EQUIVALENCE OF NONDIFFERENTIABLE METRICS

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Abstract. We study nondifferentiable metrics occurring in general relativity via the method of equivalence of Cartan adapted to the Courant algebroids. We derive new local differential invariants naturally associated with the loci of nondifferentiability and rank deficiency of the metric. As an application, we utilize the newfangled invariants to resolve the problem of causality in the interior of the black holes that contain closed timelike geodesics. Also, a no-go type theorem limits the evolution scenarios for gravitational collapse.

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2010 Mathematics Subject Classification. Primary: 53D99; Secondary: 53Z05.
1. Introduction

Our objective is to formulate the notion of invariants of nondifferentiable (pseudo-)
metrics, and to adapt Élie Cartan’s method of equivalence [4] (originally developed
to elicit differential invariants of smooth structures) so as to assimilate various metic
pathologies into differential geometry. The need for such assimilation arises
primarily in general relativity with black hole solutions of the Einstein vacuum
equations (EVE). There, genuine information is encoded via nondifferentiable ten-
sor coefficients, although several blow-up solutions of other nonlinear PDE merit
further investigation.

A trademark of metric nondifferentiability is geodesic incompleteness. Its pres-
ence in general relativity unambiguously indicates that strong gravitational fields are
beyond the scope of classical theory. Even more troublesome, the Kerr spacetime
allows closed timelike geodesics - a fact that flies in the face of causality.

So far, all attempts to exclude or domesticate the ‘wrong’ metric configurations
in a systematic way have failed. Most notably, the weak and strong cosmic censor-
ship hypotheses due to Roger Penrose [26] turned out to be of limited applicability
(see [6] for a detailed discussion as well as the recent results). Furthermore, the
hyperbolic PDE stability of solutions of EVE on the function spaces of initial data
on spacelike hypersurfaces has not been established in full generality.

In this paper we adopt a different viewpoint. Rather than trying to downplay
the undesirable pathologies by selecting generic and/or stable solutions of EVE
while retaining the desirable ones (the black hole event horizon, to name one), we
acknowledge their legitimacy and incorporate them into our framework.

Such a program necessitates building an apparatus to treat metric configurations
with or without loci of nondifferentiability uniformly. In particular, the method of
equivalence must be made to work in this formalism.

Our approach brings about a new mathematical object - a preframe, that gener-
alizes the concept of coframe onto a Courant algebroid (see Section 2.1.1), realized
as the bundle $T^*M \oplus TM$ equipped with the Courant bracket. In what follows,
$(M, \omega)$ is a symplectic manifold. With this additional structure, we can represent
the Courant algebroid as a Lie bialgebroid, such that the two complementary Dirac
structures (maximally isotropic subbundles of $T^*M \oplus TM$ with respect to the posi-
tive Courant bilinear form and closed under the Courant bracket) are related via the
bundle map assembled from the symplectic 2-form and its attendant Poisson tensor.
Subsequently, we show that for this Courant algebroid there exists a unique (up to
a symplectomorphism) connection, which we designate the symplectic connection,
that lifts preframes injectively to the cotangent bundle manifold:

$$C : T^*M \oplus TM \rightarrow TT^*M.$$  

Fortuitously, the symplectic connection acts as a morphism of Lie bialgebroids, the
image bialgebroid being an isotropic integrable subbundle of $TT^*M \oplus T^*T^*M$ (see
Section 2.1.3).

Via the canonical symplectomorphism of [1]

$$\alpha : TT^*M \rightarrow T^*TM,$$

1We deliberately avoid the term ‘singularity’ in this context. Its definition in general relativity
is narrow and does not encompass all the phenomena treated herein.
we obtain a full coframe on $T^*TM$, and thus pave the way for the application of Cartan’s equivalence method. However, this is a principally new variety of equivalence in that the resulting mapping does not descend from the level of the tangent bundle manifold down to $(M, \omega)$ in general.

It does in the case of smooth metrics, wherein our preframes turn out to be sections of a Dirac subbundle, and locally equivalent to the standard coframes viewed as sections of the trivial Dirac subbundle $T^*M$. Explicitly, the Hitchin map is the projection

$$(\operatorname{Id}^* \oplus 0) : T^*M \oplus TM \longrightarrow T^*M.$$ 

No loss of data occurs since the preframe is integrable.

Obversely, preframes not closed under the Courant bracket are not sections of some Dirac subbundle and are considered singular. Their lifts to $T^*TM$ are best characterized in terms of the Lie algebra of isometries within $\text{Diff}(T^*M)$. They are certain conjugacy classes of $\text{Diff}(T^*M)/\text{Diff}(M)$. No smooth metric configuration would yield a preframe of this type.

Having introduced preframes at a conceptual level, a more detailed account of how equivalence was established is now in order. Thus for a nondifferentiable metric tensor $[g_{ij}]$ on $M$, $\dim M = 2k$, we introduce $2k^2 + k$ local auxiliary scalar fields satisfying the Laplace-Beltrami equation on the metric background. They encapsulate the pathologies associated with the original metric, and allow for it to be reverse-engineered. To be usable at the coframe level, they (locally being at most twice differentiable) would have to be smoothed. An application of the Nash-Gromov deep smoothing operators [24, 9] outputs multiple smooth functions. Effectively, this approach trades off nondifferentiability for nonuniqueness.

Our way of coming to grips with the overflowing scalar fields involves an infinite-dimensional completely integrable Hamiltonian system that allows to organize them into parametric families united by Hamiltonian flows. Specifically, we take up the isospectral class ‘manifold’ of the Hill operator with periodic potentials, following a seminal paper of McKean and Trubowitz [20]. The auxiliary scalar fields are represented by products of an eigenfunction of some appropriate Hill operator with a potential periodic with respect to a local parameter transversal to the loci of nondifferentiability and/or rank deficiency, which we dub the blow-up parameter, and a nonvanishing function. The nondifferentiability property becomes quantifiable by comparison with the spectrum of the harmonic oscillator (the Hill operator with a constant potential). Smooth metrics produce constant Hill operator eigenfunctions, whereas those with nondifferentiability loci require oscillating ones.

The oscillating eigenfunctions are incorporated into preframes, rendering those preframes singular. Their attendant nonholonomic frames on $TT^*M$ are gotten by lifting the preframe and its complement via the symplectic connection. On the cotangent bundle manifold, we use the adjoined variables (impulse variables in the canonical phase space) to parameterize nonunique smoothed auxiliary scalar fields in a prolongation. Hence the equivalence would encompass the totality of smoothed solutions of the Laplace-Beltrami equation.

The maximal structure group for our equivalence problem has to preserve the preframes (for rotations among the base variables potentially interfere with the

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2 This version of equivalence was invented by Hitchin [14]. He constructed the moduli space of generalized Calabi-Yau manifolds by quotienting out the action of exact 2-forms (B-fields) as well as the action of conventional biholomorphic maps.
individual auxiliary scalar fields), as well as to respect the (action of the) symplectic connection $C$ and the canonical symplectomorphism $\alpha$. Therefore only a maximal torus group $\bigoplus_{i=1}^{2k} SO_i(2,\mathbb{R})$ survives. Cartan’s group reduction/frame normalization technique can only be implemented with the auxiliary scalar fields that have locally nonvanishing first and second derivatives, known as free maps - those belonging to the set of solutions of the freedom partial differential relation (9, Chapter 1). To ensure we deal with free auxiliary scalar fields, we apply the homotopy principle to deform our scalar fields into free ones. The resulting $4k$-coframes exhibit some specific intrinsic torsion, which serves as a repository of data pertaining to metric loci of nondifferentiability.

The upshot of this revamping is, new invariants emerge. They encode the minutia of metric blow-ups via holomorphic structures of the Hill surfaces (hyperelliptic Riemann surfaces of infinite genus) [21]. Using recent advances in hyperelliptic function theory [7], particularly the Torelli theorem for certain Riemann surfaces of infinite genus, these invariants can be conveniently presented as the Riemann period matrices parameterized by the base manifold variables. Their link to nonintegrable preframes involves an inversion formula for theta functions found by Its and Matveev in the finite-genus case [15], and generalized by McKean and Trubovitz to cover the Hill surfaces of infinite genus [21].

After this protracted discussion of technical matters, we at last take up the applications of our differential invariants to black hole solutions of EVE, collected in Chapter 5. First, we compute the global singular preframe of the Kerr spacetime, and delineate one open subset (we call this subset representative) of it that contains parts of all the geometrically significant loci of nondifferentiability, as well as those of rank deficiency. They are (subsets of) the event horizon, the Cauchy horizon, and the locus of curvature blow-up. That is a synopsis of Section 5.1.

Next, we offer a resolution of the nonuniqueness of $C^0$-extensions of solutions of EVE past the Cauchy horizon. The strong cosmic censorship hypothesis (in a precise formulation) has been disproved by Dafermos and Luk [6]. We find a way to select a unique extension based on the values of the germane Riemann period matrices. Thus the (internal) locus of curvature blow-up turns out to be determined by the behavior of metric coefficients at the (external) locus of rank deficiency/nondifferentiability controlled by the future development imposed by EVE. However, the apparent breakdown of causal structure still persists for dynamics on that background. To save the situation, we draw distinction between ordinary differentiable configurations and their attendant identically zero Riemann period matrices, and pathological ones. We speculate that preservation of the Riemann period matrix is inextricably linked with mass distribution, to the extent it would eliminate at least some trajectories should a mass cross the event horizon. But the door still remains half-open for an intricate mass trajectory to sneak past the horizon without disturbing the invariants. This hesitancy is informed by the effects of gravitational waves on black hole solutions. They disturb the event horizons temporarily. After the wavefront passes, the black holes return to their original state unscathed. One would expect a similar scenario to play out if the mass moves in a special way.

Section 5.4 deals with one particular implication of the invariants being there to label black holes - the quantum effect of black hole evaporation due to emission
of thermal radiation [10] [11]. Following the prescriptions of quantum field theory, we introduce absorption and emission operators acting on the eigenvalues of the Hill operator whose periods (parameterized by the manifold variables) bridge the representative set from the locus of nondifferentiability to the regular metric. Evaporation would effectively shrink the locus of nondifferentiability, and modify the discriminant so that the odd- and even-indexed eigenvalues would move closer to each other. Conversely, absorption operators would elongate the intervals of instability of the Hill operator. Hence absorption/emission operators have to be Hermitian, and, being defined on a Hilbert space of quantum states, have compact kernels with no further restrictions. The resulting configurations at the Riemann surface level would no longer encode viable geometric structures, let alone black hole solutions of EVE, but the chronological products of (combinations of) those operators may furnish a representation of the information imprinted on the event horizon and subsequently lost due to evaporation. We contrive a necessary condition for the information to be retrievable in terms of formal properties of absorption/emission operators. That is, their brackets under composition must remain noncommutative at all values of some affine time parameter. Thus our quantum operator algebra must be defined over the field of quaternions. Otherwise, the existence of invariants would not forestall the loss of information predicated on emission of thermal radiation.

In Section 5.5, we state and prove a No-Go theorem that limits the gravitational collapse of axisymmetric rotating distributed masses down to the Kerr spacetime. Once again, the time development set by EVE is reexpressed in terms of time-dependent discriminant evaluated at a fixed eigenvalue of the Hill operator. Its second partial derivative with respect to global time coordinate undergoes some anharmonic oscillations prior to coming to a stationary state. These oscillations are governed by the universal oscillator ODE with a time-dependent damping factor. Our equation establishes a link between the Kerr spacetime and the initial mass distribution informing damping.

Further exploratory steps are taken in Section 5.6. To account for multiple black holes, moving in space-time relative to each other, possibly coalescing, we need a more robust formalism. One blow-up parameter would not be enough, even locally. A logical way to overcome this limitation is to introduce multiple globally independent parameters. Hence, we argue that the spectral theory of two-dimensional Schrödinger operator,

\[ \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + Q(x_1, x_2) \right), \]

with doubly periodic potential, investigated in [17], turns out to be a substitute for the Hill operator. Complexified Fermi curves ([7], Section 16) take the place of the Hill surfaces. Torelli theorem remains, but the Its-Matveev formula no longer applies. One implication of a multiparametric theory is that the time coordinate would exist only locally in asymptotically flat space-time.

Acknowledgements. We would like to express our gratitude to Vladimir Matveev and Igor Krichever for generously sharing their insights into spectral theory and integrable systems, and Andrei Todorov for stimulating discussions.
2. Preframes

2.1. Preliminaries.

The point of departure here is an assembly of topics from differential geometry in the category of smooth manifolds and diffeomorphisms.

2.1.1. Courant algebroids. The notion of Courant algebroid, conceived as a natural outgrowth of symplectic or Poisson structures, and modeled after Lie algebroids, was first introduced by Theodore Courant [5]. Below we provide a somewhat systematic overview, tailored to our needs, following Liu, Weinstein, and Xu [18], and the original material referenced therein.

Definition 2.1. A Lie algebroid on a manifold M is a vector bundle $A \rightarrow M$, equipped with a vector bundle map, $\hat{a} : A \rightarrow TM$, over M, called the anchor of A, and a bracket $[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$, which is bilinear and antisymmetric, satisfies the Jacobi identity, and is such that
\begin{equation}
[X, uY] = u[X, Y] + (\hat{a}(X)u)Y, \quad \forall X, Y \in \Gamma(A), \quad u \in C^\infty(M),
\end{equation}
\begin{equation}
\hat{a}([X, Y]) = [\hat{a}(X), \hat{a}(Y)], \quad \forall X, Y \in \Gamma(A).
\end{equation}

The Lie algebroid A is transitive if $\hat{a}$ is fiberwise surjective, regular is $\hat{a}$ is of locally constant rank, and totally intransitive if $\hat{a} = 0$. The manifold M is the base of A.

Definition 2.2. A base-preserving morphism of Lie algebroids, referred to as simply morphism, is a vector bundle map $\varphi : (A_1, pr_1, M) \rightarrow (A_2, pr_2, M)$ such that $\hat{a}_2 \cdot \varphi = \hat{a}_1$, and
$$
\varphi[X, Y] = [\varphi(X), \varphi(Y)]. \quad \forall X, Y \in \Gamma(A_1).
$$

The anchor of a Lie algebroid encodes its geometric properties. If the algebroid is transitive, then right inverses to the anchor are connections. If the algebroid is regular, then the image of the anchor defines a foliation of the base manifold, the characteristic foliation, and over each leaf of that foliation, the Lie algebroid is transitive.

Definition 2.3. A Lie bialgebroid on a manifold M is a dual pair of Lie algebroids $(A, A^*)$ such that the coboundary operator $d_x$ on $\Gamma(\wedge^* A)$ satisfies
$$
d_x[X, Y] = [d_x X, Y] + [X, d_x Y], \quad \forall X, Y \in \Gamma(A).
$$

On classical Lie bialgebras we have the compatibility condition between $(g, [,\cdot,\cdot])$ and $(g^*, [,\cdot,\cdot]^*)$ formulated in terms of a cocycle. This definition is just an extension.

For a Poisson manifold $(M, \pi)$, there is the standard Lie bialgebroid $(TM, T^* M)$ with the coboundary operator $d_x = [\pi, \cdot]$.

Definition 2.4. A Courant algebroid is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on the bundle, a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and a bundle map $\rho : E \rightarrow TM$ such that the following properties are satisfied:
\begin{enumerate}
\item $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = D\pi(e_1, e_2, e_3) \quad \forall e_1, e_2, e_3 \in \Gamma(E);
\item $\rho(e_1, e_2) = [\rho e_1, \rho e_2] \quad \forall e_1, e_2 \in \Gamma(E);
\end{enumerate}
Coincidentally, we have differential operators introduced in (2.4):

\[ \rho(Df, e) = \frac{1}{2} \rho(e)f. \]

**Definition 2.5.** Let \( E \) be a Courant algebroid. A subbundle \( L \subset E \) is called **isotropic** if it is isotropic under the symmetric bilinear form \( (\cdot, \cdot) \). It is called **integrable** if \( \Gamma(L) \) is closed under the bracket \([\cdot, \cdot]\). A Dirac structure, or Dirac subbundle is a subbundle \( L \) which is maximally isotropic and integrable.

Any Dirac structure is trivially a Lie algebroid with anchor \( \rho|_L \). Conversely, suppose \( A \) and \( A^* \) are both Lie algebroids over the base manifold \( M \), with anchors \( \hat{a} \) and \( \hat{a}_* \) respectively. Let \( E \) denote their vector bundle direct sum \( E = A \oplus A^* \) (defined as the pullback of \( A \times_M A^* \)). On \( E \) there exist two natural nondegenerate bilinear forms, one symmetric, and the other antisymmetric, which are defined as follows:

\[ \tag{2.3} (X_1 + \xi_1, X_2 + \xi_2)_{\pm} = \frac{1}{2}(\langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle). \]

The additional structure begets some refined additional structure:

**Definition 2.6.** A Dirac structure \( L \) such that the antisymmetric bilinear form defined in (2.3) satisfies \( (X_1 + \xi_1, X_2 + \xi_2)_- = 0, \forall (X_{1,2} + \xi_{1,2}) \in \Gamma(L) \) is called a null Dirac structure.

On \( \Gamma(E) \), we introduce a bracket operation by

\[ \tag{2.4} [e_1, e_2] = ([X_1, X_2] + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - d_* (e_1, e_2)_-) 
+ ([\xi_1, \xi_2] + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + d(e_1, e_2)_-), \]

where \( e_{1,2} = X_{1,2} + \xi_{1,2}, d : C^\infty(M) \rightarrow \Gamma(A^*), d_* : C^\infty(M) \rightarrow \Gamma(A) \) are the differentials.

Now we let \( \rho : E \rightarrow M \) be the bundle map defined by \( \rho = \hat{a} + \hat{a}_* \). That is

\[ \rho(X + \xi) = \hat{a}(x) + \hat{a}_* (\xi), \ \forall X \in \Gamma(A), \ \forall \xi \in \Gamma(A^*). \]

Coincidentally, we have \( \mathcal{D} \) of Definition 2.4 expressible in terms of the exterior differential operators introduced in (2.4):

\[ \mathcal{D} = d_* + d. \]

Finally, we specialize our choice of the component algebroids. Namely, \( A = TM, \)
$A^* = T^* M$, the latter equipped with the trivial bracket. Then (2.3) takes the form:

$$[e_1, e_2] = [X_1, X_2] + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + d(e_1, e_2),$$

This is the Courant bracket that we all know and love.

There are multiple Dirac subbundles associated with the Courant algebroid $TM \oplus T^* M$. In particular, we may have a pair of transversal ones. Then $L_1 \oplus L_2 = TM \oplus T^* M$. Such a pair constitutes a Lie bialgebroid ([18], Theorem 2.6):

**Background Theorem 2.1.** In a Courant algebroid $(E, \rho, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ with a pair of transversal Dirac structures, $L_1$, $L_2$, $L_1 \oplus L_2 = E$, there exists an attendant Lie bialgebroid $(L_1, L_2)$, where $L_2 = (L_1)^*$ under the dual pairing given by 2$(\cdot, \cdot)$.

Typically, Dirac structures are obtained by interpolating the tangent and cotangent bundles via graphs: $X \mapsto X + i_X \beta$, $\beta \in \Gamma(A^2 T^* M)$, $d \beta = 0$. In ([18], Section 6) the bundle maps associated with closed 2-forms are designated ‘strong Hamiltonian operators’. However, as Liu et al. point out, there are Dirac structures that are not such graphs. Those are generated by other bundle maps, satisfying a nonlinear differential equation.

From this point on, our base manifold is symplectic, $(M, \omega)$, $\dim M = 2k$. Thus there is a bundle isomorphism induced by the symplectic form, and its inverse induced by the (nondegenerate) Poisson structure:

$$TM \xrightarrow{\omega} T^* M, \quad T^* M \xrightarrow{\omega} TM,$$

implicitly defined via

$$\langle \omega^\# (X), Y \rangle = \omega(X, Y), \quad \langle \varphi, \pi^\# (\beta) \rangle = \pi(\varphi, \beta).$$

On $TM \oplus T^* M$ they combine into a discrete automorphism of the Courant algebroid. Our choice is $\omega^\# - \pi^\#$. The formal definition involves projections and the fact that the Whitney sum of two vector bundles over the same base manifold is defined as the pullback bundle of the diagonal embedding $M \rightarrow M \times M$ establishing the isomorphism $TM \oplus T^* M \cong TM \times_M T^* M$:

$$TM \xleftarrow{(Id \oplus 0)} TM \oplus T^* M \xrightarrow{(0 \oplus Id^*)} T^* M.$$

$$\omega^\# - \pi^\#(\xi + X) \overset{\text{def}}{=} (\omega^\# \circ (Id \oplus 0) - \pi^\# \circ (0 \oplus Id^*))((\xi + X) = \omega^\#(X) - \pi^\#(\xi)$$

def for smooth sections $X \in \Gamma(TM)$, $\xi \in \Gamma(T^* M)$. This bundle automorphism satisfies

$$(\omega^\# - \pi^\#) \circ (\omega^\# - \pi^\#) = -(Id^* \oplus Id).$$

**Definition 2.7.** A prefame on a manifold $(M, \omega)$, $\dim M = 2k$, is a 2k-tuple $\{X_i + \xi_i\}$, $X_i + \xi_i \in \Gamma(TM \oplus T^* M)$ such that

$$\text{span}\{\{X_i + \xi_i\}, (\omega^\# - \pi^\#)\{X_i + \xi_i\}\} = TM \oplus T^* M, \quad \text{and}$$

$$\{X_i + \xi_i\} \cap (\omega^\# - \pi^\#)\{X_i + \xi_i\} = 0.$$

Notably, there is no differentiability provision insofar as it does not interfere with the rank requirement.
2.1.2. Hitchin spinors. The Courant bracket (2.6) is antisymmetric but does not in general satisfy the Jacobi identity. One new feature however is that unlike the Lie bracket on sections of $TM$, this bracket has non-trivial automorphisms defined by forms. Let $\beta \in \Gamma(\Lambda^2T^*M)$ be closed, and define the vector bundle automorphism of $TM \oplus T^*M$ known in literature as the B-field by

$$B_\beta(X + \xi) = X + \xi + i_X \beta.$$  

Then one may easily check that

$$B_\beta([(X_1 + \xi_1, X_2 + \xi_2)]) = [B_\beta(X_1 + \xi_1), B_\beta(X_2 + \xi_2)].$$

This interacts with a natural metric structure on $TM \oplus T^*M$.

Let $V$ be an $n$-dimensional real vector space and consider 2n-dimensional space $V \oplus V^*$. The direct sum admits a natural nondegenerate inner product of signature $(n, n)$ defined by

$$(X + \xi, X + \xi) = -\langle X, \xi \rangle,$$  

$X \in V, \xi \in V^*.$

The natural action of $GL(n, \mathbb{R})$ preserves this inner product. The Lie algebra of the orthogonal group of all transformations preserving (2.14) allows the following decomposition:

$$\mathfrak{so}(V \oplus V^*) = \operatorname{End}V \oplus \Lambda^2V \oplus \Lambda^2V^*.$$  

In particular, $\beta \in \Lambda^2V^*$ acts via

$$X + \xi \mapsto i_X \beta$$

and thus exponentiates to an orthogonal action on $V \oplus V^*$ given by

$$X + \xi \mapsto X + \xi + i_X \beta.$$  

This is the algebraic action of a closed 2-form which preserves the bracket (2.6).

Consider the exterior algebra $\Lambda^*V^*$ and the action of $X + \xi \in V \oplus V^*$ on it defined by

$$(X + \xi) \cdot \varphi \overset{\text{def}}{=} i_X \varphi + \xi \wedge \varphi.$$  

We have

$$(X + \xi)^2 \cdot \varphi = i_X(\xi \wedge \varphi) + \xi \wedge i_X \varphi = (i_X \xi)\varphi = -(X + \xi, X + \xi)\varphi.$$  

With this operation, the exterior algebra becomes a module over the Clifford algebra of $V \oplus V^*$. Hence we obtain the spin representation of the group $Spin(V \oplus V^*)$ acting on the space of differential forms with values in the canonical line bundle $\sqrt{\Lambda^nV}$. Below we use the same notation for spinors and forms, as there is a one-to-one correspondence between them, and the context makes it clear. Splitting into even and odd forms, we separate the two irreducible half-spin representations:

$$\mathcal{S}^+ = \Lambda^{\text{even}}V^* \otimes \sqrt{\Lambda^nV},$$

$$\mathcal{S}^- = \Lambda^{\text{odd}}V^* \otimes \sqrt{\Lambda^nV}.$$  

Exponentiating $\beta \in \Lambda^2V^* \subset \mathfrak{so}(V \oplus V^*)$, we end up with an element $\exp \beta \in Spin(V \oplus V^*)$ acting on spinors:

$$\exp \beta(\phi) = (1 + \beta + \frac{1}{2} \beta \wedge \beta + \cdots) \wedge \phi.$$
When \( \dim V = 2k \), there is an invariant bilinear form \( \langle \phi, \varphi \rangle \) on \( \mathbb{S}^\pm \), symmetric for \( k \) even, and antisymmetric for \( k \) odd. Using the exterior product, we expand the spinors in graded components:

\[
\langle \phi, \varphi \rangle = \sum_l (-1)^l \phi_{2l} \wedge \varphi_{2k-2l} \in \Lambda^{2k} V^* \otimes (\sqrt{\Lambda^{2k} V^*})^2
\]

for \( \phi, \varphi \in \mathbb{S}^+ \), and

\[
\langle \phi, \varphi \rangle = \sum_l (-1)^l \phi_{2l+1} \wedge \varphi_{2k-2l-1} \in \Lambda^{2k} V^* \otimes (\sqrt{\Lambda^{2k} V^*})^2
\]

for \( \phi, \varphi \in \mathbb{S}^- \).

The action (2.16) extends to spinors. Thus for \( \varphi \in \mathbb{S}^\pm \), we delineate its annihilator, the linear subspace

\[
L_\varphi = \{ X + \xi \in V \oplus V^* | (X + \xi) \cdot \varphi = 0 \}.
\]

Since \( (X + \xi) \in L_\varphi \) satisfies

\[
0 = (X + \xi) \cdot ((X + \xi) \cdot \varphi) = -(X + \xi, X + \xi) \varphi
\]

we see that \( X + \xi \) is null, and so \( L_\varphi \) is isotropic. Now we are in a position to define a different notion of integrability in Courant algebroids.

**Definition 2.8.** A spinor \( \varphi \), whose attendant annihilator subspace \( L_\varphi \) is maximally isotropic, \( \dim L_\varphi = \dim V \), is called a pure spinor.

Any two pure spinors are related by some element of \( \text{Spin}(V \oplus V^*) \). Purity is a non-linear condition.

In this article, having confined ourselves to symplectic manifolds \( (M, \omega) \), we primarily work with the pure spinors

\[
1 + \omega + \frac{1}{2} \omega \wedge \omega + \cdots + \frac{1}{k!} \omega^k,
\]

\[
1 - \omega - \frac{1}{2} \omega \wedge \omega - \cdots - \frac{1}{k!} \omega^k.
\]

Their maximal isotropic subspaces are the graph Dirac structures \( \{ X + i_X \omega | X \in \Gamma(TM) \} \), and \( (\omega^# - \pi^#) \{ X + i_X \omega | X \in \Gamma(TM) \} \). The relationship between the two, mediated by our bundle map is not coincidental. Generally, we have

**Lemma 2.1.** The bundle map \( (\omega^# - \pi^#) \) preserves integrability of Dirac structures. For every maximally isotropic integrable subbundle \( L \subset TM \oplus T^* M \), we have

\[
[(\omega^# - \pi^#) L, (\omega^# - \pi^#) L] \subset (\omega^# - \pi^#) L.
\]

**Proof.** The automorphism \( (\omega^# - \pi^#) \) preserves integrability (or lack thereof). On the other hand, the action of \( (\omega^# - \pi^#) \) admits a representation on \( \mathbb{S}^\pm \) via the symplectic star operator, first introduced by J.-L. Brylinski [3]. Recall that \( \pi^# \) extends to an operator taking differential forms into multivector fields by multiplicativity:

\[
\pi^#(\beta_1 \wedge \cdots \wedge \beta_l) = \pi^#(\beta_1) \wedge \cdots \wedge \pi^#(\beta_l),
\]

where \( \beta_i \) are 1-forms. Furthermore, the contraction \( i_{\pi^#(\beta_1, \ldots, \beta_l)} \omega^k \) is a differential form of degree \( 2k - l \). Then we set \( *\beta = i_{\pi^# \beta} \omega^k \). This is a younger sibling of the Hodge star with the symplectic volume form in place of the Riemannian volume.
form. On 0-forms, we equate \( *1 = \omega^k \) to keep the normalizations consistent. Hence in the spinorial realm we have that same operator
\[
(\phi, L\phi) \longrightarrow (*\phi, L\phi)
\]
acting according to the following rule: \( *(\beta \otimes \sqrt{\Lambda^{2k}T^*M}) = (\ast\beta) \otimes \sqrt{\Lambda^{2k}T^*M} \). Here \( L\phi \) stands for the annihilator subbundle of the spinor \( \phi \). Complementarity of the transformed spinors is manifested via \( \langle \phi, \ast\phi \rangle \neq 0 \). Now we observe that the symplectic star operator intertwines with \( B_\beta \) of (2.13): \( * B_\beta = B_{\ast\beta} \circ \ast \). Ergo so does \( (\omega^# - \pi^#) \).

While there may exist an alternative proof of this result, not involving spinors, our method highlights the fact that, on symplectic manifolds, Dirac structures’ integrability is akin to the standard Lie bracket integrability of Lagrangian submanifolds.

### 2.1.3. The symplectic connection.

Courant algebroids on symplectic manifolds enjoy some nice properties. Here we establish the concept of a connection that relates null Dirac structures on \((M, \omega)\) to Lagrangian subbundles on \((T^*M, \Omega)\). The background information on Lie algebroids, tangent bundles of bundle manifolds, and connections comes from the definitive monograph of Kirill Mackenzie [19].

Consider the manifold \( T^*M \), where \((M, \omega)\) is a symplectic manifold. Applying the tangent functor to the vector bundle operations in \( T^*M \) yields a vector bundle \( (TT^*M, dpr^*, TM) \) called the tangent prolongation of \( T^*M \). In conjunction with the standard vector bundle structure \( (TT^*M, pr_{TT^*M}, T^*M) \), this forms the tangent double vector bundle of \( T^*M \). That bundle is our basic object to work with.

In order to describe it, we need to set the stage with the various projections. Their domains and ranges are indicated on the diagrams below, undecipherable to dvi viewers, but helpful to humans. We denote elements of \( TT^*M \) by \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \), and we write \((\mathcal{A}, \beta, X, x)\) to list all the variables in a local bundle chart.

\[
\begin{array}{c}
\text{TM} \\
\downarrow \mathit{pr} \\
\text{M}
\end{array}
\begin{array}{c}
\downarrow \mathit{pr}^* \\
\text{TT}^* \text{M}
\end{array}
\begin{array}{c}
\downarrow \mathit{pr}_{TT^* \text{M}} \\
\text{T}^* \text{M}
\end{array}
\begin{array}{c}
\downarrow \mathit{pr}^* \\
\text{X, x}
\end{array}
\begin{array}{c}
\downarrow i_{X} \omega = \sum f(x)\beta \\
\text{(X, x)}
\end{array}
\begin{array}{c}
\downarrow \mathit{pr}^* \\
\text{(} \beta, x \text{)}
\end{array}
\begin{array}{c}
\downarrow \mathit{pr}_{TT^* \text{M}} \\
\text{(} \mathcal{A}, \beta, X, x \text{)}
\end{array}
\end{array}
\]

The projections are intrinsically consistent, specifically,
\[
\mathit{pr}^*(pr_{TT^*M}(\mathcal{A})) = x = \mathit{pr}(dpr^*(\mathcal{A})).
\]

There is a canonical 1-form \( \vartheta \in \Gamma(T^*T^*M) \), called the Liouville form, given by
\[
\vartheta(\mathcal{A}) \overset{\text{def}}{=} \langle pr_{TT^*M}(\mathcal{A}), dpr^*(\mathcal{A}) \rangle, \quad \mathcal{A} \in \Gamma(TT^*M).
\]

Hence the cotangent bundle comes naturally equipped with the canonical 2-form \( \Omega = -d\vartheta \), and we select our symplectic bundle manifold to be \((T^*M, \Omega)\).

In \( TM \) and \( T^*M \), we use standard notation to denote fiber addition and multiplication by scalars. The zero of \( T^*M \) over \( x \in M \) is \( 0^{T^*M}_x \), and the zero of \( TM \) over \( x \in M \) is \( 0^{TM}_x \). With respect to the standard vector bundle structure \( (TT^*M, pr_{TT^*M}, T^*M) \), we continue to use + for addition, - for subtraction, and juxtaposition for scalar multiplication. The symbol \( T^*_\beta T^*M \) will always denote the fiber \((pr_{TT^*M})^{-1}(\beta)\) for \( \beta \in \Gamma(T^*M) \) with respect to this bundle. The zero element
in $T_βT^*M$ is denoted $0_β$. Below we refer to this bundle structure as the standard tangent bundle structure.

In the prolonged tangent bundle structure, $(T^*M, dpr^*, TM)$, we use $+$ for addition, $-$ for subtraction, and $\cdot$ for scalar multiplication. The fiber over $X \in TM$ will always be denoted $(dpr^*)^{-1}(X)$, and the zero element of this fiber is $(d0)_X$. If we consider elements $A \in \Gamma(T^*M)$ as derivatives of paths in $T^*M$ and write

$$A = \frac{d}{dt}β|_0,$$

where $β_t$ denotes a path in $T^*M$ for all $0 \leq t \leq ε$, then $pr_{T^*M}(A) = β_0$, and $dpr^*(A) = \frac{d}{dt}pr^*β_t|_0$. If $A, B \in \Gamma(T^*M)$ are located on the same fiber, that is if $dpr^*(A) = dpr^*(B)$, we can arrange that $A = \frac{d}{dt}β|_0, B = \frac{d}{dt}η|_0$, where $pr^*(β_t) = pr^*(η_t)$ for all $t \leq ε$, and, as a consequence,

$$A + B = \frac{d}{dt}(β_t + η_t)|_0, \quad c \cdot A = \frac{d}{dt}cβ|_0.$$

For each $x \in M$, the tangent space $T_{0,T^*M}T^*_xM$ identifies canonically with $T^*_xM$; we denote the element of $T_{0,T^*M}T^*_xM$ corresponding to $β \in T^*_xM$ by $\bar{β}$ and call it the core element corresponding to $β$. For $β, η \in T^*_xM, c \in \mathbb{R}$,

$$\bar{β} + \bar{η} = \bar{β + η} = \bar{β + }\bar{η}, \quad c\bar{β} = \overline{cβ} = c \cdot \bar{β}.$$

Regarding $dpr^*$ as a morphism over $pr^*$, the induced map $dpr^{-1} : T^*T^*M \rightarrow pr^*TM$ is a surjective submersion. Here the target space $pr^*TM$ is the pullback bundle of $TM$. The kernel over $β \in T^*_xM$ consists of vertical tangent vectors in $T_βT^*M$. We identify $T_βT^*M$ with $\{β\} \times T^*_xM$. Thus the kernel of $dpr^{-1}$ is identified with $pr^*T^*M$. The injection $pr^*T^*M \rightarrow T^*T^*M$ takes $(β, η) \in T^*_xM \times T^*_xM$ to the vector in $T^*_xM$, denoted by $η@β$ originating at the point $β(x)$ and parallel to $η(x)$. In terms of the prolongation structure, its expression becomes $(0_β + η)(x)$. Thus, over $T^*M$, we have a short exact sequence of vector bundles

(2.19) $pr^*T^*M \rightarrow T^*T^*M \rightarrow pr^*TM$.

Analogously, there is a short exact sequence over $TM$,

(2.20) $pr^*TM \rightarrow T^*T^*M \rightarrow pr^*TM$,

where the inclusion $pr^*TM \rightarrow T^*T^*M$ is effected via $(X, η) \mapsto (d0)_X + \bar{η}$.

We refer to (2.19) as the core sequence for $pr_{T^*M}$, and to (2.20) as the core sequence for $dpr^*$.

**Definition 2.9.** A linear vector field on $T^*M$ is a pair $(A, X), A \in \Gamma(T^*M), \ X \in \Gamma(TM)$ such that

$$\begin{array}{ccc}
T^*M & \xrightarrow{pr^*} & TT^*M \\
M & \xrightarrow{\bar{X}} & TM
\end{array}$$

is a morphism of vector bundles.

To unravel this definition, $dpr^*(A) = X$, and

$$A(β + η) = A(β) + A(η), \quad A(tβ) = t \cdot A(β), \quad ∀β, η \in \Gamma(T^*M), \quad t \in \mathbb{R}.$$

The sum of two linear vector fields is a linear vector field, and a scalar multiple of a linear vector field is a linear vector field. Given a connection in $T^*M$, the
horizontal lift of any vector field on $M$ is a linear vector field. According to ([19], Section 3.4, Corollary 3.4.3, and Theorem 3.4.5), the module of linear vector fields is a Lie algebroid, and is isomorphic to the module of derivations on $T^*M$, as well as to the module of derivations on $TM$.

With the bijective correspondence between linear vector fields on $T^*M$ and $TM$ provided via the derivation modules, we now introduce a canonical pairing between $TT^*M$ and $TTM$, thought of as prolongation bundles over $TM$. Given $\tilde{X} \in \Gamma(TTM)$, $A \in \Gamma(TT^*M)$ such that $dpr(\tilde{X}) = dpr^*(A)$, and $dpr: TTM \rightarrow TM$ is the projection. Then we can find smooth paths $X_t \in \Gamma(TM) \times \mathbb{R}$, and $\beta_t \in \Gamma(T^*M) \times \mathbb{R}$ with the property

$$\tilde{X} = \frac{d}{dt} X_t \bigg|_0, \quad A = \frac{d}{dt} \beta_t \bigg|_0,$$

and $pr(X_t) = pr^*(\beta_t)$ for all $0 \leq t \leq \epsilon$.

**Definition 2.10.** The following binary operation $TTM \times TT^*M \rightarrow C^\infty(M)$ is called the tangent pairing:

$$\langle\langle \tilde{X}, A \rangle\rangle = \frac{d}{dt} \langle X_t, \beta_t \rangle \bigg|_0.$$

In ([19], Section 3.4) it is shown to be nondegenerate. Hence every smooth section of $T^*T^*M$ possesses a tangent pairing functional representative for some suitable vector field, and $\langle\langle A, \cdot \rangle\rangle \in \Gamma(T^*T^*M), \forall A \in \Gamma(TT^*M)$.

Now we adapt infinitesimal connection theory to our bundle manifold $(T^*M, \Omega)$.

**Definition 2.11.** A Koszul connection in $(T^*M, pr^*, M)$ is a map

$$\nabla: \Gamma(TM) \times \Gamma(T^*M) \rightarrow \Gamma(T^*M), \quad (X, \beta) \mapsto \nabla_X(\beta),$$

which is bilinear and satisfies the two identities

$$\nabla f X(\beta) = f \nabla_X(\beta), \quad \nabla_X(f \beta) = f \nabla_X(\beta) + X(f) \nabla_X(\beta),$$

holding for all $X \in \Gamma(TM)$, $\beta \in \Gamma(T^*M)$, $f \in C^\infty(M)$.

It is clear that $\nabla_X$, for each $X \in \Gamma(TM)$, is a derivative endomorphism of $\Gamma(T^*M)$. Put differently, for every $X \in \Gamma(TM)$, there exists a unique element $C_X \in \Gamma(TT^*M)$. We denote the linear vector field corresponding to $\nabla_X$ by $(C_X, X)$. From these vector fields we construct a map

$$C: T^*M \times TM \rightarrow TT^*M, \quad (\beta, X) \mapsto C_X(\beta).$$

From Definition 2.11 it follows that $C$ is a right inverse to

$$(pr_{T^*M}, dpr): TT^*M \rightarrow T^*M \times TM.$$

Furthermore, $\nabla_X(\beta)$ is linear in $\beta$, and $\nabla_X$ is linear in $X$, so $C$ is an arrow-reversing morphism for the two short exact sequences ([2.19], [2.20]) simultaneously. This result is formalized in ([19], Section 5.2, Proposition 5.2.4) as stating that there is a bijective correspondence between Koszul connections and maps $C$. Therefore, from now on, we will use the term ‘connection’ to designate the map as well as its attendant Koszul connection.

Koszul connections are plentiful. Hence we specify our choice - one that is fully compatible with the symplectic structure $(M, \omega)$, and Courant algebroid structure $(TM \oplus T^*M, [\cdot, \cdot], (\cdot, \cdot))$. 


Definition 2.12. A Koszul connection in \((T^*M, \Omega)\) that satisfies the conditions of isotropy and nondegeneracy, formulated as

\[
\Omega(C(X + i_X \omega), C(Y + i_Y \omega)) = 0;
\]

\[
\Omega(C(X + i_X \omega), C((\omega^\# - \pi^\#)(Y + i_Y \omega)))
= \langle i_X \omega, \pi^\#(i_Y \omega) \rangle - \langle \omega^\#(Y), X \rangle, \quad \forall X, Y \in \Gamma(TM),
\]

is called the symplectic connection.

The symplectic connection is unique up to a symplectomorphism.

On \(TT^*M \oplus T^*T^*M\) there is a natural Courant algebroid. Therefore, the symplectic connection maps pairs of transversal Dirac structures on \(TM \oplus T^*M\) into pairs of integrable subbundles of dimension \(2k\) dual with respect to the pairing \(\langle \cdot, \cdot \rangle\).

Theorem 2.1. The symplectic connection, viewed as a map \(C : T^*M \oplus TM \to TT^*M \oplus T^*T^*M\), is a morphism of Lie bialgebroids.

Proof. On \(TT^*M \oplus T^*T^*M\), the bundle isomorphism \(\Omega^\#: \text{TM} \to T^*M\) takes vector fields into differential forms. Thus the way to utilize our connection has to be

\[
C : (A, A^\ast) \mapsto (C(A), \Omega^#(C(A^\ast)));
\]

The second algebroid would always remain totally intransitive, just like \(T^*M\). We begin by tackling the case of \((C(TM), \Omega^#(C(T^*M)))\). To prove that it is indeed a Lie bialgebroid, we invoke (19, Chapter 4, Proposition 4.3.3), according to which any smooth map \(F : M \to M'\) induces a Lie algebroid morphism \(dF : TM \to T^*M\). In this case, \(F\) is the natural embedding into the cotangent bundle: \(F : M \to T^*M\).

It is easy to see that there is an orthogonal matrix \([O]\) such that \(dF\{O\} = C\{TM\}\) in view of the nondegeneracy property of \(C\) encapsulated in (2.22). We infer \(C(TM)\) is integrable. Furthermore, it is the subbundle of horizontal vector fields of \(TT^*M\).

Now, the differential forms defined by \(\langle (C(TM), \cdot) \rangle\) are in the image of the bundle diffeomorphism induced by the canonical symplectic form:

\[
\langle (C(TM), \cdot) \rangle = \Omega^#(C(T^*M)).
\]

As for arbitrary bialgebroids, we use the fact that the Hitchin spinor group action is transitive. There exist one-parameter subgroups connecting the identity of \(Spin(2k, 2k)\) with any fixed element. In particular, given a family of Dirac structures generated by a family of closed 2-forms \(t \beta \in \Lambda^2 T^*M\), \(d \beta = 0\), written as \(\{X + i_X t \beta | X \in TM\}\), \(t \in [0, 1]\), consider its image \(C\{X + i_X t \beta\}\). Now fix a nearby subbundle, \(t < \epsilon\). We can take the word 'nearby' to mean that the projection

\[
\text{pr}^{\epsilon!} : C\{X + i_X t \beta\} \to C(TM)
\]

is a bundle diffeomorphism due to the fact that \(C(TM) \subset \text{pr}^{\epsilon!} TM\), that is, all the lifts of vector fields are horizontal in \(TT^*M\). Then \(C\{X + i_X t \beta\}\) is an integrable subbundle of \(TT^*M\). It is easy to check that the requirements of Definition 2.1. are satisfied with \(\dot{a}(t) = 1|C\{X + i_X t \beta\}\), the last mapping being the restriction of the identity of \(TT^*M\). Furthermore, by virtue of \(\text{pr}^{\epsilon!}\) being a surjective submersion for all \(t \leq \epsilon\), the pullback of the characteristic foliation of the Lie algebroid \(C(TM)\) is the characteristic foliation of the Lie algebroid \(C\{X + i_X t \beta\}\). The
Proof. Let the symplectic connection maps null Dirac structures into the Lie algebra of derivations on $\mathcal{L}$. The change of coordinates required to take the horizontal characteristic foliation into the characteristic foliation of the image of the graph Dirac subbundle is

$$x_i \mapsto x_i + t \sum_j z_{ij}(x_1, \ldots, x_{2k}) \dot{x}_j, \quad z_{ij} = -z_{ji}, \quad i, j \in \{1, \ldots, 2k\},$$

where the dotted symbols are independent fiber variables, known as ‘momentum variables’ on $T^*M$. Thus the Liouville form can be written locally as $\vartheta = \sum \dot{x}_j \, dx_j$.

To cover the interval $0 \leq t \leq 1$, we use compositions of bundle diffeomorphisms projecting $\mathcal{C}(\{X + i_\mu(t + \epsilon)\beta\})$ onto $\mathcal{C}(\{X + i_\mu t\beta\})$. However, not all $\mathcal{C}(L)$ can be projected surjectively onto $\mathcal{C}(TM)$. A refinement of the above argument is in order. Thus let $\mathcal{C}(L)$ be the image of a Dirac structure such that fiberwise

$$\dim \text{pr}^*(\mathcal{C}(L)) = 2k - l, \quad l > 0,$$

$l = l(x_1, \ldots, x_{2k})$ is an integer-valued function with jump discontinuities. Then there are $l \leq 2k$ linearly independent elements of $\mathcal{C}(L)$ satisfying

$$\ker \text{pr}^* \cap \mathcal{C}(L) = \text{span}\{\mathcal{C}(\xi_1), \ldots, \mathcal{C}(\xi_l)\}.$$

The brackets involving those elements are easy to identify:

$$[\mathcal{C}(X), \mathcal{C}(\xi_i)] = \mathcal{C}(f(x_1, \ldots, x_{2k})) \mathcal{C}(\xi_i) \subset \mathcal{C}(L),$$

and $[\mathcal{C}(\xi_i), \mathcal{C}(\xi_j)] = 0$, so that the subbundle $\mathcal{C}(L)$ is integrable.

Finally, we obtain $\mathcal{C}(L), \Omega^\#(\mathcal{C}(\omega^\#, \pi^\#L)))$. In this setting, the coboundary operator is given by $d_* = d_1 = [\cdot, \cdot]$. □

Smooth functions of the base variables are sitting pretty inside $C^\infty(T^*M)$:

**Corollary 2.1.** $\mathcal{C}(C^\infty(M), \pi) \subset C^\infty(T^*M), \Pi)$ is a subalgebra of the algebra of derivations on $T^*T^*M$.

**Lemma 2.2.** The symplectic connection maps null Dirac structures into the Lagrangian subbundles of $T^*M$.

**Proof.** Let $L_0$ be a fixed null Dirac structure on $TM \oplus T^*M$. Then we have

$$L_0 \oplus (\omega^\# - \pi^\#) L_0 = TM \oplus T^*M.$$

The complementary subbundle is a null Dirac structure as well. Their images under the symplectic connection do not intersect:

$$\mathcal{C}(L_0) \cap \mathcal{C}(\omega^\# - \pi^\# L_0) = 0.$$

Hence we can take the basis of $\mathcal{C}(L_0) \oplus \mathcal{C}(\omega^\# - \pi^\# L_0)$ as a frame of $T^*T^*M$.

The action of 2-forms producing null Dirac structures refines (2.23) as

$$x_i \mapsto x_i + f_i(x_1, \ldots, x_{2k}) \dot{x}_i, \quad i \in \{1, \ldots, 2k\}$$

with arbitrary smooth $f_i$’s. Now by applying the Darboux theorem, we can recast this change of coordinates to get the simple graph Lagrangian subbundle chart:

$$x_i \mapsto y_i + \dot{x}_i, \quad i \in \{1, \ldots, 2k\}.$$
Lemma 2.3. The symplectic connection is flat: \( \nabla^C_{[X,Y]}(\beta) = \nabla^C_X \nabla^C_Y(\beta) - \nabla^C_Y \nabla^C_X(\beta) \).

Proof. Flatness is a direct consequence of Theorem 2.1. \( \square \)

2.1.4. Cartan’s method. A priori, two smooth manifolds equipped with specific tensors (i.e., pseudometrics, almost complex structures, symplectic 2-forms) are not diffeomorphic unless some precise conditions are satisfied. To formulate those conditions, we use the notion of differential invariants. To solve the problem of finding all first-order differential invariants, Élie Cartan [4] proposed a very general technique, known today as the equivalence method of Cartan. It allows one to solve the equivalence problem:

Let \( \Psi_U = (\Psi^1_U, ..., \Psi^n_U)^T \) be a coframe (an \( n \)-tuple of smooth nonvanishing 1-forms of maximal rank) on an open set \( V \subset \mathbb{R}^n \), and let \( \dot{\psi}_U = (\psi^1_U, ..., \psi^n_U)^T \) be a coframe on \( U \subset \mathbb{R}^n \), and let \( G \subset GL(n, \mathbb{R}) \) be a prescribed linear subgroup, then find necessary and sufficient conditions that there exist a diffeomorphism \( \hat{A} : U \rightarrow V \) such that for each \( u \in U \)

\[
\hat{A}^* \Psi_U \big|_{\hat{A}(u)} = [a]_{VU}(u) \dot{\psi}_U|_u,
\]

where \( [a]_{VU}(u) \in G \).

The subgroups for the aforementioned tensors are \( SO(p, n - p, \mathbb{R}) \), \( GL(n, \mathbb{C}) \), regarded as a closed real analytic subgroup of \( GL(2n, \mathbb{R}) \), and \( Sp(n) \) respectively.

There are excellent sources, the lecture notes by Robert Bryant [2] and the book by Robert Gardner [8] to name two. Even though the question we are trying to address is local, it proves advantageous to use the global coordinate-free language of bundles and connections. Thus on a manifold \( M \) we form the coframe bundle. A coframe in this context can best be thought of as a 1-jet of a coordinate system at the point. The fiber \( F^*_x \) is just the set of all coframes. The disjoint union of all such fibers is denoted by \( F^* \) and the bundle structure is determined by the projection \( \tilde{pr}^* \) built from component projections \( pr^* : T^*M \rightarrow M \) introduced in the previous section.

\[
\tilde{pr}^* : F^* \rightarrow M
\]

via \( \tilde{pr}^*(F^*_x) = x \). The group \( GL(n, \mathbb{R}) \) acts on \( F^* \) by the rule

\[
(\text{Action}) \quad \psi \cdot [a] = [a]^{-1} \psi \quad \psi \in \Gamma(F^*), \ [a] \in GL(n, \mathbb{R}).
\]

Definition 2.13. A \( G \)-structure on \( M \) is a subset \( F^*_G \) of \( GL(n, \mathbb{R}) \) acts on \( F^* \) by the rule

\[
F^*_G(G, x) = F^*_G(G, M) \cap (pr^*)^{-1}(x).
\]

By way of unraveling the last definition, we say that a \( G \)-structure is just a subbundle of the coframe bundle (which is emphasized by the use of the parenthetical subscript) with an especially nice action of \( G \subset GL(n, \mathbb{R}) \).

Cartan’s equivalence problem can now be recast in terms of \( G \)-structures. Namely, two \( G \)-structures \( 1F^*_G \) and \( 2F^*_G \) on \( M_1 \) and \( M_2 \) respectively are said to be equivalent if there is a diffeomorphism \( \hat{A} : M_1 \rightarrow M_2 \), so that it induces a bundle isomorphism \( \hat{A}^* : 1F^*_G \rightarrow 2F^*_G \).
In a standard trivialization \( U \times G \), the Maurer-Cartan structure equations take the form:

\[
d\psi = [dO] \wedge \psi_U + [O]d\psi_U = [dO][O]^{-1} \wedge [O]\psi_U + [O]d\psi_U.
\]

Here \( \psi = [O]\psi_U \), and \([dO][O]^{-1}\) is, of course, the Maurer-Cartan matrix of right invariant forms on \( G \).

Recalling that \( \psi_U \) are basic, that is, both coefficients and differentials can be expressed in terms of coordinates on \( U \), we can rewrite the structure equations in the group-fiber representation:

\[
d\psi^i = \sum a^i_j \psi^j + \frac{1}{2} \sum \gamma^i_{jm}(x,[O]) \psi^j \wedge \psi^n.
\]

The coordinates work out so that (2.24) leads to (2.25).

\[
[dO][O]^{-1} \text{ is not preserved by every equivalence between } G\text{-structures. In order to measure its deviation from equivariance, we have to analyze (2.24).}
\]

\[
\gamma^i_{jm}(x,[O]) \in \Gamma(TM \otimes \Lambda^2 T^*M) \text{ changes in step with } [dO][O]^{-1}. \text{ Gauging the connection form } [dO][O]^{-1} \mapsto [dO'][O']^{-1} \text{ leads to the replacement } \gamma \mapsto \gamma' \text{ such that } \gamma' - \gamma = \delta([M] \otimes \psi), \text{ and } \delta \text{ is the natural linear map}
\]

\[
\mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^*
\]

defined as the composition

\[
\mathfrak{g} \otimes V^* \xrightarrow{\text{Inclusion} \otimes \text{Id}} (V \otimes V^*) \otimes V^* \xrightarrow{\text{Skewsymmetrization}} V \otimes \Lambda^2 V^*.
\]

The cokernel of \( \delta \) is extremely important. It has a special name - the intrinsic torsion of \( \mathfrak{g} \) and is customarily denoted by \( H^0.2(\mathfrak{g}) \) as a reminder that the cokernel happens to be a Spencer cohomology space. However, \( \delta \) (whence its cokernel) depends on \( \mathfrak{g} \) as well as the embedding of \( \mathfrak{g} \) into \( \mathfrak{gl}(n, \mathbb{R}) \). Also, \( \ker \delta = \mathfrak{s} \subset \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}) \) consists of all symmetric matrices in \( \mathfrak{g} \), and we have the following exact sequence:

\[
0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^* \rightarrow H^0.2(\mathfrak{g}) \rightarrow 0.
\]

Now we need to build the group action into the analysis. We view \( G \subset GL(n, \mathbb{R}) \) as the identity inclusion. Now we generalize this viewpoint to fit representation theory and let

\[
\rho : G \rightarrow \text{Aut}(V), \quad V \cong \mathbb{R}^n
\]

be the representation defining \( G \subset GL(n, \mathbb{R}) \). This induces

\[
\rho_* : T_eG \rightarrow \text{Hom}(V, V)
\]

and results in

\[
\rho_* : \mathfrak{g} \rightarrow \text{Hom}(V, V).
\]

Associated to \( \rho \) is the dual representation

\[
\rho^* : \text{Aut}(V^*) \rightarrow \text{Aut}(V^*)
\]

given explicitly by \( \rho^*([O]) = (\rho([O])^T)^{-1} \). The natural \( G \)-action on \( V \otimes \Lambda^2 V^* \) is constructed out of the individual representations:

\[
\rho \otimes \Lambda^2 \rho^* : G \rightarrow \text{Aut}(V \otimes \Lambda^2 V^*).
\]
There is also a natural $G$-action on $g \otimes V^*$, given by

$$\text{Ad} \otimes \rho^\dagger : G \rightarrow \text{Aut}(g \otimes V^*).$$

All these representations tie up beautifully with the map $\delta$:

$$\rho \otimes \Lambda^2 \rho^\dagger ([O]) \circ \delta(\cdot) = \delta(\rho_\ast \circ \text{Ad} \otimes \rho^\dagger ([O])(\cdot)),$$

which shows that the map $\delta$ is actually a mapping of $G$-modules, and (2.26) is an exact sequence of $G$-modules.

Since the quotient map onto $H^0,\,2(g)$ is a $G$-module morphism as well, we see that the composition

$$\tau_U(x, [O]) : U \times G \rightarrow H^0,\,2(g)$$

acts by matrix multiplication

$$\tau_U(x, [O'|O]) = \rho \otimes \Lambda^2 \rho^\dagger ([O'])\tau_U(x, [O])$$

and deduce that the image of a fiber of $U \times G$ over $U$ is an orbit of action of $G$ on $H^0,\,2(g)$.

**Definition 2.14.** An equivalence problem is of **first-order constant type** if the image of $\tau_U(x, [O])$ is a single orbit on $H^0,\,2(g)$.

The method of equivalence guarantees that such problems admit a solution, whereas multiple orbits typically take some ingenious ad hoc steps, and may not be solvable at all. We confine our inquiry to special cases of first-order constant type. Now we choose a fixed vector, usually a particular normal form in the image of the structure map, say

$$\tau_0 = \tau_U(x_0, [O]_0) \in \tau_U(U \times G),$$

then there would be an isotropy group of that vector:

$$G_{\tau_0} = \{[O] \in G| \rho \otimes \Lambda^2 \rho^\dagger ([O]) = \tau_0\}.$$

Since $\tau_U(U \times G)$ is an orbit, and there is only one orbit, it follows that $\tau_U(U \times G)$ is the orbit of $\tau_0$. By transitivity, it can be identified with the homogeneous space $G/G_{\tau_0}$. The structure map

$$\tau_U : U \times G \rightarrow G/G_{\tau_0}$$

has constant rank, hence $\tau_U^{-1}(\tau_0)$ is a manifold. The point $\tau_U(x, e)$ is on the orbit, hence for each $x$ there is an $[O](x) \in G$ such that

$$\rho \otimes \Lambda^2 \rho^\dagger ([O](x))\tau_U(x, e) = \tau_0.$$

Therefore

$$\tau_U^{-1}(\tau_0) = \{x, [O](x)G_{\tau_0}| x \in U\}$$

is a manifold which submerses onto $U$. A $\tau_0$-modified coframe is a section of $\tau_U^{-1}(\tau_0)$.

Should there exist an equivalence $A^*$ with $A^*(x_0, [O]_0) = (y_0, [O']_0)$, it will result in

$$\tau_0 = \tau_U(x_0, [O]_0) = \tau_V \circ A^*(x_0, [O]_0) = \tau_V(y_0, [O']_0).$$

With all this preparation, we can state
Background Theorem 2.2 (Reduction of the structure group, [8], Lecture 4). A mapping
\[ A : U \rightarrow V \]
induces a \( G \)-equivalence if and only if \( \tilde{A}^* \) induces a \( G_{\tau_0} \)-equivalence between \( \tau_0 \)-modified coframes.

The question becomes how to choose a convenient \( \tau_0 \). With an eye towards equivalences with particularly amenable structure groups, typically direct products of orthogonal ones, we further narrow down the class of problems treated. Namely, we only deal with coframes that allow \( G \)-equivariant splitting of the vector space \( V \otimes \Lambda^2 V^* \).

To find a normal form for the orbit \( \tau_U(U \times G) \) in \( H^{0,2}(g) \), the easiest is to work at the Lie algebra level. The group action can be uncovered by computing the relations
\[ d^2 \psi^i = 0 \mod \psi, \]
and using linear combinations to solve for the torsion terms of (2.25). The action splits into rotations, rescalings, and translations.

\[ d\gamma_{jm} = \gamma_{jm} \sum m o^j l - \gamma_{jm} \sum a^j o^l \]
\[ + \sum a^0 o^l \gamma_{jm} - \sum a^0 o^l \gamma_{jm} \]
\[ + \sum a^0 (\phi d^2 jm) l \gamma_{jm} \mod \psi, \]
where \( d\psi^{\phi} = \gamma_{jm} (\phi d^2 jm) \psi j \wedge \psi m + \cdots \), and \( \phi \neq i \).

In his theory of Répère Mobile, Cartan called such a normalization the first-order normalization. Despite the somewhat misleading designation, this procedure can be repeated with progressively smaller groups, yielding higher-order differential invariants.

We keep track of the iterations by indexing the resulting \( G \)-structures:
\[ F^*_-(G, U) \rightarrow F^*_+(G_1, U) \rightarrow F^*_+(G_2, U) \cdots \rightarrow F^*_+(G_m, U). \]

As it turns out, preframes can be lifted to an equivalence problem of a first-order constant type. Across the board, we would be able to normalize and reduce the structure group down, hence to select a unique coframe (on \( T^*TU \)):
\[ F^*_-(G, T_U U) \rightarrow F^*_+(G_1, T_U U) \cdots \rightarrow F^*_+(\{e\}, T_U U). \]

We use the idiosyncratic symbol \( T_U U \) to single out an open subset of \( TU \) with compact closure.

After a number of iterations, the first-order normalization will produce some \( F^*_+(G_m, U) \) with the trivial action of \( G_m \) on \( H^{0,2}(g) \). If \( G_m = \{e\} \), then \( \psi = (\psi^1, ..., \psi^n) \) no longer involve group parameters and define an invariant coframe on \( U \). If \( G \) has been reduced to a group with \( s = 0 \), then \( (\psi^1, ..., \psi^n, o^1, ..., o^l) \) defines an invariant coframe on \( U \times G \).

In either case, we have an equivalence problem, one on \( U \), the other one on \( U \times G \) with an invariant coframe, and hence equivalence problems with \( G = \{e\} \). Such problems form a special class, and the underlying coframes encode the invariants of \( \text{e-structures} \) (spaces with a specified coframe to put it simply). This has a complete solution ([8], Lecture 6).
Assume we are given two coframings \((U, \psi)\) and \((V, \Psi)\), and want to find necessary and sufficient conditions that 
\[ A^* \Psi = \psi. \]
The post-reduction structure equations are
\[ d\psi^i = \sum \gamma_{jm}^i \psi^j \wedge \psi^m \quad \text{and} \quad d\Psi^i = \sum \Gamma_{jm}^i \Psi^j \wedge \Psi^m. \]
Here \(\Gamma_{jm}^i\)'s are general torsion coefficients, not to be confused with the Christoffel symbols, although those two may coincide.

If there were an equivalence, then \(s = 0\) implies
\[ \gamma_{jm}^i = \Gamma_{jm}^i \circ A. \]

Given a coframe \(\psi = (\psi^1, ..., \psi^n)^T\), there are natural ‘covariant derivatives’ for a function \(f : U \rightarrow \mathbb{R}\), defined by
\[ df = \sum f^i \psi^i. \]
Similarly, given \(\Psi = (\Psi^1, ..., \Psi^n)^T\), and a function \(h : V \rightarrow \mathbb{R}\), we define
\[ dh = \sum h^i \Psi^i. \]
Therefore,
\[ \sum \gamma_{jm}^i = d\gamma_{jm}^i = d(\Gamma_{jm} \circ A) = A^*(d\Gamma_{jm}^i) = A^*(\sum \Gamma_{jm}^i \Psi^j) = \sum \Gamma_{jm}^i \circ A \cdot A^* \Psi^j \]
and as a result,
\[ \gamma_{jm}^i = \Gamma_{jm}^i \circ A. \]

This argument is inductive for higher covariant derivatives.

Now define
\[ F_s(\psi) \overset{\text{def}}{=} \{ \gamma_{jm}^i, \gamma_{jm}^i | l_1, \cdots, \gamma_{jm}^i | l_s-1 ; 1 \leq i, j, m, l_1, \cdots, l_{s-1} \leq n \}, \]
which we view as a lexicographically ordered set. This is a set of invariants of the \(s\)-jet of the \(e\)-structure \(\psi\).

Two natural invariants of this set may be singled out. Let
\[ r_s = \text{rank}(dF_s(\psi)), \]
where we count the dimension of the closed linear span of the differentials that occur in the ordered set \(F_s(\psi)\).

Thus \(r_s\) is an integer-valued function on \(U\). The \textit{order} of the \(e\)-structure at \(x \in U\) is the smallest \(j\) such that
\[ r_j(x) = r_{j+1}(x). \]
If \(j\) is the order of the \(e\)-structure at \(x\), then the \textit{rank} of the \(e\)-structure at \(x \in U\) is \(r_j(x)\). Note that \(0 \leq j \leq n\), the lower bound occurring when the structure tensor has constant coefficients (i.e., on a Lie group), and the upper bound occurring if and only if one invariant function is adjoined at each jet level.

An \(e\)-structure is called \textit{regular of rank} \(r_j\) at \(x\) if the rank is constant in an open
neighborhood of $x$. In this case there exist functions $\{f_1, \cdots, f_r\}$ defined in an open neighborhood of $x$ such that

$$f_1, \cdots, f_r \in F_j(\psi), \quad df_1 \wedge \cdots \wedge df_r \neq 0,$$

while any function $u \in F_j(\psi)$ satisfies

$$du \wedge df_1 \wedge \cdots \wedge df_r = 0.$$

These $f_i$'s can be extended to a local coordinate system.

Now we can cite the fundamental theorem of Cartan, disposing of the restricted equivalence problem.

**Background Theorem 2.3** (Equivalence of e-structures, [8], Lecture 6). Let $\psi$ and $\Psi$ be regular e-structures of the same rank $r_j$ and order $j$. Let $h_U : U \to \mathbb{R}^m$, and $h_V : V \to \mathbb{R}^m$ be extensions of an independent set of elements of $F_j(\psi)$ and $F_j(\Psi)$ to coordinate systems constructed from identical lexicographic choices of indices. Define

$$\sigma = h_V^{-1} \circ h_U : U \to V,$$

then necessary and sufficient conditions that there exist $\tilde{A} : U \to V$ with $\tilde{A}^* \Psi = \psi$

are that

$$F_{j+1}(\Psi) \circ \sigma = F_{j+1}(\psi)$$

as lexicographically ordered sets.

2.1.5. Preternatural rotations.

In the two sections immediately preceding this one, we described (in broad terms) a version of integrability of subbundles of $TM \oplus T^*M$, elucidated via Dirac structures, and intrinsic torsion of coframes. Now we introduce a principally new concept- designed nonintegrability of preframes. In various applications of Cartan’s equivalence method one comes across a partially reduced problem, and selects the values of intrinsic torsion coefficients based on external or aesthetic considerations. We just want a linearized version of nonintegrability afforded by the Courant bracket encoding certain metric structures on the base manifold lifted up to $TT^*M$. The discrete bundle automorphism $(\omega^\# - \pi^\#)$ has its counterpart upstairs. It reshuffles symplectic-conjugate variables. We identify $2k$ smooth sections of $TT^*M$ satisfying the following conditions:

$$\mathcal{L}_{\mathcal{K}_l} \Omega = 0, \quad l \leq 2k,$$

$$[\mathcal{K}_l, \mathcal{K}_m] = 0, \quad l, m \leq 2k,$$

such that if we have a preframe with elements $X_i + \xi_i, \ X_j + \xi_j$, their bracket being

$$[X_i + \xi_i, X_j + \xi_j] = \sum_{l=1}^{2k} h_l(X_l + \xi_l) + \sum_{l=1}^{2k} f_l(\omega^\# - \pi^\#)(X_l + \xi_l), \quad \sum f_l^2 > 0,$$

lifting this bracket onto $TT^*M$ yields

$$\mathbf{C}([[(X_i + \xi_i), (X_j + \xi_j)]) = \sum_{l=1}^{2k} \mathbf{C}(h_l) \mathbf{C}(X_l + \xi_l) + \sum_{l=1}^{2k} \mathbf{C}(f_l)[\mathcal{K}, \mathbf{C}(X_l + \xi_l)].$$
where $\mathcal{K} = \sum \mathcal{K}_m$. It is easy to see, that, owing to the symplectic connection, locally we can write $\mathcal{K}_i = \dot{x}_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial \dot{x}_i}$. Those corresponding precisely to $(\omega^\# - \pi^\#)$ are discrete, constant rotations of this kind with a single parameter set at $\frac{\pi}{2}$.

We call the rotations $\mathcal{K}_m$ ‘preternatural’ as they have no precursors among the continuous symmetries associated with the Courant algebroid. The closest one gets to a precursor is, the closed 2-forms acting on the sections as Nigel Hitchin prescribed \cite{Hitchin_1986}. Once lifted, they become the ‘natural’ pseudorotations $O_l = \dot{x}_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial \dot{x}_i}$.

In terms of torsion, the coframes gotten from the nonintegrable preframes acquire additional components.

### 2.2. Regular preframes.

As the title suggests, here we enjoy the luxuries of a symplectic manifold. The equivalence method can be applied, hence smooth local coframes abound. The preframes are naturally derived from those coframes. We find it convenient to use the standard Poisson structure to effect that derivation. Thus given a fixed coframe $\{\{\psi_U\}_i\}$ we state

**Definition 2.15.** A regular preframe on an open set $U \subset (M, \omega)$, endowed with a smooth coframe $\{\{\psi_U\}_i\}$ is a 2$k$-tuple $\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\}$.

We can easily check that a regular preframe is a preframe according to Definition 2.15. Indeed we have

$$\text{span}(\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\}, (\omega_U^\# - \pi_U^\#)\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\}) = TU \oplus T^*U,$$

and

$$\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\} \cap (\omega_U^\# - \pi_U^\#)\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\} = 0.$$  

Right away, limitations show up: the structure group of the equivalence problem being solved ought to preserve the symplectic structure. That shouldn’t come as a surprise, as our regular preframes are just 2$k$-tuples of linearly independent sections of the symplectic graph Dirac subbundle. Conversely, any preframe closed under the Courant bracket is regular. Also, $\{\{\psi_U\}_i\}$ and $\{\{\psi_U\}_i + i_{\pi_U}(\psi_U)_i\}$ are equivalent under the action of the Hitchin spinor group $\text{Spin}(TU \oplus T^*U)$.

Among regular preframes, we single out a proper subset, distinguished by its simplicity:

**Definition 2.16.** A Hamiltonian preframe on an open set $U \subset (M, \omega)$, is a regular preframe obtained from a smooth coframe of exact 1-forms, $\{\{df_i + i_{\pi_U}df_i\}\}$.

Due to the fact that $\pi_U^\# = (\omega_U^\#)^{-1}$ as bundle isomorphisms, the vector fields $i_{\pi_U}df_i$ are Hamiltonian and linearly independent in $U$. One predictable property of global Hamiltonian preframes is expressed via

**Theorem 2.2.** A diffeomorphism $F : (M, \omega) \to (M, \omega)$ is symplectic if and only if it the induced bundle map $dF \oplus dF^* : TM \oplus T^*M \to TM \oplus T^*M$ preserves the set of Hamiltonian preframes.

**Proof.** The condition that the set of global Hamiltonian preframes be nonempty is topologically extremely restrictive. In particular, $(M, \omega)$ would have to be simply connected and parallelizable. Hence any symplectic action is Hamiltonian, and follows. As for $\Leftarrow$, the action that preserves Hamiltonian preframes is Hamiltonian since it splits into block-diagonal components in a fixed trivialization of $TM \oplus T^*M$, and is symplectic a fortiori. □
The regular metric case entails some positive functions of the diagonalized metric with the line element \( ds^2 = \sum g^{i j} dx_i^2 \). Abstractly,

\[
\begin{align*}
& (2.32) \begin{cases} 
  f(g^{11})(\cos \frac{\pi}{4} dx_1 + \sin \frac{\pi}{4} \frac{\partial}{\partial x_{1+k}}); \\
  \cdots \cdots \cdots \\
  f(g^{k k})(\cos \frac{\pi}{4} dx_k + \sin \frac{\pi}{4} \frac{\partial}{\partial x_{2k}}); \\
  f(g^{k+1 k+1})(\cos \frac{\pi}{4} dx_{k+1} - \sin \frac{\pi}{4} \frac{\partial}{\partial x_{2k}}); \\
  \cdots \cdots \cdots \\
  f(g^{2 k 2k})(\cos \frac{\pi}{4} dx_{2k} - \sin \frac{\pi}{4} \frac{\partial}{\partial x_{2k}}). 
\end{cases}
\end{align*}
\]

And the complementary preframe is gotten by applying \((\omega^U_{\#} - \pi^U_{\#})\) to the above preframe. Here \( f \) is a general designation for some as yet unknown standard normal form. This is just a choice of \( 2k \) linearly independent sections of the symplectic graph Dirac subbundle described earlier. Integrability follows once we see that the same coefficient appears for the forms and fields.

2.3. Singular preframes.

We plan to represent the metric tensor coefficients with some suitable expressions that capture the essence of their pathology. One has to bear in mind that rank deficiencies are to be represented without compromising linear independence. For that reason we need the trigonometric functions.

\[
\begin{align*}
& (2.33) \begin{cases} 
  \sum_j e^{w_{1j}} (\cos(\kappa_{1j}(x_1, \ldots))dx_j + \sin(\kappa_{1j}(x_1, \ldots))\frac{\partial}{\partial x_{2k+j}}); \\
  \cdots \cdots \cdots \\
  \sum_j e^{w_{kj}} (\cos(\kappa_{kj}(x_1, \ldots))dx_j + \sin(\kappa_{kj}(x_1, \ldots))\frac{\partial}{\partial x_{2k+j}}); \\
  \sum_j e^{w_{k+1j}} (\cos(\kappa_{k+1j}(x_1, \ldots))dx_{k+j} - \sin(\kappa_{k+1j}(x_1, \ldots))\frac{\partial}{\partial x_{2k}}); \\
  \cdots \cdots \cdots \\
  \sum_j e^{w_{2kj}} (\cos(\kappa_{2kj}(x_1, \ldots))dx_j - \sin(\kappa_{2kj}(x_1, \ldots))\frac{\partial}{\partial x_{2k}}). 
\end{cases}
\end{align*}
\]

The linear independence entails \( 1 \leq j \leq k \). The complementary preframe is gotten the same way as with the regular case.

3. Regularization

Our approach to nondifferentiable metrics is based on a decomposition of sterilized versions of the coefficients into products of standard nonvanishing functions and trigonometric functions. After smoothing, and overcoming technical difficulties thereof, the latter will be used to parameterize preternatural rotations.

3.1. Encoding the metric.

From this section on, we denote by \( M^N \subset M \), respectively \( M^O \subset M \) the global locus of nondifferentiability, respectively the global locus of rank deficiency. Those are assumed to be finite collections of smooth submanifolds.

The (pseudo-)metric is a symmetric tensor, locally written as

\[
g_U = \sum g^{i j} dx_i \otimes dx_j,
\]
where $U \subset M$ is open and sufficiently small to accommodate our construction. For a fixed $U$, we have $U^N = U \cap M^N$, $U^O = U \cap M^O$, the total loci of nondifferentiability and rank deficiency. At least some $g^{ij}$ are nondifferentiable. Formally, $\exists i_0$, $j_0$ such that $g^{i_0j_0} \notin C^1(M)$. We denote their loci of nondifferentiability by $U^N_{ij} \subset U^N$. We further restrict the class of such coefficients treated below. Our requirements for the metric are:

1. The attendant Christoffel symbols and Riemann tensor coefficients in the local coordinates on $U$ are either bounded away from zero or identically zero for some special triples and quadruples of indices:

$$\left| \Gamma^m_{ij} \right|_U > C_U, \quad \left| R^m_{ij} \right|_U \equiv 0, \quad \left| R^m_{i_0j_0} \right|_U \equiv 0,$$

where $C_U$ is a uniform constant depending only on the size of $U$.

2. There ought to exist a single parameter, serving as a coordinate on the base manifold, that designates all loci of nondifferentiability of the metric. Specifically, we set $x_1$ as the blow-up parameter. Hence, each $U^N_{ij}$ consists of a finite number of smooth hypersurfaces transversal to the $x_1$-direction in $U$. A locus of higher codimension would need some additional coordinates to be specified apart from a value of $x_1$.

3. Furthermore, the metric coefficients are allowed to vanish, their loci of rank deficiency, being a subset of $M$ of a finite number of smooth hypersurfaces transversal to the $x_1$-direction as well.

4. Our fourth requirement is, the metric coefficients be explicitly given by

$$g^{ij}(x_1, \ldots) = \frac{(g^{ij})^+ + (g^{ij})^-}{(g^{ij})^+}, \quad (g^{ij})^+, (g^{ij})^- \in C^0(M).$$

However, away from their respective locus of nondifferentiability/rank deficiency we need $(g^{ij})^+ \in C^2(M \setminus U^N_{ij})$, $(g^{ij})^- \in C^2(M \setminus U^O_{ij})$.

5. Finally, we also require

$$\text{rank}[g^{ij}]_{M \setminus (U^N \cup U^O)} = \dim M,$$

where $U^N = \bigcup_{i,j} U^N_{ij}$, $U^O = \bigcup_{i,j} U^O_{ij}$. The locus of rank deficiency of $g$ must consist of at most a finite number of hypersurfaces.

Our standard normal form on the tubular neighborhood

$$((U^N \cup U^O) \times (-\epsilon, \epsilon)) \cap U$$

potentially involving both loci with $\bar{U}$ compact is

$$\left( e^{w_{ij}(\kappa_{ij})} \right)^2 \mid_{((U^N \cup U^O) \times (-\epsilon, \epsilon))} = \frac{\sin((g^{ij})^-) \sin((g^{ij})^+)}{(g^{ij})^- (g^{ij})^+}. $$

The right-hand side of (3.2) is a product of the well-known sinc functions. We adopt the conventional (in electrical engineering and digital signal processing) definition:

$$\text{sinc}(0) \overset{\text{def}}{=} \lim_{x \to x_{\text{blow}}} \frac{1}{(g^{ij})^-} \sin((g^{ij})^-) = 1,$$

where

$$|g^{ij}|_{x_{\text{blow}}} = \left| g^{ij} \right|_{x_{\text{blow}}} > 0 \quad \text{and} \quad \text{rank}[g^{ij}]_{x_{\text{blow}}} = 0.$$

|Vanishing loci depending on other coordinates are acceptable, especially those associated with noncartesian coordinate systems. They just have to have sufficiently high codimension.
The weights there break the pattern: $$\text{sinc}(0) \equiv \lim_{x \to x_0} \frac{1}{(g^{ij})^+} \sin((g^{ij})^+) = 1.$$  

With that definition in place, we delineate the relationship between the weights \( w_{ij}(x_1, \ldots, x_{2k})'s \), and the trigonometric functions, implicit in (3.2). Wherever the metric coefficient blows up or vanishes, we set

\[
\cos(\kappa_{ij})|_{U^N_{ij}} = \pm 1, \quad \cos(\kappa_{ij})|_{U^O_{ij}} = \mp 1, \quad \text{and}
\]

\[
e^{2w_{ij}}|_{U^N_{ij}} = |(g^{ij})^+|^{-1} \sin((g^{ij})^+), \quad e^{2w_{ij}}|_{U^O_{ij}} = |(g^{ij})^-|^{-1} \sin((g^{ij})^-).
\]

Outside of its loci of nondifferentiability and rank deficiency, but over some other loci, \( U^N_{uv} \cap (U^N_{ij} \cup U^O_{ij}) = \emptyset, U^O_{uv} \cap (U^N_{ij} \cup U^O_{ij}) = \emptyset \), we maintain

\[
\cos(\kappa_{ij})|_{U^N_{uv}} = \pm 1, \quad \cos(\kappa_{ij})|_{U^O_{uv}} = \mp 1.
\]

The weights there break the pattern:

\[
e^{2w_{ij}}|_{U^N_{uv}} = \left| \frac{\sin((g^{ij})^-) \sin((g^{ij})^+)}{(g^{ij})^+} \right|, \quad e^{2w_{ij}}|_{U^O_{uv}} = \left| \frac{\sin((g^{ij})^-) \sin((g^{ij})^+)}{(g^{ij})^+} \right|.
\]

On \( \partial U \setminus \text{loci of nondifferentiability and rank deficiency} \), we require

\[
\cos(\kappa_{ij}(x_1, \ldots, x_{2k})|_{\partial U \setminus (U^N \cup U^O) \times (-\epsilon, \epsilon)} = \frac{\sqrt{2}}{2};
\]

\[
\frac{\partial}{\partial x_1} \kappa_{ij}(x_1, \ldots, x_{2k})|_{\partial U \setminus (U^N \cup U^O) \times (-\epsilon, \epsilon)} = 0.
\]

Informally speaking, away from \( U^N \cup U^O \), all extant dependencies are transferred onto the weights.

The tubular neighborhood of \( U^N \cup U^O \) takes more work. The intricate blow-up behavior has to be captured consistently. To this end we use analytic properties of \( g \). Specifically, we introduce an auxiliary scalar field to represent each \( g^{ij} \); depending on the number of metric coefficients present on \( U \), the total may reach \( 2k^2 + k \).

To produce that scalar field, we utilize one second-order differential equation closely allied with the metric. The Laplace-Beltrami operator commutes with all Killing vector fields, and thus carries all metric data. It is obtained by taking the (covariant) Hessian

\[
\text{Hess} f \in \Gamma(T^*M \otimes T^*M), \quad \text{Hess} f \equiv \nabla^2 f \equiv \nabla df,
\]

and contracting it with respect to the (pseudo-)metric. Thus let \( \{X_i\} \) be a basis of \( TM \) (not necessarily induced by a coordinate system). Then the components of \( \text{Hess} f \) are \( \text{(Hess} f)_{ij} = \nabla X_i \nabla X_j f - \nabla_{\nabla X_i} X_j f \). In terms of the metric, the Laplace-Beltrami equation is

\[
\Delta_g f = \sum_{ij} g^{ij} (\text{Hess} f)_{ij} = 0.
\]
For the Minkowski pseudometric, the Laplace-Beltrami operator coincides with the familiar d’Alembertian: \( \Delta f = \Box f \). In local coordinates, this becomes

\[
\Delta g f = \sum_{ij} g^{ij} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^m_{ij} \frac{\partial f}{\partial x_m} \right) = 0.
\]

Since \( g^{ij} \), \( \Gamma^m_{ij} \) are nondifferentiable on \( U^N \), we seek a solution on \( U \) (subscripted \( e_U w_{u,v}(x_1, \ldots, x_{2k}) \) \( \sin(\kappa_{uv}(x_1, \ldots, x_{2k})) \)), modified by the exponent \( l_g(U) \), understood to be the second smallest positive integer fully damping the blowup of every summand of the Laplace-Beltrami operator:

\[
\lim_{x_1 \to x_1^{\text{low}}} \left| g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} (e_U w_{u,v}(x_1, \ldots, x_{2k}) \sin(\kappa_{uv}(x_1, \ldots, x_{2k})))^{l_g-1} \right| \leq C,
\]

\[
\lim_{x_1 \to x_1^{\text{low}}} \left| g^{ij} \Gamma^m_{ij} \frac{\partial}{\partial x_m} (e_U w_{u,v}(x_1, \ldots, x_{2k}) \sin(\kappa_{uv}(x_1, \ldots, x_{2k})))^{l_g-1} \right| \leq C,
\]

\( \forall i, j, m, u, v, \) and every value of \( x_1^{\text{low}} \) uniformly in \( U \). Therefore we take the value working in both cases, specifically \( l_g = 10 \), and the equation becomes

\[
(3.12) \quad \Delta_g (e_U w_{u,v}(x_1, \ldots, x_{2k}) \sin(\kappa_{uv}(x_1, \ldots, x_{2k})))^{10} = 0,
\]

subject to the boundary conditions \( (3.10), (3.11) \). The solutions of \( (3.12) \) are obtained by a limiting process involving series expansions of offending \( g^{ij}, \Gamma^m_{ij} \), each pair diverging at its locus of nondifferentiability, as well as a power series for the exponent:

\[
(3.13) \quad g^{ij} = \sum_{s=1}^{\infty} (a_{ij})_s, \quad \Gamma_{ij}^m = \sum_{s=1}^{\infty} (b_{ij}^m)_s, \quad 10 = \sum_{s=0}^{\infty} l^s, \quad l^s = \left( \frac{9}{10} \right)^s.
\]

Then \( (3.12) \) is approximated via partial sums:

\[
(3.14) \quad \lim_{N \to \infty} \sum_{ij} \left( \sum_{s=1}^{N} (a_{ij})_s \right) \left( \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{s=1}^{N} (b_{ij}^m)_s \frac{\partial}{\partial x_m} \right) \left( e_U w_{u,v}(x_1, \ldots, x_{2k}) \sin(\kappa_{uv}(x_1, \ldots, x_{2k})) \right)^{1 + \sum_{s=1}^{N} l^s} = 0.
\]

The limiting solutions are not unique, and generically are elements of \( C^2(U) \). One bit of information captured this way is whether the trigonometric factor \( \cos(\kappa_{uv}(x_1, \ldots, x_{2k})) \) of the solution in scrutiny is a Morse-Bott function at one (or more) of hypersurface components of \( U^N_{uv} \) or \( U^O_{uv} \).

Further treatment is obviously in order. Two objectives are as yet to be achieved: smoothing, and selecting appropriate scalar fields - both crucial for building an s-structure for application of Background Theorem \( \ref{2} \). To this end we introduce one selection criterion, a far-reaching implication of \( U^N_{uv} \cap U^O_{uv} = \emptyset \): a constraint on the partial derivatives of the solution factors. Precisely, by our definitions, \( \frac{\partial w_{u,v}}{\partial x_0} = 0 \) implies \( \frac{\partial w_{u,v}}{\partial x_{10}} = 0 \) and vice versa, whenever there are manifold variables \( x_{10} \) of which \( g^{uv} \) is independent. Now, as a consequence of \( (3.1) \), for all nonvanishing Christoffel symbols, we have

\[
(3.15) \quad \left| \frac{\partial \kappa_{uv}}{\partial x_l} \right| + \left| \frac{\partial w_{u,v}}{\partial x_l} \right| > C_U, \quad \forall l \neq 0.
\]
Contrary to one’s expectation, this does not exclude the solutions with irregular rank $\nabla(e^w w(x_1, \ldots, x_{2k})) \sin(\kappa w(x_1, \ldots, x_{2k}) u)$.

3.2. Smoothing.

Presently, we construct a correspondence between our (nonunique) solutions of the boundary value problem (3.10), (3.11), (3.12) and elements of $C^\infty(U)$. The basic technique below is due to Nash [24], modified by Gromov ([9], Section 2.3).

For a function $f : \mathbb{R}^k \to \mathbb{R}$, we introduce its $C^0$-norm by $||f||_0 = \sup_{x \in \mathbb{R}^k} |f(x)|$, and for $0 < \alpha < 1$ we set its Hölder $C^\alpha$-norm to be

$$
||f||_\alpha = \max(||f||_0, \sup_{x,w} (|w|^{-\alpha} |f(x+w) - f(x)|)),
$$

where $x$ is an arbitrary point, and $w$ runs over all nonzero vectors in the unit ball centered at the origin of $\mathbb{R}^k$. For an arbitrary $\alpha = j + \theta$, $j = 0, 1, \ldots, \theta \in [0, 1)$, we put $||f||_\alpha = ||J_f^j||_\theta$, $J_f^j$ being the jet of $f$ over $U$, and if $f \notin C^j(U)$, we set $||f||_\alpha = \infty, \forall \alpha \geq j$.

We fix a sequence of linear operators $S_i : C^0(U) \to C^\infty(U)$, $i = 0, 1, \ldots$.

**Definition 3.1.** The sequences that satisfy

Locality. Every $S_i$ does not enlarge supports of elements of the domain space by more than $\varepsilon_i = (2 + 2i)^{-1}$, that is, the value $(S_if)(x_1, \ldots, x_{2k})$ depends on that $f$ within the ball of radius $\varepsilon_i$ for all $f$.

Convergence. If $f \in C^\alpha(\mathbb{R}^k)$, $\alpha = 0, 1, \ldots$, then $S_if \to f$ as $i \to \infty$ in the usual (not fine) topology. Moreover, $C^\alpha$-convergence $f_i \to f$ implies $C^\alpha$-convergence $S_if_i \to f$.

are called **local smoothing operators**.

**Definition 3.2.** A sequence of smoothing operators $S_0, S_1, \cdots$ has Nash depth $\tilde{d}_i$, if for every compact $\tilde{U} \subset \mathbb{R}^{2k}$, there are some constants $C_{\alpha}, \alpha \in [0, \infty)$, uniformly bounded on every finite interval $[0, \alpha] \subset [0, \infty)$, such that all functions satisfy the following inequalities with the norms $|| \cdot ||_\alpha = || \cdot ||_\alpha(\tilde{U})$ for all $\alpha \in [0, \infty)$, and all $i \in \mathbb{Z}^+$ (smoothing estimates):

$$
||S_{i-1}(f)||_\alpha \leq C_{\alpha} i^{2\beta} ||f||_{\alpha-\beta} \text{ for } 0 \leq \beta \leq \alpha.
$$

$$
||(S_i - S_{i-1})(f)||_\alpha \leq C_{\alpha}(i^{-2d-1} + i^{-2\beta-1}) ||f||_{\alpha+\beta} \text{ for } -\alpha \leq \beta < \infty.
$$

$$
||S_{i-1}(f) - f||_\alpha \leq C_{\alpha}(i^{-2d} + i^{-2\beta}) ||f||_{\alpha+\beta} \text{ for } \beta \geq 0.
$$

To construct `deep smoothing’, we start with a $C^\infty$-function $S : \mathbb{R}^{2k} \to \mathbb{R}$ supported in the unit ball centered at the origin, $B_0(1) \subset \mathbb{R}^{2k}$ and for an arbitrary $f \in C^0(\mathbb{R}^{2k})$ we consider the convolution

$$
(S * f)(x_1, \ldots, x_{2k}) = \int_{\mathbb{R}^{2k}} S(y_1, \ldots, y_{2k}) f(x_1 + y_1, \ldots, x_{2k} + y_{2k}) dy_1 \cdots dy_{2k}.
$$

Then we modify our original function, now set to be

$$
S_\zeta(x_1, \ldots, x_{2k}) = \zeta^{2k} S(\zeta x_1, \ldots, \zeta x_{2k}), \forall \zeta \geq 1,
$$
and observe that $S_\zeta$ is supported within a smaller ball, $B_0(\zeta^{-1})$ to be precise, and
\[
\|S_\zeta \ast f\|_\alpha \leq \int_{\mathbb{R}^{2k}} |S_\zeta(y)||f||_\alpha dy.
\]
At this point we normalize $S_\zeta$ via
\[
\int_{\mathbb{R}^{2k}} S_\zeta(y)dy = 1.
\]
Such operators $f \mapsto S_\zeta \ast f$ converge, as $\zeta \to \infty$, in all $C^j$-topologies to the identity operator.

A normalized smoothing operator has Nash depth $\bar{d}$ whenever its kernel function is orthogonal to all homogeneous polynomials of degrees $1, \cdots, \bar{d}$. One produces such kernels out of an arbitrary normalized $S$ by taking linear combinations
\[
S_{\text{deep}} = \sum_{m=0}^{\bar{d}} a_m S_\zeta^m.
\]
The normalization condition for $S_{\text{deep}}$ amounts to the identity $\sum_{m=0}^{\bar{d}} a_m = 1$, and the depth condition is encapsulated in the equation
\[
\sum_{m=0}^{\bar{d}} a_m \zeta^{-j} = 0, \quad j = 1, \cdots, \bar{d}.
\]
All such operators satisfy the estimates of Definition 3.2. Gromov ([9], section 2.3.4) goes on to show those convolutions to be infinitely differentiable functions.

To adapt the above smoothing scheme to manifolds, we embed $U$ into $\mathbb{R}^{2k}$ isometrically, and extend local scalar fields and (potential) kernels of the smoothing operators onto the ambient space using partitions of unity. Owing to the locality condition of Definition 3.1, the resulting approximating fields would depend entirely on the data originated in $U$.

Now we prove that such an essential property of the auxiliary scalar fields as living in the kernel of the Laplace-Beltrami operator is preserved after smoothing.

**Lemma 3.1.** For every solution of the boundary value problem (3.10), (3.11), (3.12) $e_{U}^{w_{vu}(x_1, \ldots, x_{2k})} \sin(\kappa_{vu}(x_1, \ldots, x_{2k}))$ we have
\[
|\Delta_y(e_{U}^{S_{\text{deep}} w_{vu}(x_1, \ldots, x_{2k})} \sin(S \ast \kappa_{vu}(x_1, \ldots, x_{2k})))|^{10} \leq \varepsilon(\bar{d}).
\]
such that $\varepsilon(\bar{d})$ is a strictly decreasing function.

**Proof.** Blowing up of the metric coefficients significantly limits our choices of the deep smoothing operators. We select them individually for each particular coefficient, and label accordingly. Thus we define the two selection criteria:
\[
\sin(S^{vu}_{\zeta_m} \ast \kappa_{vu})|_{U^N \cup U^O} = 0,
\]
and $m$ large enough to maintain
\[
|\det \text{Hess cos}(\kappa_{vu})| > 0 \implies |\det \text{Hess cos}(S^{vu}_{\zeta_m} \ast \kappa_{vu})| > 0.
\]
Our criteria are borne out by the fact that $U^N \cup U^O$ is a set of measure zero, and Definition 3.2. All other features are subordinated. Thus stationary black hole solutions of EVE take time-independent $S_{\zeta_m}$. Symmetries are allowed so long as they do not interfere with the aforementioned selection criteria.
Inside the ball of radius $\zeta_m^{-1}$ centered at $(x_1^{\text{blow}}, x_2, \cdots, x_{2k})$, we employ the following bound:

\[(3.24) \quad ||\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu}) - \sin(\kappa_{\nu})||_0 \leq C_o(m+1)^{-\tilde{d} - 1} \max\{|\kappa_{\nu}(x_1^{\text{blow}} - \zeta_m^{-1}, x_2, \cdots, x_{2k})|, |\kappa_{\nu}(x_1^{\text{blow}} - x_2, \cdots, x_{2k})|\},\]

valid for all normalized $S_{\zeta_m}^{\nu}$ of Nash depth $\tilde{d}$ and all smooth proper hypersurfaces. In the case of higher codimension, the estimate to be used is

\[(3.25) \quad ||\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu}) - \sin(\kappa_{\nu})||_0 \leq C_o(m+1)^{-\tilde{d} - 1} ||\kappa_{\nu}||_0(B(\zeta_m^{-1})).\]

Similarly,

\[(3.26) \quad ||e^{S_{\zeta_m}^{\nu} \ast w_{\nu}} - e^{w_{\nu}}||_0 \leq C_o(m+1)^{-\tilde{d} - 1} ||e^{w_{\nu}}||_0(B(\zeta_m^{-1})).\]

To estimate the derivatives we write

\[(3.27) \quad ||\frac{\partial}{\partial x_i}(e^{S_{\zeta_m}^{\nu} \ast w_{\nu}} - e^{w_{\nu}})||_0 \leq C_o(m+1)^{-\tilde{d} - 1} ||\frac{\partial}{\partial x_i}e^{w_{\nu}}||_0(B(\zeta_m^{-1})).\]

The rest of estimates is gotten in the same way.

To prove our claim we note that

\[(3.28) \quad |\Delta_g(e^{S_{\zeta_m}^{\nu} \ast w_{\nu}} \sin(S \ast \kappa_{\nu}U))^{10}| = |\Delta_g((e_{U}^{S_{\zeta_m}^{\nu} \ast w_{\nu}} \sin(S \ast \kappa_{\nu}U))^{10} - (e_{U}^{w_{\nu}} \sin(\kappa_{\nu}U))^{10})|.|\]

From this point on, the proof becomes a tedious computation. To give a taste, we tackle one summand:

\[|g^{ij}\Gamma_{ij}^a(\frac{\partial}{\partial x_a}e^{10S_{\zeta_m}^{\nu} \ast w_{\nu}})\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{10} - (\frac{\partial}{\partial x_a}e^{10w_{\nu}})\sin(\kappa_{\nu})^{10}| = \]

\[|g^{ij}\Gamma_{ij}^a(\frac{\partial}{\partial x_a}e^{10S_{\zeta_m}^{\nu} \ast w_{\nu}})\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{10} - (\frac{\partial}{\partial x_a}e^{10S_{\zeta_m}^{\nu} \ast w_{\nu}})\sin(\kappa_{\nu})^{10} + \]

\[|g^{ij}\Gamma_{ij}^a(\frac{\partial}{\partial x_a}e^{10S_{\zeta_m}^{\nu} \ast w_{\nu}} - \frac{\partial}{\partial x_a}e^{10w_{\nu}})\sin(\kappa_{\nu})^{10} + \]

\[|g^{ij}\Gamma_{ij}^a(\frac{\partial}{\partial x_a}e^{10\sin(\kappa_{\nu})^{10} - \sin(\kappa_{\nu})^{10})})| \leq \]

\[|g^{ij}\Gamma_{ij}^a(||\frac{\partial}{\partial x_a}e^{10S_{\zeta_m}^{\nu} \ast w_{\nu}} - \frac{\partial}{\partial x_a}e^{10w_{\nu}})||_0(\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{10})| + \]

\[|\frac{\partial}{\partial x_a}e^{10w_{\nu}}||\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{10} - \sin(\kappa_{\nu})^{10})| \leq \]

\[|g^{ij}\Gamma_{ij}^a||C_0(m+1)^{-\tilde{d} - 1}||\frac{\partial}{\partial x_a}e^{10w_{\nu}}||_0|\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{10}| + \]

\[|\frac{\partial}{\partial x_a}e^{10w_{\nu}}||\sum_{b=0}^9\sin(S_{\zeta_m}^{\nu} \ast \kappa_{\nu})^{9-b} \sin(\kappa_{\nu})^{b})|C_0(m+1)^{-\tilde{d} - 1}||\kappa_{\nu}||_0).\]

□
Representatives of this set of infinitely differentiable functions can, in principle, be used as coefficients of a preframe, but, unlike the genuine solutions, they are not related via the action of local isometries. Their problem of nonuniqueness is further exacerbated by the involvement of nonunique smoothing operators. With an eye on Cartan’s method, specifically, reduction of the coframe bundle down to a unique coframe, we have to effect further transformation. Geometrically, we want to demonstrate that the preframes with coefficients obtained this way cannot occur in a smooth configuration. There must exist a factorization into preternatural rotations and regular functions.

3.3. Spectral theory of the Hill operator.

3.3.1. An overview. We list all the relevant information on the Hill operator here. Our sources (ranging from very basic to recent) are the book by Magnus and Winkler [23], a seminal paper by McKean and Trubowitz [20], and the monograph by Feldman, Knörrer and Trubowitz [7].

\( Q \) denotes the Hill operator \(-\frac{d^2}{dx^2} + q(x)\) with a fixed \( q(x) \), an infinitely differentiable function of period 1. The function \( y_1(x,\lambda) \), respectively \( y_2(x,\lambda) \), is the solution of \( Qy = \lambda y \) with \( y_1(0,\lambda) = 1 \), \( y_1'(0,\lambda) = 0 \), respectively \( y_2(0,\lambda) = 0 \), \( y_2'(0,\lambda) = 1 \). These functions allow the following integral representation [23]:

\[
\begin{align*}
(3.29) \quad & y_1(x,\lambda) = \cos(\sqrt{\lambda}x) + \int_0^\xi \frac{\sin(\sqrt{\lambda}(\xi - \eta))}{\sqrt{\lambda}} q(\eta)y_1(\eta,\lambda)d\eta, \\
(3.30) \quad & y_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \int_0^\xi \frac{\sin(\sqrt{\lambda}(\xi - \eta))}{\sqrt{\lambda}} q(\eta)y_2(\eta,\lambda)d\eta.
\end{align*}
\]

The spectrum of \( Q \) acting on the class of twice differentiable functions of period 2 is called periodic. It is a sequence of real single or double eigenvalues tending to infinity:

\[
\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 \leq \cdots \to \infty.
\]

The lowest eigenvalue \( \lambda_0 \) is simple, the eigenfunction \( f_0 \) being root-free and of period 1. Then come pairs of eigenvalues \( \lambda_{2i-1} \leq \lambda_{2i}, \quad i = 1, 2, \cdots \), equality signifying that the eigenspace is of dimension 2. Both the eigenfunctions \( f_{2i-1} \) and \( f_{2i} \) have \( i \) roots in a period \( 0 \leq x < 1 \) and are themselves of period 1 or 2 according to the parity of \( i \), i.e. being of period 1 if \( i = 2, 4, 6 \cdots \), and of period 2 if \( i = 1, 3, 5 \cdots \).

The eigenfunctions are normalized by

\[
\int_0^1 f_i^2(x)dx = 1, \quad i = 0, 1, 2, \cdots .
\]

The eigenvalues obey the estimate

\[
(3.31) \quad \lambda_{2i-1}, \lambda_{2i} = i^2\pi^2 + \int_0^1 q(x)dx + O(i^{-2}) \quad \text{as} \ i \uparrow \infty.
\]

The periodic spectrum falls into two parts: the double spectrum of pairs \( \lambda_{2i-1} = \lambda_{2i} \) and the simple spectrum comprised of distinct eigenvalues. The interval \( (\lambda_{2i-2}, \lambda_{2i-1}) \) is an interval of stability; the nomenclature is suggestive since every solution of \( Qy = \lambda y \) is bounded if \( \lambda_{2i-2} < \lambda < \lambda_{2i-1} \). The complementary intervals of instability (also called lacunae) \( (-\infty, \lambda_0], [\lambda_{2i-1} \lambda_{2i}], \quad i = 1, 2, \cdots \), behave differently:

\footnote{The adjectives ‘simple’ and ‘double’ refer to the dimension of the corresponding eigenspace.}
no solution of \( Qy = \lambda y \) is bounded for \( \lambda < \lambda_0 \) or for \( \lambda_{2i-1} < \lambda < \lambda_{2i} \). The periodic spectrum is all double except \( \lambda_0 \) if and only if the potential is constant.

An alternative way to describe the spectrum makes use of the discriminant \( \star(\lambda) \) defined as

\[
\star(\lambda) \overset{\text{def}}{=} y_1(1, \lambda) + y_2'(1, \lambda).
\]

All eigenvalues lead to \( \star(\lambda_i) = \pm 2 \) with signature +1 or -1 according to whether \( i \equiv 0, 3 \pmod{4} \) or \( i \equiv 1, 2 \pmod{4} \), therefore the periodic spectrum is just the set of roots of \( \star^2(\lambda) - 4 = 0 \). The discriminant is an entire function of order \( 1/2 \) and type \( 1 \), informally, \( \star(\lambda) \sim \cos \sqrt{\lambda} \), so \( \star(\lambda) + 2 \), respectively \( \star(\lambda) - 2 \), may be expressed as a constant multiple of the canonical product of \( \lambda_0, \lambda_3, \lambda_4, \ldots \), respectively \( \lambda_1, \lambda_2, \lambda_5, \ldots \). See [20] and references therein.

Apart from the periodic spectrum, there are the roots \( \mu_i, i = 1, 2, \ldots \) of \( y_2(1, \mu_i) = 0 \) forming the spectrum of \( Q \) acting on the class of twice differentiable functions with \( f(0) = f'(1) = 0 \). They interlace the periodic spectrum \( \lambda_{2i-1} < \mu_i < \lambda_{2i} \), and fall into two classes: the trivial roots at the double periodic eigenvalues, and the remaining nontrivial roots in the nondegenerate intervals of instability. They are collectively named the tied spectrum.

Lastly, there is the reflecting spectrum \( \nu_i, i = 0, 1, 2, \ldots \) formed by the roots of \( y_1(1, \nu) = 0 \) encountered as \( Q \) acts on the class of functions \( f \in C^2([0,1]) \) obeying \( f'(0) = f'(1) = 0 \). They are similar to the tied spectrum eigenvalues with one notable exception: there is an extra root \( \nu_0 \) in the lacuna \( (-\infty, \lambda_0] \).

This classical result is known as the coexistence theorem for the Hill operator.

**Background Theorem 3.1** ([20], Theorem 7.11). **Hill equations with trigonometric polynomial potentials cannot have finite trigonometric polynomial solutions.**

One way it can be formalized is as follows:

\[
[\frac{d^2}{dx^2} + \sum_{i} a_i \cos(ix)] \sum_{i} b_i \sin(ix) \neq \lambda \sum_{i} b_i \sin(ix) \quad \forall \lambda, \ m, N < \infty.
\]

To study the spectrum via modern functional analysis, McKean and Trubowitz [20] introduce the class of smooth functions of period of length 1 having a fixed periodic spectrum of the Hill operator: \( \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots \). We designate this isospectral class \( \Sigma \subset C^\infty(\mathbb{R}/\mathbb{Z}) \). Geometrically, it is nicely nested inside a Hilbert space where normalized eigenfunctions and their derivatives constitute an orthonormal basis. While technically not being a manifold, \( \Sigma \) possesses a tangent bundle and a normal bundle viewed as subspaces of the ambient Hilbert space. These findings are summarized in

**Background Theorem 3.2** ([20], Theorem 1, Section 8). **Let \( T \Sigma \) be the span of \( \frac{\partial}{\partial x}(f_{2i}^x)^2, i = 1, \ldots, \) in \( L_1^2 \), and let \( N \Sigma \) be the span of \( (f_{2i}^x)^2, i = 1, \ldots \) supplemented by the span of \( (f_{2i-1}^x)^2, i = 1, \ldots \), and \( (f_{2i-1}^x)f_{2i}^x \) for such double eigenvalues as may exist. Then (1) \( T \Sigma \perp N \Sigma, \) (2) \( T \Sigma \oplus N \Sigma = L_1^2, \) (3) the unit function \( 1 \) belongs to \( N \), while the functions**

\[
F : [\sqrt{2}((f_{2i}^x)^2 - 1], \sqrt{2}((f_{2i}^x)^2 - 1], -2\sqrt{2} f_{2i-1}^x f_{2i}^x, -2\sqrt{2}(2\pi)^{-1} f_{2i}^x \frac{\partial}{\partial x} f_{2i}^0
\]
form an oblique base to the annihilator $1^0$ of the unit function, meaning that any $f \in 1^0$ can be uniquely written as $f = \sum c_i F_i$, its norm

$$\sqrt{\int_0^1 |f(x)|^2 dx}$$

being comparable to $\sqrt{\sum c_i^2}$.

$\mathfrak{T}$ is (generally) homeomorphic to the infinite-dimensional torus, an arbitrary $q(x) \in \mathfrak{T}$ being uniquely determined by the sequence $\mu_i$, $i = 0, 1, 2, \cdots$ and the norming constants. The former may be regarded as coordinates on the torus. Each such sequence encodes a periodic $q$.

It is an important geometrical fact that $\mathfrak{T}$ has a Poisson bracket. $\frac{d}{dx}$ maps $N\mathfrak{T}|_f^{\mathfrak{T}}$ onto $T\mathfrak{T}$. Hence there is the infinite-dimensional bilinear operation utilizing that property given by

$$(3.33) \{ F, G \} = \int_0^1 \frac{\partial F}{\partial q} \frac{d}{dx} \frac{\partial G}{\partial q} dx.$$  

This system is completely integrable. The only dissimilarity with the classical Poisson bracket is the dimension of the normal space being larger than the dimension of the tangent space by one. Thus there is a degeneracy inherent in this operation.

The periodic spectrum, ergo $\mathfrak{T}$, is preserved by the flow with Hamiltonian $H = \star(\lambda)$, $\lambda$ fixed. The flow is defined by solving $\frac{\partial q}{\partial t} = Xq = \frac{d}{dx} \frac{\partial H}{\partial q}$. $X$ can be viewed as a tangent vector field to $\mathfrak{T}$. The flows commute just as they do in finite-dimensional classical mechanics whenever the fields preserve each other Hamiltonians ([20], Section 3, Theorem 1, Amplification 2), since with the $X$ above we get $X\star(\mu) = 0$, and the flow of $H = \star(\mu)$, say $Y$, returns the favor: $Y\star(\lambda) = 0$. This allows natural representations of one-dimensional subgroups of Lie groups with Hamiltonian action.

$\mathfrak{T}$ can be considered the real part of a complex Jacobi variety of the transcendental hyperelliptic irrationality $\sqrt{\star^2(\lambda) - 4}$. The differentials of the first kind on the attendant Riemann surface $\Sigma_C$ of genus $g = \infty$ are of the form $[20, 21]

$$(3.34) \quad d\Phi = \frac{\phi}{\sqrt{\star^2(\lambda) - 4}} d\lambda.$$  

$\phi$ belongs to the class $I^{2\frac{1}{2}}$ of integral functions of order $\frac{1}{2}$ and type at most 1 controlling its growth such that

$$(3.35) \quad \int_0^\infty |\phi(\mu)|^2 \mu^\frac{1}{2} d\mu < \infty.$$  

$I^{2\frac{1}{2}}$ is naturally a separable Hilbert space. Its dual, denoted $I^{2\frac{1}{2}*}$, consists of absolutely summable sequences $\phi^* = \{\phi_1^*, \phi_2^*, \cdots\} \in l^2_{-4}$, such that

$$(3.36) \quad \sum_{i=1}^\infty |\phi_i^*|^2 i^{-4} < \infty.$$  

The Jacobi map takes the form (modulo periods):

$$(3.37) \quad \sum_{v=0}^\infty 2 \int_{\lambda_{2v-1}}^{\lambda_{2v+1}} \frac{\phi(\mu) d\mu}{\sqrt{\star^2(\mu)}} = \sum_{j=1}^\infty \phi(\lambda_{2j}) \phi_j^*.$$
The path of integration is permitted to wind around the circle \( n \) times provided
\[ \sum n^2 v^2 < \infty. \]

3.3.2. Hyperelliptic curves. The predominant source for this subsection is the mono-
graph by Trubowitz et al. \[7\]. The objects of study are the hyperelliptic Riemann
surfaces \( T_C \), appearing in conjunction with purely simple spectra of the Hill opera-
tor, colloquially known as Hill surfaces. We do not tackle a more complic ated case
of Hill operator with mixed (single and double eigenvalue) spectrum, and the lim-
iting case of purely double spectrum does not entail hyperelliptic Riemann surfaces
at all.

From now on, \( T_C \) is an open Riemann surface of infinite genus. We assume it
possesses an infinite canonical homology basis \( A_1, B_1, A_2, B_2, \ldots \). With the Hodge
decomposition, the first Hodge-Kodaira cohomology group \( H^1_{HK}(T_C) \) consists of
smooth, closed and coclosed differential forms. It is a bona fide Hilbe rt space with
the inner product
\[ \langle \eta, \beta \rangle = \int_{T_C} \eta \wedge * \beta. \]
The differential forms living in
\[ \text{Hol}(T_C) = \{ \beta \in H^1_{HK}(T_C) | * \beta = -i \beta \} \]
are declared to be holomorphic. It follows that \( \text{Hol}(T_C) \) is a closed subspace of
\( H^1_{HK}(T_C) \) with inner product
\[ \langle \eta, \beta \rangle = \int_{T_C} \eta \wedge * \beta = i \int_{T_C} \eta \wedge \beta. \]
These forms are the backbone of the holomorphic structure on noncompact complex
varieties \( S \) of dimension one, not just Riemann surfaces. In particular, \( \text{Hol}(S) \)
nail down the holomorphic structure on pairs of complex planes identified along a
discrete set of points \((\mathbb{C} \setminus \{\lambda_i\}) \times (\mathbb{C} \setminus \{\lambda_i\}) / \sim \). Such varieties will serve as limiting
cases for Hill potentials with purely double spectrum.

Now we impose an extra condition: on \( T_C \) there exists an exhaustion function
with finite charge - a proper nonnegative Morse function \( h(z) \) that satisfies
\[ \int_{T_C} |d * dh| < \infty. \]

For each \( \beta \in \text{Hol}(T_C) \), let \( D_r(z), z \in (T_C) \) be the disk of radius \( r \) centered at \( z \).
We have a representation in terms of a local coordinate \( \zeta \):
\[ \beta|_{D_r(z)} = f(\zeta(z))d\zeta, \]
One might think of this as an analogue of the Poincare lemma for holomorphic
(possibly multivalued) functions.

Now we introduce a linear functional taking the Riemann surface into the Hilbert
space dual to \( \text{Hol}(T_C) \),
\[ \delta_{z, \zeta}(\cdot) : D_r(z) \rightarrow \text{Hol}^*(T_C) \text{ via} \]
\[ \delta_{z, \zeta}(\beta) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta(z) + re^{i\alpha})d\alpha. \]
\( \delta_{z, \zeta}(\cdot) \) is a weakly analytic map that is actually holomorphic. As it stands, the
choice of local coordinates determines the local representative \( f(\zeta) \). However, there
is a way to get rid of that dependence given by the canonical map $\kappa$. It utilizes
the Cauchy-Riemann equivalence relation. Namely, $\delta z, \zeta \sim \delta z, \zeta' \iff$ if and only if the coordinates $\zeta, \zeta'$ are both holomorphic with respect to $z$:

\begin{equation}
\kappa = \delta z, \zeta / \sim : \mathcal{T}_C \to \text{Hol}^*(\mathcal{T}_C) /_{\text{Cauchy-Riemann}} = \mathbb{P}(\text{Hol}^*(\mathcal{T}_C)).
\end{equation}

**Definition 3.3.** A Riemann surface is hyperelliptic if there is a finite subset $I \in \mathbb{P}^1(\mathbb{C})$, a discrete subset $S \in \mathbb{P}^1(\mathbb{C}) \setminus I$, and a proper holomorphic map $\tau : \mathcal{T}_C \to \mathbb{P}^1(\mathbb{C}) \setminus I$ of degree 2 that ramifies over $S$.

The map $\tau$ is called the hyperelliptic projection for $\mathcal{T}_C$. One can utilize it to construct an exhaustion function with finite charge.

The canonical map $\kappa$ factors through the following commutative diagram:

\begin{equation}
\begin{tikzcd}
\mathcal{T}_C \arrow{r}{\kappa} \arrow{rd}{\tau} & \mathbb{P}(\text{Hol}^*(\mathcal{T}_C)) \\
\mathbb{P}^1(\mathbb{C}) \setminus I
\end{tikzcd}
\end{equation}

On Riemann surfaces that admit an exhaustion function with finite charge, there exists a unique canonical basis of the Hilbert space $\text{Hol}(\mathcal{T}_C)$ such that

\begin{equation}
\oint A_i \beta_j = \delta_{ij}.
\end{equation}

**Definition 3.4.** The Riemann period matrix expressed in terms of the canonical homology cycles and the canonical basis of holomorphic forms has the $ij$-entry

\begin{equation}
\mathcal{R}^{ij} = \oint_{B_i} \beta_j.
\end{equation}

$[\mathcal{R}^{ij}]$ is symmetric, and its imaginary part $\Im [\mathcal{R}^{ij}]$ is positive-definite. There is no such luxury as a unique canonical basis on $\text{Hol}(S)$ for $S = [(\mathbb{C} \setminus \{\lambda_i\}) \times (\mathbb{C} \setminus \{\lambda_i\}))/\sim$, but since the cycles $B_i$ become real line segments, two line integrals traversed in opposite directions with square integrable functions identical on the two copies of the complex plane result in $[\mathcal{R}^{ij}](S) = [0]$.

### 3.3.3. Theta functions.

Now we tackle the question of existence and naturality for theta functions on hyperelliptic Riemann surfaces of infinite genus treated in the second article by McKean and Trubowitz [21]. Here we introduce a closed linear subspace $K \subset I^2$ containing all integral functions asymptotically dropping off fast enough to satisfy

\begin{equation}
\sum_{m=1}^{\infty} \frac{|\phi(\mu_m)|^2}{(\lambda_{2m} - \lambda_{2m-1})^2} < \infty, \quad \forall \mu_m \in [\lambda_{2m} - \lambda_{2m-1}].
\end{equation}

Such a differential is completely determined by its real periods:

\begin{equation}
A_m(\phi) = \int_{\lambda_{2m-1}}^{\lambda_{2m}} \frac{\phi(\lambda)}{\sqrt{\star^2(\lambda) - 4}} d\lambda,
\end{equation}

so that $A_m(\phi) = 0, \ m \geq 1$ implies $\phi = 0$ by interpolation. $B_m(\phi)$ are dependent on the holomorphic structure at infinity.

$A_m(\cdot)$ are naturally elements of the dual Hilbert space $I^2^*$ and they furnish an orthogonal basis. Any individual element of the $l^2$ sequence $\phi^* \in I^2$ can be
decomposed into \( \phi^* = \sum a^m_n A_m(\cdot) \). To enter the realm of \( K^* \), \( \phi^* \) ought to satisfy
\[
\sum (\phi^*_m)^2 m^2 (\lambda_{2m} - \lambda_{2m-1})^2 < \infty.
\]

Our next building block is the Hilbert space \( H[\phi] \) of differentials (not just integral functions) of the transcendental irrationality \((3.34)\). The Hilbert space norm is
\[
H[\phi] = i \int_{\mathcal{C}} d\Phi \wedge \overline{d\Phi} < \infty.
\]

Sadly, \( H[\phi] \not\subseteq I^2 \). That is why we have to clutter the proofs with an extra space. Still, we would denote the elements of its dual space \( H^*[\phi] \) by the same symbol, \( \phi^* \). It is an absolutely summable sequence of complex numbers, each pertaining to a real part cycle of some hyperelliptic Riemann surface of infinite genus. \( H[\phi] \) is endowed with an orthonormal basis \( 1_j \in K \), \((j \geq 1)\) such that
\[
A_m(1_j) = \delta^m_j, \quad \int_{\mathcal{C}} d\Phi \wedge \overline{1_j} = -2B_j(\phi).
\]

Now, by analogy with the classical case, the theta function, \( \theta \), for the Hill surface \( \mathcal{C} \), is defined for \( \phi^* = \phi_{\text{real}}^* + i\phi_{\text{im}}^* \in K^* + iH^*[\phi] \) by the formula
\[
\theta(\phi^*) = \sum e^{2\pi i \phi^* (\cdot)} e^{-\frac{\pi}{2} \phi_{\text{im}}^*(\cdot)},
\]
\( \phi = \sum m_j 1_j \), for finite sums only to ensure convergence. With this expression, the analogy with the compact Riemann surfaces is complete, as we can state
\[
\theta(\phi^* + m_j A_j(1_j)) = \theta(\phi^*), \quad \theta(\phi^* + B_j(1_j)) = e^{-2\pi i (\phi^*(1_j) + \frac{1}{2}B_j(1_j))} \theta(\phi^*).
\]

An explicit connection between the holomorphic structure on hyperelliptic Riemann surfaces of finite genus (stemming from the Hill operators with a finite number of simple eigenvalues), and the elements of the isospectral class per se had been found by Its and Matveev [15] and later generalized to encompass hyperelliptic Riemann surfaces of infinite genus by McKean and Trubowitz [21].

\[
q(\xi) = -2\frac{\partial^2}{\partial \xi^2} \log \theta(\phi^* + \xi v_1), \quad 0 \leq \xi < 1,
\]

and \( v_1 \) is expressed in terms of periods \([21]\), Section 9):
\[
v_1(\phi) = \sum_{j \geq 1} 2j \int_{\lambda_{2j-1}}^{\lambda_{2j}} d\Phi.
\]

One remarkable property of this equation is that every \( q(\xi) \) inside the isospectral class corresponds to its own unique differential. Uncertainty inherent in the winding numbers of the integration contours has been absorbed into the theta function.

### 3.3.4. Torelli theorem.

The most far-reaching result in hyperelliptic function theory that we depend on in a crucial proof is the Torelli theorem for hyperelliptic Riemann surfaces of infinite genus. Without equivocation, Torelli theorem says that the Riemann period matrix is the one and only invariant of (certain class of) hyperelliptic Riemann surfaces of infinite genus. This result is applicable to a broader class of surfaces of infinite genus, so we restate it as proven in \([7]\), Chapter 2, Section 11). We sketch the geometric hypotheses (GH1) - (GH6), although exact formulations take several pages to state fully. The decomposition into a union of fragments - a compact
one with a finite number of boundary components \(T_{com}^C\), a finite number of regular components each attached to one boundary component of \(T_{com}^C\) denoted \(T_{reg}^C\), an infinite number of closed ‘handles’ \(T_{han}^C\) is used below.

1. (GH1) Regular fragments. Informally, the closure of a regular fragment is biholomorphic to a complex plane minus an open neighborhood of a discrete set.

2. (GH2) Handles. Some restrictions are placed on the deformations of cylinders that constitute handles.

3. (GH3) Gluing \(T_{reg}^C\) and \(T_{han}^C\). It deals with the possible overlaps of two handles. Some regularity conditions are imposed.

4. (GH4) Gluing in the compact fragment. Regularity of the gluing biholomorphic maps so as not to disturb the canonical homology cycles forming the basis of \(H_1(T_{com}^C, \mathbb{Z})\) to induce an inclusion into \(H_1(T^C, \mathbb{Z})\).

5. (GH5) Estimates on the Gluing maps. The handles are to be separated, and their density at infinity is to remain bounded.

6. (GH6) The discrete set of points used to guide the attachment of handles has to be distributed in a particular way.

The first four hypotheses are topological in nature. The estimates in (GH5) control the holomorphic structure of \(T^C\).

**Background Theorem 3.3. (Torelli)** Let \(T = T_{com}^C \cup T_{reg}^C \cup T_{han}^C\) and \(T' = T'_{com}^C \cup T'_{reg}^C \cup T'_{han}^C\) be Riemann surfaces that fulfill the hypotheses (GH1)-(GH6). Denote their canonical homology bases by \(A_1, B_1, A_2, B_2, \ldots\) and \(A'_1, B'_1, A'_2, B'_2, \ldots\). Let \([R]^{ij}\), respectively \([R']^{ij}\) be the associated period matrices. If \([R]^{ij} = [R']^{ij}\) for all \(i, j \in \mathbb{Z}\), then there is a biholomorphic map \(BH : T^C \rightarrow T'_C\) and \(\epsilon \in \{\pm 1\}\) such that for all \(j \in \mathbb{Z}\)

\[
BH_\ast (A_j) = \epsilon A'_j, \quad BH_\ast (B_j) = \epsilon B'_j.
\]

Trubowitz et al. ([7], Section 12) verify that Hill surfaces satisfy the hypotheses (GH1)-(GH6). In general, not all hyperelliptic surfaces do. Another well-behaved kind of Riemann surfaces of infinite genus are complexified Fermi curves that serve as spectral curves of the two-dimensional Schrödinger operator with doubly periodic potentials discussed at length in ([7], Section 16).

**3.3.5. Regularized standard potentials.** Fix an open set \(U \subset M\). Next, consider \(\partial U\) - a smooth closed hypersurface, and via Implicit function theorem set \(x_1^{-1}(x_2, \ldots, x_{2k})\) and \(x_1^{+1}(x_2, \ldots, x_{2k})\) as \(x_1\) coordinates of the hypersurface \(\partial U\). On the space of real-valued measurable functions on \(U\), define the inner product

\[
<f, h>_U \overset{\text{def}}{=} \frac{1}{\text{vol}(U)} \int_U fh \, dx_1 \ldots dx_{2k}.
\]

The associated norm would be

\[
||f||_U \overset{\text{def}}{=} \sqrt{\frac{1}{\text{vol}(U)} \int_U f^2 \, dx_1 \ldots dx_{2k}}.
\]

This is just \(L^2(U)\). From that, we extract all measurable functions periodic in \(x_1\) with periods \(\frac{2\pi}{x_1^{-1}(x_2, \ldots, x_{2k}) - x_1^{+1}(x_2, \ldots, x_{2k})}\). We denote this closed linear subspace
$L^2([x^-_1, x^+_1] \times (U/x_1))$. It allows a partial inner product:

$$< f, h >_{x_1} \eqdef \frac{1}{x^+_1(x_1, ..., x_{2k}) - x^-_1(x_1, ..., x_{2k})} \int_{x^-_1}^{x^+_1} f h \, dx_1,$$

and a partial norm

$$|| f ||_{x_1} \eqdef \sqrt{\frac{1}{x^+_1(x_1, ..., x_{2k}) - x^-_1(x_1, ..., x_{2k})} \int_{x^-_1}^{x^+_1} f^2 \, dx_1},$$

We interpret $L^2([x^-_1, x^+_1] \times (U/x_1))$ as a family of Hilbert spaces of periodic functions parameterized by $(x_2, ..., x_{2k})$. Within that family, there is a subfamily of smooth periodic functions of $x_1$ smoothly parameterized by $(x_2, ..., x_{2k})$. We call that subfamily $C^\infty((x^-_1, x^+_1] \times (U/x_1))$. It is a closed linear subspace:

$C^\infty((x^-_1, x^+_1] \times (U/x_1)) \subseteq C^\infty(U)$.

For a fixed metric coefficient, $g_{vw}$, we run through all $\kappa_{vw}$ satisfying (3.12), and all the allowable kernels of the smoothing operator to produce all the attendant periodic potentials.

The equation below is set for $U^N_{vw} \cap U = \{ x_1 = x_1^{\text{low}} \}$, and $U^O_{vw} \cap U = \emptyset$. If there are multiple hypersurfaces, some zeroes, or both, then $\lambda_0 \mapsto \lambda_{2s}$, and to determine $s$ we count the number of roots in the product $\sin g_{vw} \sin g_{ij}^T$ as $x_1$ traverses the length of $U$ through $U^N_{ij} \cap U$ and on to $U^O_{ij} \cap U$. This procedure is in agreement with numbering of the periodic spectra in ([20], Section 1).

(3.54) \[- \frac{\partial^2}{\partial x_1^2} + \cos(x_1 T_{vw})] \sin(\kappa_{vw}) = \lambda_0 \sin(\kappa_{vw});\]

As it stands, there are two unknown functions: $\lambda_0(x_2, ..., x_{2k})$, and $T_{vw}(x_1, ..., x_{2k})$. However, (3.54) is informed by the observation made in ([20], Amplification 1 of Section 7), according to which $\nu_s = \lambda_{2s}$ if and only if the potential is an even periodic function. Thus $\cos(x_1 T_{vw})$ is a correct choice. We know that potentials of the form $\cos(x_1 T_{vw})$ have purely simple periodic spectrum for almost all $T$. Hence if $\kappa = \frac{x_1^2(x_2, ..., x_{2k})^2 - x_1^2(x_2, ..., x_{2k})}{x_1^2(x_2, ..., x_{2k})^2}$, our potential must be constant by Background Theorem 3.1. Indeed, a sine polynomial cannot be an eigenfunction of the standard cosine potential. Thus in that special case, the function $T_{vw} = c x_1$. At the other extreme, as $\kappa \rightarrow \frac{\pi}{2}$ with the onset of regularity, $\cos(x_1 T_{vw}) \rightarrow 1$, and once again we arrive at $T_{vw} = c x_1$ with $c = \frac{1}{2\pi}$. Otherwise, we obtain a bona fide trigonometric potential, and $T_{vw}(x_2, ..., x_{2k})$ becomes a nonvanishing function. The second equation is given by (3.50) modified for $C^\infty((x^-_1, x^+_1] \times (U/x_1))$:

(3.55) \[
\sin(\kappa_{vw}(\xi, x_2, ..., x_{2k})) = \frac{\sqrt{2} \cos(\sqrt{\lambda_0} \xi)}{2}
+ \frac{1}{x^+_1 - x^-_1} \int_{x^-_1}^{\xi} \frac{\sin(\sqrt{\lambda_0}(\xi - \eta))}{\sqrt{\lambda_0}} \cos(\frac{\eta}{T_{vw}}) \sin(\kappa_{vw}(\eta, x_2, ..., x_{2k})) \, d\eta.
\]

Now (3.54) in conjunction with (3.55) determines a potential and the lowest eigenvalue uniquely, though not explicitly. With the potential comes the entire periodic spectrum:
there exists at least one hypersurface from uniform infinite-dimensional tori. A sufficient condition for uniformity is that the leaves with double spectra. All the same, the typical backward and forward jumps from 0 at $\lambda$ happens to be closed due to Background Theorem 3.2, even though the dimension $T$ the range within it is possible to introduce bounded linear operators having both the domain and $\partial$ the flow of $T$ agrees with the foliation downstairs. Hence there is a linear subspace $L$ spectrum in theOur potential possesses a purely simple spectrum that is as close to a purely double $\kappa$ optimized potentials ‘s stand for the periodic eigenvalues of the constant function $q(x_1) \equiv 1$. However, the optimized potentials are still not unique, since by Borg’s theorem (3.56) $\lim_{s \to \infty} (\lambda^s_0 - \lambda_0) = 0$, only the periodic spectrum, reflecting spectrum, and the normal-izations of eigenfuctions together determine the potential uniquely. Consequently, we are compelled to consider all potentials with this spectrum. At this point we have gathered enough information to introduce our main object: the isospectral class ‘manifold’ (almost)

$$\Sigma(U) \overset{\text{def}}{=} \{ q \in C^\infty([x_1^- , x_1^+] \times (U/x_1)) \mid \lambda^q_{2s} = \lambda_{2s}, \lambda^q_{2s-1} = \lambda_{2s-1} \}.$$ Here $\lambda_{2s}, \lambda_{2s-1}$ stand for the periodic eigenvalues of the Hill operator with potential $q$, and $\lambda^q_{2s}, \lambda^q_{2s-1}$ are given by (3.56) attaining (3.57). It is naturally a fibration over the foliation of codimension 2k–1 by the integral curves of $\frac{\partial}{\partial x_1}$. The $L^2$ norm $||\cdot||_{x_1}$ agrees with the foliation downstairs. Hence there is a linear subspace $T\Sigma(U)$. It happens to be closed due to Background Theorem 3.2 even though the dimension jumps from 0 at $\lambda^q_{2s-1} = \lambda^q_{2s}$ all the way to $\infty$ at the leaves with $\lambda^q_{2s-1} < \lambda^q_{2s}$. Thus it is possible to introduce bounded linear operators having both the domain and the range within $T\Sigma(U)$. They just have to slim down to the zero operator over the leaves with double spectra. All the same, the typical $\Sigma(U)$ does not stray away from uniform infinite-dimensional tori.

### 3.3.6. Isometric invariance
Theorem 3.1. The symmetry group of the nondifferentiable (pseudo-)metric preserves the periodic spectrum of the Hill operator with optimized potentials of the form \( \cos(\frac{2\pi m}{x_1 - x}) \).

Proof. On \( C^\infty(U) \) we define rotation operators \( K_{ab} \) stemming from the (smooth) Killing vector fields on \( U \). Their respective flows \( \exp(\varepsilon K_{ab}) \) preserve the blow-up parameter just as local Killing vector fields preserve the locus of nondifferentiability. Thus in terms of brackets, \( [\frac{\partial}{\partial x_1}, K_{ab}] = 0 \). They act on \( C^\infty([x_1^-, x_1^+] \times (U/x_1)) \) and the subspace \( \{ K_{ab} f | f \in C^\infty([x_1^-, x_1^+] \times (U/x_1)); a \neq b; 1 < a, b \leq 2k \} \) closed in \( C^\infty([x_1^-, x_1^+] \times (U/x_1)) \) in the standard \( C^\infty \)-topology. Via Background Theorem 3.2 adapted to \( L^2([x_1^-, x_1^+] \times (U/x_1)) \), we have for some \( s \in \mathbb{Z} \) (possibly after a permutation of basis eigenfunctions)

\[
(3.60) \quad [1 + \varepsilon K_{ab} + O(\varepsilon^2)] f_{2s}^2 = h f_{2s}^2 + \frac{\partial}{\partial x_1} f_{2s}^2 + O(\varepsilon^2);
\]

\[
(3.61) \quad [1 + \varepsilon K_{ab} + O(\varepsilon^2)] \frac{\partial}{\partial x_1} f_{2s}^2 = h \frac{\partial}{\partial x_1} f_{2s}^2 + \hat{z} f_{2s}^2 + O(\varepsilon^2).
\]

Here \( h, \hat{z}, \hat{\varepsilon} \) are smooth functions of \( \{ x_2, \ldots, x_{2k} \} \). If \( z = \hat{z} = 0 \) for all pairs \( a \neq b \), we are done as \( \exp(\varepsilon K_{ab})(T\Sigma \cos(\frac{\varepsilon}{m})) \subset T\Sigma \cos(\frac{2\pi}{m}) \), \( \exp(\varepsilon K_{ab})(N\Sigma \cos(\frac{\varepsilon}{m})) \subset N\Sigma \cos(\frac{2\pi}{m}) \). Otherwise, \( \varepsilon K_{ab} \frac{\partial}{\partial x_1} f_{2s}^2 = \hat{z} f_{2s}^2 \) on an open subset of \( U \). Hence \( z \neq 0, \hat{z} \neq 0 \) since \( K_{ab} \) is a rotation. Therefore we obtain

\[
(3.62) \quad \hat{z} f_{2s}^2 = \varepsilon K_{ab} \frac{\partial}{\partial x_1} f_{2s}^2 = \frac{\partial}{\partial x_1} \varepsilon K_{ab} f_{2s}^2 = \frac{\partial}{\partial x_1} z \frac{\partial}{\partial x_1} f_{2s}^2.
\]

The resulting ODE is

\[
(3.63) \quad \frac{\partial^2}{\partial x_1^2} f_{2s}^2 = \frac{\hat{z}}{\hat{\varepsilon}} f_{2s}^2.
\]

Now if \( \frac{\hat{z}}{\hat{\varepsilon}} > 0 \), the solutions of (3.63) would be unbounded, so our only choice is to force \( \frac{\hat{z}}{\hat{\varepsilon}} < 0 \). And this is a regular harmonic oscillator. Further constrains whittle the coefficients down to

\[
(3.64) \quad \sqrt{\frac{\hat{z}}{\hat{\varepsilon}}} = \frac{2\pi m}{x_1^+ - x_1}, \quad m \in \mathbb{Z}.
\]

Thus the only possible solutions of (3.63) are linear combinations of

\[
(3.65) \quad f_{2s}^2 = \cos(\frac{2\pi m}{x_1^+ - x_1} x_1), \quad f_{2s}^2 = \sin(\frac{2\pi m}{x_1^+ - x_1} x_1).
\]

But on the open subset \( \{ 0 < x_1 < \frac{x_1^+ - x_1}{4m} \} \) their square roots

\[
(3.66) \quad \sqrt{1 - \sin^2(\frac{2\pi m}{x_1^+ - x_1} x_1)}, \quad \sqrt{1 - \cos^2(\frac{2\pi m}{x_1^+ - x_1} x_1)}.
\]

\(^5\)The last qualifying hypothesis is unnecessary, as the gist of Theorem 3.1 holds for all optimized potentials, but that is all we need in the sequel, and the proof is streamlined as a consequence.
are not the eigenfunctions of the original Hill operator. We have reached a contradiction assuming that the flow of $K_{ab}$ shifts the periodic spectrum. \hfill \Box

To recapitulate the conclusion of Theorem 3.1, given a potential of the form $\cos(\frac{\theta}{T})$ obtained via the Laplace-Beltrami $\mapsto$ Nash-Gromov $\mapsto$ Hill equation procedure, and further selected via (3.57), its simple periodic spectrum is comprised of analytic functions of elementary symmetric polynomials in the $(x_2, \ldots, x_{2k})$ variables.

Having established a general pattern for the action of rotation operators, we proceed to delineate their modus operandi:

**Theorem 3.2.** There exists a faithful representation of $\exp tK_{ab}$ on the infinite-dimensional isospectral class $\mathfrak{I}(U)$ by the Hamiltonian flows constructed by McKean and Trubowitz.

**Proof.** By Theorem 3.1, $K_{ab}q \subset T\mathfrak{I}_q(U)$. According to Background Theorem 3.2, its tangent space is spanned by $\frac{\partial}{\partial x_1}f_{2s}$, $s \in \mathbb{Z}$, $s \geq 1$. Therefore, we formally equate

$$K_{ab}q(x_1, \ldots, x_{2k}) = \sum_s u_{2s}(x_2, \ldots, x_{2k}) \frac{\partial}{\partial x_1}f_{2s}. \tag{3.67}$$

$u_{2s}(x_2, \ldots, x_{2k}) \neq 0$ as $s \to \infty$ generically, and the question of convergence ought to be addressed. Even though $\exp tK_{ab}q \in C^\infty(U)$, that fact alone does not guarantee the existence of a decomposition into orthonormal tangent vectors. To compound our predicament, $K_{ab}$ is not a self-adjoint operator. To show convergence, we embed $U \hookrightarrow \mathbb{R}^{2k}$ isometrically, and consider some appropriate Sobolev space of compactly supported functions. Clearly, the image of $C^\infty(U)$ belongs inside. That allows for the Fourier transforms $\hat{F}(q)$ to be employed. Then by Parseval’s formula, as well as by the basic properties of differential operators we have (here $\Upsilon(K_{ab})$ denotes a unitary representation of isometries in the frequency domain realized by Hilbert-Schmidt operators):

$$||q||_{\text{Sobolev}} = ||F(q)||_{\text{Sobolev}} \quad \text{(by Parseval’s formula)}$$

$$= ||\Upsilon(K_{ab})F(q)||_{\text{Sobolev}} = ||F(K_{ab}q)||_{\text{Sobolev}} \quad \text{(by unitarity)}$$

$$= ||K_{ab}q||_{\text{Sobolev}} \quad \text{(by Parseval’s formula)}. \tag{3.68}$$

At this point we invoke the Sobolev embedding theorem ([13], Chapter 2, Section 2.3) to claim that $||K_{ab}q||_U \leq c||K_{ab}q||_{\text{Sobolev}}$. The norm on the left is the Hilbert space norm on $U$. That bound implies $tK_{ab}$, possibly after normalization, is a generator of a one-parameter contraction semigroup on $C^\infty([x_1^-, x_1^+) \times (U/x_1)) \subseteq C^\infty(U)$, provided the normalization constant $c_K$ is so chosen as to satisfy $\max_{U/x_1}||c_K K_{ab}q||_{x_1} \leq ||K_{ab}q||_U$. Then $u_{2s}^2 \leq c a 2^{-s}$ for every $s > s^K$ in (3.67) uniformly in $U$.

Now by ([20], Amplification 1 of Section 14), we have

$$Xq = \sum_{s=1}^\infty (-\frac{\partial}{\partial \lambda} \star)(\lambda_{2s})q_{2s} \frac{\partial}{\partial x_1}f_{2s}. \tag{3.69}$$

for all $q \in C^\infty([x_1^-, x_1^+) \times (U/x_1))$. By transitivity of the flows on $\mathfrak{I}(U)$, every such decomposition is a Hamiltonian flow, and to produce the desired representation we just identify $q_{2s}(x_2, \ldots, x_{2k}) = u_{2s}(x_2, \ldots, x_{2k})(-\frac{\partial}{\partial \lambda} \star)^{-1}(\lambda_{2s}). \hfill \Box
The upshot of the last result is that given a one-parameter subgroup of the isometry group, denoted by \( \exp tK_{ab} \), acting linearly on the coordinates (except \( x_1 \)), we have the following diagram (its commutativity is a direct consequence of the spectrum being fixed by the limiting process (3.57)):

\[
y_{1,2}(q, \lambda_\ast) \xrightarrow{\text{Hill operator}} y_{1,2}(\exp tK_{ab}q, \lambda_\ast)
\]

The action on the potentials is (in spite of notation) by Hamiltonian flows. This paves the way for Cartan’s method to go through, specifically, it facilitates reduction to a unique coframe. We were compelled to introduce \( T(\U) \) as a way to accommodate the regularized metric coefficients, their multitude stemming from the Laplace-Beltrami equation and Nash-Gromov smoothing. That may necessitate a prolongation.

### 3.3.7. Invariant Riemann period matrix.

Our last step is to repackage the information pertaining to the Laplace-Beltrami \( \rightarrow \) Nash-Gromov \( \rightarrow \) Hill equation procedure into a more appealing device. As the periodic spectrum is comprised of analytic functions of elementary symmetric polynomials of \((x_2, \ldots, x_{2k})\), one would expect the same to hold true for the Riemann period matrix. However, the difficulty here is to establish preservation of holomorphic structures at each point of \( U/x_1 \). They stem from hyperelliptic Riemann surfaces of infinite genus over the (tubular neighborhood of) locus of nondifferentiability, possibly comingling with adjoined complex planes identified along purely double eigenvalues, corresponding to the constant potential. In general, the fibration of Riemann surfaces over \( U/x_1 \), denoted \( \mathcal{T}_\mathbb{C}(U) \), would contain those adjoined complex planes:

\[
[(\mathbb{C}\setminus\{\lambda_i\}) \times (\mathbb{C}\setminus\{\lambda_i\})]/\sim \subset \mathcal{T}_\mathbb{C}(U).
\]

The period maps are still well-defined for those planes, with all the periods shrinking to zero. However the Hilbert spaces of holomorphic forms get depleted due to the reduction in the number of zeroes available to holomorphic forms. Nevertheless, we state

**Lemma 3.2.** The Riemann period matrices of \( \mathcal{T}_\mathbb{C}(U) \) corresponding to the periodic spectra of the Hill potentials (3.54), (3.55), (3.57) are local metric invariants.

**Proof.** The action of \( \exp tK_{ab} \) on individual infinite-dimensional tori via the Hamiltonian flows established by Theorem 3.2 trivially extends to the case where the torus collapses, \((\lambda_{2i} - \lambda_{2i-1}) \to 0\), and, as a result, \( \dim \mathcal{T}_{\mathbb{C}} = 0 \), and the action becomes trivial as there is literally no direction to flow in, and this ‘inaction’ persists on \([(\mathbb{C}\setminus\{\lambda_i\}) \times (\mathbb{C}\setminus\{\lambda_i\})]/\sim \).

On an individual proper infinite-dimensional torus, the action extends to its hyperelliptic Riemann surface via the Its-Matveev formula (3.52):

\[
\exp tK_{ab}\phi(q) = -2\frac{\partial^2}{\partial \xi^2} \log \theta(\exp tK_{ab}\phi^* + \xi(\exp tK_{ab}\nu_1)), \quad 0 \leq \xi < 1,
\]
\[ \exp tK_{ab}v_1 = \sum_{i}^{\infty} \int_{\lambda_{2i-1}}^{\lambda_{2i}} \frac{\exp tK_{ab}\phi}{\sqrt{\omega'(\lambda)} - 4} \, d\lambda, \]

and \( \exp tK_{ab}\phi^* \to H^*[\phi] \) is defined via the duality pairing:

\[ \sum_{i}^{\infty} \phi(\lambda_{2i}) \exp tK_{ab}m_iA_i \overset{\text{def}}{=} \sum_{i}^{\infty} m_iA_i(\exp tK_{ab}\phi)(\lambda_{2i}). \]

Since the Hilbert space of differentials is complete, \( \exp tK_{ab} \), acting via Hilbert-Schmidt integral operators, just reshuffles them, and the underlying holomorphic structure remains intact.

The only property left to establish is that the evolution of holomorphic structures from hyperelliptic Riemann surfaces \((g = \infty)\) to the conjoined complex planes \((\mathbb{C}\{\lambda_i\} \times \mathbb{C}\{\lambda_i\})/\sim\) is smooth despite the genus jumping from infinity to zero. To overcome the genus problem, we resort to working with universal 2-fold covers in place of Riemann surfaces, and generalize the map \( N \) of (3.46) to encompass all the Riemann surfaces as well as the conjoined complex planes.

\[ \begin{array}{ccc}
\mathbb{P}^1(\mathbb{C}^{2k-1}) \setminus I(U) & \xrightarrow{N(U)} & \mathbb{P}(\text{Hol}^*(\mathcal{F}_C(U))) \\
\uparrow & & \downarrow \\
\mathbb{P}^1(\mathbb{C}^{2k-1}) \setminus I(U) & \xrightarrow{\text{Hol}^*(\mathcal{F}_C(U))} & \mathbb{P}^1(\mathbb{C}^{2k-1}) \setminus I(U)
\end{array} \]

All elements in \([\phi^*] \in \mathbb{P}(\text{Hol}^*(\mathcal{F}))\) share the same zeroes. The evolution of the covering spaces embedded into \( \mathbb{P}^1(\mathbb{C}^{2k-1}) \setminus I(U) \) is smooth, and by the universal property, \( N(U) \) factors through. Finally, the quotient map \( \text{Hol}^*(\mathcal{F}_C(U)) \to \mathbb{P}(\mathcal{F}_C(U)) \) retains that smoothness.

To summarize, we proved the following equalities:

\[ \left[ [\mathcal{R}^{ij}]_{\beta\gamma} \right](x_2, \ldots, x_{2k}) = \exp tK_{ab}\left[ [\mathcal{R}^{ij}]_{\beta\gamma} \right](x_2, \ldots, x_{2k}) \]

\[ = \left[ [\mathcal{R}^{ij}]_{\beta\gamma} \right](\exp tK_{ab}(x_2, \ldots, x_{2k})). \]
3.4. The sheaf $\Xi(U, ds^2)$.

Given the following data: an open set $U \subset M$, endowed with closed loci of non-differentiability $\{U^N_{ij}\}$, and rank deficiency $\{U^O_{ij}\}$, a line element $ds^2$ corresponding to a fixed metric $g$ satisfying our requirements listed in Section 3.1, and the attendant Riemann period matrices $[[R^\gamma_{ij}]_{\beta\gamma}]$, we define $\Xi(U, ds^2) \subset \Gamma(TU \oplus T^*U)$ to be the sheaf of smooth 2-tuples of sections of the standard Courant algebroid that generate that particular set of Riemann period matrices on $U$. For instance, for $A_{ij} dx_j - B_{ij} \partial / \partial x_j + k$, such that $A_{ij}|_{U^N_{ij}} = 0$, we set (with appropriate boundary values)

$$\frac{A_{ij}}{\sqrt{A_{ij}^2 + B_{ij}^2}} = \sin \kappa_{ij}, \quad e^{2w_{ij}} = A_{ij}^2 + B_{ij}^2,$$

and use $\sin \kappa_{ij}$ to extract a Hill potential.

For every such $k$-tuple, its image under $(\omega^\#_{U} - \pi^\#_{U})$ would also be admissible. Together they comprise a basis of $(TU \oplus T^*U)$. The regular case is trivially included as the zero Riemann period matrix could be a part of the data set and $\sin \kappa_{ij} \equiv \frac{\sqrt{2}}{2}$.

All these local elements share the same ‘maternal’ $\Xi(U)$. On the overlaps, the transition functions give rise to conformal transformations $F_{ij} : \Xi(U_i \cap U_j) \rightarrow \Xi(U_j \cap U_i)$. One particular feature of our sheaf deserves mentioning. Namely, it is not microflexible in the sense of Gromov ([9], Section 2.2).

3.5. The algebroid $\text{Aut}(\Xi(U, ds^2))$.

Our sheaf is associated with the fixed line element $ds^2$, described by the set of Riemann period matrices $[[R^\gamma_{ij}]_{\beta\gamma}](x_2, ..., x_{2k})$. This set is an invariant of the direct product of infinite-dimensional tori $\bigotimes_{(i,j)}(ds^2) \Xi_j(U)$ parameterized by the base variables. Hence the symmetries there descend onto $\Xi(U, ds^2)$. Precisely,

$$\text{Aut}(\Xi(U, ds^2)) = \bigotimes_{(i,j)}(ds^2) j_{ij}(t_{ab} K_{ab})$$

can be delineated as follows: for each family of eigenfunctions obtained via the commutative diagram (3.70), we compute its $t_{ab}$-dependent Hamiltonian vector field $H_{\exp t_{ab} K_{ab}(e^{w_{ij} \cos \kappa_{ij}})}$ encoding auxiliary scalar fields in congruence with the original action by Hamiltonian flows $\cos(\exp t_{ab} K_{ab} \kappa_{ij})$ of McKean and Trubowitz. This congruence is based on the Laplace-Beltrami equation:

$$\Delta_g(e^{\exp t_{ab} K_{ab} w_{uv}} \sin(\exp t_{ab} K_{ab} \kappa_{uv})) = 0, \quad \forall t_{ab}.$$  

Then the vector field $j_{ij}(t_{ab} K_{ab})$ solves

$$[j_{ij}(t_{ab} K_{ab}), H_{e^{w_{ij} \cos \kappa_{ij}}}] = \frac{\partial}{\partial t_{ab}} H_{\exp t_{ab} K_{ab}(e^{w_{ij} \cos \kappa_{ij}})}.$$  

---

6We prefer to specify the line element as the metric is essentially an equivalence class of expressions defined up to a diffeomorphism, whereas the line element is less elusive.
Thus $\bigotimes_{(i,j)}(ds^2) J_{ij}(t_{ab} K_{ab})$ is a local subalgebra of $\Gamma(\mathbb{R}^{\dim(K_{ab})} \times TM)$. Its action on $\Psi_i = \sum_{j=1} (e^{w_{ij}} \cos \kappa_{ij} dx_j + e^{w_{ij}} \sin \kappa_{ij} dx_{2k+j})$, $\Psi_i \in \Xi(U, ds^2)$ is

$$\sum_{j=1} \exp t_{ab} K_{ab}(e^{w_{ij}} \cos \kappa_{ij}) dx_j + \exp t_{ab} K_{ab}(e^{w_{ij}} \sin \kappa_{ij}) dx_{2k+j}.$$

We have traded nondifferentiability/rank deficiency of the metric for families of solutions. That is the long and short of our regularization procedure. Their totality has now to be preserved in order for us to be able to reverse-engineer the original metric.

4. Equivalence

From this section on, we relabel independent variables on $T_U U \subset TT^{*} M$ to simplify notation. Set $\dot{x}_j = x_j + 2k, j \leq 2k$. The canonical symplectic form is written as $\Omega = \sum dx_j \wedge dx_{j+2k}$.

4.1. Lifting of preframes to $T^{*} TM$.

Now is the moment to take full advantage of the symplectic connection introduced in subsection 2.1.3. Given a preframe and its complement under $(\omega^u_U - \pi^u_U)$, we produce a smooth frame on $T_U U \subset TT^{*} M$ via $C$. To produce an equivalent coframe, we invoke the symplectomorphism of Abraham and Marsden (1, chapter 2):

$$\alpha : TT^{*} M \rightarrow T^{*} TM.$$

From now on, by a slight misnomer, we will tacitly assume $T_U U \subset T^{*} TM$.

Having set up the coframes, we state our objective: to formulate the necessary and sufficient conditions for the existence of local diffeomorphisms (elements of the transitive pseudogroup of diffeomorphisms $Diff T_U U \subset Diff T^{*} M$) that preserve the Riemann period matrices derived from the given (nondifferentiable) coefficients of the metric tensor on $M$.

The equivalence problem we are about to pose has to encompass both the regular metric case (2.32), and the nondifferentiable case (2.33). Thus, the structure group has to be a closed subgroup of $SU(2k) = Sp(2k) \cap SO(4k)$ in keeping with the lift provided by the symplectic connection. The freedom afforded by the choice between the preframe and its complement (and, presumably, other $2k$-tuples in between) has to be reflected in that subgroup. However, further reduction is necessary as all the off-diagonal metric coefficients are in place to manifest various rank deficiencies, and, consequently, rotations among the base variables ($x_i, x_j, i, j \leq 2k$) are out. What is left are the rotations of symplectic conjugate pairs ($x_i, x_{i+2k}, i \leq 2k$). Thus, necessarily, the Jacobians of our diffeomorphisms ought to belong to

$$(4.1) \quad \left( \bigoplus_{i=1}^{2k} SO_i(2, \mathbb{R}) \right) \times T_U U.$$

Heuristically speaking, this is the smallest subgroup of $Diff(T_U U) \subset Diff(T^{*} M)$ containing the full set of local preternatural rotations.
4.2. Prolongation.

Having delineated the structure group, we tackle the inherent nonuniqueness of the regularized preframes. Evidently, (4.1) is not spacious enough to furnish a representation, we identify the fiber variables on $TG \cong \mathfrak{g}$ with the fiber variables on $T_UU$. Consequently, our construction can only be realized for subgroups of the orthogonal group $G \subset SO(2k)$, such that $\dim R \mathfrak{g} \leq 2k - 1$. One fiber dimension (locally parameterized by $x_{2k+1}$) has already been utilized as symplectic conjugate of the singularity parameter. The most interesting case of $M = \mathbb{R}^{3+1}$ endowed with the reduced rotational group, including the Kerr spacetime and a large class of gravitational collapse scenarios, is, fortuitously, covered.

We identify

$$\exp t_{ab}K_{ab}(e^{w_{ij}} \cos \kappa_{ij}) \leftrightarrow \exp x_{2k+1+\sigma(a,b)}K_{ab}(e^{w_{ij}} \cos \kappa_{ij})$$

where the basis of the Lie algebra, consisting of anti-Hermitian matrices, is labeled via $1 \leq a < b$, and $\sigma$ is a bijection from the set of pairs of indices $(a, b)$ into natural numbers, $\sigma(a, b) \leq 2k - 1$. With the aid of Lemma 3.2 we ensure that the prolonged Hill eigenfunctions $\tilde{\kappa}_{ij}(x_1, \ldots, x_{2k}, x_{2k+1}, \ldots, x_{4k})$ generate the same Riemann period matrices. This ansatz is nothing more than an application of the third fundamental theorem of Lee. Compactness of $G$ ensures locality, i.e. $-c \leq x_{2k+1+\sigma(a,b)} \leq c$ covers all possible eigenfunctions of the Hill operator with optimized potentials.

For each $(2k-1)$-tuple $(x_{2k+2}, \ldots, x_{4k}) \in T_UU$, we let $\tilde{\omega}_{ij}(x_1, \ldots, x_{2k}, x_{2k+2}, \ldots, x_{4k})$ be the corresponding weight, stemming from the solutions of (5.12), subsequently smoothed.

4.3. Free scalar fields.

With the advent of prolonged auxiliary fields, we now need to make certain they are amenable to the method of equivalence’s requirements listed in the hypothesis of Background Theorem 2.3. More specifically, their partial derivatives (those relevant for geometric features, curvature above all) ought to remain independent throughout $T_UU$ in order for the desired $e$-structures to exist.

The fields with independent partial derivatives will be found via $h$-principle for one partial differential relation, known as the freedom relation (see 26, Section 1.1.4). Here is a more or less detailed outline.

Consider a smooth fibration $p : X \to V$, and let $J^1X$ be the bundle of germs of smooth sections $f : V \to X$. For instance, $J^1X$ consists of linear maps $L : T_v(V) \to T_x(X)$ for all $x \in X$ and $v = p(x) \in V$ such that $D_p \circ L = \text{Id} : T_v(V) \to T_v(V)$. Here $D_p : T(X) \to T(V)$ is the induced map of tangent bundles.

Given a $C^2$-map $f : V \to \mathbb{R}^c$ defined locally in coordinates $x = (x_1, \ldots, x_m)$ in the vicinity of $x \in V$, we denote by $T^2_f(V, x) \subset T_y(\mathbb{R}^c) = \mathbb{R}^c$, $y = f(x)$ the subspace spanned by the vectors

$$\frac{\partial f}{\partial x_i}(x), \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \quad 1 \leq i, j \leq m.$$

This subspace is called the (second) osculating space of $f$. The first osculating space is just $D_f(T_x(V))$. The dimension of $T^2_f(V, x)$ can vary between 0 and $\frac{1}{2}m(m + 3)$.
f is called free if \( \dim T^2_0(V,x) = \frac{1}{2}m(m+3), \ \forall x \in V. \) The freedom relation 
\( \varphi \subset J^2X \) for \( X = V \times \mathbb{R}^c \rightarrow V \) could either be an empty set if \( c < \frac{1}{2}m(m+3), \) or
an open dense subset in \( J^2X \) invariant under the natural action of diffeomorphisms of \( V \) and affine transformations of \( \mathbb{R}^c. \) Furthermore, the action of \( \text{Diff}(V) \times \text{Aff}(\mathbb{R}^c) \)
is transitive on \( \varphi. \) In fact, \( \varphi \) is the only proper open subset enjoying these properties.

The simplest example of a free map \( f : \mathbb{R}^m \rightarrow \mathbb{R}^c \) is given by the \( c \) monomials
\( x_i, x_ix_j \) on \( \mathbb{R}^m, \) where \( 1 \leq i \leq j \leq m. \) For closed manifolds, free maps are rare. The only known natural ones involve embeddings/immersions of spheres and real projective spaces into \( \mathbb{R}^c. \)

The homotopy principle (in the context of free maps) asseverates the existence of a smooth homotopy between an arbitrary twice differentiable one and a free one provided some additional conditions are met. For open manifolds, serendipously, \( f : V \rightarrow \mathbb{R}^c \) can be homotoped to a free map at the critical dimension \( c = \frac{1}{2}m(m+3) \) (see [9], Section 2.2.2 for the proof). These results have been generalized to higher order derivatives, as well as to covariant derivatives, but for our purposes, this will suffice.

We regard \( \hat{w}_{\nu\mu}(x_1, \ldots, x_{2k}, x_{2k+2}, \ldots, x_{4k}), \hat{\kappa}_{\nu\mu}(x_1, \ldots, x_{2k}, x_{2k+2}, \ldots, x_{4k}) \) as maps \( \hat{w}_{\nu\mu}, \hat{\kappa}_{\nu\mu} : T_U U \rightarrow \mathbb{R}^c, \) \( c_{\nu\mu} = \frac{1}{2}(4k - 1 - \#(l_0))(4k + 2 - \#(l_0)), \) where \( \#(l_0) \) denotes an enumeration of base variables informed by the number of identically vanishing Christoffel symbols and Riemann curvature tensor coefficients in (3.1), and their components being the partial derivatives varying over \( T_U U. \) To be punctilious, their range is a subbundle of the second osculating plane bundle, but since \( T_U U \) is contractible, we might as well take the aforementioned simplistic view. Our objective is to effect a smooth homotopy of (partial derivatives of) \( \hat{w}_{\nu\mu}^t, \hat{\kappa}_{\nu\mu}^t, \) \( t \in [0,1] \)
satisfying

\[
\hat{w}_{\nu\mu}^0 = \hat{w}_{\nu\mu}, \quad \hat{\kappa}_{\nu\mu}^0 = \hat{\kappa}_{\nu\mu}, \quad \hat{w}_{\nu\mu}^1, \hat{\kappa}_{\nu\mu}^1 \in \varphi_{\nu\mu}.
\]

Owing to \( \varphi \) being dense in \( J^2T_U U, \) we can find free \( \hat{w}_{\nu\mu}^1, \hat{\kappa}_{\nu\mu}^1 \) arbitrarily close to the original ones. In the sequel we use the same notation for the free weights and kappas. With such functions, the (generic) auxiliary scalar fields \( e^{\hat{w}_{\nu\mu}} \cos \hat{\kappa}_{\nu\mu} \) referenced in the title of this section are free.

4.4. Main theorem.

. All the seemingly disconnected pieces from the previous sections come together. So far we have accumulated the following data related to the original (pseudo-) metric \( g: \) smooth weights \( \{ e^{\hat{w}_{\nu\mu}}(x_1, \ldots, x_{2k}, x_{2k+2}, \ldots, x_{4k}) \}, \) the prolonged Hill operator eigenfunctions \( \{ \pm \sin(\hat{\kappa}_{\nu\mu}(x_1, \ldots, x_{2k})) \}, \{ \pm \cos(\hat{\kappa}_{\nu\mu}(x_1, \ldots, x_{2k})) \} \) corresponding to the loci of nondifferentiability and rank deficiency \( U^N \cup U^D, \) their attendant Riemann period matrices \( \left[ \left[ R_{\nu\mu} \right]_\beta \right]. \) Generically, \( \det(e^{w_{\nu\mu}(x_1, \ldots, x_{2k})) > 0, \) as rank deficiencies are now carried by the Hill operator eigenfunctions.

Now our regularized 4k-coframe \( \Psi \) is comprised of the image of our preframe under \( \alpha \circ C \) joined with its complement. Below we make use of the symmetries
involved $(\tilde{\kappa}_{ij} = \tilde{\kappa}_{ji}, \tilde{\omega}_{ij} = \tilde{\omega}_{ji}, \tilde{\kappa}_{i+2k} = \tilde{\kappa}_{ji}, \tilde{\omega}_{i+2k} = \tilde{\omega}_{ij})$.

\[\Psi_1 = \sum_{j=1}^{k} (e^{\tilde{\omega}_{ij}} \cos \tilde{\kappa}_{1j}dx_j + e^{\tilde{\omega}_{ij}} \sin \tilde{\kappa}_{1j}dx_{2k+j});\]

(4.4)

\[\Psi_k = \sum_{j=1}^{k} (e^{\tilde{\omega}_{kj}} \cos \tilde{\kappa}_{kj}dx_j + e^{\tilde{\omega}_{kj}} \sin \tilde{\kappa}_{kj}dx_{2k+j});\]

\[\Psi_{k+1} = \sum_{j=1}^{k} (e^{\tilde{\omega}_{kj+1}} \sin \tilde{\kappa}_{k+j}dx_j - e^{\tilde{\omega}_{kj+1}} \cos \tilde{\kappa}_{k+j}dx_{2k+j});\]

\[\Psi_{2k} = \sum_{j=1}^{k} (e^{\tilde{\omega}_{2kj}} \sin \tilde{\kappa}_{2kj}dx_j - e^{\tilde{\omega}_{2kj}} \cos \tilde{\kappa}_{2kj}dx_{2k+j}).\]

To write down the image of the complementary preframe, simply replace $\sin \tilde{\kappa}_{ij} \mapsto \cos \tilde{\kappa}_{ij}, \cos \tilde{\kappa}_{ij} \mapsto -\sin \tilde{\kappa}_{ij}$ in the coframe components advanced by $2k$.

**Theorem 4.1.** Given sets $U$ and $V$ on symplectic (pseudo-)Riemannian manifolds $(M, g, \omega)$, and $(N, g, \omega)$, there exists an isometry $\hat{A}_{UV} : T_{\Omega}U \to T_{\Omega}V$, $\hat{A}_{UV}(x_1) = y_1$, if and only if $F_{2}(\Psi)_U, F_{2}(\Psi)_V$ satisfy the hypothesis of Background Theorem 2.3.

**Corollary 4.1.** $\hat{A}^*([\mathcal{R}^u]_{\beta\gamma}) = [\mathcal{R}^u]_{\beta\gamma}$ as lexicographically ordered sets.

Should the metric turn out to be differentiable, we set $[\mathcal{R}^u]_{\beta\gamma} = [0]_{\beta\gamma}$, and the erstwhile condition becomes vacuous.

### 4.5. Proof of the Main Theorem.

To invoke Background Theorem 2.3, we have to show its hypothesis being satisfied. Our coframes were chosen on general grounds, and therein lies some uncertainty as to whether our choices do not run afoul of the Cartan's group reduction and normalization methods.

The first order of business is to delineate the intrinsic torsion coefficients. Those depend on the group structure as well as on the integrability properties of the coframe. Thus we find the requisite 2-forms for $i \leq k$. The other case can be derived mutatis mutandis.

\[d\Psi_1 = d\left( \sum_j e^{\tilde{\omega}_{ij}} \left( \cos \tilde{\kappa}_{ij}dx_j + \sin \tilde{\kappa}_{ij}dx_{j+2k} \right) \right)\]

\[= \sum_l \sum_j \frac{\partial \tilde{w}_{ij}}{\partial x_l} e^{\tilde{\omega}_{ij}} \left( \cos \tilde{\kappa}_{ij}dx_l \wedge dx_j + \sin \tilde{\kappa}_{ij}dx_l \wedge dx_{j+2k} \right)\]

\[= \sum_l \sum_j \frac{\partial \tilde{\omega}_{ij}}{\partial x_l} e^{\tilde{\omega}_{ij}} \left( \cos \tilde{\kappa}_{ij}dx_{l+2k} \wedge dx_j + \sin \tilde{\kappa}_{ij}dx_{l+2k} \wedge dx_{j+2k} \right)\]

\[+ \sum_l \sum_j \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} e^{\tilde{\omega}_{ij}} \left( -\sin \tilde{\kappa}_{ij}dx_l \wedge dx_j + \cos \tilde{\kappa}_{ij}dx_l \wedge dx_{j+2k} \right)\]

\[+ \sum_l \sum_j \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} e^{\tilde{\omega}_{ij}} \left( -\sin \tilde{\kappa}_{ij}dx_{l+2k} \wedge dx_j + \cos \tilde{\kappa}_{ij}dx_{l+2k} \wedge dx_{j+2k} \right)\]

\[= \sum_l \sum_j e^{\tilde{\omega}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_l} \cos \tilde{\kappa}_{ij} - \frac{\partial \tilde{\omega}_{ij}}{\partial x_l} \sin \tilde{\kappa}_{ij} \right)\]

\[\times \sum_{\mu} \hat{b}_{l+2k}^{u\wedge v} \Psi_\mu \wedge \Psi_v + \hat{b}_{l+2k}^{u\wedge v} \Psi_{u+2k} \wedge \Psi_v + \hat{b}_{l+2k}^{u\wedge v} \Psi_{u+2k} \wedge \Psi_{v+2k}\]
\[
\begin{align*}
&+ \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_{l+2k}} \cos \hat{\kappa}_{ij} - \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} \sin \hat{\kappa}_{ij} \right) \\
&\times \sum_l b_{l\wedge_j}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k} \\
&+ \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_{l+2k}} \sin \hat{\kappa}_{ij} + \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} \cos \hat{\kappa}_{ij} \right) \\
&\times \sum_l b_{l\wedge_j+2k}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k} \\
&+ \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_{l+2k}} \sin \hat{\kappa}_{ij} + \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} \cos \hat{\kappa}_{ij} \right) \\
&\times \sum_l b_{l\wedge_j+2k}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k}.
\end{align*}
\]

The coefficients above are entries of the lexicographically ordered compound matrix induced by $\Lambda^2 T^* M$. They are defined by the equation

\[(4.6) \quad [b_{l\wedge_j}^{u \wedge v}][\hat{\Psi}] = [dx], \quad b_{l\wedge_j}^{u \wedge v} \overset{\text{def}}{=} b_l^{u \wedge v} - b_j^{u \wedge v}.
\]

Thus $[b_{l\wedge_j}^{u \wedge v}] \in GL((\Lambda^k_2, \mathbb{R}) \times T_U U$ since the preframes have maximal rank in $U$. Similarly,

\[(4.7) \quad d\Psi_{i+2k} = d\left( \sum_j e^{\hat{w}_{ij}} \left( \cos \hat{\kappa}_{ij} dx_{j+2k} - \sin \hat{\kappa}_{ij} dx_j \right) \right)
\]

\[
\begin{align*}
&= \sum_l \sum_j \frac{\partial \hat{w}_{ij}}{\partial x_{l+2k}} e^{\hat{w}_{ij}} \left( \cos \hat{\kappa}_{ij} dx_{l+2k} \wedge dx_j - \sin \hat{\kappa}_{ij} dx_l \wedge dx_j \right) \\
&\quad + \sum_l \sum_j \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} e^{\hat{w}_{ij}} \left( \cos \hat{\kappa}_{ij} dx_{l+2k} \wedge dx_j - \sin \hat{\kappa}_{ij} dx_l \wedge dx_j \right) \\
&\quad - \sum_l \sum_j \frac{\partial \hat{\kappa}_{ij}}{\partial x_l} e^{\hat{w}_{ij}} \left( \sin \hat{\kappa}_{ij} dx_l \wedge dx_j \wedge \cos \hat{\kappa}_{ij} dx_{l+2k} \wedge dx_j \right) \\
&\quad - \sum_l \sum_j \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} e^{\hat{w}_{ij}} \left( \sin \hat{\kappa}_{ij} dx_{l+2k} \wedge dx_j \wedge \cos \hat{\kappa}_{ij} dx_{l+2k} \wedge dx_j \right) \\
&\quad = \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_{l+2k}} \cos \hat{\kappa}_{ij} - \frac{\partial \hat{\kappa}_{ij}}{\partial x_{l+2k}} \sin \hat{\kappa}_{ij} \right) \\
&\quad \times \sum_l b_{l\wedge_j}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k} \\
&\quad + \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_l} \cos \hat{\kappa}_{ij} - \frac{\partial \hat{\kappa}_{ij}}{\partial x_l} \sin \hat{\kappa}_{ij} \right) \\
&\quad \times \sum_l b_{l\wedge_j}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k} \\
&\quad - \sum_l \sum_j e^{\hat{w}_{ij}} \left( \frac{\partial \hat{w}_{ij}}{\partial x_l} \sin \hat{\kappa}_{ij} + \frac{\partial \hat{\kappa}_{ij}}{\partial x_l} \cos \hat{\kappa}_{ij} \right) \\
&\quad \times \sum_l b_{l\wedge_j}^{u \wedge v} \Psi_u \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v} \Psi_{u+2k} \wedge \Psi_v + b_{l\wedge_j+2k}^{u \wedge v+2k} \Psi_{u+2k} \wedge \Psi_{v+2k}.
\end{align*}
\]
- \sum_l \sum_j e^{\bar{\omega}_{ij}} \left( \frac{\partial \bar{\omega}_{ij}}{\partial x_{t+2k}} \cos \bar{\kappa}_{ij} + \frac{\partial \bar{\kappa}_{ij}}{\partial x_{t+2k}} \sin \bar{\kappa}_{ij} \right)
\times \sum b^{u^+}_l \frac{\partial \Psi_i}{\partial x_l} \Psi_u \wedge \Psi_v + b^{u^+}_l \frac{\partial \Psi_i}{\partial x_l} \Psi_u^+ \wedge \Psi_v^+ + b^{u^+}_l \frac{\partial \Psi_i}{\partial x_l} \Psi_u_2 \wedge \Psi_v_2 + b^{u^+}_l \frac{\partial \Psi_i}{\partial x_l} \Psi_u_2^+ \wedge \Psi_v_2^+.

Decomposing the structure group \((\text{1.1})\) into one-parameter subgroups acting on \(2 \times 2\) blocks combining the conjugate constituent 1-forms, we begin by computing the Maurer-Cartan matrix of the \(i\)-th subgroup:

\begin{equation}
(4.8) \quad \left( d \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \right) \left( \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right) = \begin{bmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{bmatrix}.
\end{equation}

Writing down the structure equation \((2.25)\), we arrive at

\begin{equation}
(4.9) \quad d \begin{bmatrix} \Psi_i \\ \Psi_{i+2k} \end{bmatrix} = \begin{bmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{bmatrix} \wedge \begin{bmatrix} \Psi_i \\ \Psi_{i+2k} \end{bmatrix} + \left( \sum (\cdot) \wedge (\cdot) \right) \end{equation}

upon modifying the Maurer-Cartan matrix via

\begin{equation}
(4.10) \quad \beta_i \rightarrow \tilde{\beta}_i
\end{equation}
Lemma 4.1.\[4.11\]

The action of the structure group on the intrinsic torsion of a 4k-coframe comprised of \[(4.4)\]

is manifested by the fact that out of 2\(k(4k - 1)\) linearly independent monomials spanning \(\Lambda^2 T^* M\), only 2\(k\) are absorbed - precisely those involving symplectic conjugate pairs of variables.

Hence the intrinsic torsion coefficients occur for the monomial 2-forms not listed in \[(4.10)\]. In terms of combinations of indices, for \(\gamma_{ms}^i\) the general expression is

\[
\gamma_{ms}^i = \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_l} \cos \tilde{\kappa}_{ij} - \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} \sin \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m\wedge s} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_{l+2k}} \cos \tilde{\kappa}_{ij} - \frac{\partial \tilde{\kappa}_{ij}}{\partial x_{l+2k}} \sin \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m+2k\wedge s} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_l} \sin \tilde{\kappa}_{ij} + \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} \cos \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m\wedge s+2k} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_{l+2k}} \sin \tilde{\kappa}_{ij} + \frac{\partial \tilde{\kappa}_{ij}}{\partial x_{l+2k}} \cos \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m+2k\wedge s+2k}.
\]

One other coefficient (to prime one’s intuition), \(\gamma_{m+2ks}^{i+2k}, s \neq i\) is given by

\[
\gamma_{m+2ks}^{i+2k} = \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_l} \cos \tilde{\kappa}_{ij} - \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} \sin \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m+2k\wedge s} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_{l+2k}} \cos \tilde{\kappa}_{ij} - \frac{\partial \tilde{\kappa}_{ij}}{\partial x_{l+2k}} \sin \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m+2k\wedge s} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_l} \sin \tilde{\kappa}_{ij} + \frac{\partial \tilde{\kappa}_{ij}}{\partial x_l} \cos \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m\wedge s+2k} + \sum_l \sum_j e^{\tilde{w}_{ij}} \left( \frac{\partial \tilde{w}_{ij}}{\partial x_{l+2k}} \sin \tilde{\kappa}_{ij} + \frac{\partial \tilde{\kappa}_{ij}}{\partial x_{l+2k}} \cos \tilde{\kappa}_{ij} \right) b_{l\wedge j}^{m+2k\wedge s+2k}.
\]

We note (without performing actual computations) that the parenthetical expressions cannot all vanish at the loci of nondifferentiability/rank deficiency. In part, this is due to the fact that all \(\sin \tilde{\kappa}_i\), \(\sin \tilde{\kappa}_m\) vanish at \(U^N \cup U^O\) and nowhere else. And the summation over \(l, j\) in conjunction with the fact that our compound matrix is invertible ensures \(|\gamma_{ms}^i|, |\gamma_{m+2ks}^{i+2k}| > 0\) on \(T_U U\).

We split the rest of the proof into smaller fragments, each stated as a subsidiary lemma.

Lemma 4.1. \textit{The action of the structure group on the intrinsic torsion of a 4k-coframe comprised of \[(4.4)\] and its complement is limited to translations involving symplectic conjugates.}

Proof. Take an arbitrary intrinsic torsion coefficient \(\gamma_{ms}^i\). All the sources of \(\Psi_m \wedge \Psi_s\) are listed here:

\[d\gamma_{ms}^i = \sum \gamma_{m+2ks}^i \beta_m - \sum \gamma_{m+2ks}^i \beta_s + \sum \gamma_{m+2ks}^i \beta_i \mod \Psi_m \wedge \Psi_s.\]
Inspecting this expression, and comparing it with the general infinitesimal group action formula (2.27), we conclude that since \( \partial^l \) is the unique anti-Hermitian matrix generating \( SO(2, \mathbb{R}) \), we have \( a_{ij}^l = a_{ij}^l = 0 \). Hence only translations are present. Now in view of \( (4.11) \), we can exclude some combinations of indices. Thus for \( \gamma^i_{ms}, \ s, m \neq i + 2k \), for those differentials are absorbed into \( \beta_i \).

**Lemma 4.2.** The equivalence problem involving \( 4k \) coframes \( (4.4) \) reduces to a regular \( e \)-structure.

**Proof.** Regularity in the context of \( e \)-structures presupposes that a) the action of the structure group on the intrinsic torsion forms a single orbit, and is amenable to normalization, and b) the rank of the set

\[
F_s(\Psi) = \{ \gamma^i_{jm}, \cdots, \gamma^i_{jm[l_1, \cdots, l_{s-1}]}, \ 1 \leq i, j, m, v, l_1, \cdots, l_{s-1} \leq 4k \}
\]

is constant on some open neighborhood of a fixed point \( (x_1, \cdots, x_{4k}) \).

Concerning the orbit unity, we can begin with \( \partial \gamma^i_{jm} \equiv 0 \mod \Psi_j \wedge \Psi_m \), which entails \( \beta_s = 0 \), \( \forall s \leq 2k \), and then gradually ‘turn on’ the \( \beta_s \). This way, judiciously choosing values of the \( 2k \) group parameters, we can continuously deform the set of intrinsic torsion coefficients to obtain any other configuration within the \( 2k \)-dimensional group parametric space.

To demonstrate the rank being constant, we expand on the remark below \( (4.11) \). Namely, the intrinsic coefficients are nonvanishing due to the standard normal form as well as the facts that the freedom relation \( \varphi \) is open and dense in the jet bundle, and, in addition, satisfies the \( h \)-principle.

The ‘next layer’ consists of first derivatives of the intrinsic torsion coefficients \( \gamma^i_{jm[l_1]} \). They involve \( \partial \gamma^i_{jm[l_1]} \), which, in turn, are products of \( \partial \gamma^i_{jm[l_1]} \), \( \partial \gamma^i_{jm[l_1]} \), \( \partial \gamma^i_{jm[l_1]} \), \( \partial \gamma^i_{jm[l_1]} \), and \( \sin \tilde{\kappa}_{iu} \), \( \sin \tilde{\kappa}_{mv} \) all summed over \( l, v, j \). It is not difficult to see that \( \gamma^i_{jm[l_1]} \) do not contribute anything to the rank of \( I_2(\Psi) \) as opposed to a scenario wherein some \( \gamma^i_{jm}(0, 0, \cdots, 0) = 0 \) but the rank remains constant due to the contribution of \( \gamma^i_{jm[l_1]}(0, 0, \cdots, 0) > 0 \).

Now we can scale all the \( \gamma^i_{jm} \) that do not vanish identically to be equal to one, and \( \beta_s \equiv 0 \mod \Psi \). \( \bigoplus_{i=1}^{2k} SO(2, \mathbb{R}) \) has been reduced to \{1\}. We fix \( 4k \) unique smooth functions \( f_i \), \( f_{i+2k} \) so that \( \beta_i = f_i \Psi_i + f_{i+2k} \Psi_{i+2k} \), and we have an \( e \)-structure. All the requirements listed in the hypothesis of Background Theorem \( 2.3 \) are satisfied. That completes the proof.

To prove Corollary \( 1.1 \) we remark that in view of Lemma \( 3.2 \) the Riemann period matrices are independent of the fiber variables. Now we claim that for an arbitrary \( A : TuU \rightarrow TV \), the induced map of cotangent bundles is on target: \( A^\times(\Psi) \in \Xi(V, ds^2) \). With the Main Theorem above, we know that \( A(U) = V \), \( A(U^N) = V^N \), \( A(U^O) = V^O \). Consequently, the periods, periodic spectra of the Hill operators for every metric tensor coefficient, and all the norming constants are identical on \( U \) and \( V \). Now we think of \( \Sigma_2(U) \) as a fibration over a foliation of codimension \( 2k - 1 \), whose fibers are Riemann surfaces (including the limiting case of \( (\mathbb{C}\backslash\{\lambda_1\}) \times (\mathbb{C}\backslash\{\lambda_1\}) \backslash \sim \)), and fix one leaf determined by the coordinates \( (x_2, x_3, \cdots, x_{2k}) \). However, the limiting fiber occurs only if the underlying leaf does not intersect the loci of nondifferentiability/rank deficiency. Formally,

\[
(U^N \cup U^O) \cap \{(x_2, x_3, \cdots, x_{2k})|\Sigma_2(x_2, x_3, \cdots, x_{2k}) = [(\mathbb{C}\backslash\{\lambda_1\}) \times (\mathbb{C}\backslash\{\lambda_1\})] \backslash \sim \} = \emptyset.
\]
Additionally, between transversality of the foliation (to the loci of nondifferentiability/rank deficiency) and the smoothness, the only possible scenario would materialize if either \( \partial(U^N \cup U^O) \neq \emptyset \), or \( \dim(U^N \cup U^O) < 2k - 1 \). At any rate, the presence of constant potentials is completely determined by the Laplace-Beltrami equation. These conditions imply that zero Riemann period matrices are mapped onto zero period matrices by an arbitrary isometry, and the induced biholomorphic map on the fiber is the identity. One such example is an axisymmetric black hole with the incomplete event horizon.

We have disposed of the limiting surfaces, so from this point on, the fiber is a bona fide Riemann surface of infinite genus \( \mathcal{T}_C \). Now we truncate it so as to have only the largest \( m \) homology cycles \( A_1, B_1, \ldots, A_m, B_m \) as the basis of its relative homology group: \( \mathcal{T}_C^m \) has \( H_1(\mathcal{T}_C^m \setminus \partial \mathcal{T}_C^m, \mathbb{Z}) \) being generated by the cycles \( (A_1, B_1, \ldots, A_m, B_m) \). This is a compact Riemann surface.

If \( \tilde{A}'(\mathcal{T}_C(x_2, x_3, \ldots, x_{2k})) \not\cong \mathcal{T}_C(y_2, y_3, \ldots, y_{2k}) \), by Background Theorem 3.3 (Torelli theorem) we have \( \tilde{A}'([R^y]^\beta(x_2, x_3, \ldots, x_{2k})) \not\cong [R^y]^\beta(y_2, y_3, \ldots, y_{2k}) \), and since all the periods are identical, there must exist a holomorphic form \( d\Phi \in \text{Hol}(\mathcal{T}_C(x_2, x_3, \ldots, x_{2k})) \) or \( d\Phi \in \text{Hol}(\mathcal{T}_C(y_2, y_3, \ldots, y_{2k})) \), responsible for the discrepancy. By Its-Matveev formula (3.52), \( d\Phi \) translates into a particular Hill potential. On the real torus \( \mathcal{T}_C^m(y_2, y_3, \ldots, y_{2k}) \) it corresponds to an \( m \)-tuple of tied spectrum eigenvalues \( (\mu_1, \ldots, \mu_m) \). But that \( m \)-tuple is already accounted for by virtue of the fact that all the lacunae are identical on \( U \) and \( V \). Hence we are compelled to look for the offending \( \mu_i \) someplace else, more specifically at the cycles with \( i > m \). Now given that \( \exists d\Phi_m \in \text{Hol}(\mathcal{T}_C^m) \) with

\[
\oint_{A_i} d\Phi = \oint_{A_i} d\Phi_m, \quad i \leq m,
\]

we can apply the Gram-Schmidt orthogonalization algorithm to produce \( d\Phi^\perp \) such that

\[
\oint_{A_i} d\Phi^\perp = 0, \quad i \leq m.
\]

Now we let \( m \to \infty \). Then by (7, Chapter 1, Theorem 1.17), \( d\Phi^\perp = 0 \) and we have reached a contradiction assuming

\[
\tilde{A}'(\mathcal{T}_C(x_2, x_3, \ldots, x_{2k})) \not\cong \mathcal{T}_C(y_2, y_3, \ldots, y_{2k}).
\]

4.6. Weak equivalence.

Having set the stage by describing in great detail the problem of equivalence of nondifferentiable metrics, we now proceed to take a closer look at an approximation problem of nondifferentiable metrics. Is there a way to tell if two such metrics expressed in terms of the same local coordinate system are so ‘close’ to each other that their first derivatives match?

Courant algebroids to the rescue! Recall (23):

\[
(X_1 + \xi_1, X_2 + \xi_2)_+ = \frac{1}{2}((\xi_1, X_2) + (\xi_2, X_1)).
\]

This is a natural nondegenerate bilinear form associated with Courant algebroids. It allows one to circumscribe the extent of nonintegrability of a preframe.

**Definition 4.1.** Given a preframe \( \mathcal{P} \), we define its annihilator, \( \mathcal{P}^\perp \) by the equation

\[
(\mathcal{P}, \mathcal{P}^\perp)_+ = (\mathcal{P}, \mathcal{P})_+ = 0.
\]
Thus an annihilator is related to the original preframe.

**Definition 4.2.** Two preframes, $\vec{\Psi}^1$, $\vec{\Psi}^2$ are weakly equivalent if there exists a nontrivial annihilator $\vec{\Psi}^\perp$ such that for some constants $\beta_1, \beta_2$ we obtain

\[
(\vec{\Psi}^1, \cos \beta_1 \vec{\Psi}^\perp + \sin \beta_1 (\omega^\# - \pi^#) \vec{\Psi}^\perp)_+ = (\vec{\Psi}^2, \cos \beta_2 \vec{\Psi}^\perp + \sin \beta_2 (\omega^\# - \pi^#) \vec{\Psi}^\perp)_+ = 0.
\]

Knowing that the structure group can be reduced explains our definition. The constant preternatural rotations should suffice. The number of independent elements of the Courant algebroid that may possibly comprise an annihilator ranges from zero to $2^k$. Maximally nonintegrable preframes are extremely picky and hard to annihilate, whereas sections of Dirac subbundles allow $2^k$-dimensional annihilators. That number is invariant under the action of the bundle map $\omega^# - \pi^#$. One instructive case is that of preframes that happen to be $2^k$-tuples of linearly independent sections of a typical Dirac subbundle. Any two of those are weakly equivalent. It makes sense since they all induce identically vanishing Riemann period matrices on open subsets.

Obviously, equivalent preframes are weakly equivalent. That readily follows from (4.1). It is unclear whether real analytic weakly equivalent preframes are equivalent.

## 5. Black Hole Solutions of EVE

Throughout this section, we work with two basic objects: The Einstein vacuum equations, under the acronym EVE\textsuperscript{7},

\[\text{Ric}(g) = 0,\]

and a special sort of open subsets of the space-time. Specifically, for a black hole solution of EVE, we make

**Definition 5.1.** An open set $U \subset \mathbb{R}^{1+3}$ is called representative if for a singular preframe encoding the black hole solution, there is an element such that $U^N_{ij} \cap U \neq \emptyset$ (an inner curvature blow-up), and $U^O_{ij} \cap U \neq \emptyset$ (the Cauchy horizon).

The significance of our choice is obvious: we intend to make use of the new metric invariants in general relativity. Admittedly, the condition of ‘general covariance’ imposed by Einstein involves the entire group of diffeomorphisms, so metric invariants, the Riemann period matrices among them, are not preserved. But the incidence relations associated with two metric configurations do persist.

### 5.1. The Kerr Preframe.

\textsuperscript{7}Strictly speaking, Sections 5.2 and 5.4 presuppose an observer with a mass crossing the event horizon, and gravitational collapse, both of which take the full-fledged Einstein equations.
The Kerr solution [16], discovered by Roy Kerr in 1963, is, perhaps, the most interesting of all exact solutions of EVE. It describes a rotating axisymmetric black hole, and provides a natural testing ground (space?) for preframe theory. Hence we work out the details, including the blowup parameter, and a representative set, explicitly. As a primary source, we utilize the book [27], predominantly Chapter 1, written by Matt Visser.

The second version (and the one we are interested in) of the Kerr line element presented in [16] was defined in terms of ‘Cartesian’ coordinates \((t,x,y,z)\):

\[
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2mr^3}{r^4 + a^2 z^2} \left[ dt + \frac{r(xdx + ydy)}{a^2 + r^2} + \frac{a(ydx - xdy)}{a^2 + r^2} + \frac{z dz}{r} \right]^2,
\]

subject to \(r(x,y,z)\), which is now a dependent function, and not a radial coordinate, being implicitly determined by:

\[
x^2 + y^2 + z^2 = r^2 + a^2 \left[ 1 - \frac{z^2}{r^2} \right].
\]

The angular momentum \(J\) is incorporated into the line element (5.1) via \(a = \frac{J}{m}\), \(m\) being the mass.

The full \(0 < a < m\) metric is now manifestly of the Kerr-Schild form:

\[
g_{ij} = \eta_{ij} + \frac{2mr^3}{r^4 + a^2 z^2} \mathcal{E}_i \mathcal{E}_j,
\]

\[
\mathcal{E}_i = \left( 1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right).
\]

Here \(\mathcal{E}_i\) is a null vector with respect to both \(g_{ij}\) and \(\eta_{ij}\).

To delineate a standard representative set \(U_{\text{Kerr}}\), we need to restrict our base manifold to a more manageble (and physically relevant) subset \(r \geq 0\). Next, we choose \(z\) as the blowup parameter. Our choice is informed by the fact that the Kerr spacetime is naturally axisymmetric, and its symmetry group is \(SO(2)\). The crucial (but not all) loci are as follows

\[
U_{\text{Kerr}}^0 = \{(t,x,y,z) \in \mathbb{R}^{1+3}\mid x^2 + y^2 = a^2, \ z = 0\},
\]

\[
U_{\text{Kerr}}^{1,2} = \{(t,x,y,z) \in \mathbb{R}^{1+3}\mid x^2 + y^2 = \frac{2m}{r_\pm}, \ z^2 = 2mr_\pm\}.
\]

All the loci have to be transversal to the \(z\)-direction. To include at least some parts of these three sets, we choose our slice of heaven to be

\[
U = \{(t,x,y,z) \in \mathbb{R}^{1+3}\mid 0 < t < 1, \ 0 < x, \ y < \frac{\sqrt{2}a}{2} + \epsilon, \ z^2 < (r_+ + \epsilon)^2\}.
\]

The remaining vanishing loci are

\[
U_{\text{Kerr}}^3 = \{(t,x,y,z) \in \mathbb{R}^{1+3}\mid rx + ay = 0\},
\]

\[
U_{\text{Kerr}}^4 = \{(t,x,y,z) \in \mathbb{R}^{1+3}\mid ry - ax = 0\},
\]

\[
U_{\text{Kerr}}^5 = \{(x,y,z) \in \mathbb{R}^{1+3}\mid x^2 + y^2 > a^2, \ z = 0\}.
\]

They are comprised of ellipses and circles, one each for a fixed value of \(r\).

In terms of individual elements, we have

\[
U_{00}^O = U_{\text{Kerr}}^{1,2}; \quad U_{00}^N = U_{\text{Kerr}}^0; \quad U_{ij}^N = U_{\text{Kerr}}^i;
\]
Hence, prior to smoothing, we find that \( \sin(\kappa_0(x, y, z)) \) is an eigenfunction of the Hill potential \( \cos(\frac{z}{T_{00}(x, y)}) \), with the designated eigenvalue indexed \( \lambda_{12}^0(x, y) \).

The function \( \sin(\kappa_0(x, y, z)) \) has exactly two roots in \( U \). Thus, we obtain \( \lambda_{1}^0(x, y) \).

\( \sin(\kappa_0(x, y, z)) \) features a coalescence of roots, \( \{ z = 0 \} \) being their common locus. That corresponds to \( \lambda_{2}^0(x, y) \). Its behavior is singularly simple:

\[
\sin(\kappa_0(x, y, z))|_{z|x_0} = \frac{\sqrt{2}}{2}; \quad \sin(\kappa_3(x, y, z))|_{z=0} = 0.
\]

\( \sin(\kappa_3(x, y, z)) \) corresponds to \( \lambda_{2}^1(x, y) \) by virtue of \( U_{01}^3 = \emptyset \).

This coefficient takes \( \lambda_{1}^2(x, y) \), as \( U_{0}^3 \cap U_{Kerr}^4 \cap U = \emptyset \).

\[
e^{2w_{00}} \cos^2 \kappa_{00} \big|_{U \cap (U_N \cup U_{00}^3)} = \frac{\sin(2mr^3 - r^4 - a^2 z^2) \sin(r^4 + a^2 z^2)}{2mr^3 - r^4 - a^2 z^2} \frac{r^4 + a^2 z^2}{r^4 + a^2 z^2}.
\]

\[
e^{2w_{01}} \cos^2 \kappa_{01} \big|_{U \cap (U_N \cup U_{01}^3)} = \frac{\sin(2mr^3 (rx + ay)) \sin((r^4 + a^2 z^2)(r^4 + a^2 z^2))}{2mr^3 (rx + ay)} \frac{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}.
\]

\[
e^{2w_{02}} \cos^2 \kappa_{02} \big|_{U \cap (U_N \cup U_{02}^3)} = \frac{\sin(2mr^3 (ry - ax)) \sin((r^4 + a^2 z^2)(r^4 + a^2 z^2))}{2mr^3 (ry - ax)} \frac{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}.
\]

\[
e^{2w_{11}} \cos^2 \kappa_{11} \big|_{U \cap (U_N \cup U_{11}^3)} = \frac{\sin(2mr^3 (rx + ay) + (r^2 + a^2 z^2))}{2mr^3 (rx + ay) + (r^2 + a^2 z^2)} \frac{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}{(r^4 + a^2 z^2)(r^4 + a^2 z^2)}.
\]

\[
e^{2w_{12}} \cos^2 \kappa_{12} \big|_{U \cap (U_N \cup U_{12}^3)} = \frac{\sin(2mr^3 (rx + ay)(ry - ax))}{2mr^3 (rx + ay)(ry - ax)} \frac{(r^4 + a^2 z^2)(a^2 + a^2 z^2)}{(r^4 + a^2 z^2)(a^2 + a^2 z^2)}.
\]

\[
e^{2w_{13}} \cos^2 \kappa_{13} \big|_{U \cap (U_N \cup U_{13}^3)} = \frac{\sin(2mr^2 (rx + ay)z)}{2mr^2 (rx + ay)z} \frac{(r^4 + a^2 z^2)(r^2 + a^2 z^2)}{(r^4 + a^2 z^2)(r^2 + a^2 z^2)}.
\]

\[
U_{01}^3 = U_{Kerr}^3; \quad U_{02}^3 = U_{Kerr}^4; \quad U_{03}^3 = U_{Kerr}^5.
\]
\[ \sin(\kappa_{22}(x, y, z)) \text{ corresponds to } \lambda^{22}_{33}(x, y). \]

\[ e^{2w_{22} \cos^2 \kappa_{22}}|_{U \cap (U^N \cup U_{S}^{33})} = \frac{\left| \sin(2mr^2z(ry - ax)) \right|}{2mr^2z(ry - ax)} \times \frac{\sin((r^4 + a^2z^2)(r^2 + a^2)^2)}{(r^4 + a^2z^2)(r^2 + a^2)^2}. \]

\[ \sin(\kappa_{23}(x, y, z)) \text{ corresponds to } \lambda^{23}_{33}(x, y) \text{ in view of the fact that } U_{S}^{33} = \emptyset. \]

\[ e^{2w_{23} \cos^2 \kappa_{23}}|_{U \cap (U^N \cup U_{S}^{33})} = \frac{\left| \sin(2mr^2z(ry - ax)) \right|}{2mr^2z(ry - ax)} \times \frac{\sin((r^4 + a^2z^2)(r^2 + a^2)^2)}{(r^4 + a^2z^2)(r^2 + a^2)^2}. \]

\[ \sin(\kappa_{33}(x, y, z)) \text{ takes } \lambda^{33}_{33}(x, y). \]

Lastly, \( \sin(\kappa_{33}(x, y, z)) \text{ takes } \lambda^{33}_{33}(x, y) \) as we realize that \( U_{S}^{33} = \emptyset. \)

Those are partially regularized functions. As yet, they are still to undergo further treatment. As \( \Box_{\text{Kerr}} \) and the smoothing operators are axisymmetric, the fully regularized functions would satisfy

\[ \mathcal{L}_{f_1, K_1 + f_2 K_2} e^{w_{ij}(x, y, z)} \cos(\kappa_{ij}(x, y, z)) = 0, \quad \mathcal{L}_{f_1, K_1 + f_2 K_2} \lambda^{ij}_{33}(x, y) = 0, \]

where \( K_1, K_2 \) are the original Killing vectors.

We fix the symplectic form: \( \omega = dt \wedge dz + dx \wedge dy. \) Then the preframe becomes

\[ \Psi_1 = e^{w_{i0}(x, y, z)} \cos(\kappa_{i0}(x, y, z))dt + e^{w_{i0}(x, y, z)} \sin(\kappa_{i0}(x, y, z)) \frac{\partial}{\partial z} + \]

\[ e^{w_{i1}(x, y, z)} \cos(\kappa_{i1}(x, y, z))dx + e^{w_{i1}(x, y, z)} \sin(\kappa_{i1}(x, y, z)) \frac{\partial}{\partial y} + \]

\[ e^{w_{i2}(x, y, z)} \cos(\kappa_{i2}(x, y, z))dy - e^{w_{i2}(x, y, z)} \sin(\kappa_{i2}(x, y, z)) \frac{\partial}{\partial x} + \]

\[ e^{w_{i3}(x, y, z)} \cos(\kappa_{i3}(x, y, z))dz - e^{w_{i3}(x, y, z)} \cos(\kappa_{i3}(x, y, z)) \frac{\partial}{\partial t}; \]

\[ \tilde{\Psi} = \begin{bmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \end{bmatrix}; \]

And the complementary preframe is gotten via the canonical bundle map:

\[ \tilde{\Psi}^* = (\omega^\# - \pi^\#)\tilde{\Psi}. \]

5.2. Strong Cosmic Censorship Conjecture.

With hyperbolic PDE on Lorenzian manifolds, there comes the attendant causal structure (a Cauchy hypersurface \( S \) with the past Cauchy development \( S^\prime \) and the future Cauchy development \( S^\prime \)). The existence of such a hypersurface is tantamount to \( S^\prime \cup S \cup S^\prime = M \), the entire manifold. If, however, \( S^\prime \cup S \cup S^\prime \neq M \), there exists a Cauchy horizon - a light-like boundary of the domain of validity of the underlying Cauchy problem for the PDE in question. One side of the horizon contains closed space-like geodesics, whereas the opposite side may contain closed
time-like geodesics. Beyond the horizon, Cauchy data no longer uniquely determines the solution. Thus the presence of such a horizon is manifestly a violation of classical determinism for all observers passing through the horizon. Some black holes, particularly the Kerr solutions with $0 < a < m$, are known to feature a distinct Cauchy horizon. Hence, to save classical determinism, Roger Penrose [26] put forth the cosmic censorship conjectures. They are classified into two categories: weak and strong ones. We focus on the strong cosmic censorship conjecture. There are several different formulations, the one we discuss below states that the maximal Cauchy development of generic compact or asymptotically flat initial data is locally inextendible as a $C^0$-Lorentzian manifold. The original formulation of the strong cosmic censorship conjecture was modelled on the prototype of the Schwarzschild solution ($a = 0$), where no Cauchy horizon is present, and the spacetime can be construed to ‘terminate’ at a spacelike curvature nondifferentiability locus, the metric being inextendible past the nondifferentiability surface, not even continuously. As Dafermos and Luk [6] put it: "The singular behavior of Schwarzschild, though fatal for reckless observers entering the black hole, can be thought of as epistemologically preferable for general relativity as a theory, since this ensures that the future, however bleak, is indeed determined". In that paper, dating back to 2017, the authors disproved the strong cosmic censorship conjecture formulation above for the Cauchy horizon of a charged, rotating black hole. Furthermore, according to [6], the $C^0$-extensions are patently nonunique. However, a modicum of uniqueness can be recovered for the black hole solutions with Cauchy horizons. It should be emphasized, this only applies to the structure of spacetime, not relativistic dynamics on the black hole background. We state

**Theorem 5.1.** Within an assertedly representative set $U$, the outer part of the tubular neighborhood of $\mathcal{U}_1 \cap U$ determines the Riemann period matrices $\left[ [R_{ij}]_{\beta \gamma} \right]_U$ of the regularized singular preframe.

**Proof.** Here we already have the complete Cauchy horizon. It is topologically a sphere. The undetermined part is the locus of nondifferentiability that hosts the curvature blow-up. However, all periods would be fixed once we settle on a representative set. Now using all conceivable extensions past the Cauchy horizon, we arrive at multiple regularized auxiliary fields for every such extension following our Laplace-Beltrami $\rightarrow$ Nash-Gromov $\rightarrow$ Hill procedure. Then we turn off the beaten path and minimize over the extensions one field at a time mimicking (3.57):

$$\lim_{N \to \infty} \lim \inf_{(\text{extensions}, S_\zeta, \kappa)} \left( ||\lambda_0^s - \lambda_0 ||^2_U + \sum_{s=1}^{N} \frac{1}{2^s} (||\lambda_2^s - \lambda_2^- ||^2_U + ||\lambda_2^s - \lambda_2^+ ||^2_U) \right).$$

This piecemeal optimization scheme may yield a configuration that is not equal to any of the existing extensions, but we still find optimized extensions by melding those suboptimal ones using partitions of unity. They are equivalent in view of Theorem 3.1.

Our eclectic optimal extension takes advantage of the fact that there no longer are relations between individual metric coefficients. $C^0$-configurations are not solutions of EVE.

The similarity between Theorem 5.1 and the holographic principle championed by some physicists is purely coincidental. We knew from the outset that the locus of curvature blow-up must be present inside.
Theorem 5.1 does not make or reinforce an argument for the validity of the strong cosmic censorship conjecture. More precisely, in conjunction with the Penrose singularity theorem [25], as the prevalence of black holes had been established, our result is a way to safeguard uniqueness. Thus, owing to Penrose, we look for a black hole solution of EVE. Then the periodicity of the Hill potentials and the structure of the Cauchy horizon determines the location and structure of $U^N \cap U$.

A much more difficult question still remains unaddressed. With closed timelike geodesics stubbornly persisting beyond the Cauchy horizon, no semblance of causality is possible. And the results of [6] only bring this unsatisfactory state of affairs into sharper focus. Our proposal to reestablish causality parts ways with general relativity. Rather than trying to finesse some ever more sophisticated formulation of the cosmic censorship, we look at the Riemann period matrices corresponding to representative sets with masses/observers moving inside the Cauchy horizon. For each instance (of an affine geodesic parameter), due to the gravitational pull of the mass/observer, we get a particular set of functions of the Riemann period matrix. Thus each instance produces a different configuration. If we compare this situation with that of a mass/observer living in a smooth metric configuration, we see that the latter (in view of identically zero Riemann period matrix) does not induce this kind of inequivalence (the curvatures do change, but the finer invariants do not). Therefore this fragmentation of physical reality resolves the problem of causality in the presence of loci of nondifferentiability. Within our paradigm, physical movement always takes place in an internal parameter-independent Riemann period matrix environment, as evidenced by dynamics on smooth Lorentzian manifolds. Such dynamics in general do not exist beyond the Cauchy horizon. Intrepid observers are the proverbial elephants rambling in the china shop of the black hole interior.

Mathematically, it makes sense: a mass moving on a regular background simply reshuffles sections of a fixed Lagrangian subbundle of $T^*\mathbb{R}^{1+3}$, whereas the same mass’ influence on a lift of some regularized preframe impinges on local integrability properties of the underlying subbundle.

5.3. Dynamics in the Interior.

As we postulate, the presence of nonzero Riemann period matrix radically alters dynamics, to the point of excluding movement of masses along generic timelike geodesics. However, there still remains one avenue open: moving masses without disturbing the invariants. Owing to Torelli theorem (Background Theorem 3.3), that amounts to conformal transformations of individual Riemann surfaces within $\mathcal{X}_C(U)$ as the affine parameter transversal to the $x_1$-direction unfolds. Now on the moduli space of the Riemann surfaces encoding the spectrum of relevant Hill operators, we can define a monodromy group of automorphisms. There would be a representation of the set of timelike geodesics on the aforementioned group. The dimension of the kernel of this representation map is what we call the effective dimension. Our conjecture is, the effective dimension is strictly less than three, and all closed geodesics are in the image of that representation. Also, the dimension may depend on the ratio of masses of the observer and the black hole being observed. The reason we cannot dismiss the possibility of nondisturbing mass movement is, absorption of gravitational waves by a black hole must induce a conformal transformation of $\mathcal{X}_C(U)$. 
5.4. Black Hole Evaporation.

We inadvertently venture into the quantum theory territory. The presence of additional differential invariants sheds some light on the process of evaporation (via emission of thermal radiation) inaugurated by Stephen Hawking in \[10, 11\]. Instead of sticking with the full-fledged Riemann surface invariants, we deconstruct the discriminant (3.32) of a pertinent Hill operator \(\star(\lambda, t, x, y)\) on a representative open set \(U = \{(t, x, y, z) \in \mathbb{R}^{1+3}\}\). For uniformity, we regard \(z\) as the singularity parameter. Specifically, \(\star(\lambda, t, x, y)\) is an integral function of \(\lambda\), growing as \(\sqrt{\lambda}\) as \(\lambda \to \infty\) (\[20, \text{Section 1}\]). As such, it allows an infinite product representation:

\[
\star(\lambda, t, x, y) = \sqrt[\infty]{v(\lambda, t, x, y)} \prod_{i=0}^{\infty} (\lambda - \lambda_{2i})(\lambda - \lambda_{2i-1}).
\]

This product converges absolutely, so by selecting a principal branch of \(\sqrt[\infty]{v(\lambda, t, x, y)}\) over some fixed ramified cover, we define

\[
\star^+(\lambda, t, x, y) \overset{\text{def}}{=} \sqrt{v(\lambda, t, x, y)} \prod_{i=0}^{\infty} (\lambda - \lambda_{2i}),
\]

\[
\star^-(\lambda, t, x, y) \overset{\text{def}}{=} \sqrt{v(\lambda, t, x, y)} \prod_{i=0}^{\infty} (\lambda - \lambda_{2i-1}).
\]

If \(\lambda_{2i-1} = \lambda_{2i}\), we call this a vacuum configuration (in conjunction with a Hamiltonian preframe on \(U\)), or a regular configuration (in conjunction with a regular but not a Hamiltonian preframe). Otherwise, this would designate a piece of some black hole if and only if \(\lambda_{2i-1} < \lambda_{2i}\). The only possibility left, \(\lambda_{2i-1} > \lambda_{2i}\), has no geometrical content, although physicists would be tempted to call such monstrosities ‘negative energy vacuum fluctuations’.

To effect a semi-rigorous derivation of black hole evaporation, Hawking \[10\] used the following reasoning: “... negative energy flux will cause the area of the event horizon to decrease and so the black hole will not, in fact, be in a stationary state. However, as long as the mass of the black hole is large compared to the Planck mass, the rate of evaporation of the black hole will be very slow compared to the characteristic time for light to cross the Schwarzschild radius. Thus it is a reasonable approximation to describe the black hole by a sequence of stationary solutions and to calculate the rate of emission in each solution.” Therefore we let the discriminant to be time-independent, and the changes be induced by some abstract time-dependent operators. Mathematically, they are Hermitian integral operators with compact kernels acting on \(\star^+(\lambda, 0, x, y)\) and \(\star^-(\lambda, 0, x, y)\):

\[
(\mathcal{A}(t_m)\star^+)\lambda, 0, x, y) = \int_{\Sigma(U)} H(t_m, x - \chi, y - v) \star^+(I, 0, \chi, v) d\ell d\chi dv,
\]

\[
(\mathcal{E}(t_m)\star^+)\lambda, 0, x, y) = \int_{\Sigma(U)} H(t_m, x - \chi, y - v) \star^+(I, 0, \chi, v) d\ell d\chi dv.
\]

Such operators must be invertible by virtue of the fact their natural extensions to the sections of the cotangent bundle induce isomorphisms of the respective Hodge-Kodaira cohomology spaces:

\[
H^1_{\text{HK}}(\mathcal{A}(0)\Sigma(U)) \cong H^1_{\text{HK}}(\mathcal{A}(t_m)\Sigma(U)), \quad H^1_{\text{HK}}(\mathcal{E}(0)\Sigma(U)) \cong H^1_{\text{HK}}(\mathcal{E}(t_m)\Sigma(U)).
\]
Their main feature is stretching/shrinking of the intervals of instability of the periodic spectrum that vary depending on the absorbed mass, charge, spin, and emitted energy (written abusing notation):

\[ A \lambda_{2i} - A \lambda_{2i-1} > \lambda_{2i} - \lambda_{2i-1}, \]

\[ E \lambda_{2i} - E \lambda_{2i-1} < \lambda_{2i} - \lambda_{2i-1}. \]

This reformulation opens a veritable Pandora box, as there is no reason to believe that all \( A(t_m)^\pm \) have an underlying stationary black hole solution of EVE. That is why it was not an empty declaration to classify this as a quantum theory.

The gist of the information loss paradox can be recast in terms of the absorption/emission operators. Namely, the chronological products

\[ \cdots \circ E(t_m) \circ E(t_{m-1}) \circ \cdots \circ E(t) \]

may have commuting operators, in which case the history path of a black hole would be nonunique, and, as a consequence, noninvertible, contradicting one of the central tenets of quantum theory. Hence, from that standpoint, their brackets under composition ought not to vanish:

\[ [A(t_1), A(t_2)] = A(t_1) \circ A(t_2) - A(t_2) \circ A(t_1) \neq 0, \]

\[ [E(t_1), E(t_2)] = E(t_1) \circ E(t_2) - E(t_2) \circ E(t_1) \neq 0. \]

That is a necessary condition. So the operator algebra of \{A(t), E(t)\} is properly defined over the field of quaternions, so that the chronological products are unique (hence invertible via substitution \( E(t_m) \mapsto A(t_{n-m}) \)). Allowing mixed chronological products would lead to negative energy vacuum fluctuations. The time degenerates into a discretely-valued parameter. Precisely, we set the time interval to be bounded below by the Planck time constant:

\[ t_{m+1} - t_m \geq t_P = \sqrt{\frac{\hbar G}{c^3}}. \]

To cram more information into the chronological products, one may want to define this operator algebra over the field of octonions thus making it nonassociative.

5.5. No-Go Theorem.

In a deterministic Universe one should be able to decide whether a gravitational collapse will result in a known stationary black hole well in advance. Any conceivable litmus test has to do with the initial mass distribution, angular momentum, charge. We present one such test based on the differential invariants of Section 4.

Since the moment in time of the collapsing matter settling down to a stationary EVE solution \( t_{\text{still}} \) depends crucially on the radius of the initial mass distribution, we can modify our representative set \( \{5,5\} \) to last long enough to accommodate a large class of gravitational collapse scenarios just by keeping the matter within a fixed radius:

\[ U = \{0 < t < t_{\text{still}} + \epsilon, \quad 0 < x, \quad y < \frac{\sqrt{2a}}{2} + \epsilon, \quad z^2 < (r_+ + \epsilon)^2 \}. \]

Furthermore, we confine those bounded mass distributions to have an \( SO(2) \) symmetry configuration, so that the Kerr solution is a distinct collapse endpoint. Specifically, we state
Theorem 5.2. For every eigenvalue of the (parametric) Hill operator $\lambda_{Kerr}^2(x, y)$, there exists a function $F_{2s}^{Kerr}(t, x, y)$ defined on the Kerr representative set \((5.30)\) such that its values antedating $t_{\text{still}}$ are sufficient to predict whether an asymptotically flat axisymmetric gravitational collapse configuration will settle down to the Kerr spacetime.

Proof. Our proof utilizes the universal oscillator equation. As matter collapses, the second time derivative of the attendant periodic spectrum discriminant (formally introduced in \((3.32)\)) undergoes some oscillations. A simple mechanical analogue would be a pendulum with some time-dependent restoring force, which grows in magnitude from zero to a maximum at the time $t_{\text{max}} < t_{\text{still}}$, then gradually returns to zero at $t_{\text{still}}$. The fact that this oscillator is anharmonic is encapsulated by the presence of the (unknown) damping function $F_{2s}^{Kerr}(t, x, y)$:

\[
(5.31) \left[ \frac{\partial^2}{\partial t^2} + 2F_{2s}^{Kerr}(t, x, y) \frac{\partial}{\partial t} + 1 \right] \frac{\partial^2 \star(t, x, y, \lambda_{2s}^{Kerr}(x, y))}{\partial t^2} = 0.
\]

We are primarily interested in the behavior of solutions (known via our Laplace-Beltrami $\rightarrow$ Nash-Gromov $\rightarrow$ Hill procedure) on the interval $[0, t_{\text{max}}]$, as the essential features of the (pending) Kerr black hole would be revealed early.

Using the properties of our ODE, we decompose solutions into the steady-state factor (de facto an amplitude), and the oscillatory part:

\[
(5.32) \frac{\partial^2 \star(t, x, y, \lambda_{2s}^{Kerr}(x, y))}{\partial t^2} = A_{\frac{\partial^2}{\partial t^2}} \cos(at + \theta).
\]

The steady-state factor is expressly a function of the damping. That allows us to apply the inverse function theorem in a neighborhood of $t = 0$, where the damping function dominates. As a result, we obtain $F_{2s}^{Kerr}(A_{\frac{\partial^2}{\partial t^2}})$. It needs to be reiterated, this procedure does not depend on $t_{\text{max}}$, a fortiori $t_{\text{still}}$.

Now we approximate by polynomials to compute first few polynomial coefficients of the expansion of $F_{2s}^{Kerr}$ in terms of the steady-state factor. Since those coefficients are linked to the eigenvalue $\lambda_{2s}^{Kerr}(x, y)$, it follows that an abstract discriminant $\star(t, x, y, \lambda_{2s}^{Kerr}(t, x, y))$ that does not produce the same polynomial coefficients will never settle down to the Kerr spacetime. \(\square\)

The alternative scenarios include, among others, precession and nutation, as well as incomplete event horizon formation in spite of the fact that all the initial data are axisymmetric. The only possible incomplete event horizons would have to be those with axisymmetric ‘bold spots’.

5.6. Beyond Monoparametric Solutions.

Our solution to the equivalence problem of nondifferentiable metrics is strictly local. On the manifold of general relativity, some ground is gained with the assumption of asymptotic flatness. The preframes are somewhat extended over a larger set in view of the representative set being constant in time. Still, our method only allows for a single black hole. To deal with several black holes, comoving or coalescing, a single global blow-up parameter is insufficient. Instead, we would need to utilize two (or more) local parameters. Hence new invariants based on relative positions may emerge. The Hill operator is not equipped to handle those. A natural
generalization would seem to be the two-dimensional Schrödinger equation
\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + Q(x_1, x_2) \right] f = \lambda f.
\]
For real \( Q(x_1, x_2) \in L^2(\mathbb{R}^2/\mathbb{Z}^2) \), and a fixed \( \vec{v} \in \mathbb{Z}^2 \), such that \( f(x_1 + v_1, x_2) = f(x_1, x_2 + v_2) = f(x_1, x_2) \), this poses a self-adjoint boundary value problem with discrete spectrum (customarily denoted by \( E_i(\vec{v}) \)), interpreted as the energy levels in solid state physics. Solutions are represented by a curve on the two-dimensional torus, called the real Fermi curve. It is complexified by modifying the original operator to add a purely imaginary term:
\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + i\vec{v} \cdot \nabla + \vec{v}^2 + Q(x_1, x_2) \right].
\]
For this operator, the Fermi curve is a one-dimensional complex analytic variety in \( \mathbb{C}^2/\mathbb{Z}^2 \). Its doubly periodic spectrum possesses the necessary complexity [17], Section 16, yet the Torelli theorem remains valid [7], Section 18.

However, that entails a radical redefinition of the time dimension. Due to the presence of two distinct event horizons, at least one of those moving, the local times in the vicinity of them are not related by a Lorentz transformation. The difference is absolute and depends on the space coordinates.

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