More convex functions by Artin’s method

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Abstract

We use Artin’s paper on the Gamma function to find more log convex functions that interpolate a sequence of natural numbers given by a recursion equation.

1 Introduction

Let us recall some ideas and results from Artin’s famous paper on the Gamma function. Therefore, let $a$ and $b$ be real numbers with $a < b$ and $f : (a, b) \to \mathbb{R}$ a function. For any $x_1, x_2 \in (a, b), x_1 \neq x_2$, we define the difference quotient

$$\varphi(x_1, x_2) := \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \varphi(x_2, x_1).$$

(1)

Then, for pairwise different $x_1, x_2, x_3 \in (a, b)$, the iterated difference quotient is defined by

$$\Phi(x_1, x_2, x_3) := \frac{\varphi(x_1, x_3) - \varphi(x_2, x_3)}{x_1 - x_3}.$$

(2)

Exercise 1 Show that $\Phi(x_1, x_2, x_3)$ does not change sign under permutation of the arguments $x_1, x_2$ and $x_3$.

Definition 1 (Convexity) The function $f : (a, b) \to \mathbb{R}$ is called convex if, for any fixed $x_3 \in (a, b)$, the difference quotient $\varphi(x_1, x_3)$ is a monotone increasing function of $x_1$, i.e. whenever $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ we have

$$\varphi(x_1, x_3) \leq \varphi(x_2, x_3).$$

(3)

Exercise 2 Show that $f : (a, b) \to \mathbb{R}$ is convex, if and only if, the iterated difference quotient $\Phi$ satisfies $\Phi(x_1, x_2, x_3) \geq 0$.

Exercise 3 Show that the sum $f + g$ of convex functions $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ is convex and the limit $\lim_{n \to \infty} f_n$ of a sequence of convex functions $f_n : (a, b) \to \mathbb{R}$ is convex.

Much is known about convex functions [2]. The following results are taken from Artin’s paper on the Gamma function [1].

Theorem 1 (Rolle) Let $f : (a, b) \to \mathbb{R}$ be continuous whose one-sided derivatives $f(x + 0)$ and $f(x - 0)$ exists for $x \in (a, b)$. Moreover, assume that $f(a) = f(b)$. Then, there is a $\xi \in (a, b)$ such that

$$f'(\xi + 0) \geq 0 \text{ and } f'(\xi - 0) \leq 0$$

(4)
Theorem 2 (Mean Value) Let $f : [a, b] \to \mathbb{R}$ be continuous with one-sided derivatives $f(x+0)$ and $f(x-0)$ exists for any $x \in (a, b)$. Then there exists $\xi \in (a, b)$ such that
\[
\frac{f(b) - f(a)}{b - a} \in (f'(\xi - 0), f'(\xi + 0))
\] (5)

Theorem 3 (Characterization of Convexity) $f : (a, b) \to \mathbb{R}$ is a convex function if, and only if, $f$ has monotonically increasing one-sided derivatives. If, in addition, $f$ is twice differentiable, convexity of $f$ is equivalent to $f'' \geq 0$ for $x \in (a, b)$.

Definition 2 (Weak Convexity) $f : (a, b) \to \mathbb{R}$ is called weakly convex if
\[
f(x_1 + x_2) \leq \frac{1}{2}(f(x_1) + f(x_2))
\] (6)
holds for all $x_1, x_2 \in (a, b)$.

Any weakly convex function is convex and the converse holds for continuous functions.

Theorem 4 (Weakly Convex plus Continuous implies Convex) $f : (a, b) \to \mathbb{R}$ is a convex function if, and only if, $f$ is weakly convex and continuous.

Exercise 4 Let $f(x) := -\log(x)$ defined on the positive real numbers $\mathbb{R}_+$. Show that $f$ is convex. How is inequality (6) called in this situation?

Definition 3 (Log-Convexity) $f : (a, b) \to \mathbb{R}$ is called log-convex (weakly log-convex) if $f$ is positive and $\log f$ is convex (weakly log-convex).

The positivity assumption on $f$ for log-convexity is, of course, a formal prerequisite because otherwise the logarithm of $f$ cannot be formed. Any logarithmically convex function is convex since it is the composite of the increasing convex function $\exp$ and the function $\log f$, but the converse generally does not hold.

Exercise 5 Verify that the function $x \mapsto x^2$ defined on $\mathbb{R}$ is convex, but not log-convex. On the other hand: given a convex function $g : \mathbb{R} \to \mathbb{R}$, that is not log-convex. Can you find some fixed other function $l : \mathbb{R} \to \mathbb{R}$ such that $l$ composed with $g$, which we call
\[
h(x) := l(g(x)),
\]
is log-convex?

Hint: Depending on your background you may call $l$ (if you pick the best/simplest $l$) the most or the the second most important function in mathematics.

As we have seen, convex functions form a vector space and so do log-convex functions, but since the product of log-convex functions is again log-convex, they even form an algebra.

Theorem 5 Let $f, g : (a, b) \to \mathbb{R}$ be two log-convex (weakly log convex) functions. Then their sum $f + g$ and their product $f \cdot g$ is log-convex (weakly log convex). The same holds for sequences of log-convex functions, if the limit function is positive.

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1In literature you will also find logarithmically convex or superconvex for what we call here log-convex.
1.1 Log-convex Integrals

The previous results can be combined to the statement on log-convexity of integrals. Suppose \( f : (a, b) \times I \to \mathbb{R} \), \( I \) some interval of \( \mathbb{R} \), is a continuous function of the two variables \( x \) and \( t \). Furthermore, for any fixed value of \( t \), suppose that \( f(t, \cdot) \) is log-convex, twice differentiable function of \( x \). For any fixed integer \( n \) we can build the function

\[
F_n(x) = h \sum_{k=0}^{n-1} f(a + kh, x)
\]

with \( h = \frac{b-a}{n} \). Being the sum of log-convex functions, \( F_n \) is also log-convex. As \( n \) approaches infinity, \( F_n \) converges to the integral

\[
\int_a^b f(t, x) \, dt,
\]

which hence is also log-convex. If \( b = \infty \), the result also holds supposed the improper integral converges. Artin is mainly interested in integral representations of Euler’s Gamma function and therefore considers integrals of the form

\[
\int_a^b \varphi(t) t^{x-1} \, dt
\]

with \( \varphi : (a, b) \to \mathbb{R} \) being a positive and continuous function for \( t \in (a, b) \). Verify that

\[
\frac{d^2}{dx^2} \left( \varphi(t) t^{x-1} \right) = 0.
\]

Sufficiently smooth log-convex functions are characterized in the next

**Theorem 6 (Characterization of Log-Convexity)** Let \( f : (a, b) \to \mathbb{R} \) be twice differentiable and without zeros. Then \( f \) is log-convex if, and only if,

\[
q(f) := \text{det} \begin{pmatrix} f & f' \\ f' & f'' \end{pmatrix} = ff'' - (f')^2 \geq 0
\]

for all \( x \in (a, b) \)

**Proof 1** Let \( f : \mathbb{R} \to \mathbb{R} \) be twice differentiable and log-convex. Thus \( f'' \) exists for all \( x \in \mathbb{R} \) and the second derivative of \( \log f \) is non-negative

\[
(\log f)'' = \left( \frac{f''}{f'} \right)' = \frac{f'' - (f')^2}{f^2} = \frac{ff'' - (f')^2}{f^2} = \frac{1}{f^2} \text{det} \begin{pmatrix} f & f' \\ f' & f'' \end{pmatrix} \geq 0
\]

and hence

\[
f \geq \frac{(f')^2}{f''}.
\]

**Theorem 7** Let \( \varphi : (a, b) \to \mathbb{R} \) be positive continuous function. Then

\[
\int_a^b \varphi(t) t^{x-1} \, dt
\]

is a log-convex function of \( x \) defined where the integral converges.
Theorem 8 Let \( f : (a, b) \to \mathbb{R} \) be a log-convex function and \( c \in \mathbb{R}, c \neq 0 \). Then the translated \( f_t : (a, b) \to \mathbb{R} \) and the scaled \( f_s : (a, b) \to \mathbb{R} \) version of \( f \) defined by \( f_t(x) := f(x + c) \) and \( f_s(x) := f(cx) \), respectively, are log-convex.

With more sophisticated words one can express the latter theorem as: the space of log-convex functions is a common invariant subspace of the uniform shift and the uniform translation operator.

2 Euler’s Gamma Function

By the famous Bohr-Mollerup theorem Euler’s Gamma function can be characterized as the unique solution to the following interpolation problem.

Theorem 9 (Bohr Mollerup) Any function \( f : \mathbb{R} \to \mathbb{R} \) that satisfies

1. \( f(1) = 1 \),
2. \( f(x + 1) = xf(x) \) for any \( x > 0 \),
3. \( f \) is log-convex,

equals the Gamma function

\[
\Gamma(x) := \int_0^\infty e^{-t}t^{x-1} \, dt. \tag{12}
\]

Because of the functional Equation in Theorem 9 (second condition) the Gamma function is well-known as an analytic continuation (from \( \mathbb{N} \) to \( \mathbb{R} \)) of the sequence of factorials \( n \mapsto n! = n \cdot (n-1)! \) with \( 1! := 1 \).

Exercise 6 Verify by partial integration that \( \Gamma(n) = (n-1)! \) for any natural number \( n \geq 2 \).

Surprising is that log-convexity is the property that characterizes the Gamma function uniquely up to some normalization (first condition in Theorem 9) as the only function that agrees at the natural number \( n \) with \( (n-1)! \).

Exercise 7 Verify that the Gamma function as defined in (12) satisfies the three properties from theorem 9.

3 Artin Type Functional Equations

We take Theorem 9 as a starting point to constructively solve multiplicative Functional Equation of the form

\[
f(x + 1) = g(x)f(x)
\]

with \( g : \mathbb{R} \to \mathbb{R} \) some continuous function. Many interesting functions satisfy a Artin functional equation (13) for some \( g : \mathbb{R} \to \mathbb{R} \). For instance, if \( g(x) = 1 \) for all real numbers \( x \), solutions \( f \) to (13) are 1-periodic functions. If \( g \) is the identity function on \( \mathbb{R} \), then \( f \) interpolates the factorials since \( f(x + 1) = xf(x) \). Under additional assumptions (for instance, if \( f \) is log-convex and \( f(1) = 1 \), see Bohr Mollerup theorem 9) we have \( f = \Gamma \).

Definition 4 Let \( g : \mathbb{R} \to \mathbb{R} \) any function and \( a > 0 \) and \( f : (0, a) \to \mathbb{R} \) a solution to the functional equation (13). Then we call \( f \) Artin function with representor \( g \). A log-convex Artin function is called Bohr Mollerup function or function of Bohr Mollerup type.
Exercise 8  Find necessary and sufficient conditions on the representer \( g \) of an Artin function \( f \) such that \( f \) is of Bohr Mollerup type.

Of course, the notion of Artin functions is merely a tautology because every function \( f : \mathbb{R} \to \mathbb{R} \) can be thought as an Artin function with representer \( g(x) := \frac{f(x+1)}{f(x)} \). We want to explore whether there is some kind of analogue to the Bohr Mollerup theorem for functional equations of the form \( [13] \) when \( g \) is not the identity function. Therefore, we have to investigate when solutions to \( [13] \) are log-convex and satisfy some interpolation property \( f(n) = a_n \) for some positive sequence \( (a_n)_{n \in \mathbb{N}} \). In the case of the Gamma function we have \( g(x) = x \) on \( \mathbb{R} \) and \( f(n) = a_n := (n-1)! \).

Theorem 10 (Bohr Mollerup Type Representation)  Let \( g : \mathbb{R} \to \mathbb{R} \) be a positive continuous function and \( f : \mathbb{R} \to \mathbb{R} \) a solution to functional equation \( [13] \), which satisfies

\[
    f(1) = g(\infty) := 1
\]  

\[
    f(n) = \prod_{k=1}^{n-1} g(k)
\]  

\[
    f \text{ is log-convex.}
\]

Then \( f : \mathbb{R} \to \mathbb{R} \) with

\[
    f(x) = \lim_{n \to \infty} g^n(n) \prod_{k=0}^{n} \frac{g(k)}{g(x+k)}
\]

is the only solution to the Bohr Mollerup functional equation \( [13] \).

Proof  

**Step 1** Find a product formula for the solution \( f : (0,1] \to \mathbb{R} \) to the Bohr Mollerup functional equation \( [13] \) by using the log-convexity \( [16] \) and the interpolation property \( [15] \).

**Step 2** Extend this solution to \( \mathbb{R} \) by iterated application of \( [13] \).

**Step 3** Show that the product representation obtained in step 1 holds on \( \mathbb{R} \).

**Step 2:** Suppose a log-convex solution \( f : (0,1] \to \mathbb{R} \) to \( [13] \) is known. Then \( f \) can be extended to the interval \((0,1]\) by iterated application of the functional equation \( [13] \):

\[
    f(x+n) = g(x+n-1)f(x+n-1) = g(x+n-1)g(x+n-2)f(x+n-2) = g(x+n-1)g(x+n-2)g(x+n-3)f(x+n-3) = \ldots
\]

\[
    = g(x+n-1)g(x+n-2)g(x+n-3) \ldots g(x+2)g(x+1)g(x)f(x) = \prod_{k=0}^{n-1} g(x+k)f(x)
\]

Now we want to extend \( f \) to include negative real numbers. Therefore, we solve the iterated functional equation \( f(x+n) = \prod_{k=0}^{n-1} g(x+k)f(x) \) for \( f(x) \) and take the expression obtained as a definition of \( f \) for negative real numbers: if \( x \) lies in the interval \((-n,-n+1)\) for some \( n \in \mathbb{N} \), we define the value of \( f \) to be

\[
    f(x) = f(x+n) \frac{1}{\prod_{k=0}^{n-1} g(x+k)}
\]

\[\text{2}\] The sequence \( (a_n)_{n \in \mathbb{N}} \) has to be positive because otherwise \( \log f \) cannot be formed and there is no hope for log-convexity of \( f \).
**Step 1:** Now, let’s use the log-convexity of $f$ to find the exact value $f(x)$ for $0 < x \leq 1$. Let $n \geq 2$. Then

$$\frac{\log f(n) - \log f(n+1)}{n(n+1)} \leq \frac{\log f(n) - \log f(n+1)}{n+1} \leq \frac{\log f(n) - \log f(n+1)}{n+1}$$

expresses the monotone increase of the difference quotients of $f$. If we use the interpolation property $f(n) = \prod_{k=1}^{n-1} g(k)$, equation (19) reads

$$\log g(n-1) \leq \log f(n) \leq \log g(n)$$

and hence

$$g^n(n-1) \leq \frac{f(n)}{f(n)} \leq g^n(n).$$

Multiplying by $f(n) = \prod_{k=0}^{n-1} g(k)$ and using the iterated functional equation $f(x+n) = f(x) \prod_{k=0}^{n-1} g(x+k)$ we obtain

$$g^n(n-1) \leq f(x) \prod_{k=0}^{n-1} \frac{g(x+k)}{g(x+k)} \leq g^n(n).$$

Setting $p_n(x) := \prod_{k=0}^{n-1} \frac{g(k)}{g(x+k)}$ with $g(0) := 1$ the latter equation reads

$$p_n(x) \cdot g^n(n-1) \leq f(x) \leq p_n(x) \cdot g^n(n).$$

Since this holds for all $n \geq 2$, we can replace $n$ by $n+1$ on the left side. Thus

$$p_{n+1}(x) \cdot g^n(n) \leq f(x) \leq p_n(x) \cdot g^n(n).$$

Now observe that $f$ is bounded from above and from below by the same function $p_{n+1}(x) \cdot g^n(n)$ up to the factor $\frac{g(x+n)}{g(n)}$ because $p_n(x) = p_{n+1}(x) \frac{g(x+n)}{g(n)}$. Hence (23) reads

$$p_{n+1}(x) \cdot g^n(n) \leq f(x) \leq p_{n+1}(x) \cdot \frac{g(x+n)}{g(n)} \cdot g^n(n).$$

Multiplying through the second part of (23) by $\frac{g(n)}{g(x+n)}$ and combining it with the first part of (23) gives

$$f(x) \frac{g(n)}{g(x+n)} \leq p_{n+1}(x) \cdot g^n(n) \leq f(x).$$

Assuming $\lim_{n \to \infty} \frac{g(n)}{g(x+n)} = 1$ gives the desired product representation of $f$:

$$f(x) = \lim_{n \to \infty} g^n(n) \prod_{k=0}^{n} \frac{g(k)}{g(x+k)}$$

on $(0, 1]$.

**Step 3:** To see that representation (27) holds even for all $x \in \mathbb{R}$, define

$$f_n(x) := g^n(n) \prod_{k=0}^{n} \frac{g(k)}{g(x+k)}$$
be the expression in (27) under the limit sign. Then we have
\[ f(x + 1) = g(x)f_n(x) \cdot \frac{g(n)}{g(x + n + 1)} \]  
(28)

and thus
\[ f_n(x) = f(x + 1) \frac{g(x + n + 1)}{g(x)g(n)} \]

We see: if the limit in (27) exists for \( x \), it also exists for \( x + 1 \) and vice versa. Hence the product representation (27) is valid for all \( x \in \mathbb{R} \).

Following the lines of Artin’s paper [1], we proceed by deriving expressions for \( \log f \) with \( f \) being a log-convex solution to the Bohr Mollerup Type functional equation (13). Assuming continuity of \( f \) in (27), we obtain
\[ \log f(x) = \lim_{n \to \infty} \left( x \log g(n) + \sum_{k=0}^{n} (\log g(k) - \log g(x+k)) \right) \]  
(29)

Now we would like to differentiate twice under the limit sign to obtain conditions on \( g \) that guarantee the log-convexity of \( f \). If the convergence in (29) is uniformly, we have
\[ (\log f(x))'' = \sum_{k=0}^{\infty} \left( \frac{(g'(x+k))^2}{g''(x+k)} - \frac{g''(x+k)}{g(x+k)} \right) \geq 0 \]  
(30)

Remark 1 For the Gamma function we have \( g(x) = x \) and condition (30) reads
\[ \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} \geq 0. \]  
(31)

We can use (30) to define an inner product.

Definition 5 Let \( g: \mathbb{R} \to \mathbb{R} \) be twice differentiable. Then
\[ a(f(x), g(x)) := \sum_{k=0}^{\infty} \det \begin{pmatrix} (f'(x+k))^2 & g''(x+k) \\ g''(x+k) & f''(x+k) \end{pmatrix} \]

Then log-convex solutions to the Artins functional equation (13) can be interpreted as functions that make the quadratic form \( q(g) := a(g, g) \) positive.

Example 1 Setting \( g = \text{id} \) you obtain the product representation of Euler’s Gamma function. Because \( g(x) = 0 \) if, and only if, \( x = 0 \), the function \( g = \text{id} \) does not satisfy the positivity assumption, that we made.

Exercise 9 (Artin functions with Artinian derivative) The derivative \( f' \) of an Artin function \( f \) is called Artinian if \( f' \) again is an Artin function, i.e. from \( f \) satisfying \( f(x + 1) = g_0(x)f(x) \) for some analytic function \( g_0 \) follows that there is an analytic function \( g_1 \) such that \( f'(x + 1) = g_1(x)f'(x) \). Show that the \( n \)-th derivative of an Artin function \( f \) with representer \( g_0 \) is Artinian if
\[ g_n(x) = \left( g_{n-1}f^{(n-1)} \right)' \]  
(32)
Example 2 Let us consider a class of Artin functions with representer \( g(x) = x^c \) with \( c \) some complex number. When are these Artin functions \( f := f_g \) satisfying \( f(x+1) = g(x)f(x) \) of Bohr Mollerup type? Condition \( \text{(30)} \) reads
\[
(\log f)'' = \sum_{k=0}^{\infty} \frac{c(x+k)^{2(c-1)}}{(x+k)^{2c}} - \frac{c(c-1)(x+k)^{c-2}}{(x+k)^c} \]
\[
= c(2-c) \sum_{k=0}^{\infty} (x+k)^{-1} \geq 0.
\]
Hence the Artin function \( f \) is of Bohr Mollerup Type, if \( \Re(c(2-c)) \geq 0 \). Consequently, \( f \) is log-convex for all real numbers \( c \).

4 Making functions log-convex

Assume \( f : \mathbb{R} \to \mathbb{R} \) is a twice differentiable, but not log-convex function. How can we modify \( f \) such that the modified version of \( f \) is log-convex?

4.1 Inner multiplication problem

Let \( f : \mathbb{R} \to \mathbb{R} \) be a twice differentiable, but not log-convex function. In the inner multiplication problem we seek to find a twice differentiable function \( m : \mathbb{R} \to \mathbb{R} \) such that \( mf \) is log-convex. Any function such that \( mf \) is log-convex is called inner multiplicator of \( f \). According to theorem \( \text{(6)} \) \( mf \) is log-convex if and only if, second derivative of \( \log (mf) \) is non-negative. This is characterized in the next

Theorem 11 Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable, but not log-convex function. Then, there is an inner multiplicator \( m : \mathbb{R} \to \mathbb{R} \) of \( f \) if and only if,
\[
mf \cdot (mf)'' \geq (mf)'^2.
\] (33)

4.2 Outer multiplication problem

Let \( f : \mathbb{R} \to \mathbb{R} \) be a twice differentiable, but not log-convex function. The outer multiplication problem consists of finding a twice differentiable function \( m : \mathbb{R} \to \mathbb{R} \) such that \( m \log f \) is convex. We use theorem \( \text{(6)} \) to characterize convexity of \( \log (mf) \) and obtain

Theorem 12 Assume \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable, but not log-convex. Then an outer multiplier of \( f \) satisfies
\[
(m \log f)'' = m'' \log f + 2m' f' f + m \left( f'' + \frac{(f')^2}{f} \right) \geq 0.
\] (34)

Theorem 13 (Curvature of Artin functions) Let \( f \) be a twice differentiable Artin function with representer \( g \):
\[
f(x+1) = g(x)f(x).
\]
Then the curvature of \( f \) is given by
\[
\kappa_f = \frac{g'' f + 2g' f' + g f''}{(1 + (g' f + g f')^2)^{3/2}}.
\]
5 Fibonacci function

The goal of this section is to find a real-valued log-convex function that interpolates the Fibonacci numbers, which are given by the following linear second order recursion equation

\[ a_n = a_{n-1} + a_{n-2} \]  

(35)

with initial values \( a_0 = a_1 = 1 \). The first Fibonacci numbers are 1, 1, 2, 3, 5, 7, 11, 18. For any linear recursion of depth two we can find a closed formula by making the ansatz \( a_n = \lambda^n \). Here this ansatz leads to the quadratic equation \( \lambda^2 - \lambda + 1 = 0 \) with solution \( \lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{5}) \). Hence the \( n \)-th Fibonacci number can be calculated directly by

\[ a_n = \frac{\varphi^n - (-1)^n \varphi^{-n}}{\sqrt{5}} \]  

(36)

with \( \varphi = \frac{1 + \sqrt{5}}{2} \) the golden ratio. If we replace \( n \) in (36) by some real number \( x \), we get an extension of the Fibonacci numbers

\[ F(x) = \frac{\varphi^x - (-1)^x \varphi^{-x}}{\sqrt{5}}. \]  

(37)

Since \((-1)^x = e^{i\pi x} = \cos(\pi x) + i \sin(\pi x)\), this extension is not always real-valued, more precisely \( F(x) \) is a real number if and only if, \( x \) is an integer. We are interested in constructing a real-valued extension of the Fibonacci numbers. Therefore, we consider the real part of \( F(x) \)

\[ f(x) := \Re F(x) = \frac{\varphi^x - \cos(\pi x) \varphi^{-x}}{\sqrt{5}}. \]  

(38)

Let us take a closer look on the behavior of \( f(x) \) at the integers. Since \( \cos \pi n = (-1)^n \) for \( n \in \mathbb{Z} \), we have

\[ f(n) = \begin{cases} 
\frac{\sqrt{5}}{2} \sinh(n \ln \varphi), & \text{n even} \\
\frac{\sqrt{5}}{2} \cosh(n \ln \varphi), & \text{n odd} 
\end{cases} \]  

(39)

Since the Fibonacci numbers form an increasing sequence of natural numbers, the real part of its canonical interpolation \( f \) is an Artin function with representer

\[ g(x) := \frac{f(x + 1)}{f(x)} = \frac{\varphi^{x+1} - \cos(\pi x) \varphi^{-(x+1)}}{\varphi^x - \cos(\pi x) \varphi^{-x}}. \]  

(40)

Is \( f : \mathbb{R} \to \mathbb{R} \), the real part of the canonical extension of the Fibonacci numbers to \( \mathbb{R} \), log-convex? No. Due to the oscillating term \( \cos(\pi x) \) the second derivative of \( f \) changes sign four times in the intervall \([0, 4]\).

References

[1] Emil Artin. The gamma function. Athena Series, 1964. Translated from the German by Michael Butler.

[2] Barry Simon. Convexity: An Analytic Viewpoint. Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont., 1973.