ADDITIVE EDGE LABELINGS

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Abstract. Let $G = (V, E)$ be a graph and $d$ a positive integer. We study the following problem: for which labelings $f_E : E \to \mathbb{Z}_d$ is there a labeling $f_V : V \to \mathbb{Z}_d$ such that $f_E(i, j) \equiv f_V(i) + f_V(j) \pmod{d}$, for every edge $(i, j) \in E$? We also explore the connections of the equivalent multiplicative version to toric ideals. We derive a polynomial algorithm to answer these questions and to obtain all possible solutions.

1. Introduction

Graph labeling is a broad subject encompassing a myriad variants. In its most general form, it involves assigning a value to each vertex or each edge of a graph, subject to some restrictions. For an extensive list of references on the subject, see the dynamic survey [2].

A classic example of graph labeling is graph coloring. Other examples are harmonious labelings [3] and felicitous labelings [7]. In the present work, we study a problem similar to these last two, but dropping the one-to-one conditions and allowing modular arithmetic over an arbitrary positive integer $d$. The particular case $d = 2$ is applied in [1] to the study of monotone dynamical systems.

Let $G = (V, E)$ be a graph and let $\mathbb{Z}_d$ denote as usual the set of integers modulo $d$. A function $f_E : E \to \mathbb{Z}_d$ is called an $e$-labeling of $G$ and a function $f_V : V \to \mathbb{Z}_d$ is called a $v$-labeling. $(G, f_E)$ denotes the graph $G$ with its edges labeled with $f_E$, and we say it is an $e$-labeled graph.

In this paper, we answer completely the question of when a given labeling of the edges of $G$ with integers modulo $d$, admits a labeling of the nodes of $G$ such that the label of each edge is the sum, modulo $d$, of the labels of its vertices. More formally, we study the following problem.

Problem 1.1. Let $(G, f_E)$ be an $e$-labeled graph. When is there a $v$-labeling $f_V$ of $G$ such that

$$f_E((v, v')) \equiv f_V(v) + f_V(v') \pmod{d}$$

holds for every edge $(v, v') \in E$?

Definition 1.2. We say that an $f_V$ satisfying (1.1) is a valid $v$-labeling of $(G, f_E)$. If such an $f_V$ exists, we say that $f_E$ is an additive $e$-labeling of $G$. We also say that $(G, f_E)$ is an additively $e$-labeled graph.

Note that we are not imposing the restriction that adjacent vertices have different labels.

Once we know that an $e$-labeling $f_E$ of a graph $G$ is additive, we can investigate how many valid $v$-labelings it admits. We denote this number by $\kappa(G, f_E)$. For example, the graph of Figure 1a with the edge labels in $\mathbb{Z}_3$, is additive and admits...
2. Characterization of additive e-labelings

In this section we show that if a given e-labeling is additive, this imposes restrictions on the cycles in $G$. Throughout this work, cycle will not necessarily mean simple cycle. Theorem 2.8 shows that these restrictions are in fact sufficient.

If $C = (V, E)$ is a cycle of length $k$ in $G$, we number its nodes “consecutively” $v_1, \ldots, v_k$ and its edges $e_1, \ldots, e_k$, where $e_i = (v_i, v_{i+1})$ for all $i < k$, and $e_k = (v_k, v_1)$.

**Definition 2.1.** We say that an e-labeled graph $(G, f_E)$ has the even cycle property if every cycle of even length in $G$, with edges $e_1, \ldots, e_{2k}$, satisfies

\begin{equation}
\sum_{l \text{ odd}} f_E(e_l) \equiv \sum_{l \text{ even}} f_E(e_l) \pmod{d}.
\end{equation}

**Definition 2.2.** Let $d$ be an even positive integer. We say that an e-labeled graph $(G, f_E)$ has the odd cycle property if every cycle of odd length in $G$, with edges $e_1, \ldots, e_{2k+1}$, satisfies

\begin{equation}
\frac{d}{2} \sum_{l=1}^{2k+1} f_E(e_l) \equiv 0 \pmod{d}.
\end{equation}

Equivalently, the odd cycle property holds if the sum

\[ \sum_{l=1}^{2k+1} f_E(e_l) \]

is an even number for all odd cycles in $G$. 

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[Figure 1: (a) An e-labeling of a graph with $\mathbb{Z}_3$ and (b) a valid v-labeling of it. (c) A non-additive e-labeling of a graph, again with $\mathbb{Z}_3$.]

A unique valid v-labeling over $\mathbb{Z}_3$, shown in Figure 1b, whereas the e-labeling of the graph of Figure 1c is not additive.

We characterize completely the existence of valid v-labelings in Theorem 2.8 and we compute $\kappa(G, f_E)$ in Theorem 2.9. In fact, we go beyond a theoretical characterization. We present a polynomial algorithm to decide the existence of valid v-labelings of an e-labeled graph $(G, f_E)$ in Theorem 5.5. We moreover show that we can enumerate all such valid v-labelings in polynomial time. We reach our complexity results in Section 5 through the computation of the Smith Normal Form (SNF) of the incidence matrix of the graph and our Theorem 5.4.

In Section 5, we comment on the equivalent multiplicative version Problem 6.1 of Problem 1.1 linking graphs and toric ideals. In particular, we obtain in Theorem 6.2 a modular version of classic results on the implicitization of toric parametrizations.

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Note that if \( d = 2 \), then both properties mean that the number of 1’s in the edges of a cycle of any length is even. This case was studied in \[1\] in a multiplicative setting as in Section 5.

**Definition 2.3.** Let \((G, f_E)\) be an e-labeled graph. We say that \((G, f_E)\) is compatible if the following conditions hold

- \( d \) is odd and \((G, f_E)\) has the even cycle property.
- \( d \) is even and \((G, f_E)\) has both the even and the odd cycle properties.

**Remark 2.4.** The preceding definitions take into account all the cycles of a graph, not just its simple cycles. The example in Figure 2 shows two simple cycles joined by a vertex. Both cycles are e-labeled, and each of them is additive in isolation. However, they assign different labels to the shared vertex. This incompatibility only appears if we check non-simple cycles too.

We show in Theorem 2.4 that the number of conditions to be checked to ensure that \((G, f_E)\) is compatible is in fact “small.”

![Figure 2: Two simple cycles joined by a vertex](image)

**Lemma 2.5.** Let \((G, f_E)\), with \( d \) even, be a connected e-labeled graph satisfying the even cycle property. Let \( C \) be any odd cycle in \( G \). Then \((G, f_E)\) satisfies the odd cycle property if and only if \((2.2)\) holds for \( C \).

**Proof.** We only need to prove one implication. Suppose that \((G, f_E)\) satisfies the even cycle property, and that \((2.2)\) holds for \( C \). Let \( C' \) be an odd cycle in \( G \). Let \( v \in C \) and \( v' \in C' \). Since \( G \) is connected, there is a path \( P \) from \( v \) to \( v' \). Let \( e_1, \ldots, e_{2k+1}, e'_1, \ldots, e'_{2s+1} \) and \( e_1', \ldots, e_r' \) be the edges of \( C \), \( C' \) and \( P \), such that \( v \) is a vertex of \( e_1 \) and of \( e'_1 \), and such that \( v' \) is a vertex of \( e'_1 \) and \( e_r' \). The even cycle property of \((G, f_E)\) applied to the even cycle made up of \( C, P \) from \( v \) to \( v' \), \( C' \) and then \( P \) from \( v' \) to \( v \), implies that

\[
\begin{align*}
&f_E(e_1) - f_E(e_2) + \cdots + f_E(e_{2k+1}) - f_E(e_1') + f_E(e_2') + \cdots + (-1)^r f_E(e_r') + \\
&(-1)^{r+1} f_E(e'_1) + \cdots + (-1)^{r+2s+1} f_E(e'_{2s+1}) + (-1)^r f_E(e_r') + \cdots - f_E(e_1') \\
&\equiv 0 \pmod{d}.
\end{align*}
\]

This is equivalent to

\[
\begin{align*}
&f_E(e_1) - f_E(e_2) + \cdots + f_E(e_{2k+1}) - 2f_E(e_1') + 2f_E(e_2') + \cdots + \\
&2(-1)^r f_E(e_r') + (-1)^{r+1} f_E(e'_1) + \cdots + (-1)^{r+2s+1} f_E(e'_{2s+1}) \equiv 0 \pmod{d}.
\end{align*}
\]

If we multiply both sides by \( d/2 \), and since \( d/2 \equiv -d/2 \pmod{d} \), we get

\[
\frac{d}{2} \left( f_E(e_1) + f_E(e_2) + \cdots + f_E(e_{2k+1}) + f_E(e'_1) + \cdots + f_E(e'_{2s+1}) \right) \equiv 0 \pmod{d}.
\]

Using the odd cycle property of \((C, f_E)\), we get

\[
\frac{d}{2} \left( f_E(e'_1) + \cdots + f_E(e'_{2s+1}) \right) \equiv 0 \pmod{d},
\]

which means that \((2.2)\) holds for \( C' \) too. \(\square\)

We now show that compatibility is a necessary condition for additivity.
Lemma 2.6. If \((G, f_E)\) is an additive \(e\)-labeled graph, then \((G, f_E)\) has the even cycle property.

Proof. Let \(e_1, \ldots, e_{2k}\) be the edges of a cycle of even length in \(G\). Recall that \(e_i = (v_i, v_{i+1})\). Let \(f_V\) be a \(v\)-labeling of \(G\) satisfying \(\text{1.1}\). We have

\[
\sum_{l \text{ even}} f_E(e_l) = \sum_{l \text{ even}} (f_V(v_l) + f_V(v_{l+1})) = \sum_{l \text{ odd}} (f_V(v_l) + f_V(v_{l+1})) = \sum_{l \text{ odd}} f_E(e_l) \quad (\text{mod } d).
\]

Lemma 2.7. If \(d\) is even, and \((G, f_E)\) is an additive \(e\)-labeled graph, then \(G\) has the odd cycle property.

Proof. Let \(e_1, \ldots, e_{2k+1}\) be the edges of a cycle of odd length in \(G\). Let \(f_V\) be a \(v\)-labeling of \(G\) satisfying \(\text{1.1}\). We have

\[
\sum_{l=1}^{2k+1} \frac{d}{2} f_E(e_l) = \sum_{l=1}^{2k+1} \left( \frac{d}{2} f_V(v_l) + \frac{d}{2} f_V(v_{l+1}) \right) = \sum_{l=1}^{2k+1} df_V(v_l) = 0 \quad (\text{mod } d).
\]

In fact, the compatibility conditions are sufficient for additivity.

Theorem 2.8. An \(e\)-labeled graph \((G, f_E)\) is additive if and only if it is compatible.

Clearly, an \(e\)-labeled graph \((G, f_E)\) is additive if and only if every connected component of \(G\) is additive with the labeling induced by \(f_E\). Also, to study the number of valid \(v\)-labelings of an \(e\)-labeled graph \((G, f_E)\), we can assume that \(G\) is a connected graph. Otherwise, if \(G_1, \ldots, G_r\) are the connected components of \(G\), we have

\[
\kappa(G, f_E) = \prod_i \kappa(G_i, f_E).
\]

Theorem 2.9. Let \((G, f_E)\) be a connected additive \(e\)-labeled graph.

If \((G, f_E)\) has no odd simple cycles, \(\kappa(G, f_E) = d\).

If \((G, f_E)\) has at least one odd simple cycle, then

- if \(d\) is odd then \(\kappa(G, f_E) = 1\)
- if \(d\) is even then \(\kappa(G, f_E) = 2\)

The proofs of these theorems occupy the next section.

3. PROOF OF THEOREMS 2.8 AND 2.9

Lemmas 2.6 and 2.7 show that compatibility is a necessary condition for additivity. We now turn our attention to sufficient conditions and to the number of valid \(v\)-labelings that an additive \(e\)-labeled graph admits, through a series of preparatory lemmas.

Lemma 3.1. Let \((G, f_E)\) be a connected additive \(e\)-labeled graph, and suppose that \(f_{V'}, f_{V''}\) are valid \(v\)-labelings of \((G, f_E)\). If there is \(v \in V\) such that \(f_V(v) = f_{V'}(v)\), then \(f_V = f_{V'}\).

Proof. Let \(v \in V\) be such that \(f_V(v) = f_{V'}(v)\). We prove our lemma by induction on the distance from \(v\). Let \(v' \in V\) be at distance 0 from \(v\). Then, \(v = v'\). Now, assume the lemma is true for all \(v'\) at distance from \(v\) smaller than \(k\). Let \(v'\) be at distance \(k\). Let \(\bar{v} \in V\) be such that \(d(v, \bar{v}) = k - 1\) and \(d(\bar{v}, v') = 1\). Then, by our inductive hypothesis, \(f_V(\bar{v}) = f_{V'}(\bar{v})\). Since \(f_V\) and \(f_{V'}\) are valid \(v\)-labelings, \(f_V(v) = f_E(v, \bar{v}) - f_V(\bar{v}) = f_E(v, \bar{v}) - f_{V'}(\bar{v}) = f_{V'}(v)\). \(\square\)
The previous lemma is important because it says that, given a connected additive e-labeled graph, once we fix the label for one vertex, the rest of the vertex labels are fixed. We use this result later on. Furthermore, it shows that $\kappa(G, f_E) \leq d$.

**Definition 3.2.** Given a simple cycle $C$ and three vertices $v, v'$ and $v''$ in $C$, we define $C[v, v', v'']$ as the simple path in $C$ from $v$ to $v'$ that contains $v''$. Conversely, $C[v, v', v'']$ is the simple path from $v$ to $v'$ in $C$ that does not contain $v''$ (see Figure 3).

![Figure 3: Two simple paths from $v$ to $v'$ in $C$.](image)

**Definition 3.3.** Let $P$ be a path with vertices $v_1, \ldots, v_k$ and edges $e_1 = (v_1, v_2), \ldots, e_{k-1} = (v_{k-1}, v_k)$. Let $f_E$ be an e-labeling of $P$. We define a function $\varphi_P : \mathbb{Z}_d \rightarrow \mathbb{Z}_d$.

\[
\varphi_P(c) = (-1)^{k-1}c + \sum_{l=1}^{k-1}(-1)^{k-1}f_E(e_l) \pmod{d}.
\]

In other words, $\varphi_P(c)$ is the label that $v_k$ would have if we assigned label $c$ to $v_1$ and propagated it through $P$.

**Remark 3.4.** Let $(C, f_E)$ be an additive e-labeled simple cycle. Let $v, v', v''$ be in $C$ and set $C_1 = C[v, v', v'']$ and $C_2 = C[v, v', v''']$. Let $f_V$ be a valid v-labeling of $(C, f_E)$. We have

\[
\varphi_{C_1}(f_V(v)) = \varphi_{C_2}(f_V(v)) = f_V(v').
\]

We now prove Theorems 2.8 and 2.9 for simple cycles of odd length.

**Lemma 3.5.** If $(C, f_E)$ is a compatible e-labeled simple cycle of odd length then it is additive. If $d$ is odd, $\kappa(C, f_E) = 1$. If $d$ is even, $\kappa(C, f_E) = 2$.

**Proof.** Let $v_1, \ldots, v_{2k+1}$ be the nodes of the cycle. Suppose that we have a valid $v$-labeling $f_V$. We want to see which are the possible values of $f_V(v_1)$. We need

\[
\varphi_{C[v_1, v_{2k+1}, v_2]}(f_V(v_1)) \equiv \varphi_{C[v_1, v_{2k+1}, v_3]}(f_V(v_1)) \pmod{d}.
\]

We have

\[
\varphi_{C[v_1, v_{2k+1}, v_2]}(f_V(v_1)) = (-1)^{2k}f_V(v_1) + \sum_{l=1}^{2k}(-1)^{2k-1}f_E(e_l) \pmod{d},
\]

and

\[
\varphi_{C[v_1, v_{2k+1}, v_3]}(f_V(v_1)) = f_E(e_{2k+1}) - f_V(v_1) \pmod{d}.
\]

Merging these two expressions with \(d\) we get

\[
(-1)^{2k}f_V(v_1) + \sum_{l=1}^{2k}(-1)^{2k-1}f_E(e_l) \equiv f_E(e_{2k+1}) - f_V(v_1) \pmod{d}.
\]
Since $2k$ is even, this expression is equivalent to
\begin{equation}
2f_V(v_1) \equiv \sum_{l=1}^{2k+1} (-1)^{l+1} f_E(e_l) \pmod{d}.
\end{equation}

If $d$ is odd, then $2$ is invertible modulo $d$ and equation (3.3) has a unique solution. That implies that there is at most one possible value for $f_V(v_1)$. Since this value gives a valid v-labeling, there is a unique valid v-labeling of $(C, f_E)$.

If $d$ is even, we use the odd cycle condition. Recall that this implies that the sum of the labels of the edges in the cycle is an even number. Since changing the sign of some summands does not alter the parity of a sum, the right side of (3.3),
\[ \ell := \sum_{l=1}^{2k+1} (-1)^{l+1} f_E(e_l), \]
is also even.

Equation (3.3) is then of the form
\[ 2X \equiv 2b \pmod{2c}. \]
This equation has exactly two solutions: $X = b$ and $X = b + c$. This means that $f_V(v_1)$ is either $\ell/2$ or $(\ell + d)/2$. Since these two values for $f_V(v_1)$ give valid v-labelings, our proof is complete.

This proof allows us to deduce the following

**Corollary 3.6.** Let $(C, f_E)$ be an additive e-labeled simple cycle of odd length, with $d$ even. If $f_V$ and $f'_V$ are its two different valid v-labelings, then $f_V(v) \equiv f'_V(v) + d/2 \pmod{d}$ for all $v \in V$.

Let $(G, f_E)$ be an e-labeled graph. In the following proofs, we abuse our notation. If $C$ is a subgraph of $G$, then $(C, f_E)$ stands for the graph $C$ labeled with the restriction of $f_E$ to the edges of $C$.

**Lemma 3.7.** Let $(G, f_E)$ be a compatible e-labeled connected graph. Let $C$ and $C'$ be two cycles of odd length in $G$. Let $e_1, \ldots, e_r$ be the edges of $C$ and $e'_1, \ldots, e'_s$ be the edges of $C'$. Assume that $C$ and $C'$ share at least one vertex $v_1$, such that both $e_1$ and $e'_1$ are incident to $v_1$. Then
\begin{equation}
\sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) \equiv \sum_{l=1}^{s} (-1)^{s-l} f_E(e'_l) \pmod{d}.
\end{equation}

**Proof.** Consider the cycle obtained by traversing $e_1, \ldots, e_r, e'_1, \ldots, e'_s$. Since $r$ and $s$ are odd, this cycle has even length. The compatibility hypothesis implies that
\begin{equation}
f_E(e_1) - f_E(e_2) + \cdots + f_E(e_r) - f_E(e'_1) + f_E(e'_2) - \cdots - f_E(e'_s) \equiv 0 \pmod{d}.
\end{equation}
But this means
\begin{equation}
\sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) - \sum_{l=1}^{s} (-1)^{s-l} f_E(e'_l) \equiv 0 \pmod{d},
\end{equation}
which is what we wanted to prove. \qed

**Proof of Theorems 2.8 and 2.9.** Let $(G, f_E)$ be a compatible e-labeled graph. Without loss of generality, we can assume that it is connected. We prove the theorems by constructing a valid v-labeling of it.

If $G$ has odd simple cycles, call one of them $C$. Choose a valid v-labeling $f$ of $(C, f_E)$. Pick a vertex $v$ in $C$ and set $\ell = f(v)$. If $G$ has no odd cycles, choose any vertex $v$ in $G$ and label it with any $\ell$ in $\mathbb{Z}_d$. \qed
We build a valid v-labeling \( f_V \) of \((G, f_E)\) by propagating the label of \( v \) to the rest of the graph. For that, set \( f_V(v) = \ell \). For any vertex \( v' \in V \), choose a path \( P \) from \( v \) to \( v' \) and set
\[
f_V(v') = \varphi_P(\ell),
\]
where \( \varphi_P \) is as in (3.11). We have to prove that \( f_V \) is well defined and that it is a valid v-labeling of \((G, f_E)\).

Given \( v' \) and two simple paths \( P_1 \) and \( P_2 \) from \( v \) to \( v' \), we have to prove that
\[
\varphi_{P_1}(\ell) = \varphi_{P_2}(\ell).
\]
Let \( e_1, \ldots, e_r \) and \( e'_1, \ldots, e'_s \) be the edges of \( P_1 \) and \( P_2 \), respectively, and assume that \( v \) is an endpoint of \( e_1 \) and \( e'_1 \). We call \( C' \) the cycle formed by the union of \( P_1 \) and \( P_2 \).

If the sum of the lengths of \( P_1 \) and \( P_2 \) is even, we can use the even cycle property of \((G, f_E)\) applied to \( C' \). That is,
\[
f_E(e_1) - f_E(e_2) + \cdots + (-1)^r f_E(e_r) + (-1)^{r+1} f_E(e'_r) + \cdots - f_E(e'_1) \equiv 0 \pmod{d}
\]
This condition is equivalent to the identity
\[
(3.7) \quad \sum_{l=1}^{r} (-1)^l f_E(e_l) \equiv \sum_{l=1}^{s} (-1)^l f_E(e'_l) \pmod{d}.
\]
We have that
\[
(3.8) \quad \varphi_{P_1}(\ell) = (-1)^r \ell + \sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) \pmod{d},
\]
and
\[
(3.9) \quad \varphi_{P_2}(\ell) = (-1)^s \ell + \sum_{l=1}^{s} (-1)^{s-l} f_E(e_l) \pmod{d}.
\]
We must prove that \( \varphi_{P_1}(\ell) = \varphi_{P_2}(\ell) \). Since the parity of \( r \) and \( s \) are the same, \((-1)^r \ell = (-1)^s \ell \), and we just need to prove that
\[
(3.10) \quad \sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) \equiv \sum_{l=1}^{s} (-1)^{r-l} f_E(e_l) \pmod{d}.
\]
If \( r \) and \( s \) are even, \((-1)^{r-l} = (-1)^{s-l} = (-1)^l \) for any integer \( l \). Therefore, (3.7) shows that (3.10) holds. If \( r \) and \( s \) are odd, \((-1)^{r-l} = (-1)^{s-l} = (-1)^{l+1} \) for any integer \( l \), and again (3.7), this time multiplied by \(-1\), shows that (3.10) holds.

If \( r \) is odd and \( s \) is even, the cycle \( C' \) has odd length. We need to prove that \( \varphi_{P_1}(\ell) = \varphi_{P_2}(\ell) \), which is equivalent to
\[
(3.11) \quad -\ell + \sum_{l=1}^{r} (-1)^{l+1} f_E(e_l) \equiv \ell + \sum_{l=1}^{s} (-1)^{l+1} f_E(e_l) \pmod{d}.
\]
This is the same as proving that
\[
(3.12) \quad 2\ell \equiv \sum_{l=1}^{r} (-1)^{l+1} f_E(e_l) + \sum_{l=1}^{s} (-1)^{l+1} f_E(e_l) \pmod{d}.
\]
The right side of (3.12) is the alternating sum of the labels of the edges of the odd cycle \( C' \), starting at \( v \). By Lemma 3.7, this sum is equal, modulo \( d \), to the alternating sum of the labels of the edges of \( C \), starting at \( v \). By Lemma 3.5, this sum is equivalent to \( 2\ell \), which is what we needed to prove.

We now know that \( f_V \) is a well-defined labeling. We must show that it is also a valid v-labeling of \((G, f_E)\). That is, for each edge \((v', v'')\),
\[
(3.13) \quad f_E((v', v'')) \equiv f_V(v') + f_V(v'') \pmod{d}.
\]
All the edges incident to \( v \) satisfy (3.13) by the previous argument. Let \( v' \) and \( v'' \) be two adjacent vertices in \( G \), both different from \( v \). Let \( e \) be the edge between \( v' \) and \( v'' \). Let \( P_1 \) and \( P_2 \) be paths from \( v \) to \( v' \) and \( v'' \), respectively. Let \( e_1, \ldots, e_r \) and \( e'_1, \ldots, e'_s \) be the edges of \( P_1 \) and \( P_2 \), respectively. We must prove that

\[
\varphi_{P_1}(\ell) + \varphi_{P_2}(\ell) \equiv f_E(e) \pmod{d}.
\]

Consider the path \( P'_2 = P_2 \cup \{e\} \). \( P'_2 \) and \( P_1 \) are two paths from \( v \) to \( v' \). If we write \( e'_{r+1} = e \), we have just proved that

\[
(3.15) \quad (-1)^r \ell + \sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) \equiv (-1)^{s+1} \ell + \sum_{l=1}^{s+1} (-1)^{s+1-l} f_E(e'_l) \pmod{d}.
\]

But the right side of (3.15) can be split

\[
(3.16) \quad \sum_{l=1}^{s+1} (-1)^{s+1-l} f_E(e'_l) = f_E(e) + \sum_{l=1}^{s} (-1)^{s+1-l} f_E(e'_l).
\]

So joining (3.15) and (3.16), we get

\[
(3.17) \quad (-1)^r \ell + \sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) + (-1)^s \ell + \sum_{l=1}^{s} (-1)^{s-l} f_E(e'_l) \equiv f_E(e) \pmod{d},
\]

which proves (3.14). \( \square \)

4. Other results on Compatibility

**Lemma 4.1.** Given a compatible e-labeled graph \((G, f_E)\), if we add any edge \( e \) to \( G \), there is an extension of \( f_E \) that assigns a label to \( e \) such that the resulting e-labeled graph is compatible.

**Proof.** If we add an edge to a graph \( G \), we can have three mutually exclusive situations:

i) We add an edge and its two endpoints. In that case, we are adding a new connected component which consists of a tree, which we know to be compatible.

ii) We add an edge \((u, v)\), and one of its two endpoints \( v \), the other one already being in \( G \). We can extend \( f_E \) by assigning any value to \( f_E((u, v)) \). We extend \( f_V \) by setting \( f_V(v) = f_E((u, v)) - f_V(u) \pmod{d} \). We then get that the augmented graph is additive, and hence compatible.

iii) We add an edge between two nodes of \( G \). Let \( f_V \) be a valid \( v \)-labeling of \((G, f_E)\). We extend \( f_E \) to the new edge \((u, v)\) by setting \( f_E((u, v)) = f_V(u) + f_V(v) \pmod{d} \). This shows that the augmented graph is additive, and hence compatible. \( \square \)

**Corollary 4.2.** Given a graph \( G \) with some of its edges labeled in \( \mathbb{Z}_d \) by a function \( f \). If the subgraph of \( G \) induced by the domain of \( f \), labeled with \( f \), is compatible, then there is an extension \( f_E \) of \( f \), such that \((G, f_E)\) is a compatible e-labeled graph.

**Proof.** The result follows from Lemma 4.1. We first decide whether the subgraph induced by \( f \) is compatible. If it is, we add the remaining edges of \( G \) one by one. \( \square \)
5. An efficient additivity test

Theorems 2.8 and 2.9 give a theoretical characterization of additive e-labeled graphs. These results are not practical per se, since they involve verifying certain conditions on all the cycles of a graph. In this section we develop a polynomial algorithm to test for additivity.

We tackle this problem by studying $A_G$, the incidence matrix of $G$. That is, $A_G \in \mathbb{Z}^{n \times m}$ such that

$[A_G]_{i,j} = \begin{cases} 1 & \text{if vertex } v_i \text{ is incident with edge } e_j, \\ 0 & \text{if not} \end{cases}$

We use the Smith Normal Form (SNF) $S$ of $A_G$ together with the left and right multipliers $U, V$. Here, $U \in \mathbb{Z}^{n \times n}, V \in \mathbb{Z}^{m \times m}, S \in \mathbb{Z}^{n \times m}$ have the following properties:

- $U$ and $V$ are unimodular,
- $S$ is a diagonal matrix, $s_{i,i} | s_{i+1,i+1}$ for all $i$, and
- $A_G = USV$.

The authors of [4] show that the SNF $S$ of $A_G$ is

$$[D \quad 0],$$

where $D = \text{diag}(1, \ldots, 1, \alpha)$ is a diagonal matrix with 1’s in every entry but the last one, which we call $\alpha$. This last entry is 0 if $G$ is bipartite (i.e. has no odd cycles) or 2 if it is not.

**Definition 5.1.** Let $G = (V, E)$ be a graph and let $C$ be any cycle of $G$. We associate a vector $\omega_C \in \mathbb{Z}^{|E|}$ with $C$. We index the coordinates of $\omega_C$ using the edges of $G$.

Label the consecutive edges of $C$

$$e_1, e_2, \ldots, e_{k-1}, e_k,$$

with $e_1$ any edge of the cycle. If $C$ is an even cycle, we adjoin $(-1)^i$ to $e_i$:

$$e_1, -e_2, \ldots, (-1)^i e_i, \ldots, e_{k-1}, -e_k.$$

If $C$ is an odd cycle and $d$ is even, we adjoin $d/2$ to each edge:

$$\frac{d}{2} e_1, \ldots, \frac{d}{2} e_i, \ldots, \frac{d}{2} e_k.$$

Since $C$ need not be a simple cycle, some edges may appear more than once in (5.2). Let $e'_1, \ldots, e'_t$ be the distinct edges of $C$. For each distinct edge $e'_i$, we define $\omega_{e'_i}$ to be the sum of the numbers adjoined to each appearance of $e'_i$ in (5.3) or (5.4). For example, if an edge $e'_i$ appears twice, both times accompanied by a 1, then the corresponding $\omega_{e'_i}$ is 2. If one of the appearances has a 1 and the other one a $(-1)$, then $\omega_{e'_i}$ is 0.

Given a cycle $C$, we define $\omega_C$ as

$$\omega_C(u,v) = \begin{cases} \omega(u,v) & \text{if } (u,v) \text{ is in } C, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that in (5.3), the choice of $e_1$ may swap the 1’s and the $-1$’s. This is not problematic, since it only changes $\omega_C$ into $-\omega_C$. The $\omega_C$, with $C$ of even length, are in the kernel of the incidence matrix of $G$ and we only use them in that context.

**Lemma 5.2.** Let $C$ be a cycle of $G$. If the length of $C$ is even, then the sum of the coordinates of $\omega_C$ is 0. If the length of $C$ is odd, then the sum of the coordinates of $\omega_C \equiv d/2 \pmod{d}$. 
Proof. Notice that we have the same number of edges accompanied by 1 as the number of edges accompanied by −1. The sum of the coordinates of \( \omega_C \) is the sum of all these 1’s and −1’s, and is therefore 0.

The sum of the coordinates of \( \omega_C \) is an odd integer multiple (i.e. the number of edges of \( C \)) of \( d/2 \).

Let \( (G, f_E) \) be an \( e \)-labeled graph, \( \omega \in \mathbb{Z}^{|E|} \). We denote
\[
(5.10) \quad S := \sum_{(u,v) \in E} \omega_{(u,v)} f_E((u,v)).
\]

Let \( \pi_d : \mathbb{Z}^{|E|} \to \mathbb{Z}_d^{|E|} \) denote the projection
\[
\pi_d(x)_{(u,v)} = r_d(x_{(u,v)}),
\]
where \( r_d \) is the remainder modulo \( d \). Finally, we denote by \( C \) the set of even cycles in \( G \).

The integer kernel of \( A_G \) is computed in [9], and is shown to be the submodule spanned by \( \{ \omega_C, C \in C \} \):
\[
(5.7) \quad \ker_Z(A_G) = \langle \omega_C, C \in C \rangle.
\]

We prove a modular version of this result. Given \( M \in \mathbb{Z}^{a \times b} \), we define \( \ker_{Z_d}(M) = \{ x \in \mathbb{Z}_d^b, Mx \equiv 0 \pmod{d} \} \).

**Proposition 5.3.** Let \( G \) be a connected graph, and let \( A_G \) be its incidence matrix. Then
i) If \( d \) is odd or if \( G \) has no odd cycles, then \( \ker_{Z_d}(A_G) = \pi_d(\ker_Z(A_G)) \).

ii) If \( d \) is even and there is an odd cycle \( C' \) in \( G \), then
\[
\ker_{Z_d}(A_G) = \pi_d(\ker_Z(A_G)) \oplus \langle \pi_d(\omega_{C'}) \rangle.
\]

Proof. In this proof, \( \{ z_1, \ldots, z_m \} \) denotes the canonical basis of \( \mathbb{Z}^m \). That is, \( (z_i)_i = 1 \) and \( (z_i)_j = 0 \), for \( j \neq i \). Analogously, \( \{ \pi_d(z_1), \ldots, \pi_d(z_m) \} \), denotes the canonical basis of \( \mathbb{Z}_d^m \).

Let \( S \) be the SNF of \( A_G \), and \( U, V \) such that \( A_G = USV \), as described in (5.1). Equivalently, \( U^{-1}A_G = SV \). Since \( U \) and \( V \) are both unimodular, they have integer inverses and they have integer inverses modulo \( d \). Therefore \( \ker_Z(A_G) = \ker_Z(SV) \) and \( \ker_{Z_d}(A_G) = \ker_{Z_d}(SV) \), implying that
\[
(5.8) \quad \ker_Z(A_G) = V^{-1}\ker_Z(S)
\]
\[
(5.9) \quad \ker_{Z_d}(A_G) = \pi_d(V^{-1}\ker_{Z_d}(S))
\]

Let \( x = (x_1, \ldots, x_m) \in \ker_{Z_d}(S) \). That means that
\[
(5.10) \quad Sx = \begin{pmatrix} x_1 \\ \vdots \\ \alpha x_m \end{pmatrix} \equiv 0 \pmod{d}
\]

If \( \alpha = 0 \) (i.e. \( G \) has no odd cycles), equation (5.10) holds if and only if \( x_1 = \cdots = x_{n-1} = 0 \). That means that
\[
\ker_{Z_d}(S) = \langle \pi_d(z_n), \ldots, \pi_d(z_m) \rangle \quad \text{and} \quad \ker_Z(S) = \langle z_n, \ldots, z_m \rangle.
\]
Therefore, we have \( \ker_{Z_d}(S) = \pi_d(\ker_Z(S)) \), whence \( \ker_{Z_d}(A_G) = \pi_d(\ker_Z(A_G)) \).

If \( \alpha = 2 \) (i.e. \( G \) has an odd cycle) and \( d \) is odd, equation (5.10) holds if and only if \( x_1 = \cdots = x_n = 0 \). That means that
\[
\ker_{Z_d}(S) = \langle \pi_d(z_{n+1}), \ldots, \pi_d(z_m) \rangle.
\]

Once more,
\[
\ker_Z(S) = \langle z_{n+1}, \ldots, z_m \rangle.
\]
And again \( \ker_{Z_d}(S) = \pi_d(\ker_{Z}(S)) \), implying \( \ker_{Z_d}(A_G) = \pi_d(\ker_{Z}(A_G)) \).

We now assume that \( \alpha = 2 \) and that \( d \) is even. From equation (5.10) we now deduce that

\[
\ker_{Z_d}(S) = \langle \pi_d(z_{n+1}), \ldots, \pi_d(z_m) \rangle \oplus \langle \frac{d}{2} \pi_d(z_n) \rangle,
\]

(5.11)

\[
\ker_{Z}(S) = \langle z_{n+1}, \ldots, z_m \rangle.
\]

Notice that

\[
\left\{ \frac{d}{2} \pi_d(z_n) \right\} = \{ 0, \frac{d}{2} \pi_d(z_n) \}.
\]

(5.13)

Combining equations (5.8), (5.9), (5.11) and (5.12), we have

\[
\ker_{Z_d}(A_G) = \langle \pi_d(V^{-1} \pi_d(z_{n+1})), \ldots, \pi_d(V^{-1} \pi_d(z_m)) \rangle \oplus \langle \pi_d(V^{-1} \frac{d}{2} \pi_d(z_n)) \rangle.
\]

(5.14)

\[
\ker_{Z}(A_G) = \langle V^{-1} z_{n+1}, \ldots, V^{-1} z_m \rangle.
\]

(5.15)

Let \( C' \) be an odd cycle of \( G \). Then \( \pi_d(\omega_{C'}) \in \ker_{Z_d}(A_G) \). To see why, recall that entry \( e_j \) of \( \omega_{C'} \) is \( d/2 \) times the number of occurrences of the edge \( e_j \) in \( C' \). For every vertex \( v_i \) of the cycle, the number of edges that enter and leave it must be the same. That means that the \( v_i \)-th entry of \( A_G \omega_{C'} \) has an even number times \( d/2 \) (if vertex \( v_i \) is in the cycle) or 0. In both cases, \( A_G \omega_{C'} \equiv 0 \) (mod \( d \)).

Now, since \( \pi_d(\omega_{C'}) \in \ker_{Z_d}(A_G) \), we must have

\[
\pi_d(\omega_{C'}) = \sum_{l=n+1}^{m} \gamma_l \pi_d(V^{-1} z_l) + \varepsilon \pi_d(V^{-1} \frac{d}{2} \pi_d(z_n)),
\]

(5.16)

where \( \varepsilon \) is 0 or 1 (see (5.13)). The first summand consists of multiples of the projections of even cycles (see (5.7)). That means that if we take the sum of the coordinates of both sides of equation (5.16), we get \( \varepsilon = 1 \) (see Lemmas 5.2 and 5.2).

If we set

\[
\gamma = \sum_{l=n+1}^{m} \gamma_l \pi_d(V^{-1} z_l),
\]

we can write

\[
\pi_d(V^{-1} \frac{d}{2} \pi_d(z_n)) = \gamma - \pi_d(\omega_{C'})
\]

(5.17)

Now, take any \( x \in \ker_{Z_d}(A_G) \). We have that

\[
x = \sum_{l=n+1}^{m} \beta_l \pi_d(V^{-1} z_l) + \beta \pi_d(V^{-1} \frac{d}{2} \pi_d(z_n)).
\]

Plugging in equation (5.17) we get

\[
x = \sum_{l=n+1}^{m} \beta_l \pi_d(V^{-1} z_l) + \beta(\gamma - \pi_d(\omega_{C'})).
\]

If we set \( \tilde{\beta}_l = \beta_l + \gamma_l \), we have

\[
x = \sum_{l=n+1}^{m} \tilde{\beta}_l \pi_d(V^{-1} z_l) + (-\beta) \pi_d(\omega_{C'}),
\]

which shows that

\[
\ker_{Z_d}(A_G) = \pi_d(\ker_{Z}(A_G)) \oplus \langle \pi_d(\omega_{C'}) \rangle.
\]

\[\square\]

The results we have discussed allow us to obtain the following
Theorem 5.4. Let \((G, f_E)\) be an e-labeled connected graph. Let \(A_G\) be the incidence matrix of \(G\). The following statements are equivalent:

i) \((G, f_E)\) is a compatible e-labeled graph.

ii) \(\langle \pi_d(\omega_C), f_E \rangle \equiv 0 \pmod{d}\), for every cycle \(C\) of \(G\).

iii) \(\langle \omega, f_E \rangle \equiv 0 \pmod{d}\), for all \(\omega \in \ker_{Z^d}(A_G)\).

iv) If \(d\) is odd or \(G\) has no odd cycles, \(\langle \omega, f_E \rangle \equiv 0 \pmod{d}\), for all \(\omega\) belonging to the projection of a finite set of generators of \(\ker_{Z^d}(A_G)\). If \(d\) is even and has an odd cycle, \(\langle \omega, f_E \rangle \equiv 0 \pmod{d}\), for all \(\omega\) belonging to a finite set of generators of \(\ker_{Z^d}(A_G)\) and for \(\omega_C\), for some odd cycle \(C\).

v) \((G, f_E)\) is an additive e-labeled graph.

Proof. Clause i) is equivalent to clause v) by Theorem 2.8. Clause ii) is a restatement of clause i) using a different notation. Clauses ii) and iii) are equivalent by Proposition 5.3. Clauses iii) and iv) also follow from that proposition: the finite sets described in clause iv) were shown to be generators of \(\ker_{Z^d}(A_G)\). ⊓⊔

The equivalence of clauses iv) and v) in the above Theorem, provides the following complexity result.

Theorem 5.5. Let \((G, f_E)\) be an e-labeled connected graph. The additivity of \((G, f_E)\) can be tested in time polynomial in the size of the graph. Furthermore, we can obtain all its valid v-labelings in polynomial time too.

Proof. We compute the Smith Normal Form (SNF) \(S\) of \(A_G\) described in the proof of Proposition 5.3, together with the left and right multipliers \(U\) and \(V\). This computation can be carried out using the polynomial algorithm presented in [9], modified to work with rectangular matrices in the way the authors of that paper suggest.

We saw in Proposition 5.3 that we can obtain generators of \(\ker_{Z^d}(A_G)\) from the columns of \(V^{-1}\). If \(G\) has no odd cycles (i.e. \(\alpha = 0\)), we use the last \(m - n + 1\) columns. If \(\alpha = 2\) and \(d\) is odd, we use the last \(m - n\) columns. If \(\alpha = 2\) and \(d\) is even, we use the last \(m - n\) columns and \(d/2\) times its \(n\)-th column. To check the additivity of \((G, f_E)\), we just need to verify that these generators satisfy the conditions stated in clause iv) of Theorem 5.4.

Once we know that \((G, f_E)\) is additive, we can efficiently obtain all its valid v-labelings. We must first know whether \(G\) has an odd cycle or not. This can be read directly off the SNF \(S\) of \(A_G\): \(G\) has an odd cycle if and only if the diagonal of \(S\) contains a 2. Having no odd cycles is classically known to be equivalent to \(G\) being bipartite (cf. for instance [5], p. 18), and can be checked in time \(O(n + m)\). We can also obtain an odd cycle in \(G\) as a byproduct of this check.

If \(G\) has no odd cycles, we can assign any of the \(d\) possible labels to an arbitrary vertex, and then propagate the label to the rest of the graph using breadth-first search (BFS). If \(G\) does have odd cycles, choose one of them and call it \(C\). Choose a vertex \(v_1\) in \(C\). Formula (3.3) shows which label (or labels, if \(d\) is even) we can assign to \(v_1\) in order to obtain valid v-labelings of \((G, f_E)\). We then propagate the label of \(v_1\) to the rest of the graph using BFS. ⊓⊔

Remark 5.6. Given a graph \(G\), consider the cycle space of \(G\) ([5]). It is the \(Z_2\)-vector space generated by the fundamental cycles of \(G\). That is, the cycles obtained when adding an edge of \(G\) to a spanning tree.

One might be tempted to think that checking the compatibility conditions on these generators suffices to verify the compatibility of a graph with labels in \(Z_d\) for any \(d\), as in the case \(d = 2\). However, consider for instance the graph in Figure 3 in which we marked the spanning tree with edges \(\{e_{14}, e_{23}, e_{24}\}\): The sum of the two fundamental triangle cycles \(C_1, C_2\) (represented by their 0,1 vectors) equals the
square cycle $C$ only when $d = 2$. This situation is depicted informally in Figure 5. However, if $d$ is odd we do not impose any conditions on $C_1$ and $C_2$, and so this

$$C_1 \oplus C_2 = C \pmod{2}$$

Figure 5: Adding two odd cycles to obtain an even one.

cannot insure the even cycle condition we need to check. When $d \neq 2$ is even, we get $d$ times the even cycle condition, which again is not sufficient to insure additivity. Consider for instance the labeling $f(e_{12}) = f(e_{24}) = f(e_{34}) = 1$, $f(e_{14}) = f(e_{23}) = 0$ and $d = 4$. The odd cycle property is verified for $C_1, C_2$ but the labeling is not additive.

6. Multiplicative version

In the previous sections, we used labelings that assigned integers modulo $d$ to the edges and vertices of a graph. But actually, everything we wrote is also valid if the labels belong to any finite cyclic group, via the isomorphism with $\mathbb{Z}_d$. In particular, we can use labelings in $G_d$, the $d$-th roots of unity. In this case, the isomorphism between $\mathbb{Z}_d$ and $G_d$ is given by

$$k \mapsto e^{2\pi ik/d}.$$

This alternate formulation is useful because it links our problem with the theory of toric ideals. As a general text on this subject, we refer the reader to [8].

Let us state this equivalent version. Let $G = (V, E)$ be a connected graph and $d$ an integer greater than 1. Let $n = |V|$ and $m = |E|$. Let $v_1, \ldots, v_n$ be the vertices of $G$ and let $e_1, \ldots, e_m$ be its edges. We work with complex variables $x_v$ for each $v_i \in V$, and $y_e$ for each $e_i \in E$. The value of $x_{v_i}$ corresponds to the label of vertex $v_i$, and the value of $y_{e_i}$ corresponds to the label of edge $e_i$. We can restate Problem 1.1 in this multiplicative setting:

**Problem 6.1.** For which $y \in G_d^m$ are there $x \in G_d^n$ such that

$$y_{e_i} = x_{u_i} x_{v_i},$$

holds for every edge $e_i = (u_i, v_i) \in E$?

According to a classic result for toric parametrizations, given a vector $y \in (\mathbb{C}^*)^m$ of complex nonzero numbers, there is an $x \in (\mathbb{C}^*)^n$ satisfying (6.2) if and only if

$$y^u = y_1^{u_1} \cdots y_m^{u_m} = 1,$$
for every \( u = (u_1, \ldots, u_m) \in \ker_\mathbb{Z}(A_G) \). Furthermore, when these conditions are satisfied, the number of such solutions is
\[
6.4 \quad g = \gcd(\{\text{maximal minors of } A_G\}),
\]
provided that \( g \neq 0 \), in which case there are infinitely many solutions. We deduce from (6.1) that \( g = 2 \) or \( 0 \), depending on whether \( G \) has an odd cycle or not, respectively. It was this result which prompted us to study the incidence matrix of \( G \) in connection with Problem 1.1.

We now state a modular version of the toric result. We impose the additional restriction that
\[
6.5 \quad x_{v_i}^d = 1,
\]
for all \( v_i \in V \). This condition, together with (6.2), implies that the \( y_e \) are also in \( G_d \).

**Theorem 6.2.** Let \( G = (V, E) \) be a connected graph. Given \( y \in G_d^m \), there exists \( x \in G_d^n \) satisfying (6.2) if and only if
\[
6.6 \quad y^u = 1,
\]
for every \( u \in \ker_\mathbb{Z}(A_G) \). If \( g \) is 0, there are \( d \) solutions to (6.2) and (6.5) simultaneously. If \( g \) is 2 and \( d \) is even, there are two solutions. Otherwise, there is a unique solution.

The result can be translated from Theorem 2.9. Alternatively, we could prove that given \( y \in G_d^m \), there are as many solutions \( x \in G_d^n \) as stated using the knowledge of \( g \) in (6.4), by checking how many of the complex solutions \( x \in (\mathbb{C}^*)^n \) consist of \( d \)-th roots of unity.

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