Quantum critical behavior of the transverse-field quantum Ising model on a fractal structure

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I study the properties of the quantum critical point of the transverse-field quantum Ising model of which the base structure is a Sierpinski carpet. Using a continuous-time quantum Monte Carlo simulation method and the finite-size scaling analysis, I identify the quantum critical point and investigate its scaling properties. The calculation of the dynamic critical exponent gives $z = 1.2 \pm 0.1$. The fact that it deviates from one is a direct consequence of the non-integer dimensionality of the system. Other critical exponents are also calculated and are compared to those of the classical critical point.

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The transverse-field quantum Ising model is one of the most widely used models for studying quantum effects on the magnetic order and critical phenomena in spin systems. In this model, quantum fluctuations are introduced by applying a magnetic field $\Delta$ perpendicular to the Ising spin direction. Starting from the zero field limit, which corresponds to the original classical Ising model, one may investigate the effect of quantum fluctuations by controlling $\Delta$. Recent theoretical studies of the ferromagnetic quantum Ising model based on various structures such as small-world networks, scale-free networks, and the Sierpinski carpet show that if one increases the transverse magnetic field, the ferromagnetic-paramagnetic phase transition temperature $T_c$ decreases monotonically. However, the transverse field is found not to affect the critical exponents $\alpha$, $\beta$, $\gamma$ and $\nu$. In addition, these studies all suggest that as the transverse magnetic field becomes strong enough, $T_c$ apparently vanishes at a critical field $\Delta_c$, but the limitation of the numerical method used in those works prohibited direct investigation of the zero temperature limit.

When $T = 0$, the phase transition is controlled solely by $\Delta$ and is governed by the quantum critical point, which belongs, in general, to a different universality class from that of the classical critical points away from $\Delta_c$. One of the characteristics of the quantum critical point is the dynamic critical exponent $z$. It determines the relative scaling of space and time which leaves the action invariant in the renormalization-group sense, in the vicinity of the quantum critical point. In particular, it is known that $z = 1$ for the transverse-field quantum Ising model in all integer dimensions. A recent study of the quantum Ising model on Watts-Strogatz small world networks also obtained results that are consistent with the above general rule for the integer dimensional models in spite of the complex nature of the network structure. It is because their model is in the mean-field limit and the upper critical dimension is an integer.

In this paper, I will study the quantum critical point of the quantum Ising model on a fractal structure. The main focus will be on the calculation of the dynamic critical exponent, because it is an interesting question whether $z$ is equal to one or not when the base structure has a non-integer dimensionality. I will also compute other critical exponents and compare them with those of the classical critical point.

The Hamiltonian of the quantum Ising model is given by

$$H = -J \sum_{\langle ij \rangle} \sigma^x_i \sigma^x_j + \Delta \sum_i \sigma^z_i$$

where $\sigma^x_i$ and $\sigma^z_i$ are Pauli matrices representing the $x$...
and \( z \) components of the spin at site \( i \). The first summation runs only over nearest neighbor pairs and the periodic boundary condition is imposed. Apart from the temperature \( T \), there are two important energy scales in this model: the ferromagnetic coupling constant \( J \) and the transverse magnetic field \( \Delta \). For simplicity I will use the energy unit in which \( J = 1 \). For \( \Delta = 0 \), this model is identical to the ordinary classical Ising model and it is straightforward to obtain all energy eigenstates of the problem. If \( \Delta \neq 0 \), however, the second term causes quantum fluctuations to the classical eigenstates because it does not commute with the first term. As a result, it tends to destroy the ferromagnetic order that may have been resulted from the Ising exchange interaction.

This model is most easily analyzed using the Suzuki-Trotter decomposition method. Writing the action as an integral in the imaginary time using the standard procedure, the temporal segments of each spin may be thought of as interacting via a nearest neighbor interaction in the time direction within each site. Therefore, a \( D \)-dimensional quantum Ising model may be mapped to a \((D + 1)\)-dimensional classical Ising model. Since the imaginary time direction of the quantum model is simply one of the spatial directions of the classical counterpart, the dynamic critical exponent is simply given by \( z = 1 \), as long as \( D \) is an integer. However, this argument does not apply to a fractal, because the spatial dimensions, being fractional, are no longer identical in nature to the temporal dimension. The quantum Ising model on a fractal structure has been studied in a previous paper in terms of the critical behavior at finite temperatures. In the current work, I will focus on the quantum critical point at \( T = 0 \).

The fractal structure I will use in this paper is the Sierpinski carpet, which is constructed in the following way. First, a two dimensional square lattice is divided into a \( 3 \times 3 \) array of equal-size square regions and then the region in the middle is removed. Each of the remaining regions is again divided into a \( 3 \times 3 \) array of smaller square regions, and this process is repeated recursively. Figure 1 shows the result of this process after five recursions. After \( n \) recursions, the total number of sites is \( 8^n \) and

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**FIG. 2:** (Color online) Binder cumulant \( U \) as a function of temperature for (a) \( \Delta = 2.395 \), (b) 2.3975, and (c) 2.4. The system size is \( L = 3^n \), where \( n \) is the number of recursions. The errorbars are smaller than the symbols.

**FIG. 3:** (Color online) Scaling function \( \tilde{U}(0, T L^z) \) for (a) \( z = 1.1 \), (b) 1.2, and (c) 1.3. The errorbars are smaller than the symbols.
the length of each side $3^n$, hence the dimension of this structure is $\log 8/\log 3 \approx 1.893$. In our quantum Ising model, each site of the Sierpinski carpet is occupied by an Ising spin.

We cannot directly access the zero temperature limit in Monte Carlo simulations, because the length in the time direction is inversely proportional to $T$ and becomes infinity. Instead, I will rely on the finite size scaling method. In order to identify the critical transverse field $\Delta_c$, I will use the fourth-order Binder cumulant $U$:

$$U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}. \quad (2)$$

Here, $m$ is the magnetization per spin and $\langle \cdots \rangle$ denotes the thermal average. In the vicinity of the quantum critical point, this quantity obeys a finite-size scaling form

$$U(T, \Delta, L) = \tilde{U} \left( (\Delta - \Delta_c) L^{1/\nu}, T L^z \right), \quad (3)$$

where $L$ is the system size and $\nu$ is the critical exponent for the correlation length. If $\Delta = \Delta_c$, the above quantity depends only on $T L^z$. Therefore, we can identify $\Delta_c$ by demanding that the peak of the scaling function $\tilde{U}$ as a function of $T$ should not depend on $L$. Then $z$ is obtained from the condition that $U(0, T L^z)$ for different system sizes should all collapse onto a single curve within the scaling regime.

I developed a quantum Monte Carlo simulation program using the Swendsen-Wang cluster algorithm. In order to handle the imaginary time dimension, I adopted a continuous-time method. The Binder cumulants calculated from the numerical simulations are shown in Fig. 2. From the condition of the maximum value being independent of $L$, we find that $\Delta_c = 2.3975 \pm 0.0025$. This is consistent with the approximate result obtained from the finite temperature analysis in Ref. 9, where it was speculated that $\Delta_c \approx 2.4$. The universal maximum value of the Binder cumulant may be simply read off from the data and is given by $U_{\text{max}} = 0.57 \pm 0.02$.

The scaling function $\tilde{U}$ at $\Delta_c$ as a function of the single parameter $T L^z$ is shown in Fig. 3. Focusing on the data close to the quantum critical point at $T = 0$, it is obvious that the middle plot [Fig. 3(b)] shows the best collapse and we obtain $z = 1.2 \pm 0.1$. We may thus conclude that the dynamic critical exponent in our fractal model is not the same as in integer dimensional models.

The critical exponent $\nu$ may be obtained using Eq. 3. If we keep $T L^z$ constant, $\tilde{U}$ must be a function of a single parameter $(\Delta - \Delta_c) L^{1/\nu}$. Figure 3(a) shows that the scaling curves collapse nicely into one single curve, from which I estimate $\nu = 0.7 \pm 0.1$. The precision in this result is estimated by tuning $\nu$ around the best fitting value and finding the limit where it is acceptable to regard all curves with different system sizes to collapse into one. Other critical exponents may also be found using the following scaling functions for the magnetization $m$ and the magnetic susceptibility $\chi$.

$$m(T, \Delta, L) = L^{-\beta/\nu} \tilde{m} \left( (\Delta - \Delta_c) L^{1/\nu}, T L^z \right), \quad (4)$$

$$\chi(T, \Delta, L) = L^{\gamma/\nu} \tilde{\chi} \left( (\Delta - \Delta_c) L^{1/\nu}, T L^z \right). \quad (5)$$

From a similar analysis as above, I obtain $\beta = 0.38 \pm 0.04$.
and $\gamma = 1.4 \pm 0.1$. Note that these values are much different from those of the classical Ising model, which are given by $\nu \approx 1.62$, $\beta \approx 0.13$, and $\gamma \approx 2.85$. This confirms that the quantum critical point indeed belongs to a different universality class than its classical counterpart.

In summary, I have studied the quantum critical point of the transverse-field quantum Ising model on a Sierpinski carpet. From the continuous-time quantum Monte Carlo simulations, I identified the critical transverse field to be $\Delta_c = 2.3975 \pm 0.0025$ and calculated the universal maximum value of the Binder cumulant to get $U_{\text{max}} = 0.57 \pm 0.02$. By fitting the scaling function to the data, I obtained the dynamic critical exponent $z = 1.2 \pm 0.1$. This is different from the known value $z = 1$ for the same model in all integer dimensions, and therefore manifests the non-integer dimensionality of the fractal structure used in this paper. I also calculated other critical exponents. The values of $\nu$, $\beta$, and $\gamma$ are different from their counterpart in the classical model, confirming that the quantum and the classical critical points belong to different universality classes.

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[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge Univ. Press, Cambridge, 1999).
[2] P. Pfeuty, Ann. Phys. (NY) 57, 79 (1970).
[3] D. Porras and J. I. Cirac, Phys. Rev. Lett. 92, 207901 (2004).
[4] G. R. Grimmett, T. J. Osborne, and P. F. Scudo, J. Stat. Phys. 181, 305 (2008).
[5] A. Friedenauer, H. Schmitz, J. T. Glueckert, D. Porras, and T. Schaetz, Nature Physics 4, 757 (2008).
[6] H. Yi and M.-S. Choi, Phys. Rev. E 67, 056125 (2003).
[7] H. Yi, Eur. Phys. J. B 61, 89 (2008).
[8] H. Yi, Phys. Rev. E 81, 012103 (2010).
[9] H. Yi, Phys. Rev. E 88, 014105 (2013).
[10] S. K. Baek, J. Um, S. D. Yi, and B. J. Kim, Phys. Rev. B 84, 174419 (2011).
[11] M. Suzuki, Prog. Theor. Phys. 56, 1454 (1976).
[12] K. Binder, Z. Phys. B 43, 119 (1981).
[13] H. Rieger and N. Kawashima, Eur. Phys. J. B 9, 233 (1999).