Gamma Comultiplication and Full Stability

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Abstract:
Let $\mathcal{R}$ be a $\Gamma$-ring with identity (not necessarily commutative), and $M$ a left $\mathcal{R}$-module. Then $M$ is called fully stable if $\theta(N) \subseteq N$ for each $\mathcal{R}$-submodule $N$ of $M$ and $\mathcal{R}$-homomorphism $\theta$ from $N$ into $M$. Equivalently, for each element $m$ in $M$ we have $R \alpha_0 m = \gamma_{\mathcal{R}}^\mathcal{M}(\alpha \Theta_0 (R \alpha_0 m))$, in other words, each $\alpha_0$-cyclic $\mathcal{R}$-submodule $R \alpha_0 m$ satisfies the double annihilator condition.

In this paper we will introduce and study the notion of $\mathcal{R}$-modules in which every $\mathcal{R}$-submodule satisfies the double annihilator condition. Many properties and characterizations of this class of $\mathcal{R}$-modules are considering, and obtain some related results, as well as, their relationship with full stability.

Keywords: Gamma modules, fully stable gamma module, gamma comultiplication, $\alpha$ –comultiplication, fully $\alpha$-stable.

1- Introduction:

In 1964, N. Nobusawa gave as a generalization of the idea of rings the thought of gamma rings [6]. In 1966 W.E.Barnes summed up this idea and obtained entirety fundamental properties of gamma rings [4].

Let $\mathcal{R}$ and $\Gamma$ be two additive abelian groups. $\mathcal{R}$ is called a $\Gamma$-ring if there is a mapping $\mathcal{R} \times \Gamma \times \mathcal{R} \rightarrow \mathcal{R}$, $(\mathcal{r}, \alpha, \mathcal{r}) \rightarrow r \alpha \mathcal{r}$ such that the following hold
(i) $$(\mathcal{r}_1 + \mathcal{r}_2) \alpha \mathcal{r}_3 = \mathcal{r}_1 \alpha \mathcal{r}_3 + \mathcal{r}_2 \alpha \mathcal{r}_3,$$
(ii) $$(\mathcal{r}_1 + \mathcal{r}_2)(\alpha + \beta) = \mathcal{r}_1 \alpha \mathcal{r}_2 + \mathcal{r}_1 \beta \mathcal{r}_2,$$
(iii) $$\mathcal{r}_1 \alpha(\mathcal{r}_2 + \mathcal{r}_3) = \mathcal{r}_1 \alpha \mathcal{r}_2 + \mathcal{r}_1 \alpha \mathcal{r}_3$$ and
(iv) $$(\mathcal{r}_1 \alpha \mathcal{r}_2)(\beta \mathcal{r}_3) = \mathcal{r}_1 \alpha(\mathcal{r}_2 \beta \mathcal{r}_3),$$ for all $\mathcal{r}_1, \mathcal{r}_2, \mathcal{r}_3, \alpha, \beta, \in \Gamma$.

In 2010, R.Ameri, R. Sadeqhi extended the idea of modules to gamma modules [2]. Let $\mathcal{R}$ be a $\Gamma$-ring. An additive abelian group $M$ is called left $\mathcal{R}$ – module, if there exist a mapping: $\mathcal{R} \times \Gamma \times M \rightarrow M$, $\mathcal{r}am$ denote the image of $(\mathcal{r}, \alpha, m)$ such that the following hold:
(i) $\mathcal{r}a(m_1 + m_2) = \mathcal{r}a m_1 + \mathcal{r}a m_2$,
(ii) $$(\mathcal{r}_1 + \mathcal{r}_2)\alpha m = \mathcal{r}_1 \alpha \mathcal{r}_2 + \mathcal{r}_2 \alpha \mathcal{r}_2 m,$$
(iii) $$\mathcal{r}(\alpha_1 + \alpha_2)m = \mathcal{r}\alpha_1 m + \mathcal{r}\alpha_2 m$$ and
(iv) $$\mathcal{r}_1 \alpha_1 (\mathcal{r}_2 \alpha_2 m) = (\mathcal{r}_1 \alpha_1 \mathcal{r}_2) \alpha_2 m,$$ for all $m, \mathcal{m}_1, \mathcal{m}_2 \in \mathcal{M}, \alpha, \alpha_1, \alpha_2 \in \Gamma$ and $\mathcal{r}_1, \mathcal{r}_2, \in \mathcal{R}$.

An $\mathcal{R}$-module $M$ is called unitary if there is $1 \in \mathcal{R}, \alpha_0 \in \Gamma$ such that $\mathcal{1}a_0 m = m$ for all $m$ in $M$. For more detail of gamma module see [2].

Ansari Toroghy, H. introduced the definition of comultiplication modules, $M$ is said to be a comultiplication $\mathcal{R}$-module if for every submodule $N$ of $M$ there exists a two sided ideal $I$ of $\mathcal{R}$ such that $N = \{0 \Gamma N \}$. [3].

In this paper, we consider the comultiplication in the category of gamma modules. A left $\mathcal{R}$-module $M$ is called comultiplication if for each $\mathcal{R}$-submodule $N$ of $M$, there is a $\Gamma$-ideal $A$ of $\mathcal{R}$ such that $N = \gamma_{\mathcal{M}}^\mathcal{M}(A)$, where $\gamma_{\mathcal{M}}^\mathcal{M}(A) = \{m \in M | A \Gamma m = 0\}$.

We give many properties and characterizations of this class of gamma modules. A left $\mathcal{R}$-module $M$ is $\Gamma$-comultiplication if and only if for each $\mathcal{R}$-submodule $N$ of $M$, $N = \{0 \Gamma \text{ann}_\mathcal{R}(N) \}$. We study the relation

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between the fully stable and $\Gamma$-comultiplication gamma module, and study the $\alpha$-comultiplication gamma module. Finally, we consider some generalization of fully stability which are related to comultiplication property.

2. Basics of $\Gamma$-comultiplication gamma modules:

Let $M$ be a left $R$-module and $\alpha$ be an arbitrary fixed element of $\Gamma$. Then $M$ is called $\Gamma$-comultiplication (resp. $\alpha$-comultiplication) if for each $R$-submodule $N$ of $M$, there is a $\Gamma$-ideal (resp. $\alpha$-ideal) $A$ of $R$ such that $N = \gamma^M_M(A)$ (resp. $N = \gamma^M_M(A)$), where $\gamma^M_M(A) = \{m \in M | A\alpha m = 0\}$ and $\gamma^M_M(A) = \{m \in M | A\alpha m = 0\}$.

This equivalent to an $R$-module $M$ is $\Gamma$-comultiplication (resp. $\alpha$-comultiplication) gamma module if and only if for each $R$-submodule $N$ of $M$, $N = (0; (\ell^n M(N)))$. (resp. $N = (0; (\ell^n M(N)))$).

In general every $R$-module of a $\Gamma$-comultiplication gamma module is a $\Gamma$-comultiplication, and every simple $R$-module is $\Gamma$-comultiplication.

Let $R$ be a $\Gamma$-ring and $I$ is an additive subgroup of $R$. We say that $I$ is right (left) $\alpha$-ideal, if for all $\alpha \in \Gamma$, where $\alpha$ arbitrary fixed element in $\Gamma$.

In the following, we give some characterizations of $\alpha$-comultiplication gamma modules.

Examples and Remarks (2.1):

(1) It is not hard matter to show that an $R$-module $M$ is $\Gamma$-comultiplication ($\alpha$-comultiplication) if and only if for each $R$-submodule $N$ of $M$,

$$N = \gamma^M_M(A).$$

Proof:
The sufficiency is clear.

Conversely, let $M$ be a $\Gamma$-comultiplication $R$-module. Then there exists two side $\Gamma$-ideal $I$ of $R$ such that $N = (0; I)$. Then we have $I \subseteq \gamma^M_M(N)$ so that $(0; \gamma^M_M(N)) \subseteq (0; I) = N$. This implies that $N = (0; \gamma^M_M(N))$. ■

(2) Every $R$-submodule of a $\Gamma$-comultiplication gamma module is a $\Gamma$-comultiplication.

(3) Every simple $R$-module is $\Gamma$-comultiplication.

(4) Let $R$ be a $\Gamma$-ring for each nonzero $R$-module $M$, $R \oplus M$ is not a $\Gamma$-comultiplication $R$-module.

Proof:
Assume that $R \oplus M$ is $\Gamma$-comultiplication $R$-module by (If $M$ has a nonzero free $R$-submodule then $M \cong R$) we get $R \oplus M \cong R \cong R \oplus 0$ since $R \oplus M$ is $\Gamma$-comultiplication $R$-module $R \oplus M = R \oplus 0$. Hence $M = 0$ contradiction. ■

(5) Let $R = Z$ and $\Gamma = S$ be arbitrary $\Gamma$-subring of $Z$. Then any abelian group $M$ is $Z$-module where $M = Z$ as $Z$-module, for an $R$-submodule $2\alpha Z$ of $Z$ we have $(0; \ell^2(2\alpha Z)) = Z$. Therefore, $Z$ is not a $\Gamma$-comultiplication $Z$-module.

In the following, our main goal is to clarify the relation between fully stable (duo) gamma module with $\Gamma$-comultiplication.

We proved in [1] that an $R$-module $M$ is fully stable if and only if $b \in R, \alpha \alpha \alpha$ implies that $\ell^\alpha R \alpha \alpha \alpha (R \alpha \alpha \alpha) \subseteq \ell^\alpha R \alpha \alpha \alpha (R \alpha \alpha \alpha b)$ for all $a$ and $b$ in $M$.

The following theorem shows that the class $\alpha$-comultiplication gamma modules is contained in that of fully stable.

Theorem (2.2):
The following are equivalent for an $R$-module $M$

(a) $M$ is $\alpha$-comultiplication.

(b) For every $R$-submodule $N$ of $M$ and every two-sided $\alpha$-ideal $A$ of $R$ with $N \subseteq \gamma^N_N(A)$, there exists a $\alpha$-ideal $B$ of $R$ such that $A \subseteq B$ and $N = \gamma^N_N(B)$.

(c) For every $R$-submodule $N$ of $M$ and every $\alpha$-ideal $A$ of $R$ with $N \subseteq \gamma^N_N(A)$, there exists a $\alpha$-ideal $B$ of $R$ with $A \subseteq B$ and $N \subseteq \gamma^N_N(B)$.
Proof:

(a) \(\Rightarrow\) (b) Let \(N\) be an \(R_G\)-submodule of \(M\) and \(\nsubseteq \gamma^G_M(A)\), where \(A\) is a two-sided \(\alpha\)-ideal of \(R\). By (a) \(N = \gamma^G_M(\ell^R_M(N))\). Put \(B = A + \ell^R_M(N)\) since \(N = \gamma^G_M(\ell^R_M(N)) \subseteq \gamma^G_M(A)\), then \(\ell^R_M(N) \nsubseteq A\) and hence \(A \nsubseteq B\). Further \(\gamma^G_M(B) = \gamma^G_M(A + \ell^R_M(N)) = \gamma^G_M(A) \cap \gamma^G_M(\ell^R_M(N)) = \gamma^G_M(A) \cap N = N\) and this complete the proof of (b).

(b) \(\Rightarrow\) (c) is clear.

(c) \(\Rightarrow\) (a) Let \(N\) be an \(R_G\)-submodule of \(M\). Consider the following family

\[\mathcal{C} = \{\mathcal{D} \mid \mathcal{D} \text{ is an } \alpha\text{-ideal of } R \text{ with } N \subseteq \gamma^G_M(D)\}\]

Clearly \(\mathcal{C}\) is a non-empty family, \(\mathcal{C}\) is partially ordered by inclusion. For each chain \(\{B_\alpha \mid \alpha \in \Lambda\}\) in \(\mathcal{C}\), there is a maximal element \(\mathcal{C} = N \subseteq \gamma^G_M(C)\). We claim that \(N = \gamma^G_M(C)\). If not then by (c) there exists two-sided \(\alpha\)-ideal \(B\) of \(R\) such that \(C \subseteq B\) and \(N \nsubseteq \gamma^G_M(B)\) which contradicts the maximality of \(\mathcal{C}\). Thus \(N = \gamma^G_M(C)\) and this shows that \(M\) is \(\alpha\)-comultiplication. ■

In the following theorem we characterize \(\alpha\)-comultiplication gamma modules.

Theorem (2.3):
The following are equivalent for an \(R_G\)-module \(M\)

(a) \(M\) is \(\alpha\)-comultiplication,
(b) For any two \(R_G\)-submodules \(N_1\) and \(N_2\) of \(M\), if \(\ell^R_M(N_1) \subseteq \ell^R_M(N_2)\), then \(N_2 \subseteq N_1\),
(c) For any \(R_G\)-submodule \(N\) of \(M\) and \(m \in M\), \(\ell^R_M(N) \subseteq \ell^R_M(m)\) implies that \(m \in N\).
(d) for all \(R_G\)-submodule \(N\) of \(M\).

Proof:

(a) \(\Rightarrow\) (b) Assume that \(\ell^R_M(N_1) \subseteq \ell^R_M(N_2)\) for some \(R_G\)-submodules \(N_1\) and \(N_2\) of \(M\). By (a) we have \(N_2 = \gamma^G_M(\ell^R_M(N_2)) \subseteq \gamma^G_M(\ell^R_M(N_1)) = N_1\).

(b) \(\Rightarrow\) (a) Let \(N\) be an \(R_G\)-submodule of \(M\) since \(\ell^R_M(\gamma^G_M(\ell^R_M(N)) = \ell^R_M(N)\), then by (b), \(N = \gamma^G_M(\ell^R_M(N))\) and hence \(M\) is \(\alpha\)-comultiplication.

(c) \(\Rightarrow\) (b) Let \(N_1\) and \(N_2\) be \(R_G\)-submodules of \(M\) such that \(\ell^R_M(N_1) \subseteq \ell^R_M(N_2)\). For \(x \in N_2\), \(\ell^R_M(N_2) \subseteq \ell^R_M(x)\). Thus by (c) we have \(x \in N_1\) and hence \(N_2 \subseteq N_1\).

(a) \(\Rightarrow\) (c) Let \(\ell^R_M(N) \subseteq \ell^R_M(m)\) for some \(m \in M\) and \(R_G\)-submodule \(N\) of \(M\). Then \(m \in \ell^R_M(\ell^R_M(m)) \subseteq \ell^R_M(\ell^R_M(N)) = N\).

(c) \(\Rightarrow\) (d) Assume that there exists \(m \in \gamma^G_M(\ell^R_M(N))\) and \(m \notin N\). By (c) \(\ell^R_M(N) \not\subseteq \ell^R_M(m)\) and hence there is \(z \in \ell^R_M(N)\), and \(z \notin \ell^R_M(m)\). So \(z \alpha m = 0\) which is a contradiction. Thus \(\gamma^G_M(\ell^R_M(N)) \subseteq N\). Thus the other inclusion is always true.

(d) \(\Rightarrow\) (a) It is obvious. ■

Corollary (2.4):
The following conditions are equivalent for quasi-injective \(\alpha_0\)-Noetherian \(R_G\)-module \(M\)

(1) \(M\) is fully stable.
(2) \(\gamma^{\alpha_0}_M(\ell^{\alpha_0}_R(N)) = N\) for each \(R_G\)-submodule \(N\) of \(M\).

Corollary (2.5):
Every \(\alpha_0\)-comultiplication \(R_G\)-module is fully stable.

Proof:
Follows from Theorem (2.2) and the fact that every \(\alpha_0\)-cyclic gamma submodule is stable. ■

For the converse of (2.5) we have the following:
Proposition (2.6):
Let \( M \) be a fully stable \( R_\Gamma \)-module if \( M \) is quasi-injective and \( \alpha_0 \)-Noetherian, then \( M \) is \( \alpha_0 \)-comultiplication.

Proof:
Follows from corollaries (2.5) and (2.4). ■

Corollary (2.7):
(1) Every \( \Gamma \)-comultiplication \( R_\Gamma \)-module is \( \alpha \)-comultiplication.
(2) Every \( \Gamma \)-comultiplication \( R_\Gamma \)-module is fully stable

Proof:
(1) Let \( N \) be an \( R_\Gamma \)-submodule of a \( \Gamma \)-comultiplication \( R_\Gamma \)-module \( M \). Then there is \( \Gamma \)-ideal \( A \) such that \( N = \gamma^M_\Gamma(A) \). It is clear that \( A \) is \( \alpha \)-ideal of \( R \) for an arbitrary fixed \( \alpha \in \Gamma \). Then we have \( N \subseteq \gamma^M_\Gamma(A) \). Then Theorem (2.1) shows that \( M \) is \( \alpha \)-comultiplication.
(2) Follows from (1) and Corollary (2.5). ■

Theorem (2.8):
The following statements are equivalent for an \( R_\Gamma \)-module \( M \).
1. \( M \) is fully stable.
2. \( X \subseteq Y \) for every \( R_\Gamma \)-submodules \( X \) and \( Y \) of \( M \) in which \( X \) is an \( R_\Gamma \)-homomorphic image of \( Y \).
3. For each \( a, b \) in \( M \), \( b \notin R_\alpha a \) implies that \( \ell^\alpha (R_\alpha a) \subseteq \ell^\alpha (R_\alpha b) \).
4. \( \gamma^M_\alpha (\ell^\alpha (R_\alpha a)) = R_\alpha a \) for all \( a \in M \).

Proof:
(1) \( \Rightarrow \) (2) Let \( X \) and \( Y \) be an \( R_\Gamma \)-submodules of \( M \) and \( \theta: Y \to X \) an \( R_\Gamma \)-epimorphism. Then \( X = \theta(Y) \subseteq Y \).
(2) \( \Rightarrow \) (3) Assume that there are \( a, b \) in \( M \) with \( b \notin R_\alpha a \) and \( \ell^\alpha (R_\alpha a) \subsetneq \ell^\alpha (R_\alpha b) \), then \( R_\alpha b \) is an \( R_\Gamma \)-homomorphic image of \( R_\alpha a \). By (2) \( R_\alpha b \subseteq R_\alpha a \), and hence \( b \in R_\alpha a \) which is a contradiction.
(3) \( \Rightarrow \) (4) Assume that there exists \( m \in \ell^\alpha (\gamma^M_\alpha (R_\alpha a)) \) and \( m \notin R_\alpha a \). By (3) \( \ell^\alpha (R_\alpha a) \subsetneq \ell^\alpha (R_\alpha m) \) and hence there is \( s \in \ell^\alpha (R_\alpha a) \), and \( s \notin \ell^\alpha (R_\alpha m) \) so \( s_\alpha m = 0 \) which is a contradiction.
Thus \( \ell^\alpha (\gamma^M_\alpha (R_\alpha a)) \subseteq R_\alpha a \). Thus the other inclusion is always true.
(4) \( \Rightarrow \) (1) It’s obvious. ■

We have proved that an \( R_\Gamma \)-module \( M \) is fully stable if and only if for each \( x, y \) in \( M \). \( \ell^\alpha (x) \subseteq \ell^\alpha (y) \) implies that \( y \in R_\alpha x \). Thus if an \( R_\Gamma \)-module \( M \) is \( \alpha_0 \)-comultiplication, then is fully stable. This motivates to introduce the following:

Definition (2.9):
Let \( M \) be an \( R_\Gamma \)-module and \( \alpha \) be an arbitrary fixed element in \( \Gamma \). Then \( M \) is called fully \( \alpha \)-stable if for each \( m \in M \) and \( R_\Gamma \)-homomorphism \( \theta: Rm \to M \) we have \( \theta(Rm) \subseteq Rm \).

Remark (2.10):
Every fully stable \( R_\Gamma \)-module is fully \( \alpha \)-stable, for each \( \alpha \in \Gamma \).

Proof:
Let \( M \) be a fully stable \( R_\Gamma \)-module and \( \alpha \) arbitrary fixed element in \( \Gamma \). Then for each \( R_\Gamma \)-homomorphism \( f: Rm \to M \) we have \( f(Rm) \subseteq Rm \). ■

The converse of above Remark is not true in general. But in the presence of the \( \alpha_0 \) it is true.
By Theorem (2.8) we get this result.

Corollary (2.11):
An \( R_\Gamma \)-module \( M \) is \( \alpha \)-comultiplication if and only if it is fully stable.
By Remark (2.10) we get this result.
An $R_1$-module $M$ is $\alpha$-comultiplication if and only if it is fully $\alpha$-stable.

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