ABSTRACT
Building upon Boiti et al’s work, a rigorous inverse scattering theory of perturbed Kadomtsev Petviashvili Grassmannian solitons is completed. It yields an $L^\infty$-stability theorem of the Kadomtsev Petviashvili multi-line solitons simultaneously.

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1. INTRODUCTION
The Korteweg-de Vries (KdV) equation
\begin{equation}
-4u_x + u_{x_1 x_1} + 6uu_x = 0
\end{equation}
is an asymptotic model for the propagation of one-dimensional small amplitude long wave surface waves. It is shown that the $L^\infty$-stability of the 1-solitons
\begin{equation}
u_c(x_1, x_3) = 2c^2 \text{sech}^2 \left( c(x_1 + c^2 x_3) \right),
\end{equation}
as solutions of the KdV equation, cannot hold but an $H^1$-orbital stability is valid [3]. To study the soliton stability under weak transversal perturbations of the KdV equation, Kadomtsev and Petviashvili derived the two-dimensional models
\begin{equation}
(-4u_x + u_{x_1 x_1} + 6uu_x)_x_1 \pm 3u_{x_2 x_2} = 0.
\end{equation}
and conjectured that solitons (1.2) are unstable for the KPI equation (corresponding to – where surface tension is present), and stable for the KPII equation (corresponding to + where surface tension is absent) [19]. The phenomena were analysed formally using the inverse scattering theory (IST) by the Gelfand-Levitan-Marchenko equation [45, 13]. Rigorous theory is obtained through PDE approaches: precisely, for the KPI equation, \( L^2(\mathbb{R}^2), L^2(\mathbb{R} \times T) \)-global well posedness of (1.2) are proved by [18, 29] respectively, \( L^2(\mathbb{R} \times T) \)-orbital stability and instability theories are derived by [26, 27]. We refer to [20] and the reference therein for more details on this topic.

The goal in this paper is to prove the following \( L^\infty \)-stability of KPII multi-line solitons:

**Theorem 1.1.** If \( u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2), \) \( u_0(x) = u_0(x_1, x_2, x_3) \) is a \( \text{Gr}(N, M)_{>0} \) KP soliton and \( \sum_{|l| \leq d+8} |\partial^l_x v_0|_{L^1 \cap L^\infty} \ll 1, d \geq 0, \) then there exists \( u : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) such that

\[
(-4u_{x_3} + u_{x_1}u_{x_1}) + 6uu_{x_1})x_1 + 3u_{x_2}x_2 = 0,
\]

\[
u(x_1, x_2, 0) = u(x_1, x_2),
\]

and

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial^l_x [u(x) - u_0(x)]|_{L^\infty} \leq C \sum_{|l| \leq d+8} |\partial^l_x v_0|_{L^1 \cap L^\infty}.
\]

Here and throughout the paper, \( C \) denotes a uniform constant which is independent of \( x, \lambda \).

To introduce the notion of \( \text{Gr}(N, M)_{>0} \) KP solitons, note that one major breakthrough in the KPII theory was given by Sato. He realized that solutions could be written in terms of points of an infinite-dimensional Grassmannian [34, 35, 36, 37]. In particular, a real finite dimensional version of the Sato theory concerns \( \text{Gr}(N, M)_{\geq 0} \) KP solitons which are regular in the entire \( x_1x_2 \)-plane with peaks localized and non decaying along certain line segments and rays. They can be constructed by

\[
u_0(x) = 2\partial^2_{x_1} \ln \tau(x)
\]

[4, 5, 24] where the \( \tau \)-function is the Wronskian determinant

\[
\tau(x) = \left| \begin{array}{ccc}
a_{11} & a_{12} & \cdots & a_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N1} & a_{N2} & \cdots & a_{NM}
\end{array} \right| \left| \begin{array}{ccc}
E_1 & \cdots & \kappa_1^{N-1}E_1 \\
E_2 & \cdots & \kappa_2^{N-1}E_2 \\
\vdots & \ddots & \vdots \\
E_M & \cdots & \kappa_M^{N-1}E_M
\end{array} \right|
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1,\ldots,j_N}(A)E_{j_1,\ldots,j_N}(x),
\]

with \( \kappa_1 < \cdots < \kappa_M, A = (a_{ij}) \in \text{Gr}(N, M)_{\geq 0} \) (full rank \( N \times M \) real matrices with non negative minors), \( E_j(x) = \exp \theta_j(x) = \exp(\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3) \), \( \Delta_{j_1,\ldots,j_N}(A) \) the \( N \times N \) minor of the
matrix $A$ whose columns are labelled by the index set $J = \{j_1 < \cdots < j_N\} \subset \{1, \cdots, M\}$, and $E_{j_1, \cdots, j_N}(x) = \prod_{l<m} (\kappa_{j_m} - \kappa_{j_l}) \exp\left(\sum_{n=1}^{N} \theta_{j_n}(x)\right)$. Gr$(N, M)_{>0}$ KP solitons of which all minors are positive form a dense subset of Gr$(N, M)_{\geq 0}$ KP solitons. Important progress in combinatoric properties, wave interaction, resonant theories, asymptotics characterization, and classification theories of Gr$(N, M)_{\geq 0}$ KP solitons have been developed [14, 21, 22].

Observe that the solitary waves (1.2) are obtained by setting $\kappa_1 = -\kappa_2 = c$ and $a = 1$ in the simplest Gr$(1, 2)_{>0}$ KP solitons

$$u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^{2}\frac{\theta_1(x) - \theta_2(x) - \ln a}{2}.$$  

Hence, Theorem [1.1] implies, in contrast to the KdV equation, under small initial perturbations, for the leading term of the KPII solution, the velocity, amplitude agree with those of the Gr$(N, M)_{>0}$ KP soliton $u_0(x)$, and no phase shift appears. This strengthens characterization in [26 (1.4)], [27 (1.4), (1.8)]. However, it demands more subtle argument to verify that perturbations result in damping by small lagging behind and going off oscillations [19, 25, 23].

Our approach to prove Theorem [1.1] is through the IST. Based on the Lax pair of the KPII equation

$$\begin{cases} 
-\partial_{x_2} + \partial_{x_1}^2 + u)\Phi(x, \lambda) = 0, \\
-\partial_{x_3} + \partial_{x_1}^2 + \frac{3}{2}u\partial_{x_1} + \frac{3}{2}u x_1 + \frac{3}{2}\partial_{x_1}^{-1} u x_2 - \lambda^3)\Phi(x, \lambda) = 0,
\end{cases}$$

a \partial formulation, i.e., a Cauchy integral equation approach, of the IST for vacuum background was completed by [14, 25, 17, 16, 11]. Pioneering research on the IST for perturbed Gr$(1, 2)_{>0}$ KP solitons were done by [39, 5]. Noval fundamental contributions on the direct problem of the IST for perturbed Gr$(N, M)_{>0}$ KP solitons, in particular, introducing the Sato theory, constructing the Green function, discovering multi-valued properties of the Green function at $\kappa_j$, deriving boundedness of the discrete part of the Green function, as well as verifying the $\mathcal{D}^\ell$-symmetry of the Sato eigenfunctions and $\mathcal{D}^{\ell}\text{-symmetry of the eigenfunctions}$ (see [22]), [31, 7, 8, 9, 10, 11, 12], have been established by Boiti, Pempinelli, Pogrebkov, and Primari. Building upon Boiti et al’s work, rigorous direct scattering theory for perturbed Gr$(N, M)_{>0}$ KP solitons is carried out in [42, 43, 44]. We sketch the theory and defer precise definitions to Section 2 here: for

$$u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2),$$

(1.8)

$$u_0(x) \text{ a Gr}(N, M)_{>0} KP \text{ soliton, } \sum_{|j|\leq d+\varepsilon} |\partial_{x_j}^d v_0|_{L^1\cap L^\infty} \ll 1, \text{ } d \geq 0,$$

$$z_1 = 0, \{z_n, \kappa_j\}_{1 \leq n \leq N, 1 \leq j \leq M} \text{ distinct reals, } \det(\frac{1}{\kappa_k - z_h})_{1 \leq k, h \leq N} \neq 0,$$

we

(1) derive, for $\forall \lambda \in \mathbb{C}/\{z_n, \kappa_j\}$, the unique solvability of the eigenfunction

$$(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x_1, x_2))m(x_1, x_2, \lambda) = 0,$$

(1.9)

$$m(x_1, x_2, \lambda) \rightarrow \tilde{\chi}(x_1, x_2, 0, \lambda) = \frac{(\lambda - z_1)^{N-1}}{\prod_{2 \leq n \leq N}(\lambda - z_n)} \chi(x_1, x_2, 0, \lambda);$$

(1.10)
(2) construct the **forward scattering transform**

\[ S(u(x_1, x_2), z_n) = \{ z_n, \kappa_j, D, s_c(\lambda) \} \]

which is continuous at each \( \text{Gr}(N, M) \geq 0 \) KP soliton and evolves linearly if \( u(x) \) solves the KPII equation;

(3) justify the system of the Cauchy integral equation and the \( D \)-symmetry constraint

\[ m(x_1, x_2, \lambda) = 1 + \sum_{n=1}^{N} \frac{m_{z_n, \text{res}}(x_1, x_2)}{\lambda - z_n} + CT_0 m, \]

\[ (e^{\kappa_1 x_1 + \kappa_1^2 x_2} m(x_1, x_2, \kappa_1^+) + \cdots + e^{\kappa_M x_1 + \kappa_M^2 x_2} m(x_1, x_2, \kappa_M^+)) D = 0. \]

In this paper we complete the IST for perturbed \( \text{Gr}(N, M) \geq 0 \) KP solitons by solving the inverse scattering problem,

**Theorem 1.2.** Given a \( d \)-admissible scattering data \( S = \{ z_n, \kappa_j, D, s_c(\lambda) \} \), there exists \( W \), such that the system of the Cauchy integral equation (CIE) and the \( D \)-symmetry constraint

\[ m(x, \lambda) = 1 + \sum_{n=1}^{N} \frac{m_{z_n, \text{res}}(x)}{\lambda - z_n} + CT m, \quad \lambda \neq z_n, \]

\[ (e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_2^2} m(x, \kappa_1^+) + \cdots + e^{\kappa_M x_1 + \kappa_M^2 x_2 + \kappa_M^3 x_2^2} m(x, \kappa_M^+)) D = 0 \]

is uniquely solved in \( W \) satisfying

\[ \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+5} |\partial_x^l [m(x, \lambda) - \tilde{\chi}_{z_n, \kappa_j, A}(x, \lambda)]| W \leq C\epsilon_0. \]

Moreover,

\[ (-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x)) m(x, \lambda) = 0, \]

\[ u(x) \equiv -2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, \text{res}}(x) + i \pi \partial_{x_1} \int T m \, d\zeta \wedge d\zeta, \]

\[ \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_x^l [u(x) - u_0(x)]|_{L^\infty} \leq C\epsilon_0, \]

and \( u : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) solves the KPII equation

\[ (-4u_{x_3} + u_{x_1} u_{x_1})_{x_1} + 3u_{x_2} u_{x_2} = 0. \]

This system of (1.14) and (1.15) represents both analytic and algebraic aspects of the IST toward an understanding of the KPII equation. Precise definition of the **eigenfunction space** \( W \), the **continuous scattering operator** \( T \), \( d \)-admissibility with relations to \( \tilde{\chi}_{z_n, \kappa_j, A}, \epsilon_0 \), and the \( \text{Gr}(N, M) \geq 0 \) KP soliton \( u_0(x) \) in (1.19) are deferred to Section 2. We shall define the **inverse scattering transform** \( S^{-1}(z_n, \kappa_j, D, s_c) \) by the representation formula (1.18). Continuity of \( S^{-1} \) at each scattering data corresponding to a \( \text{Gr}(N, M) \geq 0 \) KP soliton then follows from (1.19). Finally, for initial data satisfying (1.18), the scattering data \( S(u(x_1, x_2), z_n) \) is \( d \)-admissible and
$m(x_1, x_2, \lambda) \in W$. Hence Theorem 1.1 follows from continuities of the direct scattering transform $S$ and the inverse scattering transform $S^{-1}$.

Iteration methods are adopted to prove the system (1.14), (1.15), and the Lax equation (1.17), (1.18) of Theorem 1.2. To illustrate, applying [41], admissibility, the $D$-symmetry, the simple pole condition, and Sato theories, key estimates of the system (1.14), (1.15) reduce to deriving uniform $L^{\infty}$-estimates of the Cauchy integral operator $CT$ (CIO) near $\kappa_j$ which is a highly oscillatory, non homogeneous, asymmetric (no apparent symmetry for a cancellation) singular integral. Specially, distinct singular structures of the continuous scattering data $s_c$, arising from the multi-valued properties at $\kappa_j$ of the Green function, prevent us from using classical Fourier analysis, $L^p$ theories, and integration by parts techniques.

Stokes’ theorem and a scaling invariant property of the leading terms of the CIO at $\kappa_j$ suggest that $W_0$ near $\kappa_j$ are functions of direct sums of multi-valued functions $f^b \in L^\infty(D_{\kappa_j})$ and rescaled Hölder continuous functions $f^2 \in C^{\mu}_\tilde{\sigma}(D_{\kappa_j})$. Namely, given $f \in W_0$, denoting $\tilde{r}$, $\tilde{s}$ as the radius in the $\tilde{\sigma}$-rescaled polar coordinate neighborhood at $\kappa_j$, and $\theta$ the Heaviside function, estimates for $|\theta(1 - \tilde{r})CTf\theta(2 - \tilde{s})|_{L^\infty(\Omega^{\mu}_\tilde{\sigma})}$ can be obtained via principal integration and Hölder interior estimates. To derive estimates for $|\theta(1 - \tilde{r})CTf\theta(\tilde{s} - 2)|_{C^{\mu}_\tilde{\sigma}}$ and $|\theta(\tilde{r} - 1)CTf|_{C^{\mu}_\tilde{\sigma}}$, integrals of slow decaying kernels on non uniformly $\tilde{\sigma}$-compact domains, we apply the deformation method which is based on $\tilde{s}$-meromorphic properties of $CT$ and $f$, and a systematic stationary point analysis for the phase function $\wp(x, \lambda)$ of the continuous scattering operator $T$.

Notice that the IST fails to carry the decay information of the initial data to the KPII solution $u(x)$, but the above argument will be bootstrapped to carry the regularities to any order.

The paper is organized as: notation and definition are provided in Section 2. In Section 3, we provide estimates for the CIO on $W$, major estimates of this paper. In Section 4, we prove (1.14)-(1.16) of Theorem 1.2. In Section 5, we prove (1.17)-(1.19) of Theorem 1.2 construct the inverse scattering transform $S^{-1}$, and prove the continuity.

In Section 6, we prove (1.20) of Theorem 1.2 by the IST established in previous sections and a bilinear form computation [41, 2]. Combining with the direct scattering theory [43, 44], the Cauchy problem of the perturbed Gr($N, M_{>0}$) KP solitons is solved and a uniqueness theorem (independent of $z_n$) is proved. These amount to a proof of Theorem 1.1 simultaneously.

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2. Preliminaries

For the direct scattering problem, given an initial data satisfying (1.1), for the Lax equation is defined by \( \chi = \chi_{x}, \kappa_{j}, A(x, \lambda) \) where

\[
\chi = \chi_{x}, \kappa_{j}, A(x, \lambda) = \frac{1}{\tau(x)} \sum_{1 \leq j_{1} < \cdots < j_{N} \leq M} \Delta_{j_{1}, \ldots, j_{N}}(A)(1 - \frac{\kappa_{j_{1}}}{\lambda}) \cdots (1 - \frac{\kappa_{j_{N}}}{\lambda}) E_{j_{1}, \ldots, j_{N}}(x)
\]

is the normalized Sato eigenfunction. Here \( z_{n} \) are introduced to make poles of \( \chi \) simple. The simple pole condition is unnecessary for the direct problem but is technically important for the inverse problem. We shall prove that \( z_{n} \) are auxiliary parameters to solve the Cauchy problem of the KPII equation by showing different sets of \( z_{n} \) yield the same solution in Theorem 5.2.

The forward scattering transform \( S(u(x_{1}, x_{2}), z_{n}) \) is defined by \( z_{n} \) and \( \kappa_{j} \) which are blowing up and multi-valued points of \( m \); \( D \), satisfying (1.13), are norming constants between values of \( m \) at \( \kappa_{j}^{+} = \kappa_{j} + 0^{+} \) and can be computed by

\[
D = \bar{D} \times \left( \begin{array}{ccc} \delta_{11} & \cdots & \delta_{1N} \\ \vdots & \ddots & \vdots \\ \delta_{N1} & \cdots & \delta_{NN} \end{array} \right)^{-1} \text{diag}(\kappa_{1}^{N}, \cdots, \kappa_{N}^{N}),
\]

\[
\bar{D} = \text{diag}(\Pi_{2 \leq n \leq N}(\kappa_{1} - z_{n})^{N-1}, \cdots, \Pi_{2 \leq n \leq N}(\kappa_{M} - z_{n})^{N-1})D^{\bar{z}}
\]

\[
D^{\bar{z}} = \left( \begin{array}{c} \frac{c_{ji}}{1 - c_{jj}} \end{array} \right),
\]

\[
D^{b} = \text{diag}(\kappa_{1}^{N}, \cdots, \kappa_{M}^{N}) A^{T},
\]

where \( c_{j} = -\int \varphi_{j}(x_{1}, x_{2}, 0) v_{0}(x_{1}, x_{2}) \varphi_{j}(x_{1}, x_{2}, 0) dx_{1} dx_{2}, \varphi_{j}(x), \varphi_{j}(x) \) are residue of the adjoint eigenfunction at \( \kappa_{j} \) and values of the Sato eigenfunction at \( \kappa_{1} \) [41] Theorem 2]; \( s_{c}(\lambda) \) is the continuous scattering data, arising from the \( \bar{D} \)-characterization

\[
\partial_{\lambda} m(x_{1}, x_{2}, \lambda) = s_{c}(\lambda)e^{(\bar{\lambda} - \lambda)x_{1} + (\bar{\lambda} - \lambda^{2})x_{2}} m(x_{1}, x_{2}, \lambda), \quad \lambda \notin \mathbb{R},
\]

which is a nonlinear Fourier transform of the initial perturbation

\[
s_{c}(\lambda) = \frac{\Pi_{2 \leq n \leq N}(\bar{\lambda} - z_{n}) \text{sgn}(\lambda)}{(\bar{\lambda} - z_{1})^{N-1}} \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-[\bar{\lambda} - \lambda]x_{1} + (\bar{\lambda} - \lambda^{2})x_{2}}
\]

\[
\times \xi(x_{1}, x_{2}, 0, \bar{\lambda}) v_{0}(x_{1}, x_{2}) m(x_{1}, x_{2}, \lambda) dx_{1} dx_{2},
\]

and \( \lambda = \lambda_{R} + i\lambda_{I}, \bar{\lambda} = \lambda_{R} - i\lambda_{I}, \xi(x, \lambda) = \frac{1}{\tau(x)} \sum_{1 \leq j_{1} < \cdots < j_{N} \leq M} \Delta_{j_{1}, \ldots, j_{N}}(A) E_{j_{1}, \ldots, j_{N}}(x) \frac{E_{j_{1}, \ldots, j_{N}}(x)}{(1 - \frac{\kappa_{j_{1}}}{\lambda}) \cdots (1 - \frac{\kappa_{j_{N}}}{\lambda})}. \)

Algebraic and analytic constraints for scattering data \( S(u(x_{1}, x_{2}), z_{n}) \) are

\[
D = \left( \begin{array}{ccc} \kappa_{1}^{N} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_{M}^{N} \\ \delta_{N+1,1} & \cdots & \delta_{N+1,N} \\ \vdots & \ddots & \vdots \\ \delta_{M,1} & \cdots & \delta_{M,N} \end{array} \right),
\]
and, adapting the Fourier analysis argument of the proof of [44, Theorem 3],

\[
\left| (1 - \sum_{j=1}^{M} E_{\kappa_j}) \sum_{|l| \leq d+8} \left( |x - \lambda l_1 + |x^2 - \lambda^2 l_2| \right) s_c(\lambda) \right|_{L^\infty} \leq C \sum_{|l| \leq d+8} |\partial_{\lambda}^{2} e_0|_{L^\infty},
\]

(2.7)

\[
s_c(\lambda) = \frac{e^{\lambda x_1 + \lambda^2 x_2 m(x, \lambda)}}{s_c(\lambda)} - h_j(\lambda), \quad h_n(\lambda) = -h_n(\lambda),
\]

(2.8)

Here we have used the first \( N \times N \)-block of \( D \) is diagonal in (2.5) [43, Lemma 4.1], [44, Theorem 5]. Moreover, \( C \) is the Cauchy integral operator, \( T \) is the continuous scattering operator

\[
T \phi(x, \lambda) = s_c(\lambda) e^{(\overline{x} - \lambda) x_1 (\overline{x} - \lambda^2 x_2)} \phi(x, \overline{\lambda}),
\]

(2.12)

and \( T_0 \phi(x_1, x_2, \lambda) = T|_{(x_1, x_2, 0)} \).

For the inverse problem, introduce

**Definition 2.1.** A scattering data \( S = \{z_n, \kappa_j, D, s_c(\lambda)\} \) is called d-admissible if \( d \geq 0 \), (2.5), (2.6) hold where \( D \) are determined by some \( \bar{D}, D^b, \) and \( D^z \) satisfying

\[
\bar{D} = \bar{D} \times \left( \begin{array}{cccc}
\bar{d}_{11} & \cdots & \bar{d}_{1N} \\
\vdots & \ddots & \vdots \\
\bar{d}_{N1} & \cdots & \bar{d}_{NN}
\end{array} \right) \text{diag}(\kappa_1^N, \cdots, \kappa_N^N),
\]

(2.13)

\[
\bar{D} = \text{diag} \left( \frac{\Pi_{2 \leq n < N} (\kappa_1 - z_n)}{(\kappa_1 - z_1)^{N-1}}, \cdots, \frac{\Pi_{2 \leq n < N} (\kappa_M - z_n)}{(\kappa_M - z_1)^{N-1}} \right) D^z,
\]

(2.14)

\[
D^b = \text{diag} (\kappa_1^N, \cdots, \kappa_M^N) A^T,
\]

(2.15)

and

\[
\det \left( \frac{1}{\kappa_k - z_h} \right)_{1 \leq k, h \leq N} \neq 0, \quad z_1 = 0, \{z_n, \kappa_j\} \text{ distinct real},
\]

(2.16)

\[
\epsilon_0 \equiv (1 - \sum_{j=1}^{M} E_{\kappa_j}) \sum_{|l| \leq d+8} \left( |x - \lambda l_1 + |x^2 - \lambda^2 l_2| s_c(\lambda) \right)_{L^\infty} + \sum_{j=1}^{M} |h_j|_{C^1(D_{\kappa_j})} + \sum_{n=1}^{N} |h_n|_{C^1(D_{z_n})} + |D^z - D^b|_{L^\infty}, \quad \epsilon_0 \ll 1,
\]

(2.17)
are fulfilled. Let \( u_0 \) be the \( \text{Gr}(N, M) \) \( > 0 \) KP soliton defined by \( \kappa_j, A \) of \( D^\phi \).

**Definition 2.2.** Given \( \{z_n, \kappa_j\} \), the eigenfunction space \( W \) is the set of functions satisfying

(a) \( \phi(x, \lambda) = \phi(x, \lambda) \);
(b) \( (1 - \sum_{n=1}^{N} E_{z_n}) \phi(x, \lambda) \in L^\infty \);
(c) for \( \lambda \in D_{z_n}^\times \), \( \phi(x, \lambda) = \frac{\phi_{z_n, res}(x)}{\lambda - z_n} + \phi_{z_n, r}(x, \lambda), \phi_{z_n, res}, \phi_{z_n, r} \in L^\infty(D_{z_n}) \);
(d) for \( \lambda \in D_{\kappa_j}^\times \), \( \phi = \phi^b + \phi^s \), \( \phi^b = \sum_{l=0}^{\infty} \phi_l(X)(-\ln(1 - \gamma_j|\beta|))^l \in L^\infty(D_{\kappa_j}), \phi^s \in C^\infty_\sigma(D_{\kappa_j}), \phi^s(x, \kappa_j) = 0 \).

Here the rescaled Hölder space consists of functions satisfying

\[
\|f(s, \alpha, X)|_{H^\mu_\sigma(D_s)} \equiv \sup_{\tilde{s}_1, \tilde{s}_2 \leq \delta \tilde{s}, |\tilde{s}_1 - \tilde{s}_2| \leq 1} \frac{|f(\tilde{s}_1, \alpha_1, X) - f(\tilde{s}_2, \alpha_2, X)|}{|\tilde{s}_1 e^{i\alpha_1} - \tilde{s}_2 e^{i\alpha_2}|^\mu} < \infty
\]

for \( \lambda = \kappa_j + se^{i\alpha} = \kappa_j + \frac{\tilde{s}}{\sigma_\delta} e^{i\alpha} \in D_{\kappa_j} \) with the rescaling parameter

\[
\tilde{\sigma} = \max\{|X_1|, \sqrt{|X_2|}, \sqrt[3]{|X_3|}\},
\]

and \( X_k \) are the coefficients of the phase function of the continuous scattering operator \( T \)

\[
\phi(x, \lambda) = i[(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2 + (\lambda^3 - \lambda^3)x_3]
\]

\[= X_1 s \sin \alpha + X_2 s^2 \sin 2\alpha + X_3 s^3 \sin 3\alpha.
\]

For \( \phi \in W \), define

\[
|\phi|_W \equiv \|(1 - \sum_{n=1}^{N} E_{z_n}) \phi\|_{L^\infty} + \sum_{n=1}^{N} \|(\phi_{z_n, \text{res}})\|_{L^\infty} + \|(\phi_{z_n, r})\|_{L^\infty(D_{z_n})})
+ \sum_{j=1}^{M} \|(\phi^b)\|_{L^\infty(D_{\kappa_j})} + \|(\phi^s)\|_{C^\infty_\sigma(D_{\kappa_j})})
\]

### 3. Estimates for the Cauchy Integral Operator \( CT \)

#### 3.1. The Cauchy integral operator \( CT \) near \( \kappa_j \).

Given a \( d \)-admissible scattering data \( S = \{z_n, \kappa_j, D, s_c\} \), we shall derive uniform estimates of the CIO near \( \kappa_j \) in this section. Major difficulties arise from singularities at \( \lambda, \kappa_j \), no good symmetries for a cancellation of the leading term of \( s_c \), and the highly oscillatory non homogeneous phase function \( \varphi(x, \lambda) \) of the continuous scattering operator \( T \).

We first investigate leading terms of the CIO at \( \kappa_j \). From (2.16) and (2.12), in terms of the polar coordinates \( \lambda = \kappa_j + re^{i\alpha}, \zeta = \kappa_j + se^{i\beta}, 0 \leq r, s \leq \delta, |\alpha|, |\beta| \leq \pi \), the principal parts of \( T \) are

\[
\tilde{\gamma}_j(\zeta) = \frac{i\text{sgn}(\zeta)}{\zeta - \kappa_j} \frac{\gamma_j}{1 - \gamma_j|\beta|} = -\frac{i\partial_\beta \ln(1 - \gamma_j|\beta|)}{\zeta - \kappa_j} = -\partial_\zeta \ln(1 - \gamma_j|\beta|).
\]

Therefore, as is shown in the following lemma, leading terms of the CIO as well as the iteration at \( \kappa_j \) can be integrated and the outcomes are multi-valued functions.
Lemma 3.1. Fixed $\lambda = \kappa_j + r e^{i\alpha} \in D_{\kappa_j}$, for any non negative integer $l$, 
\[
\mathcal{C} \gamma_j \mathcal{E}_{\kappa_j} \left[ -\ln (1 - \gamma_j |\beta|) \right]^l = \left[ -\ln (1 - \gamma_j |\alpha|) \right]^{l+1} - \frac{1}{2\pi i} \oint_{|\kappa - \kappa_j| = \delta} \frac{l!}{l! - \ln (1 - \gamma_j |\beta|)} d\zeta.
\]

Proof. In view of (3.1) and applying Stokes’ theorem,
\[
\mathcal{C} \gamma_j \mathcal{E}_{\kappa_j} 1 = -\frac{1}{2\pi i} \lim_{r \to 0} \left\{ \int_{|\kappa - \kappa_j| = \delta} -\frac{\partial \ln (1 - \gamma_j |\beta|)}{\zeta - \lambda} d\zeta \right\}
+ \int_{D_{\kappa_j}, \delta} \frac{i}{2} \text{sgn}(\zeta_j) \frac{\gamma_j}{(\zeta - \kappa_j)(\zeta - \lambda)} d\zeta
+ \int_{D_{\lambda, \delta}} \frac{i}{2} \text{sgn}(\zeta_j) \frac{\gamma_j}{(\zeta - \kappa_j)(\zeta - \lambda)} d\zeta
= -\ln (1 - \gamma_j |\alpha|) + \frac{1}{2\pi i} \oint_{|\kappa - \kappa_j| = \delta} \frac{\ln (1 - \gamma_j |\beta|)}{\zeta - \lambda} d\zeta.
\]

Other identities can be proved by analogy. □

Suggested by Lemma 3.1 and, by the dilation $\zeta \mapsto \eta = \kappa_j + a(\zeta - \kappa_j)$, the scaling invariant property of the leading terms
\[
\mathcal{C} \gamma_j \mathcal{E}_{\kappa_j} f(\lambda)
= -\frac{1}{2\pi i} \int_{D_{\kappa_j}, \delta} \frac{\frac{1}{2} \text{sgn}(\zeta_j) + \gamma_j}{\zeta - \kappa_j} e^{(\zeta - \kappa_j) f(\zeta)} d\zeta
= -\frac{1}{2\pi i} \int_{D_{\kappa_j}, a\delta} \frac{\frac{1}{2} \text{sgn}(\eta_j) + \gamma_j}{\eta - \kappa_j} e^{(\eta - \kappa_j) f(\eta)} d\eta,
\]
which tames the highly oscillatory properties of the phase function, we introduce topology of $W$ at $\kappa_j$ as (d) in Definition 2.2

Lemma 3.2. If $\phi \in L^p(D_z)$, $p > 2$, then for $\nu = \frac{p-2}{p}$,
\[
|\mathcal{C} E_z \phi|_{L^\nu} \leq C|\phi|_{L^p(D_z)}, \quad |\mathcal{C} E_z \phi|_{H^\nu(D_z)} \leq C|\phi|_{L^p(D_z)}.
\]

Proof. Please see [38] Theorem 1.19 for the details. □

Hence, for $f \in W$, estimates of $|\mathcal{C} E_{\kappa_j} f|_{C^\nu_\delta(D_{\kappa_j})}$ can be reduced to that of the leading term $|\mathcal{C} \gamma_j \mathcal{E}_{\kappa_j} e^{-i\nu(s, \beta, X)} f(s, -\beta, X)|_{C^\nu_\delta(D_{\kappa_j})}$. To this aim, dilating the polar coordinates
\[
\lambda = z + re^{i\alpha} = z + \frac{r}{\delta} e^{i\alpha}, \quad r \leq \delta,
\]
\[
\tilde{\lambda} = z + \tilde{r} e^{i\alpha}, \quad \tilde{\zeta} = z + \tilde{s} e^{i\beta}, \quad \tilde{r}, \tilde{s} \leq \tilde{\delta}\delta
\]
using the scaling invariant property, and (3.1), we decompose
\[
(3.3) \quad \mathcal{C} \tilde{\gamma}_j \mathcal{E}_{\kappa_j} e^{-i\nu(s, \beta, X)} f(s, -\beta, X)
= -\frac{1}{2\pi i} \int_{D_{\kappa_j}, \delta} \frac{\tilde{\gamma}_j(\tilde{s}, \beta) e^{-i\nu(s, \beta, X)} f(s, -\beta, X)}{\tilde{\zeta} - \lambda} d\zeta \wedge d\tilde{\zeta}
\]
\[ = -\frac{1}{2\pi^2} \int_{-\pi}^{\pi} d\beta \Theta_\beta \ln(1 - \gamma_j |\beta|) \int_0^\delta \frac{e^{-ip_0^j(\hat{\sigma})^j,\beta, X) f(\hat{\sigma}, \beta, X)}{\hat{s} - \hat{r} e^{i(\alpha - \beta)}} d\hat{s} \]
\[ \equiv I_1 + I_2 + I_3 + I_4 + I_5, \]

where \( I_k = I_k(\lambda, X) \),

\begin{align}
I_1 &= -\frac{\theta(1 - \hat{r})}{2\pi i} \int \frac{\tilde{\gamma}_j(\hat{s}, \beta) f^\rho(\hat{\sigma}, -\beta, X) \bar{\beta} \bar{\lambda}}{\bar{\zeta} - \lambda} d\bar{\zeta} \wedge d\bar{\zeta}, \\
I_2 &= -\frac{\theta(1 - \hat{r})}{2\pi i} \int \frac{\tilde{\gamma}_j(\hat{s}, \beta)(e^{-ip_0^j(\hat{\sigma})^j,\beta, X) - 1] f^\rho(\hat{\sigma}, -\beta, X) \bar{\beta} \bar{\lambda}}{\bar{\zeta} - \lambda} d\bar{\zeta} \wedge d\bar{\zeta}, \\
I_3 &= -\frac{\theta(1 - \hat{r})}{2\pi i} \int \frac{\tilde{\gamma}_j(\hat{s}, \beta)e^{-ip_0^j(\hat{\sigma})^j,\beta, X) f^\rho(\hat{\sigma}, -\beta, X) \bar{\beta} \bar{\lambda}}{\bar{\zeta} - \lambda} d\bar{\zeta} \wedge d\bar{\zeta}, \\
I_4 &= -\frac{\theta(1 - \hat{r})}{2\pi i} \int \frac{\tilde{\gamma}_j(\hat{s}, \beta)e^{-ip_0^j(\hat{\sigma})^j,\beta, X) f^\rho(\hat{\sigma}, -\beta, X) \bar{\beta} \bar{\lambda}}{\bar{\zeta} - \lambda} d\bar{\zeta} \wedge d\bar{\zeta}, \\
I_5 &= -\frac{\theta(\hat{r} - 1)}{2\pi i} \int \frac{\tilde{\gamma}_j(\hat{s}, \beta)e^{-ip_0^j(\hat{\sigma})^j,\beta, X) f^\rho(\hat{\sigma}, -\beta, X) \bar{\beta} \bar{\lambda}}{\bar{\zeta} - \lambda} d\bar{\zeta} \wedge d\bar{\zeta}. 
\end{align}

In the following subsections, we shall apply Stokes’ theorem and Hölder interior estimates to derive estimates for \( I_1, I_2, \) and \( I_3 \). For \( I_4 \) and \( I_5 \), integrals with slow decaying kernels on non uniformly compact domains, we have to take advantage of the oscillatory factors \( e^{-ip_0^j(\hat{\sigma})^j,\beta, X) \). An efficient way to use oscillatory factors is the deformation method which relies on meromorphic properties in \( \hat{s} \). To carry out these estimates step by step, we introduce the following definition.

**Definition 3.1.** Let \( X \) be defined by (2.21). The phase function \( \psi \) is called \( z \)-homogeneous if \( X_2 = X_3 = 0 \) (linear), or \( X_1 = X_3 = 0 \) (quadratic), or \( X_1 = X_2 = 0 \) (cubic); \( z \)-degenerated non homogeneous by \( X_1 = 0 \) or \( X_2 = 0 \), or \( X_3 = 0 \); and \( z \)-fully non homogeneous by \( X_1X_2X_3 \neq 0 \).

We suppress the \( z \)-dependence for simplicity if no confusion is caused.

3.1. Homogeneous cases.

**Lemma 3.3.** Suppose \( S = \{ z_n, \kappa_j, D, s_c \} \) is d-admissible and \( \partial_x^l f \) are \( \lambda \)-holomorphic on \( D_{\kappa_j} \) for \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \). For quadratic or cubic homogeneous cases,

\[ \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l \text{CTE}_{\kappa_j} f| W \leq C \epsilon_0 \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l f| W. \]

**Proof.** Proofs for \( |E_{\kappa_j} \partial_x^l \text{CTE}_{\kappa_j} f| W = | \sum_{j=1}^{n+4} E_{\kappa_j} \partial_x^l C(\partial_x^l T)E_{\kappa_j}(\partial_x^l f)\}| W \) are identical. We only prove \( |E_{\kappa_j} \text{CTE}_{\kappa_j} f| W \) for simplicity. Applying Lemma 3.2 for simplicity, we shall assume

\[ \kappa_j = \kappa_1, \quad |\lambda - \kappa_1| \leq \frac{\delta}{2}, \quad X_i \geq 0, \quad |X| > 1, \]

and reduce the proof to estimating principal parts. Introduce the scaled coordinates (3.2) and use the decomposition (3.3) - (3.8).
Step 1 (Estimates for $I_1$, $I_2$, and $I_3$): We make two remarks here. Firstly, arguments in this step, for the estimates of $CTf$ restricted on the uniformly compact domains at $\kappa_1$, can be applied to either homogeneous or non-homogeneous cases. Secondly, the proof reminisces estimates of the Beltrami's equation (cf. [38, §8, Chapter I]).

Firstly, applying Lemma 3.1,

\[
I_1 = \theta(1 - \tilde{r})F^\flat(\tilde{\lambda}, X) - \frac{\theta(1 - \tilde{r})}{2\pi i} \oint_{|\zeta| = 2} \frac{F^\flat(\tilde{\zeta}, X)}{\zeta - \tilde{\lambda}} d\zeta,
\]

where

\[
f^\flat(\zeta, X) = \sum_{l=0}^\infty f_l(X)\left[-\ln(1 - \gamma|\beta)\right]^l,
\]

\[
F^\flat(\tilde{\zeta}, X) = \sum_{l=0}^\infty f_l(X)\frac{[-\ln(1 - \gamma|\beta)]^{l+1}}{l + 1},
\]

Thus

\[
P_1 = \theta(1 - \tilde{r})F^\flat(\tilde{\lambda}, X) - \frac{\theta(1 - \tilde{r})}{2\pi i} \oint_{|\zeta| = 2} \frac{F^\flat(\tilde{\zeta}, X)}{\zeta - \kappa_1} d\zeta, \quad I_1 = I_1 - P_1,
\]

\[
|I_1|_{L^\infty(D_{\kappa_1})} + |P_1|_{C^\mu(D_{\kappa_1})} \leq C \epsilon_0|f^\flat|_{L^\infty(D_{\kappa_1})}.
\]

Besides, from Lemma 3.2 and $|\tilde{\gamma}_1(\tilde{s}, \beta)[e^{-i\nu(\tilde{s}, \beta, X)} - 1]| < C$ for $\tilde{s} < 2$,

\[
|I_2|_{C^\mu(D_{\kappa_1})} \leq C \epsilon_0|f^\flat|_{L^\infty(D_{\kappa_1})}.
\]

For $I_3$, one has $|\tilde{\gamma}_1(\tilde{s}, \beta)f^\sharp(\tilde{\nu}, \beta, X)|_{L^\infty(D_{\kappa_1})} \leq C \epsilon_0|f^\sharp|_{H^\mu(D_{\kappa_1})} \tilde{s}^{\mu-1}$ from $f^\sharp \in C^\mu(D_{\kappa_1})$ and $f^\sharp(\kappa_1, X) = 0$. Therefore, an improper integral yields

\[
|I_3|_{L^\infty(D_{\kappa_1})} \leq C \epsilon_0|f^\sharp|_{H^\mu(D_{\kappa_1})}.
\]

To derive the $H^\mu$-estimate of $I_3$, from Lemma 3.2 let $\tilde{\lambda}_j = \kappa_1 + \tilde{r}_je^{i\nu_j}$, $\tilde{r}_j \leq 1$, $j = 1, 2$, and decompose

\[
I_3(\lambda_1, X) - I_3(\lambda_2, X) = \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta})(\tilde{\varphi}_f(\tilde{\nu}, \beta, X) - \varphi_f(\tilde{\nu}, \alpha_1, X)) d\tilde{z} \wedge d\tilde{\zeta}
\]

\[
- \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta})(\tilde{\varphi}_f(\tilde{\nu}, \beta, X) - \varphi_f(\tilde{\nu}, \alpha_2, X)) d\tilde{z} \wedge d\tilde{\zeta}
\]

\[
+ \frac{\varphi_f(\tilde{\nu}, \alpha_1, X)}{4\pi i} \int_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta})|\frac{1}{\tilde{\zeta} - \lambda_2} - \frac{1}{\tilde{\zeta} - \lambda_1}| d\tilde{z} \wedge d\tilde{\zeta}
\]

\[
+ \frac{\varphi_f(\tilde{\nu}, \alpha_2, X)}{4\pi i} \int_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta})|\frac{1}{\tilde{\zeta} - \lambda_2} - \frac{1}{\tilde{\zeta} - \lambda_1}| d\tilde{z} \wedge d\tilde{\zeta},
\]

where $\varphi_f(\nu, \zeta) = e^{-i\nu(\tilde{s}, \alpha_1, X)} f^\sharp(x, \zeta)$. 
Along with Lemma 3.1, yields
\[
|\varphi f_\ast(\tilde{\zeta}, \alpha, X)|_{L^\infty(D_{\lambda_1})} \leq C|f^\ast|_{H^\mu_\ast(D_{\lambda_1})} r^\mu.
\]
(3.14)

Along with Lemma 3.2 yields
\[
\left|\frac{\varphi f_\ast(\tilde{\zeta}, \alpha, X)}{4\pi i}\right| \int_{\tilde{\zeta} \leq 2} \gamma_1(\tilde{\zeta})\left[\frac{1}{\zeta - \lambda_2} - \frac{1}{\zeta - \lambda_1}\right]d\zeta \wedge d\tilde{\zeta} \leq C\epsilon_0|f^\ast|_{H^\mu_\ast(D_{\lambda_1})}|\lambda_1 - \lambda_2|^\mu, \tilde{r}_1 = \tilde{r}_2,
\]
(3.15)

In an entirely similar way,
\[
\left|\frac{\varphi f_\ast(\tilde{\zeta}, \alpha, X)}{4\pi i}\right| \int_{\tilde{\zeta} \leq 2} \gamma_1(\tilde{\zeta})\left[\frac{1}{\zeta - \lambda_2} - \frac{1}{\zeta - \lambda_1}\right]d\zeta \wedge d\tilde{\zeta} \leq C\epsilon_0|f^\ast|_{C^\mu_\ast(D_{\lambda_1})}|\lambda_1 - \lambda_2|^\mu.
\]
(3.16)

Let us now investigate the first term on the right hand side of (3.13). Applying Lemma 3.2, it suffices to derive the estimate for all \(\lambda_1, \lambda_2\), \(|\lambda_1| \leq 1\) with \(\tilde{D} \subset \{\tilde{s} \leq 2\}\) being a disk centred at \(\tilde{\lambda}_1\) with radius \(l\) and \(l = 2|\tilde{\lambda}_2 - \tilde{\lambda}_1|\) (cf. [15] 5.1)). Write
\[
\[
-\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{\zeta} \leq 2} \gamma_1(\tilde{\zeta}) \frac{\varphi f_\ast(\tilde{\zeta}, \beta, X) - \varphi f_\ast(\tilde{\zeta}, \alpha, X)}{(\tilde{\zeta} - \lambda_1)(\zeta - \lambda_2)}d\zeta \wedge d\tilde{\zeta}
\]
\[
-\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{\zeta} \leq 2} \gamma_1(\tilde{\zeta}) \frac{\varphi f_\ast(\tilde{\zeta}, \beta, X) - \varphi f_\ast(\tilde{\zeta}, \alpha, X)}{(\tilde{\zeta} - \lambda_1)(\zeta - \lambda_2)}d\zeta \wedge d\tilde{\zeta}.
\]
\]
(3.17)

Let \(\tilde{D}_0 = \{\zeta: |\tilde{\zeta} - \tilde{\lambda}_1| < \frac{|\tilde{\zeta} - \lambda_1|}{C}\}\).

- If \(\tilde{\zeta} \in \{\tilde{s} \leq 2\}/\tilde{D}\) and \(\kappa_1 \in \tilde{D}_0\), then
\[
\frac{1}{C} \leq |\tilde{\zeta} - \tilde{\lambda}_1|, |\zeta - \lambda_1|, |\zeta - \lambda_2| \leq C.
\]

In this case, using \(f^\ast \in C^\mu_\ast(D_{\lambda_1})\) and [38] Chapter 1,§6.1,
\[
\[
\leq C\epsilon_0|f^\ast|_{C^\mu_\ast(D_{\lambda_1})} |\lambda_1 - \lambda_2| \int_{\{\tilde{s} \leq 2\}/\tilde{D}} \frac{1}{|\zeta - \lambda_2||\tilde{\zeta} - \lambda_1|^2 - \mu}d\zeta \wedge d\tilde{\zeta} \leq C\epsilon_0|f^\ast|_{C^\mu_\ast(D_{\lambda_1})} |\lambda_1 - \lambda_2|^\mu.
\]
\]
(3.18)

- If \(\tilde{\zeta} \in \{\tilde{s} \leq 2\}/\tilde{D}\) and \(\kappa_1 \notin \tilde{D}_0\) then
\[
\frac{1}{C} \leq |\tilde{\zeta} - \tilde{\lambda}_1| \leq C, \ |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq \frac{1}{C} \min\{|\lambda_1 - \kappa_1|, |\lambda_2 - \kappa_1|\},
\]
In this case, using \( f^2 \in C_0^\mu(D_{\kappa_1}) \) and [38] Chapter 1, §6.1,

\[
(3.19) \quad \left| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\{\tilde{s} \leq 2\} / \tilde{D}} \tilde{\gamma}_1(\tilde{\zeta}) \varphi f^2(\tilde{\zeta}, \beta, X) - \varphi f^2(\tilde{\zeta}, \alpha_1, X) \frac{d\tilde{\zeta} \wedge d\tilde{\zeta}}{(\tilde{\zeta} - \lambda_1)(\tilde{\zeta} - \lambda_2)} \right|
\leq C\epsilon_0 |f^2|_{C_0^\mu(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2| \int_{\{\tilde{s} \leq 2\} / \tilde{D}} \frac{1}{|\tilde{\zeta} - \kappa_1||\tilde{\zeta} - \lambda_1|^{2-\mu}} d\tilde{\zeta} \wedge d\tilde{\zeta}
\leq C\epsilon_0 |f^2|_{C_0^\mu(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu.
\]

Therefore the second term on the RHS of (3.17) is done.

Let \( \tilde{L}(\zeta) = 0 \) be the line perpendicular to \( \tilde{\alpha}_1 \tilde{\alpha}_2 \) and passing through \( \frac{1}{2}(\lambda_1 + \lambda_2) \). Set

\[
\tilde{D}_{\lambda_1, \pm} = \tilde{D} \cap \{ \zeta : L(\zeta)L(\lambda_1) \geq 0 \}.
\]

Therefore, thanks to \( f^2 \in C_0^\mu(D_{\kappa_1}) \), and setting \( \eta = \frac{\tilde{\zeta} - \lambda_1}{|\tilde{\zeta} - \lambda_2|} \), \( \frac{\tilde{\zeta} - \kappa_1}{|\tilde{\zeta} - \lambda_1|} = \eta - r_0 e^{i\alpha_0}, \)

\[
(3.20) \quad \left| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{D}_{\lambda_1, +}} \tilde{\gamma}_1(\tilde{\zeta}) \varphi f^2(\tilde{\zeta}, \beta, X) - \varphi f^2(\tilde{\zeta}, \alpha, X) \frac{d\tilde{\zeta} \wedge d\tilde{\zeta}}{(\tilde{\zeta} - \lambda_1)(\tilde{\zeta} - \lambda_2)} \right|
\leq C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2| |f^2|_{C_0^\mu(D_{\kappa_1})} \int_{\{\tilde{s} \leq 2\} \cap \tilde{D}_{\lambda_1, +}} \frac{1}{|\eta - r_0 e^{i\alpha_0}|^2|\eta|^{2-\mu}|\eta - r_0 e^{i\alpha_0}|^1} d\eta d\eta
\leq C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^2|_{C_0^\mu(D_{\kappa_1})} \int_{\{\tilde{s} \leq 2\} \cap \tilde{D}_{\lambda_1, +}} \frac{1}{|\eta - r_0 e^{i\alpha_0}|^2|\eta|^{2-\mu}} d\eta d\eta
\leq C\epsilon_0 |f^2|_{C_0^\mu(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu.
\]

By analogy,

\[
\left| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{D}_{\lambda_1, -}} \tilde{\gamma}_1(\tilde{\zeta}) \varphi f^2(\tilde{\zeta}, \beta, X) - \varphi f^2(\tilde{\zeta}, \alpha, X) \frac{d\tilde{\zeta} \wedge d\tilde{\zeta}}{(\tilde{\zeta} - \lambda_1)(\tilde{\zeta} - \lambda_2)} \right|
\leq C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^2|_{C_0^\mu(D_{\kappa_1})} \int_{\{\tilde{s} \leq 2\} \cap \tilde{D}_{\lambda_1, -}} \frac{1}{|\eta - r_0 e^{i\alpha_0}|^2|\eta|^{2-\mu}} d\eta d\eta
\leq C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^2|_{C_0^\mu(D_{\kappa_1})} \int_{\{\tilde{s} \leq 2\} \cap \tilde{D}_{\lambda_1, -}} \frac{1}{|\eta - r_0 e^{i\alpha_0}|^2|\eta|^{2-\mu}} d\eta d\eta.
\]

Applying \( f^2 \in C_0^\mu(D_{\kappa_1}) \), Stokes’ theorem, and \( |\tilde{\zeta} - \tilde{\lambda}_1|, |\tilde{\zeta} - \tilde{\lambda}_2| \sim |\tilde{\lambda}_1 - \tilde{\lambda}_2| \) on the boundary of \( \tilde{D}_{\lambda_1, -} \) (assured by \( |\tilde{\lambda}| \leq 1, \tilde{D} \subset \{ \tilde{s} < 2 \} \)),

\[
\left| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{D}_{\lambda_1, -}} \tilde{\gamma}_1(\tilde{\zeta}) \varphi f^2(\tilde{\zeta}, \alpha, X) - \varphi f^2(\tilde{\zeta}, \alpha, X) \frac{d\tilde{\zeta} \wedge d\tilde{\zeta}}{(\tilde{\zeta} - \lambda_1)(\tilde{\zeta} - \lambda_2)} \right|
\leq C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^2|_{C_0^\mu(D_{\kappa_1})} \int_{\{\tilde{s} \leq 2\} \cap \tilde{D}_{\lambda_1, -}} \frac{1}{|\eta - r_0 e^{i\alpha_0}|^2|\eta|^{2-\mu}} d\eta d\eta.
\]
\[
\leq C |f^2|_{C_{s}^{\mu}(\partial D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{1+\mu} \int_{\partial \tilde{D}_{\lambda_{1}}, -} \tilde{\gamma}_{1}(\tilde{\zeta}) \frac{1}{(\tilde{\zeta} - \tilde{\lambda}_{1})(\tilde{\zeta} - \tilde{\lambda}_{2})} d\tilde{\zeta} \wedge d\tilde{\zeta}
\]
\[
= C |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{1+\mu} \int_{\partial \tilde{D}_{\lambda_{1}}, -} \partial_{\tilde{\zeta}} \left[ \ln(1 - \gamma |\beta|) \frac{1}{\tilde{\zeta} - \lambda_{1}} \right] d\tilde{\zeta} \wedge d\tilde{\zeta}
\]
\[
\leq C \epsilon_{0} |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{\mu}.
\]
Therefore the first term on the RHS of (3.17) is done. Thus
\[
(3.21) \quad \left| \frac{\tilde{\lambda}_{1} - \tilde{\lambda}_{2}}{4\pi i} \int_{\tilde{\zeta} \leq 2} \tilde{\gamma}_{1}(\tilde{\zeta}) \frac{\varphi_{f^{1}}(\tilde{\zeta}, \beta, X) - \varphi_{f^{1}}(\tilde{\zeta}, \alpha_{1}, X)}{(\tilde{\zeta} - \tilde{\lambda}_{1})(\tilde{\zeta} - \tilde{\lambda}_{2})} d\tilde{\zeta} \wedge d\tilde{\zeta} \right|
\leq C \epsilon_{0} |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{\mu}, \quad \text{for } |\tilde{\lambda}_{j} - \kappa_{1}| \leq 1, \ j = 1, 2.
\]
In an entirely similar way,
\[
(3.22) \quad \left| \frac{\tilde{\lambda}_{1} - \tilde{\lambda}_{2}}{4\pi i} \int_{\tilde{\zeta} \leq 2} \tilde{\gamma}_{1}(\tilde{\zeta}) \frac{\varphi_{f^{1}}(\tilde{\zeta}, \beta, X) - \varphi_{f^{1}}(\tilde{\zeta}, \alpha_{2}, X)}{(\tilde{\zeta} - \tilde{\lambda}_{1})(\tilde{\zeta} - \tilde{\lambda}_{2})} d\tilde{\zeta} \wedge d\tilde{\zeta} \right|
\leq C \epsilon_{0} |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{\mu}, \quad \text{for } |\tilde{\lambda}_{j} - \kappa_{1}| \leq 1, \ j = 1, 2.
\]
Plugging (3.15), (3.16), (3.21), and (3.22) into (3.13), we obtain
\[
(3.23) \quad |I_{3}(\lambda_{1}, X) - I_{3}(\lambda_{2}, X)| \leq C \epsilon_{0} |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})} |\tilde{\lambda}_{1} - \tilde{\lambda}_{2}|^{\mu}
\]
for $|\tilde{\lambda}_{j} - \kappa_{1}| \leq 1, \ j = 1, 2$.

Hence
\[
(3.24) \quad |I_{3}|_{C_{s}^{\mu}(D_{\lambda_{1}})} \leq C \epsilon_{0} |f^2|_{C_{s}^{\mu}(D_{\lambda_{1}})}.
\]

**Step 2 (Proof for $I_{4}$):** Since $I_{4}$ and $I_{5}$ have slow decaying kernels on non uniformly compact domains, thanks to holomorphic properties in $\tilde{s}$, we use the deformation method to take advantage of the oscillatory factors.

Consider the deformation
\[
\tilde{s} \mapsto \tilde{s} e^{i\tau}, \ 0 < \tilde{s} < \tilde{s} \delta = X^{1/k} \delta,
\]
with
\[
\tau = \begin{cases} 
[\epsilon_{1}, 0], & \sin k\beta > 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \\
[-\frac{\alpha}{2}, 0], & \sin k\beta > 0, \ |\alpha - \beta| \geq \frac{\alpha}{2}, \\
[0, +\epsilon_{1}], & \sin k\beta < 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \\
[0, +\frac{\alpha}{2}], & \sin k\beta < 0, \ |\alpha - \beta| \geq \frac{\alpha}{2},
\end{cases}
\]
\[
\tau_{1} = \begin{cases} 
\mp\epsilon_{1}, & \sin k\beta \geq 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \\
\mp\frac{\alpha}{2}, & \sin k\beta \geq 0, \ |\alpha - \beta| \geq \frac{\alpha}{2},
\end{cases}
\]
for $\epsilon_{1} < \frac{\pi}{2k}$. Then
\[
(3.26) \quad |\tilde{s} e^{i\tau_{1}} - \tilde{r} e^{i(\alpha - \beta)}| \geq \frac{1}{C} \max\{\tilde{r}, \tilde{s}\},
\]
In view of (3.9), $\tilde{r} < 1$.

If $f$ is holomorphic in $\lambda$, from (3.1) and a residue theorem,

$$I_4 = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)]$$

$$\times \left( \int_{S_<} + \int_{S_>} \right) e^{-i\varphi(\tilde{s} e^{i\tau}, \beta, X)} f(\tilde{s} e^{i\tau}, -\beta, X) d\tilde{s} e^{i\tau}$$

where $S_\geq = S_\geq(\beta, X, \lambda)$, $\Gamma_4 = \Gamma_4(\beta, X, \lambda)$,

$$S_<< = \{ 2e^{i\tau} : \tau \text{ is defined by (3.25)} \},$$

$$\Gamma_4 = \{ \tilde{s} e^{i\tau} : 2 \leq \tilde{s} \leq \tilde{\sigma} \delta, \tau_\uparrow \text{ is defined by (3.25)} \},$$

$$S_>> = \{ \tilde{\sigma} \delta e^{i\tau} : \tau \text{ is defined by (3.25)} \}.$$

In view of (3.9), $\tilde{r} < 1$, and (3.29),

$$\left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \right| \leq C \int_{\Gamma_4} e^{-i\varphi(\tilde{s} e^{i\tau}, \beta, X)} f(\tilde{s} e^{i\tau}, -\beta, X) d\tilde{s} e^{i\tau} |C_\delta(\beta, X)_{D_{n+1}}| \leq C_{\epsilon_0} |f|_{L^\infty(\beta, X)}.$$  

Applying (3.26), (3.27), and (3.29),

$$\left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \right| \leq C \sum_{n=1,2} \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] e^{-i(n-1)\beta} \right|$$

$$\times \int_{\Gamma_4} e^{-i\varphi(\tilde{s} e^{i\tau}, \beta, X)} f(\tilde{s} e^{i\tau}, -\beta, X) (\tilde{s} e^{i\tau} - \tilde{e} e^{i(\alpha-\beta)})^n d\tilde{s} e^{i\tau} |L^\infty(\beta, X)_{D_{n+1}}|$$

$$\leq C_{\epsilon_0} |f|_{L^\infty(\beta, X)} \int_{-\pi}^{\pi} d\beta e^{-\frac{1}{2} \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] |e^{-i(n-1)\beta} \right|}$$

$$\leq C_{\epsilon_0} |f|_{L^\infty(\beta, X)} \int_{-\pi}^{\pi} d\beta e^{-\frac{1}{2} \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] |e^{-i(n-1)\beta} \right|}$$

$$\leq C_{\epsilon_0} |f|_{L^\infty(\beta, X)} \int_{-\pi}^{\pi} d\beta e^{-\frac{1}{2} \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] |e^{-i(n-1)\beta} \right|}$$

We have to pay attention to the stationary point $\sin k\beta = 0$ of $\varphi$. If $\deg \varphi = k > 1$ then by

$$\tilde{s} \mapsto \tilde{t} = \tilde{s} \sqrt{|\sin k\beta|},$$
and improper integrals, (3.31) turns into

\begin{equation}
|\frac{\theta(1-\bar{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \int_{\Gamma_4} \frac{e^{-i\phi(\bar{s}e^{i\theta},\beta,X)} f(\frac{\bar{s}}{\sigma} e^{i\theta}, -\beta, X)}{\bar{s}e^{i\theta} - \bar{r}e^{i(\alpha-\beta)}} d\bar{s} e^{i\theta}|_{C^0_5(D_{\kappa_1})} \\
\leq C \epsilon_0 |f|_{L^\infty(D_{\kappa_1})} \int_{-\pi}^{\pi} d\beta \frac{1}{\sqrt{\sin k\beta}} \int_0^\infty e^{-t^k \sin k\beta} dt \leq C \epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

Combining (3.28), (3.30), and (3.33), we derive

\begin{equation}
|I_4|_{C^0_5(D_{\kappa_1})} \leq C \epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

**Step 3 (Proof for I_5):** Using the deformation (3.25), holomorphic property of f, and a residue theorem, one has

\begin{equation}
I_5 = -\frac{\theta(\bar{r}-1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \int_{\Gamma_5} \frac{e^{-i\phi(\bar{s}e^{i\theta},\beta,X)} f(\frac{\bar{s}}{\sigma} e^{i\theta}, -\beta, X)}{\bar{s}e^{i\theta} - \bar{r}e^{i(\alpha-\beta)}} d\bar{s} e^{i\theta} \\
\int_{S_{\tilde{r}}} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] |\sigma_{\delta}\left(\int_{\theta_{\tilde{r}}} d\beta e^{-i\kappa|\beta|} \frac{\bar{s} e^{i\theta}}{\bar{s} e^{i\theta} - \bar{r} e^{i(\alpha-\beta)}} f(\frac{\bar{s}}{\sigma} e^{i\theta}, -\beta, X)\right)|
\end{equation}

where \(\delta(\alpha)\) is defined by

\begin{equation}
\delta(\alpha) \equiv \{\beta : |\alpha - \beta| < \frac{\epsilon_1}{2}, \ (\alpha - \beta) \beta < 0\},
\end{equation}

\(S_{\tilde{r}}\) is defined by (3.29), \(\Gamma_5 = \Gamma_5(\beta, X, \lambda)\), and

\begin{equation}
\Gamma_5 = \{\bar{s} e^{i\theta_{\tilde{r}}} : 0 \leq \tilde{s} \leq \bar{s}\delta, \ \bar{s} e^{i\theta_{\tilde{r}}} is defined by (3.25)\}.
\end{equation}

From (3.29), (3.27), and (3.37), applying the same method as that for I_4 in Step 2,

\begin{equation}
|I_5|_{L^\infty(D_{\kappa_1})} \leq C \epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

To derive \(|I_5|_{H^0_{\delta}(D_{\kappa_1})}\), it reduces to studying

\begin{equation}
\int \frac{1}{2\pi i} \int_{\tilde{s} < \tilde{s}_\delta} \frac{\tilde{\gamma}_1(\tilde{\bar{s}}, \beta) e^{-i\phi(\tilde{\bar{s}}, \beta,X)} f(\frac{\tilde{\bar{s}}}{\bar{s}}, -\beta, X)}{\tilde{\bar{s}} e^{i\theta} - \tilde{\bar{s}} e^{i(\alpha-\beta)}} d\tilde{\bar{s}} \wedge d\tilde{s}
\end{equation}

\begin{equation}
-\frac{1}{2\pi i} \int_{\tilde{s} < \tilde{s}_\delta} \frac{\tilde{\gamma}_1(\tilde{\bar{s}}, \beta) e^{-i\phi(\tilde{\bar{s}}, \beta,X)} f(\tilde{\bar{s}}, -\beta, X)}{\tilde{\bar{s}} e^{i\theta} - \tilde{\bar{s}} e^{i(\alpha-\beta)}} d\tilde{\bar{s}} \wedge d\tilde{s}
\end{equation}

\begin{equation}
\leq \frac{1}{2\pi i} \int_{\tilde{s} < \tilde{s}_\delta} \theta(1/2 - |\tilde{\bar{s}} - \tilde{\bar{s}}_1|) \tilde{\gamma}_1(\tilde{\bar{s}}, \beta) e^{-i\phi(\tilde{\bar{s}}, \beta,X)} f(\tilde{\bar{s}}, -\beta, X) \left[ \frac{1}{\tilde{\bar{s}} - \tilde{\bar{s}}_1} - \frac{1}{\tilde{\bar{s}} - \tilde{\bar{s}}_2} \right] d\tilde{\bar{s}} \wedge d\tilde{s}
\end{equation}
From Fubini's theorem, the meromorphic property of the Cauchy integral, and applying (3.38),
\[
\gamma_1(\tilde{s}, \beta)e^{-i\phi(\tilde{s}, \beta, X)} \frac{1}{\tilde{s} - \tilde{\lambda}_1} - \frac{1}{\tilde{s} - \tilde{\lambda}_2} \]
for \(\tilde{\lambda}_j = \kappa_j + \tilde{r}je^{i\alpha_j}, \tilde{r}_1 \geq 1, |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq 1/4.

Applying \(f \in W, |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq 1/4,\) and Lemma 3.3 for the first term on the right hand side of (3.39); \(\tilde{r}_1 \geq 1, |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq 1/4,\) and
\[
|\theta(\tilde{s} - \tilde{\lambda}_1| - 1/2)||\frac{1}{\tilde{s} - \tilde{\lambda}_1} - \frac{1}{\tilde{s} - \tilde{\lambda}_2}| \leq C \tilde{s}^2 |\tilde{\lambda}_1 - \tilde{\lambda}_2|
\]
for the second term, and combining (3.38),
\[
|I_5|_{C^p(D_{\kappa_1})} \leq C |I_5|_{L^\infty(D_{\kappa_1})} \leq C \epsilon_0 |E_{\kappa_1}f|_{L^\infty(D_{\kappa_1})}.
\]

The lemma is proved from (3.10), (3.11), (3.24), (3.33), (3.38), and (3.40). \(\square\)

**Proposition 3.1.** Suppose \(S = \{z_n, \kappa_j, D, s_c\}\) is \(d\)-admissible and \(\partial_x f\) are \(\lambda\)-holomorphic on \(D_{\kappa_1}\) for \(0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5\). For the quadratic or cubic homogeneous cases and \(n \geq 2\),
\[
|E_{\kappa_j}(CTE_{\kappa_1})^n f|_W \leq C \epsilon_0 |E_{\kappa_1}(CTE_{\kappa_1})^{n-1} f|_W + C \epsilon_0^n |E_{\kappa_1}(CTE_{\kappa_1})^{n-2} f|_W.
\]

Consequently, for \(n \geq 0\),
\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l (CTE_{\kappa_1})^n f|_W \leq (C \epsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l f|_W.
\]

**Proof.** Without loss of generality, we assume (3.9) holds. Define \(\tilde{s}\) and \(\tilde{r}\) and rescaled coordinates by Definition 2.2 and decompose the principal part of the CIO by (3.3)-(3.8). From the proof of Lemma 3.3, it is sufficient to derive estimates of \(|I_4|_{C^p(D_{\kappa_1})}\) and \(|I_5|_{L^\infty(D_{\kappa_1})}\). To this aim, denote
\[
f^{[n]} = (CTE_{\kappa_1})^n f = CTE_{\kappa_1} f^{[n-1]}, \quad f^{[0]} = f,
\]
\[
\begin{align*}
I_4^{[n]} &= -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{s} < \tilde{\delta}} \tilde{\gamma}_1(\tilde{s}, \beta) e^{-i\phi(\tilde{s}, \beta, X)} f^{[n-1]}(\tilde{s}, -\beta, X) \tilde{d}\zeta \wedge d\tilde{\zeta}, \\
I_5^{[n]} &= -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{\tilde{s} < \tilde{\delta}} \tilde{\gamma}_1(\tilde{s}, \beta) e^{-i\phi(\tilde{s}, \beta, X)} f^{[n-1]}(\tilde{s}, -\beta, X) \tilde{d}\zeta \wedge d\tilde{\zeta},
\end{align*}
\]

**Step 1 (Proof for \(I_4^{[n]}\)) :** From Fubini’s theorem, the meromorphic property of the Cauchy kernel, and a residue theorem,
\[
\begin{align*}
I_4^{[n]} &= \frac{\theta(1 - \tilde{r})}{(2\pi i)^2} \int_{\tilde{s} < \tilde{\delta}} \int_{-\pi}^{\pi} d\beta (\partial_x \ln(1 - \gamma |\beta|)) \int_{\tilde{s} < \tilde{\delta}} \tilde{d}s \frac{e^{-i\phi(\tilde{s}, \beta, X)}}{\tilde{s} - \tilde{r} e^{i\alpha - \beta}} \\
&\times \int_{D_{\kappa_1}} \tilde{\gamma}_1(\tilde{s}', \beta') e^{-i\phi(\tilde{s}', \beta', X)} f^{[n-2]}(\tilde{s}', -\beta', X) \tilde{d}\zeta' \wedge d\tilde{\zeta}' \]
\[
= \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta (\partial_x \ln(1 - \gamma |\beta|)) \\
\times (\int_{s_<} + \int_{s_>} + \int_{s_4} + \int_{s_5}) \frac{e^{-i\phi(\tilde{s}, \beta, X)} f^{[n-1]}(\tilde{s}, -\beta, X) \tilde{d}s e^{i\tau}}{\tilde{s} e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}} \]
\[
= \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta (\partial_x \ln(1 - \gamma |\beta|)) \\
\times (\int_{s_<} + \int_{s_4} + \int_{s_5}) \frac{e^{-i\phi(\tilde{s}, \beta, X)} f^{[n-1]}(\tilde{s}, -\beta, X) \tilde{d}s e^{i\tau}}{\tilde{s} e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}}
\]
Thus introduce the deformation
\[ (3.47) \]
\[ \int_{-\pi}^{\pi} d\beta e^{-i\beta \ln(1-\gamma|\beta|)} \int_{2<\beta'<\bar{\delta}} d\zeta' \wedge d\zeta' \]
\[ \times \text{sgn}(\beta + \beta') \tilde{\gamma}_1(\tilde{s}', \beta') e^{-i\rho(\tilde{s}'(\beta+\beta'), X) e^{-i\rho(\tilde{s}', \beta', X) f^{[n-2]}(\tilde{s}', -\beta', X)}} \]
where \( \tilde{\zeta}' = \kappa_1 + \tilde{s}' e^{i\beta} \), \( S_< = S_=(\beta, X, \lambda) \), \( \Gamma_4 = \Gamma_4(\beta, X, \lambda) \) are defined by \((3.29)\).

By the same approach for \( I_4 \) in Lemma \ref{3.3} and induction,
\[ (3.45) \]
\[ | - \frac{\theta(1-\bar{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \text{sgn}(1-\gamma|\beta|) \int_{2<\beta'<\bar{\delta}} d\zeta' \wedge d\zeta' \]
\[ \times \text{sgn}(\beta + \beta') \tilde{\gamma}_1(\tilde{s}', \beta') e^{-i\rho(\tilde{s}'(\beta+\beta'), X) e^{-i\rho(\tilde{s}', \beta', X) f^{[n-2]}(\tilde{s}', -\beta', X)}} \]
\[ \leq C \epsilon_0 |f^{[n-1]}| C_0(D_{\kappa_1}). \]

For the second term, it suffices to consider
\[ (3.46) \]
\[ |\alpha - \beta| + |\beta + \beta'| \ll 1. \]

Hence if we deform \( \tilde{s}' \) then the real part of the sum of phase functions are
\[ \Re(-i[\tilde{s}'k e^{ik(\beta+\beta'+\tau)} \sin k\beta + \tilde{s}'k e^{ik\tau} \sin k\beta']) \]
\[ = \tilde{s}'k(\sin k(\beta + \beta') \sin k\beta + \sin k\beta + \sin k\beta + \sin k\beta + l.o.t.), \]
and, thanks to \((3.36)\),
\[ \sin k\beta < 0, \quad k(\beta + \beta') > 0, \quad \sin k\beta + \sin k\beta' > 0, \]
\[ \sin k\beta > 0, \quad k(\beta + \beta') < 0, \quad \sin k\beta + \sin k\beta' < 0. \]

Thus introduce the deformation
\[ \tilde{s}' \mapsto \tilde{s}' e^{i\tau}, \quad 0 < \tilde{s}' < \bar{\delta} = X_k^{1/k} \delta, \]
\[ \tau \in \left\{ \begin{array}{ll}
[-\epsilon, 0], & \beta + \beta' > 0, \quad |\alpha - 2\beta - \beta'| \leq \frac{\epsilon}{2}, \\
[-\frac{\epsilon}{2}, 0], & \beta + \beta' > 0, \quad |\alpha - 2\beta - \beta'| \geq \frac{\epsilon}{2}, \\
[0, +\epsilon], & \beta + \beta' < 0, \quad |\alpha - 2\beta - \beta'| \leq \frac{\epsilon}{2}, \\
[0, +\frac{\epsilon}{2}], & \beta + \beta' < 0, \quad |\alpha - 2\beta - \beta'| \geq \frac{\epsilon}{2},
\end{array} \right. \]

we have
\[ |\tilde{s}' e^{i(\tau_1 + \beta + \beta')} - \tilde{r} e^{i(\alpha - \beta')} | \geq \frac{1}{C} \max \{|\tilde{r}, \tilde{s}'\}, \]
\[ \Re(-i[\tilde{s}'k e^{i(\beta+\beta'+\tau)}], \beta, X) + \varphi(\tilde{s}'k e^{i(\beta+\beta'+\tau)}, \beta, X) \leq -C |\sin k\tau| \sin k\beta + \sin k\beta' |\tilde{s}'k|. \]

Consequently, by the same approach for \( I_4 \) in Lemma \ref{3.3} and induction,
\[ (3.47) \]
\[ \left| - \frac{\theta(1-\bar{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta e^{-i\beta \ln(1-\gamma|\beta|) \text{sgn}(\tau_1(\beta))} \int_{2<\beta' < \bar{\delta}} d\zeta' \wedge d\zeta' \]
\[ \times \text{sgn}(\beta + \beta') \tilde{\gamma}_1(\tilde{s}', \beta') e^{-i\rho(\tilde{s}'(\beta+\beta'), X) e^{-i\rho(\tilde{s}', \beta', X) f^{[n-2]}(\tilde{s}', -\beta', X)}} \]
\[ \leq C \epsilon_0 |f^{[n-2]}|_{L^\infty(D_{\kappa_1})}. \]
Therefore,

\[(3.48) \quad |I_4^n|_{C_\alpha^n(D_{\delta_1})} \leq C\epsilon_0 |f[n-1]|_{L_\infty(D_{\delta_1})} + C\epsilon_0^2 |f[n-2]|_{L_\infty(D_{\delta_1})}.
\]

**Step 1 (Proof for \(I_5^n\))**: From Fubini’s theorem, the meromorphic property of the Cauchy kernel, (3.36), and a residue theorem,

\[(3.49) \quad I_5^n = \frac{\theta(\bar{\gamma} - 1)}{(2\pi i)^2} \int_{-\pi}^{\pi} d\beta \frac{e^{i\beta \ln(1 - \gamma |\beta|)}}{\bar{s} e^{-i\beta} - \bar{\gamma} e^{i(\alpha - \beta)}} \times \left( \int_{\Gamma_5} + \int_{S^*} \right) \frac{\tilde{\gamma}_1 (\tilde{s}, \tilde{\beta}) e^{-i\phi(\frac{\tilde{s}}{\tilde{\beta}}, \tilde{\beta}, X)} f[n-2](\frac{\tilde{s}}{\tilde{\beta}}, -\tilde{\beta}, X) d\tilde{s} \wedge d\tilde{\beta} \right)
\]

Following similar argument as above,

\[|I_5^n|_{C_\alpha^n(D_{\delta_1})} \leq C\epsilon_0 |f[n-1]|_{L_\infty(D_{\delta_1})} + C\epsilon_0^2 |f[n-2]|_{L_\infty(D_{\delta_1})}.
\]

Along with estimates for \(I_1, I_2, I_3\), yields

\[|E_{k_1}(C_{TE_{k_1}})^n f|_W \leq \sum_{\nu=0}^{n} (C\epsilon_0)^{n} |E_{k_1}(C_{TE_{k_1}})^{n-\nu} f|_W \leq (C + 1)^{n-1} C\epsilon_0^n |E_{k_1} f|_W.
\]

\[\square\]

Estimating for linear homogeneous cases is more involved. We have to take advantage of symmetries of integrands as well as \(\lambda\)-holomorphic properties of \(E_{k_1} f\) to deal with the stationary points \(\sin \beta = 0\).

**Lemma 3.4.** Suppose \(S = \{z_n, \kappa_j, D, s_c\}\) is \(d\)-admissible and \(\partial_j f\) are \(\lambda\)-holomorphic on \(D_{\delta_1}\) for \(0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5\). For linear homogeneous cases,

\[\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{k_1} \partial_j^l C_{TE_{k_1}} f|_W \leq C\epsilon_0 \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{k_1} \partial_j^l f|_W.
\]

**Proof.** Proofs for \(|E_{k_1} \partial_j^l C_{TE_{k_1}} f|_W\) are identical. We only prove \(|E_{k_1} C_{TE_{k_1}} f|_W\) for simplicity. Without loss of generality, we assume (3.39) holds. Define \(\tilde{\sigma}\) and rescaled coordinates by Definition 2.22 and decompose the principal part of the CIO by (3.39)-(3.8). From the proof of Lemma 3.3, it is sufficient to enhance the proof of \(I_4\) and \(I_5\).
Step 1 (Proof for $I_4$): To deal with the singularity caused by (3.32), a finer decomposition can squeeze out extra $|\sin \beta|$-decay on $\Gamma_4$ or $\Gamma_5$ by the symmetry. Namely,

\begin{align}
I_4 &= -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_2^X d\tau \left( J_1 + J_2 + J_3 + J_4 + J_5 \right), \\
J_1 &= \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) \left[ e^{-i\hat{s}\sin \beta} - 1 \right] \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s}, \\
J_2 &= -\Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) \left[ e^{+i\hat{s}\sin \beta} - 1 \right] \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s}, \\
J_3 &= \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) e^{i\hat{s}\sin \beta} \left[ f(\tilde{s}, -\beta, X) \frac{1}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} - \frac{1}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} \right] d\tilde{s}, \\
J_4 &= \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) e^{i\hat{s}\sin \beta} \left[ f(\tilde{s}, -\beta, X) - f(\tilde{s}, +\beta, X) \frac{1}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} \right] d\tilde{s}, \\
J_5 &= \Theta \left( |\tilde{s} - \tilde{r}| - \frac{1}{|\sin \beta|} \right) \left( e^{-i\hat{s}\sin \beta} \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} - e^{+i\hat{s}\sin \beta} \frac{f(\tilde{s}, +\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} \right) d\tilde{s}.
\end{align}

According to the signatures of the real parts of the exponential factors in $J_j$, we deform

\begin{align}
&\left| I_4 \right|_{C^\infty(D_{\alpha_1})} = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \\
&\times \left( \int_{S_2} + \int_{\Gamma_4} + \int_{S_3} \right) \left( J_1 + J_2 + J_3 + J_4 + J_5 \right)_{C^\infty(D_{\alpha_1})},
\end{align}

where $S_2$, $\Gamma_4$ are defined by (3.27) and

\begin{align}
&J_1 = \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) \left[ e^{-i\hat{s}\sin \beta} - 1 \right] \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s} e^{i\tau}, \\
&J_2 = \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) \left[ 1 - e^{+i\hat{s}\sin \beta} \sin \beta \right] \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s} e^{-i\tau}, \\
&J_3 = \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) e^{i\hat{s}\sin \beta} \left[ f(\tilde{s}, -\beta, X) \frac{1}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha - \beta)}} - \frac{1}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha + \beta)}} \right] d\tilde{s} e^{-i\tau}, \\
&J_4 = \Theta \left( \frac{1}{|\sin \beta|} - \frac{1}{|\tilde{s} - \tilde{r}|} \right) e^{i\hat{s}\sin \beta} \left[ f(\tilde{s}, -\beta, X) - f(\tilde{s}, +\beta, X) \frac{1}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha + \beta)}} \right] d\tilde{s} e^{-i\tau}, \\
&J_5 = \Theta \left( |\tilde{s} - \tilde{r}| - \frac{1}{|\sin \beta|} \right) \left( e^{-i\hat{s}\sin \beta} \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha - \beta)}} - e^{+i\hat{s}\sin \beta} \frac{f(\tilde{s}, +\beta, X)}{\tilde{s} e^{-i\tau} - \tilde{r} e^{i(\alpha + \beta)}} \right) d\tilde{s} e^{i\tau},
\end{align}

with $\tau$ defined by (3.25) for $\beta \in [0, \pi]$. Estimates for $J_1, \ldots, J_5$ are itemized as follows.

- $J_1, J_2$: From the mean value theorem, (3.26), and (3.27),

\begin{align}
&\left| \frac{e^{i\hat{s}\sin \beta} - 1}{\tilde{s} e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}} \right|.
\end{align}
Along with (3.9), (3.25), (3.27), (3.52), (3.53), and the change of variables (3.58), yields

\[
\tilde{s} \mapsto t = \tilde{s} |\sin \beta|, 
\]

yields

\[
\sum_{j=1}^{2} \left| - \frac{\theta(1 - \bar{r})}{2\pi i} \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \left( \int_{S_{<}} + \int_{S_{>}} + \int_{\Gamma_4} \right) \mathcal{J}_{3} \mid C_{\tilde{s}}^\nu(D_{s_{1}}) \right| \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})} + C \mid f \mid_{L^{\infty}(D_{s_{1}})} \left| \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{2|\sin \beta|} X_{1} \theta(1 - |t - \bar{r}| \sin \beta) | dt \mid_{L^{\infty}(D_{s_{1}})} \right| \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})}.
\]

\[\mathcal{J}_{3} \text{: Thanks to } |\bar{r}| < 1 \text{ and } \tilde{s} > 2,\]

Together with (3.9), (3.25), (3.27), and (3.54), yields

\[
\left| - \frac{\theta(1 - \bar{r})}{2\pi i} \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \left( \int_{S_{<}} + \int_{S_{>}} + \int_{\Gamma_4} \right) \mathcal{J}_{3} \mid C_{\tilde{s}}^\nu(D_{s_{1}}) \right| \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})}.
\]

\[\mathcal{J}_{4} \text{: From holomorphic properties of } E_{s_{1}}f \text{ and (3.25)},\]

\[
\left| \frac{f\left(\frac{\tilde{s}}{s}e^{i\tau_{1}}, -\beta, X\right) - f\left(\frac{\tilde{s}}{s}e^{i\tau_{1}}, +\beta, X\right)}{s e^{i\tau_{1}} - \tilde{s} e^{i\alpha}} \right| \leq C \mid f \mid_{L^{\infty}(D_{s_{1}})} |\sin \beta|. \]

Along with (3.9), (3.25), (3.27), (3.55), and the change of variables (3.58), yields

\[
\left| - \frac{\theta(1 - \bar{r})}{2\pi i} \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \left( \int_{S_{<}} + \int_{S_{>}} + \int_{\Gamma_4} \right) \mathcal{J}_{4} \mid C_{\tilde{s}}^\nu(D_{s_{1}}) \right| \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})}.
\]

\[\mathcal{J}_{5} \text{: Using (3.9), (3.25), (3.27), (3.56), } \epsilon_{1} > 0, \text{ and rescaling (3.58), one obtains}\]

\[
\left| - \frac{\theta(1 - \bar{r})}{2\pi i} \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \left( \int_{S_{<}} + \int_{S_{>}} + \int_{\Gamma_4} \right) \mathcal{J}_{5} \mid C_{\tilde{s}}^\nu(D_{s_{1}}) \right| \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})} + C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})} \left| \int_{2|\sin \beta|} X_{1} \theta(|t - \bar{r}| \sin \beta) | - 1 \right| \frac{e^{-t \sin \frac{\pi}{2}}}{|t - \bar{r}| \sin \beta} | dt \mid_{L^{\infty}(D_{s_{1}})} \leq C\epsilon_{0} |f|_{L^{\infty}(D_{s_{1}})}.
\]
As a result,

\begin{equation}
|I_4|_{C^\mu_5(D_{\kappa_1})} \leq C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

**Step 2 (Proof for $I_5$):** Applying Lemma 3.3 if $|\beta| > \epsilon_1/8$, and following a similar argument as that in the previous step,

\begin{equation}
|I_5|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f|_{L^\infty(D_{\kappa_1})} + \left| - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta\left(\frac{\epsilon_1}{2} - |\beta|\right) \right|
\times \left( \int_{\Gamma_5} + \int_{S_>} \right) (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5)_{L^\infty(D_{\kappa_1})}
+ \theta(\tilde{r} - 1) \int_{\beta \in \mathcal{D}(\alpha)} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta\left(\frac{\epsilon_1}{2} - |\beta|\right) \operatorname{sgn}(\beta)
\times e^{-i\tilde{r}e^{i(\alpha-\beta)} \sin \beta} f \left( \frac{\tilde{r}}{\sigma} e^{i(\alpha-\beta)}, \tau, X \right)_{L^\infty(D_{\kappa_1})}
\end{equation}

where $\mathcal{D}(\alpha)$ is defined by (3.36), $S_>$ is defined by (3.29), $\Gamma_5$ defined by (3.37), and $\tilde{J}_j$, $1 \leq j \leq 5$, defined by (3.52)–(3.56).

From (3.9) and (3.27), for $1 \leq j \leq 5$,

\begin{equation}
\left| \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta\left(\frac{\epsilon_1}{2} - |\beta|\right) \int_{S_>} \tilde{J}_j_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

From $|\beta| < \epsilon_1/8$ and (3.26),

\begin{equation}
\left| \frac{1}{\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha-\beta)}} - \frac{1}{\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha+\beta)}} \right|
= \left| \frac{\tilde{r}e^{i\alpha}2\sin \beta}{(\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha-\beta)})(\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha+\beta)})} \right| \leq C|\sin \beta|.
\end{equation}

Along with (3.57), (3.62), the change of variables (3.58), and argument in previous step for $\tilde{J}_j$ on $\Gamma_4$, $j = 1, 2, 4, 5$, yields

\begin{equation}
\left| \frac{1}{\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha-\beta)}} - \frac{1}{\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha+\beta)}} \right|
= \left| \frac{\tilde{r}e^{i\alpha}2\sin \beta}{(\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha-\beta)})(\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha+\beta)})} \right| \leq C|\sin \beta|.
\end{equation}

Thus $L^\infty$ estimates for $I_5$ is complete by (3.66), (3.67), and (3.69). One can adopt the same argument as that from (3.39) to (3.40) to derive

\begin{equation}
|I_5|_{C^\mu_5(D_{\kappa_1})} \leq C|I_5|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\end{equation}

\[ \Box \]

In Lemma 3.4 we have used the $\lambda$-holomorphic property of $E_{\kappa_1}f(\lambda)$ to derive estimates for $\tilde{J}_4$ on $\Gamma_4$ or $\Gamma_5$. To generalize them for $(CTE_{\kappa_1})^n f$, one needs

**Lemma 3.5.** For $\tilde{s} > 1$ and $\tau_1$ defined by (3.25) for $\beta \in [0, \pi]$,

\begin{equation}
|\mathcal{C}_{\kappa_1 + \tilde{s}e^{-i(\tau_1+\beta)}} TE_{\kappa_1}g - \mathcal{C}_{\kappa_1 + \tilde{s}e^{-i(\tau_1-\beta)}} TE_{\kappa_1}g|
\end{equation}
Moreover, for $\Gamma_4, \Gamma_5$ are defined by (3.29), (3.37), and $\mathcal{J}_4$ defined by (3.55) with $f = \text{CTE}_{\kappa_1}g,$

\begin{equation}
\begin{aligned}
&\frac{\theta(1 - \bar{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_4} \mathcal{J}_4|c'_\sigma(D_{\kappa_1})| \\
&+ \frac{\theta(\bar{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_5} \mathcal{J}_4|c'_\sigma(D_{\kappa_1})| \leq (C\epsilon_0)^2|g|_{L^\infty(D_{\kappa_1})}.
\end{aligned}
\end{equation}

Proof. Step 1 (Proof for (3.71)) : Decompose $\text{CTE}_{\kappa_1}g = \sum_{j=1}^5 I_j$ via (3.3)-(3.8). To prove (3.71), it amounts to the investigation of $|I_5(\tilde{s}' e^{-i\tau_1}, -\beta, X) - I_5(\tilde{s}' e^{-i\tau_1}, +\beta, X)|$. Write

\begin{equation}
\begin{aligned}
&\theta(1/4 - |\bar{s} \sin \beta|)[I_5(\tilde{s}' \sigma^{-i\tau_1}, -\beta, X) - I_5(\tilde{s}' \sigma^{-i\tau_1}, +\beta, X)] \\
&= -\frac{\theta(\bar{s} - 1)\theta(1/4 - |\bar{s} \sin \beta|)}{2\pi i} \int_{-\pi}^{\pi} d\beta' [\partial_{\beta'} \ln(1 - \gamma|\beta'|)] \\
&\times \frac{\tilde{\sigma}}{\tilde{\beta}} \theta(1/2 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}|)\left|e^{-ip(\tilde{s}' \sigma^{-1}, \beta', X)} \frac{g(\tilde{s}' \sigma^{-1}, -\beta', X)}{\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}} - \frac{g(\tilde{s}' \sigma^{-1}, -\beta', X)}{\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}} \right| d\tilde{s}' \\
&- \frac{\theta(\bar{s} - 1)\theta(1/4 - |\bar{s} \sin \beta|)}{2\pi i} \int_{-\pi}^{\pi} d\beta' [\partial_{\beta'} \ln(1 - \gamma|\beta'|)] \\
&\times \frac{\tilde{\sigma}}{\tilde{\beta}} \theta(|\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}| - 1/2)\left|e^{-ip(\tilde{s}' \sigma^{-1}, \beta', X)} \frac{-2i\tilde{s} e^{i(-\tau_1 - \beta')} \sin \beta g(\tilde{s}' \sigma^{-1}, -\beta', X)}{(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta'}))(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta'))} \right| d\tilde{s}'
\end{aligned}
\end{equation}

\begin{equation}
\equiv A_1 + A_2.
\end{equation}

In view of $|\bar{s} \sin \beta| < 1/4,$

\begin{equation}
\frac{\theta(\bar{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}) - 1/2}{(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta'))}} \leq \frac{C}{(\tilde{s}')^2},
\end{equation}

one obtains

\begin{equation}
|A_2|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|\bar{s} \sin \beta||g|_{L^\infty(D_{\kappa_1})}.
\end{equation}

From $\tilde{s} > 1, |\bar{s} \sin \beta| < 1/4,$ and $|\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}| < 1/2,$ one can apply Lemma 3.2 to derive

\begin{equation}
|A_1|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|g|_{L^\infty(D_{\kappa_1})}|\bar{s} \sin \beta|^\mu.
\end{equation}

Plugging the above estimates into (3.73), we obtain

\begin{equation}
\theta(1/4 - |\bar{s} \sin \beta|)|I_5(\tilde{s}' \sigma^{-i\tau_1}, -\beta, X) - I_5(\tilde{s}' \sigma^{-i\tau_1}, +\beta, X)|
\end{equation}

\begin{equation}
\leq C\epsilon_0|g|_{L^\infty(D_{\kappa_1})}|\bar{s} \sin \beta|^\mu.
\end{equation}

On the other hand, write

\begin{equation}
\theta(|\bar{s} \sin \beta| - 1/4)(I_5(\tilde{s}' \sigma^{-i\tau_1}, -\beta, X) - I_5(\tilde{s}' \sigma^{-i\tau_1}, +\beta, X))
\end{equation}

\begin{equation}
= 2i\tilde{s} \sin \beta \frac{\theta(\bar{s} - 1)\theta(|\bar{s} \sin \beta| - 1/4)}{2\pi i} \int_{-\pi}^{\pi} d\beta' [\partial_{\beta'} \ln(1 - \gamma|\beta'|)] \int_{0}^{\tilde{\sigma} \delta} e^{-ip(\tilde{s}' \sigma^{-1}, \beta', X)} g(\tilde{s}' \sigma^{-1}, -\beta', X)
\end{equation}

\begin{equation}
\leq C\epsilon_0|g|_{L^\infty(D_{\kappa_1})}|\bar{s} \sin \beta|^\mu.
\end{equation}
\[
\times \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')}|) \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')})} d\tilde{s}' \\
+ 2i \tilde{s} \sin \beta \frac{\theta (\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}) d\tilde{s}'}{2\pi i} \\
\times \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')}|) \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')})} d\tilde{s}' \\
+ 2i \tilde{s} \sin \beta \frac{\theta (\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}) d\tilde{s}'}{2\pi i} \\
\times (1 - \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')}|) - \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}|)) \\
\times \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')})(\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')})} d\tilde{s}' \\
\equiv A_1' + A_2' + A_3'.
\]

Thanks to \( \tilde{s} > 1, |\tilde{s} \sin \beta| > 1/4, |\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')}| < 1/8 \), and Lemma 3.2, one has

\[(3.76) \quad |A_1'| \leq C \epsilon_0 |g|_{L^\infty(D_{\kappa_1})} |\tilde{s} \sin \beta|.
\]

By analogy,

\[(3.77) \quad |A_2'| \leq C \epsilon_0 |g|_{L^\infty(D_{\kappa_1})} |\tilde{s} \sin \beta|.
\]

Since

\[|1 - \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 - \beta')}|) - \theta (1/8 - |\tilde{s}' - \tilde{s} e^{i(-\tau_1 + \beta - \beta')}|)| \leq \frac{C}{(\tilde{s}')^2}.
\]

We have

\[(3.78) \quad |A_3'| \leq C \epsilon_0 |g|_{L^\infty(D_{\kappa_1})} |\tilde{s} \sin \beta|.
\]

Therefore, (3.71) is justified by (3.74)-(3.78).

**Step 2 (Proof for (3.72)):** From (3.71), for \( \tilde{s} > 1, \tau_\dagger \) defined by (3.25) for \( \beta \in [0, \pi] \),

\[(3.79) \quad \frac{C \kappa_1 e^{-i(\tau_1 + \beta)} T E_{\kappa_1} g - C \kappa_1 e^{-i(\tau_1 - \beta)} T E_{\kappa_1} g}{\tilde{s} e^{-i\tau_1} - \tilde{r} e^{i(\alpha + \beta)}} \leq C \epsilon_0 |g|_{L^\infty(D_{\kappa_1})} |\tilde{s} \sin \beta|^\mu.
\]

Along with an improper integral and the change of variables (3.58), yields

\[(3.80) \quad \left| - \frac{\theta (1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln (1 - \gamma |\beta|)] \int_{\Gamma_4} J_4 |C^\mu_{\tilde{g}}(D_{\kappa_1})| \\
\leq (C \epsilon_0)^2 |g|_{L^\infty(D_{\kappa_1})} \int_0^\pi d\beta \int_{\Gamma_4} e^{-\tilde{s} |\tilde{s}' \sin \tau_1 \sin \beta| |\sin \beta|^\mu d\tilde{s}|_{L^\infty(D_{\kappa_1})} \\
\leq (C \epsilon_0)^2 |g|_{L^\infty(D_{\kappa_1})}.
\]

Adapting the above argument,

\[(3.81) \quad \left| - \frac{\theta (\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln (1 - \gamma |\beta|)] \int_{\Gamma_5} J_4 |C^\mu_{\tilde{g}}(D_{\kappa_1})| \leq (C \epsilon_0)^2 |g|_{L^\infty(D_{\kappa_1})}.
\]
Homogeneous cases and can be applied to the remaining non homogeneous cases. By sorting (3.84) into two categories and searching for an approach which is consistent with that for fixed critical point \( \tilde{s} \), we have Proposition 3.2.

Delegated non homogeneous cases.

3.1.2. Transition 3.1 to prove this proposition. We sketch the details for simplicity.

Define \( \tilde{s} \) (3.83) for degenerated homogeneous cases.

\[
\begin{array}{|c|c|c|}
\hline
\text{Case} & \partial_{x} \psi \left( \frac{x}{\tilde{s}}, \beta, X \right) & \tilde{s} \left( \beta, X \right) \\
\hline
(21) & 2 \sin 2\beta (\bar{s} - \tilde{s}) & -\frac{X_{1}}{2X_{1}^{1/2}} \sin \beta \\
\hline
(31) & 3 \sin 3\beta (\bar{s} + \tilde{s}) (\bar{s} - \tilde{s}) & \sqrt{\frac{X_{1}}{3X_{1}^{3/2}}} \sin \beta \\
\hline
(32) & 3 \sin 3\beta \bar{s} (\bar{s} - \tilde{s}) & -\frac{2X_{2}}{3X_{3}} \sin 2\beta \\
\hline
(33) & \frac{3X_{3}}{X_{3}^{3/2}} \sin 3\beta \bar{s} (\bar{s} - \tilde{s}) & -\frac{2X_{2}}{3X_{3}} \sin 2\beta \\
\hline
(12) & \frac{2X_{2}}{X_{3}} \sin 2\beta (\bar{s} - \tilde{s}) & -\frac{X_{2}}{2X_{2}} \sin \beta \\
\hline
(13) & \frac{2X_{2}}{X_{3}} \sin 3\beta (\bar{s} + \tilde{s}) (\bar{s} - \tilde{s}) & \sqrt{\frac{X_{2}}{3X_{3}}} \sin \beta \\
\hline
\end{array}
\]

Table 3.1. Stationary points \( \tilde{s} \) for degenerated homogeneous cases

**Proposition 3.2.** Suppose \( S = \{ z_{0}, \kappa_{j}, D, s_{c} \} \) is d-admissible and \( \partial_{x}f \) are \( \lambda \)-holomorphic on \( D_{\kappa} \) for \( 0 \leq l_{1} + 2l_{2} + 3l_{3} \leq d + 5 \). For the linear homogeneous cases and \( n \geq 2 \),

\[
(3.82) \quad |E_{\kappa_{j}} (CTE_{\kappa_{j}})^{n} f|_{W} \leq C \epsilon_{0} |E_{\kappa_{j}} (CTE_{\kappa_{j}})^{n-1} f|_{W} + C \epsilon_{0} |E_{\kappa_{j}} (CTE_{\kappa_{j}})^{n-2} f|_{W}.
\]

Consequently, for \( n \geq 0 \),

\[
(3.83) \quad \sum_{0 \leq l_{1} + 2l_{2} + 3l_{3} \leq d + 5} |E_{\kappa_{j}} \partial_{x}^{l} (CTE_{\kappa_{j}}) f|_{W} \leq (C \epsilon_{0})^{n} \sum_{0 \leq l_{1} + 2l_{2} + 3l_{3} \leq d + 5} |E_{\kappa_{j}} \partial_{x}^{l} f|_{W}.
\]

**Proof.** With the help of Lemma 3.4 and 3.5, we can adapt the approach in the proof of Proposition 3.1 to derive estimates near \( \tilde{s} \). We sketch the details for simplicity.

3.1.2. Degenerated non homogeneous cases. We first have the classification

\[
(3.84) \quad \begin{align*}
(12) & \sqrt{|X_{2}|} \leq |X_{1}|, \quad X_{3} = 0; \\
(13) & \sqrt{|X_{3}|} \leq |X_{1}|, \quad X_{2} = 0; \\
(23) & \sqrt{|X_{3}|} \leq \sqrt{|X_{2}|}, \quad X_{1} = 0;
\end{align*}
\]

Define \( \tilde{s} \) by Definition 2.2 and rescaled coordinates by (3.2). Thanks to Table 3.1, besides the fixed critical point \( \tilde{s} = 0 \), we have to pay attention to movable stationary points \( \tilde{s} \). We shall sort (3.84) into two categories and search for an approach which is consistent with that for homogeneous cases and can be applied to the remaining non homogeneous cases.

- Case (21), (31), (32), (23): In this category, we shall introduce new coordinates near \( \tilde{s} \), find contours such that (3.26), (3.27) type conditions hold, and adapt argument in Proposition 3.1 to derive estimates near \( \tilde{s} \).
- Case (12), (13): Following the approach of the proof of Lemma 3.4 won’t achieve (3.26), (3.27) type estimates near \( \tilde{s} \). New dilated affine coordinates has to be introduced to overcome the difficulty.
Lemma 3.6. Suppose $S = \{ z_n, \kappa_j, D, s_c \}$ is $d$-admissible and $\partial^i_x f$ are $\lambda$-holomorphic on $D_{\kappa_j}$ for $0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5$. For the degenerated non homogeneous Case (21), (31), (32), (23), $\forall n \geq 0$,

$$\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x (C E_{\kappa_j})^n f| \leq (C \alpha)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x f|.$$  

Proof. For simplicity and WLOG, we only give a proof assuming (3.88) where if $\tilde{s} \equiv \tilde{s}_j(\beta, X)$ defined by Table 3.1. Define the essential critical points $\tilde{s}_{j*,} = \tilde{s}_{j*,}(\beta, X)$, $j = 0, 1$, defined by

$$(3.86) \quad \tilde{s}_{0*,} \equiv 0; \quad \tilde{s}_{1*,} \equiv \left\{ \begin{array}{ll} -\tilde{s}_* < 0, & \tilde{s}_*, \tilde{s}_* > 0, \end{array} \right.$$  

where $-\equiv$ means no definition. Given $\epsilon_1 < \frac{1}{2\alpha} \ll 1$, define $\tilde{U}_j^F(\beta, X) \supset \tilde{U}_j(\beta, X)$ by

$$(3.87) \quad \tilde{U}_0^\sigma \equiv \left\{ \begin{array}{ll} \{ 0 \leq \tilde{s} \leq \tilde{\sigma} \}, & \tilde{s}_* < 0, \\ \{ 0 \leq \tilde{s} \leq \frac{1}{2\cos \epsilon_1} \tilde{s}_{1*} \}, & \tilde{s}_* > 0, \end{array} \right.$$  

$$\tilde{U}_1^\sigma \equiv \left\{ \phi, \right\} \cup \left\{ \left(1 - \frac{1}{2\cos \epsilon_1}\right) \tilde{s}_{1*} \leq \tilde{s} \leq \tilde{s}_{1*} \right\} \cup \left\{ \tilde{s}_{1*} \leq \tilde{s} \leq \tilde{\sigma} \right\} \equiv \tilde{U}_{1<} \cup \tilde{U}_{1>}, \quad \tilde{s}_* > 0,$$

$$(3.88) \quad \tilde{U}_0^\sigma \equiv \{0 \leq \tilde{s} \leq \tilde{s}_{1*}\},$$  

$$\tilde{U}_1^\sigma \equiv \{0 \leq \tilde{s} \leq \tilde{s}_{1*}\} \cup \tilde{s}_{1*} \leq \tilde{s} \leq \tilde{\sigma} \tilde{\sigma} \equiv \tilde{U}_{1<} \cup \tilde{U}_{1>}.$$  

Write

$$(3.89) \quad \lambda = \kappa_1 + \frac{\tilde{r} e^{i\alpha}}{\tilde{\sigma}} = \kappa_1 + \frac{\tilde{s}_j e^{i\beta} + \tilde{r} e^{i\alpha_j}}{\tilde{\sigma}},$$  

$$\tilde{r}_j = \tilde{r}_j(\beta, X, \lambda), \quad \alpha_j = \alpha_j(\beta, X, \lambda), \quad j = 0, 1.$$  

Due to Table 3.2 and Figure 3, we define the deformation on $\tilde{U}_j^\sigma$:

$$(3.90) \quad \tilde{s} \mapsto \xi_j \equiv \tilde{s}_{j, *}, \quad \tilde{s}_j \in \tilde{U}_j^\sigma$$  

where if $\tilde{s}_* < 0$,

$$\left\{ \begin{array}{ll} \pm \epsilon_1 \leq \tau_0 \leq 0, & \text{for } \sin k\beta \geq 0, \quad |\alpha - \beta| \leq \frac{\epsilon_1}{2}, \quad \tilde{s} \in \tilde{U}_j^\sigma, \\ \pm \epsilon_1 \leq \tau_0 \leq 0, & \text{for } \sin k\beta > 0, \quad |\alpha - \beta| \geq \frac{\epsilon_1}{2}, \quad \tilde{s} \in \tilde{U}_j^\sigma, \end{array} \right.$$
| Case | $\tilde{s} \in \mathcal{U}_0$ | $\tilde{s} \in \mathcal{U}_1$ |
|------|----------------|----------------|
| (21) | $\Re(-i\psi(\frac{s_0}{2}, \beta, X))$ | $\Re(-i\psi(\frac{s_1}{2}, \beta, X))$ |
| | $\sin 2\tau_0 \sin 2\beta \tilde{s}(s - \frac{2 \sin \tau_0 \tilde{s}}{2 \sin \tau_0})$ | $\sin 2\tau_1 \sin 2\beta \tilde{s}_1$ |
| (31) | $\Re(-i\psi(\frac{s_0}{2}, \beta, X))$ | $\Re(-i\psi(\frac{s_1}{2}, \beta, X))$ |
| | $\sin 3\tau_0 \sin 3\beta \tilde{s}(s^2 - \frac{3 \sin \tau_0 \tilde{s}^2}{2 \sin \tau_0})$ | $\sin 3\tau_1 \sin 3\beta \tilde{s}_1^2(s_1 + \frac{3 \sin 2\tau_1 \tilde{s}_1}{2 \sin 3\tau_1})$ |
| (32) | $\Re(-i\psi(\frac{s_0}{2}, \beta, X))$ | $\Re(-i\psi(\frac{s_1}{2}, \beta, X))$ |
| | $\sin 3\tau_0 \sin 3\beta \tilde{s}^2(s^2 - \frac{3 \sin \tau_0 \tilde{s}^2}{2 \sin \tau_0})$ | $\sin 3\tau_1 \sin 3\beta \tilde{s}_1^2(s_1 + \frac{3 \sin 2\tau_1 \tilde{s}_1}{2 \sin 3\tau_1})$ |
| (23) | $\Re(-i\psi(\frac{s_0}{2}, \beta, X))$ | $\Re(-i\psi(\frac{s_1}{2}, \beta, X))$ |
| | $\frac{X_3}{\chi^2} \sin 3\tau_0 \sin 3\beta \tilde{s}^2(s^2 - \frac{3 \sin \tau_0 \tilde{s}^2}{2 \sin \tau_0})$ | $\frac{X_3}{\chi^2} \sin 3\tau_1 \sin 3\beta \tilde{s}_1^2(s_1 + \frac{3 \sin 2\tau_1 \tilde{s}_1}{2 \sin 3\tau_1})$ |

**Table 3.2.** Deformation for $\Re(-i\psi(\frac{s}{2}, \beta, X))$ for Case (21), (31), (32), (23) for $\tilde{s}_*>0$

![Diagram](image-url)

**Figure 3.** Signatures of $(\sin \beta, \sin 2\beta, \sin 3\beta)$ for $X_1, X_2, X_3 > 0$

$$\tau_{0,1} = \begin{cases} \pm \epsilon_1, & \text{for } \sin k\beta \geq 0, \quad |\alpha - \beta| \leq \frac{\epsilon_1}{2}, \quad \tilde{s} \in \mathcal{U}_0^s, \\ \pm \frac{\epsilon_4}{2}, & \text{for } \sin k\beta \geq 0, \quad |\alpha - \beta| \geq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_0^s, \end{cases}$$

with $k = \deg \varphi$; if $\tilde{s}_*>0$, $\tau_1 \in \begin{cases} [\pi - \epsilon_1, \pi], & \text{for } \sin k\beta > 0, \quad |\alpha_1 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, \quad \tilde{s} \in \mathcal{U}_1^s, \\ [\pi - \epsilon_4, \pi], & \text{for } \sin k\beta > 0, \quad |\alpha_1 - \beta| - \pi| \geq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^s, \\ [-\epsilon_1, 0], & \text{for } \sin k\beta > 0, \quad |\alpha_1 - \beta| \leq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \\ [-\frac{\epsilon_4}{2}, 0], & \text{for } \sin k\beta > 0, \quad |\alpha_1 - \beta| \geq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \\ [-\pi, -\pi + \epsilon_1], & \text{for } \sin k\beta < 0, \quad |\alpha_1 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \\ [-\pi, -\pi + \frac{\epsilon_4}{2}], & \text{for } \sin k\beta < 0, \quad |\alpha_1 - \beta| - \pi| \geq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \\ [0, +\epsilon_1], & \text{for } \sin k\beta < 0, \quad |\alpha_1 - \beta| \leq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \\ [0, +\frac{\epsilon_4}{2}], & \text{for } \sin k\beta < 0, \quad |\alpha_1 - \beta| \geq \frac{\epsilon_4}{2}, \quad \tilde{s} \in \mathcal{U}_1^{s_1}, \end{cases}$
\[ \tilde{r} e^{i(\alpha - \beta)} \]

**Figure 4.** A deformation of \( \tilde{s} \) in \( I_4 \) for Case (21) if \( X_1, X_2, \tilde{s}_*(\beta, X) > 0, \sin 2\beta < 0 \).

\[ \tau_0 \in \begin{cases} 
[0, +\epsilon_1], & \text{for } \sin k\beta > 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
[0, +\frac{\alpha_0}{4}], & \text{for } \sin k\beta > 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha_0}{4}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
[-\epsilon_1, 0], & \text{for } \sin k\beta < 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
[-\frac{\alpha_0}{4}, 0], & \text{for } \sin k\beta < 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha_0}{4}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon,
\end{cases} \]

and

\[ \tau_{0,1} \equiv \begin{cases} 
\pm \epsilon_1, & \text{for } \sin k\beta \geq 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
\pm \frac{\alpha_0}{4}, & \text{for } \sin k\beta \geq 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha_0}{4}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
\pm \pi \mp \epsilon_1, & \text{for } \sin k\beta \geq 0, \quad |\alpha_1 - \beta| - \pi \leq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
\pm \pi \mp \frac{\alpha_0}{4}, & \text{for } \sin k\beta \geq 0, \quad |\alpha_1 - \beta| - \pi \geq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
\mp \epsilon_1, & \text{for } \sin k\beta \geq 0, \quad |\alpha_1 - \beta| \leq \frac{\alpha_0}{2}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon, \\
\mp \frac{\alpha_0}{4}, & \text{for } \sin k\beta \geq 0, \quad |\alpha_1 - \beta| \geq \frac{\alpha_0}{4}, \quad \tilde{s} \in \mathcal{O}_1^\varepsilon
\end{cases} \]

Therefore, for \( j = 0, 1 \),

\[ |\tilde{s}_j e^{i\tau_j,1} - \tilde{r}_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max \{\tilde{r}_j, \tilde{s}_j\}, \]

and

\[ \Re(-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)) \leq -\frac{1}{C} |\sin k\tau_j \sin k\beta| \tilde{s}_j^k, \quad \tilde{s} \in \mathcal{O}_j \] for Case (21), (31), (32),

\[ \Re(-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)) \leq -\frac{1}{C} |\sin 2\tau_j \sin 2\beta| \tilde{s}_j^2, \quad \tilde{s} \in \mathcal{O}_j \] for Case (23).

**Step 2 (Proof for \( I_4, n = 1 \))**: From (3.90), the holomorphic property of \( E_{\kappa_j} f \), and a residue theorem,

\[ |I_4|_{C^\nu_\nu(D_{\kappa_1})} \leq \sum_{\nu = 1, 2} \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \left[ \partial_\beta \ln(1 - \gamma |\beta|) \right] \left( \int_{S_<} + \int_{\Gamma_{40}} + \int_{\Gamma_{41}} \right) e^{-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)} f(\xi_j, -\beta, X) d\xi_j \]

\[ + \int_{\Gamma_{41}} e^{-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)} f(\xi_j, -\beta, X) d\xi_1 + \int_{S_>} e^{-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)} f(\xi_j, -\beta, X) d\xi_{h|L^\infty(D_{\kappa_1})}, \]

with

\[ S_<(\beta, X, \lambda) = \{ \xi_0 : \tilde{s} = 2, \tau_0 \text{ defined by (3.90)} \}, \]

\[ \Gamma_{40}(\beta, X, \lambda) = \{ \xi_0 : \tilde{s} \in (2, \sigma \delta) \cap \mathcal{O}_0(\beta, X), \tau_0 = \tau_{0,1} \}, \]
\[
\Gamma_{11}(\beta, X, \lambda) = \left\{ \xi_1 : \tilde{s} \in (2, \tilde{\sigma} \delta) \cap \mathcal{U}_1(\beta, X), \quad \tau_1 = \tau_{1,1} \right\},
\]
\[
S_>(\beta, X, \lambda) = \left\{ \xi_h : h = \sup \tilde{s} = \tilde{\sigma} \delta, \quad \tau_h \text{ defined by (3.90)} \right\},
\]
with \( \xi_j, \tau_j, \tau_{j,\dagger}, \mathcal{U}_j = \mathcal{U}_j(\beta, X) \) defined by (3.90), (3.87).

From \( \tilde{r} < 1, \epsilon_1 > 0, f \in W, (3.9), (3.91)-(3.94), \) changes of variables
\[
\tilde{s}_j \mapsto t_j = \tilde{s}_j \left| \sin k \beta \right|^{1/k} \quad \text{for } \Gamma_{4j}, \text{ Case (21), (31), (32)},
\]
\[
\tilde{s}_j \mapsto t_j = \tilde{s}_j \left| \sin 2 \beta \right|^{1/2} \quad \text{for } \Gamma_{4j}, \text{ Case (23)},
\]
and improper integrals, one obtains

\[
|f|_{C^0(D_{\epsilon_1})} \leq C_{\epsilon_0} |f|_{L^\infty(D_{\epsilon_1})} + C_{\epsilon_0} |f|_{L^\infty(D_{\epsilon_1})} \sum_{h=2}^3 \int_{-\pi}^\pi \int_0^\infty \frac{1}{\sqrt{|\sin h \beta|}} e^{-th} dt |f|_{L^\infty(D_{\epsilon_1})} \leq C_{\epsilon_0} |f|_{L^\infty(D_{\epsilon_1})}.
\]

**Step 3 (Proof for \( I_5, n = 1 \)**: From (3.90), the holomorphic property of \( f \), and the residue theorem,

\[
I_5 = -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^\pi d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] \left\{ \int_{\Gamma_{50}} \frac{e^{-ip(\tilde{s}_j, \beta, X)} f(\tilde{s}_j, -\beta, X)}{\tilde{s}_1 e^{i\sigma h} - \tilde{r}_0 e^{i(\alpha_0 - \beta)}} d\xi_0 
\]
\[
+ \int_{\Gamma_{51}} \frac{e^{-ip(\tilde{s}_j, \beta, X)} f(\tilde{s}_j, -\beta, X)}{\tilde{s}_1 e^{i\sigma h} - \tilde{r}_1 e^{i(\alpha_1 - \beta)}} d\xi_1 + \int_{S_>} \frac{e^{-ip(\tilde{s}_j, \beta, X)} f(\tilde{s}_j, -\beta, X)}{\tilde{s}_h e^{i\sigma h} - \tilde{r}_h e^{i(\alpha_h - \beta)}} d\xi_h \right\}
\]
\[
- \theta(\tilde{r} - 1) \theta(\tilde{r}_1 - \frac{1}{4}) \int_{\beta \in \mathfrak{d}(\lambda)} d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] \text{sgn}(\beta)
\]
\[
\times e^{-ip(\tilde{s}_j, \beta, X)} f(\frac{\tilde{s}_1 e^{i(\alpha - \beta)}}{\tilde{\sigma}}, -\beta, X),
\]
where

\[
\mathfrak{d}(\lambda) = \left\{ \beta : \begin{array}{l}
|\alpha_0 - \beta| < \frac{\lambda}{2}, \\
|\alpha_1 - \beta| < \frac{\lambda}{2},
\end{array} \quad \text{for } \tilde{r} \in \mathcal{O}_0, \right\}
\]
\[
\mathfrak{d}(\lambda) = \left\{ \beta : \begin{array}{l}
|\alpha_0 - \beta| < \frac{\lambda}{2}, \\
|\alpha_1 - \beta| < \frac{\lambda}{2},
\end{array} \quad \text{for } \tilde{r} \in \mathcal{O}_1, \right\}
\]
\[
\mathfrak{d}(\lambda) = \left\{ \beta : \begin{array}{l}
|\alpha_0 - \beta| < \frac{\lambda}{2}, \\
|\alpha_1 - \beta| < \frac{\lambda}{2},
\end{array} \quad \text{for } \tilde{r} \in \mathcal{O}_1, \right\}
\]
\[
\mathcal{S}_> \text{ defined by (3.93), } \Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda), \quad j = 0, 1, \text{ defined by}
\]
\[
\Gamma_{50} = \left\{ \xi_0 : \tilde{s} \in \mathcal{O}_0, \tau_0 = \tau_{0,\dagger} \right\},
\]
\[
\Gamma_{51} = \Gamma_{51, \text{out}} \cup \mathcal{S}_1 \cup \Gamma_{51, \text{in}},
\]
with

\[
\Gamma_{51, \text{out}} = \left\{ \begin{array}{l}
\{ \xi_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = \tau_{1,1} \}, \\
\{ \xi_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = \tau_{1,\dagger}, \tilde{s}_1 > 1/2 \},
\end{array} \quad \tilde{r}_1 > \frac{1}{4}, \right\}
\]
\[
\Gamma_{51, \text{in}} = \left\{ \begin{array}{l}
\phi, \\
\{ \xi_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = 0 \text{ on } \mathcal{O}_1, \tau_1 = \pi \text{ on } \mathcal{O}_1, \tilde{s}_1 < 1/2 \},
\end{array} \quad \tilde{r}_1 < \frac{1}{4}, \right\}
\]
\[
\mathcal{S}_1 = \left\{ \phi, \{ \xi_1 : \tau_1 \text{ defined by (3.90), } \tilde{s}_1 = 1/2 \}, \quad \tilde{r}_1 > \frac{1}{4}, \right\}
\]
Estimates can be derived via the approach of the proof of (3.103). We skip details for simplicity.

\[ (3.102) \]

Improper integrals, and argument as that from (3.39) to (3.40) (Lemma 3.2 indeed), and \( \alpha \) 

Step 2 

Proposition 3.1 by using the deformation in 30

\[ \eta \]

Step 4 (Proof for \( I\)): Estimates can be derived via the approach of the proof of Proposition 3.1 by using the deformation in Step 2 and Step 3. We skip details for simplicity. \( \square \)
\[
\begin{array}{|c|c|}
\hline
\text{Case} & \hat{s} \in \hat{\Omega}_0 & \hat{s} \in \hat{\Omega}_1 \\
\hline
(12) & \Re(-i\varphi(\frac{\varphi}{2\pi}, \beta, X)) = \frac{X_1}{X_2} \sin 2\tau_0 \sin 2\beta \hat{s}(\tilde{s} - \frac{2\sin \tau_0}{\sin 2\pi} \tilde{s}_{1,*}) & \Re(-i\varphi(\frac{\varphi}{2\pi}, \beta, X)) = \sin 2\tau_1 \sin 2\beta \hat{s}_1^2 \\
(13) & \Re(-i\varphi(\frac{\varphi}{2\pi}, \beta, X)) = \frac{X_1}{X_2} \sin 3\tau_0 \sin 3\beta \hat{s}(\tilde{s}^2 - \frac{3\sin \tau_0}{\sin 3\pi} \tilde{s}_{1,*}^2) & \Re(-i\varphi(\frac{\varphi}{2\pi}, \beta, X)) = \sin 3\tau_1 \sin 3\beta \hat{s}_1^2 (\hat{s}_1 + \frac{3\sin 2\tau_1}{\sin 3\pi} \tilde{s}_{1,*}) \\
\hline
\end{array}
\]

Table 3.3. Deformation for \(\Re(-i\varphi(\tilde{s}, \beta, X))\) for Case (12), (13) for \(\tilde{s}_{1,*} > 0\)

Lemma 3.7. Suppose \(S = \{z_\kappa, \kappa_j, D, s_c\}\) is \(d\)-admissible and \(\partial^\tau f\) are \(\lambda\)-holomorphic on \(D_{\kappa_j}\) for \(0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5\). For the degenerated non homogeneous Case (12), (13), \(\forall n \geq 0\),

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^\tau (C T E_{\kappa_j})^n f|_W \leq (C e_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^\tau f|_W.
\]

Proof. For simplicity and without loss of generality, we only give a proof assuming \(l_1 = 1/3\), and reduce the proof to estimating principal parts. Define the parameter \(\sigma_0 = \tilde{\sigma}\) by Definition 3.2 and decompose the principal part of the CIO into (3.34)-(3.35). Thanks to (3.35), estimates on compact domains \(I_1-I_2\) can be derived by the approach in Lemma 3.3.

Step 1 (Deformation) : For \(I_4\) and \(I_5\), let \(\tilde{s}_{j,*}, \bar{\Omega}_j(\beta, X), \bar{\Omega}^\sharp_j(\beta, X), j = 0, 1, \bar{\Omega}_{1,\geq}(\beta, X), \bar{\Omega}^\sharp_{1,\geq}(\beta, X)\) defined by (3.86), (3.87), (3.88). Let \(\sigma_j\) defined by

\[
\sigma_0 = \tilde{\sigma}, \quad \sigma_1 = \sqrt{|X_k|}, \quad k = \deg(\psi).
\]

Write

\[
\lambda = \kappa_1 + \frac{\tau}{\sigma} e^{\nu_\beta} = \kappa_1 + \frac{s_{j,*} e^{\nu_\beta} + r_j e^{\nu_{\alpha_j}}}{\sigma_j},
\]

\[
s_{j,*} \equiv \tilde{s}_{j,*} \frac{\sigma_j}{\sigma_0}, \quad r_j = r_j(\beta, X, \lambda) \geq 0, \quad \alpha_j = \alpha_j(\beta, X),
\]

with \(\tilde{s}_{j,*}\) defined by (3.86). Thus

\[
\inf_{\beta} s_{1,*} \geq \frac{1}{4}
\]

for Case (12) and (13) by Table 3.1.

In view of Table 3.3 and Figure 3, define the deformation

\[
\frac{\tilde{s}}{\sigma} \mapsto \frac{\hat{\partial}_j}{\sigma_j} = \frac{s_{j,*} + s_j e^{i\tau_j}}{\sigma_j},
\]

\[
\tilde{s} \equiv (s_{j,*} \pm s_j) \frac{\sigma_0}{\sigma_j} \in \bar{\Omega}^\sharp_j(\beta, X), \text{ if } |\tau_j| \leq \frac{\pi}{2}, s_j \geq 0
\]

where \(\tau_0, \tau_{0,\dagger}\) are defined as in (3.90) for \(k \epsilon_1 \leq \frac{\pi}{2}, \deg \psi = k\). Therefore,

\[
|s_j e^{i\tau_{j,t}} - r_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max\{r_j, s_j\}
\]
and
\begin{align}
\Re(-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X)) & \leq - |\sin k\tau_0 \sin \beta| s_0, \quad s \in \mathcal{U}_0(\beta, X), \\
\Re(-i\varphi(\frac{\vartheta_1}{\sigma_1}, \beta, X)) & \leq - |\sin k\tau_1 \sin k\beta| s_1^k, \quad s \in \mathcal{U}_1(\beta, X).
\end{align}

**Step 2** (Proof for \(I_4, n = 1\)): From (3.107), the holomorphic property of \(E_{\kappa j}f\), and a residue theorem,
\begin{equation}
|I_4|_{C^0_\sigma(D_{\kappa_1})} \leq \left\{ \begin{array}{ll}
\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \left[ \partial_\beta \ln(1 - |\beta|) \right] & \int_{S_\gamma} e^{-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X)} f\left(\frac{\vartheta_0}{\sigma_0}, -\beta, X\right) \frac{d\vartheta_0}{\sigma_0 e^{i\tau_0} - \tau_0 e^{i(\alpha_0 - \beta)}} d\vartheta_0 |_{C^0_\sigma(D_{\kappa_1})} \\
+ \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{0}^{\pi} d\beta \left[ \partial_\beta \ln(1 - |\beta|) \right] & \int_{\Gamma_{40}} 3\left|_{C^0_\sigma(D_{\kappa_1})} \\
+ \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \left[ \partial_\beta \ln(1 - |\beta|) \right] & \int_{\Gamma_{41}} e^{-i\varphi(\frac{\vartheta_1}{\tau_1}, \beta, X)} f\left(\frac{\vartheta_1}{\tau_1}, -\beta, X\right) \frac{d\vartheta_1}{\tau_1 e^{i\tau_1} - \tau_1 e^{i(\alpha_1 - \beta)}} d\vartheta_1 |_{C^0_\sigma(D_{\kappa_1})} \\
+ \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \left[ \partial_\beta \ln(1 - |\beta|) \right] & \int_{S_\gamma} e^{-i\varphi(\frac{\vartheta_1}{\tau_1}, \beta, X)} f\left(\frac{\vartheta_1}{\tau_1}, -\beta, X\right) \frac{d\vartheta_1}{\tau_1 e^{i\tau_1} - \tau_1 e^{i(\alpha_1 - \beta)}} d\vartheta_1 |_{C^0_\sigma(D_{\kappa_1})},
\end{array} \right.
\end{equation}

where
\begin{align}
S_\gamma(\beta, X, \lambda) & = \{ \vartheta_0 : \tilde{s} = 2, \tau_0 \text{ defined by (3.107)} \}, \\
\Gamma_{40}(\beta, X, \lambda) & = \{ \vartheta_0 : \tilde{s} \in (2, \delta) \cap \mathcal{U}_0(\beta, X), \tau_0 = \tau_0, \}, \\
\Gamma_{41}(\beta, X, \lambda) & = \{ \vartheta_j : \tilde{s} \in (2, \delta) \cap \mathcal{U}_1(\beta, X), \tau_1 = \tau_1, \}, \\
S_\gamma(\beta, X, \lambda) & = \{ \vartheta_h : h = \sup_{\tilde{s}} j, \tilde{s} = \delta, \tau_h \text{ defined by (3.90)} \},
\end{align}

with \(\vartheta_j(\beta, X), \tau_j, \mathcal{U}_j = \mathcal{U}_j(\beta, X)\) defined by (3.107), (3.87):
\begin{align}
3 < & \left\{ \begin{array}{ll}
\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5, & \text{if } 0 < \beta < \epsilon/8, \\
0, & \text{if } -\epsilon/8 < \beta < 0, \\
e^{-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X)} f\left(\frac{\vartheta_0}{\sigma_0}, -\beta, X\right) \frac{d\vartheta_0}{\sigma_0 e^{i\tau_0} - \tau_0 e^{i(\alpha_0 - \beta)}} d\vartheta_0, & \text{if } |\beta| > \epsilon/8; \end{array} \right.
\end{align}

\begin{align}
\mathcal{J}_1 & = \theta(\frac{1}{\sin \beta}) - |\tilde{s} - \tilde{r}| f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) \frac{e^{-i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, \beta, X)}}{\tilde{s} e^{i\tau_0} - \tilde{r} e^{i(\alpha_0 - \beta)}} d\tilde{s} e^{i\tau_0}, \\
\mathcal{J}_2 & = \theta(\frac{1}{\sin \beta}) - |\tilde{s} - \tilde{r}| f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) \frac{1 - e^{i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, \beta, X)}}{\tilde{s} e^{i\tau_0} - \tilde{r} e^{i(\alpha_0 - \beta)}} d\tilde{s} e^{i\tau_0}, \\
\mathcal{J}_3 & = \theta(\frac{1}{\sin \beta}) - |\tilde{s} - \tilde{r}| e^{i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X)} f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) \\
& \times \left\{ \begin{array}{ll}
1 & \text{if } |\tilde{s} - \tilde{r}| e^{i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X)} f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) - f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, \beta, X\right)\right\} d\tilde{s} e^{i\tau_0}, \\
e^{i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X)} f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) - f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, \beta, X\right) & \text{if } |\tilde{s} - \tilde{r}| e^{i\varphi(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X)} f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, -\beta, X\right) - f\left(\frac{\tilde{s} e^{i\tau_0}}{\sigma}, \beta, X\right) \end{array} \right.
\end{align}
From (3.106), \( \tilde{r} < 1 \), similar analysis as that in Lemma 3.3 yields

\[
(3.114) \quad \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_{\beta} \ln(1 - \gamma|\beta|)] \right| \int_{S_{<}} e^{-ip(\frac{\partial_0}{\sigma_0}, \beta, X)} f(\frac{\partial_0}{\sigma_0}, -\beta, X) d\theta_0 |C^w_\sigma(D_{\kappa_1}) + e^{-ip(\frac{\partial_1}{\sigma_1}, -\beta, X)} f(\frac{\partial_1}{\sigma_1}, -\beta, X) d\theta_1 |C^w_\sigma(D_{\kappa_1}) + e^{-ip(\frac{\partial_h}{\sigma_h}, -\beta, X)} f(\frac{\partial_h}{\sigma_h}, -\beta, X) d\theta_h |C^w_\sigma(D_{\kappa_1}) \leq C_{\sigma_0} |f|_{L^\infty(D_{\kappa_1})}.
\]

On \( \Gamma_{40} \), it suffices to consider \( |\beta| \leq \epsilon_1/8 \). Note that \( J_3, J_4, J_5 \) for linear homogeneous cases in Lemma 3.4 can be adapted to derive estimates for Case (12) and (13) on \( \tilde{U}_0 \). As for \( J_1, J_2 \), in view of Table 3.1

\[
(3.115) \quad |\theta(\frac{\partial_0}{\sigma_0}, \beta, X)| \leq C |\sin \beta| \sigma_0, \quad \text{for} \quad \tilde{s} \in \tilde{U}_0(\beta, X), \quad |\beta| < \epsilon_1/8.
\]

Hence, by the mean value theorem, (3.57) can be generalized to

\[
(3.116) \quad \frac{|e^{-ip(\frac{\partial_1}{\sigma_1}, 1, \beta, X)} - 1|}{|\tilde{S} e^{\pm i\tau_1} - \tilde{r} e^{i(\alpha - \beta)}|} \leq e^{-ip(\frac{\partial_1}{\sigma_1}, \beta, X)} f(\frac{\partial_1}{\sigma_1}, -\beta, X) d\theta_1 |C^w_\sigma(D_{\kappa_1}) \leq C_{\sigma_0} |f|_{L^\infty(D_{\kappa_1})},
\]

for \( \tilde{s} \in \tilde{U}_0(\beta, X) \) and \( |\beta| < \epsilon_1/8 \).

Applying similar analysis as that in Lemma 3.3, one obtains

\[
(3.117) \quad \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{0}^{\pi} d\beta [\partial_{\beta} \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \int_{\Gamma_{40}} (J_1 + J_2 + J_3 + J_4 + J_5) |C^w_\sigma(D_{\kappa_1}) \leq C_{\sigma_0} |f|_{L^\infty(D_{\kappa_1})}.
\]

As a result,

\[
(3.118) \quad |I_4|_{C^w_\sigma(D_{\kappa_1})} \leq C_{\sigma_0} |f|_{L^\infty(D_{\kappa_1})}.
\]

**Step 3 (Proof for \( I_5 \), \( n = 1 \))**: From (3.109), the holomorphic property of \( E_{\kappa_j} f \), and the residue theorem,

\[
(3.119) \quad I_5 = \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{0}^{\pi} d\beta [\partial_{\beta} \ln(1 - \gamma|\beta|)] \int_{\Gamma_{50}} J_< e^{-ip(\frac{\partial_1}{\sigma_1}, \beta, X)} f(\frac{\partial_1}{\sigma_1}, -\beta, X) d\theta_1 - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_{\beta} \ln(1 - \gamma|\beta|)] \int_{\Gamma_{51}} J_1 e^{-ip(\frac{\partial_1}{\sigma_1}, \beta, X)} f(\frac{\partial_1}{\sigma_1}, -\beta, X) d\theta_1
\]
\[-\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] \int_{S_>} \frac{e^{-i\phi(\frac{\beta}{\pi}, \beta, X)} f(\frac{\beta}{\pi}, -\beta, X)}{\partial_\beta - r_1 e^{i(\alpha_0 - \beta)}} d\theta_1 \]
\[-\frac{\theta(\tilde{r} - 1)}{16} \int_{\alpha_0} d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] \int_{S_>} \tilde{J} \uparrow \beta, X \bigg|_{L^\infty(D_{\alpha_1})} \bigg| \leq C \epsilon_0 \bigg| f \bigg|_{L^\infty(D_{\alpha_1})}.\]

where \(\tilde{J}_<\) is defined by (3.113), \(\tilde{J}(\lambda)\) defined by (3.98), \(S_0\) defined by (3.112), and \(\Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda)\), \(j = 0, 1\), defined by
\[(3.120)\]
\[
\Gamma_{50} = \{ \theta_0 : \tilde{s} \in \mathcal{U}_0, \tau_0 = \tau_{0,\dagger}\},
\]
\[
\Gamma_{51} = \Gamma_{51,\text{out}} \cup S_{51} \cup \Gamma_{51,\text{in}},
\]

with
\[
\Gamma_{51,\text{out}} = \begin{cases} \{ \theta_1 : \tilde{s} \in \mathcal{U}_1, \tau_1 = \tau_{1,\dagger}\}, & r_1 > \frac{1}{16}, \\ \{ \theta_1 : \tilde{s} \in \mathcal{U}_1, \tau_1 = \tau_{1,\dagger}, s_1 > 1/8\}, & r_1 < \frac{1}{16}, \\ \phi, & \end{cases}
\]
\[
\Gamma_{51,\text{in}} = \begin{cases} \{ \theta_1 : \tilde{s} \in \mathcal{U}_1, \tau_1 = 0 \text{ on } \mathcal{U}_1\}, & r_1 > \frac{1}{16}, \\ \tau_1 = \pi \text{ on } \mathcal{U}_1, s_1 < 1/8\}, & r_1 < \frac{1}{16}, \\ \phi, & \end{cases}
\]
\[
S_{51} = \begin{cases} \{ \theta_1 : \tau_1 \text{ defined by (3.90), } s_1 = 1/8\}, & r_1 > \frac{1}{16}, \\ \phi, & r_1 < \frac{1}{16}, \end{cases}
\]

and \(\alpha_j, \partial_j, \tau_{j,\dagger}, \mathcal{U}_j = \mathcal{U}_j(\beta, X)\) defined by (3.87)-(3.90).

Applying (3.9), (3.108), (3.110), (3.106) (used in estimating integrals on \(\Gamma_{51,\text{in}}\) in particular), (3.120), similar analysis as that in Lemma 3.3 one obtains
\[
\bigg| \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] \int_{\Gamma_{51}} \frac{e^{-i\phi(\frac{\beta}{\pi}, \beta, X)} f(\frac{\beta}{\pi}, -\beta, X)}{s_1 e^{i\tau_1} - r_1 e^{i(\alpha_0 - \beta)}} d\theta_1 \bigg|_{L^\infty(D_{\alpha_1})} \bigg| \leq C \epsilon_0 \bigg| f \bigg|_{L^\infty(D_{\alpha_1})},
\]

On \(\Gamma_{50}\), it suffices to consider \(|\beta| \leq \epsilon_1/8\). Applying (3.108), (3.109), (3.116), similar analysis as that in Lemma 3.4 one obtains
\[
|I_5|_{C^0(D_{\alpha_1})} \leq C |I_5|_{L^\infty(D_{\alpha_1})} \leq C \epsilon_0 |f|_{L^\infty(D_{\alpha_1})}.\]

Combining (3.106), (3.121), (3.122), and Lemma 3.2
\[
|I_5|_{C^0(D_{\alpha_1})} \leq C |I_5|_{L^\infty(D_{\alpha_1})} \leq C \epsilon_0 |f|_{L^\infty(D_{\alpha_1})}.\]
\[
\begin{array}{|c|c|c|}
\hline
\text{Case} & \partial_x \varphi(\tilde{s}, \beta, X) & \text{Stationary points} \\
\hline
(F1) & \sin \beta + 2X_2 \tilde{s} \sin 2\beta + 3X_3 \tilde{s}^2 \sin 3\beta & \frac{-1 + \sqrt{1 - \Delta}}{3X_3 \sin \beta / X_1} \\
(F2) & \frac{X_1}{X_2} \sin \beta + 2\tilde{s} \sin 2\beta & \frac{-1 + \sqrt{1 - \Delta}}{3X_3 \sin \beta / X_1} \\
(F3) & \frac{X_1}{X_3} \sin \beta + 2X_2 \tilde{s} \sin 2\beta + 3\tilde{s}^2 \sin 3\beta & \frac{-1 + \sqrt{1 - \Delta}}{3X_3 \sin \beta / X_1} \\
\hline
\end{array}
\]

**Table 3.4.** Stationary points for fully non homogeneous cases

**Step 4 (Proof for I_4, I_5, n > 1):** Estimates can be derived via the approach of the proof of Proposition 3.1, 3.2 by using the deformation in Step 2 and Step 3. We skip details for simplicity.

3.1.3. Fully non homogeneous cases. Define the scaling parameter \( \tilde{\sigma} \) by Definition 2.2 and scaling coordinates by \( (3.2) \). Next, classify fully non homogeneous cases as

\[
(F1) \quad \sqrt{X_3} \leq \sqrt{X_2} \leq X_1 \quad \text{or} \quad \sqrt{X_2} \leq \sqrt{X_3} \leq X_1; \\
(F2) \quad \sqrt{X_3} \leq |X_1| \leq \sqrt{X_2} \quad \text{or} \quad |X_1| \leq \sqrt{X_3} \leq \sqrt{X_2}; \\
(F3) \quad \sqrt{X_2} \leq |X_1| \leq \sqrt{X_3} \quad \text{or} \quad |X_1| \leq \sqrt{X_2} \leq \sqrt{X_3}. 
\]

We have Table 3.4 for the stationary points where

\[
\Delta = 3X_1X_3 \sin \beta \sin 3\beta / X_2 \sin^2 2\beta .
\]

Let

\[
\Omega_1 = \{0 \leq |\beta| \leq \frac{\pi}{4}\}, \quad \Omega_2 = \{\frac{\pi}{4} \leq |\beta| \leq \frac{\pi}{2}\}, \quad \Omega_3 = \{\frac{\pi}{2} \leq |\beta| \leq \frac{3\pi}{4}\}, \quad \Omega_4 = \{\frac{3\pi}{4} \leq |\beta| \leq \pi\},
\]

as is shown in Figure 3. For any fully non homogeneous case, we classify \((\beta, X)\) into \( A, \ldots, E \) five types according to the determinant \( \Delta \). Then properties of corresponding stationary points \( \tilde{s}_\pm \) and \( \beta \)-domains are listed in Table 3.5.

**Lemma 3.8.** Suppose \( S = \{z_n, \kappa_j, D, s_c\} \) is \( d \)-admissible and \( \partial_x f \) are \( \lambda \)-holomorphic on \( D_{\kappa_j} \) for \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \). For fully non homogeneous cases \((F3), \forall n \geq 0, \)

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l (C T E_{\kappa_j})^n f|_W \leq (C\epsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l f|_W .
\]

**Proof.** For simplicity and without loss of generality, we only give a proof assuming \( l = 0, (3.9) \), and reduce the proof to estimating the principal part. Define the scaling parameter \( \tilde{\sigma} \) by Definition 2.2, scaled coordinates by \( (3.2) \), and decompose the principal part into \( (3.3) - (3.5) \). Thanks to \( (3.8) \), estimates on compact domains, i.e., \( I_j, j = 1, 2, 3 \) can be derived as in Lemma 3.3.
TABLE 3.5. Dynamics of $\tilde{s}_\pm$ and $\Delta$ for $X_1, X_2, X_3 > 0$

**Step 1 (Deformation):** According to the order of $\tilde{s}_\pm$, shown by the last column in Table 3.5, we define the essential critical points $\tilde{s}_{j,*} = \tilde{s}_{j,*}(\beta, X)$ by

$$
\tilde{s}_{0,*} = 0, \quad \tilde{s}_{1,*} = \begin{cases} 
\frac{\tilde{s}_+ + \tilde{s}_-}{2} > 0, & \text{Type } \mathfrak{A}', \\
\inf \tilde{s}_\pm > 0, & \text{Type } \mathfrak{B}', \mathfrak{C}', \\
\sup \tilde{s}_\pm > 0, & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
- & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', 
\end{cases}
\tilde{s}_{2,*} = \begin{cases} 
- & \text{Type } \mathfrak{A}'', \\
\sup \tilde{s}_\pm, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
- & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
- & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', 
\end{cases}
$$

where $-$ means no definition. Given $0 < \epsilon_1 < \frac{\pi}{2\epsilon} \ll 1$, define neighborhood $\tilde{U}_j(\beta, X) \supset \tilde{U}_j(\beta, X)$ of essential critical points $\tilde{s}_{j,*}$ by

$$
\tilde{U}_0 = \begin{cases} 
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{A}'', \\
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
[0, \tilde{\sigma}] & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', 
\end{cases}
\tilde{U}_1 = \begin{cases} 
\{(1 - \frac{1}{2\cos \epsilon_1})\tilde{s}_{1,*}, \tilde{s}_{1,*} \} \cup \{\tilde{s}_{1,*}, \tilde{s}_{1,*} \} \equiv \tilde{U}_1 < \cup \tilde{U}_1 >, & \text{Type } \mathfrak{A}'', \\
\{(1 - \frac{1}{2\cos \epsilon_1})\tilde{s}_{1,*}, \tilde{s}_{1,*} \} \cup \{\tilde{s}_{1,*}, \tilde{s}_{1,*} + \frac{\tilde{s}_+ - \tilde{s}_-}{2\cos \epsilon_1} \} \equiv \tilde{U}_1 < \cup \tilde{U}_1 >, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\{(1 - \frac{1}{2\cos \epsilon_1})\tilde{s}_{1,*}, \tilde{s}_{1,*} \} \cup \{\tilde{s}_{1,*}, \tilde{\sigma} \} \equiv \tilde{U}_1 < \cup \tilde{U}_1 >, & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
\phi & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', 
\end{cases}
\tilde{U}_2 = \begin{cases} 
\phi, & \text{Type } \mathfrak{A}'', \\
\tilde{s}_{2,*} \equiv \tilde{s}_{2,*} + \frac{\tilde{s}_+ - \tilde{s}_-}{2\cos \epsilon_1}, \tilde{\sigma} \equiv \tilde{U}_2 < \cup \tilde{U}_2 >, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\phi & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
\phi & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', 
\end{cases}
$$
\[ \mathcal{U}_0^\phi = \begin{cases} [0, \tilde{s}_{1,*}], & Type \mathcal{A}'', \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ [0, \tilde{s}_\delta], & Type \mathcal{A}', \mathcal{B}', \mathcal{C}, \\ \end{cases} \]

\[ \mathcal{U}_j^\delta = \begin{cases} [0, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_\delta], & Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ [0, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_{2,*}], & Type \mathcal{B}'', \mathcal{C}'', \\ \end{cases} \]

\[ \mathcal{U}_2^\phi = \begin{cases} \phi, & Type \mathcal{A}, \mathcal{B}', \mathcal{C}', \mathcal{D}, \mathcal{E}, \\ [\tilde{s}_{1,*}, \tilde{s}_{2,*}] \cup [\tilde{s}_{2,*}, \tilde{s}_\delta], & Type \mathcal{B}'', \mathcal{C}''. \\ \end{cases} \]

Write

\[ \lambda = \kappa_j + \frac{\tilde{r}_j e^{i\alpha}}{\sigma} = \kappa_j + \frac{\tilde{s}_{j,*} e^{i\beta} + \tilde{r}_j e^{i\alpha_j}}{\sigma}, \]

\[ \tilde{r}_j = \tilde{r}_j(\beta, X, \lambda), \quad \alpha_j = \alpha_j(\beta, X, \lambda), \quad j = 0, 1, 2. \]

Due to Figure 3.35, the deformation defined by

\[ \tilde{s} \mapsto \xi_j \equiv \tilde{s}_{j,*} e^{i\alpha_j} + \tilde{s}_{j,*}, \]

\[ \tilde{s} \equiv \pm \tilde{s}_{j,*} \in \mathcal{U}_j^\phi, |\tau_j| \leq \frac{\pi}{2}, \quad \tilde{s}_{j,*} \geq 0, \quad j = 0, 1, 2, \]

with

\[ \begin{aligned} \pm \epsilon_1 \geq \tau_0 \geq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_0 - \beta| \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_0, \ Type \mathcal{D}, \mathcal{E}, \\ \pm \frac{\epsilon_2}{2} \geq \tau_0 \geq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_0 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_0, \ Type \mathcal{D}, \mathcal{E}, \\ \mp \epsilon_1 \leq \tau_0 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_0 - \beta| \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_1, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \mp \frac{\epsilon_2}{2} \leq \tau_0 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_0 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_1, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \mp \pi \leq \tau_1 \leq \mp \pi \pm \epsilon_1, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,<}, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \mp \epsilon_1 \leq \tau_1 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,\geq}, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \mp \pi \leq \tau_1 \leq \mp \pi \pm \epsilon_1, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,<}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \pm \epsilon_1 \geq \tau_1 \geq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,<}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \pm \frac{\epsilon_2}{2} \geq \tau_1 \geq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,\geq}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \pm \pi \leq \tau_1 \leq \pm \pi \pm \frac{\epsilon_1}{2}, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,<}, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \pm \frac{\epsilon_2}{2} \leq \tau_1 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,\geq}, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \pm \pi \leq \tau_1 \leq \pm \pi \pm \frac{\epsilon_2}{2}, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,<}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \pm \frac{\epsilon_2}{2} \leq \tau_1 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_1 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{1,\geq}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \mp \epsilon_1 \leq \tau_2 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_2 - \beta| \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{2,<}, \ Type \mathcal{A}'', \mathcal{D}, \mathcal{E}, \\ \mp \pi \leq \tau_2 \leq \pm \pi \pm \epsilon_1, & \quad for \ sin 3\beta \geq 0, \ |\alpha_2 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{2,<}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \mp \epsilon_1 \leq \tau_2 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_2 - \beta| \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{2,\geq}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \mp \pi \leq \tau_2 \leq \pm \pi \pm \frac{\epsilon_2}{2}, & \quad for \ sin 3\beta \geq 0, \ |\alpha_2 - \beta| - \pi \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{2,<}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \\ \mp \frac{\epsilon_2}{2} \leq \tau_2 \leq 0, & \quad for \ sin 3\beta \geq 0, \ |\alpha_2 - \beta| \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{O}_{2,\geq}, \ Type \mathcal{B}'', \mathcal{C}'', \mathcal{D}, \mathcal{E}, \end{aligned} \]

and...
satisfies Table 3.6 and for $j = 0, 1, 2$, $\tilde{s} \in \mathcal{U}_j(\beta, X)$,

\begin{align}
(3.131) \quad |\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j - \beta)}| &\leq \frac{1}{C} \max \{\tilde{r}_j, \tilde{s}_j\}, \\
(3.132) \quad \Re(-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)) &\leq -\frac{1}{C}\sin 3\tau_j \sin 3\beta|\tilde{s}_j^3|.
\end{align}

**Step 2 (The estimates for $I_4, I_5, n = 1$):** With (3.131) and (3.132), one can adapt argument in Lemma 3.6 to derive estimates for $I_4$ and $I_5$. For illustration, a residue theorem implies

\begin{align}
(3.133) \quad I_5 &= -\frac{\theta(\tilde{r} - 1)}{2\pi i} \sum_{j=0}^{2} \int_{-\pi}^{\pi} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \int_{\Gamma_{5j}} \frac{e^{-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)} f(\frac{\xi_j}{\sigma}, -\beta, X)}{\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j - \beta)}} d\xi_j \\
&\quad - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \int_{S_{\sigma}} \frac{e^{-i\varphi(\frac{\xi_h}{\sigma}, \beta, X)} f(\frac{\xi_h}{\sigma}, -\beta, X)}{\tilde{s}_h e^{i\tau_1} - \tilde{r}_h e^{i(\alpha_1 - \beta)}} d\xi_h \\
&\quad - \theta(\tilde{r} - 1)\theta(\tilde{r}_1 - \frac{1}{4})\theta(\tilde{r}_2 - \frac{1}{4}) \int_{\beta \in \Theta(\lambda)} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \text{sgn}(\beta) \\
&\quad \times e^{-i\varphi(\frac{\tilde{r}_1(\alpha - \beta)}{\sigma}, \beta, X)} f\left(\frac{\tilde{r}_1 e^{i(\alpha - \beta)}}{\sigma}, -\beta, X\right).
\end{align}
| Case | Type $\mathfrak{A}$' |
|------|-------------------|
| $\tilde{s} \in \tilde{U}_0$ | $\mathfrak{R}(\frac{3\pi}{2} - \beta, X)$ |
| $\tilde{s} \in \tilde{U}_1$ | $\mathfrak{R}(\frac{3\pi}{2} + \beta, X)$ |

| Case | Type $\mathfrak{B}$', $\mathfrak{C}$' |
|------|-------------------|
| $\tilde{s} \in \tilde{U}_0$ | $\mathfrak{R}(\frac{3\pi}{2} + \beta, X)$ |

| Case | Type $\mathfrak{D}$, $\mathfrak{C}$ |
|------|-------------------|
| $\tilde{s} \in \tilde{U}_0$ | $\mathfrak{R}(\frac{3\pi}{2} - \beta, X)$ |

| Case | Type $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$ |
|------|-------------------|
| $\tilde{s} \in \tilde{U}_0$ | $\mathfrak{R}(\frac{3\pi}{2} - \beta, X)$ |

Table 3.6. Deformation for $\mathfrak{R}(\tilde{s}, \frac{3\pi}{2}, \beta, X)$ for Case (F1), (F2), (F3)

where $S_{\geq} = S_{\geq}(\beta, X, \lambda),

\begin{align}
\mathfrak{d}(\lambda) \equiv \{ \beta : & \begin{cases}
|\alpha_0 - \beta| < \frac{\pi}{2}, & (\alpha_0 - \beta)\beta < 0, \ \tilde{s} \in \tilde{U}_0, \\
|\alpha_0 - \beta| - \pi < \frac{\pi}{2}, & (\alpha_0 - \beta)\beta > 0, \ \tilde{s} \in \tilde{U}_n<, \ n = 1, 2, \}, \\
|\alpha_n - \beta| < \frac{\pi}{2}, & (\alpha_n - \beta)\beta < 0, \ \tilde{s} \in \tilde{U}_n>, \ n = 1, 2,
\end{cases} \\
\end{align}

and $\Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda), \ j = 0, 1, 2, \ n = 1, 2,$ defined by

\begin{align}
\Gamma_{50} &= \{ \xi_0 : \tilde{s} \in \tilde{U}_0, \ \tau_0 = \tau_0, \}, \\
\Gamma_{5n} &= \Gamma_{5n, out} \cup S_{5n} \cup \Gamma_{5n, in},
\end{align}

with
Estimates can be derived via the approach of the proof of Proposition 3.1. We shall justify the deformation (3.130) is good for Case (3.2) of Lemma 3.6, we obtain estimates for $I_n$. Thanks to (3.85), estimates for $I_n$ scaled coordinates by (3.2), and decompose the principal part into (3.3)-(3.8).

For simplicity and without loss of generality, we only give a proof assuming $\alpha_j, \tau_j, \tau_i, \tau_i, \partial_j$ are defined by (3.128)-(3.130). Using (3.131)-(3.135), applying argument as that for $I_5$ in Step 3 of Lemma 3.6, we obtain

$$|I_5|_{C_\delta (D_{\kappa_1})} \leq C \epsilon_0 |f|_{L^\infty (D_{\kappa_1})}. \tag{3.136}$$

**Step 3 (The estimates for $I_4, I_5, n > 1$):** Estimates can be derived via the approach of the proof of Proposition 3.1. We skip details for simplicity.

**Lemma 3.9.** Suppose $S = \{z_n, \kappa_j, D, s_c\}$ is d-admissible and $\partial^\lambda f$ are $\lambda$-holomorphic on $D_{\kappa_j}$ for $0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4$. For fully non homogeneous cases (F2), $\forall n \geq 0$,

$$\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4} |E_{\kappa_j} \partial^\lambda (\bar{C} T E_{\kappa_j})^n f|_W \leq (C \epsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4} |E_{\kappa_j} \partial^\lambda f|_W.$$

**Proof.** For simplicity and without loss of generality, we only give a proof assuming $l = 0$, (3.9), and reduce the proof to estimating the principal part. Define the rescaled parameter $\tilde{\sigma}$ by Definition 2.2, scaled coordinates by (3.2), and decompose the principal part into (3.3)-(3.8). Thanks to (3.85), estimates for $I_j$, $j = 1, 2, 3$ can be derived as in Lemma 3.3.

**Step 1 (Deformation):** We shall justify the deformation (3.130) is good for Case (F2). Note

$$3 \frac{X_2}{X_2^{3/2}} \sin 3\beta = \frac{X_2^{1/2}}{X_1} \sin \beta \Delta \sin 2\beta. \tag{3.137}$$

**Figure 6.** A deformation of $\tilde{s}$ for $I_5$ in Case (F3) when $\sin 3\beta, X_j > 0$, $\tilde{r}_i < \frac{1}{4}$, and $(\beta, X) \in \mathfrak{B}$. The estimates for $I_n$ are defined by (3.128)-(3.130). Using (3.131)-(3.135), applying argument as that for $I_5$ in Step 3 of Lemma 3.6, we obtain

$$|I_5|_{C_\delta (D_{\kappa_1})} \leq C \epsilon_0 |f|_{L^\infty (D_{\kappa_1})}. \tag{3.136}$$
Together with Figure 3, Table 3.4, [3.6], Δ > \frac{1}{2}, and (3.130), yields

\text{Re}\left(-i\varphi\left(\frac{\tilde{s}_j e^{i\tau_j \uparrow} + \tilde{s}_{j,\ast}}{\tilde{\sigma}}, \beta, X\right)\right) \leq -\frac{1}{C} |\sin 2\beta| \tilde{s}_j^3,

(3.138)

\tilde{s} \in \mathcal{U}_j, \ j = 0, 1, 2, \ for \ Type \ \mathfrak{A}'' , \ \mathfrak{B}'' .

Moreover, Figure 3, Table 3.4, [3.6], and (3.130) imply

\begin{align*}
|\tilde{s}_j + (-1)^j + \frac{\sqrt{1 - \Delta}}{3 \frac{X_3 \sin 3\beta}{2\tilde{\sigma} X_2} \sin 3\tau_j} | & \geq \frac{1}{C} \frac{X_3 \sin 3\beta}{X_2^3 \sin 2\beta}, \ on \ \mathcal{U}_j, \ j = 1, 2, \ for \ Type \ \mathfrak{A}'' , \\
|\tilde{s}_1 + \frac{\sqrt{1 - \Delta}}{3 \frac{X_3 \sin 3\beta}{2\tilde{\sigma} X_2} \sin 3\tau_1} | & \geq \frac{1}{C} \frac{X_3 \sin 3\beta}{X_2^3 \sin 2\beta}, \ on \ \mathcal{U}_0, \ 0 < \tilde{s}_1 \pm \ for \ Type \ \mathfrak{D}, \ \mathfrak{E} , \\
|\tilde{s}_{1,\ast} - \tilde{s}_{2,\ast} | & \geq \frac{1}{C} \frac{1}{X_2^3 \sin 2\beta}, \ for \ Type \ \mathfrak{C}'' , \ \mathfrak{D} , \ \mathfrak{E} .
\end{align*}

As a result,

\text{Re}\left(-i\varphi\left(\frac{\tilde{s}_j e^{i\tau_j \uparrow} + \tilde{s}_{j,\ast}}{\tilde{\sigma}}, \beta, X\right)\right) \leq -\frac{1}{C} |\sin 2\beta| \tilde{s}_j^2

(3.139)

for \tilde{s} \in \mathcal{U}_j, \ j = 0, 1, 2, \ for \ Type \ \mathfrak{C}'' , \ \mathfrak{D} , \ \mathfrak{E} .

Finally,

\text{Re}\left(-i\varphi\left(\varphi, \beta, X\right)\right) \leq -\frac{1}{C} |\sin 3\tau_0 \sin 3\beta| \tilde{s}_0^3, \ \tilde{s} \in \mathcal{U}_0(\beta, X), \ for \ Type \ \mathfrak{A}', \ \mathfrak{B}', \ \mathfrak{C}' .

(3.140)

Step 2: With (3.131), (3.133)-(3.140), fulfilled by the deformation (3.130), the lemma can be justified by adapting argument as that for Case (F3) in Lemma 3.8.

\begin{proof}
\end{proof}

Lemma 3.10. Suppose \( S = \{z_\ast, \kappa, \mathfrak{D}, s_\ast\} \) is d-admissible and \( \partial_x f \) is \( \lambda \)-holomorphic on \( D_{\kappa_j} \) \( for \ 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \). For fully non homogeneous cases (F1), \( \forall n \geq 0 \),

\[ \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l (C \theta E_{\kappa_j})^n f|_W \leq (C\epsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^l f|_W . \]

Proof. For simplicity and without loss of generality, we only give a proof assuming \( l = 0 \), \( (3.9) \), and reduce the proof to estimating the principal part. Define the scaling parameter \( \tilde{\sigma} \) by Definition 2.2 rescaled coordinates by \( (3.2) \), and decompose the principal part into \( (3.3) \)-(3.8). Thanks to \( (3.85) \), estimates on compact domains, i.e., \( I_j, j = 1, 2, 3 \) can be derived as in Lemma 3.8.

Step 1 (The deformation): To estimate \( I_4 \) and \( I_5 \) for the fully non homogeneous cases (F1), introduce \( \sigma_j, j = 0, 1, 2 \),

(3.141)

\[ \left\{ \begin{array}{ll}
\sigma_0 = \tilde{\sigma}, \ \sigma_1 = \sigma_2 = \sqrt{|X_3|}, & \text{for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{E}, \\
\sigma_0 = \tilde{\sigma}, \ \sigma_1 = \sigma_2 = \sqrt{|X_2|}, & \text{for Type } \mathfrak{C}, \mathfrak{D}, \\
\end{array} \right. \]

and \( s_{j,\ast} \),

(3.142)

\[ s_{j,\ast} = \frac{\tilde{s}_{j,\ast} \sigma_j}{\sigma_0}, \ \ r_j = \tilde{r}_j \frac{\sigma_j}{\sigma_0} . \]
where $\tilde{s}_{j,*}$ and $\tilde{r}_j$ are defined by (3.127) and (3.129). Define the deformation

$$\frac{\tilde{s}}{\sigma} \mapsto \tilde{\sigma}_j \equiv s_j e^{i\tau_j} + s_{j,*},$$

(3.143)

$$\tilde{s} \equiv (s_j + s_{j,*})\sigma_0 \in \tilde{U}_j, \quad s_j \geq 0, \quad \text{for } |\tau_j| \leq \frac{\pi}{2},$$

and $\tilde{U}_j$, $\epsilon_1$, $\tau_j$, are defined by (3.128), (3.130). Therefore,

$$|s_j e^{i\tau_{j,1}} - r_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max\{r_j, s_j\}.$$  

In order to derive estimates, we have to justify

(3.145) \[ \inf_{\beta} s_{1,*} = c_0, \quad 0 < c_0 < 1 \]

first (cf. (3.106)). To this aim, for Type $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, from Table 3.5 (3.141), and

$$|s_+ s_-| = \left| \frac{X_1}{X_1^{1/3} \sin 3\beta} \right| \geq 1, \quad s_+ = \frac{1 \pm \sqrt{1 - \Delta}}{3 \frac{X_2^{2/3} \sin 3\beta}{X_2^{2/3} \sin 2\beta}},$$

we derive $|s_+| \sim |s_-|$ and then (3.145) for Type $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

On the other hand, from (3.125), Table 3.5 and (3.141),

$$|s_+| = \left| \frac{-\Delta}{6 \frac{X_2 \sin 3\beta}{X_2 \sin 2\beta}} \right| + \text{l.o.t.} \geq \left| \frac{-3 X_1 X_2 \sin 3\beta \sin 3\beta}{6 X_2^{2/3} \sin 3\beta} \right| + \text{l.o.t.} \geq \frac{1}{C},$$

$$|s_-| = \left| \frac{-2}{3 \frac{X_2 \sin 3\beta}{X_2 \sin 2\beta}} \right| + \text{l.o.t.} \geq \frac{1}{C} |s_+| \geq \frac{1}{C}.$$  

Hence (3.145) is proved for Type $\mathfrak{C}$ and $\mathfrak{D}$.

Next, from (3.141)-(3.143), Table 3.6 and results of Case (F3),

$$\Re (-i\varphi(\frac{s_j e^{i\tau_{j,1}} + s_{j,*}}{\sigma_j}, \beta, X)) \leq \frac{1}{C} |\sin 3\beta| s_j^3, \quad \tilde{s} \in \tilde{U}_j, \quad j = 1, 2,$$

(3.146) \[ \text{for Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}; \]

\[ \text{Table 3.7. Deformation for } -i\varphi(\frac{s_j}{\tau}, \beta, X) \text{ for Case (F1), (F2), (F3)} \]
and from (3.141)-(3.143), Table 3.6 and results of Case (F2),
\begin{equation}
\Re(-i\varphi\frac{s_j e^{i\tau_j + i\varphi} + s_{j+1}}{\sigma_j}, \beta, X) \leq -\frac{1}{C} |\sin 2\beta| s_j^2, \ s \in \mathcal{U}_j, \ j = 1, 2, 
\end{equation}
(3.147)
for Type $\mathfrak{C}''$, $\mathfrak{D}$.

Besides, note
\begin{equation}
3 \frac{X_3}{X_1^3} \sin 3\beta = \left( \frac{X_2}{X_1^2} \sin \beta \right)^2 \sin \beta \Delta, \quad 3 \frac{X_3}{X_1^3} \sin 3\beta = \frac{X_2}{X_1^2} \sin \beta \Delta.
\end{equation}
(3.148)
Therefore, for $s \in \mathcal{U}_0$, using
\begin{equation}
\left\{ \begin{array}{ll}
\text{the second term of } \Re(-i\varphi) \text{ in Table 3.6} & \text{for } \mathfrak{A}, \Delta \geq \frac{3}{2}, \\
\text{the first term of } \Re(-i\varphi) \text{ in Table 3.6} & \text{for } \mathfrak{A}, 1 \leq \Delta \leq \frac{3}{2}, \\
(3.130), (3.148), \frac{1}{2} \leq \Delta \leq 1, \ Table 3.4, 3.6 & \text{for } \mathfrak{B}, \\
\text{Table 3.4, 3.6, (3.148), } |(\bar{s} - s_{j+1})(\bar{s} - s_{j-1})| \geq \frac{1}{C} \left( \frac{X_2}{X_1^2} \sin \beta \right)^2 |\Delta| & \text{for } \mathfrak{C}, \mathfrak{D}, \\
\text{Table 3.4, 3.6, (3.148), } |(\bar{s} - s_{j+1})(\bar{s} - s_{j-1})| \geq \frac{1}{C} \left( \frac{X_2}{X_1^2} \sin \beta \right)^2 |\Delta| & \text{for } \mathfrak{C}, \mathfrak{D},
\end{array} \right.
\end{equation}
(3.149)
we obtain
\begin{equation}
\Re(-i\varphi\frac{s_0 e^{i\tau_0}}{\sigma_0}, \beta, X) \leq -\frac{1}{C} |\sin \beta| s_0, \ s \in \mathcal{U}_0, \text{ for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}.
\end{equation}
(3.150)
Finally, in view of Table 3.4, 3.7 and adapting argument as in deriving (3.149), we have
\begin{equation}
|\varphi(\frac{\varphi_0}{\sigma_0}, \beta, X)| \leq C |\sin \beta| s_0, \ s \in \mathcal{U}_0, \text{ for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}.
\end{equation}
(3.151)
Consequently, (3.57) can be generalized to
\begin{equation}
\frac{|e^{-i\varphi(\frac{\varphi_0}{\sigma_0}, \beta, X)} - 1|}{|\bar{s} e^{i\tau_1} - \bar{s} e^{i(\alpha - \beta)}|} \leq C |\sin \beta|, \ s \in \mathcal{U}_0, \text{ for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}.
\end{equation}
(3.152)
Step 2 (The estimates for $I_4, I_5, n = 1$): With (3.144)-(3.147), (3.150), and (3.152), we can adapt argument in Lemma 3.4 and 3.7 to derive estimates for $I_4$ and $I_5$. For illustration, a residue theorem implies
\begin{equation}
I_5 = -\frac{\theta(\bar{r} - 1)}{2\pi i} \int_0^\pi d\beta \int_{\Gamma_{50}} J_3 \int_{\Gamma_{50}} \Re(-i\varphi\frac{s_0 e^{i\tau_0}}{\sigma_0}, \beta, X) \leq -\frac{1}{C} |\sin \beta| s_0, \ s \in \mathcal{U}_0, \text{ for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}.
\end{equation}
(3.153)
where \( c_0 \) is defined by (3.145), \( \vartheta(\lambda) \) is defined by (3.154), \( \tilde{J}_< \) is defined by
\[
(3.154) \quad \tilde{J}_< = \begin{cases}
\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5, & \text{if } 0 < \beta < \epsilon_1/8, \\
0, & \text{if } -\epsilon_1/8 < \beta < 0,
\end{cases}
\]
and similar analysis as that in Lemma 3.3, one obtains
\[
(3.155) \quad \tilde{J}_< \text{ with } |\beta| > \epsilon_1/8,
\]
with \( \tilde{J}_j, 1 \leq j \leq 5 \), defined by (3.113) except that \( X_1X_2X_3 \neq 0 \) here; \( S_\geq = S_\geq(\beta, \lambda, X) \), and \( \Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda), j = 0, 1, 2, n = 1, 2, \) by
\[
(3.155) \quad \Gamma_{50} = \{ \vartheta_0 : \tilde{s} \in \tilde{\mathcal{O}}_0, \tau_0 = \tau_{0,1} \},
\]
and \( \alpha_j, \tau_j, \tau_j, \tilde{\mathcal{O}}_j, \) defined by (3.128)-(3.130).

Applying (3.9), (3.141), (3.145) (in particular for estimating \( \Gamma_{5n, in} \)), (3.140), (3.147), (3.155), and similar analysis as that in Lemma 3.3 one obtains
\[
(3.156) \quad C \leq |f|_{L^\infty(D_{n1})}.
\]

On \( \Gamma_{50} \), it suffices to consider \( |\beta| \leq \epsilon_1/8 \). Applying (3.144), (3.150), (3.152), and (3.155), we can adapt argument in Lemma 3.4 and 3.7 one obtains
\[
(3.157) \quad | \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta[\partial_\beta \ln(1 - \gamma|\beta|)]\theta(\epsilon_1/8 - |\beta|) \times \int_{\Gamma_{50}} (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5) |_{L^\infty(D_{n1})} \leq C \epsilon_0 |f|_{L^\infty(D_{n1})}.
\]
Thus, along with Lemma 3.2,

\[(3.158) \quad |I_5|_{C^\mu(D_{\kappa_1})} \leq C\varepsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\]

**Step 3 (The estimates for \(I_4, I_5, n > 1\))**: Estimates can be derived via the approach of the proof of Proposition 3.1 3.2. We skip details for simplicity.

\[\square\]

### 3.1.4. Summaries

From Lemma 3.2 Proposition 3.1 3.2 and Lemma 3.6 3.10, we have

**Theorem 3.1.** Suppose \(S = \{z_n, \kappa_j, D, s_c\} \) is d-admissible and \(\partial^j_x f \) are \(\lambda\)-holomorphic on \(D_{\kappa_j}\) for \(0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5\). Then, for \(\forall n \geq 0, 1 \leq j \leq M, \)

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^j (CT\varphi_{\kappa_j})|_{L^\infty} + \sum_{j=1}^M \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j CT\varphi_{\kappa_j}|_{C^\mu(D_{\kappa_j})} \leq (C\varepsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |E_{\kappa_j} \partial_x^j f|_{L^\infty}.
\]

### 3.2. Estimates for the Cauchy integral operator \(CT\)

One can adapt the approach in Section 3.1 to derive estimates of the CIO near \(z_n\) provided \(S = \{z_n, \kappa_j, D, s_c\} \) is d-admissible.

**Lemma 3.11. (Estimates near \(z_n\))** Suppose \(S = \{z_n, \kappa_j, D, s_c\} \) is a d-admissible scattering data, \(\phi(x, \lambda) = \frac{\phi_{z_n, \text{res}}(x)}{\lambda - z_n} + \phi_{z_n, D}(x, \lambda), \partial_x^j \phi_{z_n, \text{res}}, \partial_x^j \phi_{z_n, D} \in L^\infty \) for \(0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5\). Then

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j CT\varphi_{z_n}|_{L^\infty} + \sum_{j=1}^M \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j CT\varphi_{z_n}|_{C^\mu(D_{\kappa_j})} \leq C\varepsilon_0 \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j \phi_{z_n, \text{res}}|_{L^\infty} + |\partial_x^j \phi_{z_n, D}|_{L^\infty(D_{\kappa_j})}.
\]

**Proof.** Proofs for \(\partial_x^j CT\varphi_{z_n}\) are identical. We only prove \(CT\varphi_{z_n}\) for simplicity. From (2.6), (2.12), and (3.2), decompose

\[
CT\varphi_{z_n, D} = \frac{1}{2\pi i} \int_{D_{\kappa_n, D}} \frac{\text{sgn}(\beta)h_n(\tilde{\beta}, \beta) e^{-i\varphi(\tilde{\beta}, \beta, X)} \phi_{z_n, \text{res}}(X) \overline{d\zeta} \wedge d\bar{\zeta} + C\lambda E_{z_n, D} \phi_{z_n, D}}{(\zeta - \lambda)(\zeta - z_n)}
\]

\[
\equiv I_1 + I_2 + I_3 + I_4 + I_5,
\]

where

\[
I_1 = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{r} < 2} \frac{\text{sgn}(\beta)h_n(\tilde{\beta}, \beta) \phi_{z_n, \text{res}}(X) \overline{d\zeta} \wedge d\bar{\zeta}}{(\zeta - \lambda)(\zeta - z_n)}
\]

\[
I_2 = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{r} < 2} \frac{\text{sgn}(\beta)h_n(\tilde{\beta}, \beta) e^{-i\varphi(\tilde{\beta}, \beta, X)} - 1 \phi_{z_n, \text{res}}(X) \overline{d\zeta} \wedge d\bar{\zeta}}{(\zeta - \lambda)(\zeta - z_n)}
\]

\[
I_3 = C\lambda E_{z_n, D} \phi_{z_n, D},
\]

\[
I_4 = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{2 < \tilde{r} < \tilde{\delta}} \frac{\text{sgn}(\beta)h_n(\tilde{\beta}, \beta) e^{-i\varphi(\tilde{\beta}, \beta, X)} \phi_{z_n, \text{res}}(X) \overline{d\zeta} \wedge d\bar{\zeta}}{(\zeta - \lambda)(\zeta - z_n)}
\]

\[
I_5 = -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{\tilde{r} < \tilde{\delta}} \frac{\text{sgn}(\beta)h_n(\tilde{\beta}, \beta) e^{-i\varphi(\tilde{\beta}, \beta, X)} \phi_{z_n, \text{res}}(X) \overline{d\zeta} \wedge d\bar{\zeta}}{(\zeta - \lambda)(\zeta - z_n)}.
\]
From the admissible condition and a standard Hilbert transform theory [15],

$$|II_1|_{L^\infty}, |II_2|_{L^\infty} \leq C\epsilon_0|\phi_z, \text{res}|_{L^\infty}.$$  

Writing $h_n(\zeta) = h_n(z_n) + [h_n(\zeta) - h_n(z_n)]$ and adapting argument for $I_4, I_5$ in Subsection 3.1,

$$|II_4|_{L^\infty(D_{z_n})}, |II_5|_{L^\infty(D_{z_n})} \leq C\epsilon_0|\phi_z, \text{res}|_{L^\infty}.$$  

Thanks to Lemma 3.2,

$$|II_5|_{L^\infty(D_{z_n})}, |II_5|_{C^\nu(D_{k_j})} \leq C\epsilon_0|\phi_z, \text{res}|_{L^\infty},$$

$$|II_3|_{L^\infty}, |II_3|_{C^\nu(D_{k_j})} \leq C\epsilon_0|\phi_z, r|_{L^\infty(D_{z_n})}.$$  

\[\blacksquare\]

**Lemma 3.12. (Estimates near $\infty$)** If $S = \{z_n, \kappa_j, D, s_c\}$ is $d$-admissible, $(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j})\partial_x^\nu \phi \in L^\infty$ for $0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5$. Then

$$\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^\nu \phi|_{L^\infty} 
+ \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} \sum_{j=1}^M \partial_x^\nu \phi C(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi |_{C^\nu(D_{k_j})}
\leq C\epsilon_0 \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \partial_x^\nu \phi|_{L^\infty}.$$  

**Proof.** Thanks to $d$-admissibility,

$$|(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \partial_x^\nu \phi|_{L^\infty} 
+ \sum_{j=1}^M \partial_x^\nu \phi C(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi |_{C^\nu(D_{k_j})}
\leq C\epsilon_0 \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \partial_x^\nu \phi|_{L^\infty}.$$  

Together with $\partial_x^\nu \phi(\kappa) \in W, |E_{\kappa_j} \partial_x^\nu C E_{\kappa_j} f|_W = |\sum_{l=1}^N \sum_{l'=1}^{l'} C\partial_x^\nu T E_{\kappa_j}(\partial_x^\nu f)|_W$, proofs for $D(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi$ are identical. We only prove $CT(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi$ for simplicity.

Via the coordinates change,

$$2\pi i \xi = \zeta - \zeta, \quad 2\pi i \eta = \zeta - \zeta^2, \quad \frac{d\zeta \wedge d\zeta}{\zeta - \lambda} = \frac{-2\pi^2 \text{sgn}(\xi) d\xi d\eta}{p_\lambda(\xi, \eta)},$$

(3.160)  

$$p_\lambda(\xi, \eta) = (2\pi \xi)^2 - 4\pi i \lambda + 2\pi i \eta, \quad \Omega_\lambda = \{ (\xi, \eta) \in \mathbb{R}^2 : |p_\lambda(\xi, \eta)| < 1 \}, \quad \left| \frac{1}{p_\lambda} \right|_{L^1(\Omega_\lambda, d\xi d\eta)} \leq C, \quad \left| \frac{1}{p_\lambda} \right|_{L^2(\Omega_\lambda, d\xi d\eta)} \leq C.$$  

[11] Lemma 6.31, we have

$$|CT(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi|_{L^\infty(D_{z_n})} 
\leq C |(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi|_{L^\infty} \int d\xi d\eta \int_\Gamma (1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) |s_c(\xi)| |L^\infty(D_{k_j})| p_\lambda(\xi, \eta)$$

$$\leq C |(1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) \phi|_{L^\infty} \int_\Gamma (1 - \sum_{n=1}^N E_{z_n} - \sum_{j=1}^M E_{\kappa_j}) |s_c(\xi)| |L^2(d\xi d\eta)| \left| \frac{1}{p_\lambda} \right|_{L^2(\Omega_\lambda, d\xi d\eta)}.$$
+(1 - \sum_{n=1}^{N} E_{zn} - \sum_{j=1}^{M} E_{\kappa_j}) s_c |L\infty(d\xi dn) \right| \right) \left| \frac{1}{p_\lambda} \left| L^1(\Omega, d\xi dn) \right| \right) \\
\leq C\epsilon_0 (1 - \sum_{n=1}^{N} E_{zn} - \sum_{j=1}^{M} E_{\kappa_j}) \phi |L\infty \\

Besides, \( |CT(1 - \sum_{n=1}^{N} E_{zn} - \sum_{j=1}^{M} E_{\kappa_j}) \phi)|_{C^0(D_{\kappa_j})} \leq C\epsilon_0 (1 - \sum_{n=1}^{N} E_{zn} - \sum_{j=1}^{M} E_{\kappa_j}) \phi |L\infty \) can be proved applying Lemma 3.2, (5.12), and \( \partial_x^j \phi(k) \in W \).

We conclude Section 3 by the estimate of the Cauchy integral operator on \( W \).

**Theorem 3.2.** Suppose the scattering data \( S = \{ z_n, \kappa_j, D, s_c \} \) is \( d \)-admissible, \( \partial_x^j \phi \in W \), and \( \partial_x^j \phi \) are \( \lambda \)-holomorphic on \( D_{\kappa_j} \) for \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \). Then

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j (CT)^n \phi|_W \leq (C\epsilon_0)^n \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial_x^j \phi|_W.
\]

**4. Existence of the eigenfunction**

In this section, given a \( d \)-admissible scattering data \( S = \{ z_n, \kappa_j, D, s_c \} \), via the \( d \)-admissible condition, the \( D \)-symmetry, a Sato theory, and Theorem 3.2 we shall construct a recursive sequence

\[
\phi(k)(x, \lambda) \equiv 1 + \sum_{n=1}^{N} \frac{\psi_{zn, res}(x)}{\lambda - z_n} + CT \phi(k-1)(x, \lambda),
\]

\[
\phi(0)(x, \lambda) \equiv \chi_{zn, \kappa_j, A},
\]

such that \( \partial_x^j \phi(k) \in W \), \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \), satisfy the \( D \)-symmetry (1.15), and converge to the unique solution of (1.14).

The following lemma explains that the \( D \)-symmetry plays an important closed condition to solve the inverse problem. Besides, it justifies that, unlike the IST for the KdV equation and most \( 1 + 1 \)-dimensional integrable systems, residues and poles are no longer crucial to define the scattering data in our approach. Finally, \( L^\infty \) estimates of the the residues, along with Theorem 3.2 will induce \( L^\infty \)-stability of \( \text{Gr}(N, M) > 0 \) KP solitons eventually.

**Lemma 4.1.** Suppose \( S = \{ z_n, \kappa_j, D, s_c \} \) is a \( d \)-admissible scattering data, and \( \phi(k) \) are defined by (4.1). Then for \( k > 0 \), \( \phi(k) \) satisfies the \( D \)-symmetry iff

\[
\left( \begin{array}{c}
\psi_{z_1, \text{res}}(k) \\
\vdots \\
\psi_{z_N, \text{res}}(k)
\end{array} \right) = -B^{-1} \tilde{A} \left( \begin{array}{c}
1 + C_{\kappa_1} T \phi(k-1) \\
\vdots \\
1 + C_{\kappa_M} T \phi(k-1)
\end{array} \right)
\]
Proof. By induction and applying Theorem 3.2 \( \bar{A}_z \in W \) implies \( \partial_z \psi_{z, \text{res}}^{(k)} \in W \), 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5. Next, write the \( D \)-symmetry and the evaluation at \( z_j \) of \( \phi^{(k)} \) as a linear system for \( M + N \) variables \( \{ \phi^{(k)}(x, z_j), \psi_{z, \text{res}}^{(k)}(x) \} \):

\[
\begin{align*}
A = & \begin{pmatrix}
\kappa_1 e^{\theta_1} & \cdots & 0 & D_{N+1,1} e^{\theta_{N+1}} & \cdots & D_{M,1} e^{\theta_M} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \kappa_N e^{\theta_N} & D_{N+1,N} e^{\theta_{N+1}} & \cdots & D_{M,N} e^{\theta_M} \\
-1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \\
B = & \begin{pmatrix}
\frac{1}{\kappa_1} & \cdots & \frac{1}{\kappa_N} & \cdots & \frac{1}{\kappa_{M-2}^N} & \frac{1}{\kappa_{M-1}^N} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & \cdots & \frac{1}{\kappa_{M-2}^N} & 1 \\
\end{pmatrix}, \\
(4.3)
\end{align*}
\]

and \( e^{\theta_j} = e^{\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3} \). Moreover, for \( \phi^{(k)} \) and \( \psi_{z, \text{res}}^{(k)} \) defined by (4.1) and (4.2),

\[
\begin{align*}
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial^l_x \psi_{z, \text{res}}^{(k)}|_{L^\infty} & \leq C(1 + \epsilon_0) \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial^l_x \phi^{(k-1)}|_{L^\infty}, \\
(4.4) \\
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial^l_x [\psi_{z, \text{res}}^{(k)} - \psi_{z, \text{res}}^{(k-1)}]|_{L^\infty} & \leq C\epsilon_0^k, \\
(4.5) \\
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5} |\partial^l_x [\psi_{z, \text{res}}^{(k)} - \bar{A} z_{\text{res}}]|_{L^\infty} & \leq C\epsilon_0. \\
(4.6)
\end{align*}
\]
Solving $\phi^{(k)}(x, \kappa_j^+)$ in terms of $\psi^{(k)}_{z_n, \text{res}}(x)$ and plugging the outcomes into (4.17) again yield

$$
(4.8) \quad B \begin{pmatrix} 
\psi^{(k)}_{z_1, \text{res}}(x) \\
\vdots \\
\psi^{(k)}_{z_N, \text{res}}(x)
\end{pmatrix} = -\tilde{A} \begin{pmatrix}
1 + C_{\kappa_1^{-1}} T\phi^{(k-1)} \\
\vdots \\
\vdots \\
1 + C_{\kappa_M^{-1}} T\phi^{(k-1)}
\end{pmatrix},
$$

with $B$ and $\tilde{A}$ defined by (4.3). By the $d$-admissible condition, the system (4.7) is just determined and is equivalent to (4.2).

To prove (4.4), firstly, the $d$-admissible condition implies that

$$
(4.9) \quad D^d = \text{diag}(\kappa_1^N, \ldots, \kappa_M^N) (A')^T, \quad \text{for } A' \in \text{Gr}(N, M)_{>0}.
$$

Denote $\tilde{\chi}'(x, \lambda) = \tilde{\chi}_{z_n, \kappa_j, A'}(x, \lambda)$ as the normalized Sato eigenfunction with data $\kappa_1, \ldots, \kappa_M, A'$. It satisfies the system consisting of the CIE and the $D$-symmetry

$$
\chi'(x, \lambda) = 1 + \sum_{n=1}^{N} \frac{\tilde{\chi}_{z_n, \text{res}}(x)}{\lambda - z_n},
$$

$$
(4.10) \quad (e^{\kappa_1^1 x_1 + \kappa_2^1 x_2 + \kappa_3^1 x_3} \chi'(x, \kappa_1), \ldots, e^{\kappa_M^1 x_1 + \kappa_2^1 x_2 + \kappa_3^1 x_3} \chi'(x, \kappa_M)) D = 0,
$$

and, from the $d$-admissible condition, the Sato theory, (1.5), (2.1), \forall k,

$$
\left| \tilde{\chi}'_{z_n, \text{res}}(x) - \tilde{\chi}_{z_n, \text{res}}(x) \right|_{C^k} \leq C_k \epsilon_0.
$$

Therefore, using $D$-symmetry and evaluating $\chi'$ at $\kappa_j$ and following the above argument, yield

$$
(4.13) \quad \begin{pmatrix}
\tilde{\chi}'_{z_1, \text{res}}(x) \\
\vdots \\
\tilde{\chi}'_{z_N, \text{res}}(x)
\end{pmatrix} = -B^{-1} \tilde{A} \begin{pmatrix}
1 \\
\vdots \\
\vdots \\
1
\end{pmatrix},
$$

with $B$ and $\tilde{A}$ defined by (4.3). Let $E_j = e^{\theta_j}$ and write

$$
(4.14) \quad A' = D \text{diag}(E_1, \ldots, E_M) = (D_L \ D_R) \text{ diag}(E_1, \ldots, E_M),
$$

$$
B = (D_L \ D_R) \text{ diag}(E_1, \ldots, E_M)
$$

$$
A' = (A'_L \ A'_R)
$$
where $D_L, A'_L$ are $N \times N$ matrices, $D_R, A'_R$ are $N \times (M-N)$ matrices. Then

$$
(4.15) \quad D_L = A'_L = \begin{pmatrix} \kappa_1^N & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_N^N \end{pmatrix},
$$

$$
D_R = \begin{pmatrix} \kappa_{N+1}^N \Pi_{n \neq 1}(\kappa_{n+1} - z_n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_M \Pi_{n \neq 1}(\kappa_M - z_n) \end{pmatrix} A'_L^{-1} A'_R
$$

From the Sato theory, (1.5), (2.1), (2.16), (4.13)-(4.15), rules for row and column operations of determinants, and matching the coefficients of $E_1 \times \cdots \times E_N$,

$$
B^{-1} = \frac{1}{\tau'(x)} \begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{NN} \end{pmatrix}
$$

$$
(4.16) \quad b_{kl} = \sum_{J(kl) = (j_1^{kl}, \cdots, j_{N-1}^{kl})} \Delta_{J(kl)} E_{J(kl)}(x), \quad 1 \leq j_1^{kl} < \cdots < j_{N-1}^{kl} \leq M,
$$

$\tau'(x)$ is the tau function with data $\kappa_j, A'$,

$$
|\Delta_{J(kl)}| = |\Delta_{J(kl)}(z_n, \kappa_j, A')| < C.
$$

Moreover,

$$
(4.17) \quad \tau'(x) \tilde{\chi}_{2n, \text{res}}(x)
$$

$$
= \text{the } h\text{-row of } \begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{NN} \end{pmatrix} \begin{pmatrix} \kappa_1^N E_1 + \cdots + D_{N+1,N} E_{N+1} + \cdots + D_{M,M} E_M \\ \vdots \\ \kappa_N^N E_1 + \cdots + D_{N+1,N} E_{N+1} + \cdots + D_{M,M} E_M \end{pmatrix}
$$

$$
= (\kappa_1^N E_1 + \cdots + D_{N+1,N} E_{N+1} + \cdots + D_{M,M} E_M) \sum_{|J(h1)| = N-1} \Delta_{J(h1)} E_{J(h1)}(x)
$$

$$
+ \cdots + (\kappa_N^N E_1 + \cdots + D_{N+1,N} E_{N+1} + \cdots + D_{M,M} E_M) \sum_{|J(hN)| = N-1} \Delta_{J(hN)} E_{J(hN)}(x)
$$

$$
\equiv (\tilde{a}_{11} E_1 + \cdots + \tilde{a}_{1M} E_M) \sum_{|J(h1)| < N} \Delta_{J(h1)} E_{J(h1)}(x)
$$

$$
+ \cdots + (\tilde{a}_{N1} E_1 + \cdots + \tilde{a}_{NM} E_M) \sum_{|J(hN)| = N-1} \Delta_{J(hN)} E_{J(hN)}(x),
$$

and

$$
0 = \tilde{a}_{1k} E_k \sum_{k \in J(h1), |J(h1)| = N-1} \Delta_{J(h1)} E_{J(h1)}(x) + \cdots
$$

$$
(4.18) \quad + \tilde{a}_{Nk} E_k \sum_{k \in J(hN), |J(hN)| = N-1} \Delta_{J(hN)} E_{J(hN)}(x).
$$
Using (4.2), (4.14)–(4.18), and multilinearity,
\[ \tau'(x)\psi^{(k)}_{\text{res}}(x) = \tau'(x)\chi_{\text{res}} \]
\[ + \text{the } h\text{-row of } \begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{NN} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11}E_{11}C_{\kappa_1}^{+}T\phi^{(k-1)} + \cdots + \tilde{a}_{1M}E_{1M}C_{\kappa_1}^{+}T\phi^{(k-1)} \\ \vdots \\ \tilde{a}_{N1}E_{11}C_{\kappa_1}^{+}T\phi^{(k-1)} + \cdots + \tilde{a}_{NM}E_{1M}C_{\kappa_1}^{+}T\phi^{(k-1)} \end{pmatrix} \]
\[ = \sum_{|J(h)|=N} \Delta J(h)E_{J(h)}(x), \]
with
\[ \sum_{0\leq t_1+2t_2+3t_3\leq d+5} |\partial_x^j \Delta J(h)| < C(1 + \sum_{j=1}^{M} \sum_{0\leq t_1+2t_2+3t_3\leq d+5} |\partial_x^j C_{\kappa_j}^{+}T\phi^{(k-1)}|). \]
Along with the totally positive condition of $A'$, yield
\[ \sum_{0\leq t_1+2t_2+3t_3\leq d+5} |\partial_x^j \psi^{(k)}_{\text{res}}(x)| \leq C(1 + \sum_{j=1}^{M} \sum_{0\leq t_1+2t_2+3t_3\leq d+5} |\partial_x^j C_{\kappa_j}^{+}T\phi^{(k-1)}|), \]
and (4.4)–(4.6) follow from Theorem 3.2, the admissible condition, and (4.12).

Illustrating by the simplest Gr($1,2$)$_{>0}$ KP soliton, we verify the $L^\infty$-estimate (4.6).

**Example 4.1.** Suppose $\{0, \kappa_1, \kappa_2, D, s_c\}$ is a $d$-admissible scattering data
\[ (4.19) \quad D^\delta = \begin{pmatrix} \kappa_1 \\ \kappa_2 a \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} D_{11}^\delta \\ D_{21}^\delta \end{pmatrix} = D^\delta, \quad D = \begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_1 \frac{D_{21}}{D_{11}} \end{pmatrix}, \]
\[ (4.20) \quad u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} sech^2 \theta_1(x) - \theta_2(x) - \ln a, \quad \theta_j(x) = \kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3. \]
Hence residue of the iteration sequence defined by (4.1) and (4.2) reads
\[ (4.21) \quad \psi^{(k)}_{0,\text{res}}(x) = \frac{D_{11}e^{\theta_1} + D_{21}e^{\theta_2}}{D_{11}e^{\theta_1} + D_{21}e^{\theta_2}} - \frac{D_{11}e^{\theta_1}C_{\kappa_1}^{+}T\phi^{(k-1)} + D_{21}e^{\theta_2}C_{\kappa_2}^{+}T\phi^{(k-1)}}{D_{11}e^{\theta_1} + D_{21}e^{\theta_2}}. \]
In particular, if $s_c \equiv 0$, it reduces to
\[ (4.22) \quad \tilde{\chi}_{0,\text{res}}(x) = -\frac{\kappa_1 e^{\theta_1} + \kappa_2 ae^{\theta_2}}{e^{\theta_1} + ae^{\theta_2}} \]
which also can be seen from the Sato theory $\tilde{\chi} = \frac{(1 - \kappa_1)e^{\theta_1} + (1 - \kappa_2)ae^{\theta_2}}{e^{\theta_1} + ae^{\theta_2}}$ directly.

In view of (4.21), (4.22), applying the $d$-admissible condition, and Theorem 3.2, we obtain
\[ |D - D^\delta|_{L^\infty} < \epsilon_0 \quad \text{and} \quad \sum_{0\leq t_1+2t_2+3t_3\leq d+5} |\partial_x^j \left[ \psi^{(k)}_{0,\text{res}} - \tilde{\chi}_{0,\text{res}} \right] |_{L^\infty} \leq C\epsilon_0. \]
We are ready to prove (1.14)-(1.16) of Theorem 1.2.

**Theorem 4.1.** Given a d-admissible scattering data \( S = \{ z_n, \kappa_j, D, s_c \} \), there exists uniquely \( m, \partial_x m \in W, 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \), satisfying (1.14)-(1.16). In particular, \( m \) can be constructed via the iteration sequence (4.1), (4.2), and, for \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \),

\[
\left( \partial_x^d \phi^{(k)} \right)_{z_n, \text{res}} = \partial_x^d \phi^{(k)}_{z_n, \text{res}}, \quad |\partial_x^d \phi^{(k)} - \partial_x^d \phi^{(k-1)}|_W \leq (C \epsilon_0)^k,
\]

(4.24)

\[
\lim_{k \to \infty} \partial_x^d \phi^{(k)} = \partial_x^d m \in W.
\]

**Proof.** Stipulating residues \( \psi_{z_n, \text{res}}^{(k)}(x) \) in (4.1) by (4.2), we construct the iteration sequence \( \{ \phi^{(k)} \} \) satisfying the \( D \)-symmetry for \( k > 0 \). Applying Theorem 3.2 and Lemma 4.1 we have \( \phi^{(k)} \in W \) and (1.14), (1.15).

Suppose \( m_1, m_2 \in W \) and satisfy (1.14) and (1.15). Hence

\[
m_1(x, \lambda) - m_2(x, \lambda) = \sum_{n=1}^N \frac{m_{1, z_n, \text{res}}(x) - m_{2, z_n, \text{res}}(x)}{\lambda - z_n} + CT(m_1 - m_2),
\]

(4.25)

\[
m_{1, \text{res}}(x) - m_{2, \text{res}}(x) = -B^{-1} \bar{A}
\]

Using Lemma 4.1 and Theorem 3.2

\[
|m_1 - m_2|_W \leq C \epsilon_0 |m_1 - m_2|_W.
\]

Thanks to \( \epsilon_0 \ll 1 \), \( m_1(x, \lambda) \equiv m_2(x, \lambda) \).

To prove (1.16) and (4.24), we use Theorem 3.2 the iteration sequence, and an induction procedure to bootstrap analytic properties of eigenfunctions. Proofs for \( \partial_x^d m \), \( 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \), are identical. We only prove \( \partial_x^d m \) for simplicity.

Taking \( x_1 \)-derivatives on both sides of the iteration sequence (4.1), formally

(4.26)

\[
\partial_{x_1} \phi^{(k)} = \sum_{n=1}^N \frac{\partial_{x_1} \psi_{z_n, \text{res}}^{(k)}}{\lambda - z_n} + CT(\overline{\zeta} - \zeta)\phi^{(k-1)} + CT \partial_{x_1} \phi^{(k-1)}.
\]

Using \( \partial_{x_1} \phi^{(0)} \in W \), the \( d \)-admissible condition, and Theorem 3.2

(4.27)

\[
|CT \partial_{x_1} \phi^{(0)}|_W \leq C \epsilon_0 |\partial_{x_1} \phi^{(0)}|_W, \quad |CT(\overline{\zeta} - \zeta)\phi^{(0)}|_W \leq C \epsilon_0 |\phi^{(0)}|_W.
\]

Adapting argument of Lemma 4.1 and applying (4.27), we have

\[
|\partial_{x_1} \psi_{z_n, \text{res}}^{(1)}|_{L^\infty} \leq C(1 + |CT(\overline{\zeta} - \zeta)\phi^{(0)}|_{L^\infty} + |CT \partial_{x_1} \phi^{(0)}|_{L^\infty}) < \infty.
\]
Consequently, (4.26) is valid for $k' = 1$,

$$\partial_{x_1}\phi^{(1)} \in W, \quad \left(\partial_{x_1}\phi^{(1)}\right)_{z_n,\text{res}} = \partial_{x_1}\psi^{(1)}_{z_n,\text{res}},$$

and

$$|\partial_{x_1}\phi^{(1)} - \partial_{x_1}\phi^{(0)}|_W \leq C(|CT\partial_{x_1}\phi^{(0)}|_W + |CT(\bar{\zeta} - \zeta)\phi^{(0)}|_W) \leq C\epsilon_0(|\phi^{(0)}|_W + |\partial_{x_1}\phi^{(0)}|_W).$$

Inductively, assume that (4.26) is valid for all $k' \leq k$,

$$(4.28) \quad |\partial_{x_1}\phi^{(k')} - \partial_{x_1}\phi^{(k'-1)}|_W \leq (C\epsilon_0)^{k'}(|\phi^{(0)}|_W + |\partial_{x_1}\phi^{(0)}|_W).$$

Together with the $d$-admissible condition, Theorem 3.2 and argument for proving Lemma 4.1 yields

$$|CT\partial_{x_1}\phi^{(k)}|_W \leq (C\epsilon_0)^{k+1}, \quad |CT(\bar{\zeta} - \zeta)\phi^{(k)}|_W \leq (C\epsilon_0)^{k+1},$$

and

$$|\partial_{x_1}\psi_{z_n,\text{res}}^{(k)} - \partial_{x_1}\psi_{z_n,\text{res}}^{(k)}|_{L^\infty} \leq C|CT(\bar{\zeta} - \zeta)(\phi^{(k)} - \phi^{(k-1)})|_W + |CT(\partial_{x_1}\phi^{(k)} - \partial_{x_1}\phi^{(k-1)})|_W \leq C\epsilon_0|\phi^{(k)} - \phi^{(k-1)}|_W + C\epsilon_0|\partial_{x_1}\phi^{(k)} - \partial_{x_1}\phi^{(k-1)}|_W \leq (C\epsilon_0)^{k+1}.$$  

As a result, (4.26) is valid for $k' = k + 1$ and

$$(4.29) \quad \partial_{x_1}\phi^{(k+1)} \in W, \quad \left(\partial_{x_1}\phi^{(k+1)}\right)_{z_n,\text{res}} = \partial_{x_1}\psi_{z_n,\text{res}}^{(k+1)}.$$  

Combining the $d$-admissible condition, (4.26), (4.28), (4.29), Theorem 3.2 and argument for proving Lemma 4.1 we obtain

$$|\partial_{x_1}\phi^{(k+1)} - \partial_{x_1}\phi^{(k)}|_W \leq (C\epsilon_0)^{k+1}.$$  

Hence (1.16) and (4.24) are proved by induction.  

Note that the choice of the initial map $\phi^{(0)}$ is not unique. For instance, constant functions also fulfill these criteria.

5. The inverse scattering transform

Provided the background is a vacuum, applying Fourier transform of the CIE and harmonic analysis, Wickerhauser derived a Lax pair for $\Phi(x, \lambda) = e^{\lambda x_1 + \lambda^2 x_2}m(x, \lambda)$ and Sobolev estimates of $u(x)$ [11, 40]. Such a framework for perturbed Gr$(N, M)_{>0}$ KP solitons is no longer efficient due to singular structures of the continuous scattering data. We shall rely on the iteration procedure (4.1), (4.2) to prove the Lax equation for $m(x, \lambda)$, construct the inverse scattering transform, and prove the continuity in this section.

Let

$$-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda\partial_{x_1} = -\nabla_2 + \nabla_1^2, \quad \nabla_1 = \partial_{x_1} + \lambda, \quad \nabla_2 = \partial_{x_2} + \lambda^2,$$

(5.1)
Moreover, 
\begin{equation}
(5.5) \quad \phi^{(k)} = 1 + J \phi^{(k)} + CT \phi^{(k-1)}, \quad J \phi^{(k)} = \sum_{n=1}^{N} \frac{\psi_{zn, \text{res}}^{(k)}(x)}{\lambda - z_n}.
\end{equation}

Observe formally
\begin{equation}
(5.6) \quad (-\nabla_2 + \nabla_1^2) \phi^{(k)} = [-\nabla_2 + \nabla_1^2, J] \phi^{(k)} + [-\nabla_2 + \nabla_1^2, CT] \phi^{(k-1)} + J(-\nabla_2 + \nabla_1^2) \phi^{(k)} + CT(-\nabla_2 + \nabla_1^2) \phi^{(k-1)}.
\end{equation}

Note $|\nabla_j, T| = 0$ for $j = 1, 2$, hence
\begin{equation}
(5.7) \quad \left[-\nabla_2 + \nabla_1^2, CT\right] \phi^{(k-1)} = \left[-\nabla_2 + \nabla_1^2, C\right] \phi^{(k-1)} = 2 \left[\lambda, C\right] \partial_{x_1} \left(T\phi^{(k-1)}\right)
\end{equation}
holds, and, in view of (4.21),
\begin{equation}
(5.8) \quad \left[-\nabla_2 + \nabla_1^2, J\right] \phi^{(k)} = \left[-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1}, J\right] \phi^{(k)}
\end{equation}

To carry out rigorous analysis of the above computation and investigate the asymptotics of (5.3)–(5.5) for $d$-admissible scattering data, we provide the following two lemmas.

**Lemma 5.1.** Given a $d$-admissible scattering data $S = \{z_n, \kappa_j, D, s_c\}$, let $m$ be the unique solution to (1.14), (1.15) which can be constructed via the iteration sequence (1.1), (1.2). Then $[-\nabla_2 + \nabla_1^2, J] \phi^{(k)}$ is independent of $\lambda$ and bounded, $J(-\nabla_2 + \nabla_1^2) \phi^{(k)} \in W$ satisfying
\begin{align}
(5.9) \quad [-\nabla_2 + \nabla_1^2, J] \phi^{(k)} &\to [-\nabla_2 + \nabla_1^2, J] m = 2\partial_{x_1} \sum_{n=1}^{M} m_{zn, \text{res}}(x) \quad \text{in } L^{\infty}, \\
(5.10) \quad J(-\nabla_2 + \nabla_1^2) \phi^{(k)} &\to J(-\nabla_2 + \nabla_1^2) m \quad \text{in } W.
\end{align}
Moreover,
\begin{equation}
(5.11) \quad \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_{x_{l_1}}^{l_1} \left([-\nabla_2 + \nabla_1^2, J] \phi^{(k)} + u_0\right) |_{L^{\infty}} \leq C \epsilon_0.
\end{equation}

**Proof.** Follows from (5.5) and Theorem 4.1.\hfill \qed
Lemma 5.2. If $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$ is $d$-admissible then $[-\nabla_2 + \nabla_1^2, CT] \phi^{(k-1)}$ is independent of $\lambda$, bounded, and

$$[-\nabla_2 + \nabla_1^2, CT] \phi^{(k-1)} - [-\nabla_2 + \nabla_1^2, CT] \chi_{z_n, \kappa_j, \lambda} \leq (C\epsilon_0)^k. \tag{5.9}$$

Proof. In view of (5.3), write formally

$$\partial_{x_1} \iint T \phi^{(k-1)} d\zeta \wedge d\zeta = \iint T(\zeta - \zeta) \phi^{(k-1)} d\zeta \wedge d\zeta + \iint T \partial_{x_1} \phi^{(k-1)} d\zeta \wedge d\zeta. \tag{5.10}$$

Since that singularities of $T$ and $\phi^{(k)}$ are $\kappa_j$ and $z_n$ respectively which are disjoint. Via Lemma 3.2,

$$\iint T(\zeta - \zeta) (\phi^{(k-1)} - \phi^{(k-2)}) d\zeta \wedge d\zeta | \leq \iint \left(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j} \right) T(\zeta - \zeta) (\phi^{(k-1)} - \phi^{(k-2)}) d\zeta \wedge d\zeta | \leq (C\epsilon_0)^k + \iint \left(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j} \right) T(\zeta - \zeta) (\phi^{(k-1)} - \phi^{(k-2)}) d\zeta \wedge d\zeta | \leq (C\epsilon_0)^k. \tag{5.11}$$

The $d$-admissible condition assures

$$1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j} \sum_{|l| \leq 3} ||\mathbf{x} - \lambda|^l_1 + \mathbf{x}^2 - \lambda^2|^l_2 | s_c(\lambda) \leq (C\epsilon_0)^k. \tag{5.12}$$

Together with $\phi^{(k)} \in W$, the change of variables (3.160), and Theorem 3.2 yields

$$\iint (1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j}) T(\zeta - \zeta) (\phi^{(k-1)} - \phi^{(k-2)}) d\zeta \wedge d\zeta | \leq C \iint (1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j}) |s_c| \times |\zeta| \times (\phi^{(k-1)} - \phi^{(k-2)}) | \frac{1}{|\zeta|} d\zeta | \leq C (C\epsilon_0)^{k-1} \iint (1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j}) |s_c| d\zeta | \leq C (C\epsilon_0)^{k}. \tag{5.13}$$

Consequently,

$$\iint T(\zeta - \zeta) (\phi^{(k-1)} - \phi^{(k-2)}) d\zeta \wedge d\zeta \leq (C\epsilon_0)^k. \tag{5.14}$$

Applying (3.160), (4.1), (4.24),

$$\iint (1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j}) \sum_{|l| \leq 1} ||\mathbf{x} - \lambda||_{L^2} \leq (C\epsilon_0)^k \tag{5.15}$$

assured by the $d$-admissible condition, and Theorem 3.2

$$\iint T[\partial_{x_1} \phi^{(k-1)} - \partial_{x_1} \phi^{(k-2)}] d\zeta \wedge d\zeta \leq \iint s_c(\zeta) \psi^{(k-1)}_{\zeta, n, \lambda} \partial_{x_1} \psi^{(k-2)}_{\zeta, n, \lambda} d\zeta \wedge d\zeta \tag{5.16}$$

$$\leq \iint s_c(\zeta) \psi^{(k-1)}_{\zeta, n, \lambda} \partial_{x_1} \psi^{(k-2)}_{\zeta, n, \lambda} \frac{\partial_{x_1} \psi^{(k-1)}_{\zeta, n, \lambda} \partial_{x_1} \psi^{(k-2)}_{\zeta, n, \lambda}}{\zeta - \zeta_n} d\zeta \wedge d\zeta \tag{5.17}$$
Combining with Theorem 4.1, we obtain

\begin{align}
(5.17) & \quad |\int \int s_c(\zeta)e^{(\zeta-\zeta_1)x_1+(\zeta^2-\zeta^2)x_2+(\zeta^3-\zeta^3)x_3\partial x_1}CT(\phi^{(k-2)}-\phi^{(k-3)})d\zeta \wedge d\zeta|_{L^\infty} \\
& \leq (C\epsilon_0)^k + |\int \int s_c(\zeta)e^{(\zeta-\zeta_1)x_1+(\zeta^2-\zeta^2)x_2+(\zeta^3-\zeta^3)x_3\partial x_1}CT(\phi^{(k-2)}-\phi^{(k-3)})d\zeta \wedge d\zeta|_{L^\infty}.
\end{align}

Moreover, by Fubini’s theorem, (5.12), (5.15), Theorem 3.2, and an induction,

\begin{align}
(5.18) & \quad |\int \int T[\partial x_1\phi^{(k-1)}-\partial x_1\phi^{(k-2)}]d\zeta \wedge d\zeta|_{L^\infty} \leq (C\epsilon_0)^k.
\end{align}

So the lemma follows from (5.4), (5.10), (5.14), and (5.18). □

Lemma 5.1 and 5.2 imply that (5.3)-(5.5) hold rigorously and converge to

\begin{align}
(5.19) & \quad (-\nabla_2 + \nabla_1^2)m = [-\nabla_2 + \nabla_1^2, J + CT]m + (J + CT)(-\nabla_2 + \nabla_1^2)m \in W \\
(5.20) & \quad -u(x) \equiv [-\nabla_2 + \nabla_1^2, J + CT]m = -\frac{i}{\pi} \int \int Tm d\zeta \wedge d\zeta + 2\partial x_1 \sum_{n=1}^{N} m_{z_n, res}(x)
\end{align}

with

\begin{align}
(5.21) & \quad \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial^l_x [u(x) - u_0(x)]|_{L^\infty} \leq C\epsilon_0.
\end{align}

Combining with Theorem 4.1, we obtain

\begin{align}
(5.22) & \quad (-\nabla_2 + \nabla_1^2)m = -(1 - J - CT)^{-1}u(x)1 = -u(x)(1 - J - CT)^{-1}1 = -u(x)m(x, \lambda).
\end{align}

We have proved (5.17) and (5.18) of Theorem 1.2.

**Theorem 5.1.** Given a d-admissible scattering data \( S = \{ z_n, \kappa_j, D, s_c \} \), let m be the solution to (1.14), (1.15) and \( u_0(x) \) is the Gr(N,M) KP soliton determined by \( D^0 \) of the admissible condition. Then

\[ (-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x)) m(x, \lambda) = 0, \]
with \( u : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \),

\[
(5.23) \quad u(x) \equiv -2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, res}(x) + i \frac{\partial}{\partial x_1} \int Tm \, d\zeta \wedge d\zeta,
\]

\[
(5.24) \quad \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_x^l [u(x) - u_0(x)]|_{L^\infty} \leq C \epsilon_0.
\]

**Proof.** Suffices to prove (1.19). First of all, using (5.23), (5.6), and (5.8),

\[
(5.25) \quad \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_x^l [u(x) - u_0(x)]|_{L^\infty}
\]

\[
\leq \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_x^l [-2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, res}(x) - u_0(x)]|_{L^\infty}
\]

\[
+ C \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d+4} |\partial_x^l \partial_{x_1} \int Tm \, d\zeta \wedge d\zeta|_{L^\infty}
\]

\[
\leq C \epsilon_0 + C \sum_{p_1 + q_1 = l_1 + 1, p_2 + q_2 = l_2, q_3 = l_3} \left| \int (1 - \sum_{j=1}^{M} E_{\kappa_j}) (\zeta - \zeta_1)^{p_2} (\zeta_2 - \zeta_3)^{p_3} C(\zeta_4) \right|
\]

\[
\times e^{i(\zeta_1^1 - \zeta_3) x_1 + (\zeta_2 - \zeta_3) x_2 + (\zeta_3^1 - \zeta_3^3) x_3} \partial_x^l m \, d\zeta \wedge d\zeta|_{L^\infty}.
\]

Besides, if

\[
(5.26) \quad \partial_x^l m \in W, \text{ for } 0 \leq q_1 + 2q_2 + 3q_3 \leq d + 5,
\]

\[
(5.27) \quad (1 - \sum_{j=1}^{M} E_{\kappa_j}) \sum_{0 \leq p_1 + 2p_2 + 3p_3 \leq d+4} \sum_{j=1}^{3} \left| \left( \lambda^{j}_1 - \lambda^j \right)^{p_j} C(\lambda) \right|_{L^1(|\lambda| \, d\lambda)} < C \epsilon_0,
\]

then, adapting the induction argument as that for (5.16)-(5.18), one derives

\[
(5.28) \quad \left| \int T\partial_x^l m \, d\zeta \wedge d\zeta \right|_{L^\infty} \leq C \epsilon_0.
\]

Using the change of variables (3.160), and plugging (5.26)-(5.28) into the right hand side of (5.25), we verify (1.19).

Note that (5.26) and (5.27) are implied by the \( d \)-admissible condition and Theorem 4.1.

\[\square\]

**Definition 5.1.** Given a \( d \)-admissible scattering data \( \{z_n, \kappa_j, D, s_c\} \), we define the inverse scattering transform \( S^{-1} \) by (5.23) and

\[
S^{-1}(\{z_n, \kappa_j, D, s_c(\lambda)\}) = u(x).
\]

**Theorem 5.2.** The inverse scattering transform \( S^{-1} \) is continuous at each \( d \)-admissible scattering data with trivial continuous scattering data \( \{z_n, \kappa_j, D, 0\} \) in the sense of (5.24).

Illustrating by the simplest Gr(1,2) KP soliton, we verify the \( L^\infty \)-estimate (5.21) directly.
Example 5.1. Suppose \(\{0, \kappa_1, \kappa_2, D, s_c\}\) is a d-admissible scattering data

\[
(5.29) \quad D^p = \left( \begin{array}{c} \kappa_1 \\ \kappa_2 a \end{array} \right), \quad \tilde{D} = \left( \begin{array}{c} D_{11}^2 \\ D_{21}^2 \end{array} \right) = D^2, \quad D = \left( \begin{array}{c} D_{11} \\ \kappa_1 D_{11}^2 \end{array} \right),
\]

\[
(5.30) \quad u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1(x) - \theta_2(x) - \ln a}{2}, \quad \theta_j(x) = \kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3.
\]

In view of the d-admissible condition, \((1.21) - (1.23)\) in Example 4.1 and Lemma 5.2, \(u(x) = \mathcal{S}^{-1}\{0, \kappa_1, \kappa_2, D, s_c\}\) satisfies

\[
(5.31) \quad \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4} |\partial_x^l [u(x) + 2\partial_{x_1}\tilde{\chi}_{0, \text{res}}(x)]|_{L^\infty} \leq C\epsilon_0,
\]

\[
\leq C\epsilon_0 + \sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4} |\partial_x^l \left[ -2\partial_{x_1}\psi_{0, \text{res}}^{(k)}(x) + 2\partial_{x_1}\tilde{\chi}_{0, \text{res}}(x) \right]|_{L^\infty} \leq C\epsilon_0.
\]

Instead of computing \(\partial_{x_1}\tilde{\chi}_{0, \text{res}}\) directly, via the Sato theory, one has

\[
(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u_0(x))\tilde{\chi}(x, \lambda) = 0.
\]

Taking asymptotic, \(u_0(x) = -2\partial_{x_1}\tilde{\chi}_{0, \text{res}}(x)\). Along with \((5.31)\) and \((5.30)\), we obtain

\[
\sum_{0 \leq l_1 + 2l_2 + 3l_3 \leq d + 4} |\partial_x^l [u(x) - \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1(x) - \theta_2(x) - \ln a}{2}]|_{L^\infty} \leq C\epsilon_0.
\]

6. The Evolution

In this section, applying the IST established in previous sections, \([43, 44]\), and adapting argument from \([41, 2]\), we shall

- justify the evolution equation for \(u(x) = \mathcal{S}^{-1}\{\{z_n, \kappa_j, D, s_c(\lambda)\}\}\) is the KPII equation;
- solve the Cauchy problem for perturbed \(\text{Gr}(N, M) > 0\) KP solitons;
- prove a uniqueness theorem;
- derive an \(L^\infty\)-stability theorem of \(\text{Gr}(N, M) > 0\) KP solitons.

To work out the evolution equation for \(u = \mathcal{S}^{-1}\{\{z_n, \kappa_j, D, s_c(\lambda)\}\}\), for simplicity, we introduce the following terminology

\[
(6.1) \quad \partial_{x_j} \phi = \phi^*, \quad \overline{\phi} = \partial_{x_j} \phi \quad \text{if} \ \lambda \neq z_n, \quad \int \int \int dx_1 dx_2 dx_3 = \int dx,
\]

\[
\langle \phi, \psi \rangle = \frac{i}{\pi} \int \int \phi(x, \zeta)\psi(x, \zeta) d\zeta \wedge d\zeta, \quad \langle \langle \phi, \psi \rangle \rangle = \frac{i}{\pi} \int \int \int \phi(x, \zeta)\psi(x, \zeta) dx d\zeta \wedge d\zeta,
\]

\[
T \phi(x, \lambda) = \begin{cases} T \phi, & \lambda \neq z_n, \\
\phi_{z_n, \text{res}}, & \lambda = z_n, \end{cases} \quad \text{C} \phi(x, \lambda) = \sum_{n=1}^{N} \frac{\phi(x, z_n)}{\lambda - z_n} + C_\lambda \phi,
\]

\[
\partial_{x_j} \phi = \phi^*, \quad \overline{\phi} = \partial_{x_j} \phi \quad \text{if} \ \lambda \neq z_n, \quad \int \int \int dx_1 dx_2 dx_3 = \int dx.
\]
\[
\begin{pmatrix}
\phi_{z_1,\text{res}} \\
\vdots \\
\phi_{z_N,\text{res}}
\end{pmatrix} = -B^{-1} \tilde{A}
\begin{pmatrix}
1 + C_{\kappa_1} T\phi \\
\vdots \\
1 + C_{\kappa_M} T\phi
\end{pmatrix}
\]

with $\tilde{A}$, $B$ defined by (6.3). Consequently, one has

\[
m = (1 - CT)^{-1}, \quad C^* = 0,
\]

and

\[
\frac{\pi}{i} \partial_{\mu} \left( \int \int T m \, d{\zeta} \wedge d{\zeta} \right)^* = \partial_{\mu} \langle 1, T m \rangle^* = \partial_{\mu} \langle 1, T T^* m \rangle^*
\]

\[
= \partial_{\mu} \langle 1, (T^* - CT)^{-1} T T^* m \rangle
\]

\[
= \partial_{\mu} \langle 1, T^* m + T^* (1 - CT)^{-1} C T m \rangle
\]

\[
= \partial_{\mu} \langle 1, (1 - T T^*)^{-1} T^* m \rangle
\]

\[
= \partial_{\mu} \langle \tilde{m}, T^* m \rangle = \partial_{\mu} \langle \tilde{m}, T^* m \rangle,
\]

where

\[
\tilde{m} = (1 - C t T^t)^{-1} = (1 - CT)^{-1} = 1,
\]

\[
\tilde{T} = -T^t, \quad C^t = -C, \quad \tilde{T}\phi(x, \lambda) = \begin{cases} T^t \phi, & \lambda \neq z_n, \\
\phi_{z_n,\text{res}}, & \lambda = z_n,
\end{cases}
\]

and

\[
\begin{pmatrix}
\phi_{z_1,\text{res}} \\
\vdots \\
\phi_{z_N,\text{res}}
\end{pmatrix} = +B^{-1} \tilde{A}
\begin{pmatrix}
1 + (T^t C^t) (x, \kappa_1^+) \\
\vdots \\
1 + (T^t C^t) (x, \kappa_M^+)
\end{pmatrix}
\]

Here, the superscript $t$ denotes transposition with respect to the inner product on $\mathbb{R}^2 \times \mathbb{C}$ defined by $\langle \cdot, \cdot \rangle$ in (6.1). Here $\| (1 - \sum_{j=1}^M E_{\kappa_j}) \sum_{|I| \leq 7} [ |x - \lambda||^4 + |x| - \lambda_2^2 ||^2 ] s_c(\lambda) \|_{L^\infty} < \infty$, assured by the $d$-admissible condition, is used to make integrals in (6.4) well-defined.

From (2.12), evolutions of the continuous scattering operator $T$ having the form

\[
T^* = [T, \Phi], \quad \Phi = \text{multiplication by the function } \varphi = \lambda^3.
\]
Along with (6.1), (6.3)-(6.5), yields

\[(6.8) \quad \left(\frac{i}{\pi} \partial_{x_1} \int \int T_{m} d\zeta \wedge d\zeta \right)^{\ast} \]

\[= \partial_{x_1} \langle \tilde{m}, T\varphi m \rangle - \partial_{x_1} \langle \tilde{m}, \varphi T_{m} \rangle \]

\[= - \partial_{x_1} \langle \tilde{T}_{m}, \varphi m \rangle - \partial_{x} \langle \tilde{m}, \varphi T_{m} \rangle \]

\[= - \partial_{x_1} \langle \tilde{m}, \varphi m \rangle - \partial_{x} \langle \tilde{m}, \varphi \tilde{m} \rangle \]

\[= - \frac{i}{\pi} \partial_{x_1} \int \varphi \tilde{m} d\zeta \wedge d\zeta \]

\[= - \frac{i}{\pi} \partial_{x_1} \int \zeta \tilde{m} d\zeta \wedge d\zeta \]

Moreover, denoting \( \Omega_{\epsilon,r} = D_{0,r}/\bigcup_{n=1}^{N} D_{z_n,\epsilon} \), applying Stokes’ theorem and (6.3) for \(|\zeta - z_n| = \epsilon\), applying Stokes’ theorem and (6.8) for \(|\zeta - z_n| = r\), we obtain

\[(6.9) \quad \left(\frac{i}{\pi} \partial_{x_1} \int \int T_{m} d\zeta \wedge d\zeta \right)^{\ast} \]

\[= \frac{i}{\pi} \lim_{\epsilon \to 0, r \to \infty} \partial_{x_1} \left( \int \int_{\Omega_{\epsilon,r}} T_{m} d\zeta \wedge d\zeta \right)^{\ast} \]

\[= \frac{i}{\pi} \lim_{\epsilon \to 0, r \to \infty} \partial_{x_1} \left( \int \int_{\Omega_{\epsilon,r}} \tilde{m} d\zeta \wedge d\zeta \right)^{\ast} \]

\[= - \frac{i}{\pi} \sum_{n=1}^{N} \lim_{\epsilon \to 0} \partial_{x_1} \left( \oint_{|\zeta - z_n| = \epsilon} m(x, \zeta) d\zeta \right)^{\ast} - \lim_{r \to \infty} \frac{i}{\pi} \partial_{x_1} \oint_{|\zeta| = r} \frac{[m\tilde{m}]}{\zeta} d\zeta \]

\[= \left(2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, \text{res}}(x) \right)^{\ast} + 2\partial_{x_1} [m\tilde{m}]_{4}, \]

where, as \(|\lambda| \to \infty|\),

\[(6.10) \quad [m\tilde{m}]_{4}(x) = \sum_{j+k=4} M_{j}(x)\tilde{M}_{k}(x), \quad m(x, \lambda) \sim \sum_{j=0}^{\infty} \frac{M_{j}(x)}{\lambda_j}, \quad \tilde{m}(x, \lambda) \sim \sum_{j=0}^{\infty} \frac{\tilde{M}_{j}(x)}{\lambda_j}, \]

by using uniform integrability of the CIE (1.14) for \(|\lambda| > C\).

Lemma 6.1. Given a \(d\)-admissible scattering data \(\{z_n, \kappa_j, D, s_c\}\), one has the representation

\[(6.11) \quad u^{\ast}(x) = [S^{-1}(\{z_n, \kappa_j, D, s_c\})]^{\ast} = +2\partial_{x_1} [m\tilde{m}]_{4}, \]

where

\[(6.12) \quad (-\partial_{x_2} + \partial_{x_1}^{2} + 2\lambda \partial_{x_1} + u(x)) m = 0, \]

\[(\partial_{x_2} + \partial_{x_1}^{2} - 2\lambda \partial_{x_1} + u(x)) \tilde{m} = 0, \]

\[m = (1 - CT)^{-1}1, \quad m_{z_{n, \text{res}}}(x) = (T_{m})(x, z_n), \quad \text{and if } \lambda \neq z_n \text{ then } \partial_{x_1} \tilde{m} = T_{m}, \]

\[\tilde{m} = (1 - C\tilde{T})^{-1}1, \quad \tilde{m}_{z_{n, \text{res}}}(x) = (\tilde{T}_{m})(x, z_n), \quad \text{and if } \lambda \neq z_n \text{ then } \partial_{x_1} \tilde{m} = \tilde{T}_{m}. \]
Proof. Suffices to derive the adjoint Lax equation for $\tilde{m}$ in (6.12). Denote $p_{\lambda}(D) = -\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1}$. Note $p_{\lambda}(-D)$ is dual to $p_{\lambda}(D)$ with respect to the pairing $\langle \ , \ \rangle$. Note $p_{\lambda}(-D)$ and $T$ commute. Therefore

\begin{equation}
(6.13) \quad p_{\lambda}(-D)\tilde{m}
= p_{\lambda}(-D)(1 - CT)^{-1}1 = (1 - CT)^{-1}[p_{\lambda}(-D), C]\tilde{m}
= (1 - CT)^{-1}\partial_{x_1}[(\tilde{m}\lambda, 1) - \sum_{n=1}^{N} 2m_{z_n, \text{res}}]
= (1 - CT)^{-1}\partial_{x_1}[-(1, Tm) + \sum_{n=1}^{N} 2m_{z_n, \text{res}}]
= -(1 - CT)^{-1}u
= -u\tilde{m}.
\end{equation}

\[\square\]

We completed the proof of (1.20) in Theorem 1.2.

**Theorem 6.1.** If $\{z_n, \kappa_j, D, s_c\}$ is $d$-admissible then $u = S^{-1}(\{z_n, \kappa_j, D, s_c\}) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies the KPII equation

\begin{equation}
(6.14) \quad (-4u_{x_2} + u_{x_1}x_1x_1 + 6uu_{x_1})_{x_1} + 3u_{x_2}x_2 = 0,
\end{equation}

\begin{equation}
(6.15) \quad \sum_{0 \leq l + 2k + 3m \leq d + 4} |\partial_x^l [u(x) - u_0(x)]|_{L^\infty} \leq C\epsilon_0.
\end{equation}

Proof. By Theorem 5.1 it suffices to prove (6.14). From Lemma 6.1 - (6.10),

\begin{equation}
(6.16) \quad u^*(x) = 2\partial_{x_1} [m\tilde{m}]_4,
\end{equation}

and

\begin{equation}
(6.17) \quad M_0(x) = 1, \quad \partial_{x_1}M_{j+1}(x) = (\partial_{x_2} - \partial_{x_1}^2 - u(x)) M_j(x),
\end{equation}

\[\tilde{M}_0(x) = 1, \quad \partial_{x_1}\tilde{M}_{j+1}(x) = (\partial_{x_2} + \partial_{x_1}^2 + u(x)) \tilde{M}_j(x).\]

Therefore, since $|(1 - \sum_{j=1}^{M} E_{\kappa_j}) \sum_{||| \leq 8} \left( |x - \lambda|^{l_1} + |x - \lambda|^2|^{l_2} \right) s_c(\lambda)|_{L^\infty} < \infty$ (assured by the $d$-admissible condition),

\begin{equation}
\begin{aligned}
\quad u^*(x) \quad & = 2\partial_{x_1}M_4 + 2\partial_{x_1}\tilde{M}_4 + (2\partial_{x_1}\tilde{M_1}) M_3 + (2\partial_{x_1} M_1)\tilde{M}_3
\quad + \tilde{M}_1(2\partial_{x_1} M_3) + M_1(2\partial_{x_1} \tilde{M}_3) + \tilde{M}_2(2\partial_{x_1} M_2) + M_2(2\partial_{x_1} \tilde{M}_2)
\quad = -\frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u
\quad + \frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u
\quad - \frac{1}{8} (\partial_{x_1}^{-1} u) \times (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u
\end{aligned}
\end{equation}
\[-\frac{1}{8} (\partial_{x_1}^{-1} u) \times (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[-\frac{1}{8} (\partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u) \times (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[-\frac{1}{8} (\partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u) \times (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u. \]

That is,

\[(6.18) \quad u^*(x) = -\frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2 + u) \circ (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[\blacktriangleright + \frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 - u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[-\frac{1}{8} (\partial_{x_1}^{-1} x) \circ (\partial_{x_2} - \partial_{x_1}^2 - u) \circ (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[-\frac{1}{8} (\partial_{x_1}^{-1} x) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[-\frac{1}{4} \left((\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \right) \times (\partial_{x_2} - \partial_{x_1}^2 - u) \circ u \]

\[-\frac{1}{8} (\partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u) \times (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u. \]

Using \(\partial_{x_1}^{-1} u \circ \partial_{x_1}^{-1} u = \frac{1}{2} (\partial_{x_1}^{-1} u)^2\), and

\[\blacktriangleright - \frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2 + u) \circ (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[= \left(\frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ u \right) \times \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[+ \left(\frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ (\partial_{x_2} - \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} - \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \right) \circ \left(\partial_{x_2} - \partial_{x_1}^2 - u - \frac{1}{2} (\partial_{x_1}^{-1} u)^2 \right), \]

\[\blacktriangleright + \frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 - u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[= \left(\frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ u \right) \times \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[+ \left(\frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 - u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \right) \circ \left(\partial_{x_2} + \partial_{x_1}^2 + u + \frac{1}{2} (\partial_{x_1}^{-1} u)^2 \right), \]

\[\blacktriangleright - \frac{1}{8} (\partial_{x_2} - \partial_{x_1}^2 - u) \circ (\partial_{x_2} - \partial_{x_1}^2 - u) \circ \partial_{x_1}^{-1} u \]

\[= -\frac{1}{8} \left(\partial_{x_2} (\partial_{x_2} - \partial_{x_1}^2 + u) - 2u(\partial_{x_2} - \partial_{x_1}^2) - (u_{x_2} - u_{x_1} x_1) - 2u_{x_1} x_{1x_1} + u^2 \right) \circ \partial_{x_1}^{-1} u, \]

\[\blacktriangleright - \frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[= -\frac{1}{8} \left(\partial_{x_2} (\partial_{x_2} + \partial_{x_1}^2 + u) + 2u(\partial_{x_2} + \partial_{x_1}^2) + (u_{x_2} + u_{x_1} x_1 + 2u_{x_1} x_{1x_1} + u^2 \right) \circ \partial_{x_1}^{-1} u, \]

\[\blacktriangleright - \frac{1}{8} (\partial_{x_2} + \partial_{x_1}^2 + u) \circ (\partial_{x_2} + \partial_{x_1}^2 + u) \circ \partial_{x_1}^{-1} u \]

\[= -\frac{1}{8} \left(\partial_{x_2} (\partial_{x_2} + \partial_{x_1}^2 + u) + 2u(\partial_{x_2} + \partial_{x_1}^2) + (u_{x_2} + u_{x_1} x_1 + 2u_{x_1} x_{1x_1} + u^2 \right) \circ \partial_{x_1}^{-1} u, \]
we obtain

\begin{align}
(6.19) \quad u^*(x) \\
= & +\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} (\partial_x^2 + \partial_x^4 + 2u_x \partial_x + 2u_{xx}) \circ u \\
& + \frac{1}{8} (\partial_x^2 + \partial_x^4 + 2u_x \partial_x + 2u_{xx}) \circ (\partial_x^{-1} u)^2 \\
& + \frac{1}{4} (\partial_x^{-1} u) \times (\partial_x^2 + \partial_x^4 + 2u_x \partial_x + 2u_{xx} \partial_x + u_x + u^2) \circ \partial_x^{-1} u \\
& + \frac{1}{4} (\partial_x \partial_x^{-1} u - \partial_x u - u \partial_x^{-1} u) \times (\partial_x \partial_x^{-1} u + \partial_x u + u \partial_x^{-1} u) \\
= & + \frac{3}{4} u_{xx} + \frac{1}{4} (u_{xx} + 2u_x + 2u_{xx}) \\
& + \frac{1}{8} (\partial_x^2 \partial_x^{-1} u) \times (\partial_x^{-1} u) \times (\partial_x^2 \partial_x^{-1} u + 2(\partial_x \partial_x^{-1} u)^2 + 2u_{xx} \partial_x \partial_x^{-1} u + 8uu_x + 6u^2_x) \\
& + \frac{1}{4} (2u_x \partial_x^{-1} u + u_x \partial_x^{-1} u)^2 \\
& \qquad - \frac{1}{4} (\partial_x^{-1} u) \times (\partial_x^2 \partial_x^{-1} u + u_{xx} \partial_x + u_x \partial_x^{-1} u + u^2 \partial_x^{-1} u) \\
& + \frac{1}{4} (\partial_x \partial_x^{-1} u)^2 - u_x^2 - 2u_x u \partial_x^{-1} u - u^2 (\partial_x^{-1} u)^2 \\
= & + \frac{3}{4} u_{xx} + \frac{1}{4} (u_{xx} + 2u_x + 2u_{xx}) + \frac{6}{4} u_x^2 + \frac{2}{4} u_{xx} + u_x x u.
\end{align}

Therefore the KPII equation (6.14) is verified. □

**Lemma 6.2.** Given an initial data \( u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2) \) and \( z_0, 1 \leq n \leq N \), satisfying (1.8), then \( \partial_t^l m(x_1, x_2, \lambda) \in W, 0 \leq l_1 + 2l_2 + 3l_3 \leq d + 5 \), and \( S(u(x_1, x_2), z_0) \) is \( d \)-admissible with the analytic constraint

\[ \epsilon_0 \leq C \sum_{|l| \leq d+8} |\partial_x^l v_0|_{L^1 \cap L^\infty}. \]

**Proof.** See [33] (3.9),(3.10] [44] Theorem 2] for characterization of \( \partial_t^l m(x_1, x_2, \kappa^+) \) and the proof of \( \partial_t^l m(x_1, x_2, \lambda) \in W \). To prove the \( d \)-admissible condition, we adapt the argument of the proof of [44] Theorem 3] by Fourier analysis. □

We conclude the paper by solving the Cauchy problem with a uniqueness for perturbed \( \text{Gr}(N, M)_{>0} \) KP solitons. It yields Theorem [1.1]

**Theorem 6.2.** Given \( u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2) \) and \( z_1 = 0, \{ z_n, \kappa_j \} \) satisfying (1.8), then \( u(x) = S^{-1} \circ S(u(x_1, x_2), z_n) \) satisfies, \( u : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \),

\begin{align}
(6.20) \quad (-4u_x + u_{xx})_x + 6uju_x + 3u_{xx} = 0, \\
(6.21) \quad u(x_1, x_2, 0) = u(x_1, x_2),
\end{align}
Moreover, suppose \( \det(\frac{1}{\kappa_j - z_n^\pm})_{1 \leq j,n \leq N} \neq 0 \), with \( z_n^\pm = 0 \), \( \{z_n^\pm, \kappa_j\} \) distinct real. Then

\[
S^{-1} \circ S(u(x_1, x_2), z_n^+) = S^{-1} \circ S(u(x_1, x_2), z_n^-).
\]

Proof. The Cauchy problem (6.20), (6.21), and the \( L^\infty \)-stability (6.22) are derived by applying Lemma 6.2, 1.12, 1.13, Theorem 4.1 5.1 and 6.1.

Denote

\[
u^\pm = S^{-1} \circ S(u(x_1, x_2), z_n^+) = S^{-1}(\{z_n^\pm, \kappa_j, D^\pm, s_c^\pm\}),
\]

where

\[
(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u^\pm(x)) m^\pm(x, \lambda) = 0;
\]

\[
m^\pm = 1 + \sum_{n=1}^{N} \frac{m^\pm_{+n, res}(x)}{\lambda - z_n} + CT^\pm m^\pm,
\]

\[
(e^{\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3} m^\pm(x, \kappa_1^+), \cdots, e^{\kappa_M x_1 + \kappa_M x_2 + \kappa_3 x_3} m^\pm(x, \kappa_M^+)) D^\pm = 0;
\]

\[
T^\pm \phi(x, \lambda) = \phi_c^\pm(\lambda) e^{(\lambda - \lambda_j) x_1 + (\lambda - \lambda_j) x_2 + (\lambda - \lambda_j) x_3} \phi(x, \lambda),
\]

\[
s_c^\pm(\lambda) = \frac{\Pi_{2 \leq n \leq N}(\lambda - z_n^\pm) \text{sgn}(\lambda) \Pi_{2 \leq n \leq N}(\lambda - z_n^\pm)}{(\lambda - z_1^\pm)^{N-1}} \int e^{[\lambda - z_1^\pm] x_1 + (\lambda - \lambda_j) x_2}
\]

\[
\times \tilde{\chi}(x_1, x_2, 0, \lambda) v_0(x_1, x_2) m^\pm(x_1, x_2, \lambda) dx_1 dx_2,
\]

\[
D^\pm = \tilde{D}^\pm \times \begin{pmatrix} \tilde{D}_1^\pm & \cdots & \tilde{D}_n^\pm \\ \vdots & \ddots & \vdots \\ \tilde{D}_N^\pm & \cdots & \tilde{D}_N^\pm \end{pmatrix}^{-1} \text{diag}(\kappa_1^N, \cdots, \kappa_N^N),
\]

\[
\tilde{D}^\pm = \text{diag}\left(\frac{\Pi_{2 \leq n \leq N}(\kappa_1 - z_n^\pm)}{(\kappa_1 - z_1^\pm)^N}, \cdots, \frac{\Pi_{2 \leq n \leq N}(\kappa_M - z_n^\pm)}{(\kappa_M - z_1^\pm)^N}\right) D^\pm
\]

\[
D^\pm_{ji} = \left( D^\pm_{ji} + \sum_{l=1}^{M} \frac{c_{jl} D^\pm_{li}}{1 - c_{jj}} \right),
\]

\[
D^\pm = \text{diag}(\kappa_1^N, \cdots, \kappa_M^N) A^T,
\]

and \( c_{ij} = -\int \Psi_j(x_1, x_2, 0) v_0(x_1, x_2) \varphi_1(x_1, x_2, 0) dx_1 dx_2, \Psi_j(x), \varphi_1(x) \) are residue of the adjoint eigenfunction at \( \kappa_j \) and values of the Sato eigenfunction at \( \kappa_l \) [11] Theorem 2,

\[
(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x_1, x_2)) m^\pm(x_1, x_2, \lambda) = 0,
\]

\[
m^\pm(x_1, x_2, \lambda) \to \tilde{\chi}^\pm(x_1, x_2, 0, \lambda) = \frac{(\lambda - z_1^\pm)^{N-1}}{\Pi_{2 \leq n \leq N}(\lambda - z_n^\pm)} \chi(x_1, x_2, 0, \lambda).
\]
In view of (6.26), to prove $u^+(x) \equiv u^-(x)$, it suffices to prove

$$m^+(x) = \Lambda m^-(x), \quad \Lambda = \Lambda(\lambda) = \frac{\Pi_{2 \leq n \leq N}(\lambda - z_n^-)}{\Pi_{2 \leq n \leq N}(\lambda - z_n^+)}.$$  

Applying (6.26), (6.27), and Theorem 4.1, it yields to showing

$$\partial_{\lambda} (\Lambda m^-) = T^+ (\Lambda m^-), \quad (6.36)$$

$$e^{\kappa_1 x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} (\Lambda m^-)(x, \kappa^+_1), \ldots, e^{\kappa_M x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} (\Lambda m^-)(x, \kappa_M^+) D^+ = 0, \quad (6.37)$$

Applying [43, Theorem 2], firstly one has

$$s^+_c = \Lambda^{-1} \tilde{s}^-_c.$$  

Along with (6.26), (6.28), yields

$$\partial_{\lambda} (\Lambda m^-)\big|_{(x, \lambda)} = \Lambda(\lambda)\partial_{\lambda} m^-\big|_{(x, \lambda)} = \Lambda(\lambda) [T^- m^-] (x, \lambda) \quad (6.39)$$

$$= \Lambda(\lambda) s^-_c(\lambda) e^{(\tilde{\lambda} - \lambda)x_1 + (\tilde{\lambda} - \lambda^2)x_2 + (\tilde{\lambda} - \lambda^3)x_3} m^- (x, \tilde{\lambda}) \quad (6.40)$$

$$= \Lambda(\lambda) s^-_c(\lambda) \Lambda^{-1} e^{(\tilde{\lambda} - \lambda)x_1 + (\tilde{\lambda} - \lambda^2)x_2 + (\tilde{\lambda} - \lambda^3)x_3} \Lambda(\tilde{\lambda}) m^- (x, \tilde{\lambda}) \quad (6.41)$$

$$= s^+_c(\lambda) e^{(\tilde{\lambda} - \lambda)x_1 + (\tilde{\lambda} - \lambda^2)x_2 + (\tilde{\lambda} - \lambda^3)x_3} \Lambda(\tilde{\lambda}) m^- (x, \tilde{\lambda}) \quad (6.42)$$

So (6.36) is justified.

On the other hand, from (6.30) and (6.31), proving of (6.37) is equivalent to showing

$$e^{\kappa_1 x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} (\Lambda m^-)(x, \kappa^+_1), \ldots, e^{\kappa_M x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} (\Lambda m^-)(x, \kappa_M^+) \tilde{D}^+ = 0. \quad (6.43)$$

From (6.31),

$$\tilde{D}^- = diag (\Lambda(\kappa_1), \ldots, \Lambda(\kappa_M)) \tilde{D}^+. \quad (6.44)$$

Along with (6.30) and (6.31), yields

$$e^{\kappa_1 x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} \Lambda(\kappa_1) m^- (x, \kappa^+_1), \ldots, e^{\kappa_M x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} \Lambda(\kappa_M) m^- (x, \kappa_M^+) \tilde{D}^+ = e^{\kappa_1 x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} m^- (x, \kappa^+_1), \ldots, e^{\kappa_M x_1 + \kappa_2^2 x_2 + \kappa_3^3 x_3} m^- (x, \kappa_M^+) \tilde{D}^- \quad (6.45)$$

$$= 0. \quad (6.46)$$

Therefore (6.40) is proved.

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