Tanaka’s theorem revisited

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Abstract
Tanaka (Ann Pure Appl Log 84:41–49, 1997) proved a powerful generalization of Friedman’s self-embedding theorem that states that given a countable nonstandard model \((M, A)\) of the subsystem WKL_0 of second order arithmetic, and any element \(m\) of \(M\), there is a self-embedding \(j\) of \((M, A)\) onto a proper initial segment of itself such that \(j\) fixes every predecessor of \(m\). Here we extend Tanaka’s work by establishing the following results for a countable nonstandard model \((M, A)\) of WKL_0 and a proper cut \(I\) of \(M\):

**Theorem A.** The following conditions are equivalent:
(a) \(I\) is closed under exponentiation.
(b) There is a self-embedding \(j\) of \((M, A)\) onto a proper initial segment of itself such that \(I\) is the longest initial segment of fixed points of \(j\).

**Theorem B.** The following conditions are equivalent:
(a) \(I\) is a strong cut of \(M\) and \(I \prec_{\Sigma_1} M\).
(b) There is a self-embedding \(j\) of \((M, A)\) onto a proper initial segment of itself such that \(I\) is the set of all fixed points of \(j\).

**Keywords** Peano arithmetic · WKL_0 · Nonstandard model · Self-embedding · Fixed point · Strong cut

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1 Introduction

One of the fundamental results concerning nonstandard models of Peano arithmetic (PA) is Friedman’s theorem [4, Theorem 4.4] that states every countable nonstandard model of PA is isomorphic to a proper initial segment of itself. A notable generalization of Friedman’s theorem was established by Tanaka [11] for models of the well-known subsystem WKL$_0$ of second order arithmetic, who established:

**Theorem 1.1** (Tanaka [11], see also [3]) Suppose ($\mathcal{M}, \mathcal{A}$) is a countable nonstandard model of WKL$_0$.

(a) There is a proper initial segment $J$ of $\mathcal{M}$ and an isomorphism $j$ between $(\mathcal{M}, \mathcal{A})$ and $(J, \mathcal{A}_J)$, where $\mathcal{A}_J := \{A \cap J : A \in \mathcal{A}\}$.

(b) Given any prescribed $m$ in $\mathcal{M}$, there are $J$ and $j$ as in (a) such that $j(x) = x$ for all $x \leq m$.

Tanaka’s principal motivation in establishing Theorem 1.1 was the development of non-standard methods within WKL$_0$ in the context of the reverse mathematics research program; for example Tanaka and Yamazaki [12] used Theorem 1.1 to show that the Haar measure over compact groups can be implemented in WKL$_0$ via a detour through nonstandard models. This is in contrast to the previously known constructions of the Haar measure whose implementation required the stronger subsystem ACA$_0$. Other notable applications of the methodology of nonstandard models can be found in the work of Sakamato and Yokoyama [10], who showed that over the subsystem RCA$_0$ the Jordan curve theorem and the Schönflies theorem are equivalent to WKL$_0$; and in the work of Yokoyama and Horihata [7], who established the equivalence of WKL$_0$ and Riemann’s mapping theorem for Jordan regions.

Here we continue our work [1] on the study of fixed point sets of self-embeddings of countable nonstandard models of I$\Sigma_1$ by focusing on the behavior of fixed point sets in Tanaka’s theorem. Our methodology can be generally described as an amalgamation of Enayat’s strategy for proving Tanaka’s theorem [2] with some ideas and results from [1]. Before stating our results, recall that $j$ is said to be a proper initial self-embedding of $(\mathcal{M}, \mathcal{A})$ if there is a proper initial segment $J$ of $\mathcal{M}$ and an isomorphism $j$ between $(\mathcal{M}, \mathcal{A})$ and $(J, \mathcal{A}_J)$, where $\mathcal{A}_J := \{A \cap J : A \in \mathcal{A}\}$; $I_{\text{fix}}(j)$ is the longest initial segment of fixed points of $j$; and $\text{Fix}(j)$ is the fixed point set of $j$, in other words:

\[
I_{\text{fix}}(j) := \{m \in M : \forall x \leq m \ j(x) = x\}, \quad \text{and} \quad \text{Fix}(j) := \{m \in M : j(m) = m\}.
\]

Our main results are Theorems A and B below. Note that Theorem A is a strengthening of Tanaka’s Theorem (see Sect. 3 for more detail).

**Theorem A** Suppose $(\mathcal{M}, \mathcal{A})$ is a countable nonstandard model of WKL$_0$. The following conditions are equivalent for a proper cut $I$ of $\mathcal{M}$:

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1 Enayat’s paper [2] provides a complete proof of part (a) of Theorem 1.1, and an outline of the proof of part (b) of Theorem 1.1.
(1) There is a self-embedding \( j \) of \((M, A)\) such that \( I_{\text{fix}}(j) = I \).
(2) \( I \) is closed under exponentiation.
(3) There is a proper initial self-embedding \( j \) of \((M, A)\) such that \( I_{\text{fix}}(j) = I \).

**Theorem B** Suppose \((M, A)\) is a countable nonstandard model of \( \text{WKL}_0 \). The following conditions are equivalent for a proper cut \( I \) of \( M \):

(1) There is a self-embedding \( j \) of \((M, A)\) such that \( \text{Fix}(j) = I \).
(2) \( I \) is a strong cut of \( M \) and \( I \prec_{\Sigma_1} M \).
(3) There is a proper initial self-embedding \( j \) of \((M, A)\) such that \( \text{Fix}(j) = I \).

Theorem A is established in Sect. 3, and Sect. 4 is devoted to proving Theorem B.

**2 Preliminaries**

In this section we review some definitions and basic results that are relevant to the statements and proofs of our main results.

- The language of first order arithmetic, \( L_A \), consists of \([0, 1, +, \cdot, <]\). \( \text{PA}^- \) is the \( L_A \)-theory of the non-negative part of discrete ordered rings as in [8]. \( \text{PA} \) is the \( L_A \)-theory \( \text{PA}^- \) augmented with induction scheme for all \( L_A \)-formulas.
- Throughout this paper first-order structures are denoted by \( M, N \), etc. and their universes are (respectively) denoted by \( M, N \), etc. Furthermore, we will use \( \mathbb{N} \) to denote the standard model of \( \text{PA} \), whose universe of discourse we will denote by \( \omega \).
- For any first-order language \( L \) extending \( L_A \), \( \Sigma_0(L) = \Pi_0(L) = \Delta_0(L) := \) the class of \( L \)-formulas all of whose quantifiers are of the form \( \exists x < y \varphi \) or \( \forall x < y \varphi \); for every \( n \in \omega \), \( \Sigma_{n+1}(L) \) consists of \( L \)-formulas of the form \( \exists x_0 \cdots \exists x_k \varphi \), where \( \varphi \in \Pi_n(L) \); and \( \Pi_{n+1}(L) \) consists of \( L \)-formulas of the form \( \forall x_0 \cdots \forall x_k \varphi \), where \( \varphi \in \Sigma_n(L) \) (here \( k \) ranges over \( \omega \), with the understanding that \( k = 0 \) corresponds to an empty block of quantifiers). Moreover, for an \( L \)-theory \( T \), a \( \Delta_{n+1}(L) \)-formula is an \( L \)-formula which is equivalent both to a \( \Sigma_{n+1}(L) \)-formula and a \( \Pi_{n+1}(L) \)-formula in \( T \).

When \( L = L_A \) we write \( \Delta_n \), \( \Sigma_n \), and \( \Pi_n \) instead of \( \Delta_n(L_A) \), \( \Sigma_n(L_A) \), and \( \Pi_n(L_A) \).

For \( n \in \omega \), \( \Sigma_n \) is the fragment of \( \text{PA} \) with the induction scheme limited to \( \Sigma_n \)-formulas.

Furthermore, by a \( \Sigma^0_n \)-formula \( \Delta^0_n \)-formula we mean a \( \Sigma_n \)-formula \( \Delta_n \)-formula in the language of \( L_A \cup \chi \), in which \( \chi \) is a family of unary predicates.
- For every \( L_A \)-formula \( \varphi \), \( \lnot \varphi \) denotes the Gödel number of the formula \( \varphi \).
- \( \text{WKL}_0 \) is the second order theory whose models are of the form of \((M, A)\), where \( M \models \text{PA}^- \), and \( (M, A) \) satisfies (1) Induction for all \( \Sigma^0_n \)-formulas, (2) Comprehension for \( \Delta^0_1 \)-formulas, and (3) Weak König’s Lemma (which asserts that every infinite subtree of the full binary tree has an infinite branch).
- \( \text{Exp} := \forall x \exists y \text{Exp}(x, y) \), where \( \text{Exp}(x, y) \) is a \( \Delta_0 \)-formula that expresses \( 2^x = y \) within \( I \Delta_0 \).
• The binary $\Delta_0$-formula $xEy$, known as Ackermann’s membership relation, expresses “the $x$-th bit of the binary expansion of $y$ is 1” within $\text{I} \Delta_0$.

• A subset $X$ of $M$ is coded in $\mathcal{M}$ if and only if there is some $a \in M$ such that $X = (a_E)^\mathcal{M} := \{ x \in M : \mathcal{M} \models x Ea \}$.

  Given $a \in M$, by $a$ we mean the set $\{ x \in M : x < a \}$. Note that $a$ is coded in $\mathcal{M}$, where $\mathcal{M}$ is a model of $\text{I} \Delta_0$, provided $2^a$ exists in $\mathcal{M}$.

  It is well-known that for any $n > 0$ and $M \models \text{I} \Sigma_n$, if $\varphi(x, a)$ is an unary $\Sigma_n$-formula, where $a$ is a parameter from $\mathcal{M}$, then the set $\varphi^\mathcal{M}(a) := \{ b \in M : \mathcal{M} \models \varphi(f, a) \}$ is piece-wise coded in $\mathcal{M}$; i.e. for every $c \in M$, $\varphi^\mathcal{M}(a) \cap c$ is coded in $\mathcal{M}$. More specifically, there is some element less than $2^c$ which codes $\varphi^\mathcal{M}(a) \cap c$.

  Moreover, the above statement holds for $n = 0$ if $M \models \text{I} \Delta_0 + \text{Exp}$.

• For every cut $I$ of $M$, the $I$-standard system of $\mathcal{M}$, presented by $\text{SSy}_I(\mathcal{M})$, is the family consisting of sets of the form $a_E \cap I$, where $a \in M$; in other words:

$$\text{SSy}_I(\mathcal{M}) = \{(a_E)^\mathcal{M} \cap I : a \in M\}.$$}

When $I$ is the standard cut, i.e. $I = \mathbb{N}$, we simply write $\text{SSy}(\mathcal{M})$ instead of $\text{SSy}_\mathbb{N}(\mathcal{M})$.

• Given a proper cut $I$ of $\mathcal{M}$, $I$ is called a strong cut, if for every coded function $f$ in $\mathcal{M}$ whose domain contains $I$, there exists some $s \in M$ such that for every $i \in I$ it holds that $f(i) \notin I$ if and only if $s < f(i)$.

• For every $n \in \omega$, $\text{Sat}_{\Sigma_n}$ is the arithmetical formula defining the satisfaction predicate for $\Sigma_n$-formulas within $\text{I} \Delta_0 + \text{Exp}$. It is shown in [6, Theorem 5.4 and Corollary 5.5] that within a model of $\text{I} \Delta_0 + \text{Exp}$ (with the help of a nonstandard parameter if the model is nonstandard), $\text{Sat}_{\Sigma_0}$ (which is also written as Sat$^{\Delta_0}$) is expressible as a $\Delta_1$-formula. Moreover, by an argument similar to [6, Theorem 1.75], we can see that for every $n \geq 1$, $\text{Sat}_{\Sigma_n}$ can be expressed by a $\Sigma_n$-formula in $\text{I} \Delta_0 + \text{Exp}$.

• The strong $\Sigma_n$-Collection scheme consists of formulas of the following form where $\varphi$ is a $\Sigma_n$-formula:

$$\forall w \forall v \exists z \forall x < v (\exists y \varphi(x, y, w) \rightarrow \exists y < z \varphi(x, y, w)).$$

It is well-known that the strong $\Sigma_n$-Collection scheme is provable in $\text{I} \Sigma_n$ for every $n > 0$.

• Every model $\mathcal{M}$ of $\text{I} \Delta_0 + \text{Exp}$ satisfies the Coded Pigeonhole Principle, i.e. if $b \in M$, and $f : b + 1 \rightarrow b$ is a coded function in $\mathcal{M}$, then $f$ is not injective.

• By an embedding $j$ from second order model $(\mathcal{M}, \mathcal{A})$ into $(\mathcal{N}, B)$, we mean that $j$ is an embedding from $\mathcal{M}$ into $\mathcal{N}$ such that for every $X \subseteq M$, $X \subseteq A$ if and only if $j(X) = Y \subseteq B$ for some $Y \in B$.

• When $\mathcal{M}$ and $\mathcal{N}$ are models of arithmetic and $b \in M$, we write $\mathcal{M} <_{\text{end},5_{1,\leq b}} \mathcal{N}$, when $\mathcal{N}$ is an end extension of $\mathcal{M}$ and all $\Pi_1$-formulas whose parameters are in $b + 1$ are absolute in the passage between $\mathcal{M}$ and $\mathcal{N}$, i.e., $\text{Th}_{\Pi_1}(\mathcal{M}, m)_{m \leq b} = \text{Th}_{\Pi_1}(\mathcal{N}, m)_{m \leq b}$. 

\[\text{Springer}\]
Theorem 2.1 ([2, Theorem 3.2]) Let \((\mathcal{M}, \mathcal{A})\) be a countable model of WKL\(_0\) and let \(b \in M\). Then \(\mathcal{M}\) has a countable recursively saturated proper end extension \(\mathcal{N}\) satisfying \(I\Delta_0 + \text{Exp} + B\Sigma_1\) such that \(\text{SSy}_M(\mathcal{N}) = \mathcal{A}\), and \(\mathcal{M} \prec_{\text{end}, 5_1 \leq b} \mathcal{N}\).

Remark 2.2 The proofs of our main results take advantage of the following additional features of the model \(\mathcal{N}\) constructed in Enayat’s proof of Theorem 2.1, namely: given \(b \in M\), there is an elementary chain of models \((\mathcal{N}_n : n \in \omega)\) satisfying the following three properties:

(i) \(\mathcal{N} = \bigcup_{n \in \omega} \mathcal{N}_n\);
(ii) For every \(n \in \omega\), \(\mathcal{M} \prec_{\text{end}, 5_1 \leq b} \mathcal{N}_n < \mathcal{N}\);
(iii) For every \(n \in \omega\) the elementary diagram of \((\mathcal{N}_n, a)_{a \in \mathbb{N}_n}\) is available in \((\mathcal{M}, \mathcal{A})\) via some \(\text{ED}_n \in \mathcal{A}\). Note that \(\text{Th}(\mathcal{N}_n, a)_{a \in \mathbb{N}_n}\) is a proper subset of \(\text{ED}_n\) since \(\text{ED}_n\) includes sentences of nonstandard length.

Remark 2.3 Enayat [2] noted that if \((\mathcal{M}, \mathcal{A})\) is a model of WKL\(_0\) and \(b \in M\), and there is some end extension \(\mathcal{N}\) of \(\mathcal{M}\) such that (1) \(\mathcal{N} \models I\Delta_0 + \text{Exp}\), (2) \(\text{SSy}_M(\mathcal{N}) = \mathcal{A}\), and (3) there is an initial self-embedding \(j_1\) of \(\mathcal{N}\) onto an initial segment that is bounded above by \(b\), then the restriction \(j\) of \(j_1\) to \(\mathcal{M}\) is an embedding of \(\mathcal{M}\) onto an an initial segment \(J\) of \(\mathcal{M}\) of \(J < b\) which has the important feature that \(j\) is an isomorphism between \((\mathcal{M}, \mathcal{A})\) and \((J, \mathcal{A}_j)\). Note that if \(I\) is a proper cut of \(\mathcal{M}\), then \(I_{\text{fix}}(j_1) = I\) implies that \(I_{\text{fix}}(j) = I\); and \(\text{Fix}(j_1) = I\) implies that \(\text{Fix}(j) = I\).

• The following theorem summarizes some of the results about \(I_{\text{fix}}(j)\) and \(\text{Fix}(j)\) from [1] which will be employed in this paper:

Theorem 2.4 Suppose \(\mathcal{M} \models I\Delta_0 + \text{Exp}\) and \(j\) is a nontrivial self-embedding of \(\mathcal{M}\). Then:

(a) \(I_{\text{fix}}(j) \models I\Delta_0 + B\Sigma_1 + \text{Exp}\).
(b) If \(\mathcal{M} \models I\Sigma_1\) then \(\text{Fix}(j)\) is a \(\Sigma_1\)-elementary submodel of \(\mathcal{M}\). Moreover, if \(\text{Fix}(j)\) is a proper initial segment of \(\mathcal{M}\), then it is a strong cut of \(\mathcal{M}\).

• Given two countable nonstandard model \(\mathcal{M}\) and \(\mathcal{N}\) of \(I\Sigma_1\) which share a common proper cut \(I\), the following theorem from [1, Cor. 3.3.1] provides a useful sufficient condition for existence of a proper initial embedding between \(\mathcal{M}\) and \(\mathcal{N}\) which fixes each element of \(I\):

Theorem 2.5 Let \(\mathcal{M}, \mathcal{N}\) and \(I\) be as above such that \(I\) is closed under exponentiation. The following are equivalent:

(1) There is a proper initial embedding \(f\) of \(\mathcal{M}\) into \(\mathcal{N}\) such that \(f(i) = i\) for all \(i \in I\).
(2) \(\text{Th}_{\Sigma_1}(\mathcal{M}, i)_{i \in I} \subseteq \text{Th}_{\Sigma_1}(\mathcal{N}, i)_{i \in I}\) and \(\text{SSy}_I(\mathcal{M}) = \text{SSy}_I(\mathcal{N})\).

• Another prominent subsystem of second order arithmetic is \(\text{ACA}_0\), in which the comprehension scheme is restricted to formulas with no second order quantifier. The following results of Paris and Kirby [9] and Gaifman [5, Thm. 4.9–4.11] concerning \(\text{ACA}_0\) are employed in the proof of Theorem B.\(^2\)

\(^2\) Gaifman couched his results in terms of arbitrary models of PA(\(L\)) for countable \(L\). Note that if \((\mathcal{M}, \mathcal{A}) \models \text{ACA}_0\), then the expansion \((\mathcal{M}, A)_{A \in \mathcal{A}}\) of \(\mathcal{M}\) is a model of PA(\(L\)), where \(L\) is the extension of \(L_A\) by predicate symbols for each \(A \in \mathcal{A}\). Moreover, the collection of subsets of \(M\) that are parametrically definable in \((\mathcal{M}, A)_{A \in \mathcal{A}}\) coincides with \(A\).
Theorem 2.6 (Paris and Kirby) Suppose $\mathcal{M} \models \text{I}\Delta_0$. The following are equivalent for a proper cut $I$ of $\mathcal{M}$:

(a) $I$ is a strong cut of $\mathcal{M}$.
(b) $(I, \text{SSy}(\mathcal{M})) \models \text{ACA}_0$.

Theorem 2.7 (Gaifman) Given a countable model $(\mathcal{M}, A)$ of $\text{ACA}_0$ and a linear order $\mathbb{L}$, there exists an elementary end extension $\mathcal{M}_L$ of $\mathcal{M}$ such that there is an isomorphic copy $L' = \{c_l : l \in \mathbb{L}\}$ of $\mathbb{L}$ in $\mathcal{M}_L \setminus \mathcal{M}$, and there is a composition preserving embedding $j \mapsto \hat{j}$ from the semi-group of initial self-embeddings of $\mathbb{L}$ into the semi-group of initial self-embeddings of $\mathcal{M}_L$ that satisfy the following properties:

(a) $\text{SSy}_\mathcal{M}(\mathcal{M}_L) = A$ and $\text{Fix}(\hat{j}) = \mathcal{M}$ for each initial self-embedding $j$ of $\mathbb{L}$ that is fixed point free.
(b) For each initial self-embedding $j$ of $\mathbb{L}$, $\hat{j}$ is an elementary initial self-embedding of $\mathcal{M}_L$, i.e. $\hat{j}(\mathcal{M}_L) \leq \text{end} \mathcal{N}_L$.
(c) $\mathbb{L}'$ is downward cofinal in $\mathcal{M}_L \setminus \mathcal{M}$ if $\mathbb{L}$ has no first element.
(d) For any $l_0 \in \mathbb{L}$, $l_0$ is a strict upper bound for $j(\mathbb{L})$ iff $c_{l_0}$ is a strict upper bound for $\hat{j}(\mathcal{M}_L)$.

3 The longest cut fixed by self-embeddings

This section is devoted to the proof of Theorem A. Recall that if $(\mathcal{M}, A)$ is a model of $\text{WKL}_0$ there are arbitrary large as well as arbitrary small nonstandard cuts in $\mathcal{M}$ that are closed under exponentiation. More specifically, for every nonstandard $a$ in $\mathcal{M}$, there are nonstandard cuts $I_1$ and $I_2$ (as defined below) such that $I_1 < a \in I_2$, and both are closed under exponentiation:

$I_1 := \{x \in \mathcal{M} : 2^x_n < a \text{ for all } n \in \omega\}$, where $2^x_0 := x$, and for every $n \in \omega$,

$2^x_{n+1} := 2^{2^n} ; I_2 := \{x \in \mathcal{M} : x < 2^n_a \text{ for some } n \in \omega\}.$

So Theorem A implies that $\text{I}_{\text{fix}}(j)$ can be arranged to be as high or as low in the nonstandard part of $\mathcal{M}$ as desired. In particular, Theorem A is a strengthening of Tanaka’s Theorem.

Proof of Theorem A: (1) $\Rightarrow$ (2) is an immediate consequence of Theorem 2.4.(a), and (3) $\Rightarrow$ (1) is trivial so we concentrate on establishing (2) $\Rightarrow$ (3).

Assume that $I$ is closed under exponentiation and fix some $a \in \mathcal{M} \setminus I$. We leave it as an exercise for the reader to use strong $\Sigma_1$-Collection along with the fact that $\text{Sat}_{\Delta_0}$ has a $\Sigma_1$-description in $\mathcal{M}$ to show that there is some $b \in \mathcal{M}$ such that:

(♯) $\mathcal{M} \models \forall w < a (\exists z \; \delta(z, w) \rightarrow \exists z < b \; \delta(z, w))$, for all $\Delta_0$-formulas $\delta$.

Next we invoke Theorem 2.1 to get hold of a countable recursively saturated proper end extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \models \text{I}\Delta_0 + \text{B}\Sigma_1 + \text{Exp}$, $\text{SSy}_\mathcal{M}(\mathcal{N}) = A$, and $\mathcal{M} \leq_{\text{end}, \text{st}, \leq b} \mathcal{N}$. Moreover, we will safely assume that the model $\mathcal{N}$ additionally satisfies the three properties listed in Remark 2.2. In light of Remark 2.3, in order to establish (3) it suffices to construct a proper initial self-embedding $j$ of $\mathcal{N}$ such that...
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Let \( j(N) < b \) and \( I_{\Pi^1}(j) = I \). The construction of the desired \( j \) is the novel element of the proof of Theorem A, which we now turn to.

To construct \( j \) we will employ a modification of the strategy employed in the proof of (2) \( \Rightarrow \) (3) of [1, Theorem 4.1], using a 3-level back-and-forth method. A modification is needed since we need to overcome the fact that \( \Sigma_1 \) need not hold in \( \mathcal{N} \); instead we will rely on recursive saturation of \( \mathcal{N} \) and the properties of \( \mathcal{N} \) listed in Remark 2.2.

First, note that since we need to overcome the fact that \( I \) (\( \mathcal{N} \)) is needed since we need to overcome the fact that \( I \) (\( \mathcal{N} \)) is closed under exponentiation, we can choose \( \{c_n : n \in \omega\} \) that is downward cofinal in \( \mathcal{M} \setminus I \) such that \( c_0 = a \) and \( 2^{c_{n+1}} < c_n \) for all \( n \in \omega \).

The proof will be complete once we recursively construct finite sequences \( \bar{u}_n := (u_0, \ldots, u_n) \) and \( \bar{v}_n := (v_0, \ldots, v_n) \) of elements of \( \mathcal{N} \) for all \( n \in \omega \) such that:

1. \( u_0 = 0 = v_0 \).
2. For every \( c \in \omega \) there is some \( n \in \omega \) such that \( c = u_n \).
3. For every \( n \in \omega \), \( u_n < b \), and if for some \( c \in \omega \) it holds that \( c < u_n \), then there is some \( m \in \omega \) such that \( c = v_m \).
4. For every \( m \in \omega \) the following condition holds:
   \( \forall w < c_m \exists z \delta(z, w, \bar{u}_m) \rightarrow \exists z < b \delta(z, w, v_m) \)
   for every \( \Delta_0 \)-formula \( \delta \).
5. For every \( m \in \omega \), there is some \( n \in \omega \) such that \( u_n < c_m \) and \( u_n \neq v_n \).

Note that (\( \ast_0 \)) holds thanks to (\( \ast \)) since \( c_0 = a \). Let \( \{a_n : n \in \omega\} \) and \( \{b_n : n \in \omega\} \) respectively be enumerations of elements of \( \mathcal{N} \) and \( \mathcal{B} \). By statement (i) and (\( \ast_0 \)) the first step of induction holds. Suppose for \( m \in \omega \), \( \bar{u}_m \) and \( \bar{v}_m \) are constructed such that (\( \ast_m \)) holds. In order to find suitable \( u_{m+1} \) and \( v_{m+1} \), by considering congruence modulo 3 we have three cases for \( m + 1 \): Case 0 takes care of (ii) and (iv), Case 1 takes care of (iii) and (iv), and Case 2 takes care of (v) and (iv).

**CASE 0** (\( m + 1 = 3k \), for some \( k \in \omega \)\): In this case if \( a_k \) is one of the elements of \( \bar{u}_m \), put \( u_{m+1} = u_m \) and \( v_{m+1} = v_m \). Otherwise, put \( u_{m+1} = a_k \) and define:

\[
p(y) := \{y < b\} \cup \\
\{w < c_{m+1}(\exists z \delta(z, w, \bar{u}_m, a_k) \rightarrow \exists z < b \delta(z, w, \bar{v}_m, y)) : \delta \text{ is a } \Delta_0\text{-formula}\}.
\]

Note that \( p(y) \) is a recursive type. Since \( \mathcal{N} \) is recursively saturated, it suffices to prove that \( p(y) \) is finitely satisfiable and let \( v_{m+1} \) be one of the realizations of \( p(y) \) in \( \mathcal{N} \).

Since the second set in the definition of \( p(y) \) is closed under conjunctions we only need to show that each formula in this set can be satisfied by some \( y < b \). For this purpose, suppose \( \delta \) is a \( \Delta_0 \)-formula, and let

\[
D := \{w \in c_{m+1} : \mathcal{N} \models \exists z \delta(z, w, \bar{u}_m, a_k)\}.
\]

We claim that there is some \( d < 2^{c_{m+1}} \) which codes \( D \) in \( \mathcal{N} \). To see this, we note that in the above definition, \( \mathcal{N} \) can be safely replaced by some \( \mathcal{N}_n \), where \( n \) is large enough.
to contain the parameters \( \bar{u}_m \) and \( a_k \) (thanks to properties (i) and (ii) in Remark 2.2). On the other hand, by property (iii) in Remark 2.2, there is some \( \text{ED}_n \in \mathcal{A} \) such that:

\[
D = \left\{ w \in c_{m+1} : \exists z \, \delta(z, w, \bar{u}_m, a_k) \forall \right\} \in \text{ED}_n
\]

Since \((\mathcal{M}, \mathcal{A})\) satisfies \( \Sigma_1^0 \), the above characterization of \( D \) shows that \( D \) is coded in \( \mathcal{M} \) (and therefore in \( \mathcal{N} \)) by some \( d < 2^m \) (recall that the code of each subset of \( m \) is below \( 2^m \)). Therefore we have:

1. \( \mathcal{N} \models \forall w < c_{m+1} \left( w Ed = \exists z \delta(z, w, \bar{u}_m, a_k) \right) \);

By putting (1) together with \( \mathcal{B} \Sigma_1 \) in \( \mathcal{N} \), and existentially quantifying \( a_k \) we obtain:

2. \( \mathcal{N} \models \exists t, x \forall w < c_{m+1} \left( w Ed = \exists z < t \delta(z, w, \bar{u}_m, x) \right) \).

On the other hand, coupling (2) with \((*_{m})\) yields:

3. \( \mathcal{N} \models \exists t, x < b \forall w < c_{m+1} \left( (w Ed = \exists z < t \delta(z, w, \bar{u}_m, x)) \right) \),

which makes it clear that each formula in \( p(y) \) is satisfiable in \( \mathcal{N} \).

**Case 1 (m + 1 = 3k + 1, for some k \in \omega)**: In this case if \( b_k \geq \text{Max}\{\bar{v}_m\} \) or if it is one of the elements of \( \bar{v}_m \), put \( u_{m+1} = u_m \) and \( v_{m+1} = v_m \). Otherwise, put \( v_{m+1} = b_k \) and define:

\[
q(x) := \{ \forall w < c_{m+1} (\forall z < b \neg \delta(z, w, \bar{v}_m, b_k) \rightarrow \forall z \neg \delta(z, w, \bar{u}_m, x)) : \delta \text{ is a } \Delta_0\text{-formula} \}.
\]

\( q(x) \) is clearly a recursive type and closed under conjunctions, so by recursive saturation of \( \mathcal{N} \) it suffices to verify that each formula in \( q(x) \) is satisfiable in \( \mathcal{N} \), and let \( u_{m+1} \) be one of the realizations of \( q(x) \) in \( \mathcal{N} \). Suppose some formula in \( q \) is not realizable in \( \mathcal{N} \), then for some \( \Delta_0\)-formula \( \delta \) we have:

\[
\mathcal{N} \models \forall x \left( \exists w < c_{m+1} \left( \forall z < b \neg \delta(z, w, \bar{v}_m, b_k) \land \exists z \delta(z, w, \bar{u}_m, x) \right) \right).
\]

Let:

\[
R := \left\{ w \in c_{m+1} : \mathcal{N} \models \forall z < b \neg \delta(z, w, \bar{v}_m, b_k) \right\}.
\]

Since \( R \) is \( \Delta_0\)-definable in \( \mathcal{N} \) there exists some \( r < 2^m \) which codes \( R \) in \( \mathcal{N} \). Therefore,

4. \( \mathcal{N} \models \forall x < \text{Max}\{\bar{u}_m\} \left( \exists w < c_{m+1} \left( w Er \land \exists z \delta(z, w, \bar{u}_m, x) \right) \right) \),

which by \( \Sigma_1\)-Collection in \( \mathcal{N} \) implies:

5. \( \mathcal{N} \models \exists t \forall x < \text{Max}\{\bar{u}_m\} \left( \exists w < c_{m+1} \left( w Er \land \exists z < t \delta(z, w, \bar{u}_m, x) \right) \right) \).

Putting (5) together with \((*_{m})\) yields:

6. \( \mathcal{N} \models \exists t < b \forall x < \text{Max}\{\bar{v}_m\} \left( \exists w < c_{m+1} \left( w Er \land \exists z < t \delta(z, w, \bar{v}_m, x) \right) \right) \).

By substituting \( b_k \) for \( x \) in (6) we obtain:
(7) $N \models \exists t < b \forall x < \text{Max}\{\tilde{v}_m\} (\exists w < c_{m+1} (wE r \land \exists z < t \delta(z, w, \tilde{v}_m, b_k))).$

But (7) contradicts the assumption that $r$ codes $R$. So $q(x)$ is finitely satisfiable.

Case 2 ($m + 1 = 3k + 2$, for some $k \in \omega$): Consider the type $l(x, y) := \{x \neq y, x \leq c_k\} \cup l_0(x, y)$, where:

$$l_0(x, y) := \{\forall w < c_{m+1} (\exists z \delta(z, w, \tilde{u}_m, x) \rightarrow \exists z < b \delta(z, w, \tilde{v}_m, y)) : \delta \text{ is a } \Delta_0\text{-formula}\}.$$  

Once we demonstrate that $l(x, y)$ is realized in $N$ we can define $(u_{m+1}, v_{m+1})$ as any realization in $N$ of $l(x, y)$. Since $l_0(x, y)$ is closed under conjunctions and $N$ is recursively saturated, to show that $l(x, y)$ is realized in $N$ it suffices to demonstrate that the conjunction of $x \neq y$ and $x \leq c_k$, and each formula in $l_0(x, y)$ is satisfiable in $N$. So suppose $\delta$ is a $\Delta_0$-formula and for each $s < c_k$ consider the map $F$ from $c_k$ to the power set of $c_{m+1}$ by:

$$F(s) := \{w \in c_{m+1} : N \models \exists z \delta(z, w, \tilde{u}_m, s)\}.$$  

Thanks to properties (i) through (iii) of $N$ listed in Remark 2.2, there is some $\text{ED}_n \in A$ such that:

$$F(s) = \{w \in c_{m+1} : [\exists z \delta(z, w, \tilde{u}_m, s) \rightarrow \exists \bar{z} \in \text{ED}_n]\}.$$  

Since $(M, A)$ satisfies $1\Sigma^0_1$, the above characterization of $F(s)$ together with the veracity of $1\Sigma^0_1$ in $(M, A)$ makes it clear that $F$ is coded in $M$ by some $f$ (and therefore in $N$) that codes a function from $c_k$ to $2^{c_{m+1}}$ with $f(s) := \sum_{t \in F(s)} 2^t$. On the other hand the definition of $f(s)$ and the assumption that $2^{c_{n+1}} < c_n$ for all $n \in \omega$ makes it clear that:

$$f(s) \leq \sum_{l \leq c_{m+1}} 2^l = 2^{c_{m+1}} - 1 < 2^{c_{m+1}} < c_m < c_k.$$  

So by the coded pigeonhole principle there are distinct $s, s' < c_k$ such that $f(s) = f(s')$, in other words:

$$(\ast) : N \models \forall w < c_{m+1} (\exists z \delta(z, w, \tilde{u}_m, s) \leftrightarrow \exists z \delta(z, w, \tilde{u}_m, s')).$$  

Now by repeating the argument used in Case 0 for $(\tilde{u}_m, s)$ we can find some $t < b$ such that:

$$N \models \forall w < c_{m+1} (\exists z \delta(z, w, \tilde{u}_m, s) \rightarrow \exists z < b \delta(z, w, \tilde{v}_m, t)).$$  

Note that by statement $(\ast)$ it also holds that:

$$N \models \forall w < c_{m+1} (\exists z \delta(z, w, \tilde{u}_m, s') \rightarrow \exists z < b \delta(z, w, \tilde{v}_m, t)).$$  

So since $s \neq s'$, either $s \neq t$ or $s' \neq t$. So the conjunction of $x \neq y$, $x \leq c_k$, and each formula in $l_0(x, y)$ is satisfiable in $N$, and the proof is complete.
4 Cuts which are fixed-point sets of self-embeddings

In this section we present the proof of Theorem B. But before going through the proof, let us point out that a model of WKL₀ does not necessarily carry a cut satisfying statement (2) of Theorem B (see [1, Remark 5.1.1] for an explanation). However, if \( \mathcal{M} \models \text{PA} \), there are arbitrarily high strong cuts \( I \) in \( \mathcal{M} \) such that \( I \preceq_{\Sigma_1} \mathcal{M} \). More generally, Ali Enayat pointed out to me that for every \( n \in \omega \) there are arbitrarily high strong cuts \( I \) in \( \mathcal{M} \) such that \( I \preceq_{\Sigma_n} \mathcal{M} \). To see this, note that by Gaifman’s refinement of the MacDowell–Specker Theorem ([8, Theorem 8.8]) \( \mathcal{M} \) has a conservative elementary end extension \( \mathcal{N} \). Moreover, without loss of generality by Löwenheim-Skolem Theorem we can suppose that \( \mathcal{N} \) is countable. So in one hand, since \( \text{SSy}(\mathcal{N}) \models \text{SSy}(\mathcal{M}) \) (because \( \mathcal{M} \preceq_{e} \mathcal{N} \)), by the parametric form of Friedman’s Theorem ([8, Theorem 12.3]), for every \( a \in \mathcal{M} \) there exists some embedding \( j \) from \( \mathcal{N} \) onto some \( \Sigma_n \)-elementary proper initial segment \( J \) of \( \mathcal{M} \) such that \( j(a) = a \). On the other hand, since \( \mathcal{N} \) is a conservative extension of \( \mathcal{M} \), \( \text{SSy}_{\mathcal{M}}(\mathcal{N}) \) is the collection of parametrically definable subsets of \( \mathcal{M} \). In particular, \( \mathcal{M}, \text{SSy}_{\mathcal{M}}(\mathcal{N}) \) satisfies ACA₀. Let \( A := \text{SSy}_{\mathcal{M}}(\mathcal{N}) \), and consider the restriction of \( j \) on \( \mathcal{M} \). Therefore, by Remark 2.3 if \( I := \{ j(M) \} \preceq_{e} \mathcal{M} \), then \( \mathcal{M}, A \cong (I, A_I) \) and \( I \) contains \( a \). So since \( A_I = \text{SSy}_I(\mathcal{M}) \), Theorem 2.6 together with the fact that \( \mathcal{M}, A \models \text{ACA}_0 \) implies that \( I \) is strong in \( \mathcal{M} \). Moreover, since \( \mathcal{M} \preceq \mathcal{N} \) and \( I \subset J \preceq_{\Sigma_n} \mathcal{M} \) it can be readily seen that \( I \preceq_{\Sigma_n} \mathcal{M} \).

**Proof of Theorem B.** (1) \( \Rightarrow \) (2) is an immediate consequence of Theorem 2.4(b), and (3) \( \Rightarrow \) (1) is trivial; so we concentrate on the proof of (2) \( \Rightarrow \) (3).

Suppose \( I \) is a strong cut in \( \mathcal{M} \) and \( I \preceq_{\text{end}, \Sigma_1} \mathcal{M} \). The following proof is inspired by the proof of [1, Theorem 5.1] and consists of the following four stages:

**Stage 1** Fix some \( b_0 \in \mathcal{M} \setminus I \). Using Theorem 2.1, let \( \mathcal{N} \) be a countable recursively saturated model of \( \text{IΔ}_0 + \text{BΣ}_1 + \text{Exp} \) such that \( \mathcal{M} \preceq_{\text{end}, \Sigma_1} \mathcal{N} \), \( \text{SSy}_I(\mathcal{N}) = A \), and the three conditions specified in Remark 2.2 hold for \( \mathcal{N} \).

**Stage 2** Let \( Q \) be the set of rational numbers with its natural ordering. Since \( I \) is a strong cut in \( \mathcal{M} \), by Theorem 2.6, and the case \( \mathcal{L} = Q \) of Theorem 2.7, we can find an elementary end extension \( \mathcal{I}_Q \) of \( I \) such that \( \text{SSy}_I(\mathcal{I}_Q) = \text{SSy}_I(\mathcal{M}) \) and \( \mathcal{I}_Q \setminus I \) contains a copy of \( Q' := \{ q : q \in Q \} \) of \( Q \), and there is a composition preserving embedding \( j \mapsto \widehat{j} \) from the semi-group of initial self-embeddings of \( Q \) into the semi-group of initial self-embeddings of \( \mathcal{I}_Q \) that satisfies conditions (a) through (d) of Theorem 2.7. In particular \( Q' \) is downward cofinal in \( \mathcal{I}_Q \setminus I \).

**Stage 3** An initial embedding \( k : \mathcal{N} \rightarrow \mathcal{I}_Q \) is constructed such that \( k \) fixes each element of \( I \). Note that Theorem 2.5 cannot be invoked for this purpose since \( \Sigma_1 \) need not hold in \( \mathcal{N} \); instead, we will take advantage of recursive saturation of \( \mathcal{N} \), and the properties of \( \mathcal{N} \) listed in Remark 2.2. We will go through construction of \( k \) after describing stage 4 of the proof.

**Stage 4** The desired self-embedding \( j \) satisfying (3) of Theorem B can be readily constructed as follows: Fix some \( c_{q_1} \prec k(b_0) \) in \( Q \) and let \( j_i \) be a fixed-point free initial embedding of \( Q \) such that \( j_1(q_1) \prec q_1 \). Then define \( h := k^{-1} \widehat{j}_i k \), and let \( j \) be the restriction of \( h \) to \( \mathcal{M} \). First, note that by the way \( j_i \) is chosen, \( h \) is well-defined and \( h(N) \prec b_0 \). Therefore, \( j \) is an isomorphism between \( \mathcal{M} \) and a proper cut \( J \) of \( \mathcal{M} \).
Moreover, since \( \text{Fix}(\widehat{j}_1) = I \) (as arranged in Stage 2) and \( k \) fixes each element of \( I \) (as arranged in Stage 3), by Remark 2.3 we may conclude that \( \text{Fix}(f) = I \) and \( j \) is an isomorphism between \((\mathcal{M}, \mathcal{A})\) and \((\mathcal{J}, \mathcal{A}_1)\).

The above description of the four stages of the proof should make it clear that the proof of condition (3) of Theorem B will be complete once we verify that Stage 3 can be carried out, so we focus on the construction of an initial embedding \( k \) of \( \mathcal{N} \) into \( \mathcal{I}_Q \) that fixes each element of \( I \). To do so, we first note that since (i) \( I \Sigma_1 \) holds in both \( \mathcal{M} \) and \( \mathcal{I}_Q \), (ii) \( \text{SSY}_1(\mathcal{M}) = \text{SSY}_1(\mathcal{I}_Q) \), and (iii) \( \text{Th}_{\Sigma_1}(\mathcal{M}, i)_{i \in I} = \text{Th}_{\Sigma_1}(\mathcal{I}_Q, i)_{i \in I} \) (because \( I < \Sigma_1 \mathcal{M} \), and \( I < \mathcal{I}_Q \)) by Theorem 2.5 there is a proper initial embedding \( f : \mathcal{M} \rightarrow \mathcal{I}_Q \) such that \( f(i) = i \) for each \( i \in I \). In particular, \( f(M) < e \) for some \( e \in \mathcal{I}_Q \). Moreover, since \( \text{Th}_{\Pi_1}(\mathcal{M}, x)_{x \leq b_0} = \text{Th}_{\Pi_1}(\mathcal{N}, x)_{x \leq b_0} \) we have:

\[
(*_0) : \mathcal{N} \models \exists \bar{z} \delta(z, w) \rightarrow \mathcal{I}_Q \models \exists z < e \delta(z, f(w)), \text{ for all } \Delta_0\text{-formulas } \delta \text{ and all } w < b_0.
\]

Now choose \( \{b_n : n \in \omega\} \) to be a decreasing sequence in \( M \backslash I \) such that \( b_0 \) is the element chosen in Stage 1, and \( 2^{b_{n+1}} < b_n \) for all \( n \in \omega \). In order to construct \( k \), we recursively build finite sequences \( \bar{u}_m := (u_0, \ldots, u_m) \) of elements of \( N \) and \( \bar{v}_m := (v_0, \ldots, v_m) < e \) for each \( m \in \omega \) such that:

(i) \( u_0 = 0 = v_0 \).
(ii) For every \( c \in N \) there is some \( n \in \omega \) such that \( c = u_n \).
(iii) For every \( n \in \omega \), \( v_n < b \), and if for some \( c \in \mathcal{I}_Q \) it holds that \( c < v_n \), then there is some \( m \in \omega \) such that \( c = v_m \).
(iv) For every \( m \in \omega \) the following condition holds:

\[
(*_m) : \mathcal{N} \models \exists \bar{z} \delta(z, w, \bar{u}_m) \Rightarrow \mathcal{I}_Q \models \exists z < e \delta(z, f(w), \bar{v}_m) \text{ for all } \Delta_0\text{-formulas } \delta \text{ and all } w < b_m
\]

Let \( \{a_n : n \in \omega\} \) and \( \{d_n : n \in \omega\} \) respectively be enumerations of element of \( N \) and \( e \subset \mathcal{I}_Q \), and \( \langle \delta_r : r \in M \rangle \) be a canonical enumeration of \( \Delta_0\text{-formulas in } \mathcal{M} \). The first step of induction holds thanks to \( (*_0) \) and the choice of \( u_0 \) and \( v_0 \) in statement (i). Next, suppose \( \bar{u}_m := (u_0, \ldots, u_m) \in \mathcal{N} \) and \( \bar{v}_m := (v_0, \ldots, v_m) < e \) are constructed, for given \( m \in \omega \). In order to build \( u_{m+1} \) and \( v_{m+1} \) we distinguish two cases, one handling the ‘forth’ step and the other handling the ‘back’ step of the back-and-forth construction:

**Case 0** \((m + 1 = 2k, \text{ for some } k \in \omega)\): In this case if \( a_k \) is one of elements of \( \bar{u}_m \), put \( u_{m+1} = u_m \) and \( v_{m+1} = v_m \). Otherwise, put \( u_{m+1} = a_k \) and define:

\[
A := \{ \langle r, w \rangle < b_{m+1} : \mathcal{N} \models \exists z \text{ Sat}_{\Delta_0}(\bar{u}_m, a_k) \}.
\]

Note that in the above definition, \( \mathcal{N} \) can be safely replaced by some \( \mathcal{N}_n \), where \( n \) is large enough to contain the parameters \( \bar{u}_m \) and \( a_k \) (thanks to properties (i) and (ii) in Remark 2.2). On the other hand, by property (iii) in Remark 2.2, there is some \( \text{ED}_n \in \mathcal{A} \), such that:

\[
A = \{ \langle r, w \rangle < b_{m+1} : \exists z \text{ Sat}_{\Delta_0}(\bar{u}_m, a_k) \} \subseteq \text{ED}_n.
\]

Since \((\mathcal{M}, \mathcal{A}) \) satisfies \( I \Sigma^0_1 \), the above characterization of \( A \) shows that \( A \) is coded in \( \mathcal{N} \) by some \( a < 2^{b_{m+1}} \). Therefore we have:
(1) $\mathcal{N} \models \forall (r, w) < b_{m+1}((r, w)Ea \rightarrow \exists z \text{ Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}_m, a_k)))$.

Recall that $B \Sigma_1$ holds in $\mathcal{N}$, and Sat$_{\Delta_0}$ has a $\Sigma_1$-description in $\mathcal{N}$, so (1) allows us to conclude:

(2) $\mathcal{N} \models \exists t \forall (r, w) < b_{m+1}((r, w)Ea \rightarrow \exists z < t \text{ Sat}_{\Delta_0}(\delta_r(z, w, \bar{u}_m, a_k)))$.

By quantifying out $a_k$ in (2) and coupling it with ($*_m$), we obtain:

(3) $\mathcal{I}_Q \models \exists x, t < e \forall (r, w) < f(b_{m+1})((r, w)Ef(a) \rightarrow \exists z < t \text{ Sat}_{\Delta_0}(\delta_r(z, w, \bar{v}_m, x)))$.

Clearly any element of $\mathcal{I}_Q$ that witnesses $x$ in (3) can serve as a suitable candidate for $v_{m+1}$.

**Case 1 ($m + 1 = 2k + 1$, for some $k \in \omega$)**: In this case if $d_k \geq \text{Max} \{\bar{v}_m\}$ or if it is one of the elements of $\bar{v}_m$, put $u_{m+1} = u_m$ and $v_{m+1} = v_m$. Otherwise, put $v_{m+1} = b_k$ and define:

$$B := \{ (r, w) < f(b_{m+1}) : \mathcal{I}_Q \models \forall z (\text{Sat}_{\Delta_0}(\delta_r(z, w, \bar{v}_m, d_k)) \rightarrow b < z) \}.$$  

Note that $B$ is $\Sigma_1$-definable in $\mathcal{I}_Q$, so there is some $b < 2^{f(b_{m+1})} = f(2^{b_{m+1}})$ which codes $B$ in $\mathcal{I}_Q$. Therefore $b = f(c)$ for some $c < 2^{b_{m+1}}$. Define:

$$p(x) := \{ \forall w < b_{m+1}((\lceil \bar{\delta} \rceil, w)Ec \rightarrow \forall z \neg \delta(z, w, \bar{u}_m, x)) : \delta \text{ is a } \Delta_0\text{-formula} \}.$$  

Since $\mathcal{N}$ is recursively saturated and $p(x)$ is recursive, in order to find a suitable element in $N$ which serves as $u_{m+1}$, it suffices to prove that $p(x)$ is finitely satisfiable. So suppose $p(x)$ is not finitely satisfiable. It can be readily checked that $p(x)$ is closed under conjunction, so we can safely assume there is a $\Delta_0$-formula $\delta$ such that:

(4) $\mathcal{N} \models \forall x (\exists w < b_{m+1}((\lceil \bar{\delta} \rceil, w)Ec \land \exists z \delta(z, w, \bar{u}_m, x)))$.

Clearly (4) implies:

(5) $\mathcal{N} \models \forall x < \text{Max} \{\bar{u}_m\} (\exists w < b_{m+1}((\lceil \bar{\delta} \rceil, w)Ec \land \exists z \delta(z, w, \bar{u}_m, x)))$.

We can bound variable $z$ in (5) by using $B \Sigma_1$ in $\mathcal{N}$, and next employ ($*_m$) to deduce:

(6) $\mathcal{I}_Q \models \exists t < e \forall x < \text{Max} \{\bar{v}_m\}(\exists w < f(b_{m+1})((\lceil \bar{\delta} \rceil, w)Ef(c) \land \exists z < t \delta (z, w, \bar{v}_m, x)))$.

By replacing $x$ in (6) with $d_k$, we obtain:

(7) $\mathcal{I}_Q \models \exists t < e (\exists w < f(b_{m+1})((\lceil \bar{\delta} \rceil, w)Eb \land \exists z < t \delta(z, w, \bar{v}_m, d_k)))$.

But (7) contradicts the assumption that $b$ codes $B$ in $\mathcal{I}_Q$. So $p(x)$ is finitely satisfiable.

\[ \square \]

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