Disorder fosters chimera in an array of motile particles

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Abstract

We consider an array of non-locally coupled oscillators on a ring, which for equally spaced units possesses a Kuramoto-Battogtokh chimera regime and a synchronous state. We demonstrate that disorder in oscillators positions leads to a transition from the synchronous to the chimera state. For a static (quenched) disorder we find that the probability of synchrony survival changes, in dependence on the number of particles, from nearly zero at small populations to one in the thermodynamic limit. Furthermore, we demonstrate how the synchrony gets destroyed for randomly (ballistically or diffusively) moving oscillators. We show that, depending on the number of oscillators, there are different scalings of the transition time with this number and the velocity of the units.

I. INTRODUCTION

Chimera patterns, discovered by Kuramoto and Battogtokh (KB) almost 20 years ago [1], continue to be in focus of theoretical and experimental studies (see recent reviews [2, 3]). Chimera is a spatial pattern in an oscillatory medium, where some subset of oscillators is synchronous and form an ordered patch, while other oscillators in a disordered patch are asynchronous. There is bistability in the classical KB setup of nonlocally coupled oscillators on a ring: a chimera pattern coexists with a fully synchronized, homogeneous in space state. In this bistable situation, one should specially prepare initial conditions to observe chimera, because the basin of the synchronous state is rather large. Moreover, in a finite population (i.e., for a finite number of oscillators on the ring), the synchronous state appears to be a global attractor: the chimera state is a transient, slightly irregular state, which has a lifetime exponentially growing with the number of oscillators [4]. This paper demonstrates that disorder in the KB setup fosters the opposite: synchronous state disappears, while chimera remains stable.

The effect of disorder on chimera has been explored in several recent publications. S. Sinha [5] studied different models of coupled maps and coupled oscillators, and demonstrated that with the introduction of time-varying random links to the network of interactions, a chimera is typically destroyed, and the synchronous state establishes. In paper [6], the effect of random links addition on the chimera state in coupled FitzHugh-Nagumo oscillators has been studied. It has been demonstrated, that although for a small disorder,
chimera survives, it becomes destroyed if the disorder is large.

Another way to include disorder in the setup of coupled oscillators is to assume that the units are motile particles, possibly with randomness in their motion. There are two ways in constructing such models: (i) one can assume that the oscillatory dynamics of the elements does not influence their motion, so that there is only the influence of the positions of the units on their oscillatory dynamics (see [7]), and (ii) there is a mutual interaction between motion and internal dynamics (see, e.g., [8, 9]). For example, for locally coupled phase oscillators randomly moving on one-dimensional lattice [7], motility has been shown to promote a synchronous state. For two-dimensional motions, the authors of [10] observed that there is a resonance range of random velocities, for which the transition to synchrony is extremely slow. The authors of [11] explored one-dimensional lattice with local delayed coupling, the motion of particles was modeled by random exchanges of positions of nearest neighbors; in this setup, a persistent chimera was observed in some range of parameters. Mostly close to our setup is a recent study of Wang et al. [12]. In this work, 128 diffusive particles on a line have been considered. Each particle is a phase oscillator, and the coupling is nonlocal with a cos-shaped kernel (like in the chimera studies [13]). Depending on the parameter of diffusion and coupling, both transitions from chimera to synchronous state and from synchronous state to chimera have been observed. Finally, we mention an important experimental setup where moving particles synchronize. Prindle et al. [14] realized a set of 2.5 millions of e.coli bacterial cells equipped with genetically engineered clocks, and observed their synchronization under conditions where these cells were transported in a microfluidic device, with a coupling through a chemical messenger.

In this paper, we explore the effect of disorder in particles’ positions on the properties of the “classical” KB chimera [1]. We consider quenched disorder (random fixed position of the particles on the ring), and dynamical disorder (diffusive or ballistic motion of the particles). Below we restrict our attention to the case of slow motions, which can be explored by comparing with the quenched case. We will show, that the number of particles is the essential parameter governing the dynamics, and establish scaling properties in dependence on the parameters determining the particles velocities, and on the number of them.

The paper is organized as follows. We introduce the model in Section II. The case of quenched disorder is considered in Section III. Properties of motile oscillators are considered in Section IV. We conclude and discuss the results in Section V.
II. BASIC MODELS

We introduce our basic model as a generalization of the Kuramoto-Battogtokh setup \cite{1} for a ring of coupled phase oscillators (particles). In contradistinction to \cite{1}, where equally spaced positions of the oscillators where assumed, we consider general positions \(0 \leq x_k < 1\) for \(N\) oscillators on the ring. The coupling is distance-dependent

\[
\dot{\varphi}_k = \frac{1}{N} \sum_j G(x_j - x_k) \sin(\varphi_j - \varphi_k - \alpha)
\]

according to the kernel

\[
G(y) = \frac{\kappa \cosh(\kappa(\|y\| - 0.5))}{2 \sinh \frac{\kappa}{2}},
\]

which is a generalization of the exponential kernel adopted in \cite{1} to account for periodic boundary conditions on the ring. Parameter \(\kappa\) determines the effective range of coupling; parameter \(\alpha\) is the phase shift in coupling.

For positions of the particles \(x_k\), we explore three models in this paper.

1. **Quenched disorder**: Here the positions \(x_j\) of particles are fixed, taken independently from a uniform distribution on a ring.

2. **Diffusion of the particles**: Here the particle positions are subject to independent white Gaussian noise terms, leading to their diffusion (with diffusion constant \(\sigma^2\))

\[
\dot{x}_j = \sigma \xi_j(t), \quad \langle \xi_j(t) \rangle = 0, \quad \langle \xi_j(t) \xi_k(t') \rangle = \delta_{jk} \delta(t - t') .
\]

3. **Ballistic motion of the particles**: Here the particles move with constant fixed random velocities \(v_j\). Below we consider velocities as i.i.d. Gaussian random variables with standard deviation \(\mu\).

In this paper we restrict ourselves to the cases of slow motion of the particles, i.e. to the cases of small parameters \(\sigma\) and \(\mu\).

III. QUENCHED DISORDER

A. Observation of a transition to chimera

We start with the case of quenched disorder. Here the only parameter is the number of the particles \(N\). At the thermodynamic limit \(N \to \infty\), one does not expect any deviation
of the dynamics of disordered sets from the dynamics of ordered configurations, because in both cases in the limit $N \to \infty$ one obtains a system of integro-differential equations for the distribution of phases $\varphi(x,t)$:

$$\partial_t \varphi(x,t) = \int_0^1 dy \, G(y-x) \sin(\varphi(y,t) - \varphi(x,t) - \alpha).$$

(4)

Population of phase oscillators (4), as has been first demonstrated by Kuramoto and Battogtokh [1], possesses two attracting states: (i) a fully synchronous state $\varphi(x,t) = \psi(t)$, and (ii) a spatially inhomogeneous chimera state with domain of synchrony (neighboring phases are closed to each other) and asynchrony (neighboring phases are taken from a certain probability distribution). Finite-size effects for a regular distribution of oscillators on the ring have been explored by Wolfrum and Omelchenko [4]. The synchronous state is still stable for any $N$, but the chimera state appeared to be a chaotic supertransient, which lives for a time interval exponentially growing with $N$, but eventually goes into the synchronous state.

Our main observation is that the opposite happens for an irregular distribution of oscillators on the ring. Namely, the initial synchronous state may become destroyed for finite $N$, while the chimera state is stable. We illustrate a transition from the synchronous to chimera regime in Fig. 1.

Qualitatively, destruction of the synchronous state due to disorder is similar to desynchronization in disordered oscillator lattices first described by Ermentrout and Kopell [15]. At large enough disorder a synchronous state in the lattice disappears due to a saddle-node bifurcation. In our setup we cannot directly apply theory [15], because we have a ring with long-range coupling. Furthermore, theory [15] is restricted to the case $\alpha = 0$, while in our setup parameter $\alpha$ is close to $\pi/2$.

B. Statistical evaluation

In Figure 2 we present a direct statistical evaluation of the probability for synchrony to occur. The numerical experiment has been performed as follows: for a configuration of random positions of oscillators $x_j$, equations (1) were solved starting from the state with all phases being equal $\varphi_1 = \ldots = \varphi_N$. If a steady rotating state where all the instantaneous frequencies are equal appears, the configuration is considered as a synchronous one. Otherwise, if in the set of oscillators phase slips appear, the configuration is considered as a
FIG. 1. Illustration of the transition synchrony → chimera for quenched disorder and $N = 1024$ (other parameters: $\kappa = 4$, $\alpha = 1.457$). The particles are placed randomly on the circle, and their phases are initially equal. Panels (a, b, c): snapshots of phase distributions $\varphi(x, t)$ at (a) $t = 125$, (b) $t = 375$ and (c) $t = 625$. One can see how the synchronous state is destroyed in the presence of spatial disorder. Firstly, phase slips in a certain region of space occur. Further, clusters with the highest phase gradient begin to break down, which leads to the formation of intervals with an irregular spatial distribution of the dynamic variable $\varphi(x, t)$. After that the system goes to a chimera state. Panel (d): spatio-temporal dynamics of phases $\varphi(x, t)$. Panel (e): absolute value of the local (calculated for $K = 16$ neighbors) order parameter $Z(x_n, t) = K^{-1} \sum_{k=0}^{K-1} \exp[i\varphi_{n+k-K/2}]$, additionally averaged over the time interval of 3 time units. White regions correspond to synchrony. Black dashed lines denote the moments in time for which snapshots of the phases $\varphi(x, t)$ are presented on panels (a), (b) and (c). Panel (f): the dynamics of the global order parameter $R(t) = |N^{-1} \sum_{n=0}^{N-1} \exp[i\varphi_n]|$. It is clearly seen how the transition from the initially synchronous regime with $R = 1$ to the chimera state with $R \approx 0.79$ occurs. The green dashed line shows the value $R = 0.85$, which is further taken as a criterion that determines the time of destruction of the synchronous mode.

non-synchronous (chimera). Many runs with random positions have been sampled to achieve statistical results presented in the Fig. 2. One can see that while the probability to observe synchrony is very low for relatively small $N$ (in fact, for $N = 256$ no any synchronous case out of $10^4$ runs has been observed), it becomes high for $N \gtrsim 8192$. This confirms the qual-
FIG. 2. Red dots: probabilities of existence of a synchronous state from direct numerical simulations. Curves: rescaled cumulative distributions of the maximum of field $H$, for $N = 128$ (green curve), $N = 256$ (blue) and $N = 512$ (magenta). These curves are drawn with help of expression (7) and practically overlap, what confirms the validity of the scaling $\sim N^{1/2}$.

tative picture of the local stability of the synchronous state at $N \to \infty$. We stress here that we do not consider here very small systems with a few oscillators.

C. Analytic estimate of probability of the existence of stable synchrony

Here we give a semi-analytic estimate for the probability to observe a synchronous state in a disordered array. Instead of performing a rigorous bifurcation analysis, we first estimate (approximately), at which fluctuation of the local acting field, the synchronous state disappears. If we assume that in the synchronous state all the phases are equal and rotate with the frequency $\Omega = -\sin \alpha$ (in fact, this is only true for the regular uniform distribution of the units), then from Eq. (1) it follows that an oscillator $\varphi_k$ will not be able to follow this collective synchrony if the acting field on it $H(x_k) = N^{-1} \sum G(x_j - x_k)$ exceeds the threshold $H_c = 2 - \sin \alpha$. Thus, to find the probability that this happens, we have to analyze the distribution of maxima of the field $H(x)$ defined as

$$H(x) = \frac{1}{N} \sum_{k=1}^{N} G(x - x_k), \quad (5)$$
where \( x_k \) are random positions on the interval \( 0 \leq x < 1 \) with uniform density \( w(x) = 1 \).

Statistics of the field \( H \) can be evaluated as follows. First, due to normalization \( \int_0^1 G(y)dy = 1 \), we get \( \langle H \rangle = 1 \). Next, using independence of positions \( x_k \), it is straightforward to calculate the covariance of \( H \) (this calculation is completely analogous to a calculation of the correlation function of the shot noise (sequence of independent pulses, the Campbell’s formula) \(^{16}\)):

\[
K(y) = \langle H(x)H(x+y) \rangle - 1 = N^{-1} \frac{\kappa^2 B(\kappa, y)}{4 \sinh^2 \frac{\kappa}{2}},
\]

\[
B(\kappa, y) = \frac{\cosh \kappa y}{2} + \frac{y[\cosh \kappa(y-1) - \cosh \kappa y]}{2} + \frac{\sinh \kappa y - \sinh \kappa(y-1)}{\kappa}.
\]

One can see that the variance of field \( H \) decays as expected \( \sim N^{-1} \). One can argue that for large \( N \), as a sum of \( N \) statistically independent contributions, the field \( H(x) \) is Gaussian, and this indeed is nicely confirmed by numerics (not shown). However, we are interested in the distribution of the maximum if this field, and obtaining it is a nontrivial task, because of correlations \(^6\) (cf. \(^{17}\)). These correlations, however, do not depend on \( N \) except for a factor \( N^{-1} \), and therefore one can expect that the scaled distribution of the maximum \( h = (H_{\text{max}} - 1)\sqrt{N} \) will be system-size-independent. The cumulative distribution function of maxima \( W(h) \) thus provides an estimate that for an ensemble of size \( N \) the synchronous population survives:

\[
P_s(N) = W((N)^{1/2}(1 - \sin \alpha)).
\]

In Fig. 2 we compare this estimate with direct numerical simulations, using three distributions \( W \) obtained for \( N = 128, 256, 512 \). These curves are practically indistinguishable, what is just another manifestation of validity of the scaling \( H_{\text{max}} - 1 \sim N^{-1/2} \). The curve lies below the numerical data, what means that the adopted estimate is rather crude. Nevertheless, it correctly predicts that for \( N \lesssim 1000 \) practically all configurations lead to a chimera state.

IV. TRANSITION FROM SYNCHRONY TO CHIMERA FOR MOTILE PARTICLES

In this section we consider motile particles with random trajectories. In all cases reported in this section below, we start at \( t = 0 \) with particles regularly distributed over the ring,
FIG. 3. The same as Fig. 1 but for diffusive particles with $\sigma = 10^{-3}$. Initially all the particles are placed equidistantly on the circle, and have equal phases. Developing at $t \approx 1000$ chimera pattern slowly moves along the circle, due to random rearrangement of particles positions. Panels (a, b, c): snapshots of phase distributions $\varphi(x, t)$ at (a) $t = 500$, (b) $t = 1500$, (c) $t = 2500$. Panel (d): spatio-temporal dynamics of phases $\varphi(x, t)$. Panel (e): absolute value of the local order parameter $Z(x, t)$. Panel (f): the dynamics of the global order parameter $R(t)$.

i.e. $x_k(0) = (k - 1)/N$. The phases are set to be equal, so that the initial state is the perfectly synchronized one. Because of irregular motion, disorder in the position of the particles appears. At rather large times the particles can be considered as noncorrelated, thus their positions are fully random on the ring. This, as we have seen in Section III, facilitates transition to chimera. Moreover, as in the course of time evolution different random configurations appear, eventually one which does not support synchrony will lead to a transition to chimera (we illustrate this in Fig. 3). Thus, on the contrary to the case of static configurations of Section III, we expect that a transition from synchrony to chimera will always be observed even at system sizes as large as $N = 8192$.

Our main interest below is in the dependence of the transition time (from synchrony
to chimera) on the parameters of noise and particle size. In the system of differential
equation (1), positions of the particles $x_k$ can be considered as parameters. We start with
a stable fixed point in this system, which does exist for regularly spread particles. Slow
motion of particles means slow variation of the parameters in (1), and initially the stable
steady state continues to exist. However, when the set of parameters reaches a bifurcation
point (numerical experiments show that this is a saddle-node bifurcation, like in a disordered
lattice [15]), the steady state disappears and another, chimera state, appears. Thus, what
we want to study, is the time to bifurcation.

There is also another view on the transition to chimera. In the starting configuration,
where the oscillators are equidistantly distributed, the acting field $H(x)$ (see Eq. (5)) is
constant. When the particles start moving, this field is no more constant, so one observes
roughening of $H(x)$ [18]. This roughening continues until the maximum of the field becomes
large enough to produce the bifurcation. This picture suggests that one can expect the
average time of the transition $\langle T \rangle$ to scale with parameters of the problem: characteristic
random velocities of the particles and the number of them. We explore this idea of scaling
below.

We consider two basic setups for the random motion of particles:

1. **Diffusive motion.** Here we consider diffusive motion of the particles according to (3).
The average transition times from synchrony to chimera are presented in Fig. 4(a).
As expected, the time grows with the number of particles $N$, and for small diffusion
rates $\sigma$.

2. **Ballistic motion.** Here we assume that the particles move with constant velocities
$v_j$, which are chosen from the normal distribution with standard variation $\mu$. The
average transition times are shown in Fig. 4(b).

Next, we discuss scaling properties of the time to chimera. We look for the scaling relation
in the form

$$\langle T(c, N) \rangle = N^a f \left( \frac{c}{N^b} \right)$$

where $c$ stays for one of the parameters $\mu, \sigma$, and constants $a, b$ generally depend on the setup.
We, however, could not fit all the data according to a unique law (8). As we illustrate in
Fig. 5, taking data for the interval of system sizes $128 \leq N \leq 1024$ allows to achieve a very
good collapse of data points using scaling in form \( \sigma \), with \( b = 0.45 \) and \( a = 0.15 \) for both cases (diffusive and ballistic motions). However, using these parameters for larger system sizes \( N \geq 2048 \) does not lead to a good collapse of points. Rather we use for large \( N \) values \( b = 0.3 \) and \( a = 0.6 \) for the ballistic case and \( b = 0.35 \) and \( a = 0.65 \) for the diffusive case, but they result only in an approximate collapse of data points.

We attribute this absence of a universal scaling to the properties of the quenched randomness described in section \[ \text{III} \]. As it follows from Figure [2] for \( N \lesssim 1024 \) it is enough for particles to achieve random independent positions on the circle, then the transition to chimera is nearly certain. In contradistinction, for larger populations there is a finite probability for a random quenched configuration to possess synchrony. This leads to an increase of the transition time: random motion of the particles explores different configurations, until one that does not possess synchrony is found and the transition to chimera occurs. This explains different scalings with a crossover near \( N = 1024 \). Moreover, we expect that the scaling observed for \( 2048 \leq N \leq 8192 \) will not extend to larger system sizes, because according to Figure [2] the probability of the transition in quenched configuration drastically reduces, so that the time to achieve chimera will be extremely large, if not infinite.
FIG. 5. The same data as in Fig. 4 but in scaled coordinates (diffusive particles in panels (a,b), ballistic particles in panels (c,d)). Top row (panels (a,c)): scaling for $128 \leq N \leq 1024$ with $b = 0.45$ and $a = 0.15$. Bottom row (panels (b,d)): scaling for $2048 \leq N \leq 8192$ with $b = 0.35$ and $a = 0.65$ for diffusive particles, and $b = 0.3$ and $a = 0.6$ for ballistic particles.

V. CONCLUSION

In this paper we studied the effect of the oscillators position disorder on the chimera state in the Kuramoto-Battogtokh model of nonlocally coupled phase oscillators on a ring. The level of disorder is basically determined by the number of units $N$, it disappears in the thermodynamic limit $N \to \infty$. Our main finding is that large disorder facilitates stability of chimera, and for sizes of populations below some level, it is practically impossible to observe a stable synchronous regime in a setup with a quenched disorder. For slow random motions of the particles, in the explored range of system sizes up to $N = 8192$, we observed a transition from synchronous initial configuration to a chimera in all realizations. Even when synchrony has a finite probability to exist in a quenched configuration, slow variations of positions of particles lead eventually to a configuration where synchrony state does not exist, so that a chimera develops.

We explored the scaling properties of the transition to chimera and found that for both diffusive and ballistic motions, the scaling exponents in the relation (8) are nearly the same.
Due to a nontrivial dependence of the probability of the existence of synchrony already for quenched disorder, the scaling is different for relatively small sizes $N$ (where synchrony is practically never observed) and for larger sizes, where in the quenched case there is a finite probability for synchrony to survive. We, however, have not explored very large populations $N > 8192$, because of computational restrictions.

We stress here that we studied the Kuramoto-Battogtokh model for the “standard” parameters $\kappa, \alpha$ used in [1]. The domain of existence of chimera and its basin of attraction may depend significantly on these parameters. Extension of the obtained results on other domains of parameters and on other setups where chimera patterns exist is a subject of ongoing study.

In this paper we focused on the regime of very slow motion of the particles, including the static (quenched) case. Preliminary simulations show that the regimes with fast particles can differ significantly, this is a subject of ongoing research. Another interesting case for future exploration is one close to the thermodynamic limit, where finite-size fluctuations are small. Here an analytical description based on the Ott-Antonsen reduction might be possible, to be reported elsewhere.

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