Boundary conditions for isolated asymptotically anti-de Sitter spacetimes

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We revisit the propagation of classical scalar fields in a spacetime which is asymptotically anti-de Sitter. The lack of global hyperbolicity of the underlying background gives rise to an ambiguity in the dynamical evolution of solutions of the wave equation, requiring the prescription of extra boundary conditions at the conformal infinity to be fixed. We show that the only boundary conditions that are compatible with the hypothesis that the system is isolated, as defined by the (improved) energy-momentum tensor, are of Dirichlet and Neumann types.

I. INTRODUCTION

The anti-de Sitter (AdS) spacetime is the classic solution to the vacuum Einstein equations in the presence of a negative cosmological constant. It has the highest possible degree of symmetry since it is maximally symmetric. Despite this apparent geometric simplicity, the AdS spacetime has remarkable properties that make it a particularly interesting background for the study of classical and quantum fields. In particular, it is a non-globally hyperbolic spacetime, implying that the solutions of the wave equation are not fully determined from initial data. This requires the prescription of extra boundary conditions at its spatial infinity in order to have a unique solution for the Cauchy problem. Physically, the lack of global hyperbolicity is related to the fact that information propagating in AdS can reach spatial infinity in finite time, which allows the energy to leak out of the spacetime. As a result, the AdS spacetime does not give rise, in general, to an isolated system.

This problem has been addressed in Refs. 3 and 4 within the context of supergravity in (1 + 3)-dimensions. Besides analyzing the stability of the anti-de Sitter background with respect to small scalar perturbations, these works show that the boundary conditions that make the improved energy functional positive and conserved are restricted to the Dirichlet and Neumann types.

Given the arbitrariness on the choice of the boundary condition at the conformal boundary, Wald and Ishibashi defined in Refs. 2, 5, and 6 a sensible prescription for obtaining the dynamics of a propagating field on AdS. By requiring that the field propagation respects causality and time translation/reflection invariance and, what is most important, also has a conserved energy functional, it was shown that the non-equivalent types of sensible dynamics are in one-to-one correspondence with the positive self-adjoint extensions of the spatial part of the wave operator. These self-adjoint extensions are obtained by choosing suitable boundary conditions at the conformal infinity. The resulting conserved energy functional, however, is not that extracted from the improved energy-momentum tensor $T_{\mu \nu}$. In fact, it can be shown that it arises from the subtraction of a boundary term from the energy functional coming from $T_{\mu \nu}$. This boundary term vanishes for Dirichlet or Neumann boundary conditions, and in this case, the newly defined (conserved) energy matches the usual energy, which is already conserved. For every other — generalized Robin — boundary condition, there is an effective contribution of the boundary term to the newly defined (conserved) energy functional, showing that there is an effective flux of energy through the conformal boundary of AdS.

In any event, Robin boundary conditions have recently spawned great interest in the context of Quantum Field Theory in asymptotically anti-de Sitter spacetimes and several authors analyzed the consequences implied by these boundary conditions on the quantization of the scalar field (see, for instance, Refs. 9–13 and references therein). As a matter of fact, the introduction of Robin boundary conditions is often motivated by the desire that the system be isolated, as explicitly stated in Refs. 10, 11, and 13. One of the goals of the present work is to clarify this issue and show that generic Robin boundary conditions are incompatible with the requirement that the spacetime be isolated.

More precisely, this paper is concerned with the Cauchy problem associated with the wave equation

$$\Box - m_s^2 \Phi = 0$$

in an asymptotically anti-de Sitter spacetime, where $m_s^2 \equiv \mu^2 + \xi R$ and $\xi$ is a constant which couples the field to the curvature scalar $R$. This coupling modifies the usual energy-momentum tensor obtained by the variation of the action with respect to the metric. In what follows, we use the resulting improved energy-momentum tensor to define the energy functional. Our aim is to establish the boundary conditions for which the system $\text{spacetime} + \text{field}$ can be considered as effectively isolated, a point which, as mentioned above, has occasionally been a source of confusion in the literature. It turns
out that this is equivalent to finding the boundary conditions for which the conserved energy functional defined by Wald and Ishibashi is equal to the one extracted from the improved energy-momentum tensor. We emphasize that our analysis takes into account only classical fields. In the context of quantum fields in curved spacetimes, the prescription of Wald and Ishibashi leads to a vanishing (renormalized) energy flux $\langle T_{\mu\nu} \rangle$ (see Ref. 13).

This paper is organized as follows. In Sec. II we obtain an asymptotic expression for the scalar field at spatial infinity. This is done by means of a Green function that encodes the dependence of the solution on the initial data and boundary conditions. Our analysis differs from that in Refs. 3, 4, and 8 in that we only assume that the spacetime is asymptotically AdS; we thus make no assumption (except for certain technicalities to be explained below) about its bulk structure. In Sec. III we discuss the requirements on the boundary conditions at spatial infinity for the system spacetime + scalar field to be effectively isolated. We find that the only boundary conditions that are compatible with this assumption are the (generalized) Dirichlet and Neumann boundary conditions. Finally, in Sec. IV we discuss our results and make our closing remarks.

### II. ASYMPTOTIC BEHAVIOR OF THE FIELD

Let $M$ be a stationary $n$-dimensional spacetime, which is asymptotically AdS. We choose coordinates $\{t, r, \theta_1, \ldots, \theta_{n-2}\}$ such that the metric on $M$ satisfies

$$ds^2|_{r\to\infty} \approx ds^2_{\text{AdS}} = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{n-2}^2,$$

where $ds^2_{\text{AdS}}$ is the line element in AdS$_n$ and $d\Omega_{n-2}^2$ is the metric on the $(n-2)$-dimensional unit sphere.

We separate variables for the scalar field and consider the ansatz

$$\Phi(t, r, \theta) = \sum_{\{l\}} \Phi_l(r, t) Y_l(\theta),$$

where $\{l\}$ represents the set of integer indices labeling the hyperspherical harmonics $Y_l(\theta)$. The wave equation (1) can then be written as

$$L_{\text{eff}}[\phi] = 0,$$

where $L_{\text{eff}}$ is a second order differential operator of the form

$$L_{\text{eff}} = u^{ij}(r) \partial_i \partial_j + v^i(r) \partial_i + q(r), \quad i, j = r, t.$$

When dealing with problems such as (4), it is common practice to consider a time dependence of the form $e^{-i\omega t}$ and then to solve the resulting time-independent problem. However, when considering non-conservative systems (for instance, when energy can flow through the boundaries), with $\omega$ being a complex number, such an approach leads to extra mathematical difficulties, which, in turn, make it difficult to physically interpret the resulting solutions. When the spacetime bulk contains a black hole, such an approach allows for the determination of the quasinormal mode spectrum of the system. However, the quasinormal modes do not provide a complete set of eigenfunctions, and, hence, an arbitrary initial condition cannot be expressed in terms of them.

As discussed in Ref. 14, one can overcome this difficulty by taking the initial conditions into account from the beginning. A suitable mathematical tool for implementing this strategy is the Laplace transform

$$\mathcal{L}\{\phi_l(t, r)\} = \hat{\phi}_l(\omega, r) = \int_0^\infty \phi_l(t, r)e^{i\omega t} dt.$$

Applying the Laplace transform to (4), we obtain an ordinary differential equation,

$$P_2(\omega, r) \frac{d^2 \hat{\phi}(\omega, r)}{d\omega^2} + P_1(\omega, r) \frac{d\hat{\phi}(\omega, r)}{d\omega} + P_0(\omega, r) \hat{\phi}(\omega, r) = \mathcal{F}(\omega, r),$$

for each $\omega$, with $\mathcal{F}(\omega, r)$ taking care of the initial conditions. We omitted the index $l$ to not clutter notation.

Eq. (7) can be rewritten as a Schrödinger-type equation,

$$\frac{d^2 \tilde{\psi}}{dr_s^2} - s(r_s) \tilde{\psi} = f(r_s),$$

by using a suitable change of variables,

$$\hat{\phi} \to \tilde{\psi}, \quad r \to r_s,$$

which maps $r$ into an interval $[r_s^{\text{min}}, r_s^{\text{max}}]$. This is to be determined by the specific form of the metric. The solution of Eq. (5) can then be found by the standard Green’s function method and can be expressed as

$$\psi(\omega, r_s) = \frac{\hat{\psi}_b(\omega, r_s)}{W[\hat{\psi}_b, \hat{\psi}_\infty]} \int_{r_s^{\text{min}}}^{r_s^{\text{max}}} f(\omega, r_s') \hat{\psi}_b(\omega, r_s') dr_s' + \frac{\hat{\psi}_\infty(\omega, r_s)}{W[\hat{\psi}_b, \hat{\psi}_\infty]} \int_{r_s^{\text{min}}}^{r_s^{\text{max}}} f(\omega, r_s') \hat{\psi}_\infty(\omega, r_s') dr_s'. $$

Here, $W[\hat{\psi}_b, \hat{\psi}_\infty]$ is the Wronskian of the solutions $\hat{\psi}_b$ and $\hat{\psi}_\infty$ of the homogeneous equation associated with (8).

$$W[r_s, \hat{\psi}_b, \hat{\psi}_\infty] = \psi_b \frac{\partial \hat{\psi}_\infty}{\partial r_s} - \psi_\infty \frac{\partial \hat{\psi}_b}{\partial r_s}.$$

The function $\hat{\psi}_b$ should be determined after imposing some condition at $r_s^{\text{min}}$, deep into the bulk. This could be a regularity condition at the “origin” $r = 0$ when $M = \text{AdS}$ or a condition at the event horizon when $M$ contains a black hole. On the other hand, the function $\hat{\psi}_\infty$ is determined from the boundary conditions at the conformal infinity, $r_s^{\text{max}}$.

Assuming initial data with compact support, we find the following asymptotic approximation:

$$\psi(\omega, r_s) \approx \mathcal{A}(\omega) \hat{\psi}_\infty(\omega, r_s), \quad \text{as } r_s \to r_s^{\text{max}},$$

where $\mathcal{A}(\omega)$ is a function of $\omega$. This approximation allows for the determination of the quasinormal modes of the field when the spacetime contains a black hole.

Equation (10) then becomes

$$\frac{d^2 \hat{\psi}_b}{dr_s^2} - s(r_s) \hat{\psi}_b = f(r_s).$$

The function $\hat{\psi}_b$ is determined from the boundary condition at the horizon and the initial data.

\[ \hat{\psi}_b \approx \mathcal{A}(\omega) \hat{\psi}_\infty, \quad \text{as } r_s \to r_s^{\text{max}}. \]
with \( \mathcal{S}(\omega) = (1/W[\hat{\psi}_b, \hat{\psi}_e]) \int_{r_{\text{min}}}^{1/r} f(\omega, r') \hat{\psi}_b(\omega, r') dr' \). Inverting the transformation (9) leads to
\[
\hat{\phi}(\omega, r) \approx \mathcal{S}(\omega) \hat{\phi}_\infty(\omega, r), \quad r \to \infty,
\]
where \( \hat{\phi}_\infty(\omega, r) \) is a solution of the homogeneous equation associated with \( \mathcal{S} \) obeying some boundary condition at spatial infinity. The inverse Laplace transform then yields
\[
\phi(t, r) \approx \frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \mathcal{S}(\omega) \hat{\phi}_\infty(\omega, r) e^{-i\omega t} d\omega,
\]
as \( r \to \infty \).

We note that the boundary conditions affect the resulting scalar field by means of the solutions of the homogeneous equation, \( \hat{\psi}_b(\omega, r_c) \) and \( \hat{\psi}_e(\omega, r_c) \), while the initial data are encoded in \( f(\omega, r_c) \). Equivalently, the transformation (9) allows one to interpret the dependence of the solution on the boundary conditions in terms of (the fundamental set of solutions of the homogeneous equation associated with \( \mathcal{S} \)) \( \{\hat{\phi}_b, \hat{\phi}_e\} \), while its dependence on the initial conditions is given by \( \mathcal{S}(\omega, r) \).

Since our aim here is to study the flux of energy at the conformal boundary, we will not fix any specific conditions on the field in the bulk other than requiring the usual regularity conditions, such as initial data with compact support and finiteness of the integrals associated with the asymptotic approximations. As a matter of fact, the convergence of these integrals depends on the analytical structure of the Green’s function, which, in turn, depends on the boundary conditions deep inside the spacetime bulk. Hence, the convergence of these integrals must be treated differently for each spacetime. Throughout this work, we will assume that it is always possible to find an approximation such as (14) for the spacetime at hand.

III. ENERGY FLUX IN ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES

We are now ready to study under what conditions the system spacetime + field is isolated, in the sense of having no energy flux through the timelike spatial boundary at infinity. As discussed in Sec. II, the asymptotic behavior of the solutions of (11) is encoded in \( \hat{\phi}_\infty(\omega, r) \) (we, henceforth, reinsert the \( \ell \) index for definiteness). For each value of \( \ell \), this function satisfies the homogeneous equation associated with \( \mathcal{S} \) in the limit \( r \to \infty \), which is given by
\[
\frac{\partial^2}{\partial \rho^2} \hat{\phi}_{\infty,\ell}(\omega, \rho) + (n - 2) \sec \rho \csc \rho \frac{\partial}{\partial \rho} \hat{\phi}_{\infty,\ell}(\omega, \rho)
+ \left[ \omega^2 - \frac{\ell(n - 3)}{\sin^2 \rho} - \frac{m_\pi^2}{\cos^2 \rho} \right] \hat{\phi}_{\infty,\ell}(\omega, \rho) = 0,
\]
where we have changed the radial coordinate to \( \rho \), with \( r = \tan \rho \). Multiplying the last equation by \( (\tan \rho)^{n-2} \) and performing the transformation
\[
\hat{\phi}_{\infty,\ell}(\omega, \rho) = \frac{Z_\ell(\omega, \rho)}{(\tan \rho)^{n-2}},
\]
we find
\[
\frac{\partial^2 Z_\ell(\omega, \rho)}{\partial \rho^2} + [\omega^2 - V(\rho)] Z_\ell(\omega, \rho) = 0,
\]
where the effective potential \( V \) is given by
\[
V(\rho) = \left[ \ell(n + 1) + \frac{1}{4} (n^2 - 6n + 8) \right] \csc^2 \rho
+ \left[ \frac{1}{4}(n - 2) + m_\pi^2 \right] \sec \rho.
\]
We also define
\[
d = n - 1, \quad \nu^2 = \frac{(n - 1)^2}{4} + m_\pi^2,
\]
and
\[
a = \frac{1}{2} \left( \frac{d}{2} + \ell + \nu - \omega \right), \quad b = \frac{1}{2} \left( \frac{d}{2} + \ell + \nu + \omega \right).
\]

A. A convenient fundamental set of solutions

For the sake of definiteness, let us fix a convenient set \( \{Z_{\ell}^{(D)}, Z_{\ell}^{(N)}\} \) of linearly independent solutions of (17). Following (18), we take these functions as follows.

(i) For \( \nu \) not being an integer,
\[
Z_{\ell}^{(D)}(\omega, \rho) = (\cos \rho)^{\frac{\ell}{2} + \nu} (\sin \rho)^{\frac{d - 1}{2}} \times 2F_1 \left( a, b; 1 + \nu; \cos^2 \rho \right),
\]
\[
Z_{\ell}^{(N)}(\omega, \rho) = (\cos \rho)^{\frac{\ell}{2} + \nu} (\sin \rho)^{\frac{d - 1}{2}} \times 2F_1 \left( a - \nu, b - \nu; 1 - \nu; \cos^2 \rho \right).
\]

(ii) For \( \nu = 0 \),
\[
Z_{\ell}^{(D)}(\omega, \rho) = (\cos \rho)^{\frac{\ell}{2} + \nu} (\sin \rho)^{\frac{d - 1}{2}} \times 2F_1 \left( a, b; 1; \cos^2 \rho \right),
\]
\[
Z_{\ell}^{(N)}(\omega, \rho) = (\cos \rho)^{\frac{\ell}{2} + \nu} (\sin \rho)^{\frac{d - 1}{2}} \times 2F_1 \left( a, b; 1; \cos^2 \rho \right) \ln(\cos^2 \rho)
+ \sum_{k=1}^{\infty} \left( \frac{a}{k!} \right) (\cos \rho)^{2k} \times [ \psi(a + k) - \psi(a)
+ \psi(b + k) - \psi(b) - 2 \psi(k + 1) + 2 \psi(1)] \right) .
\]
(iii) For \( v \) being a positive integer,
\[
Z^{(D)}_\ell (\omega, \rho) = (\cos \rho)^{\frac{1}{2} + v} (\sin \rho)^{1 + \frac{1}{2} - \frac{d}{2}} 
\times 2F_1 \left( a, b; 1 + v; \cos^2 \rho \right),
\]
\[
Z^{(N)}_\ell (\omega, \rho) = (\cos \rho)^{\frac{1}{2} + v} (\sin \rho)^{1 + \frac{1}{2} - \frac{d}{2}} 
\times \left\{ 2F_1 \left( a, b; 1 + v; \cos^2 \rho \right) \ln(\cos^2 \rho) 
+ \sum_{k=1}^{m} \left( a_k \right) \left( b_k \right) (1 + v)_k k! (\cos \rho)^{2k} \times [h(k) - h(0)] 
- \sum_{k=1}^{v} \left( k - 1 \right) \left( - v \right) k \left( 1 - a \right) \left( 1 - b \right) (\cos \rho)^{-2k} \right\},
\]
\[
\frac{\partial Z_{\ell}/\partial \rho}{Z_{\ell}} \Bigg|_{\rho = \frac{\pi}{2}} = -\cot \zeta.
\]

We note that \( \zeta = 0 \) and \( \zeta = \pi \) correspond to \( Z_{\ell}|_{\rho = \pi/2} = 0 \), which is the usual Dirichlet boundary condition. On the other hand, \( \zeta = \pi/2 \) corresponds to \( \partial Z_{\ell}/\partial \rho |_{\rho = \pi/2} = 0 \), the usual Neumann boundary condition. Other choices of \( \zeta \in [0, \pi] \) correspond to Robin boundary conditions. In the general case, the effective potential \( V_{\ell} \) diverges as \( \rho \) goes to \( \pi/2 \), and the ratio \( (\partial Z_{\ell}/\partial \rho)/Z_{\ell} \) is no longer well defined. Despite that, the behavior of \( G^{(N)}_{\ell} \) as \( \rho \) goes to \( \pi/2 \) is dictated by \( \sin \zeta \), while the behavior of \( \partial (G^{(N)}_{\ell} Z_{\ell})/\partial \rho \) is governed by \( \cos \zeta \), so it seems natural to define the “generalized Dirichlet boundary condition” as \( \zeta = 0 \) and the “generalized Neumann boundary condition” by \( \zeta = \pi/2 \). The other values of \( \zeta \in [0, \pi] \) parameterize the “generalized Robin boundary conditions”.

B. The flux at infinity

According to Weyl’s limit point and limit circle theory, the allowed boundary conditions at the endpoints of the interval where a Sturm-Liouville problem is defined depend on the integrability of the solutions in the vicinity of these points. In the present case, the solutions of (17) provide an approximation for the field near the point \( \rho = \pi/2 \). The integrability of these solutions depends on the parameter \( v \).

In what follows, we are going to use the improved energy-momentum tensor of the complex scalar field, to calculate the energy flux. The Killing vector field \( k = \partial /\partial t \) gives rise to the formerly conserved energy \( Q^a = |g|^{1/2} T^{\alpha \beta} k_{\beta} (\partial_{\beta} Q^{\mu} = 0) \), and the energy flux across the spatial infinity is given by
\[
\mathcal{F}_\alpha = - \lim_{\rho \to \pi/2} \int d\theta_1 \ldots d\theta_{n-2} g^{\rho \rho} Q_\rho.
\]

The case \( v^2 \geq 1 \)

This is a simplest instance to analyze. In this case, \( Z_{\ell}^{(D)} \) is square integrable near \( \rho = \pi/2 \), while \( Z_{\ell}^{(N)} \) is not. As a result, the generalized Dirichlet boundary condition must be chosen in this case. With this boundary condition the energy flux across the spatial infinity turns out to be zero. We omit the calculation since it is identical to the case \( 0 < v^2 < 1 \), treated below, once we set \( \zeta = 0 \).
The case $0 < v^2 < 1$

In this case, both solutions are square integrable near $\rho = \pi/2$. The allowed boundary conditions are therefore of Robin type.

For these values of $v_0$, (30) and (16) imply the following asymptotic expression for $\phi_\ell$:

$$\hat{\phi}_{\omega,\ell}(\omega, \rho) \approx \mathcal{N}_\ell(\omega) \left[ \cos \zeta \hat{\phi}_\ell^{(D)}(\omega, \rho) + \sin \zeta \hat{\phi}_\ell^{(N)}(\omega, \rho) \right],$$

where

$$\hat{\phi}_\ell^{(D)}(\omega, \rho) = \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}+v} + J_\ell(\omega) \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}+v+2} + O \left[ \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}+v+4} \right],$$

$$\hat{\phi}_\ell^{(N)}(\omega, \rho) = \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}-v} + K_\ell(\omega) \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}-v+2} + O \left[ \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}-v+4} \right],$$

with

$$J_\ell(\omega) = \frac{a_1(\omega)b_1(\omega)}{1 + \nu} - \frac{n - 1 + 6\ell + 2v}{12},$$

$$K_\ell(\omega) = \frac{a_2(\omega)b_2(\omega)}{1 - \nu} - \frac{n - 1 + 6\ell - 2v}{12},$$

as $\rho \to \pi/2$. Upon substitution of (38) into (14), we obtain the following asymptotic expression for $\phi$:

$$\phi(t, \rho) \approx \cos \zeta \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}+v} \mathcal{F}_\ell(t)$$

$$+ \cos \zeta \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}+v+2} \mathcal{F}_D,\ell(t)$$

$$+ \sin \zeta \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}-v} \mathcal{F}_N,\ell(t)$$

$$+ \sin \zeta \left( \frac{\pi}{2} - \rho \right)^{\frac{d}{2}-v+2} \mathcal{F}_N,\ell(t),$$

where

$$\mathcal{F}_\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_\ell(\omega) N_\ell(\omega) e^{-i\omega t} d\omega, \quad (44)$$

$$\mathcal{F}_D,\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_\ell(\omega) N_\ell(\omega) J_\ell(\omega) e^{-i\omega t} d\omega, \quad (45)$$

$$\mathcal{F}_N,\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_\ell(\omega) N_\ell(\omega) K_\ell(\omega) e^{-i\omega t} d\omega. \quad (46)$$

Using the asymptotic form (43), (56) and (37), we get

$$\mathcal{F}_\infty \sim \lim_{\rho \to \pi/2} \sin \zeta \left\{ \cos \zeta A + \sin \zeta B \left( \frac{\pi}{2} - \rho \right)^{-2v} \right\} \times \left( \sum_{\ell} \frac{d}{dt} |\mathcal{F}_\ell(t)|^2 \right), \quad (47)$$

where

$$A = \frac{d}{2} - 2\xi (d + 1), \quad (48)$$

$$B = \left( \frac{1}{4} - \xi \right) (d - 2v) - \xi. \quad (49)$$

We immediately see that by imposing the Dirichlet boundary condition ($\zeta = 0$) the flow of energy across the infinity turns out to be zero.

On the other hand, when $\zeta \neq 0$, we must choose the coupling constant so that $B = 0$ in order that the energy flux be finite. This leads to

$$\mathcal{F}_\infty \sim \sin \zeta \cos \zeta A \sum_{\ell} \frac{d}{dt} |\mathcal{F}_\ell(t)|^2. \quad (50)$$

The integrals defining $\mathcal{F}_\ell(t)$, $\ell = 0, 1, 2, \ldots$, depend on the singularity structure of the functions $\mathcal{S}_\ell$, which, in turn, depend on the boundary conditions in the spacetime bulk and on the initial conditions of the system. As a result, except for very specific field configurations, we must impose the Neumann condition ($\zeta = \pi/2$) for the system to become effectively isolated. For general Robin conditions ($\zeta \neq 0$ and $\zeta \neq \pi/2$), the energy flux across the conformal boundary is generically not zero.

There is a notable case where the energy flux (50) can be zero without imposing either $\zeta = 0$ or $\zeta = \pi/2$. For the propagation of a single mode of frequency $\omega \in \mathbb{R}$, we have $\mathcal{F}_\ell(t) \sim e^{-i\omega t}$, and then, $d|\mathcal{F}_\ell(t)|^2/dt = 0$, and the energy flux vanishes for every Robin boundary condition $\zeta \in [0, \pi]$. However, for the propagation of two field modes, this conclusion is no longer true. More generally, if the scalar field is composed of a nontrivial superposition of modes of different frequencies, then $d|\mathcal{F}_\ell(t)|^2/dt \neq 0$.

In summary, the boundary conditions that make the system scalar field + spacetime effectively isolated in this case are as follows:

(i) $\zeta = 0$ (Dirichlet).

(ii) $\zeta = \pi/2$ (Neumann), together with $\zeta$ chosen such that $B = 0$.

In particular, for a minimally coupled field ($\xi = 0$), the Dirichlet boundary condition gives zero energy flux across the spatial infinity since in this case, $B \neq 0$.

The case $v^2 = 0$

As in the previous case, both solutions are square integrable near $\rho = \pi/2$ here. The allowed boundary conditions are therefore again of Robin type.

Moreover, the behavior of both $G_{\nu^{-1}Z_0}$ and $\partial(G_{\nu^{-1}Z_0})/\partial \rho$ are governed by $\sin \zeta$ for $v = 0$. Thus, one can interpret $\zeta = 0$ (or $\zeta = \pi$) as the simultaneous imposition of generalized Neumann and Dirichlet boundary conditions. Following the same steps as in the previous case, we find that the condition of zero flux again requires $\zeta = 0$ together with $\xi = (n-1)/4n$. 


The case $v^2 < 0$

We now consider the case when $v^2 < 0$, i.e., $v = i\eta$ with $\eta > 0$. Once again, both solutions are square integrable near $\rho = \pi/2$ here. The energy flow across the spatial infinity is now given by

$$\mathcal{F} \approx \lim_{\rho \to \pi/2} \sum_{\ell}(A_{\ell} \cos 2\zeta + B_{\ell} \sin 2\zeta + C_{\ell}), \quad (51)$$

where

$$A_{\ell} = \eta \text{Im}\left\{\mathcal{T}_\ell^*(t) \frac{d \mathcal{T}_\ell(t)}{dt}\right\}, \quad (52)$$

$$B_{\ell} = \frac{1}{2} \text{Re}\left\{\left[ (n + 2i\eta)(4\xi - 1) + 1 \right] \left( \frac{\pi}{2} - \rho \right)^{2\eta}\right\} \times \text{Re}\left\{\mathcal{T}_\ell^*(t) \frac{d \mathcal{T}_\ell(t)}{dt}\right\}, \quad (53)$$

$$C_{\ell} = \frac{1}{2}[1 + n(4\xi - 1)] \text{Re}\left\{\mathcal{T}_\ell^*(t) \frac{d \mathcal{T}_\ell(t)}{dt}\right\}. \quad (54)$$

Since the functions $\sin 2\zeta$ and $\cos 2\zeta$ are linearly independent, we conclude that, in general, the system cannot be treated as isolated for $v^2 < 0$.

Once again, a notable exception is given by the propagation of a single mode with frequency $\omega \in \mathbb{R}$. In this case, we have $\mathcal{T}(t) \sim e^{i\omega t}$, and therefore, $\text{Re}\{\mathcal{T}^*(t) [d \mathcal{T}(t)/dt]\} = 0$, which implies that the coefficients $B$ and $C$ in (51) both vanish. Then, by choosing the boundary condition as $\zeta = \pi/4$, we can cancel out the energy flux through the conformal boundary.

It is worth mentioning that when $M$ is not only asymptotically AdS, but $M = AdS$, the differential operator associated with equation (17) is unbounded below for $v^2 < 0$. As a result, one cannot find positive self-adjoint extensions of $i\mathcal{F}$ so that it is not possible to define a physically “reasonable” time evolution in this case. In general, one cannot make assertions concerning the positivity of the differential operator associated with the correspondent radial equation without detailed information about the bulk structure of spacetime. Indeed, the positivity of the differential operator may be somewhat subtle to be rigorously established even when the bulk structure is fully known.

Finally, we note that the calculations in this section could be performed using the canonical (non-improved) energy-momentum tensor,

$$\mathcal{T}_{\alpha\beta} = -\frac{1}{2} \left( \partial_{\alpha} \Phi \partial_{\beta} \Phi^* + \partial_{\beta} \Phi \partial_{\alpha} \Phi^* \right) - \frac{1}{2} g_{\alpha\beta} \left( g^{\rho\sigma} \partial_{\rho} \Phi \partial_{\sigma} \Phi^* + m_0^2 \Phi \Phi^* \right). \quad (55)$$

In this case, we find that (i) for $v^2 > 0$, only the Dirichlet boundary condition yields a zero energy flux across infinity and (ii) for $v^2 \leq 0$, the flux is generically nonzero even for the Dirichlet choice. These results are also what one would obtain by formally substituting $\xi = 0$ in the above calculations for the improved energy-momentum tensor.
as soon as we consider a superposition of modes of different real frequencies (or even a single mode of complex frequency), generic Robin boundary conditions are not compatible with zero flux at infinity and the results of Subsection III B are recovered.

For the sake of completeness, we repeat this analysis for the case of a real scalar field in Appendix III. The results are essentially the same, the only difference being that general Robin boundary conditions are not compatible with zero energy flux at infinity even in the case of a single mode.

We conclude this section by noting that our results do not depend on the bulk structure of the spacetime. Regardless of the bulk, the only boundary conditions at infinity that make the system effectively isolated are those of Dirichlet and Neumann types. Some particular field configurations may, of course, have zero flux without conforming to this rule. This is the case of a single mode of a complex scalar field with real frequency, for which the flux is zero irrespective of the choice of \( \zeta \).

IV. DISCUSSION

We have studied the asymptotic behavior of scalar fields in spacetimes which are asymptotically anti-de Sitter. We determined the boundary conditions at the spatial infinity for which there is no flow of energy at the conformal boundary. We showed that the only allowed choices that are consistent with this requirement are the generalized Dirichlet and Neumann boundary conditions (the latter with a specific choice of the coupling constant). This happens regardless of the theory in the spacetime bulk. The energy flux was calculated using the improved energy-momentum tensor (36). If we had used the canonical energy-momentum tensor (55) instead, only the Dirichlet boundary conditions would be compatible with zero flux at the conformal boundary.

In particular, Robin mixed boundary conditions, as considered, for instance, in Refs. 6 and 11–13 (although physically reasonable since they provide a fully deterministic dynamics), are not compatible with the requirement that the spacetime is an isolated system.

The case of an asymptotically AdS\(_2\) spacetime can be treated in a similar manner. The fundamental difference is that in this case, the spatial infinity has two distinct components so that, in order for the system to be isolated, one must demand the energy flow to be (separately) zero at each of the boundaries. We must then impose two independent conditions at each of the two boundaries. The zero flux condition constrains those to be, again, of Dirichlet and Neumann types.

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Appendix A: Principal and non-principal solutions

For \( v \in \mathbb{R} \), the function \( Z_l^{(D)} \) defined in Sec. III A is the only solution (up to a multiplicative factor) such that

\[
\lim_{\rho \to \pi/2} \left[ Z_l^{(D)}(\rho)/Z_l(\rho) \right] = 0 
\]

for any solution \( Z_l \) not proportional to \( Z_l^{(D)} \). A solution satisfying this condition is called a principal solution (at the endpoint \( \rho = \pi/2 \)). Solutions that are not proportional to \( Z_l^{(D)} \) are called non-principal (at the endpoint \( \rho = \pi/2 \)).

We note that non-principal solutions are not unique. In fact, if \( Z_l \) is a non-principal solution, then \( \tilde{Z}_l + \alpha Z_l^{(D)} \) is also a solution of this type for any \( \alpha \in \mathbb{R} \). It is interesting to ask what would change in our analysis if we replace \( Z_l^{(N)} \) of Sec. III A by another non-principal solution \( \tilde{Z}_l^{(N)} = Z_l^{(D)} + \gamma Z_l^{(N)} \), \( \gamma \in \mathbb{R} \).

In terms of the new set \( \{ Z_l^{(D)}, \tilde{Z}_l^{(N)} \} \), the general solution of (13) can be expressed as

\[
Z_l = \mathcal{N}_l \left[ \cos \zeta Z_l^{(D)} + \sin \zeta \tilde{Z}_l^{(N)} \right],
\]

and the condition \( \zeta = \pi/2 \) no longer selects the function given in (23). The value of \( \zeta \) that selects that function is now

\[
\cot \zeta = -\gamma.
\]

The energy flux calculated in terms of the new set of solutions is given by

\[
\mathcal{F}_\infty \sim \lim_{\rho \to \pi/2} \sin \zeta \left\{ (\cos \zeta + \gamma \sin \zeta) A + \sin \zeta B \left( \frac{\pi}{2} - \rho \right)^{-2\nu} \sum_{(\ell \gamma)} \frac{d}{dt} |\mathcal{F}_\ell(t)|^2 \right\}.
\]

From (A3), we see that the boundary conditions that cancel the energy flux across the conformal boundary are \( \zeta = 0 \) (Dirichlet) and \( \zeta = \tilde{\zeta} \) (along with \( \tilde{\xi} \) chosen such that \( B = 0 \)). Therefore, regardless of how the generalized Neumann condition is defined, the boundary conditions associated with zero flux at infinity are those that select the solutions \( Z_l^{(D)} \) and \( Z_l^{(N)} \) of Sec. III A.

Appendix B: Real scalar fields

We discuss in this appendix the behavior of the energy flux across the spatial infinity for real scalar fields. The improved energy-momentum tensor in this case is given by

\[
T_{\alpha \beta} = \partial_\alpha \Phi \partial_\beta \Phi - \frac{1}{2} g_{\alpha \beta} \left[ g^{\sigma \rho} \partial_\sigma \Phi \partial_\rho \Phi + m_\gamma^2 \Phi^2 \right] + \tilde{\xi} \left( \mathcal{D}_\alpha \mathcal{D}_\beta - g_{\alpha \beta} \Box - \nabla_\alpha \nabla_\beta \right) \Phi^2.
\]

(B1)
The counterparts for real scalar fields of the real and complex frequency cases of the main text are, respectively, given as follows

(i) \( \cos(\omega t + \delta) \) when \( \omega \in \mathbb{R} \);

(ii) \( e^{\omega t} \cos(\omega t + \delta) \) when \( \omega = \omega_R + i \omega_I \in \mathbb{C} \).

Let us consider case (i) separately. Let \( \Phi_1 \) be a mode with frequency \( \omega_1 \in \mathbb{R} \),

\[
\Phi_1(t, \rho, \varphi) = \phi_{\omega_1 \ell}(\rho) \cos(\omega_1 t + \delta_1) \times [C_1 \cos \ell \varphi + D_1 \sin \ell \varphi].
\]

(B2)

This leads to

\[
\phi_{\omega_1 \ell}(\rho) \approx \cos \zeta \phi_{\omega_1 \ell}^{(D)}(\rho) + \sin \zeta \phi_{\omega_1 \ell}^{(N)}(\rho),
\]

(B3)

\( j = 1, 2, \) as \( \rho \to \pi/2 \), where

\[
\phi_{\omega_1 \ell}^{(D)}(\rho) = (\sin \rho)^\ell (\cos \rho)^{1+\nu} \, _2F_1 \left( a_1, b_1; c_1; \cos^2 \rho \right),
\]

\( (B4) \)

\[
\phi_{\omega_1 \ell}^{(N)}(\rho) = (\sin \rho)^\ell (\cos \rho)^{1-\nu} \, _2F_1 \left( a_2, b_2; c_2; \cos^2 \rho \right),
\]

\( (B5) \)

and

\[
a_1 = \frac{1}{2} (1 + \ell + \nu - \omega_1),
\]

\( (B6) \)

\[
b_1 = \frac{1}{2} (1 + \ell + \nu + \omega_1),
\]

\( (B7) \)

\[
c_1 = 1 + \nu,
\]

\( (B8) \)

\[
a_2 = \frac{1}{2} (1 + \ell - \nu - \omega_2),
\]

\( (B9) \)

\[
b_2 = \frac{1}{2} (1 + \ell - \nu + \omega_2),
\]

\( (B10) \)

\[
c_2 = 1 - \nu.
\]

\( (B11) \)

The energy flux across the spatial infinity is then given by

\[
\mathcal{F} \approx \omega_1 \sin \left[ 2(\omega_1 t + \delta_1) \right] \sin \zeta \lim_{\rho \to \pi/2} \left\{ \cos \zeta (1 - 6\xi) + \sin \zeta B \left( \frac{\pi}{2} - \rho \right)^{-2\nu} \right\},
\]

\( (B12) \)

and we see that this is zero only for the Dirichlet boundary condition (\( \zeta = 0 \)) or the Neumann boundary condition (\( \zeta = \pi/2 \)) with \( \xi \) such that \( B = 0 \). This should be compared to the corresponding result for the complex field, Eq. 57, for which the flux associated with a single mode was found to be zero even for Robin conditions.

Now, consider the superposition of two modes (still in case (i)), \( \Phi_1 \) and \( \Phi_2 \), with

\[
\Phi_1(t, \rho, \varphi) = \phi_{\omega_1 \ell}(\rho) \cos(\omega_1 t + \delta_1) \times [C_1 \cos \ell \varphi + D_1 \sin \ell \varphi],
\]

\( (B13) \)

\[
\Phi_2(t, \rho, \varphi) = \phi_{\omega_2 \ell}(\rho) \cos(\omega_2 t + \delta_2) \times [C_2 \cos \ell \varphi + D_2 \sin \ell \varphi],
\]

\( (B14) \)

with \( \omega_1, \omega_2 \in \mathbb{R} \). The energy flow across the conformal infinity is now given by

\[
\mathcal{F} \sim \left[ \cos(\omega_1 t + \delta_1) + \cos(\omega_2 t + \delta_2) \right] \times [\omega_1 \sin(\omega_1 t + \delta_1) + \omega_2 \sin(\omega_2 t + \delta_2)] \sin \zeta \times \lim_{\rho \to \pi/2} \left\{ \cos \zeta (1 - 6\xi) + \sin \zeta B \left( \frac{\pi}{2} - \rho \right)^{-2\nu} \right\}.
\]

\( (B15) \)

Since the functions \( \sin \omega_1 t \) and \( \cos \omega_1 t \) are linearly independent, the only boundary conditions that do not violate the isolated system hypothesis are again of the Dirichlet and the Neumann types (the latter with \( \xi \) such that \( B = 0 \)).

We end by considering case (ii). The real scalar field mode in this case (the counterpart of the complex mode with complex frequency) is given by

\[
\Phi_1(t, \rho, \varphi) = Re \left[ \phi_{\omega_1 \ell}(\rho) \right] e^{\beta t} \cos(\alpha t + \delta_1) \times [C_1 \cos \ell \varphi + D_1 \sin \ell \varphi],
\]

\( (B16) \)

The energy flux through infinity is now

\[
\mathcal{F} \approx e^{2\beta t} \left[ \beta_1 \cos^2(\alpha_1 t + \delta_1) - \alpha_1 \cos(\alpha_1 t + \delta_1) \sin(\alpha_1 t + \delta_1) \right] 2 \sin \zeta \times \lim_{\rho \to \pi/2} \left\{ \cos \zeta (1 - 6\xi) + \sin \zeta B \left( \frac{\pi}{2} - \rho \right)^{-2\nu} \right\},
\]

\( (B17) \)

and we see again that the only conditions compatible with the hypothesis that the system is isolated are those of Neumann and Dirichlet (the latter with \( \xi \) such that \( B = 0 \)).
By “reasonable”, we mean precisely the conditions given in Refs. 2 and 5.

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