Quarkonium Masses in a hot QCD Medium Using Conformable Fractional of the Nikiforov-Uvarov Method

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Abstract

By using conformable fractional of the Nikiforov-Uvarov (CF-NU) method, the radial Schrödinger equation is analytically solved. The energy eigenvalues and corresponding functions are obtained, in which the dependent temperature potential is employed. The effect of fraction-order parameter is studied on heavy-quarkonium masses such as charmonium and bottomonium in a hot QCD medium in the 3D and the higher dimensional space. A comparison is studied with recent works. We conclude that the fractional-order plays an important role in a hot QCD medium in the 3D and higher-dimensional space.

Keywords: Fractional derivative, Schrödinger equation, Nikiforov-Uvarov method, finite temperature, quarkonium spectra
I. INTRODUCTION

The color screening of the static chromo-electric fields is one of the most remarkable features of the quark-gluon plasma formation. The Debye screening in quantum chromodynamic theory plasma has been studied as a probe of deconfinement in a hot medium which shows a reduction in the interaction between heavy quarks and antiquarks due to color screening leading to a suppression in $J/\Psi$ yields [1, 2]. Thereby the quarkonium in the hot medium is a good tool to examine the confined/deconfined state of matter. The dissociation of heavy quarkonium states in a hot QCD medium is studied by investigating the medium modifications to a heavy-quark potential [3].

The development of the Schrödinger equation (SE) plays a major role at finite temperature. Matsui and Satz [4] have investigated the formation of a hot quark-gluon plasma by calculating the $J/\Psi$ radius of charmonium. In Ref. [5], the SE is solved by using the Funke-Hecke theorem to describe the electron and proton media. Wong [6] has studied the binding energies and wave functions of heavy quarkonia in quark-gluon plasma by using a temperature dependent potential inferred from lattice gauge calculations. Thus, the study shows that the model with the new Q-Â potential gives dissociation temperatures that agree with the spectral function analyses. The SE in a screened Coulomb heavy-quark potential is solved to study the temperature dependence of the heavy-quarkonium interaction based on the Bhanot–Peskin leading order perturbative QCD analysis in which the $1S$ charmonium thermal width is determined and compared to recent lattice QCD results [7]. Wong [8] has investigated the Q-Â potential by using the thermodynamic quantities to give spontaneous dissociation temperatures for quarkonium and has also found the quark drip lines which separate the region of bound color-singlet $Q\bar{Q}$ states from the unbound region. In Ref. [9], the authors numerically solved the SE at finite temperature by employing temperature-dependent effective potential given by a linear combination of color singlet and internal energies. By using thermodynamic laws, the SE equation is derived at finite temperature [10].

Some authors [11 – 17] focus to extend the SE to the higher-dimensional space which gives more detail about the systems under study. Moreover, the energy eigenvalues and wave functions are obtained in the higher-dimensional space and their applications on quarkonium properties are studied in the vacuum, hot and dense media.
Recently, the fractional calculus has attracted attention in the different fields of physics which the nonlinear, complex effects are included such as in Refs. [18 – 21]. In high energy physics, the description of heavy-quarkonium energy spectra and complex phenomena of the standard model as in Ref. [22], in which, the author used the conformable fractional derivative to express the fractional radial SE in the N-dimensional space for the extended Cornall potential by using generalized extended Nikiforov-Uvarov (ENU) method to the fractional domain. In Ref. [23], the fractional form of the NU method is applicable in order to solve fractional radial SE with its applications on variety of potentials such as the oscillator potential, Woods-Saxon potential, and Hulthen potential. In Ref. [24], the Caputo fractional derivative is applied on a fractional Schrödinger wave equation with using quantization of the classical nonrelativistic Hamiltonian. The free particle solutions are obtained, which are confined to a certain region of space.

Thus, The fractional calculus focuses on vacuum space without considering the effects of medium. Thus, the present study is to study the effect of fractional-order on quarkonium masses in a hot medium which are not considered in recent works such as in Refs. [18 – 24], The conformable fractional of the Nikiforov-Uvarov (CF-NU) method is applied to obtain the analytic solutions of the \( N \)-dimensional radial SE.

The paper is organized as follows: In Sec. 2, the CF method is briefly explained. In Sec. 3, The energy eigenvalue and wave function are calculated in the \( N \)-dimensional space using CF-NU method. In Sec. 4, the results are discussed. In Sec. 5, the summary and conclusion are presented.

II. 2. THEORETICAL TOOLS

A. 2.1. Conformable fractional derivative

Fractional derivative plays an important role in applied science. Riemann–Liouville and Riesz and Caputo give a good formula that allows to apply boundary and initial conditions as in Ref. [19]

\[
D^\alpha_t f (t) = \int_{t_0}^{t} K_\alpha (t - s) f^{(n)} (s) ds, \quad t > t_0
\] (1)
with

$$K_\alpha (t-s) = \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)},$$  \hspace{1cm} (2)

where $f^{(n)}$ is the $n$th derivative of the function $f(t)$, and $K_\alpha (t-s)$ is the kernel, which is fixed for a given real number $\alpha$. The kernel $K_\alpha (t-s)$ has singularity at $t = s$. Caputo and Fabrizio [25] suggested a new formula of the fractional derivative with smooth exponential kernel of the form to avoid the difficulties that found in Eq. (1)

$$D^\alpha_t f(t) = \frac{M(\alpha)}{1-\alpha} \int_{t_0}^{t} \exp \left( \frac{\alpha (t-s)}{1-\alpha} \right) \dot{y}(s) ds,$$  \hspace{1cm} (3)

where $M(\alpha)$ is a normalization function with $M(0) = M(1) = 1$. A new formula of fractional derivative, called conformable fractional derivative (CFD) is proposed by Khalil et al. [26]

$$D^\alpha_t f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon 1-\alpha) - f(t)}{\epsilon} \quad t > 0$$  \hspace{1cm} (4)

$$f(0) = \lim_{\epsilon \to 0} f(t)$$  \hspace{1cm} (5)

with $0 < \alpha \leq 1$. This new definition is simple and provides a natural extension of differentiation with integer order $n \in \mathbb{Z}$ to fractional order $\alpha \in \mathbb{C}$. Moreover, the CFD operator is linear and satisfies the interesting properties that traditional fractional derivatives do not, such as the formula of the derivative of the product or quotient of two functions and the chain rule [23]. The concept of CFD has successfully applied to the CF-NU method to obtain the eigenvalues and eigenfunctions of SE as in Refs. [22, 23].

**III. CONFORMABLE FRACTIONAL NU METHOD**

In this section, the CF-NU method is briefly given to solve the conformable fractional of differential equation which takes the following form (see Ref. [23], for details)

$$D^\alpha [D^\alpha \Psi(s)] + \frac{\tau(s)}{\sigma(s)} D^\alpha \Psi(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \Psi(s) = 0,$$  \hspace{1cm} (6)

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of maximum second degree and $\tau(s)$ is a polynomial of maximum first degree with an appropriate $s = s(r)$ coordinate transformation.

$$D^\alpha \Psi(s) = s^{1-\alpha} \Psi'(s),$$  \hspace{1cm} (7)
\[ D^\alpha [D^\alpha \Psi(s)] = (1 - \alpha) s^{1-2\alpha} \Psi'(s) + s^{2-2\alpha} \Psi''(s). \]  

(8)

Substituting by Eqs. (7) and (8) into (6), we obtain

\[ \Psi''(s) + \frac{\tilde{\tau}_f(s)}{\sigma_f(s)} \Psi'(s) + \frac{\tilde{\sigma}_f(s)}{\sigma_f^2(s)} \Psi(s) = 0, \]  

(9)

To find particular solution of Eq. (9) by separation of variables, if one deals with the transformation

\[ \Psi(s) = \Phi(s) \chi(s), \]  

(10)

it reduces to an equation of hypergeometric type as follows

\[ \sigma_f(s) \chi''(s) + \tau_f(s) \chi'(s) + \lambda \chi(s) = 0, \]  

(11)

where

\[ \sigma_f(s) = \pi_f(s) \frac{\Phi(s)}{\Phi'(s)}, \]  

(12)

\[ \tau_f(s) = \tilde{\tau}_f(s) + 2\pi_f(s); \quad \tau'_f(s) < 0, \]  

(13)

and

\[ \lambda = \lambda_n = -n \tau'_f(s) - \frac{n(n-1)}{2} \sigma''_f(s), \quad n = 0, 1, 2, \ldots \]  

(14)

\[ \chi(s) = \chi_n(s) \] which is a polynomial of \( n \) degree which satisfies the hypergeometric equation, taking the following form

\[ \chi_n(s) = \frac{B_n}{\rho_n} \int ds^n (\sigma'_f(s) \rho(s)), \]  

(15)

where \( B_n \) is a normalization constant and \( \rho(s) \) is a weight function which satisfies the following equation

\[ \frac{d}{ds} \omega(s) = \frac{\tau(s)}{\sigma_f(s)} \omega(s); \quad \omega(s) = \sigma_f(s) \rho(s), \]  

(16)

\[ \pi_f(s) = \frac{\sigma'_f(s) - \tilde{\tau}_f(s)}{2} \pm \sqrt{\left( \frac{\sigma'_f(s) - \tilde{\tau}_f(s)}{2} \right)^2 - \tilde{\sigma}_f(s) + K \sigma_f(s)}, \]  

(17)

and

\[ \lambda = K + \pi'_f(s), \]  

(18)

the \( \pi_f(s) \) is a polynomial of first degree. The values of \( K \) in the square-root of Eq. (17) is possible to calculate if the expressions under the square root are square of expressions. This is possible if its discriminate is zero.
IV. CONFORMABLE FRACTIONAL OF THE DEPENDENT TEMPERATURE
SCHRÖDINGER EQUATION

The SE for two particles interacting via a spherically symmetric (central) potential $V(r)$ in the N-dimensional space, where $r$ is inter-particle distance, is given by [11 – 13]

$$\left[\frac{d^2}{dr^2} + \left(\frac{N-1}{r}\right) \frac{d}{dr} - \frac{L(L+N-2)}{r^2} + 2\mu(E - V(r))\right]\Psi(r) = 0,$$  \hspace{1cm} (19)

where $L, N,$ and $\mu$ are the angular momentum quantum number, the dimensionality number and the reduced mass for the quarkonium particle (for charmonium $\mu = \frac{m_c}{2}$ and for bottomonium $\mu = \frac{m_b}{2}$), respectively. Setting the wave function $\Psi(r) = R(r)r^{\frac{1-N}{2}}$, the following radial SE is obtained

$$\left[\frac{d^2}{dr^2} + 2\mu(E - V(r,T)) - \left(\frac{L + (\frac{N-2}{2})^2 - \frac{1}{4}}{2\mu r^2}\right)\right]R(r) = 0.$$

(20)

where $V(r,T)$ is the Cornell potential at the finite temperature as in Ref. [15] and references therein which take following as follows

$$V(r,T) = a(T, r) r - \frac{b(T, r)}{r},$$ \hspace{1cm} (21)

where $a(T, r) = \frac{a}{m_D(T)r}(1 - e^{-m_D(T)r})$ and $b(T, r) = be^{-m_D(T)r}$ where $m_D(T)$ is the Debye mass that vanishes at $T \to 0$. $a = 0.184$ GeV$^2$ and $b$ will determine later. By substituting Eq. (21) into Eq. (20) and using approximation $e^{-m_D(T)r} = \sum_{j=0}^{\infty} \left(-\frac{m_D(T)r}{j!}\right)^j$ up to second-order, which gives good accuracy when $m_D r \ll 1$. One obtains

$$\left[\frac{d^2}{dr^2} + 2\mu(E - A + \frac{b}{r} - Cr + Dr^2 - \left(\frac{L + (\frac{N-2}{2})^2 - \frac{1}{4}}{2\mu r^2}\right)\right]R(r) = 0.$$

(22)

where,

$$A = b m_D (T), \hspace{0.5cm} C = a - \frac{1}{2}b m_D^2 (T), \hspace{0.5cm} \text{and} \hspace{0.5cm} D = \frac{1}{2}a m_D (T).$$ \hspace{1cm} (23)

By taking $r = \frac{1}{x}$, Eq. (23) takes the following form

$$\left[\frac{d^2}{dx^2} + \frac{2x}{x^2} \frac{d}{dx} + 2\mu(E - A + bx - Cx + Dx^2 - \left(\frac{L + (\frac{N-2}{2})^2 - \frac{1}{4}}{2\mu x^2}\right)\right]R(x) = 0.$$

(24)

The expansion of $\frac{C}{x}$ and $\frac{D}{x}$ in a power series around the characteristic radius $r_0$ of meson up to the second order is given Ref. [15]. The following equation is obtained

$$\left[\frac{d^2}{dx^2} + \frac{2x}{x^2} \frac{d}{dx} + \frac{2}{x^4}(-D_1 + D_2 x - D_3 x^2)\right]R(x) = 0,$$

(25)
where,
\[ D_1 = -\mu(E-A-\frac{3C}{\delta}+\frac{6D}{\delta^2}), D_2 = \mu(\frac{3C}{\delta^2} - \frac{8D}{\delta^3} + b), \text{and } D_3 = \mu \left( \frac{C}{\delta^3} - \frac{3D}{\delta^4} + \frac{(L+\frac{(N-2)}{2})^2 - \frac{1}{4}}{2\mu} \right). \]  

(26)

The transition of the conformable fractional of Eq. (25) is obtained as in Ref. [23]

\[ [D^\alpha[D^\alpha R(x)] + \frac{2x^\alpha}{x^{2\alpha}}D^\alpha R(x) + \frac{2}{x^{4\alpha}}(-D_1 + D_2 x^\alpha - D_3 x^{2\alpha})]R(x) = 0, \]  

(27)

substituting by Eqs. (7) and (8) into (27), we obtain

\[ R''(x) + \frac{\bar{\tau}_f(x)}{\sigma_f(x)}R'(x) + \frac{\bar{\sigma}_f(x)}{\sigma_f^2(x)}R(x) = 0, \]  

(28)

where

\[ \bar{\tau}_f(s) = 3x^\alpha - \alpha x^\alpha, \sigma_f(s) = x^{\alpha+1}, \text{ and } \bar{\sigma}(s) = 2(-D_1 + D_2 x^\alpha - D_3 x^{2\alpha}). \]  

(29)

Hence, the Eq. (28) satisfies Eq. (9). Therefore, Eq. (17) takes the following form after substituting by Eq. (29)

\[ \pi_f = -x^\alpha + \alpha x^\alpha \pm \sqrt{(-x^\alpha + \alpha x^\alpha)^2 - 2(-D_1 + D_2 x^\alpha - D_3 x^{2\alpha}) + Kx^{1+\alpha}}. \]  

(30)

The constant \( K \) is chosen such as the function under the square root has a double zero, i.e. its discriminant equals zero. Hence,

\[ K = \left( \frac{D_2^2}{2D_1} - (1 - 2\alpha + \alpha^2 + 2D_3) \right) x^{\alpha-1}. \]  

(31)

Substituting by Eq. (31) into Eq. (30), we obtain

\[ \pi_f(x) = -x^\alpha + \alpha x^\alpha + \frac{D_2}{\sqrt{2D_1}}x^\alpha - \sqrt{2D_1} \]  

(30)

The positive sign in Eq. (30) is determined as in Ref. [15]. By using Eq. (13), we obtain

\[ \tau_f(x) = 3x^\alpha + \alpha x^\alpha - 2 \left( \frac{D_2}{\sqrt{2D_1}}x^\alpha - \sqrt{2D_1} \right), \]  

(24)

and using Eq. (14), we obtain

\[ \lambda_n = \left( -n \left( 3\alpha - \alpha^2 \right) - \frac{2D_2}{\sqrt{2D_1}} - \frac{n(n-1)\alpha(\alpha+1)}{2} \right) x^{\alpha-1}. \]  

(25)
From Eqs. (14 and 18); $\lambda = \lambda_n$. The energy eigenvalues of Eq. (22) in the $N$-dimensional space is given

$$E_{nL}^N = A + \frac{3C}{\delta} - \frac{6D}{\delta^2} - \frac{2\mu(\frac{3C}{\delta} + b - \frac{8D}{\delta^2})^2}{[(2n + 1) \pm \sqrt{W + \frac{8\mu C}{\delta^2} + 4((L + \frac{N-2}{2})^2 - \frac{1}{4}) - \frac{24\mu D}{\delta^4})]^2}. \quad (26)$$

with

$$W = (2n\alpha + \alpha)^2 - 4 \left( n (3\alpha - \alpha^2) + \frac{1}{2} n (n - 1) \alpha (\alpha + 1) + \alpha - 1 \right). \quad (27)$$

The radial of wave function takes the following form

$$R_{nL}(r^\alpha) = C_{nL} r \left( -\frac{D^2}{\sqrt{2m}} \right)^\alpha e^{\sqrt{2m} r^\alpha} (-r^{2\alpha} D^\alpha)^n (r^{\frac{2}{\sqrt{2m}}} e^{-2\sqrt{2m} r^{\alpha}}). \quad (28)$$

$C_{nL}$ is the normalization constant that is determined by $\int |R_{nL}(r^\alpha)|^2 dr = 1$. We note that the radial wave function in Eq. (28) does not explicitly depend on the number of dimensions. Hence, $\int |R_{nL}(r)|^2 dr = 1$ remains unchanged.

V. DISCUSSION OF RESULTS

In this section, the above results are applied on quarkonium masses. the following relation is used as in Refs. [12, 27]

$$M = 2m + E_{nL}^N, \quad (29)$$

where $m$ is quarkonium bare mass for the charmonium or bottomonium mesons. By using Eq. (26), we write Eq. (29) as follows:

$$M_Q = 2m + A + \frac{3C}{\delta} - \frac{6D}{\delta^2} - \frac{2\mu(\frac{3C}{\delta} + b - \frac{8D}{\delta^2})^2}{[(2n + 1) \pm \sqrt{W + \frac{8\mu C}{\delta^2} + 4((L + \frac{N-2}{2})^2 - \frac{1}{4}) - \frac{24\mu D}{\delta^4})]^2}. \quad (30)$$

Eq. (30) represents quarkonium mass that is calculated in the N-dimensional space with considering fraction-order, Number of dimensionality, and finite temperature. One can obtain the quarkonium masses at zero temperature by taking $T = 0$ leads to $A = D = 0$ and $C = a$ and $\alpha = 1$. Therefore, Eq. (30) takes the following form

$$M_Q = 2m + \frac{3a}{\delta} - \frac{2\mu(\frac{3a}{\delta} + b)^2}{[(2n + 1) \pm \sqrt{1 + \frac{8\mu a}{\delta^2} + 4L(L + 1)}]^2}. \quad (31)$$

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Eq. (31) coincides with Ref. [28], in which the authors obtained the quarkonium mass at zero temperature and $\alpha = 1$ in 3D. At $\alpha = 1$, the following equation is obtained:

$$M_Q = 2m + A + \frac{3C}{\delta} - \frac{6D}{\delta^2} - \frac{2\mu(\frac{3C}{\delta^2} + b - \frac{8D}{\delta^2})^2}{[(2n + 1) \pm \sqrt{1 + \frac{8\mu C}{\delta^2} + 4((L + \frac{N-2}{2})^2 - \frac{1}{4}) - \frac{24\mu D}{\delta^2}]^2}. \quad (32)$$

Eq. (32) is compatible with Ref. [15], in which the author obtained the quarkonium mass at finite temperature in the N-dimensional space. To calculate quarkonium mass accordance to Eq. (30), the Debye mass is defined as in Ref. [29]

$$m_D(T) = g(T)T\sqrt{\frac{N_c}{3} + \frac{N_f}{6}}, \quad (34)$$

and

$$g^2(T) = \frac{24\pi^2}{(33 - 2N_f) \ln \left(\frac{2\pi T}{\Lambda_{MS}}\right)}, \quad (35)$$

with $N_f$ and $N_c$ as the number of flavours and colors, respectively. $\Lambda_{MS} = 0.1$ GeV and $b = \frac{g^2(T)}{3\pi}$. In Figs. (1, 2), the bottomonium mass is plotted as a function of temperature ratio $\frac{T}{T_c}$ where $T_c$ is the critical temperature equals 170 MeV as in Ref. [30]. In Fig. (1), the bottomonium mass is plotted versus temperature ratio $\frac{T}{T_c}$ in the three-dimensional space. At $\alpha = 1$ which represents the normal case for normal calculus. One note that the bottomonium mass increases up $T = 1.36T_c$ then the curve of bottomonium decreases with increasing temperature. This indicates that the bottomonium is stable up to $T = 1.36T_c$ then the bottomonium melts in hot QCD medium. In Refs. [6, 29, 31], the authors found that the bottomonium melts above the critical temperature. In Ref. [6], the dissociation of bottomonium $T_D \gg T_c$, $T_D = 3.2 T_c$ in Ref. [29], and $T_D = 1.11 T_c$ in Ref. [31]. In these works, the SE is solved for quarkonium by using different methods. In addition, the interpretation is an agreement with Ref. [27], in which the SE is solved for a nucleon in hot QCD medium. Hence, the present result is a qualitative agreement with Refs. [6, 29, 31]. To investigate the effect of fractional parameter, one takes the $\alpha = 0.4, 0.6, 0.8$. One notes that the curve of bottomonium strongly increases and then decreases in higher temperatures. In addition, the bottomonium mass is shifted to higher values by decreasing fractional parameter $\alpha$ which indicates the binding energy is more bound. Therefore, the dissociation temperature increases with increasing binding energy. Hence the effect of fraction parameter supports the bound state of bottomonium. The effect is not considered in other works.
In this work, one interests to study the effect on the dimensional number on quarkonium mass. The motivation for this as a natural consequence of the unification of the two modern theories of quantum mechanics and relativity and the emergence of the string theory, the investigation of the Standard Model particles in extra or higher-dimensional space is a hot topic of interest. From the experimental point of view, the investigation of the existence of extra dimensions is one of the primary goals of the LHC. The search for extra dimensions with the ATLAS and CMS detectors is discussed in Ref. [32]. In Fig. 2, the bottomonium mass is plotted as a function of temperature ratio for different of fractional parameter at $N = 5$. In Fig. 2, one note that the bottomonium mass decreases with increasing temperature. This indicates that the bottomonium melts around the critical temperature when the higher dimensional space is considered and also the quarkonium mass slightly shifts with increasing fractional parameter. Therefore, the fractional parameter is a little effect in the higher-dimensional space. In addition, one notes that the bottomonium mass increases with increasing dimensionality number. This finding is in agreement with Ref. [22].

**Fig. 1:** Mass Spectrum 1S of bottomonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_b = 4.823$ GeV and $a = 0.184$ GeV$^2$, at different values of fractional parameter at

\[ 3D \]
**Fig. 2:** Mass Spectrum 1S of bottomonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_b = 4.823$ GeV and $a = 0.184$ GeV$^2$ at different values of fractional parameter at 5D.

In Figs. (3, 4), the charmonium is plotted as a function of temperature ratio for different values of fractional parameter in 3D and 5D spaces, respectively. In Fig. (3), one notes that the charmonium mass decreases with increasing temperature and notes the charmonium melts in given interval of temperature. This indicates that the charmonium melts under critical temperature. In Ref. [31], the authors found the 1S state of charmonium is melts around $0.99 \ T_c$. By decreasing fractional parameter, one notes the charmonium mass increases. In Fig. (4), in 5D, the charmonium mass decreases with increasing temperature and increases with increasing dimensionality number. The effect of fractional parameter is a little sensitive on charmonium when the dimensionality number increases. Thus, the charmonium mass increases with decreasing fractional parameter and increasing dimensionality number.
Fig. 3: Mass Spectrum 1S of charmonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_c = 1.209$ GeV and $a = 0.184$ GeV$^2$ at different values of fractional parameter at 3D

Fig. 4: Mass Spectrum 1S of charmonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_c = 1.209$ GeV and $a = 0.184$ GeV$^2$ at different values of fractional parameter at 5D
VI. SUMMARY AND CONCLUSION

The CF-NU method is applied to solve N-fractional radial SE. The eigenvalues of energy and corresponding wave functions are obtained, in which they depend on the fractional parameter $0 < \alpha \leq 1$, finite temperature, and dimensionality number. Particular cases are obtained at $\alpha = 1$, $T = 0$ and $T \neq 0$ that compatible with recent works. The present results are applied on the quarkonium system such as charmonium and bottomonium masses in the hot QCD medium. The present results show that the fractional parameter plays an important role in the hot QCD medium since the binding energy of charmonium and bottomonium is more bound by decreasing fractional parameter. In addition, the effect of fractional parameter is studied in hot medium when the higher dimensional space is considered ($N = 5$). The results show that the effect of fractional parameter is a little sensitive on quarkonium masses at $N = 5$. The dissociation of charmonium and bottomonium is investigated at $\alpha = 1$ which represents a normal case and the agreement is noted in comparison with other works. One concludes that the fraction calculus plays an important role to give more information about quark-gluon plasma and expects a key for solving many theoretical problems in high energy physics.

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