Geometry of Pointwise Semi-slant Warped Products in Locally Conformal Kaehler Manifolds

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Abstract. In this paper, we study the geometry of pointwise semi-slant warped products in a locally conformal Kaehler manifold. In particular, we obtain several results which extend Chen’s inequality for CR-warped product submanifolds in Kaehler manifolds. Also, we study the corresponding equality cases. Several related results on pointwise semi-slant warped products are also proved in this paper.

Mathematics Subject Classification. 53C15, 53C40, 53C42, 53B25.

Keywords. Warped products, CR-warped product, pointwise semi-slant warped products, locally conformal Kaehler manifold, Chen inequality.

1. Introduction

The notion of slant submanifolds of an almost Hermitian manifold, which includes both classes of totally real and holomorphic submanifolds, was introduced by Chen in [11,12]. Since then many papers on slant submanifolds have been published (see, e.g., recent books [20,21]). Also, as the generalizations of totally real and holomorphic submanifolds, A. Bejancu introduced the notion of CR-submanifolds in [2]. CR-submanifolds have also been studied by many geometers (see, e.g. [3,4,6,8–10]).

As a generalization of slant submanifolds, Papaghiuc [31] introduced the notion of semi-slant submanifolds of an almost Hermitian manifold which includes the classes of CR-submanifolds and slant submanifolds. As another extension of slant submanifolds, Etayo [25] defined the notion of pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant...
submanifolds. Then, Chen and Garay studied in [19] pointwise slant submanifolds of Kaehler manifolds and proved several fundamental results for such submanifolds. Furthermore, they provided a method to construct examples of such submanifolds.

In the 1960s, Bishop and O’Neill defined warped product manifolds. It is well-known that warped product manifolds play important role in differential geometry as well as in physics.

At the beginning of this century, Chen initiated in [13–15] the study of warped products in Riemannian and Kaehler manifolds from the submanifold point of view. In particular, he proved in [13] that there do not exist warped products of the form: \( N^\perp \times_f N^T \) in any Kaehler manifold beside \( CR \)-products, where \( N^\perp \) is a totally real submanifold and \( N^T \) is a holomorphic submanifold. He also proved in [13] that there exist many \( CR \)-submanifolds which are warped products of the form \( N^T \times_f N^\perp \) by reversing the two factors \( N^T \) and \( N^\perp \). He simply called such submanifolds \( CR \)-warped products. Moreover, he proved in [13] that every \( CR \)-warped product \( N^T \times_f N^\perp \) in any Kaehler manifold satisfies the basic inequality:

\[
||h||^2 \geq 2p ||\nabla \ln(f)||^2,
\]

where \( p \) is the dimension of \( N^\perp \), \( ||h||^2 \) the squared norm of the second fundamental form, and \( \nabla \ln(f) \) the gradient of \( \ln(f) \). Since then the geometry of warped product submanifolds becomes an active research subject (for more details, we refer to Chen’s books [16,18] and his survey article [17]).

In [32], Sahin proved that there do not exist semi-slant warped products in any Kaehler manifold. Later, he investigated warped product pointwise semi-slant submanifolds of Kaehler manifolds in [33]. Further, Bonanzinga and Matsumoto [7] studied \( CR \)-warped product submanifolds and semi-slant warped product submanifolds in locally conformal Kaehler manifolds of the form \( N^T \times_f N^\perp \) or of the form \( N^T \times_f N^\theta \), where \( N^T, N^\perp \) and \( N^\theta \) are holomorphic, totally real and slant submanifolds, respectively (see also [30]). Since then such warped product submanifolds have been studied by many geometers (see, for instance, [1,18,22,23,29,34,35,37,38]).

In this paper, we investigate pointwise semi-slant warped products in locally conformal Kaehler manifolds. In particular, we obtain several results which extend Chen’s inequality (1.1). Moreover, several related results and examples are also presented in this paper.

2. Preliminaries

For a vector bundle \( \nu \) over a manifold \( M \), we denote by \( \Gamma(\nu) \) the space of smooth sections of \( \nu \).

A locally conformally Kaehler manifold \( (\tilde{M},J,g) \) (or an LCK-manifold for short) is a complex manifold \( (\tilde{M},J) \) endowed with a Hermitian metric \( g \).
which is locally conformal to a Kaehlerian metric. Equivalently, there exists an open cover \( \{ U_i \}_{i \in I} \) of \( \tilde{M} \) and a family \( \{ f_i \}_{i \in I} \) of real-valued differentiable functions \( f_i : U_i \to \mathbb{R} \) such that \( g_i = e^{-f_i} g|_{U_i} \) is a Kaehlerian metric on \( U_i \), i.e., \( \nabla^* J = 0 \), where \( J \) is the almost complex structure, \( g \) is the Hermitian metric, and \( \nabla^* \) is the covariant differentiation with respect to \( g \). A typical example of a compact \( LCK \)-manifold is a Hopf manifold which is diffeomorphic to \( S^1 \times S^{2n-1} \) and it admits no Kaehler structure (see [39]). Let \( \Omega \) and \( \Omega_i \) denote the 2-forms associated with \((J, g)\) and \((J, g_i)\), respectively (i.e., \( \Omega(X,Y) = g(X, JY) \), etc.). Then we have \( \Omega_i = e^{-f_i} \Omega|_{U_i} \).

The following result from [39] is well-known (see also [24, Theorem 1.1]).

**Theorem 2.1.** A Hermitian manifold \((\tilde{M}, J, g)\) is an \( LCK \)-manifold if and only if there exists a globally defined closed 1-form \( \alpha \) on \( \tilde{M} \) such that \( d\Omega = \alpha \wedge \Omega \).

The closed 1-form \( \alpha \) in Theorem 2.1 is called the Lee form and the vector field \( \lambda = \alpha^\# \) dual to \( \alpha \), (i.e., \( g(X, \lambda) = \alpha(X) \) for \( X \in T\tilde{M} \)), is called the Lee vector field of the \( LCK \)-manifold \( \tilde{M} \). An \( LCK \)-manifold \((\tilde{M}, J, g)\) is called a globally conformal Kaehler manifold (\( GCK \)-manifold for short) if one can choose \( U = \tilde{M} \). An \( LCK \)-manifold \( \tilde{M} \) is a \( GCK \)-manifold if and only if the 1-form \( \alpha \) is exact.

If \( \tilde{\nabla} \) denotes the Levi-Civita connection on an \( LCK \)-manifold \( \tilde{M} \), then we have

\[
(\tilde{\nabla} X J) Y = -g(\beta^\#, Y) X - g(\alpha^\#, Y) JX + g(JX, Y)\alpha^\# + g(X, Y)\beta^\#.
\]

(2.1)

for tangent vector fields \( X, Y \) on \( \tilde{M} \), where \( \alpha^\# \) is the Lee vector field, \( \beta \) is the 1-form defined by \( \beta(X) = -\alpha(JX) \) for any \( X \in T\tilde{M} \), and \( \beta^\# \) is the dual vector field of \( \beta \) (see [39]). In terms of the Lee vector field, Eq. (2.1) can be written as

\[
(\tilde{\nabla} X J) Y = [g(\lambda, JY) X - g(\lambda, Y) JX + g(JX, Y)\lambda + g(X, Y)J\lambda].
\]

(2.2)

\( LCK \)-manifolds contain rich source since their Lee forms play important roles in determining several geometric features of their submanifolds.

An \( LCK \)-manifold \((\tilde{M}, J, g)\) is called a Vaisman manifold if its Lee form \( \alpha \) is parallel, i.e., \( \tilde{\nabla} \alpha = 0 \) (see [39,40]). Vaisman manifolds form the most important subclass of \( LCK \)-manifolds.

Let \( M \) be an immersed submanifold of an \( LCK \)-manifold \((\tilde{M}, J, g, \alpha)\). Denote the induced metric tensor on \( M \) via the immersion also by \( g \). Let \( \nabla \) be the Levi-Civita connection of \((M, g)\). Then the Gauss and Weingarten formulas for \( M \) are given respectively by

\[
\tilde{\nabla} X Y = \nabla X Y + h(X, Y),
\]

(2.3)

\[
\tilde{\nabla} X \xi = -A_\xi X + \nabla^\perp_X \xi,
\]

(2.4)
for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, where $h$ is the second fundamental form, $A$ is the shape operator, $\nabla^\perp$ is the normal connection, and $T^\perp M$ is the normal bundle of the submanifold $M$.

It is well-known that the shape operator and the second fundamental form are related by

$$g(h(X,Y), \xi) = g(A\xi X,Y). \quad (2.5)$$

A submanifold $M$ is said to be totally geodesic if its second fundamental form $h$ vanishes identically on $M$, i.e., $h \equiv 0$, or equivalently, $A \equiv 0$.

For any tangent vector vector field $X \in \Gamma(TM)$, we put

$$JX = PX + FX, \quad (2.6)$$

where $PX$ and $FX$ are the tangential and normal components of $JX$, respectively. Similarly, for any normal vector field $\xi \in \Gamma(T^\perp M)$, we put

$$J\xi = t\xi + f\xi, \quad (2.7)$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $J\xi$, respectively.

It is well known that a submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is pointwise slant if and only if (see [19])

$$P^2 = - (\cos^2 \theta) I, \quad (2.8)$$

for some real-valued function $\theta$ defined on $M$, where $I$ is the identity map of the tangent bundle $TM$ of $M$. A pointwise slant submanifold is called proper if it does not contain any totally real or complex points, i.e., $0 < \cos^2 \theta < 1$.

At a given point $p \in M$, the following relations are easy consequences of (2.8):

$$g(PX, PY) = (\cos^2 \theta) g(X, Y), \quad (2.9)$$
$$g(FX, FY) = (\sin^2 \theta) g(X, Y) \quad (2.10)$$

for any $X, Y \in \Gamma(TM)$. Further, it is easy to verify that

$$tFX = - \sin^2 \theta X, \quad fFX = - FPX \quad (2.11)$$

for any $X \in \Gamma(TM)$.

**Definition 2.1.** Let $(\tilde{M}, J, g)$ be an almost Hermitian manifold and let $M$ be a submanifold of $\tilde{M}$. Then $M$ is called a pointwise semi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ on $M$ such that

(i) The tangent bundle $TM$ is the orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\theta$.

(ii) The distribution $\mathcal{D}$ is $J$-invariant or holomorphic, i.e., $J(\mathcal{D}) = \mathcal{D}$.

(iii) The distribution $\mathcal{D}^\theta$ is pointwise slant with slant function $\theta$.

The pointwise semi-slant submanifold $M$ is called proper if neither $\dim N^T = 0$ nor the slant function of $\mathcal{D}^\theta$ is $\pi/2$, i.e., $\cos^2 \theta > 0$. Otherwise, $M$ is called improper.
Definition 2.2. Let $M$ be a submanifold of an almost Hermitian manifold. Then

(i) $M$ is called $\mathcal{D}$-geodesic if $h(X,Y) = 0$,
(ii) $M$ is called $\mathcal{D}^\theta$-geodesic if $h(Z,W) = 0$,
(iii) $M$ is called mixed totally geodesic if $h(X,Z) = 0$,

for $X,Y \in \Gamma(\mathcal{D})$ and $Z,W \in \Gamma(\mathcal{D}^\theta)$.

3. Pointwise Semi-slant Submanifold of an $LCK$-Manifold

First, we prove the following results for later use.

Lemma 3.1. Let $M$ be a proper pointwise semi-slant submanifold of an $LCK$-manifold. Then we have

$$\sin^2 \theta g(\nabla_X Y, Z) = g(A_{FZ} JY - A_{FPZ} Y, X) - g(JX,Y)g(\alpha\#, FZ) - g(X,Y)g(\beta\#, FZ)$$

for any $X,Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. Under the hypothesis, for any $X,Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$, we have

$$g(\nabla_X Y, Z) = g(J\nabla_X Y, JZ) = g(J\tilde{\nabla}_X Y, FZ) + g(J\tilde{\nabla}_X Y, PZ).$$

From the covariant derivative formula of $J$, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X JY, FZ) - g(JX,Y)g(\lambda, FZ) - g(Y, \tilde{\nabla}_X \cos^2 \theta Z) - g(A_{FPZ} Y, X).$$

Then, using (2.1), (2.5) and (2.8), we arrive at

$$g(\nabla_X Y, Z) = g(A_{FZ} JY, X) - g(JX,Y)g(\lambda, FZ) - g(X,Y)g(J\lambda, FZ) - g(Y, \tilde{\nabla}_X \cos^2 \theta Z) - g(A_{FPZ} Y, X).$$

Since $M$ is a proper pointwise semi-slant submanifold, we get

$$g(\nabla_X Y, Z) = g(A_{FZ} JY - A_{FPZ} Y, X) - g(JX,Y)g(\lambda, FZ) - g(X,Y)g(J\lambda, FZ) - g(Y, \tilde{\nabla}_X \cos^2 \theta Z) - g(A_{FPZ} Y, X).$$

Thus, the lemma follows from above relations by using the orthogonality of the two distributions. □

If $\mathcal{D}$ is a totally geodesic distribution in $M$, then $g(\nabla_X Y, Z) = 0$ holds for any $X,Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Hence, Lemma 3.1 implies the following.
Lemma 3.2. Let $M$ be a proper pointwise semi-slant submanifold of an LCK-manifold $\tilde{M}$. Then the holomorphic distribution $\mathcal{D}$ defines a totally geodesic foliation if and only if
\[ g(A_{FZ}JX - A_{FPZ}X, Y) = g(X, Y)g(\beta^#, FZ) - g(JX, Y)g(\alpha^#, FZ), \]
for any $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$, where $\alpha^#$ is the Lee vector field of $\tilde{M}$, $\beta$ is the 1-form defined by $\beta(X) = -\alpha(JX)$ for any $X \in T\tilde{M}$, and $\beta^#$ is the dual vector field of $\beta$.

Corollary 3.1. Let $M$ be a proper pointwise semi-slant submanifold of an LCK-manifold $\tilde{M}$. Then the holomorphic distribution $\mathcal{D}$ defines a totally geodesic foliation if and only if
\[ A_{FZ}JX - A_{FPZ}X = g(\beta^#, FZ)X - g(\alpha^#, FZ)JX \]
holds for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$.

For leaves of the pointwise slant distribution $\mathcal{D}^\theta$, we have the following result.

Lemma 3.3. Let $M$ be a pointwise semi-slant submanifold of an LCK-manifold $\tilde{M}$ with proper pointwise slant distribution $\mathcal{D}^\theta$. Then
\[ g(\nabla ZW, X) = \csc^2 \theta g(A_{FPW}X - A_{FW}JX, Z) - g(Z, W)g(\alpha^#, X) \]
holds for any $X \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, where $\alpha^#$ is the Lee vector field of $\tilde{M}$.

Proof. For any $X \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, we find
\[ g(\nabla ZW, X) = g\left(J\tilde{\nabla} ZW, JX\right) = g\left(\tilde{\nabla} ZFW, JX\right) - g\left((\tilde{\nabla} ZJ) W, JX\right). \]
Using (2.2), we get
\[ g(\nabla ZW, X) = g\left(\tilde{\nabla} ZPW, JX\right) + g\left(\tilde{\nabla} ZFW, JX\right) - g(JZ, W)g(\lambda, JX) \]
\[ - g(Z, W)g(\lambda, JX). \]
Then, we derive
\[ g(\nabla ZW, X) = -g\left(J\tilde{\nabla} ZPW, X\right) - g(A_{FWZ} JX) - g(PZ, W)g(\lambda, JX) \]
\[ - g(Z, W)g(\lambda, X). \]
From the definition of covariant derivative of $J$ and the symmetry of the shape operator, we find
\[ g(\nabla ZW, X) = g\left((\tilde{\nabla} ZJ) PW, X\right) - g\left(\tilde{\nabla} ZJPW, X\right) - g(A_{FWJX}, Z) \]
\[ - g(PZ, W)g(\lambda, JX) - g(Z, W)g(\lambda, X). \]
Again using (2.2) and (2.6), we derive

\[ g(∇_ZW, X) = g(PZ, PW)g(λ, X) + g(Z, PW)g(Jλ, X) - g(∇_ZP^2W, X) \]
\[- g(∇_ZFPW, X) - g(A_{FW}JX, Z) - g(PZ, W)g(λ, JX) \]
\[- g(Z, W)g(λ, X). \]

From the relation (2.8), we find

\[ g(∇_ZW, X) = \cos^2 θg(∇_ZW, X) - \sin^2 θX(θ)g(W, X) + g(A_{FPW}Z, X) \]
\[ + \cos^2 θg(Z, W)g(λ, X) - g(A_{FW}JX, Z) - g(Z, W)g(λ, X). \]

By using the orthogonality of two distributions, we derive

\[ g(∇_ZW, X) = \csc^2 θ[g(A_{FPW}X, Z) - g(A_{FW}JX, Z)] - g(Z, W)g(λ, X), \]
which proves the lemma completely.

**Lemma 3.3** implies the following result.

**Corollary 3.2.** Let \( M \) be a proper pointwise semi-slant submanifold of an LCK-manifold \( \tilde{M} \). Then the slant distribution \( D^θ \) defines a totally geodesic foliation if and only if

\[ g(A_{FP}ZX - A_{FZ}JX, W) = \sin^2 θg(\alpha^#, X)g(Z, W) \]
holds for any \( X \in Γ(Ω) \) and \( Z, W \in Γ(Ω^θ) \) where \( α^# \) is the Lee vector field of \( \tilde{M} \).

**Lemma 3.4.** Let \( M \) be a proper pointwise semi-slant submanifold of an LCK-manifold \( \tilde{M} \). Then we have

\[ \sin^2 θg([Z, W], X) = g(A_{FZ}JX - A_{FPZ}X, W) - g(A_{FW}JX - A_{FPW}X, Z) \]
for any \( X \in Γ(Ω) \) and \( Z, W \in Γ(Ω^θ) \).

**Proof.** From Lemma 3.3, we have

\[ \sin^2 θg(∇_ZW, X) = g(A_{FPW}X - A_{FW}JX, Z) - \sin^2 θg(λ, X)g(Z, W) \]
(3.1)
for any \( X \in Γ(Ω) \) and \( Z, W \in Γ(Ω^θ) \). By interchanging \( Z \) and \( W \) in (3.1), we find

\[ \sin^2 θg(∇_WZ, X) = g(A_{FPZ}X - A_{FZ}JX, W) - \sin^2 θg(λ, X)g(Z, W). \]
(3.2)

Thus, after subtracting (3.1) from (3.2), we get the required result.

Now, we give the following integrability theorem.

**Theorem 3.1.** Let \( M \) be a pointwise semi-slant submanifold of an LCK-manifold \( \tilde{M} \) and \( α^# \) the Lee vector field of \( \tilde{M} \). Then we have:
Then, by interchanging $X$ Lemma 3.1 that for any $X,Y$ function $\ln$ is always totally geodesic and $\pi$ and $N$ let $NT$.

4. Pointwise Semi-slant Warped Products: $N^T \times_f N^\theta$

Let $N_1$ and $N_2$ be two Riemannian manifolds equipped with metrics $g_1$ and $g_2$, respectively, and $f$ be a positive differential function on $N_1$. Consider the product manifold $N_1 \times N_2$ with its natural projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then the warped product manifold $N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the warped product metric $g$ defined by

$$g(X,Y) = g_1(\pi_{1,*}X, \pi_{1,*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2,*}X, \pi_{2,*}Y)$$

for $X,Y \in \Gamma(TM)$, where $\pi_{i,*}$ is the tangent map of $\pi_i$. The function $f$ is called the warping function on $M$. A warped product manifold $N_1 \times_f N_2$ is called trivial if its warping function $f$ is constant.

The following lemma is well-known.

Lemma 4.1. ([5]) Let $M = N_1 \times_f N_2$ be a warped product manifold. Then we have:

(i) $\nabla_X Y \in T(N_1)$,
(ii) $\nabla_X Z = \nabla_{Z,X} X = X(\ln f)Z$,
(iii) $\nabla_Z W = \nabla^N_Z W - g(Z,W)\nabla \ln f$,

for any $X,Y \in T(N_1)$ and $Z,W \in T(N_2)$, where $\nabla$ and $\nabla^N$ denote the Levi-Civita connections on $M$ and $N_2$, respectively and $\nabla \ln f$ is the gradient of the function $\ln f$ defined as $g(\nabla f, X) = X(f)$.

Remark 4.1. It is important to note that for a warped product $N_1 \times_f N_2$, $N_1$ is always totally geodesic and $N_2$ is totally umbilical in $N_1 \times_f N_2$ (c.f., [5, 13]).
In this section, we study pointwise semi-slant warped products \( M = N^T \times_f N^\theta \) in an \( LCK \)-manifold \( \widetilde{M} \) under the assumption that the Lee vector field \( \alpha^\# \) of \( \widetilde{M} \) is tangent to \( M \).

Clearly, CR-warped products and semi-slant warped product submanifolds are special cases of pointwise semi-slant warped product submanifolds \( N^T \times_f N^\theta \) such that the slant function \( \theta \) satisfies \( \theta = \frac{\pi}{2} \) and \( \theta = \text{constant} \), respectively. For simplicity, we denote the tangent spaces of \( N^T \) and \( N^\theta \) by \( \mathfrak{D} \) and \( \mathfrak{D}^\theta \), respectively.

**Proposition 4.1.** If \( N^T \times_f N^\theta \) is a proper pointwise semi-slant warped product of an \( LCK \)-manifold \( \widetilde{M} \). Then the Lee vector field \( \lambda = \alpha^\# \) of \( \widetilde{M} \) is orthogonal to \( \mathfrak{D}^\theta \).

**Proof.** For any \( X \in \Gamma(\mathfrak{D}) \) and \( Z \in \Gamma(\mathfrak{D}^\theta) \), we have
\[
g(h(X,Y),FZ) = g(h(X,Y),JZ) = -g(J\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X Y, JZ)
\]
Using the definition of covariant derivative of \( J \) and Lemma 4.1(i), we find
\[
g(h(X,Y),FZ) = g(\left(\overline{\nabla}_X J\right)Y, Z) - g(\overline{\nabla}_X JY, Z)
\]
\[
= g(X,Y)g(J\lambda, Z) + g(JX,Y)g(\lambda, Z).
\]
(4.1)

Since \( h(X,Y) \) is symmetric with respect to \( X \) and \( Y \), we find \( g(JX,Y)g(\lambda, Z) = 0 \), which implies \( g(\lambda, Z) = 0 \) for any \( Z \in \Gamma(\mathfrak{D}^\theta) \). \( \square \)

**Remark 4.2.** Proposition 4.1 shows that in our case the Lee vector field \( \alpha^\# \) is in \( \mathfrak{D} \).

**Remark 4.3.** For a proper CR-product of an \( LCK \)-manifold, the Lee vector field \( \alpha^\# \) is normal to \( \mathfrak{D}^\perp \) (see [7]).

Now, we prove the following useful lemma.

**Lemma 4.2.** Let \( M = N^T \times_f N^\theta \) be a pointwise semi-slant warped product submanifold of an \( LCK \)-manifold \( \widetilde{M} \), where \( N^T \) and \( N^\theta \) are holomorphic and proper pointwise slant submanifolds of \( \widetilde{M} \), respectively. If the Lee vector field \( \lambda = \alpha^\# \) of \( \widetilde{M} \) is tangent to \( M \), then we have:
(i) \( g(h(X,Y),FZ) = 0 \),
(ii) \( g(h(X,Z),FW) = [g(\lambda, JX) - JX(\ln f)]g(Z,W) + [g(\lambda, X) - X(\ln f)]g(Z, PW) \) for any \( X,Y \in \Gamma(\mathfrak{D}) \) and \( Z,W \in \Gamma(\mathfrak{D}^\theta) \).

**Proof.** From (4.1) and the symmetry of \( h \), we have
\[
g(h(X,Y),FZ) = g(X,Y)g(J\lambda, Z) + g(X,JY)g(\lambda, Z),
\]
(4.2)
for any \( X,Y \in \Gamma(\mathfrak{D}) \) and \( Z \in \Gamma(\mathfrak{D}^\theta) \). Thus, it follows from (4.1) and (4.2) that
\[
g(h(X,Y),FZ) = g(X,Y)g(\beta^\#, Z).
\]
Therefore, statement (i) follows from the above relation, Proposition 4.1, and the fact that the Lee vector field \( \lambda \) is tangent to \( M \).

For the proof of statement (ii), first we notice that
\[
g(h(X, Z), FW) = g(h(X, Z), JW) = -g\left( J \left( \tilde{\nabla}_Z X - \nabla_Z X \right) , W \right),
\]
holds for any \( X \in \Gamma(\mathfrak{D}) \) and \( Z, W \in \Gamma(\mathfrak{D}^\theta) \). Hence, it follows from the covariant derivative property of \( J \) that
\[
g(h(X, Z), FW) = g\left( \left( \tilde{\nabla}_Z J \right) X, W \right) - g\left( \tilde{\nabla}_Z JX, W \right) + g(\nabla_Z X, W).
\]

Now, by applying Lemma 4.1(ii), we find
\[
g(h(X, Z), FW) = g\left( \left( \tilde{\nabla}_Z J \right) X, W \right) - JX(\ln f)g(Z, W) + X(\ln f)g(JZ, W).
\]
Thus, by using (2.2) we find
\[
g(h(X, Z), FW) = [g(\lambda, JX) - JX(\ln f)] g(Z, W) + [X(\ln f) - g(\lambda, X)] g(PZ, W),
\]
which proves statement (ii). \( \square \)

The following relations can be obtained easily by interchanging \( X \) with \( JX, Z \) with \( PZ \), and \( W \) with \( PW \) in Lemma 4.2(ii).
\[
g(h(X, PZ), FW) = [JX(\ln f) - g(\lambda, JX)] g(Z, PW) + \cos^2 \theta [g(\lambda, X) - X(\ln f)] g(Z, W), \tag{4.3}
\]
\[
g(h(X, Z), FPW) = \cos^2 \theta [X(\ln f) - g(\lambda, X)] g(Z, W) + [g(\lambda, JX) - JX(\ln f)] g(Z, PW), \tag{4.4}
\]
\[
g(h(X, PZ), FPW) = \cos^2 \theta [g(\lambda, JX) - JX(\ln f)] g(Z, W) + \cos^2 \theta [g(\lambda, X) - X(\ln f)] g(Z, PW), \tag{4.5}
\]
\[
g(h(JX, Z), FW) = [X(\ln f) - g(\lambda, X)] g(Z, W) + [g(\lambda, JX) - JX(\ln f)] g(Z, PW), \tag{4.6}
\]
\[
g(h(JX, PZ), FW) = [g(\lambda, X) - X(\ln f)] g(Z, PW) + \cos^2 \theta [g(\lambda, JX) - JX(\ln f)] g(Z, W), \tag{4.7}
\]
\[
g(h(JX, Z), FPW) = \cos^2 \theta [JX(\ln f) - g(\lambda, JX)] g(Z, W) + [X(\ln f) - g(\lambda, X)] g(Z, PW), \tag{4.8}
\]
\[
g(h(JX, PZ), FPW) = \cos^2 \theta [X(\ln f) - g(\lambda, X)] g(Z, W) + \cos^2 \theta [g(\lambda, JX) - JX(\ln f)] g(Z, PW). \tag{4.9}
\]

We need the following result for later use.

**Corollary 4.1.** Let \( M = N^T \times_f N^\theta \) be a nontrivial warped product pointwise semi-slant submanifold of an LCK-manifold \( \tilde{M} \). If the Lee field \( \lambda = \alpha^\# \) of \( \tilde{M} \) is tangent to \( M \), then we have
\[
g(h(X, PZ), FW) = -g(h(X, Z), FPW)
\]
for any \( X, Y \in \Gamma(\mathfrak{D}) \) and \( Z, W \in \Gamma(\mathfrak{D}^\theta) \).

**Proof.** The proof follows from (4.3) and (4.4). \( \square \)

For a proper pointwise semi-slant warped product \( M = N^T \times f N^\theta \) in an LCK-manifold \( \tilde{M} \), let \( \nu \) denote the invariant subbundle of \( T^\perp M \) which is the orthogonal complement of \( F \mathfrak{D}^\theta \) in \( T^\perp M \) so that

\[
T^\perp M = F \mathfrak{D}^\theta \oplus \nu.
\]

**Theorem 4.1.** Let \( M = N^T \times f N^\theta \) be a proper pointwise semi-slant warped product in an LCK-manifold \( \tilde{M} \). If we have \( h(X, Z) \in \nu \) for any \( X \in \Gamma(\mathfrak{D}) \) and \( Z \in \Gamma(\mathfrak{D}^\theta) \), then the Lee form \( \alpha \) of \( \tilde{M} \) satisfies \( \alpha(X) = X(\ln f) \).

**Proof.** By virtue of (4.2)(i) and the hypothesis of the theorem, we have

\[
\left[ g(\lambda, JX) - JX(\ln f) \right] g(Z, W) + \left[ g(\lambda, X) - X(\ln f) \right] g(Z, PW) = 0,
\]

for any \( X \in \Gamma(\mathfrak{D}) \) and \( Z, W \in \Gamma(\mathfrak{D}^\theta) \). Also, from (4.8) and the hypothesis of the theorem, we derive

\[
\cos^2 \theta \left[ JX(\ln f) - g(\lambda, JX) \right] g(Z, W) + \left[ X(\ln f) - g(\lambda, X) \right] g(Z, PW) = 0. \tag{4.12}
\]

Hence, it follows from (4.11) and (4.12) that

\[
\sin^2 \theta \left[ g(\lambda, JX) - JX(\ln f) \right] g(Z, W) = 0. \tag{4.13}
\]

Since \( M \) is proper pointwise semi-slant and \( g \) is the Riemannian metric, the desired result follows from (4.13). \( \square \)

**Corollary 4.2.** Let \( N^T \times f N^\theta \) be a mixed totally geodesic pointwise semi-slant warped product in an LCK-manifold \( \tilde{M} \). Then \( \alpha(X) = X(\ln f) \) for any \( X \in \Gamma(\mathfrak{D}) \).

By using Lemma 4.2, we deduce the following result.

**Theorem 4.2.** Let \( M = N^T \times f N^\theta \) a proper pointwise semi-slant warped product in an LCK-manifold \( \tilde{M} \). If the Lee vector field \( \alpha^\# \) of \( \tilde{M} \) is tangent to \( M \), then we have:

(i) \( g(A_FZ X, Y) = 0 \).

(ii) \( g(A_FZ JX - A_{FP} Z X, W) = \sin^2 \theta [X(\ln f) - \alpha(X)] g(Z, W) \)

for any \( X, Y \in \Gamma(\mathfrak{D}) \) and \( Z, W \in \Gamma(\mathfrak{D}^\theta) \).

**Proof.** The first part is nothing but Lemma 4.2 (i). The second part follows from (4.4) and (4.6).

**Corollary 4.3.** There do not exist a mixed totally geodesic CR-warped product submanifold of the form \( N^T \times f N^\perp \) in a Kaehler manifold \( \tilde{M} \).

**Proof.** Follows from Theorem 4.2(ii). \( \square \)
5. Characterizations Theorems

Now, we provide a characterization of pointwise semi-slant warped products.

**Theorem 5.1.** Let $M = N^T \times fN^\theta$ a proper pointwise semi-slant warped product in an LCK-manifold $\tilde{M}$. If the Lee vector field $\alpha^\#$ of $\tilde{M}$ tangent to $M$, then

\[ X(\ln f) = \alpha(X) + \tan \theta \ X(\theta) \]  

(5.1)

for any $X \in \Gamma(\mathcal{D})$.

**Proof.** For any $X \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, we have

\[ g(h(X, PZ), FW) = g\left(\tilde{\nabla}_X PZ, FW\right) = g\left(\tilde{\nabla}_X PZ, JW\right) - g\left(\tilde{\nabla}_X PZ, PW\right). \]

From the covariant derivative property of $J$, we find

\[ g\left(A_{FW} PZ, X\right) = g\left(\tilde{\nabla}_X J PZ, W\right) - g\left(\tilde{\nabla}_X JPZ, W\right) - X(\ln f) \cos^2 \theta g(Z, W). \]

Using (2.2) and orthogonality of vector fields, we obtain

\[ g\left(A_{FP} PZ, X\right) = g\left(\tilde{\nabla}_X P^2 Z, W\right) - g\left(\tilde{\nabla}_X FPZ, W\right) = X(\ln f) \cos^2 \theta g(Z, W) - 2\sin^2 \theta X(\theta) g(Z, W) + g\left(\tilde{\nabla}_X FPZ, X\right). \]

which implies

\[ g\left(A_{FP} PZ, X\right) = X(\ln f) \cos^2 \theta g(Z, W) - \sin 2\theta X(\theta) g(Z, W). \]  

(5.2)

On the other hand, from (4.3) and (4.4), we find

\[ g\left(A_{FP} PZ, X\right) - g\left(A_{FW} PZ, X\right) = 2 \cos^2 \theta \ (X(\ln f) - \alpha(X)) \ g(Z, W). \]  

(5.3)

Therefore, Eq. (5.1) follows from (5.2) and (5.3).

In order to prove another characterization for pointwise semi-slant warped products, we recall the following well-known Hiepko’s Theorem.

**Theorem 5.2.** ([26]) Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distribution on a Riemannian manifold $M$. Suppose that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then $M$ is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where $M_1$ and $M_2$ are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

Now, we are able to prove the following characterization of proper pointwise semi-slant warped product submanifolds of the form $N^T \times f N^\theta$. 

Theorem 5.3. Let $M$ be a proper pointwise semi-slant submanifold of an LCK-manifold $\tilde{M}$ with holomorphic distribution $\mathfrak{D}$ and proper pointwise slant distribution $\mathfrak{D}^0$. Then $M$ is locally a warped product submanifold of the form $N^T \times_f N^\theta$ if and only if
\[ A_{FZ}JX - A_{FPZ}X = \sin^2 \theta (X(\mu) - \alpha(X)) Z, \quad \forall X \in \Gamma(\mathfrak{D}), \ Z \in \Gamma(\mathfrak{D}^0), \] (5.4)
for some function $\mu$ on $M$ satisfying $W(\mu) = 0$ for any $W \in \Gamma(\mathfrak{D}^0)$.

Proof. Let $M = N^T \times_f N^\theta$ be a pointwise semi-slant warped product submanifold of an LCK-manifold $\tilde{M}$. Then, by Theorem 4.2(i), we have $g(A_{FZ}JX, Y) = 0$ for any $X, Y \in \Gamma(TN^T)$ and $Z \in \Gamma(TN^\theta)$. Hence, $A_{FZ}JX$ has no components in $TN^T$. Similarly, if we replace $Z$ by $PZ$ in Theorem 4.2(i), then we get $g(A_{FPZ}X, Y) = 0$. Hence, $A_{FPZ}X$ also has no components $TN^T$. Therefore, $A_{FZ}JX - A_{FPZ}X$ lies in $TN^\theta$. After applying this fact and Theorem 4.2(ii), we obtain (5.4) with $\mu = \ln f$ and $\alpha(X) = g(\lambda, X)$.

Conversely, assume that $M$ is a proper pointwise semi-slant submanifold of an LCK-manifold $\tilde{M}$ such that (5.4) holds. Then it follows from Lemma 3.1 and condition (5.4) that $\sin^2 \theta g(\nabla_Y X, Z) = 0$ for $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^0)$. Since $M$ is a proper pointwise semi-slant submanifold, $g(\nabla_Y X, Z) = 0$ holds. Therefore, the leaves of the distribution $\mathfrak{D}$ must be totally geodesic in $M$.

On the other hand, it follows from condition (5.4) and Lemma 3.4 that
\[ \sin^2 \theta g([Z, W], X) = 0 \]
holds for any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^0)$. Since $M$ is a proper pointwise semi-slant submanifold, we have $g([Z, W], X) = 0$. Thus, the slant distribution $\mathfrak{D}$ is integrable.

Next, let us consider the second fundamental form $h^\theta$ of a leaf $N^\theta$ of $\mathfrak{D}^0$ in $M$. For any $Z, W \in \Gamma(\mathfrak{D}^0)$ and $X \in \Gamma(\mathfrak{D})$, we have
\[ g(h^\theta(Z, W), X) = g(\nabla_Z W, X) = g(\tilde{\nabla}_ZW, X) = g(J\tilde{\nabla}_ZW, JX). \]
Now, using the covariant derivative property of $J$ and the LCK-structure equation, we obtain
\[ g(h^\theta(Z, W), X) = g(\tilde{\nabla}_Z PW, JX) + g(\tilde{\nabla}_Z FW, JX) - g(JZ, W)g(\lambda, JX) \\
= g(\tilde{\nabla}_Z PW, JX) + g(\tilde{\nabla}_Z FW, JX) - g(JZ, W)g(\lambda, JX) \\
- g(Z, W)g(\lambda, X). \]
Again, by applying the covariant derivative property of $J$, we find
\[ g(h^\theta(Z, W), X) = -g(\tilde{\nabla}_Z JPW, X) + g(JZ, PW, X) - g(A_{FW} Z, JX) \\
- g(PZ, W)g(\lambda, JX) - g(Z, W)g(\lambda, X). \]
Therefore, by using (2.6) and (2.2), we derive that
\[ g(h^\theta(Z, W), X) = -g\left(\tilde{\nabla}_Z P^2 W, X\right) - g\left(\tilde{\nabla}_Z FP W, X\right) + g(PZ, PW) g(\lambda, X) \]
\[ + g(Z, PW) g(J\lambda, X) - g(A_{FW} JX, Z) - g(PZ, W) g(\lambda, JX) \]
\[ - g(Z, W) g(\lambda, X). \]

Then, we obtain from (2.8) and (2.9) that
\[ g(h^\theta(Z, W), X) = \cos^2 \theta g\left(\tilde{\nabla}_Z W, X\right) + \sin 2\theta Z(\theta) g(W, X) + g(A_{FPW} Z, X) \]
\[ + \cos^2 \theta g(\lambda, X) g(Z, W) - g(A_{FW} JX, Z) - (\lambda, X) g(Z, W) \]
\[ = \cos^2 \theta g(\nabla Z W, X) + g(A_{FPW} X - A_{FW} JX, Z) \]
\[ - \sin^2 \theta g(\lambda, X) g(Z, W). \]

From condition (5.4), we find
\[ \sin^2 \theta g(h^\theta(Z, W), X) = -\sin^2 \theta X(\mu) g(Z, W). \]

Thus, we conclude from the definition of gradient that \( h^\theta(Z, W) = -\tilde{\nabla}_\mu g(Z, W), \) which implies that \( N^\theta \) is totally umbilical in \( M \) with the mean curvature vector given by \( H^\theta = -\tilde{\nabla}_\mu. \) Now, it is easy to see that the mean curvature vector \( H^\theta \) is parallel to the normal connection \( D^\# \) of \( N^\theta \) in \( M. \) For this, let us consider any \( Y \in \Gamma(D) \) and \( Z \in \Gamma(D^\theta), \) we find
\[ g\left(D_Z^\# \tilde{\nabla}_\mu, Y\right) = g\left(\nabla Z \tilde{\nabla}_\mu, Y\right) = Z g\left(\tilde{\nabla}_\mu, Y\right) - g\left(\tilde{\nabla}_\mu, \nabla Z Y\right) \]
\[ = Z(Y(\mu)) - g\left(\tilde{\nabla}_\mu, [Z, Y]\right) - g\left(\tilde{\nabla}_\mu, \nabla Y Z\right) \]
\[ = Z(Y(\mu)) - g\left(\tilde{\nabla}_\mu, Z Y\right) + g\left(\tilde{\nabla}_\mu, Y Z\right) + g\left(\nabla_Y \tilde{\nabla}_\mu, Z\right) \]
\[ = Z(Y(\mu)) - Z(Y(\mu)) + Y(Z(\mu)) = 0, \]
due to \( Z(\mu) = 0 \) for \( Z \in D^\theta. \) Hence, \( \nabla_Y \tilde{\nabla}_\mu \in D \) which means that the mean curvature of \( N^\theta \) is parallel. Thus, the leaves of \( D^\theta \) are totally umbilical with parallel mean curvature. Therefore, the spherical condition of \( D^\theta \) is satisfied. Consequently, Theorem 5.2 implies that \( M \) is a warped product submanifold \( N^T \times_f N^\theta \) with warping function \( \mu. \) \( \square \)

6. Chen Type Inequality for Pointwise Semi-slant Warped Products

In this section, we provide a sharp estimation for the length of the second fundamental form \( h \) of an \( m \)-dimensional proper pointwise semi-slant warped product \( M = N^T \times_f N^\theta \) in a \( 2k \)-dimensional LCK-manifold \( \tilde{M} \) such that the Lee vector field \( \lambda = \alpha^\# \) is tangent to \( N^T. \) Now, let us assume that \( \dim N^T = 2p \) and \( \dim N^\theta = 2q \) so that we have \( m = 2p + 2q. \)
Proof. From the definition of \( \text{Vol. 76 (2021) Pointwise Semi-slant Warped Products Page 15 of 25} \) terms of the gradient of warping function \( M \) of the holomorphic distribution \( \mathcal{D} \) and an orthonormal frame \( \{e_1, \cdots, e_p, e_{p+1} = Je_1, \cdots, e_{2p} = Je_p\} \) of the pointwise semi-slant warped distribution \( \mathcal{D}^\theta \) on \( M \). Then

\[
\{e_{m+1} = \tilde{e}_1 = \csc \theta Fe_1, \cdots, e_{m+q} = \tilde{e}_q = \csc \theta Fe_q, e_{m+q+1} = \tilde{e}_{q+1} = \csc \theta \sec \theta Fe_{q+1}, \cdots, e_{m+2q} = \tilde{e}_{2q} = \csc \theta \sec \theta Fe_{2q}\}
\]

is an orthonormal frame of \( F\mathcal{D}^\theta \). Let

\[
\{e_{m+2q+1} = \tilde{e}_{2q+1}, \cdots, e_{2k} = \tilde{e}_{2k-2q}\}
\]

be an orthonormal frames of the invariant normal subbundle \( \nu \) of \( T^\perp M \) (see (4.10)).

In the following, we will use the above frame fields to obtain a sharp estimation for the squared norm \( ||h||^2 \) of the second fundamental form \( h \) in terms of the gradient of warping function \( f \), Lee vector field \( \alpha^\#, \) and the slant function \( \theta \) of \( M \) in the LCK-manifold \( \tilde{M} \). For simplicity, we simply denote by \( \alpha^\#_{TM} \) the tangential component of the Lee vector field \( \alpha^\# \) of \( M \) on the pointwise semi-slant warped product \( M = N^T \times_f N^\theta \).

**Theorem 6.1.** Let \( M = N^T \times_f N^\theta \) be a pointwise semi-slant warped product submanifold of a locally conformal Kaehler manifold \( \tilde{M} \) where \( N^T \) and \( N^\theta \) are holomorphic and proper pointwise slant submanifolds of \( \tilde{M} \), respectively. If the Lee vector field \( \alpha^\# \) of \( \tilde{M} \) is tangent to \( M \), then we have:

(i) The squared norm of the second fundamental form \( h \) of \( M \) satisfies

\[
||h||^2 \geq 4q \left( \csc^2 \theta + \cot^2 \theta \right) \left\{ ||\nabla (\ln f)||^2 + ||\alpha^\#_{TM}||^2 - 2G^* \right\}, \quad (6.1)
\]

where \( G^* = \sum_{i=1}^{2p} g(\nabla (\ln f), e_i)g(\alpha^\#, e_i), \) \( ||\alpha^\#_{TM}||^2 \) is the squared norm of \( \alpha^\#_{TM}, \) \( q = \frac{1}{2} \dim N^T, \) \( p = \frac{1}{2} \dim N^\theta, \) and \( \nabla (\ln f) \) is the gradient of \( \ln f \).

(ii) If the equality sign of (6.1) holds identically, then \( N^T \) is totally geodesic and \( N^\theta \) is totally umbilical in \( \tilde{M} \). Furthermore, \( M \) is minimal in \( \tilde{M} \).

**Proof.** From the definition of \( h \), we have

\[
||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=m+1}^{2k} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2.
\]

We may decompose the above relation for \( F\mathcal{D}^\theta \)- and \( \nu \)-components as follows

\[
||h||^2 = \sum_{r=1}^{2q} \sum_{i,j=1}^{m} g(h(e_i, e_j), \tilde{e}_r)^2 + \sum_{r=2q+1}^{2k-m-2q} \sum_{i,j=1}^{m} g(h(e_i, e_j), \tilde{e}_r)^2. \quad (6.2)
\]
Now, by computing $F\mathcal{D}^\theta$-components terms and leaving $\nu$-components positive terms, we find

$$\|h\|^2 \geq \sum_{r=1}^{2q} \sum_{i,j=1}^{2p} g(h(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{r=1}^{2q} \sum_{i=1}^{2p} \sum_{j=1}^{2q} g(h(e_i, e_j^*), \tilde{e}_r)^2$$

$$+ \sum_{r=1}^{2q} \sum_{i,j=1}^{2p} g(h(e_i^*, e_j^*), \tilde{e}_r)^2.$$  

(6.3)

In views of Lemma 4.2(i), we know that the first term in the right hand side of (6.3) is identically zero and there is no relation for the last term $g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), F\mathcal{D}^\theta)$. Thus, we may compute just the second term of (6.3), namely,

$$\|h\|^2 \geq 2 \sum_{r=1}^{2q} \sum_{i=1}^{2p} \sum_{j=1}^{2q} g(h(e_i, e_j^*), \tilde{e}_r)^2.$$

It follows from the frame fields of $\mathcal{D}$, $\mathcal{D}^\theta$ and $F\mathcal{D}^\theta$ we chosen early in this section that

$$\|h\|^2 \geq 2 \csc^2 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(e_i, e_j^*), Fe_r^*)^2$$

$$+ 2 \csc^2 \theta \sec^4 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(e_i, Te_j^*), FTe_r^*)^2$$

$$+ 2 \csc^2 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(Je_i, e_j^*), Fe_r^*)^2$$

$$+ 2 \csc^2 \theta \sec^4 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(Je_i, Te_j^*), FTe_r^*)^2$$

$$+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(e_i, Te_j^*), Fe_r^*)^2$$

$$+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(Je_i, Te_j^*), FTe_r^*)^2$$

$$+ 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^{p} \sum_{r,j=1}^{q} g(h(e_i, e_j^*), FTe_r^*)^2.$$
Thus, we may derive from Lemma 4.2(ii) and the relations (4.3)-(4.9) that
\[
\|h\|^2 \geq 4q \left( \csc^2 \theta + \cot^2 \theta \right) \sum_{i=1}^{p} \left( g(\lambda, Je_i) - Je_i(\ln f) \right)^2 \\
+ 4q \left( \csc^2 \theta + \cot^2 \theta \right) \sum_{i=1}^{p} \left( g(\lambda, e_i) - e_i(\ln f) \right)^2 \\
= 4q \left( \csc^2 \theta + \cot^2 \theta \right) \left\{ \|\nabla \ln f\|^2 + \sum_{i=1}^{p} \left[ g(\alpha^#, e_i) + g(\alpha^#, Je_i) \right]^2 \\
-2\sum_{i=1}^{p} [e_i(\ln f)] g(\alpha^#, e_i) + (Je_i(\ln f)) g(\alpha^#, Je_i) \right\} \\
= 4q \left( \csc^2 \theta + \cot^2 \theta \right) \left\{ \|\nabla \ln f\|^2 + \sum_{i=1}^{2p} \left[ g(\alpha^#, e_i) - 2(e_i(\ln f)) g(\alpha^#, e_i) \right] \right\}.
\]

Since the Lee vector field $\alpha^#$ is in $\mathfrak{D}$ and it is orthogonal to $\mathfrak{D}^\theta$, the above inequality takes the form
\[
\|h\|^2 \geq 4q \left( \csc^2 \theta + \cot^2 \theta \right) \left\{ \|\nabla \ln f\|^2 + \|\alpha^#_{TM}\|^2 - 2\sum_{i=1}^{2p} (e_i(\ln f)) g(\alpha^#, e_i) \right\}
\]
which is exactly inequality (6.1).

If the equality sign of inequality (6.1) holds, then it follows from the omission of the $\nu$-components term in (6.2) that
\[
h(X, Y) \perp \nu, \quad \forall X, Y \in \Gamma(TM).
\]
(6.4)

Similarly, from the vanishing of the first term and the omission of the third term in the right-hand-side of (6.3), we obtain
\[
h(\mathfrak{D}, \mathfrak{D}) \perp F\mathfrak{D}^\theta, \quad h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta.
\]
(6.5)

By combining (6.4) and (6.5), we have
\[
h(\mathfrak{D}, \mathfrak{D}) = \{0\}, \quad h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) = \{0\} \quad \text{and} \quad h(\mathfrak{D}, \mathfrak{D}^\theta) \subset F\mathfrak{D}^\theta.
\]
(6.6)

Now, if $h^\theta$ denotes the second fundamental form of $N^\theta$ in $M$, then we have
\[
g(h^\theta(Z, W), X) = g(\nabla ZW, X) = -(X(\ln f)) g(Z, W)
\]
(6.7)
for any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

Since $N^T$ is totally geodesic in $M$, after applying this fact together with (6.6), we may conclude that $N^T$ is totally geodesic in $\tilde{M}$. Also, since $N^\theta$ is totally umbilical in $M$ (see [5,16]), it follows from (6.7) that $N^\theta$ is totally umbilical in $\tilde{M}$. Consequently, after applying all conditions given in (6.6) together with the facts given above, we conclude that $M$ is minimal in $\tilde{M}$. This completes the proof of the theorem. \qed
7. Some Applications

If the slant function $\theta$ of $N^\theta$ in Theorem 5.1 is a constant, then the pointwise semi-slant warped product $M = N^T \times_f N^\theta$ in the LCK-manifold $\tilde{M}$ becomes a semi-slant warped product. Such submanifolds have been studied by Tastan and Aydin in [36].

For semi-slant warped product submanifold of an LCK-manifold $\tilde{M}$, Theorem 5.1 implies the following.

**Theorem 7.1.** If $N^T \times_f N^\theta$ is a semi-slant warped product submanifold of an LCK-manifold $\tilde{M}$, then $\alpha(X) = X(\ln f)$ for any $X \in \Gamma(\mathfrak{D})$.

If $\tilde{M}$ is Kaehlerian, i.e. $\alpha^\# = 0$, Theorem 5.1 implies the following.

**Theorem 7.2.** There do not exist any proper semi-slant warped product submanifold of the form $M = N^T \times_f N^\theta$ in a Kaehler manifold.

Theorem 7.2 is the main result (Theorem 3.2) of [32]. Therefore, Theorem 5.1 generalizes Theorem 3.2 of [32].

If the slant function $\theta$ of $N^\theta$ in Theorem 5.3 satisfies $\theta = \pi/2$, then $M = N^T \times_f N^\theta$ is a CR-submanifold. Hence, in this case Theorem 5.3 implies the following.

**Theorem 7.3.** A CR-submanifold of an LCK-manifold $\tilde{M}$ with the Lee vector field $\lambda = \alpha^\#$ tangent to $M$ is a CR-warped product if and only if the Lee vector field is orthogonal to $\mathfrak{D}^\perp$ and the shape operator $A$ satisfies

$$A JZ X = -(g(J\lambda, X) + JX(\mu)) Z, \quad (7.1)$$

for $X \in \Gamma(\mathfrak{D})$, $Z \in \Gamma(\mathfrak{D}^\perp)$, where $\mu$ is a function on $M$ satisfying $W(\mu) = 0$ for $W \in \Gamma(\mathfrak{D}^\perp)$.

Theorem 7.3 is the main result (Theorem 3.5) of [28]. Hence Theorem 5.3 also generalizes the main result of [28].

If we put $\alpha^\# = 0$ and $\theta = \pi/2$ in Theorem 5.3, then the submanifold $M$ in Theorem 5.3 is a CR-submanifold of a Kaehler manifold. Such submanifolds were studied in [13]. A characterization theorem for such manifolds is the following.

**Theorem 7.4.** A proper CR-submanifold $M$ of a Kaehler manifold $\tilde{M}$ is locally a CR-warped product if and only if its shape operator $A$ satisfies

$$A JZ X = -JX(\mu) Z, \quad X \in \Gamma(\mathfrak{D}), \quad Z \in \Gamma(\mathfrak{D}^\perp), \quad (7.2)$$

for some function $\mu$ on $M$ satisfying $W(\mu) = 0$ for $W \in \Gamma(\mathfrak{D}^\perp)$, where $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are the holomorphic and totally real distributions of $M$, respectively.

In fact, this theorem is Theorem 4.2 of [13]. Further, if $\alpha^\# = 0$ and $\theta$ is a slant function in Theorem 5.3, then Theorem 5.3 is the following characterization theorem (Theorem 5.1) of [33].
Theorem 7.5. Let $M$ be a pointwise semi-slant submanifold of a Kaehler manifold $\tilde{M}$. Then $M$ is locally a non-trivial warped product manifold of the form $N^T \times_f N^\theta$ such that $N^\theta$ is a proper pointwise slant submanifold and $N^T$ is a holomorphic submanifold in $\tilde{M}$ if the following condition is satisfied

$$A_{FPW}X - A_{FW}JX = - (\sin^2 \theta) X(\mu)W$$  \hspace{1cm} (7.3)$$

where $\mu$ is a function on $M$ such that $Z(\mu) = 0$ for every $Z \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D})$.

In fact, in the relation (5.4) of Theorem 5.1 in [33], the term $(1 + \cos^2 \theta)$ should be $(1 - \cos^2 \theta)$, i.e., there is a missing term in that theorem.

If $\theta$ is constant on $M$ in Theorem 5.3, then $M$ is a semi-slant submanifold of an $LCK$-manifold. In this case, Theorem 5.3 reduces to the following.

Theorem 7.6. A semi-slant submanifold $M$ of an $LCK$-manifold $\tilde{M}$ is locally a non-trivial warped product manifold of the form $N^T \times_f N^\theta$ such that $N^\theta$ is a proper slant submanifold and $N^T$ is a holomorphic submanifold in $\tilde{M}$ if and only if the shape operator $A$ of $M$ satisfies

$$A_{FPZ}X - A_{FZ}JX = \sin^2 \theta (g(\lambda, X) - X(\mu)) Z$$  \hspace{1cm} (7.4)$$

where $\mu$ is a function on $M$ such that $W(\mu) = 0$ for every $W \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D})$.

Now, we provide the following applications of Theorem 6.1.

If the slant function $\theta$ in Theorem 6.1 is a constant on $M$, then the pointwise semi-slant warped product submanifold $M$ is a semi-slant warped product $N^T \times_f N^\theta$ in $\tilde{M}$ such that the Lee vector field of $\tilde{M}$ is tangent to $M$. In this case, inequality (6.1) remains the same as

$$\|h\|^2 \geq 4q \left( \csc^2 \theta + \cot^2 \theta \right) \left\{ \|\nabla \ln f\|^2 + \|\alpha^\#_{TM}\|^2 - 2G^* \right\},$$

where $G^* = \sum_{i=1}^{2p} (e_i \ln f) g(\alpha^\#, e_i)$.

But, if we assume $\theta = \frac{\pi}{2}$ in Theorem 6.1, then the pointwise semi-slant warped product submanifold $M$ takes the form $N^T \times_f N^\perp$ i.e., $M$ is a CR-warped product where $N^T$ and $N^\perp$ are holomorphic and totally real submanifolds of $\tilde{M}$, respectively. In this case, Theorem 6.1 reduces to the following.

Theorem 7.7. Let $M = N^T \times_f N^\perp$ be a CR-warped product submanifold of a locally conformal Kaehler manifold $\tilde{M}$. If the Lee vector field $\lambda = \alpha^\#$ of $\tilde{M}$ is tangent to $M$, then we have:

(i) The squared norm of the second fundamental form $h$ of $M$ satisfies

$$\|h\|^2 \geq 2q \left\{ \|\nabla (\ln f)\|^2 + \|\alpha^\#_{TM}\|^2 - 2G^* \right\},$$  \hspace{1cm} (7.5)$$
(ii) If the equality sign of inequality (7.5) holds identically, then $N^T$ and $N^\perp$ are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively. Furthermore, $M$ is minimal in $\tilde{M}$.

Theorem (7.7) is the main result (Theorem 4.2) of [7]. Hence, the main result of [7] is a special case of Theorem 6.1.

Since a Kaehler manifold is an $\text{LCK}$-manifold with $\alpha^# = 0$, Theorem 6.1 implies the following.

**Theorem 7.8.** Let $M = N^T \times_f N^\theta$ be non-trivial warped product pointwise semi-slant submanifold in a Kaehler manifold $\tilde{M}$. Then we have:

(i) The squared norm of the second fundamental form of $M$ satisfies

$$\|h\|^2 \geq 4q \left( \csc^2 \theta + \cot^2 \theta \right) \|\tilde{\nabla}^T (\ln f)\|^2$$

where $q = \frac{1}{2} \dim N^\theta$.

(ii) If equality sign of inequality (7.6) holds identically, then $N^T$ and $N^\theta$ are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively. Furthermore, $M$ is not mixed totally geodesic in $\tilde{M}$.

Theorem 7.8 is a special case of Theorem 6.1 which is Theorem 5.2. of [33].

If we choose $\alpha^# = 0$ and $\theta = \frac{\pi}{2}$ for Theorem 6.1, then Theorem 6.1 reduces to Theorem 5.1 of [13] which is exactly the following well-known Chen’s inequality for $\text{CR}$-warped product submanifolds in Kaehler manifolds.

**Theorem 7.9.** Let $M = N^T \times_f N^\perp$ be a $\text{CR}$-warped product submanifold in a Kaehler manifold $\tilde{M}$. Then, the squared norm of the second fundamental form $h$ of $M$ satisfies

$$\|h\|^2 \geq 2q \|\tilde{\nabla} \ln f\|^2$$

where $q = \dim N^\perp$. Furthermore, if the equality sign of (7.7) holds identically, then $N^T$ and $N^\theta$ are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively. Moreover, in this case $M$ is minimal in $\tilde{M}$.

8. Some Examples of Proper Pointwise Semi-slant Submanifolds in LCK-Manifolds

Let $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be the Cartesian coordinates of a Euclidean 2n-space $\mathbb{E}^{2n}$ equipped with the Euclidean metric $g_0$. Let $\mathbb{C}^n = (\mathbb{E}^{2n}, J_0, g_0)$ be the flat Kaehler manifold endowed with the canonical almost complex structure $J_0$ given by

$$J_0 (x_1, \cdots, x_n, y_1, \cdots, y_n) = (-y_1, \cdots, -y_n, x_1, \cdots, x_n).$$

(8.1)

The following result can be proved in the same way as Proposition 2.2 of [19].
Proposition 8.1. Let \( M = N^T \times_f N^\theta \) be a warped product pointwise semi-slant submanifold of a Kaehler manifold \((\tilde{M}, J, g)\). Then, \( M \) is also a warped product pointwise semi-slant submanifold with the same slant function in a LCK-manifold \((\tilde{M}, J, \tilde{g})\) where \( \tilde{g} = e^{-f}g \) and \( f \) is a function on \( \tilde{M} \).

Example 8.1. Let \( \mathbb{C}^3 = (E^6, J_0, g_0) \) be a flat Kaehler manifold defined above. Consider a 4-dimensional submanifold \( M \) of \( \mathbb{C}^3 \) given by

\[
x_1 = u, \quad x_2 = -ks \sin r, \quad x_3 = g(s), \quad y_1 = v, \quad y_2 = ks \cos r, \quad y_3 = h(s),
\]

where \( k \) is a positive number and \((g(s), h(s))\) is a unit speed planar curve. Clearly, the tangent bundle \( TM \) of \( M \) is spanned by

\[
X_1 = \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial v},
\]

\[
X_3 = \left( -ks \cos r \frac{\partial}{\partial x_2} - ks \sin r \frac{\partial}{\partial y_2} \right),
\]

\[
X_4 = \left( -k \sin r \frac{\partial}{\partial x_2} + g'(s) \frac{\partial}{\partial x_3} + k \cos r \frac{\partial}{\partial y_2} + h'(s) \frac{\partial}{\partial y_3} \right).
\]

Further, \( M \) is a proper semi-slant submanifold with the holomorphic distribution \( \mathcal{D} = \text{Span}\{X_1, X_2\} \) and the slant distribution \( \mathcal{D}^\theta = \text{Span}\{X_3, X_4\} \) with slant angle given by \( \theta = \cos^{-1}(k/\sqrt{1+k^2}) \). Obviously, both \( \mathcal{D} \) and \( \mathcal{D}^\theta \) are integrable and totally geodesic in \( M \). Let \( N^T \) and \( N^\theta \) be integral manifolds of \( \mathcal{D} \) and \( \mathcal{D}^\theta \), respectively. Then the metric \( \tilde{g} \) on \( M = N^T \times N^\theta \) induced from \( \mathbb{C}^3 \) is given by

\[
\tilde{g} = g_T + g_{N^\theta}, \quad g_T = du^2 + dv^2, \quad g_{N^\theta} = k^2 s^2 dr^2 + (1 + k^2)ds^2.
\]

Let \( f = f(x_1, y_1) \) be a non-constant smooth function on \( \mathbb{C}^3 \) depending only on coordinates \( x_1, y_1 \) and let us consider the Riemannian metric \( \tilde{g} = e^{-f}g_0 \) on \( \mathbb{C}^3 \) conformal to the standard metric \( g_0 \). Then \( \tilde{M} = (E^6, J_0, \tilde{g}) \) is a GCK-manifold and the metric on \( M \) induced from the GCK-manifold is the warped product metric:

\[
g_M = g_{N^T} + e^{-f}g_{N^\theta}, \quad g_{N^T} = e^{-f}g_T.
\]

Thus, it follows Proposition 8.1 that \((M, g_M)\) is a proper warped product semi-slant submanifold in \( \tilde{M} = (E^6, J_0, \tilde{g}) \). Now, since \( f = f(x_1, y_1) \) is a smooth function on \( \mathbb{C}^3 \) depending only on coordinates \( x_1, y_1 \), it follows from (8.2) that, restricted to the submanifold \( M \), the Lee form of \( \tilde{M} \) is given by

\[
\alpha = df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2.
\]

Consequently, it follows from (8.4) and (8.5) that the Lee vector field \( \alpha^\# \) is tangent to \( N^T \), and hence it is tangent to \( M \).
Example 8.2. Consider a 4-dimensional submanifold $M$ of $\mathbb{C}^3$ given by

$$x_1 = \frac{1}{2} (u_1^2 + u_2^2), \quad x_2 = u_3 \cos u_4, \quad x_3 = u_3 \sin u_4,$$

$$y_1 = \frac{1}{2} (u_1^2 - u_2^2), \quad y_2 = u_4 \cos u_3, \quad y_3 = u_4 \sin u_3,$$

(8.6)
defined on an open subset of $\mathbb{E}^4$ with $u_1, u_2 \neq 0$, $u_3 u_4 \neq 1$ and $u_3 - u_4 \in (0, \frac{\pi}{2})$. Then the tangent bundle $TM$ of $M$ is spanned by

$$X_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} \right),$$

$$X_2 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1} \right),$$

$$X_3 = \frac{1}{\sqrt{1 + u_4^2}} \left( \cos u_4 \frac{\partial}{\partial x_2} + \sin u_4 \frac{\partial}{\partial x_3} - u_4 \sin u_3 \frac{\partial}{\partial y_2} + u_4 \cos u_3 \frac{\partial}{\partial y_3} \right),$$

$$X_4 = \frac{1}{\sqrt{1 + u_3^2}} \left( -u_3 \sin u_4 \frac{\partial}{\partial x_2} + u_3 \cos u_4 \frac{\partial}{\partial x_3} + \cos u_3 \frac{\partial}{\partial y_2} + \sin u_3 \frac{\partial}{\partial y_3} \right).$$

Then $M$ is a proper pointwise semi-slant submanifold such that the holomorphic distribution is given by $\mathcal{D} = \text{Span} \{X_1, X_2\}$ and the proper pointwise slant distribution is $\mathcal{D}^\theta = \text{Span} \{X_3, X_4\}$. It is direct to show that the slant function $\theta$ of $\mathcal{D}^\theta$ satisfies

$$\cos^2 \theta = \frac{(u_3 u_4 - 1)^2 \cos^2 (u_3 - u_4)}{(1 + u_3^2)(1 + u_4^2)}.$$ 

(8.7)

Clearly, both $\mathcal{D}$ and $\mathcal{D}^\theta$ are integrable and totally geodesic in $M$. Let $N^T$ and $N^\theta$ be integral manifolds of $\mathcal{D}$ and $\mathcal{D}^\theta$, respectively. It is to see that the metric $\check{g}$ on $M = N^T \times N^\theta$ induced from $\mathbb{C}^3$ is given by

$$\check{g} = g_T + g_{N^\theta},$$

(8.8)

where

$$g_T = 2 du_1^2 + 2 du_2^2, \quad g_{N^\theta} = (1 + u_4^2) du_3^2 + (1 + u_3^2) du_4^2.$$ 

Let $f = f(x_1, y_1)$ be a non-constant smooth function on $\mathbb{C}^3$ depending only on coordinates $x_1, y_1$ and consider the Riemannian metric $\tilde{g} = e^{-f} g_0$ on $\mathbb{C}^3$ as in Example 8.1. Then the induced metric on $M$ from $\tilde{M} = (\mathbb{E}^6, J_0, \tilde{g})$ is the warped product metric:

$$g_M = g_{N^T} + e^{-f} g_{N^\theta}, \quad g_{N^T} = e^{-f} g_T.$$ 

(8.9)

Now, it follows Proposition 8.1 that $(M, g_M)$ is a proper warped product pointwise semi-slant submanifold in $\tilde{M} = (\mathbb{E}^6, J_0, \tilde{g})$. Now, since $f = f(x_1, y_1)$ is a smooth function on $\mathbb{C}^3$ depending only on coordinates $x_1, y_1$, it follows from (8.6) that, restricted to the submanifold $M$, the Lee form of $\tilde{M}$ is given by (8.5). Consequently, it follows from (8.5), (8.8), and (8.9) that the Lee vector field $\alpha^\#$ is tangent to $M$. 


Remark 8.1. By applying the same method as Example 8.2, we may construct many other examples of proper warped product pointwise semi-slant submanifolds $M$ in $LCK$-manifolds whose Lee vector fields are tangent to $M$.

Remark 8.2. A very recent paper [27] and this paper have used some similar ideas and techniques.

Acknowledgements

The authors would like to express their many thanks for reviewer’s valuable suggestions to improve the presentation of this paper.

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Received: November 9, 2020.
Accepted: September 8, 2021.

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