On Scale Space Radon Transform, Properties and Image Reconstruction

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Abstract

When developing a Filtered Backprojection (FBP) algorithm, considering the Radon transform (RT) as a line integral necessitates assuming that all elements of the Computed Tomography (CT) system, such as the detector cell, are dimensionless. It is generally the result of such inadequate CT modeling that analytical methods are sensitive to artifacts and noise. Then, to address this problem, several algebraic reconstruction techniques utilizing iterative models are suggested. The high computational cost of these methods restricts their application. In this paper, we propose the utilization of the Scale Space Radon Transform (SSRT), recognized for its good behavior in the scale space where, the detector width is already considered into the SSRT design and is controlled by the Gaussian kernel standard deviation. After depicting the basic properties and the inversion of SSRT, the FBP algorithm is used in two different ways to reconstruct the image from the SSRT sinogram: (1) Deconv-Rad-FBP: Deconvolve SSRT to estimate RT and apply FBP or (2) SSRT-FBP: Modify FBP such that RT spectrum used in FBP is replaced by SSRT, expressed in the frequency domain. Comparison of image reconstruction using SSRT and RT are performed on Shepp-Logan head and anthropomorphic abdominal phantoms by using, as quality measures, the Peak Signal to Noise Ratio (PSNR) and the Structural SIMilarity (SSIM). The first findings show that the SSRT-based image reconstruction quality is better than the one based on RT where, the SSRT-FBP method reveals to be the most accurate, especially, when the number of projections is reduced, making it more appropriate for applications requiring low-dose radiation such as medical X-ray CT. While SSRT-FBP and RT-FBP algorithm have utmost the same execution time, the former is much faster than Deconv-Rad-FBP. Furthermore, the experiments show that the SSRT-FBP method is more robust to CT data Poisson-Gaussian noise.

Keywords: Computed tomography, Scale Space Radon Transform, SSRT properties, SSRT inversion, Filtered back projection, Image reconstruction

1. Introduction

The Radon transform (RT) \cite{1} is the main mathematical tool used in Computed Tomography (CT) imaging where, the transform invertibility has made it extremely useful, involving image reconstruction from projections. Major image reconstruction algorithms fall into two categories: (1) the analytical methods based directly on the RT inverse formula, such as filtered backprojection (FBP) \cite{2,3} and (2) the algebraic reconstructions techniques which are, generally, iterative, alternating between the projection and image domains, involving the re-projection of an estimated image \cite{4}. Even if analytical methods are computationally efficient and easy to implement, they require excessive projections to reliably reconstruct high-quality images. In addition, analytical methods generally ignore measurement noise in the problem formulation and treat noise-related problems as an “afterthought” by post-filtering operations \cite{5}. This is considered as limitations of analytical reconstruction methods that impair their performance and restrict their utilization in radiation sensitive applications, including biological tissue studies \cite{6}.

To address the drawback of the analytical methods, algebraic methods based on iterative algorithms have been extensively studied. Compared to analytical reconstruction methods, iterative techniques have the major advantage
that they permit the emission and detection process to be accurately modeled and prior knowledge is relatively easily incorporated in the inverse process using penalty functions. They can also be performed with projections of limited number, or not uniformly distributed over 180 or 360 degrees [3]. A first method, called algebraic reconstruction technique (ART), which was introduced in 1970 by Gordon et al. [7], had a relatively rapid convergence speed but suffered from heavy salt and pepper noise. Other improved method versions, such as simultaneous iterative reconstruction technique (SIRT) [8, 9], simultaneous algebraic reconstruction technique (SART) [10, 11] and recent iterative approaches applying regularization terms, are proposed to reduce the noise but are, all, time consuming. To sum up over the algebraic methods, these are more robust to incomplete and noisy projections, but they have the disadvantage of high computational cost. There are many methods to model, for a discrete imaging object, the projection and backprojection procedures, such as pixel-driven model (PDM) [12], ray-driven model (RDM) [13], distance-driven model (DDM) [14, 15], and area-integral model (AIM) [16]. The last model, which is relatively recent, outperforms the other models, in terms of accuracy, but it suffers from high computational complexity. As mentioned at the beginning of this section, the Radon transform, which is a line integral, is the main tool used in CT imaging. However, in order to manage the RT mathematics in the derivation of FBP algorithm, for example, a series of assumptions are made regarding: (1) the physical dimension of the X-ray focal spot which is assumed to be infinitely small, approximating a point source (2) the detector cell and the image voxel are assumed to be dimensionless [17]. However, these assumptions do not describe the reality where, for most clinical CT scanners, the dimensions of the evoked parameters are non-zero.

With the aim to reduce the gap between the assumed and the real underlined CT system parameters, the authors in [17] attempt to model the "shadow" cast by each image voxel onto the detector and rely on point response functions to perform the optics modeling. In this approach, only one ray is cast between the center of the focal spot and the center of the image voxel to locate the point of intersection on the detector, and a bell-shaped function is used to distribute the contribution of the image voxel to different detector cells during the forward-projection process. We imagine that if the ray source and/or detectors are moved according to a certain geometric relationship, we obtain the projections equivalent to the Radon projections convolved with a bell-shaped function such as a Gaussian, permitting thus to attenuate the high frequency components. Still, to deal with this situation, the projection and backprojection operations are based almost on the same approximations about the detector thickness. The AIM [16] is one of the most performing models of the state-of-the-art that considers an X-ray as a narrow fan-beam, which covers a region connecting the X-ray source and the two endpoints of a detector cell. The weighting coefficient of AIM is a normalized area defined as the ratio between the overlapped area and a normalized factor of beam width at the pixel location, which can be approximated by the product of the distance from the source to the pixel center and the fan angle of the ray path. The computational cost of the AIM-based method is high because the computation of the overlapped area and the normalization factor needs many multiplication operations [15]. For 2D parallel geometry, the normalization factor becomes the distance from the source to the pixel center [1]. From the definition of the weighting coefficient, we can observe that the greater the detector width, the greater the coefficient value and then, the smoother the projection. In addition, the weighting coefficient is larger for the pixels of the detector center and are smaller for the detector edges. These two examples of the state-of-the-art recall the behavior of the Scale Space Radon Transform (SSRT), introduced recently in [18], which is a matching of an embedded shape in an image and the Gaussian Kernel. This permits to deal with the real physical dimension of CT system components including the detector width. Contrary to RT, SSRT does not require the use of a zero-width detector because the detector width is taken into account by the SSRT-based projection process, where it is adjusted by the Gaussian kernel standard deviation.

The choice of this integral transform satisfies several requirements such as nice behavior in scale space, robustness to noise, uniqueness of transform maximum, etc. [18]. It follows that RT is a particular case of SSRT where, the Gaussian kernel is replaced by a Dirac distribution δ. As for the Radon transform, several properties could be derived from SSRT and which could be very useful in computer vision applications such as industrial and medical CT. In this paper, some basic properties of SSRT will be presented. Furthermore, the approaches of SSRT inversion will be developed and discussed where, a new version of Filtered Backprojection (FBP) dedicated to SSRT will be derived in detail and applied on reconstruction of test and real CT scan images.

The remainder of the paper is organized as follows. The definition of SSRT and its relationship with Radon transform are depicted in Section 2. Section 3 is devoted to the SSRT properties including its inversion. Image
reconstruction using SSRT is detailed in Section 4 and main experiments and results are reported in Section 5. Lastly, main conclusions and perspectives are drawn in Section 6.

2. Scale Space Radon Transform

2.1. Definition

Let \((x, y)\) the Cartesian coordinates of a point in a 2-D space, \(f(x, y)\) a continuous function with a compact support and \(g_{\Delta, \theta}\) a set of parallel linear Gaussians \(g_\theta(r - \rho, \sigma)\), parameterized by the orientation angle \(\theta\), the standard deviation \(\sigma\) and the centerline \((\Delta)\): \(x\cos \theta + y\sin \theta - \rho = 0\), as illustrated in Fig. 1. By referring to Fig. 1, Scale Space Radon transform (SSRT), noted \(\mathcal{R}_\sigma f(\rho, \theta)\), is defined by \([18]\)

\[
\mathcal{R}_\sigma f(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g(x \cos \theta + y \sin \theta - \rho) \, dx \, dy
\]

where, \(g(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)}\) is the Gaussian function.

![Figure 1: Definition of the Scale Space Radon Transform](image)

2.2. Its relationship with Radon transform

Let consider a \((r, s)\) coordinate system, as in Fig. 1, a rotated version of the original \((x, y)\) system, expressed as

\[
\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

It follows that the SSRT is rewritten as

\[
\mathcal{R}_\sigma f(\rho, \theta) = \int_r \int_s f(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) g(r - \rho) \, dr \, ds
\]

Equivalently,

\[
\mathcal{R}_\sigma f(\rho, \theta) = \int_r g(r - \rho) \left[ \int_s f(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) \, ds \right] \, dr
\]
Note that the expression between brackets is nothing but the expression of the Radon transform $Rf(\rho, \theta)$. It follows that
\[ R_\sigma f(\rho, \theta) = \int g(r - \rho) Rf(\rho, \theta) dr = (g \ast^\rho Rf)(\rho, \theta) \tag{5} \]
where $\ast^\rho$ is the convolution operation with regards to (w.r.t.) the variable $\rho$.

The formulae (5) constitutes a straightforward way to compute the SSRT which is nothing but the convolution of Radon transform with a Gaussian kernel tuned by a scale parameter $\sigma$. Consequently, the SSRT implementation is reduced to a convolution operation.

3. SSRT Properties

3.1. Basic Properties

In order to better handle the equations dealing with the SSRT properties, let us define the unitary vectors in the $(r-s)$ coordinates system, as $\xi = [\cos \theta \ \sin \theta]^T$ and $\xi^\perp = [-\sin \theta \ \cos \theta]^T$, the position vector as $z = [x \ y]^T$ and $dz = dxdy$. First, observe that the equation of the line $(\Delta)$, in Fig. 1 may be written $\rho = \xi^T z = x \cos \theta + y \sin \theta$ and then the SSRT can be rewritten as
\[ R_\sigma f(\rho, \xi) = \int f(z) g_\sigma (\xi^T z - \rho) dz \tag{6} \]

Several elementary properties of the SSRT of a function $f$ defined in $\mathbb{R}^2$ follow directly from the definition.

3.1.1. Linearity

Let us consider two functions $f_1$ and $f_2$, two real numbers $a$ and $b$ and a function $h$ such as $h = af_1 + bf_2$, we have then
\[ R_\sigma h(\rho, \xi) = \int [af_1(z) + bf_2(z)] g_\sigma (\xi^T z - \rho) dz = aR_\sigma f_1(\rho, \xi) + bR_\sigma f_2(\rho, \xi) \tag{7} \]

3.1.2. Homogeneity

If $a$ is a non-zero real number, then we have
\[ R_\sigma f(a\rho, a\xi) = \int f(z) g_\sigma (a\xi^T z - a\rho) dz = \frac{1}{|a|} \int f(z) g_{\sigma/|a|} (\xi^T z - \rho) dz = \frac{1}{|a|} R_\sigma f(\rho, \xi) \tag{8} \]

Because of the presence of the amounts $\sigma/|a|$ in the result, the property of homogeneity is not satisfied for the SSRT, except when $|a| = 1$, i.e. $a = \pm 1$. For $a = 1$, it consists in the identity transform whilst $a = -1$ relates the case of symmetry in regards to the axes system origin. The last property will be detailed in the next paragraph.

3.1.3. Symmetry

Setting $a = -1$ in (8) yields
\[ R_\sigma f(-\rho, -\xi) = R_\sigma f(\rho, \xi) \tag{9} \]
In another notation, it means: $R_\sigma f(-\rho, \theta \pm \pi) = R_\sigma f(\rho, \theta)$. 

4
3.1.4. Geometric transformation

Let \( z \) and \( z' \), two vectors representing two points in \( \mathbb{R}^2 \) space so that
\[
z' = Az
\]
where \( A \) is a nonsingular matrix with real elements representing the transfer matrix of geometric transformations such as scaling, rotation, shearing, etc.

\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = \int f(\mathbf{Az}) \cdot g_\sigma(\xi \cdot z - \rho) \, dz
\]

Since \( z = A^{-1}z' \) and \( dz = |A^{-1}| \, dz' \) where \( |\cdot| \) denotes the matrix determinant, then, we obtain
\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = |A^{-1}| \int f(z') \cdot g_\sigma(\xi' A^{-1}z' - \rho) \, dz'
= |A^{-1}| \mathcal{R}_\sigma[f(z)](\rho, (\xi A^{-1})')
= |A^{-1}| \mathcal{R}_\sigma[f(z)](\rho, A^{-1} \xi)
\]

**Case of Rotation:** If \( A \) is a matrix of rotation, i.e. \( A' = A^{-1} \) and \( |A^{-1}| = 1 \), we have
\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = \mathcal{R}_\sigma[f(z)](\rho, A^T \xi)
\]

In other words, if \( A = M_{\text{Rot}}^{\theta_0} = \begin{bmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{bmatrix} \) then \( A' \xi = \begin{bmatrix} \cos(\theta + \theta_0) \\ \sin(\theta + \theta_0) \end{bmatrix} \). In former notation, this means that
\[
\mathcal{R}_\sigma[f(\mathbf{A})](\rho, \theta + \theta_0)
\]

**Case of Scaling:** Let \( A = \alpha I \) where, \( \alpha (\alpha \in \mathbb{R}^+) \) a scale factor, then \( A^{-1} = \frac{1}{\alpha} \) and \( |A^{-1}| = 1/\alpha^2 \). Hence, we obtain
\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = \frac{1}{\alpha^2} \mathcal{R}_\sigma[f(z)] \left( \alpha \rho, \frac{\xi}{\alpha} \right)
\]

On the other hand,
\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = \frac{1}{\alpha^2} \int f(z') \cdot g_\sigma \left( \frac{\xi}{\alpha} z' - \rho \right) \, dz'
= \frac{1}{\alpha^2} \int f(z') \cdot g_\sigma \left( \frac{1}{\alpha} (\xi' z' - \alpha \rho) \right) \, dz'
= \frac{1}{\alpha^2} \int f(z') \cdot [\alpha g_\sigma(\xi' z' - \alpha \rho)] \, dz'
= \frac{1}{\alpha} \mathcal{R}_{\alpha \sigma}[f(z)](\alpha \rho, \xi)
\]

In terms of former notation, we have
\[
\mathcal{R}_{\alpha \sigma}^{\alpha}[f(\mathbf{A})](\rho, \theta) = \frac{1}{\alpha} \mathcal{R}_{\sigma \alpha}[f(\alpha \rho, \theta)]
\]

3.1.5. Translation

Let \( z, z' \) and \( b \) three vectors representing the coordinates of 3 points in \( \mathbb{R}^2 \) space such as \( z' = z + b \). In order to find a relationship between the SSRTs of the functions \( f(z) \) and \( f(z') \), it suffices to write
\[
\mathcal{R}_\sigma[f(z')](\rho, \xi) = \int f(z + b) \cdot g_\sigma(\xi' z - \rho) \, dz
= \int f(z') \cdot g_\sigma(\xi' z - \xi' b - \rho) \, dz'
\]
which yields
\[ \mathcal{R}_\sigma[f(z')](\rho, \xi) = \mathcal{R}_\sigma[f(z)](\rho + \xi \xi) \]  
(19)

Using another notation, if \( b = [b_x, b_y]' \) then the translation property can be expressed as
\[ \mathcal{R}_\sigma^b f(\rho, \theta) = \mathcal{R}_\sigma f(\rho + b_x \cos \theta + b_y \sin \theta, \theta) \]  
(20)

3.2. Analytical Inversion

According to (5), we can write
\[ Rf(\rho, \theta) = F^{-1}_\rho \left[ F[\mathcal{R}_\sigma f(\rho, \theta)](\omega) \right] (\rho, \theta) \]  
(21)

where \( F_\rho \) and \( F^{-1}_\rho \) are, respectively, the 1D Fourier transform and its inverse with regards to the variable \( \rho \). Let us recall that the inverse Radon transform theorem states that
\[ f(x, y) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{\partial Rf(\rho, \theta)}{\partial \rho} \frac{1}{\rho x \cos \theta + y \sin \theta - \rho} d\rho d\theta \]  
(22)

for \( 0 < \theta < \pi \) and \( -\infty < \rho < \infty \).

By substituting (21) in (23), we obtain
\[ f(x, y) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{\partial F^{-1}_\rho \left[ F[\mathcal{R}_\sigma f(\rho, \theta)](\omega) \right] (\rho, \theta)}{\rho x \cos \theta + y \sin \theta - \rho} d\rho d\theta \]  
(23)

4. Image reconstruction using Scale Space Radon Transform

4.1. SSRT inversion through image deconvolution

The computation of the inverse SSRT (iSSRT) through (23), evoked in §3.2, is straightforward but it is just given as an indication, since the deconvolution defined by the division in Fourier space is hazardous. On the basis of the link between the SSRT and the RT, provided in (5), the inversion can be implemented in two steps:

• Step 1: Estimate the Radon transform \( Rf(\cdot) \) given the SSRT \( \mathcal{R}_\sigma f(\cdot) \) and the Gaussian function \( g(\cdot) \) using an existing deconvolution algorithm.

• Step 2: Estimate the image function \( f(\cdot) \) by using an existing RT inversion algorithm.

For Step 1, efficient deconvolution methods to estimate the Radon transform could be applied such as the image deconvolution method using the regularized filter algorithm [19]. About Step 2., dealing with image reconstruction from Radon projections, various approaches are investigated and developed by the researchers, among them, the popular Filtered Backprojection reconstruction method which is based on the Fourier Slice Theorem (FST). With the aim to compare the reconstruction methods in terms of performance, the deconvolution scheme in [19], followed by FBP, will be used later in the Experiments.

In the following two subsections, we will go over FST and FBP, which are frequently utilized with RT, in order to address SSRT inversion.
4.2. Fourier Slice Theorem and Frequency-Domain Reconstruction for SSRT

The 1-D Fourier transform of Radon transform w.r.t. to the variable $\rho$, noted $P_\theta(\omega)$ is given by

$$P_\theta(\omega) = \mathcal{F}_\rho (\mathcal{R}f(\rho, \theta)) = \int_{-\infty}^{+\infty} \mathcal{R}f(\rho, \theta)e^{-i2\pi\omega \rho} d\rho$$

(24)

where, $\omega$ is the frequency.

By substituting the RT expression into (24), we obtain

$$P_\theta(\omega) = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho) dx dy \right] e^{-i2\pi\omega \rho} d\omega$$

(25)

where, $\delta$ is the Dirac function.

Rearranging the order of integrals and using the sifting property of Dirac’s delta function, gives

$$P_\theta(\omega) = \int_{-\infty}^{+\infty} f(x, y) e^{-i2\pi\omega(x \cos \theta + y \sin \theta)} dx dy$$

(26)

Comparing now this equation with the expression of the 2-D Fourier transform of $f(x, y)$, given by

$$\mathcal{F}(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-i2\pi(ux + vy)} dx dy$$

(27)

We remark that the right side of the expression $P_\theta(\omega)$ represents the 2D Fourier transform of $f(x, y)$ expressed in polar coordinates system of the Fourier plan, which results in the proposed statement

$$P_\theta(\omega) = \mathcal{F}(u, v)|_{u=\omega \cos \theta, v=\omega \sin \theta} = \mathcal{F}_\text{polar}(\omega, \theta)$$

(28)

Thus, in parallel-beam geometry, the FST states that the 1-D Fourier transform $P_\theta(\omega)$ of a projection $\mathcal{R}f(\rho, \theta)$, obtained from the image $f(x, y)$, is identical to the central slice, at the angle $\theta$, of the 2-D image spectrum $\mathcal{F}(u, v)$.

Now, how could we apply the FST on the SSRT? For this purpose, for given $\theta$ and $\sigma$, let us compute the 1-D Fourier of the SSRT w.r.t. the variable $\rho$. Using (28) and Fourier transform of the convolution we can write

$$P_{\theta, \sigma}(\omega) = \mathcal{F}_\rho[\mathcal{R}_\sigma f(\rho, \theta)](\omega) = \mathcal{F}_\rho[g(\rho)](\omega) \cdot \mathcal{F}_\rho[\mathcal{R}f(\rho, \theta)](\omega)$$

$$= G(\omega)P_\theta(\omega)$$

(29)

where, $G(\omega) = e^{-2\pi^2\sigma^2\omega^2}$ is the Fourier transform of the Gaussian function $g$.

Since, the FST deals with the Radon projections, the spectrum of the latter, which is linked to the 2-D Fourier transform of $f(x, y)$ as seen above, can be deduced from the SSRT projections as

$$P_\theta(\omega) = P_{\theta, \sigma}(\omega)/G(\omega) = P_{\theta, \sigma}(\omega)/e^{-2\pi^2\sigma^2\omega^2}$$

(30)

We can see, for small values of $G(\omega)$ i.e., for very large values of $\sigma$ and/or $\omega$, that computing $P_\theta(\omega)$ by (30) becomes problematic, since, divisions by a value close to zero will greatly increase high-frequency noise contribution.

To address this issue, among others, a Wiener filter could be used. Before going further, let us consider a discrete $M \times N$ image $f$, defined on the image domain $D_f$ and corrupted by additive white noise $\eta$ with zero mean and variance $\sigma^2_\eta$. For each pixel $(x, y)$, we define the observed image $f_n$ as $f_n(x, y) = f(x, y) + \eta(x, y)$. Using the linearity property in (7), the SSRT of the observed image $f_n$ is given by

$$\mathcal{R}_\sigma f_n(\rho, \theta) = \mathcal{R}_\sigma f(\rho, \theta) + \mathcal{R}_\sigma \eta(\rho, \theta)$$

$$= (RF \circ g)(\rho, \theta) + \mathcal{R}_\sigma \eta(\rho, \theta)$$

(31)
and, for a given $\theta$, we can compact (31) to obtain

$$\mathbb{R}_\theta f_n(\rho) = (\mathbb{R}_\theta f * g)(\rho) + \mathbb{R}_\theta \eta(\rho)$$

(32)

The formulae (32) represents a typical mathematical model for many imaging systems including CT, in which $g$ is commonly known as the point spread function (PSF). This function being known, which consists, in our case, in Gaussian function, deconvolution operations permit to reverse the process in order to improve the resolution of the imaging system. In our case, the deconvolution consists in the estimation of the Radon projection from SSR T projection in presence of noise and Gaussian smoothing. Numerous algorithms could be developed to recover $\mathbb{R}_\theta f$ from (32). Examples include Wiener deconvolution [21, 22], Lucy-Richardson algorithm [23] and variational model [24, 25].

Wiener deconvolution applies a Wiener filter inherent in the deconvolution to the observed noisy samples, enabling simultaneous reconstruction and denoising [22]. It is designed so that the expectation value of the Mean Square Error (MSE)

$$\epsilon = \mathbb{E}\{|(\mathbb{R}_\theta f(\rho) - \hat{\mathbb{R}}_\theta f(\rho))|^2\}$$

is minimized where, the hat symbol is used to indicate the estimate of the noise-free Radon projection. By deriving the MSE minimization in the frequency domain, the 1-D Fourier transform, w.r.t. $\rho$, of the noise-free Radon projection estimate is given by [19]

$$\hat{P}_\theta(\omega) = H_W(\omega)P_{f*}(\omega)$$

(33)

where, $P_{f*}(\omega)$ is the SSR T projection spectrum of the observed noisy image and $H_W(\omega)$ is the frequency response of the Wiener filter, expressed as

$$H_W(\omega) = \frac{1}{G(\omega)} \frac{|G(\omega)|^2}{|G(\omega)|^2 + K}$$

(34)

where, $K = P_\theta(\omega)$ with $P_\eta(\omega) = \mathbb{E}|P_\theta(\omega)|^2$ and $P_\eta^2(\omega) = \mathbb{E}|\hat{P}_{\eta,\sigma}(\omega)|^2$ are the mean power spectral densities of $\mathbb{R}_\theta f(\rho)$ and $\mathbb{R}_{\theta,\sigma} \eta(\rho)$, respectively.

When there is zero noise, i.e. $K = 0$, the Wiener filter is simply the inverse filter. In practice, the quantities $P_w(\omega)$ and $P_\eta(\omega)$ are not known; so $K$ is set to a constant scalar which is determined empirically.

Figure 2: Illustration of Fourier Slice Transform in case of SSR

In case of SSRT, the FST flowchart can be illustrated in Fig[2]. So, by rotating the SSRT detector for 180°, the entire 2D Fourier transform $\mathcal{F}(u,v)|_{u = \omega \cos \theta, v = \omega \sin \theta}$ is “measured”. Once the 2D frequency space of the object function is
filled, a 2-D inverse Fourier transform then yields the reconstructed object function, i.e.

\[ f(x, y) = \int_{-\infty}^{+\infty} \mathcal{F}(u, v)e^{2\pi i (ux + vy)} du dv \tag{35} \]

The method of image reconstruction, depicted until now, is called Frequency-Domain Reconstruction. As ascertained in [3], in practice, only a finite number of projections of an object can be taken. Then, the function \( \mathcal{F}(u, v) \) is only known along a finite number of radial lines such as in Fig. 3. To overcome this limitation and fill completely the 2D frequency space of the object function, very time-consuming operations of interpolation are required. Furthermore, since the density of the radial points becomes sparser as one gets farther away from the center, the interpolation error also becomes larger. These errors, implied in the high frequency domain, result in image degradation [3]. Consequently, the Frequency-Domain Reconstruction method is not used in practice.

![Sampling of the 2-D Fourier space of the object function.](image)

**4.3. Filtered Backprojection (FBP) using SSRT for Parallel-Beam Geometry**

As with Radon transform, the FST will be applied on the SSRT projections to derive the FBP algorithm. This derivation is based on the reformulation of the FST into polar coordinates, yielding a 2-step reconstruction method: a projection filtering and a backprojection onto the image domain.

The 2D inverse Fourier transform of \( \mathcal{F}(u, v) \) is given by

\[ f(x, y) = \mathcal{F}^{-1}\{\mathcal{F}(u, v)\} = \int_{-\infty}^{+\infty} \mathcal{F}(u, v)e^{2\pi i (ux + vy)} du dv \tag{36} \]

Expressing this equation in a polar coordinate system \((\omega, \theta)\) with \( u = \omega \cos \theta, v = \omega \sin \theta \) and \( du dv = \omega d\omega d\theta \), yields

\[ f(x, y) = \int_0^{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}_{polar}(\omega, \theta)e^{2\pi i \omega (x \cos \theta + y \sin \theta)} \omega d\omega d\theta \tag{37} \]

Splitting the outer integral domain into \([0, \pi)\) and \([\pi, 2\pi)\), and using \( \mathcal{F}_{polar}(\omega, \theta + \pi) = \mathcal{F}_{polar}(-\omega, \theta) \) yields

\[ f(x, y) = \int_0^{\pi} \left[ \int_{-\infty}^{+\infty} \mathcal{F}_{polar}(\omega, \theta) |\omega|e^{2\pi i \omega (x \cos \theta + y \sin \theta)} \omega d\omega \right] d\theta \tag{38} \]
According to the FST proposal (28) and the RT/SSRT relationship in frequency domain provided in (33), we obtain

\[ f(x, y) = \int_{0}^{\pi} \left[ \int_{-\infty}^{+\infty} P_{\omega, \sigma}^{\theta} (\omega) H_{\omega} (\omega) |\omega| e^{i2\pi \omega (x\cos \theta + y\sin \theta)} d\omega \right] d\theta \] \tag{39}

The integral in (39) may be expressed as

\[ f(x, y) = \int_{0}^{\pi} Q_{\omega, \sigma} (x\cos \theta + y\sin \theta) d\theta \] \tag{40}

where

\[ Q_{\omega, \sigma} (\rho) = \int_{-\infty}^{+\infty} P_{\omega, \sigma}^{\theta} (\omega) H_{\omega} (\omega) |\omega| e^{i2\pi \omega \rho} d\omega \] \tag{41}

The formulae (41) represents a filtering operation, where the frequency response of the filter is given by \( H_{\omega} (\omega) = |\omega| H_{W} (\omega) \) whilst, (40) represents the backprojection of the filtered projections. The resulting projections for the noise-free Radon projection, as in (33). According to the reasoning developed until now, we have considered the idealized case where, there is a continuum of projection views. In practice, since the image function is discrete and bounded, the sinograms have only a finite number of angular and radial samples. Then, the formula (41) is not absolutely integrable, it is not possible to derive directly the corresponding impulse response by inverse Fourier transform. To address this issue, we have to window the filter so that it becomes zero outside a defined frequency interval.

Suppose that \( H_{\omega} (\omega) \) is band-limited with maximum frequency less than \( \omega_{\text{max}} = \frac{1}{\tau} \) where, \( \tau \) is the sampling interval with which the projection data are measured. Then,

\[ H_{\omega} (\omega) = \begin{cases} |\omega| H_{W} (\omega), & |\omega| < \omega_{\text{max}} \\ 0, & \text{otherwise} \end{cases} \] \tag{42}

An example of frequency response for \( H_{\omega} (\omega) \) is given in Fig. 4 where, the constant \( K \) and the Gaussian kernel \( std \), for Wiener filter, are set to 0.02 and 1.5, respectively. It is known that the ramp filter \( H(\omega) = |\omega| \) amplifies high frequency noise, so the low-pass Wiener filter \( H_{W} \) is used, here, as a smoothing filter.

The steps of the FBP algorithm applied on SSRT projections are summarized as follows.

- **Step 1:** Compute the 1-D Fourier transform \( P_{\theta, \sigma}^{\omega} (\omega) \) of each SSRT projection for the observed image.
- **Step 2:** Multiply each SSRT projection spectrum by the Wiener filter \( H_{W} (\omega) \) to obtain the spectrum estimate \( \hat{P}_{0} (\omega) \) of the noise-free Radon projection, as in (33).
- **Step 3:** Multiply \( \hat{P}_{0} (\omega) \), for each \( \theta \), by the 1-D windowed ramp filter \( H(\omega) = |\omega| \).
- **Step 4:** Perform a 1-D inverse Fourier transform on the filtered projection using (41).
- **Step 5:** Perform the backprojection by integrating, over \( \theta \), all the 1-D inverse transforms from Step 4 using (40).
5. Experiments

The proposed Filtered Backprojection method using SSRT projections, of which the steps are summarized in §4.3, is denoted SSRT-FBP. In order to show the effectiveness of this method in CT image reconstruction, let us compare its performance to: (1) the SSRT inversion scheme given at §4.1, denoted Deconv-Rad-FBP and (2) the classical FBP-based inverse Radon transform, denoted Radon-FBP. Because of its widespread utilization in the evoked domain, a Shepp-Logan phantom of size $M \times N = 512 \times 512$, coded on 256 gray levels, is used as test image in the first experimental analysis (see Fig. 5).

In order to compare between the image reconstruction performance of each of the used methods, the influence of three factors are depicted: (1) the number of CT projections, (2) the value of the Gaussian kernel standard deviation $\sigma$ and (3) the type and the level of CT data noise.

For the first factor, the test image is transformed by SSRT and RT to provide sinograms on 180°, used as input projections, with angular steps of $\Delta \theta = 0.5°, 1°, 2°, 5°, 10°, 20°, 30°$.

As seen, previously, the SSRT projections magnitude depends on the standard deviation $\sigma$ of the Gaussian function involved in SSRT. That is why, to see the influence of the second factor on the outcomes, a range of $\sigma$ values, varying from $\approx 0$ to $\sigma = 2.8$ with a step of $\Delta \sigma = 0.2$, is employed in these first experiments.

If $S$ denotes the sinogram data $Rf(\rho, \theta)$ or $\mathcal{R}_\sigma f(\rho, \theta)$ of an input object $f$, with $(\rho, \theta)$ as sinogram coordinates, produced, respectively, by the RT-based or SSRT-based CT scans, then, the one-dimensional (1-D) version of $S$, denoted $S^1$, will represent the sinogram as a vector with elements $S^1_i, i = 1, \ldots, N_S$, where $N_S$ is the resolution of the sinogram $S$ [26]. This resolution is nothing but the number of measurements in the scan which can be calculated as $N_S = m_S \times n_S$, where $m_S$ and $n_S$ are the number of discrete values of $\rho$ and $\theta$, representing the number of detector elements and the number of projection views, respectively.

Using Lambert-Beer’s law under the assumption of a monoenergetic X-ray beam, the mean of the transmission or the projection data can be expressed as $\bar{Y}_i = I_0 \exp\{-S^1_i\}$, where $I_0$ denotes the incident flux representing the amount of photons along the projection path acquired by the detector bin $i$.

In practice, the CT data are noisy and the measurements are often modeled as a sum of a Poisson distribution representing photon-counting statistics and an independent Gaussian distribution with mean $m_n$ and variance $\sigma_n^2$ representing additive electronic noise [27,28,29,30], where $m_n$ is assumed to be zero because the data acquisition system (DAS) offsets can be measured by a dark scan and subtracted from the CT data, which does not modify the variance of the data [30]. The quanta noise is due to the limited number of X-ray photons collected by the detector and the electronic noise is the result from electronic fluctuation in the detector photodiode and other electronic components [31]. That is said, by taking into account the noise, the CT measurements are expressed as $Z_i = Y_i + \eta_i$, where: (1) $Y_i \sim \text{Poisson}(\lambda)$ is the quanta of rays received by detector bin $i$ with $\lambda = \bar{Y}_i$ is, as above-mentioned, the mean number of photons or the noise-free transmission datum for detector bin $i$ at a projection view; and (2) $\eta_i \sim \mathcal{N}(0, \sigma_n^2)$, with $\sigma_n^2$
denoting the variance of the electronic noise which has been converted to photons for detector bin \( i \).

In these first experiments, to compare the performance of the used image reconstruction methods in presence of Poisson-Gaussian noise, we generated noisy projections of Shepp-Logan phantom with two incident exposure levels \( I_0 = \{10^5, 2 \times 10^3\} \) photons per ray and two values for Gaussian noise variance \( \sigma_n^2 = \{0.01, 1\} \).

The reconstructed images by the methods Radon-FBP, Deconv-Rad-FBP and SSRT-FBP are denoted by the 2D functions \( f^{Rad} \), \( f^{deconv+Rad} \) and \( f^{SSRT} \), respectively. Well-known metric of Peak Signal-to-Noise Ratio (PSNR) and Structural SIMilarity (SSIM) are used to evaluate the quality of the above-mentioned image reconstruction methods. Here, \( PSNR_k = 10 \log_{10} \frac{255^2}{MSE_k} \) where, \( MSE_k = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( f_{ij} - f'_{ij}\right)^2 / MN \) with, MSE is the acronym of the mean square error, \( f \) is the free-noise input test image and \( k \in \{ iRad, deconv+Rad, iSSRT \} \). The higher the PSNR, the better the image reconstruction method; while the values of SSIM \([32]\) belong to the range \([0, 1]\) where, high values express better results.

To assure the statistical validity of the experiments since the noise is generated randomly, the experiments are carried out \( N_t \) times (\( N_t = 20 \)) and the average result is taken. For the Wiener filter, a value of \( K \) equal to 0.02 reveals to be sufficient to make a trade-off between the effects of sharpening and smoothing. For all methods, the running time is expressed in seconds \( (s) \) and the implementations are performed in Matlab environment on a Dell/OptiPlex Workstation with 2.9 GHz CPU and 32 GB RAM.

Fig. [6] and Fig. [7] depict, respectively, the PSNR and the SSIM values for Shepp-Logan image reconstruction by browsing \( \sigma \) and \( \Delta \theta \) on their value interval when the CT data noise is: (1) absent and (2) modeled by Poisson/Gaussian distributions with above-mentioned noise levels. So, the interpretation and analysis of the PSNR and SSIM curves provided in Fig. [6] and Fig. [7] will depend on the variables: (1) the SSRT scale parameter \( \sigma \), (2) the number of projection which is linked with \( \Delta \theta \) (#projections = \( 180 / \Delta \theta \)) and (3) the levels of CT data noise which is, in turn, expressed as a mixture of: (3a) Poisson noise, parameterized by the number of incident photons \( I_0 \) and (3b) Gaussian white noise, parameterized by zero-mean electronic noise with variance \( \sigma_n^2 \). It is worth to note that, although the Radon-FBP method is not concerned by \( \sigma \), the latter is just included in its PSNR and SSIM graphics for the sake of comparison.

At first, let examine the PSNR and SSIM curves, provided in Figs. [5] and [7], in case of free-noise CT data. It appears, clearly, that the image reconstruction, for all methods, is better as \( \Delta \theta \) decreases; which is not surprising, since the quality of the reconstructed input Shepp-Logan phantom is increasingly improved as the number of projections increases. We can remark by running the \( \sigma \) values, for each of the angle steps \( \Delta \theta \), that the values of \( \sigma \) less than 0.4 ~ 0.5 give almost the same performance for all the methods. This is expectable since, for small \( \sigma \)’s, the behavior of SSRT-based methods tends to that of RT, as given in (2.1). However, for greater \( \sigma \) values, the SSRT-FBP method outperforms Radon-FBP and Deconv-Rad-FBP methods. As illustrated in Figs. [5]-[8] and Figs. [7]-[10], these observations are also valid when the input CT data is corrupted by Poisson-Gaussian noise with various levels where, the difference between the PSNR/SSIM curves, for the three methods, in the \( \sigma - \Delta \theta \) space, is more pronounced. In fact, we observe that the PSNR/SSIM values dropped, drastically, for Radon-FBP and, to a lesser extent, for Deconv-Rad-FBP, as the noise level increases, i.e. when \( I_0 \) decreases and \( \sigma \) increases. Inversely, the SSRT-FBP method is shown to be robust to the noise inherent to CT data. As first observation, the image reconstruction scores of the SSRT-based methods (Deconv-Rad-FBP and SSRT-FBP) are better than those obtained by Radon-FBP method where, particularly, the SSRT-FBP method is the best. The claimed outstanding performance is reasonable given that, in one hand, the SSRT-based approaches are more resistant to CT data noise thanks to the inclusion of a Gaussian kernel in their structural design which acts as a noise filter and, in the other hand, the SSRT-FBP method assumes that the X-ray beam projections are collected by detectors with a certain width, which match better with SSRT than with RT. In fact, as explained in Sect. 1 and shown in Fig. [1], the projection made by the SSRT-based CT beam at \( \rho \) along the line \( (\Delta) \) actually realizes a Gaussian weighting sum of the gray levels of the pixels generating an image band with \( (\Delta) \) as centerline. The thickness of this band which depends on the \( \sigma \) value (see [18]) is, in reality, the thickness of the X-ray beam projection on the CT detector. That is said, for RT-based CT, the displacement of the point \((0, \rho)\) in the \( r - s \) plan is so small when the projection axis is rotated by smaller angles, i.e. \( \Delta \theta \sim 0.5 - 0.7° \), that practically all of the projections displayed on the RT sinogram are produced by all the pixels forming the input CT data. However, more noise and artifacts are added to the RT sinogram when the rotation angle increment is sufficiently large, i.e. when low-dose CT is used, because the missing projections result in interpolation errors on the reconstructed image space. As previously mentioned, the SSRT, on the other hand, manages a non-zero line projection which play a fil-
tering role that enables it to cover the input image within a number of projections. This SSRT characteristic could be advantageous for a variety of CT applications, such as medical CT scanning, where the decrease of radiation doses is essential since an excessive patient exposure to penetrating radiation can result in major health harm.

To better examine, for the tested methods, the variations of PSNR and SSIM in terms of variables $\Delta \theta$ and $\sigma$, let us display, in Fig. 8, the same pictures given in Figs. 6a, 7a, 6e and 7e, according to another angle of view. In case of noise-free CT data (see Fig. 8a and 8b), it can be seen that, for $\Delta \theta = 0.5^\circ$, higher values of PSNR and SSIM are obtained by SSRT-FBP for $\sigma \sim 0.8$ whereas, for $\Delta \theta = 1^\circ$ and $\Delta \theta = 2^\circ$, the best results are still given by the last method when $\sigma \sim 1.6$ and $\sigma > 2.8$, respectively. The same remarks on the increase of $\sigma$ values could be made for larger $\Delta \theta$ values of over 2°. In presence of pronounced noise as is the case in Fig. 8c and 8d, highest quality reconstructions are obtained with SSRT-FBP when specific values of $(\Delta \theta, \sigma)$ pairs are taken such as $(\Delta \theta, \sigma) = \{(0.5^\circ, 1.8), (1^\circ, 2), (2^\circ, > 2.8)\}$ and the same observations as in noise-free case can be made for larger $\Delta \theta$. We remark that, in presence of noise, the values of $\sigma$, for the same values of $\Delta \theta$, are a bit higher than those of the noise-free version. That is can be explained by the fact that $\sigma$, should be enough large, in presence of noise, so that the SSRT-FBP can achieve the noise filtering task, in addition of its role in minimizing the effects of small number of CT projections.

To support this explanation, we can observe, in noise-free case, that the Radon-FBP PSNR is slightly higher than the Deconv-Rad-FBP and SSRT-FBP PSNRs when $\Delta \theta$ is in the range [0.5° 0.7°] and $\sigma$ is higher than 1.8 as shown, in Fig. 8. This observation could be justifiable since, due the line integral principal, the smaller $\Delta \theta$’s are, the bigger the accuracy of RT-FBP is in CT image reconstruction. So, in this case, beyond 1.8, $\sigma$ increases the smoothing effect of SSRT-FBP at the expense of the details preservation. However, for the same $\Delta \theta$, when the input CT data is corrupted by a strong composite noise, the RT-FBP performance deteriorates substantially, as illustrated in Fig. 8, which highlights the advantage of the proposed method, namely SSRT-FBP, in CT image reconstruction.

The abovementioned observations ascertain the good behavior of SSRT-FBP in X-ray CT image reconstruction by handling correctly: (1) the missing data due to the reduction of the number of projections and (2) the presence of Poisson-Gaussian in CT sinograms. Consequently, the best recovery results could be obtained by a fine tuning of the Gaussian kernel standard deviation $\sigma$ in function of the number of body X-ray exposures.

Fig. 9a displays the noise-free image of Shepp-Logan phantom and Figs 9b-9d depicts its RT, Deconv-RT and SSRT sinograms and their corresponding reconstructed images with methods of Radon-FBP, Deconv-Rad-FBP and SSRT-FBP, respectively, where $(\sigma, \Delta \theta) = (1.2, 2^\circ)$ and the Poisson-Gauss noise has as parameters $b = 10^4$ and $\sigma_x = 0.5$. The gray level profiles of the noise-free image and the abovementioned methods-based recovery results are depicted in Fig. 10. The gray level profiles are located, in the x – y plan, at the pixel positions $y = 280$ and $x$ from 50 to 300, as illustrated by the yellow line segment in Fig. 9a. It can be observed that the profiles of the results from SSRT-FBP are closer to that of the ground truth image than those of Deconv-Rad-FBP and Radon-FBP methods, where the fitting mean absolute errors of the profiles in the reconstructed images, compared to those of the ground truth, are 22.34, 15.14 and 7.35 for Radon-FBP, Deconv-Rad-FBP and SSRT-FBP, respectively. These last findings
(a) Without noise

(b) $I_0 = 10^5, \sigma_n = 0.1$

(c) $I_0 = 10^5, \sigma_n = 1$

(d) $I_0 = 2 \times 10^4, \sigma_n = 0.1$

(e) $I_0 = 2 \times 10^4, \sigma_n = 1$

Figure 6: PSNR values for the reconstructed Shepp-Logan head phantom image in function of $\sigma$ and $\Delta \theta$ and noise parameters.
Figure 7: SSIM values for the reconstructed Shepp-Logan head phantom image in function of $\sigma$ and $\Delta\theta$ and noise parameters.
ascertain again the high noise reduction capability of SSRT-FBP, in addition of its ability to minimize the projections number reduction effects, as previously explained.

The SSRT-FBP method has a runtime of about 0.03 s, which is equivalent to Radon-FBP in terms of execution speed, as shown in Table 1. In fact, the SSRT-FBP routine in Matlab, called issrt, consists in adapting iradon so that the SSRT scale parameter $\sigma$ is introduced in the various functions composing iradon program and where, Wiener filter with specific constant $K$ is used in issrt while, only ramp filter, designated by ram – lak, is used in iradon. So, both routines iradon and issrt follow the same algorithmic complexity. However, Deconv-Rad-FBP method takes around 0.09 s where, the time execution difference, compared to Radon-FBP, is spent on the deconvolution operation.

| Table 1: Shepp-Logan image reconstruction runtime scores |
|-------------|-------------|-------------|
| Runtime (s) | Radon-FBP   | Deconv-Rad-FBP | SSRT-FBP   |
|-------------|-------------|-------------|-------------|
|             | 0.029       | 0.092       | 0.027       |

In order to evaluate the effectiveness of the suggested reconstruction approaches in relation to the noise level and the number of projections, an anthropomorphic abdominal phantom was utilized in this work. The experimental phantom data were acquired from the work in [33] where, the irradiated specimen consists in an anthropomorphic

(a) Without noise

(b) Without noise

(c) $I_0 = 2 \times 10^4$, $\sigma_n = 1$

(d) $I_0 = 2 \times 10^4$, $\sigma_n = 1$

Figure 8: PSNR and SSIM values of a sample of reconstructed Shepp-Logan images in function of $\sigma$ and $\Delta \theta$.
Figure 9: Image reconstruction results for Shepp-Logan phantom corrupted by Poisson-Gaussian noise with $I_0 = 10^4$ and $\sigma_n = 0.5$. (a) Unnoisy input Shepp-Logan phantom with $(\sigma, \Delta \theta) = (1.2, 2^\circ)$. (b)-(d) Corrupted sinograms and image reconstruction results by methods of Radon-FBP, Decom-Rad-FBP and SSRT-FBP, respectively.
abdominal phantom with a removable liver insert, designed by the research team and custom manufactured by QRM (Moehrendorf, Germany).

Two liver inserts, each containing 19 embedded synthetic lesions with known volumes of varying diameter (6-40 mm), shape (spherical, ellipsoidal and lobulated), contrast and density (homogenous and mixed) were designed to have liver parenchyma and lesion CT values simulating arterial phase (AP phantom) and portal venous phase (PVP phantom) imaging, respectively. The schematic of the phantom are illustrated by Fig. 1 in [33].

The phantom was imaged at the Columbia University Medical Center with two 64-slice multi-detector helical CT scanners: GE 750HD (GE Healthcare, Chicago, IL, US) and Siemens Biograph mCT (Siemens Healthcare, Erlangen, Germany). The data were acquired at 120 kVp and at three dose levels of approximately 3.8, 7.6, and 19 mGy corresponding to 50, 100, and 250 effective tube current time product (mAs), respectively. Effective mAs is defined as (total mAs)/pitch for the scan. There are small differences between the acquired pitch and associated slice thicknesses between the two CT systems as the available setting of these two imaging parameters is different for these two CT scanners. For the GE systems, the pitch factor, defined as the ratio of the table increment over the detector collimation, has, as values, 1.375 and 0.983, while, for the Siemens systems, it takes the values of 1.35 and 1.0. The mAs was adjusted for each pitch to reach the three pre-set effective mAs. CT acquisitions for the Siemens image data acquired at a pitch of 1.0 were collected for only a single dose of 250 effective mAs. The GE image data were reconstructed at slice thicknesses of 5.0, 2.5, 1.25 and 0.625 mm and the Siemens data were reconstructed at 5.0, 3.0, 1.5 and 0.6 mm. Filtered backprojection (FBP) reconstructions were used with two different convolution kernels (GE/Siemens: Standard and Soft/B20f and B30f). Two repeated scans were performed for each protocol and for each phantom [33].

In Fig. 11, serving as input ground truth example, we have utilized a 512×512 reconstructed image of the CT scan issued from the anthropomorphic abdominal phantom data and which depicts the 29th slice image of a total of 49 slices stacked from bottom to top, corresponding to plane A-A simulating liver arterial phase insert lesions (see Fig. 1 in [33]). It consists in the reconstructed anthropomorphic liver phantom with the following CT imaging protocols: (1) Acquisition parameters: (a) Scanner: 64-slice multi-detector helical CT GE 750HD , (b) peak voltage: 120 kVp, (c) effective current time product: 250 mAs, (d) pitch: 1.375; (2) Reconstruction parameters: (a) Slice thickness: 5 mm, (b) Reconstruction algorithm : FBP, (c) Convolution kernel: Standard.
The thick red line depicts the optimal value of $\sigma$ methods and the number of CT projections used in image reconstruction. The results are depicted in Fig. 11b-11e.

As image quality indicators, the PSNR and SSIM were calculated on the reconstructed images from the CT projections of the ground truth image shown in Fig. 11a. In one case, these projections are used “as is” without any noise altering while, in another case, the projections were corrupted by Poisson-Gaussian noise with the above-mentioned parameters. The reconstructions are done by each of the methods Radon-FBP, Deconv-Rad-FBP and SSRT-FBP and for each value of the pair $(\sigma, \Delta \theta)$ ranging in the intervals $R_\sigma = [0.6]$ and $R_{\Delta \theta} = [0.5^\circ, 10^\circ]$, respectively. As explained in this paper, the parameters $\sigma$ and $\Delta \theta$ describe, respectively, the kernel width used for the SSRT-based methods and the number of CT projections used in image reconstruction. The results are depicted in Fig. 11a-11e. The thick red line depicts the optimal value of $\sigma$ for each $\Delta \theta$ for which the PSNR and SSIM, obtained for SSRT-FBP method, have maximal values, i.e.

$$\sigma_{\text{opt}}(\Delta \theta) = \arg\max_{\sigma \in R_\sigma} \text{PSNR}^{\text{SSRT-FBP}}(\sigma, \Delta \theta), \ \forall \Delta \theta \in R_{\Delta \theta}$$

Similar reasoning was followed for SSIM.

At the light of the results given in Fig. 11, we remark that, below a certain value of $\sigma(\sim 0.4)$, the performances of all methods are equivalent, as for the Shepp-logan phantom, where the reasons are already evoked. Above this value of $\sigma$, the image reconstruction, performed by SSRT-FBP method, is better quality, in terms of PSNR and SSIM, than the two other methods whatever the value of $\sigma$. However, for small $\Delta \theta (< 2^\circ)$(i.e. high number of CT projections), the PSNR ans SSIM values, for SSRT-FBP are very sensitive to the variation of $\sigma$ where, maximum values could be reached by taking $\sigma$ less than 3. For SSRT-FBP method, it is clear, in this case, that the proportionality between $\sigma_{\text{opt}}$ and $\Delta \theta$ is established. This can be predictable and which can be explained as follows: when the number of the projections decreases, the lack of information can be compensated by a certain thickness of the X-ray beam projection on the CT non-zero width detector. Such a thickness is obtained by taking an appropriate $\sigma$ value of the Gaussian kernel, embedded by SSRT.

When Poisson-Gaussian noise is added to the input CT data (see Figs 11 and 11b), the similar tendency for PSNR and SSIM variations in the $\sigma – \Delta \theta$ space, as in the noiseless case (see Figs 11 and 11a), may be seen. However, in this case, the methods Radon-FBP and Deconv-Rad-FBP have low performance, especially for PSNR, compared with SSRT-FBP, even with very small $\Delta \theta$s. This relates the sensitivity of the Radon transform to noise, in comparison to SSRT, where Gaussian kernel acts as a noise filter.

From the last results, we provide in Fig. 12 and Fig. 13 exhaustive reconstruction outcomes for abdominal phantom CT data (of which the ground truth recovery is given in Fig. 11a) with the following parameters: $(\sigma, \Delta \theta) = (2, 1^\circ)$ for data acquisition and $I_0 = 5 \times 10^4$ and $\sigma_\epsilon = 0.5$ for Poisson-Gaussian noise.

Figs. 12a, 12b and 13 depict the reconstructed images with methods of Radon-FBP, Deconv-Rad-FBP and SSRT-FBP, respectively, where the image binarization results are also given for region of interests (ROIs), inside the reconstructed images, simulating liver lesions. The gray level profiles of the ground truth image and the abovementioned methods-based recovery results are depicted in Fig. 13. The gray level profiles are located, in the $x – y$ plan, at the pixel positions (1) $y = 240$ and $x$ from 1 to 200 and (2) $y = 180$ and $x$ from 201 to 350, as illustrated by the yellow line segments in Fig. 11a.

It can be observed from Fig. 12 that visual inspection of the reconstructed images confirmed the difficulty in detecting the liver lesions or identifying the boundary of the lesions for Radon-FBP, Deconv-Rad-FBP, in comparison with SSRT-FBP where, for the latter, the lesion shape is better highlighted and the noise effect is negligible. This can be confirmed by the image profiles (see Fig. 13) where, the SSRT-FBP-based profiles are closer to that of the ground truth image than those of Deconv-Rad-FBP and Radon-FBP methods. The execution time and quality measures for each of the tested methods are summarized in Table 2.
Table 2: Some numerical results from abdominal phantom image reconstructions displayed in Figs[12] and[13]

|                | Radon-FBP | Deconv-Rad-FBP | SSRT-FBP |
|----------------|-----------|----------------|----------|
| PSNR           | 18.02     | 21.42          | 27.21    |
| SSIM           | 0.817     | 0.867          | 0.923    |
| Profile MAE*   | 28.24     | 15.32          | 13.52    |
| Runtime (s)    | 0.051     | 0.126          | 0.042    |

* Fitting mean absolute errors (MAE) of the reconstructed image profiles

For medical image reconstruction from the anthropomorphic abdominal phantom CT data, these experiments ascertained again the outstanding performance of the proposed SSRT-based FBP method which meets the double challenge of (1) low recovery quality for low-dose CT and (2) data noise inherent to CT systems.

6. Conclusion

In this paper, we first presented the basic properties, the inversion approach of SSRT and its application in FBP-based image reconstruction. The use of SSRT is motivated by the fact that RT, which is a line integral, does not take into account the non-zero dimensions of the CT system components, such as the X-ray detector which has a certain width. The solutions provided by some iterative ARTs such as DDM and AIM have limited practical utility due to their high computational cost. Therefore, due to its fast implementation, SSRT which allows accurate modeling of the non-zero width X-ray beam could be applied.

For image reconstruction from projections, we have originally modified the FBP algorithm to be applied on SSRT where, the noise-free Shepp-Logan phantom image and its versions, contaminated by additive white noise, are taken as test input images. Still using the FBP procedure, the obtained results are compared with two other sinograms: (1) the original RT sinogram and (2) the RT sinogram derived via SSRT deconvolution. The preliminary results demonstrate the great performance of SSRT-based image reconstruction techniques in comparison to the RT-based one using PSNR and SSIM as image quality metric where, the SSRT-based FBP technique outperforms Radon-FBP and Deconv-Rad-FBP, especially, with moderate and high levels of CT data noise while it require less computation time, comparable to the one of Radon-FBP.

Additionally, when the number of projections is reduced, the SSRT-FBP approach reveals on Shepp-Logan and anthropomorphic abdominal phantoms, taken as test data in experiments, to be more accurate than the two other methods, making it more appropriate for applications requiring managed radiation dose, as in medical X-ray CT.

As future work, we plan to compare the SSRT-based CT image reconstruction with some recent algorithms of algebraic reconstruction techniques known to generate less-noisy images than the analytical FBP algorithm with the same datasets.

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(a) Abdominal phantom CT slice

(b) Without noise

(c) Without noise

(d) $I_0 = 5 \times 10^4, \sigma_n = 0.5$

(e) $I_0 = 5 \times 10^4, \sigma_n = 0.5$

Figure 11: PSNR and SSIM values of a sample of reconstructed abdominal phantom images in function of $\sigma$ and $\Delta \theta$ where, the thick red line depicts the relationship between these two parameters for which the PSNR and SSIM have maximal values.
Figure 12: Image reconstruction results of the abdominal phantom ground truth, illustrated in Fig. [11], with parameters $(\sigma, \Delta \theta) = (2, 1^\circ)$ and corrupted by Poisson-Gaussian noise with $k_0 = 5 \times 10^4$ and $\sigma_n = 0.5$, for the methods: (a) Radon-FBP, (b) Deconv-Rad-FBP and (c) SSSR-FBP. The results include, for each method, the ROI outcomes representing two low-contrast liver lesions and the results of their binarization.
Figure 13: A portion of gray-level profiles of reconstructed images, depicted in Fig. [22], corresponding to line segments in Fig. [11], from noise-free and noisy CT data for abdominal phantom.