A relation between deformed superspace and Lee–Wick higher-derivative theories

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Abstract

We propose a non-anticommutative superspace that relates to the Lee–Wick type of higher-derivative theories, which are known for their interesting properties and have led to proposals of phenomenologically viable higher-derivative extensions of the Standard Model. The deformation of superspace we consider does not preserve supersymmetry or associativity in general, but, we show that a non-anticommutative version of the Wess–Zumino model can be properly defined. In fact, the definition of chiral and antichiral superfields turns out to be simpler in our case than in the well known supersymmetric case. We show that when the theory is truncated at the first nontrivial order in the deformation parameter, supersymmetry is restored, and we end up with a well-known Lee–Wick type of higher-derivative extension of the Wess–Zumino model. Thus, we show how non-anticommutativity could provide an alternative mechanism for generating these higher-derivative theories.

Keywords: supersymmetry, noncommutative theories, higher-derivative theories

1. Introduction

The fundamental nature of spacetime at the Planck scale is still considered to be an open question, and the emergence of some type of noncommutativity between coordinates has been studied in different contexts for some time [1–6]. One of the nontrivial aspects of these studies is the fate of spacetime symmetries when noncommutativity is considered. Poincaré
symmetry can be broken by the noncommutativity, such as in the low-energy description of some string dynamics with a fixed background [5]. It can also be preserved in some adequate sense [7, 8], or deformed in an algebraically consistent way, such as in the context of kappa deformed spacetimes [2, 3]. Poincaré symmetry is one of the conceptual cornerstones of the current understanding of elementary interactions. In addition, in recent decades the possibility of small deviations of Lorentz symmetry has been studied in a systematic way [9], providing increasingly stronger phenomenological constraints on Lorentz violation [10]. These facts reinforce the interest in studying the fundamental symmetries of proposed deformations of spacetime, including those involving noncommuting coordinates.

Supersymmetry is known to be the essentially unique, consistent extension of the Poincaré symmetry [11], and not surprisingly, the possibility of implementing non-commutative deformations of superspace has also been studied for some time. Following the discovery of the connection between noncommutative geometry and superstring theory, a string configuration where a supersymmetric field theory with spacetime noncommutativity appears as a low-energy limit was identified in [12]. In this case, supersymmetry is left untouched by the deformation, which affects only the spacetime coordinates \( x^\mu \), while fermionic coordinates \( \theta^\alpha \), \( \bar{\theta}^\dot{\alpha} \) satisfy the usual anticommutation relations. Because of this, all the superspace formalism for studying the classical and quantum aspects of supersymmetric field theories can be preserved in the study of its noncommutative counterparts [13–16].

In [17], a general discussion of nontrivial (anti)commutation relations between spacetime (and spinorial) coordinates of superspace was presented, and not surprisingly, the discussion becomes more complicated if the algebra of the fermionic coordinates \( \theta^\alpha \), \( \bar{\theta}^\dot{\alpha} \) is modified. It was pointed out that in the non-anticommutative case, the supersymmetry algebra is generally deformed, in addition to presenting the associativity of the products of superfields is lost. The authors of [17] discussed a special case where associativity is maintained, in addition to presenting a more general study of non-anticommutative deformations of superspace that preserve the essential properties of supersymmetry and associativity, which can be developed using the formalism of twist deformations of Hopf algebras [18–20].

In [21], another interesting connection between string theory and non-anticommutative theory was unveiled. In this case, the anticommutation relation involving fermionic coordinates, \( \theta^\alpha \), is the primary source of deformation, and the algebra of half of the supersymmetry generators is broken, thus giving the name \( N = 1/2 \) supersymmetry. The notion of chirality has to be dealt with properly, and a consistent interacting theory involving gauge and scalar superfields has been defined. The spurion technique can be used to treat these theories at the quantum level, and their renormalizability has been studied [22–27]. We will review some aspects of the \( N = 1/2 \) supersymmetric field theories in section 2, since they will be relevant to our work.

In this work, we propose a new type of deformed superspace, in which the anticommutator involving \( \theta^\alpha \) and \( \bar{\theta}^\dot{\alpha} \) is deformed. We argue that this possibility has very interesting properties, mainly that at the second order in the deformation parameter, the scalar superfield theory defined in this non-anticommutative superspace exactly coincides with the higher-derivative supersymmetric Wess–Zumino model studied, for example, in [28]. This is interesting because Lee–Wick-type theories [29, 30] have recently been revived in higher-derivative extensions of the Standard Model, which attempts to be phenomenologically viable and free of hierarchy problems [31–34]. Supersymmetric models with higher-derivative operators have also attracted some attention at a more formal level in recent years [35–37]. The connection found in this work, between a particular form of deformation of superspace and the higher-derivative supersymmetric model studied in [28, 35] is even more interesting.
because the former breaks supersymmetry in general. However, we will show how, at least at some finite order of the deformation parameter, supersymmetry is fully restored in our model.

This work is organized as follows. In section 2 we review some aspects of the $N = 1/2$ supersymmetry construction, and we propose our deformed superspace, discussing its main properties. The supersymmetric algebra in this deformed superspace is studied in section 3, and we show that it is also deformed, signaling that supersymmetry is not preserved by our construction in general. In section 4, however, we show that we can define an action for scalar superfields in this deformed superspace, and we show how it becomes equivalent, at the second order of the deformation parameter, to a higher-derivative, supersymmetric invariant Wess–Zumino model. Section 5 contains our conclusions and perspectives.

2. Deformed superspace

We start by reviewing some of the basic aspects of the non-anticommutative superspace discussed in [21].

The four-dimensional superspace has bosonic coordinates, $x^{μ}$, which are spacetime vectors, and fermionic coordinates, $θ^{α}$, $\bar{θ}^{\dot{α}}$, which are two-component Weyl spinors, whose indices can be lowered and raised with the antisymmetric symbols $\epsilon^{αβ}$ and $\epsilon_{αβ}$, normalized according to $\epsilon_{12} = -\epsilon^{12} = 1$, and similarly for dotted indices. It is also useful to introduce the symbols

$$(\sigma^{μ})_{αα} = (1, σ_{1}, σ_{2}, σ_{3}),$$  \hfill (1a)

$$(\bar{σ}^{μ})_{αα} = (1, -σ_{1}, -σ_{2}, -σ_{3}),$$  \hfill (1b)

$$(σ^{μν})_{αα} = (σ^{μν} - σ^{νμ})_{αα},$$  \hfill (1c)

with $σ_{i}$ being the Pauli matrices. Finally, supersymmetry generators and supercovariant derivatives are defined by

$$Q_{α} = \partial_{α} - iσ^{μ}_{αα} \bar{θ}^{α} \partial_{μ}, \quad \bar{Q}_{α} = -\partial_{α} + iθ^{α}σ^{α}_{α} \partial_{μ},$$  \hfill (2a)

$$D_{α} = \partial_{α} + iσ^{μ}_{αα} \bar{θ}^{α} \partial_{μ}, \quad \bar{D}_{α} = -\partial_{α} - iθ^{α}σ^{α}_{α} \partial_{μ}.$$  \hfill (2b)

We refer the reader to [38] for other notations and definitions that are used in this work.

The deformation in superspace is introduced by assuming that the coordinates, $θ^{α}$, no longer anticommute, but instead satisfy

$$\{θ^{α}, θ^{β}\} = ε^{αβ},$$  \hfill (3)

where $ε^{αβ}$ is a two-dimensional symmetric matrix that plays the role of a deformation parameter. The $\bar{θ}^{α}$ coordinates anticommute with themselves and $θ^{α}$ as usual, but commutation relations involving $x^{μ}$ and $θ^{α}$ are changed to

$$[x^{μ}, x^{ν}] = \partial_{μ}ε^{αβ}ε_{βν}(σ^{α})_{αα},$$  \hfill (4)

$$[x^{μ}, θ^{α}] = iε^{αβ}(σ^{μ})_{αβ} \bar{θ}^{β}.$$  \hfill (5)

This particular algebra is interesting because, if written in terms of chiral coordinates,

$$y^{μ} = x^{μ} + iθ^{α}σ^{μ} \bar{θ}^{α},$$  \hfill (6)
it implies that
\[
[y^\mu, y^\nu] = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^\dot{\alpha}] = 0,
\]
which simplifies the definition of chiral superfields.

Due to equation (3), products of the $\theta$ coordinate should be properly ordered, and when multiplying two functions of $\theta$, the result should be reordered; this can be accomplished by means of a star product,
\[
f(\theta) \star g(\theta) = f(\theta) \exp \left( -\frac{\epsilon c^{\alpha\dot{\beta}}}{2} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} \right) g(\theta),
\]
which implements the Weyl (symmetric) ordering. It is noteworthy that the exponential in equation (8) is finite due to the anticommuting nature of the derivatives, $\partial/\partial \theta^\alpha$. Chiral superfields are defined to be functions of $y$ and $\theta$ alone, and when multiplying two chiral superfields via the star product, the result is still a chiral superfield.

Finally, when studying the anticommutation relations between the supersymmetry generators $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$, one finds the standard relations
\[
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^\mu_\alpha \partial_\mu, \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 0.
\]

On the other hand,
\[
\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = -4c^{\alpha\dot{\beta}}\sigma^\mu_\alpha \sigma^{\dot{\beta}}_\dot{\alpha} \frac{\partial^2}{\partial y^\mu \partial \bar{y}^{\dot{\mu}}},
\]
signaling that half of the supersymmetry generators are broken—thus the name $N = 1/2$ supersymmetry for this construction.

One can continue to define antichiral and vector superfields, as done in [21], but at this point we will present our proposal for a different non-anticommutative superspace.

Our proposal is defined via a different star product that will be responsible for reordering functions of $\theta$ and $\bar{\theta}$ that are multiplied together,
\[
f(z) \star g(z) = f(z) \exp \left[ \frac{\epsilon}{2} \mathcal{C}^{\alpha\dot{\beta}} \left( \bar{D}_\alpha \bar{D}_{\dot{\beta}} + \bar{D}_{\alpha} \bar{D}_{\dot{\beta}} \right) \right] g(z)
\]
\[
= fg + \frac{\epsilon}{2} (-1)^{y_{\bar{\mu}}} \left[ (D_{\nu}D_{\bar{\mu}})(D_{\bar{\nu}}g) + (D_{\bar{\nu}}D_{\nu})g \right] \\
- \frac{\epsilon^2}{16} \mathcal{C} \left[ (D_{\bar{\nu}}D_{\nu})(D_{\bar{\nu}}g) + (D_{\bar{\nu}}D_{\nu})g \right] \\
- \frac{\epsilon^2}{8} \mathcal{C}^{\alpha\dot{\beta}} \left[ (D_{\beta}D_{\dot{\alpha}})(D_{\bar{\nu}}D_{\nu}g) + (D_{\beta}D_{\dot{\alpha}})(D_{\bar{\nu}}D_{\nu}g) \right] + O(\bar{\epsilon}^3),
\]
where
\[
|\mathcal{C}| = \frac{1}{2} C_{\alpha\dot{\beta}} \epsilon_{\alpha\dot{\beta}}
\]
(13)
is the determinant of $C^{\alpha\alpha}$,

$$\xi = \frac{1}{M^2},$$

(14)

with $M$ being a very large mass scale, whose significance will become clear later, and $s_f$ being the parity of the function, $f(z)$ (i.e., $s_f = 0$ if $f$ is bosonic and $s_f = 1$ if $f$ is fermionic). Note that, unlike the star product considered in the $N = 1/2$ supersymmetry, (equation (8)), the star product above is not finite due to a mixture of the two different types of covariant derivatives, $D$ and $\bar{D}$. We will discuss this fact, in connection with our particular model, later on.

Based on the definition in equation (12), we can calculate the deformed anticommutator between the fermionic variables $\theta$ and $\bar{\theta}$, obtaining

$$\{ \theta^\alpha, \bar{\theta}^\beta \}_\star = \bar{\theta}^\beta \star \theta^\alpha + \bar{\theta}^\beta \star \theta^\alpha$$

$$= \bar{\theta}^\beta \theta^\alpha - \frac{\xi}{2} C^{\beta \beta} (D_\beta \theta^\alpha)(D_\bar{\beta} \bar{\theta}^\beta)$$

$$+ \bar{\theta}^\beta \theta^\alpha - \frac{\xi}{2} C^{\bar{\beta} \bar{\beta}} (\bar{D}_\bar{\beta} \bar{\theta}^\beta)(\bar{D}_\beta \theta^\alpha)$$

$$= \xi C^{\alpha\alpha}.$$  

(15)

In the same fashion, we can calculate the complete algebra of the variables $x^\mu, \theta^\alpha, \bar{\theta}^\alpha$, and the chiral coordinates $y^\mu$ (see equation (6)), the and antichiral coordinates,$$
\bar{y}^\mu = x^\mu - i \theta \sigma^\mu \bar{\theta}.
$$

The result follows:

$$\{ \theta^\alpha, \bar{\theta}^\alpha \}_\star = \xi C^{\alpha\alpha},$$

(17a)

$$\{ \theta^\alpha, \theta^\beta \}_\star = 0, \quad \{ \bar{\theta}^\alpha, \bar{\theta}^\beta \}_\star = 0,$$

(17b)

$$[x^\mu, \theta^\alpha]_\star = \xi C^{\mu\alpha} \left( \alpha_{\alpha \beta} \sigma_{\beta \bar{\beta}} - \alpha_{\alpha \bar{\beta}} \sigma_{\beta \bar{\beta}} \right) \theta^\beta,$$

(18a)

$$[x^\mu, \bar{\theta}^\alpha]_\star = -i \xi C^{\mu\alpha} \sigma_{\beta \bar{\beta}} \theta^\beta,$$

(18b)

$$[x^\mu, \bar{\theta}^\alpha]_\star = -i \xi C^{\mu\alpha} \sigma_{\beta \bar{\beta}} \theta^\beta,$$

(18c)

$$[x^\mu, y^\nu]_\star = -2 \xi C^{\mu\nu} \alpha_{\alpha \beta} \theta^\beta,$$

(19a)

$$[x^\mu, y^\nu]_\star = 2 \xi C^{\mu\nu} \alpha_{\alpha \beta} \theta^\beta,$$

(19b)

$$[y^\mu, y^\nu]_\star = [\bar{y}^\mu, \bar{y}^\nu]_\star = 0,$$

(19c)

$$[y^\mu, \bar{y}^\nu]_\star = 4 \xi C^{\mu\nu} \alpha_{\alpha \beta} \theta^\beta,$$

(19d)

and

$$[y^\mu, \theta^\alpha]_\star = 0,$$

(20a)
\[
\begin{align*}
\left[ \gamma^\mu, \bar{\Theta}^\alpha \right]_\ast &= -2i \xi C^{\beta \delta} \sigma^\mu_{\beta \rho} \bar{\Theta}^\rho = 2 \left[ \gamma^\mu, \bar{\Theta}^\alpha \right], \\
\left[ \bar{\gamma}^\mu, \Theta^\alpha \right]_\ast &= -2i \bar{\xi} C^{\alpha \beta} \sigma^\mu_{\alpha \rho} \Theta^\rho = 2 \left[ \bar{\gamma}^\mu, \Theta^\alpha \right], \\
\left[ \bar{\gamma}^\mu, \bar{\Theta}^\alpha \right]_\ast &= 0.
\end{align*}
\]

Equations 17 define the essential non-anticommutative properties of this superspace. Most interesting are equations (19c), (20a), and (20d), which show that the commutation relations involving chiral coordinates \( \gamma^\mu \) and \( \Theta^\alpha \), and those involving antichiral coordinates \( \bar{\gamma}^\mu \) and \( \bar{\Theta}^\alpha \), are not changed. This greatly simplifies the definition of chiral and antichiral superfields. For example, chiral superfields are of the form \( \Phi (\gamma^\mu, \Theta^\alpha) \), and since \( \gamma^\mu \) and \( \Theta^\alpha \) exhibit trivial (anti)commutation relations, chiral superfields do not have to be reordered; clearly, the (usual) product of chiral superfields is a chiral superfield. That means the holomorphic potential, \( W (\Phi) \), of the Wess–Zumino model is not affected by the non-anticommutativity. The same happens for antichiral superfields, and consequently, for the antiholomorphic potential. Any nontrivial modification brought by the superspace deformation to our model will be in the Kähler part of the action, \( \int d^dz K (\Phi, \bar{\Phi}) \), which in the standard Wess–Zumino action at the classical level, reduces to the kinetic term, \( \int d^dz \Phi \bar{\Phi} \).

This last observation is also very important since the star product in equation (12) is nonassociative. Indeed, it can be regarded as a particular case of the general star product discussed in [17]. (See section 4 of that paper, where the deformation considered here corresponds to setting \( P^{\alpha \beta} \neq 0 \).) There, a general discussion on nonanticommutativity was presented, and it was shown that for very limited cases—not including ours—it leads to an associative star product. Another way to see the nonassociativity of equation (12) is briefly stated at the end of the next section. The nonassociativity implies that defining star products of more than two superfields could introduce an ambiguity in the deformed action. However, since the star product does not appear in the potentials, we face no ambiguity in defining a Wess–Zumino model in our deformed superspace, at least at the classical level.

3. Operators algebra

In this section we present the algebra of the supersymmetric generators and the covariant derivatives in our deformed superspace, which can be calculated directly from the definitions contained in the last section.

The only nonvanishing anticommutators are,

\[
\begin{align*}
\{ Q_\alpha, \bar{Q}_\beta \}_\ast &= 2i \sigma^\mu_{\alpha \beta} \partial_\mu + \xi C^{\mu \beta} \sigma^\mu_{\alpha \rho} \sigma^\rho_{\beta \lambda} \partial_\lambda, \\
\{ D_\alpha, \bar{D}_\beta \}_\ast &= -2i \sigma^\mu_{\alpha \beta} \partial_\mu + \bar{\xi} C^{\alpha \beta} \sigma^\mu_{\alpha \rho} \sigma^\rho_{\beta \lambda} \partial_\lambda, \\
\{ Q_\alpha, D_\beta \}_\ast &= -\bar{\xi} C^{\mu \beta} \sigma^\mu_{\alpha \rho} \sigma^\rho_{\beta \lambda} \partial_\lambda, \\
\{ \bar{Q}_\alpha, D_\beta \}_\ast &= -\xi C^{\alpha \beta} \sigma^\mu_{\alpha \rho} \sigma^\rho_{\beta \lambda} \partial_\lambda.
\end{align*}
\]

In a general way, defining \( D_A = (D_\alpha, \bar{D}_\alpha, \partial_\mu) \) and \( Q_A = (Q_\alpha, \bar{Q}_\alpha, -\partial_\mu) \), the deformed algebra will have the following structure [17]
\[ [Q_A, Q_B]_* = T_{AB}^C Q_C + R_{AB}^{CD} Q_D, \]
\[ [D_A, D_B]_* = T_{AB}^C D_C + R_{AB}^{CD} D_D, \]
\[ [Q_A, D_B]_* = R_{AB}^{CD} Q_C D_D, \]

(22)

where \( T_{AB}^C \) is the torsion and \( R_{AB}^{CD} = -\frac{1}{8} \epsilon^{MN} T_{M(A} C_{T_B)N} D \) is the curvature tensor. In our case, the only non vanishing tensors of this type are

(23)

\[ T_{\alpha\beta}^{\mu} = -2 \sigma_{\alpha\beta}^{\mu}, \]

\[ R_{\alpha\beta}^{\mu
u} = -\frac{1}{8} \xi C^{\beta
u} \left( T_{\mu\nu}^\rho T_{\rho\alpha} + T_{\rho\alpha}^\rho T_{\mu\nu}^\rho \right) = \frac{1}{2} \xi C^{\beta
u} \left( \sigma_{\alpha\rho}^{\mu} \sigma_{\rho\alpha}^{\nu} + \sigma_{\rho\alpha}^{\nu} \sigma_{\rho\alpha}^{\mu} \right). \]

(24)

Therefore, the algebra in equation (21) can be written as

(25)

\[ \{ Q_\alpha, Q_\alpha \}_* = T_{\alpha\beta}^{\mu} Q_\mu + R_{\alpha\beta}^{\mu\nu} Q_\mu Q_\nu, \]

(26)

\[ \{ D_\alpha, D_\alpha \}_* = T_{\alpha\beta}^{\mu} D_\mu + R_{\alpha\beta}^{\mu\nu} D_\mu D_\nu, \]

(27)

\[ \{ Q_\alpha, D_\alpha \}_* = R_{\alpha\beta}^{\mu\nu} Q_\mu D_\nu. \]

From this result, we see that the supersymmetry algebra is broken in general in our deformed superspace. This could be seen as a setback in our construction, but in the next section, we will see how this problem can be solved, at least if the parameter \( \xi \) is considered to be so small that the theory can be truncated at a certain order in \( \xi \).

It is worth mentioning that, similar to the three-dimensional non-anticommutative superspace discussed in [39], one could define nonlinear generators \( \tilde{Q}_\alpha, \tilde{Q}_\bar{\alpha} \) and supercovariant derivatives \( \tilde{D}_\alpha, \tilde{D}_{\bar{\alpha}} \), which respect the usual supersymmetry algebra, in the form

(28a)

\[ \tilde{Q}_\alpha = Q_\alpha + \frac{i}{2} \xi C^{\beta\rho} \partial_\rho \partial_{\alpha\beta} \tilde{Q}_\alpha = \tilde{Q}_\alpha - \frac{i}{2} \xi C^{\beta\rho} \partial_\rho \partial_{\alpha\beta}, \]

\[ \tilde{D}_\alpha = D_\alpha - \frac{i}{2} \xi C^{\beta\rho} \partial_\rho \partial_{\alpha\beta}. \]

(28b)

These generators do not respect the standard Leibniz rule and, being nonlinear, they could be naturally incorporated in the formalism of Hopf algebras. This observation raises the question of whether the deformation we defined in equation (12) could be understood as a supersymmetric Drinfeld twist, as studied, for example, in [18, 19]. However, the answer to this question is negative. Indeed, explicit calculation shows that the ‘twist element’ that would correspond to equation (12) does not satisfy the two-cocycle condition that must be satisfied to define a Drinfeld twist. This two-cocycle condition is what guarantees the associativity of the star product, so this last observation is another way to state the fact that the deformed product defined in equation (12) is not associative in general, as discussed at the end of section 2.

4. Wess–Zumino model in the deformed superspace

To construct a Wess–Zumino action in the deformed superspace, we must first calculate the products between superfields with different chiralities. Due to equation (15), they do not
commute, and they should be multiplied using the star product (12) to properly reorder the non-anticommuting coordinates. The star products between superfields $\Phi$ and $\bar{\Phi}$, truncated at the order $\xi^2$, are given by

$$\Phi \star \Phi = \Phi \Phi + \frac{\xi^2}{2} \mathcal{C}^{\alpha \alpha} (D_\alpha \Phi)(\bar{D}_\alpha \Phi)$$

$$- \frac{\xi^2}{16} |\mathcal{C}| \left( \left( D^2 \Phi \right) \left( \bar{D}^2 \Phi \right) \right) + \mathcal{O} \left( \xi^3 \right).$$

(29)

$$\Phi \star \Phi = \Phi \Phi + \frac{\xi^2}{2} \mathcal{C}^{\alpha \alpha} (\bar{D}_\alpha \Phi)(D_\alpha \Phi)$$

$$- \frac{\xi^2}{16} |\mathcal{C}| \left( \left( \bar{D}^2 \Phi \right) \left( D^2 \Phi \right) \right) + \mathcal{O} \left( \xi^3 \right).$$

(30)

These two expressions are different, so we have an ambiguity in the generalization of the Wess–Zumino kinetic term, $\int d^8z \Phi \bar{\Phi}$, to the deformed superspace. We adopt a symmetric prescription,

$$\Phi \star \Phi + \Phi \star \Phi = 2\Phi \Phi - \frac{\xi^2}{8} |\mathcal{C}| \left( \left( D^2 \Phi \right) \left( \bar{D}^2 \Phi \right) \right) + \mathcal{O} \left( \xi^3 \right),$$

(31)

where in obtaining the right-hand side, we used the fact that $D_\alpha \Phi \bar{D}_\alpha \bar{\Phi} = -\bar{D}_\alpha \bar{\Phi} D_\alpha \Phi$.

We use equation (31) to define the free Wess–Zumino model in the deformed superspace as

$$S = \frac{1}{2} \int d^8z \left( \Phi \star \Phi + \Phi \star \Phi \right)$$

$$= \int d^8z \left[ \Phi \Phi - \frac{\xi^2}{16} |\mathcal{C}| \left( \left( D^2 \Phi \right) \left( \bar{D}^2 \Phi \right) \right) + \mathcal{O} \left( \xi^3 \right) \right].$$

(32)

Integrating by parts and using the relation $D^2 \bar{D}^2 \Phi = 16 \square \bar{\Phi}$, we obtain,

$$S = \int d^8z \left[ \Phi \left( 1 - \xi^2 \right) |\mathcal{C}| \bar{\Phi} + \mathcal{O} \left( \xi^3 \right) \right].$$

(33)

In terms of component fields, chiral and antichiral superfields are written as,

$$\Phi = A(x) + i \theta \sigma^\mu \partial_\mu A(x) + \frac{1}{4} \theta^2 \partial^2 \Box A(x) + \sqrt{2} \theta \psi(x) - \frac{i}{\sqrt{2}} \theta^2 \partial^\mu \psi(x) \sigma^\mu \partial_\nu + \theta^2 F(x)$$

$$= A(y) + \sqrt{2} \theta \psi(y) + \theta^2 F(y),$$

(34)

$$\Phi = \tilde{A}(x) - i \theta \sigma^\mu \partial_\mu \tilde{A}(x) + \frac{1}{4} \theta^2 \partial^2 \Box \tilde{A}(x) + \sqrt{2} \partial \psi(x) + \frac{i}{\sqrt{2}} \theta^2 \partial^\mu \psi(x) \sigma^\mu \partial_\nu + \theta^2 \tilde{F}(x)$$

$$= \tilde{A}(y) + \sqrt{2} \partial \psi(y) + \theta^2 \tilde{F}(y),$$

(35)

so the component form of equation (33) can be cast as

$$S = \int d^4x \left[ A \left( 1 - \xi^2 \right) |\Box| A + i \partial_\mu \psi^\mu \left( 1 - \xi^2 \right) |\Box| \bar{\psi} \right.$$

$$\left. + F \left( 1 - \xi^2 \right) |\Box| \tilde{F} + \mathcal{O} \left( \xi^3 \right) \right].$$

(36)

This is the same action considered in [28].
Equations (33) and (36), together with the fact (pointed out in section 2) that the (anti)holomorphic potentials are not modified by the star product, mean that the net effect of the superspace deformation is the introduction of Lee–Wick-type terms in the action. Most interestingly, the action in equation (33) is supersymmetric invariant. This is a result of the symmetric prescription adopted in equation (31), which cancels the linear terms present in equations (29) and (30); these linear terms are the only nonsupersymmetric terms up to order $\xi^2$. Indeed, the $\mathcal{O}(\xi^2)$ term in equation (30), for example, is written in terms of component fields as follows,

$$S^{(3)}_{\bar{D}\partial\Phi} = \int d^4x \frac{\xi}{2} C^{\alpha\alpha} \left[ i \sigma^\mu_{\alpha\alpha} \left( \partial_\mu A \cdot \bar{A} - \Box A \partial_\mu A + \bar{F} \partial_\mu F - F \partial_\mu \bar{F} \right) + \frac{1}{2} \left( \psi_\alpha \Box \bar{\psi}_\alpha + \Box \psi_\alpha \cdot \bar{\psi}_\alpha \right) \right] \left( \epsilon^\mu_{\alpha\alpha} \left( \partial_\mu \psi_\beta \bar{\psi}_\beta + \partial_\mu \bar{\psi}_\beta \cdot \partial_\mu \bar{\psi}_\beta \right) \right),$$

(37)

where

$$\zeta^\mu_{\alpha\alpha} = \sigma^\mu_{\alpha\alpha} \sigma^\nu_{\alpha\alpha} + \sigma^\nu_{\alpha\alpha} \sigma^\mu_{\alpha\alpha}.$$  

(38)

By integrating by parts, $S^{(3)}_{\bar{D}\partial\Phi}$ can be cast as

$$S^{(3)}_{\bar{D}\partial\Phi} = \int d^4x \frac{\xi}{2} C^{\alpha\alpha} \left[ i \sigma^\mu_{\alpha\alpha} \left( 2 \partial_\mu \Box A \cdot \bar{A} + 2 \bar{F} \partial_\mu F \right) + \left( \Box \psi_\alpha \cdot \bar{\psi}_\alpha \right) - \zeta^\mu_{\alpha\alpha} \partial_\mu \psi_\beta \bar{\psi}_\beta \right].$$

(39)

It is a straightforward yet cumbersome calculation to verify that equation (39) is not invariant under the standard supersymmetric transformations.

Therefore, despite the broken supersymmetric algebra discussed in section 3, if the model is truncated at order $\xi^2$, we can use the ordering ambiguity introduced by the deformation itself to cancel the nonsymmetric parts, thus restoring supersymmetry. As a conclusion, we can state that at the first nontrivial order in the deformation parameter, $\xi$, the particular form of the non-anticommutativity that we considered presents itself as an alternative mechanism for the generation of a very important class of higher-derivative theories, which are usually considered to appear as a result of the integration of fields with very large mass [28]. From equation (33), we see that the parameter, $\xi$, is indeed inversely proportional to the square of the mass scale where the higher-dimensional operators are relevant (see the discussion in [28] for example); this justifies our definition (14).

As we commented earlier, the star product we are considering does not present a finite expansion in the parameter $\xi$, so we expect our result to hold only to some finite order in $\xi$. Calculating the $\mathcal{O}(\xi^3)$ terms in equations (29) and (30), we obtained

$$\Phi \star \Phi + \bar{\Phi} \star \bar{\Phi} \big|_{\mathcal{O}(\xi^3)} = -\frac{\zeta^3}{48} C^{\alpha\alpha} C^{\beta\beta} C^{\gamma\gamma} \left[ D_\gamma \bar{D}_\gamma D_\alpha \Phi \right] \left( \bar{D}_\alpha D_\beta \bar{D}_\beta \Phi \right) + \left( \bar{D}_\beta \bar{D}_\alpha \Phi \right) \left( \bar{D}_\beta \bar{D}_\alpha \Phi \right).$$

(40)

which, due to the antisymmetry,

$$\left( D_\beta \bar{D}_\beta D_\alpha \Phi \right) \left( \bar{D}_\alpha D_\beta \bar{D}_\beta \Phi \right) = -\left( \bar{D}_\alpha \bar{D}_\beta \bar{D}_\beta \Phi \right) \left( D_\beta \bar{D}_\beta D_\alpha \Phi \right),$$

(41a)

$$\left( \bar{D}_\beta \bar{D}_\beta \bar{D}_\alpha \Phi \right) \left( \bar{D}_\alpha \bar{D}_\beta D_\beta \Phi \right) = -\left( \bar{D}_\alpha \bar{D}_\beta \bar{D}_\alpha \Phi \right) \left( \bar{D}_\gamma \bar{D}_\beta \bar{D}_\gamma \Phi \right),$$

(41b)
and after a proper relabelling of dummy indices, can be shown to vanish,
\[ \Phi \star \Phi + \Phi \star \Phi \big|_{(v^i)} = 0. \] (42)

We expect the same to happen for all odd orders of \( \xi \) due to the antisymmetry of an odd number of covariant derivatives, as in equation 41. As for the \( \mathcal{O}(\xi^3) \) terms, we found
\[
\Phi \star \Phi \big|_{(v^i)} = \frac{\xi^4}{4!} \frac{e}{2^3} |C| C^\gamma C^\alpha \left[ D_{\beta} D_{\gamma} D^2 \Phi D^2 D_{\gamma} D_{\beta} \Phi + D^2 D_{\gamma} D_{\beta} \Phi D_{\gamma} D_{\beta} D^2 \Phi \right.
\]
\[ + D_{\beta} D^2 D_{\gamma} D \Phi D_{\gamma} D_{\beta} D^2 \Phi \bigg]. \] (43)

One may verify that \( \Phi \star \Phi \big|_{\Theta(v^i)} = \Phi \star \Phi \big|_{\Theta(v^i)} \), so these terms sum up instead of canceling each other, resulting in
\[
\Phi \star \Phi + \Phi \star \Phi \big|_{(v^i)} = -\frac{\xi^4}{96} |C| C^\gamma C^\alpha \left( \sigma_{\mu}^\nu \sigma_{\alpha}^{\nu} + \sigma_{\mu}^\nu \sigma_{\nu}^{\mu} \right) \left( \partial_\mu D^2 \Phi \right) \left( \partial_\nu D^2 \Phi \right). \] (45)

The resulting fourth-order action is
\[
S \big|_{(v^i)} = \frac{1}{2} \int d^4 z \left( \Phi \star \Phi + \Phi \star \Phi \bigg| \big|_{(v^i)} \right)
\]
\[ = -\frac{\xi^4}{192} B^{\mu \nu} \int d^8 z \left( \partial_\mu D^2 \Phi \right) \left( \partial_\nu D^2 \Phi \right). \] (46)

where
\[
B^{\mu \nu} = |C| C^\gamma C^\alpha \left( \sigma_{\mu}^\nu \sigma_{\alpha}^{\nu} + \sigma_{\nu}^\mu \sigma_{\mu}^{\nu} \right). \] (47)

In component form, these \( \mathcal{O}(\xi^3) \) terms can be cast as
\[
S \big|_{(v^i)} = -\frac{\xi^4}{12} B^{\mu \nu} \int d^4 x \left( \partial_\mu \Box \partial_\nu \Box \Box + i \partial_\mu \partial_\nu A \partial_\sigma \Box \Box + \partial_\mu F \partial_\nu \Box \Box \right). \] (48)

which can be shown to be invariant under standard supersymmetric transformation. We see again, at the fourth order in \( \xi \), the cancellation of any non invariant terms in the action. One might conjecture that the same pattern holds for higher orders, but we do not pursue a general proof here, since our main result is the exact equivalence of our model up to \( \mathcal{O}(\xi^3) \) order with the supersymmetric Lee–Wick model studied in the literature. Terms like the ones in equation (45) represent higher-dimensional corrections to this model, whose effects we assume to be very small due to the smallness of the scale, \( \xi \).

5. Conclusions and perspectives

In this work, we proposed an alternative deformed superspace, which has the interesting property of being connected with higher-derivative theories that are well studied in the literature and are relevant even to phenomenological models based on the Standard Model.
We defined a non-anticommutative version of the Wess–Zumino model in this super-
space, showing that despite the supersymmetry algebra being broken in general, super-
symmetry is restored by a proper ordering prescription if the theory is truncated at the first non
trivial order in the deformation parameter, $\xi$. These results leave us with some open
questions that deserve further investigation. First, since nonassociative structures have already
appeared in the context of string theory and AdS/CFT correspondence [40], one might
wonder whether it is possible to obtain the kind of non-anticommutativity considered in this
work from some more fundamental theory, such as what was done for $N = 1/2$
supersymmetry in [21]. A second perspective is to investigate quantum corrections and study the
consistency of the model at the quantum level. Actually, this question has already received
some attention in the literature: the one-loop scalar potential in the Wess–Zumino model with
higher-derivatives was first studied with an explicit soft supersymmetry breaking term in [41].
A similar discussion without the soft breaking term was presented in [35], which actually
starts with the same action obtained in this work (equation (33)), and ends up calculating the
one-loop Kähler potential, which agrees with the work done in [42]. Due to the presence of
ghosts, the quantum consistency of such higher-dimensional models still is an interesting
question. Finally, it would be interesting to study the extension of this work to gauge theories.

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