DERIVATIVE ESTIMATES FOR SCREENED VLASOV-POISSON SYSTEM AROUND PENROSE-STABLE EQUILIBRIA

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Abstract. In this paper, we establish derivative estimates for the Vlasov-Poisson system with screening interactions around Penrose-stable equilibria on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$, with dimension $d \geq 3$. In particular, we establish the optimal decay estimates for higher derivatives of the density of the perturbed system, precisely like the free transport, up to a log correction in time. This extends the recent work [13] by Han-Kwan, Nguyen and Rousset to higher derivatives of the density. The proof makes use of several key observations from [13] on the structure of the forcing term in the linear problem, with induction arguments to classify all the terms appearing in the derivative estimates.

1. Introduction.

1.1. The system. In this paper, we consider the screened Vlasov-Poisson system on the phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with the dimension $d \geq 3$:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0$$

(1)

Here $f = f(t, x, v) \geq 0$ is the probability distribution of charged particles in plasma,

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$$

is the electric charge density, and

$$E(t, x) = -\nabla_x (1 - \Delta_x)^{-1}(\rho - 1)$$

is electric field. This model has been investigated in physical literatures [6, 7, 2] and also recent mathematical works [5, 13]. We refer the readers to [19, 20, 15, 9, 17] for global existence and regularity results. The system (1) has

$$(f, \rho, E) = (\mu(v), 1, 0)$$

as a steady solution for any smooth and decaying function $\mu(v) \geq 0$ with the normalized condition

$$\int_{\mathbb{R}^d} \mu(v) dv = 1.$$
We assume that $\mu(v)$ satisfies the Penrose stability condition ($\nabla v \mu$ denotes the Fourier transform):

$$\inf_{\tau \leq 0} \inf_{\xi \in \mathbb{R}^d} \left| 1 - \int_0^\infty e^{-i\tau s} \frac{1}{1 + |\xi|^2} t^\xi \cdot \nabla_v \mu(s \xi) ds \right| \geq \kappa \quad \text{for} \quad \kappa > 0,$$

(2)

which is classical in the study of Landau damping [18, 4], where the authors justify the asymptotic stability for homogeneous equilibria that satisfies (2) on the phase space $\mathbb{T}^d \times \mathbb{R}^d$, under analytic and Gevrey perturbations for the Vlasov-Poisson system. The stability condition (2) is also natural in studying the quasineutral limit of the Vlasov-Poisson equations [14, 11, 10] and the long time estimates for the Vlasov-Maxwell equations [12].

1.2. Previous works. In [5], Bedrossian, Masmoudi and Mouhot justify the asymptotic stability of the equilibria when $\mu(v)$ satisfies the condition (2). The proof is inspired by [4] for Landau damping on the confined phase space $\mathbb{T}^d \times \mathbb{R}^d$. Using the dispersive mechanism in Fourier space to control the plasma echo resonance, the authors in [5] prove that the Fourier mode of the density $\hat{\rho}(t, \xi)$ decays like $1 / t^\delta$ if the initial perturbation is in Sobolev space of high regularity order $N$ and some $\delta \in (0, N)$. The decay is far from being optimal, as the dispersion in the physical space was not taken into account.

In the recent work [13], Han-Kwan, Nguyen, and Rousset revisit the asymptotic stability of equilibria that satisfy the condition (2). They obtain the decay estimates for the perturbed electric charge density $\rho(t)$ as follows:

$$\|\rho(t)\|_{L^1} + \langle t \rangle \|\nabla_x \rho(t)\|_{L^1} + \langle t \rangle^d \|\rho(t)\|_{L^\infty} + \langle t \rangle^{d+1} \|\nabla_x \rho(t)\|_{L^\infty} \lesssim \varepsilon_0 \log(1 + t),$$

$$\langle t \rangle = \sqrt{t^2 + 1}.$$

Unlike [5], which relies on the nonlinear energy estimates, the authors in [13] use direct dispersive mechanism in the physical space, which is like the free transport up to $\log(t)$. This is achieved by a pointwise dispersive estimate, directly on the resolvent kernel for the linearized system around non-zero stable equilibria $\mu(v)$. This is followed by solving the equations by characteristic methods, inspired from the classical work of Bardos and Degond [3]. At the same time, the authors in [13] are able to propagate $C^1$ norm for the initial data and thus allow more general perturbations. We note that the dispersive mechanism of the free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}^d \times \mathbb{R}^d$ is also one of the key ingredients in the classical results of Bardos and Degond [3] in 1985, where they study the asymptotic stability of Vlasov-Poisson around vacuum ($\mu(v) = 0$).

Regarding the stability of vacuum (when $\mu(v) = 0$) for the unscreened Vlasov-Poisson systems, we refer the readers to the work [16] for the extension of Bardos-Degond results for optimal decays of higher derivatives. In [21], Smulevici obtains the spatial decay by the vector field methods. In [22], Wang justifies the stability of vacuum for Vlasov-Poisson by Fourier methods. In [8], Choi, Ha and Lee justify the same result for 2D screened Vlasov-Poisson.

2. Main results.

2.1. Main theorem. In this paper, we will give decay estimates for higher derivatives of $\rho(t)$, under the assumption that the initial perturbation $f_0$ is small in suitable Sobolev space and for $\mu(v)$ satisfying decaying assumption: Given any $m \in \mathbb{N}$
and $M > 0$, there exists $C_{m,M} > 0$ such that
\[|\nabla^n_x \mu(v)| \leq C_{m,M} (v)^{-M} \text{ for all } v \in \mathbb{R}^d.\] (3)

The equation for the perturbed quantities around the equilibria of (1) reads
\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu &= -E \cdot \nabla_v f, \\
E &= -\nabla_x (1 - \Delta_x)^{-1} \rho,
\end{aligned}
\] (4)

Our main theorem is as follows:

**Theorem 2.1.** Let $N > 1$ be an integer. Let $\mu(v) \geq 0$ be sufficiently smooth and satisfy the Penrose stability condition (2), the decaying bound (3) and the normalized condition $\int_{\mathbb{R}^d} \mu(v) dv = 1$. There exists $\varepsilon_0 > 0$ such that for all $f_0(x,v)$ satisfying the smallness assumption
\[
\max_{0 \leq k \leq N} \left(\|\nabla^k_x f_0\|_{L^1_x} + \|\nabla^k_x \rho_0\|_{L^\infty_x}\right) \leq \varepsilon_0,
\]
the solution $f(t,x,v)$ to the equations (4) with initial data $f|_{t=0} = f_0(x,v)$ satisfies
\[
\max_{0 \leq k \leq N} \left(\|\nabla^k_x \rho(t)\|_{L^1} + \|\nabla^k_x \rho(t)\|_{L^\infty}\right) \lesssim \varepsilon_0 (1 + t)\]
where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v) dv$. Here the norm $\| \cdot \|_{L^p_t L^q_x}$ is defined by
\[
\|g\|_{L^p_t L^q_x} = \left(\int_{\mathbb{R}^d} \|g(x,v)\|_{L^q_x}^p dx\right)^{1/p}.
\]

2.2. **Motivation.** The improved decays for higher derivatives of $\rho(t)$ can be seen from the free transport equation
\[
\partial_t f_{\text{free}} + v \cdot \nabla_x f_{\text{free}} = 0, \quad f|_{t=0} = f_0.
\]

The solution is given by
\[
f_{\text{free}} = f_0(x - tv,v), \quad \rho_{\text{free}}(t,x) = \int_{\mathbb{R}^d} f_0(x - tv,v) dv.
\]

Making the change of variables $w = x - tv$, we obtain
\[
\rho_{\text{free}}(t,x) = \int_{\mathbb{R}^d} f_0 \left( w, \frac{x - w}{t} \right) t^{-d} dw.
\] (5)

Hence
\[
\|\rho_{\text{free}}(t)\|_{L^\infty} \leq t^{-d} \|f_0\|_{L^1_x L^\infty_v} \quad \text{and} \quad \|\rho_{\text{free}}(t)\|_{L^1} \leq \|f_0\|_{L^1_x L^1_v}.
\]

Thus for the free transport equations, $\rho_{\text{free}}(t)$ decays like $t^{-d}$ in $L^\infty$. This dispersive mechanism for the free transport (which can be seen as linearized Vlasov-Poisson around zero) is one of the key ingredients in the classical work [3] by Bardos and Degond.

Now we discuss the decay for one derivative of $\rho_{\text{free}}(t)$. Taking $\nabla_x$ on both sides of (5), we get
\[
\nabla_x \rho_{\text{free}}(t) = t^{-d-1} \int_{\mathbb{R}^d} \nabla_{v} f_0 \left( w, \frac{x - w}{t} \right) dw,
\]
and hence
\[
\|\nabla_x \rho_{\text{free}}(t)\|_{L^\infty} \leq t^{-(d+1)} \|\nabla_v f_0\|_{L^1_x L^\infty_v} \quad \text{and} \quad \|\nabla_x \rho_{\text{free}}(t)\|_{L^1} \leq t^{-1} \|\nabla_v f_0\|_{L^1_x L^1_v}.
\]

This implies that $\nabla_x \rho_{\text{free}}(t)$ decays like $t^{-d-1}$ in $L^\infty$ and $t^{-1}$ in $L^1$. This decaying mechanism is in fact extended to the Vlasov-Poisson system around vacuum by
Hwang, Rendall and Velazquez [16]. In particular, the authors in [16] establish the improved decay estimates
\[ \| \nabla^k \rho(t) \|_{L^\infty} \lesssim (1 + t)^{-d-k} \quad \text{for any} \quad k \geq 0 \] (6)
for small initial data near vacuum.

The natural question is whether the estimate (6) still holds for solution to the screened Vlasov-Poisson under small perturbation around nonzero homogeneous equilibria \( \mu(v) \) that satisfies Penrose condition (2). In this paper, we prove that this is essentially the case, namely \( \nabla^k \rho(t) \) decays like \( \nabla^k \rho_{\text{free}}(t) \), up to a log in time correction.

2.3. Outline of the proof. The set up: Using the characteristics
\[
\begin{align*}
\frac{d}{dt} X_{s,t}(x,v) &= V_{s,t}(x,v), \quad X_{t,t}(x,v) = x, \\
\frac{d}{dt} V_{s,t}(x,v) &= E(s, X_{s,t}(x,v)), \quad V_{t,t}(x,v) = v,
\end{align*}
\] (7)
the solution \( f(t,x,v) \) to the perturbation equation (4) can be written as
\[ f(t,x,v) = f_0(X_0,t(x,v),V_0,t(x,v)) - \int_0^t E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v)) ds. \]
Integrating both sides in \( v \in \mathbb{R}^d \) and using the fact that \( E = -\nabla_x (1 - \Delta_x)^{-1} \rho \), we get
\[ \rho(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x (1 - \Delta_x)^{-1} \rho(s, x - (t-s)v) \cdot \nabla_v \mu(v) dv ds + S(t,x) \] (8)
where \( S(t,x) \) is given by
\[ S(t,x) = \int_{\mathbb{R}^d} f_0(X_0,t(x,v),V_0,t(x,v)) dv + \int_0^t \int_{\mathbb{R}^d} E(s, x - (t-s)v) \cdot \nabla_v \mu(v) dv ds - \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v)) dv ds. \] (9)
Taking spacetime Fourier transform, one can express \( \tilde{\rho}(\tau,\xi) \) as
\[ \tilde{\rho}(\tau,\xi) = \frac{1}{1 - \tilde{K}(\tau,\xi)} \tilde{S}(\tau,\xi), \quad \text{where} \quad \tilde{K}(\tau,\xi) = \int_{0}^{\infty} e^{-i\tau t} \frac{i\xi}{1 + |\xi|^2} \cdot \nabla_v \mu(t\xi) dt. \] (10)
The Penrose stability condition (2) is equivalent to
\[ \inf_{\exists (\tau) \leq 0} \inf_{\xi \in \mathbb{R}^d} |1 - \tilde{K}(\tau,\xi)| \geq \kappa \quad \text{for some} \quad \kappa > 0, \]
which is to avoid the singularity in (10). Following [13], we can write in the original \((t,x)\) variables as follows:
\[ \rho(t) = S(t) + \int_0^t G(t-s) *_x S(s) ds \] (11)
where
\[ G(t,x) = \mathcal{F}_{(\tau,\xi) \to (t,x)}^{-1} \left( \frac{\tilde{K}(\tau,\xi)}{1 - \tilde{K}(\tau,\xi)} \right). \] (12)
One of the main results in [13] was to derive pointwise estimates for the resolvent kernel \( G \).
Sketch of the proof of the main theorem:

Our first step is to derive higher derivative estimates for the Green kernel $G(t)$, by using Paley-Littlewood decomposition and localized frequency bounds of $\nabla^k$. Making use of the decay bounds for $\nabla^k G(t)$, we are able to propagate the decay of $\nabla^k \rho(t)$ by the forcing term $S(t)$:

$$\|\nabla^k \rho(t)\|_{L^1} \lesssim (t)^{-k} \log(1 + t) \|S\|_{Y^N} \quad \text{and} \quad \|\nabla^k \rho(t)\|_{L^\infty} \lesssim (t)^{-d-k} \log(1 + t) \|S\|_{Y^N},$$

(13)

where

$$\|S\|_{Y^N} = \max_{0 \leq k \leq N} \sup_{0 \leq s \leq t} \left( \langle s \rangle^k \|\nabla^k_x S(s)\|_{L^1} + \langle s \rangle^{d+k} \|\nabla^k_x S(s)\|_{L^\infty} \right).$$

Thus, it suffices to bound the derivatives of the forcing term $S(t, x)$, defined in (9). Inspired by [13, 3], we show that the trajectories $(X_{s,t}(x, v), V_{s,t}(x, v))$ are closed to the characteristics of the free transport $(x-(t-s)v, v)$. To bound $\nabla^k_x S(t, x)$, we also need to use induction argument to decompose $\nabla^k_x S(t, x)$ into quantities involving $\nabla^k_x E, \nabla^k \rho$ and lower order derivative terms involving characteristic trajectories.

2.4. Organization of the paper. The paper is organized as follows: In Section 3, we justify the estimate (13) for the density $\rho(t)$. In Section 4, we prove the decay estimates for the characteristics. In Section 5, we bound the forcing term $S(t, x)$, by determining the forms of its derivatives (Lemma 5.2 and 5.7), and then justify the decay estimates for each of the terms (Proposition 5.3 and Proposition 5.9).

2.5. Notations. For any complex numbers $A, B$, we write $A \lesssim B$ or $A = O(B)$, to mean that there exists a universal constant $C_0 > 0$ such that $|A| \leq C_0 |B|$.

For a vector field $F(x) = (F_1(x), \ldots, F_q(x)) \in \mathbb{R}^q$ with $x \in \mathbb{R}^m$, we denote $\nabla_x F$ to be the set of derivatives

$$\{\partial^a F_j : \quad 1 \leq j \leq q, \quad |\alpha| = k \}.$$

Moreover, for two vector functions $F, G$ defined on $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we denote $(\nabla_x F)(\nabla_y G)$ to be the set of all products $XY$, where $X \in \nabla_x F$ and $Y \in \nabla_y G$.

For any two vectors $a = (a_i)_{1 \leq i \leq p}, b = (b_i)_{1 \leq i \leq p}$, we denote $a \leq b$ if $a_i \leq b_i$ for all $1 \leq i \leq d$.

We also denote $\langle t \rangle = (1 + t^2)^{1/2}$. It is obvious that $\langle t \rangle \lesssim 1 + t \lesssim \langle t \rangle$ for all $t \geq 0$.

For a function $f(x)$ with $x \in \mathbb{R}^d$, we denote $\hat{f}(\xi)$ to be the standard Fourier transform of $f$, given by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

For a function $h(t, x)$ with $x \in \mathbb{R}^d$ and $t \geq 0$, the space-time Fourier transform of $h$ is denoted by $\widetilde{h}(\tau, \xi)$, and is given by

$$\widetilde{h}(\tau, \xi) = \int_0^\infty \int_{\mathbb{R}^d} h(t, x) e^{-i\tau t} e^{-ix\cdot\xi} dx dt.$$

We also use the Paley-Littlewood decomposition on $\mathbb{R}^d$. Let $\chi \in [0, 1]$ be a smooth compactly supported function on the annulus $\frac{1}{2} \leq |\xi| \leq 4$ and equal to one on the annulus $\frac{1}{4} \leq |\xi| \leq 2$. Define

$$\chi_q(\xi) = \chi \left( \frac{\xi}{2^q} \right) \quad \text{for} \quad q \in \mathbb{Z}. $$
For \( u \in S'(\mathbb{R}^d) \), we have the Paley-Littlewood decomposition
\[
u = \sum_{q \in \mathbb{Z}} u_q, \quad \text{where} \quad \hat{u}_q(\xi) = \hat{u}(\xi) \chi_q(\xi).
\]

3. Linear estimates.

3.1. Dispersive estimates for the Green kernel. Now let \( G \) be the kernel defined as in (12). Using Paley-Littlewood decomposition, we can decompose \( G \) as
\[
G = \sum_{q \in \mathbb{Z}} G_q.
\]

Now, we recall the following decaying bounds on the localized-in-frequency Green kernel \( G_q \) in [13].

**Lemma 3.1.** Let \( \mu(v) \) be smooth, satisfy the bound (3) and the Penrose stability condition (2). For any \( K > 0 \), there exists \( A \geq 1 \) such that for every \( \delta \in (0, 1] \), and for every \( q \in \mathbb{Z} \) with \( 2^q \geq A \):
\[
\|G_q(t)\|_{L^1} \lesssim \frac{2^q(1+\delta)}{1+2^q} \frac{1}{(1+2\delta)K} \quad \text{and} \quad \|G_q(t)\|_{L^\infty} \lesssim \frac{2^q(d+1+\delta)}{1+2^q} \frac{1}{(1+2\delta)K}.
\]

Moreover, for \( 2^q \leq A \), one has
\[
\|G_q(t)\|_{L^1} \lesssim \frac{2^q}{(1+2^q)K} \quad \text{and} \quad \|G_q(t)\|_{L^\infty} \lesssim \frac{2^q(d+1)}{(1+2^q)K}.
\]

Making use of the decay bounds on the Green kernels \( \{G_q\}_{q \in \mathbb{Z}} \), we establish the following bound on the derivatives of \( G(t) \):

**Theorem 3.2.** For any integer \( k \geq 0 \), there holds
\[
\|\nabla_x^k G(t)\|_{L^1} \lesssim t^{-k-1} \quad \text{and} \quad \|\nabla_x^k G(t)\|_{L^\infty} \lesssim t^{-d+1-k}
\]
for \( t > 0 \), where \( G \) is the kernel defined in (12).

**Proof.** We take \( K \) so that \( K > k + d + 1 \). We bound \( \|\nabla_x^k G(t)\|_{L^1} \) as follows:
\[
\|\nabla_x^k G(t)\|_{L^1} \lesssim \sum_{2^q \leq A} 2^{kq} \|G_q(t)\|_{L^1} + \sum_{2^q \geq A} 2^{kq} \|G_q(t)\|_{L^1}
\]
\[
\lesssim \sum_{2^q \leq A} \frac{2^{q(k+1)}}{(1+2^q)K} + \sum_{2^q \geq A} \frac{2^{q(k+1)+\delta}}{(1+2^q)(1+2^q)K}.
\]

For the first term, we see that
\[
\sum_{2^q \leq A} \frac{2^{q(k+1)}}{(1+2^q)K} \lesssim \left( \sum_{2^q \leq t^{-1}} + \sum_{t^{-1} \leq 2^q \leq 1} + \sum_{1 \leq 2^q \leq A} \right) \frac{2^{q(k+1)}}{(1+2^q)K}
\]
\[
\lesssim \sum_{2^q \leq t^{-1}} 2^{q(k+1)} + t^{-K} \sum_{2^q \leq 1} 2^{q(k+1-K)} + t^{-K} \sum_{1 \leq 2^q \leq A} 2^{q(k+1-K)}
\]
\[
\lesssim t^{-(k+1)} + t^{-K} \lesssim t^{-(k+1)}.
\]

The second term is treated as follows:
\[
\sum_{2^q \geq A} \frac{2^{q(k+1)+\delta}}{(1+2^q)(1+2^q)K} \lesssim t^{-K} \sum_{2^q \geq A} 2^{q(k+1+\delta)-2-K} \lesssim t^{-K} \lesssim t^{-k-1}
\]
since $N > k + 1 + d$. The decay bound for $\|\nabla_x^k G(t)\|_{L^1}$ is complete. Now we bound $\|\nabla_x^k G(t)\|_{L^\infty}$ as follows:

$$\|\nabla_x^k G(t)\|_{L^\infty} \lesssim \sum_{2^v \leq A} 2^{kv} \|G_q(t)\|_{L^\infty} + \sum_{2^v \geq A} 2^{kv} \|G_q(t)\|_{L^\infty}$$

$$\lesssim \sum_{2^v \leq A} \frac{2^{q(d+1+k)}}{(1 + 2qt)^K} + \sum_{2^v \geq A} \frac{2^{q(d+1+\delta+k)}}{1 + 2^{2q}} \cdot \frac{1}{(1 + 2qt)^K}.$$  

For the first term, we see that

$$\sum_{2^v \leq A} \frac{2^{q(d+1+k)}}{(1 + 2qt)^K} \lesssim \left( \sum_{2^v \leq t^{-1}} + \sum_{t^{-1} \leq 2^v \leq 1} + \sum_{1 \leq 2^v \leq A} \right) \frac{2^{q(d+1+k)}}{(1 + 2qt)^K}$$

$$\lesssim \sum_{2^v \leq t^{-1}} 2^{q(d+1+k)} + t^{-K} \sum_{t^{-1} \leq 2^v \leq 1} 2^{q(d+1+k-K)} + t^{-K} \sum_{1 \leq 2^v \leq A} 2^{q(d+1+k-K)}$$

$$\lesssim t^{-(d+1+k)} + t^{-K} \lesssim t^{-(d+1+k)}.$$

For the second term, we see that

$$\sum_{2^v \geq A} \frac{2^{q(d+1+\delta+k)}}{(1 + 2qt)^K} \cdot \frac{1}{(1 + 2^{2q})} \lesssim t^{-K} \sum_{2^v \geq A} 2^{q(d+1+\delta+k-K-2)}$$

$$\lesssim t^{-K} \lesssim t^{-(d+1+k)}.$$

The proof is complete. \qed

3.2. Decay estimates for the density and electric field of the linearized problem. In this section, we drive decay estimates for higher derivatives of the density $\rho(t)$, defined in (11) and the electric field $E = -\nabla_x (1 - \nabla_x)^{-1} \rho$. For $N > 0$, we define

$$\|S\|_{Y^N} = \max_{0 \leq k \leq N} \sup_{0 \leq s \leq t} \left( \langle s \rangle^k \|\nabla_x^k S(s)\|_{L^1} + \langle s \rangle^{d+k} \|\nabla_x^k S(s)\|_{L^\infty} \right).$$  

(14)

By definition, we see that

$$\|\nabla_x^k S(s)\|_{L^1} \leq \langle s \rangle^{-k} \|S\|_{Y^N} \quad \text{and} \quad \|\nabla_x^k S(s)\|_{L^\infty} \leq \langle s \rangle^{-(k+d)} \|S\|_{Y^N}$$

for $0 \leq k \leq N$ and $0 \leq s \leq t$.

**Theorem 3.3.** Let $N \geq 1$ be an integer. The density $\rho(t)$ defined in (11) satisfies the bound

$$\max_{0 \leq k \leq N} \left\{ (t)^k \|\nabla_x^k \rho(t)\|_{L^1} + (t)^{d+k} \|\nabla_x^k \rho(t)\|_{L^\infty} \right\} \lesssim \log(1 + t) \|S\|_{Y^N} \quad \text{for} \quad t > 0.$$

(15)

**Proof.** Fix $k$ so that $0 \leq k \leq N$. First we estimate $\nabla_x^k \rho$ in $L^1$. Applying $\nabla_x^k$ to both sides of (11), we get

$$\nabla_x^k \rho(t) = \nabla_x^k S(t) + \int_0^{t/2} \nabla_x^k G(t-s) \ast_x S(s) ds + \int_{t/2}^t G(t-s) \ast_x \nabla_x^k S(s) ds.$$
This implies
\[
\| \nabla_x \rho(t) \|_{L^1} \leq \| \nabla_x^k S(t) \|_{L^1} + \int_0^{t/2} \| \nabla_x^k G(t-s) \|_{L^1} \| S(s) \|_{L^1} ds
\]
\[
+ \int_{t/2}^t \| G(t-s) \|_{L^1} \| \nabla_x^k S(s) \|_{L^1} ds
\]
\[
\lesssim \langle t \rangle^{-k} \| S \|_{Y_t^N}^N
\]
\[
+ \| S \|_{Y_t^N} \left( \int_0^{t/2} \frac{1}{(t-s)^{k+1}} ds + \int_{t/2}^t \frac{1}{(1+t-s)} \langle s \rangle^{-k-d} ds \right)
\]
\[
\lesssim \| S \|_{Y_t^N} \left( \langle t \rangle^{-k} + k^{-d} + \log(1 + t) \right).
\]
Hence
\[
\langle t \rangle^k \| \nabla_x^k \rho(t) \|_{L^1} \lesssim \log(1 + t) \| S \|_{Y_t^N}.
\]

Now we estimate the $L^\infty$ norm of $\nabla_x^k \rho$. We have
\[
\| \nabla_x^k \rho(t) \|_{L^\infty} \leq \| \nabla_x^k S(t) \|_{L^\infty} + \int_0^{t/2} \| \nabla_x^k G(t-s) \|_{L^\infty} \| S(s) \|_{L^1} ds
\]
\[
+ \int_{t/2}^t \| G(t-s) \|_{L^1} \| \nabla_x^k S(s) \|_{L^\infty} ds
\]
\[
\lesssim \| S \|_{Y_t^N} \langle t \rangle^{-k-d}
\]
\[
+ \| S \|_{Y_t^N} \left( \int_0^{t/2} \frac{1}{(t-s)^{k+d+1}} ds + \int_{t/2}^t \frac{1}{(1+t-s)} \langle s \rangle^{-k-d} ds \right)
\]
\[
\lesssim \frac{\log(1 + t)}{(1 + t)^{d+k}} \| S \|_{Y_t^N},
\]
and hence
\[
\langle t \rangle^{d+k} \| \nabla_x^k \rho(t) \|_{L^\infty} \lesssim \log(1 + t) \| S \|_{Y_t^N}.
\]
The proof is complete. \(\Box\)

**Theorem 3.4.** Assume that
\[
\max_{0 \leq k \leq N} \{ \langle t \rangle^{d+k} \| \nabla_x^k \rho(t) \|_{L^\infty} + \langle t \rangle^k \| \nabla_x^k \rho(t) \|_{L^1} \} \lesssim \varepsilon \log(1 + t),
\]
there holds
\[
\max_{0 \leq k \leq N} \{ \langle t \rangle^{d+k} \| \nabla_x^k E(t) \|_{L^\infty} + \langle t \rangle^k \| \nabla_x^k E(t) \|_{L^1} \} \lesssim \varepsilon \log(1 + t).
\]

**Proof.** Since $E = -\nabla_x (1 - \Delta_x)^{-1} \rho$, we have
\[
\nabla_x^k E = -\nabla_x^k (1 - \Delta_x)^{-1} (\nabla_x^k \rho)
\]
By the standard elliptic estimate, we get
\[
\begin{align*}
\| \nabla_x^k E(t) \|_{L^\infty} & \lesssim \| \nabla_x^k \rho(t) \|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-d-k} \log(1 + t),
\| \nabla_x^k E(t) \|_{L^1} & \lesssim \| \nabla_x^k \rho(t) \|_{L^1} \lesssim \varepsilon \langle t \rangle^{-k} \log(1 + t).
\end{align*}
\]
The proof is complete. \(\Box\)
4. Decay estimates for the characteristics.

4.1. Main theorem. From the equations (7), the characteristics $X_{s,t}(x,v)$ and $V_{s,t}(x,v)$ can be written as follows:

$$
X_{s,t}(x,v) = x - (t - s)v + \int_s^t (\tau - s)E(\tau, X_{\tau,t}(x,v))d\tau,
$$
$$
V_{s,t}(x,v) = v - \int_s^t E(\tau, X_{\tau,t}(x,v))d\tau.
$$

Following [13], we define $(Y_{s,t}(x,v), W_{s,t}(x,v))$ so that

$$
X_{s,t}(x,v) = x - (t - s)v + Y_{s,t}(x - tv, v),
$$
$$
V_{s,t}(x,v) = v + W_{s,t}(x - tv, v).
$$

Hence, we get

$$
Y_{s,t}(x,v) = \int_s^t (\tau - s)E(\tau, x + \tau v + Y_{\tau,t}(x,v))d\tau,
$$
$$
W_{s,t}(x,v) = -\int_s^t E(\tau, x + \tau v + Y_{\tau,t}(x,v))d\tau.
$$

Our main theorem in this section is as follows:

**Theorem 4.1.** Assume that

$$
\max_{0 \leq k \leq N} \left< s \right>d_k \left\| \nabla^k_x E(t) \right\|_{L^\infty} \lesssim \varepsilon \log(1 + t) \quad \text{for all} \quad t > 0,
$$

there holds, for all $s \in [0, t]$, the following inequalities:

$$
\max_{0 \leq k \leq N} \left\| \nabla^k_v Y_{s,t} \right\|_{L^\infty} \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-2}}
$$

and

$$
\max_{0 \leq k \leq N} \left\| \nabla^k_v W_{s,t} \right\|_{L^\infty} \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}}.
$$

Before giving the proof, we recall the Faa di Bruno’s formula in [1], which allows us to compute higher order derivatives of $Y_{s,t}$ and $W_{s,t}$ by the generalized chain rule:

**Lemma 4.2.** Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^m$ and $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be smooth functions. For each multi-index $\alpha \in \mathbb{N}^d$, we have

$$
\partial^\alpha (F \circ u) = \sum_{\mu, \nu} C_{\mu, \nu} \partial^\mu F \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq m} (\partial^j u)^{\nu_j}
$$

where $C_{\mu, \nu}$ are non negative integers, and the sum is taken over $\mu, \nu$ such that $1 \leq |\mu| \leq |\alpha|$, $\nu_j \in \mathbb{N}^*$,

$$
\sum_{1 \leq |\beta| \leq |\alpha|} \nu_j = \mu_j \quad \text{for} \quad 1 \leq j \leq m, \quad \text{and} \quad \sum_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq m} \beta \nu_j = \alpha.
$$

**Proof of Theorem 4.1:** Fix $k \in \{0, 1, \cdots, N\}$. We shall prove that

$$
\left< s \right>d_k \left\| \nabla^k_v Y_{s,t} \right\|_{L^\infty} + \left< s \right>d_k \left\| \nabla^k_v W_{s,t} \right\|_{L^\infty} \lesssim \varepsilon \log(1 + s).
$$

In [13], the authors prove the above statement when $k \in \{0, 1\}$, thus we shall only consider $k \geq 2$. Let us first justify the bound for the case $k = 2$. Applying $\partial^2_{v_i v_j}$ to
both sides of (17), we have
\[
\partial^2_{v_i v_j} Y_{s,t}(x,v) \\
= \int_s^t (\tau - s) \partial^2_{x_i x_j} E(\tau, x + \tau v + Y_{\tau,t}(x,v)) \left( (\tau + \partial_{v_i} Y_{\tau,t}(x,v)) \partial_v Y_{\tau,t}(x,v) \right) d\tau \\
+ \int_s^t (\tau - s) \partial_{x_i} E(\tau, x + \tau v + Y_{\tau,t}(x,v)) \partial^2_{v_i v_j} Y_{\tau,t}(x,v) d\tau.
\]
This implies
\[
|\partial^2_{v_i v_j} Y_{s,t}(x,v)| \leq \int_s^t (\tau - s) \|\nabla^2_x E(\tau)\|_{L^\infty} (\tau + \|\nabla_v Y_{\tau,t}\|_{L^\infty})^2 d\tau \\
+ \int_s^t (\tau - s) \|\nabla_x E(\tau)\|_{L^\infty} \sup_{0 \leq s \leq t} \|\nabla^2_y Y_{s,t}\|_{L^\infty} d\tau \\
\lesssim \int_s^t (\tau - s) \cdot \varepsilon \log(1 + \tau) (\tau)^{-d-2} (\tau + \varepsilon)^2 d\tau \\
+ \sup_{0 \leq s \leq t} \|\nabla^2_y Y_{s,t}\|_{L^\infty} \int_s^t (\tau - s) \cdot \varepsilon \log(1 + \tau) (\tau)^{-d-1} d\tau.
\]
Hence we get
\[
\|\nabla^2_y Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \log(1 + s) \frac{\log(1 + s)}{(1 + s)^d-2} + \varepsilon \log(1 + s) \frac{\log(1 + s)}{(1 + s)^d-1} \sup_{0 \leq s \leq t} \|\nabla^2_y Y_{s,t}\|_{L^\infty}.
\]
Thus, as long as \( \varepsilon \frac{\log(1+s)}{(1+s)^d-2} \) is small, we have
\[
\|\nabla^2_y Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1 + s)}{(1 + s)^d-2}.
\]
Now we estimate \( \nabla^2_y W_{s,t} \). Applying \( \partial^2_{v_i v_j} \) to both sides of (17), we get
\[
\partial^2_{v_i v_j} W_{s,t}(x,v) \\
= -\int_s^t \left( \partial^2_{x_i x_j} E(\tau, x + \tau v + Y_{\tau,t}(x,v)) \left( (\tau + \partial_{v_i} Y_{\tau,t}(x,v)) \partial_v Y_{\tau,t}(x,v) \right) \partial_{v_j} Y_{\tau,t}(x,v) \right) d\tau \\
- \int_s^t \left( \partial_{x_i} E(\tau, x + \tau v + Y_{\tau,t}(x,v)) \left( \partial^2_{v_i v_j} Y_{\tau,t}(x,v) \right) \right) d\tau.
\]
This implies
\[
|\partial^2_{v_i v_j} W_{s,t}(x,v)| \leq \int_s^t \|\nabla^2_x E(\tau)\|_{L^\infty} (\tau + \|\nabla_v Y_{\tau,t}\|_{L^\infty})^2 + \int_s^t \|\nabla_x E(\tau)\|_{L^\infty} \|\nabla^2_y Y_{\tau,t}\|_{L^\infty} d\tau
\]
Using the fact that \( \langle t \rangle^{d+1} \|E(t)\|_{L^\infty} \lesssim \varepsilon \log(1+t) \) and \( \|\nabla^2_y Y_{\tau,t}\|_{L^\infty} \lesssim \varepsilon \), we get
\[
|\partial^2_{v_i v_j} W_{s,t}(x,v)| \lesssim \varepsilon \int_s^t \frac{\log(1 + \tau)}{(1 + \tau)^{d+2}} (\tau + \varepsilon)^2 d\tau + \varepsilon \int_s^t \frac{\log(1 + \tau)}{(1 + \tau)^{d+1}} d\tau \\
\lesssim \varepsilon \int_s^t \frac{\log(1 + \tau)}{(1 + \tau)^d} d\tau + \varepsilon \int_s^t \frac{\log(1 + s)}{(1 + s)^d} d\tau \lesssim \varepsilon \frac{\log(1 + s)}{(1 + s)^d}.
\]
Now for a general multi-index $\alpha$ with $k = |\alpha| \geq 3$, we proceed by induction on $k$, by assuming that the decay estimates (19) and (20) are true for all index with length less than $k$. Applying the chain rule (4.2) for $Y_{s,t}$, we get

$$
\partial_x^\alpha Y_{s,t}
= \int_s^t (\tau - s) \sum_{(\mu, \nu) \in I} \mathcal{C}_{\mu, \nu} \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \left( \partial_x^\beta (x_j + \tau v_j + Y_{\tau, t}(x, v)) \right)^{\nu_{\beta_j}} d\tau 
$$

$$
= \int_s^t (\tau - s) \sum_{(\mu, \nu) \in I} \mathcal{C}_{\mu, \nu} \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \left( \partial_x^\beta (\tau v_j + Y_{\tau, t}(x, v)) \right)^{\nu_{\beta_j}} d\tau 
$$

$$
= \int_s^t (\tau - s) \sum_{(\mu, \nu) \in I} \mathcal{C}_{\mu, \nu} \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \tau^{\nu_{\beta_j}} \left( \partial_x^\beta v_j \right)^{\nu_{\beta_j}} d\tau 
$$

$$
+ \int_s^t (\tau - s) \sum_{(\mu, \nu) \in I} \mathcal{C}_{\mu, \nu} \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \left( \partial_x^\beta Y_{\tau, t}(x, v) \right)^{\nu_{\beta_j}} d\tau.
$$

(21)

where

$$
I = \left\{ (\mu, \nu) \in \mathbb{N}^{2d} : \sum_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \nu_{\beta_j} = \mu_j \text{ for } 1 \leq j \leq d, \sum_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \beta \nu_{\beta_j} = \alpha \right\}.
$$

Now we fix $\mu, \nu$ and estimate each term appearing in (21). The first term can be estimated as follows:

$$
\int_s^t (\tau - s) \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \tau^{\nu_{\beta_j}} \left( \partial_x^\beta v_j \right)^{\nu_{\beta_j}} d\tau \lesssim \int_s^t (\tau - s) \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d + |\mu|}} \tau^{\sum_{j=1}^d \nu_{\beta_j}} d\tau 
$$

$$
\lesssim \int_s^t \tau^{\mu_j + 1} \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d + |\mu|}} d\tau 
$$

$$
\leq \int_s^t \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d - 1}} d\tau \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d - 2}}.
$$

Now we estimate the second term in (21), which is

$$
\int_s^t (\tau - s) \partial_x^\mu E \prod_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq d} \left( \partial_x^\beta Y_{\tau, t}(x, v) \right)^{\nu_{\beta_j}} d\tau.
$$

(22)

We consider two cases:

**Case 1:** $\beta_j < |\alpha|$ for all $1 \leq j \leq d$.

In this case, we use the induction hypothesis on $\partial_x^\beta Y_{\tau, t}$, which is $|\partial_x^\beta Y_{\tau, t}(x, v)| \lesssim \varepsilon$.

Hence (22) can be bounded by

$$
\int_s^t (\tau - s) \mathcal{C}_{\mu, \nu} \partial_x^\mu E(\tau) \|L^\infty \varepsilon \sum_{j=1}^d \nu_{\beta_j} d\tau \lesssim \int_s^t \tau \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d + |\mu|}} \varepsilon|\mu| d\tau \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d - 2}}.
$$

**Case 2:** There exists $j_0 \in \{1, 2, \cdots, d\}$ such that $\beta_{j_0} = |\alpha|$.

In this case, since $|\beta| \leq |\alpha|$, we have $\beta_j = 0$ for all $j \neq j_0$. Hence the term is reduced to

$$
\int_s^t (\tau - s) \partial_x^\mu E \cdot (\partial_{\beta_{j_0}} Y_{\tau, t}^{(\alpha)})^{\nu_{\beta_{j_0}}} O(\varepsilon) d\tau.
$$
where we use the fact that \( \|Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \). Moreover, since \( \sum_{j=1}^{d} \beta \nu_{j} = \alpha \), we have \( \nu_{|\alpha|} \leq 1 \). Hence

\[
\int_{s}^{t} (\tau - s) \partial_{s}^{\alpha} E \cdot (\partial_{s}^{\beta} Y_{s,t} Y_{s,t}) \nu_{\beta} O(\varepsilon) \lesssim \varepsilon \int_{s}^{t} \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+|\beta|}} \sup_{0 \leq \nu \leq \tau} \|\partial_{s}^{\beta} Y_{s,t}\|_{L^\infty} d\tau
\]

Thus, we get the inequality of the form

\[
\|\nabla_{x}^{\alpha} Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1 + s)}{(1 + s)^{d-1}} + \varepsilon \sup_{0 \leq s \leq t} \|\nabla_{v}^{\alpha} Y_{s,t}\|_{L^\infty}.
\]

Hence for \( \varepsilon \) small, we get

\[
\|\nabla_{x}^{\alpha} Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1 + s)}{(1 + s)^{d-1}} \quad \text{for all } 0 \leq s \leq t.
\]

The proof is complete.

4.2. Straightening the characteristics. Next, we recall the following lemma about straightening the characteristics from [13].

**Lemma 4.3.** Let \( Y_{s,t} \) be defined as in (16) and (17). Assume that \( E \) satisfies the decay estimates (18), there holds

\[
\|\nabla_{x} Y_{s,t}\|_{L^\infty} \lesssim \varepsilon \frac{\log(1 + s)}{(1 + s)^{d-1}}.
\]

**Proof.** By the definition of \( Y_{s,t} \), we have

\[
Y_{s,t}(x,v) = \int_{s}^{t} (\tau - s) E(\tau, x + \tau v + \Psi_{s,t}(x,v)) d\tau.
\]

Hence we get

\[
|\nabla_{x} Y_{s,t}(x,v)| \leq \int_{s}^{t} (\tau - s) \|\nabla_{x} E(\tau)\|_{L^\infty} (1 + \|\nabla_{x} Y_{s,t}\|_{L^\infty}) d\tau
\]

\[
\lesssim \int_{s}^{t} (\tau - s) \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}} d\tau + \varepsilon \sup_{0 \leq \tau \leq t} \|\nabla_{x} Y_{s,t}\|_{L^\infty} \int_{s}^{t} (\tau - s) \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}} d\tau.
\]

\[
\lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} + \left\{ \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} \right\} \sup_{0 \leq s \leq t} \|\nabla_{x} Y_{s,t}\|_{L^\infty}.
\]

Hence as long as \( \varepsilon \) is small enough, we have

\[
|\nabla_{x} Y_{s,t}(x,v)| \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}}.
\]

The proof is complete.

**Lemma 4.4.** For \( 0 \leq s \leq t \), there exists a \( C^1 \) map \( (x,v) \rightarrow \Psi_{s,t}(x,v) \) such that

\[
X_{s,t}(x, \Psi_{s,t}(x,v)) = x - (t - s)v
\]

for all \( x,v \in \mathbb{R}^d \). Moreover, if \( E \) satisfies the estimates (18), there holds

\[
(s)^d \|\Psi_{s,t}(x,v) - v\| + (s)^{d-1} |\nabla_{v}(\Psi_{s,t}(x,v) - v)| \lesssim \varepsilon \log(1 + s)
\]

for all \( x,v \in \mathbb{R}^d \) and \( 0 \leq s \leq t \).
Proof. We write
\[ X_{s,t}(x,v) = x - (t-s)(v + \Phi_{s,t}(x,v)) \]
and will show that \((x,v) \to (x,v + \Phi_{s,t}(x,v))\) is a \(C^1\) diffeomorphism. To this end, we prove that
\[ \langle s \rangle^d \| \Phi_{s,t} \|_{L^\infty} + \langle s \rangle^d \| \nabla_x \Phi_{s,t} \|_{L^\infty} + \langle s \rangle^{d-1} \| \nabla_v \Phi_{s,t} \|_{L^\infty} \lesssim \varepsilon \log(1 + s) \]
We have
\[ \Phi_{s,t}(x,v) = -\frac{1}{t-s}(X_{s,t}(x,v) - (t-s)v) \]
\[ = -\frac{1}{t-s}\int_s^t (\tau-s)E(\tau, x-(t-\tau)v + Y_{\tau,t}(x-vt,v))d\tau \quad (23) \]
\[ \lesssim \frac{1}{t-s}\int_s^t (\tau-s)\| E(\tau)\|_{L^\infty}d\tau \lesssim \int_s^t \| E(\tau)\|_{L^\infty}d\tau. \]
Since \(E = -\nabla_x(1 - \Delta_x)^{-1}\rho\), we have
\[ \| E(\tau)\|_{L^\infty} \lesssim \| \nabla_x \rho(\tau)\|_{L^\infty} \lesssim \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}}. \]
Thus we have
\[ |\Phi_{s,t}(x,v)| \lesssim \int_s^t \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}}d\tau \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^d}. \]
Thus \(\langle s \rangle^d \| \Phi_{s,t} \|_{L^\infty} \lesssim \varepsilon \log(1 + s)\). Next, applying \(\nabla_x\) to both sides of \((23)\), we have
\[ \| \nabla_x \Phi_{s,t} \|_{L^\infty} \lesssim \frac{1}{t-s}\int_s^t (\tau-s)\| \nabla_x E(\tau)\|_{L^\infty} (1 + \| \nabla_x Y_{\tau,t} \|_{L^\infty}) d\tau. \]
Now using lemma 4.3, we get
\[ \| \nabla_x \Phi_{s,t} \|_{L^\infty} \lesssim \int_s^t \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}}d\tau \lesssim \frac{\varepsilon \langle s \rangle^{-d} \log(1 + s)}{(1 + s)^{d-1}}. \]
Now we bound \(\| \nabla_v \Phi_{s,t} \|_{L^\infty}\). Applying \(\nabla_v\) to both sides of \((23)\), we get
\[ \| \nabla_v \Phi_{s,t} \|_{L^\infty} \lesssim \frac{1}{t-s}\int_s^t (\tau-s)\| \nabla_v E(\tau)\|_{L^\infty} \{ (t-\tau) + t \| \nabla_x Y_{\tau,t} \|_{L^\infty} \} d\tau. \]
Again, using Lemma 4.3, we have
\[ \| \nabla_v \Phi_{s,t} \|_{L^\infty} \lesssim \frac{1}{t-s}\int_s^t (\tau-s) \frac{\varepsilon \log(1 + \tau)}{(1 + \tau)^{d+1}} (t-\tau) + \frac{\varepsilon t \log(1 + \tau)}{(1 + \tau)^{d-1}} d\tau \]
\[ \lesssim \frac{\log(1 + s)}{(1 + s)^{d-1}} + \varepsilon \int_s^t \frac{\log(1 + \tau)^2}{t-s} d\tau \]
\[ \lesssim \frac{\log(1 + s)}{(1 + s)^{d-1}} + \varepsilon \int_s^t \frac{\log(1 + \tau)^2}{(1 + \tau)^{d-1}} d\tau \]
Hence, the map \((x,v) \to (x,v + \Phi_{s,t}(x,v))\) is a \(C^1\) diffeomorphism. Thus there exists a \(C^1\) diffeomorphism \(v \to \Psi_{s,t}(x,v)\) such that
\[ X_{s,t}(x,\Psi_{s,t}(x,v)) = x - (t-s)v. \]
Combining this with $X_{s,t}(x,v) = x - (t-s)(v + \Phi_{s,t}(x,v))$, we have
\[
\begin{cases}
|\Psi_{s,t}(x,v) - v| \lesssim \|\Phi_{s,t}\|_{L^{\infty}} \lesssim \varepsilon(s)^{-d} \log(1+s), \\
|\nabla_v(\Psi_{s,t}(x,v) - v)| \lesssim \|\nabla_v \Phi_{s,t}\|_{L^{\infty}} \lesssim \varepsilon(s)^{-(d-1)} \log(1+s).
\end{cases}
\]

The proof is complete. \qed

5. Decay estimates for the forcing term. In this section, we derive decay estimates for the derivatives of the forcing term, appearing in the equation (8). The forcing term $S(t,x)$ is given by
\[
S(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v),V_{0,t}(x,v))dv + \int_0^t \int_{\mathbb{R}^d} E(s,x-(t-s)v) \cdot \nabla_v \mu(v)dvds \\
- \int_0^t \int_{\mathbb{R}^d} E(s,X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))dvds \\
= \mathcal{I}(t,x) + \mathcal{R}_L(t,x) - \mathcal{R}_{NL}(t,x),
\]
where
\[
\begin{align*}
\mathcal{I}(t,x) &= \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v),V_{0,t}(x,v))dv, \\
\mathcal{R}_L(t,x) &= \int_0^t \int_{\mathbb{R}^d} E(s,x-(t-s)v) \cdot \nabla_v \mu(v)dvds, \\
\mathcal{R}_{NL}(t,x) &= \int_0^t \int_{\mathbb{R}^d} E(s,X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))dvds.
\end{align*}
\]

Fix $N \geq 1$, we will give decay estimates for $\|S\|_{Y^N}$, under the decaying assumptions on derivatives of $E$, $Y_{s,t}$ and $W_{s,t}$. We also recall the smallness assumption for the initial perturbation $f_0(x,v)$ as follows:
\[
\max_{0 \leq k \leq N} \|\nabla_{x,v} f_0\|_{L^1_x L^\infty_v} \leq \varepsilon_0.
\]

Our main theorem is as follows:

**Theorem 5.1.** Let $N > 1$ be an integer. Assume that $E, W_{s,t}, Y_{s,t}$ satisfy the decay estimates (18), (20) and (19) and $f_0$ satisfies the smallness assumption (25), there holds
\[
\sup_{0 \leq k \leq N} (t)^k \|\nabla_x^k \mathcal{I}(t)\|_{L^1} + (t)^{d+k} \|\nabla_x^k \mathcal{I}(t)\|_{L^{\infty}} \lesssim \varepsilon_0,
\]
and
\[
\sup_{0 \leq k \leq N} (t)^k \|\nabla_x^k \mathcal{R}(t)\|_{L^1} + (t)^{d+k} \|\nabla_x^k \mathcal{R}(t)\|_{L^{\infty}} \lesssim \varepsilon^2.
\]

5.1. Decay estimates for the initial data term. First, we estimate $\mathcal{I}(t,x)$, under suitable smallness assumption (25) on the initial data $f_0$. In [13], the authors prove that
\[
\|\mathcal{I}(t)\|_{L^1} + (t)^d \|\mathcal{I}(t)\|_{L^{\infty}} + \|\nabla_x \mathcal{I}(t)\|_{L^1} + (t)^{d+1} \|\nabla_x \mathcal{I}(t)\|_{L^{\infty}} \lesssim \varepsilon_0.
\]

Hence, we shall establish the bound (26) for $k \geq 2$. First, we establish the following lemma

**Lemma 5.2.** Let $k \geq 2$ and
\[
\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v),V_{0,t}(x,v))dv.
\]
The term $\nabla_x^k I(t, x)$ can be written as a sum of many terms, which are all in the form

$$\int_{\mathbb{R}^d} \nabla_\alpha f_0 \cdot (\nabla^\beta Y_{0,t})^{k_1} \cdot (\nabla^\gamma W_{0,t})^{s_1} \cdots (\nabla^\gamma W_{0,t})^{s_t} \, dw \quad (27)$$

where $\alpha, (\beta_1, k_1), \ldots, (\beta_r, k_r)$, and $(\gamma_1, s_1), \ldots, (\gamma_t, s_t)$ satisfy

$$1 \leq |\alpha| \leq k \quad \text{and} \quad (\beta_1 k_1 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_t s_t) \leq k. \quad (28)$$

**Proof.** From (16), we get

$$I(t, x) = \int_{\mathbb{R}^d} f_0(x - tv + Y_{0,t}(x - tv, v), v + W_{0,t}(x - tv, v)) \, dv.$$  

Let $w = x - tv$, we get

$$I(t, x) = \int_{\mathbb{R}^d} f_0 \left( w + Y_{0,t}(w, \frac{x - w}{t}, \frac{x - w}{t} + W_{0,t}(w, \frac{x - w}{t}) \right) \, dw \quad (29)$$

By a direct calculation, we have

$$\partial^2_{x,v} I(t, x) = \int_{\mathbb{R}^d} \left( \partial^2_{x,v} f_0 \partial_{v,j} Y_{0,t} + \partial_{x,v} f_0 (1 + \partial_{v,j} W_{0,t}) \right) (\partial_{v,j} Y_{0,t}) \, dw \quad (29)$$

It is clear from the above that (29) that

$$\nabla_x^2 I(t, x) = \int_{\mathbb{R}^d} \left\{ (\nabla^2 f_0) (\nabla_v Y_{0,t}) + (\nabla_{x,v} f_0) (\nabla_{v}^2 Y_{0,t}) + (\nabla^2_{x,v} f_0) (\nabla_v Y_{0,t}) \right. \left. + (\nabla^2_{x,v} f_0) (\nabla_v W_{0,t}) + (\nabla^2_{x,v} f_0) (\nabla_v W_{0,t})^2 + (\nabla_{v} f_0) (\nabla^2_{v} W_{0,t}) \right\} \, dw \quad (29)$$

which satisfies the hypothesis for $k = 2$. Now by induction, we assume that this statement is true for $k$, and we shall prove it for $k + 1$. Applying $\partial_{x,i}$ to the term (27) and using the product rules, we have three types of terms that appear, namely:

$$I_1 = \int_{\mathbb{R}^d} \frac{d}{dx_i} (\nabla_\alpha f_0) (\nabla^\beta Y_{0,t})^{k_1} \cdots (\nabla^\gamma W_{0,t})^{s_1} \cdots (\nabla^\gamma W_{0,t})^{s_t} \, dw \quad (29)$$

$$I_2 = \int_{\mathbb{R}^d} (\nabla^\alpha_{x,v} f_0) (\nabla_{v}^\beta Y_{0,t})^{k_1-1} \cdots (\nabla^\beta Y_{0,t})^{k_r} \cdot (\nabla^\gamma W_{0,t})^{s_1} \cdots (\nabla^\gamma W_{0,t})^{s_t} \, dw \quad (29)$$

$$I_3 = \int_{\mathbb{R}^d} (\nabla_{x,v}^\alpha f_0) (\nabla^\beta Y_{0,t})^{k_1} \cdots (\nabla^\beta Y_{0,t})^{k_r} \cdot (\nabla_v W_{0,t}) (\nabla^\gamma W_{0,t})^{s_1-1} \cdots (\nabla^\gamma W_{0,t})^{s_t} \, dw \quad (29)$$
Here, $I_1, I_2, I_3$ appear when $\partial_x$ hits $\nabla^\alpha_{x,v} f_0, (\nabla^\beta_v Y_{0,t})^{k_1}$ and $(\nabla^\gamma_v W_{0,t})^{s_1}$ respectively. Note that we assume that the derivative hits the above terms on just the pairs $(\beta_1, k_1)$ or $(\gamma_1, s_1)$, as this is up to a permutation of indices.

**Treating $I_1$:**

By a direct calculation, we see that $I_1$ can be written as

$$\int_{\mathbb{R}^d} (\nabla^\alpha_{x,v} f_0 \cdot \nabla_v Y_{0,t} + \nabla^\alpha_{x,v} f_0 (1 + \nabla_v W_{0,t})) \cdot (\nabla^\beta_v Y_{0,t})^{k_1} \cdots (\nabla^\beta_v Y_{0,t})^{k_r} \cdot (\nabla^\gamma_v W_{0,t})^{s_1} \cdots (\nabla^\gamma_v W_{0,t})^{s_r} \frac{dw}{t^{d+k+1}}$$

which satisfies the induction hypothesis for $|\alpha| + 1 = k + 1$.

**Treating $I_2$:**

For $I_2$, we check the condition (28) for the new indices and multi-indices, which is

$$|\alpha| \leq k+1 \quad \text{and} \quad \beta_1+1 + \beta_1(k_1-1)+(\beta_2 k_2 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_l s_l) \leq k+1$$

The statement $|\alpha| \leq k + 1$ is true, as $|\alpha| \leq k$. The second condition follows directly from the induction hypothesis:

$$(\beta_1 k_1 + \beta_2 k_2 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_l s_l) \leq k.$$ 

The proof is complete. Finally, the term $I_3$ is treated exactly as $I_2$, and we skip the details. \hfill \Box

**Theorem 5.3.** Assume that

$$\max_{0 \leq k \leq N} \|\nabla^k_{x,v} f_0\|_{L^1_x L^\infty_v} \leq \epsilon_0$$

and $E, Y_{s,t}, W_{s,t}$ satisfy the decay estimates (18), (19), and (20) for $0 \leq k \leq N$. Then

$$\mathcal{I}(t,x) = \int_{\mathbb{R}^d} f_0(X_{0,t}(x,v), V_{0,t}(x,v)) dv$$

satisfies the following decay estimate:

$$\sup_{0 \leq k \leq N} (\langle t \rangle^k \|\nabla^k_x \mathcal{I}(t)\|_{L^1} + (\langle t \rangle^{d+k} \|\nabla^k_x \mathcal{I}(t)\|_{L^\infty}) \lesssim \epsilon_0.$$ 

**Proof.** By the above lemma, it suffices to prove that

$$\langle t \rangle^k \|\nabla^k_x \mathcal{J}(t)\|_{L^1} + (\langle t \rangle^{d+k} \|\nabla^k_x \mathcal{J}(t)\|_{L^\infty}) \lesssim \epsilon_0,$$

where

$$\mathcal{J} = \int_{\mathbb{R}^d} \nabla^\alpha_{x,v} f_0 \cdot (\nabla^\beta_v Y_{0,t})^{k_1} \cdots (\nabla^\beta_v Y_{0,t})^{k_r} \cdot (\nabla^\gamma_v W_{0,t})^{s_1} \cdots (\nabla^\gamma_v W_{0,t})^{s_r} \frac{dw}{t^{d+k}}$$

and

$$1 \leq |\alpha| \leq k \quad \text{and} \quad (\beta_1 k_1 + \cdots + \beta_r k_r) + (\gamma_1 s_1 + \cdots + \gamma_l s_l) \leq k.$$ 

We have

$$\mathcal{J}(t,x) \lesssim t^{-k} \int_{\mathbb{R}^d} |\nabla^\alpha_{x,v} f_0|(X_{0,t}(x,v), V_{0,t}(x,v)) dv$$

Hence, we get

$$t^k \|\mathcal{J}(t)\|_{L^1} + t^{d+k} \|\mathcal{J}(t)\|_{L^\infty} \lesssim \epsilon_0.$$ 

The proof is complete. \hfill \Box
5.2. Decay estimates for the reaction term. In this section, we estimate the derivatives of the reaction term

\[ \mathcal{R} = \mathcal{R}_L - \mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(v) dv ds \]

appearing as a forcing term in (24). For a general time-dependent vector field \( E(s) \) and a smooth decaying function \( \mu \), we also denote \( T \) to be

\[ T(E,\mu) = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(v) dv ds \]

Our main theorem is as follows:

**Theorem 5.4.** Let \( N > 1 \) be an integer. Assume that \( E, W_{s,t}, Y_{s,t} \) satisfy the decay estimates (18), (20) and (19) for all \( 0 \leq k \leq N \), there holds

\[ \max_{0 \leq k \leq N} (\langle t \rangle^k \| \nabla_x^k R(t) \|_{L^1} + \langle t \rangle^{d+k} \| \nabla_x^k R(t) \|_{L^\infty}) \lesssim \varepsilon^2. \]

First, we recall the following proposition from [13]. We also give a detailed proof for the readers convenience.

**Proposition 5.5.** Assuming that \( E, Y_{s,t}, W_{s,t} \) satisfies the decaying estimates (18), (19) and (20) respectively, we have

\[ \| T(E,\mu) \|_{L^1} + \langle t \rangle^d \| T(E,\mu) \|_{L^\infty} \lesssim \varepsilon^2. \]

*Proof.* Making the change of variables \( v \rightarrow \Psi_{s,t}(x,v) \) so that

\[ X_{s,t}(x,\Psi_{s,t}(x,v)) = x -(t-s)v \]

(see Lemma 4.4), we have

\[ T(E,\mu) = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(v) dv ds \]

\[ - \int_0^t \int_{\mathbb{R}^d} E(s,X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v)) dv ds \]

\[ = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(v) dv ds \]

\[ - \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(V_{s,t}(x,\Psi_{s,t}(x,v)) det(\nabla_v \Psi_{s,t}(x,v))) dv ds. \]

Hence, one can rewrite \( T \) as \( T_1 + T_2 \), where

\[ T_1 = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \{ \mu(v) - \mu(V_{s,t}(x,\Psi_{s,t}(x,v))) \} dv ds, \]

\[ T_2 = \int_0^t \int_{\mathbb{R}^d} E(s,x -(t-s)v) \cdot \nabla_v \mu(V_{s,t}(x,\Psi_{s,t}(x,v)) \{1 - det(\nabla_v \Psi_{s,t}(x,v)) \} dv ds. \]

Bounding \( T_1(t) \):
We have
\[ T_1(t, x) = \int_0^t \int_{\mathbb{R}^d} E(s, x - (t - s)v) \cdot \nabla_v \{ \mu(v) - \mu(V_{s,t}(x, \Psi_{s,t}(x, v))) \} \, dv \, ds \]
\[ \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)| \cdot |\langle v \rangle - M - |v - V_{s,t}(x, \Psi_{s,t}(x, v))|dvd\,ds \]
\[ \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)|\langle v \rangle^{-M} \]
\[ \{ |v - V_{s,t}(x, v)| + |V_{s,t}(x, v) - V_{s,t}(x, \Psi_{s,t}(x, v))| \} \, dv \, ds \]
\[ \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)|\langle v \rangle^{-M} \]
\[ \{ |v - V_{s,t}(x, v)|_{L^\infty} + \|\nabla_v V_{s,t}\|_{L^\infty} |v - \Psi_{s,t}(x, v)| \} \, dv \, ds. \]

Using the fact that
\[ |v - V_{s,t}(x, v)| \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}}, \quad |v - \Psi_{s,t}(x, v)| \]
\[ \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^d} \quad \text{and} \quad \|\nabla_v V_{s,t}\|_{L^\infty} \lesssim 1, \quad (31) \]
we get
\[ T_1(t, x) \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)|\langle v \rangle^{-M} \left( \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} + \frac{\varepsilon \log(1 + s)}{(1 + s)^d} \right) \, dv \, ds. \]

Hence
\[ \|T_1(t)\|_{L^1} \lesssim \int_0^t \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} \int_{\mathbb{R}^d} \|E(s)\|_{L^1} \langle v \rangle^{-M} \, dv \]
\[ \lesssim \int_0^t \varepsilon^2 \frac{\log(1 + s)^2}{(1 + s)^{d-1}} ds \lesssim \varepsilon^2, \]
and
\[ \|T_1(t)\|_{L^\infty} \lesssim \int_0^t \int_{\mathbb{R}^d} \|E(s)\|_{L^\infty} \langle v \rangle^{-M} \left( \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} + \frac{\varepsilon \log(1 + s)}{(1 + s)^d} \right) \, dv \, ds \]
\[ \lesssim \int_0^t \varepsilon^2 \frac{\log(1 + s)^2}{(1 + s)^{2d-1}} ds \lesssim \varepsilon^2 \log(1 + t) \int_0^t \frac{\log(1 + s)}{(1 + s)^{2d-1}} ds \]
\[ \lesssim \varepsilon^2 \frac{(1 + t)^2}{(1 + t)^{2d-2}} \lesssim \varepsilon^2 (t)^{-d}. \]
Thus
\[ \|T_1(t)\|_{L^1} + \langle t \rangle^d \|T_1(t)\|_{L^\infty} \lesssim \varepsilon^2. \]

Bounding $T_2(t)$:
Since $v \rightarrow \Psi_{s,t}(x, v)$ is a differomorphism, making the change of variables
\[ \Psi_{s,t}^{-1} : \Psi_{s,t}(x, v) \rightarrow v \]
This implies that

There holds Lemma 5.6.

gives

\[ T_2(t, x) = \int_0^t \int_{\mathbb{R}^d} E(s, x - (t - s)v) \cdot \nabla_v (V_{s,t}(x, v)) \cdot \left( 1 - \det(\nabla_v \Psi_{s,t}(x, v)) \right) dvds \]

\[ \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)| \cdot \left| \nabla_v (V_{s,t}(x, v)) \cdot \det(\nabla_v \Psi_{s,t}(x, v)) \right|^{-1} - 1 | dvds. \]

Using the inequality \(|\nabla_v (\Psi_{s,t}(x, v) - v)| \lesssim \frac{\varepsilon \log(1+s)}{(1+s)^{d+1}}\), we have

\[ T_2(t, x) \lesssim \int_0^t \int_{\mathbb{R}^d} |E(s, x - (t - s)v)| \cdot \left| \nabla_v (V_{s,t}(x, v)) \cdot \frac{\varepsilon \log(1+s)}{(1+s)^{d+1}} \right| dvds. \]

Hence we have

\[ \| T_2(t) \|_{L^1} \lesssim \int_0^t \left\{ \varepsilon \frac{\log(1+s)}{(1+s)^{d+1}} \| E(s) \|_{L^1} \right\} ds \]

\[ \lesssim \int_0^t \varepsilon^2 \frac{\log(1+s)^2}{(1+s)^{d+1}} ds \lesssim \varepsilon^2. \]

and

\[ \| T_2(t) \|_{L^\infty} \lesssim \int_0^t \| E(s) \|_{L^\infty} \cdot \left( \frac{\varepsilon \log(1+s)}{(1+s)^{d+1}} \sup_{x \in \mathbb{R}^d} |\nabla_v (V_{s,t}(x, v))| dvds \right) \]

\[ \lesssim \int_0^t \varepsilon^2 \frac{\log(1+s)^2}{(1+s)^{2d+1}} ds \lesssim \varepsilon^2 \log(1+t) \int_0^t \frac{\log(1+s)}{(1+s)^{2d+1}} ds \lesssim \varepsilon^2 \frac{\log(1+t)}{(1+t)^{2d-2}} \]

\[ \lesssim \varepsilon^2 (t)^{-d}. \]

This implies that

\[ \| T_2(t) \|_{L^1} + \langle t \rangle^d \| T_2(t) \|_{L^\infty} \lesssim \varepsilon^2. \]

The proof is complete. \(\square\)

**Lemma 5.6.** There holds

\[ \partial_3 T(E, \mu) = \frac{1}{t} (T(s \partial_3 E, \mu) + T(E, \partial_3 \mu)) \]

\[ + \frac{1}{t} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} \partial_3 Y_{s,t}(x - tv, v) \cdot \nabla_x E_k(s, X_{s,t}(x, v)) \cdot \partial_3 \mu(V_{s,t}(x, v)) dvds \]

\[ + \frac{1}{t} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k(s, X_{s,t}(x, v)) \cdot \partial_3 W_{s,t}(x - tv, v) \cdot \nabla_v (\partial_3 \mu(V_{s,t}(x, v))) dvds. \]

**Proof.** We recall that

\[ \mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x, v)) \cdot \nabla_v (V_{s,t}(x, v)) dvds. \]

Using the identities (16), we have

\[ \mathcal{R}_{NL} = \int_0^t \int_{\mathbb{R}^d} E(s, x - (t - s)v + Y_{s,t}(x - tv, v)) \cdot \nabla_v (v + W_{s,t}(x - tv, v)) dvds. \]
Lemma 5.7. Let 

\[ R_{NL} = \int_0^t \int_{\mathbb{R}^d} E \left( s, w + \frac{s}{t} (x - w) + Y_{s,t}(w, \frac{x - w}{t}) \right) \]

\[ \cdot \nabla_v \mu \left( \frac{x - w}{t} + W_{s,t}(w, \frac{x - w}{t}) \right) t^{-d} dw \, ds \]

= \left[ t^{-d} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k \left( s, w + \frac{s}{t} (x - w) + Y_{s,t}(w, \frac{x - w}{t}) \right) \right. 

\left. \partial_k \mu \left( \frac{x - w}{t} + W_{s,t}(w, \frac{x - w}{t}) \right) dw \, ds \right].

Similarly, for \( R_L \) we have

\[ R_L = \int_0^t \int_{\mathbb{R}^d} E(s, x - (t - s)v) \cdot \nabla_v \mu(v) dv \, ds \]

\[ = \left[ t^{-d} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} E_k \left( s, w + \frac{s}{t} (x - w) \right) \partial_k \mu \left( \frac{x - w}{t} \right) dw \, ds \right]. \]

The lemma follows by a direct calculation. The proof is complete. \( \square \)

Now we establish the following lemma, by induction on the degree of derivatives.

Lemma 5.7. Let \( n \geq 2 \), then \( \nabla_x^2 R(t, x) \) can be written as a sum of terms, are all either of the form

\[ \frac{1}{t^n} T(s^k \nabla_x^k E, f(\mu)) \]

or the form

\[ \frac{1}{t^n} \int_0^t \int_{\mathbb{R}^d} \left\{ (\nabla_v^{m_1} Y_{s,t})^{k_1} \cdots (\nabla_v^{m_a} Y_{s,t})^{k_a} \right\} \cdot \left\{ (\nabla_v^{n_1} W_{s,t})^{l_1} \cdots (\nabla_v^{n_b} W_{s,t})^{l_b} \right\} \]

\[ \times \left\{ (s^t \nabla_x u^1 E) \cdots (s^t \nabla_x u^e E) \right\} f(\mu) dv \, ds \]

where \( f(\mu) \) is some expression only depending on \( \mu(v) \) or its derivatives. Moreover, the indices satisfy the following set of conditions:

- No loss of derivative condition: \( k \leq n \), \( \max\{m_1, k_1, \cdots, m_a, k_a\} \cup \{n_1, l_1, \cdots, n_b, l_b\} \cup \{t_1, u_1, \cdots, t_e, u_e\} \leq n \).
- E-decay condition: \( (t_1, t_2, \cdots, t_e) \leq (u_1, u_2, \cdots, u_e) \).
- Y-W show-up condition: \( \min\{a, b\} \geq 1 \).
- E-show up condition: \( c \geq 1 \).
- W-E decay condition: If \( b = 0 \) then \( t_h + 1 \leq u_h \) for some \( 1 \leq h \leq b \).

Let us call the first form to be type-I and the second one is type-II.

Remark 5.8. The conditions on the indices are important for the decay estimates. The first condition means that we do not lose derivatives in the estimates. The second condition requires a good decay for the quantities \( s^t \nabla_x^u E \). The third condition means that at least \( Y_{s,t} \) or \( W_{s,t} \) (or their derivatives) shows up in the expression. The forth condition means that \( E \) or its derivatives must show up in the expression. Finally, the last condition means that if \( W_{s,t} \) (or its derivatives) does not show up, then we have more gradient of \( E \) to control the power of \( s \). The reason for the last condition will be clear when we estimate (34) later in the paper.
Proof. The lemma is proved by induction on n. Assuming the lemma is true for n, we justify the above claim for n + 1.

**Gradient of type-I terms:**
Applying $\partial_j$ to the term $T(s^k \nabla^k E, f(\mu))$ and using Lemma 5.6, we have

$$\frac{1}{t^n} \partial_j T(s^k \nabla^k E, f(\mu)) = \frac{1}{t^{n+1}} \left( T(s^{k+1} \nabla^{k+1} E, f(\mu)) + T(s^k \nabla^k E, \partial_j f(\mu)) \right) + \frac{1}{t^{n+1}} \left( \int_0^t \int_{B^d} \left( (\partial_{sv} \mathcal{V}_x(x-w, v)) \cdot \nabla_x (s^k \nabla^k E) (s, \mathcal{V}_x(x,v)) (\nabla_x f(\mu)) (\mathcal{V}_x(x,v)) \right) \, d\mathcal{V}_s \, d\mathcal{V}_x \right)
$$

We can see that all of the above terms are either type-I or type-II, and satisfy the induction hypothesis with order $n + 1$.

**Gradient of type-II form:**
Now we consider type-II terms:

$$\mathcal{D} = \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
$$

where $(\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1, (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1$ is evaluated at $(w, x - w)$ and $\nabla_x^w E$ is evaluated at $(s, \mathcal{V}_x(x,v))$. Making the change of variables $w = x - tw$, we can rewrite $\mathcal{D}$ as

$$\mathcal{D} = t^{-d} \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
$$

Applying $\nabla_x$ to $\mathcal{D}$ and using the product rules, we have

$$\begin{align*}
\mathcal{D}_1 &= \mathcal{D}_2 + \mathcal{D}_3
\end{align*}
$$

Now making the change of variables $v = \frac{x-w}{t}$, we get

$$\begin{align*}
\mathcal{D}_1 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$

$$\begin{align*}
\mathcal{D}_2 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$

$$\begin{align*}
\mathcal{D}_3 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$

with

$$\begin{align*}
\mathcal{D}_1 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$

$$\begin{align*}
\mathcal{D}_2 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$

$$\begin{align*}
\mathcal{D}_3 &= \frac{1}{t^n} \int_0^t \int_{B^d} \left\{ (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_1}^1 \cdots (\nabla_{v}^m \mathcal{Y}_{s,t})_{k_n}^n \right\} \cdot \left\{ (\nabla_{v}^n \mathcal{V}_{s,t})_{l_1}^1 \cdots (\nabla_{v}^n \mathcal{V}_{s,t})_{l_n}^n \right\} \times \left\{ (s^t \nabla_x E) \cdots (s^t \nabla_x E) \right\} f(\mu) \, d\mathcal{V}_s \, d\mathcal{V}_x \right\}
\end{align*}
$$
From the above expressions, it is straightforward that the new terms are of type-II, which satisfy the induction hypothesis with \( n + 1 \). The proof is complete.

**Proposition 5.9.** Let \( \mathcal{R} \) be defined as in (30). There holds, for \( n \geq 2 \), the decaying estimates

\[
\langle t \rangle^n \| \nabla_v^m \mathcal{R}(t) \|_{L^1} + \langle t \rangle^{d+n} \| \nabla_x^k \mathcal{R}(t) \|_{L^\infty} \lesssim \varepsilon^2.
\]

**Proof.** By Lemma 5.7, \( \nabla_v^2 \mathcal{R}(t,x) \) can be decomposed as a sum of many terms, all are either of the form \( \mathcal{R}_1 \) or \( \mathcal{R}_2 \), where

\[
\begin{align*}
\mathcal{R}_1 &= \frac{1}{m} \mathcal{T}(s^k \nabla_v^k E, f(\mu)), \\
\mathcal{R}_2 &= \frac{1}{m} \int_0^\infty \int_{\mathbb{R}^d} \left\{ \left( \nabla_v^{m_1} Y_{s,t} \right)^{k_1} \cdots \left( \nabla_v^{m_s} Y_{s,t} \right)^{k_s} \right\} \cdot \left\{ \left( \nabla_v^{n_1} W_{s,t} \right)^{l_1} \cdots \left( \nabla_v^{n_s} W_{s,t} \right)^{l_s} \right\} \\
&\cdot \left\{ (s^t \nabla_v^1 E) \cdots (s^n \nabla_v^n E) \right\} f(\mu) dv ds
\end{align*}
\]

where \( k \leq n \) and \( \max\{\{m_1, k_1, \cdots, m_s, k_s\} \cup \{n_1, l_1, \cdots, n_s, l_s\} \cup \{t_1, u_1, \cdots, t_c, u_c\} \} \leq n \).

**Bounding** \( \mathcal{R}_1(t) \):

We have

\[
\mathcal{R}_1(t) = \frac{1}{m} \mathcal{T}(s^k \nabla_v^k E, f(\mu))
\]

with \( k \leq n \) and \( f(\mu) \) is a decay function in \( v \) which only depends on \( \mu \) or its derivatives. Applying Proposition 5.5, we only need to check the assumption

\[
\| s^k \nabla_v^k E(s) \|_{L^\infty} \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^d},
\]

which is true, since \( \| \nabla_v^k E(s) \|_{L^\infty} \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d+\delta}} \). The proof for \( \mathcal{R}_1 \) is complete.

**Bounding** \( \mathcal{R}_2(t) \):

Now we bound \( \mathcal{R}_2(t) \), where

\[
\mathcal{R}_2(t,x) = \frac{1}{m} \int_0^t \int_{\mathbb{R}^d} \left\{ \left( \nabla_v^{m_1} Y_{s,t} \right)^{k_1} \cdots \left( \nabla_v^{m_s} Y_{s,t} \right)^{k_s} \right\} \cdot \left\{ \left( \nabla_v^{n_1} W_{s,t} \right)^{l_1} \cdots \left( \nabla_v^{n_s} W_{s,t} \right)^{l_s} \right\} \\
\cdot \left\{ (s^t \nabla_v^1 E) \cdots (s^n \nabla_v^n E) \right\} f(\mu) dv ds
\]

Here \( (\nabla_v^{m_t} Y_{s,t})^{k_t}, (\nabla_v^{m_s} W_{s,t})^{l_s} \) is evaluated at \( (x - tv, v) \) and \( \nabla_v^n E \) is evaluated at \( (s, X_{s,t}(x,v)) \).

We also recall the following set of conditions, proved in Lemma 5.7:

\[
\begin{align*}
&\left\{ \begin{array}{l}
k \leq n \quad \text{and} \quad \max\{\{m_1, k_1, \cdots, m_s, k_s\} \cup \{n_1, l_1, \cdots, n_s, l_s\} \cup \{t_1, u_1, \cdots, t_c, u_c\} \} \leq n, \\
\{t_1, u_1, \cdots, t_c, u_c\} \leq n, \\
\{t_1, t_2, \cdots, t_c\} \leq \{u_1, u_2, \cdots, u_c\}, \\
\min\{a, b\} \geq 1,
\end{array} \right. \\
&c \geq 1,
\end{align*}
\]

If \( b = 0 \) then \( t_h + 1 \leq u_h \) for some \( 1 \leq h \leq c \).

First, we will show that \( \| \mathcal{R}_2(t) \|_{L^1} \lesssim \varepsilon^2 \langle t \rangle^{-n} \).

Using the third condition in (32) and the fact that

\[
\max_{0 \leq k \leq n} \left( \| \nabla_v^k Y_{s,t} \|_{L^\infty} + \| \nabla_v^k W_{s,t} \|_{L^\infty} \right) \lesssim \varepsilon \frac{\log(1+s)}{(1+s)^{d-2}},
\]

we get
\[ R_2(t, x) \leq t^{-n} \int_0^t \int_{\mathbb{R}^d} \frac{\log(1 + s)}{(1 + s)^{d-2}} \prod_{i=1}^c \left( s^{h_i} \| \nabla_x^{u_i} E(s) \|_{L^\infty} \right) |f(\mu)| (V_{s,t}(x,v)) dvds. \] (33)

Now using the second and the forth condition in (32), and the decay of $\nabla_x^{u_i} E$, we get
\[ s^{h_i} \| \nabla_x^{u_i} E(s) \|_{L^\infty} \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^{d+n_i-1}} \lesssim \frac{\varepsilon \log(1 + s)}{(1 + s)^d}. \] Applying the above inequality to (33), we obtain
\[ R_2(t, x) \leq t^{-n} \int_0^t \int_{\mathbb{R}^d} \frac{\varepsilon^2 \log(1 + s)}{(1 + s)^{d-2}} \cdot \frac{\log(1 + s)}{(1 + s)^d} |f(\mu)| (V_{s,t}(x,v)) dvds \]
\[ \lesssim \varepsilon^2 t^{-n} \int_0^t \int_{\mathbb{R}^d} \frac{(1 + s)^2}{(1 + s)^{2d-2}} |f(\mu)| (V_{s,t}(x,v)) dvds. \]

Hence, using the decaying assumption of $\mu$, we obtain
\[ \| R_2(t) \|_{L^1} \lesssim \varepsilon^2 t^{-n} \int_0^t \int_{\mathbb{R}^d} \frac{(1 + s)^2}{(1 + s)^{2d-2}} |f(\mu)| (V_{s,t}(x,v)) dvds \] \[ \lesssim \varepsilon^2 t^{-n}. \]

The decaying bound for $\| R_2(t) \|_{L^1}$ is complete.

We split the integral in $R_2$ into $\int_{t/2}^t + \int_{t/2}^1$, so that $R_2 = R_3 + R_4$, where
\[
\begin{align*}
R_3(t, x) &= \frac{1}{\pi} \int_{t/2}^t \int_{\mathbb{R}^d} \left\{ \left( \nabla_{y_v}^{m_{1,Y_v,t}} \right)^{k_1} \cdots \left( \nabla_{y_v}^{m_{n,Y_v,t}} \right)^{k_n} \right\} |f(\mu)| (V_{s,t}(x,v)) dvds, \\
R_4(t, x) &= \frac{1}{\pi} \int_{t/2}^t \int_{\mathbb{R}^d} \left\{ \left( \nabla_{y_v}^{m_{1,Y_v,t}} \right)^{k_1} \cdots \left( \nabla_{y_v}^{m_{n,Y_v,t}} \right)^{k_n} \right\} |f(\mu)| (V_{s,t}(x,v)) dvds.
\end{align*}
\]

For $R_3(t, x)$, by the same argument for $R_1(t, x)$, we have the pointwise bound
\[ R_3(t, x) \lesssim \varepsilon^2 t^{-n} \int_{t/2}^t \frac{(1 + s)^2}{(1 + s)^{2d-2}} |f(\mu)| (V_{s,t}(x,v)) dvds \]
\[ \lesssim \varepsilon^2 t^{-n-d} \int_{t/2}^t \frac{(1 + s)^2}{(1 + s)^{2d-2}} ds \lesssim \varepsilon^2 t^{-(n+d)}. \]

Thus $\| R_3(t) \|_{L^\infty} \lesssim \varepsilon^2 t^{-(n+d)}$.

Now to bound $\| R_4(t) \|_{L^\infty}$, we use the inequalities
\[ \max_{0 \leq k \leq n} (s)^{d-2} \| \nabla_x^{k} Y_{s,t} \|_{L^\infty} + (s)^{d-1} \| \nabla_v^{b} W_{s,t} \|_{L^\infty} \lesssim \varepsilon \log(1 + s) \]
to get
\[ R_4(t, x) \lesssim t^{-n} \int_{t/2}^t \int_{\mathbb{R}^d} \left[ \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-2}} \right]^{a} \left[ \frac{\varepsilon \log(1 + s)}{(1 + s)^{d-1}} \right]^{b} \]
\[ \times \prod_{i=1}^c \left( s^{h_i} \| \nabla_x^{u_i} E|(s, X_{s,t}(x,v)) \|_{L^\infty} \right) |f(\mu)| (V_{s,t}(x,v)) dvds \]
\[ \lesssim t^{-n} \varepsilon^{a+b} \int_{t/2}^t \int_{\mathbb{R}^d} \frac{(\log(1 + s))^{a+b}}{(1 + s)^{(d-2)+b(d-1)}} \]
\[ \times \prod_{i=1}^c \left( s^{h_i} \| \nabla_x^{u_i} E|(s, X_{s,t}(x,v)) \|_{L^\infty} \right) |f(\mu)| (V_{s,t}(x,v)) dvds. \]
Using the change of variable $v \to \Psi_{s,t}(x,v)$ so that $\Psi_{s,t}(x,\Psi_{s,t}(x,v)) = x - (t-s)v$, we have

$$R_4(t, x) \lesssim t^{-n} e^{a+b} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{(\log(1+s))^{a+b}}{(1+s)^{a(d-2)+b(d-1)}} \prod_{i=1}^c \left\{ s^{t_i} |\nabla_x^i E| (s, x - (t-s)v) \right\} \cdot |f(\mu)| \left( V_{s,t}(x, \Psi_{s,t}(x,v)) \right) dv ds.$$  

Now making the change of variables $w = x - (t-s)v$, we have

$$R_4(t, x) \lesssim t^{-n} e^{a+b} \int_0^{t/2} (t-s)^{-d} \int_{\mathbb{R}^d} \frac{(\log(1+s))^{a+b}}{(1+s)^{a(d-2)+b(d-1)}} \prod_{i=1}^c \left\{ s^{t_i} |\nabla_x^i E| (s, w) \right\} \cdot |f(\mu)| \left( V_{s,t}(x, \Psi_{s,t}(x,x-w/t)) \right) dw ds \lesssim t^{-(a+d)} e^{a+b} \int_0^{t/2} \frac{(1+s)^{a+b}}{(1+s)^{a(d-2)+b(d-1)}} \cdot \min_{1 \leq i \leq c} \left\{ s^{t_i} \log(1+s) \right\} ds \lesssim t^{-(a+d)} e^{a+b+1} \int_0^{t/2} \frac{(1+s)^{a+b+1}}{(1+s)^{a(d-2)+b(d-1)+\max_{1 \leq i \leq c}(u_i-t_i)}} ds.$$

Now using the third condition in (32), we get $\varepsilon^{a+b+1} \lesssim \varepsilon^2$ as long as $\varepsilon$ is small. Hence we get

$$\|R_4(t)\|_{L^\infty} \lesssim \varepsilon^2 t^{-(a+d)} \int_0^{t/2} \frac{(1+s)^{a+b+1}}{(1+s)^{a(d-2)+b(d-1)+\max_{1 \leq i \leq c}(u_i-t_i)}} ds \lesssim 1. \quad (34)$$

Thus, it suffices to prove that

$$C = \int_0^{t/2} \frac{(1+s)^{a+b+1}}{(1+s)^{a(d-2)+b(d-1)+\max_{1 \leq i \leq c}(u_i-t_i)}} ds \lesssim 1. \quad (35)$$

We consider two cases:

**Case 1:** $b = 0$.

In this case, we use use the last condition listed in (32) to get $\max_{1 \leq i \leq c}(u_i-t_i) \geq 1$, and hence

$$C \lesssim \int_0^{t/2} \frac{(1+s)^{a+1}}{(1+s)^{a(d-2)+1}} \lesssim \int_0^{t/2} \frac{(1+s)^{a+1}}{(1+s)^{a(d-2)+1}} ds \lesssim 1.$$  

**Case 2:** $b \geq 1$.

In this case, we estimate $C$ as follows:

$$C \lesssim \int_0^{t/2} \frac{(1+s)^a}{(1+s)^{a(d-2)}} \cdot \frac{(1+s)^{b+1}}{(1+s)^{b(d-1)}} ds \lesssim \int_0^{t/2} \frac{(1+s)^{b+1}}{(1+s)^{b(d-1)}} ds \lesssim 1.$$  

The proof is complete. \qed

Finally, we give the proof for the main theorem 2.1.

**Proof of Theorem 2.1.** Let

$$N(t) = \sup_{0 \leq s \leq t} \max_{0 \leq k \leq N} \left( \langle s \rangle^k \|\nabla_x^k \rho(s)\|_{L^1} + \langle s \rangle^{k+d} \|\nabla_x^k \rho(s)\|_{L^\infty} \right)/\log(1+s).$$
Now we fix a constant $M_0 > 0$, which will be chosen later. We prove that $\mathcal{N}(t) \leq M_0 \varepsilon_0$ for all $t \geq 0$, as long as $\varepsilon_0$ is small enough. Let

$$T_* = \sup \{ T > 0 : \mathcal{N}(t) \leq M_0 \varepsilon_0 \text{ for all } 0 \leq t \leq T \} \quad (35)$$

We shall prove that $T_* = \infty$ by contradiction argument. Assuming that $T_* < \infty$, we have

$$\mathcal{N}(T_*) = \varepsilon_0 \text{ and } \sup_{0 \leq t \leq T} \mathcal{N}(t) < \varepsilon_0 \text{ for all } T < T_*.$$

For $t \in (0, T_*)$, by Theorem 3.3 and 5.3, there exists $C_0, C_1 > 0$ such that

$$\mathcal{N}(t) \leq C_0 \| S \|_{Y^N} \leq C_0 C_1 (\varepsilon_0 + \varepsilon_0^2)$$

for $\varepsilon_0$ small enough. Let $t \to T_*$, we have

$$M_0 \varepsilon_0 \leq C_0 C_1 (\varepsilon_0 + \varepsilon_0^2)$$

which is false, as long as $M_0 > 2C_0 C_1$ and $\varepsilon_0$ small. The proof is complete.

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