STUDY OF FRACTIONAL POINCARÉ INEQUALITIES ON UNBOUNDED DOMAINS

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Abstract. The aim of this paper is to study (regional) fractional Poincaré type inequalities on unbounded domains satisfying the finite ball condition. Both existence and nonexistence type results on regional fractional inequality are established depending on various conditions on domains and on the range of \( s \in (0,1) \). The best constant in both regional fractional and fractional Poincaré inequality is characterized for strip like domains \((\omega \times \mathbb{R}^{n-1})\), and the results obtained in this direction are analogous to those of the local case. This settles one of the natural questions raised by K. Yeressian in \([\text{Asymptotic behavior of elliptic nonlocal equations set in cylinders, Asymptot. Anal. 89, (2014), no 1-2}\].

1. Introduction. The well known Poincaré inequality states that for a bounded domain \( D \subset \mathbb{R}^n \) and \( 1 \leq p < \infty \) there exists a constant \( C = C(p,D) \) such that

\[
\int_D |u|^p \, dx \leq C \int_D |\nabla u|^p \, dx, \quad \text{for all } u \in W^{1,p}_0(D),
\]

where the space \( W^{1,p}_0(D) \) is the closure of the space of smooth functions with compact support \((C_0^\infty(D))\) in the norm \( ||u||_{1,p,D} := \left( \int_D |u|^p \, dx + \int_D |\nabla u|^p \, dx \right)^{1/p} \). This inequality remains true if \( D \) is bounded in one direction and also if \( D \) has finite measure. Also, it is easy to see that the finite ball condition (which means that \( D \) cannot contain arbitrarily large balls, see Definition 2.2) is necessary for

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Poincaré inequality to hold. A natural question is what are the analogues of Poincaré inequalities in fractional Sobolev spaces, in particular for unbounded domains.

In the present paper we only deal with the case \( p = 2 \) for simplicity. However, most of the results generalize to any \( 1 < p < \infty \), and a paper dealing with this generalization is in progress by the fourth author. The difficulties are mainly of technical nature.

Let \( \Omega \) be any open set in \( \mathbb{R}^n \), \( 0 < s < 1 \) and let us define the Gagliardo semi-norm of \( u \) as

\[
[u]_{s,2,\Omega} := \left( \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

Taking into account the constant \( C_{n,s} \) will be crucial to study the best constants in fractional Poincaré inequality (c.f. Theorem 1.1). We refer to Section 2 for details on the constant \( C_{n,s} \). Further, the fractional Sobolev Space \( W^{s,2}(\Omega) \), is defined as

\[
W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < \infty \right\},
\]

endowed with the norm

\[
\|u\|_{s,2,\Omega} := \left( \|u\|_{L^2(\Omega)}^2 + [u]_{s,2,\Omega}^2 \right)^{\frac{1}{2}}.
\]

The spaces \( W^{s,2}_0(\Omega) \) and \( H^s_0(\mathbb{R}^n) \) denote the closure of \( C_c^\infty(\Omega) \) with respect to the norms \( \|\cdot\|_{s,2,\Omega} \) and \( \left( \|u\|_{L^2(\Omega)}^2 + [u]_{s,2,\mathbb{R}^n}^2 \right)^{\frac{1}{2}} \) respectively. For more details about the fractional Sobolev space we refer to [1], [3], [19], [29] and [31]. These spaces play an important role in studying the Dirichlet problems involving fractional (and regional fractional) Laplace operators, see [6], [18], [19] and [35] for related works close to this direction.

To discuss the results regarding fractional Poincaré inequality, let us define

\[
P^1_{n,s}(\Omega) := \inf_{u \in W^s_0(\Omega), u \neq 0} \frac{[u]_{s,2,\Omega}^2}{\int_{\Omega} u^2} \quad \text{and} \quad P^2_{n,s}(\Omega) := \inf_{u \in H^s(\mathbb{R}^n), u \neq 0} \frac{[u]_{s,2,\mathbb{R}^n}^2}{\int_{\Omega} u^2}.
\]

If \( P^1_{n,s}(\Omega) > 0 \), then we say regional fractional Poincaré inequality holds true. Whereas if \( P^2_{n,s}(\Omega) > 0 \) we say fractional Poincaré inequality holds true. \( P^1_{n,s} \) and \( P^2_{n,s} \) differ significantly in many properties, see for instance Proposition 2.3, where we have summarized the known results on bounded domains. One of the relevant differences concerns the domain monotonicity: \( P^2_{n,s}(\Omega_1) \geq P^2_{n,s}(\Omega_2) \) if \( \Omega_1 \subset \Omega_2 \). However no domain monotonicity is known for \( P^1_{n,s} \). But they have in common that the finite ball condition is required for both \( P^1_{n,s}(\Omega), P^2_{n,s}(\Omega) > 0 \) to hold. Another known difference is the behavior with respect to Schwarz symmetrization \( u \mapsto u^* \) and \( \Omega \mapsto \Omega^* \): it is well know that \( P^2_{n,s}(\Omega^*) \leq P^2_{n,s}(\Omega) \), see for example Frank and Seiringer [22], but this is not true for \( P^1_{n,s} \), see [25].

To the best of our knowledge, the study of regional fractional Poincaré and fractional Poincaré inequalities for general domains, in particular unbounded ones, is largely open. Some literature is only available on fractional Poincaré inequalities in bounded domains and with zero mean value condition, see for instance the papers [16], [24], and references therein.

It is easy to see that—as in the local case—the finite ball condition (see Definition 2.2) is necessary for fractional Poincaré inequality to hold, regional or not.
In particular the inequality cannot hold in $\mathbb{R}^n$ (see however [30] for fractional Poincaré inequality on $\mathbb{R}^n$ with zero mean value condition and with positive integrable weights). One of the simplest unbounded domain satisfying the finite ball condition is the strip $\Omega_\infty := \mathbb{R}^{n-1} \times (-1, 1)$. For the local case, it is well known that the best Poincaré constant of $\Omega_\infty$ is same as the best Poincaré constant of the cross section $(-1, 1)$. The proof is elementary, as we illustrate in 2 dimension: if $\mu_1(\Omega) = \inf\{\int_\Omega |\nabla u|^2 : u \in C_0^\infty(\Omega), \|u\|_{L^2(\Omega)} = 1\}$ then for any $v \in C_0^\infty(\Omega_\infty)$ one has the estimate, using the Poincaré inequality in $(-1, 1)$,

$$\|v\|^2_{L^2(\Omega_\infty)} \leq \frac{1}{\mu_1((-1, 1))} \int_{\mathbb{R}} \int_{-1}^1 u_{x_2}^2 \leq \frac{1}{\mu_1((-1, 1))} \int_{\Omega_\infty} (u_{x_1}^2 + v_{x_2}^2).$$

(1)

This shows that $\mu_1(\Omega_\infty) \geq \mu_1((-1, 1))$. Inverse inequality $\mu_1(\Omega_\infty) \leq \mu_1((-1, 1))$ is easily shown by taking a sequence of functions $w_\ell(x) = \varphi(x_2) v_\ell(x_1)$, where $\{v_\ell\}_{\ell \in \mathbb{N}} \subset C_0^\infty(\mathbb{R})$ is an approximation of the function identically equal to 1 in $\mathbb{R}$ and $\varphi$ satisfies $\int_{-1}^1 |\varphi'|^2 = \mu_1((-1, 1)) \int_{-1}^1 \varphi^2$. One of the goal of this article is to obtain similar characterization for both $P_{n,s}^1(\Omega_\infty)$ and $P_{n,s}^2(\Omega_\infty)$. In [38], the author proved that $P_{n,s}^2(\Omega_\infty) > 0$ and asked whether $P_{n,s}^2(\Omega_\infty) = P_{1,s}^1((-1, 1))$ [see [38] section 4.2]. Theorem 1.2 answers this question, even for more general domain. However, in the first theorem we deal with this issue for regional fractional Poincaré inequality. Note that, the proofs are much more involved in our case, as there is no analogy to the estimate $\ell^2(x) \leq u_{x_1}^2 + v_{x_2}^2$ of (1).

**Theorem 1.1.** For $\Omega_\infty = \mathbb{R}^{n-1} \times (-1, 1) \subset \mathbb{R}^n$ the following statements hold:

1. $P_{n,s}^1(\Omega_\infty) = P_{1,s}^1((-1, 1)) = 0$, if $0 < s \leq \frac{1}{2}$.

2. $P_{n,s}^1(\Omega_\infty) > 0$, if $\frac{1}{2} < s < 1$. More precisely: the best constant $P_{n,s}^1(\Omega_\infty)$ is equal to the best constant of the cross section of the strip $\Omega_\infty$, i.e.

$$P_{n,s}^1(\Omega_\infty) = P_{1,s}^1((-1, 1)).$$

Next we deal with the characterization of $P_{n,s}^2(\Omega_\infty)$. In the class of simply connected domains in two dimensions, it was shown in [28] that the finite ball condition [see definition 2.2] is a necessary and sufficient condition for local Poincaré inequality to hold true. As already mentioned above, the finite ball condition is still necessary in the fractional case, but fails to be sufficient. For the range $s \in (0, \frac{1}{2}]$ this is trivial for $P_{n,s}^1$ by Theorem 1.1 (1). Regarding $P_{n,s}^2$, examples of domains have been constructed in [11] to establish that the finite ball condition is not sufficient for $P_{n,s}^2 > 0$ in the range $s \in (0, \frac{1}{2}]$. The case $s > 1/2$ is more subtle and a satisfactory characterization of the domains for which $P_{n,s}^2 > 0$ is still unknown.

**Theorem 1.2.** Consider the strip $\Omega_\infty = \mathbb{R}^m \times \omega$ in $\mathbb{R}^n$ with $1 \leq m < n$, where $\omega$ is a bounded open subset of $\mathbb{R}^{n-m}$. Then for $0 < s < 1$, we have

$$P_{n,s}^2(\Omega_\infty) = P_{n-m,s}^2(\omega).$$

Let us compare Theorems 1.1 and 1.2. The class of domains considered in Theorem 1.2 is wider to the one of Theorem 1.1 due to the completely different method of proof. We are able to give a more general result in Theorem 1.2 thanks to the characterization of $P_{n,s}^2(\Omega)$ as the first Dirichlet eigenvalue of the fractional Laplace operator on $\Omega$. This does not work in the setting of Theorem 1.1, whose proof is much more elementary and is based on a geometric idea and a rescaling argument. There is little hope of extending this geometric proof to sets of the form $\mathbb{R}^m \times \omega$. One of the main obstructions is, taking for example $m = 1$ and $n = 3$, that intersecting
The proof of Theorem 1.2 follows by two main steps. First, as an application of discrete Picone identity we prove that \( P_{2,n,s}^2(\Omega_{\infty}) \geq P_{2,n-m,s}^2(\omega) \). The other inequality is obtained by constructing suitable test functions on truncated domains \( \Omega_{\ell} = B_m(0, \ell) \times \omega \) and then finally letting \( \ell \) tend to infinity. Very recently, and after our result has appeared, an independent proof of the above theorem was also given by [2]. The proof of [2] is completely different from ours and is based on Fourier space methods. Moreover they require \( \omega \) to have Lipschitz boundary. For general bounded open sets, whose boundary is not smooth enough, the notion of the Poincaré constant might not be uniquely defined, see (33) and Example 6.3 in the Appendix where we explain this in more detail. Independently, various kinds of problems (mainly PDEs) on \( \Omega_{\ell} \) have been considered, and their asymptotic behavior as \( \ell \to \infty \) is studied. Such kind of theories are now well studied in the local case and for more details on this subject we refer [7], [8], [9], [17], [38] and the references therein.

The rest of the paper discusses the existence and non-existence issues regarding regional fractional Poincaré inequality. The lack of any known domain monotonicity property for \( P_{n,s}^1(\Omega) \) makes the study of regional fractional Poincaré inequality more interesting, even for specific domains, or any special class of domains. Our next theorem provides a sufficient conditions on the domain \( \Omega \) for which regional fractional Poincaré inequality remains true. At the end of Section 3 we will give some examples of domains which satisfy the hypothesis of Theorem 1.3. Here \( S^{n-1} \subset \mathbb{R}^n \) denotes the unit sphere and \( \mathcal{H}^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^n \) be a measurable set and \( \frac{1}{2} < s < 1 \). Suppose there exists \( \Sigma \subset S^{n-1} \) with \( \mathcal{H}^{n-1}(\Sigma) > 0 \) and such that for all \( w \in \Sigma \) and all \( x \in \mathbb{R}^n \) the one dimensional intersections with \( \omega \)

\[
A_{x,w} := \{ t \in \mathbb{R} : x + tw \in \Omega \}
\]

satisfy uniformly one dimensional finite ball condition, that is

\[
\sup \{ \text{length}(I) : I \text{ interval}, I \subset A_{x,w}, x \in \mathbb{R}^n, w \in \Sigma \} = m < \infty.
\]

Then regional fractional Poincaré inequality holds, more precisely

\[
P_{n,s}^1(\Omega) \geq \frac{C_{n,s}}{2C_{1,s}} \mathcal{H}^{n-1}(\Sigma) \frac{P_{1,s}^1((0,1))}{m^{2s}}.
\]

An example of a domain which does not satisfy the hypothesis of Theorem 1.3 (but satisfies the finite ball condition) is an infinite union of concentric annuli, see Example 3.4 (v):

\[
D = \bigcup_{k=1}^{\infty} B_{2k}(0) \setminus B_{2k-1}(0).
\]

On the other hand, we know \( P_{n,s}^1(\Omega) = 0 \) if \( \Omega \) is bounded and \( s \in (0, \frac{1}{2}) \) [see Proposition 2.3]. It has been observed by Frank, Jin and Xiong [21] that the same proof works also for domains with finite measure whose boundary does not have “too big tubular neighborhoods” in some sense, see (8) and Remark 1 for the precise
definition. For example, if we consider the domain
\[ \Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, \ 0 < x_2 < 1/x_1^{1+\epsilon}\} \text{ where } \epsilon > 0, \]
then by [21] it follows that \( P_{1,n,s}^\epsilon(\Omega) = 0 \) whenever \( 0 < s \leq \frac{\epsilon}{2(1+\epsilon)} \), but the same result is inconclusive for the range \( \frac{\epsilon}{2(1+\epsilon)} < s \leq \frac{1}{2} \). In our Theorem 5.5 we deal with similar kinds of domains as \( \Omega_\epsilon \), but of much greater generality and prove among others that \( P_{1,n,s}^\epsilon(\Omega) = 0 \) for the whole range of \( s \in (0, \frac{1}{2}) \). Noteworthy, applying Theorem 1.3 to \( \Omega_\epsilon \), we see that \( P_{1,n,s}^\epsilon(\Omega) > 0 \) for \( s > \frac{1}{2} \).

In our next theorem, we prove a very general condition on domains (not only of finite measure) for which regional fractional Poincaré inequality does not hold.

**Theorem 1.4.** Let \( 0 < s < \frac{1}{2} \) and suppose that \( \Omega \subset \mathbb{R}^n \) is a Lipschitz set. Suppose there exists a bounded open Lipschitz set \( U \) and a sequence \( \lambda_k \) tending to infinity such that
\[
\lim_{k \to \infty} \frac{1}{\mathcal{L}^n(\lambda_k U \cap \Omega)} \int_{\lambda_k U \cap \Omega} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} = 0,
\]
where \( \lambda_k U = \{\lambda_k x : x \in U\} \). Then \( P_{1,n,s}^\epsilon(\Omega) = 0 \).

In section 5, we discuss the application of this result in detail by providing several scenarios through examples. For instance, the result applies to any domain which satisfies the growth condition \( \mathcal{L}^n(\Omega \cap B_R) \geq cR^n \) [see, Corollary 1], the domain of type (2) is such an example.

The proof of Theorem 1.4 fails for \( s = \frac{1}{2} \). Whether \( P_{1,n,s}^\epsilon(\Omega) = 0 \) for every unbounded domain if \( s \in (0, \frac{1}{2}] \) is still an open problem to the best of our knowledge.

We end the paper by a last result in this direction and give a sufficient condition for unbounded domains and \( s \in (0, \frac{1}{2}) \), but with finite measure, for \( P_{1,n,s}^\epsilon(\Omega) = 0 \) to hold. See Theorem 5.5 in Section 5.2.

2. Some elementary properties and known results. We briefly fix the notation that we will use throughout this paper. For an integer \( n \) and a measurable set \( \Omega \subset \mathbb{R}^n \) we write \( \mathcal{L}^n \) to denote the Lebesgue measure, or shortly \( |\Omega| \) if there is no ambiguity concerning \( n \). \( B_R(x) \) denotes a ball of radius \( R \) centered at \( x \). We omit the center if \( x = 0 \).

We will use that \( \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \), where \( \Gamma \) is the standard Gamma function.

We will use the Beta function, which is defined for \( x, y > 0 \) by
\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt,
\]
and its properties:
\[
B(x, y) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2x-1}(\cos \theta)^{2y-1} d\theta, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

The constant \( C_{n,s} \) in the definition of the \( [\cdot]_{s,2,\Omega} \) seminorm is uniquely defined if we want consistency with fractional partial integration and with the fractional Laplacian in the following sense
\[
[u]_{s,2,\Omega}^2 = \int_{\mathbb{R}^n} |(-\Delta)^s u|^2 dx,
\]
and \((-\Delta_n)^s u = F^{-1}(\xi^{2s}(F u)(\xi))\), where \(F\) is the Fourier transform, see [31] for details. The fractional Laplace operator can be equivalently defined as
\[
(-\Delta_n)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.
\]
(6)

Here \(P.V.\) denotes the principal value and the above integral is defined for \(u \in C^2_c(\mathbb{R}^n)\). We refer to [10],[20],[31],[37] and [36] for related works concerning the fractional Laplace operator. The constant \(C_{n,s}\) is explicitly given by
\[
C_{n,s} = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}} dz \right)^{-1} = \frac{s^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^s \Gamma(1-s)}.
\]
(7)

We first list some simple and known properties regarding the \(\|\cdot\|_{2,\Omega}\) seminorm and the fractional Poincaré inequality. The regional fractional Poincaré inequality on bounded domains is deduced from the fractional Hardy inequality, which we recall here, stated only in the setting that we will require it.

The next theorem holds true for any \(s\) bigger than \(\frac{1}{2}\). So we state it in that form, although we will use it only in the range \(\frac{1}{2} < s < 1\).

**Theorem 2.1.** (Dyda [13], see also [14]) Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set with Lipschitz boundary \(\partial \Omega\) and \(\frac{1}{2} < s < 1\). Then there exists a constant \(C\) depending only on \(\Omega, n, s\) such that
\[
\int_{\Omega} \frac{|u(x)|^2}{(\text{dist}(x, \partial \Omega))^{2s}} dx \leq C(\Omega, n, s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \quad \text{for all } u \in C^\infty_c(\Omega).
\]

We mention that the validity of fractional Hardy type inequalities on unbounded domains has been characterized in [23, Theorem 6.1]. Their characterization is in terms of a uniform inverse subadditivity inequality for capacity involving the intersection of any compact subset of \(\Omega\) with a Whitney decomposition of \(\Omega\). Such a condition is however difficult to verify in practice.

We recall the following definition.

**Definition 2.2 (Finite ball condition).** We say that a set \(\Omega \subset \mathbb{R}^n\) satisfies the finite ball condition if \(\Omega\) does not contain arbitrarily large balls, that is
\[
\sup \{r : B_r(x) \subset \Omega, \ x \in \Omega\} < \infty.
\]

The finite ball condition plays an important role in the characterization of domains for which the local Poincaré inequality holds. It is immediate to see (by domain monotonicity and scaling) that the finite ball condition is necessary for local Poincaré to hold. But it is actually equivalent for simply connected domains in dimension 2, see Sandee-Mancini [28].

**Proposition 2.3.** Let \(n \geq 1\) be a positive integer and \(0 < s < 1\).

(i) Let \(\Omega \subset \mathbb{R}^n\) be a bounded open Lipschitz set. Then
\[
P^1_{n,s}(\Omega) > 0 \quad \text{if } \frac{1}{2} < s < 1, \quad \text{and} \quad P^1_{n,s}(\Omega) = 0 \quad \text{if } 0 < s \leq \frac{1}{2},
\]
whereas (this does not require \(\Omega\) to be Lipschitz)
\[
P^2_{n,s}(\Omega) > 0 \quad \text{if } 0 < s < 1.
\]

(ii) Let \(\Omega \subset \mathbb{R}^n, t > 0\) and \(u \in W^{s,2}(\Omega)\). Define \(v_t \in W^{s,2}(\Omega)\) by \(v_t(x) = u(tx)\). Then \([u]^{2}_{s,2,t\Omega} = t^{n-2s}[v_t]^{2}_{s,2,\Omega}\), and moreover
\[
P^1_{n,s}(t\Omega) = \frac{P^1_{n,s}(\Omega)}{t^{2s}}, \quad P^2_{n,s}(t\Omega) = \frac{P^2_{n,s}(\Omega)}{t^{2s}}.
\]
(iii) $P_{n,s}^2$ has the domain monotonicity property, i.e. if $\Omega_1 \subset \Omega_2$ then $P_{n,s}^2(\Omega_2) \leq P_{n,s}^2(\Omega_1)$.

(iv) If $\Omega \subset \mathbb{R}^n$ does not satisfy the finite ball condition then $P_{n,s}^1(\Omega) = P_{n,s}^2(\Omega) = 0$.

**Remark 1.** The hypothesis $\Omega$ bounded and Lipschitz in part (i) for $P_{n,s}^1(\Omega) = 0$ in the case $0 < s < \frac{1}{2}$ can be weakened significantly, see [21] Lemma A.2. It is sufficient to require that $\Omega$ has finite measure and that

$$L^n(\{x \in \Omega : \text{dist}(x, \Omega^c) < \delta\}) = o(\delta^{2s}) \quad \text{as } \delta \to 0.$$  \hspace{1cm} (8)

For a bounded Lipschitz domain one has the estimate $L^n(\{x \in \Omega : \text{dist}(x, \Omega^c) < \delta\}) \leq C\mathcal{H}^{n-1}(\partial \Omega)\delta$. Such an estimate remains true if $\partial \Omega$ is only piecewise Lipschitz and in this case the condition (8) is satisfied as long as $\mathcal{H}^{n-1}(\partial \Omega) < \infty$. We will need this observation to use the result for the intersection of two Lipschitz domains. The proof of [21, Lemma A.2] is identical to the proof we present below and is based on taking a smooth test function $u_\delta$ which is equal to 1 inside $\Omega$ except in a small tubular neighborhood of the boundary.

**Remark 2.** It is immediate to see that in 1 dimension it holds that

$$P_{1,s}^1(\Omega) > 0 \iff \Omega \text{ satisfies finite ball condition.}$$

The necessity of the condition is (iv) of the previous proposition. For the sufficiency, assume that $\Omega \subset \mathbb{R}$ is an open set satisfying the finite ball condition. Hence there exists a countable number of open intervals $I_k$ such that

$$\Omega = \bigcup_{k=1}^{\infty} I_k, \quad I_k \cap I_j = \emptyset \text{ if } k \neq j, \quad \sup_k(\text{length}(I_k)) = m < \infty.$$  

Hence, using (ii), we get for $u \in C_c^\infty(\Omega)$

$$[u]_{s,2,\Omega}^2 \geq \frac{C_{1,s}}{2} \sum_{k=1}^\infty \int_{I_k} \int_{I_k} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \geq \sum_{k=1}^\infty P_{1,s}(I_k) \int_{I_k} u^2$$

$$= \sum_{k=1}^\infty \frac{1}{(\text{length}(I_k))^{2s}} P_{1,s}((0,1)) \int_{I_k} u^2 \geq \frac{P_{1,s}((0,1))}{m^{2s}} \int_\Omega u^2,$$

which gives

$$P_{1,s}(\Omega) \geq \frac{P_{1,s}((0,1))}{m^{2s}}. \hspace{1cm} (9)$$

**Proof of Proposition 2.3.** (i) Case $\frac{1}{2} < s < 1$: This follows from Theorem 2.1 and the estimate $\text{dist}(x, \partial \Omega) \leq \frac{1}{2} \text{diam}(\Omega)$ for any $x \in \Omega$. Hence

$$P_{n,s}^1(\Omega) \geq \left(\frac{1}{2} \text{diam}(\Omega)\right)^{-2s} C^{-1},$$

where $C = C(\Omega, n, s)$ is the constant in Theorem 2.1.

**Case** $0 < s \leq \frac{1}{2}$: It is well known [see, Theorem 11.1 in [26]] that $C_c^\infty(\Omega)$ is dense in $W^{s,2}(\Omega)$ if and only if $s \leq \frac{1}{2}$. Notice that the constant function $1 \in W^{s,2}(\Omega)$ and $[1]_{s,2,\Omega} = 0$. Hence the claim follows.

However, we present a detailed proof for the case $s \in (0, \frac{1}{2})$, as it will be useful later. Let $T_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) \leq \delta\}$ be some small interior tubular neighborhood of $\partial \Omega$ for $\delta > 0$. $u_\delta$ shall be an approximation, as $\delta \to 0$, of the characteristic
function of $\Omega$:

$$u_{\delta} \in C_c^\infty(\Omega), \quad 0 \leq u_{\delta} \leq 1, \quad u_{\delta} = 1 \text{ in } \Omega \setminus T_{\delta}, \quad |\nabla u_{\delta}| \leq \frac{2}{\delta} \text{ in } \Omega. \quad (10)$$

Hence for $\delta \to 0$ one has $\|u_{\delta}\|_{L^2(\Omega)}^2 \to |\Omega|$ and it is sufficient to show that $[u_{\delta}]_{s,2,\Omega} \to 0$. As $|u_{\delta}(x) - u_{\delta}(y)| = 0$ for $x, y \in \Omega \setminus T_{\delta}$ we get

$$[u_{\delta}]_{s,2,\Omega}^2 \leq 2 \int_{T_{\delta}} dx \int_{\Omega} dy \frac{|u_{\delta}(x) - u_{\delta}(y)|^2}{|x - y|^{n+2s}} = A + B,$$

where

$$A = \int_{T_{\delta}} dx \int_{\{y \in \Omega; |x - y| < \delta\}} dy \frac{|u_{\delta}(x) - u_{\delta}(y)|^2}{|x - y|^{n+2s}}$$

and

$$B = \int_{T_{\delta}} dx \int_{\{y \in \Omega; |x - y| \geq \delta\}} dy \frac{|u_{\delta}(x) - u_{\delta}(y)|^2}{|x - y|^{n+2s}}.$$

In view of the last property of (10) we find

$$\frac{|u_{\delta}(x) - u_{\delta}(y)|}{|x - y|} \leq \frac{2}{\delta} \text{ for any } x \in T_{\delta} \text{ and } y \in \Omega \cap B_{\delta}.$$

Therefore we can estimate the inner integral of $A$ as

$$\int_{\{y \in \Omega; |x - y| < \delta\}} \frac{1}{|x - y|^{n+2s}} \leq \int_{|x - y| < \delta} \frac{1}{|x - y|^{n+2s}} = \int_{B_\delta(0)} \frac{dy}{|y|^{n+2s}}.$$

After radial integration and using that $|T_{\delta}|$ is of the order $\mathcal{H}^{n-1}(\partial \Omega)\delta$ one gets that $A \leq C\delta^{1-2s}$ for some constant $C = C(n, s, \Omega)$. For $B$ one uses the estimates $|u_{\delta}(x) - u_{\delta}(y)| \leq 1$ and

$$\int_{\{y \in \Omega; |x - y| \geq \delta\}} \frac{1}{|x - y|^{n+2s}} \leq \int_{|x - y| \geq \delta} \frac{1}{|x - y|^{n+2s}} = \int_{B_\delta(x)\setminus B_\delta(0)} \frac{1}{|x - y|^{n+2s}} dy.$$

After radial integration, one proceeds as for estimating $A$, and concludes that $B$ is also of order $\delta^{1-2s}$.

The last statement concerning $P^2_{n,s}(\Omega)$ follows from Lemma 2.4 in [4] if $\Omega$ is a bounded open set. Alternatively, in the case $2s < n$, one could prove the statement using the fractional Sobolev embedding. More precisely, if $2 \leq 2^* := 2n/(n - 2s)$ then

$$\|u\|_{L^{2^*}(\Omega)} \leq C\|u\|_{L^{2^*}(\Omega)} \leq C[u]_{s,2,R^n} \quad \text{if } 2s < n.$$ 

(ii) This is immediate by change of variables.

(iii) Follows directly from the definition.

(iv) Let $x \in \Omega$ and $r > 0$ be such that $B_r(x) \subset \Omega$. Then by (iii) and (ii) we get that

$$P^2_{n,s}(\Omega) \leq P^2_{n,s}(B_r(x)) = P^2_{n,s}(B_r(0)) = \frac{P^2_{n,s}(B_1(0))}{r^{2s}}.$$ 

As $r$ can be chosen arbitrarily big this proves that $P^2_{n,s}(\Omega) = 0$. By definition $P^1_{n,s}(\Omega) \leq P^2_{n,s}(\Omega)$, so we also have that $P^1_{n,s}(\Omega) = 0$. \qed

The following lemma relates the constants $C_{n,s}$ for different values of the dimension $n$. It is a generalization of the Lemma 3.1 in [10]. We will use these algebraic relations several times.
Lemma 2.4. For each \(m,n \in \mathbb{N} \) with \(1 \leq m < n \) and \(0 < s < 1\), let \(C_{n,s}\) be the constant (7) appearing in the definition of the \([\cdot]_{n,2} \) seminorm. The following two identities hold:

(i) \(C_{n,s} \Theta_{m,n} = C_{n-m,s}\), where \(\Theta_{m,n} = \mathcal{H}^{m-1}(S^{n-1}) \int_0^\infty \frac{t^{m-1}}{(1+t^2)^{n+2s}} \, dt\)

where \(S^{n-1}\) is the unit sphere in the Euclidean space \(\mathbb{R}^n\).

(ii) If \(a > 0\) and \(z \in \mathbb{R}^m\) then

\[
\int_{\mathbb{R}^m} \frac{dx}{\left(1 + \frac{|x-z|^2}{a^2}\right)^{n+2s}} = a^m \Theta_{m,n}.
\]

Proof. Using the change of variables \(t = \tan \theta\) in the expression \(\Theta_{m,n}\), we obtain

\[
\Theta_{m,n} = \int_{S^{n-1}} \text{d}\sigma \int_0^\infty (\sin \theta)^{m-1} (\cos \theta)^{n-m+2s-1} \, d\theta = \frac{1}{2} \beta \left( \frac{m}{2}, \frac{n-m+2s}{2} \right) \frac{2\pi^m}{\Gamma \left( \frac{m}{2} \right)}.
\]

From the definition of \(C_{n,s}\) in (7), and formulas (4) and (5) we get the desired result.

(ii) The integral clearly does not depend on \(z\). So taking \(z = 0\), the identity follows immediately by change of variables and radial integration. \(\square\)

In what follows we will use the following abbreviations: for \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) we write \(x = (x_1, x')\). We next prove a lemma regarding domain symmetrization.

In addition to its usefulness in proving Theorem 5.5, this may have an independent interest when dealing with regional fractional Poincaré and fractional Poincaré inequalities for general domains.

Definition 2.5. Let \(\Omega \subset \mathbb{R}^n\) be a measurable set. We define \(\Omega^*\), its cylindrical Schwarz symmetrization, as the set which is rotationally symmetric with respect to the \(x_1\) axis and \(\mathcal{H}^{n-1}(\Omega \cap \{x_1 = R\}) = \mathcal{H}^{n-1}(\Omega^* \cap \{x_1 = R\})\) for all \(R \in \mathbb{R}\). More precisely,

\(\Omega^* = \{(x_1, x') \in \mathbb{R}^n : x' \in (\Omega_{x_1})^*_{n-1}\}\), where \(\Omega_{x_1} = \{x' \in \mathbb{R}^{n-1} : (x_1, x') \in \Omega\}\) and \((A)_{n-1}^*\) is the standard Schwarz symmetrization of \(A\) in \(\mathbb{R}^{n-1}\), i.e. \(A\) is replaced by a ball of same \(\mathcal{L}^{n-1}\) measure and centered at the origin.

The next lemma holds true for any positive \(s\). So we state it in this general form, although we will use it only in the range \(0 < s < 1\).

Lemma 2.6. Let \(\Omega \subset \mathbb{R}^n\) be a measurable set and \(0 < s\). Then for any two disjoint sets \(I, J \subset \mathbb{R}\)

\[
\int_{\Omega \cap \{x_1 \in I\}} \int_{\Omega \cap \{x_1 \in J\}} \frac{1}{|x-y|^{n+2s}} \leq \int_{\Omega \cap \{x_1 \in I\}} \int_{\Omega \cap \{x_1 \in J\}} \frac{1}{|x-y|^{n+2s}}.
\]

Proof. We write the left hand side of (11) as

\[
\int_I \int_{\Omega_{x_1}} \int_J \int_{\Omega_{y_1}} \frac{dy}{|x-y|^{n+2s}} = \int_I \int_J \int_{\Omega_{x_1}} \int_{\Omega_{y_1}} \frac{dy'}{|x-y'|^{n+2s}}.
\]

Let \(\chi_A\) denote the characteristic function of a set \(A\) and abbreviate for each fixed \((x_1, y_1)\) the function

\[
h_{(x_1, y_1)}(z) = \frac{1}{((x_1 - y_1)^2 + |z|^{2(n+2s)})^{n/2}}, \quad z \in \mathbb{R}^{n-1}.
\]
$h$ is a radially decreasing symmetric function of $z$, so that $h^* = h$. Then the left
hand side of (11) can be written as

$$
\int_I dx \int_J dy \left( \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \chi_{\Omega_1}(x') \chi_{\Omega_2}(y') h(x_1, y_1) |(x' - y')| dx' dy' \right).
$$

In the same way the right hand side of (11) can be expressed, by replacing $\Omega$ by
$\Omega^*$. Thus the lemma follows from the Riesz rearrangement inequality.

3. Proof of Theorem 1.1 and Theorem 1.3. We start by giving some propo-
sitions and lemmas that will be useful in the proof of Theorem 1.1 and Theorem
1.3.

Proposition 3.1. Let $1 \leq m < n$ be two integers, $0 < s < 1$ and $\Omega_\infty = \mathbb{R}^m \times \omega$ be
a strip in $\mathbb{R}^n$ where $\omega \subset \mathbb{R}^{n-m}$ is a bounded open set. Then we have

$$
P_{n,s}^1(\Omega_\infty) \leq P_{n-m,s}^1(\omega) \quad \text{and} \quad P_{n,s}^2(\Omega_\infty) \leq P_{n-m,s}^2(\omega).
$$

Proof. We only prove the Proposition for $P_{n,s}^1$. The proof for $P_{n,s}^2$ is almost identical,
except that in the domains of integration (of $I_1, I_2, I_3$ in the proof below) one has
to do the replacements $\Omega_\infty \mapsto \mathbb{R}^n \setminus \omega \mapsto \mathbb{R}^{n-m}$.

Step 1: It is sufficient to show that for any $W \in C_\infty^c(\omega)$ and $\epsilon > 0$ there exists
$u \in C_\infty^c(\Omega_\infty)$ such that

$$
\frac{|u|_{L^2(\Omega_\infty)}}{|u|_{L^2(\omega)}} \leq \frac{|W|_{L^2(\omega)}}{|W|_{L^2(\omega)}^n} + \epsilon.
$$

Take a function $v \in C_\infty^c(\mathbb{R}^m)$ such that

$$
\int_{\mathbb{R}^m} |v|^2 = 1 \quad \text{and define for } \ell > 0 \quad v_\ell(x) = \ell^{-m} v \left( \frac{x}{\ell} \right).
$$

Then $v_\ell \in C_\infty^c(\mathbb{R}^m)$ and

$$
\int_{\mathbb{R}^m} |v_\ell|^2 = 1 \quad \text{for all } \ell.
$$

We shall use the notation $x = (X_1, X_2) \in \mathbb{R}^n$, $X_1 \in \mathbb{R}^m$ and $X_2 \in \mathbb{R}^{n-m}$. At last
we define

$$
u_\ell(X_1, X_2) = v_\ell(X_1)W(X_2),
$$

and claim that for $\ell$ big enough $u_\ell$ has the desired property. Without loss of
generality we can assume that $|W|_{L^2(\omega)} = 1$. Therefore, using (12) we get that

$$
|u_\ell|^2_{L^2(\Omega_\infty)} = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} v_\ell^2(X_1)W^2(X_2)\,dX_2\,dX_1 = |W|_{L^2(\omega)} = 1.
$$

We therefore have to prove that, for some $\ell$ big enough

$$
|u_\ell|^2_{L^2(\Omega_\infty)} \leq |W|_{L^2(\omega)}^n + \epsilon.
$$

Using that

$$|u_\ell(x) - u_\ell(y)|^2 + v_\ell(Y_1)W(X_2) - v_\ell(Y_1)W(X_2) + v_\ell(Y_1)W(X_2) - v_\ell(Y_1)W(Y_2) |^2

= v_\ell^2(Y_1) |W(X_2) - W(Y_2)|^2 + |v_\ell(Y_1) - v_\ell(Y_1)|^2 W^2(X_2)

+ 2v_\ell(Y_1)W(X_2) (W(X_2) - W(Y_2)) (v_\ell(X_1) - v_\ell(Y_1)),
$$

we write $|u_\ell|^2_{L^2(\Omega_\infty)}$ as $|u_\ell|^2_{L^2(\Omega_\infty)} := I_1 + I_2 + I_3$, where

$$I_1 = \frac{C_{n,s}}{2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|v_\ell(Y_1)(W(X_2) - W(Y_2))|^2}{|x - y|^{n+2s}} dx\,dy,
$$

and
We will show that

\[ I_1 = |W|^{2, \omega}_{s, 2} \quad \text{and} \quad I_2, I_3 \to 0 \quad \text{as} \quad \ell \to \infty. \quad (13) \]

**Step 2 (Calculation of \( I_1 \)):** We obtain from Lemma 2.4 (ii), whenever \( X_2 \neq Y_2 \),

\[
\int_{\mathbb{R}^m} \frac{dX_1}{(1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2})^{\frac{n+2s}{2}}} = |X_2 - Y_2|^m \Theta_{m,n} \quad \text{for any} \quad Y_1 \in \mathbb{R}^m.
\]

Plugging this identity into the definition of \( I_1 \), using Lemma 2.4 (i) and then (12) gives

\[
I_1 = \frac{C_{n,s}}{2} \int_{\Omega_\omega} \int_{\Omega_\omega} \frac{|v_\ell(Y_1)(W(X_2) - W(Y_2))|^2}{|X_2 - Y_2|^{n+2s}} dX dY
\]

\[
= \int_{\omega} \int_{\omega} \left( C_{n,s} \frac{W(X_2) - W(Y_2)}{2 |X_2 - Y_2|^{n+2s}} \left[ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \frac{dX_1}{(1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2})^{\frac{n+2s}{2}}} \right) |v_\ell(Y_1)|^2 dY_1 \right] dX_2 dY_2 \right)
\]

\[
= \frac{C_{n,s} \Theta_{m,n}}{2} \int_{\omega} \int_{\omega} \frac{|W(X_2) - W(Y_2)|^2}{|X_2 - Y_2|^{n-m+2s}} dX_2 dY_2 \int_{\mathbb{R}^m} |v_\ell(Y_1)|^2 dY_1
\]

\[
= \frac{C_{n,m,s}}{2} \int_{\omega} \int_{\omega} \frac{|W(X_2) - W(Y_2)|^2}{|X_2 - Y_2|^{n-m+2s}} dX_2 dY_2 = |W|^{2, \omega}_{s, 2, \omega}.
\]

This proves the first statement of (13).

**Step 3 (Estimates for \( I_2 \) and \( I_3 \)):** We write \( I_2 \) as

\[
I_2 = \frac{C_{n,s}}{2} \int_{\Omega_\omega} \int_{\Omega_\omega} \frac{|(v_\ell(X_1) - v_\ell(Y_1))W(X_2)|^2}{|X_1 - Y_1|^{n+2s}} dX dY
\]

\[
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left( \frac{C_{n,s} |v_\ell(X_1) - v_\ell(Y_1)|^2}{2 |X_1 - Y_1|^{n+2s}} \right) \left( \int_{\omega} \left( \frac{dY_2}{(1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2})^{\frac{n+2s}{2}}} \right) |W(X_2)|^2 dX_2 \right) dX_1 dY_1.
\]

Using Lemma 2.4 (ii) we get

\[
\int_{\omega} \frac{dY_2}{(1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2})^{\frac{n+2s}{2}}} \leq \int_{\mathbb{R}^{n-m}} \frac{dY_2}{(1 + \frac{|X_2 - Y_2|^2}{|X_1 - Y_1|^2})^{\frac{n+2s}{2}}} = |X_1 - Y_1|^{n-m} \Theta_{n,m,n}.
\]
Plugging this into the definition of $I_2$ and using once more that $\|W\|_{L^2} = 1$ gives

$$I_2 \leq \frac{C_{n,s}}{2} \frac{\Theta_{n-m,n}}{2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|v_t(X_1) - v_t(Y_1)|^2}{|X_1 - Y_1|^{m+2s}} \, dX_1 \, dY_1 = |v_t|^2_{s,2,\mathbb{R}^m}.$$  

By Proposition 2.3 (ii) and definition of $v_t$

$$|v_t|^2_{s,2,\mathbb{R}^m} = \frac{\ell^{m-2s}}{\ell^m} |v|^2_{s,2,\mathbb{R}^m} = \frac{1}{\ell^{2s}} |v|^2_{s,2,\mathbb{R}^m} \quad \Rightarrow \quad I_2 \leq \frac{|v|^2_{s,2,\mathbb{R}^m}}{\ell^{2s}},$$

which proves (13) for $I_2$. For $I_3$ we use Hölder inequality and estimate it as

$$I_3 \leq 2\sqrt{1 + I_2} \leq 2|W|_{s,2,\omega} \frac{|v|_{s,2,\mathbb{R}^m}}{\ell^s}.$$  

This shows (13) also for $I_3$. 

For the proofs of Theorem 1.1 and 1.3 we will use the following lemma, which follows from an appropriate integration in spherical coordinates and an application of the change of variables formula.

**Lemma 3.2** (Loss and Sloane [27], Lemma 2.4). Let $p > 0$, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ be a measurable set. Then for any $u \in C^\infty_c(\Omega)$

$$2 \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} = \frac{\mathcal{H}^{n-1}(w)}{\mathcal{H}^{n-1}(x)} \int_{\mathbb{R}^n} d\mathcal{H}^{n-1}(w) \int_{\{x: x \cdot w = 0\}} \frac{|u(x + \ell t) - u(x + t w)|^p}{|\ell - t|^{1+sp}} dtd\ell.$$

**Lemma 3.3.** Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^n$ be a measurable set. Suppose that there exists a $\mathcal{H}^{n-1}$ measurable function $f : \mathbb{S}^{n-1} \to [0, \infty)$ such that

$$P_{1,s}^1(\{t \in \mathbb{R} : x + tw \in \Omega\}) \geq f(w)$$

for a.e. $w \in \mathbb{S}^{n-1}$ and a.e. $x \in \{y \in \mathbb{R}^n : y \cdot w = 0\}$. Then it holds that

$$P_{1,s}^1(\Omega) \geq \frac{C_{n,s}}{2C_{1,s}} \int_{\mathbb{S}^{n-1}} f(w) \, d\mathcal{H}^{n-1}(w).$$

**Proof.** Let $w \in \mathbb{S}^{n-1}$ and $x \in L_w := \{y \in \mathbb{R}^n : y \cdot w = 0\}$. Define $A_{x,w} := \{t \in \mathbb{R} : x + tw \in \Omega\}$. Then by hypothesis

$$\frac{C_{1,s}}{2} \int_{\{t : x + tw \in \Omega\}} \int_{\{t : x + tw \in \Omega\}} \frac{|u(x + \ell t) - u(x + tw)|^2}{|\ell - t|^{1+2s}} dtd\ell \geq \frac{P_{1,s}^1(A_{x,w})}{P_{1,s}^1(A_{x,w})} \int_{A_{x,w}} |u(x + tw)|^2 dt \geq f(w) \int_{A_{x,w}} |u(x + tw)|^2 dt.$$

Note that for any $w \in \mathbb{S}^{n-1}$, by Fubini (or change of variables)

$$\int_{L_w} d\mathcal{H}^{n-1}(x) \int_{A_{x,w}} |u(x + tw)|^2 dt = \int_{\Omega} |u|^2.$$

Therefore it follows from Lemma 3.2 that

$$|u|^2_{W^{s,2} (\Omega)} \geq \frac{C_{n,s}}{2C_{1,s}} \left( \int_{\mathbb{S}^{n-1}} f(w) \, d\mathcal{H}^{n-1}(w) \right) \int_{\Omega} |u|^2,$$

which proves the lemma.  

$\square$
We will use the explicit form of the following hyperspherical coordinates and their properties. Let us define $Q_{n-1} \subset \mathbb{R}^{n-1}$ by

$$Q_{n-1} = (0, \pi)^{n-2} \times (0, 2\pi).$$

The hyper spherical coordinates $H = (H_1, \ldots, H_n) : Q_{n-1} \rightarrow S^{n-1}$ are defined as: for $k = 1, \ldots, n$ and $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$

$$H_k(\varphi) = \cos \varphi_k \prod_{l=0}^{k-1} \sin \varphi_l \quad \text{with convention} \quad \varphi_0 = \frac{\pi}{2}, \quad \varphi_n = 0.$$

A calculation shows that $d_i(\varphi) := \left(\frac{\partial H}{\partial \varphi_i}, \frac{\partial H}{\partial \varphi_i}\right) = \prod_{l=0}^{i-1} \sin^2 \varphi_l > 0$. One verifies that the metric tensor in these coordinates is diagonal $g_{ij}(\varphi) = \left(\frac{\partial H}{\partial \varphi_i}, \frac{\partial H}{\partial \varphi_j}\right) = \delta_{ij} d_i(\varphi)$, $\delta_{ij} = 1$ if $i = j$ and 0 else) and hence the surface element $g_{n-1}$ is given by

$$g_{n-1}(\varphi) = \sqrt{\det g_{ij}(\varphi)} = \prod_{k=1}^{n-1} d_k(\varphi) = \prod_{k=1}^{n-2} (\sin \varphi_k)^{n-k-1}.$$

Note that for any function $f$ depending only of $\varphi_1$ we have that

$$\int_{Q_{n-1}} f(\varphi_1) g_{n-1}(\varphi) d\varphi$$

$$= \int_0^\pi f(\varphi_1)(\sin \varphi_1)^{n-2} \left(\int_{Q_{n-2}} (\sin \varphi_2)^{n-3} \cdots \sin \varphi_{n-2} d\varphi_2 \cdots d\varphi_{n-2}\right) d\varphi_1$$

$$= \int_0^\pi f(\varphi_1)(\sin \varphi_1)^{n-2} \left(\int_{Q_{n-2}} g_{n-2}(\theta) d\theta\right) d\varphi_1$$

$$= \mathcal{H}^{n-2}(S^{n-2}) \int_0^\pi f(\varphi_1)(\sin \varphi_1)^{n-2} d\varphi_1.$$

In particular for $f(\varphi) = |\cos \varphi_1|^{2s}$ we obtain, using (5), that

$$\int_{Q_{n-1}} |\cos \varphi_1|^{2s} g_{n-1}(\varphi) d\varphi = 2\mathcal{H}^{n-2}(S^{n-2}) \int_0^\pi (\cos \varphi_1)^{2s} (\sin \varphi_1)^{n-2} d\varphi_1$$

$$= \frac{2\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} B\left(\frac{n-1}{2}, \frac{2s+1}{2}\right). \quad (14)$$

**Proof of Theorem 1.1.** Part (1): We apply the above Proposition 3.1 by choosing $m = n-1$ and $\omega = (-1, 1) \subset \mathbb{R}$. In particular we have $P_{1,s}(\Omega_\infty) \leq P_{1,s}((-1,1))$. Now use that $P_{1,s}((-1,1)) = 0$ for any $s \in (0, \frac{1}{2}]$, see Proposition 2.3 (i).

Part (2): By Proposition 3.1 we know that $P_{1,s}(\Omega_\infty) \leq P_{1,s}((-1,1))$. So it is sufficient to show that

$$P_{1,s}(\Omega_\infty) \geq P_{1,s}((-1,1)). \quad (15)$$

We will deduce this inequality from Lemma 3.3. Let $w = (w_1, \ldots, w_n) \in S^{n-1}$ be such that $w_1 \neq 0$. By the special form of $\Omega_\infty = (-1, 1) \times \mathbb{R}^{n-1}$ we have that the length of the intersection $\Omega_\infty \cap \{x + tw : t \in \mathbb{R}\}$ does not depend on $x$. So we obtain
It is immediate to verify, using (5), that bounded continuous function such that
\( f(x) = t_0(w) \) where \(-1 + t_0(w)w_1 = 1 \Rightarrow t_0(w) = \frac{2}{w_1} \).

From Proposition 2.3 (ii) we obtain that
\[
P_{1,s}^1 \left( \{ t \in \mathbb{R} : x + tw \in \Omega_\infty \} \right) = \left( \frac{|w_1|}{2} \right)^{2s} P_{1,s}^1((0,1)) = |w_1|^{2s} P_{1,s}^1((-1,1)).
\]

Using Lemma 3.3, the hyperspherical coordinates and (14) gives
\[
P_{n,s}(\Omega_\infty) \geq P_{1,s}^1((-1,1)) \frac{C_{n,s}}{2C_{1,s}} \int_{g^{n-1}} |w_1|^{2s} d\mathcal{H}^{n-1} \]
\[
= P_{1,s}^1((-1,1)) \frac{C_{n,s}}{2C_{1,s}} \int_{Q_{n-1}} |\cos \varphi_1|^{2s} g_{n-1}(\varphi) d\varphi \]
\[
= P_{1,s}^1((-1,1)) \frac{C_{n,s}}{2C_{1,s}} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma \left( \frac{n+1}{2} \right)} B \left( \frac{n-1}{2}, \frac{2s+1}{2} \right).
\]

It is immediate to verify, using (5), that
\[
\frac{C_{n,s}}{C_{1,s}} \frac{\pi^{\frac{n+1}{2}}}{\Gamma \left( \frac{n+1}{2} \right)} B \left( \frac{n-1}{2}, \frac{2s+1}{2} \right) = 1,
\]
which concludes the proof of (15).

**Proof of Theorem 1.3.** By hypothesis, Remark 2, and (9) we obtain that
\[
P_{1,s}^1 \left( \{ t \in \mathbb{R} : x + tw \in \Omega_\infty \} \right) \geq \frac{P_{1,s}^1((0,1))}{m^{2s}} \text{ for all } w \in \Sigma, x \in \mathbb{R}^n.
\]

Thus if we define
\[
f(w) = \begin{cases} \frac{P_{1,s}^1((0,1))}{m^{2s}} & \text{if } w \in \Sigma \\ 0 & \text{if } w \in S^{n-1} \setminus \Sigma \end{cases}
\]
then \( f \) satisfies the hypothesis of Lemma 3.3 and we get that
\[
P_{n,s}^1(\Omega) \geq \frac{C_{n,s}}{2C_{1,s}} \mathcal{H}^{n-1}(\Sigma) \frac{P_{1,s}^1((0,1))}{m^{2s}}.
\]

This completes the proof of the theorem.

We will provide some examples of domains which satisfy the hypothesis of Theorem 1.3 and one which does not.

**Example 3.4.** (i) **Domain between Graphs:** Let \( f_1, f_2 : \mathbb{R}^{n-1} \to [m, M] \) be two bounded continuous function such that \( f_1 < f_2 \). \( \Omega \) is defined as
\[
\Omega = \{ (x,y) \in \mathbb{R}^n : f_1(x) < y < f_2(x) \}.
\]

(ii) **Finite union of strips:** The domain \( \Omega \) is the finite union of the strips in \( \mathbb{R}^n \). Let \( O(n) \) denote the set on \( n \times n \) orthogonal(rotation) matrices. Given \( A_i \in O(n), z_i \in \mathbb{R}^n, b_i \in \mathbb{R} \) for \( i = 1, \ldots, M \) and \( \Omega_\infty = (-1,1) \times \mathbb{R}^{n-1} \), then define
\[
\Omega = \bigcup_{i=1}^M (b_i A_i(\Omega_\infty) + z_i).
\]
(iii) *Infinitely many parallel strips:* $\Omega = (\bigcup_{i=1}^{\infty} I_i) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, where $I_i$ are disjoint open intervals and length $(I_i) \leq m$ for some uniform upper bound $m$.

(iv) *Infinite “L” type domain:*

$$\Omega = ((0, 1) \times (0, \infty)) \bigcup ((0, \infty) \times (0, 1)) \subset \mathbb{R}^2.$$

(v) *Concentric Annulus:* The following domain satisfies the finite ball condition but does not satisfy the hypothesis of Theorem 1.3:

$$\Omega = \bigcup_{k=1}^{\infty} B_{2k}(0) \setminus B_{2k-1}(0).$$

4. **Proof of Theorem 1.2.** The main tool to prove Theorem 1.2 is to use discrete version of fractional Picone identity. There are various version of Picone identity, we refer to [[8], equation no 6.12] for the version which is particularly helpful for problems involving second order elliptic operators.

**Lemma 4.1** (Discrete Picone inequality). Let $u$, $v$ be two measurable functions with $u > 0$ and $v \geq 0$. Then

$$\left( u(x) - u(y) \right) \left[ \frac{v^2(x) - v^2(y)}{u(x)} \right] \leq |v(x) - v(y)|^2.$$

Expanding the left side of this inequality and by using Young’s inequality we get the lemma. For further generalization, we refer to the Proposition 4.2 in [5]. We will use the following abbreviation $x = (X_1, X_2) \in \mathbb{R}^n, X_1 \in \mathbb{R}^m, X_2 \in \mathbb{R}^{n-m}$. $\Omega_\infty = \mathbb{R}^m \times \omega, \omega \subset \mathbb{R}^{n-m}$ shall always denote the sets defined in Theorem 1.2.

We assume that $W$ is the first Eigenfunction of the fractional Laplace operator in $\omega$. By Proposition 6.1 (see Appendix) $W$ is smooth and strictly positive in $\omega$ and satisfies for some $P_{n-m,s}^2(\omega) > 0$

$$\left\{ (-\Delta_{n-m})^s W = P_{n-m,s}^2(\omega)W \right\} \text{ in } \omega,$$

$$W = 0 \text{ in } \mathbb{R}^{n-m} \setminus \omega,$$

$$W > 0 \text{ in } \omega.$$

**Lemma 4.2.** Let $x = (X_1, X_2) \in \Omega_\infty$ and define $u^*(x) := W(X_2)$. Then we have

$$(-\Delta_n)^s u^* = P_{n-m,s}^2(\omega) u^* \text{ in } \Omega_\infty.$$

**Proof.** Using first Lemma 2.4 (ii) and then (i) we get

$$(-\Delta_n)^s u^*(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u^*(x) - u^*(y)}{|x-y|^{n+2s}} dy = C_{n,s} \int_{\mathbb{R}^n} \frac{W(X_2) - W(Y_2)}{|x-y|^{n+2s}} dy$$

$$= C_{n,s} \int_{\mathbb{R}^{n-m}} dY_2 \frac{W(X_2) - W(Y_2)}{|X_2 - Y_2|^{n+2s}} \int_{\mathbb{R}^m} \frac{dY_1}{\left(1 + \frac{|X_1 - Y_1|^2}{|X_2 - Y_2|^2}\right)^{1+2s}}$$

$$= C_{n,s} \Theta_{n,m} \int_{\mathbb{R}^{n-m}} \frac{W(X_2) - W(Y_2)}{|X_2 - Y_2|^{n-m+2s}} dY_2 = (-\Delta_{n-m})^s W(X_2)$$

$$= P_{n-m,s}^2(\omega) W(X_2) = P_{n-m,s}^2(\omega) u^*(x).$$

**Proof of the Theorem 1.2.** Thanks to Proposition 3.1 we know that $P_{n,s}^2(\Omega_\infty) \leq P_{n-m,s}^2(\omega)$. So we only have to check the converse inequality.
Since \( u^*(x) := W(X_2) \in C^\infty(\Omega_\infty) \) is strictly positive in \( \Omega_\infty \). We have for any \( v \in C^\infty_c(\Omega_\infty) \) the function \( \phi = v^2 \) belongs to \( \phi \in C^\infty_c(\Omega_\infty) \). Then by Discrete Picone inequality we have

\[
(u^*(x) - u^*(y))(\phi(x) - \phi(y)) = (u^*(x) - u^*(y))\left[\frac{v^2(x)}{u^*(x)} - \frac{v^2(y)}{u^*(y)}\right] \leq |v(x) - v(y)|^2.
\]

Integrating two times over \( \mathbb{R}^n \) we obtain

\[
\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u^*(x) - u^*(y))(\phi(x) - \phi(y))}{|x-y|^{n+2s}} \, dx \, dy \leq \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]

Writing the left hand side as sum of two integral, one containing \( \phi(x) \) and the other \( \phi(y) \) and making a change of variables in the second \( x \mapsto y \) gives

\[
C_{n,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u^*(x) - u^*(y))\phi(x)}{|x-y|^{n+2s}} \, dx \, dy \leq \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]

It follows that

\[
C_{n,s} \int_{\Omega_\infty} \frac{v^2(x)}{u^*(x)} \int_{\Omega_\infty} \frac{u^*(x) - u^*(y)}{|x-y|^{n+2s}} \, dy \, dx \leq \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]

As this is true for any \( v \in C^\infty_c(\Omega_\infty) \) and using Lemma 4.2, we get that \( P_{n-m,s}^2(\omega) \leq P_{n,s}^2(\Omega_\infty) \). \( \square \)

5. **Sufficient conditions for** \( s \in (0, \frac{1}{2}) \).

5.1. **Proof of Theorem 1.4 and related discussions.** In this section we will denote \( x' \in \mathbb{R}^{n-1}, \{ x_1 = R \} = \{(x_1, x') \in \mathbb{R}^n : x_1 = R \} \), and similar notations for \( x_1 > R \). Theorem 1.4 is having a general and relatively abstract condition on domains. We discuss various examples satisfying the condition later. First, we prove this main theorem.

**Proof of Theorem 1.4.** Define \( \Omega_k = \Omega \cap \lambda_k U \). Let \( \epsilon > 0 \) be given. By hypothesis there exists \( k \in \mathbb{N} \) such that

\[
\frac{1}{E^n(\Omega_k)} \int_{\Omega_k} \frac{dx}{\text{dist}(x,(\lambda_k U)^c)^2} \leq \epsilon.
\]

By hypothesis \( \Omega_k \) is bounded. \( \Omega_k \) might not be Lipschitz, but \( H^{n-1}(\partial \Omega_k) \leq H^{n-1}(\partial \Omega \cap \lambda_k U) + C \lambda_k^{n-1} \), \( C < \infty \). Therefore by Proposition 2.3 (i) and Remark 1 there exists a \( v_k \in C^\infty_c(\Omega_k) \) such that

\[
\frac{1}{\|v_k\|_{L^2(\Omega_k)}^2} \int_{\Omega_k} \int_{\Omega_k} \frac{|v_k(x) - v_k(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \leq \epsilon.
\]

Recall, by the proof of Proposition 2.3 (i) or see [21] Lemma A.2, that \( v_k \) is an approximation of the characteristic function of \( \Omega_k \) by cutting it off near the boundary. Thus we can also assume that

\[
|v_k| \leq 1 \quad \text{and} \quad \|v_k\|_{L^2(\Omega_k)}^2 \geq \frac{E^n(\Omega_k)}{2}.
\]

Finally define \( u_k := v_k \) in \( \Omega_k \) and \( u_k := 0 \) in \( \Omega \setminus \Omega_k \). Then \( u_k \in C^\infty_c(\Omega) \) and

\[
\|u_k\|_{L^2(\Omega_k)}^2 = \|v_k\|_{L^2(\Omega_k)}^2 \geq \frac{E^n(\Omega_k)}{2}.
\]
Therefore, we get that
\[
\int_{\Omega_k} dx \int_{\Omega_k} dy \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} \leq \int_{\Omega_k} dx \int_{\Omega_k} dy \frac{1}{|x - y|^{n+2s}}.
\]
Now use that for \( x \in \Omega_k \)
\[
\int_{\Omega \setminus \Omega_k} dy \frac{1}{|x - y|^{n+2s}} \leq \int_{\{y : |x-y| > \text{dist}(x, (\lambda_k U)^c)\}} |x - y|^{n+2s} = c(n) \int_{\text{dist}(x, (\lambda_k U)^c)} x^{n-1} \frac{dr}{r^{n+2s}} = \frac{c(n, s)}{\text{dist}(x, (\lambda_k U)^c)^{2s}}.
\]
Plugging this into the previous inequality and using (18), (16) gives that
\[
\frac{1}{\|u_k\|_{L^2(\Omega_k)}^2} \int_{\Omega_k} dx \int_{\Omega_k} dy \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} \leq 2c(n, s) \frac{\mathcal{L}^n(\Omega)}{\text{dist}(x, (\lambda_k U)^c)^{2s}} \quad (19)
\]
Using (17) and (19) we conclude that
\[
\frac{1}{\|u_k\|_{L^2(\Omega)}^2} \int_{\Omega} dx \int_{\Omega} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} dx dy
\]
\[
= \frac{1}{\|v_k\|_{L^2(\Omega)}^2} \int_{\Omega_k} dv_k(x) - v_k(y)\|^2 |x - y|^{n+2s} dx dy
\]
\[
+ \frac{2}{\|u_k\|_{L^2(\Omega)}^2} \int_{\Omega_k} \int_{\Omega_k} |u_k(x) - u_k(y)|^2 \frac{x - y}{|x - y|^{n+2s}} dx dy
\]
\[
\leq \epsilon (1 + 4c(n, s)).
\]
This completes the proof of the theorem. 

\[\square\]

**Example 5.1.** Let \( \Omega_{\infty} \) is the union of two perpendicular infinite strips in \( \mathbb{R}^n \), i.e.
\[
\Omega_{\infty} = (\mathbb{R}^{n-1} \times (-1, 1)) \cup ((-1, 1) \times \mathbb{R}^{n-1}).
\]
Then \( P_{n,s}(\Omega_{\infty}) = 0 \) whenever \( s \in (0, \frac{1}{2}) \).

The idea is to apply Theorem 1.4 with \( \Omega = \Omega_{\infty} \) and \( U = (-1, 1)^n \). Let \( A := \mathbb{R}^{n-1} \times (-1, 1) \), \( B := (-1, 1) \times \mathbb{R}^{n-1} \). Then clearly,
\[
\int_{\lambda_k U \cap \Omega} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} \leq \int_{\lambda_k U \cap A} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} + \int_{\lambda_k U \cap B} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}}.
\]
Using the symmetric property of the domain \( \Omega_{\infty} \), it is sufficient to show the following:
\[
\lim_{k \to \infty} \frac{1}{\lambda_k U \cap A} \int_{\lambda_k U \cap A} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} = 0. \quad (20)
\]
We verify this result for dimension 3. Consider \( \lambda_k = k \in \mathbb{N} \). Therefore, \( \mathcal{L}^3(\Omega \cap \lambda_k U) = Ck^2 \) for some constant \( C > 0 \) independent of \( k \). Further, we write \( \lambda_k U \cap A = A_1 \cup A_2 \cup A_3 \), where
\[
A_1 = (A \cap \lambda_k U) \cap (-1, 1)^3, \quad A_2 = ((A \cap \lambda_k U) \setminus (-1, 1)^3) \cap \{|x_1| > |x_2|\}, \quad A_3 = ((A \cap \lambda_k U) \setminus (-1, 1)^3) \cap \{|x_2| > |x_1|\}.
\]
For, \( x \in A_1 \) we get that \( \text{dist}(x, (\lambda_k U)^c) \geq k - 1 \), and thus
\[
\frac{1}{\mathcal{L}^3(\lambda_k U \cap \Omega)} \int_{A_1} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} \leq \frac{C}{k^2} \cdot \frac{\mathcal{L}^3(A_1)}{(k-1)^{2s}} \leq \frac{C}{k^{2s+2}}.
\] (21)

Observe that
\[
\text{dist}(x, (\lambda_k U)^c) = \begin{cases} 
  k - |x_1|, & \text{if } x \in A_2, \\
  k - |x_2|, & \text{if } x \in A_3.
\end{cases}
\]

Therefore we obtain that
\[
\frac{1}{\mathcal{L}^3(\lambda_k U \cap \Omega)} \left( \int_{A_2} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} + \int_{A_3} \frac{dx}{\text{dist}(x, (\lambda_k U)^c)^{2s}} \right)
\leq \frac{C}{k^2} \left( \int_{|x_3|<1} \int_{|x_2|<k} dx_3 dx_2 \int_{|x_1|<k} \frac{dx_1}{(k - |x_1|)^{2s}} 
+ \int_{|x_1|<1} \int_{|x_2|<k} dx_3 dx_1 \int_{|x_2|<k} \frac{dx_2}{(k - |x_2|)^{2s}} \right)
\leq \frac{Ckk^{1-2s}}{k^2} \leq Ck^{-2s}.
\] (22)

Therefore, combining (21) and (22) we see that (20) holds.

**Corollary 1.** Assume \( \Omega \) is a Lipschitz set such that for some \( c = c(\Omega) > 0 \), it holds that \( \mathcal{L}^n(\Omega \cap B_R) \geq c R^n \) for all \( R > 0 \). Then \( P_{n,s}(\Omega) = 0 \) for \( 0 < s < \frac{1}{2} \).

**Proof.** Take \( U = B_1 \) and note that \( \text{dist}(x, B_R^n) = R - |x| \) for \( x \in \Omega \cap B_R \). Hence we get that
\[
\frac{1}{\mathcal{L}^n(B_R \cap \Omega)} \int_{B_R \cap \Omega} \frac{dx}{\text{dist}(x, B_R^n)^{2s}} \leq \frac{1}{c R^n} \int_0^R \frac{r^{n-1}}{(R - r)^{2s}} \leq \frac{C(n, s, \Omega)}{R^{2s}},
\]
which tends to 0 as \( R \) goes to \( \infty \). Therefore, the result follows by applying Theorem 1.4.

**Example 5.2.** Let \( \Omega \) be the domain as in Example 3.4(v) (Concentric Annulus). One can easily verify that \( \mathcal{L}^n(\Omega \cap B_R) \geq c R^n \) for any \( R > 0 \). By Corollary 1 we get \( P_{n,s}(\Omega) = 0 \) for \( 0 < s < \frac{1}{2} \).

### 5.2. Sufficient conditions for domains with finite measure.

The main result of this section is Theorem 5.5, which deals with \( P_{n,s}(\Omega) \) for unbounded domains \( \Omega \) of finite measure and for the range \( s \in (0, \frac{1}{2}) \). The theorem covers for instance the example (3) of \( \Omega \) in the introduction and shows that \( P_{n,s}(\Omega) = 0 \) for \( s \in \left(0, \frac{1}{2}\right) \).

**Definition 5.3 (Decay condition in one direction).** We say that \( \Omega \subset \mathbb{R}^n \) satisfies the decay condition in one direction if for both \( \Omega_+ = \Omega \) and \( \Omega_- := \{(-x_1, x') : (x_1, x') \in \Omega\} \) the following holds: there exists some function \( h : [0, \infty) \rightarrow [0, \infty) \) such that
\[
\mathcal{H}^{n-1}(\Omega_+ \cap \{x_1 = R\}) \leq h(R), \quad \forall R > 0,
\]
and there exists \( a > 0 \) and an infinite sequence \( \{R_k\}_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to \infty} R_k = \infty, \quad \lim_{k \to \infty} h(R_k) = 0, \quad h(R_k + \eta) \leq h(R_k) \quad \text{for all } k \text{ and } \eta \in [0, a).
\] (23)

Here are some examples of sets which have finite measure and satisfy the decay condition in one direction.
Example 5.4. (i) Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be two Lipschitz functions, $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : f_1(x_1) \leq x_2 \leq f_2(x_1)\}$ and assume $\mathcal{L}^n(\Omega) < \infty$. To see that $\Omega$ satisfies the decay condition in one direction, take $h = f_2 - f_1$ and $R_k$ such that

$$\sup_{x_1 \in [k, \infty)} h(x_1) = h(R_k).$$

Using that $h$ is Lipschitz and $\mathcal{L}^n(\Omega) < \infty$, it can be easily checked that this supremum is attained at some $R_k \in [k, \infty)$ and that $h(R_k) \to 0$ as $k \to \infty$. Finally $a$ can be taken to be any positive real number.

(ii) Let $\epsilon > 0$ and for $\ell \in \mathbb{N}$, define $A(\ell) := (\ell, \ell + \frac{1}{n+\epsilon}) \times (0, \ell) \subset \mathbb{R}^2$. Set $\Omega = \bigcup_{\ell = 1}^{\infty} A(\ell)$. Here one can take $h$ as

$$h(x_1) = \begin{cases} \ell & \text{whenever } x_1 \in (\ell, \ell + \ell^{-2-\epsilon}) \text{ for } \ell \in \mathbb{N}, \\
0 & \text{else}, \end{cases}$$

and a positive number $0 < a < \frac{1}{2}$, which implies $\operatorname{dist}(A(\ell), A(\ell + 1)) > a$ for all $\ell$.

Theorem 5.5. Let $0 < s < \frac{1}{2}$ and $\Omega \subset \mathbb{R}^n$ be a measurable Lipschitz set of finite measure $\mathcal{L}^n(\Omega) < \infty$. Assume that $\Omega$ satisfies the decay condition in one direction and that for any $K > 0$ the set $\Omega \cap \{x_1 < K\}$ is bounded. Then $P_{s,n}(\Omega) = 0$.

Example 5.6. The theorem applies to (i)-(ii) of Example 5.4 and also to $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, 0 < x_2 < 1/x_1^{1+\epsilon}\}$, where $\epsilon > 0$.

Remark 3. It is immediate to see that the technique for the proof of the theorem, without modifications, shows that also for the domain, for $\epsilon > 0$,

$$\Omega = \left\{ |x_1| \geq 1, x_2^2 + x_3^2 < \frac{1}{|x_1|^{1+\epsilon}} \right\} \cup \left\{ |x_2| \geq 1, x_1^2 + x_3^2 < \frac{1}{|x_2|^{1+\epsilon}} \right\} \cup B_2(0) \subset \mathbb{R}^3,$$

it holds that $P_{s,n}(\Omega) = 0$. Or any kind of domain where only a finite number of “tentacles” at infinity are unbounded in at most only one direction, which could be along a curve going to infinity, and that these tentacles satisfies the decay condition in one direction. An example of a domain to which the methods of Theorem 5.5 would not apply is the set $(\epsilon > 0$ so that $\Omega$ has finite measure).

$$\Omega = \left\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 > 1, 0 < x_3 < \frac{1}{(x_1^2 + x_2^2)^{1+\epsilon}} \right\}.$$
Lemma 5.7. Let $0 < s < 1/2$ and $a > 0$. There exists a constant $C(n,s) > 0$ such that
\[ L_s(A_-(R), A_+(R)) \leq C(n,s)a^{1-2s}R^{n-1} \quad \text{for all } R > 0 \]
where
\[ A_-(R) = \{ x = (x_1, x') \in \mathbb{R}^n : x_1 \in (-a, 0), \ |x'| < R \} \]
and
\[ A_+(R) = \{ x \in \mathbb{R}^n : x_1 \in (0, a) \}. \]

Proof. We write $L_s(A_-(R), A_+(R))$ as
\[ \int_{-a}^{0} dx_1 \int_{B_0(0)} dx' \int_{0}^{a} dy_1 \int_{\mathbb{R}^{n-1}} dy' \frac{1}{|x_1 - y_1|^{n+2s}} \frac{1}{(1 + |x' - y'|^2/|x_1 - y_1|^2)^{n+2s}}. \]

Now we proceed as in Step 2 of the proof of Proposition 3.1, using Lemma 2.4. In this way we get
\[ \int_{\mathbb{R}^{n-1}} \frac{dy'}{(1 + |x' - y'|^2/|x_1 - y_1|^2)^{n+2s}} = \int_{\mathbb{R}^{n-1}} \frac{dy'}{(1 + |y'|^2/|x_1 - y_1|^2)^{n+2s}} = \mathcal{H}^{n-2} S^{n-2} |x_1 - y_1|^{n-1} \int_{0}^{\infty} \frac{r^{n-2}}{(1 + r^2)^{n+2s}} dr = C(n,s) |x_1 - y_1|^{n-1}. \]

Thus we get that, renaming the constant $C(n,s)$ that
\[
L_s(A_-(R), A_+(R)) = \int_{-a}^{0} dx_1 \int_{B_0(0)} dx' \int_{0}^{a} dy_1 \frac{|x_1 - y_1|^{n-1}}{|x_1 - y_1|^{n+2s}} \]
\[ = C(n,s) R^{n-1} \int_{-a}^{0} \int_{0}^{a} \frac{dy_1 dx_1}{|x_1 - y_1|^{1+2s}} \leq C(n,s)a^{1-2s}R^{n-1}. \]
\[
\square
\]

Lemma 5.8. Suppose $0 < s < 1/2$ and that $\Omega$ has finite measure and that it decays in one direction (Definition 5.3). Then for the $R_k$ defined in (23) it holds that
\[
\lim_{k \to \infty} \int_{\Omega \cap \{|x| < R_k\}} \int_{\Omega \cap \{|x| > R_k\}} \frac{1}{|x - y|^{n+2s}} dx \, dy = 0. \tag{25}
\]

Proof. Note that the property of decaying in one direction is not affected by cylindrical Schwarz symmetrization. Hence, by Lemma 2.6, we can assume that $\Omega$ is rotationally symmetric with respect to $x_1$ axis and we will show the Lemma for $\Omega^*$. Thus we can assume that for some function $f : (-\infty, \infty) \to [0, \infty)$
\[
\Omega = \{ (x_1, x') : |x'| < f(x_1) \}, \quad \mathcal{H}^{n-1}(\Omega \cap \{ x_1 = R \}) = \alpha_{n-1} f(R)^{n-1},
\]
where $\alpha_{n-1} = \mathcal{L}^{n-1}(B_1')$ is the measure of the unit ball in $\mathbb{R}^{n-1}$. By Definition 5.3 we can assume that
\[
f(R)^{n-1} \leq \begin{cases} \frac{1}{\alpha_{n-1}} h(R) & \text{if } R > 0, \\ \frac{1}{\alpha_{n-1}} h(-R) & \text{if } R < 0, \end{cases} \quad \text{and } h \text{ satisfies (23).} \tag{26}
\]

We now write the left hand side of (25) as
\[ L_s(\Omega \cap \{|x| < R_k\}, \Omega \cap \{|x| > R_k\}) = A_k(\Omega) + D_k(\Omega), \]
where

\[ A_k(\Omega) = L_s(\Omega \cap \{ |x_1| < R_k - a \}, \Omega \cap \{ x_1 > R_k \}) + L_s(\Omega \cap \{ R_k - a < x_1 < R_k \}, \Omega \cap \{ x_1 < R_k \}) + L_s(\Omega \cap \{ R_k - a < x_1 < R_k \}, \Omega \cap \{ x_1 > R_k \}) \]

and

\[ D_k(\Omega) = L_s(\Omega \cap \{ |x_1| < R_k - a \}, \Omega \cap \{ x_1 < R_k \}) + L_s(\Omega \cap \{ -R_k < x_1 < -R_k + a \}, \Omega \cap \{ x_1 > R_k \}) + L_s(\Omega \cap \{ -R_k < x_1 < -R_k + a \}, \Omega \cap \{ x_1 < R_k \}) \]

We only have to show that \( A_k \to 0 \) as \( k \to \infty \) and then the result follows by symmetry also for \( D_k \), as \( D_k(\Omega) = A_k(\Omega) = A_k(\{ (x_1, x) : x \in \Omega \}) \). We now write \( A_k(\Omega) \) as

\[ A_k(\Omega) = A^1_k(\Omega) + A^2_k(\Omega) + A^3_k(\Omega) + A^4_k(\Omega) \]

where

\[ A^1_k(\Omega) = L_s(\Omega \cap \{ |x_1| < R_k - a \}, \Omega \cap \{ x_1 > R_k \}) \]

\[ A^2_k(\Omega) = L_s(\Omega \cap \{ R_k - a < x_1 < R_k \}, \Omega \cap \{ x_1 < R_k \}) \]

\[ A^3_k(\Omega) = L_s(\Omega \cap \{ R_k - a < x_1 < R_k \}, \Omega \cap \{ x_1 > R_k + a \}) \]

\[ A^4_k(\Omega) = L_s(\Omega \cap \{ R_k - a < x_1 < R_k \}, \Omega \cap \{ R_k < x_1 < R_k + a \}) \]

**Estimates for \( A^1_k, A^2_k \) and \( A^3_k \).** Note that

\[ |x - y| \geq a, \quad \text{for } x \in \Omega \cap \{ |x_1| < R_k - a \}, \quad y \in \Omega \cap \{ x_1 > R_k \} \]

Hence

\[ A^1_k(\Omega) \leq a^{-(n+2s)} L^n(\Omega \cap \{ |x_1| < R_k - a \}) \] \[ \times L^n(\Omega \cap \{ x_1 > R_k \}) \]

\[ \leq a^{-(n+2s)} L^n(\Omega) L^n(\Omega \cap \{ x_1 > R_k \}) \]

As \( R_k \) tends to infinity and the measure of \( \Omega \) is finite, we obtain \( \lim_{k \to \infty} A^1_k(\Omega) = 0 \). The same type of estimates apply also to \( A^2_k \) and \( A^3_k \), because also in these cases \( |x - y| \geq a \) if \( x \) and \( y \) are contained in the sets defining \( A^2_k \), \( A^3_k \). Thus we get that

\[ \lim_{k \to \infty} A^1_k(\Omega) = \lim_{k \to \infty} A^2_k(\Omega) = \lim_{k \to \infty} A^3_k(\Omega) = 0. \]

**Estimate for \( A^4_k \).** Using (26) and (23) we get that

\[ \Omega \cap \{ R_k < x_1 < R_k + a \} = \{(x_1, x') : R_k < x_1 < R_k + a, \ |x'| < f(x_1)\} \]

\[ \subset \{(x_1, x') : R_k < x_1 < R_k + a, \ |x'| < C h(R_k)^{1/(n-1)}\} \]

\[ \subset \{(x_1, x') : R_k < x_1 < R_k + a, \ |x'| < C h(R_k)^{1/(n-1)}\} \]

for some constant \( C = C(n) \). Thus we obtain that

\[ A^4_k(\Omega) \leq L_s(\{ R_k - a < x_1 < R_k \}, \{ R_k < x_1 < R_k + a, \ |x'| < C h(R_k)^{1/(n-1)}\}) \]

\[ = L_s(\{ x_1 \in (0, a) \}, \{ x_1 \in (-a, 0), \ |x'| < C h(R_k)^{1/(n-1)}\}) \]

It follows from Lemma 5.7 that \( A^4_k(\Omega) \leq C(n, s)a^{1-2s}h(R_k) \). Using the second equation in (23) proves that \( A^4_k(\Omega) \) goes to zero as \( k \to \infty \).
Proof of Theorem 5.5. If $\epsilon > 0$ then there exists by Lemma 5.8 a $R_k > 0$ such that

$$\int_{\Omega_k} \int_{\Omega \cap \Omega_k} \frac{1}{|x-y|^{n+2s}} dx
dy \leq \epsilon. \quad (27)$$

By hypothesis $\Omega_k$ is bounded and therefore contained in a cylinder of some radius $M_k$, i.e. $\Omega_k \subset \{x_1 \in \mathbb{R}, |x'| \leq M_k\} =: D_{M_k}$. Thus, even if $\Omega_k$ might not be Lipschitz, we still have that it is piecewise Lipschitz and

$$\mathcal{H}^{n-1}(\partial \Omega_k) \leq \mathcal{H}^{n-1}(\partial \Omega \cap \{x_1 \leq R_k\}) + \mathcal{H}^{n-1}(D_{M_k} \cap \{x_1 = ±R_k\})$$

$$= \mathcal{H}^{n-1}(\partial \Omega \cap \{x_1 \leq R_k\}) + 2\alpha_{n-1}M_k^{n-1} < \infty.$$

By the proof of Proposition 2.3 (i) and Remark 1, there exists $v_k \in C_c^\infty(\Omega_k)$ (which is an approximation of the characteristic function of $\Omega_k$ by cutting it off near the boundary) such that $0 \leq v_k \leq 1$,

$$\frac{1}{\|u_k\|^2_{L^2(\Omega_k)}} \int_{\Omega} \int_{\Omega_k} \frac{|v_k(x) - v_k(y)|^2}{|x-y|^{n+2s}} dxdy \leq \epsilon, \quad \text{and} \quad \|v_k\|_{L^2(\Omega_k)} \geq \mathcal{L}^n(\Omega_k)/2.$$

Define now $u_k \in C_c^\infty(\Omega)$ by $u_k := v_k$ in $\Omega_k$ and $u_k := 0$ in $\Omega \setminus \Omega_k$. We now obtain from (27) that

$$\frac{1}{\|u_k\|^2_{L^2(\Omega)}} \int_{\Omega} \int_{\Omega_k} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{n+2s}} dxdy$$

$$= \frac{1}{\|u_k\|^2_{L^2(\Omega)}} \int_{\Omega_k} \int_{\Omega_k} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{n+2s}} dxdy$$

$$+ \frac{2}{\|u_k\|^2_{L^2(\Omega)}} \int_{\Omega_k} \int_{\Omega \setminus \Omega_k} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{n+2s}} dxdy$$

$$\leq \epsilon \left(1 + \frac{4}{\mathcal{L}^n(\Omega_k)}\right).$$

For $k$ big enough $\mathcal{L}^n(\Omega_k) \geq \mathcal{L}^n(\Omega)/2$, and hence the theorem is proven. \hfill \Box

If the domain $\Omega$ is already cylindrically symmetric with respect to the direction in which it decays, then there is a slightly simpler proof of Theorem 5.5, which relies on Theorem 1.4.

Proof of Theorem 5.5 (for symmetric domains). In this proof we assume that $\Omega$ is rotationally symmetric about $x_1$ axis and the domain is parametrized by a function $f$, i.e. $\Omega = \{(x_1, x') \in \mathbb{R}^n | |x'| < f(x_1)\}$. We would like apply Theorem 1.4 and thus have to show that there exist a bounded Lipschitz set $V \subset \mathbb{R}^n$ and a sequence $\{\lambda_k\}$ tending to infinity as $k \to \infty$ such that

$$\lim_{k \to \infty} \frac{1}{\mathcal{L}^n(\Omega \setminus \lambda_k V)} \int_{\Omega \setminus \lambda_k V} \frac{dx}{\text{dist}(x, (\lambda_k V) c)^{2s}} = 0.$$ 

We prove this result for dimension 2 only to keep the argument relatively simple. Choose $V = (-1, 1) \times (-1, 1)$ and $\lambda_k = R_k + a$, where $R_k$ and $a$ are as in Definition 5.3. As $\mathcal{L}^2(\Omega) < \infty$, we get

$$\mathcal{L}^2\left(\Omega \cap \{ |x_1| \geq \frac{\lambda_k}{2} \}\right) \to 0 \quad \text{as} \quad k \to \infty. \quad (28)$$
We write $\Omega \cap \lambda_k V = A_k \cup B_k \cup C_k$, where,

$$A_k = \{(x_1, x_2) \in \Omega : |x_1| \leq \frac{\lambda_k}{2}, \ |x_2| \leq \lambda_k\},$$

$$B_k = \{(x_1, x_2) \in \Omega : \frac{\lambda_k}{2} \leq |x_1| \leq R_k, \ |x_2| \leq \lambda_k\},$$

$$C_k = \{(x_1, x_2) \in \Omega : R_k < |x_1| < \lambda_k, \ |x_2| \leq \lambda_k\}.$$

We will show that as $k \to \infty$

$$\int_{A_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} + \int_{B_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} + \int_{C_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \to 0.$$

We first estimate $I_3$. We claim that

$$\text{dist}(x, (\lambda_k V)^c) = \lambda_k - |x_1|$$

for any $x \in C_k$ whenever $\lambda_k$ is large enough. The claim follows if we show that $|x_1| \geq |x_2|$ for all $x \in C_k$, if $\lambda_k$ is big enough. Here we use in a crucial way that $\Omega$ is cylindrically symmetric and thus by Definition 5.3 and the last inequality in (23) $\Omega \cap \{|x_1| \in (R_k, R_k + a)\} \subset \{|x_1| \in (R_k, R_k + a), \ |x_2| < f(x_1) \leq f(R_k)\}$. Now use that $R_k \to \infty$ and $h(R_k) \to 0$, which shows that $|x_1| \geq |x_2|$. Further, for large $\lambda_k$, by assumption (23) we get

$$\int_{C_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \leq \int_{\{x_1 < h(R_k)\}} \int_{\{R_k \leq |x_1| < \lambda_k\}} \frac{dx_2 dx_1}{(\lambda_k - |x_1|)^{2s}} \leq Kh(R_k) a^{1-2s} \to 0.$$

Next we estimate $I_2$, we split $B_k = E_k \cup F_k$ where $E_k = \{x \in B_k : \ |x_2| \leq |x_1|\}$ and $F_k = \{x \in B_k : \ |x_2| > |x_1|\}$. Therefore, we get that

$$\text{dist}(x, (\lambda_k V)^c) = \begin{cases} \lambda_k - |x_1| & \text{if } x \in E_k, \\ \lambda_k - |x_2| & \text{if } x \in F_k. \end{cases}$$

If $x \in E_k$ we get $\text{dist}(x, (\lambda_k V)^c) \geq a$, also note $L^2(E_k) \to 0$ as $k \to 0$ by (28). Then

$$\int_{x \in E_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \leq \frac{L^2(E_k)}{a^{2s}} \to 0. \quad (29)$$

If $x \in F_k$, we define, $Q_k = \left\{x \in \mathbb{R} : \frac{\lambda_k}{2} \leq |x_1| \leq R_k \text{ and } f(x_1) > |x_1|\right\}$. By definition of $Q_k$ we have

$$F_k = \left\{x \in \mathbb{R}^2 : x_1 \in Q_k, \ |x_1| < |x_2| < \min\{f(x_1), \lambda_k\}\right\} \subset \left\{x \in \mathbb{R}^2 : x_1 \in Q_k, \ \frac{\lambda_k}{2} < |x_2| < \lambda_k\right\}.$$

Then clearly,

$$\int_{F_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \leq \int_{x_1 \in Q_k} \int_{\left\{\frac{\lambda_k}{2} < |x_2| < \lambda_k\right\}} \frac{dx_1 dx_2}{(\lambda_k - |x_2|)^{2s}} \leq K_2 L^1(Q_k) \left(\frac{\lambda_k}{2}\right)^{1-2s},$$

$$\quad (30)$$
where $K_2$ is a constant independent of $k$. We also notice that $Q_k \subset \{ x_1 \in \mathbb{R} : f(x_1) > \frac{\lambda_k}{2} \}$, then by Chebyshev’s inequality we see
\[
\mathcal{L}^1(Q_k) \leq \mathcal{L}^1\left\{ x_1 : f(x_1) > \frac{\lambda_k}{2} \right\} \leq \frac{\|f\|_{L^1(\mathbb{R})}}{\lambda_k / 2} \leq 2\mathcal{L}^2(\Omega) / \lambda_k.
\] (31)
Hence, using (30) and (31) we get
\[
\int_{R_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \leq \frac{K_2 \mathcal{L}^2(\Omega)}{\lambda_k^{2s}}.
\] (32)
Combining (29) and (32) we get that $I_2 \to 0$ as $k \to \infty$. Next, to estimate $I_1$ we define $A_k := M_k \cup N_k$, where
\[
M_k = \left\{ x \in \Omega : |x_1| \leq \frac{\lambda_k}{2}, |x_2| \leq \frac{\lambda_k}{2} \right\},
\]
\[
N_k = \left\{ x \in \Omega : |x_1| \leq \frac{\lambda_k}{2}, \frac{\lambda_k}{2} < |x_2| < \lambda_k \right\}.
\]
We get
\[
\int_{M_k} \frac{dx}{\text{dist}(x, (\lambda_k V)^c)^{2s}} \leq \frac{\mathcal{L}^2(M_k)}{(\lambda_k^{2s})} \leq \frac{\mathcal{L}^2(\Omega)}{(\lambda_k^{2s})} \to 0.
\]
Finally, the estimate on $N_k$ follows by a similar argument as in $F_k$. \qed

6. Appendix: Existence and regularity of the first eigenfunction. In this appendix we collect some known results on the minimizer for $P^2_{n,s}(\Omega)$. Since we merely assume in Theorem 1.2 that $\omega$ is a bounded open set, without any further regularity assumptions on the boundary, the references are not entirely complete and we fill in the missing links between these references. Recall that $H^s_\Omega(\mathbb{R}^n)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\left( ||u||^2_{L^2(\Omega)} + ||u||^2_{s,2,\mathbb{R}^n} \right)^{1/2}$ and emphasize that for a general bounded open set $\Omega$ we only have the strict inclusion
\[
H^s_\Omega(\mathbb{R}^n) \subset X^s_0(\Omega) := \{ u \in L^2(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega, [u]_{s,2,\mathbb{R}^n} < \infty \}.
\]
The two spaces coincide if $\Omega$ has continuous boundary (which includes Lipschitz boundary) by Theorem 6 in [19]. A simple example for the strict inclusion is the set $U = (-1,0) \cup (0,1)$ with $s \in (1/2,1)$, see Remark 7 in [19]. However, if $s < 1/2$, then $H^s_\Omega(\mathbb{R}) = X^s_0(\mathbb{U})$, see Example 6.4 below. In view of the inclusion the following inequality holds between the different Poincaré constants, depending over which set of functions we take the infimum:
\[
\lambda^2_{n,s}(\Omega) := \inf_{u \in X^s_0(\Omega)} \frac{[u]^2_{s,2,\mathbb{R}^n}}{\||u||^2_{L^2(\Omega)}} \leq P^2_{n,s}(\Omega).
\] (33)
There can be strict inequality in general, if $\Omega$ does not have continuous boundary, see Example 6.3 at the end of this section.

We now state the main properties that we need concerning existence and regularity of the minimizers of $P^2_{n,s}(\Omega)$. We from now on assume that the infima defining $\lambda^2_{n,s}$ and $P^2_{n,s}$ are taken only over non-negative functions. It is well known that this does not change the value of the Poincaré constants, see for instance [4, Section 3].

**Proposition 6.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then the following statements hold true:
Theorem 6.2

(a) There exists a non-negative minimizer \( u \in H^s_\Omega(\mathbb{R}^n) \cap L^\infty(\mathbb{R}) \) of \( P^2_{n,s}(\Omega) \) and every minimizer satisfies

\[
\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n + 2s}} dxdy = P^2_{n,s}(\Omega) \int_{\mathbb{R}^n} u(x) \varphi(x) dx \quad \forall \varphi \in H^s_\Omega(\mathbb{R}^n).
\]

(b) Every minimizer \( u \) is smooth and strictly positive in \( \Omega \).

Remark 4. If one minimizes in \( X^s_0(\Omega) \) instead of \( H^s_\Omega(\mathbb{R}^n) \) (which is the case if \( \Omega \) is Lipschitz, as then the spaces are equal) then the proposition is an immediate consequence of the following references: (a) follows from [35, Proposition 9], [34, Proposition 4], or [2, Proposition 2.1], whereas for the regularity statement in (b) see for instance [33, Theorem 1.1] and [32]. The strict positivity of \( u \) is contained for instance in [36, Corollary 8].

Indeed, the proof of Proposition 6.1 (b) follows from the references mentioned in the remark. We carry out this in detail for completeness. We start by recalling the following regularity result.

Theorem 6.2 ([33] Thm. 1.1 and [32] Section 6.1). Let \( v \in W^{s,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) satisfy

\[
\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(\psi(x) - \psi(y))}{|x - y|^{n + 2s}} dxdy = \int_{\mathbb{R}^n} f(x) \psi(x) dx \quad \forall \psi \in X^s_0(B_1).
\]

(i) If \( f \in L^\infty(B_1) \) and \( s \neq 1/2 \), then \( v \in C^{2s}(B_{1/2}) \) and

\[
\|v\|_{C^{2s}(B_{1/2})} \leq C \left( \|f\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)} \right).
\]

If \( s = 1/2 \), then the same estimate holds true with the \( C^{2s}-\)norm on the left hand side replaced by \( C^{2s-\epsilon} \) for every \( \epsilon > 0 \).

(ii) If \( f \in C^\alpha(B_1) \) and \( \alpha + 2s \) is not an integer, then \( v \in C^{\alpha+2s}(B_{1/2}) \) and

\[
\|v\|_{C^{\alpha+2s}(B_{1/2})} \leq C \left( \|f\|_{C^\alpha(B_1)} + \|v\|_{L^\infty(B_1)} \right).
\]

We now prove Proposition 6.1.

Proof. (a) can be found in [4, Section 3]. So we only have to prove (b). It is sufficient to show that for every \( x_0 \in \Omega, \epsilon > 0 \) with \( B_\epsilon(x_0) \subset \Omega \), and \( \alpha \geq 0 \) one has the estimate

\[
\|u\|_{C^{\alpha+2s}(B_{\epsilon/2}(x_0))} \leq C \left( \|u\|_{C^\alpha(B_{\epsilon}(x_0))} + \|u\|_{L^\infty(\mathbb{R}^n)} \right),
\]

whenever \( \alpha + 2s \) is not an integer. The case \( \alpha = 0 \) has to be understood as in Theorem 6.2 (i), by omitting the \( C^\alpha \)-norm on the right hand side of the inequality and replacing \( 2s \) by \( 2s - \epsilon \) in the case \( s = 1/2 \).

Once the estimate (34) has been shown, then the smoothness of \( u \) follows from (a) and a bootstrapping argument. In a first step we obtain that \( u \in C^{2s}(\Omega) \) or \( C^{2s-\epsilon}(\Omega) \). Applying again (34) gives that \( u \in C^{4s} \) or \( C^{4s-\epsilon} \), and so on. Now by induction it follows that \( u \in C^\alpha \) for every \( \alpha > 0 \). In case \( \alpha + 2s \) is an integer, then simply continue the iteration with \( u \in C^{\alpha-\epsilon} \).

We now show (34). Set \( v(x) = u(x_0 + \epsilon x) \) and note that \( v \in W^{s,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) by (a). Let \( \psi \in X^s_0(B_1) \) and define \( \varphi(x) = \psi((x - x_0)/\epsilon) \). As \( \psi = 0 \) in \( \mathbb{R}^n \setminus B_1 \), we obtain that \( \varphi = 0 \) in \( \mathbb{R}^n \setminus B_\epsilon(x_0) \) and therefore \( \varphi \in X^s_0(B_\epsilon(x_0)) = H^s_{B_\epsilon(x_0)}(\mathbb{R}^n) \subset \).
$H^s_\Omega(\mathbb{R}^n)$, because $B_r(x_0)$ is a smooth set. We abbreviate $\gamma = 2P^2_{n,s}/C_{n,s}$. Then we obtain by change of variables and (a) that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y)) (\psi(x) - \psi(y))}{|x-y|^{n+2s}} dxdy = \epsilon^{-n+2s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy$$

$$= \epsilon^{-n+2s} \gamma \int_{\mathbb{R}^n} u(x) \varphi(x) dx = \epsilon^{-2s} \gamma \int_{\mathbb{R}^n} v(x) \psi(x) dx.$$ 

So if $u \in C^\alpha(B_r(x_0))$, then $v \in C^\alpha(B_1)$ and (34) is a consequence of Theorem 6.2 applied to $f = \epsilon^{-2s} \gamma v$.

It is left to show the strict positivity of $u$ in $\Omega$. As $u$ is smooth it follows from (a) that

$$(-\Delta_n)^s u(x) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2 u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy = P^2_{n,s}(\Omega) u(x) \quad \forall x \in \Omega.$$ 

In this expression the integrand is in $L^1(\mathbb{R})$ and we do not need the principle value of the integral as in (6). We argue by contradiction: if $u(x) = 0$ for some $x \in \Omega$, then $2u(x) - u(x+y) - u(x-y) \leq 0$, because $u \geq 0$. Since $u$ does not vanish identically, we obtain that $P^2_{n,s}(\Omega) u(x) = (-\Delta_n)^s u(x) < 0$, which is a contradiction to the non-negativity of $u$.

Using Proposition 6.1 we now give an example for the strict inequality in (33).

**Example 6.3.** Let $1/2 < s < 1$ and consider the set $\mathcal{U} = (-1,0) \cup (0,1)$. Since integrals are not affected by sets of measure zero we have $X^s_0(\mathcal{U}) = X^s_0((-1,1))$. The set $(-1,1)$ is now smooth and we get $X^s_0((-1,1)) = H^s((-1,1))$. We therefore have to show that

$$P^2_{1,s}((1,1)) < P^2_{1,s}((-1,0) \cup (0,1)).$$

Let $u^* \in H^s_\mathcal{U}(\mathbb{R})$ be the minimizer for $P^2_{1,s}(\Omega)$. Since $s > 1/2$ we obtain from the embedding of fractional Sobolev spaces into Hölder spaces (see for instance [31] Theorem 8.2) and the definition of $H^s_\mathcal{U}(\mathbb{R})$ that $u^*(0) = 0$. Assume by contradiction that there is equality in (35). Then $u^* \in H^s_\mathcal{U}(\mathbb{R}) \subset H^s((-1,1))$ is also a minimizer for $P^2_{1,s}((-1,1))$ and has to be strictly positive in $(-1,1)$ by Proposition 6.1 (b). This is a contradiction to $u^*(0) = 0$.

**Example 6.4.** The previous example cannot work for $0 < s < 1/2$. It actually holds that $H^s_\mathcal{U}(\mathbb{R}) = X^s_0(\mathcal{U})$ as can be seen by the following simple argument and calculation. Recall that $X^s_0(\mathcal{U}) = X^s_0((-1,1))$ and that $C^\infty_c((-1,1))$ is dense in $X^s_0((-1,1))$. So it is sufficient to show that for any $u \in C^\infty_c((-1,1))$ there exists $u_\delta \in C^\infty_c(\mathcal{U})$ such that $|u - u_\delta|_{s,2,\mathcal{U}} \to 0$ as $\delta \to 0$. Define $u_\delta$ by cutting off $u$ on both sides of the origin in a $\delta$-neighborhood of 0. Then $v_\delta = u - u_\delta \in C^\infty_c(\mathbb{R})$ and has support in $(-\delta, \delta)$. Using that $|v_\delta| \leq C(u)$, and $|v_\delta|^s \leq C(u)/\delta$, for some constant $C(u)$ depending only on $u$, one easily obtains by a similar argument as in the proof of Proposition 2.3 (i) that

$$|u - u_\delta|^2_{s,2,\mathcal{U}} \leq C(u,s) \delta^{1-2s},$$

for some constant $C(u,s)$ independent of $\delta$. Letting $\delta \to 0$ gives the desired claim.
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