FLOW MONOTONICITY AND STRICHARTZ INEQUALITIES

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This talk is largely based on joint work [5] with Jonathan Bennett and Marina Iliopoulou.

BACKGROUND

A number of important inequalities from geometric analysis may be understood via the monotonicity of an appropriate functional, often referred to as a Lyapunov functional, as the input evolves under a well-chosen flow. For instance, consider the sharp Young convolution inequality on euclidean space, due to Beckner [1] and Brascamp–Lieb [8], which states that whenever \( d \geq 1 \) and \( p_1, p_2, p \geq 1 \) satisfy the scaling hypothesis

\[
1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]

then

\[
\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \leq C(p_1, p_2) \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)}
\]

for each \( f_1 \in L^{p_1}(\mathbb{R}^d) \) and \( f_2 \in L^{p_2}(\mathbb{R}^d) \). Here, the constant \( C(p_1, p_2) \) is given by

\[
C(p_1, p_2) = \|H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2}\|_{L^p(\mathbb{R}^d)}
\]

where the parameters \( (\sigma_1, \sigma_2) \in (0, \infty)^2 \) satisfy \( p_1 p'_1 \sigma_1 = p_2 p'_2 \sigma_2 \), and \( H_t \) is the heat kernel on \( \mathbb{R}^d \) given by

\[
H_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.
\]

Written in this way, the constant \( C(p_1, p_2) \) is easily seen to be best possible by taking

\[
(f_1, f_2) = (H_{\sigma_1}^{1/p_1}, H_{\sigma_2}^{1/p_2})
\]

and using the fact that the heat kernel has unit mass for each time. The left-hand side of (1) with

\[
(f_1, f_2) \to ((e^{\sigma_1 t \Delta} f_1^{p_1})^{1/p_1}, (e^{\sigma_2 t \Delta} f_2^{p_2})^{1/p_2})
\]

gives rise to the quantity

\[
Q(t) = \|(e^{\sigma_1 t \Delta} f_1^{p_1})^{1/p_1} * (e^{\sigma_2 t \Delta} f_2^{p_2})^{1/p_2}\|_{L^p(\mathbb{R}^d)}
\]

which generates the sharp inequality (1). In particular, for (sufficiently nice) non-negative \( f_1 \) and \( f_2 \), we have

\[
\lim_{t \to 0^+} Q(t) = \|f_1 * f_2\|_{L^p(\mathbb{R}^d)}, \quad \lim_{t \to \infty} Q(t) = C(p_1, p_2) \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)}
\]

and it was shown in [2] that \( Q \) is nondecreasing.

This is an example of an inequality with certain gaussian input functions as extremisers. In fact, it can be shown that all extremisers must be gaussian, and thus
the above example indicates that when the objective is to establish an inequality in sharp form, the choice of flow is constrained by the class of extremisers.

We also note that under the above assumptions on \((p_1, p_2, \sigma_1, \sigma_2)\), except now \(0 < p_1, p_2 \leq 1\), the Lyapunov functional \(Q\) is nonincreasing for (sufficiently nice) \(f_1 \in L^{p_1}(\mathbb{R}^d)\) and \(f_2 \in L^{p_2}(\mathbb{R}^d)\) (see \([2]\)). This monotonicity yields the sharp reverse form of the Young convolution inequality

\[
\|f_1 * f_2\|_{L^p(\mathbb{R}^d)} \geq C(p_1, p_2) \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)}
\]

for each nonnegative \(f_1 \in L^{p_1}(\mathbb{R}^d)\) and \(f_2 \in L^{p_2}(\mathbb{R}^d)\). With the best constant (in this case, the largest), this inequality was first established by Brascamp–Lieb \([8]\) who also observed a fundamental link with geometry by showing that the Prékopa–Leindler inequality follows in the limiting case as \(p\) tends to zero. The Prékopa–Leindler inequality is a functional form of the Brunn–Minkowski inequality, a core inequality in geometric analysis which, for example, quickly implies the classical isoperimetric inequality.

Lieb \([18]\) also observed that the sharp (forward) Young convolution inequality implies the Shannon entropy power inequality from information theory, which states that

\[
e^{2H(X+Y)} \geq e^{2H(X)} + e^{2H(Y)}
\]

for independent random variables \(X\) and \(Y\) in \(\mathbb{R}\), with equality when \(X\) and \(Y\) are gaussian random variables. Here, for a random variable \(X\) in \(\mathbb{R}\) with appropriate probability density function \(f\), \(H(X)\) is the entropy of \(X\) and is given by

\[
H(X) = -\int_{\mathbb{R}} f \log f.
\]

An early proof of the entropy power inequality was given by Stam \([21]\) (and Blachman \([7]\), including higher dimensions) using heat-flow monotonicity. Crucially, the time derivative of the entropy functional \(H\) along heat-flow coincides with the Fisher information \(I\) along heat-flow (de Bruijn’s identity), the latter being of quadratic nature and thus more accessible. Interestingly, the Blachman–Stam inequality, a certain subadditivity of the Fisher information, is a crucial ingredient in the heat-flow monotonicity approach to the entropy power inequality, and a stronger version of this inequality is key in establishing the monotonicity of the above functional \(Q\) for the Young convolution inequalities (see Toscani \([22]\) for this observation).

Switching focus, we turn our attention to the main purpose of the talk which is to discuss monotonicity phenomena in the context of Strichartz inequalities for various evolution equations.

**The Schrödinger equation**

It is natural to begin with the free Schrödinger propagator whose classical Strichartz estimates

\[
\|e^{is\Delta}f\|_{L^{2}\left(\mathbb{R}^{d}\right)} \lesssim \|f\|_{L^{2}\left(\mathbb{R}^{d}\right)}
\]
are conjectured to have gaussian extremisers for each $d \geq 1$; see Foschi [15] and also Hundertmark–Zharnitsky [16], where the conjecture is stated and established for $d = 1, 2$.

It was proved in [3] that a monotonicity phenomenon exists in this context, where the initial data evolves under the quadratic heat-flow

$$f \to (e^{t\Delta}|f|^2)^{1/2}. $$

**Theorem 1.** ([3]) Let $d = 1, 2$ and $f \in L^2(\mathbb{R}^d)$. Then

$$Q(t) = \|e^{i\Delta}(e^{t\Delta}|f|^2)^{1/2}\|_{L^6_{x,t}(\mathbb{R} \times \mathbb{R}^d)}$$

is nondecreasing for each $t > 0$.

Underpinning this is the monotonicity of the Cauchy–Schwarz functional

$$\int_{\mathbb{R}^d} f_1 f_2$$

under such quadratic heat-flow; see work of Bennett–Carbery–Christ–Tao [6] and Carlen–Lieb–Loss [10].

A limitation of Theorem 1 is that it appears to be rather rigid and does not seem to extend in several desirable directions, including higher dimensions, and to a broader class of Sobolev–Strichartz inequalities for other dispersive or wave-like propagators, including initial data measured in the scale of classical Sobolev spaces. Here, we shall see that we can achieve this, to some extent, by considering linear flows rather than quadratic. Additionally, in Theorems 2, 3, 5 and 6, stated in terms of a Lyapunov functional $Q(t)$, the monotonicity may be strengthened to complete monotonicity in the sense that the $k$th derivative of $Q$ has sign $(-1)^k$ for every $k \in \mathbb{N}_0$.

To begin to describe these results, we state a linear heat-flow counterpart to Theorem 1.

**Theorem 2.** ([5]) Suppose $f \in L^2(\mathbb{R}^d)$. Then, for $d = 1$,

$$Q(t) = \frac{1}{2\sqrt{3}} \|e^{t\Delta}f\|_{L^6(\mathbb{R}^d)}^6 - \|e^{i\Delta}e^{t\Delta}f\|_{L^6_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^6$$

is nonincreasing for $t > 0$, and for $d = 2$,

$$Q(t) = \frac{1}{4} \|e^{t\Delta}f\|_{L^4(\mathbb{R}^2)}^4 - \|e^{i\Delta}e^{t\Delta}f\|_{L^4_{x,t}(\mathbb{R} \times \mathbb{R}^2)}^4$$

is nonincreasing for $t > 0$.

Theorems 1 and 2 both recover the sharp Strichartz inequalities in [15] and [16] by comparing $Q(t)$ as $t \to 0^+$ and $t \to \infty$. In the former case, this is analogous to the earlier recovery of the Young convolution inequalities, with $\lim_{t \to 0^+} Q(t)$ giving the left-hand side of (2) and $\lim_{t \to \infty} Q(t)$ giving the sharp form of the right-hand side of (2). In the latter case, the quantities $Q(t)$ in Theorem 2 tend to zero as $t \to \infty$ and the sharp form of (2) is equivalent to the nonnegativity of $\lim_{t \to 0^+} Q(t)$. The idea of flowing the difference of two sides of an inequality (first raised to an appropriate
power) is not new, and can be seen, for example, in work of Carlen–Carrillo–Loss [9] in the context of the sharp Hardy–Littlewood–Sobolev inequality.

Theorem 2 is, in fact, a special case of a result which holds in all spatial dimensions. If we write

\[ \mathcal{I}(f) = \int_{(\mathbb{R}^d)^m} |\hat{f} \otimes \cdots \otimes \hat{f}(\xi)|^2 \left( \sum_{1 \leq i < j \leq m} |\xi_i - \xi_j|^2 \right)^{\frac{1}{2}} \frac{1}{2} \frac{1}{(d(m-1)-2)} \, d\xi \]

then whenever \( d \geq 1, m \geq 3, \) or \( d \geq 2, m \geq 2, \) and whenever \( \mathcal{I}(f) < \infty, \) the quantity

\[ Q(t) = \frac{\|e^{it\Delta}f\|_{L^4_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^4}{2m \frac{1}{2(d-1)}} \mathcal{I}(e^{it\Delta}f) - \|e^{it\Delta}e^{it\Delta}f\|_{L^2_{x,t}((\mathbb{R} \times \mathbb{R}^d))}^{2m} \]

is nonincreasing for each \( t > 0. \) To help ground this, by looking at the case \( m = 2, \) we see that the implied inequality, by comparing initial and eternal times, \( t, \)

\[ \|e^{is\Delta}f\|_{L^4_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^4 \leq \frac{|S|^{d-1}}{2^{d}(2\pi)^{d-1}} \int_{(\mathbb{R}^d)^2} |\hat{f}(\xi_1)|^2 |\hat{f}(\xi_2)|^2 |\xi_1 - \xi_2|^{-d} \, d\xi. \]

This sharp estimate is due to Carneiro [11], holds for each \( d \geq 2, \) and has gaussian extremisers. One may view this sharp inequality as a relative of

\[ \|e^{is\Delta}f\|_{L^4_{x,t}(\mathbb{R} \times \mathbb{R}^d)} \leq \|f\|_{C^{d-2}}(\mathbb{R}^d), \]

which is a classical Sobolev–Strichartz estimate.

Ozawa–Tsutsumi [19] also proved a sharp relative of (3) where no additional regularity on the initial data beyond square-integrability is assumed, and this is compensated for by measuring the modulus square of the solution in a classical homogeneous Sobolev space with nonpositive order; specifically

\[ \|(-\Delta)^{\frac{d}{2}} e^{it\Delta}f\|_{L^2_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^2 \leq \frac{|S|^{d-1}}{4(2\pi)^{d-1}} \|f\|_{L^2(\mathbb{R}^d)}^2 \]

for each \( f \in L^2(\mathbb{R}^d). \) This inequality is valid for each \( d \geq 2 \) and the given constant is sharp with gaussian extremisers (when \( d = 1, \) (4) is in fact an identity). Furthermore, it may be seen as a consequence of the following monotonicity.

**Theorem 3.** ([5]) Let \( d \geq 2 \) and \( f \in L^2(\mathbb{R}^d). \) Then

\[ Q(t) = \frac{|S|^{d-1}}{4(2\pi)^{d-1}} \|e^{it\Delta}f\|_{L^4_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^4 - \|(-\Delta)^{\frac{d}{2}} e^{it\Delta}f\|_{L^2_{x,t}(\mathbb{R} \times \mathbb{R}^d)}^2 \]

is nonincreasing for each \( t > 0. \)

Our proofs of Theorems 2 and 3 proceed via Fourier analysis. The linear nature of the flows in these theorems allows such an approach, in stark contrast to the quadratic flow in Theorem 1. A significant advantage of using Fourier analysis is that Strichartz inequalities for other dispersive and wave-like equations may be handled in a similar manner; see the forthcoming section.
Up to now, we have stated various monotonicity phenomena, each of which generates the sharp constant in the underlying inequality. These results are, however, restricted to inequalities where the Lebesgue space exponent on the solution is an even integer. Our next result demonstrates that if one is willing to sacrifice the sharpness of the constant, then the monotonicity phenomenon in Theorem 2 may be extended to the full range of admissible exponents, including the mixed space-time norm regime.

**Theorem 4.** ([5]) Suppose \((p, q, d) \neq (2, \infty, 2)\) is such that \(2 \leq p, q \leq \infty\) and
\[
\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.
\]
Then there exists a constant \(C_{p, q}\) such that
\[
Q(t) = C_{p, q} \| e^{t\Delta} f \|_{L^p_2(\mathbb{R}^d)}^p - \| e^{is\Delta} e^{t\Delta} f \|_{L^p L^q_2(\mathbb{R} \times \mathbb{R}^d)}^p
\]
is nonincreasing for each \(t > 0\).

In the above generality, a Fourier analytic approach appears to be difficult to implement, and we prove Theorem 4 using PDE methods.

**The wave and Klein–Gordon equations**

Using the Fourier analytic approach, we may prove the following analogous results to Theorem 2 for the one-sided wave and Klein–Gordon propagators.

**Theorem 5.** ([5]) Suppose \(f \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)\). Then for \(d = 2\),
\[
Q(t) = \frac{1}{2\pi} \| e^{-t\sqrt{1-\Delta}} f \|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}^4 - \| e^{is\sqrt{1-\Delta}} e^{-t\sqrt{1-\Delta}} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4
\]
is nonincreasing for each \(t > 0\), and for \(d = 3\),
\[
Q(t) = \frac{1}{2\pi} \| e^{-t\sqrt{1-\Delta}} f \|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^4 - \| e^{is\sqrt{1-\Delta}} e^{-t\sqrt{1-\Delta}} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^3)}^4
\]
is nonincreasing for each \(t > 0\).

Regarding notation, \(\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)\) denotes the homogeneous Sobolev space on \(\mathbb{R}^d\) of order \(\frac{1}{2}\). The extremisers for the inequalities generated by the monotone quantities in Theorem 5 were found by Foschi [15] and include initial data \(f\) such that
\[
\hat{f}(\xi) = e^{-|\xi|},
\]
from which we see the relevance of the flow \(e^{-t\sqrt{1-\Delta}}\).

For the Klein–Gordon equation, we obtain the following monotonicity phenomena under the flow \(e^{-t\sqrt{1-\Delta}} f\).

**Theorem 6.** ([5]) Suppose \(f \in H^{\frac{1}{2}}(\mathbb{R}^d)\). Then for \(d = 2\),
\[
Q(t) = \frac{1}{2} \| e^{-t\sqrt{1-\Delta}} f \|_{H^{\frac{1}{2}}(\mathbb{R}^2)}^4 - \| e^{is\sqrt{1-\Delta}} e^{-t\sqrt{1-\Delta}} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4
\]

is decreasing for each $t > 0$, and for $d = 3$,

$$Q(t) = \frac{1}{2\pi} \left\| e^{-t\sqrt{1-\Delta}} f \right\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}^4 - \left\| e^{is\sqrt{1-\Delta}} e^{-t\sqrt{1-\Delta}} f \right\|_{L^{4}_{s,x}(\mathbb{R}^3)}^4$$

is decreasing for each $t > 0$.

Here, $H^{\frac{1}{2}}(\mathbb{R}^d)$ denotes the inhomogeneous Sobolev space of order $\frac{1}{2}$. The Strichartz inequalities generated by the strict monotonicity of $Q$ in Theorem 6 do not have extremisers. However, if we write

$$\hat{f}_a(\xi) = e^{-a \sqrt{1+|\xi|^2}}$$

then $(f_a)$ is an extremising sequence as $a \to \infty$ for $d = 2$, and $a \to 0^+$ for $d = 3$ (Quilodrán [20] first established the underlying sharp inequalities, the lack of extremisers and identified such extremising sequences). Despite the lack of extremisers, we see the relevance of the flow $e^{-t\sqrt{1-\Delta}} f$.

**THE KINETIC TRANSPORT EQUATION**

We conclude with some observations regarding flow monotonicity in the context of the kinetic transport equation

$$\partial_s u(s, x, v) + v \cdot \nabla_x u(s, x, v) = 0, \quad u(0, x, v) = f(x, v)$$

for $(s, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. The Strichartz estimates for the solution of this equation are of the form

$$(5) \quad \left\| \rho(f) \right\|_{L^q_s L^p_{s,x}((\mathbb{R} \times \mathbb{R}^d) \times \mathbb{R}^d)} \lesssim \left\| f \right\|_{L^{2}_{s} L^{2}_{v}(\mathbb{R} \times \mathbb{R}^d)},$$

where the velocity averaging operator $\rho$, often called the macroscopic density, is given by

$$\rho(f)(s, x) = \int_{\mathbb{R}^d} f(x - vs, v) \, dv.$$

Based on results in [12], [17] and [4], it is now known that (5) holds if and only if $(a, p, q)$ satisfies

$$q > a, \quad p \geq a, \quad \frac{2}{q} = d \left( 1 - \frac{1}{p} \right), \quad \frac{1}{a} = \frac{1}{2} \left( 1 + \frac{1}{p} \right).$$

Given Theorem 1 for the Schrödinger equation, it is natural to look for monotonicity in the context of (5) in the particular case

$$(6) \quad \left\| \rho(f) \right\|_{L^{\frac{d+2}{d+1}}_s L^{\frac{d+2}{d}}_{s,x}((\mathbb{R} \times \mathbb{R}^d) \times \mathbb{R}^d)} \lesssim \left\| f \right\|_{L^{\frac{d+2}{d+1}}_s L^{\frac{d+2}{d}}_{s,v}(\mathbb{R} \times \mathbb{R}^d)}$$

corresponding to $(a, p, q) = \left( \frac{d+2}{d+1}, \frac{d+2}{d}, \frac{d+2}{d} \right)$. The dual estimate to (6) is

$$(7) \quad \left\| \rho^*(g) \right\|_{L^{\frac{d+2}{d+1}}_s L^{\frac{d+2}{d}}_{s,x}(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \left\| g \right\|_{L^{\frac{d+2}{d+1}}_s L^{\frac{d+2}{d}}_{s,v}(\mathbb{R} \times \mathbb{R}^d)}$$

where the operator $\rho^*$ is given by

$$\rho^*(g)(x, v) = \int_{\mathbb{R}} g(s, x + vs) \, ds.$$
The sharp constant in the space-time X-ray transform estimate (7) was identified by Drouot [13], who observed that the function
\[ g(s, x) = \frac{1}{1 + s^2 + |x|^2} \]
is amongst the class of extremisers. Thanks to work of Flock [14], it is known that, modulo symmetries of the inequality, all extremisers are of this type and thus rules out a heat-flow monotonicity phenomena in this context.

We conclude by presenting some evidence in the form of the following result (for the 2-plane transform rather than the X-ray type transform above) to suggest that it is reasonable to expect monotonicity under certain fast diffusion flows.

**Theorem 7.** ([5]) Suppose \( d \geq 2, m = \frac{d+1}{d+3} \) and let \( g \in L^2_{d+1} (\mathbb{R}^{d+1}) \) be nonnegative and of compact support. If \( u : [0, \infty) \times \mathbb{R}^{d+1} \to [0, \infty) \) satisfies
\[ \partial_t u = \Delta (u^m); \quad u(0, \cdot) = g, \]
then
\[ Q(t) = C_d^2 \| u(t, \cdot) \|_{L^2_{\frac{d+4}{d+1}} (\mathbb{R}^{d+1})}^2 - \| T_{2,d+1} (u(t, \cdot)) \|_{L^2 (\mathbb{M}_{2,d+1}^{d+1})}^2 \]
is nonincreasing for each \( t > 0. \)

Here, \( T_{2,d+1} \) denotes the classical 2-plane transform on \( \mathbb{R}^{d+1} \) and \( \mathbb{M}_{2,d+1} \) the Grassmann manifold of all affine 2-planes in \( \mathbb{R}^{d+1} \). The constant \( C_d \) is the sharp constant in the inequality
\[ \| T_{2,d+1}(g) \|_{L^2(\mathbb{M}_{2,d+1}^{d+1})} \leq C_d \| g \|_{L^2_{\frac{d+4}{d+1}} (\mathbb{R}^{d+1})} \]
whose extremisers include the function
\[ g(s, x) = \frac{1}{(1 + s^2 + |x|^2)^{-\frac{d+1}{2}}} \]
This explains the appearance of the fast diffusion flow in (8) whose asymptotic profiles (Barenblatt profiles) take this shape for the given value of \( m \). Theorem 7 is simply proved by combining an identity of Drury which connects \( \| T_{2,d+1}(g) \|_{L^2(\mathbb{M}_{2,d+1}^{d+1})} \) with the diagonal Hardy–Littlewood–Sobolev functional, and the recent observation of Carlen–Carrillo–Loss [9] that certain cases of the sharp Hardy–Littlewood–Sobolev inequality may be established via fast diffusion monotonicity. It is reasonable to hope that other versions of Drury’s identity may lead to monotonicity phenomena under fast diffusion in the context of the kinetic transport equation for the Strichartz inequalities (6) and (7).

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