Control parameter optimization of a nonlinear vehicle suspension system with time-delayed acceleration feedback

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Abstract
A time-delayed acceleration feedback control is proposed to improve the vibration performance of a nonlinear vehicle suspension system. First, the harmonic balance method is applied to obtain the vertical acceleration amplitude of the system excited by simple harmonic road excitation. Then, taking the amplitude of the sprung mass acceleration and control force into account, the single-objective and multiple-objective optimization problems of time-delayed feedback control parameters, respectively, are discussed. Finally, the mathematical simulation is provided to verify the correctness of the optimization results. It is concluded that the nonlinear suspension with optimal time-delayed feedback control has better vibration control performance compared to passive one. The acceleration amplitude of the sprung mass is significantly reduced by the single-objective optimization of the control parameters. Moreover, when the optimal time delay is introduced, the active control force input is fewer than that without time delay. The phenomenon of energy transfer between the sprung mass and the unsprung mass is observed in some road-excitation frequencies.

Keywords
Time-delayed feedback control, parameter optimization, vibration suppression, nonlinear suspension

Introduction
The suspension system is a key component which affects the ride comfort, handling stability, and safety of the vehicle. Typically, the vehicle suspension is divided into three categories: passive, active, and semi-active suspension. Owing to the advantages of simple structure, easy realization, and low cost, the passive suspension is commonly used in vehicles. However, the passive suspension loses efficacy in the case of the complex road conditions. According to the driving conditions, the real-time adjustment of control parameters in active and semi-active suspensions may be realized to achieve satisfactory performance. Hence, a great deal of effort has been devoted to the studies of active and semi-active suspensions in recent years. Various control strategies are presented to improve the performance of the suspension such as sliding mode control, fuzzy control, and so on.

It should be pointed out that time delay is inherent in active and semi-active suspensions, which comes from the process of the acquisition and transmission of signals, the implementation of control algorithms, and the operation of the actuator. Previous studies have shown that time delay has a great influence on the performance of vehicle suspension, which may induce chaos, weaken the vibration control effect, or even unstabilize the system. There have been extensive studies and applications of time-delay compensation technologies to reduce or counteract the negative influence of time delays. With the in-depth study of time-delay problem, it is found that the proper introduction of time delay in the control loop may improve the vibration control effect and prevent the...
complex dynamical behavior of the suspension. Therefore, the time-delayed feedback control is suggested and applied to improve the performance of the active and semi-active suspensions, in which time delay and feedback gain coefficient are taken as adjustable control parameters. Abdelhafez and Omara\textsuperscript{12} developed the proportion derivative controller and positive position feedback controller with time delay in a quarter-car suspension system to achieve stability and comfort. Li et al.\textsuperscript{13} designed a delayed positive feedback controller with a controllable-damping and a variable stiffness for a single degree of freedom (SDOF) semi-active suspension on in-vehicle networks. Afshar et al.\textsuperscript{14} applied a delay-dependent memory state-feedback $H_{\infty}$ controller to a quarter-car model of an active suspension considering the actuator time delay. Kucukefe and Kaypmaz\textsuperscript{15} presented a time-delay auto-synchronization controller to improve damping of natural mode oscillations excited by terrain disturbance in a 7-DOF full-vehicle model. Wang and Song\textsuperscript{16} proposed an analytical computation method of steady-state probability density function of a nonlinear automotive system with time-delayed displacement and speed feedbacks. Yan et al.\textsuperscript{17} theoretically and experimentally studied the time-delayed optimal control of vehicle suspension system. The influence of time delay on the acceleration amplitude of sprung mass, suspension deflection, and rode holding is analyzed. Taffo et al.\textsuperscript{18} investigated the dynamics of a parametrically excited SDOF quarter-car with time-delayed position and velocity feedbacks. The effect of parametric excitation, time delay, and feedback gain coefficients on the stability of the system is discussed. After that, Taffo et al.\textsuperscript{19} studied the stability switches and bifurcation of a harmonically excited 2-DOF nonlinear quarter-car with time-delayed position feedback control. Aiming at an SDOF nonlinear quarter-car forced by simple harmonic road excitation, Naik and Singru\textsuperscript{20} analyzed the effect of time delay and feedback gains on the steady-state responses of primary, super-harmonic, and sub-harmonic resonances when the time-delayed state feedback is activated.

To maximize the performance of the suspension, a key issue is the control parameter optimization of time-delayed feedback control. However, limited work has been done in this respect. The main purpose of this paper is to solve the optimization problem of the time-delayed feedback control parameters of a nonlinear suspension in order to minimize the amplitudes of the sprung mass vertical acceleration and active control force.

This paper is organized as follows. After an introduction, the mechanical and mathematical models of the nonlinear suspension system under time-delayed acceleration feedback control are established in “System modeling” section. Single-objective and multiple-objective optimization problems of control parameters are, respectively, discussed in “Parameter optimization of time-delayed feedback control” section. Conclusions are given in the final section.

System modeling

Since the vertical vibration of the vehicle is the key factor that affects the ride comfort, the model under consideration is presented in Figure 1. The model consists of sprung mass $m_1$ and unsprung mass $m_2$. $x_1$ and $x_2$ denote the vertical displacement of the sprung mass and the unsprung mass, respectively. The road-displacement excitation is represented by a simple harmonic function $x_3 = r\sin(\omega t)$. The sprung mass is supported by a linear damper and a nonlinear spring. $c_1$ is the linear damping coefficient of the damper. $k_1$ and $\delta_1$ are the linear and cubic nonlinear stiffness coefficients of the spring. $c_2$ is the linear damping coefficient of the tyre. $k_2$ and $\delta_2$ are the linear and cubic nonlinear stiffness coefficients of the tyre. An actuator is mounted between the sprung mass and

![Figure 1. Quarter vehicle suspension model.](image-url)
the unsprung mass to provide the time-delayed feedback control force $u = g \frac{d^2 x_1(t-\tau)}{dt^2}$, $g$ is feedback gain coefficient, and $\tau$ is time delay.

The dynamic equations of the quarter vehicle suspension model are given by

$$m_1 \frac{d^2 x_1}{dt^2} + c_1 \left( \frac{dx_1}{dt} - \frac{dx_2}{dt} \right) + k_1(x_1 - x_2) + \delta_1(x_1 - x_2)^3 + g \frac{d^2 x_1(t-\tau)}{dt^2} = 0$$  \hspace{1cm} (1)

$$m_2 \frac{d^2 x_2}{dt^2} - c_1 \left( \frac{dx_1}{dt} - \frac{dx_2}{dt} \right) + c_2 \left( \frac{dx_2}{dt} - \frac{dx_3}{dt} \right) - k_1(x_1 - x_2) + k_2(x_2 - x_3)$$

$$-\delta_1(x_1 - x_2)^3 + \delta_2(x_2 - x_3)^3 - g \frac{d^2 x_1(t-\tau)}{dt^2} = 0$$ \hspace{1cm} (2)

Applying the harmonic balance method,\textsuperscript{21} the approximate solution of $x_1$ and $x_2$ is sought to be

$$x_1 = p_1 \sin(\omega t) + q_1 \cos(\omega t)$$ \hspace{1cm} (3)

$$x_2 = p_2 \sin(\omega t) + q_2 \cos(\omega t)$$ \hspace{1cm} (4)

where $p_1$, $q_1$, $p_2$, and $q_2$ are unsolved coefficients.

Substituting equations (3) and (4) into equations (1) and (2) and then extracting the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$ from both sides of the equations, the nonlinear algebraic equations about $p_1$, $q_1$, $p_2$, and $q_2$ are obtained

$$\begin{bmatrix}
expr_1 
expr_2 
expr_3 
expr_4
\end{bmatrix}^T = \begin{bmatrix} 0 \end{bmatrix}$$ \hspace{1cm} (5)

The expressions of $\text{expr}_1$–$\text{expr}_4$ refer to Appendix 1. The values of $p_1$, $q_1$, $p_2$, and $q_2$ can be calculated by numerically solving equation (5). Hereinafter, $A$, $B$, and $U$ are defined as the amplitudes of the sprung mass acceleration, the unsprung mass acceleration, and the control force, respectively

$$A = \omega^2 \sqrt{p_1^2 + q_1^2}$$ \hspace{1cm} (6)

$$B = \omega^2 \sqrt{p_2^2 + q_2^2}$$ \hspace{1cm} (7)

$$U = \left| g \omega^2 \sqrt{p_1^2 + q_1^2} \right|$$ \hspace{1cm} (8)

**Parameter optimization of time-delayed feedback control**

In this section, the effect of the nonlinear stiffness on the amplitude-frequency characteristic of the passive suspension is first discussed, and then the single-objective and multi-objective optimization problems of time-delayed feedback control parameters are solved.

**Nonlinear passive suspension**

The physical parameter values of the quarter vehicle suspension model are given in Table 1.

The influence of nonlinear stiffness on the amplitude-frequency characteristic of the passive suspension system is shown in Figure 2, where $r = 0.01m$ and $\Omega = \omega / (2\pi)$. It is observed from Figure 2 that in comparison with those in linear suspension system, the amplitudes of the sprung mass and the unsprung mass in nonlinear suspension system obviously decrease when $\Omega > 6$ Hz. It is straightforward to show that the introduction of nonlinear stiffness improves the vibration control effect of the suspension system. In addition, although there are quantitative differences, the numerical results are in agreement with the theoretical ones qualitatively.
In this section, taking time delay and feedback control coefficient as optimized parameters and minimum sprung mass acceleration amplitude as optimization objective, the single-objective optimization of the control parameters is carried out.

The optimization process is as follows.

Step 1: For convenience, the excitation frequency, feedback gain coefficient, and time delay are discretized in different ranges. Discrete frequency points \( \Omega_i \) (\( i = 1, 2, \ldots, 40 \)) are obtained at intervals of 0.5 Hz within the range of 0.5–20 Hz. Discrete values \( g_m \) (\( m = 1, 2, \ldots, 800 \)) of feedback gain coefficient are determined at intervals of 1 kg within the range of [–400 kg, 0 kg \( \cup \) 0 kg, 400 kg]. Discrete values \( \tau_n \) (\( n = 1, 2, \ldots, 1400 \)) of time delay are taken at intervals of 0.001 s within the range of (0 s, 1.4 s]. Hence, \( 1.12 \times 10^6 \) sets \( (g_m, \tau_n) \) of control parameters are obtained.

Step 2: For \( \Omega = \Omega_i \), each set of control parameters is successively substituted into equation (6), and thus, \( 1.12 \times 10^6 \) values of sprung mass acceleration amplitude are computed.

Step 3: Minimum amplitude \( A_{\text{min}} \) of the sprung mass acceleration is recorded, and the corresponding values of control parameters are taken as the optimal ones denoted by \( g_{\text{op}} \) and \( \tau_{\text{op}} \).

Step 4: Numerical simulation of the original equations (1) and (2) is conducted to verify the stability of the system. If the system is stable, \( A_{\text{min}} \), corresponding \( g_{\text{op}} \) and \( \tau_{\text{op}} \) are outputted. Otherwise, these three values are removed, and the above process is repeated until the system is stable.

According to the optimization process, the minimum sprung mass acceleration amplitude and the optimal values of control parameters at each frequency point \( \Omega_i \) are obtained.

The optimization results are shown in Appendix 2. It is noted that for each excitation frequency, there are two optimal values of feedback gain coefficient. In addition, multiple optimal values of time delay exist for some optimal feedback gain coefficient values, which provide flexibility for the choice of time delay. Figure 3 shows how
the acceleration amplitudes of the system change as a function of excitation frequency. Dashed line represents the theoretical results with passive control. The dotted and solid line, respectively, represent the numerical and the theoretical results with optimal time-delayed feedback control. From Figure 3(a), the acceleration amplitude of the sprung mass under optimal time-delayed feedback control significantly decreases compared to that under passive control. Especially, the acceleration amplitude of the sprung mass is reduced by 50.00% and 20.20% for \( \Omega = 1 \) and 6 Hz, respectively.

Figures 4 and 5 illustrate the acceleration time histories of the system for \( \Omega = 1 \) and 6 Hz, respectively, where \( r = 0.01 \text{m} \). Obviously, the numerical results agree well with the corresponding theoretical ones as shown in Figure 3. It is indicated that the optimization results of the control parameters are correct.
Multi-objective optimization of control parameters

To balance the vibration control effect and control force input, multi-objective optimization of time delay and feedback gain coefficient is considered. Thus, linear-weighted combination of the amplitude of the sprung mass and feedback control force is determined to be the objective function

\[
\min \quad J = z_{11}z_{21}A + z_{12}z_{22}U
\]

s. t.

\[
\begin{align*}
&g \in [-400kg, 0kg) \cup (0kg, 400kg] \\
&\tau \in (0s, 1.4s]
\end{align*}
\]

Table 2. Values of \( \Phi_A, \Phi_{A,op}, \Phi_B, \Phi_U \) and \( \Phi_U \) for different excitation frequencies.

| \( \Omega \) (Hz) | \( \Phi_A \) | \( \Phi_B \) | \( \Phi_U \) |
|------------------|----------------|----------------|----------------|
| 0.5              | -49.61         | -55.81         | -49.61         |
| 1                | -72.44         | -52.17         | -72.44         |
| 1.5              | -78.05         | -27.71         | -78.05         |
| 2                | -80.06         | -15.27         | -80.06         |
| 2.5              | -76.53         | 8.73           | -76.53         |
| 3                | -72.49         | 31.70          | -72.49         |
| 3.5              | -67.95         | 55.24          | -67.95         |
| 4                | -63.00         | 79.51          | -63.00         |
| 4.5              | -57.45         | 107.15         | -57.45         |
| 5                | -51.30         | 136.35         | -51.30         |
| 5.5              | -42.42         | 145.69         | -42.42         |
| 6                | -35.22         | 173.87         | -35.22         |
| 6.5              | -42.16         | 188.59         | -42.16         |
| 7                | -52.17         | 162.54         | -52.17         |
| 7.5              | -52.39         | 146.92         | -52.39         |
| 8                | -54.00         | 135.31         | -54.00         |
| 8.5              | -54.96         | 124.43         | -54.96         |
| 9                | -57.71         | 94.07          | -57.71         |
| 9.5              | -58.87         | 95.78          | -58.87         |
| 10               | -59.78         | 91.55          | -59.78         |
| 10.5             | -60.69         | 88.63          | -60.69         |
| 11               | -61.56         | 84.88          | -61.56         |
| 11.5             | -62.32         | 82.40          | -62.32         |
| 12               | -63.08         | 78.58          | -63.08         |
| 12.5             | -63.79         | 75.72          | -63.79         |
| 13               | -64.46         | 72.60          | -64.46         |
| 13.5             | -65.12         | 70.24          | -65.12         |
| 14               | -65.68         | 67.64          | -65.68         |
| 14.5             | -66.29         | 65.04          | -66.29         |
| 15               | -66.85         | 62.67          | -66.85         |
| 15.5             | -67.37         | 59.36          | -67.37         |
| 16               | -67.88         | 58.16          | -67.88         |
| 16.5             | -68.34         | 55.93          | -68.34         |
| 17               | -68.80         | 52.65          | -68.80         |
| 17.5             | -69.21         | 51.32          | -69.21         |
| 18               | -69.62         | 49.71          | -69.62         |
| 18.5             | -70.02         | 48.36          | -70.02         |
| 19               | -70.38         | 46.01          | -70.38         |
| 19.5             | -70.74         | 44.07          | -70.74         |
| 20               | -71.10         | 42.52          | -71.10         |
where \( z_{11} \) and \( z_{12} \) are intrinsic weight factors, and \( z_{21} \) and \( z_{22} \) are correction weight factors. \( z_{11} = 0.6, z_{12} = 0.4, z_{21} = 1.623, \) and \( z_{22} = 0.0012. \) The optimization process is the same as that shown in “Single-objective optimization of control parameters.” The optimization results are shown in Appendix 3.

To facilitate quantitative analysis, \( \Phi_A, \Phi_B, \) and \( \Phi_U \) are defined to denote the percentage change in amplitudes of the sprung mass acceleration, the unsprung mass acceleration, and the active control force, respectively. Table 2 shows the values of \( \Phi_A, \Phi_B, \) and \( \Phi_U \) for different excitation frequencies. From Table 2, the amplitude of the sprung mass acceleration and feedback control force with optimal time delay significantly decrease compared to those without time delay. For \( \Omega = 1 \) and 6 Hz, the amplitudes of the sprung mass acceleration and control force are reduced by 72.44% and 35.22%, respectively. Obviously, once the optimal time delay is activated, the superior vibration control effect and less control force are simultaneously achieved

\[
\Phi_i = \frac{Y_i[g=g_{op},r=r_{op}]-Y_i[g=g_{op},r=0]}{Y_i[g=g_{op},r=0]} \times 100\%, \quad (i = A, B, U) \tag{10}
\]

In Figure 6, the comparison of the amplitude-frequency curves of the system with and without optimal time delay is shown. For \( \Omega \in [2.5 \text{ Hz}, 20 \text{ Hz}], \) the acceleration amplitude of the sprung mass decreases while that of the unsprung mass increases when the optimal time delay is introduced. It follows that the vibration energy is transferred from the sprung mass to the unsprung mass. Figure 7 demonstrates how the amplitude of the feedback control force changes with the variation of excitation frequency.

To verify the correctness of the multi-objective optimization results, Figures 8 and 9 show the time histories of the system acceleration and feedback control force for \( \Omega = 1 \) and 6 Hz, where \( r = 0.01 \text{m}. \) Clearly, the numerical results are consistent with the theoretical ones shown in Figures 6 and 7.
The vibration control effect of the nonlinear vehicle suspension is greatly improved by the single-objective optimization of control parameters. Compared to that under passive control, the acceleration amplitude of the sprung mass is reduced by 50% for $\Omega = 1$ Hz. For fixed excitation frequency, there are multiple values of $\tau$, including $\tau = 0$ and $\tau = 0.056s$, under optimal time-delayed feedback control. Moreover, in some excitation frequencies, there is an increase in the amplitude of the sprung mass acceleration and control force due to the presence of time delay, the amplitude of the sprung mass acceleration and control force is reduced by 72.44% and 35.22%, respectively, under optimal time-delayed feedback control. Additionally, for $\Omega = 6$ Hz: (a) sprung mass, (b) unsprung mass, and (c) feedback control force.

Conclusions

The single-objective and the multi-objective optimizations of time-delayed feedback control parameters in a nonlinear vehicle suspension are carried out. The conclusions are summarized as follows:

1. The vibration control effect of the nonlinear vehicle suspension is greatly improved by the single-objective optimization of control parameters. Compared to that under passive control, the acceleration amplitude of the sprung mass is significantly reduced.
sprung mass is reduced by 50% for $\Omega = 1$ Hz. For fixed excitation frequency, there are multiple values of optimal control parameters, which benefit the flexible choice of control parameters.

2. The multiple-objective optimization of control parameters reduces the control force amplitude while improving the vibration control effect. Especially, for $\Omega = 1$ and $6$ Hz, compared to those under feedback control without time delay, the amplitude of the sprung mass acceleration and control force is reduced by 72.44% and 35.22%, respectively, under optimal time-delayed feedback control. Moreover, in some excitation frequencies, there is energy transfer between the sprung mass and the unsprung mass.

**Declaration of conflicting interests**

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**Appendix**

**Notation**

\[ A \text{ amplitude of sprung mass acceleration (m/s}^2) \]
\[ B \text{ amplitude of unsprung mass acceleration (m/s}^2) \]
\[ c_1 \text{ linear damping coefficient of the damper (N·s/m)} \]
\[ c_2 \text{ linear damping coefficient of the tyre (N·s)} \]
\[ g \text{ feedback gain coefficient (kg)} \]
\[ g_{op} \text{ optimal value of feedback gain coefficient (kg)} \]
\[ k_1 \text{ linear stiffness coefficient of the spring (N/m)} \]
\[ k_2 \text{ linear stiffness coefficient of the tyre (N/m)} \]
\[ m_1 \text{ sprung mass (kg)} \]
\[ m_2 \text{ unsprung mass (kg)} \]
\[ r \text{ amplitude of road displacement excitation (m)} \]
\[ t \text{ time (s)} \]
\[ u \text{ time-delayed feedback control force (N)} \]
\[ U \text{ amplitude of active control force (N)} \]
\[ x_1 \text{ vertical displacement of sprung mass (m)} \]
\[ x_2 \text{ vertical displacement of unsprung mass (m)} \]
\[ x_3 \text{ road displacement excitation (m)} \]
\[ \delta_1 \text{ cubic nonlinear stiffness coefficient of the spring (N/m}^3) \]
\[ \delta_2 \text{ cubic nonlinear stiffness coefficient of the tyre (N/m}^3) \]
\[ \tau \text{ time delay (s)} \]
\[ \tau_{op} \text{ optimal value of time delay (s)} \]
\[ \Phi_A \text{ percentage change in amplitude of sprung mass acceleration} \]
\[ \Phi_B \text{ percentage change in amplitude of unsprung mass acceleration} \]
\[ \Phi_U \text{ percentage change in amplitude of active control force} \]
\[ \omega \text{ angular frequency of road displacement excitation (rad/s)} \]
\[ \Omega \text{ frequency of road displacement excitation (Hz)} \]

**Appendix 1**

\[
expr_1 = k_1 p_1 - m_1 \omega^2 p_1 + 0.75 \delta_1 p_1^3 - c_1 \omega q_1 + 0.75 \delta_1 q_1^2 - k_1 p_2 - 2.25 \delta_1 p_1^2 p_2 - 0.75 \delta_1 q_1^2 p_2 + 2.25 \delta_1 p_1^3 p_2 - 0.75 \delta_1 p_2^2 + 1.5 \delta_1 q_1 q_2 + 1.5 \delta_1 q_1 q_2 + 0.75 \delta_1 p_1 q_2^2 - 0.75 \delta_1 p_2 q_2^2 - g \omega^2 p_1 \cos(\omega t) - g \omega^2 q_1 \sin(\omega t)
\]

\[
expr_2 = c_1 \omega p_1 + k_1 q_1 - m_1 \omega^2 q_1 + 0.75 \delta_1 p_1^2 q_1 + 0.75 \delta_1 q_1^2 - c_1 \omega p_3 - 1.5 \delta_1 q_1 p_3 - 0.75 \delta_1 q_1 p_3^3 - k_1 q_2 - 0.75 \delta_1 p_1 q_2 + 2.25 \delta_1 q_1 q_2^2 + 1.5 p_1 p_3 q_2 - 0.75 \delta_1 p_1 q_2 + 2.25 \delta_1 q_1 q_2^2 - 0.75 \delta_1 q_2^3 - g q_1 \cos(\omega t) + g \omega^2 p_1 \sin(\omega t)
\]
expression

\[ expr3 = -k_2 r - 0.75 \delta_2 r^3 - k_1 p_1 - 0.75 \delta_1 p_1^3 + c_1 \omega q_1 - 0.75 \delta_1 p_1 q_1^2 + k_1 p_2 + k_2 q_2 + 2.25 \delta_2 r^2 p_2 \\
- m_2 \omega^2 p_2 + 2.25 \delta_1 p_1^2 p_2 + 0.75 \delta_1 q_1^2 p_2 - 2.25 \delta_2 r p_2^2 - 2.25 \delta_1 p_1 p_2^2 + 0.75 \delta_1 p_2^3 + 0.75 \delta_2 p_2^3 \\
- c_1 \omega q_2 - c_2 \omega p_2 + 1.5k_1 p_1 q_1 - 1.5k_1 \delta_2 p_2 q_2 - 0.75 \delta_2 r q_2^2 - 0.75 \delta_1 p_1 q_2^2 + 0.75 \delta_1 p_2 q_2^2 \\
+ 0.75 \delta_2 p_2 q_2^2 + g \omega^2 p_1 \cos(\omega t) + g \omega^2 q_1 \sin(\omega t) \]

\[ expr4 = -c_2 \omega - 0.75 \delta_1 p_1^2 q_1 - 0.75 \delta_1 q_1^3 + c_1 \omega p_2 + c_2 \omega p_2 + 1.5 \delta_1 p_1 q_1 p_2 \\
- 0.75 \delta_1 q_1 p_2^2 + k_2 q_2 + k_2 q_2 + 0.75 \delta_2 r^2 q_2 - m_2 \omega^2 p_2 + 0.75 \delta_1 p_2^2 q_2 + 2.25 \delta_1 q_1^2 q_2 \\
- 1.5 \delta_2 r p_2 q_2 - 1.5 \delta_1 p_1 p_2 q_2 + 0.75 \delta_1 p_2^2 q_2 + 0.75 \delta_2 p_2^2 q_2 - 2.25 \delta_1 q_1 q_2^2 + 0.75 \delta_1 q_2^3 \\
+ 0.75 \delta_2 q_2^3 + g \omega^2 q_1 \cos(\omega t) - g \omega^2 p_1 \sin(\omega t) \]

**Appendix 2**

**Table 3. Results of single-objective optimization.**

| \( \Omega \) (Hz) | \( g_{op} \) (kg) | \( \tau_{op} \) (s) | \( \Omega \) (Hz) | \( g_{op} \) (kg) | \( \tau_{op} \) (s) |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0.5               | -400              | 1.388             | 10.5              | -400              | 1.309             |
| 400               | 0.588             | 0.887             | 0.962             |
| 1                 | -400              | 0.668             | 11                | -400              | 0.873             | 0.964             | 1.055             |
| 400               | 0.168             | 0.827             | 0.918             | 1.009             |
| 1.5               | -400              | 1.043             | 11.5              | -400              | 0.747             | 0.834             |
| 400               | 0.043             | 0.959             | 0.182             | 0.269             |
| 2                 | -400              | 0.765             | 12                | -400              | 0.716             | 0.799             |
| 400               | 0.015, 0.515      | 0.091             | 0.174             | 0.258             |
| 2.5               | -400              | 0.604             | 12.5              | -400              | 0.607             | 0.687             | 0.767             |
| 400               | 0.404             | 0.087             | 0.167             | 0.247             | 0.327             |
| 3                 | -400              | 0.832             | 13                | -400              | 0.045             |
| 400               | 0.138, 0.332      | 0.160             | 0.237             | 0.314             | 0.391             |
| 3.5               | -400              | 0.709             | 13.5              | -400              | 0.113             |
| 400               | 0.567, 0.853      | 0.080             | 0.154             | 0.228             | 0.302             |
| 4                 | -400              | 0.618             | 14                | -400              | 0.078             | 0.149             | 0.220             | 0.292             |
| 400               | 0.241, 0.491      | 0.109             |
| 4.5               | -400              | 0.769             | 14.5              | -400              | 0.074             | 0.143             | 0.212             | 0.281             | 0.350             |
| 5                 | -400              | 0.214, 0.436      | 15               | -400              | 0.105             |
| 400               | 0.290, 0.890      | 0.063             | 0.122             | 0.239             | 0.298             | 0.357             |
| 5.5               | -400              | 0.590, 0.990      | 15.5              | -400              | 0.071             | 0.138             | 0.205             | 0.271             |
| 400               | 0.608, 0.790      | 0.101             |
| 0.154, 0.336, 0.518 | -400              | 0.069             | 0.134             | 0.198             | 0.263             | 0.327             |
| 0.389, 0.556      | 16                | -400              | 0.099             |
| 0.143, 0.309, 0.476 | 400               | 0.067             | 0.129             | 0.192             | 0.254             | 0.317             |
| 6.5               | -400              | 0.542, 0.696      | 16.5              | -400              | 0.095             |
| 400               | 0.309, 0.463, 0.617 | 0.057             | 0.111             | 0.165             |
| 7                 | -400              | 0.647             | 17                | -400              | 0.151             |
| 400               | 0.019             | 0.063             | 0.122             | 0.239             | 0.298             | 0.357             |
| 7.5               | -400              | 1.140             | 17.5              | -400              | 0.147             |
| 400               | 0.142, 0.542      | 0.061             | 0.118             | 0.175             | 0.232             |
| 0.936             | 18                | -400              | 0.142             |
| 0.016             | 18.5              | -400              | 0.138             | 0.192             |
| 0.122, 0.592      | 19                | -400              | 0.057             | 0.111             | 0.165             |
| 400               | 0.016             | 0.135             | 0.188             |
| 0.575v0.012       | 19.5              | -400              | 0.108             | 0.161             | 0.214             | 0.266             |
| 400               | 0.106, 0.157, 0.208, 0.259, 0.311 | 0.131             | 0.182             |
| 10                | -400              | 0.061             | 20                | -400              | 0.478             | 0.528             | 0.578             |
| 400               | 0.011, 0.111      | 0.103             | 0.153             | 0.203             | 0.253             | 0.303             |
### Appendix 3

**Table 4.** Results of multiple-objective optimization.

| Ω (Hz) | $g_{op}$ (kg) | $\tau_{op}$ (s) | Ω (Hz) | $g_{op}$ (kg) | $\tau_{op}$ (s) |
|--------|----------------|-----------------|--------|----------------|-----------------|
| 0.5    | -400           | 1.388           | 10.5   | -400           | 1.309           |
| 1      | -400           | 0.668           | 11     | -400           | 0.691, 0.782    |
| 1.5    | -400           | 0.376           | 11.5   | -400           | 0.573, 0.660    |
| 2      | -400           | 0.265           | 12     | -400           | 0.549, 0.633    |
| 2.5    | -400           | 0.204           | 12.5   | -400           | 0.047, 0.527    |
| 3      | -400           | 0.165           | 13     | -400           | 0.045           |
| 3.5    | -400           | 0.138           | 13.5   | -400           | 0.043           |
| 4      | -400           | 0.118           | 14     | -400           | 0.041           |
| 4.5    | -400           | 0.102           | 14.5   | -400           | 0.040           |
| 5      | -400           | 0.090           | 15     | -400           | 0.039           |
| 5.5    | -400           | 0.426           | 15.5   | -400           | 0.037           |
| 6      | -400           | 0.056           | 16     | -400           | 0.099           |
| 6.5    | -400           | 0.388           | 16.5   | -400           | 0.095           |
| 7      | -400           | 0.504           | 17     | -400           | 0.092           |
| 7.5    | -400           | 0.740           | 17.5   | -400           | 0.090           |
| 8      | -400           | 0.811           | 18     | -400           | 0.086           |
| 8.5    | -400           | 0.535           | 18.5   | -400           | 0.084           |
| 9      | -400           | 0.497           | 19     | -400           | 0.082           |
| 9.5    | -400           | 0.364           | 19.5   | -400           | 0.080           |
| 10     | -400           | 0.061           | 20     | -400           | 0.378, 0.428    |