Equivalence of Constrained Models

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Abstract: We study two constrained scalar models. While there seems to be equivalence when the partially integrated Feynman path integral is expanded graphically, the dynamical behaviours of the two models are different when quantization is done using Dirac constraint analysis.
INTRODUCTION

There are many models in the literature which claim that they are equivalent to Quantum Chromodynamics (QCD) in some sense. We can cite a recent work by Hasenfratz and Hasenfratz /1/, or the work of Akdeniz et al /2/ among many papers concerning this topic. We mention these two papers since in these papers the starting lagrangians look very different, although the effective lagrangians obtained after some manipulations are made, are exactly the same. There are many other papers in the same spirit. One example is reference 3, which, in some sense, gave rise to reference 1. Another example is reference 4, which was actually a pioneering paper in this endeavor of finding models whose effective lagrangian looks like the standard model.

It was always a puzzle to us how similar manipulations made on very differently looking Lagrange functions resulted in completely the same effective lagrangian. In this note we try to investigate this phenomena using scalar models. We think that our results in the scalar case may give additional information on this phenomenon.

We will use two constrained scalar models. In reference 1, the authors imposed the constraint $J_{\mu}^f = \bar{\psi}i\gamma_{\mu}\tau_{\alpha}\psi = 0$ on the fields of the free spinor lagrangian. To resemble this model here we first study the case where we write a lagrangian which is essentially equivalent to $L = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi$. We will impose a different constraint, though, since a constraint $\phi\partial_{\mu}\phi = 0$ results in a truly trivial model. We can also introduce inner symmetry to the theory and make a $O(N)$ model along similar lines. For the time being we do not pursue this.

The authors in reference 2 impose the condition that their current $J_{\mu}^f$ equals product of vector fields instead of zero, $J_{\mu}^f = A_{\mu}A^2$, which differs from the constraint used in reference 1. This complicates the problem, but all of additional fields introduced to the model decouple and at the end only one vector field survives. The propagator for this field, and only for this field, is generated in the one loop calculation. At this point the resulting effective theory looks exactly like that of reference 1.
We have doubts whether these two models are actually equivalent to QCD in all aspects. One may refer to an old work of Wilson \textsuperscript{5} and to a more recent work of Zinn-Justin \textsuperscript{6}, and using the calculations made in reference 7, claim that these two models are actually examples of trivial models \textsuperscript{8}.

We will not dwell on these points here. We will only investigate in what sense two models are equivalent when the effective lagrangians derived from them seem so. In the next section we present two constrained scalar models. We get a theory which is totally trivial if we impose the current made out of scalar fields equal to zero, the analogous case as given in reference 1. We instead use two models where the current is equal to one and two auxiliary fields, thus introducing eight and sixteen new degrees of freedom respectively plus constraints that will eliminate these. We study the Dirac bracket relations satisfied by the respective fields. We see that the new introduced vector fields via the constraint equations somehow replace the canonical momentum of the scalar field.

In Section III we derive effective lagrangians for these two cases and show why do they seem to be equivalent on this level. We end with some remarks.

II. Quantization of the Models using Dirac Constraint Analysis

II. A

We start with

\[ L_A = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + ig \lambda_\mu \phi \partial^\mu \phi + \frac{1}{2} g^2 \phi^2 \lambda^2. \]  

We are in four dimensional Minkowski space and \( \mu \) takes the values zero to three. Here \( \lambda_\mu \) is an auxiliary field with no kinetic term. \( g \) is a coupling constant.

The equations of motion are

\[ \partial_\mu \partial^\mu \phi + ig \partial_\mu (\lambda^\mu \phi) = ig \lambda_\mu \partial^\mu \phi + g^2 \lambda^2 \phi, \]  

\[ ig \phi \partial_\mu \phi = -g^2 \lambda_\mu \phi^2 \]
which can be shown to be equivalent to
\[ \partial_\mu \partial^\mu \phi = 0. \]  

In this calculation we use the methods given in Dirac’s book /9/. The canonical momenta are
\[ \pi_\phi = (\partial_0 + ig\lambda_0)\phi \]  
and
\[ \pi_{\lambda_\mu} = 0 \]  
which gives us four primary constraints.

The canonical hamiltonian is
\[ H = \frac{1}{2} (\pi_\phi - ig\lambda_0\phi)^2 + ig\lambda_i\phi\partial_i\phi + \frac{1}{2} \partial_i\phi\partial_i\phi - \frac{1}{2} g^2 \lambda^2 \phi^2. \]  
We get secondary constraints when we set the Poisson bracket of \( H \) with \( \pi_{\lambda_\mu} \) equal to zero.

\[ K_\mu = \frac{d}{dt} \pi_{\lambda_\mu} = (ig\phi \pi_\phi + g^2 \lambda_0 \phi^2)g_{\mu_0} - ig\phi \partial_i\phi g_{i\mu} + g^2 \lambda_\mu \phi^2 = 0. \]  
We take
\[ H_E = H + c_\mu \pi_{\lambda_\mu} \]  
and further calculate
\[ \frac{d}{dt} K_\mu = [H_E, K_\mu] \]  
where square brackets mean Poisson brackets. Note that \( \lambda_0 \) appears in \( K_0 \), and the \( \lambda_i \) in \( K_i \). We get one equation with \( c_0 \) when \( [H_E, K_0] \) is calculated, which fixes the value of \( c_0 \) and does not give additional constraints on the system. \( [H_E, K_i] \) give us equations which fix \( c_i \). We do not get any additional constraints.

We can calculate the Poisson brackets between the different constraints.

\[ [\pi_{\lambda_0}, K_0] = 2g^2 \phi^2 g_{00}, \quad [\phi_{\lambda_i}, K_j] = -g^2 \phi^2 g_{ij}, \]
\[ [K_0, K_i] = g^2 \phi \partial_i \phi - 2ig^3 \lambda_i \phi^2 \]

all the other brackets of the constraints with each other are zero. We see that all these brackets are second class. We calculate Dirac brackets between different fields.

\[ [\phi(x), \lambda_0(y)]^D = \frac{i}{2g\phi} \delta^3(x - y), \]

\[ [\phi(x), \lambda_i] = 0, \]

which shows that \( \lambda_0 \) is like \( \pi_\phi \), and \( \lambda_i \) decouples. We can set \( \lambda_i \) equal to zero.

II.B

We propose another model where

\[ L_B = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + ig(\lambda_\mu + A_\mu)\phi \partial^\mu \phi - g A_\mu \lambda^\mu A^2 \]

The primary constraints are

\[ \pi_{A_\mu} = 0 \]

\[ \pi_{\lambda_\mu} = 0 \]

The Hamiltonian reads

\[ H = \frac{1}{2}(\pi_\phi - ig(\lambda_0 + A_0)\phi)^2 + \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + ig(\lambda_i + A_i)\phi \partial_i \phi + g \lambda_\mu A^\mu A^2 \]

where

\[ \pi_\phi = \partial_0 \phi + ig(\lambda_0 + A_0)\phi \]

\[ H_E = H + c^\mu_1 \pi_{A_\mu} + c^\mu_2 \pi_{\lambda_\mu} \]

Secondary constraints

\[ [H_E, \pi_{A_\mu}] = Q^1_\mu = 0 \]

\[ [H_E, \pi_{\lambda_\mu}] = Q^2_\mu = 0 \]

are given as

\[ Q^1_\mu = ig\phi(\pi_\phi - ig(\lambda_0 + A_0)\phi)g_{\mu 0} - ig\phi \partial_\mu \phi g_{i\mu} - g \lambda_\mu A^2 - 2g \lambda_\nu A^\nu A_\mu \]
\[
Q^2_{\mu} = ig\phi(\pi_\phi - ig(\lambda_0 + A_0)\phi) g_{\mu0} - ig\phi \partial_i \phi g_{i\mu} - gA_\mu A^2
\]

We see that the system is closed, since the Poisson brackets of \(Q^1_{\mu}, Q^2_{\mu}\) with \(H_E\) involve eight coupled equations for \(c^1_{\mu}\) and \(c^2_{\mu}\). We get no further constraints.

When we calculate the Poisson brackets of the constraints with each other, we see that they are all of second class. We have sixteen second class constraints and eighteen degrees of freedom. We have traded some of our fields in terms of others, but we did not change the number of independent variables.

Now we can calculate the Dirac brackets between different fields. We are particularly interested in the brackets between \(\phi\) and \(A_\mu, \lambda_\mu\), since the Poisson brackets between the same fields are zero. The effect of the constraints in the system are reflected to the Dirac brackets; hence, they do not vanish when they are taken between \(\phi\) and the auxiliary fields. We give below the result of some sample calculations.

\[
[\phi(x), A_1(y)]^D \phi(y) = \frac{i\phi^2(-2A_0A_1)[g\phi^2(A^2 - 2A^2) - 3A^4]}{\Delta} \delta^{(3)}(x - y), \quad 21a
\]

\[
[\phi(x), A_0(y)]^D \phi(y) = \frac{i\phi^2(A^2 - 2A^2)[g\phi^2(A^2 - 2A^2) - 3A^4]}{\Delta} \delta^{(3)}(x - y), \quad 21b
\]

\[
[\phi(x), \lambda_0(y)]^D \phi(y) = \frac{i\phi^2(A^2 - 2A^2)[2g\phi^2(A^2 - 2A^2) - 3A^4 - 4A^2\lambda_\mu A^\mu]}{\Delta} \delta^{(3)}(x - y), \quad 21c
\]

\[
[\phi(x), \lambda_2(y)]^D \phi(y) = 2i \frac{\phi^2[-3A_0A_2(A^4 - 4A^2A_\mu A^\mu) - 3A^4(A_0\lambda_2 + A_2\lambda_0)]}{\Delta} \delta^{(3)}(x - y) \quad 21d
\]

where

\[
\Delta = [g^2\phi^4(A^2 - 2A^2)^2 - (6A^2 + 4\lambda_\mu A^\mu)(A^2 - 2A^2)A^2 g\phi^2 + 9A^8]
\]

Here \(A^2\) means the three vector \(A\) squared.

Upon quantization we see that \(A_\mu\) and \(\lambda_\mu\) seem to contain part of \(\pi_\phi\). We expect only \(\pi_\phi\) to have nonzero commutation relations with \(\phi\) and in this model both \(A_\mu\) and \(\lambda_\mu\) also will have non zero commutations with \(\phi\). The constraints \(Q^1_{\mu} = 0\) and \(Q^2_{\mu} = 0\) relates \(\pi_\phi\) to these fields.

The model we studied seems to be considerably different from the one studied in the first Section. The fields in the model has non zero Dirac brackets; so, we can not set them
equal to zero, as in the previous model. The space components of the vector fields do not
decouple and can not be set to zero.

Note that in both of these models the degrees of freedom is two. In Model A we start
with ten degrees of freedom, two for the $\phi$ and eight for the $\lambda$ field and their respective
momenta. Eight constraints reduce these to two. In Model B we start with eighteen
degrees of freedom since we have two vector particles. Sixteen constraint equations reduce
this number to two. As far as the equations of motion are considered these two models do
not seem to be alike.

III. Feynman Rules using the Path Integral

Here we study the two models using Feynman diagram expansions of the path integral
after the integral is partially integrated. We start by studying model A, then contrast our
results with that of Model B.

III.A Here the path integral is written as

$$Z = \int d\phi d\pi d\lambda d\pi d\lambda \delta(\pi_\mu)\delta(\lambda_\mu)\delta(K_\mu)\det M_{\mu\nu} \exp iS,$$

where

$$M_{\mu\nu} = \frac{\partial K_\mu}{\partial \lambda_\nu},$$

$$S = \int d^4x [\pi_\phi \partial_0 \phi + \pi_\lambda \partial_\mu \lambda_\mu - H_E].$$

We write the Dirac delta functions in the integral form, introducing new variables $A_\mu$ and
express the determinant in the exponential form using ghost fields.

$$\delta(K_\mu) = \frac{1}{2\pi} \int dA_\mu e^{-i A_\mu K_\mu},$$

$$\det M_{\mu\nu} = \int dc_\mu dc_\nu e^{ic_\mu M_{\mu\nu} c_\nu}.$$

The integrations over the momenta and $\phi$ are performed easily and we end up with

$$Z = N \int dA_\mu d\lambda d\mu d\lambda dc_\alpha dc_\beta e^{-\frac{i}{2} \text{tr} \log \left[ -\partial_\mu \partial_\nu + ig N_\mu \partial_\nu - ig \partial_\mu N_\nu + g^2 \left( \frac{1}{2} \lambda^2 - A_\mu \lambda_\mu - \frac{1}{2} A_\alpha^+ c_\alpha + c_\alpha^+ c_\alpha + c_\mu^+ c_\mu \right) \right]}. $$
where we define $N_\mu = \lambda_\mu - A_\mu$. We can calculate the inverse propagator, $D_{\mu\nu}^{-1}$ for the $N_\mu$ field by taking two derivatives of eq. (25) with respect to the $N_\mu$ field. In the momentum representation we get

$$D_{\mu\nu}^{-1}(q) = -g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{(p_\mu + q_\mu)(p_\nu + 2q_\nu)}{p^2(p + q)^2}$$

$$= -g^2 \frac{\Gamma(\epsilon)}{6(4\pi)^2} \left( g_{\mu\nu} q^2 - 10q_\mu q_\nu \right)$$

which looks like the massless vector boson propagator, at least in a particular gauge. Note that all the components of the vector field have non-zero propagation.

All other fields have zero propagators if we use dimensional regularization. Here we set $\int d^4 \frac{1}{p^2} = 0.$ When we drop all the fields with zero propagators we end up with

$$S_{eff} = -\frac{1}{2} Tr \log (-\partial^2 + igN_\mu \partial^\mu - ig \partial^\mu N_\mu).$$

Upon expanding the logarithm we can evaluate the multi-point functions for the $N^\mu$ fields. Eq. 26 dictates a necessary condition on the coupling constant $g$, though, to have a well defined expression for the propagator function given by this equation, which reads

$$g^2 \frac{\Gamma(\epsilon)}{6(4\pi)^2} = 1.$$  

This condition makes the model asymptotically free in the ultraviolet regime.

By taking all the non vanishing terms we see that for the composite field $\lambda_\mu$ the effective lagrangian can be written as

$$L_{eff} = \frac{1}{2} \partial_\mu N_\nu \partial^\mu N_\nu + \partial_\mu N_\nu \partial^\nu N_\mu + gf^{\mu\nu\rho} N_\mu N^\nu N^\rho + g^2 V_{\mu\nu\rho\sigma} N_\mu N^\nu N^\rho N^\sigma.$$  

Here $f_{\mu\nu\rho}$ is proportional to momentum and Kronecker deltas and $V_{\mu\nu\rho\sigma}$ is made out of Kronecker deltas. Higher order functions, starting with the fifth point function, drop with higher powers of $g$. For example the five point function goes as $g^5$. They do not fit into this scheme of effective lagrangian and are calculated as loop corrections.

Here we calculated the Feynman rules for this model and showed that apart from the restriction dictated by eq. 28, we get rules similar to those as a gauge theory. One can
calculate physical processes using these rules and will get free parton model results, as is
the case in a similar model due to the restriction dictated by eq. 28. All the physical
processes that involve interactions will involve powers of the coupling constant which goes
to zero. Any possible divergences due to loops will be canceled by the zeroes coming from
extra powers of the coupling constant. Only terms which do not involve any interactions
are finite. These terms are the same as those given in the free field case.

III.B The path integral for Model B, in the hamiltonian formalism, is written as

\[ \int dA_\mu d\pi_{A_\mu} d\pi_{\lambda_\mu} d\lambda_\mu d\phi d\pi_\phi \delta(\pi_{\lambda_\mu}) \delta(\pi_{A_\mu}) \delta(Q^1_\nu) \delta(Q^2_\nu) (detM) \exp iS \]

Here

\[ S = \int d^4x [\pi_\phi \partial_0 \phi + \pi_{A_\mu} \partial_0 A_\mu + \pi_{\lambda_\mu} \partial_0 \lambda_\mu - \frac{1}{2} [\pi_\phi + ig(\lambda_0 + A_0) \phi]^2 - \frac{1}{2} \partial_i \phi \partial_i \phi \]

\[ -i g(\lambda_i + A_i) \phi \partial_i \phi - g \lambda_\mu A^\mu A^2] \]

\[ Q^1_\mu = \phi(\pi_\phi - (\lambda_0 + A_0) \phi) g_{\mu 0} - \phi \partial_i \phi g_{i\mu} - \lambda_\mu A^2 - 2 \lambda_\nu A^\nu A_\mu \]

\[ Q^2_\mu = \phi(\pi_\phi - (\lambda_0 + A_0) \phi) g_{\mu 0} - \phi \partial_i \phi g_{i\mu} - A_\mu A^2 \]

M is an eight by eight matrix whose entities are made out of derivatives of \(Q^1_\mu\) and \(Q^2_\mu\) with
respect to the fields \(A_\mu\) and \(\lambda_\mu\).

We can use the integral representation of the Dirac delta functions.

\[ \delta(Q^1_\mu) = \frac{1}{2\pi} \int dB_\mu \exp -i B^\mu Q^1_\mu \]

\[ \delta(Q^2_\mu) = \frac{1}{2\pi} \int dE_\mu \exp -i E^\mu Q^2_\mu \]

Using ghost, i.e. Grassmann valued fields \(c_\mu, e_\mu, c^\dagger_\mu, e^\dagger_\mu\), we can raise \(detM\) to the exponential.

\[ detM = \int dc^\dagger_\mu dc_\nu de^\dagger_\sigma de_\rho \exp iN \]

where

\[ N = (c^\dagger_\mu + e^\dagger_\mu)(g^2 \phi^2 g_{\mu 0} g_{\nu 0})(c_\nu + e_\nu) + c^\dagger_\mu(-2gA_\mu \lambda_\nu - 2gg_{\mu \nu} \lambda_\kappa A^\kappa - 2g\lambda_\mu A_\nu - 2g\lambda_\nu A_\mu) c_\nu \]
+ c_\mu^\dagger (\nabla^\mu A^2 - 2 g A^\mu A^\nu) e_\nu + e_\mu^\dagger (\nabla^\mu A^2 - 2 g A^\mu A^\nu) c_\nu  

When the momentum integrals are performed we get

\begin{align*}
L_{\text{eff}} &= i \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + ig G_\mu \phi \partial^\mu \phi - \frac{g^2}{2} (B_0 + E_0)^2 \phi^2 - g \lambda A^\mu A^2 + g B^\mu \lambda A^2 \\
&+ 2 g B_{\mu \nu} \lambda A^\nu + g E_{\mu A^\mu A^2} + g^2 f_0^\dagger \phi f_0 + c_\mu^\dagger (\nabla^\nu A^\mu - 2 g A^\mu A^\nu) f_\nu \\
&+ f_\mu^\dagger (\nabla^\nu A^\mu - 2 g A^\mu A^\nu) e_\nu + c_\mu^\dagger [2 g A^2 g^\mu A^\nu - 2 g \lambda A^\nu A^\nu - 2 g g^\mu A_\kappa A^\kappa - 2 g A^\mu A^\nu + 4 g A_\mu A^\nu] c_\nu 
\end{align*}

Here $f_\mu = c_\mu + e_\mu$. We set $G_\mu = A_\mu + \lambda - B_\mu - E_\mu$.

We perform the integration over $\phi$ and obtain

\begin{align*}
S_{\text{eff}} &= - \frac{\text{Trlog}}{2} [-\partial^2 + ig G_\mu \partial^\mu - ig \partial_\mu G - \frac{g^2}{2} (B_0 + E_0)^2 + g^2 f_0^\dagger f_0] \\
&+ \int d^4 x \left[ -2 g \lambda A_\mu A^2 + g B_{\mu \nu} \lambda A^2 + 2 g \lambda A_{\mu A^\nu} + g E_{\mu A^\mu A^2} + 4 g A^\nu A^\mu + c_\mu^\dagger (\nabla^\nu A^\mu - 2 g A^\mu A^\nu) e_\nu \\
&+ f_\mu^\dagger (\nabla^\nu A^\mu - 2 g A^\mu A^\nu) e_\nu + c_\mu^\dagger [2 g A^2 g^\mu A^\nu + 4 g A_\mu A^\nu - 2 g \lambda A^\nu A^\nu - 2 g \lambda A_\mu A^\nu + 2 g \lambda A^\mu A^\nu] c_\nu 
\end{align*}

Note that only $G_\mu$ propagates among all the fields given above. To find the propagator we take two derivatives with respect to the respective fields.

\begin{align*}
\frac{\partial^2 S_{\text{eff}}}{\partial G_\mu(x) \partial G_\nu(y)} |_0 &= - \frac{1}{2} \frac{g^2}{(2\pi)^4} \int d^4 p \frac{(p^\mu + q^\mu)(p^\nu + 2 q^\nu)}{p^2 (p + q)^2} 
\end{align*}

Subscript zero on the derivative means that all the fields are put to zero after the differentiation is performed.

Note that this is the same expression for the propagator of the $\lambda_\mu$ field as given in eq.(26) We also see that

\begin{align*}
\frac{\partial^2 S_{\text{eff}}}{\partial B_0^2} &= \frac{\partial^2 S_{\text{eff}}}{\partial E_0^2} = \frac{\partial^2 S_{\text{eff}}}{\partial g_0 \partial g_0} = \frac{1}{(2\pi)^4} \int d^4 p \frac{p^2}{p^2}
\end{align*}

This expression is zero by dimensional regularization. All the other fields also have zero propagators since the effective lagrangian does not have any terms which are only bilinear.
in these fields. All these terms involve quartic interactions of these fields. When we drop all the field with zero propagators we end up with

\[ S''_{\text{eff}} = -\frac{\text{Tr} \log 2}{2} (-\partial^2 - igG_\mu \partial^\mu + ig\partial_\mu G^\mu) \]

This is the same expression we found for Model A. Therefore all the results obtained for Model A from this expression are also true for Model B. We can not differentiate Model A from Model B as far as perturbative expansion in terms of Feynman diagrams are concerned.

**Conclusion**

Here we have studied two very dissimilar models which have the same Feynman expansions. A complete constrained Hamiltonian analysis shows that the two models are different. One reason we have studied this problem is to be able to clarify the behaviour of many Nambu-Jona-Lasinio\textsuperscript{10,3}like models which are claimed to be similar to QCD\textsuperscript{11}. There are people\textsuperscript{12} who disagree with this equivalence. The claim in reference 12 is that after an investigation of a lattice Nambu-Jona-Lasinio model both by the Monte Carlo method and Schwinger-Dyson equations, studying renormalization group flows in the neighborhood of the critical coupling where the chiral symmetry breaking phase transition takes place, in no region of the bare parameter space renormalizability of the model is found. We propose that, in addition to the standard methods of looking at the renormalization flow and fixed point structure of two models to show equivalence, their constrained analysis may be another check. Still another method is to study the predictions of these models for different physical processes. An old calculation\textsuperscript{7} and old paper\textsuperscript{5} seem to suggest that the Nambu-Jona-Lasinio type models may indeed be trivial at four dimensional space-time, perhaps like the $\phi^4$ model is.

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