Rényi Information flow in the Ising model with single-spin dynamics

Zehui Deng  
Physics Department, Beijing Normal University, Beijing 100875, China

Jinshan Wu   
School of Systems Science, Beijing Normal University, Beijing 100875, China

Wenan Gu$^\ddagger$  
Physics Department, Beijing Normal University, Beijing 100875, China and  
State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics,  
Chinese Academy of Science, Beijing 100190, China  
(Dated: November 4, 2014)

The $n$-index Rényi mutual information and transfer entropies for the two-dimensional kinetic Ising model with arbitrary single-spin dynamics in the thermodynamic limit are derived as functions of ensemble averages of observables and spin-flip probabilities. Cluster Monte Carlo algorithms with different dynamics from the single-spin dynamics are thus applicable to estimate the transfer entropies. By means of Monte Carlo simulations with the Wolff algorithm, we calculate the information flows in the Ising model with the Metropolis dynamics and the Glauber dynamics, respectively. We find that, not only the global Rényi transfer entropy, but also the pairwise Rényi transfer entropy peaks in the disorder phase.

PACS numbers: 05.20.-y, 89.70.Cf, 89.75.Fb, 75.10.Hk

I. INTRODUCTION

Information theory has found its fruitful applications recently [1–3] in the study of phase transitions and critical phenomenon which, traditionally, are studied by using measures based on two-point correlation functions. This may not be surprising considering the concept of entropy, which was first used by Shannon to quantify information [4], has its roots in thermodynamics. Mutual information (MI) has proved to be a powerful tool for determining the thermal and quantum phase transitions and their universality classes without knowledge of order parameter [5, 11]. (In the context of quantum critical phenomena, the classical Shannon entropy is related to the Von Neumann entropy, and the MI is related to the entanglement entropy [5].)

Besides physical systems, there are other complex systems with interacting agents, such as stock markets, crowd dynamics or traffic flow, also have phase-transition-like phenomena. In such more general complex systems, there might not be a well-defined order parameter, or even a well-defined driven parameter as temperature to the Ising model. To predict, or even identify, phase-transition-like behaviors in these systems is very important, but hard. MI is therefore very useful in studying such systems, e.g., Vicsek’s particle swarm model [12], random Boolean networks [13] and financial markets [14]. Rényi entropy [15] and corresponding mutual entropy, as the extensions of the Shannon entropy and mutual entropy, also play important roles in these methods.

On the other hand, with human civilization dives deeper and deeper into the era of big data, time-series data in the complex systems become more readily accessible. In principle, time-series data before and after the critical point should have different qualitative features. Methodologies to identify and predict critical points from time-series data in these systems, if established, will be an essential step of progress to research works. Unfortunately, mutual information does not contain dynamical information. However, an alternative information theoretic measure, transfer entropy, that shares some of the desired properties of mutual information but takes the information flow into account has been introduced [16]. For example, consider two Ising spins $s_1$ and $s_2$ coupled by exchange interaction. Let $s_1(t)$ and $s_2(t)$, $t = 1, 2, \cdots$, denote sequences of states of the two spins. If the state of $s_2$ has no influence on the transition of $s_1$, e.g., in the high temperature limit, we have the Markov property $P(s_1(t) \mid s_1(t-1)) = P(s_1(t) \mid s_2(t-1), s_2(t-1))$ with $p$ denoting the transition probability of $s_1$ from $t-1$ to $t$. This means that there is no information flow from $s_2$ to $s_1$. The deviation from this relation is thus quantified as the transfer entropy [16]. The transfer entropy detects the directed exchange of information between two systems and thus might have more potential applications in the study of dynamic systems with time-series data available [17, 18].

For a complex dynamic system, it is known that information flow between elements always peak in an intermediate order regime. However, the peak may not coincide with phase transition. It was recently conjectured that, by contrast, information flow in such systems generally may peak strictly at the disordered side of a phase transition [15]. This conjecture was verified for the ferromag-
netic two-dimensional (2D) kinetic Ising model with the Glauber dynamics \[20\], in which a global transfer entropy measure attains a maximum in the disordered phase. However, a pairwise transfer entropy does not show such a maximum in the disordered phase \[19\]. The numerically observed peak of global transfer entropy at the disorder side can be practically very valuable. In a stock market, ordered phase, where lots of stocks move in the same direction, in a sense corresponds to a large bubble or a big crash. Peaks in the disorder side implies that it might be used as an indicator of a critical region in the near future before stocks in the market really start to act in the same direction.

The MI and information flow discussed in Ref. \[19\] are based on the Shannon entropy. It is natural to extend the theory to include the general Rényi entropy \[15\] and to verify other single-spin dynamics than the Glauber dynamics. In present work, we define the Rényi pairwise and global MI measures and the corresponding transfer entropy measures. For the 2D kinetic Ising model with general single-spin dynamics, these measures are derived as functions of ensemble averages of observables, including those related to the single-spin flipping probabilities, in the thermodynamic limit. We further numerically calculate these measures by using Monte Carlo simulations with the Wolff algorithm \[21\]. We find that the Shannon transfer entropies for the Ising model with the Metropolis dynamics \[22\] behave similarly as those for the Glauber dynamics. The Rényi pairwise and global MI measures are also found to have similar behaviors as the Shannon counterparts for both dynamics. However, the Rényi pairwise and global transfer entropies show different behaviors from the Shannon counterparts. The most evident difference is that the Rényi pairwise transfer information measure peaks in the disordered phase, which is absent for the Shannon pairwise information flow.

The paper is organized as follows: In Sec II, we define and derive the Rényi entropy based MI and flow measures in the thermodynamic limit. In Sec III, we calculate the measures for the 2D kinetic Ising model with Glauber and Metropolis dynamics. We conclude in Sec IV.

II. RÉNYI MUTUAL INFORMATION AND FLOW

We consider the ferromagnetic 2D Ising model on the square lattice with periodic boundary conditions. The Hamiltonian is given by

\[ \mathcal{H}(\mathbf{S}) = -J \sum_{\langle i,j \rangle} S_i S_j, \]  

where \( \mathbf{S} = (S_1, \ldots, S_N) \), \( S_i \in \{+1, -1\} \), denotes the spin configuration and \( \langle i,j \rangle \) the nearest neighbors. \( J = 1 \) sets the energy unit. The Boltzmann-Gibbs probability of a configuration \( \mathbf{S} \) is

\[ P(\mathbf{S}) = \frac{1}{Z} e^{-\beta \mathcal{H}(\mathbf{S})}, \]

where \( \beta = 1/T \) is the inverse temperature with the Boltzmann constant \( k_B = 1 \), and \( Z = \sum_{\mathbf{S}} e^{-\beta \mathcal{H}(\mathbf{S})} \) is the partition function.

The model is largely solved in the thermodynamic limit \[22\] \[23\]. We quote the main exact results here for later use:

The critical inverse temperature

\[ \beta_c = \frac{1}{T_c} = \frac{1}{2} \log(1 + \sqrt{2}), \]  

the magnetization

\[ m = \begin{cases} \pm (1 - \sinh^{-4} 2\beta)^{\frac{1}{2}}, & T < T_c; \\ 0, & T \geq T_c \end{cases}, \]  

the free energy per site

\[ -2\beta f = \log(2 \cosh^2 2\beta) + \frac{2}{\pi} \int_0^{\pi/2} \log(1 + \sqrt{1 - \kappa^2 \sin^2 \theta}) d\theta, \]  

and the internal energy per site

\[ u = -\operatorname{coth} 2\beta [1 + \frac{2}{\pi} (\kappa \sinh 2\beta - 1) \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}}], \]

where \( \kappa = \frac{2 \sinh 2\beta}{\cosh^2 2\beta} \).

Mutual information between random variables is the essential information-theoretic quantity, which can be framed in terms of statistical dependence. Based on the Shannon entropy \( H(X) \) of a random variable \( X \), the mutual information \( I(X : Y \mid Z) \) between two random variables \( X \) and \( Y \), optionally conditional on a third variable \( Z \), is defined as

\[ I(X : Y \mid Z) = H(X \mid Z) - H(X \mid Y, Z), \]

which is equivalent to

\[ I(X : Y \mid Z) = H(X \mid Z) + H(Y \mid Z) - H(X, Y \mid Z). \]

Barnett et al. \[19\] thus define the pairwise MI measure

\[ I_{pw} = \frac{1}{2N} \sum_{(i,j)} I(S_i : S_j) = \frac{1}{2N} \sum_{(i,j)} (2H(S_i) - H(S_i, S_j)), \]  

and the global MI measure as the multi-information

\[ I_{gl} = \sum_i H(S_i) - H(\mathbf{S}). \]

The parametric family of entropies so called Rényi entropy were introduced by Alfred Rényi as a mathematical generalization of the Shannon entropy. The definition of Rényi’s entropy of index \( n \) is given by \[15\]

\[ H_n(X) = \frac{1}{1 - n} \log(\sum_{i \in X} p^*_i^n), \]

where...
where $X$ represents a random variable and $p_i$ is the probability of outcome $i \in X$. Shannon entropy is the special case at the limit $n \to 1$.

There are alternatives to define the Rényi MI $I_n(X : Y)$ between two random variables $X, Y$ \cite{23}, e.g., $I_n(X : Y) \equiv H_n(X) - H_n(X | Y)$, or, $I_n(X : Y) \equiv H_n(X) + H_n(Y) - H_n(X, Y)$. Following Iaconis et al. \cite{9}, we adopt the latter and extend $I_{p\sigma}$ to the Rényi pairwise MI measure $I_{p\sigma}$ and $I_{\sigma}^{\text{gl}}$ to the Rényi global MI measure $I_{\sigma}^{\text{gl}}$ by replacing the Shannon entropy $H$ to the Rényi entropy $H_{\rho}$ in Eq. (9) and (10), respectively.

Following Barnett et al. \cite{19}, we express them in the thermodynamic limit

$$I_{p\sigma} = \frac{2}{1-n} \log\left(\sum_{\sigma} p_{\sigma}^{\alpha}\right) - \frac{1}{1-n} \log\left(\sum_{\sigma, \sigma'} p_{\sigma \sigma'}^{\alpha}\right) \tag{12}$$

and

$$\frac{1}{N} I_{\sigma}^{\text{gl}} = \frac{1}{1-n} \log\left(\sum_{\sigma} p_{\sigma}^{\alpha}\right) + \frac{n\beta}{1-n} (f(T/n) - f(T)) \tag{13}$$

where the sums are over $\sigma, \sigma' = \pm 1$, with

$$p_{\sigma} = \frac{1}{2} (1 + \sigma m), \quad p_{\sigma \sigma'} = \frac{1}{4} (1 + \sigma \sigma' m - \frac{1}{2} \sigma \sigma' u) \tag{14}$$

and $n$ is the index of the Rényi entropy, $m, f$ and $u$ is the magnetization, the free energy persist and the internal energy persist, respectively. Note that for $T < T_c$, the sign of the magnetization $m$ does not affect these two and any subsequent quantities, which is to say that the information measures are invariant under symmetry breaking.

$I_{p\sigma}^R$ and $I_{\sigma}^{\text{gl}}$, at the thermodynamic limit, can be computed directly by substituting the exact results as in Eq. (14) into their analytic expressions (12) and (13). Also we note that the second derivative of $I_{p\sigma}^R$ has singular points at $T_c$ and $nT_c$ due to the singular behavior of the free energy.

To study the information flow between stationary stochastic processes $X(t)$ and $Y(t)$, the transfer entropy $T_{Y \to X} \equiv I(X(t) : Y^{(t)}(t) | X^{(t)}(t))$ with l-length history is useful. Here $X^{(t)} \equiv X(t-1), \ldots, X(t-l)$. Barnett et al. \cite{19} considered the $l = 1$ history pairwise transfer entropy measure and global transfer entropy measure based on the Shannon entropy:

$$T_{p\sigma} = \frac{1}{2N} \sum_{i,j} T_{S_i \to S_j} \tag{15}$$

$$= \frac{1}{2N} \sum_{i,j} (H(S_i(t) | S_j(t-1)) - H(S_i(t) | S_j(t-1)), \tag{16}$$

and

$$T_{\sigma}^{\text{gl}} = \sum_i (H(S_i(t) | S_i(t-1)) - H(S_i(t) | S(t-1))) \tag{17}$$

where $S_i(t)$ denotes the spin $i$ at time $t$, $S_j(t-1)$ represents the neighboring spin $j$ at time $t - 1$, $H(S_i(t) | S_i(t-1))$ is the Shannon entropy of $S_i(t)$ conditional on $S_i(t-1)$ and similarly for the others. Starting from these definitions, these measures are calculated for an arbitrary single-spin dynamics of the Ising model in the thermodynamic limit \cite{12}, where exact results (Eq. (11), (12)) in the thermodynamic limit are used. For the sake of completeness, we quote the their results as follows:

$$N T_{p\sigma} = -q \sum_{\sigma} \log \left(\frac{q_{\sigma}}{p_{\sigma}}\right) - \sum_{\sigma, \sigma'} \log \left(\frac{q_{\sigma \sigma'}}{p_{\sigma \sigma'}}\right) \tag{17}$$

and

$$N T_{\sigma}^{\text{gl}} = -q \sum_{\sigma} \log \left(\frac{q_{\sigma}}{p_{\sigma}}\right) + \left(P_i(S) \log P_i(S)\right), \tag{18}$$

where

$$q = \frac{1}{2} (P_i(S)), \quad q_{\sigma \sigma'} = \frac{1}{4} (P_i(S) + \sigma' (S_j P_i(S))), \tag{19}$$

with $i, j$ arbitrary nearest neighbors and $(S_j P_i(S)) \equiv 0$ for $T \geq T_c$; $P_i(S)$ is the flipping probability of spin $S_i$ in a given spin configuration $S$. which describes any single-spin process as long as this process satisfies the detailed balance. It is important to notice that the MI measures are independent of the dynamics, while the transfer entropy measures do depend on the dynamics.

We can also generalize the pairwise and global transfer (Shannon) entropy measures to the Rényi pairwise and Rényi global transfer entropy measures. For two stationary stochastic processes $X(t)$ and $Y(t)$, we define the $l = 1$-length history Rényi transfer entropy

$$T_{p\sigma}^R \to X \equiv H_n(X(t) | X(t-1) - H_n(X(t) | X(t-1), Y(t-1)), \tag{20}$$

which reduces to the Shannon transfer entropy $T_{Y \to X}$ at the limit $n \to 1$. The Rényi pairwise and Rényi global transfer entropy measures are thus defined by replacing $H$ to $H_{\rho}$ in Eq. (15) and (16), respectively. The expressions, at thermodynamic limit, are found to be

$$T_{p\sigma}^R = \frac{1}{1-n} \sum_{\sigma} p_{\sigma} \log ((1 - \frac{q}{N p_{\sigma}})^n + (\frac{q}{N p_{\sigma}})^n)$$

$$- \frac{1}{1-n} \sum_{\sigma, \sigma'} p_{\sigma \sigma'} \log ((1 - \frac{q_{\sigma \sigma'}}{N p_{\sigma \sigma'}})^n + (\frac{q_{\sigma \sigma'}}{N p_{\sigma \sigma'}})^n) \tag{21}$$

and

$$T_{\sigma}^{\text{gl}} = \frac{1}{1-n} \sum_{\sigma} p_{\sigma} \log ((1 - \frac{q}{N p_{\sigma}})^n + (\frac{q}{N p_{\sigma}})^n)$$

$$- \frac{1}{1-n} \log ((1 - \frac{P_i(S)}{N})^n + (\frac{P_i(S)}{N})^n), \tag{22}$$

respectively. Here, $p_{\sigma}, p_{\sigma \sigma'}, q$ and $q_{\sigma \sigma'}$ are defined in Eq. (14) and (13).
For large system $N \to \infty$, we obtain the index $n = 2$ Rényi $T_{\text{pw}}^R$ and $T_{\text{gl}}^R$, by applying Taylor expansion, to the order $O(\frac{1}{N^4})$, respectively:

$$T_{\text{pw}}^R = -\sum_{\sigma} p_\sigma (-2 \frac{q}{N p_\sigma} + \frac{4}{3} \sum_{\sigma'} q_{\sigma'}^3 p_{\sigma'}) + \sum_{\sigma,\sigma'} p_{\sigma}\sigma' (-2 \frac{q_{\sigma'}}{N p_{\sigma'}} + \frac{4}{3} \sum_{\sigma''} q_{\sigma''}^3 p_{\sigma''} + O(1))$$

and

$$T_{\text{gl}}^R = -\sum_{\sigma} p_\sigma (-2 \frac{q}{N p_\sigma} + \frac{4}{3} \sum_{\sigma'} q_{\sigma'}^3 p_{\sigma'}) + O(1)$$

(23)

and

(24)

One great advantage of these two formula is that they are expressed in terms of ensemble averages of observables, based only on the Boltzmann-Gibbs distribution. The nature of the transfer entropies which are sensitive to the update scheme is represented in ensemble averages of quantities like $\langle P_i(S) \rangle$ and $\langle S_j P_i(S) \rangle$, which can be calculated using efficient MC method with dynamics other than the single-spin flip dynamics involved in the kinetic model, given that the update probability $P_i(S)$ is specified. Simulation results of these two quantities for two dynamics are presented in Section III A.

III. NUMERICAL RESULTS

The Metropolis algorithm is the first MC algorithm to simulating lattice models.

The discrete-time Metropolis spin-flip dynamics is defined as follows: at each time step, an arbitrary spin $i$ is chosen randomly. Consider the energy difference between the state that spin $i$ is flipped and the original state: $\Delta E_i = 2 s_i \sum_{j \in v(i)} s_j$, $v(i)$ denotes the nearest neighbors of spin $i$. The spin-flipped state will be accepted with probability 1, if $\Delta E_i \leq 0$; otherwise, the state will be accepted with the probability

$$P_i(S) = e^{-\Delta E_i/T}.$$  

(25)

The discrete-time Glauber spin-flip dynamics is slightly different from the Metropolis dynamics: The randomly chosen spin $i$ flips with the probability

$$P_i(S) = [1 + e^{\Delta E_i/T}]^{-1}.$$  

(26)

These processes satisfy detailed balance.

Since not every term of the transfer entropies has an analytic expression, we make use of MC method to obtain their behavior. The Wolff cluster algorithm is used to generate microscopic states. The ensemble average of an observable is calculated as means in the samples. This algorithm is much more efficient than other single-spin flip algorithms, such as the Metropolis algorithm and its variations. In particular, it suppresses critical slowing down. In our simulations, typically $10^5$ samples are used to obtain ensemble averages and statistical errors after equilibrating the systems. It is worthy to note that we do not study the transfer entropies of the kinetic Ising model with the dynamics of the Wolff algorithm. Instead, the Wolff MC method is used to calculate the information flows, according to Eqs. (17), (18), (20), and (24), in the kinetic Ising model with the Metropolis and the Glauber dynamics, respectively.

A. Shannon entropy based information flow for the Metropolis dynamics

To further verify the conjecture raised in 19 that MI flows peak in the disordered phase, we study the MI flows in the Ising model with the Metropolis dynamics.

![Graph showing data](https://via.placeholder.com/150)

FIG. 1. (color online) Plot of $NT_{\text{pw}}$ for the Metropolis dynamics against temperature for several system sizes. The statistical errors are much smaller than the symbol sizes. The inset shows the maximum of $T_{\text{pw}}$ as a function of $1/L$, in which the horizontal dashed line indicates $T_c = 2.2692$.

We simulate the Ising model on the square lattice of size $N = L \times L$ for $L = 8, 16, 32, 64, 128, 256$. Figure 1 and 2 shows $T_{\text{pw}}$ and $T_{\text{gl}}$ as functions of temperature $T$ and linear size $L$ for the Metropolis dynamics, respectively. The results are very similar to those found for the Glauber dynamics in Ref. 19. As the system size grows, the finite-size effects are reducing. Kinks turn to appear in the $T_{\text{pw}}$ and $T_{\text{gl}}$ versus temperature curves at the ex-
The maximum keeps sitting in the disorder region when $T \to \infty$. Compared with Ref. [19] in which $T_{gl}$ is found max at $T = 2.354 \pm 0.003$ for the Glauber dynamics, $T_{gl}$ has a maximum at $T = 2.44 \pm 0.01$.

The conclusion is the same as Barnett et al. [19], namely, $T_{pw}$ peaks at $T_c$ while $T_{gl}$ has a maximum at the disordered phase. Similar results on a measure related to $T_{pw}$ have been obtained by direct simulating the Ising model with the Metropolis dynamics and Glauber dynamics [18].

It is worthy to mention that, in Ref. [19], the authors stressed that the dynamics of the spin updating in their MC algorithm is necessarily to be the same as the dynamics under discussing. By contrast, we come to a conclusion that it is irrelevant to use which MC algorithm or dynamics to update configurations in the simulations, as far as the MC means is equal to the ensemble averages. This is because that the entropy flows have been expressed as ensemble averages of observables in the equilibrated system (see Eq. [17], [18], [23], and [24]). There’s no need to extract the time series of the entropies appearing in the definitions of the flows. For example, $\langle P_i \rangle$ and $\langle S_j P_i \rangle$ are two essential quantities related to the specific dynamics in above expressions, which can be determined by MC simulations with spin update algorithms different from the dynamics studied, as far as the Boltzmann-Gibbs distribution are realized by the simulations. Figure 3 and 4 show these two quantities as functions of temperature for the kinetic Ising model with the Metropolis dynamics and with the Glauber dynamics, respectively. The results are obtained by MC simulations with the Wolff algorithm. It is seen that singularities develop in the two quantities closing to the critical point when system size turns large. We have verified this conclusion by repeating Barnett’s results using the Wolff algorithm (not shown here).

**B. Rényi entropy based mutual information and flow for the Glauber and Metropolis dynamics**

We now study the generalized index 2 Rényi MI measures and transfer entropy measures for the Ising model with the Metropolis dynamics and Glauber dynamics, respectively.

The Rényi MI measures do not depend on the dynamics, thus can be calculated analytically. By substituting the exact $m$, $f$ and $u$ into Eq. [12] and Eq. [13], we obtain the results, which are plotted against temperature.
As expected, the Rényi pairwise MI and global MI bear singularities at the exactly known critical point. We also expect singular behavior of $I_{R,v}^R$ at $2T_c$ due to the singularity in the free energy (see Eq. (13)). Such singularity is not visible directly in Fig. 5 (lower panel), but should appear as a logarithmic divergence in the second order derivative.

By contrast, the Rényi pairwise MI flow $T_{pw}^R$ and global MI flow $T_{gl}^R$ depend on the dynamics of the kinetic Ising model. MC simulations with the Wolff algorithm are used to calculate the two measures. According to [23] and [24], the leading terms in $T_{pw}^R$ and $T_{gl}^R$ scale as $1/N^3$. We therefore evaluate $N^3T_{pw}^R$ and $N^3T_{gl}^R$.

Figure 6 illustrates the Rényi transfer entropy $T_{pw}^R$ and global transfer entropy $T_{gl}^R$ for the Metropolis dynamics as functions of temperature for system sizes $L = 8, 16, 32, 64, 128, 256, 512$, while Figure 7 shows those for the Glauber dynamics. The two dynamic processes have similar behavior in $T_{pw}^R$ and $T_{gl}^R$: as system size turns large, they all develop a kink around $T_c$; the curve for each size has a hump in the disorder region. For Rényi pairwise transfer entropy, the maximum point in the curve of the largest size $L = 512$ is at $T = 2.89 \pm 0.05$ for the Metropolis dynamics and $T = 2.70 \pm 0.05$ for the Glauber dynamics, respectively; For Rényi global transfer entropy, the maximum point in the curve of the largest size $L = 512$ is at $T = 3.21 \pm 0.05$ and $T = 2.93 \pm 0.05$ for the two dynamics, respectively. All these curves are rather flat around the maximum points, which lead to large errors in the estimates of maximum points.

All these measures peak in the disordered regime, regardless the type of the single-spin dynamics, is remarkable. In particular, the behavior of the Rényi global transfer entropy $T_{gl}^R$ is similar to the Shannon global transfer entropy $T_{gl}$ for the Glauber dynamics, however, the Rényi pairwise transfer entropy $T_{pw}^R$ shows a completely different behavior from the Shannon $T_{pw}$, namely, $T_{pw}^R$ peaks in the disordered region, while $T_{pw}$ does not.

FIG. 5. (color online) Plot of $I_{pw}^R$ (upper panel) and $I_{gl}^R/N$ (lower panel) against temperature.

FIG. 6. (color online) $N^3T_{pw}^R$ (upper panel) and $N^3T_{gl}^R$ (lower panel) plotted against temperature for the 2D Ising model with Metropolis dynamics for several system sizes $L$. The statistical errors are much smaller than the symbol sizes.

FIG. 7. (color online) $N^3T_{pw}^R$ (upper panel) and $N^3T_{gl}^R$ (lower panel) as a function of temperature for the 2D Ising model with the Glauber dynamics for several system sizes. The statistical errors are much smaller than the symbol sizes.

IV. CONCLUSIONS AND DISCUSSION

We have extended the Shannon pairwise, global MI and $l = 1$ history transfer entropies to the Rényi counterparts. Expressions related to thermodynamic quantities and ensemble averages of dynamic probability are derived in the thermodynamic limit for the 2D kinetic Ising model with arbitrary single-spin dynamics. Cluster Monte Carlo algorithms with different dynamics from the single-spin dynamics are thus applicable to estimate the transfer entropies. As a result, much larger system sizes and numerical accuracy can be reached in simulations.

By using Wolff cluster Monte Carlo simulations, we
have calculated the transfer entropies for both the Shannon and the Rényi entropy, for the kinetic Ising model with the Glauber and the Metropolis dynamics.

The Shannon global transfer entropy is shown to have a maximum point in the disordered regime for the Metropolis dynamics, similar to that found \cite{19} for the Glauber dynamics. Also, the Shannon pairwise transfer entropy for the Metropolis dynamics behaves similarly as that for the Glauber dynamics \cite{19}: $T_{pw}$ peaks at $T_C$, but does not max in the disordered regime.

For the Rényi transfer entropies with index 2, we have found that, in additional to the global transfer entropy $T_{gl}$, the Rényi pairwise transfer entropy $T_{Rpw}$ peaks in the disordered phase for both the Metropolis and the Glauber dynamics. This is different from the behavior of the Shannon pairwise transfer entropies. $T_{gl}$ is regarded as measure of collective information transfer \cite{27}, capturing both pairwise and higher-order (multivariate) correlations of a site. Its peak is interpreted \cite{19} in terms of conflicting tendencies amongst these components as the level of disorder in the system increases when the system is further away from the phase transition. This might also explain the postcritical peak in our Rényi global transfer entropy $T_{Rgl}$. However, we don’t have an intuitive explanation for the postcritical peak in our Rényi pairwise transfer entropy $T_{Rpw}$ which is absent in the Shannon counterpart. Further investigation is required.

Acknowledgment This work is supported by the National Science Foundation of China (NSFC) under Grant 11175018 (Guo) and 11205014 (Wu).

\begin{thebibliography}{99}
\bibitem{1} H. Matsuda, K. Kudo, R. Nakamura, O. Yamakawa, and T. Murata, Int. J. Theor. Phys. \textbf{35}, 839 (1996).
\bibitem{2} S.-J. Gu, C.-P. Sun, and H.-Q. Lin, J. Phys. A \textbf{41}, 025002 (2008).
\bibitem{3} R. G. Melko, A. B. Kallin, and M. B. Hastings, Phys. Rev. B \textbf{82}, 100409 (2010).
\bibitem{4} C. E. Shannon and W. Weaver, \textit{The Mathematical Theory of Information} (University of Illinois Press, Urbana, IL, 1949).
\bibitem{5} L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. \textbf{80}, 517 (2008).
\bibitem{6} P. Calabrese and J. Cardy, J. Stat. Mech. (2004) P06002.
\bibitem{7} M. A. Metlitski, C. A. Fuertes, and S. Sachdev, Phys. Rev. B \textbf{80}, 115122 (2009).
\bibitem{8} R. R. P. Singh, M. B. Hastings, A. B. Kallin, and R. G. Melko, Phys. Rev. Lett. \textbf{106}, 135701 (2011).
\bibitem{9} J. Iaconis, S. Inglis, A. B. Kallin, and R. G. Melko, Phys. Rev. B \textbf{87}, 195134 (2013).
\bibitem{10} S. Inglis and R. G. Melko, Phys. Rev. E \textbf{87}, 013306 (2013).
\bibitem{11} A. B. Kallin, K. Hyatt, R. R. P. Singh, and R. G. Melko, Phys. Rev. Lett. \textbf{110}, 135702 (2013).
\bibitem{12} R. T. Wicks, S. C. Chapman, and R. O. Dendy, Phys. Rev. E \textbf{75}, 051125 (2007).
\bibitem{13} A. S. Ribeiro, S. A. Kauffman, J. Lloyd-Price, B. Samuelsson, and J. E. S. Socolar, Phys. Rev. E \textbf{77}, 011901 (2008).
\bibitem{14} M. Harré and T. Bossomaier, Europhys. Lett. \textbf{87}, 18009 (2009).
\bibitem{15} A. Rényi, \textit{Proc. of the 4-th Berkeley Symposium on Mathematics, Statistics and Probability, 1960}, 547 (1961).
\bibitem{16} T. Schreiber, Phys. Rev. Lett. \textbf{85}, 461 (2000).
\bibitem{17} M. Prokopenko, J. T. Lizier, and D. C. Price, Entropy \textbf{15}, 524 (2013).
\bibitem{18} D. Marinazzo, M. Pellicoro, G. Wu, L. Angelini, J. M. Cortés, and S. Stramaglia, Plos One. \textbf{9}, e93616 (2014).
\bibitem{19} L. Barnett, J. T. Lizier, M. Harré, A. K. Seth, and T. Bossomaier, Phys. Rev. Lett. \textbf{111}, 177203 (2013).
\bibitem{20} R. J. Glauber, J. Math. Phys. (N. Y.) \textbf{4}, 294 (1963).
\bibitem{21} U. Wolff, Phys. Rev. Lett. \textbf{62}, 361 (1989).
\bibitem{22} N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. \textbf{21}, 1087 (1953).
\bibitem{23} L. Onsager, Phys. Rev. \textbf{65}, 117 (1944).
\bibitem{24} C. N. Yang, Phys. Rev. \textbf{85}, 808 (1952).
\bibitem{25} B. M. McCoy and T. T. Wu, \textit{The Two-Dimensional Ising Model} (Harvard University Press, Cambridge, MA, 1973).
\bibitem{26} J. Principe, \textit{Information Theoretic Learning: Rényi’s entropy and Kernel perspectives} (Springer Science+Bussiness Media. LLC2010).
\bibitem{27} J. T. Lizier, M. Prokopenko, and A. Y. Zomaya, Chaos \textbf{20}, 037109 (2010).
\end{thebibliography}