Perfect State Transfer in Laplacian Quantum Walk

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Abstract

For a graph $G$ and a related symmetric matrix $M$, the continuous-time quantum walk on $G$ relative to $M$ is defined as the unitary matrix $U(t) = \exp(-itM)$, where $t$ varies over the reals. Perfect state transfer occurs between vertices $u$ and $v$ at time $\tau$ if the $(u,v)$-entry of $U(\tau)$ has unit magnitude. This paper studies quantum walks relative to graph Laplacians. Some main observations include the following closure properties for perfect state transfer:

• If a $n$-vertex graph has perfect state transfer at time $\tau$ relative to the Laplacian, then so does its complement if $n\tau \in 2\pi \mathbb{Z}$. As a corollary, the double cone over any $m$-vertex graph has perfect state transfer relative to the Laplacian if and only if $m \equiv 2 \pmod{4}$. This was previously known for a double cone over a clique (S. Bose, A. Casaccino, S. Mancini, S. Severini, Int. J. Quant. Inf., 7:11, 2009).

• If a graph $G$ has perfect state transfer at time $\tau$ relative to the normalized Laplacian, then so does the weak product $G \times H$ if for any normalized Laplacian eigenvalues $\lambda$ of $G$ and $\mu$ of $H$, we have $\mu(\lambda - 1)\tau \in 2\pi \mathbb{Z}$. As a corollary, a weak product of $P_3$ with an even clique or an odd cube has perfect state transfer relative to the normalized Laplacian. It was known earlier that a weak product of a circulant with odd integer eigenvalues and an even cube or a Cartesian power of $P_3$ has perfect state transfer relative to the adjacency matrix.

As for negative results, no path with four vertices or more has antipodal perfect state transfer relative to the normalized Laplacian. This almost matches the state of affairs under the adjacency matrix (C. Godsil, Discrete Math., 312:1, 2011).

Keywords: quantum walk, perfect state transfer. Laplacian (combinatorial, signless, normalized), equitable and almost-equitable partitions, join, weak product, line graph.

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1 Introduction

Given a graph $G = (V, E)$, we may associate a matrix $M$ with $G$. For example, in graph theory, common choices for $M$ include the adjacency matrix $A$ and the Laplacian $D - A$, where $D$ is the diagonal degree matrix of $G$. On the other hand, in probability theory, natural choices for $M$ include a simple random walk matrix $P = AD^{-1}$ and a lazy random walk matrix $W = \frac{1}{2}I + \frac{1}{2}P$. If $M$ is Hermitian, then we may define a continuous-time quantum walk on $G$ relative to $M$ as the time-dependent unitary matrix

$$U_G(t) = \exp(-itM),$$

where $t \in \mathbb{R}$. This definition is motivated by Schrödinger’s equation where $M$ is viewed as the Hamiltonian of the underlying system. Continuous-time quantum walk on graphs is a useful method for designing efficient quantum algorithms (see Childs et al. [8] and Farhi et al. [14]) and is a universal model for quantum computation (see Childs [7]).

In an early seminal work, Farhi and Gutmann [15] used the infinitesimal generator matrix to define their quantum walk. The latter matrix is a weighted Laplacian matrix used commonly to define a continuous-time random walk (see Grimmett and Stirzaker [24]). This Laplacian matrix provides arguably the most natural connection between the continuous-time classical random walk and its quantum counterpart. As pointed out by Bose et al. [5], from a physics viewpoint, the quantum walks relative to the adjacency and Laplacian matrices are intimately related to quantum spin chains in the XY and XYZ interaction models, respectively. The XYZ interaction model is also known as the isotropic Heisenberg model.

The literature on graph Laplacians is vast and has a strong focus on the following three different Laplacians. The aforementioned standard (or combinatorial) Laplacian $D - A$ is closely related to Laplace’s heat equation and has beautiful algorithmic applications (see Spielman [31]). The signless Laplacian $D + A$ of a graph $G$ shares a strong spectral correspondence with the line graph $\ell(G)$ through the incidence matrix of $G$. The normalized Laplacian $L = D^{-1/2}(D - A)D^{-1/2}$ has an interesting connection to the Heat Kernel random walk which is defined as $e^{-tL}$ (see Chung’s monograph [11]). Since $L = I - D^{-1/2}PD^{1/2}$, where $P$ is the simple random walk matrix, the normalized Laplacian is similar to $I - P$. But even though $e^{-itL}$ is a well-defined quantum walk, the “quantum walk” $e^{-it(I-P)}$ is illegal since $P$ might not be symmetric. The latter is related to the Heat Kernel random walk via an imaginary time transformation $t \leftrightarrow it$.

A quantum walk on a graph $G$ relative to a matrix $M$ has perfect state transfer between vertices $u$ and $v$ at time $\tau$ if the $(u, v)$-entry of the unitary matrix $U(\tau)$ has unit magnitude; that is:

$$|\langle e_v | e^{-i\tau M} e_u \rangle| = 1.$$  

(2)

The notion of state transfer was introduced by Bose [4] in the context of information transfer in quantum spin chains. In his work, Bose considered perfect state transfer in the XYZ model. This notion was further studied by Christandl et al. [10, 9] for paths and hypercubes in the XY (adjacency matrix) model. They observed that the $n$-cube has antipodal perfect state transfer at time $\pi/2$, for any $n$ (which is counter-intuitive since the diameter of the $n$-cube

\footnote{This is apparently a common technique in statistical and quantum physics. See [26, 25] for an application of this method to continuous-time random and quantum walks.}
| Graph family       | PST time          | Laplacian      | Source                      |
|-------------------|-------------------|----------------|-----------------------------|
| $Q_n$             | $\pi/2$          | standard/signless | Christandl *et al.* [9]     |
| $Q_n$             | $n\pi/2$         | normalized     | Moore and Russell [28]      |
| $K_2 + G_{4n-2}$  | $\pi/2$          | standard       | this work                   |
| $K_2 + G_{2n,n-1}$| $\pi/2/\sqrt{n}$ | signless       | this work                   |
| $P_3 \times \{K_{2n}, Q_{2n-1}\}$ | $(2n-1)\pi$   | normalized     | this work                   |
| $P_{n \geq 3}$, $T_{n \geq 3}$ | $\infty$       | standard/signless | Godsil [17], Coutinho and Liu [12] |
| $P_{n \geq 4}$    | $\infty$         | normalized     | this work                   |

Figure 1: Summary of some results on Laplacian perfect state transfer (PST): a PST time of $\infty$ denotes no perfect state transfer; $n \geq 1$ is a positive integer; $G_n$ denotes any family of $n$-vertex graphs; $G_{n,k}$ denotes any family of $(n,k)$-regular graphs; $P_n$, $T_n$, $K_n$ denote a path, tree, and complete graph on $n$ vertices, respectively; $Q_n$ is the $n$-dimensional hypercube.

increases with $n$). In contrast, relative to the normalized Laplacian, the antipodal perfect state transfer time of the $n$-cube is $n\pi/2$.

Our main goal is to understand how these graph Laplacians affect state transfer on graphs and how they compare with the adjacency matrix model. For regular graphs, the quantum walks relative to the adjacency and Laplacian matrices are equivalent (up to irrelevant phase factors, time dilations, and time reversal). On bipartite graphs, the quantum walks relative to the standard and signless Laplacians are equivalent. So, our primary focus will be on irregular and/or nonbipartite graphs. We describe some of our results in what follows.

For the standard Laplacian, we show that perfect state transfer is closed under complementation with some mild assumptions. As a corollary, we characterize perfect state transfer on double cones: $K_2 + H$ has perfect state transfer relative to the standard Laplacian if and only if $|V(H)| \equiv 2 \pmod{4}$. This generalizes a result of Bose *et al.* [5] where $H$ is the complete graph. We also compare this to the $XY$ model where the double cone has perfect state transfer if $H$ is a $(n,k)$-regular graph, provided both $k$ and $\sqrt{k^2 + 8n}$ are integers divisible by four and the largest powers of two which divide them are distinct (see Angeles-Canul *et al.* [2]). Thus, perfect state transfer on double cones is less complicated in the $XYZ$ model compared to the $XY$ model. In contrast, we also find double cones with perfect state transfer relative to the signless Laplacian, but not the standard Laplacian. In particular, we show $K_2 + H$ has perfect state transfer relative to the signless Laplacian if $|V(H)|$ is even and it is densely regular (more precisely, regular with degree $\frac{1}{2}|V(H)| - 1$). Most of our proofs employ the machinery of quotient graphs under equitable and almost-equitable partitions.

By exploiting the spectral connection between the signless Laplacian and line graphs, we show there is no perfect state transfer on a family of odd unicyclic graphs relative to the signless Laplacian. The latter family of graphs is obtained by attaching two pendant paths to a three-cycle. Our proof uses the idea of controllable subsets in graphs (see Godsil [19]). Using the same technique, we can also show there is no perfect state transfer on paths with five or more vertices relative to the signless (and standard, by switching equivalence) Laplacian. But, a better result (with optimal proof) is known to Godsil who showed that paths on at least three vertices have no perfect state transfer under the standard Laplacian.
Recently, Coutinho and Liu [12] improved this considerably and showed there is no perfect state transfer on trees with at least three vertices relative to the standard Laplacian.

So, we know $P_3$ has no perfect state transfer relative to the standard/signless Laplacians since it is a double cone over a single vertex (also, of course, from the results of Godsil, Coutinho and Liu mentioned above). Interestingly, $P_3$ has perfect state transfer under the normalized Laplacian at time $\pi$ (as opposed to $\pi/\sqrt{2}$ in the $XY$ model). We use this to show that a weak product of $P_3$ with either an even clique or an odd cube has perfect state transfer relative to the normalized Laplacian. This is a consequence of another closure property: if $G$ has perfect state transfer under the normalized Laplacian at time $\tau$, then so does the weak product $G \times H$ provided that for any normalized Laplacian eigenvalues $\lambda$ of $G$ and $\mu$ of $H$, $\mu(\lambda - 1)\tau$ is an integer multiple of $2\pi$. In comparison, relative to the adjacency matrix, it is known that a weak product of a circulant which has odd integer eigenvalues with either $Q_{2n}$ or $P_{3}^{\leq n}$, where $n$ is a positive integer, has perfect state transfer (see Ge et al. [16]).

Finally, we show that no path on four or more vertices has antipodal perfect state transfer relative to the normalized Laplacian. The proof is based on a reduction to even cycles in the adjacency matrix model. This almost matches the strong negative result for paths in the $XY$ model, where there is no perfect state transfer between any pair of vertices (see Godsil [18]).

We summarize some of the known results on Laplacian state transfer along with our contributions in Figure 1. A survey on state transfer from a graph-theoretic perspective is given by Godsil [18].

## 2 Preliminaries

For a logical statement $S$, we use the Iversonian bracket $[S]$ to denote 1 if $S$ is true, and 0 otherwise (see [23]). The $n$-dimensional all-one vector is denoted $1_n$. The identity matrix of order $n$ is denoted $I_n$. The $m \times n$ all-one matrix is denoted $J_{m,n}$ or simply $J_n$ whenever $m = n$. We omit dimensions if the context is clear. For a matrix $A$, $A^T$ and $A^*$ denote its transpose and Hermitian transpose, respectively. The inner product of vectors $\vec{u}$ and $\vec{v}$ is denoted $\langle \vec{u} | \vec{v} \rangle$. Given an index $u$, let $e_u$ denote the unit vector that is 1 at position $u$ and zero elsewhere. We often consider the inner product $\langle x | Ay \rangle$, and in the form of $\langle e_u | A e_v \rangle$, it is simply the $(u, v)$-entry of $A$.

For two sets $A, B$ of numbers, we denote their sum as $A + B = \{a + b : a \in A, b \in B\}$, their product as $AB = \{ab : a \in A, b \in B\}$, and the scalar product of $A$ with a constant $c$ as $cA = \{ca : a \in A\}$.

Let $G = (V, E)$ be a graph that is simple, undirected, and (mostly) connected. Two vertices $u$ and $v$ are adjacent, or $u \sim v$, if $(u, v) \in E$. The degree of a vertex $u \in V$, which we denote $\deg(u)$, is the number of vertices adjacent to $u$; that is, $\deg(u) = \sum_{v \in V} |u \sim v|$. A graph $G$ is called $(n, k)$-regular if $G$ has $n$ vertices and each vertex has degree $k$. As is customary, we let $P_n$ and $K_n$ denote a path and a complete graph on $n$ vertices, respectively, and $Q_n$ denote the $n$-dimensional hypercube.

For a graph $G = (V, E)$, its adjacency matrix $A$ is defined as $A_{u,v} = [u, v \in E]$ and its diagonal degree matrix $D$ is defined as $D_{u,v} = [u = v \deg(u)]$. We focus on the following graph Laplacians. The standard Laplacian is given by $L = D - A$, the signless Laplacian by
Figure 2: Small examples of graphs with Laplacian perfect state transfer (between vertices marked white): (i) $P_3 = K_2 + K_1$ has perfect state transfer at time $\pi_1$ relative to the normalized Laplacian (but not the standard/signless Laplacian); (ii) $\overline{K_2} + K_2$ has perfect state transfer at time $\pi/2$ relative to the standard/signless Laplacians; (iii) $\overline{K_2} + 2K_2$ has perfect state transfer at time $\pi/\sqrt{8}$ relative to the signless Laplacian (but not the standard Laplacian).

$Q = D + A$, and the normalized Laplacian is $L = I - D^{-1/2}AD^{-1/2}$. For a matrix $M$ related to a graph $G$, the $M$-spectrum of $G$, denoted $\text{Spec}_M(G)$, is the set of eigenvalues of $M(G)$.

The complement of a graph $G$, denoted $\overline{G}$, is a graph whose vertex set is $V(G)$ with edge set $\{(u, v) : (u, v) \notin E(G), u \neq v\}$. For two graphs $G$ and $H$, their disjoint union $G \cup H$ is a graph whose vertex set is $V(G) \cup V(H)$ and edge set is $E(G) \cup E(H)$, respectively. Here, we assume $V(G)$ and $V(H)$ are disjoint sets. The join of $G$ and $H$, denoted $G + H$, is defined as $G + H = G \cup H$. We also consider products of $G$ and $H$ where the vertex set is $V(G) \times V(H)$ and the edge set is defined by an adjacency rule on the pairs $(g_1, h_1)$ and $(g_2, h_2)$:

- weak product $G \times H$: $(g_1, h_1) \sim (g_2, h_2)$ if $g_1 \sim g_2$ and $h_1 \sim h_2$. The adjacency matrix is given by $A(G \times H) = A(G) \otimes A(H)$.
- Cartesian product $G \square H$: $(g_1, h_1) \sim (g_2, h_2)$ if $g_1 \sim g_2$ and $h_1 = h_2$, or $g_1 = g_2$ and $h_1 \sim h_2$. The adjacency matrix is given by $A(G \square H) = A(G) \otimes I_H + I_G \otimes A(H)$.

The line graph of a graph $G$, denoted $\ell(G)$, is a graph whose vertex set is $E(G)$ where two edges are adjacent in $\ell(G)$ if they share a common vertex; that is, $E(\ell(G)) = \{(e_1, e_2) : e_1, e_2 \in E(G), |e_1 \cap e_2| = 1\}$.

A vertex partition $\pi$ of a graph $G = (V, E)$ given by $V = V_1 \cup \ldots \cup V_m$ is called equitable if there are constants $d_{j,k}$, for $1 \leq j, k \leq m$, so that

$$\forall j, k \in \{1, \ldots, m\})(\forall u \in V_j) \ |N(u) \cap V_k| = d_{j,k}. \quad (3)$$

If condition (3) is only required to hold for $j \neq k$, the partition is called almost equitable.

The (normalized) partition matrix $\mathcal{P}$ of $\pi$ is given by

$$\mathcal{P} = \sum_{u \in V, k \in [m]} \frac{[u \in V_k]}{\sqrt{|V_k|}} e_u e_k^T \quad (4)$$

We state the following well-known properties of equitable partitions.

\footnote{We follow a convention used by Mike Newman [29].}
Fact 1. (Godsil [18])
Let $G = (V,E)$ be a graph with an equitable partition $\pi$ given by $V = \bigcup_{k=1}^{m} V_k$ with constants $d_{j,k}$, for $j,k = 1, \ldots , m$. Let $\mathcal{P}$ be the (normalized) partition matrix of $\pi$, where $\mathcal{P}^T\mathcal{P} = I_m$. Then:

1. $\mathcal{P}\mathcal{P}^T = \text{diag}(\{J_{|V_k|} : k \in [m]\})$ which commutes with $A(G)$.
2. $A(G)\mathcal{P} = \mathcal{P}B$, where $B$ is a $m \times m$ matrix defined as

$$B_{j,k} = \sqrt{d_{j,k}d_{k,j}}, \quad (5)$$

where $j,k = 1, \ldots , m$.

Thus, $B = A(G/\pi)$ is the adjacency matrix of the quotient graph $G/\pi$.

We will need the following lemma which relates perfect state transfer in quantum walks on a graph and on its quotient under an equitable partition.

Lemma 1. (Bachman et al. [3])
Let $G = (V,E)$ be a graph with equitable partition $\pi$. Suppose $u,v \in V(G)$ belong to singleton partitions under $\pi$. Then,

$$\langle e_u | e^{-itA(G)}e_v \rangle = \langle e_{\pi(u)} | e^{-itA(G/\pi)}e_{\pi(v)} \rangle. \quad (6)$$

Therefore, perfect state transfer occurs between $u$ and $v$ in $G$ if and only if it occurs between $\pi(u)$ and $\pi(v)$ in $G/\pi$.

Further background on algebraic graph theory can be found in Godsil and Royle [21].

3 Basic observations

In this section, we state some basic facts about Laplacian quantum walk on graphs.

Definition 1. (Equivalence under quantum walk)
Given a graph $G$ and two matrices $M_1(G)$ and $M_2(G)$ associated with $G$, the quantum walks based on $M_1(G)$ and $M_2(G)$ are equivalent if for every time $t \in \mathbb{R}$, we have

$$|\langle e_u | e^{-itM_1(G)}e_v \rangle| = |\langle e_u | e^{-i(\alpha t)M_2(G)}e_v \rangle|, \quad (7)$$

for each $u,v \in V(G)$ and for some $\alpha \in \mathbb{R}$.

Here, we consider two quantum walks equivalent if their entry-wise complex magnitudes are the same at all times. The global phase factors (of the form $e^{i\theta}$ for some real $\theta$) may be safely ignored since they are undetectable by quantum measurements.
3.1 Regular graphs

Fact 2. For any regular graph $G$, the quantum walks based on the adjacency matrix, the standard and signless Laplacians, and the normalized Laplacian are all equivalent.

Proof. Let $G$ be a $k$-regular graph. The standard, signless and normalized Laplacians of $G$, respectively, are given by $L(G) = kI - A(G)$, $Q(G) = kI + A(G)$, and $L(G) = I - \frac{1}{k}A(G)$. Therefore, their quantum walks are defined as

\[
\exp(-itL(G)) = e^{-itA(G)} \\
\exp(-itQ(G)) = e^{-itA(G)} \\
\exp(-itL(G)) = e^{-it\frac{1}{k}A(G)},
\]

which are all equivalent to $e^{-itA(G)}$ up to phase factors, time reversal and time dilations. □

3.2 Bipartite graphs

Fact 3. For any bipartite graph $G$, the quantum walks based on the standard and signless Laplacians are equivalent.

Proof. If $G$ is bipartite, then $-A(G) = DA(G)D^{-1}$ for some nonsingular diagonal matrix $D$ with $\pm 1$ entries along its diagonal (see Godsil and Royle [21], for example). This implies that $Q(G) = DL(G)D^{-1}$ and, moreover, $e^{-itQ(G)} = De^{-itL(G)}D^{-1}$. □

3.3 Cartesian products

To construct infinite families of graphs with perfect state transfer in the $XY$ model, the Cartesian product is a useful closure operator. The seminal works of Christandl et al. [10, 9] showed that both $K_2 \square n$ and $P_3 \square n$ have perfect state transfer (since each of $K_2$ and $P_3$ have such property). We state a similar observation for the standard/signless Laplacians.

Fact 4. Let $M$ denote the standard or signless Laplacian. Suppose $G$ has perfect state transfer at time $t$ between $g_1$ and $g_2$ relative to $M$. Suppose $H$ has perfect state transfer at time $t$ between $h_1$ and $h_2$ relative to $M$. Then, $G \square H$ has perfect state transfer at time $t$ between $(g_1, h_1)$ and $(g_2, h_2)$ relative to $M$.

Proof. Note that $D(G \square H) = D(G) \otimes I_H + I_G \otimes D(H)$. Thus, $M(G \square H) = M(G) \otimes I_H + I_G \otimes M(H)$. This shows that

\[
\exp(-itM(G \square H)) = e^{-itM(G)} \otimes e^{-itM(H)},
\]

which implies the claim. □

Later, we apply Fact 4 to some examples of graphs with Laplacian perfect state transfer. For example, $Q_n \square (K_2 + G)$ has perfect state transfer at time $\pi/2$ relative to the standard Laplacian, for any $n$-cube $Q_n$ and any graph $G$ with $|V(G)| \equiv 2 \pmod{4}$ (see Corollary 4). See Figure 6.
Figure 3: The weighted graph $P_3(\alpha)$: antipodal perfect state transfer occurs at time $\tau$ relative to the adjacency matrix if and only if $\cos(\tau \alpha / 2) \cos(\tau \Delta) = -1$, where $\Delta^2 = (\alpha/2)^2 + 2$.

3.4 Three-vertex path

We show that $P_3$ has perfect state transfer relative to the normalized Laplacian.

**Fact 5.** $P_3$ has antipodal perfect state transfer relative to the normalized Laplacian at time $\pi$.

**Proof.** Note that $L(P_3) = I - \frac{1}{\sqrt{2}} A(P_3)$ with eigenvalues $\lambda = 0, 1, 2$. Therefore, we have

$$e^{-itL(P_3)} = \exp \left( -i t \left[ I - \frac{1}{\sqrt{2}} A(P_3) \right] \right) = e^{-it} e^{i(t/\sqrt{2})A(P_3)}.$$  

Since $A(P_3)$ has antipodal perfect state transfer at time $\pi/\sqrt{2}$ (see Godsil [18]), $P_3$ has antipodal perfect state transfer at time $\pi$ relative to the normalized Laplacian. □

In what follows, we consider a path on three vertices with a weighted self-loop on the middle vertex. We describe a necessary and sufficient condition on the weight of the self-loop that yields antipodal perfect state transfer. This simple graph will be useful later when we analyze the double cone $K_2 + G$ on Laplacians.

**Fact 6.** For a real number $\alpha \in \mathbb{R}$, let $G(\alpha)$ be a graph on the vertex set $\{0, 1, 2\}$ with the following adjacency matrix:

$$A(G(\alpha)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then, $G(\alpha)$ has antipodal perfect state transfer relative to the adjacency matrix at time $\tau$ if and only if

$$e^{-i\alpha \tau / 2} \cos(\Delta \tau) = -1,$$

where $\Delta = \sqrt{(\alpha/2)^2 + 2}$.

**Proof.** Let $\tilde{\alpha} = \alpha/2$ and $\Delta = \sqrt{\tilde{\alpha}^2 + 2}$. The eigenvalues of $A(G(\alpha))$ are $\lambda_0 = 0$ and $\lambda_{\pm} = \tilde{\alpha} \pm \Delta$, with the following corresponding eigenvectors:

$$\vec{z}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{z}_{\pm} = \frac{1}{\sqrt{2\Delta(\Delta \pm \tilde{\alpha})}} \begin{bmatrix} 1 \\ \lambda_{\pm} \end{bmatrix}$$

where we have used $2 + \lambda_{\pm}^2 = 2\Delta(\Delta \pm \tilde{\alpha})$. 

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Thus, the antipodal fidelity of the quantum walk $e^{-itA(G(\alpha))}$ is given by

$$\langle e_2 | e^{-itA(G(\alpha))} e_0 \rangle = -\frac{1}{2} + \frac{1}{2\Delta} \sum_{\pm} e^{-it\lambda_{\pm}} \frac{(\Delta \pm \bar{\alpha})}{(\Delta \pm \bar{\alpha})}$$  \hspace{1cm} (16)

$$= -\frac{1}{2} + \frac{e^{-it\bar{\alpha}}}{2} \left[ \cos(\Delta t) + i \left( \bar{\alpha} \Delta \right) \sin(\Delta t) \right]$$  \hspace{1cm} (17)

Let $g(\alpha, t) = \cos(\Delta t) + i(\bar{\alpha}/\Delta) \sin(\Delta t)$. Note that $|g(\alpha, t)| = 1$ if and only if $\Delta t \in \mathbb{Z} \pi$, since $\bar{\alpha}/\Delta < 1$.

We show that there is a time $\tau$ so $|\langle e_2 | e^{-i\tau A(G(\alpha))} e_0 \rangle| = 1$ if and only if $e^{-i\tau \bar{\alpha}} \cos(\Delta \tau) = -1$. By inspecting (17), the condition is clearly sufficient. To show it is necessary, suppose there are $\tau, \theta \in \mathbb{R}$ so that $\langle e_2 | e^{-i\tau A(G(\alpha))} e_0 \rangle = e^{i\theta}$. From (17), we have

$$e^{i\theta} = -\frac{1}{2} + \frac{1}{2} e^{-i\tau \bar{\alpha}} g(\alpha, \tau).$$  \hspace{1cm} (18)

This implies that $|g(\alpha, \tau)| = 1$ (by taking the complex conjugate and multiplying). Note that if a convex combination of numbers of the form $e^{i\beta_k}$ lies on the complex unit circle, then all $\beta_j$ are congruent modulo $2\pi$. Hence, $e^{-i\tau \bar{\alpha}} \cos(\Delta \tau) = -1$. \hfill \Box

## 4 Standard Laplacian

### 4.1 Complements

We show that perfect state transfer relative to the Laplacian is closed under complementation. Relative to the adjacency matrix, this only holds for regular graphs.

**Theorem 2.** If $G$ is a graph with perfect state transfer between vertices $u$ and $v$ at time $t$ relative to the standard Laplacian, where

$$|V(G)|t \in 2\pi\mathbb{Z},$$  \hspace{1cm} (19)

then $\bar{G}$ has perfect state transfer between vertices $u$ and $v$ at time $t$ relative to the standard Laplacian.

**Proof.** Let $G$ be a graph on $n$ vertices. The standard Laplacian of $\bar{G}$ is given by

$$L(\bar{G}) = [(n - 1)I - D(G)] - [J - I - A(G)] = nI - J - L(G).$$  \hspace{1cm} (20)

Since $L(G)$ commutes with $J$, we get

$$e^{-itL(\bar{G})} = e^{-int} e^{itJ} e^{itL(G)}. \hspace{1cm} (21)$$

By the spectral theorem, $e^{itJ} = e^{int} J/n + I - J/n$, which implies

$$e^{-itL(\bar{G})} = e^{-int} \left[ (e^{int} - 1)\frac{1}{n} J + I \right] e^{itL(G)}. \hspace{1cm} (22)$$

Thus, if $nt \in 2\pi\mathbb{Z}$, we obtain $e^{-itL(\bar{G})} = e^{itL(G)}$. \hfill \Box
We show applications of Theorem 2 to perfect state transfer on graph joins and on double cones relative to the standard Laplacian.

**Corollary 3.** Let $\overrightarrow{G}$ be a graph which has perfect state transfer between vertices $u$ and $v$ at time $t$ relative to the standard Laplacian. For any graph $H$, the join $G + H$ has perfect state transfer between vertices $u$ and $v$ at time $t$ relative to the standard Laplacian provided

$$t(|V(G)| + |V(H)|) \in 2\pi\mathbb{Z}. \quad (23)$$

**Proof.** We note that $G + H = \overrightarrow{G} \cup H$ and apply Theorem 2. \hfill $\square$

**Corollary 4.** The join $\overrightarrow{K}_2 + H$ has perfect state transfer at time $\pi/2$ between the vertices of $\overrightarrow{K}_2$ relative to the standard Laplacian if $|V(H)| \equiv 2 \pmod{4}$.

**Proof.** We apply Corollary 3 with $G = \overrightarrow{K}_2$ which has perfect state transfer at time $t = \pi/2$. Thus, $\overrightarrow{K}_2 + H$ has perfect state transfer if $(2 + |V(H)|)\pi/2 \in 2\pi\mathbb{Z}$, which proves the claim. \hfill $\square$

**Remark:** Corollary 4 is a generalization of the main result due to Bose, Casaccino, Mancini and Severini [5] which studied Laplacian perfect state transfer in complete graphs with a missing edge. By viewing $K_n \setminus e$ as a double cone and using closure under complementation for Laplacian perfect state transfer, we found a simpler proof for a more general result. Also, contrast Corollary 4 with a similar result in the adjacency matrix model due to Angeles-Canul et al. [2]. They showed that perfect state transfer occurs on $\overrightarrow{K}_2 + H$ under a more complicated number-theoretic conditions and only when $H$ is regular.

### 4.2 Double cones

In this section, we show a tighter version of Corollary 4 using the machinery of quotient graphs relative to almost equitable partitions. This provides a characterization of perfect state transfer on the double cones relative to the standard Laplacian. First, we state a symmetric quotient graph within the framework developed by Cardoso et al. [6].

**Fact 7.** *(based on Cardoso et al. [6]*)

Let $G = (V, E)$ be a graph with an almost equitable partition $\pi$ given by $V = \bigcup_{k=1}^m V_k$ with constants $d_{j,k}$, for $1 \leq j < k \leq m$. Let $\mathcal{P}$ be the (normalized) partition matrix of $\pi$, where $\mathcal{P}^T \mathcal{P} = I_m$. Then:

1. $\mathcal{P} \mathcal{P}^T = \text{diag}\{ J_{|V_k|} : k \in [m] \}$ which commutes with $L(G)$.

2. $L(G) \mathcal{P} = \mathcal{P} B$, where $B$ is a $m \times m$ matrix defined as

$$B_{j,k} = \begin{cases} -\sqrt{d_{j,k}d_{k,j}} & \text{if } j \neq k; \\ \sum_{\ell \neq j} d_{j,\ell} & \text{if } j = k \end{cases} \quad (24)$$

where $j, k = 1, \ldots, m$.

Thus, $B = L(G/\pi)$ is the standard Laplacian of the quotient graph $G/\pi$. 

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Proof. Follows immediately from Cardoso et al. [6] via the normalized partition matrix. □

Using Fact 7, we provide the following tight version of Corollary 4.

**Corollary 5.** The join \( K_2 + H \) has perfect state transfer at time \( \pi/2 \) between the vertices of \( K_2 \) relative to the standard Laplacian if and only if \( |V(H)| \equiv 2 \pmod{4} \).

**Proof.** Let \( G = K_2 + H \), where \( H \) is a \( n \)-vertex graph. By Fact 7, the Laplacian quotient \( B \) of \( G \) is given by

\[
B = \begin{bmatrix}
  n & -\sqrt{n} & 0 \\
-\sqrt{n} & 2 & -\sqrt{n} \\
 0 & -\sqrt{n} & n
\end{bmatrix}
\]  

In a quantum walk, up to phase factors, time reversal, and time dilatations, \( B \) is equivalent to

\[
\tilde{B} = -\frac{1}{\sqrt{n}}(B - nI) = \begin{bmatrix}
  0 & 1 & 0 \\
  1 & \frac{1}{\sqrt{n}}(n - 2) & 1 \\
  0 & 1 & 0
\end{bmatrix}
\]  

We apply Fact 6 with \( \alpha = \frac{1}{\sqrt{n}}(n - 2) \). Note also that \( \tilde{\alpha} = \alpha/2 \) and \( \Delta = \sqrt{\tilde{\alpha}^2 + 2} \). Thus, there is antipodal perfect state transfer using \( \tilde{B} \) at time \( t \) if and only if

\[
\cos(\tilde{\alpha}t) \cos(\Delta t) = -1.
\]  

This implies that \( \tilde{\alpha}t, \Delta t \in \mathbb{Z}\pi \) and \( \tilde{\alpha}/\Delta \) is a rational number \( p/q \) of distinct parities (either \( p \) is odd and \( q \) is even, or \( p \) is even and \( q \) is odd). But, note that

\[
\frac{\tilde{\alpha}}{\Delta} = \frac{n - 2}{\sqrt{(n - 2)^2 + 8n}} = \frac{n - 2}{n + 2}.
\]  

Since the parities of the numbers in the fraction \( p/q \) must be distinct, it is clear that \( n \) must be even and must satisfy \( n \equiv 2 \pmod{4} \). By Lemma 1 (lifting), we obtain the claim on the double cone \( K_2 + H \). □

### 4.3 Joins

We revisit perfect state transfer on graph joins relative to the standard Laplacian and show a negative result on connected double cones.

**Fact 8.** Let \( G \) and \( H \) be graphs on \( m \) and \( n \) vertices, respectively. For vertices \( u \) and \( v \) of \( G \), the quantum walk on \( G + H \) relative to the standard Laplacian satisfies

\[
\langle e_u | e^{-itL(G+H)} | e_v \rangle = e^{-itn} \langle e_u | e^{-itL(G)} | e_v \rangle + \frac{(e^{-it(m+n)} - e^{-itn})}{m} + \frac{(1 - e^{-it(m+n)})}{m + n}.
\]  

Moreover, if perfect state transfer occurs between vertices \( u \) and \( v \) in \( G + H \) at time \( t \) relative to the standard Laplacian, then \( t(m + n) \in 2\pi\mathbb{Z} \).
Proof. Let the spectral decompositions of the standard Laplacians of $G$ and $H$ be

$$L(G) = \sum_{k=0}^{m-1} \lambda_k E_k, \quad L(H) = \sum_{\ell=0}^{n-1} \mu_\ell F_\ell.$$  

(30)

Then, the join $G + H$ has the following spectral decomposition:

$$L(G + H) = 0 \cdot z_0 z_0^T + (m + n) z_1 z_1^T + \sum_{k=1}^{m-1} (n + \lambda_k) \begin{bmatrix} E_k & O \\ O & O \end{bmatrix} + \sum_{\ell=1}^{n-1} (m + \mu_\ell) \begin{bmatrix} O & O \\ O & F_\ell \end{bmatrix}.$$  

(31)

where

$$z_0 = \frac{1}{\sqrt{m+n}} 1_{m+n}, \quad z_1 = \frac{1}{\sqrt{mn(m+n)}} \begin{bmatrix} n 1_m \\ -m 1_n \end{bmatrix}.$$  

(32)

Using this, the quantum walk on $G + H$ relative to the standard Laplacian is

$$e^{-itL(G+H)} = \frac{J_{m+n}}{m+n} + e^{-it(m+n)} z_1 z_1^T + \sum_{k=1}^{m-1} e^{-it(n+\lambda_k)} \begin{bmatrix} E_k & O \\ O & O \end{bmatrix} + \sum_{\ell=1}^{n-1} e^{-it(m+\mu_\ell)} \begin{bmatrix} O & O \\ O & F_\ell \end{bmatrix}.$$  

(33)

For the vertices $u$ and $v$ of $G$, we have

$$\langle e_u | e^{-itL(G+H)} e_v \rangle = \frac{1}{m+n} + \left[ \frac{1}{m} - \frac{1}{m+n} \right] e^{-it(m+n)} + e^{-itn} \sum_{k=1}^{m-1} e^{-it\lambda_k} \langle e_u | E_k e_v \rangle.$$  

(34)

$$= \frac{(e^{-it(m+n)} - e^{-itn})}{m} + \frac{(1 - e^{-it(m+n)})}{m+n} + e^{-itn} \langle e_u | e^{-itL(G)} e_v \rangle.$$  

(35)

where we have used the fact $E_0 = J_m/m$.

To show the second claim, let $y$ be a vertex of $H$. Then, by (33), we have

$$\langle e_u | e^{-itL(G+H)} e_y \rangle = \frac{1}{m+n} (1 - e^{-it(m+n)}).$$  

(36)

This expression is zero if perfect state transfer occurs between $u$ and $v$ in $G + H$. Therefore, $e^{-it(m+n)} = 1$, which implies $t(m+n) \in 2\pi\mathbb{Z}$. 

We show that connected double cones $K_2 + G$ have no perfect state transfer relative to the standard Laplacian, unlike its counterpart $\overline{K_2} + G$.

**Corollary 6.** For any graph $G$, there is no perfect state transfer on $K_2 + G$ between the two vertices of $K_2$ relative to the standard Laplacian.

**Proof.** Let $G$ be a graph on $n$ vertices. Suppose there is perfect state transfer on $K_2 + G$ at time $t$ between the vertices $u$ and $v$ of $K_2$ relative to the standard Laplacian. By Fact 8, we have

$$\langle e_u | e^{-itL(K_2+G)} e_v \rangle = e^{-itn} \left( \langle e_u | e^{-itL(K_2)} e_v \rangle + \frac{1}{2} (e^{-2it} - 1) \right),$$  

(37)

since $e^{-it(2+n)} = 1$. But, note that $\langle e_u | e^{-itL(K_2)} e_v \rangle = \frac{1}{2} (1 - e^{-2it})$. So, the right-hand side of (37) is zero, which is a contradiction since we assume $K_2 + G$ has perfect state transfer at time $t$ between $u$ and $v$. 

□
5 Signless Laplacian

5.1 Double cones

By Fact 3, the quantum walks relative to the Laplacians $D \pm A$ are equivalent for regular and/or bipartite graphs. We describe a family of graphs with perfect state transfer relative to the signless Laplacian $D + A$, but not under the standard Laplacian $D - A$. First, we state some facts about quotient graphs relative to the signless Laplacian.

**Fact 9.** Let $G = (V, E)$ be a graph with an equitable partition $\pi$ given by $V = \bigcup_{k=1}^{m} V_k$ with constants $d_{j,k}$, for $j, k = 1, \ldots, m$. Let $\mathcal{P}$ be the (normalized) partition matrix of $\pi$, where $\mathcal{P}^T \mathcal{P} = \mathbb{I}_m$. Then:

1. $\mathcal{P} \mathcal{P}^T = \text{diag}\{\{J_{|V_k|} : k \in [m]\}\}$ which commutes with $Q(G)$.
2. $Q(G) \mathcal{P} = \mathcal{P}B$, where $B$ is a $m \times m$ matrix defined as

$$B_{j,k} = \begin{cases} \sqrt{d_{j,k}d_{k,j}} & \text{if } j \neq k \\ 2d_{j,j} + \sum_{\ell \neq j} d_{j,\ell} & \text{if } j = k \end{cases}$$

where $j, k = 1, \ldots, m$.

Thus, $B = Q(G/\pi)$ is the signless Laplacian of the quotient graph $G/\pi$.

**Proof.** To show that $\mathcal{P} \mathcal{P}^T$ commutes with $Q(G)$, it suffices to show it commutes with $D(G)$. Note that $D(G)$ is a diagonal matrix over $m$ blocks with the following form:

$$D(G) = \text{diag} \left( \left\{ \sum_{\ell=1}^{m} d_{k,\ell} \right\} \mathbb{I}_{|V_k|} : k \in [m] \right).$$

Since $\mathcal{P} \mathcal{P}^T = \text{diag}\{\{J_{|V_k|} : k \in [m]\}\}$, it commutes with $D(G)$ and hence with $Q(G)$.

Next, we show that $Q(G) \mathcal{P} = \mathcal{P}B$, where $B$ is given by (38). If we let $B = \mathcal{P}^T Q(G) \mathcal{P}$,

$$\langle e_j | B e_k \rangle = \langle e_j | \mathcal{P}^T Q(G) \mathcal{P} e_k \rangle$$

$$= \frac{1}{\sqrt{|V_j||V_k|}} \sum_{u \in V_j, v \in V_k} \langle e_u | Q(G) e_v \rangle$$

$$= \frac{1}{\sqrt{|V_j||V_k|}} \sum_{u \in V_j, v \in V_k} \langle e_u | (D(G) + A(G)) e_v \rangle$$

$$= [j = k] \left[ 2d_{j,j} + \sum_{\ell \neq j} d_{j,\ell} \right] + [j \neq k] \sqrt{d_{j,k}d_{k,j}}.$$  

The above case for $j \neq k$ follows from Fact 1.

From $B = \mathcal{P}^T Q(G) \mathcal{P}$, by multiplying both sides by $\mathcal{P}$ from the left, we get

$$\mathcal{P}B = \mathcal{P} \mathcal{P}^T Q(G) \mathcal{P} = Q(G) \mathcal{P} \mathcal{P}^T \mathcal{P} = Q(G) \mathcal{P},$$

since $\mathcal{P} \mathcal{P}^T$ commutes with $Q(G)$ and $\mathcal{P}^T \mathcal{P} = \mathbb{I}_m$. This proves the second claim. \[\square\]
Theorem 7. For an integer \( m \geq 1 \), if \( H \) is a \((2m, m−1)\)-regular graph, then \( K_2 + H \) has perfect state transfer relative to the signless Laplacian.

Proof. Let \( H \) be a \((n, k)\)-regular graph and consider the double cone \( G = K_2 + H \). By Fact 9, the signless Laplacian quotient \( B \) of \( G \) is given by

\[
B = \begin{bmatrix}
    n & \sqrt{n} & 0 \\
    \sqrt{n} & 2k + 2 & \sqrt{n} \\
    0 & \sqrt{n} & n
\end{bmatrix}
\] (45)

If \( k = (n−2)/2 \), we have \( B = nI + \sqrt{n}A(P_3) \). Let \( a \) and \( b \) be the conical vertices of \( G \) (which are also the antipodal vertices of the quotient). Therefore,

\[
\langle e_b | e^{-itB} e_a \rangle = e^{-int} \langle e_b | e^{-i(\sqrt{n}t)A(P_3)} e_a \rangle,
\]

which implies that the signless Laplacian quotient of \( G \) has perfect state transfer at time \( t = \pi/\sqrt{2n} \).

Using Lemma 1, we lift the perfect state transfer from the quotient to the original graph:

\[
\langle e_b | e^{-itQ(G)} e_a \rangle = \langle e_b | P^T e^{-itQ(G)} P e_a \rangle = \langle e_b | e^{-itB} e_a \rangle,
\]

since \( e^{-itB} = P^T e^{-itQ(G)} P \). Finally, we set \( n = 2m \) to complete the proof. □

Example: For completeness, we describe a simple family \( \{G_m\} \) of \((2m, m−1)\)-regular graphs, where the double cone \( K_2 + G_m \) has perfect state transfer between the two conical vertices relative to the signless Laplacian (by Theorem 7). Each graph \( G_m \) is a circulant over \( \mathbb{Z}_{2m} \) with the following generating set:

\[
S_m = \begin{cases} 
\{\pm 1, \ldots, \pm \frac{1}{2}(m−1)\} & \text{if } m−1 \text{ is even} \\
\{\pm 1, \ldots, \pm \frac{1}{2}(m−2)\} \cup \{\pm m\} & \text{if } m−1 \text{ is odd}
\end{cases}
\] (48)

It is clear that the circulant \( G_m = \text{Circ}(\mathbb{Z}_{2m}, S_m) \) is \((2m, m−1)\)-regular. Moreover, note that the double cone \( K_2 + G_m \) has no perfect state transfer relative to the standard Laplacian, whenever \( 2m \equiv 0 \pmod{4} \) (by Corollary 4).

5.2 Line graphs

The (normalized) incidence matrix \( B \) of \( G \) is a \( n \times m \) matrix defined as \( B_{u,e} = \frac{1}{\sqrt{2}}[u \in e] \), for each vertex \( u \in V(G) \) and each edge \( e \in E(G) \). We state a well-known connection between the signless Laplacian and line graphs.

Fact 10. For any graph \( G = (V, E) \) with (normalized) incidence matrix \( B \), we have:

\[(i) \ B^T B = \frac{1}{2}Q, \text{ and is nonsingular if } G \text{ is connected and nonbipartite.} \]
\[(ii) \ B^T B = \frac{1}{2}A(\ell(G)) + I, \text{ and is nonsingular if } G \text{ is a tree.} \]

Proof. See Godsil and Royle [21], Theorem 8.2.1 for example. □
We observe the following connections between the quantum walk on a graph $G$ relative to the signless Laplacian and the quantum walk on the line graph $\ell(G)$ relative to the adjacency matrix. The third observation in Lemma 8 was suggested by Ada Chan.

**Lemma 8.** Let $G$ be a graph with (normalized) incidence matrix $B$. Then:

a) $B^T e^{-itQ(G)} = e^{-2it} e^{-itA(\ell(G))} B^T$.

b) $e^{-itQ(G)} B = e^{-2it} B e^{-itA(\ell(G))}$.

c) $B^T e^{-itQ(G)} B = e^{-2it} e^{-itA(\ell(G))} B^T B$.

**Proof.** For the first identity, we have

$$B^T e^{-itQ(G)} = B^T \sum_{k=0}^{\infty} \frac{(-2it)^k}{k!} (BB^T)^k = \sum_{k=0}^{\infty} \frac{(-2it)^k}{k!} (B^T B)^k B^T = e^{-2it} B B^T,$$

and apply $2BB^T = A(\ell(G)) + 2\mathbb{I}$ to get the result. The proof of the second identity is similar.

For the third identity, we have

$$B^T e^{-itQ(G)} B = B^T \left[ \sum_{k=0}^{\infty} \frac{(-2it)^k}{k!} (B^T B)^k \right] B = \left[ \sum_{k=0}^{\infty} \frac{(-2it)^k}{k!} (B^T B)^k \right] B B^T,$$

and again apply $2BB^T = A(\ell(G)) + 2\mathbb{I}$.

**Theorem 9.** Let $G$ be a graph which has perfect state transfer at time $t$ from a vertex $u_1$ of degree one to another vertex $u_2$ relative to the signless Laplacian. Then, $u_2$ must have degree one and the line graph $\ell(G)$ has perfect state transfer at time $t$ between the unique edges incident to $u_1$ and $u_2$ relative to the adjacency matrix.

**Proof.** Let $B$ be the normalized incidence matrix of $G$. Suppose $G$ has perfect state transfer from vertex $u_1$ of degree one to another vertex $u_2$ at time $t$ relative to the signless Laplacian.

Let $e_1$ be the unique edge incident to $u_1$ and let $e_2$ be any edge incident to $u_2$. Say, $e_2 = (u_2, z)$, for some vertex $z$. By Lemma 8, we have

$$e^{-2it} e^{-itA(\ell(G))} B^T = B^T e^{-itQ(G)}.$$

Therefore, up to phase factors, we have

$$\frac{1}{\sqrt{2}} \langle e_{e_2} | e^{-itA(\ell(G))} e_{e_1} \rangle = \langle e_{e_2} | e^{-itA(\ell(G))} B^T e_{u_1} \rangle = \langle e_{e_2} | B^T e^{-itQ(G)} e_{u_1} \rangle = \frac{1}{\sqrt{2}} \langle e_{u_2} + e_z | e^{-itQ(G)} e_{u_1} \rangle.$$

We have $\langle e_z | e^{-itQ(G)} e_{u_1} \rangle = 0$ since $|\langle e_{u_2} | e^{-itQ(G)} e_{u_1} \rangle| = 1$. Thus,

$$\langle e_{e_2} | e^{-itA(\ell(G))} e_{e_1} \rangle = \langle e_{u_2} | e^{-itQ(G)} e_{u_1} \rangle,$$

which completes the proof.
We show that Theorem 9 may be used as a tool for showing the absence of perfect state transfer relative to the signless Laplacian in some graphs. It would be interesting if we can apply this similarly in the other direction.

**Corollary 10.** For \( n \geq 5 \), there is no antipodal perfect state transfer on \( P_n \) relative to the signless Laplacian.

**Proof.** For \( n \geq 5 \), suppose there is a time \( t \) so \( |\langle e_1 | e^{-itQ(P_n)} e_n \rangle| = 1 \). By Theorem 9, since \( \ell(P_n) = P_{n-1} \), we have \( |\langle e_{e_1} | e^{-itA(P_{n-1})} e_{e_{n-1}} \rangle| = 1 \), where \( e_1 \) and \( e_{n-1} \) are the edges incident to vertices 1 and \( n \), respectively. But, there is no perfect state transfer on paths \( P_m \), for \( m \geq 4 \) (see Christandl et al. [10]).

**Remark:** A better version of Corollary 10 (with optimal proof) is due to Godsil who showed that there is no perfect state transfer on \( P_n \), for \( n \geq 3 \), relative to the standard Laplacian. So, a minor novelty of Corollary 10 is in using the spectral link between the unnormalized Laplacians and the adjacency matrix of the line graph. The latest breakthrough result by Coutinho and Liu [12] showed that Godsil’s result holds for trees with at least three vertices.

### 5.3 Odd unicyclic graphs

We describe an application of Theorem 9 to nonbipartite graphs. To this end, we consider a family of graphs obtained from paths by adding a unique odd-cycle (here, we focus on the three-cycle \( C_3 \)). We show that this family of odd unicyclic graphs has no antipodal perfect state transfer relative to the signless Laplacian. Our proof exploits a connection between perfect state transfer and controllability described by Godsil [19, 20] (see also Godsil and Severini [22]).

We formally define our family of odd unicyclic graphs. For an integer \( m \geq 1 \), let \( U_m \) be the graph obtained by attaching two pendant paths \( P_{m+1} \) (with \( m \) edges) to a three-cycle \( C_3 \) (see Figure 5(a)). The line graph of \( U_m \) is a graph which has two pendant paths \( P_m \) attached to the pair of vertices of degree two in the cone \( K_1 + P_4 \).

In what follows, we briefly describe the machinery of controllable subsets on graphs. Let \( G = (V, E) \) be a graph on \( n \) vertices with adjacency matrix \( A \). For a subset of vertices \( S \subseteq V \), the walk matrix \( W_S \) on \( G \) with respect to \( S \) is defined as

\[
W_S = [e_S \ A e_S \ A^2 e_S \ \ldots \ A^{n-1} e_S]
\]

where \( e_S \) denotes the characteristic vector of \( S \). We say that the pair \((G, S)\) is controllable if \( W_S \) has full rank. A vertex \( u \) of \( G \) is called controllable if \((G, \{u\})\) is controllable.

The following theorems of Godsil on controllability and state transfer will prove useful.

**Theorem 11.** (Godsil [20], Theorem 7.4)
If \( G \) has perfect state transfer (relative to the adjacency matrix) between vertices \( u \) and \( v \), then neither \( u \) nor \( v \) is controllable.

**Theorem 12.** (Godsil [19])
Let \( G = (V, E) \) be a graph and \( S \subseteq V \) be a subset of vertices. Let \( \tilde{G}_S \) be a graph obtained from \( G \) and a path with endpoints \( u \) and \( v \) (which may be identical) whereby we connect \( u \) to all vertices in \( S \). If \((G, S)\) is controllable, then \( v \) is controllable in \( \tilde{G}_S \).
A main ingredient of our proof is the next lemma on the controllability of $K_1 + P_4$.

**Lemma 13.** Let $\hat{P}_4 = K_1 + P_4$ and let $u$ and $v$ be vertices of degree two in $\hat{P}_4$. For $m \geq 0$, let $G_m$ be the graph obtained by attaching a pendant path $P_{m+1}$ (see Figure 5(b)) to vertex $v$. Then, $u$ is controllable in $G_m$ if and only if $m \not\equiv 2 \pmod{3}$.

**Proof.** Let $A$ be the adjacency matrix of $G_m$ whose spectral decomposition is given by

$$A = \sum_{\ell=1}^{n} \lambda_{\ell} \vec{z}_{\ell}\vec{z}_{\ell}^\dagger$$

where $\{ \vec{z}_1, \ldots, \vec{z}_n \}$ is the set of orthonormal eigenvectors of $A$ which satisfies $A\vec{z}_{\ell} = \lambda_{\ell} \vec{z}_{\ell}$, for every $\ell = 1, \ldots, n$. Consider the walk matrix $W_u$ relative to vertex $u$:

$$W_u = \sum_{k=0}^{n-1} A^k e_u e_k^T$$

The rank of $W_u$ is equal to the cardinality of the set $\{ \ell : \langle \vec{z}_{\ell}|e_u \rangle \neq 0 \}$. To see this, note that

$$A^k e_u = \sum_{\ell=1}^{n} \lambda_{k,\ell} \langle \vec{z}_{\ell}|e_u \rangle \vec{z}_{\ell}$$

which shows that the columns of $W_u$ is spanned by the vectors $\vec{z}_{\ell}$ satisfying $\langle \vec{z}_{\ell}|e_u \rangle \neq 0$.

In what follows, we label the vertices of $\hat{P}_4$ as $\{0, 1, 2, 3, 4\}$ where 0 is the conical vertex, 1 and 4 are the endpoints of $P_4$, 2 and 3 are the middle vertices with 2 adjacent to 1 and 3 adjacent to 4. The vertices of the pendant path $P_{m+1}$ will be labeled consecutively as 4 followed by $4+k$, for $k = 1, \ldots, m$; see Figure 5(b). We show $\langle \vec{z}_{\ell}|e_1 \rangle = 0$, for some $\ell$, if and only if $m \equiv 2 \pmod{3}$.

Suppose $\vec{z}$ is an eigenvector of $A$ with eigenvalue $\lambda$ where $\langle \vec{z}|e_1 \rangle = 0$. Assume, without loss of generality, that $\langle \vec{z}|e_2 \rangle = a$, for some $a \neq 0$. Using the fact that $\lambda \langle \vec{z}|e_u \rangle = \sum_{v \sim u} \langle \vec{z}|e_v \rangle$ and following the chain of implications, we obtain:

$$\langle \vec{z}|e_0 \rangle = -a$$
$$\langle \vec{z}|e_3 \rangle = (1 + \lambda)a$$
$$\langle \vec{z}|e_4 \rangle = \lambda(1 + \lambda)a.$$
Since $\lambda \langle \vec{z} | e_0 \rangle = \sum_{k=1}^{4} \langle \vec{z} | e_k \rangle$, we have $(1 + \lambda)(2 + \lambda) = 0$, which implies $\lambda = -1$ or $\lambda = -2$. We consider these two cases separately.

**Case:** $\lambda = -1$. We have $\langle \vec{z} | e_3 \rangle = \langle \vec{z} | e_4 \rangle = 0$. For $k \geq 0$, this forces the three-step sequence along the pendant path:

$$
\langle \vec{z} | e_{4+k} \rangle = \begin{cases} 
0 & \text{if } k \equiv 0 \pmod{3} \\
+a & \text{if } k \equiv 1 \pmod{3} \\
-a & \text{if } k \equiv 2 \pmod{3}
\end{cases} \quad (63)
$$

The last vertex on the pendant path must have index $k \equiv 2 \pmod{3}$ for the eigenvector $\vec{z}$ to be well-defined.

**Case:** $\lambda = -2$. We have $\langle \vec{z} | e_3 \rangle = -a$ and $\langle \vec{z} | e_4 \rangle = 2a$. For $k \geq 0$, this forces the following pattern along the pendant path:

$$
\langle \vec{z} | e_{4+k} \rangle = (-1)^k 2a, \quad (64)
$$

which must continue indefinitely. Hence, such an eigenvector $\vec{z}$ does not exist.

This proves the claim.

We state our other corollary of Theorem 9 for the family of odd unicyclic graphs $U_m$. This provides a nonbipartite generalization of Corollary 10.

**Corollary 14.** For $m \geq 1$, the family of graphs $U_m$ has no antipodal perfect state transfer under the signless Laplacian whenever $m \not\equiv 0 \pmod{3}$.

**Proof.** Let $\widehat{P}_4$ be the cone $K_1 + P_4$ and let $u$ and $v$ be the vertices of degree two in $\widehat{P}_4$. Also, let $G_m$ denote the graph obtained by attaching to $v$ a pendant path with $m$ edges. Then, the line graph of $U_m$, that is $\ell(U_m)$, is obtained from $G_m$ by attaching to $u$ a pendant path with $m$ edges.

To prove the claim, we show that $\ell(U_m)$ does not have antipodal perfect state transfer under the adjacency matrix and then apply Theorem 9. By Theorem 11, it suffices to show that the two “antipodal” vertices of minimum degree in $\ell(U_m)$ are controllable. By Lemma 13, the two vertices of degree two in the cone $\widehat{P}_4$ are controllable. Using Theorem 12, we conclude that the unique vertex of degree one in $G_m$ is controllable. Since vertex $u$ in $G_m$ is controllable (by Lemma 13 again), if we attach a pendant path with $m$ edges to $u$, the other endpoint of this path is controllable by Theorem 12.

We state our other corollary of Theorem 9 for the family of odd unicyclic graphs $U_m$. This provides a nonbipartite generalization of Corollary 10.

**Corollary 14.** For $m \geq 1$, the family of graphs $U_m$ has no antipodal perfect state transfer under the signless Laplacian whenever $m \not\equiv 0 \pmod{3}$.

**Proof.** Let $\widehat{P}_4$ be the cone $K_1 + P_4$ and let $u$ and $v$ be the vertices of degree two in $\widehat{P}_4$. Also, let $G_m$ denote the graph obtained by attaching to $v$ a pendant path with $m$ edges. Then, the line graph of $U_m$, that is $\ell(U_m)$, is obtained from $G_m$ by attaching to $u$ a pendant path with $m$ edges.

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Remark: Our argument in Corollary 14 allows pendant paths of different lengths attached to a three-cycle provided the length of one of the paths is not divisible by three. It would be interesting to show a similar result to Corollary 14 for arbitrary odd unicyclic graphs. These graphs are interesting since $B^T B$ is nonsingular.

6 Normalized Laplacian

A quantum walk on the hypercube $Q_n$ relative to the adjacency matrix has antipodal perfect state transfer at time $\pi/2$ for any $n$. This might contradict the postulate that the speed of light is constant. In contrast, a quantum walk on $Q_n$ relative to the normalized Laplacian has antipodal perfect state transfer at time $n\pi/2$. Thus, the normalized Laplacian takes into account the diameter of the $n$-cube whereas the adjacency matrix does not. This motivates a closer study of quantum walks relative to normalized Laplacians.

6.1 Weak products

We show that the weak product is a useful operation for constructing classes of graphs with perfect state transfer relative to the normalized Laplacian. First, we observe that the normalized Laplacian of a weak product $G \times H$ has a strong resemblance in form to the adjacency matrix of a strong product $G \circledast H$.

Fact 11. For graphs $G$ and $H$, we have

$$\mathcal{L}(G \times H) = \mathcal{L}(G) \otimes I_H + I_G \otimes \mathcal{L}(H) - \mathcal{L}(G) \otimes \mathcal{L}(H).$$

(65)

Proof. Note that the degree matrix of $G \times H$ is given by $D(G \times H) = D(G) \otimes D(H)$. The normalized Laplacian is (also) defined as $\mathcal{L} = I - \mathcal{A}$, where $\mathcal{A} = D^{-1/2}AD^{-1/2}$ is the normalized adjacency matrix. In our case, we have $\mathcal{A}(G \times H) = \mathcal{A}(G) \otimes \mathcal{A}(H)$. Therefore,

$$\mathcal{L}(G \times H) = I_G \otimes I_H - \mathcal{A}(G) \otimes \mathcal{A}(H)$$

(66)

$$= I_G \otimes I_H - (I_G - \mathcal{L}(G)) \otimes (I_H - \mathcal{L}(H))$$

(67)

$$= \mathcal{L}(G) \otimes I_H + I_G \otimes \mathcal{L}(H) - \mathcal{L}(G) \otimes \mathcal{L}(H).$$

(68)

This proves the claim.

We apply Fact 11 to derive a useful form on the quantum walk on a weak product relative to the normalized Laplacian.

Lemma 15. Let $G$ and $H$ be graphs whose normalized Laplacians have spectral decompositions given by $\mathcal{L}(G) = \sum_k \lambda_k E_k$ and $\mathcal{L}(H) = \sum_\ell \mu_\ell F_\ell$. Then, the quantum walk on $G \times H$ relative to the normalized Laplacian is given by

$$\exp(-it\mathcal{L}(G \times H)) = \sum_{k,\ell} \exp[-it(\lambda_k + \mu_\ell - \lambda_k\mu_\ell)] E_k \otimes F_\ell.$$  

(69)

\[\text{Doob}[13]\] showed that $-2 \in \text{Spec}(\ell(G))$ if and only if $G$ contains an even cycle or two odd cycles in the same component.
Proof. Follows from Fact [11] since $\mathcal{L}(G \times H)$ consists of three commuting matrices $\mathcal{L}(G) \otimes I_H$, $I_G \otimes \mathcal{L}(H)$, and $\mathcal{L}(G) \otimes \mathcal{L}(H)$. \hfill $\square$

We show a closure property for perfect state transfer under weak products relative to the normalized Laplacian.

**Theorem 16.** Let $G$ be a graph with perfect state transfer between vertices $g_1$ and $g_2$ at time $t_G$ relative to the normalized Laplacian. Suppose that $H$ is a graph where

$$t_G \text{Spec}_\mathcal{L}(H)(\text{Spec}_\mathcal{L}(G) - 1) \subseteq 2\pi\mathbb{Z}. \tag{70}$$

Then, $G \times H$ has perfect state transfer between vertices $(g_1, h)$ and $(g_2, h)$ at time $t_G$ relative to the normalized Laplacian.

**Proof.** Suppose $\mathcal{L}(G) = \sum_k \lambda_k E_k$ and $\mathcal{L}(H) = \sum_\ell \mu_\ell F_\ell$ are the spectral decompositions of the normalized Laplacians of $G$ and $H$. By Lemma [15] we have

$$\langle e_{(g_2,h_2)} | e^{-it\mathcal{L}(G \times H)} e_{(g_1,h_1)} \rangle = \sum_k e^{-it\lambda_k} \langle e_{g_2} | E_k e_{g_1} \rangle \sum_\ell e^{-it\mu_\ell (1-\lambda_k)} \langle e_{h_2} | F_\ell e_{h_1} \rangle. \tag{71}$$

Note we have used $e_{(g,h)} = e_g \otimes e_h$. Suppose at time $t_G$, we have $|\langle e_{g_2} | e^{-it\mathcal{L}(G)} e_{g_1} \rangle| = 1$. Since $t_G \text{Spec}_\mathcal{L}(H)(\text{Spec}_\mathcal{L}(G) - 1) \subseteq 2\pi\mathbb{Z}$, we have

$$\langle e_{(g_2,h_2)} | e^{-it\mathcal{L}(G \times H)} e_{(g_1,h_1)} \rangle = \sum_k e^{-it\lambda_k} \langle e_{g_2} | E_k e_{g_1} \rangle \sum_\ell \langle e_{h_2} | F_\ell e_{h_1} \rangle = \langle e_{g_2} | e^{-it\mathcal{L}(G)} e_{g_1} \rangle \langle e_{h_2} | e_{h_1} \rangle, \tag{72}$$

which proves the claim. \hfill $\square$

**Corollary 17.** For any integer $m \geq 1$, the weak product $P_3 \times K_{2m}$ has perfect state transfer at time $t = (2m - 1)\pi$ relative to the normalized Laplacian.

**Proof.** The normalized Laplacian spectrum of the clique $K_{2m}$ is given by

$$\text{Spec}_\mathcal{L}(K_{2m}) = \left\{ 0, 1 + \frac{1}{(2m-1)} \right\}. \tag{74}$$

Let $t_G = (2m - 1)\pi$. By Fact [3] the spectrum of $P_3$ is given by $\text{Spec}_\mathcal{L}(P_3) = \{0, 1, 2\}$ and it has perfect state transfer at time $t_G$ relative to the normalized Laplacian. Note that

$$(2m - 1)\pi \times \left\{ 0, 1 + \frac{1}{(2m-1)} \right\} \times \{0,1,2\} \subseteq 2\pi\mathbb{Z}. \tag{75}$$

Thus, by Theorem [10] $P_3 \times K_{2m}$ has perfect state transfer at time $t_G$ relative to the normalized Laplacian. \hfill $\square$

**Corollary 18.** For any integer $m \geq 1$, the weak product $P_3 \times Q_{2m-1}$ has perfect state transfer at time $t = (2m - 1)\pi$ relative to the normalized Laplacian.
Figure 6: Some graph products with perfect state transfer (between vertices marked white):
(a) the weak product $P_3 \times K_4$ has perfect state transfer at time $3\pi$ relative to the normalized Laplacian. (b) the Cartesian product $K_2 \Box (K_2 + K_2)$ has perfect state transfer at time $\pi/2$ relative to the standard Laplacian.

**Proof.** The normalized Laplacian spectrum of the cube $Q_{2m-1}$ is given by
\[
\text{Spec}_L(Q_{2m-1}) = \left\{ \frac{2k}{2m-1} : k = 0, \ldots, 2m-1 \right\}.
\] (76)
Let $t_G = (2m - 1)\pi$. By Fact 5, the spectrum of $P_3$ is given by $\text{Spec}_L(P_3) = \{0, 1, 2\}$ and it has perfect state transfer at time $t_G$ relative to the normalized Laplacian. Note that
\[
(2m - 1)\pi \times \left\{ \frac{2k}{2m-1} : k = 0, \ldots, 2m-1 \right\} \times \{0, 1, 2\} \subseteq 2\pi\mathbb{Z}.
\] (77)
Thus, by Theorem 16, $P_3 \times Q_{2m-1}$ has perfect state transfer at time $t_G$ relative to the normalized Laplacian. \square

We show another closure property for perfect state transfer under weak products relative to the normalized Laplacian.

**Theorem 19.** Let $G$ and $H$ be graphs with perfect state transfer between vertices $g_1, g_2$ and $h_1, h_2$, respectively, both at time $\tau$ relative to the normalized Laplacian. Suppose that
\[
\tau \text{Spec}_L(G) \text{Spec}_L(H) \subseteq 2\pi\mathbb{Z}.
\] (78)
Then, $G \times H$ has perfect state transfer between vertices $(g_1, h_1)$ and $(g_2, h_2)$ at time $\tau$ relative to the normalized Laplacian.

**Proof.** By Lemma 15, we have
\[
\langle e_{(g_2, h_2)} | e^{-itL(G \times H)} e_{(g_1, h_1)} \rangle = \sum_k e^{-it\lambda_k} \langle e_{g_2} | E_k e_{g_1} \rangle \sum_\ell e^{it\lambda_k \mu_\ell} e^{-it\mu_\ell} \langle e_{h_2} | F_\ell e_{h_1} \rangle.
\] (79)
where $L(G) = \sum_k \lambda_k E_k$ and $L(H) = \sum_\ell \mu_\ell F_\ell$ are the spectral decompositions of the normalized Laplacians of $G$ and $H$. Since $\tau \text{Spec}_L(G) \text{Spec}_L(H) \subseteq 2\pi\mathbb{Z}$, we have
\[
\langle e_{(g_2, h_2)} | e^{-itL(G \times H)} e_{(g_1, h_1)} \rangle = \sum_k e^{-ir\lambda_k} \langle e_{g_2} | E_k e_{g_1} \rangle \langle e_{h_2} | e^{-irL(H)} e_{h_1} \rangle
\] (80)
\[
= \langle e_{g_2} | e^{-irL(G)} e_{g_1} \rangle \langle e_{h_2} | e^{-irL(H)} e_{h_1} \rangle,
\] (81)
which proves the claim. \square
Remark: Examples of graphs which are realizations of Theorem 19 have proved elusive.

6.2 Paths

We show that paths of length at least four have no antipodal perfect state transfer relative to the normalized Laplacian. This nearly matches the situation in the adjacency matrix model (see Christandl et al. [10, 9] and Godsil [18]). We show a connection between paths under the normalized Laplacian and even cycles under the adjacency matrix. This connection seems well-known (see Aldous and Fill [1]), but we state a version useful for quantum walks.

Lemma 20. Let \( n \geq 2 \) be an integer. The path \( P_n \) has antipodal perfect state transfer relative to the normalized Laplacian if and only if the cycle \( C_{2(n-1)} \) has antipodal perfect state transfer relative to the adjacency matrix.

Proof. Let \( m = n - 1 \). Consider the cycle \( C_{2m} \) with the vertex set \( \{0, 1, \ldots, 2m - 1\} \) where vertex \( j \) is adjacent to vertex \( k \) whenever \( j - k \equiv \pm 1 \) (mod \( 2m \)). Let \( \pi \) be an equitable partition of \( C_{2m} \) with \( m + 1 \) cells where \( V_0 = \{0\}, V_m = \{m\}, \) and \( V_k = \{k, 2m - k\} \), for \( k = 1, \ldots, m - 1 \). Then, \( C_{2m}/\pi \) is a weighted path \( \tilde{P}_{m+1} \) with adjacency matrix \( A(\tilde{P}_{m+1}) \) defined as:

\[
\langle e_j | A(\tilde{P}_{m+1}) e_k \rangle = (\sqrt{2})^{[\beta(j,k)]} \langle e_j | A(P_{m+1}) e_k \rangle
\]

where \( \beta(j,k) \) holds if either \( j \) or \( k \) is a boundary vertex in \( \{0, m\} \). We note that

\[
A(\tilde{P}_{m+1}) = 2D^{-1/2}A(P_{m+1})D^{-1/2},
\]

where \( D \) is the degree matrix of \( P_{m+1} \). Therefore, we have

\[
\mathcal{L}(P_{m+1}) = I - \tfrac{1}{2}A(C_{2m}/\pi).
\]

This shows that

\[
\langle e_m | e^{-it\mathcal{L}(P_{m+1})} e_0 \rangle = e^{-it} \langle e_{V_m} | e^{itA(C_{2m}/\pi)} e_{V_0} \rangle = e^{-it} \langle e_m | e^{itA(C_{2m})} e_0 \rangle,
\]

where the last equality follows by lifting (see Lemma 1).

We will need the following results for our main theorem in this section.

Proposition 21. (Godsil, Corollary 8.2.2. in [17])

If perfect state transfer occurs on a connected vertex-transitive graph \( G \), then the eigenvalues of \( G \) are integers.

Fact 12. (Olmstead, see Corollary 3.12 in Niven [30])

If \( \theta \in 2\pi \mathbb{Q} \), then the only rational values of \( \cos(\theta) \) are \( 0, \pm\tfrac{1}{2}, \pm 1 \).

Theorem 22. For \( n \geq 4 \), there is no antipodal perfect state transfer on \( P_n \) relative to the normalized Laplacian.
Proof. Let \( n \geq 4 \). By Lemma 20, if \( P_n \) has antipodal perfect state transfer relative to the normalized Laplacian, then \( C_{2(n-1)} \) has antipodal perfect state transfer relative to the adjacency matrix. By Proposition 21 since \( C_{2(n-1)} \) is connected and vertex-transitive, if it has perfect state transfer, then its eigenvalues must be integers. But the eigenvalues of the cycles \( C_{2(n-1)} \) are given by \( \{2 \cos(2\pi k/2(n-1)) : 0 \leq k < 2(n-1)\} \). By Fact 12, these are integers only at 0, \( \pm 1 \) which implies that \( k/(n-1) \in \mathbb{Z}/2 \). This implies that \( n = 2, 3 \), which is a contradiction.

Remark: The application of Fact 12 in the proof of Theorem 22 followed similar ideas used by Godsil in the context of standard Laplacians (see [17]).

7 Conclusions

In this work, we studied perfect state transfer in quantum walk relative to graph Laplacians. As pointed out by Bose, Casaccino, Mancini and Severini [5], a quantum walk relative to the standard Laplacian is related to quantum spin networks in the isotropic Heisenberg (XYZ interaction) model whereas a quantum walk with the adjacency matrix is connected to the XY model. In their seminal work, Farhi and Gutmann [15] used a weighted Laplacian matrix to define continuous-time quantum walks to underscore the close connection with continuous-time random walks. In the first work which introduced perfect state transfer, Bose [4] studied quantum spin chains in the XYZ (or “Laplacian”) model.

Our main goal in this work is to understand perfect state transfer in quantum walks relative to the standard, signless and normalized Laplacians. Our focus was on irregular graphs (since all Laplacian quantum walks are equivalent otherwise) and nonbipartite graphs (since the standard/signless Laplacian quantum walks are equivalent otherwise). To the best of our knowledge, the signless and normalized Laplacians have not been studied extensively in the context of quantum walks. Although the signless Laplacian of a graph \( G \) has no clear “physical” motivation, it shares a strong spectral bond with the line graph \( \ell(G) \). So, it provides a method for analyzing quantum walk on line graphs in the XY model. This is a direction which merits closer study. In contrast, the normalized Laplacian has a clear “physical” meaning, albeit in a more classical sense. It has been closely studied in connection with the Heat Kernel random walk in spectral graph theory (see Chung [11]) and in machine learning (see Kondor and Lafferty [27]).

We observed a useful closure property relative to the standard Laplacian: complementation preserves perfect state transfer. Relative to the adjacency matrix (and perhaps the other two Laplacians), this property holds only for regular graphs. This closure property allowed us to characterize Laplacian perfect state transfer on double cones. In turn, we generalized a known result of Bose et al. [5] and found a much simpler proof. We also found families of double cones with perfect state transfer relative to the signless Laplacian, but not relative to the standard Laplacian. Our proofs relied on ideas from the theory of equitable and almost-equitable partitions.

By exploiting the connection between signless Laplacians and line graphs, we showed some negative results for perfect state transfer relative to the signless Laplacian. Using a reduction to the adjacency matrix model, we observed that paths with five or more vertices
have no antipodal perfect state transfer relative to the signless Laplacian (also standard, by switching equivalence). But, a better negative result is known for paths (due to Godsil [17]) and, recently, for trees (due to Coutinho and Liu [12]). We applied our techniques to nonbipartite graphs and showed this for the simplest family of odd unicyclic graphs (two pendant paths attached to a three cycle). Our proof made heavy use of Godsil’s results on controllable subsets of graphs [19].

A paradoxical lore of quantum walk on the $n$-cube relative to the adjacency matrix is that its (antipodal) perfect state transfer time is $\pi/2$, for any $n$. This striking statement seems to violate the constant speed of light postulate. In contrast, relative to the normalized Laplacian, the $n$-cube has antipodal perfect state transfer at time $n\pi/2$. This example suggests that quantum walks relative to the normalized Laplacian might be closer to reality. Here, we proved another closure property for perfect state transfer but under the weak product. This is not too surprising given that the normalized Laplacian spectrum behaves well under weak product (and not under, say, Cartesian product). As a corollary, we showed that a weak product of $P_3$ with either an even clique or odd cube has perfect state transfer. It is curious that $P_3$ has perfect state transfer under the normalized Laplacian but not relative to the standard/signless Laplacians. To complete the picture, we showed that paths with four or more vertices do not have (antipodal) perfect state transfer under the normalized Laplacian. This almost matches the state of affairs under the adjacency matrix, where no perfect state transfer exists between any pair of vertices (see Godsil [18]). It is unclear if this stronger result holds relative to the normalized Laplacian.

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