On the Maximal Output Admissible Set for a Class of Bilinear Discrete-time Systems

Youssef Benfatah, Amine El Bhiih, Mostafa Rachik, and Abdessamad Tridane*

Abstract: Given a discrete-time controlled bilinear systems with initial state $x_0$ and output function $y_i$, we investigate the maximal output set $\Theta(\Omega) = \{x_0 \in \mathbb{R}^n, y_i \in \Omega, \forall i \geq 0\}$ where $\Omega$ is a given constraint set and is a subset of $\mathbb{R}^q$. Using some stability hypothesis, we show that $\Theta(\Omega)$ can be determined via a finite number of inequations. Also, we give an algorithmic process to generate the set $\Theta(\Omega)$. To illustrate our theoretical approach, we present some examples and numerical simulations. Moreover, to demonstrate the effectiveness of our approach in real-life problems, we provide an application to the SI epidemic model and the SIR model.

Keywords: Asymptotic stability, bilinear systems, constraint set, discrete-time systems, output admissible set.

1. INTRODUCTION

Output admissible sets arise in many essential practical applications such as stability analysis and control of constrained systems (see [1–6]). This concept has been considerably investigated (see [7, 8] and the references therein), but not in the case of the bilinear systems. Consequently, its applicability is limited.

Bilinear systems were introduced into control theory in the 1960s. They were a distinct type of nonlinear systems, in which nonlinear terms are developed by way of multiplication of control vector and state vector. These type of systems has attracted a large community of researchers in almost half a century (see [9]). Such systems’ significance lies in the fact that many necessary engineering processes can be modeled through bilinear systems [11].

As we know, a nonlinear discrete-time finite-dimensional system can be represented by the following dynamic equation:

$$x_{i+1} = f(x_i, u_i),$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^q$, for $i \geq 0$, are respectively the state and the control.

If we assume that $f$ is linear with respect to the state or the control, we obtain, respectively.

$$x_{i+1} = h_2(x_i)u_i + f_2(x_i),$$

and

$$x_{i+1} = h_2(x_i)u_i + f_2(x_i),$$

we obtain another form of bilinear systems given as follows:

$$x_{i+1} = Ax_i + Bx_{i}u_i + Bu_i,$$

where for each $i \geq 0$, $A$ is a $n \times n$ matrix, $B$ is a $n \times q$ matrix. $Bx_{i}u_i$ is a bilinear form in the $x_i, u_i$ variables that can be expressed as

$$Bx_{i}u_i = \sum_{j=1}^{q} u_j B_j x_i,$$

where $u_i = \{u_j\}_{j=1}^{q}$ and $B_j$ is a $n \times n$ matrix.

The use of bilinear systems has two significant advantages: First, they provide better modeling of a nonlinear phenomenon than a linear system [12]). Second, many real-life problems can be model using bilinear systems.

1.1. Related work

The maximal output admissible (MOA) sets are an important concept for analyzing controlled systems with constraints. The MOA set has been well studied particularly for linear systems with state and control constraints [13, 14]. This concept provides an understanding of the analysis of constrained control systems. Besides, it is widely used in control system design methods [15].
The MOA set for a class of nonlinear systems has recently been studied by Rachik et al. in [16]. Later on, Hirata and Ohta [17] considered a special class of nonlinear systems, the so-called polynomial systems, and discussed the finite determinability of the MOA set.

Some other resources that provide information about the study of the concept MOA sets are given in the references [18–24]. In [18], Ossareh considered a Lyapunov-stable periodic system and gave the development of the MOA set theory. He also investigated some of its geometric and algebraic properties. In [19], Yamamoto presented MOA’s computational procedure for a trajectory tracking control of biped robots.

Various algorithms have been added in the literature for determining the maximal state constraint sets [13,25]. In [25] the authors considered autonomous linear discrete-time systems subject to linear constraints, and proposed an effective procedure to determine the maximal set of admissible initial states. Other models from population dynamics and optimal control problems can be found in [26,27].

1.2. Problem statement

This work aims to present a new approach to studying the MOA set for a class of bilinear discrete-time systems. To the best of our knowledge, the MOA sets for this type of system have not been investigated before. More specifically, we characterize the initial states of a controlled bilinear system whose resulting trajectory verify a pointwise constraint. Moreover, our approach is applicable to epidemiological, sociological, and physiological models. In these examples, the MOA set represents the set of initial data that guarantee for these systems do not violate certain constraints.

This paper considers discrete-time controlled bilinear systems. More precisely, the system has the following form:

\[
\begin{aligned}
x_{i+1} &= Ax_i + \sum_{j=1}^{q} u^j_i B_j x_i + B v_i, \quad i \geq 0, \\
x_0 &\in \mathbb{R}^n,
\end{aligned}
\]

(6)

the corresponding output is

\[
y_i = C x_i, \quad i \in \mathbb{N},
\]

(7)

where \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) is the matrix of the state, \( \mathbb{R} \) is the set of real numbers, \( \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) is the set of real matrices of order \( n \times n \), \( B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p) \) is the matrix of the input, \( C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p) \) is the matrix of the output, \( x_i \in \mathbb{R}^n \) is the state variable, \( B_j \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) for all \( j \in \{1, 2, \ldots, q\} \), \( \mathbb{N} \), is the set of nonnegative integers, \( n, m, p \) and \( q \) are the nonnegative integer and \( u^j_i, v_i \) are the feedback controls defined by

\[
u^j_i = K^j y_i \quad \text{and} \quad v_i = L y_i
\]

(8)

with \( L \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m) \) and \( K^j \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}) \), \( j \in \{1, \ldots, q\} \) are matrices with single row, and are given by

\[
K^j = [k^j_1 k^j_2 k^j_3 \ldots k^j_{p-1} k^j_p].
\]

(9)

Our purpose right here is to determine all vectors \( x_0 \) (the initial states) such that the corresponding signal output \( y_i \), verify the condition \( y_i \in \Omega, \forall i \in \mathbb{N} \), where \( \Omega \subset \mathbb{R}^p \) is a given constraint set. The set of all such vectors is the maximal output admissible set \( \Theta(\Omega) \).

For technical consideration we restrict our own selves in the current work to the study of \( \Theta^k(\Omega) \), the maximal output \( \lambda \)-admissible set, which defined by

\[
\Theta^k(\Omega) = \Theta(\Omega) \cap B(0, \lambda),
\]

(10)

where \( B(0, \lambda) \) is the ball of center 0 and radius \( \lambda > 0 \), this means we study the set of all vectors \( x_0 \in \mathbb{R}^n \) such that \( \|x_0\| \leq \lambda \) and \( y_i \in \Omega, \forall i \in \mathbb{N} \), where \( \lambda \) is any positive number.

Moreover, we give some useful properties of \( \Theta^k(\Omega) \), and we investigate its determination by way of a finite number of functional inequalities.

The rest of this paper is structured as follows: In Section 2, we present some preliminary results. In Section 3, we characterize the maximal output \( \lambda \)-admissible set. Moreover, in Section 4 we give some sufficient conditions to ensure the finite determination of the maximal output \( \lambda \)-admissible set. In the following section, we propose an algorithm for determining the output admissibility index and, consequently the set \( \Theta^k(\Omega) \). Various numerical examples to illustrate our results are given in Section 6. An application to a controlled SI epidemic model and SIR model is given, to demonstrate the effectiveness of our approach in Section 7. The last section includes conclusion.

In the next section, we will give some necessary mathematical preliminaries that will be used in this paper.

2. PRELIMINARY RESULTS

In this section we consider the discrete-time controlled bilinear systems described by (6)-(7).

We replace \( u^j_i \) and \( v_i \) by its values in system (6) we obtain

\[
\begin{aligned}
x_{i+1} &= (A + BLC)x_i + \sum_{j=1}^{q} K^j C x_i B_j x_i, \quad i \geq 0, \\
x_0 &\in \mathbb{R}^n.
\end{aligned}
\]

(11)

**Notations:** In what follows we will denote the matrix \( A + BLC \) by \( \tilde{A} \), \( K^j C \) by \( \tilde{K}^j \) and \( \{i, i+1, \ldots, k\} \), \( i \leq k \), the finite subset of \( \mathbb{N} \) by \( \sigma^k \).

**Remark 1:** In case \( q = 1 \) the system (11) becomes

\[
\begin{aligned}
x_{i+1} &= \tilde{A} x_i + \tilde{\delta} x_i B_1 x_i, \quad i \geq 0 \\
x_0 &\in \mathbb{R}^n
\end{aligned}
\]

(12)
where \( \delta = K^1C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \).

Hence and by taking into account the previous notations, we consider in this paper the class of discrete systems defined by

\[
\begin{align*}
x_{i+1} &= \tilde{A}x_i + \sum_{j=1}^{q} \tilde{K}^j x_j B_j x_i, \quad i \geq 0, \\
x_0 &\in \mathbb{R}^n,
\end{align*}
\]

(13)

increased by the output

\[
y_i = Cx_i, \quad \forall i \geq 0,
\]

(14)

and we are interested in the theoretical and numerical characterization of the set

\[
\Theta^k(\Omega) = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid y_i \in \Omega, \forall i \in \mathbb{N}\}. \quad (15)
\]

Indeed, let \( \psi \) be the function defined as follows:

\[
\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n \\
x \mapsto \psi(x) = \sum_{j=1}^{q} \tilde{K}^j x B_j x.
\]

(16)

Let \( K \) be the matrix defined by

\[
K = \begin{pmatrix}
k_1^1 & k_2^1 & \cdots & k_p^1 \\
& k_1^2 & \cdots & k_p^2 \\
& & \ddots & \cdot \\
& & & k_1^q & k_2^q & \cdots & k_p^q
\end{pmatrix}
\]

(17)

**Definition 1:** Let \( \Omega \subset \mathbb{R}^p \) and \( x_0 \in \mathbb{R}^n \). The initial state \( x_0 \) is said to be \( \Omega \)-output admissible if

\[
y_i \in \Omega, \quad \forall i \in \mathbb{N}.
\]

The set of all such initial states is defined by

\[
\Theta(\Omega) = \{x_0 \in \mathbb{R}^n \mid y_i \in \Omega, \forall i \in \mathbb{N}\}.
\]

(18)

The general solution of system (6) is given by

\[
x_i = \Phi^i(x_0), \quad \forall i \in \mathbb{N},
\]

(19)

where the function \( \Phi \) is defined by

\[
\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \\
x \mapsto \Phi(x) = \tilde{A}x + \psi(x),
\]

and

\[
\Phi^i = \Phi \circ \Phi \circ \cdots \circ \Phi.
\]

Using (18) the set \( \Theta(\Omega) \) could be rewritten as follows:

\[
\Theta(\Omega) = \{x_0 \in \mathbb{R}^n \mid C\Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\}.
\]

In this paper we restrict our-selves to the study of the set \( \Theta^k(\Omega) \) defined by

\[
\Theta^k(\Omega) = \Theta(\Omega) \cap B(0, \lambda) = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid C\Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\}. \quad (19)
\]

The restriction to the set \( \Theta^k(\Omega) \) will not reduce the value of the work and this for the following reason. Having an initial state \( x_0 \in \mathbb{R}^n \), we would like to know if \( x_0 \) is \( \Omega \)-output admissible or not. In order to give an answer to such a question, we first identify the set \( \Theta^k(\Omega) = \Theta(\Omega) \cap B(0, \lambda) \) where \( \lambda \) is a real that verifies \( \|x_0\| \leq \lambda \) and we check if \( x_0 \in \Theta^k(\Omega) \) or not.

**Remark 2:** 1) The general solution of system (6) is given by

\[
x_i = \prod_{k=0}^{i-1} \left( \tilde{A} + \sum_{j=1}^{q} u_k^j B_j \right) x_0, \quad \forall i \geq 1.
\]

(20)

2) \( \forall x, y \in \mathbb{R}^n \),

\[
\|\psi(x) - \psi(y)\| \\
\leq \left\{ \sum_{j=1}^{q} \|\tilde{K}^j\|^j \|B_j\| (\|x\| + \|y\|) \right\} \times \|x - y\|,
\]

where \( \top \) denotes the transpose.

3) For \( x_0 \in B(0, \lambda) \), we have

\[
\|x_i\| \leq F(\lambda), \quad \forall i \in \mathbb{N}
\]

(22)

where \( F \) is defined by

\[
F(t) = \alpha t + Cx \tau^2, \quad \forall t \in \mathbb{R},
\]

(23)

with

\[
\alpha = \|\tilde{A}\| \quad \text{and} \quad C_K = \sum_{j=1}^{q} \|\tilde{K}^j\|^j \|B_j\|.
\]

Now, we give some conditions which are sufficient to ensure, for convenient initial states \( x_0 \), the asymptotic stability of system (6). Our main result in this direction is the following.

**Proposition 1:** Suppose the following hypothesis to hold.

1) There exists \( \tau \geq 1 \) and \( \theta \in [0, 1] \) such that \( \|\tilde{A}\|^k \leq \tau \theta^k, \forall k \in \mathbb{N} \) (the matrix \( L \) can be chosen such that the hypothesis (1) holds),

2) \( \theta + Cx \tau^2 < 1 \), where

\[
C_K = \sum_{j=1}^{q} \|\tilde{K}^j\|^j \|B_j\|. \quad (24)
\]
Then the system (6) is asymptotically stable in the ball of center 0 and radius \( \lambda \), i.e.,
\[
\lim_{t \to \infty} \|x(t)\| = \lim_{t \to \infty} \|P^t(x_0)\| = 0, \quad \forall x_0 \in B(0, \lambda).
\] (25)

**Proof:** Let \( x_0 \in B(0, \lambda) \). Then the general solution of system (6) can be written as
\[
x_i = \tilde{A}^i_0 x_0 + \sum_{k=0}^{i-1} \tilde{A}^{i-k-1}_i \psi(x_k), \quad \forall i \geq 1
\] (26)

\[
\Rightarrow \|x_i\| \leq \|\tilde{A}^i\| \|x_0\| + \sum_{k=0}^{i-1} \|\tilde{A}^{i-k-1}\| \|\psi(x_k)\|
\] (27)

\[
\Rightarrow \|x_i\| \leq \tau \theta^i \|x_0\| + \sum_{k=0}^{i-1} \tau \theta^{i-k-1} \|\psi(x_k)\|
\] (28)

On the other hand, we have
\[
\|\psi(x_k)\| = \|\sum_{j=1}^q \tilde{K}^j x_k B_j x_k\|
\] (29)

\[
\leq \sum_{j=1}^q \|\tilde{K}^j\| \|x_k\| \|B_j\| \|x_k\|
\] (30)

we have used Cauchy-Schwarz inequality. This leads to
\[
\|x_i\| \leq \tau \theta^i \|x_0\|
\] (31)

Let \( z_i = \frac{x_i}{\tau} \). Then
\[
\sum_{k=0}^{i-1} \theta^{-k} \|x_k\|^2, \quad \forall i \geq 1
\] (32)

Since \( \|z_0\| \leq \tau \), we prove that
\[
\|z_i\| \leq \tau, \quad \forall i \in \mathbb{N}.
\] (33)

Indeed, if we assume that
\[
\|z_i\| \leq \tau, \quad \forall i \in \sigma_i^N
\] (34)

then using (31) it follows
\[
\|z_i\| \leq \tau \theta^i \|z_0\|
\] (35)

\[
+ \lambda C_K \tau^2 \theta^{i-1} \sum_{k=0}^{i-1} \theta^{-k} \|z_k\|^2, \quad \forall i \in \sigma_i^{N+1}.
\] (36)

\[
\Rightarrow y_i = \theta^{-i} \|z_i\| \quad \text{and} \quad \Gamma_i = \alpha + \beta \sum_{k=0}^{i-1} y_k,
\] (37)

where
\[
\alpha = \tau \|z_0\| \quad \text{and} \quad \beta = \lambda C_K \tau^2 \theta^{-1}.
\] (38)

From (34) we conclude
\[
y_i \leq \Gamma_i, \quad \forall i \in \sigma_i^{N+1}.
\] (39)

We have \( y_i = \frac{\Gamma_{i+1} - \Gamma_i}{\beta} \). Then
\[
\frac{\Gamma_{i+1} - \Gamma_i}{\beta} \leq \Gamma_i, \quad \forall i \in \sigma_i^{N+1},
\] (40)

which implies
\[
y_i \leq \Gamma_i \leq (1 + \beta)^{-1} \Gamma_1, \quad \forall i \in \sigma_i^{N+1}.
\] (41)

If one replaces \( y_1 \) and \( \Gamma_1 \) by their values, we deduce from (38) that \( \forall i \in \sigma_i^{N+1} \),
\[
\|z_i\| \leq \frac{(\theta + C_K \lambda \tau^2)^{i} \left(\tau + C_K \lambda \tau^2 \theta^{-1}\right)}{1 + C_K \lambda \tau^2 \theta^{-1}} \|z_0\|.
\] (42)

Because, \( \theta + C_K \lambda \tau^2 < 1 \) and \( \|z_0\| \leq 1 \) it follows
\[
\|z_i\| \leq \tau, \quad \forall i \in \sigma_i^{N+1},
\] (43)

and
\[
\|z_i\| \leq \tau, \quad \forall i \in \mathbb{N}.
\] (44)

Using (39) for all \( i \geq 0 \), we deduce that
\[
\lim_{i \to \infty} \|z_i\| = 0 \quad \text{since} \quad \theta + C_K \lambda \tau^2 \leq 1.
\] (45)

Therefore
\[
\lim_{i \to \infty} \|P^i(x_0)\| = 0.
\] (46)

This completes the proof. □

The following result assumes that \( 0 \in \bar{\Omega} \) (\( \bar{\Omega} \) denoted the interior of \( \Omega \)), this assumption is satisfied in any reasonable application and has nice consequences (see [13]). In practice, the set \( \Omega \) has the form
\[
\Omega = \{y \in \mathbb{R}^p \mid f_i(y) \leq 0, \quad f_i(y) \leq 0, \quad \ldots, \quad f_i(y) \leq 0\},
\] (47)

where \( s \in \mathbb{N} \) and \( f_i, i \in \sigma_i^N \) are continuous functions such that \( f_i(0) \leq 0, \forall i \in \sigma_i^N \) implies \( 0 \in \Theta^i(\bar{\Omega}) \) and then the set \( \Theta^i(\bar{\Omega}) \) nonempty. Indeed, the set \( \Theta^i(\bar{\Omega}) \) is given by
\[
\Theta^i(\bar{\Omega}) = \{x_0 \in B(0, \lambda) \cap \mathbb{R}^n \mid C \Phi^i(x_0) \in \Omega, \forall i \in \mathbb{N}\}.
\] (48)
\[ \mathcal{X}_0 = \{ x_0 \in B(0, \lambda) \mid f_j(C\Phi^i(x_0)) \leq 0, \forall j \in \Theta_i^k, \forall i \in \mathbb{N} \}, \]
and we have
\[ f_j(C\Phi^i(0)) = f_j(0) \leq 0, \forall j \in \Theta_i^k, \forall i \in \mathbb{N}, \quad (43) \]
since \(\Phi(0) = 0\).
Hence \(0 \in \Theta_i^k(\Omega)\) because \(0 \in B(0, \lambda)\).
Imposing special conditions on \(\bar{A}, \bar{R}, \bar{B}, j \in \{1, \ldots, q\}\) and \(\Omega\) which imposes corresponding conditions on \(\Theta_i^k(\Omega)\).

**Proposition 2:** 1) If \(\Omega\) is closed, then the set \(\Theta_i^k(\Omega)\) is also closed.
2) If we assume the following assumptions to hold:
(a) there exists \(\tau \geq 1\) and \(\theta \in [0, 1]\) such that \(\|A^\tau\| \leq \tau \theta^\tau, \forall k \in \mathbb{N}\),
(b) \(\theta + C_k \lambda \tau^2 \leq 1\), \(C_k = \sum_{j=1}^q \| (\bar{R}^j)^\top \| \|B_j\|\),
(c) \(0_{2p} \in \hat{\Omega}\),
then \(0_{2p} \in \Theta_i^k(\Omega)\).

**Proof:** 1) Let \(i \in \mathbb{N}\). Then we define the functions
\[ \Psi_i : B(0, \lambda) \cap \mathbb{R}^p \to \mathbb{R}^p, \quad x_0 \mapsto C\Phi^i(x_0). \quad (44) \]
Using these functions, the set \(\Theta_i^k(\Omega)\) can be written as
\[ \Theta_i^k(\Omega) = \bigcap_{i \in \mathbb{N}} \Psi_i^{-1}(\Omega). \quad (45) \]
Since \(\Omega\) is closed and \(\Psi_i, i \geq 0\) are continuous functions (since \(\psi(x) = \sum_{j=1}^q K_jx^jB_jx\) is continuous), \(\Psi_i^{-1}(\Omega)\) is closed and then \(\Theta_i^k(\Omega)\) is closed.
2) By hypothesis (a), (b), and (c), and from Proposition 1 we have
\[ \forall z_0 \in B(0, \lambda), \lim_{i \to \infty} \|\Phi^i(z_0)\| = 0. \quad (46) \]
This leads to
\[ \forall z_0 \in B(0, \lambda), \forall \epsilon > 0, \exists i_0 \in \mathbb{N} : \|\Phi^i(z_0)\| \leq \epsilon, \forall i \geq i_0. \]
i.e.,
\[ \forall z_0 \in B(0, \lambda), \forall \epsilon > 0, \exists i_0 \in \mathbb{N} : \Phi^i(z_0) \in B(0, \epsilon), \forall i \geq i_0. \]
On the other hand
\[ 0 \in \hat{\Omega} \implies \exists \eta > 0 : B(0, \eta) \subset \Omega. \quad (47) \]
For \(\epsilon = \frac{\eta}{\|C\|}\), we have
\[ \forall z_0 \in B(0, \lambda), \exists i_0 \in \mathbb{N} \]
such that
\[ C\Phi^i(z_0) \in B(0, \eta) \subset \Omega, \forall i \geq i_0. \quad (48) \]
It remains to show that
\[ C\Phi^i(z_0) \in \Omega, \forall i \in \sigma_0^{i_0-1}. \quad (49) \]
\(\Phi^{i_0-1}\) continuous in 0 implies
\[ \exists \eta_{i_0-1} > 0, \forall z_0 \in B(0, \eta_{i_0-1}), \quad \|\Phi^{i_0-1}(z_0) - \Phi^{i_0-1}(0)\| \leq \eta \frac{\|C\|}{\|C\|} \]
\[ \implies \exists \eta_{i_0-1} > 0, \forall z_0 \in B(0, \eta_{i_0-1}), \quad C\Phi^{i_0-1}(z_0) \in \Omega. \]
\(\Phi^{i_0-2}\) continuous in 0 implies
\[ \exists \eta_{i_0-2} > 0, \forall z_0 \in B(0, \eta_{i_0-2}), \quad C\Phi^{i_0-2}(z_0) \in \Omega \]
by a similar way, using the continuity of \(\Phi^j, j \in \sigma_1^{i_0-3}\) in 0, we obtain
\[ \forall j \in \sigma_1^{i_0-3}, \exists \eta_j > 0, \forall z_0 \in B(0, \eta_j), \quad C\Phi^j(z_0) \in \Omega. \quad (50) \]
We have also
\[ \forall z_0 \in B \left( \frac{\eta}{\|C\|}, \Omega \right), \quad C\Phi^i(z_0) \in \Omega. \quad (51) \]
If we take \(\zeta = \inf(\eta_{i_0-1}, \ldots, \eta_{i_0-1}, \eta\frac{\|C\|}{\|C\|})\), we get
\[ \forall z_0 \in B(0, \zeta), \quad C\Phi^i(z_0) \in \Omega, \forall i \in \sigma_0^{i_0-1}. \quad (52) \]
If we choose this time \(\zeta = \inf(\lambda, \zeta)\), it follows
\[ \forall z_0 \in B(0, \zeta), \quad C\Phi^i(z_0) \in \Omega, \forall i \in \mathbb{N}. \quad (53) \]
Thus \(B(0, \zeta) \subset \Theta_i^k(\Omega)\), and consequently \(0 \in \Theta_i^k(\Omega)\).
This completes the proof. \qed

### 3. Characterization of the Maximal Output \(\lambda\)-Admissible Set

It is more difficult to characterize the set \(\Theta_i^k(\Omega)\) defined in (19), for that reason we define for each \(k \in \mathbb{N}\) the set
\[ \Theta^k_i(\Omega) = \{ x_0 \in B(0, \lambda) \cap \mathbb{R}^n : C\Phi^i(x_0) \in \Omega, \forall i \in \sigma_0^{i_0} \}, \quad (54) \]
and we introduce the following definition.

**Definition 2:** The set \(\Theta_i^k(\Omega)\) is said to be finitely determined if there exists an integer \(k_0\) such that \(\Theta_i^{k_0}(\Omega) = \Theta_i^{k_0}(\Omega)\).

**Remark 4:** 1) For \(r, s \in \mathbb{N}\) such that \(r \leq s\), we have
\[ \Theta_i^k(\Omega) \subset \Theta_i^k(\Omega) \subset \Theta_i^k(\Omega). \quad (55) \]
2) If $\Theta^k(\Omega)$ is finitely determined, and $k_0$ the smallest $k$ such that $\Theta_{k_0}^k(\Omega) = \Theta_{k+1}^k(\Omega)$, then we have

$$\Theta^k(\Omega) = \Theta_{k_0}^k(\Omega), \forall k \geq k_0. \quad (56)$$

Conditions that demonstrate finite determinability, are examined in the following proposition.

**Proposition 3:** 1) If $\Theta^k(\Omega)$ is finitely determined then there exists $k_0 \in \mathbb{N}$ such that $\Theta_{k_0}^k(\Omega) = \Theta_{k+1}^k(\Omega)$.

2) If $\Phi(B(0, \lambda)) \subset B(0, \lambda)$ and $\Theta_k^k(\Omega) = \Theta_{k_0}^k(\Omega)$ for some $k_0 \in \mathbb{N}$ then $\Theta^k(\Omega)$ is finitely determined.

**Proof:** 1) Assume that the set $\Theta^k(\Omega)$ is finitely determined. Then by Definition 2 it follows

$$\exists k_0 \in \mathbb{N} \text{ such that } \Theta^k(\Omega) = \Theta_{k_0}^k(\Omega). \quad (57)$$

Using Remark 4 we have $\Theta_{k+1}^k(\Omega) \subset \Theta_{k_0}^k(\Omega)$ since $k + 1 \geq k_0$. On the other hand,

$$\Theta^k(\Omega) \subset \Theta_{k+1}^k(\Omega). \quad (58)$$

We deduce from (57) and (58) that

$$\Theta_{k_0}^k(\Omega) \subset \Theta_{k_0+1}^k(\Omega). \quad (59)$$

Therefore

$$\Theta_{k_0}^k(\Omega) = \Theta_{k_0+1}^k(\Omega) \text{ for some } k_0 \in \mathbb{N}. \quad (60)$$

2) Let $x_0 \in \Theta_{k_0}^k(\Omega)$, then $x_0 \in \Theta_{k_0+1}^k(\Omega)$.

This leads

$$x_0 \in B(0, \lambda), \quad C\Phi(x_0) \in \Omega, \forall i \in \sigma_{k_0+1}.$$ 

it follows

$$C\Phi^j(\Phi(x_0)) \in \Omega, \forall i \in \sigma_{k_0}. \quad (62)$$

Thus $\Phi(x_0) \in \Theta_{k_0}^k(\Omega)$ since

$$\Phi(x_0) \in \Phi(B(0, \lambda)) \subset B(0, \lambda). \quad (63)$$

By iteration

$$x_0 \in \Theta_{k_0}^k(\Omega) \implies \Phi(x_0) \in \Theta_{k_0}^k(\Omega), \forall j \in \mathbb{N}$$

$$\implies C\Phi^j(\Phi(x_0)) \in \Omega, \forall i \in \sigma_{k_0}$$

$$\implies C\Phi(x_0) \in \Omega, \forall i \in \mathbb{N}$$

$$\implies x_0 \in \Theta^k(\Omega) \text{ since } x_0 \in B(0, \lambda).$$

Hence

$$\Theta_{k_0}^k(\Omega) \subset \Theta^k(\Omega). \quad (64)$$

Using Remark 4, we conclude

$$\Theta_{k_0}^k(\Omega) = \Theta^k(\Omega) \text{ for some } k_0 \in \mathbb{N}, \quad (65)$$

which completes the proof. \qed

4. **SUFFICIENT CONDITIONS FOR FINITE DETERMINATION OF $\Theta^k(\Omega)$**

The following two theorems are the main results that give simple conditions to guarantee the finite determination of the set $\Theta^k(\Omega)$.

**Theorem 1:** Assume the following hypothesis to hold

(i) There exists $T \geq 1$ and $\theta \in [0, 1]$ such that $\|z\| \leq T\theta^k$, $\forall k \in \mathbb{N}$,

(ii) $0 \in \Omega$,

(iii) $\Phi(\Omega) \subset B(0, \lambda)$.

Then, $\Theta^k(\Omega)$ is finitely determined.

**Theorem 2:** If we assume that

(i) $\|\Phi(z)\| \leq \nu\|z\|$, $\forall z \in \mathbb{R}^p$ and $\nu \in [0, 1]$.

(ii) $0 \in \Omega$.

Then, $\Theta^k(\Omega)$ is finitely determined.

5. **ALGORITHMIC DETERMINATION**

In this section we note $k^*_{\Omega}$ the smallest integer such that

$$\Theta^{k^*_{\Omega}}(\Omega) = \Theta_{k^*_{\Omega}}(\Omega), \quad (66)$$

we call $k^*_{\Omega}$ the admissibility index. We shall provide a useful algorithm (see [13]) for identifying this admissibility index $k^*_{\Omega}$ and therefore the characterization of the set $\Theta^k(\Omega)$.

**Remark 5:** 1) Algorithm 1 produce $k^*_{\Omega}$ and $\Theta^k(\Omega)$ if and only if $\Theta^k(\Omega)$ is finitely determined.

2) If Algorithm 1 does not converge then the set $\Theta^k(\Omega)$ is not finitely determined.

Algorithm 1 is not practical due to the fact it does not describe how the test $\Theta^k(\Omega) = \Theta_{k_0}^k(\Omega)$ is implemented. To deal with this difficulty we will consider the following approach.

Let $\Omega$ be defined as

$$\Omega = \{y \in \mathbb{R}^p/ f_1(y) \leq 0, f_2(y) \leq 0, \ldots, f_q(y) \leq 0\}. \quad (67)$$

**Algorithm 1:** Determination of $k^*_{\Omega}$.

**Require:** $\lambda > 0$, $n$, $p$, $q \in \mathbb{N}^*$, $A$, $C$, $\psi$, $\Omega \subset \mathbb{R}^p$

$k ← 0$

if $\Theta^k(\Omega) = \Theta_{k+1}^k(\Omega)$ then

let $k^*_{\Omega} ← k$ and stop

else

let $k ← k + 1$ and continue (return to previous test)

end if
Require: \( \lambda > 0, n, p, q, s \in \mathbb{N}, \tilde{A}, C, \Psi, f_i, i = 1, \ldots, s \)
\( k \leftarrow 0 \)
Repeat
for \( i = 1, \ldots, s \) do
Maximize \( J_i(x) = f_i(C\Phi^{k+1}(x)) \)
Subject to the constraints
\[
\begin{align*}
&f_j(C\Phi^k(x)) \leq 0, \ x \in B(0, \lambda), \\
&\forall j \in \{1, \ldots, s\}, \ \forall l \in \{0, \ldots, k\}.
\end{align*}
\]
end for
\( J_i^* \leftarrow \max\{J_i(x)\} \)
if \( J_i^* \leq 0, \forall i = 1, 2, \ldots, s \) then
\( k^* \leftarrow k \)
break
else
\( k \leftarrow k + 1 \)
end if

Thus, for all \( k \in \mathbb{N} \), the set \( \Theta_k^2(\Omega) \) can be written as
\[
\Theta_k^2(\Omega) = \{x \in B(0, \lambda) \mid f_j(C\Phi^k(x)) \leq 0, \ j = 1, \ldots, s, \ i = 0, \ldots, k\}. \tag{68}
\]

On the other hand,
\[
\Theta_{k+1}^2(\Omega) = \{x \in \Theta_k^2(\Omega, K) \mid f_j(C\Phi^{k+1}(x)) \leq 0, \ j = 1, \ldots, s\}. \tag{69}
\]

Now, since \( \Theta_{k+1}^2(\Omega) \subset \Theta_k^2(\Omega) \) for all \( k \in \mathbb{N} \), we deduce
\[
\Theta_{k+1}^2(\Omega) = \Theta_k^2(\Omega, K).
\]

Therefore, the test \( \Theta_k^2(\Omega) = \Theta_{k+1}^2(\Omega) \) leads to a set of mathematical programming problems. We will suggest another version of Algorithm 1, this new algorithm is given by Algorithm 2.

**Remark 6:** If the matrix \( \tilde{A} \) is Lyapunov stable then the supremum in Algorithm 2 is defined for all \( x \in \mathbb{R}^n \). Indeed, using the notation defined in Proposition 1 it follows
\[
\|C\Phi^k(x)\| = \left\| C\tilde{A}x + C \sum_{j=1}^{q} R^jxBx \right\| \leq \|C\tilde{A}x\| + \|C\| \|K\| \|x\|^2. \tag{70}
\]

On the other hand, the assumption of Lyapunov stability of \( \tilde{A} \) implies \( \|C\tilde{A}x\| \leq \gamma \|x\| \) for some positive \( \gamma \). This leads
\[
\|C\Phi^k(x)\| \leq (\gamma + \|C\| \|K\| \|x\|) \|x\|. \tag{71}
\]

Let \( k \in \mathbb{N} \) and define the function \( F \) by
\[
F(\alpha) = (\gamma + \|C\| \|K\| \|x\|) \alpha, \ \forall \alpha \in \mathbb{R}. \tag{73}
\]

Then
\[
\|C\Phi^{k+1}(x)\| \leq F^{k+1}(\|x\|), \tag{74}
\]
and
\[
C\Phi^{k+1}(x) \in B(0, F^{k+1}(\|x\|)), \tag{75}
\]
using the compactness of \( B \) and the continuity of the functions \( f_i \), the sequence \( f_i(C\Phi^{k+1}(x)) \) is bounded from above.

**Remark 7:** 1) This algorithm can not be effectively used if there is not a feasible way to find global optima. When \( \Omega \) is a polyhedron (i.e., the functions \( f_j \) are affine for all \( j \)), the difficulty becomes less.

2) Assumptions of our two previous results (Theorem 1 and 2 of Section 4) are sufficient but not necessary. If these assumptions are not established, then the convergence of Algorithm 2 is not guaranteed.

To illustrate our results, we should give several examples, which will be given in the upcoming section.

### 5.1. Algorithm 2 steps
In case \( k = 0 \) we have
for \( i = 1 \)
max \( f_1(C\Phi(x)) \)
\[
s.\text{c.} f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0,
\]
\[
|x_1| \leq \lambda, \ldots, |x_n| \leq \lambda,
\]
for \( i = 2 \)
max \( f_2(C\Phi(x)) \)
\[
s.\text{c.} f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0,
\]
\[
|x_1| \leq \lambda, \ldots, |x_n| \leq \lambda,
\]
we continue like this until \( i = s \)
max \( f_s(C\Phi(x)) \)
\[
s.\text{c.} f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0,
\]
\[
|x_1| \leq \lambda, \ldots, |x_n| \leq \lambda.
\]
If \( f_s(C\Phi(x)) \leq 0, \forall i \in \{1, \ldots, s\} \) then we stop and \( k^* = 0 \) else we continue.

In case \( k = 1 \) we have
for \( i = 1 \)
max \( f_1(C\Phi^2(x)) \)
\[
s.\text{c.} f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0,
\]
\[
|x_1| \leq \lambda, \ldots, |x_n| \leq \lambda,
\]
for $i = 2$

$$\max f_i(C\Phi^2(x))$$

\[
\begin{cases}
  f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0, \\
  f_1(C\Phi(x)) \leq 0, \ldots, f_s(C\Phi(x)) \leq 0, \\
  |x_1| \leq \lambda, \ldots, |x_n| \leq \lambda,
\end{cases}
\]

we continue until $i = s$

$$\max f_s(C\Phi^3(x))$$

\[
\begin{cases}
  f_1(Cx) \leq 0, \ldots, f_s(Cx) \leq 0, \\
  f_1(C\Phi(x)) \leq 0, \ldots, f_s(C\Phi(x)) \leq 0, \\
  f_1(C\Phi^2(x)) \leq 0, \ldots, f_s(C\Phi^2(x)) \leq 0, \\
  |x_1| \leq \lambda, \ldots, |x_n| \leq \lambda.
\end{cases}
\]

If $f_i(C\Phi^k(x)) \leq 0$, $\forall i \in \{1, \ldots, s\}$ then we stop and $k^*_0 = k^*$ else we continue. We can use a predefined output function to be called at each iteration.

**Remark 8:** Let us suppose that the cost of function maximization is $\text{Cost}_{\max}$, the cost of repeat loop is $\text{Cost}_{\text{repeat}}$ and the cost associated to the computation of $C\Phi^i, i \in \{0, 1, \ldots, \text{Cost}_{\text{repeat}}\}$ is $\text{Cost}_{\text{for}}$. Then the computational complexity of Algorithm 2 is given by

$$\text{Cost}_{\text{algo}} = (s \times \text{Cost}_{\max} + 2s) \times \text{Cost}_{\text{repeat}} + \text{Cost}_{\text{for}}.$$  \hfill (76)

### 6. NUMERICAL EXAMPLES

To illustrate our results, we present some examples in two and three dimensional cases ($n = 2$ and $n = 3$). For this reason, we construct the matrices $K^j, j \in \sigma_i^0$ (in the case when $p = 1$) by the following way

$$K^j = \frac{1}{Nb \times \|B_j\| \times |C| \times q \times \lambda \times \tau^2},$$

where the choice of the number $Nb > 1$ depend on the value of $\theta$ which satisfies the first hypothesis of Proposition 1. This choice of $K^j$ guarantee the verification of the second hypothesis of Proposition 1. The matrix $\bar{A}$ is chosen such that $\|\bar{A}\| < 1$. Even if $\|A\| > 1$, this choice is possible by selecting some matrix $L$ since $\bar{A} = A + BLC$.

Using these matrices, we guarantee the asymptotic stability of the considered systems. Assumption $\Phi(B(0, \lambda)) \subset B(0, \lambda)$ is also taken into consideration. In all examples the dotted region will denote the set $\Theta^0(\Omega)$. For Example 4, we have $k^*_0 = \infty$, which means that Algorithm 2 does not converge. Various selections of matrices determining the systems are considered.

In Examples 1-7, we construct $\Omega$ as follows:

$$\Omega = \{ y \in \mathbb{R}^p \mid f_i(y) \leq 0, \ i = 1, \ldots, 2p \},$$

where $f_i: \mathbb{R}^p \rightarrow \mathbb{R}$ are defined for all $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$ by

\[
\begin{cases}
  f_{2m-1}(x) = x_m - \epsilon, \ m \in \{1, 2, \ldots, p\}, \\
  f_{2m}(x) = -x_m - \epsilon, \ m \in \{1, 2, \ldots, p\},
\end{cases}
\]

clearly $0 \in \Omega$, $f_i, i \in \sigma_i^0$ are continuous functions such that $f_i(0) \leq 0$, $\forall i \in \sigma_i^0$.

**Example 1:** Let $\lambda, q$ and $n$ defined by

$$\lambda = \frac{1}{2}, \ q = 1, \ n = 2.$$  

Let the matrix $\bar{A}, B_1, C$ and $\bar{K}^1$ defined by

$$\bar{A} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ $$

$$\bar{K}^1 = \begin{pmatrix} \frac{1}{64} \\ 0 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$  

Then the functions $\Phi(x), C\Phi(x)$ and $C\Phi^2(x)$ are defined by

\[
\Phi \left( \begin{array}{c}
  x \\
  y
\end{array} \right) = \left( \frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{8}y \right),
\]

$$C\Phi \left( \begin{array}{c}
  x \\
  y
\end{array} \right) = \frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6},$$

$$C\Phi^2 \left( \begin{array}{c}
  x \\
  y
\end{array} \right) = \frac{1}{36}x^2 + \frac{1}{36}y + \frac{1}{64} \left( \frac{1}{6}x^2 + \frac{1}{6}x + \frac{1}{6} \right)^2 + \frac{1}{384}x^2.$$

Using Algorithm 2 with $\epsilon = 0.2$ we get $k^*_0 = 0$

$$\Theta^0(\Omega) = \left\{ \left( \begin{array}{c}
  x \\
  y
\end{array} \right) \in \mathbb{R}^2 \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}, |x| \leq \epsilon \right\},$$

and using the same algorithm with $\epsilon = 0.1$ we get $k^*_0 = 1$

$$\Theta^0(\Omega) = \left\{ \left( \begin{array}{c}
  x \\
  y
\end{array} \right) \in \mathbb{R}^2 \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}, \left| \frac{1}{64}x^2 + \frac{1}{6}x + \frac{1}{6} \right| \leq \epsilon \right\}.$$
Example 2: Let $\lambda, q, n$ and $\epsilon$ be defined as

$\lambda = 1$, $q = 1$, $n = 2$, $\epsilon = 0.3$.

Let the matrix $\tilde{A}$, $C$, $\tilde{K}$ and $B_1$ defined as

$$
\tilde{A} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{8} \\
\frac{1}{2} & \frac{1}{8}
\end{pmatrix},
B_1 = \begin{pmatrix}
1 \\
2
\end{pmatrix},
\tilde{K} = \begin{pmatrix}
\frac{1}{8} \\
0
\end{pmatrix},
C = \begin{pmatrix}
1 & 0
\end{pmatrix}.
$$

Then the functions $\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, $C\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and $C\Phi^2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ are defined by

$$
\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \begin{pmatrix}
\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2}y \\
\frac{1}{2}x + \frac{1}{2}y
\end{pmatrix},
C\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{1}{8}x^2 + \frac{1}{6}x + \frac{1}{4}y,
C\Phi^2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{1}{512}x^4 + \frac{1}{192}x^3 + \frac{1}{128}x^2y + \frac{25}{288}x^2 + \frac{1}{96}xy + \frac{11}{72}x + \frac{1}{128}y^2 + \frac{7}{96}y.
$$

Using Algorithm 2 we obtain $k_0^1 = 1$, and then the set $\Theta^1(\Omega)$

$$
\Theta^1(\Omega) = \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\} \in \mathbb{R}^2 | \begin{array}{c} |x| \leq 1, |y| \leq 1, |x| \leq \epsilon, \\
|x| \leq \epsilon, \epsilon \leq 1, \epsilon \leq 1
\end{array} \right\}. \epsilon
$$

Example 3: Let $\lambda, q, n$ and $\epsilon$ be defined as

$\lambda = 2$, $q = 2$, $n = 2$ and $\epsilon = 0.1$.

Let the matrix $\tilde{A}$, $C$, $\tilde{K}$, $\tilde{K}^2$, $B_1$ and $B_2$ be defined as

$$
\tilde{A} = \begin{pmatrix}
\frac{4}{3} & \frac{1}{3} \\
\frac{4}{3} & \frac{1}{3}
\end{pmatrix},
B_1 = \begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix},
B_2 = \begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix},
\tilde{K} = \begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix},
\tilde{K}^2 = \begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix},
C = \begin{pmatrix}
1 & 0
\end{pmatrix}.
$$

Then

$$
\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \begin{pmatrix}
\frac{1}{2}x + \frac{1}{2}y \\
\frac{1}{2}x + \frac{1}{2}y
\end{pmatrix},
C\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{1}{2}x - \frac{1}{2}y,
C\Phi^2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{1}{12}x - \frac{5}{12}y,
C\Phi^3\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{23}{72}x - \frac{7}{72}y.
$$

Using Algorithm 2 we obtain $k_0^1 = 2$, and then the set $\Theta^1(\Omega)$

$$
\Theta^1(\Omega) = \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right\} \in \mathbb{R}^2 | \begin{array}{c} |x| \leq \lambda, |y| \leq \lambda, |x| \leq \epsilon, \\
\frac{1}{2}x - \frac{1}{2}y \leq \epsilon, \\
\frac{1}{12}x - \frac{5}{12}y \leq \epsilon
\end{array} \right\}. \epsilon
$$
Example 4: Let \( \lambda, q, n \) and \( \varepsilon \) defined by

\[
\lambda = 2, \quad q = 2, \quad n = 2, \quad \varepsilon = 0.3.
\]

Let the matrix \( \bar{A}, B_1, B_2, C, \bar{K}^1 \) and \( \bar{K}^2 \) defined as

\[
\bar{A} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \bar{K}^1 = \begin{pmatrix} \frac{1}{64} \\ 0 \end{pmatrix}, \quad \bar{K}^2 = \begin{pmatrix} \frac{1}{125} \\ 0 \end{pmatrix}.
\]

Then

\[
\Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{pmatrix}, \quad C\Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{pmatrix},
\]

\[
C\Phi^2 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{5}{12}x + \frac{7}{12}y \\ \frac{5}{12}x + \frac{7}{12}y \end{pmatrix}, \quad C\Phi^3 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{29}{72}x + \frac{43}{72}y \\ \frac{29}{72}x + \frac{43}{72}y \end{pmatrix},
\]

\[
C\Phi^4 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{173}{432}x + \frac{259}{432}y \\ \frac{173}{432}x + \frac{259}{432}y \end{pmatrix},
\]

\[
C\Phi^5 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1037}{2592}x + \frac{1555}{2592}y \\ \frac{1037}{2592}x + \frac{1555}{2592}y \end{pmatrix},
\]

\[
C\Phi^6 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{6221}{1552}x + \frac{9331}{1552}y \\ \frac{6221}{1552}x + \frac{9331}{1552}y \end{pmatrix},
\]

\[
C\Phi^7 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{37325}{93312}x + \frac{55987}{93312}y \\ \frac{37325}{93312}x + \frac{55987}{93312}y \end{pmatrix}.
\]

Using Algorithm 2 we get \( k_0^* = \infty \).

Example 5: Let \( \lambda, q, n \) and \( \varepsilon \) defined by

\[
\lambda = 2, \quad q = 2, \quad n = 2, \quad \varepsilon = 0.1.
\]

Let the matrix \( \bar{A}, B_1, B_2, C, \bar{K}^1 \) and \( \bar{K}^2 \) defined as

\[
\bar{A} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \bar{K}^1 = \begin{pmatrix} \frac{1}{64} \\ 0 \end{pmatrix}, \quad \bar{K}^2 = \begin{pmatrix} \frac{1}{125} \\ 0 \end{pmatrix}.
\]

Then, the functions \( \Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right), C\Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) \) and \( C\Phi^2 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) \) are defined by

\[
\Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{16}x^2 + \frac{1}{8}x + \frac{1}{4}y \\ \frac{1}{8}x + \frac{1}{4}y \end{pmatrix},
\]

\[
C\Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{16}x^2 + \frac{1}{8}x + \frac{1}{4}y \\ \frac{1}{8}x + \frac{1}{4}y \end{pmatrix},
\]

\[
C\Phi^2 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{4096}x^4 + \frac{1}{1024}x^3 + \frac{1}{512}x^2y + \frac{19}{1024}x^2 \\ \frac{1}{256}x^3y + \frac{19}{128}x^2y + \frac{1}{256}y^2 + \frac{1}{24}y \end{pmatrix}.
\]

Using Algorithm 2 we get \( k_0^* = 1 \).

\[
\Theta^4 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{array}{c} |x| \leq 2, |y| \leq 2, |x| \leq \varepsilon, \\ |y| \leq \varepsilon \end{array} \right\}.
\]

Example 6: Let \( \bar{A}, B_1, \bar{K}^1, C, q, \varepsilon \) and \( \lambda \) defined as

\[
C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \lambda = 2, \quad q = 1, \quad \varepsilon = 0.05,
\]

\[
B_1 = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & 0 & 4 \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 0 & \frac{1}{10} & 0 \end{pmatrix}, \quad \bar{K}^1 = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}.
\]

Then, we have

\[
\Phi \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \end{pmatrix},
\]

\[
\Phi \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) + \frac{1}{40}x \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z + \frac{1}{5}xz + \frac{1}{5}x^2 \end{pmatrix} + \frac{1}{40}x \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.
\]
Example 7: To show the efficiency of the stability we take the following example.

Let \( \lambda, q, n, \varepsilon \) and \( \tau \) defined as

\( \lambda = 1, \quad q = 1, \quad n = 2, \quad \varepsilon = 0.1, \quad \tau = 1. \)

Using Algorithm 2 we obtain \( k_0 = 2 \), and then

\[
\Theta^2(\lambda) = \begin{bmatrix} \frac{3}{160}y + \frac{13}{600}z + \frac{13}{600}z \end{bmatrix}.
\]

Let \( B_1, \tilde{A}, \tilde{K}^1, C, C_k \) and \( \theta \) defined by

\[
B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0.1 + \gamma & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \tilde{K}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_k = \frac{1}{10}, \quad \theta = \| \tilde{A} \|.
\]

Now, we use Algorithm 2 we obtain

\[
\gamma = \theta + C_k \lambda \tau^2 \quad k_0
\]

6.1. Comment

In Examples 1-7, we have determined the index of admissibility \( k_0 \) using the predefined function \texttt{fmincon} in
Matlab [28], which allow us to solve the maximization problems in Algorithm 2. In Figs. 1-5, we have plotted all the constraints forming the sets $\Theta^k(\Omega)$, in order to visualize such sets of Examples 1, 2, 3, and 5.

**Example 8:** In this example we take $\Omega$ in the form

$$\Omega = \{ y \in \mathbb{R} \mid f(y) = y^2 + y - 2 \leq 0 \}.$$  

It is easy to see that $0 \notin \Omega$. $f$ is continuous function such that $f(0) \leq 0$ and that $\Omega$ is closed since $\Omega = f^{-1}([0,0])$.

Now, we use the results of Example 3 and Algorithm 2 it follows

Maximize

$$f(C\Phi(x,y)) = \left(\frac{1}{2}x - \frac{1}{2}y\right)^2 + \left(\frac{1}{2}x - \frac{1}{2}y\right) - 2$$

s.c

$$\begin{cases}
  x^2 + x - 2 \leq 0, \\
  |x| \leq 2, \ |y| \leq 2,
\end{cases}$$

the solution of this problem gives: 1.7500.

Maximize

$$f(C\Phi^2(x,y)) = \left(\frac{1}{12}x - \frac{5}{12}y\right)^2 + \left(\frac{1}{12}x - \frac{5}{12}y\right) - 2$$

s.c

$$\begin{cases}
  x^2 + x - 2 \leq 0, \\
  |x| \leq 2, \ |y| \leq 2,
\end{cases}$$

the solution of this problem gives: $-1.2000e^{-06}$.

Therefore, the value of index of admissibility is $k^*_0 = 1$.

Hence

$$\Theta^k(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| \leq 2, \ |y| \leq 2, \ x^2 + x - 2 \leq 0, \right\}$$

Using results of Example 5 and Algorithm 2 we obtain $k^*_0 = 0$, and then the maximal output $\lambda$-admissible set $\Theta^k(\Omega)$ is given by

$$\Theta^k(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| \leq 2, \ |y| \leq 2, \ x^2 + x - 2 \leq 0 \right\}$$

6.2. Comment

In Examples 8 and 9, we have modified the set $\Omega$ which affects the mathematical programming problems that arise in Algorithm 2, we have plotted again the sets $\Theta^k(\Omega)$ in Figs. 6 and 7.

In order to obtain the numerical computational (elapsed time in seconds (s)) of Examples 1, 2, 3, 5, and 6 using Algorithm 2 the predefined output function fmincon (see [28]) is applied with different values for the coefficient $\lambda$, $\varepsilon$, $k^*_0$, $n$ to solve the maximization problems which arise in such algorithm. The obtained results are presented in Table 1.

**Remark 9:** In case $p \geq 2$ we have to reconstruct the matrices $K^i$ such that the condition $\theta + \lambda C_k r^2 < 1$ of Proposition 1 is verified.

| Table 1. Parameters values and elapsed time of Examples 1-3, 5, and 6. |
|---|---|---|---|---|---|---|
| Examples | $\hat{A}$ | $\lambda$ | $s$ | $k^*_0$ | $\varepsilon$ | Time (s) |
| Examples 1 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 0 | 0.2 | 1.660970 |
| Examples 1 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 2 | 0.1 | 3.1711 |
| Examples 2 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 1 | 0.3 | 3.2184 |
| Examples 3 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 2 | 0.1 | 3.0743 |
| Examples 5 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 2 | 0.1 | 3.5962 |
| Examples 6 | $\begin{pmatrix} \hat{A} & 1 \\ \hat{A} & 1 \end{pmatrix}$ | $\frac{1}{2}$ | 2 | 1 | 0.05 | 8.5662 |

Example 9: In this example the set $\Omega$ is represented by

$$\Omega = \{ y \in \mathbb{R} \mid f_1(y) = y^2 + y - 3 \leq 0, \ f_2(y) = y^2 + 2y - 4 \leq 0 \}.$$  

It is easy to see that $f_i(0) \leq 0, \forall i \in \{1, 2\}$ and the functions $f_i, i = 1, 2$ are continuous.
7. APPLICATIONS

This section aims to apply the approach developed in this paper to epidemiological and sociological models that includes control terms.

7.1. Application in epidemiological model (App 1)

We consider a deterministic model of susceptible-infected (SI), where $S_i$ represents the number of individuals susceptible to infection of the disease but not yet infected at the time $i$. $I_i$ denotes the number of individuals who have been infected with the disease, and they are infectious. These types of models are used to model many infectious diseases such as HIV. The model is based on the assumption that disease transmission happens as a result of an “effective” contact between a person who is susceptible and an infectious person. Because of the nonlinear contact dynamics within the population, the $\alpha SI$ incidence function is used to reflect successful disease transmission, where $\alpha$ is the rate of effective contact with infected individuals in compartment $I$.

The treatment and awareness campaigns are well known as efficient strategies to control and prevent the spread of such infectious diseases. Consequently, we introduce in the following system with two controls $u_i$ and $v_i$.

- $u_i$: refers to the effort to protect individuals who are susceptible to becoming infected. This effort to provide awareness.
- $v_i$: refers to the effort to provide treatment.

Therefore, we use the following model:

$$
\begin{align*}
S_{i+1} &= S_i - \alpha S_i I_i - u_i S_i, \\
I_{i+1} &= I_i + \alpha S_i I_i - v_i I_i, \\
S(0) &= S_0, I(0) = I_0.
\end{align*}
$$

(77)

The corresponding output is

$$
y_i = \gamma S_i + \beta I_i = (\gamma \beta) \begin{pmatrix} S_i \\ I_i \end{pmatrix},
$$

(78)

where $\gamma$ and $\beta$ can be the cost of treatment).

Take $x_i = \begin{pmatrix} S_i \\ I_i \end{pmatrix}$ and $u_i$, $v_i$ feedback controls defined by

$$
u_i = m + k y_i, \quad v_i = m + \tilde{k} y_i.
$$

(79)

We assume that our control depends on the level of endemicity in the population (given the observation times $k$ and $\tilde{k}$) and the available resource to control the spread of the disease ($m$ and $\tilde{m}$, respectively). In fact, the public health authorities are all-time facing limited resources to contain the infection. Therefore, our control is a function of the observation $y_i$, which gives the decision-maker an optimal way to treat and vaccinate the population.

Then, the system (77) can be rewritten as follows:

$$
\begin{align*}
S_{i+1} &= (1 - m) S_i - \alpha S_i I_i - k S_i I_i, \\
I_{i+1} &= (1 - \tilde{m}) I_i + \alpha S_i I_i - \tilde{k} S_i I_i, \\
i &\geq 0,
\end{align*}
$$

or equivalently

$$
\begin{align*}
\begin{pmatrix} S_{i+1} \\ I_{i+1} \end{pmatrix} &= \begin{pmatrix} 1 - m & 0 \\ 0 & 1 - \tilde{m} \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} \\
&+ (\alpha 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} \\
&+ (\alpha 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} \\
&+ (k 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix} \\
&+ (\tilde{k} 0) \begin{pmatrix} S_i \\ I_i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \end{pmatrix}.
\end{align*}
$$

This model can be written as

$$
\begin{align*}
x_{i+1} &= \tilde{A} x_i + \sum_{j=1}^{q} \tilde{K}^j x_i B_j x_i, \\
x_0 &\in \mathbb{R}^{2}, \\
y_i &= (\gamma \beta) x_i = C x_i,
\end{align*}
$$

(81)

where

$$
\tilde{A} = \begin{pmatrix} 1 - m & 0 \\ 0 & 1 - \tilde{m} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},
$$

$$
B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
B_4 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad K^1 = (\alpha 0),
$$

$$
K^2 = (\alpha 0), \quad K^3 = (k 0),
$$

$$
\tilde{K}^4 = (\tilde{k} 0), \quad q = 4.
$$

Now we select $\tilde{K}^3$ and $\tilde{K}^4$ as follows:

$$
\tilde{K}^3 = \frac{1}{4}, \quad \tilde{K}^4 = \frac{1}{3},
$$

and the parameter $\alpha$ is taken as

$$
\alpha = \frac{1}{2}.
$$

The second hypothesis of Proposition 1 is verified. Indeed,

$$
\theta + C_k \lambda \tau^2 = 0.3 + \frac{19}{12} \lambda \tau^2 = 0.99271 < 1,
$$

with $\theta = \|\tilde{A}\| = 0.3$, $\lambda = \frac{42}{96}$, $\tau = 1$ ($m = \tilde{m} = 0.7$).

Therefore,

$$
\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
$$
\[
\begin{align*}
\Phi &= \begin{pmatrix}
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{array}
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
x \\
y
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathcal{C}\Phi \left( \begin{array}{c}
x \\
y
\end{array} \right) &= \left( \frac{2}{3} \right) \begin{pmatrix}
0.3x - \frac{1}{2}xy - \frac{1}{4}x^2 \\
0.3y + \frac{1}{6}xy
\end{pmatrix} \\
&= 0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2,
\end{align*}
\]

\[
\Phi^2 \left( \begin{array}{c}
x \\
y
\end{array} \right) = \begin{pmatrix}
0.09x + \frac{1}{2} \left( 0.3y + \frac{1}{6}xy \right) \\
\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2
\end{pmatrix} - \frac{9}{40} \begin{pmatrix}
\frac{1}{2}xy - 0.3x + \frac{1}{4}x^2 \\
0.3y + \frac{1}{6}xy
\end{pmatrix}^2 - 0.11xy - 0.0675x^2.
\]

We have \( \Phi (B \left( \frac{0.42}{96} \right)) \subset B \left( \frac{0.42}{96} \right) \) since

\[
0.3 \frac{42}{96} + \frac{142}{96} + \frac{42}{496} = 0.2748 < 42 \frac{96}{96},
\]

and

\[
0.3 \frac{42}{96} + \frac{142}{96} + \frac{42}{96} = 0.16315 < 42 \frac{96}{96}.
\]

**Algorithm 2**

In case \( k = 0 \), we have

for \( i = 1 \)

\[
\begin{align*}
\text{s.c.} & \quad \begin{cases}
\frac{9}{10}x + \frac{1}{2}y \leq 0, \\
|x| \leq \frac{42}{96}, |y| \leq \frac{42}{96},
\end{cases}
\end{align*}
\]

we have \( J_1^1, J_2^1 > 0 \), we then go to the next step \( (k = 1) \).

For \( i = 1 \)

\[
\begin{align*}
\text{s.c.} & \quad \begin{cases}
0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 \leq 0.03, \\
|x| \leq \frac{42}{96}, |y| \leq \frac{42}{96},
\end{cases}
\end{align*}
\]

since \( J_1^1, J_2^2 \leq 0 \), we stop.

The computational complexity of applying this algorithm (see Remark 8) is described as follows:

\[
\text{Cost}_{\text{algo}} = (s \times \text{Cost}_{\text{max}} + 2s) \times \text{Cost}_{\text{Repeat}} + \text{Cost}_{\text{fct}},
\]

where \( s = 2 \), \( \text{Cost}_{\text{Repeat}} = 2 \), \( \text{Cost}_{\text{fct}} \) is the number of operations to compute \( C \left( \begin{array}{c} x \\ y \end{array} \right), \mathcal{C}\Phi \left( \begin{array}{c} x \\ y \end{array} \right) \) and \( \mathcal{C}\Phi^2 \left( \begin{array}{c} x \\ y \end{array} \right) \) and \( \text{Cost}_{\text{max}} \) is the number of operations \((\times\text{and}+)\) applying the predefined output function \textbf{fmincon} from the Matlab Optimization Toolbox [28] to solve the maximization problems which arise in each iteration.

Thus, \( k_0^1 = 1 \) and the set \( \Theta^4(\Omega) \) is given by

\[
\Theta^4(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \begin{align*}
0 & \leq x \leq \frac{42}{96}, \\
0 & \leq y \leq \frac{42}{96}, \\
0.27x + 0.15y - \frac{11}{30}xy - \frac{9}{40}x^2 & \leq 0.03,
\end{align*} \right\}
\]

The plot of the set \( \Theta^4(\Omega) \) is given in Fig. 8.

7.2. Application in rumor spreading model (App 2)

Our next application is to consider a SIR type model of rumor [29], where the population is divided into three categories: ignorants (people who never heard the rumor), spreaders (people who have heard the rumor and transmitted it to others), and stiflers (having heard the rumor but
have not passed it to others). The variables \( S_i, I_i \), and \( R_i \) to denote the population densities of ignorant users, spreading users, and stifler users, respectively. This model’s dynamic is based on the following assumption: Rumor transmission happens as a result of adequate contact between a person who is ignorant and a person who is a spreader. The ignorant person accepts the rumor and becoming a spreading at a rate \( \beta \). We assume that the propagation of a constant rumor in a population with a constant emigration rate \( \mu \).

Our model is described by

\[
\begin{aligned}
S_{i+1} &= S_i - \beta S_i I_i - \mu S_i - u S_i, \\
I_{i+1} &= I_i + \beta S_i I_i - \mu I_i, \quad i \geq 0, \\
R_{i+1} &= R_i - \mu R_i + u S_i,
\end{aligned}
\]

where \( u_i \) represent the awareness and refers to the effort to protect individuals who are ignorants against becoming spreaders.

The input \( u_i \) is taken as feedback of \( y_i \), i.e.,

\[
u_i = k y_i,
\]

where \( y_i \) is the corresponding output, i.e.,

\[
y_i = I_i = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} S_i \\ I_i \\ R_i \end{pmatrix} = C x_i.
\]

Using the same approach developed before it follows

\[
\begin{aligned}
x_{i+1} = \tilde{A} x_i + \sum_{j=1}^{q} \tilde{K}^j x_j B x_i, \quad i \geq 0, \\
x_0 \in \mathbb{R}^3,
\end{aligned}
\]

where

\[
\tilde{A} = \begin{pmatrix} 1 - \mu & 0 & 0 \\ 0 & 1 - \mu & 0 \\ 0 & 0 & 1 - \mu \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\tilde{K}^1 = (0 \ \beta \ 0), \quad \tilde{K}^2 = (0 \ \beta \ 0), \quad \tilde{K}^3 = (0 \ \beta \ 0), \quad \tilde{K}^4 = (0 \ \beta \ 0), \quad q = 4.
\]

We choose \( \lambda, \mu, \beta \) and \( k \) as

\[
\lambda = 1, \quad \mu = 0.2, \quad \beta = k = \frac{1}{18}.
\]

Therefore

\[
\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.8x - \frac{1}{18} xy \\ 0.8y + \frac{1}{18} xy \\ 0.8z + \frac{1}{18} xy \end{pmatrix},
\]

\[
C \Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.8y + \frac{1}{18} xy.
\]

**Algorithm 2**

In case \( k = 0 \), we have

for \( i = 1 \)

\[
\max 0.8y + \frac{1}{18} xy - 0.9
\]

\[
s.t. \begin{cases}
|y| \leq 0.9, \\
|x| \leq 1, \ |y| \leq 1,
\end{cases}
\]

for \( i = 2 \)

\[
\max - \left( 0.8y + \frac{1}{18} xy \right) - 0.9
\]

\[
s.t. \begin{cases}
|y| \leq 0.9, \\
|x| \leq 1, \ |y| \leq 1,
\end{cases}
\]

since \( J_1, J_2 \leq 0 \), we stop.

The computational complexity of applying this algorithm (see Remark 8 ) is described as follows:

\[
Cost_{algo} = (s \times Cost_{max} + 2s) \times Cost_{repeat} + Cost_{fct},
\]

where \( Cost_{repeat} = 1 \) and \( s = 2 \).

Therefore \( k_0 = 0 \) and \( \Theta^k(\Omega) \) is given by

\[
\Theta^k(\Omega) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 0 \leq x \leq 1, \ 0 \leq y \leq 0.9, \ 0 \leq z \leq 1 \right\}.
\]

The plot of the set \( \Theta^k(\Omega) \) is given in Fig. 9. In order to obtain the numerical computational (elapsed time in seconds (s)) of two applications (Apps) using Algorithm 2, the predefined output function \texttt{fmicon} (see [28]) is applied with different values for the coefficient \( \lambda, \epsilon, k_0, n \) to solve the maximization problems which arise in such algorithm. The obtained results are presented in Table 2.
Table 2. Parameters values and elapsed time of two applications.

| Apps | $\hat{A}$          | $\lambda$ | $s$ | $k_0$ | $\varepsilon$ | Time (s) |
|------|---------------------|-----------|-----|-------|---------------|----------|
| App 1 | $\begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}$ | $\frac{\pi}{18}$ | 2   | 1     | 0.03          | 5.444    |
| App 2 | $\begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}$ | 2         | 0   | 0.9   | 2.856         |

Fig. 9. The set $\Theta^k(\Omega)$.

Remark 10: The computational complexity of Algorithm 2 depend on the following factors:
1) The maximization problems which occur in such algorithm,
2) the value of index of admissibility $k_0$,
3) the dimension $n$, 
4) the parameters $\lambda$, $\varepsilon$ and $s$.

In the following section, we will present a conclusion of our work.

8. CONCLUSION

This paper presented a new problem of maximal output admissible set for discrete-time bilinear controlled systems. First, we gave a new sufficient condition that assures the asymptotic stability of the system. Then, we characterized the maximal output $\lambda$-admissible set $\Theta^k(\Omega)$, and we showed that, under certain specific conditions, the finite determinations of $\Theta^k(\Omega)$. Moreover, we proposed an algorithmic approach to determine the admissibility index $k_0$ and, consequently, the set $\Theta^k(\Omega)$. Our approach was illustrated with several numerical examples with different dimensions and parameters. In these examples, we were able to determine the maximal output and to sketch it in the 2 and 3 dimensions.

To give more applications to our approach, we applied our technique to an SI epidemic model. Our goal was to find the initial data that guarantees the control of the spread of the epidemic via awareness and treatment. In addition, we also used our approach of a rumor spread model. To contain the spread of a rumor, the control term aimed to increase the awareness of the population. In the future and inspired by the work of Larrache et al. [30], we investigate the problem of the maximal output set for discrete-time bilinear distributed systems.

APPENDIX A: PROOFS OF THEOREMS

A.1. Proof of Theorem 1

By hypothesis (i), (ii), (iii), we can show from Proposition 2 that

$$\forall z_0 \in B(0, \lambda), \exists \eta_0 \in \mathbb{N} : C\Phi_i(z_0) \in B(0, \eta) \subset \Omega, \forall i \geq i_0.$$ (A.1)

For $i = i_0$, we have

$$\forall z_0 \in B(0, \lambda), C\Phi_i(z_0) \in \Omega.$$ (A.1)

Let $z_0 \in \Theta^k_{\lambda_i}(\Omega)$, then

$$z_0 \in B(0, \lambda) \text{ and } C\Phi_i(\Omega) \in \Omega, \forall i \in \sigma^{h_i}_{\lambda_i-1},$$ (A.2)

using (A.1) we conclude that $z_0 \in \Theta^k_{\lambda_i}(\Omega)$. Thus $\Theta^k_{\lambda_i-1}(\Omega)$ is finitely determined (using this time hypothesis (iv) and Proposition 3).

A.2. Proof of Theorem 2

It is easy to verify that

$$\|\Phi_i(z_0)\| \leq \|v^i\|z_0\|, \forall i \in \mathbb{N}.$$ (A.3)

Using limit definition ($\lim_{v^i \rightarrow 0} v^i = 0$) it follows

$$\exists \eta_0 \in \mathbb{N}, \forall i \geq i_0.$$ (A.4)

For $z_0 \in B(0, \lambda)$ we have

$$\|C\Phi_i(z_0)\| \leq \eta, \forall i \geq i_0.$$ (A.5)

Let $z_0 \in \Theta^k_{\lambda_i-1}(\Omega)$. Then

$$z_0 \in B(0, \lambda) \text{ and } C\Phi_i(z_0) \in \Omega, \forall i \in \sigma^{h_i}_{\lambda_i-1}.$$ (A.6)

From (A.5) we have $C\Phi_i(z_0) \in B(0, \eta) \subset \Omega$. Therefore $z_0 \in \Theta^k_{\lambda_i}(\Omega)$.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.
REFERENCES

[1] D. I. Martinez, J. de J. Rubio, T. M. Vargas, V. Garcia, G. Ochoa, R. Balcazar, D. R. Cruz, A. Aguilar, J. F. Novoa, and C. Aguilar-Ibáñez, “Stabilization of robots with a regulator containing the sigmoid mapping,” IEEE Access, vol. 8, pp. 89479-89488, 2020.

[2] J. de J. Rubio, G. Ochoa, D. Mujica-Vargas, E. Garcia, R. Balcazar, I. Elias, D. R. Cruz, C. F. Juarez, A. Aguilar, and J. F. Novoa, “Structure regulator for the perturbations attenuation in a quadrotor,” IEEE Access, vol. 7, pp. 138244-138252, 2019.

[3] J. O. Escobedo-Alva, E. C. Garcia-Estrada, A. Paramo-Carranza, J. A. Meda-Campana, and R. Tapia-Herrera, “Theoretical application of a hybrid observer on altitude tracking of quadrotor losing GPS signal,” IEEE Access, vol. 6, pp. 76900-76908, 2018.

[4] J. de J. Rubio, D. I. Martinez, V. Garcia, G. J. Gutierrez, T. M. Vargas, G. Ochoa, R. Balcazar, J. Pacheco, J. A. Medina-Camah, and D. Mujica-Vargas, “The perturbations estimation in two gas plants,” IEEE Access, vol. 8, pp. 83081-83091, 2020.

[5] C. Aguilar-Ibáñez and M. S. Suarez-Castanon, “A trajectory planning based controller to regulate an uncertain 3D overhead crane system,” Int. J. Appl. Math. Comput. Sci., vol. 29, pp. 693-702, 2019.

[6] J. H. Pérez-Cruz, P. A. Tamayo-Meza, M. Figueroa, R. Silva-Ortigoza, M. Ponce-Silva, R. Rivera-Blas, and M. Aldape-Pérez, “Exponential synchronization of chaotic Xian system using linear feedback control,” Complexity, vol. 2019, Article ID 4706491, 2019.

[7] P. O. Gutman and P. Hagander, “A new design of constrained controllers for linear systems,” IEEE Trans. Automat. Contr., vol. 30, no. 1, pp. 22-23, 1985.

[8] M. Rachik, A. Abdelhak, and J. Karrakchou, “Discrete systems with delays in state, control and observation: The maximal output sets with state and control constraints,” Optimization, vol. 42, pp. 169-183, 1997.

[9] M. Lhous, M. Rachik, and El M. Magri, “Ideal observability for bilinear discrete-time systems with and without delays in observation,” Archives of Control Sciences, vol. 28, no. 4, pp. 601-616, 2018.

[10] R. R. Mohler, Bilinear Control Processes, vol. 106 of Mathematics in Science and Engineering, Academic Press, New York, 1973.

[11] M. Ekman, Modeling and Control of Bilinear Systems: Applications to the Activated Sludge Process (in English), Acta Universitatis Upsaliensis, Upsala Dissertations from the Faculty of Science and Technology 65, pp. 231, Upsala, Sweden, 2005.

[12] P. A. Marchi, L. dos S. Coelho, and A. A. R. Coelho, “Comparative study of parametric and structural methodologies in identification of an experimental nonlinear process,” Proceedings of the IEEE International Conference on Control Applications (Cat. No.99CH36328), vol. 2, pp. 1062-1067, Kohala Coast, HI, USA, 1999.

[13] E. G. Gilbert and K. T. Tan, “Linear systems with state and control constraints: The theory and application of maximal output admissible sets,” IEEE Transactions on Automatic Control, vol. 36, no. 9, pp. 1008-1020, Sep. 1991.

[14] I. Kolmanovsky and E. G. Gilbert, “Theory and computation of disturbance invariant sets for discrete-time linear systems,” Mathematical Problems in Engineering, vol. 4, pp. 317-367, 1998.

[15] I. Kolmanovsky and E. G. Gilbert, “Multimode regulators for systems with state and control constraints and disturbance inputs,” Control Using Logic-based Switching, LNCS, vol. 222, pp. 104-117, Springer 1997.

[16] M. Rachik, M. Lhouss, and A. Tridane, “On the maximal output admissible set for a class of nonlinear discrete systems,” Systems Analysis Modelling Simulation, vol. 42, no. 11, pp. 1639-1658, 2002.

[17] K. Hirata and Y. Ohta, “Exact determinations of the maximal output admissible set for a class of nonlinear systems,” Proceedings of the 44th IEEE Conference on Decision and Control, pp. 8276-8281, 2005.

[18] H. R. Ossareh, “Reference governors and maximal output admissible sets for linear periodic systems,” International Journal of Control, vol. 93, pp. 113-125, 2019.

[19] K. Yamamoto, “Maximal output admissible set for trajectory tracking control of biped robots and its application to falling avoidance control,” Proc. of IEEE/RSJ International Conference on Intelligent Robots and Systems, Tokyo, pp. 3643-3648, 2013.

[20] J. Osorio and H. R. Ossareh, “A stochastic approach to maximal output admissible sets and reference governors,” Proc. of IEEE Conference on Control Technology and Applications (CCTA), pp. 704-709, 2018.

[21] A. El Bhih, Y. Benfatah, and M. Rachik, “Exact determination of maximal output admissible set for a class of semi-linear discrete systems,” Archives of Control Sciences, vol. 30, no. 3, pp. 523-552, 2020.

[22] A. Abdelhak and M. Rachik, “Model reduction problem of linear discrete systems: Admissibles initial states,” Archives of Control Sciences, vol. 29, no. 1, pp. 41-55, 2019.

[23] M. S. Darup and M. Mönnigmann, “Computation of the largest constraint admissible set for linear continuous-time systems with state and input constraints,” Proceedings of the 19th World Congress, The International Federation of Automatic Control, Cape Town, South Africa, August 24-29, 2014.

[24] K. Yamamoto, “Time-variant feedback controller based on capture point and maximal output admissible set of a humanoid,” Advanced Robotics, vol. 33, no. 18, pp. 944-955, 2019.

[25] C. E. T. Dórea and J. C. Hennet, “Computation of maximal admissible sets of constrained linear systems,” Proceedings of the 4th IEEE Mediterranean Symposium on New Directions on Control and Automation, Maleme, Greece, pp. 286-291, 1996.
[26] A. El Bhih, Y. Benfatah, S. B. Rhila, M. Rachik, and A. El A. Llaaroussi, “A spatiotemporal prey-predator discrete model and optimal controls for environmental sustainability in the multifishing areas of Morocco,” Discrete Dynamics in Nature and Society, vol. 2020, Article ID 2780651, pp. 1–18, 2020.

[27] A. El Bhih, Y. Benfatah, A. Kouidere, and M. Rachik, “A discrete mathematical modeling of transmission of COVID-19 pandemic using optimal control,” Commun. Math. Biol. Neurosci., vol. 2020, Article ID 75, 2020.

[28] Optimisation Toolbox Matlab.

[29] A. El Bhih, R. Ghazzali, S. B. Rhila, M. Rachik, and A. El A. Llaaroussi, “A discrete mathematical modeling and optimal control of the rumor propagation in online social network,” Discrete Dynamics in Nature and Society, vol. 2020, Article ID 4386476, pp. 1-12, 2020.

[30] A. Larrache, M. Lhous, S. B. Rhila, M. Rachik, and A. Tridane, “An output sensitivity problem for a class of linear distributed systems with uncertain initial state,” Archives of Control Sciences, vol. 30, no. 1, pp. 139-155, 2020.

Youssef Benfatah is a Ph.D. student. He received his M.S. degree in applied mathematics in 2017 from Faculty of Sciences and Techniques of Settat, Hassan I University, Morocco. Since 2019, he is a member of the Analysis, Modeling and Simulation Laboratory at Faculty of Sciences Ben M'sik, University of Hassan II. His research interests include optimal control, dynamical systems, mathematical modeling and distributed systems.

Amine El Bhih is a Ph.D. student. He received his M.S. degree in applied mathematics in 2017 from Faculty of Sciences Ben M'sik, Hassan II University, Casablanca, Morocco. Since 2019, he is a member of the Analysis, Modeling and Simulation Laboratory at the same faculty. His research interests include optimal control, dynamical systems, mathematical modeling and distributed systems.

Mostafa Rachik is a Professor of Mathematics and Computer Science at Faculty of Sciences Ben M'sik, Casablanca (Morocco). He received his Ph.D. degree in control systems. Professor Rachik wrote many papers in the area of systems analysis and control. Now he is the Head of the research team LAMS (Analysis, Modeling and Simulation Laboratory) at the same faculty. His main research interests are dynamical systems, mathematical modelling, stochastic epidemic systems, robotics, optimization, distributed systems, optimal control, and its applications.

Abdessamad Tridane is an Associate Professor in the Department of Mathematical Sciences at United Arab Emirates University, Al Ain, UAE. He received his Ph.D. degree in control systems theory. Dr. Tridane wrote many papers in the area of control systems, mathematical epidemiology and mathematical medicine. His main research interests are dynamical systems, control systems, mathematical epidemiology and medicine.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.