On weighted estimates for a class of Volterra integral operators

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Abstract
Volterra integral operators $A = \sum_{k=0}^{m} A_k$, $(A_k f)(x) = a_k(x) \int_0^x t^k f(t) \, dt$, are studied acting between weighted $L_2$ spaces on $(0, +\infty)$. Under certain conditions on the weights and functions $a_k$, it is shown that $A$ is bounded if and only if each $A_k$ is bounded. This result is then applied to describe spaces of pointwise multipliers in weighted Sobolev spaces on $(0, +\infty)$.

1. In the theory of functions, it is rather common to study the possibility of weighted estimates of the kind
\[ \| v A f \|_p \leq c \| u f \|_p \] (1)
for Volterra integral operators $A$, defined by the formula
\[ (A f)(x) = \int_0^x A(x, t) f(t) \, dt. \]

Here the function $f$ and its image $Af$ are defined on the half-line $\mathbb{R}^+ = (0, +\infty)$; $u, v$ are nonnegative on $\mathbb{R}^+$ functions (weights); $\| \cdot \|_p = \| \cdot \|_{L_p(\mathbb{R}^+)}$, $1 < p < +\infty$; constant $c > 0$ is independent of $f$.

The answer (whether (1) takes place or not) must be given in terms of the kernel $A(x, t)$ and the weights $u, v$. At present this answer is known only for certain classes of non-negative kernels close in some sense to the kernel of the Riemann-Liouville operator $(x - t)^\alpha$, $\alpha \geq 0$, and imposing at the same time minimal restrictions on the weights $u, v$. It seems that the most general class of such kernels was pointed out by R. Oinarov [1].

This work considers a new class of kernels of the kind
\[ A(x, t) = \sum_{k=0}^{m} a_k(x) t^k, \quad m \in \mathbb{N}. \] (2)

Restrictions on functions $a_k$ are minimal or absent. In particular, these functions are not assumed related and can take values of either sign. Therefore, our class

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contains sign-changing kernels. On the other hand, estimate (1) for operators with kernels of the kind (2) is studied only for \( p = 2 \). Some rather strong condition is also imposed on the weight \( u \).

2. Denote by \( B_\delta, \delta \geq 0 \), the set of positive locally integrable on \( \mathbb{R}^+ \) functions \( w \) satisfying with some constant \( C_w \) the integral doubling condition

\[
\int_\Delta w(x) \, dx \leq C_w \int_{\frac{1}{2}\Delta} w(x) \, dx
\]

for any interval \( \Delta \subset \mathbb{R}^+ \) of length \( |\Delta| \geq \delta \), where \( \frac{1}{2}\Delta \) is a twice smaller interval centered at the same point.

The main result of this work is the following theorem.

**Theorem 1.** Let \( u^{-2} \in B_\delta \) for some \( \delta \geq 0 \). If \( \delta > 0 \), then assume in addition \( a_k v \in L^2(0, r) \forall r > 0, k = 0 \ldots m - 1 \). Then to have the estimate

\[
\| vA f \|_2 \leq c \| u f \|_2,
\]

where \( A \) is an operator with the kernel (2), it is necessary and sufficient that

\[
s_k = \sup_{r>0} \| a_k v \|_{L^2(r, +\infty)} \cdot \| x^k u^{-1} \|_{L^2(0, r)} < +\infty, \quad k = 0 \ldots m.
\]

**Remarks.** 1) Denote by \( \mathcal{L}^2, u \) the weighted space of functions \( f \) on \( \mathbb{R}^+ \) with norm \( \| f u \|_2 \). Having (3) now means that \( A : \mathcal{L}^2, u \to \mathcal{L}^2, v \). Represent \( A \) as a sum

\[
A = \sum_{k=0}^m A_k,
\]

where

\[
(A_k f)(x) = a_k(x) \int_0^x t^k f(t) \, dt.
\]

Inequality (3) for the operator \( A_k \) instead of \( A \) reduces, via the substitution \( f_1(x) = x^k f(x), u_1(x) = x^{-k} u(x), v_1(x) = a_k(x)v(x) \), to the well-studied weighted Hardy inequality

\[
\left\| v_1 \int_0^x f_1(t) \, dt \right\|_2 \leq c \| u_1 f_1 \|_2,
\]

criterion for whose validity is known since a long time (see [2]) and takes the form

\[
\sup_{r>0} \| v_1 \|_{L^2(r, +\infty)} \cdot \| u_1^{-1} \|_{L^2(0, r)} < +\infty.
\]

From here it follows that

\[
s_k < +\infty \iff A_k : \mathcal{L}^2, u \to \mathcal{L}^2, v.
\]

Therefore, Theorem 1 is equivalent to the statement that (under its assumptions) we have “splitting” for the operator \( A = \sum_{k=0}^m A_k \), in the sense that

\[
A : \mathcal{L}^2, u \to \mathcal{L}^2, v \iff A_k : \mathcal{L}^2, u \to \mathcal{L}^2, v, \quad k = 0 \ldots m.
\]
2) Our method of proof of Theorem 1 gives the following estimate for the smallest constant $c$ in inequality (3) (or, what is the same, the norm of the operator $A$ acting from $L_{2,u}$ to $L_{2,v}$):

$$c_1 \sum_{k=0}^{m} s_k \leq \|A\|_{L_{2,u} \rightarrow L_{2,v}} \leq c_2 \sum_{k=0}^{m} s_k.$$ 

Constant $c_2$ here is universal. As for $c_1$, this constant depends on $m$, on $C_{n-2}$ (the doubling constant for $u^{-2}$), as well as (if $\delta > 0$) on the quantity $\sum_{k=0}^{m-1} \|a_k v\|_{L_2(0,r_0)} \cdot \|x^k u^{-1}\|_{L_2(0,r_0)}$, where $r_0$ is determined by $m, C_{n-2}, \delta$.

3. The following statement plays the main role in the proof of Theorem 1.

**Lemma 1.** Let $u^{-2} \in B_\delta, m \in \mathbb{N}$. Then there exists such $r_0 \geq 0$ (for $\delta = 0$), that

$$G(u^{-1} \chi_R, xu^{-1} \chi_R, \ldots, x^m u^{-1} \chi_R) \geq \varepsilon \|u^{-1} \chi_R\|_2 \|xu^{-1} \chi_R\|_2 \cdots \|x^m u^{-1} \chi_R\|_2, \quad r \geq r_0,$$

with a constant $\varepsilon > 0$ independent of $r \geq r_0$.

Here $G$ is the Gram determinant of a system of functions in $L_2$, $\chi_R = \chi_{(0,r)}$ is a characteristic function of the interval. This statement therefore asserts the uniform in $r$ non-degeneration of the parallelepiped with edges $u^{-1} \chi_R, \ldots, x^m u^{-1} \chi_R$.

Let us show how to derive Theorem 1 from this Lemma. From the above discussion it’s clear that we only need to prove the necessity of conditions (4).

Inequality (3) written for a function $f_r$ supported on $[0, r]$ implies the inequality

$$\left\|v(x)a_0(x) \int_0^r f_r(t) dt + v(x) \sum_{k=1}^{m} a_k(x) \int_0^r t^k f_r(t) dt \right\|_{L_2(r, +\infty)} \leq c \|uf_r\|_{L_2(0, r)}.$$  

(5)

Suppose $uf_r$ belongs to the orthogonal complement of the linear span $E_r$ of the set of functions $xu^{-1} \chi_R, \ldots, x^m u^{-1} \chi_R$. By Lemma 1, the angle between the vector $u^{-1} \chi_R$ and the subspace $E_r$ is separated from zero uniformly in $r \geq r_0$. Therefore we can choose $f_r$ so that the angle between $uf_r$ and $u^{-1} \chi_R$ is uniformly separated from $\pi/2$, i.e.

$$\int_0^r f_r(t) dt \geq \alpha \|uf_r\|_{L_2(0, r)} \cdot \|u^{-1}\|_{L_2(0, r)}, \quad r \geq r_0 \quad (\alpha > 0).$$

In this case (5) implies

$$\|a_0 v\|_{L_2(r, +\infty)} \cdot \|u^{-1}\|_{L_2(0, r)} \leq \frac{c}{\alpha}, \quad r \geq r_0,$$

from where $s_0 < +\infty$. Finiteness of the other constants $s_k$ is shown by induction.

We see that the argument uses essentially the geometry of the Hilbert space $L_2$. 

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4. Let us consider some applications. Consider on $\mathbb{R}^+$ the weighted Sobolev space $W = W_{2,u}^{(l)}$ with the norm $\|f\|_W = \|f\|_{L^2(0,1)} + \|f^{(l)}u\|_2$. Particularity of this norm is that by taking the norm of the function itself only on an initial interval of $\mathbb{R}^+$ allows to include into this space polynomials of degree $\leq l - 1$. Spaces $W_{2,u}^{(l)}$ were introduced and studied by L.D. Kudryavtsev in [3], using an equivalent norm $\sum_{k=0}^{l-1} |f^{(k)}(0)| + \|f^{(l)}\|_2$.

Function $\varphi$ is called (pointwise) multiplier from $1W$ into $2W$, if $\varphi f \in 2W \forall f \in 1W$. The space of multipliers is denoted by $M(1W \rightarrow 2W)$.

Various aspects of the theory of multipliers in unweighted spaces of differentiable functions were studied in the book [4]. G.A. Kalyabin [5] described multipliers in Sobolev spaces on $\mathbb{R}^n$ with the norm $\|f\|_{L^p(B(0,1))} + \|\nabla f\|_p$ in the case $p > n$; he also posed the question of describing multipliers in the considered here weighted case.

Obtained in Theorem 1 criterion of boundedness in weighted spaces of operators with kernels of the kind (2) is decisive to prove the following result.

**Theorem 2.** Let $u^{-2} \in B_{\delta}$, $v^{-1} \in L_2(0,r) \forall r > 0$. Then the space $M(W_{2,u}^{(l)} \rightarrow W_{2,v}^{(m)}), m \leq l$, consists of those and only those $\varphi$ satisfying the following two conditions:

$$\|(\varphi x^k)^{(m)}v\|_2 < +\infty, \quad k = 0 \ldots l - 1 \quad (6)$$

$$\sup_{r>0} \|(\varphi x^k)^{(m)}v\|_{L^2(r, +\infty)} \cdot \|x^j - k - 1u^{-1}\|_{L^2(0,r)} < +\infty, \quad k = 0 \ldots l - 1, \quad (7)$$

to which in the case $m = l$ one more condition is added, namely

$$\|\varphi uv^{-1}\|_{L^\infty(\mathbb{R}^+)} < +\infty. \quad (8)$$

**Remark.** For $m = l, u = v, (1 + x^{l-1})u^{-1} \in L_2(\mathbb{R}^+)$ the considered space of multipliers was described by the author in [6]. The obtained result had the form of the combination of two conditions (6),(8), which is natural, since in that case (6)$\Rightarrow$(7). Theorem 3 allows to widen significantly the class of weights for which a description of multipliers is available. Interestingly, multipliers in spaces with exponentially decreasing at $\infty$ weights remain not studied: their description should become the subject of future investigations.

Two formulas from the following lemma show how operators with kernels of the form (2) appear in the problem about multipliers.

**Lemma 2.** Let function $g$ on $\mathbb{R}^+$ be such that

$$g^{(k)}(0) = 0, \quad k = 0 \ldots l - 1. \quad$$

Then

$$(\varphi g)^{(l)}(x) = \varphi(x)g^{(l)}(x) + \frac{1}{(l-1)!} \sum_{k=0}^{l-1} C_{l-1}^{k} (\varphi x^k)^{(l)} \int_0^x (-t)^{l-k-1} g^{(l)}(t) \, dt; \quad (1)$$

$$(\varphi g)^{(m)}(x) = \frac{1}{(l-1)!} \sum_{k=0}^{l-1} C_{l-1}^{k} (\varphi x^k)^{(m)} \int_0^x (-t)^{l-k-1} g^{(l)}(t) \, dt, \quad m < l. \quad (2)$$
Detailed proofs of all statements will be published in Proceedings of Steklov Institute of Mathematics.

**Note added (September 1997):** In the time since submission of this article, proofs of the given results have appeared in [7]. The authors have also obtained generalizations for \( p \neq 2 \) [8].

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