MORSE THEORY WITH THE NORM-SQUARE OF A HYPERKÄHLER MOMENT MAP

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Abstract. We prove that the norm-square of a moment map associated to a linear action of a compact group on an affine variety satisfies a certain gradient inequality. This allows us to bound the gradient flow, even if we do not assume that the moment map is proper. We describe how this inequality can be extended to hyperkähler moment maps in some cases, and use Morse theory with the norm-squares of hyperkähler moment maps to compute the Betti numbers and cohomology rings of all toric hyperkähler orbifolds.

1. Introduction

Let $M$ be a symplectic manifold acted upon by a compact group $K$ with moment map $\mu : M \to \mathfrak{k}^*$. If $0$ is a regular value of $\mu$, then the reduced space $M//K := \mu^{-1}(0)/G$ is a symplectic orbifold. Following the ideas of Atiyah and Bott [1], Kirwan showed that the topology of $M//K$ can be understood via $K$-equivariant Morse theory with $|\mu|^2$ [14]. Let $H^*_K(M)$ denote the $K$-equivariant cohomology of $M$ (we take cohomology with rational coefficients unless otherwise specified). There is a natural map $\kappa : H^*_K(M) \to H^*(M//K)$ called the Kirwan map, and one of the most important consequences of Kirwan’s analysis is that (under mild assumptions) this map is surjective. Thus we have an isomorphism $H^*(M//K) \cong H^*_K(M)/\ker \kappa$, so that if $H^*_K(M)$ is known then the problem of computing the cohomology ring of $M//K$ reduces to that of computing the kernel of $\kappa$, a problem which has been well-studied (see, for example, [12] [25]).

If $M$ is a hyperkähler manifold then a hyperkähler moment map is a certain triple of moment maps $\mu_{\text{HK}} = (\mu_1, \mu_2, \mu_3)$, and the hyperkähler quotient [10] is defined to be $M///K := \mu_{\text{HK}}^{-1}(0)/K$ (we review this construction in Section 2). As in the symplectic case, we have a natural map $H^*_K(M) \to H^*(M///K)$ which we call the hyperkähler Kirwan map. It has been an outstanding question whether the hyperkähler Kirwan map is surjective. There are several families of examples for which surjectivity is known (see for example [16] [17]): however there is currently no general result. Recently, it was proved [11] that surjectivity of the hyperkähler Kirwan map would follow from Morse theory with the function $|\mu_C|^2 := |\mu_2|^2 + |\mu_3|^2$, provided that its gradient flow satisfies certain analytic conditions. In principle, this provides a general method for proving surjectivity of the hyperkähler Kirwan map.

However, the analytic conditions needed to apply these methods appear to be very difficult to check—in [11], these conditions are verified only for linear $S^1$ actions, but are conjectured to hold for all linear actions. The difficulty is that unlike the symplectic case, we cannot assume that the moment map is proper, as virtually all interesting hyperkähler quotients are non-compact.

Date: May 1, 2014.
Motivated by this problem, our goal is to provide tools to study the gradient flow of $|\mu|^2$ without making any compactness assumptions. To control the gradient flow, we introduce a certain gradient inequality, motivated by the works [6, 20, 22]. The main result in this direction is Theorem 4.7 which gives a precise gradient inequality for $|\mu|^2$ when $\mu$ is a moment map associated to a linear action. We introduce the relevant gradient inequalities in Section 4 and discuss some immediate applications. We prove Theorem 4.7 in Section 5. Combined with Proposition 4.3, this gives us rather good quantitative control on the gradient trajectories of $-|\mu|^2$.

When the group $K$ is abelian, we also obtain analogous results for $|\mu_{HK}|^2$ and $|\mu_C|^2$. We use these results to show that the analytic conditions of [11] are satisfied by all linear actions by tori. In Section 6 we study this case in detail and show how Morse theory with the hyperkähler moment map can be used to compute the cohomology rings of all toric hyperkähler orbifolds, reproducing several known results [3, 8, 16] in a rather uniform way.

Acknowledgements. The author would like to thank Lisa Jeffrey for her careful guidance and supervision, as well as Andrew Dancer, Pierre Milman, and Nicholas Proudfoot for helpful conversations. This research was supported by an NSERC Canada Graduate Scholarship (Doctoral).

2. Kähler and Hyperkähler Quotients

Let $K$ be a compact Lie group acting on a symplectic manifold $(M,\omega)$. Let $\mathfrak{k}$ denote the Lie algebra of $K$. The action is said to be Hamiltonian if there is a $K$-equivariant map $\mu : M \to \mathfrak{k}^*$ which satisfies

$$d\langle \mu, \xi \rangle = i(\nu_\xi)\omega \quad \forall \xi \in \mathfrak{k},$$

where $i(\cdot)$ denotes the interior product and $\nu_\xi$ is the fundamental vector field

$$\nu_\xi(x) = \left. \frac{d}{dt} \right|_{t=0} e^{t\xi} \cdot x,$$

associated to any $\xi \in \mathfrak{k}$. Since $K$ is compact, we can choose an invariant inner product on $\mathfrak{k}$ to identify it with $\mathfrak{k}^*$, and via this identification we will often think of moment maps as taking values in $\mathfrak{k}$.

If $\alpha$ is a regular central value of $\mu$, then the symplectic reduction

$$M//_\alpha K := \mu^{-1}(\alpha)/K,$$

is a symplectic orbifold, and a manifold if $K$ acts freely on $\mu^{-1}(0)$. We will sometimes denote this by $M//K$ when we do not wish to emphasize the parameter $\alpha$. If in addition $M$ admits a $K$-invariant Kähler metric $g$ compatible with $\omega$, the Kähler structure descends to $M//K$ in a natural way. In this case we call $M//K$ a Kähler quotient to emphasize this fact.

A manifold $M$ is hyperkähler if it has a metric $g$ and a triple of symplectic forms $(\omega_1, \omega_2, \omega_3)$ which are all Kähler forms with respect to $g$ (i.e. compatible and parallel), and such that the respective complex structures $(I_1, I_2, I_3)$ satisfy the quaternion relations $(I_1 I_2 = I_3, \text{etc.})$. A hyperkähler moment map for an action of $K$ on $M$ is a triple $\mu_{HK} := (\mu_1, \mu_2, \mu_3)$ such that each $\mu_i$ is a moment map for the action of $K$ with respect to $\omega_i$. Let us introduce

$$\omega_R = \omega_1,$$

$$\omega_C = \omega_2 + i\omega_3.$$
with corresponding moment maps $\mu_R := \mu_1$ and $\mu_C := \mu_2 + i\mu_3$. We call $\mu_R$ and $\mu_C$ the real and complex moment maps, respectively. We will often think of $\mu_{K^*}$ as the pair $(\mu_R, \mu_C)$. For a pair $(\alpha, \beta) \in \mathfrak{t}^* \oplus \mathfrak{t}_C^*$ that is regular and central, we define the hyperkähler quotient \cite{10} to be

$$M///_{(\alpha, \beta)} K := (\mu_R^{-1}(\alpha) \cap \mu_C^{-1}(\beta))/K.$$  

This is a hyperkähler orbifold, and a manifold if $K$ acts freely on $\mu_R^{-1}(\alpha) \cap \mu_C^{-1}(\beta)$.

We now recall the most important source of examples of hyperkähler quotients. Let $V$ be a Hermitian vector space. Its cotangent bundle $T^*V$ is naturally a hyperkähler vector space, with metric $g$ and complex structure $I_1$ induced from $V$, with $I_2$ defined by $I_2(x, y) = (-\bar{y}, \bar{x})$, and with $I_3 := I_1I_2$. Any unitary representation of $K$ on $V$ induces a linear action on $T^*V$ which preserves this hyperkähler structure. Since $T^*V$ is a vector space, a moment map always exists, and we may take the hyperkähler quotient $T^*V/// K$. For a general discussion of hyperkähler analogues of Kähler quotients, see \cite{23}.

\textbf{Example 2.1.} Let $V = \mathbb{C}^{N+1}$ with the standard diagonal action of $S^1$. The moment map $\mu : V \to \mathbb{R}$ is given by $x \mapsto |x|^2/2$, and the Kähler quotient is $\mathbb{C}P^N$ equipped with the Fubini-Study metric. For the induced action of $S^1$ on $T^*V$, we have

$$\mu_R = \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2,$$

$$\mu_C = x \cdot y,$$

and the hyperkähler quotient $T^*V/// S^1$ is $T^*\mathbb{C}P^N$ with the Calabi metric. Note that while the moment map for the $S^1$ action on $\mathbb{C}^{N+1}$ is a square, the moment map for the induced action on $T^*\mathbb{C}^{N+1}$ is a difference of squares.

\textbf{Remark 2.2.} In the above example, we found that

$$T^*\mathbb{C}^{N+1///} S^1 \cong T^*(\mathbb{C}^{N+1///}/S^1).$$

It is always true that $T^*(V/// K)$ embeds into $T^*V/// K$ as an open dense set (as long as it is not empty), but in general this is not an isomorphism.

3. Equivariant Morse Theory

We now recall some of the main ideas of Kirwan \cite{14}. Let $M$ be a symplectic manifold with a Hamiltonian action of a compact group $K$, and assume that $0$ is a regular value of the moment map. Let $f = |\mu|^2$, and for the moment assume that $f$ is proper.

Kirwan shows that the components $C$ of the critical set of $f$ can be understood in terms of the fixed-point sets of subtori of a maximal torus of $K$. Using a local description of $|\mu|^2$ near each component $C$, she then proves that $|\mu|^2$ is \textit{minimally degenerate} (a weaker condition than being Morse-Bott). This allows her to conclude that the stable manifolds $S_C$ (the set of points $x \in M$ such that the gradient flow of $-|\mu|^2$ through $x$ has a limit point in $C$) form a smooth $K$-invariant stratification of $M$. Thus for each $C$ we obtain the equivariant Thom-Gysin sequence

$$\cdots \to H^*_{K^*} (\lambda_C) \to H^*_{K} (M_C^+) \to H^*_K (M_C^-) \to \cdots$$

where $\lambda_C$ is the Morse index of $C$ and $M_C^+ = f^{-1}((-\infty, f(C) + \epsilon])$, for $\epsilon > 0$ sufficiently small. The map $H^*_{K^*} (\lambda_C) \to H^*_K (M_C^+)$ is, after composition with restriction $H^*_K (M_C^+) \to H^*_K (C)$, multiplication by the equivariant Euler class of
the negative normal bundle to $C$. By the Atiyah-Bott lemma \cite{2}, the equivariant Euler class is not a zero divisor, hence the equivariant Thom-Gysin sequence splits into short exact sequences

$$0 \to H^*_K(M^+_{\infty}) \to H^*_K(M^-_{\infty}) \to H^*_K(M) \to 0.$$  

Thus the restriction $H^*_K(M^+_{\infty}) \to H^*_K(M^-_{\infty})$ is surjective, and its kernel consists of those classes whose restriction to $C$ is a multiple of the equivariant Euler class of the negative normal bundle to $C$. An inductive argument then yields surjectivity of the restriction $H^*_K(M) \to H^*_K(\mu^{-1}(0)) \cong H^*(M//K)$.

We are interested in the case when $|\mu|^2$ is not proper. To aid the discussion, we introduce the following definition.

**Definition 3.1.** A function $f : M \to \mathbb{R}$ is said to be flow-closed if every (positive time) trajectory of $-\nabla f$ is contained in a compact set.

As remarked in \cite[§9]{14}, the above results still hold under the weaker condition that $|\mu|^2$ is flow-closed. We summarize these results as follows.

**Theorem 3.2** (Kirwan \cite{14}). Suppose $f = |\mu|^2$ is flow-closed and that 0 is a regular value of $\mu$. Then the stable manifolds $S_C$ form a smooth $K$-invariant stratification of $M$, and the equivariant Thom-Gysin sequence splits into short exact sequences. Consequently, the function $|\mu|^2$ is equivariantly perfect, and the Kirwan map $\kappa : H^*_K(M) \to H^*(M//K)$ is surjective. The kernel of $\kappa$ is the ideal in $H^*_K(M)$ generated by those classes whose restriction to some component $C$ of the critical set of $f$ is the equivariant Euler class of the negative normal bundle to $C$.

One might expect analogous results to hold for $|\mu_{HK}|^2$. However, this appears not to be the case (except when $K$ is a torus \cite{15}). The problem is that in the nonabelian situation, the norm of the gradient of $|\mu_{HK}|^2$ contains a term proportional to the structure constants of the group that is difficult to understand (see Remark \cite[5.2]{15}). In \cite{11}, it was found instead that the function $|\mu_C|^2$ is better behaved, owing to the fact that $\mu_C$ is $I_1$-holomorphic.

**Theorem 3.3** (Jeffrey-Kiem-Kirwan \cite{11}, Kirwan \cite{15}). The function $f = |\mu_C|^2$ is minimally degenerate. If $f$ is flow-closed, then the stable manifolds $S_C$ form a $K$-invariant stratification of $M$ and the equivariant Thom-Gysin sequence splits into short exact sequences. Consequently, the restriction $H^*_K(M) \to H^*_K(\mu^{-1}(0))$ is surjective. If $K$ is a torus, the same conclusions hold for $|\mu_{HK}|^2$ provided that it is flow-closed.

Note that $\mu^{-1}(0)$ is a $K$-invariant complex submanifold of $M$, so that $M///K \cong \mu^{-1}(0)//K$. Hence surjectivity of map $H^*_K(\mu^{-1}(0)) \to H^*(M///K)$ can be studied by the usual methods (but see Corollary \cite{11.13}). In principle, this reduces the question of Kirwan surjectivity for hyperkähler quotients to the following conjecture.

**Conjecture 3.4** (\cite{11}). If $\mu_C$ is a complex moment map associated to a linear action by a compact group $K$ on a vector space, then $|\mu_C|^2$ is flow-closed.

This was proved for the special case of $S^1$ actions in \cite{11}, but the method of proof does not admit any obvious generalization. We will prove the following.

**Theorem 3.5.** Conjecture \cite[3.4]{11} is true when $K$ is a torus.
This is an immediate consequence of Proposition 4.3 and Theorem 4.7. To appreciate why this result requires some effort, let us contrast it with the analogous statement for $|\mu|^2$, where $\mu$ is an ordinary moment map associated to a linear action on a Hermitian vector space $V$. It is immediate from the definitions that $\nabla |\mu|^2 = 2Iv_\mu$. Hence, the gradient trajectory through a point $x$ always remains in the $K_C$ orbit through $x$. Thus it suffices to restrict attention to the $K_C$ orbits. Flow-closedness is then a consequence of the following proposition.

**Proposition 3.6** (Sjamaar [24] Lemma 4.10). Suppose $K_C$ acts linearly on $V$, let $\{g_n\}$ be a sequence of points in $K_C$, and let $\{x_n\}$ be a bounded sequence in $V$. Then the sequence $\{g_n x_n\}$ is bounded if $\{\mu(g_n x_n)\}$ is bounded.

Now consider the gradient flow of $|\mu_C|^2$ on $T^*V$. Since $|\mu_C|^2 = |\mu_2|^2 + |\mu_3|^2$, we see that $\nabla |\mu_C|^2 = 2I_2v_{\mu_2} + 2I_3v_{\mu_3}$. The problem is immediate: due to the simultaneous appearance of $I_2$ and $I_3$, the gradient trajectories appear to lie on the orbits of a “quaternification” of $K$, but in general, no suitable quaternionization of $K$ exists. More precisely, if we let $D$ be the distribution on $V$ generated by the $K$ action (i.e. the integrable distribution whose leaves are the $K$-orbits), then $D_C := D + I_2D$ is integrable, whereas the distribution $D_3 := D + I_1D + I_2D + I_3D$ on $T^*V$ is not integrable in general.

In a few specific examples, a detailed study of $D_3$ leads to definite conclusions about the gradient flow, but at present we cannot prove a general result in this direction. In any case, we will not pursue this approach in the present work (but see Remarks 5.2 and 5.3 which are related to this problem).

With these considerations in mind, it would be illuminating to have a proof of the flow-closedness of $|\mu|^2$ that relies neither on arguments involving $K_C$ nor on properness, as such a proof might generalize to the hyperkähler setting. This is exactly the content of our main result, Theorem 4.7. The key idea is to relax the assumption of properness to the weaker condition of satisfying a certain gradient inequality, which we introduce next.

### 4. Łojasiewicz Inequalities

We begin by giving a precise definition of the type of inequality we wish to consider, as well as its most important consequence.

**Definition 4.1.** Let $(M, g)$ be a complete Riemannian manifold. A smooth real-valued function $f$ on $M$ is said to satisfy a global Łojasiewicz inequality if for any real number $f_\epsilon$ in the closure of the image of $f$, there exist constants $\epsilon > 0$, $k > 0$, and $0 < \alpha < 1$, such that

$$|\nabla f(x)| \geq k|f(x) - f_\epsilon|^\alpha,$$

for all $x \in M$ such that $|f(x) - f_\epsilon| < \epsilon$.

**Remark 4.2.** The term global Łojasiewicz inequality is borrowed from [13]; however, we use it in a different way, as we are concerned specifically with bounding the gradient of $f$.

**Proposition 4.3.** Suppose $f$ satisfies a global Łojasiewicz inequality and is bounded below. Then $f$ is flow-closed.

**Proof.** Let $x(t)$ be a trajectory of $-\nabla f$. Since $f(x(t))$ is decreasing and bounded below, $\lim_{t \to \infty} f(x(t))$ exists. Call this limit $f_\epsilon$. Let $\epsilon, k,$ and $\alpha$ be the constants
appearing in the global Lojasiewicz inequality for $f$ with limit $f_c$. For large enough $T$, we have $|f(x(t)) - f_c| < \epsilon$ whenever $t > T$. Consider $t_2 > t_1 > T$, and let $f_1 = f(x(t_1))$ and $f_2 = f(x(t_2))$. Since $\dot{x} = -\nabla f$, we have that
\[
d(x(t_1), x(t_2)) \leq \int_{t_1}^{t_2} |\nabla f(x(t))| \, dt,
\]
where $d(x, y)$ denotes the Riemannian distance. By the change of variables $t \mapsto f(x(t))$, we obtain
\[
d(x(t_1), x(t_2)) \leq \int_{f_2}^{f_1} |\nabla f|^{-1} \, df \leq k^{-1} \int_{f_2}^{f_1} |f - f_c|^{-\alpha} \, df = k^{-1}(1 - \alpha)^{-1} \left( |f_1 - f_c|^{1-\alpha} - |f_2 - f_c|^{1-\alpha} \right) < k^{-1}(1 - \alpha)^{-1} |f(x(T)) - f_c|^{1-\alpha}.
\]
Since $\alpha < 1$, the last expression can be made arbitrarily small by taking $T$ sufficiently large, so we see that $\lim_{t \to \infty} x(t)$ exists, and in particular the gradient trajectory is contained in a compact set. □

This argument establishes flow-closedness directly from the Lojasiewicz inequality, without appealing to compactness. Thus it would be sufficient to show that $|\mu|^2$ satisfies such an inequality. To motivate why we might expect this to be case, we recall the classical Lojasiewicz inequality.

**Theorem 4.4 (Lojasiewicz Inequality)**. Let $f$ be a real analytic function on an open set $U \subseteq \mathbb{R}^N$, and let $c$ be a critical point of $f$. Then on any compact set $K \subseteq U$, there are constants $k > 0$ and $0 < \alpha < 1$ such that the inequality
\[
|\nabla f(x)| \geq k |f(x) - f(c)|^\alpha
\]
holds for all $x \in K$.

If $f$ is a proper real analytic function, then this immediately implies that $f$ satisfies a global Lojasiewicz inequality as defined above. Since our primary concern is Morse theory, we can relax the assumption of analyticity as follows.

**Proposition 4.5.** Suppose $f$ is a proper Morse function. Then $f$ satisfies a global Lojasiewicz inequality.

**Proof.** By the Morse lemma, near each critical point we can choose coordinates in which $f$ is real analytic. Hence $f$ satisfies the classical Lojasiewicz inequality near each critical point, and since $f$ is proper this can be extended to a global inequality. □

For a moment map $\mu$, the function $|\mu|^2$ is in general neither Morse nor Morse-Bott, but minimally degenerate (in the sense of [14]). Nonetheless, we can still obtain a global Lojasiewicz inequality whenever $\mu$ is proper.

**Proposition 4.6.** Suppose $\mu$ is a moment map associated to an action of a compact Lie group, and suppose furthermore that $\mu$ is proper. Then $|\mu|^2$ satisfies a global Lojasiewicz inequality.
Proof. Lerman \cite{20} uses local normal forms to show that $|\mu|^2$ is real analytic in a neighborhood of each component of its critical set, and hence satisfies the classical Lojasiewicz inequality. Since $\mu$ is proper, this can be extended to a global inequality.

Since we would like to drop the assumption of properness, it is natural to ask whether there are examples of moment maps which are not proper but nevertheless satisfy a global Lojasiewicz inequality. The answer to this question is in the affirmative, at least when the action is linear. The following is our main theorem, which we prove in Section 5.

**Theorem 4.7.** Let $\mu$ be a moment map associated to a unitary representation of a compact group $K$ on a Hermitian vector space $V$. Then $f = |\mu|^2$ satisfies a global Lojasiewicz inequality. In detail, for every $f_c \geq 0$, there exist constants $k > 0$ and $\epsilon > 0$ such that

$$|\nabla f(x)| \geq k|f(x) - f_c|^\frac{4}{3}$$

whenever $|f(x) - f_c| < \epsilon$. If $K$ is a torus, then the same holds for the functions $|\mu_C|^2$ and $|\mu_{HK}|^2$ associated to the action of $K$ on $T^*V$.

**Remark 4.8.** This theorem is a generalization of \cite{22} Theorem A.1. However, in \cite{22}, it is assumed that the constant term in the moment map is chosen so that $f$ is homogeneous (see equation 9), leading to an inequality of the form

$$|\nabla f| \geq kf^\frac{4}{3},$$

which holds on all of $V$. Both sides are homogeneous of the same degree, so that it suffices to prove the inequality on the unit sphere, allowing the use of compactness arguments. In Theorem 4.7, we make no such assumption, and this complicates several steps of the proof.

**Remark 4.9.** If $X \subset V$ is a complex subvariety, then it is easy to see that Theorem 4.7 implies that the restriction of $f$ to $X$ satisfies a global Lojasiewicz inequality. Similarly, if $X \subset T^*V$ is a hyperkähler subvariety, we deduce the inequality for the restrictions of $|\mu_C|^2$ and $|\mu_{HK}|^2$.

**Remark 4.10.** In \cite{6} and \cite{18} it is shown that certain classes of functions satisfy similar global Lojasiewicz inequalities; indeed this was the motivation to consider such inequalities. However, these general theorems cannot rule out the possibility $\alpha \geq 1$, which is not sharp enough to prove the boundedness of all gradient trajectories. In this sense, the real content of Theorem 4.7 is the bound on the exponent.

In what follows, let $K$ be a compact group acting unitarily on a Hermitian vector space $V$, with moment map $\mu$. Note that since the action is linear, we have $H^*_K(V) = H^*_K(\text{point}) =: H^*_K$. For $\alpha \in \mathfrak{t}^*$ and $\beta \in \mathfrak{t}_c^*$, we denote

$$X(\alpha) := V/\alpha K,$$

$$M(\alpha, \beta) := T^*V/\alpha, \beta K.$$

**Corollary 4.11.** Let $f = |\mu|^2$. Then for each component $C$ of the critical set of $f$, the gradient flow defines a $K$-equivariant homotopy from the stable manifold $S_C$ to the critical set $C$. If $K$ is abelian, then the same holds for the functions $|\mu_C|^2$ and $|\mu_{HK}|^2$. 

Corollary 4.13. If \( \mu \) is assumed to be proper then this is a special case of the main theorem of [20]; in the present case we make no such assumption. However, the essential ingredient of the proof is the Łojasiewicz inequality. We give the proof below for completeness, but the details do not differ significantly from [20].

**Proof.** We define a continuous map \( F : S_C \times [0, \infty) \to S_C \) by \( (x, t) \mapsto x(t) \), where \( x(t) \) is the trajectory of \( -\nabla f \) beginning at \( x \), evaluated at time \( t \). By Proposition 4.13 we can extend this to a map \( F : S_C \times [0, \infty) \to S_C \) by \( (x, \infty) \mapsto \lim_{t \to \infty} x(t) \).

This map is the identity when restricted to \( C \), and maps \( S_C \times \{ \infty \} \) to \( C \). We must verify that this extended map is continuous.

We must show that for any \( x_0 \in S_C \) and any sequence \( \{(x_n, t_n)\} \) of points in \( S_C \times [0, \infty] \) satisfying \( \lim_{n \to \infty} x_n = x_0 \in S_C \) and \( \lim_{n \to \infty} t_n = \infty \) that \( \lim_{n \to \infty} x_n(t_n) \) exists. We will show that it is equal to \( x_c := \lim_{t \to \infty} x_0(t) \). Let \( f_c \) be the value of \( f \) on \( C \) and let \( \epsilon, k \) be the constants appearing in Theorem 4.14. Given such a sequence, let \( \eta > 0 \) and assume \( \eta < \epsilon \) but is otherwise arbitrary; let \( T > 0 \) be chosen large enough so that \( |f(x_0(t)) - f_c| < \eta \) for \( t > T \); and let \( N > 0 \) be chosen to that \( t_n > T \) for \( n > N \). The map \( S_C \to S_C \) given by \( x \mapsto x(T) \) is continuous, so we can find \( \delta > 0 \) such that

\[
|x - x_0| < \delta \implies |x(T) - x_0(T)| < \eta.
\]

Since \( x \mapsto f(x(T)) \) is continuous, we can shrink \( \delta \) if necessary so that

\[
|x - x_0| < \delta \implies |f(x(T)) - f(x_0(T))| < \eta.
\]

Choose \( N \) larger if necessary such that \( |x_n - x_0| < \delta \) for \( n > N \). We would like to show that \( |x_n(t_n) - x_c| \to 0 \) as \( n \to \infty \). For \( n > N \) we have

\[
|x_n(t_n) - x_c| = |x_n(t_n) - x_n(T) + x_n(T) - x_0(T) + x_0(T) - x_c| \\
\leq |x_n(t_n) - x_n(T)| + |x_n(T) - x_0(T)| + |x_0(T) - x_c|.
\]

By our choice of \( N \), the second term is bounded by \( \eta \), and we may apply the argument in the proof of Proposition 4.13 to show that the third term is bounded by \( k^{-1}|f(x_0(T)) - f_c| < 4k^{-1}\eta \). Finally, to bound the first term we again apply the argument of Proposition 4.13 to obtain the bound

\[
4k^{-1} \left( |f(x_n(T)) - f_c| + |f(x_n(t_n)) - f_c| \right) < 4k^{-1}|f(x_0(T)) - f_c| \leq 8k^{-1}\eta.
\]

Since \( |x_n - x_0| < \delta \), we have

\[
|f(x_n(T)) - f_c| = |f(x_n(T)) - f(x_0(T)) + f(x_0(T)) - f_c| \\
\leq |f(x_n(T)) - f(x_0(T))| + |f(x_0(T)) - f_c| \\
< 2\eta.
\]

Hence the first term is bounded by \( 4k^{-1}(2\eta) \leq 8k^{-1}\eta \), and we obtain

\[
|x_n(t_n) - x_c| \leq 12k^{-1}\eta + \eta.
\]

Since we may take \( \eta \) arbitrarily small, we see that \( |x_n(t_n) - x_c| \to 0 \).

**Corollary 4.13.** If \( \beta \in T_C^* \) is regular central, then for any central \( \alpha \in T^* \), the set \( \mu_{k_{-1}}^{-1}(\alpha) \cap \mu_C^{-1}(\beta) \) is a \( K \)-equivariant deformation retract of \( \mu_C^{-1}(\beta) \), and in particular

\[
H^*_K(\mu_C^{-1}(\beta)) \cong H^*(M(\alpha, \beta)).
\]
Proof. Since $\beta$ is regular central, $\mu_\mathbb{C}^{-1}(\beta)$ is a $K$-invariant complex submanifold of $T^*V$, and furthermore $K$ acts on $\mu_\mathbb{C}^{-1}(\beta)$ with at most discrete stabilizers. Hence the only component of the critical set of $|\mu_\mathbb{R} - \alpha|^2$ that intersects $\mu_\mathbb{C}^{-1}(\beta)$ is the absolute minimum, which occurs on $\mu_\mathbb{R}^{-1}(\alpha)$. By Corollary 4.11 the gradient flow of $-|\mu_\mathbb{R} - \alpha|^2$ gives the desired $K$-equivariant deformation retract from $\mu_\mathbb{C}^{-1}(\beta)$ to $\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(\beta)$. Thus

$$H^*_K(\mu_\mathbb{C}^{-1}(\beta)) \cong H^*_K(\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(\beta)) \cong H^*(M(\alpha, \beta)).$$

\[ \square \]

Corollary 4.14. If $\alpha$ is a regular central value of $\mu$, then the Kirwan map $H^*_K \to H^*(X(\alpha))$ is surjective. If $K$ is a torus and $(\alpha, \beta)$ is a regular value of the hyperkähler moment map, then the hyperkähler Kirwan map $H^*_K \to H^*(M(\alpha, \beta))$ is surjective.

Proof. Theorem 4.7 and Proposition 4.3 show that $|\mu - \alpha|^2$ is flow-closed, so we obtain surjectivity of $H^*_K \to H^*(X(\alpha))$ by Theorem 3.2.

In the hyperkähler case, if $K$ is a torus then $|\mu_\mathbb{C} - \beta|^2$ is flow-closed, so by Theorem 3.3 we obtain surjectivity of $H^*_K \to H^*_K(\mu_\mathbb{C}^{-1}(\beta))$. By Corollary 4.13 the map $H^*_K(\mu_\mathbb{C}^{-1}(\beta)) \to H^*(M(\alpha, \beta))$ is an isomorphism, so we have surjectivity of $H^*_K \to H^*(M(\alpha, \beta))$.

Remark 4.15. Konno proved surjectivity of the map $H^*_K \to H^*(M(\alpha, \beta))$ when $K$ is a torus using rather different means [16]. Konno also computed the kernel of the Kirwan map, giving an explicit description of the cohomology ring $H^*(M(\alpha, \beta))$. We will study this case in detail in Section 6, and we will see that Morse theory allows us to compute the kernel very easily.

5. PROOF OF THE MAIN THEOREM

Let $K$ be a compact Lie group with Lie algebra $\mathfrak{t}$, and suppose $K$ acts unitarily on a Hermitian vector space $V$. Without loss of generality we will regard $K$ as a subgroup of $U(V)$ and identify $\mathfrak{t}$ with a Lie subalgebra of $\mathfrak{u}(V)$, which we identify with the Lie algebra of skew-adjoint matrices. We will use the trace norm on $\mathfrak{u}(V)$ to induce an invariant inner product on $\mathfrak{t}$, and use this to identify $\mathfrak{t}$ with its dual. For any $\xi \in \mathfrak{t}$ there is a fundamental vector field $v_\xi$ which is given by $v_\xi(x) = \xi x$. We denote by stab$(x)$ the Lie algebra of the stabilizer of a point $x \in V$; i.e.

$$\text{stab}(x) = \{ \xi \in \mathfrak{t} \mid v_\xi(x) = 0 \}. \quad \text{(8)}$$

If we fix an orthonormal basis $\{e_a\}$ of $\mathfrak{t}$, then a moment map is given by

$$\mu(x) = \sum_a \frac{1}{2} \langle i e_a x, x \rangle - \alpha, \quad \text{where } i = \sqrt{-1}$$

is the complex structure on $V$ and $\alpha$ is any central element of $\mathfrak{t}$. Then for $f = |\mu|^2$, we have

$$\nabla f(x) = 2 i v_{\mu(x)}(x) = \sum_a 2 i \mu^a(x) e_a x, \quad \text{where } i \text{ is unitary},$$

and since $i$ is unitary, we have that

$$|\nabla f| = 2|v_\mu|. \quad \text{(11)}$$
Lemma 5.1. Suppose $K$ is abelian, and consider its action on $T^*V$. Let $f_i = |\mu_i|^2$ for $i = 1, 2, 3$. Then $\langle \nabla f_i, \nabla f_j \rangle = 0$ for $i \neq j$.

Proof. We compute:
\[
\langle \nabla f_2, \nabla f_3 \rangle = 4 \langle Jv_{\mu_2}, K v_{\mu_3} \rangle = 4 \langle I v_{\mu_2}, v_{\mu_3} \rangle = 4 \omega_1 (v_{\mu_2}, v_{\mu_3}) = 4 \langle \mu_2, d\mu_1(v_{\mu_3}) \rangle = 4 \langle \mu_2, [\mu_3, \mu_1] \rangle = 0.
\]

Similar computations show that the other two cross terms vanish. \qed

Remark 5.2. The proof of this lemma makes it clear why the assumption that $K$ is abelian is so useful in the hyperkähler setting. The cross term
\[
\langle \nabla f_j, \nabla f_k \rangle = \pm 4 \langle \mu_1, [\mu_2, \mu_3] \rangle
\]
is exactly the obstruction to proving an estimate in the general nonabelian case. The function on the right hand side is very natural, and seems to be genuinely hyperkähler, having no analogue in symplectic geometry. Numerical experiments suggest that it is small in magnitude compared to $|\nabla f_2| + |\nabla f_3|$, but we do not know how to prove this. A theorem in this direction might be enough to prove flow-closedness (and hence Kirwan surjectivity) in general. It certainly warrants further study.

Remark 5.3. In light of the discussion following Proposition 5.1, the cross term $\langle \mu_1, [\mu_2, \mu_3] \rangle$ should have an interpretation in terms of the geometry of the non-integrable distribution $D_H$. Subriemannian geometry may have a key role to play in proving Kirwan surjectivity for hyperkähler quotients by nonabelian groups.

Proposition 5.4. Let $T \subset K$ be a maximal torus of $K$, and let $\mu_K$ and $\mu_T$ be the corresponding moment maps. Let $f_K = |\mu_K|^2$ and $f_T = |\mu_T|^2$. Suppose that for any $f_c \geq 0$, $f_T$ satisfies a global Lojasiewicz inequality. Then $f_K$ satisfies a global Lojasiewicz inequality with the same constants and exponent.

Proof. Since $K$ is compact, for each $x \in V$ we can find some $k \in K$ so that $Ad_k \mu_K(x) \in t$. By equivariance of the moment map, we have $Ad_k \mu_K(x) = \mu_K(kx) \in t$. Hence $\mu_K(kx) = \mu_T(kx)$, so that $|v_{\mu_K(kx)}| = |v_{\mu_T(kx)}|$. Using equality \ref{eq:muKmuT}, this tells us that $|\nabla f_K(kx)| = |\nabla f_T(kx)|$. Since $f_K(kx) = |\mu_K(kx)|^2 = |\mu_T(kx)|^2 = f_T(kx)$, we deduce the Lojasiewicz inequality for $f_K$ from the inequality for $f_T$. \qed

We assume for the remainder of this section that $K$ is a torus.

Proposition 5.5. Fix $f_c \geq 0$, and suppose that for each $\mu_c \in t$ satisfying $|\mu_c|^2 = f_c$, there exist constants $c' > 0$ and $c'' > 0$ (depending on $\mu_c$) such that
\[
|\nabla f(x)| \geq c' |f(x) - f_c|^{\frac{1}{2}}
\]
whenever $|\mu(x) - \mu_c| < c'$. Then $f$ satisfies a global Lojasiewicz inequality, i.e., there exist constants $\epsilon > 0$ and $c > 0$ so that the inequality \ref{eq:globalLojasiewicz} holds whenever $|f(x) - f_c| < \epsilon$. 

(12)
Proof. Suppose that for each $\mu_\xi$ as above we can find constants $c(\mu_\xi)$ and $c(\mu_\xi)$ so that inequality \([12]\) holds. Let $U(\mu_\xi)$ be the $c(\mu_\xi)$-ball in $\mathfrak{t}$ centered at $\mu_\xi$. These open sets cover the sphere $S$ of radius $\sqrt{|\mu_\xi|}$ in $\mathfrak{t}$, and by compactness we can choose a finite subcover. Denote this finite subcover by $\{U_i\}_{i=1}^n$, with centers $\mu_i$ and constants $c_i$, and let $c = \min_i c_i$. The finite union $\bigcup_i U_i$ contains an $\epsilon$-neighbourhood of $S$ for some sufficiently small $\epsilon$. Since $f(x) = |\mu(x)|^2$, if we choose $\epsilon' > 0$ sufficiently small then $|f(x) - f_\epsilon| < \epsilon'$ implies that $|\mu(x)| - \sqrt{|\mu_\xi|} < \epsilon$, so that $\mu(x) \in U_i$. In particular, there is some $j$ such that $\mu(x) \in U_j$, and by inequality \([12]\) we have

$$|\nabla f(x)| \geq c_j |f(x) - f_\epsilon|^\frac{1}{2} \geq c|f(x) - f_\epsilon|^\frac{1}{2},$$

as desired. \(\Box\)

Before giving the proof of Theorem 4.7, we isolate some of the main steps in the following lemmas. Let us introduce the following notation. For $x \in V \setminus \{0\}$, let $\hat{x}$ denote its projection to the unit sphere, i.e. $\hat{x} = x/|x|$ or equivalently $x = |x|\hat{x}$. Since the action is linear, we have that $v_\xi(x) = |x|v_\xi(\hat{x})$ and stab($\hat{x}$) = stab($x$).

**Lemma 5.6.** Fix $\hat{y}$ in the unit sphere in $V$. Let $P$ be the orthogonal projection from $\mathfrak{t}$ to stab($\hat{y}$)$^\perp$, and $Q = 1 - P$. Then there is a neighbourhood $U$ of $\hat{y}$ such that for any $\xi \in \mathfrak{t}$, inequalities

$$|v_\xi(x)| \geq c|x||P\xi|$$

$$|v_\xi(x)| \geq c'|v_P\xi(x)|$$

$$|v_\xi(x)| \geq c''(|v_P\xi(x)| + |v_Q\xi(x)|)$$

hold for all $x$ such that $\hat{x} \in U$. The constants $c, c', c''$ are positive and depend only on $\hat{y}$ and $U$ but not on $x$ or $\xi$.

**Remark 5.7.** A version of this lemma appears as part of the proof of [22, Theorem A.1], though it is not stated exactly as above. We repeat the argument below so that our proof of Theorem 4.7 is self-contained.

Proof. Fix $\hat{y}$ and let $P$ and $Q$ be as above. Let $W$ be the smallest $K$ invariant subspace of $V$ containing $\hat{y}$, and let $P_W : V \to W$ be the orthogonal projection. Note that $W$ is generated by vectors of the form $\xi_1 \cdots \xi_l \hat{y}$, with $\xi_i \in \mathfrak{k}$. Since $P_W$ is a projection, $|v_\xi| \geq |P_Wv_\xi|$, so to establish inequality \([13]\) it suffices to show that $|P_Wv_\xi| \geq c|P\xi|$. Note that $P_W$ is equivariant, i.e. $\xi P_W = P_W \xi$ for all $\xi \in \mathfrak{t}$. Note also that since $K$ is abelian, if $\xi \in \text{stab}(\hat{y})$ then $\xi \in \text{ann}(W)$, since $\xi_1 \cdots \xi_l \hat{y} = \xi_1 \cdots \xi_l \xi \hat{y} = 0$. For any orthonormal basis $\{e_a\}_{a=1}^d$ of $\mathfrak{t}$ chosen so that $\{e_a\}_{a=1}^d$ is an orthonormal basis of stab($\hat{y}$)$^\perp$ and $\{e_a\}_{a=n+1}^d$ is an orthonormal basis of stab($\hat{y}$), we have $P \xi = \sum_{a=1}^{d} \xi^a e_a = \sum_{a=1}^{n} \xi^a e_a$. Similarly, we find

$$P_Wv_\xi(x) = \sum_{a=1}^{d} P_W \xi^a e_a x = \sum_{a=1}^{d} \xi^a e_a P_W x = \sum_{a=1}^{n} \xi^a e_a P_W x = P_Wv_P \xi(x).$$

Taking norms, we see that

$$|P_Wv_\xi(x)|^2 = \sum_{a=1}^{n} \sum_{b=1}^{n} \xi^a \xi^b \langle P_W e_a x, P_W e_b x \rangle = (P\xi)^T G(x) (P\xi),$$

where $G(x)$ is the matrix with entries $G_{ab}(x) = \langle P_W e_a x, P_W e_b x \rangle$ for $a, b = 1, \ldots, n$. By construction, this matrix is is positive definite at $\hat{y}$, so for a sufficiently small
neighbourhood $U$ of $\hat{y}$, we obtain $|P_Wv_ξ(\hat{x})|^2 \geq c|Pξ|^2$, with the constant $c$ depending only on $\hat{y}$ and the choice of neighbourhood $U$. For any $x$ with $\hat{x} \in U$, we obtain $|v_ξ(x)| = |x||v_ξ(\hat{x})| \geq c|x||Pξ|$, which is inequality (13).

To establish inequality (15), first note the following consequence of the triangle inequality (14) above, we obtain

$$|v_Pξ(\hat{x})| = |\sum_{a=1}^n ξ^ae_a\hat{x}| \leq |Pξ| |\sum_{a=1}^n |e_a\hat{x}|.$$ 

Shrinking $U$ if necessary, we can assume that the functions $|e_a\hat{x}|$ are bounded on $U$, and so we obtain $|v_Pξ(\hat{x})| \leq c'|Pξ| \leq c|v_ξ(\hat{x})|$. Both sides are homogeneous of the same degree in $\hat{x}$, so the inequality holds for any $x$ with $\hat{x} \in U$. This establishes inequality (14).

To establish inequality (16), first note the following consequence of the triangle inequality. If $v, w$ are vectors in some normed vector space, and $|v + w| \geq a|v|$ with $a > 0$, then we have

$$|v| + |w| = |v| + |v + w - v| \leq 2|v| + |v + w| \leq \left(1 + \frac{2}{a}\right)|v + w|.$$ 

Since $ξ = Pξ + Qξ$, we have $v_ξ(x) = v_Pξ(x) + v_Qξ(x)$, so applying inequality (14) together with the inequality (16) above, we obtain

$$|v_Qξ(x)| \geq c''\left(|v_Pξ(x)| + |v_Qξ(x)|\right),$$ 

with the constant $c''$ depending only on $\hat{y}$ and the neighbourhood $U$.

**Lemma 5.8.** Let $f = |μ|^2$ and fix $μ_c \in \kappa$. Then there exist constants $c > 0$ and $ε > 0$ (depending on $μ_c$) such that whenever $|μ(x) - μ_c| < ε$, we have

$$|x|^2|f(x)| \geq c|f(x) - f_c|^2.$$ 

**Proof.** Fix some particular $μ_c$. Recall that $μ$ is quadratic in the coordinate $x$ with no linear terms. Thus $μ$ is affine in the coordinates $v_{ij} = x_i\bar{x}_j$, and we may write $μ(x) = \phi(v)$, where $\phi(v) = Av - α$, for some linear transformation $A : V \otimes V \rightarrow \kappa$. We have $|v_{ij}| = |x_i|x_j| \leq \frac{1}{2}(|x_i|^2 + |x_j|^2)$, so that $|v| \leq \frac{1}{2}\sum_{i,j} |x_i|^2 + |x_j|^2 = N|x|^2$, where $N = \text{dim } V$. Thus $|x|^2|f(x)| = |x|^2|\phi(v)|^2 \geq N^{-1}|v|^2|\phi(v)|^2$, hence it suffices to show that

$$|v||\phi(v)|^2 \geq c||\phi(v)|^2 - f_c|^2,$$

whenever $|ϕ(v) - μ_c| < ε$. This follows immediately from Lemma 5.9 which we state and prove below. 

**Lemma 5.9.** Let $V_1$ and $V_2$ be inner product spaces, and consider an affine map $ϕ : V_1 \rightarrow V_2$ given by $ϕ(v) = Av - α$, for some linear map $A : V_1 \rightarrow V_2$ and constant $α \in V_2$. Then for any $ϕ_c$ in the image of $ϕ$, there exist constants $c > 0$ and $ε > 0$ such that

$$|v||ϕ(v)|^2 \geq c||ϕ(v)|^2 - |ϕ_c|^2|^2,$$

whenever $|ψ(v) - ϕ_c| < ε$.

**Proof.** To avoid unnecessary clutter, we sometimes use the shorthand $v^2 := |v|^2$ below. First note that if $P$ is a projection such that $AP = A$, then we have $ϕ(v) = ϕ(Pv)$, and so

$$|v||ϕ(v)|^2 = |v||ϕ(Pv)|^2 \geq |Pv||ϕ(Pv)|^2.$$
Taking $P$ to be the orthogonal projection from $V_1$ to $(\ker A)^{\perp}$, without loss of generality we can assume that $A$ is injective. Similarly, without loss of generality we may assume that $A$ is surjective. Suppose that $\phi_c \in V_2$ is fixed. Pick $v_c \in V_1$ so that $\phi(v_c) = \phi_c$. There are four possible cases.

Case 1: $v_c = 0, \phi_c = 0$. In this case, $\alpha = 0$, and $\phi(v) = Av$, so that $|\phi(v)| \leq |A||v|$. Thus $|v||\phi(v)|^2 \geq |A|^{-1}|\phi(v)|^3$, as desired. We may take $\epsilon$ to be any positive number, and $c = |A|^{-1}$.

Case 2: $v_c = 0, \phi_c \neq 0$. Take $\epsilon \leq |\phi_c|/2$. Then

$$|\phi(v)^2 - \phi_c^2| = |\phi(v) - \phi_c, \phi(v) + \phi_c|$$

$$\leq |\phi(v) - \phi_c||\phi(v) + \phi_c|$$

$$\leq |\phi(v) - \phi_c|(2|\phi_c| + \epsilon)$$

$$\leq \frac{5}{2}|\phi(v) - \phi_c||\phi_c|,$$

so we have $|\phi(v)^2 - \phi_c^2| \leq c_1|\phi(v) - \phi_c||\phi_c|$, where $c_1$ is a numerical constant independent of $\phi_c$. Since $|\phi(v) - \phi_c| < |\phi_c|/2$, we also have $|\phi(v)^2 - \phi_c^2| \leq c_2|\phi_c|^2$. Combining these two inequalities, we have $|\phi(v)^2 - \phi_c^2|^2 \leq c_3|\phi(v) - \phi_c||\phi_c|^2$. Since $\phi(v) - \phi_c = (v - v_c) = Av$, we have $|\phi(v) - \phi_c| \leq |A||v|$. Putting this back into the previous inequality, we obtain $|\phi(v)^2 - \phi_c^2|^2 \leq |A||v||\phi_c|^2$. On the other hand, with our choice of $\epsilon$ we have $|\phi| \geq \frac{1}{2}|\phi_c|$, so that $|\phi(v)^2 - \phi_c^2|^2 \leq 4|A||v||\phi|^2$, as desired.

Case 3: $v_c \neq 0, \phi_c = 0$. Take $\epsilon = |A|^{-1}|\alpha|/2$. Since in this case $Av_c = \alpha$, we have $\epsilon \leq |A|^{-1}|v_c|/2$, and

$$|v_c| = |v - (v - v_c)|$$

$$\leq |v| + |v - v_c|$$

$$= |v| + |A^{-1}\phi(v)|$$

$$\leq |v| + |A^{-1}|\epsilon$$

$$\leq |v| + \frac{|v_c|}{2}.$$ 

Thus $|v| \geq |v_c|/2 \geq |A|^{-1}|\alpha|/2$. Then

$$|\phi(v)|^3 \leq c|\phi(v)|^2 = \frac{1}{2}|A^{-1}|^{-1}|A|^{-1}|\alpha||\phi(v)|^2 \leq |A|^{-1}|v||\phi(v)|^2,$$

which is the desired inequality.

Case 4: $v_c \neq 0, \phi_c \neq 0$. Let $\epsilon'$ be chosen as in case (3), and let $\epsilon = \min\{\epsilon', |\phi_c|/2\}$. As in the previous cases, this choice of $\epsilon$ guarantees that $|v| \geq |v_c|/2$, and that $|\phi(v)| \geq |\phi_c|/2$. As before,

$$|\phi(v)^2 - \phi_c^2| \leq |\phi(v) - \phi_c||\phi_c + \phi_c|$$

$$\leq \epsilon(2|\phi_c| + \epsilon)$$

$$\leq \frac{5}{4}|\phi_c|^2.$$

Similarly, since $|v - v_c| \leq |v_c|/2$ and $|\phi(v) - \phi_c| \leq |A||v - v_c|$, we also have

$$|\phi(v)^2 - \phi_c^2| \leq (3/4)|A||v_c||\phi_c|.$$ 

Putting these together, we have that

$$|\phi(v)^2 - \phi_c^2|^2 \leq c|v_c||\phi_c|^2 \leq c'|v||\phi(v)|^2,$$
where $c$ and $c'$ are numerical constants independent of $\phi_c$. □

Lemmas 5.6 and 5.8 allow us to prove the following local estimates, which are essential in the proof of Theorem 4.7.

**Proposition 5.10.** Let $\mu_c \in \mathfrak{k}$ be fixed and $y$ is some point in $V$ with discrete stabilizer. Then there exists an open neighbourhood $U$ of $y$ and constants $c > 0$ and $\epsilon > 0$ such that

$$|\nabla f(x)| \geq k|f(x) - f_c|^2$$

for all $x \in V \setminus \{0\}$ such that $\hat{x} \in U$ and $|\mu(x) - \mu_c| < \epsilon$.

**Proof.** Suppose $y \in V$ and $\text{stab}(y) = 0$. Then by Lemma 5.6 there is a neighbourhood $U$ of $y$ so that

$$|v_{\xi}(x)|^2 \geq c|x|^2|\xi|^2$$

for all $x$ such that $\hat{x} \in U$. Take $\xi = \mu(x)$ and apply Lemma 5.8 to find $c'$ and $\epsilon$ so that

$$|v_{\mu(x)}(x)|^2 \geq c' |\mu(x)|^2 - |f_c|^2,$$

whenever $|\mu(x) - \mu_c| < \epsilon$. Since $|\nabla f_o(x)| = 2|v_{\mu(x)}(x)|$, this gives the desired inequality. □

**Proposition 5.11.** Let $y \in V$ and suppose $\text{stab}(y)$ is a proper nontrivial subspace of $\mathfrak{k}$. Then there are proper subtori $K_1, K_2$ of $K$ such that $K \cong K_1 \times K_2$, a neighbourhood $U$ of $\hat{y}$, and a constant $c > 0$ such that

$$|\nabla f(x)| \geq c (|\nabla f_{K_1}(x)| + |\nabla f_{K_2}(x)|)$$

for all $x \in V \setminus \{0\}$ with $\hat{x} \in U$, where $f_{K_i} = |\mu_{K_i}|^2$ and $f_{K_2} = |\mu_{K_2}|^2$.

**Proof.** Let $\mathfrak{t}_1 = \text{stab}(y)$ and $\mathfrak{t}_2 = \mathfrak{t}_1^\perp$. Since $\mathfrak{k}$ is abelian, both $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are Lie subalgebras, and $\mathfrak{k} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$. Then $\mu_K = \mu_{K_1} \oplus \mu_{K_2}$, and $|\mu_K|^2 = |\mu_{K_1}|^2 + |\mu_{K_2}|^2$, so the result follows immediately from Lemma 5.6 and inequality 15. □

**Proof of Theorem 4.7.** By Proposition 5.10 we will assume that $K$ is a torus. By Proposition 5.5 it suffices to show that for each $\mu_c \in \mathfrak{k}$, there is some $\epsilon > 0$ so that inequality 15 holds when $|\mu(x) - \mu_c| < \epsilon$. Additionally, since 0 is always a critical point of $f$, it suffices to prove the estimate only on $V \setminus \{0\}$. Furthermore, it suffices to show that each point $\hat{y}$ of the unit sphere has a neighbourhood $U$ such that the estimate holds for all $x$ with $\hat{x} \in U$, since by compactness we can choose finitely many such neighbourhoods to cover the unit sphere, and this yields the inequality on $V \setminus \{0\}$.

We will prove the estimate by induction on the dimension of $K$. First suppose $\dim K = 1$. Then we may assume without loss of generality that $K$ acts locally freely on $V \setminus \{0\}$, since otherwise the fundamental vector field $v_\xi(x)$ vanishes on a nontrivial subspace and we can restrict our attention to its orthogonal complement. Then Proposition 5.10 yields the desired neighbourhoods and estimates.

Now assume that $\dim K > n$ and we have proved the estimate for tori of dimension $\leq n$. Without loss of generality, we can assume that there is no nonzero vector in $V$ which is fixed by all of $K$ (since we can restrict to its orthogonal complement). Let $\hat{y}$ be some point in the unit sphere, and let $\mathfrak{t}_1 = \text{stab}(\hat{y})$. If $\mathfrak{t}_1 = 0$, we may apply Proposition 5.10 to get a neighbourhood $U_{\hat{y}}$ and a constant $k_{\hat{y}}$ such that the estimate holds on $U_{\hat{y}}$. Otherwise, let $\mathfrak{t}_2 = \mathfrak{t}_1^\perp$ so that $\mathfrak{k} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, corresponding to
subtori \( K_1 \) and \( K_2 \). Then we may apply Proposition 5.11 to find a neighbourhood \( U_\hat{y} \) so that
\[
\|\nabla f(x)\| \geq k \left( |\nabla f_{K_1}(x)| + |\nabla f_{K_2}(x)| \right),
\]
holds for all \( x \) with \( \hat{x} \in U_\hat{y} \). Let \( P_i : \mathfrak{k} \to \mathfrak{k}_i \), \( i = 1,2 \) be the orthogonal projections, \( \mu_{c,i} = P_i \mu_c \), and \( f_{c,i} = |\mu_{c,i}|^2 \). Since \( \mu_{K_i} \) are the moment maps for the action of \( K_i \), which are tori of dimension \( \leq n \), we may apply the induction hypothesis to find a neighbourhood \( U \) of \( \hat{y} \) and constants \( \epsilon > 0 \), \( c > 0 \) so that
\[
|\nabla f_{K_1}(x)| \geq k' |f_{K_1}(x) - f_{c,1}|^{3/4},
\]
\[
|\nabla f_{K_2}(x)| \geq k' |f_{K_2}(x) - f_{c,2}|^{3/4},
\]
for all \( x \) that \( \hat{x} \in U \) and \( |\mu_{K_i}(x) - \mu_{c,i}| < \epsilon \). For any non-negative numbers \( a, b \), we have \( a^{3/4} + b^{3/4} \geq (a + b)^{3/4} \), so we obtain
\[
|\nabla f(x)| \geq k'' \left( |f_{K_1}(x) - f_{c,1}| + |f_{K_2}(x) - f_{c,2}| \right)^{3/4} \geq k'' |f(x) - f_c|^{3/4},
\]
whenever \( |\mu(x) - \mu_c| < \epsilon \), as desired.

To obtain the estimate for the functions \( |\mu_c|^2 \) and \( |\mu_{HK}|^2 \), we simply note that by Lemma 5.1, the norm of the gradient is bounded below by a sum of terms of the form \( |\nabla |\mu_i|^2| \), and since we can bound each term individually we obtain a bound for the sum.

**Remark 5.12.** As pointed out in Remark 4.8, Theorem 4.7 is a generalization of Theorem A.1 and much of the argument is similar. The main new ingredients are Lemma 5.8 and Proposition 5.10 which are absolutely essential in handling the general case of a nonzero constant term in the moment map.

### 6. Toric Hyperkähler Orbifolds

#### 6.1. Notation and Definitions

Let \( T \) be a subtorus of the standard \( N \)-torus \((S^1)^N\), with quotient \( K := (S^1)^N/T \). We have a short exact sequence
\[
1 \to T \xrightarrow{i} (S^1)^N \xrightarrow{\pi} K \to 1.
\]
Taking Lie algebras, we have
\[
0 \to \mathfrak{t} \xrightarrow{i} \mathbb{R}^N \xrightarrow{\pi} \mathfrak{k} \to 0,
\]
\[
0 \to \mathfrak{t}^* \xrightarrow{\pi^*} \mathbb{R}^N \xrightarrow{i^*} \mathfrak{t}^* \to 0.
\]
Recall the standard Hamiltonian action of \((S^1)^N\) on \( \mathbb{C}^N \). This restricts to a Hamiltonian action of \( T \) on \( \mathbb{C}^N \), and hence induces an action on \( T^* \mathbb{C}^N \). For a generic \((\alpha, \beta) \in \mathfrak{t}^* \oplus \mathfrak{t}_C^*\) we define \( M(\alpha, \beta) \) to be the quotient
\[
M(\alpha, \beta) := T^* \mathbb{C}^N \sslash\sslash_T (\alpha, \beta),
\]
which is a toric hyperkähler orbifold \([3]\). There is a residual Hamiltonian action of \( K \) on \( M \). The homeomorphism type of \( M(\alpha, \beta) \) is independent of \((\alpha, \beta)\) as long as \((\alpha, \beta)\) is generic, so we will often write \( M \) instead of \( M(\alpha, \beta) \).

We can organize the data determining \( M \) as follows. We will assume for the moment that \( M \) is taken to be the reduction at \((\alpha, 0)\) with \( \alpha \) generic. Let \( \{e_i\} \) be the standard basis of \( \mathbb{R}^N \). Then we obtain a collection \( A := \{u_j\} \) of weights defined by \( u_j := i^*(e_i) \in \mathfrak{t}^* \), as well as a collection of normals \( \{n_i\} \) defined by \( n_i = \pi(e_i) \in \mathfrak{k} \). Note that we allow repetitions, i.e. \( u_i \) and \( u_j \) are considered to be
distinct elements of $A$ for $i \neq j$ even if $u_i = u_j$ as elements of $\mathfrak{t}^*$, and similarly for the normals. Using the inner product on $\mathfrak{t}$ induced by the embedding $\mathfrak{t} \hookrightarrow \mathbb{R}^N$, we can identify $\mathfrak{t} \cong \mathfrak{t}^*$ and we will think of the weights $u_j$ as elements of $\mathfrak{t}$ rather than $\mathfrak{t}^*$ whenever it is convenient to do so. Pick some $d \in \mathbb{R}^N$ such that $i^*(d) = \alpha$. Then we can define affine hyperplanes $H_i$ by
\begin{equation}
H_i = \{ x \in \mathfrak{t}^* \mid \langle n_i, x \rangle - d_i = 0 \},
\end{equation}
as well as half-spaces
\begin{equation}
H_i^\pm = \{ x \in \mathfrak{t}^* \mid \pm (\langle n_i, x \rangle - d_i) \geq 0 \}.
\end{equation}
This arrangement of hyperplanes will be denoted by $\mathcal{A}$. It is shown in [3] that the arrangement $\mathcal{A}$ plays a role in toric hyperkähler geometry analogous to that of the moment polytope in symplectic toric geometry. In particular, the arrangement $\mathcal{A}$ determines $M$ up to equivariant hyperkähler isometry.

**Definition 6.1.** Let $J \subseteq \{1, \cdots, N\}$, and define a subspace
\begin{equation}
t_J := \text{span}\{ u_j \mid j \in J \} \subset \mathfrak{t}
\end{equation}
with corresponding subtorus $T_J \subset T$. We will call the set $J$ critical if the following condition is satisfied: $u_j \in t_J$ if and only if $j \in J$.

If $J$ is critical, we define a subspace $V_J \subset \mathbb{C}^N$ by
\begin{equation}
V_J := \text{span}\{ e_j \mid j \in J \}.
\end{equation}
The action of $T_J$ preserves $V_J$, and we may take the hyperkähler quotient
\begin{equation}
M_J := T^*V_J /// T_J.
\end{equation}
We will always assume that the reduction is taken at a generic regular value.

**Remark 6.2.** The critical sets $J$ are precisely the flats of the matroid associated to the collection of vectors $\{u_j\}$. However, we do not wish to assume familiarity with matroids, so we choose to avoid using this language. See [9] for a detailed discussion of the relation between the geometry of toric hyperkähler varieties and the combinatorics of matroids, and [5] for matroids in general.

The inclusion $i : T \hookrightarrow (S^1)^N$ induces a surjective map of rings $i^* : H^*_T(\mathbb{C}^N) \to H^*_T$, so that $H^*_T \cong \mathbb{Q}[u_1, \ldots, u_N]/\ker i^*$ as a graded ring. By abuse of notation we will write $u_j$ to denote its image in $H^*_T$. (Note that we also use the symbol $u_j$ to denote the vectors $i^*(e_j)$, but no confusion should arise as it should be clear from context which of the two meanings is intended.)

If $J$ is a subset of $\{1, \ldots, N\}$, then we define a class $u_J \in H^*_T$ by
\begin{equation}
u_J := \prod_{i \in J^c} u_i,
\end{equation}
and note that the product is taken over the complement of $J$. 

6.2. Analysis of the Critical Sets. We now consider a quotient of the form $M(\alpha, \beta)$ with $\beta$ a regular value of $\mu_C$. Note that this includes quotients of the form $M(\alpha, 0)$ as a special case, since we can always rotate the hyperkähler frame. We identify $T^*\mathbb{C}^N$ with $\mathbb{C}^N \times \mathbb{C}^N$ and use coordinates $(x, y)$. Shifting $\mu_C$ by $\beta$, we can take it to be

\begin{equation}
\mu_C(x, y) = \sum_{i=1}^{N} x_i y_i u_i - \beta,
\end{equation}

and we will consider Morse theory with the function $f = |\mu_C|^2$.

**Proposition 6.3.** For a generic parameter $\beta$, the critical set of $f$ is the disjoint union of sets $C_J$, where the union runs over all critical subsets $J \subseteq \{1, \ldots, N\}$, and the sets $C_J$ are defined by

\begin{equation}
C_J = \left( \bigcap_{i \in J} \{ u_i \cdot \mu_C = 0 \} \right) \cap \left( \bigcap_{j \notin J} \{ (x_j, y_j) = 0 \} \right).
\end{equation}

The Morse index of $C_J$ is given by $\lambda_J = 2(N - \# J)$. Up to a nonzero constant, the $T$-equivariant Euler class of the negative normal bundle to $C_J$ is given by the restriction of the class $u_J$ to $H^*_T(C_J)$, where $u_J$ is defined by (27). The $T$-equivariant Poincaré series of $C_J$ is equal to $(1 - t^2)^{-r} P(M_J)$, where $r$ is the codimension of $T_J$ in $T$ and $M_J$ is the quotient defined by (29).

**Proof.** Using equation (28), we see that

\[ |\nabla f(x, y)|^2 = 4 \sum_J \left( |x_j|^2 + |y_j|^2 \right) |u_j \cdot \mu_C(x, y)|^2. \]

Since this is a sum of non-negative terms, if $\nabla f(x, y) = 0$, each term in the sum must be 0. Thus for each $J$, we must have either that $x_j = y_j = 0$ or that $u_j \cdot \mu_C = 0$.

Let us fix some particular critical point $(x_c, y_c) \in T^*\mathbb{C}^N$, and let $J$ be the set of indices $j \in \{1, \ldots, N\}$ for which $u_j \cdot \mu_C(x_j, y_j) = 0$. By construction $J$ is critical and $(x_c, y_c) \in C_J$. Hence every critical point is contained in $C_J$ for some critical set $J$. Conversely, $\nabla f = 0$ on $C_J$ by construction, so we see that $\text{Crit}f = \bigcup_J C_J$, where the union runs over critical sets $J$. Note that $C_J \neq \emptyset$ when $J$ is critical.

To see that the union is disjoint, write $\mu_C = \mu_J + \mu_{J^c}$, where

\begin{equation}
\mu_J(x, y) = \sum_{i \in J} x_i y_i u_i - \beta_J,
\end{equation}

\begin{equation}
\mu_{J^c}(x, y) = \sum_{i \notin J} x_i y_i u_i - \beta_J^+,\n\end{equation}

$\beta_J$ is the projection of $\beta$ to $t_J$, and $\beta_J^+ = \beta - \beta_J$. Then at $(x_c, y_c)$, we have $\mu_J(x_c, y_c) = 0$ and $\mu_{J^c}(x_c, y_c) = -\beta_J^+$. Thus on $C_J$, $\mu_C$ takes the value $-\beta_J^+$. For generic $\beta$, we have $\beta_J^+ \neq \beta_J^+$ for $J \neq J'$, hence $C_J \cap C_{J'} = \emptyset$ for $J \neq J'$.

To determine the Morse index of $C_J$, we compute

\[ |\mu_C(x, y)|^2 = |\mu_J(x, y)|^2 + |\mu_{J^c}(x, y)|^2 + 2\text{Re}(\mu_J(x, y), \mu_{J^c}(x, y)). \]

The term $|\mu_J(x, y)|^2$ has an absolute minimum at $(x_c, y_c)$, and so does not contribute to the Morse index. Since $\beta_J^+$ is orthogonal to $\mu_J(x, y)$ for all $(x, y)$, the third term can be rewritten as

\[ 2\text{Re}(\mu_J(x, y), \mu_{J^c} + \beta_J^+). \]
Looking at the expressions (30) and (31) for \( \mu_J \) and \( \mu_{J^c} \), we see that at \((x_c,y_c)\), \( \mu_J \) vanishes to first order, whereas \( \mu_{J^c} + \beta^c_j \) vanishes to second order. Hence the inner product of these terms vanishes to third order and does not affect the Morse index. Thus the Morse index is determined solely by the second term, which is
\[
|\mu_{J^c}(x,y)|^2 = |\beta^c_j|^2 - 2\Re \sum_{i \in J^c} \langle \beta^c_j, u_i \rangle x_i y_i + \text{fourth order.}
\]

For generic \( \beta \), we have \( \langle \beta^c_j, u_i \rangle \neq 0 \) for all \( i \in J^c \), and since each term in the sum is the real part of the holomorphic function \( \langle \beta^c_j, u_i \rangle x_i y_i \) it must contribute 2 to the Morse index. Hence the Morse index is \( \lambda_J := 2\# J^c = 2(N - \# J) \). Since the \( j \)th factor of \((S^1)^N\) acts on \((x_j, y_j)\) with weight \((1, -1)\), this also shows that the equivariant Euler class is given by a nonzero multiple of \( u_J \) (defined by (27)), as claimed.

Finally, we compute the equivariant Poincaré series of \( C_J \). Let \( V_J \in \mathbb{C}^\mathbb{C} \) be defined as above, and let \( K_J \subset T \) be the subtorus that acts trivially on \( V_J \). Then we have an isomorphism \( T \cong T_J \times K_J \). Let \( r \) be the dimension of \( K_J \), which is the codimension of \( T_J \) in \( T \). The moment map for the action of \( T_J \) on \( T^* V_J \) is given by the restriction of \( \mu_J \) (as defined by equation (30)) to \( T^* V_J \). Hence \( C_J = \mu_J^{-1}(0) \cap T^* V_J \), and
\[
P_T(C_J) = P_{T_J \times K_J}(C_J) = (1 - i^2)^{-r} P_{T_J}(C_J) = (1 - i^2)^{-r} P(M_J).
\]

\[\square\]

**Remark 6.4.** Note that the critical sets \( C_J \) are all nonempty, and that the Morse indices do not depend on \( \beta \) (as long as it is generic). This is due to the fact that \( \mu_C \) is holomorphic. In the real case, i.e. \( |\mu_R - \alpha|^2 \), the critical sets and Morse indices have a much more sensitive dependence on the level \( \alpha \).

By Theorem 3.3 and Corollary 4.14 the hyperkähler Kirwan map \( \kappa : H^*_J \to H^*(M) \) is surjective, and its kernel is the ideal generated by the equivariant Euler classes of the negative normal bundles to the components of the critical set. Since we described these explicitly in Proposition 6.3, we immediately obtain the following description of \( H^*(M) \).

**Theorem 6.5.** The cohomology ring \( H^*(M) \) is isomorphic to \( H^*_J / \ker \kappa \), where \( \ker \kappa \) is the ideal generated by the classes \( u_J \), for every proper critical set \( J \subset \{1, \ldots, N\} \).

\[\square\]

**Remark 6.6.** The cohomology ring \( H^*(M) \) was first computed by Konno [16, Theorem 3.1]. The relations defining \( \ker \kappa \) obtained by Konno are not identical to those in Theorem 6.5, but it is not difficult to see that they are equivalent. It was pointed out to us by Proudfoot that this equivalence is a special case of Gale duality [5].

**Remark 6.7.** Under the assumption that \( M \) is smooth (and not just an orbifold), the same result holds with \( \mathbb{Z} \) coefficients [16]. In certain cases, the Kirwan method can be extended to handle cohomology with \( \mathbb{Z} \) coefficients, provided that the group action satisfies certain additional hypotheses [25].

Quotients of the form \( M(\alpha,0) \) inherit an additional \( S^1 \)-action induced by the \( S^1 \) action on \( T^* \mathbb{C}^N \) given by \( t \cdot (x,y) = (x, ty) \). This action preserves the Kähler structure and rotates the holomorphic symplectic form (i.e. \( t^* \omega_C = t \omega_C \)). Let us
fix some particular $\alpha$ and denote $M := M(\alpha, 0)$. We would like to understand the $S^1$-equivariant cohomology $H^*_S(M)$ (which, unlike the ordinary cohomology of $M$, does depend on the choice of $\alpha$). To compute the $S^1$-equivariant cohomology, it is more convenient to work directly with $|\mu_{HK}|^2 = |\mu_R|^2 + |\mu_C|^2$, where

\begin{align*}
\mu_R &= \sum_i (|x_i|^2 - |y_i|^2) u_i - \alpha, \\
\mu_C &= \sum_i x_i y_i u_i.
\end{align*}

By Theorems 3.3 and 4.7 it is minimally degenerate and flow-closed, and since it is also $S^1$-invariant we can consider the $T \times S^1$-equivariant Thom-Gysin sequence. The usual arguments of the Kirwan method extend to the $S^1$-equivariant setting, so we obtain surjectivity of map $\kappa_{S^1} : H^*_{T \times S^1} \rightarrow H^*_S(M)$, and its kernel is generated by the $T \times S^1$-equivariant Euler classes of the negative normal bundles to the critical sets of $|\mu_{HK}|^2$.

To find the critical sets of $|\mu_{HK}|^2$ and to compute the equivariant Euler classes, we can repeat the arguments of Proposition 6.3 almost without modification. The components of the critical set are again indexed by critical subset $J$. The only important difference is that since we now work with $T \times S^1$-equivariant cohomology, we have to be more careful in computing the equivariant Euler classes. Let us make the identification $H^*_{T \times S^1} \cong H^*_T[u_0]$. When we expand $|\mu_{HK}|^2$ about a critical point as in the proof of Proposition 6.3, the relevant term is now (cf. equation (32))

\[ -2 \sum_{i \in J^+} \langle \alpha^+_J, u_i \rangle \left( |x_i|^2 - |y_i|^2 \right), \]

where $\alpha^+_J$ is defined in a manner analogous to $\beta^+_J$ as in the proof of Proposition 6.3. We see that the $x_i$ term appears with an overall negative sign if $\langle \alpha^+_J, u_i \rangle > 0$, otherwise it is the $y_i$ term that appears with a negative sign. Since $S^1$ acts on $x$ with weight 0 and acts on $y$ with weight 1, and since $T$ acts on $x$ and $y$ with oppositely signed weights, we find that the equivariant Euler class is given (up to an overall constant) by

\begin{equation}
\tilde{u}_J := \prod_{i \in J^+} u_i \prod_{j \in J^-} (u_0 - u_j),
\end{equation}

where

\begin{equation}
J^\pm := \{ i \in J^c \mid \pm \langle \alpha^+_J, u_i \rangle > 0 \}.
\end{equation}

Thus we obtain the following.

**Theorem 6.8.** The $S^1$-equivariant cohomology $H^*_S(M)$ is isomorphic to

\[ H^*_{T \times S^1} / \ker \kappa_{S^1}, \]

where $\ker \kappa_{S^1}$ is the ideal generated by the classes $\tilde{u}_J$, for every proper critical subset $J$.

\[ \square \]

**Remark 6.9.** The $S^1$-equivariant cohomology rings were first computed by Harada and Proudfoot [8]. As in Theorem 6.5 our description of the kernel is dual (but equivalent). In this instance, the duality is for oriented matroids.
Remark 6.10. Note that unlike the ordinary cohomology ring, the $S^1$-equivariant cohomology ring depends explicitly on the parameter $\alpha$.

6.3. Hyperkähler Modifications. There is a natural operation called modification \cite{2} defined on hyperkähler manifolds (or orbifolds) equipped with a Hamiltonian $S^1$ action, which is the hyperkähler analogue of symplectic cutting \cite{19}. We will show in this section that the critical sets $C_J$ can be understood inductively in terms of modifications and quotients.

Let $M := T^*C^N /// T$ be a toric hyperkähler orbifold. This has a residual Hamiltonian action of a torus $K$. Fix some particular $S^1$ subgroup of $K$. Then we can consider $M \times T^*C$, which has two commuting $S^1$ actions, diagonal and anti-diagonal. We define the modification $\tilde{M}$ of $M$ with respect to this $S^1$ action to be the hyperkähler quotient
\begin{equation}
\tilde{M} := (M \times T^*C) /// S^1 = T^*(C^{N+1} /// (T \times S^1)),
\end{equation}
where the quotient is by the anti-diagonal $S^1$. We also consider the quotient
\begin{equation}
\tilde{M} := M /// S^1 = T^*C^{N+1 /// (T \times S^1)}.
\end{equation}
We will use the notation $\tilde{T} := T \times S^1$ so that $\tilde{M}$ and $\tilde{M}$ are quotients by $\tilde{T}$. Let $\mu$, $\tilde{\mu}$, and $\hat{\mu}$ denote the respective (complex) moment maps, and let $A$, $\tilde{A}$, and $\hat{A}$ denote the respective collections of weights. The critical sets of $|\mu|^2$, $|\tilde{\mu}|^2$, and $|\hat{\mu}|^2$ are defined with respect to $A$, $\tilde{A}$, and $\hat{A}$. We can relate the weights $\tilde{A}$ and $\hat{A}$ corresponding to a modification and quotient of $M$ to the weights $A$ as follows.

Lemma 6.11. Let $\tilde{A} = \{\tilde{u}_j\}_{j=1}^{N+1}$. Then $A = \{u_j\}_{j=1}^N$ and $\hat{A} = \{\hat{u}_j\}_{j=1}^N$, where $u_j$ is the image of $\tilde{u}_j$ after quotienting by $\text{span}\{\tilde{u}_{N+1}\}$.

Proof. The weights are determined by the embeddings $t \hookrightarrow \mathbb{R}^N$, $\tilde{t} \hookrightarrow \mathbb{R}^{N+1}$, and $\tilde{t} \hookrightarrow \mathbb{R}^N$. Note that $\tilde{t} \cong t \oplus \mathbb{R}$. If we pick a basis of $t$ (and use the standard basis of $\mathbb{R}^N$), we can represent the embedding $t \hookrightarrow \mathbb{R}^N$ by some matrix $B$. The $S^1$ action on $M$ is determined by specifying its weights on $\mathbb{C}^N$; this is equivalent to adjoining a column to $B$. This gives the matrix $\tilde{B}$ determining $\tilde{t} \hookrightarrow \mathbb{R}^N$. Finally, to obtain the modification $\tilde{M}$, we let the $S^1$ act on an additional copy of $\mathbb{C}$ with weight $-1$. This amounts to adjoining the row $(0, \ldots, 0, -1)$ to $\tilde{B}$, to obtain $\hat{B}$ determining $\tilde{t} \hookrightarrow \mathbb{R}^{N+1}$. Since the weights $u_j$, $\tilde{u}_j$, and $\hat{u}_j$ correspond to the rows of $B$, $\tilde{B}$, and $\hat{B}$, respectively, the result follows from this description. \hfill $\square$

Now that we understand the relationship between the weights, we describe the relationship between the critical subsets $J$, which are defined with respect to the weights. We will say that $J \subseteq \{1, \ldots, N+1\}$ is critical for $A$ (respectively $\tilde{A}, \hat{A}$) if it indexes a component of the critical set of $|\mu|^2$ (respectively $|\tilde{\mu}|^2, |\hat{\mu}|^2$). If $N+1 \in J$ then we will not consider it to be critical for $A$ (cf. Definition \ref{def:critical_subset}).

Lemma 6.12. Let $J \subseteq \{1, \ldots, N\}$. Then

1. $J$ is critical for $A$ if and only if $J \cup \{N+1\}$ is critical for $\tilde{A}$.
2. If $J$ is critical for $A$, then $J$ is critical for $\hat{A}$.
3. $J$ is critical for $\tilde{A}$ if and only if at least one of $J$ or $J \cup \{N+1\}$ is critical for $\hat{A}$.

Proof. Let $\tilde{A} = \{\tilde{u}_j\}_{j=1}^{N+1}$. By the previous lemma, $\tilde{A} = \{\tilde{u}_j\}_{j=1}^{N+1}$ and $A = \{u_j\}_{j=1}^N$, where $u_j$ is the image of $\tilde{u}_j$ after quotienting by $\text{span}\{\tilde{u}_{N+1}\}$.
To prove (1), first suppose that $J$ is critical for $A$. Suppose that there is some $\tilde{u}_i \in \tilde{t}_J$. If $i = N + 1$ then $i \in J \cup \{N + 1\}$ and there is nothing to check, so suppose that $i \neq N + 1$. Applying the quotient map, we see that $u_i \in t_J$ (since $\tilde{u}_{N+1}$ goes to 0), and since $J$ was critical for $A$ we see that $i \in J \subset J \cup \{N + 1\}$. Hence $J \cup \{N + 1\}$ is critical for $A$.

Conversely, suppose that $J \cup \{N + 1\}$ is critical for $A$. Suppose that there is some $u_i \in t_J$. Then if $u_i = \sum_{j \in J} a_j u_j$, we see that $\tilde{u}_i - \sum_{j \in J} a_j \tilde{u}_j$ is in the kernel of the projection, and so is some multiple of $\tilde{u}_{N+1}$. Hence $u_i \in \tilde{t}_{J \cup \{N + 1\}}$. Since $J \cup \{N + 1\}$ is critical for $A$, we must have $i \in J \cup \{N + 1\}$. But $i \neq N + 1$ by assumption, so we have $i \in J$.

To prove (2), suppose that $J$ is critical for $A$. If $\tilde{u}_i \in \tilde{t}_J$, then applying the quotient map we find $u_i \in t_J$. Hence $i \in J$.

To prove (3), first suppose that $J \cup \{N + 1\}$ is critical for $A$. Then by (1), $J$ is critical for $A$, and by (2) $J$ is critical for $A$. On the other hand, if $J$ is critical for $A$, then it is certainly critical for $A$. This establishes one direction.

Conversely, suppose that $J$ is critical for $A$. If $\tilde{u}_{N+1} \notin \tilde{t}_J$ then $J$ is critical for $\tilde{A}$; otherwise $\tilde{u}_{N+1} \in \tilde{t}_J$, and thus $J \cup \{N + 1\}$ is critical for $\tilde{A}$. □

We rephrase the preceding lemma as the following trichotomy:

**Lemma 6.13.** Let $J \subseteq \{1, \ldots, N\}$ be a critical subset with respect to $\tilde{A}$. Then exactly one of the following cases occurs:

1. $J$ is critical for $\tilde{A}$ and $J$ is not critical for $A$.
2. $J$ is critical for $A$ and both $J$ and $J \cup \{N + 1\}$ are critical for $\tilde{A}$.
3. $J$ is critical for $A$ and $J \cup \{N + 1\}$ is critical for $\tilde{A}$, while $J$ is not.

Moreover, every critical subset for $A$ and $\tilde{A}$ occurs as exactly one of the above. □

**Theorem 6.14.** If $\tilde{M}$ is a hyperkähler modification of $M$ and $M$ is the corresponding quotient, then

$$P(\tilde{M}) = P(M) + t^2 P(\tilde{M}).$$

*Proof.* We will prove this by induction on $N = \#A$, the number of the number of weights (equivalently, the number of hyperplanes in $A$). The base case can be verified easily, so we assume that the result is true for modifications $(\tilde{M}', M', M')$, where $M'$ is a quotient of $T^*\mathbb{C}^{N'}$, with $N' < N$.

By Theorems 3.3 and 4.7, the functions $f = |\mu_C|^2$, $\tilde{f} = |\tilde{\mu}_C|^2$, and $\tilde{f} = |\tilde{\mu}_C|^2$ are equivariantly perfect. $\tilde{M}$ is a quotient of $T^*\mathbb{C}^N$ by a torus of rank $d$, while $M$ and $\tilde{M}$ are quotients of $T^*\mathbb{C}^{N+1}$ and $T^*\mathbb{C}^N$, respectively, by a torus of rank $d+1$. Hence we have

$$\frac{1}{(1-t^2)^d} = \sum_C t^{\lambda_C} P_T(C),$$

$$\frac{1}{(1-t^2)^{d+1}} = \sum_C \tilde{t}^{\lambda_C} P_{\tilde{T}}(\tilde{C}),$$

$$\frac{1}{(1-t^2)^{d+1}} = \sum_C \tilde{t}^{\lambda_C} P_{\tilde{T}}(\tilde{C}).$$
Since 0 is the absolute minimum in each case, we obtain
\[
\frac{1}{(1-t^2)^d} = P(M) + \sum_{C,f(C)>0} t^{\lambda_c} P_T(C),
\]
and similarly for \(\tilde{M}\) and \(\tilde{M}\). Since
\[
\frac{1}{(1-t^2)^{d+1}} = \frac{1}{(1-t^2)^d} + \frac{t^2}{(1-t^2)^{d+1}} = 0,
\]
we just have to show that
\[
\sum_{\tilde{C},f(C)>0} t^{\tilde{\lambda}_c} P_T(\tilde{C}) = \sum_{C,f(C)>0} t^{\lambda_c} P_T(C) + \sum_{\tilde{C},f(C)>0} t^{\tilde{\lambda}_c+2} P_T(\tilde{C})
\]
to obtain the desired recurrence relation among the Poincaré polynomials of \(M\), \(\tilde{M}\), and \(\tilde{M}\). By Lemma 6.13, there is a trichotomy relating the critical sets of \(\hat{f}\) to the critical sets of \(f\) and \(\hat{f}\). We will consider each case separately.

Case (1): We have \(\tilde{C} = \tilde{C}_J\) where \(J\) is critical with respect to \(\hat{A}\) and \(\hat{A}\) but not with respect to \(A\). From Proposition 6.3, we have \(\tilde{\lambda}_J = \lambda_J + 2\) and \(\tilde{C}_J = \tilde{C}_J \times (0, 0)\). Hence \(t^{\tilde{\lambda}_J} P_T(\tilde{C}_J) = t^{\lambda_J+2} P_T(\tilde{C}_J)\).

Case (2): We have \(\tilde{C} = \tilde{C}_J\) where \(J\) is critical with respect to \(\hat{A}\) and \(J \cup \{N+1\}\) is also critical for \(\hat{A}\). Then \(\tilde{\lambda}_J = \lambda_J + 2\) and \(\tilde{C}_J = \tilde{C}_J \times (0, 0)\). The terms involving \(\tilde{C}_J\) and \(\tilde{C}_J\) are equal as in case (1). Since both \(J\) and \(J \cup \{N+1\}\) are critical for \(\hat{A}\), it must be that \(\tilde{u}_{N+1} \notin \mathfrak{I}_J\). Hence \(\tilde{T}_{J \cup \{N+1\}} \simeq \tilde{T}_J \times S^1\), where the last factor is generated by \(\tilde{u}_{N+1}\), and we find that \(\tilde{M}_{J \cup \{N+1\}} \simeq M_J\). Hence \(P_T(\tilde{C}_{J \cup \{N+1\}}) = P_T(C_J)\).

Case (3): \(\tilde{C} = \tilde{C}_{J \cup \{N+1\}}\), where \(J\) is critical for \(A\) and \(\hat{A}\) but not for \(\hat{A}\). By Proposition 6.3, we have \(P_T(\tilde{C}_{J \cup \{N+1\}}) = (1-t^2)^{-r} P(M_{J \cup \{N+1\}})\), \(P_T(C_J) = (1-t^2)^{-r} P(M_J)\), and \(P_T(\tilde{C}_J) = (1-t^2)^{-r} P(M_J)\), where \(r\) is the codimension of \(T_J \cap T\). Thus we have to show that
\[
\sum_{\tilde{C}_{J \cup \{N+1\}}} t^{\tilde{\lambda}_c} P_T(\tilde{C}) = \sum_{C \cup \{N+1\}} t^{\lambda_c} P_T(C) + \sum_{\tilde{C}_{J \cup \{N+1\}}} t^{\tilde{\lambda}_c+2} P_T(\tilde{C})
\]
But \(\tilde{M}_{J \cup \{N+1\}}\) is a modification of \(M_J\), and \(\tilde{M}_J\) is the corresponding quotient of \(M_J\). Since \(J\) is a proper subset of \(\{1, \ldots, N\}\), the relation \([30]\) is true by induction.

The relation \([30]\) is equivalent to the following recurrence relation among the Betti numbers of \(M\), \(\tilde{M}\), and \(\tilde{M}\):
\[
\tilde{b}_{2k} = b_{2k} + \hat{b}_{2k} + \hat{b}_{2k-2}.
\]
If we let \(d_k\) denote the number of \(k\)-dimensional facets of the polyhedral complex generated by the half-spaces \(H_i^\pm\) (with \(\tilde{d}_k\) and \(\tilde{d}_k\) defined similarly), it turns out \([31]\) that these satisfy the same relation:
\[
\tilde{d}_k = d_k + \hat{d}_k + \hat{d}_{k-1}.
\]
Since any toric hyperkähler orbifold can be constructed out of a finite sequence of modifications starting with \(T^*\mathbb{C}^n\), an easy induction argument then yields the following explicit description of \(P(M)\).
Corollary 6.15 (Bielawski-Dancer [3, Theorem 7.6]). The Poincaré polynomial of $M$ is given by

$$P(M) = \sum_k d_k (t^2 - 1)^k.$$ 

□

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