BOUNDARY VALUE PROBLEMS ON PRODUCT DOMAINS

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Abstract. We consider the inhomogeneous Dirichlet problem on product domains. The main result is the asymptotic expansion of the solution in terms of increasing smoothness up to the boundary. In particular, we show the exact nature of the singularities of the solution at singularities of the boundary by constructing singular functions which make up an asymptotic expansion of the solution.

1. Introduction

In [2], we examined the $\bar{\partial}$-Neumann problem for $(0, 1)$-forms on product domains in $\mathbb{C}^n$ of the form $D = D_1 \times \cdots \times D_n$ where $D_i \subset \mathbb{C}$. In our analysis we related the solution of the $\bar{\partial}$-Neumann problem to the inhomogeneous Dirichlet problem, and as a corollary obtained specific information on the nature of the singularities. We were able to construct singular functions which we used to write an asymptotic expansion of the solution, each successive term in the expansion exhibiting a higher degree of differentiability up to the boundary. This paper is written to generalize this corollary result of [2]. Although we focus on the inhomogeneous Dirichlet problem, the methods used can be applied to other boundary value problems as well.

We let $\Omega \subset \mathbb{R}^n$ be a product of $q$ smooth bounded domains, $\Omega = \Omega_1 \times \cdots \times \Omega_q$, where $\Omega_i \subset \mathbb{R}^{1+j_i}$, for $1 \leq j_i$ and $\sum_{i=1}^{q} j_i = n - q$. Locally, in a neighborhood, $U$ of $x_0$, a point in the distinguished boundary of $\Omega$, $\partial \Omega_1 \times \cdots \times \partial \Omega_q$, $\rho_i$ is a defining function for $\Omega_i$, which we shall assume to be of the form

$$\rho_i = \phi_i(t_{i1}, \ldots, t_{ij_i}) - x_i$$

for $1 \leq i \leq q$, and $\phi_i \in C^\infty(\mathbb{R}^{j_i})$. Thus, $\Omega \cap U$ is the set of all $x \in \Omega$ such that $\rho_i(x) < 0$ for all $1 \leq i \leq q$.

We consider the inhomogeneous Dirichlet problem on the product domain, $\Omega$

$$\begin{align*}
\triangle u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

(1.1)

and the singular behavior of $u$ in a neighborhood, $U$, of a point, $x_0$, of the distinguished boundary, at which $\partial \Omega$ is not smooth. In our analysis $f$ will be in the class of $C^\infty(\Omega)$. Let $H^\alpha(\Omega)$ denote the Sobolev $\alpha$ space. Existence and uniqueness of a solution follows from Jerison-Kenig [4]:

**Theorem 1.1** (Jerison-Kenig). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Suppose that $\frac{1}{2} < \alpha < \frac{n}{2}$. Then the inhomogeneous Dirichlet problem (1.1) has a unique solution $u \in H^\alpha(\Omega)$ and

$$\|u\|_{H^\alpha(\Omega)} \lesssim \|f\|_{H^{\alpha-2}(\Omega)}$$

2000 Mathematics Subject Classification. Primary 35B65; Secondary 35G15.
for every \( f \in H^{\alpha - 2}(\Omega) \).

In the same paper [4] (see also [5]) Jerison and Kenig also proved that in our situation, \( u \in H^{3/2}(\Omega) \). In contrast to the case of smooth boundary, the non-smooth case exhibits singularities. Thus the classical \( L^2 \) treatment (see Lions-Magenes [6]) in which gains in derivatives are obtained has to be modified. In this paper it is our purpose to write an explicit solution as a sum of terms with increasing degrees of differentiability up to the boundary, and thus give an analysis of the behavior of the solution near the singular parts of the boundary.

In [2], the case of \( \Omega_j \in \mathbb{R}^2 \) for each \( j \) was considered and a conformal mapping was used to reduce the problem to one on the product of half-planes. Instead of a conformal mapping, in this paper we use a change of coordinates, so that, locally, the domain looks like a product of half-spaces.

We organize our paper as follows. In Section 2, we setup the problem, and transform it to one on a product of half-planes. We begin the process of localizing the problem, so that we only concern ourselves with function behavior in a neighborhood of a singular point of the boundary. In Section 3 we write down an infinite sum, which represents a solution modulo terms smooth up to the boundary for the problem setup in Section 2. In Section 4 we construct explicit singular functions which comprise the terms in the asymptotic expansion of our solution.

2. A problem on a product of half-planes

Recall from above that \( \Omega = \Omega_1 \times \cdots \times \Omega_q \), and \( \Omega_i \) in a neighborhood of a point on the distinguished boundary has as a defining function

\[
\phi_i(t^1_i, \ldots, t^l_i) = x_i.
\]

We thus use a transformation of coordinates

\[
y_j = x_j - \phi_j \quad 1 \leq j \leq q
\]
\[
y_{q+m} = t^l_k \quad 1 \leq m \leq n - q, \quad j_1 + \cdots + j_{l-1} + k = m, \quad k \leq j_i
\]

where \( x_0 \) corresponds to \( y = 0 \), and the related matrix \( A = [a_{ij}] \), where

\[
a_{ij} = \begin{cases} 1 & i = j \\ -\phi_{j,k} & i \geq q, j = l \\ 0 & i \neq j, j > q, \end{cases}
\]

and \( \phi_{j,k} = \frac{\partial \phi_j}{\partial t^l_k} \), where \( k \) and \( l \) are determined by the unique representation of \( i \) as \( i = j_1 + \cdots + j_{l-1} + k \), and \( k \leq j_i \).

The transformation (2.1) leads to a Dirichlet problem on the domain \( H^n_q \) in \( \mathbb{R}^n \) given by \( H^n_q = \mathbb{H}^1 \times \cdots \times \mathbb{H}^q \times \mathbb{R}^{n-q} \), where \( \mathbb{H}^i = \{(y_1, \ldots, y_n) : y_i > 0\} \) for \( i = 1, \ldots, q \) in which the differential operator, \( \Delta \), is replaced with

\[
\Delta' = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{ki} \frac{\partial}{\partial y_i} \right) \left( \sum_{j=1}^{n} a_{kj} \frac{\partial}{\partial y_j} \right)
\]
\[
= \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{k=1}^{n} \sum_{i,j=1}^{n} a_{ki} \frac{\partial a_{kj}}{\partial y_i} \frac{\partial}{\partial y_j}
\]
Lemma 2.1. Let the singularities must be along intersections of boundaries of more than one
where \( g^{ij} = \sum_{k=1}^{n} a_{ki} a_{kj} \) is the metric tensor given by the \((i, j)\) entries of the matrix
\( A^t A \). Because of the invertibility of \( A \), and from the smoothness up to the boundary
of the \( \rho_i \), we see that \( g^{ij} \in C^\infty(\mathbb{R}^n) \) as are the terms \( a_{ki} \frac{\partial \rho_i}{\partial y_i} \). Calculating \( g^{ij} \) gives
\[
\Delta' = \sum_{i=1}^{q} (1 + |\nabla \phi_i|^2) \frac{\partial^2}{\partial y_i^2} + 2 \sum_{i=1}^{q} \sum_{k=1}^{j_i} \left( -\frac{\partial \phi_i}{\partial y_k} \right) \frac{\partial}{\partial y_{q+j_1+\cdots+j_{i-1}+k}} \frac{\partial}{\partial y_i} + \sum_{i=1}^{q} \sum_{j,k=1}^{n} a_{kj} \frac{\partial a_{ki}}{\partial y_j} \frac{\partial}{\partial y_i} + \sum_{i=q+1}^{n} \frac{\partial^2}{\partial y_i^2} + \sum_{i=q+1}^{n} \sum_{j,k=1}^{n} a_{kj} \frac{\partial a_{ki}}{\partial y_j} \frac{\partial}{\partial y_i}.
\]
Therefore, using the change of coordinates \( (2.1) \) we examine the problem
\[
\varphi \Delta' u = \varphi f \quad \text{in} \quad \mathbb{H}^n,
\]
\[
u \varphi \quad \text{on} \quad y_i = 0, \quad i = 1, \ldots, q,
\]
where \( \varphi \in C^\infty_0(\Omega) \) is cutoff function such that \( 0 \leq \varphi \leq 1 \) and \( \text{supp} \varphi \subset U \), and
\( \varphi \equiv 1 \) near \( x_0 \).
Commuting \( \varphi \) with the operator \( \Delta' \) gives
(2.2) \[
\Delta' \varphi u = \varphi f + [\Delta', \varphi] u.
\]
We begin our discussion of the singularities of the solution by noting the location
of the singularities must be along intersections of boundaries of more than one \( \partial \Omega_i \).

**Lemma 2.1.** Let \( f \in C^\infty(\Omega) \), \( u \in L^2(\Omega) \) be the unique solution to the Dirichlet
problem, \( (1.1) \), on \( \Omega \). Let \( V \subset \Omega \) such that \( V \cap \partial \Omega_j \neq \emptyset \) for at most one \( j \). Then
\( u \in C^\infty(V) \).

**Proof.** The lemma is a consequence of regularity at the boundary of the Dirichlet
problem \( 3 \). \( \square \)

3. **AN ASYMPTOTIC CONSTRUCTION OF A SOLUTION**

We set \( v = \varphi u \) and \( h \) to be the right hand side of (2.2), and note that \( h \equiv f \)
in a neighborhood of \( x_0 \). Without loss of generality we can assume \( h \) is just \( \varphi f \)
since the error term \( [\Delta', \varphi] u \) leads to terms smooth up to the boundary in a small
neighborhood of our chosen point \( x_0 \). This can be seen by writing the solution
in terms of Green’s function. Let \( \phi(y) \) be a function which is equivalently 0 near
\( x_0 \), and let \( G(x, y) \phi(y) dV(y) \) be the Green’s function for the Dirichlet problem on \( \Omega \),
then \( \int_{\Omega} G(x, y) \phi(y) dV(y) \) can be extended smoothly across the boundary near \( x_0 \).

We shall make use of odd reflections along \( y_i = 0 \) for \( 1 \leq i \leq q \). We take odd
reflections of (2.2) with respect to the variables \( y_i \) for \( 1 \leq i \leq q \) and we denote the extension by a superscript \( o \). We also denote by a superscript \( o_j \) an even extension
in variables \( y_i \) for \( 1 \leq i \leq q \), \( i \neq j \) and an odd extension in \( y_j \), and a superscript \( e \)
is used to denote even extensions in all variables \( y_i \) for \( 1 \leq i \leq q \). We write (2.2),
after reflections, in the form
(3.1)
\[
\sum_{i=1}^{q} \left( a_{i}^e \frac{\partial^2}{\partial y_i^2} + b_{i}^o \frac{\partial}{\partial y_i} + c_{i}^e \right) v^o = h^o.
\]
Due to our discussion above on the error term in (2.2) leading to terms which are $C^\infty$ smooth up to the boundary of $\Omega$ in a neighborhood of $x_0$, we look to solve (3.1) with $h^o$ replaced by $(\varphi f)^o$.

We make the note here that the difficulty in following the process in the case of smooth domains to attempt to construct a parametrix lies in the existence of singularities in the symbol along $y_i = 0$ for $1 \leq i \leq q$ due to the reflections. This difficulty can be resolved by referring to Lemma 2.1, and we could then proceed to show the error terms of the parametrix construction yield an error term in the solution which is also smooth up to the boundary as we do in Theorem 3.2 below. Since this examination of the error terms is independent of the method of parametrix construction, we choose here to follow our analysis in [2] from the beginning, which nonetheless has the flavor of a parametrix construction.

Here $\eta_i$ will be the Fourier variable corresponding to $y_i$ for $i \leq q$, and $\xi_i$ the Fourier variable corresponding to $y_i$ for $q + 1 \leq i \leq n$. We denote the symbols of $b_i$ and $c_i$ by $B_i$ and $C_i$, respectively, for $1 \leq i \leq q$. Let $(q, i) = q + j_1 + \cdots + j_{i-1}$ and from the symbol of the operator in (3.1),

$$P(y, \xi, \eta) = -\sum_{i=1}^{q} (1 + |\nabla \phi_i|^2)\eta_i^2 + \sum_{i=1}^{q} B_i^c (y_{q+1}, \ldots, y_n, \xi)\eta - \sum_{i=1}^{q} C_i (y_{q+1}, \ldots, y_n, \xi),$$

we define the operator $K$ by

$$Ku = -\sum_{i=1}^{q} \int (B_i^c \eta_i - C_i) \hat{u} e^{i\eta \cdot (\xi, \eta)} d\xi d\eta,$$

and with

$$v_0 = \frac{1}{(2\pi)^q} \int \chi(\eta) \frac{(\varphi f)^o (\eta, y_{q+1}, \ldots, y_n)}{-\sum_{i=1}^{q} (1 + |\nabla \phi_i|^2)\eta_i^2} e^{i(y_{q+1}, \ldots, y_n) \cdot \eta} d\eta,$$

where $\hat{\cdot}$ refers to a partial Fourier transform in the variables $y_1, \ldots, y_q$, we inductively define

$$v_{j+1} = \frac{1}{(2\pi)^q} \int \chi(\eta) \frac{Kv_j (\eta, y_{q+1}, \ldots, y_n)}{-\sum_{i=1}^{q} (1 + |\nabla \phi_i|^2)\eta_i^2} e^{i(y_{q+1}, \ldots, y_n) \cdot \eta} d\eta, \quad j \geq 0.$$

We write the solution $v^o$ to (3.1) in the form

$$v^o = v_0 + \cdots + v_N + v_{R_N}$$

for any integer $N \geq 0$ where the remainder term, $v_{R_N}$, satisfies

$$\sum_{i=1}^{q} \left( a_i^e \frac{\partial^2}{\partial y_i^2} + b_i^o \frac{\partial}{\partial y_i} + c_i^e \right) v_{R_N} =$$

$$K v_N + F.T.^{-1}_{\eta} \left( (1 - \chi) \left( \sum_{j=1}^{N-1} K v_j + (\varphi f)^o \right) \right).$$

We write $F.T.^{-1}_{\eta}$ to refer to an inverse Fourier transform in the $\eta$ variables only. The forms of the $v_j$ are written in the following lemma.

**Lemma 3.1.** Each $v_j$ is of the form

$$v_j = \int_{R^n} \chi^{j+1} (\eta) \frac{p_j (y_{q+1}, \ldots, y_n, \xi, \eta)}{(\sum_{i=1}^{q} (1 + |\nabla \phi_i|^2)\eta_i^2)^{3j+1}} \hat{\varphi f}^o (\xi, \eta) e^{i\eta \cdot (\xi, \eta)} d\xi d\eta.$$
where \( p_j(y, \xi, \eta) \) is \( C^\infty \) smooth as a function of \( y_{q+1}, \ldots, y_n \) and is a polynomial in \( \xi \) and \( \eta \), of order \( 5j \) in the \( \eta \) variables.

**Proof.** The lemma would be clear if it were not for the odd reflections in the symbols, \( B_k \). We thus examine terms of the form \( H(y_k)\eta_k \hat{u}^\circ \), where \( H(t) \) is defined to be 1 for \( t \geq 0 \) and \(-1\) for \( t < 0 \). We have

\[
F.T^{-1}_{\eta} (\eta_k \hat{u}^\circ) = i \frac{\partial u}{\partial y_k} e_k + i \delta(y_k) u^\circ,
\]

where \( e_k \) denotes an even reflection in the \( y_k \) variable, and odd reflections in \( y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q \). And thus

\[
H(y_k)F.T^{-1}_{\eta} (\eta_k \hat{u}^\circ) = i \left( \frac{\partial u}{\partial y_k} \right)^\circ + i \delta(y_k) u^\circ.
\]

Using

\[
\tilde{\partial u} = -i\eta_k \hat{u}^e_k
\]

in (3.3), and noting that, since \( \tilde{v}_j \) is odd in all \( \eta \) variables by our construction, \( v_j(y_k = 0) = 0 \) for \( k = 1, \ldots, q \), we can write

\[
- \sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2 \tilde{v}_{j+1}^e = \sum_{k=1}^{q} \left( B_k(y_{q+1}, \ldots, y_n, \xi) (i\eta_k \tilde{v}_j^e_k + iF.T.\eta_k v_j(y_k = 0)) + C_k(y_{q+1}, \ldots, y_n, \xi) \tilde{v}_j \right)
\]

\[
= \sum_{k=1}^{q} \left( B_k(y_{q+1}, \ldots, y_n, \xi) i\eta_k \tilde{v}_j^e + C_k(y_{q+1}, \ldots, y_n, \xi) \tilde{v}_j \right),
\]

(3.7)

where \( F.T.\eta_k \) refers to a Fourier transform in all \( y_1, \ldots, y_q \) variables but \( y_k \). Now

\[
- \sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2 (-i\eta_k \tilde{v}_j^e) = -i\eta \chi(\eta) \sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2 \tilde{v}_j^e - i \eta_k \tilde{v}_j^e + i F.T.\eta_k \partial y_j (y_l = 0),
\]

and so

\[
\tilde{v}_j^e = -\frac{\chi(\eta) \sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2 \tilde{v}_j^e}{\sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2} + \frac{F.T.\eta_k \partial y_j (y_l = 0)}{\sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2} \quad j \geq 1,
\]

while for \( e_0 \) we have

\[
\tilde{v}_0^e = -\frac{\chi(\eta) \sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2 \tilde{v}_0^e}{\sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2} + \frac{F.T.\eta_k \partial y_0 (y_l = 0)}{\sum_{k=1}^{q} (1 + |\nabla \phi_k|^2) \eta_k^2}
\]

\( v_0 \), from (3.2), has the form of (3.6) and taking an inverse transform with respect to \( \eta_l \) of the last term on the right side of (3.3) shows that it also has a Fourier transform in \( y_1, \ldots, y_q \) of the form in (3.6). We use an induction argument and assume \( \tilde{v}_j \) is of the form in (3.6).

The proof will be complete, by (3.7), if we show \( v_j^e \) is also of the form (3.6). An inverse transform with respect to \( \eta_k \) of the last term on the right side of (3.8)
shows that it also of the form in (3.6). For the first term on the right of (3.8) we use

$$\hat{K} v_{j-1}^{e_l} = - \sum_{i=1}^{q} F.T_{\eta} \left( (B_{i}^{\eta} \eta_{l} - C_{i}) \hat{v}_{j-1}^{e_l} e^{iy(\eta,\xi)} d\eta d\xi \right)$$

where $e_l$ means an even extension in both $y_l$ and $y_i$, and we use the fact that an even extension in $y_l$ of a function $H(y_l) F.T_{\eta}^{-1} (\eta \tilde{u}^o)$ restricted to $y_l > 0$ has a transform $\eta \tilde{u}^o$ if $u(y_l) = 0$. The same arguments above showing $\hat{v}_0^e$ is of the form (3.6) shows we may apply the induction hypothesis to $\hat{v}_{j-1}^e$ as well. The induction hypothesis then shows that $\hat{v}_j^e$ also has the form in (3.6), and with $\hat{v}_j^e$ inserted into (3.7) we have $\hat{v}_{j+1}$ of the form in (3.6).

We show that (3.4) gives an asymptotic expansion of our solution $v$.

**Theorem 3.2.** Each successive term in (3.4) is of increasing differentiability up to the boundary, and the remainder terms $v_{R_N}$ are also of increasing differentiability up to the boundary. The representation (3.4) is therefore an asymptotic expansion of the solution to the Dirichlet problem.

**Proof.** From our definition of the $v_j$, an induction argument shows each $v_j$ is infinitely differentiable with respect to the variables $y_{q+1}, \ldots, y_n$. Furthermore, it follows by our construction of the $v_j$, and by (3.6), that if $v_j \in H^k(\mathbb{R}^n)$, the Sobolev-$k$ space, then $v_{j+1} \in H^{k+1}(\mathbb{R}^n)$. The proof of Lemma 3.1 also shows that restricting $v_j$ to $y_i > 0$ for $1 \leq i \leq q$ and then reflecting about $y_i = 0$ in an even manner to form $v_j^e$ gives $v_j^e \in H^k(\mathbb{R}^n)$. To see the remainder terms, $v_{R_N}$ are of increasing smoothness up to the boundary with increasing $N$, we take Fourier transforms of (3.4) with respect to the variables $y_1, \ldots, y_q$ to obtain

$$F.T_{\xi}^{-1} (P(y_{q+1}, \ldots, y_n, \xi, \eta)v_{R_N}) = \hat{K} v_N + (1 - \chi) \left( \sum_{j=1}^{N-1} \hat{K} v_j + (\varphi f)^o \right).$$

From the proof of Lemma 3.1 we have

$$\hat{K} v_N = \sum_{k=1}^{q} \left( B_k(y_{q+1}, \ldots, y_n, \xi) \hat{v}_N^{e_k} + C_k(y_{q+1}, \ldots, y_n, \xi) \hat{v}_N \right).$$

If we assume $v_N \in H^{k(N)}(\mathbb{R}^n)$, and hence $v_N^{e_k} \in H^{k(N)}(\mathbb{R}^n)$ from above, for some $k(N)$ strictly increasing in $N$, with the property that $\frac{\partial^\alpha}{\partial y^\alpha} v_N \in H^{k(N)}(\mathbb{R}^n)$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 = \cdots = \alpha_q = 0$, then $K v_N \in H^{k(N)-1}(\mathbb{R}^n)$ by definition of the operator $K$, and $p_{k(N)-1}(\eta) v_N \in L^2$ for $p_{k(N)-1}(\eta)$ a polynomial in the $\eta$ variables of degree $k(N) - 1$. Also, since any polynomial of any degree in the $\eta$ variables multiplied by the last term on the right hand side of (3.10) is in $L^2$
due to the compact support in \( \eta \) of the term \( 1 - \chi \), we see, by multiplying \( [3.10] \) by \( p_k(N)\eta_1(\eta) \), that

\[
p_{k(N) - 1} \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q} \right) v_{RN}
\]

is the solution to a problem with mixed Dirichlet and Neumann conditions, and by \( [4] \), its restriction to \( \mathbb{H}^q \) is in \( H^\alpha(\mathbb{H}^q) \) for \( \alpha < 3/2 \). Therefore, \( v_{RN}|_{\mathbb{H}^2} \in H^{k(N) + \alpha}(\mathbb{H}^q) \) for \( \alpha < 1/2 \), and so is of increasing smoothness up to the boundary for increasing \( N \).

\[\square\]

4. AN EXPLICIT CALCULATION OF THE SINGULARITIES

In order to determine the singularities in each term in the expansion \( (3.4) \), we take inverse Fourier transforms of the expression \( (3.6) \) with respect to the \( \eta \) variables.

Without loss of generality we suppose the cutoff function \( \chi \) is of the form

\[
\chi_{\eta_1} \cdots \chi_{\eta_q},
\]

where \( \chi_{\eta_i} \) is a cutoff function in the variable \( \eta_i \) only, equal to 0 in a neighborhood of \( \eta_i = 0 \). We can do this because

\[
\chi^{j+1}(\eta) - \chi_{\eta_1} \cdots \chi_{\eta_q}
\]

can be written as a sum of terms, each of which has support contained in large \( \eta_k \) for at most one \( k \). Hence,

\[
(\chi^{j+1}(\eta) - \chi_{\eta_1} \cdots \chi_{\eta_q}) \frac{p_j(\eta_{q + 1}, \ldots, \eta_q, \xi, \eta)}{2(\eta_{q + 1} + \ldots + \eta_q)} (\hat{\varphi f})^o,
\]

where \( a_i = 1 + |\nabla \phi_i|^2 \), is a sum of terms which have infinite decay in all but one \( \eta_j \), and thus is the transform of a function which is \( C^\infty \) in all variables but one \( y_j \). As such a term is odd in \( y_j \) and as the denominator \( (\sum_{i=1}^q a_i \eta_i^2)^{3j+1} \) is the symbol of an elliptic operator, such a term is the solution to a Dirichlet problem on a half-plane, and is therefore \( C^\infty \) smooth up to the boundary of the half-plane.

In summary the difference between \( (3.6) \) and

\[
(4.1)
\]

\[
\chi_{\eta_1} \cdots \chi_{\eta_q} \frac{p_j(\eta_{q + 1}, \ldots, \eta_q, \xi, \eta)}{(\sum_{i=1}^q a_i \eta_i^2)^{3j+1}} (\hat{\varphi f})^o
\]

is the transform of a term which, when restricted to \( \mathbb{R}^{n_q} \), is in \( C^\infty(\mathbb{R}^{n_q}) \).

We then take \( [4.1] \) and integrate by parts with respect to the variables \( y_i \) for \( 1 \leq i \leq q \) in the Fourier integral of \( \varphi f \( \eta, \xi \), starting with \( y_1 \) and proceeding to \( y_q \), keeping as remainder terms those which have decay in one \( \eta \) variable to the order \( -2(N + q) - 1 \). Such terms are of the form

\[
(4.2)
\]

\[
\chi_{\eta_1} \cdots \chi_{\eta_q} \frac{p_j(\eta_{q + 1}, \ldots, \eta_q, \xi, \eta)}{(\sum_{i=1}^q a_i \eta_i^2)^{3j+1}} \frac{1}{(i\eta)^\alpha (i\eta_{k+1})^{2(N+q)+1}} \times F.T._{k+1, \ldots, n} \left( \frac{\partial^{2(N+q)+1}}{\partial y_{k+1}^{2(N+q)+1}} \frac{\partial^\alpha (\varphi f)}{\partial y^{\alpha}} (0, \ldots, 0, y_{k+1}, \ldots, y_n) \right)
\]

in which \( \alpha \) is a \( q \) index for which \( \alpha_{k+1} = \cdots = \alpha_n = 0 \). We shall show below in Theorem \( [4.2] \) that such remainder terms are sufficiently continuous up to the boundary of \( \mathbb{H}^q \).
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Setting aside the remainder terms \(4.2\), we analyze those other terms which result from the expansion of \((\varphi f)^{\leftarrow}\ (\eta, \xi)\), and we are led to study terms of the form

\[
\chi_{\eta_1} \cdots \chi_{\eta_q} \frac{1}{y^k} \frac{\phi_{k,j}(y_{q+1}, \ldots, y_n, \xi, \eta)}{\left(\sum_{i=1}^q a_i \eta_i^2\right)^{3j+1}},
\]

where \(\phi_{k,j}(y, \xi, \eta)\) takes the form of odd reflections along \(y_i = 0\) for each \(i = 1 \ldots q\) of functions of \(y\) smooth up to the boundary, and is a polynomial in \(\xi\) and \(\eta\) of order \(5j\) in the \(\xi\) and \(\eta\) variables, and a polynomial of degree \(j\) in the \(\eta\) variables.

We use the notation \(\eta^k = \eta_1^{k_1} \ldots \eta_q^{k_q}\) for \(k = (k_1, \ldots, k_q)\).

Up to multiplication by a constant the following relation holds for \(0 < l < \frac{q}{2}\)

\[
\int_{\mathbb{R}^q} e^{i(y_1, \ldots, y_q) \cdot \eta} d\eta = \frac{1}{\prod_{i=1}^q \sqrt{a_i}} \left(\frac{1}{\sum_{i=1}^q \eta_i^2}\right)^{\frac{q}{2} - l}.
\]

Let

\[
\Phi_q^l = \frac{1}{\left(\sum_{i=1}^q \frac{\eta_i^2}{a_i}\right)^{\frac{q}{2} - l}}, \quad 0 < l < \frac{q}{2}.
\]

For \(l \geq \frac{q}{2}\) and \(q\) even, we define \(\Phi_q^l\) to be the unique solution of the form

\[
p_1(y) \log \left(\sum_{i=1}^q \frac{\eta_i^2}{a_i}\right) + p_2(y),
\]

where \(p_2(y_i = 0) = 0\), \(p_1\) and \(p_2\) are polynomials of degree \(2l - q\) in the variables \(y_i\) for \(1 \leq i \leq q\), and are \(C^\infty\) smooth with respect to variables \(y_i\) for \(q + 1 \leq i \leq n\), to the equation

\[
\frac{\partial \Phi_q^l}{\partial y_1} = y_1 \Phi_q^{l-1}.
\]

For the case \(q = 2\), we take

\[
\Phi_2^l = -\frac{i}{2} \log \left(\frac{y_1^2}{a_1} + \frac{y_2^2}{a_2}\right).
\]

For \(q\) odd we define \(\Phi_q^l\) for \(l \geq \frac{q}{2}\) as in \(4.4\), the Fourier transform being calculated in the sense of distributions.

In the sense of distributions, we take transforms of \(4.5\) and write, for \(q > 2\)

\[
i \eta_1 \hat{\Phi}_2^l = -\frac{\partial}{\partial \eta_1} \hat{\Phi}_2^{l-1}
\]

\[
= -2i(l - 1)a_1 \frac{\eta_1}{\left(\sum_{i=1}^q a_i \eta_i^2\right)^{l}},
\]

which implies

\[
\chi_{\eta_1} \hat{\Phi}_2^l = -2(l - 1)a_1 \chi_{\eta_1} \frac{1}{\left(\sum_{i=1}^q a_i \eta_i^2\right)^{l}}
\]

for \(q > 2\). For the case \(q = 2\), we use \(1\) to write

\[
\chi_{\eta_1} \chi_{\eta_2} \hat{\Phi}_2^l = \chi_{\eta_1} \chi_{\eta_2} \frac{\sqrt{a_1 a_2}}{a_1 \eta_1^2 + a_2 \eta_2^2} + s,
\]

where \(\chi\) is a cutoff function such that \(\chi \equiv 1\) in a neighborhood of the origin as in Section \(2\) and here and below we use \(s\) to denote terms which after taking inverse
transforms give \( C^\infty \) functions in some neighborhood of the origin in \( \mathbb{R}^n \). We have therefore established the property for \( l > 0 \)
\[
\chi \eta \Phi^q_l = \chi \eta \prod_{i=1}^{q} \sqrt{a_i} + s,
\]
up to multiplication by a constant.

For \( k = (k_1, \ldots, k_q) \) we further define
\[
\Phi^q_{lk} = \int_{\mathbb{R}^n} \chi \Phi^q_l e^{-i(y_1, \ldots, y_n)} dy = \frac{1}{i^{k_1} q^{k_2} \cdots \int_{\mathbb{R}^n} \chi \Phi^q_l \frac{\partial^{k_1}}{\partial y_1} e^{-i(y_1, \ldots, y_n)} dy}
\]
where the term \( s \) in the last line comes from taking derivatives of the cutoff \( \chi \). By definition
\[
\frac{\partial^{k_1}}{\partial y_1} \Phi^q_{lk} = \Phi^q_{l1}.
\]
Therefore,
\[
\chi \Phi^q_{lk} = \frac{(-1)^{k_1}}{i^{k_1} q^{k_2} \cdots \int_{\mathbb{R}^n} \chi \Phi^q_l + (1 - \chi) \Phi^q_l + s,
\]
and from the definitions and discussion above,
\[
\chi \Phi^q_{lk} = \Phi^q_{l1} + (1 - \chi) \Phi^q_l
\]
\[
\chi \Phi^q_{lk} = \Phi^q_{l1} + s
\]
\[
\chi \eta \Phi^q_{lk} = \chi \eta \prod_{i=1}^{q} \sqrt{a_i} \eta^2 + s,
\]
Inserting (4.8) into (4.7) finishes the proof.

For a multi-index \( p = (p_1, \ldots, p_l) \), in which \( p_j \leq q \), we define \( \Phi^q_{lk} \) for \( 2 \leq i \leq q - 1 \) in the same fashion as we did \( \Phi^q_{lk} \), but with respect to the \( i \) variables \( y_1, \ldots, y_{p_i} \), in particular \( k \) is a multi-index of length \( i \). Thus
\[
\chi \eta \Phi^q_{lk} = \chi \eta \prod_{i=1}^{q} \sqrt{a_i} \eta^2 + s,
\]
where $\chi_{\eta_1} = \chi_{\eta_2} \cdots \chi_{\eta_p}$. We also use the notation $\Phi_{lk}^p = \Phi_{lk}^q$ when $p = (1, \ldots, q)$.

From (4.6), (4.9), and the construction of the $\Phi_{lk}^p$ we see that
\[
\left( \sum_{i=1}^q a_i \eta_i^2 \right)^m \eta_1^{m_1} \cdots \eta_q^{m_q} \left( \chi_{\eta_1} (\hat{\chi} \Phi_{lk}^p) - \chi \Phi_{lk}^p \right) \in s
\]
for large enough $m, m_1, \ldots, m_n$. Now $\chi_{\eta_1} - 1$ has support for large $\eta_k$ for at most one $k$, and thus $\eta_1^{m_1} \cdots \eta_q^{m_q} \left( \chi_{\eta_1} (\hat{\chi} \Phi_{lk}^p) - \chi \Phi_{lk}^p \right)$ is the transform of a solution to a Dirichlet problem on a half-plane with data smooth up to the boundary. Then, by the ellipticity of the operator whose symbol is $\left( \sum_{i=1}^q a_i \eta_i^2 \right)^m$, we obtain derivatives of the inverse transform of
\[
\chi_{\eta_1} (\hat{\chi} \Phi_{lk}^p) - \chi \Phi_{lk}^p
\]
is a $C^\infty$ function on a half-plane. By inverting derivatives, by integration, and invoking Lemma 2.1 to show the constants of integrations are smooth up to the boundary, we see the terms described by $\chi_{\eta_1} (\hat{\chi} \Phi_{lk}^p)$ and $\chi \Phi_{lk}^p$ differ by functions smooth up to the boundary, and $\chi \Phi_{lk}^p$ will thus be seen to describe the singularities of the solution.

With a slight abuse of notation we also use the notation $\Phi_{lk}^p$ even after a change of variables back to the domain $\Omega$. We are now ready to prove the

**Theorem 4.2.** Let $f \in C^\infty(\bar{\Omega})$ and let $u \in L^2(\Omega)$ be the unique solution to the inhomogeneous Dirichlet problem on $\Omega$. Then near the distinguished boundary, $\partial \Omega_1 \times \cdots \times \partial \Omega_q$, $u$ is of the form
\[
(4.10) \quad u = \sum_{|k| \geq 0, \ell \geq 1, \sum_{\ell \neq m} \eta_i^{\ell} \neq 0} c_{klp} \Phi_{lk}^p
\]
where $c_{klp} \in C^\infty(\bar{\Omega})$, and where $\Phi_{lk}^p$ are defined as above.

**Proof.** (4.10) is obtained by following the procedure outlined above, matching terms, (4.3), in Fourier space to the appropriate function $\Phi_{kl}^p$. We thus need to study the effect $\phi_{k,j}$ has as a polynomial in the $\xi$ and $\eta$ variables in (4.3) on the functions $\Phi_{kl}^p$. We also have to treat the remainder terms (4.2).

We first show the remainder terms given by (4.2) are described by the functions $\Phi_{lk}^p$. We consider the case in which (4.2) is given by
\[
\chi_{\eta_1} \cdots \chi_{\eta_q} \left( \frac{1}{\eta_1^{\ell} \cdots \eta_{q-1}^{\ell}} \frac{1}{\eta_q^{2(N+q)+1}} \sum_{i=1}^q a_i \eta_i^{\ell} \frac{F.T._m}{(\partial \eta_m^{2(N+q)+1})^a} \right)
\]
restricted to $y_i = 0$ for $1 \leq i \leq q$ and $i \neq m$, where $F.T._m$ denotes the partial Fourier transform with respect to $y_m$. The other such remainder terms are handled in a similar manner. The fraction
\[
\frac{1}{\sum_{i=1}^q a_i \eta_i^{\ell}}
\]
is expanded in a geometric series in
\[
\sum_{\ell \neq m} a_m \eta_m^{\ell}
\]
up to \((N + q)/2\) terms (we assume \((N + q)\) is even), the first terms leading to the functions \(\Phi^p_{lk}\) in which \(m \notin p\), while the last term is, up to multiplication by a \(C^\infty\) smooth function of the variables \(y_i\) for \(q + 1 \leq i \leq n\), given by

\[
(4.11) \quad \left( \chi_{\eta_1} \cdots \chi_{\eta_q} \frac{F.T. m}{\eta_m^{N+q+1}} \left( \sum_{i=1}^{q} \frac{a_i \eta_i^2}{\sum_{i \neq m} a_i} \right)^{(N+q)/2+1} \right)
\]

The term in parentheses is the transform of a function odd in the variable \(y_m\) and in \(C^N(\mathbb{R}^{q-1} \times \{y_m > 0\})\), and so all of (4.11) may be viewed as the solution to a Dirichlet problem depending \(C^\infty\) smoothly on the parameters \(y_i\) for \(q + 1 \leq i \leq n\) on a half-space, and by regularity of the Dirichlet problem, the term in (4.11), after an inverse transform, is in \(C^N(\mathbb{R}^n)\).

We finish the proof of Theorem 4.2 by showing that the polynomials of the \(\eta\) and \(\xi\) variables in the numerator of (4.3) still preserve the form of the functions \(\Phi^p_{lk}\).

Lastly, we restrict our expansion (3.4) to the product of upper half-spaces, \(\mathbb{H}^q\), and obtain an expression of the solution modulo terms in \(C^\infty(\mathbb{H}^q)\) in terms of the functions \(\Phi^p_{lk}\). After changing coordinates back to \(\Omega\), we obtain the expansion in (4.10).

Our results here are comparable to those in [2], in which \(\Omega_i \subset \mathbb{R}^2\), and we note that an increase in the dimensions of the \(\Omega_i\) do not affect the form of singularities occurring. See also [1] for the specific case of \(\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^4\), where \(\Omega_i \subset \mathbb{R}^2\) for \(i = 1, 2\), in which logarithmic and arctangent singularities are found along the corner.

Lastly, it is important to mention that there are cases in which the solution to the Dirichlet problem does exhibit singularities. The sum (4.10) is not trivial; the coefficients \(c_{klp}\) are not always 0. The example \(f \equiv 1\) on \(\Omega\) reveals this to be the case.

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