HOLOMORPHIC DISCS, SPIN STRUCTURES AND
FLOER COHOMOLOGY OF THE CLIFFORD TORUS

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ABSTRACT. We compute the Bott-Morse Floer cohomology of the Clifford torus in \( \mathbb{P}^n \) with all possible spin-structures. Each spin structure is known to determine an orientation of the moduli space of holomorphic discs, and we analyze the change of orientation according to the change of spin structure of the Clifford torus. Also, we classify all holomorphic discs with boundary lying on the Clifford torus by establishing a Maslov index formula for such discs. As a result, we show that in odd dimensions there exist two spin structures which give non-vanishing Floer cohomology of the Clifford torus, and in even dimensions, there is only one such spin structure. When the Floer cohomology is non-vanishing, it is isomorphic to the singular cohomology of the torus (with a Novikov ring as its coefficients). As a corollary, we prove that any Hamiltonian deformation of the Clifford torus intersects with it at least at \( 2^n \) distinct intersection points, when the intersection is transversal.

We also compute the Floer cohomology of the Clifford torus with flat line bundles on it and verify the prediction made by Hori using a mirror symmetry calculation.

1. INTRODUCTION

The Floer homology of Lagrangian intersection was first defined by Floer [Fl] and since then, it is emerging as a powerful technique in symplectic geometry. It has received much more attention after Kontsevich [Ko] proposed a homological Mirror symmetry conjecture to use Floer homology in the context of \( A_{\infty} \) category that Fukaya introduced [Fuk]. The construction of Floer homology has been generalized and applied to the problems in symplectic geometry by Oh [O1, O2, O4], and recently, it was studied in complete generality and its obstruction to the well-definedness of Floer homology was established by Fukaya, Oh, Ohta and Ono in [FOOO]. But computing actual Floer homologies is still a difficult task, since one has to analyze all holomorphic strips with boundary lying on two Lagrangian submanifolds. The construction of Floer homology in the Bott-Morse setting is a big step forward in this respect as in Morse theory.

In this paper, we compute Floer cohomology of the Clifford torus \( T^n \) in \( \mathbb{P}^n \) in the Bott-Morse setting. There are two main issues in the computation. The first one is an orientation problem. Floer and Oh defined Floer homology with \( \mathbb{Z}/2\mathbb{Z} \)-coefficients. In [FOOO], Fukaya, Oh, Ohta and Ono developed a way to give an orientation of the moduli space of holomorphic discs (and strips). This orientation depends on the (relative) spin structure of a Lagrangian submanifold. Hence, one can define Floer homology with \( \mathbb{Z} \) or \( \mathbb{Q} \)-coefficients. A spin structure of an oriented vector bundle \( E \) over \( X \) can be understood as a homotopy class of a trivialization of \( E \) over the 1-skeleton of \( X \) which can be extended to the 2-skeleton of \( X \). It is...
already observed in [FOOO] that different homotopy classes of the trivialization of a certain bundle will reverse orientation of the moduli space, with an example in the case of Maslov index 0 disc. Here we give a proof of this observation in general by using the Index theorem of Silva in [S] (see Theorem 6.2).

The Clifford torus is an interesting example since it has $2^n$ different spin structures. These $2^n$ spin structures can possibly give rise to $2^n$ different Floer cohomologies. Or more generally, we can consider Floer cohomology of $T^n$ with flat line bundles on it. It may be considered as an advantage in that we may exploit this freedom to define non-vanishing Floer homology, if possible.

But for the Clifford torus, one can choose a natural spin structure which we will call the standard spin structure. Under the standard spin structure, it is not hard to determine the orientation of the moduli space as described in [FOOO] (see section 8). For the other spin structures, we will determine the orientation of the moduli space by studying the change of orientation with respect to the change of spin structures. Hence, we can determine orientations needed to define Floer boundary operator for any spin structures of the Clifford torus.

The second issue is to classify all the holomorphic discs with boundary lying on a Lagrangian submanifold. For that purpose, we prove the Maslov index formula (Theorem 9.1) and classify all the holomorphic discs with boundary on the Clifford torus with any Maslov index. In this case, any non-constant holomorphic disc has positive Maslov index, in which case Bott-Morse Floer homology is rather easy to define.

By the classification theorem, Theorem [CHO], we can explicitly calculate Bott-Morse Floer boundary operators for Floer cohomology. It turns out that among $2^n$ spin structures, for $n = \text{dim}(L)$ even, there is only one spin structure which gives non-vanishing Floer cohomology. And for $n$ odd, there exists two spin structures which gives non-vanishing Floer cohomology. And when it is non-vanishing, $HF(T^n, T^n; \Lambda_{\text{nov}})$ as a $\Lambda_{\text{nov}}$-module is isomorphic to the singular cohomology with $\Lambda_{\text{nov}}$-coefficient.

One immediate corollary of the latter result is that intersection between the Clifford torus and its Hamiltonian deformation must have at least $2^n$ distinct points when the intersection is transversal. In particular, the Clifford torus must intersect any Hamiltonian deformations thereof. While we are in the preparation of the thesis [Cho], we have learned from Oh that this latter intersection result was also proved by Biran-Entov-Polterovich [BEP] using a completely different method without using the Floer homology.

As an application to physics, one can compute a D-brane Floer cohomology (Floer cohomology with flat line bundle on the Lagrangian submanifold). The homological mirror symmetry conjecture is about Calabi-Yau manifolds, but, its extension to Fano case has been studied by Hori [H]. With minor modification from our calculation of Floer cohomology, we can compute compute D-brane Floer cohomology. As a result, we found $(n + 1)$ flat line bundles with specific holonomies over the Clifford torus whose Floer cohomology is non-vanishing, which has been predicted by Hori [H], Hori-vafa [HV] by B-model calculation.

More generally, the prediction by K. Hori about the Floer cohomology of Lagrangian torus fibers of Fano toric manifolds is that the Floer cohomology of all the fibers vanish except at a finite number of base points in the momentum polytope that are critical points of the super-potential of the Landau-Ginzburg mirror to the
toric manifold. We generalize the scheme used in the paper to this case and will prove the exact correspondence in [CO].

This is the simplified version of the author’s Ph. D. thesis in the University of Wisconsin-Madison.

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2. The Maslov index

In this section, we recall basic definitions including the Maslov index of a map and its generalization in the case that the domain of a map is a smooth Riemann surface with boundary.

Let \((M, \omega)\) be a \(2n\)-dimensional compact symplectic manifold. Let \(L\) be a Lagrangian submanifold. There are two homomorphisms \(I_\omega, \mu\) on \(\pi_2(M, L)\) defined as follows. The symplectic energy \(I_\omega: \pi_2(M, L) \to \mathbb{R}\) is defined as

\[
I_\omega = \int_{D^2} w^*\omega.
\]

The Maslov index \(\mu: \pi_2(M, L) \to \mathbb{Z}\) is defined as follows: We first consider the Lagrangian Grassmannian \(\Lambda(C^n)\) consisting of all \(n\)-dimensional \(\mathbb{R}\) linear subspaces \(V\) of \(\mathbb{C}^n\) such that the standard symplectic form \(\omega_0\) of \(\mathbb{C}^n\) vanishes on \(V\). The unitary group \(U(n)\) acts transitively on \(\Lambda(C^n)\) and the isotropy group is \(O(n)\). Therefore, we have \(\Lambda(C^n) \cong U(n)/O(n)\).

Each Lagrangian plane can be written as \(A \cdot \mathbb{R}^n\) for some \(A \in U(n)\) and two such matrices \(A_1, A_2\) define the same plane if and only if \(A_1 \cdot A_1^{-1} = A_2 \cdot A_2^{-1}\). By Proposition 4.2 [O3], the map \(B: \Lambda(C_n) \to \widetilde{\Lambda(C_n)}: A \mapsto A \cdot A^{-1} = A \cdot A^t\) is a diffeomorphism for \(\widetilde{\Lambda(C_n)} = \{D \in GL(n; \mathbb{C})|D^*D = Id, D = D^t\}\).

Now, for any loop \(\gamma: S^1 \to \Lambda(C_n)\), the Maslov index of a loop \(\gamma\) is defined to be the degree of the map \(\phi = \text{det} \circ B \circ \gamma: S^1 \to U(1)\).

Now let \(w: (D^2, \partial D^2) \to (M, L)\) be a smooth map representing the homotopy class \(\beta \in \pi_2(M, L)\). Then we can find a unique trivialization (up to homotopy) of the pull-back bundle \(w^*(TM) \simeq D^2 \times \mathbb{C}^n\) as a symplectic vector bundle. The trivialization defines a map from \(\gamma: \partial D^2 \to \Lambda(C^n)\) and we define

\[
\mu(w) := \mu(\gamma) \in \mathbb{Z}.
\]

It is independent of the trivialization. We will call \(\mu(\beta) = \mu(w)\) the Maslov index of \(\beta\). The minimal Maslov number \(\Sigma_L\) is the positive generator of the abelian subgroup \([\mu|_{\pi_2(M,L)}] \subset \mathbb{Z}\).

**Definition 2.1.** A Lagrangian submanifold \(L\) is said to be monotone if there exists \(c > 0\) independent of \(\beta \in \pi_2(M, L)\) such that

\[
\mu(\beta) = cI_\omega(\beta).
\]

Let \(\Sigma\) be a smooth Riemann surface with boundary. We will denote by \(R_0, \cdots, R_h\) the connected components of \(\partial \Sigma\), with orientation induced by the orientation of \(\Sigma\). We assume that the number of boundary components \(h\) is nonzero. Let \(w: (\Sigma, \partial \Sigma) \to (M, L)\) be a smooth map with \(w(\partial \Sigma) \subset L\). Then we can also define the Maslov index of the map \(w\) as follows (see [KLu]).
Let $E$ be the complex vector bundle $w^*TM$, and let $E_R$ be the Lagrangian subbundle $w|_{\mathbb{C}^n}^*TL$. Since any complex vector bundle over a Riemann surface with nonempty boundary is trivial, we may fix the trivialization of $E$ as $\Phi : E \cong \Sigma \times \mathbb{C}^n$. Then, for each boundary component $R_i$, we have a map $\gamma_i : S^1 \to \Lambda(\mathbb{C}^n)$. Let $\mu(\Phi, R_i) = \mu(\gamma_i)$. We define the Maslov index of the map $w$ as

$$\mu(w) = \mu(\Phi, w) = \sum_{i=0}^{\delta} \mu(\Phi, R_i)$$

**Proposition 2.1 (Katz-Liu \[KL\] Proposition 3.3.6).** The Maslov index defined above is independent of the choice of trivialization $\Phi : E \cong \Sigma \times \mathbb{C}^n$.

**3. The Clifford torus**

We follow the description of the Clifford torus given in [O1]. Consider the isometric embedding

$$T^{n+1} := S^1(\frac{1}{\sqrt{n+1}}) \times \cdots \times S^1(\frac{1}{\sqrt{n+1}}) \hookrightarrow S^{2n+1}(1) \subset \mathbb{C}^{n+1}$$

This embedding is Lagrangian in $\mathbb{C}^{n+1}$, and the standard action by $S^1$ on $\mathbb{C}^{n+1}$ restricts to both the above torus and $S^{2n+1}(1)$. By taking the quotients by this action, the torus $T^n := T^{n+1}/S^1$ in $\mathbb{P}^n$ is Lagrangian submanifold. This torus is a minimal submanifold in Riemannian geometry; it is called the Clifford torus in $\mathbb{P}^n$. For the case $n=1$, $T^1$ is nothing but the great circle in $\mathbb{P}^1$.

**Proposition 3.1 (Oh [O1] Proposition 2.4).** The above Clifford torus $T^n \subset \mathbb{P}^n$ is monotone with respect to the standard symplectic structure on $\mathbb{P}^n$.

**Proof.** We first describe the homotopy classes in $\pi_2(\mathbb{P}^n, T^n)$. We have the homotopy exact sequence,

$$\rightarrow \pi_2(T^n) \to \pi_2(\mathbb{P}^n) \to \pi_2(\mathbb{P}^n, T^n) \to \pi_1(T^n) \to \pi_1(\mathbb{P}^n) \to$$

with $\pi_2(T^n) \cong 0$ and $\pi_1(\mathbb{P}^n) \cong 0$. We have

$$\pi_2(\mathbb{P}^n, T^n) \cong \pi_2(\mathbb{P}^n) \oplus \pi_1(T^n)$$

since the boundary map has an obvious right inverse.

For $0 \leq i \leq n$, let $b_i$ be the holomorphic disc

$$b_i = [1 : \cdots : 1 : z : 1 : \cdots : 1]$$

We will denote their homotopy classes as $\beta_i = [b_i] \in \pi_2(\mathbb{P}^n, T^n)$. These are discs with the Maslov index 2, and we will call them standard discs. Later, we will show that any holomorphic disc of Maslov index 2 with boundary lying on $T^n$ is in fact one of the standard discs up to an automorphism of a disc.

Now we want to show that the spherical homotopy class $i(\pi_2(\mathbb{P}^n)) \subset \pi_2(\mathbb{P}^n, T^n)$ can be obtained as a sum of $b_i$. For the generator $\alpha \in \pi_2(\mathbb{P}^n)$ with $c_1(\alpha) = n + 1$ where $c_1$ is the first chern class of the tangent bundle of $\mathbb{P}^n$, it is known that the Maslov index of $i(\alpha)$ is actually $2c_1(\alpha) = 2(n + 1)$. Now, note that

$$j(\beta_0 + \beta_1 + \cdots + \beta_n) = 0 \in \pi_1(T^n)$$
and $\mu(\beta_0 + \cdots + \beta_n) = 2n + 2$. Since the homotopy sequence is exact, $(\beta_0 + \cdots + \beta_n)$ lies in the image of the map $i : \pi_2(\mathbb{P}^n) \to \pi_2(\mathbb{P}^n, T^n)$. Hence, we have

$$i(\alpha) = \beta_0 + \beta_1 + \cdots + \beta_n.$$  

Then, it is easy to show that the Lagrangian submanifold $T^n$ is monotone: If $I_\omega(\beta_i) = c\mu(\beta_i)$ for all $i$ for fixed $c$, then

$$I_\omega(i(\alpha)) = (n+1)I_\omega(\beta_i) = (n+1)c\mu(\beta_i) = c\mu(i(\alpha)).$$

This proves that the Clifford torus is monotone. □

4. Bott-Morse Floer cohomology

We review the construction of Bott-Morse Floer cohomology $HF(L, L)$ following [FOOO]. There is a canonical isomorphism $HF(L, L) \to HF(L, \phi(L))$.

**Definition 4.1** (Novikov ring). We consider the formal (countable) sum $\sum_{i=0}^{\infty} c_i e^{d_i}$, such that $c_i \in \mathbb{Q}$, $d_i \in \mathbb{Z}$, $\lim_{i \to \infty} d_i = \infty$.

The totality of such formal sums becomes a ring, and we denote this ring by $\Lambda_{nov}$. We consider $\sum_{i} c_i e^{d_i}$ with $d_i \geq 0$ in addition and denote it by $\Lambda_{0,nov}$. Here we set the degree of $e$ to be 2.

**Remark 4.2.** Since we only consider monotone Lagrangian submanifolds, we do not need to include the energy term here.

To construct Floer cohomology in this case, we need a cochain complex which represents cohomology theory of $L$. For a given $(n-k)$-dimensional geometric chain $[P, f]$, we consider the current $T([P, f])$ which is defined as follows: The current $T([P, f])$ is an element in $D^k(M; \mathbb{R})$ where $D^k(M; \mathbb{R})$ is the set of distribution valued $k$-forms on $M$: For any smooth $(n-k)$-form $\omega$, we put

$$\int_M T([P, f]) \wedge \omega = \int_P f^* \omega \quad (4.1)$$

This defines a homomorphism $T : S_{n-k}(M; \mathbb{Q}) \to D^k(M; \mathbb{R})$ where $S_{n-k}(M; \mathbb{Q})$ is the set of all $(n-k)$-dimensional geometric chains with $\mathbb{Q}$-coefficient. Let $S^k(M, \mathbb{Q})$ be the image of the homomorphism $T$. Then we take a countably generated subcomplex $C(L; \mathbb{Q})$ of $S^k(M, \mathbb{Q})$ such that the cohomology of $C$ is isomorphic to the cohomology of $H^*(M, \mathbb{Q})$. Since we consider the elements in the image of $T$, if the image of the map $f$ of the geometric chain $[P, f]$ is smaller than expected dimension, then it gives 0 as a current. This fact will be used crucially later on.

We recall the definition of the compactified moduli space of holomorphic discs (See [FOOO] for details).

**Definition 4.3** ([FOOO]). Let $\beta \in \pi_2(M, L)$ and denote by $M_m(\beta)$ the set of all isomorphism classes of genus 0 stable maps from open curve with $(m,0)$ marked points $((\Sigma, \vec{z}), w)$ such that $w_*([\Sigma]) = \beta$. Also denote by $M_m^{reg}(\beta)$ the subset of $M_m(\beta)$ with $\Sigma = D^2$.

For the analysis of orientations, we define the moduli space of holomorphic discs without compactification.
Definition 4.4. For a given homotopy class $\beta \in \pi_2(M, L; x)$, we define
$$\tilde{M}(L, J : \beta) = \{ w : D^2 \to M | \partial Jw = 0, w(\partial D^2) \subset L, [w] = \beta \}.$$ 
We also similarly define $\tilde{M}_n(L, J : \beta)$ to be the moduli space of holomorphic discs with $n$ marked points. We will abbreviate $\tilde{M}_n(L, J : \beta)$ as $\tilde{M}_n(\beta)$ from now on. The group $PSL(2 : \mathbb{R}) = Aut(D^2, j_{D^2})$ acts on $\tilde{M}(\beta)$ by $\phi \cdot w = w \circ \phi^{-1}$ for $\phi \in PSL(2 : \mathbb{R})$ and $w \in \tilde{M}(\beta)$ and it acts on a marked point $z_i$ as $\phi(z_i)$. Then, we have $\tilde{M}_n(\beta)/PSL(2 : \mathbb{R}) \cong \mathcal{M}^{reg}_n(\beta)$.

Now we recall the definition of Floer coboundary operator.

Definition 4.5. For a geometric chain $[P, f] \in C^*(L : Q)$, define
$$\delta_{\beta}([P, f]) = (\mathcal{M}_2(\beta)_{ev_1} \times f P, ev_0) \text{ for } \beta \neq 0,$$
$$\delta_0([P, f]) = (-1)^n[\partial P, f]$$

Remark 4.6. Well-definedness of this fiber product is rather technical, because the moduli space possibly has codimension 1 corners and the product is defined in the chain level. One need Smooth-correspondence developed in [FOOO] appendix A. However, in our later calculation, only non-trivial fiber product occurs for the moduli space of Maslov index 2 discs, in which case, the moduli space is closed (without boundary) since the homotopy class is minimal. And we use the spectral sequence to compute the Floer cohomology, therefore, after the first step, we can work on the homology level.

Theorem 4.1 ([FOOO] Proposition 13.16). For $[P, f] \in C^k(L : Q)$,
$$\delta_{\beta}([P, f]) \in C^{k-\mu(\beta)+1}(L : Q).$$

Now we define our coboundary operator $\delta$ on $C^*(L; \Lambda_{nov})$ by extending the following boundary operator linearly over $\Lambda_{nov}$
$$\delta([P, f]) = \sum_{\beta \in \pi_2(M, L)} \delta_{\beta}([P, f]) \otimes e^{\frac{\mu(\beta)}{2}}$$

Theorem 4.2. $\delta \circ \delta = 0$.

Remark 4.7. This is a combination of arguments used in [FOOO] Theorem 6.24 and [addenda,O1]. The proof in [FOOO] deals with the case when $L$ is un-obstructed, while in this case obstruction cycle is a multiple of the fundamental class $[L]$.

Proof. It is enough to show that $\delta \circ \delta([P, f]) = 0$. Now,
$$\delta \circ \delta([P, f]) = \sum_{A \in \pi_2(M, L)} \sum_{A_1 + A_2 = A} \delta_{A_1} \circ \delta_{A_2} [P, f] \otimes e^{\frac{\mu(A)}{2}}$$

We consider the geometric chain $(\mathcal{M}_2(A)_{ev_1} \times f P, ev_0)$. Note that we consider not the moduli space itself but its image under evaluation map.

Now, we can describe the boundary components of the image of the chain as in Fig. First there is a component corresponding to boundary of the chain $P$, and a splitting of the moduli space of $A$, and also there is a component with a disc or sphere bubble. A component with sphere bubble has at least codimension 2, hence it causes no trouble. But a component with a disc bubble has codimension 1.
But in this Morse-Bott setting, in the case of monotone Lagrangian submanifolds, image of such a component with a disc bubble does not appear as a codimension 1 boundary as follows: (Generally, disc bubbling phenomenon causes trouble defining \( HF(L_0, L_1) \) for two different Lagrangian submanifolds \( L_0, L_1 \). But in the case \( L_1 = \phi(L_0) \), or \( L_0 = L_1 \) disc bubbling with positive maslov index does not cause much trouble defining Floer cohomology. See Proposition 7.3 [FOOO]). Basically, we only consider the image under the evaluation map and we claim that the image is always of codimension 2 or more.

As in Fig 1 if the disc \( A \) splits, we call the component meeting the chain \( P \) as \( A_1 \) and the other component as \( A_2 \).

First, consider the case that \( \mu(A_1) \neq 0 \). Note that \( \mu(A_2) \geq 2 \) since homotopy class of a bubble is always non-trivial and the Lagrangian submanifold is orientable. Then the image under the evaluation map of such a component is contained in \( (\mathcal{M}_2(A_1)_{ev_1} \times f P, ev_0) \) whose chain dimension is \( (n-k) + \mu(A_1) - 1 \). But the original chain \( (\mathcal{M}_2(A)_{ev_1} \times f P, ev_0) \) has chain dimension \( (n-k) + \mu(A) - 1 \). Since \( \mu(A) \geq \mu(A_1) + 2 \), the image is of codimension 2 or more as claimed.

Now, we consider the case that \( \mu(A_1) = 0 \). Then, actually there should be pairs of bubbles occur as in the Fig 2. Since \( A_1 \) is a constant holomorphic disc, its image under the evaluation maps are the same. But these two components have different orientation because of the ordering of the marked points. Therefore these two bubbles cancel out each other. This proves that the image of the component with a disc bubble will not give a codimension 1 boundary.
The remaining boundary components can be written as follows.

\[
\delta_0 \circ \delta_A [P, f] = \delta_0 (\mathcal{M}_2(A)_{ev_1} \times_f P)
\]
\[
= (-1)^n \partial (\mathcal{M}_2(A)_{ev_1} \times_f P)
\]
\[
= (-1)^n (\partial \mathcal{M}_2(A)_{ev_1} \times_f P) \bigcup (-1)^n (-1)^{\dim \mathcal{M}_2(A) + n} (\mathcal{M}_2(A)_{ev_1} \times_f \partial P)
\]
\[
= (-1)^n (\partial \mathcal{M}_2(A)_{ev_1} \times_f P) \bigcup (-1)^n (\mathcal{M}_2(A)_{ev_1} \times_f \partial_0 P)
\]
\[
= (-1)^n + n + 1 ((\mathcal{M}_2(A)_{ev_1} \times_{ev_0} \mathcal{M}_2(A)_{ev_2} \times_f \partial_0 P)
\]
\[
= (-1) \delta_{A_1} \circ \delta_{A_2} [P, f] \bigcup (-1) (\mathcal{M}_2(A)_{ev_1} \times_f \delta_0 P)
\]

The third equality is from the formula \[7.1\] in section \[7\] and the fifth equality is from the formula \[7.2\] in the same section. Hence it proves that

\[
\delta \circ \delta ([P, f]) = \delta_0 \circ \delta_A [P, f] + \delta_{A_1} \circ \delta_{A_2} [P, f] + \delta_A \circ \delta_0 [P, f] = 0.
\]

\[\Box\]

We also recall the construction of the spectral sequence which converges to \(HF(L, L)\) for the monotone Lagrangian submanifold \(L\). Existence of the spectral sequence was first observed by Oh \[O4\]. When the Lagrangian submanifold is monotone, we have the minimal Maslov index \(\Sigma_L\) of \(L\). Let \(\delta_i\) be the formal sum of \(\delta_\beta\) with the Maslov index \(i\). Then, we have

\[
\delta = \delta_0 + \delta_\Sigma_L \otimes e^{\Sigma_L/2} + \delta_2 \Sigma_L \otimes e^{\Sigma_L} + \cdots
\]

This filtration basically gives the spectral sequence of the Floer cohomology. The spectral sequence will start from cochain complex and the boundary maps in \(E_i^{p, q}\) will be \(\delta_{2i-2}\). For \(L\) monotone, filtration by energy and that by Maslov index are equivalent. But in general, one should consider filtration by energy (see \[FOOO\]).

**Theorem 4.3** (\[FOOO\] Theorem 6.13). There exists a spectral sequence with

\[
E_2^{p, q} \cong \bigoplus_{i=0}^{\infty} (H^{p-2i}(L : \mathbb{Q}) \otimes e^i) \cong (H^*(L : \mathbb{Q}) \otimes e^q)^p
\]
converging to $HF(L, L)$ where $(\ )^p$ means the total degree $p$. Moreover it collapses after a finite number of steps.

Proof. We only prove the last statement. For the holomorphic disc $\beta$ with $\mu(\beta) \geq n + 2$, the boundary $\delta_\beta$ is a zero map because of Theorem 4.1. In monotone case, there exists only finitely many homotopy classes $\beta$ with $\mu(\beta) < n + 2$. So, spectral sequence collapes at a finite step, say $r_0$. In fact, $r_0$ may be taken as the smallest number which satisfies

$$(2r_0 - 2) \geq (n + 2).$$

□

5. Orientation

We consider an orientation on the moduli space of holomorphic discs with Lagrangian boundary condition. It is well-known that moduli space of $J$-holomorphic discs is not always orientable. (For example, consider the Lagrangian submanifold $\mathbb{R}P^2 \subset \mathbb{P}^2$. The moduli space of constant discs with boundary in $\mathbb{R}P^2$ is non-orientable.)

In this section, we recall how to orient the moduli space of $J$-holomorphic discs with a given spin structure. In section 5, we analyze how the change of spin-structure of a Lagrangian submanifold will result in the change of orientation described in this section. In section 7, we introduce necessary orientation conventions and formulae, which will be used for the explicit computation for the case of the Clifford torus. In section 8, we show that there exists a standard spin-structure for the Clifford torus and we describe how it determines the orientation of moduli spaces of holomorphic discs.

We first recall the following theorem about orientability of the moduli space in [FOOO]

**Theorem 5.1** ([FOOO] Theorem 21.1). The moduli space of $J$-holomorphic discs is orientable, if $L \subset (M, \omega)$ is a (relatively) spin Lagrangian submanifold. Furthermore the choice of (relative) spin structure on $L$ determines an orientation on $M(L, \beta)$ canonically for all $\beta \in \pi_2(P, L)$.

**Remark 5.1.** For simplicity, we will sketch the proof only when $L$ is a spin manifold. For the relative spin case, see [FOOO].

**Proof.** It suffices to show that the index of the linearized operator is oriented. The linearized operator $D\bar{\partial}$ for the $J$-holomorphic curve equation is a first order elliptic differential operator with the same symbol as the Dolbeault operator: for the $J$-holomorphic map $w : (D^2, \partial D^2) \rightarrow (M, L)$ with $(p > 2)$,

$$D\bar{\partial}_w : W^{1,p}(D^2, \partial D^2; w^*TM, (w|_{\partial D^2})^*TL) \rightarrow L^p(D^2; w^*TM).$$

It suffices to show that the index of the linearized operator is oriented. Since the zero order term does not affect the index problem, we assume that the operator is the Dolbeault operator $\bar{\sigma}_w$

$$\bar{\sigma}_{(wTM, (w|_{\partial D^2})^*TL)} : W^{1,p}(D^2, \partial D^2; w^*TM, (w|_{\partial D^2})^*TL) \rightarrow L^p(D^2; w^*TM).$$

We recall how to determine a pointwise orientation of the index bundle from [FOOO]
Proposition 5.2 ([FOOO] Proposition 21.3). Let $E$ be a complex vector bundle over a disc $D^2$. Let $F$ be a totally real subbundle of $E|_{\partial D^2}$ over $\partial D^2$. We denote by $\overline{\partial}_{(E,F)}$ the Dolbeault operator on $D^2$ with coefficient $(E,F)$,

$$\overline{\partial}_{(E,F)} : W^{1,p}(D^2, \partial D^2; E, F) \to L^p(D^2; E)$$

Assume $F$ is trivial and take a trivialization of $F$ over $\partial D^2$. Then the trivialization gives an orientation of the virtual vector space $\ker \overline{\partial}_{(E,F)} - \operatorname{Coker} \overline{\partial}_{(E,F)}$

Proof. Here is a proof of the Proposition given in [FOOO]. Suppose that the totally real subbundle $F$ is trivially flat and the connection is product in a collar neighborhood of $\partial D$. Assume $F$ is trivial and take a trivialization of $F$ over $\partial D$. Let $C$ be a concentric circle in the collar neighborhood of $\partial D^2$. If we pinch $C$ to a point, we have the union of a disc $D^2$ and a 2-sphere $\mathbb{P}^1$ with the center $O \in D^2$ and $S \in \mathbb{P}^1$ identified. By the parallel translation along radials, the trivial vector bundle $F$ extends up to $C$ and its complexification gives a trivialization of $E|_C$. Thus the bundle descends to $D^2 \cup \mathbb{P}^1$. We also denote this vector bundle by $E$. Then one can show that the indices of the following two operators are isomorphic to each other with the argument given in [MS] Appendix A.

$$\overline{\partial}_{(E,F)} : W^{1,p}(D^2, \partial D^2; E, F) \to L^p(D^2; E \otimes T^n_{0,1} D^2)$$

$$\nabla : \{ (\xi_0, \xi_1) \in W^{1,p}(D^2, \partial D^2; E, F) \times W^{1,p}(\mathbb{P}^1, E) \mid \xi_0(O) = \xi_1(S) \} \to L^p(D^2; E \otimes T^n_{0,1} D^2) \times L^p(\mathbb{P}^1; E \otimes T^n_{0,1} \mathbb{P}^1)$$

If we have an element of the second index bundle, then use a cut-off function to define an approximate element in the kernel of the first operator. Then it projects onto the kernel of the first operator. Hence we will get an orientation preserving isomorphism.

Since the real vector bundle $F$ is trivialized, and by the above construction, the kernel of the second operator is the kernel of the homomorphism:

$$(\xi_0, \xi_1) \in \text{Hol}(D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n) \times \text{Hol}(\mathbb{P}^1, E) \to \xi_0(O) - \xi_1(S) \in \mathbb{C}^n \cong E_S \quad (5.1)$$

Note that the kernel can be oriented by the orientation of $\mathbb{R}^n \cong \text{Hol}(D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n)$ since $\text{Hol}(\mathbb{P}^1, E)$, and $\mathbb{C}^n$ carries a complex orientation. This proves the Proposition. \qed

Hence, we set $E = w^* TM, F = (w|_{\partial D^2})^* TL$ and apply Proposition 5.2 to determine the pointwise orientation of the index bundle of $\overline{\partial}_{(w^* TM, (w|_{\partial D^2})^* TL)}$.

After fixing an orientation at one disc, say $w_0$, we can extend this orientation to any disc $w$ in the path component of the moduli space containing $w_0$. We consider a path $w_t : (D^2, \partial D^2) \to (M, L)$ for $t \in [0, 1]$ starting from $w_0$ and ending at $w$. Since $[0, 1]$ is contractible, we have a trivialization of $(w_t|_{\partial D^2})^* TL$, which gives an orientation for $\text{Index}(\overline{\partial}_{w_t})$ for all $t \in [0, 1]$, i.e the orientation for $w$.

Now under the assumption that $L$ is spin, we can show that this assignment of orientation described above is independent of a choice of paths. If there is a loop of holomorphic discs

$$w_0 : (D^2 \times S^1, \partial D^2 \times S^1) \to (M, L),$$

$(w_0|_{\partial D^2})^* TL$ will be (stably) trivial over $\partial D^2 \times S^1$ because $L$ is spin. So, it implies that we will get a consistent orientation. This finishes the proof. \qed
6. The changes of spin structures

In this section, we analyze how the change of spin structures affects the orientation of the moduli space. First we recall the definition of a spin-structure given by Milnor [M].

**Definition 6.1.** A spin structure of an oriented vector bundle $E$ over $X$ is a homotopy class of a trivialization of $E$ over the 1-skeleton of $X$ which can be extended to the 2-skeleton of $X$.

**Remark 6.2.** Note that orientation is a homotopy class of a trivialization over the 0-skeleton which can be extended to the 1-skeleton.

The above definition is equivalent to the usual definition of the spin structure (for example, the definition in [LM]). Recall that an oriented vector bundle $E$ over a manifold is called spin if its second Stiefel-Whitney class of $E$ is zero. Here are basic properties of spin-ness

**Theorem 6.1** ([LM] Theorem 2.1.3). Let $E$ be an oriented vector bundle of rank $\geq 3$ over $X$. Then $E$ is spin if and only if for any compact surface $\Sigma$ and any continuous map $f : \Sigma \to X$, the bundle $f^*E$ is trivial. Furthermore, if $E$ is spin, then the distinct spin structures on $E$ are in one to one correspondence with the elements of $H^1(M; \mathbb{Z}/2\mathbb{Z})$.

In case the rank of the bundle $E$ is two or less, we add trivial vector bundle to the bundle $E$, and we will get a stable trivialization instead of a trivialization by the previous Theorem. But stable trivialization is good enough to deal with the orientation problem.

Let $w : (D^2, \partial D^2) \to (M, L)$ be a holomorphic disc with Lagrangian boundary condition. Recall that orientation of the moduli space of holomorphic discs is determined by the trivialization of $(w|_{\partial D^2})^*T L$. We will prove the following Theorem which is crucial to understand the change of orientation.

**Theorem 6.2.** If we reverse the homotopy class of a trivialization of $(w|_{\partial D^2})^*T L$, then orientation given on the index bundle of $\tilde{\mathcal{M}}(\beta)$ will be reversed.

**Remark 6.3.** Note that for any orientable vector bundle over $S^1$, there exist only two homotopy classes of a (stable) trivializations. The above theorem was already stated without proof as a remark 21.6 of [FOOO] with an example.

With this Theorem in hand, one can analyze the change of orientation as follows: First we fix a spin structure of a Lagrangian submanifold $L$. Hence, for any holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$, this determines the homotopy type of the trivialization of $(w|_{\partial D^2})^*T L$. By [FOOO] Proposition 21.3, it determines the orientation of the moduli space of holomorphic discs $\tilde{\mathcal{M}}(\beta)$. To analyze the orientation of $\tilde{\mathcal{M}}(\beta)$ for a different spin structure, we note that a change of a spin structure will result in a change of homotopy type of a trivialization of $T L$ where both change correspond to $H^1(L; \mathbb{Z}/2\mathbb{Z})$. Hence for any holomorphic disc $w$, if the homotopy class of a trivialization of $(w|_{\partial D^2})^*T L$ is reversed due to the change of the spin structure, then the induced orientation for $\tilde{\mathcal{M}}(\beta)$ will be reversed. This will be exactly the way we will calculate Floer cohomology of the Clifford torus with various spin-structures.

To prove the Theorem we need an index theorem for the holomorphic discs proved by Silva [S]. We will state it briefly here.
Definition 6.4. A bundle pair \((T, \lambda)\) is a complex vector bundle \(T\) over \(D^2\) and a real vector bundle \(\lambda\) over \(\partial D^2\) such that \(\lambda \otimes \mathbb{C}\) is identified with \(T|_{\partial D^2}\).

For such a pair, we define \(\text{ind}(T, \lambda)\) as follows. To incorporate boundary conditions, we restrict the domain of \(\overline{\partial}\) :

\[
\Gamma_\lambda(T) := \{ s \in \Gamma(T) | s : \partial D^2 \to \lambda \subset T|_{\partial D^2} \}
\]

then, a canonical Cauchy-Riemann operator with additional boundary condition gives an elliptic boundary value problem.

\[
\overline{\partial} : \Gamma_\lambda(T) \to \Gamma(T \otimes T^{0,1} D^2)
\]

We define \(\text{ind}(T, \lambda)\) to be the index bundle of this operator. It depends only on the homotopy type of \((T, \lambda)\) and is additive under taking direct sums.

Now we consider a family of discs, parametrized by a compact space \(X\).

Definition 6.5. A bundle data \((T, \lambda)\) is a unitary bundle \(T\) over \(D^2 \times X\) and an orthogonal bundle \(\lambda\) over \(\partial D^2 \times X\) such that \(\lambda \otimes \mathbb{C}\) is identified with \(T|_{\partial D^2 \times X}\).

By choosing a continuous family \(\{\partial_x\}_{x \in X}\) of Cauchy-Riemann operators in the fibres, we obtain an index bundle

\[
\text{ind}(T, \lambda) \in KO(X)
\]

Again, the index depends only on the homotopy type of \((T, \lambda)\) and is additive under taking direct sums.

For a bundle data \((T, \lambda)\), assume \(T\) is trivial of rank \(n\). Then by fixing a trivialization of \(T\) over \(D^2 \times X\), we can specify \(\lambda\) as a map \(\phi_\lambda : \partial D^2 \times X \to U(n)/O(n)\). If \(n\) is large compared to the dimension of \(X\), we can replace \(U(n)/O(n)\) by its stable limit \(U/O\). Let \(x_0\) be a basepoint for \(X\) and also assume that \(\phi_\lambda\) is constant on \(\partial D^2 \times \{x_0\} \cup \{1\} \times X\). The index bundle then necessarily has rank \(n\) since it is trivial over \(x_0\). Subracting \(n\) from it, we obtain an element of \(KO(X) \cong [X, BO]\).

Theorem 6.3 (Silva [S]). The above construction gives an isomorphism of abelian groups

\[
\text{ind} : [\Sigma X, U/O] \to [X, BO]
\]

where the addition is given by taking direct sum. Here, \(\Sigma X\) denotes a reduced suspension of \(X\). \([\cdot, \cdot]_*\) denotes homotopy class of based maps

Remark 6.6. Note that the left hand side is only defined under the assumption that the bundle \(T\) is trivial and \(\phi_\lambda\) is constant on \(\partial D^2 \times \{x_0\} \cup \{1\} \times X\). Hence one can not define this isomorphism for every bundle pair. But one can take the direct sum of the given vector bundle with a certain bundle pair to define such a map (for details, see [S]).

We will apply this index Theorem for the case \(X = S^1\). Then we have

\[
[S^1, U/O]_* = [S^2, U/O]_* = \pi_2(U/O) \cong \overline{KO}(S^1) \cong \pi_0(O) \cong \mathbb{Z}/2\mathbb{Z}
\]

So a non-trivial generator of \(\pi_2(U/O)\) will give rise to the non-orientable index bundle over \(S^1\). But in view of the homotopy exact sequence

\[
\pi_2(U) \to \pi_2(U/O) \to \pi_1(O)
\]

with \(\pi_2(U) \cong 0\), the non-trivial element of \(\pi_2(U/O)\) corresponds to the non-trivial element of \(\pi_1(O)\). This correspondence is the main reason for the Theorem [S].
The change of the homotopy class of a trivialization of the real vector bundle \( \lambda \), which in fact comes from the twisting of a trivialization by non-trivial element of \( \pi_1(O) \), will reverse the orientation of the index bundle given by Proposition 26.2 \[ \text{[FOOO]} \].

**Proof.** We will construct a bundle pair which contains both homotopy classes of trivializations and we will show that its index bundle is non-orientable using Theorem 6.3. We start with the case of Maslov index 0.

Consider the trivial bundle \( (D^2 \times [0,1]) \times \mathbb{C}^N \) over \( (D^2 \times [0,1]) \). On the base, by identifying \( D^2 \times \{0\}, D^2 \times \{1\}, \) we get \( D^2 \times S^1 \). And we glue the fibers \( \partial D^2 \times \{0\} \times \mathbb{R}^n, \partial D^2 \times \{1\} \times \mathbb{R}^n \) by homotopically non-trivial loop \( \gamma : \partial D^2 \to SO(n) \) with \( \gamma(1) = Id \in SO(n) \). i.e. for \( z \in D^2, x \in \mathbb{R}^n \), we identify

\[
(z,0,x) \sim (z,1,\gamma(z)x)
\]

The inclusion of \( \pi_1(SO(n)) \to \pi_1(SU(n)) \) is trivial since \( \pi_1(SU(n)) \cong 0 \). So we can extend the map \( \gamma \) to \( \Gamma : D^2 \to SU(n) \). Then we identify \( D^2 \times \{1\} \times \mathbb{C}^n \) with \( D^2 \times \{0\} \times \mathbb{C}^n \) by the map \( \Gamma \). Note that this identification matches with the one given on \( \mathbb{R}^n \) before. We denote the resulting bundle data as \( (T, \lambda) \). We can give a trivialization of the bundle \( T \) as follows: Let \( C_{Id} : D^2 \to U(n) \) be the constant map \( C_{Id}(z) = Id \in U(n) \) for \( z \in D^2 \).

First, there is a homotopy \( H : D^2 \times [0,1] \to U(n) \) between the two maps \( C_I \) and \( \Gamma \) such that for \( z \in D^2, t \in [0,1] \)

\[
\begin{align*}
H(z,0) &= C_{Id}(z) \\
H(z,1) &= \Gamma(z) \\
H(1,t) &\equiv Id \in U(n)
\end{align*}
\]

Then, we define

\[
\Psi : D^2 \times [0,1] \times \mathbb{C}^n \to D^2 \times [0,1] \times \mathbb{C}^n
\]

\( (z,t,x) \to (z,t,H(t,z)x) \)

This map \( \Psi \) defines a trivialization of \( T \): We identified \((z,1,x)\) with \((z,0,\Gamma(z)(x))\) and

\[
\Psi(z,1,x) = (z,1,H(1,z)x) = (z,1,\Gamma(z)x)
\]

\[
\Psi(z,0,\Gamma(z)(x)) = (z,0,H(0,z)\Gamma(z)(x)) = (z,1,\Gamma(z)x)
\]

Hence, the trivialization \( \Psi : T \to D^2 \times S^1 \times \mathbb{C}^n \) is well-defined where \( S^1 \) is given by \( \mathbb{R}/\mathbb{Z} \). Under the trivialization \( \Psi \), we define a map \( \phi : \partial D^2 \times S^1 \to U(n)/O(n) \) as follows: For \( z \in \partial D^2, t \in S^1 \)

\[
(z,t) \mapsto [\lambda_{(z,t)}] \in U(n)/O(n)
\]

where \([\lambda_{(z,t)}]\) is an element in the Lagrangian Grassmanian corresponding to the Lagrangian subspace \( \lambda_{(z,t)} \subset \Psi(T_{(z,t)}) \cong \mathbb{C}^n \).

Then from the construction of \( \Psi \), we have

\[
\phi|_{\partial D^2 \times 0} \equiv \phi|_{\partial D^2 \times 1} \equiv Id \in U(n)/O(n)
\]

\[
\phi|_{[1] \times S^1} \equiv Id
\]

By modding out \( \partial D^2 \times \{0\} \cup \{1\} \times S^1 \) from \( \partial D^2 \times S^1 \), we may consider \( \phi \) to be a map from \( S^2 \) to \( U(n)/O(n) \).

It is not hard to see that \( \phi \) gives an non-trivial element in \( \pi_2(U(n)/O(n)) \). So the bundle data \((T, \lambda)\) has non-orientable Cauchy Riemann index bundle by the
Silva’s index Theorem. And it implies that orientations given by these two different homotopy classes of trivializations cannot be same.

Now we study the case of Maslov index 2. Basically we will use the same technique as Maslov index 0 case. In $\mathbb{C}^n$, for $z \in \partial D^2$, our Lagrangian subspaces will be $z \cdot \mathbb{R} \times \mathbb{R}^{n-1}$. Take the trivial bundle $(D^2 \times [0,1]) \times \mathbb{C}^n$. By identifying $D^2 \times \{0\}$, $D^2 \times \{1\}$, we get $D^2 \times S^1$.

We glue the Lagrangian subspaces of $\mathbb{C}^n$ fibers, $\partial D^2 \times \{0\} \times z \cdot \mathbb{R} \times \mathbb{R}^{n-1}$, $\partial D^2 \times \{1\} \times z \cdot \mathbb{R} \times \mathbb{R}^{n-1}$ with non-trivial loop in $SO(n)$ as follows: We define a map $R : S^1 \to U(n)$ as $R(e^{i\theta}) = \text{diag}(e^{-i\theta}, 1, \cdots, 1) \in U(n)$. Along $\partial D^2$, we identify two Lagrangian fibres by $R^{-1} \circ \gamma \circ R$ where $\gamma$ is the non-trivial loop in $SO(n)$ used in Maslov index 0 case. This identification can be extended to the whole fiber $\mathbb{C}^n$ and also over $D^2$, since $R^{-1} \circ \gamma \circ R$ is a loop in $SU(n)$. Let $(\tilde{T}, \tilde{\lambda})$ be the bundle data obtained by gluing each end with this identification.

To show that the bundle data $(\tilde{T}, \tilde{\lambda})$ has non-orientable Cauchy-Riemann index bundle, we will show that $(\tilde{T}, \tilde{\lambda})$ still gives the non-trivial element of $\pi_2(U/O)$ after some modification.

First, we will make a direct sum $(\tilde{T}', \tilde{\lambda}')$ with a bundle pair with Maslov index $(-2)$:

$$i.e. (D^2 \times I \times \mathbb{C}, \partial D^2(z) \times I \times z^{-1} \cdot \mathbb{R})$$

Denote the resulting bundle data as $(\tilde{T}', \tilde{\lambda}')$. And when we glue two ends 0 and 1, we glue this extra $\mathbb{C}$-fiber without twisting. Since this extra bundle pair has an orientable index bundle, and index bundle is additive upon direct sums. So it is enough to show that the bundle data $(\tilde{T}', \tilde{\lambda}')$ obtained this way is non-orientable.

Now we will find a trivialization of $\tilde{T}'$ as Maslov index 0 case. We will regard $\gamma(z) \in SU(n)$ and $\Gamma(z) \in SU(n)$ as in $SU(n+1)$ extending by 0 except the last diagonal entry where we extend it by 1.

Now, let

$$F(e^{i\theta}) = \text{diag}(e^{-i\theta}, 1, \cdots, 1, e^{+i\theta}) \in SU(n+1)$$

The map $F : \partial D^2 \to SU(n+1)$ can be extended over $D^2$ and we will denote the extension again by $F : D^2 \to SU(n+1)$.

Again let $C_{Id} : D^2 \to SU(n+1)$ be a constant map to $Id \in SU(n+1)$. We get a homotopy $\tilde{H}(z, t) : D^2 \times [0, 1] \to U(n+1)$ between $C_{Id}$ and $F^{-1} \circ \gamma \circ F$ with the following properties: For $z \in D^2, t \in S^1$,

\[
\begin{aligned}
\tilde{H}(z, 0) &= C_{Id}(z) \equiv Id \in SU(n+1) \\
\tilde{H}(z, 1) &= F^{-1} \circ \Gamma \circ F(z) \\
\tilde{H}(1, t) &= Id \in U(n+1)
\end{aligned}
\]

(6.3)

Then, we define

$$\tilde{\Psi} : D^2 \times [0, 1] \times \mathbb{C}^{n+1} \to D^2 \times [0, 1] \times \mathbb{C}^{n+1}$$

$$(z, t, x) \to (z, t, F(z)\tilde{H}(t, z)x)$$

One can check that this map $\tilde{\Psi}$ gives a well-defined trivialization of the bundle $\tilde{T}'$ as before. Note that since we composed $F(z)$ in the trivialization $\tilde{\Psi}$, the real vector bundle $\tilde{\lambda}'$ will map to constant Lagrangian subspace $\mathbb{R}^n \subset \mathbb{C}^n$ over $\partial D^2 \times \{0\}$. 

Under this trivialization, we can similarly define a map by considering the image the real bundle \( \tilde{\lambda}' \) in the Lagrangian Grassmannian,

\[
\phi : \partial D^2 \times S^1 \to U(n+1)/O(n+1)
\]

Then,

\[
\phi|_{\partial D^2 \times 0} \equiv \phi|_{\partial D^2 \times 1} \equiv \text{Id} \in U(n+1)/O(n+1)
\]

\[
\phi|_{\{1\} \times S^1} \equiv \text{Id}
\]

By moding out \( \partial D^2 \times \{0\} \cup \{1\} \times S^1 \) from \( \partial D^2 \times S^1 \), we may consider \( \phi \) to be map from \( S^2 \) to \( U(n)/O(n) \).

As before, it defines a nontrivial element in \( \pi_2(U/O) \). Hence the bundle pair \((\tilde{T}', \tilde{\lambda}')\) has non-orientable Cauchy Riemann index bundle by Silva’s index Theorem, which implies that original bundle pair \((\tilde{T}, \tilde{\lambda})\) also has non-orientable index bundle.

Other cases can be done similarly. This finishes the proof of the Theorem \( \Box \)

7. Orientation conventions and formulae

In this section, we will fix some basic conventions concerning orientations. These conventions agree with the ones defined in [FOOO].

- We will assume that all circles are oriented counter-clockwise.
- We orient the Clifford torus \( T^n \) as a torus \( (S^1)^n \subset U_0 \sim= C^n \) where \( U_0 = \{ z_0 \neq 0 \} \subset \mathbb{P}^n \), where each \( S^1 \subset \mathbb{R}^2 \sim= \mathbb{C} \) is oriented counter-clockwise.
- For a Clifford torus, we have a torus action \((S^1)^n\) action on it given by

\[
(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot [z_0; \cdots; z_n] \mapsto [z_0; e^{i\theta_1}z_1; \cdots; e^{i\theta_n}z_n].
\]

- The elements of \( PSL(2 : \mathbb{R}) \) can be written as \( e^{i\theta}(\frac{z-\alpha}{\bar{z}-\overline{\alpha}}) \) for \( \alpha \in D^2 \). We will orient \( PSL(2 : \mathbb{R}) \) as \( S^1 \times D^2 \) where the latter carries a complex orientation.
- Let \( X \) be an oriented smooth manifold with boundary \( \partial X \). Then we define an orientation on \( \partial X \) so that

\[
T_*X \cong \mathbb{R}_{out} \times T_*(\partial X).
\]

is an isomorphism of oriented vector spaces. Here \( \mathbb{R}_{out} \) is an \( \mathbb{R} \) oriented by outer normal vector.
- Let \( G \) be a Lie group given an orientation. When \( G \) acts on an oriented manifold \( X \) smoothly and freely, then we define an orientation of the quotient space \( X/G \) so that

\[
T_*X \cong T_*(X/G) \times \text{Lie } G
\]

is an isomorphism of oriented vector spaces. Here \( \text{Lie } G \) is the Lie algebra of \( G \).
- We orient the moduli space \( \tilde{\mathcal{M}}_m(\beta) \) as \( \tilde{\mathcal{M}}(\beta) \times (\partial D^2)^m \).
- In [FOOO], the orientation of the fibre product \( X_1 \times_Y X_2 \) is given for the case when the maps \( f_i : X_i \to Y \) are submersions. Here we specify the orientation for the case that the map \( f_2 : X_2 \to Y \) is an embedding. This will be used throughout our computation.

Let \( X, L \) and \( P \) be oriented smooth submanifolds and let \( f : X \to L \) be a submersion and \( i : P \to L \) be an embedding. Here we will regard \( P \) as a submanifold of \( L \). By \( x, l, p \) we denote the dimension of

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Lemma 7.1. In this setup, it is easy to see that

Recall that oriented basis of $X$. From our convention for the boundary orientation, we have

$$X = \bigcup (1)^{x_{i=1}} X \times \partial P$$

Hence, we write

$$[X] = [\mathbb{R}_X] \times [\partial X].$$

Hence, we write

$$[P] = [\mathbb{R}_P] \times [\partial P].$$

The orientation of $X \times_L P$ can be written as

$$[X \times_L P] = [X^0] \times [P].$$

Hence

$$[\mathbb{R}_X \times_L P] \times [\partial (X \times_L P)] = [X \times_L P] = [X^0] \times [P]$$

$$= [\mathbb{R}_X] \times [(\partial X)^0] \times [P] \bigcup [X^0] \times [\mathbb{R}_P] \times [\partial P]$$

$$= [\mathbb{R}_X] \times [(\partial X)^0] \times [P] \bigcup (-1)^{\dim X^0} [\mathbb{R}_P] \times [X^0] \times [\partial P]$$

$$= [\mathbb{R}_X] \times [(\partial X) \times_L P] \bigcup (-1)^{x_{i=1}} [\mathbb{R}_P] \times [X \times_L \partial P].$$

Another formula we use in the proof of Theorem 7.2 is the orientation formula for the gluing from Proposition 23.2 in [FOOO].

Proposition 7.2 ([FOOO] Proposition 23.2).

$$\partial \mathcal{M}_2(A + B) = (-1)^{\dim L + 1} \mathcal{M}_2(A)_{ev_1} \times_{ev_0} \mathcal{M}_2(B).$$

8. Orientation of the moduli space for the Clifford torus

In this section, we show that there exists a natural spin structure of the Clifford torus, which we denote by standard spin structure. Under the standard spin structure, it is rather easy to determine the orientation of the moduli space as described in Theorem 7.2.

Proposition 8.1. There exists a standard spin structure of the Clifford torus. Or equivalently, there exists a natural homotopy class of a trivialization of the tangent bundle $T(T^n)$ of the Clifford torus.
Proof. Let $S^1 := e^{i\theta}$ be the unit circle embedded in $\mathbb{C}$. The tangent bundle of $S^1$ has a natural trivialization given by $S^1 \times \mathbb{R} \cdot \frac{d}{d\theta}$. Similarly there is a natural trivialization of the tangent bundle of $(S^1)^n \subset \mathbb{C}^n$. The Clifford torus $T^n$ sits inside the intersection of $n + 1$ standard open covers $U_i(\cong \mathbb{C}^n) = \{z_i \neq 0\} \subset \mathbb{P}^n$. So, each open cover induces a trivialization of tangent bundle of $T^n$. One can check that the trivializations of $T(T^n)$ obtained with each open set $U_i$ are in the same homotopy class: Because the transition matrices between these trivializations are constant matrices, which implies that there is no twisting of frames. By permuting coordinates to have positive determinants, if necessary, the trivializations induced in each open set are in the same homotopy class. This is what we mean by the standard spin structure of $T^n$.

But we need to fix a trivialization in this homotopy class to fix an orientation. We will fix the trivialization to be the one obtained from the open set $U_0 \subset \mathbb{P}^n$.

Now, we discuss the orientation of the moduli space of holomorphic discs with boundary on $T^n$. The discussion is based on the classification Theorem of such holomorphic discs in section 10. Let $\tilde{M}(\beta)$ be the space of holomorphic discs representing the homotopy class $\beta \in \pi_2(M, L)$ as defined in Definition 4.4. The orientation of $\tilde{M}(\beta)$ can be determined by the Proposition 5.2 after we fix the spin structure. We start with an example.

For a homotopy class $\beta_0 \in \pi_2(\mathbb{P}^n, T^n)$, we will see that the moduli space $\tilde{M}(\beta)$ is

$$\{ \frac{z - \alpha}{\overline{\alpha}z} : e^{i\theta_1}, \cdots, e^{i\theta_n} | \alpha \in \text{int } D^2, \theta_i \in S^1 \}$$

Since $\alpha \in D^2 \subset \mathbb{C}$ carries a complex orientation, the orientation of $\tilde{M}(\beta)$ is determined by the orientation of $(\theta_1, \cdots, \theta_n) \in (S^1)^n$. With the standard spin structure, it will oriented as $(\frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_n})$.

From now on we fix the standard spin structure. Let $w : (D^2, \partial D^2) \to (\mathbb{P}^n, T^n)$ be a holomorphic disc. Recall that in Proposition 5.2 we had a decomposition of the tangent space of $\tilde{M}(\beta)$ as a kernel of the homomorphism

$$(\xi_0, \xi_1) \in \text{Hol}(D^2, \partial D : \mathbb{C}^n, \mathbb{R}^n) \times \text{Hol}(\mathbb{P}^1, E) \to \xi_0(O) - \xi_1(S) \in \mathbb{C}^n.$$ 

Here $\text{Hol}(D^2, \partial D : \mathbb{C}^n, \mathbb{R}^n)$ is in fact just $\mathbb{R}^n$ and this $\mathbb{R}^n$ comes from the trivialization $T(T^n)$ along $w|_{\partial D^2}$. It is not hard to see that this $\text{Hol}(D^2, \partial D : \mathbb{C}^n, \mathbb{R}^n)$ corresponds to the subspace of tangent space $T_w(\tilde{M}(\beta))$ which is given by the translation of disc $w$ along the tangent directions of the Lagrangian submanifold $T^n$ under the standard spin structure. Therefore, $\text{Hol}(D^2, \partial D : \mathbb{C}^n, \mathbb{R}^n)$ is oriented by our choice of natural trivialization in the previous Proposition. All other factors in the above decomposition carries complex orientations. This gives the orientation of $\tilde{M}(\beta)$.

Under non-standard spin structures, such a direct analysis is not possible, but we can still assign the orientation of the moduli space as described in the paragraph after Theorem 6.2.

Now we compute the orientation of $\mathcal{M}_1(\beta)$ with our orientation convention. Recall that we orient the moduli space with marked points $\mathcal{M}_m(\beta)$ as $[\tilde{M}(\beta)] \times [\partial D^2]^m$. where $[,]$ means the oriented basis of the tangent space. The moduli space
\( \mathcal{M}^{reg}_m(\beta) \) is oriented as \( \tilde{\mathcal{M}}(\beta) \times [(\partial D^2)^m]/\text{PSL}(2 : \mathbb{R}) \).

\[
[\tilde{\mathcal{M}}(\beta)] \times [(\partial D^2)^m] = [\mathcal{M}^{reg}_m(\beta)] \times [\text{PSL}(2 : \mathbb{R})]
\]

where \( [\text{PSL}(2 : \mathbb{R})] \) represents a frame at the tangent space of each disc in \( \tilde{\mathcal{M}}_m(\beta) \) which is given by \( [\text{PSL}(2 : \mathbb{R})] \) action on \( [\tilde{\mathcal{M}}(\beta)] \times [(\partial D^2)^m] \). If we only consider the holomorphic discs with Maslov index 2, then the homotopy classes \( \beta_i \)'s are minimal, hence \( \mathcal{M}^{reg}_1(\beta_i) = \mathcal{M}_1(\beta_i) \). Hence, we have

\[
\tilde{\mathcal{M}}(\beta_i)/\text{PSL}(2 : \mathbb{R}) \cong \mathcal{M}_1(\beta_i).
\]

**Proposition 8.2.** Let \( \beta_i \in \pi_2(\mathbb{P}^n, T^n) \) be the homotopy class described in Proposition 8.1 for \( i = 0, 1, \cdots, n \). Then the evaluation map \( ev_0 : \mathcal{M}_1(\beta_i) \rightarrow T^n \) is an orientation preserving homeomorphism for all \( i = 0, 1, \cdots, n \).

**Proof.** It is easy to see that \( ev_0 \) is a homeomorphism by the classification Theorem 8.1. To find out the orientation of \( \mathcal{M}_1(\beta_i) \), we need to specify the orientation of \( \tilde{\mathcal{M}}(\beta_i) \) and \( \text{PSL}(2 : \mathbb{R}) \). Since \( \tilde{\mathcal{M}}_1(\beta_i) \cong \tilde{\mathcal{M}}(\beta_i) \times [\partial D^2_0] \), we have

\[
[M_1(\beta_i)] = ([\tilde{\mathcal{M}}(\beta_i)] \times [(\partial D^2_0)])//\text{PSL}(2 : \mathbb{R}).
\]

Recall that \( T(\tilde{\mathcal{M}}(\beta_i)) \) have a decomposition as \( [T^n] \times [D^2] \) (see the expression (1)). Hence, by taking a quotient of \( D^2 \subset \text{PSL}(2 : \mathbb{R}) \) which carries a complex orientation,

\[
[M_1(\beta_i)] = ([T^n] \times [D^2] \times [\partial D^2_0])//[S^1] \times [D^2]
\]

\[
= ([T^n] \times [\partial D^2_0])//[S^1]
\]

Here an element \( e^{it} \in S^1 \subset \text{PSL}(2 : \mathbb{R}) \) acts on a holomorphic disc \( w \) as \( e^{it} \cdot w(z) = w(e^{-it}z) \) and it acts on a marked point \( z_0 \) as \( e^{it}z_0 \). So under the evaluation map \( ev_0 \), we obtain

\[
[(M_1(\beta_i), ev_0)] \cong [T^n].
\]

This finishes the proof. \( \square \)

9. **Maslov index formula for discs in \( \mathbb{P}^n \) with boundary in \( T^n \)**

The most challenging part in computing the Floer cohomologies is to classify all the holomorphic(J-holomorphic) discs with Maslov index \( \leq n + 1 \). In the case of the Clifford torus, or generally for torus fiber in compact toric manifolds, the following formula is a fundamental tool. Later on, by using this index formula, we will classify all holomorphic discs with boundary on the Clifford torus for any Maslov indices.

**Theorem 9.1.** For a holomorphic disc \( w : (D^2, \partial D^2) \rightarrow (\mathbb{P}^n, T^n) \), the Maslov index of the disc \( w \) is twice the sum of intersection multiplicities between the image of the disc \( w \) with hyperplanes \( H_i \) for \( i = 1, \ldots, n \), where \( H_i \)'s are hyperplanes defined by \( z_i = 0 \) in \( \mathbb{P}^n \).

**Proof.** We first prove the following elementary lemma regarding the Maslov index of a map.

**Lemma 9.2.** Let \( L \) be a Lagrangian submanifold whose tangent bundle \( TL \) is trivial. Let \( \alpha_L \) be the zero section of the cotangent bundle of \( T^*L \). Let \( \Sigma \) be a smooth Riemann surface with boundary. Then, for any smooth map \( w : (\Sigma, \partial \Sigma) \rightarrow (T^*L, \alpha_L) \), the Maslov index of \( w \) is zero.
Proof. First, consider the case that the image of $w(\Sigma)$ is entirely contained in $o_L$. At the zero section $o_L$ of the cotangent bundle, there exists a canonical splitting of $T(T^*L) \cong TL \oplus T^*L$. From the trivialization of $TL$, we obtain a trivialization of $TM|_{o_L} \cong TL \oplus J_0(TL)$. Hence for the pull back the above trivialization by $w$, the Maslov index of the map $w$ is zero.

For general cases, we can homotope $w$ to $pr(w)$ where $pr : T^*L \to o_L$ is the projection map of the cotangent bundle of $L$, and we may replace the homotopy by a smooth one. Hence then $\mu(w) = \mu(pr(w)) = 0$. □

Note that $\mathbb{P}^n \setminus (H_0 \cup H_1 \cup \cdots \cup H_n)$ can be identified with the cotangent bundle of $T^n$, which will be used crucially later in the proof.

First, consider the case that a disc $w$ does not meet any hyperplanes $H_i$s at all. Then the disc is in fact in the cotangent bundle of $T^n$. From the Lemma 9.2 its Maslov index is zero, hence the Theorem holds in this case.

To consider the general discs, we first write the map in terms of the homogeneous coordinate functions.

**Lemma 9.3.** For a holomorphic disc $w : (D^2, \partial D^2) \to (\mathbb{P}^n, T^n)$, we can write the map as

$$[\gamma_0(z) : \gamma_1(z) : \cdots : \gamma_n(z)]$$

where $\gamma_i(z) : D^2 \to \mathbb{C}$ is a holomorphic function for $i = 0, \cdots, n$ with $\cap_{j=0}^n \text{Zero}(\gamma_j) = \emptyset$.

**Proof.** There is the holomorphic line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$ whose global sections are generated by $z_0, z_1, \cdots, z_n$. Now consider the pull-back bundle $w^*\mathcal{O}(1)$ and over the disc, we fix its holomorphic trivialization $\Psi : w^*\mathcal{O}(1) \to D^2 \times \mathbb{C}$. Let $\gamma_i(z) = \Psi(w^*z_i)$. □

Now we assume that there exists at least one intersection between the image of the map $w$ and the given hyperplanes, where one of the $\gamma_i(z)$ becomes zero. We label by $p_1, p_2, \cdots, p_m \in D^2$ every point where one of the $\gamma_i(z)$ becomes zero. We find disjoint open balls $B_i(\epsilon) \subset D^2$ centered at $p_i$ with fixed radius $\epsilon$ for sufficiently small $\epsilon$ for all $i = 1, 2, \cdots, m$. Our strategy is to deform the map $w$ inside the ball $B_i$ so that we can decompose the disc into several regions whose boundary satisfies Lagrangian boundary condition and then we will compute the Maslov index using the decomposition.

At $p_1$, we may assume without loss of generality that

$$\gamma_0(p_1) = \gamma_1(p_1) = \cdots = \gamma_s(p_1) = 0$$

for $0 \leq s < n$. Denote by $d_i$ the order of zero of $\gamma_i$ at $p_1$ for $0 \leq i \leq n$. Then, $d_i > 0$ for $0 \leq i \leq s$ and $d_i = 0$ for $s < i \leq n$. We may further assume that $p_1 = 0 \in D^2$ for simplicity. Recall that $B_1(\epsilon) \subset D^2$ is a ball centered at 0 of radius $\epsilon$. Since, $\gamma_i$ does not have common zero, $\gamma_n(B_1)$ is away from zero. Hence the image of the map $w$ is contained in the open set $U_n$.

Define $f_i : B_1 \to U_n(\cong \mathbb{C}^n)$ for $i = 0, 1, \cdots, n - 1$ as

$$f_i(z) = \frac{\gamma_i(z)}{\gamma_n(z)}.$$

These $f_i$’s are holomorphic functions. For $i = 0, 1, \cdots, n - 1$, we can choose $a_i \in \mathbb{C}$ and holomorphic functions $R_i(z) : B_1 \to \mathbb{C}^n$ with

$$f_i(z) = a_i z^{d_i} + R_i(z)$$
where \( R_i(z) = O(|z|^{d_i+1}) \).

Basically, we want to deform the map \( f_i(z) = a_i z^{d_i} + R_i(z) \) to \( (a_i z^{d_i} / |a_i(t/2)^{d_i} + 0) \) inside the ball \( B_1(t/2) \) without changing the map near the boundary of the ball \( B_1(\epsilon) \).

Note that the constants are chosen to map \( \partial B_1(t/2) \) to the Clifford torus.

Here are two cut-off type functions \( \xi, \eta : \mathbb{R} \rightarrow \mathbb{R} \).

\[
\xi_c(x) = \begin{cases} 
1 & \text{if } |x| \geq \frac{c}{2} \\
\frac{1}{c} & \text{if } |x| \leq \frac{c}{2}
\end{cases}
\]

\[
\eta(x) = \begin{cases} 
1 & \text{if } |x| \geq \frac{c}{2} \\
0 & \text{if } |x| \leq \frac{c}{2}
\end{cases}
\]

We extend \( \xi, \eta \) smoothly over \( \mathbb{R} \) with \( \frac{1}{2} \leq \xi_c \leq 1, 0 \leq \eta \leq 1 \).

We also define the deformation between \( \xi, \eta \) and the constant function 1 as

\[
\xi_c(t) = (1 - t) \cdot 1 + t\xi_c(x)
\]

\[
\eta(t) = (1 - t) \cdot 1 + t\eta(x)
\]

Now we choose the constants \( c_i = |a_i(t/2)^{d_i}|. \) Let

\[
f_i^c(z) = \xi_c^c(|z|) a_i z^{d_i} + \eta(|z|) B_i(z)
\]

Then, \( f_i^c(z) = f_i(z) \) and \( f_i^c \) gives the smooth deformation of the original map \( w \) to a new map, say \( w_1 : (D^2, \partial D^2) \rightarrow (\mathbb{P}^n, T^n) \), where \( w|_{B_1(t/2)} \) can be written as

\[
\begin{bmatrix} a_0 z^{d_0} \\ \vdots \\ a_s z^{d_s} \\ \vdots \\ a_{s+1} z^{d_{s+1}} \\ \vdots \\ a_{n-1} z^{d_{n-1}} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ \frac{a_s z^{d_s}}{|a_s(t/2)^{d_s}|} \\ \vdots \\ \frac{a_{s+1} z^{d_{s+1}}}{|a_{s+1}|} \\ \vdots \\ \frac{a_{n-1} z^{d_{n-1}}}{|a_{n-1}|} \\ 1 \end{bmatrix}. \tag{9.1}
\]

We perform the same deformation for \( p_2, p_3, \ldots, p_m \) inside the ball \( B_2, \ldots, B_m \) and write the resulting map as \( \tilde{w} \). Over the punctured disc

\[
\Sigma = D^2 \setminus (B_1(t/2) \cup \cdots B_m(t/2)),
\]

the deformed map \( \tilde{w} \) does not intersect with the hyperplanes, and it intersects with the Clifford torus along the boundaries of the punctured disc. Recall that the Maslov index is a homotopy invariant. Hence, we have \( \mu(\tilde{w}|_{\Sigma}) = \mu(w) \).

Now we will compute the the Maslov index of the map \( \tilde{w} \). Note that the boundary \( \partial \Sigma \) is \( \partial D^2 \cup (\cup_{i} \partial B_i(\epsilon/2)) \).

Since the image of the map \( \tilde{w} \) on the boundaries of the ball \( B_i(t/2) \) lies on the Lagrangian submanifold \( T^n \), the map \( \tilde{w} : (\Sigma, \partial \Sigma) \rightarrow (\mathbb{P}^n, T^n) \) satisfies the Lagrangian boundary condition. Furthermore, since every intersection with the hyperplane occurs inside the balls \( B_i(\epsilon/2) \), \( \tilde{w}|_{\Sigma} \) does not meet the hyperplanes. Hence, it can be considered as a map into the cotangent bundle of \( T^n \), since \( \mathbb{P}^n \setminus (H_0 \cup H_1 \cup \cdots \cup H_n) \) can be identified with the cotangent bundle of \( T^n \). From the Lemma 9.2

\[
\mu(\tilde{w}|_{\Sigma}) = 0. \tag{9.2}
\]

By the Definition in section 9.2 the Maslov index of the map \( \tilde{w}|_{\Sigma} \) is given by the sum of the Maslov indices of \( \partial \Sigma \) after fixing the trivialization.

Now consider the map \( \tilde{w} : D^2 \rightarrow \mathbb{P}^n \) and we fix a trivialization \( \Phi \) of the pullback bundle \( \tilde{w}^* T\mathbb{P}^n \). It gives a trivialization \( \Phi_{\Sigma} \) of the pullback bundle \( (\tilde{w}|_{\Sigma})^* T\mathbb{P}^n \) restricted over \( \Sigma \). In this trivialization, it is easy to see that

\[
\mu(\Phi_{\Sigma}, \partial D^2) = \mu(\Phi, \partial D^2) = \mu(\tilde{w}) = \mu(w).
\]
Since the boundary of the balls $B_i$ are oriented in the opposite way, and from the explicit description (9.1) of the deformed map on the ball $B_i$, we have

$$\mu(\Phi \Sigma, \partial B_i) = -2(\text{sum of intersection multiplicities in } B_i).$$

From the equation (9.2), we have

$$\mu(w) - 2(\text{sum of intersection multiplicities}) = 0.$$

□

10. Classification and regularity of the holomorphic discs

With the Maslov index formula in Theorem 9.1, we can completely classify all holomorphic discs with boundary lying on the Clifford torus. Here is our classification Theorem.

**Theorem 10.1.** Let $w : (D^2, \partial D^2) \to (\mathbb{P}^n, T^n)$ be a holomorphic disc. Then, homogeneous coordinate functions of the map $w$ can be chosen so that they are a finite Blaschke products.

i.e. the map $w$ has homogeneous coordinates $[\gamma_0(z) : \cdots : \gamma_n(z)]$ such that for all $i = 0, 1, \cdots, n$, there exists $\mu_i \in \mathbb{Z}_+$, $\alpha_{i,j} \in \text{int}(D^2)$ for $j = 1, 2, \cdots, \mu_i$ and we can write

$$\gamma_i(z) = e^{\theta_i} \prod_{j=1}^{\mu_i} \frac{z - \alpha_{i,j}}{1 - \overline{\alpha}_{i,j}z},$$

where $\bigcap_{i=0}^n \cup_{j=1}^{\mu_i} \{\alpha_{i,j}\} = \emptyset$. And the Maslov index of $w$ is $\sum_{i=0}^n \mu_i$.

**Proof.** Any map with these coordinate functions are obviously holomorphic, But we need to prove that there does not exist any other holomorphic discs.

First, consider the case that the Maslov index of the holomorphic disc is less than $2n+2$. By Theorem 9.1, any disc which intersects all $(n+1)$ hyperplanes $H_i$'s will have at least $2(n+1)$. Hence the image should miss at least one hyperplane, say $H_0$. Then the map $w$ can be considered as a holomorphic map from $D^2$ to $U_0 \cong \mathbb{C}^n$ with boundary in $(S^1)^n \subset \mathbb{C}^n$. Let $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection map onto $i$-th coordinate. The composition $p_i \circ w : D^2 \to \mathbb{C}$ maps the boundary $\partial D^2$ to $S^1 \subset \mathbb{C}$. But we have a complete classification of such maps. Namely, they are given by a finite Blaschke products. This proves the Theorem in the case that the Maslov index of the disc is less than $2n + 2$.

Now, we consider the case that the Maslov index of the disc is bigger or equal to $2n + 2$. If such a disc misses at least one of the hyperplanes $H_i$, then one can argue similarly as above. So we assume that the image of the map $w$ intersects with all $n + 1$ hyperplanes. We label every point of the domain of the intersection of the map $w$ with the fixed hyperplane $H_0$ as $p_1, p_2, \cdots, p_m \in D^2$. And we denote the intersection multiplicity at the point $p_i$ as $d_i$. Let $u : (D^2, \partial D^2) \to (\mathbb{P}^n, T^n)$ be the map given by

$$\prod_{i=1}^m \frac{1 - \overline{z}}{z - p_i}^{d_i}, \gamma_0(z) : \cdots : \gamma_n(z)$$

Note that the multiplication preserves the boundary condition, and resulting map is still holomorphic. But the map $u$ no longer intersects with the hyperplane $H_0$, hence previous arguments can be applied. This proves the theorem and the statement about the Maslov index follows from Theorem 9.1.
Now, we show the regularity of $J_0$ which justifies that we may use the standard complex structure to compute the Floer cohomology.

**Theorem 10.2.** In the case of the Clifford torus, the standard complex structure $J_0$ is regular for the holomorphic discs with Maslov index less than $2n+2$. i.e. $\text{Coker} \, D\overline{D} J_0 = 0$.

**Remark 10.1.** For the regularity of discs with the Maslov index $\geq 2n+2$, see [CO].

**Proof.** Let $w : (D^2, \partial D^2) \rightarrow (\mathbb{P}^n, T^n)$ be a holomorphic disc with Maslov index less than $2n+2$. Because of Theorem 9.1, we consider it as a map $w : (D^2, \partial D^2) \rightarrow (\mathbb{C}^n, (S^1)^n)$. If we linearize at $w$, we obtain a Riemann-Hilbert Problem (see [O3] or [O2]).

\[
\begin{cases}
\frac{\partial \xi}{\partial z} = 0 & \text{in } D^2 \\
\xi(z) \in T_{w(z)}(S^1)^n & \text{for } z \in \partial D^2
\end{cases}
\]  

where $\xi : D^2 \rightarrow \mathbb{C}^n$ is a smooth map.

Actually, problem (10.1) is completely separable into $n$ equations of one variable of the type: for the projection map onto $i$-th coordinate $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}$,

\[
\begin{cases}
\frac{\partial \eta}{\partial z} = 0 & \text{in } D^2 \\
\eta(z) \in T_{\pi_i(w)}S^1 & \text{for } z \in \partial D^2
\end{cases}
\]  

Now the theorem immediately follows from the study of 1-dimensional Riemann-Hilbert problem with this Lagrangian loop: Oh ([O3]) proved the regularity of holomorphic discs with partial indices $\geq -1$, and in the 1-dimensional problem, partial index equals the Maslov index which is non-negative in our case. This finishes the proof. \hfill \Box

For certain elements, for example, the holomorphic disc given by $w(z) = [z^2 : 1 : \cdots : 1]$, has a nontrivial automorphism if we do not put any marked point to the moduli space. i.e. $z \rightarrow e^{\pi i} z$ gives rise to an $\mathbb{Z}/2\mathbb{Z}$-automorphism group for the element $((D^2, 0), w(z)) \in \mathcal{M}(\beta_0)$. After putting one or more marked point, the moduli spaces of holomorphic discs always have trivial automorphisms if it does not contain sphere bubbles as shown in Lemma 9.1 in [FOOO]. In our case, when we are considering moduli space of holomorphic discs with Maslov index $< 2n+2$, these discs cannot bubble off a sphere since sphere bubbling can only occurs for discs with Maslov index at least $2n+2$.

11. Computation of Floer cohomology with the standard spin structure

Since the standard complex structure $J_0$ is regular, we can compute explicitly the Floer boundary operators using the classification Theorem 10.1.

Here we work with the spectral sequence described in Theorem 4.3. Then the $E_2$ term of the spectral sequence is given by $(H^*(L : \mathbb{Q}) \otimes e^{i})^p$. The boundary map on $E_2$ is given by

\[ \delta_2 = \sum_{\beta, \mu(\beta) = 2} \delta_\beta. \]

Hence, we compute the boundary map $\delta_2$ for cohomology generators of $T^n$. 

Now we choose the generators of the singular cohomology of the Clifford torus. We denote by $L_i$ the boundary cycle of the standard disc

$$b_i = \left[ \frac{1}{z} : \cdots : \frac{1}{z} : \frac{1}{z} : \cdots : 1 \right]$$

for $i = 0, 1, \cdots, n$. Then, the cycles $L_i$ for $i = 1, 2, \cdots, n$ generates $H_1(T^n)$, and we write $L_i \times L_j$ ($i \neq j$) for the cycle given from the boundary of

$$\left[ \frac{1}{z} : \cdots : \frac{1}{z} : \frac{1}{z} : \cdots : 1 \right].$$

We also define products $L_{i_1} \times \cdots \times L_{i_k}$ similarly. These products will give all the generators of $H_*(T^n)$. Recall that we identify these cycles as an element of cohomology by the relation (4.1).

Now we can compute the boundary operator $\delta_{\beta_i}$ for $\mu(\beta_i) = 2$. First, we have

$$\delta_{\beta_i} \circ pt = (\mathcal{M}_2(\beta_i) \times \times L_i < pt >, ev_0)$$

Here

$$[\mathcal{M}_2(\beta_i)] = \left( [\mathcal{M}(\beta)] [\partial D_0^2] [\partial D_0^2] / \text{PSL}(2 : \mathbb{R}) \right) = (-1)^n (\mathcal{M}(\beta) [\partial D_0^2] / \text{PSL}(2 : \mathbb{R}) = (-1)^n [\partial D_0^2] [T^n]$$

where $[\cdot]$ means oriented basis of the tangent space at any element $((D^2, z_0, z_1), w) \in \mathcal{M}_2(\beta_i)$. Here $\partial D_0^2$ denotes $i$-th marked point. So by the definition of the orientation of the fibre product in section 7,

$$\delta_{\beta_i} \circ pt = (\mathcal{M}_2(\beta_i) \times \times L_i < pt >, ev_0) = (-1)^n L_i$$

where we obtained $L_i$ as we evaluate the marked point $z_0$ along the boundary of the disc $D_0^2$. And similarly for $i \neq j$ we have

$$\delta_{\beta_i} \circ L_i = (\mathcal{M}_2(\beta_i) \times \times L_i, ev_0) = (-1)^n (L_j \times L_i) \quad (11.1)$$

For $i = j$, one can easily see that $\delta_{\beta_i} \circ L_i = 0$. Hence we will use the equation (11.1) even for $i = j$ with the convention $L_i \times L_i = 0$.

If we take a sum of $\delta_{\beta_i}$ for all $i = 0, 1, \cdots, n$, then

$$\delta_2 \circ pt = (-1)^n (L_0 + \cdots + L_n) \equiv (-1)^n ((-L_1 - \cdots - L_n) + L_1 + \cdots + L_n) = 0 \quad \text{in } H^*(T^n : \mathbb{Q})$$

For the higher dimensional generators, we can proceed similarly. For any generator of the singular cohomology of $T^n$ represented as $L_{i_1} \times L_{i_2} \times \cdots \times L_{i_k}$, we compute the boundary of it as

$$\delta_2 (L_{i_1} \times L_{i_2} \times \cdots \times L_{i_k}) = (-1)^n (L_0 + L_1 + \cdots + L_n) \times (L_{i_1} \times L_{i_2} \times \cdots \times L_{i_k}) \equiv 0 \quad \text{in } H^*(T^n : \mathbb{Q})$$

We can see that under the standard spin-structure, the boundary operator $\delta_2 \equiv 0$ for all cohomology generators of $T^n$.

For the boundary operators $\delta_k$ for $k \geq 4$, we have the following Proposition, which follows from the description of the moduli space of holomorphic discs.

**Proposition 11.1.** For the Bott-Morse Floer cohomology of the Clifford torus, the boundary operator $\delta_\beta \equiv 0$ for $\mu(\beta) \geq 4$.
Proof. We will show that when $\mu(\beta) \geq 4$, the dimension of the image under the evaluation map $ev_0$ of the moduli space $(M_2(\beta)_{ev_0} \times_f P)$ is always less than the dimension of moduli space itself. Hence it will prove that $\delta_\beta \equiv 0$ as we consider them as currents.

Consider any homotopy classes $\beta \in \pi_2(\mathbb{P}^n, T^n)$ with $\mu(\beta) = 4$. The dimension of the moduli space $(M_2(\beta)_{ev_1} \times_f P)$ is $\dim(P) + 3$. But we claim that, for any point $< pt > \in P$,

$\dim(ev_0(M_2(\beta)_{ev_1} \times_f < pt >)) \leq 2$.

The claim easily follows from the classification theorem: We argue by example. Consider the homotopy class $(\beta_0 + \beta_1)$, which is the homotopy class of the map

$$[\frac{z - \alpha_1}{1 - \alpha_1 z} : \frac{z - \alpha_2}{1 - \alpha_2 z} : 1 : \cdots : 1]$$

where $\alpha_i \in D^2$ for $i = 1, 2$.

$ev_0((M_2(\beta_0 + \beta_1)_{ev_1} \times_f < pt >) \subset [p_0 e^{i\theta_1} : p_1 e^{i\theta_2} : p_2 : \cdots : p_n]$ where $[p_0 : \cdots : p_n]$ represents point $f(p)$ in $T^n$ and $0 \leq \theta_i \leq 2\pi$ for $i = 1, 2$. Hence the dimension of the chain $(M_2(\beta) \times P, ev_0)$ is at most $\dim(P) + 2$. Hence, it gives zero as a current. i.e. $\delta_\beta \equiv 0$.

The above argument can be easily generalized for homotopy classes with higher Maslov indices.

Since all quantum boundary operators are zero, the spectral sequence degenerates at $E_2$. Hence we have

**Theorem 11.2.** For the standard spin structure, we have an isomorphism of $\Lambda_{nov}$-modules with $\mathbb{Z}$-grading.

$$HF^*(T^n, T^n; \Lambda_{nov}) \cong H^*(T^n) \otimes \Lambda_{nov}.$$

**Remark 11.1.** This isomorphism does not preserve the product structure. It is only a module isomorphism.

### 12. The Floer cohomology with non-standard spin structures.

In this section, we compute the Floer cohomology with non-standard spin structures. It will be done by describing the change of sign in the Floer coboundary operator according to the change of spin structure. As we have described in Theorem 6.2, the change of spin-structure of $T^n$ results in the change of orientations of certain moduli spaces $M(\beta)$ for $\beta \in \pi_2(M, L)$. There exist $|H^1(T^n; \mathbb{Z}/2\mathbb{Z})| = 2^n$ spin structures of the Clifford torus. We denote elements of $\mathbb{Z}/2\mathbb{Z}$ as 0 and 1. We label the standard spin structure as $(0, 0, \cdots, 0) \in (\mathbb{Z}/2\mathbb{Z})^n$. Let $I$ be a subset of $\{1, 2, \ldots, n\}$. Then $\text{Spin}_I$ will denote the spin structure corresponding to $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ where $a_i = 1$ for $i \in I$ and $a_i = 0$ for $i \notin I$. Let $\text{Spin}_0$ denote the standard spin structure.

But we will give a different labeling of the spin structures as follows. Let $\varepsilon_i \in \{-1, +1\}$ for $i \in \{0, 1, \cdots, n\}$

**Definition 12.1.** Consider $(\varepsilon_0, \cdots, \varepsilon_n) \in \{-1, +1\}^{n+1}$ which satisfies

$$\varepsilon_0 \cdot \varepsilon_1 \cdots \varepsilon_n = 1$$

And let

$$I := \{i | \varepsilon_i = -1, i \neq 0\}$$
Then we will denote the spin structure $\text{Spin}_I$ by $(\varepsilon_0, \cdots, \varepsilon_n)$.

This labeling is more convenient because $\varepsilon_i$ will be the orientation change of the moduli space $\mathcal{M}_2(\beta_i)$, for each $i = 0, 1, \cdots, n$ when we change the standard spin structure to the spin structure $\text{Spin}_I$:

$$[\mathcal{M}(\beta_i)]_{\text{Spin}_I} = \varepsilon_i [\mathcal{M}(\beta_i)]_{\text{Spin}_0} \quad (12.1)$$

The reason is that for $i \neq 0$, if $i \in I$, the spin structure $\text{Spin}_I$ will twist the trivialization of tangent bundle of $T^n$ along the $L_i$ from the standard trivialization, which will change the orientation of the moduli space $\mathcal{M}_2(\beta_i)$. So $\varepsilon_i$ is exactly the sign change of the moduli space $\mathcal{M}_2(\beta_i)$, or the sign change of the boundary operator $\delta_{\beta_i}$. When $i = 0$, orientation change of $\mathcal{M}_2(\beta_0)$ will depend on the product $\varepsilon_1 \cdots \varepsilon_n = \varepsilon_0$ since the boundary of $\beta_0$ disc has homology class $L_0 \cong -L_1 - L_2 - \cdots - L_n$ in $\pi_1(T^n)$.

Similar sign changes occur for homotopy classes with higher Maslov indices according to their boundary elements. (according to the map $(\pi_2(M, L) \to \pi_1(L)$)

But because of Proposition 11.1, it will be irrelevant to the Floer cohomology.

Now we calculate the Floer cohomology of $T^n$ with these spin structures. We fix our spin structure by $(\varepsilon_0, \cdots, \varepsilon_n)$ with $\varepsilon_0 \cdot \varepsilon_1 \cdots \varepsilon_n = 1$ or Spin$_I$. We consider the homotopy classes with Maslov index 2. Recall that they are indexed as $\beta_i$ for $i = 0, \cdots, n$. From the sign change rule $(12.1)$, we have

$$\delta_2 < pt > = \sum_{i=0}^{n} \delta_{\beta_i} < pt > = (-1)^n \sum_{i=0}^{n} \varepsilon_i L_i$$

$$= (-1)^n \sum_{i=1}^{n} (\varepsilon_i - \varepsilon_0) L_i \quad \text{in } H^*(T^n; \mathbb{Q}) \quad (12.2)$$

Last equality can be obtained by writing $L_0$ as $(-L_1 - L_2 - \cdots - L_n)$. Hence, $\delta_2 < pt > = 0$ if and only if $\varepsilon_i = \varepsilon_0$ for all $i$. There exist at most two such spin structures. The case $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_n = 1$ is the standard spin structure case, and the case $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_n = -1$ can occur only when $n$ is an odd integer because it has to satisfy $\varepsilon_0 \cdot \varepsilon_1 \cdots \varepsilon_n = 1$. Except these two possibilities, $\delta_2 < pt > \neq 0$. Therefore, $< pt >$ is no longer in the kernel of $\delta$.

Later, we will consider Floer cohomology with flat line bundle $L$ on $T^n$. Let $h_i$ be the holonomy of $L$ along the cycle $L_i$. Then Floer cohomology with spin structure $(\varepsilon_0, \cdots, \varepsilon_n)$ is same as that with flat line bundle $L$ whose holonomy is given as

$$h_i = \varepsilon_i \quad (12.3)$$

(Compare (12.2) with (13.2) for example).

**Remark 12.2.** These various spin-structures can be realized as being given a flat complex line bundle with holonomy $e^{2\pi i}$ for the corresponding generators. More precisely, $H^1(L, \mathbb{Z}/2\mathbb{Z})$ which characterizes the spin structures, also gives the flat real line bundles over $L$. We get the corresponding flat complex line bundle by tensoring $\mathbb{C}$ to this real line bundle.

Then, the Theorem 13.1 can be interpreted in terms of spin structures as follows:

**Theorem 12.1.** For $n$ even, with any non-standard spin structure, Floer cohomology $H^*(T^n, T^n; \Lambda_{nov})$ vanishes.
For $n$ odd, let $(0, 0, 0) \in (\mathbb{Z}/2)^n$ be the standard spin structure. Then Floer cohomology for other spin structures vanishes except the spin structure $(1, 1, \ldots, 1) \in (\mathbb{Z}/2)^n$, in which case

$$HF^*(T^n, T^n; \Lambda_{\text{nov}}) \cong H^*(T^n) \otimes \Lambda_{\text{nov}}.$$  

**Proof.** When $n$ is even, non-standard spin structures do not give specified holonomies whose Floer cohomology is non-vanishing. But when $n$ is odd, for $k = (n + 1)/2$,

$$e^{\frac{2\pi i}{n}} = e^{\pi i}.  \quad (12.4)$$

Then the theorem follows from Theorem 13.1. □

**Remark 12.3.** We may define Floer cohomology with Novikov ring with $\mathbb{Z}$-coefficient since $T^n$ is monotone. In this coefficient ring, with non-standard spin structure, it gives a non-vanishing Floer cohomology because it will have a torsion element. (We can not divide by 2 in (13.6) for example).

### 13. D-branes and Floer cohomology

The following definitions are from [Fuk2].

**Definition 13.1.** [Fuk2] Let $(M, \omega)$ be a symplectic manifold. A pair $(L, L)$ of Lagrangian submanifold $L$ of $M$ and a flat complex line bundle $L$ on $L$ is a brane (in a classical sense) of $A$-model compactified by $(M, \omega)$.

Floer cohomology of the above pairs $(L_0, L_0), (L_1, L_1)$ was proposed by Konsevich [Ko]. One can define a Bott-Morse Floer cohomology of the pair $(L_0, L_0)$ by modifying the boundary operator as follows. And we use the Novikov ring with $\mathbb{C}$-coefficient $\Lambda_{\text{C,nov}}$ instead of $\mathbb{Q}$-coefficient (see Definition 13.1).

**Definition 13.2.** We define Bott-Morse D-brane Floer cohomology of the cochain complex $C^*(L, \Lambda_{\text{C,nov}})$ by defining the coboundary map as

$$\delta \beta ( [P, f] ) = (M_2(\beta)_{ev_1 \times f, ev_0} \cdot (\text{hol}_{\beta} \mathcal{L}) \text{ for } \beta \neq 0, \quad (13.1)$$

where $\text{hol}_{\beta} \mathcal{L}$ is the holonomy of the flat line bundle along the closed curve $\partial \beta$. And define the coboundary map $\delta$ as

$$\delta ([P, f]) = \sum_{\beta \in \pi_2(M, L)} \delta \beta ([P, f]) \otimes e^{\frac{\nu(\beta)}{2}}.$$  

Then, $\delta \circ \delta = 0$ follows from Theorem 13.2. Note that $\delta_0$ does not change from definition of Bott-Morse Floer cohomology. It corresponds to the fact that thinline trajectories in Floer cohomology between $L$ and $\phi(L)$ will not have any holonomy factor if we consider the flat line bundle induced on $\phi(L)$ from $\mathcal{L}$.

Now, let $h_j \in S^1$ denote the holonomy of $\mathcal{L}$ along the generators $L_j$. Then we can see that

$$\delta \beta_j < pt > = ((-1)^n L_j) \cdot h_j$$

$$\delta \beta_0 < pt > = ((-1)^n L_0) \cdot h_1^{-1} \cdot h_2^{-2} \cdots h_n^{-1}$$

Consider the case that

$$h_j = e^{2\pi i j/k} \text{ for all } j = 1, 2, \cdots, n, \text{ for a fixed } k \in \mathbb{Z}$$
Then
\[ h_0 = h_1^{-1} \cdot h_2^{-2} \cdots h_n^{-1} = e^{-\frac{2\pi ki}{n+1}} = e^{\frac{2\pi ki}{n+1}} = h_j. \]

Therefore, in this case, boundary operators \( \delta_{\beta_j}(<pt>) \) are multiplied by the factor \( e^{\frac{2\pi ki}{n+1}} \) for all \( j \), so under the standard spin-structure, the boundary operator \( \delta_2 \) is still zero. Hence we get the same Floer cohomology as in Theorem 11.2.

**Theorem 13.1.** Let the following
\[ (1, \cdots, 1), (\alpha, \cdots, \alpha), \cdots, (\alpha^n, \cdots, \alpha^n) \]
for \( \alpha = e^{\frac{2\pi ki}{n+1}} \) represent the holonomies of the flat line bundles along the generators of the Clifford torus \( T^n \) for \( n \geq 1 \). Under the standard spin-structure, the above A-branes are the only ones which give non-trivial Floer cohomology, which are isomorphic to the singular cohomology as in Theorem 11.2.

**Remark 13.3.** The above Theorem confirms the prediction of Hori [H]. See Kapustin-Li [KLi] for explicit statement on the product structure for the case \( n = 2 \).

**Proof.** To finish the proof of the above Theorem, we need to prove that for flat bundles with other holonomies, D-brane Floer cohomology vanishes. It will be similar to the calculation of Floer cohomology with a non-standard spin structure. Again, we work with the spectral sequence and we start with \( E_2 \) as before. Recall that \( h_j \) denotes the holonomy along the generator \( L_j \).

Define
\[
S := \{ j \in \{1, 2, \cdots, n\} | h_j = h_0 \}
\]
\[
S^c := \{ j \in \{1, 2, \cdots, n\} | h_j \neq h_0 \} \tag{13.2}
\]

First, the case when \( S^c \) is empty is exactly the case when \( h_j = e^{\frac{2\pi ki}{n+1}} \) for all \( j = 0, 1, \cdots, n \) for some \( k \). Then \( \delta \) is always zero, hence Floer cohomology has the same generator as the singular cohomology of \( T^n \) as in the Theorem.

So let us assume that \( S^c \) is not empty. Then
\[
\delta_2 <pt> = \sum_{j=0}^{n} \delta_{\beta_j} <pt> = (-1)^n \sum_{j=0}^{n} h_j L_j
\]
\[
= (-1)^n \sum_{j=1}^{n} (h_j - h_0) L_j = (-1)^n \sum_{j \in S^c} (h_j - h_0) L_j \tag{13.3}
\]

So \(<pt>\) is not in the kernel of \( \delta_2 \). For 1 dimensional cycles,
\[
\delta_2 \left( \sum_{j=1}^{n} a_j L_j \right) = \sum_{i=0}^{n} \sum_{j=1}^{n} a_j (\delta_{\beta_i} L_j)
\]
\[
= (-1)^n \sum_{i=0}^{n} \sum_{j=1}^{n} a_j (h_i L_i \times L_j)
\]
\[
= (-1)^n \sum_{i=1}^{n} \sum_{j=1}^{n} a_j (h_i - h_0) L_i \times L_j
\]
\[
= (-1)^n \sum_{1 \leq i < j \leq n} L_i \times L_j (a_j (h_i - h_0) - a_i (h_j - h_0))
\]

So to be in the kernel,
\[
(a_j (h_i - h_0) - a_i (h_j - h_0)) = 0 \tag{13.4}
\]
for all $i, j$ with $i < j$. Now, for $i, j \in S^c$, the equation becomes

$$\frac{a_j}{(h_j - h_0)} = \frac{a_i}{(h_i - h_0)}$$

So for all $j \in S^c$, $a_j$ are same and we denote it by $a$. For $i \in S$, $j \in S^c$ with $i < j$, the equation becomes $a_i = 0$. Similarly, for $i \in S$, $j \in S^c$ with $i > j$, the equation implies $a_i = 0$.

Now the elements in the kernel can be written as

$$\sum_{i \in S^c} a_i L_i = \sum_{i \in S^c} a(h_i - h_0) L_i$$

In fact this element is in the image of $\delta_2$: One can check that

$$\delta_2((-1)^n a < pt>) = \sum_{i \in S^c} a(h_i - h_0) L_i$$

Similarly, for higher dimensional cycles, we show that the elements in the kernel of $\delta_2$ lies in the image of $\delta_2$.

First, we will set up our notation for the indices

**Definition 13.4.** By $G, I, J$ we will denote subsets of $\{1, 2, \cdots, n\}$ with number of elements $|G| = k - 1, |I| = k, |J| = k + 1$. We will also denote its elements as $G = \{g_1, \cdots, g_{k-1}\}$ with $g_1 < g_2 < \cdots < g_{k-1}$. And we denote $G_s = G \setminus \{g_s\}$. This notation will be applied to any index set.

We need the following elementary lemma, which states that the $k$-th simplicial cohomology of the standard $(n - 1)$-simplex is zero. Let $R$ be a coefficient ring which will be either $\mathbb{Q}$ or $\mathbb{C}$.

**Lemma 13.2.** We fix $1 \leq k \leq n$. Suppose we are given numbers $A_I \in R$ for every subset $I$ with $|I| = k$. And suppose those numbers satisfy the following equation: For any $J$ with $|J| = k + 1$,

$$\sum_{s=1}^{k+1} (-1)^{s-1} A_J = 0$$

Then, there exists $B_G \in R$ for all $G$ with $|G| = k - 1$ so that each $A_I$ can be written as

$$A_I = \sum_{s=1}^{k} (-1)^{s-1} B_{G_s}.$$  

When $k = 1$, there exist a number $B_G$ for the empty set $G$ with $|G| = 0$ so that

$$A_I = \sum_{s=1}^{k} (-1)^{s-1} B_G.$$  

**Proof.** There is an obvious correspondence between the index set, say $I \subset \{1, 2, \cdots, n\}$, and the simplicial chain, say $C_I$, of the standard $(n - 1)$-simplex. Then, consider $A_*$ as a simplicial $k$ cochain which assigns the number $A_I$ to the chain $C_I$. Then the hypothesis is equivalent to the fact that $\delta A_* = 0$. Hence, there exists $k - 1$ dimensional cochain $B_*$ with

$$\delta B_* = A_*.$$

\qed
Now we denote an arbitrary element of \( k \) dimensional cycles as
\[
\sum_{I, |I| = k} A_I L_I
\]
The boundary of this element is
\[
\delta_2 \left( \sum_{I, |I| = k} A_I L_I \right) = \sum_I A_I (\delta L_I)
\]
\[
= \sum_I A_I (h_0 L_0 + \cdots + h_n L_n) \times L_I
\]
\[
= \sum_I \sum_{s=1}^n (h_s - h_0) A_I L_s \times L_I
\]
\[
= \sum_{J, |J| = k+1} \sum_{s=1}^{k+1} A_{J_s} (-1)^{s-1} (h_{J_s} - h_0) L_J
\]
Hence, the element \( \delta_2 \left( \sum_{I, |I| = k} A_I L_I \right) \) is in the kernel if for all \( J \) with \( |J| = k+1 \),
\[
\sum_{s=1}^{k+1} A_{J_s} (-1)^{s-1} (h_{J_s} - h_0) = 0 \tag{13.5}
\]
Now we show that any element in the kernel lies in the image of the boundary map. First, we consider the case that the set \( S = \{ i \in \{1, 2, \cdots, n\} | h_i = h_0 \} \) is empty, i.e. \( h_0 - h_i \neq 0 \) for all \( i = 1, \cdots, n \). For all \( I, |I| = k \), we set
\[
B_I = \prod_{i \in I} \frac{A_I}{(h_i - h_0)}.
\tag{13.6}
\]
Then, the equation (13.5) is nothing but
\[
\sum_{s=1}^{k+1} (-1)^{s-1} B_{J_s} = 0
\]
By the Lemma 13.2 there exists \( C_G \) for all \( G \subset \{1, 2, \cdots, n\} \) with \( |G| = k - 1 \) such that
\[
B_I = \sum_{s=1}^k (-1)^{s-1} C_{I_s}
\]
Then we take the boundary of \( \sum_G (\prod_{i \in G} (h_i - h_0)) C_G L_G \), where the sum is taken over all \( G \subset \{1, 2, \cdots, n\} \) with \( |G| = k - 1 \).
\[
\delta_2 \left( \sum_G (\prod_{i \in G} (h_i - h_0)) C_G L_G \right) = \sum_G \sum_{s=0}^n C_G (\prod_{i \in G} (h_i - h_0))(h_s - h_0) L_s L_G
\]
\[
= \sum\left( \prod_{i \in I} (h_i - h_0) \right) B_I L_I
\]
\[
= \sum_I A_I L_I
\]
This shows that codimension k kernels are coboundaries.
Now when $S$ is not empty, we carry out the same argument, but it will be more complicated to prove it. Without loss of generality, we may set

$$S = \{r + 1, r + 2, \cdots, n\}, \quad S^c = \{1, 2, \cdots, r\}$$

From now on, we write

$$A_I = A_I^{I \cap S^c}.$$

For example, we write $A_{I,2,\cdots,r+2}$ as $A_{I,r_1+1,\cdots,r_2}$. This is to distinguish elements in $S$ and $S^c$. Now the codimension $k$ element in the kernel can be written as $(\sum_{I,|I|=k} A_I L_I)$, where $A_I$ satisfies the equation (13.5). But if $j_s \in S$, we have $h_{j_s} - h_0 = 0$. Hence we may write, for each fixed $J$ with $|J| = k + 1$,

$$\sum_{j_s \in S^c \cap J} A_J^{J \cap S^c} (-1)^{s-1}(h_{j_s} - h_0) = 0 \quad (13.7)$$

Now, for $P \subset S^c, T \subset S$, we let

$$B_T^P = \frac{A_T^T}{\prod_{p \in P}(h_p - h_0)}$$

Then, the equation (13.7) is equivalent to

$$\sum_{s=1}^{|J \cap S|} (-1)^{s-1} B_T^{J \cap S} = 0$$

We collect all such equations with respect to the same index set $J \cap S$. By the Lemma 13.2, there exists $C_{J \cap S}^T$ for all $Q \subset S^c$ with $|Q| = |J \cap S^c| - 2$ such that for any $P \subset S^c$ with $|P| = |J \cap S^c| - 1$,

$$B_T^{J \cap S} = \sum_{s=1}^{|P|} (-1)^{s-1} C_{P_s}^{J \cap S}$$

Then we may rewrite the above as

$$B_T^{J \cap S} = \sum_{s=1}^{|P|} (-1)^{s-1} C_{P_s}^{J \cap S}$$

Now we will show that $\sum_I A_I L_I$ is exactly in the image of the following element. In the following, we take the sum over any subset $T \subset S$, and over any subset $Q \subset S^c$ with $|T| + |Q| = |I| - 1$ and for any subset $P \subset S^c$ with $|T| + |P| = |I|$.

$$\delta_2(\sum_T \sum_Q (\prod_{q \in Q} (h_q - h_0)) C_Q^T L_Q L_T) = \sum_T \sum_Q (\prod_{q \in Q} (h_q - h_0)) C_Q^T (\sum_{s=1}^r (h_s - h_0) L_s L_Q L_T) = \sum_T \sum_{p \in P} (h_p - h_0) B_P^T L_P L_T = \sum_T \sum_P A_P^T L_P L_T = \sum_T \sum_P A_{P,T} L_{P,T} = \sum_T A_I L_I$$
This finishes the proof.

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