Unbiased estimators for correlation measurements

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Abstract

Higher order correlation measurements involve multiple event averages which must run over unequal events to avoid statistical bias. We derive correction formulas for small event samples, where the bias is largest, and utilize the results to achieve savings in CPU time consumption for the star integral. Results from a simple model of correlations illustrate the utility and importance of these corrections. Single-event correlation measurements such as in galaxy distributions and envisaged at RHIC must take great care to avoid this unnecessary pitfall.

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I. INTRODUCTION

In the hope of obtaining new insights into the old problem of soft interactions in high energy physics, there has been much interest in multiparticle correlations in the last few years, spurred by new theoretical perspectives and a large amount of multiparticle data in hadronic and nuclear collisions [1,2]. While various Monte Carlo codes and analytical models often yield very similar behavior in rapidity and $p_{\perp}$ distributions, they predict widely differing particle correlations. Experimentally measured correlations are therefore becoming an important and severe test of such theoretical models.

Experience has shown, however, that correlation measurements require considerably more subtle and sophisticated understanding of statistics than single-particle quantities do, and there has been much improvisation in methodology and interpretation of data. A clean and consistent statistical basis for such methodology has become a matter of urgency.

Recently, we have shown how, through the use of the correlation integral, the measurement of multiparticle correlations can be greatly improved, both in conventional variables such as rapidity and azimuthal angle [3] and in terms of relative momenta used in pion interferometry [4]. By deriving all quantities from first principles, our techniques, besides greatly improving the accuracy of correlation measurements, permit for the first time the direct measurement of cumulants. Moments, while easily measured, contain lower-order correlations. Cumulants, testing the actual correlations, are to be preferred, but they are hard to implement for at least two reasons: they contain a hidden statistical bias and are expensive in terms of CPU time.

The mentioned bias is present in all correlation measurements; it is large for small data samples and strong correlations while becoming negligible for large samples and weak correlations. Our analysis provides the framework for understanding and dealing with this bias in any present or future data set.

Secondly, correlation integral algorithms, while much superior to conventional methods, run at least as the square of the event multiplicity and the sample size $N_{ev}$. In understanding this bias, we point the way to huge reductions in computer time also. Defining for inner event averages a “reduced sample average” containing only $A$ events, and correcting for the resulting bias, we obtain, compared to full event mixing, savings of a factor $N_{ev}/A$ for the star integral. For a typical case with $N_{ev} = 10^5$ events and $A = 100$, the savings amount to a factor 1000 over full event mixing.

Besides the bias under discussion, there clearly are other biases, both statistical and systematic, which greatly influence multiparticle correlations. Typical unwanted but often important effects include the “empty bin effect” [3] and contamination by trivial sources of particle correlations such as Dalitz decays and gamma conversion [3] or the misidentification of pieces of a single track as two (highly correlated) particles [7]. All these have been shown to be capable of drowning other correlations in the background. Eliminating such biases is therefore a sine qua non of multiparticle correlations. We take here a simple model of such correlations, the split track model [8], to illustrate both the use of the reduced event average with bias correction and the effect such contamination may have on correlation data.

In Section II, we first explain the use and significance of unbiased estimators and find a general form for unbiased estimators of products of densities. We develop the general formalism in Section III and apply these in Section IV to the star integrals. An example
of behavior of the star integral as applied to the split track model is given in Section V, followed by an outline of steps needed to measure unbiased correlations in truly small samples and a brief discussion of corrections for other correlation methods. We conclude with some comments on small samples and single-event measurements. First results regarding unbiased estimators can be found in Ref. [9]. More recently, this formalism has been applied to the problem of normalization in a fixed-bin context [10].

II. UNBIASED ESTIMATORS FOR PRODUCTS OF DISTRIBUTIONS

We briefly remind the reader of some basics of statistical theory. Suppose we have a random variable $U$ which for a given trial (or “event” in the parlance of high energy physics) takes on a value $\hat{U}$. For a finite number of events $N_{ev}$, the set of values of $\hat{U}$ make up a sample, for which the sample average of $U$ can be found, $\langle U \rangle_s = \sum_e \hat{U}_e/N_{ev}$. By carrying out an infinite number of trials (the population), one can theoretically determine the “true” behavior $\bar{U}$ of the random variable. The expectation value $E[U]$ of a quantity $U$ is the value found over an infinite number of trials, $\bar{U} = E[U] = \lim_{N_{ev} \to \infty} \sum_{e=1}^{N_{ev}} \hat{U}_e/N_{ev}$.

An experimental sample invariably consists of a finite number of events, so that $E[U]$ cannot be found directly. A large part of statistics occupies itself with the question how the information contained in a limited sample can be extrapolated to estimate its true behavior over the whole population. Rather than taking the limit $N_{ev} \to \infty$, one imagines that there are $N$ samples, each with $N_{ev}$ events and a particular value of $\langle U \rangle_s$ for each. These sample averages themselves form a distribution, the sampling distribution. For infinite $N_{ev}$, the sampling distribution of course narrows to a delta function centered on $\bar{U}$, but for finite $N_{ev}$ the sampling distribution has a nonzero width independent of the number of samples $N$.

There is no way to ascertain where the $\langle U \rangle_s$ obtained for one experimental sample will fall in this distribution, i.e. one can never claim with certainty that $\langle U \rangle_s = \bar{U}$. All that can be achieved is to make sure that, even for finite $N_{ev}$, the sampling average of the sampling distribution

$$\{U\} \equiv \lim_{N \to \infty} \frac{1}{N} \sum_s \langle U \rangle_s$$

equals the true value $\bar{U}$. Surprisingly, this is not generally true: for finite $N_{ev}$, $\{U\}$ is not necessarily equal to $\bar{U}$. When it is not, $U$ is termed a biased estimator of $\bar{U}$, and one attempts to find a corresponding unbiased estimator $e(\bar{U})$ which does fulfil the condition

$$\{e(\bar{U})\} = \bar{U} \quad \text{for all finite } N_{ev}.$$  

Note: Here and throughout this paper, we use the shortened notation $e(\bar{U})$ to denote the unbiased estimator for the true value $\bar{U}$, i.e. the $\bar{U}$ inside the brackets is not the argument of $e$ but the desired result. The set of $\hat{U}_e$ of the experimental sample make up the arguments of $e$, which in full notation should be written as $e_U(\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_{N_{ev}})$. 

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For the case of multiparticle physics, the basic random variables \( U \) correspond to the one-particle inclusive density of event \( e \),
\[
\hat{\rho}_e^e(x) = \sum_{i=1}^{N} \delta(x - X_e^{i}),
\]
where \( X_i \) are the set of measured coordinates of the \( N \) particles of the event, and the corresponding \( q \)-th order densities for the event \( e \),
\[
\hat{\rho}_e^q(x_1, \ldots, x_q) = \sum_{i_1 \neq i_2 \neq \cdots \neq i_q} \delta(x_1 - X_e^{i_1}) \cdots \delta(x_q - X_e^{i_q}).
\]
These yield the sample average \( \rho_q = \langle \hat{\rho}_e^q \rangle_s = \sum_e \hat{\rho}_e^q / N_{\text{ev}} \), identified with the usual experimental inclusive density
\[
\rho_q(x_1, x_2, \ldots, x_q) \equiv \frac{1}{\sigma_I} \frac{d^3\sigma_{\text{incl}}}{dx_1 dx_2 \ldots dx_q}
\]
which is normalized to the factorial moment of the event multiplicity \( N \), \( \int \rho_q = \langle N[q] \rangle = \langle N(N-1) \cdots (N-q+1) \rangle \).

It has long been known that the inclusive density is an unbiased estimator for the true value, \( \{\rho_q\} = \bar{\rho}_q \), and so little attention has been paid to the theory of estimators in high energy physics. Unlike a single inclusive density, however, a product of two or more densities is a biased estimator. This we illustrate for the simple example of the product of two single-particle densities \( \rho_1(x_1)\rho_1(x_2) \) before considering the general case. The sampling average of \( \rho_1\rho_1 \) is
\[
\{\rho_1\rho_1\} = \left\{ N_{\text{ev}}^{-2} \sum_{e_1, e_2} \hat{\rho}_1^{e_1}\hat{\rho}_1^{e_2} \right\} = \left\{ N_{\text{ev}}^{-2} \sum_{e_1 \neq e_2} \hat{\rho}_1^{e_1}\hat{\rho}_1^{e_2} \right\} + \left\{ N_{\text{ev}}^{-2} \sum_{e_1} \hat{\rho}_1^{e_1}\hat{\rho}_1^{e_1} \right\},
\]
i.e. there are \( N_{\text{ev}} \) out of the total \( N_{\text{ev}}^2 \) terms in which the two \( \hat{\rho}_1 \)'s refer to the same event and thus effectively introduce a correlation. Because the densities of different events are independent, the sampling average of their product factorizes, yielding the true inclusive densities\footnote{The true value \( \bar{\rho}_q \) can be written as the sampling average of either the sample-averaged density or of the single-event density, \( \bar{\rho}_q = \{\rho_q\} = \{\hat{\rho}_q\} \), because Eq. (3) is valid for single-event “samples” \( N_{\text{ev}} = 1 \) also.}
\[
\{\hat{\rho}_1^{e_1}\hat{\rho}_1^{e_2}\} = \{\hat{\rho}_1^{e_1}\} \{\hat{\rho}_1^{e_2}\} = \hat{\rho}_1 \bar{\rho}_1 \quad \text{if } e_1 \neq e_2
\]
so that
\[
\{\rho_1\rho_1\} = (1 - N_{\text{ev}}^{-1}) \bar{\rho}_1 \bar{\rho}_1 + N_{\text{ev}}^{-1} \{\hat{\rho}_1^{e_1}\hat{\rho}_1^{e_1}\},
\]

1
meaning that $\{\rho_1\rho_1\}$ is not equal to $\bar{\rho}_1\bar{\rho}_1$ and thus $\rho_1\rho_1$ is a biased estimator for the latter. The culprit is clearly the equal-event part in Eq. (7). For just one available sample, the needed unbiased estimator for the true value $\bar{\rho}_1\bar{\rho}_1$ is the unequal-event sum

$$e(\bar{\rho}_1\bar{\rho}_1) = \frac{1}{N^{[2]}_{\text{ev}} \sum_{e_1 \neq e_2} \bar{\rho}^{e_1}_{\rho_1} \bar{\rho}^{e_2}_{\rho_1}}. \quad (10)$$

The above simple example generalizes to the following result: Given a product of $K$ inclusive densities of order $q_1, q_2, \ldots, q_K$, respectively, the unbiased estimator for the product of true values is given by

$$e(\bar{\rho}_{q_1}\bar{\rho}_{q_2}\cdots\bar{\rho}_{q_K}) = \frac{1}{N^{[K]}_{\text{ev}} \sum_{e_1 \neq e_2 \neq \cdots \neq e_K} \bar{\rho}^{e_1}_{q_1} \bar{\rho}^{e_2}_{q_2} \cdots \bar{\rho}^{e_K}_{q_K}}; \quad (11)$$

for example, the unbiased estimator for $\bar{\rho}_2\bar{\rho}_1\bar{\rho}_1$ will be given by $\sum_{e_1 \neq e_2 \neq e_3} \bar{\rho}^{e_1}_{2} \bar{\rho}^{e_2}_{1} \bar{\rho}^{e_3}_{1}/N^{[3]}_{\text{ev}}$. This equation is the most important point of our paper. In the following sections, we explore the consequences for various correlation measurements of taking only unequal events in products of densities.

Products such as in Eq. (11) can be written in terms of event mixing, a procedure used heuristically before to normalize correlation measurements. From here on, we distinguish three different kinds of event mixing: Denoting the first event average by the index $a$ and subsequent averages by $b, c$, full event mixing is given by running all indices over the full sample with $N_{\text{ev}}$ terms,

$$\frac{1}{N_{\text{ev}}} \sum_{a=1}^{N_{\text{ev}}} \frac{1}{N_{\text{ev}}-1} \sum_{b=1 \neq a}^{N_{\text{ev}}} \frac{1}{N_{\text{ev}}-2} \sum_{c=1 \neq a, b}^{N_{\text{ev}}} \cdots, \quad (12)$$

the reduced event average runs the inner event averages over $A$ events only,

$$\frac{1}{N_{\text{ev}}} \sum_{a=1}^{N_{\text{ev}}} \frac{1}{A} \sum_{b=a-A}^{a-1} \frac{1}{A-1} \sum_{c=a-A \neq b}^{a-1} \cdots, \quad (13)$$

while fake event mixing selects randomly a track from each of $N$ different events (where $N$ itself must follow a Poisson distribution) and does the standard analysis on a sample of such fake events $[11]$. While full event mixing is exact, it is feasible only for small samples, so that in practice the reduced average or fake event procedures are chosen. The latter is easy to understand and implement for the normalization $\rho^q_1$, but hard to implement for the cumulant expansions introduced below. We shall concentrate therefore on using the reduced event average.

III. CORRECTION TERMS FOR $K$-FOLD PRODUCTS

Before going into the details of unbiased estimators for the various correlation measurements in current use, we establish the general framework for these corrections which will be applicable for all occurrences of products of random variables. To simplify notation, we write
for the single-event inclusive densities \( \hat{\rho}_q \) the variables \( \hat{U}, \hat{V}, \hat{W}, \ldots \) and \( \langle \hat{U} \rangle_s = N_{av}^{-1} \sum_e \hat{U}_e \) the sample averages; the desired true values are \( \bar{U}, \bar{V}, \bar{W}, \ldots \). As in Section [II], the desired unbiased estimator for a given product is obtained when the single factors come from different events, as then the sampling average factorizes,

\[
\{ \hat{U}^{e_1} \hat{V}^{e_2} \hat{W}^{e_3} \ldots \}_{e_1 \neq e_2 \neq e_3 \ldots} = \{ \hat{U}^{e_1} \} \{ \hat{V}^{e_2} \} \{ \hat{W}^{e_3} \} \ldots = \bar{U} \bar{V} \bar{W} \ldots ,
\]

and, to make full use of all events in the sample, the sums over all (unequal) events are introduced. Products of experimentally measured inclusive densities, on the other hand, have unrestricted sums, so that it is necessary to expand the unequal-event sums in terms of unrestricted ones. Writing the Kronecker delta \( \delta_{e_1 e_2} \) as \( \delta_{12} \) for short, \( \delta_{123} \equiv \delta_{e_1 e_2} \delta_{e_2 e_3} \) and so on, we have for the double sum

\[
\sum_{e_1 \neq e_2} = \sum_{e_1 e_2} - \sum_{e_1 = e_2} = \sum_{e_1 e_2} (1 - \delta_{12}) .
\]

i.e. the factor \( (1 - \delta_{12}) \) forces the unrestricted sum to the unequal-event sum. In third order, the corresponding combinatorics are

- \( e_1 = e_2 \neq e_3 \) \( \delta_{12} (1 - \delta_{23}) \)
- \( e_1 = e_3 \neq e_2 \) \( \delta_{13} (1 - \delta_{23}) \)
- \( e_2 = e_3 \neq e_1 \) \( \delta_{23} (1 - \delta_{13}) \)
- \( e_1 = e_2 = e_3 \) \( \delta_{123} \)
- \( e_1 \neq e_2 \neq e_3 \) \( 1 - \delta_{12} - \delta_{13} - \delta_{23} + 2 \delta_{123} \),

where the last line is obtained from the previous ones by requiring that all cases have to add up to 1, so that

\[
\sum_{e_1 \neq e_2 \neq e_3} = \sum_{e_1 e_2 e_3} \left[ 1 - \delta_{12} - \delta_{13} - \delta_{23} + 2 \delta_{123} \right] ,
\]

while in fourth order,

\[
\sum_{e_1 \neq e_2 \neq e_3 \neq e_4} = \sum_{e_1 e_2 e_3 e_4} \left[ 1 - \delta_{12} + 2 \sum_{(6)} \delta_{123} + \sum_{(4)} \delta_{12} \delta_{34} - 6 \delta_{1234} \right] ;
\]

the brackets under the sums indicating the number of permutations to be taken.

These expansions are utilized as follows. Let \( A \) be the number of events over which an average is performed, \( \langle \hat{U} \rangle = \sum_e \hat{U}_e / A \) (this differs from the full sample average \( \langle \hat{U} \rangle_s \) when doing reduced event mixing). The unbiased estimator for \( \bar{U} \bar{V} \) is expanded in second order to

\[
e(\bar{U} \bar{V}) = \frac{1}{A^2} \sum_{e_1 \neq e_2} \hat{U}^{e_1} \hat{V}^{e_2} = \frac{A^2}{A^2} \langle \hat{U} \rangle \langle \hat{V} \rangle - \frac{A}{A^2} \langle \hat{U} \hat{V} \rangle ,
\]

and with \( A^2 / A^2 = 1 + 1/(A-1) \),

\[
e(\bar{U} \bar{V}) = \langle \hat{U} \rangle \langle \hat{V} \rangle - \frac{1}{A-1} \kappa_2 (\bar{U}, \bar{V}) ,
\]

\( \kappa_2 (\bar{U}, \bar{V}) \) being the bias of the biased estimator.
where
\[ \kappa_2(\hat{U}, \hat{V}) \equiv \langle \hat{U} \hat{V} \rangle - \langle \hat{U} \rangle \langle \hat{V} \rangle, \tag{20} \]
i.e. we get a correction consisting of a second-order correlation, suppressed by a factor \((A-1)\). Using Eq. (16) and expanding \(A^3/A[3] = 1 + 3/(A-1) + 4/(A-1)^2\) etc., we get for the third order unbiased estimator
\[
e(\hat{U}\hat{V}\hat{W}) = \frac{1}{A[3]} \sum_{e_1 \neq e_2 \neq e_3} \hat{U}e_1 \hat{V}e_2 \hat{W}e_3
\]
\[= \langle \hat{U} \rangle \langle \hat{V} \rangle \langle \hat{W} \rangle - \frac{1}{A-1} \sum_{(3)} \kappa_2(\hat{U}, \hat{V}) \langle \hat{W} \rangle + \frac{2}{(A-1)^2} \kappa_3(\hat{U}, \hat{V}, \hat{W}), \tag{21} \]
where
\[ \kappa_3(\hat{U}, \hat{V}, \hat{W}) \equiv \langle \hat{U} \hat{V} \hat{W} \rangle - \langle \hat{U} \hat{V} \rangle \langle \hat{W} \rangle - \langle \hat{W} \hat{U} \rangle \langle \hat{V} \rangle - \langle \hat{V} \hat{W} \rangle \langle \hat{U} \rangle + 2\langle \hat{U} \rangle \langle \hat{V} \rangle \langle \hat{W} \rangle \tag{22} \]
is a third-order correlation, suppressed in Eq. (21) by a factor \(1/(A-1)^2\). In fourth order, with
\[ \kappa_4(\hat{U}, \hat{V}, \hat{W}, \hat{X}) \equiv \langle \hat{U} \hat{V} \hat{W} \hat{X} \rangle - \sum_{(4)} \langle \hat{U} \hat{V} \hat{W} \rangle \langle \hat{X} \rangle
\]
\[= \sum_{(3)} \langle \hat{U} \hat{V} \rangle \langle \hat{W} \hat{X} \rangle + 2 \sum_{(6)} \langle \hat{U} \hat{V} \rangle \langle \hat{W} \rangle \langle \hat{X} \rangle - 6 \langle \hat{U} \rangle \langle \hat{V} \rangle \langle \hat{W} \rangle \langle \hat{X} \rangle, \tag{23} \]
we have
\[
e(\hat{U}\hat{V}\hat{W}\hat{X}) = \langle \hat{U} \rangle \langle \hat{V} \rangle \langle \hat{W} \rangle \langle \hat{X} \rangle
\]
\[= \frac{1}{A-1} \sum_{(6)} \kappa_2(\hat{U}, \hat{V}) \langle \hat{W} \rangle \langle \hat{X} \rangle
\]
\[+ \frac{1}{(A-1)^2} \left( 2 \sum_{(4)} \kappa_3(\hat{U}, \hat{V}, \hat{W}) \langle \hat{X} \rangle + \sum_{(3)} \kappa_2(\hat{U}, \hat{V}) \kappa_2(\hat{W}, \hat{X}) \right)
\]
\[= \frac{1}{(A-1)^3} \left( 6\kappa_4(\hat{U}, \hat{V}, \hat{W}, \hat{X}) + 3 \sum_{(3)} \kappa_2(\hat{U}, \hat{V}) \kappa_2(\hat{W}, \hat{X}) \right). \tag{24} \]

IV. BIAS CORRECTIONS FOR THE STAR INTEGRAL

As stressed previously, the quantity underlying all correlation measurements is the inclusive density \(\rho_q\): Bose-Einstein measurements \([4]\), fixed-bin factorial moments \([1]\) and cumulants \([12]\), as well as correlation integrals \([3]\) all sample \(\rho_q\) in the form of (unnormalized) factorial moments
\[ \xi_q(\Omega) = \int_\Omega dx_1 dx_2 \ldots dx_q \rho_q(x_1, x_2, \ldots, x_q). \tag{25} \]
The only difference between these different correlation measurements lies in the different choice of integration domain $\Omega$.

To explore the utility of unbiased estimators, let us look at the so-called star integral, a particular method for measuring multiparticle correlations \[3\]. The domain $\Omega$ for the star integral is given by the sum of all spheres of radius $\epsilon$ centered around each of the $N$ particles in the event. The number of particles (“sphere count”) within each of these spheres is, not counting the particle at the center $X_{i_1}$,

$$\hat{n}(X_{i_1}, \epsilon) \equiv \sum_{i_2=1, i_2 \neq i_1}^{N} \Theta(\epsilon - |X_{i_1} - X_{i_2}|),$$  \hspace{1cm} (26)

and the factorial moment of order $q$ is

$$\xi_{\text{star}}^q(\epsilon) = \left< \sum_{i_1} \hat{n}(X_{i_1}, \epsilon)[q-1] \right>_s.$$ \hspace{1cm} (27)

This can be derived rigorously \[3\] from Eq. (5) using for $\Omega$ the equivalent definition

$$\xi_{\text{star}}^q(\epsilon) = \int \rho_q(x_1, \ldots, x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q,$$  \hspace{1cm} (28)

with the theta functions $\Theta_{ij} \equiv \Theta(\epsilon - |x_1 - x_j|)$ restricting all $q-1$ coordinates $x_j$ to within a distance $\epsilon$ of $x_1$.

For various reasons, it has become customary in high energy physics to measure normalized factorial moments \[1\]. Dividing by the integral of the uncorrelated background $\bar{\rho}_q$ over the same domain, the normalized star integral factorial moment is

$$F_{\text{star}}^q(\epsilon) \equiv \frac{\xi_{\text{star}}^q(\epsilon)}{\xi_{\text{norm}}^q(\epsilon)} = \frac{\int \rho_q(x_1, \ldots, x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q}{\int \bar{\rho}_1(x_1) \ldots \rho_1(x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q},$$  \hspace{1cm} (29)

where the denominator $\xi_{\text{norm}}^q$ is given by the double event average

$$\xi_{\text{norm}}^q(\epsilon) = \left< \sum_{i_1} \left< \sum_{i_2} \Theta(\epsilon - X_{i_1i_2}^{ab}) \right>^{q-1} \right>_s \equiv \left< \sum_{i_1} \left< \hat{n}_b(X_{i_1}^{a}, \epsilon) \right>^{q-1} \right>_s,$$  \hspace{1cm} (30)

with $X_{i_1i_2}^{ab} \equiv |X_{i_1}^a - X_{i_2}^b|$ measuring the distance between two particles taken from different events $a$ and $b$. The (full $N_{ev}$) outer event average and sum over $i_1$ are taken over the center particle taken from event $a$, each of which is used as the center of sphere counts $\hat{n}_b(X_{i_1}^a, \epsilon)$ taken over all events $b$ in the (reduced) inner event average.

Having defined our terms, let us now analyse them from the point of view of estimators. Because $\rho_q$ is an unbiased estimator for the true $\bar{\rho}_q$, the numerator $\xi_{\text{star}}^q$ is also unbiased and

\[2\] When a particle is closer than $\epsilon$ to the overall domain boundaries, the sphere around it is truncated by the latter, so that this definition is rigorous only for an infinite domain. Boundary effects are, of course, the scourge of many correlation measurements, even in astronomy \[13\]. Eq. (28) is rigorous for all domain sizes.
does not need correction. The denominator $\xi^n_{\text{norm}}$, however, is an integral over the biased estimator $\rho_1(x_1) \cdots \rho_1(x_q)$. To shorten notation, we abbreviate the sphere counts introduced previously by

$$
a \equiv \sum_j \Theta(\epsilon - X_{ij}^{aa}) = \hat{n}(X_i^a, \epsilon), \quad j \neq i
$$

$$
b \equiv \sum_j \Theta(\epsilon - X_{ij}^{ab}) = \hat{n}_b(X_i^a, \epsilon),
$$

so that the uncorrected normalization is $\xi^n_{\text{norm}} = \langle \sum_i \langle b \rangle_q^{-1} \rangle_s$ for short. The term inside the outer event average we write as $\hat{\xi}_q = \langle b \rangle_q^{-1}$, a $(q-1)$-fold product. Inserting these $b$’s into Eqs. (19), (21) and (24), unbiased estimators for the normalization moments are found to be

$$
e(\hat{\xi}^2_{\text{norm}}) = \langle b \rangle,
$$

$$
e(\hat{\xi}^3_{\text{norm}}) = \langle b \rangle^2 - \frac{\kappa_2(b, b)}{(A-1)},
$$

$$
e(\hat{\xi}^4_{\text{norm}}) = \langle b \rangle^3 - \frac{3\langle b \rangle \kappa_2(b, b)}{(A-1)} + 2\frac{\kappa_3(b, b, b)}{(A-1)^2},
$$

$$
e(\hat{\xi}^5_{\text{norm}}) = \langle b \rangle^4 - \frac{6\langle b \rangle^2 \kappa_2(b, b)}{(A-1)} + \frac{8\langle b \rangle \kappa_3(b, b, b) + 3\kappa_2^2(b, b)}{(A-1)^2}
- \frac{6\kappa_4(b, b, b, b) + 9\kappa_2^2(b, b)}{(A-1)^3},
$$

where the definitions of $\kappa_q$ are given in Eqs. (20), (22) and (23). In other words, the naive normalization $\langle b \rangle^{q-1}$ is corrected by correlations of order $q-1$ and lower, suppressed by powers of $A$. The sample-averaged unbiased estimator for the normalization is then $e(\hat{\xi}^{\text{norm}}_q) = \langle \sum_i e(\hat{\xi}^{\text{norm}}_q) \rangle_s$, and the bias-corrected normalized star integral is the ratio

$$
e(F_q) = \frac{\hat{\xi}^a_q}{e(\hat{\xi}^{\text{norm}}_q)}.
$$

A further possible bias must be tested, and, if necessary, corrected for. Both numerator $\hat{\xi}^a_q$ and normalization use the same sample, and thus will also contain a residual correlation by referring to the same event during their respective averages. The most obvious (but probably not the most elegant) way to remove this correlation is to demand that the denominator explicitly exclude each event $a$ currently under consideration in the numerator. The bottom-line unbiased estimator for the normalized moment is therefore

$$
e(F_q) \equiv \frac{1}{N_{\text{ev}}} \sum_{a=1}^{N_{\text{ev}}} \frac{\hat{\xi}^a_q}{D_q^a}.
$$

3 To avoid unnecessarily complicated notation, we omit here and below the bar over “hatted” quantities inside the brackets.
where \( \hat{D}_q^a \) must now be found from a product of \( q \) single-particle densities restricted additionally by the condition that all sums must exclude event \( a \).

Consigning the details to the appendix, we here merely state the results. Defining the “correction function” \( \hat{g}_q^a \) implicitly by

\[
\hat{D}_q^a \equiv e(\xi_{q_{\text{norm}}}) \left[ 1 - \frac{\hat{g}_q^a}{N_{\text{ev}} - 1} \right],
\]

the corrected normalized moment can be written as a geometric series

\[
e(\hat{F}_q) = \frac{1}{N_{\text{ev}}} \sum_{a=1}^{N_{\text{ev}}} e(\xi_{q_{\text{norm}}}) \sum_{p=0}^{\infty} \left( \frac{\hat{g}_q^a}{N_{\text{ev}} - 1} \right)^p.
\]

Suppressed by powers of \((N_{\text{ev}}-1)\), this series converges rapidly except for very small values of \( N_{\text{ev}} \). This means that the correction due to correlation between numerator and denominator can probably be neglected and only the \( p=0 \) term corresponding to Eq. (37) need be kept. Should doubt arise as to the importance of this correlation, the \( p=1 \) term \( \langle \hat{c}_q^a \hat{g}_q^a \rangle_s / (N_{\text{ev}}-1) e(\xi_{q_{\text{norm}}}) \) should be evaluated for the sample in question and compared to the lowest-order term.

**Cumulants** are combinations of correlation functions constructed in such a way as to become zero whenever any one or more of the points \( x \) becomes statistically independent of the others.

\[
C_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\rho_1(x_2),
\]

\[
C_3(x_1, x_2, x_3) = \rho_3(x_1, x_2, x_3) - \rho_1(x_1)\rho_2(x_2, x_3) - \rho_1(x_2)\rho_2(x_3, x_1) - \rho_1(x_3)\rho_2(x_1, x_2) + 2 \rho_1(x_1)\rho_1(x_2)\rho_1(x_3),
\]

etc. Using combinations of conventional moments, they have been measured for various experiments [12,14,15]. Integrating to get the unnormalized star integral cumulants \( f_q \), defined by

\[
f_q(\epsilon) \equiv \int C_q(x_1, \ldots, x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q,
\]

we obtained previously [3], with \( f_q = \langle \sum_i \hat{f}_q(i) \rangle_s \),

\[
\hat{f}_2 = a - \langle b \rangle, \quad \hat{f}_3 = a[2] - \langle b[2] \rangle - 2a\langle b \rangle + 2\langle b \rangle^2,
\]

and so on for higher orders (see below). The second order \( \hat{f}_2 \) has only a single event average and so is unbiased, \( e(\hat{f}_2) = \hat{f}_2 \). Correcting according to Section [11] the last term for \( \hat{f}_3 \) which involves a double event average, the unbiased version becomes (again omitting the bars)

\[
e(\hat{f}_3) = a[2] - \langle b[2] \rangle - 2a\langle b \rangle + 2\langle b \rangle^2 - \frac{2}{A-1} \kappa_2(b, b);
\]
similarly, the unbiased estimators for $\hat{f}_4$ and $\hat{f}_5$ are found to be
\[
\begin{align*}
\epsilon(\hat{f}_4) &= a^{[3]} - \langle b^{[3]} \rangle - 3a^{[2]}\langle b \rangle - 3a\langle b^{[2]} \rangle + 6\langle b \rangle\langle b^{[2]} \rangle + 6a\langle b \rangle^2 - 6\langle b \rangle^3 \\
&\quad + \frac{6}{A - 1} \left[ (3\langle b \rangle - a) \kappa_2(b, b) - \kappa_2(b, b^{[2]}) \right] - \frac{12}{(A - 1)^2}\kappa_3(b, b, b), \quad (47)
\end{align*}
\]
\[
\epsilon(\hat{f}_5) = a^{[4]} - \langle b^{[4]} \rangle - 4a^{[3]}\langle b \rangle - 4a\langle b^{[3]} \rangle \\
- 6a^{[2]}\langle b^{[2]} \rangle + 8\langle b \rangle\langle b^{[3]} \rangle + 12a^{[2]}\langle b \rangle^2 + 6\langle b^{[2]} \rangle\langle b^{[2]} \rangle \\
+ 24a\langle b \rangle\langle b^{[2]} \rangle - 36\langle b \rangle^2\langle b^{[2]} \rangle - 24\langle b \rangle^3 + 24\langle b \rangle^4 \\
- \frac{2}{A - 1} \left[ \kappa_2(b, b) \left( 6a^{[2]} - 18\langle b^{[2]} \rangle - 36\langle b \rangle + 72\langle b \rangle^2 \right) \right. \\
+ 4\kappa_2(b, b^{[3]}) + 3\kappa_2(b^{[2]}, b^{[2]}) + (12a - 36\langle b \rangle) \kappa_2(b, b^{[2]}) \\
+ \frac{24}{(A - 1)^2}\left[ 3\kappa_2(b, b) + (8\langle b \rangle - 2a)\kappa_3(b, b, b) - 3\kappa_3(b, b, b^{[2]}) \right] \\
- \frac{72}{(A - 1)^3}\left[ 2\kappa_4(b, b, b, b) + 3\kappa_2(b, b) \right]. \quad (48)
\]
\]
The normalized cumulants are estimated by
\[
e(\bar{K}_{q}^{\text{star}}(\epsilon)) \equiv \frac{e(\hat{f}_q)}{e(\xi_{\text{norm}})} = \frac{\langle \sum_i e(\hat{f}_q)_i \rangle_s}{e(\xi_{\text{norm}})} \quad (49)
\]
and must therefore be corrected for bias in both numerator and denominator. For cumulants, too, the residual correlation between numerator and denominator can be tested and corrected for; as for the moments, we expect this correction to be negligible. See the appendix for details.

A second useful form for star moments and cumulants are the so-called differential moments: Here, one defines not only a maximum distance $\epsilon_t$ but a minimum also, $\epsilon_{t-1}$ (t can define a sequence of such distances). For a given combination of $q-1$ particles around a center particle at $X_{i_1}$, at least one of these must lie inside the spherical shell bounded by radii $\epsilon_{t-1}$ and $\epsilon_t$, while the others are restricted only by the maximum distance $\epsilon_t$. This definition leads rigorously \textbf{[8]} to the simple and efficient prescriptions for measurement of the normalized differential moments and cumulants
\[
\Delta F_q(t) = \frac{\left\langle \sum_i a_t^{[q-1]} - a_{t-1}^{[q-1]} \right\rangle_s}{\left\langle \sum_i \langle b_t \rangle^{q-1} - \langle b_{t-1} \rangle^{q-1} \right\rangle_s}, \quad (50)
\]
\[
\Delta K_q(t) = \frac{\left\langle \sum_i \hat{f}_q(i, \epsilon_t) - \hat{f}_q(i, \epsilon_{t-1}) \right\rangle_s}{\left\langle \sum_i \langle b_t \rangle^{q-1} - \langle b_{t-1} \rangle^{q-1} \right\rangle_s}, \quad (51)
\]
using the shorthand $a_t \equiv \hat{n}(X_{i_1}, \epsilon_t)$ and $b_t \equiv \hat{n}_b(X_{i_1}, \epsilon_t)$. Unbiased estimators are found by correcting individual terms as set out for moments and cumulants above.

\section{V. AN EXAMPLE: THE SPLIT TRACK MODEL}

To illustrate the effect of bias and the use of the reduced inner event averages for the star integral, we make use of a simple but effective model, invented previously \textbf{[8]} to simulate
the effects of spurious correlations introduced by Dalitz decays, gamma conversion and the mismatching of tracks by detectors [7].

For each “event”, the split track model generates $P$ “points” distributed uniformly inside a one-dimensional window, with $P$ itself following a poisson distribution. Each of these $P$ points is then either with probability $g$ split up into $k$ “particles”, all situated at exactly the same position, or with probability $(1-g)$ becomes a single “particle”. The average multiplicity is thus $\langle N \rangle = (1-g)\langle P \rangle + gk\langle P \rangle$. Clearly, the $k$ particles in a cluster are maximally correlated, since they always fall within the same sphere, no matter how small the radius $\epsilon$.

This simple model can be solved analytically and is known to yield scaling cumulants $K_q$ for $q \leq k$, while cumulants of order greater than $k$ are zero exactly [8].

We created $N_{ev} = 10,000$ events with average total number of points 20 and setting $g = 0.1$ and $k = 3$. This translates to an average total multiplicity $\langle N \rangle = 24$. Doing the reduced event averages for the inner ($b$-)event average, only $A = 11$ events rather than the full $N_{ev}$ were used. This means a savings of CPU time of about a factor 1000 compared to full event mixing. Since there are only three particles per cluster, the true cumulants of fourth and fifth order are zero exactly. Both second and third order cumulants should be nonzero and scaling.

In Figure 1, we show the effect of bias corrections on the factorial moments $F_q$ and cumulants $K_q$ (note the different $y$-scales, both linear!). $F_2$ and $K_2$ have no bias corrections; for the higher orders, the difference grows with increasing order $q$ and smaller sphere radius $\epsilon$. As expected, the unbiased $K_4$ and $K_5$ are zero to within statistical errors, while the biased $K_4$ and $K_5$ rise strongly. The rise is due entirely to the equal-event bias which is the subject of this paper. $F_4$ and $F_5$ contain contributions from second- and third-order correlations [12] and therefore are not zero.

Note also that the biased estimate lies below the unbiased one for the moments, while for the cumulants, it lies above the unbiased estimator. The reason is that the $F_q$ are corrected only through the normalization $\xi_q^{\text{norm}}$, which in Eqs. (33)–ff. are all seen to be corrected downwards; the numerator $\xi_q^{\text{star}}$ is unbiased. For $K_q$, on the other hand, both the numerator and denominator require bias corrections.

The corresponding differential moments and cumulants are shown in Figure 2. The most important feature is that only the data point corresponding to the smallest $\epsilon$ contains the split track contributions to $\Delta K_2$ and $\Delta K_3$. This must be so because all three particles belonging to a given cluster are by construction separated by zero distance.

Secondly, the difference between biased and unbiased differentials is much smaller than for the corresponding integral quantities $F_q$ and $K_q$. This is because in Eqs. (50)–(51) the subtraction of terms ($\langle b_t \rangle^{q-1} - \langle b_{t-1} \rangle^{q-1}$ etc) means that corresponding corrections also largely cancel. The only exception is the smallest-$\epsilon$ bin where there are no terms $f_q(i, \epsilon_{t-1})$ and $b_{t-1}$ to subtract, so that the bias corrections for this data point remain uncancelled.

It may be tempting to use a large value $A$ for inner event averages while neglecting bias corrections rather than implementing them. That this is usually not helpful is shown by Figure 3, where we have plotted the dependence of the (biased, uncorrected) $K_5$ on the number of events $A$ taken for the inner event average. Again, the “true” value is $K_5 \equiv 0$. Clearly, the resulting curves converge rather slowly to zero even for large $A$. The unbiased $K_5$, however, are virtually indistinguishable for all values of $A$ shown here, meaning that, for
the present parameters, even the smallest value $A = 11$ is sufficient to obtain good results if the bias corrections are implemented. The factor 10 in CPU time needed for the $A = 101$ case shown is thus largely wasted. The only remaining advantage of using a larger $A$ is that statistical errors become smaller (but the mean value remains the same).

The curves shown here are for the one-dimensional model; for higher dimensions, the effect of split tracks on the correlation is much larger since the rise of the cumulants goes roughly like $\epsilon^{-d}$, where $d$ is the dimension of the phase space.

At this point we also comment on the use of different random number generators. As can be seen in Figures 1–3, the fifth order cumulant $K_5$ and differential $\Delta K_5$ show some deviation from the theoretical value of zero for small $\epsilon$. We have tested various available random number generators with the split track model, using exactly the same parameters quoted above. It turns out that the different generators produce substantially different results for $K_5$ at small $\epsilon$, with some deviating above zero, others below, with varying sizes of error bars. The calculation of cumulants in the split track model is clearly a very sensitive test of the quality of a random number generator, just as it has proven itself in ferreting out statistical and systematic experimental biases. A really good random number generator should yield results for $K_5$ within the split track model which are compatible with zero.\textsuperscript{4}

We therefore recommend that, before any experimental measurements of correlations are attempted and compared to so-called “random” number data, all random number generators first be tested whether they produce truly zero cumulants of higher orders. Only when they do can any further conclusions as to correlations in the data be drawn.

VI. VERY SMALL SAMPLES

When the number of events in the experimental sample becomes very small, of the order of 100 or less, full event mixing may become unavoidable. In this case, of course, it becomes mandatory to avoid the equal-event bias, otherwise the measurement is simply wrong. Because for small samples CPU time is not an issue, the best and most transparent method is directly to implement the full unequal-event estimator of Eq. (11) for all products in cumulants and normalization.

If for higher $q$ it does become advantageous to avoid direct implementation of unequal-event algorithms, our procedures can be used in modified form as follows.

Whereas the above bias corrections assumed that events $b, c, \ldots$ were always unequal to the “outer” event $a$, full event mixing must allow and correct for all possible combinations of equal and unequal events. Therefore, the simple procedure of using the expansions of $\hat{U} \hat{V}$ etc. of Section [11] cannot be applied directly; rather, one must start from first principles and apply the sum combinatorics to all sums. For example, the unbiased estimator for the second order normalization becomes, after rearrangement

$$e(\xi_2^{\text{norm}}) = \frac{1}{N_{ev}^{[2]} \sum_{a \neq b} \sum_{i,j} \Theta(\epsilon - X_{ij}^{ab})}$$

\textsuperscript{4} For the present examples, the generator RAN4 from Ref. [16] was used.
\[ \frac{1}{N_{ev}^2} \sum_{a,b} \sum_{i,j} \Theta(\epsilon - X_{ij}^{ab}) \]
\[ - \frac{1}{N_{ev}^2} \sum_{a} \left( \sum_{i,j} \Theta(\epsilon - X_{ij}^{aa}) - \frac{1}{N_{ev}} \sum_{b} \sum_{i,j} \Theta(\epsilon - X_{ij}^{ab}) \right) \]
\[ = \left\langle \sum_{i} \langle b \rangle_{s} \right\rangle_{s} - \frac{1}{N_{ev} - 1} \left\langle \sum_{i} [a + 1 - \langle b \rangle_{s}] \right\rangle_{s}, \]

(52)

so that one can infer

\[ e(\xi_{2}^{\text{norm}}) = \langle b \rangle_{s} - \frac{1}{N_{ev} - 1} [a + 1 - \langle b \rangle_{s}]. \]

(53)

The extra “1” stems from the fact that the \( i, j \) sums are not restricted to unequal particles, so that the count always includes the center particle also. Unlike the reduced event mixing case of Eq. (33), which run only over the \( A \) events following \( a \), the event averages here are performed over all \( N_{ev} \) events, including the \( a = b \) case.

Using similar first-principle combinatorics, we find for the full event mixing cumulant

\[ e(\hat{f}_{2}) = a - \langle b \rangle_{s} + \frac{1}{N_{ev} - 1} [a + 1 - \langle b \rangle_{s}]. \]

(54)

Higher order normalizations and cumulants are derived analogously.

**VII. CORRECTIONS FOR BOSE-EINSTEIN AND OTHER CORRELATIONS**

The prescription that only unequal events be used is of course true for any kind of correlation measurement. In the case of Bose-Einstein correlations, most experimental measurements to date are for second order only, where the double event average in the normalization is found through fake event mixing. Very few higher order measurements exist, and these are in the form of moments rather than cumulants, so that the problem did not arise either.

Recently, we have derived formulae for the direct measurement of cumulants in Bose-Einstein correlations. The particular definition used for the \( q \)-particle relative four-momentum,

\[ Q^2 = \sum_{\alpha < \beta = 1}^{q} -(p_{\alpha} - p_{\beta})^2, \]

(55)

while convenient because it is directly related to the \( q \)-particle invariant mass \( M^2 = (p_1 + \ldots + p_q)^2 \), does not allow for a factorization of the multiple sums as was the case for the star integral. For this reason, there is little sense in deriving corresponding correction formulas; rather, one simply must enforce all event sums to refer to unequal events as in Eq. (11) and do the full \( q \)-times event average (or the corresponding reduced version).

There is one choice of the \( q \)-particle four-momentum that does allow for factorization of the sums as in the star integral, namely
\[ Q^2 \equiv \sum_{i=2}^{q} -(p_1 - p_i)^2. \]  

(56)

For this case, corresponding correction formulae can be derived and the savings in CPU time achieved. It is unclear, however, whether such choice of variable is preferable to the original choice of Eq. (55) for reasons other than convenience.

The situation is quite different for the traditional (bin-based) factorial moments \( F_q = (1/M) \sum_m \langle n^q_m \rangle / \langle n_m \rangle^q \) of Bialas and Peschanski \(^1\) and their cumulants \(^12\). Here, the multiple event averages must be corrected in the same way as the star integral; for example

\[ \langle n_m \rangle^2 \rightarrow \langle n_m \rangle^2 - \frac{\langle n^2_m \rangle - \langle n_m \rangle^2}{N_{ev} - 1} \]  

(57)

and so on for higher order normalizations and cumulants. The inherent instability and large error bars found for these moments and cumulants, however, make it doubtful that these corrections will make a discernible difference.

**VIII. CONCLUSIONS**

The statistical bias arising through the need for multiple event averages must be understood and corrected for. We have shown how the theory of unbiased estimators leads to correction formulas for the star integral, thereby making it possible to run it under fast algorithms without loss in accuracy. For the envisaged large data samples, this savings in CPU time may prove the difference between viability and impossibility of correlation measurements in future.

For truly small samples, the correction for this bias is not a tool for faster analysis but constitutive for a correct measurement. Typical small samples are found in cosmic ray data and in galaxy correlations as well as the subdivision of inclusive data samples into fixed-multiplicity subsamples. All these must take cognizance of the bias and correct for it.

This brings us to the subject of single-event measurements: event mixing is, of course, not possible when there is just one available. For the proposed measurement of Bose-Einstein correlations in single events in nuclear collisions at RHIC and LHC, the solution is clearly to normalize by event mixing based on a sample of similar events. Most notably, this mixing sample should have the same multiplicity and general characteristics; such requirements will necessarily restrict the sample to relatively few events, so that the bias corrections may become important.

Galaxy distributions, on the other hand, present a much more difficult task: there is no pool of big bang events to make up the uncorrelated background. So far, the preferred solution was to assume a uniform distribution on a sufficiently large scale. Recent results on the large-scale structure of the universe, however, make this assumption increasingly untenable. The only alternative route would appear to be to select a number of windows in the sky (with about the same overall galaxy count as the window used for the numerator) and, neglecting the long-range correlations, count these as different “events”. In this way, no assumption of overall uniformity need be made.
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APPENDIX: UNBIASED ESTIMATORS FOR NORMALIZED MOMENTS

In this appendix, we derive the correction functions $\hat{g}_q$ to be used for checking for residual correlations between numerator and denominator of the normalized factorial moment and cumulant. Our notation will be as follows: we use roman letters $a, b, \ldots$ for the event indices of the numerator of the normalized moments, and greek letters $\alpha, \beta, \ldots$ for the denominator.

1. Reduced event mixing

The numerator of the biased uncorrected $F_2$ is given by

$$\xi_2 = \frac{1}{N_{ev}} \sum_{a=1}^{N_{ev}} \xi_2^a , \quad (A1)$$

$$\hat{\xi}_2 = \sum_{i \neq j} \Theta(\epsilon - X_{ij}^{aa}) , \quad (A2)$$

while the denominator is

$$e(\xi_2^{\text{norm}}) = \frac{1}{N_{ev} A} \sum_{a=1}^{N_{ev}} \sum_{\beta=a-A}^{a-1} T_{\alpha\beta} , \quad (A3)$$

with

$$T_{\alpha\beta} = \sum_{i,j} \Theta(\epsilon - X_{ij}^{a\beta}) . \quad (A4)$$

Note that the inner $\beta$-average $\langle T_{\alpha\beta} \rangle$ is equal to $\sum_i \langle b \rangle$ in the shortened notation of Eq. (32).

In order to get an unbiased estimator $e(F_2)$ of the normalized second order factorial moment, we exclude explicitly from the denominator the event $a$ used in the numerator event sum:

$$e(F_2) = \frac{1}{N_{ev}} \sum_{a=1}^{N_{ev}} \hat{\xi}_2^a \quad (A5)$$

where the denominator is now $a$-dependent:
\[ \hat{D}_2^a = \frac{1}{(N_{ev} - 1)} \sum_{\alpha=1}^{N_{ev}} (1 - \delta_{a\alpha}) \frac{1}{A} \sum_{\beta} T_{\alpha\beta} \]

\[ = \frac{1}{(N_{ev} - 1)} \sum_{\alpha=1}^{N_{ev}} (1 - \delta_{a\alpha}) \left[ \frac{1}{A} \sum_{\beta=\alpha-A}^{\alpha-1} (1 - C_{a\alpha})T_{\alpha\beta} + \frac{1}{A} \sum_{\beta=\alpha-A+1}^{\alpha-1} (1 - \delta_{\beta\alpha})C_{a\alpha}T_{\alpha\beta} \right], \quad (A6) \]

and the condition

\[ C_{a\alpha} = \sum_{u=\alpha+1}^{a+A} \delta_{\alpha u} \quad (A7) \]

is unity whenever \( \alpha \) is in the range \([a+1, \ldots, a+A]\) and zero otherwise (note that \( C_{a\alpha} \delta_{a\alpha} = 0 \)). The reason for the splitting of the \( \beta \)-sum is that whenever \( \alpha \) is in this range, the index \( \beta \) must “jump” the \( a \)-event, meaning that the count must start at \( \alpha - A - 1 \). The form (A6) thus explicitly excludes the currently-used numerator event \( a \).

To find the relation between \( e(\mathcal{F}_2) \) and \( \mathcal{F}_2 \), we factor out of \( \hat{D}_2^a \) the usual normalization and write the remainder in terms of a function \( \hat{g}_2^a \) which is to be determined:

\[ \hat{D}_2^a \equiv e(\xi_2^{\text{norm}}) \left[ 1 - \frac{\hat{g}_2^a}{N_{ev} - 1} \right]. \quad (A8) \]

The moment estimator is then a geometric series

\[ e(\mathcal{F}_2) = \frac{1}{N_{ev}} \sum_{\alpha=1}^{N_{ev}} \left[ e(\xi_2^{\text{norm}}) \right] \sum_{p=0}^{\infty} \left( \frac{\hat{g}_2^a}{N_{ev} - 1} \right)^p \]

\[ = \frac{\langle \xi_2^{\text{norm}} \rangle_s}{e(\xi_2^{\text{norm}})} + \frac{1}{(N_{ev} - 1)} \frac{\langle \hat{g}_2^a \hat{g}_2^a \rangle_s}{e(\xi_2^{\text{norm}})} + \cdots \quad (A9) \]

which usually converges rapidly. The correction function \( \hat{g}_2^a \) is found as follows. The quantity in the square brackets of Eq. (A6) yields, on rearrangement,

\[ \frac{C_{a\alpha}}{A} (T_{\alpha,a-A-1} - T_{\alpha,a}) + \frac{1}{A} \sum_{\beta=\alpha-A}^{\alpha-1} T_{\alpha\beta}, \quad (A10) \]

so that

\[ \hat{D}_2^a - e(\xi_2^{\text{norm}}) = \sum_{\alpha=1}^{N_{ev}} \frac{1}{A} \left[ \frac{C_{a\alpha}(T_{\alpha,a-A-1} - T_{\alpha,a})}{N_{ev} - 1} + \sum_{\beta=\alpha-A}^{\alpha-1} \left( \frac{(1 - \delta_{a\beta})T_{\alpha\beta}}{N_{ev} - 1} - \frac{T_{\alpha\beta}}{N_{ev}} \right) \right]. \quad (A11) \]

After changing to index \( u = \beta + A - 1 \), the \( \delta_{a\beta} \) term becomes \( \sum_{u=\alpha+1}^{a+A} \delta_{a\beta} T_{\alpha\beta} = \sum_{u=\alpha+1}^{a+A} T_{a,u-A-1} \), which yields, using Eqs. (A3) and (A7),

\[ \hat{D}_2^a - e(\xi_2^{\text{norm}}) = \frac{1}{N_{ev} - 1} \left[ e(\xi_2^{\text{norm}}) - \frac{1}{A} \sum_{u=\alpha+1}^{a+A} (T_{a,u-A-1} - T_{u,u-A-1} + T_{a,u}) \right], \quad (A12) \]

so that we can identify

\[ \hat{g}_2^a = -1 + \frac{1}{e(\xi_2^{\text{norm}})A} \sum_{u=\alpha+1}^{a+A} (T_{a,u-A-1} - T_{u,u-A-1} + T_{a,u}) \]. \quad (A13) \]

Implementing this type of correction thus involves keeping the sphere counts of events mixed within a range \([a - A, \ldots, a + A]\).
2. Full event mixing

Correcting for bias in the case of full event mixing is somewhat easier than for the reduced event mixing above because the $\beta$-sum does not have to be split up. The equivalent definitions are

\[
e(\xi_{\text{norm}}^2) = \frac{1}{N_{\text{ev}}(N_{\text{ev}} - 1)} \sum_{\alpha,\beta=1}^{N_{\text{ev}}} (1 - \delta_{\alpha\beta})T_{\alpha\beta}
\]  

(A14)

and

\[
\hat{D}_2^a = \frac{1}{(N_{\text{ev}}-1)(N_{\text{ev}}-2)} \sum_{\alpha,\beta=1}^{N_{\text{ev}}} (1 - \delta_{aa})(1 - \delta_{a\beta})(1 - \delta_{\alpha\beta})T_{\alpha\beta},
\]  

(A15)

giving, using the symmetry of $T_{\alpha\beta}$,

\[
\hat{D}_2^a - e(\xi_{\text{norm}}^2) = \frac{1}{N_{\text{ev}}^{[3]}} \sum_{\alpha,\beta} (1 - \delta_{\alpha\beta})T_{\alpha\beta}[N_{\text{ev}}(1 - \delta_{aa} - \delta_{a\beta} + \delta_{aa\beta}) - (N_{\text{ev}} - 2)]
\]

\[
= \frac{2}{N_{\text{ev}} - 2} \left[ \frac{1}{N_{\text{ev}}^{[2]}} \sum_{\alpha,\beta} (1 - \delta_{a\beta})T_{a\beta} - \frac{1}{N_{\text{ev}} - 1} \sum_{\beta} (1 - \delta_{a\beta})T_{a\beta} \right]
\]

\[
= \frac{2}{N_{\text{ev}} - 2} \left( e(\xi_{\text{norm}}^2) - \sum_i \langle b \rangle_a \right),
\]  

(A16)

giving, the second term being an event mixing average performed around tracks $i$ of (numerator) event $a$ only, and hence

\[
\hat{g}_2^a = 2 \left( \frac{\sum_i \langle b \rangle_a}{e(\xi_{\text{norm}}^2) - 1} \right).
\]  

(A17)

The difference between this and the reduced event mixing case is that the former keeps the “mixing tail” to strictly $A$ events, so that even for the maximal $A = N_{\text{ev}} - 2$ it always leaves out one event in the mixing. The full event mixing outlined above, on the other hand, changes from mixing with $N_{\text{ev}} - 2$ events in $\hat{D}_2^a$ to $N_{\text{ev}} - 1$ in $e(\xi_{\text{norm}}^2)$. The power series expansion now reads

\[
e(F_2) = \frac{\hat{\xi}_q^a}{e(\xi_{\text{norm}}^2)^s} + \frac{1}{(N_{\text{ev}} - 2)} \frac{(\hat{\xi}_q^a \hat{\xi}_q^a)^s}{e(\xi_{\text{norm}}^2)} + \cdots
\]  

(A18)

3. Corrections for higher order

For higher orders, a similar prescription would be followed in eliminating bias arising from numerator-denominator correlations. The unbiased form is

\[
e(F_q) = \frac{1}{N_{\text{ev}}} \sum_{a=1}^{N_{\text{ev}}} \hat{\xi}_q^a \hat{D}_q^a.
\]  

(A19)
With the understanding that all indices \(\alpha_i\) are kept strictly unequal to each other throughout,

\[
\hat{D}_q = \frac{1}{(N_{ev} - 1)} \sum_{\alpha_1 \neq a}^{N_{ev}} \left[ \frac{1}{A^{[q-1]}} \sum_{a_2 \ldots a_q}^{(A-A)} (1 - C_{a\alpha_1}) T_{\alpha_1,\ldots,\alpha_q} + \frac{1}{A^{[q-1]}} \sum_{a_2 \ldots a_q}^{(A-A)} C_{a\alpha_1} T_{\alpha_1,\ldots,\alpha_q} \right] \tag{A20}
\]

where

\[
T_{\alpha_1,\alpha_2,\ldots,\alpha_q} = \sum_{i_1,i_2,\ldots,i_q} \Theta_{i_1 i_2}^{\alpha_1 \alpha_2} \Theta_{i_1 i_3}^{\alpha_1 \alpha_3} \cdots \Theta_{i_1 i_q}^{\alpha_1 \alpha_q} \tag{A21}
\]

with \(\Theta_{i_1 i_2}^{\alpha_1 \alpha_2} = \Theta(\epsilon - X_{i_1 i_2}^{\alpha_1 \alpha_2})\) as usual. Note that \(T\) is symmetric in all indices except \(\alpha_1\).

The idea is then to expand \(\hat{D}_q\) in terms of the corresponding \(a\)-independent normalization \(e(\xi_{q}^{\text{norm}})\) and a correction function \(\hat{g}_q^a\). The unbiased normalized moment is then given by an expansion of the form of Eq. (A9) in powers of \(N_{ev} - 1\).

By excluding one event from the sum, we are explicitly breaking the symmetry of the sphere counts that permitted factorization of the multiple sums, so that the correction function \(\hat{g}_q^a\) becomes rapidly more complex with \(q\). Here, we merely outline the results for third order. We have from Eq. (A20), after going through similar steps as for \(q = 2\),

\[
\hat{D}_3 = e(\xi_{3}^{\text{norm}}) = \frac{1}{N_{ev} - 1} \left[ e(\xi_{3}^{\text{norm}}) + \frac{1}{A^{[2]}} \sum_{u=a+1}^{a+A} \left( 2 \sum_{\beta=u-A}^{u-1} (T_{u,u-A-1,\beta} - T_{u,a,\beta}) \right. \right. \\
\left. \left. - 2T_{u,a,a} - \sum_{\beta=a-A}^{a-1} (1 - \delta_{\beta,u-A-1}) T_{u,u-A-1,\beta} \right) \right]. \tag{A22}
\]

Note that \(e(\xi_{3}^{\text{norm}})\) itself is an unbiased estimator obtained from the biased form through Eq. (34). The correction function is then

\[
\hat{g}_3^a = -1 + \frac{1}{A^{[2]}} \sum_{u=a+1}^{a+A} \left[ 2 \sum_{\beta=u-A}^{u-1} (T_{u,a,\beta} - T_{u,u-A-1,\beta}) \right. \\
\left. - 2(T_{u,a,a} - T_{u,a,u-A-1}) + \sum_{\beta=a-A}^{a-1} (1 - \delta_{\beta,u-A-1}) T_{u,u-A-1,\beta} \right]. \tag{A23}
\]

Because the correction \(\hat{g}_q^a\) applies to the denominator only, cumulants of higher order are immediately found from

\[
e(\bar{K}_q) = \frac{1}{N_{ev}} \sum_{\alpha=1}^{N_{ev}} \frac{e(\hat{f}_q^a)}{D_q} \]

\[
= \frac{\langle \sum_i e(\hat{f}_q^a) \rangle_s}{e(\xi_{q}^{\text{norm}})} + \frac{1}{(N_{ev} - 1)} \frac{\langle \hat{g}_q^a \sum_i e(\hat{f}_q^a) \rangle_s}{e(\xi_{q}^{\text{norm}})} + \cdots \tag{A24}
\]

For the case of full event mixing, we obtain for the moments

\[
e(\bar{F}_q) = \frac{\langle \xi_q \rangle_s}{e(\xi_{q}^{\text{norm}})} + \frac{1}{(N_{ev} - q)} \frac{\langle \hat{g}_q^a \xi_q^a \rangle_s}{e(\xi_{q}^{\text{norm}})} + \cdots \tag{A25}
\]
where for third order

\[
\hat{g}_3^a = -3 + \frac{1}{(N_{ev} - 1)(N_{ev} - 2)} \sum_{\alpha \neq \beta = 1 \atop \alpha \neq a}^{N_{ev}} (T_{\alpha a \beta} + 2T_{aa \beta}) .
\]  
(A26)
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APPENDIX: LIST OF FIGURES

Figure 1: Normalized factorial moments $F_q$ and cumulants $K_q$ for $q = 2, \ldots, 5$ for the split
track model, with 10% of the points split up into 3 tracks. For the inner event average
to calculate the $\hat n$ sphere counts, only $A = 11$ events were used rather than the full event
mixing of $N_{ev} = 10,000$ events (i.e. shortening the CPU time by a factor $\sim 1000$). The
biased moments and cumulants are clearly wrong ($K_4$ and $K_5$ should be zero), while the
unbiased version are fine.

Figure 2: Differential moments and cumulants. As the three split tracks are all at the
same point, the correlation due to their presence is always contained in the smallest bin; this is clearly visible as the single point in the unbiased $\Delta K_2$ and $\Delta K_3$.

Figure 3: Full event mixing (using all $N_{ev}$ events for the inner event averages) is not a useful alternative to bias corrections. As shown here, one needs upwards of $A = 101$ events in the inner loop to make the biased estimate approach that of the true value $K_5 = 0$; the unbiased estimators (filled circles) of $K_5$, on the other hand, all lie close to zero even for $A = 11$ so that this small number is sufficient for a good estimate. CPU time is roughly proportional to $A$, i.e. a factor 9 larger for $A = 101$ than for $A = 11$. 
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