SYMPLECTIC CAPACITIES OF HERMITIAN SYMMETRIC SPACES

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Abstract. Inspired by the work of G. Lu on pseudo symplectic capacities we obtain several results on the Gromov width and the Hofer–Zehnder capacity of Hermitian symmetric spaces of compact type. Our results and proofs extend those obtained by Lu for complex Grassmannians to Hermitian symmetric spaces of compact type. We also compute the Gromov width and the Hofer–Zehnder capacity for Cartan domains and their products.

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1. Introduction

Consider the open ball of radius \( r \),

\[
B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 < r^2 \}
\]

in standard symplectic space \((\mathbb{R}^{2n}, \omega_0)\), where \( \omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j \).

The Gromov width of a \( 2n \)-dimensional symplectic manifold \((M, \omega)\), introduced in [17], is defined as

\[
c_G(M, \omega) = \sup \{ \pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega) \}.
\]

By Darboux’s theorem \( c_G(M, \omega) \) is a positive number. Computations and estimates of the Gromov width for various examples can be found in [3], [4], [5], [7], [17], [18], [23], [28], [34], [35], [36], [37], [45], [50].

Gromov’s width is an example of symplectic capacity introduced in [21] (see also [22]). A map \( c \) from the class \( C(2n) \) of all symplectic manifolds of dimension \( 2n \) to \([0, +\infty)\) is called a symplectic capacity if it satisfies the following conditions:

1. **Monotonicity** if there exists a symplectic embedding \((M_1, \omega_1) \to (M_2, \omega_2)\) then \( c(M_1, \omega_1) \leq c(M_2, \omega_2) \);
2. **Conformality** \( c(M, \lambda \omega) = |\lambda| c(M, \omega) \), for every \( \lambda \in \mathbb{R} \setminus \{0\} \);
3. **Nontriviality** \( c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0) \).

Here \( B^{2n}(1) \) and \( Z^{2n}(1) \) are the open unit ball and the open cylinder in the standard \((\mathbb{R}^{2n}, \omega_0)\), i.e.

\[
Z^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2 \}.
\]

Note that the monotonicity property implies that \( c \) is a symplectic invariant. The existence of a capacity is not a trivial matter. It is easily seen that the Gromov width is the smallest symplectic capacity, i.e. \( c_G(M, \omega) \leq c(M, \omega) \) for any capacity \( c \). Note that the nontriviality property for \( c_G \) comes from the celebrated Gromov’s nonsqueezing theorem stating that the existence of a symplectic embedding of \( B^{2n}(r) \) into \( Z^{2n}(R) \) implies \( r \leq R \). Actually it is easily seen that the existence
of any capacity implies Gromov’s nonsqueezing theorem. H. Hofer and E. Zehnder [21] prove the existence of a capacity, denoted by $c_{HZ}$ which plays, for example, an important role in the study of Hofer geometry on the group of symplectomorphisms of a symplectic manifold and in establishing the existence of closed characteristics on or near an energy surface. However, it is difficult to compute or estimate $c_{HZ}$ even for closed symplectic manifolds. So far the only examples are closed surfaces, for which $c_{HZ}$ has been computed as the area [47], and complex projective spaces and their products. H. Hofer and C. Viterbo [20] proved that $c_{HZ}(\mathbb{C}P^n, \omega_{FS}) = \pi$. This has been extended by G. Lu to the product of projective spaces (see Theorem 1.21 in [34] or (10) below). Lu’s ingenious idea was that of defining and studying the concept of pseudo symplectic capacity, more flexible than that of symplectic capacity, and its link with Gromov-Witten invariants (see Section 3 below). This allows him to obtain several important results, e.g. the Gromov width of Grassmannians and their products and a lower bound for the Hofer–Zehnder capacity for the product of any closed symplectic manifold with a Grassmannian. One of the aims of this paper is to extend Lu’s results to the case of Hermitian symmetric spaces of compact type (denoted by HSSCT in the sequel). Moreover, we compute the Gromov width and Hofer–Zehnder capacity of Cartan’s domains and their products. In the next section we provide a description of our results and the ideas of their proofs.

2. Statements of the main results

The following three theorems describe our results about the Gromov width and the Hofer-Zehnder capacity of HSSCT.

**Theorem 1.** Let $(M, \omega_{FS})$ be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form $\omega_{FS}$ normalized so that $\omega_{FS}(A) = \pi$ for the generator $A \in H_2(M, \mathbb{Z})$. Then

$$c_G(M, \omega_{FS}) = \pi. \quad (4)$$

**Theorem 2.** Let $(M_i, \omega_{FS}^i), i = 1, \ldots, r$, be irreducible HSSCT of complex dimension $n_i$ endowed with the canonical symplectic (Kähler)
forms $\omega_{FS}$ normalized so that $\omega_{FS}(A_i) = \pi$ for the generator $A_i \in H_2(M_i, \mathbb{Z})$, $i = 1, \ldots, r$. Then

$$c_G (M_1 \times \cdots \times M_r, \omega_1^{\times} \oplus \cdots \oplus \omega_r^{\times}) = \pi. \tag{5}$$

Moreover, if $a_1, \ldots, a_r$ are nonzero constants, then

$$c_G (M_1 \times \cdots \times M_r, a_1 \omega_1^{\times} \oplus \cdots \oplus a_r \omega_r^{\times}) \leq \min\{|a_1|, \ldots, |a_r|\} \pi \tag{6}$$

and

$$c_{HZ} (M_1 \times \cdots \times M_r, a_1 \omega_1^{\times} \oplus \cdots \oplus a_r \omega_r^{\times}) \geq (|a_1|+\cdots+|a_r|) \pi. \tag{7}$$

**Theorem 3.** Let $(M, \omega_{FS})$ be an irreducible HSSCT and $(N, \omega)$ be any closed symplectic manifold. Then, for any nonzero real number $a$,

$$c_{G}(N \times M, \omega \oplus a\omega_{FS}) \leq |a| \pi. \tag{8}$$

Formulas (4) and (5) extend Theorem 1.15 and formula (22) in [34] respectively (valid for the Grassmannians) to the case of HSSCT. The lower bounds $c_G(M, \omega_{FS}) \geq \pi$ in Theorem 1 and

$$c_G (M_1 \times \cdots \times M_r, \omega_1^{\times} \oplus \cdots \oplus \omega_r^{\times}) \geq \pi$$

in Theorem 2 are obtained by using the results in [11] which imply the existence of a symplectic embedding of the noncompact dual $(\Omega, \omega_0)$ of $(M, \omega_{FS})$ into $(M, \omega_{FS})$ (where $\omega_0$ is the standard symplectic form of $\Omega \subset \mathbb{C}^n$, being $n$ the complex dimension of $M$) and by the existence of a symplectic embedding of $B^{2n}(1)$ into $(\Omega, \omega_0)$ (see Sections 4 and 5 below for details). The upper bounds $c_G(M, \omega_{FS}) \leq \pi$ and

$$c_G (M_1 \times \cdots \times M_r, \omega_1^{\times} \oplus \cdots \oplus \omega_r^{\times}) \leq \pi$$

follow by the use of Lu’s pseudo symplectic capacities and their estimation in terms of Gromov-Witten invariants. The key ingredient to obtain these upper bounds is the non vanishing of some genus-zero three-points Gromov-Witten invariants (cfr. Lemma 15 in Section 6).
below). Inequality (6), which extends (21) in [34] to HSSCT, is a consequence of (8) in Theorem 3, which in turn extends [34, Corollary 1.31].

When $M_j = \mathbb{C}P^1$ for all $j = 1, \ldots, r$, inequality (6) is indeed an equality, i.e.

$$c_G(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, a_1 \omega_{FS} \oplus \cdots \oplus a_r \omega_{FS}) = \min\{|a_1|, \ldots, |a_r|\} \pi.$$  

(9)

(see [39, Example 12.5] for a proof). We do not know the exact value of

$$c_G(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}, a_1 \omega_{FS}^1 \oplus \cdots \oplus a_r \omega_{FS}^r)$$

if $n_i > 1$ or $a_j \neq 1$ for some $i = 1, \ldots, r$ or $j = 1, \ldots, r$.

When $M$ and the $M_j$’s are projective spaces inequality (7) is an equality, namely

$$c_{HZ}(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}, a_1 \omega_{FS}^1 \oplus \cdots \oplus a_r \omega_{FS}^r) = (|a_1| + \cdots + |a_r|) \pi.$$  

(10)

In fact, Lu [34] was able to prove that

$$c_{HZ}(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}, a_1 \omega_{FS}^1 \oplus \cdots \oplus a_r \omega_{FS}^r) \leq (|a_1| + \cdots + |a_r|) \pi$$  

(11)

which, combined with (7), yields (10). To the authors’ best knowledge no upper bound of $c_{HZ}(M, \omega_{FS})$ is known for HSSCT $(M, \omega_{FS})$, even for the case of the complex Grassmannians (different from the projective space). In Remark 22 below we sketch the idea of Lu’s proof of the upper bound (11) and explain why his argument cannot be used to achieve a similar upper bound for HSSCT.

The following two theorems summarize our results on Gromov width and Hofer–Zehnder capacity of Cartan domains.
Theorem 4. Let \((\Omega, \omega_0)\) be a Cartan domain. Then
\[
c_G(\Omega, \omega_0) = \pi
\] (12)
and
\[
c_{HZ}(\Omega, \omega_0) = \pi.
\] (13)
Moreover, if \(\Omega_i \subset \mathbb{C}^{n_i}, i = 1, \ldots, r\) are Cartan domains of complex dimension \(n_i\) equipped with the standard symplectic form \(\omega_0^i\) of \(\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}\), then
\[
c_G(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r) = \pi.
\] (14)
If \(a_1, \ldots, a_r\) are nonzero constants, then
\[
c_G(\Omega_1 \times \cdots \times \Omega_r, a_1\omega_0^1 \oplus \cdots \oplus a_r\omega_0^r) \leq \min\{|a_1|, \ldots, |a_r|\}\pi
\] (15)

Theorem 5. Let \((\Omega, \omega_0)\) be a Cartan domain and let \((N, \omega)\) be any closed symplectic manifold. Then
\[
c_{HZ}(N \times \Omega, \omega \oplus \omega_0) = \pi.
\] (16)

The proof of Theorem 4 which extends the results in [35] valid for classical Cartan domains to the product of Cartan domains (including the exceptional ones), is based (together with the inclusion \(B^{2n}(1) \subset (\Omega, \omega_0)\)) on the fact that any \(n\)-dimensional Cartan domain \((\Omega, \omega_0)\) symplectically embeds into the cylinder \((Z^{2n}(1), \omega_0)\) (see Sections 4 and 5 for details).

The organization of the paper is as follows. In Section 3 we summarize the above mentioned Lu’s work and some of his results needed in this paper. In Section 4 we briefly recall some tools on Hermitian positive Jordan triple systems which will be used in Section 5 to construct the above mentioned embeddings of a Cartan domain into its compact dual, of the unit ball into a Cartan domain and of a Cartan domain into the unitary cylinder. Moreover in Subsection 5.1 we show how these symplectic embeddings could be used to get estimate and computation of the minimal number of Darboux charts needed to cover a HSSCT.
Finally, Section 6 is dedicated to the (conclusion of the) proofs of our theorems.

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3. Lu’s pseudo symplectic capacities and Gromov–Witten invariants

G. Lu [34] defines the concept of *pseudo symplectic capacity* by weakening the requirements for a symplectic capacity in such a way that this new concept depends on the homology classes of the symplectic manifold in question (the reader is referred to [34] for more details.). More precisely, if one denotes by $C(2n, k)$ the set of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold $(M, \omega)$ and $k$ nonzero homology classes $\alpha_i \in H_*(M; Q)$, $i = 1, \ldots, k$, a map $c^{(k)}$ from $C(2n, k)$ to $[0, +\infty]$ is called a $k$-pseudo symplectic capacity if it satisfies the following properties:

**(pseudo monotonicity)** if there exists a symplectic embedding $\varphi : (M, \omega_1) \rightarrow (M, \omega_2)$ then, for any $\alpha_i \in H_*(M_1; Q)$, $i = 1, \ldots, k$,

$$c^{(k)}(M_1, \omega_1; \alpha_1, \ldots, \alpha_k) \leq c^{(k)}(M_2, \omega_2; \varphi(\alpha_1), \ldots, \varphi(\alpha_k));$$

**(conformality)** $c^{(k)}(M, \lambda \omega; \alpha_1, \ldots, \alpha_k) = |\lambda|c^{(k)}(M, \omega; \alpha_1, \ldots, \alpha_k)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$ and all homology classes $\alpha_i \in H_*(M; Q) \setminus \{0\}$, $i = 1, \ldots, k$;

**(nontriviality)** $c(B^{2n}(1), \omega_0; pt, \ldots, pt) = \pi = c(Z^{2n}(1), \omega_0; pt, \ldots, pt)$, where we denote by $pt$ the homology class of a point.

Note that if $k > 1$ a $(k-1)$-pseudo symplectic capacity is defined by

$$c^{(k-1)}(M, \omega; \alpha_1, \ldots, \alpha_{k-1}) := c^{(k)}(M, \omega; pt, \alpha_1, \ldots, \alpha_{k-1})$$
and any $c^{(k)}$ induces a true symplectic capacity

$$c^{(0)}(M, \omega) := c^{(k)}(M, \omega; pt, \ldots, pt).$$

Observe also that (unlike symplectic capacities) pseudo symplectic capacities do not define symplectic invariants.

In [34] G. Lu was able to construct two 2-pseudo symplectic capacities (christened by Lu as pseudo symplectic capacities of Hofer–Zehnder type) denoted by $C^{(2)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2)$ and $C^{(2o)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2)$ respectively (see Definition 1.3 and Theorem 1.5 in [34]), where $\alpha_1$ and $\alpha_2$ are homology classes in $H_*(M; \mathbb{Q})$. Denote by

$$C_{\text{HZ}}(M, \omega) := C^{(2)}_{\text{HZ}}(M, \omega; pt, pt)$$

(resp. $C^{(2o)}_{\text{HZ}}(M, \omega) := C^{(2o)}_{\text{HZ}}(M, \omega; pt, pt)$) the corresponding true symplectic capacities associated to Lu’s pseudo symplectic capacities. In the next lemma we summarize some properties of the concepts involved so far.

**Lemma 6.** Let $(M, \omega)$ be any symplectic manifold. Then, for arbitrary homology classes $\alpha_1, \alpha_2 \in H_*(M; \mathbb{Q})$ and for a nonzero homology class $\alpha$, with $\dim \alpha \leq \dim M - 1$, the following inequalities hold true:

$$C^{(2)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2) \leq C^{(2o)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2)$$

(17)

$$C^{(2)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2) \leq C_{\text{HZ}}(M, \omega) \leq c_{\text{HZ}}(M, \omega)$$

(18)

$$C^{(2o)}_{\text{HZ}}(M, \omega; \alpha_1, \alpha_2) \leq C^{(2o)}_{\text{HZ}}(M, \omega) \leq c^{(o)}_{\text{HZ}}(M, \omega)$$

(19)

$$c_{G}(M, \omega) \leq C^{(2)}_{\text{HZ}}(M, \omega; pt, \alpha),$$

(20)

where $c^{(o)}_{\text{HZ}}(M, \omega)$ is the $\pi_1$-sensitive Hofer–Zehnder capacity introduced in [46], $c_{\text{HZ}}(M, \omega)$ is the Hofer-Zehnder capacity and $c_{G}(M, \omega)$ is the

1In the notations of [34] the generic classes $\alpha_1$ (resp. $\alpha_2$) are called $\alpha_0$ (resp. $\alpha_\infty$). The reason for this notation comes from the concept of hypersurface $S \subset M$ separating the homology classes $\alpha_0$ and $\alpha_\infty$ (see Definition 1.3 and the $(\alpha_0, \alpha_\infty)$-Weinstein conjecture at p.6 of [34]).
Gromov width of \((M, \omega)\). Furthermore, if \(M\) is closed then
\[
C_{HZ}(M, \omega) = c_{HZ}(M, \omega)
\]
and
\[
C^o_{HZ}(M, \omega) = c^o_{HZ}(M, \omega).
\]

**Proof.** See Lemma 1.4 and (12) in [34]. □

**Remark 7.** It follows by (17) and by the last two equalities that for a closed symplectic manifold \((M, \omega)\)
\[
c_{HZ}(M, \omega) \leq c^o_{HZ}(M, \omega).
\]
Thus inequality (7) in Theorem 2 holds true also when we replace \(c_{HZ}\) with \(c^o_{HZ}\).

When the symplectic manifold is closed the pseudo symplectic capacities \(C_{HZ}^{(2)}(M, \omega; \alpha_1, \alpha_2)\) and \(C_{HZ}^{(2o)}(M, \omega; \alpha_1, \alpha_2)\) can be estimated by other two pseudo symplectic capacities \(GW(M, \omega; \alpha_1, \alpha_2)\) and \(GW_0(M, \omega; \alpha_1, \alpha_2)\) defined in terms of Liu–Tian type Gromov-Witten invariants as follows. Let \(A \in H_2(M, \mathbb{Z})\): the Liu–Tian type Gromov–Witten invariant of genus \(g\) and with \(k\) marked points is a homomorphism
\[
\Psi^M_{A,g,k} : H_*(\overline{M}_{g,k};\mathbb{Q}) \times H_*(M;\mathbb{Q})^k \to \mathbb{Q}, \ 2g + k \geq 3
\]
where \(\overline{M}_{g,k}\) is the space of isomorphism classes of genus \(g\) stable curves with \(k\) marked points. When there is no risk of confusion, we will omit the superscript \(M\) in \(\Psi^M_{A,g,k}\). Roughly speaking, one can think of \(\Psi^M_{A,g,k}(C; \alpha_1, \ldots, \alpha_k)\) as counting, for suitable generic \(\omega\)-tame almost complex structure \(J\) on \(M\), the number of \(J\)-holomorphic curves of genus \(g\) representing \(A\), with \(k\) marked points \(p_i\) which pass through cycles \(X_i\) representing \(\alpha_i\), and such that the image of the curve belongs to a cycle representing \(C\) (the reader is referred to the Appendix in [34] and references therein for details).

In fact, there are several different constructions of Gromov-Witten invariants in the literature and the question whether they agree is not
trivial (see [34] and also Chapter 7 in [38]). The most commonly used are the Gromov–Witten invariants described in the book of D. McDuff and S. Salamon [38] which are homomorphisms
\[ \Psi_{A,g,m+2} : H_*(M; \mathbb{Q})^{m+2} \to \mathbb{Q}, \ m \geq 1 \]
and which play an important role in the proofs of this paper. The Lemma 9 below gives conditions under which these invariants agree with the ones considered by Lu.

Let \( \alpha_1, \alpha_2 \in H_*(M, \mathbb{Q}) \). Following [34] one defines
\[ GW_g(M, \omega; \alpha_1, \alpha_2) \in (0, +\infty] \]
as the infimum of the \( \omega \)-areas \( \omega(A) \) of the homology classes \( A \in H_2(M, \mathbb{Z}) \) for which the Liu–Tian Gromov–Witten invariant \( \Psi_{A,g,m+2}(C; \alpha_1, \alpha_2, \beta_1, \ldots, \beta_m) \neq 0 \) for some homology classes \( \beta_1, \ldots, \beta_m \in H_*(M, \mathbb{Q}) \) and \( C \in H_*(M_{g,m+2}; \mathbb{Q}) \) and integer \( m \geq 1 \) (we use the convention \( \inf \emptyset = +\infty \)). The positivity of \( GW_g \) reflects the fact that \( \Psi_{A,g,m+2} = 0 \) if \( \omega(A) < 0 \) (see, for example, Section 7.5 in [38]). Set
\[ GW(M, \omega; \alpha_1, \alpha_2) := \inf \{ GW_g(M, \omega; \alpha_1, \alpha_2) | g \geq 0 \} \in [0, +\infty]. \]

(Lemma 8. Let \( (M, \omega) \) be a closed symplectic manifold. Then
\[ 0 \leq GW(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2). \]
Moreover \( GW(M, \omega; \alpha_1, \alpha_2) \) and \( GW_0(M, \omega; \alpha_1, \alpha_2) \) are pseudo symplectic capacities and, if the dimension \( \dim M \geq 4 \) then, for nonzero homology classes \( \alpha_1, \alpha_2 \), we have
\[ C_{HZ}^{(2)}(M, \omega; \alpha_1, \alpha_2) \leq GW(M, \omega; \alpha_1, \alpha_2) \]
\[ C_{HZ}^{(2o)}(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2). \]
In particular, for every nonzero homology class \( \alpha \in H_*(M, \mathbb{Q}) \),
\[ C_{HZ}^{(2)}(M, \omega; pt, \alpha) \leq GW(M, \omega; pt, \alpha) \]
\[ C_{HZ}^{(2o)}(M, \omega; pt, \alpha) \leq GW_0(M, \omega; pt, \alpha). \]
Proof. See Theorems 1.10 and 1.13 in [34]. □

We end this section with the following lemmata fundamental for the proof of our results. Recall that a closed symplectic manifold is monotone if there exists a number \( \lambda > 0 \) such that \( \omega(A) = \lambda c_1(A) \) for \( A \) spherical (a homology class is called spherical if it is in the image of the Hurewicz homomorphism \( \pi_2(M) \to H_2(M, \mathbb{Z}) \)). Further a homology class \( A \in H_2(M, \mathbb{Z}) \) is indecomposable if it cannot be decomposed as a sum \( A = A_1 + \cdots + A_k, k \geq 2 \), of classes which are spherical and satisfy \( \omega(A_i) > 0 \) for \( i = 1, \ldots, k \).

**Lemma 9.** Let \( (M, \omega) \) be a closed monotone symplectic manifold. Let \( A \in H_2(M, \mathbb{Z}) \) be an indecomposable spherical class, let \( pt \) denote the class of a point in \( H_*(M_{g,m+2}; \mathbb{Q}) \) and let \( \alpha_i \in H_*(M, \mathbb{Z}), i = 1, 2, 3 \). Then the Liu–Tian Gromov–Witten invariant \( \Psi_{A,0,3}(pt; \alpha_1, \alpha_2, \alpha_3) \) agrees with the Gromov–Witten invariant \( \Psi_{A,0,3}(\alpha_1, \alpha_2, \alpha_3) \).

Proof. See [34, Proposition 7.6]. □

**Lemma 10.** Let \( (N_1, \omega_1) \) and \( (N_2, \omega_2) \) be two closed symplectic manifolds. Then for every integer \( k \geq 3 \) and homology classes \( A_2 \in H_2(N_2; \mathbb{Z}) \) and \( \beta_i \in H_*(N_2; \mathbb{Z}), i = 1, \ldots, k \),
\[
\Psi^{N_1 \times N_2}_{A_2,0,k}(pt; [N_1] \otimes \beta_1, \ldots, [N_1] \otimes \beta_{k-1}, pt \otimes \beta_k) = \Psi^{N_2}_{A_2,0,k}(pt; \beta_1, \ldots, \beta_k).
\]

Proof. See [34, Proposition 7.4]. □

4. Hermitian positive Jordan triple system

We refer the reader to [42] (see also [33]) for more details on Hermitian symmetric spaces of noncompact type (HSSNT) and Hermitian positive Jordan triple systems (HPJTS).

**Definitions and notations.** A Hermitian Jordan triple system is a pair \( (\mathcal{M}, \{\cdot,\cdot\}) \), where \( \mathcal{M} \) is a complex vector space and \( \{\cdot,\cdot\} \) is a map
\[
\{\cdot,\cdot\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}
\]
\[
(u, v, w) \mapsto \{u, v, w\}
\]
which is $\mathbb{C}$-bilinear and symmetric in $u$ and $w$, $\mathbb{C}$-antilinear in $v$ and such that the following *Jordan identity* holds:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} =$$

$$= \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}. $$

For $x, y, z \in \mathcal{M}$ consider the operators

$$T(x, y) z = \{x, y, z\}$$

$$Q(x, z) y = \{x, y, z\}$$

$$Q(x, x) = 2Q(x)$$

$$B(x, y) = \text{id}_\mathcal{M} - T(x, y) + Q(x)Q(y).$$

The operator $B(x, y)$ is called the *Bergman operator*. A Hermitian Jordan triple system is called *positive* if the sesquilinear form

$$(u \mid v) = \frac{1}{\gamma} \text{tr} T(u, v)$$

is a Hermitian product, where $\gamma$ is a positive constant called the *genus* of $(\mathcal{M}, \{\cdot, \cdot\})$.

**HSSNT associated to HPJTS.** M. Koecher ([25], [26]) discovered that to every HPJTS $(\mathcal{M}, \{\cdot, \cdot\})$ one can associate an Hermitian symmetric space of noncompact type, in its realization as circled² bounded symmetric domain $\Omega_\mathcal{M}$ centered at the origin $0 \in \mathcal{M}$. More precisely, $\Omega_\mathcal{M}$ is defined as the connected component containing the origin of the set of all $u \in \mathcal{M}$ such that $B(u, u)$ is positive definite with respect to the Hermitian product (24).

**HPJTS associated to HSSNT.** The HPJTS $(\mathcal{M}, \{\cdot, \cdot\})$ can be recovered from its associated HSSNT $\Omega_\mathcal{M}$ by defining $\mathcal{M} = T_0\Omega_\mathcal{M}$ (the tangent space to the origin of $\Omega_\mathcal{M}$) and

$$\{u, v, w\} = -\frac{1}{2} (R_0(u, v) w + J_0 R_0(u, J_0 v) w),$$

²The domain $\Omega \subset \mathcal{M}$ is circled if $e^{i\theta} \cdot \Omega = \Omega$
where $R_0$ (resp. $J_0$) is the curvature tensor of the Bergman metric (resp. the complex structure) of $\Omega_M$ evaluated at the origin. The reader is referred to Proposition III.2.7 in [2] for details. More informations on the correspondence between HPJTS and HSSNT can also be found at p. 85 of Satake’s book [44].

**Spectral decomposition.** Let $(\mathcal{M}, \{ , , \})$ be a HPJTS. An element $c \in \mathcal{M}$ is called *tripotent* if $\{c, c, c\} = 2c$. Two tripotents $c_1$ and $c_2$ are called *(strongly) orthogonal* if $T(c_1, c_2) = 0$. Each element $v \in \mathcal{M}$ has a unique *spectral decomposition*

$$ v = \lambda_1 c_1 + \cdots + \lambda_s c_s \quad (\lambda_1 > \cdots > \lambda_s > 0), $$

where $(c_1, \ldots, c_s)$ is a sequence of pairwise orthogonal (with respect to (24)) tripotents and the $\lambda_j$’s are real numbers called eigenvalues of $v$.

The integer $s$ is called the *rank* of $v$ and is denoted by $	ext{rk}(v)$. The *rank* of $\mathcal{M}$ is the positive integer $r$ defined as $r = \max \{\text{rk}(z) \mid z \in \mathcal{M}\}$. The elements $z \in \mathcal{M}$ such that $	ext{rk}(z) = r$ are called *regular*.

Let us denote by $\|v\|_{\text{max}}$ the largest eigenvalue of $v$. Due to the convexity of $\Omega_M$, $\|v\|_{\text{max}}$ is a norm on $\mathcal{M}$, called the *spectral norm*. The following proposition provides a description of the domain $\Omega_M$ in terms of its spectral norm.

**Proposition 11.** Let $\Omega_M \subset \mathcal{M}$ be the HSSNT associated to $(\mathcal{M}, \{ , , \})$. Then

$$ \Omega_M = \{v \mid \|v\|_{\text{max}} < 1\}. \quad (25) $$

**Proof.** See [33, Corollary 3.15].

5. **Cartan domains, their compact duals and some symplectic embeddings**

Let $(\mathcal{M}, \{ , , \})$ be a HPJTS and $\Omega_M$ be its associated HSSNT. Let $n$ be the complex dimension of $\mathcal{M}$. By fixing an orthonormal basis $\mathcal{C} = \{e_1, \ldots, e_n\}$ of $(\mathcal{M}, (\cdot | \cdot))$ we get the identification

$$ \mathcal{M} \to \mathbb{C}^n, \quad v \mapsto z = (z_1, \ldots, z_n), \quad v = z_1 e_1 + \cdots + z_n e_n, \quad (26) $$
which induces an isometry between \( (\mathcal{M}, \langle \cdot | \cdot \rangle) \) and \((\mathbb{C}^n, h_0)\), where \( h_0 \) is the canonical Hermitian product on \( \mathbb{C}^n \). Under the identification

\[
(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)
\]

between \( \mathbb{C}^n \) and \( \mathbb{R}^{2n} \) we have \( h_0 = g_0 + i\omega_0 \), where \( g_0 = \sum_{j=1}^n dx_j^2 + dy_j^2 \) is the standard scalar product on \( \mathbb{R}^{2n} \) and \( \omega_0 \) is the canonical symplectic form \( \omega_0 = \sum_{j=1}^n dx_j \wedge dy_j \) on \( \mathbb{C}^n = \mathbb{R}^{2n} \). From now on we assume \( \mathcal{M} \) is simple which is equivalent to the irreducibility of \( \Omega_\mathcal{M} \). Then, under the previous identification, the HSSNT \( \Omega_\mathcal{M} \) corresponds to a bounded symmetric domain \( \Omega = \overline{\mathbf{e}(\Omega_\mathcal{M})} \subset \mathbb{C}^n \). The complex and Riemannian geometry of these domains, also called Cartan domains, is well-known (see, e.g. [24]). Below, we describe some symplectic geometric aspects of these domains and their compact duals needed in this paper (for the concept of compact dual see [19] or [11] and references therein).

Let \( \Omega \subset \mathbb{C}^n \) be a Cartan domain and let \( M \) be its compact dual. Then \( M \) is an \( n \)-dimensional HSSCT. Denote by

\[
BW : M \rightarrow CP^N
\]  

(27)

the Borel–Weil (holomorphic) embedding. It is well-known (see e.g. [49]) that the pull-back \( BW^*\omega_{FS} \) of the Fubini–Study form \( \omega_{FS} \) of \( CP^N \) is a homogeneous Kähler-Einstein form on \( M \) (\( \omega_{FS} \) is the Kähler form which, in the homogeneous coordinates \([z_0, \ldots, z_N]\) on \( CP^N \), is given by \( \omega_{FS} = 2i\partial\bar{\partial} \log(|z_0|^2 + \cdots + |z_N|^2) \)). In this paper, we denote (with a slight abuse of notation and terminology) by \( \omega_{FS} \) the form \( BW^*\omega_{FS} \) and call it the Fubini–Study form on \( M \). The symplectic form \( \omega_{FS} \) can be equivalently described as the symmetric or canonical form on \( M \) normalized so that \( \omega_{FS}(A) = \pi \) for the generator \( A \in H_2(M, \mathbb{Z}) \).

The domain \((\Omega, \omega_0)\) can be embedded into \((M, \omega_{FS})\).
Let \((\Omega, \omega_0)\), \( \Omega \subset \mathbb{C}^n \), be a Cartan domain equipped with the canonical symplectic form \( \omega_0 \) of \( \mathbb{R}^{2n} \) and let \((M, \omega_{FS})\) be its compact dual. In [11] the first author in collaboration with A. J. Di Scala, by using HPJTS, construct an embedding

\[
\Phi_\Omega : \Omega \rightarrow M
\]  

(28)
such that $\Phi^*_\Omega \omega_{FS} = \omega_0$.

Actually in [11] much more is proved, namely that the embedding $\Phi_\Omega$ induces a global symplectomorphism

$$\Phi_\Omega : (\Omega, \omega_0) \to (M \setminus \text{Cut}_0(M), \omega_{FS})$$

where $\text{Cut}_0(M)$ is the cut locus of $(M, \omega_{FS})$ with respect to a fixed point $0 \in M$ (see [11, Theorem 1.1]). This diffeomorphism has been christened in [11] as a \textit{symplectic duality} due to the fact that, amongst other properties, it also satisfies $\Phi^*_\Omega \omega_0 = \omega_{hyp}$, where $\omega_0$ denotes the standard form on $\mathbb{C}^n \cong M \setminus \text{Cut}_0(M)$ and $\omega_{hyp}$ is the hyperbolic metric on $\Omega$ (see either [11] or [12] for details and also [9], [13], [29], [31], [32] and [40] for the construction of explicit symplectic coordinates.

\textbf{Remark 12.} In [34, Lemma 4.1 in Section 4] it is shown the existence of a symplectic embedding

$$\Phi_{\Omega[k,n]} : \Omega_I[k,n] \to G(k,n)$$

from the first Cartan domain $\Omega_I[k,n] \subset \mathbb{C}^{k(n-k)}$ into its compact dual $G(k,n)$, namely the complex Grassmannian of $k$ dimensional subspaces of $\mathbb{C}^n$. Our result (28) extends Lu’s results to all HSSCT.

\textbf{The unitary ball} $(B^{2n}(1), \omega_0)$ \textbf{can be embedded into} $(\Omega, \omega_0)$.

Let $v = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular point $v \in \Omega_M \subset \mathcal{M}$, then the distance $d_0(0,v)$ from the origin $0 \in \mathcal{M}$ to $v$ is given by

$$d_0(0,v) = (v \mid v)^{1/2} = \sqrt{\sum_{j=1}^r \lambda_j^2},$$

(see [42, Proposition VI.3.6] for a proof). Since the set of regular points of $\mathcal{M}$ is dense ([42, Proposition IV.3.1]) we conclude, by (25) and by the identification $\Omega_\mathcal{M} \cong \Omega$ (induced by $(\mathcal{M}, (\cdot \mid \cdot)) \cong (\mathbb{C}^n, h_0)$) that

$$(B^{2n}(1), \omega_0) \subset (\Omega, \omega_0).$$

\textbf{Remark 13.} The inclusion (31) has been obtained in [34, Lemma 4.2, Section 4] for the case of the first Cartan domain, namely $B^{2k(n-k)}(1) \subset$
\[ \Omega_I[k, n] \] (see also \cite{33} for the case of classical Cartan domains). Combining this with the symplectic embedding \cite{29} Lu was able see \cite{34, Theorem 1.35} to obtain the upper bound

\[ F(G(k, n), \omega_{FS}) \leq \left\lfloor \frac{n}{k} \right\rfloor, \]

where \( F(N, \omega) \) denotes the Fefferman invariant of a closed symplectic manifold \((N, \omega)\), namely the largest integer \( p \) for which there exists a symplectic packing by \( p \) open unit balls, and \( \left\lfloor \frac{n}{k} \right\rfloor \) is the largest integer less than or equal to \( \frac{n}{k} \). The authors believe it is an intriguing problem (by using the techniques of this paper) to give a similar upper bound for all HSSCT.

The domain \((\Omega, \omega_0)\) can be embedded into \((Z^{2n}(1), \omega_0)\).

Let \( Z^{2n}(1) = \{ (x, y) \mid x_1^2 + y_1^2 < 1 \} \) be the unitary cylinder in \( \mathbb{R}^{2n} \).

Let \( v = \lambda_1 c_1 + \cdots + \lambda_r c_r \) be the spectral decomposition of a regular point \( v \in \Omega_M \subset M \). By \cite{30} and by the continuity of \( d_0 \) (the distance function from the origin \( 0 \in M \)) we see that \( d_0(0, c_1) = 1 \). Set \( c := c_1 \), by \cite{33} Corollary 4.8 \( c \in \partial \Omega_M \). As \( \Omega_M \) is convex \cite{33} Corollary 4.7, by the supporting hyperplane property there exists a real hyperplane \( \pi \) of \( M \) through \( c \) not intersecting \( \Omega_M \). Denote by \( p = \overline{\pi}(c) \in \partial \Omega \) the image of the tripotent \( c \) by the isometry \cite{26}. Hence \( p \in S^{2n-1} \), where \( S^{2n-1} = \partial B^{2n}(1) \) is the \((2n-1)\)-dimensional unit sphere centered at the origin of \( \mathbb{R}^{2n} \). By \cite{31}, \( B^{2n}(1) \subset \Omega = \overline{\pi}(\Omega_M) \) and hence \( \overline{\pi}(\pi) = T_p S^{2n-1} \). By applying the same argument to any tripotent \( c_\theta := e^{i\theta} \cdot c \), we see that \( \Omega \) is contained in the cylinder \( \tilde{Z} \) bounded by the envelope of the family of real hyperplanes \( \{ T_p S^{2n-1}, p_\theta = \overline{\pi}(c_\theta) \}_{\theta \in \mathbb{R}} \). Let \( W \in U(n) \) such that

\[ W \cdot p = (z_1, 0, \ldots, 0) \]

for some \( z_1 \in \mathbb{C}, \| z_1 \| = 1 \). It follows that \( W \cdot \tilde{Z} = Z^{2n}(1) \) and the desired symplectic embedding of \((\Omega, \omega_0)\) into \((Z^{2n}(1), \omega_0)\) is given by

\[ \Omega \subset \tilde{Z} \xrightarrow{W} Z^{2n}(1). \]  

(32)

Remark 14. A similar (symplectic) embedding \((\Omega, \omega_0) \hookrightarrow (Z^{2n}(1), \omega_0)\) has been considered in \cite{33} for the classical Cartan domains.
5.1. **Minimal symplectic atlases of HSSCT.** Consider a closed symplectic manifold \((M, \omega)\). In [43] Yu. B. Rudyak and F. Schlenk have introduced the *symplectic Lustermik-Schnirelmann category* \(S(M, \omega)\), defined as
\[
S(M, \omega) = \min \{ k \mid M = \cup_i U_i \}
\]
where each \(U_i\) is the image \(\Phi_i(U_i)\) of a symplectic embedding \(\Phi_i : U_i \to M\) of a bounded subset \(U_i\) of \((\mathbb{R}^{2n}, \omega_0)\) diffeomorphic to an open ball in \(\mathbb{R}^{2n}\). From our results one obtains the upper bound
\[
S(M, \omega_{FS}) \leq N + 1 \tag{33}
\]
for Lustermik-Schnirelmann category of a Hermitian symmetric space of compact type \((M, \omega_{FS})\), where \(N\) is the dimension of the complex projective space \(\mathbb{C}P^N\) where the manifold can be Kähler embedded via the Borel–Weil embedding \(BW : M \to \mathbb{C}P^N\) (see (27)). Indeed, as in the case of the complex Grassmannian \(G(k, n)\) (where the Borel–Weil embedding is given by the Plücker embedding \(P : G(k, n) \to \mathbb{C}P^{C(n)}\)), one can define a *canonical atlas* on \((M, \omega_{FS})\) using the \(N + 1\) holomorphic charts \(\Omega_0, \ldots, \Omega_N\) defined as \(\Omega_j = M \setminus \{BW^{-1}(Z_j = 0)\}\), and \(Z_j = 0, j = 0, \ldots, N\), is the standard hyperplane of \(\mathbb{C}P^N\). Each \(\Omega_j \subset \mathbb{C}^n, j = 0, \ldots, N\), is biholomorphic to the noncompact dual \(\Omega\) of \(M\). It follows by (28) that \((\Omega_j, \omega_0)\) can be symplectically embedded into \((M, \omega_{FS})\) for \(j = 1, \ldots, N\). On the other hand, each \(\Omega\) is a bounded domain diffeomorphic to the ball in \(\mathbb{R}^{2n}\) and so (33) follows.

Our knowledge of the Gromov width of any HSSCT \((M, \omega_{FS})\) can be used to estimate and compute the minimal numbers of Darboux charts needed to cover \(M\). This number, introduced in [43] and denoted there by \(S_B(M, \omega)\), has been computed and estimated for various symplectic manifolds including the complex Grassmannian (see [43, Corollary 5.10]). Similar computations and related problems (which will appear in a forthcoming paper) can be done for all HSSCT using the results of this section.
6. The proofs of Theorems 1, 2, 3, 4 and 5

The following lemma is the key ingredient to achieve the upper bound of Gromov width in Theorems 1, 2 and 3.

Lemma 15. Let \((M, \omega_{FS})\) be an irreducible HSSCT of complex dimension \(n\) and let \(A = [CP^1]\) be a generator of \(H_2(M, \mathbb{Z})\) such that \(\omega_{FS}(A) = \pi\). Then there exist \(\alpha(M, \omega_{FS})\) and \(\beta(M, \omega_{FS})\) in \(H_*(M, \mathbb{Z})\) such that

\[
\dim \alpha(M, \omega_{FS}) + \dim \beta(M, \omega_{FS}) = 4n - 2c_1(A)
\]

and

\[
\Psi_{A,0,3}(pt; \alpha(M, \omega_{FS}), \beta(M, \omega_{FS}), pt) \neq 0.
\] (34)

Proof. Since the canonical symplectic form \(\omega_{FS}\) is Kähler-Einstein, it follows that \((M, \omega_{FS})\) is monotone, so that Lemma 9 applies under our assumptions. We need then to show the existence, for every irreducible HSSCT, of a non-vanishing Gromov-Witten invariant \(\Psi_{A,0,3}(\alpha(M, \omega_{FS}), \beta(M, \omega_{FS}), pt)\). This follows from the results about the quantum cohomology of these spaces proved in [1], [8], [27], [41], [48]. Let us recall that the quantum cohomology ring of \(M\) is the product \(H_*(M) \otimes \mathbb{Z}[q]\) endowed with the quantum cup product, defined for any two homology classes \(\alpha, \beta \in H_*(M)\) as

\[
\alpha \ast \beta = \sum_{\gamma, d} \Psi_{dA,0,3}(\alpha, \beta, \gamma) \gamma^* q^d,
\] (35)

the sum running over \(d \in \mathbb{Z}\) and \(\gamma\) such that \(\dim(\alpha)+\dim(\beta)+\dim(\gamma) = 4n - 2dc_1(A)\), where \(\gamma^*\) denotes the dual class of \(\gamma\).

Looking at the formulas for the quantum product proved in the above-mentioned references, it is not hard to find a Gromov-Witten invariant \(\Psi_{A,0,3}(\alpha, \beta, pt)\) which does not vanish for some classes \(\alpha, \beta\). More in detail, when \(M\) is the Grassmannian \(G(k, n)\), by [48] there exist \(\alpha \in H_{2k(n-1)}(M)\) and \(\beta \in H_{2n(k-1)}(M)\) such that this holds; by [41] the same is true for suitable \(\alpha = \beta \in H_{(n-1)(n-2)}(SO(2n)/U(n))\); by Corollary 8 in [27] \(\alpha\) and \(\beta\) can be taken of codimension \(n\) and 1
when $M$ is the Lagrangian Grassmannian $LG(n, 2n)$; in [8] (see the formulas in Sections 5.1 and 5.2) it is shown that for the Cayley plane (resp. for the Freudenthal variety) one can take for example $\alpha$ and $\beta$ of codimensions 8 and 4, (resp. of codimensions 13 and 5). Finally, in [1] is studied the quantum cohomology of complete intersections, which in particular gives a non-vanishing Gromov-Witten invariant for the complex quadric.

Remark 16. Formulas for quantum products in the homogeneous spaces, expressed in terms of the combinatorial invariants of the Lie algebra of the symmetry group of the space (Dynkin diagram and Weyl group), can be found in [16] (see also [14], [15]) and could be also used to prove the above Lemma.

We are now in the position to prove Theorem 1.

Proof of Theorem 1 In order to use Lemma 8 we can assume, without loss of generality, that $\dim M \geq 4$. Indeed the only irreducible HSSCT of dimension $\leq 4$ are either $(\mathbb{C}P^1, \omega_{FS})$ or $(\mathbb{C}P^2, \omega_{FS})$ whose Gromov width is well-known to be equal to $\pi$. Let $A = [\mathbb{C}P^1]$ be the generator of $H_2(M, \mathbb{Z})$ as in the statement of Theorem 1. Then the value $\omega_{FS}(A) = \pi$ is clearly the infimum of the $\omega_{FS}$-areas $\omega_{FS}(B)$ of the homology classes $B \in H_2(M, \mathbb{Z})$ for which $\omega_{FS}(B) > 0$.

By Lemma 15 we have $\Psi_{A,0,3}(pt; pt, \alpha, \beta) \neq 0$, with $\alpha = \alpha(M, \omega_{FS})$ and $\beta = \beta(M, \omega_{FS})$, and hence, by definition of $GW_g$, $GW(M, \omega_{FS}; pt, \gamma) = GW_0(M, \omega_{FS}; pt, \gamma) = \pi$ \hspace{1cm} (36)

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. It follows by the inequalities (17), (20), (22) and (23) that

\[ c_G(M, \omega_{FS}) \leq C^{(2)}_{HZ}(M, \omega_{FS}; pt, \gamma) \leq C^{(2o)}_{HZ}(M, \omega_{FS}; pt, \gamma) \leq \pi \] \hspace{1cm} (37)

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. Combining this with the lower bound $c_G(M, \omega_{FS}) \geq \pi$ coming from the inclusion $B^{2n}(1) \subset (\Omega, \omega_0)$ (cfr. (31)), the symplectic embedding $\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$ (cfr.
(28)) and the monotonicity and nontriviality of $c_G$, one gets:

$$c_G(M, \omega_{FS}) = C_{HZ}^{(2)}(M, \omega_{FS}; pt, \gamma) = C_{HZ}^{(2o)}(M, \omega_{FS}; pt, \gamma) = \pi$$

(38)

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. This concludes the proof of Theorem 1. □

Remark 17. Observe that we have proven more than stated in Theorem 1. Indeed, we have computed the value of Lu’s pseudo symplectic capacities evaluated at the homology class of a point and at $\alpha(M, \omega_{FS})$ (or $\beta(M, \omega_{FS})$), namely

$$c_G(M, \omega) = C_{HZ}^{(2)}(M, \omega; pt, \alpha(M, \omega_{FS})) = C_{HZ}^{(2o)}(M, \omega; pt, \alpha(M, \omega_{FS})) =$$

$$= C_{HZ}^{(2)}(M, \omega; pt, \beta(M, \omega_{FS})) = C_{HZ}^{(2o)}(M, \omega; pt, \beta(M, \omega_{FS})) = \pi.$$ 

This extends the result obtained by G. Lu for the complex Grassmannian (cfr. [34, Theorem 1.15] for details) to HSSCT.

Remark 18. An alternative proof of the upper bound $c_G(M, \omega_{FS}) \leq \pi$ in Theorem 1 can be achieved by combining Lemma 15 with [23, Proposition 4.1] which asserts that if $(M, \omega)$ is a symplectic manifold of (real) dimension $2n$, $A \in H_2(M, \mathbb{Z})$ is an indecomposable spherical class and $\Phi_{A,0,3}(pt, \alpha_0, \beta_0) \neq 0$, for suitable $\alpha_0$ and $\beta_0$ in $H_*(M, \mathbb{Z})$ (which necessarily satisfy $\dim \alpha_0 + \dim \beta_0 = 4n - 2c_1(A)$) then $c_G(M, \omega) \leq \omega(A)$.

Using the same idea, A. C. Castro ([11]) recently found an upper bound for the Gromov width of homogeneous spaces $G/K$, with $G = SU(n)$, endowed with a general $SU(n)$-invariant symplectic structure $\omega$, provided a technical assumption on $\omega$ aimed to assure that some homology classes are $\omega$-indecomposable. When $G/K$ is the Grassmannian manifold, endowed with the Kähler-Einstein structure normalized as in our paper, we recover our result. The same method was used also in [50] to bound from above the Gromov width of $G/K$ when $G$ is any simple compact group and $K$ a maximal torus in $G$ (notice that the only HSSCT satisfying these assumptions is $\mathbb{C}P^1$).
In [23] the exact value of the Gromov width of the Grassmannian manifold is in fact calculated by giving also the lower bound. This follows from the existence of Hamiltonian circle actions on the manifold with an isolated fixed point and isotropy weights equal to 1, which by a refined version of the equivariant Darboux theorem allows the author to find an open set of the manifold equivariantly symplectomorphic to the Euclidean ball. This approach could probably be adapted to prove the lower bound at least for the quotients of the classical groups.

In order to prove Theorem 2 we need the following lemma, interesting on its own sake, which extends Lu’s formula (20) in [34, Theorem 1.16] (for the Grassmannian) to the case of HSSCT.

**Lemma 19.** Let $(M, \omega_{FS})$ be a HSSCT and let $(N, \omega)$ be any closed symplectic manifold. Then

\[
C_{HZ}^{(2o)}(N \times M, \omega \oplus a\omega_{FS}; pt, [N] \times \gamma) \leq |a|\pi
\]

for any $a \in \mathbb{R} \setminus \{0\}$ and $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$, with $\alpha(M, \omega_{FS})$ and $\beta(M, \omega_{FS})$ given by Lemma 15.

**Proof.** Since by [34] we have $\Psi^{M}_{A,0,3}(pt; \alpha, \beta, pt)) \neq 0$, with $\alpha = \alpha(M, \omega_{FS})$ and $\beta = \beta(M, \omega_{FS})$, it follows by Lemma 10 that

\[
\Psi^{N \times M}_{B,0,3}(pt; [N] \times \alpha(M, \omega_{FS}), [N] \times \beta(M, \omega_{FS}), pt) \neq 0
\]

for $B = 0 \times A$, where 0 denotes the zero class in $H_2(N, \mathbb{Z})$ and $A$ the generator of $H_2(M, \mathbb{Z})$. Hence (39) easily follows from (23) in Lemma 8. □

**Proof of Theorem 2.** To see (5) we assume $r > 1$ because of the result in Theorem 1. It immediately follows from (17) and (20) in Lemma 6 and by (39) that

\[
c_G (M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r) \leq \pi.
\]

On the other hand, we have the symplectic embeddings

\[
\times_{j=1}^r B^{2n_j}(1) \subset \times_{j=1}^r \Omega_j \xrightarrow{\Phi_{\Omega_1} \times \cdots \times \Phi_{\Omega_r}} \times_{j=1}^r M_j
\]
(induced by (31) and (28) respectively) and the natural inclusion
\[ B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^r B^{2n_j}(1). \]

Thus, it follows by the monotonicity and nontriviality of \( c_G \) that
\[ c_G \left( \bigtimes_{j=1}^r M_j, \bigoplus_{j=1}^r \omega_{FS}^j \right) \geq \pi. \]
Hence (5) follows. As we have already pointed out in the Introduction, inequality (6) is a straightforward consequence of (8) in Theorem 3.

Inequality (7) follows by (4), by the monotonicity of \( c_{HZ} \) and from the fact that for two compact symplectic manifolds \((N_1, \omega_1)\) and \((N_2, \omega_2)\)
\[ c_{HZ}(N_1 \times N_2, \omega_1 \oplus \omega_2) \geq c_{HZ}(N_1, \omega_1) + c_{HZ}(N_2, \omega_2) \]
(see [34, Lemma 4.3, p. 43] for a proof). This concludes the proof of Theorem 2. \(\square\)

**Remark 20.** The upper bound
\[ c_G \left( \bigtimes_{j=1}^r M_j, \bigoplus_{j=1}^r \omega_{FS}^j \right) \leq \pi \]
obtained in the proof of Theorem 2 can also be achieved by using the fact that HSSCT and their products are uniruled manifolds (see Definition 1.14, Theorem 1.27 in [34] and the remark following this theorem).

**Remark 21.** Note that in [34, Theorem 1.16] another interesting result is proven namely formula (21). Using the techniques developed so far one can prove the analogous of this formula, namely
\[ C_{HZ}^{(2o)}(X_{j=1}^r M_j, \bigoplus_{j=1}^r a_j \omega_{FS}^j; pt, X_{j=1}^r \alpha_j) \leq (|a_1| + \cdots + |a_r|)\pi, \]
for all \(a_j \in \mathbb{R} \setminus \{0\}\) and \(\alpha_j = \alpha_j(M_j, \omega_{FS}^j)\) or \(\beta_j = \beta_j(M_j, \omega_{FS}^j)\).

**Remark 22.** We do not know if the inequality
\[ c_{HZ} \left( \bigtimes_{j=1}^r M_j, \bigoplus_{j=1}^r a_j \omega_{FS}^j \right) \leq (|a_1| + \cdots + |a_r|)\pi. \]
holds true. Unfortunately, the proof given by Lu in the case of product of projective spaces and [34, Theorem 1.21]) do not extend to the general case of HSSCT. Indeed the Gromov–Witten invariant
\( \Psi_{A,0,m+2}(pt, pt, \beta_1, \ldots, \beta_m) \) of \( M = M_1 \times \cdots \times M_r \) does not vanish (for some homology classes \( \beta_1, \ldots, \beta_m \)) if and only if all the \( M_j \)'s are projective spaces, since it is easily checked that the dimension condition 
\[ \sum_{j=1}^{m} \deg(\beta_j) = 2(c_1(A) - \dim(M) - 1 + m), \] necessary for the Gromov-Witten invariant to be nonzero [38], p. 11, is satisfied only in this case. The reader is also referred to [34, Corollary 1.19 and Example 1.20]) for some comments and conjectures related to this problem.

**Proof of Theorem 3.** It follows from (17) and (20) in Lemma 6 and by (39) that 
\[ c_{\text{G}}(N \times M, \omega \oplus a \omega_{FS}) \leq C_{\text{HZ}}(N \times M, \omega \oplus a \omega_{FS}; pt, [N] \times \gamma) \leq |a| \pi, \]
where \( \gamma = \alpha(M, \omega_{FS}) \) (or \( \gamma = \beta(M, \omega_{FS}) \)), yielding the desired inequality (8).

**Proof of Theorem 4.** By (B2n(1), \( \omega_0 \)) \( \subset (\Omega, \omega_0) \overset{W}{\rightarrow} (Z^{2n}(1), \omega_0) \),
(given by (31) and (32) respectively) and the monotonicity and non-triviality of \( c_{\text{G}} \) and \( c_{\text{HZ}} \) we get \( c_{\text{G}}(\Omega, \omega_0) = c_{\text{HZ}}(\Omega, \omega_0) = \pi \), namely (12) and (13). Analogously, let us denote \( M_j \) the compact dual of \( \Omega_j \): by (6) and by the symplectic embedding \( \times \overset{r}{\times} \)
induced by (28) one obtains (15) which, together with the symplectic embedding \( \times \overset{r}{\times} B^{2n}(1) \subset \times \overset{r}{\times} \Omega_j \) (induced by (31)) and (40) yields (14).

In order to prove Theorem 5 we need the following interesting result of Lu.

**Lemma 23.** Let \( (N, \omega) \) be any closed symplectic manifold. Then, for any \( r > 0 \) one has 
\[ c_{\text{HZ}}(N \times B^{2n}(r), \omega \oplus \omega_0) = c_{\text{HZ}}(N \times Z^{2n}(r), \omega \oplus \omega_0) = \pi r^2. \]
where \( Z^{2n}(r) \) is given by (5).
Proof. See [34, Theorem 1.17, p.14].

Proof of Theorem 5. By \((B^{2n}(1), \omega_0) \subset (\Omega, \omega_0) \overset{W}{\to} (Z^{2n}(1), \omega_0)\) one has the embeddings

\[(N \times B^{2n}(1), \omega \oplus \omega_0) \subset (N \times \Omega, \omega \oplus \omega_0) \overset{id_N \times W}{\to} (N \times Z^{2n}(1), \omega \oplus \omega_0)\]

and so the desired \((16)\), i.e. \(c_{HZ}(N \times \Omega, \omega \oplus \omega_0) = \pi\), follows by Lemma 23 and the monotonicity of \(c_{HZ}\).

Final remarks on Seshadri constants

Our knowledge of the Gromov width of a HSSCT allows us to obtain an upper bound of the Seshadri constant of an ample line bundle over a HSSCT \((M, \omega_{FS})\). Recall that given a compact complex manifold \((N, J)\) and an holomorphic line bundle \(L \to N\) the Seshadri constant of \(L\) at a point \(x \in N\) is defined as the nonnegative real number

\[\epsilon(L, x) = \inf_{C \ni x} \frac{\int_C c_1(L)}{\text{mult}_x C},\]

where the infimum is taken over all irreducible holomorphic curves \(C\) passing through the point \(x\) and \(\text{mult}_x C\) is the multiplicity of \(C\) at \(x\) (see [10] for details). The (global) Seshadri constant is defined by

\[\epsilon(L) = \inf_{x \in M} \epsilon(L, x).\]

Note that Seshadri’s criterion for ampleness says that \(L\) is ample if and only if \(\epsilon(L) > 0\). P. Biran and K. Cieliebak [6, Prop. 6.2.1] have shown that

\[\epsilon(L) \leq c_G(M, \omega_L),\]

where \(\omega_L\) is any Kähler form which represents the first Chern class of \(L\), i.e. \(c_1(L) = [\omega_L]\). Consider now an irreducible HSSCT \((M, \omega_{FS})\) and the line bundle \(L \to M\) such that \(c_1(L) = \left[\frac{\omega_{FS}}{\pi}\right]\) (\(L\) can be taken as the pull-back via the Borel–Weil embedding \([27]\) of the universal bundle of \(\mathbb{C}P^N\)). Therefore, by using the upper bound \(c_G(M, \omega_{FS}) \leq \pi\) and the conformity of \(c_G\) we get:

Corollary 24. Let \((M, \omega_{FS})\) be an irreducible HSSCT and let \(L \to M\) as above. Then \(\epsilon(L) \leq 1.\)
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