DICHOTOMY RESULTS FOR THE $L^1$ NORM OF THE DISCREPANCY FUNCTION

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Abstract. It is a well-known conjecture in the theory of irregularities of distribution that the $L^1$ norm of the discrepancy function of an $N$-point set satisfies the same asymptotic lower bounds as its $L^2$ norm. In dimension $d = 2$ this fact has been established by Halász, while in higher dimensions the problem is wide open. In this note, we establish a series of dichotomy-type results which state that if the $L^1$ norm of the discrepancy function is too small (smaller than the conjectural bound), then the discrepancy function has to be large in some other function space.

1. Introduction

1.1. Preliminaries. For integers $d \geq 2$, and $N \geq 1$, let $\mathcal{P}_N \subset [0,1]^d$ be a finite point set with cardinality $\# \mathcal{P}_N = N$. Define the associated discrepancy function by

$$D_N(x) = \#(\mathcal{P}_N \cap [0, x)) - N||[0, x)||,$$

where $x = (x_1, \ldots, x_d)$ and $[0, x) = \prod_{j=1}^d [0, x_j)$ is a rectangle with antipodal corners at $0$ and $x$, and $|\cdot|$ stands for the $d$-dimensional Lebesgue measure. The dependence upon the selection of points $\mathcal{P}_N$ will be suppressed, as we are interested in bounds that are only a function of $N = \#\mathcal{P}_N$. The discrepancy function $D_N$ measures equidistribution of $\mathcal{P}_N$: a set of points is well-distributed if $D_N$ is small in some appropriate function space.

It is a basic fact of the theory of irregularities of distribution that relevant norms of this function in dimensions 2 and higher must tend to infinity as $N$ grows. The classic results are due to Roth [10] in the case of the $L^2$ norm and Schmidt [11] for $L^p$, $1 < p < 2$:

Theorem 1.1. For $1 < p < \infty$ and any collection of points $\mathcal{P}_N \subset [0,1]^d$, we have

$$(1.1) \quad \|D_N\|_p \gtrsim (\log N)^{(d-1)/2}.$$ 

Moreover, we have the endpoint estimate

$$(1.2) \quad \|D_N\|_{L(\log L)^{(d-2)/2}} \gtrsim (\log N)^{(d-1)/2}. $$

In dimension $d = 2$ the $L^1$ endpoint estimate above was established by Halász [8], while its Orlicz space generalization for dimensions $d \geq 3$ is due to the last author [9] (notice that, when $d = 2$, we have $L(\log L)^{(d-2)/2} = L^1$).

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The symbol “≳” in this paper stands for “greater than a constant multiple of”, and the implied constant may depend on the dimension, the function space, but not on the configuration \(\mathcal{P}_N\) or the number of points \(N\). \(A \sim B\) means \(A \lesssim B \lesssim A\).

Estimate (1.1) is sharp, i.e. there exist sets \(\mathcal{P}_N\) that meet the \(L^p\) bounds (1.1) in all dimensions. This remarkable fact is established by beautiful and quite non-trivial constructions of point distributions \(\mathcal{P}_N\). We refer the reader to one of the very good references [1, 6, 7] for more information about low-discrepancy sets, which is an important complement to the theme of this note.

The subject of our paper is the \(L^1\) endpoint. Halász’s original argument yields the following very weak extension to higher dimensions.

**Theorem 1.2.** In all dimensions \(d \geq 3\), we have

\[
\|D_N\|_{L^1([0,1]^d)} \gtrsim (\log N)^{(d-1)/2}.
\]

No improvements of (1.3) have been obtained thus far – embarrassingly, it is not even known whether the \(L^1\) norm of \(D_N\) should grow as the dimension increases. It is widely believed that the correct bound for the \(L^1\) norm matches Roth’s \(L^2\) estimates (1.1).

**Conjecture 1.3.** In all dimensions \(d \geq 3\), the following estimate holds

\[
\|D_N\|_{L^1([0,1]^d)} \gtrsim (\log N)^{(d-1)/2}.
\]

Observe that (1.2) supports this conjecture.

1.2. Main results. While the conjectural bound (1.4) does not seem accessible at this point, we shall prove several dichotomy-type results for the \(L^1\) norm, which essentially say that either the \(L^1\) norm is large, or some larger norm has to be very large.

We start with a very simple result, valid in all dimensions, which states that if a point distribution has optimally small (according to (1.1)) \(L^p\) norm of the discrepancy, then it has to satisfy the conjectured \(L^1\) estimate (1.4). In other words, if there exist sets with \(L^1\)-discrepancy so small as to violate Conjecture 1.3, they cannot simultaneously have low \(L^p\)-discrepancy.

**Theorem 1.4.** Let \(p \in (1, \infty)\). For every constant \(C_1 > 0\), there exists \(C_2 > 0\) such that whenever \(\mathcal{P}_N \subset [0,1]^d\) satisfies \(\|D_N\|_p \leq C_1 (\log N)^{(d-1)/2}\), it implies that

\[
\|D_N\|_{L^1([0,1]^d)} \geq C_2 (\log N)^{(d-1)/2}.
\]

The next theorem, also true for general dimensions, amplifies this effect. It states that if the \(L^1\)-discrepancy fails Conjecture 1.3 by a small exponent, then the \(L^2\)-discrepancy is not just suboptimal, but huge.

**Theorem 1.5.** For all dimensions \(d \geq 3\), there is an \(\epsilon = \epsilon(d) > 0\) and \(c = c(d) > 0\) such that for all integers \(N \geq 1\), every \(\mathcal{P}_N \subset [0,1]^d\) satisfies either

\[
\|D_N\|_{L^1([0,1]^d)} \geq (\log N)^{(d-1)/2-\epsilon} \quad \text{or} \quad \|D_N\|_{L^2([0,1]^d)} \geq \exp(c(\log N)^\epsilon).
\]
Thus a putative example of a distribution $\mathcal{P}_N$ with $D_N$ very small in the $L^1$ norm must be very far from extremal in the $L^2$-norm. The proof will show that one can take $\epsilon(d)$ as large as a fixed multiple of $1/d$. Specializing to the case of dimension $d = 3$, we can replace the $L^2$ norm above by a much smaller norm.

**Theorem 1.6.** In dimension $d = 3$, there holds

$$\|D_N\|_1 \cdot \|D_N\|_{L(\log L)} \gtrsim (\log N)^2.$$

Unfortunately, this estimate is consistent with a putative distribution $\mathcal{P}_N$, for which $\|D_N\|_1 \lesssim (\log N)^{1/2}$. The last theorem of this series addresses possible examples, where $D_N$ is less than $(\log N)^{1/2}$ in the $L^1$ norm.

**Theorem 1.7.** For all dimensions $d \geq 3$ and all $C_1 > 0$, there is a $C_2 > 0$ so that if $\|D_N\|_1 \leq C_1 \sqrt{\log N}$, then $\|D_N\|_2 \gtrsim N^{C_2}$.

Finally, the dichotomies above are of an essentially optimal nature in light of the examples in this next result.

**Theorem 1.8.** For all dimensions $d \geq 2$, there is a distribution such that

$$\|D_N\|_1 \lesssim (\log N)^{(d-1)/2} \quad \text{and} \quad \|D_N\|_2 \gtrsim N^{1/4}.$$

The proofs are based upon the detailed information used to obtain non-trivial improvement in the $L^\infty$ endpoint estimates in [3, 4]. We recall the required estimates in the next section and then turn to the proofs of Theorems 1.4–1.8 in §3.

2. The Orthogonal Function Method

All progress on these universal lower bounds has been based upon the orthogonal function method, initiated by Roth [10], with the modifications of Schmidt [11], as presented here. Denote the family of all dyadic intervals $I \subset [0,1]$ by $\mathcal{D}$. Each dyadic interval $I$ is the union of two dyadic intervals $I_-$ and $I_+$, each of exactly half the length of $I$, representing the left and right halves of $I$ respectively. Define the Haar function associated to $I$ by $h_I = -\chi_{I_-} + \chi_{I_+}$. Here and throughout we will use the $L^\infty$ (rather than $L^2$) normalization of the Haar functions.

In dimension $d$, the $d$-fold product $\mathcal{D}^d$ is the collection of dyadic intervals in $[0,1]^d$. Given $R = R_1 \times \cdots \times R_d \in \mathcal{D}^d$, the Haar function associated with $R$ is the tensor product

$$h_R(x_1, \ldots, x_d) = \prod_{j=1}^d h_{R_j}(x_j).$$

These functions are pairwise orthogonal as $R \in \mathcal{D}^d$ varies.

For a $d$-dimensional vector $r = (r_1, \ldots, r_d)$ with non-negative integer coordinates let $\mathcal{D}_r$ be the set of those $R \in \mathcal{D}^d$ that for each coordinate $1 \leq j \leq d$, we have $|R_j| = 2^{-r_j}$. These rectangles partition $[0,1]^d$. We call $f_r$ an $r$-function (a generalized Rademacher function) if for some choice of signs $\{\varepsilon_R : R \in \mathcal{D}_r\}$, we have

$$f_r(x) = \sum_{R \in \mathcal{D}_r} \varepsilon_R h_R(x).$$
The following is the crucial lemma of the method, see \cite{2,10,11}. Given an integer $N$, we set $n = \lceil 1 + \log_2 N \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

**Lemma 2.1.** In all dimensions $d \geq 2$ there is a constant $c_d > 0$ such that for each $r$ with $|r| := \sum_{j=1}^{d} r_j = n$, there is an $r$-function $f_r$ with $\langle D_N, f_r \rangle \geq c_d$. Moreover, for all $r$-functions there holds $|\langle D_N, f_r \rangle| \lesssim N^{-|r|}$.

Heuristically, this lemma quantifies the fact that most of the information about the discrepancy function is encoded by the Haar coefficients corresponding to boxes $R \in \mathcal{D}^d$ with volume $|R| \approx 1/N$. The proofs of most known lower bounds for the discrepancy function have been guided by this idea. We briefly outline the argument leading to (1.1).

For integer vectors $\vec{r} \in \mathbb{N}^d$, let $f_{\vec{r}}$ be an $\vec{r}$-function as in the previous lemma. Set

$$Z := \frac{1}{n^{(d-1)/2}} \sum_{\vec{r} : |\vec{r}| = n} f_{\vec{r}}.$$

It is easy to see that, due to orthogonality and the fact that the number of vectors $\vec{r} \in \mathbb{N}^d$ with $|\vec{r}| = n$ is of the order $n^{d-1}$, we have $\|Z\|_2 \approx 1$. Moreover, it also satisfies $\|Z\|_p \lesssim 1$ for all $1 < p < \infty$. This extension can be derived using Littlewood–Paley theory or, as originally done in \cite{11}, using combinatorial arguments if $p$ is an even integer. This is enough to establish (1.1): Hölder inequality and Lemma 2.1 yield

$$n^{d-1} \lesssim \langle D_N, Z \rangle \lesssim \|D_N\|_p \cdot \|Z\|_p \lesssim \|D_N\|_p.$$

The following is a deep exponential-squared distributional estimate for $Z$ – indeed, it is a key estimate behind the main theorems of \cite{4} on the $L^\infty$ norm of the discrepancy function.

**Theorem 2.2.** \cite[Theorem 6.1]{4} There is an absolute constant $0 < c < 1$, such that in all dimensions $d \geq 3$, for $\epsilon = c/d$ we have

$$|\{x : |Z(x)| > t\}| \lesssim \exp(-ct^2), \quad 0 < t < cn^{\frac{1-2\epsilon}{b-2}}$$

3. **Proofs**

We now proceed to the proofs of the main theorems.

**Proof of Theorem 1.4.** Assume that for a given $1 < p < \infty$ we have $\|D_N\|_p \leq C_1 (\log N)^{\frac{d-1}{2}}$.

The Roth–Schmidt bound (1.1) states that $\|D_N\|_{2p/(p+1)} \geq c_{2p/(p+1)} (\log N)^{\frac{d-1}{2}}$. Interpolating between 1 and $p$ using Hölder’s inequality we find that $\|D_N\|_{2p/(p+1)} \leq \|D_N\|_1^{1/2} \|D_N\|_p^{1/2}$.

Therefore

$$\|D_N\|_1 \geq \frac{\|D_N\|_{2p/(p+1)}}{\|D_N\|_p} \geq \frac{c^2_{2p/(p+1)} (\log N)^{d-1}}{C_1 (\log N)^{\frac{d-1}{2}}} = C_2 (\log N)^{\frac{d-1}{2}},$$

which proves (1.5) with $C_2 = \frac{c^2_{2p/(p+1)}}{C_1}$.

\hfill \Box
Proof of Theorem 1.5. Set \( q = n^\varepsilon \), where \( \varepsilon \simeq 1 / d \), and define
\[
Y := \frac{1}{n^{(d-1)/2}q} \sum_{\vec{r} : |\vec{r}| = n} f_{\vec{r}}.
\]
Then \( \|Y\|_p \lesssim q^{-1} \) for \( 1 < p < \infty \). Besides, one has \( \langle D_N, Y \rangle \geq c n^{d-1/2}/q \). But unfortunately \( Y \) is not bounded, preventing an immediate conclusion about the \( L^1 \) norm of \( D_N \).

On the other hand, from Theorem 2.2 we get
\[
|\{ |Y| > 1 \}| \lesssim \exp(-cq^2).
\]
Using a trilinear Hölder’s inequality, we obtain
\[
\int |D_N \cdot Y| \, dx \leq |\{ |Y| > 1 \}|^{1/4} \|Y\|_4 \|D_N\|_2 \\
\lesssim \exp(-c'q^2) \cdot q^{-1} \|D_N\|_2.
\]
This last quantity will be at most \( \frac{1}{2} \langle D_N, Y \rangle \), if \( \|D_N\|_2 \lesssim \exp(c''q^2) \). Then
\[
\|D_N\|_1 \geq |\langle D_N, Y \cdot 1_{\{|Y| \leq 1\}} \rangle| \\
\geq \langle D_N, Y \rangle - \int_{\{ |Y| > 1 \}} |D_N \cdot Y| \, dx \geq \frac{1}{2} \langle D_N, Y \rangle \geq n^{d-1/2 - \varepsilon}
\]
and this proves Theorem 1.5.

Proof of Theorem 1.6. Define
\[
Y = \frac{1}{\sqrt{n}} \sum_{j=1}^{n/2} \sin \left( cn^{-1/2} \sum_{\vec{r} : r_1 = j} f_{\vec{r}} \right)
\]
where \( 0 < c < 1 \) is a sufficiently small constant.

Lemma 3.1. The following two estimates hold. First, \( \langle D_N, Y \rangle \gtrsim n \), and second,
\[
\mathbb{P}(|Y| > \alpha) \lesssim \exp(-ca^2) \quad \alpha > 1.
\]
Proof. Modify, in a straightforward way, [9, §3] to see that for \( c \) sufficiently small,
\[
\left\langle D_N, \sin \left( cn^{-1/2} \sum_{\vec{r} : r_1 = j} f_{\vec{r}} \right) \right\rangle \gtrsim \sqrt{n}, \quad 1 \leq j \leq n/2.
\]
Sum this over \( j \) to prove the first claim of the Lemma.

The second claim, the distributional estimate, is equivalent to the bound \( \|Y\|_p \lesssim C \sqrt{p} \) for \( 2 \leq p < \infty \). This is estimate (4.1) in [9].

Set \( E = \{|Y| > \alpha\} \), where \( \alpha > 1 \) is to be chosen. We consider the inner product
\[
cn \leq \langle D_N, Y \rangle \leq \langle D_N, Y 1_{E^c} \rangle + \langle D_N, Y 1_E \rangle \\
\leq \alpha \|D_N\|_1 + \|D_N\|_{L(\log L)} \|Y 1_E\|_{\exp(L)}
\]
\[ \leq \alpha \|D_N\|_1 + \alpha^{-1} \|D_N\|_{L(\log L)}, \]

where we have used the duality of the spaces \( L(\log L) \) and \( \exp(L) \). The last estimate depends upon the calculation

\[ \|Y^{1E}\|_{\exp(L)} \simeq \sup_{t \geq 1} t \cdot |\log\{|Y| > \max\{t, \alpha\}\}|^{-1} \lesssim \sup_{t \geq 1} \left\{ \frac{1}{t^\alpha}, \frac{t}{\alpha^2} \right\} \simeq \alpha^{-1}. \]

Choose \( \alpha^2 \simeq \|D_N\|_{L \log L}/\|D_N\|_1 \geq 1 \). We then have

\[ n \lesssim \|D_N\|_{L \log L}^{1/2} \|D_N\|_1^{1/2}, \]

and this proves Theorem 1.6. \( \square \)

**Proof of Theorem 1.7.** Assume that \( \|D_N\|_1 \leq C_1 \log N \). We shall utilize the main result of [9], namely (1.2). Consider the probability measure \( \mathbb{P}_N \) which is the normalized \( |D_N| \, dx \), i.e.

\[ d\mathbb{P}_N(x) = \frac{|D_N(x)|^{d-2}}{\|D_N\|_1^{d-2}} \, dx. \]

We see that

\[ \int (\log_+ |D_N|)^{d-2} d\mathbb{P}_N(x) \geq \frac{\|D_N\|_{L(\log L)}^{d-2}}{\|D_N\|_1^{d-2}} \geq Cn^{d-2}. \]

It is obvious that \( |D_N(x)| \leq N \), therefore \( \log |D_N| \leq n \). It follows from a Paley–Zygmund-type inequality that for some \( c > 0 \)

\[ (3.1) \quad \mathbb{P}_N\{\log_+ |D_N| > cn\} \gtrsim 1. \]

Indeed, denoting \( f = \log_+ |D_N| \) and \( \alpha = (d - 2)/2 \), using Cauchy–Schwarz inequality we get

\[ Cn^\alpha \leq \mathbb{E}|f|^\alpha \leq \mathbb{E}|f|^\alpha 1_{\{|f| > cn\}} + \mathbb{E}|f|^\alpha 1_{\{|f| \leq cn\}} \]
\[ \leq (\mathbb{E}|f|^{2\alpha})^{1/2} \cdot \mathbb{P}^{1/2}_{N}\{ |f| > cn \} + c^\alpha n^\alpha \]
\[ \leq n^\alpha \cdot \left( \mathbb{P}^{1/2}_{N}\{ |f| > cn \} + c^\alpha \right), \]

which yields (3.1) if \( c \) is small enough. From this, using the fact that \( \|D_N\|_1 \gtrsim \sqrt{n} \) (Theorem 1.2), we deduce that

\[ \|D_N\|_2^2 \gtrsim \int_{\{\log D_N > cn\}} D_N^2(x) \, dx \gtrsim \sqrt{n} \cdot \int_{\{\log D_N > cn\}} |D_N(x)| \, d\mathbb{P}_N(x) \gtrsim N^{C'}, \]

which is the conclusion of Theorem 1.7. \( \square \)

For the last proof we need an additional definition.

**Definition 3.1.** A distribution \( \mathcal{P}_N \) of \( N = p^s \) points is called a \( p \)-adic net, if any \( p \)-adic rectangle

\[ \Delta = \prod_{j=1}^d [m_j p^{-a_j}, (m_j + 1)p^{-a_j}), \quad 0 \leq m_j < a_j \]

of volume \( \frac{1}{N} \) contains exactly one point of \( \mathcal{P}_N \).
For any dimension $d \geq 2$ and a prime $p \geq d - 1$, there exist nets with $p^s$ points for all values of $s \geq 2$. One can show that if $\mathcal{P}_N$ is a $p$-adic net of $N = p^s$ points, then for any rectangle $R \subset [0,1]^d$

$$|\sharp(\mathcal{P}_N \cap R) - |R||N| \leq s^{d-1}.$$ 

A similar inequality can be obtained for arbitrary $N$.

**Proof of Theorem 1.8.** Let us take a net $\mathcal{P}_N$ with small $L_2$ discrepancy, i.e.

$$\|D_N\|_2 \lesssim (\log N)^{(d-1)/2}.$$ 

The existence of such nets is well-known [5,6]. Then clearly we also have $\|D_N\|_1 \lesssim (\log N)^{(d-1)/2}$. For $\delta > 0$ we define the cube $Q = [1-N^{-\delta},1]^d$, which lies at the top right corner of $[0,1]^d$. As $|Q| = N^{-\delta d}$ and the distribution $\mathcal{P}_N$ is a net, it follows that $Q$ contains about $N^{1-\delta d}$ points of $\mathcal{P}_N$.

Let $\mathcal{P}'_N$ be a new distribution obtained from $\mathcal{P}_N$ by replacing the points inside $Q$ with $(1,1,\ldots,1)$ and keeping the points outside $Q$ unchanged. Let $D'_N$ be the associated discrepancy function. Then $D_N(x) = D'_N(x)$ for $x \notin Q$, and $D'_N$ has no contribution from the distribution of points inside $Q$. Hence for $x \in Q$

$$|D_N(x) - D'_N(x)| \lesssim N^{1-\delta d}.$$

Because $\mathcal{P}_N$ is a net, in a positive proportion of $Q$ we will also have

$$|D_N(x) - D'_N(x)| \gtrsim N^{1-\delta d}.$$

Therefore we have

$$\|D_N - D'_N\|_1 \simeq N^{1-2\delta d} \quad \text{and} \quad \|D_N - D'_N\|_2^2 \simeq N^{2-3\delta d}.$$ 

If we take $\delta = \frac{1}{2d}$, we obtain

$$\|D_N - D'_N\|_1 \simeq 1 \quad \text{and} \quad \|D_N - D'_N\|_2 \simeq N^{1/4},$$ 

which implies that

$$\|D'_N\|_1 \lesssim (\log N)^{(d-1)/2} \quad \text{and} \quad \|D'_N\|_2 \gtrsim N^{1/4}.$$

\[\square\]

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References

[1] J. Beck and W. W. L. Chen, *Irregularities of distribution*, Cambridge Tracts in Mathematics, vol. 89, Cambridge University Press, Cambridge, 1987. ↑2

[2] D. Bilyk, *On Roth’s orthogonal function method in discrepancy theory*, Unif. Distrib. Theory 6 (2011), no. 1, 143–184. ↑4

[3] D. Bilyk and M. T. Lacey, *On the small ball inequality in three dimensions*, Duke Math. J. 143 (2008), no. 1, 81–115. ↑3

[4] D. Bilyk, M. T. Lacey, and A. Vagharshakyan, *On the small ball inequality in all dimensions*, J. Funct. Anal. 254 (2008), no. 9, 2470–2502. ↑3, 4

[5] W. W. L. Chen and M. M. Skriganov, *Explicit constructions in the classical mean squares problem in irregularities of point distribution*, J. Reine Angew. Math. 545 (2002), 67–95. ↑7

[6] J. Dick and F. Pillichshammer, *Digital nets and sequences*, Cambridge University Press, Cambridge, 2010. Discrepancy theory and quasi-Monte Carlo integration. ↑2, 7

[7] M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Mathematics, vol. 1651, Springer-Verlag, Berlin, 1997. ↑2

[8] G. Halász, *On Roth’s method in the theory of irregularities of point distributions*, Recent progress in analytic number theory, Vol. 2 (Durham, 1979), Academic Press, London, 1981, pp. 79–94. ↑1

[9] M. T. Lacey, *On the discrepancy function in arbitrary dimension, close to L^1*, Anal. Math. 34 (2008), no. 2, 119–136 (English, with English and Russian summaries). ↑1, 5, 6

[10] K. F. Roth, *On irregularities of distribution*, Mathematika 1 (1954), 73–79. ↑1, 3, 4

[11] W. M. Schmidt, *Irregularities of distribution. X*, Number theory and algebra, Academic Press, New York, 1977, pp. 311–329. ↑1, 3, 4

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