Inflation and tachyon model of Barrow holographic dark energy

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We study the connection between the tachyon dark energy model and Barrow holographic dark energy (BHDE). The latter is a modified scenario based on the application of the holographic principle with Barrow entropy instead of the usual Bekenstein-Hawking one. We reconstruct the dynamics of the tachyon scalar field both in the presence and absence of interactions between dark energy and matter. For a flat Friedmann-Robertson-Walker Universe, it is shown that the time derivative of the tachyon field $T^2$ vanishes. This leads to the condition $\omega_D = -1$ for the equation of state parameter of BHDE, providing a potential explanation for the origin of the cosmological constant. On the other hand, the tachyon field exhibits a non-trivial dynamics for a non-flat Universe. In particular, in a closed Universe $T^2$ decreases monotonically for increasing $\cos(R_h/a)$, where $R_h$ and $a$ are the future event horizon and the scale factor. The same behavior occurs in an open Universe, but in this case $T^2 < 0$, revealing that BHDE cannot model the tachyon dark energy. We then study the inflation mechanism. We find an analytical solution for the characteristic slow-roll parameters, the scalar spectral index and the tensor-to-scalar ratio. We finally comment on the Trans-Planckian Censorship Conjecture in BHDE.

I. INTRODUCTION

Experimental evidences from Supernova SNIa, Baryon acoustic oscillations and gravitational waves have definitely proved that our Universe is expanding at an accelerated rate. In spite of enormous effort, a fully consistent explanation for the origin of this behavior is missing. Among the various mechanisms, the existence of an unknown form of energy (Dark Energy, DE) affecting the Universe on large scales is the most widely accepted proposal. But yet, the nature of DE remains quite elusive. The possibility that DE is modeled by the cosmological constant acting as source of vacuum energy has been originally considered as natural way out of the DE puzzle [1, 2]. However, this scenario is at odds with our field theoretical understanding of the quantum properties of vacuum, thus requiring further investigation. Along this line, a plethora of DE models have been put forward over the years [3–21].

An interesting model to account for the nature of DE is the so called Holographic Dark Energy (HDE) [22–28], which emerges within quantum gravity framework. The main ingredient of this approach is the holographic principle, according to which the description of a volume of space can be thought of as encoded on a lower-dimensional boundary surface to the region. In [29, 30] it has been pointed out that effective local quantum field theories over-count the number of independent degrees of freedom, predicting that entropy scales extensively ($S \sim L^3$) for systems of size $L$ with UV cutoff $\Lambda$. Later on, a solution to this problem has been provided in [22], where it has been argued that the total energy of a system with size $L$ should not exceed that of an equally sized black hole, i.e. $L^3 \rho_L \leq L M_p^2$. Here, $M_p = (8\pi G)^{-1/2}$ is the reduced Planck mass, while $\rho_L$ denotes the quantum zero-point energy density caused by the UV cutoff $\Lambda$ (we are working in natural units $\hbar = c = 1$). The inequality is saturated for the largest value of $L$. In this context, the holographic dark energy density is obtained as

$$\rho_L = \frac{3c^2 M_p^2}{L^2},$$  \hspace{1cm} (1)

where $c$ is a dimensionless constant and the factor 3 has been introduced for mere convenience.

Cosmological applications of the holographic principle and HDE have been largely considered in literature. As an example, it was analyzed by [31] that the consequence of excluding those degrees of freedom of the system that will never be observed by the effective field theory results into an IR cutoff $L$ at the future event horizon. In a DE dominated Universe, such an horizon is then predicted to tend toward a constant value of the order $H_0^{-1}$, with $H_0$ being the present Hubble radius [32]. Furthermore, the issue of assuming the apparent (Hubble) horizon $R_A = 1/H$ as IR cutoff in a flat Universe has been examined in [33].

Despite the intensive study, the shortcomings of the HDE in describing the history of a flat Friedmann-Robertson-Walker (FRW) Universe have prompted tentative changes to this approach. For instance, HDE has been used to address the DE problem in Brans-Dicke Cosmology [34–39] by considering different IR cutoffs [27, 28, 38] or and/or deformed entropy-area laws [40–45]. In par-

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ticular, the latter path has led to promising models, such as Tsallis \[40, 41\], Barrow \[42\] and Kaniadakis \[43, 45\] holographic dark energy. Recently, motivated by the analysis of \[32\], it has been shown that Tsallis holographic description of DE (THDE) is non-trivially intertwined with tachyon dark energy model \[46\], in which the tachyon scalar field is considered as a source of DE. Specifically, a correspondence between the two scenarios has been established based on the reconstruction of the dynamics of the tachyon field in THDE.

Starting from the above premises, in this work we explore more in-depth the connection between the tachyon dark energy model and HDE. We frame our analysis in the context of HDE based on Barrow entropy \[42\], which arises from the attempt to incorporate quantum gravity effects on the horizon surface \[47\]. In this sense, our study must be intended as a preliminary effort toward extending the paradigm of \[46\] to a fully quantum gravity picture. We analyze the cases of flat and non-flat FRW Universe, for both interacting and non-interacting DE. Since scalar fields are generally conjectured to have driven inflation in the very early Universe, we then study the inflation mechanism in our BHDE model. We find an analytical solution for the slow-roll parameters, the scalar spectral index and the tensor-to-scalar ratio. We also compare our findings with recent results in the literature.

The remainder of the work is structured as follows: in the next Section we introduce BHDE. Section III and IV are devoted to analyze the correspondence between the tachyon dark energy and BHDE in a flat and non-flat FRW Universe, respectively. In Sec. V we discuss inflation in BHDE, while conclusions and outlook are summarized in Sec. VI.

II. BARROW HOLOGRAPHIC DARK ENERGY

Let us briefly review the basics of BHDE. We consider the four-dimensional Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),$$

of scale factor \(a(t)\) and spatial curvature \(k = 0, 1, -1\) for a flat, closed and open Universe, respectively.

We use the definition (1) for the holographic dark energy in standard Cosmology and assume

$$L(t) = a(t)r(t),$$

where \(r(t)\) is the (time-dependent) radius that is relevant to the future event horizon of the Universe \[32\]. Since

$$\int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \frac{1}{\sqrt{|k|}} \sin^{-1}(\sqrt{|k|r_1})$$

we easily obtain

$$L(t) = a(t) \sin \left[ \frac{\sqrt{|k|} R_h(t)/a(t)}{\sqrt{|k|}} \right],$$

where \(R_h\) is the future event horizon given by

$$R_h = a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2}, \quad H = \frac{\dot{a}}{a}. \quad (6)$$

HDE relies on the holographic principle, which asserts that the number of degrees of freedom describing the physics of any quantum gravity system i) scales as the bounding surface (rather than the volume) of the system and ii) should be constrained by an infrared cutoff \[29, 30\]. This is in tune with Bekenstein-Hawking (BH) relation \(S_{BH} = A/4\pi \) for black holes, where \(S_{BH} \) and \(A \) denote the entropy and area of the black hole, respectively, while \(A_0 = 1/(4G) \) its Planck area. Recently, deformations of this relation have been proposed to take account of quantum \[47–49\] and/or relativistic \[50\] effects. In particular, in \[47\] it has been argued that quantum gravity may introduce intricate, fractal features on the black hole horizon, leading to the modified area law

$$S_{\Delta} = \left( \frac{A}{A_0} \right)^{1+\Delta/2}. \quad (7)$$

Deviations from BH entropy are quantified by the exponent \(0 \leq \Delta \leq 1\), with \(\Delta = 0\) giving the BH limit, while \(\Delta = 1\) corresponding to the maximal horizon deformation. We emphasize that although this relation resembles Tsallis entropy in non-extensive statistical thermodynamics \[48, 49\], the origin and motivation underlying Eq. (7) are completely different. Cosmological implications of Barrow entropy have been recently studied in the context of Big Bang Nucleosynthesis \[51\] and Baryogenesis \[52\], among others. The possibility of a running \(\Delta\) has also been considered in \[53\].

Strictly speaking, Eq. (7) has been formulated for black holes. However, it is known that in any gravity theory one can consider the entropy for the Universe horizon in the same form as the black hole entropy, the only adjustment being the replacement of the black hole horizon radius with the apparent horizon radius. This is at the heart of the various generalizations of HDE with modified entropy laws (see, e.g. \[40–42, 44, 45\]).

Now, in \[22\] Cohen et al. have proposed the following inequality between the entropy, the IR (\(L\)) and UV (\(\Lambda\))
We also introduce the three fractional energy densities
\[ \rho_D = C L^{\Delta - 2}, \]  
where \( C \) is an unknown parameter with dimensions \([L]^{-2-\Delta}\). It is worth noticing that for \( \Delta = 0 \), the above relation reduces to the standard HDE \((1)\), provided that \( C = 3c^2 M_p^2 \). On the other hand, in the case where deformations effects switch on \((\Delta \neq 0)\), BHDE departs from the standard HDE, leading to different cosmological scenarios \([42]\).

Following the standard literature, we now define the critical energy density \( \rho_{cr} \) and the curvature energy density \( \rho_k \) as
\[ \rho_{cr} = 3M_p^2 H^2, \quad \rho_k = \frac{3k}{8\pi Ga^2}. \]  
We also introduce the three fractional energy densities
\[ \Omega_m = \frac{\rho_m}{\rho_{cr}} = \frac{\rho_m}{3M_p^2 H^2}, \]  
\[ \Omega_D = \frac{\rho_D}{\rho_{cr}} = \frac{C}{3M_p^2 H^2} L^{\Delta - 2}, \]  
\[ \Omega_k = \frac{\rho_k}{\rho_{cr}} = \frac{k}{H^2 a^2}, \]  
where \( \rho_m = 3(1-e^2) M_p^2 H^2 \) is the matter energy density. In particular, by setting \( L = H^{-1} \) we obtain
\[ \Omega_D = \frac{C}{3M_p^2 H^\Delta}. \]  

From Eqs. \((3)\) and \((4)\), one can derive the following expression for the time derivative of \( L \) \([46]\)
\[ \dot{L} = H L + a^2 = 1 - \frac{1}{\sqrt{|k|}} \cos n(\sqrt{|k|} R_k/a), \]  
where we have defined \([32]\)
\[ \frac{1}{\sqrt{|k|}} \cos n(\sqrt{|k|} x) = \begin{cases} \cos(x), & k = 1, \\ 1, & k = 0, \\ \cosh(x), & k = -1. \end{cases} \]  
Now, for a flat FRW Universe filled by non-interacting BHDE and pressureless DM, the first Friedmann equation takes the form
\[ H^2 = \frac{1}{3M_p^2} (\rho_D + \rho_m), \]  
which, by use of Eqs. \((12)\) and \((13)\), can be rewritten as
\[ \Omega_m + \Omega_D = \Omega_D (1 + u) = 1, \]  
where
\[ u = \frac{\rho_m}{\rho_D} = \frac{\Omega_m}{\Omega_D}. \]  
Since BHDE does not interact with other parts of cosmos (DM), the conservation equations of dust and THDE read
\[ \dot{\rho}_m + 3H \rho_m = 0, \]  
\[ \dot{\rho}_D + 3H \rho_D (1 + \omega_D) = 0, \]  
where we have denoted by \( \omega_D = p_D/\rho_D \) and \( p_D \) the equation of state parameter and pressure of THDE, respectively.

Deriving Eq. \((18)\) respect to time and using the continuity equations \((21)\) and \((22)\), after some algebra we are led to
\[ \frac{\dot{H}}{H^2} = -\frac{3}{2} (1 + \omega_D + u) \Omega_D. \]  
Likewise, by plugging Eq. \((10)\) into \((22)\), we find
\[ \frac{\dot{H}}{H^2} = (1 + \omega_D) \frac{3}{\Delta - 2}, \]  
which gives, by comparison with Eq. \((23)\)
\[ \omega_D = \frac{u(2 - \Delta) \Omega_D}{2 - (2 - \Delta) \Omega_D} - 1. \]  
With the aid of Eq. \((19)\), this finally yields
\[ \omega_D = \frac{\Delta}{(2 - \Delta) \Omega_D - 2}. \]  

\[ ^1 \text{There are several choices for the IR cutoff } L. \text{ Following } [40], \text{ here we resort to the simplest one } L = H^{-1}. \text{ Other possible choices are the particle horizon, the future event horizon, the GO cutoff } [53] \text{ or combination thereof. However, in these cases one must generally resort to numerical evaluation to study the cosmological evolution of the model } [41]. \text{ Since we are interested in extracting analytical solutions and given the degree of arbitrariness in the selection of the best dark energy description, we leave the analysis of dark energy models with different IR cutoffs for future investigation.} \]  

\[ ^2 \text{Unlike THDE model, where } \omega_D \text{ is divergent for } \delta_T < 1 \text{ and } \Omega_D = 1(2 - \delta_T) \text{ (} \delta_T \text{ is the Tsallis exponent), BHDE is well-defined for any value of } 0 \leq \Delta \leq 1. \]
On the other hand, if there exists an interaction

\[ Q = 3b^2 H (\rho_m + \rho_D) \]  

(27)

between BHDE and matter, the continuity equations (21) and (22) become

\[ \dot{\rho}_m + 3H \rho_m = Q, \]  

(28)

\[ \dot{\rho}_D + 3H \rho_D (1 + \omega_D) = -Q. \]  

(29)

Following similar calculations as above, one can show that Barrow holographic energy equation of state takes the form

\[ \omega_D = \frac{\Delta + 2b^2 / \Omega_D}{(2 - \Delta) \Omega_D - 2}, \]  

(30)

where \( b \) is the coupling parameter that quantifies the interaction.

### A. The age of the Universe

We can now estimate the age of the present Universe as

\[
t = \int dt \frac{dH}{dH} = \int \frac{H^2}{H} dH,
\]

\[
= 2 \left( \frac{C}{3M_p^2} \right)^{1/\Delta} \int \frac{1 - (1 - \Delta/2) \Omega_D}{\Omega_D^{1 - 1/\Delta}} \Omega_D dz, \quad (31)
\]

where we have used Eqs. (24) and (26) along with

\[
\frac{dH}{H^2} = -3 \left( \frac{C}{3M_p^2} \right)^{1/\Delta} \frac{1}{\Delta \Omega_D^{1/\Delta}} \Omega_D dz. \quad (32)
\]

By integrating the above relation, it follows that

\[
t = \frac{2 - \Delta}{3H} \left[ 1 + \left( \frac{\Delta}{2 - \Delta} \right) 2F_1 \left( 1, \frac{\Delta}{1}; \frac{1}{\Delta}; \frac{C}{3M_p^2 H^\Delta} \right) \right], \quad (33)
\]

where \( 2F_1(a, b; c; d) \) is the hypergeometric function of first kind. This equation can be used to estimate the order of the age of the current Universe \( (z = 0) \) in our model. By exploiting Eqs. (24) and (31), we get

\[
t \approx \left( \frac{H^2}{H} \right) \bigg|_{z=0} \int \frac{dH}{H^2} = \frac{2 - \Delta}{3H_0} \left( 1 - \frac{\omega_D(z = 0)}{1 + \omega_D(z = 0)} \right). \quad (34)
\]

For \( \omega_D(z = 0) = -2/3 \), we then have \( t = 1/H_0 \) for \( \Delta = 1 \), corresponding to the maximal deformation of the Bekenstein-Hawking area law. As observed in [40], further corrections to Eq. (34) may arise due to either different modifications of the horizon entropy or other IR cutoffs.

### III. TACHYON SCALAR FIELD AS BARROW HOLOGRAPHIC DARK ENERGY: FLAT FRW UNIVERSE

In this Section we analyze the correspondence between the tachyon dark energy model and the BHDE scenario in a flat FRW Universe. We consider the non-interacting and interacting case separately. A preliminary study along this direction has been carried out in [56].

#### A. Non-interacting case

In [32] it has been shown that the energy density \( \rho_T \) and pressure \( p_T \) for the tachyon scalar field take the form

\[
\rho_T = \frac{V(T)}{\sqrt{1 - T^2}}, \quad (35)
\]

\[
p_T = -V(T) \sqrt{1 - T^2}, \quad (36)
\]

where \( V(T) \) is the tachyon potential energy. From these relations, we derive the equation of state parameter for the tachyon as

\[
\omega_T = p_T / \rho_T = T^2 - 1. \quad (37)
\]

We now aim at describing the dynamics of the tachyon scalar field in BHDE. Toward this end, we assume that the tachyon energy density (35) can be modeled by Barrow holographic dark energy density (10) evaluated at the future event horizon \( R_h \). We then obtain

\[
CR_h^{\Delta - 2} = \frac{V(T)}{\sqrt{1 - T^2}}. \quad (38)
\]

On the other hand, by equating Eqs. (26) and (37), it follows that

\[
\frac{\Delta}{(2 - \Delta) \Omega_D - 2} = T^2 - 1, \quad (39)
\]

which gives

\[
\dot{T} = \sqrt{\frac{(2 - \Delta) \Omega_D + \Delta - 2}{(2 - \Delta) \Omega_D - 2}}. \quad (40)
\]

Let us now invert Eq. (38) respect to the potential \( V(T) \)

\[
V(T) = CR_h^{\Delta - 2} \sqrt{1 - T^2}. \quad (41)
\]

Inserting Eq. (40), we infer

\[
V(T) = CR_h^{\Delta - 2} \frac{\Delta}{2 - (2 - \Delta) \Omega_D}. \quad (42)
\]

Similarly, from Eq. (36) we get for the tachyon pressure

\[
p_T = p_D = -CR_h^{\Delta - 2} \frac{\Delta}{2 - (2 - \Delta) \Omega_D}. \quad (43)
\]
The above relations can be further elaborated by taking the time derivative of $\rho_D = \rho_T$ in Eq. (10)

$$\dot{\rho}_D = C(\Delta - 2)L^{\Delta - 3}\dot{L},$$  \hspace{1cm} (44)

and observing that $\dot{L} = 0$ in a flat Universe (see Eq. (16)), which implies $\dot{\rho}_D = 0$. In the absence of interactions between holographic dark energy and matter, from Eq. (22) we obtain $\omega_D = -1$ and, thus, $T = 0$. This is satisfied, provided that

$$(2 - \Delta)\Omega_D + \Delta - 2 = 0,$$  \hspace{1cm} (45)

which admits as solution

$$\Omega_D = 1.$$  \hspace{1cm} (46)

By using the above results in Eqs. (41) and (43), we reconstruct the tachyon potential and pressure as

$$V(T) = CR_h^{\Delta - 2},$$  \hspace{1cm} (47)

$$\rho_T = -CR_h^{\Delta - 2},$$  \hspace{1cm} (48)

We notice that the above picture is consistent with the cosmological constant scenario, since $\omega_D = -1$ as discussed above.

### B. Interacting case

Let us now extend the above analysis to the interacting case. In this context, the continuity equations for dark energy and matter are provided by Eqs. (28) and (29), respectively, with the coupling term being given by Eq. (27). As in the previous case, the condition $\dot{\rho}_D = 0$ holds true. However, by combining with Eq. (29), we now obtain

$$\omega_D = -b^2\left(\frac{\Omega_m}{\Omega_D} + 1\right) - 1$$

$$= -\frac{b^2}{\Omega_D} - 1,$$  \hspace{1cm} (49)

where in the second step we have used Eq. (19). By equating to Eq. (37), we get

$$\dot{T}^2 = -\frac{b^2}{\Omega_D},$$  \hspace{1cm} (50)

which makes only sense for $b = 0$. As in [46], we thus find that in a flat FRW Universe, $\dot{T}^2$ must always be vanishing. We can also extract the following relations

$$V(T) = \rho_D,$$  \hspace{1cm} (51)

$$p_D = -\rho_D,$$  \hspace{1cm} (52)

which are consistent with the condition $\omega_D = -1$ discussed in the non-interacting case.

From Eqs. (51) and (52) we can also explore the stability of our BHDE model against perturbation. To this end, we compute the squared sound speed $v_s^2$. Clearly, for $v_s^2 > 0$, the model is stable, otherwise it is unstable. The squared sound speed is defined as

$$v_s^2 = \frac{dp_D}{d\rho_D},$$  \hspace{1cm} (53)

which gives in our case $v_s^2 = -1 < 0$. This means that the model is unstable. Therefore, we cannot conclude that a Barrow holographic dark energy dominated Universe will be the fate of the future Universe.

Moreover, we have $\omega_D = -1$, then $\omega_D = 0$, where the prime denotes the derivative with respect to $\ln a$. Therefore, the $\omega$-$\omega'$ analysis is meaningless for this model.

### IV. Tachyon Scalar Field as Barrow Holographic Dark Energy: Non-Flat FRW Universe

Let us explore how the connection between BHDE and tachyon dark energy model appears in a non-flat FRW Universe. Toward this end, we consider the time derivative of BHDE (10) and use Eq. (16) to get

$$\dot{\rho}_D = C(\Delta - 2)L^{\Delta - 3}\left[1 - \frac{1}{\sqrt{|k|}}\cos n(\sqrt{|k|}x)\right],$$  \hspace{1cm} (54)

where $x \equiv R_h/a$. By means of the continuity equation (29), this can be cast as

$$-3H\rho_D(1 + \omega_D) - 3b^2H(\rho_m + \rho_D)$$

$$= C(\Delta - 2)L^{\Delta - 3}\left[1 - \frac{1}{\sqrt{|k|}}\cos n(\sqrt{|k|}x)\right].$$  \hspace{1cm} (55)

We can now resort to Eq. (37) (we remind that we are imposing $\omega_T = \omega_D$ in our model) to obtain

$$-3H\rho_D\dot{T}^2 - 3b^2H(\rho_m + \rho_D)$$

$$= C(\Delta - 2)L^{\Delta - 3}\left[1 - \frac{1}{\sqrt{|k|}}\cos n(\sqrt{|k|}x)\right].$$  \hspace{1cm} (56)

This relation can be further manipulated by dividing both sides by $3H\rho_D$ and using Eq. (20) to give

$$-\dot{T}^2 - b^2(u + 1)$$

$$= \frac{C(\Delta - 2)L^{\Delta - 3}}{3H\rho_D}\left[1 - \frac{1}{\sqrt{|k|}}\cos n(\sqrt{|k|}x)\right].$$  \hspace{1cm} (57)

After employing Eq. (10) and the condition $L = H^{-1}$, we finally reach

$$-\dot{T}^2 - b^2(u + 1)$$

$$= \frac{\Delta - 2}{3}\left[1 - \frac{1}{\sqrt{|k|}}\cos n(\sqrt{|k|}x)\right].$$  \hspace{1cm} (58)
which can be equivalently written as
\[
\dot{T}^2 = \frac{2}{3} - b^2(u + 1) - \frac{\Delta}{3} + \frac{\Delta - 2}{3} \cos(x).
\]

In order for \(\dot{T}\) to be zero (i.e. \(\omega_D = -1\)), we must have
\[
\cos(x) = \frac{3b^2(u + 1)}{\Delta - 2} + 1.
\]

This provides the condition for which tachyon model of BHDE can be used to explain the origin of the cosmological constant in a closed FRW Universe. By contrast, in [46] it is shown that \(T^2\) cannot be zero in Tsallis holographic dark energy in a non-flat Universe. The evolution trajectory of \(T^2\) in Eq. (60) is plotted in Fig. 1 for fixed \(b\) and \(u\) and various \(\Delta\). One can see that \(T^2\) is positive and decreases monotonically for increasing \(\cos(x)\).

Similarly, we can consider the dynamics of the tachyon field in an open \((k = -1)\) Universe. Following the same reasoning as above, we now obtain
\[
\dot{T}^2 = \frac{2}{3} - b^2(u + 1) - \frac{\Delta}{3} + \frac{\Delta - 2}{3} \cosh(x),
\]

which vanishes provided that
\[
\cosh(x) = \frac{3b^2(u + 1)}{\Delta - 2} + 1.
\]

However, since \(\cosh(x) > 1\), we infer that this model cannot explain the cosmological constant, in agreement with the result of [46]. Indeed, there is no value of \(0 \leq \Delta \leq 1\), such that the right side of Eq. (63) is higher than unity. The evolution of \(T^2\) in Eq. (62) is plotted in Fig. 2 for different \(\Delta\). As before, we notice that \(T^2\) decreases monotonically for increasing \(\cosh(x)\), but in this case it is always smaller than zero, which is not a physically valid situation. In passing, we mention that the same behavior has been found in [46] in the context of HDE based on Tsallis entropy. Therefore, we conclude that BHDE is not suitable to model the tachyon dark energy in an open Universe.

V. INFLATION IN BARROW HOLOGRAPHIC DARK ENERGY

In this Section we discuss inflation in BHDE. For reasons that will appear clear below and following [57], here we consider the more general expression for the length scale \(L = \alpha H^2 + \beta H\), where \(\alpha\) and \(\beta\) are dimensionless constant. Assuming that the expansion of the Universe is driven by BHDE (10) and neglecting the matter contribution due to the rapid inflationary expansion, Eq. (18) becomes
\[
H^2 = \frac{C}{3M_p^2} \left( \alpha H^2 + \beta H \right)^{1-\Delta/2},
\]

from which we infer
\[
\dot{H} = \frac{H^2}{\beta} \left[ \left( \frac{3M_p^2}{C} \right)^{\frac{2-\Delta}{2-\alpha}} (H^2)^{\frac{\Delta}{2-\alpha}} - \alpha \right].
\]

From this relation, it is clear that setting the IR cutoff \(L \approx H^{-1}\) (i.e. \(\beta = 0\)) as in the previous study would give rise to technical issues in the present framework.

To simplify the resolution of Eq. (65), we introduce the e-folds variable \(N = \log(a/a_i)\), where \(a_i\) is the initial
value of the scale factor $a$. By observing that $dN = H dt$ and $\dot{H} = \frac{1}{2} \frac{dH^2}{dN}$, integration of Eq. (65) gives

$$
\log \left\{ \tilde{H}^2 \left[ \frac{1}{\gamma} \left( \tilde{H}^2 \right)^{1 - 2/\Delta} + \alpha \right] \right\} = \frac{2\alpha N}{\beta},
$$

where $\tilde{H} = H/M_p$ is the dimensionless Hubble parameter and

$$
\gamma = \left( \frac{3M_p^2}{C} \right) \frac{\tilde{H}}{M_p^{2+\Delta/\alpha}}.
$$

Here we have denoted the Hubble parameter at the end of inflation by $\tilde{H}_f$.

From Eq. (64) we can now compute the characteristic parameters of slow-roll inflation. Specifically, the first slow-roll parameter is given by

$$
\epsilon_1 = -\frac{\dot{H}}{H} = -\frac{1}{\beta} \left[ \gamma \left( \tilde{H}^2 \right)^{1 - 2/\Delta} - \alpha \right].
$$

The other slow-roll parameters can be derived by using the definition $\epsilon_{n+1} = d\log(\epsilon_n)/dN$. For the second parameter $\epsilon_2$ we get

$$
\epsilon_2 = -\frac{\dot{\epsilon}_1}{H \epsilon_1} = \frac{2\gamma}{\beta} \left( \frac{\Delta}{2 - \Delta} \right) \left( \tilde{H}^2 \right)^{1 - 2/\Delta}.
$$

Let us now evaluate the Hubble parameter at the end of inflation. This phase is characterized by $\epsilon_1 = 1$. By straightforward calculations, we obtain

$$
\tilde{H}^2_f = \left( \frac{\gamma}{\alpha - \beta} \right)^{1 - 2/\Delta}.
$$

On the other hand, at the beginning of inflation (including the horizon crossing time) Eq. (66) gives

$$
\tilde{H}^2_i = \left( \frac{\gamma}{\alpha} \left( 1 + \frac{\beta e^{2\alpha N/\beta}}{\alpha - \beta} \right) \right)^{1 - 2/\Delta},
$$

which can be used to calculate the slow-roll parameters for earlier time by direct substitution in Eqs. (68) and (69).

In order to derive the scalar spectral index $n_s - 1$ and the tensor-to-scalar ratio $r$, we follow [57, 58] and make use of the usual perturbation procedure. We are led to

$$
n_s - 1 = -2\epsilon_1 - 2\epsilon_2, \quad r = 16\epsilon_1.
$$

Clearly, a full perturbation analysis is needed to obtain the exact expressions of $n_s - 1$ and $r$.

Two comments are in order here: first, we notice that the constant $\gamma$ does not intervene in the calculation of the slow-roll parameters at the horizon crossing time, which means that neither $n_s - 1$ nor $r$ depend on it. As explained in [57], this constant can be estimated by considering the amplitude of the scalar perturbation. Furthermore, it is worth mentioning that a similar analysis of inflation and correspondence between BHDE and tachyon field has been proposed in [56]. However, in that case the authors consider values of Barrow parameter $\Delta$ higher than unity, which are actually forbidden in Barrow model. This somehow questions the results exhibited in [56].

A. Trans-Planckian Censorship Conjecture

The large-scale structures we currently see in the Universe originated from matter and energy quantum fluctuations produced during inflation. Such fluctuations cross the Hubble radius during the early phase, are stretched out and classicalize, and finally re-enter the Hubble horizon to produce the CMB anisotropies. The key point is that if inflation lasted longer than the supposed minimal period, then it would be possible to observe length scales originated from modes smaller than the Planck length at inflation [59]. This problem is usually referred to as “Trans-Planckian problem”. To avoid inconsistencies, it has been conjectured that this problem cannot arise in any consistent model of quantum gravity (“Trans-Planckian Censorship Conjecture”, TCC) [60].

The TCC states that no length scales which cross the Hubble horizon could ever have had a wavelength smaller than the Planck length. This is imposed by requiring that

$$
L_p/a_i < \frac{H_f^{-1}}{a_f},
$$

where $L_p = 1/M_p$ is the Planck length and we have denoted by $a_f$ the scale factor at the end of inflation. By using Eq. (70) for the Hubble parameter at the final time, the TCC (73) becomes

$$
\left( \frac{\gamma}{\alpha - \beta} \right)^{1 - 2/\Delta} < (8\pi e^N)^2,
$$

the validity of which can be examined by comparison with future observational data.

VI. CONCLUSIONS AND OUTLOOK

The origin of the accelerated expansion of the Universe is an open problem in modern Cosmology. To date, the most reliable explanation is provided by the existence of an enigmatic form of energy - the Dark Energy - affecting the Universe on large scales. Several candidates have been considered to account for this phenomenon. In particular, the holographic dark energy has been largely studied, also in connection with different real scalar field theories, such as quintessence [3–8], K-essence [9], phantom [13–15], braneworld [10], interacting models [20] and tachyon [21, 32]. Recently, the interest has been extended to the Tsallis holographic dark energy [40–42, 44, 45] and the possibility of using it to describe the dynamics of the tachyon field [46].
In this work we have considered the further scenario of tachyon model as Barrow holographic dark energy. Barrow entropy arises from the effort to include quantum gravity effects on the black hole horizon. In this sense, the present analysis must be intended as a preliminary step toward a fully quantum gravity extension of [46]. We have established a correspondence between BHDE and the tachyon field model in a FRW Universe, both in the presence and absence of interactions between dark energy and matter. For a flat FRW Universe, we have found that $T^2 = 0$, i.e. $\omega_D = -1$, providing a possible explanation for the cosmological constant. On the other hand, the dynamics of the tachyon field turns out to be much more complicated for a curved Universe. Specifically, for a closed Universe $T^2$ decreases monotonically for increasing $\cos(R_3/a)$ and vanishes when Eq. (61) is satisfied. By contrast, $T^2$ is always negative for an open Universe, which shows that BHDE cannot be used to model tachyon field dynamics in this case.

We have finally investigated an inflationary scenario described by a Universe filled with BHDE and commented on the Trans-Planckian Censorship Conjecture. Comparison with future observational data will enable us to establish whether such model is empirically consistent.

Further aspects remain to be addressed. For instance, we can look at the correspondence between the tachyon field and other dark energy scenarios, in particular stable dark energy models. Furthermore, it is interesting to extend the above framework to the case of Kaniadakis holographic dark energy [45], which is based on a relativistic self-consistent generalization of the classical Boltzmann-Gibbs entropy [50]. Finally, it is essential to examine to what extent our effective model reconciles with predictions of more fundamental candidate theories of quantum gravity, such as String Theory or Loop Quantum Gravity. Work along these directions is presently under active investigation and will be presented elsewhere.

References

[1] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000).
[2] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003).
[3] P. J. E. Peebles and B. Ratra, Astrophys. J. Lett. 325, L17 (1988).
[4] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[5] C. Wetterich, Nucl. Phys. B 302, 668 (1988).
[6] J. A. Frieman, C. T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. 75, 2077 (1995).
[7] M. S. Turner and M. J. White, Phys. Rev. D 56, R4439 (1997).
[8] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).
[9] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000).
[10] L. Amendola, Phys. Rev. D 62, 043511 (2000).
[11] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. D 63, 103510 (2001).
[12] A. Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B 511, 265 (2001).
[13] R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
[14] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. 91, 071301 (2003).
[15] S. Nojiri and S. D. Odintsov, Phys. Lett. B 562, 147 (2003).
[16] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B 607, 35 (2005).
[17] Z. K. Guo, Y. S. Piao, X. M. Zhang and Y. Z. Zhang, Phys. Lett. B 608, 177 (2005).
[18] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, 043539 (2004).
[19] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D 71, 063004 (2005).
[20] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D 65, 044023 (2002).
[21] A. Sen, JHEP 07, 065 (2002).
[22] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Phys. Rev. Lett. 82, 4971 (1999).
[23] P. Horava and D. Minic, Phys. Rev. Lett. 85, 1610 (2000).
[24] S. D. Thomas, Phys. Rev. Lett. 89, 081301 (2002).
[25] M. Li, Phys. Lett. B 603, 1 (2004).
[26] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. 342, 155 (2012).
[27] S. Ghaffari, M. H. Dehghani and A. Sheykhi, Phys. Rev. D 89, 123009 (2014).
[28] S. Wang, Y. Wang and M. Li, Phys. Rept. 696, 1 (2017).
[29] G. ’t Hooft, Conf. Proc. C 930308, 284 (1993).
[30] L. Susskind, J. Math. Phys. 36, 6377 (1995).
[31] K. Enqvist, S. Hannestad and M. S. Sloth, JCAP 02, 004 (2005).
[32] M. R. Setare, Phys. Lett. B 653, 116 (2007).
[33] S. D. H. Hsu, Phys. Lett. B 594, 13 (2004).
[34] Y. g. Gong, Phys. Rev. D 61, 043505 (2000).
[35] N. Banerjee and D. Pavon, Class. Quant. Grav. 18, 593 (2001).
[36] H. Kim, H. W. Lee and Y. S. Myung, Phys. Lett. B 628, 11 (2005).
[37] M. R. Setare, Phys. Lett. B 644, 99 (2007).
[38] L. Xu and J. Lu, Eur. Phys. J. C 60, 135 (2009).
[39] A. Khodam-Mohammadi, E. Karimkhani and A. Sheykhi, Int. J. Mod. Phys. D 23, 1450081 (2014).
[40] M. Tavayef, A. Sheykhi, K. Bamba and H. Moradpour, Phys. Lett. B 781, 195 (2018).
[41] E. N. Saridakis, K. Bamba, R. Myrzakulov and F. K. Anagnostopoulos, JCAP 12, 012 (2018).
[42] E. N. Saridakis, Phys. Rev. D 102, 123525 (2020).
[43] A. Hernández-Almada, G. Leon, J. Magaña, M. A. García-Aspeitia, V. Motta, E. N. Saridakis and K. Yesmakhanova, Mon. Not. Roy. Astron. Soc. 511, 4147 (2022).
[44] H. Moradpour, A. H. Ziaie and M. Kord Zangeneh, Eur. Phys. J. C 80, 732 (2020).
[45] N. Drepanou, A. Lyperis, E. N. Saridakis and K. Yesmakhanova, Eur. Phys. J. C 82, no.5, 449 (2022).
[46] Y. Liu, Eur. Phys. J. Plus 136, 579 (2021).
[47] J. D. Barrow, Phys. Lett. B 808, 135643 (2020).
[48] C. Tsallis, J. Statist. Phys. 52, 479 (1988).
[49] C. Tsallis and L. J. L. Cirto, Eur. Phys. J. C 73, 2487 (2013).
[50] G. Kaniadakis, Phys. Rev. E 66, 056125 (2002).
[51] J. D. Barrow, S. Basilakos and E. N. Saridakis, Phys. Lett. B 815, 136134 (2021).
[52] G. G. Luciano and E. N. Saridakis, [arXiv:2203.12010 [gr-qc]].
[53] S. Di Gennaro and Y. C. Ong, [arXiv:2205.09311 [gr-qc]].
[54] B. Guberina, R. Horvat and H. Nikolic, JCAP 01, 012 (2007).
[55] L. Granda and A. Oliveros, Phys. Lett. B 669, 275 (2008).
[56] S. Maity and P. Rudra, JHAP 2, 1 (2022).
[57] A. Mohammadi, T. Golanbari, K. Bamba and I. P. Lobo, Phys. Rev. D 103, 083505 (2021).
[58] S. Nojiri, S. D. Odintsov, and E. N. Saridakis, Phys. Lett. B 797, 134829 (2019).
[59] J. Martin and R. H. Brandenberger, Phys. Rev. D 63, 123501 (2001).
[60] A. Bedroya and C. Vafa, JHEP 09, 123 (2020).