Quantum lower bounds for the set equality problems

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Abstract

The set equality problem is to decide whether two sets $A$ and $B$ are equal or disjoint, under the promise that one of these is the case. Some other problems, like the Graph Isomorphism problem, is solvable by reduction to the set equality problem. It was an open problem to find any $w(1)$ query lower bound when sets $A$ and $B$ are given by quantum oracles with functions $a$ and $b$.

We will prove $\Omega\left(\frac{n^{1/3}}{\log^{1/3} n}\right)$ lower bound for the set equality problem when the set of the preimages are very small for every element in $A$ and $B$.

1 Introduction, Motivation and Results

The Shor’s integer factoring quantum algorithm provides exponential speed-up over the best known classical algorithm. This motivates to search other quantum algorithms with great speed-up. However, proving quantum lower bounds for such problems is not trivial, for example, proving the exponential quantum lower bound for NP-Complete problems will imply $P \neq NP$.

One of the problems quantum computer could have an exponential speed-up over classical computer is the Graph Isomorphism problem. One way to attack this problem could be by the reduction to the set equality problem. Notice the sets of all permutations over vertexes for given graphs. If these sets are equal, then there is an isomorphism between the graphs, but if there is not isomorphism between graphs, then these sets are strictly disjoint.

Denote the set $\{1, 2, ..., n\}$ by $[n]$.

Definition 1 Let $a : [n] \mapsto [m]$ and $b : [n] \mapsto [m]$ be a functions. Let $A$ be a set of all $a$’s images $A = \{a(1), a(2), ..., a(n)\}$ and $B = \{b(1), b(2), ..., b(n)\}$. There is a promise that either $A = B$ or $A \cap B = \emptyset$.

Call the general set equality problem to distinguish these two cases.

Finding quantum query lower bound for general set equality problem was posed an open problem by Shi[12].
We will show that Ambainis’ [2] adversary method imply $\Omega(\sqrt{n})$ lower bound for the general set equality problem. The proof uses the possibility to have many preimages for some image. However, graph theorists think that the Graph Isomorphism problem, when graphs are promised not to be equal with themselves by any nonidentical permutation, still is very complex task. Now reduction lead us to the set equality where $a$ and $b$ are one-to-one functions.

**Definition 2** Call the general set equality problem to be a one-to-one set equality problem if $a(i) \neq a(j)$ and $b(i) \neq b(j)$ for all $i \neq j$.

The proof that worked for the general set equality problem does not work for one-to-one set equality problem, because it uses that fact that there can be very many preimages for any element of the sets. However, we will prove lower bound for a problem between these problems.

**Definition 3** Call the general set equality problem to be a $f(n)$ set equality problem if $|a^{-1}(x)| = O(f(n))$ and $|b^{-1}(x)| = O(f(n))$ for all images $x \in a([n]) \cup b([n])$ and for some function $f$.

We will prove $\Omega\left(\frac{n^{1/3}}{(\log n)^{1/3}}\right)$ lower bound for the $\log(n)$ set equality problem.

The first result for lower bounds of the set equality like problem was done by Aaronson [1]. He showed $\Omega(n^{1/6})$ lower bound for so called set comparison problem: to decide whether two sets are equal or disjoint on a constant fraction of elements. He also assumed that both $a$ and $b$ are one-to-one functions. In this paper, we will study lower bound of problem when these sets $A$ and $B$ are strictly disjoint or equal, however $a$ and $b$ is not a one-to-one.

## 2 Preliminaries

### 2.1 Quantum Query algorithms

The most popular model of quantum computing is a query (oracle) model where the input is given by a black box. For more details, see a survey by Ambainis [3] or textbook by Gruska [8]. In this paper we are able to skip them because our proof will be built on reduction to solved problems.

One of the most amazing quantum algorithms is a Grover’s search algorithm. It shows how a given $x_1 \in \{0, 1\}, x_2 \in \{0, 1\}, ..., x_n \in \{0, 1\}$ to find the $i$ that $x_i = 1$ with $O(\sqrt{n})$ queries.

This algorithm can be generalized to so called amplitude amplification [7]. Using amplitude amplification one can make good quantum algorithms for many problems till the quadratic speed-up over classical algorithms.

By straightforward use of amplitude amplification we get a quantum algorithm with $O(\sqrt{n})$ queries for the general set equality problem and a quantum algorithm with $O(n^{1/3})$ queries for the one-to-one set equality problem.
2.2 Quantum query lower bounds

There are two main approaches to get good quantum lower bounds. The first is Ambainis’ [2] quantum adversary method. The other is lower bound by polynomials introduced by Beals et al. [5] and substantially generalized by Aaronson [1] and Shi [12]. Although explicitly we will use only Ambainis’ method, main result we will get by a reduction to problem, solved by polynomials’ method.

The basic idea of adversary method is that, if we can construct relation $R \subseteq A \times B$, where $A$ and $B$ consisting of 0-instances and 1-instances and there is a lot of ways how to get from an instance in $A$ to an instance in $B$ that is in the relation and back by flipping various variables, then query complexity must be high.

**Theorem 1** [2] Let $f(x_1, \ldots, x_N)$, be a function of $n \{0, 1\}$-valued variables and $X, Y$ be two sets of inputs such that $f(x) \neq f(y)$ if $x \in X$ and $y \in Y$. Let $R \subset X \times Y$ be such that

- For every $x \in X$, there exist at least $m$ different $y \in Y$ such that $(x, y) \in R$.
- For every $y \in Y$, there exist at least $m'$ different $x \in X$ such that $(x, y) \in R$.
- For every $x \in X$ and $i \in \{1, \ldots, n\}$, there are at most $l$ different $y \in Y$ such that $(x, y) \in R$ and $x_i \neq y_i$.
- For every $y \in Y$ and $i \in \{1, \ldots, n\}$, there are at most $l'$ different $x \in X$ such that $(x, y) \in R$ and $x_i \neq y_i$.

Then, any quantum algorithm computing $f$ uses $\Omega(\sqrt{mm'l'})$ queries.

2.3 The collision problem

Finding $w(1)$ quantum lower bound for the collision problem was an open problem since 1997. In 2001 Scott Aaronson [1] solved it showing polynomial lower bound. Later his result was improved by Yaoyun Shi [12]. Newfy Shi’s result was extended by Samuel Kutin [10] and by Andris Ambainis [4] in another directions.

Below is exact formulation of collision problem due to Shi[12].

**Definition 4** Let $n > 0$ and $r \geq 2$ be integers with $r|n$, and let a function of domain size $n$ be given as an oracle with the promise that it is either one-to-one or $r$-to-one. Call the $r$-to-one collision problem the problem to distinguishing these two cases.

Shi [12] showed quantum lower bound for $r$-to-one collision problem.

**Theorem 2** [12] Any error-bounded quantum algorithm to solve $r$-to-one collision must evaluate the function $\Omega((n/r)^{1/3})$ times.
3 Results

3.1 Lower bound for the general set equality problem

Theorem 3 Any quantum algorithm which solves the general set equality problem makes $\Omega(\sqrt{n})$ queries.

Proof. Simple use of Ambainis’ Theorem 1. Since Ambainis’ Theorem 1 deals with boolean functions, we will modify any quantum algorithm that solves the general set equality problem to an algorithm, that computes boolean function.

We will prove this theorem even in a restricted case, when functions returns only two values, let say 0 and 1. So we have a problem, given two functions $a : n \mapsto \{0, 1\}$ and $b : n \mapsto \{0, 1\}$ answer either the sets $A = \{a(1), ..., a(n)\}$ and $B = \{b(1), ..., b(n)\}$ are equal or disjoin under the promise that one of this is the case.

Let $f : \{0, 1\}^{2n} \mapsto \{0, 1\}$ be partially defined function, such that

$$f(a_1, a_2, ..., a_n, b_1, b_2, ..., b_n) = \begin{cases} 1, & \text{if } \{a_1, ..., a_n\} = \{b_1, ..., b_n\}; \\ 0, & \text{if } \{a_1, ..., a_n\} \cap \{b_1, ..., b_n\} = \emptyset. \end{cases}$$

It is easy to see, that if we can solve a general set equality problem, we can compute this function with constant slowdown, too.

Let construct the relation $R$ from Ambainis’ Theorem 1 with $X = \{0^n1^n\}$ and $Y = \{0^i1^{n-i-1}1^i01^{n-i-1} : 0 \leq i < n\}$ as follows:

$$R = X \times Y = \{(0^n1^n, 0^i1^{n-i-1}1^i01^{n-i-1}) : 0 \leq i < n\}.$$ 

One can check that $R$ is well defined and $m = n$, $m' = 1$, $l = 1$ and $l' = 1$. Thus any quantum algorithm computing $f$ makes $\Omega(\sqrt{n})$ queries. ✷

3.2 Lower bound for the log(n) set equality problem

Now we will prove the main result in this paper.

Theorem 4 Any error-bounded quantum algorithm which solves the log(n) set equality problem makes $\Omega(\frac{n^{1/3}}{\log(n)} \log^{1/3} n)$ queries.

Proof: To prove Theorem 4 we will reduce r-to-one collision problem to the log(n) set equality problem. We are given function $f : [n] \mapsto [m]$, under promise to be either r-to-one or one-to-one and $r = \lceil \log n \rceil$ and $r|n$. We randomly choose two sets $A$ and $B$ such that $|A| = |B| = n/2$ and $A \cup B = [n]$ and $A \cap B = \emptyset$. Denote $A' = f(A)$ and $B' = f(B)$. It is obviously that if $f$ is one-to-one then $A' \cap B' = \emptyset$.

If $f$ is r-to-one then the situation is more complicate. In the next subsection we will prove that with big probability holds that $A'$ and $B'$s includes all images of f, thus $A' = B' = f([n])$.

Let the functions $a$ and $b$ from Theorem 4 be the same as $f$ but domain for $a$ is $A$ and domain for $b$ is $B$. 

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Denote the set of all preimages of x in the set A by \( f^{-1}_A(x) = f^{-1}(x) \cap A \). Since \( |f^{-1}(x)| = r \) for every \( x \in f([n]) \), it is clear that also \( |a^{-1}(x)| = O(\log n) \) and \( |b^{-1}(x)| = O(\log n) \) for every \( x \in f([n]) \).

So with constant probability we get the \( \log(n) \) set equality problem with domain size \( n/2 \). Now Theorem 2 implies Theorem 4. \( \square \)

3.3 Reduction

Lemma 1 From all possible divisions of a set \([n]\) into two equal sized parts A and B such that \( A \cap B = \emptyset \), only few of them are such that for some \( x \in f([n]) \) there is no preimage either in A or B.

Proof:
Total count of all (possibly uniform) divisions are

\[
\mathcal{C}_{n/2}^n = \frac{n!}{(n/2)!(n/2)!}.
\]

Total count of such divisions is at most the count of images \( (n/r) \) multiplied by count of divisions where one fixed \( x \in f([n]) \) has no preimage either in A or B.

Assume that all preimages of \( x \) is in A, thus B has not any of them. Number of ways how we can choose residual elements is

\[
\mathcal{C}_{n/2-r}^{n/2-r} = \frac{(n-r)!}{(n/2-r)!(n/2)!}.
\]

Analysis of an opposite assumption is similar, so probability to choose division which is bad on \( x \) is

\[
\frac{2\mathcal{C}_{n/2-r}^{n/2-r}}{\mathcal{C}_{n/2}^n} = \frac{2(n-r)!(n/2)!(n/2)!}{(n/2-r)!(n/2)!n!} = \frac{2(n/2)(n/2-1)(n/2-2)...(n/2-r+1)}{n(n-1)(n-2)...(n-r+1)} = \\
= \frac{n/2-1}{n-1} \frac{n/2-2}{n-2} ... \frac{n/2-r+1}{n-r+1} \leq \left(\frac{1}{2}\right)^{r-1}.
\]

So probability to choose bad division for any \( x \in f([n]) \) is at most

\[
\left(\frac{1}{2}\right)^{r-1} \frac{n}{r} = \frac{2n}{2^r r}.
\]

Since \( r = \lceil \log n \rceil \)

\[
\frac{2n}{2^r r} \leq \frac{2}{\lceil \log n \rceil}
\]

which is small for large \( n \).

\( \square \)
4 Conclusion

Finding lower bound for the set equality problem is one of the most challenging today’s task in theory of quantum query lower bounds. We have solved this problem partially. One can argue that to solve the set equality problem can be easier when functions are promised to be with a small range of preimages for all images. Our paper shows that the difference between the general set equality problem and the \( \log n \) set equality problem is very small, respectively \( \Omega(\sqrt{n}) \) and \( \Omega(\frac{n^{1/3}}{\log^{1/3}n}) \). This enforce opinion that quantum computer probably cannot solve the one-to-one set equality problem with only polylogarithmic number of questions.

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