Optimality Conditions for Variational Problems in Incomplete Functional Spaces

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Abstract
This paper develops a novel approach to necessary optimality conditions for constrained variational problems defined in generally incomplete subspaces of absolutely continuous functions. Our approach consists of reducing a variational problem to a (nondynamic) problem of constrained optimization in a normed space and then applying the results recently obtained for the latter class by using generalized differentiation. In this way, we derive necessary optimality conditions for nonconvex problems of the calculus of variations with velocity constraints under the weakest metric subregularity-type constraint qualification. The developed approach leads us to a short and simple proof of first-order necessary optimality conditions for such and related problems in broad spaces of functions including those of class $C^k$ as $k \geq 1$.

Keywords  Calculus of variations · Constrained optimization · Optimal control · Necessary optimality conditions · Variational analysis · Generalized differentiation

Mathematics Subject Classification 49K24 · 49J52 · 49J53 · 90C48

Dedicated to Professor Franco Giannessi in the occasion of his 85th birthday.

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1 Introduction

The classical calculus of variations primarily deals with minimizing integral functionals on classes of smooth curves that mainly belong to the spaces of continuously differentiable or twice continuously differentiable functions; see the fundamental monographs by Bolza [3], Tonelli [19], and Bliss [2] as well as extensive further developments on the subject. Although the aforementioned and other spaces used in the calculus of variations are incomplete, this does not create any obstacles in deriving necessary conditions for optimal solutions to such problems due to the employed (Lagrangian) method of variation.

Modern variational analysis offers powerful techniques to derive necessary optimality conditions in problems of dynamic optimization including those in the calculus of variations and optimal control. This machinery is based on advanced variational principles and approximation/limiting procedures, which are applied to general problems governed, in particular, by differential inclusions where the method of variations and its modifications are not applicable. For various techniques in this vein, we refer the reader to the books [5,10,15,18,20] with the bibliographies and commentaries therein. However, there is a price to pay: all such methods unavoidably require the completeness of the space in question and thus cannot be applied to optimization problems in spaces of smooth functions that have been traditionally considered in the calculus of variations.

This paper is devoted to developing a novel approach of variational analysis and generalized differentiation to derive necessary optimality conditions in constrained problems of dynamic optimization over curves belonging to a prescribed normed space located between the collections of absolutely continuous and infinitely differentiable functions. The suggested approach is based on the reduction of a given dynamic optimization problem to an infinite-dimensional nondynamic problem of constrained optimization for which necessary optimality conditions have been recently obtained in our paper [11] in arbitrary normed spaces under weak constraint qualifications. Although this approach works in more general frameworks of constrained dynamic optimization, for simplicity we concentrate here on an extended Bolza problem of the calculus of variations considered in the aforementioned (generally incomplete) spaces of curves subject to endpoint and hard/pointwise constraints on velocity functions that depend on the current state position. Pointwise velocity constrains have been recognized as the most challenging ones in the calculus of variations. Even in a modern setting with the usage of an advanced variational technique largely different from the method of variations, the necessary optimality conditions for strong local minimizers in problems of the calculus of variations with pure velocity constraints in the complete space of absolutely continuous functions are obtained under the restrictive “interiority hypothesis” in the most recent Clarke’s book [5, Theorem 18.1].

The reduction method developed in this paper allows us to represent the original variational problem in an equivalent form of nondynamic infinite-dimensional constrained optimization and then apply the necessary optimality conditions to the latter problem established in [11]. In this way, we present a rather simple derivation of necessary optimality conditions for strong local (in a generalized sense) minimizers of the extended Bolza problem under consideration defined in generally incomplete
subspaces of absolutely continuous functions. The obtained necessary optimality conditions consist of the *Euler–Lagrange equation*, the *Weierstrass–Pontryagin maximization condition*, and the *transversality inclusion* in the qualified/normal/KKT form established under the weakest constraint qualification of the *metric subregularity* type.

The rest of the paper is organized as follows. Section 2 contains the required definitions and preliminaries from variational analysis and generalized differentiation used in the formulations and proofs of the subsequent results. We present here the underlying theorem from [11] giving us necessary optimality conditions for infinite-dimensional constrained optimization problems to which we reduce the extended Bolza problem of our study.

Section 3 starts with the formulation and discussion of this extended version of the Bolza problem with endpoint and pointwise velocity constraints. Then, we formulate the aforementioned necessary optimality conditions for the extended Bolza problem that are proved in the remaining part of the paper by the reduction to constrained optimization.

All the reduction steps are furnished in Sect. 4, which is the most technical part of the paper while containing results of their own interest. Using this reduction and the obtained optimality conditions in nondynamic constrained optimization, we complete in Sect. 5 the derivation of the necessary optimality conditions for the extended Bolza problem that are formulated in Sect. 3. Section 6 summarizes the main achievements of the paper and discusses some topics of our future research.

### 2 Basic Definitions and Preliminaries

First we recall some standard notation of variational analysis used in the paper. Unless otherwise stated, \( \mathbb{X} \) and \( \mathbb{Y} \) stand for *normed spaces* with the generic symbol \( \| \cdot \| \) for norms and \( \langle \cdot, \cdot \rangle \) for scalar products between the spaces in question and their topological duals.

Given an extended-real-valued function \( \varphi : \mathbb{X} \to (-\infty, \infty] \) with the domain \( \text{dom} \varphi := \{ x \in \mathbb{X} \mid \varphi(x) < \infty \} \), the *(Dini–Hadamard)* *subderivative* of \( \varphi \) at \( \bar{x} \in \text{dom} \varphi \) is the function \( d\varphi(\bar{x}) : \mathbb{X} \to [-\infty, \infty] \) defined by

\[
d\varphi(\bar{x})(\bar{u}) := \liminf_{t \downarrow 0, \bar{u} \to \bar{u}} \frac{\varphi(\bar{x} + tu) - \varphi(\bar{x})}{t}, \quad \bar{u} \in \mathbb{X},
\]

where the limit \( u \to \bar{u} \) in (2.1) can be equivalently omitted if \( \varphi \) is locally Lipschitzian around \( \bar{x} \). The latter form reduces to the *directional derivative* of \( \varphi \) in the direction \( \bar{u} \) provided that the full limit in (2.1) exists. It is well known that the *Gâteaux differentiability* of \( \varphi \) at an interior point \( \bar{x} \) of the domain corresponds to the existence of the directional derivative in any direction and its linearity with respect to the direction variable.

Turning next to sets, we associate with any nonempty subset \( \Omega \subset \mathbb{X} \) the *indicator function* \( \delta_{\Omega}(x) \) of \( \Omega \) that equals 0 if \( x \in \Omega \) and \( \infty \) otherwise, and the *distance function*
dist($x; \Omega$) of $\Omega$ defined as usual by
$$
\text{dist}(x; \Omega) := \inf \{ ||x - u|| \mid u \in \Omega \}.
$$
The latter function is Lipschitz continuous on $X$ with Lipschitz constant $\ell = 1$. Since the main goal of this paper is to illuminate the suggested approach to deriving necessary optimality conditions in problems of dynamic optimization by reducing them to nondynamic constrained optimization without much of technical complications, we are not going to involve here tangent and normal cone constructions for nonconvex sets. The only normal cone used in what follows is the classical one for convex sets $\Omega \subset X$ defined by
$$
N_{\Omega}(\bar{x}) := \{ x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega \}.
$$
(2.2)
If $\bar{x} \in \Omega$ with $N_{\Omega}(\bar{x}) := \emptyset$ otherwise. The set of normals in (2.2) is obviously convex and closed in the weak* topology of the dual space $X^*$.

Considering further a set-valued mapping $F : X \to Y$ with the graph $\text{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$, recall that $F$ is metrically regular around $(\bar{x}, \bar{y}) \in \text{gph} F$ if there exist a constant $\kappa > 0$ and neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that we have
$$
\text{dist}(x; F^{-1}(y)) \leq \kappa \text{ dist}(y; F(x)) \text{ for all } (x, y) \in U \times V.
$$
(2.3)
If $y = \bar{y}$ in (2.3), the mapping $F$ is called to be metrically subregular at $(\bar{x}, \bar{y})$. The reader is referred to the books [10,15,17] for more information about these and equivalent properties of set-valued mappings with their broad applications in variational analysis.

Now, we formulate a class of (nondynamic) constrained optimization problems, which was studied in our previous paper [11] with deriving various types of primal and dual necessary optimality conditions for their local minimizers. Given a cost function $J : X \to \mathbb{R}$, a constraint mapping $f : X \to Y$ between arbitrary normed spaces, and a constraint set $\Theta \subset Y$, the basic constrained optimization problem is defined as follows:
$$
\text{minimize } J(x) \text{ subject to } f(x) \in \Theta
$$
(2.4)
with the set of feasible solutions denoted by $\Omega := \{ x \in X \mid f(x) \in \Theta \}$. Among the necessary optimality conditions obtained for (2.4) in [11], we select the dual one established in the refined KKT form under the following constraint qualification.

**Definition 2.1 (metric subregularity constraint qualification).** Let $\bar{x}$ be a feasible solution (2.4). Then, we say that the **metric subregularity constraint qualification** (MSCQ) holds at $\bar{x}$ if the set-valued mapping $x \mapsto f(x) - \Theta$ is metrically subregular at $(\bar{x}, 0)$, i.e., there exist a constant $\kappa > 0$ and a neighborhood $U$ of $\bar{x}$ such that
$$
\text{dist}(x; \Omega) \leq \kappa \text{ dist}(f(x); \Theta) \text{ for all } x \in U.
$$
(2.5)
Note that the replacement of the metric subregularity of the mapping $x \mapsto f(x) - \Theta$ at $(\overline{x}, 0)$ in Definition 2.1 by the metric regularity of this mapping around the pair $(\overline{x}, 0)$ brings us to a significantly more restrictive constraint qualification, which reduces to the well-known ones for particular classes of optimization problems (e.g., the Mangasarian–Fromovitz constraint qualification in nonlinear programming, the Robinson constraint qualification in conic programming, etc.). This follows from applying the Mordukhovich coderivative criterion to the mapping $f - \Theta$ around $(\overline{x}, 0)$ for sets $\Theta$ that appear in particular constraint systems; see [14,15,17]. Regarding the more subtle MSCQ, its relationships with other constraint qualifications, and various applications, we refer the reader to, e.g., [6–8,11,13] with the additional details and discussions.

Finally in this section, we present the necessary optimality conditions for the constrained problem (2.4) in normed spaces used in what follows. Note that the following theorem is a special case of [11, Theorem 7.3], where the optimality conditions are established under more general assumptions. However, we confine ourselves to the ones below to simplify the subsequent derivation of necessary conditions for the extended Bolza problem formulated in the next section. Recall that $A^*$ indicates the adjoint operator of the linear operator $A$, which reduces to the matrix transposition in finite dimensions.

**Theorem 2.2** (necessary conditions for constrained optimization). Let $\mathbb{X}$ and $\mathbb{Y}$ be arbitrary normed spaces, and let $\overline{x} \in \Omega$ be a local (in the norm of the space $\mathbb{X}$) minimizer of problem (2.4), where $\Theta \subset \mathbb{Y}$ is convex and locally closed around $f(\overline{x})$, where $J : \mathbb{X} \to \mathbb{R}$ is Gâteaux differentiable at $\overline{x}$ and locally Lipschitzian around this point with Lipschitz constant $\ell > 0$ and where $f : \mathbb{X} \to \mathbb{Y}$ is continuously Fréchet differentiable around $\overline{x}$. Assume in addition that MSCQ (2.5) holds at $\overline{x}$ with some constant $\kappa > 0$. Then, we have the following necessary optimality conditions:

there exists $\lambda \in \mathbb{Y}^*$ such that

$$
\begin{align*}
0 &= \nabla J(\overline{x}) + \nabla f(\overline{x})^*\lambda, \\
\lambda &\in N_{\Theta}(f(\overline{x})), \quad \|\lambda\| \leq \kappa \ell,
\end{align*}
$$

(2.6)

where the same symbol $\nabla$ is used for both Gâteaux and Fréchet derivatives.

Note that if $J$ is also continuously Fréchet differentiable around $\overline{x}$, then we can set $\ell := \|\nabla J(\overline{x})\|$; see [11, Corollary 7.5].

### 3 Extended Bolza Problem with Velocity Constraints

In this section, we define a constrained variational problem written in an extended form of the Bolza problem of the calculus of variations, while in the presence of pointwise velocity constraints depending on the current curve position. Feasible curves in this problem belong to a prescribed generally incomplete subspace of functions situated between the spaces of infinite differentiable and absolutely continuous ones.

To formulate the problem of our study, consider the terminal cost $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the running cost $\vartheta : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the constraint mappings $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, the dynamic constraint set $\Omega_1 \subset \mathbb{R}^n$, and the endpoint constraint set $\Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$. 
where the time $T > 0$ is fixed. Let $\mathbb{X}$ be an arbitrary normed space such that

$$C^\infty([0, T]; \mathbb{R}^n) \subset \mathbb{X} \subset AC([0, T]; \mathbb{R}^n),$$

(3.1)

where $C^\infty([0, T]; \mathbb{R}^n)$ stands for the standard space of functions $x : [0, T] \rightarrow \mathbb{R}^n$ infinite differentiable on $[0, T]$, and where $AC([0, T]; \mathbb{R}^n)$ indicates the space of all functions $x(\cdot)$ that are absolutely continuous on $[0, T]$ with the norm

$$\|x(\cdot)\|_{ac} := \|x(0)\| + \int_0^T \|\dot{x}(t)\| \, dt.$$  

(3.2)

Both inclusions in (3.1) can be nonstrict, i.e., the extreme cases of $\mathbb{X} = C([0, T]; \mathbb{R}^n)$ and $\mathbb{X} = AC([0, T]; \mathbb{R}^n)$ are also acceptable. Unless otherwise stated, the norm on the space $\mathbb{X}$ is given by (3.2). In particular, the choice of $\mathbb{X}$ in (3.1) includes the incomplete spaces $C^k([0, T]; \mathbb{R}^n)$ of $k$-times differentiable vector functions with $k = 1, 2, \ldots$, which are typically encountered in the classical calculus of variations.

Our basic extended Bolza problem is formulated as follows:

$$\text{minimize } J(x(\cdot)) := \varphi(x(0), x(T)) + \int_0^T \theta(t, x(t), \dot{x}(t)) \, dt \text{ over } x(\cdot) \in \mathbb{X}$$

subject to $\dot{x}(t) + g(t, x(t)) \in \Omega_1$ a.e. $t \in [0, T]$, $(x(0), x(T)) \in \Omega_2$.  

(3.3)

We say as usual that $x(\cdot) \in \mathbb{X}$ is a feasible solution to problem (3.3) if $x(\cdot)$ satisfies all the constraints in this problem and gives a finite value of the cost functional therein. The set of feasible solutions to (3.3) is denoted by $\mathcal{S}$. A feasible solution $\bar{x} = \bar{x}(\cdot)$ is said to be an $\mathbb{X}$-strong local minimizer of (3.3) if there exists $\varepsilon > 0$ such that

$$J(x(\cdot)) \leq J(\bar{x}(\cdot)) \text{ for any } x(\cdot) \in \mathcal{S} \text{ with } \|x(\cdot) - \bar{x}(\cdot)\|_{ac} < \varepsilon.$$  

(3.4)

Note that the notion of $\mathbb{X}$-strong local minimizers defined in (3.4) is different (even for $\mathbb{X} = AC([0, T]; \mathbb{R}^n)$) from the standard notion of strong minimizers in the calculus of variations, where the $\mathbb{X}$-closeness in (3.4) is replaced by the closeness in the uniform topology of the space $C([0, T]; \mathbb{R}^n)$; cf. [2,3,5]. In fact, $\mathbb{X}$-strong local minimizers occupy a (proper) intermediate position between weak and strong minimizers of the calculus of variations; see [15, Section 6.1] for more discussions, examples, and references.

As mentioned above, our intention is to reduce the extended Bolza problem of dynamic optimization (3.3) in the normed space $\mathbb{X}$ to the nondynamic one (2.4) in a suitable functional space in order to apply to the latter the necessary optimality conditions established in Theorem 2.2. To proceed in this way, we have to formulate appropriate assumptions on the given data of (3.3) that ensure the fulfillment of the required assumptions of Theorem 2.2 for the reduced constrained optimization problem (2.4). Let us impose the following assumptions on the initial data $\varphi, \theta, g, \Omega_1, \Omega_2$ of (3.3) around the reference optimal solution ($\mathbb{X}$-strong local minimizer) $\bar{x} = \bar{x}(\cdot) \in \mathbb{X}$ of this problem.
\textbf{(H1)} The terminal cost \( \varphi(x_0, x_T) \) is continuously differentiable around \((\overline{x}(0), \overline{x}(T))\).

\textbf{(H2)} The running cost \( \vartheta(t, x, v) \) is measurable in \( t \) on \([0, T]\), continuously differentiable in \((x, v)\) around \((\overline{x}(t), \overline{\dot{x}}(t))\) for a.e. \( t \in [0, T]\), and locally Lipschitzian with respect to \((x, v)\) around \((\overline{x}(-), \overline{\dot{x}}(-))\) in the \( X \)-norm (3.2) with a summable Lipschitz modulus \( \ell(t) \) on \([0, T]\). This means that there exists \( \varepsilon > 0 \) such that for all \( x_1(-), x_2(-) \in X \) near \( \overline{x}(t) \) we have

\[
\left\| \vartheta\left(t, x_1(t), \dot{x}_1(t)\right) - \vartheta\left(t, x_2(t), \dot{x}_2(t)\right) \right\| \\
\leq \ell(t)\left(\|x_1(t) - x_2(t)\| + \|\dot{x}_1(t) - \dot{x}_2(t)\|\right) \quad \text{a.e. } t \in [0, T]
\]

provided that \( \|x_1(-) - x_2(-)\|_{ac} \leq \varepsilon \). For simplicity we suppose that \( \ell(t) \equiv \ell \) on \([0, T]\).

\textbf{(H3)} The constraint sets \( \Omega_1 \) and \( \Omega_2 \) are convex and locally closed around \( \overline{x}(-) \) in the spaces \( \mathbb{R}^n \) and \( \mathbb{R}^n \times \mathbb{R}^n \), respectively. This means that there exist closed balls around \( \overline{\dot{x}}(t) + g(t, \overline{x}(t)) \) for a.e. \( t \in [0, T]\) and \((\overline{x}(0), \overline{\dot{x}}(T))\) in the corresponding finite-dimensional spaces such that the intersections of these balls with \( \Omega_1 \) and \( \Omega_2 \) are closed.

\textbf{(H4)} The constraint mapping \( g(t, x) \) is continuously differentiable in \( x \) and measurable in \( t \) together with its derivative \( \nabla_x g(t, x) \). Furthermore, both \( g(t, x) \) and \( \nabla_x g(t, x) \) are essentially bounded on \([0, T]\) for all \( x(-) \) around \( \overline{x}(-) \), where the localization is understood similarly to the description in (H3).

\textbf{(H5)} There exists a constant \( \kappa > 0 \) such that for all \( x(-) \in X \) sufficiently close to \( \overline{x}(-) \) we have the constraint qualification

\[
\text{dist}(x(-); \mathcal{S}) \leq \kappa \int_0^T \text{dist}(\dot{x}(t) + g(t, x(t)); \Omega_1) \, dt + \kappa \text{dist}((x(0), x(T)); \Omega_2).
\]

\text{(3.5)}

As observed by one of the referees, in the case where \( \Omega_2 = \Omega_0 \times \mathbb{R}^n \), assumption (H5) follows from the Filippov–Gronwall inequality; see, e.g., [1, Proposition 1 of Chapter 2]. Furthermore, we will see below that the local Lipschitz continuity assumption on the running cost in the \( X \)-norm imposed in (H2), which is generally different from the standard local Lipschitz continuity of \( \vartheta(t, x, v) \) in \((x, v) \in \mathbb{R}^n \times \mathbb{R}^n\), allows us to deal with \( X \)-strong local minimizers \( \overline{x}(-) \in X \) of the extended Bolza problem (3.3).

Here is the formulation of necessary optimality conditions for strong local minimizers of (3.3), which are proved in the next two sections.

**Theorem 3.1** (necessary optimality conditions for the extended Bolza problem) \textit{Let} \( \overline{x}(-) \in X \) \textit{be an} \( X \)-\textit{strong local minimizer} (3.4) \textit{of the extended Bolza problem} (3.3) \textit{under the fulfillment of the assumptions (H1)–(H5) around} \( \overline{x}(-) \). \textit{Then, there exists an adjoint arc} \( p(-) \in AC ([0, T]; \mathbb{R}^n) \) \textit{for which the following conditions are satisfied:}

\[
The \text{EULER-} \text{ -} \text{LAGRANGE} \ \text{EQUATION} \text{ for a.e. } t \in [0, T]:
\]

\[
\dot{p}(t) - \nabla_x g(t, \overline{x}(t))^* p(t) = \nabla_x \vartheta\left(t, \overline{x}(t), \overline{\dot{x}}(t)\right) - \nabla_x g\left(t, \overline{x}(t)\right)^* \nabla_v \vartheta\left(t, \overline{x}(t), \overline{\dot{x}}(t)\right).
\]

\text{(3.6)}
The Weierstrass–Pontryagin maximization condition for a.e. \( t \in [0, T] \):
\[
\{ p(t) - \nabla \vartheta (t, x(t), \dot{x}(t)), \dot{x}(t) + g(t, x(t)) \} = \max_{w \in \Omega_1} \{ p(t) - \nabla \vartheta (t, x(t), \dot{x}(t)), w \}.
\]
(3.7)

The transversality inclusion:
\[
(p(0), -p(T)) \in \nabla \varphi (x(0), x(T)) + N_{\Omega_2} (x(0), x(T)).
\]
(3.8)

Observe that if \( g \equiv 0 \) in (3.3), i.e., we have the problem of Bolza with pure velocity constraints and if \( X = AC([0, T]; \mathbb{R}^n) \), then the maximization condition (3.7) reduces to the Weierstrass condition obtained [5, Theorem 18.1] under the “Interiority Hypothesis” that is much more restrictive than the qualification condition (3.5). On the other hand, condition (3.7) corresponds to the extensions of the Pontryagin maximum principle [16] to variational problems governed by differential inclusions \( \dot{x} \in F(t, x) \) with \( F(t, x) := \Omega_1 - g(t, x) \); see, e.g., [5,10,15,18,20] for various results and proofs in complete spaces of functions. Our simple reduction proof of Theorem 3.1 is given in the next two sections.

4 Reduction to Constrained Optimization

First we rewrite the extended Bolza problem (3.3) in the form of the constrained optimization problem (2.4) in the normed space \( X \) of functions \( x = x(\cdot) \) taken from (3.1) and endowed with the norm (3.2). The data of this problem are defined in terms of (3.3) by
\[
J(x) := \varphi (x(0), x(T)) + \int_0^T \vartheta (t, x(t), \dot{x}(t)) \, dt,
\]
\[
f(x) := (\dot{x} + g(\cdot, x(\cdot)), (x(0), x(T))),
\]
\[
\Theta := \{ y(\cdot) \in L^1 ([0, T]; \mathbb{R}^n) \mid y(t) \in \Omega_1 \text{ a.e. } t \in [0, T] \} \times \Omega_2.
\]
(4.1)

It is easy to see that the set \( \Theta \) in (4.1) is convex and locally closed around \( f (\bar{x}(\cdot)) \) in the space under consideration. To proceed with the applications of Theorem 2.2, we need to check that the mappings \( J \) and \( f \) from (4.1) satisfy the Lipschitz continuity and differentiability assumptions imposed in the latter theorem.

Let us begin by observing that the space \( AC([0, T]; \mathbb{R}^n) \), which contains \( X \) and is equipped with norm (3.2), is isometric to the space \( L^1 ([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \) via the isometry \( x \mapsto (\dot{x}, x(0)) \), and hence the dual space of \( AC([0, T]; \mathbb{R}^n) \) can be identified with \( L^\infty ([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \). This tells us that the space \( X \) is densely embedded into \( L^1 ([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \), which tells us that the dual space of \( X \) can be identified with \( L^\infty ([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \). Furthermore, using the integral representation
\[
T x(0) = \int_0^T x(t) \, dt - \int_0^T \left( \int_0^s \dot{x}(t) \, dt \right) \, ds
\]
implies that the norm \( \| \cdot \|_{ac} \) in (3.2) is equivalent to
\[
\| x \|_{1,1} := \int_0^T \| x(t) \| \, dt + \int_0^T \| \dot{x}(t) \| \, dt. \tag{4.2}
\]
In fact, we have the precise equivalence relationships
\[
\frac{1}{1+T} \| x \|_{1,1} \leq \| x \|_{ac} \leq \frac{2T+1}{T} \| x \|_{1,1} \quad \text{for all} \quad x \in AC([0, T]; \mathbb{R}^n). \tag{4.3}
\]
Moreover, it follows from the inequality
\[
\left\| \int_0^T x(t) \, dt \right\| \leq \int_0^T \| x(t) \| \, dt
\]
that
\[
\| x \|_{\infty} := \sup \left\{ |x_i(t)| \mid i = 1, \ldots, n, \; 0 \leq t \leq T \right\} \leq \frac{2 + 2T}{T} \| x \|_{1,1} \tag{4.4}
\]

The next two theorems of their own interest verify the Lipschitz continuity and differentiability assumptions of Theorem 2.2 in the case where \( J \) and \( f \) are taken from (4.1) under the assumptions imposed in (H1)–(H4). Observe that both theorems do not require that \( \bar{x}(\cdot) \) is an \( X \)-strong local minimizer of (3.3) as formulated in latter assumptions while being hold for broader classes of curves \( \bar{x}(\cdot) \in \mathcal{X} \) satisfying the corresponding properties.

We start with verifying the required properties of the cost functional \( J \) in (3.3).

**Theorem 4.1** (properties of the cost functional). Let the assumptions (H1) and (H2) be satisfied around a given curve \( \bar{x} = \bar{x}(\cdot) \in \mathcal{X} \) with \( J(\bar{x}) < \infty \). Then, the cost functional in (3.3) is locally Lipschitzian around \( x \) as well as the Gâteaux differentiable at \( x \) with the following calculation of its Gâteaux derivative at \( \bar{x} \) in any direction \( u = u(\cdot) \in \mathcal{X} \):
\[
\nabla J(\bar{x})(u) = \left\langle \nabla \varphi(\bar{x}(0), \bar{x}(T)), (u(0), u(T)) \right\rangle
\]
\[
+ \int_0^T \left[ \left\langle \nabla_x \vartheta(t, \bar{x}(t), \dot{x}(t)), u(t) \right\rangle + \left\langle \nabla_v \vartheta(t, \bar{x}(t), \dot{x}(t)), \dot{u}(t) \right\rangle \right] \, dt. \tag{4.5}
\]

**Proof** First, we consider only the integral part
\[
I(x) := \int_0^T \vartheta(t, x(t), \dot{x}(t)) \, dt \tag{4.6}
\]
of the cost functional in (4.1) and establish its local Lipschitz continuity around \( \bar{x} \) as well as the Gâteaux differentiability at \( \bar{x} \) with the Gâteaux derivative representation
\[
\nabla I(\bar{x})(u) = \int_0^T \left[ \left\langle \nabla_x \vartheta(t, \bar{x}(t), \dot{x}(t)), u(t) \right\rangle + \left\langle \nabla_v \vartheta(t, \bar{x}(t), \dot{x}(t)), \dot{u}(t) \right\rangle \right] \, dt
\]
for all \( u(\cdot) \in \mathbb{X} \). To proceed, pick any \( x(\cdot) \in \mathbb{X} \) near \( \overline{x}(\cdot) \) and deduce from (H2) that
\[
|\vartheta(t, x(t), \dot{x}(t))| \leq |\vartheta(t, \overline{x}(t), \dot{\overline{x}}(t))| + \ell \|x(t) - \overline{x}(t)\| + \ell \|\dot{x}(t) - \dot{\overline{x}}(t)\|.
\]
Taking the integral over \([0, T]\) from both sides of the above inequality and using the equivalent norm description (4.2) implies that \( I(x) \) is finite, i.e., the integral functional (4.6) is real-valued around \( \overline{x}(\cdot) \) in the \( \mathbb{X} \)-norm.

Next, we verify that the integral functional (4.6) is Lipschitz continuous around \( \overline{x}(\cdot) \) in the \( \mathbb{X} \)-norm. Take any \( x_1(\cdot), x_2(\cdot) \in \mathbb{X} \) from the \( \varepsilon \)-neighborhood of \( \overline{x}(\cdot) \) in the \( \mathbb{X} \)-norm where (H2) holds. Combining this assumption with (4.3) gives us the estimates
\[
|I(x_1) - I(x_2)| \leq \int_0^T \|\vartheta(t, x_1(t), \dot{x}_1(t)) - \vartheta(t, x_2(t), \dot{x}_2(t))\| \, dt \\
\leq \int_0^T \left( \ell \|x_1(t) - x_2(t)\| + \ell \|\dot{x}_1(t) - \dot{x}_2(t)\| \right) \, dt \\
\leq \ell \|x_1 - x_2\|_{1,1} \leq \ell (1 + T) \|x_1 - x_2\|_{ac},
\]
which ensure the claimed local Lipschitz continuity of the functional \( I \) in \( \mathbb{X} \) around \( \overline{x}(\cdot) \).

To show now that \( I \) is Gâteaux differentiable at \( \overline{x}(\cdot) \), fix any direction \( u(\cdot) \in \mathbb{X} \). Using definition (2.1) of the Dini–Hadamard subderivative and the established local Lipschitz continuity of \( I \) at \( \overline{x}(\cdot) \) in the space \( \mathbb{X} \) under consideration, we find a decreasing sequence of positive number \( \tau_k \downarrow 0 \) as \( k \to \infty \) such that
\[
dI(\overline{x})(u) = \liminf_{\tau \downarrow 0} \frac{I(\overline{x} + \tau u) - I(\overline{x})}{\tau} = \lim_{k \to \infty} \frac{I(\overline{x} + \tau_k u) - I(\overline{x})}{\tau_k} \\
= \lim_{k \to \infty} \int_0^T \left( \frac{\vartheta(t, \overline{x} + \tau_k u(t), \dot{\overline{x}}(t) + \tau_k \dot{u}(t)) - \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t))}{\tau_k} \right) \, dt \\
= \int_0^T \left( \lim_{k \to \infty} \frac{\vartheta(t, \overline{x}(t) + \tau_k u(t), \dot{\overline{x}}(t) + \tau_k \dot{u}(t)) - \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t))}{\tau_k} \right) \, dt \\
= \int_0^T \left( \nabla_x \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t)), u(t) \right) + \left( \nabla_v \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t)), \dot{u}(t) \right) \, dt,
\]
where in the third line we interchange the limit and integral signs by using the Lebesgue dominated convergence theorem with taking into account the integrand function in the second line is dominated by \( \ell (\|u(t)\| + \|\dot{u}(t)\|) \) for a.e. \( t \in [0, T] \) due to Lipschitzian assumption in (H2). The last line above comes from the smoothness assumption on \( \vartheta \) imposed in (H2). By similar arguments, we arrive at the upper limit representation
\[
\limsup_{\tau \downarrow 0} \frac{I(\overline{x} + \tau u) - I(\overline{x})}{\tau} = \int_0^T \left( \nabla_x \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t)), u(t) \right) + \left( \nabla_v \vartheta(t, \overline{x}(t), \dot{\overline{x}}(t)), \dot{u}(t) \right) \, dt.
\]
Unifying the latter with the previous one for the lower limit proves the existence of the classical directional derivative of \( I \) at \( \overline{x} \) given by

\[
dI(\overline{x})(u) = \lim_{\tau \downarrow 0} \frac{I(\overline{x} + \tau u) - I(\overline{x})}{\tau}
\]

for any \( u(\cdot) \in \mathbb{X} \), which is clearly linear and continuous with respect to the direction variable with \( |dI(x)(u)| \leq \ell\|u\| \) for each \( u \in \mathbb{X} \). This shows that the integral functional (4.6) is Gâteaux differentiable at \( \overline{x} \) with the Gâteaux derivative representation (4.7).

To complete the proof of the theorem, it remains to show that the mapping \( x(\cdot) \mapsto \varphi(x(0), x(T)) \) associated with the terminal cost in (3.3) is continuously differentiable at \( \overline{x} \) in the space \( \mathbb{X} \). To this end, observe that this mapping can be represented in the composition form \( \varphi \circ h \) with \( h(x) := (x(0), x(T)) \), which is a linear operator on \( \mathbb{X} \). Combining the inequalities in (4.3) and (4.4) tells us that \( h: \mathbb{X} \to \mathbb{R}^n \times \mathbb{R}^n \) is a bounded linear mapping on \( \mathbb{X} \) satisfying the estimates

\[
|h(x)| \leq \sqrt{2}\|x\|_{\infty} \leq 2\sqrt{2}\frac{1 + T}{T}\|x\|_{1,1} \leq 2\sqrt{2}\frac{(1 + T)^2}{T}\|x\|_{ac} \text{ for all } x \in \mathbb{X}.
\]

This ensures the continuous differentiability of \( h \) on \( \mathbb{X} \) with the derivative \( \nabla h(\overline{x})(u) = (u(0), u(T)) \) whenever \( u(\cdot) \in \mathbb{X} \). Applying finally the classical chain rule verifies that the composition \( \varphi \circ h \) is differentiable on \( \mathbb{X} \) with the derivative

\[
\nabla[\varphi \circ h](\overline{x})(u) = \{\nabla\varphi(\overline{x}(0), \overline{x}(T)), (u(0), u(T))\}, \quad u(\cdot) \in \mathbb{X},
\]

which justifies together with (4.7) the claimed formula (4.5) and thus ends the proof.

\[\square\]

The next theorem deals with the constraint mapping \( g: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) from (3.3) and verifies that the assumptions in (H3) imposed on \( g \) ensure the fulfillment of the smoothness assumption on the mapping \( f: \mathbb{X} \to \mathbb{X} \times \mathbb{R}^n \times \mathbb{R}^n \) defined in (4.1), which is required by the necessary optimality conditions of Theorem 2.2.

Prior to the formulation and proof of this result, we recall the notion of continuous embedding. Let \( \mathbb{X} \) and \( \mathbb{Z} \) be two normed spaces. Then, \( \mathbb{X} \) is said to be continuously embedded into \( \mathbb{Z} \), with the notation \( \mathbb{X} \hookrightarrow \mathbb{Z} \), if \( \mathbb{X} \subset \mathbb{Z} \) and the identity mapping \( i: \mathbb{X} \to \mathbb{Z} \) is continuous. For example, \( L^\infty([0, T]; \mathbb{R}^n) \hookrightarrow L^1([0, T]; \mathbb{R}^n) \), and by using (4.4) we have that \( AC[0, T] \hookrightarrow L^\infty[0, T] \). It is straightforward to check by definition that if \( F: \overline{\mathbb{X}} \to \overline{\mathbb{Z}} \) is a (Fréchet) continuously differentiable mapping and if \( \mathbb{X} \hookrightarrow \overline{\mathbb{X}} \) and \( \mathbb{Z} \hookrightarrow \overline{\mathbb{Z}} \), then \( F: \mathbb{X} \to \mathbb{Y} \) is continuously differentiable as well.

**Theorem 4.2** (Fréchet differentiability of the constraint mapping). Let \( g: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) satisfy the assumptions imposed in (H4) around a given curve \( \overline{x}(\cdot) \in \mathbb{X} \). Then, the mapping \( f: \mathbb{X} \to L^1([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \) defined in (4.1) is continuously Fréchet differentiable around \( \overline{x} = \overline{x}(\cdot) \), and its Fréchet derivative operator at \( \overline{x} \) is calculated by the following formula, which is valid for a.e. \( t \in [0, T] \):

\[
\nabla f(\overline{x})u(t) = (\dot{u}(t) + \nabla_x g(t, \overline{x}(t))u(t), \ (u(0), u(T)) \text{ whenever } u(\cdot) \in \mathbb{X}.
\]
We are going to verify that $G$ is continuously differentiable around $\bar{x}(\cdot)$ in the Fréchet sense with its Fréchet derivative at $\bar{x}(\cdot)$ calculated by

$$\nabla G(\bar{x})u(t) = \nabla_x g(t, \bar{x}(t))u(t) \quad \text{a.e. } t \in [0, T] \quad (4.10)$$

for all $u(\cdot) \in X$. Consider the mapping $G^\infty : L^\infty([0, T]; \mathbb{R}^n) \to L^\infty([0, T]; \mathbb{R}^n)$ given by (4.9) but acting between different spaces in comparison with $G$. Also define the corresponding derivative mapping $D : L^\infty([0, T]; \mathbb{R}^n) \to L^\infty([0, T]; \mathbb{R}^n)$ by

$$D(x)(t) := \nabla_x g(t, x(t)) \quad \text{whenever } x(\cdot) \in L^\infty([0, T]; \mathbb{R}^n).$$

We deduce from (H4) that both $G^\infty$ and $D$ are well-defined for all $x(\cdot)$ near $\bar{x}(\cdot)$.

All of this ensures that the assumptions in [9, Theorem 7] are satisfied for the case where $p = q = \infty$ therein, and thus we get by the latter result that the above mapping $G^\infty$ is continuously Fréchet differentiable around $\bar{x}(\cdot)$. The aforementioned embeddings $L^\infty([0, T]; \mathbb{R}^n) \hookrightarrow L^1([0, T]; \mathbb{R}^n)$ and $\mathbb{X} \hookrightarrow L^\infty([0, T]; \mathbb{R}^n)$ combined with [9, Theorem 7] tell us therefore that the mapping $G$ from (4.9) is continuously Fréchet differentiable around $\bar{x}(\cdot)$ with its Fréchet derivative at $\bar{x}(\cdot)$ calculated by formula (4.10).

Considering further the constraint mapping $f : \mathbb{X} \to L^1([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ defined in (4.1) via $\dot{x}(\cdot)$, $g(\cdot, x(\cdot))$, and $(x(0), x(T))$. Observe that $x(\cdot) \mapsto \dot{x}(\cdot)$ is a bounded linear mapping from $\mathbb{X}$ to $L^1([0, T]; \mathbb{R}^n)$ due to the obvious inequality $\|\dot{x}\|_{L^1} \leq \|x\|_{ac}$. Thus it is continuously Fréchet differentiable together with the mapping $x(\cdot) \mapsto (x(0), x(T))$ as shown in the proof of Theorem 4.1. Combining all of this with the above result for the mapping $G$ from (4.9) tells us that the constraint mapping $f$ is Fréchet differentiable around $\bar{x}(\cdot)$ with its derivative at $\bar{x}(\cdot)$ calculated by (4.8). This completes the proof.

The last result of this section concerns the calculation of the normal cone $N_{\Theta}$ in the necessary optimality conditions of Theorem 2.2 for the convex set $\Theta$ defined in (4.1) via the initial data of the extended Bolza problem (3.3). In fact, the structure of the set $\Theta$ in (4.1) suggests that it suffices to calculate the normal cone to the set

$$\Theta_1 := \{ y \in L^1([0, T]; \mathbb{R}^n) \mid y(t) \in \Omega_1 \text{ a.e. } t \in [0, T] \}, \quad (4.11)$$

where $\Omega_1$ is a closed and convex subset of $\mathbb{R}^n$. Indeed, from the calculation of $N_{\Theta_1}$ we immediately come to the required formula for the normal cone to the set $\Theta$ in question by the elementary calculus rule for normals to set products in convex analysis.
Theorem 4.3 (normal cone calculation for the constraint set). Let \( \bar{y} = \bar{y}(\cdot) \in L^1([0, 1]; \mathbb{R}^n) \) be such that \( \bar{y}(t) \in \Omega_1 \), where the set \( \Omega_1 \subset \mathbb{R}^n \) is convex and locally closed around \( \bar{y}(t) \) for a.e. \( t \in [0, T] \). Then, we have the calculation formula

\[
N_{\Theta_1}(\bar{y}) = \left\{ p \in L^\infty([0, T]; \mathbb{R}^n) \mid p(t) \in N_{\Omega_1}(\bar{y}(t)) \text{ a.e. } t \in [0, T] \right\}
\] (4.12)

for the normal cone to the set \( \Theta_1 \) defined in (4.11).

Proof Pick any \( p \in L^\infty([0, T]; \mathbb{R}^n) \) such that \( p(t) \in N_{\Theta_1}(\bar{y}(t)) \) for a.e. \( t \in [0, T] \). Taking an arbitrary function \( y(\cdot) \in \Theta_1 \) and using the normal cone definition (2.2) we get that \( \langle p(t), y(t) - \bar{y}(t) \rangle \leq 0 \) for a.e. \( t \in [0, T] \). This leads us, by using the canonical pairing between \( L^1([0, T]; \mathbb{R}^n) \) and the dual space \( L^\infty([0, T]; \mathbb{R}^n) \), to

\[
\langle p, y - \bar{y} \rangle = \int_0^T \langle p(t), y(t) - \bar{y}(t) \rangle \, dt \leq 0,
\]

which yields \( p \in N_{\Theta_1}(\bar{y}) \). To verify the opposite inclusion, fix \( p \in N_{\Theta_1}(\bar{y}) \) and \( t \in (0, T) \). Taking a countable dense subset \( Q := \{c_i\}_{i=1}^\infty \) of \( \Omega_1 \), pick any \( c_i \in Q \) and choose \( r = r(t) > 0 \) to be so small that \( (t - r, t + r) \subset (0, T) \). Define now

\[
y^i : [0, T] \to \mathbb{R}^n \text{ by}
\]

and easily observe that \( y^i(\cdot) \in \Theta_1 \). Thus we get

\[
0 \geq \langle p, y^i - \bar{y} \rangle = \int_0^T \langle p(s), y^i(s) - \bar{y}(s) \rangle \, ds = \int_{t-r}^{t+r} \langle p(s), c_i - \bar{y}(s) \rangle \, ds.
\]

Then, basic real analysis tells us that

\[
\frac{1}{2r} \int_{t-r}^{t+r} \langle p(s), c_i - \bar{y}(s) \rangle \, ds \to \langle p(t), c_i - \bar{y}(t) \rangle \text{ as } r \downarrow 0
\]

for all \( t \in [0, T] \setminus S_i \) with \( S_i \) being of zero measure. This implies that for all \( i \) we have

\[
\langle p(t), c_i - \bar{y}(t) \rangle \leq 0 \text{ whenever } t \notin S_i.
\]

Consider further the set \( S := \bigcup_{i=0}^\infty S_i \), which is also of zero measure, where \( S_0 := \{t \in [0, T] \mid \bar{y}(t) \notin \Omega_1 \} \). Hence, we have

\[
\langle p(t), c_i - \bar{y}(t) \rangle \leq 0 \text{ for all } i \in \{1, 2, \ldots\} \text{ and } t \notin S.
\]

Since the set \( Q \) is dense in \( \Omega_1 \), it follows from the above that \( p(t) \in N_{\Omega_1}(\bar{y}(t)) \) for a.e. \( t \in [0, T] \), which completes the proof of the theorem. \( \square \)
5 Derivation of Necessary Conditions

Having in hand the above results supporting the reduction of the extended Bolza problem (3.3) to the nondynamic constrained optimization (2.4), we derive in this section the necessary optimality conditions for (3.3) formulated in Theorem 3.1 from those obtained in Theorem 2.2 for problem (2.4) in general normed spaces.

In our derivation, we need the following extended version of the fundamental lemma of the calculus of variations that we were not able to find in the literature.

**Lemma 5.1** (extended fundamental lemma of the calculus of variations). Let \(a, b \in \mathbb{R}^n\), and let \(l(\cdot), q(\cdot) \in L^1([0, T]; \mathbb{R}^n)\). Assume that

\[
\int_0^T \langle l(t), h(t) \rangle \, dt + \int_0^T \langle q(t), \dot{h}(t) \rangle \, dt + \langle h(0), a \rangle + \langle h(T), b \rangle = 0 \tag{5.1}
\]

for all \(h(\cdot) \in C^\infty([0, T]; \mathbb{R}^n)\). Then, there exists a unique function \(\overline{q}(\cdot) \in AC([0, T]; \mathbb{R}^n)\) such that \(q(t) = \overline{q}(t)\) for a.e. \(t \in [0, T]\) with

\[
\overline{q}(0) = a, \quad \overline{q}(T) = -b, \quad \text{and} \quad l(t) = \dot{\overline{q}}(t) \quad \text{a.e.} \ t \in [0, T].
\]

Moreover, the function \(\overline{q}(t)\) can be determined by

\[
\overline{q}(t) = \frac{d}{dt} \int_0^t q(s) \, ds \quad \text{whenever} \quad t \in [0, T]. \tag{5.2}
\]

**Proof** We may assume for simplicity that all the functions under consideration are real-valued (not vector-valued), because in the vector setting the same arguments can be applied to the components of these functions. To begin with, let us first use (5.1) for smooth functions \(h(\cdot)\) with \(h(0) = h(T) = 0\). Since \(l(\cdot) \in L^1([0, T]; \mathbb{R})\), the integral

\[
L(t) := \int_0^t l(s) \, ds
\]

is absolutely continuous on \([0, T]\). Integrating the first term in (5.1) by parts gives us

\[
\int_0^T \langle (q - L)(t), \dot{h}(t) \rangle \, dt = 0 \quad \text{for all such} \  h(\cdot).
\]

This implies that \(q(\cdot) - L(\cdot)\) is a constant function a.e. on \([0, T]\), which allows us to find a real number \(\gamma \in \mathbb{R}\) such that \(q(t) = L(t) + \gamma\) for a.e. \(t \in [0, T]\). Defining \(\overline{q}(t) := L(t) + \gamma\) for all \(t \in [0, T]\), we immediately get \(\overline{q}(\cdot) \in AC([0, T]; \mathbb{R})\) with \(\overline{q}(t) = l(t)\) and \(\overline{q}(t) = q(t)\) a.e. on \([0, T]\). The latter allows us to conclude that (5.1) holds for all \(h(\cdot) \in C^\infty([0, T]; \mathbb{R})\) if we replace \(q(\cdot)\) by \(\overline{q}(\cdot)\) therein. This leads us to the equality

\[
\langle h(T), (b + \overline{q}(T)) \rangle + \langle h(0), (a - \overline{q}(0)) \rangle = 0 \quad \text{for all} \  h(\cdot) \in C^\infty([0, T]; \mathbb{R}),
\]
which yields \( q(0) = a \) and \( q(T) = -b \). Observe that the integral mapping \( t \mapsto \int_0^t q(s) \, ds \) is differentiable by the fundamental theorem of calculus. Thus, for all \( t \in [0, T] \) we have

\[
q(t) = \frac{d}{dt} \int_0^t q(s) \, ds = \frac{d}{dt} \int_0^t q(s) \, ds,
\]

which verifies (5.2) and completes the proof of the lemma. □

It is worth mentioning that if the functions \( l(\cdot) \) and \( q(\cdot) \) satisfy the assumptions of Lemma 5.1, then any pointwise perturbations of them on a measure zero subset of \([0, T]\) also satisfy these assumptions. This tells us that there are many functions \( q(\cdot) \) satisfying (5.1), which are not absolutely continuous on \([0, T]\), and thus they differ from their (unique) absolute continuous representatives.

Now we are in a position to prove Theorem 3.1 by combining the above results on the reduction of the extended Bolza problem (3.1) to problem (2.4) of nondynamic optimization in normed spaces with the usage of some other tools of variational analysis.

**Proof of Theorem 3.1.** Fix any normed space \( \mathbb{X} \) of functions \( x : [0, T] \to \mathbb{R}^n \) satisfying the inclusions in (3.1), and let \( \mathbb{Y} := L^1([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \). As discussed in Section 4, the extended Bolza problem (3.3) can be written in the nondynamic form (2.4) of constrained optimization with \( J : \mathbb{X} \to \mathbb{R}, f : \mathbb{X} \to \mathbb{Y}, \) and \( \Theta \subset \mathbb{Y} \) defined in (4.1). Let us confirm that all the assumptions of Theorem 2.2 hold for the initial data (4.1) generated by the extended Bolza problem under the assumptions in (H1)–(H5). Indeed, the convexity of \( \Theta = \Theta_1 \times \Omega_2 \), with \( \Theta_1 \) defined in (4.11), immediately follows from the convexity of \( \Omega_1 \) and \( \Omega_2 \) imposed in (H3). The local closedness of \( \Theta \subset \mathbb{Y} \) around \( f(\mathbb{X}(\cdot)) \) follows from the closedness of \( \Omega_1, \Omega_2 \) in (H3) and the structure of \( \Theta \) in (4.1) due to the classical result of real analysis telling us that the (norm) convergence of the sequence in \( L^1([0, T]; \mathbb{R}^n) \) yields the a.e. convergence of a subsequence on \([0, T]\). The Gâteaux differentiability of the cost functional \( J \) at \( \mathbb{X} \) and its local Lipschitz continuity around this point under the assumptions in (H1) and (H2) follow from Theorem 4.1 applied to the \( \mathbb{X} \)-strong minimizer \( \mathbb{X} = \mathbb{X}(\cdot) \in \mathbb{X} \). Furthermore, the continuous Fréchet differentiability of the constraint mapping \( f \) from (4.1) is proved in Theorem 4.2 under the assumptions imposed in (H4) on the \( \mathbb{X} \)-strong local minimizer \( \mathbb{X}(\cdot) \) of (3.1).

To finish checking the assumptions of Theorem 2.2 with the data taken from (4.1), let us show that the imposed qualification condition in (H5) is equivalent to the metric subregularity constraint qualification (2.5) for the reduced problem (2.4) with the same modulus \( \kappa > 0 \). To proceed, pick \( x(\cdot) \in \mathbb{X} \) and get the equalities

\[
\text{dist}(\dot{x}(\cdot) + g(\cdot, x(\cdot)); \Theta_1) = \inf \left\{ \int_0^T \| \dot{x}(t) + g(t, x(t)) - y(t) \| \, dt \mid y(\cdot) \in L^1([0, T]; \mathbb{R}^n), y(t) \in \Omega_1 \text{ a.e. } t \in [0, T] \right\}
\]

\[
= \inf \left\{ \int_0^T \| \dot{x}(t) + g(t, x(t)) - y(t) \| \, dt \mid y(\cdot) \in L^1([0, T]; \mathbb{R}^n), y(t) \in \Omega_1 \text{ for all } t \in [0, T] \right\}
\]
\[
\begin{align*}
&= \inf_{y(\cdot) \in L^1([0,T];\mathbb{R}^n)} \int_0^T \|\dot{x}(t) + g(t, x(t)) - y(t) + \delta_{\Omega_1}(y(t))\| \, dt \\
&= \int_0^T \inf_{y \in \mathbb{R}^n} \|\dot{x}(t) + g(t, x(t)) - y + \delta_{\Omega_1}(y)\| \, dt \\
&= \int_0^T \inf_{y \in \mathcal{C}_1} \|\dot{x}(t) + g(t, x(t)) - y\| \, dt = \int_0^T \text{dist}\left(\dot{x}(t) + g(t, x(t)); \Omega_1\right) \, dt,
\end{align*}
\]

where we interchange the integral and infimum signs by using [17, Theorem 14.60]. This tells us that (H5) can be equivalently written as

\[
\text{dist}(x(\cdot); S) \leq \kappa \int_0^T \text{dist}\left(\dot{x}(t) + g(t, x(t)); \Omega_1\right) \, dt + \kappa \text{dist}\left((x(0), x(T)); \Omega_2\right) \\
\leq \kappa \text{dist}\left(\dot{x}(\cdot) + g(\cdot, x(\cdot)); \Theta_1\right) + \kappa \text{dist}\left((x(0), x(T)); \Omega_2\right) = \kappa \text{dist}\left(f(x(\cdot)); \Theta\right).
\]

Therefore, all the assumptions of Theorem 2.2 are satisfied for the nondynamic version (2.4) of the extended Bolza problem (3.3), and now we can apply to the local minimizer \( \bar{x} = \bar{x}(\cdot) \) of (2.4) the necessary optimality conditions (2.6) in the case where the data of (2.4) are given by (4.1). Note that the local minimizer \( \bar{x} \) of (2.4) corresponds to the \( X \)-strong local minimizer \( \bar{x}(\cdot) \) in the sense of (3.4).

According to (2.6) in our setting, there exists a multiplier \( \lambda(\cdot) = (\mu(\cdot), s_1, s_2) \in \mathbb{Y}^* = L^\infty([0, T]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \) such that for all \( u(\cdot) \in \bar{X} \) we have

\[
\begin{align*}
\nabla J(\bar{x}(\cdot))(u(\cdot)) + \{ \nabla f(\bar{x}(\cdot))(u(\cdot)), (\mu(\cdot), s_1, s_2)\} &= 0, \\
\mu(\cdot) &\in N_{\Omega_1}(\bar{x}(\cdot) + g(\cdot, \bar{x}(\cdot))), \text{ and } (s_1, s_2) \in N_{\Omega_2}(\bar{x}(0), \bar{x}(T)).
\end{align*}
\]

(5.3)

where the second line follows from the application to the product set \( \Theta = \Theta_1 \times \Omega_2 \) the normal cone product formula

\[
N_{\Theta}(y(\cdot), z) = N_{\Theta_1}(y(\cdot)) \times N_{\Omega_2}(z) \text{ for all } y(\cdot) \in \Theta_1 \text{ and } z \in \Omega_2.
\]

Using the calculation of the normal cone to \( \Theta_1 \) given in Theorem 4.3 tells us that the condition \( \mu(\cdot) \in N_{\Theta_1}(\bar{x}(\cdot) + g(\cdot, \bar{x}(\cdot))) \) is equivalent to

\[
\langle \mu(t), \dot{x}(t) + g(t, \bar{x}(t)) \rangle \geq \langle \mu(t), w \rangle \text{ for all } w \in \Omega_1 \text{ and a.e. } t \in [0, T].
\]

(5.4)

Furthermore, using the calculations of the Gâteaux derivative of \( J \) and the Frécher derivative of \( f \) obtained in Theorem 4.1 and Theorem 4.2, respectively, allows us to rewrite the first line of (5.3) in the form

\[
\begin{align*}
&\int_0^T \left( \left( \nabla_x \vartheta(t, \bar{x}(t), \dot{x}(t)) + \nabla_x g(t, \bar{x}(t))^* \mu(t), u(t) \right) + \left( \nabla_v \vartheta(t, \bar{x}(t), \dot{x}(t)) + \mu(t), \dot{u}(t) \right) \right) dt \\
&+ [u(0), s_1 + \nabla_x \varphi_1(\bar{x}(0), \bar{x}(T))] + [u(T), s_2 + \nabla_x \varphi(\bar{x}(0), \bar{x}(T))] = (0, 0).
\end{align*}
\]
Since the above relationships hold for all \( u(\cdot) \in \mathcal{X} \), appealing to Lemma (5.1) leads us to the Euler–Lagrange equation (3.6) together with the endpoint conditions

\[
p(0) = s_1 + \nabla_{x_0} \varphi(\bar{x}(0), \bar{x}(T)), \\
p(T) = -s_2 - \nabla_{x_T} \varphi(\bar{x}(0), \bar{x}(T)),
\]

where \( p : [0, T] \to \mathbb{R}^n \) is the unique absolutely continuous representative of the mapping \( t \mapsto \nabla_v \varphi(t, x(t), \dot{x}(t)) + \mu(t) \). Remembering that \((s_1, s_2) \in N_{\Omega_2}(\bar{x}(0), \bar{x}(T))\), the transversality inclusion (3.8) is implied by (5.5). Observe furthermore that

\[
\mu(t) = p(t) - \nabla_v \varphi(t, \bar{x}(t), \dot{\bar{x}}(t)) \text{ for a.e. } t \in [0, T].
\]

Replacing \( \mu(\cdot) \) in (5.4) by the latter expression, we arrive at the Weierstrass–Pontryagin maximization condition (3.7) and thus complete the proof of the theorem. □

**Remark 5.2 (quantitative relationships in optimality conditions).** The necessary optimality conditions of Theorem 3.1, while being based on the results of Theorem 2.2 for nondynamic constrained optimization in normed spaces, do not explore the novel quantitative condition of the latter theorem that gives us an efficient estimate of the multiplier \( \lambda \) in terms of the problem data of (2.4). Our intention is to utilize this estimate in deriving explicit qualitative relationships of this type for the extended Bolza problem (3.3). This seems to be important and implementable within our approach, but requires some technical work, which will be done in our future research.

Finally, we present an example that contains a simple class of variational problems, where all the assumptions of Theorem 3.1 are satisfied and the obtained necessary optimality conditions are explicitly formulated.

**Example 5.3 [illustrating optimality conditions]** We consider the following problem of the calculus of variations:

\[
\begin{align*}
\text{minimize} & \quad \varphi(x(0), x(T)) + \int_0^T \vartheta(t, x(t)) \, dt \\
\text{subject to} & \quad x(0) \in \Theta_1, \ x(T) \in \Theta_2, \\
& \quad x(\cdot) \in AC([0, T], \mathbb{R}^n),
\end{align*}
\]

where \( \varphi \) and \( \vartheta \) are continuously differentiable, thus they satisfy assumption (H1) and (H2) for all \( x \in AC([0, T], \mathbb{R}^n) \). Further, we assume \( \Theta_1 \) and \( \Theta_2 \) are polyhedral convex sets. It is easy to see that the set of feasible solutions to this problem is also polyhedral in \( AC([0, T], \mathbb{R}^n) \). Using an appropriate infinite-dimensional extension of the seminar Hoffman’s lemma from [4, Theorem 6a] tells us that assumption (H5) holds for \( \Omega_2 := \Theta_1 \times \Theta_2 \). Then the necessary optimality conditions of Theorem 3.1 ensure the existence of a dual arc \( p(\cdot) \in AC([0, T], \mathbb{R}^n) \) satisfying the following relationships:

The **Euler–Lagrange equation** for a.e. \( t \in [0, T] \):

\[
\dot{p}(t) = \vartheta_x(t, x(t)).
\]
The transversality inclusion:

$$(p(0), -p(T)) \in \nabla \varphi(x(0), x(T)) + N_{\Omega_2}(x(0), x(T)).$$

The Euler-Lagrange equation reads as $p(t) = \int_0^t \vartheta_x(s, x(s)) ds + p(0)$ for all $t \in [0, T]$. Plugging there $t = T$, we get the equality

$$p(0) - p(T) = -\int_0^T \vartheta_x(t, x(t)) dt.$$ 

Employing finally the transversality inclusion together with the fact that $N_{\Omega_2}(\overline{x}(0), \overline{x}(T)) = N_{\Omega_1}(\overline{x}(0)) \times N_{\Omega_2}(\overline{x}(T))$ confirm that any minimizer $\overline{x}(\cdot)$ of the above problem has to satisfy the following explicit condition

$$\int_0^T \vartheta_x(t, \overline{x}(t)) dt + \nabla_{x_0} \varphi(\overline{x}(0), \overline{x}(T)) + \nabla_{x_T} \varphi(\overline{x}(0), \overline{x}(T)) \in -N_{\Omega_1}(\overline{x}(0)) - N_{\Omega_2}(\overline{x}(T))$$

which allows us to eliminate nonoptimal solutions and eventually calculate local minimizers for specified initial data of the problem under consideration.

### 6 Conclusions

This paper develops a new approach to study problems of dynamic optimization in generally incomplete spaces by reducing them to nondynamic problems of constrained optimization in normed spaces with subsequent applications of the refined necessary optimality conditions recently obtained [11]. We implement here this approach to deriving first-order necessary optimality conditions in the extended Bolza problem of the calculus of variations with pointwise velocity constraints depending on current state positions. In contrast to the previously used methods of modern variational analysis that require the completeness of spaces of feasible solutions, we are now able to deal with both complete space frameworks as well as with incomplete spaces of type $C^k$ for $k = 1, 2, \ldots$. Investigating strong local minimizers in the spaces under consideration, we confined ourselves for simplicity to variational problems with smooth data although suitable constructions of generalized differentiation in normed spaces are used even in such settings. Proceeding in this way, we derive necessary optimality conditions for $X$-strong local minimizers of the extended Bolza problem in generally incomplete spaces that contain the appropriate Euler–Lagrange, Weierstrass–Pontryagin, and transversality relations.

In our future research, we plan to develop this approach with covering nonsmooth and nonconvex extended Bolza problems as well as optimal control problems for constrained differential inclusions. One of the important novel features of this approach is the possibility to obtain some quantitative estimates for adjoint functions as discussed in Remark 5.2. We also plan to implement this approach to deriving second-order optimality conditions for variational problems by extending to infinite dimensions and further developing the recent results of second-order variational analysis achieved in [12,13].
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