A note on pseudofinite dimensions and forking

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February 24, 2014

Abstract

In this paper we show that an instance of forking in pseudofinite structures can be wit-
nessed by a drop of the pseudofinite dimension. As an application of this result we give new
proofs of known results for asymptotic classes of finite structures.

1 Introduction

Dimension theory (so-called) has become one of the most important concepts in model theory
and has been used to give a combinatorial description of the definable sets of first order
structures. Even more, it is possible to get structural properties of the models of a first order
theory $T$ by assuming some bound on the different ranks associated to $T$.

One of the recurrent themes in the notions of rank is their relationship with forking. It
is often desired that any instance of forking (on types or formulas) can be detected by a
decrease of the dimension.

In [7], Hrushovski and Wagner defined the notion of quasidimension on some structure $M$
as a way to generalize the concept of dimension allowing values different from the integers.
The main example is what I call “logarithmic pseudofinite dimension” which is defined on
ultraproducts of finite structures by taking the logarithm of the cardinality of nonstandard
finite sets and factor it out by the convex hull of the nonstandard reals. In the later papers
[5, 6], Hrushovski states some properties of this pseudofinite dimension and used it to get
asymptotic results in additive combinatorics.

We present a similar connection between forking and the logarithmic pseudofinite dimen-
sion: any instance of forking in a pseudofinite structure is witnessed by a decrease of the
dimension. This connection is used to get some known results in asymptotic classes of finite
structures, as defined in [8] and [2].

The paper is organized as follows: in section 2 we present the definition of quasi-
dimensions and the construction of the logarithmic pseudofinite dimension. In section 3
we present the main result of this paper: any instance of forking can be witnessed by a
decrease of the pseudofinite dimension. We start the proof recalling some results in combi-
natorics and measure theory (section 3.1) and using them to prove the result in section 3.2.

Section 4 contains a calculation of the possible pseudofinite dimensions for the ultrapro-
ducts of 1-dimensional asymptotic classes, which can be easily generalized to the context of
$N$-dimensional asymptotic classes. As a corollary, we obtain new proofs of the following
known results from [8, 2]: Every infinite ultraproduct of the members of a 1-dimensional
asymptotic class (resp. $N$-dimensional asymptotic class) is supersimple of $U$-rank 1. (resp.
$U$-rank less than or equal to $N$).

Acknowledgements: I would like to thank my advisors Thomas Scanlon and Alf Onshuus
for all their help and their valuable comments and discussions through the development of
this paper.
2 The logarithmic pseudofinite dimension

In this section we present the definition of the logarithmic pseudofinite dimension as presented in [7] and give the construction of the logarithmic pseudofinite dimension, also proving that it define a quasi-dimension on the ultraproducts of finite structures.

Definition 2.1. Let $M$ be any structure. A quasi-dimension on $M$ is a map $\delta$ from the class of definable sets into an ordered abelian group $G$, together with a formal element $-\infty$, satisfying:

1. $\delta(\emptyset) = -\infty$, and $\delta(X) \geq -\infty$ implies $\delta(X) \geq 0$.
2. $\delta(X \cup Y) = \max\{\delta(X), \delta(Y)\}$
3. For every $g \in G \cup \{-\infty\}$ the following holds: If $X$ is a definable subset of $M^k$, $\pi$ is the projection to some of the coordinates and $\delta(\pi^{-1}(\mathfrak{a})) \leq g$ for all $\mathfrak{a} \in \pi(X)$, then $\delta(X) \leq \delta(\pi(X)) + g$.

We will focus in the logarithmic pseudofinite dimension, which is a quasidimension defined on ultraproducts of finite structures. Consider the following construction:

Assume $M$ is an infinite ultraproduct of finite structures $\langle M_i : i \in I \rangle$, with $|M_i| \to \infty$.

For a definable set $X$, there is a map

$$
\log_i : \text{Def}(M_i) \to \mathbb{R} \cup \{-\infty\}
$$

$$
X(M_i) \to \log(|X(M_i)|)
$$

where log is the usual natural logarithm and $|X(M_i)|$ represents the size of the definable set $X(M_i)$.

It is possible to take the ultraproduct of such functions and obtain a map

$$
\log = \prod_{i \in I} \log_i : \text{Def}(M) \to \mathbb{R}^* \cup \{-\infty\}
$$

$$
X \mapsto \log(|X|) := [\log(|X(M_i)|)]_{i \in I}
$$

where $\mathbb{R}^*$ is a non-standard real closed field. Let $C$ be the convex hull of $Z$ in $\mathbb{R}^*$ (a convex subgroup of $\mathbb{R}^*$) and $\pi : \mathbb{R}^* \cup \{-\infty\} \to \mathbb{R}^*/C \cup \{-\infty\}$ the natural quotient map (with $\pi(-\infty) = -\infty$).

For a definable subset $X$ of $M$, define

$$
\delta(X) = \pi(\log(|X|))
$$

This is a way to measure “bigness” of the definable sets in $M$. For instance, note that $\delta(X) = 0$ if and only if $\log(|X|) \in C$, which (by compactness) implies that $|X_i|$ is uniformly bounded by a fixed $M$ on a $\mathcal{U}$-large set.

Proposition 2.2. $\delta(X) = \pi(\log(|X|))$ is a quasi-dimension on $M = \prod_{i \in I} M_i$.

Proof. (1) Clearly $\delta(\emptyset) = -\infty$, and for any definable $X$ we have that, if $X$ is non-empty in the ultraproduct, then

$$
\{i \in I : |X_i| \geq 1\} = \{i \in I : \log_i(|X_i|) \geq 0\} \in \mathcal{U}
$$

which implies $\delta(X) \geq 0$.

(2) Let $X, Y$ be definable subsets of $M$, and assume without loss of generality that $\delta(X) \geq \delta(Y)$. By the construction of $\delta$, this implies in particular that $|X(M_i)| \geq |Y(M_i)|$ for $\mathcal{U}$-almost all $i \in I$. In those indices, we have

$$
|X(M_i)| \leq |X(M_i) \cup Y(M_i)| \leq 2|X(M_i)|
$$

$$
\log(|X(M_i)|) \leq \log(|X(M_i) \cup Y(M_i)|) \leq \log(2) + \log(|X(M_i)|)
$$

$$
0 \leq \log(|X(M_i) \cup Y(M_i)|) - \log(|X(M_i)|) \leq \log(2)
$$

So, $\delta(X \cup Y) = \pi(\log(|X \cup Y|)) = \pi(\log(|X|)) = \delta(X)$. 

3. Pseudodimension and forking

The purpose of this section is to show the relationship between the pseudofinite dimension defined in section 2 and the forking relation inside the structure \( M \). This relationship can be viewed as a generalization of the notion of rank in stable or simple theories, in the sense that forking can be witnessed by a drop in the dimension.

To prove this, we will use some more or less known results in enumerative combinatorics and measure theory. We include the proofs here for completeness.

3.1 Some little lemmas from combinatorics and measure theory

We start with the following lemma:

**Lemma 3.1.** If \( m \geq 2i + 1 \) are integers, then

\[
\binom{m}{i} - \binom{m}{i+1} \leq 0
\]

**Proof.** This is a straightforward calculation:

\[
\begin{align*}
\binom{m}{i} - \binom{m}{i+1} &= \frac{m!}{i!(m-i)!} - \frac{m!}{(i+1)!(m-i-1)!} \\
&= \frac{m!}{i!(m-i-1)!} \left( \frac{1}{m-i} - \frac{1}{i+1} \right) \\
&= \frac{m!}{i!(m-i-1)!} \left( \frac{2i+1-m}{(m-i)(i+1)} \right) \\
&\leq \frac{m!}{(i+1)!(m-i)!}(2i+1-2i-1) = 0
\end{align*}
\]

Now we present the measure theoretic lemma. Assume we have a measure space \((X, B, \mu)\). Given measurable sets \( A_1, \ldots, A_n \), we can define \( S_k \) to be the sum of the measures of all \( k \)-intersections of \( A_1, \ldots, A_n \), namely,

\[
S_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mu(A_{i_1} \cap \cdots \cap A_{i_k})
\]

We know from the inclusion-exclusion principle that the measure of \( \bigcup_{i=1}^n A_i \) is an alternating sum of \( S_k \)'s. What we will prove now is that starting from positive term of this sum, the result is positive.
Proposition 3.2. [Truncated inclusion-exclusion principle] Let \( X \) be a measure space and \( A_1, \ldots, A_n \) be measurable sets, and let \( S_1, \ldots, S_n \) as defined above. Then for every \( k \leq n/2 \),
\[
\sum_{i=2k+1}^{n} (-1)^{i-1}S_i \geq 0
\]

Proof. By the inclusion-exclusion principle we know that
\[
\mu \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{m=1}^{n} (-1)^{m-1} S_m
\]

For every non-empty \( W \subseteq \{1, \ldots, n\} \), define \( E_W = \bigcap_{i \in W} A_i \cap \bigcap_{i \notin W} A_i^c \). These are the non-empty atoms of the algebra of sets generated by \( A_1, \ldots, A_n \) that are contained in \( \bigcup_{i=1}^{n} A_i \).

They are disjoint and we have the following easy identities:
\[
\mu \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{W \subseteq \{1, \ldots, n\}} \mu(E_W), \quad \mu(A_{i_1} \cap \cdots \cap A_{i_m}) = \sum_{i_1, \ldots, i_m \in W} \mu(E_W)
\]

So, the inclusion-exclusion principle states that
\[
\sum_{W \subseteq \{1, \ldots, n\}} \mu(E_W) := \sum_{m=1}^{n} (-1)^{m-1} S_m
\]
\[
= \sum_{m=1}^{n} (-1)^{m-1} \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \mu(A_{i_1} \cap \cdots \cap A_{i_m}) \right)
\]
\[
= \sum_{m=1}^{n} (-1)^{m-1} \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \left( \sum_{i_1, \ldots, i_m \in W} \mu(E_W) \right) \right)
\]
\[
= \sum_{m=1}^{n} (-1)^{m-1} \left( \sum_{W \subseteq \{1, \ldots, n\}} \alpha_W^m \mu(E_W) \right)
\]

where the coefficient \( \alpha_W^m \) is the number of times that the sum mand \( \mu(E_W) \) appears in \( S_m \).

Thus, we know that for every \( W \subseteq \{1, \ldots, n\} \), \( \sum_{m=1}^{n} (-1)^{m-1} \alpha_W^m = 1 \)

So we have three cases:

1. If \(|W| < 2k+1\), all the possible summands \( \mu(E_W) \) are already in the sum \( \sum_{i=1}^{2k} (-1)^{i-1}S_i \).

2. If \(|W| = 2k+1\), the coefficient of \( \mu(E_W) \) in \( \sum_{i=1}^{2k} (-1)^{i-1}S_i \) is 0, because there is exactly one more term \( \mu(E_W) \) appearing in \( S_{2k+1} \) (the measure of the intersection \( \bigcap_{i \in W} A_i \)).

3. If \(|W| > 2k+1\): Note that in every \( S_i \) appear \( \binom{|W|}{i} \) summands of the form \( \mu(E_W) \) for a fixed \( W \subseteq \{1, \ldots, n\} \). So, the coefficient of \( \mu(E_W) \) in \( \sum_{i=1}^{2k} (-1)^{i-1}S_i \) is:

\[
\beta_W = \sum_{m=1}^{2k} (-1)^{m-1} \binom{|W|}{i} = \sum_{j=1}^{k} \left[ \binom{|W|}{2j} - \binom{|W|}{2j-1} \right] \leq 0
\]

using the previous lemma (note that \( j \leq k \) implies \( 2j - 1 \leq 2k - 1 < |W| \).
Therefore, we obtain

\[
\sum_{W \subseteq \{1, \ldots, n\}} \mu(E_W) = \sum_{m=1}^{n} (-1)^{m-1} S_m
\]

\[
= \sum_{m=1}^{2k} (-1)^{m-1} S_m + \sum_{m=2k+1}^{n} (-1)^{m-1} S_m
\]

\[
= \sum_{|W| < 2k+1} \mu(E_W) + 0 + \left( \sum_{|W| = 2k+1} \beta_W \cdot \mu(E_W) \right) + \sum_{m=2k+1}^{n} (-1)^{m-1} S_m
\]

So,

\[
\sum_{|W| > 2k+1} \mu(E_W) = \sum_{|W| = 2k+1} \beta_W \cdot \mu(E_W) + \sum_{m=2k+1}^{n} (-1)^{m-1} S_m,
\]

and we conclude that

\[
\sum_{m=2k+1}^{n} (-1)^{m-1} S_m = \sum_{|W| > 2k+1} \mu(E_W) - \sum_{|W| = 2k+1} \beta_W \mu(E_W) \geq 0
\]

because all the coefficients \(\beta_W\) are less than or equal to 0. \(\square\)

The following measure-theoretic proposition will play a key role in the proof that forking implies a decrease of pseudofinite dimension.

**Proposition 3.3.** Let \(X\) be a measure space with \(\mu(X) = 1\) and fix \(0 < \epsilon \leq \frac{1}{2}\). Let \(\langle A_i : i < \omega \rangle\) be a sequence of measurable subsets of \(X\) such that \(\mu(A_i) \geq \epsilon\) for every \(i\). Then, for every \(k < \omega\) there are \(i_1 < i_2 < \ldots < i_k\) such that

\[
\mu \left( \bigcap_{j=1}^{k} A_{i_j} \right) \geq \epsilon^{k-1}
\]

**Proof.** The proof will be by induction on \(k\).

- \(k = 1\): By hypothesis we have \(\mu(A_i) \geq \epsilon = \epsilon^{1-1}\). \(\checkmark\)

- \(k = 2\): Assume not, then \(\mu(A_i \cap A_j) < \epsilon^{2-1} = \epsilon^3\) for every \(i \neq j\). By the truncated inclusion-exclusion principle we know that for every \(N \in \mathbb{N}\),

\[
1 \geq \mu \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \mu(A_i) - \sum_{1 \leq i < j \leq N} \mu(A_i \cap A_j) \quad \text{(by Proposition 3.2)}
\]

\[
\geq N\epsilon - \frac{N(N-1)}{2} \epsilon^3 \quad \text{(i)}
\]

Define the quadratic function given by \(f(x) = x \cdot \epsilon - x(x-1) \epsilon^3 = \frac{x^2}{2} \epsilon^3 + x \left( \epsilon + \frac{\epsilon^3}{2} \right)\)

This function achieve its maximum value at \(x_0 = \frac{1}{\epsilon^2} + \frac{1}{2} > 0\), and by taking any integer
$N \in [x_0 - 1, x_0]$ we have that

$$f(N) \geq f(x_0 - 1) = \left(\frac{1}{\epsilon^2} - \frac{1}{2}\right) \epsilon - \frac{\left(\frac{1}{\epsilon^2} - \frac{1}{2}\right) \left(\frac{1}{\epsilon^2} - \frac{3}{2}\right)}{2} \cdot \epsilon^3$$

$$= \frac{1}{\epsilon} - \frac{\epsilon}{2} - \frac{2}{\epsilon^2} + \frac{3}{2} \cdot \epsilon^3$$

$$= \frac{1}{\epsilon} - \frac{\epsilon}{2} - \frac{1}{2\epsilon} + \epsilon - \frac{3}{8} \epsilon^3$$

$$= \frac{1}{2\epsilon} + \frac{\epsilon}{2} - \frac{3}{8} \epsilon^3$$

$$\geq 1 + \epsilon \left(\frac{1}{2} - \frac{3}{8} \epsilon^2\right) \quad [\text{because } \epsilon \leq \frac{1}{2}]$$

$$> 1.$$  

contradicting the inequality (†). ✓

Now, assume the induction hypothesis, which is that there is a tuple $(i_1, \ldots, i_k)$ satisfying

$$i_1 < \ldots < i_k \quad \text{and} \quad \mu \left(\bigcap_{j=1}^{k} A_{i_j}\right) \geq \epsilon^{3^{k-1}} \quad (\ast)$$

**Claim:** There are infinitely many such tuples.

**Proof of the Claim:** Assume not, and take $\ell$ to be the maximum of all indices appearing in the tuples $(i_1, \ldots, i_k)$ which satisfies (\ast). The sequence $\langle A_j : j \geq \ell + 1 \rangle$ would contradict the induction hypothesis. ✓

Now, let $(\alpha_j : j < \omega)$ be an enumeration of all tuples satisfying (\ast) and define $B_j = \bigcap_{i \in \alpha_j} A_i$.

By construction, $\mu(B_j) \geq \epsilon^{3^{k-1}}$.

By the $k = 2$ case, there are indices $j_1 \neq j_2$ such that

$$\mu(B_{j_1} \cap B_{j_2}) \geq (\epsilon^{3^{k-1}})^3 = \epsilon^{3^{k-1} \cdot 3} = \epsilon^{3^k}$$

where $j_1, j_2$ are indices corresponding to two different tuples $\alpha_{j_1} \neq \alpha_{j_2}$. In particular, there are (at least) $k + 1$ indices $i_1 < i_2 < \cdots < i_k < i_{k+1}$ such that

$$\mu \left(\bigcap_{j=1}^{k+1} A_{i_j}\right) \geq \mu(B_{j_1} \cap B_{j_2}) \geq \epsilon^{3^k} = \epsilon^{3^{(k+1)-1}}$$

3.2 Forking and drop of the pseudofinite dimension

With the results of the previous subsection, we are now able to give a proof of the main result of this note. The setting, as in the definition of logarithmic pseudofinite dimension, is the following: $\langle M_i : i \in I \rangle$ is a family of finite structures, $M$ is an infinite ultraproduct of the family and $\delta$ denotes the logarithmic pseudofinite dimension defined on definable subsets of $M$.

**Proposition 3.4.** Let $X = \psi(x, \overline{a})$ be a definable subset of $M$ and $\phi(x, \overline{b})$ a formula implying $\psi(x, \overline{a})$. If $\phi(x, \overline{b})$ forks over $\overline{a}$, then there exists $\overline{b}' \models tp(\overline{b}/\overline{a})$ such that $\delta(\phi(x, \overline{b}')) < \delta(X)$. 

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Proof. Note that it is enough to show the proposition for formulas which divide over \( \bar{\pi} \), because a forking formula implies a disjunction of dividing formulas. So, if \( \phi(x, \bar{b}) \models \bigvee_{i=1}^{n} \psi_i(x, c_i) \) then
\[
\delta(\phi(x, \bar{b})) \leq \delta \left( \bigvee_{i=1}^{n} \psi_i(x, c_i) \right) = \max_{i \leq n} \{ \delta(\psi_i(x, c_i)) \} < \delta(X).
\]

Assume that \( \phi(x, \bar{b}) \) divides over \( \bar{\pi} \). Towards a contradiction, assume that for every \( b' = tp(\bar{b}/\bar{\pi}) \) we have \( \delta(\phi(x, b')) = \delta(X) \). Then for each \( \bar{b} \) there is \( n_{b'} \in \mathbb{N} \) such that
\[
\log(|X|) - \log(|\phi(x, b')|) < n_{b'}.
\]

Thus,
\[
\log \left( \frac{|X|}{|\phi(x, b')|} \right) < n_{b'}
\]
\[
\frac{|X|}{|\phi(x, b')|} < e^{n_{b'}}
\]
\[
\frac{|\phi(x, b')|}{|X|} \geq e^{-n_{b'}}
\]
\[
e^{n_{b'}} |\phi(x, b')| \geq |X|
\]

In particular, there is \( M_{b'} \in \mathbb{N} \) such that \( M_{b'}|\phi(x, b)| \geq |X| \).

Claim: There is an uniform bound \( M \) such that \( M|\phi(x, b')| \geq |X| \) for every \( b' = tp(\bar{b}/\bar{\pi}) \).

Proof of the Claim: If not, for every \( n < \omega \) there is \( \bar{b}_n \models \bar{\pi} \) such that \( \log(|X|) - \log(|\phi(x, \bar{b}_n)|) > n \). Consider the multi-sorted structures given by
\[
\mathcal{M}_i = \langle M_i, \mathbb{R}, \log_\varphi \rangle_{\varphi \in \mathcal{L}}
\]
where \( \log_\varphi \) is an function between different sorts interpreted as
\[
\log_\varphi : M_i \rightarrow \mathbb{R}
\]
\[
\bar{b} \mapsto \log(|\varphi(x, \bar{b})|)
\]

Now, take the ultraproduct \( \mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i \). In this structure, consider the type
\[
p(y) = tp(\bar{b}/\bar{\pi}) \cup \{ \log_\varphi(\bar{\pi}) - \log_\varphi(\bar{b}) > n : n \in \mathbb{N} \}
\]
This type is finitely satisfiable in \( \mathcal{M} \), and by \( \aleph_1 \)-saturation of the ultraproduct, there is \( \bar{b}' \models p(y) \) which means \( \bar{b}' \models tp(\bar{b}/\bar{\pi}) \) and \( \delta(\phi(x, \bar{b}')) < \delta(X) \), a contradiction. \( \checkmark \)

Therefore, there is \( M \in \mathbb{N} \) such that \( M|\phi(x, \bar{b}')| \geq |X| \) for every \( \bar{b} \models \bar{\pi} \). Since \( \phi(x, \bar{b}) \) divides over \( \bar{\pi} \), there is an indiscernible sequence \( \langle \bar{b}_j : j < \omega \rangle \) (which can be assumed to be in \( M \) by \( \aleph_1 \)-saturation) such that:
\begin{itemize}
  \item \( \bar{b}_i \models tp(\bar{b}/A) \).
  \item \( \{ \phi(x, \bar{b}_j) : j < \omega \} \) is \( k \)-inconsistent for some \( k < \omega \).
\end{itemize}

Assume \( \bar{b}_j = [\bar{b}_j]_{\mathcal{U}} \). By the claim, \( M|\phi(x, \bar{b}_j)| \geq |X| \) which implies that \( \frac{|\phi(x, \bar{b}_j)|}{|X|} \geq \frac{1}{M} \) for \( \mathcal{U} \)-almost all \( i \).

Consider the normalized counting measure localized on \( X(M_i) \) in each finite structure \( M_i \), and the Loeb measure induced on \( M \) by these measures. Therefore, \( \mu(X) = 1 \) and
\( \phi(M, \bar{b}_j) : j < \omega \) is a sequence of measurable sets with \( \mu(\phi(M, \bar{b}_j)) \geq \frac{1}{M} \) for every \( j < \omega \). By the Proposition 3.3 there are \( j_1 < \ldots < j_k < \omega \) such that

\[
\mu \left( \bigcap_{l=1}^{k} \phi(M, b_{j_l}) \right) \geq \frac{1}{M^{k-1}} > 0,
\]

in particular, \( \bigcap_{l=1}^{k} \phi(M, b_{j_l}) \) is non-empty, contradicting \( k \)-inconsistency.

This proposition allows us to conclude that the number of possible different values for pseudofinite dimensions of definable sets is a bound for the length of forking chains, providing also a bound for the U-rank in types. We will explore this idea in the following section.

4 Pseudofinite dimension and 1-dimensional asymptotic structures

In general, the logarithmic pseudofinite dimension can take infinitely many different values on the definable sets of \( M = \prod_{i} M_i \). For instance, consider the class \( C_{ord} \) of finite linear orders. If \( M_n = ([1, n], <) \) and \( \alpha, \beta \) are elements in the interval \([0, 1]\) with \( \alpha < \beta \), we may define \( X(M_n) = [1, [n^\alpha]] \) and \( Y(M^n) = [1, [n^\beta]] \). Weshow that \( \delta(X) < \delta(Y) \).

Otherwise, \( \delta(X) \) and \( \delta(Y) \) are equal, which means there is a natural number \( N \) such that

\[
\log |X| + N \geq \log |Y|.
\]

Therefore,

\[
\alpha \log n + N \geq \beta \log n
\]

\[
N \geq (\beta - \alpha) \log n
\]

which is not true since \( n \) tends to infinity.

The main feature of this example is that an infinite linear order can be defined on the ultraproducts, implying they are unstable non-simple and thus they have arbitrarily long forking chains.

On the other hand, there are classes of finite structures with a better behavior of their ultraproducts. That is the case of the 1-dimensional asymptotic classes (and more generally of the N-dimensional asymptotic classes) whose definition appear in [8] and [2]. These classes are known to have supersimple ultraproducts, which implies a finite bound on the length of forking chains in their ultraproducts.

The purpose of this section is to show that the supersimplicity of these classes can be detected by the logarithmic pseudofinite dimension. For instance, we will show that for 1-dimensional classes (which ultraproducts are supersimple of U-rank 1) the only possible values for the pseudofinite dimension are \(-\infty, 0\) and \( \alpha = \delta(M) \).

First, recall the definition of these classes:

**Definition 4.1.** Let \( \mathcal{L} \) be a first order language, and \( C \) be a collection of finite \( \mathcal{L} \)-structures. Then \( C \) is a 1-dimensional asymptotic class if the following hold for every \( m \in \mathbb{N} \) and every formula \( \varphi(x, \bar{y}) \), where \( \bar{y} = (y_1, \ldots, y_m) \).

1. There is a positive \( C \) and a finite set \( E \subseteq \mathbb{R}^0 \) such that for every \( M \in C \) and \( \bar{y} \in M^m \), either \( |\varphi(M, \bar{y})| \leq C \) of for some \( \mu \in E \),

\[
||\varphi(M, \bar{y})| - \mu| |M|| \leq C|M|^{1/2}.
\]

2. For every \( \mu \in E \), there is an \( \mathcal{L} \)-formula \( \varphi_\mu(\bar{y}) \), such that, for all \( M \in C \), \( \varphi_\mu(M^m) \) is precisely the set of \( \bar{y} \in M^m \) with

\[
||\varphi(M, \bar{y})| - \mu| |M|| \leq C|M|^{1/2}.
\]
**Proposition 4.2.** Let $C$ be a class of finite structures. If $C$ satisfies the condition (1) in definition [4.7] then for every infinite ultraproduct $M$ of elements in $C$ there are only two possible values for $\delta(X)$ while $X$ varies among the non-empty definable subsets of $M^1$.

**Proof.** Let $\varphi(x,\bar{a})$ be a definable set in the ultraproduct and take $\mu_1, \ldots, \mu_k > 0$ the possible measures in $E$ satisfying (1). Assume $M = \prod_{i \in I} M_i$ with $i \in I$ and $\bar{\pi} = [\bar{a}]_i \in I$.

For every $M_i \in C$ one of the following hold:

- $|\varphi(M_i, \bar{a})| \leq C$
- $|\varphi(M_i, \bar{a})| - \mu_j |M_i| \leq C|M_i|^{1/2}$ for some $j = 1, 2, \ldots, k$.

Consider the sets

$$A_0 = \{ i \in I : |\varphi(M_i, \bar{a})| \leq C \}$$

$$A_1 = \{ i \in I : |\varphi(M_i, \bar{a})| - \mu_1 |M_i| \leq C|M_i|^{1/2} \}$$

$$\vdots$$

$$A_k = \{ i \in I : |\varphi(M_i, \bar{a})| - \mu_k |M_i| \leq C|M_i|^{1/2} \}$$

Since $A_0 \cup A_1 \cup \cdots \cup A_k = I$, one of these sets belongs to $U$ because $U$ is an ultrafilter on $I$.

We have to consider two cases:

- If $A_0 \in U$, then $|\varphi(M_i, \bar{a})| \leq C$ (a.e. in $U$) for some fixed $C > 0$, which implies $|\varphi(M, \bar{a})| \leq C$ and therefore
  $$\delta(\varphi(M, \bar{a})) = \pi(\log(|\varphi(x, \bar{a})|)) \leq \pi(\log(C)) = 0$$

- If $A_j \in U$ for some $j = 1, 2, \ldots, k$ then we obtain
  $$\mu_j |M_i| - C|M_i|^{1/2} \leq |\varphi(M_i, \bar{a})| \leq \mu_j |M_i| + C|M_i|^{1/2}$$

Put $\mu_* = \min\{\mu_1, \ldots, \mu_k\}$ and $\mu^* = \max\{\mu_1, \ldots, \mu_k\}$. Then for every definable set $X = \prod_{i \in I} X_i$, either $\delta(X) = 0$ (because the corresponding set $A_0$ belong to $U$) or

$$\mu_* |M_i| - C|M_i|^{1/2} \leq |X_i| \leq \mu^* |M_i| + C|M_i|^{1/2}$$

So,

$$|M_i|^{1/2} \left( \mu_* |M_i|^{1/2} - C \right) \leq |X_i| \leq |M_i|^{1/2} \left( \mu^* |M_i|^{1/2} + C \right)$$

$$|M_i|^{1/2} \left( \mu_* |M_i|^{1/2} - \frac{\mu_*}{2} |M_i|^{1/2} \right) \leq |X_i| \leq |M_i|^{1/2} \left( \mu^* |M_i|^{1/2} + \frac{\mu_*}{2} |M_i|^{1/2} \right)$$

( asymptotically, because $|M_i| \to \infty$)

$$|M_i|^{1/2} \left( \frac{\mu_*}{2} |M_i|^{1/2} \right) \leq |X_i| \leq |M_i|^{1/2} \left( \frac{3\mu_*}{2} |M_i|^{1/2} \right)$$

$$\frac{1}{2} \log(|M_i|) + \log \left( \frac{\mu_*}{2} \right) + \frac{1}{2} \log(|X_i|) \leq \log(|M_i|) \leq \frac{1}{2} \log(|M_i|) + \log \left( \frac{3\mu_*}{2} \right) + \frac{1}{2} \log(|M_i|)$$

$$\log(|M_i|) + \log \left( \frac{\mu_*}{2} \right) \leq \log(|X_i|) \leq \log(|M_i|) + \log \left( \frac{3\mu_*}{2} \right)$$

$$\pi(\log(|M_i|)) = \pi \left( \log(|M_i|) + \log \left( \frac{\mu_*}{2} \right) \right) \leq \pi(\log(|X_i|)) \leq \pi \left( \log(|M_i|) + \log \left( \frac{3\mu_*}{2} \right) \right) = \pi(\log(|M_i|))$$

$$\delta(M) \leq \delta(X) \leq \delta(M)$$

as we desired.

\[\square\]
Now, we present a new proof of the following known result, that appears in \cite{8} (Lemma 4.1).

**Corollary 4.3.** Let $C$ be a class of finite structures satisfying the condition (1) of Definition 4.1, and let $M$ be an infinite ultraproduct of members of $C$. Then $Th(M)$ is supersimple of $SU$-rank 1.

**Proof.** Assume $SU(M) \geq 2$. Then there is an increasing chain of types $p_0 \subset p_1 \subset p_2$ such that $p_{i+1}$ is a forking extension of $p_i$ for $i = 0, 1$. In particular, there are formulas $\phi_1(x, a_1) \in p_1$ which forks over $A_0$ and $\phi_2(x, a_2) \in p_2$ which forks over $A_1$. By Proposition 3.4 there are tuples $a_1' \equiv_{A_0} a_1$ and $a_2' \equiv_{A_1} a_2$ such that

$$\delta(\phi(x, a_2')) < \delta(\phi(x, a_1')) < \delta(M)$$

This contradicts Proposition 4.2 which states there are only two possible values for $\delta$ on non-empty definable subsets of $M$. \qed

**Remark 4.4.** The proof above can be easily generalized to the context of $N$-dimensional asymptotic classes, defined in \cite{2}. Namely, we can prove using similar calculations that the pseudofinite dimension $\delta$ only admits $N$ values in a $N$-dimensional classes, obtaining as a corollary that the ultraproducts of $N$-dimensional classes are supersimple of $U$-rank at most $N$.

**References**

[1] G. Cherlin, E. Hrushovski. *Finite structures with few types*. Princeton University Press. Princeton and Oxford. 2003

[2] R. Elwes. *Asymptotic classes of finite structures*. Journal of Symbolic Logic. Volume 72, Issue 2 (2007), 418-438.

[3] I. Goldbring, H. Towsner. *An approximate logic for measures*. Israel Jornal of Mathematics (to appear). Preprint. \texttt{arXiv:1106.2854v1}. June 2011.

[4] W. Hodges. *Model Theory*. Encyclopedia of mathematics and its applications. Cambridge University Press. 1994

[5] E. Hrushovski. *Stable group theory and approximate subgroups*. Journal of the American Mathematical Society. Volume 25. Number 1. January 2012. Pages 189-243.

[6] E. Hrushovski. *On Pseudo-Finite Dimensions*. Notre Dame Journal of Formal Logic. Volume 54 (2013), no. 3-4, 463–495.

[7] E. Hrushovski, F. Wagner. *Counting and dimensions*. Model Theory with applications to Algebra and Analysis. 2008

[8] D. Macpherson, C. Steinhorn. *One-dimensional asymptotic classes of finite structures*. Transactions of the American Mathematical Society. Volume 360, pages 411-448. 2007.

[9] T. Tao, V.H. Vu *Additive Combinatorics*. Cambridge studies in advances mathematics [105]. Cambridge University Press. 2006