Experimentally testable geometric phase of sequences of Everett’s relative quantum states

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Abstract. - Everett’s concept of relative state is used to introduce a geometric phase that depends nontrivially on entanglement in a pure quantum state. We show that this phase can be measured in multiparticle interferometry. A correlation-dependent generalization of the relative state geometric phase to mixed quantum states is outlined.

Introduction. – Pancharatnam’s geometric phase [1] is a property of a discrete set of polarization states obtained by sending light beams through a polarization analyzer. This geometric phase has a natural counterpart in quantum mechanics, namely the phase that arises when a quantal system is exposed to a sequence of filtering measurements [3–5]. Geometric phases associated with discrete sequences of quantum states have been considered in various contexts such as quantum Zeno effect [6], quantum jumps [7], and weak measurements [8].

Here, we wish to consider another physical context in which discrete geometric phases may occur. We propose a notion of geometric phase associated with sequences of relative quantum states [9]. The essential property of this geometric phase is that it relates directly to correlations and entanglement; as such it connects to earlier studies of geometric phases time evolving entangled systems [10–16]. We wish to examine the geometric phase of relative quantum states and demonstrate how it can be unveiled experimentally.

Geometric phase of relative states. – For some composite quantum system, consider a state $\Psi$, represented by the normalized vector $|\Psi\rangle \in \mathcal{H}$. Assume the partitioning into subsystems, given by the tensor product decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and the corresponding ray (state) spaces $P(\mathcal{H})$, $P(\mathcal{H}_1)$, and $P(\mathcal{H}_2)$. Let $|\phi\rangle \in \mathcal{H}_1$ represent a state of the first subsystem. Physically, this state corresponds to a certain outcome of a local projective measurement on this subsystem. The resulting state $\Psi(\phi) \in P(H_2)$ of the second subsystem relative $\phi$ [9], i.e., the state in $P(H_2)$ conditioned upon the measurement outcome corresponding to $\phi$, can be represented by the partial scalar product $|\Psi(\phi)\rangle = \langle \phi|\Psi \rangle \in \mathcal{H}_2$. If $\phi$ is normalized, the squared norm of $\Psi(\phi)$ is the maximal probability to obtain $\phi$ given $\Psi$, i.e., $||\Psi(\phi)||^2 = \langle \phi|\rho_1|\phi \rangle$ with $\rho_1 = \text{Tr}_2|\Psi\rangle\langle\Psi|$ the marginal state on $\mathcal{H}_1$ (Tr is partial trace over system 2). The relative state may be formulated in a compact way in terms of the antilinear map $\mathcal{L}_\Psi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $|\phi\rangle \mapsto \langle \phi|\Psi \rangle$ [17].

Pancharatnam’s [1] notion of “in-phase” is the underlying principle of the geometric phase. For two nonorthogonal states $\phi_a$ and $\phi_b$ in $P(\mathcal{H}_1)$, the vector representatives $|\phi_a\rangle$ and $e^{i\gamma}|\phi_b\rangle$ in $\mathcal{H}_1$ are in-phase if they produce a maximum in intensity when superposed. This corresponds to the condition $e^{-i\gamma}|\phi_b\rangle|\phi_a\rangle > 0$. Solving for $\gamma$ yields the Pancharatnam phase $\gamma = \arg\langle \phi_b|\phi_a \rangle$. Now, the superposition $|\phi_b\rangle + e^{i\gamma}|\phi_b\rangle$ is mapped by $\mathcal{L}_\Psi$ to $|\Psi(\phi_a)\rangle + e^{i\gamma}|\Psi(\phi_b)\rangle$. Provided $\Psi(\phi_a)$ and $\Psi(\phi_b)$ are nonorthogonal, the in-phase condition yields the Pancharatnam phase

$$f = \arg\langle \Psi(\phi_a)|\Psi(\phi_b) \rangle = \arg\langle \phi_b|\rho_1|\phi_a \rangle$$

over the relative states $\Psi(\phi_a)$ and $\Psi(\phi_b)$. Note the antilinearity-induced interchange $\phi_a \leftrightarrow \phi_b$ in Eq. (1).

We are now prepared to introduce the geometric phase for a sequence of relative states for a given bipartite state $\Psi \in P(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let $L : \phi_1, \ldots, \phi_N$ be an ordered sequence of states in $P(\mathcal{H}_1)$ and let $|\phi_1\rangle, \ldots, |\phi_N\rangle \in \mathcal{H}_1 - \{0\}$ be vectors over $L$. $L$ defines the sequence $\Psi(L) : \Psi(\phi_1), \ldots, \Psi(\phi_N)$ of relative states in $P(\mathcal{H}_2)$. Assume all adjacent pairs in $\Psi(L)$ are nonorthogonal. We may assign a geometric phase $\gamma(\Psi(L))$ to the sequence $\Psi(L)$ of relative states by using the Bargmann prescrip-
tion [18] and Eq. (1), yielding
\[
\gamma[\Psi(L)] = \arg \left( \langle \Psi(\phi_1) | \Psi(\phi_N) \rangle \times \langle \Psi(\phi_N) | \Psi(\phi_{N-1}) \rangle \cdots \langle \Psi(\phi_2) | \Psi(\phi_1) \rangle \right) \\
= -\arg \left( \langle \phi_1 | \rho_1 | \phi_N \rangle \times \langle \phi_N | \rho_1 | \phi_{N-1} \rangle \cdots \langle \phi_2 | \rho_1 | \phi_1 \rangle \right). \tag{2}
\]

Similarly, we can define the geometric phase associated with \( L \) as
\[
\gamma[L] = \arg \left( \langle \phi_1 | \rho_N | \phi_{N-1} \rangle \cdots \langle \phi_2 | \rho_1 | \phi_1 \rangle \right) \tag{3}
\]
provided the nonorthogonality between adjacent pairs is satisfied also along \( L \). The antilinearity-induced interchange \( \phi_a \leftrightarrow \phi_b \) mentioned above, results in an effective reversal of the path in \( P(H_2) \), which is the origin of the sign difference in the right-hand side of Eqs. (3) and (2).

\( \gamma[\Psi(L)] \) and \( \gamma[L] \) are invariant under the local gauge transformation \( |\phi_j\rangle \rightarrow c_j|\phi_j\rangle, c_j \in \mathbb{C} - \{0\} \) for \( j = 1, \ldots, N \).

Let us now see how the geometric phase of \( \Psi(L) \) depends on entanglement. For a product state (no entanglement), \( \rho_1 \) is a pure projector, which applied to Eq. (2) implies that \( \gamma[\Psi(L)] \) vanishes. Thus, the phase \( \gamma[\Psi(L)] \) needs entanglement to be nontrivial. For a maximally entangled state \( \Psi \) (assuming \( K = \dim H_2 \leq \dim H_2 \) finite), \( \rho_1 = \frac{1}{K} I \) and we obtain \( \gamma[\Psi(L)] = -\gamma[L] \). This relation between \( \gamma[\Psi(L)] \) and \( \gamma[L] \) is a consequence of the antiunitary (“time-reversal”) nature of \( \sqrt{KL} \) for maximally entangled \( \Psi \) [17].

We may put \( \gamma[\Psi(L)] \) on integral form by using the concept of null phase curves [19]. These are defined as curves that have vanishing geometric phase for any portion of them. A free geodesics is always a null phase curve, but the converse is not true, in general. Free geodesics and null phase curves fully coincide only in the two-dimensional (qubit) case. Now, let the relative states \( \Psi(\phi_j), j = 1, \ldots, N \) be connected by null phase curves forming a continuous path \( \bar{L} \) path in ray space. Then, following Ref. [19], we may write
\[
\gamma[C] = \int_{\bar{L}} A \tag{4}
\]
with connection one-form
\[
A = \text{Im} \frac{\langle \phi | \rho_1 | d\phi \rangle}{\langle \phi | \rho_1 | \phi \rangle}. \tag{5}
\]

**Physical realization.** – We demonstrate how the geometric phase \( \gamma[\Psi(L)] \) can be measured (Fig. 1 shows a circuit version of the setup). The key step is to implement the Pancharatnam phase of adjacent pairs of relative states. This can be done by consuming entangled pairs of extra ancillary qubit pairs.

Prepare the pure state \( |\Xi\rangle = |\Psi\rangle |\Phi_+\rangle \), where \( |\Phi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) is a Bell state of two ancilla qubits. Alice (Bob) possesses the first (second) half of \( \Psi \) and one of the ancilla qubits. Alice performs a measurement \( M \) that realizes the projection operator
\[
\Pi_{j+1,j} = \frac{1}{2} \left( |\phi_j\rangle \langle 0 + e^{if_j} |\phi_{j+1} \mod N \rangle \langle 1 \right) \\
\times \left( |\phi_j\rangle \langle 0 + e^{-if_j} |\phi_{j+1} \mod N \rangle \langle 1 \right), \tag{6}
\]
which swaps the entanglement into the state
\[
\Pi_{j+1,j} \otimes \frac{1}{\sqrt{2}} |\Xi\rangle = \frac{1}{2} \left( |\phi_j\rangle \langle 0 + e^{if_j} |\phi_{j+1} \mod N \rangle |1 \right) \\
\times \left( |\Psi(\phi_j)\rangle \langle 0 + e^{-if_j} |\Psi(\phi_{j+1} \mod N \rangle |1 \right). \tag{7}
\]

Alice and Bob can do independent interference experiments on the post-selected ensemble described by \( \Pi_{j+1,j} \otimes \frac{1}{\sqrt{2}} |\Xi\rangle \). Bob’s experiment relates to \( \gamma[\Psi(L)] \) as follows. He first applies a phase shift \( f : |x\rangle \rightarrow e^{ixf} |x\rangle, x = 0, 1 \), followed by a Hadamard \( H : |x\rangle \rightarrow \frac{1}{\sqrt{2}} (|x\rangle + (-1)^x |x + 1\rangle) \), and finally performs a measurement \( m \) of the 0 state, say. This intensity is maximal for \( f = \arg(\Psi(\phi_{j+1} \mod N) \langle \Psi(\phi_j)\rangle + f_j \equiv f_{j+1} \) if \( f_{j+1} \) yields \( f_{N+1} = \gamma[\Psi(L)] \). We note that Alice and Bob must communicate classically in order to measure \( \gamma[\Psi(L)] \). In this sense, \( \gamma[\Psi(L)] \) is a nonlocal quantity that reflects entanglement in pure quantum states. On the other hand, the marginal subsystem geometric phase is a local quantity since no classical communication is needed to measure it (see, e.g., Fig. 1 of Ref. [12]). Furthermore, note that one may replace the ancillary Bell state \( \Phi_\pm \) by a nonmaximally entangled state \( |\Phi\rangle = \sqrt{a} |00\rangle + \sqrt{1-a} |11\rangle, a \in [0, 1] \), at the expense of reducing...
the interference visibility by a factor $2\sqrt{a(1-a)}$. This factor varies from 0 (product states $a = 0, 1$) to the optimal value 1 (Bell state $a = \frac{1}{2}$).

Fig. 2: Sequence of qubit states $L : \phi_1, \phi_2, \phi_3$ and their relative states $\Psi_\lambda(L) : \Psi_\lambda(\phi_1), \Psi_\lambda(\phi_2), \Psi_\lambda(\phi_3)$ for the entangled two-qubit state $\Psi_\lambda$ in Eq. (8). The two discrete sets of states are joined by null phase curves, which coincide with geodesics (great circle segments) on the Bloch sphere. The sequences $L$ and $\Psi_\lambda(L)$ enclose the (signed) solid angles $\varphi$ and $-\varphi + 2\arctan(\cos \lambda \tan \frac{\gamma}{2})$, respectively.

Examples. – First, consider a two-qubit state

$$|\Psi_\lambda\rangle = \cos \frac{\lambda}{2}|00\rangle + \sin \frac{\lambda}{2}|11\rangle, \lambda \in [0, \pi] \tag{8}$$

with maximal entanglement for $\lambda = \frac{\pi}{2}$ and product state for $\lambda = 0$ or $\pi$. This state may be prepared experimentally in the polarization of two photons using spontaneous down-conversion technique [20]. Consider the sequence $L : \phi_1, \phi_2, \phi_3$ connected by null phase curves [19] (here parts of great circles) on the Bloch sphere, as shown in Fig. 2. Explicitly, $\phi_1, \phi_2, \phi_3$ are represented by the vectors $|0\rangle, \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\varphi}|1\rangle) \in H_1$, respectively, with $0 \leq \varphi < \pi$. The geometric phase becomes $\gamma(L) = -\frac{\varphi}{2}$. The vectors $\cos \frac{\lambda}{2}|0\rangle, \frac{1}{\sqrt{2}}(\cos \frac{\lambda}{2}|0\rangle + \sin \frac{\lambda}{2}|1\rangle), \frac{1}{\sqrt{2}}(\cos \frac{\lambda}{2}|0\rangle + e^{i\varphi}\sin \frac{\lambda}{2}|1\rangle) \in H_2$ represent the members of the sequence $\Psi_\lambda(L) : \Psi_\lambda(\phi_1), \Psi_\lambda(\phi_2), \Psi_\lambda(\phi_3)$ relative $L$ (again see Fig. 2). The corresponding geometric phase reads

$$\gamma[\Psi_\lambda(L)] = \frac{\varphi}{2} - \arctan \left( \cos \lambda \tan \frac{\varphi}{2} \right), \tag{9}$$

which varies from 0 to $\frac{\pi}{2} = -\gamma(L)$ when $\lambda$ increases from 0 (product state) and $\frac{\pi}{2}$ (maximally entangled state). Note that $\gamma[\Psi_\lambda(L)]$ is undefined for this specific $L$ if $\lambda = \pi$.

Our second example concerns two-mode squeezed states with $r \geq 0$ the squeezing parameter, which determines the amount entanglement in $\Psi_r$ ($r = 0$ corresponds to no entanglement and $\Psi_r$ tends to a maximally entangled state when $r \to \infty$). This class of states may be prepared experimentally by sending a down-converted photon-pair through optical fibers [22]. Consider a sequence $L : z_1, \ldots, z_N$ of coherent states of the first oscillator mode, where each $z_j$ defines a point $(q_j, p_j)$ in phase space. Explicitly, $z_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$. The null phase curves that join adjacent $z_j$’s become straight lines, defining an $N$-vertex polygon $\tilde{L}$ in phase space, as shown in Fig. 3. Note that these null phase curves correspond to constrained geodesics in the infinite dimensional Hilbert space of the oscillator mode. The geometric phase $\gamma[L]$ is the area $\int_{\tilde{L}} dq dp$ enclosed by $\tilde{L}$. The corresponding sequence $\Psi_r(L)$ of relative states of the second oscillator mode are coherent states $\Psi_r(z_j) = \tanh(r)z_j^*$, related to $z_j$ by a scale factor $\tanh(r)$ and a mirror reflection through the $q$-axis (see Fig. 3). It follows that

$$\gamma[\Psi_r(L)] = -\tanh^2(r)\int_{\tilde{L}} dq dp, \tag{10}$$

Fig. 3: Sequence of coherent states $L : z_1, \ldots, z_N$ (solid line) and their relative states $\Psi_r(L) : \Psi_r(z_1), \ldots, \Psi_r(z_N)$ (dotted line) for the entangled two-mode squeezed state $\Psi_r$ in Eq. (10). The two discrete sets of states are joined by null phase curves, which are straight lines in the complex plane. Each coherent state $z_j$ defines a point $(q_j, p_j)$ in phase space via the relation $z_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$. We find that $\Psi_r(z_j) = \tanh(r)z_j^*$, i.e., scaled coherent states mirror reflected through the $q$-axis. The sequences $L$ and $\Psi_r(L)$ enclose the (signed) phase space areas $\int_{\tilde{L}} dq dp$ and $-\tanh^2(r)\int_{\tilde{L}} dq dp$, respectively, where $\tilde{L}$ is the geodesic polygon defined by $L$. [21]:

$$|\Psi_r\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \tanh^n(r)|nn\rangle \tag{10}$$
which interpolates between 0 for product state $r = 0$ and $-\gamma |L|$ when $r \to \infty$.

**Mixed state case.** – The above analysis can be extended to mixed states by using the Uhlmann holonomy [23]. Consider a mixed state represented by the density operator $\rho$ that acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$. This $\rho$ is a product state if $\rho = \text{Tr}_2 \varrho \otimes \text{Tr}_1 \varrho$; $\varrho$ is separable (classically correlated) if it can be written as a convex sum of product states; $\rho$ is nonseparable (entangled) otherwise. The members of the ordered sequence $\rho(L) : \rho(\phi_1), \ldots, \rho(\phi_N)$ relative $L : \phi_1, \ldots, \phi_N \in \mathcal{P}(\mathcal{H}_1)$ are defined as $\rho(\phi_j) = \text{Tr}_1 \left[ |\phi_j \rangle \langle \phi_j | \otimes 1|\varrho] = |\phi_j | \langle \varrho | \phi_j \rangle$. For simplicity, suppose all $\rho(\phi_j)$ are faithful, i.e., full rank. The Uhlmann holonomy for $\rho(L)$ reads

$$U_{\rho}(L) = \left( \sqrt{\rho(\phi_N) \rho(\phi_{N-1}) \sqrt{\rho(\phi_N)}} \right)^{-1/2} \times \sqrt{\rho(\phi_N) / \sqrt{\rho(\phi_{N-1})} \cdots \left( \sqrt{\rho(\phi_2)} \sqrt{\rho(\phi_1)} \right)^{-1/2} \times \sqrt{\rho(\phi_2) / \sqrt{\rho(\phi_1)}}. \quad (12)$$

This quantity becomes trivially equal to the identity for product states, but is in general nontrivial for separable and nonseparable states. Thus, $U_{\rho}(L)$ depends on both classical and quantum correlations in the state; as such $U_{\rho}(L)$ should be regarded a correlation-dependent rather than entanglement-dependent quantity of mixed bipartite states. It can in principle be measured experimentally (see Ref. [24] for an explicit interferometric scheme to observe the Uhlmann holonomy for an arbitrary discrete sequence of faithful density operators).

**Conclusions.** – Geometric phases for discrete sequences of quantum states have been considered in the past. Here, we have introduced such a concept for sequences of relative states with respect to a bipartite tensor product decomposition of a pure quantum state. This phase depends on the amount of entanglement in the state, in particular it vanishes for product states. We have demonstrated that the geometric phase of sequences of relative quantum states can be tested in multiparticle interferometry, by means of local projective measurements and classical communication assisted post-selection. We have argued that a mixed state generalization based on the Uhlmann holonomy leads to a multidimensional holonomy that depends on both classical and quantum correlations in a noisy bipartite system.

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