N = 1 Finite Unified Theories
Predictions and Dualities

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Abstract: N = 1, all-loop Finite Unified Theories (FUTs) are very interesting not only since they realize an old theoretical dream, but also due the remarkable predictive power of particular models as well as for providing candidates that might shed light in non-perturbative Physics. Here we discuss (a) the recent developments concerning the soft supersymmetry breaking (SSB) sector of these theories and the resulting predictions in very interesting realistic models, and (b) the results of a recent search for duals of N = 1, all-loop FUTs.

1 Introduction

Finiteness is an outstanding issue in the various theoretical endeavors that attempt to achieve a deeper understanding of Nature. Most theorists, quite independently from the school of thinking they belong to, tend to believe that the divergencies of ordinary field theory are not of fundamental nature, but rather they are signaling the existence of New Physics at higher scales. Therefore, it is very natural to believe that the final theory, which hopefully unifies all interactions, should be completely finite. Given the searches of strings, non-commutative geometry and q-groups that aim, among others, to achieve finiteness, it is worth stressing that finiteness does not require necessarily gravity. The latter statement is well established in gauge theories with extended supersymmetry. For instance, it is well known that all N = 4 and some N = 2 supersymmetric gauge theories are free from ultraviolet (UV) divergencies at all-orders in perturbation theory (PT). Of particular interest is the existence of

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N = 1 supersymmetric gauge theories \[2, 5\], which are finite at all-orders in PT and the construction in that framework of a realistic finite SU(5) GUT, which successfully has predicted, among others, the top quark mass \[1\].

In the following we shall first report on the recent progress that has been done, and which increases appreciably the predictive power of the models that can be constructed and provides us with new predictions for the Higgs particles masses and the s-spectrum, which were not available in the previous attempts.

The discussion about finite N = 1 gauge theories till recently was limited to perturbative aspects of these theories while non-perturbative problems, like their bound state spectrum, were left open. The knowledge of the perturbative properties were sufficient for the study of GUTs. However, the recent progress that has been done might permit also studies at strong couplings. The basic idea that could allow us to address non-perturbative problems in these theories is the hope that there exist some kind of electric-magnetic, strong-weak coupling duality, like the one that is believed to be exhibited in N = 4 supersymmetric gauge theories. Moreover, the low energy effective action of N = 2 supersymmetric gauge theories can be even solved using duality and holomorphy \[8\]. Finally, N = 1 supersymmetric gauge theories, which can describe realistically the so far observed world, exhibit a weaker version of duality symmetry \[9\], namely theories differing in the UV can describe the same Physics in the infrared (IR).

In the following we shall also report on a systematic search for duals of N = 1, all-loop finite gauge theories \[7\]. This search covers most of the chiral finite gauge theories including the realistic SU(5) model mentioned earlier and several vector-like models. Surprisingly, we found that the duals of the N = 1, all-loop finite gauge theories are usually not finite, with the exception of an SO(10) theory which emerges from this search as a candidate to exhibit also S-duality.

## 2 Finiteness and Reduction of Couplings in N = 1 SUSY Gauge Theories

Consider a chiral, anomaly free, N = 1 globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

\[ W = \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k \, , \tag{1} \]

where \( m^{ij} \) and \( C^{ijk} \) are gauge invariant tensors and the matter field \( \Phi_i \) transforms according to the irreducible representation \( R_i \) of the gauge group G.

The one-loop \( \beta \)-function of the gauge coupling g is given by

\[ \beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[ \sum_i l(R_i) - 3 C_2(G) \right] \, , \tag{2} \]

where \( l(R_i) \) is the Dynkin index of \( R_i \) and \( C_2(G) \) is the quadratic Casimir of the adjoint representation of the gauge group G. The \( \beta \)-functions of \( C^{ijk} \), by virtue of
the non-renormalization theorem, are related to the anomalous dimension matrix $\gamma^j_i$ of the matter fields $\Phi_i$ as:

$$\beta_{ij}^{jk} = \frac{d}{dt} C^{ijp} = C^{ijp} \sum_{n=1}^{\infty} \left( \frac{1}{(16\pi^2)^n} \gamma_{p}^{k(n)} + (k \leftrightarrow i) + (k \leftrightarrow j) \right).$$  \hspace{1cm} (3)

At one-loop level $\gamma^j_i$ is

$$\gamma^{j(1)}_i = \frac{1}{2} C_{i pq} C^{ijpq} - 2 g^2 C_{2}^{(R)} \delta^{ij}_i,$$  \hspace{1cm} (4)

where $C_{2}^{(R)}$ is the quadratic Casimir of the representation $R$, and $C^{ij} = C^{*}_{ij}$.  

As one can see from Eqs. (2) and (4) all the one-loop $\beta$-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma^{j(1)}_i$ vanish, i.e.

$$\sum_i \ell(R_i) = 3C_{2}^{(G)}, \quad \frac{1}{2} C_{i pq} C^{ijpq} = 2 \delta^{ij}_i g^2 C_{2}^{(R_i)}.$$  \hspace{1cm} (5)

A very interesting result is that the conditions (5) are necessary and sufficient for finiteness at the two-loop level.

A natural question to ask is what happens at higher loop orders. The finiteness conditions (5) impose relations between gauge and Yukawa couplings. We would like to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs)

$$\beta_g \frac{dC^{ijp}}{dg} = \beta_{ij}^{jk}$$  \hspace{1cm} (6)

and hold at all-orders. Remarkably the existence of all-order power series solutions to (6) can be decided at the one-loop level (4).

A very interesting theorem [2] guarantees the vanishing of the $\beta$-functions to all-orders in perturbation theory, if we demand reduction of couplings, and that all one-loop anomalous dimensions of the matter fields in the completely and uniquely reduced theory vanish identically.

### 3 All-Loop Finite Unified Theories

A predictive Finite Unified $SU(5)$ model which is finite to all-orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma^{(1)j}_i \propto \delta^{ij}_i$.
2. Three fermion generations, $\Phi_i \ (i = 1, 2, 3)$, obviously should not couple to 24.

   This can be achieved for instance by imposing $B - L$ conservation.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model.

A: The model of ref. [1].

B: A slight variation of the model A.

The superpotential which describe the two models takes the form [1, 6]

\[ W = \sum_{i=1}^{3} \left[ \frac{1}{2} g_i^u 10_i 10_i H_i + g_i^d 10_i \overline{5}_i \overline{H}_i \right] + g_{23}^u 10_2 10_3 H_4 \]

\[ + g_{23}^d 10_2 \overline{5}_3 \overline{H}_4 + g_{32}^d 10_3 \overline{5}_2 \overline{H}_4 + \sum_{a=1}^{4} g_a^f H_a 24 \overline{H}_a + \frac{g^\lambda}{3} (24)^3, \]

where \( H_a \) and \( \overline{H}_a \) (\( a = 1, \ldots, 4 \)) stand for the Higgs quintets and anti-quintets.

The non-degenerate and isolated solutions to \( \gamma^{(i)} = 0 \) for the models \{A, B\} are:

\[(g_1^u)^2 = \left\{ \frac{8}{5}, \frac{8}{5} \right\} g^2, \quad (g_1^d)^2 = \left\{ \frac{6}{5}, \frac{6}{5} \right\} g^2, \quad (g_2^u)^2 = (g_2^d)^2 = \left\{ \frac{8}{5}, \frac{4}{5} \right\} g^2, \quad (g_3^u)^2 = (g_3^d)^2 = \left\{ \frac{6}{5}, \frac{3}{5} \right\} g^2, \quad (g^\lambda)^2 = \left\{ \frac{15}{7} g^2, \right\} \]

\[(g_4^f)^2 = (g_3^f)^2 = \left\{ 0, \frac{1}{2} \right\} g^2, \quad (g_1^f)^2 = 0, \quad (g_4^f)^2 = \left\{ 1, 0 \right\} g^2. \]

According to the theorem of ref. [4] these models are finite to all-orders. After the reduction of couplings the symmetry of \( W \) is enhanced [5].

The main difference of the models A and B is that three pairs of Higgs quintets and anti-quintets couple to the 24 for B so that it is not necessary to mix them with \( H_4 \) and \( \overline{H}_4 \) in order to achieve the triplet-doublet splitting after the symmetry breaking of \( SU(5) \).

4 Supersymmetry Breaking and Predictions of Low Energy Parameters

The above models are completed as realistic theories by adding SSB terms as follows:

\[ - \mathcal{L}_{SB} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)^i_j \phi^* \phi_j + \frac{1}{2} M \lambda \lambda + \text{h.c.}, \]

where the \( \phi_i \) are the scalar parts of the chiral superfields \( \Phi_i \), \( \lambda \) are the gauginos and \( M \) their unified mass.

Concerning the supersymmetry breaking sector of the theory it has been recently shown that the requirement of two-loop finiteness of SSB in a finite gauge theory leads to
a. the relation \( h^{ijk} = -MY^{ijk} \) and,

b. the soft scalar-mass sum rule \( (m_i^2 + m_j^2 + m_k^2)/MM^\dagger = 1 + g^2/16\pi^2 \Delta^{(1)} \) for \( i,j,k \) with \( C^{ijk} \neq 0 \), where \( \Delta^{(1)} \) is the two-loop correction, which vanishes for the universal choice \( \mathbb{R} \), but also in the considered models without universal masses.

Since the gauge symmetry is spontaneously broken below \( M_{\text{GUT}} \), the finiteness conditions do not restrict the renormalization property at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings \( \mathbb{R} \), the \( h = -MC \) relation and the soft scalar-mass sum rule at \( M_{\text{GUT}} \). We have examined the evolution of these parameters according to their renormalization group equations at two-loop for dimensionless parameters and at one-loop for dimensional ones with these boundary conditions. Below \( M_{\text{GUT}} \) their evolution is assumed to be governed by the MSSM. It is further assumed a unique supersymmetry breaking scale \( M_s \) so that below \( M_s \) the SM is the correct effective theory.

The predictions for the top quark mass \( M_t \) are \( \sim 183 \) and \( \sim 174 \) GeV in models A and B respectively. Comparing these predictions with the most recent experimental value \( M_t = (175.6 \pm 5.5) \) GeV, and recalling that the theoretical values for \( M_t \) may suffer from a correction of less than \( \sim 4\% \) \( \mathbb{R} \), we see that they are consistent with the experimental data.

Using the sum rule we can now determine the spectrum of realistic models in terms of just a few parameters. In addition to the successful prediction of the top quark mass the characteristic features of the spectrum are that 1) the lightest Higgs mass is predicted \( \sim 120 \) GeV and 2) the s-spectrum starts above 200 GeV. Therefore, the next important test of these Finite Unified Theories will be given with the measurement of the Higgs mass, for which the models show an appreciable stability in their prediction.

5 Dualities of Finite Gauge Theories

In ref. \[7\] we have done an extensive search for dual gauge theories of all-loop finite, \( N = 1 \) supersymmetric gauge theories. It is shown how to find explicitly the dual gauge theories of almost all chiral, \( N = 1 \), all-loop finite gauge theories, while several models have been discussed in detail, including the realistic finite \( SU(5) \) unified theory of ref. \[4\]. As we have seen one- and two-loop finiteness of a gauge theory is guaranteed by, first, choosing the particle content such that the one-loop gauge \( \beta \)-function vanishes, and subsequently, by adding a superpotential such that all one-loop matter field anomalous dimensions are zero. Furthermore, the all-loop finiteness requires that the relations among gauge and Yukawa couplings, obtained by demanding the vanishing of the one-loop anomalous dimensions, should be unique solutions of the reduction equations, which in turn guarantees that they can be uniquely determined to all-orders in PT. Therefore, using established methods for searching for duals \( \text{à la Seiberg} \), we have examined almost all known \( N = 1 \) supersymmetric gauge
theories (with the exception of $E_6$ models and $SO(10)$ containing anti-spinors) and vanishing one-loop $\beta$-functions. These theories have, first, been promoted to all-loop finite ones, by adding appropriate superpotential and then by meeting the requirements of all-loop finiteness. In addition, certain vector-like, all-loop finite, $N = 1$ gauge theories and their duals have been discussed in the standard field theory framework but also by using the derivation of gauge theories from branes. However, the brane picture still encounters several difficulties in the corresponding hunting for finite gauge theories.

The result of our search is that the duals à la Seiberg of all-loop finite gauge theories are asymptotically free. In certain cases looking to the IR limit of the theories, where both theories of the dual pair are describing the same physics, we found that the spontaneously broken theories seem superficially asymptotically non-free (as the $U(1)$ of the Standard Model embedded in a GUT).

There is only one model which is one-loop finite in the IR. The model is based on the gauge group $SO(10)$ and has matter content consisting of 8 vector and 8 spinor superfields. The dual of this theory is based on the gauge group $SU(17) \times SP(14)$. The first factor is asymptotically free, while the second is one-loop finite. However, in the IR, after spontaneous symmetry breaking of the $SU(17)$ down to $SU(9)$, both gauge factors are one-loop finite. We found evidence that the IR dual is all-loop finite and therefore the dual pair is a candidate to exhibit $N = 1$ S-duality.

The dual of the realistic finite unified theory based on $SU(5)$ has been determined and discussed in some detail. However, the resulting dual theory is rather complicated and it does not give, so far, any hint for any useful use of it. On the other hand, we should note that the dual was constructed by using the deconfinement method, which does not lead to unique results. Therefore, we should not exclude the possibility that more interesting duals can be constructed.

6 Conclusions

The search for realistic Finite Unified theories started a few years ago with the successful prediction of the top quark mass, and it has now been complemented with a new important ingredient concerning the finiteness of the SSB sector of the theory. Specifically, a sum rule for the soft scalar masses has been obtained which guarantees the finiteness of the SSB parameters up to two-loops, avoiding at the same time serious phenomenological problems related to the previously known “universal” solution. It is found that this sum rule coincides with that of a certain class of string models in which the massive string modes are organized into $N = 4$ supermultiplets.

Motivated from the recent developments on dualities of gauge theories with extended supersymmetry we have been searching for candidates with S-duality among the $N = 1$ supersymmetric gauge theories, which have richer dynamics and are much more promising in describing the real world, as the one discussed above. The strategy was to look for duals à la Seiberg of all-loop, $N = 1$ FUTs which are FUTs too and therefore exhibiting S-duality. From our search, one chiral, $N = 1$, all-loop
FUT has been singled out giving promises of having S-duality as the $N = 4$ gauge theories.

Added notes: Since the submission of the present contribution there exist some interesting related developments, which we would like to mention. The first concerns the soft scalar-mass rule presented in sect. 4, which has been extended recently to all-orders [11]. In the case of finite unified theories, the sum rule ensures the all-loop finiteness in the soft supersymmetry breaking sector. In addition the exact \( \beta \)-function for the soft scalar masses in the NSVZ scheme [12] for softly-broken supersymmetric QCD has been obtained [11]. The second concerns the construction of a $N = 1$, one-loop finite $SU(4)^3$ gauge theory with 3 chiral generations in three dual ways from strings [13]. In addition, motivated by the connection among gauge theories realized on branes and their supergravity realizations at large $N$ [14], a generalization of the above model has been constructed, by studying D3 branes at orbifolds, based on $U(N)^3$ gauge group with three chiral generations [13]. The latter models appear together with a superpotential and it can be easily shown that are all-loop finite using the methods that have been reviewed here. For both models naturally it is conjectured that they have improved properties such as S-duality since they are directly connected to or hoped to have inherited properties from $N = 4$ gauge theories. It is very interesting to see what can really be proven.

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