Magnetic fluctuations in 2D metals close to the Stoner instability

B. N. Narozhny and I. L. Aleiner

Department of Physics and Astronomy, SUNY at Stony Brook, Stony Brook, NY 11794

A.I. Larkin

Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
L.D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia

We consider the effect of potential disorder on magnetic properties of a two-dimensional metallic system (with conductance \( g \gg 1 \)) when interaction in the triplet channel is so strong that the system is close to the threshold of the Stoner instability. We show, that under these conditions there is an exponentially small probability for the system to form local spin droplets which are local regions with non zero spin density. Using a non-local version of the optimal fluctuation method we find analytically the probability distribution and the typical spin of a local spin droplet (LSD). In particular, we show that both the probability to form a LSD and its typical spin are independent of the size of the droplet (within the exponential accuracy). The LSDs manifest themselves in temperature dependence of observable quantities. We show, that below certain cross-over temperature the paramagnetic susceptibility acquires the Curie-like temperature dependence, while the dephasing time (extracted from magneto-resistance measurements) saturates.

I. INTRODUCTION

The itinerant ferromagnetism in metals has been the subject of extensive theoretical and experimental studies. In clean systems the ferromagnetic instability is described by the Stoner criterion, which defines the critical value of the spin-exchange interaction constant (at which the system becomes unstable with respect to ferromagnetic ordering). The value of the constant depends on the material. In most simple metals it is small (so that the electronic system exhibits paramagnetic response), while in palladium it approaches the critical value. Theoretically, the Stoner criterion is most easily obtained within the framework of the Hubbard model or in Landau Fermi liquid theory.

In two dimensions disorder tends to localize the electronic system. However, if either the sample size \( L \) or the dephasing length \( L_D \) is smaller than the localization length, the sample can still be considered as metallic. However, the interaction constants are renormalized from their clean values. In particular, the spin-exchange interaction constant was shown to flow towards strong coupling. It is possible therefore to have a sample with the renormalized constant approaching the critical value while at the same time far from the Anderson localization. In this case a mean-field structure of the ferromagnetically ordered ground state was conjectured recently in Ref. 1.

An important issue in the physics of disordered systems is the role of mesoscopic fluctuations. Indeed, the Stoner criterion in its usual form reflects the tendency of the system to acquire uniform non zero magnetization (if the interaction constant happens to reach the critical value). However, for each realization of disorder the spin exchange interaction is non-local and random. The effective interaction constant entering the Stoner criterion is essentially the interaction kernel averaged over the whole system. At the same time, averaging over some small part of the system might produce a value of the effective interaction constant different from the system-wide average. In particular, some rare impurity configurations can lead to the locally averaged interaction constant in some region to satisfy the Stoner criterion, while the system-wide average does not. This would mean the appearance of non zero spin polarization in such regions. The similar effect in finite-size mesoscopic systems was considered in Ref. 10.

In this paper we investigate the plausibility of such a scenario and its effect on magnetic and transport properties of the system. We consider a good metal (characterized by a large dimensionless conductance \( g = 2\pi \hbar / e^2 R \gg 1 \)) in 2D close to the instability, but still in the paramagnetic phase (at the mean field level), so that the naive mean-field value of the total magnetization of the system is zero. We describe the spin exchange interaction by the effective averaged constant \( F_0 \) with all the renormalizations already included (as shown in Ref. 8) and by the random, sample specific non-local susceptibility, which, when averaged over the area of a small region, gives the local effective interaction constant. Using the “optimal fluctuation” method we find that there is exponentially small but non zero probability to find such region with non zero spin polarization which we call a local spin droplet (LSD). With the exponential accuracy this probability does not depend on the size of a LSD, therefore droplets of all sizes (up to the size of the order of the thermal length \( L_T = \sqrt{\hbar D / T} \) (\( D \) is the diffusion constant and \( T \) is the temperature) can appear. The total spin of a LSD is also independent of its size. The effective interaction between LSDs

\( D \) is the diffusion constant and \( T \) is the temperature.)
is determined by the correlations of the same non-local susceptibility. Since LSDs are extended objects, the correlation functions which determine the effective interaction constant have to be averaged over the area of both interacting LSDs. The oscillating parts of the correlation functions (which usually lead to the RKKY interaction) do not survive this averaging. Instead, we find that the average value of the effective interaction is ferromagnetic and decays with the distance between LSDs only as a power law. However, at large distances fluctuations of the effective interaction constant exceed the average and the sign of the interaction becomes random.

The contribution of LSDs to physical observables manifests itself at low temperatures. Indeed, as any system of weakly interacting moments at temperatures, higher than the point of magnetic ordering, the system of LSDs exhibit the Curie-like susceptibility. We show, that at not so low temperatures, this contribution exceeds the (temperature-independent) Pauli susceptibility of the electron system. Likewise, the LSDs contribute to the dephasing time $\tau_\phi$ (extracted from magnetoresistance measurements, see Ref. [4]). Only, while usually the dephasing time in two dimensional electron systems behaves like $\tau_\phi^{-1} \sim T$, the contribution of LSDs is temperature independent. Thus at temperatures lower than certain cross-over temperature the dephasing time saturates. The cross-over temperature is roughly the same for both quantities. When $T \to 0$, interactions of LSDs with each other or with itinerant electrons should lead either to the screening of the local spins or to forming of some spin glass state (due to the randomness of the interaction). We do not consider such regime in this paper.

The paper is organized as follows. In Section II we review the basic physical description of the Stoner instability, establish notations and, in subsection II C we outline the effect of mesoscopic fluctuations. Next, in Section II we discuss the formation of LSDs qualitatively. The subsection IV A is devoted to the calculation of the probability to form a LSD. Then, in subsection IV B we find the distribution of the total spin of the LSD by means of the “optimal fluctuation” method [5], which we adapt to the non-linear problem. Section V describes the interaction of LSDs, and in Section VI we discuss the contribution of LSDs to physical observables. Our results are summarized in Conclusions. Some mathematical details of the non-local susceptibility correlations are relegated to the Appendix.

II. STONER INSTABILITY

The purpose of this Section is to recall the basic ideas leading to the Stoner criterion (subsections II A and II B) and to contrast the situation in clean systems to that in disordered metals. In subsection II C we demonstrate the role of mesoscopic fluctuations and discuss how they can lead to formation of LSDs. For further details on Stoner ferromagnetism the reader is referred to the standard textbooks Ref. [4,5].

A. Renormalized paramagnetic susceptibility

The paramagnetic response of a system of non-interacting electrons is described by the Pauli susceptibility which depends only on the electronic density of states at the Fermi level $\nu$. This can be seen from the following observation. The electron energy $\epsilon$ enters all thermodynamic functions in combination $\epsilon - \mu$ with the chemical potential $\mu$. The interaction energy of the electron spin $\hat{s}$ with the external magnetic field $\hat{h}$ (which is proportional to $-\hat{\sigma} \hat{h}$) can thus be considered as a shift of the chemical potential. Since it is proportional to $\hat{s}$, the number of electrons which spin is aligned with $\hat{h}$ exceeds the number of electrons with the opposite spin, resulting in the total magnetization proportional to $\nu \hat{h}$.

Although quite general, the above argument relies on the fact that weak magnetic field does not change the energy spectrum of the electron system. Taking into account the electron-electron interactions, however, changes the distribution function of electrons and thus the electronic energy spectrum. As a result, some physical quantities become renormalized from their bare values. In particular, the paramagnetic susceptibility is renormalized by the spin exchange interaction [6] (we choose the units with Bohr magneton equal to unity)

$$\chi = \frac{\nu}{1 + F_0}.$$  

Here the parameter $F_0$ is the effective dimensionless coupling constant of magnetic interaction between electron spins. Within the phenomenology of the Landau Fermi-liquid theory, it can be obtained by averaging the spin-exchange part of the Landau function over the Fermi surface. In the case when $F_0 < 0$, the interaction tends to align the electrons spins, competing with the Pauli exclusion principle. If the interaction is strong enough, the gain in the magnetic energy exceeds the kinetic energy cost needed to realign the spins and the ground state of the system changes to the one with non zero total spin - it becomes ferromagnetic. According to Eq. (1) the instability occurs when $F_0 = -1$, which is known as the Stoner criterion [5].

More formally, the full susceptibility tensor is given by the commutator of the spin density operators $\hat{\sigma}_\alpha(x,t)$ (hereafter $x_i$ denotes the two-component coordinate vector)

$$\chi_{\alpha\beta}(x_1,x_2;t_1-t_2) = i\theta(t_1-t_2) \langle [\hat{\sigma}_\alpha(x_1,t_1), \hat{\sigma}_\beta(x_2,t_2)] \rangle,$$  

where $\alpha, \beta = x, y, z$. In the paramagnetic state of an isotropic system the tensor $\chi_{\alpha\beta}$ is diagonal and isotropic.
and can be expressed in terms of the transverse susceptibility $\chi$,

$$\chi_{\alpha\beta} = 2\chi \delta_{\alpha\beta},$$

which is determined in terms of the commutator of the spin raising and lowering operators $\hat{\sigma}^+$ and $\hat{\sigma}^-$ similarly to Eq. (2).

The transverse susceptibility can be evaluated in the generalized Hartree-Fock approximation\[^4\], which amounts to summation of ladder diagrams in the particle-hole channel. In the Galilean invariant system the susceptibility depends only on the coordinate difference and in the momentum representation is given by

$$\chi(q, \omega) = \frac{\Pi(q, \omega)}{1 + U\Pi(q, \omega)}.$$  \hspace{1cm} (3)

Here $\Pi(q, \omega)$ is the electron polarization operator, which represents the susceptibility of the non-interacting electron gas. In the limit $\omega = 0$ and $q = 0$ it gives the Pauli susceptibility [since $\Pi(q = 0, \omega = 0) = \nu$]. The parameter $U$ is the spin exchange coupling constant. In the context of the Hubbard model it appears as a phenomenological parameter of the Hamiltonian. In the microscopic Fermi-liquid theory it is the spin exchange part of the vertex function $\Gamma^\omega$ averaged over the Fermi surface (compare with the scattering amplitude $\Gamma_2$ in Ref. \[^3\]).

The instability in the ground state of the system corresponds to the singularity in the static limit of the response function $\chi(q, \omega = 0)$. The instability criterion is thus $U\Pi(q, \omega = 0) = -1$. At $q = 0$ this corresponds to a tendency of the system to acquire spontaneously a uniform (or ferromagnetic) spin density. The criterion for this instability

$$U
\nu = -1,$$ \hspace{1cm} (4)

corresponds to the Stoner criterion if one identifies the Landau parameter $F_0$ with the spin exchange coupling constant.

B. Magnetization energy

The transition to the ferromagnetic state can also be described similarly to the Landau description of the phase transitions with the induced spin density as the order parameter. Close to the instability point, the thermodynamic potential $\Omega$ of the system can be expanded in powers of the spin density $\bar{\sigma}$

$$\Omega = \frac{\int dV}{2\nu} \left[ (1 + F_0)\bar{\sigma}^2 + a^2|\nabla \bar{\sigma}|^2 + \frac{1}{2} B(\bar{\sigma}^2)^2 + \ldots \right],$$ \hspace{1cm} (5)

where $B > 0$. The value of $\bar{\sigma}$ of any particular state of the system (as described by the constants $F_0$, $a$, and $B$) can be found by minimizing the thermodynamic potential $\Omega$. In the limit $q = 0$ (uniform spin potential $\Omega$) the minimum condition is

$$(1 + F_0 + B\bar{\sigma}^2)\bar{\sigma} = 0.$$ \hspace{1cm} (6)

Here we have neglected the gradient term relative to the linear term, since the coefficient $a \sim k_F^{-2}$ and under our assumptions, $g \gg 1$ and $g(1 + F_0) \gg 1$ (the latter assumption will be elaborated on in Section III). Therefore the linear term dominates, $1 + F_0 \gg k/k_F$, even for large momenta $k \sim 1/l$ ($l$ is the mean free path). As usual, in order to have a non-trivial solution $\bar{\sigma} \neq 0$, one should have $1 + F_0 < 0$, which is again the Stoner criterion. In the ferromagnetic phase the solution to Eq. (6) gives the total value of the induced spin density $\bar{\sigma}^2 = -(1 + F_0)/B$.

C. Mesoscopic fluctuations

In the presence of disorder the local spin density $\sigma(x)$ depends on the particular impurity distribution and before averaging can be taken as random. In the paramagnetic susceptibility \[^3\] the coupling $U$ is determined by small distances and thus does not depend on disorder. On the contrary, the polarization operator $\Pi(x_1, x_2, \omega)$ includes large distances (since we are interested in the limit $q = 0$) and is thus strongly affected by disorder. Consequently, the paramagnetic susceptibility is random. Moreover, it depends on both coordinates since translational invariance is lost and it is also non-local. However, if disorder is not too strong, then non-local, random part of the susceptibility can be separated from the uniform term, which is independent of disorder and represents the susceptibility of the clean system. Preserving the form of Eq. (6), the susceptibility of the disordered system can be found as a solution to the integral equation

$$\int d^2x_2 \left[ (1 + F_0)\delta(x_1 - x_2) + F_1(x_1, x_2) \right] \chi(x_2, x_3) = \nu \delta(x_2 - x_3),$$ \hspace{1cm} (7)

where $F_1(x_1, x_2)$ is a random quantity with zero mean. Similarly, the thermodynamic potential \[^3\] becomes

$$\Omega = \frac{1}{2\nu} \int d^2x (1 + F_0)\bar{\sigma}^2(x)$$

$$+ \frac{1}{2\nu} \int d^2x_1 d^2x_2 F_1(x_1, x_2) \left( \bar{\sigma}(x_1)\bar{\sigma}(x_2) \right)$$ \hspace{1cm} (8)

$$+ \frac{1}{4\nu} \int \prod_{i=1}^4 d^2x_i B \{x_j\} \left( \bar{\sigma}(x_1)\bar{\sigma}(x_2) \right) \left( \bar{\sigma}(x_3)\bar{\sigma}(x_4) \right).$$

Again, we are interested in the limit $q \to 0$ (neglecting the gradient term; see the previous subsection and an
estimate below). In the second order term we have neglected the possibility of the presence of spin-orbit coupling. While the spin-orbit interaction can be taken into account, its presence does not affect our main results (see Section V for discussion). In addition to the fourth-order term written in Eq. (9) there is a term with different spin structure, namely $[\vec{\sigma}(x_1) \times \vec{\sigma}(x_2)] [\vec{\sigma}(x_3) \times \vec{\sigma}(x_4)]$ (see Appendix for details). In what follows we assume the simplest spin structure for a LSD $\vec{\sigma} = (0, 0, \sigma)$. In this case the additional cross product term vanishes and we can treat the spin density as a scalar.

Similarly to Eq. (9) the minimum of $\Omega$ can be found from the (now non-local) integral equation

$$\begin{align*}
(1+F_0)\sigma(x_1) + \int d^2x_2 F_1(x_1, x_2)\sigma(x_2) \\
+ \prod_{i=2}^4 d^2x_i B[\{x_j\}] \sigma(x_2)\sigma(x_3)\sigma(x_4) = 0. 
\end{align*}$$

(9)

The coefficient $B[\{x_j\}]$ in the thermodynamic potential Eq. (9) is also random. In this paper we take both $F_1$ and $B$ to be Gaussian random matrices, with the distribution, which in compactified notation is given by

$$w[F_1, B] \propto \exp \left[ - \int (F_1; B) \hat{K}^{-1} \left( \begin{array}{c} F_1 \\ B \end{array} \right) \right].$$

(10)

The Gaussian approximation is valid while the expression in the exponent does not exceed the dimensionless conductance $g$, where the log-normal tail appears. The integration is over all the variables of $F_1$ and $B$. The distribution $w[F_1, B]$ should be understood in the operator sense and will be used to evaluate the functional integrals below. The weight operator $\hat{K}$ is constructed from the correlators

$$\hat{K} = \left( \begin{array}{cc} \langle F_1 F_1 \rangle \\ \langle B F_1 \rangle \end{array} \right) \left( \begin{array}{cc} \langle F_1 B \rangle \\ \langle B B \rangle \end{array} \right),$$

(11)

which are discussed in detail in Appendix, where we give their explicit form.

For a given realization of disorder, the equation (9) might allow for some non-trivial solution $\sigma^{(0)}(x)$. To estimate the value of the total spin corresponding to such a solution, we write the spin density as

$$\sigma^{(0)}(x) = \sigma_0 \psi(x),$$

(12)

where $\psi(x)$ is normalized to unity, therefore both $\psi(x)$ and $\sigma_0$ have dimension of inverse length. The total value of the spin is determined by

$$S = \sigma_0 \int d^2 x \psi(x),$$

(13)

while $\sigma_0$ can be found from Eq. (9) (in the case when it allows a non-trivial solution) in the form

$$\sigma_0^2 = -\frac{1 + F_0 + F_1^{(0)}}{B^{(0)}}.$$  

(14)

Here the constants $F_1^{(0)}$ and $B^{(0)}$ are the “matrix elements” of the non-local operators $F_1(x_1, x_2)$ and $B[\{x_j\}]$ (if $\psi(x)$ is interpreted as a “wave function”)

$$F_1^{(0)} = \int d^2x_1 d^2x_2 F_1(x_1, x_2)\psi(x_1)\psi(x_2)$$

(15a)

$$B^{(0)} = \int \prod_{i=1}^4 d^2x_i B[\{x_j\}] \prod_{k=1}^4 \psi(x_k).$$

(15b)

Consider now a metal close to the Stoner instability, so that $0 < 1 + F_0 \ll 1$. While, as follows from Eq. (9), the averaged, uniform spin density is zero, solutions (14) to the non-local equation (9) might exhibit non-zero spin density in some rare regions, where due to a particular configuration of impurities the non-local part of the susceptibility $F_1$ is negative. If there are several regions with non-zero total spin, then the spin-spin or magnetic interaction between them will contribute to the ground state energy of the system and correspondingly to the magnetic susceptibility. If such interaction favors some kind of ordering of the spins, then the appearance of these regions can change the magnetic response of the system, in other words change the ground state.

Fluctuation effects in systems close to a phase transition have been studied extensively (see, for instance, Ref. [14–18]). In particular, the picture of smearing the transition point by formation of fluctuation regions with non-zero value of an order parameter was considered in Ref. [14]. In order to determine a value of the order parameter in a fluctuation region a solution of the non-linear Ginsburg-Landau equation in the presence of disorder was needed. Two issues make this case different from ours. First, in Ref. [14] the fourth-order term $B$ was not random. Second, the fluctuations of the order parameter were assumed to be local. Therefore, the white noise approximation (i.e. approximating the correlation functions by delta-functions) for the disorder was appropriate. As a result, the fluctuation regions did not interact and the percolation scenario of the phase transition was needed. In our case the non-locality of fluctuations (expressed in terms of the correlation functions beyond the delta-functions) leads to interaction between LSDs which results in a change of behavior of observable quantities as discussed in Section VI.

### III. Qualitative Discussion

In the previous subsection we indicated how LSDs - local regions with non zero spin polarization - could appear in a metal close to the Stoner instability due to fluctuations in impurity distribution. Here we estimate
qualitatively the probability to find a LSD and the value of its total spin.

Treating a LSD as an open region of the size $R$ we can characterize it by the Thouless energy $E_T = D/R^2$ (where $D$ is the diffusion constant). The inverse of the Thouless energy is the “escape time” $\tau_{esc} = E_T^{-1}$, which is the time it takes for the diffusing particle to leave the LSD. This time scale serves as the infrared cut-off for the correlation function, which describes the mesoscopic fluctuations of the density of states (DoS)

\[
\langle \rho(\epsilon) \rho(\epsilon + \omega) \rangle \approx \text{Re} \int \frac{R^2 d^2 Q}{(-i\omega + DQ^2 + \tau_{esc})^2} = \frac{\pi}{E_T} \text{Re} \frac{1}{-i\omega + \tau_{esc}}. \tag{16}
\]

The magnetic energy of the LSD, written in terms of its total spin $S$ is

\[
E(S) = \delta_1 (1 + F_0) S^2 + \delta E(S), \tag{17}
\]

where $\delta_1$ is the mean level spacing and $\delta E(S)$ denotes contribution of all non-linear terms in Eq. (9). All the terms contributing to $\delta E(S)$ are random and can be expressed in terms of the fluctuating (random) DoS

\[
\delta E(S) = \int ds_1 \int ds_2 \rho(s_2) + \rho(-s_2). \tag{18}
\]

The averaged $\delta E(S)$ equal to zero, but the average of its square $\langle (\delta E(S))^2 \rangle$ is not and it is determined by the correlator

\[
\langle (\delta E(S))^2 \rangle \simeq \begin{cases} \frac{\delta_1^2 S^4}{g}, & S \ll g; \\ \frac{\delta_2^2 S^4}{g}, & S \gg g. \end{cases} \tag{19}
\]

In this paper we restrict ourselves to consideration of metals close to the instability point, where the overall spin of the LSD is small $S \ll g$. In this case we can treat $\delta E(S)$ as a Gaussian random quantity. Moreover, we can expand it in powers of $S/g$ so that the magnetic energy of the LSD becomes

\[
E(S) = \delta_1 (1 + F_0) S^2 + \zeta \left[ \frac{S^2}{g} - \frac{S^4}{g^3} \right], \tag{20}
\]

where $\zeta$ is a random Gaussian variable with the distribution

\[
P(\zeta) \propto \exp \left(-\zeta^2\right). \tag{21}
\]

This equation allows for non-trivial solutions $S > 0$ when $\zeta < -g(1 + F_0)$. As was noted above, we are working in the regime where $g(1 + F_0) \gg 1$, so that the fluctuation that creates the LSD is rare indeed. To determine the distribution of spin values, we solve the equation (22) for $\zeta$ and substitute in the Gaussian distribution Eq. (21). As a result, we estimate the distribution (up to numerical coefficients)

\[
P(S) = \exp \left( \frac{g^2(1 + F_0)^2}{(1 - \frac{S^2}{g^2})^2} \right). \tag{23}
\]

This distribution is only valid when $g(1 + F_0) \gg 1$ and $S \ll g$, therefore we can expand the denominator in the exponent without loss of accuracy

\[
P(S) = \exp \left[-g^2(1 + F_0)^2 - (1 + F_0)^2 S^2 \right]. \tag{24}
\]

Integrating over the spin $S$ we estimate the probability to find a LSD

\[
P \propto \exp \left(-g^2(1 + F_0)^2 \right). \tag{25}
\]

It is determined by the first term in the exponent Eq. (24). The second term determines the typical value of the spin of the LSD

\[
S \simeq \frac{1}{1 + F_0}. \tag{26}
\]

Remarkably, this value and the probability Eq. (25) do not depend on the size $R$ of the LSD, which is the main qualitative result of this Section. Note that it is similar to the result for zero-dimensional grains. Also the spin value is independent of the dimensionless conductance $g$. These facts determine the contribution of LSDs to the physical observables considered below in Section IV.

The applicability of the consideration of this Section and quantitative results of Section IV is limited by two requirements. First, the probability Eq. (24) must be exponentially small, so that $g(1 + F_0) \gg 1$, second, the Gaussian approximation Eq. (21) [similarly to the distribution Eq. (44)] is valid while the expression in the exponent is smaller than the dimensionless conductance of the system, so that in Eq. (23) $g^2(1 + F_0)^2 < g$. Combining the two limits, we obtain the region of applicability of the results Eqs. (23) - (26) as

\[
1/g \ll 1 + F_0 < 1/\sqrt{g}. \tag{27}
\]
IV. LOCAL SPIN DROPLETS

In this section we calculate the probability to find a local region with non zero spin, which we call a local spin droplet (LSD) and the value of the total spin of the LSD. The spin and the spatial profile of the LSD can be found from the non-linear equation (1). In subsection IV.A we show that to the exponential accuracy the probability to find the LSD Eq. (25) is captured by the linear part of Eq. (1), while the non-linear term fixes the spin value, as shown in subsection IV.B.

A. Probability to form a LSD

The calculation of the probability to find a rare fluctuation leading to formation of a LSD can be performed along the lines of the argument used in Ref. [1] to calculate the exponentially small tail in the density of states (DoS) of a particle in a random potential [4]. In the quantum mechanical problem, considered in Ref. [1], one looks for such fluctuation of the random potential that creates a low energy bound state, thus leading to non zero DoS at that energy. The probability to form the bound state is determined by the distribution of the matrix elements of the random potential. While being exponentially small, the probability should be maximized by choosing the “optimal” fluctuation of the potential.

To gain some intuition about how the optimal fluctuation method can be applied to the problem at hand, in this section we consider the linear part of the equation (1), disregarding for a moment the higher order $B$ term. Such an approach can be justified by observing that close to the instability the non-linear term in Eq. (1) is small compared to the linear ones, since the induced spin density on average is equal to zero. The non-linear term stabilizes a non-trivial solution and fixes its amplitude, while the existence of such a solution can be uncovered at the level of the linear problem. Thus the linear equation captures the main contribution to the probability and at the same time demonstrates the similarity of our problem to the problem of tails in the DoS as well as the peculiar differences.

We can write the linear equation in the operator form

$$\delta F_1 \psi(x) = E \psi(x).$$

We write $\psi(x)$ instead of $\sigma(x)$ to stress the point that the linear equation does not allow us to determine the value of the spin but only the spatial profile of the LSD. Therefore, Eq. (28) is simply the eigenproblem for the operator $F_1$ and as such does not fix the normalization of eigenfunctions $\psi(x)$, which we are free to normalize to unity for convenience. Since the eigenvalue problem Eq. (28) is similar to the quantum mechanical problem of Ref. [1], we can adopt the language of the Schrödinger equation, with the (now integral) operator $F_1$ playing the role of the “Hamiltonian”, $E$ the “energy” and $\psi(x)$ the wave-function.

For some particular realizations of the random potential in Eq. (25) there is a low energy bound state with the energy $E_0\{\{F_1\}\}$ [given by Eq. (15a)], resulting in non zero DoS at this energy. For energies close to $E_0\{\{F_1\}\}$ only the bound state contributes to the DoS, which before averaging over disorder is given by the single delta-function

$$\rho(E) = \delta(E - E_0\{\{F_1\}\}).$$

Averaging this DoS over disorder takes into account contributions of all possible realizations of the random potential leading to such bound states and thus results in an exponentially small but smooth function of energy $E$. This function is proportional to the probability to find the bound state at energy $E$. In particular, for the special value $E = -(1 + F_0)$, it would give the probability to find the non-trivial solution to the linear part of Eq. (1) (or to find the LSD).

The random quantity in the linear problem Eq. (28) is $F_1$ itself. Its distribution is obtained from Eq. (1). The averaged probability is then

$$\langle \rho(E) \rangle = \int \mathcal{D}[F_1] \delta(E - E_0\{\{F_1\}\}) \times \exp \left[ - \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 A \right],$$

where $K_{FF}^{-1} \{\{x_j\}, \{y_j\}\} F_1(y_1, y_2)$, i.e.

$$\int d^2y_1 d^2y_2 K_{FF}^{-1} \{\{x_j\}, \{y_j\}\} K_{FF} \{\{y_j\}, \{z_j\}\} = \delta(x_1 - z_1)\delta(x_2 - z_2).$$

In the “optimal fluctuation” approach one has to evaluate the integral Eq. (28) in the saddle point approximation. To find the saddle point one has to minimize the exponent $A$ of the Gaussian probability with respect to all functions $F_1(x_1, x_2)$ subjected to the condition

$$E = E_0\{\{F_1\}\},$$

represented by the delta-function in Eq. (31). This involves solving the equations

$$\frac{\delta}{\delta F_1(x_1, x_2)} [A + \lambda E_0\{\{F_1\}\}] = 0,$$

where $\lambda$ is the Lagrange multiplier to be found from the condition Eq. (31).

The saddle point solution for $F_1$ represents the optimal fluctuation of the random potential,
\[
\tilde{F}_1(x_1, x_2) = \\
\lambda \int d^2z_1 d^2z_2 \psi_{KFF} \{x_j, \{z_j\} \} \psi(z_1) \psi(z_2), 
\] (32)
given in terms of the eigenfunction \(\psi(x)\) corresponding to the eigenstate \(E_0\). Substituting Eq. (12) into Eq. (28) we obtain

\[
E \psi(x_1) = \lambda \int d^2x_2 d^2z_1 d^2z_2 \psi(x_2) \times \psi(z_1) \psi(z_2), 
\] (33)
which, together with the normalization condition \(\int d^2x |\psi(x)|^2 = 1\), constitute the analogue of the non-linear Schrödinger equation of Ref. 1.

Both the saddle point value of \(\lambda\) and \(\psi(x)\) should be found from the non-linear equation (33) and the normalization condition. The averaged DoS is given by the Gaussian probability in Eq. (30), evaluated at the saddle point,

\[
\langle \rho(E) \rangle \sim \exp \left( -\frac{E^2}{2I_{FF}} \right), 
\] (34)
where

\[
I_{FF} = \int d^2z_1 d^2z_2 d^2x_1 d^2x_2 \times \psi(x_1) \psi(x_2) \psi(z_1) \psi(z_2). 
\] (35)
The integral \(I_{FF}\) is dimensionless (independent of any length scale) since we require the eigenfunction \(\psi(x)\) to be normalized, so that \(\psi(x) \sim 1/R\), where \(R\) is roughly the size of the LSD. As we are discussing the single LSD, all four of the eigenfunctions in Eq. (35) are centered around approximately the same point, therefore \(R\) is the only scale in Eq. (35). In this case the dependence of the correlator \(K_{FF}\) on \(R\) is given by Eq. (12), \(K_{FF} \propto g^{-2}R^{-4}\). Thus the integral Eq. (35) and the probability Eq. (34) are independent of the size of the LSD \(R\).

The probability to find the LSD is given by Eq. (34), evaluated at the point \(E = -1 + F_0\),

\[
\rho \sim e^{-\gamma g \psi^2(1+F_0)^2}, 
\] (36)
where \(\gamma\) is the numerical factor which is given by the dimensionless counterpart of the integral Eq. (35). This result is valid while the number in the exponent is large, which corresponds to the lower limit of applicability of our consideration \(1 + F_0 \gg 1/g\) (see the last paragraph of Section 1).

To determine the numerical coefficient \(\gamma\) in Eq. (36) we need to know the precise form of the eigenfunction \(\psi(x)\). We could not solve the non-linear integral equation (33) analytically and used a variational approach. Since the kernel \(K_{FF}\) in the integral Eq. (35) decays as \(R^{-4}\) at large distances, the optimal \(\psi(x)\) is a limited-range function (i.e. its normalization integral is determined on a limited interval of \(x\)). To estimate the upper limit of \(\gamma\) we take the variational function of the Gaussian form \(\psi(x) = \pi^{-1/2} R^{-1} \exp(-x^2/R^2)\), substitute in Eq. (33) and evaluate the integral numerically. The resulting estimate is

\[
I_{FF} \approx \frac{1}{0.8\pi^2 g^2}, 
\] (37)
and the numerical factor in Eq. (36) is thus \(\gamma \approx 3.9\).

The most important feature of the result Eq. (36) is its independence on the size \(R\) of the LSD in agreement with the qualitative results of Section 11. This means that LSDs of small and large sizes can appear with equal (in the exponential sense) probability, given the suitable fluctuation of the impurity configuration. We shall return to this point below, when we consider the interaction between LSDs.

B. Spin distribution

The argument that led us to the probability to find an LSD Eq. (36) is not complete because it does not help us to determine the value of the spin of the LSD. This follows from our consideration of the linear part of Eq. (4) only. To determine the value of the spin we must solve the full non-linear equation.

The single-mode approximation to the solution of the non-linear problem was outlined in subsection 11C. The formal solution for the amplitude of the spin of a LSD is given by Eq. (14). Again, as we did above in the case of the linear problem, we employ the optimal fluctuation method to find the probability to form a LSD, which is characterized by the total spin \(S\). However, this time our task is simplified since the probability to form the LSD (regardless of its spin) has already been found. We now need to find how that probability depends on the spin value \(S\). Therefore we take the function \(\psi(x)\), which describes the spatial profile of the LSD from the linear problem and focus on the distribution of the spin amplitude \(\sigma\).

Similar to Eq. (23), the probability to find an LSD characterized by the spin density amplitude \(\sigma\) is given by the delta-function

\[
\rho(\sigma^2) = \delta(\sigma^2 - \sigma_0^2|F_1, B|) 
\] (38)
which we write in terms of \(\sigma^2\) for convenience.

Averaging over disorder is performed as it was done for the linear problem [see Eq. (83)]. Only now we have two random quantities, \(F_1\) and \(B\), therefore we need to average with the distribution Eq. (10)

\[
\langle \rho(\sigma^2) \rangle = \int D[F_1, B] \delta(\sigma^2 - \sigma_0^2) |w| [F_1, B], 
\] (39)
To find the saddle point (or the optimal fluctuation) we have to minimize the exponent

\[ A_{nl} = \lambda \left( \sigma^2 + \frac{1 + F_0 + F_1(0)}{B(0)} \right) - \int \left( \frac{F_1}{B} \right) \hat{K}^{-1} \left( \frac{F_1}{B} \right) \]  

(40)

where \( F_1(0) \) and \( B(0) \) are the integrals Eq. (15) and \( \lambda \) is again the Lagrange multiplier. The saddle point equations are given by

\[ \hat{K}^{-1} \left( \frac{F_1}{B} \right) = \frac{\lambda}{2B(0)} \left( \frac{\psi(x_1)\psi(x_2)}{\sigma^2\psi(y_1)\psi(y_2)\psi(y_4)} \right), \]  

(41)

where the functions \( \psi(x) \) appeared after differentiating the integrals Eq. (15) with respect to \( F_1 \) and \( B \). Multiplying both the left-hand and the right-hand sides of Eq. (41) by \( \hat{K} \) we obtain two equations with \( F_1 \) and \( B \) in the left-hand side only. We then multiply the first equation by two functions \( \psi(x) \) and the second by four functions \( \psi(x) \) and integrate over their variables to obtain the algebraic equations, with integrals Eq. (15) as the unknowns

\[ F_1(0) = \frac{\lambda}{2B(0)} \left( I_{FF} + \sigma^2 I_{FB} \right), \]  

(42a)

\[ B(0) = \frac{\lambda}{2B(0)} \left( I_{FB} + \sigma^2 I_{BB} \right), \]  

(42b)

where \( I_{FF} \) is given by Eq. (22) and \( I_{FB} \) and \( I_{BB} \) are similarly defined as

\[ I_{FB} = \int d^2x_1 d^2x_2 \prod_{i=1}^4 d^2y_i \psi(x_1)\psi(x_2) \times K_{FB} \{ \{ x \} \}, \]  

\[ I_{BB} = \int \prod_{i=1}^4 d^2x_i d^2y_i \prod_{k=1}^4 \psi(x_k) \times K_{BB} \{ \{ x \} \}, \]  

(43a)

Equations (42) can be easily solved and we have for the optimal fluctuation

\[ \bar{F}_1(0) = \sqrt{\frac{I_{FF} + \sigma^2 I_{FB}}{2(I_{FB} + \sigma^2 I_{BB})}}, \]  

(44a)

\[ \bar{B}(0) = \frac{1}{2} \sqrt{\frac{\lambda}{2(I_{FB} + \sigma^2 I_{BB})}}. \]  

(44b)

Now we only need to find \( \lambda \) from the constraint \( \sigma^2 = -(1 + F_0 + F_1(0))/B(0) \) or, equivalently, to find the saddle point solution for \( \lambda \). Substituting the saddle point solutions (44) into the constraint, we find

\[ \sqrt{\lambda} = -(1 + F_0) \frac{\sqrt{2(I_{FB} + \sigma^2 I_{BB})}}{I_{FF} + 2\sigma^2 I_{FB} + \sigma^4 I_{BB}}. \]  

(45)

The exponent \( A_{nl} \) at the saddle point Eq. (44) is given by

\[ A_{nl}(\lambda) = \frac{\lambda}{2B(0)} \left[ \sigma^2 B(0) + \bar{F}_1(0) + 2(1 + F_0) \right]. \]  

(46)

Finally, evaluating the exponent (46) for the optimal value of \( \lambda \) Eq. (44), we obtain the probability to find an LSD with the spin density amplitude \( \sigma \)

\[ \rho(\sigma^2) \sim \exp \left( -\frac{(1 + F_0)^2}{2(I_{FB} + 2\sigma^2 I_{FB} + \sigma^4 I_{BB})} \right). \]  

(47)

which for \( \sigma = 0 \) coincides with the result Eq. (36) of the linear problem, just as the result Eq. (24) of the qualitative argument above.

The final step is to rewrite the distribution Eq. (47) in terms of the spin \( S \) of the LSD. Converting the spin density amplitude \( \sigma \) in Eq. (47) to the spin \( S \) by means of Eq. (13), we obtain the final expression for the spin distribution

\[ \rho(S^2) \sim \exp \left( -\frac{\gamma g^2(1 + F_0)^2}{1 - 2\alpha \frac{S^2}{g^2} + \beta \frac{S^2}{g^2}} \right), \]  

(48)

where the numerical factors are

\[ \alpha = \frac{g^2 I_{FB}}{I_{FF} \left[ \int d^2x \psi(x) \right]^2} \approx 1.92, \]  

(49a)

\[ \beta = \frac{g^4 I_{BB}}{I_{FF} \left[ \int d^2x \psi(x) \right]^4} \approx 7.36, \]  

(49b)

\[ \gamma = \frac{1}{2g^2 I_{FF}} \approx 3.9. \]  

(49c)

The factor \( \gamma \) is the same as in the linear problem and is listed here for completeness.

The factors Eq. (49) are now independent of the size \( R \) of the LSD. To see that, one needs to notice that all the correlators \( \hat{K} \) depend on this scale in the same way (when a single LSD is considered, so that \( R \) is the only scale in the integral) \( \hat{K} \sim R^{-4} \) [see Eq. (A16)]. The wave function \( \psi(x) \) is inverse in \( R \). Therefore, the integrals Eq. (43), unlike the integral Eq. (33), which appears in
the solution to the linear problem, do depend on $R$, since they contain different number of functions $\psi(x)$

$$I_{FB} \propto -\frac{R^2}{g^4}, \quad (50a)$$

$$I_{BB} \propto \frac{R^4}{g^6}. \quad (50b)$$

The different size dependence of the integrals Eq. (35) and $I_{FB}$ is compensated in Eq. (49) by additional factors of $\int d^2x \psi(x)$.

The distribution Eq. (48) is thus the same as the qualitative result Eq. (23) only now with coefficients Eq. (49). The coefficients were evaluated numerically using the solution $\psi(x)$, which follows from the linear problem. The coefficients are positive [the negative sign of $I_{FB}$ being taken into account explicitly in Eq. (15)]. The coefficient $\beta$ is positive, ensuring convergence of the expansion Eq. (23), which again is the way to interpret the distribution Eq. (48). In the calculation leading to Eq. (48) the limitation to small spins follows from the separation procedure Eq. (12) since it is valid only close to the instability, where typical spins are small.

The typical value of spin of the LSD is still given by Eq. (26). The spin $S$ turns out to be large, $1 \ll S < g$, so on length scales larger than $R$ or at high enough temperature, LSDs behave as classical moments. However, LSDs are extended objects and can have any size with equal probability (with the exponential accuracy). Thus their spins can not be considered as local moments, especially when discussing their interactions.

V. INTERACTIONS BETWEEN LSDS

The importance of LSDs is that their appearance can dramatically change the magnetic response of the system. At high enough temperatures, we can consider them as independent, classical moments, thus we expect the system to be the usual paramagnet with the susceptibility described by the Curie law (but with Curie constant different form that of the free electron gas). As the temperature becomes smaller, the system might change its ground state in a way that depends on the interaction between LSDs, in particular on its sign and typical range.

Let us first recall the basic physics of local moments in a metallic system. In a system of local moments there are two competing types of interaction. First, there is the direct contact interaction, which sign does not change with distance between the moments, but the amplitude decays exponentially beyond the correlation length. This interaction tends to turn the system into a ferromagnet. Then there is the RKKY interaction, which as a function of distance oscillates and decays only as a power law ($\sim R^{-2}$ in 2D). The RKKY interaction between local moments tends to form a spin glass at low temperatures.

Our case is different. As we have shown in this paper, in a metal close to Stoner instability there is a non zero probability for LSDs to be spontaneously formed. This probability is independent of the size of LSDs, so that droplets of all sizes can appear. Therefore interaction between the LSDs can not be described in the same way as interaction between local moments. Rather, it is given by the non-local susceptibility $F_1(x_1,x_2)$, averaged over the area of the interacting LSDs. Calculation of the average, previously denoted as $F_1^{(0)}$, involves integration of $F_1(x_1,x_2)$ with two functions $\psi(x-a)$ which describe the spatial profile of the LSDs. In this section we use the same Gaussian functions we used to evaluate the integrals Eq. (35) and Eq. (49). Only now these functions carry explicitly the dependence on the coordinate $a$ of the center of the LSD.

Consider now two LSDs separated by distance $L$ much larger than the size of both LSDs $L \gg R$, see Fig. 1. In this argument we take both LSDs to be of the same size $R$, but it can be easily generalized for the case where the interacting LSDs differ in size substantially. Choosing different combinations of the wavefunctions $\psi(x-a)$ and $\psi(y-b)$ describing the two LSDs, we can form three different averages

$$F_1^{(1)} = \int d^2x_1 d^2x_2 F_1(x_1,x_2)\psi(x_1-a)\psi(x_2-a), \quad (51a)$$

$$F_1^{(2)} = \int d^2y_1 d^2y_2 F_1(y_1,y_2)\psi(y_1-b)\psi(y_2-b), \quad (51b)$$

FIG. 1. Two LSDs located around points $a$ and $b$ and the three averages of the non-local susceptibility $F_1$. The two averages $F_1^{(1)}$ and $F_1^{(2)}$ describe correlations within each single LSD, while $F_1^{(12)}$ describes correlations between different LSDs and thus determines the interaction
The integrals Eq. (54) were calculated in the leading order in $R/L$. Clearly, $J_2 \neq J_3$ exactly, but the difference comes in the numerical factor under the logarithm, which we here neglect. This does not have any bearing on our conclusions. The $1/L^4$ dependence of all the integrals follows from the frequency integral in Eq. (A11), which is determined by the Thouless energy corresponding to the largest length in the problem, which is now $L$.

The integral $I_{FF}$ is independent of all length scales and therefore is much larger than any of $J_i$. Thus the weight matrix $\hat{L}^{-1}$ in Eq. (52) to the leading order in $R/L$ is

$$\hat{L}^{-1} \approx 0.8\pi^2 g^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & f^{-1} \end{pmatrix},$$

where the dimensionless function $f(R/L)$ is given by

$$f(R/L) \approx 0.72 \frac{R^4}{L^4} \ln^2 \frac{R}{L}.$$  

To describe the interaction between LSDs we need to find the distribution of $F_{1(12)}^{(12)}$ under the condition that the two LSDs exist, namely that $F_{1(1)}^{(1)} < 0$ and $F_{1(1)}^{(2)} < 0$. This is given by the conditional probability distribution

$$W[F_{1(12)}] = \frac{1}{N^2} \int D[F_{1(1)}^{(1)}, F_{1(1)}^{(2)} \theta(-1 - F_0 - F_{1(1)}) \times \theta(-1 - F_0 - F_{1(1)}) w[F_{1(1)}^{(1)}, F_{1(1)}^{(2)}, F_{1(12)}^{(12)}],$$

$$N = \int D[F_{1(1)}^{(1)}] e^{-\frac{(F_{1(1)}^{(1)})^2}{2}} \theta(-1 - F_0 - F_{1(1)}),$$

where $w$ is the distribution Eq. (52). Since the weight matrix in Eq. (52) is a c-number, the integrals in Eq. (57) are usual Gaussian integrals and not functional integrals. The $\theta$-functions in Eq. (57) make the exact integration in terms of elementary functions impossible, but we can use the small parameter $f(R/L) \ll 1$ to estimate $W[F_{1(12)}]$ with exponential accuracy, which is all we need to describe interaction between LSDs. Up to the pre-exponential factor

$$W[F_{1(12)}] \propto \exp \left[ -\frac{0.8\pi^2 g^2}{f} \left( F_{1(12)}^{(12)} + 2(1 + F_0) f \right)^2 \right],$$

so that the distribution is a Gaussian (as it should be since we considered $F_1$ to be a Gaussian random quantity from the very beginning).

The average $F_{1(12)}^{(12)}$ given by the distribution Eq. (58) is shifted from zero to the negative value $-2f(R/L)(1 + F_0)$, i.e. the average interaction appears to be ferromagnetic.
However, the distribution Eq. [58] also allows for strong fluctuations of $F_1^{(12)}$. These fluctuations can be estimated as
\[
\frac{\langle (\delta F_1^{(12)})^2 \rangle}{\langle F_1^{(12)} \rangle^2} \sim \frac{1}{g^2(1+F_0)^2} f.
\]
(59)

The function $f$ [see Eq. (56)] decreases with the distance between LSDs. Therefore at large enough distances the fluctuations of $F_1^{(12)}$ exceed the average and the sign of $F_1^{(12)}$ becomes random.

The cross-over distance $L^\ast$ can be estimated as (from the condition $\langle (\delta F_1^{(12)})^2 \rangle/\langle F_1^{(12)} \rangle^2 \sim 1$)
\[
L^\ast \sim R \sqrt{g(1+F_0)} \gg R.
\]
(60)

This distance has to be compared with the typical distance between LSDs. The latter can be estimated as follows. The concentration of LSDs has to be proportional to the probability Eq. (36)
\[
n \propto \frac{1}{R^2} e^{-\gamma g^2(1+F_0)^2},
\]
(61)

where $R$ is the characteristic length of a LSD. Therefore the typical distance between LSDs is exponentially large in the parameter $g(1+F_0) \gg 1$, while the cross-over length $L^\ast$ is large only as a power of the same parameter. Thus, the interaction between typical LSDs has random sign.

The energy of interaction between typical LSDs decays as the second power of the distance between them
\[
U_k = \frac{1}{\nu g L^2} \sim \frac{1}{\nu g R^2} e^{-\gamma g^2(1+F_0)^2},
\]
(62)

where $\nu$ is the density of states and we have estimated the typical distance between LSDs from Eq. (31).

The results of this Section resemble the results of Ref. 17, where the electron-mediated interaction between magnetic moments in two ferromagnets separated by a disordered metal was also found to have a random sign. The difference between our model and that of Ref. 17 is that in our case the ferromagnetic regions (LSDs) are created by the very same impurity configurations as those responsible for the interaction between LSDs. Therefore, the interaction between LSDs can be considered only under the condition of their existence and thus is characterized by the conditional probability distribution Eq. (58).

As a result, the average $L_1^{(12)}$ is negative, i.e. ferromagnetic, and dominates the fluctuations at distances smaller than $L^\ast$. However, since the typical distance between LSDs is exponentially large, the typical interaction has random sign.

This fact is not surprising, since we are dealing with a disordered system where one could expect to find some spin glass phase (at $T = 0$, similar to the case of a superconductor in weak magnetic field, considered in Ref. 18). In the model considered in Ref. 18 mesoscopic fluctuations become uncorrelated beyond the magnetic length. In our case the role similar to that of the magnetic field in Ref. 18 would have been played by the spin-orbit coupling, which we have so far neglected. However, taking the spin-orbit coupling into account does not change the main results of this paper. As we have shown above, the typical interaction between LSDs is random due to large fluctuations in Eq. (58). Should we include the spin-orbit coupling, we would need to compare the spin-orbit length to $L^\ast$ in order to determine the length beyond which the interaction becomes random. But since the typical distance between LSDs is exponentially larger than $L^\ast$ the exact value of such cross-over length is not very important.

VI. CONTRIBUTION TO PHYSICAL QUANTITIES

In this section we estimate the contribution of the LSDs to observable physical quantities. We consider the examples of paramagnetic susceptibility and the dephasing time. To be observable, the contribution of LSDs should exceed the regular contribution of the electron system. We show that, for both quantities, it happens at the same temperature $T^\ast$, below which the temperature dependence of both quantities changes. The Pauli susceptibility crosses over to a Curie-type $1/T$ dependence, while the dephasing time saturates and becomes temperature independent.

Since the typical interaction energy Eq. (12) is exponentially small, LSDs behave as weakly interacting moments and their contribution to the paramagnetic susceptibility is given by the Curie law $\chi_{LSD} = C/T$, where the Curie constant $C$ is proportional to the square of the spin $S$ of the LSD [see Eq. (29)] and the density (or concentration) of LSDs Eq. (61). We have calculated the probability Eq. (58) with the exponential accuracy, therefore the pre-exponential factor in the density of LSDs should be given by the typical length characterizing the distribution of LSDs. This dependence could be elucidated from the following dimensional argument. The probability to find an LSD in unit volume (or, rather, area, which we still denote by $V$) of size in between $R$ and $R + dR$, with spin in between $S$ and $S + dS$ is given by the distribution
\[
dW = \rho(S^2)dVdSd^2R
\]
(63)

Clearly, the dimensionality of $\rho(S^2)$ is $L^{-4}$, where $L$ is the characteristic length. Since the exponential in $\rho(S^2)$ [see Eq. (18)] is independent of length, the size of a LSD can not be pinned to any length in the system. Since individual LSDs contribute to Curie susceptibility independently, we need to sum over all possible sizes and given the $L^{-4}$ dependence, the main contribution comes from LSDs of the smallest possible size, namely the mean-free path $l$. Thus, contribution to the susceptibility (up to a numerical coefficient) is
where $\tau$ is the scattering time. Here we used the typical value $13$ of the spin of a LSD. This contribution has to be compared with the Pauli susceptibility Eq. (1), which is temperature independent. As a result, the contribution Eq. (64) dominates at temperatures smaller than $T^*$, which up to numerical factors is

$$
T^* \sim \frac{1}{\tau g(1 + F_0)} e^{-\gamma g^2(1 + F_0)^2}.
$$

This temperature has to be compared with the typical interaction energy Eq. (62). Substituting the mean-free path $l$ for the characteristic size $R$ in Eq. (62) we find the ratio $U_t/T^*$ to be

$$
\frac{U_t}{T^*} \sim \frac{(1 + F_0)^2}{g(1 + F_0)} \ll 1.
$$

Therefore, the Curie behavior Eq. (64) persists over a wide temperature range $U_t < T < T^*$. Similarly, we estimate the contribution of LSDs to the dephasing time $\tau_\varphi$. LSDs are regions where the impurity configuration makes it energetically favorable for the electrons to align their spins. Some other electron entering such region will “feel” the overall polarization as if it were magnetic field. The corresponding dephasing time can be estimated as the square of the Zeeman energy divided by the Thouless energy (at the size of the LSD). More formally, since the interaction with the polarization stems from the exchange interaction, we can find $\tau_\varphi$ from the perturbation theory

$$
\frac{1}{\tau_\varphi} = \frac{F_0^2}{\nu^2} \int \frac{d^2r_1}{V} \int \frac{d^2r_2}{V} \langle \sigma(r_1)\sigma(r_2) \rangle D(r_1 - r_2),
$$

where $D(r_1 - r_2)$ is the (electron) diffusion propagator and $\langle \sigma(r_1)\sigma(r_2) \rangle$ is the correlation function of the spins of LSDs. We can now estimate $\tau_\varphi$ as

$$
\frac{1}{\tau_\varphi} \sim \frac{F_0^2}{\nu^2} \frac{nS^2}{\rho} \sim \frac{F_0^2}{\tau} \frac{1}{g^2(1 + F_0)^2} e^{-\gamma g^2(1 + F_0)^2},
$$

where again the dominant contribution comes from LSDs of smallest size $n$ is the concentration of LSDs, see Eq. (12). This should be compared with the contribution of the usual Gaussian spin fluctuation

$$
\frac{1}{\tau_{\varphi}} \sim \frac{2F_0^2}{(1 + F_0)(2 + F_0)} \frac{T}{g} \ln(g(1 + F_0)).
$$

Again, up to the numerical factors, the contribution of LSDs dominates at temperatures lower than $T^*$ [given by Eq. (65)].

The actual crossover temperatures for different physical quantities might differ by a factor of order $\ln g(1 + F_0)$, but such difference is beyond the accuracy of our treatment. However, the discussed contribution of LSDs to physical quantities suggests that our scenario of the magnetic fluctuations in a metal close to the Stoner instability can be experimentally observed. The LSDs lead to the saturation of the dephasing time at low temperatures. If such saturation is observed, one should look at the behavior of the paramagnetic susceptibility in the same temperature range. If the LSDs are present in the system, then the onset of the Curie-like temperature dependence should also be detected.

VII. CONCLUSIONS

In this paper we have considered the effect of disorder on magnetic properties of the ground state of a metal close to the Stoner instability. We have shown that even though on the mean field level the ground state of the metal is paramagnetic ($1 + F_0 > 0$), there is non zero (exponentially small) probability to form local spin droplets, i.e. domains of non zero spin polarization. The probability to form a LSD is independent of its size $R$, thus LSDs of any size can appear. The total spin of the LSD is also independent of its size and obeys the distribution Eq. (24), with the typical value $S \simeq (1 + F_0)^{-1}$, which is large $1 \ll S \ll g$, so that LSDs can be considered as classical spins.

Considered as independent moments, LSDs contribute to the observables, changing the temperature dependence of both the paramagnetic susceptibility and the dephasing time at temperatures lower than certain cross-over temperature Eq. (65). When $T < T^*$, the dephasing time saturates to the temperature-independent value Eq. (64), while the susceptibility acquires the Curie-like $1/T$ dependence.

Both the Curie susceptibility Eq. (64) and the dephasing time Eq. (62) were obtained in approximation of non-interacting LSDs. This approach is valid at temperatures larger than the typical value $12$ of the interaction between LSDs. Since the cross-over temperature $T^*$ is much larger than the interaction $12$, there is a parametrically wide temperature regime [by our large parameter $g(1 + F_0)$, see Eq. (60)], $U_t < T < T^*$, where the Curie behavior of the susceptibility and the saturation of the dephasing time can be observed. At smaller temperatures $T < U_t$, however, the LSDs can not be considered as non-interacting moments and the behavior of the system changes. Interaction of LSDs with each other or with itinerant electrons should lead either to screening of the local spins or to forming some spin glass state (due to the random sign of the interaction). Such regime was not considered in this paper.

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APPENDIX:

Here we discuss the correlation functions that describe the mesoscopic fluctuations in the system, i.e. give the weight in the distribution Eq. (31) of the random quantities \( F_1 \) and \( B \). Both quantities are the coefficients in the expansion of the thermodynamic potential Eq. (2) in powers of the spin density. Therefore the diagrams for \( F_1 \) and \( B \) can be obtained by differentiating the diagram for the exact thermodynamic potential. To calculate the correlators one has to multiply the random quantities and then average over the disorder. As a result we get three different correlators, depicted diagrammatically on Figs. 2 - 4.

In terms of exact electronic Green’s functions the thermodynamic potential can be written as

\[
\Omega = \int \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr} \ln \frac{G^R(\epsilon)}{G^A(\epsilon)},
\]

where the Green’s function is defined in the fluctuating field \( \delta \), i.e.

\[
G^{R/A}(\epsilon) = \left[ \epsilon - \hat{H} - \xi \delta \pm i0 \right]^{-1}.
\]

Here \( \hat{H} \) is the Hamiltonian of the system, \( \tau^j \) are the Pauli matrices. The symbol \( \text{Tr} \) denotes the trace over spin indices and the integration over all spatial coordinates. For brevity we do not explicitly indicate that \( G^{R/A}(\epsilon) \) depend on spatial coordinates (due to disorder in \( \hat{H} \)).

The expansion (2) can be achieved by taking variational derivatives of the thermodynamic potential (A1) with respect to \( \delta \). The second variational derivative of the thermodynamic potential corresponds to the second order term in Eq. (5),

\[
\frac{\delta^2 \Omega}{\delta \sigma^\alpha(x_1) \delta \sigma^\beta(x_2)} = \int \frac{d\epsilon}{2\pi i} f(\epsilon) \times \text{Tr}' \left[ G^{R R \tau^\alpha} G^{R R \tau^\beta} - G^{A R \tau^\alpha} G^{A R \tau^\beta} \right].
\]

Here the prime in \( \text{Tr}' \) means that there is no integration over the coordinates in the left-hand side (in this case \( x_1 \) and \( x_2 \)). The trace over the Pauli matrices gives (in the absence of spin-orbit coupling) \( \text{Tr}[\tau^\alpha \tau^\beta] = 2\delta^{\alpha \beta} \). Upon subtracting the average, the derivative (A3) is proportional to the non-local quantity \( F_1(x_1, x_2) \):

\[
F_1(x_1, x_2) = \frac{\delta^2 \Omega}{\delta \sigma^\alpha(x_1) \delta \sigma^\beta(x_2)} - \langle \frac{\delta^2 \Omega}{\delta \sigma^\alpha(x_1) \delta \sigma^\beta(x_2)} \rangle.
\]

Similarly, the coefficient \( B \) in Eq. (3) is given by the fourth variational derivative of the thermodynamic potential (A4) (only the average is zero in this case):

\[ B[x_1] = \int \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr}' \left[ G^{R R \tau^\alpha} G^{R R \tau^\beta} G^{R R \tau^\mu} G^{R R \tau^\nu} - G^{A R \tau^\alpha} G^{A R \tau^\beta} G^{A R \tau^\mu} G^{A R \tau^\nu} \right]. \]  

(A5)

The trace over the Pauli matrices has two parts, \( \text{Tr}[\tau^\alpha \tau^\beta \tau^\mu \tau^\nu] = 2[\delta^{\alpha \beta} \delta^{\mu \nu} - (\delta^{\alpha \mu} \delta^{\beta \nu} - \delta^{\alpha \nu} \delta^{\beta \mu})] \). The first part corresponds to the fourth order term in Eq. (5), while the second part is the cross-product term, which disappears since we consider only the singlet fluctuations \( \delta = (0, 0, \sigma) \).

We are now ready to calculate correlation functions of the mesoscopic fluctuations \( F_1 \) and \( B \). Averaging over disorder is performed in a standard manner (see Ref. [7] for details). The correlator of the fluctuation \( F_1(x_1, x_2) \)

\[
K_{\text{FF}}[\{x_j\}, \{y_j\}] = K_{\text{FF}}(x_1, x_2, y_1, y_2) = \langle F_1(x_1, x_2) F_1(y_1, y_2) \rangle
\]

(A6)

is given by a sum of four diagrams, one of which is depicted on Fig. 2.

FIG. 2. Diagrams for the averaged correlation function of the non-linear part of the paramagnetic susceptibility.

All four diagrams contribute equally to the integral Eq. (33) (as will be clear from the explicit form of \( K_{\text{FF}} \)), therefore we shall proceed evaluating the contribution of the diagrams on Fig. 2. After averaging, each diagram corresponds to a product of four diffusion propagators (or diffusons) and four vertex blocks.

The diffusion in momentum representation is given by

\[
D(\omega; q) = \frac{1}{-i\omega + Dq^2},
\]

where \( D \) is the diffusion constant, and \( q \) is a 2D momentum vector. Each vertex block (see Fig. 4) contains...
precisely the same factors $2\pi\nu\tau^2$ as each diffuson, thus
in momentum representation the diagrams on Fig. 2 are expressed in terms of the integral

$$K_{FF}[\{k_j\}] = \int_{-\infty}^{0} \frac{de_1}{\pi \nu} \int_{-\infty}^{0} \frac{de_2}{\pi \nu} \int d^2Q \ \mathcal{D}(\omega; Q)$$

$$\times \mathcal{D}(\omega; Q-k_1) \mathcal{D}(\omega; Q-k_3) \mathcal{D}(\omega; Q-k_1-k_3)$$

$$\times \delta(k_1+k_4-k_2-k_3), \quad (A7)$$

where $\omega = \epsilon_1 - \epsilon_2$.

For the purposes of this paper it is more convenient to write the correlator Eq. (A6) in the coordinate representation as (we have also evaluated one of the frequency integrals)

$$K_{FF}[\{x_j\}, \{y_j\}] = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi \nu \tau^2} \|\omega\| \mathcal{D}(\omega; y_1 - x_1)$$

$$\times \mathcal{D}(\omega; x_1 - y_2) \mathcal{D}(\omega; x_2 - y_1) \mathcal{D}(\omega; y_2 - x_2), \quad (A8)$$

where $x_i$ and $y_i$ are 2D coordinate vectors and

$$\mathcal{D}(\omega; x) = \int \frac{d^2q}{(2\pi)^2} \mathcal{D}(\omega; q) e^{-iqx}$$

is the Fourier transform of the diffusion denominator to the position space.

The remaining frequency integral can be evaluated by using the following integral representation for $\mathcal{D}(\omega; x)$. First, we represent the diffusion denominator as an integral over an auxiliary variable

$$\frac{1}{-i\omega + Dq^2} = \int_{0}^{\infty} dt \exp \left[ -t(-i\omega + Dq^2) \right].$$

The momentum integral in Eq. (A9) becomes Gaussian and we obtain

$$\mathcal{D}(\omega; x) = \frac{1}{4\pi D} \int_{0}^{\infty} \frac{dt}{t} \exp \left[ i\omega t - \frac{x^2}{4tD} \right].$$

Substituting the integral representation Eq. (A10) into the correlator Eq. (A8), we find the final expression for the contribution of the diagram on Fig. 2 to the correlator Eq. (A6)

$$K_{FF}[\{x_j\}, \{y_j\}] = \frac{1}{(2\pi \nu D)^2} \frac{1}{4\pi^4} \int \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3} \frac{dt_4}{t_4}$$

$$\exp \left[ -\frac{(x_1-y_1)^2}{t_1} - \frac{(x_2-y_2)^2}{t_2} - \frac{(x_1-y_2)^2}{t_3} - \frac{(x_2-y_1)^2}{t_4} \right] \quad (A11)$$

To estimate the size of the fluctuation regions with non zero spin (LSD) we need the explicit dependence of the correlator Eq. (A11) on the parameters of the problem. To do that we introduce a length scale $R$ which characterizes the size if the LSD and write Eq. (A11) as

$$K_{FF}[\{x_j\}, \{y_j\}] = \frac{1}{g^2 R^4} \tilde{K}_{FF}$$

where $\tilde{K}_{FF}$ is the dimensionless counterpart of the integral Eq. (A11).

FIG. 3. Typical diagram for the averaged correlation function of $(F_i B)$. The rest of the diagrams are obtained by shifting positions of the spin vertices on the inside line relative to those on the outside line and interchanging coordinate indices similar to the diagrams on Fig. 2. There are 3 topologically different diagrams.

FIG. 4. Typical diagram for the averaged correlation function of $(BB)$. The rest of the diagrams are obtained by shifting positions of the spin vertices on the inside line relative to those on the outside line and interchanging coordinate indices similar to the diagrams on Fig. 2. There are 8 topologically different diagrams.
The remaining correlators in Eq. (11) are constructed in the same manner as Eq. (A11), the only difference being the number of diffusons. The corresponding diagrams are given in Fig. 3 and 4. Again, the total number of diagrams is large, therefore we give expressions for the typical contributions depicted on Figs. 3 and 4. The rest of the diagrams are obtained by interchanging coordinate indices.

The diagram on Fig. 3 contains six diffusons and six vertex blocks. Since each of the vertex blocks carries a factor of $i$, the overall sign of the diagram is negative. Using the integral representation Eq. (A10) for diffusion denominators, we find

$$
K_{FB}([x_i], [y_j]) = -\frac{1}{2(2\pi)^4} \frac{1}{(2\pi D)^6} \times \int_0^\infty \prod_{k=1}^6 \frac{dt_k \exp[-T_{FB}]}{t_k} \prod_{i=1}^6 (t_i)^2
$$

Similarly to Eq. (A13), the dimensional analysis gives the size dependence in terms of the scale $R$

$$
\tilde{K} = \frac{1}{g^2 R^4} \left( \tilde{K}_{FF} - \frac{1}{g^2} \tilde{K}_{FB} - \frac{1}{g^2} \tilde{K}_{BB} \right). 
$$

The sign of the correlator follows from the diagrams, so that the dimensionless integrals $\tilde{K}$ are positive numbers.

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