Unconventional interaction between vortices in a polarized Fermi gas

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Abstract

Recently, a homogeneous superfluid state with a single gapless Fermi surface was predicted to be the ground state of an ultracold Fermi gas with spin population imbalance in the regime of molecular Bose-Einstein condensation. We study vortices in this novel state using a symmetry-based effective field theory, which captures the low-energy physics of gapless fermions and superfluid phase fluctuations. This theory is applicable to all spin-imbalanced ultracold Fermi gases in the superfluid regime, regardless of whether the original fermion pairing interaction is weak or strong. We find a remarkable, unconventional form of the interaction between vortices. The presence of gapless fermions gives rise to a spatially oscillating potential, akin to the RKKY indirect-exchange interaction in non-magnetic metals. We compare the parameters of the effective theory to the experimentally measurable quantities and further discuss the conditions for the verification of the predicted new feature. Our study opens up an interesting question as to the nature of the vortex lattice resulting from the competition between the usual repulsive logarithmic (2D Coulomb) and predominantly attractive fermion-induced interactions.

Key words: Polarized Fermi gas, Superfluidity, Effective field theory, Vortices

PACS:
1 Introduction

The appearance of quantized vortices is a hallmark of superfluid flow. Vortices have been studied for decades, experimentally and/or theoretically, in a variety of systems as diverse as type-II superconductors, superfluid Helium liquids, rotating ultracold atomic Bose and Fermi gases, and neutron stars. Among these systems, the quantum gas of resonantly interacting fermionic atoms with equal populations of both (hyperfine) spin components, a prototype system for the interesting BEC-BCS crossover physics, has been intensively studied over the past several years. The first experimental observation of vortices in 2005 by the MIT group provided a definitive evidence for superfluidity in atomic Fermi gases. In addition, several theoretical studies have analyzed the possible new properties of vortices across a Feshbach resonance from the BCS to the BEC side.

The physics of atomic Fermi gases is also of fundamental interest beyond the standard BCS/BEC physics, owing to the new tuning flexibility in the atomic gas systems. Under the condition of density imbalance (hence mismatched Fermi surfaces) between the spin-up and -down fermions, a modulated Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) superfluid phase, has long been theoretically anticipated. The theoretical interest in pairing with mismatched Fermi surfaces has been revitalized by the proposal of breached-pairing superfluidity, with a number of exotic superfluid states being proposed or revisited. Breached-pairing superfluid phase with two gapless Fermi surfaces (BP2), related to the unstable Sarma phase, was found to be stable under the introduction of new effects, such as the mass imbalance and/or momentum-dependent pairing interaction. Important developments in the subject are recent studies by various groups investigating the Feshbach-resonant regime of strong interactions. The first experiments on ultracold fermionic gases with spin population imbalance have recently been carried out and thereby brought the subject to the forefront of the cold atom physics. The imbalanced Fermi gas is presently the subject of fervent research activities.

One of the states commonly found in various theoretical approaches is a homogeneous superfluid with a single gapless Fermi surface on the molecular (BEC) side of the Feshbach resonance (the BEC regime). This state, which consists of coexisting molecular superfluid and fully-polarized Fermi gas of the majority-spin component, is closely related to the BP2 phase, but differs from the latter in the number of gapless Fermi surfaces. We will refer to this phase as BP1 (breached-pairing state with a single gapless Fermi surface).

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after Ref. [25] and [28] (also dubbed as “magnetized superfluid” and denoted SF$_M$ in Ref. [19] and [21]). The BP1 phase is predicted to robustly exist in a relatively large area in the phase diagram of the spin imbalanced Fermi gases (also called “polarized Fermi gases”). Additionally, several theoretical works have found the analog of BP1 in a trap, the so-called superfluid-normal-mixture phase. [29] It is then of great interest to examine properties of this novel superfluid.

When fermionic excitations are fully gapped, the physics of vortices belongs to the universality class of the XY model where the phase of the superfluid order parameter plays the dominant role. The vortex sector of this model is described by a 2D Coulomb gas of “charges” with a repulsive logarithmic interaction. This paradigm is challenged in a fundamental fashion in the BP1 state. The presence of gapless fermionic quasiparticles is the distinguishing feature of this superfluid phase and is expected to have important consequences for its physical properties. Since the observation of a vortex lattice is perhaps the only unambiguously signature of superfluidity in ultracold fermionic gases, [7] it is of interest to examine possible ramifications of the presence of gapless fermions on the interaction between vortices in this system.

In this work, we determine the effective interaction potential between the vortices in the BP1 phase. Because we are solely concerned with the intrinsic effect of the gapless fermions on the interaction between vortices in the BP1 state, we consider only the homogeneous case, disregarding the effects of traps. We exploit the method of effective field theory, [30] based explicitly on broken continuous symmetry. [31] This method is particularly suitable for problems involving strongly-coupled systems in the long-wavelength limit and has already proven to be fruitful in treatments of strongly-interacting regimes of ultracold atomic gases. [32,33,34,35] In the present context, the relevant degrees of freedom for a low-energy effective field theory are the superfluid phase field and the field describing the gapless fermionic excitations.

We show that the resulting interaction between vortices in the BP1 phase is not of the pure Coulomb form, but contains an additional fermion-induced contribution that oscillates on a length scale set by the spin polarization, closely resembling the Ruderman-Kittel-Kasuya-Yosida (RKKY) indirect-exchange interaction in non-magnetic metals. [36] In order to show that such an unusual vortex interaction is perfectly compatible with the BP1 phase, we calculate the superfluid density from a microscopic model in the parameter regime relevant for the BP1 state. We demonstrate that the superfluid density in BP1 is positive throughout, which corroborates the dynamical stability of this phase and warrants its further investigation.

The outline of the remainder of this paper is as follows. In Sec. II we introduce notation and conventions for fermion quasiparticles and superfluid phase to be
used throughout. In Sec. II we present self-contained field-theoretical derivation of the superfluid density, followed by the calculation of this quantity in the parameter regime where the BP1 phase is realized. Sec. IV starts with a low-energy effective field theory for gapless fermions and superfluid phase field, from which we derive an effective theory for phase fluctuations by integrating out the fermionic degrees of freedom. In Sec. V we first derive the effective theory for vortices and their effective interaction in momentum space. Then we present the calculation of the effective vortex interaction potential in real space, accompanied by the discussion of its physical significance. Finally, we summarize in Sec. VI. Some mathematical details are relegated to the Appendices.

2 Notation and conventions

2.1 Gapless branch of fermionic quasiparticles

The Bogoliubov-quasiparticle energy spectrum of the system containing two fermion gases with equal masses \( m \) and unequal chemical potentials \( \mu_\uparrow \neq \mu_\downarrow \) is given by (in what follows \( \hbar = 1 \), unless stated otherwise)

\[
E_k^\pm = \sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + \Delta^2 \pm \delta^2}, \tag{1}
\]

where \( \mu = (\mu_\uparrow + \mu_\downarrow)/2 \) is the average chemical potential (thermodynamically conjugate to the overall atomic number of two species combined) and \( \delta = \mu_\downarrow - \mu_\uparrow \) measures the mismatch between the two chemical potentials (conjugate to the relative density imbalance). Since our treatment concerns the spin-polarized homogeneous superfluid realized deeply on the BEC side of the Feshbach resonance, the average chemical potential is assumed to be negative \((\mu < 0)\) in what follows. For definiteness, we hereafter also assume that \( \delta > 0 \).

For sufficiently large mismatch \( \delta \) (more precisely, for \( \delta/2 > \sqrt{\mu^2 + \Delta^2} \)), the lower branch \( E_k^- \) of the above quasiparticle dispersion is gapless, with a single effective Fermi surface. Hereafter, for the sake of brevity, we denote it as

\[
\varepsilon_k = \sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + \Delta^2 - \frac{\delta^2}{2}}. \tag{2}
\]

The effective Fermi surface is defined by \( \varepsilon_k = 0 \) for \( |k| = k_b \), where \( k_b \) is the radius of the “breached-pairing Fermi ball” in momentum space. The latter is controlled by the density imbalance \( n_b = n_\downarrow - n_\uparrow \) between the two pseudospin components (here \( n_\downarrow > n_\uparrow \), as a consequence of the fact that \( \delta > 0 \)),
which, as implied by the Luttinger theorem \[37\], is equal to the volume of the
“breached-pairing Fermi ball” in momentum space
\[
\int_{|\mathbf{k}| \in [0, k_b]} d^3\mathbf{k} \frac{k_b^3}{(2\pi)^3} = \frac{k_b^3}{6\pi^2}.
\]
(3)
This leads to a simple expression for \(k_b\):
\[
k_b = [6\pi^2(n\downarrow - n\uparrow)]^{1/3}.
\]
(4)
An example of gapless dispersion given by Eq. (2) is depicted in Fig. D.1.

2.2 Superfluid phase field and its decomposition

The superfluid phase field \(\theta(\mathbf{x}, \tau)\) represents the phase of the complex Cooper-pair amplitude:
\[
\langle \psi_\uparrow \psi_\downarrow \rangle = |\langle \psi_\uparrow \psi_\downarrow \rangle| e^{i\theta},
\]
(5)
where \(\psi_\uparrow\) and \(\psi_\downarrow\) are the fields describing fermions of opposite spins. In the following, we use the standard decomposition of the superfluid phase field into a static (classical) contribution \(\theta_v(\mathbf{x})\) (singular part) and a quantum-fluctuating contribution \(\phi(\mathbf{x}, \tau)\) (regular part):
\[
\theta(\mathbf{x}, \tau) = \theta_v(\mathbf{x}) + \phi(\mathbf{x}, \tau).
\]
(6)
\(\phi(\mathbf{x}, \tau)\) is a non-compact (unbounded) field describing “spin-wave” (smooth, i.e., non-topological) phase fluctuations; \(\theta_v(\mathbf{x})\) is a multi-valued field pertaining to the topological defects of broken global \(U(1)\) symmetry—vortices.

Superfluid vortices can effectively be treated as classical point-objects in two dimensions. In reality, a two-dimensional theory is applicable to rotating superfluids as long as the rotation frequency is not too high and the flow is everywhere confined to a plane perpendicular to the rotation axis. \[38\] These conditions allow formation of straight vortex lines parallel to the axis of rotation. The arrangement is then essentially two-dimensional, equivalent to an array of point vortices with circulation of the same sign.

The gradient of the singular part of the phase field can conveniently be expressed as \[39\]
\[
\nabla \theta_v = \kappa_0 (\hat{\mathbf{e}}_z \times \nabla) \int d^2\mathbf{x}' G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') ,
\]
(7)
where \(\kappa_0 = \hbar/2m = \pi \hbar/m\) (\(\hbar\) restored for the sake of clarity) is the circulation quantum, \(\rho(\mathbf{x})\) stands for the vortex “charge density”, and \(G(\mathbf{x}, \mathbf{x}') = G(|\mathbf{x} - \mathbf{x}'|)\) is the Green’s function of the two-dimensional Laplacian:
\[
\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta^{(2)}(\mathbf{x} - \mathbf{x}')\).
\]
(8)
The vortex charge density is defined as

$$\rho(x) = 2\pi \sum_{\alpha} N_\alpha \delta^{(2)}(x - x_\alpha) ,$$  \hspace{1cm} (9)

where $N_\alpha$ is the winding number, viz. topological charge, of a vortex located at position $x_\alpha$ in the xy-plane and $\hat{e}_z$ is the unit vector perpendicular to this plane. Following the standard prescription, \[31,40\] the gradient of $\theta_v$ can be associated to a vortex gauge (vector) field $a(x)$ through $a = -\nabla \theta_v$, which by virtue of Eq. (7) becomes

$$a = -\kappa_0 (\hat{e}_z \times \nabla) \int d^2x' G(x,x') \rho(x') .$$  \hspace{1cm} (10)

Consequently, the vortex gauge field obeys the condition

$$\nabla \times a = -\kappa_0 \rho(x) \hat{e}_z ,$$  \hspace{1cm} (11)

whose momentum-space version (obtained by spatial Fourier transformation) reads

$$-i\mathbf{q} \times a_{\mathbf{q}} = \kappa_0 \tilde{\rho}(\mathbf{q}) \hat{e}_z ,$$  \hspace{1cm} (12)

where $\mathbf{q}$ is a two-dimensional wave-vector ($\mathbf{q} \cdot \hat{e}_z = 0$) of phase fluctuations and $\tilde{\rho}(\mathbf{q})$ is the Fourier transform of the vortex charge density $\rho(x)$.

We now make use of the decomposition $a_{\mathbf{q}} = a_{\mathbf{q}}^\parallel + a_{\mathbf{q}}^\perp$ of the vortex gauge field into the longitudinal and transverse components with respect to the momentum $\mathbf{q}$, respectively, and they are given by

$$a_{\mathbf{q}}^\parallel = \frac{(\mathbf{q} \cdot a_{\mathbf{q}}) \mathbf{q}}{q^2} , \quad a_{\mathbf{q}}^\perp = \frac{(\mathbf{q} \times a_{\mathbf{q}}) \times \mathbf{q}}{q^2} .$$  \hspace{1cm} (13)

Because $\mathbf{q} \times a_{\mathbf{q}}^\parallel = 0$, Eq. (12) implies that

$$\tilde{\rho}(\mathbf{q}) \hat{e}_z = -\frac{i}{\kappa_0} \mathbf{q} \times a_{\mathbf{q}}^\perp , \quad \tilde{\rho}(-\mathbf{q}) \hat{e}_z = \frac{i}{\kappa_0} \mathbf{q} \times a_{-\mathbf{q}}^\perp ,$$  \hspace{1cm} (14)

where the second equation in (14) has been obtained from the first one by a simple replacement $\mathbf{q} \rightarrow -\mathbf{q}$. The scalar product of the last two equations yields

$$\tilde{\rho}(\mathbf{q}) \tilde{\rho}(-\mathbf{q}) = \frac{1}{\kappa_0^2} (\mathbf{q} \times a_{\mathbf{q}}^\perp) \cdot (\mathbf{q} \times a_{-\mathbf{q}}^\perp) .$$  \hspace{1cm} (15)

Now using the fact that $\mathbf{q} \cdot a_{\mathbf{q}}^\perp = \mathbf{q} \cdot a_{-\mathbf{q}}^\perp = 0$, together with the identity of vector algebra

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) ,$$  \hspace{1cm} (16)
it is straightforward to obtain a useful relation
\[ \tilde{\rho}(q)\tilde{\rho}(-q) = \frac{q^2}{\kappa_0^2} (a^+_{q} \cdot a^-_{-q}) . \]  

(17)

For convenience, we henceforth adopt the Coulomb gauge \( \nabla \cdot a = 0 \). In momentum space this reads \( q \cdot a_q = 0 \), which means that in this gauge the vector field \( a_q \) is purely transverse \( (a^\parallel_q = 0, \text{viz. } a_q = a^\perp_q) \). Consequently, Eq. (17) can be rewritten as
\[ \tilde{\rho}(q)\tilde{\rho}(-q) = \frac{q^2}{\kappa_0^2} (a^\perp_q \cdot a^-_{-q}) , \]

(18)
a form that will be used in the following sections.

### 3 Superfluid density calculation

The superfluid density is a quantity of paramount importance in the realm of quantum liquids. This macroscopic observable has the nature of a transport coefficient and describes the response of a superfluid system to a Galilean boost transformation. Its low-temperature behavior reflects the key properties of the ground state. In what follows, we calculate the superfluid density in the BP1 state, to be subsequently used as an input to the effective field theory in the second part of this paper. In order to make the presentation self-contained, we start from a microscopic fermion-pairing model and derive a general expression for the superfluid density by following the standard field-theoretic method of Ref. [42]. This approach has proved to yield equivalent results as that of Ref. [13]. We then specialize to the case of equal mass fermions and evaluate the superfluid density in the relevant parameter regime for the realization of the BP1 phase.

Our starting point is the microscopic Lagrangian
\[ \mathcal{L}_0 = \sum_{\sigma = \uparrow, \downarrow} \psi^*_\sigma \left( \partial_\tau - \frac{\nabla^2}{2m_\sigma} - \mu_\sigma \right) \psi_\sigma + g \psi^*_\uparrow \psi^*_\downarrow \psi_\downarrow \psi_\uparrow . \]

(19)
describing pairing of two species (denoted by a formal pseudo-spin variable \( \sigma = \uparrow, \downarrow \)) of fermions with masses \( m_\sigma \), chemical potentials \( \mu_\sigma \), and attractive inter-species contact interaction with coupling constant \( g \). In the mean-field approximation, the thermodynamic potential for this system is given by
\[ \Omega = -\frac{\Delta^2}{g} - \beta^{-1} \sum_{\omega \in \nu} \int \frac{d^3k}{(2\pi)^3} \text{tr} \ln \mathcal{G}^{-1} , \]

(20)
where
\[
\mathcal{G}^{-1} = \begin{pmatrix} i\omega_n - \epsilon_{k,\uparrow} & \Delta \\ \Delta & i\omega_n + \epsilon_{k,\downarrow} \end{pmatrix}
\]
(21)
is the inverse of the fermion propagator in the Nambu space, with free fermion dispersions \(\epsilon_{k,\sigma} = k^2/(2m_\sigma) - \mu_\sigma\) and \(\Delta\) set to be real. This fermion propagator has the matrix form
\[
\mathcal{G} = \begin{pmatrix} \mathcal{G}_{\uparrow\uparrow} & \mathcal{G}_{\uparrow\downarrow} \\ \mathcal{G}_{\downarrow\uparrow} & \mathcal{G}_{\downarrow\downarrow} \end{pmatrix}
\]
(22)
with
\[
\begin{align*}
\mathcal{G}_{\uparrow\uparrow} &= \frac{i\omega_n - \epsilon^+_k - \epsilon^-_k}{(i\omega_n - \epsilon^-_k)^2 - \epsilon^2_\Delta}, \\
\mathcal{G}_{\downarrow\downarrow} &= \frac{i\omega_n - \epsilon^-_k + \epsilon^+_k}{(i\omega_n - \epsilon^-_k)^2 - \epsilon^2_\Delta}, \\
\mathcal{G}_{\uparrow\downarrow} &= -\frac{\Delta}{(i\omega_n - \epsilon^-_k)^2 - \epsilon^2_\Delta}, \\
\mathcal{G}_{\downarrow\uparrow} &= -\frac{\Delta}{(i\omega_n - \epsilon^-_k)^2 - \epsilon^2_\Delta},
\end{align*}
\]
(23-26)
and \(\epsilon^\pm_k, \epsilon_\Delta\) defined as
\[
\epsilon^\pm_k = \frac{1}{2} (\epsilon_{k,\uparrow} \pm \epsilon_{k,\downarrow}) \quad , \quad \epsilon_\Delta = \sqrt{\epsilon^+_k + \Delta^2}.
\]
(27)
Quasiparticle energy spectrum is determined by the poles of propagator (22), i.e., by the solution of equation \(\det \mathcal{G}^{-1} = 0\):
\[
E^+_k = \epsilon_\Delta + \epsilon^-_k \quad , \quad E^-_k = \epsilon_\Delta - \epsilon^-_k.
\]
(28)
The occupation numbers of two species of fermions can be calculated from the diagonal elements of propagator (22):
\[
\begin{align*}
n_{\uparrow}(k) &= \beta^{-1} \lim_{\eta \to 0} \sum_{i\omega_n} \mathcal{G}_{\uparrow\uparrow} e^{i\omega_n \eta}, \\
n_{\downarrow}(k) &= \beta^{-1} \lim_{\eta \to 0} \sum_{i\omega_n} \mathcal{G}_{\downarrow\downarrow} e^{-i\omega_n \eta}.
\end{align*}
\]
(29-30)
Upon performing Matsubara frequency summations we obtain
\[
\begin{align*}
n_{\uparrow}(k) &= u_k^2 n_F(E^+_k) + v_k^2 n_F(-E^-_k), \quad (31) \\
n_{\downarrow}(k) &= u_k^2 n_F(E^-_k) + v_k^2 n_F(-E^+_k), \quad (32)
\end{align*}
\]
where \( n_F(z) \equiv (\exp(\beta z) + 1)^{-1} \) is the Fermi distribution function and

\[
u_k^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_k^+}{\epsilon_\Delta} \right) , \quad \nu_k^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_k^+}{\epsilon_\Delta} \right) ,
\]

are the coherence factors (squared Bogoliubov amplitudes).

Under Galilean boost with velocity \( \mathbf{v}_s \), \( \Delta \) transforms as \( \Delta \rightarrow \Delta e^{i(m_1 + m_\downarrow) \mathbf{v}_s \cdot \mathbf{x}} \), while fermion fields transform as \( \psi_\sigma \rightarrow \psi_\sigma e^{i \mathbf{q}_\sigma \cdot \mathbf{x}} \), where \( \mathbf{q}_\sigma = m_\sigma \mathbf{v}_s \). The superfluid (mass) density tensor \( \rho_{ij} \) is defined through

\[
\Omega(v_s) = \Omega(0) + j_s \cdot \mathbf{v}_s + \frac{1}{2} \rho_{ij}(v_s)(v_s)_i(v_s)_j + O(v_s^3) .
\]

For a homogeneous and isotropic superfluid this tensor is diagonal, viz. \( \rho_{ij} = \delta_{ij} \rho_s/3 \), where \( \rho_s \) is the superfluid mass density. Accordingly, the last formula reduces to

\[
\Omega(v_s) = \Omega(0) + j_s \cdot \mathbf{v}_s + \frac{1}{6} \rho_s v_s^2 + O(v_s^3) .
\]

By transforming the fermion fields and \( \Delta \) according to the rules stated above, the thermodynamic potential becomes

\[
\Omega(v_s) = -\Delta^2 g - \beta^{-1} \sum_{i\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{tr} \ln G_s^{-1} ,
\]

where \( G_s^{-1}(i\omega_n, \mathbf{k}) \) is the \( \mathbf{v}_s \)-dependent fermion inverse propagator

\[
G_s^{-1} = \begin{pmatrix} \Delta & i\omega_n - \epsilon_{k+m_\downarrow} \mathbf{v}_s, \uparrow \\ i\omega_n + \epsilon_{k-m_\downarrow} \mathbf{v}_s, \downarrow & \Delta \end{pmatrix} .
\]

It is easy to check that the latter can be expressed as

\[
G_s^{-1} = G^{-1} - (\mathbf{k} \cdot \mathbf{v}_s) 1_{2\times2} - \frac{1}{2} v_s^2 \Sigma_m ,
\]

where \( \Sigma_m = \text{diag}(m_\uparrow, -m_\downarrow) \), i.e., as

\[
G_s^{-1} = G^{-1} \left\{ 1 - (\mathbf{k} \cdot \mathbf{v}_s) G - \frac{1}{2} v_s^2 (G \Sigma_m) \right\} .
\]

By making use of the well-known expansion formula \( \ln(1-z) = z + \frac{1}{2} z^2 + O(z^3) \), we obtain

\[
\text{tr} \ln G_s^{-1} = \text{tr} \ln G^{-1} + (\mathbf{k} \cdot \mathbf{v}_s) \text{tr}(G) - \frac{v_s^2}{2} \text{tr}(G \Sigma_m) - \frac{1}{2} (\mathbf{k} \cdot \mathbf{v}_s)^2 \text{tr}(G^2) + O(v_s^3) .
\]

The last result, combined with Eq. (36), enables us to expand the thermodynamic potential \( \Omega(v_s) \) in powers of \( \mathbf{v}_s \) and read off the superfluid density from
the quadratic term:

\[ \rho_s = m_\uparrow n_\uparrow + m_\downarrow n_\downarrow + \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{3} (\sigma_{\uparrow\downarrow} + \sigma_{\downarrow\uparrow} + 2\sigma_{\uparrow\uparrow}) , \]  

(41)

where

\[ \sigma_{\uparrow\downarrow} = \beta^{-1} \sum_{\omega_{\uparrow\downarrow}} G_{\uparrow\downarrow} G_{\downarrow\uparrow} , \]  

(42)

\[ \sigma_{\downarrow\uparrow} = \beta^{-1} \sum_{\omega_{\downarrow\uparrow}} G_{\downarrow\uparrow} G_{\uparrow\downarrow} , \]  

(43)

\[ \sigma_{\uparrow\uparrow} = \beta^{-1} \sum_{\omega_{\uparrow\uparrow}} G_{\uparrow\uparrow} G_{\uparrow\uparrow} . \]  

(44)

By carrying out these Matsubara frequency summations we get

\[ \sigma_{\uparrow\downarrow} = \frac{\epsilon \Delta}{\rho} \left( n_F(E_\uparrow^+) + n_F(E_\downarrow^-) \right) \]  

(45)

\[ \sigma_{\downarrow\uparrow} = \frac{\epsilon \Delta}{\rho} \left( n_F(E_\downarrow^+) + n_F(E_\uparrow^-) \right) \]  

(46)

\[ \sigma_{\uparrow\uparrow} = \frac{1}{\epsilon \Delta} \left[ n_F(E_\uparrow^+) - n_F(E_\downarrow^-) \right] \]  

(47)

where \( n_F'(x) \equiv dn_F(x)/dx \). Using the last three equations and identity \( u_k^2 + v_k^2 = 1 \), we then obtain

\[ \rho_s = m_\uparrow n_\uparrow + m_\downarrow n_\downarrow + \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{3} \left[ n_F(E_\uparrow^+) + n_F(E_\downarrow^-) \right] \]  

(48)

Here \( n_\uparrow \) and \( n_\downarrow \) are momentum-space integrals of \( n_\uparrow(k) \) and \( n_\downarrow(k) \), respectively. In what follows, we employ formula (48) to determine the superfluid (number) density \( n_s = \rho_s/(m_\uparrow + m_\downarrow) \) at zero temperature in the special case of an equal mass system of interest in the present work.

At zero temperature \( n_F(x) = \theta(-x) \), whereby for \( m_\uparrow = m_\downarrow = m \) we readily obtain

\[ \rho_s = m(n_\uparrow^0 + n_\downarrow^0) - \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{3} \left[ \delta(E_\uparrow^-) + \delta(E_\downarrow^+) \right] \]  

(49)

where \( n_\uparrow^0 \) and \( n_\downarrow^0 \) are the zero-temperature values of \( n_\uparrow \) and \( n_\downarrow \), i.e., the respective momentum-space integrals of

\[ n_\uparrow^0(k) = u_k^2 \theta(-E_\uparrow^+) + v_k^2 \theta(E_\uparrow^-) \]  

(50)

\[ n_\downarrow^0(k) = u_k^2 \theta(-E_\downarrow^-) + v_k^2 \theta(E_\downarrow^+) \]  

(51)
The squared Bogoliubov amplitudes (coherence factors) in this special case are given by

\[ u_k^2 = \frac{1}{2} \left( 1 + \frac{k^2 - \mu}{\sqrt{(k^2 - \mu)^2 + \Delta^2}} \right), \]  

(52)

\[ v_k^2 = \frac{1}{2} \left( 1 - \frac{k^2 - \mu}{\sqrt{(k^2 - \mu)^2 + \Delta^2}} \right). \]  

(53)

After performing a trivial angular integration, on account of the fact that the upper branch \( E_k^+ \) is always positive in our case and that \( E_k^- = \varepsilon_k \), we obtain an expression for the superfluid density \( n_s = \rho_s/(2m) \):

\[ n_s = \frac{1}{2} (n^0_\uparrow + n^0_\downarrow) - \frac{1}{12\pi^2 m} \int_0^\infty |k|^4 \delta(\varepsilon_k) d|k|. \]  

(54)

We now invoke the property of Dirac’s \( \delta \) function

\[ \delta(f(x)) = \frac{1}{|f'(x)|} \sum_i \delta(x - x_i), \]  

(55)

\( x_i \) being the simple zeros of the function \( f(x) \) (i.e., \( f(x_i) = 0, f'(x_i) \neq 0 \)). This property can be equivalently stated as

\[ \int_D h(x) \delta(f(x)) dx = \sum_i \frac{h(x_i)}{|f'(x_i)|}, \]  

(56)

where the last sum extends over all the simple zeros of \( f(x) \) within the domain of integration \( D \). On account of the fact that \( |k| = k_b \) is the only zero of the function \( \varepsilon(|k|) \), simple transformations lead to

\[ n_s = \frac{1}{2} (n^0_\uparrow + n^0_\downarrow) - \frac{k^3_b}{12\pi^2} \sqrt{\frac{(k^2_b - \mu)^2 + \Delta^2}{k^2_b - \mu}}. \]  

(57)

As can straightforwardly be derived, \( n^0_\uparrow \) and \( n^0_\downarrow \) are given by

\[ n^0_\uparrow = \frac{1}{2\pi^2} \int_{k_b}^\infty |k|^2 v^2_k d|k|, \]  

(58)

\[ n^0_\downarrow = \frac{1}{2\pi^2} \left( \frac{k^3_b}{3} + \int_{k_b}^\infty |k|^2 v^2_k d|k| \right). \]  

(59)

Their calculation requires numerical evaluation of the integral

\[ \int_{k_b}^\infty |k|^2 v^2_k d|k| = \frac{1}{2} \int_{k_b}^\infty |k|^2 \left( 1 - \frac{|k|^2}{\sqrt{(|k|^2 - \mu)^2 + \Delta^2}} \right) d|k|. \]  

(60)
Finally, we eliminate $k_b$ from Eq. (57) (in favor of parameters $\mu$, $\delta$, and $\Delta$) using the identity $\sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + \Delta^2} = \delta/2$ and thereby obtain:

$$n_s = \frac{1}{2} \left( n_0^0 + n_0^\uparrow \right) - \frac{\delta}{24\pi^2} \frac{\left\{2m\left(\mu + \sqrt{\left(\frac{\delta}{\tau}\right)^2 - \Delta^2}\right)\right\}^{3/2}}{\sqrt{\left(\frac{\delta}{\tau}\right)^2 - \Delta^2}}. \quad (61)$$

The superfluid density is calculated numerically based on the derived expressions. Some typical results thereby obtained for the superfluid density as a function of the pairing gap are presented in Fig. D.2. In Fig. D.3 superfluid density is plotted as a function of the spin-polarization $P = (n_\downarrow - n_\uparrow) / (n_\downarrow + n_\uparrow)$ for different values of the pairing gap. In contrast to the related BP2 state, no anomalous negative value of the superfluid density in the BP1 phase is found, thus corroborating the dynamical stability of this phase.

4 Low-energy effective field theory

In this Section, we start from a symmetry-based low-energy effective Lagrangian for the gapless branch of fermions and the superfluid phase field. As is widely accepted, deep in the superfluid regime the dominant role is played by the superfluid phase fluctuations, while the fluctuations of the amplitude of the order parameter can be neglected (the London limit). We then derive an effective phase-only action by integrating out the fermionic degrees of freedom. The upper cutoff for the wave vector $q$ of the phase fluctuations is set by $k_\Delta = (2m\Delta)^{1/2}$, a momentum scale corresponding to the pairing gap $\Delta$. This is a consequence of the BP1 superfluid phase being realized in the strong-coupling regime on the BEC side of a Feshbach resonance, where the pairing gap is related to the binding energy of a Feshbach molecule. In more common examples of fermionic pairing (e.g., the weak-coupling regime on the atomic side of a Feshbach resonance) the momentum cutoff would have been set by $\xi^{-1} \sim \Delta/v_F$ (the inverse of the coherence length), where $v_F$ is the average Fermi velocity of the pairing fermions. It is important to point out that the magnitude $|q|$ of the wave vector of phase fluctuations is not necessarily small as compared to $k_b$. Namely, as already stated in Sec. 2.1 (recall Eq. (11)), $k_b$ is controlled by the spin population imbalance and can therefore (by tuning the population imbalance) be made arbitrarily small.

The form of our effective theory will be chosen so as to obey the Galilean invariance. In a Galilean invariant system the momentum density $T_{0i}$ (the off-diagonal part of the stress tensor) has to be equal to the mass carried by the
particle number current $J_i$, i.e.,

$$T_{bi} = mJ_i.$$  (62)

This is an example of algebraic identity between operators implementing symmetries that hold in the microscopic theory and must be retained in the effective theory. [43]

### 4.1 Parametrization of the effective theory

Throughout the analysis in this work, we shall use $k_b$ (or, alternately, the spin-polarization $P$) and the pairing gap $\Delta$ as free input parameters that will be compared to experiments. As to this choice of free parameters, a remark is in order. In a truly microscopic theory, formulated in terms of the original fermions, the pairing gap would be determined by solving the gap equation [44] together with equations specifying conservation of the total number of atoms and the population imbalance. The strength of coupling between fermions, naturally, shows up in these equations. Our theory, however, is not microscopic: we here assume the existence of the BP1 phase and construct an effective field theory for this phase. Being formulated in terms of collective rather than microscopic degrees of freedom, our effective theory does not explicitly have the coupling strength between original fermions and instead uses the pairing gap as an independent parameter. This choice is also motivated by the recent experimental developments in the field of atomic Fermi gases: it was demonstrated that using the rf-spectroscopy it is possible to measure the pairing gap by breaking fermion pairs. [45,46] Alternative methods of detecting a long-range pairing order in a degenerate Fermi gas have also been theoretically proposed, where the pairing function is directly measured in real space via a matter-wave interferometric techniques. [47]

### 4.2 Symmetry-based effective Lagrangian

The effective Lagrangian of the system ought to obey two global $U(1)$ symmetries, one of which corresponds to the total atom number conservation (to be denoted as $U_n(1)$), and the other one to the conservation of the difference in the number of atoms of spin-up and spin-down species (denoted as $U_s(1)$). Our low-energy effective Lagrangian for the gapless branch of fermions (Bogoliubov quasiparticles), described by the field $\chi(x)$, and the superfluid phase
field \( \theta(x) \) (in the imaginary-time path-integral formalism, with \( \tau = it \)) reads

\[
\mathcal{L} = \chi^* \partial_{\tau} + \varepsilon(-i\nabla)\chi + c_1 (\partial_{\tau} \theta)^2 + c_2 (\nabla \theta)^2 + c_3 \chi^* \chi \left[ i\partial_{\tau} \theta + \frac{1}{2m_p} (\nabla \theta)^2 \right] + \nabla \theta \cdot j + \ldots,
\]

and represents an extension of the theory derived by Son and Stephanov [20] to the case of an arbitrary spin population imbalance. In (63) the ellipses stand for possible higher-order derivative terms of the \( \theta \) field; \( j = (\chi^* \nabla \chi - \nabla \chi^* \chi) / (2m_p i) \) is the “paramagnetic” fermion (mass) current with \( m_p = 2m \) being the total mass of the Cooper pair; \( \varepsilon(-i\nabla) \) is the operator form of the gapless fermion dispersion (2), written in the coordinate representation. The Lagrangian has the shift symmetry \( \theta \to \theta + \alpha \), due to the \( U_c(1) \) particle number symmetry. Consequently, it contains the coordinate and time derivatives of \( \theta \), but not \( \theta \) itself.

The phenomenological parameters \( c_1, c_2 \) and \( c_3 \) are not constrained by the \( U(1) \) symmetries. While \( c_1 = \partial n / \partial \mu \) (\( n \) being the total atomic density), \( c_2 \) and \( c_3 \) are constrained by the superfluid density \( n_s \). In this regard, an important difference between the bosonic (phase) sector of our theory and the effective low-energy theories of bosonic superfluids or neutral fully-gapped superconductors (with equal spin population) ought to be pointed out. Namely, in theories of the present type, in order to satisfy the Galilean invariance represented by the constraint (62), the low-energy effective Lagrangian can depend on the phase field only through the Galilean-invariant combination \( U_\theta \equiv \partial_{\tau} \theta + \frac{1}{2m_0} (\nabla \theta)^2 \) (\( m_0 \) being the mass of an elementary superfluid constituent, e.g., the mass of a single atom in the case of \( ^4 \text{He} \) or the mass of a Cooper pair in case of fermionic superfluids), that is,

\[
\mathcal{L}_\theta = P \left( i\partial_{\tau} \theta + \frac{1}{2m_0} (\nabla \theta)^2 \right),
\]

where \( P \) stands for an arbitrary polynomial. Keeping only the terms of the lowest order in the derivatives of \( \theta \), \( \mathcal{L}_\theta \) reduces to the form \( c_1 (\partial_{\tau} \theta)^2 + c_2 (\nabla \theta)^2 \), where coefficients \( c_1 \) and \( c_2 \) are fixed by the requirement that this Lagrangian correctly describes the dynamics of the gapless Goldstone mode (Anderson-Bogoliubov mode in the case of a neutral superconductor) associated with the spontaneously broken global \( U(1) \) symmetry. [Note that the term linear in \( \partial_{\tau} \theta \) is omitted, despite being of the lowest order, since it is a total derivative and the time-dependent topological configurations are not considered.] In our case, however, with additional low-energy degrees of freedom (gapless fermion excitations), the coefficient \( c_2 \) is renormalized at every order of the effective theory and is constrained together with \( c_3 \) by an additional requirement that the superfluid density matches the one calculated from the microscopic theory. This identification will be made in the following section.

The Galilean invariance of the fermion-dependent part of this Lagrangian is ex-
plicitly demonstrated in Appendix A. An important consequence of this invar-
ance is that the coefficient of the term $\nabla \theta \cdot j$ must be unity. As a prerequisite for proving Galilean invariance, we have shown that the Bogoliubov-quasiparticle field remains invariant under Galilean transformations. The transformation law for this quasiparticle field is thus essentially different from that of the original fermions, used as a basis for an alternative effective field theory of a polarized Fermi gas in Ref. [35]. This is consistent with a quite general argument that the transformation properties for quasiparticles in the low-energy effective theories should not depend on the quantities such as the bare particle mass $m$. [48] As a by-product of this transformation law, the Bogoliubov-quasiparticle current $j$ is invariant under Galilean boosts, which is also consistent with the invariance of the quasiparticle momentum.

4.3 Effective action for phase fluctuations

Using the $\phi$ and $a$ fields via Eq.(6), the Lagrangian (63) can be rewritten as

$$\mathcal{L} = \chi^*[\partial_\tau + \varepsilon(-i\nabla)]\chi + c_1(\partial_\tau \phi)^2 + c_2(\nabla \phi - a)^2 + c_3\chi^*\chi\left[i\partial_\tau \phi + \frac{(\nabla \phi - a)^2}{2m_p}\right] + (\nabla \phi - a) \cdot j.$$ (65)

In order to arrive at an effective phase-only action $S[\theta] \equiv S[\phi, a]$, we integrate out the fermion field $\chi$:

$$e^{-S[\theta]} = \int D(\chi^*, \chi) e^{-S[\chi, \theta]},$$ (66)

where $S[\chi, \theta] \equiv S[\chi, \phi, a] = \int_0^\beta d\tau \int d\mathbf{x} \mathcal{L}$ is the Euclidean action corresponding to Lagrangian (63) (with $\beta \equiv (k_B T)^{-1}$ the inverse temperature). To this end, we first note that the fermion field enters Lagrangian (65) through a quadratic form $\chi^* K \chi = \chi^*(-G_0^{-1} + X)\chi$, where

$$G_0 = [-\partial_\tau - \varepsilon(-i\nabla)]^{-1}$$ (67)

is the noninteracting fermion propagator, and $X = X^{(1)} + X^{(2)}$ where

$$X^{(1)} = i\partial_\tau \phi + \frac{1}{2m_p} (\nabla \phi - a) \cdot \nabla,$$ (68)

$$X^{(2)} = \frac{1}{2m_p} (\nabla \phi - a)^2,$$ (69)

are respectively of the first and second orders in fields $\phi$ and $a$. 

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Integrating out the fermionic degrees of freedom gives rise to a contribution
\[ S_F[\phi, a] = -\text{tr} \ln K \] to the effective action \( S[\phi, a] \), where
\[ -\text{tr} \ln K = -\text{tr} \ln(-G_0^{-1}) - \text{tr} \ln(1 - G_0 X) . \] (70)

The contribution of the self-energy \( X \) to the effective phase-only action is evaluated by employing the usual loop-expansion of the trace: by Taylor-expanding the second term on the right-hand side of the last equation (using \( \ln(1 - z) = -\sum_{n=1}^{\infty} z^n/n \)) we obtain
\[ -\text{tr} \ln K = \text{const.} + \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}[(G_0 X)^n] . \] (71)

Using diagonality of the noninteracting fermion propagator in the momentum-frequency space \( (G_0(k, k') \equiv G_0(k)\delta_{kk'} \) with \( G_0(k) = (i\omega_n - \varepsilon_k)^{-1} \), here displayed using compact four-momentum notation : \( k \equiv (k, i\omega_n) \)), it is straightforward to show that
\[ \text{tr}(G_0 X) = \frac{1}{\beta V} \sum_k G_0(k)X_{k,k} , \] (72)
\[ \text{tr}[(G_0 X)^2] = \frac{1}{(\beta V)^2} \sum_{k,q} G_0(k)X_{k,k+q}G_0(k+q)X_{k+q,k} , \] (73)

where \( X_{k,k'} \) stands for the Fourier transform of \( X \). In order to obtain the effective action \( S[\phi, a] \) to second order in fields \( \phi \) and \( a \), we employ the above expansion to first order in \( X^{(2)} \) (tree level) and to second order in \( X^{(1)} \) (one-loop order).

The tree-level contribution of \( X^{(2)} \) to \( S_F[\phi, a] \) (and therefore to the effective phase-only action) is obtained by replacing \( \chi^*\chi \) by its average value \( \langle \chi^*\chi \rangle = n_b \). It can easily be demonstrated that
\[ X^{(1)}_{k,k'} = (\omega_n - \omega_{n'})\phi_{k-k'} + \frac{1}{2m_p}[(k+k') \cdot \{a_{k-k'} - i(k-k')\phi_{k-k'} \} , \] (74)

where \( a_k \equiv a_k\delta_{\omega_n,0} \) (the vortex gauge field is time-independent, i.e. classical). As a special case of the last equation, in the previously adopted Coulomb gauge (in which \( \mathbf{q} \cdot \mathbf{a}_{\pm q} = 0 \), hence \( \mathbf{q} \cdot \mathbf{a}_{+q} = 0 \)) we obtain
\[ X^{(1)}_{k+q,k} = -\left\{ \omega_l - \frac{i}{m_p} \mathbf{q} \cdot \left( \mathbf{k} + \frac{\mathbf{q}}{2} \right) \right\} \phi_q - \frac{1}{m_p} \mathbf{k}_\perp \cdot \mathbf{a}_q , \] (75)
\[ X^{(1)}_{k,k+q} = -\left\{ \omega_l - \frac{i}{m_p} \mathbf{q} \cdot \left( \mathbf{k} + \frac{\mathbf{q}}{2} \right) \right\} \phi_{-q} - \frac{1}{m_p} \mathbf{k}_\perp \cdot \mathbf{a}_{-q} , \] (76)

where \( q \equiv (\mathbf{q}, i\omega_l) \) and \( \mathbf{k}_\perp \equiv \{ (\mathbf{q} \times \mathbf{k}) \times \mathbf{q} \}/q^2 \) is the transverse component of the three-dimensional vector \( \mathbf{k} \) with respect to \( \mathbf{q} \). While it is easy to show that
the first order contribution (tree level) of $X^{(1)}$ is equal to zero, by inserting
the last two equations into Eq. (73) we find its contribution to $S_F[\phi, a]$ at
one-loop order.

The effective action for $\phi$ and $a$ is obtained by gathering $S_F[\phi, a]$ and the
fermion-independent terms of the original action:

$$S[\phi, a] = \int_0^\beta d\tau \int d^4 x \left[ c_1 (\partial_\tau \phi)^2 + c_2 (\nabla \phi - a)^2 \right] + S_F[\phi, a].$$  \hspace{1cm} (77)

In the momentum-frequency space, to second order in fields $\phi$ and $a$, it is
represented by the quadratic form

$$S[\phi, a] = \sum_q \left\{ \left( c_2 + \frac{m_g}{2m_p} c_3 \right) q^2 + \frac{1}{2m_p^2} R_{ij}(q) q_i q_j + \left( c_1 - \frac{\Pi(q)}{2} \right) \omega_i^2 \right\} \phi_q \phi_{-q}$$

$$+ \sum_q \left\{ c_2 + \frac{m_g}{2m_p} c_3 + \frac{P(q)}{2m_p^2} \right\} a_q \cdot a_{-q}. \hspace{1cm} (78)$$

The first term corresponds to the propagating Goldstone modes of broken $U(1)$ symmetry, and the second one to its corresponding topological defects - vortices. [Summation over repeated indices in the last equation is implicit.]

Here

$$\Pi(q) = \frac{1}{\beta V} \sum_k g_0(k) g_0(k + q)$$  \hspace{1cm} (79)

is the fermion density polarization bubble, while

$$R_{ij}(q) = \frac{1}{\beta V} \sum_k g_0(k) g_0(k + q) \left( k_i + \frac{q_i}{2} \right) \left( k_j + \frac{q_j}{2} \right),$$  \hspace{1cm} (80)

$$P(q) = \frac{1}{\beta V} \sum_k g_0(k) g_0(k + q) \frac{k^2}{2} \hspace{1cm} (81)$$

represent the longitudinal and transverse current-current correlation functions, respectively. In obtaining the form of the latter, we have made use of the identity

$$\sum_k k_{\perp \perp} k_{\perp k} F(|k|) = \frac{\delta_{ij}}{2} \sum_k k_i^2 F(|k|),$$  \hspace{1cm} (82)

valid for any rotationally-invariant function $F(|k|)$. The Matsubara frequency
sum that is implicit in all of these response functions evaluates to

$$\frac{1}{\beta} \sum_{k \neq \pm q} g_0(k) g_0(k + q) = \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+q})}{i\omega_l + \epsilon_k - \epsilon_{k+q}}. \hspace{1cm} (83)$$

It is important to point out that there is no RPA-type correction from the
interaction vertex $j \cdot \nabla \phi$ to the transverse current-current correlation function;
this is manifest in our choice of the Coulomb gauge for the topological gauge field \(a\).

As proven in Appendix D, \(R_{ij}(q) = R(q)\delta_{ij}\). Consequently, the phase-only action in Eq. (78) in the zero-temperature static limit reduces to

\[
S[\phi, a] = \sum_q \left( c_2 + \frac{n_b}{2m_p} c_3 + \frac{R^0_q}{2m_p^2} \right) q^2 \phi q \phi_{-q} \\
+ \sum_q \left( c_2 + \frac{n_b}{2m_p} c_3 + \frac{P^0_{q=0}}{2m_p^2} \right) a_q \cdot a_{-q},
\]  

where \(P^0_q\) and \(R^0_q\) are the zero-temperature static limits of \(P(q)\) and \(R(q)\), respectively.

The superfluid mass density \(\rho_s\), which plays the role of rigidity in the present problem (“spin-wave” stiffness in the XY-model terminology), \([49]\) can be identified from the long-wavelength (\(q \to 0\)) limit through the relation

\[
\frac{\rho_s}{2} = c_2 + \frac{n_b}{2m_p} c_3 + \frac{R^0_{q=0}}{2m_p^2}.
\]  

This constraint on \(c_2\) and \(c_3\) can be equivalently stated as

\[
c_2 + \frac{n_b}{2m_p} c_3 = \frac{n_s}{2m_p} - \frac{R^0_{q=0}}{2m_p^2},
\]  

and implies that the phase-only action in Eq. (84) adopts the form

\[
S[\phi, a] = \sum_q \frac{n_s}{2m_p} q^2 \phi q \phi_{-q} + \sum_q \left( \frac{n_s}{2m_p} + \frac{P^0_{q} - R^0_{q=0}}{2m_p^2} \right) a_q \cdot a_{-q},
\]

which is free of the phenomenological parameters of the original theory.

5 Effective theory for vortices and the interaction potential

Starting from the effective theory of phase fluctuations (described by (87)) and integrating out the regular (spin-wave) part of the phase field, we derive the effective action \(S_{\text{eff}}[a]\) for vortices:

\[
e^{-S_{\text{eff}}[a]} = \int D(\tilde{\phi}, \phi) e^{-S[\phi, a]}.
\]

Along these lines, a straightforward Gaussian functional integration yields the result

\[
S_{\text{eff}}[a] = \sum_q \left( \frac{n_s}{2m_p} + \frac{P^0_{q} - R^0_{q=0}}{2m_p^2} \right) a_q \cdot a_{-q}.
\]
(Because the vortex gauge field belongs to the classical sector of the theory, the derived effective action contains only the \( \omega_l = 0 \) part). With the aid of identity (18), the last result can be conveniently recast as

\[
S_{\text{eff}} = \sum_q \tilde{\rho}(q) \kappa_0^2 \left\{ \frac{n_s}{2m_p} \frac{1}{q^2} + \frac{1}{2m_p^2} \frac{P_0^0 - R_0^0}{q^2} \right\} \tilde{\rho}(-q) .
\] (90)

From the last equation we read off the momentum-space form of the effective interaction potential between the vortices :

\[
V_{\text{eff}}(q) = \kappa_0^2 \left( \frac{n_s}{2m_p} \frac{1}{q^2} + \frac{1}{2m_p^2} \frac{P_0^0 - R_0^0}{q^2} \right) .
\] (91)

In addition to the long-range component proportional to \( 1/q^2 \) (logarithmic interaction in the real space, i.e., 2D Coulomb potential), characteristic of the conventional two-dimensional charge-neutral superfluids, we have an additional component

\[
V_{\text{ind}}(q) = \kappa_0^2 \frac{P_0^0 - R_0^0}{2m_p^2} q^2
\] (92)
due to the presence of gapless fermions.

### 5.1 Properties of \( P_0^0 \) and \( R_0^0 \)

To calculate \( P_0^0 \) one has to resort to a numerical evaluation. Yet, before embarking on numerical work we can put \( P_0^0 \) into a convenient analytical form. In Appendix B we demonstrate that \( P_0^0 \) can be reduced to a two-dimensional principal-value integral

\[
P_0^0 = \frac{k_b^3}{(2\pi)^2} \mathcal{P} \int_0^1 |k|^4 \, d|k| \int_{-1}^1 dx \frac{1 - x^2}{\sqrt{\xi_k^2 + \left( \frac{\Delta_k}{\xi_k^2} \right)^2} - \sqrt{\xi_k + \frac{|q|^2}{2m} + \frac{|q||q|}{m} x^2 + \left( \frac{\Delta_k}{\xi_k^2} \right)^2}}
\] (93)

where momenta \( k \) and \( q \) are expressed in units of \( k_b \) and \( \xi_k \equiv |k|^2/2m - (\mu/k_b^2) \). The presence of the prefactor \( k_b^3 \propto (n_1 - n_l) \) indicates that in the thermodynamic limit the induced potential is proportional to the density of gapless fermions, which could have been expected on physical grounds.

In the regime of small \( |q| \) \( (|q| < 0.1k_b) \) numerical evaluation becomes rather troublesome due to the strongly singular character of the integrand in Eq. (93). However, as demonstrated in Appendix C by replacing dispersion \( \varepsilon_k \) with its linearized form \( (\varepsilon_k \rightarrow v_b(|k| - k_b)) \), for \( |q| \ll k_b \) we can derive an expression
for $P_0^q$ in the form of a controlled expansion in powers of $|q|/k_b$:

$$
P_0^q = -\frac{k_b^4}{6\pi^2 v_b} - \frac{k_b^4}{10\pi^2 v_b} \left( \frac{|q|}{k_b} \right)^2 + \mathcal{O}\left( \left( \frac{|q|}{k_b} \right)^4 \right) \quad (|q| \ll k_b).
$$

(94)

Thus in the $|q| \to 0$ limit we obtain:

$$
P_0^q \to -\frac{k_b^4}{6\pi^2 v_b} \quad (|q| \to 0).
$$

(95)

The last result can be given in a more concrete form. Applying the general expression

$$
v_b = \left| \frac{\partial \varepsilon_k}{\partial k} \right|_{|k|=k_b},
$$

(96)

to the case of dispersion (2), we find

$$
v_b = \frac{k_b}{m} \frac{k_b^2 - \mu}{\sqrt{(k_b^2 - \mu)^2 + \Delta^2}}.
$$

(97)

Inserting the last result into Eq. (95) gives

$$
P_0^q \to -\frac{mk_b^3 k^2}{6\pi^2} \sqrt{\frac{(k_b^2 - \mu)^2 + \Delta^2}{k_b^2 - \mu}} \quad (|q| \to 0).
$$

(98)

Some typical results of numerical evaluation of the response function $P_0^0$ for $0.1k_\Delta \leq |q| \leq k_\Delta$ are displayed in Fig. D.4 (where $2k_b < k_\Delta$). The salient characteristic of these results is a knee-like feature at $|q| = 2k_b$, which reflects the existence of an effective Fermi surface with diameter $2k_b$. It bears analogy to the $2k_F$-feature of the paramagnetic spin susceptibility in 3D, responsible for the RKKY indirect-exchange interaction between magnetic impurities in non-magnetic metals, [36] albeit the $2k_b$-feature found here comes from the current-current correlator so that it is different from the RKKY interaction in its dynamical origin. The values of $P_0^q$ obtained analytically in $|q| \to 0$ limit, based on Eq. (98), differ just slightly from numerical values at $|q| = 0.1k_\Delta$, indicating that $P_0^q$ can be approximated as a constant in this numerically-inaccessible region $0 < |q| < 0.1k_\Delta$. The fact that $P_0^q$ has very weak momentum dependence at small $q$ can be inferred from the coefficients in the controlled expansion of $P_0^q$ given by Eq. (94).

In Appendix D, using methodology analogous to the one employed in Appendix C, we show that $|q| \to 0$ limit of $R_0^q$ is equal to that of $P_0^q$:

$$
R_{q=0}^0 = P_{q=0}^0 = -\frac{k_b^4}{6\pi^2 v_b},
$$

(99)
whereby Eq. (92) can be recast as

$$V_{\text{ind}}(q) = \frac{\kappa_0^2 P_q^0 - P_{q=0}}{2m_p^2} q^2.$$  (100)

Now, by virtue of controlled expansion (94), we obtain that

$$V_{\text{ind}}(q) = -\frac{\kappa_0^2}{2m_p^2} k_b^2 10\pi^2 v_b + \mathcal{O}\left(\frac{|q|^2}{k_b^2}\right) \quad (|q| \ll k_b),$$  (101)

and, in particular,

$$V_{\text{ind}}(q = 0) = \int V_{\text{ind}}(r) d^2 r = 2\pi \int_0^{\infty} r V_{\text{ind}}(r) dr$$  (102)

is finite:

$$V_{\text{ind}}(q = 0) = -\frac{\kappa_0^2}{2m_p^2} k_b^2 10\pi^2 v_b.$$  (103)

### 5.2 Effective vortex interaction potential in real space

Let $F(|q|)$ be a rotationally-invariant function in momentum space and $\Lambda$ the upper momentum cutoff. The inverse two-dimensional Fourier transform of $F(|q|)$ is given by

$$F(r) = \frac{1}{(2\pi)^2} \int_{|q| \leq \Lambda} F(|q|) e^{iq \cdot r} d^2 q.$$  (104)

Using the identity

$$\int_0^{2\pi} e^{i|q|r \cos \varphi} d\varphi = 2\pi J_0(|q|r),$$  (105)

where $J_0(x)$ is the zeroth-order Bessel function of the first kind, the last equation becomes

$$F(r) = \frac{1}{2\pi} \int_{|q| \leq \Lambda} |q| F(|q|) J_0(|q|r) d|q|.$$  (106)

In our effective theory, the upper momentum cutoff is set by $k_\Delta$, thus the induced potential in real space is given by

$$V_{\text{ind}}(r) = \frac{1}{2\pi} \int_{0}^{k_\Delta} |q| V_{\text{ind}}(|q|) J_0(|q|r) d|q|,$$  (107)

viz.,

$$V_{\text{ind}}(r) = \frac{\kappa_0^2}{4\pi m_p^2} \int_{0}^{k_\Delta} \frac{P_q^0 - P_{q=0}}{|q|} J_0(|q|r) d|q|.$$  (108)

Our numerical calculations of $V_{\text{ind}}(r)$ for different values of relevant parameters $(k_b, \Delta)$ show that the induced potential has damped oscillatory character,
closely resembling the spatial dependence of the RKKY exchange integral. As can be seen from Fig. D.5 this induced potential has alternating attractive \( (dV_{\text{ind}}/dr > 0) \) and repulsive \( (dV_{\text{ind}}/dr < 0) \) parts. At short distances the induced potential is always attractive, and the first repulsive branch appears at the length scale \( r \sim (10-25)k_\Delta^{-1} \), depending on the polarization. Spatial period of the observed oscillations is set by the spin polarization, but is not so simply related to the radius of the effective Fermi surface as in the case of genuine RKKY or Friedel oscillations.

The total (effective) vortex-vortex interaction potential in real space is given by the sum of the induced potential and the conventional repulsive logarithmic potential. The latter is given by

\[
V_0(r) = -\kappa_0^2 \frac{n_s}{2m_p} \ln(k_\Delta r) ,
\]

where the superfluid density \( n_s \) is calculated in Sec. 3. As our calculations demonstrate, the effective vortex-vortex interaction shows three characteristic types of behavior, i.e. three polarization-dependent regimes. The critical polarizations corresponding to the boundaries between these different regimes are not universal but depend on the actual location in the part of the phase diagram pertaining to the BP1 phase.

In the regime of relatively low polarization, the total potential is dominated by the conventional repulsive logarithmic part; the effective vortex interaction is repulsive \( (dV_{\text{eff}}/dr < 0) \) at all distances. The resulting vortex phase is accordingly expected to be conventional, with triangular vortex arrangement. An example is shown in Fig. D.6.

In the other extreme - the regime of high polarization, the induced potential plays a dominant role at short and intermediate distances. This renders the total potential attractive at short distances, with pronounced oscillating features resembling the RKKY interaction, as illustrated in Fig. D.7. Comparison of the induced and the total vortex-vortex interaction potential in the high polarization regime is depicted in Fig. D.8.

The attractive nature of two-body interaction already at short distances suggests an instability of the vortex lattice. However, whether this instability really occurs is still an open question for the following reasons. The physics at distances shorter than the healing length \( \xi \) (to be discussed in the next section) is not captured by our effective theory; also, the multi-vortex interactions, not considered here but certainly allowed as higher orders in the effective vortex action, may support unusual vortex phases. This regime thus requires more elaborate further investigation.

Apart from the two extreme regimes already described, in a narrow window
of parameters the total potential is repulsive at short distances \((r \approx (2 - 3) k_\Delta^{-1})\) and becomes attractive at intermediate ones. This intermediate regime is illustrated in Fig. [D.9].

Due to the finite range of the RKKY-like induced potential, the truly long-distance dependence of the effective potential is governed by the infinite-range repulsive logarithmic interaction. However, for sufficiently large polarization, the effective potential is non-monotonous function of the distance between two vortices, a behavior that could potentially give rise to some exotic vortex-lattice structure. As is well known, the triangular-lattice configuration minimizes the energy of a system of point vortices interacting through a repulsive logarithmic (2D Coulomb) potential. [50] Physically, this is a consequence of the fact that triangular vortex arrangement provides maximum nearest-neighbor distance at fixed vortex density per unit area, which is a natural tendency with purely repulsive interactions (at least in the continuum, i.e., in the absence of a vortex-pinning lattice structure). Examples of such behavior can be found even in physical situations unrelated to vortices, such as the low-density limit of an electron gas, where a triangular Wigner crystal is formed. Interestingly, as can be inferred from Ref. [50], with the conventional logarithmic interaction the total energy of the triangular configuration of point-vortices is only around 0.8\% smaller than that of the square-lattice configuration. Such a small difference, however, is not very surprising given that the lattice periods of these two configurations (for the same aerial vortex densities) are also not very different, namely \(a_{tr} = \sqrt{2/\sqrt{3}} a_{sq} \simeq 1.0746 a_{sq}\). For our modified potential between vortices, which is not repulsive at all distance scales, the structure of the vortex lattice is an open issue. It is worth mentioning that a non-monotonous interaction potential between vortices (albeit without oscillating character) has recently been found in multicomponent superconductors by Babaev and Speight. [51] The authors have also predicted the existence of exotic (non-triangular) vortex-lattice structures.

In general, the interactions between topological defects mediated by the environment in which they are embedded is an important subject of current interest. Very interesting in this regard was the study of nodal-quasiparticle-induced interaction between vortices in d-wave superconductors performed by Nikolić and Sachdev. [52] They have found that the effect of quasiparticles on the effective vortex-vortex interaction (and, for that matter, some other properties of vortices) is not very dramatic. This can probably be ascribed to the nodal character of quasiparticle spectrum in d-wave superconductors, as compared to the fully-gapless situation that we are concerned with in the present work.
5.3 Experimental parameters and conditions

In order to elucidate the realm of validity of our effective theory and make contact with experiments, it is useful to estimate the physical healing (coherence) length and compare it with the inverse of the momentum scale \( k_\Delta \). In this section, we discuss different regimes where our effective theory applies or may not be relevant.

To that end, we analyze the bosonic sector of the theory. It is known from the BEC studies [53] that the healing length can be expressed as \( \xi = (8\pi n a)^{-1/2} \), where \( n \) is bosonic density and \( a \) the corresponding scattering length. Generically, this is the length scale set by the chemical potential of bosons \( (\hbar^2/(2m\xi^2) = \mu_B) \), expressed to lowest order in \( \sqrt{na^3} \). Therefore, in our case the healing length can be expressed as

\[
\xi = (8\pi n_s a_m)^{-1/2},
\]

where \( n_s \) is the superfluid density is the density of bosonic Feshbach molecules. Starting from expressions for \( \xi \) and \( k_\Delta \) we obtain

\[
\frac{\xi}{k_\Delta^{-1}} = \sqrt{\frac{2m\Delta}{8\pi n_s a_m}} = \sqrt{\frac{2m\epsilon_F(\Delta/\epsilon_F)}{8\pi n a_f(n_s/n)(a_m/a_f)}},
\]

that is,

\[
\frac{\xi}{k_\Delta^{-1}} = \frac{1}{\sqrt{8\pi}} \left( \frac{n_F}{n} \right)^{1/2} \left( \frac{a_m}{a_f} \right)^{1/2} \frac{(3\pi^2 n)^{1/3}}{\sqrt{\kappa}},
\]

with \( \epsilon_F = k_F^2/(2m) \), where \( k_F = (3\pi^2 n)^{1/3} \) is the momentum scale set by the total fermion density. The last equation can be conveniently recast as

\[
\frac{\xi}{k_\Delta^{-1}} = \frac{(3\pi^2)^{1/3}}{\sqrt{8\pi}} \left( \frac{n_F}{n} \right)^{1/2} \left( \frac{a_m}{a_f} \right)^{1/2} |\kappa|^{1/6},
\]

where

\[
\kappa \equiv -\frac{1}{na_f^3}
\]

is a dimensionless diluteness parameter. [20] [Recall the familiar results in three important limits: \( \kappa \to -\infty (+\infty) \) in the BEC (BCS) limit and \( \kappa = 0 \) at unitarity.] Eq. (112) is equivalent to

\[
\frac{\xi}{k_\Delta^{-1}} = 0.6171 \times \left( \frac{n_F}{n} \right)^{1/2} \left( \frac{a_m}{a_f} \right)^{1/2} |\kappa|^{1/6}
\]

and implies that \( \xi/k_\Delta^{-1} \) depends on three dimensionless ratios and the diluteness parameter.
To provide a quantitative estimate of the ratio $\xi/k_\Delta^{-1}$ in the parameter regime relevant for realization of the BP1 state, it is useful to recall the relevant details of the mean-field phase-diagram of a polarized Fermi gas, based on the two-channel (i.e., boson-fermion) model. This mean-field theory yields quantitatively reliable results in the narrow-resonance regime, being exact in the limit of a vanishing resonance-width. According to this phase diagram, for intermediate negative Feshbach-resonance detuning ($\nu$) BP1 (SF$_M$) exists in the region between lines $\delta_m \approx |\nu|$ (boundary to unpolarized BEC superfluid) and $\delta_\text{c1} \approx 1.3|\nu|$ (the boundary to a phase separated state – the superfluid-normal coexistence region). [Note the following difference in notation: here, the chemical potential difference is denoted as $\delta$, whereas in Ref. [19,55] it is $2\hbar$ while the detuning is denoted as $\delta$.]

It is known that in the strong-coupling BEC regime of a superfluid Fermi gas with equal populations of two hyperfine spin components (balanced Fermi mixtures) the molecular scattering length is given by $a_m = 0.6 a_f$ ($a_f$ being the scattering length between fermionic atoms). For a polarized Fermi gas, however, as shown by Sheehy and Radzihovsky (see Eq. 16 and Fig. 2 of Ref. [21]), the molecular scattering length decreases monotonously as a function of $\delta/|\nu|$ (or, equivalently, of the polarization) and vanishes at the aforementioned boundary of first-order phase transition to the phase separated state. Therefore, as follows from Eq. (114), right at the boundary to phase separation and in the immediate vicinity of it the coherence length becomes much greater than $k_\Delta^{-1}$, thus making the quantitative implications of our theory not directly applicable in this special case.

Taking $a_m(\delta = 0)$ in place of $a_m$, together with typical values of $n_s/n$ and $\Delta/\epsilon_F$ ($\epsilon_F = k_F^2/(2m)$, where $k_F = (3\pi^2 n)^{1/3}$ is the momentum scale set by the total fermion density) in the BEC regime, we estimate that $\xi$ is of the same order as $k_\Delta^{-1}$ when $|\kappa| \sim 1-100$. From numerical results for $a_m(|\nu|, \delta)/a_m(|\nu|, \delta = 0)$ obtained in Ref. [21], on account of the fact that $\xi/k_\Delta^{-1} \propto (a_m/a_f)^{-1/2}$, we can infer that the above estimate is just slightly modified as a result of $a_m$ decreasing as a function of $\delta/|\nu|$: for example, for $\delta/|\nu| = 1.2$ the true molecular scattering length is an order of magnitude smaller than that of the unpolarized system, but the ratio $\xi/k_\Delta^{-1}$ is modified only by a factor of $\sqrt{10} \approx 3.16$. For smaller values of $\delta/|\nu|$ this factor is even smaller, i.e., it is of the order of unity. Thus, this estimate confirms that our choice of $k_\Delta$ as the upper momentum cutoff of the theory is physically pertinent.

Moreover, using the expression of $\Delta/\epsilon_F$ in Ref. [5], we can straightforwardly infer that in the BEC limit $\xi/k_\Delta^{-1} \sim |\kappa|^{1/4}$. It follows that $\xi/k_\Delta^{-1} \to \infty$ in the BEC limit, which seems to suggest that this limit is out of the application scope of our theory, since the latter is intrinsically valid for physics at distances longer than $\xi$. It is, however, important to emphasize that the effect of gapless fermions on the interaction between vortices is not even expected
to bear any physical relevance in the BEC limit, where the system at hand essentially becomes a Bose-Fermi mixture akin to the $^3\text{He}-^4\text{He}$ mixture. Technically speaking, this point is manifest in the Nishida-Son formulation of the effective Lagrangian for the imbalanced Fermi gas, through a vanishing coupling coefficient between the fermion current and the gradient of phase field (supercurrent) in this limit. In our case, largeness of the physically-allowed inter-vortex distance scale ($r \gtrsim \xi$) compared to $k_\Delta^{-1}$ in the BEC limit and the fact that at very long distances (compared to $k_\Delta^{-1}$) the fermion-induced part of the vortex-vortex interaction is quantitatively unimportant compared to the conventional (infinitely-ranged) repulsive logarithmic contribution are indeed suggesting that the physical effect under consideration is absent in this limit. This is an important consistency check of our results.

6 Summary and conclusions

In summary, starting from a Lagrangian for the superfluid phase field and the gapless branch of fermionic quasiparticles, we have obtained the effective action for vortices in a spin-polarized homogeneous superfluid state with a single gapless Fermi surface. We have demonstrated that besides the conventional repulsive logarithmic part (2D Coulomb potential) the effective vortex interaction potential has an additional, predominantly attractive, component induced by the presence of gapless fermions. This fermion-induced potential has oscillating character analogous to the RKKY indirect-exchange interaction. Interactions between defects mediated by the continuum they are immersed in (either bosonic or fermionic) have been studied quite recently in several different physical contexts and different dimensionalities. Our work, however, constitutes the first study of this kind that concerns the interaction between vortices in superfluids. It shows that besides the Friedel oscillations (charge sector) and the RKKY (spin sector), an analogous oscillating phenomenon appears in the vortex sector.

Our study opens up a question as to the nature of the vortex lattice in gapless fermionic superfluids. Due to the partly attractive nature of the effective vortex potential that we have found, the resulting vortex lattice structure in BP1 superfluid phase could be different than the triangular lattice, which would be a spectacular experimental signature. The complexity of the problem, however, calls for an elaborate future study. Even when the potential has a unique distance dependence of a known analytical form (such as, for example, the conventional logarithmic interaction), calculation of the resulting lattice structure is quite a nontrivial task, since lattice-summation-methods for long-range potentials are strongly dependent on the actual form of the potential. The new vortex-vortex interaction potential is not obtained, due to the complexity of the problem, in a closed analytical form and has both attractive
and repulsive parts. This unusual, non-monotonous, distance dependence of the effective potential implies that vortex lattice structure may in fact not be unique, but also depend on the geometrical constraints on the system, for example, the range of distances between individual vortices realized for a given size of the superfluid container. The standard lattice summation methods may not be applicable and more sophisticated strategies need to be employed, for instance Monte Carlo calculations.

In the present work we have studied the intrinsic effect of gapless fermionic excitations on the interaction between vortices in the BP1 state and have therefore considered only the homogeneous case. Our results are expected to be also valid for a trapped system as long as the trap potential varies smoothly on the scale of the Fermi wavelength (or, more generally, the longest physical length-scale in the problem), that is, in the regime of validity of the local density approximation. However, an important problem yet to be explored is the possible influence of strong spatial inhomogeneities caused by the presence of the trap on the form of the vortex lattice, as studied by Sheehy and Radzihovsky in the context of trapped Bose gases. [60] They have provided an explanation for the striking uniformity of the vortex lattices seen in experiments in spite of the strong spatial variation of the local superfluid density imposed by the trap. Moreover, they have shown that an interplay of an inhomogeneous trap potential and vortex discreteness leads to a vortex density that is largest in the center of the trap, a counterintuitive result from the energetic point of view because both the kinetic energy cost and the repulsive interaction between vortices are proportional to the local superfluid density and are therefore largest in the center of the trap. As we have shown in the present study of a spin-polarized Fermi gas, for sufficiently high polarization the effective interaction between vortices in this system is attractive at short distances and could therefore bring about some completely new effects, such as the competition between this attractive interaction and the kinetic energy cost. Further investigation of the properties of “vortex matter” in spin-polarized Fermi gases is thus clearly called for.

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A Galilean invariance

In this Appendix, we explicitly demonstrate the Galilean invariance of the fermion-dependent part of Lagrangian (63). We shall first establish an explicit relation of the quasiparticle field in the present effective field theory to the fermion particle field in a microscopic model, and then derive the Galilean transformation properties of the quasiparticle field from that of the (microscopic) fermion fields. Subsequently, an alternative approach will be given to provide a further justification and understanding.

A.1 Microscopic relation of the quasiparticle field

To examine the Galilean transformation of the Bogoliubov quasiparticle field, let us consider a microscopic model of Lagrangian (19). As a result of the Hubbard-Stratonovich transformation in the Cooper channel, introducing the auxiliary pair field $\Delta(x)$, this Lagrangian changes to

$$\tilde{\mathcal{L}} = \psi^*_{\sigma} \left( \partial_{\tau} - \frac{\nabla^2}{2m_\sigma} - \mu_\sigma \right) \psi_{\sigma} + (\psi^*_\uparrow \psi^*_\downarrow \Delta(x) + c.c) + \frac{1}{g} |\Delta(x)|^2 .$$

(A.1)

[Summation over repeated pseudo-spin indices in the last equation is implicit.]

Ignoring fluctuations of the amplitude of the order parameter, i.e., assuming that $\Delta(x) = \Delta e^{i\theta(x)}$, it is advantageous to transform the fermion fields at each space-time point as

$$\psi_{\sigma}(x) = \tilde{\psi}_{\sigma}(x) e^{i\frac{\theta(x)}{2}} , \quad \psi^*_{\sigma}(x) = \tilde{\psi}^*_{\sigma}(x) e^{-i\frac{\theta(x)}{2}} .$$

(A.2)

This local (gauge) transformation is designed to transform away the phase-fluctuation dependence from the off-diagonal pairing potential terms to the diagonal (kinematic) terms in the fermion sector of the theory. As a result, the $\tilde{\psi}_{\sigma}$ fermion fields are locally stripped off of any dependence on the $U(1)$ phase $\theta(x)$. The transformed Lagrangian can be written as

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 + \mathcal{L}_{\tilde{\psi},\theta} ,$$

where

$$\tilde{\mathcal{L}}_0 = \tilde{\psi}^*_{\sigma} \left( \partial_{\tau} - \frac{\nabla^2}{2m_\sigma} - \mu_\sigma \right) \tilde{\psi}_{\sigma} + (\Delta \tilde{\psi}^*_\uparrow \tilde{\psi}^*_\downarrow + c.c)$$

(A.3)

is the mean-field Lagrangian for $\tilde{\psi}_{\sigma}$ fermions, and

$$\mathcal{L}_{\tilde{\psi},\theta} = \tilde{\psi}^*_\sigma \tilde{\psi}_\sigma \left( i \partial_{\tau} \theta + \frac{1}{2m_\sigma} (\nabla \theta)^2 \right) - \frac{i}{2m_\sigma} \left( \tilde{\psi}_\sigma \nabla \tilde{\psi}_\sigma - \nabla \tilde{\psi}^*_\sigma \tilde{\psi}_\sigma \right) \cdot \nabla \theta .$$

(A.4)

With $\tilde{\mathcal{L}}_0$ naturally giving rise to the Bogoliubov quasiparticles as its elementary excitations, $\mathcal{L}_{\tilde{\psi},\theta}$ essentially contains, in an implicit form, all the couplings of these excitations to the superfluid phase fluctuations.
A cautious remark is needed for the Lagrangian derived above. It appears that we have just provided a derivation for the postulated effective Lagrangian \((63)\). One may be tempted to determine the “phenomenological” coefficients of this Lagrangian in this way. For weak coupling, this can indeed be done. For a strongly interacting Fermi gas, the derivation from the microscopic model cannot be done in a controlled approximation, once the pairing amplitude and density fluctuations are included. The symmetry-based Lagrangian of the postulated form \((63)\) describes the same physics, albeit from a more phenomenological point of view. Moreover, it does not suffer from the difficulty in strong coupling. In summary, the above derivation is understood to provide an example of how to separate the low energy Goldstone bosons (the phase fluctuation) from other degrees of freedom, but not a rigorous proof of the effective Lagrangian itself on a microscopic level.

### A.2 Galilean transformation for quasiparticles

Let us denote the laboratory frame as \(K\) and the corresponding spatial and time coordinates as \(x\) and \(t\). We shall also denote a frame moving with velocity \(u\) relative to \(K\) as \(K'\), and its spatial and time coordinates as \(x'\) and \(t'\). Under Galilean boost transformation with velocity \(u\) the space-time coordinates transform as:

\[
x 
\rightarrow
\begin{align*}
x' & = x - ut, \\
t & = t'
\end{align*}
\]

The spatial and time derivatives transform as

\[
\nabla \rightarrow \nabla' = \nabla, \quad \partial_\tau \rightarrow \partial_\tau' = \partial_\tau - i(u \cdot \nabla).
\]

Under the above Galilean transformation, the (microscopic) fermion field of mass \(m\) transforms in the standard way,

\[
\psi(x) \rightarrow \psi'(x') = e^{i(-mu \cdot x + \frac{1}{2}mu^2t)}\psi(x)
\]

where \(x \equiv (x, \tau = it)\). Being locally stripped off of any ‘charge’ \(U_c(1)\) phase dependence, the \(\tilde{\psi}_\sigma\) fermion fields are by construction invariant under Galilean transformation:

\[
\tilde{\psi}_\sigma(x) \rightarrow \tilde{\psi}'_\sigma(x') = \tilde{\psi}_\sigma(x),
\]

and the ‘charge’ \(U_c(1)\) phase transforms

\[
\theta(x) \rightarrow \theta'(x') = \theta(x) - 2mu \cdot x - imu^2\tau.
\]

The Bogoliubov quasiparticles (the two branches being denoted by \(\chi_\uparrow\) and \(\chi_\downarrow\)) can now be introduced as

\[
\chi_\uparrow(k, \tau) = u_k \tilde{\psi}_1(k, \tau) + v_k \tilde{\psi}_1^*(-k, \tau),
\]

\[
\chi_\downarrow(k, \tau) = u_{-k} \tilde{\psi}_1(k, \tau) + v_{-k} \tilde{\psi}_1^*(-k, \tau)
\]
where \( u_k, v_k \) are Bogoliubov amplitudes. For the lower branch quasiparticle field, which is simply denoted by \( \chi \) (i.e., \( \chi \equiv \chi_\downarrow \)), this becomes in real space

\[
\chi(x, \tau) = \int dy \left[ u(y - x) \bar{\psi}_\downarrow(y, \tau) - v(x - y) \bar{\psi}^*_\uparrow(y, \tau) \right]
\]

(A.11)

By using (A.7), we find

\[
\chi(x) \rightarrow \chi'(x') = \chi(x).
\]

(A.12)

From the transformation properties of \( \chi \) and \( \theta \), it is straightforward to prove the Galilean-invariance of the term

\[
\chi^* \chi \left\{ i \partial_\tau \theta + \frac{1}{2mp} (\nabla \theta)^2 \right\} = \chi^* U_\theta
\]

in Lagrangian (63). Besides, using relations (A.8) and (A.5) (the latter implies that \( \varepsilon(-i\nabla) \) is an invariant), we can readily prove that the combination

\[
\chi^* [\partial_\tau + \varepsilon(-i\nabla)] \chi + \nabla \theta \cdot \mathbf{j}
\]

is Galilean invariant up to an unimportant total derivative. This proves the Galilean invariance for the fermion-dependent part of Lagrangian (63).

A.3 The Doppler shift

An alternative check of the Galilean invariance of the quasiparticle field can be obtained by starting from the requirement that the quasiparticle energy is Doppler shifted under a Galilean boost.

We first review a standard derivation of the Galilean transformation [62]. Recall how momentum and energy of particles with quadratic dispersion (e.g., bare fermions) transform under this Galilean boost:

\[
p \rightarrow p' = p - mu, \quad E \rightarrow E' = E - p \cdot u + \frac{1}{2}mu^2.
\]

(A.15)

Using these rules, it is straightforward to show that the combination \( p \cdot x - Et \) shifts by a factor \( -mu \cdot x + \frac{1}{2}mu^2 t \), which depends on the parameters of the transformation \((m, u)\) but does not depend on \( p \). Accordingly, every plane wave

\[
\varphi_p(x, t) = \text{const} \times e^{i(p \cdot x - Et)}
\]

(A.16)

acquires the same phase factor \( \exp[i(-mu \cdot x + \frac{1}{2}mu^2 t)] \) under the Galilean boost, regardless of \( p \). Moreover, since an arbitrary single particle wavefunction can be expanded in plane-waves (A.16), we conclude that each wave function picks up that same phase factor under this boost. Because Galilean
transformations are space-time symmetry transformations, the transformation property of the single-particle wave-function carries over to the field operators

$$\hat{\psi}(x, t) = \sum_n \hat{a}_n \phi_n(x, t) \ , (A.17)$$

where $\phi_n(x, t)$ form an arbitrary complete orthonormal set of single particle states.

Bearing in mind the definition (5), as a by-product of the transformation rule found above, we conclude that the superfluid phase field is transformed as

$$\theta(x, t) \longrightarrow \theta'(x', t') = \theta(x, t) - m_p u \cdot x + \frac{1}{2} m_p u^2 t \ , (A.18)$$

that is, we recover transformation (A.8) when changing over to the imaginary time. By making use of the transformation properties (A.5) it is straightforward to show that the combination

$$U_\theta = i \partial_\tau \theta + \frac{1}{2m_p} (\nabla \theta)^2$$

(A.19)

is invariant under the Galilean transformation.

In order to determine how the Bogoliubov quasiparticle field transforms under the Galilean transformation, we recall that the momentum of a quasiparticle is invariant under the Galilean transformation while the quasiparticle energy is Doppler-shifted (to leading order in the boost velocity):

$$p \longrightarrow p' = p \ , \quad E \longrightarrow E' = E - p \cdot u \ . \ (A.20)$$

Based on these properties, it is easy to demonstrate that for Bogoliubov quasiparticles the combination $p \cdot x - Et$ remains invariant under the Galilean transformations (independent of $p$ and $E$), which using analogous reasoning as above implies that an arbitrary single-particle wave-function and the field operator $\hat{\chi}(x, t)$ of a Bogoliubov quasiparticle is invariant under Galilean transformations:

$$\hat{\chi}(x, t) \longrightarrow \hat{\chi}'(x', t') = \hat{\chi}(x, t) \ . \ (A.21)$$

This is the equivalent form of (A.12) in operator formalism.

B Expression for the transverse current response function

Because $\varepsilon_k > 0$ for $|k| > k_b$ and $n_F(\varepsilon) \rightarrow \theta(-\varepsilon)$ as $T \rightarrow 0$, in the zero-temperature static limit the response function $P(q)$ (defined by Eq. (81))
By virtue of the Sohatsky-Plemelj formula, we can now demonstrate that $\text{Im}\{P\}$ reduces to
\[ P^0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\epsilon_k - \epsilon_{k+q} + i\eta}{\epsilon_k - \epsilon_{k+q} + i\eta} \right] k_{\perp}^2 , \]
where $\eta \to 0+$ and the momentum sum in $\text{(B.1)}$ has been replaced by an integral. We now undertake the change of variables $k' = -k - q$ in the second term of the last equation. Because $k_{\perp}' = -k_{\perp}$, we have that $k_{\perp}' = k_{\perp}^2$. In other words, $k_{\perp}^2$ is invariant under this change of variables. Consequently, we arrive at
\[ P^0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_b)\theta(k_b - |k|) \left[ \frac{1}{\epsilon_k - \epsilon_{k+q} + i\eta} - \frac{1}{\epsilon_{k+q} - \epsilon_k - i\eta} \right] k_{\perp}^2 , \]
where the superfluous prime has been omitted (i.e., we have returned to the initial integration variable $k$). The last equation can obviously be simplified to
\[ P^0 = \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_b)\theta(k_b - |k|) \frac{k^2_{\perp}}{\epsilon_k - \epsilon_{k+q} + i\eta} \quad (\eta \to 0+) \, . \]

By virtue of the Sohatsky-Plemelj formula
\[ \lim_{\eta \searrow 0} \frac{1}{x \pm i\eta} = \mathcal{P} \frac{1}{x} \mp i\pi\delta(x) \, , \]
we can now demonstrate that $\text{Im}\{P^0\} = 0$ and that $\text{Re}\{P^0\} = P^0$ is given by
\[ P^0 = \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \theta(|k + q| - k_b)\theta(k_b - |k|) \frac{k^2_{\perp}}{\epsilon_k - \epsilon_{k+q}} \, , \]
where $\mathcal{P}$ stands for the Cauchy principal value. Using the identity $\theta(x) = 1 - \theta(-x)$ for $x = |k + q| - k_b$, the last equation becomes
\[ P^0 = \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \left[ 1 - \theta(k_b - |k + q|) \right] \theta(k_b - |k|) \frac{k^2_{\perp}}{\epsilon_k - \epsilon_{k+q}} \, . \]

The term that contains the product of two step functions vanishes identically after the integration, because this product is even under the interchange $k \leftrightarrow k + q$, while the fraction $k^2_{\perp}/(\epsilon_k - \epsilon_{k+q})$ is odd under the same transformation (while $\epsilon_k - \epsilon_{k+q}$ is obviously odd, the fact that $k_{\perp} = (k + q)_{\perp}$ implies that $k^2_{\perp}$ is even); accordingly, we have
\[ P^0 = \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \theta(k_b - |k|) \frac{k^2_{\perp}}{\epsilon_k - \epsilon_{k+q}} \, . \]

With the aid of identity $k^2_{\perp} = |k|^2(1 - \cos^2 \theta)$ and momentum re-scaling $k/k_b \to k$ (such that all momenta are expressed in units of $k_b$) we express $P^0$ as a
principal-value integral over the dimensionless momentum:

$$P^0_q = \frac{k_b^3}{(2\pi)^2} \mathcal{P} \int_0^1 |k|^4 d|k| \int_0^\pi \frac{1 - \cos^2 \theta}{\varepsilon_k - \varepsilon_{k+q}} \sin \theta d\theta . \quad (B.8)$$

Finally, upon inserting dispersion (\ref{B.2}) and making substitution $x = \cos \theta$, this integral leads to

$$P^0_q = \frac{k_b^3}{(2\pi)^2} \mathcal{P} \int_0^1 |k|^4 d|k| \int_{-1}^1 dx \frac{1 - x^2}{\sqrt{\xi_k^2 + \left(\frac{x}{k_b}\right)^2} - \sqrt{\left(\xi_k + \frac{|q|^2}{2m} + \frac{|k||q|}{m} x\right)^2 + \left(\frac{x}{k_b}\right)^2}}, \quad (B.9)$$

where $\xi_k \equiv |k|^2/2m - (\mu/k_b^2)$.

\section*{C Behavior of $P^0_q$ for $|q| \ll k_b$ and the $|q| \to 0$ limit}

The most general expression for $P(q) \equiv P(q, i\omega_l)$ reads

$$P(q, i\omega_l) = \frac{1}{2V} \sum_k \frac{n_F(\varepsilon_k) - n_F(\varepsilon_{k+q})}{i\omega_l + \varepsilon_k - \varepsilon_{k+q}} k_\perp^2 , \quad (C.1)$$

that is

$$P(q, i\omega_l) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{n_F(\varepsilon_k) - n_F(\varepsilon_{k+q})}{i\omega_l + \varepsilon_k - \varepsilon_{k+q}} k_\perp^2 . \quad (C.2)$$

In order to calculate $P^0_q$ for $|q| \ll k_b$, we start from the expansion

$$n_F(\varepsilon_k) - n_F(\varepsilon_{k+q}) = \frac{\partial n_F(\varepsilon_k)}{\partial \varepsilon_k} (\varepsilon_k - \varepsilon_{k+q}) + O(|q|^2) , \quad (C.3)$$

valid for $|q| \ll k_b$. At zero temperature $n_F(\varepsilon) = \theta(-\varepsilon)$, implying that $\partial n_F(\varepsilon)/\partial \varepsilon = -\delta(\varepsilon)$. For linearized dispersion $\varepsilon_k = v_b(|k| - k_b)$, using the fact that $\delta(cx) = \delta(x)/|c|$, we find

$$n_F(\varepsilon_k) - n_F(\varepsilon_{k+q}) = (|k + q| - |k|) \delta(|k| - k_b) + O(|q|^2) . \quad (C.4)$$

Here $k \cdot q = |k||q| \cos \theta$, and consequently $|k + q| = (|k|^2 + 2|k||q| \cos \theta + |q|^2)^{1/2}$.

On account of result (\ref{C.4}), together with $k_\perp^2 = |k|^2(1 - \cos^2 \theta)$, Eq. (\ref{C.2}) leads to an integral (trivial integration over the azimuthal angle yields factor $2\pi$)

$$P(q, i\omega_l) \simeq \frac{1}{2(2\pi)^2} \int_0^\infty |k|^4 d|k| \int_{-1}^1 d(\cos \theta) \frac{(1 - \cos^2 \theta)(|k + q| - |k|) \delta(|k| - k_b)}{i\omega_l - v_b(|k + q| - |k|)} . \quad (C.5)$$
Upon executing the integral over $|k|$ and introducing substitution $x = \cos \theta$, we arrive at

$$P(q, i\omega_l) \simeq \frac{k_b^4}{2(2\pi)^2} \int_{-1}^{1} \left(1 - x^2\right) \left(\frac{\sqrt{k_b^2 + 2k_b|q|x + |q|^2} - k_b}{\sqrt{k_b^2 + 2k_b|q|x + |q|^2} - k_b}\right) dx . \quad (C.6)$$

Another variable substitution $t = \sqrt{k_b^2 + 2k_b|q|x + |q|^2}$ turns the last integral into

$$P(q, i\omega_l) \simeq \frac{k_b}{8(2\pi)^2v_b|q|^3} \int_{k_b - |q|}^{k_b + |q|} \frac{t(t - k_b)\{(t^2 - k_b^2 - |q|^2)^2 - (2k_b|q|)^2\}}{t - k_b - i\frac{\omega_l}{v_b}} dt . \quad (C.7)$$

By carrying out this integral and taking the static limit $\omega_l \to 0$, we obtain the result (without the prefactor) $-\frac{16\pi^3}{3}k_b^3|q|^3$, implying that the first order term in expansion (C.3) yields the $q$-independent contribution

$$- \frac{k_b^4}{6\pi^2v_b} \quad (C.8)$$

to $P^0_q$. In a similar manner, lengthy but otherwise straightforward calculation shows that the next (second-order) term in expansion (C.3), namely

$$\frac{1}{2} \frac{\partial^2 n_F(\varepsilon_k)}{\partial \varepsilon_k^2} (\varepsilon_k - \varepsilon_{k+q})^2 = -\frac{1}{2}(|k + q| - |k|)^2 \delta'(|k| - k_b) , \quad (C.9)$$

adds the contribution

$$- \frac{k_b^2}{10\pi^2v_b} |q|^2 + \frac{1}{420\pi^2v_b} |q|^4 . \quad (C.10)$$

Therefore, for $|q| \ll k_b$ this response function is given by

$$P^0_q = - \frac{k_b^4}{6\pi^2v_b} - \frac{k_b^2}{10\pi^2v_b} |q|^2 + \mathcal{O}(|q|^4) , \quad (C.11)$$

implying that

$$P^0_q \to - \frac{k_b^4}{6\pi^2v_b} \quad ( |q| \to 0 ) . \quad (C.12)$$

D Calculation of $R_{ij}^0(q)$ in the $|q| \to 0$ limit

The most general expression for $R_{ij}(q) \equiv R_{ij}(q, i\omega_l)$ reads

$$R_{ij}(q, i\omega_l) = \frac{1}{V} \sum_k \frac{n_F(\varepsilon_k) - n_F(\varepsilon_{k+q})}{i\omega_l + \varepsilon_k - \varepsilon_{k+q}} \left(k_i + \frac{q_i}{2}\right) \left(k_j + \frac{q_j}{2}\right) , \quad (D.1)$$
that is

\[ R_{ij}(\mathbf{q}, i\omega_l) = \int \frac{d^3k}{(2\pi)^3} \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+\mathbf{q}})}{i\omega_l + \epsilon_k - \epsilon_{k+\mathbf{q}}} \left( k_i + \frac{q_i}{2} \right) \left( k_j + \frac{q_j}{2} \right) . \]  

(D.2)

We first show that \( R_{ij}(\mathbf{q}, i\omega_l) = 0 \) for \( i \neq j \). To that end, we perform a rotation of the coordinate system around the \( z \)-axis that maps the \( x \)-axis onto the \( y \)-axis and the \( y \)-axis onto \(-x\). Knowing that the module of the jacobian of this transformation (rotation) is unity and that \( \epsilon_{\mathbf{k}} \) depends only on \(|\mathbf{k}|\) (which is invariant under this transformation) we obtain that \( R_{xy}(q) = -R_{yx}(q) \) and \( R_{yz}(q) = -R_{yz}(q) \), which implies that \( R_{xy}(q) = R_{yx}(q) = 0 \).

In order to calculate

\[ R_{ii}(\mathbf{q}, i\omega_l) = \int \frac{d^3k}{(2\pi)^3} \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+\mathbf{q}})}{i\omega_l + \epsilon_k - \epsilon_{k+\mathbf{q}}} \left( k_i + \frac{q_i}{2} \right)^2 \]  

(D.3)

we perform a rotation of the coordinate system that maps the \( i \)-axis onto the \( z \)-axis, while leaving the remaining axis invariant. \( R_{ii}(\mathbf{q}, i\omega_l) \) then becomes

\[ R_{ii}(\mathbf{q}, i\omega_l) = \int \frac{d^3k'}{(2\pi)^3} \frac{n_F(\epsilon_{k'}) - n_F(\epsilon_{k'+\mathbf{q}})}{i\omega_l + \epsilon_{k'} - \epsilon_{k'+\mathbf{q}}} \left( k_z' + \frac{|\mathbf{q}|}{2} \right)^2 \]  

(D.4)

for both \( i = x \) and \( i = y \). Thus \( R_{xx}(q) = R_{yy}(q) = R(q) \), and since the last integral can depend only on \(|\mathbf{q}|\) we can choose \( \mathbf{q} \) to lie along the \( z \)-axis, in which case \( R(q) \) can be expressed as

\[ R(\mathbf{q}, i\omega_l) = \int \frac{d^3k}{(2\pi)^3} \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+\mathbf{q}})}{i\omega_l + \epsilon_k - \epsilon_{k+\mathbf{q}}} \left( k_z + \frac{|\mathbf{q}|}{2} \right)^2 , \]  

(D.5)

i.e. as

\[ R(\mathbf{q}, i\omega_l) = \int \frac{d^3k}{(2\pi)^3} \frac{n_F(\epsilon_k) - n_F(\epsilon_{k+\mathbf{q}})}{i\omega_l + \epsilon_k - \epsilon_{k+\mathbf{q}}} \left( |\mathbf{k}| \cos \theta + \frac{|\mathbf{q}|}{2} \right)^2 . \]  

(D.6)

By employing transformations analogous to (C.3)-(C.7) in the calculation of \( P_q^0 \) we arrive at the expression for \( R(\mathbf{q}, i\omega_l) \) in the zero-temperature limit:

\[ R(\mathbf{q}, i\omega_l) \simeq \frac{k_b}{8(2\pi)^2 v_b |\mathbf{q}|^3} \int_{k_b-|\mathbf{q}|}^{k_b} \frac{t(t-k_b)(t^2-k_b^2)^2}{t-k_b-i\omega_l/v_b} dt . \]  

(D.7)

By carrying out this integral and taking the static limit \( \omega_l \to 0 \), we obtain

\[ R_q^0 \to -\frac{k_b^4}{6\pi^2 v_b} \left( |\mathbf{q}| \to 0 \right) . \]  

(D.8)
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FIG. D.1. An example of gapless fermion quasiparticle dispersion $\varepsilon_k$. Values of parameters $\delta$ and $\Delta$ are indicated (expressed in units of $|\mu|$).
FIG. D.2. Superfluid density in units of the total atomic density $n = n_\uparrow + n_\downarrow$ for three different values of the chemical potential mismatch $\delta$. Both $\delta$ and $\Delta$ are expressed in units of $|\mu|$. 
FIG. D.3. Superfluid density as a function of spin polarization for different values of the pairing gap $\Delta$ (expressed in units of $|\mu|$).
FIG. D.4. Transverse current response function $P_{q}^{0}$ as a function of dimensionless momentum, for $m = 1.0$ and $\Delta / |\mu| = 2.0$. Values of $k_b$ are given in units of $k_\Delta$. 
FIG. D.5. Induced vortex interaction potential in real space (in units of $\Delta$), for $m = 1.0$ and $\Delta/|\mu| = 1.0$. Values of $k_b$ are given in units of $k_\Delta$: $k_b = 1.352; 1.640$ correspond to polarizations $P = 0.702; 0.808$, respectively.
FIG. D.6. Effective vortex interaction potential in real space (in units of $\Delta$), for $m = 1.0$ and $\Delta/|\mu| = 2.0$. Values of $k_b$ are given in units of $k_\Delta$: $k_b = 0.623; 0.724; 0.826$ correspond to polarizations $P = 0.155; 0.227; 0.314$, respectively.
FIG. D.7. Effective vortex interaction potential in real space (in units of $\Delta$), for $m = 1.0$ and $\Delta/|\mu| = 1.0$. Values of $k_b$ are given in units of $k_\Delta$: $k_b = 1.500; 1.352; 1.205$ correspond to polarizations $P = 0.763; 0.702; 0.626$, respectively.
FIG. D.8. Comparison of the induced and the effective vortex interaction potential in real space (in units of $\Delta$), for $m = 1.0$, $\Delta/|\mu| = 2.0$. Values of $k_b$ (in units of $k_\Delta$) and $P$ are indicated in the plot.
FIG. D.9. Effective vortex interaction potential in real space (in units of $\Delta$), for $m = 1.0$ and $\Delta/|\mu| = 1.0$. $k_b = 1.022$ (in units of $k_\Delta$) corresponds to $P = 0.504$. 

$\Delta = 1.0$

$k_b = 1.022$

$n_s / n = 0.219$