Exact results for the orbital angular momentum of magnons on honeycomb lattices

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Abstract
We obtain exact results for the orbital angular momentum (OAM) of magnons at the high symmetry points of ferromagnetic (FM) and antiferromagnetic (AF) honeycomb lattices in the presence of Dzyalonshinskii–Moriya (DM) interactions. For the FM honeycomb lattice in the absence of DM interactions, the values of the OAM at the corners of the Brillouin zone (BZ) \((k_1^* = (0, 2\sqrt{3}/9)2\pi/a, \ k_2^* = (1/3, \sqrt{3}/9)2\pi/a, \ldots)\) are alternately \(\pm 3\hbar/16\) for both magnon bands. The presence of DM interactions dramatically changes those values by breaking the degeneracy of the two magnon bands. The OAM values are alternately \(3\hbar/8\) and 0 for the lower magnon band and \((-3\hbar/8\) and 0 for the upper magnon band. For the AF honeycomb lattice, the values of the OAM at the corners of the BZ are \(\pm (3\hbar/16)\) on one of the degenerate magnon bands and \((3\hbar/8)(1 + \kappa/2)\) on the other, where \(\kappa\) measures the anisotropy and the result is independent of the DM interaction.

Keywords: spin–waves, orbital angular momentum, spin-orbit coupling

1. Introduction
The conversion of spin into orbital angular momentum (OAM) and back has been of great interest since the early measurements of Einstein and de Hass [1] and Barnett [2]. One recently posed question [3–6] is whether spin excitations, commonly called magnons, can have both spin and OAM. While the magnon produced by a ferromagnet (FM) with moments up has spin \(S = -\hbar\), its OAM \(L\) is not known. The observation of the OAM associated with magnons would have wide-ranging physical ramifications. Since the total angular momentum \(\mathcal{J} = S + L\) is conserved [7, 8], magnons with \(S\) parallel to \(L\) and those with \(S\) antiparallel to \(L\) will behave differently under collisions and in external fields. Consequently, memory storage devices may be able to utilize the OAM of magnons in the developing field of magnonics [9]. It is likely that the magnon Hall [10–12] and Nernst [13] effects will sensitively depend on the OAM of the magnons. Another fascinating possibility is that magnons with OAM can be created and controlled by the interaction with photons [14] and phonons [15, 16] that also carry OAM.

In a recent paper [17], two of us demonstrated that the magnons of collinear magnets may exhibit OAM even for simple collinear magnets. Surprisingly, the OAM of magnons on FM and antiferromagnetic (AF) zig-zag and honeycomb lattices in two dimensions depends on the network of exchange interactions and not on spin–orbit (SO) coupling, as was proposed in [5]. Rather, we found that the confinement of magnons can generate an effective OAM by disrupting the locally symmetric environment at each lattice site. In that sense, our previous efforts followed earlier studies of the OAM of magnons in confined spherical [18–20] and cylindrical [21–23] geometries. At least qualitatively, our

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predictions agreed with the well-known results of Matsumoto and Murakami [3, 4] for the self-rotation of magnon packets in the case of FM systems at low excitation energies.

Our previous work was based almost entirely on numerical simulations. In this paper, we obtain exact results for the OAM of magnons on FM and AF honeycomb lattices at the corners \( \mathbf{k}^* \) of the Brillouin zone (BZ) in reciprocal space (see figure 3) in the presence of Dzyaloshinskii–Moriya (DM) interactions, which are induced by SO interactions. In addition to providing checks on our numerics, this work also provides deeper insight into the profound and unexpected effects of the DM interaction on the OAM of honeycomb lattices. For a FM honeycomb lattice, we find that DM interactions can dramatically change the amplitude of the OAM near the BZ corners.

Section 2 summarizes the OAM formalism for the magnons of FM and AF honeycomb lattices. Section 3 presents our results for the FM honeycomb lattice in the absence of DM interactions. We then add DM interactions in section 4. Section 5 treats the AF honeycomb lattice. Finally, section 6 contains a conclusion.

2. OAM formalism

As sketched in figure 1, honeycombs are non-Bravais lattices with two unique sites 1 and 2. The total magnetization \( \mathbf{M}_i \) at site \( i \) can be written in terms of the dynamical magnetization \( \mu_i \) as

\[
\mathbf{M}_i = \mu_i + \mathbf{n} \cdot \mathbf{M}_0 - \mu_i^2,
\]

where the static magnetization \( \mathbf{M}_0 = 2 \mu_B \mathbf{S} \) lies along \( \mathbf{n} = \pm \mathbf{z} \) and \( \mu_i \cdot \mathbf{n} = 0 \). The ‘pseudo-momentum’ \( \mathbf{p} \) is derived [17] from the energy–momentum tensor \( T_{\alpha\beta} \) [24] for a model spin Lagrangian

\[
L = \frac{1}{4 \mu_B \mathbf{M}_0} \sum_{i=1}^{N} (\mu_i \times \mathbf{n}_i) \cdot \mu_i - H_2,
\]

where \( H_2 \) is obtained by expanding the Hamiltonian

\[
H = -\sum_{i=1}^{N} J_{\alpha\beta} \mathbf{M}_i \mathbf{M}_{i+1,\beta} - \sum_{i=1}^{N} \mathbf{M}_i \cdot \mathbf{h}_i
\]

\[
-\frac{K}{4 \mu_B} \sum_{i=1}^{N} \mathbf{M}_i^2 - \frac{1}{8 \pi} \sum_{i=1}^{N} \mathbf{h}_i^2,
\]

to second order in \( \mu_i \). Here, the exchange \( J_{\alpha\beta} \) includes possible antisymmetric terms like the DM interaction, the magnetic dipole field \( \mathbf{h}_i \) satisfies \( \nabla \times \mathbf{h}_i = 0 \) and \( \nabla \cdot \mathbf{h}_i = -4 \pi \nabla \cdot \mathbf{M}_i \), and \( K \) is the easy-axis anisotropy along \( \mathbf{z} \). It is straightforward to show [7, 8] that the Hamiltonian equations of motion for \( \mu_i \) can also be obtained from the Euler–Lagrange equations for \( L \). Consequently,

**Figure 1.** A honeycomb lattice with unique sites 1 and 2 and the translation vectors \( \mathbf{R}_1 = (3/2, -\sqrt{3}/2)a \), \( \mathbf{R}_2 = (3/2, \sqrt{3}/2)a \), and \( \mathbf{R}_3 = (0, \sqrt{3})a \) coupling neighboring sites 1.
are the OAM operators in real and momentum space.

Converting \( a^{(r)}_k \) and \( a^{(i)}_k \) to the Boson operators \( b^{(n)}_k \) and \( b^{(n)}_{-k} \) that diagonalize the Hamiltonian in terms of the eigenmodes \(|k, n\rangle = b^{(n)}_k |0\rangle\) with frequencies \(\omega_n(k) (n = 1, 2)\), we define [25]

\[
\begin{align*}
\hat{a}^{(r)}_k &= \sum_{n=1}^2 \left\{ X^{-1}(k)_m b^{(n)}_m X^{-1}(k^\ast)_m \right\}, \\
\hat{a}^{(i)}_{-k} &= \sum_{n=1}^2 \left\{ X^{-1}(k)_m b^{(n)}_{m+1} X^{-1}(k^\ast)_m \right\}.
\end{align*}
\]

(10)

The zero-temperature expectation value of \(\mathcal{L}_c\) in eigenmode \(n\) is then given by

\[
\mathcal{L}_{cn}(k) = \langle k, n| \mathcal{L}_c| k, n \rangle
\]

\[
= \hbar \sum_{r=1}^2 \left\{ X^{-1}(k)_m \hat{I} X^{-1}(k^\ast)_m - X^{-1}(k)_{m+1} \hat{I} X^{-1}(k^\ast)_{m+1} \right\}.
\]

(11)

For collinear spin states without symmetry-breaking DM interactions, \( X^{-1}(-k) = X^{-1}(k)\) so that \(\mathcal{L}_{cn}(k) = -\mathcal{L}_{cn}(-k)\) is an odd function of \(k\). The \(4 \times 4\)-dimensional matrix \(X(k)\) will feature prominently in our subsequent discussion.

Due to the linear terms \(k_i\) and \(k_j\) in the OAM operator \(\hat{I}_k\) of equation (9), the resulting \(\mathcal{L}_{cn}(k)\) is not a periodic function of \(k\) in reciprocal space. This shortcoming arises because the ‘pseudo-momentum’ \(p_i\) in \(\mathcal{L}_c = \sum_{r} (\mathbf{R}_i \times \mathbf{p}_i) \cdot \mathbf{z}\) contains the derivatives \(\partial/\partial x_i\) and \(\partial/\partial y_i\), from equation (4) that enter the real-space OAM operator \(\hat{I}_c\) of equation (8). On a discrete lattice, those continuous derivatives should be replaced by finite differences.

The finite difference of a discrete function \(f(r)\) produced by the translation vector \(\mathbf{R}_i\) is

\[
\delta f(r) = \frac{1}{2|\mathbf{R}_i|} \left\{ f(r + \mathbf{R}_i) - f(r - \mathbf{R}_i) \right\}.
\]

(12)

Transforming the continuous derivative \(\partial f(r)/\partial r_\alpha\) by summing over distinct translation vectors \(\mathbf{R}_i\), we obtain

\[
\Delta_\alpha f(r) = \sum_i \frac{R_{i\alpha}}{|\mathbf{R}_i|} \delta f(r)
\]

\[
= \sum_i \frac{R_{i\alpha}}{2|\mathbf{R}_i|^2} \left\{ f(r + \mathbf{R}_i) - f(r - \mathbf{R}_i) \right\},
\]

(13)

where \(\alpha\) is the projection of \(\mathbf{R}_i\) along the \(\alpha\) axis. Note that \(\Delta_\alpha f(r) \rightarrow \partial f(r)/\partial r_\alpha\) when the lattice translation vectors \(\mathbf{R}_i\) are orthogonal and their sizes vanish.

The Fourier transform of a magnon annihilation operator contains the factor \(\exp(i\mathbf{k} \cdot \mathbf{r})\), which has the discrete difference \(\Delta_\alpha \exp(i\mathbf{k} \cdot \mathbf{r}) = i\exp(i\mathbf{k} \cdot \mathbf{r}) \sum_i \frac{R_{i\alpha}}{|\mathbf{R}_i|^2} \sin(k_i\mathbf{R}_i)\) with periodic function

\[
\tilde{k}_\alpha = \sum_i \frac{R_{i\alpha}}{|\mathbf{R}_i|} \sin(k_i\mathbf{R}_i).
\]

(15)

For the honeycomb lattice in figure 1, summing over the three translation vectors \(\mathbf{R}_i\) that couple site 1 to three neighboring sites of type 1 produces

\[
\tilde{k}_\alpha = \sin(3k_i\mathbf{a}/2) \cos(\sqrt{3} k_i\mathbf{a}/2),
\]

\[
\tilde{k}_\alpha = \frac{1}{\sqrt{3}} \left\{ \sin(\sqrt{3} k_i\mathbf{a}/2) \cos(3k_i\mathbf{a}/2) \right\}.
\]

(16)

In the limit of small \(k_i\) and \(k_j\), \(\tilde{k}_\alpha \rightarrow 3k_i/2\) and \(\tilde{k}_\alpha \rightarrow 3k_j/2\).

The revised OAM operator in momentum space is then given by

\[
\hat{I}_k = -i \left( \tilde{k}_\alpha \frac{\partial}{\partial k_\alpha} - \tilde{k}_\beta \frac{\partial}{\partial k_\beta} \right).
\]

(17)

In the limit \(a \rightarrow 0\), there is no need to replace the momentum space derivatives \(\partial/\partial k_\alpha\) and \(\partial/\partial k_\beta\) by finite differences. Using the periodic functions \(\tilde{k}_\alpha\) and \(\tilde{k}_\beta\) instead of \(k_i\) and \(k_j\) imposes a natural bound on the OAM. As seen below, this replacement has important consequences for the OAM at the high-symmetry points of the honeycomb lattice.

For a FM state in the limit of small energies and momenta, our results for the OAM (equations (11) and (17)) reduce to the expression of Matsumoto and Murakami [3, 4], which was parameterized in terms of a density-of-states and an effective mass. Since we are interested in the OAM of both FM and AF magnons throughout the BZ, we prefer to work with the more general expressions given above.

3. FM honeycomb with \(D = 0\)

We first consider the FM honeycomb lattice shown in figure 2(a) with spins up, exchange coupling \(J > 0\), and no DM interaction. The Hamiltonian is then

\[
H = -J \sum_{(i,j)} \mathbf{S}_i \cdot \mathbf{S}_j,
\]

(18)

where the sum runs over all nearest neighbors of the honeycomb lattice. Second order in the vector operator \(\mathbf{v}_k = (a^{(1)}_k, a^{(2)}_k, a^{(1)}_{-k}, a^{(2)}_{-k})\), the Hamiltonian \(H_2 = \sum_k \mathbf{v}_k \cdot \mathbf{L}(k) \cdot \mathbf{v}_k\) is given by

\[
\mathbf{L}(k) = \frac{3JS}{2} \begin{pmatrix}
1 & -\Gamma_k^x & 0 & 0 \\
-\Gamma_k^x & 1 & 0 & 0 \\
0 & 0 & 1 & -\Gamma_k^x \\
0 & 0 & -\Gamma_k^x & 1
\end{pmatrix},
\]

(19)
where
\[
\Gamma_k = \frac{1}{3} \left\{ e^{ik_x} + e^{-i(k_x + \sqrt{3}k_y)/2} + e^{-i(k_x - \sqrt{3}k_y)/2} \right\}. \tag{20}
\]

The magnon dynamics is determined by diagonalizing \( L \cdot \mathbf{N} \) where
\[
\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{21}
\]
and \( I \) is the two-dimensional identity matrix. Using the relation \( \mathbf{N} \left( \mathbf{X} \right)^{-1}(\mathbf{k}) \cdot \mathbf{N} = \mathbf{X}^{-1}(\mathbf{k}) \) to normalize the eigenvectors [25], we find
\[
\mathbf{X}^{-1}(\mathbf{k}) = \frac{1}{\sqrt{2\Gamma_k}} \begin{pmatrix} -\Gamma_{k_x} & \Gamma_{k_x} & 0 \\ \Gamma_{k_x} & \Gamma_{k_x} & 0 \\ 0 & 0 & -\Gamma_{k_y} \end{pmatrix}. \tag{22}
\]

It is then simple to show that
\[
L_{\text{cn}}(\mathbf{k}) = \frac{\hbar}{2} \frac{\Gamma_{k_x}}{\left| \Gamma_{k_x} \right|} \frac{\Gamma_{k_y}}{\left| \Gamma_{k_y} \right|} \tag{23}
\]
is the same for magnon bands \( n = 1 \) and 2 with frequencies \( \omega_{1,2}(\mathbf{k}) = 3J\mathbf{S}(1 \pm |\Gamma_k|) \) (1 for +, 2 for −).

The upper and lower mode frequencies \( \omega_1(\mathbf{k}) \) and \( \omega_2(\mathbf{k}) \) cross at wavevectors \( \mathbf{k}_1^* = (0,2\sqrt{3}/9)2\pi/a \) and \( \mathbf{k}_2^* = (1,3/\sqrt{3})2\pi/a \), and symmetry-related points \( \mathbf{k}^* \) at the corners of the BZ where \( \Gamma_k = 0 \). As seen in figure 2(b), \( L_{\text{cn}}(\mathbf{k}^*) \) has alternating values of \( \sim 0.185\hbar \) at these crossing points with the positive value \( \sim 0.185\hbar \) at \( \mathbf{k}_1 \).

To analytically evaluate the OAM at \( \mathbf{k}^* \), we construct the unit vector \( \mathbf{k}^* \).

\[
\hat{u}(\mathbf{k}) = \frac{\Gamma_{k_x}}{|\Gamma_{k_x}|} \frac{\Gamma_{k_y}}{|\Gamma_{k_y}|} A_+ \left( \mathbf{k} \right) \cos(3k_xa/4) + iA_- \left( \mathbf{k} \right) \sin(3k_xa/4) \tag{24}
\]

is singular at \( \mathbf{k}_1^* \); we calculate the OAM at fixed \( k_y = 0 \) while \( k_y = (2\sqrt{3}/9)2\pi/a \) is approached from below, as sketched in figure 3. (Of course, the OAM can also be evaluated by approaching \( \mathbf{k}_1^* \) from above or from any other direction.) We shall then evaluate the OAM at other BZ corners \( \mathbf{k}_m \) by applying a rotation operator.

When \( k_x = 0 \) but \( k_y < (2\sqrt{3}/9)2\pi/a \) and \( \mathbf{k} = 1 + 2\cos(\sqrt{3}k_xa/2) \) and \( \hat{u}(\mathbf{k}) = 1 \). It is then straightforward to show that
\[
\lim_{k_x \to 0} \frac{\partial \hat{u}(\mathbf{k})^*}{\partial k_x} = -\frac{3ia}{2} \frac{1}{1 + 2\cos(\sqrt{3}k_xa/2)}. \tag{27}
\]

Since \( k_x \partial \hat{u}(\mathbf{k})/\partial k_x \) vanishes as \( k_x \to 0 \), we obtain
\[
L_{\text{cn}}(\mathbf{k}_1^*) = \frac{ih}{4} \lim_{k_x \to k^*} \frac{\partial \hat{u}(\mathbf{k})^*}{\partial k_x} = \frac{\sqrt{3}h}{8} \lim_{k_x \to 4\pi/3} \frac{\sin(\sqrt{3}k_xa/2) + \sin(\sqrt{3}k_xa)}{1 + 2\cos(\sqrt{3}k_xa/2)} = \frac{3h}{10} \approx 0.1875\hbar. \tag{28}
\]

which uses \( k_x = \sqrt[3]{\sin(\sqrt{3}k_xa/2) + \sin(\sqrt{3}k_xa)} \)/\( \sqrt{3} \). Notice that it is essential to use the periodic function \( k_x \) in the OAM operator \( \hat{t}_k \) and in equation (28) to produce a finite result for \( L_{\text{cn}}(\mathbf{k}_1^*) \).

Suppose wavevector \( \mathbf{k}' = \mathbf{R}(\theta) \cdot \mathbf{k} \) is obtained from wavevector \( \mathbf{k} \) by applying the rotation operator
\[
\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \tag{29}
\]
about the $z$ axis. For $\theta = \pi/3$, it is easy to show that $\Gamma_{k'} = \Gamma_{k} - \Gamma_{-k}$ and, using equation (22), that $X^{-1}(k') = X(k)^*$ = $X(-k)$. So with high-symmetry points $k'_m = R_{\pi/3} \cdot k_{m-1}$ and $k'_{m-1}$ rotated with respect to one another by $\pi/3$, $L_{\omega m}(k_m) = L_{\omega m}(-k'_{m-1}) = -L_{\omega m}(k_{m-1})$. Hence, the OAM changes sign around the reciprocal-space hexagon in figure 3.

4. FM honeycomb with DM interaction

We next consider the FM honeycomb lattice shown in the inset to figure 4 with spins up and DM interaction $D$ between nearest neighbors. The Hamiltonian is then given by

$$H = -J \sum_{\langle ij \rangle} S_i \cdot S_j - D \sum_{\langle ij \rangle} z \cdot (S_i \times S_j)$$

$$+ D \sum_{\langle ij \rangle} z \cdot (S_i \times S_j) - K \sum_{i} S_i^2,$$

where the first DM sum runs over the first triangle shown in the inset to figure 4 and the second sum with opposite sign runs over the second triangle. The anisotropy $K > 0$ is assumed to be sufficiently large to prevent the spins from tilting away from the $z$ axis.

We then obtain

$$L(k) = \frac{3JS}{2} \left( \begin{array}{cccc} 1 - G_k & -\Gamma_k^+ & 0 & 0 \\ -\Gamma_k^- & 1 + G_k & 0 & 0 \\ 0 & 0 & 1 + G_k & -\Gamma_k^+ \\ 0 & 0 & -\Gamma_k^- & 1 - G_k \end{array} \right),$$

(31)

where $G_k = d \Theta_k$ with $d = -2D/3J$ and

$$\Theta_k = 4 \cos(3k_d/2) \sin(\sqrt{3}k_d/2) - 2 \sin(\sqrt{3}k_d).$$

(32)

Because the anisotropy $\kappa = 2K/3J$ merely shifts the magnon energies

$$\hbar \omega_{1,2}(k) = \frac{3JS}{2} \left( 1 + \kappa \pm \sqrt{1+\kappa^2} \right),$$

(33)

but does not affect the OAM, we neglect its contribution to $L(k)$.

It follows that

$$X^{-1}(k) = \frac{1}{\sqrt{2}\Gamma_k} \left( \begin{array}{cccc} 1 - \Gamma_k^+ & \Gamma_k^- & 0 & 0 \\ \Gamma_k^+ & 1 - \Gamma_k^+ & 0 & 0 \\ 0 & 0 & 1 + \Gamma_k & -\Gamma_k^- \\ 0 & 0 & -\Gamma_k^+ & 1 - \Gamma_k \end{array} \right),$$

(34)

where

$$\Gamma_k^\pm = 1 \pm \frac{G_k}{\sqrt{1+\kappa^2 + G_k^2}}.$$  

(35)

Since $F_k^\pm$ is real, it is simple to show that

$$L_{1,2}(k) = \frac{\hbar}{4} F_k \frac{\Gamma_k^+}{|\Gamma_k|} \Gamma_k^+ \frac{\Gamma_k}{|\Gamma_k|}.$$  

(36)

for the lower and upper magnon bands, respectively. Because $G_k = -G_k$ and $F_k^\pm = F_k^\mp$, the OAM satisfies the relation $L_{1,2}(k) = L_{1,2}(-k)$ and is no longer an odd function of $k$ within a single band.

Numerical results for the OAM at $d = 0.001$ and 0.01 are plotted in figure 4. While the OAM for the lower band on the bottom of figure 4 is biased towards positive values, the OAM for the upper band on the top is biased towards negative values. Nevertheless, the sum of the OAM over the two bands vanishes, as guaranteed by the symmetry relation above.

These results are verified by the one-dimensional slice along the edge of the BZ plotted in figure 5 with $k_d/2\pi = 1/3$ and $k_d/2\pi$ running from $-\sqrt{3}/9$ to $\sqrt{3}/9$. The corners $k_1^*$ and $k_2^*$ of the BZ lie at the beginning and end of this path. For $d = 0$, the OAM equals $\pm 0.1875\hbar$ at the corners of the BZ for both bands. But even a small DM interaction changes that result dramatically.

This can be readily seen from an exact solution. Since $\Theta_{k_m} = 3\sqrt{3} > 0$ and $\Theta_{k_{m+1}} = -3\sqrt{3} < 0$, it follows that $F_{k_m}^+ = 2$, $F_{k_{m+1}}^+ = 0$, $F_{k_m}^- = 0$, and $F_{k_{m+1}}^- = 2$. Using the results of the previous section, we then find that for $d \neq 0$,

$$L_{1,2}(k_m^*) = \frac{3\hbar}{8} = 0.375\hbar,$$

(38)

$$L_{1,2}(k_{m+1}^*) = 0,$$

(39)

$$L_{1,2}(k_m^*) = 0,$$

(40)

$$L_{1,2}(k_{m+1}^*) = -\frac{3\hbar}{8} = -0.375\hbar.$$  

(41)
in agreement with the numerical results of figure 5. Remarkably, these exact results are satisfied in the presence of infinities of the DM interaction. But as can be easily seen from figures 4 and 5, the region around $k^*$ with a large or vanishingly small value of $L_{\text{av}}(k)$ shrinks as $d \to 0$. Notice that figures 5(a) and (b) satisfy the symmetry relation $L_{\text{av}}(k) = -L_{\text{av}}(-k)$.

5. AF honeycomb

Finally, we consider the AF honeycomb lattice in figure 6(a) with alternating up and down spins, exchange coupling $J < 0$, easy-axis anisotropy $K$, and DM interaction $D$. Then

$$L(k) = \frac{-3JS}{2} \begin{pmatrix} 1 + \kappa & 0 & 0 & -\Gamma_k^* \\ 0 & 1 + \kappa & -\Gamma_k & 0 \\ 0 & -\Gamma_k & 1 + \kappa & 0 \\ -\Gamma_k & 0 & 0 & 1 + \kappa \end{pmatrix}$$

(42)

with corresponding

$$X^{-1}(k) = \frac{1}{\sqrt{2k^* k_k}} \begin{pmatrix} -h_k \Gamma_k^* & 0 & 0 & h_k^* \Gamma_k \\ 0 & h_k \Gamma_k & -h_k^* g_k & 0 \\ 0 & -h_k \Gamma_k & h_k^* g_k & 0 \\ -h_k^* g_k & 0 & 0 & h_k \Gamma_k \\ 
\end{pmatrix}$$

(43)

where $f_k = \sqrt{(1 + \kappa)^2 - |\Gamma_k|^2}$, $g_k^\pm = 1 + \kappa \pm f_k$, and $h_k^\pm = \sqrt{g_k^2}$. Because the DM interactions shift the degenerate mode frequencies

$$\omega_{1,2}(k) = 3JS \left\{ \sqrt{(1 + \kappa)^2 - |\Gamma_k|^2} + G_k \right\}$$

(44)

through the $G(k)$ term but do not affect the OAM, we neglect their contribution to $L(k)$ above. Surprisingly, the degenerate magnon bands exhibit distinct OAM with

$$L_{\text{av}}(k) = \frac{h \Gamma_k^*}{4f_k} \frac{\Gamma_k}{|\Gamma_k|} \frac{\hat{l}_k}{|\hat{l}_k|}$$

(45)

As seen in figure 6(b), the major ($n = 1$) and minor ($n = 2$) bands have different OAM patterns but are both threefold symmetric. Notice that $L_{\text{av}}(k) = (L_{\text{av}}(k) + L_{\text{av}}(k^*))/2$ for the two bands of the AF honeycomb lattice equals the OAM $L_{\text{av}}(k)$ of the FM honeycomb lattice with $D = 0$, given by equation (23) and plotted in figure 2(b).

At the BZ corners $k^*_n$, $\Gamma_k = 0$, $f_k = 1$, and the magnon frequencies reach maxima of $\omega_n(k^*) = 3/8(1 + \kappa \pm 3\sqrt{3}\ell D)$ at $k^*_{2n}$ and $k^*_{2n+1}$, respectively. Comparing equations (45) and (46) with equation (23), we find that $L_{\text{av}}(k^*) = \pm (3\ell/8)(1 + \kappa/2)$ and $L_{\text{av}}(k^*) = \mp (3\ell/16)\kappa$. Hence, the average amplitude $|L_{\text{av}}(k^*)|$ of the two bands of the AF honeycomb lattice equals $3\ell/16$, as expected.

6. Conclusion

The theory developed in this paper can be easily extended to treat many real three-dimensional materials that contain honeycomb lattices lying within their $ab$ planes. Sivadas et al. [26] reviewed the magnetic phase diagrams of honeycomb systems with chemical formula $\text{ABX}_3$. Some examples of FM honeycomb lattices within that class are CrSiTe$_3$ and CrGeTe$_3$ [27–29]. Two other Cr-based FM honeycomb systems are CrI$_3$ [30, 31] and CrCl$_3$ [32]. Two examples of AF honeycomb lattices are MnPS$_3$ and MnPSe$_3$ [33].

The exact calculations in this paper reveal several pertinent points about the OAM of magnons on honeycomb lattices. First, finite results for the OAM at high-symmetry points $k^*$ are only reached if periodic functions $\hat{l}_k$ and $\hat{S}_k^z$ are used in the OAM operator $\hat{L}_k$. If non-periodic terms $k_i$ and $k_z$ were retained in the OAM operator, then the OAM would diverge at the high-symmetry points $k^*$. Second, the OAM of the FM and AF honeycomb lattices are closely connected. The average OAM of the major and minor AF honeycomb bands equals the OAM of the FM bands. Third, the largest OAM for the pure honeycomb lattice with amplitude $3\ell/8$ is still substantially less than $h$. So the spin angular momentum of a single spin flip is bigger than the largest OAM. Fourth, the DM interaction produced by SO coupling controls the behavior of the OAM near the corners of the BZ, no matter how small the size of $D$. We conclude that once the network of exchange interactions generates an OAM by confining the magnons, SO
coupling can still have important effects. Experimentally, the maximum OAM expected for a FM honeycomb material with even a small to moderate value of the DM interaction should be about 0.375 $\hbar$. Although performed at zero temperature, these calculations indicate the available magnon states that can be filled at nonzero temperatures. For the FM honeycomb lattice with $D = 0$, the symmetry relation $\mathbb{L}_{\theta n}(\mathbf{k}) = -\mathbb{L}_{\theta n}(-\mathbf{k})$ for magnon bands $n = 1$ and 2 implies that the thermal average $\langle \mathbb{L}_z \rangle$ vanishes due to the integral over wavevectors $\mathbf{k}$. As discussed above, the DM interaction $D$ breaks that symmetry relation leaving only $\mathbb{L}_{\theta 1}(\mathbf{k}) = -\mathbb{L}_{\theta 2}(-\mathbf{k})$. Consequently, the lower or upper magnon band will carry positive or negative OAM, respectively. This implies that $\langle \mathbb{L}_z \rangle$ peaks at a temperature $T$ of order $D^2/J$. We will examine this behavior more carefully in future work.

Even for the simple case of a honeycomb lattice with a single exchange interaction, nearest-neighbor DM interactions, and easy-axis anisotropy, many questions remain unanswered. What is the coupling between the spin and OAM: do they tend to lie parallel or antiparallel? What is the effect of next-neighbor exchange couplings for both FM and AF honeycomb lattices? Of magnetic fields? Is there some easy way to understand the qualitative difference between the OAM of the major and minor bands of the AF honeycomb lattice? Since the major and minor bands lie at the same energy and momentum, can OAM be easily exchanged between the magnons of the two bands? We hope that future research on the OAM of magnons in other non-Bravais lattices will be motivated by the exact results presented in this work.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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