QUANTITATIVE RECURRENT OF SOME DYNAMICAL SYSTEMS WITH AN INFINITE MEASURE IN DIMENSION ONE

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Abstract. We are interested in the asymptotic behaviour of the first return time of the orbits of a dynamical system into a small neighbourhood of their starting points. We study this quantity in the context of dynamical systems preserving an infinite measure. More precisely, we consider the case of $Z$-extensions of subshifts of finite type. We also consider a toy probabilistic model to enlighten the strategy of our proofs.

1. Introduction

The quantitative recurrence properties of dynamical systems preserving a probability measure have been studied by many authors since the work of Hirata [6]. Some properties are defined by estimating the first return time of a dynamical system into a small neighbourhood of its starting point. Results in this concern have been described in [13], let us mention works in this situation [11]. This question has been less investigated in the context of dynamical systems preserving an infinite measure. In [3], Bressaud and Zweimüller have established first results of quantitative recurrence for piecewise affine maps of the interval with infinite measure. The case of $Z^2$-extension of mixing subshifts of finite type has been investigated in [11]. Results have been also established for random walks on the line [12], for billiards in the plane [10] and for null-recurrent Markov maps in [13].

A measure-preserving dynamical system is given by $(X, \mathcal{B}, \mu, T)$ where $(X, \mathcal{B})$ is a measurable set, $\mu$ is a finite or $\sigma$-finite positive measure and $T : X \to X$ is a measurable transformation preserving the measure $\mu$ (i.e. $\mu(T^{-1}A) = \mu(A)$, for every $A \in \mathcal{B}$). We are interested in the case where $\mu$ is $\sigma$-finite. We assume that $X$ is endowed with some metric $d_{X}$ and that $\mathcal{B}$ contains the open balls $B(x, r)$ of $X$. Our interest is in the first time the orbit comes back close to its initial position. For every $y \in X$, we define the first return time $\tau_{y}$ of the orbit of $y$ in the ball $B(y, \epsilon)$ as:

$$
\tau_{y}(y) := \inf\{n \geq 1 : T^{n}(y) \in B(y, \epsilon)\} \in \mathbb{N} \cup \{+\infty\}.
$$

We consider conservative dynamical systems, that is dynamical systems for which the conclusion of the poincaré theorem is satisfied. This ensures that, for every $\epsilon > 0$, $\tau_{y} < \infty$, $\mu$ almost everywhere. The main goal of this article is to study the behavior of $\tau_{y}$ as $\epsilon \to 0$. A classical example of dynamical systems preserving an infinite measure is given by $Z$-extensions of a probability-preserving dynamical system. Given a probability-preserving dynamical system $(\bar{X}, \bar{\mathcal{B}}, \bar{\nu}, \bar{T})$ and a measurable function $\varphi : \bar{X} \to \mathbb{Z}$, we construct the $Z$-extension $(X, \mathcal{B}, \mu, T)$ of $(\bar{X}, \bar{\mathcal{B}}, \bar{\nu}, \bar{T})$ by setting $X := \bar{X} \times \mathbb{Z}$, $\mathcal{B} := \bar{\mathcal{B}} \otimes \mathcal{P}(\mathbb{Z})$, $\mu := \bar{\nu} \otimes \sum_{z \in \mathbb{Z}} \delta_{z}$ and $T(x, l) = (\bar{T}(x), l + \varphi(x))$. We endow $X$ with the product metric given by $d_{X}((x, l), (x', l')) := \max\{d_{\bar{X}}(x, x'), \|l - l'\|\}$. Hence $T^{n}(x, l) = (\bar{T}^{n}(x, l) + S_{n}\varphi(x))$, where $S_{n}\varphi$ is the ergodic sum $S_{n}\varphi := \sum_{k=0}^{n-1} \varphi \circ \bar{T}^{k}$. Therefore, for $\epsilon$ small enough,

$$
T^{n}(x, l) \in B((x, l), \epsilon) \iff T^{n}(x) \in B_{\bar{X}}(x, \epsilon) \text{ and } S_{n}\varphi(x) = 0.
$$

Our main results concern the case when $(\bar{X}, \bar{\mathcal{B}}, \bar{\nu}, \bar{T})$ is a mixing subshift of finite type (see Section 3 for precise definition), which are classical dynamical systems used to model a wide class of dynamical systems such as geodesic flows in negative curvature, etc.

Consider $(\bar{X}, \bar{\mathcal{B}}, \bar{\nu}, \bar{T})$ a mixing subshift of finite type and $\nu$ a Gibbs measure associated to a Hölder continuous potential. Moreover we have a $\nu$-centered Hölder continuous function $\varphi$. Then we get

$$
\lim_{\epsilon \to 0} \frac{\log \tau_{y}}{\log \epsilon} = -2d,
$$

$\mu$-almost everywhere, where $d$ is the Hausdorff dimension of $\nu$. Moreover the following convergence holds in distribution with respect to any probability measure absolutely continuous with respect to $\mu$:

$$
\mu(B(\cdot, \epsilon)) \sqrt{\tau_{y}(\cdot)} \xrightarrow{\epsilon \to 0} \frac{\epsilon}{|X|},
$$

for random walks on the line [12], for billiards in the plane [10] and for null-recurrent Markov maps in [13].
where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables with respective exponential distribution of mean 1 and standard normal distribution (see Theorem 2.1 and Theorem 2.2 for precise statements).

Roughly speaking the strategy of our proof is that there is a large scale (corresponding to $S_n \varphi(x)$) and a small scale (corresponding to $\overline{T^n(x)}$), which behave independently asymptotically. To enlight this strategy, we start this paper with the study of the toy probabilistic model $(Y_n, S_n)$, where $(S_n)_n$ is the simple symmetric random walk and $(Y_n)_n$ is a sequence of independent random variables, with uniform distribution on $(0,1)^d$ and where $S_n$ and $Y_n$ are independent. For this simple model, we obtain the same results. More precisely, we prove that (1.1) holds almost surely and that (1.2) holds in distribution.

2. TOY PROBABILISTIC MODEL

Let $d \in \mathbb{N}$. In this section, we give a real random walk $(M_n)_{n \geq 0}$ with values in $\mathbb{R} \times ]0,1[^{d-1} \subset \mathbb{R}^d$.

2.1. Description of the model and statement of the results. The random process $M_n$ is given by $M_n = (S_n, 0) + Y_n$, $(S_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ are independent such that:

- $Y_n$ is uniformly distributed on $(0,1)^d$.
- $S_n$ is the simple symmetric random walk on $\mathbb{Z}$ given by $S_0 = 0$, i.e. $S_n = \sum_{k=1}^n X_k$, where $(X_k)_k$ is a sequence of independent random variables such that: $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = 1/2$.

We want to study the asymptotic behavior, as $\varepsilon$ goes to 0, of $\tau_{\varepsilon}$ for the metric associated to some norm on $\mathbb{R}^d$. Let $c$ be the Lebesgue measure of the unit ball in $\mathbb{R}^d$. We will prove the following:

**Theorem 2.1.** Almost surely, $\frac{\log \tau_{\varepsilon}}{\log \varepsilon}$ converges to $2d$ as $\varepsilon$ goes to 0.

For this constant $c > 0$, we have the following result:

**Theorem 2.2.** The sequence of random variables $((c^d \sqrt{\varepsilon}))_n$ converges in distribution to $\frac{\mathcal{E}}{\mathcal{N}}$, where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables, $\mathcal{E}$ having an exponential distribution of mean 1 and $\mathcal{N}$ having a standard Gaussian distribution.

2.2. Proof of the pointwise convergence of the recurrence rate to the dimension. $M_0$ is in $]0,1[^d$, let $\varepsilon$ so small that $B(M_0; \varepsilon)$ is contained in $]0,1[^d$. Note that $\text{Leb}(B(x, \varepsilon)) = c \varepsilon^d$.

We define for any $p \geq 0$ the $p^{\text{th}}$ return time $R_p$ of $(M_n)_n$ in $]0,1[^d$, setting $R_0 = 0$, by induction:

$$R_{p+1} := \inf \{ m > R_p : S_m = 0 \}.$$

We have the relation:

$$\tau_{\varepsilon} = R_{T_{\varepsilon}} \text{ with } T_{\varepsilon} := \min \{ l \geq 1 : Y_{R_l} \in B(Y_0, \varepsilon) \}$$

We will study the asymptotic behavior of the random variables $R_n$ and $T_{\varepsilon}$ and use the relation between them to prove Theorem 2.1.

2.2.1. Study the return of the random variable $R_n$.

**Proposition 2.3** (Feller [3]). There exists $C > 0$ such that:

$$\mathbb{P}(R_1 > s) \sim \frac{C}{\sqrt{s}} \text{ as } s \to \infty$$

**Remark 2.4.** Due to the strong Markov property, the delays $U_p := R_p - R_{p-1}$ between successive return times are independent and identically distributed.

**Lemma 2.5.** Almost surely, $\frac{\log \sqrt{R_n}}{\log n}$ converges to 1 as $n$ goes to $\infty$.

**Proof.** The proof of the lemma directly holds, once the following inequality is proved:

$$\forall \alpha \in (0,1), \exists n_0, \forall n \geq n_0, \quad n^{1-\alpha} \leq \sqrt{R_n} \leq n^{1+\alpha}$$

Let $\alpha \in (0,1)$, by independence (using Remark 2.2.1), we have:

$$\mathbb{P}(\sqrt{R_n} \leq n^{1-\alpha}) \leq \mathbb{P}(\forall p \leq n, \sqrt{R_p - R_{p-1}} \leq n^{1-\alpha}) \leq \mathbb{P} \left( \sqrt{R_1} \leq n^{1-\alpha} \right)^n.$$
Due to the asymptotic formula given in Proposition 2.1, for \( n \) sufficiently large
\[
\mathbb{P}(\sqrt{R_1} \leq n^{1-\alpha}) \leq \left(1 - \frac{C}{2n^{1-\alpha}}\right)^n \leq \exp\left(-\frac{C n^\alpha}{2}\right).
\]
This allows us to get the first inequality of (2.2) by using the Borel Cantelli lemma. Again, using proposition 2.1 we have
\[
\mathbb{P}\left(R_1^{1-\alpha} > s\right) \leq \frac{C'}{s^{1-\alpha}}
\]
for some \( C' > 0 \), implying obviously that \( \mathbb{E}\left(R_1^{1-\alpha}\right) < \infty \).

Note that one can see,
\[
R_n = \sum_{i=1}^{n} U_i \leq n^{2+2\alpha} \left(\frac{1}{n} \sum_{i=1}^{n} U_i^{\frac{1}{2+2\alpha}}\right)^{2+2\alpha}.
\]
But \( \frac{1}{n} \sum_{i=1}^{n} U_i^{\frac{1}{2+2\alpha}} \) converges almost surely to \( \mathbb{E}\left(R_1^{\frac{1}{2+2\alpha}}\right) < \infty \) due to the strong law of large numbers. Hence \( R_n = O(n^{2+2\alpha}) \) almost surely, from which we get the second inequality.

2.2.2. Study the return of the random variable \( T_\epsilon \). In this subsection the asymptotic behavior of the random variable \( T_\epsilon \) is illustrated in the following lemma.

**Lemma 2.6.** Almost surely, \( \frac{\log T_\epsilon}{\log \epsilon} \to d \) as \( \epsilon \to 0 \).

**Proof.** Given \( Y_0 \), let \( \epsilon > 0 \) be such that \( B(Y_0, \epsilon) \subset (0,1)^d \). The random variable \( T_\epsilon \) has a geometric distribution with parameter \( \lambda_\epsilon := c\epsilon^d \).

For any \( \alpha > 0 \), a simple decomposition gives:
\[
\mathbb{P}\left(\frac{\log T_\epsilon}{\log \epsilon} - d > \alpha\right) = \mathbb{P}(T_\epsilon > \epsilon^{-d-\alpha}) + \mathbb{P}(T_\epsilon < \epsilon^{-d+\alpha}).
\]
The first term is handled by the Markov inequality:
\[
\mathbb{P}(T_\epsilon > \epsilon^{-d-\alpha} | Y_0) \leq \epsilon^\alpha \frac{\epsilon^d}{\lambda_\epsilon} = O(\epsilon^\alpha).
\]

While the second term using the geometric distribution:
\[
\mathbb{P}(T_\epsilon < \epsilon^{-d+\alpha}) = 1 - (1 - \epsilon^d)\epsilon^{-d+\alpha} \leq 1 - \exp[\epsilon^{-d+\alpha} \log(1 - \epsilon^d)] \leq (-\epsilon)^{-d+\alpha} \log(1 - \epsilon^d) = O(\epsilon^\alpha).
\]

Let us define \( \epsilon_n := n^{-\frac{2}{d}} \). Thus \((\epsilon_n)_{n \geq 1}\) is a decreasing sequence of real numbers, and \( T_\epsilon \) is monotone in \( \epsilon \), so that:
\[
\sum_{n \geq 1} \mathbb{P}(\frac{\log T_{\epsilon_n}}{\log \epsilon_n} - d > \alpha) < +\infty.
\]

According to Borel Cantelli lemma \( \frac{\log T_{\epsilon_n}}{\log \epsilon_n} \to d \) almost surely as \( n \to +\infty \).

Hence the proof follows since \( \lim_{n \to +\infty} \epsilon_n = 0 \) and \( \lim_{n \to +\infty} \epsilon_{n+1} = 1. \)

**Proof of Theorem 2.1** The theorem follows from the two previous lemmas 2.5 and 2.6 since:
\[
\frac{\log \sqrt{\tau_\epsilon}}{-\log \epsilon} = \frac{\log \sqrt{R_{T_\epsilon}}}{\log T_\epsilon} \frac{\log T_\epsilon}{\log \epsilon} \to 1 \times d = d \quad \text{a.s.}
\]

Hence, we get:
\[
\frac{\log \tau_\epsilon}{-\log \epsilon} \to 2d \text{ as } \epsilon \to 0 \quad \text{a.s.}
\]
2.3. Proof of the convergence in distribution of the rescaled return time.

**Proposition 2.7.** The sequence of random variables \( \left( \frac{R_n}{\sqrt{n}} \right)_n \) converges in distribution to \( N^{-2} \) where \( N \) is a standard Gaussian random variable.

The proof of this proposition follows from the two following successive lemmas; the proof of which is straightforward and is omitted.

**Lemma 2.8.** \( \sum_{n \geq 0} \mathbb{P}(S_{2n} = 0)s^{2n} = \frac{1}{\sqrt{s}} \) and \( \mathbb{P}(S_{2n} = 0) \) \( \rightarrow \frac{1}{\sqrt{s}} \).

Note that \( \mathbb{P}(S_{2n} = 0) = \sum_{k=0}^{n} \mathbb{P}(S_k = 0)\mathbb{P}(R_1 = 2n-2k) \). Hence, \( \sum_{n \geq 1} \mathbb{P}(S_{2n} = 0)s^{2n} = \left( \sum_{n \geq 0} \mathbb{P}(S_{2n} = 0)s^{2n} \right) \mathbb{E}(s^{R_1}) \).

And so \( \mathbb{E}[s^{R_1}] = 1 - \sqrt{1 - s^2} \).

**Lemma 2.9.** The moment generating function of \( N^{-2} \) is \( \mathbb{E}\left[e^{-tN^{-2}}\right] = e^{-\sqrt{2t}} \), \( \forall t \geq 0 \), where \( N \) is standard Gaussian random variable.

**Proof of Proposition 2.7.** Knowing that \( R_1, (R_2 - R_1), \ldots, (R_n - R_{n-1}) \) are i.i.d., and the fact that \( \mathbb{E}[s^{R_1}] = 1 - \sqrt{1 - s^2} \), we get:

\[
\mathbb{E}[e^{-\frac{t}{n}R_n}] = \left( \mathbb{E}[e^{-\frac{t}{n}R_1}] \right)^n = \left[ 1 - \sqrt{1 - e^{-\frac{2}{n}}} \right]^n
\]

and from Lemma 2.9, we have:

\[
\forall t \geq 0, \lim_{n \to \infty} \mathbb{E}[e^{-\frac{t}{n}R_n}] = e^{-\sqrt{2t}} = \mathbb{E}[e^{-tN^{-2}}].
\]

Hence, \( \left( \frac{R_n}{\sqrt{n}} \right)_n \) converges in distribution to \( N^{-2} \).

**Lemma 2.10.** \( (\lambda_e T_e) \) converges in distribution to an exponential random variable \( \mathcal{E} \) of mean 1.

**Proof.** Given \( Y_0, T_e \) has a geometric distribution of parameter \( \lambda_e = \lambda(B(Y_0, \epsilon)) \). Let \( t > 0 \),

\[
\mathbb{P}(\lambda_e T_e \leq t \mid Y_0) = \sum_{n=1}^{\left\lfloor \frac{t}{\lambda_e} \right\rfloor} \lambda_e(1 - \lambda_e)^{n-1} = 1 - \exp\left( \left\lfloor \frac{t}{\lambda_e} \right\rfloor \log(1 - \lambda_e) \right),
\]

it follows that, for \( \mathcal{E} \) a random variable which follows \( \exp(1) \),

\[
\lim_{\epsilon \to 0} \mathbb{P}(\lambda_e T_e \leq t \mid Y_0) = 1 - e^{-t} = \mathbb{P}(\mathcal{E} \leq t), \quad a.s.
\]

**Proof of Theorem 2.8.** Let us prove that the family of couples \( \left( \lambda_e T_e, \frac{R_n}{\lambda_e} \right)_{\epsilon > 0} \) converges in distribution, as \( \epsilon \to 0 \), to \( (\mathcal{E}, N^{-2}) \), where \( \mathcal{E} \) and \( N^{-2} \) are assumed to be as above and independent.

Let \( s > 0 \) and \( t \in \mathbb{R} \), then using the independence of \( (T_e)_{\epsilon} \) and \( (R_n)_{n} \), we get:

\[
\left| \mathbb{P}\left( \lambda_e T_e > s, \frac{R_n}{\lambda_e} > t \right) - \mathbb{P}(\lambda_e T_e > s)\mathbb{P}(N^{-2} > t) \right| \leq \sum_{n \geq \frac{s}{\lambda_e}} \lambda_e(1 - \lambda_e)^{n-1}\left| \mathbb{P}\left( \frac{R_n}{n^2} > t \right) - \mathbb{P}(N^{-2} > t) \right|
\]

\[
\leq \sup_{n > \frac{s}{\lambda_e}} \left| \mathbb{P}\left( \frac{R_n}{n^2} > t \right) - \mathbb{P}(N^{-2} > t) \right|
\]

This latter goes to 0 as \( \epsilon \) goes to 0, due to Proposition 2.7. Moreover by Lemma 2.10 \( \mathbb{P}(\lambda_e T_e > s) \to \mathbb{P}(\mathcal{E} > s) \) as \( \epsilon \to 0 \), hence:

\[
\forall s > 0, \forall t \lim_{\epsilon \to 0} \mathbb{P}(\lambda_e T_e > s, \frac{R_n}{\lambda_e} > t) - \mathbb{P}(\mathcal{E} > s, N^{-2} > t) = 0.
\]

This proves that the couple \( \left( \lambda_e T_e, \frac{R_n}{\lambda_e} \right) \) converges in distribution, as \( \epsilon \) goes to 0, to \( (\mathcal{E}, N^{-2}) \).

Knowing that \( \tau_e = R_{T_e} \), we thus find that:

\[
(ce^d)^2 \tau_e = \left( \frac{ce^d}{\lambda_e} \right)^2 \lambda_e^2 T_e^2 \frac{R_n}{T_e^2}
\]
Since \((x, y) \mapsto x^2 y\) is continuous, \(\lambda^2 T^2 u \frac{B_\varepsilon}{\varepsilon} \overset{d}{\to} E^2 N^{-2}\) as \(\varepsilon \to 0\). Observe that \((\frac{\varepsilon}{\lambda})^2 \overset{a.s.}{\to} 1\), hence by Slutzky’s Lemma, we end up with:

\[
(\varepsilon^d)^2 \tau_n \overset{d}{\to} E^2 N^{-2}, \quad \text{as } \varepsilon \to 0.
\]

\[
\square
\]

3. \(\mathbb{Z}\)-extension of a mixing subshift of finite type

Let \(\mathcal{A}\) be a finite set, called the alphabet, and let \(M\) be a matrix indexed by \(\mathcal{A} \times \mathcal{A}\) with 0-1 entries. We suppose that there exists a positive integer \(n_0\) such that each component of \(M^{n_0}\) is non zero. The subshift of finite type with alphabet \(\mathcal{A}\) and transition matrix \(M\) is \((\Sigma, \theta)\), with

\[
\Sigma := \{ w := (w_n)_{n \in \mathbb{Z}} : \forall n \in \mathbb{Z}, M(w_n, w_{n+1}) = 1 \}
\]

together with the metric \(d(w, w') := e^{-m}\), where \(m\) is the greatest integer such that \(w_i = w'_i\) whenever \(|i| < m\), and the shift \(\theta : \Sigma \to \Sigma, \theta((w_n)_{n \in \mathbb{Z}}) = (w_{n+1})_{n \in \mathbb{Z}}\). Let \(\nu\) be the Gibbs measure on \(\Sigma\) associated to some Hölder continuous potential \(h\), and denote by \(\sigma^2_h\) the asymptotic variance of \(h\) under the measure \(\nu\). Recall that \(\sigma^2_h\) vanishes if and only if \(h\) is cohomologous to a constant, and in this case \(\nu\) is the unique measure of maximal entropy.

For any function \(f : \Sigma \to \mathbb{R}\) we denote by \(S_n f := \sum_{i=0}^{n-1} f \circ \theta^i\) its ergodic sum. Let us consider a Hölder continuous function \(\varphi : \Sigma \to \mathbb{Z}\), such that \(f \varphi \) is integrable. We consider the \(\mathbb{Z}\)-extension \(F\) of the shift \(\theta\) by \(\varphi\). Recall that

\[
F : \Sigma \times \mathbb{Z} \to \Sigma \times \mathbb{Z}
\]

\[
(x, m) \to (\theta x, m + \varphi(x)).
\]

Recall that \(\Sigma \times \mathbb{Z}\) is endowed with distance \(d_0((w, l), (w', l')) := \max \{ d(w, w'), |l - l'| \}\). Note that, if \(\varepsilon < 1\), for every \((w, l) \in \Sigma \times \mathbb{Z}\), we have \(\mu(B_{\Sigma \times \mathbb{Z}}((w, l), \varepsilon)) = \mu(B_{\Sigma}(w, \varepsilon))\). We want to know the time needed for a typical orbit starting at \((x, m) \in \Sigma \times \mathbb{Z}\) to return \(\varepsilon\)-close to the initial point after iterations of the map \(F\). By the translation invariance we can assume that the orbit starts in the cell \(m = 0\). Recall that

\[
\tau_\varepsilon(x) = \min \{ n \geq 1 : F^n(x, 0) \in B(x, \varepsilon) \times \{0\} \}
\]

\[
= \min \{ n \geq 1 : S_n \varphi(x) = 0 \ \text{and} \ d(\theta^n x, x) < \varepsilon \}.
\]

We know that there exists a positive integer \(m_0\) such that the function \(\varphi\) is constant on each \(m_0\)-cylinders. Let us denote by \(\sigma^2_{\varphi}\) the asymptotic variance of \(\varphi\):

\[
\sigma^2_{\varphi} = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[(S_n \varphi)^2].
\]

We assume that \(\sigma^2_{\varphi} \neq 0\) (otherwise \((S_n \varphi)_n\) is bounded). We reinforce this by the following non-arithmeticity hypothesis on \(\varphi\): We suppose that, for any \(u \in [-\pi, \pi]\setminus\{0\}\) the only solutions \((\lambda, g)\), with \(\lambda \in \mathbb{C}\) and \(g : \Sigma \to \mathbb{C}\) measurable with \(|g| = 1\), of the functional equation

\[
(3.1) \quad g \circ \theta g = \lambda e^{iu \varphi}
\]

is the trivial one \(\lambda = 1\) and \(g = \text{const.}\) The fact that there is no non constant \(g\) satisfying \((3.1)\) for \(\lambda = 1\) ensures that \(\varphi\) is not a coboundary and so that \(\sigma^2_{\varphi} \neq 0\). The fact that there exists \((\lambda, g)\) satisfying \((3.1)\) with \(\lambda \neq 1\) would mean that the range of \(S_n \varphi\) is essentially contained in a sub-lattice of \(\mathbb{Z}\); in this case we could just do a change of basis and apply our result to the new reduced \(\mathbb{Z}\)-extension. We emphasize that this non-arithmeticity condition is equivalent to the fact that all the circle extensions \(T_u\) defined by \(T_u(x, t) = (\theta(x), t + u, \varphi(x))\) are weakly mixing for \(u \in [-\pi, \pi]\setminus\{0\}\).

In this section we obtain the following results:

**Theorem 3.1.** The sequence of random variables \(\frac{\log \sqrt{\tau_\varepsilon}}{-\log \varepsilon}\) converges \(\nu\)-almost everywhere as \(\varepsilon \to 0\) to the Hausdorff dimension \(d\) of the measure \(\nu\).
Theorem 3.2. The sequence of random variables \( \nu((B_\epsilon(\cdot)))\sqrt{\tau_\epsilon(\cdot)} \) converges in distribution with respect to every probability measure absolutely continuous with respect to \( \nu \) as \( \epsilon \to 0 \) to \( E \), where \( E \) and \( \mathcal{N} \) are independent random variables, \( E \) having an exponential distribution of mean 1 and \( \mathcal{N} \) having a standard Gaussian distribution.

Corollary 3.3. If the measure \( \nu \) is not the measure of maximal entropy, then the sequence of random variables \( \frac{\log \sqrt{\tau_\epsilon(\cdot)} + d\log \epsilon}{\sqrt{-\log \epsilon}} \) converges in distribution as \( \epsilon \to 0 \) to a centered Gaussian random variable of variance \( 2\sigma_h^2 \).

3.1. Spectral theory of the transfer operator and Local Limit Theorem. In this subsection, we follow [11] to adapt our results. To begin with, let us define:

\[
\hat{\Sigma} := \{ w := (w_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N}, M(w_n, w_{n+1}) = 1 \},
\]

the set of all one-sided infinite sequences of elements of \( \mathcal{A} \), endowed with the metric \( \hat{d}( (w_n)_{n \geq 0}, (w_n')_{n \geq 0}) := e^{-\inf\{ m \geq 0 : w_m \neq w'_m \}} \), and the one-sided shift map \( \hat{\theta}( (w_n)_{n \geq 0}) = (w_{n+1})_{n \geq 0} \). The resulting topology is generated by the collection of cylinders:

\[
C_{a_0, \ldots, a_n} = \{ (w_n)_{n \in \mathbb{N}} \in \hat{\Sigma} : w_0 = a_0, \ldots, w_n = a_n \}.
\]

Let us introduce the canonical projection \( \Pi : \Sigma \to \hat{\Sigma} \), \( \Pi((w_n)_{n \in \mathbb{Z}}) = (w_n)_{n \geq 0} \). Denote by \( \hat{\nu} \) the image probability measure (on \( \hat{\Sigma} \)) of \( \nu \) by \( \Pi \). There exists a function \( \psi : \hat{\Sigma} \to \mathbb{C} \) such that \( \psi \circ \Pi = \varphi \circ \theta^m \).

let us denote by \( P : L^2(\hat{\nu}) \to L^2(\hat{\nu}) \) the Perron-Frobenius operator such that:

\[
\forall f, g \in L^2(\hat{\nu}), \int_{\hat{\Sigma}} Pf(x)g(x)d\hat{\nu}(x) = \int_{\hat{\Sigma}} f(x)g \circ \hat{\theta}(x)d\hat{\nu}(x).
\]

Let \( \eta \in [0; 1] \). Let us denote by \( B \) the set of bounded \( \eta \)-Hölder continuous function \( g : \hat{\Sigma} \to \mathbb{C} \) endowed with the usual Hölder norm:

\[
||g||_B := ||g||_\infty + \sup_{x \neq y} \frac{|g(y) - g(x)|}{\hat{d}(x, y)^\eta}.
\]

We denote by \( B^* \) the topological dual of \( B \). For all \( u \in \mathbb{R} \), we consider the operator \( P_u \) defined on \( (B, ||.||_B) \) by:

\[
P_u(f) := P(e^{iu\hat{\psi}}f).
\]

Note that the hypothesis of non-arithmeticity of \( \varphi \) is equivalent to the following one on \( \psi \): for any \( u \in [-\pi; \pi] \setminus \{0\} \), the operator \( P_u \) has no eigenvalue on the unit circle.

We will use the method introduced by Nagaev in [3] and [8], adapted by Guivarch and Hardy in [5] and extended by Hennion and Hervé in [2]. It is based on the family of operators \( (P_u)_u \) and their spectral properties expressed in the two next propositions.

Proposition 3.4. (Uniform Contraction). There exist \( \alpha \in (0; 1) \) and \( C > 0 \) such that, for all \( u \in [-\pi; \pi] \setminus [-\beta; \beta] \) and all integer \( n \geq 0 \), for all \( f \in B \), we have:

\[
||P_u^n(f)||_B \leq C\alpha^n ||f||_B.
\]

This property easily follows from the fact that the spectral radius is smaller than 1 for \( u \neq 0 \). In addition, since \( P \) is a quasicompact operator on \( B \) and since \( u \to P_u \) is a regular perturbation of \( P_0 = P \), we have:

Proposition 3.5. (Perturbation Result). There exist \( \alpha > 0, \beta > 0, C > 0, c_1 > 0, \theta \in [0; 1] \) such that: there exists \( u_0 \) belonging to \( C^3([-\beta; \beta] \to \mathbb{C}) \), there exists \( u \to u_0 \) belonging to \( C^3([-\beta; \beta] \to \mathbb{B}) \), there exists \( u \to \varphi_u \) belonging to \( C^3([-\beta; \beta] \to B^*) \) such that, for all \( u \in [-\beta; \beta] \), for all \( f \in B \) and for all \( n \geq 0 \), we have the decomposition:

\[
P_u^n(f) = \lambda_u^n \varphi_u(f)u_n + N_u^n(f),
\]

with

\[
(1) \ ||N_u^n(f)||_B \leq C\alpha^n ||f||_B,
(2) \ |\lambda_u| \leq e^{-c_1|u|^2} \text{ and } c_1|u|^2 \leq \sigma^2_u u.u,
(3) \text{ with initial values } : v_0 = 1, \varphi_0 = \hat{\nu}, \lambda_{u=0} = 0 \text{ and } \lambda_{u=0} = -\sigma^2_u.
\]
Lemma 3.6. There exist $\gamma' > 0$ and $C_\eta > 0$ such that, $\forall q \geq m_0$ and all $2q$-cylinder $\hat{A}$ of $\hat{\Sigma}$, we have:  
\begin{equation}
(3.3) \quad \forall u \in (-\pi, \pi], \quad ||P_\eta^q P^q(1_{\hat{A}} \circ \hat{\theta}^m_\eta)||_B \leq C_\eta e^{-\gamma'(2q-m_0)}.
\end{equation}
In particular, we have $\hat{\nu}(\hat{A}) \leq C_\eta e^{-\gamma'(2q-m_0)}$.

Proof.

Thus, from these computations, we verify that:

\[ ||P_\eta^q P^q(1_{\hat{A}} \circ \hat{\theta}^m_\eta)(y) || = \sum_{w: \hat{\theta}^{2q-m_0}_w y \neq y} e^{\gamma h(w)} 1_{\hat{A}}(w) e^{i\mu \hat{S}_\eta \psi \hat{\theta}^{2q-m_0}(w)} \]

where $w_y \in \hat{A}$ is the unique element such that $\hat{\theta}^{2q-m_0} w_y = y$ (it exists if $y \in \hat{\theta}^{2q-m_0} \hat{A}$). From this later formula, we can obtain that $||P_\eta^q P^q(1_{\hat{A}} \circ \hat{\theta}^m_\eta)||_B \leq e^{\max h(2q-m_0)}$, where $\max h < 0$.

Now, a step to compute the norm $||.||_B$ is to estimate the $\eta$–Hölder coefficient. Let $x \neq y \in \hat{\Sigma}$, we know that $\hat{d}(x, y) = e^{-\eta}$, for some $n \in \mathbb{N}^*$. We will consider two cases: the first case when $n > m_0$, we note the equivalence $x \in \hat{\theta}^{2q-m_0} \hat{A} \iff y \in \hat{\theta}^{2q-m_0} \hat{A}$. Thus, either $x, y \in \hat{\theta}^{2q-m_0} \hat{A}$ and hence

\[ ||P_\eta^q P^q(1_{\hat{A}})(y) - P_\eta^q P^q(1_{\hat{A}})(x)|| = 0. \]

Or $x, y \in \hat{\theta}^{2q-m_0} \hat{A}$, so that $\hat{d}(w_x, w_y) = 2q - m_0 + n$. Let us denote for simplicity $F_{h, \psi} = S_{2q-m_0} h(\cdot) + iu \hat{S}_\eta \psi \circ \hat{\theta}^{2q-m_0}(\cdot)$. Introducing the ergodic sum formula, we get:

\[ |S_{2q-m_0} h(w_y) - S_{2q-m_0} h(w_x)| \leq \sum_{i=0}^{2q-m_0} |h| \eta e^{-\eta(2q-m_0+n-i)} \leq c|h| \eta \hat{d}(x, y), \]

where $c$ is a constant such that $\sum_{j \geq 1} e^{-\alpha j} \leq c < \infty$. And in the same way for $S_{\psi}(\hat{\theta}^{2q-m_0}(\cdot))$, we can see that:

\[ |S_{\psi}(\hat{\theta}^{2q-m_0}(w_y)) - S_{\psi}(\hat{\theta}^{2q-m_0}(w_x))| \leq c|\psi| \eta \hat{d}(x, y). \]

Thus, from these computations, we verify that:

\[ ||P_\eta^q P^q(1_{\hat{A}})(y) - P_\eta^q P^q(1_{\hat{A}})(x)|| = |e^{F_{h, \psi}(w_y)} - e^{F_{h, \psi}(w_x)}| \leq e^{\max h(2q-m_0)} c(|h| \eta + |\psi| \eta) \hat{d}(x, y). \]

Now, we treat the second case where $n \leq m_0$. Here, if $x \in \hat{\theta}^{2q-m_0} \hat{A}$, then $y \notin \hat{\theta}^{2q-m_0} \hat{A}$,

\[ ||P_\eta^q P^q(1_{\hat{A}})(y) - P_\eta^q P^q(1_{\hat{A}})(x)|| \leq \sup_{w \in \hat{A}} |e^{S_{2q-m_0} h(w) + iu \hat{S}_\eta \psi \hat{\theta}^{2q-m_0}(w)}| e^{\eta m} e^{-\eta m} \leq e^{\max h(2q-m_0)} \eta k_{\eta} \hat{d}(x, y). \]

From all this process, setting $\gamma' := \min(\eta, -\max h) > 0$, we get an estimation for the $\eta$–Hölder coefficient, $\forall n \geq 0$:

\[ ||P_\eta^q P^q(1_{\hat{A}})||_B \leq e^{-\gamma'(2q-m_0)} \max(\eta m, c(|h| \eta + |\psi| \eta)) \]

Hence, for $C_\eta := (1 + \max(\eta m, c(|h| \eta + |\psi| \eta)))$, we deduce that

\[ ||P_\eta^q P^q(1_{\hat{A}})||_B \leq C_\eta e^{-\gamma'(2q-m_0)}. \]

\[ \square \]

Next proposition is a two-dimensional version of Proposition 13 in [11]. We give a more precise error term in order to accomodate the one-dimensional case. It may be viewed as a doubly local version of the central limit theorem: first, it is local in the sense that we are looking at the probability that $S_n \varphi = 0$ while the classical central limit theorem is only concerned with the probability that $|S_n \varphi| \leq \epsilon \sqrt{n}$; second, it is local in the sense that we are looking at this probability conditioned to the fact that we are starting from a set $A$ and landing on a set $B$ on the base.
Proposition 3.7. There exist real numbers $C_1 > 0$ and $\gamma > 0$ such that, for all integers $n$, $q$, $k$ such that $n - 2k > 0$ and all $m_0 < q \leq k$, all two-sided $q$-cylinders $A$ of $\Sigma$ and all measurable subset $B$ of $\Sigma$, we have:

$$
\left| \nu \left( A \cap \{ S_n \phi = 0 \} \cap \theta^{-n} (\theta^k (\Pi^{-1} (B))) \right) - \frac{\nu(A) \hat{\nu}(B)}{\sqrt{n - k} \sigma} \right| \leq C_1 \frac{\hat{\nu}(B) k^2 e^{-\gamma q}}{n - 2k}.
$$

Proof. Set $Q := A \cap \{ S_n \phi = 0 \} \cap \theta^{-n} (\theta^k (\Pi^{-1} (B)))$. The proof of the proposition will be illustrated in estimating the measure of the set $Q$.

Since $\phi \circ \theta^{m_0} = \psi \circ \Pi$ and using the semi-conjugacy $\theta \circ \Pi = \Pi \circ \theta$, we have the identity: $\{ S_n \phi \circ \theta^{m_0} = 0 \} = \{ S_n \psi \circ \Pi = 0 \}$. In addition, $I m(\psi) \in \mathbb{Z}$, thus we have:

$$
1_{\theta^{-q} = 0} Q = \left( \hat{\nu}^{m_0} \circ \hat{\theta} \circ \nu \right) \left( \nu \left( \theta^{-n} (\theta^k (\Pi^{-1} (B))) \right) \right) \circ \Pi,
$$

with $A := \Pi^{\theta^{-q}} \hat{A}$ (indeed $\theta^{-q} A = \Pi^{-1} \hat{A}$ since $A$ is a $q$-cylinder). Since the measure $\nu$ is $\theta$-invariant, then we can verify that:

$$
\nu(Q) = \frac{1}{2\pi} \int_{[-\pi, \pi]} \mathbb{E}_{\nu} \left( \left( 1_{\hat{A}} \circ \hat{\theta}^{m_0} \circ \hat{\theta} \circ \nu \right) e^{iu \cdot S_n \psi} \right) du.
$$

Now we want to estimate the expectation $a(u) = \mathbb{E}_{\nu}(\ldots)$. Introducing the Perron-Frobenius operator $P$, and using the fact that it is the dual of $\hat{\theta}$, we get:

$$
a(u) = \mathbb{E}_{\nu} \left( Pu \left( \hat{\theta}^{m_0} \circ \hat{\theta} \circ \nu \right) e^{iu \cdot S_n \psi} \right),
$$

We will use two cases concerning the values of $u$. Let us denote for simplicity $l := n - (k - m_0 - q)$. First, using the contraction inequality given in Proposition 3.4 applied to $P_u^l (1_B)$, the fact that $\| P_u^l \| \leq e^{-\gamma (2l - m_0)}$ from Lemma 3.6, and the fact that $\mathbb{E} (P_u^{k-m_0} (1_B g)) \leq \hat{\nu}(B) \| g \|_B$, we will show that $a(u)$ is negligible for large values of $u$, so $u \notin [-\beta, \beta]$ we get for $q = 2\gamma$:

$$
|a(u)| = \mathbb{E}_{\nu} \left( P_u^{k-m_0} (B P_u^{l-m_0} q P_u^l) \right) = O \left( \hat{\nu}(B) \alpha \epsilon^{-\gamma} \right).
$$

We now use the decomposition in 3.3 to obtain an estimation of the main term coming from small values of $u$. Indeed, whenever $u \in [-\beta, \beta]$, we have:

$$
a(u) = \mathbb{E}_{\nu} \left( P_u^{k-m_0} (1_B P_u^l P_u^l P_u^l) \right) = \lambda u \lambda_u (P_u^{k-m_0} (1_B v_u)) + \mathbb{E}_{\nu} \left( P_u^{k-m_0} (1_B v_u) \right) = a_1 (u) + a_2 (u).
$$

Using inequality (1) in Proposition 3.6, one can see that the second term is of order

$$
a_2 (u) = O(\hat{\nu}(B) \alpha \epsilon^{-\gamma}).
$$

The mappings $u \mapsto v_u$ and $u \mapsto \phi_u$ are $C^1$-regular with $v_0 = 1$ and $\phi_0 = \hat{\nu}$, from which we find that:

$$
a_1 (u) = \lambda u \lambda_u (P_u^{k-m_0} (1_B \hat{\theta}^{m_0})) + \lambda u \lambda_u (P_u^{k-m_0} (1_B \hat{\theta}^{m_0})) + \lambda u \lambda_u (P_u^{k-m_0} (1_B O(u))) + \lambda u \lambda_u (P_u^{k-m_0} (1_B O(u)))
$$

To obtain an approximation of the first term $a_1 (u)$, we introduce the formula of $P$ in $P_u$:

$$
\left| \mathbb{E}_{\nu} \left( P_u^{k-m_0} (1_B) \right) - \hat{\nu}(B) \right| = \left| \mathbb{E}_{\nu} \left( e^{iu \cdot S_{k-m_0} \psi} - 1 \right) 1_B \right| \leq \left| e^{iu \cdot S_{k-m_0} \psi} - 1 \right| \| 1_B \| \| \psi \|_{L^1 (\nu)} \leq |u| (k - m_0) \| \psi \|_{L^1 (\nu)} \hat{\nu}(B).
$$
so that, from this approximation, we get:
\[
\begin{align*}
 a_1(u) &= \lambda_u^i \nu(P_u^i P_u^i(1_A \circ \hat{\theta}^{m_0})) \mathbb{E}_\phi(P_u^{k-m_0}(1_B)) + O(\lambda_u^i |u| \nu(B)e^{-\eta q}) \\
 &= \lambda_u^i \nu(A) \nu(B) + O(1 + O(|u| q)) + O(1 + O(|u| (k - m_0)) + O(\lambda_u^i |u| \nu(B)e^{-\eta q}) \\
 &= \lambda_u^i \nu(\hat{A}) \nu(B) + O(\lambda_u^i |u| \nu(B)k^2 e^{-\eta q}).
\end{align*}
\]

Using Proposition 3.5 and that \( u \mapsto \lambda_u \) belongs to \( C^3([-\beta; \beta] \to \mathbb{C}) \), hence applying the intermediate value theorem gives:
\[
|\lambda_u^i - e^{-\frac{1}{2} \sigma_\phi^2 u^2}| \leq \lambda_u^i - e^{-\frac{1}{2} \sigma_\phi^2 u^2} = l(e^{-c_1 l |u|^2} - 1)|u| + O(|u|^3) = C_0 (l|u|^2 e^{-c_1 l |u|^2}) e^{c_1 |u|^2} |u| = O(e^{-c_2 l |u|^2} |u|), \text{ for the constant } c_2 = c_1/2.
\]

As a consequence, an estimate for \( a_1(u) \) is:
\[
a_1(u) = e^{-\frac{1}{2} \sigma_\phi^2 u^2} \nu(\hat{A}) \nu(B) + O(e^{-c_2 l |u|^2} |u| \nu(B)k e^{-\eta q}),
\]
since \( \nu(\hat{A}) = O(e^{-\gamma (2q-m_0)}) \). A final step to reach an estimation of \( \nu(Q) \) is to integrate the approximated quantity of \( a_1(u) \) obtained above. Using the Gaussian integral, a change of variable \( v = u\sqrt{l} \) gives:
\[
\int_{[-\beta, \beta]} e^{-\frac{1}{2} \sigma_\phi^2 u^2} du = \frac{2\pi}{\sqrt{l} \sigma_\phi} + O\left(\frac{1}{l}\right).
\]

In the same way we treat the error term to get:
\[
\int_{[-\beta, \beta]} |u| e^{-c_2 l |u|^2} du = \int_{[\beta, \beta]} |v| e^{-c_2 |v|^2} dv = O\left(\frac{1}{l}\right).
\]

From these computations, it follows that:
\[
\int_{[-\beta, \beta]} a_1(u) du = \frac{2\pi}{\sqrt{l} \sigma_\phi} \nu(A) \nu(B) + O\left(\frac{\nu(B)k^2 e^{-\eta q}}{l}\right).
\]

From this main estimate and (3.1) and (3.4) we conclude that:
\[
\nu(Q) = \frac{1}{2\pi} \int_{[-\pi, \pi]} a_1(u) du = \frac{1}{\sqrt{n-k \sigma_\phi}} \nu(\hat{A}) \nu(B) + O\left(\frac{\nu(B)k^2 e^{-\eta q}}{n-k^2}\right).
\]

3.2. Proof of the pointwise convergence of the recurrence rate to the dimension. Let us denote by \( G_n(\epsilon) \) the set of points for which \( n \) is an \( \epsilon \)-return:
\[
G_n(\epsilon) := \{ x \in \Sigma \colon S_n \varphi(x) = 0 \text{ and } d(\theta^n(x), x) < \epsilon \}.
\]

Let us consider the first return time in an \( \epsilon \)-neighborhood of a starting point \( x \in \Sigma \):
\[
\tau_\epsilon(x) := \inf\{ m \geq 1 \colon S_m \varphi(x) = 0 \text{ and } d(\theta^m(x), x) < \epsilon \} = \inf\{ m \geq 1 \colon x \in G_m(\epsilon) \}.
\]

Proof of Theorem 3.1 Let us denote by \( C_k \) the set of two-sided \( k \)-cylinders of \( \Sigma \). For any \( \delta > 0 \) denote by \( C_k^\delta \subset C_k \) the set of cylinders \( C \in C_k \) such that \( \nu(C) \in (e^{-(d+k)}, e^{-(d-k)}) \). For any \( x \in \Sigma \), let \( C_k(x) \in C_k \) be the \( k \)-cylinder which contains \( x \). Since \( d \) is the entropy of the ergodic measure \( \nu \), by the Shannon-Breiman theorem, the set \( K_N = \{ x \in \Sigma \colon \forall k \geq N, C_k(x) \in C_k^\delta \} \) has a measure \( \nu(K_N^\delta) > 1 - \delta \) provided \( N \) is sufficiently large.

- First, let us prove that, almost surely:
\[
\liminf_{\epsilon \to 0} \frac{\log \sqrt{\tau_\epsilon}}{- \log \epsilon} \geq d.
\]
Let $\alpha > \frac{1}{q}$ and $0 < \delta < d - \frac{1}{\alpha}$. Let us take $\epsilon_n := n^{-\frac{\alpha}{2}}$ and $k_n := \lceil -\log \epsilon_n \rceil$. In view of Proposition 3.7, whenever $k_n \geq N$, we have:

$$\nu(K_N^{\delta} \cap G_n(\epsilon_n)) = \nu \left( \{ x \in K_N^{\delta} : S_n \varphi(x) = 0 \text{ and } \theta^n(x) \in C_{k_n}(x) \} \right)$$

$$= \sum_{C \in C_{k_n}} \nu(C \cap \{ S_n \varphi = 0 \} \cap \theta^{-n}(C))$$

$$= \sum_{C \in C_{k_n}} \nu(C)^2 + O \left( \frac{\nu(C)k_n^2 e^{-\gamma k_n}}{n - 2k_n} \right).$$

Notice that for $\epsilon_n$ and $k_n$ taken as above, one can verify that the term $\frac{k_n^2 e^{-\gamma k_n}}{n - 2k_n} = O(n^{-1 - \frac{\alpha}{2}}(\log n)^2)$.

In addition, for $C \in C_{k_n}$, $\nu(C) \leq n^{-\frac{\alpha(d-\delta)}{2}}$, from which it follows that

$$\nu(K_N^{\delta} \cap G_n(\epsilon_n)) = O \left( \frac{(\log n)^2}{n \min \{1 + \frac{\alpha}{2}, 1 + \frac{\alpha(d-\delta)}{2} \}} \right)$$

but $\frac{1 + \frac{\alpha(d-\delta)}{2}}{2} > 1$, so $\sum_n \nu(K_N^{\delta} \cap G_n(\epsilon_n)) < \infty$.

Hence by the Borel Cantelli lemma, for a.e. $x \in K_N^{\delta}$, if $n$ is large enough, we have $\tau_{\epsilon_n} > n$, which in turn implies that:

$$\liminf_{n \to \infty} \frac{\log \tau_{\epsilon_n}}{-\log \epsilon_n} \geq \frac{1}{\alpha} \text{ a.e.,}$$

and this proves the lower bound on the lim inf, since $(\epsilon_n)_n$ decreases to zero and $\liminf_{n \to +\infty} \frac{\tau_{\epsilon_{n+1}}}{\tau_{\epsilon_n}} = 1$, and since we have taken an arbitrary $\alpha > \frac{1}{2}$.

Next, we will prove the upper bound $\nu$ on the lim sup:

$$\limsup_{\epsilon \to 0} \frac{\log \sqrt{\tau_{\epsilon}}}{-\log \epsilon} \leq d.$$

Let $\alpha \in (0, \frac{1}{d})$ and $\delta > 0$ such that $1 - \alpha d - \alpha \delta > 0$. Take $\epsilon_n := n^{-\frac{\alpha}{2}}$ and $k_n := \lceil -\log \epsilon_n \rceil$. We define for all $l = 1, \ldots, n$, the sets $A_l(\epsilon) := G_l(\epsilon) \cap \theta^{-l}(\tau_{\epsilon_n} > n - l)$ which are pairwise disjoint. Setting $L_n := \lceil n^a \rceil$ with $a > \alpha(d + \delta - \gamma)$, we then realize that:

$$1 \geq \sum_{l=0}^{n} \nu(A_l(\epsilon_n)) \geq \sum_{l=L_n}^{n} \sum_{C \in C_{k_n}} \nu(C \cap A_l(\epsilon_n)).$$

But due to Proposition 3.7 for any $C \in C_{k_n}$ and $l \geq L_n$, whenever $k_n \geq N$, we have:

$$\nu(C \cap A_l(\epsilon_n)) = \nu(C \cap \{ S_l \varphi = 0 \} \cap \theta^{-l}(C \cap \{ \tau_{\epsilon_n} > n - l \}))$$

$$= \nu(C) \nu(C \cap \{ \tau_{\epsilon_n} > n - l \}) + O \left( \frac{\nu(C \cap \{ \tau_{\epsilon_n} > n - l \})k_n^2 e^{-\gamma k_n}}{l} \right)$$

$$\geq c \epsilon_n^{d+\delta} \frac{1}{\sqrt{l}} \nu(C \cap \{ \tau_{\epsilon_n} > n - l \}).$$

Indeed, the error is negligible, because for $a > \alpha(d + \delta - \gamma)$, $\frac{k_n^2 e^{-\gamma k_n}}{\sqrt{l}} = O(\epsilon_n^{d+\delta})$.

Now, note that:

$$\nu \left( K_N^{\delta} \cap \{ \tau_{\epsilon_n} > n \} \right) \leq \sum_{C \in C_{k_n}} \nu \left( C \cap \{ \tau_{\epsilon_n} > n \} \right).$$

Next, we will work to prove that $\nu \left( K_N^{\delta} \cap \{ \tau_{\epsilon_n} > n \} \right)$ is summable. Observe that:

$$\sum_{l=L_n}^{n} \nu(C \cap A_l(\epsilon_n)) \geq c \epsilon_n^{d+\delta} \nu(C \cap \{ \tau_{\epsilon_n} > n \}) \left( \sqrt{n} - \sqrt{L_n} \right).$$

But, from (3.5), it follows immediately that

$$1 \geq \sum_{C \in C_{k_n}} \sum_{l=L_n}^{n} \nu(C \cap A_l(\epsilon_n)) \geq \sum_{C \in C_{k_n}} c \epsilon_n^{d+\delta} \nu(C \cap \{ \tau_{\epsilon_n} > n \}) \left( \sqrt{n} - \sqrt{L_n} \right).$$
from which one gets
\[
\nu \left( K_N^c \cap \{ \tau_{e_n} > n \} \right) \leq \sum_{C \in \mathcal{C}_{e_n}} \nu(C \cap \{ \tau_{e_n} > n \}) = O \left( \frac{1}{n \left( 1-\nu(x,y) \right)} \right).
\]

Now let us take \( n_p := \frac{1}{\log \alpha} \). We have: \( \sum_{p \geq 1} \nu(K_N^c \cap \{ \tau_{e_{n_p}} > n_p \}) \) is finite, revealing that, using Borel Cantelli lemma, almost surely \( x \in K_N^c, \tau_{e_{n_p}}(x) \leq n_p \), which implies that:
\[
\limsup_{n \to +\infty} \frac{\log \tau_{e_{n_p}}}{\log n} \leq \frac{1}{\alpha}.
\]

This gives the estimate \( \limsup_{n \to +\infty} (e_{n_p})_p \) decreases to 0 and since \( \lim_{p \to +\infty} \frac{e_{n_p}}{\epsilon_{n_p}} = 1 \).

\[\square\]

### 3.3. Fluctuations of the rescaled return time.
Throughout this subsection, we adapt the general strategy of [12,13]. Recall that \( C_k(x) = \{ y \in \Sigma : d(x,y) < e^{-k} \} \). Let \( R_k(y) := \min\{ n \geq 1 : \theta^n(y) \in C_k(y) \} \) denote the first return time of a point \( y \) into its \( k \)-cylinder \( C_k(y) \), or equivalently the first repetition time of the first \( k \) symbols of \( y \). We recall that \( C_k(x) = \{ y \in \Sigma : d(x,y) < e^{-k} \} \). There have been a lot of studies on the quantity \( R_k \), among all the results we will use the following.

**Proposition 3.8.** (Hirata [9]) For \( \nu \)-almost every point \( x \in \Sigma \), the return time into the cylinders \( C_k(x) \) are asymptotically exponentially distributed in the sense that
\[
\lim_{k \to +\infty} \nu(C_k(x)) \left( \frac{R_k(.)}{\nu(C_k(x))} \right) = e^{-t}
\]
for a.e. \( x \), where the convergence is uniform in \( t \).

**Lemma 3.9.**
\[
\forall t > 0, \quad \limsup_{k \to +\infty} \nu \left( \tau_{e-k} > \left( \frac{t}{\nu(C_k(x))} \right)^2 | C_k(x) \right) \leq \frac{1}{1 + \beta t},
\]
with \( \beta := \frac{1}{\delta} \).

**Proof.** Let \( k \geq m_0 \) and \( n \) be some integers. We make a partition of a cylinder \( C_k(x) \) according to the value \( l \leq n \) of the last passage in the time interval \( 0, \ldots, n \) of the orbit of \((x,0)\) by the map \( F \) into \( C_k(x) \times \{0\} \). This gives the following equality:
\[
(3.6) \quad \nu(C_k(x)) = \sum_{l=0}^{n} \nu \left( C_k(x) \cap \{ S_l = 0 \} \cap \theta^{-l}(C_k(x) \cap \{ \tau_{e-k} > n - l \}) \right).
\]
Let \( n_k := \left( \frac{t}{\nu(C_k(x))} \right)^2 \). We claim that:
\[
\lim_{k \to +\infty} \nu(\{ \tau_{e-k} > n_k \} | C_k(x)) \leq \frac{1}{1 + \beta t}
\]

According to the decomposition (3.6) and to Proposition 3.7, there exists \( c_1 > 0 \) such that we have:
\[
\nu(C_k(x)) \geq \nu(C_k(x) \cap \{ \tau_{e-k} > n_k \}) \left( 1 + \beta \nu(C_k(x)) \sum_{l=2^{k+1}}^{n_k} \frac{1}{\sqrt{l - k}} \right) - c_1 \nu(C_k(x)) k^2 e^{-\gamma k} \sum_{l=2^{k+1}}^{n_k} \frac{1}{l - 2^k} \leq \beta t \text{ and the term } k^2 e^{-\gamma k} \sum_{l=2^{k+1}}^{n_k} \frac{1}{l - 2^k} \leq 1. \]
\[\square\]

**Corollary 3.10.** The family of conditional distributions of the random variables \( \nu(C_k(x) \sqrt{\tau_{e-k}} | C_k(x))_{k \geq 0} \) is tight.

Hence it will be enough to prove that the advertised limit law is the only possible accumulation point of our destination. We hence abbreviate
\[
X_k := \nu(C_k(x) \sqrt{\tau_{e-k}})
\]
Lemma 3.11. Suppose that the sequence of conditional distributions of \((X_{k_p} \mid C_{k_p}(x))_p\) converges to the law of some random variable \(X\). Then the limit satisfies the integral equation:

\[
1 = \mathbb{P}(X > t) + \beta t \int_0^1 \mathbb{P}(X > t\sqrt{1-u}) \sqrt{u} \, du \quad \forall t > 0.
\]

Proof. To begin with, let us set \(f(t) := \mathbb{P}(X > t)\)

- First, we will prove that

\[
\forall t > 0 \quad 1 \geq f(t) + \beta t \int_0^1 u^{-1/2} f(t(1-u)^{1/2}) \, du.
\]

The decomposition in (3.6) and Proposition 3.7 implies that there exist \(c > 0\) such that we have:

\[
1 \geq \nu(\tau_{e-\nu} > n_k \mid C_k(x)) + \beta \nu(C_k(x)) \sum_{l=1}^{n_k} \frac{\nu(\tau_{e-\nu} > n_k - l \mid C_k(x))}{\sqrt{l-k}} - c \sum_{l=1}^{n_k} k^2 e^{-\gamma k} \nu(C_k(x))
\]

We want to estimate the formula of this inequality when \(k \to \infty\). We note that through the proof of Lemma 3.9, it has been proved that \(\lim \sum_{l=1}^{n_k} k e^{-\gamma k} = 0\). Thus, if we set \(B_{nk} := \sum_{l=1}^{n_k} \frac{\nu(\tau_{e-\nu} > n_k - l \mid C_k(x))}{\sqrt{N-k}}\), we are left to estimate mainly the lower bound on the lim inf of \(\nu(C_k(x))B_{nk}\) as \(p \to \infty\). Now, by monotonicity, we have

\[
B_{nk} \geq \sum_{l=0}^{N-1} \frac{\nu(\tau_{e-\nu} > n_k - l \mid C_k(x))}{\sqrt{l+r[n_k/N]}}.
\]

Observe that the term:

\[
\nu(\tau_{e-\nu} > n_k - l \mid C_k) \geq \nu(\tau_{e-\nu} > (1-r/N)n_k \mid C_k(x)) = \mathbb{P}(X_k > (\sqrt{1-r/N}) \mid C_k(x)).
\]

Thus, now by evaluating the following sum:

\[
\sum_{l=0}^{N-1} \frac{1}{\sqrt{l+r[n_k/N]}} \geq \sqrt{\frac{n_k}{N}} \frac{1}{\sqrt{r+1}}
\]

We obtain

\[
B_{nk} \geq \sum_{r=1}^{N-1} \sqrt{\frac{n_k}{N}} \frac{1}{\sqrt{r+1}} \mathbb{P}(X_k > t\sqrt{1-r/N} \mid C_k(x))
\]

But, from the hypothesis that \(\mathbb{P}(X_{k_p} > t\sqrt{1-r/N}) \to f(t\sqrt{1-r/N})\), we get:

\[
\liminf_{p \to \infty} \nu(C_{k_p}(x))B_{nk} \geq t \sum_{r=1}^{N-1} \frac{\mathbb{P}(X > t\sqrt{1-r/N})}{\sqrt{(r+1)/N}}
\]

Combining these estimates and taking the limit when \(k_p \to \infty\), we establish the desired inequality:

\[
1 \geq f(t) + \beta t \int_0^1 \frac{f(t(1-u)^{1/2})}{\sqrt{u}} \, du.
\]

- In the same way we treat the converse inequality, using the other half of Proposition 3.7, then there exists \(c' > 0\), such that:

\[
1 \leq \nu(\tau_{e-\nu} > n_k \mid C_k(x)) + \beta \nu(C_k(x)) \sum_{l=1}^{n_k} \frac{\nu(\tau_{e-\nu} > n_k - l \mid C_k(x))}{\sqrt{l-k}} + c' \sum_{l=1}^{n_k} k^2 e^{-\gamma k} \frac{1}{l-2k}
\]
Let \( m_k = o \left( \frac{1}{\nu(C_k(x))} \right) \), then using Proposition 3.8, we have
\[
\nu(\tau_{e-k} \leq 1 - \nu \left( \frac{m_k \nu(C_k(x))}{\nu(C_k(x))} \right) \xrightarrow{k \to \infty} 1 - e^0 = 0.
\]
from which we observe that we can forget the first \( m_k \) term of the following sum, because
\[
\sum_{l=1}^{m_k} \nu(C_k(x) \cap \{ S_l = 0 \} \cap \theta^{-l}(C_k(x) \cap \{ \tau_{e-k} > m_k - l \}))
\]
\[
= \nu \left( \bigcup_{l=1}^{m_k} C_k(x) \cap \{ S_l = 0 \} \cap \theta^{-l}(C_k(x) \cap \{ \tau_{e-k} > m_k - l \}) \right)
\]
\[
\leq \nu(\{ \tau_{e-k} \leq m_k \}) \cap C_k(x)
\]
\[
o(\nu(C_k(x))).
\]
Furthermore, one verifies that this sum of terms between \( m_k \) and \( \lfloor n_k/N \rfloor \) is bounded above by \( \frac{2k}{\nu(C_k(x))\sqrt{N}} \). Hence, we get:
\[
1 \leq \nu(\tau_{e-k} > n_k \mid C_k(x)) + \beta \nu(C_k(x)) \sum_{l=\lfloor \frac{n_k}{N} \rfloor}^{n_k} \frac{\nu(\tau_{e-k} > n_k - l \mid C_k(x))}{\sqrt{l-k}}
\]
\[
+ C \sum_{l=\lfloor \frac{n_k}{N} \rfloor}^{n_k} \frac{k^2 e^{-\gamma k}}{l - 2k} + o(\nu(C_k(x))) + \beta \frac{2t}{\sqrt{N}},
\]
where \( N \) is so large that the last three terms goes to 0 as \( k \to 0 \). Moreover, if we set
\[
B'_{n_k} := \sum_{l=\lfloor \frac{n_k}{N} \rfloor}^{n_k} \frac{\nu(\tau_{e-k} > n_k - l \mid C_k(x))}{\sqrt{l-k}} \leq (1 + \varepsilon_k) \left( B'_{n_k} + N \frac{1}{\sqrt{\lfloor n_k/N \rfloor}} \right).
\]
We now proceed to show the bound on the lim sup of \( \nu(C_k(x))B'_{n_kp} \) as \( p \to 0 \)
\[
B'_{n_k} = \sum_{r=1}^{N-1} \sum_{l=\lfloor \frac{n_k}{N} \rfloor}^{\lfloor \frac{n_k}{N} \rfloor - 1} \frac{\nu(\tau_{e-k} > n_k - l \mid C_k(x))}{\sqrt{l-r\lfloor n_k/N \rfloor}}
\]
\[
\leq \sum_{r=1}^{N-1} \sum_{l=0}^{\lfloor \frac{n_k}{N} \rfloor - 1} \frac{\nu(\tau_{e-k} > n_k - l - (r+1)\lfloor n_k/N \rfloor \mid C_k(x))}{\sqrt{l+r\lfloor n_k/N \rfloor}}.
\]
It can be easily seen that
\[
\sum_{l=0}^{\lfloor \frac{n_k}{N} \rfloor - 1} \frac{1}{\sqrt{l+r\lfloor n_k/N \rfloor}} \leq \sqrt{\frac{\lfloor n_k/N \rfloor}{\sqrt{r}}},
\]
\[
\text{hence, it follows immediately that}
\]
\[
B'_{n_k} \leq \sum_{r=1}^{N-1} \sqrt{\frac{\lfloor n_k/N \rfloor}{\sqrt{r}}} \nu(\tau_{e-k} > (1-(r+1)/N)n_k \mid C_k(x)).
\]
Applying \( \limsup \) when \( p \to \infty \), then
\[
\limsup_{p \to \infty} \nu(C_k(x)) \sum_{l=\lfloor n_kp/N \rfloor}^{N-1} \frac{\nu(\tau_{e-kp} > n_kp - l \mid C_kp(x))}{\sqrt{l}} \leq t \int_0^1 f(t \sqrt{1-u}) \frac{1}{\sqrt{u}} du.
\]
Taking the limit when \( k_p \to \infty \), and combining all these estimates, we get the second inequality:
\[
1 \leq f(t) + \beta t \int_0^1 f(t \sqrt{1-u}) \frac{1}{\sqrt{u}} du.
\]
\[\square\]
Lemma 3.12. We know that the conditional distributions of the $X_{k_{\mu}}$ converge to a random variable $X$ iff the conditional distributions of the $X^2_{k_{\mu}}$ converge to $X^2$. The later then satisfies
\[
1 = \mathbb{P}(X^2 > t) + \int_0^t \frac{\mathbb{P}(X^2 > t - v)}{\sqrt{v}} dv \quad \forall t > 0.
\]

Lemma 3.13. Let $W$ be a random walk variable with values in $[0, \infty]$ satisfying
\[
\mathbb{P}(W \leq t) = \int_0^\infty \frac{\mathbb{P}(W > t - v)}{\sqrt{v}} dv \quad \forall t > 0,
\]
then
\[
\mathbb{E}[e^{-s W}] = \frac{1}{1 + c_\beta \sqrt{s}} \quad \forall s > 0.
\]
with $c_\beta := (\beta E(\frac{1}{\nu}))^{-1}$. In particular, the distribution of $W$ coincides with that of $\frac{\beta^2 E^2}{N^2}$, where the independent variables $\mathcal{E}$ and $\mathcal{N}$ are the exponential distribution of mean 1 and the standard Gaussian distribution respectively.

Proof. Let $s > 0$. We have
\[
\mathbb{E}[e^{-s W}] = \int_0^\infty \mathbb{P}(e^{-s W} \geq u) du = \int_0^\infty \mathbb{P}(W \leq v) e^{-sv} dv.
\]
Hence, for any $s > 0$, we find
\[
\mathbb{E}[e^{-s W}] = \int_0^\infty \left[ \int_0^v \frac{\mathbb{P}(W \geq v - w)}{\sqrt{w}} e^{-sv} dw \right] dv
= \int_0^\infty \frac{1}{\sqrt{w}} \left[ \beta \int_w^\infty \mathbb{P}(W \geq v - w) e^{-sv} dv \right] dw
= \beta \int_0^\infty e^{-sw} dw \cdot [1 - \mathbb{E}[e^{-s W}]],
\]
and our claim about the Laplace transform of $W$ follows, because up to a change of variable ($v^2 = 2sw$), we have
\[
\int_0^\infty \frac{e^{-sw}}{\sqrt{w}} dw = \int_0^\infty \frac{e^{-\frac{v^2}{2\sqrt{s}^2}}}{\sqrt{2\pi s}} dv = \frac{\sqrt{\beta}}{\sqrt{s}}.
\]
Hence, as a consequence of the previous computations, we end up with $\mathbb{E}[e^{-s W}] = \frac{1}{1 + c_\beta \sqrt{s}}$. Then $W$ has the same Laplace transform of $\frac{\beta^2 E^2}{N^2}$.

Proof of theorem 3.2. According to Lemma 3.10, the family of distributions of $X_k$ is tight. By Lemmas 3.11, 3.12, and 3.13, the law of $c_\beta \frac{\mathcal{E}}{|\mathcal{N}|}$ is the only possible accumulation point of the family of distributions of $(\nu(C_k(x)) \sqrt{\tau_{C_k(x)}} | C_k(x))_{k \geq 0}$. Let $\mathcal{P}$ be a probability measure absolutely continuous with respect to $\mu$, with density $h$. Set $H(x) := \sum_{k \in \mathbb{Z}} h(x, t)$. Note that by $\mathbb{Z}$-periodicity, the distribution of $\tau_e$ under $\mathcal{P}$ is the same of that under the probability measure with density $(x, t) \mapsto H(x)$ with respect to $\nu \otimes \delta_0$.

Assume first that the density $H$ is continuous. Denote by $A_k := \{ y : \nu(C_k(y) \sqrt{\tau_{C_k(y)}} > t) \}$, then we have:
\[
\mathbb{P}(A_k | C_k(x)) \sim_{k \to \infty} \nu(A_k > t | C_k(x)) \sim_{k \to \infty} \mathbb{P}(c_\beta \frac{\mathcal{E}}{|\mathcal{N}|} | C_k(x)) = \mathbb{P}(c_\beta \frac{\mathcal{E}}{|\mathcal{N}|}).
\]
And so, by the dominated Lebesgue theorem, we get:
\[
\mathbb{P}(A_k) = \int \mathbb{P}(A_k | C_k(x)) H(x) d\nu(x)
\sim \mathbb{P}(c_\beta \frac{\mathcal{E}}{|\mathcal{N}|}).
Now, take in general the density $H$ in $L^1(\nu)$. We use the fact that the set of the continuous functions is dense in $L^1(\nu)$, so that there exists $H_n$ continuous such that $H_n \xrightarrow{L^1(\nu)} H$.

\[
\mathbb{P}(A_k) = \int 1_{A_k} H(x) d\nu(x) \\
\leq \int 1_{A_k} H_n(x) d\nu(x) + \|H_n - H\|_{L^1(\nu)}.
\]

We know that there is $n$ such that $\forall \epsilon > 0 \|H_n - H\|_{L^1(\nu)} < \frac{\epsilon}{2}$. Moreover $H_n$ is continuous, then there is $k$ such that $\forall \epsilon > 0 \left| \int 1_{A_k} H_n(x) d\nu(x) - \mathbb{P}\left(c \frac{\epsilon}{\|\sigma^j\|} \right) \right| < \frac{\epsilon}{2}$. Hence the conclusion follows.

\[
\square
\]

**Proof of Corollary 3.3.** Let us set:

\[
Y_k := \log \frac{\sqrt{\tau_{e+1} \cdots \tau_{e+k}} - kd}{\sqrt{k}}.
\]

We have the case that $\nu$ is a Gibbs measure with a non degenerate Hölder potential $h$. There is a constant $c_h > 0$ such that $\log \nu(C_k(x)) = \sum_{j=-k}^{k} h \circ \sigma^j(x)$. This Birkhoff sum follows a central limit theorem (e.g. [2]), which implies that:

\[
\frac{\log \nu(C_k(.)) + kd}{\sqrt{k}} \xrightarrow{dist} N(0, 2\sigma_h^2).
\]

Observe that $Y_k$ has the following decomposition:

\[
Y_k = \frac{\log \left( \frac{\nu(C_k(.))\sqrt{\tau_{e+1} \cdots \tau_{e+k}}}{\sqrt{k}} \right)}{\sqrt{k}} - \frac{\log \nu(C_k(.)) + kd}{\sqrt{k}}.
\]

Hence, it will be enough to prove that the first term of $Y_k$ converges in distribution to 0, which is true due to Theorem 3.2.

\[
\square
\]

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