i)t_t = \text{rk}((B_\chi(j, i))_{tt}) \text{ for all } i \neq j,
\]
and \( B_\chi(j, i)_{mk} = A(k, m)_{ij} \).
\( \Leftarrow \) This follows from Theorem 8.21, since by Remark 8.3 we know that \( F(A(m, m)) = F_0(\mathcal{A}, m) \) and \( A(i, i) = \text{id}_n \) implies that \( F_0(\mathcal{A}, i) = \mathbb{N}_n^* \) for all \( i < m \).

9 Reduced Rank 1

In [12] the case of twisting maps \( \chi \) in which all the columns of \( A_\chi \) have reduced rank less than or equal to 1 (see Definition 8.14) is analysed. In this section we use our tools, that are completely different to the ones used in [12], in order to describe these twisting maps.

**Proposition 9.1** Let \( \chi : K^m \otimes K^n \rightarrow K^n \otimes K^m \) be a twisting map. Assume that \( \text{rrank}_\chi(l) = 1 \) and \( A_\chi(i, l) \neq 0 \) where \( i \neq l \). The following facts hold:

1. If \( A_\chi(l, i) = 0 \), then the \( l \)-th column of \( A_\chi \) is standard. Moreover, if \( A_\chi(l, l)_{kk} = 0 \), then \( A_\chi(i, l)_{kj} = \delta_{kj} \) for all \( j \).

2. If \( A_\chi(l, i) \neq 0 \) and \( \text{rrank}_\chi(i) = 1 \), then there is a twisting map \( \psi : K^2 \otimes K^n \rightarrow K^n \otimes K^2 \) with \( A_\psi(a, b) := A_\chi(f(a), f(b)) \), where \( a, b \in \{1, 2\}, f(1) := i \) and \( f(2) := l \).

**Proof** (1) By Proposition 5.10 we know that \( A_\chi(l, l) \) is a 0,1-matrix, and clearly
\[
A_\chi(l, l) + A_\chi(i, l) = \text{id} \Rightarrow \text{Supp}(A_\chi(i, l)) \subseteq \text{Supp}(A_\chi(l, l)) \cup \text{Supp}(\text{id}).
\]
So the \( l \)-th column of \( A_\chi \) is standard. The last assertion follows from Proposition 7.15, since \( A_\chi(l, l)_{kk} = 0 \) implies that \( k \in F(A_\chi(i, l)) \).

(2) The family of matrices \( (A_\psi(a, b))_{1 \leq a, b \leq 2} \) satisfies the conditions of Corollary 3.6. In fact, this is clear for the three first conditions, whereas the last one follows easily from the fact that
\[
\sum_{h=1}^m A_\chi(u, h)_{kj} A_\chi(h, v)_{k,j'} = A_\chi(u, i)_{kj} A_\chi(i, v)_{k,j'} + A_\chi(u, l)_{kq} A_\chi(l, v)_{k,j'},
\]
if \( v \in \{i, l\} \).

**Proposition 9.2** Let \( \mathcal{A} = (A(i, l))_{1 \leq i, l \leq m} \) be a pre-twisting of \( K^m \) with \( K^n \). For each \( l \) whose reduced rank is 1, let \( i(l) \) denote the unique \( i(l) \neq l \) such that \( A(i(l), l) \neq 0 \). If \( \text{rrank}_\mathcal{A}(l) \leq 1 \) for all \( l \), then there exists a twisting map \( \chi : K^m \otimes K^n \rightarrow K^n \otimes K^m \) with
\[ A_\chi = \mathcal{A} \text{ if and only if for each } l \in \mathbb{N}_m^* \text{ such that } \text{rank}_A(l) = 1, \text{ the following facts hold:} \]

1. If \( A(l, i(l)) = 0 \), then:
   a. \( A(l, l) \) is equivalent to a standard idempotent \( 0,1 \)-matrix via identical permutations in rows and columns,
   b. \( A(i(l), i(l))_{kk} = \delta_{kj} \) for all \( j \), whenever \( A(j, j)_{kk} = 0 \).

2. If \( A(l, i(l)) \neq 0 \), then there exists a twisting map \( \psi : \mathbb{K}^2 \otimes \mathbb{K}^n \rightarrow \mathbb{K}^n \otimes \mathbb{K}_2 \) with \( A_\psi(a, b) := A_\chi(f(a), f(b)) \), where \( a, b \in \{1, 2\} \), \( f(1) := i \) and \( f(2) := j \).

**Proof** The conditions are necessary by Proposition 9.1 and Corollary 5.9. On the other hand, it is straightforward to check that if \( \mathcal{A} \) satisfies items (1) and (2), then it also fulfills Condition (4) of Corollary 3.6.

We associate a quiver \( Q_\chi \) with a twisting map \( \chi : \mathbb{K}^m \otimes \mathbb{K}^n \rightarrow \mathbb{K}^n \otimes \mathbb{K}_m \) in the following way. The vertices are \( 1, \ldots, m \) and the adjacency matrix of \( Q_\chi \) is the \( 0,1 \)-matrix with 1 in the entry \( (i, l) \) if and only if \( A_\chi(i, l) \neq 0 \).

**Remark 9.3** Proposition 9.2 allows to construct all the twisting maps \( \chi : \mathbb{K}^m \otimes \mathbb{K}^n \rightarrow \mathbb{K}^n \otimes \mathbb{K}_m \) of reduced rank 1 (this means that each column of \( A_\chi \) has reduced rank less than or equal to 1, and at least one of its columns has reduced rank 1). For this it suffices to consider twisting maps with connected quivers, since every twisting map is the direct sum of the twisting maps restricted to the connected components. Each connected component of the quiver \( Q_\chi \) has at most one proper oriented cycle. This follows from the fact that each vertex of the quiver is the head of at most one arrow from another vertex, since the reduced rank of \( \chi \) is 1. So, in order to construct such a twisting map \( \chi \), take a quiver \( Q \) fulfilling this condition and fix a connected component. There are three possible cases, which were treated in [12]: the connected component is a 2-cycle, the connected component contains no 2-cycle or the connected component contains properly a 2-cycle. The two first cases are very easy to describe in our setting: In the first one \( \chi \) is obtained from a twisting map \( \psi : \mathbb{K}^2 \otimes \mathbb{K}^n \rightarrow \mathbb{K}^n \otimes \mathbb{K}_2 \), as in Proposition 9.2(2). In the second one by Proposition 9.1 all columns are standard, so it suffices to consider standard twisting maps compatible with the chosen quiver in the sense that \( A_\chi(i, l) \neq 0 \) if and only if the adjacency matrix of \( Q \) has 1 in the entry \( (i, l) \). In the third case assume that the 2-cycle is at the vertices \( i, j \). Suppose that there is a reduced rank 1 twisting map \( \chi \) such that \( Q_\chi = Q \). By Proposition 9.1 we know that the \( l \)-th columns of \( A_\chi \) is standard for all \( l \neq \{i, j\} \). This implies that if \( \chi \) has an arrow from \( i \) to \( l \), then \( F_0(A_\chi, i) \neq 0 \) (and similarly for \( j \)). In fact, we have

\[ \emptyset \neq F(A(i, l)) \subseteq F_0(A_\chi, i), \]

where inequality holds since \( A(i, l) = \text{id} - A(l, l) \) and \( A(l, l) \) is an idempotent \( (0,1) \)-matrix, while the inclusion is true by Proposition 7.15. Thus, in order to obtain such a twisting map \( \chi \) we first construct a twisting map

\[ \psi : \mathbb{K}^2 \otimes \mathbb{K}^n \rightarrow \mathbb{K}^n \otimes \mathbb{K}_2, \]

such that

- \( A_\psi(1, 2) \neq 0 \neq A_\psi(2, 1) \),
- \( F_0(A_\chi, i) \neq 0 \) if \( Q \) has an arrow that starts at \( i \) and does not end at \( j \),
- \( F_0(A_\chi, j) \neq 0 \) if \( Q \) has an arrow that starts at \( j \) and does not end at \( i \).
Then we set $A_X(h, i) := 0$ and $A_X(h, j) := 0$ for $h \notin \{i, j\}$, and $A_X(f(a), f(b)) := A_\psi(a, b)$, where $f(1) := i$ and $f(2) := j$. After that, for each vertex $l \in Q_0 \setminus \{i, j\}$, we take a standard column $(A_X(u, l))_{u \in Q_0}$ such that

- $A_X(u, l) \neq 0$ if and only $Q$ has an arrow from $u$ to $l$,
- $F(A(v, l)) \subseteq F_0(A, v)$ for all $v \in Q_0$ and $l \in Q_0 \setminus \{i, j\}$.

By Proposition 7.15, Corollary (4) is satisfied for all $l \notin \{i, j\}$. Since a straightforward computation shows that it is also satisfied for $i$ and $j$, this method produces all the twisting maps of reduced rank 1 with quiver $Q$.

## 10 Quiver Associated with Standard and Quasi-Standard Twisting Maps

In this section we will construct quivers that characterize completely the standard twisting maps. Moreover, the quiver indicates how one could possibly generate quasi-standard twisting maps out of a standard one.

### 10.1 Characterization of Standard Twisted Tensor Products

The aim of this subsection is to completely characterize the standard twisted tensor products of $K^n$ with $K^m$. In particular we will prove that they are algebras with square zero Jacobson radical. Our main result, Theorem 10.4, generalizes [6, Theorem 4.2]. Let

$$\chi : K^m \otimes K^n \rightarrow K^n \otimes K^m$$

be a standard twisting map. As in Remark 3.18, for each $j \in \mathbb{N}_m^*$ and $i \in \mathbb{N}_n^*$ we let $x_{ji}$ denote $f_j \otimes e_i$. In that remark we saw that $x_{ki}x_{jl} = A_X(i, l)_{kj}x_{kl}$.

**Remark 10.1** By Remark 7.4 we know that

$$A_X(i, l)_{kj} = \begin{cases} 1 & \text{if } k \in J_i(l), i = l \text{ and } j = k, \\
1 & \text{if } k \notin J_i(l), i = l \text{ and } j = c_k(A_X(l, l)), \\
1 & \text{if } k \notin J_i(l), i = k(l, l, A_X) \text{ and } j = k, \\
-1 & \text{if } k \notin J_i(l), i = k(l, l, A_X) \text{ and } j = c_k(A_X(l, l)), \\
0 & \text{otherwise}, \end{cases}$$

which implies that

$$x_{ki}x_{jl} = \begin{cases} x_{kl} & \text{if } k \in J_i(l), i = l \text{ and } j = k, \\
x_{kl} & \text{if } k \notin J_i(l), i = l \text{ and } j = c_k(A_X(l, l)), \\
x_{kl} & \text{if } k \notin J_i(l), i = k(l, l, A_X) \text{ and } j = k, \\
x_{kl} & \text{if } k \notin J_i(l), i = k(l, l, A_X) \text{ and } j = c_k(A_X(l, l)), \\
x_{kl} & \text{if } k \notin J_i(l), i = k(l, l, A_X) \text{ and } j = c_k(A_X(l, l)), \\
0 & \text{otherwise}. \end{cases}$$

**Remark 10.2** By Remark 10.1 we know that $x_{ki}x_{jl} = 0$ if $j \notin J_i(l), k = j$ and $i = l$, if $j \notin J_i(l)$ and $k \neq j$, or if $k \notin J_i(l)$ and $l \neq i$. Consequently $I := \bigoplus_{j \in \mathbb{N}_m^*} \bigoplus_{j \notin J_i(l)} Kx_{ji}$ is a square zero two-sided ideal of $K^n \otimes_K K^m$. Furthermore, by Remark 3.18 each two-sided ideal including properly $I$ has an idempotent element $x_{ji}$, and therefore it is not a nilpotent ideal. Hence, $I$ is the Jacobson radical $J(K^n \otimes_K K^m)$ of $K^n \otimes_K K^m$. 

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Let \( X Q \) be the quiver with set of vertices \( X Q_0 := \{(j, i) \in \mathbb{N}_n^* \times \mathbb{N}_m^* : j \in J_l(i)\} \), set of arrows \( X Q_1 := \{\alpha_{jl} : l \in \mathbb{N}_m^* \text{ and } j \in \mathbb{N}_n^* \setminus J_l(l)\} \), and source and target maps \( s, t : X Q_1 \longrightarrow X Q_0 \), given by

\[
 s(\alpha_{jl}) := (j, i(j, l, A_{\chi}(j, j))), t(\alpha_{jl}) := (i(l, j, B_{\chi}(j, j)), l).
\]

Note that \( s \) and \( t \) are well defined by Remark 5.8 applied to \( B(j, j) \) with \( k = l \) and \( A(l, l) \) with \( k = j \), respectively.

**Remark 10.3** By Remark 7.4(4), in the quiver \( X Q \) there is an arrow from \((k, i)\) to \((j, l)\) if and only if \( A_{\chi}(i, l) kj = -1 \).

Now we compute the products \( x_{ki}x_{jl} \) in terms of the maps \( s \) and \( t \). By the computations made in Remark 10.1, the following facts hold:

(a) If \( k \in J_i(i) \) and \( j \in J_l(l) \), then \( x_{ki}x_{jl} = x_{jl} \) if \((k, i) = (j, l)\), \(-x_{kl} \) if \((k, i) = s(\alpha_{kl}) \) and \((j, l) = t(\alpha_{kl})\), 0 otherwise.

(b) If \( k \notin J_i(i) \) and \( j \in J_l(l) \), then \( x_{ki}x_{jl} = x_{ki} \) if \((j, l) = t(\alpha_{ki})\), 0 otherwise.

(c) If \( k \in J_i(i) \) and \( j \notin J_l(l) \), then \( x_{ki}x_{jl} = x_{jl} \) if \((k, i) = s(\alpha_{jl})\), 0 otherwise.

(d) If \( k \notin J_i(i) \) and \( j \notin J_l(l) \), then \( x_{ki}x_{jl} = 0 \).

**Theorem 10.4** The twisted tensor product \( K^n \otimes_{\chi} K^m \) is isomorphic to the radical square zero algebra \( K^X Q / (X Q_1^2) \).

**Proof** For each \((j, l) \in X Q_0 \) set \( \text{In}(j, l) := \{\alpha_{ki} \in X Q_1 : (j, l) = t(\alpha_{ki})\} \). A straightforward computation using (a)–(d) shows that the map \( \phi : X Q_0 \cup X Q_1 \longrightarrow K^n \otimes_{\chi} K^m \), defined by

\[
 \phi(j, l) := x_{jl} + \sum_{\text{In}(j, l)} x_{ki} \quad \text{for } (j, l) \in X Q_0 \quad \text{and} \quad \phi(\alpha_{jl}) := x_{jl} \quad \text{for } \alpha_{jl} \in X Q_1,
\]

extends to an algebra morphism \( \phi : K^X Q \longrightarrow K^n \otimes_{\chi} K^m \). Since the elements \( x_{jl} \)'s generate linearly \( K^n \otimes_{\chi} K^m \), the morphism \( \phi \) is surjective. Moreover, by item (d) above, a path of length two of \( X Q \) has zero image. Since both algebras \( K^X Q / (X Q_1^2) \) and \( K^n \otimes_{\chi} K^m \) have the same dimension, the induced map

\[
 \overline{\phi} : K^X Q / (X Q_1^2) \longrightarrow K^n \otimes_{\chi} K^m
\]

is an algebra isomorphism, as desired. \( \square \)

The following remark generalizes the correct version of [6, Theorem 4.6].

**Remark 10.5** The quiver \( X Q = (X Q_0, X Q_1, s, t) \) associated with a standard twisting map \( \chi \) of \( K^m \) with \( K^n \) fulfill the following properties:

1. \( X Q_0 \subseteq \mathbb{N}_n^* \times \mathbb{N}_m^* \) and for all \( l \in \mathbb{N}_m^* \) there exists \( j \in \mathbb{N}_n^* \) such that \((j, l) \in X Q_0 \),
2. \( X Q_1 = \{\alpha_{jl} : (j, l) \in (\mathbb{N}_n^* \times \mathbb{N}_m^*) \setminus X Q_0\} \),
for all \((j, l) \in (\mathbb{N}^*_m \times \mathbb{N}^*_m) \setminus X\) \(Q_0\) there exist \(i \in \mathbb{N}^*_m\) and \(k \in \mathbb{N}^*_n\) such that \(s(\alpha_{ji}) = (j, i)\) and \(t(\alpha_{ji}) = (k, l)\).

Conversely if \(Q = (Q_0, Q_1, s, t)\) is a quiver that satisfies Conditions (1), (2) and (3), then there exists a unique standard twisting map \(\chi\) of \(K^m\) with \(K^n\), such that \(Q = X\). Indeed, by Theorem 7.11 in order to construct a such \(\chi\) it suffices to determine families \((A(l))_{l \in \mathbb{N}^*_m}\) and \((B(k))_{k \in \mathbb{N}^*_n}\) of idempotent 0,1-matrices \(A(l) \in M_n(K)\) and \(B(k) \in M_m(K)\) such that

1. \[((j, l) \in \mathbb{N}^*_m \times \mathbb{N}^*_m : A(l)_{jj} = 1) = Q_0,\]
2. if \((j, l) \notin Q_0\) and \(t(\alpha_{ji}) = (k, l)\), then \(A(l)_{jk} = 1,\)
3. if \((j, l) \notin Q_0\) and \(s(\alpha_{ji}) = (j, i)\), then \(B(j)_{li} = 1,\)
4. if \((j, l) \notin Q_0\), then \(B(j)_{li} = 1\) and \(A(l)_{jj} = B(j)_{li}\), for all \(l\) and \(j\).

For this we define the \(j\)-th row of \(A(l)\) and the \(l\)-th row of \(B(j)\) as follows:

1. If \((j, l) \in Q_0\), then we set \(A(l)_{jh} := \delta_{jh},\)
2. if \((j, l) \notin Q_0\), then we set \(A(l)_{jh} := \delta_{kh}\), where \(k\) is defined by \(t(\alpha_{ji}) = (k, l)\),
3. if \((j, l) \in Q_0\), then we set \(B(j)_{ih} := \delta_{ih},\)
4. if \((j, l) \notin Q_0\), then we set \(B(j)_{ih} := \delta_{ih}\), where \(i\) is defined by \(s(\alpha_{ji}) = (j, i)\).

### 10.2 Iterative Construction of Quasi-Standard Twisted Tensor Products

The aim of this subsection is to give a method to construct the quasi-standard twisting tensor products of \(K^m\) with \(K^n\). Through it we use the notation of the previous sections, specially those introduced in the fifth one. By Theorem 7.11 we can associate in an evident way a standard twisting map \(\hat{\chi}\) to each quasi-standard twisting map \(\chi\). This allows us to associate a quiver \(\chi Q := \hat{\chi} Q\) with each quasi-standard twisting tensor product \(K^n \otimes_{\chi} K^m\). By the construction in Theorem 7.11, we have

\[
A_{\hat{\chi}}(l, l) = A_{\chi}(l, l) \quad \text{and} \quad A_{\hat{\chi}}(i, l)_{kk} = B_{\hat{\chi}}(k, k)_{li} = B_{\chi}(k, k)_{li} = A_{\chi}(i, l)_{kk}
\]

for all \(i, l\) and \(k\). Consequently \(J_1(l)\) of \(\hat{\chi}\) coincides with \(J_1(l)\) of \(\chi\) for all \(i\) and \(l\), and so the construction of \(A_{\hat{\chi}}\) can be performed easily using Remark 7.5.

**Proposition 10.6** Let \(\chi\) be a quasi-standard twisting map of \(K^m\) with \(K^n\) and \(u, l \in \mathbb{N}^*_m\) and \(k, d \in \mathbb{N}^*_n\). Assume that \(A_{\chi}(u, l)_{kd} \neq 0\) and let \(v \in \mathbb{N}^*_n\) be such that \(d \in J_v(l)\). If \(A_{\chi}(u, l)_{kk} = 1\) and \(A_{\chi}(u, l)_{kd} = 0\), then \(u, v\) and \(l\) are three different elements of \(\mathbb{N}^*_m\), \(k \in J_u(l)\), \(d \notin \{k, c_k\}\), where \(c_k := c_d(A_{\chi}(l, l))\), and

\[
A_{\chi}(u, l)_{kd} = -A_{\chi}(v, l)_{kd} = A_{\chi}(v, l)_{kdk}.
\]  

Moreover \(c_d = c_k\), where \(c_d := c_d(A_{\chi}(l, l))\), and there are the following arrows in the quiver of \(\hat{\chi}\):

- \(\alpha_{kk}\), from \((k, u)\) to \((d, v)\),
- \(\alpha_{kl}\), from \((k, u)\) to \((c_k, l)\),
- \(\alpha_{dl}\), from \((d, v)\) to \((c_d, l)\).

**Proof** The equality \(A_{\chi}(u, l)_{kk} = 1\) says that \(k \in J_u(l)\). Moreover \(u \neq l\), because otherwise

\[
0 \neq A_{\chi}(l, l)_{kd} = \delta_{kd} \Rightarrow d = k \Rightarrow A_{\chi}(l, l)_{kd} = A_{\chi}(l, l)_{kk} = 1,
\]

which contradicts \(A_{\hat{\chi}}(u, l)_{kd} = 0\). Also by this equality, in order to prove that \(d \notin \{k, c_k\}\) it suffices to note that

\[
A_{\hat{\chi}}(u, l)_{kk} = B_{\hat{\chi}}(k, k)_{lu} = B_{\chi}(k, k)_{lu} = A_{\chi}(u, l)_{kk} = 1,
\]
and that $A_{\hat{\chi}}(u, l)_{kck} = -1$ by Theorem 7.11. Since $u \neq l$ and $d \notin \{k, c_k\}$, by Lemma 8.11(3) we have $v \neq l$ and $v \neq u$. Furthermore, since $A_{\hat{\chi}}(u, l)_{kd} \neq 0$, by Definition 8.2(4) we know that $c_d = c_k$. The first equality in Eq. 10.1 follows now from Remark 8.6 while the second one follows from Lemma 8.11(2) and Corollary 3.6(2). Moreover $\alpha_{kd}$, $\alpha_{dl}$ and $\alpha_{kv}$ are arrows of $\chi Q$, since $k \notin J_I(l)$, $d \notin J_I(l)$ and $k \notin J_v(v)$ by Theorem 8.13(2b). Furthermore the starting and target vertices of these arrow are those ones given in the statement because

- $c_v(B_{\hat{\chi}}(k, k)) = u$, since $B_{\hat{\chi}}(k, k)_{vv} = B_{\chi}(k, k)_{vv} = A_{\chi}(u, v)_{kk} = 1,$
- $c_k(A_{\hat{\chi}}(v, v)) = d$, since $A_{\chi}(v, v)_{kv} = A_{\chi}(v, v)_{kd} = 1$,
- $c_l(B_{\hat{\chi}}(k, k)) = u$, since $B_{\hat{\chi}}(k, k)_{lu} = B_{\chi}(k, k)_{lu} = A_{\chi}(u, l)_{kk} = 1$,
- $c_k(A_{\hat{\chi}}(l, l)) = c_k$, since $A_{\chi}(l, l) = A_{\chi}(l, l)$,
- $c_d(B_{\hat{\chi}}(d, d)) = v$, since $B_{\hat{\chi}}(d, d)_{dv} = B_{\chi}(d, d)_{dv} = A_{\chi}(v, l)_{dd} = 1$,
- $c_d(A_{\hat{\chi}}(l, l)) = c_d$, since $A_{\chi}(l, l) = A_{\chi}(l, l)$,

where in the first and second item the last equality hold by Theorem 8.13(2).

**Definition 10.7** Let $\chi : K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a quasi-standard twisting map, $u, v, l \in \mathbb{N}^*_m$, $k \in J_u(l)$ and $d \in J_v(l)$. Assume that $u \neq l \neq v$ and that there are the following arrows in the quiver of $\hat{\chi}$:

- $\alpha_{kv}$, from $(k, u)$ to $(d, v)$,
- $\alpha_{kl}$, from $(k, u)$ to $(c_k, l)$, where $c_k := c_k(A_{\chi}(l, l))$,
- $\alpha_{dl}$, from $(d, v)$ to $(c_d, l)$, where $c_d := c_d(A_{\chi}(l, l))$.

If $c_d = c_k$ and $\text{Supp}(D_{(u,l)}^{uv}) = \emptyset$, then for each $\lambda \in K$ we define the map

$$\chi_1 : K^m \otimes K^n \longrightarrow K^n \otimes K^m,$$

by

$$A_{\chi_1}(u, l)_{kd} := \lambda, \quad A_{\chi_1}(v, l)_{kd} := -\lambda, \quad A_{\chi_1}(v, l)_{ck} := \lambda, \quad A_{\chi_1}(u, l)_{cck} := A_{\chi}(u, l)_{kck} - \lambda, \quad A_{\chi_1}(i, t)_{js} := A_{\chi}(i, t)_{js} \quad \text{if } (i, t, j, s) \notin \{(u, l, k, d), (v, l, k, d), (v, l, k, c_k), (u, l, k, c_k)\}.$$

If necessary to be more precise the map $\chi_1$ will be denoted by $\Lambda_{(k,u),(d,v),(c_k,l)}^\chi(\chi)$.

**Remark 10.8** By the very definition of $\chi Q$, the existence of the arrows $\alpha_{kv}$, $\alpha_{kl}$ and $\alpha_{dl}$ implies that $k \in J_u(u) \setminus (J_v(v) \cup J_I(l))$, $d \in J_v(v) \setminus J_I(l)$ and $c_k \in J_I(l)$. Hence $u$, $v$ and $l$ are three different elements of $\mathbb{N}^*_m$ and $k$, $c_k$ and $d$ are three different elements of $\mathbb{N}^*_n$. Moreover

$$A_{\chi}(u, l)_{kd} = 0, \quad A_{\chi}(v, l)_{kd} = 0 \quad \text{and} \quad A_{\chi}(v, l)_{cck} = 0.$$

In fact, the first two equalities hold since $(D_{(u,l)}^{uv})_{kd} = 0$ and $(D_{(v,l)}^{uv})_{kd} = -(D_{(u,l)}^{uv})_{kd}$ by Remark 8.6; and the third equality holds because, on the contrary, by Lemma 8.10 there exists $j \in J_I(l)$ such that $(D_{(v,l)}^{uv})_{kj} \neq 0$ (which is impossible since, by Remark 8.6, it implies $(D_{(u,l)}^{uv})_{kj} = -(D_{(v,l)}^{uv})_{kj} \neq 0$). Hence if $\lambda = 0$, then $\chi_1 = \chi$.

**Remark 10.9** Assume that the twisting map $\chi$ satisfies the assumptions made in Definition 10.7. Since $k$, $c_k$ and $d$ are three different elements of $\mathbb{N}^*_n$, we know that $A_{\chi_1}(i, t)_{jj} =$
$A_X(i,t)_{jj}$ for all $i$, $t$ and $j$. So $F(A_X(i,l)) = F(A_X(i,l))$ for all $i$ and $t$. In other words, $J_i(t)$ does not depend on $\lambda$. Moreover, if $X_1$ is a quasi-standard twisting map, then clearly $\tilde{\chi}_1 = \tilde{\chi}$. Furthermore, $A_{\tilde{\chi}}(u,l)_{kk} = 0$, because $\text{Supp}(A_{\tilde{\chi}}(u,l)_{kk}) = \{k, c_k\}$, while by Remark 10.3, we have $A_{\tilde{\chi}}(u,l)_{kk} = -1$, which by Remark 7.10 implies that $A_{\tilde{\chi}}(u,l)_{kk} = A_{\tilde{\chi}}(u,l)_{kk} = 1$. So, if $\lambda \neq 0$, then the hypothesis of Proposition 10.6 are satisfied by $\chi_1$, provided that it is a quasi-standard twisting map.

**Remark 10.10** Assume that the twisting map $\chi$ satisfies the assumptions made in Definition 10.7. If $\chi_1$ is a (quasi-standard) twisting map, then $\Gamma_{\chi_1} = \Gamma_{\chi}$ and $\tilde{\Gamma}_{\chi_1} = \tilde{\Gamma}_{\chi}$.

**Proposition 10.11** Let $\chi$, $\chi_1$, $u$, $v$, $l$, $k$, $d$, $\alpha_{ku}$, $\alpha_{kl}$, $\alpha_{kd}$ and $\lambda$ be as in Definition 10.7. Assume that $\lambda \neq 0$. If $\chi_1$ is a quasi-standard twisting map, then

1. $A_{\chi_1}(i, u)_{kj} = A_{\tilde{\chi}}(i, u)_{kj}$ and $A_{\chi_1}(i, v)_{kj} = A_{\tilde{\chi}}(i, v)_{kj}$ for all $i$, $j$,
2. $A_{\chi_1}(u, l)$, $A_{\chi_1}(v, l)$, $B_{\chi_1}(d, k)$ and $B_{\chi_1}(c_k, k)$ are idempotent matrices.

Reciprocally, Condition (2) implies that $\chi_1$ is a (quasi-standard) twisting map.

**Proof** By Remark 10.9 we know that $A_{\chi_1}(u, l)_{kk} = A_{\chi}(u, l)_{kk} = 1$, and so, by Theorem 8.13(1),

$$A_{\chi_1}(i, u)_{kj} = \delta_{iu} \delta_{kj} = A_{\tilde{\chi}}(i, u)_{kj} \quad \text{for all } i, j.$$  

On the other hand, by Theorem 7.11,

$$A_{\tilde{\chi}}(i, v)_{kj} := \begin{cases} A_{\chi_1}(v, v)_{kj} & \text{if } i = v, \\ A_{\chi_1}(i, v)_{kk} & \text{if } k = j, \\ -1 & \text{if } i \neq v, j \neq k \text{ and } A_{\chi_1}(v, v)_{kj} = A_{\chi_1}(i, v)_{kk} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in order to finish the proof of item (1) it suffices to show that for all $i \neq v$ and $j \neq k$,

$$A_{\chi_1}(v, v)_{kj} = \delta_{jd}, \quad A_{\chi_1}(i, v)_{kk} = \delta_{iu} \quad \text{and} \quad A_{\chi_1}(i, v)_{kj} = -\delta_{iu} \delta_{jd}.$$  

But, since $A_{\chi_1}(u, l)_{kk} \neq 0$, this follows easily from by Theorem 8.13(2). Item (2) follows immediately from items (1) and (2) of Proposition 3.3. Reciprocally, by Propositions 3.5 and 3.11, Condition (2) is sufficient for $\chi_1$ to be a twisting map, because $\sum_i A_{\chi_1}(i, h) = \text{id}_X$ for all $h$ and $A_{\chi_1}(i, h) \mathbb{I} = \mathbb{I}$ for all $i$ and $h$. Finally, it is easy to see that Conditions (1)–(4) of Definition 8.2 are satisfied.

**Corollary 10.12** Under the assumptions made in Definition 10.7, if $\chi$ is standard, then $\chi_1$ is a quasi-standard twisting map.

**Proof** When $\lambda = 0$ this is evident, whereas when it is different from $0$ a straightforward computation shows that $A_{\chi_1}(u, l)$, $A_{\chi_1}(v, l)$, $B_{\chi_1}(d, k)$ and $B_{\chi_1}(c_k, k)$ are idempotent matrices.

**Remark 10.13** A straightforward computation shows that if the map $\chi_1$, introduced in Definition 10.7, is a twisting map, then the construction $\chi_1$ out of $\chi$ corresponds to a formal deformation in the sense of Gerstenhaber. To be more precise, the multiplication map $\mu_{\chi_1}$ of $\chi_1$ is given by

$$\mu_{\chi_1}(a \otimes b) = \mu_0(a \otimes b) + \lambda \mu_1(a \otimes b),$$

where $\lambda$ is as in Definition 10.7.
where $\mu_0$ is the multiplication in $D := K^n \otimes X K^m$ and $\mu_1 : D \otimes D \rightarrow D$ is the map defined by

$$\mu_1(x_{ku} \otimes x_{dl}) = x_{kl},$$

$$\mu_1(x_{kv} \otimes x_{dl}) = -x_{kl},$$

$$\mu_1(x_{kv} \otimes x_{cl}) = x_{kl},$$

$$\mu_1(x_{ku} \otimes x_{cl}) = -x_{kl},$$

and

$$\mu_1(x_{pq} \otimes x_{rs}) = 0 \quad \text{if} \quad (x_{pq}, x_{rs}) \notin \{(x_{ku}, x_{dl}), (x_{kv}, x_{dl}), (x_{ku}, x_{cl}), (x_{kv}, x_{cl})\}.$$

**Proposition 10.14** Assume that $\chi_1 : K^m \otimes K^n \rightarrow K^n \otimes K^m$ is a quasi-standard twisting map such that $A_{\chi_1}(u, l)_{kd} \neq 0$, where $k \in J_u(l)$ and $d \in J_v(l)$ with $u \neq v$, $u \neq l$ and $v \neq l$. Define a new map $\hat{\chi}$, from $K^m \otimes K^n$ to $K^n \otimes K^m$, by setting

$$A_{\hat{\chi}}(u, l)_{kd} := 0,$$

$$A_{\hat{\chi}}(v, l)_{kd} := 0,$$

$$A_{\hat{\chi}}(v, l)_{kk^c} := 0,$$

$$A_{\hat{\chi}}(u, l)_{kk^c} := A_{\chi_1}(u, l)_{kk^c} + \lambda,$$

$$A_{\hat{\chi}}(i, t)_{js} := A_{\chi_1}(i, t)_{js} \text{ if } (i, t, j, s) \notin \{(u, l, k, d), (v, l, k, d), (v, l, k, c_k), (u, l, k, c_k)\}$$

where $c_k := c_k(A_{\chi_1}(l, l))$ and $\lambda := A_{\chi_1}(u, l)_{kd}$. Then $\hat{\chi}$ is a quasi-standard twisting map.

**Proof** It is easy to see that if $\chi$ is a twisting map, then it is quasi-standard. So we must prove that the conditions in Proposition 3.11 are satisfied. For all $i, u, v \in \mathbb{N}_m^*$, set $D_{(i)}^{uv} := A_{\chi_1}(i, l)|_{J_u \times J_v}$, where $J_u := J_u(l)$ and $J_v := J_v(l)$. By Remark 8.6 and Lemma 8.11(2), we have

$$A_{\chi_1}(v, l)_{kk^c} = -A_{\chi_1}(u, l)_{kk^c} := -(D_{(i)}^{uv})_{kd} = (D_{(i)}^{uv})_{kd} = A_{\chi_1}(u, l)_{kd} = \lambda.$$

It is easy to see, using these facts, that Conditions (2) and (3) of Proposition 3.11 hold. We must prove that Conditions (1) and (4) are also satisfied. By Proposition 3.5(3) this is equivalent to show that $A_{\hat{\chi}}(u, l), A_{\hat{\chi}}(v, l), B_{\chi}(d, k)$ and $B_{\chi}(c_k, k)$ are idempotent matrices. We first prove that $A_{\hat{\chi}}(u, l)$ is idempotent. Let

$$X := A_{\chi_1}(u, l)|_{J_i \times J_i}, \quad Y := A_{\chi_1}(u, l)|_{J_i \times J_i^c}, \quad U := A_{\chi_1}(u, l)|_{J_i^c \times J_i}, \quad \text{and} \quad W := A_{\chi_1}(u, l)|_{J_i^c \times J_i^c},$$

and let

$$\hat{X} := A_{\hat{\chi}}(u, l)|_{J_i \times J_i}, \quad \hat{Y} := A_{\hat{\chi}}(u, l)|_{J_i \times J_i^c}, \quad \hat{U} := A_{\hat{\chi}}(u, l)|_{J_i^c \times J_i}, \quad \text{and} \quad \hat{W} := A_{\hat{\chi}}(u, l)|_{J_i^c \times J_i^c},$$

where $J_i := J_i(l)$ and $J_i^c := \mathbb{N}_m^* \setminus J_i$. Since $\chi_1$ is a quasi-standard twisting map, it follows from the definition of $\chi$ and Remark 8.3 that $X = \hat{X} = 0$ and $Y = \hat{Y} = 0$. Using this it is easy to see that $A_{\hat{\chi}}(u, l)$ is idempotent if and only if $W^2 = W$ and $WU = U$. Let $\overline{U} := A_{\chi_1}(l, l)|_{J_i^c \times J_i}$. Since $A_{\chi_1}(u, l)A_{\chi_1}(l, l) = 0$, we have $\hat{U} = -\hat{W} \overline{U}$. We claim that $U = -W \overline{U}$. That is

$$\sum_{s \in J_i^c} W_{is} \overline{U}_{sj} = -U_{ij} \quad \text{for all } i \in J_i^c \text{ and } j \in J_i.$$

If $i \neq k$, then clearly

$$\sum_{s \in J_i^c} W_{is} \overline{U}_{sj} = \sum_{s \in J_i^c} \hat{W}_{is} \overline{U}_{sj} = -\hat{U}_{ij} = -U_{ij},$$
as desired. Assume that $i = k$. Since $(D^{\mu\nu}_{(u)})_{kd} = \lambda \neq 0$, by Definition 8.2(4) we have $c_k = c_d$. Consequently $\hat{U}_{dj} = \delta_{jck}$, which combined with the fact that $W_{kd} = 0$, gives

$$
\sum_{s \in J_{i}^c} W_{ks} \hat{U}_{sj} = \sum_{s \in J_{i}^c \setminus \{d\}} \hat{W}_{ks} \hat{U}_{sj} = -\hat{U}_{kj} - \hat{W}_{kd} \hat{U}_{dj} = -\hat{U}_{kj} - \lambda \delta_{jck} = -U_{kj},
$$

as desired. Hence we have $U = -W \hat{U}$. Therefore $W^2 = W$ implies $WU = -W^2 \hat{U} = -W \hat{U} = U$. Thus, we only must prove that $W^2 = W$, or, equivalently, that

$$
\sum_{s \in J_{i}^c} W_{is} W_{sj} = W_{ij} \quad \text{for all } i, j \in J_{i}^c \tag{10.2}
$$

We will use that $\hat{W}^2 = \hat{W}$ (which follows using that $\hat{X} = 0$, $\hat{Y} = 0$ and $A_{X_1}(u, l)$ is idempotent) and that $W_{ij} = \hat{W}_{ij}$ if $i \in J_{i}^c \setminus \{k\}$ or $j \in J_{i}^c \setminus \{d\}$.

We first consider Equality (10.2) with $i \neq k$ and $j \neq d$. We have

$$
\sum_{s \in J_{i}^c} W_{is} W_{sj} = \sum_{s \in J_{i}^c} \hat{W}_{is} \hat{W}_{sj} = \hat{W}_{ij} = W_{ij},
$$

as desired. We consider now Equality (10.2) with $i = k$ and $j = d$. Since $\hat{W}_{kk} = 1$, $\hat{W}_{dd} = 0$ and $W_{kd} = 0$, we have

$$
\sum_{s \in J_{i}^c} W_{ks} W_{sd} = \sum_{s \in J_{i}^c \setminus \{k, d\}} \hat{W}_{ks} \hat{W}_{sd} = \hat{W}_{kd} - \hat{W}_{kk} \hat{W}_{kd} - \hat{W}_{kd} \hat{W}_{dd} = W_{kd},
$$

as desired. Next we consider Equality (10.2) with $i = k$ and $j \neq d$. Since $W_{kd} = 0$, we have

$$
\sum_{s \in J_{i}^c} W_{ks} W_{sj} = \sum_{s \in J_{i}^c \setminus \{d\}} \hat{W}_{ks} \hat{W}_{sj} = \hat{W}_{kj} - \hat{W}_{kd} \hat{W}_{dj} = W_{kj} - \hat{W}_{kd} \hat{W}_{dj}.
$$

Consequently, in order to finish the proof of Equality (10.2) in this case, it is sufficient to show that $\hat{W}_{kd} \hat{W}_{dj} = 0$. By Definition 8.2(3) if $j \in J_s$ with $s \neq u$, then $\hat{W}_{dj} = (D^{uv}_{(u)})_{dj} = 0$. On the other hand, we have

$$
D^{\mu\nu}_{(u)} D^{\mu\nu}_{(u)} = D^{\mu\nu}_{(v)} D^{\mu\nu}_{(v)} = \sum_i D^{\mu\nu}_{(v)} D^{\mu\nu}_{(v)} = D^{\mu\nu}_{(v)} = 0,
$$

where the first equality follows from Remark 8.6; the second one, since by Definition 8.2(3) we have $D^{\mu\nu}_{(v)} = 0$ and $D^{\mu\nu}_{(v)} = 0$ for all $i \neq \{v, l\}$; and the last one, again by Definition 8.2(3). Thus, if $j \in J_u$, then, by Definition 8.2(4),

$$
\hat{W}_{kd} \hat{W}_{dj} = \sum_i (D^{\mu\nu}_{(u)})_{kj} (D^{\mu\nu}_{(u)})_{lj} = 0,
$$

as desired. Finally, we consider Equality (10.2) with $i \neq k$ and $j = d$. Using again that $W_{kd} = 0$, we obtain that

$$
\sum_{s \in J_{i}^c} W_{is} W_{sd} = \sum_{s \in J_{i}^c \setminus \{k\}} \hat{W}_{is} \hat{W}_{sd} = \hat{W}_{id} - \hat{W}_{ik} \hat{W}_{kd} = W_{id} - \hat{W}_{ik} \hat{W}_{kd}.
$$

Consequently, in order to finish the proof of Equality (10.2) in this case, it is sufficient to show that $\hat{W}_{ik} \hat{W}_{kd} = 0$. By Remark 8.5, if $i \in J_u \setminus \{k\}$, then $\hat{W}_{ik} = (D^{uv}_{(u)})_{ik} = 0$. On the other hand, for every $s \in I_n^m \setminus \{u, l\}$, we have

$$
D^{\mu\nu}_{(u)} D^{\mu\nu}_{(u)} = \sum_i D^{\mu\nu}_{(u)} D^{\mu\nu}_{(u)} = D^{\mu\nu}_{(u)} = 0,
$$
where the first equality follows since, by Definition 8.2(3) we have $D_{(u)}^{l_{v}} = 0$ and $D_{(u)}^{s_{i}} = 0$ for all $i \notin \{u, l\}$; and the last one, again by Definition 8.2(3). Hence, if $i \in J_{u}$ with $s \notin \{u, l\}$, then, by Definition 8.2(4),

$$\hat{W}_{ik} \hat{W}_{kd} = \sum_{t} (D_{(u)}^{su})_{it}(D_{(u)}^{ut})_{td} = 0,$$

as desired. This ends the proof of Eq. 10.2.

A similar argument shows that $A_{\chi}(v, l)$ is idempotent.

We next prove that $B_{\chi}(d, k)$ and $B_{\chi}(c_{k}, k)$ are idempotent using duality. Let

$$\tilde{\chi}_{1} : K^{m} \otimes K^{n} \rightarrow K^{n} \otimes K^{m} \quad \text{and} \quad \tilde{\chi} : K^{m} \otimes K^{n} \rightarrow K^{n} \otimes K^{m}$$

be the dual maps of $\chi_{1}$ and $\chi$ respectively. By Proposition 8.20 we know that $\tilde{\chi}_{1}$ is a quasi-standard twisting map. Moreover, by Remark 3.2

$$A_{\tilde{\chi}}(c_{k}, k)_{lu} = B_{\chi}(c_{k}, k)_{lu} = A_{\chi}(v, l)_{kck} = 0,$$

$$A_{\tilde{\chi}}(d, k)_{lu} = B_{\chi}(d, k)_{lu} = A_{\chi}(v, l)_{kd} = 0,$$

$$A_{\tilde{\chi}}(c_{k}, k)_{lu} = B_{\chi}(c_{k}, k)_{lu} = A_{\chi}(u, l)_{kck} = 0,$$

$$A_{\tilde{\chi}}(c_{k}, k)_{lu} + \lambda = A_{\tilde{\chi}_{1}}(c_{k}, k)_{lu} + \lambda$$

and

$$A_{\tilde{\chi}}(i, t)_{js} = B_{\chi}(i, t)_{js} = A_{\chi}(s, j)_{ti} = A_{\chi_{1}}(s, j)_{ti} = B_{\tilde{\chi}_{1}}(i, t)_{js} = A_{\tilde{\chi}_{1}}(i, t)_{js},$$

for $(i, t, j, s) \notin \{(c_{k}, k, l, v), (d, k, l, v), (d, k, l, u), (c_{k}, k, l, u)\}$. Note that $c_{k} \neq d, c_{k} \neq k, d \neq k$ and

$$A_{\tilde{\chi}_{1}}(c_{k}, k)_{lu} = B_{\chi_{1}}(c_{k}, k)_{lu} = A_{\chi_{1}}(v, l)_{kck} = \lambda.$$

Let $\tilde{J}_{c_{k}} = \tilde{J}_{c_{k}}(k) := F(A_{\tilde{\chi}_{1}}(c_{k}, k))$ and $\tilde{J}_{d} = \tilde{J}_{d}(k) := F(A_{\tilde{\chi}_{1}}(d, k))$. We claim that

$$A_{\tilde{\chi}_{1}}(c_{k}, k)_{ll} = B_{\chi_{1}}(c_{k}, k)_{ll} = A_{\chi_{1}}(l, l)_{kck} = 1;$$

the second one, since

$$A_{\tilde{\chi}_{1}}(k, k)_{lu} = B_{\chi_{1}}(k, k)_{lu} = A_{\chi_{1}}(u, l)_{kk} = 1,$$

where the last equality holds because $k \in J_{u}(l)$; and the last one, since

$$A_{\tilde{\chi}_{1}}(d, k)_{vv} = B_{\chi_{1}}(d, k)_{vv} = A_{\chi_{1}}(v, v)_{kd} = 1,$$

where the last equality follows from Theorem 8.13. Hence by duality we know that

$$B_{\chi}(d, k) = A_{\tilde{\chi}}(d, k) \quad \text{and} \quad B_{\chi}(c_{k}, k) = A_{\tilde{\chi}}(c_{k}, k)$$

are idempotent matrices. \hfill \square

**Corollary 10.15** For each quasi-standard twisting map $\chi : K^{m} \otimes K^{n} \rightarrow K^{n} \otimes K^{m}$ there exists a family $(\chi_{i} : K^{m} \otimes K^{n} \rightarrow K^{n} \otimes K^{m})_{0 \leq i \leq r}$, of quasi-standard twisting maps, such that $\chi_{0}$ is a standard twisting map; for $0 \leq i < r$, the twisting map $\chi_{i+1}$ is obtained from $\chi_{i}$ as in Definition 10.7; and $\chi = \chi_{r}$.

**Remark 10.16** Let $\chi : K^{m} \otimes K^{n} \rightarrow K^{n} \otimes K^{m}$ be a quasi-standard twisting map. For each $u, v, l \in \mathbb{N}^{m}_{n}$ and $k, d \in \mathbb{N}_{n}^{m}$ such that $u, v$ and $l$ are three different elements of $\mathbb{N}^{m}_{n}$, $k \in J_{u}(l), d \in J_{v}(l)$ and $A_{\chi}(u, l)_{kd} \neq 0$, the quiver of $\tilde{\chi}$ has a triangle with vertices $(k, u)$, $(d, v)$ and $(c_{k}, l)$, and arrows $\alpha_{k_{v}}, \alpha_{k_{l}}$ and $\alpha_{d_{l}}$, from $(k, u)$ to $(d, v)$, $(k, u)$ to $(c_{k}, l)$, and
(d, v) to (c_\ell, l), respectively (recall that c_k = c_d). In fact, this follows from the previous results.

### 10.3 Jacobson radical of Quasi-Standard Twisted Tensor Products

Let \( \chi : K^m \otimes K^n \rightarrow K^n \otimes K^m \) be a quasi-standard twisting map. For each \( j \in \mathbb{N}_n^* \) and \( l \in \mathbb{N}_m^* \), let \( x_{jl} \) be as in Remark 3.18. In this subsection we prove that, as in the case when \( \chi \) is standard, the Jacobson radical \( J(C) \) of \( C := K^n \otimes \chi K^m \) is the ideal \( I = \bigoplus_{l \in \mathbb{N}_m^*} \bigoplus_{j \notin J_l(l)} Kx_{jl} \) of \( C \) (however unlike what happens in the standard case, when \( \chi \) is not standard \( I \) can be not a square zero ideal). We also prove that there exists a subalgebra \( A \simeq \frac{C}{J(C)} \) of \( C \) such that \( C = A \bigoplus J(C) \).

**Theorem 10.17** Let \( \chi, C \) and \( I \) be as above. Then \( I \) is the Jacobson radical of \( C \).

**Proof** For each \( j, k, i \in \mathbb{N}_n^* \) and \( i, l \in \mathbb{N}_m^* \), if \( i \neq l \) or \( k \neq j \), then

\[ x_{ki}x_{jl} = A_{\chi}(i, l)_{kj}x_{kl} \in I. \]

In fact, if \( k \notin J_l(l) \), then this is true by the very definition of \( I \), and if \( k \in J_l(l) \), then it is true because \( A_{\chi}(i, l)_{kj} = 0 \) by Theorem 8.13(1). Using this it is easy to see that \( I \) is a two-sided ideal of \( C \). To finish the proof it suffices to show that

\[ x_{jl}x_{j_1l_1} \cdots x_{j_nl_n} = 0 \quad \text{for each } x_{jl}, \ldots, x_{j_nl_n} \in I. \quad (10.3) \]

Since the \( l_i \)'s belong to \( \mathbb{N}_m^* \), there exist \( u < v \) such that \( l_u = l_v \). Clearly we can assume that if \( v > u + 1 \), then \( l_{u+1} \neq l_v \). An inductive argument shows that

\[ x_{j_1l_1} \cdots x_{j_nl_n} = \prod_{h=u}^{v-1} A_{\chi}(l_h, l_v)_{j_h, j_{h+1}} x_{j_{h+1}l_v}. \]

Since \( j_u \notin J_{l_u}(l_u) \) we know that

\[ A_{\chi}(l_u, l_v)_{j_u, j_{u+1}} = \delta_{j_u, j_{u+1}}, \quad \text{where } c_{j_u} = c_k(A_{\chi}(l_u, l_v)). \]

Consequently, if \( j_{u+1} \neq j_u \), then Equality (10.3) holds. On the other hand, if \( j_{u+1} = j_u \in J_{l_u}(l_u) \), then \( v > u + 1 \) (because \( j_u \neq J_{l_{u+1}}(l_{u+1}) \)), and Equality (10.3) holds since

\[ A_{\chi}(l_{u+1}, l_v)_{j_u, j_{u+1}, j_{u+2}} = \delta_{l_{u+1}, l_v} \delta_{j_{u+1}, j_{u+2}} = 0 \]

by Theorem 8.13(1). \( \square \)

**Corollary 10.18** Under the hypothesis of Theorem 10.7, the quotient algebra \( \frac{C}{J(C)} \) is isomorphic to the direct product \( \prod_{l \in \mathbb{N}_m^*} \prod_{j \in J_l(l)} Kx_{jl} \) of fields, and there exists a subalgebra \( A \simeq \frac{C}{J(C)} \) of \( C \) such that \( C = A \bigoplus J(C) \).

**Proof** The first assertion follows from the fact that for each \( i, l \in \mathbb{N}_m^* \), \( k \in J_l(i) \) and \( j \in J_l(l) \),

\[ x_{jl}x_{jl} = A_{\chi}(l, l)_{jj}x_{jl} = x_{jl}, \]

\[ x_{ki}x_{jl} = A_{\chi}(i, l)_{kj}x_{kl} \in I \text{ if } k \notin J_l(l) \]

and

\[ x_{ki}x_{jl} = A_{\chi}(i, l)_{kj}x_{kl} = 0 \text{ if } k \in J_l(l) \text{ and } (k, i) \neq (j, l). \]
Twisted tensor products of $K^n$ with $K^m$...  

The second assertion follows now by a direct application of the Principal Theorem of Wedderburn-Malcev [19, Chapter 11].

**Remark 10.19** It is easy to check that if $\chi : K^m \otimes K^n \to K^n \otimes K^m$ is a quasi-standard twisting map that it is not standard, then $J(C)^2 \neq 0$.

### Appendix: Quasi-Standard Twisting Maps of $K^3$ with $K^3$

There is a bijection between the set of quivers that satisfy Conditions (1), (2) and (3) of Remark 10.5 with $n = m = 3$ and the set $\text{Stm}$ of standard twisting maps of $K^3$ with $K^3$. Moreover, by Remark 7.8 we know that $\text{Stm}$ splits in classes of isomorphic twisting maps. The first, second, fourth and fifth columns of the following table are self explanatory. In the third column we list a quiver for each one of the equivalent classes in $\text{Stm}$. In the sixth column we list the number of the standard twisting maps equivalent to the one determined by the quiver listed in the third column. Finally, in the last column we list all the quasi-standard twisting maps (which are not standard) associated with the standard twisting map determined in the third column. For this we use recursively the construction in Definition 10.7, verifying in each step that the conditions in item (2) of Proposition 10.11 are satisfied.

**Table 1** Quasi-standard twisting maps of $K^3$ with $K^3$

| # | $\sum \text{Tr}$ | quiver | $\Gamma_x$ | $\tilde{\Gamma}_x$ | $\#$ equiv. | quasi-st. |
|---|------------------|--------|------------|-----------------|-------------|----------|
| 1. | 9                | …·…   | (3 0 0)    | (3 0 0)        | 1           | –        |
| 2. | 8                | …·…   | (3 0 0)    | (3 0 0)        | 36          | –        |
| 3. | 7                | …·…   | (3 0 0)    | (3 1 1)        | 18          | –        |
| 4. | 7                | …·…   | (3 0 0)    | (3 1 1)        | 18          | –        |
| 5. | 7                | …·…   | (3 0 2)    | (3 0 2)        | 18          | –        |
| #  | $\sum \text{Tr}$ | quiver | $\Gamma_\chi$ | $\check{\Gamma}_\chi$ | # equiv. | quasi-st. |
|----|-----------------|--------|--------------|-----------------|----------|---------|
| 6. | 7               | ![Quiver Diagram](image1) | (3, 1, 1) | (3, 0, 1) | 18       | –       |
| 7. | 7               | ![Quiver Diagram](image2) | (3, 1, 1) | (3, 1, 1) | 18       | –       |
| 8. | 7               | ![Quiver Diagram](image3) | (3, 1, 0) | (3, 1, 1) | 36       | –       |
| 9. | 7               | ![Quiver Diagram](image4) | (3, 0, 0) | (3, 1, 1) | 18       | –       |
| 10. | 7             | ![Quiver Diagram](image5) | (3, 1, 1) | (3, 1, 0) | 36       | –       |
| 11. | 7            | ![Quiver Diagram](image6) | (3, 1, 0) | (3, 1, 0) | 36       | –       |
| 12. | 7            | ![Quiver Diagram](image7) | (3, 1, 0) | (3, 0, 1) | 36       | –       |
| 13. | 7            | ![Quiver Diagram](image8) | (3, 0, 0) | (3, 1, 0) | 36       | –       |
| 14. | 7            | ![Quiver Diagram](image9) | (3, 1, 1) | (3, 0, 0) | 18       | –       |
Table 1  (continued)

| #  | $\Sigma \text{Tr}$ | quiver | $\Gamma_x$   | $\hat{\Gamma}_x$ | # equiv. | quasi-st. |
|----|-------------------|--------|--------------|-------------------|----------|----------|
| 15 | 7                 |        | $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ | 36       | –        |
| 16 | 7                 |        | $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ | 18       | –        |
| 17 | 6                 |        | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | 36       | –        |
| 18 | 6                 |        | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | 36       | –        |
| 19 | 6                 |        | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | 36       | –        |
| 20 | 6                 |        | $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | 36       | $\chi_1 = \Lambda_{(1,1),(2,2),(1,3)}(\chi)$ |
| 21 | 6                 |        | $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ | 36       | –        |
| 22 | 6                 |        | $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ | 36       | –        |
Table 1  (continued)

| #  | $\sum \text{Tr}$ | quiver | $\Gamma_\chi$ | $\tilde{\Gamma}_\chi$ | # equiv. | quasi-st. |
|----|------------------|--------|---------------|------------------------|----------|----------|
| 23. | 6                | ![Quiver 23](image.png) | \((3,1,1)\) | \((2,0,1)\) | 36 | – |
| 24. | 6                | ![Quiver 24](image.png) | \((3,1,0)\) | \((2,0,1)\) | 36 | – |
| 25. | 6                | ![Quiver 25](image.png) | \((3,0,2)\) | \((2,0,1)\) | 36 | – |
| 26. | 6                | ![Quiver 26](image.png) | \((3,0,1)\) | \((2,0,1)\) | 36 | – |
| 27. | 6                | ![Quiver 27](image.png) | \((3,0,1)\) | \((2,0,1)\) | 36 | – |
| 28. | 6                | ![Quiver 28](image.png) | \((3,0,0)\) | \((2,0,1)\) | 36 | – |
| 29. | 6                | ![Quiver 29](image.png) | \((2,0,1)\) | \((3,1,2)\) | 36 | – |
| 30. | 6                | ![Quiver 30](image.png) | \((2,0,1)\) | \((3,1,1)\) | 36 | – |
| 31. | 6                | ![Quiver 31](image.png) | \((2,0,1)\) | \((3,1,1)\) | 36 | – |
Table 1 (continued)

| #  | \( \sum \text{Tr} \) | quiver | \( \Gamma_{\chi} \) | \( \hat{\Gamma}_{\chi} \) | # equiv. | quasi-st. |
|----|-----------------|--------|-----------------|-----------------|-----------|----------|
| 32. | 6               | ![Quiver Diagram](image1) | \( \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \) | 36        | –        |
| 33. | 6               | ![Quiver Diagram](image2) | \( \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \) | 36        | –        |
| 34. | 6               | ![Quiver Diagram](image3) | \( \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \) | 36        | –        |
| 35. | 6               | ![Quiver Diagram](image4) | \( \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \) | 36        | –        |
| 36. | 6               | ![Quiver Diagram](image5) | \( \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \) | 36        | –        |
| 37. | 6               | ![Quiver Diagram](image6) | \( \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \) | 36        | –        |
| 38. | 6               | ![Quiver Diagram](image7) | \( \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \) | 36        | –        |
| 39. | 6               | ![Quiver Diagram](image8) | \( \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) | 36        | –        |
| 40. | 6               | ![Quiver Diagram](image9) | \( \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \) | \( \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \) | 36        | –        |
| #  | $\sum \text{Tr}$ | quiver | $\Gamma_X$ | $\tilde{\Gamma}_X$ | # equiv. | quasi-st. |
|----|-----------------|--------|----------|-----------------|---------|---------|
| 41 | 6               | ![Diagram](image1) | (2 1 0)   | (2 1 0)         | 36      | –       |
| 42 | 6               | ![Diagram](image2) | (2 1 1)   | (2 1 0)         | 36      | –       |
| 43 | 6               | ![Diagram](image3) | (2 0 1)   | (2 0 0)         | 36      | –       |
| 44 | 6               | ![Diagram](image4) | (2 0 0)   | (2 0 0)         | 36      | –       |
| 45 | 6               | ![Diagram](image5) | (2 0 1)   | (2 0 0)         | 36      | –       |
| 46 | 6               | ![Diagram](image6) | (2 0 1)   | (2 1 0)         | 12      | –       |
| 47 | 6               | ![Diagram](image7) | (2 1 0)   | (2 1 0)         | 12      | –       |
| 48 | 6               | ![Diagram](image8) | (2 1 1)   | (2 1 0)         | 36      | –       |
| 49 | 5               | ![Diagram](image9) | (3 2 2)   | (3 2 2)         | 9       | –       |
Table 1 (continued)

| #  | \(\sum \text{Tr} \) | quiver | \(\Gamma_X\) | \(\hat{\Gamma}_X\) | \# equiv. | quasi-st. |
|----|----------------|--------|-------------|----------------|----------|----------|
| 50. | 5 | ![Quiver](image1) | \((3, 2, 2)\) | \((2, 1, 1)\) | 18 | – |
| 51. | 5 | ![Quiver](image2) | \((3, 2, 1)\) | \((2, 1, 1)\) | 36 | \(X_1 = \Lambda_{3,1}^{(3,1)(1,2)(2,3)}(X)\) |
| 52. | 5 | ![Quiver](image3) | \((3, 1, 1)\) | \((2, 1, 1)\) | 18 | \(X_1 = \Lambda_{3,1}^{(3,1)(2,3)(1,2)}(X)\) |
| 53. | 5 | ![Quiver](image4) | \((2, 1, 1)\) | \((3, 2, 2)\) | 18 | – |
| 54. | 5 | ![Quiver](image5) | \((2, 1, 1)\) | \((3, 2, 1)\) | 36 | \(X_1 = \Lambda_{3,2}^{(3,2)(1,2)(1,3)}(X)\) |
| 55. | 5 | ![Quiver](image6) | \((2, 1, 1)\) | \((3, 1, 1)\) | 18 | \(X_1 = \Lambda_{3,2}^{(2,1)(1,3)(2,1)}(X)\) |
| 56. | 5 | ![Quiver](image7) | \((2, 0, 0)\) | \((2, 0, 0)\) | 36 | – |
| 57. | 5 | ![Quiver](image8) | \((2, 0, 0)\) | \((2, 0, 0)\) | 36 | – |
| 58. | 5 | ![Quiver](image9) | \((2, 0, 0)\) | \((2, 0, 0)\) | 36 | – |
| #  | $\sum \text{Tr}$ | quiver | $\Gamma_x$ | $\tilde{\Gamma}_x$ | # equiv. | quasi-st. |
|----|-----------------|--------|-----------|-----------------|-----------|----------|
| 59.| 5               |        | (2 0 1)   | (2 0 1)         | 18        | –        |
| 60.| 5               |        | (0 2 1)   | (0 2 1)         | 18        | –        |
| 61.| 5               |        | (1 1 1)   | (1 1 1)         | 36        | –        |
| 62.| 5               |        | (2 1 0)   | (2 1 0)         | 36        | –        |
| 63.| 5               |        | (1 2 2)   | (1 2 2)         | 36        | –        |
| 64.| 5               |        | (0 0 1)   | (0 0 1)         | 36        | –        |
| 65.| 5               |        | (2 0 0)   | (2 1 0)         | 36        | –        |
| 66.| 5               |        | (1 2 2)   | (1 2 1)         | 36        | $X_i = \Lambda^{(3,2)(1,1)(2,3)}(\chi)$ |
| 67.| 5               |        | (0 0 1)   | (0 0 1)         | 36        | –        |
| #  | $\sum \text{Tr}$ | quiver | $\Gamma_{\chi}$ | $\tilde{\Gamma}_{\chi}$ | # equiv. | quasi-st. |
|----|----------------|--------|----------------|----------------|---------|----------|
| 68. | 5              |        | (2 0 1)       | (2 0 1)       | 36      | $\chi_i = \Lambda_{(1,1)}^{(3,2),(1,2)}(\chi)$ |
|     |                |        | (1 2 1)       | (1 2 1)       |         |          |
|     |                |        | (0 1 1)       | (0 0 1)       |         |          |
| 69. | 5              |        | (2 1 0)       | (2 0 0)       | 36      |          |
|     |                |        | (1 2 2)       | (1 2 2)       |         |          |
|     |                |        | (0 0 1)       | (0 1 1)       |         |          |
| 70. | 5              |        | (2 1 1)       | (2 0 0)       | 36      |          |
|     |                |        | (1 2 1)       | (1 2 2)       |         |          |
|     |                |        | (0 0 1)       | (0 1 1)       |         |          |
| 71. | 5              |        | (2 0 0)       | (2 0 0)       | 36      |          |
|     |                |        | (1 2 2)       | (1 2 2)       |         |          |
|     |                |        | (0 1 1)       | (0 1 1)       |         |          |
| 72. | 5              |        | (2 0 1)       | (2 0 0)       | 36      |          |
|     |                |        | (1 2 1)       | (1 2 2)       |         |          |
|     |                |        | (0 1 1)       | (0 1 1)       |         |          |
| 73. | 5              |        | (2 1 0)       | (2 0 1)       | 36      |          |
|     |                |        | (1 2 2)       | (1 2 1)       |         |          |
|     |                |        | (0 0 1)       | (0 1 1)       |         |          |
| 74. | 5              |        | (2 1 1)       | (2 0 1)       | 36      | $\chi_i = \Lambda_{(1,1)}^{(3,2),(1,2)}(\chi)$ |
|     |                |        | (1 2 1)       | (1 2 1)       |         |          |
|     |                |        | (0 0 1)       | (0 1 1)       |         |          |
| 75. | 5              |        | (2 0 0)       | (2 0 1)       | 36      |          |
|     |                |        | (1 2 2)       | (1 2 1)       |         |          |
|     |                |        | (0 1 1)       | (0 1 1)       |         |          |
| 76. | 5              |        | (2 0 1)       | (2 0 1)       | 36      | $\chi_i = \Lambda_{(1,1)}^{(3,2),(1,2)}(\chi)$ |
|     |                |        | (1 2 1)       | (1 2 1)       |         |          |
|     |                |        | (0 1 1)       | (0 1 1)       |         |          |
| #  | \(\sum \text{Tr} \) | quiver | \(\Gamma_x\) | \(\bar{\Gamma}_x\) | \# equiv. | quasi-st. |
|----|----------------|--------|-------------|-------------|------------|----------|
| 77. | 4              | ![Quiver_77](image) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\) | 36 | \(x_1 = \Lambda_{(2,1)(3,3)(1,2)}^4(x)\) |
| 78. | 4              | ![Quiver_78](image) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\) | 36 | \(x_1 = \Lambda_{(2,1)(3,3)(1,2)}^4(x)\) \(x_2 = \Lambda_{(2,1)(1,2)(3,3)}^4(x)\) |
| 79. | 4              | ![Quiver_79](image) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | 36 | \(x_1 = \Lambda_{(2,1)(3,3)(1,2)}^4(x)\) \(x_2 = \Lambda_{(3,3)(2,1)(1,2)}^4(x)\) |
| 80. | 4              | ![Quiver_80](image) | \(\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | 36 | \(x_1 = \Lambda_{(2,1)(3,3)(1,2)}^4(x)\) \(x_2 = \Lambda_{(3,3)(2,1)(1,2)}^4(x)\) \(x_3 = \Lambda_{(2,1)(1,2)(3,3)}^4(x)\) |
| 81. | 4              | ![Quiver_81](image) | \(\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\) | 36 | – |
| 82. | 3              | ![Quiver_82](image) | \(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | \(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) | 6 | \(x_1 = \Lambda_{(3,3)(2,1)(1,2)}^3(x)\) \(x_2 = \Lambda_{(2,2)(3,3)(1,1)}^3(x)\) \(x_3 = \Lambda_{(2,2)(1,1)(3,3)}^3(x)\) \(x_4 = \Lambda_{(3,3)(1,1)(2,2)}^3(x)\) \(x_5 = \Lambda_{(3,3)(2,1)(1,2)}^3(x)\) \(x_6 = \Lambda_{(3,3)(2,1)(1,2)}^3(x)\) \(x_7 = \Lambda_{(3,3)(2,1)(1,2)}^3(x)\) \(x_8 = \Lambda_{(2,2)(1,1)(3,3)}^3(x)\) \(x_9 = \Lambda_{(2,2)(3,3)(1,1)}^3(x)\) \(x_{10} = \Lambda_{(2,2)(1,1)(3,3)}^3(x)\) |
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