Multifractal analysis of the fat-tail PDFs observed in fully developed turbulence

T Arimitsu\(^1\), N Arimitsu\(^2\)

\(^1\) Graduate School of Pure and Applied Sciences, University of Tsukuba, Ibaraki 305-8571, Japan
\(^2\) Graduate School of Environment and Information Sciences, Yokohama Nat’l. University, Yokohama 240-8501, Japan

E-mail: tarimtsu@sakura.cc.tsukuba.ac.jp

Abstract. The fundamentals of the multifractal analysis (MFA) is given, which is a unified statistical mechanical theory that treats the systems containing intermittent phenomena and representing fat-tail probability density functions (PDFs) for appropriate observables. MFA utilizes two distinct Tsallis-type MaxEnt distribution functions, one for the tail part of PDF and the other for its center part. It is shown that A&A model within MFA can explain the recently observed PDFs of turbulence in the highest accuracy superior to the analyses based on other multifractal models such as the log-normal model and the p model.

1. Introduction
The theoretical research on fully developed turbulence (we simply call it turbulence in the following unless it is confusing) starts with Kolmogorov’s dimensional analysis (K41) \([1]\) based on the assumption of the self-similarity of fluctuating velocity field in the inertial range. After Landau’s criticism against K41 and the preliminary research by Heisenberg, the methods dealing with the breakdown of the self-similarity and the intermittency of velocity fields develop, mainly, into two directions. One is the dynamical approach (the indented items in Table 1), the other is the ensemble approach (the no-indented items in Table 1). Within the dynamical approach one treats the stochastic Navier-Stokes equation, perturbationally, whereas within the ensemble approach one performs statistical mechanical analysis of turbulence under the assumption that eddies make up energy cascade. It has been, gradually, revealed that, among the ensemble methods, new theoretical framework called multifractal analysis (MFA) and A&A model proposed within the framework can analyze in a high precision the data extracted out from the recent experiments and simulations conducted with high accuracy \([2, 3, 4, 5, 6, 7]\).

The velocity derivatives and the fluid particle accelerations have some singularities due to the scaling invariance of the basic equation of turbulence, i.e., the Navier-Stokes (N-S) equation, in the situation where turbulence realizes (in high Reynolds numbers). MFA is a statistical mechanical theory of an ensemble \([2, 3, 4, 5, 6, 7]\), constructed by following the assumption \([8]\) that the strengths of the singularities distribute, multifractally, in real physical space. This distribution of singularities determines the tail part of the probability density function (PDF). The parameters appeared in the theory are determined, uniquely, by the intermittency exponent representing the strength of intermittency. On the other hand, observed PDF should include
2. Cascade model and the Kolmogorov spectrum

Let us consider the velocity difference (fluctuation)

$$\delta u_n = |u(\bullet + \ell_n) - u(\bullet)|$$

(1)

of a component $u$ of the velocity field $\vec{u}$ between two points separated by the distance $\ell_n$. Looking at, carefully, the sketch of stream lines crossing a grid in Fig. 1, we notice that when the laminar
Figure 1. Drawing of stream lines crossing a grid (Sketched from the photograph in [9] taken by T. Corke and H. Nagib).

Flow (the white lines from the left) with a constant velocity passes the grid, there appear the stagnating part (it looks dark) behind the crosspieces of the grid and the laminar-like part (the white lines) momentarily after passing through the openings between the crosspieces (the length of the space between two adjacent crosspieces is put to be $\ell$).

The shear friction between these two distinct formations in the flow creates, in a little bit downstream, big eddies with the diameter about $\ell$ (the largest size of eddies behind the grid). Let us put the rotation velocity of the biggest eddies to be $\delta u = |u(\bullet + \ell) - u(\bullet)|$. The eddies flow downstream with the mean current producing smaller eddies one after another. As a result, there forms turbulent state as a mixture of various sizes of eddies from large to small (fully developed turbulence) at about one forth of the photograph from the right end. In order to describe this situation, one assigns

$$\ell_n = \ell \delta_n, \quad \delta_n = \delta^{-n} (n = 0, 1, 2, \cdots), \quad \delta > 1$$

for the diameters of eddies generated one after another. Here, $n$ is the number of steps in the cascade producing smaller eddies in a stream. The length $\ell_{n=0} = \ell$ is the distance between the adjacent crosspieces of the grid, i.e., the diameter of the largest eddies. This picture of turbulence is called the cascade model that was proposed first by Richardson [10]. We are astonished to see that Leonardo da Vinci had already found out through his precise observation that turbulence ("la turbolenza") is composed of various sizes of eddies about 400 years before Richardson, i.e., back to the beginning the 16th century. Turbulence is not fluctuating in the same rhythm but with different rhythms, intermittently, which allow one to observe the intermittent burst of fluctuations (see Fig. 5). The cascade model was introduced in order to represent these intermittent phenomena in the form matching to this observed fact.

The kinetic energy of eddies with the diameter $\ell_n \sim \ell_n + d\ell_n$ per unit mass is defined by

$$E_n = \int_{k_n}^{k_{n+1}} dk \ E_k = \frac{\delta u_n^2}{2},$$

1 We put $\delta = 2$ in the following. It means that the diameters of eddies being produced one after another become one half at each generation of smaller eddies from a bigger one. In order to see if it is actually the case or not, analyses of experimental data are required.
where we put $k_n = \ell_n^{-1}$. $\delta u_n$ is the order of the rotational velocity of the eddy. $E_k$ is called the energy spectrum. There are two characteristic times (relaxation times) for the eddy. One is the time necessary for the eddy to rotate once, i.e.,

$$t_n = \frac{\ell_n}{\delta u_n}. \quad (4)$$

This can be interpreted as the time (life-time) for the eddy with the diameter $\ell_n$ to pass its kinetic energy to eddies with the diameter $\ell_{n+1}$. Then, the energy transfer rate $\epsilon_n$ from the eddies with the diameter $\ell_n$ to the eddies with $\ell_{n+1}$ may be estimated as

$$\epsilon_n \sim \frac{E_n}{t_n} \sim \frac{(\delta u_n)^3}{\ell_n}. \quad (5)$$

We interpret it as follows: At each step in the cascade, say the $n$th step, where an eddy breaks up into two eddies ($\delta = 2$), the energy is delivered from the eddy with the diameter $\ell_n$ to the one with $\ell_{n+1}$ with the energy transfer rate $\epsilon_n$ per unit mass. Another characteristic time is the time required for the energy of eddies to dissipate into heat, i.e.,

$$t_{\text{diss}}^n \sim \frac{\ell_n^2}{\nu}. \quad (6)$$

This is given as a quantity having the dimension of time by making use of the fact that the kinematic viscosity $\nu$ has the dimension $\nu = L^2/T$. For the condition $t_n \ll t_{\text{diss}}^n$, the dissipative effect to eddies can be neglected, whereas for $t_n \gg t_{\text{diss}}^n$ no eddy can survive as its rotational energy is transferred into heat almost all at once.

Kolmogorov assumed in 1941 in his explanation of the universal slope $-5/3$ observed in the energy spectrum of turbulence that, in the region where the condition $t_n \ll t_{\text{diss}}^n$ is satisfied and the effect of dissipation is safely neglected, $\epsilon_n$ is constant and does not depend on $n$, i.e., $(\epsilon_n = \epsilon)$. Substituting this into (5), we have $\delta u_n \sim (\epsilon \ell_n)^{1/3}$, $E_n \sim (\epsilon \ell_n)^{2/3}$. Then, we see that $t_n \sim (\ell_n^2/\epsilon)^{1/3}$, and that the smaller the eddies become, the shorter their life-times become in the delivery of their energy. The lifetime and the diameter $\eta$ of the eddy satisfying the condition $t_n = t_{\text{diss}}^n (\equiv \tau_\eta)$ are, respectively, estimated as

$$\tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}, \quad \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \quad (7)$$

They are called the Kolmogorov time and the Kolmogorov length, respectively. Finally, we understand that the shape of the energy spectrum of turbulence can be treated by dividing it into three ranges with respect to the wave number $k$ given by the inverse of the diameter of eddies ($k_n = \ell_n^{-1}$) as an estimate (see Fig. 2). The range in the wave number of the order of the mesh size $\ell_{\text{in}}$ of the grid (the range I in Fig. 2) is called the energy input range since it is the region where the energy is fed into turbulent system (with the energy input rate $\epsilon$). The range where the diameter of eddies satisfies $\ell_{\text{in}} \gg \ell_n \gg \eta$ (the range II in Fig. 2) is called the inertial range since the effect of dissipation can be neglected, where the energy of eddies are delivered, consecutively, to smaller eddied. In the range, from (3), the energy spectrum becomes

$$E_k \sim E_n k_n^{-1} \sim \epsilon^{2/3} k^{-5/3}, \quad (8)$$

which has the desired slope. This is called the Kolmogorov spectrum. For the eddies whose diameters are much smaller and are about the order of $\eta$, the effect of dissipation grows and the
energy of turbulent system expels out of the system in the form of heat. The region is called the dissipation range (the range III in Fig. 2). The Reynolds number of the system can be expressed as
\[
Re = \frac{\delta u_{\text{in}} \ell_{\text{in}}}{\nu} = \left(\frac{\ell_{\text{in}}}{\eta}\right)^{4/3},
\]
which tells us that the inertial range is wide \((\ell_{\text{in}} \gg \eta)\) for the turbulence with high Reynolds numbers, i.e., \(Re \gg 1\).

The scaling exponent \(\zeta_m\) of the \(m\)th order velocity structure function (the \(m\)th moment of velocity fluctuations) defined through
\[
\left\langle \left(\frac{\delta u_n}{\delta u_{\text{in}}}\right)^m \right\rangle = \left(\frac{\ell_n}{\ell_{\text{in}}}\right)^{\zeta_m}
\]
is one of the quantities which characterizes turbulence. Here, \(\langle \cdots \rangle\) indicates to take an appropriate time average, spatial average or ensemble average. Actually, the present main issue is to search for an appropriate probability density function for the average. The scaling exponents for K41 are given by \(\zeta_m = m/3\).

3. Basic Equation of Fully Developed Turbulence and Singularities
3.1. Scale invariance and the appearance of singularities
In this paper, we consider an incompressible fluid where the mass density \(\rho = \rho(\vec{x}, t)\) of the fluid is constant in time and space.\(^2\) In this case, the N-S equation for the velocity field \(\vec{u} = \vec{u}(\vec{x}, t)\) reduces to
\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u},
\]
where we introduced \(p = \tilde{p}/\rho\), and the kinematic viscosity \(\nu = \tilde{\eta}/\rho\). \(\tilde{p} = \tilde{p}(\vec{x}, t)\) is the pressure of fluid, and \(\tilde{\eta}\) is the viscosity. The condition for incompressibility reduces to the equation representing that there is no divergence in velocity field, i.e.,
\[
\nabla \cdot \vec{u} = 0.
\]
\(^2\) When the fluid velocity does not exceed the sound velocity and when there is no large temperature difference in the fluid, one can regard the mass density of the fluid as constant. When the fluid velocity exceeds the sound velocity, there appears a shock wave. At the front of the shock wave, the change of the mass density is quite large.
Putting the characteristic magnitude of velocity field to be $U$, and assigning the characteristic length of change of the velocity field to be $L$, the second term in the left-hand side (the drift term) of the N-S equation (11) representing convection and the second term on the right-hand side (the dissipative term) are estimated, respectively, as $U^2/L$ and $\nu U/L^2$. The ratio between them, i.e., $\text{Re} = UL/\nu$, is called the Reynolds number that is a non-dimensional quantity specifying the state of fluid. Fully developed turbulences are the phenomena observed at high Reynolds numbers ($\text{Re} \gg 1$).

The N-S equation (11) is invariant under the Galilei transformation

$$t \rightarrow t' = t, \quad \vec{x} \rightarrow \vec{x}' = \vec{x} + \vec{U} t, \quad \vec{u} \rightarrow \vec{u}' = \vec{u} + \vec{U}. \quad (13)$$

It means that the system is invariant even when one moves to the inertial coordinates traveling with a velocity $\vec{U}$ to the original coordinates. The N-S equation is also invariant under the scale transformation

$$\vec{x} \rightarrow \lambda \vec{x}, \quad \vec{u} \rightarrow \lambda^{\alpha/3} \vec{u}, \quad t \rightarrow t' = \lambda^{1-\alpha/3} t, \quad p \rightarrow p' = \lambda^{2\alpha/3} p, \quad \nu \rightarrow \nu' = \lambda^{1+\alpha/3} \nu. \quad (14)$$

for an arbitrary real number $\alpha$. The invariance of the Reynolds number under the scale transformation is utilized for a trick shot in making a film using miniatures. If one floats a miniature battleship on seawater, the audience easily notices the trick since the characteristic length $L$, say, of the span of the miniature battleship is different from the real one, which causes the change of the Reynolds number. Therefore, one floats the miniature ship on a fluid, e.g., oil etc., in order to keep the Reynolds number unchanged, and shots a scene in the film. With this trick, the audience may not distinguish the miniature from a real battleship.

By the way, in the region where the intermittency of turbulence is conspicuous, the effect of the dissipative term in the N-S equation is very small compared with those of other terms (especially, the drift term). Therefore, let us try to extract the phenomena which is invariant under the scale transformation (without changing the kinematic viscosity)$^3$

$$\vec{x} \rightarrow \lambda \vec{x}, \quad \vec{u} \rightarrow \lambda^{\alpha/3} \vec{u}, \quad t \rightarrow t' = \lambda^{1-\alpha/3} t, \quad p \rightarrow p' = \lambda^{2\alpha/3} p. \quad (15)$$

with an arbitrary real number $\alpha$, which is valid when one neglects the dissipative term [8, 12]. We utilize this scale transformation in order to introduce at the zero-th order approximation the fact that the appearance of the velocity field for fully developed turbulence is invariant even if we change the scale (or the distance) of observation. However, we should keep in mind that the dissipative term can become effective depending on the region under consideration since the term does exist, i.e., non zero (see the discussions in the following).

Let us now find out what character the system has when it is invariant under the transformation (15). The scale transformation gives us$^4$

$$\frac{\delta u_n}{\delta u_0} = \delta_n^{\alpha/3}, \quad \frac{\delta p_n}{\delta p_0} = \delta_n^{2\alpha/3}. \quad (16)$$

From (5) and (16), we also have

$$\frac{\epsilon_n}{\epsilon} = \delta_n^{\alpha-1}. \quad (17)$$

$^3$ Strictly speaking, the N-S equation (11) is invariant under this transformation only the case $\nu = 0$, i.e., when the Reynolds number is infinite.

$^4$ The pressure difference divided by the mass density $\delta p_n = |p(\bullet + \ell_n) - p(\bullet)|$ between two points separated by the distance $\ell_n$ is also an important observable.
where we put $\epsilon = \epsilon_{in} = \epsilon_0$. The velocity derivative and the fluid particle acceleration are described, respectively, by

$$|u'| = \lim_{n \to \infty} u'_n, \quad |\vec{a}| = \lim_{n \to \infty} a_n.$$  \hspace{1cm} (18)

Here, we introduced the velocity derivative and acceleration corresponding to the characteristic length $\ell_n$ by

$$u'_n = \frac{\delta u_n}{\ell_n}, \quad a_n = \frac{\delta p_n}{\ell_n},$$ \hspace{1cm} (19)

respectively. We see that the velocity derivative and the fluid particle acceleration have singularities, respectively, for $\alpha < 3$ and $\alpha < 1$, i.e.,

$$|u'| \propto \lim_{\ell_n \to 0} \ell_n^{(\alpha/3) - 1} \to \infty, \quad |\vec{a}| \propto \lim_{\ell_n \to 0} \ell_n^{(2\alpha/3) - 1} \to \infty.$$ \hspace{1cm} (20)

The energy transfer rate also has singularities for $\alpha < 1$, i.e.,

$$\lim_{n \to \infty} \frac{\epsilon_n}{\epsilon_0} = \lim_{n \to \infty} \ell_n^{\alpha - 1} \to \infty.$$ \hspace{1cm} (21)

The exponent $\alpha$ plays the role of an index representing the degree of singularities.

As $\epsilon_n$ is assumed to be constant independent of $n$ within the treatment of K41, we see from (17) that K41 is the case corresponding to $\alpha = 1$. If we look at this way, the arbitrariness of $\alpha$, appeared in the scale transformation (15), indicates that $\epsilon_n$ can be viewed as a stochastic variable, i.e., one can introduce fluctuations in $\epsilon_n$. It means that there is a possibility to give an answer to the criticism against K41 raised by Landau. In other words, the energy transfer rate can take various values even for the eddies with the same diameter. The distribution of the values, i.e., the distribution of $\alpha$, is determined by a delicate balance between the non-linear convective term and the dissipative term in the N-S equation. To extract what is the appropriate distribution for $\alpha$ leads us to a resolution for the origin of intermittency in turbulence. MFA provides us with a systematic framework to make a connection between the distribution of $\alpha$ and observed data.

### 3.2. Looking at singularities

In the epoch-making experiment conducted by Bodenschatz and others [11], a direct trace of a test particle (a measurement within the Lagrangian picture) is performed (see Fig. 3). The spheres lined in the figure represent the track of the test particle at regular time interval ($4.3$ msec). The tone painted on each sphere shows the acceleration of fluid particle at the point (see the scale in Fig. 3 which gives the relationship between the tones and the values of acceleration). At the left below in the figure where the track shows a small circle, the acceleration of the fluid particle is raised all of a sudden from zero to $12000 \text{ m/sec}^2$. Note however that we do not know the shape of the singularity itself if it is point-like or linear, etc.. This experiment can be interpreted as the one that captures, directly, the existence of the singularities in the real space, which is the basis of MFA. The Reynolds number for the experiment is estimated as $\text{Re} = 31400$ by substituting the energy input length $\ell_{in} = 0.071 \text{ m}$ and the Kolmogorov scale $\eta = 30.3 \mu\text{m}$ into its formula. The probability density function (PDF) obtained by the observations for long time will be analyzed later in section 6.

### 4. Multifractal Distribution of Singularities

#### 4.1. Fractal dimension

In order to understand what is multifractal, one needs to know first the view of fractal.

The fluid particle acceleration $\vec{a}$ is given by $\vec{a} = \partial\vec{u}/\partial t + (\vec{u} \cdot \nabla)\vec{u}$.

In practice, as the resolutions in experiments or numerical simulations are finite, it may be appropriate to
Figure 3. A test particle in a turbulent fluid is abnormally accelerated [11].

Figure 4. Sierpinski’s triangular gaskets (upper row) and carpet (lower row).

Let us derive the fractal dimension of the space occupied by Sierpinski’s triangular gaskets given in the upper row in Fig. 4. Putting the length of the first \((n = 0)\) side of Sierpinski’s triangular gasket to be 1, the length of the side of the triangle at the \(n\)th step \((n = 0, 1, 2, \cdots)\) is given by

\[
\delta_n = \delta^{-n}
\]

\((\delta = 2)\), and its number by

\[
N_n = 3^n.
\]

Then, we see that the area \(S_n\) of the black triangles is

\[
S_n = \frac{\sqrt{3}}{4} N_n \delta_n^2 = \frac{\sqrt{3}}{4} \left(\frac{3}{4}\right)^n.
\]

interpret that the term \textit{singularity} here means to take abnormally large values.

\(^7\) From (16), we interpret that the velocity fluctuation and the pressure fluctuation are also stochastic variables.
In the limit \( n \to \infty \), the area reduces to zero. The area is the \textit{volume} in 2 dimensional space.

Now, let us derive the dimension \( D \) of the space in which the black triangles can occupies whole the space without a vacancy. As the \textit{volume} of one black triangle at the \( n \)th step in this space is given by \( \delta_n^{D_n} \), with the condition that the \( N_n \) black triangles cover the \textit{volume} \( 1^D \) at \( n = 0 \), i.e.,

\[
N_n = \frac{1^D}{\delta_n^D} = 2^{nD},
\]

we obtain

\[
D = \frac{\ln 3}{\ln 2} \sim 1.58.
\]

The reason why \( S_n \) goes to zero in the limit \( n \to \infty \) can be attributed to the wrong use of formula for 2 dimensional area in the calculation of the \textit{volume} of the black triangles living in the space whose dimension is less than 2.

For Sierpinski’s carpet in the lower row in Fig. 4, we obtain

\[
D = \frac{\ln 8}{\ln 3} \sim 1.89
\]

by the same process as the case of triangular gaskets. We understand the usefulness of the fractal dimension, when we notice that the darkness of the Sierpinski’s gaskets and that of carpet at a glance can be distinguished, reasonably, by the difference of the fractal dimensions.

### 4.2. Multifractal spectrum

For \( \epsilon_n \neq 0 \), (17) shows that \( \alpha \) and \( \epsilon_n \) are related with each other by \( \alpha = [\ln(\epsilon_n/\epsilon)]/\ln \delta_n] + 1 \). We need \( \delta_n^{-d} \) boxes (the volume of each box is \( \ell^d_0 \)) to cover whole the space of the volume \( \ell_0^d \) in \( d \) dimensional space without a vacancy. Let us pay attention to one of these boxes, and assume that the probability \( P^{(n)}(\epsilon_n/\epsilon)d(\epsilon_n/\epsilon) \) to find in the box a non-zero value \( \epsilon_n/\epsilon \) between the domain \( \epsilon_n/\epsilon \sim \epsilon_n/\epsilon + d(\epsilon_n/\epsilon) \) is given by

\[
P^{(n)}(\epsilon/\epsilon) d(\epsilon/\epsilon) P^{(n)}(\epsilon_n/\epsilon \neq 0) = P^{(n)}(\alpha)d\alpha.
\]

Here, \( P^{(n)}(\epsilon_n/\epsilon \neq 0) \) is the probability that the selected box satisfies the condition \( \epsilon \neq 0 \), and is given by the proportion of the number \( \delta_n^{-D_0} \) of boxes satisfying \( \epsilon_n \neq 0 \) in the space with the fractal dimension \( D_0 \) to the number \( \delta_n^{-d} \) of all the boxes, i.e.,

\[
P^{(n)}(\epsilon_n/\epsilon \neq 0) = c_1 \delta_n^{d-D_0}.
\]

On the other hand, the eddies specified by \( \alpha \) occupy the space with the fractal dimension \( f_\alpha(d) \). If we think that the probability to find a value within the domain \( \alpha \sim \alpha + d\alpha \) in the box under consideration is given by the proportion of the number \( \delta_n^{-D_0} \) of boxes occupying whole the space with the fractal dimension \( f_\alpha(d) \) without a vacancy to the number \( \delta_n^{-d} \) of all the boxes, we have the expression

\[
P^{(n)}(\alpha)d\alpha = c_2(\alpha)\delta_n^{d-f_\alpha(d)}d\alpha.
\]

Substituting (29) and (30) into (28), we, finally, obtain [12]

\[
P^{(n)}(\epsilon_n/\epsilon) = \frac{c_2(\alpha)}{c_1 \ln \delta_n} \delta_n^{1+D_0-d-f_\alpha(d)}.
\]

In the following, we will proceed the investigation assuming that the \( \alpha \) dependence of the normalization coefficient \( c_2(\alpha) \) is negligible.
Here, let us introduce the mass exponent $\tau_d(q)$ through the relation
\[
Z_d^{(n)} = \sum_{\text{boxes}} \left( \frac{\ell_n \ell_d}{\ell_0 \ell_0} \right)^q = \sum_{\text{boxes}} \delta_n^{(\alpha-1+d)\bar{q}} \propto \delta_n^{-\tau_d(q)}.
\] (32)
The summation with respect to the number of boxes can be translated into the integration with respect to $\alpha$ as
\[
Z_d^{(n)} = \int d\alpha \rho(\alpha) \delta_n^{(\alpha-1+d)\bar{q}} f_d(\alpha).
\] (33)
Evaluating the integration in the limit $\delta_n \to 0 (n \to \infty)$ with the help of the method of the steepest descent, we obtain the relation
\[
f_d(\alpha) - (\alpha - 1 + d)\bar{q} = \tau_d(\bar{q}), \quad \bar{q} = \frac{df_d(\alpha)}{d\alpha}.
\] (34)
With
\[
\alpha - 1 + d = -d \frac{\tau_d(\bar{q})}{\bar{q}},
\] (35)
(34) constitutes the Legendre transformation between $f_d(\alpha)$ and $\tau_d(\bar{q})$. $f_d(\alpha)$ is called the multifractal spectrum. The generalized dimension (the Renyi dimension), $D_{\bar{q}}$, is introduced by the relation
\[
\tau_d(\bar{q}) = (1 - \bar{q})D_{\bar{q}}.
\] (36)
Using the mass exponent, the $\bar{q}$th moment of the energy transfer rate is given by
\[
\left\langle \left( \frac{\epsilon_n}{\epsilon} \right)^\bar{q} \right\rangle = \int_0^\infty d \left( \frac{\epsilon_n}{\epsilon} \right)^\bar{q} P^{(n)}(\epsilon) \left( \frac{\epsilon_n}{\epsilon} \right) \sim \delta_n^{-\tau_d(\bar{q}) + D_0 - \bar{q}d}.
\] (37)
Since the condition for the normalization of probability, i.e., $\langle 1 \rangle = 1$, reduces to
\[
\tau_d(0) = D_0 = f_d(\alpha_0),
\] (38)
and the energy conservation law, i.e., $\langle \epsilon_n \rangle = \epsilon$, to
\[
\tau_d(1) = D_0 - d,
\] (39)
we obtain the dimension $D_0$ as
\[
D_0 = d.
\] (40)
Note that, generally, $\tau_d(1) = 0$ when $D_{\bar{q}}$ is finite. The definition of the intermittency exponent $\mu$, i.e., $\langle \epsilon_n^2 \rangle = \epsilon^2 \delta_n^{-\mu}$, gives
\[
\mu = \tau_d(2) - D_0 + 2d.
\] (41)
Summarizing all the relations, we have
\[
\tau_d(0) = d, \quad \tau_d(1) = 0, \quad \mu = d + \tau(2) = d - D_2.
\] (42)
Note that the first equation in (42) is always satisfied since the number of boxes with the side length $\ell_n$ necessary to cover the $d$ dimensional space is given by $\sum_{\# \text{ of boxes}} 1 \propto \delta_n^{-d}$. The scaling exponent of the $m$th order velocity structure function defined by (10) is related to the mass exponent as
\[
\zeta_m = 1 - \tau_d \left( \frac{m}{3} \right).
\] (43)
In the following, we put $d = 1$ and introduce the simplified notations $f_{d-1}(\alpha) = f(\alpha)$ and $\tau_{d-1}(\bar{q}) = \tau(\bar{q})$, since we are, mainly, analyzing in this paper the one dimensional time-series data for a component of velocity field.

---

8 We are neglecting the $\alpha$ dependence of the density $\rho(\alpha)$ introduced in the translation \[ \sum_{\# \text{ of boxes}} = \int d\alpha \rho(\alpha) \delta_n^{-f_d(\alpha)}. \]
Figure 5. The time dependence of the quantity proportional to $\nu u'^2$ (cm$^2$/sec$^4$) (Utilized the time-series data for the velocity field observed by Mouri and others in a wind tunnel.).

Figure 6. Comparison between the generalized dimension $D_q$ extracted from Fig. 5 (dots) and the one obtained by A&A model with $\mu = 0.240$ (solid line).

4.3. Capture the singularities

We display in Fig. 5 the time dependence of the quantity$^9$ $\nu u'^2 \propto (\partial u/\partial t)^2$ closely related to the energy transfer rate $\epsilon_n$, obtained by making use of the time-series data of the grid turbulence measured by Mouri and others in the large wind tunnel whose scales of the measuring space are 18 m in length, 3 m in width and 2 m in height.$^{10}$ We put in the figure the time dependence of the quantity closely related to the energy dissipation rate. We see a strong intermittency in its time evolution. Making use of this time-series, we obtain (32) for various $\ell_n$ by translating spatial width to time width with the help of Taylor’s frozen hypothesis. Then, we display in Fig. 6 by closed circles the generalized dimension obtained through the relation $D_q \ln \delta_n = (\bar{q} - 1)^{-1} \ln Z^{(n)}_{d=1}$. The solid line represents the generalized dimension derived by the Legendre transformation from the multifractal spectrum $f(\alpha)$ belonging to the Tsallis-type distribution function (54) adopted in A&A model (see subsection 5.3). We capture singularities passing by the measuring point one after another, and extract the information of their spatial

$^9$ In replacing the time derivative to the spatial derivative, we used Taylor’s frozen hypothesis, i.e., there is a given relation between the time and spatial differences since the turbulent velocity field moves downstream riding on the mean flow (see Fig. 1).

$^{10}$ The average wind velocity is 10 m/sec. The spatial resolution is estimated as 0.67 mm which is about three times longer than the Kolmogorov length.
distribution. From this analysis, one can interpret that the basis of multifractal analysis and the validity of A&A model are guaranteed. The origin of the slight discrepancy for $\bar{q} < -5$ will be stated in subsection 6.1.

The Reynolds number for this experiment in the wind tunnel is estimated as $Re = 6500$ by putting into the formula the length of the space between two adjacent crosspieces $\ell_{in} = 16$ cm (a rough estimate of the energy-input scale) and the Kolmogorov length $\eta = 0.22$ mm.

5. Multifractal Model of Turbulence

5.1. Log-normal model

Within K41, it is assumed that the energy transfer rate $\epsilon_n$ takes a constant value $\epsilon$ independent of $n$. It is the log-normal model [13, 14, 15] that was proposed to take in the intermittency of turbulence by introducing fluctuation of $\epsilon_n$, in the form of the first reply to the criticism raised by Landau. The quantities $\epsilon_n/\epsilon_{n-1}$ ($n = 1, 2, \cdots$), which measures how much energy received by the eddy with the diameter $\ell_{n-1}$ is delivered to the eddy of the size $\ell_n$, are regarded as stochastic variables defined within the domain $[0, \infty]$. Let us assume that the variables specified by $n$ are, mutually, stochastically independent, and that they have an identical distribution function. In this case, the distribution of the stochastic variable

$$\frac{1}{\sqrt{n}\sigma^2} \sum_{j=1}^{n} \ln \left( \frac{\epsilon_j}{\epsilon_{j-1}} \right) = \frac{\sqrt{n}}{\sigma} (1 - \alpha) \ln \delta$$

must be the canonical distribution (Gaussian distribution)

$$P^{(n)}(\alpha) \propto e^{-n(\alpha - \alpha_0)^2/2\sigma^2}$$

because of the central limit theorem for $n \gg 1$. The domain of $\alpha$ is $[-\infty, \infty]$.

With the help of the two independent relations, i.e., the energy conservation law given by the second equation in (42) and the definition of the intermittency exponent $\mu$ given by the third equation in (42), the two parameters $\alpha_0$ and $\sigma$ are determined as the functions of $\mu$ in the forms

$$\alpha_0 = 1 + \frac{\mu}{2}, \quad \sigma^2 = \frac{\mu}{\ln \delta},$$

respectively.

Substituting (45) into the left-hand side of (30) for $d = 1$, we obtain $f(\alpha)$ which provides us with the mass exponent

$$\tau(\bar{q}) = (1 - \bar{q}) \left(1 - \frac{\mu}{2} \bar{q}\right)$$

through the Legendre transformation (34). Therefore, the generalized dimension becomes

$$D_{\bar{q}} = 1 - \frac{\mu}{2} \bar{q}.$$  

5.2. $p$ model

Within $p$ model, it is assumed that, when an eddy breaks up into two half-size eddies, the energy is delivered with the constant ratio of $p$ to $1 - p$ ($p > 1/2$) [16, 12]. By making use of the binomial distribution function

$$P_B^{(n)}(\xi) = \delta_n \left( \frac{n}{\xi n} \right),$$

the distribution function for $p$ model is given by

$$P^{(n)}(\alpha) d\alpha = P_B^{(n)}(\xi) d\xi = [2\xi^{\xi} (1 - \xi)^{1-\xi}]^{-n} d\xi.$$
In the derivation, the use has been made of the Stirling formula \( n! = (2\pi)^{1/2}n^{n+1/2}e^{-n} \quad (n \gg 1) \). The relation between \( \xi \) and \( \alpha \) is given by

\[
\alpha = -[\xi \log_2 p + (1 - \xi) \log_2 (1 - p)].
\]

(51)

The domain of the variable is \(-\log_2 p \leq \alpha \leq -\log_2 (1 - p) \quad (1 \geq \xi \geq 0)\).

In the same way as in the previous subsection, we can obtain the mass exponent as

\[
\tau(\bar{q}) = \log_2 [p\bar{q} + (1 - p)\bar{q}^2].
\]

(52)

By making use of the definition of the intermittency exponent \( \mu \) given by the third equation in (42), we have the parameter \( p \) as a function of \( \mu \) in the form

\[
p = \frac{1}{2} \left( 1 + \sqrt{2^\mu - 1} \right).
\]

(53)

The energy conservation law given by the second equation in (42) is satisfied, automatically, by virtue of the characteristics of the model.

When \( p = 1/2 \) (\( \mu = 0 \)), \( p \) model reduces to K41. Therefore, the generalized dimension turns out to be \( D_{\bar{q}} = 1 \) (\( d = 1 \)) for K41 as it should be.

5.3. A&\( \alpha \) model

Within A&\( \alpha \) model, for \( P^{(n)}(\alpha) \), the Tsallis-type distribution

\[
P^{(\alpha)}(\alpha) \propto \left[ 1 - \frac{(1 - q) \log_2 2}{2X} (\alpha - \alpha_0)^2 \right]^{\alpha/(1 - q)}
\]

(54)

with the entropy index \( q \) is adopted [2, 3, 4, 5, 6, 7]. The domain of \( \alpha \) is \( \alpha_{\min} \leq \alpha \leq \alpha_{\max} \) where \( \alpha_{\min} \) and \( \alpha_{\max} \) are given by \( \alpha_{\max} - \alpha_0 = \alpha_0 - \alpha_{\min} = \sqrt{2X/(1 - q)} \log_2 2 \).

Following the same way as the previous two models, the Legendre transformation of \( f(\alpha) \) gives us the mass exponent in the form

\[
\tau(\bar{q}) = 1 - \alpha_0 \bar{q} + \frac{2X \bar{q}^2}{1 + \sqrt{C_{\bar{q}}}} + \frac{1}{1 - q} \left[ 1 - \log_2 \left( 1 + \sqrt{C_{\bar{q}}} \right) \right]
\]

(55)

with

\[
C_{\bar{q}} = 1 + 2\bar{q}^2(1 - q)X \log_2 2.
\]

(56)

For large \( |\bar{q}| \), there appears the log term, \( \log_2 |\bar{q}| \), in the \( \bar{q} \) dependence of \( \tau(\bar{q}) \), which is the one of the characteristics of A&\( \alpha \) model.

The three parameters \( \alpha_0 \), \( X \) and \( q \) are determined as the functions of \( \mu \) with the help of the three conditions, i.e., the energy conservation law given by the second equation in (42), the definition of the intermittency exponent, i.e., the third equation in (42), and the scaling law

\[
\frac{1}{1 - q} = \frac{1}{\alpha_-} - \frac{1}{\alpha_+}
\]

(57)

with \( \alpha_\pm \) satisfying \( f(\alpha_\pm) = 0 \).

\[\text{Regardless if the fundamental entropy is the extensive Rényi entropy or the non-extensive Tsallis entropy, the MaxEnt distribution functions which give the extremum of these entropies have a common structure. Within the present approach, one cannot determine which is the background entropy for turbulence.}\]
5.4. Competition among scaling exponents

We display in Fig. 7 the competition among the scaling exponents $\zeta_m$ of the $m$th order velocity structure function derived by various theories and those obtained by the numerical experiment reported in [17] (Re = 32 000).\textsuperscript{12} We see that the scaling exponents within A&A model succeeded to explain the observed results with high precision.

6. Multifractal PDF Analysis

6.1. Probability Density Function (PDF)

The scaling exponents is the quantity reflecting the characteristics of the PDF producing them. Since we succeeded the precise reproduction of the scaling exponents as has been seen in the previous subsection, let us now try to obtain the expression for PDF itself.

For this purpose, we introduce the fluctuation

$$\delta x_n = |x(\bullet + \ell_n) - x(\bullet)|$$

of a physical quantity related to $\alpha$ by the relation

$$|x_n| = \left| \frac{\delta x_n}{\delta x_0} \right| = \delta^{\alpha n}/3.$$  \hspace{1cm} (59)

Then, the spatial derivative defined by

$$|x'| = \lim_{\ell_n \to 0} \frac{\delta x_n}{\ell_n}$$

diverges when $\alpha < 3/\phi$. $|x'|$ reduces to the velocity derivative and fluid particle acceleration for $\phi = 1$ and $\phi = 2$, respectively, and formally to the energy transfer rate (17) for $\phi = 3$. Now, we

\textsuperscript{12}The explanations of the $\beta$ model [18] and the log-Poisson model [19, 20] are omitted because of the lacking of space for them. Please refer, for example, to [4].
assume that the probability $\Pi^{(n)}_{\phi}(x_n)dx_n$ to find the physical quantity $x_n$ taking a value in the domain $x_n \sim x_n + dx_n$ can be, generally, divided into two parts as

$$\Pi^{(n)}_{\phi}(x_n)dx_n = \Pi^{(n)}_{\phi, S}(x_n)dx_n + \Delta\Pi^{(n)}_{\phi}(x_n)dx_n. \quad (61)$$

Here, the first term describes the contribution from the abnormal part of the physical quantity $x_n$ due to the fact that its singularities distribute themselves, *multifractally*, in real space. It is the part given by

$$\Pi^{(n)}_{\phi, S}(|x_n|)dx_n \propto P^{(n)}(\alpha)d\alpha \quad (62)$$

through the variable translation (59) between $|x_n|$ and $\alpha$. On the other hand, the second term $\Delta\Pi^{(n)}_{\phi}(x_n)dx_n$ represents the contributions from the dissipative term in the N-S equation and/or from the errors in measurements, etc. The dissipation term violates the invariance based on the scale transformation, and, therefore, the effect has been neglected in the above consideration for the distribution of singularities. The second term is the correction term to the first one. The values of $|x_n|$ representing the part originated from the singularities are describing the large deviations due to intermittency. The values of $|x_n|$ for the part contributing to the correction term is smaller than its standard deviation. The normalization of PDF is given by

$$\int_{-\infty}^{\infty} dx_n \Pi^{(n)}_{\phi}(|x_n|) = 1. \quad (63)$$

Each term is composed of the product of two PDFs, i.e., 1) the PDF that determines from which the contribution originated out of the two independent origins, and 2) the conditional PDF to find a value $x_n$ in the domain $x_n \sim x_n + dx_n$. The observed PDFs are, usually, asymmetric, however, in the following, we are investigating symmetrized PDFs assuming that the asymmetry is not originated from the fundamental process for intermittency.

The $m$th order structure function (moments) of the variable $|x_n|$ is given by

$$\langle|\langle x_n \rangle|^m \rangle_{\phi} \equiv \int_{-\infty}^{\infty} dx_n |x_n|^m \Pi^{(n)}_{\phi}(x_n) = 2\gamma^{(n)}_{\phi, m} + \left(1 - 2\gamma^{(n)}_{\phi, 0}\right) a_{\phi m} \delta_{\phi m}. \quad (64)$$

The main cause of the discrepancy of the generalized dimension $D_\phi$ based on A&A model (solid line in Fig. 6) for $\bar{q} < -5$ in the comparison with the generalized dimension extracted out from the time-series data can be attributed to the existence of the first term in (64) due to the correction term of PDF. We expect that the inclusion of the effect of this term in the analysis of the time-series data may dissolve the discrepancy.

In order to compare with observed PDFs, we introduce the PDF, $\hat{\Pi}^{(n)}_{\phi}(\xi_n)$, of the new variable

$$\xi_n = \frac{x_n}{\langle x_n^2 \rangle^{1/2}} \quad (65)$$

scaled by its standard deviation $\langle x_n^2 \rangle^{1/2}$ through the relation

$$\hat{\Pi}^{(n)}_{\phi}(\xi_n)d\xi_n = \Pi^{(n)}_{\phi}(x_n)dx_n. \quad (66)$$

We assume that, for the large fluctuations satisfying $\xi_n^* \leq \xi_n$ (the tail part of PDF), one can neglect the contribution from the second correction term in (61). In other words, $P^{(n)}(\alpha)$

---

13 Here, $2\gamma^{(n)}_{\phi, m} = \int_{-\infty}^{\infty} dx_n |x_n|^m \Delta\Pi^{(n)}_{\phi}(x_n)C a_{\phi m} = [2C_{\phi m}^{1/4}(1 + C_{\phi m}^{1/3})]^{1/2}$ and $\zeta_{\phi m} = 1 - \tau(\phi m/3)$. Note that the independence of $\zeta_{\phi m}$ on the multifractal depth $n$ is a manifestation of the invariance under the scale transformation.
Figure 8. Composition of PDF. (a) log scale, (b) linear scale.

determines this part through the relation (62). The solid line in Fig. 8 represents the contribution of $P^{(n)}(\alpha)\alpha$ to $\hat{\Pi}^{(n)}(\xi_n)\frac{d\xi_n}{\xi_n}$, originated from the multifractal distribution of singularities. We see that the contribution from the singularities becomes small, drastically, for $\xi_n \leq 1$ (smaller than the standard deviation of $x_n$). As there is no theory at present that determines $\hat{\Pi}^{(n)}(\xi_n)$ for the domain $\xi_n \leq \xi_n^*$ (central part), we proceed the following discussion by putting, here, a trial PDF

$$
\hat{\Pi}^{(n)}(\xi_n) \propto \left\{ 1 - \frac{1 - q'}{2} \left( 1 + \frac{3f'(\alpha^*)}{\phi} \right) \left[ \left( \frac{\xi_n}{\xi_n^*} \right)^2 - 1 \right] \right\}^{1/(1-q')}
$$

(67)
of the Tsallis-type with the entropy index $q'$. The part of the dotted line in Fig. 8 represents this contribution. The part between the dotted line and the solid line for $\xi_n \leq \xi_n^*$ gives the contribution from the correction term in (61). The two PDFs of the center part and the tail part are connected at $\xi_n = \xi_n^*$ in the manner to have the same value and slope there. The tail part of PDF is mainly determined by the intermittency exponent $\mu$ and the multifractal depth $n$ (or, equivalently, the distance $\ell_n$), and the central part by the entropy index $q'$.

6.2. Competition among PDFs

In order to perform a competition among the various PDFs derived by the multifractal PDF analysis for three multifractal models of turbulence introduced in the previous section, let us analyze the PDFs obtained by the following two experiments (or numerical experiments).

(i) Direct numerical simulation (DNS) conducted by Gotoh et al. [21]: PDFs of fluid particle accelerations and of velocity fluctuations at $Re = 13\,000$, which are obtained by solving the N-S equation with the largest lattice mesh size $1024^3$ at that time.

(ii) Experiment performed in the Lagrangian picture by Bodenschatz et al. [11]: PDF of fluid particle accelerations at $Re = 31\,400$. The direct measurement of fluid particle accelerations is realized by raising, dramatically, the spatial and temporal measuring resolutions by making use of the silicon strip detectors. This is the experiment introduced in subsection 3.2.

We performed the competition among PDFs by displaying in Fig. 9 and in Fig. 10, respectively, the PDF $\hat{\Lambda}^{(n)}(\omega_n)$ of fluid particle accelerations (the variable $\omega_n$ is the fluid particle

---

14 We chose $\xi_n^*$ at the point where $\hat{\Pi}^{(n)}(\xi_n^*)$ has the least $n$ dependence for $n \gg 1$. Then, the value of $\alpha^*$ corresponding to $\xi_n^*$ is given by the smaller solution of $\frac{\zeta_2}{2} - \phi \alpha / 3 + 1 - f(\alpha) = 0$. 

accelerations normalized by its standard deviation) and the PDF $\tilde{\Pi}^{(n)}(\xi_n)$ of velocity fluctuations (the variable $\xi_n$ is the velocity fluctuations normalized by its standard deviation). The log scale (a) is good to see the tail part, whereas the linear scale (b) is appropriate to study the central part. In both figures, PDFs are displayed in a set of two lines from the top, with closed circles representing the observed PDF, broken lines the PDF for log-normal mode, dotted lines for p model and solid lines for A&A model. The pair of solid lines in each set are the same PDF derived by A&A model. For better visibility, each PDF is shifted, vertically, by $-2$ in (a), and by $-0.4$ in (b). In order to secure impartiality, the observed PDFs are analyzed by the least square method with the theoretically derived PDF for each model. The top pair in Fig. 9 is the competition of the PDF of accelerations extracted by Gotoh et al., whereas the bottom pair is the comparison of the PDF of fluid particle accelerations measured by Bodenschatz et al.. Since

Figure 9. Comparison of PDFs of fluid particle accelerations [7]. (a) log scale, (b) linear scale.

Figure 10. Comparison of PDFs of velocity fluctuations [7]. (a) log scale, (b) linear scale.
the PDFs of fluid particle accelerations is almost symmetric, we analyzed them with their raw data, i.e., without symmetrized them. The distances $r/\eta$ of two measuring points for three sets of pair in Fig. 10 are, from the top pair to the bottom one, 2.38, 19.0 and 1220, respectively. For both Fig. 9 and Fig. 10, $\mu = 0.240$, therefore, $q = 0.391$, $\alpha_0 = 1.14$ and $X = 0.285$.

The connection points of the center part and the tail part within A&A model for the top and the bottom pair in Fig. 9 are, respectively, $\omega_n = 0.539$ and 0.547 ($\alpha^* = 1.01$; common to both pairs). The connection points for three pairs in Fig. 10 are, from the top to the bottom, $\xi_n = 1.10$, 1.23 and 1.43 ($\alpha^* = 1.08$; common to three pairs), respectively. Note that $\alpha^* \approx 1$ for every cases. Since $\alpha < \alpha^*$ for the tail part, we confirm the consistency of the multifractal PDF analysis. Remember the fact that the fluid particle accelerations, the velocity fluctuations and also the energy transfer rates become singular for $\alpha < 1$. The connection points for log-normal model and p model are almost the same as that for A&A model. As for other parameters please refer to original papers [7].

Either competition shows the superiority of A&A model. The PDFs within log-normal model have rather higher tail part than the observed PDFs. This is a manifestation of the fact that the values of the scaling exponents become smaller for larger $m$ and, finally, take negative values as was seen in Fig. 7. Their discrepancies at the center part are also observable. The PDFs within p model are very close to those of A&A model, and explain the observed PDFs quite well, but the domain of variables are, regrettably, too small and the tail of PDFs of p model terminate before the observed PDFs end. Since the accuracy of the observations were improved, one can say that the function of p model had finished. Note that the domains of variables for the PDFs within A&A model are, usually, about 300 times larger than their standard deviations. Observation of such a large fluctuation is, practically, impossible.

7. Summary and Prospects
The various PDFs observed in experiments of turbulence are analyzed with high precision by the theoretical formulae for PDFs derived by A&A model within MFA. The analyses reveal that there exist two distinct elements for the mechanism to administer the flows in turbulence. One is the mechanism to administrate the tail part of PDF, and the other is the one to control the center part of PDF. The shape of the tail part is a manifestation of the global structure which is the origin of intermittency taken on the flows in turbulence. The structure is the outcome of the multifractal distribution of singularities in real space. One knows the relations between observables and $\alpha$ through the scale transformation. The entropy index $q$ is the quantity which does not depend on the distance of two measuring points, and its value is determined once a turbulent system is specified (or the intermittency exponent is given). On the other hand, the shape of the central part is a reflection of the difference due to the violation of scale invariance. The origin of this difference may be attributed to the results of a dissipative structure in turbulence and/or of errors in measurement, etc.. The entropy index $q'$ determining the central shape of PDF is expected to be specified by a local structure of flow fields, and its value may be dependent on the distance of two measuring points. The word, local structure, here is referring the wave and oscillation of vortex in turbulence and/or the interaction between vortices, etc.. Anyway, it is one of the attractive future problems to find out two different dynamics one of which determines the tail part of PDF controlled by $q$, and the other of which determines the central part of PDF controlled by $q'$. When the underlying dynamics of MFA is revealed, theoretically, by starting the consideration with the N-S equation including an energy input term, the mechanism for the appearance of the two distinct contributions to PDF will be revealed.

By the success of MFA, it has become possible to investigate the ensemble theoretical aspect of turbulence with high precision. It can be said that one has gotten a clue to search for the fundamental process of intermittency, i.e., the origin of singularities and the reason why
the singularities distribute themselves multifractally, etc.. One of the most important systems appropriate to search for the fundamental process may be the system of vortex tangle (quantum turbulence) [22] which occurs in superfluid $^4$He and $^3$He, because the vorticity is quantized, and because the mutual friction between the superfluid and normal fluid components (in the sense of the two-fluid model) is negligibly small at very low temperatures. We can imagine at least two cases [5], i.e., 1) If the singularity originates from the core structure of vortex, the multifractality of turbulence in normal fluid can be related to various values of vorticities in the fluid. In this case, the vortex tangle may be uni-fractal, and the appearance of intermittency in quantum turbulence can be different from that in classical turbulence. 2) On the other hand, if the fundamental process of singularity originates from the reconnection of vortices, the multifractal distribution of singularities may be corresponding to the distribution of the reconnection points in real space. In this case, the distribution of singularities in quantum turbulence becomes also multifractal, and the quantum turbulence reveals the same intermittency as the classical turbulence. The multifractality of turbulence in normal fluid is related to the distribution of reconnection points in the fluid. Then, the vortex tangle may be also multifractal, and does exhibit intermittency. A proposal to give an answer to this problem is to study the spatial distribution (the multifractal spectrum) of the magnitudes of velocity and pressure derivatives with the help of a snapshot of DNS for classical turbulence at a certain time. As for quantum turbulence, it is good to study a snapshot at a certain time extracted out from a simulation within the vortex tangle model [23].

Tabeling and others [24] observed PDF of velocity fluctuations of quantum turbulence in superfluid $^4$He.\textsuperscript{15} We symmetrized the PDF and displayed it by closed circles in Fig. 11. The difference of two measuring times of the top PDF is 1 msec, and that of the bottom PDF is 100 msec. For better visibility of the figure, each PDF is shifted, vertically, by $-1$ in (a), and $-0.1$ in (b). Solid lines represent the PDFs obtained by A&A model. The agreement is satisfactory. By this analysis, we found that the value of the intermittency exponent is given by $\mu = 0.326$. Note that this value is much larger than the almost universally observed value $\mu \approx 0.24$ in classical turbulences (see Fig. 6, Fig. 7, Fig. 9 and Fig. 10). It seems that we are capturing the difference between quantum and classical turbulences, but more precise experiments and their analyses are required.

\textsuperscript{15} By measuring pressure with the Pitot tube, they observed the flow velocities of superfluid $^4$He at 1.4 K. Note that the mass density ratio of the superfluid component to the normal fluid component is 9 to 1 at this temperature.
Let us close this paper by referring that the observed PDF of velocity fluctuations of granular turbulence (granulence [25]) and the stationary PDF of movements in stocks in econophysics [26] are analyzed with the help of the multifractal PDF analysis. The agreements in both cases are quite satisfactory.

Acknowledgments
The authors are grateful to Dr. Mouri at Meteorological Research Institute for his kindness to tell them various important information related to his experiment, and to Dr. Yoshida at University of Tsukuba for fruitful discussions. One of the authors, N.A., acknowledges the partial support from the Grant-in-Aid for Scientific Research (15360495) of MEXT Japan.

References
[1] Kolmogorov A N 1941 Dokl. Akad. Nauk SSSR 30 301; 1941 ibid 31 538
[2] Arimitsu T and Arimitsu N 2000 Phys. Rev. E 61 3237
[3] Arimitsu T and Arimitsu N 2000 J. Phys. A: Math. Gen. 33 L235 [CORRIGENDUM: 2001 ibid 34 673]
[4] Arimitsu T 2003 "Bussei Kennkyu" 81-3 334 (in Japanese) and the references therein (available at http://www.px.tsukuba.ac.jp/home/tcm/arimitsu/MathPhys03.pdf)
[5] Arimitsu T and Arimitsu N 2003 AIP Conf. Proc. 695 135
[6] Arimitsu T and Arimitsu N 2004 Physica D 193 218
[7] Arimitsu T and Arimitsu N 2004 Physica A 340 347
[8] Frisch U and Parisi G 1985 Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics (New York: North-Holland) p 84
[9] Van Dyke M 1982 An Album of Fluid Motion (The Parabolic Press, Stanford, California)
[10] Richardson L F 1922 Weather Prediction by Numerical Process (Cambridge: Cambridge Univ. Press)
[11] La Porta A, Voth G A, Crawford A M, Alexander J and Bodenschatz E 2001 Nature 409 1017
[12] Meneveau C and Sreenivasan K R 1987 Nucl. Phys. B (Proc. Suppl.) 2 49
[13] Oboukhov A M 1962 J. Fluid Mech. 13 77
[14] Kolmogorov A N 1962 J. Fluid Mech. 13 82
[15] Yaglom A M 1966 Sov. Phys. Dokl. 11 26
[16] Meneveau C and Sreenivasan K R 1987 Phys. Rev. Lett. 59 1424
[17] Meneveau C and Sreenivasan K R 1991 J. Fluid Mech. 224 429
[18] Frisch U, Sulem P-L and Nelkin M 1978 J. Fluid Mech. 87 719
[19] She Z-S and Leveque E 1994 Phys. Rev. Lett. 72 336
[20] She Z-S and Wyamire E 1995 Phys. Rev. Lett. 74 262
[21] Gotoh T, Fukayama D and Nakano T 2002 Phys. Fluids 14 1065
[22] Feynman R P 1955 Progress in Low Temperature Physics, (Amsterdam: North Holland) p 17
[23] Schwartz K W 1988 Phys. Rev. B38 2398
[24] Maurer J and Tabeling P 1998 Europhys. Lett. 43 29
[25] Radjiiri P and Roux S 2002 Phys. Rev. Lett. 89 064302
[26] Mantegna R N and Stanley H E 1995 Nature 376 46