Evaluation of weighted Fibonacci sums of a certain type

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Abstract

We derive a formula for the evaluation of weighted generalized Fibonacci sums of the type

\[ S^k_n(w, r) = \sum_{j=0}^{k} w^j j^r G_j^n. \]

Several explicit evaluations are presented as examples.

1 Introduction

The generalized Fibonacci numbers \( G_i, i \geq 0 \), are defined through the second order recurrence relation \( G_{i+1} = G_i + G_{i-1} \), with arbitrary seeds \( G_0 \) and \( G_1 \). In particular, when \( G_0 = 0 \) and \( G_1 = 1 \), we have the Fibonacci numbers, denoted \( F_i \), and when \( G_0 = 2 \) and \( G_1 = 1 \), we have the Lucas numbers, \( L_i \). Extension to negative index is provided through \( G_{-i} = (-1)^i (F_{i+1} G_0 - F_i G_1) \), Vajda [12], Formula (9).

In this paper we will derive a formula (Theorem 3.2) for the evaluation of weighted generalized Fibonacci sums of the type

\[ S^k_n(w, r) = \sum_{j=0}^{k} w^j j^r G_j^n, \]  

where \( w \) is a real or complex parameter and \( r, k \) and \( n \) are nonnegative integers. For brevity, \( S^1_k(w, r) \) will be denoted \( S_k(w, r) \) and \( S^1_k(1, r) \) will often be denoted simply by \( S_k(r) \). Similarly, we will write \( \overline{S_k(r)} \) for \( S^1_k(-1, r) \). The evaluation of \( S_k(r) = \sum_{j=0}^{k} j^r G_j \) has received some attention in the literature, see for example references [10, 9, 2, 5, 11].

Since \( S^0_k(w, 0) \) in (1.1) is to give \( \sum_{j=0}^{k} w^j G_j^n \), we shall adopt the convention that \( j^0 = 1 \) for all \( j \), including \( j = 0 \).

For evaluating sums with \( a > b \) we shall use

\[ \sum_{j=a}^{b} f_j \equiv - \sum_{j=b+1}^{a-1} f_j. \]
In particular
\[
\sum_{j=a+1}^{a-1} f_j = -f_a \quad \text{and} \quad \sum_{j=a}^{a-1} f_j = 0.
\]

Following Gauthier [4], we introduce the differential operator
\[
D = w \frac{d}{dw},
\]
and note the following properties:
1. \(D^0 u(w) = u(w)\).
2. \(D(au(w) + bv(w)) = aD(u(w)) + bD(v(w))\), for complex numbers \(a\) and \(b\).
3. \(D^p (u(w)v(w)) = \sum_{r=0}^{p} \binom{p}{r} D^r u(w) D^{p-r} v(w)\) for \(p\) times differentiable functions \(u(w)\) and \(v(w)\).
4. If \(u(w) = \sum_{r=0}^{n} a_r w^r\), where the coefficients \(a_r\) are real or complex numbers and \(n\) is an integer, then, \(D^p u(w) = a_0 \delta_{p0} + \sum_{r=1}^{n} r^p a_r w^r\), where \(\delta_{ij}\) is the usual Kronecker delta.
5. \(D^p S^n_k (w, 0) = S^n_k (w, p)\).

The strength of our analysis lies in the properties [4] [4] and [5] which are easily verified by induction on \(p\).

Here is a couple of anticipated evaluations to whet the reader’s appetite for reading on.

\[
\sum_{j=1}^{k} j^5 G_j = 2671 G_0 + 4322 G_1 + (k^5 - 5 k^4 + 50 k^3 - 310 k^2 + 1285 k - 2671) G_k
+ (k^5 - 10 k^4 + 80 k^3 - 500 k^2 + 2080 k - 4322) G_{k+1},
\]

\[
\sum_{j=1}^{k} (-1)^j j G_j = 3 G_0 - 2 G_1 - (-1)^k (k + 3) G_k + (-1)^k (k + 2) G_{k+1},
\]

\[
\sum_{j=1}^{k} j G_j
= 6 G_0 + 10 G_1 - (k + 6) \frac{G_k}{2k} - (k + 5) \frac{G_{k+1}}{2k-1},
\]

\[
\sum_{j=1}^{k} 2^j j G_j
= \frac{2}{5} G_1 + \frac{2^{k+2}}{5} k G_k + \frac{2^{k+1}}{5} (k - 1) G_{k+1},
\]

\[
\sum_{j=1}^{k} (-1)^j G_{2j} = \frac{3}{5} G_0 - \frac{1}{5} G_1 + (-1)^k \frac{2}{5} G_{2k} + (-1)^k \frac{1}{5} G_{2k+1},
\]

\[
\sum_{j=1}^{k} (-1)^{j-1} G_{2j-1} = -\frac{1}{5} G_0 + \frac{2}{5} G_1 + (-1)^k \frac{1}{5} G_{2k} - (-1)^k \frac{2}{5} G_{2k+1}.
\]
\[
\sum_{j=0}^{\infty} \frac{G_j^2}{4j} = \frac{3552}{3553}G_0^2 + \frac{224}{3553}G_1^2 + \frac{16}{3553}G_2^2,
\]
\[
\sum_{j=0}^{k} (-1)^j G_{2j}^3 = \frac{23}{25}G_0^3 - \frac{9}{50}G_1^3 - \frac{13}{50}G_2^3 + \frac{3}{50}G_3^3 - \frac{3}{50}(-1)^k G_{2k+1}^3 + \frac{37}{50}(-1)^k G_{2k+2}^3 + \frac{9}{50}(-1)^k G_{2k+3}^3 - \frac{2}{25}(-1)^k G_{2k+4}^3.
\]

2 Evaluation of \( S_k(w, r) = \sum_{j=0}^{k} w^j j^r G_j \)

**Lemma 1.** If \( k \) is a positive integer and \( w \) is a parameter, real or complex, then
\[
S_k(w, 0) = \sum_{j=0}^{k} w^j G_j = \frac{-(w-1)G_0 + wG_1 - w^{k+2}G_k - w^{k+1}G_{k+1}}{1 - w - w^2}.
\]

In particular, we have
\[
S_k(1, 0) = \sum_{j=0}^{k} G_j = G_{k+2} - G_1.
\]

**Proof.** Multiply through the recurrence relation \( G_j = G_{j-1} + G_{j-2} \) by \( w^j \) and sum over \( j \), obtaining
\[
\sum_{j=0}^{k} w^j G_j = \sum_{j=0}^{k} w^j G_{j-1} + \sum_{j=0}^{k} w^j G_{j-2} = w \sum_{j=-1}^{k-1} w^j G_j + w^2 \sum_{j=-2}^{k-2} w^j G_j = w \left( \frac{G_{-1}}{w} + \sum_{j=0}^{k} w^j G_j - w^k G_k \right) + w^2 \left( \frac{G_{-2}}{w^2} + \frac{G_{-1}}{w} + \sum_{j=0}^{k} w^j G_j - w^{k-1}G_{k-1} - w^k G_k \right),
\]
so that we have
\[
S_k(w, 0) = G_{-1} + wS_k(w, 0) - w^{k+1}G_k + G_{-2} + wG_{-1} + w^2S_k(w, 0) - w^{k+1}G_{k-1} - w^{k+2}G_k,
\]
and the result follows.

Note that in the identity of Lemma 1 if we let \( k \) approach infinity and require \( w^k G_k \) to vanish as \( k \) approaches infinity, then we obtain the generating function for \( G_i \), namely,
\[
S_\infty(w, 0) = \sum_{j=0}^{\infty} w^j G_j = \frac{(1 - w)G_0 + wG_1}{1 - w - w^2}.
\]
Lemma 2. The rational function

\[ A(w; m) = D^m \frac{1}{1 - w - w^2} = D^m A(w; 0) , \]

satisfies the following recursion relation

\[ (1 - w - w^2) A(w; m) = \delta_{m0} + w \sum_{j=0}^{m-1} \binom{m}{j} (2^{m-j} w + 1) A(w; j), \quad m = 0, 1, 2, \ldots . \]

Proof. Write \((1 - w - w^2) A(w; 0) = 1\) and apply \(D^m\) to both sides.

Theorem 2.1. If \(k\) and \(r\) are nonnegative integers and \(w\) a real or complex parameter, then

\[ S_k(w, r) = \sum_{j=0}^{k} w^j j^r G_j = -G_0 \sum_{m=0}^{r} \binom{r}{m} (w - \delta_{rm}) A(w; m) + wG_1 \sum_{m=0}^{r} \binom{r}{m} A(w; m) \]

\[ - w^{k+2} G_k \sum_{m=0}^{r} \binom{r}{m} (k + 2)^m A(w; r - m) \]

\[ - w^{k+1} G_{k+1} \sum_{m=0}^{r} \binom{r}{m} (k + 1)^m A(w; r - m) , \]

where the \(A(w; p), p = 0, 1, 2, \ldots, r,\) are given recursively by

\[ (1 - w - w^2) A(w; p) = \delta_{p0} + w \sum_{j=0}^{p-1} \binom{p}{j} (2^{p-j} w + 1) A(w; j) . \]

Proof. Apply \(D^r\) to both sides of the identity of Lemma 1 and make use of Lemma 2 and the properties 3, 4 and 5 of the \(D\) operator.

Numerous interesting summation formulas can be obtained directly from Theorem 2.1. By setting \(w = 1/2, r = 0, 1, 2,\) respectively, we already have some examples:

\[ \sum_{j=0}^{k} \frac{G_j}{2^j} = 2G_0 + 2G_1 - \frac{G_k}{2^{k-1}} - \frac{G_{k+1}}{2^{k-1}} , \quad (2.2) \]

corresponding to Vajda [12], Formula (37), proved by induction, which in the limit as \(k \to \infty\) gives

\[ \sum_{j=0}^{\infty} \frac{G_j}{2^j} = 2G_0 + 2G_1 . \]

\[ \sum_{j=0}^{k} \frac{jG_j}{2^j} = 6G_0 + 10G_1 - (k + 6) \frac{G_k}{2^k} - (k + 5) \frac{G_{k+1}}{2^{k-1}} , \quad (2.4) \]

\[ \sum_{j=0}^{\infty} \frac{jG_j}{2^j} = 6G_0 + 10G_1 . \]

(2.5)
\[
\sum_{j=0}^{k} \frac{j^2 G_j}{2^j} = 58G_0 + 94G_1 - (k^2 + 12k + 58) \frac{G_k}{2^k} - (k^2 + 10k + 47) \frac{G_{k+1}}{2^{k-1}},
\]

\[
\sum_{j=0}^{\infty} \frac{j^2 G_j}{2^j} = 58G_0 + 94G_1.
\]

Note that the identity (2.5) subsumes Formulas (A3.52) and (A3.53) of Dunlap [3].

If \( i = \sqrt{-1} \) is the imaginary unit, then using \( f_j = i^j j^r G_j \) in the summation identity

\[
\sum_{j=0}^{2k} f_j = \sum_{j=0}^{k} f_{2j} + \sum_{j=1}^{k} f_{2j-1}
\]

allows us to write

\[
\sum_{j=0}^{2k} i^j j^r G_j = 2^r \sum_{j=0}^{k} (-1)^j j^r G_{2j} + i \sum_{j=1}^{k} (-1)^{j-1}(2j - 1)^r G_{2j-1}.
\]

Thus, setting \( w = i \) in Theorem 2.1 and making use of identity (2.8), we have the following results for \( r = 0, 1 \).

\[
\sum_{j=0}^{k} (-1)^j G_{2j} = \frac{3}{5} G_0 - \frac{1}{5} G_1 + (-1)^k \frac{2}{5} G_{2k} + (-1)^k \frac{1}{5} G_{2k+1},
\]

\[
\sum_{j=1}^{k} (-1)^{j-1} G_{2j-1} = -\frac{1}{5} G_0 + \frac{2}{5} G_1 + (-1)^k \frac{1}{5} G_{2k} - (-1)^k \frac{2}{5} G_{2k+1},
\]

\[
\sum_{j=0}^{k} (-1)^j j G_{2j} = -\frac{1}{5} G_0 + \frac{1}{5} (-1)^k (2k + 1) G_{2k} + \frac{1}{5} (-1)^k k G_{2k+1},
\]

\[
\sum_{j=1}^{k} (-1)^{j-1} j G_{2j-1} = -\frac{1}{5} G_0 + \frac{1}{5} G_1 + \frac{1}{5} (-1)^k (k + 1) G_{2k}
\]

\[
- \frac{1}{5} (-1)^k (2k + 1) G_{2k+1}.
\]

Identities (2.9) and (2.10) are the alternating counterparts of Formulas (34) and (35) of Vajda [12].

**Corollary 2.2.** If \( k \) is a nonnegative integer and \( w \) is a real or complex parameter, then

\[
\sum_{j=0}^{k} w^j j G_j = \frac{2 - w}{(1 - w - w^2)^2} w^2 G_0 + \frac{w^2 + 1}{(1 - w - w^2)^2} w G_1
\]

\[
+ \frac{k w^2 + (k + 1) w - (k + 2)}{(1 - w - w^2)^2} w^{k+2} G_k
\]

\[
+ \frac{(k - 1) w^2 + k w - (k + 1)}{(1 - w - w^2)^2} w^{k+1} G_{k+1}.
\]
In particular, we have

\[
\sum_{j=0}^{k} 2^j jG_j = \frac{2}{5} G_1 + \frac{2^{k+2}}{5} kG_k + \frac{2^{k+1}}{5} (k-1)G_{k+1}
\]  

(2.13)

and

\[
\sum_{j=0}^{k} 3^j jG_j = -\frac{9}{121} G_0 + \frac{30}{121} G_1 + \frac{3^{k+2}}{121} (11k + 1)G_k
\]

\[+ \frac{3^{k+1}}{121} (11k - 10)G_{k+1}.
\]  

(2.14)

**Corollary 2.3.** If \( r \) is a nonnegative integer and \( w \) is a real or complex parameter such that \( w^k G_k \to 0 \) as \( k \to \infty \) for all positive integers \( k \), then

\[
S_\infty(w, r) = \sum_{j=0}^{\infty} w^j j^r G_j = -G_0 \sum_{m=0}^{r} \binom{r}{m} (w - \delta_{rm}) A(w; m)
\]

\[+ wG_1 \sum_{m=0}^{r} \binom{r}{m} A(w; m).
\]

Thus, Corollary 2.3 provides the evaluation of generating functions for \( j^r G_j \). A strategy for using probability to evaluate \( S_\infty(1/2, r) \) was discussed in reference [1]. Dropping the last two terms in Corollary 2.2, we have

\[
S_\infty(w, 1) = \sum_{j=0}^{\infty} w^j jG_j = \frac{2-w}{(w^2 + w - 1)^2} w^2 G_0 + \frac{w^2 + 1}{(w^2 + w - 1)^2} wG_1,
\]  

(2.15)

a result that was also obtained and whose convergence was exhaustively discussed by Glast-ter [6, 7] for the Fibonacci case, \((G=F)\).

**Corollary 2.4.** If \( k \) and \( r \) are nonnegative integers, then

\[
S_k(r) = \sum_{j=0}^{k} j^r G_j = -G_0 \sum_{m=0}^{r} \binom{r}{m} (1 - \delta_{rm}) A(m) + G_1 \sum_{m=0}^{r} \binom{r}{m} A(m)
\]

\[- G_k \sum_{m=0}^{r} \binom{r}{m} (k + 2)^m A(r - m)
\]

\[- G_{k+1} \sum_{m=0}^{r} \binom{r}{m} (k + 1)^m A(r - m),
\]

where the \( A(m) \) satisfy the recurrence relation

\[
A(m) = -\delta_{m0} - \sum_{j=0}^{m-1} \binom{m}{j} (2^{m-j} + 1) A(j), \quad m = 0, 1, 2, \ldots
\]

Here is a little list from Corollary 2.4

\[
\sum_{j=0}^{k} G_j = -G_1 + G_k + G_{k+1},
\]  

(2.16)
\[
\sum_{j=0}^{k} jG_j = G_0 + 2G_1 + (k - 1)G_k + (k - 2)G_{k+1}, \quad (2.17)
\]
\[
\sum_{j=0}^{k} j^2G_j = -5G_0 - 8G_1 + (k^2 - 2k + 5)G_k + (k^2 - 4k + 8)G_{k+1}, \quad (2.18)
\]
\[
\sum_{j=0}^{k} j^3G_j = 31G_0 + 50G_1 + (k^3 - 3k^2 + 15k - 31)G_k + (k^3 - 6k^2 + 24k - 50)G_{k+1},
\]
\[
\sum_{j=0}^{k} j^4G_j = -257G_0 - 416G_1 + (k^4 - 4k^3 + 30k^2 - 124k + 257)G_k
\]
\[
+ (k^4 - 8k^3 + 48k^2 - 200k + 416)G_{k+1},
\]
\[
\sum_{j=0}^{k} j^5G_j = 2671G_0 + 4322G_1 + (k^5 - 5k^4 + 50k^3 - 310k^2 + 1285k - 2671)G_k
\]
\[
+ (k^5 - 10k^4 + 80k^3 - 500k^2 + 2080k - 4322)G_{k+1},
\]
\[
\sum_{j=0}^{k} j^6G_j = -33305G_0 - 53888G_1
\]
\[
+ (k^6 - 6k^5 + 75k^4 - 620k^3 + 3855k^2 - 16026k + 33305)G_k
\]
\[
+ (k^6 - 12k^5 + 120k^4 - 1000k^3 + 6240k^2 - 25932k + 53888)G_{k+1}.
\]

**Corollary 2.5.** If \( k \) and \( r \) are nonnegative integers, then

\[
\overline{S}_k(r) = S_k(-1, r) = \sum_{j=0}^{k} (-1)^j j^r G_j
\]
\[
= G_0 \sum_{m=0}^{r} \binom{r}{m} (\delta_{rm} + 1) \overline{A}(m) - G_1 \sum_{m=0}^{r} \binom{r}{m} \overline{A}(m)
\]
\[
- (-1)^k G_k \sum_{m=0}^{r} \binom{r}{m} (k + 2)^m \overline{A}(r - m)
\]
\[
+ (-1)^k G_{k+1} \sum_{m=0}^{r} \binom{r}{m} (k + 1)^m \overline{A}(r - m),
\]

where the \( \overline{A}(p) \) are given recursively by

\[
\overline{A}(p) = \delta_{p0} + \sum_{j=0}^{p-1} \binom{p}{j} (2^{p-j} - 1) \overline{A}(j), \quad p = 1, 2, \ldots
\]

Here are a few evaluations.

\[
\sum_{j=0}^{k} (-1)^j G_j = 2G_0 - G_1 - (-1)^k G_k + (-1)^k G_{k+1}, \quad (2.23)
\]
\[
\sum_{j=0}^{k} (-1)^j G_j = 3G_0 - 2G_1 - (-1)^k (k + 3)G_k + (-1)^k (k + 2)G_{k+1}, \tag{2.24}
\]
\[
\sum_{j=0}^{k} (-1)^j j^2 G_j = 13G_0 - 8G_1 - (-1)^k (k^2 + 6k + 13)G_k + (-1)^k (k^2 + 4k + 8)G_{k+1}, \tag{2.25}
\]
\[
\sum_{j=0}^{k} (-1)^j j^3 G_j = 81G_0 - 50G_1 - (-1)^k (k^3 + 9k^2 + 39k + 81)G_k + (-1)^k (k^3 + 6k^2 + 24k + 50)G_{k+1}. \tag{2.26}
\]

3 Evaluation of \( S_k^n(w, r) = \sum_{j=0}^{k} w^j G_j^n \)

**Lemma 3.** If \( k \) and \( n \) are nonnegative integers and \( w \) is a parameter, real or complex, then

\[
S_k^n(w, 0) = \sum_{j=0}^{k} w^j G_j^n = \left( \sum_{s=0}^{n+1} \binom{n+1}{s} \left( \frac{(-1)^{(n-s+1)/2} \sum_{j=0}^{s-1} w^{j+n-s+1} G_j^n} {F_{n+1}} \right) \right) \sum_{s=0}^{k} w^j G_{n+s} = 0
\]

where \([u]\) is the smallest integer greater than \( u \) and the Fibonomial coefficients are defined by

\[
\binom{p}{q}_F = \frac{F_p F_{p-1} \cdots F_{p-q+1}} {F_q F_{q-1} \cdots F_1} = \prod_{j=1}^{q} \frac{F_{p-q+j}} {F_j}.
\]

**Proof.** Multiply through the following identity [8, section 1.2.8, Exercise 30] by \( w^j \)

\[
\sum_{s=0}^{n+1} \binom{n+1}{s} \left( \frac{(-1)^{(n-s+1)/2} \sum_{j=0}^{s-1} w^{j+n-s+1} G_j^n} {F_{n+1}} \right) = 0
\]

and sum over \( j \), obtaining

\[
\sum_{s=0}^{n+1} \binom{n+1}{s} \left( \frac{(-1)^{(n-s+1)/2} \sum_{j=0}^{k} w^j G_{n+s}^n} {F_{n+1}} \right) = 0,
\]

which after shifting the index \( j \) becomes

\[
\sum_{s=0}^{n+1} \binom{n+1}{s} \left( \frac{(-1)^{(n-s+1)/2} \sum_{j=0}^{k+s} w^j G_j^n} {F_{n+1}} \right) = 0.
\]

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Since
\[ \sum_{j=s}^{k+s} w^j G_j^n = \sum_{j=0}^{k} w^j G_j^n + \sum_{j=k+1}^{k+s} w^j G_j^n - \sum_{j=0}^{s-1} w^j G_j^n, \]
we have
\[ \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{(n-s+1)/2} w^{-s} \left( \sum_{j=0}^{k} w^j G_j^n + \sum_{j=k+1}^{k+s} w^j G_j^n - \sum_{j=0}^{s-1} w^j G_j^n \right) = 0. \]

Thus,
\[ \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{(n-s+1)/2} w^{-s} \left( S_n^k(w, 0) + \sum_{j=k+1}^{k+s} w^j G_j^n - \sum_{j=0}^{s-1} w^j G_j^n \right) = 0. \]

Clearing brackets, multiplying through by \( w^{n+1} \) and making \( S_n^k(w, 0) \) the subject, the lemma is proved.

In particular, we have
\[ \sum_{j=0}^{k} w^j G_j^2 = -\frac{(2w^2 + 2w - 1)G_0^2}{w^3 - 2w^2 - 2w + 1} - \frac{(2w^2 - w)G_1^2}{w^3 - 2w^2 - 2w + 1} + \frac{w^2 G_2^2}{w^3 - 2w^2 - 2w + 1} + \frac{(2w^2 + 2w - 1)w^{k+1}G_{k+1}^2}{w^3 - 2w^2 - 2w + 1} \]
\[ + \frac{(2w - 1)w^{k+2}G_{k+2}^2}{w^3 - 2w^2 - 2w + 1} - \frac{w^{k+3}G_{k+3}^2}{w^3 - 2w^2 - 2w + 1} \] (3.1)

and
\[ \sum_{j=0}^{k} w^j G_j^3 = \frac{(3w^3 - 6w^2 - 3w + 1)G_0^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} - \frac{(6w^3 + 3w^2 - w)G_1^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} - \frac{(3w^3 - w^2)G_2^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} + \frac{w^3 G_3^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} \]
\[ - \frac{(3w^3 - 6w^2 - 3w + 1)w^{k+1}G_{k+1}^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} + \frac{(6w^2 + 3w - 1)w^{k+2}G_{k+2}^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} \]
\[ + \frac{(3w - 1)w^{k+3}G_{k+3}^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} - \frac{w^{k+4}G_{k+4}^3}{w^4 + 3w^3 - 6w^2 - 3w + 1}. \] (3.2)

Interesting evaluations can already be obtained from (3.1) and (3.2). For example, at \( w = 1 \), we have
\[ \sum_{j=0}^{k} G_j^2 = \frac{3}{2} G_0^2 + \frac{1}{2} G_1^2 - \frac{1}{2} G_2^2 - \frac{3}{2} G_{k+1}^2 - \frac{1}{2} G_{k+2}^2 + \frac{1}{2} G_{k+3}^2 \] (3.3)

and
\[ \sum_{j=0}^{k} G_j^3 = \frac{5}{4} G_0^3 + 2G_1^3 + \frac{1}{2} G_2^3 - \frac{1}{4} G_3^3 \]
\[ - \frac{5}{4} G_{k+1}^3 - 2G_{k+2}^3 - \frac{1}{2} G_{k+3}^3 + \frac{1}{4} G_{k+4}^3, \] (3.4)
while \( w = 1/2 \) gives

\[
\sum_{j=0}^{k} \frac{G_j^2}{2^j} = \frac{4}{3} G_0^2 - \frac{2}{3} G_2^2 - \frac{1}{3} G_{k+1}^2 + \frac{1}{3} G_{k+3}^2
\]  

(3.5)

and

\[
\sum_{j=0}^{k} \frac{G_j^3}{2^j} = \frac{26}{25} G_0^3 + \frac{16}{25} G_1^3 + \frac{2}{25} G_2^3 - \frac{2}{25} G_3^3 - \frac{13}{25} G_{k+1}^3 + \frac{8}{25} G_{k+2}^3 - \frac{1}{25} G_{k+3}^3 + \frac{1}{25} G_{k+4}^3.
\]  

(3.6)

At \( w = 2 \), we have

\[
\sum_{j=0}^{k} 2^j G_j^2 = \frac{11}{3} G_0^2 + 2 G_1^2 - \frac{4}{3} G_2^2 - \frac{11}{3} 2^{k+1} G_{k+1}^2 - 2^{k+2} G_{k+2}^2 + \frac{1}{3} 2^{k+3} G_{k+3}^2
\]  

(3.7)

and

\[
\sum_{j=0}^{k} 2^j G_j^3 = -\frac{5}{11} G_0^3 - \frac{58}{11} G_1^3 - \frac{20}{11} G_2^3 + \frac{8}{11} G_3^3 + \frac{5}{11} 2^{k+1} G_{k+1}^3 + \frac{29}{11} 2^{k+2} G_{k+2}^3 + \frac{5}{11} 2^{k+3} G_{k+3}^3 - \frac{1}{11} 2^{k+4} G_{k+4}^3.
\]  

(3.8)

At \( w = 1/16 \), we have

\[
\sum_{j=0}^{k} \frac{G_j^2}{2^j} = \frac{3552}{3553} G_0^2 + \frac{224}{3553} G_1^2 + \frac{16}{3553} G_2^2 - \frac{222}{3553} G_{k+1}^2 - \frac{7}{3553} G_{k+2}^2 - \frac{1}{3553} G_{k+3}^2 + \frac{1}{3553} G_{k+4}^2.
\]  

(3.9)

giving

\[
\sum_{j=0}^{\infty} \frac{G_j^2}{2^j} = \frac{3552}{3553} G_0^2 + \frac{224}{3553} G_1^2 + \frac{16}{3553} G_2^2.
\]  

(3.10)

Evaluation at \( w = \sqrt{-1} = i, n = 2, 3 \) gives

\[
\sum_{j=0}^{k} (-1)^j G_j^2 = \frac{5}{6} G_0^2 + \frac{1}{6} G_1^2 - \frac{1}{6} G_2^2 + \frac{(-1)^k}{6} G_{2k+1}^2 - \frac{(-1)^k}{2} G_{2k+2}^2 - \frac{(-1)^k}{6} G_{2k+3}^2,
\]  

(3.11)

\[
\sum_{j=1}^{k} (-1)^{j-1} G_{2j-1}^2 = \frac{1}{6} G_0^2 + \frac{1}{2} G_1^2 - \frac{1}{6} G_2^2 - \frac{5}{6} (-1)^k G_{2k+1}^2 - \frac{(-1)^k}{6} G_{2k+2}^2 + \frac{(-1)^k}{6} G_{2k+3}^2,
\]  

(3.12)
\[
\sum_{j=0}^{k} (-1)^j G_{2j}^3 = \frac{23}{25} G_0^3 - \frac{9}{50} G_1^3 - \frac{13}{50} G_2^3 + \frac{3}{50} G_3^3 - \frac{3}{50} (-1)^k G_{2k+1}^3.
\]  \tag{3.13}

and
\[
\sum_{j=1}^{k} (-1)^{j-1} G_{2j-1}^3 = -\frac{3}{50} G_0^3 + \frac{37}{50} G_1^3 + \frac{9}{50} G_2^3 - \frac{2}{25} (-1)^k G_{2k+1}^3
\]  \tag{3.14}

**Corollary 3.1** (Generating function for \( G_j^n \)). If \( n \) is a nonnegative integer and \( w \) is a parameter, real or complex, such that \( w^k G_k^n \) vanishes as \( k \) approaches infinity, then
\[
S_n^j(w, 0) = \sum_{j=0}^{\infty} w^j G_j^n = \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[(n+s)/2]} \sum_{j=0}^{s-1} w^{j+n-s+1} G_j^n \left( \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[(n+s)/2]} w^{n-s+1}. \right)
\]

In particular, the generating functions of \( G_j^2 \) and \( G_j^3 \) are, respectively,
\[
S_2^j(w, 0) = \sum_{j=0}^{\infty} w^j G_j^2 = \frac{w^2 G_2^2 - (2w^2 + 2w - 1) G_0^2 - (2w^2 - w) G_1^2}{w^4 - 2w^2 - 2w + 1} \tag{3.15}
\]

and
\[
S_3^j(w, 0) = \sum_{j=0}^{\infty} w^j G_j^3 = \frac{(3w^3 - 6w^2 - 3w + 1) G_0^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} - \frac{(6w^3 + 3w^2 - w) G_1^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} - \frac{(3w^3 - w^2) G_2^3}{w^4 + 3w^3 - 6w^2 - 3w + 1} + \frac{w^3 G_3^3}{w^4 + 3w^3 - 6w^2 - 3w + 1}. \tag{3.16}
\]

**Lemma 4.** The rational function
\[
A_n(w; m) = D^m A_n(w; 0) = D^m \frac{1}{\sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[(n+s)/2]} w^{n-s+1}}
\]
satisfies the following recurrence relation
\[
A_n(w; m) \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[(n+s)/2]} w^{n-s+1} = \delta_{m0} - \sum_{j=0}^{m-1} \binom{m}{j} A_n(w; j) \sum_{s=0}^{n} \binom{n+1}{s} (-1)^{[(n+s)/2]} (n - s + 1)^{m-j} w^{n-s+1}.
\]

**Proof.** Write
\[
A_n(w; 0) \left( 1 + \sum_{s=0}^{n} \binom{n+1}{s} (-1)^{[(n+s)/2]} w^{n-s+1} \right) = 1.
\]

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Apply $D^m$ to both sides to obtain
\[
\sum_{j=0}^{m} \binom{m}{j} D^j A_n(w; 0) \left( \delta_{jm} + \sum_{s=0}^{n} \binom{n+1}{s} (-1)^{[n-s+1]}/2 (n-s+1)^{m-j} w^{n-s+1} \right) = \delta_{m0},
\]
from which the result follows.

\[\text{Theorem 3.2.}\] If $k$, $r$ and $n$ are nonnegative integers and $w$ is a real or complex parameter, then
\[
S^n_k(w, r) = \sum_{j=0}^{k} w^j G_j^n
\]
\[
= A_n(w; r) G_0^n + \sum_{m=0}^{r} \binom{r}{m} A_n(w; m) \sum_{j=1}^{n} j^{r-m} w^j G_j^n
\]
\[
+ \sum_{m=0}^{r} \binom{r}{m} A_n(w; m) \sum_{s=0}^{n} \binom{n+1}{s} (-1)^{[n-s+1]/2} \sum_{j=0}^{s-1} (j+n-s+1)^{r-m} w^{j+n-s+1} G_j^n
\]
\[
+ \sum_{m=0}^{r} \binom{r}{m} A_n(w; m) \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[n-s+1]/2} \sum_{j=k+1}^{s} (j+n-s+1)^{r-m} w^{j+n-s+1} G_j^n,
\]
where the functions $A_n(w, m)$ are given recursively by
\[
A_n(w; m) \sum_{s=0}^{n+1} \binom{n+1}{s} (-1)^{[n-s+1]/2} w^{n-s+1}
\]
\[
= \delta_{m0} - \sum_{j=0}^{m-1} \binom{m}{j} A_n(w; j) \sum_{s=0}^{n} \binom{n+1}{s} (-1)^{[n-s+1]/2} (n-s+1)^{m-j} w^{n-s+1},
\]
for $m = 0, 1, 2, \ldots, r$.

\[\text{Proof.}\] Apply $D^r$ to both sides of the identity of Lemma 3 and make use of Lemma 4 and the properties 3, 4 and 5 of the $D$ operator.

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