Light-Cone Approach to Random Surfaces
Embedded in Two Dimensions

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ABSTRACT

We review the recently proposed light-cone quantization of the matrix model which is expected to have a critical point describing 2-d quantum gravity coupled to \( c = 2 \) matter. In the \( N \to \infty \) limit, we derive a linear Schroedinger equation for the free string spectrum. Numerical study of this equation suggests that the spectrum is tachyonic, and that the string tension diverges at the critical point.

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1. Introduction

Recently there has been considerable renewed interest in large-$N$ matrix models. The one-dimensional hermitian matrix chain models have been solved in the double-scaling limit and identified with the $c < 1$ minimal models coupled to two-dimensional quantum gravity\cite{1,2}. In the same fashion the hermitian matrix quantum mechanics, which describes the $c = 1$ theory, has been solved both for non-compact \cite{3} and for circular target space \cite{4}. The $c = 1$ model can be interpreted in terms of $D = 2$ string theory \cite{5}, where the role of the extra dimension is played by the conformal factor of the world sheet quantum gravity. One should keep in mind, however, that the models with $c > 1$ are of much greater interest because they are expected to correspond to string theories in $D > 2$. These theories should have a much richer structure than in $D = 2$ because strings can exhibit transverse oscillations. In spite of some interesting new developments \cite{6,7}, there is little solid information about the $c > 1$ theories.

From the existing analytical and Monte Carlo studies there is some evidence that the discretized $c > 1$ models are in the branched polymer phase \cite{8} and do not lead to acceptable bosonic string theories. On the other hand, the continuum approach indicates that these models are tachyonic \cite{9}. It is often suggested that the branch polymer behavior is indicative of the presence of the tachyon. However, some of the discretized models should not be tachyonic due to reflection positivity \cite{8}. We believe that a deeper understanding of the $c > 1$ models is needed because, even though the simplest such theories are probably unacceptable, there may be modifications that lead to new interesting string models. With this idea in mind, Dalley and one of the authors \cite{10} have recently attempted a new approach to the theory of random surfaces embedded in two dimensions ($c = 2$). The corresponding matrix model is a two-dimensional super-renormalizable scalar field theory which can be quantized using the light-cone coordinates. The $N \rightarrow \infty$ limit naturally leads to a free string light-cone Schroedinger equation \cite{11}. In these notes we will review the light-cone quantization of this model, and will report on new numerical studies of the free string spectrum which suggest a different kind of behavior than what was anticipated in ref. 10.
2. Random Surfaces Embedded in One and Two Dimensions

Before proceeding to the \( c = 2 \) model, we will briefly review some of the lessons learned from the exactly soluble \( c = 1 \) matrix model for the purpose of comparison. The discretized approach to 2-d quantum gravity coupled to \( c = 1 \) matter \cite{12} is generated by the euclidean matrix quantum mechanics with the action

\[
\int dx \, \text{Tr} \left( \frac{1}{2} \left( \frac{\partial M}{\partial x} \right)^2 + \frac{1}{2\alpha'} M^2 - \frac{\lambda}{3\sqrt{N}} M^3 \right),
\]

where \( M(x) \) is an \( N \times N \) hermitian matrix. The connection of this matrix model with triangulated random surfaces follows, as usual, after identifying the Feynman graphs with the graphs dual to triangulations. The lattice link factor is the one-dimensional scalar propagator,

\[
G(x_i, x_j) = \frac{\sqrt{\alpha'}}{2} \exp \left( -\frac{|x_i - x_j|}{\sqrt{\alpha'}} \right)
\]

so that the parameter \( \alpha' \) sets the scale in the embedding dimension. The model (1) possesses a global \( SU(N) \) symmetry under \( M \rightarrow \Omega M \Omega^\dagger \). It is well-known that the \( SU(N) \) singlet spectrum is described by \( N \) free fermions moving in the cubic potential, with the Planck constant \( \sim 1/N \). In the leading WKB approximation the ground state energy is \( \sim N^2 \), which corresponds to the sum over surfaces of spherical topology. This sum is singular as \( \Delta \sim \lambda_c - \lambda \rightarrow 0 \), which is the value of the coupling where the Fermi level reaches the local maximum of the cubic potential,

\[
Z_0 = \frac{1}{\sqrt{\alpha'}} \frac{N^2 \Delta^2}{\ln \Delta}.
\]

The geometrical meaning of this singularity is that the Feynman graphs with very large numbers of vertices are becoming important. Near this singularity the universal continuum limit of the 2-d quantum gravity coupled to \( c = 1 \) matter can be defined. In this limit, the gaps in the low-lying spectrum go to zero as \( 1/|\ln \Delta| \). The underlying dispersion relation \cite{5}

\[
\alpha' E^2 - p^2_\phi = 0
\]

is characteristic of a massless two-dimensional field theory. The hidden \( \phi \)-dimension derives from the fluctuating conformal factor of the world sheet. It is worth emphasizing that
the string scale $\alpha'$ does not undergo any singular renormalization in the $\Delta \to 0$ limit and essentially corresponds to the “bare” string scale defined by the link factor (2).

An important fact about the $c = 1$ matrix model is the decoupling of the $SU(N)$ non-singlet states, whose energies diverge as $|\ln \Delta|$. As a result, only the singlet states are important for random surfaces embedded in a non-compact dimension, or in a circle of radius $R > R_c$ [4]. The decoupling of the non-singlet states corresponds physically to the confinement of Kosterlitz-Thouless vortices [13, 14].

In ref. 10 a generalization of the matrix model (1) to two dimensions was studied,

$$S_E = \int d^2 x \, \text{Tr} \left( \frac{1}{2} (\partial_\alpha M)^2 + \frac{1}{2} \mu M^2 - \frac{1}{3\sqrt{N}} \lambda M^3 \right),$$

where $M(x^0, x^1)$ is an $N \times N$ hermitian matrix field. Just as for $c = 1$, the Feynman graphs of this theory are identified with the graphs dual to triangulations. The only change is that now the lattice link factor is the two-dimensional scalar propagator,

$$G(\vec{x}_i, \vec{x}_j) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{i\vec{p} \cdot (\vec{x}_i - \vec{x}_j)}}{p^2 + \mu} = \frac{1}{2\pi} K_0(\sqrt{\mu} |\vec{x}_i - \vec{x}_j|),$$

where $K_0$ is a modified Bessel function. Thus, at the leading order in $N$, we obtain a sum over the planar triangulated random surfaces embedded in two dimensions. The logarithmic divergence of $G$ at small separations is very mild. If the tadpole graphs are discarded, as they should be because they do not correspond to good triangulations, then each separate Feynman graph is finite. This is similar to what we find in the matrix models for $c \leq 1$. For these types of theories there are general arguments [15] indicating that, for sufficiently small $\lambda/\mu$, the sum over planar graphs is finite. Therefore, we may look for singularities in various physical quantities at some critical value of the dimensionless parameter $\lambda/\mu$. The crucial question is whether, as for $c = 1$, a sensible string theory can be defined near this critical coupling. Although the answer is almost certainly negative, it is interesting to study the nature of the critical point. In these notes we will describe our recent numerical results which suggest that the spectrum of the theory is tachyonic, and that the string tension

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* Note that, as for $c = 1$, the coordinates are specified at the centers of the triangles.

† This is essentially because the number of planar graphs grows only exponentially with the number of vertices, and one can put exponential bounds on the value of each graph.
actually diverges at the critical point. We will also speculate on how to find a cure for this undesirable behavior.

3. Light-Cone Quantization

Although the \( c = 2 \) matrix model of eq. (3) is certainly more complex than the matrix quantum mechanics, we should still be able to take advantage of the simplifications that distinguish two-dimensional field theories. The method to pursue this that seems particularly convenient, is to continue \( x^0 \to ix^0 \) and to carry out light-cone quantization [16] of the resulting \((1 + 1)\)-dimensional field theory, treating \( x^+ = (x^0 + x^1)/\sqrt{2} \) as the time and \( x^- = (x^0 - x^1)/\sqrt{2} \) as the spatial variable. From the Minkowski signature action,

\[
S = \int dx^+ dx^- \Tr \left( \partial_+ M \partial_- M - \frac{1}{2} \mu M^2 + \frac{1}{3\sqrt{N}} \lambda M^3 \right),
\]

we derive the light-cone components of the total momentum

\[
P^+(x^+) = \int dx^- \Tr (\partial_- M)^2,
\]

\[
P^-(x^+) = \int dx^- \Tr \left( \frac{1}{2} \mu M^2 - \frac{\lambda}{3\sqrt{N}} M^3 \right).
\]

Upon quantization, the commutation relations are imposed at equal \( x^+ \):

\[
[M_{ij}(x^-), \partial_- M_{kl}(\tilde{x}^-)] = \frac{1}{2} i \delta(x^- - \tilde{x}^-) \delta_{il} \delta_{jk}.
\]

These commutation relations differ from the canonical by the extra factor of \( \frac{1}{2} \), which follows from the appropriate constrained quantization. This extra factor ensures, for instance, that \([P^+, M] = \partial_- M\). Expanding in Fourier modes\(\dagger\),

\[
M_{ij} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left( a_{ij}(k^+) e^{-ik^+x^-} + a_{ji}^\dagger(k^+) e^{ik^+x^-} \right)
\]

\(\dagger\) The symbol \(\dagger\) is always understood to have purely quantum meaning and does not act on indices.
we find the conventional oscillator algebra

\[ [a_{ij}(k^+), a_{lk}^\dagger(\tilde{k}^+)] = \delta(k^+ - \tilde{k}^+)\delta_{il}\delta_{jk}. \]  

(9)

After substituting eq. (8) into eq. (6) and normal ordering, we obtain

\[ P^+ = \int_0^\infty dk^+ k^+ a_{ij}^\dagger(k^+)a_{ij}(k^+), \]
\[ P^- = \frac{1}{2\mu}\int_0^\infty \frac{dk^+}{k^+}a_{ij}^\dagger(k^+)a_{ij}(k^+) - \frac{\lambda}{4\sqrt{N\pi}} \]
\[ \times \int_0^\infty \frac{dk_1^+ dk_2^+}{\sqrt{k_1^+ k_2^+(k_1^++k_2^+)^2}} \left\{ a_{ij}^\dagger(k_1^++k_2^+)a_{ik}(k_2^+)a_{kj}(k_1^+) + a_{ik}^\dagger(k_1^+)a_{kj}^\dagger(k_2^+)a_{ij}(k_1^++k_2^+) \right\} \]

(10)

(repeated indices are summed over). The normal ordering is equivalent to removing the tadpole graphs in the Lagrangian approach. \( P^+ \) commutes with \( P^- \) so that they can be diagonalized simultaneously. As usual in light-cone quantization, the vacuum \(|0\rangle\), which is defined by

\[ a_{ij}(k^+)|0\rangle = 0, \]  

(11)

is an eigenstate of the fully interacting light-cone Hamiltonian \( P^- \) with eigenvalue zero. This is related to the fact that the light-cone momentum \( k^+ \) is positive for all quanta. We will find that, for sufficiently large \( \lambda/\mu \), \(|0\rangle\) is not the ground state.

Our goal is to study the ground state and the low-lying excitations. Due to the global \( SU(N) \) symmetry, \( M \rightarrow \Omega^\dagger M\Omega \), the eigenstates can be classified according to their transformation properties. We expect the low-lying states to transform as singlets under the \( SU(N) \). A general singlet state, carrying light-cone momentum \( P^+ \), can be written as

\[ |\Psi(P^+)\rangle = \sum_{b=1}^{\infty} \int_0^{P^+} dk_1 dk_2 \ldots dk_b \delta \left( \sum_{i=1}^{b} k_i - P^+ \right) \]
\[ f_b(k_1, k_2, \ldots, k_b) N^{-b/2} \text{Tr}[a_{ij}^\dagger(k_1) \ldots a_{ij}^\dagger(k_b)]|0\rangle, \]

(12)

where we have dropped the superscripts + on \( k_i \) for brevity. Due to the cyclic property of
the trace, the functions $f_b$ trivially obey the cyclic symmetry

$$f_b(k_1, k_2, \ldots, k_b) = f_b(k_b, k_1, \ldots, k_{b-1}) = \ldots = f_b(k_2, k_3, \ldots, k_b, k_1)$$

From the normalization condition

$$\langle \Psi(P^+) | \Psi(P^+) \rangle = \delta(P^+ - P'^+)$$

we obtain

$$\sum_{b=1}^{\infty} \int_{0}^{P^+} dk_1 dk_2 \ldots dk_b \delta \left( \sum_{i=1}^{b} k_i - P^+ \right) |f_b(k_1, k_2, \ldots, k_b)|^2 = 1.$$  \hspace{1cm} (14)

The singlet states play a special role in this theory: they have no $SU(N)$ degeneracy factors and can be thought of as closed strings. Each $a^\dagger_{ij}(k)$ creates a string bit carrying longitudinal momentum $k$, and a state of the form

$$\int_{0}^{P^+} dk_1 dk_2 \ldots dk_b \delta \left( \sum_{i=1}^{b} k_i - P^+ \right) f_b(k_1, k_2, \ldots, k_b) N^{-b/2} \text{Tr}[a^\dagger_{1}(k_1) \ldots a^\dagger_{b}(k_b)]|0 \rangle.$$ 

has the interpretation of a closed string which is a bound state of $b$ string bits. The function $f_b(k_1, k_2, \ldots, k_b)$ superposes states with different distributions of longitudinal momentum along the string. If a large-$N$ model is to correspond to a reasonable string theory, it should be energetically favorable for all the oscillator indices to be contracted, so that the quanta of the matrix field are bound into strings. In fact, if there are finite energy states with free indices, then at sufficiently high energy the composite strings are unstable with respect to disintegration into bits. As we will discuss in the conclusion, this disease is likely to occur in any model with no local gauge symmetry.

For now, we restrict ourselves to the study of $SU(N)$ singlets, which are the good string states. When acting on such states, the light-cone Hamiltonian $P^-$ has an important property: in the limit $N \to \infty$, $P^-$ takes single closed string states into single closed string states. One can easily check that the terms that convert one closed string into two closed strings
(two oscillator traces acting on the vacuum) are of order $1/N$. Thus, as expected, the string coupling constant is $\sim 1/N$, and sending it to zero allows us to study the spectrum of free closed string states. The resulting linear light-cone Schrödinger equation can be expressed as a set of integral equations for the functions $f_b$,

$$P^- f_b(k_1, k_2, \ldots, k_b) = \frac{\mu}{2} f_b(k_1, k_2, \ldots, k_b) \sum_{i=1}^{b} \frac{1}{k_i}$$

$$- \frac{\lambda}{4 \sqrt{\pi}} \left\{ \int_{0}^{k_1} dk'_1 f_{b+1}(k'_1, k_1 - k'_1, k_2, \ldots, k_b) \sqrt{k_1 k'_1 (k_1 - k'_1)} + \int_{0}^{k_2} dk'_2 f_{b+1}(k_1, k_2, k_2 - k'_2, k_3, \ldots, k_b) \sqrt{k_2 k'_2 (k_2 - k'_2)} + \cdots \right. \right.$$

$$+ \left. \frac{f_{b-1}(k_1 + k_2, k_3, \ldots, k_b)}{\sqrt{k_1 k_2 (k_1 + k_2)}} + \frac{f_{b-1}(k_1, k_2 + k_3, \ldots, k_b)}{\sqrt{k_2 k_3 (k_2 + k_3)}} + \cdots \right\} .$$

(15)

The terms proportional to $\lambda$ arise either from replacing two neighboring bits by one or from dividing a bit into two neighboring bits. The Lorentz invariance of eq. (15) can be made explicit if we introduce the longitudinal momentum fractions $x_i = k_i/P^+$, and rewrite the equation in terms of the functions

$$\tilde{f}_b(x_1, x_2, \ldots, x_b) = (P^+)^{(b-1)/2} f_b(x_1 P^+, x_2 P^+, \ldots, x_b P^+) .$$

From eq. (15) it is not hard to derive

$$2P^+ P^- \tilde{f}_b(x_1, x_2, \ldots, x_b) = \mu \tilde{f}_b(x_1, x_2, \ldots, x_b) \sum_{i=1}^{b} \frac{1}{x_i}$$

$$- \frac{\lambda}{2 \sqrt{\pi}} \left\{ \int_{0}^{x_1} dx'_1 \tilde{f}_{b+1}(x'_1, x_1 - x'_1, x_2, \ldots, x_b) \sqrt{x_1 x'_1 (x_1 - x'_1)} + \int_{0}^{x_2} dx'_2 \tilde{f}_{b+1}(x_1, x_2, x_2 - x'_2, x_3, \ldots, x_b) \sqrt{x_2 x'_2 (x_2 - x'_2)} + \cdots \right. \right.$$

$$+ \left. \frac{\tilde{f}_{b-1}(x_1 + x_2, x_3, \ldots, x_b)}{\sqrt{x_1 x_2 (x_1 + x_2)}} + \frac{\tilde{f}_{b-1}(x_1, x_2 + x_3, \ldots, x_b)}{\sqrt{x_2 x_3 (x_2 + x_3)}} + \cdots \right\} ,$$

(16)

and from eq. (14) we find that the functions $\tilde{f}_b$ are normalized according to

$$\sum_{b=1}^{\infty} b \int_{0}^{1} dx_1 dx_2 \ldots dx_b \delta \left( \sum_{i=1}^{b} x_i - 1 \right) |\tilde{f}_b(x_1, x_2, \ldots, x_b)|^2 = 1 .$$

(17)

$2P^+ P^-$ is manifestly Lorentz invariant, and its eigenvalues are the squared masses of the closed string states.
Eqs. (16) and (17) define a rather unusual eigenvalue problem which involves an infinite number of unknown functions $\tilde{f}_b$. Each of these functions depends on a number of real variables $x_i$ confined to the interval $[0, 1]$, with the constraint $\sum_{i=1}^{b} x_i = 1$. An important issue is the behavior of the functions as $x_i \to 0$. For $\mu > 0$ the behavior

$$\lim_{x_1 \to 0} \tilde{f}_b(x_1, x_2, \ldots, x_b) \neq 0$$

is inconsistent with a finite energy eigenstate because the contribution $\sim \mu$ blows up near $x_1 = 0$ faster than the other terms. Guided by this consideration, we will assume that the functions $\tilde{f}_b$ approach zero as the variables approach the end-points of the interval.

The system of equations (16) is rather difficult, and we do not know its exact solution. For that reason, we will attempt to estimate the spectrum numerically, after introducing a cut-off. In light-cone quantization, a particularly convenient method is to replace the continuous momentum fractions $x$ by a discrete set $n/K$, where the positive integer $K$ is sent to infinity as the cut-off is removed [17,18]. Thus, we replace the functions $\tilde{f}_b(x_1, x_2, \ldots, x_b)$ by

$$g_b(n_1, n_2, \ldots, n_b) = \left(K\right)^{(1-b)/2} \tilde{f}_b(n_1/K, n_2/K, \ldots, n_b/K)$$

and

$$\int_{0}^{1} dx \to \frac{1}{K} \sum_{n=1}^{K}.$$

According to our discussion of the boundary conditions, we assume that $g_b$ is non-vanishing only if all its arguments are positive integers ($n = 0$ is excluded). The cut-off eigenvalue problem assumes the form

$$\frac{2P^+ P^-}{\mu} g_b(n_1, n_2, \ldots, n_b) = Kg_b(n_1, n_2, \ldots, n_b) \sum_{i=1}^{b} \frac{1}{n_i}$$

$$-Ky \left\{ \sum_{n'_1=1}^{n_1-1} \frac{g_{b+1}(n'_1, n_1 - n'_1, n_2, \ldots, n_b)}{\sqrt{n_1 n'_1 (n_1 - n'_1)}} + \sum_{n'_2=1}^{n_2-1} \frac{g_{b+1}(n_1, n'_2, n_2 - n'_2, n_3, \ldots, n_b)}{\sqrt{n_2 n'_2 (n_2 - n'_2)}} + \ldots 
+ \frac{g_{b-1}(n_1 + n_2, n_3, \ldots, n_b)}{\sqrt{n_1 n_2 (n_1 + n_2)}} + \frac{g_{b-1}(n_1, n_2 + n_3, \ldots, n_b)}{\sqrt{n_2 n_3 (n_2 + n_3)}} + \ldots \right\} ,$$

(18)
where \( y = \lambda / (2\sqrt{\pi \mu}) \) is the dimensionless coupling. The normalization condition becomes

\[
\sum_{b=1}^{\infty} \sum_{n_1=1}^{K-1} \ldots \sum_{n_{b-1}=1}^{K-1} |g_b(n_1, n_2, \ldots, n_{b-1}, K - \sum_{i=1}^{b-1} n_i)|^2 = 1.
\]  

(19)

The cut-off eigenvalue problem reduces to matrix diagonalization because the \( g \)'s contain only a finite number of degrees of freedom, which is equal to the number of partitions of \( K \) into positive integers, modulo cyclic permutations. All such partitions can be constructed recursively with the help of a computer. We find that for \( K = 10, 11, 12, 13, 14, 15, 16, 17, 18 \) the number of degrees of freedom is 107, 187, 351, 631, 1181, 2191, 4115, 7711, 14601 respectively. As could have been expected, it grows roughly exponentially with \( K \). The maximum number of bits is equal to \( K \). As the cut-off \( K \) is taken to infinity, the possible values of the longitudinal momentum fractions \( x_i \) densely populate the interval \((0, 1)\), which corresponds to the continuum limit, i.e. the eigenvalue problem of eq. (16).

If we regard all the independent components of \( g_b(n_1, n_2, \ldots, n_b) \), with \( b = 1, 2, \ldots, K \), as components of a vector, then the eigenvalue problem of eq. (18) is equivalent to diagonalizing the matrix

\[
\frac{2P^+P^-}{\mu} = K (V - yT).
\]  

(20)

The matrix \( V \) is diagonal, while \( T \) is off-diagonal, connecting string states whole length (number of bits) differs by 1. The entries of \( V \) and \( T \) can be easily read off from eqs. (18) and (19). Alternatively, as shown in ref. 10, \( V, T \), and a basis of normalized states can be constructed in terms of creation and annihilation operators.

4. Numerical Work

The objective of our numerical approach is to compute the low-lying spectrum of the matrix (20). First we fix the parameter \( y \), and vary the cut-off \( K \) from 9 to 15, which is the highest value accessible to us at present. Then, for each \( y \), an extrapolation of the data to \( K \to \infty \) gives us an estimate of the spectrum of the continuum eigenvalue problem of eq. (16). Finally we plot the estimated continuum eigenvalues versus \( y \) looking for critical behavior at some \( y_c \).
What kind of singularity can we expect? Based on our experience with the $c = 1$ matrix model, we might expect that the spectrum becomes dense at $y = y_c$, indicating the appearance of continuous Liouville momentum $p_\phi$. The resulting string theory would be 3-dimensional, with its spectrum given by

$$\frac{2P^+P^-}{T} = p_\phi^2 + 4r - \frac{1}{6}$$  \hspace{1cm} (21)

where $r$ runs over non-negative integers, and the tachyonic ground state energy is found from the standard formula $(2 - D)/6$. $T$ is the string tension in the embedding dimensions, and $\mu$ can be thought of as the “bare” string tension. It is possible that $T$ exhibits some dependence on $y$, and it can even become singular as $y \to y_c$. Depending on the behavior of $T$ near the critical point, there are two very different possibilities:

1) The string tension $T$ is finite at the critical point $y_c$. Then eq. (21) suggests that the spectrum of eq. (16) should become continuous at $y = y_c$, starting at a tachyonic value. This is the possibility suggested in ref. 10.

2) $T$ diverges at the critical point, so that all the low-lying eigenvalues of eq. (16) tend to $-\infty$. This possibility was not discussed in ref. 10, but the divergence of $T$ has in fact been advocated in earlier literature [8].

Now we present our current numerical results. Unfortunately, the nature of the numerical problem makes it hard to decide with certainty whether 1) or 2) is correct, unless values of $K$ considerably higher than 15 can be reached. This is impossible with our present means, but may be feasible on a supercomputer. Judging by the available data, 2) is the more likely possibility.

In fig. 1 we plot the lowest three eigenvalues of eq. (18) versus $K$ for $y = 0.2$. Here the convergence to the continuum limit is found to be quite fast. In general, we expect the dependence on $K$ to be of the form

$$m^2_i(y, K) = m^2_i(y, \infty) + \sum_{n=1}^{\infty} c_{i,n}(y) K^{-n}$$

In order to estimate the continuum limit, we fitted the dependence on $K$ at fixed $y$ to a ratio of two polynomials in $1/K$. This fit allows us to extrapolate fig. 1 to arbitrarily large values of $K$. This extrapolation saturates rapidly, showing intuitively expected behavior.
In fig. 2 we plot the lowest eigenvalue, $m_1^2/\mu$, versus $K$ for $y = 0.2, 0.52, 0.65$. These plots show the dependence of the rate of convergence on $y$. For $y = 0.52$ the convergence is much slower than for $y = 0.2$. For $y = 0.65$, instead of being concave, the graph becomes convex, so that the points no longer appear to converge to a finite limit as $K \to \infty$.

In fig. 3 we show extrapolations of the lowest eigenvalue to large $K$. For $y = 0.50$ (fig. 3a) the extrapolation converges to a value which is far below that found for $K = 15$. In fact, the continuum value is tachyonic, as anticipated from a $D = 3$ string theory. For $y = 0.55$ (fig. 3b) the extrapolation fails to converge. It follows that there should be a critical value of $y$, which lies between 0.50 and 0.55, where the extrapolated value of $m_1^2/\mu$ begins to diverge towards $-\infty$. This is indeed what happens.

In fig. 4 we plot the lowest three eigenvalues, extrapolated to infinite $K$, versus $y$. While for $y = 0$ the second and third eigenvalues are degenerate, for $y > 0$ they are separated by a finite gap, which is at first too small to be visible on the figure. The discreteness of the spectrum for $y > 0$ indicates that the quanta are indeed bound into strings. The lowest eigenvalue, $m_1^2/\mu$, begins to dip rapidly for $y$ beyond 0.45. First it becomes tachyonic, and then diverges as $y$ approaches 0.53. If we identify this value of $y$ with $y_c$, then the divergence of the lowest eigenvalue seems consistent with the divergence of the string tension suggested in the possibility 2). Of course, it may be that the gaps in the spectrum, $m_2^2 - m_1^2$, $m_3^2 - m_2^2$, etc. vanish before the eigenvalues themselves diverge, but fig. 4 indicates otherwise. We find that the extrapolated $m_2^2/\mu$ plotted vs. $y$ exhibits behavior similar to $m_1^2/\mu$, but diverges at a higher value of $y$, and the gap $m_2^2 - m_1^2$ does not vanish anywhere. If possibility 2) is correct, then all the eigenvalues should diverge at the same critical value of $y$. The fact that this is not the case in fig. 4 may be attributed to the lack of precision of our extrapolation. We find, however, that the places where $m_1^2$ and $m_2^2$ diverge become closer as the extrapolation includes the data for higher $K$. In general, we have to add a cautionary note that the precise shape of fig. 4 is not reliable. As $y$ approaches $y_c$, the rate of convergence towards the continuum limit becomes slower and slower, so that higher values of $K$ are needed for a reliable estimate. Further numerical work is crucial for testing our conjectures. We hope, however, that our numerical studies provide a clue about the universal features of the continuum limit.

5. Discussion
The possibility 2) suggested in the previous section and supported, to some extent, by the available data can be interpreted as an infinite multiplicative renormalization of the string tension. In other words, the effective string tension at the critical point $y = y_c$ is infinitely larger than the bare string tension $\mu$. A very similar scenario was advocated in ref. 8 and interpreted as the branched polymer phase of random surfaces. Our results are different, though, in that the spectrum at the critical point is tachyonic, as expected from the continuum reasoning in eq. (21).

The divergence of the string tension at $y = y_c$ could probably be cured by appropriately scaling the bare string tension $\mu$ to zero. However, the resulting string theory with finite spectrum would still be tachyonic.

Another problem, which is perhaps even more severe, comes from the states that transform non-trivially under $SU(N)$, such as the adjoint representation states of the form

$$\sum_{b=1}^{\infty} \int_0^{P^+} dk_1 \ldots dk_b \delta \left( \sum_{i=1}^b k_i - P^+ \right) f_b(k_1, \ldots, k_b) N^{(1-b)/2} a^\dagger_{ij}(k_1) a^\dagger_{jk}(k_2) \ldots a^\dagger_{rs}(k_{b-1}) a^\dagger_{st}(k_b) |0\rangle.$$  

(22)

As discussed previously, the $SU(N)$ non-singlet states cannot be identified with closed strings. One indication of that is their diverging degeneracy factors in the $N \rightarrow \infty$ limit. Perhaps the non-singlets can be thought of as closed strings that have disintegrated into separate bits. Of course, one way of preventing this is through a confinement phenomenon which would push their energies to infinity in the continuum limit. In fact, confinement is at work in the $c = 1$ string theory, where the non-singlet energies diverge logarithmically in the cut-off [13, 14].

Could the $c = 2$ model of eq. (3) also expel the non-singlet states to infinite energy? The answer is negative, as can be seen from a simple variational calculation. Taking $a^\dagger_{ij}(P^+) |0\rangle$ as the variational state, and evaluating the matrix element of $2P^+P^-/\mu$, gives an upper bound of 1 for the lowest eigenvalue. Therefore, there is no confinement, and the non-singlet states probably cause additional problems for the interacting string theory. Some heuristic arguments with similar conclusions were made in the context of $c > 1$ models with discrete target spaces [13]. The proliferation of the non-singlets was associated with the Kosterlitz-Thouless vortices which wind around the target space plaquettes of lattice size. An advantage of the light-cone approach is that the non-singlet spectrum can be found with
the same technique as the singlet spectrum. It would be quite useful to carry out a more
detailed numerical study of the spectrum in the adjoint representation.

The lack of confinement of the non-singlet states can be traced to the fact that the action
(3) has only the global $SU(N)$ symmetry, which is not gauged. Gauging the $SU(N)$ leads
to the confinement, and the non-singlets are pushed to infinite energy [19]. The effect of
gauging the $SU(N)$ symmetry on the two-dimensional theory of eq. (3) is to give the action

$$S_{gauged} = \int d^2x \, \text{Tr} \left( \frac{1}{4g^2} F_{\alpha\beta}^2 + \frac{1}{2} (\partial_\alpha M + i[A_\alpha, M])^2 + \frac{1}{2} \mu M^2 - \frac{1}{3} \frac{\lambda}{\sqrt{N}} M^3 \right). \tag{23}$$

Light-cone quantization of this action was investigated in ref. 19. There a linear light-
cone Schroedinger equation for the $N \to \infty$ limit, similar to eq. (15), was derived. At the
moment we see no major obstacles to studying this equation using the discretized longitudinal
momentum cut-off. Perhaps, the theory (23) has new types of critical behavior which lead
to more interesting string theories than those originating from the model (3).

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FIGURE CAPTIONS
1. The lowest three eigenvalues of eq. (18) versus $K$ for $y = 0.2$.
2. The plots of the lowest eigenvalue of eq. (18) versus $K$ for $y = 0.2$, 0.52, 0.65.
3. Extrapolation to large $K$ of the lowest eigenvalue for a) $y = 0.50$; b) $y = 0.55$.
4. The lowest three eigenvalues, extrapolated to infinite $K$, versus $y$. 