A class of continua that are not attractors of any IFS

Marcin Kulczycki¹*, Magdalena Nowak¹,²†

¹ Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland
² Institute of Mathematics, Jan Kochanowski University, Świętokrzyska 15, 25-406 Kielce, Poland

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Abstract: This paper presents a sufficient condition for a continuum in \(\mathbb{R}^n\) to be embeddable in \(\mathbb{R}^n\) in such a way that its image is not an attractor of any iterated function system. An example of a continuum in \(\mathbb{R}^2\) that is not an attractor of any weak iterated function system is also given.

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1. Introduction

The notion of an iterated function system (abbrev. IFS), introduced by John Hutchinson in 1981 [4], has proven to be a fertile field of research as well as a versatile and useful tool in lossy data compression (especially where image data is concerned). This paper is a study in one specific aspect of the theory — the possibility of encoding a particular set as an attractor of an IFS. We now recall some basic terminology.

Let \((X, d)\) be a complete metric space. A map \(f: X \to X\) is called a contraction if there exists a constant \(\lambda \in [0, 1)\) such that for every \(x, y \in X\) we have \(d(f(x), f(y)) \leq \lambda d(x, y)\). A map \(f: X \to X\) is called contractive if for every \(x, y \in X\), \(x \neq y\), we have \(d(f(x), f(y)) < d(x, y)\), see [2]. A family \(F = \{f_1, \ldots, f_n\}\) of (contractive maps) contractions \(f_i: X \to X\) is called a (weak) iterated function system, see [1]. Given a compact \(B \subset X\), define

\[
F(B) = \bigcup_{i=1}^{n} f_i(B).
\]

* E-mail: Marcin.Kulczycki@im.uj.edu.pl
† E-mail: magdalena.nowak805@gmail.com
This transformation, acting on the space of nonempty compact subsets of $X$ with the Hausdorff metric, is called the
Barnsley–Hutchinson operator. A fixed point of $F$ is called the attractor of (weak) IFS.

It is shown in [4] that every IFS has a unique attractor (an analogous fact may not be true for a weak IFS, unless
compactness of $X$ is assumed). M. Hata proved in [3] that if the attractor of some IFS is connected, then it is also locally
connected. M. J. Sanders showed in [7] that every arc of finite length is an attractor for some IFS. Additionally, he has
proved that if a is an endpoint of some arc $A \subset \mathbb{R}^n$ which has the properties:

1. for all $x, y \in A \setminus \{a\}$ the length of the subarc of $A$ with endpoints $x$ and $y$ is finite,
2. for every $x \in A \setminus \{a\}$ the length of the subarc of $A$ with endpoints $x$ and $a$ is infinite,

then $A$ is not an attractor of any IFS acting on $\mathbb{R}^n$. One example of such an arc is the harmonic spiral [6]. The example
of M. Kwieciński from [5] may also be easily modified to satisfy these assumptions.

It is elementary to check that every continuum in $\mathbb{R}$ is an attractor of some IFS. Moreover, any embedding of such
continuum in $\mathbb{R}$ still is an attractor of some IFS. In dimension two and higher, however, the situation becomes more
complex. Our results provide a sufficient condition for a continuum to be embeddable in $\mathbb{R}^n$ so that its image is not an
attractor of any IFS.

2. Main results

**Definition 2.1.**
Let $(X, d)$ be a metric space, $A \subset X$, $x, y \in A$, and $\varepsilon > 0$. Consider all the sequences $x_1, \ldots, x_k$ such that $k \in \mathbb{N}$, $x_1 = x$, $x_k = y$, $x_i \in A$, $d(x_i, x_{i+1}) < \varepsilon$. Denote by $\overline{d}(x, y, A, \varepsilon)$ the infimum of the sums $\sum_{i=1}^{k-1} d(x_i, x_{i+1})$ for these sequences. Define $\overline{d}(x, y, A) = \lim_{\varepsilon \searrow 0} \overline{d}(x, y, A, \varepsilon)$. This limit may be infinite.

It is elementary that if $A \subset B$ then $\overline{d}(x, y, A) \geq \overline{d}(x, y, B)$.

**Theorem 2.2.**
Let $n \geq 2$. Let $C \subset \mathbb{R}^n$ be a continuum. Assume that there exists an $(n-1)$-dimensional hyperplane $B \subset \mathbb{R}^n$ such that $B \cap C = \{p\}$ and $C \setminus \{p\}$ is connected. Assume additionally that for every $x, y \in C \setminus \{p\}$ there exists $U_{xy}$ which is a neighbourhood of $p$ such that $\overline{d}(x, y, C \setminus U_{xy}) < +\infty$. Then there exists an embedding $h : C \to \mathbb{R}^n$ such that $h(C)$ is not an attractor of any IFS.

**Proof.** By applying an affine transformation we may assume without loss of generality that $B = \{0\} \times \mathbb{R}^{n-1}$, $p = (0, \ldots, 0)$, and $C \subset [0,1] \times [-1,1]^{n-1}$. Next define $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$h_1(x_1, \ldots, x_n) = \left( x_1, \frac{x_1}{100} x_2, \ldots, \frac{x_1}{100} x_n \right), \quad h_2(x_1, \ldots, x_n) = \left( x_1, \sqrt{x_1} \sin x_1^{-1} + x_2, x_3, \ldots, x_n \right).$$

Then define the embedding $h : C \to \mathbb{R}^n$ as the composition $h_2 \circ h_1$.

Roughly speaking, $h_1$ transforms $C$ into a sharp needle, while $h_2$ bends that needle to fit into a thickened-up graph of the function $\sqrt{x} \sin x^{-1}$. As a result of the second transformation the needle becomes, speaking imprecisely, of infinite length. Figure 1 illustrates the process for $n = 2$.

The map $h_1$ does not increase distance, and therefore for every $x, y \in h_1(C \setminus \{p\})$ there exists $U_{xy}$ which is a neighbourhood of $h_1(p)$ such that $\overline{d}(x, y, h_1(C) \setminus U_{xy}) < +\infty$. Note that, outside of any neighbourhood $U$ of $h_1(p)$, the expansivity constant of $h_2 \circ h_1(U)$ is bounded from above. This implies that for every $x, y \in h(C \setminus \{p\})$ there exists $U_{xy}$ which is a neighbourhood of $h(p)$ such that $\overline{d}(x, y, h(C) \setminus U_{xy}) < +\infty$.  

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union of $A/R$ to our best knowledge, open. We shall now give an example of a subcontinuum of some IFS. The example of Kwieciński [5] provided a negative answer, but the same question for weak IFS’s remains, not increase one-dimensional measure). Consequently, only finitely many of the points $\lambda$ and almost all of the sets $\lambda \in C$. Also note that, by the assumptions, $\lambda$ is infinite. If, on the other hand, $f(h(p)) \neq h(p)$ then there exist $x \in h(C)$ such that $f(x) = h(p)$ and $y \in h(C) \setminus \{h(p)\}$ such that $f(y) \neq h(p)$. Then $d(x, y, h(C))$ would be finite and $\tilde{d}(f(x), f(y), h(C))$ would be infinite, which contradicts the contractiveness of $f$, completing the proof that if $f$ takes value $h(p)$ on at least one argument then it has to be constant. Consequently, if $F$ is the Barnsley–Hutchinson operator for some IFS and $F(h(C)) \subset h(C)$, then $F(h(C))$ may comprise of $\{h(p)\}$ and possibly also finitely many other closed sets not containing $h(p)$. But then $F(h(C)) \neq h(C)$, proving that $h(C)$ is not an attractor of $F$. 

**Remark 2.3.**

The assumptions of Theorem 2.2 are technical and may seem very restrictive. Its assertion, however, is true not only for the continua that satisfy them directly, but also for the continua that are homeomorphic to subsets of $R$ as a subset of $R$. For any two points in the continuum $A \subset R$ can be connected in $A$ by a path of finite length, then it can be easily seen that any one-point union of $A$ and $[0, 1]$ is homeomorphic to a subset of $R^{n+1}$ for which the assumptions of Theorem 2.2 are satisfied.

After the result of Hata [3] it has been an open problem whether every locally connected continuum in $R^n$ is an attractor of some IFS. The example of Kwieciński [5] provided a negative answer, but the same question for weak IFS’s remains, to our best knowledge, open. We shall now give an example of a subcontinuum of $R^2$ that is not an attractor of any weak IFS.

**Definition 2.4.**

In this definition we switch to the standard polar coordinate system $(r, \theta)$ on $R^2$, that is $(x, y) = (r \cos \theta, r \sin \theta)$. Put $p_0 = (0, 0)$ and $p_n = (2^{-n}, 2^{-n})$ for $n \geq 1$. For any $n \geq 1$ choose a broken line segment $l_n$ without self-intersections, consisting of finitely many intervals, that starts at $p_0$, ends at $p_n$, has the total length of $2^n$, and is contained in $[(0, 2^{-n}) \times (2^{-n} - 2^{-n-2}, 2^{-n} + 2^{-n-2})] \cup \{p_n\}$. Define $P = \bigcup_{n=1}^\infty l_n$.

**Theorem 2.5.**

The space $P$ as a subset of $R^2$ with the Euclidean metric is not an attractor of any weak IFS.

**Proof.** Suppose that $f: P \to P$ is contractive. We shall examine how many of the points $p_i$ can belong to $f(P)$. If $f(p_0) \neq p_0$ then there is a neighbourhood $U$ of $f(p_0)$ such that the distance $d(p_0, U) > 0$ and $U$ contains finitely many points $p_i$ and almost all of the sets $f(l_i)$. Note that only finitely many of the sets $f(l_i)$ may reach the outside of $U$. Also observe that each $f(l_i)$ covers at most finitely many points $p_i$ because the lengths of $l_i$ are not increased by $f$ (this elementary property of contractions can be proven either by using $\delta$-chains, or, as in [5], by using the fact that $f$ does not increase one-dimensional measure). Consequently, only finitely many of the points $p_i$ belong to $f(P)$.

**Figure 1.** The map $h$ for $n = 2.
A class of continua that are not attractors of any IFS

If, on the other hand, \( f(p_0) = p_0 \), then, given \( n \geq 1 \), note that \( p_n \) may not belong to \( f(l_i) \) for \( i < n \), because the lengths of these sets are too small to traverse the whole \( l_0 \). But no other point in \( P \) can be mapped onto \( p_n \) by \( f \), because \( f \) decreases the distance between \( p_0 \) and any other point. Therefore, the only point \( p_i \) present in \( f(P) \) is \( p_0 \). In conclusion, if \( F \) is a weak IFS, then only finitely many of the points \( p_i \) can belong to \( F(P) \), and therefore \( P \) is not an attractor of \( F \).

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