Computation of the conformal algebra of 1+3 decomposable space-times

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Abstract

The conformal algebra of a 1+3 decomposable spacetime can be computed from the Conformal Killing Vectors (CKV) of the 3-space. It is shown that the general form of such a 3-CKV is the sum of a gradient CKV and a Killing or homothetic 3-vector. It is proved that spaces of constant curvature always admit such conformal Killing vectors. As an example the complete conformal algebra of a Gödel-type spacetime is computed. Finally it is shown that this method can be extended to compute the conformal algebra of more general non-decomposable spacetimes.

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1 Introduction

A spacetime is called 1+3 decomposable if it admits a covariantly constant non-null vector field. These spacetimes have a metric of the form:

\[ ds^2 = \varepsilon(dx^1)^2 + g_{\mu\nu}(x^\rho)dx^\mu dx^\nu \]

where \( \mu, \nu = 2, 3, 4 \), \( \partial/\partial x^1 \) is the constant vector field and \( \varepsilon = \text{sign}(\partial/\partial x^1) \). For \( \varepsilon = 1 \) \((-1)\) we refer to 1+3 spacelike (timelike) spacetimes.

A complete classification of decomposable spacetimes in terms of the holonomy group can be found in a recent paper by Capocci and Hall [1]. The 1+3 spacetimes (timelike or spacelike) are only two of the possible 15 types of spacetimes resulting from this classification.

Although the 1+3 decomposable spacetimes are rather special they are important because many well known spacetimes are of this form. For example 1+3 timelike spacetimes are all static spacetimes with vanishing vorticity (which include many classical spacetimes). Concerning the 1+3 spacelike spacetimes we refer the Gödel-type spacetimes which have attracted significant interest in recent years [2]. These spacetimes are generalizations of the Gödel spacetime.
Another less well known application of decomposable spacetimes is their relation with the Affine Conformal Vectors (ACV). The existence of an ACV is equivalent to the existence of a covariantly constant symmetric tensor field $K_{ab}$ which, as it has been shown by Hall and da Costa [3] leads to a decomposable 1+3 spacetime or to a further 1+1+2 decomposition or, finally, to a pp-wave spacetime which in general it is not decomposable.

Coley and Tupper [4] have studied in depth ACVs in spacetime and they have shown that these vectors can be found from the Conformal Killing Vectors (CKV) of a 1+3 decomposable spacetime or of a pp-wave spacetime. They have found all 1+3 decomposable spacetimes which admit a CKV and they have also given a general form for the CKV for the timelike and the spacelike cases. Finally Hall and Capocci [1] have studied CKVs in decomposable spacetimes in a different context and they have shown how their results can be related to the general results of [4].

Although both works referred above have done the major work on CKVs in 1+3 decomposable spacetimes, their methods are difficult to use in the computation of the conformal algebra in a given 1+3 decomposable spacetime. Indeed Coley and Tupper present the general solution in a special coordinate system adapted to the conformal factor $\psi$ which is not known and Capocci and Hall work on a more general level which is appropriate for producing general results but difficult to work with in actual applications. The purpose of this work is to move between these two approaches and present a theory which will systematize the computation of the conformal algebra of a 1+3 decomposable spacetime irrespectively of the coordinate system used.

In Section II we present the theory and show that the CKVs of the four dimensional 1+3 spacetimes are computed from the CKVs of the invariant 3-spaces provided they are gradient vector fields in these spaces. We also show that if the invariant 3-space is of constant curvature then there exist always such gradient vector fields. The results are in complete agreement with the results of Coley and Tupper [4]. In Section 3 we apply the method and compute the complete conformal algebra of the RT (Rebouças-Tiommo) spacetime [5] which is an interesting homogeneous spacetime of the Gödel-type with 7 KVs and no causal anomalies. Finally we show how the method can be generalized to compute the algebra of certain important non-decomposable spacetimes.

## 2 The CKVs of a 1+3 decomposable spacetime

Let $M$ a 1+3 decomposable spacetime with metric $ds^2 = \varepsilon(dx^1)^2 + g_{\mu\nu}(x^\sigma)dx^\mu dx^\nu$ ($\rho = 2, 3, 4$). Let $\zeta^a = \delta^a_1$ be the constant vector field defining the preferred direction. The projection tensor $h_{ab} = g_{ab} - \varepsilon\zeta_a\zeta_b$ is the metric of the 3-surface $x^1 = \text{const}$ (due to the obvious isometry between these 3-spaces it is enough to work in one of them). Covariant derivatives with respect to the metric $g_{ab}$ (respectively $h_{ab}$) shall be denoted with “;” (respectively “|”). The covariant derivative of any smooth vector field $X^a$ on $M$ can be decomposed uniquely as follows:

$$X_{a:b} = \psi(X)g_{ab} + H_{ab}(X) + F_{ab}(X)$$  \quad (1)

where $H_{ab}(X)$ is symmetric and traceless and $F_{ab}(X) = -F_{ba}(X)$. Obviously:
Thus $X^a$ is a CKV iff $H_{ab}(X) = 0$. A similar decomposition holds for the vectors $\xi_\alpha$ in the 3-space $x^1 = \text{const.}$:

$$\xi_{\alpha|\beta} = \lambda(\xi)g_{\alpha\beta} + \mathcal{H}_{\alpha\beta}(\xi) + \mathcal{F}_{\alpha\beta}(\xi)$$

and

$$\mathcal{L}_\xi g_{\alpha\beta} = 2\lambda(\xi)g_{\alpha\beta} + 2\mathcal{H}_{\alpha\beta}(\xi)$$

so that $\xi^\alpha$ is a CKV of the 3-metric iff $\mathcal{H}_{\alpha\beta}(\xi) = 0$.

In order to relate the CKVs of the 4-metric $g_{ab}$ with those of the 3-metric $g_{\alpha\beta}$ we decompose $X^a$ along and normally to $\zeta^a$:

$$X^a = f(x^i)\zeta^a + X'_a$$

where $X'_a = h^b_aX_b$. We define the vector $K_\alpha$ in the 3-space $x^1 = \text{const}$ by the requirement:

$$X'_a = K_\alpha\delta^a_\alpha$$

so that:

$$X^a = f(x^i)\zeta^a + K_\alpha\delta^a_\alpha.$$  

Using standard analysis we prove easily the following result:

$$X_{a;b} = f_{,b}\zeta_a + \varepsilon K_\alpha^* \delta^a_\alpha \zeta_b + K_{a|\beta}\delta^a_\alpha \delta^\beta_b$$

where $\cdot,\cdot$ denotes partial derivative and an asterisk over a letter denotes partial differentiation w.r.t. $x^1$ i.e. $K_{\alpha} = K_{\alpha,1}$. Using the irreducible decompositions (1) and (3) and projecting equation (7) along $\zeta^a\zeta^b$, $\zeta^a h^b_c\zeta^b_c$, $h^a_c h^b_d$ we are able to express the quantities $\psi(X), H_{ab}(X), F_{ab}(X)$ of $X^a$ in terms of the corresponding quantities of the 3-vector $K_\alpha$ and the partial derivatives of $f$. Using an obvious block matrix notation we have the following result:

$$4\psi(X) = f^* + 3\lambda(K)$$

$$H_{ab}(X) = \begin{pmatrix}
\varepsilon(f^* - \psi(X)) & \frac{1}{2}(f_{,\mu} + \varepsilon K^*_\mu) \\
\frac{1}{2}(f_{,\mu} + \varepsilon K^*_\mu) & \mathcal{H}_{\mu\nu}(K) + \frac{1}{4}(\lambda - f)^* g_{\mu\nu}
\end{pmatrix}$$

$$F_{ab}(X) = \begin{pmatrix}
\circ & \frac{3}{2}(f_{,\mu} - \varepsilon K^*_\mu) \\
\frac{3}{2}(f_{,\mu} - \varepsilon K^*_\mu) & \mathcal{F}_{\mu\nu}(K)
\end{pmatrix}.$$  

The condition that $X^a$ is a CKV, $H_{ab}(X) = 0$, is equivalent to the following equations:

$$f^* = \psi(X)$$
\[ f_\alpha = -\varepsilon K_\alpha \]  \hfill (12)

\[ \mathcal{H}_{\alpha\beta}(K) = 0 \quad \text{and} \quad \psi(X) = \lambda(K). \]  \hfill (13)

Thus (11) can be written:

\[ f^* = \lambda(K). \]  \hfill (14)

Relations (11) - (14) have to be supplemented with the integrability conditions of \( f \) i.e.

\[ f_{,\alpha} = (f_\alpha)^* \quad \text{and} \quad f_{,\alpha\beta} = f_{,\beta\alpha}. \]

Thus we obtain the additional equations:

\[ \lambda(K),\alpha = -\varepsilon K_\alpha \]  \hfill (15)

\[ \mathcal{F}_{\alpha\beta}(K) = 0. \]  \hfill (16)

Equations (12)-(16) must be considered in two sets: equations (12), (14) which calculate the function \( f(x^i) \) in terms of \( K_\alpha \) and equations (13)-(15) which characterize the vector field \( K_\alpha \) and in fact (as Coley and Tupper have shown in [4]) define completely the form of the 3-metric \( g_{\alpha\beta}(X^\rho) \). Obviously \( K_\alpha \) allows the computation of \( X^a \) thus we concentrate on the relations (13), (15), (16) concerning \( K_\alpha \). Equation (13) means that \( K^\alpha \) is a CKV of \( g_{\alpha\beta} \) (already shown in [1]). Equation (13) can then be written:

\[ \lambda(K)_{,\alpha} \mid_\beta = -\varepsilon \lambda^*(K) g_{\alpha\beta} \]  \hfill (17)

which is the fundamental equation (cf equation (A10) in the Appendix of [4]) found by Coley and Tupper. The new result is equation (16) which states that either the bivector \( \mathcal{F}_{\alpha\beta}(K) \) is independent of the coordinate \( x^1 \) or \( \mathcal{F}_{\alpha\beta}(K) = 0 \). We have the following Proposition.

**Proposition 2.1.**

All proper CKVs \( X^a \) of the four metric \( g_{ab} \) are calculated from the CKVs \( K^\alpha \) of the 3-metric \( g_{\alpha\beta} \) of the form:

\[ K^\alpha = \frac{1}{p} m(x^1) \xi^\alpha + L^\alpha(x^\rho) \]

where

(a) \( \xi_\alpha = A_\alpha \) is a gradient CKVs of the 3-metric \( g_{\alpha\beta} \) whose conformal factor satisfies the relation

\[ \lambda(\xi)_{\alpha\beta} = p\lambda(\xi) g_{\alpha\beta} \]

(b) the function \( m(x^1) \) satisfies equation (26) (bellow).

(c) \( L_\alpha \) is a KV or a HKV of the 3-metric \( g_{\alpha\beta} \) which is not a gradient vector field and its bivector equals \( F_{ab}(K) \). This implies that the non-gradient KVs of the 3-metric are identical with those of the 4-metric.

**Proof.**
Consider first the KVs and HKVs of the 3-metric \( g_{\alpha\beta} \). These are defined by the condition \( \lambda(K)_{,\alpha} = 0 \) which by (13) and (16) implies that \( K_{\alpha}(x^i) = K^i_{,\alpha} \) otherwise the spacetime becomes \( \{1+1+2\} \) decomposable. We conclude that the KVs of the 3-metric \( g_{\alpha\beta} \) are KVs (recall that \( \psi(X) = \lambda(K) \)) of the full metric \( g_{ab} \). We consider next the proper CKVs of the 3-metric. Due to the form of equation (17) and the decomposability of spacetime, we are looking for solutions of the form:

\[
\lambda(K) = m(x^1)A(x^\rho) + B(x^\rho).
\]

(18)

Differentiating and using equation (13), we find:

\[
m(x^1)A_{,\alpha} + B_{,\alpha} = -\varepsilon \dddot{m} K_{\alpha}.
\]

(19)

Differentiating again and using equation (17), we get:

\[
m(x^1)A_{[\alpha\beta]} + B_{[\alpha\beta]} = -\varepsilon \dddot{m} m(x^1)A(x^\rho)g_{\alpha\beta}.
\]

(20)

Because \( A(x^\rho) \), \( B(x^\rho) \) are functions of \( x^\rho \) only, equation (20) implies that:

\[
m(x^1)A_{[\alpha\beta]} + \varepsilon \dddot{m} A(x^\rho)g_{\alpha\beta} = C_1
\]

(21)

\[
B_{[\alpha\beta]} = -C_{1\alpha\beta}
\]

(22)

where \( C_{1\alpha\beta} \) is a constant tensor. Integrating (19) it follows:

\[
\dot{K}_{\alpha} = -\varepsilon \int m(x^1)dx^1A_{,\alpha} - \varepsilon B_{,\alpha}x^1 + D_{\alpha}(x^\rho).
\]

(23)

Differentiating and taking the antisymmetric part we find that \( D_{\alpha}(x^\rho) \) is a gradient vector field which we denote by \( E_{,\alpha}(x^\rho) \). Differentiating (21) w.r.t. \( x^1 \) we find:

\[
A_{[\alpha\beta]} = pA(x^\rho)g_{\alpha\beta}
\]

(24)

\[
\dddot{m} + \varepsilon pm = C_2
\]

(25)

where \( p, C_2 \) are constants. Equation (24) says that when \( p \neq 0 \) then \( A_{,\alpha} \) is a gradient CKV of \( g_{\alpha\beta} \) with conformal factor \( pA(x^\rho) \) and when \( p = 0 \) then \( A_{,\alpha} \) is a gradient KV. Because as we shall see the gradient KVs do not interest us we require \( p \neq 0 \). We compute easily \( (A_{,\alpha}A^\alpha)_{[\beta} = pA^2_{,\beta} \) thus \( A_{,\alpha} \) is not a null vector field. Combining (24) and (25) with (21) we find \( C_{1\alpha\beta} = 0, C_2 = 0 \). Thus \( B_{,\alpha} \) is a gradient KV of the metric \( g_{\alpha\beta} \) and the function \( m(x^1) \) satisfies the equation:

\[
\dddot{m} + \varepsilon pm = 0
\]

(26)

which means that \( ( p \neq 0 ) m(x^1) = \sin(\sqrt{|p|}x^1), \cos(\sqrt{|p|}x^1), \sinh(\sqrt{|p|}x^1), \cosh(\sqrt{|p|}x^1) \). Replacing these results in (23) and integrating we find:

\[
K_{\alpha} = \frac{1}{p}m(x^1)A_{,\alpha} - \frac{\varepsilon}{2}B_{,\alpha}(x^1)^2 + E_{,\alpha}x^1 + L_{\alpha}(x^\rho).
\]

(27)
Differentiating \((\ref{eqn:27})\) and using the fact that \(K_{\alpha}\) is a CKV of \(g_{\alpha\beta}\) with conformal factor \(\lambda(K)\) we find:

\begin{align}
E_{[\alpha\beta} = 0 \quad & (28) \\
L_{\alpha]\beta} = B g_{\alpha\beta} + F_{\alpha\beta}(K) \quad & (29)
\end{align}

Equation \((\ref{eqn:28})\) means that the vector \(E_{[\alpha}\) is also a gradient KV of the three metric \(g_{\alpha\beta}\) and equation \((\ref{eqn:29})\) that the vector \(L_{\alpha}\) is a Special CKV (or KV or HKV) of \(g_{\alpha\beta}\) which is not a gradient vector field. Thus we conclude that the only part of the proper CKV \(K_{\alpha}\) which is a proper CKV, is the gradient CKV \(A_{[\alpha}\) of the metric \(g_{\alpha\beta}\). Equation \((\ref{eqn:27})\) is the most general solution of the conditions \((\ref{eqn:13}), (\ref{eqn:15}), (\ref{eqn:16})\).

However if one of the gradient KVs \(B_{\alpha}\) and \(E_{\alpha}\) is not equal to zero the spacetime decomposes further to a 1+1+2 spacetime and Coley and Tupper \([4]\) have shown that these spacetimes do not admit any CKVs except in the trivial case they degenerate to Minkowski spacetime. Thus these vectors must vanish and then the remaining vector \(L_{\alpha}\) is a KV or a HKV of the 3-metric whose bivector is equal to the bivector of \(K_{\alpha}\). □

This result is fundamental and allows us to compute the conformal algebra of the four metric \(g_{ab}\). We note that this result is consistent with equations \((2.6)\) and \((3.6)\) of \([4]\). Coley and Tupper have used equation \((\ref{eqn:17})\) to calculate all possible 1+3 spacetimes. We shall use \((\ref{eqn:13})\) and \((\ref{eqn:16})\) to calculate the CKVs in any 1+3 spacetime.

The conformal factor \(\lambda(K)\) of the CKV \(K^\alpha\) equals:

\[\lambda(K) = m(x^1)\lambda(\xi) + b\]  

where \(b(=0\) or 1) is the conformal factor of the vector \(L_{\alpha}\). Taking into account these results the function \(f(x^a)\) is computed from equations \((\ref{eqn:12}), (\ref{eqn:14})\) as follows:

\[f(x^a) = -\frac{\varepsilon}{p} \cdot \frac{\lambda(x^1)}{m} \lambda(\xi) + bx^1.\]  

Finally it is useful to note how one can use \(\xi_\alpha\) and determine all 1+3 metrics in a simple manner. Indeed equation \((\ref{eqn:24})\) states that \(A_{[\alpha}(x^\rho)\) is a (non-null) gradient CKV of the 3-metric \(g_{\alpha\beta}\) hence Petrov’s result \([3]\) quoted in \([4]\) applies and means that there exist coordinates \((x^2, x^A)\) \((A, B, C = 3, 4)\) in which:

(i) \(A_{\alpha} = e\delta_{\alpha}^2\), i.e. \(A(x^2) = ex^2 + C\) where \(e\) is the sign of \(A_{\alpha}\)

(ii) The 3-metric \(g_{\alpha\beta}\) is written as:

\[g_{\alpha\beta} = eg_{22}(x^2)(dx^2)^2 + \frac{1}{g_{22}(x^2)p_{AB}(x^C)}dx^A dx^B\]  

where \((p =\text{constant})\):

\[g_{22}(x^2) = \frac{1}{2p \int A(x^2)dx^2}\]  

where (without loss of generality) we take \(g_{22} > 0\). Defining the new coordinate \(x'^2\) by the equation \(\sqrt{g_{22}(x^2)}dx^2 = dx'^2\) the metric \(g_{\alpha\beta}\) is written:
\[ g_{\alpha\beta} = e(dx^\alpha)^2 + M^2(x^\nu)p_{AB}(x^C)dx^Adx^B \]  

(34)

where

\[ M^2(x^\nu) = 1/g_{22} = 2p \int A(x^2)dx^2. \]  

(35)

Differentiating twice (35) w.r.t. \( x^2 \) and using the definition of the coordinate \( x^2 \) one shows that \( M^2(x^\nu) \) satisfies the equation

\[ \frac{d^2M}{dx^2} - epM = 0. \]

Equation (34) coincides formally with equation (A19) of Coley and Tupper [4] but now the coefficient \( M(x^\nu) \) has a direct geometrical meaning because according to (35) it is related to the conformal factor of the CKV of the 3-metric \( g_{\alpha\beta} \). Furthermore depending on the value of the product \( ep \) it has solutions \( \sin, \cos, \sinh, \cosh, ax^2 + b \) \((a, b\) constants). We note that the case \( p = 0 \) corresponds to SCKVs of the 3-metric \( g_{\alpha\beta} \) [7].

3 The case of spaces of constant curvature

The fact that the CKVs of the 3-metric which produce the CKVs of the 4-metric must be of the form \( K^\alpha = \frac{1}{p}m(x^1)\xi^\alpha + L^\alpha(x^\rho) \) is a strong condition and one wonders that perhaps there are very few 3-dimensional metrics which admit such vector fields. We show the following result. 

**Proposition 3.1.**
The metrics of spaces of constant curvature of dimension \( n \) admit \( n + 1 \) gradient CKVs. 

**Proof**
Consider two metrics \( g_{ij} \) and \( \bar{g}_{ij} \) which are conformally related:

\[ g_{ij} = N^2(x^k)\bar{g}_{ij}. \]  

(1)

Taking the Lie derivative of (1) w.r.t. the vector field \( X^a \) and using (1) and (2), we find:

\[ \psi(X) = X(\ln N) + \phi(X) \]  

(2)

\[ H_{ij}(X) = N^2\bar{H}_{ij}(X) \]  

(3)

\[ F_{ij}(X) = N^2\bar{F}_{ij}(X) - 2NN_{[i}\bar{X}_{j]}]. \]  

(4)

We specialize the metric \( \bar{g}_{ij} \) to be the flat metric (with Euclidean or Lorentzian character) \( \eta_{ij} \) so that \( g_{ij} \) is conformally flat. Then \( X \) is a vector of the flat conformal algebra whose in generic form in Cartesian coordinates is [8][9]:

\[ X^i = a^i + a^i_{\ j}x^j + bx^i + 2(b \cdot x)x^i - b'(x \cdot x) \]  

(5)
where $a^i$, $a^i_{..j}$, $b$, $b^i$ are constants and $(x \cdot x) = \eta_{ij}x^i x^j$. KVs are defined by the constants $a^i$, $a^i_{..j}$, there exists only one HKV defined by the constant $b$ and the constants $b^i$ define the remaining four Special CKVs.

Using (5) we compute for the generic vector field:

$$F_{ij}(X) = X_{[ij]} = \frac{1}{4} \left( \eta^{ij} + 4b_{[ij]}x^k \right)N^2 + 2 \left( \ln N \right)_{[j}x_{i]}.$$  \hfill (6)

Assume now that the metric $g_{ij}$ is a metric of constant curvature. Then the conformal factor:

$$N(x^i) = \frac{1}{1 + \frac{K}{4}(x \cdot x)}$$

where $K = \frac{R}{n(n-1)}$ and $R$ is the Gaussian curvature of space. Introducing $N(x^i)$ in (6) we compute:

$$F_{ij}(X) = a_{ij} - K N x_{[ij]} - K N x_{[ij]} x^r + 4N b_{[ij]} x^r.$$  \hfill (7)

Following the notation of [9] we write the CKVs of the flat space as follows:

$$P_i = \partial_i, \quad M_{ij} = x_i \partial_j - x_j \partial_i \quad \text{(KVs)}$$

$$H = x^i \partial_i \quad \text{(HKV)}$$

$$K_i = 2x_i H - (x \cdot x) P_i \quad \text{(SCKVs)}.$$  \hfill (8)

Then (7) implies:

$$F_{ij}(P_r) = -KN x_{[ij]} \delta^r_i$$

$$F_{ij}(M_{rs}) = a_{ij}(M_{rs}) - K N x_{[ij]} x^r x^s + 4N b_{[ij]} x^s.$$  \hfill (9)

$$F_{ij}(H) = 0$$

$$F_{ij}(K_r) = 4 N x_{[ij]} \delta^r_i.$$  \hfill (10)

We note that the only vector whose bivector vanishes identically (hence it is a gradient vector field) is the proper CKV $H$. However we also note that the bivectors of the proper CKVs $P_i - \frac{K}{4} K_i$ also vanish. Thus we have proved that in an n-dimensional space (of Euclidean or Lorentzian character) of constant curvature there exist always $(n + 1)$ gradient CKVs. These vectors can be used to compute the conformal algebra of any 1+3 spacetime whose 3-d subspaces are spaces of constant curvature. \hfill \Box

The above result is important because:

a) It says that there are at most four (non flat) spacetimes whose 3-d subspaces are spaces of constant curvature. Two spacetimes are of the 1+3 timelike type and are the Einstein spacetime ($K < 0$) and anti-Einstein spacetime ($K > 0$). The other two spacetimes are
of the 1+3 spacelike type and have been found by Rebouças and Tiommo (RT metric) \cite{3} and Rebouças and Texeira (ART metric) \cite{10}. The RT spacetime is of Gödel type and has constant curvature $R < 0$ whereas the ART spacetime is not of Godel type and has $R > 0$. Both spacetimes can be found from the general types described in \cite{4} (Note that in \cite{4} only the spacetimes which satisfy the strong energy condition have been considered).

b) The four space–times mentioned above have 7 KVs a fact that makes them interesting and very rare. These KVs have been computed in Refs \cite{3},\cite{10}. However the computation is not necessary and one can arrive at this result without any explicit computation. Indeed we have shown that the KVs and the HKVs of the 3-space are also KVs and HKVs of the full 1+3 spacetime. Because there are no HKVs we have $\frac{3 \cdot 4}{2} = 6$ KVs for the 3-spaces of constant curvature. If we include in these the constant vector field decomposing the spacetime (i.e. $\partial/\partial x^1$) the total number of KVs becomes seven.

c) All these spacetimes are conformally flat. This can be established either by showing directly that their Weyl tensor vanishes or by showing that they admit a Conformal Algebra of 15 CKVs. The first method is straightforward once one has the metric. The second method is interesting because it does not require a knowledge of the metric. Indeed we have shown that in a three dimensional space there are four proper CKVs (the $H, P_i - \frac{4}{3} K_i$) with vanishing bivector. From (24), (26) we note that for each gradient CKV we obtain two functions $m(x^1)$ (depending on the sign of $p$) thus two CKVs of spacetime. Hence the four CKVs of the 3-metric produce eight CKVs of the 1+3 metric which implies that its algebra consists of 15 CKVs, hence spacetime is conformally flat.

Finally using (2) it is easy to prove that the conformal factor of the generic CKV $X^i$ in (4) is given by the following formula:

$$\psi(K) = -\frac{1}{2}KNa^r\bar{x}_r + (2N - 1)b + 2N(b \cdot x).$$ \hfill (12)

From (12) we conclude that the vectors, $M_{ij}$ are KVs whereas the vectors $P_i, H$ and $K_i$ are proper CKVs of the 3-metric $g_{\alpha\beta}$.

## 4 Application

We apply the results of Section 3 to compute the CKVs of the RT spacetime \cite{3} whose metric is:

$$ds^2 = - [dt + H(r)d\Phi]^2 + dr^2 + D^2(r)d\Phi^2 + dz^2$$ \hfill (1)

where

$$H(r) = -\frac{1}{\Omega} \sinh^2(\Omega r), \quad D(r) = \frac{1}{2\Omega} \sinh(2\Omega r).$$ \hfill (2)

As we have already mentioned the KVs of this metric have been computed in \cite{3} and there remain the eight proper CKVs to be found for the completion of the conformal algebra. It is easy to show that the 3D RT spacetime $z=$constant is a space of constant
curvature with \( R = -6\Omega^2 \), hence it is possible to apply the results of the previous section and compute the CKVs of the full RT spacetime.

The method we follow in the computation is the following: The 3-d RT metric \( ds_{RT}^2 \) has the same CKVs with the 3-dimensional Lorentzian space \(-d\tau^2 + dx^2 + dy^2\). Thus we need the CKVs of the 3-d Lorentz metric whose bivector vanishes. These vectors are the vectors \( H, P_i - \frac{K_i}{4}K \) which are known in the Lorentz coordinates \( \{\tau, x, y\} \) but not in the coordinates \( \{t, r, \Phi\} \). If we find the transformation equations:

\[
\tau(t, r, \Phi), \quad x(t, r, \Phi), \quad y(t, r, \Phi)
\]

then we shall be able to express these vectors in the new coordinates and have the required CKVs. In fact it is enough to find only one proper CKV which is a gradient vector and then compute the rest three by taking the commutators of this vector with the seven KVs already known. Obviously the convenient vector to use is the proper CKV \( H \).

Because the 3-dimensional RT space is of constant curvature \( R = -6\Omega^2 \) we have \( K = -\Omega^2 \) and:

\[
ds_{RT}^2 = \frac{1}{\Omega^2 \left[ 1 - \frac{1}{4}(x^2 + y^2 - \tau^2) \right]^2} ds_L^2
\]

where \( ds_L^2 = -d\tau^2 + dx^2 + dy^2 \). Using the transformation:

\[
\Phi = \phi - \Omega t, \quad \Omega = \frac{1}{a}
\]

we get:

\[
ds_{RT}^2 = -\cosh^2(\frac{r}{a}) dt^2 + dr^2 + a^2 \sinh^2(\frac{r}{a}) d\phi^2.
\]

After enough trial and error we have managed to show that the transformation \((t, r, \phi) \rightarrow (\tau, x, y)\) which makes \((3)\) valid is \((0 < \phi < 2\pi, 0 < t < \infty)\):

\[
\tau = -\frac{2 \sin(\frac{\phi}{2}) \cosh(\frac{r}{a})}{1 - \cosh(\frac{r}{a}) \cos(\frac{\phi}{2})}, \\
x = \frac{2 \sin(\frac{\phi}{2}) \cos \phi}{1 - \cosh(\frac{r}{a}) \cos(\frac{\phi}{2})}, \\
y = \frac{2 \sin(\frac{\phi}{2}) \sin \phi}{1 - \cosh(\frac{r}{a}) \cos(\frac{\phi}{2})}
\]

In these coordinates the conformal factor \( \frac{1}{\left[ 1 - \frac{1}{4}(x^2 + y^2 - \tau^2) \right]^2} \) becomes:

\[
N(r, t) = \frac{1}{2} \left[ 1 - \cosh(\frac{r}{a}) \cos(\frac{t}{a}) \right].
\]

Using the transformation \((5)\) we find that in the coordinates \((t, r, \phi)\) the vector \( H \equiv \xi_1 \) is:
\[ \xi_1 = -a \frac{\sin(\frac{r}{a})}{\cosh(\frac{r}{a})} \partial_t - a \cos(\frac{t}{a}) \sinh(\frac{r}{a}) \partial_r. \]  

(7)

The conformal factor of \( \xi_1 \) is read from (12):

\[ \lambda(\xi_1) = 2N - 1 = -\cosh(\frac{r}{a}) \cos(\frac{t}{a}). \]  

(8)

Following the same procedure we recover the six KVs of the 3-dimensional RT metric (these are the three vectors \( \mathbf{M}_{AB} \) and the three vectors \( \mathbf{P}_A - \frac{1}{4} \mathbf{K}_A \) \( K = -1 \)) where \( A, B = \tau, x, y \) which have been already computed in [3]. To find the rest three gradient proper CKVs of the 3-dimensional RT metric we take the Lie bracket of the proper CKV \( \mathbf{H} \) with these KVs. We find:

\[ \xi_2 = -a \frac{\cos(\frac{r}{a})}{\cosh(\frac{r}{a})} \partial_t + a \sin(\frac{t}{a}) \sinh(\frac{r}{a}) \partial_r \]  

(9)

\[ \lambda(\xi_2) = \cosh(\frac{r}{a}) \sin(\frac{t}{a}) \]  

(10)

\[ \xi_3 = -a \cosh(\frac{r}{a}) \cos(\phi) \partial_r + \frac{\sin(\phi)}{\sinh(\frac{r}{a})} \partial_\phi \]  

(11)

\[ \lambda(\xi_3) = -\sinh(\frac{r}{a}) \cos(\phi) \]  

(12)

\[ \xi_4 = a \cosh(\frac{r}{a}) \sin(\phi) \partial_r + \frac{\cos(\phi)}{\sinh(\frac{r}{a})} \partial_\phi \]  

(13)

\[ \lambda(\xi_4) = \sinh(\frac{r}{a}) \sin(\phi). \]  

(14)

For each conformal factor we calculate the constant \( p \) by using equation (24) and subsequently the function \( m(z) \) using (26). Finally \( f(x^i) \) is computed from equation (31) and we write the CKVs of the 4-metric using the formula:

\[ X^a = f(x^i) \zeta^a + m(z) \xi^a. \]

The results of these calculations are the following eight proper CKVs of the RT spacetime \( k = 1, 2, 3, 4 \) and \( \alpha = 0, 1, 2 \):

\[ X_{(k)\alpha} = a^2 A_{k,\alpha}, X_{(k)3} = -a^2 A_{k,3}. \]

with conformal factors:

\[ \psi_k = A_k \]

and
\[ X_{(k+4)\alpha} = a^2 B_{k,\alpha}, X_{(k+4)3} = -a^2 B_{k,3}. \]

with conformal factors:

\[ \psi_{k+4} = B_k \]

where

\[ A_k = \cosh\left( \frac{r}{a} \right) \left[ \cos\left( \frac{t}{a} \right) \cos\left( \frac{z}{a} \right), \sin\left( \frac{t}{a} \right) \cos\left( \frac{z}{a} \right), \cos\left( \frac{t}{a} \right) \sin\left( \frac{z}{a} \right), \sin\left( \frac{t}{a} \right) \sin\left( \frac{z}{a} \right) \right] \]

\[ B_k = \sinh\left( \frac{r}{a} \right) \left[ \sin \phi \cos\left( \frac{z}{a} \right), \cos \phi \cos\left( \frac{z}{a} \right), \cos \phi \sin\left( \frac{z}{a} \right), \sin \phi \sin\left( \frac{z}{a} \right) \right]. \]

To write the conformal algebra of RT spacetime in a compact form it is necessary to define a new basis of CKVs. For the Killing Vectors we define:

\[ C_0 = K_1 + K_3 \quad M_{12} = K_1 - K_3 \]
\[ C_1 = K_7 - K_4 \quad M_{10} = K_5 + K_6 \]
\[ C_2 = K_5 - K_6 \quad M_{20} = K_4 + K_7 \]

\[ K_2 = a \partial_z \]

where:

\[ K_1 + K_3 = a \partial_t \quad K_1 - K_3 = \partial \phi \]
\[ K_5 - K_6 = a \tanh\left( \frac{r}{a} \right) \sin \phi \sin\left( \frac{z}{a} \right) \partial_t - a \sin \phi \cos\left( \frac{z}{a} \right) \partial_r + \coth\left( \frac{r}{a} \right) \cos \phi \cos\left( \frac{z}{a} \right) \partial_\phi \]
\[ K_7 - K_4 = a \tanh\left( \frac{r}{a} \right) \cos \phi \sin\left( \frac{z}{a} \right) \partial_t - a \cos \phi \cos\left( \frac{z}{a} \right) \partial_r + \coth\left( \frac{r}{a} \right) \sin \phi \cos\left( \frac{z}{a} \right) \partial_\phi \]
\[ K_5 + K_6 = a \tanh\left( \frac{r}{a} \right) \cos \phi \cos\left( \frac{z}{a} \right) \partial_t + a \cos \phi \sin\left( \frac{z}{a} \right) \partial_r - \coth\left( \frac{r}{a} \right) \sin \phi \sin\left( \frac{z}{a} \right) \partial_\phi \]
\[ K_4 + K_7 = a \tanh\left( \frac{r}{a} \right) \sin \phi \cos\left( \frac{z}{a} \right) \partial_t + a \sin \phi \sin\left( \frac{z}{a} \right) \partial_r + \coth\left( \frac{r}{a} \right) \cos \phi \sin\left( \frac{z}{a} \right) \partial_\phi \]

For the CKVs we introduce the following notation:

\[ I_0 = X_2 \quad Z_0 = X_4 \]
\[ I_1 = X_6 \quad Z_1 = X_7 \]
\[ I_2 = X_5 \quad Z_2 = X_8 \]

and:

\[ H_1 = X_1 \quad H_2 = X_3. \]

Then the conformal algebra is written as follows:
\[
[M_{\alpha\beta}, H_1] = [M_{\alpha\beta}, H_2] = [M_{\alpha\beta}, K_2] = [C_{\alpha}, K_2] = 0
\]

\[
[M_{\alpha\beta}, I_\gamma] = \eta_{\alpha\gamma} I_\beta - \eta_{\beta\gamma} I_\alpha \quad [M_{\alpha\beta}, Z_\gamma] = \eta_{\beta\gamma} Z_\alpha - \eta_{\alpha\gamma} Z_\beta
\]

\[
[I_\alpha, C_\beta] = \eta_{\alpha\beta} H_1 \quad [Z_\alpha, C_\beta] = \eta_{\alpha\beta} H_2
\]

\[
[Z_\alpha, I_\beta] = \eta_{\alpha\beta} K_2 \quad [H_1, H_2] = K_2
\]

\[
[H_1, K_2] = \frac{H_2}{2} \quad [K_2, H_2] = \frac{H_1}{2}
\]

\[
[I_\alpha, K_2] = \eta_{\alpha\beta} Z_\beta \quad [K_2, Z_\alpha] = \eta_{\alpha\beta} I_\beta
\]

\[
[M_{\alpha\beta}, C_\gamma] = \eta_{\beta\gamma} C_\alpha - \eta_{\alpha\gamma} C_\beta
\]

\[
[M_{\alpha\beta}, M_\gamma\delta] = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma}
\]

where \(\eta_{\alpha\beta} = \text{diag}(-1, 1, 1)\) and \(\alpha, \beta, \gamma, \delta = 0, 1, 2\) plus the three relations:

\[
[M_{12}, I_1] = -I_2 \quad [M_{12}, I_2] = I_1 \quad [Z_0, C_0] = H_2
\]

### 5 Conclusions

We have shown that it is possible to develop a systematic method for the computation of the conformal algebra of any 1+3 decomposable spacetime. The key step of the method, in the special case of 3-d metrics of constant curvature, is to find a coordinate transformation between the coordinates in which the metric is given and the Lorentz coordinates of 3-d Minkowski spacetime. This transformation is not trivial but it is possible to be found by a trial and error procedure. The method we have developed is systematic and exhibits the geometrical character of the various crucial quantities involved and, furthermore, it lays down the ground for the systematic computation of higher collineations in these spacetimes.

One can extend this method to calculate the conformal algebra of non-decomposable spacetimes which are conformally related to 1+3 decomposable spacetimes. For example let us consider the important case of Friedman-Robertson-Walker (FRW) spacetimes. In conformal coordinates the line element of these spaces can be written as:

\[
ds^2 = R^2(\tau)[-d\tau^2 + d\sigma^2]
\]

where \(\tau\) is the proper time along the comoving observers, \(R(\tau)\) is the scale factor and \(d\sigma^2\) is the metric of a 3-d space of constant curvature. Thus essentially the FRW
spacetimes are conformally related to a 1+3 decomposable spacetimes whose 3-d invariant subspaces are spaces of constant curvature. Having shown that we can always compute the CKVs of these spaces and taking into account that the algebra of conformally related spaces is the same we compute easily and systematically the CKVs of the FRW spacetimes in the coordinates used. These vectors have been computed in [9] and there is no point to repeat the computation here. Instead using a similar procedure we give in the Appendix the CKVs of the anti-DeSitter spacetime which, to our knowledge, they have not been published before.

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7 Appendix

The line element of anti-DeSitter spacetime is:

\[ ds^2 = -\cosh^2\left(\frac{r}{a}\right)dt^2 + dr^2 + a^2 \sinh^2\left(\frac{r}{a}\right)(d\theta^2 + \sin^2 \theta d\phi^2). \]

The proper CKVs of this spacetime are the following:

\[ \xi_1 = a \frac{\sin(\frac{r}{a})}{\cosh(\frac{r}{a})} \partial_t + a \cos(\frac{r}{a}) \sinh(\frac{r}{a}) \partial_r \]

\[ \psi_1 = \cos(\frac{r}{a}) \cosh(\frac{r}{a}) \]

\[ \xi_2 = -a \frac{\cos(\frac{r}{a})}{\cosh(\frac{r}{a})} \partial_t + a \sin(\frac{r}{a}) \sinh(\frac{r}{a}) \partial_r \]

\[ \psi_2 = \sin(\frac{r}{a}) \cosh(\frac{r}{a}) \]

\[ \xi_3 = a \cosh(\frac{r}{a}) \sin \theta \cos \phi \partial_r - \frac{\sin \phi}{\sinh(\frac{r}{a}) \sin \theta} \partial_\phi + \frac{\cos \theta \cos \phi}{\sinh(\frac{r}{a})} \partial_\theta \]

\[ \psi_3 = \sinh(\frac{r}{a}) \sin \theta \cos \phi \]

\[ \xi_4 = a \cosh(\frac{r}{a}) \sin \theta \sin \phi \partial_r + \frac{\cos \phi}{\sinh(\frac{r}{a}) \sin \theta} \partial_\phi + \frac{\cos \theta \sin \phi}{\sinh(\frac{r}{a})} \partial_\theta \]

\[ \psi_4 = \sinh(\frac{r}{a}) \sin \theta \sin \phi \]

\[ \xi_5 = a \cosh(\frac{r}{a}) \cos \theta \partial_r - \frac{\sin \theta}{\sinh(\frac{r}{a})} \partial_\theta \]

\[ \psi_5 = \sinh(\frac{r}{a}) \cos \theta \]
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