Scaling behavior of the quantum fisher information in the Dicke model

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Quantum Fisher information to the X states of the collective atom ensemble and the scaled quantum fisher information of the atomic subsystem are evaluated in the Dicke Hamiltonian. Critical behavior of the two kinds of Fisher information are in accordant with the analytical results in the thermodynamic limit. Finite-size scaling is analyzed with the large accessible system size due to the effective bosonic coherent-state technique proposed previously. Scaling exponents of the quantum Fisher information are obtained exactly.

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I. INTRODUCTION

During the last decade a plentiful contamination between condensed-matter physics and quantum-information theory has been exploited. Quantum Fisher information (QFI), one of the quantum-information-based tools, is a basic concept in quantum estimation theory, which depicts the theoretical bound for the variance of an estimator \[1\] \[5\]. Latterly, a new emphasis has emerged in which QFI is related to properties of interacting many-body systems. This approach is being pursued most vigorously in connection with quantum phase transition (QPT) \[6\], as it is hoped that the QFI may shed light upon the dramatic effects occurring in critical systems which involve complex collective quantum mechanical behavior. QPT and quantum-critical phenomena, which are induced by the change of parameters and are accompanied by a dramatic change of physical properties, occur at zero temperature in many-body quantum systems. And quantum critical systems have been shown to provide a resource for quantum estimation and metrology, offering precision in the characterization of coupling parameters \[7\] \[9\].

Consider the problem of estimating a parameter \(\theta\) from a quantum state \(\rho(\theta)\). The precision of the estimation of \(\theta\), which is limited by unavoidable measurement errors, is determined by the QFI. The geometrical approach to QPT has shown how to improve estimation strategies for experimental inaccessible parameters by driving the system toward critical points, where a sudden change in the ground-state structure takes place \[7\] \[10\]. In this framework, systems described by the Dicke model \[11\] provide nontrivial examples to assess quantum criticality as a resource for quantum estimation.

The Dicke model describes the interaction of \(N\) two-level atoms with a single bosonic mode. The QPT in a radiation-matter interacting system was explored based in the Dicke model, exhibiting a superradiant phase transition in the thermodynamics limit \[12\] \[15\]. Although the Dicke model cannot be solved analytically, an extended bosonic coherent state approach can solve the Dicke model numerical accurately for large size systems \[15\]. For finite-size atoms Dicke model has been characterized in terms of entanglement of its ground states \[13\] \[15\] \[17\] \[18\]. Fidelity susceptibility \[19\] and the Berry phase \[20\]. However, the QFI has not been well analyzed, except preliminarily results for the QFI of the field mode and atoms in the ground state \[21\]. A convincing scaling exponents of the QFI for the finite-size Dicke model is still lacking. To the best of our knowledge, the finite-size studies are limited to numerical diagonalization in the bosonic Fock state in small-size systems \(N \leq 35\) \[22\] \[23\].

Our paper is intended to solve the QFI to X states of the collective atom ensemble and the corresponding scaling exponents by our bosonic coherent-state technique.

In this paper, we study the QFI of the collective atom ensemble and the scaled QFI of the atomic subsystem in the Dicke mode both in the thermodynamics limit and for finite-size systems. And the scaling exponents are calculated. The paper is organized ad follows. In Sec. II we review the definition of the QFI and its physical signatures by the parameter estimation theory. In Sec. III we analyze the QFI to the X states of the collective atom ensemble by the pairwise reduced density matrix, and show the scaling behavior. Moreover, the finite-size scaling exponent for the QFI of the atomic subsystem is studied. Finally, we summarize our work in Sec. IV.

II. GENERAL FORMALISM FOR THE QFI

To begin with, we briefly review the parameter estimation theory and the quantum information tools it provides to evaluate bounds to precision of any estimation process involving quantum systems. The generalized quantum Fisher information (QFI) \(F_Q(\rho(\theta))\) is defined as \[5\] \[24\]

\[
F_Q(\rho(\theta)) = \text{Tr}[\rho(\theta)L^2].
\]

(1)

where the density operator \(\rho(\theta)\) is dependent on the parameter \(\theta\). The symmetric logarithmic derivative operator \(L\) is determined by

\[
\partial_\theta \rho(\theta) = \frac{1}{2}[L\rho(\theta) + \rho(\theta)L].
\]

(2)

Assume that the spectral decomposition of the density operator is given by \(\rho(\theta) = \sum_{i=1}^{s} p_{i} |\phi_{i}\rangle\langle \phi_{i}|\) with eigenvalues \(p_{i}\) and eigenvectors \(|\phi_{i}\rangle\) of \(\rho(\theta)\). Then the QFI can

\[
\rho(\theta) = \sum_{i=1}^{s} p_{i} |\phi_{i}\rangle\langle \phi_{i}|.
\]
be obtained as

\[
F_Q(\rho(\theta)) = \sum_{i=1}^{s} \frac{(\partial \rho_{p_i})^2}{p_i} - \sum_{i \neq j} \frac{8p_ip_j}{p_i + p_j} |\langle \varphi_i | \partial \varphi_j \rangle|^2
+ 4\sum_{i=1}^{s} p_i (|\langle \varphi_i | \partial \varphi_i \rangle|^2 - |\langle \varphi_i | \partial \varphi_i \rangle|^2)
\]

In Eq. (3), the first term is the classical Fisher information, which vanishes for pure states and for the unitary parametrization. By a unitary transformation \(\exp(-i\theta \hat{O})\) with the phase-shift generator \(\hat{O}\), the QFI obtained can be written as

\[
F_Q(\rho(\theta), \hat{O}) = 4\sum_{i=1}^{s} p_i (|\langle \varphi_i | \partial \varphi_i \rangle|^2 - |\langle \varphi_i | \partial \varphi_i \rangle|^2)
- \sum_{i \neq j} \frac{8p_ip_j}{p_i + p_j} |\langle \varphi_i | \partial \varphi_j \rangle|^2.
\] (4)

For pure states, it reduces to \(F_Q(\rho(\theta), \hat{O}) = 4\sum_{i=1}^{s} p_i (|\langle \varphi_i | \partial \varphi_i \rangle|^2 - |\langle \varphi_i | \partial \varphi_i \rangle|^2)\). From the QFI, we obtain the lower bound of the variance of the estimator for the parameter \(\theta\), given by the quantum Cramer-Rao (QCR) theorem: \((\Delta \theta)^2 \geq \frac{1}{F_Q(\rho(\theta), \hat{O})}\). The QFI \(F_Q(\rho(\theta), \hat{O})\) depends on the density matrix state \(\rho(\theta)\) and the choice of the phase-shift generator \(\hat{O}\).

III. SCALING OF QFI IN THE DICKE MODEL

We study the QFI and its scaling behavior for \(N\) two-level atoms system in the Dicke model. The Dicke Hamiltonian can be written in terms of the collective momentum form \(16, 29\)

\[
H = \omega a^\dagger a + \Delta J_z + \frac{2\lambda}{\sqrt{N}} (a^\dagger + a) J_x,
\]

where \(a^\dagger\) and \(a\) are the bosonic annihilation and creation operators of the single-mode cavity, \(\Delta\) and \(\omega\) are the transition frequency of the qubit and the frequency of the single bosonic mode, \(\lambda\) is the coupling constant. \(J_x\) and \(J_z\) are the collective spin operators. It exhibits a "superradiant" phase, where the atomic ensemble spontaneously emits with an intensity proportional to \(N^2\) \(25\). Thus, this model undergoes a second order QPT from the normal phase to the super-radiant phase, separated by the critical point \(\lambda_c = \sqrt{\omega \Delta}/2\).

A. Scaling behavior of the QFI to the X states of the atom ensemble

For the Dicke model, the reduced pairwise matrix in the standard basis \(\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}\) (with \(\sigma_z|\uparrow\rangle = |\uparrow\rangle\) and \(\sigma_z|\downarrow\rangle = |\downarrow\rangle\) \(26\)) can be derived as

\[
\rho = \begin{pmatrix}
   v_+ & w^* & x_+ & u^* \\
   w & v & x_\dagger & w \dagger \\
x_\dagger & x_\dagger & y & w \dagger \\
u & u & w & v_\dagger
\end{pmatrix}.
\] (5)

The detailed expressions for these elements are

\[
v_\pm = \frac{N^2 - 2N + 4(J^2_\pm)\pm 4(N - 1)|J_z|}{4N(N - 1)},
\]

\[
x_\pm = \frac{(N - 1)^2 |J_z| \pm (J_+^2 + J_\dagger^2)}{2N(N - 1)},
\]

\[
w = \frac{N^2 - 4(J^2_\pm) - N/2}{4N(N - 1)},
\]

\[
u = \frac{(J^2_\pm)}{N(N - 1)},
\]

where \([A, B]_+ = AB + BA\). \(w = y\) for \(\sum_{\alpha=x,y,z} J^2_\alpha = J^2 = N/4(N^2 + 1)\). For the symmetric states with parity conservation, we find \(x_\pm = 0\) from Refs. \(26, 27\). Hence the pairwise reduced density matrix is shown in \(X\) form as

\[
\rho = \begin{pmatrix}
v_+ & 0 & 0 & u^* \\
0 & w & y & 0 \\
0 & y & w & 0 \\
u & 0 & 0 & v_\dagger
\end{pmatrix}.
\] (7)

For the two-level collective atom ensemble states in \(X\) form, the QFI may be derived analytically.

We consider an estimation of the parameter \(\theta\) introduced by the following unitary operation \(U = \exp(-i\theta \sigma_0^2)\) with \(\sigma_0^2 = \sigma_z \otimes I\) \(28\). Here \(I\) is the \(2 \times 2\) identity matrix and \(\sigma_z\) is the pauli matrix. From the definition in Eq. (4), the QFI are obtained as

\[
F_Q(\rho, \sigma_0^2) = 16(\frac{u^2}{v_+ + v_\dagger} + \frac{w}{2})
\] (8)

which are evaluated in detail in Appendix.

We first apply the Holstein-Primakoff transformation to change the collective angular operators to the boson operators \(b(b^\dagger)\) by \(J_+ = b^\dagger \sqrt{N} - b^\dagger b, J_\dagger = \sqrt{N} - b^\dagger bb, J_z = b^\dagger b - N/2\), and then the displacements of the boson operators are introduced to depict the behaviors of super-radiation phase as \(c^\dagger = a^\dagger + \sqrt{N}b\) and \(d^\dagger = b^\dagger - \sqrt{N}\beta\). By using large \(N\) expansions of \(H_{\text{Dicke}}\) with respect to the new boson operators \(c^\dagger\) and \(d^\dagger\) up to the \(1/N\), we obtain the ground state energy as

\[
\frac{E_G(\alpha, \beta)}{N} = \omega\alpha^2 - 4\alpha\beta \sqrt{1 - \beta^2} + \Delta(\beta^2 - 1).
\]

Minimizing the ground state energy gives

\[
\omega\alpha - 2\beta \sqrt{1 - \beta^2} = 0 \quad (9)
\]

\[
2\alpha\lambda \sqrt{1 - \beta^2} - \frac{2\alpha^2 \beta^2}{\sqrt{1 - \beta}} - \beta\Delta = 0.
\]
then we have

$$\beta^2 = \max\{0, \frac{1}{2}(1-\mu)\}, \quad \alpha = \frac{2\lambda}{\omega} \beta \sqrt{1-\beta^2}. \quad (10)$$

where $\mu = 1$ in the normal phase and $\mu = (\lambda_c/\lambda)^2$ in the superradiant phase with the critical point $\lambda_c = \sqrt{\omega \Delta}/2$. Next we can derive the matrix elements of the pairwise reduced density in Eq. (7) up to $O(1)$

$$v_+ = \beta^4, \quad v_- = (1-\beta^2)^2, \quad w = y = \beta^2(1-\beta^2), \quad u = \beta^2(1-\beta^2). \quad (11)$$

From Eq.(4), the QFI to the X state in the thermodynamics limit $F_{Q,\infty}$ is obtained as

$$F_{Q,\infty} = \frac{8\beta^2(1-\beta^2)}{\beta^4 + (1-\beta^2)^2}. \quad (13)$$

Thus, at the critical point, one has the QFI in the thermodynamics limit $F_{Q,\infty}(\lambda_c) = 0$.

We plot the QFI in the thermodynamics limit for different detunings $D = \Delta/\omega$ in Fig.(1)(a). Without loss of generality, we mainly focus on the resonant case $D = 1$ and off-resonance $D = 0.5$ in the following. The QFI in the normal phase equals to zero for $\beta = 0$, which indicates that the variance of our estimation is large. As the coupling strength enters into the super-radiant phase $\lambda > \lambda_c$, the QFI shows monotonous increasing behaviors, demonstrating the existence of the QPT at the critical point $\lambda_c = \sqrt{2}/2$ for $D = 1$. As $\lambda$ approaches $\lambda \rightarrow \infty$ limit, giving $\beta \rightarrow \frac{1}{2}$, the QFI tends to the constant value $F_{Q,\infty} \rightarrow 4$.

We next investigate the QFI in the finite size Dicke model. An extended coherent state technique $^{[16]}$ has been proposed to the Dicke model up to very huge size. This effective approach has been confirmed recently by comparing with the results in terms of basis of the Fock states $^{[30]}$. It was demonstrated that it is very difficult to obtain convergent results for large number of atoms based on usual basis of the Fock states $^{[30]}$.

In the numerically exact approach $^{[16]}$, the wave function can be expressed in terms of the basis $\{|\varphi_n\rangle_b \otimes |j, n\rangle\}$, where $|j, n\rangle$ is the Dicke state with $j = N/2$ and $|\varphi_n\rangle_b$ is the bosonic extended coherent state

$$|\varphi_n\rangle_b = \sum_{k=0}^{N_{tr}} c_{n,k} \frac{1}{\sqrt{k!}} (a^\dagger + g_n)^k e^{-g_n a^\dagger a - g_n^2/2} |0\rangle_a, \quad (14)$$

where $g_n = 2\lambda n/(\omega \sqrt{N})$. $N_{tr}$ is the truncated bosonic number in the space of the new operator $A_n = a + g_n$, $|0\rangle_a$ is the vacuum as $a|0\rangle_a = 0$, and the coefficient $c_{n,k}$ can be determined through the exact Lanczos diagonalization. Then, we can derive the elements of pairwise density matrix in Eq. (7) and calculate the QFI $F_Q$ in Eq.(5) in finite number $N$ of atoms.

Fig.(1)(a) displays the QFI $F_Q$ to the X states of the Dicke model as a function of the atom-cavity coupling strength for different detunings $D = 0.5$ and 1 for several system sizes $N = 20, 256$. In the normal phase, it tends to zero with the increase of the atomic number, which agrees well with that in the thermodynamic limit. While in the super-radiant phase, $\lambda > \lambda_c$, the QFI increases from zero to the maximum value 4. For the large system size $N = 256$, there is nearly no deviation of the behavior of the QFI from that in the thermodynamic limit, locating the accurate critical point at $\lambda_c = \sqrt{2}/2$ for detuning parameters $D = 1$. Because there is absence of the finite-size scaling for the QFI, it facilitates the calculation of the scaling behavior at the critical point.

Next, we illustrate the scaling behavior of the QFI to the X state of the collective atom ensemble $F_{Q,N}$ in the vicinity of the critical point $\lambda_c$ $^{[13]}$.

$$F_{Q,N}(\lambda) \simeq \frac{(\lambda - \lambda_c)^k}{N^n} f[N(\lambda - \lambda_c)^{3/2}], \quad (15)$$

where $f$ is a function depending on the scaling variable $N(\lambda - \lambda_c)^{3/2}$ and $k$, $n$, $\xi$ are exponents. To cure the sin-
gularity coming from \((\lambda - \lambda_c)^\xi\), one has \(f(x) \sim x^{-2\xi/3}\), which leads to \(F_{Q,N}^{\text{fin}}(\lambda_c) \sim N^{-(n+2\xi/3)}\).

The QFI \(F_Q\) in Eq. (14) can be expressed explicitly in terms of collective operators \(\langle J_z^2 \rangle\) and \(\langle J_z^2 \rangle\)
\[
F_Q = \frac{32\langle J_z^2 \rangle^2}{N(N-1)(N^2 - 2N + 4\langle J_z^2 \rangle)} + 2N^2 - 4\langle J_z^2 \rangle \frac{N(N-1)}{N(N-1)}
\]

Since finite-size scaling exponents at the critical point for several observable in the Dicke model has been derived by Vidal and Dusuel [13], such as the scaling behaviors of the two-point correlation function \(\langle J_z^2 \rangle/N^2 \sim N^{-2/3}\), \(\langle J_z^2 \rangle/N^2 \sim N^{-4/3}\) and \(\langle J_z^2 \rangle/N^2 \sim N^{-2/3}\). From those, it is easily to obtain the finite-size scaling behavior of QFI exactly as
\[
F_Q \sim N^{-2/3}.
\]

It is very interesting to observe a power law scaling \(F_Q(\lambda - \lambda_c) \propto N^\nu\) at the critical point by the bosonic coherent state in Eq. (14) for different detunings \(D = 0.5\) and 1, as shown in Fig. (1)(b). The asymptotic slope in the log-log scale for the finite size systems gives a exponent \(\nu = -0.65 \pm 0.01\) with the atom number up to \(N \sim 4000\), which agrees well with Eq. (17). To the best of our knowledge, such a finite size scaling for the QFI itself has never been reported in Dicke model.

### B. Scaling behavior of the QFI of the atomic subsystem

As addressed in [21], the QFI of the atoms in the ground state of the Dicke model has been reported as a signature of the superradiant phase transition. But there is absence of the finite-size scaling behavior of the QFI of the atoms. It is interesting to study the scaling of the QFI of the atomic at the critical point. We investigate the QFI of the atomic subsystem with respect to the unitary transformation \(\rho_A(\phi) = e^{-i\phi J_z} \rho_A e^{i\phi J_z}\), where \(\rho_A\) is the atom reduced density matrix and \(\phi\) is a parameter for an estimation. In general, the atom reduced state \(\rho_A = \text{Tr}_{\text{ph}}(\langle G \rangle \langle G \rangle)\) is obtained by tracing over the field degree of the freedom, where \(\langle G \rangle\) is the ground state wavefunction. And the QFI of the atomic subsystem \(F_A\) can be evaluated from Eq. (14) as
\[
F_A(\rho_A, J_z) = 4 \sum_n p_n \langle \phi_n | J_z^2 | \phi_n \rangle - |\langle \phi_n | J_z | \phi_n \rangle|^2
- \sum_{m \neq n} 8 p_m p_n |\langle \phi_m | J_z | \phi_n \rangle|^2,
\]

where the weights \(\{p_n\}\) are nonzero eigenvalues of \(\rho_A\), and \(\{|\phi_n\}\) are the corresponding eigenvectors.

In the thermodynamics limit, the analytical results of the scaled QFI for the atomic subsystem is [21]
\[
F_{A,\infty}/N = \frac{2\mu \Delta}{\varepsilon_+ + \varepsilon_- + (\Delta^2/\mu^2 - \omega^2)/(\varepsilon_+ + \varepsilon_-)}
\]

where the excitation energies is given by
\[
\varepsilon_{\pm}^2 = \frac{1}{2}(\omega^2 + \Delta^2/\mu^2) \pm \frac{1}{2} \sqrt{(\omega^2 - \Delta^2/\mu^2)^2 + 16\lambda^2\omega^2}\Delta\mu.
\]

At the critical point \(\lambda_c\), the QFI in the thermodynamics limit is given analytically
\[
F_{A,\infty}(\lambda_c)/N = \frac{\sqrt{\omega^2 + \Delta^2}}{\Delta}
\]

We calculate the scaled QFI \(F_A/N\) for the atomic subsystem for different values of \(D\) in Fig. (2)(a). In the thermodynamic limit, when \(\lambda = 0\), we obtain \(F_{A,\infty}/N = 1\) for \(D = 1\). The scaled QFI \(F_A/N\) displays non-monotonous behavior as the coupling increases, which is different from the QFI to X states in Fig. (1)(a). There is a maximum at the critical point, then it decreases to zero in the super-radiant phase, which agrees with the results in the thermodynamic limit \(F_{A,\infty}/N \to 0\) for \(\lambda \to \infty\).
note that the maximum of the scaled QFI \( F_A \) approaching the critical point is consistent with analytical results in Eq. (21). The singularity of \( F_A/N \) demonstrates the phase transition at the critical point, which agrees well with results in Ref. [21].

Next, we study the finite-size scaling of the scaled QFI for the atomic subsystem. The QFI per atom \( F_A/N \) near the critical point is evaluated as

\[
F_A(N \to \lambda_c) \approx \frac{\sqrt{\omega^2 + \Delta^2}}{\Delta} + \sqrt{\frac{32\omega^2\lambda_c^3}{\Delta^2(16\lambda_c^4 + \omega^4)}}|\lambda_c - \lambda|^{1/2}.
\]

(22)

The vanishing of the scaled QFI \( (F_A - F_{A,\infty})/N \) at \( \lambda_c \) reveals this to be a second-order phase transition. Using the scaling hypothesis (16), the expression of \( F_A/N \) allow us to indentify \( \xi = 1/2 \) and \( n = 0 \), leading to the scaling behavior

\[
(F_A - F_{A,\infty})/N \sim N^{-1/3}.
\]

(23)

Moreover, we calculate \( (F_{A,\infty} - F_A)/N \) numerically at the critical point \( \lambda_c \) as a function of \( N \) for different values of \( D = 0.5 \) and 1 using the bosonic coherent-state technique [13] on a log-log scale, as shown in Fig. (2)(b). A power-law behavior exists at large \( N \). Due to the advantage of the bosonic coherent state technique [16], we are able to study the system up to \( N = 2000 \sim 3000 \) atoms. One can see that the finite-size exponents extracted from all curves tend to a converging values \(-0.33 \pm 0.01\), which consistent with exact results in Eq. (22). The scaling behavior has not been reported in the critical systems of Dicke model. And the precise estimate of the scaling exponent for the QFI is very significant to help clarify the universality of the QPT.

IV. CONCLUSION

In summary, we have proposed the QFI to the X states of the collective atom ensemble by the pairwise reduced density matrix and the scaled QFI of the atomic subsystem by the atom reduced matrix of the Dicke Hamiltonian. It allowed us to quantify the variance of the estimator across a QPT. The QFI in the thermodynamic limit have been evaluated analytically. We find distinguished difference behaviors of the two kinds of Fisher information. Especially, the scaled QFI of the atomic subsystem displays a singularity at the critical point. Finite-size scaling of the QFI are calculated up to large atoms number \( N = 3000 \sim 4000 \). Power law scaling behavior at the critical point is observed in both Fisher information. Such a scaling behavior has not been reported in the critical systems of Dicke model, as far as we know. These salient features might be used for quantum metrology and quantum estimation in some experimentally realized systems to the quantum information science and the quantum computing.

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Appendix: the QFI to the X states of the collective atom ensemble

For the pairwise atom reduced density matrix in Eq. (17), the corresponding eigenvalues are given by

\[
p_1 = 2w, p_2 = 0, p_{\pm} = \frac{1}{2}(v_+ + v_- + \sqrt{\Delta}),
\]

(A.1)

where \( \Delta = (v_+ - v_-)^2 + 4|u|^2 \). The corresponding eigenstates to \( p_1 \) and \( p_{\pm} \) are

\[
|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]

(A.2)

\[
|\phi_{\pm}\rangle = \epsilon_{\pm} \begin{pmatrix} v_+ - v_- \pm \sqrt{\Delta}/2u \\ 0 \\ 1 \end{pmatrix},
\]

(A.3)

with \( \epsilon_{\pm}^2 = 2u/\sqrt{\Delta \pm (v_+ - v_-)\sqrt{\Delta}} \). For the unitary operation \( U = \exp(-i\theta\sigma_z) \), the QFI to the X states of the collective atom ensemble can be evaluated as

\[
F_Q = 4p_{\pm}(\Delta^2\sigma_z^0)^{\pm} + 4p_1(\Delta^2\sigma_z^1)
- \frac{16p_{\pm}p_1}{p_+ + p_-}|\langle \phi_+|\sigma_z^0|\phi_+\rangle|^2
- \sum_{i=\pm}^{1} \frac{16p_{\pm}p_1}{p_i + p_1}|\langle \phi_i|\sigma_z^0|\phi_i\rangle|^2,
\]

(A.4)

where the variance of operator \( \sigma_z^0 \) is \( \langle \Delta^2\sigma_z^0 \rangle_i = \langle \phi_i|\sigma_z^0|^2|\phi_i\rangle - |\langle \phi_i|\sigma_z^0|\phi_i\rangle|^2 \). Substituting the values of \( p_{\pm,1} \) and \( |\phi_{\pm,1}\rangle \) into the equation, the QFI can be given analytically.

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