FINITE-GAP SOLUTIONS OF THE FUCHSIAN EQUATIONS

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Abstract. We find a new class of the Fuchsian equations, which have an algebraic geometric solutions with the parameter belonging to a hyperelliptic curve. Methods of calculating the algebraic genus of the curve, and its branching points, are suggested. Numerous examples are given.

Introduction

Integrability of the Heun equation with half-odd characteristic exponents

\[
\frac{d^2y}{dz^2} + P(z) \frac{dy}{dz} + Q(z)y = 0, \tag{0.1}
\]

where

\[
P(z) = \frac{1}{2} \left( \frac{1 - 2m_1}{z} + \frac{1 - 2m_2}{z - 1} + \frac{1 - 2m_3}{z - a} \right), \tag{0.2}
\]

\[
Q(z) = \frac{N(N - 2m_0 - 1)z + \lambda}{4z(z - 1)(z - a)}, \tag{0.3}
\]

\[
N = m_0 + m_1 + m_2 + m_3, \quad m_i \in \mathbb{Z}_{\geq 0}, \quad \lambda, z \in \mathbb{C}, \tag{0.4}
\]

was probably discovered by Darboux more then 100 years ago [1]. But only recently so-called finite-gap solutions

\[
Y_{1,2}(m; \lambda; z) = \sqrt{\Psi_{g,N}(\lambda, z)} \exp \left( \pm \frac{i\nu(\lambda)}{2} \int \frac{z^{m_1}(z - 1)^{m_2}(z - a)^{m_3} dz}{\Psi_{g,N}(\lambda, z) \sqrt{z(z - 1)(z - a)}} \right) \tag{0.5}
\]

of this Heun equation were wrote out and analyzed [2, 3]. Here \(i^2 = -1\),

\[
\Gamma: \quad \nu^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j),
\]

\(\Psi_{g,N}(\lambda, z)\) is some polynomial of the degree \(N\) in \(z\) and of the degree \(g\) in \(\lambda\).

In works [1, 2, 3] the connection between Heun equation (0.1)–(0.4) and Treibich-Verdier equation

\[
\psi_{xx} - u(x)\psi = E\psi, \tag{0.6}
\]
\[ u(x) = m_0(m_0 + 1) \varphi(x) + \sum_{i=1}^{3} m_i(m_i + 1) \varphi(x - \omega_i), \quad (0.7) \]

was used. Here \( \varphi(x) \) is the Weierstrass function \( \wp \),

\[ [\varphi'(x)]^2 = 4\varphi^3(x) - g_2\varphi(x) - g_3 = 4 \prod_{j=1}^{3} (\varphi(x) - e_j), \quad (0.8) \]

\[ \varphi(\omega_i) = e_i, \quad \varphi(x - 2\omega_i) \equiv \varphi(x). \]

The equation \( (0.6), (0.7) \) is generalization of well-known Lamé equation

\[ \psi_{xx} - N(N + 1)\varphi(x)\psi = E\psi. \quad (0.9) \]

At beginning of 90-th author \( [9, 10] \) investigated Shrödinger operator with finite-gap elliptic potentials and proposed the next finite-gap elliptic generalization of potential \( (0.7): \)

\[ u(x) = m_0(m_0 + 1)\varphi(x) + \sum_{i=1}^{3} m_i(m_i + 1)\varphi(x - \omega_i) + \]

\[ + \sum_{k=1}^{M} n_k(n_k + 1) \{ \varphi(x - \delta_k) + \varphi(x + \delta_k) \} \quad (0.10) \]

Treibich later \( [14, 15] \) proved that for \( M = 1, n_1 = 1 \) and for any system of numbers \( m_i \in \mathbb{Z}_{\geq 0} \) there exists a point \( \delta_1 \) such that potential \( (0.10) \) is finite-gap potential of Shrödinger operator \( (0.6) \). On the other hand the formula\(^1\)

\[ \varphi \left( x \left| \frac{2\omega}{k}, \frac{2\omega'}{l} \right. \right) = \sum_{i=0}^{l-1} \sum_{j=0}^{k-1} \varphi \left( x + \frac{2j\omega}{k} + \frac{2i\omega'}{l} \right| \right) 2\omega, 2\omega' -
\]

\[ - \sum' \varphi \left( \frac{2j\omega}{k} + \frac{2i\omega'}{l} \right| \right) 2\omega, 2\omega' , \quad (0.11) \]

where

\[ \omega = \omega_1, \quad \omega' = \omega_3, \quad \omega_2 = \omega + \omega', \]

and examples in works \( [9, 10] \) show us what Treibich cases do not exhaust all the set of even elliptic finite-gap potentials (see for example remark \( [5] \)).

Since parallel using of properties of algebraic polynomials and of elliptic meromorphic functions make possible a big progress in study Heun and Treibich-Verdier equations (in particular simple methods of obtaining spectral curves and monodromy matrix \( [2, 3] \)), we

\(^1\)prime after sign of summation as ever indicate what term with \( i^2 + j^2 = 0 \) is omitted.
decide to apply this method to analysis of Shrödinger equation (0.6) with potentials (0.10) and of appropriate Fuchsian equations.

In present work we introduce concept of ‘finite-gap’ Fuchsian equation, and prove necessary and sufficient conditions of ‘finite-gapness’ of Fuchsian equation. Also we generalize the formula (0.5) for the case ‘finite-gap’ Fuchsian equation with five and more singular points, and derive the equation of appropriate spectral curve. Algebraic genus of spectral curve is bounded and calculated. The equation with five singular point is examined in detail. For equation with five singular points \((M = 1, n_1 = 1)\) the condition for position of fifth point is found.

The needed facts from the theory of finite-gap elliptic potentials for the Schrödinger operator are collected without proof in the first paragraph.

In the Appendix we give several simplest solutions of the ‘finite-gap’ Fuchsian equation and appropriate finite-gap potentials and their spectral curves.

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1. Schrödinger operator with finite-gap elliptic potential

Proposition 1 ([10] [17] [18] [19]). Any \(g\)-gap potential \(u(x)\) of the Schrödinger operator (0.6) satisfies the Novikov equation:

\[
J_g + \sum_{m=1}^{g} c_m J_{g-m} = d,
\]

or, which is the same, the stationary ‘higher’ Korteweg-de Vries (KdV) equation:

\[
\partial_x \left( J_g + \sum_{m=1}^{g} c_m J_{g-m} \right) = 0.
\]

Here \(c_m, d\) are constants and the functions \(J_m\) are the flows of the ‘higher’ KdV equations \(\partial_t u = \partial_x J_m\).

The expressions for the flows \(J_m\) are found from the relations

\[
L\psi = \psi_{xx} - 4u\psi + 2u_x \int_{x}^{\infty} \psi(\tau)d\tau,
\]

\[
(J_n)_x = L^n(u_x),
\]

where \(u(x)\) is a potential decreasing fast at \(\infty\).

In particular,

\[
J_0 = u, \quad J_1 = u_{xx} - 3u^2, \quad J_2 = u_{xxxx} - 10u_{xx}u - 5u_x^2 + 10u^3,
\]
\[ J_3 = u_{xxxxx} - 14u u_{xxxx} - 28u_x u_{xxx} - 21u_{xx}^2 + 70u^2 u_{xx} + 70u u_x^2 - 35u^4. \]

**Remark 1.** In the case of decreasing fast at \( \infty \) potential \( u(x,t) \) the variables
\[ C_j(t) = \int_{-\infty}^{\infty} J_j(x,t) \, dx \]
constitute an infinite set of integrals of motion for the KdV equation
\[ \partial_t u = \partial_x J_1. \]

**Proposition 2** ([20, 19, 17]). The function
\[ \tilde{\Psi}(x, E) = E^g + \sum_{j=1}^{g} \gamma_j(x) E^{g-j}, \]  \hspace{1cm} (1.3)
where
\[ \gamma_j(x) = -\frac{2}{4^j} \left( J_{j-1} + \sum_{m=1}^{j-1} c_m J_{j-m-1} - \frac{c_j}{2} \right), \]  \hspace{1cm} (1.4)
obey the equation
\[ \tilde{\Psi}_{xxx} = 4(u + E) \tilde{\Psi}_x + 2u_x \tilde{\Psi}, \]
the solutions of which are the products of any two solutions of the equation \( (0.6) \). Here \( c_j, J_j \) are the same as in \( (1.1) \) and \( u(x) \) is a \( g \)-gap potential of the Schrödinger operator \( (0.6) \).

**Definition** ([21, 22]). Let \( u(x) \) be an elliptic function. If the general solution of equation \( (0.6) \) is meromorphic for each complex number \( E \) then \( u(x) \) is called a Picard potential.

**Proposition 3** ([21, 22]). Function \( u(x) \) is an elliptic finite-gap potential if and only if \( u(x) \) is a Picard potential.

\section{2. Even elliptic potentials of Schrödinger operator and Fuchsian equation}

Let us consider the Schrödinger equation \( (0.6) \) with even elliptic potential \( (0.10) \). It is not difficult to check that with the help of appropriate change of variable this equation can be transformed into partial case of Fuchsian equation with \( M + 4 \) singular points.

For \( M = 0 \) equation of change variable \([2]^{2}\)
\[ \psi(x) = y(z) z^{-m_1/2} (z-1)^{-m_2/2} (z-a)^{-m_3/2}, \quad \phi(x) = e_1 + (e_2 - e_1) z, \]
\[ e_2 = \frac{a - 2}{a + 1} e_1, \quad e_3 = \frac{1 - 2a}{a + 1} e_1, \quad E = (e_1 - e_2) \lambda + \text{const}. \]  \hspace{1cm} (2.1)
transform equation \( (0.6), (0.7) \) into Heun equation \( (0.1)-(0.4) \).

\cite{2}In [11, 8] another changes are used
For \( M = 1 \) following equation of change variable

\[
\psi(x) = y(z)z^{-m_1/2(z-1)} - m_2/2(z-a) - m_3/2(z-b)^{-n_1},
\]

\[
\varphi(x) = e_1 + (e_2 - e_1)z, \quad E = (e_1 - e_2)\lambda + \text{const},
\]

\[
e_2 = \frac{a - 2}{a + 1}e_1, \quad e_3 = \frac{1 - 2a}{a + 1}e_1, \quad \varphi(\delta_1) = \frac{a + 1 - 3b}{a + 1}e_1,
\]

transform equation (0.6), (0.10) into Fuchsian equation (0.1) with five singular points, where

\[
P(z) = \frac{1}{2} \left( \frac{1 - 2m_1}{z} + \frac{1 - 2m_2}{z - 1} + \frac{1 - 2m_3}{z - a} \right) - 2n_1 \frac{z}{z - b},
\]

(2.2)

\[
Q(z) = \frac{N(N - 2m_0 - 1)z + \lambda + 2n_1\rho(z - b)^{-1}}{4z(z - 1)(z - a)},
\]

(2.3)

\[
\rho = 2(b - 1)(b - a)m_1 + 2b(b - a)m_2 + 2b(b - 1)m_3 + (3b^2 - 2(a + 1)b + a)n_1, \quad (2.4)
\]

\[
N = m_0 + m_1 + m_2 + m_3 + 2n_1, \quad m_i, n_1 \in \mathbb{Z}_{\geq 0}, \quad \lambda, z \in \mathbb{C}.
\]

(2.5)

It is not difficult to generalize this change of variables for greater number of singular points

\[
\psi(x) = y(z)z^{-m_1/2(z-1)} - m_2/2(z-a) - m_3/2 \prod_{k=1}^{M}(z - b_k)^{-n_k},
\]

\[
\varphi(x) = e_1 + (e_2 - e_1)z, \quad E = (e_1 - e_2)\lambda + \text{const}
\]

\[
e_2 = \frac{a - 2}{a + 1}e_1, \quad e_3 = \frac{1 - 2a}{a + 1}e_1, \quad \varphi(\delta_k) = \frac{a + 1 - 3b_k}{a + 1}e_1.
\]

(2.7)

Coefficients of corresponding Fuchsian equation (2.11) with \( M + 4 \) singular points has a next form

\[
P(z) = \frac{1}{2} \left( \frac{1 - 2m_1}{z} + \frac{1 - 2m_2}{z - 1} + \frac{1 - 2m_3}{z - a} \right) - 2 \sum_{k=1}^{M} \frac{n_k}{z - b_k},
\]

(2.8)

\[
Q(z) = \frac{N(N - 2m_0 - 1)z + \lambda + R_M(z)}{4z(z - 1)(z - a)}, \quad R_M(z) = 2 \sum_{k=1}^{M} \frac{n_k\rho_k}{z - b_k},
\]

(2.9)

\[
\rho_k = 2(b_k - 1)(b_k - a)m_1 + 2b_k(b_k - a)m_2 + 2b_k(b_k - 1)m_3 +
\]

\[
+ (3b_k^2 - 2(a + 1)b_k + a)n_k + 4 \sum_{j \neq k} \frac{b_k(b_k - 1)(b_k - a)}{b_k - b_j}n_j,
\]

(2.10)

\[
N = \sum_{j=0}^{3} m_j + 2 \sum_{k=1}^{M} n_k, \quad m_j, n_k \in \mathbb{Z}_{\geq 0}, \quad \lambda, z \in \mathbb{C}.
\]

(2.11)
3. Finite-gap Fuchsian equations

Let us introduce the concept of ‘finite-gap’ solutions of the Fuchsian equation.

**Definition.** The Fuchsian equation \((0.1), (2.8) – (2.11)\) will be called ‘finite-gap’ if it can be obtained from Shrödinger equation with finite-gap potential \((0.10)\) via the change of variables \((2.7)\). Solutions of ‘finite-gap’ Fuchsian equations will be called ‘finite-gap’ solutions.

**Theorem 1.** The Fuchsian equation \((0.1), (2.8) – (2.11)\) is ‘finite-gap’ if and only if points \(z = b_k\) are false singular points.

**Proof.** Let us calculate characteristic exponents at singular points of equation

\[
\begin{align*}
\rho_1(0) &= \frac{1}{2} + m_1, \quad \rho_2(0) = 0, \\
\rho_1(1) &= \frac{1}{2} + m_2, \quad \rho_2(1) = 0, \\
\rho_1(a) &= \frac{1}{2} + m_3, \quad \rho_2(a) = 0, \\
\rho_1(\infty) &= -\frac{N}{2}, \quad \rho_2(\infty) = -\frac{N - 2m_0 - 1}{2}, \\
\rho_1(b_k) &= 2n_k + 1, \quad \rho_2(b_k) = 0.
\end{align*}
\]

Assume that Fuchsian equation is ‘finite-gap’ but points \(z = b_k\) are not false singular points. The a general solution will have at those points logarithmic singularities (see for example \([23, 24]\)). Hence a general solution of Shrödinger equation \((0.6)\) with finite-gap potential \((0.10)\) will have logarithmic singularities at points \(x = \pm \delta_k\). That is in contradiction with proposition \(3\).

If we now assume that points \(z = b_k\) are false singular points of equation \((0.1), (2.8) – (2.11)\) then from properties of general solution of this equation and from the equation of the change of variable \((2.2)\) it follows that singularities of general solution \(\psi(x)\) of equation \((0.6), (0.10)\) are only poles:

\[
\begin{align*}
\psi(x) &= c_1 (x - \omega_1)^{-m_1} + o((x - \omega_1)^{-m_1}), \quad x \to \omega_1; \\
\psi(x) &= c_2 (x - \omega_2)^{-m_2} + o((x - \omega_2)^{-m_2}), \quad x \to \omega_2; \\
\psi(x) &= c_3 (x - \omega_3)^{-m_3} + o((x - \omega_3)^{-m_3}), \quad x \to \omega_3; \\
\psi(x) &= d_k (x \mp \delta_k)^{-n_k} + o((x \mp \delta_k)^{-n_k}), \quad x \to \pm \delta_k.
\end{align*}
\]

I.e. a general solution \(\psi(x)\) is a meromorphic function and potential \((0.10)\) is Picard potential. Therefore potential \((0.10)\) is finite-gap and Fuchsian equation is ‘finite-gap’ equation. \(\square\)
Now let us follow \[2\] and give the definition of Novikov’s equation for the Fuchsian equation.

**Definition.** By *Novikov’s equation of order* \(g\) *for the Fuchsian equation* we will call the following equality:

\[
I_g + \sum_{j=1}^{g} \tilde{c}_j I_{g-j} = \tilde{d},
\]

where \(\tilde{c}_j, \tilde{d}\) are some constants,

\[
I_{j+1} = \mathcal{L}(I_j), \tag{3.2a}
\]

\[
\mathcal{L}(f) = z(z-1)(z-a) \frac{d^2 f}{dz^2} + \frac{3z^2 - 2(a+1)z + a}{2} \cdot \frac{df}{dz} - \int \left( 4I_0 \frac{df}{dz} + 2f \frac{dI_0}{dz} \right) dz, \tag{3.2b}
\]

\[
I_0(z) = \frac{m_0(m_0 + 1)}{4} z^4 + \frac{m_1(m_1 + 1)}{4} \cdot \frac{a}{z} + \frac{m_2(m_2 + 1)}{4} \cdot \frac{z - a}{z - 1} + \frac{m_3(m_3 + 1)}{4} \cdot \frac{a(z - 1)}{z - a} + \sum_{k=1}^{M} n_k(n_k + 1) \left( \frac{b_k(b_k - 1)(b_k - a)}{(z - b_k)^2} + \frac{3b_k^2 - 2(a + 1)b_k + a}{2(z - b_k)} \right). \tag{3.3}
\]

**Theorem 2.** The function \(I_0\) satisfies Novikov’s equation of order \(g\) for the Fuchsian equation if and only if the potential \(u(x)\) satisfies Novikov’s equation of the same order.\(^3\)

**Remark 2.** The constant of integration in the definition of the function \(I_n\) is not fixed. Because of that, all the functions \(I_n\) are defined modulo a linear combination of lower order functions \(I_k\). However, it is easily seen that the property of ‘finite-gapness’ of the Fuchsian equation does not depend on the concrete values of integration constants in the definition of \(I_n\). The latter affect only the values of the constants \(\tilde{c}_m\) and \(\tilde{d}\) in the equation (3.1).

**Corollary 1.** The Fuchsian equation (0.1), (2.8)–(2.11) is ‘finite-gap’ if and only if the function \(I_0\) satisfies Novikov’s equation of order \(g\) for the Fuchsian equation.

**Theorem 3.** For any \(m_i \in \mathbb{Z}_{\geq 0}\) there exists number \(b\) such that the Fuchsian equation with five singular points (0.1), (2.3)–(2.6) for \(n_1 = 1\) is ‘finite-gap’.

\(^3\)cf. \[2\]
Proof. From the properties of the flows $J_n$ (see, e.g., [18]), from the properties of elliptic functions [13] and from the equation of the change of variable (2.2) it follows that all functions $I_n$ are rational functions of the variable $z$ (i.e. these functions do not have logarithmic singularities).

From equation (3.3) it follows what $I_0$ has at the point $z = b$ a pole of second order at the point $z = b$

$$I_0 = n_1(n_1 + 1) \left( \frac{b(b - 1)(b - a)}{(z - b)^2} + \frac{3b^2 - 2(a + 1)b + a}{2(z - b)} \right) + O(1), \quad z \to b. \quad (3.4)$$

If we now assume that the function $I_j(z)$ has at the point $z = b$ a pole of order $2\alpha \leq 2n_1$:

$$I_j(z) = \frac{A}{(z - b)^{2\alpha}} + \frac{B}{(z - b)^{2\alpha - 1}} + O((z - b)^{-2\alpha}), \quad z \to b,$$

then we obtain that the function $I_{j+1}(z)$,

$$I_{j+1} = \frac{2(2\alpha + 1)(\alpha + n_1 + 1)(\alpha - n_1)}{\alpha + 1} \cdot \frac{b(b - 1)(b - a)A}{(z - b)^{2\alpha + 2}} +$$

$$+ \frac{(4\alpha + 1)(2\alpha^2 + \alpha - n_1 - n_1^2)}{2\alpha + 1} \cdot \frac{(3b^2 - 2(a + 1)b + a)A}{(z - b)^{2\alpha + 1}} +$$

$$+ \frac{2\alpha(2\alpha + 2n_1 + 1)(2\alpha - 2n_1 - 1)}{2\alpha + 1} \cdot \frac{b(b - 1)(b - a)B}{(z - b)^{2\alpha + 1}} +$$

$$+ O((z - b)^{-2\alpha}), \quad z \to b,$$

has at the same point a pole of order $\alpha' \leq 2n_1 + 1$.

In particular

$$I_1 = -3n_1(n_1 - 1)(n_1 + 1)(n_1 + 2)b(b - 1)(b - a) \times$$

$$\times \left( \frac{b(b - 1)(b - a)}{(z - b)^4} - \frac{3b^2 - 2(a + 1)b + a}{(z - b)^3} \right) + O((z - b)^{-2}).$$

Let now $n_1 = 1$. Then the function $I_1$ has a pole of second order at the point $z = b$ while $I_0$ has also a pole of second order at the same point. Hence, there exist a linear combination $\tilde{I}_1 = I_1 + cI_0$ with a pole of first order in $s = b$. Let us find $b$ from condition

$$\text{Res}_{z=b} \tilde{I}_1 = 0. \quad (3.5)$$

Let us take functions $\tilde{I}_j$,

$$\tilde{I}_j = \mathcal{L}(\tilde{I}_{j-1}), \quad j > 1. \quad (3.6)$$

Firstly this functions do not have poles at the point $z = b$. Secondly, for any $j$ the order of poles of functions $\tilde{I}_j(z)$ in singular points $z = z_k$ ($z_1 = 0, z_2 = 1, z_3 = a$) of the Fuchsian equation does not exceed corresponding characteristic $m_k$ (see for example [2]).
Hence, for any $j$ the dimension of the linear span of rational functions $1, \tilde{I}_1, \ldots, \tilde{I}_j$ does not exceed $N$ and therefore there exists a number $g$,

$$\max_{0 \leq i \leq 3} m_i \leq g \leq N - 1,$$

such that the equality (3.1) is fulfilled. We remind that functions $\tilde{I}_j$ are linear combinations of functions $I_i$ of lesser orders (Remark 2). \hfill \square

The condition (3.5) can be found in explicit form. Direct calculation gives us next polynomial equation of sixth order on $b$ (position of false singular point):

$$k_0^2b^6 - 2k_0^2(a + 1)b^5 + h_4b^4 + 2ah_3b^3 + ah_2b^2 + 2k_1^2a^2(a + 1)b - k_1^2a^3 = 0, \quad (3.8)$$

where $k_j = m_j - 1/2$

$$h_4 = (k_2^2 - k_3^2)a^2 + (4k_0^2 + k_2^2 + k_3^2 - k_1^2)a + k_0^2 - k_2^2,$$

$$h_3 = (k_2^2 + k_3^2 - k_0^2 - k_2^2)a + k_1^2 + k_3^2 - k_0^2 - k_2^2,$$

$$h_2 = (k_2^2 - k_1^2)a^2 + (k_2^2 - 4k_0^2 - k_3^2 - k_2^2)a + k_3^2 - k_2^2.$$

**Remark 3.** Any value $b$ satisfying the sixth-order polynomial equation (3.8) gives a ‘finite-gap’ Fuchsian equation and a finite-gap elliptic potential of Shrödinger operator. Hence, in general cases for any set of $m_i \in \mathbb{Z}_{\geq 0}$ there exist six different finite-gap elliptic potentials ($s = 1, \ldots, 6$):

$$u(x) = m_0(m_0 + 1)\varphi(x) + \sum_{i=1}^{3} m_i(m_i + 1)\varphi(x - \omega_i) + 2\varphi(x - \delta_s) + 2\varphi(x + \delta_s). \quad (3.9)$$

For example, if

$$m_0 = m_1 = m_2 = m_3 = 0, \quad n_1 = 1$$

then

$$u(x) = 2\varphi(x - \delta_s) + 2\varphi(x + \delta_s),$$

where

$$\delta_1 = \frac{\omega}{2}, \quad \delta_2 = \frac{\omega}{2} + \omega', \quad \delta_3 = \frac{\omega'}{2}, \quad \delta_4 = \omega + \frac{\omega'}{2}, \quad \delta_5 = \frac{\omega + \omega'}{2}, \quad \delta_6 = \frac{\omega - \omega'}{2}.$$
the potential
\[ u(x) = g(g + 1)\varphi(x - \omega j/2) + g(g + 1)\varphi(x + \omega j/2) \]
is \( g \)-gap potential because it represents a \( g \)-gap Lamé potential with changed period of lattice
\[ u(x) = g(g + 1)\varphi(x) + g(g + 1)\varphi(x + \omega j) \]
which shifted on one fourth of new period. With the help of formula (0.11) and with the shift of argument one can construct from Treibich-Verdier potentials (0.7) more complicated even finite-gap potentials. Hence, there it exists more complicated (but more special) ‘finite-gap’ Fuchsian equations.

In order to find finite-gap solutions of Fuchsian equation (0.1), (2.8)–(2.11) let us use the results of the theory of finite-gap elliptic potentials for the Schrödinger operator and consider the equation
\[
\frac{d^3\Psi}{dz^3} + 3P(z)\frac{d^2\Psi}{dz^2} + (P'(z) + 4Q(z) + 2P^2(z)) \frac{d\Psi}{dz} + (2Q'(z) + 4P(z)Q(z)) \Psi = 0, \tag{3.10}
\]
solutions of which are the products of any two solutions of Heun’s equation (0.1).

**Theorem 4.** If the equation (0.1), (2.8)–(2.11) is finite-gap then the equation (3.10), (2.8)–(2.11) with nonnegative integer characteristics \( m_i, n_j \) has as its solution the function \( \Psi_{g,N}(\lambda, z) \), which is a polynomial in \( \lambda \) of the degree \( g \) and in \( z \) of the degree \( N \) (2.11)
\[
\Psi_{g,N}(\lambda, z) = a_0(\lambda)z^N + a_1(\lambda)z^{N-1} + \ldots + a_N(\lambda) =
\tilde{a}_0(z)^g + \tilde{a}_1(z)^{g-1} + \ldots + \tilde{a}_g(z). \tag{3.11}
\]
The leading coefficient of this function considered as a polynomial in \( \lambda \) is equal to
\[
\tilde{a}_0(z) = z^{m_1}(z - 1)^{m_2}(z - a)^{m_3} \prod_{k=1}^{M} (z - b_k)^{2n_k}. \tag{3.12}
\]

**Proof.** The product \( \hat{\Psi}(x, E) \) of eigenfunctions of the Schrödinger operator with the potential \( u(x) \) (0.10) is an elliptic meromorphic function in the variable \( x \), because \( u(x) \) is a Picard potential (Proposition 3). As a function of the variable \( x \) the function \( \hat{\Psi}(x, E) \) has poles of multiplicity \( 2m_j \) at the points \( x = \omega j, (\omega_0 \equiv 0) \) and \( 2n_k \) at the points \( x = \pm \delta_k \).

From equation (1.3) and from evenness \( u(x) \) (0.10) it follow what \( \hat{\Psi}(x, E) \) is rational function in \( \varphi(x) \) and a polynomial of degree \( g \) in the spectral parameter \( E \). Hence, the function
\[
\Psi_{g,N}(\lambda, z) = \text{const} \cdot \hat{\Psi}(x, E) \prod_{j=1}^{3}(\varphi(x) - \epsilon_j)^{m_j} \prod_{k=1}^{M}(\varphi(x) - \varphi(\delta_k))^{2n_k}, \tag{3.13}
\]where \( \lambda \) and \( z \) are related with \( E \) and \( x \) by equalities (2.7), is a polynomial in \( \lambda \) of degree \( g \) and in \( z \) of degree \( N \) (i.e. it is a rational function in the variable \( z \) with the unique pole
of order $N$ at the point $z = \infty$). The constant in the equality (3.13) is chosen such that the leading coefficient of $\Psi_{g,N}(\lambda, z)$, considered as a polynomial in $\lambda$, is equal (3.12). □

**Corollary 2.** Coefficients $\tilde{a}_j(z)$ of the polynomial $\Psi_{g,N}(\lambda, z)$ have the form:

$$\tilde{a}_j(z) = \tilde{a}_0(z)\tilde{I}_{j-1}, \quad j = 1, \ldots, g,$$

where $\tilde{I}_j$ is a linear combination of the rational functions $I_j, \ldots, I_0, 1$ having poles in the singularities of Fuchsian equation.

Proof follows from the equalities (1.4), (3.13) and from the change of variable (2.7). □

**Corollary 3.** If the equation (3.10) does not have solution in polynomial form (3.11), then the Fuchsian equation (0.1), (2.8)–(2.11) is not finite-gap.

Knowing the product of the solutions of Heun’s equation it is not difficult to find the solutions themselves (see, for instance, [25, §19.53, §23.7, §23.71]).

**Theorem 5.** Finite-gap solutions of Fuchsian equation (0.1), (2.8)–(2.11) with $m_i, n_k \in \mathbb{Z}_{\geq 0}$ have the form

$$Y_{1,2}(m, n; \lambda; z) = \sqrt{\Psi_{g,N}(\lambda, z)} \exp\left(\pm \frac{i\nu(\lambda)}{2} \int_{z_0}^{z} \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3} \prod_{k=1}^{M} (z-b_k)^{2n_k} dz}{\Psi_{g,N}(\lambda, z)\sqrt{z(z-1)(z-a)}}\right).$$

(3.15)

Here $i^2 = -1$,

$$i^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j), \quad \lambda_j = \lambda(E_j),$$

(3.16)

$E_j$ are the gap edges of the finite-gap elliptic potential $u(x)$ (0.10).

Proof. From the Liouville formula it follows that the Wronskian of two linearly independent solutions of the linear homogeneous differential equation,

$$y'' + P(z)y' + Q(z)y = 0,$$

has the following dependence from $z$:

$$W[y_1(\lambda, z), y_2(\lambda, z)] = W_0(\lambda) \exp\left\{-\int_{z_0}^{z} P(t) dt\right\},$$

where $W_0(\lambda)$ is Wronskian’s value at $z_0$. Hence, the Wronskian of two linearly independent solutions of Fuchsian equation (0.1), (2.8)–(2.11) is equal to

$$W[y_1(\lambda, z), y_2(\lambda, z)] = -i\nu(\lambda) \cdot \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3} \prod_{k=1}^{M} (z-b_k)^{2n_k}}{\sqrt{z(z-1)(z-a)}},$$

(3.17)
where
\[ \nu(\lambda) = iW_0(\lambda) \cdot \frac{\sqrt{z_0(z_0 - 1)(z_0 - a)}}{z_0^{m_1}(z_0 - 1)^{m_2}(z_0 - a)^{m_3} \prod_{k=1}^{M}(z_0 - b_k)^{2n_k}}. \]

If we now divide the Wronskian of two solutions (3.17) by their product,
\[ y_1(\lambda, z) \cdot y_2(\lambda, z) = \Psi_{g,N}(\lambda, z), \] (3.18)
we obtain a simple differential equation of the first order:
\[ \frac{y_2' - y_1'}{y_2} = -i\nu(\lambda) \cdot \frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3} \prod_{k=1}^{M}(z-b_k)^{2n_k}}{\Psi_{g,N}(\lambda, z)\sqrt{z(z-1)(z-a)}}. \] (3.19)

The equation (3.19) can be easily integrated:
\[ \frac{y_2(\lambda, z)}{y_1(\lambda, z)} = C(\lambda) \cdot \exp \left[ -i\nu(\lambda) \int_{z_1}^{z} \frac{z^{m_1}(t-1)^{m_2}(t-a)^{m_3} \prod_{k=1}^{M}(t-b_k)^{2n_k}}{\Psi_{g,N}(\lambda, t)\sqrt{t(t-1)(t-a)}} \, dt \right], \] (3.20)
where \( C(\lambda) = y_2(\lambda, z_1)/y_1(\lambda, z_1) \).

If we now consider solutions \( Y_{1,2}(m, n; \lambda, z) \) with the same Wronskian (3.17) and product (3.18),
\[ Y_1(m, n; \lambda, z) = \sqrt{C(\lambda)} \cdot y_1(\lambda, z), \quad Y_2(m, n; \lambda, z) = \frac{y_2(\lambda, z)}{\sqrt{C(\lambda)}}, \] (3.21)
then from (3.18) and (3.20) we get (3.15).

Substituting the ansatz (3.15) into the Fuchsian equation (0.1), (2.8)–(2.11) we get
\[ \nu^2(\lambda) = \frac{2\Psi\Psi'' - (\Psi')^2 + 2P(z)\Psi' + 4Q(z)\Psi^2}{z^{2m_1-1}(z-1)^{2m_2-1}(z-a)^{2m_3-1} \prod_{k=1}^{M}(z-b_k)^{4n_k}}, \] (3.22)
where
\[ \Psi = \Psi_{g,N}(\lambda, z), \quad \Psi' = \frac{d\Psi}{dz}, \quad \Psi'' = \frac{d^2\Psi}{dz^2}. \]

From (3.15), (3.22) it follows that \( \nu^2(\lambda) \) is a polynomial in \( \lambda \) of the degree \( 2g + 1 \) with the leading coefficient equal to 1.

It is not difficult to show that under the change (2.7) the solutions \( Y_{1,2}(m, n; \lambda, z) \) of Fuchsian equation (0.1), (2.8)–(2.11) turn into Floquet solutions of the equation (0.7), (0.10). Therefore, zeros \( \lambda_j \) \( (j = 1, \ldots, 2g + 1) \) of the polynomial \( \nu(\lambda) \) or, which is the same, zeros of the Wronskian of the solutions \( Y_{1,2}(m, n; \lambda, z) \) correspond to zeros of the Wronskian of the Floquet solutions of equation (0.7), (0.10), i.e. they correspond to the gap edges \( E_j \) \( (j = 1, \ldots, 2g + 1) \) of spectrum of the potential (0.10). Hence, the hyperelliptic curve \( \Gamma \) (3.16) is isomorphic to the spectral curve \( \tilde{F} \)
\[ w^2 = \prod_{j=1}^{2g+1} (E - E_j) \] (3.23)
of the finite-gap elliptic potential \( u(x) \) \( (0.10) \).

**Remark 6.** Knowing the product of the eigenfunctions of the Schrödinger operator \( \hat{\Psi}(x, E) \) \( (1.3) \) it is possible to write the formulae for these eigenfunctions

\[
\psi_{1,2}(x, E) = \sqrt{\hat{\Psi}(x, E)} \exp \left( \mp \int \frac{dx}{\hat{\Psi}(x, E)} \right), \quad w(E) = \frac{1}{2} W[\psi_1, \psi_2] \quad (3.24)
\]

and for equation of spectral curve \( (3.23) \)

\[
w^2(E) = (u + E) \hat{\Psi}^2 + \frac{1}{4} \hat{\Psi}_x^2 - \frac{1}{2} \hat{\Psi}_{xx} \hat{\Psi}. \quad (3.25)
\]

**Concluding remarks**

There is one important difference between finite-gap Heun equation \( [2] \) and finite-gap Fuchsian equation. In case of the Heun equation the spectral curve and its algebraic genus \( g \) are completely determined by characteristics \( m_i \):

1. In the case of even \( N = \sum m_i \)

\[
g = \max \left\{ \max_{0 \leq i \leq 3} m_i, \frac{N}{2} - \min_{0 \leq i \leq 3} m_i \right\};
\]

2. In the case of odd \( N \)

\[
g = \max \left\{ \max_{0 \leq i \leq 3} m_i, \frac{N + 1}{2} \right\}.
\]

On the other hand in case of the Fuchsian equation the spectral curve and its algebraic genus \( g \) depend not only on characteristics \( m_i, n_k \) but also on positions of singular points \( b_k \). For example, the potential

\[
u(x) = 2\varphi(x) + 2\varphi(x + \delta) + 2\varphi(x - \delta)
\]

for \( \varphi(2\delta) = -2\varphi(\delta) \) is two-gap, but for \( \varphi(2\delta) = \varphi(\delta) \) is one-gap potential. Therefore, knowing only characteristics \( m_i, n_k \) we can only estimate (see \( (3.7) \)) algebraic genus of spectral curve \( (3.16) \). Examples of simple ‘finite-gap’ solutions of Fuchsian equation and of finite-gap elliptic potentials that are not Lamé or Treibich-Verdier potentials, are discussed in Appendix.

It is easy to see that equation \( (0.6), (0.10) \) is invariant with respect to transformation \( m_i \rightarrow -m_i - 1, n_k \rightarrow -n_k - 1 \). Therefore it is not difficult to transform ‘finite-gap’ solutions of equation \( (0.1), (2.8)-(2.11) \) with non-negative characteristics \( m_i, n_k \) into solutions with negative characteristics. Corresponding transformations for Heun equation one can find, for example, in \( [24, 26, 27, 28, 2] \).
Appendix A. Simplest ‘finite-gap’ solutions

At the end of this paper we would like to list equations for $b$, polynomials $\Psi(\lambda, z)$ and canonical equations (3.16) of the hyperelliptic curves $\Gamma = \{(\nu, \lambda)\}$ for some simplest ‘finite-gap’ solutions (3.15) of Fuchsian equation with five singular points $(0.1), (2.3)–(2.6)$ with characteristics $n_1 = 1, \quad m_i \in \mathbb{Z}_{\geq 0} (i = 0, 1, 2, 3)$. There will be given also finite-gap elliptic potentials $\tilde{u}(x) = u(x) + \text{const}$ and their spectral curves (3.23). Potentials $\tilde{u}(x)$ are normalized by condition

$$\sum_{j=1}^{2g+1} \tilde{E}_j = 0.$$ 

Our examples are indexed by the characteristics $(m_0, m_1, m_2, m_2)$.

$(0,0,0,0)$:

$$(b^2 - a)(b^2 - 2b + a)(b^2 - 2ab + a) = 0. \quad (A.1)$$

$b^2 - a = 0$:

$$\Psi_{1,2}(\lambda, z) = (z - b)^2 \lambda + (3 + 3a - 4b)z^2 - 2(5b + 5ab - 8a)z + a(3 + 3a - 4b),$$

$$\nu^2 = (\lambda - 4b + 3a + 3)(\lambda^2 + 7(1 + a - 2b)\lambda + 2(6a^2 + 36a + 6 - 25ab - 25b)).$$

The potential $\tilde{u}(x)$ is one-gap potential with changed period of lattice.

$$\tilde{u}(x) = 2\varphi(x - \delta) + 2\varphi(x + \delta) - 2e_1, \quad \varphi(2\delta) = e_1,$$

$$w^2 = (E + 2e_1)(E + e_1 - 2\varphi(\delta))(E - 3e_1 + 2\varphi(\delta)).$$

$b^2 - 2b + a = 0$:

$$\Psi_{1,2}(\lambda, z) = (z - b)^2 \lambda + (3a - 4b)z^2 - 2(6a + 5ab - 12b)z - 3a^2 + 12a - 24b + 14ab,$$

$$\nu^2 = (\lambda - 4b + 3a)(\lambda^2 + (4 + 7a - 14b)\lambda + 2(6a^2 - 16a - 25ab + 36b)),$$

The potential $\tilde{u}(x)$ is one-gap potential with changed period of lattice.

$$\tilde{u}(x) = 2\varphi(x - \delta) + 2\varphi(x + \delta) - 2e_2, \quad \varphi(2\delta) = e_2,$$

$$w^2 = (E + 2e_2)(E + e_2 - 2\varphi(\delta))(E - 3e_2 + 2\varphi(\delta)).$$

$b^2 - 2ab + a = 0$:

$$\Psi_{1,2}(\lambda, z) = (z - b)^2 \lambda + (3 - 4b)z^2 - 2(6a + 5b - 12ab)z - 3a + 12a^2 - 24a^2b + 14ab,$$

$$\nu^2 = (\lambda - 4b + 3)(\lambda^2 + (4a + 7 - 14b)\lambda + 2(6 - 16a - 25b + 36ab)),$$

The potential $\tilde{u}(x)$ is one-gap potential with changed period of lattice.

$$\tilde{u}(x) = 2\varphi(x - \delta) + 2\varphi(x + \delta) - 2e_3, \quad \varphi(2\delta) = e_3,$$

$$w^2 = (E + 2e_3)(E + e_3 - 2\varphi(\delta))(E - 3e_3 + 2\varphi(\delta)).$$
(1.0,0.0):

\[(3b^2 - 2(a + 1)b + a)(3b^4 - 4(a + 1)b^3 + 6ab^2 - a^2) = 0.\]  
\[3b^2 - 2(a + 1)b + a = 0;\]

\[\Psi_{2,3}(\lambda, z) = (z - b)^2 \lambda^2 + \left( z^3 + 3(1 + a - 4b)z^2 - (7a - 8ab - 8ab)z + \right.\]
\[\left. + \frac{1}{3}(5a^2 + 5a - 10a^2b - 10b - 2abb) \right) \lambda - \]
\[- \frac{4}{3}(a^3 - 4a^2 + a - 2a^3b + 3a^2b + 3a^2b - 2b),\]

\[\nu^2 = \lambda(\lambda^4 + 10(a + 1 - 3b)\lambda^3 + 33(a^2 - a + 1)\lambda^2 +\]
\[+ 3(12a^3 + a^2 + a + 12 - 38ab + 38b - 38b)\lambda -\]
\[- 12(a^3 - 4a^2 + a - 2a^3b + 3a^2b + 3a^2b - 2b)),\]

The potential \(\tilde{u}(x)\) is an isospectral deformation of two-gap Lamé potential.

\[u = 2\varphi(x) + 2\varphi(x - \delta) + 2\varphi(x + \delta), \quad \varphi(2\delta) = -2\varphi(\delta), \quad \varphi^2(\delta) = \frac{g_2}{12} \]

\[w^2 = \left( E^3 - \frac{9g_2}{4}E + \frac{27g_3}{4} \right) (E - 6\varphi(\delta))(E + 6\varphi(\delta)).\]

(3b^4 - 4(a + 1)b^3 + 6ab^2 - a^2 = 0;\]

\[\Psi_{1,3}(\lambda, z) = (z - b)^2 \lambda + z^3 + 3(1 + a - 4b)z^2 +\]
\[+ (2a - 10ab - 10b + 27b^2)z + 2ab + 3ab^2 + 3b^2 - 12b^3,\]

\[\nu^2 = \lambda^3 + 10(1 + a - 3b)\lambda^2 + 3(11a^2 + 19a + 90b^2 - 60ab - 60a + 11)\lambda +\]
\[+ 36a^3 + 73a^2 + 73a + 36 - 254b - 376ab - 254ab^2 + 630b^2 + 630a^2b - 630a^2b.\]

The potential \(\tilde{u}(x)\) is one-gap potential with changed period of lattice.

\[u = 2\varphi(x) + 2\varphi(x - \delta) + 2\varphi(x + \delta) - 4\varphi(\delta), \quad \varphi(2\delta) = \varphi(\delta), \quad \varphi^2(\delta) = \frac{g_2}{12} \]

\[w^2 = E^3 + \left( \frac{9g_2}{4} - 30\varphi^2(\delta) \right) E - 70\varphi^3(\delta) + \frac{21g_2}{2}\varphi(\delta) + \frac{27g_3}{4}.\]

(1,1,0,0):

\[(b^2 - a)(3b^2 - 2(a + 2)b + 3a)(3b^2 - 2(2a + 1)b + 3a) = 0.\]

\[b^2 - a = 0;\]

\[\Psi_{1,4}(\lambda, z) = z(z - b)^2 \lambda + z^4 + 8(1 + a - 2b)z^3 +\]
\[\nu^2 = \lambda^2 + (25a + 25 - 42b)\lambda^2 + 8(26a^2 + 120a + 26 - 85b - 85ab)\lambda + 144(4a^3 + 41a^2 + 41a + 4 - 19a^2b - 52ab - 19b).\]

The potential \(\widetilde{u}(x)\) is one-gap potential with changed period of lattice.

\[\widetilde{u}(x) = 2\varphi(x) + 2\varphi(x - \omega_1) + 2\varphi(x - \delta) + 2\varphi(x + \delta) - 4\varphi(\delta) - 2e_1, \quad \varphi(2\delta) = e_1,\]

\[w^2 = E^3 - (75e_1^2 + 60e_1\varphi(\delta) - 11g_2)E - 2e_1(137e_1^2 - 15g_2) - 28(15e_1^2 - g_2)\varphi(\delta).\]

\[3b^2 - 2(a + 2)b + 3a = 0:\]

\[\Psi_{2,4}(\lambda, z) = z(z - b)^2\lambda^2 + \left(z^4 + (13a + 8 - 20b)z^3 - \frac{4}{3}(30a + ab - 28b)z^2 + \frac{1}{9}(3a^2 + 168a - 2a^2b + 64ab - 224b)z - \frac{a}{3}(3a - 2ab - 4b)\right)\lambda + \frac{1}{27}(15a^3 + 492a - 480a - 10a^3b + 39a^2b - 696ab + 640b)z^2 + \frac{a}{9}(15a - 120a + 10a^2b - 44ab - 160b),\]

\[\nu^2 = (\lambda + 5a - 4b)\left(\lambda^4 + (30a + 25 - 46b)\lambda^3 + (333a^2 - 199a + 208 - 506ab + 188b)\lambda^2 + \frac{4}{9}(3636a^3 - 7212a^2 + 3612a - 1296 - 5449a^2b + 9065ab - 4912b)\lambda + \frac{8}{27}(9720a^4 - 32019a^3 + 26751a^2 - 3912a^2) - 14555a^3b + 44451a^2b - 40836ab + 10400b\right).\]

The potential \(\widetilde{u}(x)\) is an isospectral deformation of two-gap 4-elliptic Treibich-Verdier potential with additional pole at the point \(x = \omega_2\) (see, for example, [9] [10] [11] [12]).

\[\widetilde{u}(x) = 2\varphi(x) + 2\varphi(x - \omega_1) + 2\varphi(x - \delta) + 2\varphi(x + \delta) - 2e_1, \quad \varphi(2\delta) = \frac{e_1 + 5e_2}{3},\]

\[w^2 = (E - 6e_2)(E^2 + 2E(e_2 - e_3) - 3(13e_2^2 + 2e_2e_3 - 3e_3^2)\times\]
\( \times (E - 3(e_1 - e_2) - 6\varphi(\delta))(E + e_1 - e_2 + 6\varphi(\delta)). \)

\[ 3b^2 - 2(2a + 1)b + 3a = 0: \]

\[
\Psi_{2,4}(\lambda, z) = z(z - b)^2 \lambda^2 + \left( z^4 + (13 + 8a - 20b)z^3 - \frac{4}{3}(30a + b - 28ab)z^2 + \right.
\]
\[ + \frac{1}{9}(3a + 168a^2 - 2b + 64ab - 224a^2b)z - \frac{a}{3}(3a - 2b - 4ab) \lambda + \]
\[ + (5a - 4b)z^4 + \frac{8}{3}(15 - 9a - 26b + 20ab)z^3 - \]
\[ - \frac{2}{9}(660a - 480a^2 - 35b - 776ab + 640a^2b)z^2 + \]
\[ + \frac{4}{27}(15a + 492a^2 - 480a^3 - 10b + 39ab - 696a^2b + 640a^3b)z + \]
\[ + \frac{a}{9}(15a + 120a^2 - 10b + 44ab - 160a^2b), \]

\[ \nu^2 = (\lambda + 5 - 4b) \left( \lambda^4 + (30 + 25a - 46b)\lambda^3 + (333 - 199a + 208a^2 - 506b + 188ab)\lambda^2 + \right. \]
\[ + \frac{4}{9}(3636 - 7212a + 3612a^2 + 1296a^3 - 5449b + 9065ab - 4912a^2b)\lambda + \]
\[ + \frac{8}{27}(9720 - 32019a + 26751a^2 - 3912a^3 - \]
\[ - 14555b + 44451ab - 40836a^2b + 10400a^3b) \right). \]

The potential \( \tilde{u}(x) \) is an isospectral deformation of two-gap 4-elliptic Treibich-Verdier potential [9, 10, 11, 12] with additional pole at the point \( x = \omega_3 \)

\[ \tilde{u}(x) = 2\varphi(x) + 2\varphi(x - \omega_1) + 2\varphi(x - \delta) + 2\varphi(x + \delta) - 2e_3, \quad \varphi(2\delta) = \frac{e_1 + 5e_3}{3}, \]

\[ w^2 = (E - 6e_3)(E^2 - 2E(e_2 - e_3) - 3(13e_3^2 + 2e_2e_3 - 3e_2^2) \times \]
\[ \times (E - 3(e_1 - e_3) - 6\varphi(\delta))(E + e_1 - e_3 + 6\varphi(\delta)). \]

\( (2,0,0,0): \)

\[ 25b^6 - 50(a + 1)b^5 + (24a^2 + 101a + 24)b^4 - \]
\[ - 48a(a + 1)b^3 + 19a^2b^2 + 2a^2(a + 1)b - a^3 = 0. \quad (A.4) \]

\[ \Psi_{2,4}(\lambda, z) = (z - b)^2 \lambda^2 + (3z^3 + (3a + 3 - 26b)z^2 + \]
\[ + (2a - 10ab - 10b + 49b^2)z + b(2a + 3ab + 3b - 22b^2))\lambda + 9z^4 - 48bz^3 - \]
\[\begin{align*}
\nu^2 &= \lambda^5 + 10(a + 1 - 5b)\lambda^4 + (33a^2 + 17a + 33 - 260ab - 260b + 790b^2)\lambda^3 + \\
&\quad + (36a^3 - 135a^2 - 135a + 36 - 234a^2b + \\
&\quad + 624ab + 1170ab^2 - 234b + 1170b^2 - 4330b^3)\lambda^2 - \\
&\quad - (216a^3 - 189a^2 + 216a - 144b + 108ab + 108a^2b - 144a^3b - \\
&\quad - 828b^2 - 924ab^2 - 828a^2b^2 + 4300b^3 + 4300ab^3 - 11885b^4)\lambda - \\
&\quad - 4(27a^2 + 27a^3 + 108ab - 351a^2b + 108a^3b - 36b^2 - 405ab^2 - \\
&\quad - 36a^3b^2 + 528b^3 + 3008ab^3 + 582a^2b^3 - 2870b^4 - 2870ab^4 + 4099b^5).
\end{align*}\]

The potential \(\tilde{u}(x)\) is a finite-gap elliptic potential which can not be transformed into Lamé or Treibich-Verdier potential by shifting of argument or by transformation of lattice of periods \([0,1]\). It seems that this potential first appeared in [9,10].

\[\tilde{u}(x) = 6\varphi(x) + 2\varphi(x - \delta) + 2\varphi(x + \delta) - 4\varphi(\delta), \quad \varphi'(2\delta) = -3\varphi'(\delta),\]

\[w^2 = E^5 - \left(210\varphi^2(\delta) - \frac{49g_2}{4}\right) E^3 + \left(630\varphi^3(\delta) - \frac{189g_2}{2}\varphi(\delta) - \frac{225g_3}{4}\right) E^2 + \\
+ \left(12285\varphi^4(\delta) - \frac{3213g_2}{2}\varphi^2(\delta) - 297g_3\varphi(\delta) + \frac{621g_2^2}{16}\right) E - \\
- 59454\varphi^5(\delta) + 11421g_3\varphi^3(\delta) + 3240g_3\varphi^2(\delta) - \frac{4239g_2^2}{8}\varphi(\delta) - \frac{1107g_2g_3}{4}.
\]

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FINITE-GAP SOLUTIONS OF THE FUCHSIAN EQUATIONS

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