Hamiltonian Formalism of General Bimetric Gravity

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Abstract: We perform the Hamiltonian analysis of general bimetric gravity. We determine four first class constraints that are generators of the diagonal diffeomorphism. We further analyze the remaining constraints and we present an evidence that these constraints should be the second class constraints in order to have theory with the Hamiltonian constraint as the first class constraint. We also discuss the case of the non-linear bimetric gravity and argue that it is very difficult to eliminate the ghost mode.

Keywords: Massive Gravity
1. Introduction

The bimetric theory of gravity were introduced in [1] in order to describe the interaction of gravity with a massive spin 2–meson. These theories were also investigated recently due to their cosmological solutions [2, 3, 4, 5, 6, 7, 8, 9, 10]. Generally, bimetric theories contain one massive and one massless spin–2 field and one scalar ghost mode. As a result the bimetric theories of gravity are plagued by the same problems as the general massive theories of gravity.

On the other hand recent formulation of the massive gravity known as the non-linear massive gravity seems to be free of ghosts [11, 12]; for further improvement, see [14, 13]. This theory was further extended in [15] where the theory was formulated with general reference metric. The Hamiltonian analysis of given theory was performed in several papers [16, 17, 18, 19, 20, 21, 22] with the most important results derived in [23, 24] with the outcome that this non-linear massive theory possesses one additional constraint and the resulting constraint structure is sufficient for the elimination of the ghost degree of freedom.

Due to the fact that the non-linear massive gravity with general reference metric is free of ghost it is tempting to generalize this theory by introducing the kinetic term for the reference metric which now becomes dynamical. As a result \( \hat{g}_{\mu\nu} \) and \( \hat{f}_{\mu\nu} \) come in the symmetric way in the action which means that the non-linear massive gravity was generalized to the bimetric theory of gravity. This important step was performed in [25]. Then it was argued in [23] that the resulting theory is the ghost free formulation of the bimetric theory of gravity using the very detailed proof of the absence of ghosts in the non-linear massive gravity performed here. However the question of the absence of the ghosts in the new formulation of the bimetric theory of gravity was questioned recently in
This analysis was based on the careful Hamiltonian analysis of the bimetric gravity with redefined shift function \cite{25}. We have argued that due to the fact that the lapse function of the metric \( \hat{f}_{\mu\nu} \) is now Lagrange multiplier whose value is determined by the dynamics of the theory it is not possible to find another additional constraint so that the ghost mode is still present. On the other hand we showed that there are no first class constraints corresponding to the generators of the diagonal diffeomorphism and we argued that this is a consequence of the redefinition of the shift function performed in \cite{25}.

The goal of this paper is to extend the analysis presented in \cite{30} to the case of the Hamiltonian analysis of the bimetric theory of the gravity with general potential between two metric \( \hat{g}_{\mu\nu} \) and \( \hat{f}_{\mu\nu} \). We identify four the first class constraints corresponding to the diagonal diffeomorphism and determine the Poisson brackets among them in order to show that they obey the right form of the algebra. These results immediately solve the issue found in \cite{28} since we are now able to identify the four first class constraints that generate the diagonal diffeomorphism in case of non-linear bimetric gravity. As the next step in our analysis we try to answer the question whether it is possible to find an additional constraint in bimetric gravity which could eliminate the ghost mode. In other words we analyze the time development of the four remaining constraints. We present some evidence that it is very difficult to have such a form of the potential that will lead to the emergence of the additional constraint. Then we analyze the square root form of the potential and we argue that it is difficult to find an additional constraint which suggests that the ghost mode is still presented in given theory which also confirms the observation presented in \cite{28}.

The structure of this paper is as follows. In the next section (2) we introduce the bimetric theory of gravity and find its Hamiltonian formulation. Then in section (3) we calculate the Poisson brackets between constraints and find the constraints that are generators of the diagonal diffeomorphism. In section (4) we analyze the time evolution of the second class constraints. Finally in conclusion (5) we outline our result and suggests possible extension of this work.

2. Hamiltonian Formulation of Bigravity with General Potential

Let us consider bimetric theory of gravity defined by the action

\[
S = M^2_L \int d^4x \sqrt{-\hat{g}} R(\hat{g}) + M^2_R \int d^4x \sqrt{-\hat{f}} R(\hat{f}) - \mu \int d^4x (\det \hat{g} \det \hat{f})^{1/4} V(H_{\mu\nu}^{\rho\sigma}) , \tag{2.1}
\]

where we presume that the potential term is the general function of

\[
H_{\nu}^{\mu} = \hat{g}_{\mu\rho} \hat{f}_{\rho\nu} . \tag{2.2}
\]

When we require that the action (2.1) is invariant under following diffeomorphism transformations

\[
\hat{g}_{\mu\nu}(x') = \hat{g}_{\rho\sigma}(x) \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}}, \quad \hat{f}_{\mu\nu}(x') = \hat{f}_{\rho\sigma}(x) \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \tag{2.3}
\]

\[1\] This issue was also addressed in the recent paper \cite{32}.
we have to demand that the potential term has to be functions of various powers of \( H^\mu_\nu \) and their traces. The goal of this paper is to extend the Hamiltonian analysis of the particular bimetric gravity performed in [30] to the case of the general form of the potential \( \mathcal{V} \).

To begin with we introduce the 3 + 1 decomposition of the four dimensional metric \( \hat{g}_{\mu\nu} \)

\[
\begin{align*}
\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \\
\hat{g}_{0i} &= N_i, \\
\hat{g}_{ij} &= g_{ij}, \\
\end{align*}
\]

(2.4)

together with the metric \( \hat{f}_{\mu\nu} \)

\[
\begin{align*}
\hat{f}_{00} &= -M^2 + L_i f^{ij} L_j, \\
\hat{f}_{0i} &= L_i, \\
\hat{f}_{ij} &= f_{ij}, \\
\end{align*}
\]

(2.5)

to proceed further we use the well known relation \(^2\)

\[
\begin{align*}
^{(4)} R[\hat{g}] &= K_{ij} \hat{G}^{ijkl} K_{kl} + R^{(g)}, \\
^{(4)} R[\hat{f}] &= \tilde{K}_{ij} \tilde{G}^{ijkl} \tilde{K}_{kl} + R^{(f)},
\end{align*}
\]

(2.6)

where \( R^{(g)} \) and \( R^{(f)} \) are three dimensional scalar curvatures evaluated using the spatial metric \( g_{ij} \) and \( f_{ij} \) respectively and where the extrinsic curvatures \( K_{ij} \) and \( \tilde{K}_{ij} \) are defined as

\[
K_{ij} = \frac{1}{2N} \left( \partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right), \\
\tilde{K}_{ij} = \frac{1}{2M} \left( \partial_t f_{ij} - \tilde{\nabla}_i L_j - \tilde{\nabla}_j L_i \right),
\]

(2.7)

and where \( \nabla_i \) and \( \tilde{\nabla}_i \) are covariant derivatives evaluated using the metric components \( g_{ij} \) and \( f_{ij} \) respectively. Finally note that \( \hat{G}^{ijkl} \) and \( \tilde{G}^{ijkl} \) are de Witt metrics defined as

\[
\hat{G}^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - g^{ij} g^{kl}, \\
\tilde{G}^{ijkl} = \frac{1}{2} \left( f^{ik} f^{jl} + f^{il} f^{jk} \right) - f^{ij} f^{kl}
\]

(2.8)

with inverse

\[
\begin{align*}
\hat{G}_{ijkl} &= \frac{1}{2} \left( g_{ik} g_{jl} + g_{il} g_{jk} \right) - \frac{1}{2} g_{ij} g_{kl}, \\
\tilde{G}_{ijkl} &= \frac{1}{2} \left( f_{ik} f_{jl} + f_{il} f_{jk} \right) - \frac{1}{2} f_{ij} f_{kl}
\end{align*}
\]

(2.9)

that obey the relation

\[
G_{ijkl} G^{klmn} = \frac{1}{2} \left( \delta^m_i \delta^o_j + \delta^o_i \delta^m_j \right), \\
\tilde{G}_{ijkl} \tilde{G}^{klmn} = \frac{1}{2} \left( \delta^m_i \delta^o_j + \delta^o_i \delta^m_j \right). 
\]

(2.10)

Using (2.6) we rewrite the action (2.11) into the form that is suitable for the Hamiltonian analysis

\[
\begin{align*}
S &= \int dt L = M_g^2 \int d^3x dt \sqrt{g} N [K_{ij} \hat{G}^{ijkl} K_{kl} + R^{(g)}] + \\
+ M_f^2 \int d^3x dt \sqrt{f} M \left[ \tilde{K}_{ij} \tilde{G}^{ijkl} \tilde{K}_{kl} + R^{(f)} \right] - \mu \int d^3x dt g^{1/4} f^{1/4} \sqrt{NM} \mathcal{V}.
\end{align*}
\]

(2.11)

\(^2\)We ignore the boundary terms.
Further, using (2.15) we find following form of the matrix $H$

\[ H_{ij} = \frac{\delta L}{\delta q_{ij}} = M^{-2}g^{ijkl}K_{kl}, \quad \rho_{ij} = \frac{\delta L}{\delta q_{ij}} = M^{-2}\tilde{g}^{ijkl}\tilde{K}_{kl}, \]

\[ \pi_i = \frac{\delta L}{\delta q_i N_i} \approx 0, \quad \rho_i = \frac{\delta L}{\delta q_i L_i} \approx 0, \]

\[ \pi_N = \frac{\delta L}{\delta q_i N} \approx 0, \quad \rho_M = \frac{\delta L}{\delta q_i M} \approx 0 \]

and then using the standard procedure we derive following Hamiltonian

\[ H = \int d^3x (N\mathcal{R}_0^{(g)} + M\mathcal{R}_0^{(f)} + N^i\mathcal{R}_i^{(g)} + L^i\mathcal{R}_i^{(f)} + \mu\sqrt{NMg^{1/4}f^{1/4}V}), \]

where

\[ \mathcal{R}_0^{(g)} = \frac{1}{M^2\sqrt{g}} \pi^{ij}g_{ijkl}\pi^{kl} - M^2\sqrt{g}R^{(g)}, \quad \mathcal{R}_0^{(f)} = \frac{1}{M^2\sqrt{f}} \pi^{ij}\tilde{g}_{ijkl}\rho^{kl} - M^2fR^{(f)}, \]

\[ \mathcal{R}_i^{(g)} = -2g_{ij}\nabla_k\pi^{kj}, \quad \mathcal{R}_i^{(f)} = -2f_{ij}\nabla_k\rho^{kj}. \]

An important point is to identify four constraints that are generators of the diagonal diffeomorphism. In order to do this we proceed as in [10] and introduce following variables

\[ \bar{N} = \sqrt{NM}, \quad n = \sqrt{\frac{N}{M}}, \quad \bar{N}^i = \frac{1}{2}(N^i + L^i), \quad n^i = \frac{N^i - L^i}{\sqrt{NM}}, \]

\[ N = \bar{N}n, \quad M = \bar{N}/n, \quad L^i = \bar{N}^i - \frac{1}{2}n^i\bar{N}, \quad N^i = \bar{N}^i + \frac{1}{2}n^i\bar{N}, \]

where again clearly their conjugate momenta are the primary constraints of the theory

\[ P_N \approx 0, \quad p_n \approx 0, \quad P_i \approx 0, \quad p_i \approx 0. \]

Note that the canonical variables have following non-zero Poisson brackets

\[ \{g_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(x - y), \quad \{f_{ij}(x), \rho^{kl}(y)\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(x - y), \]

\[ \{\bar{N}(x), P_N(y)\} = \delta(x - y), \quad \{n(x), P_n(y)\} = \delta(x - y), \]

\[ \{\bar{N}^i(x), P_j(y)\} = \delta_j^i\delta(x - y), \quad \{n^i(x), p_j(y)\} = \delta_j^i\delta(x - y). \]

Further, using (2.13) we find following form of the matrix $H_{\mu}^{\nu}$

\[ H_{00}^0 = a + \bar{N}^i v_i, \quad H_{0j}^0 = \frac{1}{N} v_j, \]

\[ H_{0i}^j = -\bar{N}^i a - \bar{N}^i v_k N^k + a^i_j N^j + \bar{N} w^i, \]

\[ H_{0j}^i = a_i^j - \frac{1}{N} \bar{N}^i v_j, \]

(2.18)
where
\[ v_i = \frac{f_{ij}n^i}{n^2}, \quad w^j = \frac{1}{4n^2} n^i (n^m f_{mn} n^n) - \frac{n^i}{2n^4} - \frac{1}{2} g^{im} f_{mk} n^k, \]
\[ a = \frac{1}{n^4} - \frac{n^i f_{ij} n^j}{2n^2}, \quad a^i_j = g^{ik} f_{kj} - \frac{1}{2n^2} n^i n^k f_{kj}. \]
(2.19)

The crucial point for the existence of four constraints that generate the diagonal diffeomorphism is to show that the potential \( V \) does not depend on \( \bar{N} \) and \( \bar{N}^i \). To do this we will argue that the matrix \( H^\mu_\nu \) is similar to the matrix \( A^\mu_\nu \) defined as
\[ A = \begin{pmatrix} a & v_j \\ w^i & a^i_j \end{pmatrix}. \]
(2.20)

Our arguments are as follows. It is known that matrices are characterized by their characteristic polynomials. The characteristic polynomial of an \( n \times n \) matrix \( X \) is given by
\[ p(\lambda) = \det(X - \lambda I) = (-1)^n [\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-1} \lambda + c_n], \]
(2.21)
where \( c_1, c_2, \ldots, c_n \) are expressed using the powers of the traces of \( X \). Explicitly, if we introduce the notation
\[ t_k = \text{Tr}(X^k) \]
(2.22)
we find that in case of \( 4 \times 4 \) matrix the characteristic polynomial is determined by following coefficients
\[ c_1 = -t_1, \]
\[ c_2 = \frac{1}{2} (t_1^2 - t_2), \]
\[ c_3 = -\frac{1}{6} t_1^3 + \frac{1}{2} t_1 t_2 - \frac{1}{3} t_3, \]
\[ c_4 = \frac{1}{24} t_1^4 - \frac{1}{4} t_1^2 t_2 + \frac{1}{3} t_1 t_3 + \frac{1}{8} t_2^2 - \frac{1}{4} t_4. \]
(2.23)

Let us now determine \( t_k \) for the matrix \( H^\mu_\nu \)
\[ t_1 = \text{Tr}H = H^\mu_\mu = a + a^i_i = A^\mu_\mu, \]
\[ t_2 = \text{Tr}H^2 = H^\mu_\nu H^\nu_\mu = a^2 + v_i w^i + a^i_j a^j_i = A^\mu_\nu A^\nu_\mu, \]
\[ t_3 = \text{Tr}H^3 = H^\mu_\nu H^\nu_\rho H^\rho_\mu = a^3 + 3 v_i a^i j w^j + a^i_j a^j_i, \]
\[ = A^\mu_\nu A^\nu_\rho A^\rho_\mu, \]
\[ t_4 = \text{Tr}H^4 = H^\mu_\nu H^\nu_\rho H^\rho_\sigma H^\sigma_\mu = \]
\[ = a^4 + 4a^2 v_i w^i + 2(v_i w^i)^2 + 4a v_i a^i j w^j + 4v_i a^i_j a^j_i + a^i_j a^j_i a^k k a^l l = \]
\[ = A^\mu_\nu A^\nu_\rho A^\rho_\sigma A^\sigma_\mu. \]
(2.24)
We see that the characteristic polynomials of $H^\mu_\nu$ and $A^\mu_\nu$ are the same which also implies that corresponding minimal polynomials are the same. Further, by presumption $H^\mu_\nu$ is diagonalizable which means that the minimal polynomial is a product of distinct linear factors. However since the minimal polynomial of $H^\mu_\nu$ is the same as the minimal polynomial of $A^\mu_\nu$ we immediately find that $A^\mu_\nu$ is diagonalizable to the same diagonal matrix which also implies that $H^\mu_\nu$ and $A^\mu_\nu$ are similar. In other words there exist the matrix $T^\mu_\nu$ so that

$$H^\mu_\nu = T^\mu_\rho A^\rho_\sigma (T^{-1})^\sigma_\nu.$$ (2.25)

Then we find

$$\mathcal{V}(H) = \mathcal{V}(A)$$ (2.26)

since by definition the potential is given as the traces of various powers of $H^\mu_\nu$. Now the equation (2.26) is the desired result which shows that $\mathcal{V}$ does not depend on $\bar{N}$ and $\bar{N}^i$ and hence they appear in the action linearly. Using these calculations we find the Hamiltonian in the form

$$H = \int d^3x (\bar{N} \bar{R} + \bar{N}^i \bar{R}_i) ,$$ (2.27)

where

$$\bar{R} = n \mathcal{R}^{(g)}_0 + \frac{1}{n} \mathcal{R}^{(f)}_0 + \frac{1}{2} n^i \mathcal{R}^{(g)}_i - \frac{1}{2} n^i \mathcal{R}^{(f)}_i + \mu^2 g^{1/4} f^{1/4} \mathcal{V}(A) , \quad \bar{R}_i = \mathcal{R}^{(g)}_i + \mathcal{R}^{(f)}_i .$$ (2.28)

Now we proceed to the analysis of the requirement of the preservation of the primary constraints (2.16)

$$\partial_t P_N = \{ P_N , H \} = - \bar{R} \approx 0 ,$$

$$\partial_t P_i = \{ P_i , H \} = - \bar{R}_i \approx 0 ,$$

$$\partial_t p_n = \{ p_n , H \} = - \mathcal{R}^{(g)}_0 + \frac{1}{n^2} \mathcal{R}^{(f)}_0 - \mu^2 g^{1/4} f^{1/4} \frac{\delta \mathcal{V}}{\delta \bar{n}} \approx \mathcal{G}_n \approx 0 ,$$

$$\partial_t p_i = \{ p_i , H \} = - \frac{1}{2} \mathcal{R}^{(g)}_i + \frac{1}{2} \mathcal{R}^{(f)}_i - \mu^2 g^{1/4} f^{1/4} \frac{\delta \mathcal{V}}{\delta \bar{n}^i} \approx \mathcal{G}_i \approx 0 .$$ (2.29)

As a result we have following total Hamiltonian

$$H_T = \int d^3x (\bar{N} \bar{R} + \bar{N}^i \bar{R}_i + V_N P_N + V^i P_i + v_n p_n + v^i p_i + u^n \mathcal{G}_n + u^i \mathcal{G}_i) ,$$ (2.30)

where $V_N , V^i , v_n , v^i , u^n , u^i$ are Lagrange multipliers corresponding to the constraints $P_N \approx 0 , P_i \approx 0 , p_n \approx 0 , p_i \approx 0 , \mathcal{G}_n \approx 0 , \mathcal{G}_i \approx 0$. In the next section we determine the algebra of constraints $\bar{R} , \bar{R}_i$. 

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3. Algebra of Constraints $\hat{R}$, $\hat{R}_i$

For the consistency of the theory it is important to show that the constraints $\hat{R}$ and $\hat{R}_i$ are the first class constraints. To proceed it is useful to introduce the smeared form of the constraint $\hat{R}$

$$T_T(N) = \int d^3x N(x) \hat{R}(x) .$$

(3.1)

Instead of the constraint $\hat{R}_i$ we introduce the constraint $\hat{R}_i$ that is given as an extension of the constraints $\hat{R}_i$ by appropriate combinations of the primary constraints $p_n, p_i$

$$\hat{R}_i = R_i^{(g)} + R_i^{(f)} + \partial_i n p_n + \partial_i n^j p_j + \partial_j (n^j p_i)$$

(3.2)

with equivalent smeared form

$$T_S(N^i) = \int d^3x N^i \hat{R}_i .$$

(3.3)

Then using (2.17) we determine the Poisson brackets between $T_S(N^i)$ and the canonical variables

$$\{ T_S(N^i), g_{ij} \} = -N^k \partial_k g_{ij} - \partial_i N^k g_{kj} - g_{ik} \partial_j N^k ,$$

$$\{ T_S(N^i), \pi^j \} = -\partial_k (N^k \pi^j) + \partial_k N^i \pi^k + \pi^k \partial_k N^j ,$$

$$\{ T_S(N^i), f_{ij} \} = -N^k \partial_k f_{ij} - \partial_i N^k f_{kj} - f_{ik} \partial_j N^k ,$$

$$\{ T_S(N^i), \rho^{ij} \} = -\partial_k (N^k \rho^{ij}) + \partial_k N^i \rho^{kj} + \rho^{ik} \partial_k N^j ,$$

$$\{ T_S(N^i), n \} = -N^i \partial_k n ,$$

$$\{ T_S(N^i), \pi_n \} = -\partial_i (N^i \pi_n) ,$$

$$\{ T_S(N^i), n^j \} = -N^k \partial_k n^j + \partial_j N^i n^j ,$$

$$\{ T_S(N^i), \pi_i \} = -\partial_k (N^k \pi_i) - \partial_i N^k \pi_k .$$

(3.4)

With the help of these results we easily find how various components $A^\mu_\nu$ transform under spatial diffeomorphism

$$\{ T_S(N^i), a \} = -N^i \partial_\alpha a ,$$

$$\{ T_S(N^i), v_\gamma \} = -N^j \partial_j v_\gamma - \partial_i N^k v_k ,$$

$$\{ T_S(N^i), w^j \} = -N^k \partial_k w^j + w^k \partial_k N^j ,$$

$$\{ T_S(N^i), a^j \} = -N^k \partial_k a^j + \partial_k N^i a^k - a^k \partial_j N^k .$$

(3.5)

and also

$$\{ T_S(N^i), T_S(M^j) \} = T_S(\{N^j \partial_j M^i - M^j \partial_j N^i\}) .$$

(3.6)

This result suggests that $T_S(N^i)$ is the generator of the spatial diffeomorphism.
For further purposes it is convenient to introduce the smeared forms of the constraints \( \mathcal{R}_0^{(f),(g)} \) and \( \mathcal{R}_i^{(f),(g)} \)

\[
\begin{align*}
T^g_f(N) &= \int d^3x N(x) \mathcal{R}_0^{(g)}(x) , & T^g_f(N) &= \int d^3x N(x) \mathcal{R}_0^{(f)}(x) ,
T^g_f(N^i) &= \int d^3x N^i(x) \mathcal{R}_1^{(g)}(x) , & T^g_f(N^i) &= \int d^3x N^i(x) \mathcal{R}_1^{(f)}(x) .
\end{align*}
\]

(3.7)

It is well known that these smeared constraints have following non-zero Poisson brackets \(^3\)

\[
\begin{align*}
\{ T^g_f(N), T^g_f(M) \} &= T^g_f((N \partial_i M - M \partial_i N) g^{ij}) ,
\{ T^g_f(N), T^g_f(M) \} &= T^g_f((N \partial_i M - M \partial_i N) f^{ij}) ,
\{ T^g_f(N^i), T^g_f(M^i) \} &= T^g_f(N^i \partial_i M) ,
\{ T^g_f(N^i), T^f_f(M) \} &= T^f_f(N^i \partial_i M) ,
\{ T^g_f(N^i), T^g_f(M^j) \} &= T^g_f(N^j \partial_j M^i - M^j \partial_j N^i) ,
\{ T^g_f(N^i), T^f_f(M^j) \} &= T^f_f(N^j \partial_j M^i - M^j \partial_j N^i) .
\end{align*}
\]

(3.8)

As the next step we determine the Poisson bracket between \( T_S(N^i) \) and \( T_T(N) \). We firstly determine following Poisson bracket

\[
\begin{align*}
\{ T_S(N), g^{1/4} f^{1/4} \mathcal{V} \} &= -N^k \partial_k [g^{1/4} f^{1/4} \mathcal{V}] - \partial_k N^k g^{1/4} f^{1/4} \mathcal{V} +
+ g^{1/4} f^{1/4} \left[ \frac{\delta \mathcal{V}}{\delta n^i} \partial_j N^i n^j - 2 \frac{\delta \mathcal{V}}{\delta f_{kl}} \partial_k N^m f_{ml} + 2 \frac{\delta \mathcal{V}}{\delta g^{ml}} \partial_m N^k g^{ml} \right] .
\end{align*}
\]

(3.9)

On the other hand we know that \( \mathcal{V} \) depends on \( A^\mu \), through the trace. As a result we find that \( \mathcal{V} \) depends on the eigenvalues \( \lambda_i \) that are solutions of the characteristic polynomials. Further, we know that the coefficients of this polynomial are functions of \( t_1, \ldots, t_4 \). As a result if we show that \( t_i \) transform as scalars under spatial diffeomorphism we find that \( \lambda_i \) and consequently \( \mathcal{V} \) transform as the scalar as well. In fact, using the explicit form of \( t_i \) given in (2.24) and also using the Poisson brackets (3.7) we easily find that \( t_i \) transform as scalars under spatial diffeomorphism. These results imply following Poisson bracket

\[
\{ T_S(N), g^{1/4} f^{1/4} \mathcal{V} \} = -N^k \partial_k [g^{1/4} f^{1/4} \mathcal{V}] - \partial_k N^k g^{1/4} f^{1/4} \mathcal{V} .
\]

(3.10)

Finally using the Poisson brackets (3.8) we find

\[
\begin{align*}
\{ T_S(N^i), T_T(M) \} &= T_T(N^i \partial_i M) , \\
\{ T_S(N^i), G_n(M) \} &= G_n(N^i \partial_i M) .
\end{align*}
\]

(3.11)

\(^3\)See, for example [3].
In case of $G_i$ the situation is more complicated since we have to determine the variation of $t_i$ with respect to $n^j$ and then its Poisson bracket with $T_S(N^i)$. For example, in case $t_1$ and $t_2$ we obtain
\[
\frac{\delta t_1}{\delta n^i} = -2 \frac{f_{ij} n^j}{n^2}, \quad \frac{\delta t_2}{\delta n^i} = - \frac{a}{n^2} f_{ij} n^j + \frac{1}{n^2} f_{ij} n^j - \frac{v_i}{2} - \frac{1}{2} f_{ik} g^{kj} v_j .
\] (3.12)

In the same way we can calculate $\frac{\delta t_3}{\delta n^i}$ and $\frac{\delta t_4}{\delta n^i}$ and then we find
\[
\left\{ T_S(N^i), \frac{\delta t_j}{\delta n^i} \right\} = - N^k \partial_k \left[ \frac{\delta t_j}{\delta n^i} \right] - \frac{\delta t_j}{\delta n^i} \partial_i N^k .
\] (3.13)

With the help of these results we easily obtain
\[
\left\{ T_S(N^i), \frac{\delta V}{\delta n^i} \right\} = \frac{\delta V}{\delta t_j} \left\{ T_S(N^i), \frac{\delta t_j}{\delta n^i} \right\} = - N^k \partial_k \left[ \frac{\delta V}{\delta n^i} \right] - \frac{\delta V}{\delta n^i} \partial_i N^k .
\] (3.14)

Finally using (3.8) and (3.14) we find
\[
\left\{ T_S(N^i), G_i \right\} = - N^k \partial_k G_i - G_j \partial_i N^j .
\] (3.15)

Collecting all these results we find that $\tilde{R}_i$ are the first class constraints whose smeared forms correspond to the generator of the diagonal spatial diffeomorphism.

On the other hand more interesting is to determine the Poisson bracket between smeared forms of the Hamiltonian constrains (3.1). Following the same calculations as in [30] we find
\[
\left\{ T_T(N), T_T(M) \right\} = T_S((N \partial_i M - M \partial_i N) n^2 g^{ij}) + T_S((N \partial_i M - M \partial_i N) \frac{1}{n^2} f^{ij}) - \left[ T_S((N \partial_i M - M \partial_i N) n^2 g^{ij}) - G_T((N \partial_i M - M \partial_i N) n^i) \right] + \left[ T_S((N \partial_i M - M \partial_i N) \frac{1}{n^2} f^{ij}) + \int d^3x (N \partial_i M - M \partial_i N) \Sigma^i[V] \right] ,
\] (3.16)

where we defined the smeared forms of the constraints $G_i$ and $G_n$
\[
G_T(N) = \int d^3x N(x) G_n(x) , \quad G_S(N^i) = \int d^3x N^i(x) G_i(x) ,
\] (3.17)

and where $\Sigma^i[V]$ is defined as
\[
\Sigma^i[V] = \mu g^{1/4} f^{1/4} \left[ -n^2 g^{ij} \frac{\delta V}{\delta n^j} + f^{ij} \frac{\delta V}{\delta n^j} \frac{1}{n^2} - \frac{\delta V}{\delta g^{ij}} n^k g^{ij} - \frac{\delta V}{\delta f^{ij}} n^k f_{kj} - \frac{1}{2} n^i n^j \frac{\delta V}{\delta n^j} \right] .
\] (3.18)
We see that the Poisson bracket between the smeared forms of the Hamiltonian constraints \( (3.16) \) vanish on the constraint surface on condition that \( \Sigma^i[V] \) is zero. In order to calculate \( \Sigma^i[V] \) we proceed in the similar way as in case when we calculated the Poisson bracket between \( T^T(N^i) \) and \( V \). Explicitly, we know that \( A^\mu_\nu \) is similar to diagonal matrix with eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). In other words we have

\[
V(A) = V(\lambda_i) . \tag{3.19}
\]

On the other hand we know that \( \lambda_i \) are solutions of the characteristic polynomial when the coefficients are functions of \( t_1, \ldots, t_4 \) so that

\[
\lambda_i = \lambda_i(t_1, \ldots, t_4) . \tag{3.20}
\]

Then if we show that

\[
\Sigma^i[t_j] = 0 , j = 1, 2, 3, 4 \tag{3.21}
\]

we find that

\[
\Sigma^i[\lambda_j] = 0 \tag{3.22}
\]

and consequently

\[
\Sigma^i[V(\lambda)] = \Sigma^i[V(A)] = 0 . \tag{3.23}
\]

We have already found in \([8]\) that \( \Sigma^i[t_1] = \Sigma^i[t_2] = 0 \). The case of \( t_3 \) and \( t_4 \) is more intricate. For example, the explicit form of \( t_3 \) is equal to

\[
t_3 = \frac{1}{n_{12}} - \frac{3}{n_{10}} n^i f_{ij} n^j + \frac{3}{n_{5}} (n^i f_{ij} n^j)^2 - \frac{1}{n_{6}} (n^i f_{ij} n^j)^3 - \frac{3}{n_{6}} n^i f_{ik} g^{kl} f_{lm} n^m + \frac{3}{n_{4}} (n^i f_{ik} g^{kl} f_{lm} n^m)(n^r f_{rs} n^s) - \frac{3}{n_{2}} n^m f_{mij} g^{ip} f_{pj} g^{jr} f_{rkn} + g^{im} f_{mij} g^{jk} f_{kl} g^{lp} f_{pi} . \tag{3.24}
\]

Then using the explicit form of \( \Sigma^i \) and after some calculations we find

\[
\Sigma^i[t_3] = 0 . \tag{3.25}
\]

In the same way we find \( \Sigma^i[t_4] = 0 \). In summary we have

\[
\Sigma^i[t_j] = 0 , \text{ for } j = 1, 2, 3, 4 . \tag{3.26}
\]

With the help of this result we obtain the fundamental result that the Poisson bracket between Hamiltonian constraint \( (3.16) \) vanishes on the constraint surface. This result together with \( (3.6) \) and \( (3.16) \) shows that \( T^T(N) \) and \( T^S(N^i) \) are the first class constraints that are generators of the diagonal diffeomorphism.
4. Time Evolution of Remaining Constraints

In this section we analyze the time evolution of the constraints $p_n, p_i, G_n, G_i$. Note that the total Hamiltonian is given in (2.30). For further purposes we calculate following Poisson brackets

$$\{p_n(x), G_i(y)\} = \mu^2 \frac{\delta^2 \mathcal{V}}{\delta n^i \delta n^j} (x) \delta(x-y) \equiv \triangle_{ni}(x) \delta(x-y),$$

$$\{p_n(x), G_n(y)\} = \frac{1}{n} G_n(x) \delta(x-y) +$$

$$+ n(x) \left[ \mathcal{R}_0^{(g)} + \frac{1}{n^2} \mathcal{R}_0^{(f)} + \mu^2 g^{1/4} f^{1/4} \frac{\delta \mathcal{V}}{\delta n} + \mu^2 g^{1/4} f^{1/4} \frac{\delta^2 \mathcal{V}}{\delta n^i \delta n^j} \right] \delta(x-y) \approx$$

$$\approx n(x) \left[ \frac{1}{n} \mathcal{R} + \frac{n^i}{n} \mathcal{G}_i + \frac{n^i}{n^j} \mu^2 g^{1/4} f^{1/4} \frac{\delta \mathcal{V}}{\delta n^i} - \frac{\mu^2}{n} g^{1/4} f^{1/4} \mathcal{V} +$$

$$+ \mu^2 g^{1/4} f^{1/4} \frac{\delta \mathcal{V}}{\delta n^i} + \mu^2 g^{1/4} f^{1/4} \frac{\delta^2 \mathcal{V}}{\delta n^i \delta n^j} \right] \delta(x-y) \approx$$

$$\approx \mu^2 g^{1/4} f^{1/4} \left[ \frac{n^i}{n} \frac{\delta \mathcal{V}}{\delta n^i} - \mathcal{V} + \frac{\delta \mathcal{V}}{\delta n^i} + \frac{\delta^2 \mathcal{V}}{\delta n^i \delta n^j} \right] \delta(x-y) \equiv \triangle_{nn}(x-y),$$

$$\{p_i(x), G_n(y)\} = \mu^2 g^{1/4} f^{1/4} \frac{\delta^2 \mathcal{V}}{\delta n^i \delta n^j} (x) \delta(x-y) \equiv \triangle_{in}(x) \delta(x-y),$$

(4.1)

We introduce following common notation $p_a \equiv (p_n, p_i)$ and $G_a \equiv (G_n, G_i)$. Then the requirement of the preservation of the primary constraints $p_a$ takes the form

$$\partial_t p_a = \{p_a, H_T\} = G_n + \int d^3x u^b(x) \{p_a, G_b(x)\} \approx$$

$$\approx \int d^3x u^b(x) \{p_a, G_b(x)\} = \triangle_{ab} u^b = 0 .$$

(4.2)

We have four equations for four unknown $u^a$. We have to distinguish two cases. The case when $\det \triangle_{ab} \neq 0$ and the case when $\det \triangle_{ab} = 0$.

4.1 The case $\det \triangle_{ab} \neq 0$

This situation is particular simple and we believe that this is in fact the situation in all examples of the bimetric theories of gravity that are invariant under diagonal diffeomorphism. In this case we find that the solution of the equations (4.2) is $u^b = 0$. Then we can proceed to the analysis of the preservation of the constraints $G_a$

$$\partial_t G_a = \{G_a, H_T\} \approx \int d^3x \left( N(x) \{G_a, \ddot{R}(x)\} + v^b \{G_a, p_b(x)\} \right) =$$

$$= \int d^3x N(x) \{G_a, \ddot{R}(x)\} - \triangle_{ab} v^b = 0$$

(4.3)
that again using the fact that $\Delta_{ab} \neq 0$ can be solved for $v^a$ as functions of the canonical variables and $N$. In other words we find that $\mathcal{G}_a, p_a$ are the second class constraints that can be solved for $n^a, n^i$ at least in principle. Then four first class constraints $P_N, P_i$ can be gauge fixed and hence we can eliminate $P_N, P_i$ and $\tilde{N}, \tilde{N}^i$ as dynamical variables. Then remaining four first class constraints $\tilde{\mathcal{R}}, \tilde{\mathcal{R}}_i$ can be again gauge fixed so that we can eliminate eight degrees of freedom from 24 degrees of freedom $g_{ij}, \pi^{ij}, f_{ij}, \rho^{ij}$ so that we have 16 physical degrees of freedom where 4 correspond to the massless graviton, 10 to the massive one and 2 corresponding to the scalar mode at least at the linearized level. Of course, this Hamiltonian analysis cannot answer an important question which is the identification of the physical metric that couples to the matter fields, for recent detailed analysis, see [3].

To conclude this section we have to mention the form of the symplectic structure on the reduced phase space spanned by $z^\alpha \equiv (g_{ij}, \pi^{ij}, f_{ij}, \rho^{ij})$. Due to the fact that we have eight second class constraints $p_n, p_i, \mathcal{G}_n, \mathcal{G}_i$ we should replace the Poisson brackets with corresponding Dirac brackets. Explicitly, let us introduce the common notation for all second class constraints $\chi_A \equiv (P_a, \mathcal{G}_a)$. Then we find following matrix of the Poisson brackets of the second class constraints $\chi_A$

$$\{\chi_A(x), \chi_A(y)\} \equiv \Theta_{AB} = \begin{pmatrix} 0 & \Delta(x) \delta(x - y) \\ -\Delta(y) \delta(x - y) & \Omega(x, y) \end{pmatrix}, \quad (4.4)$$

where the matrix $\Omega_{ab}(x, y)$ is defined as

$$\{\mathcal{G}_a(x), \mathcal{G}_b(y)\} = \Omega_{ab}(x, y). \quad (4.5)$$

Since $\Delta$ is invertible matrix we can find the matrix inverse to $\Theta_{AB}$ in the form

$$(\Theta^{-1})^{AB} = \begin{pmatrix} \Delta^{-1}(x) \Omega(x, y) \Delta^{-1}(y) & -\Delta^{-1}(x) \delta(x - y) \\ -\Delta^{-1}(y) \delta(x - y) & 0 \end{pmatrix}. \quad (4.6)$$

Following the standard analysis of the system with the second class constraints we replace the Poisson bracket by corresponding Dirac bracket defined as

$$\{F, G\}_D = \{F, G\} - \int d^3 x d^3 y \{F, \chi_A(x)\} (\Theta^{-1})^{AB}(x, y) \{\chi_B(y), G\}. \quad (4.7)$$

Now we see that for the variables on the reduced phase space spanned by $z^\alpha \equiv (g_{ij}, \pi^{ij}, f_{ij}, \rho^{ij})$ the Dirac bracket coincides with the Poisson bracket. Explicitly, we have

$$\left\{z^\alpha(x), z^\beta(y)\right\}_D = \left\{z^\alpha(x), z^\beta(y)\right\} - \int d^3 z d^3 z' \left\{z^\alpha(x), \chi_A(z)\right\} (\Theta^{-1})^{AB}(z, z') \left\{\chi_B(z'), z^\beta(x)\right\} = \left\{z^\alpha(x), z^\beta(y)\right\} -$$

$$- \int d^3 z d^3 z' \sum_{A=4}^{7} \sum_{B=4}^{7} \left\{z^\alpha(x), \chi_A(z)\right\} (\Theta^{-1})^{AB}(z, z') \left\{\chi_B(z'), z^\beta(x)\right\} = \left\{z^\alpha(x), z^\beta(y)\right\} \quad (4.8)$$

due to the fact that $(\Theta^{-1})^{AB} = 0$ for $A, B = 3, \ldots, 7$ are zero.
4.2 The case $\det \Delta_{ab} = 0$

In this case it is possible to find the vector $u_0^a$ such that

$$\Delta_{ab} u_0^b = 0 .$$

(4.9)

Then we define new constraints $\tilde{P}, \tilde{G}$ defined as

$$\tilde{P} = u_0^a P_a , \tilde{G} = u_0^a G_a$$

(4.10)

together with

$$\tilde{P}_a = P_a - \frac{1}{u_{0a} u_0^a} u_{0a} \tilde{P} , \quad \tilde{G}_a = G_a - \frac{1}{u_{0a} u_0^a} u_{0a} \tilde{G} , \quad u_{0a} = \delta_{ab} u_0^b$$

(4.11)

that by definition obey the conditions

$$u_0^a \tilde{P}_a = 0 , \quad u_0^a \tilde{G}_a = 0$$

(4.12)

which means that we have 6 independent constraints. As a result we consider the total Hamiltonian in the form

$$H_T = \int d^3 x (N \tilde{R} + N^i \tilde{R}_i + V_N P_N + V^i P_i + \tilde{v} \tilde{P} + \tilde{u}^a \tilde{P}_a + \tilde{u} \tilde{G} + \tilde{u}^a \tilde{G}_a) ,$$

(4.13)

where

$$\tilde{v}^a = v^a - \frac{u_0^a (v^a u_{0a})}{u^a u_0^a} , \quad \tilde{u}^a = u^a - \frac{(u^a u_{0a}) u_0^a}{u^a u_0^a} ,$$

(4.14)

where $v^a, u^a$ are arbitrary four dimensional vectors and we clearly see that $\tilde{v}^a, \tilde{u}^a$ are orthogonal to $u_0^a$ so that they have 3 independent components.

Now we proceed to the analysis of the time evolution of the constraints $\tilde{P}, \tilde{P}_a, \tilde{G}, \tilde{G}_a$. Note that the constraint $\tilde{P}$ has vanishing Poisson bracket with $\tilde{G}$ and $\tilde{G}_a$ since

$$\{ \tilde{P}(x), \tilde{G}(y) \} = u_0^a \Delta_{ab} u_0^b \delta(x - y) = 0 ,$$

$$\{ \tilde{P}(x), \tilde{G}_a(y) \} = u_0^b \Delta_{ba} \delta(x - y) = 0$$

(4.15)

so that it is trivial to see that $\tilde{P}$ is preserved during the time evolution of the system and that it is the first class constraint. Let us now analyze the time evolution of the constraint $\tilde{P}_a$

$$\partial \tilde{P}_a = \{ \tilde{P}_a, H_T \} \approx \int d^3 x (\tilde{u}(x) \left\{ \tilde{P}_a, \tilde{G}(x) \right\} + \tilde{u}^b(x) \left\{ \tilde{P}_a, \tilde{G}_b(x) \right\}) \approx \Delta_{ab} \tilde{u}^b = 0$$

(4.16)

using

$$\{ \tilde{P}_a(x), \tilde{G}(y) \} = \{ P_a(x), G_b(y) \} u_0^b = \Delta_{ab} u_0^b \delta(x - y) = 0 .$$

(4.17)
Now due to the fact that $\tilde{u}^a$ is orthogonal to $u_0^a$ we find that the only solution of (4.16) is given by $\tilde{u}^a = 0$.

We proceed to the analysis of the time evolution of the constraint $\tilde{G}, \tilde{G}_a$. We start with the constraints $\tilde{G}_a$

$$\partial_t \tilde{G}_a = \left\{ \tilde{G}_a, H_T \right\} \approx \int d^3 x \left( N(x) \left\{ \tilde{G}_a, \bar{R}(x) \right\} + \tilde{u}(x) \left\{ \tilde{G}_a, \tilde{G}(x) \right\} \right) + \tilde{v}^b \Delta_{ba} = 0$$

(4.18)

that again using the definition of $\tilde{v}^b$ given (4.14) can be solved for $\tilde{v}^a$ as function of the canonical variables and $N$ and $\tilde{u}$. This result implies that $\tilde{P}_a, \tilde{G}_b$ are the second class constraints.

Now we come the most intricate analysis which is the time preservation of the constraint $\tilde{G}$

$$\partial_t \tilde{G} = \left\{ \tilde{G}, H_T \right\} \approx \int d^3 x \left( N(x) \left\{ \tilde{G}, \bar{R}(x) \right\} + \tilde{u}(x) \left\{ \tilde{G}, \tilde{G}(x) \right\} \right) = 0 .$$

(4.19)

Note that generally $\left\{ \tilde{G}(x), \tilde{G}(y) \right\} \neq 0$. However we see that we have to analyze the time evolution of the constraint $\bar{R}$ as well 4 so that

$$\partial_t \bar{R} = \left\{ \bar{R}, H_T \right\} \approx \int d^3 x \tilde{u}(x) \left\{ \bar{R}, \tilde{G}(x) \right\} = 0 .$$

(4.20)

To proceed further we have to determine the Poisson bracket between $\bar{R}$ and $\tilde{G}$. This is very complicated due to the complex form of these constraints. As an example we determine the Poisson bracket between $T_T(N)$ and $G_T(M)$

$$\{ T_T(N), G_T(M) \} = - T^g_S (n(N \partial_i M - M \partial_i N) g^{ij} - MN \partial_i n g^{ij}) +$$

$$+ T^f_S (n^{-3} (N \partial_i M - M \partial_i N) f^{ij} - n^{-4} \partial_i n M f^{ij}) -$$

$$- \frac{1}{2} T^g_f (n^i \partial_i M) - \frac{1}{2} T^f_f (n^i \partial_i (n^{-2} M)) +$$

$$+ 2 \mu^2 \int d^3 x MN \left( \frac{1}{M^2 g} n \pi^{kl} G_{ijkl} G^{kl} - \frac{1}{M^2 g} \rho^{kl} G_{ijkl} F^{ij} \right) +$$

$$+ \frac{1}{2} \mu^2 \int d^3 x G_{ij}^{ij} \left( N n^k \partial_k g_{ij} + \partial_i (N n^i) g_{kj} + g_{ik} \partial_j (N n^k) \right) -$$

$$- \frac{1}{2} \mu^2 \int d^3 x F_{ij}^{ij} \left( N n^k \partial_k f_{ij} + \partial_i (N n^i) f_{kj} + f_{ik} \partial_j (N n^k) \right) -$$

$$- 2 \mu^2 \int d^3 x MN \left( \frac{1}{M^2 g} n \pi^{kl} G_{ijkl} G^{kl} - \frac{1}{M^2 g} \rho^{kl} G_{ijkl} F^{ij} \right) ,$$

(4.21)

4Note that this was not necessary in case when $\det \triangle_{ab} \neq 0$ since in this case we found that $u^a = 0$ and hence the constraint $\bar{R}$ is trivially preserved.
where
\[
G^{kl} = \delta \left( \frac{\delta (g^{1/4} f^{1/4} V)}{\delta g_{kl}} \right), \quad F^{kl} = \delta \left( \frac{\delta (g^{1/4} f^{1/4} V)}{\delta f_{kl}} \right),
\]
\[
G^n_k = \delta \left( \frac{\delta (g^{1/4} f^{1/4} \delta V)}{\delta g_{kl}} \right), \quad F^n_k = \delta \left( \frac{\delta (g^{1/4} f^{1/4} \delta V)}{\delta f_{kl}} \right).
\]

(4.22)

In the same way we calculate the Poisson bracket between \(T(N)\) and \(G_S(N^i)\) and we derive similar result. Then in principle we could express the Poisson bracket between \(T(N)\) and \(\int d^3x M(x) \tilde{G}(x)\) in the form
\[
\{ T_T(N), \int d^3x M(x) \tilde{G}(x) \} = \int d^3x N(x) M(x) F(x) + \partial_x N(x) V^i(x) M(x) + N(x) \partial_x M(x) W^i(x),
\]

(4.23)

where the explicit form of \(F, V^i, W^i\) can be derived from the Poisson brackets between \(T(N)\) and \(G_T(N^i), G_S(N^i)\) and from the known vector \(u_0^a\). For our purposes it is useful the local form of the Poisson bracket (4.23)
\[
\{ \tilde{R}(x), \tilde{G}(y) \} = \delta(x - y) F(x) + \frac{\partial}{\partial y^i} \delta(x - y) V^i(y) + \frac{\partial}{\partial x^i} \delta(x - y) W^i(x).
\]

(4.24)

With the help of this result we find that (4.20) takes the form
\[
\partial_t \tilde{R} = \tilde{u} (F - \partial_i V^i) + \frac{\partial \tilde{u}}{\partial x^i} (W^i - V^i) = 0.
\]

(4.25)

In order to find the meaning of this result we can argue in the same way as in [28] where more details can be found. Briefly, as we can deduce from (4.21) we have that \(W^i \neq V^i\) so that we should interpret the equation (4.25) as the differential equation for \(\tilde{u}(x)\) and this equation could be solved for \(\tilde{u}(x)\) at least in principle. Further, using (4.24) we find that (4.19) becomes differential equation for \(N(x)\)
\[
\partial_t G_n = N(-F + \partial_i W^i) + \frac{\partial N}{\partial y^i} (W^i - V^i) + \int d^3y \tilde{u}(y) \{ G_n(x), G_n(y) \} = 0.
\]

(4.26)

This equation can be now solved for \(N(x)\) as function of the canonical variables since the Lagrange multiplier \(\tilde{u}\) has been already determined by the equation (4.20). These results imply that \(\tilde{R}\) and \(\tilde{G}\) are the second class constraints. On the other hand we could say that by consistency the theory should have the first class Hamiltonian constraint as a consequence of the fact that the theory is invariant under diagonal diffeomorphism. In
other words we are tempting to say that in bimetric theories that are invariant under diagonal diffeomorphism the case when \( \det \Delta_{ab} = 0 \) cannot occur \(^5\).

### 4.3 Square root potential

Recently it was shown that the non-linear massive gravity with the square root form of the potential is free from ghosts. The extension to the bimetric gravity was performed in \(^[25]\) where it was also argued that resulting theory should be ghost free. This fact was questioned in \(^[28]\) where the Hamiltonian analysis of the bimetric gravity introduced in \(^[25]\) was performed. We argued that the theory with redefined shift function lost the manifest diffeomorphism invariance and so that it difficult to find the generator of the diagonal diffeomorphism. We also argued that the number of the physical degrees of freedom is 16 so that the scalar mode cannot be eliminated.

In order to resolve the apparent issue of the impossibility to identify the generators of the diagonal diffeomorphism in the non-linear bimetric theory of gravity with redefined shift function \(^[25]\) we can certainly use the analysis of the general bimetric theory of gravity presented in this paper. Obviously we can identify four first class constraints \( \bar{\mathcal{R}}, \bar{\mathcal{R}}_i \) corresponding to the diagonal diffeomorphism. On the other hand we can ask the question whether there is possibility to identify an additional constraint that could eliminate the scalar mode. To be more concrete, let us consider following square root potential

\[
\mathcal{V} = \text{Tr} \sqrt{H_{\mu}^\nu} = \text{Tr} \sqrt{A_{\mu}^\nu}.
\]

The problem is that it is not easy task to find the square root of the matrix \( A_{\mu}^\nu \) since we were not able to find such redefinition of the canonical fields as in \(^[13, 14]\) that could allow us to find the explicit form of the square root of \( A_{\mu}^\nu \). For that reason we try to find such a matrix in the following way. We temporally write \( n^i \) with the parameter \( \epsilon \) so that \( n^i \to \epsilon n^i \) when we take \( \epsilon = 1 \) in the end. Then we can write

\[
A = \epsilon^0 E^{(0)} + \epsilon^1 E^{(1)} + \epsilon^2 E^{(2)} + \epsilon^3 E^{(3)},
\]

where

\[
E^{(0)} = \begin{pmatrix}
\frac{1}{n^i} & 0 \\
0 & g^{ik} f_{kj}
\end{pmatrix}, \quad
E^{(1)} = \begin{pmatrix}
0 & \frac{n^k f_{kj}}{n^i} \\
-\frac{n^i}{2n^u} & 0
\end{pmatrix},
\]

\[
E^{(2)} = \begin{pmatrix}
\frac{n^i f_{kj} n^j}{2n^u} & 0 \\
0 & -\frac{n^i n^k f_{kj}}{2n^u}
\end{pmatrix}, \quad
E^{(3)} = \begin{pmatrix}
0 & 0 \\
\frac{1}{4n^u n^i (n^m f_{mn} n^n)} & 0
\end{pmatrix}.
\]

\(^5\)The situation is different in the miracle case when \( W^i = V^i \). In this case it is more natural to interpret \( G^{(3)} \equiv F - \partial_i V^i = 0 \) as the new constraint rather than impose the condition \( \bar{u} = 0 \). Then of course the equation \( (4.19) \) vanishes on the constraint surface \( G^{(3)} \approx 0 \) and the requirement of the preservation of the constraint \( G^{(3)} \approx 0 \) leads to the emergence of the new constraint \( G^{(4)} \approx 0 \) on condition that the Poisson bracket between \( G^{(3)} \) and \( \bar{\mathcal{R}} \) does not contain the derivative of the delta function. However due to the complex form of the constraints \( \bar{\mathcal{R}} \) and \( G_a \) and corresponding Poisson brackets we believe that such a case is highly improbable.
We can solve \( \sqrt{A} \) as the expansion with respect to \( \epsilon \)

\[
\sqrt{A} = \sum_{n=0}^{\infty} \epsilon^n B^{(n)} .
\]  

(4.30)

Taking square of given expression and comparing expressions of given order in \( \epsilon \) we find

\[
E^{(0)} = B^{(0)} B^{(0)} , \\
E^{(1)} = B^{(0)} B^{(1)} + B^{(1)} B^{(0)} , \\
E^{(2)} = B^{(0)} B^{(2)} + B^{(2)} B^{(0)} + B^{(1)} B^{(1)} , \\
E^{(3)} = B^{(0)} B^{(3)} + B^{(3)} B^{(0)} + B^{(1)} B^{(2)} + B^{(2)} B^{(1)} + B^{(3)} B^{(0)} , \\
0 = B^{(1)} B^{(3)} + B^{(2)} B^{(2)} + B^{(3)} B^{(1)} + B^{(0)} B^{(4)} + B^{(4)} B^{(0)} ,
\]

(4.31)

From the first equation in (4.31) we find

\[
B^{(0)} = \begin{pmatrix} \frac{1}{n^2} & 0 \\ 0 & \gamma^i_j \end{pmatrix} ,
\]

(4.32)

where

\[
\gamma^i_j = \sqrt{g^{ik} f_{kj}} , \quad \gamma^i_k \gamma^k_j = g^{ik} f_{kj} .
\]

(4.33)

Further, the second equation in (4.31) can be solved as

\[
B^{(1)} = \begin{pmatrix} 0 & \frac{n^k f_{ij}}{n^2} (M^{-1})^i_j \\ (M^{-1})^i_k (-\frac{n^k}{2n^2} - \frac{1}{2} g^{km} f_{mn} n^n) & 0 \end{pmatrix}
\]

(4.34)

where

\[
M^i_k = \frac{1}{n^2} \delta^i_k + \gamma^i_k .
\]

(4.35)

Repeating this procedure we could determine all matrices \( B^{(n)} \) at least in principle. On the other hand we see that the resulting expression contains all powers of \( n \) and \( n^i \). Then from (4.1) we can deduce that \( \Delta_{ab} \neq 0 \) and hence \( p_a, G_a \) are the second class constraints without possibility to eliminate additional scalar mode.

5. Conclusion

Let us outline the results derived in this paper. We performed the Hamiltonian analysis of bimetric theory of gravity with general form of the potential. We found the first class constraints that are generators of the diagonal diffeomorphism. Then we analyzed the requirement of the preservation of remaining constraints during the time development of the system. We show that there are two possibilities. In the first case when the determinant \( \det \Delta_{ab} \neq 0 \) we found that there are eight second class constraints. Unfortunately these second class constraints are not sufficient for the elimination of the scalar mode. On the
other hand we showed that in the second case $\det \Delta_{ab} = 0$ we find the first class constraint $\tilde{P}$. However then we argued that in this case the Hamiltonian constraint $\tilde{\mathcal{R}}$ together with the constraint $\tilde{\mathcal{G}}$ are the second class constraints. On the other hand since we know that the theory is invariant under diagonal diffeomorphism we suggests that the Hamiltonian constraint should be the first class constraint and consequently the case when $\det \Delta_{ab} = 0$ should not occur. All these results could also imply that the ghost mode is still presented in the non-linear bimetric theory of gravity.

As the extension of this work we suggest the Hamiltonian analysis of the multimetric theories of gravity that were introduced in paper [31] and we hope to perform such analysis soon.

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