Spontaneous formation of a non-uniform chiral spin liquid in a "moat" lattice

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A honeycomb lattice allowing hops between nearest- and next-nearest neighbors hosts "moat" bands with degenerate energy minima attained along closed lines in Brillouin zone. If populated with hard-core bosons, the degeneracy prevents their condensation. At half-filling, the system is equivalent to $s = 1/2$ XY model at zero magnetic field; absence of condensation means no spontaneous polarization in XY plane. However, our consideration indicates formation of a state spontaneously breaking the time-reversal symmetry. This state has a bulk gap and chiral gapless edge excitations, and is similar to the one in Haldane’s "quantum Hall effect without Landau levels".

The applications of the developed analytical theory include an explanation of recent unexpected numerical findings and a suggestion of a chiral spin liquid realization in experiments with cold atoms in optical lattices.

Recent experiments with Herbertsmithite and other zinc paratacamites [1–6] indicated a possible observation of a long-sought quantum spin liquid. Importantly, in practice quantum many-body systems endowed with the spin-liquid properties can also be engineered in experiments with cold atoms [7–11]. These developments added motivation for theoretical investigation of frustrated spin Hamiltonians on a variety of lattices [12–30], including triangular, honeycomb, and Kagome ones. In a majority of models, an increase of the frustration could lead to a formation of $\mathbb{Z}_2$ quantum spin liquid preserving time-reversal symmetry (TRS). In recent numerical studies [28–30] of a frustrated Kagome lattice, however, a chiral spin liquid ground state was found.

Chiral spin liquid (CSL) suggested by Kalmeyer and Laughlin [31], is a possible ground state of frustrated spin systems, resembling quantum Hall states. CSL is gapped in the bulk, but supports gapless chiral excitations along edges, owing to a nontrivial Chern number. Such a ground state is described by an incompressible bosonic wave function $|\Psi\rangle$. The concept was further developed in Ref. [32], where a set of order parameters was identified and an explicit solvable example of a CSL was constructed. A convenient choice for a CSL order parameter is the chirality, $\chi = \langle S_i \cdot (S_j \times S_k) \rangle$, where $S$ is the spin-$1/2$ operator and $i, j, k$ label lattice cites forming a triangular plaquet. Finite expectation value, $\chi$, violates P and T symmetries, leaving the combined PT symmetry intact.

The crucial observation is that once the frustration is strong enough, the lattice dispersion exhibits the moat shape - i.e. the degenerate energy minimum along a closed line in the Brillouin zone [33]. In such a case, as in one dimension, the single particle density of states diverges at the bottom of the band. In analogy with the Tonks-Girardeau gas, it suggests to transform the bosons into spinless fermions [33, 34], which automatically satisfy the hard core condition. In two dimensions it may be achieved with the help of the Chern-Simons flux attachment, similar to the one employed in the theory of the fractional Quantum Hall effect [25, 37]. We found that its lattice realization leads to fermions subject to fluxes staggered within the unit cell. As first realized by Haldane [35], such staggered fluxes bring about topologically non-trivial fermionic ground state with a non-zero Chern number and a chiral edge states. In our opinion, this is the most direct and intuitive root to identify CSL state.

As an example we consider the $XY$ model on a honeycomb lattice with nearest-neighbor (NN) spin coupling, $J_1$, and next-NN coupling, $J_2$. Finite $J_2$ leads to frustration and hence the model is expected to have a rich phase diagram. Numerical studies, performed using exact diagonalization [12], variational Monte Carlo [13], and the density matrix renormalization group (DMRG) [14] techniques suggest that the AF order in X-Y plane survives up to a critical value $J_2/J_1 \simeq 0.2$, where the system undergoes a phase transition. Reference [12] reported exact diagonalization study, suggesting existence of a new phase in the parameter range $0.2 \lesssim J_2/J_1 \lesssim 0.36$. The authors suggested that it is a "Bose liquid" phase in which the spin ordering is absent down to zero temperature. Recently Zhu, Huse, and White (ZHW) [14] employed DMRG technique in cylindrical geometry for the same range of parameters (throughout this paper we refer to it as intermediate frustration regime). They found that while there is indeed no order vis-a-vis the X-Y plane, there is a weak antiferromagnetic Ising order in z direction. It breaks the symmetry between A and B sublattices with $\langle S_A^z - S_B^z \rangle$ taking values in between 0.27 and 0.28.

Here we study the model by reformulating it as a
Chern-Simons (CS) fermionic field theory on a lattice. We found that in the intermediate frustration regime the AF Ising order of ZHW is stabilized by appearance of staggered CS fluxes within the unit cell. The zero-average modulated fluxes, induced by the lattice CS field, are exactly the same as postulated in a celebrated Haldane model [33] of quantum Hall effect without net magnetic field. Solution of self-consistent mean-field equations puts the model into its topologically non-trivial sector with Chern number \( C = \pm 1 \). This allows us to identify the \( \mathcal{Z} \)-modulated state of ZHW with CSL state, which supports gapless spinon excitations along the edges. It would be extremely interesting to see if DMRG studies can check this prediction. Moreover we predict a relation between the AF Ising order parameter \( \phi = \frac{2}{3} (S_A^z - S_B^z) \) and chirality \( \chi = \langle S_A \cdot (S_B \times S_A) \rangle \propto \sin \phi \), reflected in instets of Fig. [2] which may be also directly checked in simulations.

To quantify the aforementioned ideas, we start with the Hamiltonian of frustrated spin-1/2 XY model on a honeycomb lattice with nearest and next to nearest neighbor interaction terms

\[
H = J_1 \sum_{\mathbf{r},j} S_\mathbf{r}^+ S_{\mathbf{r}+\mathbf{e}_j}^- + J_2 \sum_{\mathbf{r},j} S_\mathbf{r}^+ S_{\mathbf{r}+\mathbf{a}_j}^- + H.c. \quad (1)
\]

Here the spin-1/2 operators are related to Pauli matrices as \( S_\mathbf{r}^x = \sigma_\mathbf{r}^x \) and \( S_\mathbf{r}^z = \sigma_\mathbf{r}^z/2 \). Vectors \( \mathbf{e}_j \) and \( \mathbf{a}_j \), \( j = 1, 2, 3 \), shown in Fig. [4] are connecting nearest and next-to-nearest neighbor cites of the honeycomb lattice.

The XY model [4] is equivalent to the model of hard-core bosons, as one may rewrite the spin 1/2 operators \( S_\mathbf{r}^\pm \) in terms of bosonic creation and annihilation operators. When \( J_2/J_1 > 1/6 \), the corresponding single particle dispersion relation undergoes dramatic changes: it becomes infinitesimally degenerate and exhibits an energy minimum along a closed line in the reciprocal space surrounding the \( \Gamma \) point [33] – the moat. The single particle density of states diverges near the moat bottom as \( (E - E_\mathbf{g})^{-1/2} \), highlighting similarities with one-dimensional systems, where the ground state of hard-core bosons is given by the Tonks-Girardeau gas of free fermions. This observation supports the idea that spinless fermion representation might be an effective description of 2D boson systems in a moat, as was suggested in Refs. [33] [34]. Advantage of spinless fermions is that they automatically satisfy the hard-core condition and thus do not suffer from a repulsive interaction energy.

We proceed with the lattice version of the CS transformation (its continuum analog was employed in e.g. Refs. [35] [37] [39] [40])

\[
S_\mathbf{r}^\pm = \tilde{c}_\mathbf{r}^{(\pm)} e^{\pm i \sum_{\mathbf{r}' \neq \mathbf{r}} \arg(\mathbf{r}' - \mathbf{r}) n_{\mathbf{r}'}, \quad (2)
\]

where the summation runs over all sites of the lattice. Since the bosonic operators on different sites commute, the newly defined operators \( \tilde{c}_\mathbf{r} \) and \( \tilde{c}_\mathbf{r}^{\dagger} \) obey fermionic commutation relations. Also notice that the number operator is given by \( n_\mathbf{r} = c_\mathbf{r}^{\dagger} c_\mathbf{r} = S_\mathbf{r}^z + 1/2 \).

Substitution of transformation Eq. (2) into the Hamiltonian [1] yields

\[
H = J_1 \sum_{\mathbf{r},j} \tilde{c}_\mathbf{r}^{\dagger} \tilde{c}_{\mathbf{r}+\mathbf{e}_j} e^{i A_{\mathbf{r},\mathbf{r}+\mathbf{e}_j}} + J_2 \sum_{\mathbf{r},j} \tilde{c}_\mathbf{r}^{\dagger} \tilde{c}_{\mathbf{r}+\mathbf{a}_j} e^{i A_{\mathbf{r},\mathbf{r}+\mathbf{a}_j}} + H.c.
\]

where \( A_{\mathbf{r},\mathbf{r}'} = \sum_\mathbf{r} [\arg(\mathbf{r}_1 - \mathbf{r}) - \arg(\mathbf{r}_2 - \mathbf{r})] n_\mathbf{r} \) with summation running over all lattice sites. Due to (spinless) fermionic nature of the operators the hard core condition is automatically satisfied. One can remove exponential string operators by introducing CS magnetic field, \( \mathcal{B}_r = A_{\mathbf{r}+\mathbf{e}_1,\mathbf{r}+\mathbf{e}_2} + A_{\mathbf{r}+\mathbf{e}_1,\mathbf{r}+\mathbf{e}_3} + A_{\mathbf{r}+\mathbf{e}_2,\mathbf{r}+\mathbf{e}_3} = 2\pi n_\mathbf{r} \), which is the lattice analog of \( \mathcal{B} \) (see Fig. [1]). To this end one introduces the \( \delta \)-function, \( 2\pi \delta(\mathcal{B}_\mathbf{r}/(2\pi) - n_\mathbf{r}) = \int \prod_\mathbf{r} [dA_0^r] \exp [i A_0^r(\mathcal{B}_\mathbf{r}/(2\pi) - n_\mathbf{r})] \). The corresponding functional integration with respect to the CS vector potential \( A_{\mathbf{r},\mathbf{r}'} \) is also implied. The Lagrange multiplier \( A_0^r \) plays the role of the zero component of the vector potential.

These notations enable one to represent the model as a fermion system coupled to the fluctuating CS gauge field. In analogy with the continuum case [37], we write

\[
S = \int dt \left[ \sum_\mathbf{r} \bar{c}_\mathbf{r} (i \partial_t - A_0^\mathbf{r}) c_\mathbf{r} + \frac{1}{2\pi} \sum_\mathbf{r} A_0^\mathbf{r} \mathcal{B}_\mathbf{r} \right. \quad (3)
\]

\[
- J_1 \sum_{\mathbf{r},j} \bar{c}_\mathbf{r} \bar{c}_{\mathbf{r}+\mathbf{e}_j} e^{i A_{\mathbf{r},\mathbf{r}+\mathbf{e}_j}} - J_2 \sum_{\mathbf{r},j} \bar{c}_\mathbf{r} \bar{c}_{\mathbf{r}+\mathbf{a}_j} e^{i A_{\mathbf{r},\mathbf{r}+\mathbf{a}_j}} + H.c. \right].
\]

Here the fermions are Gaussian and one may integrate them out obtaining the fermionic free energy functional, \( \mathcal{W}[A_0^\mathbf{r}, \mathcal{B}_\mathbf{r}] \). Along with the CS term the latter defines the formally exact effective action, \( S_{eff} = \int dt \left( \mathcal{W}[A_0^\mathbf{r}, \mathcal{B}_\mathbf{r}] + \frac{1}{2\pi} \sum_\mathbf{r} A_0^\mathbf{r} \mathcal{B}_\mathbf{r} \right) \). We shall treat it in the mean-field approximation, looking for solutions of the equations of motion: \( \delta S_{eff} = 0 \), \( \delta \mathcal{B}_\mathbf{r} S_{eff} = 0 \). Non-trivial solutions of these equations, if exist, determine expectation values of CS fields \( \mathcal{B}_\mathbf{r} \) and \( A_0^\mathbf{r} \) in a self-consistent
way. While the mean-field is uncontrolled, it will result in a solution with the fermionic spectrum gapped in the bulk. The fluctuation corrections are thus finite and appear to be numerically small, giving some confidence in the validity of the mean-field. In this sense situation here is more favorable as compared to the theory of the half filled Landau level[33,37], where the spectrum is gapless.

We shall look for spatially homogeneous solutions of the mean-field equations, allowing for broken symmetry between $A$ and $B$ sub-lattices. Such choice is motivated by recent numerical results of ZHW where an asymmetry between sublattices $A$ and $B$ was observed. Therefore, we will look for solution $\{A^0_{r_A}, B_{r_A}\}$ and $\{A^0_{r_B}, B_{r_B}\}$ that are independent of $r_A$ and $r_B$ respectively. It is convenient to separate symmetric and antisymmetric combinations of gauge fields between $A$ and $B$ sites, belonging to the same unit cell: $(A_{r_A} \pm A_{r_B})$ and $(B_{r_A} \pm B_{r_B})$. It is easy to see that the homogeneous symmetric components may be gauged out at half filling[33]. The reason is that the corresponding total flux threading the unit cell is $2\pi$, which may be disregarded due to the periodicity. This leads to the conclusion that the total CS flux threading the unit cell is gauge equivalent to zero. Hence one can gauge out the phases $A_{r_A + r_B}$ residing on NN links in Eq. (3). As a result only antisymmetric gauge fields residing on the next-NN links (see Fig. 1) remain. These fields give rise to the flux threading the large equilateral triangle with a site in it, depicted in Fig. 1. It is given in terms of antisymmetric fields as $3\phi \equiv \langle B_{r_A} - B_{r_B} \rangle$. The flux threading the neighboring "empty" equilateral triangle (with no site in it), is thus $-3\phi$, which is consistent with the fact that the total flux through the unit cell is zero. The antisymmetric component of the gauge field thus precisely gives raise to the staggered Haldane flux configuration[33] in the unit cell.

Introducing notation, $A^0 = (A^0_{r_A} - A^0_{r_B})/2$, the equations of motion for the asymmetric components take the form

$$\partial_{A^0} W(A^0, \phi) + \frac{3}{2\pi} \phi = 0, \quad \partial_{\phi} W(A^0, \phi) + \frac{3}{2\pi} A^0 = 0.$$  

The field $A^0$ here plays the role of the inversion symmetry breaking mass term. Indeed, it appears in action [3] as $A^0 \sum_{\{A, r\}} \frac{1}{2\pi} \phi \left( n_{r_A} - n_{r_B} \right)$, where the summation is performed over NN dimer pairs $\{r_A, r_B\}$. Then, the first equation of motion [4] yields the self-consistency condition $3\phi = 2\pi (n_{r_A} - n_{r_B})$.

The above consideration shows that the fermionic mean-field theory with the staggered CS flux configuration depicted in Fig. 1 naturally gives raise to the Haldane’s model[33], studied in connection with parity anomaly in a honeycomb lattice. The spectrum of the Haldane Hamiltonian is gapped, and consist of two bands:

$$E_p(A^0, \phi) = J_2 \cos \phi \left[ G_p^2 - 3 \right] + \sqrt{m_p^2 + J_2^2 G_p^2},$$

where $m_p = A^0 - 2J_2 \sin \phi \sum_{\{p\}} \sin(p a_p)$ is the gap, and $G_p = |\sum_{i} e^{ip a_i}|$. Fermionic ground state energy of the model is given by $W(A^0, \phi) = E_p(A^0, \phi)$, where the sum runs over occupied states of the half-filled system. For $J_2/J_1 \lesssim 1/3$, only the lowest band is filled, Fig. 3. Substituting $W(A^0, \phi)$, along with the spectrum [5], into first Eq. (3), one obtains a self consistent equation relating parameters $\phi$ and $A^0$:

$$\frac{3}{2\pi} \phi = \sum_p \frac{m_p}{\sqrt{m_p^2 + J_2^2 G_p^2}}.$$  

Numerical solution of Eq. (6) is shown in the Haldane’s phase diagram in Fig. 2 for two values $J_2/J_1 = 0.2, 0.36$. The second of Eqs. (4) minimizes the ground state energy $F = W(A^0, \phi) + \frac{3}{2\pi} \phi A^0$ with respect to $\phi$ given the self-consistency relation (6). Minimization yields finite values, shown by dashed line in Fig. 2 for both, $\phi$ and $A^0$ in the intermediate frustration regime $0.2 \lesssim J_2/J_1 \lesssim 0.36$. At the boundaries of this regime $\phi$ and $A^0$ vanish simultaneously, indicating two distinct phase transitions. In between these two transitions there is a broad regime where $\phi \approx 0.55$ is almost independent of the ratio $J_2/J_1$.\
FIG. 3: (Color online) The spectrum $E_{\theta,0,p}$ plotted from Eq. (6) along $K' - \Gamma - K$ line of the Brillouin zone in the absence of the CS field modulation, $m_p = 0$. Lowest bands correspond to $J_2/J_1 = 0.3; 1/3; 0.35; 0.37$ from bottom up. Horizontal lines represent chemical potentials at half filling. For $J_2/J_1 \leq 1/3$, the chemical potential is at $\mu = 0$, while two upper lines correspond to $J_2/J_1 = 0.37, 0.35$, from up to bottom. Notice that for $J_2/J_1 = 1/3$ the maximum of the lowest branch at $\Gamma$ point reaches the height of Dirac points $K$ and $K'$. A phase transition is thus expected for $J_2/J_1 > 1/3$, when the lowest band still exhibits a moat capable of carrying CSL, while a divergent density of states corresponds to a nested Fermi surface in the effective fermion description. This results in an instability towards spin density wave state [11].

All these values fall in the topologically nontrivial region of the phase diagram depicted in Fig. 2 resulting thus in topological phase of fermions with Chern number $C = \pm 1$. On the other hand, finite $\phi$ means stabilization of Ising antiferromagnetic ordering of the spins for $0.2 \lesssim J_2/J_1 \lesssim 0.36$, in agreement with the DMRG results of ZHW [14]. Nevertheless, we identify this phase as a non-uniform CSL, rather than an Ising antiferromagnet (AF). The AF polarization is merely a consequence of the staggered gauge flux emerging upon the TRS breaking. While reproducing the found numerically AF polarization, the state we suggest bears the hallmarks of a CSL. It is characterized by a finite gap for the $S = 1$ excitations in the bulk and gapless chiral edge state having $S = 1/2$ spinon excitations. In a finite-length cylinder geometry, spin transfer between the two edge states should reveal the fractional nature of the spinon excitations. In a numerical simulation, this can be checked by realizing Laughlin’s flux insertion construction as in Ref. [26].

We now briefly discuss the order of the two transitions. Within our mean-field treatment, transition from X-Y AF into CSL at $J_2/J_1 \approx 0.2$ is continuous. To see this one may expand the ground-state energy functional $F$ in terms of the order parameters $A^0$ and $\phi$ in the spirit of the Landau free energy

$$F(A^0, \phi) = F_0 + 2a_{12}A^0 + a_{11}(A^0)^2 + a_{22}\phi^2 \ldots$$

where coefficients $a_{ij}, i, j = 1, 2$ are functions of the ratio $J_2/J_1$, which may be easily evaluated from the fermionic spectrum [3]. The finite values of the two order parameters appear, when the determinant of the quadratic form $\det[a_{ij}] = a_{11}a_{22} - a_{12}^2$ changes its sign. We have checked that the determinant indeed goes through zero at $J_2/J_2 \approx 0.2$, signaling the second order transition towards a state with finite $A^0$ and $\phi$.

The nature of the transition at $J_2/J_1 \approx 0.36$ is rather different. It is associated with the change of the underlying band structure. Indeed, at $J_2/J_1 = 1/3$ the maximum at $\Gamma$ point reaches the energy of the Dirac points at $K$ and $K'$, Fig. 3. Thus at $J_2/J_1 > 1/3$ the half-filled system exhibits hole fermi pocket at $\Gamma$ and particle fermi pockets at $K$ and $K'$. Presence of the occupied states in the upper band near $K$ and $K'$ is a strong disincentive against formation of the gap (since the upper band moves up in energy, the total energy is not lowered). In our mean-field treatment the non-trivial self-consistent solution disappears (see insets to Fig. 2) in the first order way at $J_2/J_1 \approx 0.36$, giving rise to the phase diagram depicted in Fig. 4. The question remains, though, whether the fluctuations may turn this transition into a continuous one.

Our mean-field theory of intermediate frustration regime is in a satisfactory agreement with the numerical data. In the bulk of this range $\phi \approx 0.55$ (see upper inset in Fig. 2) which corresponds $\langle S^z_\Delta - S^z_{\Delta'} \rangle \approx 0.26$. This agrees (within small error bars) with the numerical observation reported in Ref. [14]. The numerical value of the ground state energy per spin for e.g. $J_2/J_1 = 0.3$ is $F/J_1 \approx -0.311$. This is lower than the corresponding exact diagonalization result [12] by only about 5%.

The theory also provides predictions, which may be tested against numerics. The finite order parameter $\phi$ implies time reversal symmetry broken chiral ground state. One may thus expect a finite value of chirality $\chi = \langle S_A \cdot (S_B \times S_A') \rangle$, where the spins reside on the sites of the $120^\circ$ triangle shown in Fig. 1. Within our approximations one maps spin operators onto non-interacting fermions with CS phases. The phases lead to $\chi \propto \sin \phi$, where the proportionality coefficient involves summation of the product of the fermionic Green functions over the Brillouin zone. The result is plotted in the lower inset in Fig. 2, showing that one is expected to find $\chi \approx \pm 0.03$ in the bulk of the regime. The sign is fixed by spontaneous symmetry breaking between sub-lattices $A$ and $B$. Topological nature of the CSL phase on the other hand can be directly revealed numerically in a cylindrical geometry with introducing an additional flux through it. The flux threading the cylinder affects spins on $A$, $B$ sub-lattices differently, leading to asymmetry between the edges of the cylinder. This is the so called topological pump charge effect [42], which gives a possibility to directly measure the Chern number [43, 44]. The latter can be obtained by applying $2\pi$ flux and calculating the polarization.

The sign of the chirality determines direction for the
gapless chiral edge states. One way to detect the gapless state is to calculate equal time spin-spin correlation function. Since the effective low-energy theory describing these edge excitations is the chiral Luttinger liquid\cite{10} \cite{11}, one expects

$$\langle S^z_l S^z_{l'} \rangle \sim \frac{1}{(l-l')^2},$$  \hspace{1cm} (8)$$

where $\langle S^z_l \rangle \approx 0.017$, and spin $l$ and $l'$ are located along the edge. This should be compared with the exponential decay of correlations in the bulk. One may also check that the equal time correlation function $\langle S^z_l S^z_{l'} \rangle$ exhibits the power law behavior with the same exponent $\alpha = 2$.

In an experimental cold-atoms implementation, the chiral pattern of bulk currents in principle can be detected with the recently developed technique of the sudden decoupling \cite{16}. Other properties of the CSL can be revealed in time of flight experiments. We predict that boson density-density correlations on the periphery of the gas follow Eq. (8) exhibiting power-law behavior, whereas correlations in the bulk decay exponentially. Finally, the momentum distribution function manifests itself in the form of the Bose surface\cite{12}. This property, together with real space density asymmetry on sublattices $A$ and $B$ can also be probed in cold atom settings.

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\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 AF & CSL & SDW & $120^\circ$ order \\
$O(2)$ & $Z_2 \cdot T$ & $Z_2 \cdot O(2)$ & $Z_2$ \\
$C=0$ & $C=1$ & $C=0$ & \\
\hline
0 & 0.2 & 0.36 & 1.1 \\
\hline
\end{tabular}
\end{table}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{(Color online) Schematic phase diagram of the XY model across the range of the ratio $J_2/J_1$. Phases are marked by broken symmetries and corresponding Chern numbers, $C$.}
\end{figure}

\begin{thebibliography}{99}
\bibitem{1} J. S. Helton, K. Matan, M. P. Shores, E. A. Nytko, B. M. Bartlett, Y. Yoshida, Y. Takano, A. Guslov, Y. Qiu, J.-H. Chung, D. G. Nocera and Y. S. Lee, Phys. Rev. Lett. 98, 107204 (2007).
\bibitem{2} M. A. de Vries, K. V. Kamanov, W. A. Kockelmann, J. Sanchez-Benitez and A. Harrison, Phys. Rev. Lett. 100, 157205 (2008).
\bibitem{3} J. S. Helton, K. Matan, M. P. Shores, E. A. Nytko, B. M. Bartlett, Y. Qiu, D. G. Nocera and Y. S. Lee, Phys. Rev. Lett. 104, 147201 (2010).
\bibitem{4} T. H. Han, J. S. Helton, S. Chu, A. Prodi, D. K. Singh, C. Mazzoli, P. Muller, D. G. Nocera and Y. S. Lee, Phys. Rev. B 83, 100402(R) (2011).
\bibitem{5} T. H. Han, J. S. Helton, S. Chu, D. G. Nocera, J. A. R.-Rivera, C. Broholm and Y. S. Lee Nature 492, 406 (2012).
\bibitem{6} D. V. Pilon, C. H. Lui, T. -H. Han, D. Shrekenhamer, A. J. Frenzel, W. J. Padilla, Y. S. Lee, and N. Gedik, Phys. Rev. Lett. 111, 127401 (2013).
\bibitem{7} R. Jordens, N. Strohmaier, K. Gunter, H. Moritz, and T. Esslinger, Nature 455, 204 (2008).
\bibitem{8} L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Nature (London) 483, 302 (2012).
\bibitem{9} K. K. Gomes, W. Mar, Wonhee Ko, F. Guinea, and H. C. Manoharan, Nature (London) 483, 306 (2012).
\bibitem{10} J. Simon and M. Greiner, Nature (London) 483, 282 (2012).
\bibitem{11} M. Hermele, V. Gurarie, and A. M. Rey, Phys. Rev. Lett. 103, 135301 (2009); ibid. 107, 059901 (2011).
\bibitem{12} C. N. Varney, K. Sun, V. Galitski and M. Rigol, Phys. Rev. Lett. 107, 077201 (2011).
\bibitem{13} J. Carrasquilla, A. Di Ciolo, F. Becca, V. Galitski, M. Rigol, Phys. Rev. B 88, 241109(R) (2013), A. Di Ciolo, J. Carrasquilla, F. Becca, M. Rigol, V. Galitski, Phys. Rev. B 89, 094413 (2014).
\bibitem{14} Z. Zhu, D. Huse and S.R. White, Phys. Rev. Lett. 111, 257201 (2013).
\bibitem{15} L. Wang, D. Poilblanc, Z. Gu, X. Wen, and F. Verstraete, Phys. Rev. Lett. 111, 037202 (2013).
\bibitem{16} S. Gong, W. Zhu, D. N. Sheng, O. I. Motrunich, and M. P. A. Fisher, Phys. Rev. Lett. 113, 027201 (2014).
\bibitem{17} H. C. Jiang, H. Yao, and L. Balents, Phys. Rev. B 86, 024424 (2012).
\bibitem{18} S. Gong, D. N. Sheng, O. I. Motrunich, and M. P. A. Fisher, Phys. Rev. B 88, 165138 (2013).
\bibitem{19} R. V. Mishmash, I. Gonzales, R. G. Melko, O. I. Motrunich, and M. P. A. Fisher, arXiv:1403.4258v1.
\bibitem{20} Z. Zhu, D. A. Huse, and S. R. White, Phys. Rev. Lett 110, 127205 (2013).
\bibitem{21} S. Yan, D. A. Huse and S. R. White, Science 332, 1173 (2011).
\bibitem{22} Y. M. Lu, Y. Ran, and P. A. Lee, Phys. Rev. B 83, 224413 (2011).
\bibitem{23} L. Balents, Nature 464, 199 (2010).
\bibitem{24} S. Sachdev, Phys.Rev. B 45, 12377 (1992).
\bibitem{25} C. Xu and S. Sachdev, Phys. Rev. B 79, 064405 (2009).
\bibitem{26} F. Wang and A. Vishwanath, Phys. Rev. B 74, 174423 (2006).
\bibitem{27} A. Mulder, R. Ganesh, L. Capriotti, and A. Paramekanti, Phys. Rev. B 81, 214419 (2010).
\bibitem{28} Y.-C. He, D. N. Sheng, and Y. Chen, Phys. Rev. Lett. 112, 137202 (2014).
\bibitem{29} S.-S. He, W. Zhu, and D. N. Sheng, arXiv:1312.4519
\bibitem{30} B. Bauer, L. Cincio, B. P. Keller, M. Dolfi, G. Vidal, S. Trebst, and A. W. W. Ludwig, arXiv:1401.3017.
\bibitem{31} V. Kalmyev and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987).
\bibitem{32} X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B 39, 11413 (1989).
\bibitem{33} T. A. Sedrakyan, L. I. Glazman, and A. Kamenev, Phys. Rev. B 89, 201112(R) (2014).
\bibitem{34} T. A. Sedrakyan, A. Kamenev, and L. I. Glazman, Phys. Rev. A 86, 063639 (2012).
\bibitem{35} J. K. Jain, Phys. Rev. Lett. 63, 199 (1989).
\end{thebibliography}
[36] A. Lopez and E. Fradkin, Phys. Rev. B 44, 5246 (1991).
[37] B. Halperin, P. A. Lee and N. Read, Phys. Rev. B 47, 7312 (1993).
[38] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
[39] R. Shankar, Ann. Phys. (Berlin) 523, 751 (2011).
[40] T. S. Jackson, N. Read, and S. H. Simon, Phys. Rev. B 88, 075313 (2013).
[41] R. Nandkishore, Gia-Wei, and A. Chubukov, Phys. Rev. Lett. 108, 227204 (2012).
[42] D. J. Thouless, Phys. Rev. B 27, 6083 (1983); Q. Niu and D. J. Thouless, J. of Phys. A 17, 2453 (1984).
[43] D. Sheng, Z. Weng, L. Sheng, and F. Haldane, Phys. Rev. Lett. 97, 036808 (2006).
[44] L. Wang, H. Hung, and M. Troyer, arXiv:1402.6704
[45] X. G. Wen, Phys. Rev. B 41, 12838 (1990).
[46] D. Varjas, M. P. Zaletel, and J. E. Moore, Phys. Rev. B 88, 155314 (2013).
[47] M. Atala, M. Aidelsburger, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Nature Physics 10, 588 (2014).