Lorentz invariant nonabelian gauge theory on noncommutative space-time and BRST symmetry

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Abstract

Lorentz covariance is the fundamental principle of every relativistic field theory which insures consistent physical descriptions. Even if the space-time is noncommutative, field theories on it should keep Lorentz covariance. In this paper, the nonabelian gauge theory on noncommutative spacetime is defined and its Lorentz invariance is maintained based on the idea of Carlson, Carone and Zobin. The deviation from the standard model in particle physics has not yet observed, and so any model beyond standard model must reduce to it in some approximation. Noncommutative gauge theory must also reproduce standard model in the limit of noncommutative parameter \( \theta^{\mu\nu} \rightarrow 0 \). Referring to Jurčo et. al., we will construct the nonabelian gauge theory that deserves to formulate standard model. BRST symmetry is very important to quantize nonabelian gauge theory and construct the covariant canonical formulation. It is discussed about the fields in noncommutative gauge theory without considering those components. Scale symmetry of ghost fields is also discussed.

1 Introduction

In the past several years, field theories on the noncommutative spacetime have been extensively studied from many different aspects. The motivation comes from the string theory which makes obvious that end points of the open strings trapped on the D-brane in the presence of two form B-field background turn out to be noncommutative [1] and then the noncommutative supersymmetric gauge theories appear as the low energy effective theory of such D-brane [2], [3]. Though this is a driving force of recent prevalence of noncommutative field theories, noncommutative spacetime has long history. Especially, Snyder [4] proposed Lorentz invariant algebra between spacetime \( \hat{x}^\mu \) and the generator of Lorentz transformation \( \hat{M}^{\mu\nu} \) and showed the existence of a noncommutative spacetime with a fundamental length. Based on the Snyder’s algebra, the relativistic field theory was developed [5] though it was not so successful. The noncommutativity of spacetime in recent surge is characterized by the algebra \( [x^\mu, x^\nu] = i\theta^{\mu\nu} \) where \( \theta^{\mu\nu} \) is a real, anti-symmetric constant with dimension 2, which is reflected as the Moyal star product in field theories. According to this prescription, one can build noncommutative version of scalar, Dirac and gauge theories. Thus, apart from the string theory, the studies of noncommutative field theories have been proposing very interesting as well as reversely serious outcomes. The noncommutative scalar field theories are investigated in [6], [7],[8], [17]. It was showed that the noncommutative scalar theory with \( \Phi^4 \) interaction is renormalizable and the parameter \( \theta^{\mu\nu} \) doesn’t receive the quantum corrections up to two loop order. However, since the Moyal star product contains an infinite series of \( x^n \) derivatives, noncommutative field theories are nonlocal. The nonlocality especially in timelike noncommutativity \( \theta^{0i} \neq 0 \) leads to the unitarity violation [17] and the difficulty of renormalizability. All these investigations are carried out under the condition of constant \( \theta^{\mu\nu} \), which means that Lorentz covariance of theory is violated.

Doplicher, Fredenhagen and Roberts (DFR) proposed [9] a new algebra of noncommutative spacetime through consideration of the spacetime uncertainty relations derived from quantum mechanics and general relativity. In their algebra, \( \theta^{\mu\nu} \) is promoted to an anti-symmetric tensor operator, which leads to the Lorentz covariant noncommutative spacetime and enables one to construct the Lorentz invariant noncommutative field theories. Carlson, Carone and Zobin (CCZ) [10] formulated the noncommutative gauge theory by referring to the DFR algebra in the Lorentz invariant way. In their formulation, fields in the theory depend on spacetime \( x^\mu \) and the noncommutative parameter \( \theta^{\mu\nu} \). The action is obtained.
by integrating Lagrangian over spacetime $x^\mu$ as well as the noncommutative parameter $\theta^{\mu\nu}$. The characteristic features of CCZ [10] are to set up the 6-dimensional $\theta$ space in addition to $x$ space and define the action as the integration of Lagrangian over the $\theta$ space as well as $x$ space. In this article, we will construct Lorentz invariant noncommutative field theory by taking over the idea of the 6-dimensional $\theta$ space, but not the integration over $\theta$ space. We can choose the spacelike noncommutativity ($\theta^{0i} = 0$) after the appropriate Lorentz transformation and so any serious outcomes such as quantization and unitarity violation don’t appear.

U(1) noncommutative gauge theory requires the matter field to have charge 0 or $\pm 1$ in order to keep the gauge invariance and define the covariant derivative [11]. Armoni [12] indicated that U(N) gauge theory has the consistency in calculations of gluon propagator and three gluons vertex to one loop order, whereas SU(N) gauge theory is not consistent. These problems have been overcome by Jurčo, et al Jurco who constructed nonabelian gauge theory on the enveloping Lie algebra stemming from the Moyal star product. No extra fields other than fields in ordinary commutative gauge field theory appear in their formulation after performing the Seiberg-Witten map. This approach has allowed them to construct the noncommutative standard model as well as SO(10) GUT [14]. Referring to their consideration, we will formulate the noncommutative nonabelian gauge theory by use of the enveloping group SU(N)∗ with enveloping Lie algebra.

BRST symmetry is very important to quantize nonabelian gauge theory and construct the covariant canonical formulation of it [15]. This subject in noncommutative U(N) gauge theory was recently studied by Soroush [18] by determining the BRST transformations of the components of gauge and related fields. Let us in this article introduce the BRST transformations of fields as the representations of gauge group, not component of fields, which leads us to transparent view of BRST symmetry. The scale transformation of ghost fields under which full Lagrangian of gauge theory is invariant is also discussed. The algebra of BRST charge and FP ghost charge which may serve the classification of asymptotic fields and the proof of S matrix unitarity is displayed.

## 2 Nonabelian gauge theory on noncommutative spacetime

Let us first consider the nonabelian gauge theory on the commutative spacetime with the symmetry SU(N) given by the Lagrangian

$$L = -\frac{1}{4} \text{Tr} [F_{\mu\nu}(x)F^{\mu\nu}(x)] + \bar{\Psi}(x) \{ i\gamma^\mu (\partial_\mu - igA_\mu(x)) - m \} \Psi(x),$$

where we omit the gauge fixing and FP ghost terms and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig [A_\mu(x), A_\nu(x)]$$

with the configuration

$$A_\mu(x) = \sum_{a=1}^{N^2-1} A^a_\mu(x) T^a.$$  \(2.3\)

Lagrangian (2.1) is invariant under the gauge transformations

$$A^a_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g} U(x)\partial_\mu U^{-1}(x),$$

$$\Psi^a(x) = U(x)\Psi(x),$$

where the gauge transformation function $U(x)$ is written as

$$U(x) = e^{i\alpha(x)}$$

with the Lie algebra valued function

$$\alpha(x) = \sum_{a=1}^{N^2-1} \alpha^a(x) T^a.$$  \(2.7\)
The commutator between two Lie algebra valued functions $\alpha(x)$ and $\beta(x)$

$$[\alpha(x), \beta(x)] = \sum_{a,b=1}^{N^2-1} \alpha^a(x)\beta^b(x) [T^a, T^b]$$

$$= \sum_{c=1}^{N^2-1} \left( \sum_{a,b=1}^{N^2-1} if^{abc} \alpha^a(x)\beta^b(x) \right) T^c$$

is also Lie algebra valued function. Thus, they constitute the closed Lie algebra.

Then, we introduce the nonabelian gauge theory on the noncommutative spacetime by employing the Moyal *product

$$f(x) \ast g(x) = e^{\frac{i}{2} \theta_{\mu\nu} \partial_\mu \partial_\nu} f(x_1)g(x_2) \big|_{x_1 = x_2 = x},$$

where $\theta^{\mu\nu}$ is two rank tensor to characterize the noncommutativity of spacetime. Though $\theta^{\mu\nu}$ is usually seemed to be a constant not to transform corresponding to Lorentz transformation, $\theta^{\mu\nu}$ is regarded as two rank tensor in this article as discussed in the next section. Because of the Moyal *product, *commutators between the Lie algebra valued functions don’t close within themselves and extend to the enveloping Lie algebra since Eq. changes to

$$[\alpha(x), \beta(x)]_* = \sum_{a,b=1}^{N^2-1} \left( \alpha^a(x) \ast \beta^b(x)T^a T^b - \beta^b(x) \ast \alpha^a(x)T^b T^a \right)$$

$$= \sum_{c=1}^{N^2-1} \sum_{a,b=1}^{N^2-1} \left( if^{abc} \frac{1}{2} \{\alpha^a(x), \beta^b(x)\}_* T^c + d^{abc} \frac{1}{2} \{\alpha^a(x), \beta^b(x)\}_* \right) T^c$$

$$+ \sum_{a,b=1}^{N^2-1} d^{abc} \frac{1}{2} \{\alpha^a(x), \beta^b(x)\}_* T^0,$$

where

$$[T^a, T^b] = \sum_{c=1}^{N^2-1} if^{abc} T^c, \quad \{T^a, T^b\} = \sum_{c=0}^{N^2-1} d^{abc} T^c.$$  \hspace{1cm} (2.11)

Thus, in general, the enveloping Lie algebra valued functions are written as

$$\alpha(x, \theta) = \sum_{a=0}^{N^2-1} \alpha^a(x, \theta) T^a,$$  \hspace{1cm} (2.12)

where the condition

$$\lim_{\theta \to 0} \alpha^0(x, \theta) = 0$$  \hspace{1cm} (2.13)

should be satisfied according to Eq. (2.11). In terms of the enveloping Lie algebra, we are able to construct the enveloping nonabelian group $\text{SU}(N)^*$ which elements are defined by

$$U(x, \theta) = e^{i\alpha(x, \theta)}$$  \hspace{1cm} (2.14)

with Eq. (2.12). It should be noted that

$$\lim_{\theta \to 0} U(x, \theta) = U(x) = e^{i\sum_{a=1}^{N^2-1} \alpha^a(x) T^a} \in \text{SU}(N),$$  \hspace{1cm} (2.15)

which indicates that the enveloping group $\text{SU}(N)^*$ reduces to nonabelian group $\text{SU}(N)$ when $\theta^{\mu\nu}$ becomes to 0.

Lagrangian of the nonabelian gauge theory on the noncommutative spacetime is simply obtained by replacing the ordinary product with the Moyal *product in Eq.(2.1),

$$\mathcal{L} = -\frac{1}{4} \text{Tr} [F_{\mu\nu}(x) \ast F^{\mu\nu}(x)] + \bar{\Psi}(x) \ast \{i \gamma^\mu \partial_\mu - m\} \ast \Psi(x).$$  \hspace{1cm} (2.16)
where $\mathcal{D}_\mu = \partial_\mu - igA_\mu(x)$ is covariant derivative and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]_s$$

(2.17)

with the definition of *commutator

$$[A(x), B(x)]_s = A(x) * B(x) - B(x) * A(x).$$

(2.18)

Gauge transformations of the fields are given by

$$A^\theta_\mu(x) = U(x, \theta) * A_\mu(x) * U^{-1}(x, \theta),$$

(3.2)

$$\Psi^\theta(x) = U(x, \theta) * \Psi(x),$$

(3.3)

which leads to the commutativity between $\hat{A}^{\mu\nu}$.

Owing to the algebraic rules of Moyal *product, $F_{\mu\nu}(x)$ and $\mathcal{D}_\mu \Psi(x)$ are transformed covariantly

$$F^{\theta}_{\mu\nu}(x) = U(x, \theta) * F_{\mu\nu}(x) * U^{-1}(x, \theta),$$

(2.21)

$$\{\mathcal{D}_\mu \Psi(x)\}^\theta = U(x, \theta) * \mathcal{D}_\mu \Psi(x)$$

(2.22)

Under the gauge transformation of $F_{\mu\nu}$ in Eq. (2.21), the gauge field term in Eq. (2.16) is transformed as in

$$\text{Tr} \left[ F^{\theta}_{\mu\nu}(x) * F^{\mu\nu}\Psi(x) \right] = \text{Tr} \left[ U(x, \theta) * F_{\mu\nu}(x) * F^{\mu\nu}(x) * U^{-1}(x, \theta) \right]$$

(2.23)

which shows the gauge term itself is not gauge invariant because of the Moyal *product but the action is invariant thanks to the rule

$$\int d^4x \ f(x) * g(x) = \int d^4x \ g(x) * f(x).$$

(2.24)

On the other hand, the fermion term in Eq. (2.16) is invariant under gauge transformations Eqs. (2.20) and (2.22).

In order to construct the nonabelian gauge theory on the noncommutative spacetime we start from Eq. (2.16) where the gauge field $A_\mu(x)$ doesn’t contain the 0 component $A^0_\mu(x)$, but it is induced in the gauge transformation. However, the 0 component of $A^\theta_\mu(x)$ in Eq. (2.19) depends on $A_\mu(x)$ in Eq. (2.16) and so, it is not independent field. Moreover it vanishes owing to Eq. (2.15) when the noncommutative parameter $\theta^{\mu\nu}$ approaches to 0. Thus, even if the 0 component of $A^\theta_\mu(x)$ appears in Eq. (2.16), it is out of our considerations. The existing of the enveloping SU(N)* group insures the construction of nonabelian gauge theory on the noncommutative spacetime.

3 Lorentz invariance of noncommutative field theory

Lorentz invariance is the fundamental principle of every relativistic field theory which insures consistent physical descriptions such as causality, unitarity and so on. However, it hasn’t been respected in the study of noncommutative field theory so far. Doplicher, Fredenhagen and Roberts (DFR) first addressed this problem to propose a new algebra of noncommutative spacetime operator $\hat{x}_\mu$.

$$[\hat{x}^\mu, \hat{x}^\nu] = i \hat{\theta}^{\mu\nu},$$

(3.1)

where $\hat{\theta}^{\mu\nu}$ is an antisymmetric tensor operator, not a constant considered so far. They further assumed

$$[\hat{x}^\mu, \hat{\theta}^{\mu\nu}] = 0,$$

(3.2)

which leads to the commutativity between $\hat{\theta}^{\mu\nu}$ through the Jacobi identity

$$[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\sigma\rho}] = 0.$$

(3.3)

Equation (3.3) enables us to simultaneously diagonalize the operator $\hat{\theta}^{\mu\nu}$.

$$\hat{\theta}^{\mu\nu} | \theta > = \theta^{\mu\nu} | \theta >,$$

(3.4)
where \( |\theta> \) is an eigenstate and \( \theta^{\mu \nu} \) is its specific eigenvalue.

Carlson, Carone and Zobin (CCZ) \cite{10} formulated the noncommutative gauge theory by referring to the DFR algebra in the Lorentz invariant way. In their formulation, fields in the theory depend on spacetime \( x^\mu \) and the noncommutative parameter \( \theta^{\mu \nu} \). The action is obtained by integrating Lagrangian over spacetime \( x^\mu \) as well as the noncommutative parameter \( \theta^{\mu \nu} \).

\[
S = \int d^4x d^6\theta \, W(\theta) \mathcal{L}(\phi(x, \theta), \partial^\mu \phi(x, \theta)),
\]

(3.5)

where Lorentz invariant function \( W(\theta) \) is a weight function to render the \( \theta \) integral finite. Following to CCZ, Kase, Morita, Okumura and Umezawa \cite{16} reconsidered the Lorentz invariant noncommutative field theory by pointing out the inconsistency of the c-number \( \theta \)-algebra and indicated that the normalizability of the weight function in Lorentz metric leads to the division of the \( \theta \) space into two disjoint regions not connected by any Lorentz transformation, so that the CCZ covariant moments formula holds in each region separately.

The characteristic features of CCZ \cite{10} are to set up the 6-dimensional \( \theta \) space in addition to \( x \) space and define the action as the integration of Lagrangian over the \( \theta \) space as well as \( x \) space. In this article, we take over the idea of the 6-dimensional \( \theta \) space, but don’t the integration over \( \theta \) space. Let us pick up one specific point \( \theta^{\mu \nu} \) in the 6-dimensional \( \theta \) space that follows from Eq. (3.4). If we denote the Lorentz transformation operator to be \( U(\Lambda) \), the equation

\[
U(\Lambda) \hat{\theta}^{\mu \nu} U^{-1}(\Lambda) = \hat{\theta}'^{\mu \nu}
\]

(3.6)

holds. Since the Lagrangian is invariant under Lorentz transformation, the nontrivial equation

\[
\mathcal{L}(\hat{x}, \hat{\theta}^{\mu \nu}) = \mathcal{L}(\hat{x}', \hat{\theta}'^{\mu \nu})
\]

(3.7)

follows. In order to obtain the Lorentz invariant Lagrangian from this equation, we derive several useful equations. When the Lorentz transformation operator \( U(\Lambda) \) works on Eq. (3.4) the equation

\[
\mathcal{L}(\hat{x}, \hat{\theta}^{\mu \nu}) = \mathcal{L}(\hat{x}', \hat{\theta}'^{\mu \nu})
\]

(3.8)

follows, where

\[
|\theta'\rangle = U(\Lambda) |\theta\rangle, \quad \langle \theta' | = = <\theta| U^{-1}(\Lambda),
\]

\[
\hat{\theta}'^{\mu \nu} = U(\Lambda) \hat{\theta}^{\mu \nu} U^{-1}(\Lambda).
\]

(3.9)

Since the Lorentz transformation for operator \( \hat{\theta}^{\mu \nu} \) is

\[
U(\Lambda) \hat{\theta}^{\mu \nu} U^{-1}(\Lambda) = \Lambda_\rho^\mu \Lambda_\sigma^\nu \hat{\theta}^{\rho \sigma} = \hat{\theta}'^{\mu \nu}
\]

(3.10)

the Lorentz transformation for its eigenvalue \( \theta^{\mu \nu} \) is

\[
\Lambda_\rho^\mu \Lambda_\sigma^\nu \theta^{\rho \sigma} = \hat{\theta}'^{\mu \nu} \iff \Lambda^\mu_\rho \Lambda^\nu_\sigma \theta^{\rho \sigma} = \theta^{\mu \nu}.
\]

(3.11)

where \( \theta^{\mu \nu} \) is defined as

\[
\hat{\theta}^{\mu \nu} |\theta'\rangle = \theta^{\mu \nu} |\theta'\rangle.
\]

(3.12)

Operating \( U^{-1}(\Lambda) \) on the above equation and using Eq. (3.10), we derive

\[
\hat{\theta}'^{\mu \nu} |\theta\rangle = \theta^{\mu \nu} |\theta\rangle.
\]

(3.13)

Sandwiching Eq. (3.7) with \( |\theta\rangle \), we obtain the equation

\[
<\theta| \mathcal{L}(\hat{x}, \hat{\theta}^{\mu \nu}) |\theta\rangle = <\theta| \mathcal{L}(\hat{x}', \hat{\theta}'^{\mu \nu}) |\theta\rangle.
\]

(3.14)

which owing to Eq. (3.13) leads to

\[
\mathcal{L}(\hat{x}, \hat{\theta}^{\mu \nu}) = \mathcal{L}(\hat{x}', \hat{\theta}'^{\mu \nu}).
\]

(3.15)

after the normalization factor \( <\theta|\theta\rangle \) is scaled out.
In this stage, the choice of $\theta^{\mu\nu}$ is arbitrary. However, quantization restricts the allowable region of $\theta^{\mu\nu}$ because when the timelike noncommutativity $\theta^{0i} \neq 0$ exists, the conjugate momentum of a field $\phi$ defined by

$$\Pi = \frac{\partial L}{\partial \dot{\phi}(\mu\nu)} \tag{3.16}$$

is not qualified as an appropriate momentum owing to the infinite series of time derivatives in the Moyal *products. Thus, we can restrict the region of $\theta^{\mu\nu}$ in such a way that we can render $\theta^{0i}$ to be 0 by making an appropriate Lorentz transformation of $\theta^{\mu\nu}$. The unitarity problem pointed out by Gomis and Mehen [17] might vanish by considering the Lorentz invariance of the theory and the proper choice of $\theta^{\mu\nu}$ as discussed above. This problem will be elaborated in another article.

4 BRST symmetry of noncommutative gauge theory

BRST symmetry is very important to quantize the nonabelian gauge theory through which the physical states are normally defined. It also plays an indispensable role in deriving the Ward-Takahashi identity used to prove the renormalization of nonabelian gauge theory. Similarly, it could be important in the case of the nonabelian gauge theory on noncommutative spacetime. This subject was recently studied by Sorouch [18] by determining the BRST transformations of the components of gauge and related fields. Let us here introduce the BRST transformations of fields as the representations of gauge group, not component of fields, which leads us to transparent view of BRST symmetry.

The total Lagrangian of noncommutative gauge theory consists of

$$L = L_G + L_D + L_{GF} + L_{FP}, \tag{4.1}$$

where

$$L_G = -\frac{1}{4} \text{Tr} [F_{\mu\nu}(x) * F^{\mu\nu}(x)], \tag{4.2}$$

$$L_D = \bar{\psi}(x) * \{ i\gamma^\mu (\partial_\mu - igA_\mu(x)) - m \} * \psi(x) = \bar{\psi}(x) * \{ i\gamma^\mu \bar{D}_\mu - m \} * \psi(x), \tag{4.3}$$

$$L_{GF} = \text{Tr} \left( -\partial^\mu B(x) * A_\mu(x) + \frac{\alpha}{2} B(x) * B(x) \right), \tag{4.4}$$

$$L_{FP} = \text{Tr} \left\{ -i\partial^\mu c(x) * (\partial_\mu c(x) - ig[A_\mu(x), c(x)]) \right\} = \text{Tr} \left\{ -i\partial^\mu c(x) * D_\mu c(x) \right\}. \tag{4.5}$$

$L_G$ is the gauge field term with $F_{\mu\nu}(x)$ given in Eq. (4.15). $L_D$ is the Dirac fermion term in the fundamental representation of the gauge group. $L_{GF}$ is the gauge fixing term with the auxiliary field $B(x)$ called the Nakanishi-Lautrup field. $L_{FP}$ is the Faddeev-Popov ghost term. Ghost fields $c(x), \bar{c}(x)$ and Nakanishi-Lautrup field $B(x)$ are denoted with components fields by

$$c(x) = \sum_{a=1}^{N^2-1} c^a(x) T^a, \tag{4.6}$$

$$\bar{c}(x) = \sum_{a=1}^{N^2-1} \bar{c}^a(x) T^a, \tag{4.7}$$

$$B(x) = \sum_{a=1}^{N^2-1} B^a(x) T^a. \tag{4.8}$$

BRST transformations of gauge field $A_\mu(x)$ and fermion field $\Psi(x)$ are derived by replacing $\alpha(x, \theta)$ by $ig\lambda c(x)$ in Eqs. (4.19) and (4.20), respectively

$$\delta_B A_\mu(x) = \lambda (\partial_\mu c(x) - ig[A_\mu(x), c(x)]) = \lambda D_\mu c(x), \tag{4.9}$$

$$\delta_B \Psi(x) = ig\lambda c(x) * \Psi(x). \tag{4.10}$$
where the parameter $\lambda$ is a Grassmann variable not depending on spacetime coordinate $x_\mu$. This parameter $\lambda$ is usually eliminated in the formula. Such BRST transformation is written by $\delta_B$ which satisfies the rule

$$
\delta_B(F * G) = (\delta_B F) * G + (-1)^{|F|} F * (\delta_B G), \tag{4.11}
$$

where $|F|$ is the ghost number of field $F$.

The BRST transformations of fields $c(x)$, $\bar{c}(x)$, and $B(x)$ are determined in order for the BRST operator $\delta_B$ to satisfy the nilpotency such as

$$
\begin{align*}
\delta_B c(x) &= igc(x) * c(x), \tag{4.12} \\
\delta_B \bar{c}(x) &= iB(x), \tag{4.13} \\
\delta_B B(x) &= 0. \tag{4.14}
\end{align*}
$$

Let us show the nilpotency of the BRST operator $\delta_B$, that is $\delta_B^2 = 0$. For the gauge field $A_\mu(x)$, it is proved as

$$
\delta_B^2 A_\mu(x) = \delta_B(\delta_B A_\mu(x)) = \delta_B(\delta_B c(x) - ig[A_\mu(x), c(x)]_\ast) = ig\delta_B c(x) - ig\delta_B A_\mu(x) + igA_\mu(x) * \delta_B c(x).
$$

Owing to the rule (4.11). For other fields,

$$
\begin{align*}
\delta_B^2 \bar{c}(x) &= \delta_B(iB(x)) = 0, \tag{4.15} \\
\delta_B^2 B(x) &= 0. \tag{4.15}
\end{align*}
$$

It should be noted that these BRST transformations of fields expressed in the form of representation of nonabelian group such as in Eqs. (4.12), (4.13) and (4.14) corresponds with those introduced by Soroush [8] if they are represented in components of fields.

The parameter of BRST transformation is the Grassmann variable $\lambda$ which doesn’t depend on $x_\mu$, that means the BRST symmetry is global. In such a case, the conserved current $j_\mu(x)$ exists in commutative field theory according to the Noether theorem. However, in the case of noncommutative field theory, it was deduced by Micu and Sheikh-Jabbari [8] that the divergence of the current is equal to the Moyal *product of the some functions.

$$
\partial^\mu j_\mu(x) = [f(x), g(x)]_\ast, \tag{4.20}
$$

where functions $f(x)$ and $g(x)$ are specific to the symmetry. Here, we would like to indicate the *commutator to be rewritten in the form of the total derivative. It follows that

$$
[f(x), g(x)]_\ast = \left(\epsilon^{\mu_1 \mu_2 \mu_3} \partial_\mu_1 \delta_\mu_2^2 - \epsilon^{-\mu_1 \mu_2 \mu_3} \partial_\mu_1 \delta_\mu_2^2\right) f(x_1)g(x_2) \big|_{x_1 = x_2 = x}
$$

$$
= \partial_\mu \left(\theta^{\mu_1} f(x) \partial_\mu g(x) + \frac{1}{3!} \theta^{\mu_1 \mu_2} \frac{1}{2} \theta^{\mu_2 \mu_3} \partial_\mu \partial_\mu \partial_\mu f(x) \partial_\mu \partial_\mu \partial_\mu g(x) + \cdots\right)
$$

$$
\begin{align*}
= \partial_\mu & \left(\theta^{\mu_1} f(x) \partial_\mu g(x) + \frac{1}{3!} \theta^{\mu_1 \mu_2} \frac{1}{2} \theta^{\mu_2 \mu_3} \partial_\mu \partial_\mu \partial_\mu f(x) \partial_\mu \partial_\mu \partial_\mu g(x) + \cdots\right) \tag{4.21}
\end{align*}
$$

because of the antisymmetric $\theta^{\mu_1 \mu_2}$. Thus, if the current is redefined to be $J_\mu(x) = j_\mu(x) - h_\mu(x)$, $J_\mu(x)$ is the conserved current. As a result, the Noether theorem may cover the noncommutative field theory. This is the case also in the nonabelian gauge theory on noncommutative spacetime.

Let us investigate the BRST current which results from the BRST invariance of the Lagrangian of the nonabelian gauge theory given by Eq. (4.21). From Eq. (2.23),

$$
\mathcal{L}(A'_\mu(x), \Psi'(x), c'(x), \bar{c}'(x), B'(x))
$$
\[-\mathcal{L}(A_\mu(x), \Psi(x), c(x), \bar{c}(x), B(x)) = ig\lambda \text{Tr}(c(x), F_{\mu\nu} * F^{\mu\nu}), \tag{4.22}\]

where the prime signs on fields denote BRST transformed fields. Owing to the equation of motions, the left-hand side of the equation changes to

\[
LHS = -\partial_\mu \left[ \sum_{a=0}^{N^2-1} \left\{ F^a_{\mu\nu}(x, \theta) * \delta_B A^a_\mu(x, \theta) - i\partial^\mu c^a(x, \theta) * \delta_B c^a(x, \theta) \right. \right.
+ i(D^\mu c(x, \theta))^a * \delta_B \bar{c}^a(x, \theta) \left. \right\} + \bar{\Psi}(x, \theta) i\gamma^\mu * \delta_B \Psi(x, \theta) \right]
+ \sum_k [f_k(x, \theta), g_k(x, \theta)]_s, \tag{4.23}\]

where the last term as well as equations of motion are explicitly written in Appendix. Removing the Grassmann parameter \(\lambda\) from the left-hand side, we may define the BRST current

\[
j_\mu = -F^a_{\mu\nu}(x, \theta) * \delta_B A^a_\mu(x) + \bar{\Psi}(x) i\gamma^\mu * \delta_B \Psi(x) \nonumber
- i\partial^\mu c^a(x) * \delta_B c^a(x) + i(D^\mu c(x))^a * \delta_B \bar{c}^a(x), \tag{4.24}\]

where the symbol of the sum over superscript \(a\) is abbreviated. From Eq. (4.23), the equation of continuity of BRST current

\[
\partial^\mu j_\mu = -\frac{1}{\lambda} \sum_k [f_k(x), g_k(x)]_s \tag{4.25}\]

follows.

If the last term of Eq. (4.25) and the right-hand side of Eq. (4.22) are rewritten as in Eq. (4.21), we may define the current \(J_\mu(x)\) which is conserved without any restrictions. However, it is not necessary to do so because our formulation is Lorentz covariant as in Section 3 and the noncommutative parameter \(\theta^{\mu\nu}\) may be taken to be \(\theta^{0\mu} = 0\) if the appropriate Lorentz transformation is carried out. The corresponding BRST charge

\[
Q_B = \int d^3x j_0(x) \tag{4.26}\]

is conserved since

\[
\int d^3x [f(x), g(x)]_s = 0 \tag{4.27}\]

owing to only space-like noncommutativity \(\theta^{0\mu} = 0\).

Space-noncommutativity enables us to quantize the fields without difficulties. The conjugate momentums of fields are defined as

\[
\pi^{a\mu} = \partial \mathcal{L} / \partial \dot{A}^a_\mu = F^{a\mu\alpha}_\mu, \tag{4.28}\]
\[
\pi^a_B = \partial \mathcal{L} / \partial \dot{B}^a = -A^a_0, \tag{4.29}\]
\[
\pi^k_\Psi = \partial \mathcal{L} / \partial \dot{\Psi}_k = \bar{\Psi}^k \delta^0_\beta, \tag{4.30}\]
\[
\pi^a_c = \partial \mathcal{L} / \partial \dot{c}^a = -i\dot{c}_a^a \tag{4.31}\]
\[
\pi^c_\bar{c} = \partial \mathcal{L} / \partial \dot{\bar{c}} = i(D_0 c)^a, \tag{4.32}\]

where \((D_\mu c)^a = \partial_\mu c^a + g f^{abc} A_\mu^b * c^c - g h^{abc} [A_\mu^b, c^c]_s\) and dots on fields denote time derivative. Canonical commutation relations between fields and those conjugate momentums follows from the ordinary quantization

\[
[A^a_j(x, t), \pi^{bk}(y, t)] = i\delta^{ab} \delta^k_\delta \delta(x - y), \tag{4.33}\]
\[
[A^a_0(x, t), \pi^b_\Psi(y, t)] = i\delta^{ab} \delta^3(x - y), \tag{4.34}\]
\[
[\bar{\Psi}^k(x, t), \pi^{ij}_c(y, t)] = i\delta^{kj} \delta^{i\alpha} \delta^3(x - y), \tag{4.35}\]
\[
\{c^a(x, t), \pi^{bj}_c(y, t)\} = \{\bar{c}^c(x, t), \pi^{a}_c(y, t)\} = i\delta^{ab} \delta^3(x - y), \tag{4.36}\]

which yield together with Eqs. (4.28) - (4.32)

\[
[i\lambda Q_B, \Phi_k(x)] = \lambda \delta_\alpha \Phi_k(x), \tag{4.37}\]

where \(\Phi_k(x)\) represents every related field. Equation (4.37) confirms that the charge \(Q_B\) is a generator of BRST symmetry.
5 Scale symmetry of ghost fields

The Lagrangian (4.1) is invariant under the scale transformation
\[ c^a(x) \to e^s c^a(x), \quad \bar{c}^a(x) \to e^{-s} \bar{c}^a(x), \tag{5.1} \]
where \( s \) is a real parameter. It should be noted that ghost fields \( c^a \) and \( \bar{c}^a \) are real fields, so that they haven’t the phase transformation. From the invariance of (4.1) under the transformation (5.1), it follows that
\[ \partial_\mu (\epsilon^* (D_\mu c)^a - \partial_\mu \bar{c}^a \epsilon^a) = gh_{abc} [\partial_\mu \epsilon^a, A^b_\mu]^c, \tag{5.2} \]
from which the Noether current and its charge are derived
\[ j_\mu = i (\bar{c}^a (D_\mu c)^a - \partial_\mu \bar{c}^a \epsilon^a), \tag{5.3} \]
\[ Q_c = i \int d^3 x (\bar{c}^a (D_0 c)^a - \partial_0 \bar{c}^a \epsilon^a). \tag{5.4} \]

\( Q_c \) is called the FP ghost charge and conserved owing to the space-like noncommutativity (\( \theta^{0i} = 0 \)) and Eq. (5.2). \( Q_c \) is also a generator of scale transformation Eq. (5.1) and satisfies that
\[ [i Q_c, c^a(x)] = c^a(x), \quad [i Q_c, \bar{c}^a(x)] = -\bar{c}^a(x), \tag{5.5} \]
which deduces that the number operator of ghost field \( N_{FP} \) is defined as
\[ N_{FP} = i Q_c. \tag{5.6} \]

According to the definitions Eqs. (5.2) and (5.4), the BRST charge and FP ghost charge satisfy the following simple algebra.
\[ \{ Q_B, Q_B \} = 2 Q_B^2 = 0, \tag{5.7} \]
\[ [i Q_c, Q_B] = Q_B, \tag{5.8} \]
\[ [Q_c, Q_c] = 0, \tag{5.9} \]
which may play an important role in the classification of asymptotic fields and the proof of S matrix unitarity.

6 conclusions

The deviation from the standard model in particle physics has not yet observed, and so any model beyond standard model must reduce to it in some approximation. Noncommutative gauge theory must also reproduce standard model in the limit of noncommutative parameter \( \theta^{\mu\nu} \to 0 \). In order to meet this condition, we have formulated following to Jurčo et al. Jurčo the nonabelian gauge theory on the noncommutative spacetime which reduces to the ordinary gauge theory in the limit \( \theta^{\mu\nu} \to 0 \). This is because our noncommutative gauge theory depends on the enveloping group \( SU(N)^* \) that is made of the enveloping Lie algebra with the condition Eq. (2.13). However, It should be investigated if the noncommutative gauge theory thus constructed may clear the allegations pointed out by Armoni[12], which is our future work.

Lorentz covariance is the fundamental principle of every relativistic field theory which insures consistent physical descriptions. Even if the space-time is noncommutative, field theories on it should keep Lorentz covariance. In this paper, we defined the nonabelian gauge theory on noncommutative spacetime where its Lorentz covariance is maintained since one specific eigen state of the noncommutative operator \( \hat{\theta}^{\mu\nu} \) is picked up in the 6 dimensional \( \theta \) space. We can choose the spacelike noncommutativity (\( \theta^{0i} = 0 \)) after the appropriate Lorentz transformation and so any serious outcomes such as quantization and unitarity violation don’t appear.

We also investigated the BRST symmetry of noncommutative nonabelian gauge theory. Our formulation is more transparent that that by Soroush[18] because we consider the BRST transformations of the fields as the representations of gauge group, not component of fields as in [18]. The scale transformation of ghost fields is also investigated. BRST symmetry as well as the scale transformation are
very important to quantize the ordinary gauge theory and achieve the covariant canonical formulation of it. The Ward-Takahashi identity derived from BRST symmetry plays an important role in the proof of renormalizability of the gauge theory. It could be certainly important in the case of the nonabelian gauge theory on noncommutative spacetime. The detailed study about these interesting subjects is our future work.

A equation of motion

\[
\partial_{\mu} F^{\alpha \mu \nu} + g f^{abc} A^b_{\mu} * F^{\alpha \mu \nu} + g \bar{\Psi} \gamma^{\nu} T^a * \Psi - ig f^{abc} \partial^\nu c^b * c^c - \partial^\nu B^a \\
= h^{abc} [A^a_{\mu}, F^{\alpha \mu \nu}] - ig f^{abc} \{ \partial^\nu c^b, c^c \} + g \{ \bar{\Psi} \gamma^{\nu}, \Psi \} \gamma_{\mu \beta} T^a_{ij},
\]

(A.1)

\[
(i \gamma^\mu D^\mu - m) \Psi = 0,
\]

(A.2)

\[
\partial^\mu A^a_{\mu} = 0,
\]

(A.3)

\[
(D^\mu \partial_\mu c)^a + gh^{abc} \{ \partial^\nu c^b, A^c_{\mu} \} = 0,
\]

(A.4)

\[
\partial^\mu A^a_{\mu} + B^a = 0,
\]

(A.5)

where space term coordinate \( x_\mu \) is abbreviated and \( h^{abc} = \frac{1}{2} (f^{abc} + id^{abc}) \).

B *Commutator in Noether theorem

\[
\sum_k \left[ f_k(x), g_k(x) \right] = -\frac{1}{2} \left[ \delta_{\mu} \left( \partial_{\mu} A^a_{\nu} \right), F^{\alpha \mu \nu} \right]_* \\
\quad - \frac{1}{8} g f^{abc} - id^{abc} \left\{ \left[ A^b_{\mu} * \delta_B A^c_{\nu}, F^{\alpha \mu \nu} \right]_* + \left[ F^{\alpha \mu \nu} * \delta_B A^b_{\mu}, A^c_{\nu} \right]_* \right\} \\
\quad - \frac{1}{4} gh^{abc} \left\{ \left[ A^c_{\nu} * \delta_B A^b_{\mu}, F^{\alpha \mu \nu} \right]_* + \left[ F^{\alpha \mu \nu} * \delta_B A^c_{\nu}, A^b_{\mu} \right]_* + g \{ \bar{\Psi} \gamma^{\nu} i \gamma_{\mu \beta}, \delta_B A^b_{\mu}, T^a_{ij} * \Psi \} \gamma_{\mu \beta} \}_* \\
\quad - ig f^{abc} - h^{abc} \{ \partial^\mu c^b, c^c \} + g \delta_B \partial^\mu c^a, A^c_{\mu} \}_* \\
\quad - i\delta_B \partial^\mu \bar{c^a}, (D_\mu c)^a \}_* - ig h^{abc} \{ \partial^\mu c^a, \delta_B c^c, A^b_{\mu} \}_*
\]

(B.1)

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