ABJM Baryon Stability and Myers effect

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Based on work with:

- Y. Lozano, M. Picos and K. Sfetsos, JHEP 07(2011)03, arXiv:1105.093[hep-th].
AdS/CFT and motivation

- The AdS/CFT correspondence is a tool to extract information for gauge theories at strong coupling from gravity. Prototype was the $\text{AdS}_5 \times S^5$ dual to $\mathcal{N}=4$ SYM [Maldacena 99]. Extensions: temperature, velocity, Coulomb branch, marginally deformed backgrounds.

- $\text{AdS}_4/\text{CFT}_3$: Type IIA string theory on $\text{AdS}_4 \times \text{CP}^3$ with an $\mathcal{N}=6$ quiver CS matter theory with gauge group $\text{U}(N)_k \times \text{U}(N)_{-k}$ and marginal superpotential [ABJM Model]. Being the superpotential coupling proportional to $k^{-2}$, $N^{1/5} \ll k$ allows for a weak coupling regime ($\lambda = N/k$). The Type IIA theory is then weakly curved when $k \ll N$.

- Bound states of quarks are dual to classical string/brane probe solutions. Discrepancies arise in many examples between field theory/experimental expectations and their gravitational description, baryons: colorless states.
Plan of the talk:

- Construction of baryons within the gravity/gauge theory duality.

- Macroscopical calculation of binding energy and charges.

- Stability analysis:
  - Based on general statements concerning the perturbative stability of such string solutions (transcendental equation), rather than (heavy) numerics.
  - Applications and resolutions of the discrepancies, non-sinlet solutions.

- Microscopical calculation of binding energy and charges.

- Conclusions and future directions.
**Baryon potential within AdS/CFT**

- Heavy baryon potential $E(L)$ is extracted from Wilson loop expectation values $\langle W(C) \rangle$.
- Within AdS/CFT, the interaction potential energy of the baryon is given by [prototype by Witten 1998]

$$e^{-iET} = \langle W(C) \rangle \simeq \exp(iS[C]),$$

$$S[C] = S_{NG} + S_{DBI} + S_{CS}, \textbf{Note: Quarks are external.}$$

*Figure:* Baryon Configuration.
**Dp-Brane energy**

The DBI action of a $Dp$-brane in string units reads

$$S_p = -T_p \int d^{p+1} \zeta \ e^{-\phi} \sqrt{|\text{det}(P[g + 2\pi \mathcal{F}])|} , \quad T_p = \frac{1}{(2\pi)^p} ,$$

where $g$ is the induced metric and $\mathcal{F} = F + \frac{1}{2\pi} B_2$ is the magnetic flux. The metric of a $Dp$-brane wrapping on $\mathbb{CP}^{p/2}$ cycles (gauge choice is time and the angles of the $\mathbb{CP}^{p/2}$ cycles) reads

$$ds^2_{\text{ind}} = -\frac{16 \rho^2}{L^2} d\tau^2 + L^2 ds^2_{\mathbb{CP}^{p/2}} ,$$

where $F = \mathcal{N} J$ and $B_2 = -2\pi J$, $J$ is the Kähler form (equations of motion are satisfied) and $\mathcal{N} \in 2\mathbb{Z}$. The energy of the Dp-brane is [Lozano et. al. 2010].

$$E_{DBI}^{Dp} = -Q_p \frac{4\rho_0}{L} , \quad Q_p = \frac{T_p}{g_s} \ Vol(\mathbb{CP}^{p/2}) \left( L^4 + (2\pi)^2 (\mathcal{N} - a_p)^2 \right)^{\frac{p}{4}} ,$$

$$a_{2,6} = 1, \quad a_4 = 0, \quad \text{due Freed-Witten anomaly of } \mathbb{CP}^2, \text{ not spin-manifold.}$$

Note: $\mathcal{N}^2$ is comparable to $L^4 \gg 1$. 
**Dp-brane Charges**

The CS action for a Dp-brane reads

\[
S_{CS} = T_p \int d^{p+1} \xi \left[ P \left( \sum_q C_q e^{B_2} \right) e^{2\pi F} \right]_p.
\]

Both the D4 and D6-branes have $CP^1$ D2-branes dissolved. Therefore in the presence of a magnetic flux they capture the $F_2$ flux and develop a tadpole with charge $q = k \frac{N_{p-1}}{2^{p-1}(p-1)!}$, \[Lozano et al 2010\].

There are three more couplings for $D6$:

- The $\int_{D6} F_2 \wedge B_2 \wedge B_2 \wedge A$ which cancels \[Aharony et al 09\] from higher curvature terms \[Green et al 96, Cheung et al 97, Bachas et al 99\].

- The $\int_{D6} F_2 \wedge F \wedge B_2 \wedge A$ which contributes to its $k$ charge,

  \[
  q_{D6} = N + k \frac{N(N-2)}{8}, \text{ where the } N \text{ units induced by the } F_6 \text{ flux } S_{CS}^{D6} = 2\pi T_6 \int_{R \times \mathbb{R}^3} P[F_6] \wedge A = N T_{F1} \int dtA_t.
  \]
**Classical solution**

We consider a classical configuration consisting on a $Dp$-brane wrapped on $CP^p_2$, located at $\rho = \rho_0$, $l$ strings stretching from $\rho_0$ to the boundary of $AdS_4$ and $(q-l)$ straight strings that go from $\rho_0$ to 0 (Figure). Solving the e.o.m. and imposing the b.c. at the boundary and the baryon vertex we find that the length and the energy of the distribution reads

$$\ell = \frac{L^2 \rho_1^2}{12 \rho_0^3} 2F1 \left( \frac{1}{2}, \frac{3}{4}, \frac{7}{4}; \frac{\rho_1^4}{\rho_0^4} \right), \quad \frac{\rho_1^4}{\rho_0^4} = 4x (1-x), \quad x = \frac{l_{\text{min}}}{l},$$

$$E_{bin} = E_{Dp} + E_{IF1} + E_{(q-l)F1} =$$

$$= l T_{F1} \rho_0 \left\{ -2F1 \left( -\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 4x (1-x) \right) + 2x - 1 \right\} .$$

$$l \geq \frac{q}{2} \left( 1 + \sqrt{1 - \beta^2} \right) = l_{\text{min}}, \quad \sqrt{1 - \beta^2} \equiv \frac{2Q_p}{L q T_{F1}}.$$

Thus, the binding energy reads

$$E_{bin} = -f(x) \frac{(g_s N)^{2/5}}{\ell} \leq 0, \quad f(x) \geq 0 .$$

[Conformal dependence, non-pertubative and concavity].
Instabilities can emerge only from the longitudinal fluctuations of the $l$ strings [Sfetsos, K.S. 2008]. Perturbing the embedding according to $r = r_{cl} + \delta r(\rho)$ and expanding the Nambu-Goto action to quadratic order in the fluctuations, the zero mode solution vanishing in the UV reads

$$\delta r = A \int_{\rho}^{\infty} \frac{d\rho}{(\rho^4 - \rho_{1}^4)^{3/2}} = \frac{A}{3\rho^3} \frac{\rho^2}{2F_1\left(\frac{3}{2}, \frac{3}{4}; \frac{7}{4}; \frac{\rho_{1}^4}{\rho^4}\right)}.$$ 

imposing the boundary condition at the baryon vertex $\rho_0$ we find

$$2F_1\left(\frac{3}{2}, \frac{3}{4}; \frac{7}{4}; 1 - \gamma^2\right) = \frac{3}{2\gamma(1 + \gamma^2)}, \quad \gamma \equiv \sqrt{1 - \frac{\rho_{1}^4}{\rho_0^4}}.$$ 

The solution of the transcendental equation is $\gamma_c \simeq 0.538$, thus the bound of $F$-strings coming from the stability is more restrictive $l \geq \frac{q}{1 + \gamma_c}(1 + \sqrt{1 - \beta^2})$. As for the brane fluctuations they prove to be stable. Note that there are no boundary conditions for these fluctuations, the reason being that the $\mathbb{R} \times CP^\frac{p}{2}$ space has no boundary.
Microscopical energy

D0-brane charge on the D$p$-branes wrapped on (fuzzy) $CP^{p/2}$ suggests a close analogy with the dielectric effect [Emparan 97, Myers 99]. The DBI action describing the dynamics of $n$ coincident D0-branes expanded into fuzzy $CP^{p/2}$

$$S_{nD0}^{DBI} = - \frac{1}{g_s} \int d\tau \frac{4\rho}{L} \text{Str} \sqrt{\det(Q)} , \quad Q^i{}_j = \delta^i{}_j + \frac{i}{2\pi} [X^i, X^k] E_{kj}.$$  

where $E = g + B_2$. It is a difficult task to compute $\det(Q)$ in general, since $Q^i{}_j = \delta^i{}_j + M^i{}_j$ with $M^i{}_j$ given by

$$M^i{}_j = - \frac{1}{p^2 + 1} \Lambda(m) f_{ikl} X^l \left( \frac{pL^2}{8\pi} \delta^k{}_j - \sqrt{\frac{p}{4(p^2 + 1)}} f_{kjm} X^m \right),$$

and one has to compute traces of powers of $M$ using the constraints of the construction of $CP^{p/2}$ space.
For example, for $B_2 = 0$ we find

\[
\text{Tr}(M) = 0, \quad \text{Tr}(M^2) = -\frac{p}{2^4 \pi^2} r \mathbb{I}, \quad \text{Tr}(M^3) = -i \frac{p(p + 1)}{2^7 \pi^3 L^2} r^2 \mathbb{I}, \\
\text{Tr}(M^4) = \frac{p}{2^8 \pi^4} r^2 \mathbb{I} + \frac{p}{2^{10} \pi^4 L^4} \left( \left( \frac{p}{2} + 1 \right)^2 - 4 \right) r^3 \mathbb{I},
\]

where $r = \frac{L^4}{m(m+p/2+1)}$. However, in the limit

\[
L \gg 1, \quad m \gg 1, \quad \text{with} \quad r \simeq \frac{L^4}{m^2} = \text{finite},
\]

\[
\text{Tr}(M^{2n}) \simeq p (-1)^n \left( \frac{r}{16 \pi^2} \right)^n \mathbb{I}, \quad \text{Tr}(M^{2n+1}) \simeq 0.
\]

Thus the energy of $n$ D0-branes expanding into a fuzzy $CP_{2}^{p}$ is then given to leading order in $m$ by

\[
E_{nD0} \simeq -\frac{n}{g_s} \left( 1 + \frac{L^4}{16 \pi^2 m^2} \right) \frac{p}{4} 4\rho_0 L \quad (L \gg 1, L^p \ll n),
\]

where $n = \text{dim}(m, 0)$. For $m \sim \frac{N}{2}$ the leading order in $m$ coincides with the macroscopic result. For $B_2 \neq 0$ the discussion is more technical and would not be presented here. It turns out, that redefinition of $m$ gets corrected: $N^c = 2m + 2$, $p = 2, 4$ and $N^c = 2m + 4$, $p = 6$. 
Microscopical charges

Next we shall show how fundamental strings that stretch from the Dp-brane to the boundary of AdS4 strings arise in the microscopic setup. The CS action for $n$ coincident D0-branes reads

$$S_{CS} = \int_{\mathbb{R}} S\text{Tr} \left\{ P \left( e^{\frac{i}{2\pi} (iX iX)} \sum_q C_q e^{B_2} \right) e^{2\pi F} \right\}.$$

The relevant CS terms in the AdS$_4 \times CP^3$ are

$$S_{CS} = S_{CS_1} + S_{CS_2} + S_{CS_3} + S_{CS_4}$$

$$= i \int d\tau \ S\text{Tr} \left\{ \left[ (iX iX) F_2 - \frac{1}{(2\pi)^2} (iX iX)^3 F_6 + \frac{i}{2\pi} (iX iX)^2 F_2 \wedge B_2 - \frac{1}{2} \frac{1}{(2\pi)^2} (iX iX)^3 F_2 \wedge B_2 \wedge B_2 \right] A_\tau \right\}.$$

Where we have expanded the background potentials on the non-Abelian scalars occurs through the Taylor expansion [Garousi et al 1998] and the pull-backs into the worldline are given in terms of gauge covariant derivatives, $D_\tau X^\mu = \partial_\tau X^\mu + i [A_\tau, X^\mu]$. 
In the large $m$ limit we find:

- $S_{CS_1} = k \left( m(m + \frac{p}{2} + 1) \right)^{-1/2} \frac{(m + \frac{p}{2})!}{m!(\frac{p}{2})!} \int d\tau A_\tau \simeq q \int d\tau A_\tau$,  
  $q = \frac{2}{p} k \frac{N\frac{p-1}{2}}{2^{\frac{p}{2}}(\frac{p}{2}-1)!}$, the number of fundamental string charge in each $CP^1$, in agreement with the macroscopical result.

- $S_{CS_2} \simeq N \int d\tau A_\tau$, in agreement with the macroscopical result.

- The third and fourth terms contribute when we take into account the $B_2$ field that is necessary to compensate the Freed-Witten worldvolume field of the D4-brane.
  
  - $S_{CS_3} \simeq -k \frac{m_{\frac{p}{2}-2}}{(\frac{p}{2})!} \int d\tau A_\tau$
  
  - $S_{CS_4} \simeq \frac{3!}{8} k \frac{m_{\frac{p}{2}-3}}{(\frac{p}{2})!} \int d\tau A_\tau$

To find the total $k$ charge we add the subleading contributions in the large $m$ expansion of $S_{CS_1}$. Next we use the corrected redefinition of $m$ we recover precisely the units of F-charge for D2, D4 and D6 brane plus a $k/8$ contribution for the D6, coming from $S_{CS_4}$. Stability analysis goes along the same lines than in the macroscopical set-up; non-singlet classical stable solutions.
**Dielectric higher-curvature terms**

Generalizing the Chern–Simons action for multiple Dp-branes [Myers 1999] to include higher curvature terms we find for our background

\[
S_{h.c.} = -\frac{1}{2(2\pi)^2} \int_{\mathbb{R}} STr\left\{ P[(iX_i X_i)^2 C_1 \wedge \Omega_4] \right\}
\]

\[
= -\frac{i}{(2\pi)^2} \int_{\mathbb{R}} STr[(iX_i X_i)^3 (F_2 \wedge \Omega_4)] A ,
\]

where \( \Omega_4 \) is given in term of the Pontryagin classes of the normal and the tangent bundle of the three \( CP^2 \) circles of the \( CP^3 \) manifold [Eguchi et al 1980, Bergman et al 2009];

\[
\Omega_4 = 3(1 - 3) \frac{(2\pi)^4}{48\pi^2} J \wedge J .
\]

Substituting \( F_2 \) and \( \Omega_4 \) we find:

\[
S_{h.c.} = -\frac{\kappa}{8} (m(m+4))^{-3/2} \frac{(m+3)!}{m!} \int_{\mathbb{R}} d\tau A_\tau \simeq -\frac{\kappa}{8} \int_{\mathbb{R}} d\tau A_\tau .
\]

Thus this higher curvature coupling cancels the \( S_{CS4} \) contribution as in the macroscopical case.
Summary—future directions

- We have constructed macroscopically various configurations of magnetically charged particle-like branes in ABJM with reduced number of quarks. The stability analysis increases the classical lower bound for each value of the magnetic flux.

- We have given an alternative description in terms of D0-branes expanded into fuzzy $CP^{\frac{p}{2}}$ spaces that allows to explore the finite 't Hooft coupling region, $L^p \ll n$.

- We have constructed dielectric higher curvature couplings that to the best of our knowledge have not been considered before in the literature. This new coupling exactly cancels the $k/8$ contribution to the D6-brane tadpole.

- It would be interesting to extend to theories with reduced supersymmetry, like the Klebanov–Strassler backgrounds, where the internal geometry is the $T^{1,1}$ conifold. Non-singlet baryon vertex???
\textbf{Review of the AdS_4 \times CP^3 background}

In our conventions the $AdS_4 \times CP^3$ metric reads

$$ds^2 = L^2 \left( \frac{1}{4} ds^2_{AdS_4} + ds^2_{CP^3} \right),$$

with $L$ the radius of curvature in string units

$$L = \left( \frac{32\pi^2 N}{k} \right)^{1/4}, \quad g_s = \frac{L}{k},$$

and where we have normalized the two factors such that $R_{\mu\nu} = -3g_{\mu\nu}$ and $8g_{\alpha\beta}$ for $AdS_4$ and $CP^3$, respectively. The explicit parameterization of $AdS_4$ we use is

$$ds^2_{AdS_4} = \frac{16\rho^2}{L^2} d\vec{x}^2 + L^2 \frac{d\rho^2}{\rho^2}, \quad d\vec{x}^2 = -d\tau^2 + dx_1^2 + dx_2^2.$$

For the metric on $CP^3$ we use the parameterization in [Pope 1984, Warner 1985]

$$ds^2_{CP^3} = \quad d\mu^2 + \sin^2 \mu \left[ d\alpha^2 + \frac{1}{4} \sin^2 \alpha \left( \cos^2 \alpha (d\psi - \cos \theta \, d\phi)^2 + d\theta^2 

+ \sin^2 \theta \, d\phi^2 \right) + \frac{1}{4} \cos^2 \mu \left( d\chi + \sin^2 \alpha (d\psi - \cos \theta \, d\phi) \right)^2 \right],$$
where
\[ 0 \leq \mu, \alpha \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi, \chi \leq 4\pi. \]

Inside \( CP^3 \) there is a \( CP^1 \) for \( \mu = \alpha = \pi / 2 \) and fixed \( \chi \) and \( \psi \) and also a \( CP^2 \) for fixed \( \theta \) and \( \phi \).

In these coordinates the connection in \( ds^2_{S^7} = (d\tau + \mathcal{A})^2 + ds^2_{CP^3} \) reads
\[ \mathcal{A} = \frac{1}{2} \sin^2 \mu \left( d\chi + \sin^2 \alpha \left( d\psi - \cos \theta \, d\phi \right) \right). \]

The Kähler form
\[ J = \frac{1}{2} d\mathcal{A}, \]

is then normalized such that
\[ \int_{CP^1} J = \pi, \quad \int_{CP^2} J \wedge J = \pi^2, \quad \int_{CP^3} J \wedge J \wedge J = \pi^3. \]

Therefore,
\[ \frac{1}{6} J \wedge J \wedge J = d\text{Vol}(\mathbb{P}^3) \quad \text{and} \quad \text{Vol}(\mathbb{C}P^3) = \frac{\pi^3}{6}. \]
The $AdS_4 \times CP^3$ background fluxes can then be written as

$$F_2 = \frac{2L}{g_s} J,$$
$$F_4 = \frac{3L^3}{8g_s} \, d\text{Vol}(AdS_4),$$
$$F_6 = -(*_{10}F_4) = \frac{6L^5}{g_s} \, d\text{Vol}(\mathbb{P}^3),$$

where $g_s = \frac{L}{k}$. The flux integrals satisfy

$$\int_{CP^3} F_6 = 32\pi^5 N, \quad \int_{CP^1} F_2 = 2\pi k.$$

The flat $B_2$-field that is needed to compensate for the Freed–Witten worldvolume flux in the D4-brane is given by [Aharony et al 2009]

$$B_2 = -2\pi J.$$
Fuzzy $\text{CP}^{\frac{p}{2}}$ manifold

$\text{CP}^{\frac{p}{2}}$ is the coset manifold $SU(\frac{p}{2} + 1)/U(\frac{p}{2})$, and can be defined by the submanifold of $\mathbb{R}^{\frac{p^2}{4} + p}$ determined by the set of $p^2/4$ constraints

$$\sum_{i=1}^{\frac{p^2}{4} + p} x^i x^i = 1,$$
$$\sum_{j,k=1}^{\frac{p^2}{4} + p} d^{ijk} x^j x^k = \frac{\frac{p}{2} - 1}{\sqrt{\frac{p^2}{4}(\frac{p}{2} + 1)}} x^i$$

where $d^{ijk}$ are the components of the totally symmetric $SU(\frac{p}{2} + 1)$-invariant tensor. The Fubini–Study metric of the $\text{CP}^{\frac{p}{2}}$ is given by

$$ds^2_{\text{CP}^{\frac{p}{2}}} = \frac{p}{4(\frac{p}{2} + 1)} \sum_{i=1}^{\frac{p^2}{4} + p} (dx^i)^2.$$

A fuzzy version of $\text{CP}^{\frac{p}{2}}$ can then be obtained by imposing the conditions at the level of matrices. This is achieved with a set of coordinates $X^i$ ($i = 1, \ldots, \frac{p^2}{4} + p$) in the irreducible totally symmetric representation of order $m$, $(m, 0)$, satisfying:
\[ [X^i, X^j] = i\Lambda(m)f_{ijk}X^k, \quad \Lambda(m) = \frac{1}{\sqrt{\frac{pm^2}{4\left(\frac{p}{2}+1\right)} + \frac{p}{4}m}} \]

with \( f_{ijk} \) the structure constants in the algebra of the generalized Gell-Mann matrices of \( SU\left(\frac{p}{2} + 1\right) \). The dimension of the \((m, 0)\) representation is given by

\[
\dim(m, 0) = \frac{(m + \frac{p}{2})!}{m!(\frac{p}{2})!}.
\]

The Kähler form of the fuzzy \( CP_{\frac{p}{2}} \) is given by:

\[
J_{ij} = \frac{1}{\frac{p}{2} + 1}\sqrt{\frac{p}{4\left(\frac{p}{2} + 1\right)}}f_{ijk}X^k.
\]