Zitterbewegung and symmetry switching in Klein’s four-group

L. Chotorlishvili\textsuperscript{1}\textsuperscript{,} P. Zi\˛eba\textsuperscript{2}, I. Tralle\textsuperscript{2} and A. Ugulava\textsuperscript{3}

\textsuperscript{1} Institut für Physik, Martin-Luther Universität Halle-Wittenberg, D-06120 Halle/Saale, Germany
\textsuperscript{2} Faculty of Mathematics and Natural Sciences, University of Rzeszow, Pigatoria str. 1, 35-310 Rzeszow, Poland
\textsuperscript{3} Faculty of Mathematics and Natural Sciences, Tbilisi State University, Chavchavadze av.3, 0128 Tbilisi, Georgia

E-mail: levan.chotorlishvili@gmail.com

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Abstract

Zitterbewegung is the exotic phenomenon associated either with relativistic electron–positron rapid oscillation or to electron–hole transitions in narrow gap semiconductors. In the present work, we enlarge the concept of Zitterbewegung and show that trembling motion may occur due to dramatic changes in the symmetry of the system. In particular, we exploit a paradigmatic model of quantum chaos, the quantum mathematical pendulum (universal Hamiltonian). The symmetry group of this system is Klein’s four-group that possesses three invariant subgroups. The energy spectrum of the system parametrically depends on the height of the potential barrier, and contains degenerate and non-degenerate areas, corresponding to the different symmetry subgroups. Change in the height of the potential barrier switches the symmetry subgroup and leads to trembling motion. We analyzed mean square fluctuations of the velocity operator and observed that trembling is enhanced in highly excited states. We observed a link between the phenomena of trembling motion and the uncertainty relations of noncommutative operators of the system.

Keywords: quantum nonlinear resonance, Zitterbewegung, statistical physics, dynamical system, quantum chaos

(Some figures may appear in colour only in the online journal)
1. Introduction

Zitterbewegung (ZB) is the trembling motion, a phenomenon discovered by Schrödinger [1]. For a long time, ZB was associated solely with the Dirac equation. Due to its purely relativistic nature, oscillation frequency of the relativistic ZB $2mc^2/h \approx 10^7$ THz is far beyond experimentally detectable frequencies. Therefore, interest in ZB dwindled soon after its discovery. Nevertheless, in the last decade we have witnessed renewed interest in this phenomena.

The reason for the renewed interest of ZB is the application of the Dirac equation in non-relativistic condensed matter physics. In particular, in systems with a spin-orbit coupling (graphene, 2D electron gas, topological insulators) light velocity is replaced by the velocity of electrons at the Fermi surface. Naturally, this lowers the threshold frequency of the ZB towards the experimentally detectable scale. The relativistic ZB concerns the electron–positron rapid transition oscillations, while in the narrow-gap semiconductors one could talk about oscillations due to the mixing of the conductance and hole band states [2]. ZB of electronic wave packets has been studied in semiconductor quantum wells [3] and optical traps [4].

A fascinating relativistic effect of condensed matter physics is the Klein paradox in graphene [5]. Due to the matching of the electron and positron wave functions across the barrier, the transmission probability is large even for high barriers. This effect cannot occur in the non-relativistic case because of exponential decay of the transmission probability with the barrier height. The Hamiltonian of a single ion trapped in the Paul trap bears a striking resemblance to the Dirac Hamiltonian [6]. This allows experimental observation of ZB in cold atom physics [7, 8]. Note that in case of the ‘nonrelativistic’ ZB particle, trembling motion is not necessarily performed by a free particle but can be a particle trapped in the external potential [9, 10]. The role of the truncated Coulomb potential for 1D Dirac materials has been studied recently [11].

In the broader sense, ZB may occur in an arbitrary system characterized by anomalous velocity relevant to the case when the momentum operator commutes with the Hamiltonian but the velocity operator does not commute. In particular, the Hamiltonian of the relativistic ZB reads:

$$\hat{H} = c(\alpha \hat{p}) + \gamma_0 mc^2, \quad \frac{d\hat{p}}{dt} = 0, \quad \frac{d\hat{r}}{dt} = c\hat{a},$$

(1)

$$\alpha = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -i\sigma^0 \\ i\sigma^0 & 0 \end{pmatrix},$$

(2)

leading to the time dependent linear and trembling terms

$$x(t) = c^2 \hat{p}, \hat{H}^{-1} t - \frac{ch^2}{4} \hat{c}_1(0) \exp \left( -\frac{2i\hat{H}t}{h} \right) \hat{H}^{-1}.$$  

(3)

Here $\sigma^0$ is the vector of Pauli matrices and $\hat{H}$ is the free particle Dirac’s Hamiltonian. Thus, in general, motion may have a trembling character if the following commutator is nonzero

$$\frac{d\hat{r}}{dt} = (1/sh) [\hat{r}, \hat{H}].$$

(4)

Typically, the phase space of nonintegrable dynamical systems contains different areas with topologically different characteristic phase trajectories. A paradigmatic model of a complex, chaotic system with minimal chaos is the perturbed mathematical pendulum. The mathematical pendulum is exactly integrable in the absence of time-dependent external driving. However, when a time dependent perturbation is applied, dynamics in the vicinity of the separatrix become chaotic. For more details, one can refer to [12].
2. Quantum parametrical resonance and Mathieu–Schrödinger equation

An atom in the external electric field can be described via the driven nonlinear oscillator model (Lorentz’s model):

\[ H(x, p, t) = H_0(x, p) + H_{NL} + \varepsilon V(x, t), \]  

where

\[ H_0 = \frac{1}{2} \left( \frac{\dot{p}^2}{m} + \omega_0^2 m x^2 \right), \quad H_{NL} = \beta x^3 + \mu x^4 + \ldots, \]

\[ V(x, t) = V_0 \cos \Omega t, \quad \varepsilon V_0 = e_f_0, \quad \varepsilon \ll 1. \]

Here \( x \) and \( p \) are the position and momentum of the electron, \( \omega_0 \) is the frequency of the oscillations, \( \beta \) and \( \mu \) are constants of the nonlinear terms. We note that in the regime of moderate nonlinearity, in the nonlinear term, \( H_{NL} \) is enough to retain \( \beta x^3, \mu x^4 \) terms only [12]. By means of the transformation to the canonical action–angle variables \( x = (2I/m\omega_0)^{1/2} \cos(\theta) \) \( p = -(2Im\omega_0)^{1/2} \sin(\theta) \) and assuming that the resonance condition holds \( \Omega = \omega_0 \) one can deduce the transformed Hamiltonian

\[ H = H_0(I) + \varepsilon V(I) \cos(\varphi), \]

where

\[ H_0(I) = \omega_0 I + H_{NL}, \quad H_{NL} = \frac{3\pi}{4} \frac{I^2}{m\omega_0^2} \mu, \]

and

\[ \varphi = \theta - \omega t, \quad \varepsilon V(I) = V_0 \sqrt{I/m\omega_0}. \]

We assume that deviation of the action \( \Delta I = I - I_0 \) from the nonlinear resonance condition \( \omega_0 + \omega_{NL}(I_0) = \Omega \), \( \omega_{NL} = (3\pi/2)(I\mu/m\omega_0^2) \) is small. After implementing the series expansion finally we obtain

\[ H = \frac{\omega'}{2} (\Delta I)^2 + U \cos \varphi. \]

Here \( \omega' = \left. \frac{d\omega_{NL}(I)}{dI} \right|_{I=I_0}, U = \varepsilon V(I_0). \)

The classical phase space of the Hamiltonian (11) consists of the two topologically different domains: domains of the closed and open phase trajectories divided by the area of separatrix. Thus, the solution of the classical problem shows a bifurcation tendency. Namely, the solution drastically depends on the total energy of the system \( E \) and in the explicit form reads:

\[ \Delta I = \sqrt{(E + U)\omega'\text{dn} \left( \omega' \sqrt{(E + U)\omega' t, k} \right)}, E > U \]

\[ \Delta I = \sqrt{(E + U)\omega'\text{cn} \left( \omega' \sqrt{(E + U)\omega' t, 1/k} \right)}, E < U. \]

Here \( \text{dn}(u, \varphi) \) and \( \text{cn}(u, \varphi) \) are the Jacobian delta amplitude and Jacobian elliptic cosine, respectively. Frequency of the system \( \omega(I) = \pi/\ln \left( 32/(1 - E) \right) \) diverges logarithmically in the vicinity of the separatrix \( k = \sqrt{2U/(E + U)} = 1 \). Equilibrium points
are defined via condition \( p_s = 0, \frac{dU(q_s)}{dq} = 0 \). In the vicinity of the equilibrium point \( p - p_s = \pm (E - E_s - \frac{1}{2} \frac{d^2U(q_s)}{dq^2}(q_s)(q - q_s)^2)^{1/2} \). Our particular interest concerns hyperbolic equilibrium points where motion is unstable \( \frac{d^2U(q_s)}{dq^2} < 0 \). When a time dependent perturbation is applied; the stochastic layer and the complex homoclinic structure appear in the vicinity of the separatrix. The width of the layer is proportional to the perturbation strength.

Due to the fundamental principle of the correspondence, one could expect to see nontrivial behavior of the system in the quantum case as well. Transition to the quantum case can be performed through the substitution \( \Delta l \rightarrow -i\hbar \partial / \partial \phi \) and after a little algebra we deduce the Mathieu–Schrödinger equation:

\[
\frac{d^2\psi_n}{d\phi^2} + (E_n - V(l, \phi))\psi_n = 0. \tag{14}
\]

Here \( V(l, \phi) = 2l \cos 2\phi \) and we rescaled energy, potential barrier and angle, respectively: \( E_n \rightarrow \frac{8E_n}{\hbar^2}, l \rightarrow \frac{8l}{\hbar^2}, \phi \rightarrow 2\phi \).

We note that the Mathieu–Schrödinger equation can be derived in a less formal way by considering a model of a quantum nonlinear oscillator interacting with a strong electric field [13]. A detailed analysis of the Mathieu–Schrödinger equation (14) can be found in references [15, 16]. The energy spectrum of the Mathieu–Schrödinger equation parametrically depends on the potential barrier \( E_n(l) \) and contains two degenerate and one non-degenerate domain. The main discovery of [16] is the link between the quantum parametric resonance and Klein’s four-group. Namely, transformation operations

\[
G(\phi \rightarrow -\phi) = a, \quad G(\phi \rightarrow \pi - \phi) = b, \\
G(\phi \rightarrow \pi + \phi) = c, \quad G(\phi \rightarrow \phi) = e, \tag{15}
\]

of the Mathieu functions \( ce_n(\phi), se_n(\phi) \) form Klein’s four-group \( G \) with the following three invariant subgroups

\[
G_0 \subset e, a, \\
G_+ \subset e, b, \\
G_- \subset e, c, \tag{16}
\]

Irreducible representation basis functions of the subgroup \( G_0 \) formed by Mathieu functions correspond to the non-degenerate energy spectrum \( \xi_n(\phi, l) = ce_n(l, \phi), se_n(l, \phi) \), while irreducible representation basis functions \( \phi_n^\pm(\phi, l) = \frac{\sqrt{2}}{l}(ce_n(l, \phi) \pm ise_n(l, \phi)) \) and \( \psi_n^\pm(\phi, l) = \frac{\sqrt{2}}{l}(ce_n(l, \phi) \pm ise_{n+1}(l, \phi)) \) of the two other subgroups \( G_-, G_+ \) correspond to the degenerate energy spectrum.

These three domains \( G_-, G_+ \) and \( G_0 \) on the parametric space \( (E_n(l), l) \) are divided by splitting and merging points of the Mathieu characteristics (see figure 1). The slight variation of the barrier’s height \( l(t) = \tilde{l}_n \pm \Delta l \cos(\omega t) \) in the vicinity of the splitting and merging points \( \tilde{l}_n \) leads to abrupt changes in the symmetry of the system. The key issue is that the values \( l = \tilde{l}_n - \Delta l \) and \( l = \tilde{l}_n + \Delta l \) belong to different symmetry subgroups \( G_-, G_+, G_0 \). Note that the values of the splitting and merging points can be defined precisely for each quantum level (see tables 1 and 2).
Our primary interest concerns the question of whether the switching of the symmetry subgroup may lead to trembling motion. To answer this question we explore expectation values of the commutator

\[ \mathcal{v}_\phi = \dot{\phi} = \frac{1}{i} [\phi, H], \]

that means:

\[ \langle \phi_n^\pm (\varphi, \ell) | \dot{\varphi} | \phi_n^\pm (\varphi, \ell) \rangle, \]

\[ \langle \xi_n (\varphi, \ell) | \dot{\varphi} | \xi_n (\varphi, \ell) \rangle, \]

\[ \langle \xi_n (\varphi, \ell) | \dot{\varphi} | \xi_n (\varphi, \ell) \rangle, \]

\[ \langle \psi_n^\pm (\varphi, \ell) | \dot{\varphi} | \psi_n^\pm (\varphi, \ell) \rangle. \]

Here \( \ell_n \) in equation (17) are the splitting points corresponding to the transitions \( G_- \rightarrow G_0 \), while the merging points \( \ell_n \) in equation (18) correspond to the transition \( G_0 \rightarrow G_+ \). In short, the following notations of the irreducible basis functions are adopted:

\[ G^- \rightarrow \phi_n^\pm (\varphi, \ell) = \frac{\sqrt{2}}{2} (c_{\ell n}(l, \varphi) \pm i s_{\ell n}(l, \varphi)), \]

\[ G^0 \rightarrow \xi_n (\varphi, \ell) = c_{\ell n}(l, \varphi), \text{ or } \eta_n (\varphi, \ell) = s_{\ell n}(l, \varphi), \]

3. Switching of the symmetry subgroup and observable quantities

Figure 1. Parametric dependence of the energy spectrum \( E_n(l) \) of the Mathieu–Schrödinger equation (Mathieu characteristics) on the barrier height \( l \). Splitting and merging points define the boundaries of the \( G_-, G_+ \) and \( G_0 \) subgroups.

Table 1. Splitting points of the parameter \( l \) for the transition from the \( G_- \) subgroup to the \( G_0 \) subgroup for different energy levels \( E_n \).

| \( n \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \ell \) | 0.0 | 0.2 | 1.14| 3.17| 6.42| 10.95| 16.78| 23.93|

Table 2. Merging points of the parameter \( l \) for the transition from the \( G_0 \) subgroup to the \( G_+ \) subgroup for different energy levels \( E_n \).

| \( n \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \ell \) | 3.42| 7.51| 13.93| 18.4| 24.69| 32.23| 40.96| 50.84|
and

\[ G^+ \to \psi_n^+(\varphi, l) = \frac{\sqrt{2}}{2} \left( ce_n(l, \varphi) \pm i se_{n+1}(l, \varphi) \right). \]  (21)

In order to explore the effect of symmetry switching, we evaluate equations (17) and (18) in the limit of \( \Delta l \to 0 \).

The expressions for the expectation value of the velocity operator can be further simplified using a trigonometric representation of Mathieu functions [17]:

\[ ce_{2m}(l, \phi) = \sum_{r=0}^{\infty} A_{2r}^{(2m)}(l) \cos(2r\phi), \]  (22)
\[ ce_{2m+1}(l, \phi) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}(l) \cos((2r + 1)\phi), \]  (23)
\[ se_{2m+1}(l, \phi) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)}(l) \sin((2r + 1)\phi), \]  (24)
\[ se_{2m+2}(l, \phi) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)}(l) \sin((2r + 2)\phi). \]  (25)

Here \( A_{2r}^{(2m)}(l) \), \( A_{2r+1}^{(2m+1)}(l) \) and \( B_{2r+1}^{(2m+1)}(l) \), \( B_{2r+2}^{(2m+2)}(l) \) are the Fourier coefficients that depend on the quantum number \( m \) and the barrier height \( l \). We note that trembling should occur directly at the bifurcation (splitting/merging) points. We are interested in the estimation of the velocity increment in the bifurcation point \( \Delta v_\varphi = v_\varphi(G_0, l = l_c + \Delta l)_{\Delta l \to 0} - v_\varphi(G_0, l = l_c - \Delta l)_{\Delta l \to 0} \), where \( v_\varphi(G_0, l = l_c - \Delta l)_{\Delta l \to 0} \) is the expectation value of velocity before passing the bifurcation point and \( v_\varphi(G_0, l = l_c + \Delta l)_{\Delta l \to 0} \) is the expectation value of velocity in the subgroup \( G_0 \) after passing the bifurcation point.

Taking into account (22)–(25) we derive analytical expressions of the expectation values of the velocity operator at the point \( l_c \) corresponding to the switching of the symmetry subgroups. In particular, for the symmetry switching \( G_- \to G_0 \), in the limit \( l \to l_c \) we deduce that:

- For the states \( \phi_{2n+1}^+(l, \varphi) \)

\[ \langle \phi_1 \rangle_{l \to l_c^+} = -2i \sum_{r=0}^{\infty} (2r + 1) A_{2r+1}^{(2n+1)}(l_c) B_{2r+1}^{(2n+1)}(l_c), \]  (26)

- For the states \( \xi_{2n+1}(l, \varphi) \)

\[ \langle \xi_1 \rangle_{l \to l_c^-} = -2i \sum_{r=0}^{\infty} (2r + 1) A_{2r+1}^{(2n+1)}(l_c) B_{2r+1}^{(2n+1)}(l_c), \]  (27)

- For the states \( \eta_{2n+1}(l, \varphi) \)
After straightforward calculations in the limit $\lim \frac{\Delta}{\epsilon}$ of the high potential barrier and this naturally suppresses kinetic effects. The above expressions allow us to determine the jumps in the value of velocity at the bifurcation point for different transitions between $G_-$ and $G_0$ states. Results of the calculations are presented in table 3. We note that trembling occurs only because of the fact that the separatrix line is the border between different symmetry subgroups of the Mathieu–Schrödinger equation. Eigenfunctions are smooth functions of the barrier height and therefore within the subgroups the effect of trembling is absent.

Table 3. Jump in velocity for the transitions $G_- \rightarrow G_0$ between the states $\phi^\pm_n(l, \varphi) \rightarrow c\phi_n(l, \varphi)$:

| $n$ | $\phi_+ \rightarrow \xi$ | $\phi_+ \rightarrow \eta$ | $\phi_- \rightarrow \xi$ | $\phi_- \rightarrow \eta$ |
|-----|----------------|----------------|----------------|----------------|
| 1   | 2.            | 2.             | -2.            | -2.            |
| 2   | 3.981         | 3.981          | -3.981         | -3.981         |
| 3   | 5.929         | 5.929          | -5.929         | -5.929         |
| 4   | 7.927         | 7.927          | -7.927         | -7.927         |
| 5   | 9.815         | 9.815          | -9.815         | -9.815         |
| 6   | 11.665        | 11.665         | -11.665        | -11.665        |
| 7   | 13.437        | 13.437         | -13.437        | -13.437        |
| 8   | 13.884        | 13.884         | -13.884        | -13.884        |

\(\langle \hat{v} \rangle_{l \rightarrow l^+} = -2i\)
\times \langle \eta_{2n+1}(l_c + \Delta l, \varphi) | \frac{\partial}{\partial \varphi} \eta_{2n+1}(l_c + \Delta l, \varphi) \rangle_{\Delta l \rightarrow 0}
= 0. \hspace{1cm} (28)

The results for the quantum states $\Psi_2^{\pm}(l, \varphi), \Psi_2^{-}(l, \varphi)$ can be obtained in a similar way (not shown for brevity). An interesting fact is the absence of trembling during transitions between subgroups $G_0 \rightarrow G_+$. The reason is quite clear. Transition $G_0 \rightarrow G_+$ occurs in the limit of the high potential barrier and this naturally suppresses kinetic effects.

To infer the increment in the squared velocity that occurs in the bifurcation point, we calculate the expectation value of the squared velocity operator \((\Delta \hat{v}_\varphi^2)_{l^+}\). In particular, we estimate the jump that occurs due to symmetry switching \(\Delta v^2_\varphi = v^2_\varphi(G_- \rightarrow G_0) - v^2_\varphi(G_-)\). After straightforward calculations in the limit $l \rightarrow l_c$ we deduce:

- for the states $\phi^\pm_{2n+1}(l, \varphi)$
\(\langle \hat{v}^2 \rangle_{l \rightarrow l^+} = -4\)
\times \langle \phi^\pm_{2n+1}(l_c - \Delta l, \varphi) | \frac{\partial^2}{\partial \varphi^2} \phi^\pm_{2n+1}(l_c - \Delta l, \varphi) \rangle_{\Delta l \rightarrow 0}
= -8 \sum_{r=0}^{\infty} (2r + 1)^2 \left( (A_{2r+1}^{2n+1}(l_c))^2 + (B_{2r+1}^{2n+1}(l_c))^2 \right). \hspace{1cm} (29)

- for the states $\xi_{2n+1}(l, \varphi)$
The results of the calculations for the jumps in the expectation values of the squared velocity at the bifurcation point for different transitions between symmetry subgroups $G_-$ and $G_0$ are summarized in table 4.

It is easy to see that the mean increment of the squared velocity (equations (29)–(31)) $\Delta v^2_{\phi}$ is not equal to the square of the increment of mean velocity (equation (26)) $(\Delta v_{\phi})^2$. Taking into account equations (26)–(31), we calculate the mean square fluctuations $\mathcal{F} = \sqrt{\Delta v^2_{\phi} - (\Delta v_{\phi})^2}$ of the velocity increment (see tables 3 and 4). Apparently, fluctuations increase with the quantum number $n$ (see figures 2 and 3). Our results confirm that not only free particles experience trembling motion, but also particles trapped in the potential well in excited quantum states.

4. Uncertainty relations and crossover with trembling motion

Overwhelmingly, under the ‘uncertainty’ of arbitrary quantity $\hat{A}$, the mean square deviation is meant $\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$. This formulation was established by the works of Heisenberg. In the present work, we are interested in the question of whether a crossover exists between the quenching of symmetry and the principle of uncertainty? Note that the derivation of...
uncertainty relations implicitly implies that the self-adjoint operators are defined on the same set of the basis function. For an angular momentum operator and an angular variable, this is not the case. The operator adjoint to the angular momentum operator should be a periodic function of the angular variable. This problem is precisely studied in the literature, see [14] and references therein. Here, we will follow a formalism described in [14] in detail and try to find the crossover between the uncertainty relations and trembling in the context of the quenching symmetry. On an intuitive level, albeit this crossover is predictable. However, our promise is to provide more rigorous arguments in support of this. Uncertainty relations between the z-component of the angular momentum operator \(L_z = -i \frac{\partial}{\partial \phi}\) and angular variable \(\phi\) can be quantified as follows [14]

\[
ur_a = (\Delta L_z)^2(\Delta \sin \phi)^2 - \frac{1}{4}(\Delta \cos \phi)^2 \geq 0,
\]

and

\[
ur_b = (\Delta L_z)^2(\Delta \cos \phi)^2 - \frac{1}{4}(\Delta \sin \phi)^2 \geq 0.
\]

Here we introduced the following notations:

\[
(\Delta \sin \phi)^2 = (\sin^2 \phi) - (\sin \phi)^2,
\]

\[
(\Delta \cos \phi)^2 = (\cos^2 \phi) - (\cos \phi)^2,
\]

\[
(\Delta L_z)^2 = \langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2.
\]

Taking into account that \(L_z = -i \frac{\partial}{\partial \phi} = \frac{1}{2} \tilde{v}_\phi\), \(\tilde{L}_z^2 = \frac{1}{2} \tilde{v}_\phi^2\) and using equations (22)–(25) it is easy to determine the analytical expressions for the expectation values of the operators required to calculate \(ur_a\) and \(ur_b\)

\[
\langle \phi_{n}^{\pm}(l, \phi) \mid \sin \phi \mid \phi_{n}^{\pm}(l, \phi) \rangle_{t \rightarrow t^+} = 0,
\]

\[
\langle \phi_{n}^{\pm}(l, \phi) \mid \cos \phi \mid \phi_{n}^{\pm}(l, \phi) \rangle_{t \rightarrow t^+} = 0,
\]
\begin{align}
\langle \xi_n(l, \varphi) | \sin \varphi | \xi_n(l, \varphi) \rangle_{l \rightarrow l_c^-} &= 0, \\
\langle \eta_n(l, \varphi) | \sin \varphi | \eta_n(l, \varphi) \rangle_{l \rightarrow l_c^-} &= 0, \\
\langle \xi_n(l, \varphi) | \cos \varphi | \xi_n(l, \varphi) \rangle_{l \rightarrow l_c^-} &= 0, \\
\langle \eta_n(l, \varphi) | \cos \varphi | \eta_n(l, \varphi) \rangle_{l \rightarrow l_c^-} &= 0,
\end{align}

Further simplification of equations (32) and (33) relies on the fact that wave functions have a certain maximum for particular values of the angle \( \varphi = n \pi, \ n = 0, 1, 2, \ldots N. \) This fact mimics the quantum counterpart of classical dynamical systems. Namely, during motion, a particle spends a major part of its time in the vicinity of the hyperbolic equilibrium points. As we can see, this effect is more profound for the high exited states and is absent in the ground state (see figure 4). Thus, when studying uncertainty relations, we are interested in the excited states. After expanding equations (32) and (33) in the vicinity of the maximum points, we immediately see that the second equation holds automatically while the first is reduced to the following form:

\[
(D_{l_c}) \langle \varphi^2 \rangle \geq \frac{1}{4} \left(1 - \frac{1}{2} \langle \varphi^2 \rangle \right).
\]
**Figure 3.** Jump in the square of velocity for the transition between symmetry subgroups \(G_{-} \rightarrow G_{0}\) and particular between the states \(\phi_{n}^{\pm}(l, \varphi) \rightarrow \xi_{n}(l, \varphi)\) (blue dots) and states \(\phi_{n}^{\pm}(l, \varphi) \rightarrow \eta_{n}(l, \varphi)\) (yellow dots).

**Figure 4.** Probability density for \(\phi_{n}^{+}(l, \varphi), n = 1, 2, ..., 8\).
in the uncertainty relations. In particular, we see a jump in the expectation values (32) and (33).

5. Torsional oscillation in polyatomic molecules

The results obtained in the previous sections have a certain physical application in polyatomic molecules. It is well-known that polyatomic molecules can perform an internal rotational motion of two types: torsional oscillation and free rotation of one part of the molecule with respect to the other part [18]. On the phase plane, these two types of motions are separated by the separatrix line. The Hamiltonian of the polyatomic molecule related to the internal rotation has the following form [18]:

\[ \hat{H} = -\frac{\hbar^2}{2I} \frac{d^2}{d\phi^2} + U(\phi). \]  

(50)

Here \( I = I_1 I_2 / (I_1 + I_2) \) is the reduced moment of inertia, \( I_1 \), \( I_2 \) are the moments of inertia of the rotating parts of the molecule with respect to the symmetry axis. Potential energy has the form:

\[ U(\phi) = \frac{V_0}{2} (1 - \cos(n\phi)). \]  

(51)

Here \( V_0 \) defines the height of the potential barrier that separates the torsional oscillations from the rotation of one part of the molecule with respect to the other part, and \( n \) defines the quantity of equilibrium orientations of one part of the molecule with respect to the other part [18].

| Table 5. Values of the expression ura for \( \phi_+ \), \( \phi_- \) (region \( G_- \)), \( \xi \) and \( \eta \) (region \( G_0 \)). |
|---|---|---|---|---|
| \( n \) | \( \phi_+ \) | \( \phi_- \) | \( \xi \) | \( \eta \) |
| 1 | -0.125 | -0.125 | 0.0625 | 0.6875 |
| 2 | -0.119738 | -0.119738 | 1.59984 | 1.92984 |
| 3 | -0.0795267 | -0.0795267 | 3.81139 | 4.15142 |
| 4 | -0.0617743 | -0.0617743 | 7.20129 | 7.31649 |
| 5 | 0.0777876 | 0.0777876 | 10.8688 | 11.0587 |
| 6 | 0.303495 | 0.303495 | 15.1676 | 15.4109 |
| 7 | 0.68743 | 0.68743 | 19.8053 | 20.2087 |
| 8 | 2.69405 | 2.69405 | 17.8697 | 23.1462 |

| Table 6. Values of the expression urb for \( \phi_+ \), \( \phi_- \) (region \( G_- \)), \( \xi \) and \( \eta \) (region \( G_0 \)). |
|---|---|---|---|---|
| \( n \) | \( \phi_+ \) | \( \phi_- \) | \( \xi \) | \( \eta \) |
| 1 | -0.125 | -0.125 | 0.6875 | 0.0625 |
| 2 | -0.106602 | -0.106602 | 2.11391 | 1.82766 |
| 3 | -0.0514835 | -0.0514835 | 4.79486 | 4.55658 |
| 4 | -0.0343079 | -0.0343079 | 8.39221 | 8.31676 |
| 5 | 0.147406 | 0.147406 | 13.3947 | 13.2956 |
| 6 | 0.454449 | 0.454449 | 19.5324 | 19.4411 |
| 7 | 1.00283 | 1.00283 | 26.8837 | 26.7593 |
| 8 | 4.80609 | 4.80609 | 35.0808 | 35.2863 |
paradigmatic model of organic molecules characterized by the property of internal rotation is the molecule of ethane $C_2H_6$ with the corresponding parameters: $I_1 = I_2 = 5.3 \cdot 10^{-47}$ kg m$^2$, $V_0 = 2.1 \cdot 10^{-20}$ J. Using transformations $\varphi \mapsto n\varphi/2$, $E \mapsto \frac{n^2}{4I} (E - V_0/2)$ equation (50) can be easily mapped into the Mathieu–Schrödinger equation (14). For the experiment we propose to use a monochromatic pumping field with the frequency $\Omega \ll V_0/\hbar$. Such a pumping field can cause a slow modulation of swift electron motion in a molecule. The formation of an energy barrier is the result of averaging over swift electron motion and due to the pumping effect the value of the barrier becomes time-dependent $V(t) = V_0 + \Delta V \cos \Omega t$.

6. Conclusions

The aim of the present work is to generalize the concept of Zitterbewegung for systems that possess complex symmetry properties. In particular, we discovered that trembling motion might occur due to dramatic changes in the symmetry of the system. For this purpose, we exploit the paradigmatic model of quantum chaos, the quantum mathematical pendulum. The symmetry group of this system is Klein’s four-group with three invariant subgroups. The energy spectrum of the system parametrically depends on the height of the potential barrier and contains degenerate and non-degenerate areas corresponding to the different symmetry subgroups. We observed that the changes in the potential barrier height switches the symmetry subgroup and lead to the trembling motion. We have shown that trembling is enhanced for higher excited states, which in turn is proved by the analysis of the mean square fluctuations of the velocity operator.

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ORCID iDs

L Chotorlishvili https://orcid.org/0000-0001-7042-9273

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