THE EXPANSION OF WRONSKIAN HERMITE POLYNOMIALS IN THE HERMITE BASIS

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ABSTRACT. We express Wronskian Hermite polynomials in the Hermite basis and obtain an explicit formula for the coefficients. From this we deduce an upper bound for the modulus of the roots in the case of partitions of length 2. We also derive a general upper bound for the modulus of the real and purely imaginary roots. These bounds are very useful in the study of irreducibility of Wronskian Hermite polynomials.

1. INTRODUCTION

Let \{H_n(x)\}_{n \geq 0} be the classical Hermite polynomials, solutions to the equation \(y''(x) - 2xy'(x) + 2ny(x) = 0\). In this paper we study the Wronskian of such polynomials: if \(n_1 < n_2 < \ldots < n_r\) is a sequence of non-negative integers, we can define the Wronskian \(\text{Wr}[H_{n_1}(x), H_{n_2}(x), \ldots, H_{n_r}(x)]\) as the determinant

\[
\begin{vmatrix}
H_{n_1}(x) & H_{n_2}(x) & \cdots & H_{n_r}(x) \\
H'_{n_1}(x) & H'_{n_2}(x) & \cdots & H'_{n_r}(x) \\
\vdots & \vdots & \ddots & \vdots \\
H^{(r-1)}_{n_1}(x) & H^{(r-1)}_{n_2}(x) & \cdots & H^{(r-1)}_{n_r}(x)
\end{vmatrix}
\]

Wronskians of Hermite polynomials appear in the study of rational potentials admitted by the Schrödinger operator \(L = -\frac{\partial^2}{\partial x^2} + V(x)\). Oblomkov ([10]) characterized rational potentials of monodromy-free Schrödinger operators that grow as \(x^2\) at infinity. In this case, the potentials have the form

\[
V(x) = -2 \frac{\partial^2}{\partial x^2} \log \text{Wr}[H_{n_1}(x), H_{n_2}(x), \ldots, H_{n_r}(x)] + x^2 + c. \tag{1}
\]

From (1) the zeros of \(\text{Wr}[H_{n_1}(x), H_{n_2}(x), \ldots, H_{n_r}(x)]\) are precisely the poles of the potential. Because of this relationship it is important to understand the geometry of the zeros of the Wronskian ([4], [6]). Unfortunately not much is known about the set of zeros. Veselov (see [6]) conjectured that all the zeros are simple, except possibly at the origin. This conjecture is known to be true in a few special cases, but in general it is still open. In contrast, for the Hermite polynomials \(H_n(x)\) it is well known that the zeros are real and simple.

It turns out it is useful to define the Wronskian polynomials in terms of partitions.

Let \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)\) be any partition. We define the degree sequence of \(\lambda\) as \(n_\lambda := (\lambda_r, \lambda_{r-1} + 1, \ldots, \lambda_1 + r - 1)\). Furthermore, let

\[
\Delta(x_1, x_2, \ldots, x_r) := \det[x_i^{j-1}]_{1 \leq i, j \leq r} = \prod_{j > i}(x_j - x_i)
\]

be the Vandermonde determinant, with \(\Delta(x_1) := 1\).

We shall consider a rescaled version of \(H_n(x)\), defined by \(\text{He}_n(x) = 2^{-\frac{n}{2}}H_n\left(\frac{x}{\sqrt{2}}\right)\). These are the probabilistic Hermite polynomials, solution to the equation \(y''(x) - xy'(x) + ny(x) = 0\).

**Definition 1.** For any partition \(\lambda \vdash n\) we define the Wronskian Hermite polynomial associated to \(\lambda\) as

\[
\text{He}_\lambda(x) := \frac{\text{Wr}[\text{He}_{n_1}(x), \text{He}_{n_2}(x), \ldots, \text{He}_{n_r}(x)]}{\Delta(n_\lambda)}, \tag{2}
\]

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where \( n_\lambda = (n_1, n_2, \ldots, n_r) \) is the degree sequence of \( \lambda \).

Then \( H_\lambda(x) \) is a monic polynomial of degree \( n \). Furthermore, \( H_\lambda(x) \) has up to scaling the same set of zeros as the Wronskian of \( \{H_{n_1}(x), H_{n_2}(x), \ldots, H_{n_r}(x)\} \).

Recently, Bonneux, Dunning and Stevens ([1]), following earlier work ([2]), found an explicit formula for the coefficients of \( H_\lambda(x) \) in terms of the characters of the symmetric group.

**Theorem 1** ([1], Theorem 4.2). Let \( \lambda \vdash n \). Then

\[
H_\lambda(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k H(\lambda)}{2^k k!(n-2k)!} x^{n-2k},
\]

where \( H(\lambda) := \frac{n!}{\lambda' \lambda} \).

Our main contribution in this paper is to establish a dual version of Theorem 1, where we determine the coefficients of \( H_\lambda(x) \) in the Hermite basis.

**Theorem 2.** Let \( \lambda \vdash n \). Then

\[
H_\lambda(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{H(\lambda)}{k!(n-2k)!} K_{\lambda' \mu_k} H_{n-2k}(x),
\]

where \( \mu_k := 2^k 1^{n-2k} \), \( \lambda' \) is the conjugate partition of \( \lambda \), and \( K_{\lambda' \mu} \) are the Kostka numbers.

Our interest in Theorem 2 stems from the need to find good bounds for the modulus of zeros, in the case when \( \lambda = (\lambda_1, \lambda_2) \) and \( \lambda_2 \) is fixed.

**Corollary 4.** Let \( \lambda = (\lambda_1, \lambda_2) \vdash n \). If \( z \) is a root of \( H_n(x) \) then \(|z| \leq \sqrt{\frac{2(n-1)}{n+2}}\).

We made no effort to optimize the constant in Corollary 4, as the correct bound is likely close to \( 2\sqrt{\lambda_1} \).

**Corollary 5.** Let \( \lambda \vdash n \). If \( z \) is a real or purely imaginary root of \( H_\lambda(x) \) then \(|z| \leq x_n \), where \( x_n \) is the largest root of \( H_n(x) \).

However, Corollary 5 does not give the full picture. By exploiting the Schrödinger equation it is possible to obtain a better bound.

**Proposition 6.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \). If \( z \) is a real root of \( H_\lambda(x) \) then \(|z| \leq x_{\lambda_1+r-1} \), where \( x_{\lambda_1+r-1} \) is the largest root of \( H_{\lambda_1+r-1}(x) \).

This result is already apparent in the work of García-Ferrero and Gómez-Ullate ([8], Section 2). However, the proof in [8] is done under the additional assumption of semi-degeneracy, which is so far unproven for Wronskian Hermite polynomials. Therefore, we include a proof of Proposition 6.

These bounds are very useful in the study of irreducibility of \( H_\lambda(x) \). We will treat this topic in a forthcoming paper.

The rest of this note is organized as follows. In Section 2, we fix the notation and state the auxiliary results that we will need. In Section 3, we prove Theorem 2. In Section 4, we derive the two corollaries. Finally, in Section 5, we prove Proposition 6.

2. **Notation and auxiliary results**

In this section we define the notation that we use throughout the paper. We also state several results that will be needed for the proof.
2.1. **Partitions and the symmetric group.** If \( n \geq 0 \) is an integer, a partition \( \lambda \) of \( n \), denoted \( \lambda \vdash n \), is a sequence of non-negative integers \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \) such that \( \sum_{i=1}^{r} \lambda_i = n \). We denote \(|\lambda| = n\) and call \( \ell(\lambda) := r \) the length of the partition \( \lambda \). We say that \( \lambda_i \) are the *parts* of the partition.

Note that we allow parts of size 0. This will simplify many of our statements and proofs.

We shall frequently use the notation \((\lambda_1, \lambda_2, \ldots, \lambda_r)\) for \( \lambda \). Sometimes we will also use the notation \( \mu = 1^e_1 \cdot 2^e_2 \cdot 3^e_3 \cdot \ldots \), meaning that the partition \( \mu \) has \( r \) parts of size \( i \).

The *degree sequence* of the partition \( \lambda \) is defined as \( n_\lambda := (\lambda_r, \lambda_{r-1} + 1, \ldots, \lambda_1 + r - 1) \).

The **Ferrers diagram** of \( \lambda \) is \( D_\lambda = \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq \lambda_i\} \). This can be represented as a collection of unit squares arranged in rows, with the \( i \)-th row having \( \lambda_i \) squares. For example,

\[
D_{(3,2,2,1)} = \begin{array}{|c|c|c|}
\hline
& & \\
& & \\
& & \\
\hline
\end{array}
\]

The *conjugate partition* \( \lambda' \) is obtained from \( \lambda \) by transposing the Ferrers diagram: \( \lambda_j' \) is the largest index \( i \) such that \( \lambda_i \geq j \).

If \( \lambda \) and \( \mu \) are two partitions, a semistandard Young tableau of shape \( \lambda \) and type \( \mu \) is a filling of the Ferrers diagram of \( \lambda \) with the numbers \( 1, 2, \ldots, \ell(\mu) \) such that the number \( i \) appears \( \mu_i \) times, the numbers weakly increase along rows, and strictly increase along columns. The **Kostka number** \( K_{\lambda \mu} \) is the number of semistandard Young tableaux of shape \( \lambda \) and type \( \mu \).

Clearly \( K_{\lambda \mu} \geq 0 \). Furthermore, \( K_{\lambda \mu} > 0 \) if and only if \( \lambda \) dominates \( \mu \), written \( \lambda \trianglerighteq \mu \), that is when \(|\lambda| = |\mu|\) and \( \lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i \) for \( i = 1, \ldots, \ell(\mu) \).

We denote by \( \chi^\lambda \) the irreducible character of \( S_n \) associated to \( \lambda \). Let \( F_\lambda := \chi^\lambda(1) \) be the degree of the irreducible representation. Then this is given by the formula (see [7], (4.11)):

\[
F_\lambda = \frac{|\lambda|!}{H(\lambda)}, \quad \text{where} \quad H(\lambda) := \frac{n_{\lambda,1}!n_{\lambda,2}! \cdots n_{\lambda,r}!}{\Delta(n_\lambda)}.
\]

(3)

2.2. **Schur polynomials.** Let \( x_1, x_2, \ldots, x_k \) be \( k \) variables. We let \( e_i(x_1, \ldots, x_k) \) be the elementary symmetric polynomials:

\[
e_i(x_1, \ldots, x_k) = \sum_{1 \leq j_1 < \ldots < j_i \leq k} x_{j_1}x_{j_2}\cdots x_{j_i}.
\]

By convention \( e_i(x_1, \ldots, x_k) = 0 \) when \( i > k \).

Similarly, we let \( h_i(x_1, \ldots, x_k) \) be the complete symmetric polynomials:

\[
h_i(x_1, \ldots, x_k) = \sum_{1 \leq j_1 \leq \ldots \leq j_i \leq k} x_{j_1}x_{j_2}\cdots x_{j_i}.
\]

Finally, we let \( p_i(x_1, \ldots, x_k) \) be the power-sum polynomials:

\[
p_i(x_1, \ldots, x_k) = x_1^i + \ldots + x_k^i.
\]

We are not going to write the variables if they are clear from the context. For our purposes we will also use the convention \( e_0 = h_0 = p_0 = 1 \).

If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is a partition, we define

\[
e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_r},
\]

\[
h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_r},
\]

\[
p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_r}.
\]

Now assume that \( k \geq r \). By adding parts of size 0, we can further assume that \( r = k \). In this case we define \( W_\lambda(x_1, \ldots, x_k) := \det[x_{i,j}^{\lambda_i+k-j}]_{i,j=1}^{k} \). Then \( W_\lambda \) is an alternating polynomial, and hence it is divisible by the Vandermonde determinant \( W_0(x_1, \ldots, x_k) := \det[x_i^{k-j}]_{i,j=1}^{k} \). Set \( s_\lambda(x_1, \ldots, x_k) := \frac{W_\lambda}{W_0} \). We call \( s_\lambda \)
the Schur polynomial for $\lambda$. Then $s_\lambda$ is a symmetric polynomial, and is defined for any partition $\lambda$ with $\ell(\lambda) \leq k$. In the case $\ell(\lambda) > k$ we define $s_\lambda(x_1, \ldots, x_k) := 0$.

We will need the following consequence of Pieri’s rule (see [7], Appendix A.1).

**Theorem 7.** For any partition $\mu$ we have

$$e_\mu = \sum_\lambda K_{\lambda \mu} s_\lambda,$$

where the sum is taken over all partitions $\lambda$ with non-zero parts.

We shall also need the following (see [7], Lecture 4, (4.10), with $C_i$ the conjugacy class of $\mu$).

**Theorem 8** (Frobenius character formula). Let $\lambda, \mu \vdash n$ be any partitions such that $\ell(\lambda) = n$. Then $\chi_\lambda(\mu)$ is the coefficient of the monomial $x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^n$ in the polynomial $p_\mu(x_1, \ldots, x_n)W_0(x_1, \ldots, x_n)$.

2.3. **Bounds for the roots of polynomials.** To obtain effective bounds on the modulus of the roots, we will use a result which is essentially due to Turán ([12]).

**Theorem 9.** Suppose $P(x) = \sum_{k=0}^n a_k H_k(x)$ is a polynomial of degree $n$. If $z$ is a zero of $P(x)$ and $x_{n,n}$ is the largest root of $H_n(x)$ then

$$|z| \leq x_{n,n} + \sum_{k=0}^{n-1} \left|\frac{a_k}{a_n}\right|^{1/(n-k)},$$

where $z$ is a root of $P(x)$ and $z$ is not a root of $H_n(x)$.

However, Turán only proved an inequality for $|\text{Im } z|$ for a decomposition in the base of classical Hermite polynomials. For convenience, we explain how to adapt the proof to obtain Theorem 9.

**Proof.** Let $x_{k,1} \leq x_{k,2} \leq \ldots \leq x_{k,k}$ be the roots of $H_k(x)$.

If $z$ is a complex number, we define $D(z) = \min |z - x_{k,i}|$, where the minimum is taken over all $1 \leq i \leq k \leq n$. The main step of the proof is showing that

$$D(z) \leq \sum_{k=0}^{n-1} \left|\frac{a_k}{a_n}\right|^{1/(n-k)},$$

if $z$ is a root of $P(x)$ and $z$ is not a root of $H_n(x)$.

The inequality (4) follows verbatim from Turán’s proof by replacing $|y|$ with $D(z)$ everywhere (see also [9] for an exposition of the proof). Therefore we shall not reproduce the proof here.

Using (4) we will show the inequality in the theorem.

If $z$ is a root of $H_n(x)$ then the inequality is trivially true, as $x_{n,n} \geq 0$.

So we may assume that $z$ is not a root of $H_n(x)$. As the roots of Hermite polynomials interlace, we have $x_{n,1} \leq x_{k,i} \leq x_{n,n}$. The roots of Hermite polynomials are also symmetric around the origin, i.e. $x_{n,1} = -x_{n,n}$. Therefore the root with largest absolute value among $x_{k,i}$ is $x_{n,n}$. Then

$$|z - x_{k,i}| \geq |z| - |x_{k,i}| \geq |z| - x_{n,n}.$$

Therefore $D(z) \geq |z| - x_{n,n}$, and the inequality follows from (4).

3. **Proof of the main result.**

We start by writing the base-change formula.

**Lemma 10.** Let $\lambda \vdash n$ and write $H_\lambda(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} a_j x^{n-2j}$. Then $H_\lambda(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_k H_{n-2k}(x)$ where

$$b_k = \sum_{j=0}^{k} \frac{(n-2j)!}{2^{k-j}(k-j)!(n-2k)!} a_j, 0 \leq k \leq \lfloor n/2 \rfloor.$$
Lemma 11. Let \( x \in \mathbb{Z} \) and 0 ≤ \( k \leq \left\lfloor \frac{n}{2} \right\rfloor \). Set \( \mu_k := 2^k \lambda^{2n-2k} \). Then \( S^\lambda_k = 2^k K_{\lambda^\mu_k} \).

Proof. By adding parts of size 0, we may assume without lack of generality that \( \lambda \) has length \( n \), and that we have \( n \) variables \( x_1, \ldots, x_n \).

Set \( \mu_j := 2^j 1^{n-2j} \), 0 ≤ \( j \leq k \), and define \( M_\lambda \) as the monomial \( x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \).

From Theorem 8, we know that \( \chi^\lambda(\mu_j) \) is the coefficient of \( M_\lambda \) in the polynomial \( p_{\mu_j}(x_1, \ldots, x_n)W_0(x_1, \ldots, x_n) \). Therefore \( S^\lambda_k \) must be the coefficient of \( M_\lambda \) in the polynomial

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} p_{\mu_j} W_0 = \sum_{j=0}^{k} (-1)^j \binom{k}{j} p_2^j p_1^{n-2j} W_0
\]

The last equality follows from the binomial theorem.

However, \( p_1^2 - p_2 = 2e_2 \). Furthermore, \( p_1 = e_1 \). Hence the above is the polynomial

\[
2^k e_1^{n-2k} e_2^k W_0 = 2^k e_{\mu_k} W_0
\]

by Theorem 7, as \( s_\rho = \frac{W_\rho}{W_0} \) when \( \ell(\rho) \leq n \), and \( s_\rho = 0 \) otherwise,
where in the last sum the appropriate number of zero parts were added to $\rho$ such that $W_{\rho}$ is defined.

Recall that we need the coefficient of $M_{\lambda}$. We argue that this can only come from $\rho = \lambda$.

Let $\rho$ be a partition such that the determinant $W_{\rho} = \det[x_i^{\rho} + n - j]$ contains the monomial $M_\lambda$. Assume for a contradiction that $\rho \neq \lambda$. By comparing powers of each $x_i$, there must exist a permutation $\sigma$ such that $\rho_{\sigma(i)} + n - \sigma(i) = \lambda_i + n - i$. Hence $\rho_{\sigma(i)} = \lambda_i + \sigma(i) - i$ for $1 \leq i \leq n$.

Let $i$ be minimal such that $\sigma(i) \neq i$. Then $\sigma(i) > i$. Let $j > i$ such that $\sigma(j) = i$. As $\rho_i \geq \rho_{\sigma(i)}$ we have

$$\rho_i = \lambda_j + i - j \geq \rho_{\sigma(i)} = \lambda_i + \sigma(i) - i,$$

So $\lambda_j \geq \lambda_i + \sigma(i) - i + (j - i) > \lambda_i + \sigma(i) - i > \lambda_i$, a contradiction to the fact that $\lambda_i \geq \lambda_j$ for $j \geq i$.

Therefore the coefficient of $M_{\lambda}$ comes from $\rho = \lambda$ only, and so it is $2^k K_{\lambda' \mu_k}$. This finishes the proof. □

Theorem 2 now follows directly by replacing $S_k^{\lambda}$ in (5):

$$b_k = \frac{H(\lambda)}{2^k k!(n-2k)!} S_k^{\lambda} = \frac{H(\lambda)}{k!(n-2k)!} K_{\lambda' \mu_k}.$$

4. AN UPPER BOUND FOR THE MODULUS OF THE ROOTS

In this section we derive the bounds on the absolute value of the roots. We again write $H(\lambda)(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_k H_{n-2k}(x)$.

Proof of Corollary 4. The plan is to compute the coefficients $b_k$ exactly and then use Theorem 9.

Let $\lambda = (\lambda_1, \lambda_2) \vdash n$. The conjugate of the partition $\lambda$ is $\lambda' = 2\lambda_2 1^{\lambda_1 - \lambda_2}$. We will determine $K_{\lambda' \mu_k}$, where recall that $\mu_k = 2^k 1^{n-2k}$.

If $k > \lambda_2$ then $K_{\lambda' \mu_k} = 0$, as $\lambda'$ does not dominate $\mu_k$. So $b_k = 0$.

If $k \leq \lambda_2$ then $K_{\lambda' \mu_k} = F_{\rho}$, where $\rho = 2^{\lambda_2-k} 1^1 \lambda_1 - \lambda_2$. Using the fact that $F_{\rho} = F_{\rho'}$ we get $K_{\lambda' \mu_k} = F_{\lambda_1-k, \lambda_2-k}$. So

$$b_k = \frac{H(\lambda)}{k!(n-2k)!} F_{\lambda_1-k, \lambda_2-k} = \frac{(\lambda_1 + 1)! \lambda_2!}{(\lambda_1 - \lambda_2 + 1)! (\lambda_1 - k + 1)! (\lambda_2 - k)!} (\lambda_1 - \lambda_2 + 1)$$

$$\quad = (\lambda_1 + 1)! \lambda_2! / (\lambda_1 - k + 1)! (\lambda_2 - k)! k!.$$

From the inequality $k! \geq 2\pi k^{k+\frac{1}{2}} e^{-k}$ we get

$$b_k \leq \frac{(\lambda_1 + 1)^k \lambda_2^k}{2\pi k^{k+\frac{1}{2}} e^{-k}}.$$

Let $z$ be a root of $H(\lambda)(x)$. From Theorem 9 and the fact that $b_0 = 1$, we obtain $|z| \leq x_{n,n} + \sum_{k=1}^{\lambda_2} 2^k b_k$, where $x_{n,n}$ is the largest root of $H_n(x)$. But

$$\sum_{k=1}^{\lambda_2} 2^k b_k \leq \sqrt{(\lambda_1 + 1)! \lambda_2!} \sum_{k=1}^{\lambda_2} \frac{1}{\sqrt{k}} \sqrt{2\pi k} < \sqrt{(\lambda_1 + 1)! \lambda_2!} \sum_{k=1}^{\lambda_2} \frac{1}{\sqrt{k}}.$$

Using the inequality $\sum_{k=1}^{\lambda_2} \frac{1}{\sqrt{k}} < 2\sqrt{\lambda_2}$ we get

$$\sum_{k=1}^{\lambda_2} 2^k b_k < \sqrt{(\lambda_1 + 1)! \lambda_2!} 2 \sqrt{\lambda_2} = 2 \sqrt{e} \lambda_2 \sqrt{\lambda_1 + 1}.$$

Furthermore, Theorem 3 shows that $x_{n,n} \leq 2(n-1) \frac{2(n-1)}{\sqrt{n+2}} \leq 2\sqrt{n-1}$ (the constant 2 comes from the rescaling $H_n(x) = 2^{-\frac{\pi}{2}} H_n(\frac{x}{\sqrt{2}})$). Hence

$$|z| \leq 2\sqrt{n-1} + 2 \sqrt{e} \lambda_2 \sqrt{\lambda_1 + 1}.$$

Now $n-1 = \lambda_1 + \lambda_2 - 1 < 2(\lambda_1 + 1)$, so

$$|z| < 2\sqrt{2(\lambda_1 + 1)} + 2 \sqrt{e} \lambda_2 \sqrt{\lambda_1 + 1} = 2(\sqrt{e} \lambda_2 + \sqrt{2}) \sqrt{\lambda_1 + 1}.$$
This finishes the proof. □

Turán’s theorem is modelled after Walsh’s theorem ([13], see also [9]), which gives a similar bound, but in terms of the expansion in the usual basis \(\{x^n\}_{n \geq 0}\). However, applying Walsh’s theorem to the coefficients in Theorem 1 only gives a bound of the order \(O(n\sqrt{n})\). The bound does not change for partitions of length 2: for example, for \(H_{n,2}(x)\) it is possible to compute the character values exactly and show that the upper bound in Walsh’s theorem is at least \(\Omega(n\sqrt{n})\). Hence changing to the Hermite basis and relying on Theorem 2 is necessary.

Let us now look at the real and purely imaginary roots of \(\text{He}_\lambda(x)\).

**Proof of Corollary 5.** Let \(x_{k,1} \leq x_{k,2} \leq \ldots \leq x_{k,k}\) be the roots of \(\text{He}_\lambda(x)\).

We show that \(\text{He}_\lambda(x)\) has no real root in the interval \((x_{n,n}, +\infty)\). The roots of Hermite polynomials interlace, so \(x_{n,1} \leq x_{k,i} \leq x_{n,n}\) for all \(1 \leq i \leq k \leq n\). Therefore if \(z \in \mathbb{R}\) is greater than \(x_{n,n}\), then \(\text{He}_k(z) > 0\) for all \(k \leq n\). Furthermore, \(b_k \geq 0\) for all \(k\) and \(b_0 = 1\). Hence \(\text{He}_\lambda(z) > 0\), so \(z\) is not a root.

On the other hand, \(\text{He}_\lambda(-z) = (-1)^{|\lambda|}\text{He}_\lambda(z)\) (see Lemma 3.6, [3]). Therefore \(\text{He}_\lambda(x)\) has no roots in the interval \((-\infty, -x_{n,n})\). This shows that any real root \(z\) of \(\text{He}_\lambda(x)\) satisfies \(|z| \leq x_{n,n}\).

Now let \(z \in i\mathbb{R}\) be a purely imaginary root of \(\text{He}_\lambda(x)\). Then \(\text{He}_\lambda(z) = i^{|\lambda|}\text{He}_{\lambda}(-iz)\) (see Proposition 3.8, [3]). Hence \(iz\) is a real root of \(\text{He}_\lambda(x)\). But \(|\lambda| = n\), so \(|z| = |iz| \leq x_{n,n}\) by the above argument. This finishes the proof. □

## 5. Proof of Proposition 6

In this section we will work with the unnormalized Hermite functions. For any \(n \geq 0\), define

\[
\varphi_n(x) = e^{-\frac{x^2}{2}}H_n(x).
\]

The functions \(\varphi_n(x)\) satisfy the Schrödinger equation:

\[
-\varphi_n''(x) + x^2 \varphi_n(x) = (2n + 1)\varphi_n(x).
\]

Furthermore, for any non-negative integers \(n_1, n_2, \ldots, n_r\), we have (see [8], Proposition 3.1):

\[
\text{Wr}[\varphi_{n_1}(x), \varphi_{n_2}(x), \ldots, \varphi_{n_r}(x)] = e^{-\frac{x^2}{2}}\text{Wr}[H_{n_1}(x), H_{n_2}(x), \ldots, H_{n_r}(x)]. \tag{6}
\]

**Proposition 12.** Let \(n_1, n_2, \ldots, n_r\) be non-negative integers. Suppose \(R \geq 0\) is such that any root \(x_0\) of \(\varphi_{n_i}(x)\) verifies \(|x_0| \leq R\), for all \(1 \leq i \leq r\). If \(z\) is a real root of \(\text{Wr}[\varphi_{n_1}(x), \varphi_{n_2}(x), \ldots, \varphi_{n_r}(x)]\) then \(|z| \leq R\).

Proposition 12 is equivalent to Proposition 6. This follows from Definition 1, the fact that \(H_{n}(x)\) is a rescaling of \(H_n(x)\), and the fact that the roots of Hermite polynomials interlace.

**Proof of Proposition 12.** For any \(R \geq 0\), define \(I_R := (-\infty, -R) \cup (R, +\infty)\). We may assume without loss of generality that the numbers \(n_1, \ldots, n_r\) are distinct, otherwise the determinant vanishes and the claim is trivially true. We will show that for \(x \in I_R\), \(\text{Wr}[\varphi_{n_1}(x), \varphi_{n_2}(x), \ldots, \varphi_{n_r}(x)] \neq 0\). We can order the numbers in increasing order, i.e. \(n_1 < n_2 < \ldots < n_r\).

The proof relies on the following simple observation.

**Observation 13.** Suppose \(\psi_1(x)\) and \(\psi_2(x)\) verify the Schrödinger equation

\[
-\psi_1''(x) + V(x)\psi_1(x) = E_1\psi_1(x)
\]

on \(I_R\), and do not vanish in \(I_R\). Let \(w(x) := \text{Wr}[\psi_1(x), \psi_2(x)]\). If \(E_1 \neq E_2\) and \(\lim_{x \to \pm\infty} w(x) = 0\) then \(w(x)\) has no zeros in \(I_R\).

**Proof.** Note that

\[
w(x) = \psi_1(x)\psi_2'(x) - \psi_1'(x)\psi_2(x)
\]

\[
w'(x) = \psi_1(x)\psi_2''(x) - \psi_1''(x)\psi_2(x) \tag{7}
\]

As \(E_1 \neq E_2\) and \(\psi_1(x)\psi_2(x) \neq 0\) on \(I_R\), \(w'(x)\) has constant sign on each interval \((R, +\infty)\) and \((-\infty, -R)\). As \(\lim_{x \to \pm\infty} w(x) = 0\), it follows that on each interval \((R, +\infty)\) and \((-\infty, -R)\), either \(w(x)\) strictly decreases...
from a positive value to 0, or strictly increases from a negative value to 0. Hence \( w(x) \) is never 0 in these intervals.

We now prove the statement by induction on \( r \geq 1 \).

If \( r = 1 \), the Wronskian is just \( \varphi_n(x) \), so the claim is trivially true.

If \( r = 2 \), notice that

\[
\text{Wr}[\varphi_n(x), \varphi_{n+1}(x)] = e^{-x^2} \text{Wr}[H_n(x), H_{n+1}(x)].
\]

Therefore \( \lim_{x \to \pm \infty} \text{Wr}[\varphi_n(x), \varphi_{n+1}(x)] = 0 \). By the choice of \( R \), \( \varphi_n(x) \) and \( \varphi_{n+1}(x) \) have no roots in \( I_R \).

Then the statement follows from Observation 13.

Now assume \( r \geq 3 \) and the induction hypothesis holds. Define

\[
\begin{align*}
\psi_{n-r}(x) &= \text{Wr}[\varphi_{n-r}(x), \varphi_{n-r+1}(x)], \\
\psi_{n-r+1}(x) &= \frac{\text{Wr}[\varphi_{n-r+1}(x), \varphi_{n-r+2}(x), \varphi_{n-r-1}(x)]}{\psi_{n-r-1}(x)}, \\
\psi_{n-r+2}(x) &= \frac{\text{Wr}[\varphi_{n-r+2}(x), \varphi_{n-r-1}(x)]}{\psi_{n-r-1}(x)}.
\end{align*}
\]

From the induction hypothesis it follows that \( \psi_{n-r-1}(x), \psi_{n-r+1}(x) \) and \( \psi_{n-r+2}(x) \) do not vanish in \( I_R \). Then from the definition they are repeatedly differentiable in \( I_R \). In this situation, Crum ([5]) showed that for \( i \in \{n_{r-1}, n_r\} \), \( \psi_i \) verifies the Schrödinger equation

\[
-\psi''_i(x) + V(x)\psi_i(x) = (2i+1)\psi_i(x),
\]

where

\[
V(x) = x^2 - 2\frac{\partial^2}{\partial x^2} \log \psi_{n-r-2}(x).
\]

Crum proved this in the case when \( n_1, n_2, \ldots, n_{r-2} \) are consecutive integers starting from 0, and only for the interval \((0, 1)\) with boundary conditions. However, the proof of (8) remains valid for a sequence \( n_1 < n_2 < \ldots < n_{r-2} \) of non-consecutive integers, in a neighborhood of \( x \) where the Wronskians do not vanish.

Let \( w(x) := \text{Wr}[\psi_{n-r-1}(x), \psi_{n-r+2}(x)] \). Jacobi’s identity for Wronskians tells us that

\[
\text{Wr}[\varphi_{n-r-1}(x), \varphi_{n-r+2}(x), \varphi_{n-r+1}(x)] = \psi_{n-r-1}(x)w(x).
\]

Hence

\[
\begin{align*}
w(x) &= \frac{\text{Wr}[\varphi_n(x), \varphi_{n+1}(x), \varphi_{n-1}(x)]}{\text{Wr}[\varphi_n(x), \varphi_{n+1}(x), \varphi_{n-2}(x)]} \\
&= e^{-\frac{x^2}{2}} \frac{\text{Wr}[H_n(x), H_{n+1}(x), H_{n-1}(x)]}{\text{Wr}[H_n(x), H_{n+1}(x), H_{n-2}(x)]}.
\end{align*}
\]

Then \( \lim_{x \to \pm \infty} w(x) = 0 \). From this and (8), we may apply Observation 13 to deduce that \( w(x) \) has no zeros in \( I_R \). As \( \psi_{n-r-1}(x) \neq 0 \) on \( I_R \), the right-hand side of (9) does not vanish in \( I_R \). Hence the left-hand side Wronskian in (9) does not vanish in \( I_R \) either.

\[\square\]

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