NORMAL FORM AND NEKHOROSHEV STABILITY FOR NEARLY-INTEGRABLE HAMILTONIAN SYSTEMS WITH UNCONDITIONALLY SLOW APERIODIC TIME DEPENDENCE

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ABSTRACT. The aim of this paper is to extend the results of Giorgilli and Zehnder for aperiodic time dependent systems to a case of general nearly-integrable convex analytic Hamiltonians. The existence of a normal form and then a stability result are shown in the case of a slow aperiodic time dependence that, under some smallness conditions, is independent of the size of the perturbation.

1. INTRODUCTION

The study of the solutions of a near-integrable Hamiltonian system goes back to Poincaré [Poi92], who emphasized the relevance of this model by describing it as “General problem of the Dynamics”. Motivated by problems arising from Celestial Mechanics, stability would have been among the most interesting (and urgent) questions to be addressed. The persistence of invariant tori of an integrable Hamiltonian under small perturbations, initially faced by Kolomogorov [Kol54], gave a powerful answer to this problem: the perpetual stability of certain invariant tori. A different proof, due to Arnold [Arn63], showed that the invariant tori persisted on a very special subdomain of the phase space \( \mathcal{D} \), a Cantor set whose (Lebesgue) measure is “close” to the (Lebesgue) measure of the phase space \( \mathcal{D} \). As a drawback of Arnold’s method of proof, the construction of an “exact” normal form (i.e. by an infinite number of steps) provided by this result, holds in a very special subdomain of the phase space \( \mathcal{D} \) (Cantor set), which measure is close to the \( \mathcal{D} \) one, but it is completely different from a topological point. This “high probability” [Pös01] to lie on an invariant torus is not adequate for certain applications.

The possibility of a weaker statement on a more suitable domain from the applications point of view, consisted of the so-called effective stability. After the initial contributions by Moser [Mos55], Littlewood [Lit59b], [Lit59a] and subsequently by Glimm [Gli64], it was realized in a general setting by Nekhoroshev [Nek77]. The starting point changed the KAM point of view: by keeping only a order \( r \) – truncated (resonant) normal form, it is possible to preserve an open subset of the phase space, then a careful choice of \( r \) can be made in order to obtain a stability time as large as possible. Obviously this result, as it is, is only of local nature. The decisive contribution of Nekhoroshev was the so-called geometric part, in which the entire phase space is covered by using suitable subsets having known resonance properties (geometry of resonances) to which the normal form result can be applied.

The relevance of this result has rapidly raised the interest of the scientific community outside the Russian school, especially in Italy with Benettin, Galgani, Gallavotti and Giorgilli, e.g. [BGG85], [Gal86] and subsequent papers, then in France with Lochak, e.g. [Loc92], who developed a new approach (simultaneous Diophantine approximations) able to enlarge the exponent of the stability bound. The steepness feature of the unperturbed Hamiltonian initially considered by Nekhoroshev, was profitably replaced by the (slightly) less general but remarkable convexity in the ‘80, and then weakened to quasi-convexity in [Pös93]. As it was reasonable to expect, solutions “close” to a KAM torus would possess special stability.
properties. This aspect was made precise in [MG95] and [PW94], with the former reference showing that solutions starting exponentially close to a KAM torus are indeed super-exponentially stable.

Meanwhile, the paper [GZ92] proposed a different direction, by considering a model of the form \( H(x, y, t) = |\dot{y}|^2/2 + f(x, y, t) \) (i.e. convex). The dependence of \( f \) is quasi-periodic on \( x \) but only analytic on \( t \), introducing in this way, for the first time, an aperiodic time dependence. A Nekhoroshev type result is shown for motions with “high” kinetic energy, i.e., after a time rescaling \( t = \varepsilon \tau \), for small \( \varepsilon \) and bounded energies. As a side effect, the dependence on \( \tau \) turns out to be slow with \( \varepsilon \).

The key property used in the perturbative setting of [GZ92], consists of the possibility to disregard the dependence on \( \tau \) in the solution of the homological equation. This has the irrelevant effect of losing \( t \) control of the variable canonically conjugate to \( \tau \), say \( \eta \), the latter being a fictitious variable\(^1\). This argument of partial normal form substantially simplifies the discussion and, as it will be shown, allows an immediate interfacing with the standard quasi-periodic case.

Despite these innovative features, the mentioned result has not been widely known for more than twenty years. Recently, the problem has been reconsidered in [Bou13], giving an outline of the elements necessary to adapt previous results by the same author to a system of the form \( H(I, \theta, t) = h(I) + \varepsilon f(I, \theta, \varepsilon t) \), \( c \in \mathbb{R}^+ \). A slow time dependence similar to [GZ92] is considered.

Also in the light of the applications of this kind of result pointed out in [WM13], the aim of this paper is to extend the results by [GZ92] to more general systems, along the lines of a proof of the Nekhoroshev theorem described in the comprehensive paper [Gio02]. The fully constructive setting given by the Lie transform method allowed a deeper analysis of the slow time dependence problem. As also stressed in [Bou13], an hypothesis of slow time dependence is completely reasonable. Otherwise, it is natural to expect the existence of ad hoc perturbations able to “drive” the solutions along some resonance channel. Roughly, the role of the small parameter is to create a safe separation between the frequencies of the unperturbed system and those produced by the perturbation. Nevertheless, by considering a two-parameters system (1), we show that the “speed” of the time dependence and the size of the perturbation, under some smallness hypothesis, should not be necessarily related. This is the feature behind the unconditionally slow dependence. It leads to a great advantage from the applications point of view and is an extension of the results obtained in [GZ92] and [Bou13].

The partial normal form, whose existence is shown in sec. 3, allows to use exactly the same geometric arguments of a standard Nekhoroshev result. For this reason, more general hypothesis on the unperturbed Hamiltonian (used only in the geometric part) than the convexity are not addressed here and a brief outline of the argument described in [Gio02], is given in sec. 4 for the sake of completeness.

2. Set-up and main result

Let \( \mathcal{G} \) be an open subset of \( \mathbb{R}^n \) and consider the nearly integrable system described by the following Hamiltonian function

\[
H(I, \varphi, t) := h(I) + \varepsilon f(I, \varphi, \mu t),
\]

where \( I = (I_1, \ldots, I_n) \in \mathcal{G}, \varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n \) is a set of action-angle variables and \( t \in \mathbb{R} \) is an additional variable (time) on which \( f \) does not need to depend quasi-periodically.

As usual, by setting \( \xi := \mu t \) and denoting by \( \eta \in \mathbb{R} \) the variable conjugate to \( t \), Hamiltonian (1) can be seen as autonomous in the extended phase space \( \mathcal{D} := \mathcal{G} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R} \ni (I, \eta, \varphi, \xi) \) in the form

\[
H(I, \varphi, \xi, \eta) := h(I) + \mu \eta + \varepsilon f(I, \varphi, \xi).
\]

\(^1\)Furthermore if \( \mu \) is a scalar variable as in our case, bounded variations of the actions imply that \( \eta \) is bounded as well, simply by the conservation of energy.
Given two control parameters $\rho, \sigma \in (0, 1]$, consider the complex neighbourhood of $D$, defined as $D_{\rho, 2\sigma} := \mathcal{G}_\rho \times R_\rho \times T^n_{2\sigma} \times R_{2\sigma}$ where
\[
\mathcal{G}_\rho := \bigcup_{I \in \mathcal{G}} \Delta_\rho(I), \quad \Delta_\rho(I) := \{ \hat{I} \in \mathbb{C}^n : |\hat{I} - I| < \rho \}
\]
\[
\mathcal{R}_\rho := \{ \hat{\eta} \in \mathbb{C} : |\hat{\eta}| < \rho \}
\]
\[
T^n_{2\sigma} := \{ \hat{\varphi} \in \mathbb{C}^n : |\hat{\varphi}| < 2\sigma \}
\]
\[
R_{2\sigma} := \{ \xi \in \mathbb{C} : |\xi| < 2\sigma \}
\]
The space $D_{\rho, 2\sigma}$ is endowed with the usual supremum norm
\[
|F|_{\rho, \sigma} := \sup_{z \in D_{\rho, 2\sigma}} |F(z)|.
\]
For any analytic functions $F = F(I, \varphi, \xi) \in D_{\rho, 2\sigma}$, admitting a Fourier expansion of the form
\[
F(I, \eta, \xi) = \sum_{k \in \mathbb{Z}^n} f_k(I, \xi)e^{ik\cdot\varphi},
\]
the Fourier norm is defined as
\[
\|F\|_{\rho, \sigma} := \sum_{k \in \mathbb{Z}^n} |f_k|_{\rho, \sigma} e^{|k|\sigma},
\]
where $|k| = |k_1| + \ldots + |k_n|$. System (1) will be studied under the following

**Hypothesis 2.1.** $h(I)$ and $f(I, \varphi, \xi)$ are holomorphic and bounded functions on the space $D_{\rho, 2\sigma}$, in particular
\[
|f|_{\rho, \sigma} =: C_f < +\infty.
\]

**Hypothesis 2.2.** The unperturbed Hamiltonian $h(I)$ is a convex function, i.e. there exists two constants $M \geq m > 0$ such that, for all $I \in \mathcal{G}_\rho$
\[
|\partial^2 h(I)v| \leq M|v|, \quad |\langle \partial^2 h(I)v, v \rangle| \geq m|v|^2,
\]
for all $v \in \mathbb{R}^n$.

The set of parameters $\rho, \sigma, M, m, C_f$ are characterized by a given Hamiltonian and will be supposed fixed once and for all. Let us define
\[
\tilde{F} = C_f \left(1 + e^{-\frac{\varphi}{\delta}}\right)^n, \quad \lambda_{\varepsilon, \mu} := \mu + \varepsilon \tilde{F} \varepsilon.
\]

In this framework, the main result is stated as follows

**Theorem 2.3** (Aperiodic Nekhoroshev). Assume hypotheses 2.1 and 2.2. Then there exists constants $\Delta^*$ and $T$ depending on $\rho, \sigma, M, m, C_f$ and $n$ such that, if $\varepsilon$ and $\mu$ satisfy
\[
\lambda_{\varepsilon, \mu} < \frac{1}{3|\Delta^*|},
\]
then orbits $(I(t), \varphi(t))$ of (1) starting in $\mathcal{G} \times T^n$ at $t_0$, satisfy
\[
|I(t) - I(t_0)| < (\Delta^* \lambda_{\varepsilon, \mu})^{\frac{1}{2}} \rho, \quad \text{for} \quad |t - t_0| < \frac{T}{\varepsilon} \exp \left[ \left( \frac{1}{\Delta^* \lambda_{\varepsilon, \mu}} \right)^{\frac{1}{2n}} \right],
\]
where $a = n^2 + n$. We remark that $\Delta^*$ is defined in sec. 4. The main feature of this formulation is that the smallness condition (6) allows a certain freedom in the choice of $\varepsilon$ and $\mu$. Essentially, within the described threshold, parameters $\varepsilon$ and $\mu$ can be treated as independent. In principle, $\mu$ is even allowed to be increased, as $\varepsilon$ tends to 0, still preserving a normal form result (see Theorem 3.3) and a stability estimate, despite the fact that, in such case, it is a worse estimate as the bound is only $O(\varepsilon^{-1})$ as $\varepsilon$ vanishes. In any case, this
3. NORMAL FORM

3.1. Basic notions and statement.

Definition 3.1. A subset $\mathcal{M}$ of $\mathbb{Z}^n$ is said to be a resonance module if it satisfies
\[ \text{span}(\mathcal{M}) \cap \mathbb{Z}^d = \mathcal{M}. \tag{7} \]

If $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ is a basis for $\mathcal{M}$ (i.e. $\mathcal{M} = \{\alpha_1 k_1 + \cdots + \alpha_n k_n : \alpha_j \in \mathbb{Z}\}$), with $\text{span}(\mathcal{M})$ we denote the set $\{\gamma_1 k_1 + \cdots + \gamma_n k_n : \gamma_j \in \mathbb{R}\} \subseteq \mathbb{R}^n$. Hence, the purpose of condition (7), is to exclude subspaces of $\mathbb{Z}^d$ which contain less points of the lattice than the real space $\text{span}(\mathcal{M})$.

Definition 3.2. Let $\mathcal{M}$ be a resonance module, $\alpha \in \mathbb{R}^+$ and $N \in \mathbb{N}$. A subspace $\mathcal{V}$ of the action space $\mathcal{G}$ is said to be a non-resonance domain of the type $(\mathcal{M}, \alpha, \delta, N)$ if, for all $k \in \mathbb{Z}^n \setminus \mathcal{M}$ with $|k| < N$, the following condition holds
\[ |\langle \omega(I), k \rangle| > \alpha, \quad \forall I \in \mathcal{G}_\delta, \]
where $\omega(I) := \partial_1 h(I)$.

It will be denoted by $\tilde{D}_{\delta, \sigma} \subseteq D_{\delta, \sigma}$, the complex extension of $D$ with $\mathcal{G}$ is replaced with $\mathcal{V}$.

Theorem 3.3 (Existence of a normal form). Consider the Hamiltonian (2) with the regularity assumptions of hypothesis 2.1 and the associated parameters $\delta, \sigma, C_f$. Given $\mathcal{M}$ a resonance module, $r, K \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, suppose that $\mathcal{V} \subseteq \mathcal{G}$ is a non-resonance domain of the type $(\mathcal{M}, \alpha, \delta, N)$ with $N = rK$, and that $r, K, \varepsilon$ and $\mu$ are such that
\[ \Delta := \frac{2^8 \rho}{\alpha \delta \sigma} \lambda_{c; \mu} + 4 e^{-K \frac{\rho^2}{2}} \leq \frac{1}{2}, \tag{8a} \]
\[ e^{-K \frac{\rho^2}{2}} \geq (8 + 3 \varepsilon^2)^{-1}. \tag{8b} \]

Then there exists a symplectic, $\varepsilon-$close to identity, analytic change of variables $z \to C_r(z)$ defined on $\tilde{D}_n^{(\delta, \sigma)}$ such that
\[ D_{\rho^{(\delta, \sigma)}} \subseteq C_r \tilde{D}_{\rho^{(\delta, \sigma)}} \subseteq \tilde{D}_{\rho^{(\delta, \sigma)}}, \]
(the same for holds for $C_r^{-1}$), and casting the Hamiltonian (2) in resonant normal form up to order $r$, i.e.
\[ H(C_r(z)) = h(I) + \eta + Z^{(r)} + R^{(r+1)}, \tag{9} \]
with $Z^{(r)} = \sum_{k \in \mathcal{M}} z_k(I, \xi) e^{ik \cdot \varphi}$ for all $|k| \leq N$ and
\[ \|R^{(r+1)}\|_{\tilde{H}^{(\delta, \sigma)}} \leq 8 \varepsilon \Delta^r. \tag{10} \]

Remark 3.4. As mentioned before, by condition (8a) it is evident that the normal form exists as long as $\lambda_{c; \mu}$ is sufficiently small, no matter if there is relation or not between $\varepsilon$ and $\mu$. The technical hypothesis (8b) does not appear in [Gio02], anyway, it will be shown that it can be assumed without loss of generality.
The proof of the already stated result goes along the lines of [GZ92] and [Gio02], and can be achieved in two steps. In the first one, a suitable perturbative algorithm is built up in order to remove the effect of the perturbation up to a pre-fixed order $r$ (except for a particular set of harmonics given by $\mathcal{M}$). The perturbative scheme is initially discussed at a formal level (i.e. disregarding the problem of the series convergence) and it is based on the Lie transform method. The subsequent step consists of giving quantitative estimate on the convergence of the scheme by using standard analytic tools (see [Gio02]).

3.2. Perturbative setting: formal scheme. Given $K$, define, for all $j \geq 1$, the following class of functions on $\mathcal{D}_{\delta,\sigma}$

$$\mathcal{P}_j := \{ g : g(I, \varphi, \xi) = \sum_{|k| < jK} g_k(I, \xi)e^{ik\varphi} \}.$$ 

By setting, for all $s = 1, 2, \ldots$,

$$H_s := \varepsilon \sum_{(s-1)K \leq |k| < sK} f_k(I, \xi)e^{ik\varphi},$$

the Fourier expansion of $f$ had been split in such a way $H_s \in \mathcal{P}_s$ for all $s \geq 1$, and the Hamiltonian (2) reads as

$$H(I, \varphi, \xi, \eta) = h(I) + \eta + H_1 + H_2 + \ldots \quad (11)$$

The aim is to find a local, $\varepsilon-$close to the identity, symplectic diffeomorphism casting the Hamiltonian (2) into the form (9). This is achieved via a suitable choice of a (finite) sequence of functions $\chi^{(r)} := \{ \chi_s \}_{s=1,\ldots,r}$ (generating sequence), and setting $C_r := T_{\chi^{(r)}}$, where $T_{\chi^{(r)}}$ is the Lie transform operator associated to $\chi^{(r)}$

$$T_{\chi^{(r)}} := \sum_{s=1}^{r} E_s, \quad E_s := \begin{cases} \text{Id} & s = 0 \\ \frac{1}{s} \sum_{j=1}^{s} j\mathcal{L}_{\chi_s}E_{s-j} & s \geq 1 \end{cases} \quad (12)$$

and $\mathcal{L}_{\chi_s}g := \{ g, \chi_s \}$ stands for the Lie derivative. Note that if $f, g$ are two functions independent of $\eta$ (as the objects involved in the above perturbative scheme), the parenthesis $\{ f, g \}$ reduces to $\sum_{i=1}^{n}(\partial_{\varphi_i}f\partial_{\varphi_i}g - \partial_{\varphi_i}f\partial_{\varphi_i}g)$. In particular, $\{ \xi, f \} = 0$ for all $f = f(I, \varphi, \xi)$, i.e. $T_{\chi^{(r)}}(\xi) = \xi$. Hence the considered transformation does not act on time.

Taking into account of (11) and writing $Z^{(r)} = Z_1 + \ldots + Z_r$ and one gets the following

**Proposition 3.5.** Equation (9) is equivalent to the following hierarchy of homological equations

$$\mathcal{L}_h \chi_s + Z_s = \psi_s, \quad (13)$$

for $s = 1, \ldots, r$ with $Z_s \in \mathcal{P}_s$ and

$$\psi_s := \begin{cases} H_1 & s = 0 \\ H_s + \mu E_{s-1}\eta + \frac{1}{s} \sum_{j=1}^{s-1} j[\mathcal{L}_{\chi_j}H_{s-j} + E_{s-j}H_j] & 2 \leq s \leq r \end{cases} \quad (14)$$

**Proof.** (Sketch). Use the well known identity $H(T_{\chi^{(r)}}) = T_{\chi^{(r)}}H$, then substitute into (9) the involved objects in form of (finite) sums, equating, at the $s-$th stage, terms on the same level $\mathcal{P}_s$. See [Gio02] for the details. □

**Remark 3.6.** Note that the operator $\mathcal{L}_h$ is exactly the same as for the standard (quasi-periodic) case. This is the main advantage in considering a *partial normal form* and the solution of the homological equation (13) can be done by a standard comparison of Fourier coefficient. Note that in this case, at each stage, the averaged term does not depend only on $I$ but also on $\xi$. The key fact used in [GZ92] is that one only needs a partial normal form and this term can be anyway included in $Z$ as this does not affect the evolution of the variables $I$, but only of $\eta$. Consistently, if the aperiodic dependence on $\xi$ is supposed to be quasi-periodic, the dependence on the “angle” $\xi$ is annihilated by averaging and the partial normal
form becomes “full”.
Hence, the argument is reduced to the control of the extra-term \( E_{s-1} \eta \) arising from the aperiodic time dependence.

As for the solution of (13), denote by \( z_k^{(s)}, c_k^{(s)} \) and \( \psi_s \) the Fourier coefficients of \( Z_s, \chi_s \) and \( \psi_s \) respectively. As the resonance module \( \mathcal{M} \) is fixed, if \( k = 0 \) or \( k \in \mathcal{M} \) one sets \( c_k^{(s)} = 0 \) and \( z_k^{(s)} = \psi_k^{(s)} \). Otherwise, if only \( I \in \mathcal{V} \) are considered, the quantity \( \langle k, \omega(I) \rangle \) is bounded away from zero and it is possible to set \( c_k^{(s)} = i(\langle k, \omega(I) \rangle)^{-1} \psi_k^{(s)} \) then \( z_k^{(s)} = 0 \). This yields immediately the following two inequalities

\[
\|Z_s\|_{(1-d)(\delta,\sigma)} \leq \|\psi_s\|_{(1-d)(\delta,\sigma)}, \quad \|\chi_s\|_{(1-d)(\delta,\sigma)} \leq \frac{1}{\alpha} \|\psi_s\|_{(1-d)(\delta,\sigma)},
\]

valid for all \( d \in (0, 1) \).

3.3. Convergence.

Lemma 3.7. Assume hypothesis 2.1, then the following sequence of “nested” statements holds:

1. There exists \( h > 0 \) and \( \mathcal{F} \geq 0 \) such that

\[
\|H_s\|_{(\delta,\sigma)} \leq h^{s-1} \mathcal{F}, \quad s \geq 1.
\]

2. Supposing (16), holds

\[
\|\psi_s\|_{(1-d)(\delta,\sigma)} \leq \frac{h^{s-1}}{s} \mathcal{F}, \quad s \geq 1,
\]

for all \( d < 1/4 \) and some \( b \geq 0 \). Hence, by (15), the truncated series \( \sum_{j=1}^r \chi_s \) and \( \sum_{j=1}^r Z_s \) are well defined on \( \mathcal{V} \), yielding respectively \( \chi^{(r)} \) and \( Z^{(r)} \) as a solution of (9).

3. Assume (17) and that for all \( d \in (0, \frac{1}{4}) \) the condition

\[
\frac{2e\mathcal{F}}{d^2\alpha\delta\sigma} + b \leq \frac{1}{2},
\]

is satisfied, then the operator \( T_{\chi^{(r)}} \) (and its inverse \( T_{\chi^{(r)}}^{-1} \)) define a canonical transformation on the domain \( \tilde{D}_{(1-d)(\delta,\sigma)} \) with the following properties

\[
\tilde{D}_{(1-2d)(\delta,\sigma)} \subset T_{\chi^{(r)}} \tilde{D}_{(1-d)(\delta,\sigma)} \subset \tilde{D}_{(\delta,\sigma)}
\]

\[
\tilde{D}_{(1-2d)(\delta,\sigma)} \subset T_{\chi^{(r)}}^{-1} \tilde{D}_{(1-d)(\delta,\sigma)} \subset \tilde{D}_{(\delta,\sigma)}.
\]

The proof of 1 can be found in [GZ92, Pag. 851] where in addition it is shown that

\[
\mathcal{F} = e\tilde{\mathcal{F}}, \quad h = e^{-K^{(r)}},
\]

while the third one is straightforward from a general result on the convergence of Lie transform, see [Gio02]. The statement 2 requires a further analysis with respect to the existing case, due to the presence of the extra-term \( E_{s-1} \eta \) related to the aperiodic time dependence.

In order give an estimate for \( \|\psi_s\|_{(1-d_s)(\delta,\sigma)} \), Lie operators appearing in (14) can be treated via standard Cauchy tools, whereas a domain restriction is provided. For, let \( \tilde{D}_{(1-d_s)(\delta,\sigma)} \) be the domains sequence, where a convenient choice is given by

\[
d_s := d \sqrt{\frac{s-1}{r-1}},
\]

for all \( s = 1, \ldots, r \) and where \( d \), appearing in (18), will be determined later.
In order to control terms appearing in (14) for all \( s \), three sequences are considered that are implicitly
defined by the following inequalities
\[
\|\psi_s\|_{(1-d_s)(\delta,\sigma)} \leq \beta_s \mathcal{F},
\]
\[
\|E_s H_j\|_{(1-d_{s+j})(\delta,\sigma)} \leq \theta_{s,j} \mathcal{F} \quad j = 1, \ldots, r - s
\]
\[
\|E_{s-1} \eta\|_{(1-d_s)(\delta,\sigma)} \leq \gamma_{s-1} \mathcal{F}
\]
Taking into account of (16) and the definition of \( \psi_1 \), inequality (21a) computed for \( s = 0 \) immediately gives \( \beta_1 = 1 \). On the other hand, recalling that \( E_0 \) is \( 1 \)d, by (21b) one gets \( \theta_{0,j} := h^{j-1} \). As for (21c), a Cauchy estimate, the second of (15) and then (21a), yield
\[
\|E_1 \eta\|_{(1-d_2)(\delta,\sigma)} \leq \left\| \frac{\partial \chi_1}{\partial \xi} \right\|_{(\delta,\sigma)} \leq \frac{1}{d\sigma} \|\chi_1\|_{(\delta,\sigma)} \leq \frac{\mathcal{F}}{\alpha d\sigma},
\]
i.e., defining
\[
\Gamma := (\alpha d\sigma)^{-1},
\]
one obtains \( \gamma_1 = \Gamma \). Setting \( m = s - 1 \), it is straightforward by definition that for all \( m \geq 1 \)
\[
\|E_m \eta\|_{(1-d_{m+1})(\delta,\rho)} \leq \frac{1}{m} \sum_{l=1}^{m-1} l \|\mathcal{L}_{\chi_1} E_{m-l} \eta\|_{(1-d_{m+1})(\delta,\rho)} + \|\mathcal{L}_{\chi_m} \eta\|_{(1-d_{m+1})(\delta,\rho)}.
\]
The first term of the right hand side can be estimated by using the following result (see [GZ92, Pag. 853] for the proof)

**Lemma 3.8.** Let \( d', d'' \in \mathbb{R}^+ \) such that \( d' + d'' < 1 \) and

1. \( G(1, \varphi, \xi) \) be analytic and bounded in \( \tilde{D}_{(1-d')(\delta,\sigma)} \).
2. \( F(1, \varphi, \xi) \) be analytic and bounded in \( \tilde{D}_{(1-d'')(\delta,\sigma)} \).

Then, for all \( 0 < d < 1 - d' - d'' \) the following inequality holds
\[
\|\mathcal{L}_{\xi} F\|_{(1-d-d'-d'')(\delta,\sigma)} \leq C \|F\|_{(1-d'')(\delta,\sigma)} \|G\|_{(1-d')(\delta,\sigma)},
\]
where \( C = 2e(d + d')(d + d'')d\sigma^{-1} \).

By construction \( \chi_l \) is analytic on \( \tilde{D}_{(1-d_l)(\delta,\sigma)} \) while \( E_{m-l} \eta \) is analytic on \( \tilde{D}_{(1-d_{m-l})(\delta,\sigma)} \) hence by lemma (3.8) with \( d' = d_l \) and \( d'' = d_{m-l} \), one gets on \( \tilde{D}_{(1-d_{m+1})(\delta,\sigma)} \)
\[
\|\mathcal{L}_{\chi_l} E_{m-l} \eta\|_{(1-d_{m+1})(\delta,\sigma)} \leq \frac{2}{e\delta(d_{m+1} - d_l)(d_{m+1} - d_{m-l})} \beta_l \mathcal{F} \gamma_{m-l},
\]
having used the second of (15) then (21a) and (21c).
In conclusion, the first term of the r.h.s. of (24), can be bounded by (26), recalling (20) and the elementary inequality
\[
(\sqrt{m} - \sqrt{l-1})(\sqrt{m} - \sqrt{m-l-1}) \geq \frac{1}{2}, \quad 1 \leq l \leq m - 1.
\]
As for the second term, the same argument of (22) can be used, yielding, for all \( m \geq 1 \)
\[
\|E_m \eta\|_{(1-d_{m+1})(\delta,\rho)} \leq \frac{C_r}{m} \sum_{l=1}^{m-1} l\beta_l \gamma_{m-l} + \beta_m \Gamma \mathcal{F},
\]
where
\[
C_r := \frac{4(r - 1) \mathcal{F}}{\alpha e d^2 \delta \sigma}.
\]

\(^2\)Recalling (20), one checks immediately that \( d_l + d_{m-l} < 1 - d \) as required by Lemma 3.8 if \( d \leq 1/3 \), justifying in this way the assumption on \( d \) in Lemma 3.7. Further estimates (29a) and (29b) hold under the same assumption.
This estimate can be clearly written in the form (21c) provided
\[ \gamma_{s-1} := \frac{C_r}{s-1} \sum_{l=1}^{s-2} l \beta_l \gamma_{s-l-1} + \Gamma \beta_{s-1} \] (28)

The remaining estimates (see [Gio02] for the details), can be found in the same way and take the form
\[ \| L_{\lambda_l} H_{s-l} \|_{(1-d_s)(\delta, \sigma)} \leq C_r \beta_l h^{s-l-1} F \] (29a)
\[ \| E_s H_j \|_{(1-d_s+j)(\delta, \sigma)} \leq \frac{C_r}{s} \sum_{l=1}^{s-1} \beta_l \bar{\theta}_{s-l,j} F \] (29b)

Collecting the estimates, one obtains the following system of recurrence equations
\[
\begin{align*}
\beta_s &:= h^{s-1} + \mu \gamma_{s-1} + \frac{1}{s} \sum_{l=1}^{s-1} lh^{l-1} \theta_{s-l} + \frac{C_r}{s} \sum_{l=1}^{s-1} \beta_l h^{s-l-1} \\
\theta_s &:= \frac{C_r}{s} \sum_{l=1}^{s-1} \beta_l \theta_{s-l} \\
\gamma_s &:= \frac{C_r}{s} \sum_{l=1}^{s-1} \beta_l \gamma_{s-l} + \Gamma \beta_s
\end{align*}
\] (30)
(see below for the definition of \( \theta_s \)) with the following initial conditions
\[ \beta_1 = 1, \quad \theta_0 = 1, \quad \gamma_1 = \Gamma. \]

Indeed, a comparison between (29b) and (21b) gives \( \theta_{s,j} = (C_r/s) \sum_{j=1}^{s} j \beta_j \bar{\theta}_{s-j,j} \). From the latter, by using \( \bar{\theta}_{0,j} = h^{j-1} \) and defining \( \theta_s := \bar{\theta}_{s,1} \), it is easy to check that \( \bar{\theta}_{s,j} = h^{j-1} \theta_s \). This leads to the second equation. As for the first one, starting from (14), use hypothesis (16), (21c) compared with (28), (21b) and again the already obtained expression for \( \bar{\theta}_{s,j} \).

The following result provides the control of the behaviour of \( \beta_s \).

**Lemma 3.9.** Suppose \( 2C_r \leq 3h \). Then condition (17) holds provided
\[ b := 4(h + C_r + \mu \Gamma) < 1. \] (31)

**Remark 3.10.** If \( \mu = 0 \) this choice coincides with \( b \) found in [Gio02].

**Proof.** Condition (17) is proven if
\[ \beta_s \leq \frac{x^{s-1}}{s}, \] (32)
for all \( s \) and some \( \tau \leq b \). Note that this is trivially true for \( s = 1 \), hence suppose it for all \( s \leq r - 1 \) and proceed by induction.

**Proposition 3.11.** Suppose (32), then
\[ \theta_s \leq \frac{C_r}{s} (\tau + C_r)^{s-1}, \quad \gamma_s \leq \frac{\Gamma}{s} (\tau + C_r)^{s-1}. \] (33)

**Proof.** Define \( \hat{\theta}_s := C_r \sum_{j=1}^{s} \tau^{j-1} \hat{\theta}_{s-j} \) with \( \hat{\theta}_0 = 1 \). By (32), follows from (30) that \( \theta_s \leq \hat{\theta}_s/s \) for all \( s \).

By a slight variant of the argument used for Catalan numbers (see e.g. [?]), define \( f(z) := C_r \sum_{j=0}^{\infty} \hat{\theta}_j z^j \) and \( g(z) := \sum_{j=0}^{\infty} \tau^{j-1} z^j \). It is immediate to see that these functions satisfy
\[ C_r z f(z) g(z) + C_r \hat{\theta}_0 = f(z). \]

Now it is sufficient to expand \( f(z) \) in power of \( z \), checking by induction that coefficients of the expansion (i.e. \( \hat{\theta}_j \)) are exactly \( C_r (\tau + C_r)^{j-1} \). The argument for \( \gamma_s \) is analogous. \( \square \)
Now set $\tau := b - 2C_r$. In this way, $\tau + C_r < 1$ and due to assumption (31), sequences (33) are monotonically decreasing. By using their expressions, the last two terms of the first of (30), computed for $s + 1$ can be respectively bounded in the following way
\[
\frac{1}{s + 1} \sum_{l=1}^{s} \theta_l \leq C_r \left( \frac{\tau + C_r}{\tau(s + 1)} \right)^l \leq C_r \left( \frac{\tau + C_r}{s + 1} \right),
\] (34)
moreover,
\[
\frac{C_r}{s + 1} \sum_{l=1}^{s} l \beta_l h^{s-l} \leq C_r \left( \frac{h^s}{\tau(s + 1)} \right)^l \leq C_r \left( \frac{\tau^s}{3h(s + 1)} \right).
\] (35)
as $\tau \geq 4h$ by hypothesis. Recalling $\tau + C_r < 1$, the expression for $\beta_{s+1}$ can be estimated as follows for $s \geq 1$
\[
\beta_{s+1} \leq h^s + \mu \frac{\tau^s}{s + 1} + \frac{C_r}{(s + 1)^2} \left( \frac{\tau^s}{s + 1} \right).
\]
The inductive step is achieved once the r.h.s. of the latter is shown to be not greater than $\tau^s/(s + 1)$. As $C_r/(3h) \leq 1/2$ by hypothesis, one gets $\tau^s \geq 2C_r + 4\mu + 2(s + 1)h^s$, satisfied by $\tau$ chosen as above for all $s \geq 1$. This justifies the choice of $b$ as in (31).

By using expression (31) with (27) and (23) and recalling that $\delta < 1$, the quantity appearing in (18) can be bounded as follows
\[
\frac{2e^2 \mathcal{F}}{d^2 \alpha \delta \sigma} + b \leq \frac{4r \mu}{d^2 \alpha \delta \sigma} + \frac{4e^2 + 16(r - 1)}{d^2 \alpha \delta \sigma} \mathcal{F} + 4h \leq \frac{4r}{d^2 \alpha \delta \sigma} \lambda_{\mu, \varepsilon} + 4h,
\]
in which the inequality $4e^2 + 16(r - 1) \leq 4e^2 r$ for all $r \geq 1$ in which the second of (5) have been used. It is sufficient to set $d = 1/8$ and recall (19) to show that (8a) implies (18) and then (31). On the other hand, by (27), (8a) and finally by (8b) compared with (19)
\[
C_r \leq \frac{1}{e^2} \frac{2b \epsilon r \mathcal{F}}{\alpha \delta \sigma} < \frac{1}{e^2} \frac{2b \epsilon r \lambda_{\mu, \varepsilon}}{\alpha \delta \sigma} \leq \frac{1}{e^2} \left( \frac{1}{2} - 4h \right) \leq \frac{3}{2} h,
\]
as required by Lemma 3.9. Hence (17) holds and hypotheses of Lemma 3.7, statement 3, are satisfied. A further restriction of the domain by $1/8$ completes the proof of Theorem 3.3.

The remainder estimate (10) is described in detail in [Gio02].

4. NEKHOROSEV STABILITY

The Nekhoroshev type estimate contained in Theorem 2.3 can be straightforwardly obtained combining the analytic part given by Theorem 3.3 and the geometric part described in [Gio02]. The main steps leading to the wanted estimate can be summarized as follows.

A direct consequence of Theorem 3.3 on a non-resonance domain, is the existence of $n - \dim \mathcal{M} \approx \lambda_1 \mathcal{F}$ first integrals. More precisely, for each independent unit vector $\lambda \in \mathcal{M}^\perp$ it is easy to check (see [Gio02]) that the function $\Phi = T_{\lambda}(\psi) \Phi_0$ with $\Phi_0 = \langle \lambda, I \rangle$ is a first integral for the Hamiltonian in the normal form (9), up to the remainder $\mathcal{R}^{(r+1)}$. Given an initial condition $I(0) \in \mathcal{V}$ these first integrals determine invariant regions in phase space whose intersection is the so called plane of fast drift $\Pi_\mathcal{M}(I(0)) := I(0) + \text{span}(\mathcal{M})$. This means that a solution starting in $I(0)$ may undergo a variation due to the resonant “residual” $Z$ in the normal form, which size is not controlled by the normalization order $r$, as depending only on $\mathcal{M}$. Hence, the action of the remainder can be interpreted as a deviating effect from this plane. However, this deviation is “small”, as quantitatively stated in the following

**Corollary 4.1.** Assume the same hypothesis of Theorem 3.3 and consider a trajectory $(I(t), \varphi(t))$ for (1) such that $I(t) \in \mathcal{V}$ for all $t \in [a, b]$, $ab < 0$. Then this solution satisfies $\text{dist}(I(t), \Pi_\mathcal{M}(I(0))) \leq \delta/2$ for all $t \in [a, b] \cap [-t^*, t^*]$ with
\[
t^* = \frac{e^2 \delta}{C_1 \varepsilon \Delta r}.
\] (36)
Proof. Given in [Gio02].

Unfortunately, the result above holds only as long as the trajectory remains in $V$. Roughly, the aim of the geometric part is to show that, given $I(0) \in G$, there exists a suitable domain containing it, for which the above result can be used, then finally providing a parameters choice in a way the solution starting at $I(0)$ remains in this set for an exponentially long time. These sets, called extended block, cover the entire action space and are shown to be non-resonance domains of the type $(\mathcal{M}, \beta_s/2, \delta_s, N)$, where $\beta_0 < \ldots < \beta_n$ and $\delta_0 < \ldots < \delta_n < \delta/2$ are suitable sequences of parameters (see giorgilli02).

Here hypothesis 2.2 plays a key role. In this way it is possible to show that Corollary 4.1 acquire global validity and is enforced as follows.

By (8a) and (36), define $\Delta_0 := \Delta|_{\beta=\beta_0, \delta=\delta_0}$ then $t_0^* := t^*|_{\delta=\delta_0, \Delta=\Delta_0}$.

**Proposition 4.2.** Assume Hypothesis 2.2. Given $\rho > 0$ suppose $\Delta_0 \leq 1/2$ for all $\delta < \rho/3$ and $K$ satisfying (8b). Then every trajectory for (1) with $I(0) \in G$ satisfies $\text{dist}(I(t), I(0)) < \delta$ for all $|t| < t_0^*$.

By using the above mentioned values, we have

$$\beta_0 \delta_0 = K \frac{\delta^2 m^2 r}{\delta^2 M r^n}, \quad \delta_s := (n + 2)! \left( \frac{4M}{m} \right)^{n+1} K \frac{\rho}{e},$$

with $a = n^2 + n$. Substituting in $\Delta_0$ one gets

$$\Delta_0 = \Delta_0^* \frac{r^a \rho^2}{2e \delta} \lambda_{\epsilon, \mu} + 4e^{-K} \frac{\rho}{2}, \quad \Delta_0^* := \frac{211 e \delta^s M}{\sigma K M^2 \rho^2}.$$

First hypothesis of Proposition 4.2 is satisfied if each addend of $\Delta_0$ is smaller than $(2e)^{-1}$ which leads to

$$r = \left[ \frac{\lambda_{\epsilon, \mu}}{(\Delta_0^* \rho^2 \lambda_{\epsilon, \mu})^2} \left( \frac{\delta^2}{\lambda_{\epsilon, \mu}} \right)^{\frac{1}{\pi}} \right], \quad K = [\Sigma], \quad \Sigma := \frac{2(1 + 3 \log 2)}{\sigma},$$

where $[\cdot]$ and $\lceil \cdot \rceil$ denote the rounding to the lower and to the greater integer respectively. The choice $\delta = (\Delta_0^* \rho)^{1/2}$ ensures that $r > 1$ then condition $\delta < \rho/3$ is true provided (6) is satisfied. As a consequence of the above described choice for $K$, one has $K \leq 1 + \Sigma$, which satisfies (8b) as $\sigma \leq 1$.

The exponential estimate is straightforward from Proposition 4.2, by replacing the already determined expression for $r$ and $\Delta_0 \leq 1/e$ in $t_0^*$. This completes the proof of Theorem 2.3.

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