Singular Perturbation Approximations for a Class of
Linear Quantum Systems

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Abstract

This paper considers the use of singular perturbation approximations for a class of linear quantum systems arising in the area of linear quantum optics. The paper presents results on the physical realizability properties of the approximate system arising from singular perturbation model reduction.

I. INTRODUCTION

The modelling and control of quantum linear systems is an important emerging application area which is motivated by the fact that quantum mechanical features emerge as the systems being controlled approach sub-nanometer scales and as the required levels of accuracy in control and estimation approach quantum noise limits. In recent years, there has been considerable interest in the feedback control and modeling of linear quantum systems; e.g., see [1]–[17]. Such linear quantum systems commonly arise in the area of quantum optics; e.g., see [18]–[20]. The feedback control of quantum optical systems has applications in areas such as quantum communications, quantum teleportation, and gravity wave detection. In particular, the papers [8], [15]–[17] have been concerned with a class of linear quantum systems in which the system can be defined in terms of a set of linear quantum stochastic differential equations (QSDEs)

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expressed purely in terms of annihilation operators. Such linear quantum systems correspond to optical systems made up of passive optical components such as optical cavities, beam-splitters, and phase shifters. The main results of this paper apply to this class of linear quantum systems for the square case in which the number of outputs is equal to the number of inputs.

This paper is concerned with the use of singular perturbation approximations in order to obtain reduced dimension models for the class of linear quantum systems under consideration. Singular perturbation approximations are is widely used for obtaining reduced dimension models for classical systems; e.g., see [21]. In the case of quantum systems, a reduced dimension model may be desired for a quantum plant to be controlled in order to simplify the controller design process which can be very complicated using existing quantum controller design methods such as the quantum LQG method of [6]. Another application of model reduction for linear quantum systems arises in the case of controller reduction where a reduced dimension controller is obtained from a high order synthesized controller. In the case of coherent quantum control such as considered in [5], [6], [17], the controller is required to be a quantum system itself and thus the reduced dimension system must be physically realizable.

In the physics literature, a commonly used technique in the modeling of quantum systems is the method of adiabatic elimination, which is closely connected to the singular perturbation method in linear systems theory; e.g, see [22]–[25]. The papers [22]–[25] also consider the issue of convergence of these singular perturbation approximations. In this paper, we consider the properties of the singular perturbation approximation to a linear quantum system from a linear systems point of view; e.g., see [26] for a detailed description of singular perturbation methods in linear systems theory including error characterization in both the time and frequency domains. In particular, we are concerned with the physical realizability properties of the singular perturbation approximation to a linear quantum system. The issue of physical realizability for linear quantum systems was considered in the papers [5], [6], [16], [17]. This notion relates to whether a given QSDE model represents a physical quantum system which obeys the laws of quantum mechanics. In particular, the results of the papers [5]–[7], [16] show that the notion of physical realizability enables a direct connection between results in quantum linear systems theory and linear systems theory. In applying singular perturbation methods to obtain approximate models of quantum systems, it is important that model obtained is a physically realizable quantum system so that it retains the essential features of a quantum system. Also, if the approximate model of a quantum
plant is to be used for controller synthesis, the controller synthesis procedure may need to exploit
the physical realizability of the plant model. In addition, if model order reduction is applied to a
coherent feedback controller which is to be implemented as a quantum system, then this reduced
order controller model must be physically realizable.

In the paper [16], the notion of physical realizability is shown to be equivalent to the lossless
bounded real property for the class of square linear quantum systems under consideration. This
property requires that the system matrix is Hurwitz and that the system transfer function is unitary
for all frequencies. The main result of this paper shows that if a singularly perturbed linear
quantum system is physically realizable for all values of the singular perturbation parameter,
then the corresponding reduced dimension approximate system has the property that all of its
poles are in the closed left half of the complex plane and its transfer function is unitary for all
frequencies. These properties indicate that in all but pathological cases, the singular perturbation
approximation method will yield a physically realizable reduced dimension system. In addition,
an example is given showing one such pathological system in which the singular perturbation
approximation is not strictly Hurwitz.

The paper also presents a result for a special case of the singularly perturbed linear quantum
systems considered in this paper. This special case corresponds to singular perturbations which
arise physically from a perturbation in the system Hamiltonian. In this case, the result shows
that the corresponding reduced dimension approximate system is always physically realizable.
This result can in fact be derived from the nonlinear quantum system results presented in the
papers [23], [25]. However, we have included this result, along with a straightforward proof, for
the sake of completeness. We have also included an example of a singularly perturbed linear
quantum optical system which fits into the subclass of singularly perturbed quantum systems
for which this result applies. This example illustrates how such singularly perturbed quantum
systems can arise naturally in physical quantum optical systems.

The remainder of this paper proceeds as follows. In Section II we define the class of
linear quantum systems under consideration and recall some preliminary results on the physical
realizability of such systems. In Section III we consider the singular perturbation approximation
to a linear quantum system. We first present a result for the class of singularly perturbed linear
quantum systems under consideration which relates to the lossless bounded real property. We then
consider a special class of singular perturbations which is related to corresponding perturbations
of the quantum system coupling operator and Hamiltonian operator. We present a result which relates to this class of singular perturbations and shows that the corresponding approximate reduced dimension system is guaranteed to be physically realizable. In Section IV, we present a simple example from the field of quantum optics to illustrate the proposed theory. In Section V, we present some conclusions.

II. A Class of Linear Quantum Systems

We consider a class of linear quantum systems described in terms of the annihilation operator by the following quantum stochastic differential equations (QSDEs):

\[ da(t) = Fa(t)dt + Gu(t); \]
\[ dy(t) = Ha(t)dt + Ku(t) \]

where \( F \in \mathbb{C}^{n \times n} \), \( G \in \mathbb{C}^{n \times m} \), \( H \in \mathbb{C}^{m \times n} \) and \( K \in \mathbb{C}^{m \times m} \); e.g., see [5], [16], [17], [19], [20]. Here \( a(t) = [a_1(t) \cdots a_n(t)]^T \) is a vector of (linear combinations of) annihilation operators. The vector \( u(t) \) represents the input signals and is assumed to admit the decomposition:

\[ du(t) = \beta_u(t)dt + \tilde{u}(t) \]

where \( \tilde{u}(t) \) is the noise part of \( u(t) \) and \( \beta_u(t) \) is an adapted process (see [27] and [28]). The noise \( \tilde{u}(t) \) is a vector of quantum noises. The noise processes can be represented as operators on an appropriate Fock space (for more details see [27]). The process \( \beta_u(t) \) represents variables of other systems which may be passed to the system (1) via an interaction. More details concerning this class of quantum systems can be found in the references [16], [5].

Definition 1: (See [7], [16], [17].) A linear quantum system of the form (1) is said to be *physically realizable* if there exists a commutation matrix \( \Theta = \Theta^\dagger > 0 \), a coupling matrix \( \Lambda \), a Hamiltonian matrix \( M = M^\dagger \), and a scattering matrix \( S \) such that

\[ F = -\Theta \left( iM + \frac{1}{2} \Lambda^\dagger \Lambda \right); \]
\[ G = -\Theta \Lambda^\dagger S; \]
\[ H = \Lambda; \]
\[ K = S \]

and \( S^\dagger S = I \).
Here, the notation $\dagger$ represents conjugate transpose. In this definition, if the system (1) is physically realizable, then the matrices $S$, $M$ and $\Lambda$ define a open harmonic oscillator with scattering matrix $S$, coupling operator $L = \Lambda a$ and a Hamiltonian operator $H = a^\dagger Ma$; e.g., see [19], [27], [29] and [5]. This definition is an extension of the definition given in [5], [16], [17] to allow for a general scattering matrix $S$; e.g., see [7].

The following theorem is a straightforward extension of Theorem 5.1 of [16] to allow for a general scattering matrix $S$.

**Theorem 1:** (See [16].) A linear quantum system of the form (1) is physically realizable if and only if there exists a matrix $\Theta = \Theta^\dagger > 0$ such that

$$F \Theta + \Theta F^\dagger + GG^\dagger = 0;$$

$$G = -\Theta H^\dagger K;$$

$$K^\dagger K = I.$$  \hspace{1cm} (3)

In this case, the corresponding Hamiltonian matrix $M$ is given by

$$M = \frac{i}{2} \left( \Theta^{-1} F - F^\dagger \Theta^{-1} \right),$$  \hspace{1cm} (4)

the corresponding coupling matrix $\Lambda$ is given by

$$\Lambda = H$$  \hspace{1cm} (5)

and the corresponding scattering matrix is given by $S = K$.

Note that $M$ is a Hermitian matrix.

**Definition 2:** The linear quantum system (1) is said to be *lossless bounded real* if the following conditions hold:

i) $F$ is a Hurwitz matrix; i.e., all of its eigenvalues have strictly negative real parts;

ii) The transfer function matrix $\Phi(s) = H(sI - F)^{-1}G + K$ satisfies $\Phi(i\omega)^\dagger \Phi(i\omega) = I$ for all $\omega \in \mathbb{R}$.

The following definition extends the standard linear systems notion of minimal realization to linear quantum systems of the form (1); see also [16].

**Definition 3:** A linear quantum system of the form (1) is said to be *minimal* if the following conditions hold:

i) *Controllability.* $x^\dagger F = \lambda x^\dagger$ for some $\lambda \in \mathbb{C}$ and $x^\dagger G = 0$ implies $x = 0$;
ii) **Observability.** $Fx = \lambda x$ for some $\lambda \in \mathbb{C}$ and $Hx = 0$ implies $x = 0$.

The following theorem is an straightforward extension of Theorem 6.6 of [16] to allow for a general scattering matrix $S$.

**Theorem 2:** A minimal linear quantum system of the form (I) is physically realizable if and only if the system is lossless bounded real.

### III. **Singularly Perturbed Linear Quantum Systems**

#### A. General Singular Perturbations

We now consider a class of quantum systems of the form (I) dependent on a parameter $\epsilon > 0$ which are referred to as singularly perturbed quantum systems:

$$
\begin{align*}
\dot{a}_1(t) &= F_{11}a_1(t)dt + F_{12}a_2(t)dt + G_1du(t); \\
\dot{a}_2(t) &= \frac{1}{\epsilon}F_{21}a_1(t)dt + \frac{1}{\epsilon}F_{22}a_2(t)dt + \frac{1}{\epsilon}G_2du(t); \\
\dot{y}(t) &= H_1a_1(t)dt + H_2a_2(t)dt + Kdu(t). \\
\end{align*}
$$

This system can be re-written in the more standard singularly perturbed form (e.g., see [26]):

$$
\begin{align*}
\dot{a}_1(t) &= F_0a_1(t)dt + G_0du(t); \\
\epsilon \dot{a}_2(t) &= F_2a_1(t)dt + F_2a_2(t)dt + G_2du(t); \\
\dot{y}(t) &= H_0a_1(t)dt + H_2a_2(t)dt + Kdu(t).
\end{align*}
$$

If the matrix $F_{22}$ is non-singular, we can define the corresponding reduced dimension slow subsystem (e.g., see [26]) by formally setting $\epsilon = 0$ in (7) to obtain

$$
\begin{align*}
\dot{a}_1(t) &= F_0a_1(t)dt + G_0du(t); \\
\dot{y}(t) &= H_0a_1(t)dt + K_0du(t)
\end{align*}
$$

where

$$
\begin{align*}
F_0 &= F_{11} - F_{12}F_{22}^{-1}F_{21}; \\
G_0 &= G_1 - F_{12}F_{22}^{-1}G_2; \\
H_0 &= H_1 - H_2F_{22}^{-1}F_{21}; \\
K_0 &= K - H_2F_{22}^{-1}G_2.
\end{align*}
$$
This is the singular perturbation approximation to the system (6). We are interested in whether the reduced dimension quantum system (8), (9) is physically realizable if the singularly perturbed quantum system (7) is physically realizable for all $\epsilon > 0$. One approach to addressing this question might be to apply Theorem 2 and indeed, we can obtain the following theorem which is the main result of the paper:

**Theorem 3:** If the singularly perturbed linear quantum system (7) is physically realizable for all $\epsilon > 0$ and the matrix $F_{22}$ is non-singular, then the corresponding reduced dimension quantum system (8), (9) is such that the matrix $F_0$ has all of its eigenvalues in the closed left half of the complex plane and the transfer function matrix $\Phi_0(s) = H_0(sI - F_0)^{-1}G_0 + K_0$ satisfies

$$\Phi_0(i\omega)^\dagger\Phi_0(i\omega) = I$$

for all $\omega \in \mathbb{R}$.

The proof of this theorem is given in the appendix.

Note that this result is not sufficient to prove the physical realizability of the reduced dimension quantum system (8), (9) since the application of Theorem 2 requires that the system realization be minimal and hence the conditions of Theorem 2 will only be satisfied if the matrix $F_0$ is Hurwitz. However, the properties established in this theorem indicate that in all but pathological cases, the singular perturbation approximation will yield a physically realizable reduced dimension system. These pathological cases can be detected by testing the eigenvalues and minimality of the reduced order system. In addition, the following example shows one such pathological system in which the singular perturbation approximation is not strictly Hurwitz and not minimal.

**Example** We consider a singularly perturbed quantum linear system of the form (7) where

$$F_{11} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} ; F_{12} = I, F_{21} = \frac{1}{2} I, F_{22} = -I,$$

$$G_1 = -I, G_2 = I, H_1 = I, H_2 = -2I, K = I.$$

For each $\epsilon > 0$, we calculate the characteristic polynomial of the matrix $F_\epsilon = \begin{bmatrix} F_{11} & F_{12} \\ F_{21}/\epsilon & F_{22}/\epsilon \end{bmatrix}$ to be

$$p(s) = s^4 + \left(1 + \frac{2}{\epsilon}\right)s^3 + \left(\frac{5}{4} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)s^2 + \frac{2}{\epsilon}s + \frac{1}{\epsilon^2}.$$
From this, it follows using the Routh-Hurwitz criterion that the matrix $F_\epsilon$ is Hurwitz for all $\epsilon > 0$. Furthermore, it is straightforward to verify that the matrix $\Theta_\epsilon = \begin{bmatrix} I & 0 \\ 0 & I/\epsilon \end{bmatrix} > 0$ satisfies the conditions

$$F_\epsilon \Theta_\epsilon + \Theta_\epsilon F_\epsilon^\dagger + G_\epsilon G_\epsilon^\dagger = 0;$$

$$G_\epsilon + \Theta_\epsilon H^\dagger = 0$$

(11)

where $G_\epsilon = \begin{bmatrix} G_1 \\ G_2/\epsilon \end{bmatrix}$ and $H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$. Hence, it follows from Theorem 1 that this singularly perturbed quantum system is physically realizable for all $\epsilon > 0$. Furthermore, it follows from (11) that this system is in fact minimal for all $\epsilon > 0$. However, when we consider the reduced order approximate system, we calculate $F_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which is not Hurwitz. Also, $G_0 = 0$, $H_0 = 0$, $K_0 = -I$ and thus, the reduced order system is not minimal.

This example shows that a stronger result than Theorem 3, which guarantees minimality and Hurwitzness of the approximate system, cannot be obtained in the general case. In the next subsection, we consider a special class of singular perturbations for which the physical realizability of the reduced dimension system can be guaranteed.

**B. A Special Class of Singular Perturbations**

We now consider a special class of singularly perturbed physically realizable quantum systems of the form (6) defined in terms of the matrices $S$, $\Lambda$ and $M$ in Definition 1. Indeed, we consider the case in which $\Theta = I$,

$$\Lambda = \begin{bmatrix} \Lambda_1 & \frac{1}{\sqrt{\epsilon}} \Lambda_2 \end{bmatrix} ; \quad M = \begin{bmatrix} M_{11} & \frac{1}{\sqrt{\epsilon}} M_{12} \\ \frac{1}{\sqrt{\epsilon}} M_{12}^\dagger & \frac{1}{\epsilon} M_{22} \end{bmatrix}$$
for all $\epsilon > 0$ where $S^\dagger S = I$, and $M_{11}$ and $M_{22}$ are Hermitian matrices. Then, substituting these values into (2), we obtain the following linear quantum system of the form (1):

$$
\begin{align*}
da_1(t) &= -\left( \frac{1}{2} \Lambda_1^\dagger \Lambda_1 + i M_{11} \right) a_1(t) dt \\
&\quad - \frac{1}{\sqrt{\epsilon}} \left( \frac{1}{2} \Lambda_1^\dagger \Lambda_2 + i M_{12} \right) a_2(t) dt - \Lambda_1^\dagger S du(t); \\
da_2(t) &= -\frac{1}{\sqrt{\epsilon}} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_1 + i M_{12}^\dagger \right) a_1(t) dt \\
&\quad - \frac{1}{\epsilon} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right) a_2(t) dt - \frac{1}{\sqrt{\epsilon}} \Lambda_2^\dagger S du(t); \\
dy(t) &= \Lambda_1 a_1(t) dt + \frac{1}{\sqrt{\epsilon}} \Lambda_2 a_2(t) dt + S du(t).
\end{align*}
$$

(12)

If we make the change of variables $\bar{a}_2(t) = \frac{1}{\sqrt{\epsilon}} a_2(t)$, this leads to the following singularly perturbed quantum system of the form (6):

$$
\begin{align*}
da_1(t) &= -\left( \frac{1}{2} \Lambda_1^\dagger \Lambda_1 + i M_{11} \right) a_1(t) dt \\
&\quad - \frac{1}{\epsilon} \left( \frac{1}{2} \Lambda_1^\dagger \Lambda_2 + i M_{12} \right) \bar{a}_2(t) dt - \Lambda_1^\dagger S du(t); \\
d\bar{a}_2(t) &= -\frac{1}{\epsilon} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_1 + i M_{12}^\dagger \right) a_1(t) dt \\
&\quad - \frac{1}{\epsilon} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right) \bar{a}_2(t) dt - \frac{1}{\epsilon} \Lambda_2^\dagger S du(t); \\
dy(t) &= \Lambda_1 a_1(t) dt + \Lambda_2 \bar{a}_2(t) dt + S du(t).
\end{align*}
$$

(13)

Note that even though we formally let $\epsilon \to 0$ in the singular perturbation approximation, this state space transformation can be applied for each fixed $\epsilon > 0$. Then, for the singularly perturbed linear quantum system (13), we can obtain the corresponding reduced dimension approximate system according to equations (8), (9).

The following result is obtained for singularly perturbed linear quantum systems of the form (13). This result can also be derived from the general nonlinear results presented in the papers [23], [25]. However, this result for the linear case is included here for the sake of completeness.

**Theorem 4:** Consider a singularly perturbed linear quantum system (13) which is physically realizable for all $\epsilon > 0$ and suppose that the matrix $-\left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)$ is nonsingular. Then the corresponding reduced dimension approximate system defined by equations (8), (9) is physically realizable.
The proof of this theorem is given in the appendix.

IV. ILLUSTRATIVE EXAMPLE

We consider an example from quantum optics involving the interconnection of two optical cavities as shown in Figure 1. Each optical cavity consists of two partially reflective mirrors which are spaced at a specified distance to give a cavity resonant frequency which corresponds to the frequency of the driving laser; e.g., see [18], [20]. In practice, optical isolators would also need to be included in the optical connections between the cavities to ensure that the light traveled only in one direction.

![Diagram of a linear optical quantum system](image)

Fig. 1. A linear optical quantum system.

Here $K_1$ and $K_2$ are the coupling parameters of the first cavity and $\tilde{\gamma}$ is the coupling parameter of the second cavity. These parameters are determined by the physical characteristics of each cavity including the mirror reflectivities. The QSDE of the form (1) describing this quantum system is as follows:

$$
\begin{align*}
\frac{da}{dt} &= \begin{bmatrix}
-K_1 & K_2 - \sqrt{K_1 K_2} & -\sqrt{K_1 \tilde{\gamma}} & -\frac{\tilde{\gamma}}{2}
\end{bmatrix}
\begin{bmatrix}
a \\
\tilde{a}
\end{bmatrix}
\quad dt - \begin{bmatrix}
\sqrt{K_1} & \sqrt{K_2} & \sqrt{\tilde{\gamma}}
\end{bmatrix}
\begin{bmatrix}
a \\
\tilde{a}
\end{bmatrix}
\quad du_2;

dy_1 &= \begin{bmatrix}
\sqrt{K_1} & \sqrt{K_2} & \sqrt{\tilde{\gamma}}
\end{bmatrix}
\begin{bmatrix}
a \\
\tilde{a}
\end{bmatrix}
\quad dt + du_2.
\end{align*}
$$

(14)

We wish to consider the reduced dimension approximation to this system which is obtained by letting $\tilde{\gamma} \to \infty$. This corresponds to the case in which the mirrors in Cavity 2 are perfectly...
reflecting and so there is a direct optical feedback from the output $y_2$ of Cavity 1 into the input $u_1$ of Cavity 1. If we let $\epsilon = \frac{1}{\gamma}$, it is straightforward to verify that this system is a system of the form (12) with $\Theta = I$, $S = I$, $\Lambda_1 = \sqrt{K_1} + \sqrt{K_2}$, $\Lambda_2 = 1$, $M_{11} = 0$, $M_{22} = 0$, $M_{12} = \frac{i}{2} \left( \sqrt{K_1} - \sqrt{K_2} \right)$. With the change of variables $\bar{a} = \sqrt{\gamma} \tilde{a} = \frac{1}{\sqrt{\epsilon}} \tilde{a}$, the system becomes

\[
\frac{d}{dt} \begin{bmatrix} a \\ e\bar{a} \end{bmatrix} = \begin{bmatrix} -\frac{K_1 + K_2}{2} - \sqrt{K_1} K_2 \\ -\sqrt{K_2} \\ \sqrt{K_1} + \sqrt{K_2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ \tilde{a} \end{bmatrix} dt - \begin{bmatrix} \sqrt{K_1} + \sqrt{K_2} \\ 1 \end{bmatrix} du_2;
\]

\[
dy_1 = \left( \sqrt{K_1} + \sqrt{K_2} \right) \begin{bmatrix} a \\ \tilde{a} \end{bmatrix} dt + du_2
\]  

which is a singularly perturbed quantum system of the form (7). Hence, the corresponding reduced dimension slow subsystem (8), (9) is given by

\[
da = \left( -\frac{K_1 + K_2}{2} + \sqrt{K_1 K_2} \right) dt + \left( \sqrt{K_1} - \sqrt{K_2} \right) du_2
\]

\[
dy_1 = \left( \sqrt{K_1} - \sqrt{K_2} \right) dt - du_2.
\]  

(16)

Since the system (15) satisfies the conditions of Theorem 4, it follows from this theorem that the system (16) will be physically realizable. This can also be verified directly by noting that the system (16) satisfies the conditions of Theorem 1 with $\Theta = 1$.

Note that for this example, if $K_1 = K_2$, then the reduced dimension quantum system is uncontrollable, unobservable and has a pole at the origin.

V. CONCLUSIONS

In this paper, we have considered the physical realizability properties of the singular perturbation approximation to a class of singularly perturbed linear quantum systems. These results may be useful in the modeling of linear quantum systems such as gravity wave detectors where a simplified model is required without sacrificing physical realizability.

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APPENDIX

Proof of Theorem

If the singularly perturbed quantum system (7) is physically realizable for all $\epsilon > 0$, then it follows from Theorem 1 that for all $\epsilon > 0$, there exists a matrix $\Theta > 0$ such that the matrices

$$ F_\epsilon = \begin{bmatrix} F_{11} & F_{12} \\ \frac{1}{\epsilon} F_{21} & \frac{1}{\epsilon} F_{22} \end{bmatrix}, \quad G_\epsilon = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon} G_2 \end{bmatrix}; $$

$$ H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}; \quad K $$

satisfy the conditions (3). Hence, it follows from the first of these equalities and Fact 12.21.3 of [30] that matrix $F_\epsilon$ has all of its eigenvalues in the closed left half of the complex plane for all $\epsilon > 0$. Then, using a standard result on singularly perturbed linear systems (e.g., see Theorem 3.1 on page 57 of [26]) it follows that the matrix $F_0$ has all of its eigenvalues in the closed left half of the complex plane.

With the matrices $F_\epsilon, G_\epsilon, H$ and $K$ defined as above, it follows by a straightforward but tedious calculation that we can write the transfer function $\Phi_\epsilon(s) = H(sI - F_\epsilon)^{-1}G_\epsilon + K$ in the form:

$$ \Phi_\epsilon(s) = \left( H_0 + H_2 F_{22}^{-1} \left( I - \frac{F_{22}}{\epsilon s} \right)^{-1} F_{21} \right) \left[ sI - F_0 - F_{12} \left( I - \frac{F_{22}}{\epsilon s} \right)^{-1} F_{21} \right]^{-1} $$

$$ \times \left( G_0 + F_{12} \left( I - \frac{F_{22}}{\epsilon s} \right)^{-1} G_2 \right) $$

$$ + K_0 + H_2 F_{22}^{-1} \left( I - \frac{F_{22}}{\epsilon s} \right)^{-1} G_2 $$

where the matrices $F_0, G_0, H_0, K_0$ are defined as in (9).

Now for small values of $\epsilon > 0$, we can approximate the term $\left( I - \frac{F_{22}}{\epsilon s} \right)^{-1}$ in the above expression as follows:

$$ \left( I - \frac{F_{22}}{\epsilon s} \right)^{-1} = -\epsilon s F_{22}^{-1} + O(\epsilon^2). $$

From this, it follows that we can write

$$ \Phi_\epsilon(s) = \left( H_0 - \epsilon s H_2 F_{22}^{-2} F_{21} \right) (sI - F_0)^{-1} \left[ I + \epsilon s F_{12} F_{22}^{-1} F_{21} (sI - F_0)^{-1} \right]^{-1} $$

$$ \times \left( G_0 - \epsilon s F_{12} F_{22}^{-1} G_2 \right) $$

$$ + K_0 - \epsilon s H_2 F_{22}^{-2} G_2 + O(\epsilon^2). $$
From this and some further straightforward manipulations and simplifications, we can obtain

\[ \Phi_\epsilon(s) = \Phi_0(s) - \epsilon s \left( H_0 (sI - F_0)^{-1} F_{12} + H_2 F_{22}^{-1} \right) F_{22}^{-1} \left( F_{21} (sI - F_0)^{-1} G_0 + G_2 \right) + O(\epsilon^2). \]  

(17)

Now using the fact that the matrices \( F_\epsilon, G_\epsilon, H \) and \( K \) satisfy the conditions (3), we will show that transfer function matrix \( \Phi_\epsilon(s) \) is unitary at all frequencies. Indeed using (3), we have for all \( \omega \in \mathbb{R} \)

\[ H (i\omega I - F_\epsilon)^{-1} G_\epsilon G_\epsilon^\dagger (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger \]

\[ = H (i\omega I - F_\epsilon)^{-1} [(i\omega I - F_\epsilon) \Theta + \Theta (-i\omega I - F_\epsilon^\dagger)] (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger \]

\[ = H \Theta (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger + H (i\omega I - F_\epsilon)^{-1} \Theta H^\dagger \]

\[ = -K G_\epsilon^\dagger (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger - H (i\omega I - F_\epsilon)^{-1} G_\epsilon K^\dagger. \]

Now using the third equation of (3) and the fact that \( K \) is square, we have for all \( \omega \in \mathbb{R} \)

\[ 0 = I - KK^\dagger - KG_\epsilon^\dagger (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger - H (i\omega I - F_\epsilon)^{-1} G_\epsilon K^\dagger \]

\[ -H (i\omega I - F_\epsilon)^{-1} G_\epsilon G_\epsilon^\dagger (-i\omega I - F_\epsilon^\dagger)^{-1} H^\dagger \]

\[ = I - \Phi_\epsilon(i\omega) \Phi_\epsilon(i\omega)^\dagger. \]

Therefore, since \( \Phi_\epsilon(i\omega) \) is square we have

\[ \Phi_\epsilon(i\omega)^\dagger \Phi_\epsilon(i\omega) = I \quad \forall \omega \in \mathbb{R} \]  

(18)

for all \( \epsilon > 0 \). Hence, it follows from (17) and the fact that (18) holds for all \( \epsilon > 0 \) that we must have

\[ \Phi_0(i\omega)^\dagger \Phi_0(i\omega) = I \]

for all \( \omega \in \mathbb{R} \). This completes the proof of the theorem. \( \square \)

**Proof of Theorem**

For the singularly perturbed linear quantum system (13), it is straightforward but tedious to verify that the corresponding reduced dimension slow subsystem (3), (9) is given by

\[ da_1(t) = -\left( i\bar{M} + \frac{1}{2} \bar{\Lambda}^\dagger \bar{\Lambda} \right) a_1(t) dt - \bar{\Lambda}^\dagger \bar{S} du(t); \]

\[ dy(t) = \bar{\Lambda} a_1(t) dt + \bar{S} du(t) \]  

(19)
where
\[ \tilde{\Lambda} = \Lambda_1 - \Lambda_2 \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_1 + i M_{12}^\dagger \right); \]
\[ \tilde{S} = S - \Lambda_2 \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} \Lambda_2^\dagger S; \]
\[ \tilde{M} = M_{11} + \frac{1}{4} \Lambda_1^\dagger \Lambda_2 \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 - i M_{22} \right)^{-1} M_{22} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} \Lambda_2^\dagger \Lambda_2 \]
\[ + M_{12} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 = i M_{22} \right)^{-1} M_{22} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} M_{22}^\dagger \]
\[ - \frac{1}{4} M_{12} \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 - i M_{22} \right)^{-1} \Lambda_2^\dagger S \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} \Lambda_2^\dagger \Lambda_1 \]
\[ - \frac{1}{4} \Lambda_1^\dagger \Lambda_2 \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 - i M_{22} \right)^{-1} \Lambda_2^\dagger \Lambda_2 \left( \frac{1}{2} \Lambda_2^\dagger \Lambda_2 + i M_{22} \right)^{-1} \Lambda_2^\dagger M_{12}^\dagger. \] (20)

Furthermore, it is straightforward to verify that \( \tilde{S}^\dagger \tilde{S} = I \) and the matrix \( \tilde{M} \) is Hermitian. Hence, it follows from Definition [1] that the system (19) is physically realizable with the matrices \( \tilde{\Lambda}, \tilde{S}, \tilde{M} \) defined as above and with \( \Theta = I \). This completes the proof of the theorem. ☐

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