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On unique parametrization of the linear group GL(4,C) and its subgroups by using the Dirac matrix algebra basis
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A unifying overview of the ways to parameterize the linear group GL(4,C) and its subgroups is given. As parameters for this group there are taken 16 coefficients \(G = G(A, B, A_k, B_k, F_{kl})\) in resolving matrix \(G \in GL(4,C)\) in terms of 16 basic elements of the Dirac matrix algebra. Alternatively to the use of 16 tensor quantities, the possibility to parameterize the group GL(4,C) with the help of four 4-dimensional complex vectors \((k, m, n, l)\) is investigated. The multiplication rules \(G'G\) are formulated in the form of a bilinear function of two sets of 16 variables. The detailed investigation is restricted to 6-parameter case \(G(A, B, F_{kl})\), which provides us with spinor covering for the complex orthogonal group \(SO(3.1,C)\). The complex Euler’s angles parametrization for the last group is also given. Many different parametrizations of the group based on the curvilinear coordinates for complex extension of the 3-space of constant curvature are discussed. The use of the Newmann-Penrose formalism and applying quaternion techniques in the theory of complex Lorentz group are considered. Connections between Einstein-Mayer study on semi-vectors and Fedorov’s treatment of the Lorentz group theory are stated in detail. Classification of fermions in intrinsic parities is given on the base of the theory of representations for spinor covering of the complex Lorentz group.

Key words: Dirac matrices, orthogonal groups, covering group, discrete transformations, intrinsic parity, fermion, semi-vectors, Newmann-Penrose formalism, Majorana basis.

1 Introduction

Physical applications especially in connection with relativity theory and quantum mechanics extensively employ the real orthogonal groups: \(SO(3.1,R), SO(4,R)\), and \(SO(2.2,R)\). Also 3-rotation groups \(SO(3,R)\) and \(SO(2,1)\) are widely used. Much attention was given to the complex orthogonal groups \(SO(4,C)\), including all previous as sub-groups. Surely, from general mathematical viewpoint, the theory of these orthogonal groups can be considered to have been successfully solved many years ago and one should not expect to obtain new facts. At the same time, everyday employing of these groups in many different contexts requires some elaborate and desirably unified apparatus in parameterizing these groups. In practice, all calculation with those groups involve some parametrization of them. It is the more so when calculation is done with finite transformations of the groups. Special and accurate parametrization for any orthogonal group becomes a point of first significance when turning to the problem of its covering spinor groups.

Impressive progress had been reached in the original treatment of the theory of the Lorentz group given by Fedorov F.I. (1979). Though the author (with co-authors) had given a self-closed theory of the real Lorentz group and extension of the methods used to deal with other orthogonal groups, this treatment stays some distance away from the standard and widely used ways of treatment of the Lorentz group.

In Fedorov approach, an arbitrary finite Lorentz transformation in 4-vector space from the very beginning is constructed as a product of two mutually commuting matrices, each of those is a linear function of a complex 3-vector (parameter). Then it is demonstrated that in this way one can obtain all proper and orthochronous Lorentz transformations in 4-vector space. Six independent parameters are obtained as a result of imposing additional restrictions on two complex vectors, involving complex conjugation. Such approach had been extended to the real groups \(SO(4,R), SO(2,2)\) and complex orthogonal group \(SO(3.1,C) \sim SO(4,C)\). The use of this vector-based parametrization of the Lorentz group has permitted the solving in full details many of particular problems where substantial role is given to relativistic invariance. The use of the fixed parameters associated with relativistic transformations, including arbitrarily oriented Euclidean 3-rotation and any Lorentz boost, enables the possibility of overcoming many technical difficulties and bring calculations to final analytical results in closed mathematical form.

We will now discuss our list of references as they apply to this paper. The groups under consideration are now part of everyday physics practice. So a list of references will be quite long. Since a discussion of the references in relation to each other is a rather daunting task, we have chosen an historical principle and use classifying by date. In the present paper we give only several guideline comments and some indications on the
content of only part of works from the reference list, the most directly referring to subject under consideration.

After establishing the fundamental role of the Lorentz group in physics: Lorentz (1904), Poincaré (1904, 1905), Einstein (1905), and after Minkowski (1908-1909) elaboration of 4-dimensional space-time geometry, the relativistic 4-tensor apparatus had been entered into usage in physics. Because the main relativistic object was the electromagnetic field, the most of the work was given to this field: Silberstein (1907); Minkowski (1908-1909); Sommerfeld (1910).

Usually, invention of spinors as mathematical objects is given to Cartan (1913); however it should be noted that before Cartan other French mathematician Darboux had examined (1900,1004,1905, 1914) particular geometric properties of the objects that in essence represent spinors. These mathematical achievements in spinors had been ignored by physics. Mathematicians became interested in the theory of continuous transformation groups: Weyl (1924); Schreier (1925-1926).

There arises the physical concept of the spin: Uhlenbeck and Goudsmit (1925); Thomas (1926); Frenkel (1926); Pauli (1927). In the context of quantum mechanic, Dirac (1929) had invented a relativistic wave equation for a particle with spin half-one. In this connection the interest in 2-spinors and 4-spinors had been grown much: Möglich (1928); Neuman (1929); Van der Waerden (1929); Laporte and Uhlenbeck (1931); Rumer (1931). Weyl (1931) and Wigner (1931) had written their great books on applying the group theory in quantum physics. In the context of Schrödinger equation, Wigner (1932) had proposed to use the operation of inverting the time which included the complex conjugation. Einstein and Mayer (1932-1933) gave the original mathematical treatment for Lorentz group and its simplest representations, introduced the concept of semi-vector, the 4-dimensional objects closely connected with 2-spinors.

Steady interest to spinors retained: Mic (1933); Veblen (1933); Pauli (1933); Brauer and Weyl (1935); Sommerfeld (1936); Runer (1936). Majorana had written his great paper on neutral 4-spinors (1937). This problem gave rise to a new interest to the role of complex imaginary unit in quantum theory: Dirac (1937); Racah (1937); Kramers (1937); Hettner (1938); Scherzer (1938); Furay (1938). There appeared the reviewing physical and mathematical works: Cartan (1938); Dirac (1939); Brillouin (1938); Wigner (1939); Pauli and Belinfante (1940); Pauli (1941) Einstein and Bargmann (1944,1947) had written two papers on possible use of bivectors in the context of a unified theory.

After evident stopping investigation, the role of relativistic invariance in the quantum theory is again of special interest: Gel'fand and Yaglom (1948); Wigner (1948). Yang and Tjonmo (1950) had written their paper in which firstly the problem of classifying fermions in intrinsic parities had been posed. In the same time there appeared many others works studying the same problem: Zharkov (1950); Wick, Wigner, Wightmann (1952); Pais and Jost (1952); Shapiro (1952,1954). Interest to the Lorentz group theory and discrete symmetries is steady: Bade and Jehle (1953); Lüders (1954); Umezava, Kamelfuchi, Tanaka (1954); Naimark (1954); Rashevski (1955); Good (1955); Case (1955); Pauli (1955); Watanabe (1955); Gel'fand, Minlos, Shapiro (1956); Umezava (1956).

Non-conservation of parity: Lee and Yang (1956,1957); Lee, Oehme, Yang (1957); Landau (1957) gave an impulse for many new works: Golfand (1956); Wigner (1957); Sokolik (1957); Case (1957); Heine (1957); Schrep (1957); Feinberg and Weinberg (1959).

Practically in the same time, and independently from each other, there appeared general surveys on the Lorentz group invariance: Corson (1953); Winogradski (1957-1959); Van Winter (1957); Shirakov (1957-1959); Naimark (1958); Grawerts, Lüders, Rolnik (1959), Halbwachs, Hillion, Vigier (1959); Wightmann (1960), Jost (1960,1962); Fedorov and Bogush (1858, 1961, 1962); Macfarlane (1962); Wigner (1962,1964). We can see several tendencies in the manner of working with the Lorentz group invariance: to analyze the relativistic invariance in the frames of the corresponding Lie algebra (formalism of infinitesimal transformations); to parameterize the finite Lorentz group transformations with the help of: 1) three Euler’s complex angles; 2) bi-vector; 3) a 4-vector on complex sphere; 4) a complex 3-dimensional vector. All these techniques are in use up to present time.
Brumby, Foot, Volkas (1996); Annandan (1998); Erdem (1998); Erber (2004). The same can be noted for many other aspects of the Lorentz group theory; Penrose (1974); Good (1995); Buchbinder, Gitman, Shelepin (2001); much attention is given the Majorana object (for instance see: Hu et and Neuberger (1996,1998); Dvoeglazov (1977); on applying of quaternions see: Stefano, Rodrigues (1998).

Several words must be added in connection with general spinor approach in physics and the idea on spinor structure of space-time: Newman and Penrose (1962); Frolov (1977); Alexeev and Khlebnikov (1978); Penrose and Rindler (1984,1986) - in the last books see comprehensive list of references on the subject.

It is well-known that the space-time vector $x^a = (t, x, y, z)$ can be identified with the explicit realization of the one-valued simplest representation of the (restricted) Lorentz group $L^{↑+}$. But in nature we face particles of integer and half-integer spin, bosons and fermions. Among respective sets of Lorentz group representations, entities describing bosons and fermions, there exists a clear-cut distinction: boson-based representations $T_{bos.}$ are single-valued whereas fermion-based representations $T_{ferm.}$ are double-valued on the group $L^{↑+}$. In other words, $T_{bos.}$ are global representations, whereas $T_{ferm.}$ are just local ones. One might have advanced a number of theoretical arguments to neglect such a slight trouble. However, the fact of prime importance is that this global-local difficulty cannot be cleared up – in the frames of the orthogonal group $L^{↑+} = SO(3,1)$ it is insurmountable. It has long been known that to work against the global-local problem one must investigate and employ one-valued representations (boson-based as well as fermion-based) of the covering group $SL(2,C)$. The group $SL(2,C)$ is other group, different from $L^{↑+} = SO(3,1)$, but it is linked to the latter by a quite definite homomorphic mapping. At this every local representation of the orthogonal group has its counterpart – global representation of the covering group: evidently, it is a quite usual procedure.

In essence, a strong form of this changing $L^{↑+} = SO(3,1) \implies SL(2,C)$, when instead of the orthogonal Lorentz group $L^{↑+} = SO(3,1)$ we are going to employ the covering group $SL(2,C)$ and its representations throughout, and in addition we are going to work in the same manner at describing the space-time structure itself, is the known Penrose-Rindler spinor approach.

The idea on spinor structure of space-time can be of crucial significance in our attempts to solve the problem of classifying the fermions in intrinsic parities. Indeed, it is natural requirement: if one adopts the space-time with spinor structure, then one must consider the problem of classifying particles in intrinsic parities in terms of exact linear representations of the covering for the full Lorentz group $L^{↑↓}_+$, including $P$ and $T$- reflections. This problem was considered previously by one of authors: Red’kov (1996, 2002). Now we extend the results to spinor covering for complex Lorentz group.

The main goal of the present paper is to give a brief account of the ways to use some parameters while working with the Lorentz group and the groups related to it. In particular, we are interested in extending all the apparatus on the complex Lorentz group – on group theoretical ground for twistors idea see Penrose and Rindler (1986). And finally one other idea was of value while preparing this paper: we have presented one special way to treat the theory of general linear complex group $GL(4,C)$ in the line given by the Dirac matrix algebra. On one hand, we obtain extension of the methods used and developed for Lorentz group theory; on the other hand, this linear group includes unitary sub-groups, $SU(4)$ and related to it, which may be interesting in many physical applications.

The paper is organized as follows:

1. Introduction
2. On linear (tensor and vector) parametrization of the group $GL(4,C)$
3. Parametrization of the complex Lorentz group and its subgroups
4. Complex Euler’s angles, coordinates on the sphere
5. On connection between Fedorov’s construction for the Lorentz group theory and Einstein-Mayer concept of semi-vectors
6. Complex Lorentz group and quaternions
7. On the use of Newman-Penrose formalism
8. The covering for $SO(3.1,C)$ and intrinsic fermion parity
9. On the structure of Majorana bases
10. Conclusions
2 On linear (tensor and vector) parametrization of the group $GL(4,C)$

We have started from simple idea: any 4-matrix can be resolved in terms of 16 basic elements of Dirac matrix algebra, so we represent a matrix $G \in GL(4,C)$ as linear combination of Dirac ones

$$G = A \, I + i B \, \gamma^5 + i A_0 \, \gamma^0 + i A_i \, \gamma^i + B_l \, \gamma^l \gamma^5 + F_{mn} \, \sigma^{mn}.$$  \hfill (1)

We can determine a natural way to parameterize the linear group $GL(4,C)$ in terms of 16 basic elements of independent complex (tensor) quantities $G = G(A, B, A_k, B_k, F_{kl})$. With the use of algebraic properties of the Dirac matrices we have derived the composition rule for the introduced parameters of the group $GL(4,C)$

$$A'' = A' A - B' B - A'_l A^l - B'_l B^l - \frac{1}{2} F_{kl} F^{kl},$$

$$B'' = A' B + B' A + A'_l B^l - B'_l A^l + \frac{1}{4} F_{mn} F_{cd} \epsilon^{mncd},$$

$$A''_l = A' A_l - B' B_l + A'_i A_l + B'_i B_l + A'' A_l + B'' B_l +$$

$$+ F_{kl} A_k + \frac{1}{2} B_k F_{mn} \epsilon_l k m n + \frac{1}{2} F_{mn} B_k \epsilon_l m n k,$$

$$B''_l = A' B_l + B' A_l - A'_l B^l + B'_l A^l + A'' A_l + B'' B_l +$$

$$+ F_{kl} B_k + \frac{1}{2} A_k F_{mn} \epsilon_l k m n + \frac{1}{2} F_{mn} A_k \epsilon_l m n k,$$

$$F''_{mn} = A' F_{mn} + F'_{mn} A - (A'_m A_n - A'_n A_m) - (B'_m B_n - B'_n B_m) +$$

$$+ A'_l B_k \epsilon^{lkmn} - B'_l A_k \epsilon^{lkmn} + \frac{1}{2} F_{kl} F_{mn} \epsilon_{kl mn} + \frac{1}{2} F_{n m} F_{k l} \epsilon_{kl mn} +$$

$$(F_{nk} F_{nl} - F_{nk} F_{nl}).$$  \hfill (2)

For the following, it is convenient to use $(3+1)$-splitting:

$$G = A \, I + i B \, \gamma^5 + i A_0 \, \gamma^0 + i A_i \, \gamma^i +$$

$$+ B_0 \, \gamma^0 \gamma^5 + B_l \, \gamma^l \gamma^5 + a_i \, K^i + b_i \, J^i$$  \hfill (3)

where the notation is used

$$a = (a_i) = (F_{hi}) , \quad b = (b_i) = \left( \frac{1}{2} \epsilon_{ikl} F_{kl} \right),$$

$$K = (K^i) = (2 \sigma^0), \quad J = (J^i) = (\epsilon_{ikl} \sigma^{kl}).$$

In spinor basis of Dirac matrices [113]

$$\gamma^a = \begin{pmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \sigma^a = (I, \sigma^j), \bar{\sigma}^a = (I, -\sigma^j), \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix},$$

the matrix $G$ will take the form

$$G = \begin{pmatrix} k_0 + k \, \bar{\sigma} & n_0 - n \, \bar{\sigma} \\ -l_0 - l \, \bar{\sigma} & m_0 - m \, \bar{\sigma} \end{pmatrix},$$  \hfill (4)

where the notation is used:

$$k_0 = a_0 - i b_0, \quad k = a - i b,$$

$$m_0 = a_0 + i b_0, \quad m = a + i b,$$

$$l_0 = B_0 - i A_0, \quad l = B - i A,$$

$$n_0 = B_0 + i A_0, \quad n = B + i A.$$  \hfill (5)

In these parameters $(k, m, n, l)$ the composition rule \hfill (2)

$$(k'', m'', n'', l'') = <(k', m', n', l'), (k, m, n, l) >;$$
Transition from covering 4-spinor transformations to 4-vector ones is performed through the known relationship

\[ G\gamma^a G^{-1} = \gamma^c L^a_c \]

The formulas are true for any matrices from the group \( GL(4.C) \). This parametrization could be the base for theory of the special linear groups \( SL(4.C) \) and \( SL(4.R) \), and also for the theory of the unitary groups \( SU(4), SU(3.1), SU(2.2) \); they will be considered separately elsewhere.

### 3 Parametrization of the complex Lorentz group and its subgroups

Below we will restrict ourselves to the case of \( 4 \times 4 \) matrices when

\[ G(A, B, A_k = 0, B_k = 0, F_{kl}), \quad \text{or} \quad G(k, m; 0, 0), \]

i.e. consider the 8-parametric \( 4 \times 4 \) matrices in the quasi diagonal form

\[ G = \begin{bmatrix} k_0 + k \bar{\sigma} & 0 \\ 0 & m_0 - m \bar{\sigma} \end{bmatrix}. \]  

(7)

The composition rules are reduced respectively to

\[ A'' = A' A - B' B - \frac{1}{2} F'_{kl} F^{kl}, \]

\[ B'' = A' B + B' A + \frac{1}{4} F'_{mn} F_{cd} \epsilon^{mncd}, \]

\[ F''_{mn} = (A' F_{mn} + F'_{mn} A) + \left( \frac{1}{2} B' F_{kl} \epsilon^{kl mn} + \frac{1}{2} F'_{kl} B \epsilon^{kl mn} \right) + (F'_{mk} F^k_n - F'_{nk} F^k_m) ; \]

and

\[ k'' = k'_0 k_0 + k' k, \quad k'' = k'_0 k_0 + i k' \times k ; \]

\[ m'' = m'_0 m_0 + m' m, \quad m'' = m'_0 m_0 + i m' \times m. \]

(9)

With two additional constraints on 8 quantities \( (A, B, F_{kl}) \):

\[ k_0^2 - k^2 = +1, \quad m_0^2 - m^2 = +1, \quad \text{or} \]

\[ A B - a b = 0, \quad (A^2 - B^2) - (a^2 - b^2) = +1 \]

(10)

we will arrive at a definite way to parameterize spinor covering for complex Lorentz group \( SO(3.1.C) \sim SO(4.C) \), treated here as a sub-group of the linear group \( GL(4.C) \). At this problem of inverting the \( G \) matrices is easily solved: in tensor and vector forms:

\[ G = G(A, B, F_{kl}), \quad G^{-1} = G(A, B, -F_{kl}), \]

\[ G = G(k_0, k, m_0, m), \quad G^{-1} = G(k_0, -k, m_0, -m). \]

(11)

Transition from covering 4-spinor transformations to 4-vector ones is performed through the known relationship

\[ G\gamma^a G^{-1} = \gamma^c L^a_c \]
which determine $2 \implies 1$ map from $\pm G$ to $L$. At this an explicit form of the Lorentz vector matrices is derived, they are bilinear functions of $(A, B, F_{kl})$ or $(k_a, m_a)$:

$$L^0_0 = k_0 m_0 + k_j m_j,$$

$$(L^0_j) = -k_0 m_j - m_0 k_j - i \epsilon_{jln} k_l m_n,$$

$$(L^0_l) = -k_0 m_j - m_0 k_j + i \epsilon_{jln} k_l m_n,$$

$$L^1_l = (k_0 m_0 - k_n m_n) \delta_{lj} + (m_l k_j + m_k l_j) + i \epsilon_{ljn} (k_0 m_n - m_0 k_n).$$

(12)

There exist the direct connection between the above 4-dimensional vector parametrization of the spinor group $G(k_a, m_a)$ and the Fedorov's parametrization [100, 137] of the group of complex orthogonal Lorentz transformations with the help of 3-dimensional vectors (to the case of real Lorentz group corresponds imposing an additional constrain $m_a = k'_a$ or $M = Q^*$):

$$Q = \pm\frac{k}{\pm k_0}, \quad M = \pm\frac{m}{\pm m_0}, \quad k_0 = \sqrt{1 + k^2}, \quad m_0 = \sqrt{1 + m^2};$$

(13)

with the simple composition rules for vector parameters$^3$

$$Q'' = \frac{Q + Q' + i Q' \times Q}{1 + Q' \cdot Q}; \quad M'' = \frac{M + M' - i M' \times M}{1 + M' \cdot M};$$

(14)

Evidently, this pair $(Q, M)$ provides us with possibility to parameterize correctly orthogonal matrices only. Instead, the $(A, B, F_{kl})$ or $(k_a, m_a)$ represent correct parameters for the spinor covering group.

It should be noted that there exists one other way to parameterize 4-spinor (covering) matrices by means of two complex vectors without any additional restrictions on them – it was given in [139]:

$$p = \frac{k}{1 + k_0}, \quad s = \frac{m}{1 + m_0}.$$  

(15)

It is readily verified that the sign distinction in pure spinor matrices transforms into the form

$$(k_0, k) \implies p, \quad (-k_0, -k) \implies \frac{p}{p^2} = p',$$

$$(m_0, m) \implies s, \quad (-m_0, -m) \implies \frac{s}{s^2} = s'.$$

(16)

Sometimes, when we are interested only in local properties of the spinor representations then no substantial differences between orthogonal groups and their spinor coverings exist. However, in opposite cases global difference between orthogonal and spinor groups may be very substantial as well as correct parametrization of them.

One may compile the table in which all parametrizations used are listed:

- **complex tensors** $A, B, F_{kl}$  
  (two subsidiary conditions are imposed; spinor and orthogonal groups are parameterized);

- **complex bi-vector** $f_{kl}$  
  (no subsidiary conditions are impose; orthogonal group only are parameterized);

- two complex 4-vectors $k_a$ and $m_a$  
  (two subsidiary conditions are imposed; spinor and orthogonal group are parameterized);

- two complex 3-vectors $Q$ and $M$  
  (no subsidiary conditions imposed; orthogonal group only is parameterized);

- two complex 3-vector $p$ and $s$  
  (no subsidiary condition are imposed; spinor and orthogonal group are parameterized);

- complex Euler angles $(\alpha, \beta, \gamma)$

$^3$ Slight difference with formulas from [137] can be eliminated by the formal change $Q \implies iQ, M \implies iM$. 

6
complex curvilinear coordinates \((u^1, u^2, u^3)\) on \(SO(4.C)\)

Restrictions specifying the spinor coverings for orthogonal sub-groups

\[
SO(3.1.R), SO(2.2.R), SO(4.R), SO(3.C), SO(3.R), SO(2.1.R)
\]

are well known [118,123]. In particular, restriction to real Lorentz group \(O(3,1,R)\) is achieved by imposing one condition (including complex conjugation)

\[
(k, m) \implies (k, k^*) .
\]

The case of real orthogonal group \(O(4.R)\) is achieved by a formal change (transition to real parameters)

\[
(k_0, k) \implies (k_0, i k) , \quad (m_0, m) \implies (m_0, i m) ,
\]

and the real orthogonal group \(O(2.2,R)\) corresponds transition to real parameters according to

\[
(k_0, k_1, k_2, k_3) \implies (k_0, k_1, k_2, ik_3) ,
\]

\[
(m_0, m_1, m_2, m_3) \implies (m_0, m_1, m_2, im_3) .
\]

Expressions for vector and spinor (covering) matrices concurrently for all these groups can be readily written in explicit form.

4 Complex Euler’s angles, coordinates on the sphere

To parameterize 4-spinor and 4-vector transformations of the complex Lorentz group one may use curvilinear coordinates. The simplest and widely used ones are Euler’s complex angles [64, 70, 90-93, 137, 144]. This possibility is closely connected with cylindrical coordinates on the complex 3-sphere ⁴. Such complex cylindrical coordinates can be introduced by the following relations:

\[
k_0 = \cos \rho \cos z , \quad k_3 = i \cos \rho \sin z , \quad k_1 = i \sin \rho \cos \phi , \quad k_2 = i \sin \rho \sin \phi ,
\]

\[
m_0 = \cos R \cos Z , \quad m_3 = i \cos R \sin Z , \quad m_1 = i \sin R \Phi , \quad m_2 = i \sin R \sin \Phi .
\]

Here 6 complex variables are independent, \((\rho, z, \phi) , (R, Z, \Phi)\), additional restrictions are satisfied identically by definition. Instead of cylindrical coordinates \((\rho, z, \phi) , (R, Z, \Phi)\) one can introduce Euler’s complex variables \((\alpha, \beta, \gamma)\) and \((A, B, \Gamma)\) through the simple linear formulas:

\[
\begin{align*}
\alpha &= \phi + z , & \beta &= 2 \rho , & \gamma &= \phi - z , \\
A &= \Phi + Z , & B &= 2 R , & \Gamma &= \Phi - Z .
\end{align*}
\]

⁴In general, on the base of the analysis given by Olewski [162] on coordinates in the Lobachevski space, one can propose 34 different complex coordinate systems appropriate to parameterize the complex Lorentz group and its 4-spinor covering.
Euler’s angles \((\alpha, \beta, \gamma)\) and \((A, B, \Gamma)\) are referred to \(k_a, m_a\)-parameters by the formulas

\[
\begin{align*}
\cos \beta &= k_0^2 - k_2^2 + k_4^2, \\
\sin \beta &= 2 \sqrt{k_0^2 - k_3^2} \sqrt{-k_1^2 - k_2^2}, \\
\cos \alpha &= \frac{-ik_0k_1 + k_2k_3}{\sqrt{k_0^2 - k_3^2} \sqrt{-k_1^2 - k_2^2}}, \\
\sin \alpha &= \frac{-ik_0k_2 - k_1k_3}{\sqrt{k_0^2 - k_3^2} \sqrt{-k_1^2 - k_2^2}}, \\
\cos \gamma &= \frac{-ik_0k_1 - k_2k_3}{\sqrt{k_0^2 - k_3^2} \sqrt{-k_1^2 - k_2^2}}, \\
\sin \gamma &= \frac{-ik_0k_2 + k_1k_3}{\sqrt{k_0^2 - k_3^2} \sqrt{-k_1^2 - k_2^2}}.
\end{align*}
\]

\[
\begin{align*}
\cos B &= m_0^2 - m_2^2 + m_1^2 + m_2^2, \\
\sin B &= 2 \sqrt{m_0^2 - m_3^2} \sqrt{-m_1^2 - m_2^2}, \\
\cos A &= \frac{+im_0m_1 + m_2m_3}{\sqrt{m_0^2 - m_3^2} \sqrt{-m_1^2 - m_2^2}}, \\
\sin A &= \frac{+im_0m_2 - m_1m_3}{\sqrt{m_0^2 - m_3^2} \sqrt{-m_1^2 - m_2^2}}, \\
\cos \Gamma &= \frac{+im_0m_1 - m_2m_3}{\sqrt{m_0^2 - m_3^2} \sqrt{-m_1^2 - m_2^2}}.
\end{align*}
\]

(22)

Explicit forms of \(A, B, \Gamma\) are referred to the known Pauli matrices:

\[
B(k) = e^{-i\sigma^2 \alpha/2} e^{i\sigma^1 \beta/2} e^{i\sigma^3 \gamma/2}, \quad B(\bar{m}) = e^{-i\sigma^3 \gamma/2} e^{i\sigma^1 B/2} e^{i\sigma^3 A/2}.
\]

(23)

5 On connection between Fedorov’s construction for the Lorentz group theory and Einstein-Mayer concept of semi-vectors

The main relations serving the initial points for systematically constructing the Lorentz group theory in the Fedorov approach and its generalization now can be straightforwardly written down – the origin of these is evident when turning to the spinor covering matrices specified in Weyl basis, where they have a quasi diagonal structure \(\otimes\). For the complex Lorentz group it looks as follows

\[
B(k) = \sigma^a k_a, \quad B(m) = \sigma^a m_a, \quad \bar{m} = (m_0, -m_i),
\]

\[
G(k, m) = \begin{vmatrix} B(k) & 0 & 0 \\ 0 & B(\bar{m}) & 0 \end{vmatrix},
\]

\[
G(k, I) = \begin{vmatrix} B(k) & 0 & I \\ 0 & I & B(\bar{m}) \end{vmatrix}, \quad G(I, m) = \begin{vmatrix} I & 0 & 0 \\ 0 & B(m) & 0 \end{vmatrix},
\]

\[
G(k, I)G(I, m) = G(I, m)G(k, I) = G(k, m), \quad L(k, I)L(I, m) = L(I, m)L(k, I) = L(k, m).
\]

(24)

Explicit forms of \(L(k, I)\) and \(L(I, m)\) are

\[
L(k, I) = \begin{vmatrix} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -i k_3 & i k_2 \\ -k_2 & i k_3 & k_0 & -i k_1 \\ -k_3 & -i k_2 & i k_1 & k_0 \end{vmatrix}, \quad L(k, I) = \begin{vmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & m_0 & i m_3 & -i m_2 \\ -m_2 & -i m_3 & m_0 & i m_1 \\ -m_3 & i m_2 & -i m_1 & m_0 \end{vmatrix}.
\]

(25)

In the case of real Lorentz group, when \(m = k^*\), the above relations will take the form

\[
G(k, I)G(I, k^*) = G(I, k^*)G(k, I) = G(k, k^*), \quad L(k, I)L(I, k^*) = L(I, k^*)L(k, I) = L(k, k^*), \quad L(k, k^*) = [L(k, I)]^*.
\]

(26)

It should especially be noted that existence of such a factorized structure in the theory of Lorentz group has long been known: it appeared in 1932 when Einstein and Mayer had published a paper of a series of these [26-28] on the so called semi-vectors, special 4-dimensional complex quantities, 1-st and 2-nd types, transformed by means of complex matrices \(L(k, I)\) or \(L(I, k^*)\) respectively.
Complex 4-dimension quantities transformed with the help of the matrices \( L(k, I) \) and \( L(I, m) \) by definition are called semi-vectors of 1-st and 2-nd type (respectively \( U \) and \( V \) semi-vector):

\[
U' = L(k, I)U, \quad V' = L(I, m)V.
\]

With the help of special linear transformations, the matrices defining the transformation laws for semi-vector can be changed to quasi diagonal form. Indeed (see [137])

\[
\begin{pmatrix}
U_0 \\
U_1 \\
U_2 \\
U_3
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 \\
i & 0 & i & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{pmatrix}, \quad U = F \Psi, \quad U' = L(k, I)U: \Rightarrow \Psi' = (F^{-1}LF)\Psi, \quad F^{-1}L(k, I)F = \begin{pmatrix}
B(k) & 0 \\
0 & \sigma^2 B(k) \sigma^2
\end{pmatrix}.
\]

This means that the semi-vector \( U \) is equivalent to a direct sum of the following 2-spinors (with differently positioned indices):

\[
U \Rightarrow (\xi^1, \xi^2) \oplus (\Xi_1, \Xi_2).
\]

Now analogously one should consider \( V \) semi-vector transformed by the matrix \( L(I, m) \):

\[
\begin{pmatrix}
V_0 \\
V_1 \\
V_2 \\
V_3
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 \\
i & 0 & -i & 0 \\
0 & -1 & 0 & -1
\end{pmatrix} \begin{pmatrix}
\eta_1 \\
\eta_2 \\
H^1 \\
H^2
\end{pmatrix}, \quad V = L(I, m)V: \Rightarrow \Phi' = (\varphi^{-1}L\varphi)\Phi, \quad \varphi^{-1}L(I, m)\varphi = \begin{pmatrix}
B(\bar{m}) & 0 \\
0 & \sigma^2 B(\bar{m}) \sigma^2
\end{pmatrix}.
\]

In other words, the semi-vector \( V \) is equivalent to a direct sum of two following spinors:

\[
V \Rightarrow (\eta_1, \eta_2) \oplus (H^1, H^2).
\]

Restriction to real Lorentz group is achieved by imposing \( m = k^* \), therefore the spinor \( \eta \) transforming with the help of the law

\[
\eta = \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}, \quad \eta' = B^+(\bar{k}) \eta;
\]

which corresponds to a standard convention on spinor (doted and non-doted) indices [113].

6 Complex Lorentz group and quaternions

Lorentz matrices (25) can be represented in a short algebraic form in the frame of quaternion notation (see for example in [90-93, 130, 138, 144, 156, 157]):

\[
L(k, I) = k_0 e_0 + k_i e_i = k_a e_a, \quad L(I, m) = m_0 e_0^* + m_i e_i^* = m_a e_a^*,
\]

where (the asterisk symbol * stands for complex conjugation)

\[
e_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad e_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0
\end{pmatrix},
\]

\[
e_2 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & i \\
1 & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{pmatrix}, \quad e_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
The quaternion quantities \( e_a e^*_a \) obey the rules:
\[
e_k e_l = \delta_{kl} + i \epsilon_{klm} e_m, \quad e_k^* e_l^* = \delta_{kl} - i \epsilon_{klm} e_m^*, \quad e_i e_i^* = e_i^* e_i.
\] (34)

The complex Lorentz matrix can be resolved in terms of (double) quaternion units \( e_a e_b^* \):
\[
L(k, m) = (k_0 + k_i e_i)(m_0 + m_j e_j^*) = k_0 m_0 + (k_0 m_j - m_0 k_j) e_i + k_i m_j e_i e_j^* ;
\] (35)
in case of real Lorentz group, when \( m_a = k_a^* \), the previous relation will look
\[
L(k, k^*) = k_0 k_0^* + k_0 k_j^* e_j + k_i k_0^* e_i + k_i k_j^* e_i e_j^*.
\] (36)

Let us briefly discuss one special application of the quaternions in the Lorentz group theory. If the vector Lorentz matrix is done explicitly, one should find a corresponding 4-spinor matrix, that is two parameters \( (k, k^*) \) of \( G \). The above structure of real matrix \( L(k, k^*) \) points out another simple way to treat the same problem, which is based on looking at the coefficients at quaternion combinations \( e_a e_b^* \).

From known
\[
| k_0 |^2 I, \quad | k_1 |^2 e_1 e_1^*, \quad | k_2 |^2 e_2 e_2^*, \quad | k_3 |^2 e_3 e_3^* ;
\]
on one can determine the modulus \( | k_a | \), and then from
\[
(k_0 k_1^*) e_1^*, \quad (k_0^* k_1) e_1, \quad (k_0 k_2^*) e_2^*, \quad (k_0^* k_2) e_2, \quad (k_0 k_3^*) e_3^*, \quad (k_0^* k_3) e_3;
\]
one may find the phases of \( k_a \).

The same problem may be posed for the complex Lorentz group as well. Starting from a given complex matrix \( L(k, m) \) one may wish to find two parameters \( k \) and \( m \). Let an algorithm to establish the numerical matrix consisting of the coefficient at quaternion combinations \( e_a e_b^* \) be known:
\[
M_{ab} = k_a m_b .
\] (37)

Having the matrix \( M \) be given, one may solve the problem as follows.

1) From elements of first line (with \( \pm \) sign ambiguity) one finds the \( k_0 \):
\[
k_0 = \pm \sqrt{(k_0 m_0)^2 - (k_0 m_1)^2 - (k_0 m_2)^2 - (k_0 m_3)^2}
\] (38)

2) From elements of the first row one finds remaining three components \( k_1, k_2, k_3 \):
\[
k_1 = k_0 \frac{M_{10}}{M_{00}}, \quad k_2 = k_0 \frac{M_{20}}{M_{00}}, \quad k_3 = k_0 \frac{M_{30}}{M_{00}} .
\] (39)

3) From elements of first row one finds \( m_0 \):
\[
m_0 = \pm \sqrt{(k_0 m_0)^2 - (k_1 m_0)^2 - (k_2 m_0)^2 - (k_3 m_0)^2} , \quad m_0 k_0 = M_{00}
\] (40)
the sign \((+ \text{ or } -)\) of \( m_0 \) must be consistent with the sign of \( k_0 \).

4) From elements of the first line one finds three components \( m_1, m_2, m_3 \):
\[
m_1 = m_0 \frac{M_{01}}{M_{00}} \quad m_2 = m_0 \frac{M_{02}}{M_{00}} \quad m_3 = m_0 \frac{M_{03}}{M_{00}} .
\] (41)

The matrix \( M \) has a diad-structure \( \{ k, k^* \} \) that leads to the equations (one should take into account the identities: \( k k = 1, \bar{k} = \delta k, m \bar{m} = 1, \bar{m} = \delta m \)):
\[
M_{ab} m_b = k_a , \quad M_{cb} \bar{k}_c = m_b
\] (42)

From these one can easily reduce the task to determine \( k \) and \( m \) from given \( L(k, m) \) to a pair of eigen-value problems. Indeed, eqs. \( \{ \} \) can be rewritten in matrix form
\[
\begin{align*}
M \bar{k}_m k &= \bar{k}_m ,
& \bar{M} \bar{k_k} m &= \bar{k}_k .
\end{align*}
\]
from where
\[
(\bar{M} \bar{M}) k = k , \quad (\bar{M} \bar{M}) m = m .
\] (43)
Thus, we have constructed the matrices for which the 4-vector parameters \( k \) and \( m \) are eigen-vectors.
7 On the use of Newman-Penrose formalism

In the real Lorentz group theory, especially in connection to the problems particle fields in general relativity the wide use is found the so-called isotropic basis (also commonly known as Newman-Penrose formalism of the light tetrad [163-168]). This basis can be effectively employed in the theory of complex Lorentz group as well. Let us turn again to the factorized form (24) of the complex Lorentz matrices

\[ L(k, m) = L(k, I)L(I, m) = L(I, m)L(k, I), \]

and perform a special similarity transformation

\[
\begin{align*}
L(k, I) & \quad \rightarrow \quad L_{\text{isotr}}(k, I) = SL(k, I)S^{-1}, \\
L(I, m) & \quad \rightarrow \quad L_{\text{isotr}}(I, m) = SL(I, m)S^{-1}, \\
L(k, m) & \quad \rightarrow \quad U(k, m) = L(k, m)_{\text{isotr}} = SL(k, m)S^{-1},
\end{align*}
\]

where

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -i \\
0 & 1 & -i & 0 \\
0 & 1 & i & 0
\end{pmatrix}, \quad S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & i & -i \\
1 & -1 & 0 & 0
\end{pmatrix},
\]

in the isotropic basis we obtain

\[
L_{\text{isotr}}(k, I) = \begin{pmatrix}
(k_0 - k_3) & 0 & -(k_1 + ik_2) & 0 \\
0 & (k_0 + k_3) & 0 & -(k_1 - ik_2) \\
-(k_1 - ik_2) & 0 & (k_0 + k_3) & 0 \\
0 & -(k_1 + ik_2) & 0 & (k_0 - k_3)
\end{pmatrix},
\]

\[
L_{\text{isotr}}(I, m) = \begin{pmatrix}
(m_0 - m_3) & 0 & 0 & -(m_1 - im_2) \\
0 & (m_0 + m_3) & -(m_1 + im_2) & 0 \\
0 & -(m_1 - im_2) & (m_0 - m_3) & 0 \\
-(m_1 + im_2) & 0 & 0 & (m_0 + m_3)
\end{pmatrix}.
\]

It is convenient to employ the other notation referring to initial spinor matrices:

\[
B(k) = \begin{pmatrix}
a & d \\
c & b
\end{pmatrix}, \quad B(\bar{m}) = \begin{pmatrix}
B & -C \\
-D & A
\end{pmatrix},
\]

then the previous relations will take the form

\[
L_{\text{isotr}}(k, I) = \begin{pmatrix}
b & 0 & -c & 0 \\
0 & a & 0 & -d \\
-d & 0 & a & 0 \\
0 & -c & 0 & b
\end{pmatrix}, \quad L_{\text{isotr}}(I, m) = \begin{pmatrix}
B & 0 & 0 & -C \\
0 & A & -D & 0 \\
0 & -C & B & 0 \\
-D & 0 & 0 & A
\end{pmatrix}.
\]

The Lorentz complex vector matrix \( L_{\text{isotr}}(k, m) \) in isotropic basis looks as

\[
L_{\text{isotr}}(k, m) = \begin{vmatrix}
bB & cC & -cB & -bC \\
dD & aA & -aD & -dA \\
-dB & -aC & aB & dC \\
-bD & -cA & cD & bA
\end{vmatrix} \in SO(3.1.C).
\]

Restriction to real Lorentz group is achieved by a formal substitution:

\[
A \rightarrow a^*, \quad B \rightarrow b^*, \quad C \rightarrow c^*, \quad D \rightarrow d^*,
\]

\[
L_{\text{isotr}}(k, k^*) = \begin{vmatrix}
bb^* & cc^* & -cb^* & -bc^* \\
\bar{d}d^* & \bar{a}a^* & -\bar{a}d^* & -\bar{d}a^* \\
-d\bar{b}^* & -\bar{a}c^* & ab^* & \bar{d}c^* \\
-bd^* & -ca^* & cd^* & ba^*
\end{vmatrix} \in SO(3.1.R).
\]

In contrast to the usual (non isotropic) basis, here the Lorentz matrix consists of 16 elements each of these is constructed as a single multiple of two quantities. In the case of ordinary basis each matrix element represent a sum of four such combinations. It is this feature that makes the isotropic basis so helpful in applications.
8 The covering for $SO(3.1.C)$ and intrinsic fermion parity

The problem of intrinsic parity for a fermion has a long history (see Introduction), the problem seems to be unsolved till now. The main difficulty consists in double valuedness of spinors considered in the frames of orthogonal groups. In our opinion, the problem should be studied on the basis of the representation theory for spinor covering of the orthogonal groups. Below we propose the way to solve this problem through extension of the covering for complex Lorentz group $SO(3.1.C)$ by adding two discrete 4-spinor operations, $P = i\gamma^0$ and $T = \gamma^0\gamma^5$ can be quite easily solved. One may search pure spinor representations of the covering group $G_{\text{cover}} = \{G \oplus P \oplus T\}$ in the form

$$T(g) = s(g) g, \quad g \in G_{\text{cover}}, \quad s(g_1) s(g_2) = s(g_1 g_2),$$

(48)

where $s(g)$ is a numerical function on the group. There exist four different solutions $s_i(g)$:

$$s_1(g) = s_2(g) = s_3(g) = s_4(g) = G(k, m),$$

$$P: \quad +1 \quad +1 \quad +1 \quad +1,$$

$$T: \quad +1 \quad -1 \quad +1 \quad -1.$$

(49)

Correspondingly we arrive at four type of representations $S_i(g)$ of the group $G_{\text{cover}}$. With the use of the relations

$$F = \text{const} \begin{vmatrix} -1 & 0 \\ 0 & +1 \end{vmatrix}, \quad F G(k, m) F^{-1} = G(k, m),$$

$$F P F^{-1} = -P, \quad F T F^{-1} = -T,$$

one can conclude that representation $S_2(g)$ is equivalent to $S_1(g)$, and $S_4(g)$ is equivalent to $S_3(g)$:

$$S_2(g) = F S_1(g) F^{-1}, \quad S_4(g) = F S_3(g) F^{-1}.$$

(50)

Thus, there exist only two different types of the pure spinor representations of the covering group

$$S_1(g) \sim S_2(g), \quad S_3(g) \sim S_4(g).$$

(51)

This doubling can be connected with intrinsic space-time parity for a fermion. Special attention should be given to the following: no separate $P$-parity or $T$-parity for a fermion exist in the group-theoretical sense, instead the group theory leads to a unified intrinsic characteristic as a fermion’s parity.

Manifestation of these two discrete spinor operations in tensor representation one can examine by looking at 2-rank 4-spinor:

$$\Psi' = G \Psi, \quad \Phi' = G \Phi, \quad \Psi \otimes \Phi = U, \quad U' = GU \tilde{G},$$

$$U = [\varphi I + i\varphi \gamma^5 + i\varphi_I \gamma^1 + \varphi_I \gamma^1 \gamma^5 + \varphi_{mn} \sigma_{mn}] \gamma^0 \gamma^2;$$

(52)

the symbol of tilde over $G$ stands for the matrix transposition. $\gamma^0 \gamma^2$ is a 4-spinor metric matrix. The transformations properties of irreducible tensor components of 2-rank 4-spinor are given by

$$\begin{align*}
\mathbf{G}: & \quad \varphi' = \varphi, \quad \varphi' = \varphi, \\
& \quad \varphi' = L_k^l \varphi_I, \quad \varphi' = L_k^l \varphi_I, \\
& \quad \varphi_{kl} = L_k^m L_l^n \varphi_{mn},
\end{align*}$$

$$\begin{align*}
\mathbf{P}: & \quad \varphi' = +\varphi, \quad \varphi' = -\varphi, \\
& \quad \varphi' = +L_k^{(P)} \varphi_I, \quad \varphi' = -L_k^{(P)} \varphi_I, \\
& \quad \varphi_{kl} = +L_k^{(P)} L_l^{(P)} \varphi_{mn},
\end{align*}$$

$$\begin{align*}
\mathbf{T}: & \quad \varphi' = -\varphi, \quad \varphi' = +\varphi, \\
& \quad \varphi' = -L_k^{(T)} \varphi_I, \quad \varphi' = +L_k^{(T)} \varphi_I, \\
& \quad \varphi_{kl} = -L_k^{(T)} L_l^{(T)} \varphi_{mn},
\end{align*}$$

(53)

\footnote{In essence, a strong form of this changing orthogonal groups to their covering groups and its representations throughout, and when we are going to work in the same manner at describing the space-time structure itself, is the known Penrose-Rindler spinor approach [163-167].}

\footnote{The choice of phase factors at these two matrices is in accordance with Racah choice [39]. In any of Majorana bases these discrete operations are given by real matrices.}
where

\[
L^{(P)a}_c = \delta^a_c = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{vmatrix}, \quad L^{(T)a}_c = -\delta^a_c = \begin{vmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}.
\]

In order to construct tensors with different properties under discrete operations one should take into account the possibility to distinguish 4-spinor with respect to intrinsic parity.

The 2-rank 4-spinor of the type \( U_{13} = \Psi_1 \otimes \Phi_3 \sim \Psi_3 \otimes \Phi_1 \) contains irreducible tensors with the following properties:

\[
\begin{align*}
G: \quad & \varphi' = \varphi, \quad \varphi' = \tilde{\varphi}, \\
& \varphi'_k = L_k^l \varphi_l, \quad \varphi'_k = L_k^l \tilde{\varphi}_l, \\
& \varphi'_{kl} = L_k^m L_l^n \varphi_{mn},
\end{align*}
\]

\[
\begin{align*}
P: \quad & \varphi' = +\varphi, \quad \varphi' = -\tilde{\varphi}, \\
& \varphi'_k = +L_k^l \varphi_l, \quad \varphi'_k = -L_k^l \tilde{\varphi}_l, \\
& \varphi'_{kl} = +L_k^m L_l^n \varphi_{mn},
\end{align*}
\]

\[
\begin{align*}
T: \quad & \varphi' = +\varphi, \quad \varphi' = -\tilde{\varphi}, \\
& \varphi'_k = +L_k^l \varphi_l, \quad \varphi'_k = -L_k^l \tilde{\varphi}_l, \\
& \varphi'_{kl} = +L_k^m L_l^n \varphi_{mn},
\end{align*}
\]

Thus all four type of scalars and vectors, (+, +), (−, −), (+, −), (−, +) have been constructed.

Now we consider the problem of linear representations of the spinor groups that are supposedly cover the partly extended Lorentz groups \( L_{\perp}^- \) and \( L_{\perp}^+ \) (improper orthochronous and proper non-orthochronous respectively). Such covering of partly extended Lorentz group can be constructed by adding only one matrix discrete operation: \( P \) or \( T \). The result obtained for simplest representations of these groups is as follows: all above representations \( S_l(g) \) at confining them to sub-groups \{\( G \oplus P \)\} or \{\( G \oplus T \)\} lead to representations changing into each other by a similarity transformation. In other words, in fact there exists only one representation of these partly extended spinor groups. This may be understood as impossibility to determine any group-theoretical parity concept (\( P \) or \( T \)) within the limits of partly extended spinor groups\(^7\).

## 9 On the structure of Majorana bases

Additionally, several points in connection to the Majorana bases in the context of the real Lorentz group should be added. In particular, any Lorentz transformation in bispinor space is resolved in terms of the Dirac matrices [101]:

\[
S(k, k^*) = \frac{1}{2} \left( k_0 + k_0^* \right) - \frac{1}{2} \left( k_0 - k_0^* \right) \gamma^5 + k_1 (\sigma^{01} + i \sigma^{23}) + k_1^* (\sigma^{01} - i \sigma^{23}) + k_2 (\sigma^{02} + i \sigma^{31}) + k_2^* (\sigma^{02} - i \sigma^{31}) + k_3 (\sigma^{03} + i \sigma^{12}) + k_3^* (\sigma^{03} - i \sigma^{12}) .
\]

Because in Majorana basis \( (i \gamma^5_M)^* = i \gamma^5_M \), \( (\sigma^{ab}_M)^* = + \sigma^{ab}_M \), the Lorentzian 4-spinor matrix \( S(k, k^*) \) is real.

Now we are to describe all possible Majorana’s bases. To this end we should find all transformations \( A \) in 4-spinor space that change all the Dirac matrices to the imaginary form

\[
\Psi_M(x) = A \Psi(x), \quad \gamma^a_M = A \gamma^a A^{-1}, \quad (\gamma^a_M)^* = -\gamma^a_M ;
\]

here \( \gamma^a \) stand for the Dirac matrices in spinor representation. One should note that the problem must have a many of solutions. Indeed, if \( A \) satisfies eq. \( 55 \) then any matrix of the form \( A' = e^{i \alpha} R A \), where \( R \) is real, will satisfy eq. \( 56 \) as well:

\[
\text{if} \quad \left( A \gamma^a A^{-1} \right)^* = - \left( A \gamma^a A^{-1} \right),
\]

\[\text{then} \quad \left[ (e^{i \alpha} R A) \gamma^a (e^{i \alpha} R A)^{-1} \right]^* = - \left[ (e^{i \alpha} R A) \gamma^a (e^{i \alpha} R A)^{-1} \right].\]

\(^7\)This is in evident contradiction with the analysis of fermion parity given by Yang and Tiomno [54] and many others in the frames of orthogonal groups theory.
Simplest of the Majorana bases can be taken as follows (only different from zero parameters are specified):

\[
A^* \gamma^a (A^*)^{-1} = -A \gamma^a A^{-1}, \quad \text{or} \quad (A^{-1} A^*) (\gamma^a)^* (A^{-1} A^*)^{-1} = -\gamma^a.
\]

With the use of notation \(A^{-1} A^* = U\), the latter reads

\[
U (\gamma^a)^* U^{-1} = -\gamma^a. \tag{57}
\]

In spinor representation four identities hold

\[
(\gamma^0)^* = +\gamma^0, \quad (\gamma^1)^* = +\gamma^1, \quad (\gamma^2)^* = -\gamma^2, \quad (\gamma^3)^* = +\gamma^3,
\]

therefore, solution of eq. 57 looks as

\[
U = \text{const} \gamma^2, \quad \det (\gamma^2) = +1. \tag{58}
\]

Because \(\det U = (\det A)^*/(\det A)\), one must conclude that \(\text{const}\) is equal to a phase factor \(e^{i\alpha}\). Thus, the problem is reduced to

\[
A^{-1} A^* = e^{i\alpha} \gamma^2, \quad \text{or} \quad A^* = e^{i\alpha} A \gamma^2. \tag{59}
\]

Any 4-dimensional matrix can be decomposed into sixteen Dirac elementary matrices

\[
A = \left[\begin{array}{cc}
(M_0 \gamma^2 + m_0) & \gamma^1 (N_0 \gamma^2 + n_0) \\
+ \gamma^5 [(M_1 \gamma^2 + m_1) & \gamma^1 (N_1 \gamma^2 + n_1)] \\
+ \gamma^0 [(M_2 \gamma^2 + m_2) & \gamma^1 (N_2 \gamma^2 + n_2)] \\
+ \gamma^5 \gamma^0 [(M_3 \gamma^2 + m_3) & \gamma^1 (N_3 \gamma^2 + n_3)]
\end{array}\right]. \tag{60}
\]

With the notation

\[
\Gamma_0 = I, \quad \Gamma_1 = \gamma^5, \quad \Gamma_2 = \gamma^0, \quad \Gamma_3 = \gamma^5 \gamma^0,
\]

eq 60 \text{ is written in the abridged form}

\[
A = \Gamma_i \left[ (M_i \gamma^2 + m_i) + \gamma^1 (N_i \gamma^2 + n_i) \right]. \tag{61}
\]

Now, let us substitute 62 into eq. 59:

\[
\Gamma_i \left[ (-M_i \gamma^2 + m_i) + \gamma^1 (-N_i \gamma^2 + n_i) \right] = e^{i\alpha} \Gamma_i \left[ (m_i \gamma^2 - M_i) + \gamma^1 (n_i \gamma^2 - N_i) \right];
\]

further it follow equations for unknown parameters:

\[
M_i = -e^{-i\alpha} m_i^*, \quad N_i = -e^{-i\alpha} n_i^*.
\]

Therefore, expression for a matrix \(A\), relating spinor basis with any one Majorana's is given by \(^8\)

\[
A = \Gamma_i \left[ (m_i - e^{-i\alpha} m_i^* \gamma^2) + \gamma_1 (n_i - e^{-i\alpha} n_i^* \gamma^2) \right], \\
\Psi_M(x) = A(m_i, n_i, e^{i\alpha}) \Psi(x).
\]

Evidently, 17 arbitrary real parameters enter these formulas; there exists one additional restriction, \(\det A \neq 0\). Simplest of the Majorana bases can be taken as follows (only different from zero parameters are specified):

\[
m_0 = 1/\sqrt{2}, \quad e^{i\alpha} = +1, \quad A = \frac{1 - \gamma^2}{\sqrt{2}}, \quad A^{-1} = \frac{1 + \gamma^2}{\sqrt{2}},
\]

\[
\gamma_M^0 = +\gamma^0 \gamma^2, \quad \gamma_M^1 = +\gamma^1 \gamma^2, \quad \gamma_M^2 = \gamma^2, \quad \gamma_M^3 = +\gamma^3 \gamma^2. \tag{64}
\]

There exist 16 such possibilities, they can be listed in the two tables:

\(^8\)All the matrices in the right-hand side are referred to spinor representation.
\[ e^{i\alpha} = +1 \]

\[
\begin{array}{c|cccc|cccc}
1/\sqrt{2} &=& m_0 & m_1 & m_2 & m_3 & n_0 & n_1 & n_2 & n_3 \\
\gamma^0_M &=& \gamma^0 \gamma^2 \times & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\
\gamma^1_M &=& \gamma^1 \gamma^2 \times & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
\gamma^2_M &=& \gamma^2 \times & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\
\gamma^3_M &=& \gamma^3 \gamma^2 \times & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\
\end{array}
\]

(65)

\[ e^{i\alpha} = -1 \]

\[
\begin{array}{c|cccc|cccc}
1/\sqrt{2} &=& m_0 & m_1 & m_2 & m_3 & n_0 & n_1 & n_2 & n_3 \\
\gamma^0_M &=& \gamma^0 \gamma^2 \times & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\
\gamma^1_M &=& \gamma^1 \gamma^2 \times & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\
\gamma^2_M &=& \gamma^2 \times & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\
\gamma^3_M &=& \gamma^3 \gamma^2 \times & -1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 \\
\end{array}
\]

(66)

These Majorana bases are similar to each other: \( \gamma^2_M = \pm \gamma^2 \), the remaining three Dirac matrices are multiplied by \( \pm \gamma^2 \).

10 Conclusions

A unifying overview of the ways to parameterize the linear group \( GL(4, \mathbb{C}) \) and its subgroups is given. As parameters for this group there are taken 16 coefficients \( G = G(A, B, A_k, B_k, F_{kl}) \) in resolving matrix \( G \in GL(4, \mathbb{C}) \) in terms of 16 basic elements of the Dirac matrix algebra. The multiplication rules \( G'G \) are formulated in the form of a bilinear function of two sets of 16 variables. The detailed investigation is restricted to 6-parameter case \( G(A, B, F_{kl}) \), which provides us with spinor covering for the complex orthogonal group \( SO(3.1, C) \). Lorentz matrices \( L \) in complex space are determined through the formula \( G \gamma^a G^{-1} = \gamma^c L_c^a \), \( L = L(A, B, F_{kl}) \). Restrictions to parameters corresponding to spinor coverings of sub-groups \( SO(3.1, R), SO(2.2, R), \) \( SO(4, R), SO(3, C), SO(3, R), SO(2.1, R) \) are formulated explicitly. The use of the Newman-Penrose formalism and applying quaternion techniques in the theory of complex Lorentz group are discussed. The complex Euler’s angles parametrization for complex Lorentz group is elaborated as underlaid by the orthogonal complex cylindrical coordinates in the complex extension for Riemannian 3-space of constant curvature. The Majorana bases are studied in detail. In an explicit form 17-parametric formula referring spinor (Weyl’s) basis to any Majorana’s one has been established: \( \Psi_M(x) = A(m_i, n_i, e^{i\alpha}) \Psi(x) \). The most simple choices of \( (m_i, n_i, e^{i\alpha}) \) associated with widely used Majorana representations are given.

The problem of extending the group \( SO(3.1, C) \) by two additional discrete spinor operations, \( P \) and \( T \) reflections: \( G_{cover} = \{ G \oplus P \oplus T \} \) is solved. It is shown that the extended covering group \( G_{cover} \) has only two types of irreducible fundamental 4-spinor representations, which can serve the group-theoretical base for the concept of intrinsic parity of 4-spinor field. Resolving the 2-rank spinor \( \Psi \oplus \Phi \) into a set of irreducible components under the covering group \( G_{cover} \) is given; four types of scalars and 4-vectors are constructed. The problem of extending the group looks the same for \( O(3.1, C), SO(3.1, R), SO(2.2), SO(4, R) \). Connections between Einstein-Mayer study on semi-vectors and Fedorov’s treatment of the Lorentz group theory are stated in detail.

The work is dedicated to the 95-th anniversary of birthday of Academician F.I. Fedorov.

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