Complete spin structure of the pion-nucleon-loop delta self-energy

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Abstract

The complete spin structure of the pion-nucleon-loop contribution to the delta self-energy and dressed propagator is calculated in vacuum, with the most general form of the pion-nucleon-delta vertex. The imaginary parts of the ten Lorentz-scalar coefficients are calculated in closed form, while the real parts are obtained numerically from a dispersion relation. The effect of the pion-nucleon-delta coupling constants and form-factor on the pion-nucleon phase-shift in the spin-3/2 isospin-3/2 channel is studied.

I. INTRODUCTION

The delta resonance plays a significant role in a number of processes due to the fact that it is easily excited from a nucleon, especially by pions. The spin-3/2 value of the delta is responsible for its somewhat cumbersome treatment in a fully relativistic approach. As a consequence, either a nonrelativistic approximation is used [1], or the (dressed) delta propagator is approximated by using a definite scheme for calculating its self-energy [2–4].

There is a general ambiguity in the Rarita-Schwinger form of the propagator of a spin-3/2 particle [5], which allows one to take for the free propagator the usual form:

$$G_{0}^{\mu\nu}(p) = \frac{\not{p} + M_\Delta}{p^2 - M_\Delta^2 + i\varepsilon} \left[ g^{\mu\nu} - \frac{\gamma^\mu \gamma^\nu}{3} - \frac{2p^\mu p^\nu}{3M_\Delta^2} + \frac{p^\nu\gamma^\mu - p^\mu\gamma^\nu}{3M_\Delta} \right], \quad (1)$$

where $M_\Delta$ is the (bare) mass of the delta. The most general form of the pion-nucleon-delta coupling is described by the interaction Lagrangian density

$$\mathcal{L}_{\pi N \Delta} = \frac{g}{M} \partial_\alpha \pi \bar{\Delta}_\beta (g^{\alpha\beta} + a\gamma^\beta \gamma^\alpha) N + \text{h.c.}, \quad (2)$$

where $g$ is the (dimensionless) coupling, $M$ the physical mass of nucleon (introduced in the coupling to make explicit its dimension), $a$ an arbitrary (the so called off-shell) parameter and the isospin coefficients and indices are not written out. In some treatments the value of the parameter $a$ is taken to be zero for reasons of simplicity, but we do not want to make such an ad hoc restriction.

The large value of the coupling $g$ means that the properties of the nucleon, pion and delta are mutually strongly influenced. For the pion and the nucleon in vacuum the modification of the spectral function occurs at rather large momenta, where finite-size (i.e. form-factor) effects are expected to suppress the correction. For the delta, however, the pion-nucleon-loop
correction determines the vacuum spectral function in a decisive way, as one can see from the delta decay properties. Such a dressed delta plays an important role in a number of processes with nucleon (or nuclear) targets, motivating a fully relativistic calculation of its self-energy and dressed propagator. The pion-nucleon loop is expected to give the dominant contribution at low and intermediate energies, which is the region most readily probed by experiments.

It is probably even of greater importance to study the properties of the delta (and nucleon and pion) in the hot nuclear medium, as created for example in collisions of heavy ions, than in the relatively simple case of free space. The in-medium calculations at present usually involve a number of approximations, one of which is to assume a simple spin-structure for the delta self-energy (achieved most simply by using the nonrelativistic limit). As a prelude to a more complete treatment we consider here the vacuum case which may hopefully serve as starting point for the much more involved in-medium computations.

The organization of the presentation is the following. In section 2 the calculation of the self-energy and dressed propagator is presented. The results are applied in section 3 to computing the pion-nucleon phase shift in the spin-3/2 isospin-3/2 channel and fitting the parameters to observed data. Conclusions and outlook are discussed in section 4.

II. CALCULATION OF THE SELF-ENERGY AND PROPAGATOR

From expression (2) the (unregularized) one-loop delta self-energy has the following form:

$$\Sigma^\mu\nu(p) = -i \frac{g^2}{M^2} \int \frac{d^4k}{(2\pi)^4} \left( k^\mu + \alpha \gamma^\mu k \right) \frac{(\not{p} + \not{k} + M)}{(p + k)^2 - M^2 + i\epsilon} \left( k^\nu + \alpha \gamma^\nu k \right) \frac{1}{k^2 - m^2 + i\epsilon},$$

where $M$ is the mass of the nucleon and $m$ that of the pion. The above expression is divergent and for regularization we use a Lorentz-scalar form-factor in the vertex. The form-factor is used only when calculating the imaginary part of the self-energy, whose real part is then obtained from a dispersion relation. In this way the correct analytic properties of the self-energy and thus causality are assured.

The self-energy can be written in the following general form, which contains ten independent second-order tensor forms which can be constructed from the delta’s four-momentum $p$, the gamma matrices and the metric tensor:

$$\Sigma^\mu\nu(p) = g^{\mu\nu}(\beta_1 + \beta_2) + p^\mu p^\nu(\beta_3 + \beta_4) + \gamma^\mu \gamma^\nu(\beta_5 + \beta_6) + p^\mu \gamma^\nu \beta_7 + p^\nu \gamma^\mu \beta_8 + p^\mu \gamma^\nu \beta_9 + p^\nu \gamma^\mu \beta_{10}.$$  (4)

The functions $\beta_i \equiv \beta_i(p^2)$ are Lorentz-scalars which depend only on $p^2$.

When calculating imaginary parts of $\beta_i$ we start from the expression for the imaginary part of a loop diagram [3]:

$$\text{Im} \Sigma^\mu\nu = 2 \frac{g^2}{M^2} \int \frac{d^4k}{(2\pi)^4} \Gamma^\mu \text{Im} G(p + k) \text{Im} D(k) \Gamma^\nu \theta(p_0 + k_0) \theta(-k_0),$$

with $\Gamma^\mu$ standing for the spin structure of the coupling (including also the form-factor) and the imaginary parts of the (free) nucleon and pion propagators given by
\[ \text{Im } D(k) \, \theta(-k_0) = -\frac{\pi}{2\sqrt{m^2 + k^2}} \delta(k_0 + \sqrt{m^2 + k^2}), \]
\[ \text{Im } G(p + k) \, \theta(p_0 + k_0) = -\frac{\pi}{2\sqrt{M^2 + (p + k)^2}} (\hat{p} + \hat{k} + M) \]
\[ \times \delta(p_0 + k_0 - \sqrt{M^2 + (p + k)^2}). \]  
(6)

The presence of the two delta-functions allows for complete evaluation of the integral. The imaginary parts of functions \( \beta_i(p^2) \) are nonzero only if \( p^2 > (M + m)^2 \) and in the calculation it is convenient to use \( p = 0 \). Taking into account the presence of the form-factor \( F(p^2) \) (with explicit form specified below), the imaginary parts of coefficients \( \beta_i(p^2) \) all have the following form

\[ \text{Im } \beta_i(p^2) = \alpha_i(p^2) \frac{g^2 F(p^2)^2}{16\pi M^2 p^2} \sqrt{(p^2 - M^2 + m^2)^2 - 4m^2p^2}. \]  
(7)

Introducing the notation

\[ k_+^2 \equiv \frac{1}{4p^2}[(p^2 - M^2 + m^2)^2 - 4m^2p^2], \]
\[ p_+^2 \equiv p^2 - M^2 + m^2, \]  
(8)

for the functions \( \alpha_i(p^2) \) we obtain:

\[ \alpha_1 = -\frac{Mk_+^2}{3}, \]
\[ \alpha_2 = -\frac{k_+^2}{6p^2}(p^2 + M^2 - m^2), \]
\[ \alpha_3 = \frac{(1 + 2a)M}{3p^4} [p_+^4 - m^2 p^2], \]
\[ \alpha_4 = \frac{1}{12p^6} [p_+^4(p^2 + 3M^2 - 3m^2) + 2m^2p^2(p^2 - 3M^2 + 3m^2)], \]
\[ \alpha_5 = -\frac{aM}{3}(2k_+^2 - 3am^2), \]
\[ \alpha_6 = -\frac{2}{3}ak_+^2 - \frac{a^2}{2p^2} [p_+^4 - m^2(3p^2 - M^2 + m^2)], \]
\[ \alpha_7 = -\alpha_8 = -\frac{aM}{3p^4} (p_+^4 - m^2 p^2), \]
\[ \alpha_9 = \frac{1}{6p^2} [k_+^2 p_+^2 + a(p_+^4 + 2m^2 p^2 - 3m^2 p_+^2)], \]
\[ \alpha_{10} = \frac{1}{6p^2} \left\{ k_+^2 p_+^2 + 3a(1 + 2a) [p_+^4 - 2m^2 p^2 - m^2 p_+^2] \right\}. \]  
(9)

The real part of \( \beta_i(p^2) \) is then calculated numerically using the dispersion relation

\[ \text{Re } \beta_i(p^2) = \frac{\mathcal{P}}{\pi} \int_{(M + m)^2}^{\infty} \frac{d\sigma^2 \text{Im } \beta_i(\sigma^2)}{\sigma^2 - p^2}, \]  
(10)
assuring the correct analytic properties of the self-energy.

For the form-factor we used an exponential form:

\[
F(p^2) = \exp \left[ -\frac{p^2 - (M + m)^2}{\Lambda^2} \right],
\]

which provides suppression in the kinematical range of interest, since the imaginary part of the self-energy in the considered model is nonzero only if \( p^2 > (M + m)^2 \).

In order to be able to solve the Dyson equation for the dressed delta propagator

\[
G^\mu_\nu(p) = G^\mu_\nu_0 + G^\mu_\alpha_0 \Sigma_\alpha_\beta G^\beta_\nu,
\]

we have to rewrite the self-energy and the free delta propagator in terms of the following spin-projection operators [4] (omitting on the left side the two Lorentz-vector indices):

\[
\begin{align*}
P^{3/2} &= g_{\mu\nu} - \frac{2p^\mu p^\nu}{p^2} - \frac{\gamma^\mu \gamma^\nu}{3} + \frac{1}{3p^2}(\gamma^\mu p^\nu - p^\mu \gamma^\nu)\hat{p}, \\
P^{1/2}_{11} &= \frac{\gamma^\mu \gamma^\nu}{3} - \frac{p^\mu p^\nu}{3p^2} - \frac{1}{3p^2}(\gamma^\mu p^\nu - p^\mu \gamma^\nu)\hat{p}, \\
P^{1/2}_{22} &= \frac{p^\mu p^\nu}{p^2}, \\
P^{1/2}_{21} &= \frac{1}{\sqrt{3p^2}}(-p^\mu p^\nu + \gamma^\mu p^\nu\hat{p}), \\
P^{1/2}_{12} &= \frac{1}{\sqrt{3p^2}}(p^\mu p^\nu - p^\nu \gamma^\mu \hat{p}).
\end{align*}
\]

The coefficient of each projector has the form \( a(p^2) + ib(p^2) \), where \( a(p^2) \) and \( b(p^2) \) are Lorentz scalars. From Eq. (12) the dressed propagator can be obtained through

\[
G^{-1} = G^{-1}_0 - \Sigma,
\]

where from (11)

\[
G^{-1}_0 = (P^{3/2} - 2P^{1/2}_{11})(\hat{p} - M_\Delta) + \sqrt{3}M_\Delta(P^{1/2}_{12} + P^{1/2}_{21}).
\]

Using the properties of the projectors (13) one can invert expression (14) to obtain the dressed delta propagator in the following form:

\[
G(p) = \frac{a_1\hat{p} - b_1}{p^2a_1^2 - b_1^2}P^{3/2} + (a_2\hat{p} + b_2)P^{1/2}_{11} + (a_3\hat{p} + b_3)P^{1/2}_{22} + (a_4\hat{p} + b_4)P^{1/2}_{12} + (a_5\hat{p} + b_5)P^{1/2}_{21}.
\]

The coefficients \( a_i \) and \( b_i \) are functions of \( p^2 \) and in general have a real and imaginary part. The factor of the projector \( P^{3/2} \) in (14) is written explicitly in terms of \( a_1 \) and \( b_1 \) which multiply (in the form \( a_1\hat{p} + b_1 \)) that projector from the right in the expression for \( G^{-1} \). The other eight functions \( a_i, b_i, \ i = 2 \ldots 5 \) are obtained from an algebraic system of eight equations containing the eight coefficients of the other four projectors in the expression for
As a partial check on the correctness of the calculation, for every set of parameters used, we numerically computed the sum-rule for the $g^{\mu\nu}\gamma^0$ term of the delta spectral-function. From expression (18) it follows that the free propagator satisfies

$$\int_{-\infty}^{\infty} \rho_0(p_0, p) dp_0 = 1,$$

where $\rho_0$ is the imaginary part (divided by $-\pi$) of the coefficient of the $\gamma^0g^{\mu\nu}$ term of the propagator. Numerical evaluation of the corresponding sum-rule for the dressed propagator, which in view of (13) and notation (14) can be written as

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} dp_0 p_0 \text{Im} \frac{a_1}{p^2 a_1^2 - b_1^2} = 1,$$

showed it to be satisfied to accuracy better than 1%.

III. PIÖN-NUCLEON PHASE-SHIFT IN THE (3/2,3/2) CHANNEL

The pion-nucleon phase-shift in the spin-3/2 isospin-3/2 channel (more precisely, the $P_{33}$ phase-shift) shows a resonance behavior which can be explained very well with a diagram (see Fig. 1) involving the delta intermediate state [7].

Fig. 1. The pion-nucleon scattering in the spin-3/2 isospin-3/2 channel can be described well through a delta in intermediate state.

We want to establish whether the same good quality fit can be achieved with the present complete relativistic treatment of the delta. The usual treatment [7,8] involves the nonrelativistic limit, while in Ref. [3] a relativistic treatment was used, but with an approximation for the self-energy and dressed delta propagator.

In the nonrelativistic limit the pion-nucleon-delta coupling has only a p-wave ($\ell = 1$) component, while a fully relativistic vertex includes also a d-wave ($\ell = 2$) term. This means that apart from the $P_{33}$ phase-shift the diagram in Fig. 1 contains the $D_{33}$ phase-shift too. The general expression for the S-matrix element for scattering of spin-zero particle by spin-one particle in the center-of-mass system [9] is:

$$< \hat{e}_f, \frac{1}{2}, \nu'|S|\hat{e}_i, \frac{1}{2}, \nu > = \sum_{\ell m'm'JM} Y_{\ell}^{m'}(\hat{e}_f) Y_{\ell}^{m*}(\hat{e}_i) < \ell, \frac{1}{2}; m', \nu'|JM >$$

$$\times < \ell, \frac{1}{2}; m, \nu|JM > T_{\ell}',$$

where $T_{\ell}'$ is the phase-shift.
where \( \hat{e}_i \) (\( \hat{e}_f \)) gives the direction of the incoming (outgoing) momentum, \( \nu \) (\( \nu' \)) is the spin projection on the momentum for the incoming (outgoing) state and

\[
T_{\ell}^{\prime} \equiv \frac{1}{2\pi i \rho} \left( 1 - e^{2i \delta_{\ell}^{\prime}} \right).
\] (20)

\( \delta_{\ell}^{\prime} \) is the phase-shift in the considered channel and \( \rho \) is the energy density of states. Taking \( \nu = \nu' = 1/2 \) and \( \hat{e}_i \) to point in the direction of the z-axis, only terms with \( m = m' = 0 \) (and \( M = 1/2 \)) survive in the sum on the right-side of (19). Since the delta intermediate state means that \( J = 3/2 \), the two possible \( \ell \) values are 1 and 2. We can eliminate the term involving the \( \ell = 2 \) phase-shift by choosing a scattering angle such that \( Y_2^0(\hat{e}_f) = 0 \). This gives for the polar angle \( \vartheta = 54.7^\circ \).

![Graph](image)

Fig. 2. Phase-shift in the spin-3/2 isospin-3/2 channel. The points represent measurement results from Ref. [10], while the line is our calculated result with values of parameters given in text.

The scattering amplitude of the diagram in Fig. 1 was calculated numerically, using the dressed delta propagator and the pion-nucleon-delta vertex corresponding to interaction (2). The \( P_{33} \) pion-nucleon phase-shift was reproduced with excellent agreement (shown in Fig. 2) up to pion laboratory momentum of 500 MeV. The parameters used were the following: coupling \( g = 19 \), cut-off \( \Lambda = 0.97 \) GeV, bare mass of the delta \( M_\Delta = 1.279 \) GeV. Based on the considered \( P_{33} \) phase-shift it is not possible to fit uniquely the value of the off-shell parameter \( a \) in the interaction (2). Values of \( a \) in the range from \(-1\) to \(0\) give excellent fits, but much larger or much smaller values completely destroy the agreement. The cut-off \( \Lambda \) is seemingly much larger than the one obtained in Ref. [8], where a non-relativistic approach was used. A direct comparison of the two values is, however, not possible because of the different arguments used (in Ref. [8] it was the pion’s three-momentum which appeared in the form-factor). An approximate correspondence of the two functional forms is established for \( \Lambda \frac{3}{5} = \Lambda/\sqrt{2} = 0.69 \) GeV, still a considerably larger value than obtained in Ref. [8]. The form-factor in the present approach, nevertheless, is rather soft, similarly to the result of Ref. [3]. We remark that a very good fit could be obtained by using a somewhat larger value of the coupling \( g = 20 \) with a slightly softer form-factor with \( \Lambda = 0.92 \) GeV, and a bare mass of the delta \( M_\Delta = 1.273 \) GeV. These values of \( g \) and \( \Lambda \) give a similar value for the on-mass-shell coupling as the values \( g = 19 \) and \( \Lambda = 0.97 \) GeV.
IV. CONCLUSIONS

Although the full tensor structure of the pion-nucleon-loop delta self-energy in the relativistic treatment is rather involved, it can be calculated in a straightforward way. It is convenient to use the dispersion relation satisfied by the self-energy and calculate first its imaginary part (more precisely the imaginary parts of the Lorentz-scalar coefficients determining it). These were calculated in an analytic form, using the most general pion-nucleon-delta coupling, involving also the off-shell parameter. Using an exponential form-factor to regularize the self-energy the real parts of the ten Lorentz-scalar functions were calculated numerically, as well as the Lorentz-scalar coefficients of the dressed propagator, obtained from solving the Schwinger-Dyson equation. A sum-rule for the dressed delta propagator was checked numerically.

The dressed delta propagator was used to calculate the pion-nucleon P_{33} phase-shift in the spin-3/2 isospin-3/2 channel. An excellent fit was obtained using a rather soft pion-nucleon-delta form-factor and a coupling whose on-shell value is close to the one determined from the decay width of the delta. The off-shell parameter $a$ could not be determined uniquely from a fit to this process, since values between $-1$ and $0$ give practically the same result (much larger or smaller values destroy the fit). It would be of interest to examine how observables in some other processes involving the delta (e.g. Compton scattering on a nucleon, pion electroproduction, etc.) are affected by the off-shell parameter and in general by the present more complete treatment of the delta propagator.

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