ON FREE GELFAND–DORFMAN–NOVIKOV SUPERALGEBRAS AND
A PBW TYPE THEOREM†

ZERUI ZHANG∗, YUQUN CHEN♯, AND LEONID A. BOKUT†

Abstract. We construct linear bases of free GDN superalgebras. As applications, we
prove a Poincaré–Birkhoff–Witt type theorem, that is, every GDN superalgebra can be
embedded into its universal enveloping associative differential supercommutative algebra.
An Engel theorem is given.

1. Introduction

We recall that a superalgebra over a field $k$ is a vector space $A$ with a direct sum
decomposition $A = A_0 \oplus A_1$ together with a bilinear multiplication $\circ: A \times A \to A$
such that $A_i \circ A_j \subseteq A_{i+j}$, where the subscripts are elements of $\mathbb{Z}_2$. The parity $|x|$ of every
element $x$ in $A_0$ is 0, and the parity $|x|$ of every nonzero element $x$ in $A_1$ is 1. If a
superalgebra $A$ satisfies the following two identities

$$(x \circ (y \circ z)) - ((x \circ y) \circ z) = (-1)^{|x||y|}((y \circ (x \circ z)) - ((y \circ x) \circ z)) \quad (\text{left supersymmetry}),$$

and

$$((x \circ y) \circ z) = (-1)^{|y||z|}((x \circ z) \circ y) \quad (\text{right supercommutativity})$$

for all elements $x, y, z$ in $A_0 \cup A_1$, then $A$ is called a (left) Novikov superalgebra [17]. (There
is a “right” version of using right supersymmetry and left supercommutativity.) Moreover,
if a (left) Novikov superalgebra $A$ equals to its even part, i.e., $A = A_0$, then $A$ is just an ordinary (left) Novikov algebra [3, 11, 13]. Since Novikov algebras were invented by Balinskii and Novikov [3], and independently by Gelfand and Dorfman [11], we also call
Novikov algebra as Gelfand–Dorfman–Novikov algebra (GDN algebra) and call Novikov superalgebra as Gelfand–Dorfman–Novikov superalgebra (GDN superalgebra).

A rich structure and combinatorial theory of GDN algebras have been done up to
now. Zelmanov solved Novikov’s problem on classification of simple GDN algebras over
an algebraically closed field: There are no such algebras besides trivial [18]. Osborn
and Zelmanov classified simple GDN algebras $A$ over an algebraically closed field of

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♯Corresponding author.
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characteristic 0 with a maximal subalgebra $H$ such that $A/H$ has a finite dimensional irreducible $H$-submodule\cite{14}. Xu gave a complete classification of finite dimensional simple GDN algebras and their irreducible modules over an algebraically closed field with prime characteristic\cite{15}, and he introduced some quadratic GDN superalgebras connecting with Gelfand–Dorfman ($\Omega$-bi) algebras\cite{16} (Gelfand–Dorfman ($\Omega$-bi) algebras were invented in\cite{13}). See also, for example, Bai and Meng\cite{1,2}, Burde and Dekimpe\cite{6}, Chen, Niu and Meng\cite{7}, Kang and Chen\cite{12}, Zhu and Chen\cite{19}, Bokut, Chen and Zhang\cite{4,5}.

Dzhumadildaev and L"ofwall proved that the set of all the Novikov tableaux (we call them GDN tableaux because of the above reason) over a well-ordered set $X$ forms a linear basis of a free GDN algebra generated by $X$ by using trees and by appealing to the connection with free commutative associative differential algebra\cite{8} (the idea of this connection was given by S.I. Gelfand, see\cite{11}). And we wonder what would a basis of a free GDN superalgebra be like. The method of using trees developed in\cite{8} can not be directly applied for GDN superalgebras, but the idea of tracing a root of a tree can be modified to define the root number of a term. Moreover, the definition of GDN tableau can be easily extended to a definition of GDN supertableau, see Definition\cite{2.6}. One of the results we prove below is as follow:

**Theorem A.** The set of all the GDN supertableaux over a well-ordered set $X = X_0 \cup X_1$ forms a linear basis of the free GDN superalgebra $\text{GDN}_s(X)$ generated by $X$, where every element of the set $X_0$ is of parity 0 and every element of the set $X_1$ is of parity 1.

We also prove a Poincaré–Birkhoff–Witt (PBW) type theorem for GDN superalgebras:

**Theorem B.** Every GDN superalgebra can be embedded into its universal enveloping associative differential supercommutative algebra.

As a corollary, we show that every GDN superalgebra generated by a finite set of elements of parity 1 is nilpotent. Several results concerning the nilpotency of certain GDN algebras have been found up to now. Zelmanov proved that, if $\mathcal{A}$ is a left-nilpotent finite dimensional (right) GDN algebra over a field of characteristic zero, then $\mathcal{A}^2$ is nilpotent\cite{18}. Filippov proved that a right-nil algebra of bounded index over a field of characteristic zero is right nilpotent provided that it is right symmetric and is nilpotent provided that it is a right GDN algebra\cite{11}. Dzhumadildaev and Tulenbaev proved that if a (right) GDN algebra $\mathcal{A}$ over a field of characteristic $p$ is left-nil of bounded index $n$ and $p = 0$ or $p > n$, then $\mathcal{A}^2$ is nilpotent\cite{9}. Again, to some extent, this kind of result can be extended to the case of GDN superalgebras, and we prove the following Engel theorem:

**Theorem C.** Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a (left) GDN superalgebra over a field of characteristic 0 generated by $X = X_0 \cup X_1$, where every element of the set $X_0$ is of parity 0 and every element of the set $X_1$ is of parity 1. If for some integer $n > 0$, the even part $\mathcal{A}_0$ is right-nil of bounded index $n$ and $X_1$ is a finite set, then $\mathcal{A}^2$ is nilpotent.

The paper is organized as follows. In section\cite{2} we construct a linear generating set $\text{Tab}_s(X)$ for a free GDN superalgebra generated by a well-ordered set $X$ over a field of characteristic $\neq 2$. (For the case of characteristic 2, a GDN superalgebra is the same as
Moreover, if \( \mu \) length \( n \) all terms based on \([8]\), in which the authors appealed to the tool of trees (and roots of trees). For into a linear combination of some specified terms (hereafter called GDN supertableaux). nonnegative integers. Finally, we develop several handy properties of the root map. They of all terms over \( X \) claims: (i) \(|\mu| = 0 \) if \( \mu \) lies in \( X_0 \), and \(|\mu| = 1 \) if \( \mu \) lies in \( X_1 \). (ii) \(|\mu| = |\mu_1| + |\mu_2| \) modulo 2 if \( \mu = (\mu_1 \circ \mu_2) \).

**Definition 2.1.** We define a root map \( r \) from the set \( X^{(*)} \) to the set \( \mathbb{Z}_{\geq 0} \) of nonnegative integers defined inductively as follows:

(i) \( r(a) = 0 \) for every element \( a \) in \( X \);
(ii) \( r((\mu \circ \nu)) = r(\mu) + 1 \) if \( \nu \) lies in \( X \), and \( r((\mu \circ \nu)) = r(\mu) + r(\nu) \) if \( \nu \) does not lie in \( X \).

For every term \( \mu \) in \( X^{(*)} \), we call \( r(\mu) \) the root number of \( \mu \) to indicate that our idea is based on \([8]\), in which the authors appealed to the tool of trees (and roots of trees). For all terms \( \mu_1, \ldots, \mu_n \) in \( X^{(*)} \), to make the notations shorter, define

\[
\begin{align*}
[\mu_1, \ldots, \mu_n]_L &= ((\mu_1 \circ \mu_2) \circ \mu_3) \circ \ldots \circ \mu_n \quad \text{(left-normed bracketing)}, \\
[\mu_1, \ldots, \mu_n]_R &= (\mu_1 \circ \ldots \circ (\mu_{n-1} \circ \mu_n)) \quad \text{(right-normed bracketing)}.
\end{align*}
\]

Moreover, if \( \mu_1, \ldots, \mu_n \) are elements of \( X \), then we call \( [\mu_1, \ldots, \mu_n]_R \) a simple term over \( X \) of length \( n \).

Below we offer an instance of counting the root number of a term in \( X^{(*)} \).
Proof. The lemma follows immediately from Lemma 2.4.

Lemma 2.5. If the equality \(r\) holds, then the induction hypothesis forces \(\ell(\mu_1) = 1\). The claim follows by induction hypothesis.

The following lemma offers a formula for counting the root number of a term.

Lemma 2.4. For every term \(\mu = (\mu_1, \mu_2)\) in \(X^{(*)}\), we have

\[ r(\mu) = r(\mu_1) + \max(1, r(\mu_2)) \geq 1, \]

with \(r(\mu) = 1\) only if \(\mu = [a_1, \ldots, a_{\ell(\mu)}]_R\) for some elements \(a_1, \ldots, a_{\ell(\mu)}\) in \(X\).

Proof. Use induction on \(\ell(\mu)\). For \(\ell(\mu) = 1\), we have \(r(\mu) = 0 = \ell(\mu) - 1\). For \(\ell(\mu) > 1\), we have \(\mu = (\mu_1 \circ \mu_2)\) for some terms \(\mu_1\) and \(\mu_2\) in \(X^{(*)}\). If \(\ell(\mu_2) > 1\), then by induction hypothesis, we have

\[ r(\mu) = r(\mu_1) + 1 \leq r(\mu_1) - 1 + \ell(\mu_2) - 1 < \ell(\mu) - 1. \]

On the other hand, if \(\ell(\mu_2) = 1\), then by induction hypothesis, we have

\[ r(\mu) = r(\mu_1) + 1 \leq r(\mu_1) - 1 + 1 = \ell(\mu_1) = \ell(\mu) - 1, \]

with the equality only if \(r(\mu_1) = \ell(\mu_1) - 1\). The claim follows by induction hypothesis.

The following lemma shows that the root map is compatible with the right supercommutativity, and to some extent, the root map is also compatible with the product \(\circ\).

Lemma 2.5. For all terms \(\mu_1, \mu_2\) and \(\mu_3\) in \(X^{(*)}\), we have

(i) \(r([\mu_1, \mu_2, \mu_3]_R) = r([\mu_1, \mu_3, \mu_2]_R)\);
(ii) If \(r(\mu_1) > r(\mu_2)\), then \(r((\mu_1 \circ \mu_3) > r((\mu_2 \circ \mu_3))\);
(iii) If \(r(\mu_1) > r(\mu_2)\) and \(r(\mu_1) > 1\), then \(r((\mu_3 \circ \mu_1)) > r((\mu_3 \circ \mu_2))\);
(iv) If \(r(\mu_1) > r(\mu_2)\) and \(r(\mu_1) = 1\), then \(r((\mu_3 \circ \mu_1)) = r((\mu_3 \circ \mu_2))\).

Proof. The lemma follows immediately from Lemma 2.4.
2.2. GDN supertableaux. Now we are ready to define the notion of a GDN supertableau, which is directly reminiscent of the notation of a GDN tableau. Our aim in this subsection is to show that, if \( \text{char}(k) \neq 2 \), then the set of all GDN supertableaux over \( X \) forms a linear generating set of the free GDN superalgebra \( \text{GDN}_s(X) \).

**Definition 2.6.** We call a term \( \mu \) a Gelfand–Dorfman–Novikov supertableau (GDN supertableau) over a well-ordered set \( X = X_0 \cup X_1 \) if, for some letter \( a \) in \( X \), for some nonnegative integer \( n \), and for some simple terms \( \mu_i = [a_i, r_i, \ldots, a_i, 1] \) (1 \( \leq i \leq n \)) over \( X \) of length \( r_i \geq 1 \), we have

\[
\mu = [a, \mu_1, \ldots, \mu_n],
\]

such that the following conditions hold:

(i) The integers \( r_1, \ldots, r_n \) satisfy that \( r_1 \geq \cdots \geq r_n \);

(ii) If \( r_i = r_i^{i+1} \), then \( a_i, 1 \geq a_i, 1+1 \) holds;

(iii) The inequality \( a \geq a_1, r_1 \geq \cdots \geq a_1, 2 \geq a_2, r_2 \geq \cdots \geq a_2, 2 \geq \cdots \geq a_n, n \geq \cdots \geq a_n, 2 \) holds;

(iv) If for some integers \( i, t, j \) and \( l \) satisfying \( i \leq n, t \leq n, 2 \leq j \leq r_j \) and \( 2 \leq l \leq r_t \), the letters \( a_i, j \) and \( a_i, l \) lie in \( X_1 \), then the inequality \( a_i, j \neq a_i, l \) holds;

(v) If for some integer \( i \leq n - 1 \), the elements \( a_i, 1 \) and \( a_i, 1+1 \) lie in \( X_1 \), and \( r_i = r_i^{i+1} \), then the inequality \( a_i, 1 \neq a_i, 1+1 \) holds.

Every term of the form (2.1) satisfying Points (i)-(iii) is called a Novikov tableau \( \tilde{X} \) over \( X \), and we call it a GDN tableau because of the reason explained in the introduction. Denote by \( \text{Tab}_s(X) \) the set of all the GDN supertableaux over \( X \). It is quite easy to show that every term in \( X^{(s)} \subseteq \text{GDN}_s(X) \) of the form (2.1) can be written as a linear combination of terms satisfying Points (i) and (ii) by the right supercommutativity, but what remains becomes complicated and we will need the notion of root number of a term.

The strategy for rewriting is to apply the right supercommutativity and the left supersymmetry. Unfortunately, whenever we apply the left supersymmetry to a term, we shall get three other terms in return, and thus this process becomes complicated. So a simplified notation is needed. Because of this reason, we introduce the following notation.

**Definition 2.7.** For all terms \( \mu \) and \( \nu \) in \( X^{(s)} \) such that \( r(\mu) = r(\nu) \) and \( \ell(\mu) = \ell(\nu) \), for all nonzero elements \( \alpha \) and \( \beta \) in the field \( k \), the polynomials \( \alpha \mu \) and \( \beta \nu \) in \( \text{GDN}_s(X) \) are said to be equivalent, denoted by

\[
\alpha \mu \sim \beta \nu,
\]

if \( \alpha \mu - \beta \nu = \sum_i \alpha_i \mu_i \) in \( \text{GDN}_s(X) \) for some elements \( \alpha_i \) in \( k \) and terms \( \mu_i \) in \( X^{(s)} \) such that \( r(\mu) < r(\mu_i) \) and \( \ell(\mu) = \ell(\mu_i) \) for every \( i \).

It is clear that if \( \alpha \mu \sim \beta \nu \) and \( \beta \nu \sim \alpha' \mu' \), then we get \( \alpha \mu \sim \alpha' \mu' \).

Recall that for every element \( a \) of \( X \), the parity \( |a| \) of \( a \) is \( i \) if \( a \) lies in \( X_i \) with \( i = 0, 1 \). Moreover, for all elements \( a_1, \ldots, a_n \) (\( n \geq 1 \)) of \( X \), we define the parity \( |a_1 \ldots a_n| \) of the string \( a_1 \ldots a_n \) to be \( |a_1| + \cdots + |a_n| \) modulo 2, extended with \( |\varepsilon| = 0 \) for the empty string \( \varepsilon \).

The following lemma shows that, for every simple term \( \mu[a_r, \ldots, a_1] \) with \( r \geq 3 \), we can rearrange \( a_r, \ldots, a_2 \) at the expense of adding a linear combination of terms of length \( r \).
and with root numbers > 1. We shall see in Lemma 2.10 that the added terms do not increase the difficulty of rewriting an arbitrary term into a linear combination of GDN supertableaux.

**Lemma 2.8.** For all elements \(a_1, ..., a_r\) in \(X\), for every simple term \(\mu = [a_r, ..., a_1]_R\), the following claims hold:

(i) For every integer \(j\) such that \(2 \leq j < r\), we have

\[
\mu \sim (-1)^{|a_j||a_{j+1}..a_r|}[a_j, a_r, ..., a_{j+1}, a_{j-1}, ..., a_1]_R;
\]

(ii) For all integers \(i\) and \(j\) such that \(2 \leq j < i \leq r\), we have

\[
\mu \sim (-1)^{|a_i|a_{i+1}..a_{i-1}+|a_j||a_{j+1}..a_{i-1}|}[a_r, ..., a_{i+1}, a_j, a_{i-1}, ..., a_{j+1}, a_i, a_{j-1}, ..., a_1]_R.
\]

**Proof.** We shall just prove Point (i), because Point (ii) can be proved in a similar way. Assume \(\nu = [a_{j-1}, ..., a_1]_R\). Then by the left supersymmetry, we have

\[
\mu = (-1)^{|a_j||a_{j+1}||a_{j+2}..a_{j+1}, a_j, a_{j+1}+1, a_j, a_{j+1}, a_{j-1}, ..., a_1]_R.
\]

Since

\[
\mu = [a_r, ..., a_{j+2}, (a_{j+2} \circ a_j), \nu]_R = (a_r, ..., a_{j+2}, (a_j \circ a_{j+1}), \nu]_R = 2
\]

we have

\[
\mu \sim (-1)^{|a_j||a_{j+1}||a_{j+2}..a_{j+1}, a_j, a_{j+1}, a_{j-1}, ..., a_1]_R.
\]

By induction on \(r - j\), we obtain

\[
\mu \sim (-1)^{|a_j||a_{j+1}||a_{j+2}..a_{j+1}, a_j, a_{j+1}, a_{j-1}, ..., a_1]_R.
\]

The proof is completed. \(\square\)

For every simple term \(\mu = [a_r, ..., a_1]_R\) in \(X^{(s)}\), for every integer \(i\) such that \(2 \leq i \leq r\), we define

\[
\mu_{a_i} = [a_r, ..., a_{i-1}, a_{i+1}, ..., a_1]_R
\]

and

\[
\mu_{a_i \rightarrow b_j} = [a_r, ..., a_{i-1}, b_j, a_{i+1}, ..., a_1]_R.
\]

The following lemma is crucial to the construction of a linear basis of the free GDN superalgebra \(\text{GDN}_s(X)\). It shows that, for the product of two simple terms, we can “interchange” certain letters of the two simple terms in the sense of adding a linear combination of some nonessential terms. We shall see that, as a result of the following lemma, the set of all the GDN tableaux over \(X\) is not linearly independent in \(\text{GDN}_s(X)\) provided that \(X_1\) is nonempty and the characteristic of the field is not 2.
Lemma 2.9. For all elements \(a_1, \ldots, a_{r+1}, b_1, \ldots, b_m\) \((r \geq 2, m \geq 2)\) in \(X\), for all integers \(i, j\) such that \(2 \leq i \leq r + 1\) and \(2 \leq j \leq m\), for all simple terms \(\mu = [a_{r+1}, \ldots, a_1]_R\) and \(\nu = [b_m, \ldots, b_1]_R\), we can interchange \(a_i\) and \(b_j\) in \((\mu \circ \nu)\) in the sense that
\[
(\mu \circ \nu) \sim -1^{a_i|a_{i-1} \cdots a_1 b_m \cdots b_j| + b_j|a_i|a_{i-1} \cdots a_1 b_m \cdots b_j+1|}(\mu_{a_i \rightarrow b_j} \circ \nu_{b_j \rightarrow a_i}).
\]
In particular, if \(r = m\), \(a_1 = b_1\) and \(|a_1| = 1\), then we get \((\mu \circ \nu) \sim - (\mu \circ \nu)\).

Since \(\text{char}(k) \neq 2\), the term \((\mu \circ \nu)\) can be written as a linear combination of terms that are of root numbers \(> r(\mu) + r(\nu)\) and with lengths \(\ell(\mu) + \ell(\nu)\).

Proof. By Lemmas 2.5 and 2.8 we get
\[
(\mu \circ \nu) \sim -1^{a_i|a_{i-1} \cdots a_1 b_m \cdots b_j| + b_j|a_i|a_{i-1} \cdots a_1 b_m \cdots b_j+1|}(\mu_{a_i \rightarrow b_j} \circ \nu_{b_j \rightarrow a_i})
\]
In particular, if \(r = m\), \(a_1 = b_1\) and \(|a_1| = 1\), then we obtain
\[
(\mu \circ \nu) \sim -1^{a_i|a_{i-1} \cdots a_1 b_m \cdots b_j| + b_j|a_i|a_{i-1} \cdots a_1 b_m \cdots b_j+1|}(\mu_{a_i \rightarrow b_j} \circ \nu_{b_j \rightarrow a_i})
\]

Now we are in a position to show that the set of all the GDN supertableaux over a well-ordered set \(X = X_0 \cup X_1\) forms a linear generating set of the free GDN superalgebra \(GDN_k(X)\) generated by \(X\).

Lemma 2.10. For every term \(\lambda\) in \(X^{(s)}\), we have \(\lambda = \sum \alpha_i \lambda_i\) for some elements \(\alpha_i\) in the field \(k\) and for some GDN supertableaux \(\lambda_i\) such that \(\ell(\lambda_i) = \ell(\lambda)\) and \(r(\lambda_i) \geq r(\lambda)\).

Proof. We use induction on \(\ell(\lambda)\). For \(\ell(\lambda) \leq 2\), it is clear. For \(\ell(\lambda) > 2\), we use a second (downward) induction on \(r(\lambda)\). For \(r(\lambda) = \ell(\lambda) - 1\), by Lemma 2.3 and by the right supercommutativity, we may assume \(\lambda = [a_1, \ldots, a_{\ell(\lambda)}]_k\) and \(a_2 = \cdots = a_{\ell(\lambda)}\). If Condition Definition 2.6 is not satisfied, then \(\lambda = 0\), otherwise, \([a_1, \ldots, a_{\ell(\lambda)}]_k\) is already a GDN supertableau. For \(r(\lambda) < \ell(\lambda) - 1\), we may assume that \(\lambda = (\mu \circ \nu)\) for some terms \(\mu, \nu\) with \(\ell(\mu) < \ell(\lambda)\) and \(\ell(\nu) < \ell(\lambda)\). By induction hypothesis, both \(\mu\) and \(\nu\) can be written as linear combinations of GDN supertableaux, say \(\mu = \sum \alpha_i \mu_i^\prime\) and \(\nu = \sum \beta_j \nu_j^\prime\). Then
for all \( i, j \), we have 
\[
\ell((\mu_i' \circ \nu_j')) = r(\mu_i') + \max(1, r(\nu_j')) \geq r(\mu) + \max(1, r(\nu)) = r((\mu \circ \nu))
\]
and 
\[
\ell((\mu_i' \circ \nu_j')) = \ell((\mu \circ \nu)).
\]

Now we can assume that \( \mu \) and \( \nu \) are GDN supertableaux. Suppose that \( \mu = [a, \mu_1, ..., \mu_p]_L \) and \( \nu = [b, \nu_1, ..., \nu_q]_L \), where all the \( \mu_i \) and \( \nu_j \) are simple terms, and \( a, b \) are elements in \( X \).

For \( \ell(\mu) > 1 \), we have \( p > 0 \) and 
\[
\lambda = (\mu \circ \nu) = (-1)^{|\nu||\mu_p|}([a, \mu_1, ..., \mu_{p-1}, \nu]_L \circ \mu_p).
\]
By induction hypothesis again, we can write the term \( [a, \mu_1, ..., \mu_{p-1}, \nu]_L \) as a linear combination of GDN supertableaux. Therefore, we may assume that \( \ell(\mu) > 1 \) and \( \nu \) is a simple term. In other words, we can assume that \( \lambda = [a, \lambda_1, ..., \lambda_n]_L \) (\( n \geq 1 \)), where \( a \) is an element in \( X \) and each \( \lambda_i = [a, i, r, ..., a_i]_L \) is a simple term. We first show that in this case \( \lambda \) can be written as a linear combination of GDN supertableaux with the claimed conditions. This is the main case with which we should deal.

Applying the right supercommutativity whenever necessary, we may assume that the conditions of Definition 2.6(i)-(ii) are satisfied. By Lemmas 2.5, 2.8 and 2.9 and by induction hypothesis, the conditions of Definition 2.6(iii) can also be obtained. For instance, say \( a_{2,2} < a_{4,3} \). Then we need to interchange \( a_{2,2} \) and \( a_{4,3} \) in the sense of Equation (2.2).
Since \( [a, \lambda_1, ..., \lambda_n]_L = (-1)^{|\lambda_2|+|\lambda_3|+|\lambda_4|}a, \lambda_2, \lambda_1, \lambda_3, \lambda_5, ..., \lambda_n]_L \), we can apply Lemma 2.9 to the term \( (a \circ \lambda_2) \circ \lambda_4 \).

Suppose that the conditions of Definition 2.6(iv) are destroyed. If \( a_{i,j} = a_{i,j+1} \in X_1 \) for some integers \( i \) and \( j \) such that \( 1 \leq i \leq n \) and \( 2 \leq j < r_i \), then by left supersymmetry, we obtain
\[
\lambda_i - [a, i, r_i, ..., a, i, j+2, (a, i, j+1 \circ a, i, j), a, i, j-1, ..., a, i]_R
= -1(\lambda_i - [a, i, r_i, ..., a, i, j+2, (a, i, j \circ a, i, j+1), a, i, j-1, ..., a, i]_R).
\]
Since \( \text{char}(k) \neq 2 \), we have \( \lambda_i = [a, i, r_i, ..., a, i, j+2, (a, i, j+1 \circ a, i, j), a, i, j-1, ..., a, i]_R \) in \( \text{GDN}_s(X) \) and \( r([a, i, r_i, ..., a, i, j+2, (a, i, j+1 \circ a, i, j), a, i, j-1, ..., a, i]_R) = 2 > 1 = r(\lambda_i) \). By the second induction hypothesis on root numbers, we are done. Therefore, we may assume that, for every integer \( j \geq 2 \), we have \( a_{i,j} \neq a_{i,j+1} \) if \( a_{i,j} \) lies in \( X_1 \). Similarly, we may assume that \( a \neq a_{1,r_1} \) if \( a \) lies in \( X_1 \). On the other hand, if \( a_{i,2} = a_{i+1, r_{i+1}} \), then we have
\[
\lambda = \alpha_1[a, \lambda_i, \lambda_i+1, ..., \lambda_i+1, \lambda_i-1, ..., \lambda_i-1, \lambda_i+2, ..., \lambda_i]_L
\sim \alpha_2[a, i, 2, (\lambda_i)_{a, i, 2 \rightarrow a}, \lambda_i+1, ..., \lambda_i+1, \lambda_i-1, ..., \lambda_i-1, \lambda_i+2, ..., \lambda_i]_L
\sim \alpha_3[a, i, 2, (\lambda_i)_{a, i, 2 \rightarrow a}, \lambda_i, ..., \lambda_i, \lambda_i+1, ..., \lambda_i]_L
\]
for some elements \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) in \( k \). The claim follows by the above reasoning.

Finally, by Lemma 2.9 right supercommutativity and by induction hypothesis, the conditions of Definition 2.6(v) can also be satisfied.

For \( \ell(\mu) = 1 \), we have \( \ell(\nu) \geq 2 \) and thus \( q \geq 1 \). For \( q = 1 \), the term \( \lambda = (\mu \circ \nu) \) is a simple term. By Lemma 2.8 and induction hypothesis on root number, we are done. For \( q > 1 \), we shall resort to the case of \( \ell(\mu) > 1 \). By left supersymmetry, we have
\[
\lambda = (\mu \circ [b, \nu_1, ..., \nu_q]_L) = ((\mu \circ [b, \nu_1, ..., \nu_{q-1}]_L) \circ \nu_q)
\]
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\[ + (-1)^{|[\mu]|} |[\nu_1, ..., \nu_{q-1}]| (([\mu, \nu_1, ..., \nu_{q-1}] \circ (\mu \circ \nu_q)) - (([\mu, \nu_1, ..., \nu_{q-1}] \circ \nu_q)). \]

By induction hypothesis and the above reasoning for the case of \( \ell(\mu) > 1 \), the result follows. \qed

3. A linear basis of \( \text{GDN}_s(X) \) and a Poincaré-Birkhoff-Witt type Theorem

Our aim in this section is to show that the set \( \operatorname{Tab}_s(X) \) of all the GDN supertableaux over a well-ordered set \( X = X_0 \cup X_1 \) forms a linear basis of the free GDN superalgebra \( \text{GDN}_s(X) \). We already know that it is a linear generating set of \( \text{GDN}_s(X) \), so what remains is to prove the linear independence. We shall also prove a PBW type theorem for GDN superalgebra, that is, every GDN superalgebra can be embedded into its universal enveloping associative differential supercommutative algebra.

3.1. Associative differential supercommutative algebra. In this subsection, we shall first construct the free associative differential supercommutative algebra generated by \( X \). It will be instrumental in proving the linear independence of the set \( \operatorname{Tab}_s(X) \) of all the GDN supertableaux over \( X \).

Recall that a supercommutative algebra is a superalgebra \( A \) satisfying the following identity:

\[ x \cdot y = (-1)^{|x||y|} y \cdot x \]

for all elements \( x, y \) in \( A_{0} \cup A_{1} \), and an associative differential supercommutative algebra is an associative supercommutative algebra \( (A, \cdot, D) \) with a linear derivation \( D \) of parity 0 satisfying that \( D(A_i) \subseteq A_i \) (\( i = 0, 1 \)) and the identity:

\[ D(x \cdot y) = D(x) \cdot y + x \cdot D(y), \]

for all elements \( x, y \) in \( A_{0} \cup A_{1} \).

S.I. Gelfand [11] pointed out that, every associative differential commutative algebra \( (A, \cdot, D) \) becomes a GDN algebra under the new operation \( \circ \) defined by \( x \circ y := x \cdot D(y) \), and with the help of this discovery, Dzhumadil'daev and L"{o}fwall proved that the set of all the GDN tableaux over a well-ordered set \( X \) forms a linear basis of the free GDN algebra generated by \( X \). This idea motivates us to establish the connection of GDN superalgebra and associative differential supercommutative algebra. The proof for the following observation is straightforward and thus omitted.

**Lemma 3.1.** For every associative differential supercommutative algebra \( (A, \cdot, D) \), if we define a new bilinear operation on \( A \) by the rule:

\[ x \circ y = x \cdot D(y) \]

for all elements \( x \) and \( y \) in \( A \), then \( (A, \circ) \) becomes a GDN superalgebra.

Let \( A \) be an associative differential supercommutative algebra over \( k \) generated by a set \( X = X_0 \cup X_1 \). We say \( A \) is free on \( X \) if, for every map \( \psi \) of \( X \) into an associative differential supercommutative algebra \( B = B_0 \oplus B_1 \) such that \( \psi(X_i) \subseteq B_i \) (\( i = 0, 1 \)), there exists a unique homomorphism \( \varphi : A \to B \) extending \( \psi \). We shall construct the free associative differential supercommutative algebra \( k_s\{X\} \) generated by a set \( X \) directly.
Define $D^0(a) = a$ for every $a$ in $X$. Define $Y = \{D^n(a) \mid a \in X, n \geq 0, n \in \mathbb{N}\}$ and let $Y^+$ be the free semigroup (without unit) generated by $Y$. For every $u = D^{i_1}(a_1) \ldots D^{i_n}(a_n)$ in $Y^+$, define the parity $|u|$ of $u$ to be $|a_1| + \ldots + |a_n|$ modulo $2$, and define $D^i(a) < D^j(b)$ if $(i, a) < (j, b)$ lexicographically. Finally, define

$$D_s[X] := \{D^{i_1}(a_1) \ldots D^{i_n}(a_n) \in Y^+ \mid D^{i_1}(a_1), \ldots, D^{i_n}(a_n) \in Y, D^{i_1}(a_1) \leq \ldots \leq D^{i_n}(a_n),$$

if $a_p = a_q \in X_1$ for some integers $p \neq q \leq n$, then $i_p \neq i_q$.

Let $kD_s[X]$ be the $k$ linear space with a $k$-basis $D_s[X]$. Define a bilinear operation $\cdot$ on the space $kD_s[X]$ as follows: For all

$$u = D^{i_1}(a_1) \ldots D^{i_n}(a_n), \quad v = D^{j_1}(b_1) \ldots D^{j_m}(b_m) \quad \text{and} \quad D^i(b) \quad \text{in} \quad D_s[X],$$

if $b$ lies in $X_1$ and $D^j(b) = D^{i_1}(a_1)$ for some integer $t \leq n$, then $u \cdot D^j(b)$ is defined to be $0$. Otherwise, assume that $D^{i_1}(a_1) \ldots D^{i_{t-1}}(a_{t-1})D^j(b)D^{i_t}(a_t) \ldots D^{i_n}(a_n)$ lies in $D_s[X]$ for some integer $t$ satisfying $1 \leq t \leq n + 1$, where $t = 1$ (or $t = n + 1$, resp.) means $D^{i_1}(a_1) \ldots D^{i_{t-1}}(a_{t-1})$ (or $D^{i_t}(a_t) \ldots D^{i_n}(a_n)$, resp.) is an empty sequence. Then define $u \cdot D^j(b)$ to be

$$(-1)^p \sum_{p \leq l \leq m} |a_p| |b_l| D^{i_1}(a_1) \ldots D^{i_{t-1}}(a_{t-1})D^j(b)D^{i_t}(a_t) \ldots D^{i_n}(a_n),$$

where the sum is over all the integer $p$ such that $D^{i_p}(a_p) > D^j(b)$. Next, the product $u \cdot v$ is defined inductively as follows:

$$u \cdot v := (u \cdot D^{j_1}(b_1)) \cdot D^{j_2}(b_2) \ldots D^{j_m}(b_m).$$

Finally, define a unary linear operation $D$ on $kD_s[X]$ as follows:

$$D(u) = \sum_{1 \leq i \leq n} (D^{i_1}(a_1) \ldots D^{i_{t-1}}(a_{t-1}) \cdot D^{i_t+1}(a_t)) \cdot D^{i_{t+1}}(a_{t+1}) \ldots D^{i_n}(a_n).$$

The following lemma offers an explicit formula for calculating the product of arbitrary two elements in $D_s[X]$.

**Lemma 3.2.** Let $u$ and $v$ be as in Equation \((3.1)\). If $u \cdot v \neq 0$, then we have

$$u \cdot v = (-1)^{p} \sum_{p \leq q \leq m} |a_p| |b_q| D^{i_1}(d_1) \ldots D^{i_{n+m}}(d_{n+m}),$$

where $D^{i_1}(d_1), \ldots, D^{i_{n+m}}(d_{n+m})$ is a reordering of $D^{i_1}(a_1), \ldots, D^{i_n}(a_n), D^{j_1}(b_1), \ldots, D^{j_m}(b_m)$ such that $D^{i_1}(d_1) \ldots D^{i_{n+m}}(d_{n+m})$ lies in $D_s[X]$, and the sum is over all the pairs $(p, q)$ such that $D^{i_p}(a_p) > D^{j_q}(b_q)$. Moreover, the equality $u \cdot v = 0$ holds if and only if, for some integers $t \leq n$ and $l \leq m$, we have $i_t = j_t$ and $a_t = b_t \in X_1$.

**Proof.** The second claim is clear, so we just prove the first one. Use induction on $m$. For $m = 1$, the claim follows by the definition of the operation $\cdot$. For $m > 1$, since the inequality $D^{j_1}(b_1) \leq D^{j_2}(b_2) \leq \ldots \leq D^{j_m}(b_m)$ holds, we obtain

$$u \cdot v = (u \cdot D^{j_1}(b_1)) \cdot D^{j_2}(b_2) \ldots D^{j_m}(b_m)$$

$$= (-1)^{p} \sum_{p \leq q \leq m} |a_p| |b_q| D^{i_1}(a_1) \ldots D^{i_{t-1}}(a_{t-1})D^{i_t}(a_t) \ldots D^{i_n}(a_n) \cdot D^{j_2}(b_2) \ldots D^{j_m}(b_m)$$

$$= (-1)^{p} \sum_{p \leq q \leq m} |a_p| |b_q| D^{i_1}(d_1) \ldots D^{i_{n+m}}(d_{n+m}).$$
with the desired properties. \hfill \Box

Now we are in a position to show that, endowed with the defined operations \( \cdot \) and \( D \), the vector space \( kD_s[X] \) becomes a free associative differential supercommutative algebra.

**Lemma 3.3.** The algebra \( (kD_s[X], \cdot, D) \) is isomorphic to the free associative differential supercommutative algebra \( k_s\{X\} \) generated by \( X \). In particular, if we define a linear operation \( \circ \) on \( (kD_s[X], \cdot, D) \) by the rule: \( u \circ v = u \cdot D(v) \) for all \( u \) and \( v \) in \( D_s[X] \), then \( (kD_s[X], \circ) \) becomes a GDN superalgebra.

**Proof.** We first show that \( (kD_s[X], \cdot, D) \) is an associative differential supercommutative algebra. By Lemma 3.2 the associativity is straightforward. As for the supercommutativity, let \( u \) and \( v \) be as in Equation (3.1). For \( u \cdot v = 0 \), it is clear that \( v \cdot u = 0 \). For \( u \cdot v \neq 0 \), with the same notation of Lemma 3.2 we have

\[
u \cdot v = \left(-1\right)^{\sum_{i=1}^{\ell} |a_{i-1}| \cdot |b_l|} D^{i_\ell}(a_{i_\ell}) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n),
\]

where the sum is over all the pairs \( (p, q) \) such that \( D^{i_p}(a_p) > D^{i_q}(b) \). Similarly, we get

\[
v \cdot u = \left(-1\right)^{\sum_{i=1}^{\ell} |a_{i-1}| \cdot |b_l|} D^{i_\ell}(a_{i_\ell}) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n),
\]

where the sum is over all the pairs \( (p', q) \) such that \( D^{i_p}(a_p) < D^{i_q}(b) \). Combining the above two formulas, we get

\[
u \cdot v = \left(-1\right)^{\sum_{i=1}^{\ell} |a_{i-1}| \cdot |b_l|} v \cdot u,
\]

where the sum is over all the pairs \( (p, q) \) such that \( D^{i_p}(a_p) \neq D^{i_q}(b) \). Moreover, if \( i_p = j_q \) and \( a_p = \overline{b_q} \) for some integers \( p, q \) such that \( 1 \leq p \leq n \) and \( 1 \leq q \leq m \), then \( a_p \) lies in \( X_0 \) and thus \( (-1)^{|a_p||b_q|} = 1 \). So we obtain \( v \cdot u = \left(-1\right)^{|u||v|} u \cdot v \).

To show that \( D(u \cdot v) = D(u) \cdot v + u \cdot D(v) \), we use induction on \( m \). For \( m = 1 \), we have

\[
D(u \cdot D^{j_1}(b_1)) = \left(-1\right)^{\sum_{i=1}^{\ell} |a_{i-1}| \cdot |b_l|} D(D^{i_\ell}(a_{i_\ell}) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n))
\]

\[
= \left(-1\right)^{\sum_{i=1}^{\ell} |a_{i-1}| \cdot |b_l|} \left(D^{i_\ell}(a_{i_\ell}) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n)\right)
\]

\[
+ \sum_{1 \leq q \leq t-1} \left(D^{i_\ell}(a_{i_\ell}) \cdots D^{i_q-1}(a_{i_q-1}) \cdot D^{i_q+1}(a_q)\right) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n)
\]

\[
+ \sum_{t \leq q \leq n} \left(D^{i_\ell}(a_{i_\ell}) \cdots D^{i_{q-1}}(a_{i_{q-1}}) \cdot D^{i_q+1}(a_q)\right) \cdots D^{i_2}(a_{i_2}) D^{i_1}(b_1) D^{i_0}(a_n)
\]

\[
= D(u) \cdot D^{j_1}(b_1) + u \cdot D^{j_1+1}(b_1) \quad \text{(by applying associativity and supercommutativity)}.
\]

For \( m > 1 \), we obtain

\[
D(u \cdot v) = D((u \cdot D^{j_1}(b_1)) \cdot \cdots) D^{j_m}(b_m))
\]

\[
= (D(u) \cdot D^{j_1}(b_1)) \cdot \cdots \cdot D^{j_m}(b_m) + (u \cdot D^{j_1}(b_1)) \cdot D(D^{j_2}(b_2) \cdots D^{j_m}(b_m))
\]

\[
= (D(u) \cdot D^{j_1}(b_1)) \cdot \cdots \cdot D^{j_m}(b_m) + (u \cdot D^{j_1+1}(b_1)) \cdot D^{j_2}(b_2) \cdots D^{j_m}(b_m)
\]

\[
+ (u \cdot D^{j_1}(b_1)) \cdot \cdots \cdot D^{j_m}(b_m) = D(u) \cdot v + u \cdot D(v).
\]
Therefore, \((kD_{s}[X], \cdot, D)\) is an associative differential supercommutative algebra.

It remains to show that \((kD_{s}[X], \cdot, D)\) is free on \(X\). By applying associativity and supercommutativity in \(k_{s}\{X\}\), it is easy to see that the set of all the monomials of the form:

\[
(((\ldots (D^{i_{1}}(a_{1}) \cdot D^{i_{2}}(a_{2})) \cdot \ldots) \cdot D^{i_{n-1}}(a_{n-1})) \cdot D^{i_{n}}(a_{n})) \quad \text{(left-normed bracketting)}
\]

such that \(D^{i_{1}}(a_{1}) \cdots D^{i_{n}}(a_{n})\) lies in \(D_{s}[X]\) forms a linear generating set of \(k_{s}\{X\}\). Define a map \(\psi: X \to kD_{s}[X]\) by \(\psi(a) = a\) for every \(a\) in \(X\), and extend \(\psi\) to a superalgebra homomorphism \(\tilde{\psi}: k_{s}\{X\} \to kD_{s}[X]\). Then

\[
\tilde{\psi}(((\ldots (D^{i_{1}}(a_{1}) \cdot D^{i_{2}}(a_{2})) \cdot \ldots) \cdot D^{i_{n-1}}(a_{n-1})) \cdot D^{i_{n}}(a_{n})) = D^{i_{1}}(a_{1}) \cdots D^{i_{n}}(a_{n}).
\]

Since the set \(D_{s}[X]\) is linearly independent in \(kD_{s}[X]\), the homomorphism \(\tilde{\psi}\) is an isomorphism. \(\square\)

Thanks to Lemma 3.3, we can identify \(k_{s}\{X\}\) with \(kD_{s}[X]\).

### 3.2. The linear independence of the set \(\text{Tab}_{s}(X)\)

Our aim in this subsection is to show that the set of all the GDN supertableaux \(\text{Tab}_{s}(X)\) over \(X\) is linearly independent. Our strategy is to construct an GDN superalgebra homomorphism from \((\text{GDN}_{s}(X), \circ)\) to \((kD_{s}[X], \cdot)\), and show that the image of \(\text{Tab}_{s}(X)\) is linearly independent in \(kD_{s}[X]\), where the operation \(\circ\) is defined in Lemma 3.1.

We define an ordering \(<\) on \(D_{s}[X]\) as follows: For all \(u\) and \(v\) as be in Equation (3.1), we define

\[
(*) \quad u < v \iff (n, i_{n}, a_{n}, \ldots, i_{1}, a_{1}) < (m, j_{m}, b_{m}, \ldots, j_{1}, b_{1}) \text{ lexicographically},
\]

and define the length \(\ell(u)\) of \(u\) to be \(n\).

For every element \(f = \sum_{1 \leq i \leq n} \alpha_{i}u_{i}\) with each \(\alpha_{i} \neq 0\) in \(k\) and \(u_{1} > u_{2} > \ldots > u_{n}\) in \(D_{s}[X]\), we call \(\overline{f} := u_{1}\) the leading monomial of \(f\), and call \(\text{lc}(f) := \alpha_{1}\) the leading coefficient of \(f\).

Now we are ready to show that the set \(\text{Tab}_{s}(X)\) is linearly independent in \(\text{GDN}_{s}(X)\). Recall that by Lemma 3.3, \((kD_{s}[X], \circ)\) is a GDN superalgebra.

**Theorem 3.4.** Let \(\varphi: (\text{GDN}_{s}(X), \circ) \to (kD_{s}[X], \cdot)\) be a GDN superalgebra homomorphism induced by \(\varphi(a) = a\) for every element \(a\) in \(X\). Then \(\varphi\) is injective. Moreover, the set \(\text{Tab}_{s}(X)\) of all the GDN supertableaux over \(X\) forms a linear basis of the free GDN superalgebra \(\text{GDN}_{s}(X)\).

**Proof.** We first show that \(\varphi\) is injective. Let \(\mu\) be a GDN supertableau as in Equation (2.1). Then it is easy to see that

\[
\varphi(\mu) = \varphi(a) \cdot D(\varphi(\mu_{1})) \cdots D(\varphi(\mu_{n})).
\]

Therefore, it is straightforward to show that

\[
\varphi(\mu) = a_{n,2} \cdots a_{n,r_{n}}a_{n-1,2} \cdots a_{n-1,r_{n-1}} \cdots a_{1,2} \cdots a_{1,r_{1}} aD^{r_{n}}(a_{n,1}) \cdots D^{r_{1}}(a_{1,1}),
\]

and \(\text{lc}(\varphi(\mu)) = 1\) or \(\text{lc}(\varphi(\mu)) = -1\).
Therefore, for all GDN supertableaux $\mu$ and $\nu$, if $\mu \neq \nu$, then we obtain $\overline{\varphi(\mu)} \neq \overline{\varphi(\nu)}$. Suppose that for some pairwise different GDN supertableaux $\mu_1, \ldots, \mu_n$ in $\text{Tab}_s(X)$, for some nonzero elements $\alpha_1, \ldots, \alpha_n$ in $k$, we have $\sum \alpha_i \mu_i = 0$. Then the equality $\sum \varphi(\alpha_i \mu_i) = 0$ contradicts to the fact that $\varphi(\mu_i)$ are pairwise different. Therefore, the set $\text{Tab}_s(X)$ is linearly independent and the homomorphism $\varphi$ is injective. In particular, by Lemma 2.10, the set $\text{Tab}_s(X)$ is a linear basis of $\text{GDN}_s(X)$.

3.3. A Poincaré-Birkhoff-Witt type Theorem. We call an associative differential supercommutative algebra $B = B_0 \oplus B_1$ a universal enveloping algebra of a GDN superalgebra $A = A_0 \oplus A_1$ if, there is a linear map $\psi : A \to B$ satisfying $\varphi(A_i) \subseteq B_i$ ($i = 0, 1$) and

$$\psi(x \circ y) = \psi(x) \cdot D(\psi(y))$$

for all $x$ and $y$ in $A$, and the following holds: for an arbitrary associative differential supercommutative algebra $C = C_0 \oplus C_1$, for every linear map $\psi' : A \to C$ satisfying the equation $\psi'(x \circ y) = \psi'(x) \cdot D(\psi'(y))$ for all $x$ and $y$ in $A$, and $\psi'(A_i) \subseteq C_i$ ($i = 1, 2$), there exists a unique homomorphism of associative differential supercommutative algebras $\varphi : B \to C$ such that $\varphi \circ \psi = \psi'$. It is easy to see that whenever such an universal enveloping algebra $B$ exists, then it is unique up to isomorphism.

Let $A = A_0 \oplus A_1$ be a superalgebra and let $S$ be a subset of $A$. We call $S$ a homogeneous set if $S$ is a subset of $A_0 \cup A_1$. For every homogenous subset $S$ of $\text{GDN}_s(X)$, the notation $\text{GDN}_s(X|S)$ means the quotient superalgebra $\text{GDN}_s(X)/\text{Id}(S)$, where $\text{Id}(S)$ means the ideal of $\text{GDN}_s(X)$ generated by $S$. Let $\varphi$ be as that in Theorem 3.3, and denote by $\text{Id}_D[\varphi(S)]$ the associative differential supercommutative algebra generated by $X$ with the set $\varphi(S)$ of defining relations, that is, the quotient superalgebra $kD_s[X]/\text{Id}_D[\varphi(S)]$. Then it is easy to see that, for every GDN superalgebra $\text{GDN}_s(X|S)$, the associative differential supercommutative algebra $kD_s[X|\varphi(S)]$ is the universal enveloping algebra of $\text{GDN}_s(X|S)$.

Our aim in this subsection is to show that every GDN superalgebra can be embedded into its universal enveloping associative differential supercommutative algebra. We first consider the subalgebra of $(kD_s[X], \circ)$ (as GDN superalgebra) generated by $X$.

For every monomial $u = D^{i_1}(a_1)...D^{i_n}(a_n)$ in $D_s[X]$, define the weight $\text{wt}(u)$ of $u$ to be $(\sum_{1 \leq j \leq n} i_j) - n + 1$. Then it is easy to see that $\text{wt}(u \cdot v) = \text{wt}(u) + \text{wt}(v) - 1$ for all $u$ and $v$ in $D_s[X]$ such that $u \cdot v \neq 0$.

The following lemma offers another linear basis of the free GDN superalgebra generated by $X$, that is, the set $D^0_s[X]$ of all the monomials of weight 0 in $D_s[X]$.

Lemma 3.5. Let $kD^0_s[X]$ be the subspace of $kD_s[X]$ spanned by all the monomials of weight 0 in $D_s[X]$. Then $(kD^0_s[X], \circ)$ is the subalgebra of $(kD_s[X], \circ)$ generated by $X$. Moreover, let $\varphi : \text{GDN}_s(X) \to kD^0_s[X]$ be the GDN superalgebra homomorphism induced by $\varphi(a) = a$ for every $a$ in $X$. Then $\varphi$ is an isomorphism.

Proof. We first show that $(kD^0_s[X], \circ)$ is a GDN superalgebra. It is enough to show that, for all $u$ and $v$ in $D^0_s[X]$, the product $u \cdot D(v)$ lies in $kD^0_s[X]$. Assume that $u$ and $v$ are as
Corollary 3.6. Proof. It is enough to show that there is some positive integer \(k_D\) such that for every GDN superalgebra \(A\), the inequality \(\ell(\mu) \geq n\) implies that \(\mu = 0\). Let \(\varphi\) be as in Lemma 3.5. For every GDN superalgebra \(A\), we have \(\varphi(\mu) = \sum_{1 \leq p \leq q} a_p u_p\) for some nonzero elements \(a_i\) in \(k\), and for some monomials \(u_p\) in \(D_n[X]\) such that \(\varphi(u_p) = 0\) and \(\ell(u_p) = \ell(\mu)\). Say \(u = u_p\) for some integer \(p \leq q\). Then we may assume that

\[ u = a_{r_1} \cdots a_{r_n} D(b_1) \cdots D(b_l) D^{r_1}(a_1) \cdots D^{r_n}(a_n) \]

for some elements \(a_i, b_j, c_l\) in \(X\) such that \(2 \leq r_1 \leq \cdots \leq r_n\). Then we have

\[ n \leq (r_1 - 1) + (r_2 - 1) + \cdots + (r_n - 1) = m - 1. \]

Therefore, we obtain \(\ell(\mu) = n + m + t \leq 2m + t - 1\). So if \(\ell(\mu) > 3(2X_1)\), where \(2X_1\) is the cardinality of \(X_1\), then \(t > (2X_1)\) or \(m > (2X_1)\), both of which imply that \(\varphi(\mu) = 0\). Since \(\varphi\) is an isomorphism, we get \(\mu = 0\).}

Let \(S\) be a homogeneous subset of \(\text{GDN}_s(X)\) and let \(\varphi\) be as that in Lemma 3.5. Denote by \(I^{(D)}_D[\varphi(S)]\) the GDN superalgebra ideal of \((kD^n_s[X],\circ)\) generated by \(\varphi(S)\) and denote by \(kD^n_s[X|\varphi(S)]\) the quotient of \((kD^n_s[X],\circ)\) and \(I^{(D)}_D[\varphi(S)]\).

By Lemma 3.5, it is clear that \(kD^n_s[X|\varphi(S)]\) is isomorphic to \(\text{GDN}_s(X|S)\). Therefore, for the embedding, it is enough to prove that \((kD^n_s[X|\varphi(S)],\circ)\) can always be embedded
into \((kD_s[X] \varphi(S)), \circ)\). We shall first investigate the elements of \(\text{Id}_D[\varphi(S)]\) and investigate those of \(\text{Id}_D^0[\varphi(S)]\).

Since \(S\) is homogeneous, it is easy to see that \(\varphi(S)\) is also homogeneous, in particular, for every element \(s \in S\), the parity \(|\varphi(s)|\) of \(\varphi(s)\) is well-defined. Therefore, for every monomial \(u \in D_s[X]\), we have \(u \cdot \varphi(s) = (-1)^{|\varphi(s)||u|} \varphi(s) \cdot u\). By applying right supercommutativity, we obtain

\[
\text{Id}_D[\varphi(S)] = \operatorname{span}_k \{u \cdot D^t(\varphi(s)) \mid u \in D_s[X], t \in \mathbb{Z}_{\geq 0}, s \in S\},
\]

where \(D^0(\varphi(s))\) is defined to be \(\varphi(s)\). We are now ready to describe the ideal of \((kD_s^0[X], \circ)\) generated by the set \(\varphi(S)\).

**Lemma 3.7.** Let \(S\) be a homogeneous subset of \(\text{GDN}_s(X)\) and let \(\varphi\) be as in Lemma 3.5. Suppose that \(\text{Id}_D^0[\varphi(S)]\) is the ideal of the GDN superalgebra \((kD_s^0[X], \circ)\) generated by \(\varphi(S)\). Then we have

\[
(3.3) \quad \text{Id}_D^0[\varphi(S)] = \operatorname{span}_k \{u \cdot D^t(\varphi(s)) \mid u \in D_s[X], t \in \mathbb{Z}_{\geq 0}, s \in S, \operatorname{wt}(u \cdot D^t(\varphi(s))) = 0\}.
\]

**Proof.** Since \(S\) is homogeneous, it is clear that the right part of Equation (3.3) is an ideal including \(\varphi(S)\). So to prove the lemma, it is enough to show that \(u \cdot D^t(\varphi(s))\) lies in \(\text{Id}_D^0[\varphi(S)]\) whenever \(\operatorname{wt}(u \cdot D^t(\varphi(s))) = 0\). Since every monomial in the expansion of \(D^t(\varphi(s))\) has weight \(t\), we may suppose that

\[
u = a_1 \cdots a_mb_1 \cdots b_tD_{r_1}(c_1) \cdots D_{r_n}(c_n)
\]

lies in \(D_s[X]\) such that \(m = r_1 + \cdots + r_n - n\) and \(r_n \geq r_{n-1} \geq \cdots \geq r_1 \geq 1\). Then in \(kD_s^0[X]\), we have

\[
u \cdot D^t(\varphi(s)) = \alpha D^t(s) \cdot b_1 \cdots b_t \cdot a_1 \cdots a_m D_{r_1}(c_1) \cdots D_{r_n}(c_n)
\]

for some integer \(\alpha\). So the lemma will be clear if we show that the following two claims hold:

(i) The polynomial \(D^t(\varphi(s)) \cdot b_1 \cdots b_t\) lies in \(\text{Id}_D^0[\varphi(S)]\) if \(s\) lies in \(S\);

(ii) The polynomial \(f \cdot a_1 \cdots a_{r-1} \cdot D^r(s)\) lies in \(\text{Id}_D^0[\varphi(S)]\) if \(f\) is a homogeneous polynomial in \(\text{Id}_D^0[\varphi(S)]\).

To prove (i), we use induction on \(t\). For \(t = 0\), we get \(D^0(\varphi(s)) \cdot b_1 \cdots b_t = \varphi(s) \in \text{Id}_D[\varphi(S)]\). For \(t > 0\), the polynomial

\[
D^t(s) \cdot b_1 \cdots b_t = (-1)^{|b_1||s|}b_1 \cdot D^t(s) \cdot b_2 \cdots b_t
\]

\[
= (-1)^{|b_1||s|}((b_1 \circ (D^{t-1}(s) \cdot b_2 \cdots b_t)) - (-1)^{|b_1||s|}b_1 \cdot \sum_{2 \leq i \leq t} (D^{t-1}(s) \cdot b_2 \cdots b_{i-1} \cdot (Db_i) \cdot b_{i+1} \cdots b_t))
\]

\[
= (-1)^{|b_1||s|}((b_1 \circ (D^{t-1}(s) \cdot b_2 \cdots b_t)) - \sum_{2 \leq i \leq t} (-1)^{|b_i||b_{i+1} \cdots b_t|}((D^{t-1}(s) \cdot b_1 \cdots b_{i-1} b_{i+1} \cdots b_t) \circ b_i)
\]

lies in \(\text{Id}_D^0[\varphi(S)]\) by induction hypothesis.

To prove (ii), we use induction on \(r\). For \(r = 1\), we obtain \(f \cdot D(c) = f \circ c \in \text{Id}_D[\varphi(S)]\). For \(r > 1\), the polynomial
\[ f \cdot a_1 \ldots a_{r-1} D^r(c) = (f \circ (a_1 \ldots a_{r-1} D^{r-1}(c))) - \sum_{1 \leq i \leq r-1} (f \cdot a_1 \ldots a_{i-1} \cdot Da_i \cdot a_{i+1} \ldots a_{r-1} D^{r-1}(c)) \]

\[ = (f \circ (a_1 \ldots a_{r-1} D^{r-1}(c))) - \sum_{1 \leq i \leq r-1} (-1)^{|a_i|} |a_{i+1} \ldots a_{r-1}| c_{|a_i| a_{i+1} \ldots a_{r-1} D^{r-1}(c) } a_i \]

lies in \( \text{ld}_D^r[\varphi(S)] \) by induction hypothesis. \( \square \)

We then have the following Poincaré-Birkhoff-Witt type theorem.

**Theorem 3.8.** Every GDN superalgebra \( \text{GDN}_k(X) \) can be embedded into its universal enveloping associative differential supercommutative algebra \( kD_\ast[X|\varphi(S)] \), where

\[ \varphi : \text{GDN}_k(X) \rightarrow kD_\ast[X] \]

is the GDN superalgebra homomorphism induced by \( \varphi(a) = a \) for every \( a \) in \( X \).

**Proof.** By Lemmas 3.5 and 3.7 we obtain

\[ \text{GDN}_k(X) \cong kD_\ast[X|\varphi(S)] = \frac{\left( \text{kD}_\ast[X], \circ \right)}{\text{ld}_D^s[\varphi(S)]} = \frac{\left( \text{kD}_\ast[X], \circ \right)}{\text{ld}_D^s[\varphi(S)]} \cap kD_\ast[X] \]

\[ \cong \frac{kD_\ast[X]}{\text{ld}_D^s[\varphi(S)]} \leq \frac{\left( kD_\ast[X], \circ \right)}{\text{ld}_D^s[\varphi(S)]} = kD_\ast[X|\varphi(S)]. \]

The lemma follows. \( \square \)

### 4. Engel Theorem

Our aim in this section is to prove an Engel theorem for GDN superalgebras (Theorem 4.4), which is based on what was done for GDN algebras 39. In this section, we assume that the characteristic \( \text{char}(k) \) of the field \( k \) is 0, and assume that \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) is a GDN superalgebra.

For every \( x \) in \( (\mathcal{A}, \circ) \), let \( \rho_x \) be the right multiplication operator

\[ \rho_x : \mathcal{A} \rightarrow \mathcal{A}, \quad \rho_x(y) = (y \circ x) \text{ for every } y \in \mathcal{A}. \]

Then \( \mathcal{A} \) is called right-nil of bound index if, for some positive integer \( n \), for every \( x \in \mathcal{A} \), we have \( \rho_x^n(x) = 0 \). We use the notation \( x^n_l \) for \( \rho_x^n(x) \). For all \( x_1, \ldots, x_n \) in \( \mathcal{A} \), define

\[ [x_1, \ldots, x_n]_l = \left( (\cdots ((x_1 \circ x_2) \circ x_3) \circ \cdots ) \circ x_n \right) \text{ (left-normed bracketing)}. \]

For all subspace \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) of \( \mathcal{A} \), define

\[ [\mathcal{V}_1, \ldots, \mathcal{V}_n]_l = \text{span}_k \{ [x_1, \ldots, x_n]_l \mid x_i \in \mathcal{V}_i, 1 \leq i \leq n \}. \]

In particular, we obtain

\[ \mathcal{V}_1^n_l = [\mathcal{V}_1, \ldots, \mathcal{V}_1]_l \text{ and } [\mathcal{V}_1, \mathcal{V}_2]_l = (\mathcal{V}_1 \circ \mathcal{V}_2) = \text{span}_k \{ (x_1 \circ x_2) \mid x_1 \in \mathcal{V}_1, x_2 \in \mathcal{V}_2 \}. \]

We call an algebra \( \mathcal{A} \) right-nilpotent if \( \mathcal{A}_1^n_l = 0 \) for some positive integer \( n \). Finally, for every subspace \( \mathcal{V} \) of \( \mathcal{A} \), for every integer \( n \geq 1 \), define the subspace \( \mathcal{V}^n \) of \( \mathcal{A} \) inductively as follows:
(i) \( \mathcal{V}^1 = \mathcal{V} \) and \( \mathcal{V}^2 = (\mathcal{V} \circ \mathcal{V}) \);
(ii) \( \mathcal{V}^n = \sum_{1 \leq i \leq n-1} (\mathcal{V}^i \circ \mathcal{V}^{n-i}) \).

We call an algebra \( A \) nilpotent if \( A^n = 0 \) for some positive integer \( n \).

Since \( A_0 \) is an ordinal GDN algebra, by Lemmas 6 and 7 in \[9\], we get the following lemma, which shows that every right-nil GDN algebra of bound index is right nilpotent. For the convenience of the readers, we quickly repeat the argument.

**Lemma 4.1.** \[9\] Let \( A = A_0 \oplus A_1 \) be a GDN superalgebra over a field of characteristic 0. If for some positive integer \( n \), for every \( x \in A_0 \), we have \( x^n = 0 \), then \( (A_0)^{n+1} = 0 \).

**Proof.** For all \( x_1, \ldots, x_t \) in \( A_0 \), define
\[
S(x_1, x_2, \ldots, x_t) = \sum_{\sigma \in S_t} [x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(t)}]_L,
\]
where \( S_t \) is the symmetric group of order \( t \). Then for every term \( \mu \) occurred in the polynomial \( (x_1 + x_2 + \cdots + x_t)^t = S(x_1, x_2, \ldots, x_t) \), there is some integer \( i \leq t \) such that the letter \( x_i \) does not occur in \( \mu \). By the inclusion-exclusion properties, we get
\[
(x_1 + x_2 + \cdots + x_t)^t - S(x_1, x_2, \ldots, x_t) = \sum_{\varnothing \neq \{i_1, i_2, \ldots, r\} \subseteq \{1, \ldots, t\}} (-1)^{t-r+1}(x_{i_1} + \cdots + x_{i_r})^t.
\]
Therefore, for \( t \geq n \), we get \( S(x_1, x_2, \ldots, x_t) = 0 \). Moreover, using right (super)commutativity, it is straightforward to show that
\[
S(x_1, \ldots, x_{t+1}) = t(S(x_2, \ldots, x_{t+1}) \circ x_1) + t! [x_1, \ldots, x_{t+1}]_L.
\]
Since \( \text{char}(k) = 0 \), we have \( [x_1, \ldots, x_{t+1}]_L = 0 \) for every \( t \geq n + 1 \).

The following lemma shows that, if a GDN superalgebra \( A \) is right nilpotent, then \( A^2 \) is nilpotent. This result is directly reminiscent of that for GDN algebras \[9\].

**Lemma 4.2.** Let \( A \) be a GDN superalgebra. Then for every positive integer \( n \), the space \( A^n_L \) is an ideal of \( A \), and we have \( (A^2)^n_L \subseteq A_L^{n+1} \).

**Proof.** We first use induction on \( n \) to show that \( A_L^n \) forms an ideal of \( A \). For \( n = 1 \), it is clear. For \( n \geq 2 \), we have
\[
(A_L^2 \circ A) = A_L^{n+1} \subseteq A_L^n,
\]
and by induction hypothesis, we also have
\[
(A \circ A_L^n) \subseteq ((A \circ A_L^{n-1}) \circ A) + (A_L^{n-1} \circ (A \circ A)) + ((A_L^{n-1} \circ A) \circ A) \subseteq A_L^n.
\]

Now we use induction on \( n \) to show \( (A^2)^n_L \subseteq A_L^{n+1} \). For \( n = 1 \), it is clear. For \( n \geq 2 \), we obtain
\[
(A^2)^n = \sum_{1 \leq i \leq n-1} ((A^2)^i \circ (A^2)^{n-i}) \subseteq \sum_{1 \leq i \leq n-1} (A_L^{i+1} \circ A_L^{n-i+1})
\]
\[
\subseteq \sum_{1 \leq i \leq n-1} [A, A_L^{i+1}, A_L^i, \ldots, A_L^0]_L \subseteq \sum_{1 \leq i \leq n-1} [A_L^{i+1}, A_L^i, \ldots, A_L^0]_L = A_L^{n+1}.
\]

The proof is completed. \( \square \)
We want to show that, under certain conditions, every right-nil GDN superalgebra of bounded index is right nilpotent. And the main difficulty lies in how to deal with the space \([A_0, A_1, A_1, ..., A_1]_1\). The idea is to “split” \(A_1\) in the following sense.

**Lemma 4.3.** Let \(A = A_0 \oplus A_1\) be a GDN superalgebra generated by \(X = X_0 \cup X_1\). Then for every integer \(q \geq 1\), we have

\[
[A_0, A_1, ..., A_1]_L \subseteq \sum_{q \text{ times}} [A_0, A_0, ..., A_0, kX_1, ..., kX_1, A_1, ..., A_1]_L,
\]

where \(kX_1\) is the subspace of \(A_1\) spanned by \(X_1\).

**Proof.** We use induction on \(q\). For \(q \leq 2\), it is clear. For \(q = 3\), we shall show that

\[
[A_0, A_1, A_1, A_1]_L \subseteq [A_0, A_1, A_0]_L + [A_0, A_1, A_1, kX_1]_L.
\]

It is enough to show that, for every term \(\mu\) over \(X\) of parity 0, for all terms \(\mu_1, \mu_2\) and \(\mu_3\) over \(X\) of parity 1, we have

\[
[\mu, \mu_1, \mu_2, \mu_3]_L \in [A_0, A_1, A_0]_L + [A_0, A_1, A_1, kX_1]_L.
\]

We use induction on \(\ell(\mu_1)\). For \(\ell(\mu_1) = 1\), the claim follows by right supercommutativity. For \(\ell(\mu_1) > 1\), suppose that \(\mu_1 = (\mu_{11} \circ \mu_{12})\). If \(\mu_{12}\) lies in \(A_1\), then \(\mu_{11}\) lies in \(A_0\), and by induction hypothesis, we have \([\mu, \mu_{12}, (\mu_{11} \circ \mu_{12}), \mu_3]_L \in [A_0, A_1, A_0]_L + [A_0, A_1, A_1, kX_1]_L\). Therefore, we obtain

\[
[\mu, \mu_{11}, \mu_2, \mu_3]_L = [\mu, (\mu_{11} \circ \mu_{12}), \mu_2, \mu_3]_L
\]

If \(\mu_{12}\) lies in \(A_0\), then \(\mu_{11}\) lies in \(A_1\), and we obtain

\[
[\mu, \mu_{11}, \mu_2, \mu_3]_L = [\mu, (\mu_{11} \circ \mu_{12}), \mu_2, \mu_3]_L
\]

So \([\mu, \mu_{11}, \mu_2, \mu_3]_L\) lies in \([A_0, A_1, A_0]_L\). For \(q \geq 4\), by right supercommutativity and the case \(q = 3\), we have

\[
[A_0, A_1, ..., A_1]_L \subseteq [A_0, A_1, ..., A_1]_L + [A_0, A_1, ..., A_1, kX_1]_L
\]

with \(q\) times \(q - 2\) times \(q - 1\) times.
The claim follows. □

We are now in a position to prove the following Engel theorem.

**Theorem 4.4.** Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a GDN superalgebra generated by $X = X_0 \cup X_1$ over a field of characteristic 0, where every element of the set $X_0$ is of parity 0 and every element of the set $X_1$ is of parity 1. If $X_1$ is a finite set and the even part $\mathcal{A}_0$ is right-nil of bounded index $n$, then $\mathcal{A}$ is right nilpotent, in particular, the ideal $\mathcal{A}^2$ of $\mathcal{A}$ is nilpotent.

**Proof.** By Lemma 4.11, we have $(\mathcal{A}_0)^{n+1}_L = 0$. Let $n_0 = \max(\sharp(X_1), n+1)$ and let $q = 3n_0 + 1$. We shall show that $\mathcal{A}^q_L = 0$. It is enough to show the following two claims:

\[
[A_0, A_1, \ldots, A_0, \underbrace{A_0, \ldots, A_0}_j, \underbrace{A_1, \ldots, A_1}_{q - 2 - j} \ldots ]_L = 0 \quad \text{for every integer } j \text{ such that } 0 \leq j \leq q - 2,
\]

and

\[
[A_1, A_1, \ldots, A_1, \underbrace{A_0, \ldots, A_0}_j, \underbrace{A_1, \ldots, A_1}_{q - 1 - j} \ldots ]_L = 0 \quad \text{for every integer } j \text{ such that } 0 \leq j \leq q - 1.
\]

For the first claim, if $j = 0$, then by Lemma 4.11, we get $(A_0)_{L}^{q-1} = 0$. For every integer $j$ such that $1 \leq j \leq q - 2$, by Lemma 4.13, we have

\[
[A_0, A_1, \ldots, A_1, \underbrace{A_0, \ldots, A_0}_j, \underbrace{A_1, \ldots, A_1}_{q - 2 - j} \ldots ]_L
\]

\[
\subseteq \sum_{2t + m + p = q, t, m \geq 0, p \in \{1, 2\}} [A_0, \underbrace{A_0, \ldots, A_0}_{t + q - 2 - j}, \underbrace{kX_1, \ldots, kX_1}_m, \underbrace{A_1, A_1}_p]_L.
\]

If $m > n_0$, then we obtain

\[
[A_0, \underbrace{A_0, \ldots, A_0}_{t + q - 2 - j}, \underbrace{kX_1, \ldots, kX_1}_m, \underbrace{A_1, A_1}_p]_L = 0.
\]

If $m \leq n_0$, then we obtain $t = \frac{1}{2}(j - m - p)$ and

\[
t + q - 2 - j = \frac{1}{2}(j - m - p) + q - j - 2
\]

\[
= \frac{1}{2}(-m - p + q) + \frac{1}{2}(q - j) - 2 \geq \frac{1}{2}(-n_0 - 2 + 3n_0 + 1) + \frac{1}{2} \times 2 - 2 = n_0 - \frac{3}{2}.
\]
Since $t + q - 2 - j$ is an integer, we have $t + q - 2 - j \geq n_0 - 1$. So we get

$$[A_0, \overbrace{A_0, \ldots, A_0}^{t + q - 2 - j \text{ times}}, \overbrace{kX_1, \ldots, kX_1}^{m \text{ times}}, \overbrace{A_1, A_1}^{p \text{ times}}]_L = 0.$$ 

The first claim follows.

For the second claim, if $q = 0$, then by the first claim, we get

$$[A_1, \overbrace{A_1, \ldots, A_1}^{j \text{ times}}, \overbrace{A_0, \ldots, A_0}^{q - 1 \text{ times}}, \overbrace{A_0, \ldots, A_0}^{j - 1 \text{ times}}, \overbrace{A_0, \ldots, A_0}^{q - 2 - (j - 1) \text{ times}}]_L = 0.$$ 

If $j = 0$, then we first use induction on $q$ to show that, for every $q \geq 3$, we have

$$[A_1, \overbrace{A_0, \ldots, A_0}^{j \text{ times}}]_L \subseteq (A_1 \circ (A_0)^{q - 1}_L) + ((A_0)^{q - 1}_L \circ A_1) + ((A_0)^{q - 2}_L \circ (A_1 \circ A_0)).$$ 

For $q = 3$, by left supersymmetry and right supercommutativity, we get

$$[A_1, A_0, A_0]_L \subseteq (A_1 \circ (A_0 \circ A_0)) + ((A_0 \circ A_1) \circ A_0) + (A_0 \circ (A_1 \circ A_0))$$

$$\subseteq (A_1 \circ (A_0 \circ A_0)) + ((A_0 \circ A_0) \circ A_1) + (A_0 \circ (A_1 \circ A_0)).$$

For $q > 3$, by induction hypothesis, we get

$$[A_1, A_0, \ldots, A_0]_L = ([A_1, A_0, \ldots, A_0]_L \circ A_0)$$

$$\subseteq (A_1 \circ ((A_0)^{q - 2}_L \circ A_0)) + (((A_0)^{q - 2}_L \circ A_1) \circ A_0) + (((A_0)^{q - 3}_L \circ (A_1 \circ A_0)) \circ A_0)$$

$$\subseteq (A_1 \circ (A_0)^{q - 1}_L) + ((A_0)^{q - 1}_L \circ A_1) + ((A_0)^{q - 2}_L \circ (A_1 \circ A_0)).$$

Finally, since $(A_0)^{q - 2}_L = 0$, the second claim follows.

\[ \square \]

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Z.Z., SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU 510631, P. R. CHINA
E-mail address: 295841340@qq.com

Y.C., SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU 510631, P. R. CHINA
E-mail address: yqchen@scnu.edu.cn

L.A.B., SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY GUANGZHOU 510631, P. R. CHINA; SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, 630090, RUSSIA; NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK 630090, RUSSIA
E-mail address: bokut@math.nsc.ru