EIGENPOLYTOPES, SPECTRAL POLYTOPES AND EDGE-TRANSITIVITY

MARTIN WINTER

Abstract. Starting from a finite simple graph $G$, for each eigenvalue $\theta$ of its adjacency matrix one can construct a convex polytope $P_G(\theta)$, the so called $\theta$-eigenpolytope of $G$. For some polytopes this technique can be used to reconstruct the polytopes from its edge-graph. Such polytopes (we shall call them spectral) are still badly understood. We give an overview of the literature for eigenpolytopes and spectral polytopes.

We introduce a geometric condition by which to prove that a given polytope is spectral (more exactly, $\theta_2$-spectral). We apply this criterion to the edge-transitive polytopes. We show that every edge-transitive polytope is $\theta_2$-spectral, is uniquely determined by this graph, and realizes all its symmetries. We give a complete classification of distance-transitive polytopes.

1. Introduction

Eigenpolytopes are a construction in the intersection of combinatorics and geometry, using techniques from spectral graph theory. Eigenpolytopes provide a way to associate several polytopes to a finite simple graph, one for each eigenvalue of its adjacency matrix. A formal definition can be found in Section 2.2.

Eigenpolytopes can be applied from two directions: for the first, one starts from a given graph, computes its eigenpolytopes, and tries to deduce, from the geometry and combinatorics of these polytopes, something about the original graph. For the other direction, one starts with a polytope, asks whether it is an eigenpolytope, and if so, for which graphs, which eigenvalues, and how these relate to the original polytope. Eigenpolytopes have several interesting geometric and algebraic properties, and establishing that a family of polytopes consists of eigenpolytopes opens up their study to the techniques of spectral graph theory.

For some graphs the connection to their eigenpolytopes is especially strong: it can happen that a graph is the edge-graph of one of its eigenpolytopes, or equivalently, that a polytope is an eigenpolytope of its edge-graph. Such graphs/polytopes are

Date: September 7, 2020.

2010 Mathematics Subject Classification. 51M20, 52B05, 52B11, 52B12, 52B15, 05C50, 05C62.

Key words and phrases. Eigenpolytopes, spectral polytopes, edge-transitive polytopes, spectral graph realization.
quite special and we shall call them spectral. For example, all regular polytopes are spectral, but there are many others. Their properties are not well-understood.

We survey the literature of eigenpolytope and spectral polytopes. We establish a technique with which to prove that certain polytopes are spectral polytopes and we apply it to edge-transitive polytopes. That are polytopes for which the Euclidean symmetry group Aut(\(P\)) \(\subset\) O(\(R^d\)) acts transitively on the set of edge \(\mathcal{F}_1(P)\). As we shall explain, this characterization suffices to proves that an edge-transitive polytope is uniquely determined by its edge-graph, and also realizes all its combinatorial symmetries. A complete classification of edge-transitive polytopes is not known as of yet. However, using results on eigenpolytopes, we are able to give a complete classification of a sub-class of the edge-transitive polytopes, namely, the distance-transitive polytopes.

1.1. Outline of the paper. Section 2 starts with a motivating example for directing the reader towards the definition of the eigenpolytope as well as the phenomenon of spectral graphs and polytopes. We include a literature overview for eigenpolytopes and spectral polytopes.

In Section 3 we give a first rigorous definition for the notion “spectral polytope” via balanced polytopes. The latter is a notion related to the rigidity theory.

In Section 4 we introduce the, as of yet, most powerful tool for proving that certain polytopes are spectral.

In the final section, Section 5, we apply this result to edge-transitive polytopes. It is a simple corollary of the previous section that these are \(\theta_2\)-spectral. We explore the implications of this finding: edge-transitive polytopes (in dimension \(d \geq 4\)) are uniquely determined by the edge-graph and realize all of its symmetries. We discuss sub-classes, such as the arc-, half- and distance-transitive polytopes. We close with a complete classification of the latter (based on a result of Godsil).

2. Eigenpolytopes and spectral polytopes

2.1. A motivating example. Let \(G = (V, E)\) be the edge-graph of the cube, with vertex set \(V = \{1, ..., 8\}\), numbers assigned to the vertices as in the figure below.

![Graph of the cube](image)

The spectrum of that graph (i.e., of its adjacency matrix) is \((-3)^1, (-1)^3, 1^3, 3^1\). Most often, one denotes the largest eigenvalue by \(\theta_1\), the second-largest by \(\theta_2\), and so on. In spectral graph theory, there exists the general rule of thumb that the most exciting eigenvalue of a graph is not its largest, but its second-largest eigenvalue \(\theta_2\) (which is related to the algebraic connectivity of \(G\)).

For the edge-graph of the cube, we have \(\theta_2 = 1\), of multiplicity three. And here are three linearly independent eigenvectors to \(\theta_2\):
We can write these more compactly in a single matrix $\Phi \in \mathbb{R}^{8 \times 3}$:

$$
\Phi = 
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & 1 \\
1 & -1 & -1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & -1 & -1
\end{pmatrix}
$$

We now take a look at the rows of that matrix, of which it has exactly eight. These rows are naturally assigned to the vertices of $G$ (assign $i \in V$ to the $i$-th row of $\Phi$), and each row can be interpreted as a vector in $\mathbb{R}^3$.

If we place each vertex $i \in V$ at the position $v_i \in \mathbb{R}^3$ given by the $i$-th row of $\Phi$, we find that this embeds the graph $G$ exactly as the skeleton of a cube (see the figure above). In other words: if we compute the convex hull of the $v_i$, we get back the polyhedron from which we have started. What a coincidence, isn’t it?

This example was specifically chosen for its nice numbers, but in fact, the same works out as well for many other polytopes, including all the regular polytopes in all dimension. One probably learns to appreciate this magic when suddenly in need for the vertex coordinates of some not so nice polytope, say, the regular dodecahedron or 120-cell. With this technique in the toolbox, these coordinates are just one eigenvector-computation away (we included a short Mathematica script in Appendix A). Note also, that we never specified the dimension of the embedding, but it just so happened, that the second-largest eigenvalue has the right multiplicity. This phenomenon definitely deserves an explanation.

On the choice of eigenvectors. One might object that the chosen eigenvectors $u_1, u_2$ and $u_3$ look suspiciously cherry-picked, and we may not get such a nice result if we would have chosen just any eigenvectors. And this is true. For an appropriate choice of these vectors, we can, instead of a cube, get a cuboid, or a parallelepiped.

In fact, we can obtain any linear transformations of the cube. But, we can also get only linear transformations, and nothing else. The reason is the following well know fact from linear algebra:

**Theorem 2.1.** Two matrices $\Phi, \Psi \in \mathbb{R}^{n \times d}$ have the same column span, i.e., $\text{span } \Phi = \text{span } \Psi$, if and only if their rows are related by an invertible linear transformation, i.e., $\Phi = \Psi T$ for some $T \in \text{GL}(\mathbb{R}^d)$.
In our case, the column span is the $\theta_2$-eigenspace, and the rows are the coordinates of the $v_i$. We say that any two polytopes constructed in this way are \textit{linearly equivalent}.

The only notable property of the chosen basis in the example is, that the vectors $u_1, u_2$ and $u_3$ are orthogonal and of the same length. Any other choice of such a basis of the $\theta_2$-eigenspace (\textit{e.g.} an orthonormal basis) would also have given a cube, but reoriented, rescaled and probably with less nice coordinates. For details on how this choice relates to the orientation, see \textit{e.g.} [21, Theorem 3.2].

### 2.2. Eigenpolytopes

We compile our example into a definition.

**Definition 2.2.** Start with a graph $G = (V, E)$, an eigenvalue $\theta \in \text{Spec}(G)$ thereof, as well as an orthonormal basis $\{u_1, ..., u_d\} \subset \mathbb{R}^n$ of the $\theta$-eigenspace. We define the \textit{eigenpolytope matrix} $\Phi \in \mathbb{R}^{n \times d}$ as the matrix in which the $u_i$ are the columns:

\begin{equation}
\Phi := \begin{pmatrix}
| & | \\
u_1 & \cdots & u_d \\
| & | \\
\end{pmatrix} = \begin{pmatrix}
\vdots \\
v_1^T \\
\vdots \\
v_n^T \\
\end{pmatrix}.
\end{equation}

Let $v_i \in \mathbb{R}^d$ denote the $i$-th row of $\Phi$. The polytope

$$P_G(\theta) := \text{conv}\{v_i \mid i \in V\} \subset \mathbb{R}^d$$

is called $\theta$-\textit{eigenpolytope} (or just eigenpolytope) of $G$.

For later use we define the \textit{eigenpolytope map}

\begin{equation}
\phi : V \ni i \mapsto v_i \in \mathbb{R}^d
\end{equation}

that to each vertex $i \in V$ assigns the $i$-th row of the eigenpolytope matrix.

Note that the basis $\{u_1, ..., u_d\} \subset \text{Eig}_G(\theta)$ in Definition 2.2 is explicitly chosen to be an \textit{orthonormal basis}. This is not strictly necessary, but this choice is convenient from a geometric point of view: a different choice for this basis gives the same polytope, but with a different orientation rather than, say, transformed by a general linear transformation. This preserves metric properties and is closer to how polytopes are usually consider up to rigid motions. We can also reasonably speak of the $\theta$-eigenpolytope, as any two differ only by orientation.

With this terminology in place, our observation in the example of Section 2.1 can be summarized as “the cube is the $\theta_2$-eigenpolytope of its edge-graph”, or alternatively as “the cube-graph is the edge-graph of its $\theta_2$-eigenpolytope”. Here is a depiction of all the eigenpolytopes of the cube-graph, one for each eigenvalue:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{cube_graph_eigenpolytopes.png}
\caption{Eigenpolytopes of the cube-graph.}
\end{figure}

We observe that the phenomenon from Section 2.1 only happens for $\theta_2$. In general, the $\theta_1$-eigenpolytope of a regular graph will always be a single point (which is, why we rarely care about the largest eigenvalue). Also, whenever a graph is bipartite, the eigenpolytope to the smallest eigenvalue is 1-dimensional, hence a line segment.

We are now free to compute the eigenpolytopes of all kinds of graphs, including graphs which are not the edge-graph of any polytope (so-called \textit{non-polytopal} graphs).
It is then little surprising that no edge-graph of any of its eigenpolytope gives the original graph again.

But even if we start from a polytopal graph, one is not guaranteed to find an eigenpolytope that has the initial graph as its edge-graph (e.g. the edge-graph of the triangular prism has no eigenvalue of multiplicity three, hence no eigenpolytope of dimension three, see also Example 3.4). Equivalently, if one starts with a polytope, it is not guaranteed that this polytope is the eigenpolytope of its edge-graph (or even combinatorially equivalent to it).

Example 2.3. A neighborly polytope is a polytope whose edge-graph is the complete graph $K_n$. The spectrum of $K_n$ is $\{(-1)^{n-1}, (n-1)\}$. One checks that the eigenpolytopes are a single point (for $\theta_1 = n-1$) and the regular simplex of dimension $n-1$ (for $\theta_2 = -1$).

Consequently, no neighborly polytope other than a simplex is combinatorially equivalent to an eigenpolytope of its edge-graph.

That a graph and its eigenpolytope translate into each other as well as in the case of the cube in Section 2.1 is a very special phenomenon, to which we shall give a name: a polytope (or graph) for which this happens, will be called spectral\(^1\). We cannot formalize this definition right away, as there is some subtlety we have to discuss first (we give a formal definition in Section 3, see Definition 3.5).

Example 2.4. The image below shows two spectral realizations of the 5-cycle $C_5$\(^2\).

The left image shows the realization to the second-largest eigenvalue $\theta_2$, the right image shows the realization to the smallest eigenvalue $\theta_3$. In both cases, the convex hull (the actual eigenpolytope) is a regular pentagon, whose edge-graph is $C_5$ again. But we see that only in the case of $\theta_2$ the edges of the graphs get properly mapped into the edges of the pentagon.

While it is true that the 5-cycle $C_5$ is the edge-graph of its $\theta_3$-eigenpolytope, the adjacency informations gets scrambled in the process: while, say, vertex 1 and 2 are adjacent in $C_5$, their images $v_1$ and $v_2$ do not form an edge in the $\theta_3$-eigenpolytope.

We do not want to call this “spectral”, as the adjacency information is not preserved.

The same can happen in higher dimensions too, e.g. with $G$ being the edge-graph of the dodecahedron:

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\(^1\)There was at least one previous attempt to give a name to this phenomenon, namely, in [14], where it was called self-reproducing.

\(^2\)Spectral realizations are essentially defined like eigenpolytopes, assigning coordinates $v_i \in \mathbb{R}^d$ to each vertex $i \in V$ (as in Definition 2.2), but without taking the convex hull. Instead, one draws the edges between adjacent vertices.
Observation 2.5. From studying many examples, there are two interesting observations to be made, both concern $\theta_2$, none of which is rigorously proven:

(i) It appears as if only $\theta_2$ can give rise to spectral polytopes/graphs. At least, all known examples are $\theta_2$-spectral (see also Question 6.2). Some considerations on nodal domains make this plausible, but no proof is known in the general case (a proof is known in certain special cases, see Theorem 5.7).

(ii) If $i \in V$ is a vertex of $G$, then $v_i$ is not necessarily a vertex of every eigenpolytope ($v_i$ might end up in the interior of $P_G(\theta)$ or one of its faces). And even if $v_i, v_j \in F_0(P_G(\theta))$ are distinct vertices and $ij \in E$ is an edge of $G$, it is still not necessarily true that $\text{conv}\{v_i, v_j\}$ is also an edge of the eigenpolytope (as seen in Example 2.4).

However, this seems to be no concern in the case $\theta_2$. It appears as if all edges of $G$ become edges of the $\theta_2$-eigenpolytope, even if $G$ is not spectral (under mild assumptions on the end vertices of the edge). In other words, the adjacency information of $G$ gets imprinted on the edge-graph of the $\theta_2$-eigenpolytope, whether $G$ is spectral or not. This is known to be true only in the case of distance-regular graphs [10, Theorem 3.3 (b)], but unproven in general (see also Question 6.3).

2.3. Literature. Eigenpolytope were first introduced by Godsil [9] in 1978. Godsil proved the existence of a group homomorphism $\text{Aut}(G) \to \text{Aut}(P_G(\theta))$, i.e., any combinatorial symmetry of the graph translates into a Euclidean symmetry of the polytope. From that, he deduces results about the combinatorial symmetry group of the original graph.

We say more about the group homomorphism: for every $\theta \in \text{Spec}(G)$ we have

\textbf{Theorem 2.6} ([9], Theorem 2.2). If $\sigma \in \text{Aut}(G) \subseteq \text{Sym}(n)$ is a symmetry of $G$, and $\Pi_\sigma \in \text{Perm}(\mathbb{R}^n)$ is the associated permutation matrix, then

$$T_\sigma := \Phi^T \Pi_\sigma \Phi \in \text{O}(\mathbb{R}^d),$$

($\Phi$ is the eigenpolytope matrix)

is a Euclidean symmetry of the eigenpolytope $P_G(\theta)$ that also permutes the $v_i$ as prescribed by $\sigma$, i.e., $T_\sigma \circ \phi = \phi \circ \sigma$, or $T_\sigma v_i = v_{\sigma(i)}$ for all $i \in V$.

This result is also proven (more generally for spectral graph realizations) in [23, Corollary 2.9].

\textbf{Theorem 2.6} explicitly uses that eigenpolytopes are defined using an orthonormal bases rather than any basis of the eigenspace, to conclude that the symmetries $T_\sigma$ are orthogonal matrices. Also, the statement of \textbf{Theorem 2.6} is not too satisfying in general, as it can happen that non-trivial symmetries of $G$ are mapped to the identity transformation. We not necessarily have $\text{Aut}(G) \cong \text{Aut}(P_G(\theta))$. 
Several authors construct the eigenpolytopes of certain famous graphs or graph families. Powers [18] computed the eigenpolytopes of the Petersen graph, which he termed the Petersen polytopes (one of which will appear as a distance-transitive polytope in Section 5.4). The same author also investigates eigenpolytopes of general distance-regular graphs in [19]. In [15], Mohri described the face structure of the Hamming polytopes, the $\theta_2$-eigenpolytopes of the Hamming graphs. Seemingly unknown to the author, these polytopes can also be described as the cartesian powers of regular simplices (also distance transitive, see Section 5.4).

There exists a wonderful enumeration of the eigenpolytopes (actually, spectral realizations) of the edge-graphs of all uniform polyhedra in [3]. Sadly, this write-up was never published formally. This provides empirical evidence that every uniform polyhedron has a spectral realization. The same question might then be asked for uniform polytopes in higher dimensions.

Rooney [20] used the combinatorial structure of the eigenpolytope (the size of their facets) to deduce statements about the size of cocliques in a graph.

In [16], the authors investigate how common graph operations translate to operations on their eigenpolytopes.

Particular attention was given to the eigenpolytopes of distance-regular graphs [8,10,19]. It was shown that in a $\theta_2$-eigenpolytope of a distance-regular graph $G$, every edge of $G$ corresponds to an edge of the eigenpolytope [10]. Consequently, $G$ is a spanning subgraph of the edge-graph of the eigenpolytope. It remains open if the same holds for less regular graphs, e.g. 1-walk regular graphs or arc-transitive graphs (see also Question 6.3).

The observation that some polytopes are the eigenpolytopes of their edge-graph (i.e., they are spectral in our terminology) was made repeatedly, e.g. in [8] and [14]. In the latter, this was shown for all regular polytopes, excluding the exceptional 4-dimensional polytopes, the 24-cell, 120-cell and 600-cell. This gap was filled in [23] via general considerations concerning spectral realizations of arc-transitive graphs.

In sum, all regular polytopes are known to be $\theta_2$-spectral.

The next major result for spectral polytopes was obtained by Godsil in [10], where he was able to classify all $\theta_2$-spectral distance-regular graphs (see also Section 5.4):

**Theorem 2.7** ([10], Theorem 4.3). Let $G$ be distance-regular. If $G$ is $\theta_2$-spectral, then $G$ is one of the following:

(i) a cycle graph $C_n$, $n \geq 3$,
(ii) the edge-graph of the dodecahedron,
(iii) the edge-graph of the icosahedron,
(iv) the complement of a disjoint union of edges,
(v) a Johnson graph $J(n,k)$,
(vi) a Hamming graph $H(d,q)$,
(vii) a halved $n$-cube $\frac{1}{2}Q_n$,
(viii) the Schl"{a}fli graph, or
(ix) the Gosset graph.

A second look at this list reveals a remarkable “coincidence”: while the generic distance-regular graph has few or no symmetries, all the graphs in this list are highly symmetric, in fact, distance-transitive (a definition will be given in Section 5.4).

It is a widely open question whether being spectral is a property solely reserved for highly symmetric graphs and polytopes (see also Question 6.4). There is only
3. BALANCED AND SPECTRAL POLYTOPES

In this section we give a second approach to spectral polytopes that circumvents the mentioned subtleties.

For the rest of the paper, let $P \subset \mathbb{R}^d$ denote a full-dimensional polytope in dimension $d \geq 2$ with vertices $v_1, ..., v_n \in F_0(P)$. We distinguish the skeleton of $P$, which is the graph with vertex set $F_0(P)$ and edge set $F_1(P)$, from the edge-graph $G_P = (V, E)$ of $P$, which is isomorphic to the skeleton, but has vertex set $V = \{1, ..., n\}$. The isomorphism will be denoted

$$\psi : V \ni i \mapsto v_i \in F_0(P),$$

and we call it the skeleton map.

3.1. Balanced polytopes.

Definition 3.1. The polytope $P$ is called $\theta$-balanced (or just balanced) for some real number $\theta \in \mathbb{R}$, if

$$\sum_{j \in N(i)} v_j = \theta v_i,$$

for all $i \in V$,

where $N(i) := \{ j \in V \mid ij \in E \}$ denotes the neighborhood of a vertex $i \in V$.

One way to interpret the balancing condition (3.2) is as a kind of self-stress condition on the skeleton of $P$ (the term “balanced” is motivated from this). For each edge $ij \in E$, the vector $v_j - v_i$ is parallel to the edge $\text{conv}\{v_i, v_j\}$. If $P$ is $\theta$-balanced, at each vertex $i \in V$ we have the equation

$$\sum_{j \in N(i)} (v_j - v_i) = \sum_{j \in N(i)} v_j - \deg(i)v_i = (\theta - \deg(i))v_i.$$

This equation can be interpreted as two forces that cancel each other out: on the left, a contracting force along each edge (proportion only to the length of that edge), and on the right, a force repelling each vertex away from the origin (proportional to the distance of that vertex from the origin, and proportional to $\theta - \deg(i)$).

A second interpretation of (3.2) is via spectral graph theory. Define the matrix

$$\Psi := \begin{pmatrix} v_1^\top \\ \vdots \\ v_n^\top \end{pmatrix}$$

in which the $v_i$ are the rows. This matrix will be called the arrangement matrix of $P$. Note that the skeleton map $\psi$ assigns $i \in V$ to the $i$-th row of $\Psi$. Since we use that $P \subset \mathbb{R}^d$ is full-dimensional, we have rank $\Psi = d$.

Observation 3.2. Suppose that $P$ is $\theta$-balanced. The defining equation (3.2) can be equivalently written as the matrix equation $A\Psi = \theta \Psi$. In this form, it is apparent that $\theta$ is an eigenvalue of the adjacency matrix $A$, and the columns of $\Psi$ are $\theta$-eigenvectors, or span $\Psi \subseteq \text{Eig}_{G_P}(\theta)$.

We have seen that for a balanced polytope, the columns of $\Psi$ must be eigenvectors. But they are not necessarily a complete set of $\theta$-eigenvectors, i.e., they not necessarily span the whole eigenspace.
Example 3.3. Every centered neighborly polytope $P$ is balanced, but except if it is a simplex, it is not spectral (the latter was shown in Example 2.3). Centered means that
\[ \sum_{i \in V} v_i = 0. \]
Since $P$ is neighborly, we have $G_P = K_n$ and $N(i) = V \setminus \{i\}$ for all $i \in V$. Therefore
\[ \sum_{j \in N(i)} v_j = \sum_{j \in V} v_j - v_i = -v_i, \quad \text{for all } i \in V. \]
And indeed, $K_n$ has spectrum $\{(-1)^{n-1}, (n-1)^1\}$. So $P$ is $(-1)$-balanced.

The last example shows that every neighborly polytopes can be made balanced by merely translating it. More generally, many polytopes have a realization (of their combinatorial type) that is balanced. But other polytopes do not:

Example 3.4. Let $P \subset \mathbb{R}^3$ be a triangular prism.

The spectrum of the edge-graph of $P$ is $\{(-2)^2, 0^2, 1^1, 3^1\}$. Note that there is no eigenvalue of multiplicity greater than two. In particular, we cannot choose three linearly independent eigenvectors to a common eigenvalue. But if $P$ were balanced, then Observation 3.2 tells us that the columns of the arrangement matrix $\Psi$ would be three eigenvectors to the same eigenvalue (linearly independent, since rank $\Psi = 3$), which is not possible. And so, no realization of $P$ can be balanced.

3.2. Spectral graphs and polytopes. In the extreme case, when the columns of $\Psi$ span the whole eigenspace, we can finally give a compact definition of what we want to consider as spectral:

Definition 3.5.

(i) A polytope $P$ is called $\theta$-spectral (or just spectral), if its arrangement matrix $\Psi$ satisfies $\text{span } \Psi = \text{Eig } G_P(\theta)$.

(ii) A graph is said to be $\theta$-spectral (or just spectral) if it is (isomorphic to) the edge-graph of a $\theta$-spectral polytope.

This definition is now perfectly compatible with our initial motivation for the term “spectral” in Section 2.2.

Lemma 3.6.

(i) If a polytope $P$ is $\theta$-spectral, then $P$ is linearly equivalent to the $\theta$-eigendynlotope of its edge-graph (see also Proposition 3.7).

(ii) If a graph $G$ is $\theta$-spectral, then $G$ is (isomorphic to) the edge-graph of its $\theta$-eigendynlotope (see also Proposition 3.8).

In both cases, the converse is not true. This is intentional, to avoid the problems mentioned in Example 2.4. Both statement will be proven below by formulating a more technical condition that is then actually equivalent to being spectral.

Proposition 3.7. A polytope $P$ is $\theta$-spectral if and only if it is linearly equivalent to the $\theta$-eigendynlotope of its edge-graph via some linear map $T \in \text{GL}(\mathbb{R}^d)$ for which the following diagram commutes:

\[
\begin{array}{c}
P \xrightarrow{T} P_{G_P}(\theta) \\
\downarrow_{\psi} \downarrow_{\phi} \\
G_P
\end{array}
\]
where φ and ψ denote the eigenpolytope map and skeleton map respectively.

Proof. By definition, the θ-eigenpolytope of GP satisfies span Φ = EigGP(θ), where Φ is the corresponding eigenpolytope matrix.

Now, by definition, P is θ-spectral if and only if span Ψ = EigGP(θ), where Ψ is its arrangement matrix. But by Theorem 2.1, Φ and Ψ have the same span if and only of their rows are related by some invertible linear map T ∈ GL(Rd), that is, ΨT = Φ, or T ◦ ψ = φ. The latter expresses exactly that (3.4) commutes. □

This also proves Lemma 3.6 (i).

Proposition 3.8. A graph G is θ-spectral if and only if the eigenpolytope map φ: V(G) → Rd provides an isomorphism between G and the skeleton of its θ-eigenpolytope PG(θ).

Proof. Suppose first that G is θ-spectral. Then there is a θ-spectral polytope Q with edge-graph GQ = G and skeleton map ψ: V(GQ) → F0(Q). By Lemma 3.6 (i), Q is linearly equivalent to PG(θ) via some linear map T ∈ GL(Rd). By Proposition 3.7, the eigenpolytope map satisfies φ = T ◦ ψ. Since T induces an isomorphism between the skeleta of Q and PG(θ), and ψ is an isomorphism between G and the skeleton of Q, we find that φ must be an isomorphism between G and the skeleton of PG(θ).

This shows one direction.

For the converse, suppose that φ is an isomorphism. Set P := PG(θ) and let GP be its edge-graph with skeleton map ψ: V(GP) → F0(P). Then σ := ψ −1 ◦ φ is a graph isomorphism between G and GP. So, since G ∼= GP, each eigenpolytope of G is also an eigenpolytope of GP. We can therefore choose PG,θ = PG(θ), with corresponding eigenpolytope map φ′ := σ −1 ◦ φ. In sum, the outer square in the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\sigma} & GP \\
\downarrow{\phi} & & \downarrow{\phi'} \\
P := PG(\theta) & \xrightarrow{\text{Id}} & PG,\theta(\theta)
\end{array}
\]

Also, by construction of σ, the upper triangle commutes. In conclusion, the lower triangle must commute as well, which is exactly (3.4) with T = Id. This proves that P is θ-spectral via Proposition 3.7. Since G is isomorphic to GP, G is θ-spectral. □

This also proves Lemma 3.6 (ii).

It is also possible to give a definition of spectral graphs purely in terms of graph theory, without any explicit reference to polytopes:

Lemma 3.9. A graph G is θ-spectral if and only if it satisfies both of the following:

(i) for each vertex i ∈ V exists a θ-eigenvector u = (u1, ..., un) ∈ EigG(θ) whose single largest component is ui, or equivalently,

\[\text{Argmax}_{k \in V} u_k = \{i\}.\]

(ii) any two vertices i, j ∈ V form an edge ij ∈ E in G if and only if there is a θ-eigenvector u = (u1, ..., un) ∈ EigG(θ) whose only two largest components are ui and uj, or equivalently,

\[\text{Argmax}_{k \in V} u_k = \{i, j\}.\]
This characterization of spectral graphs can be interpreted as follows: a spectral graph can be reconstructed from knowing a single eigenspace, rather than, say, all eigenspaces and their associated eigenvalues.

**Proof of Lemma 3.9.** Let \( P_G(\theta) \subset \mathbb{R}^d \) be the \( \theta \)-eigenpolytope of \( G \) with eigenpolytope matrix \( \Phi \) and eigenpolytope map \( \phi: V \ni i \mapsto v_i \in \mathbb{R}^d \).

Since \( \text{span} \Phi = \text{Eig}_G(\theta) \), the eigenvectors \( u = (u_1, \ldots, u_n) \in \text{Eig}_G(\theta) \) are exactly the vectors that can be written as \( u = \Phi x \) for some \( x \in \mathbb{R}^d \). If then \( e_k \in \mathbb{R}^n \) denotes the \( k \)-th standard basis vector, we have

\[
 u_k = \langle u, e_k \rangle = \langle \Phi x, e_k \rangle = \langle x, \Phi^\top e_k \rangle = \langle x, e_k \rangle.
\]

Therefore, there is a \( \theta \)-eigenvector \( u = (u_1, \ldots, u_n) \in \text{Eig}_G(\theta) \) with \( \text{Argmax}_{k \in V} u_k = \{i_1, \ldots, i_m\} \) if and only if there is a vector \( x \in \mathbb{R}^d \) with

\[
 \text{Argmax}_{k \in V} \langle x, v_k \rangle = \{i_1, \ldots, i_m\}.
\]

But this last line is exactly what it means for \( \text{conv}\{v_1, \ldots, v_{m}\} \) to be a face of \( P_G(\theta) = \text{conv}\{v_1, \ldots, v_n\} \) (and \( x \) is a normal vector of that face).

In this light, we can interpret (i) as stating that \( v_1, \ldots, v_n \) form \( n \) distinct vertices of \( P_G(\theta) \), and (ii) as stating that \( \text{conv}\{v_i, v_j\} \) is an edge of \( P_G(\theta) \) if and only if \( ij \in E \). And this means exactly that \( \phi \) is a graph isomorphism between \( G \) and the skeleton of \( P_G(\theta) \). By Proposition 3.8, this is equivalent to \( G \) being \( \theta \)-spectral. \( \square \)

In practice, to reconstruct a spectral graph from an eigenspace, the steps could be the following: given a subspace \( U \subset \mathbb{R}^n \) (the claimed eigenspace), then

(i) choose any basis \( u_1, \ldots, u_d \in \mathbb{R}^n \) of \( U \),
(ii) build the matrix \( \Phi = (u_1, \ldots, u_d) \in \mathbb{R}^{n \times d} \) in which the \( u_i \) are the columns,
(iii) define \( v_i \) as the \( i \)-th row of \( \Phi \),
(iv) define \( P := \text{conv}\{v_1, \ldots, v_n\} \subset \mathbb{R}^d \) as the convex hull of the \( v_i \),
(v) the reconstructed graph \( G = GP \) is then the edge-graph of \( P \).

### 3.3. Properties of spectral polytopes.

We discuss two properties of spectral polytopes that make them especially interesting in polytope theory.

**Reconstruction from the edge-graph.** The edge-graph of a general polytope carries little information about that polytope i.e., given only its edge-graph, we can often not reconstruct the polytope from this (up to combinatorial equivalence). Often, one cannot even deduce the dimension of the polytope from its edge-graph. Reconstruction might be possible in certain special cases, as e.g., for 3-dimensional polyhedra, simple polytopes or zonotopes. The spectral polytopes provide another such class.

**Theorem 3.10.** A \( \theta_k \)-spectral polytope is uniquely determined by its edge-graph up to invertible linear transformations.

The proof is simple: every \( \theta_k \)-spectral polytope is linearly equivalent to the \( \theta_k \)-eigenpolytope of its edge-graph (by Lemma 3.6 (i)). Our definition of the \( \theta_k \)-eigenpolytope already suggests an explicit procedure to construct it (a script for this is included in Appendix A). This property of spectral polytopes appears more exciting when applied to graph classes that are not obviously spectral (see Section 5).
Realizing symmetries of the edge-graph. Every Euclidean symmetry of a polytope induces a combinatorial symmetry on its edge-graph. The converse is far from true. Think, for example, about a rectangle that is not a square. Even worse, it can happen that a polytope does not even have a realization that realizes all the symmetries of its edge-graph (e.g. the polytope constructed in [4]).

We have previously discussed (in Theorem 2.6) the existence of a homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(P_G(\theta))$ between the symmetries of a graph $G$ and the symmetries of its eigenpolytopes. There are two caveats:

(i) this is not necessarily an isomorphism, and

(ii) it says nothing about the symmetries of the edge-graph of $P_G(\theta)$, as this one needs not to be isomorphic to $G$.

Still, it suffices to makes statement of the following form: if $G$ is vertex-transitive, then so are all its eigenpolytopes. This might not work with other transitivities, as for example edge-transitivity.

This is no concern for spectral graphs/polytopes:

**Theorem 3.11.**

(i) If $G$ is $\theta$-spectral, then $P_G(\theta)$ realizes all its symmetries, which includes $\text{Aut}(G) \cong \text{Aut}(P_G(\theta))$ via the map $\sigma \mapsto T_\sigma$ given in Theorem 2.6, as well as that $T_\sigma$ permutes the vertices and edges of $P_G(\theta)$ exactly as $\sigma$ permutes the vertices and edges of the graph $G$.

(ii) If $P$ is $\theta$-spectral, then $P$ has a realization that realizes all the symmetries of its edge-graph, namely, the $\theta$-eigenpolytope of its edge-graph.

This is mostly straightforward, with large parts already addressed in Theorem 2.6. The major difference is that for spectral graphs $G$ the eigenpolytope has exactly the distinct vertices $v_1, \ldots, v_n \in \mathbb{R}^d$. The statement from Theorem 2.6 that $T_\sigma$ permutes the $v_i$ as prescribed by $\sigma$, then becomes, that $T_\sigma$ permutes the vertices as prescribed by $\sigma$, and hence also the edges. Also, since the $v_i$ are distinct, no non-trivial symmetry $\sigma$ can result in trivial $T_\sigma$, making $\sigma \mapsto T_\sigma$ into a group isomorphism.

For part (ii) merely recall that the eigenpolytope $P_{G_{\theta}}(\theta)$ is indeed a realization of $P$ by Lemma 3.6 (i).

The major consequence of this is, that for spectral graphs/polytopes also more complicates types of symmetries translate between a polytope and its graph, as e.g. edge-transitivity (see also Section 5).

4. The Theorem of Izmestiev

We introduce our, as of yet, most powerful tool for proving that certain polytopes are $\theta_2$-spectral. For this, we make use of a more general theorem by Izmestiev [13], first proven in the context of the Colin de Verdière graph invariant. The proof of this theorem requires techniques from convex geometry, most notably, mixed volumes, which we not address here. We need to introduce some terminology.

As before, let $P \subset \mathbb{R}^d$ denote a full-dimensional polytope of dimension $d \geq 2$, with edge-graph $G_P = (V, E), V = \{1, \ldots, n\}$ and vertices $v_i \in \mathcal{F}_0(P), i \in V$. Recall, that the polar dual of $P$ is the polytope

$$P^o := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq 1 \text{ for all } i \in V\}.$$

The proof of Izmestiev's theorem is beyond the scope of this introduction. However, we will use its consequences to prove spectral properties of certain graphs.
We can replace the 1-\s in this definition by variables \( c = (c_1, ..., c_n) \), to obtain

\[
P^o(c) := \{ x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq c_i \text{ for all } i \in V \}.
\]

The usual polar dual is then \( P^o = P^o(1, ..., 1) \).

\[
\begin{align*}
&\text{Figure 1. Visualization of } P^o(c) \text{ for different values of } c \in \mathbb{R}^n. \\
&P^o(1, ..., 1) \quad P^o(1.25, 1, ..., 1) \quad P^o(0.75, 1, ..., 1)
\end{align*}
\]

In the following, \( \text{vol}(\cdot) \) denotes the volume of convex sets in \( \mathbb{R}^d \) (w.r.t.

Theorem 4.1 (Izmestiev [13], Theorem 2.4). Define a matrix \( X \in \mathbb{R}^{n \times n} \) with components

\[
X_{ij} := -\frac{\partial^2 \text{vol}(P^o(c))}{\partial c_i \partial c_j} \bigg|_{c=(1,\ldots,1)}.
\]

The matrix \( X \) has the following properties:

(i) \( X_{ij} < 0 \) whenever \( ij \in E(G_P) \),

(ii) \( X_{ij} = 0 \) whenever \( ij \notin E(G_P) \),

(iii) \( X\Psi = 0 \) (where \( \Psi \) is the arrangement matrix of \( P \)),

(iv) \( X \) has a unique negative eigenvalue, and this eigenvalue is simple,

(v) \( \dim \ker X = d \).

One can view the matrix \( X \) as some kind of adjacency matrix of a vertex- and edge-weighted version of \( G_P \). Part (iii) states that \( v \) satisfies a weighted form of the balancing condition (3.2) with eigenvalue zero. Since \( \text{rank } \Psi = d \), part (v) states that \( \text{span } \Psi \) is already the whole 0-eigenspace. And part (iv) states that zero is the second smallest eigenvalue of \( X \).

Theorem 4.2. Let \( X \in \mathbb{R}^{n \times n} \) be the matrix defined in Theorem 4.1. If we have

(i) \( X_{ii} \) is independent of \( i \in V(G_P) \), and

(ii) \( X_{ij} \) is independent of \( ij \in E(G_P) \),

then \( P \) is \( \theta_2 \)-spectral.

Proof. By assumption there are \( \alpha, \beta \in \mathbb{R}, \beta > 0 \), so that \( X_{ii} = \alpha \) for all vertices \( i \in V(G_P) \), and \( X_{ij} = \beta < 0 \) for all edges \( ij \in E(G_P) \) (we have \( \beta < 0 \) by Theorem 4.1 (i)). We can write this as

\[
X = \alpha \text{Id} + \beta A \implies (*) A = \frac{\alpha}{\beta} \text{Id} + \frac{1}{\beta} X,
\]

where \( A \) is the adjacency matrix of \( G_P \). By Theorem 4.1 (iv) and (v), the matrix \( X \) has second smallest eigenvalue zero of multiplicity \( d \). By Theorem 4.1 (iii), the columns of \( M \) are the corresponding eigenvectors. Since \( \text{rank } \Psi = d \) we find that these are all the eigenvectors and \( \text{span } \Psi \) is the 0-eigenspace of \( X \).
By (\star) the eigenvalues of $A$ are the eigenvalues of $X$, but scaled by $1/\beta$ and shifted by $\alpha/\beta$. Since $1/\beta < 0$, the second-smallest eigenvalue of $X$ gets mapped onto the second-largest eigenvalue of $A$. Therefore, $A$ (and also $GP$) has second-largest eigenvalue $\theta_2 = \alpha/\beta$ of multiplicity $d$, and span $\Psi$ is the corresponding eigenspace. By definition, $P$ is then the $\theta_2$-eigenpolytope of $GP$ and is therefore $\theta_2$-spectral. □

It is unclear whether Theorem 4.2 already characterizes $\theta_2$-spectral polytopes, or even spectral polytopes in general (see also Question 6.1).

5. Edge-transitive polytopes

We apply Theorem 4.2 to edge-transitive polytopes, that is, to polytopes for which the Euclidean symmetry group $\text{Aut}(P) \subset \text{O}(\mathbb{R}^d)$ acts transitively on the edge set $F_1(P)$. No classification of edge-transitive polytopes is known. Some edge-transitive polytopes are listed in Section 5.2.

Despite the name of this section, we are actually going to address polytopes that are simultaneously vertex- and edge-transitive. This is not a huge deviation from the title: as shown in [22], edge-transitive polytopes in dimension $d \geq 4$ are always also vertex-transitive, and the exceptions in lower dimensions are few (a continuous family of $2n$-gons for each $n \geq 2$, and two exceptional polyhedra).

Theorem 4.2 can be directly applied to simultaneously vertex- and edge-transitive polytopes, and so we have

Corollary 5.1. A simultaneously vertex- and edge-transitive polytope is $\theta_2$-spectral.

We collect all the notable consequences in the following theorem:

Theorem 5.2. If $P \subset \mathbb{R}^d$ is simultaneously vertex- and edge-transitive, then

(i) $\text{Aut}(P) \subset \mathbb{R}^d$ is irreducible as a matrix group.
(ii) $P$ is uniquely determined by its edge-graph up to scale and orientation.\(^3\)
(iii) $P$ realizes all the symmetries of its edge-graph.
(iv) If $P$ has edge length $\ell$ and circumradius $r$, then

\begin{equation}
\frac{\ell}{r} = \sqrt{\frac{2\lambda_2}{\deg(G_P)}} = \sqrt{2\left(1 - \frac{\theta_2}{\deg(G_P)}\right)},
\end{equation}

where $\deg(G_P)$ is the vertex degree of $G_P$, and $\lambda_2 = \deg(G_P) - \theta_2$ denotes its second smallest Laplacian eigenvalue.

(v) If $\alpha$ is the dihedral angle of the polar dual $P^\circ$, then

\begin{equation}
\cos(\alpha) = -\frac{\theta_2}{\deg(G_P)}.
\end{equation}

Proof. The complete proof of (i) and (ii) has to be postponed until Section 5.1 (see Theorem 5.4). Concerning (ii), from Corollary 5.1 and Theorem 3.10 already follows that $P$ is determined by its edge-graph up to invertible linear transformations, but not necessarily only up to scale and orientation.

Part (iii) follows from Theorem 3.11. Part (iv) and (v) were proven (in a more general setting) in [23, Proposition 4.3]. This applies literally to (iv). For (v), note the following: if $\sigma_i \in F_{d-1}(P^\circ)$ is the facet of the polar dual $P^\circ$ that corresponds

\(^3\)This shows that $P$ is perfect, i.e., is the unique maximally symmetric realization of its combinatorial type. See [7] for an introduction to perfect polytopes.
to the vertex \(v_i \in \mathcal{F}_1(P)\), then the dihedral angle between \(\sigma_i\) and \(\sigma_j\) is \(\pi - \angle(v_i, v_j)\).

The latter expression was proven in [23] to agree with (5.2).

It is worth emphasizing that large parts of Theorem 5.2 do not apply to polytopes of a weaker symmetry, as e.g. vertex-transitive polytopes. Prisms are counterexamples to both (i) and (ii). There are vertex-transitive neighborly polytopes (other than simplices) and they are counterexamples to (ii) and (iii).

**Remark 5.3.** There are two edge-transitive polyhedra that are not vertex-transitive: the **rhombic dodecahedron** and the **rhombic triacontahedron** (see also Figure 2). Only the former is \(\theta_2\)-spectral, and the latter is not spectral for any eigenvalue (this was already mentioned in [14]). Since the rhombic dodecahedron is not vertex-transitive, nothing of this follows from Corollary 5.1. However, this polytope satisfies the conditions of Theorem 4.2, which seems purely accidental. It is the only known spectral polytope that is not vertex-transitive.

### 5.1. Rigidity and irreducibility

The goal of this section is to prove the missing part of Theorem 5.2:

**Theorem 5.4.** If \(P \subset \mathbb{R}^d\) is simultaneously vertex- and edge-transitive, then

1. \(\text{Aut}(P) \subset O(\mathbb{R}^d)\) is irreducible as a matrix group, and
2. \(P\) is determined by its edge-graph up to scale and orientation.

To prove Theorem 5.4, we make use of **Cauchy’s rigidity theorem** for polyhedra (with its beautiful proof listed in [2, Section 12]). It states that every polyhedron is uniquely determined by its combinatorial type and the shape of its faces. This was generalized by Alexandrov to general dimensions \(d \geq 3\) (proven e.g. in [17, Theorem 27.2]):

**Theorem 5.5 (Alexandrov).** Let \(P_1, P_2 \subset \mathbb{R}^d, d \geq 3\) be two polytopes, so that

1. \(P_1\) and \(P_2\) are combinatorially equivalent via a face lattice isomorphism \(\phi: \mathcal{F}(P_1) \to \mathcal{F}(P_2)\), and
2. each facet \(\sigma \in \mathcal{F}_{d-1}(P_1)\) is congruent to the facet \(\phi(\sigma) \in \mathcal{F}_{d-1}(P_2)\).

Then \(P_1\) and \(P_2\) are congruent, i.e., are the same up to orientation.

**Proposition 5.6.** Let \(P_1, P_2 \subset \mathbb{R}^d\) be two combinatorially equivalent polytopes, each of which has

1. all vertices on a common sphere (i.e., is inscribed), and
2. all edges of the same length \(\ell_i\).

Then \(P_1\) and \(P_2\) are the same up to scale and orientation.

**Proof.** W.l.o.g. assume that \(P_1\) and \(P_2\) have the same circumradius, otherwise rescale \(P_2\). It then suffices to show that \(P_1\) and \(P_2\) are the same up to orientation.

We proceed with induction by the dimension \(d\). The induction base is given by \(d = 2\), which is trivial, since any two inscribed polygons with constant edge length are regular and thus completely determined (up to scale and orientation) by their number of vertices.

Suppose now that \(P_1\) and \(P_2\) are combinatorially equivalent polytopes of dimension \(d \geq 3\) that satisfy (i) and (ii). Let \(\phi\) be the face lattice isomorphism between them. Let \(\sigma \in \mathcal{F}_{d-1}(P_1)\) be a facet of \(P_1\), and \(\phi(\sigma)\) the corresponding facet in

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4This proof was proposed by the user Fedor Petrov on MathOverflow [3].
In particular, $\sigma$ and $\phi(\sigma)$ are combinatorially equivalent. Furthermore, both $\sigma$ and $\phi(\sigma)$ are of dimension $d - 1$ and satisfy (i) and (ii). This is obvious for (ii), and for (i) recall that facets of inscribed polytopes are also inscribed. By induction hypothesis, $\sigma$ and $\phi(\sigma)$ are then congruent. Since this holds for all facets $\sigma \in F_{d-1}(P_1)$, Theorem 5.5 tells us that $P_1$ and $P_2$ are congruent, that is, the same up to orientation.

We can now prove the main theorem of this section:

Proof of Theorem 5.4. By Theorem 5.2 the combinatorial type of $P$ is determined by its edge-graph. By vertex-transitivity, all vertices are on a sphere. By edge-transitivity, all edges are of the same length. We can then apply Proposition 5.6 to obtain that $P$ is unique up to scale and orientation. This proves (ii).

Suppose now, that $\text{Aut}(P)$ is not irreducible, but that $\mathbb{R}^d$ decomposes as $\mathbb{R}^d = W_1 \oplus W_2$ into non-trivial orthogonal $\text{Aut}(P)$-invariant subspaces. Let $T_\alpha \in \text{GL}(\mathbb{R}^d)$ be the linear map that acts as identity on $W_1$, but as $\alpha \text{Id}$ on $W_2$ for some $\alpha > 1$. Then $T_\alpha P$ is a non-orthogonal linear transformation of $P$ (in particular, combinatorially equivalent), on which $\text{Aut}(P)$ still acts vertex- and edge-transitively. By (ii), this cannot be. Hence $\text{Aut}(P)$ must be irreducible, which proves (i). □

5.2. A word on classification. Despite the simple appearance of the definition of an edge-transitive polytope, no classification was obtained so far.

There exists a classification of the 3-dimension edge-transitive polyhedra: besides the Platonic solids, these are the ones shown in Figure 2 (nine in total).

![Figure 2](image_url)

Figure 2. From left to right, these are: the cuboctahedron, the icosidodecahedron, the rhombic dodecahedron, and the rhombic triacontahedron.

There are many known edge-transitive polytopes in dimension $d \geq 4$ (so we are not talking about a class as restricted as the regular polytopes). There are 15 known edge-transitive 4-polytopes (and an infinite family of duoprisms\(^5\)), but already here, no classification is known. It is known that the number of irreducible\(^6\) edge-transitive polytopes grows at least linearly with the number of dimensions. For example, there are $\lfloor d/2 \rfloor$ hyper-simplices in dimension $d$. These are edge-transitive (even distance-transitive, see Section 5.4).

It is the hope of the author, that the classification of the edge-transitive polytopes can be obtained using their spectral properties. Their classification can now be

---

\(^5\)The $(n, m)$-duoprism is the cartesian product of a regular $n$-gon and a regular $m$-gon. Those are edge-transitive if and only of $n = m$. Technically, the 4-cube is the $(4, 4)$-duoprism but is usually not counted as such, because of its exceptionally large symmetry group.

\(^6\)Being not the cartesian product of lower dimensional edge-transitive polytopes.
stated purely as a problem in spectral graph theory: the classification of the edge-
transitive polytopes (in dimension $d \geq 4$) is equivalent to the classification of $\theta_k$-
spectral edge-transitive graphs, and since Lemma 3.9, we have a completely graph
theoretic characterization of spectral graphs.

**Theorem 5.7.** Let $G$ be an edge-transitive graph. If $G$ is $\theta_k$-spectral, then

(i) $k = 2$, and

(ii) if $G$ is not vertex-transitive, then $G$ is the edge-graph of the rhombic dodec-
ahedron (see Figure 2).

**Proof.** We first prove (ii). As shown in [22] all edge-transitive polytopes in dimen-
sion $d \geq 4$ are vertex-transitive. If $G$ is edge-transitive, not vertex-transitive and
$\theta_k$-spectral, then its $\theta_k$-eigenpolytope is also edge-transitive but not vertex-transitive,
hence of dimension $d \leq 3$. One checks that the 2-dimensional spectral polytopes are
regular polygons, hence vertex-transitive. The remaining polytopes are polyhedra,
and we mentioned in Remark 5.3 that among these, only the rhombic dodecahedron
is spectral, in fact $\theta_2$-spectral. This proves (ii).

Equivalently, if $G$ is vertex- and edge-transitive, then so is its eigenpolytope. By
Corollary 5.1 this is a $\theta_2$-eigenpolytope. Together with part (ii), we find $k = 2$ in all
cases, which proves (i). \qed

5.3. **Arc- and half-transitive polytopes.** In a graph or polytope, an *arc* is an
incident vertex-edge-pair. A graph or polytope is called *arc-transitive* if its sym-
metry group acts transitively on the arcs. Being arc-transitive implies both, being
vertex-transitive, and being edge-transitive. In addition to that, in an arc-transitive
graph, every edge can be mapped, not only onto every other edge, but also onto
itself with flipped orientation.

There exist graphs that are simultaneously vertex- and edge-transitive, but not
arc-transitive. Those are called *half-transitive* graphs, and are comparatively rare.
The smallest one has 27 vertices and is known as the *Holt graph* (see [5,12]).

For polytopes on the other hand, it is unknown whether there exists a distinction
being arc-transitive and being simultaneously vertex- and edge-transitive. No *half-
transitive polytope* is known. Because of Theorem 5.2 (i), we know that the edge-
graph of a half-transitive polytope must itself be half-transitive. Since such graphs
are rare, the existence of half-transitive polytopes seems unlikely.

**Example 5.8.** The Holt graph is not the edge-graph of a half-transitive polytope:
the Holt graph is of degree four, and its second-largest eigenvalue is of multiplicity
six, giving rise to a 6-dimensional $\theta_2$-eigenpolytope. But a 6-dimensional polytope
must have an edge-graph of degree at least six, and so the Holt graph is not spectral.

The lack of examples of half-transitive polytopes means that all known edge-
transitive polytopes in dimension $d \geq 4$ are in fact arc-transitive. Likewise, a clas-
sification of arc-transitive polytopes is not known.

5.4. **Distance-transitive polytopes.** Our previous results about edge-transitive
polytopes already allow for a complete classification of a particular subclass, namely,
the *distance-transitive polytopes*, thereby also providing a list of examples of edge-
transitive polytopes in higher dimensions.

The distance-transitive symmetry is usually only considered for graphs, and the
distance-transitive graphs form a subclass of the distance-regular graphs. The usual
reference for these is the classic monograph by Brouwer, Cohen and Neumaier [6].
For any two vertices $i, j \in V$ of a graph $G$, let $\text{dist}(i, j)$ denote the graph-theoretic distance between those vertices, that is, the length of the shortest path connecting them. The diameter $\text{diam}(G)$ of $G$ is the largest distance between any two vertices in $G$.

**Definition 5.9.** A graph is called distance-transitive if $\text{Aut}(G)$ acts transitively on each of the sets  
$$D_\delta := \{(i, j) \in V \times V \mid \text{dist}(i, j) = \delta\}, \quad \text{for all } \delta \in \{0, ..., \text{diam}(G)\}.$$ 

Analogously, a polytope $P \subset \mathbb{R}^d$ is said to be distance-transitive, if its Euclidean symmetry group $\text{Aut}(P)$ acts transitively on each of the sets  
$$D_\delta := \{(v_i, v_j) \in F_0(P) \times F_0(P) \mid \text{dist}(i, j) = \delta\}, \quad \text{for all } \delta \in \{0, ..., \text{diam}(G_P)\}.$$ 

Note that the distance between the vertices is still measured along the edge-graph rather than via the Euclidean distance.

Being arc-transitive is equivalent to being transitive on the set $D_1$. Hence, distance-transitivity implies arc-transitivity, thus edge-transitivity.

By our considerations in the previous sections, we know that the classification of distance-transitive polytopes is equivalent to the classification of the $\theta_2$-spectral distance-transitive graphs. Those were classified by Godsil (see Theorem 2.7).

In the following theorem we translated each such $\theta_2$-spectral distance-transitive graph into its respective eigenpolytope. This gives a complete classification of the distance-transitive polytopes.

**Theorem 5.10.** If $P \subset \mathbb{R}^d$ is distance-transitive, then it is one of the following:

1. a regular polygon ($d = 2$),
2. the regular dodecahedron ($d = 3$),
3. the regular icosahedron ($d = 3$),
4. a cross-polytopes, that is, $\text{conv}\{\pm e_1, ..., \pm e_d\}$ where $\{e_1, ..., e_d\} \subset \mathbb{R}^d$ is the standard basis of $\mathbb{R}^d$,
5. a hyper-simplex $\Delta(d, k)$, that is, the convex hull of all vectors $v \in \{0, 1\}^{d+1}$ with exactly $k$ 1-entries,
6. a cartesian power of a regular simplex (also known as the Hamming polytopes; this includes regular simplices and hypercubes),
7. a demi-cube, that is, the convex hull of all vectors $v \in \{-1, 1\}^d$ with an even number of 1-entries,
8. the $2_{21}$-polytope, also called Gosset-polytope ($d = 6$),
9. the $3_{21}$-polytope, also called Schlafli-polytope ($d = 7$).

The ordering of the polytopes in this list agrees with the ordering of graphs in the list in Theorem 2.7. The latter two polytopes where first constructed by Gosset in [11].

We observe that the list in Theorem 5.10 contains many polytopes that are not regular, and contains all regular polytopes excluding the 4-dimensional exceptions, the 24-cell, 120-cell and 600-cell. The distance-transitive polytopes thus form a distinct class of remarkably symmetric polytopes which is not immediately related to the class of regular polytopes.

Another noteworthy observation is that all the distance-transitive polytopes are Wythoffian polytopes, that is, they are orbit polytopes of finite reflection groups. Figure 3 shows the Coxeter-Dynkin diagrams of these polytopes.
In this paper we have studied eigenpolytopes and spectral polytopes. The former are polytopes constructed from a graph and one of its eigenvalues. A polytope is spectral if it is the eigenpolytopes of its edge-graph. These are of interest because spectral graph theory then ensures a strong interplay between the combinatorial properties of the edge-graph and the geometric properties of the polytope.

The study of eigenpolytopes and spectral polytopes has left us with many open questions. Most notably, how to detect spectral polytopes purely from their geometry. We introduced a tool (Theorem 4.2), which was sufficient to prove that (most) edge-transitive polytopes are spectral. We do not know how much more general it can be applied.

**Question 6.1.** Does Theorem 4.2 already characterize $\theta_2$-spectral polytopes (or even spectral polytopes in general)?

If the answer is affirmative, this would provide a geometric characterization of polytopes that are otherwise defined purely in terms of spectral graph theory. The result of Izmestiev suggests that polytopes with sufficiently regular geometry are $\theta_2$-spectral: the entry of the matrix $X$ in Theorem 4.1 at index $ij \in E$ can be expressed as

$$X_{ij} = \frac{\operatorname{vol} (\sigma_i \cap \sigma_j)}{\|v_i\| \|v_j\| \sin \angle (v_i, v_j)}$$

where $\sigma_i$ and $\sigma_j$ are the facets of the polar dual $P^\circ$ that correspond to the vertices $v_i, v_j \in F_0(P)$. Because of this formula, it might be actually easier to classify the polar duals of $\theta_2$-spectral polytopes.

An affirmative answer to **Question 6.1** would also mean a negative answer to the following:

**Question 6.2.** Is there a $\theta_k$-spectral polytope/graph for some $k \neq 2$?

The answer is known to be negative for edge-transitive polytopes/graphs (see Theorem 5.7), but unknown in general.

The second-largest eigenvalue $\theta_2$ is special for other reasons too. Even if a graph is not $\theta_2$-spectral, it seems to still imprint its adjacency information onto the edge-graph of its $\theta_2$-eigenpolytope.
Question 6.3. Given an edge $ij \in E$ of $G$, if $v_i$ and $v_j$ (as defined in Definition 2.2) are distinct vertices of the $\theta_2$-eigenpolytope $P_G(\theta_2)$, is then also $\text{conv}\{v_i, v_j\}$ an edge of $P_G(\theta_2)$?

This was proven for distance-regular graphs in [10], and is not necessarily true for eigenvalues other than $\theta_2$.

All known spectral polytopes are exceptionally symmetric. It is unclear whether this is true in general.

Question 6.4. Are there spectral polytopes with trivial symmetry group?

An example for Question 6.4 must be asymmetric, yet with a reasonably large eigenspaces. Such graphs exist among the distance-regular graphs, but all spectral distance-regular graphs were determined in [10] (see also Theorem 2.7) and turned out to be distance-transitive, i.e., highly symmetric.

A clear connection between being spectral and being symmetric is missing. To emphasize our ignorance, we ask the following:

Question 6.5. Can we find more spectral polytopes that are not vertex-transitive? What characterizes them?

The single known spectral polytope that is not vertex-transitive is the rhombic dodecahedron (see Figure 2). The fact that it is spectral appears purely accidental, as there seems to be no reason for it to be spectral, except that we can explicitly check that it is. For comparison, the highly related rhombic triacontahedron is not spectral.

On the other hand, vertex-transitive spectral polytopes might be quite common.

Question 6.6. Let $P \subset \mathbb{R}^d$ be a polytope with the following properties:

(i) $P$ is vertex-transitive,
(ii) $P$ realizes all the symmetries of its edge-graph, and
(iii) $\text{Aut}(P)$ is irreducible.

Is $P$ (combinatorially equivalent to) a spectral polytope?

No condition in Question 6.6 can be dropped. If we drop vertex-transitivity, we could take some polytope whose edge-graph has trivial symmetry and only small eigenspaces. Dropping (ii) leaves vertex-transitive neighborly polytopes, for which we know that these are mostly not spectral (except for the simplex). Dropping (iii) leaves us with the prisms and anti-prisms, the eigenspaces of their edge-graphs are rarely of dimension greater than two.

Finally, we wonder whether these spectral techniques can be any help in classifying the edge-transitive polytopes.

Question 6.7. Can we classify the edge-transitive graphs that are spectral, and by this, the edge-transitive polytopes?

Question 6.8. Can the existence of half-transitive polytopes be excluded by using spectral graph theory (see Section 5.3)?

Acknowledgements. The author gratefully acknowledges the support by the funding of the European Union and the Free State of Saxony (ESF).
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APPENDIX A. IMPLEMENTATION IN MATHEMATICA

The following short Mathematica script takes as input a graph $G$ (in the example below, this is the edge-graph of the dodecahedron), and an index $k$ of an eigenvalue. It then computes the $v_i$ (or `vert` in the code), i.e., the vertex-coordinates of the $\theta_k$-eigenpolytope. If the dimension turns out to be appropriate, the spectral embedding of the graph, as well as the eigenpolytope are plotted.

(* Input: 
* the graph G, and 
* the index k of an eigenvalue (k = 1 being the largest eigenvalue). *)
G = GraphData["DodecahedralGraph"];
k = 2;

(* Computation of vertex coordinates 'vert' *)
n = VertexCount[G];
A = AdjacencyMatrix[G];
eval = Tally[Sort[Sort[Eigenvalues[A//N]]], Round[#1 - #2, 0.00001] == 0 &];
d = eval[[-k, 2]]; (* dimension of the eigenpolytope *)
vert = Transpose[Orthogonalize[NullSpace[eval[[-k, 1]] * IdentityMatrix[n] - A]]];

(* Output: 
* the graph G, 
* its eigenvalues with multiplicities, 
* the spectral embedding, and 
* its convex hull (the eigenpolytope). *)
G
Grid[Join[{{"\theta", "mult"}}, eval], Frame -> All]
Which[
  d < 2 , Print["Dimension too low, no plot generated."],
  d == 2 , GraphPlot[G, VertexCoordinates -> vert],
  d == 3 , GraphPlot3D[G, VertexCoordinates -> vert],
  d > 3 , Print["Dimension too high, 3-dimensional projection is plotted."],
    GraphPlot3D[G, VertexCoordinates -> vert[[;;, 1;;3]] ]
]
If[d == 2 || d == 3,
  RegionMesh[ConvexHullMesh[vert]]
]

Faculty of Mathematics, University of Technology, 09107 Chemnitz, Germany
E-mail address: martin.winter@mathematik.tu-chemnitz.de