Creation of planar charged fermions in Coulomb and Aharonov-Bohm potentials

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The creation of charged fermions from the vacuum by a Coulomb field in the presence of an Aharonov–Bohm (AB) potential are studied in 2+1 dimensions. The process is governed by a (singular) Dirac Hamiltonian that requires the supplementary definition in order for it to be treated as a self-adjoint quantum-mechanical operator. By constructing a one-parameter self-adjoint extension of the Dirac Hamiltonian, specified by boundary conditions, we describe the (virtual bound) quasistationary states with “complex energy” emerging in an attractive Coulomb potential, derive for the first time, complex equations (depending upon the electron spin and the extension parameter) for the quasistationary state “complex energy”. The constructed self-adjoint Dirac Hamiltonians in Coulomb and AB potentials are applied to provide a correct description to the low-energy electron excitations, as well as the creation of charged quasiparticles from the vacuum in graphene by the Coulomb impurity in the presence of AB potential. It is shown that the strong Coulomb field can create charged fermions for some range of the extension parameter.

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I. INTRODUCTION

The creation of electron–positron pairs from the vacuum by a Coulomb field is an important effect of quantum electrodynamics exhaustively studied in [1–8]. We recall that the energy spectrum of an electron in the Coulomb four-potential \( A^0(r) = a/e_0 r, A = 0, a > 0, e_0 > 0, (−e_0)\) is a charge with \( 1 > a \) consists of a continuous spectrum \( |E| \geq m \) separated by a gap \( 2m \) (\( m \) is the electron mass, and we use the system of units with \( c = \hbar = 1 \)) and a discrete spectrum \( 0 < E < m \) inside the gap. The formula for the lowest energy level \( E = m \sqrt{1 − \alpha^2} \) formally gives imaginary eigenvalues for the Dirac Hamiltonian with \( a > 1 \). As for the Dirac equation with \( a > 1 \), it was considered inconsistent and physically meaningless. The difficulty of the imaginary spectrum in thecase of \( a > 1 \) are solved (see, [8]) by replacing the singular potential \( a/e_0 r \) with a Coulomb potential cut off at short distance \( R \) for which the Dirac equation has physically meaningful solutions. In a (cut off) Coulomb potential, as \( a \) increases, the lowest electron energy level becomes negative for \( a > 1 \), and descends to the upper boundary \( E = −m \) of the lower continuum at \( a = \alpha_{cr} \), and can dive into it for \( a > \alpha_{cr} \), signaling the instability of the quantum electrodynamic vacuum in the overcritical Coulomb field. The lowest state then turn into resonance with a finite lifetime which can be described as a quasistationary state with “complex energy”. The so-called critical charge \( \alpha_{cr} \) is defined as the condition for the appearance of the imaginary part of “the energy”. The latter is related to the total probability of the creation of electron-positron pairs by the overcritical Coulomb field: the positron goes to infinity and the electron is coupled to the Coulomb center. Thus, the problem can no longer be considered a one-particle one.

In the problem on a massive charged fermion in a strong (cut off) Coulomb field in 2+1 dimensions, the picture is similar, but the ground-state energy vanishes at \( a = 1/2 \) [9, 10]. The case of massless charged fermions also is of great interest. Close to the so-called Dirac points, charged quasiparticle excitations in the potential of graphene lattice are massless Dirac-like fermions characterized by a linear dispersion relation [11, 13] and so a single electron dynamics in graphene is described by a massless two-component Dirac equation [12, 14, 15]. This allows to consider graphene as the condensed matter analog for relativistic quantum field theory [16] and massless charged quasiparticles in graphene [20] can provide an interesting realization of quantum electrodynamics in 2+1 dimensions [21, 22]. Since, the “effective fine structure constant” in graphene is large, there appears a new possibility to study a strong-coupling version of the quantum electrodynamics (QED) and the existence of charged Fermi quasiparticles in graphene makes experimentally feasible to observe the creation of quasiparticles by static electric fields [19].

For massless fermions, there are no discrete levels in the cut offCoulomb potential due to scale invariance of the massless Dirac equation, nevertheless for \( a > 1 \) quasistationary states emerge [13, 17, 23, 25]. It should be noted that the induced current in graphene in the field of solenoid was found to be a finite periodical function of the magnetic flux [26] and Coulomb impurity problems, such as the vacuum polarization and screening, in graphene were studied in [14, 17, 27]. The creation of graphene quasiparticles from vacuum by the space homogeneous static electric field was studied in [19] by means of the methods of planar quantum electrodynamics developed in [28, 31].

The above-mentioned difficulties do not arise if the Dirac Hamiltonian with the no cutoff Coulomb field (and with arbitrary \( a \)) is correctly defined as a self-adjoint operator. By constructing of the self-adjoint Dirac Hamiltonians using the so-called form asymmetry method developed in [32, 33], here we investigate the creation of charged fermions from the vacuum by a Coulomb field in the presence of AB potential in 2+1 dimensions. We show that there exists a family of self-adjoint Dirac Hamiltonians parameterized by an extension parameter (and specified by boundary conditions at the singular point) and a set of quasistationary states with “complex energies” can be evaluated for each Hamiltonian (see, also [33]). The different boundary conditions on the wave functions imposed at the origin are of importance leading to inequivalent physical cases in the relevant two spatial dimensions. The presence of AB potential allows us to study the influence of the particle spin on the physical effects, which is due to the interaction between the electron spin magnetic moment and the Aharonov-Bohm magnetic field [34]. It will be noted that the self-adjoint Dirac Hamiltonians in 2+1 dimensions were constructed in [35, 37] for the AB problem, analyzed in the nonrelativistic limit in [38] for the so-called Aharonov–Casher problem [39] of the motion of a neutral fermion with an anomalous magnetic moment in the electric field of an electrically charged conducting long straight thin thread oriented perpendicularly to the plane of fermion motion; particle creation in a moving cosmic string, governed by the Dirac Hamiltonian with AB potential in 2+1 dimensions, was discussed in [40]. The problems of self-adjointness of the Dirac Hamiltonians with Aharonov-Bohm and magnetic-solenoid fields were studied in [41, 42].
II. SOLUTIONS AND SPECTRAS OF THE RADIAL DIRAC HAMILTONIAN. SELF-ADJOINT BOUNDARY CONDITIONS

The space of particle quantum states in two spatial dimensions is the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$ of square-integrable functions $\Psi(r), r = (x, y)$ with the scalar product

$$\langle \Psi_1, \Psi_2 \rangle = \int \Psi_1^*(r) \Psi_2(r) dr, \quad dr = dx dy.$$  \hspace{1cm} (1)

The Dirac equation for a fermion in a given external field can be obtained just as in 3+1 dimensions.

First, we consider the massive case. The Dirac $\gamma^\mu$-matrix algebra is known to be represented in terms of the two-dimensional Pauli matrices $\sigma_j$ and the parameter $s = \pm 1$ can be introduced to label two types of fermions in accordance with the signature of the two-dimensional Dirac matrices $[43]$ and is applied to characterize two states of the fermion spin (spin “up” and “down”) $[44, 45]$. Then, the Dirac Hamiltonian for a fermion of the mass $m$ of fermions in accordance with the signature of the two-dimensional Dirac matrices $[43]$ and is applied to $e$

A case $B$ quantity $e$ “spin” potential is invariant under the changes of the two-dimensional Pauli matrices $\gamma^{01}$ and leads to the interaction potential of the electron spin magnetic moment with the magnetic field in strength of the impurity is a three-dimensional (not two-dimensional) vector. Therefore, the potential of the pointlike charge of an impurity occurs in a physical (three-dimensional) space and the electric field in graphene), although the electrons move in a plane, their Coulomb interaction with the external field decreases as $1/r$ that decrease as $1/r$ with the distance from the source, having in mind that in a physical situation (e.g., in graphene), although the electrons move in a plane, their Coulomb interaction with the external field of the pointlike charge of an impurity occurs in a physical (three-dimensional) space and the electric field strength of the impurity is a three-dimensional (not two-dimensional) vector. Therefore, the potential

$$A_0(r) \sim 1/r \quad \text{and not} \quad A_0(r) \sim \log r, \quad \text{as would be the case in 2+1 dimensions) does not satisfy the two-dimensional Poisson equation with a pointlike source at the origin. Similarly, in real physical space, the quantity $B$ characterizes the flux of the Aharonov–Bohm magnetic field $\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} = \pi B \delta(r)$ and leads to the interaction potential of the electron spin magnetic moment with the magnetic field in the form $-seB \delta(r)/r$, which is singular and must influence the behavior of solutions at the origin. The “spin” potential is invariant under the changes $e \to -e, \quad s \to -s$, and it hence suffices to consider only the case $e = -e_0 < 0$ and $eB \equiv -\mu < 0$. Then, the potential is attractive for $s = -1$ and repulsive for $s = 1$.

Eigenfunctions of the Hamiltonian (2) are (see, $[46, 47]$)

$$\Psi(t, r) = \frac{1}{\sqrt{2\pi r}} \begin{pmatrix} f_1(r) \\ f_2(r) e^{i\varphi} \end{pmatrix} \exp(-iEt + il\varphi),$$  \hspace{1cm} (3)

where $E$ is the fermion energy, $l$ is an integer. The wave function $\Psi$ is an eigenfunction of the operator $\hat{J}$ with eigenvalue $j = l + s/2$ and

$$\hat{h} F = EF, \quad F = \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix},$$  \hspace{1cm} (4)

where

$$\hat{h} = i\sigma_x \frac{d}{dr} + \sigma_1 \frac{l + \mu + s/2}{r} + \sigma_3 m - \frac{a}{r}, \quad \mu \equiv e_0 B.$$  \hspace{1cm} (5)

Thus, the problem is reduced to that for the radial Hamiltonian $\hat{h}$ in the Hilbert space of doublets $F(r)$ square-integrable on the half-line.

As was shown in $[45, 47]$ any correct doublets $F(r), G(r)$ of the Hilbert space $\mathcal{H} = L^2(0, \infty)$ must satisfy

$$\lim_{r \to 0} \hat{G}^\dagger(r)i\sigma_2 F(r) = 0.$$  \hspace{1cm} (6)

Then, the needed solution of (1) is

$$F = e^{-x^2/2} x^{\gamma s} \left[v^+ \Phi(a^s, c_s ; x) + v^- m^+_s \Phi(a^s + s, c_s ; x)\right] \equiv Y(r, \gamma_s, E), \quad x = 2\lambda r, \lambda = \sqrt{m^2 - E^2}.$$  \hspace{1cm} (7)
Here $\gamma_s = \pm \sqrt{(l + \mu + s/2)^2 - a^2} \equiv \gamma_s^\pm$, $a^* = \gamma_s + \frac{1}{2} - \frac{aE}{\lambda}$, $c_s = 2\gamma_s + 1$, $m_s^\pm = [s\gamma_s^\pm - Ea/\lambda]/[\nu + am/\lambda]$,

$$v^+ = \left( \frac{1}{p} \right), \quad v^- = \left( -\frac{1}{p} \right), \quad p = \sqrt{\frac{m - E}{m + E}}. \quad (8)$$

$\Phi(a, c; x)$ is the confluent hypergeometric function $48$

We denote $\gamma_s^+ = \sqrt{\nu^2 - a^2} \equiv \gamma$ for $a^2 \leq \nu^2$, $\gamma_s^- = i\sqrt{a^2 - \nu^2} \equiv i\sigma$ for $a^2 > \nu^2$ and we note that

$$\gamma(\pm l, s = 1, \mu, a) = \gamma(\pm l + 1, s = -1, \mu, a). \quad (9)$$

Then, for $\gamma \neq n/2, n = 1, 2, \ldots$, the needed linear independent solutions are:

$$U_1(r; E) = Y(r, \gamma_s, E)|_{\gamma_s = \gamma},$$

$$U_2(r; E) = Y(r, \gamma_s, E)|_{\gamma_s = -\gamma} \quad (10)$$

with the asymptotic behavior at $r \to 0$

$$U_1(r; E) = r^{\gamma}u_+ + O(r^{\gamma+1}),$$

$$U_2(r; E) = r^{-\gamma}u_- + O(r^{-\gamma+1}), \quad (11)$$

where

$$u_\pm = \left( (\pm s\gamma + \nu)/a \right).$$

as well as

$$V_1(r; E) = U_1(r; E) + \frac{a}{2s\gamma} \omega(E)U_2(r; E), \quad (12)$$

where $\omega(E) = \text{Wr}(U_1, V_1)$ is the Wronskian:

$$\omega(E) = \frac{\Gamma(2\gamma)\Gamma(-\gamma + (1 - s)/2 - aE/\lambda)}{\Gamma(-2\gamma)\Gamma(\gamma + (1 - s)/2 - aE/\lambda)} \frac{(2\lambda)^{-2\gamma} (1 - m_\gamma^-)}{m^{-2\gamma}} \frac{2s\gamma}{(1 - m_\gamma^+ a)} \equiv \tilde{\omega}(E). \quad (13)$$

Any doublet of the domain $D(h)$ must satisfy

$$(F^1(r)i\sigma_2 F(r))|_{r=0} = (\tilde{f}_1 f_2 - \tilde{f}_2 f_1)|_{r=0} = 0. \quad (14)$$

The quantities $q = \sqrt{\nu^2 - a^2}$ and $q_\gamma = \nu \leftrightarrow \gamma = 0$ are called the effective and critical charge, respectively; it is helpful also to determine $q_u = \sqrt{\nu^2 - 1/4}$, $\gamma = 1/2$. As was shown in $47$ for $q \leq q_u, \gamma \geq 1/2$, the domain $D(h)$ is the space of absolutely continuous doublets $F(r)$ regular at $r = 0$ with $hF(r)$ belonging to $L^2(0, \infty)$.

For $0 < \gamma < 1/2$ ($q_u < q < q_\gamma$) there is one-parameter $U(1)$-family of the operators $h_\xi \equiv h_\xi$, $\xi = \tan(\theta/2), -\infty \leq \xi \leq +\infty, -\infty \sim +\infty$, with the domain $D_\xi$

$$h_\xi: \left\{ \begin{array}{l} F(r) : F(r) \text{ is absolutely continuous in } [0, \infty), \\ F, hF \in L^2(0, \infty), \\ F(r) = c[r^\gamma u_+ - \xi r^{-\gamma} u_-] + O(r^{1/2}), |\xi| < \infty, \\ F(r) = c e^{-\gamma} u_- + O(r^{1/2}), r \to 0, \xi = \infty, \\ h_\xi F = hF, \end{array} \right. \quad (15)$$

where $c$ is an arbitrary constant. The operator $h^0$ is not determined as an unique self-adjoint operator and so the additional specification of its domain, given with the real parameter $\xi$ (the self-adjoint extension parameter) is required in terms of the self-adjoint boundary conditions. Physically, the self-adjoint boundary conditions show that the probability current density is equal to zero at the origin.

$$\frac{d\sigma(E)}{dE} = \frac{1}{\pi} \lim_{\varepsilon\to 0} \text{Im} \frac{1}{\omega_\xi(E + i\varepsilon)}, \quad (15)$$

where the generalized function $\omega_\xi(E + i\varepsilon)$ is obtained by the analytic continuation of the corresponding Wronskian in the complex plane of $E$; on the real axis of $E$ it is just the function $\omega(E)$ determined by $48$ for $\xi = 0$. For $0 < \gamma < 1/2$ the doublet $U_\xi(r; E) = U_1(r; E) - \xi U_2(r; E)$ and at $r \to 0$ $U_\xi(r; E) = -\xi U_1(r; E)$, $\lim_{r \to 0} U_1^\prime(r; E)$ are
of the complex energy spectrum of bound states is determined by roots of this equation. The Wronskians as a function so the function where \( V \) is finite in the interval \( 0 < \theta < \pi \). Thus, there is one-parameter family of the operators where \( \lambda = \sqrt{m^2 - E^2} \), have two cuts \( (-\infty, -m) \) and \( [m, \infty) \) in the complex plane of \( E \) and two sheets: Re\( \lambda > 0 \), the first (physical) sheet and Re\( \lambda < 0 \), the second (unphysical) sheet. Bound states are situated on the physical sheet of \( \lambda \). For \( \gamma \geq 1/2 \) the discrete spectrum is not invariant under the replacement \( r \to -r \). We have taken into account that \( c_2 = e^{i\theta} c_1, \) \( 0 \leq \theta \leq 2\pi \) is equivalent to \( c_1 = e^{i\theta} c_2, \) \( 0 \leq \theta \leq \pi \) with replacement \( \theta \to 2\pi - 2\theta \). For \( \gamma = i\sigma \) the doublets \( U_0(r; E) \) and \( V_0(r; E) \) should be chosen in the form

\[
U_0(r; E) = e^{i\theta} U_1(r; E) + e^{-i\theta} U_2(r; E),
\]

\[
V_0(r; E) = U_0(r; E) + \frac{4i\sigma}{4s\sigma} [e^{i\theta} U_1(r; E) - e^{-i\theta} U_2(r; E)],
\]

where \( U_1(r; E), U_2(r; E) \) are determined by eq. \( 11 \) with \( \gamma = i\sigma \), the Wronskian \( \omega_0(E) \equiv \text{Wr}(U_0, V_0) \) is

\[
\omega_0(E) = -\frac{4is\sigma}{a} \frac{1 - \tilde{\omega}(E)e^{2i\theta}}{1 + \tilde{\omega}(E)e^{2i\theta}}, \quad \tilde{\omega}(E) = \frac{a}{2is\sigma} \omega(E)
\]

and \( \omega(E) \) is given by eq. \( 13 \) with \( \gamma = i\sigma \). In the range \( |E| > m \) the function \( \omega_0(E) \) is continuous, complex-valued and not equal to zero for real \( E \); the spectral function \( \sigma(E) \) exists and is absolutely continuous and the energy spectrum is continuous. In the range \( |E| < m \), let us write \( \tilde{\omega}(E) \equiv e^{-2\sigma \Gamma(E)} \), so the function \( \omega_1(E) = 4s\sigma \tan(\Omega(E) - \theta)/a \) is real and the spectrum is implicitly determined by

\[
\sigma \ln \frac{2\lambda}{m} + \arg \Gamma(2i\sigma) + \arg \Gamma \left( \frac{1 - s}{2} + \frac{aE}{\lambda} + i\sigma \right) + \arg \left[ \frac{s\sigma}{\nu + a(m + E)/\lambda} \right] + \theta = k\pi, \quad k = 0, \pm1,
\]

One can show that \( E = m \) is a spectrum accumulation point and the number of discrete energy levels is finite in the interval \( 0 < E < m \). For \( \mu > 0 \) the lowest bound state is the state with \( s = -1 \). For \( 0 < \sigma \ll 1 \) eq. \( 10 \) has real solution \( E = -m \) for \( k = 0 \) and \( \theta = \pi/2 - \sigma \ln 2a - \arctan(s\sigma\epsilon)/\nu + \sigma\epsilon \),
where $\mathcal{C} = -\psi(1) = 0.57721$ is the Euler constant and $\psi(z)$ is the logarithmic derivative of Gamma function $\Gamma(z)$.

As we introduce a small change in $\sigma$ such that $\sigma > \sigma_c$, there is a sudden change in spectrum: there are no solutions of Eq. (19) for real $E$. Therefore, one of the bound state poles disappears from the physical sheet: for $E < -m$ only the continuous spectrum exists, but below $\text{Re}\lambda > 0, \text{Im}\lambda > 0$ there is a second (unphysical) sheet $\text{Re}\lambda < 0, \text{Im}\lambda < 0$ on which the virtual bound state pole resides at $\sigma > \sigma_c$. The key difference of the case $\sigma > \sigma_c$ is that the vacuum charge spatial distribution has complex energies $E = |E| \text{e}^{\ii \tau}$, which are determined by complex equation $\omega_0 = 0$:

$$
\frac{\Gamma(2i\sigma)}{\Gamma(-2i\sigma)} \frac{\Gamma(-i\sigma + (1-s)/2 - iaE/p)}{\Gamma(i\sigma + (1-s)/2 - iaE/p)} \frac{\Gamma(2i\sigma) - (2p)^{-2i\sigma} \nu + i[a(E + m)/p + s\sigma]}{\Gamma(2i\sigma) - (2p)^{-2i\sigma} \nu + i[a(E + m)/p - s\sigma]} = e^{-2i\theta},
$$

$$
p = \sqrt{E^2 - m^2}.
$$

(20)

For $\text{Re}E = -(m + \epsilon), \epsilon \to +0$, $1 \gg \sigma > 0$, one obtains $[1 + \ii 2\sigma \text{Im}\psi(-i\tau)]e^{-\pi\sigma + 2\sigma \alpha} = 1$ and $\alpha \approx \pi + (\pi/2)e^{-\sqrt{2m\pi a^2}/r}$, as well as

$$
\arg\Gamma(2i\sigma) - \sigma \text{Re}\psi(-i\tau) - (\sigma/2) \ln(8\epsilon/m) + (1/2) \arctan[s\sigma(1 - a^2\epsilon/2ma^2)/|\nu|] = -\theta + \pi n,
$$

(21)

where $z = \sqrt{ma^2/2}\epsilon$. The first quasistationary state emerges when the ground bound state with $s = -1, l = 0$ "dives" into the lower continuum. There appears the pole on the unphysical sheet, counted now as a "positron" state.

Putting $\text{Re}\psi(-i\tau) \approx -\mathcal{C} + \ln z$, $n = 0$, we obtain:

$$
\frac{s\sigma a^2 \epsilon}{2m|\nu|^2} = -[\mathcal{C} + \ln(2a_c) + \sigma/(2\sigma_c) - s/|\nu|](\sigma - \sigma_c).
$$

(22)

The physical picture can be seen as follows. When $\sigma > \sigma_c$, the lowest energy level dives into the negative energy continuum and becomes a resonance. It is spread out over an energy range of the order $\Gamma_g \sim me^{-\sqrt{2m\pi a^2}/r}$ and strongly distort around the impurity. The width $\Gamma_g$ is the doubled probability of the creation of the electron-positron pair by the Coulomb potential in the presence of AB potential. It is exponentially small in this case. The additional distortion of the negative energy continuum (due to the diving bound state) leads to a negative charge density due to the "real vacuum polarization", since its origin is not a fluctuating pair or the K-shell bound electron state (for $a < a_c$), but the structured vacuum of supercritical QED $\mathcal{R}$. The diving point for the energy level defines and depends upon the parameter $\theta$.

The resonance is not usual bound level diluted inside a continuum, where its lifetime essentially disappears. The overcritical level remains sharply defined with diverging lifetime $\tau \sim e^{\sqrt{2m\pi a^2}/r}/m$. The resonance is practically a bound state. This diving of bound levels entails a complete restructuring of the vacuum. If the emergent level was empty, an electron–positron pair will be created: the electron from the Dirac sea occupies this level and shields the charge of the source, while the positron (hole) escapes to infinity $\mathcal{R}, \mathcal{S}$. As a result, when $\sigma > \sigma_c$, the QED vacuum acquires the charge $e$ $\mathcal{R}$, thus leading to the concept of a charged vacuum in overcritical fields due to the real vacuum polarization $\mathcal{R}, \mathcal{S}$. An essential detail is that the vacuum charge spatial distribution is similar to the modulus squared of the fermion wave function in the lowest bound state. However, the modulus squared of the fermion wave function is the probability of finding the charge (equal to $e$) at a given spatial point $r$ while the vacuum charge density characterizes the spatial distribution of the real electric charge appearing in the vacuum. The spatial distribution of the real vacuum charge is at $r \to 0$

$$
e|\Psi(r)|^2 \sim cm[2(\ln mr - \xi)^2 - 2s(\ln mr - \xi)/a + 1/a^2]
$$

and at $r \to \infty$

$$
e|\Psi(r)|^2 \sim ee^{-2\sqrt{r/l}}/r, \quad l = 1/\sqrt{2me},
$$

where $e$ depends upon $\xi, \gamma, \mu, a$. In 2+1 dimensions, the QED vacuum can also acquire a magnetic moment equal to the spin magnetic moment of the electron. Other levels will sequentially follow at higher $\sigma_c$. 

IV. QUASISTATIONARY STATES AND CREATION OF MASSLESS FERMIONS

The massless fermions do not have spin degree of freedom in 2+1 dimensions. Nevertheless, the Dirac Hamiltonians in the AB potential for charged massless fermions in 2+1 dimensions keep the introduced spin parameter. So, all obtained solutions (doublets) are valid for the case \( m = 0 \) in the corresponding charge ranges if we put: \( m = 0 \), \( x = -2i|E|r \), \( a^* = \gamma_a + (1 - s)/2 - ie'a \), \( e' = E/|E| \), \( m_0^2 = (s\gamma - ie'a)/\nu \). The main Wronskian \( \omega(E) = \text{Wr}(U_1, V_1) \) at \( m = 0 \) takes the form

\[
\omega_0(E) = \frac{\Gamma(2\gamma)\Gamma(-\gamma + (1 - s)/2 - ia)}{\Gamma(-2\gamma)\Gamma(\gamma + (1 - s)/2 - ia)}(-2i)E^{-2\gamma}e^{x + ia + s\gamma}e^{-\gamma}a = \frac{\omega_0(E)}{\Gamma(-2\gamma)},
\]

(23)

For \( 0 < \gamma < 1/2 \) now the energy spectrum is determined by \( \omega_0(E) = \text{Wr}(U_1, V_2) = \omega_0(E) + 2s\gamma\xi/a \) with \( \omega_0(E) \) (23). It can be verified that in the range \( |E| > 0 \) the functions \( \omega_0(E) \) and \( \omega_0(E) \) are continuous, complex-valued and not equal to zero for real \( E \); the spectral function \( \sigma(E) \) exists and is absolutely continuous. The energy spectrum in the range \( |E| > 0 \) is continuous and the quantum system under discussion does not have bound states. Nevertheless, \( E(a, \nu, s, \xi) \) determined by equation \( \text{Re}\omega_0^2(E) = 0 \)

\[
E = \frac{c'}{2} \left[ \frac{\Gamma(1 + 2\gamma)\Gamma(-\gamma - ia)}{\Gamma(1 - 2\gamma)\Gamma(\gamma - ia)} \right]^{1/2} \sqrt{\nu + s^2 / \nu - s^2},
\]

(24)

may characterize some kind of accumulation points of fermion states and the corresponding values \( a, \nu, s, \xi \) for these points must satisfy equation \( \text{Im}\omega_0^2(E) = 0 \)

\[
\pi \left( e'^p - \frac{1}{2} \right) - 3 + s^2 \arctan \frac{4a\gamma}{4\gamma^2 - (1 + \nu^2)(1 - s)} + \sum_{n=1}^{\infty} \arctan \frac{8a\gamma}{(2n + 1 - s)^2 + 4(a^2 - \gamma^2)} = (p - 1)\pi / 2,
\]

(25)

where \( p = \xi/|\xi| = \pm 1, p = 1(-1) \) for \( \infty > \xi \geq 0(0 \geq \xi > -\infty) \).

We shall put \( \mu > 0 \). The case \( \mu < 0 \) can be discussed similarly with the signs of \( l \) and \( s \) flipped. The range near \( |E| = 0 \) is of interest. For \( \gamma \to 1/2 \)

\[
E = \frac{c'}{2} \left[ \frac{\Gamma(1 - 2\gamma)\Gamma(-1 - 2\gamma - ia)}{\Gamma(1 - 2\gamma - ia)} \right]^{1/2} \sqrt{\nu + s^2 / \nu - s^2},
\]

(26)

hence \( |E| = 0 \) and eq. (25) is satisfied by \( \gamma = 1/2 \) for \( e' = 1, p = 1(\pi \geq \theta \geq 0) \) and for \( e' = -1, p = -1(2\pi \geq \theta \geq \pi) \) only when \( a^2 = \nu^2 - 1/4 \), i.e. at \( \mu = 0 \) only for \( a = 0 \) (compare with claim in 16). There is the particle-hole symmetry in free particle case \( (a, \mu = 0) \).

For \( \gamma \to 0 \), \( |E| \) tends to 0 as \( 2E \approx e'(1/|\xi|)^{1/2} \) and (25) is satisfied by \( e' = \pm 1, \gamma = 0 \) only for \( p = -1(0 \geq \xi > -\infty, 2\pi > \theta \geq 3/2\pi) \). This means that the fermion states heap up close to the point \( E = 0 \) for \( E \) tends to 0, for \( E < 0 \) only when \( |\xi| > 1 \) (see also 10) but no fermion states will cross it as well as no virtual bound states exist when \( q < q_c \). For \( \gamma = \theta \) the point \( E = 0 \) is the branch point of the Wronskians in the complex plane of \( E \); the quasistationary states situate on the unphysical sheet. For \( m = 0 \) the main Wronskian has the form (18) in which \( \omega(E) \) is given by (23) with \( \gamma = \theta \). One can verify again that \( \omega_0^2(E) \) are continuous, complex-valued and is not equal to zero for real \( E \), so no bound states exist. Physically, this is because there is no natural length scale in the problem to characterize bound states. Nevertheless, the virtual bound (resonant) states can emerge when \( q > q_c \); their complex “energies” \( E = |E|e^{i\alpha} \) are determined by:

\[
\frac{\Gamma((1 - s)/2 - i(a + \sigma))}{\Gamma((1 - s)/2 - i(a - \sigma))} \sqrt{\frac{a + s\sigma}{a - s\sigma}} e^{-\pi\sigma + 2\sigma\alpha} = 1,
\]

(27)

and the equation for the energy spectrum

\[
2\sigma \ln(|E|/E_0) = 2\theta - \pi(1 + 2k) - 2\sigma C + \arctan \frac{s\sigma}{\nu} + \sum_{n=1}^{\infty} \left( \frac{2\sigma}{n} - 2 \arctan \frac{2\sigma}{n} + \arctan \frac{2\sigma^2n}{n^2 + \nu^2} \right).
\]

(28)
increase $k$ and decrease the energy. This has to do with the fact that, in reality, the Dirac point is an accumulation point of infinitely many resonances \[16\].

For $\sigma \ll 1$, eq. \[27\] has approximate solution $\alpha \approx -(1 + s)/4\alpha + \text{Im}(ia) + \pi/2$ and $\alpha \approx (1 + \coth(\pi/2))\pi/2 \approx (1 + 0.04)\pi$ for $a = 1/2, s = 1$. Eqs. \[27\] and \[28\] are approximately satisfied near $|E| = 0$ only for hole region $E < 0$. Indeed, for $a > \nu, \sigma > 0$ eq. \[27\] is satisfied only at $c' = -1, \tau > \pi$ for which the right hand side of \[28\] is negative. Then, for $\sigma \ll 1$ the energy spectrum is

$$E_{k,\theta,s} = E_0 \cos(\alpha) \exp \left[-\pi(1 + 2k)/2\sigma + \theta/\sigma - (C + (1 - s)/2 + \pi^2/6 - (\pi \coth \pi a)/2a)\right]. \quad (29)$$

These energies have an essential singular point at $\sigma = 0$ \[13, 17, 24\]. The infinite number of quasistationary levels is related to the long-range character of the Coulomb potential \[16, 17, 24\]. These quasi-localized resonances have negative energies, thus they are situated in the hole sector. The resonances are directly associated with the positron production in the QED \[50\].

The imaginary part of $E_{k,\theta,s}$ defines the width of virtual resonant level $\Gamma_{k,\theta,s}$ or the inverse lifetime (decay rate) of particle resonance. For $\sigma \ll 1$ this width $\sim |E_{k,\theta,s}|$ is very small, hence, the resonances are practically stationary states.

The spectrum in the case of charged massless fermions is continuous everywhere, and so there is no restructuring of negative energy (hole) continuum in overcritical fields due to the real vacuum polarization as described for the massive case. The physical picture can be seen as follows. If the emergent virtual level was empty, a quasiparticle pair will be created: the fermion (particle) of the filled valence band occupies this level and shields the center, while the emergent (in the valence band) hole is escaped to infinity. Now the quantity $\Gamma_{k,\theta,s}$ is the doubled probability of the creation of the quasiparticle pair by the Coulomb potential in the presence of AB potential.

The physically meaningful quantity is the number of pairs created per unit area of graphene per unit time. So, the creation of massless charged fermions can be studied by means of the local density of states (LDOS) in the hole continuum. The LDOS per unit area is determined as a function of energy and distance from origin by \[16\]

$$N(E,r) = \sum_l |\Psi(t, r)|^2 = \sum_l n_l(E,r), \quad n_l(E,r) = \frac{|f_1(r, E,l)|^2 + |f_2(r, E,l)|^2}{2|A_l(E)|^2 \pi r}, \quad (30)$$

where $f_1(r, E,l)/A_l(E)$ and $f_2(r, E,l)/A_l(E)$ are the doublets normalized (on the half-line with measure $dr$) by imposing orthogonality on the energy scale and $A_l(E)$ is the normalization constant.

The LDOS is: 1. $N_{reg}(E,r) = \sum_l n_l(E,r)$ with $n_l(E,r)$ constructed by regular solutions of \[10\] and with the sum taken over $l$ of $\sqrt{(l + \mu + s/2)^2 - a^2} \geq 1/2$ for $\gamma \geq 1/2$. For $a = 0, \mu = 0$ the free density of states is recovered from $N_{reg}(E,r) = |E|/2\pi$; 2. $N_{\xi}(E,r) = \sum_l n_l(\xi,E,r)$ with the sum taken over $l$ of $1/2 > \sqrt{(l + \mu + s/2)^2 - a^2} > 0$ for $1/2 > \gamma > 0$; 3. $N_{\theta}(E,r) = \sum_l n_l(\theta,E,r)$ with $n_l(\theta,E,r)$ constructed by \[17\] and the sum taken over $l$ of $a^2 > (l + \mu + s/2)^2$ for the overcritical range $\gamma = i\sigma, 0 \geq \theta \geq \pi$. The total LDOS is $N(E,r) = N_{reg}(E,r) + N_{\xi}(E,r) + N_{\theta}(E,r)$.

The LDOS exhibits resonances of the width $\sim |E_{k,\theta,s}|$ at the negative energies \[29\], which decay away from the impurity (Figs. 1 for $s = 1$ and 2 for $s = -1$); strong resonances signal the presence of quasistationary states, i.e. the creation of charged fermions. It should be commented that, according to \[9\], the families of the curves for the LDOS with another sign $s$ are qualitatively like to the ones given in Figs. 1 and 2 at the same values of $a, \mu, \xi, \theta$ except to the shift $\pm l \rightarrow \pm l + 1, s \rightarrow -s$.

**FIG. 1.** LDOS $N_0(E,r)$ with $l = -2, -1, 0$ for $a = 1.5, \mu = 0.1, s = 1 (\sigma \approx 0.539, 1.446, 1.375)$ and $r = 0.3$ (a), $r = 1$ (b); the insets are magnifications for $E \approx 0$.

Increasing the effective charge will cause energy quasiparticles to decrease and their number to increase.
of quasiparticles creation, are obtained \[51\].

Some range of extension parameter there reveals an infinite number of quasistationary states. (i) there is no restructuring of the hole (lower) continuum, (ii) in the presence of magnetic flux with \(\mu > 0\), there exists a single resonance (with \(k = 0\) at \(\theta = \pi/2\), and only for \(s = 1\)), which is in good accord with \[20\].

\[ \Gamma_g \sim mc^{-\sqrt{2m\pi a^2}/\epsilon}. \]

The critical charge, respectively, decreases (increases) at fixed \(a\) in the presence of magnetic flux with \(\mu > 0\) for \(s = -1\) (\(s = 1\)). This means that the vacuum of the quantum electrodynamics becomes unstable, which results in positron creation; it is reconstructing; a new state with the energy \(E < -m\) emerges and is spread out over an energy range of the order \(\Gamma_g \sim mc^{-\sqrt{2m\pi a^2}/\epsilon}\). The critical charge, respectively, decreases (increases) at fixed \(a\) in the presence of magnetic flux with \(\mu > 0\) for \(s = -1\) (\(s = 1\)). This means that the vacuum of the quantum electrodynamics in 2+1 dimensions in Coulomb and AB potentials with \(\mu > 0\) becomes less stable with respect to the creation of positrons with the spin \(s_p = 1\) and more stable with respect to the creation of positrons with the spin \(s_p = -1\).

The creation of massless charged quasiparticles in Coulomb and AB fields in graphene differs with the case of massive particles as follows: (i) there is no restructuring of the hole (lower) continuum, (ii) in some range of extension parameter there reveals an infinite number of quasistationary states at \(\sigma > \sigma_c\) in the lower continuum, (iii) when the mass \(m = 0\) there is no natural length scale to characterize such quasistationary states.

It will be noted that at the moment graphene single crystals with characteristics (such as dimensions, electron mobility or concentration of impurities), which is favorable enough for observation of the effect of quasiparticles creation, are obtained \[51\].
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