Structure of the Chiral Scalar Superfield in Ten Dimensions

P. S. Kwon
Department of Physics, Kyungsung University, Pusan 608-736, Korea
and
M. Villasante
Department of Physics, University of California, Los Angeles, CA 90024-1547

Abstract

We describe the tensors and spinor-tensors included in the $\theta$-expansion of the ten-dimensional chiral scalar superfield. The product decompositions of all the irreducible structures with $\theta$ and the $\theta^2$ tensor are provided as a first step towards the obtention of a full tensor calculus for the superfield.

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I Introduction

The field structure of higher dimensional supergravities as well as of \( N \geq 3 \) extended supergravities is still an open problem. It is an old problem whose general solution was deemed impossible for a while due to some “no-go theorems” [1] establishing the impossibility of writing quadratic Lagrangians for the linearized (free) theory. The underlying problem was the so-called “self-duality counting paradox” [2] which was subsequently resolved [3] by the discovery of the fact that the Lagrangian for the linear theory is not quadratic when is dealing with fields having self-dual field strength.

In particular one would really like to know the auxiliary field structure of 10-dimensional supergravity [4], a theory unaffected by the above mentioned no-go theorems, due to its relevance for string theory applications.

Traditionally the auxiliary field structures for supergravities that are known have always been found in a rather \textit{ad hoc} manner by counting degrees of freedom and trying to add suitable new fields in order to match the bosonic and fermionic degrees of freedom off-shell [5]. It was only later, after the answer was known, that more systematic ways of deriving the result were found. However, for the more complicated theories the auxiliary field structure becomes so complex that it has been impossible to guess. Complicating matters further is the above-mentioned self-duality counting paradox, and we are finally bound to use a systematic approach to solve the problem.

A fruitful approach in 4 dimensions is the use of the superconformal framework in which the different Poincaré supergravities correspond to using different compensators to fix the extra degree of freedom [6]. However, while the super-Poincaré algebra remains essentially the same in higher dimensions, the same is not true for the superconformal one which acquires a multitude of new generators [7], which complicates enormously this gauge-fixing procedure. In fact, even though the complete off-shell structure of ten-dimensional conformal supergravity was obtained long ago in [8], a satisfactory off-shell Poincaré version is still lacking (see [9, 10]).

In ref. [10] it was proposed a linearized off-shell 10-dimensional supergravity adding to the conformal supergravity multiplet a set of 2 full-fledged chiral scalar superfields. However this is in all likelihood a reducible version since each chiral scalar superfield contains 3 irreducible pieces [11]. Furthermore, the tensorial structure and transformation rules of the component fields were not provided, even at the linearized level.

A second more promising approach is the irreducible superfield method, which has been successfully used in the \( N = 1 \) [12] and \( N = 2 \) [13] cases. In working with superfields [14], one is automatically assured that the numbers of fermionic and bosonic degrees of freedom will match, but general superfields are usually objects too large to handle, containing many more fields that one is interested in, especially in higher dimensions (though some interesting four-dimensional results have been obtained using unconstrained superfields in the so called harmonic superspace approach [14]). That is why the importance of irreducible superfields, which are much simpler objects satisfying additional supersymmetric constraints. These subsidiary conditions are usually differential equations involving the superspace covariant derivatives, and can be obtained by applying appropriate projection operators for the corresponding eigenvalues of the Casimir [12]. The Casimir operators for the super-Poincaré algebras in all dimensions are known and they have been used to decompose the 11-dimensional [16] and 10-dimensional massive scalar superfields. In the 10-dimensional case, there is an additional interesting complication, namely that the lowest (quadratic) Casimir operator \( C_2 \) does not distinguish between the 3 irreducible
pieces since it has the same eigenvalue for the corresponding representation \([11]\). Therefore one would have to construct projection operators using the second lowest (quartic) Casimir operator \(C_4\), which does distinguish among those representations, but the resulting differential equations are so complicated as to render the method impractical. However, this difficulty was circumvented by resorting to the Cartan subalgebra in order to obtain simple differential equations which were used to characterize the irreducible pieces of the massless and massive 10-dimensional scalar superfield in \([17]\) and \([18]\) respectively. The irreducible superfields were then obtained as expansions in Grassmann-Hermite polynomials, but the field components of these non-covariant expressions remained to be sorted out, though in principle it can be done.

In all this one final basic stumbling block remains though: while it is known from group theory methods what are the fields contained in scalar superfield \([19]\), it is not known in what form they appear. In other words, while it is trivial to write the scalar superfield in multispinor language:

\[
\Phi(x, \theta) = \sum_{j=0}^{16} \chi_{\alpha_1...\alpha_j}(x) \theta^{\alpha_1} \ldots \theta^{\alpha_j},
\]

it is a rather different proposition to extract the irreducible fields with their tensor (non-spinor) indices out of the \(\chi_{\alpha_1...\alpha_j}(x)\) fields. The latter is equivalent to decompose into irreducible pieces all the possible powers of the anticommuting variable \(\theta^\alpha\), and that is what we will do in this paper. The irreducible \(SO(10)\) representations contained in the corresponding powers of \(\theta^\alpha\) are reproduced in Table 1. The list is for increasing powers of one of the basic spinorial representations \([1111]\) corresponding to the positive chirality projection \(\theta^+(+)\). For the negative chirality case \(\theta^(-)\) one just needs to read Table 1 upside down. In either case the representations corresponding to the fields \(\chi_{\alpha_1...\alpha_j}(x)\) are the same but with opposite chirality and duality when they apply. In other words, the representations for the fields accompanying a certain power of \(\theta^+(+)\) are given by the same power of \(\theta^(-)\) and viceversa.
Table 1: Decomposition of the totally antisymmetrized Kronecker (wedge) powers of the basic spinor representation of SO(10), as given by their highest weights.

| $j$ | $\theta^{\alpha_1} \ldots \theta^{\alpha_j}$ | Dimension |
|-----|---------------------------------|-----------|
| 0   | [0]                             | 1         |
| 1   | $[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 16        |
| 2   | [1 1 1]                         | 120       |
| 3   | $[\begin{smallmatrix} 3 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 560       |
| 4   | $[2 2] \oplus [2 1 1 1 - 1]$    | 770 + 1050|
| 5   | $[\begin{smallmatrix} 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{smallmatrix}] \oplus [\begin{smallmatrix} 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 3696 + 672|
| 6   | [3 1 1] $\oplus [2 2 1 1 - 1]$  | 4312 + 3696|
| 7   | $[\begin{smallmatrix} 7 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}] \oplus [\begin{smallmatrix} 5 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 2640 + 8800|
| 8   | [4] $\oplus [3 1 1 1] \oplus [2 2 2]$ | 660 + 8085 + 4125|
| 9   | $[\begin{smallmatrix} 7 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}] \oplus [\begin{smallmatrix} 5 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 2640 + 8800|
| 10  | [3 1 1] $\oplus [2 2 1 1 1]$    | 4312 + 3696|
| 11  | $[\begin{smallmatrix} 5 & 3 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}] \oplus [\begin{smallmatrix} 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 3696 + 672|
| 12  | [2 2] $\oplus [2 1 1 1 1]$      | 770 + 1050|
| 13  | $[\begin{smallmatrix} 3 & 3 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{smallmatrix}]$ | 560       |
| 14  | [1 1 1]                         | 120       |
| 15  | $[\begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{smallmatrix}] - [\begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{smallmatrix}]$ | 16        |
| 16  | [0]                             | 1         |
II Fierz Identity

The 10-dimensional Fierz identity for strictly anticommuting $\theta$’s can be put in a very simple form

$$\bar{\theta}^{(\pm)} O_1 \theta^{(\pm)} \bar{\theta}^{(\pm)} O_2 \theta^{(\pm)} = \frac{1}{96} \bar{\theta}^{(\pm)} O_1 \Pi^{(\pm)} \Gamma_{B_1B_2B_3} \theta^{(\pm)} \Gamma_{B_1B_2B_3} \theta^{(\pm)}$$

(2.1)

where $\Pi^{(\pm)} = \frac{1}{2} (I \pm \Gamma_{(11)})$ are the Weyl projection operators (see Appendix A for our conventions).

Then one obtains immediately the vanishing of the triple contraction:

$$\bar{\theta}^{(\pm)} \Gamma_{B_1B_2B_3} \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{B_1B_2B_3} = 0$$

(2.2)

since, in 10 dimensions, $\Gamma_{B_1B_2B_3} C_1 C_2 C_3 \Gamma_{B_1B_2B_3} = -48 \Gamma_{C_1 C_2 C_3}$. Likewise, using the properties of the Dirac algebra, it is relatively simple to show that the following double contraction vanishes:

$$\bar{\theta}^{(\pm)} \Gamma_A B_1 B_2 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{B_1B_2C} \theta^{(\pm)} = \bar{\theta}^{(\pm)} \Gamma_A \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{B_1B_2C} \theta^{(\pm)} = 0.$$  

(2.3)

For the single trace we get a non-trivial result:

$$\bar{\theta}^{(\pm)} \Gamma_A B_1 A_2 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{C_1 C_2} = 2 \bar{\theta}^{(\pm)} \Gamma_B \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_A [C_1 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_A] C_2 \theta^{(\pm)}.$$  

(2.4)

In particular, (2.4) implies the vanishing of the antisymmetric combination:

$$\bar{\theta}^{(\pm)} \Gamma_A A_2 \Gamma_B \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{C_1 C_2} \Gamma_B \theta^{(\pm)} = 0.$$  

(2.5)

In fact, (2.4) implies the more powerful and useful result

$$\bar{\theta}^{(\pm)} \Gamma_A A_2 \Gamma_B \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{A_3} C B \theta^{(\pm)} = 0.$$  

(2.6)

Therefore we conclude that $\bar{\theta}^{(\pm)} \Gamma_A A_2 \Gamma_B \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{C_1 C_2} \Gamma_B \theta^{(\pm)}$ is a traceless tensor which contains no antisymmetric parts of more than 2 indices, and must therefore correspond to the representation

$$\Box$$  

or $[2 \ 2]$.

Finally we are ready to tackle the uncontracted product, and we obtain:

$$\frac{9}{8} \bar{\theta}^{(\pm)} \Gamma_A A_2 A_3 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{C_1 C_2 C_3} \theta^{(\pm)} =$$

$$= \frac{1}{32} \bar{\theta}^{(\pm)} \Gamma_A A_2 A_3 D_1 D_2 D_3 D_4 D_5 [C_1 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{C_3}] D_1 D_2 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{D_3 D_4 D_5} \theta^{(\pm)}$$

$$- \frac{9}{8} \bar{\theta}^{(\pm)} \Gamma_A A_2 D_4 [C_1 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{A_3}] C_2 C_3] \theta^{(\pm)}$$

$$+ \frac{9}{4} \bar{\theta}^{(\pm)} \Gamma_A A_2 \theta^{(\pm)} \bar{\theta}^{(\pm)} \Gamma_{A_3 C_2 C_3} \theta^{(\pm)}$$

(2.7)

where one has to make use of the Dirac algebra and in particular
Before we can make sense of Eq. (2.7), let us note that if we call:

$$X(\pm)_{C;D_1...D_5} = \bar{\theta}(\pm) \Gamma^{C[D_1 D_2 \theta(\pm)] \Gamma^{D_3 D_4 D_5} \theta}$$

we get

$$X^{(\pm)}_{[C_1;C_2 C_3]} A_1 A_2 A_3 = \frac{1}{10} (\bar{\theta}(\pm) \Gamma^{A_1 A_2 A_3 \theta(\pm)} \Gamma^{C_1 C_2 C_3 \theta(\pm)} - 3 \bar{\theta}(\pm) \Gamma^{[A_1 A_2 \theta(\pm)] \Gamma^{A_3 C_1 C_2 C_3 \theta(\pm)}})$$

(2.1)

$X^{(\pm)}$ is clearly traceless by virtue of (2.5) and trivially satisfies

$$X^{(\pm)}_{[A;B_1...B_5]} = 0$$

(2.1)

And, since $X^{(\pm)}$ has five totally antisymmetric indices, it is a good candidate for the other irreducible piece of the $\theta^4$ sector. This will be confirmed shortly. Then we can rewrite (2.7) as

$$\bar{\theta}(\pm) \Gamma^{A_1 A_2 A_3 \theta(\pm)} \Gamma^{C_1 C_2 C_3 \theta(\pm)} =$$

$$+ \frac{1}{48} \epsilon^{A_1 A_2 A_3 D_1 D_2 D_3 D_4 D_5} [C_1 C_2 \bar{\theta}(\pm) C_3] D_1 D_2 \theta(\pm) \bar{\theta}(\pm) \Gamma^{D_3 D_4 D_5} \theta(\pm) +$$

$$+ 5 \epsilon^{X(\pm)[A_1;A_2 A_3] C_1 C_2 C_3}$$

$$+ 3 \bar{\theta}(\pm) \Gamma^B_{[A_2 A_3 \theta(\pm)] \eta^A_1 [C_1 \bar{\theta}(\pm)] \Gamma^{C_2 C_3} B \theta(\pm)}$$

(2.1)

This equation implies the (anti-) self-duality of $X^{(\pm)A;B_1...B_5}$:

$$X^{(\pm)A;B_1...B_5} = \frac{1}{5!} \epsilon^{B_1...B_5 D_1...D_5} X^{(\pm)A;D_1...D_5}$$

$$X^{(\pm)A;B_1...B_5} = \pm \frac{1}{5!} \epsilon^{B_1...B_5 D_1...D_5} X^{(\pm)A;D_1...D_5}$$

(2.1)

thus confirming that it is the missing irreducible piece from the $\theta^4$ sector.

Therefore, the basic identity (2.12) gives the decomposition of the general $\theta^4$ tensor in irreducible pieces. It is the basic identity from which all the higher order decompositions must necessarily follow by appropriate iterative use of it.

In the remainder of the paper we are going to concentrate only on the positive chirality case $\theta^{(+)}$. To obtain the corresponding results for $\theta^{(-)}$ one just has to remember that all the chiral and duality properties are reversed.
III $\theta^6$ Decompositions

In order to simplify notation let us call

\[ M^{ABC} = \tilde{\theta}(+) \Gamma^{ABC} \theta(+). \]  

(3.1)

Also in the remainder of the paper we are going to use the following letter convention: \( u \) contracted indices labeled by the same letter with different subindex are understood to be antisymmetrized except if the letter involved is \( S \) or \( X \) in which case they are understood to be symmetrized. For instance:

\[ \begin{aligned}
F^{CA_1 A_2 A_3} G^{A_4 A_5 D} &\equiv F^{C[A_1 A_2 A_3} G^{A_4 A_5] D} \\
N^{C D S_1 S_2 X_1 P^{S_3} A B}_X &\equiv N^{C D (S_1 S_2 (P^{S_3} A B)_X)}
\end{aligned} \]  

(3.2)

where the square and round brackets are the by now standard notations denoting normalized total antisymmetrization and symmetrization respectively. This notation will dramatically reduce the need for brackets which would make some formulae otherwise practically impossible to write.

Then, Eq. (2.12) becomes:

\[ M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} = \frac{5}{2} \left( M^{A_1[A_2 A_3} M^{B_1 B_2 B_3]} - \frac{1}{5!} \epsilon^{A_1 A_2 A_3} B_1 B_2 B_3 D_1 D_2 D_3 D_4 D_5 M^{B_1 B_2 D_1 D_2 D_3 D_4 D_5} \right) \\
+ \frac{3}{2} \eta^{A_1 B_1} M^{A_2 A_3} D M^{B_2 B_3 D}. \]  

(3.3)

Eq. (3.3) is equivalent to the following two statements:

\[ M^{C A_1 A_2} M^{A_3 A_4 A_5} = -\frac{1}{5!} \epsilon^{A_1 ... A_5} B_1 ... B_5 M^{C} B_1 B_2 M^{B_3 B_4 B_5} \]  

(3.4)

\[ M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} = 5 M^{A_1[A_2 A_3} M^{B_1 B_2 B_3]} + \frac{3}{2} \eta^{A_1 B_1} M^{A_2 A_3} D M^{B_2 B_3 D}. \]  

(3.5)

Eqs. (3.3) or (3.5) clearly give the decomposition of \( M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} \) into its irreducible parts, the anti-selfdual \( [2111 - 1] \) piece:

\[ \mathcal{M}^{A_1 B_1 ... B_5} = M^{A B_1 B_2} M^{B_3 B_4 B_5} \]  

(3.6)

and the \( [22] \) piece:

\[ \mathcal{M}^{A_1 A_2 B_1 B_2} = M^{A_1 A_2} E M^{B_1 B_2 E}. \]  

(3.7)

From their definitions and the results of this and the previous section, we get the following properties:

\[ \begin{aligned}
\mathcal{M}_4^{A; B_1 ... B_5} &\equiv 0 \quad \mathcal{M}_4 E^{EB_1 ... B_4} = 0 \\
\mathcal{M}_4^{A; B_1 ... B_5} &\equiv -\frac{1}{5!} \epsilon^{B_1 ... B_5 D_1 ... D_5} \mathcal{M}_4^{A; D_1 ... D_5}
\end{aligned} \]  

(3.8)
\[
M_{A_1 A_2}^{B_1 B_2} = M_{A_1 B_2}^{B_1 A_2} \quad M_{A B}^{E A E B} = 0
\]

(3.9)

In order to decompose the next product \( M_{A_1 A_2 A_3}^{B_1 B_2 B_3} M_{C_1 C_2 C_3}^{D_1 D_2 D_3} \) one can proceed to iterate (3.3) for the different binary products. After several iterations and a lot of algebra it is possible to obtain the following decomposition:

\[
M_{A_1 A_2 A_3}^{B_1 B_2 B_3} M_{C_1 C_2 C_3}^{D_1 D_2 D_3} = S(A, B, C) \left\{ 18 \eta^{B_1 C_1} M_{[A_1 A_2 A_3}^{B_1 B_2 B_3]} D M_{B_3 C_3] D} + \frac{18}{5} \eta^{A_1 C_1} \eta^{B_1 C_2} M_{C_3 D E M A_2 A_3}^{D} M_{B_1 B_2 B_3}^{E} \right. \\
\left. \frac{-9}{5} \eta^{B_1 C_1} \eta^{B_2 C_2} M_{C_3 D E M B_3 A_1}^{D} M_{A_2 A_3}^{E} \right\} + \frac{1}{20} \epsilon_{B_1 B_2 B_3 A_1 A_2 C_2 D_1 D_2 D_3} M_{A_3 E_1}^{D_1} M_{D_2 D_3}^{E_1} M_{E_1 E_2}^{C_3}
\]

(3.10)

where \( S(A, B, C) \) is the normalized operator that fully symmetrizes on the letters \( A, B, C \). The last term in (3.10) is automatically symmetric upon interchange of the three letters, as can be easily proven by using the fact that a complete antisymmetrization of 11 indices must necessarily vanish.

In deriving (3.10) one has to make use of many identities (see Appendix A) which are all consequences of (3.3), specially

\[
M_{A D E M B E F}^{B F} M_{C}^{D} = 0
\]

which follows almost immediately from (2.10) and (2.3). Eq. (3.11) means that all triple contractions of \( M^3 \) vanish, as it should be since there are no objects with 3 indices in the \( \theta^6 \) sector.

The amount of effort required to obtain (3.10) by iteration of (3.3) makes it clear that an alternative way is needed if one hopes to decompose all the higher order products. Nevertheless it illustrates the fact that all the necessary product decompositions are direct consequences of the Fierz identity (2.12).

There is a much simpler way to obtain the decomposition (3.10), by systematically removing traces (since the irreducible pieces are traceless) and using the appropriate Young projectors on the traceless parts. This is possible because we already know beforehand what are the irreducible representations involved (see Table 1).

Let us begin by removing all the traces from the object:

\[
M_{A_1 A_2 A_3}^{B_1 B_2} M_{D B_1 B_2}^{B_1 B_2} M_{D C_1 C_2}^{D C_1 C_2} = Traceless \left( M_{A_1 A_2 A_3}^{B_1 B_2} M_{D B_1 B_2}^{B_1 B_2} M_{D C_1 C_2}^{D C_1 C_2} \right) \\
+ \frac{2}{5} (2 \eta^{A_1 A_1} M_{A_1 A_2 A_3}^{D E} M_{D B_1 B_2}^{B_1 B_2} M_{D C_1 C_2}^{D C_1 C_2} + 2 \eta^{A_1 A_1} M_{A_1 A_2 A_3}^{D B_1 B_2} M_{B_1 B_2}^{D E} M_{D C_1 C_2}^{D C_1 C_2} + \eta^{B_1 C_1} M_{A_1 A_2 A_3}^{D C_1 C_2} M_{D B_1 B_2}^{D C_1 C_2} M_{D C_1 C_2}^{D C_1 C_2})
\]

(3.12)

Next we decompose \( Traceless \left( M_{A_1 A_2 A_3}^{B_1 B_2} M_{D B_1 B_2}^{B_1 B_2} M_{D C_1 C_2}^{D C_1 C_2} \right) \) using the Young projectors corresponding to the representation \( \square \) (see Table 1) whose construction is detailed in Appendix C:
To obtain the traceless part in (3.15), we apply the Young projector corresponding to the representation \(\begin{array}{c}
\end{array}\) for \(SO(10)\). Explicitly displaying the equivalence of the traces:

\[
 M^{A_1A_2A_3}M^{[B_1B_2]}M^{DC_1C_2} = Y \left( \begin{array}{c}
\end{array}\right) M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2D} \]
\[
= \frac{2}{3}(M^{[A_1A_2A_3}M^{B_1B_2]}D)M^{C_1C_2D} + M^{[A_1A_2A_3}M^{C_1}C_2]D)M^{B_1B_2D} + 2M^{[A_1A_2A_3}M^{B_1C_1]}D)M^{B_2D}).
\] (3.1)

Now we do the same for the uncontracted product \(M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}\), first removing the traces:

\[
 M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} = Traceless (M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3})
+ \frac{9}{5}S(A, B, C)\left\{ \frac{3}{2}M^{D_2B_3M^{C_1C_2C_3} - M^{D_2A_3}M^{C_1B_2B_3}M^{D_2C_2C_3} + 2M^{D_2A_3}M^{D_2C_2M^{A_3C_3}} \right\}. \] (3.1)

Using some of the identities in Appendix A and the decomposition (3.12)-(3.13) we get

\[
 M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} = Traceless (M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3})
+ 9S(A, B, C)\left\{ 2\eta^{A_1B_1}M^{C_1C_2C_3}M^{A_2B_2]}D)M^{A_3B_3D} + \frac{2}{5}\eta^{A_1B_1}\eta^{C_1B_2}M^{B_3DE}M^{A_2A_3D}M^{C_2C_3E} \right\}. \] (3.1)

To obtain the traceless part in (3.15), we apply the Young projector corresponding to the representation \(\begin{array}{c}
\end{array}\) for \(SO(10)\)

\[
 Traceless (M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}) =
= Y \left( \begin{array}{c}
\end{array}\right) M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} \]
\[
= 21M^{B_1[B_2B_3]A_1A_2A_3M^{C_2C_3]}C_1 \]
\[
= \frac{1}{20}\partial^{A_1A_2A_3B_1B_2C_1C_2E_1E_3F}M^{FDB_3M^{DE}E_1E_2E_3}M^{E_3F}C_3
\] (3.1)

where the last equality follows from the anti-selfduality of \(M^{A[B_1B_2B_3B_4]}\) by rotating indices and explicitly displays the aforementioned equivalence of \(SO(10)\) representations.

Eq. (3.16) together with (3.15) reproduces for us the decomposition (3.10). We will delay the study of the irreducible pieces of the \(\theta^6\) sector until the next section.
IV Irreducible Bosonic Structures

The difficulty in proceeding along the lines of the previous section is that one needs to know beforehand what are the irreducible pieces of the higher $\theta$ powers in order to decompose the products into irreducible pieces. That is why we are now going to proceed backwards, starting from the scalar corresponding to $\theta^{16}$ and come down from there.

To construct the above scalar we first notice that it is easy to identify the totally symmetric tensor of $\theta^8$ sector corresponding to the representation $[4]$:

$$\mathcal{M}^{ABCD}_{8} = M^{A}_{E} M^{B}_{F} M^{C}_{G} M^{D}_{H} M^{E}_{E}.$$  \hfill (4.1)

It is obviously traceless (see (3.11)) and cyclically symmetric:

$$\mathcal{M}^{ABCD}_{8} = \mathcal{M}^{BACD}_{8}.$$  \hfill (4.2)

and the antisymmetrization of any two neighboring indices vanishes

$$\mathcal{M}^{[AB]CD}_{8} = \frac{1}{2} M^{BA}_{E} M^{E}_{F} M^{C}_{G} M^{D}_{H} M^{E}_{E} = \frac{1}{4} M^{BA}_{E} M^{E}_{F} M^{C}_{G} M^{D}_{H} M^{E}_{E} = 0.$$  \hfill (4.3)

where we have twice made use of (2.6) and then (2.3). Thus

$$\mathcal{M}^{ABCD}_{8} = \mathcal{M}^{BACD}_{8}.$$  \hfill (4.4)

Properties (4.2) and (4.4) imply that $\mathcal{M}^{ABCD}_{8}$ is completely symmetric in all four indices.

The scalar we are looking for is the square of (4.1)

$$\mathcal{M}_{16} = \mathcal{M}^{S_{1}S_{2}S_{3}S_{4}}_{8} \mathcal{M}^{S_{1}S_{2}S_{3}S_{4}}_{8} \mathcal{M}^{S_{1}S_{2}S_{3}S_{4}}_{8} = \mathcal{M}^{E_{1}E_{2}} M^{S_{1}S_{2}F_{1}} M^{S_{3}F_{1}F_{2}} M^{S_{4}F_{1}} M^{G_{1}G_{2}} M^{S_{2}G_{2}H_{1}} M^{S_{3}H_{1}G_{1}} M^{S_{4}H_{1}G_{1}}.$$  \hfill (4.5)

where all the $M$-factors are equivalent.

Since all the factors in (4.5) are equivalent, there is only one possible expression to be obtained by removing any one of them and that must be our irreducible piece:

$$\mathcal{M}^{ABC} = \mathcal{M}^{S_{1}S_{2}S_{3}S_{4}}_{8} M^{S_{1}} E_{1} E_{2} M^{S_{2}E_{2}F_{1}} M^{S_{3}} F_{1} F_{2} M^{S_{4}E_{1}} M^{G_{1}G_{2}} M^{S_{2}G_{2}H_{1}} M^{S_{3}H_{1}H_{2}} M^{S_{4}H_{1}G_{1}}.$$  \hfill (4.6)
which is obviously antisymmetric in $B, C$:

$$M^{ABC} = -M^{ACB}$$  \hspace{1cm} (4.7)

but must be totally antisymmetric because it must belong to $\equiv [1 1 1]$. In order to prove this, we first put it in a more appealing form using the symmetry of the $\equiv$ part as well as (2.6):

$$M^{ABC} = M^{A_{S_{1}D_{1}}M^{S_{2}D_{2}}M^{F_{1}F_{2}M^{F_{3}G_{1}M^{S_{2}G_{1}G_{2}}M^{C_{E}{G_{2}}}}}}. \hspace{1cm} (4.8)$$

Then, reordering factors and using (2.6) once more we obtain

$$M^{ABC} = M^{B_{S_{1}D_{1}}M^{S_{2}D_{1}D_{2}}M^{F_{1}F_{2}M^{F_{3}G_{1}M^{S_{2}G_{1}G_{2}}M^{A_{E}{G_{2}}}}}} = M^{BCA}. \hspace{1cm} (4.9)$$

Properties (4.7) and (4.9) imply that $M^{ABC}$ is completely antisymmetric in all 3 indices.

From (4.6) and (4.5) we note that

$$M^{A_{1}A_{2}A_{3}M^{A_{1}A_{2}A_{3}}} = -M^{16} \hspace{1cm} (4.10)$$

and therefore we have the product decomposition

$$M^{A_{1}A_{2}A_{3}M^{B_{1}B_{2}B_{3}}} = -\frac{1}{120} \eta^{A_{1}B_{1}}\eta^{A_{2}B_{2}}\eta^{A_{3}B_{3}}M^{16}. \hspace{1cm} (4.11)$$

Not all the factors in (4.6) are equivalent, so now we get two possible structures by removing one factor from $M^{ABC}$. One is:

$$\hat{M}_{12}^{AB,CD} = M^{A_{1}D_{1}M^{S_{2}D_{2}}M^{F_{1}F_{2}M^{F_{3}G_{1}M^{S_{2}G_{1}G_{2}}M^{C_{E}{G_{2}}}}}}. \hspace{1cm} (4.12)$$

which is clearly traceless and, by virtue of (2.6), (2.3), has the symmetry properties:

$$\hat{M}_{12}^{AB,CD} = \hat{M}_{12}^{BA,CD} = \hat{M}_{12}^{AB,DC} = \hat{M}_{12}^{BA,DC}. \hspace{1cm} (4.13)$$

By using (2.6) in a different way we can also derive

$$\hat{M}_{12}^{AB,CD} + \hat{M}_{12}^{AC,DB} + \hat{M}_{12}^{CB,AD} = 0 \hspace{1cm} (4.14)$$

$$\hat{M}_{12}^{AB,CD} + \hat{M}_{12}^{DB,AC} + \hat{M}_{12}^{AD, CB} = 0. \hspace{1cm} (4.15)$$

Combining (4.14) with (4.13) we get

$$\hat{M}_{12}^{A[B,C]D} + \hat{M}_{12}^{D[B,C]A} = 0 \hspace{1cm} (4.16)$$

while combining (4.14) and (4.15),
\[
\hat{M}_{12}^{AB,CD} = \hat{M}_{12}^{CD,AB}.
\]  

(4.17)

Once we have obtained (4.17) we see that (4.14) and (4.15) simply mean:

\[
\hat{M}_{12}^{A(B,CD)} = 0.
\]  

(4.18)

Eq. (4.16) tells us that antisymmetrizing on two indices on opposite sides of the comma automatically makes the other pair also antisymmetric. Thus we recognize the object that displays the symmetry of the Young pattern \[\Box\] :

\[
\hat{M}_{12}^{A_1A_2;B_1B_2} = \hat{M}_{12}^{A_1B_1,B_2A_2} = \hat{M}_{12}^{A_1B_1,A_2B_2}.
\]  

(4.19)

However it is interesting to note for reference, the more interesting properties of the \[\hat{M}_{12}\] tensor.

From the definition (4.19) it is clear that \([\hat{M}_{12}^{A_1A_2,B_1B_2}]\) is traceless and that it satisfies:

\[
\hat{M}_{12}^{A_1A_2;B_1B_2} = 0.
\]  

(4.20)

Thus it has the same properties as the tensor \(M_{12}^{A_1A_2,B_1B_2}\) except for nilpotency.

Even though \(\hat{M}_{12}\) and \(M_{12}\) have apparently different symmetry properties they both have the same number of degrees of freedom, 770, i.e. the dimension of the irrep. [22] of \(SO(10)\), and they both can be expressed in terms of the other. The inverse of (4.19) is

\[
\hat{M}_{12}^{AB,CD} = \frac{2}{3}(M_{12}^{AD;BC} + M_{12}^{BD;AC})
\]  

(4.21)

as can be easily seen by using (4.18).

From (4.8) and (4.12) we see that

\[
\hat{M}_{12}^{AE,BF}M_{FE}^{C} = M_{12}^{AB,EF}M_{FE}^{C} = M_{ABC}
\]  

(4.22)

and then we have for the decomposition of the single contraction:

\[
\hat{M}_{12}^{S_1S_2,XE}M_{A_1A_2E} = \frac{1}{11\times 7}\left[36S_1^A\hat{M}^{S_2X}A_2 - \frac{1}{3}S_1^X\hat{M}^{S_2}A_1A_2 + \frac{1}{3}S_1^{S_2}\hat{M}^{X}A_2A_1\right].
\]  

(4.23)

Eq. (4.23) is easily obtained since it must have that general form and the coefficients are given by the traces of the left-hand side, either zero or (4.22). For the other object we have

\[
\hat{M}_{12}^{B_1B_2;CE}M_{E}^{A_1A_2} = \frac{1}{14}(3\eta^{A_1C}M^{A_2B_1B_2} - 3\eta^{A_1B_1}M^{A_2B_2C} + \eta^{C_B_1}M^{B_2A_1A_2}).
\]  

(4.24)

Using (4.23) and following the same procedure one derives for the full product

\[
\hat{M}_{12}^{S_1S_2,X_1X_2}M_{A_1A_2A_3} = -\frac{3}{11 \times 7} \left[8\eta^{S_1A_1}\eta^{X_1A_2}M^{S_2X_2A_3} - \eta^{S_1X_1}\eta^{S_2A_1}M^{X_2A_2A_3} - \eta^{S_1X_1}\eta^{S_2A_1}M^{S_2X_2A_3} + \eta^{X_1X_2}\eta^{S_1A_2}M^{S_2X_2A_3} + \eta^{S_1S_2}\eta^{X_1A_1}M^{X_2A_2A_3} - \eta^{X_1X_2}\eta^{S_1S_2} - \eta^{X_1S_1}\eta^{X_2S_2}\right]M_{A_1A_2A_3}.
\]  

(4.25)
\[
\mathcal{M}_{12}^{B_1B_2;C_1C_2}M^{A_1A_2A_3} = -\frac{6}{11 \times 7} \left( \eta^{A_1B_1} \eta^{A_2B_2} \mathcal{M}^{A_2C_1C_2} + 2\eta^{A_1B_1} \eta^{A_2C_2} \mathcal{M}^{A_3B_2C_2} \right.
+ \eta^{A_1C_1} \eta^{A_2C_2} \mathcal{M}^{A_3B_1B_2} - \frac{3}{4} \eta^{B_1C_1} \eta^{C_2A_1} \mathcal{M}^{A_2A_3B_2}\left.ight)
- \frac{3}{4} \eta^{B_1C_1} \eta^{B_2A_1} \mathcal{M}^{A_2A_3C_2} + \frac{1}{12} \eta^{B_1C_1} \eta^{B_2C_3} \mathcal{M}^{A_1A_2A_3} \right) .
\] (4.2)

If we remove a different factor from \(\mathcal{M}^{ABC}\) we extract the new structure
\[
\hat{\mathcal{M}}_{12}^{XABY;E_1E_2} = M_{D_1}X_{D_2}M_{D_3}^{A} M_{D_4}^{B} D_3 M_{D_4}^{C} D_2 \mathcal{M}_{D_5}^{E} D_1 M_{E_1}^{F} E_1 .
\] (4.27)

It has the obvious property
\[
\hat{\mathcal{M}}_{12}^{XABY;C_1C_2} = -\hat{\mathcal{M}}_{12}^{YBA;C_1C_2}
\] (4.2)

and by applying (2.6) it is also easy to prove
\[
\hat{\mathcal{M}}_{12}^{X[ABY;C_1C_2]} = \hat{\mathcal{M}}_{12}^{[AXBY;C_1C_2]}
\] (4.2)

which in turn implies:
\[
\hat{\mathcal{M}}_{12}^{[XABY;C_1C_2]} = 0.
\] (4.3)

However, this object is not irreducible because it is not completely traceless, but rather has two non-vanishing traces:

\[
\hat{\mathcal{M}}_{12}^{E;ABY;EC} = -\hat{\mathcal{M}}_{12}^{AC;BY}
\]
\[
\hat{\mathcal{M}}_{12}^{E;XAB;EC} = \hat{\mathcal{M}}_{12}^{BC;AX}
\] (4.3)

In order to decompose it one removes the traces and applies the appropriate Young projector:

\[
\hat{\mathcal{M}}_{12}^{XABY;C_1C_2} = Traceless \left( \hat{\mathcal{M}}_{12}^{XABY;C_1C_2} \right) \\
+ \frac{1}{9 \times 21} \left\{ -46(\eta^{XC_1} \hat{\mathcal{M}}_{12}^{AC_2,BY} - \eta^{YC_1} \hat{\mathcal{M}}_{12}^{BC_3,AX}) \\
-3(\eta^{XB} \hat{\mathcal{M}}_{12}^{AC_1,YC_2} - \eta^{YA} \hat{\mathcal{M}}_{12}^{BC_1,XY}) \\
-5(\eta^{XC_1} \hat{\mathcal{M}}_{12}^{BC_2,AY} - \eta^{YC_1} \hat{\mathcal{M}}_{12}^{AC_1,BX} + \eta^{AC_1} \hat{\mathcal{M}}_{12}^{BC_2,XY}) \\
-\eta^{BC_1} \hat{\mathcal{M}}_{12}^{AC_2,XY} - \eta^{AC_1} \hat{\mathcal{M}}_{12}^{SC_2,BY} + \eta^{BC_1} \hat{\mathcal{M}}_{12}^{YC_1,AX}) \\
+2(\eta^{XA} \hat{\mathcal{M}}_{12}^{BC_1,YC_2} - \eta^{XB} \hat{\mathcal{M}}_{12}^{AC_1,XY}) \\
+4\eta^{XY} \hat{\mathcal{M}}_{12}^{AC_1,BC_2} - \eta^{AB} \hat{\mathcal{M}}_{12}^{X_C_1,YX_2} \right\} ,
\] (4.3)
\[ \text{Traceless } (\mathcal{M}_{12}^{XABY;C_1C_2}) = Y \left( \mathcal{M}_{12}^{XABY;C_1C_2} \right) \]
\[ = \frac{5}{6} (\mathcal{M}_{12}^{X[ABY;C_2C_3]} + \mathcal{M}_{12}^{A[BYX;C_1C_2]} + \mathcal{M}_{12}^{B[YXA;C_1C_2]} + \mathcal{M}_{12}^{Y[XAB;C_1C_2]}) \]  
\[ (4.3) \]

From (4.33) it is apparent that the second irreducible structure is
\[ \mathcal{M}_{12}^{B;A_1...A_5} = \mathcal{M}_{12}^{B;A_1A_2A_3;A_4A_5} \]
\[ = M^B D_1 D_2 F D_3 M^{A_1} D_3 D_4 M^{A_2} D_4 D_5 M^{A_3} D_5 D_1 M^{A_4 A_5}, \]  
\[ (4.34) \]

whose tracelessness is confirmed by (3.11). Eq. (4.30) implies the property
\[ \mathcal{M}_{12}^{[B;A_1...A_5]} = 0 \]  
\[ (4.3) \]

and in Appendix A we prove the duality property
\[ \mathcal{M}_{12}^{B;A_1...A_5} = \frac{1}{5!} e^{A_1...A_5 E_1...E_5} \mathcal{M}_{12}^{B;E_1...E_5} \]  
\[ (4.34) \]

which is opposite to the one satisfied by \( \mathcal{M}_{4}^{[B;A_1...A_5]} \). The definitions (4.34), (4.6) give the result for the triple contractions
\[ \mathcal{M}_{12}^{B;A_1A_2E_1E_2E_3 M_{E_1E_2E_3}} = -\frac{1}{5} \mathcal{M}^{B;A_1A_2} \]
\[ \mathcal{M}_{12}^{E_1;E_2E_3A_1A_2A_3 M_{E_1E_2E_3}} = -\frac{1}{5} \mathcal{M}^{A_1A_2A_3}. \]  
\[ (4.4) \]

and by simple detracing,
\[ \mathcal{M}_{12}^{B;A_1A_2A_3E_1E_2 M_{CE_1E_2}} = -\frac{1}{70} (\delta^B_C \mathcal{M}^{A_1A_2A_3} - \eta^{BA_1} \mathcal{M}^{A_2A_3} + 5 \delta^{A_1} \mathcal{M}^{BA_2A_3}) \]
\[ \mathcal{M}_{12}^{E_1;E_2A_1...A_1 M_{CE_1E_2}} = -\frac{4}{35} \delta^{A_1} \mathcal{M}^{A_2A_3A_4}. \]  
\[ (4.5) \]

From (4.38) and Young-projecting
\[ \mathcal{M}_{12}^{B;A_1...A_4 E M_{C_1C_2E}} = \text{Traceless } (\mathcal{M}_{12}^{B;A_1...A_4 E M_{C_1C_2E}}) \]
\[ + \frac{2}{3 \times 35} (\delta^B_C \mathcal{M}^{A_2A_3A_4} - 2 \delta^{A_1} \delta^{A_2} \mathcal{M}^{BA_3A_4} - \eta^{BA_1} \delta^{A_2} \mathcal{M}^{C_2A_3A_4}) \]  
\[ (4.3) \]
\[ \text{Traceless } (\mathcal{M}_{12}^{B:A_1\ldots A_4 E M^{C_1 C_2}_E}) = Y \left( \begin{array}{c} B \\ A_1 \ldots A_4 E \\ C_1 \\ C_2 \\ E \end{array} \right) \mathcal{M}_{12}^{B:A_1\ldots A_4 E M^{C_1 C_2}_E} \]

\[ = \mathcal{M}_{12}^{[B:A_1\ldots A_4 E M^{C_1 C_2}_E]} = \frac{1}{5} \mathcal{M}_{12}^{E:[BA_1\ldots A_4 M^{C_1 C_2}_E]} \]

\[-\frac{4}{5} \frac{1}{7!} \epsilon^{BA_1\ldots A_4 C_1 C_2 E_1 E_2 E_3} \mathcal{M}_{E_1 E_2 E_3} \]  

Eqns. (4.39), (4.40) and (4.35) then give

\[ \mathcal{M}_{12}^{E:A_1\ldots A_5 M^{C_1 C_2}_E} = -\frac{2}{21} \left( \frac{1}{5!} \epsilon^{A_1\ldots A_5 C_1 C_2 E_1 E_2 E_3} \mathcal{M}_{E_1 E_2 E_3} + \eta^{A_1 C_1} \eta^{A_2 C_2} \mathcal{M}^{A_3 A_4 A_5}_E \right). \]  

Finally for the full product

\[ \mathcal{M}_{12}^{B:A_1\ldots A_5 M^{C_1 C_2 C_3}_E} = -\frac{2}{7!} \left\{ \eta^{BC_1} \epsilon^{A_1\ldots A_5 C_1 C_2 C_3 E_1 E_2 E_3} - \eta^{BA_1} \epsilon^{A_2\ldots A_5 C_1 C_2 C_3 E_1 E_2 E_3} \right. \]

\[ + \eta^{C_1 A_1} \epsilon^{BA_2\ldots A_5 C_1 C_2 C_3 E_1 E_2 E_3} \} \mathcal{M}_{E_1 E_2 E_3} \]

\[-\frac{1}{35} \left[ \eta^{BC_1} \eta^{A_1 C_2} \eta^{A_2 C_3} \mathcal{M}^{A_3 A_4 A_5} + \eta^{A_1 C_1} \eta^{A_2 C_2} \eta^{A_3 C_3} \mathcal{M}^{BA_4 A_5} \right. \]

\[-\eta^{BA_1} \eta^{A_2 C_1} \eta^{A_3 C_2} \mathcal{M}^{A_4 A_5 C_3} \].

The first structure we encounter by removing a factor from \( \hat{\mathcal{M}}_{12}^{AB,CD} \) is

\[ \hat{\mathcal{M}}_{10}^{S_1 S_2 S_3; A_1 A_2} = \mathcal{M}_{8}^{S_1 S_2 S_3 E} M^{A_1 A_2}_E \]  

whose symmetry properties are manifest. Its tracelessness follows from these symmetries and from the tracelessness of \( \mathcal{M}_{8}^{S_1 S_2 S_3 S_4} \). The object in (4.43) also satisfies

\[ \hat{\mathcal{M}}_{10}^{(S_1 S_2 S_3; A)B} = 0, \]

and its product decompositions can be derived as before and we just list them:

\[ \hat{\mathcal{M}}_{10}^{EDA;BF} M^{C}_{EF} = -\hat{\mathcal{M}}_{12}^{AD,BC} \]

\[ \hat{\mathcal{M}}_{10}^{ABD;EF} M^{C}_{EF} = \hat{\mathcal{M}}_{10}^{EFA;BD} M^{C}_{EF} = 0 \]

\[ \hat{\mathcal{M}}_{10}^{S_1 S_2 S_3; AE} M^{B_1 B_2}_{E} = \frac{4}{7} \eta^{S_1 B_1} \hat{\mathcal{M}}_{12}^{S_2 S_3, AB_2} - \frac{2}{21} \eta^{S_1 S_2} \hat{\mathcal{M}}_{12}^{S_3 B_1, AB_2} \]
\[
\begin{align*}
\hat{M}_{10}^{S_1S_2E;A_1A_2} M_{B_1B_2}^E &= -\frac{10}{3} \hat{M}_{12}^{S_1S_2A_1A_2B_1B_2} \\
&+ \frac{2}{63} \left[ \frac{10}{3} \left( \eta^{S_1A_1} \hat{M}_{12}^{A_1B_2A_2S_2} + \eta^{S_1A_1} \hat{M}_{12}^{A_2B_1B_2S_2} \right) \\
&+ 17 \eta^{A_1B_1} \hat{M}_{12}^{S_1S_2A_2B_2} + \frac{2}{3} \eta^{S_1S_2} \hat{M}_{12}^{A_1B_1A_2B_2} \right] \\
\hat{M}_{10}^{S_1S_2S_3;A_1A_2} M_{B_1B_2B_3} &= -\frac{15}{11} \left\{ \frac{18}{7} \eta^{S_1B_1} \hat{M}_{12}^{S_2S_3A_1A_2B_2B_3} - \eta^{S_1S_2} \hat{M}_{12}^{B_2S_3A_1A_2B_2B_3} \right\} \\
&+ \frac{4}{7} \left( \eta^{S_1A_1} \hat{M}_{12}^{S_2S_3A_2B_1B_2B_3} - \eta^{S_1S_2} \hat{M}_{12}^{A_1S_3A_2B_1B_2B_3} \right) \\
&+ \frac{2}{35} \left[ 2 \eta^{S_1B_1} \eta^{S_2A_1} \hat{M}_{12}^{A_2B_2B_3S_3} + 9 \eta^{S_1B_1} \eta^{S_2A_1} \hat{M}_{12}^{S_2S_3A_2B_2B_3} \right] \\
&- \frac{1}{21} \left[ 2 \eta^{S_1S_2} \eta^{A_1B_1} \hat{M}_{12}^{A_2B_2B_3S_3} - \frac{1}{5} \eta^{S_1S_2} \eta^{S_3B_1} \hat{M}_{12}^{A_2B_2A_2B_3} \right]. \\
(4.46)
\end{align*}
\]

However, the symmetry properties of the tensor \( \hat{M}_{10}^{S_1S_2S_3;A_1A_2} \) are not the ones of the Young pattern \( \boxtimes \) as it is conventionally understood, but it is easy to construct a new tensor which corresponds to \( \boxtimes \):

\[
\hat{M}_{10}^{S_1S_2A_1A_2A_3} = \hat{M}_{10}^{S_1S_2[A_1A_2A_3]}.
\]

But, just like we had in the \( \theta^{12} \) case, both of these objects are equivalent, both are irreducible and carry the same number of degrees of freedom (4312) and they can be expressed in terms of each other. The inverse of (4.46) is:

\[
\hat{M}_{10}^{S_1S_2B;A_1A_2} = \frac{3}{5} (\hat{M}_{10}^{S_1S_2;BA_1A_2} + 2 \hat{M}_{10}^{S_1B;S_2A_1A_2}).
\]

From the definition (4.46) we get the property

\[
\hat{M}_{10}^{S[B;A_1A_2A_3]} = 0.
\]

The new products are immediately obtained from (4.45):

\[
\hat{M}_{10}^{S_1S_2E;A_1A_2E} M_{B_1E_2}^C = -\frac{2}{3} \hat{M}_{12}^{A_1A_2;SC}
\]

\[
\hat{M}_{10}^{S_1S_2;AE_1E_2} M_{B_1E_2}^B = \frac{2}{3} \hat{M}_{12}^{S_1S_2;AB} = \frac{8}{9} \hat{M}_{12}^{S_1B;S_2A}
\]

\[
\hat{M}_{10}^{E_1E_2;A_1A_2A_3} M_{B_1E_2}^B = 0
\]
\[ \mathcal{M}_{10}^{S;A_1A_2A_3} M^{B_1B_2}_E = -\frac{10}{9} (\mathcal{M}_{12}^{S;B_1B_2A_1A_2A_3} + \mathcal{M}_{12}^{B_2SA_1A_2A_3}) \\
+ \frac{2}{27} (8\eta^{A_1B_1} \mathcal{M}_{12}^{A_2A_3B_2S} + \eta^{SA_1} \mathcal{M}_{12}^{A_2A_3B_1B_2}) \]

\[ \mathcal{M}_{10}^{S_1S_2;A_1A_2E} M^{B_1B_2}_E = -\frac{10}{9} \mathcal{M}_{12}^{S_1;S_2A_1A_2B_1B_2} \\
+ \frac{4}{7 \times 81} \left\{ 58\eta^{A_1B_1} \mathcal{M}_{12}^{S_1B_2S_2A_2} + 31\eta^{S_1B_1} \mathcal{M}_{12}^{A_1A_2S_2B_2} \\
+ 11\eta^{S_1A_1} \mathcal{M}_{12}^{A_2S_2B_1B_2} - 2\eta^{S_1S_2} \mathcal{M}_{12}^{A_1A_2B_1B_2} \right\} \]

\[ \mathcal{M}_{10}^{S_1S_2;A_1A_2A_3} M^{B_1B_2B_3} = \mathcal{M}_{10}^{S_1S_2[A_1;A_2A_3]} M^{B_1B_2B_3} \]

\[ = -\frac{15}{11} \left\{ \frac{6}{7} (\eta^{S_1B_1} \mathcal{M}_{12}^{S_2;A_1A_2A_3B_2B_3} + \eta^{S_1B_1} \mathcal{M}_{12}^{A_1A_2S_2A_3B_2B_3} + \eta^{A_1B_1} \mathcal{M}_{12}^{S_1;S_2A_1A_2A_3B_2B_3}) \\
- \frac{26}{63} \eta^{S_1A_1} \mathcal{M}_{12}^{S_2A_2A_3B_1B_2B_3} - \frac{8}{63} \eta^{S_1A_1} \mathcal{M}_{12}^{A_2S_2A_3B_1B_2B_3} - \frac{1}{7} \eta^{S_1S_2} \mathcal{M}_{12}^{A_1A_2A_3B_1B_2B_3} \right\} \\
+ \frac{2}{45} \eta^{S_1A_1} \eta^{S_2B_1} \mathcal{M}_{12}^{A_2A_3B_1B_2B_3} - \frac{32}{21 \times 15} \eta^{S_1A_1} \eta^{A_2B_1} \mathcal{M}_{12}^{B_2B_3A_3S_2} \\
+ \frac{12}{35} \eta^{S_1B_1} \eta^{A_1B_2} \mathcal{M}_{12}^{A_2A_3B_1B_2B_3} + \frac{8}{35} \eta^{A_1B_1} \eta^{A_2B_2} \mathcal{M}_{12}^{A_2A_3S_1S_2B_3} \\
- \frac{1}{35} \eta^{S_1S_2} \eta^{A_1B_1} \mathcal{M}_{12}^{A_2A_3B_1B_2B_3} \right\} \]

(4.4)

The second irreducible piece has 7 indices; we can extract a seven-index object by removing one of the factors from \(\mathcal{M}_{12}^{XY;C_1C_2} \) to obtain the structure

\[ \hat{\mathcal{M}}_{10}^{XYZ;A_1A_2B_1B_2} = M^{XED_1} M^{YD_1D_2} M^{ZD_2F} M^{A_1A_2} M^{F} B_1B_2. \]

(4.5)

It is clear that

\[ \hat{\mathcal{M}}_{10}^{[XYZ;A_1A_2]} B_1B_2 = 0 \quad \hat{\mathcal{M}}_{10}^{XY[Z;A_1A_2]} B_1B_2 = 0 \]

(4.5)

aside from the obvious antisymmetry in \(A_1, A_2\) and \(B_1, B_2\).

The object in (4.50) is not irreducible because it is not traceless. Its only non-vanishing traces are

\[ \hat{\mathcal{M}}_{10}^{YZ;A_1A_2 EB} = -\hat{\mathcal{M}}_{10}^{YZB;A_1A_2} \]

\[ \hat{\mathcal{M}}_{10}^{XY ;EAB_1B_2} = \hat{\mathcal{M}}_{10}^{YXA;B_1B_2} \]

\[ \hat{\mathcal{M}}_{10}^{YZ;AEB} = -\hat{\mathcal{M}}_{10}^{YZBA} \]
\[ N^{YXZBA} = M^{XD_1D_2}M^{YD_3D_4}M^{ZD_5D_6} \] (4.5)

For \( N^{XYZBA} \) we have
\[ N^{XYZBA} = N^{AXYB} \]
\[ N^{XYZBA} = -N^{ABZYX} \] (4.5)
as well as, by using (2.6) in the last two factors,
\[ N^{YXZAB} = N^{XYZBA} + \hat{M}^{XYZ;BA} \] (4.5)

Iterating (4.54) and using (4.44) one can derive the decomposition
\[ N^{YXZAB} = -\frac{1}{2} \hat{M}^{ZAB;XY} - \hat{M}^{BA(X,Y)Z} + \frac{1}{2} \hat{M}^{XYZ;BA} \] (4.5)
and therefore
\[ N^{[XY]ZAB} = -\frac{1}{2} \hat{M}^{ZAB;XY} \]
\[ N^{[XY]ZAB} = -\hat{M}^{AB[X;Y]Z} - \frac{1}{2} \hat{M}^{ABY;XZ} \] (4.5)
expressions that will be needed later.

In order to obtain the second irreducible piece of this \( \theta^{10} \) sector we can just project (4.51) according to the pattern \( \boxed{A} \). One obtains the structure
\[ M^{CD;A_1...A_5} = \hat{M}^{CA_1D_1;A_2...A_5} \]
\[ = M^{CEG}M^{A_1G_1}M^{DG_2}M^{A_2A_3}M^{F}M^{A_4A_5}, \] (4.5)
which is completely antisymmetric in \( A_1, ..., A_5 \) (we remind the reader of our letter convention) and by virtue of (4.51) it is also antisymmetric in \( C, D \):
\[ M^{CD;A_1...A_5} = -M^{DC;A_1...A_5} \] (4.5)
Its tracelessness is immediate from (3.11), (2.6), (2.3) and (4.3), and it also satisfies
\[ M^{C[D_1A_1...A_5]} = 0 \]
\[ M^{[C_1C_2;B_1...B_5]} = 0 \] (4.5)
and it is self-dual:
\[ M^{C_1C_2;B_1...B_5} = \frac{1}{5!} \epsilon^{B_1...B_5D_1...D_5} M^{C_1C_2;D_1...D_5} \] (4.6)

The list of decompositions is:
\[ M^{A_1A_2;B_1B_2E_1E_2E_3}M_{E_1E_2E_3} = \frac{2}{5} M^{A_1A_2;B_1B_2} \]
\( \mathcal{M}_{10}^{E_1 A_1 B_1 E_3 E_2 B_3 E_4 E_3 M_{E_1 E_2 E_3} = 0 } \)

\( \mathcal{M}_{10}^{E_1 A_2 B_2 ... B_4 E_2 M^C E_1 E_2 = \mathcal{M}_{12}^{[C_1 A_1 B_1 ... B_4} \]

\( \mathcal{M}_{10}^{E_1 A_2 B_2 ... B_3 E_1 E_2 M^C E_1 E_2 = 2 \mathcal{M}_{12}} \)

\( \mathcal{M}_{10}^{A_1 A_2 B_1 B_2 B_3 E_1 E_2 M^C E_1 E_2 = \mathcal{M}_{12}^{[C_1 A_1 B_1 B_2 B_3} + \frac{2}{45} \eta A_1 B_1 \mathcal{M}_{12}^{A_2 C_2 B_2 B_3} + \frac{7}{45} \eta \mathcal{M}_{12}^{B_1 A_2 B_2 B_3} \)

\( \mathcal{M}_{10}^{A_1 A_2 B_1 ... B_4 E M^C_{C_1 C_2 E} = \frac{1}{2} \left[ \frac{5}{11} \eta A_1 C_1 \left( \mathcal{M}_{12}^{A_2 C_2 B_1 ... B_4} - \mathcal{M}_{12}^{C_2 A_2 B_1 ... B_4} \right) \right. \]

\( \left. + \frac{2}{3} \eta A_1 B_1 \left( 7 \mathcal{M}_{12}^{C_1 C_2 A_2 B_2 B_3 B_4} + 5 \mathcal{M}_{12}^{A_2 C_2 B_2 B_3 B_4} \right) \right) + 2 \eta B_1 C_1 \left( 5 \mathcal{M}_{12}^{A_1 A_2 C_2 B_2 B_3 B_4} + 2 \mathcal{M}_{12}^{C_2 A_1 B_2 B_3 B_4} \right) \]

\( + \frac{1}{9 \times 25} \left( \eta A_1 B_1 \eta A_2 B_2 \mathcal{M}_{12}^{B_3 B_4 C_1 C_2} + 21 \eta B_1 C_1 \eta B_2 C_2 \mathcal{M}_{12}^{A_1 A_2 B_3 B_4} - 14 \eta A_1 B_1 \eta C_1 B_2 \mathcal{M}_{12}^{A_2 C_2 B_2 B_3 B_4} \right) \)

\( \mathcal{M}_{10}^{E_1 A_1 B_3 ... E_5 M^C_{C_1 C_2 E} = \frac{5}{11} \left[ \eta A_1 C_1 \mathcal{M}_{12}^{C_2 B_1 ... B_5} \right. \]

\( \left. + \frac{3}{2} \eta A_1 B_1 \mathcal{M}_{12}^{C_2 B_2 ... B_5} \right) + \frac{3}{2} \eta B_1 C_1 \mathcal{M}_{12}^{C_2 A_2 B_2 B_3 B_4} \]

\( \mathcal{M}_{10}^{A_1 A_2 B_1 ... B_5 M^C_{C_1 C_2 C_3} = \)

\( - \frac{1}{6!} \left( \frac{1}{10} \epsilon B_1 ... B_5 A_2 C_2 C_1 E_1 E_2 \mathcal{M}_{12}^{C_1 C_2 E_1 E_2} \right) + \frac{1}{2 \epsilon B_1 ... B_5 C_2 C_1 E_1 E_2} \mathcal{M}_{12}^{A_1 A_2 E_1 E_2} \]

\( - \frac{3}{5} \epsilon B_1 ... B_5 A_2 C_2 C_3 E_1 E_2 \mathcal{M}_{12}^{A_1 C_1 E_1 E_2} \)

\( + \frac{1}{11} \left[ 2 \eta A_1 C_1 \eta A_2 C_2 \mathcal{M}_{12}^{C_3 B_1 ... B_5} + 5 \eta A_1 B_1 \eta A_2 B_2 \mathcal{M}_{12}^{C_1 C_2 B_3 B_4 B_5} \right. \]

\( - \frac{15}{2} \eta A_1 B_1 \eta A_2 C_1 \mathcal{M}_{12}^{C_2 C_3 B_2 ... B_5} - 15 \eta B_1 A_1 \eta B_2 C_1 \left( \mathcal{M}_{12}^{A_2 C_2 C_3 B_3 B_4 B_5} + \mathcal{M}_{12}^{C_2 C_3 A_2 B_3 B_4 B_5} \right) \]

\( + 5 \eta B_1 C_1 \eta B_2 C_1 \left( 3 \mathcal{M}_{12}^{A_1 A_2 C_3 B_3 B_4 B_5} + \mathcal{M}_{12}^{C_3 C_1 A_2 B_3 B_4 B_5} \right) \]

\( + \frac{5}{4} \eta C_1 A_1 \eta C_2 B_1 \left( 5 \mathcal{M}_{12}^{A_2 B_2 B_3 B_4 B_5} - 9 \mathcal{M}_{12}^{A_2 C_3 B_3 B_4 B_5} \right) \]

\( + \frac{1}{12} \eta B_1 C_1 \eta B_2 C_2 \eta B_3 C_3 \mathcal{M}_{12}^{A_1 A_2 B_2 B_4 B_5} + \frac{1}{10} \eta B_1 A_1 \eta B_2 C_1 \eta B_3 C_2 \mathcal{M}_{12}^{A_2 C_3 B_3 B_4 B_5} \]

\( + \frac{1}{60} \eta B_1 A_1 \eta B_2 A_2 \eta B_3 C_1 \mathcal{M}_{12}^{A_2 C_3 B_1 B_4 B_5} \)
At the beginning of this section we introduced one of the irreducible parts of the $\theta^8$ sector, namely the totally symmetric tensor in (4.1):

$$\mathcal{M}_8^{S_1 S_2 S_3 S_4} = M^{S_1 E} M^{S_2 F} M^{S_3 G} M^{S_4 H}.$$  \hspace{1cm} (4.61)

Its products with $M^{A_1 A_2 A_3}$ are particularly easy to decompose using (4.43):

$$\mathcal{M}_8^{S_1 S_2 S_3 E} M^{A_1 A_2 A_3} = \hat{\mathcal{M}}_8^{S_1 S_2 S_3}.$$

This sector contains two additional irreducible pieces (see Table 1). In order to isolate them, first we remove one factor from $\hat{\mathcal{M}}_8^{S_1 S_2 S_3 A_1 A_2}$ to get the structure

$$\hat{\mathcal{M}}_8^{XY A_1 A_2 B_1 B_2} = M^{XED} M^Y D^F M^{A_1 A_2} E M^{B_1 B_2 F}.$$  \hspace{1cm} (4.62)

with the following properties

$$\hat{\mathcal{M}}_8^{XY A_1 A_2 B_1 B_2} = \hat{\mathcal{M}}_8^{XY B_1 B_2 A_1 A_2}$$

$$\hat{\mathcal{M}}_8^{[XY] A_1 A_2} B_1 B_2 = 0 \quad \hat{\mathcal{M}}_8^{[XY] A_1 A_2 B_1 B_2} = 0.$$  \hspace{1cm} (4.63)

It is reducible,

$$\hat{\mathcal{M}}_8^{XY EA B} = -\hat{\mathcal{M}}_8^{XY AB}.$$  \hspace{1cm} (4.64)

but easy to detrace:

$$\hat{\mathcal{M}}_8^{XY A_1 A_2 B_1 B_2} = Traceless(\hat{\mathcal{M}}_8^{XY A_1 A_2 B_1 B_2}) - \frac{1}{2} \eta^{A_1 B_1} \mathcal{M}_8^{XY A_2 B_2}.$$  \hspace{1cm} (4.65)

The traceless part is going to contain 2 irreducible pieces corresponding to the patterns \(\square\) and \(\square\). First,

$$Y\left(\begin{array}{c}
\square
\end{array}\right) \hat{\mathcal{M}}_8^{XY A_1 A_2 B_1 B_2} = \hat{\mathcal{M}}_8^{XY [A_1 A_2] B_1 B_2} + 2 \hat{\mathcal{M}}_8^{A_1 B_1 [X A_2 Y B_2]}.$$  \hspace{1cm} (4.66)

and thus the \(\square\) irreducible structure is

$$\mathcal{M}_8^{XY; B_1 B_2 B_3 B_4} = \hat{\mathcal{M}}_8^{XY; B_1 B_2 B_3 B_4}.$$  \hspace{1cm} (4.67)

which is completely antisymmetric in $B_1, ..., B_4$ and, by (4.65), symmetric in $X, Y$:

$$\mathcal{M}_8^{XY; B_1 B_2 B_3 B_4} = \mathcal{M}_8^{XY; B_1 B_2 B_3 B_4}.$$  \hspace{1cm} (4.68)
and satisfying

\[ \mathcal{M}_8^{X;B_1B_2B_3B_4} = 0. \]  

(4.7)

Second,

\[
Y \left( \hat{\mathcal{M}} \right) \mathcal{M}_8^{XY, A_1A_2B_1B_2} = -\frac{1}{2} \left( \hat{\mathcal{M}}_8^{[X[A_1B_1B_2]YA_2]} + \hat{\mathcal{M}}_8^{[X[A_1YA_2]B_1B_2]} \right.
+ \hat{\mathcal{M}}_8^{[X[A_1B_1A_2]YB_2]} + \hat{\mathcal{M}}_8^{[X[A_1B_1A_2]YB_2]} \right) \]  

(4.7)

giving as the irreducible structure the object

\[ \mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} = \hat{\mathcal{M}}_8^{A_1B_1B_2B_3A_2A_3} \]  

(4.7)

Of course it is completely antisymmetric in the \( A \) and \( B \) indices separately and, from (4.65), we see that it is symmetric upon interchange of both groups of indices

\[ \mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} = \mathcal{M}_8^{B_1B_2B_3;A_1A_2A_3} \]  

(4.7)

The remaining important property of this tensor can be derived from the definitions (4.73), (4.64) by using once more the properties of the \( \theta^4 \) sector,

\[ \mathcal{M}_8^{A_1A_2[C;B_1B_2B_3]} = 0 \]  

(4.7)

that implies also

\[ \mathcal{M}_8^{A[B_1B_2;B_3C]} = 0 \]  

(4.7)

By using the properties in (4.72) we can write finally for the decomposition in (4.67):

\[
\hat{\mathcal{M}}_8^{XY, A_1A_2B_1B_2} = \mathcal{M}_8^{XY;A_1A_2B_1B_2} + 2\mathcal{M}_8^{A_1B_1;XA_2YB_2}
+ \frac{3}{8} \left( 3\mathcal{M}_8^{Y[A_1A_2;XB_1B_2} - \mathcal{M}_8^{X[A_1A_2;YB_1B_2]} \right)
- \frac{1}{2} \eta^{A_1B_1} \mathcal{M}_8^{XYA_2B_2} \]  

(4.7)

The lists of products decompositions for these irreducible pieces are

\[ \mathcal{M}_{10}^{A_1A_2;B_1B_2E_1E_2E_3} M_{E_1E_2E_3} = \frac{2}{5} \mathcal{M}_{12}^{A_1A_2;B_1B_2} \]

\[ \mathcal{M}_8^{S_1S_2;BE_1E_2E_3} M_{E_1E_2E_3} = 0 \]

\[ \mathcal{M}_8^{SE_1;B_1B_2E_2E_3} M_{E_1E_2E_3} = 0 \]

\[ \mathcal{M}_8^{SE_1;B_1B_2B_3E_2} M_{E_1E_2E_3} = 0 \]

\[ \mathcal{M}_8^{SE_1;B_1B_2B_3} M_{E_1E_2} = -\frac{1}{2} \mathcal{M}_{10}^{SC;B_1B_2B_3} \]

\[ \mathcal{M}_8^{SE_1;B_1B_2B_3} M_{E_1E_2} = 0 \]
\begin{align*}
\mathcal{M}_{8}^{S_{1}S_{2};B_{1}B_{2}E_{1}E_{2}}M^{C}E_{1}E_{2} &= \frac{1}{3} \left( 2\mathcal{M}_{10}^{S_{1}S_{2}B_{1};B_{2}C} - \mathcal{M}_{10}^{S_{1}S_{2}C;B_{1}B_{2}} \right) \\
&= \frac{1}{5} \left( 3\mathcal{M}_{10}^{S_{1}S_{2};B_{1}B_{2}C} - \mathcal{M}^{CS_{1};S_{2}B_{1}B_{2}} \right)
\end{align*}

\begin{align*}
\mathcal{M}_{8}^{S_{1}S_{2};B_{1}B_{2}B_{3}E}MC_{1}C_{2} &= \frac{10}{9} \mathcal{M}_{10}^{S_{1}C_{1};S_{2}C_{2}B_{1}B_{2}B_{3}} \\
&\quad + \frac{1}{6} \eta_{B_{1}C_{1}} \left( S_{1}C_{1} \mathcal{M}_{10}^{S_{2}C_{2}B_{1};B_{2}B_{3}} + \eta_{B_{1}B_{2}} \mathcal{M}_{10}^{S_{1}C_{2}B_{1};B_{2}B_{3}} \right) \\
&\quad + \frac{1}{4} \eta_{B_{1}C_{1}} \left( - \frac{7}{3} \mathcal{M}_{10}^{S_{1}S_{2}B_{1};B_{2}C_{2}} + \mathcal{M}_{10}^{S_{1}S_{2}B_{1};C_{2}B_{3}} \right) \\
&= \frac{10}{9} \mathcal{M}_{10}^{S_{1}C_{1};S_{2}C_{2}B_{1}B_{2}B_{3}} \\
&\quad + \frac{1}{6} \eta_{S_{1}C_{1}} \mathcal{M}_{10}^{S_{2}C_{2}B_{1};B_{2}B_{3}} + 1 \frac{1}{4} \eta_{S_{1}B_{1}} \mathcal{M}_{10}^{S_{1}C_{2}B_{1};B_{2}B_{3}} \\
&\quad + \frac{1}{20} \eta_{B_{1}C_{1}} \left( - 11 \mathcal{M}_{10}^{S_{1}S_{2}B_{1};B_{2}B_{3}C_{2}} + 13 \mathcal{M}_{10}^{S_{1}S_{2}C_{2};B_{1}B_{2}B_{3}} \right)
\end{align*}

\begin{align*}
\mathcal{M}_{8}^{S_{1}S_{2};B_{1}...B_{4}}MC_{1}C_{2} &= \frac{5}{9} \left( \mathcal{M}_{10}^{C_{1}C_{2};S_{1}B_{1}...B_{4}} + 5 \mathcal{M}_{10}^{S_{1}C_{1};C_{2}B_{1}...B_{4}} \right) \\
&\quad + 2 \frac{3}{3} \eta_{B_{1}C_{1}} \mathcal{M}_{10}^{S_{2}C_{2};B_{2}B_{3}B_{4}}
\end{align*}

\begin{align*}
\mathcal{M}_{8}^{S_{1}S_{2};B_{1}...B_{4}MC_{1}C_{2}C_{3} &= - \frac{2}{3 \times 5!} \eta_{B_{1}...B_{4}}^{C_{1}C_{2}C_{3}E_{1}E_{2}E_{3}} \mathcal{M}_{10}^{S_{1}S_{2};E_{1}E_{2}E_{3}} \\
&\quad + \frac{4}{21} \left[ \eta_{S_{1}C_{1}} \left( 10 \mathcal{M}_{10}^{S_{2}C_{2};C_{3}B_{1}...B_{4}} + 2 \mathcal{M}_{10}^{C_{2}C_{3};S_{2}B_{1}...B_{4}} \right) \\
&\quad - \eta_{S_{1}B_{1}} \left( 6 \mathcal{M}_{10}^{S_{2}C_{1};C_{2}C_{3}B_{2}B_{4} + C_{1}C_{2}B_{2}B_{3}} + \mathcal{M}_{10}^{C_{1}C_{2};C_{2}B_{3}B_{4}} \right) \\
&\quad - 8 \eta_{B_{1}C_{1}} \mathcal{M}_{10}^{S_{2}C_{2}C_{3}B_{2}B_{3}B_{4} - \eta_{S_{1}C_{2}} \mathcal{M}_{10}^{C_{1}C_{2};C_{2}B_{1}...B_{4}} \right] \\
&\quad + \frac{1}{3} \left[ 2 \eta_{S_{1}C_{1}} \eta_{B_{1}C_{2}} \mathcal{M}_{10}^{S_{2}C_{3}B_{1}B_{2}B_{3}} + 3 \eta_{S_{1}B_{1}} \eta_{C_{1}B_{2}} \mathcal{M}_{10}^{S_{2}C_{3}B_{3}B_{4}} \\
&\quad + 3 \eta_{B_{1}C_{1}} \eta_{B_{2}C_{2}} \left( \mathcal{M}_{10}^{S_{1}S_{2};C_{3}B_{2}B_{4} - \mathcal{M}_{10}^{S_{1}C_{3};C_{2}B_{3}B_{4}} \right) \right] \quad (4.7)
\end{align*}

and

\begin{align*}
\mathcal{M}_{8}^{A_{1}A_{2}A_{3};E_{1}E_{2}E_{3}}M_{E_{1}E_{2}E_{3}} &= 0 \quad \mathcal{M}_{8}^{A_{1}A_{2}E_{1};B_{2}E_{2}E_{3}}M_{E_{1}E_{2}E_{3}} = 0 \\
\mathcal{M}_{8}^{A_{1}A_{2}A_{3};B_{1}E_{1}E_{2}}M_{C}E_{1}E_{2} &= \frac{2}{3} \mathcal{M}_{10}^{B_{C};A_{1}A_{2}A_{3}} \quad \mathcal{M}_{8}^{A_{1}A_{2}E_{1};B_{1}E_{2}E_{3}}M_{C}E_{1}E_{2} = - \frac{2}{3} \mathcal{M}_{10}^{C_{A_{1}};A_{2}B_{1}B_{2}}
\end{align*}
\[ M_8^{A_1A_2A_3;B_1B_2E}M^{C_1C_2}_{\ E} = \]
\[ -\frac{8}{9} \left( \frac{2}{3} M_{10}^{B_1B_2;A_1A_2A_3C_1C_2} + M_{10}^{A_1A_2;A_3B_1B_2C_1C_2} - \frac{1}{6} M_{10}^{C_1C_2;A_1A_2A_3B_1B_2} \right) \]
\[ + \frac{1}{15} \left[ \frac{7}{3} \eta B_1C_1 M_{10}^{B_2C_2;A_1A_2A_3} + \frac{7}{2} \eta A_1C_1 M_{10}^{C_2A_2;A_3B_1B_2} - \frac{3}{2} \eta A_1B_1 M_{10}^{A_2B_2;A_3C_1C_2} \right] \]

\[ M_8^{A_1A_2A_3;B_1B_2B_3}M^{C_1C_2C_3} = \]
\[ + \frac{4}{15} \left\{ \eta B_3C_3 \left( -4 M_{10}^{B_1B_2;A_1A_2A_3C_1C_2} + M_{10}^{C_1C_2;A_1A_2A_3B_1B_2} - 6 M_{10}^{A_1A_2;A_3B_1B_2C_1C_2} \right) \right. \]
\[ + \eta A_3C_3 \left( -4 M_{10}^{A_1A_2;B_1B_2B_3C_1C_2} + M_{10}^{C_1C_2;B_1B_2B_3A_1B_2} - 6 M_{10}^{B_1B_2;B_3A_1A_2C_1C_2} \right) \]
\[ + \left. 3 \eta A_3B_3 \left( M_{10}^{A_1A_2;B_1B_2B_3C_1C_2} + M_{10}^{B_1B_2;A_1A_2C_1C_2C_3} - M_{10}^{C_1C_2;A_1A_2B_1B_2} \right) \right\} \]
\[ + \frac{4}{15} \eta B_1C_1 \eta B_2C_2 M_{10}^{B_3C_3;A_1A_2A_3} + \eta A_1C_1 \eta A_2C_2 M_{10}^{A_3C_3;B_1B_2B_3} \]
\[ - \frac{3}{2} \eta B_1C_1 \eta B_2A_1 M_{10}^{B_3A_2A_3C_2C_3} - \frac{3}{2} \eta A_1C_1 \eta A_2B_1 M_{10}^{A_3B_2B_3C_2C_3} \]
\[ - 3 \eta A_1C_1 \eta B_1C_2 M_{10}^{C_3A_2A_3B_2B_3} + \frac{1}{7} \eta A_1B_1 \eta A_2B_2 M_{10}^{A_3B_3;C_1C_2C_3} \]  \( (4.7) \)

\( \psi_6 \).

In the decomposition (3.10) of the product of three \( M^{A_1A_2A_3} \) we have two types of irreducible structures:

\[ \hat{M}_6^{A_1B_1B_2C_1C_2} = M^{ADE} M^{B_1B_2 D} M^{C_1C_2}_{\ E} \]  \( (4.8) \)

and

\[ M_6^{A_1A_2;B_1\ldots B_5} = M^{A_1A_2} E M^{B_1B_2} M^{B_3B_4B_5} \]  \( (4.8) \)

The expression (4.80) trivially satisfies

\[ \hat{M}_6^{A_1B_1B_2C_1C_2} = -\hat{M}_6^{A_1C_1C_2 B_1B_2} \]  \( (4.8) \)

and (2.6) implies

\[ \hat{M}_6^{[A_1B_1B_2]C_1C_2} = 0 \]  \( (4.8) \)

The tensor \( \hat{M}_6^{[A_1B_1B_2]C_1C_2} \) must belong to the representation \( \overline{\rho} \) and in order to make the corresponding Young symmetry obvious, we define the new tensor

\[ \hat{M}_6^{XY;B_1B_2B_3} = \hat{M}_6^{XYB_1B_2B_3} \]  \( (4.8) \)

Both tensors are completely equivalent though, the inverse of (4.84) being

\[ \hat{M}_6^{A_1B_1B_2C_1C_2} = 3 \hat{M}_6^{A_1B_1B_2C_1C_2} \]  \( (4.8) \)
It is easy to see that $\mathcal{M}_{6}^{XY:B_1B_2B_3}$ must be symmetric in $X, Y$:

$$
\mathcal{M}_{6}^{[XY]:B_1B_2B_3} = M^{DE[X} M^{Y]B_1} D M^{B_2B_3}_E = -\frac{1}{2} M^{XY} D M^{DE[B_1} M^{B_2B_3]}_E = 0
$$

(4.89)

where we have used (2.3) twice. Thus

$$
\mathcal{M}_{6}^{XY:B_1B_2B_3} = \mathcal{M}_{6}^{YX:B_1B_2B_3}
$$

(4.89)

The remaining important property of this tensor is

$$
\mathcal{M}_{6}^{[XY]:B_1B_2B_3} = 0
$$

(4.89)

as we have come to expect and can be immediately seen from (4.84) and (4.80). This time we have the following product decompositions:

$$
\begin{align*}
\mathcal{M}_{6}^{S_1S_2;E_1E_2E_3} &{} M_{E_1E_2E_3} = 0 &{} \mathcal{M}_{6}^{S_E;B_1E_2E_3} &{} M_{E_1E_2E_3} = 0 \\
\mathcal{M}_{6}^{S_E;B_1B_2E_2} &{} M_{C_1C_2E} = 0 &{} \mathcal{M}_{6}^{S_E;B_1B_2E_2} &{} M_{C_1E} = -\frac{2}{3} \mathcal{M}_{8}^{S_1S_2BC} \\
\mathcal{M}_{6}^{S_1S_2;B_1B_2E} &{} M_{C_1C_2} = \frac{4}{3} \mathcal{M}_{8}^{S_1C_1C_2S_2B_1B_2} + \frac{2}{3} \mathcal{M}_{8}^{S_1S_2B_1B_2C_1C_2} \\
&{} - \frac{1}{2} \mathcal{M}_{8}^{B_1B_2C_1C_2S_2} + \frac{1}{3} \mathcal{M}_{8}^{S_1S_2B_1C_2C_2B_2} \\
\mathcal{M}_{6}^{S_E;B_1B_2B_3} &{} M_{C_1C_2} = \frac{4}{3} \mathcal{M}_{8}^{S_1C_1C_2B_1B_2B_3} - \mathcal{M}_{8}^{S_1C_1C_2B_1B_2B_3} \\
\mathcal{M}_{6}^{S_1S_2;B_1B_2B_3} &{} M_{B_4B_5B_6} = \frac{1}{2 \times 5!} \epsilon^{B_1...B_6E_1...E_4} \mathcal{M}_{8}^{S_1S_2;E_1...E_4}
\end{align*}
$$

$$
\begin{align*}
\mathcal{M}_{6}^{S_1S_2;B_1B_2B_3} &{} M_{C_1C_2C_3} = \\
&{} \frac{3}{8 \times 5!} \left( \epsilon_{B_1B_2B_3C_1C_2C_3E_1...E_4} \mathcal{M}_{8}^{S_1S_2;E_1...E_4} + 2 \epsilon_{B_1B_2C_1C_2C_3E_1...E_4} \mathcal{M}_{8}^{S_2B_3;E_1...E_4} \right) \\
&{} + 9\eta_{B_1C_1} \left( -\frac{1}{5} \mathcal{M}_{8}^{S_1C_1C_2C_2B_3B_3} + \mathcal{M}_{8}^{S_1S_2B_2B_3C_1C_2} \right) \\
&{} + \frac{9}{10} \eta_{B_1C_1} \mathcal{M}_{8}^{S_1S_2C_1C_2B_3B_3} + \frac{1}{2} \eta_{B_1C_1} \mathcal{M}_{8}^{S_1B_2B_3B_3C_1C_2} \\
&{} + \frac{3}{56} \left[ -9\eta_{B_1C_1} \mathcal{M}_{8}^{S_1B_2B_3;C_1C_2C_3} - 12\eta_{B_1C_1} \mathcal{M}_{8}^{S_1C_1C_2C_3B_1B_2B_3} \\
&{} + 2\eta_{B_1C_1} \mathcal{M}_{8}^{S_1B_2B_3C_1C_2C_3} + \eta_{B_1C_1} \mathcal{M}_{8}^{S_1B_2B_3C_1C_2C_3} \right] \\
&{} - \frac{3}{14} \eta_{B_1C_1} \mathcal{M}_{8}^{S_1S_2B_2B_3C_3}
\end{align*}
$$

(4.89)
Turning our attention to (4.81), we get the duality property

$$\mathcal{M}_{6}^{A_1A_2;B_1\ldots B_5} = -\frac{1}{5!} \epsilon^{B_1\ldots B_5 D_1\ldots D_5} \mathcal{M}_{6}^{A_1A_2;D_1\ldots D_5}$$

(4.9)

as a direct consequence of the one for $\mathcal{M}_{4}^{C;D_1\ldots D_5}$ (eq. (3.8)). The following bracket property is also immediate

$$\mathcal{M}_{6}^{A[B_1\ldots B_5]} = 0$$

(4.9)

Finally, to complete this section we have the following list of decompositions:

$$\mathcal{M}_{6}^{A_1A_2;B_1B_2E_1E_2E_3} M_{E_1E_2E_3} = 0 \quad \mathcal{M}_{6}^{A_1E_1;B_1B_2B_3E_2E_3} M_{E_1E_2E_3} = 0$$

$$\mathcal{M}_{6}^{E_1E_2;B_1\ldots B_5} M_{E_1E_2} = 0$$

$$\mathcal{M}_{6}^{A_1;B_1B_2E_1E_2} M_{E_1E_2} = -\frac{3}{5} \mathcal{M}_{6}^{A;B_1\ldots B_4}$$

$$\mathcal{M}_{6}^{A_1A_2;B_1B_2B_3E_1E_2} M_{E_1E_2} = \frac{1}{5} \left( 4 \mathcal{M}_{8}^{CA_1A_2;B_1B_2B_3} - 3 \mathcal{M}_{8}^{CA_1A_2;B_1B_2B_3} \right)$$

$$\mathcal{M}_{6}^{B_1B_2;AB_3B_4B_5} M_{CB_6}^{E_1} = \frac{1}{20 \times 5!} \epsilon^{B_1\ldots B_5 E_1\ldots E_4} \mathcal{M}_{8}^{CA_1;E_1\ldots E_4}$$

$$\mathcal{M}_{6}^{AB_1;B_2B_3} M_{CB_6}^{E_1} = -2 \mathcal{M}_{6}^{B_1B_2;AB_3B_4B_5} M_{CB_6}^{E_1}$$

$$\mathcal{M}_{6}^{A;E_1\ldots B_5} M_{C_1C_2}^{E_1E_2} = \frac{6}{5} \left( \frac{1}{5!} \epsilon^{B_1\ldots B_5 C_1E_1\ldots E_4} \mathcal{M}_{8}^{AC_2;E_1\ldots E_4} - \eta B_3C_1 \mathcal{M}_{8}^{AC_2;B_2\ldots B_5} \right)$$

$$\mathcal{M}_{6}^{A_1A_2;B_1\ldots B_4E} M_{C_1C_2}^{E_1E_2} = -\frac{3}{5 \times 5!} \epsilon^{B_1\ldots B_4 A_1C_1E_1\ldots E_4} \mathcal{M}_{8}^{A_2C_2;E_1\ldots E_4}$$

$$+ \frac{24}{25} \left[ \eta B_1C_1 \mathcal{M}_{8}^{C_2A_1A_2B_3B_4} - \frac{1}{4} \eta A_1C_1 \mathcal{M}_{8}^{A_2C_1A_2B_3B_4} - \frac{1}{3} \eta A_1B_3 \mathcal{M}_{8}^{A_2C_1A_2B_3B_4} \right]$$

$$- \frac{3}{10} \left( 3 \eta B_1C_1 \mathcal{M}_{8}^{C_2A_1A_2B_3B_4} + \eta A_1B_1 \mathcal{M}_{8}^{A_2C_1A_2B_3B_4} \right)$$

$$\mathcal{M}_{6}^{C_1C_2;B_1\ldots B_5} M_{C_3C_4}^{B_7} = \frac{2}{7 \times 5!} \epsilon^{B_1\ldots B_7 E_1E_2E_3} \mathcal{M}_{8}^{C_1C_2C_3;E_1E_2E_3}$$
\[ M^{A_1 A_2; B_1 \ldots B_5}_{6} M^{C_1 C_2 C_3}_{\cdots} = \]
\[ = \frac{1}{32 \times 35} \left( \epsilon^{B_1 \ldots B_5 A_1 A_2 E_1 E_2 E_3} M^{C_1 C_2 C_3}_{8; E_1 E_2 E_3} + 15 \epsilon^{B_1 \ldots B_5 C_1 C_2 E_1 E_2 E_3} M^{A_1 A_2 C_3}_{8; E_1 E_2 E_3} \right) \]
\[ - 12 \epsilon^{B_1 \ldots B_5 A_1 C_1 E_1 E_2 E_3} M^{A_2 C_3}_{8; E_1 E_2 E_3} \]
\[ - \frac{1}{16 \times 5} \left( \frac{4}{5} \eta^{A_1 C_1} \epsilon^{B_1 \ldots B_5 C_1 E_1 \ldots E_4} M^{A_2 C_3}_{8; E_1 \ldots E_4} + \eta^{B_1 C_1} \epsilon^{B_2 \ldots B_5 A_1 C_2 E_1 \ldots E_4} M^{A_2 C_3}_{8; E_1 \ldots E_4} \right) \]
\[ - \eta^{A_1 B_1} \epsilon^{B_2 \ldots B_5 C_1 C_2 E_1 \ldots E_4} M^{A_2 C_3}_{8; E_1 \ldots E_4} \]
\[ + \frac{6}{5} \left( \eta^{B_1 C_1} \eta^{B_2 C_2} M^{C_3 A_1; A_2 B_3 B_4 B_5}_{8} + \frac{3}{4} \eta^{A_1 C_1} \eta^{B_1 C_2} M^{C_3 A_2; B_2 \ldots B_5}_{8} \right) \]
\[ + \frac{1}{4} \eta^{B_1 A_1} \eta^{B_2 C_1} M^{A_2 C_3; C_1 B_3 B_4 B_5}_{8} \]
\[ - \frac{3}{7} \left( \frac{15}{4} \eta^{B_1 C_1} \eta^{B_2 C_2} M^{C_3 A_1; A_2 B_3 B_4 B_5}_{8} - 3 \eta^{B_1 A_1} \eta^{B_2 C_1} M^{A_2 C_3; B_3 B_4 B_5}_{8} \right) \]
\[ + \frac{1}{8} \eta^{A_1 B_1} \eta^{A_2 B_2} M^{B_3 B_4 B_5; C_1 C_2 C_3}_{8} \right) \]
V \ \theta^3\text{-Fierz Identity and } \Gamma\text{-tracelessness.}

The basic Fierz identity does not need to have four \( \theta \)'s but only three. Thus, (2.1) can be derived from

\[ \theta(\pm)\bar{\theta}(\pm)O\theta(\pm) = \frac{1}{96}\Pi(\pm)\Gamma B_1B_2B_3\bar{\theta}(\pm)\Gamma B_1B_2B_3\theta(\pm) \]  \hspace{1cm} (5.1)

An immediate consequence of (5.1) is

\[ \Gamma B_1B_2B_3\theta(\pm)\bar{\theta}(\pm) = 0 \]  \hspace{1cm} (5.2)

and using (5.1) and (5.2) one easily obtains

\[ \Gamma B_1B_2\theta(\pm)\bar{\theta}(\pm) = \frac{1}{2}\Theta A_1A_2A_3\theta(\pm) \]  \hspace{1cm} (5.3)

Then one can finally Fierz the general uncontracted product to obtain

\[ \theta(\pm)\bar{\theta}(\pm) = \frac{1}{2}\Theta A_1A_2A_3\theta(\pm) \]  \hspace{1cm} (5.4)

after using (5.1-5.3) and the properties of the Dirac algebra. Eq. (5.4) gives us the decomposition of the product \( \mathcal{M}^{A_1A_2A_3}\theta \) into irreducible pieces, and we see that the \( \theta^3 \) irreducible spinor-tensor corresponding to \( \begin{pmatrix} 3 & 3 & 1 \ \ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \) is

\[ \Theta_{3}^{A_1A_2} = \Gamma E M^{A_1A_2E}\theta \]  \hspace{1cm} (5.5)

which is obviously traceless and by (5.3) also \( \Gamma \)-traceless. Thus, (5.4) means

\[ M^{A_1A_2A_3}\theta = \frac{1}{2}\Gamma A_1\Theta_{3}^{A_2A_3} \]  \hspace{1cm} (5.6)

Of course, this decomposition can be obtained easily by detracing and Young-projecting,

\[ M^{A_1A_2A_3}\theta = \text{Traceless}(M^{A_1A_2A_3}\theta) + a\Gamma [A_1] \Gamma E M^{A_2A_3]}E\theta \]  \hspace{1cm} (5.7)

where “Traceless” now means both \( \eta \)- and \( \Gamma \)-traceless and there are no \( \eta \) terms on the r.h.s. because the l.h.s. is trivially \( \eta \)-traceless. But the Traceless term in (5.7) vanishes because there are no irreducible objects with 3 tensor indices in the \( \theta^3 \) sector. The constant \( a \) is easily determined by contracting (5.7) with \( \Gamma_{A_1} \), to get \( a = \frac{1}{2} \) and therefore reobtaining (5.6). The fermionic version of the Young-projector mentioned in the previous paragraph is straightforward enough, but it can become quite complicated for higher order decompositions. In order to simplify things, the general way to proceed is as follows. First, we figure out the irreducible objects by contracting as many indices as possible in the product \( \mathcal{M}^{A_1A_2A_3}\Theta_n \) so that the number of remaining tensor indices are equal to the number of boxes of the corresponding Young-pattern, and then we apply the Young-projector to the resulting object. Next, we decompose the \( \mathcal{M}_{n+1}\theta \) products in terms of those irreducible pieces instead of decomposing \( M^{A_1A_2A_3}\Theta_n \) since the former is much easier than the latter in general. Finally, we may use the results of the bosonic decompositions to obtain the decomposition of \( M^{A_1A_2A_3}\Theta_n \), since every fermionic irreducible object \( \Theta_n \) is expressed as some \( \Gamma \)-contraction of \( \mathcal{M}_{n-1}\theta \). The procedure will be illustrated in the first few examples of the next section.
VI Irreducible Spinor-Tensors.

Unlike in the bosonic case, this time we will proceed forward.

\[ \Theta^{A_1 \ldots A_5} = M^{A_1 A_2 A_3} \Theta_3^{A_4 A_5} = M^{A_1 A_2 A_3} M^{A_4 A_5} C \Gamma C \theta = \Gamma_C M_4^{C : A_1 \ldots A_5} \theta \]

Evidently it is traceless, but it is also \( \Gamma \)-traceless:

\[ \Gamma_D \Theta_5^{D A_1 \ldots A_4} = \frac{1}{5} \Gamma_D \left( 3 M^{D A_1 A_2} M^{A_3 A_4 C} + 2 M^{A_1 A_2 A_3} M^{A_4 D C} \right) \Gamma C \theta = \frac{3}{5} M^{D A_1 A_2} M^{A_3 A_4 C} \Gamma D \theta = 0 \]

where we have used (5.3) as well as (2.6). The anti-selfduality

\[ \Theta^{A_1 \ldots A_5} = -\frac{1}{5!} \epsilon^{A_1 \ldots A_5 B_1 \ldots B_5} \Theta_5^{B_1 \ldots B_5} \]

together with (6.2) imply the property

\[ \Gamma^{[B} \Theta_5^{A_1 \ldots A_5]} = 0 \]

The second irreducible \( \theta^5 \) piece is:

\[ \Theta_5^{A ; B_1 B_2} = M^{B_1 B_2 E} \Theta_3^{A E} = M^{B_1 B_2 E} M^{A E D} \Gamma_D \theta = \mathcal{M}_4^{D A ; B_1 B_2} \Gamma_D \theta \]

Usual tracelessness is also obvious here, while

\[ \Gamma_D \Theta_5^{D ; B_1 B_2} = 0 \]

follows again from (5.3). The other \( \Gamma \)-trace also vanishes:

\[ \Gamma_D \Theta_5^{A ; D B} = \Gamma_D M^{D B E} M^{A E F} \Gamma_F \theta = M^{D B E} M^{A E F} \Gamma_D \theta = 0 \]

where we used our old friend (2.6) and (5.3) once more. Lastly, a property inherited from \( \mathcal{M}_4^{A_1 A_2 ; B_1 B_2} \) is

\[ \Theta_5^{[A ; B_1 B_2]} = 0 \]
Next we proceed to decompose products. By detracing one readily arrives at

\[ M_{A_1 A_2; B_1 B_2} = \frac{1}{5} \left[ \Gamma A_1 \Theta_{A_2; B_1 B_2} + \Gamma B_1 \Theta_{B_2; A_1 A_2} \right] \quad (6.9) \]

\[ M_{A B_1 \ldots B_5} = \frac{1}{10} \left( \Gamma A B_1 \ldots B_5 + \Gamma B_1 B_2 B_3 \Theta_{A; B_4 B_5} \right) \quad (6.10) \]

With (6.9), (6.10) and (3.5) one can write the more general product

\[ M^{A_1 A_2 A_3 \ldots B_1 \ldots B_5} = \frac{1}{2} \Gamma A_1 \Theta_{A_2; A_3 B_1 B_2} + \frac{3}{20} \left[ \Gamma A_1 \Gamma B_1 B_2 \Theta_{B_3; A_2 A_3} + \Gamma B_1 \Gamma A_1 A_2 \Theta_{A_3 B_1 B_2} \right] \quad (6.11) \]

from which in turn we get

\[ M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} = \frac{1}{2} \Gamma A_1 \Theta_{A_2; A_3 B_1 B_2} + \frac{3}{10} \Gamma A_1 \Gamma B_1 B_2 \Theta_{B_3; A_2 A_3} + \frac{3}{10} \Gamma A_1 A_2 \Theta_{A_3 B_1 B_2} - \frac{6}{10} \eta A_1 B_1 B_2 \Theta_{A_3; A_2 B_3} \quad (6.12) \]

\[ \hat{\Theta}^7 \]

For the representation \( \left[ \begin{array}{c} 5 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{array} \right] \) we need an object with 4 tensor indices, so consider

\[ \hat{\Theta}_7^{A_1 A_2; B_1 B_2} = M^{A_1 A_2} E \Theta_{E; B_1 B_2} = M^{A_1 A_2} D M^{B_1 B_2 E} = \Gamma C M^{A_1 A_2} E M_{A_3 B_3} = 3 \Gamma C M_{A_1 A_2 B_1 B_2} \theta \quad (6.13) \]

This object is evidently antisymmetric in \( A_1, A_2 \) and in \( B_1, B_2 \), but it is also antisymmetric upon interchange of both sets of indices:

\[ \hat{\Theta}_7^{A_1 A_2; B_1 B_2} = -\hat{\Theta}_7^{B_1 B_2; A_1 A_2} \quad (6.14) \]

Normal tracelessness is obvious and \( \Gamma \)-tracelessness follows from that of \( \Theta_5^{A; B_1 B_2} \):

\[ \Gamma E \hat{\Theta}_7^{E; B_1 B_2} = \Gamma E \hat{\Theta}_7^{B_1 B_2; A E} = 0 \quad (6.15) \]

Also, from the definition we extract the properties

\[ \hat{\Theta}_7^{A [B; C] D} = \hat{\Theta}_7^{D [B; C] A} \]

\[ \hat{\Theta}_7^{\{A_1 A_2; B_1 B_2\}} = 0 \quad (6.16) \]

Clearly, this object must be irreducible; however, the corresponding Young pattern symmetry does not manifest, so we define the new object

\[ \Theta_7^{\{B; A_1 A_2 A_3\}} = \hat{\Theta}_7^{\{B; A_1 A_2 A_3\}} = \Gamma C M_6^{B; A_1 A_2 A_3} \theta \quad (6.17) \]
Eq. (6.16) implies

\[ \Theta_7^{[B:A_1A_2A_3]} = 0 \] (6.16)

Again, these two spinor-tensors are equivalent and the inverse of (6.17) is

\[ \hat{\Theta}_7^{A_1A_2B_1B_2} = -3\Theta_7^{[B_1B_2]A_1A_2} \] (6.18)

For the representation \([\frac{7}{2}22222]\) we need an object with 3 tensor indices, so try

\[ \Theta_7^{ABC} = \Gamma_D M^{DAE} \Theta_5^{C;BE} = \Gamma_D M^{DAE} M^B_{FE} \Theta_3^{FC} = \Gamma_D M^{DAE} \Gamma_E \Theta_3^{FC} \] (6.2)

From (6.20), (5.5), (4.80), (4.85) and the properties of \(M_{S_1S_2}^{S_1S_2}B_1B_2B_3\) one can also obtain

\[ \Theta_7^{ABC} = -\frac{3}{2} \Gamma_{D_1D_2} M_6^{(A;C)D_1D_2} \theta \] (6.2)

which shows that \(\Theta_7^{ABC}\) is symmetric in \(A, C\). In order to show that it is completely symmetric, we need to prove symmetry in \(A, B\):

\[ \Theta_7^{[AB]C} = -\frac{1}{2} \Gamma_D M^{ABE} M^B_{FE} \Theta_3^{FC} = -\frac{1}{2} M^{AB} \Gamma_D \Theta_3^{C;DE} = 0 \] (6.2)

Thus:

\[ \Theta_7^{ABC} = \Theta_7^{BAC} = \Theta_7^{CBA} = \Theta_7^{ACB} \] (6.2)

Next let us show that it vanishes upon contraction with \(\Gamma_A\),

\[ \Gamma_C \Theta_7^{ABC} = \Gamma_C \Gamma_D M^{DAE} M^B_{FE} \Theta_3^{FC} = 2M_C^{A\Theta} M^B_{FE} \Theta_3^{FC} = -M_BAE M^B_{FE} \Theta_3^{FC} = 0 \] (6.2)

as it is clear from (5.5) and (2.3).

Now we proceed to list the \(\theta^6 \times \theta^4\) decompositions. First, by Young projection we get

\[ \Gamma_{E_1E_2} M_6^{S_1S_2;C_1C_2E_1E_2} \theta = -\frac{2}{3} \Theta_7^{S_1S_2C} \] (6.2)

which can also be obtained from (6.21) plus (6.23). For the remaining \(M_6^{S_1S_2;B_1B_2B_3}\) products we have, together with (6.17),

\[ \Gamma_{E_1E_2} M_6^{S_1S_2;E_1E_2B_1B_2} \theta = 0 \]

\[ \Gamma_E M_6^{S_1S_2;C_1C_2E_2C_1C_2} \theta = \frac{1}{2} \Theta_7^{S_1S_2C_1C_2} + \frac{1}{6} \Gamma_{C_1C_2} \Theta_7^{C_1C_2S_1S_2} \]

\[ M_6^{S_1S_2;B_1B_2B_3} \theta = \frac{1}{7} \Gamma_{B_1B_2B_3} \Theta_7^{S_1S_2B_1B_2B_3} + \frac{3}{28} \Gamma_{B_1B_2B_3} \Theta_7^{S_1S_2B_1B_2B_3} + \frac{1}{28} \Gamma_{B_1B_2B_3} \Theta_7^{S_1S_2B_1B_2B_3} \] (6.2)
For $\mathcal{M}_6^{S_1S_2B_1\ldots B_5}\theta$ we have instead:

$$
\Gamma_{E_1\ldots E_4} \mathcal{M}_6^{A_1A_2;CE_1\ldots E_4}\theta = 0 \quad \Gamma_{E_1\ldots E_4} \mathcal{M}_6^{A_1B_1B_2E_3E_4}\theta = 0
$$

$$
\Gamma_{E_1E_2E_3} \mathcal{M}_6^{AE_1;E_2E_3B_1B_2B_3}\theta = \frac{-6}{5}\Theta_7^{A_1B_1B_2B_3} \quad \Gamma_{E_1E_2E_3} \mathcal{M}_6^{A_1A_2;B_1B_2E_1E_2E_3}\theta = \frac{-18}{5}\Theta_7^{A_1A_2B_1B_2}
$$

$$
\Gamma_{E_1E_2} \mathcal{M}_6^{E_1E_2;B_1\ldots B_5}\theta = 0
$$

$$
\Gamma_{E_1E_2} \mathcal{M}_6^{AE_1;E_2B_1B_2B_3B_4}\theta = \frac{-6}{5}\Gamma_{B_1}\Theta_7^{A_1B_2B_3B_4} \quad \Gamma_{E_1E_2} \mathcal{M}_6^{A_1A_2;B_1B_2E_1E_2E_3}\theta = \frac{-9}{5}\Gamma_{B_1}\Theta_7^{A_1A_2B_1B_2}
$$

$$
\Gamma_{E} \mathcal{M}_6^{AEB_1\ldots B_5}\theta = \Gamma_{B_1B_2}\Theta_7^{A_1B_3B_4B_5}
$$

$$
\Gamma_{E} \mathcal{M}_6^{A_1A_2;B_1\ldots B_4E}\theta = \frac{2}{35}[\Gamma^{A_1}\Gamma_{B_1}\Theta_7^{A_2B_2B_3B_4} + 12\Gamma_{B_1B_2}\Theta_7^{A_1A_2B_3B_4} + 2\Theta_7^{A_1B_1}\Theta_7^{A_2B_2B_3B_4}]
$$

$$
\mathcal{M}_6^{A_1A_2;B_1\ldots B_5}\theta = \frac{-1}{7}\Gamma_{A_1}\Gamma_{B_1B_2}\Theta_7^{A_2B_3B_4B_5} + \frac{3}{14}\Gamma_{B_1B_2B_3}\Theta_7^{A_1A_2B_4B_5}
$$

(6.29)

$\hat{\Theta}^0$

In this sector, we have the same representations than in the previous ($\hat{\Theta}^7$) one. Inspired by (6.16), one defines

$$
\Theta_9^{ABC} = M^{A}_{DE}\hat{\Theta}_7^{BD;EC} = \frac{3}{2}\Gamma_FM^{A}_{DE}\mathcal{M}_6^{F(B:C)DE}\theta
$$

$$
= -\Gamma_F\mathcal{M}_8^{FABC}\theta
$$

(6.28)

Its tracelessness and total symmetry have become obvious in the last equality in (6.28) hence this is the irreducible spinor-tensor corresponding to $[\frac{711111-1}{2222222}]$. By projecting the product $M^{A_1A_2D}\Theta_7^{S_1S_2D}$ one realizes that the other irreducible structure must be

$$
\Theta_9^{B;A_1A_2A_3} = M^{A_1A_2 D}\Theta_7^{A_3 BD}
$$

$$
= \frac{-3}{2}\Gamma_{E_1E_2}M^{A_1A_2 D}\mathcal{M}_6^{A_3D;B E_1E_2}\theta = \frac{3}{2}\Gamma_{E_1E_2}\mathcal{M}_8^{B E_1E_2A_1A_2A_3}\theta.
$$

(6.29)

Exploiting the symmetry of $\Theta_9^{ABC}$ we can interchange the roles of $A_3$ and $B$ in (6.29a) and using the properties of $\mathcal{M}_8^{S_1S_2D_1\ldots D_4}$ as well as the last equality in (6.29a), one can equally derive

$$
\Theta_9^{B;A_1A_2A_3} = 2\Gamma_{EF}\mathcal{M}_8^{B E;F A_1A_2A_3}\theta.
$$

(6.29)

The ordinary trace vanishes manifestly as does the first $\Gamma$-trace:
\[ \Gamma_E \Theta_9^{E;A_1A_2A_3} = 0 \] 

The other one also vanishes:

\[ \Gamma_E \Theta_9^{B;E A_1A_2} = \frac{2}{3} \Gamma_E M^{EA_1}_D \Theta_9^{A_2BD} = \frac{2}{3} \Gamma_E M^{EA_1}_D \Gamma_F M^{FA_2}_C \Theta_9^{B;DC} = \frac{2}{3} M^{EA_1}_D M^{EA_2}_C \Theta_9^{B;DC} = -\frac{1}{3} M^{EA_2}_D M^{E DC}_5 \Theta_9^{B;DC} = 0 \] 

as implied by (6.5) and (2.3). The remaining property inherited from (4.75) is

\[ \Theta_9^{[B;A_1A_2A_3]} = 0 \] 

Turning to the \( \theta^8 \times \theta \) decompositions, the first one is trivially inferred from (6.28)

\[ M_8^{S_1S_2S_3S_4} \theta = -\frac{1}{4} \Gamma_9 \theta_9^{S_2S_3S_4} \theta \]

From (6.29a) one successively derives the set:

\[ \Gamma_E \mathcal{M}_8^{A_1A_2A_3;B_1B_2E} \theta = -\frac{2}{15} \Gamma_9 \theta_9^{E3A_1A_2A_3} + \frac{2}{10} \Gamma_9 \theta_9^{B_1B_2A_1A_2A_3} \]

\[ \mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} \theta = -\frac{1}{45} \left( \Gamma_9 \theta_9^{A_3B_1B_2B_3} + \Gamma_9 \theta_9^{B_3A_1A_2A_3} \right) \]

while from (6.29b) instead, the set

\[ \Gamma_E \mathcal{M}_8^{E;B_1\ldots B_4} \theta = -\frac{1}{2} \Gamma_9 \theta_9^{A_1B_2B_3B_4} \]

\[ \mathcal{M}_8^{A;B_1\ldots B_4} \theta = -\frac{2}{63} \left( \Gamma_9 \theta_9^{C;B_2B_3B_4} + \Gamma_9 \theta_9^{C;B_1B_2B_3B_4} \right) \]

\[ + \frac{1}{126} \left( \eta^{AB_1} \theta_9^{C;B_2B_3B_4} + \eta^{CB_1} \theta_9^{A;B_2B_3B_4} \right) \]

\[ + \frac{1}{42} \Gamma_9 \theta_9^{A;C B_1B_2B_3B_4} + \frac{1}{42} \Gamma_9 \theta_9^{A;C B_1B_2B_3B_4} \]

\[ \theta_{11}^{11} \]

For the representation \[ \begin{bmatrix} 5 & 3 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix} \] we first construct the object with 3 indices by contracting \( M_8^{A_1A_2A_3} \) with \( \Theta_9^{ABC} \). We define

\[ \hat{\Theta}_1^{ABC} = \Gamma_D M_D^{DAE} \Theta_9^{BC} E = \Gamma_D \mathcal{M}_8^{DCBE} \Theta_9^{A} E \]

\[ = \Gamma_{E_1E_2} \mathcal{M}_{10}^{BC} \Theta_9^{AE_1E_2} \theta \]

\[ \hat{\Theta}_1^{ABC} \]
Then, we see that the tracelessness of $\hat{\Theta}^{A;BC}_{11}$ is trivially satisfied and the $\Gamma$-tracelessness is also immediate from (6.36):

$$\Gamma_A \hat{\Theta}^{A;BC}_{11} = \Gamma_A \Gamma_D M_{8}^{DBC}E \Theta^{A}_{E} = 2 \eta_{AD} M_{8}^{DBC}E \Theta^{A}_{E} = 0$$

$$\Gamma_B \hat{\Theta}^{A;BC}_{11} = \Gamma_B \Gamma_D M_{DAE}^{DAE} \Theta^{BC}_{E} = 2 \eta_{BD} M_{DAE}^{DAE} \Theta^{BC}_{E} = 0$$

(6.3)

So $\hat{\Theta}^{A;BC}_{11}$ is irreducible, and a useful property of $\hat{\Theta}^{A;BC}_{11}$ can be inferred from the group theory, i.e., we must have

$$\hat{\Theta}^{(A;BC)}_{11} = 0,$$

(6.3)

which reflects the fact that we can not have an irreducible object with totally symmetrized 3 indices in $\theta^{11}$-sector (see Table 1). In fact, (6.38) can be readily verified from the definition (6.36):

$$\hat{\Theta}^{(A;BC)}_{11} = -\Gamma_{E_1E_2} \hat{\Theta}^{E_1(BC;A)E_2}_{E} = \frac{1}{3} \Gamma_{E_1E_2} \hat{\Theta}^{E_1BCA;E_1E_2}_{E} \theta$$

$$= \frac{1}{3} \mathcal{M}_{8}^{BCAF} \Gamma_{E_1E_2} M_{E_1E_2}^{E_1E_2} \theta = 0$$

(6.4)

Even though $\hat{\Theta}^{A;BC}_{11}$ is irreducible, its Young symmetry is not manifest, so we need to define a new object for $[\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \\ -2 & -2 & -2 & -2 & -2 & -2 \end{array}]$:

$$\Theta^{B;CD}_{11} = \hat{\Theta}^{[C;D]B}_{11} = \frac{3}{2} \Gamma_{E_1E_2} M_{10}^{BE_1;E_2CD} \theta.$$ 

(6.4)

Then, it is obvious from the definition (6.39) and (4.48) that $\Theta^{B;A_1A_2}_{11}$ satisfies

$$\Theta^{[B;A_1A_2]}_{11} = 0,$$

(6.4)

and the inverse of (6.39) is

$$\Theta_{11}^{A_1S_1S_2} = -\frac{4}{3} \Theta_{11}^{S_1S_2A}.$$ 

(6.4)

Turning to the representation $[\begin{array}{ccccccc} 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -2 & -2 & -2 & -2 & -2 \end{array}]$, we need an object with 5 totally antisymmetric tensor indices. Naturally, we define

$$\Theta_{11}^{A_1...A_5} = M_{E}^{E_1A_2} \Theta_{9}^{E_1A_3A_4A_5} = \Gamma_{E_1E_2} M_{10}^{E_1E_2;A_1...A_5} \theta.$$ 

(6.4)

Again, the tracelessness is trivial, but for the $\Gamma$-tracelessness we need a little work:

$$\Gamma_A \Theta_{11}^{A_1...A_5} = \Gamma_A M_{E}^{E_1A_2} \Theta_{9}^{E_1A_3A_4A_5} = \frac{2}{5} \Gamma_D M_{DA}^{DA_2} \Theta_{9}^{E_1A_3A_4A_5}$$

$$= \frac{2}{5} M_{A_2}^{A_1F} \Gamma_D M_{DA}^{DA_1} \Theta_{7}^{A_1F} = \frac{3}{5} M_{A_2}^{A_1F} \Gamma_D \Theta_{9}^{E_1A_3A_4A_5} = 0.$$ 

(6.4)

The irreducible object $\Theta_{11}^{A_1...A_5}$ satisfies similar properties to those of $\Theta_{9}^{A_1...A_5}$. First, it is self-dual:
\[ \Theta^{A_1 \ldots A_5}_{11} = \frac{1}{5!} \epsilon^{A_1 \ldots A_5 B_1 \ldots B_5} \Theta_{11 B_1 \ldots B_5} \]  
\[ \text{(6.4)} \]

and it satisfies

\[ \Gamma[B \Theta^{A_1 \ldots A_5}_{11}] = 0. \]  
\[ \text{(6.4)} \]

While the self-duality (6.44) is obvious from (4.60) and (6.42), eq. (6.45) may be obtained from (6.43) and (6.44) similarly to the case of \( \Theta^{A_1 \ldots A_5}_{11} \). In fact, the property (6.45) as well as (6.4) may be also justified by the fact that: (1) \( \Gamma[A_1 \Theta^{A_2 \ldots A_6}_{11}] \) and \( \Gamma[A_1 \Theta^{A_2 \ldots A_6}_{11}] \) are irreducible and, (2) we cannot have an irreducible object with 6 fully antisymmetrized indices in the \( \theta^{11} \)- and \( \theta^5 \)-sectors.

\[ \Gamma[A_1 \Gamma^{A_1 \Theta^{A_2 \ldots A_6}_{11}}] = 0, \]  
\[ \text{(6.4)} \]

as can be seen by expanding the bracket.

Now let us list the \( \theta^{10} \times \theta \) decompositions. For \( \mathcal{M}^{S_1 S_2; A_1 A_2 A_3}_{10} \) products we first have (6.36), (6.39) and

\[ \Gamma_{E_1 E_2} \mathcal{M}_{10}^{S_1 S_2; A E_1 E_2} \theta = - \frac{8}{9} \Theta^{S_1; S_2}_{11}. \]  
\[ \text{(6.4)} \]

Then from these two we successively obtain the remaining decompositions:

\[ \Gamma_{E} \mathcal{M}_{10}^{E A: B_1 B_2 B_3} \theta = - \frac{1}{3} \Theta^{A_1; B_2 B_3}_{11} \]

\[ \Gamma_{E} \mathcal{M}_{10}^{S_1 S_2; A_1 A_2 E} \theta = \frac{4}{63} \left( \Gamma^{S_1 \Theta_{11}^{S_2; A_1 A_2}} + 3 \Gamma^{A_1 \Theta_{11}^{S_1; S_2 A_2}} \right) \]

\[ \mathcal{M}_{10}^{S_1 S_2; A_1 A_2 A_3} \theta = \frac{1}{210} \left( -9 \Gamma^{S_1 \Gamma \Theta_{11}^{S_2; A_2 A_3}} + 4 \Gamma^{S_1 \Theta_{11}^{S_2; A_2 A_3}} + 6 \Gamma^{A_1 A_2 \Theta_{11}^{S_1; S_2 A_3}} \right). \]  
\[ \text{(6.4)} \]

On the other hand, for \( \mathcal{M}_{10}^{A_1 A_2; B_1 \ldots B_5} \theta \) we have (6.42) and

\[ \Gamma_{E_1 \ldots E_4} \mathcal{M}_{10}^{A_1 A_2; B E_1 \ldots E_4} \theta = - \frac{8}{5} \Theta^{B; A_1 A_2}_{11} \]

\[ \Gamma_{E_1 \ldots E_4} \mathcal{M}_{10}^{B E_1; E_2 E_3 E_4 A_1 A_2} \theta = - \frac{2}{5} \Theta^{B; A_1 A_2}_{11} \]

\[ \Gamma_{E_1 E_2 E_3} \mathcal{M}_{10}^{A_1 A_2; B_1 B_2 E_1 E_2 E_3} \theta = \frac{2}{5} \Gamma^{B_1 \Theta_{11}^{A_1 A_2}} \]

\[ \Gamma_{E_1 E_2} \mathcal{M}_{10}^{A E_1; E_2 E_3 B_1 B_2 B_3} \theta = \frac{1}{5} \Gamma^{B_1 \Theta_{11}^{A_1 B_2 B_3}} \]
\[ \Gamma_{E_1E_2E_3}M_{i_0}^{E_1E_2;E_3A_1\ldots A_4} \theta = 0 \]

\[ \Gamma_{E_1E_2}M_{i_0}^{AE_1;E_2B_1\ldots B_4} \theta = \frac{1}{5} \Theta_{11}^{AB_1\ldots B_4} + \frac{4}{50} \Gamma_{B_1B_2} \Theta_{11}^{A_1B_1B_2} \]

\[ \Gamma_{E_1E_2}M_{i_0}^{A_1A_2;B_1B_2B_3E_1E_2} \theta = - \frac{1}{10} \Theta_{11}^{A_1A_2B_1B_2B_3} \\
+ \frac{1}{200} \left( 17 \Gamma_{B_1B_2} \Theta_{11}^{B_3A_1A_2} - \Gamma_{A_1} \Gamma_{B_1} \Theta_{11}^{A_1A_2B_1B_2B_3} - 4 \eta_{A_1B_1} \Theta_{11}^{A_2B_1B_2} \right) \]

\[ \Gamma_{E}M_{i_0}^{E,A_1A_2;B_1\ldots B_5} \theta = \frac{1}{10} \Gamma_{B_1\ldots B_5} A_{11} + \frac{1}{30} \Gamma_{B_1B_2B_3} \Theta_{11}^{A_1A_4B_4} \]

\[ \Gamma_{E}M_{i_0}^{A_1A_2;B_1\ldots B_4} \theta = - \frac{1}{20} \Gamma_{A_1} \Theta_{11}^{A_1B_1\ldots B_4} \\
- \frac{1}{600} \left( 3 \Gamma_{A_1} \Gamma_{B_1B_2} \Theta_{11}^{B_3A_1A_2} + 11 \Gamma_{B_1B_2B_3} \Theta_{11}^{B_3A_1A_2} + 6 \eta_{A_1B_1} \Gamma_{B_2} \Theta_{11}^{A_2B_1B_2B_3} \right) \]

\[ \Theta_{13}^{AB} = M_{i_0}^{A_1A_2;B_1\ldots B_5} \theta = \frac{1}{88} \left( \Gamma_{A_1} \Gamma_{B_1\ldots B_5} - 2 \eta_{A_1B_1} \Theta_{11}^{B_5A_1A_2} \right) \\
+ \frac{1}{240} \left( \Gamma_{A_1} \Gamma_{B_1B_2B_3} \Theta_{11}^{B_3B_4} - \Gamma_{B_1\ldots B_4} \Theta_{11}^{B_5A_1A_2} \right) \quad (6.51) \]

The only representation we have in this sector is just \( \{3, 3, 1, 1, 1\} \) like in the \( \theta^3 \)-sector and this means that we need an object with 2 antisymmetric tensor indices again. Let us define

\[ \Theta_{13}^{AB} = M_{i_0}^{A_1E_1} \Theta_{11}^{B_1E_1} \theta \quad (6.52) \]

Then the antisymmetry property of \( \Theta_{13}^{AB} \) is automatically insured as soon as we obtain the following identity. That is, if we use (6.39), (4.48) and the first equation of (4.49), eq. (6.52) becomes

\[ \Theta_{13}^{AB} = \frac{3}{2} \Gamma_{D_1D_2} M_{i_0}^{BD_1;D_2E_1E_2} M_{i_0}^{A_1E_1E_2} \theta \\
= - \frac{3}{2} \Gamma_{D_1D_2} M_{i_0}^{BE_1;E_2D_1D_2} M_{i_0}^{A_1E_1E_2} \theta \\
= - \Gamma_{D_1D_2} M_{i_0}^{D_1D_2;AB} \theta \quad (6.53) \]

Further, the other expression for \( \Theta_{13}^{AB} \) is also immediately obtained from (6.51) if we use the first equation of (4.61), and (6.42):
\[ \Theta_{13}^{AB} = -\frac{5}{2} \Gamma_{D1D2} \mathcal{M}_{10D1D2;ABE1E2E3} M_{E1E2E3} \Theta_{11}^{E1E2E3AB} \quad (6.52) \]

On the other hand, the normal tracelessness of this antisymmetric spinor-tensor is trivial and

\[ \Gamma_D \Theta_{13}^{DA} = 0 \quad (6.53) \]

is also obvious from the last equality in (6.52). So \( \Theta_{13}^{A_1A_2} \) is the irreducible object corresponding to the representation \( \left[ \begin{array}{cccc} 3 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{array} \right] \). Now, for the \( \theta^{12} \times \theta \) decompositions we have

\[ \Gamma_E \mathcal{M}_{12}^{E;A_1B_1B_2} \theta = \frac{1}{12} \left( \Gamma^{A_1} \Theta_{13}^{B_1B_2} - \Gamma^{B_1} \Theta_{13}^{A_1B_2} \right) \]

\[ \mathcal{M}_{12}^{S_1S_2;X_1X_2} \theta = \frac{1}{33} \Gamma^{S_1X_1} \Theta_{13}^{S_2X_2} \]

\[ \mathcal{M}_{12}^{A_1A_2;B_1B_2} \theta = \frac{1}{132} \left( \Gamma^{A_1A_2} \Theta_{13}^{B_1B_2} + \Gamma^{B_1B_2} \Theta_{13}^{A_1A_2} + 2 \Gamma^{A_1B_1} \Theta_{13}^{A_2B_2} \right) \quad (6.54) \]

and

\[ \Gamma_{E_1...E_5} \mathcal{M}_{12}^{A;E_1...E_5} \theta = 0 \]

\[ \Gamma_{E_1...E_4} \mathcal{M}_{12}^{A;B_1E_1...E_4} \theta = -\frac{4}{5} \Theta_{13}^{AB} \]

\[ \Gamma_{E_1...E_4} \mathcal{M}_{12}^{E_1;E_2E_3A_1A_2} \theta = \frac{2}{5} \Theta_{13}^{A_1A_2} \]

\[ \Gamma_{E_1E_2E_3} \mathcal{M}_{12}^{B;A_1A_2E_1E_2E_3} \theta = -\frac{3}{15} \Gamma^{A_1} \Theta_{13}^{A_2B} \]

\[ \Gamma_{E_1E_2E_3} \mathcal{M}_{12}^{E_1;E_2E_3A_1A_2A_3} \theta = -\frac{1}{5} \Gamma^{A_1} \Theta_{13}^{A_2A_3} \]

\[ \Gamma_{E_1E_2E_3} \mathcal{M}_{12}^{B;A_1A_2A_3E_1E_2} \theta = -\frac{1}{450} \left( 19 \Gamma^{A_1A_2} \Theta_{13}^{A_3B} + \Gamma^{B} \Gamma^{A_1} \Theta_{13}^{A_2A_3} + 4 \eta^{BA_1} \Theta_{13}^{A_2A_3} \right) \]

\[ \Gamma_{E_1E_2} \mathcal{M}_{12}^{E_1;E_2E_3...A_4} \theta = -\frac{2}{25} \Gamma^{A_1A_2} \Theta_{13}^{A_3A_4} \]

\[ \Gamma_E \mathcal{M}_{12}^{E;A_1...A_5} \theta = \frac{1}{30} \Gamma^{A_1A_2A_3} \Theta_{13}^{A_4A_5} \]
\[ \Gamma_E \mathcal{M}_{12}^{B:A_1 \ldots A_4} \theta = \frac{1}{450} \left( 4 \Gamma^A_{A_1 A_2 A_3} \Theta_{13}^{A_4 B} - \Gamma^B_{A_1 A_2} \Theta_{13}^{A_3 A_4} - 2 \eta^{B A_1} \Gamma^A_{A_2} \Theta_{13}^{A_3 A_4} \right) \]

\[ \mathcal{M}_{12}^{B:A_1 \ldots A_5} \theta = \frac{1}{540} \left( \Gamma^B_{B A_1 A_2 A_3} \Theta_{13}^{A_4 A_5} + \Gamma^A_{A_1 \ldots A_4} \Theta_{13}^{A_5 B} \right) \] (6.55)

\( \theta^{15} \)

Finally, for \( \theta^{15} \)-sector we have again only one representation, which is \( \left[ \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{array} \right] \) and the corresponding irreducible object is a spinor with no tensor indices just like \( \theta \), but with opposite chirality in this case. So the only possible candidate for \( \Theta_{15} \) is:

\[ \Theta_{15} \equiv \Theta = \Gamma_D M^{DE_1 E_2} \Theta_{13 E_1 E_2} = \Gamma_{E_1 E_2 E_3} M^{E_1 E_2 E_3} \theta. \] (6.5)

For the decompositions we have

\[ \Gamma_{E_1 E_2} M^{E_1 E_2 A} \theta = \frac{1}{10} \Gamma^A \Theta \]

\[ \Gamma_E M^{E A_1 A_2} \theta = -\frac{1}{90} \Gamma^{A_1 A_2} \Theta \]

\[ \mathcal{M}^{A_1 A_2 A_3} \theta = -\frac{1}{720} \Gamma^{A_1 A_2 A_3} \Theta \] (6.58)
VII Products of $M^{A_1 A_2 A_3}$ with Spinor-Tensors

In this section we list the products of $M^{A_1 A_2 A_3}$ with all the $\Theta_n$ of section VI, since they are another necessary ingredient in the development of the tensor calculus. Other more esoteric product identities are given in Appendix B.

\[ \begin{align*}
\theta_6
M^{A_1 A_2 A_3} \Theta_3^{B_1 B_2} &= \Theta_3^{A_1 A_2 A_3 B_1 B_2} \\
&= \Theta_3^{A_1 A_2 A_3 B_1 B_2} \\
&- \frac{3}{10} \Gamma^{A_1} \Gamma^{B_1} \Theta_5^{B_2: A_2 A_3} + \frac{3}{10} \Gamma^{A_1} \Gamma^{A_2} \Theta_5^{A_3: B_1 B_2} - \frac{6}{10} \eta^{A_1 B_1} \Theta_5^{B_2: A_2 A_3} 
\end{align*} \] (7.3)

\[ \begin{align*}
\theta_7
M^{A_1 A_2 A_3} \Theta_5^{B_1 \ldots B_5} &= \frac{1}{14} \Gamma^{B_1 \ldots B_5} \Theta_7^{B_2: A_1 A_2 A_3} + \frac{15}{14} \left( \Gamma^{A_1} \Gamma^{B_1} \Gamma^{B_2 B_3} - 2 \eta^{A_1 B_1} \Gamma^{B_2 B_3} \right) \Theta_7^{B_2: B_5 A_2 A_3} \\
&- \frac{5}{7} \left( \Gamma^{A_1} \Gamma^{B_1} \Gamma^{B_2} + 2 \eta^{A_1 B_1} \Gamma^{A_2} \Gamma^{B_2} - 2 \eta^{A_1 B_1} \eta^{A_2 B_2} \right) \Theta_7^{A_3: B_3 B_4 B_5} 
\end{align*} \] (7.4)

\[ \begin{align*}
M^{A_1 A_2 A_3} \Theta_7^{C_1 B_1 B_2} &= \frac{5}{14} \left( \Gamma^{C_1} \Gamma^{A_1} + 4 \eta^{C A_1} \right) \Theta_7^{A_2: A_3 B_1 B_2} + \frac{1}{21} \left( \Gamma^{C_1} \Gamma^{B_1} - 4 \eta^{C B_1} \right) \Theta_7^{B_2: A_1 A_2 A_3} \\
&- \frac{5}{56} \left( \Gamma^{A_1} \Gamma^{B_1} - 10 \eta^{A_1 B_1} \right) \Theta_7^{B_1: C_2 A_2 A_3} - \frac{5}{56} \left( 5 \Gamma^{A_1} \Gamma^{B_1} - 2 \eta^{A_1 B_1} \right) \Theta_7^{C_1 B_2 A_2 A_3} \\
&+ \frac{5}{28} \Gamma^{A_1} \Gamma^{A_2} \Theta_7^{A_3: C B_1 B_2} - \frac{15}{28} \Gamma^{A_1} \Gamma^{A_2} \Theta_7^{C_2 A_1 B_1 B_2} - \frac{1}{21} \Gamma^{B_1} \Gamma^{B_2} \Theta_7^{C_1 A_1 A_2 A_3} \\
&+ \frac{1}{28} \left( \Gamma^{A_1} \Gamma^{B_1} - 4 \eta^{A_1 B_1} \Gamma^{A_2} \right) \Theta_7^{A_1 B_2 C} 
\end{align*} \] (7.5)

\[ \begin{align*}
M^{A_1 A_2 A_3} \Theta_7^{B_3 C_1 C_2 C_3} &= \frac{1}{60} \Gamma^{A_1} \Gamma^{A_2} \Theta_9^{B_3 C_1 C_2 C_3} + \frac{1}{140} \Gamma^{C_1} \Gamma^{C_2} \Theta_9^{B_3 A_1 A_2 A_3} \\
&+ \frac{1}{140} \left( \Gamma^{B_3} \Gamma^{C_1 C_2} + \frac{2}{3} \eta^{B C_1} \Gamma^{C_2} \right) \Theta_9^{C_3: A_1 A_2 A_3} + \frac{1}{10} \eta^{B A_1} \Gamma^{A_2} \Theta_9^{A_3: C_1 C_2 C_3} \\
&- \frac{9}{280} \left( \Gamma^{A_1} \Gamma^{C_1 C_2} + \frac{2}{3} \eta^{A_1 C_1} \Gamma^{C_2} \right) \Theta_9^{C_3: B A_2 A_3} - \frac{1}{280} \left( 23 \Gamma^{A_1} \Gamma^{C_1 C_2} - 22 \eta^{A_1 C_1} \Gamma^{C_2} \right) \Theta_9^{B_3: C_2 A_3 A_3} \\
&+ \frac{3}{20} \Gamma^{A_1} \Gamma^{A_2} \Gamma^{C_1} \Theta_9^{B_3 C_2 C_3} - \frac{1}{20} \left( \Gamma^{C_1} \Gamma^{A_1 A_2} - 6 \eta^{A_1 A_1} \Gamma^{A_2} \right) \Theta_9^{A_3: B C_2 C_3} \\
&+ \frac{3}{20} \left( \Gamma^{B_3} \Gamma^{A_1} \Gamma^{C_1} + \eta^{A_1 C_1} \Gamma^{B} + \eta^{B A_1} \Gamma^{C_1} - \eta^{B C_1} \Gamma^{A_1} \right) \Theta_9^{A_2: A_3 C_2 C_3} \\
&- \frac{1}{140} \left( \Gamma^{A_1} \Gamma^{A_2} \Gamma^{C_1 C_2} - 6 \eta^{A_1 C_1} \Gamma^{A_2} \Gamma^{C_2} - 22 \eta^{A_1 C_1} \eta^{A_2 C_2} \right) \Theta_9^{A_3 C_3 B} 
\end{align*} \] (7.6)
\[ M^{A_1 A_2 A_3} \theta_1^{S_1 S_2 S_3} = -\frac{1}{16} \Gamma^{A_1 A_2 A_3} \theta_9^{S_1 S_2 S_3} + \frac{3}{112} \left( 5 \Gamma^{A_1 A_2} \Gamma - 8 \eta^{S_1 A_1} \Gamma^{A_2} \right) \theta_9^{S_2 S_3} \] 

\[ - \frac{1}{7} \eta^{S_1 S_2} \theta_9^{A_2 A_3} - \frac{3}{28} \left( \Gamma^{S_1} \Gamma^{A_1} - 14 \eta^{S_1} \Gamma^{A_2} \right) \theta_9^{S_2 S_3} A_3 \]

\[ \theta^{11} \]

\[ M^{A_1 A_2 A_3} \theta_9^{S_1 S_2 S_3} = \frac{2}{70} \left( \Gamma^{A_1 A_2} \Gamma^{S_1} - 10 \eta^{S_1 A_1} \Gamma^{A_2} \right) \theta_9^{S_2 S_3} A_3 \]

\[ - \frac{3}{70} \left( \eta^{S_1 S_2} \Gamma^{A_1} - 2 \eta^{S_1 A_1} \Gamma^{S_2} \right) \theta_9^{S_3 A_2 A_3} \]

\[ \theta^{13} \]

\[ M^{A_1 A_2 A_3} \theta_9^{C; B_1 B_2 B_3} = \frac{26}{330} \Gamma^{A_1 A_2} \theta_9^{A_3 B_1 B_2 B_3 C} + \frac{31}{330} \Gamma^{B_1 B_2 B_3} \theta_9^{A_1 A_2 A_3 C} \]

\[ + \frac{1}{330} \left( 21 \Gamma^{C} \Gamma^{A_1} + 109 \eta^{C A_1} \right) \theta_9^{B_2 A_3 B_1 B_2 B_3} - \frac{91}{330} \eta^{C B_1} \theta_9^{B_2 A_3 B_1 A_2 A_3} + \frac{213}{330} \eta^{B_1 A_1} \theta_9^{A_2 A_3 B_1 B_2 B_3} \]

\[ + \frac{1}{1680} \left( 11 \Gamma^{B_1 B_2 B_3} \Gamma^{A_1} - 50 \eta^{A_1 B_1} \Gamma^{B_2 B_3} \right) \theta_9^{C; A_2 A_3} - \frac{1}{40} \Gamma^{A_1 A_2 A_3} \Gamma^{B_1} \theta_9^{C; B_1 B_3} \]

\[ + \frac{1}{1680} \left( -11 \Gamma^{A_1} \Gamma^{C B_1 B_2} - 30 \eta^{C B_1} \Gamma^{B_2} \Gamma^{A_1} + 24 \eta^{A_1 B_1} \Gamma^{C} \Gamma^{B_2} + 140 \eta^{B_1 B_1} \eta^{C B_1} \right) \theta_9^{B_3 A_2 A_3} \]

\[ - \frac{1}{840} \left( \Gamma^{B_1 B_2 \Gamma^{A_1} A_2} + 42 \eta^{A_1 B_1} \Gamma^{A_2} \Gamma^{B_2} + 168 \eta^{A_1 B_1} \eta^{A_2 B_2} \right) \theta_9^{B_3 A_2 A_3} \]

\[ - \frac{1}{840} \left( 29 \Gamma^{A_1 A_2} \Gamma^{B_1 B_2} + 22 \eta^{A_1 B_1} \Gamma^{A_2} \Gamma^{B_2} - 64 \eta^{A_1 B_1} \eta^{A_2 B_2} \right) \theta_9^{C; A_3 B_3} \]

\[ \theta^{13} \]

\[ M^{A_1 A_2 A_3} \theta_9^{C; B_1 B_2} = \frac{1}{11} \left[ -\frac{1}{36} \left( \Gamma^{C} \Gamma^{A_1 A_2 A_3} - 12 \eta^{C A_1} \Gamma^{A_2 A_3} \right) \theta_9^{B_1 B_2} \right] \]

\[ + \frac{1}{60} \left( 5 \eta^{C B_1} \Gamma^{B_2} \Gamma^{A_1} - 18 \eta^{C B_1} \eta^{B_2 A_1} + 3 \eta^{C A_1} \Gamma^{B_1 B_2} + 3 \eta^{A_1 B_1} \Gamma^{C} \Gamma^{B_2} \right) \theta_9^{A_3 A_3} \]

\[ + \frac{1}{60} \left( 2 \Gamma^{C} \Gamma^{B_1} \Gamma^{A_1 A_2} - 8 \eta^{C B_1} \Gamma^{A_2} \Gamma^{B_1} - 30 \eta^{C A_1} \Gamma^{A_2} \Gamma^{B_1} - 20 \eta^{C A_1} \eta^{A_2 B_1} \right) \theta_9^{A_3 B_2} \]

\[ - \frac{1}{36} \left( \Gamma^{A_1 A_2 A_3} \Gamma^{B_1} + 6 \eta^{A_1 B_1} \Gamma^{A_2 A_3} \right) \theta_9^{C; B_2} \]

\[ - \frac{1}{60} \left( 2 \Gamma^{B_1 B_2} \Gamma^{A_1 A_2} + 30 \eta^{A_1 B_1} \Gamma^{B_2} \Gamma^{A_2} - 80 \eta^{A_1 B_1} \eta^{A_2 B_2} \right) \theta_9^{A_3 C} \]

\[ \theta^{13} \]

\[ M^{A_1 A_2 A_3} \theta_9^{B_1 \ldots B_5} = \frac{1}{72} \left( -\Gamma^{A_1 A_2 A_3} \Gamma^{B_1 B_2 B_3} + 6 \eta^{B_1 A_1} \Gamma^{A_2 A_3} \Gamma^{B_2 B_3} + 12 \eta^{A_1 B_1} \eta^{A_2 B_2} \Gamma^{A_3 B_3} \right) \theta_9^{B_3 B_4} \]

\[ + \frac{1}{60} \left( \Gamma^{A_1 A_2} \Gamma^{B_1 B_4} + 6 \eta^{A_1 B_1} \Gamma^{A_2} \Gamma^{B_2 B_3} - 8 \eta^{A_1 B_1} \eta^{A_2 B_2} \Gamma^{B_3 B_4} \right) \theta_9^{B_5 A_3} \]

\[ + \frac{2}{720} \left( \Gamma^{B_1 B_2 \Gamma^{A_3 A_1} - 6 \eta^{A_1 B_1} \Gamma^{B_2 B_3} \right) \theta_9^{A_2 A_3} \]

\[ \theta^{13} \]
\[ M^{A_1 A_2 A_3} \Theta_{13}^{B_1 B_2} = -\frac{1}{7 \times 720} \left( \Gamma^{A_1 A_2 A_3} \Gamma^{B_1 B_2} + 6\eta^{B_1 A_1} \Gamma^{A_2 A_3} \Gamma^{B_2} - 24\eta^{B_1 A_1} \eta^{B_2 A_2} \Gamma^{A_3} \right) \Theta \] (7.10)

**VIII Conclusions**

We have presented here in detail the irreducible tensors and spinor-tensors contained in a scalar superfield of definite chirality, \( \Phi(x, \theta^{(+)}) \) in particular but the results for \( \Phi(x, \theta^{(-)}) \) are trivially obtained making the changes explained in the introduction. The results for the most basic products of these irreducible structures have also been presented as a first step towards a full tensor calculus. The remaining products can be derived by iteration of the formulae here and will appear elsewhere.

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Appendix A. Conventions and Bosonic Identities

Our conventions are \( \eta^{AB} = \eta_{AB} = \text{diag}(+ - \ldots -) \), \( \epsilon^{01\ldots9} = \epsilon_{01\ldots9} = 1 \) and the Dirac algebra is

\[
\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \quad A, B = 0, 1, \ldots 9.
\]

(A.1)

Our definition for \( \Gamma_{(11)} \) is

\[
\Gamma_{(11)} = \Gamma_0 \Gamma_1 \ldots \Gamma_9
\]

which satisfies

\[
\Gamma_{(11)}^2 = I \quad \Gamma_{(11)}^\dagger = \Gamma_{(11)}
\]

Then \( \theta^{(+)} = \Pi^{(+)}\theta = \frac{1}{2}(I + \Gamma_{(11)})\theta \) belongs to the \([\frac{11}{11}] \) representation of \( \text{SO}(10) \) while \( \theta^{(-)} = \Pi^{(-)}\theta = \frac{1}{2}(I - \Gamma_{(11)})\theta \) belongs to \([\frac{11}{11-\frac{1}{2}}] \).

In 10 dimensions the Majorana and Weyl condition can be implemented simultaneously and therefore our Majorana-Weyl spinors \( \theta^{(\pm)} \) satisfy

\[
\bar{\theta}^{(\pm)}\Gamma_{A_1\ldots A_n}\theta^{(\pm)} = 0 \quad \text{for} \quad n \neq 3, 7
\]

(A.2)

The only independent bilinear in \( \theta^{(\pm)} \) is then \( M^{(\pm)}_{ABC} = \bar{\theta}^{(\pm)}\Gamma_{ABC}\theta^{(\pm)} \) since we have the identity (2.8). Powers of this bilinear satisfy many identities, implied by the basic Fierz one, that are used in the straightforward derivation of the decomposition of the \( \theta^6 \) product in section III. Here is a list,

\[
M_{A_1A_2} C_1 M_{A_3C_2C_3} = -\frac{1}{3} M_{A_1A_2A_3} M_{C_1C_2C_3} + \delta_{A_1}^{C_1} M_{A_2A_3E} M_{C_2C_3E}
\]

\[
M^{E[A_1A_2} M_{B_1B_2B_3]} = \frac{2}{5} M^{E}_{A_1A_2} M_{B_1B_2B_3} + \frac{3}{5} M^{EB_1B_2} M_{B_3A_1A_2}
\]

\[-\frac{3}{10} \eta^{EB_1} M_{A_1A_2D} M_{B_2B_3 D} - \frac{3}{5} \eta^{A_1B_1} M^{A_2DE} M_{B_2B_3 D}
\]

\[
M^{A_1C_1}_{D} M^{A_2C_2}_{B_3} M^{B_2B_3 D} = \]

\ [-\frac{1}{2} M^{C_1}_{D} M^{A_1A_2B_1}_{B_2B_3} + \frac{1}{2} M^{D}_{C_1C_2} M^{A_1A_2}_{D} M^{B_1B_2B_3} - \frac{1}{2} M^{B_1C_1C_2}_{D} M^{A_1A_2}_{D} M^{B_2B_1B_3}]

\[-\frac{1}{2} \eta^{A_1B_1} M^{A_2DE}_{D} M^{C_1C_2}_{B_3} M^{B_2B_3 E} - \eta^{A_1C_1}_{D} M^{C_2DE}_{D} M^{A_2B_1}_{D} M^{B_2B_3 E} - \frac{1}{2} \eta^{B_1C_1}_{D} M^{C_2DE}_{D} M^{A_1A_2}_{D} M^{B_2B_3 E}
\]

\[
M^{A_3C_3}_{B_3} M^{A_1A_2B_1}_{B_3} M^{C_2C_2}_{B_2} = \frac{1}{6} M^{B_1B_2B_3 D} M^{A_1A_2A_3}_{B_3} M^{C_1C_2C_3} - \frac{1}{2} M^{A_3B_1B_2}_{B_3} M^{B_3C_1C_2}_{B_2} M^{C_3A_1A_2}
\]

\[-\frac{1}{2} \eta^{B_1C_1}_{D} M^{A_1A_2A_3}_{D} M^{B_2B_3D} M^{C_2C_3} - \frac{1}{2} \eta^{A_1B_1}_{D} M^{A_2A_3D} M^{B_2B_3D} M^{C_1C_2}
\]

\[-\frac{1}{2} \eta^{A_1B_1}_{D} M^{A_2A_3}_{D} M^{B_2B_3C_1}_{D} M^{C_2C_3D} + \frac{1}{2} \eta^{A_1C_1}_{D} M^{A_2A_3D} M^{B_2B_3C_1}_{D} M^{C_2C_3D}
\]

\[-\frac{1}{2} \eta^{A_1B_1}_{D} M^{A_2A_3}_{D} M^{B_2B_3C_1}_{D} M^{C_2C_3D} + \eta^{B_1B_2B_3}_{D} M^{C_2DE} M^{A_1A_2A_3}_{D} M^{B_2B_3E}
\]

\[-\frac{1}{2} \eta^{A_1B_1}_{D} M^{A_2A_3}_{D} M^{B_2B_3C_1}_{D} M^{C_2C_3D} + \eta^{B_1B_2B_3}_{D} M^{C_2DE} M^{A_1A_2A_3}_{D} M^{B_2B_3E}
\]
\[ \epsilon^{AB_1\ldots B_7 CE_1\ldots E_4} M^{DB_1 B_2} M_{DE_1 E_2} M_{E_3 E_4} B_3 = 0 \]

\[ M^{DE}[A M_{DF_1 F_2} M_{F_3 E} C] = 0 \]

\[ M^{AB_1 B_2 M B_3 B_4 B_5 M B_6 B_7 C} = -\frac{2}{7 \times 5!} \epsilon^{B_1\ldots B_7 E_1 E_2 E_3} M^{DAF} M_{DE_1 E_2} M_{E_3 F} C \]

\[ M^{AD_1 D_2} M^{D_3 D_4 E} M^{C_1 C_2} E = \frac{5}{6} M^{AE_1 D_1 D_2} M^{D_3 D_4} M^{C_1 C_2} E + \frac{5}{3} M^{[AD_1 D_2} M^{C_1 C_2]} E M^{D_3 D_4} E + \frac{2}{3} \eta^{D_1 C_1} M^{AE_1 D_1} M^{D_2 D_3 E_2} M_{E_1 E_2} C_2 \]

\[ \frac{1}{5!} \epsilon^{B_1\ldots B_6 D_1\ldots D_4} M^{A} D_1 D_2 M^{D_3 D_4 E} M^{C_1 C_2} E = \]

\[ \frac{2}{3} \frac{1}{5!} \epsilon^{B_1\ldots B_6 D_1\ldots D_3} M^{AE_1} D_1 M^{D_2 E_2} M_{E_1 E_2} C_2 \]

\[ + \frac{4}{3} \eta^{A} B_1 M^{B_2 B_3} M^{B_4 B_5} B_6 M^{C_1 C_2} E + \frac{2}{3} \eta^{C_1} B_1 M^{B_2 B_3} M^{B_4 B_5} B_6 M^{C_2 A} E \]

A curious identity in the \( \theta^{10} \) sector that is easy to prove is

\[ \epsilon_{F_1 F_2\ldots F_{10}} M^{A F_1 F_2} M^{B F_3 F_4} M^{C F_5 F_6} M^{D F_7 F_8} M^{E F_9 F_{10}} = 0 \]

as it should be since no such symmetric object is allowed to exist.

Next we give a summary of how eq.(3.10) is derived directly from (2.12) or (3.3-3.5). We start with

\[ M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} M^{C_1 C_2 C_3} = \]

\[ = \left( -\frac{1}{4!} \epsilon^{B_1 B_2 B_3 D_1\ldots D_5 A_1 A_2} M^{A_3} D_1 D_2 M_{D_3 D_4 D_5} + \frac{3}{2} \eta^{A_1 B_1} M^{A_2 A_3} E M^{B_2 B_3 E} \right) M^{C_1 C_2 C_3} = \]

\[ = -\frac{1}{4!} \epsilon^{B_1 B_2 B_3 D_1\ldots D_5 A_1 A_2} M^{A_3} D_1 D_2 \left( -\frac{1}{4!} \epsilon_{D_3 D_4 D_5} E_1\ldots E_5 C_1 C_2 M^{C_3} E_1 E_2 M_{E_3 E_4 E_5} + \frac{3}{2} \eta^{C_1} D_3 M^{D_4 D_5 E} M_{E_3 E_4 E_5} \right) \]

\[ + \frac{3}{2} \eta^{A_1 B_1} M^{A_2 A_3} E M^{B_2 B_3 E} M^{C_1 C_2 C_3} \]

Expanding the product of the Levi-Civita symbols and using heavily the identities above, one gets after a lot algebra

\[ M^{A_1 A_2 A_3} M^{B_1 B_2 B_3} M^{C_1 C_2 C_3} = \frac{9}{8} M^{A_3 C_1 C_2} M^{B_3 A_1 A_2} M^{C_3 B_1 B_2} + I^{A_1 A_2 A_3} B_1 B_2 B_3 C_1 C_2 C_3 \]

with
with solution

$I^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} = $

\[= \frac{3}{8} \left( 6\eta^{A_1B_1} M^{A_1A_2A_3} M^{B_2B_3D} M^{C_2C_3D} + 4\eta^{A_1C_1} M^{A_2A_3D} M^{B_1B_2B_3} M^{C_2C_3D} + \frac{9}{2} \eta^{A_1B_1} M^{A_2A_3D} M^{B_2B_3D} M^{C_1C_2C_3} - 6\eta^{A_1C_1} M^{A_2A_3B_1} M^{B_2B_3D} M^{C_2C_3D} + \frac{3}{2} \eta^{A_1B_1} \left( M^{A_2A_3C_1} M^{B_2B_3D} M^{C_2C_3D} - M^{A_2A_3D} M^{B_2B_3C_1} M^{C_2C_3D} \right) \right) \]

Iterating this equation, we arrive at

\[M^{A_3C_1C_2} M^{B_3A_1A_2} M^{C_3B_1B_2} - I^{C_1C_2A_3B_1B_2C_3A_1A_2} = \frac{9}{8} M^{A_3[A_1A_2]B_3[B_1B_2]C_3C_1C_2C_3} \]

\[= \frac{5}{24} M^{A_1A_2A_3} M^{B_1B_2B_3} M^{C_1C_2C_3} - \frac{1}{6} \left( M^{A_3C_1C_2} M^{B_3A_1A_2} M^{C_3B_1B_2} + \frac{1}{2} M^{A_3B_1B_2} M^{B_3C_1C_2} M^{C_3A_1A_2} \right) + II^{A_1A_2B_3B_2B_1C_2C_3} \]

with

\[II^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} = \]

\[-\frac{1}{4} \left( \eta^{B_1C_1} M^{A_1A_2A_3} M^{B_2B_3D} M^{C_2C_3D} + \eta^{A_1C_1} M^{A_2A_3D} M^{B_1B_2B_3} M^{C_2C_3D} \right) + \eta^{A_1B_1} M^{A_2A_3D} M^{B_2B_3D} M^{C_1C_2C_3} \]

\[+ \frac{1}{12} \left( \eta^{A_1C_1} M^{A_2A_3B_1} M^{B_2B_3D} M^{C_2C_3D} + \eta^{B_1C_1} M^{A_1A_2D} M^{B_2B_3D} M^{C_2C_3A_3} \right) + \frac{1}{6} \left( \eta^{A_1B_1} \eta^{A_2C_1} M^{A_3D} M^{B_2B_3D} M^{C_2C_3E} + \eta^{A_1C_1} \eta^{B_1C_2} M^{C_3D} M^{A_2A_3D} M^{B_2B_3E} \right) + \eta^{A_1B_1} \eta^{B_2C_1} M^{B_3D} M^{A_2A_3D} M^{C_2C_3E} \]

Applying the (normalized) operator $S(A, B, C)$ that fully symmetrizes upon interchange of the letters $A, B, C$, to the equations we have just obtained, we get a system of two equations with solution

\[M^{A_1A_2A_3} M^{B_1B_2B_3} M^{C_1C_2C_3} = \frac{16}{65} S(A, B, C) \left[ 5 I^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} + \frac{9}{2} \left( I^{C_1C_2A_3B_1B_2C_3A_1A_2B_3} + II^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} \right) \right] \]
Let us now proceed to prove the duality properties of the tensors $\mathcal{M}_{12}^{A_1\ldots A_5}$ and $\mathcal{M}_{10}^{CD_1B_1\ldots B_5}$.

From (4.34) and (2.6) we can also write

$$\mathcal{M}_{12}^{B_5; A_1\ldots A_5} = \frac{1}{2} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M_F A_4 A_5$$  \hspace{1cm} (A.1)$$

But:

$$M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5} =$$

$$= M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 \frac{1}{5} \left(3M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5} + 2M^{F A_2}_{D_5} M^{A_3 A_4 A_5}_{D_5}\right)$$

$$= \frac{3}{5} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5},$$

so

$$\mathcal{M}_{12}^{B_5; A_1\ldots A_5} = \frac{5}{6} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5}$$

$$= \frac{5}{6} \frac{1}{5} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 \epsilon F A_2 A_3 A_5 E_1 \ldots E_5 M_{D_3 E_1 E_2 M_{E_3 E_4 E_5}}$$

Now we have to “rotate” indices; that is, from the identity:

$$M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5},$$

we see that

$$M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5}$$

$$= \frac{3}{5} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5},$$

the second and third term vanish identically because of (2.3) and (2.11) respectively, and we obtain

$$M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5} =$$

$$= \frac{3}{5} M^B_{D_1} M^B_{D_2} D_3 M^A_{D_3} D_4 M^{D_1}_{D_4} D_5 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5},$$

Therefore

$$\mathcal{M}_{12}^{B_5; A_1\ldots A_5} = \frac{1}{2 \times 5!} \epsilon A_1 \ldots A_5 F E_1 \ldots E_4 M^B_{D_1} M^B_{D_2} D_3 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5}$$

$$= -\frac{1}{2 \times 5!} \epsilon A_1 \ldots A_5 F E_1 \ldots E_4 M^B_{D_1} M^B_{D_2} D_3 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5}$$

$$= \frac{1}{2 \times 5!} \epsilon A_1 \ldots A_5 F E_1 \ldots E_4 M^B_{D_1} M^B_{D_2} D_3 M^{A_2 A_3}_{D_5} M^{F A_4 A_5}_{D_5}$$

$$= \frac{1}{5!} \epsilon A_1 \ldots A_5 F E_1 \ldots E_4 \mathcal{M}_{12}^{B_5; A_1\ldots A_5},$$
the desired result. Notice the opposite sign with respect to the $\theta^4$ piece, whose duality was explicitly used. For $\mathcal{M}_{10}^{CD;B_1...B_5}$ the derivation proceeds similarly and again one obtains a result opposite to the $\theta^4$ one.
Appendix B. Fermionic Identities

In this Appendix we list identities involving some products of powers of $M^{ABC}$ with the spinor-tensors.

\[ M_{E_1E_2}^{C} \Theta_5^{A_1A_2A_3E_1E_2} = \frac{3}{5} \Theta_7^{C;A_1A_2A_3} \]  
(B.1)

\[ M_{E_1E_2}^{EB:A_1A_2} \Theta_3^C_E = \frac{1}{2} \left( \hat{\Theta}_7^{A_1A_2;BC} - \hat{\Theta}_7^{BA_1;A_2C} \right) + \frac{1}{4} \Gamma A_1 \Theta_7^{A_2BC} \]  
(B.2)

\[ M_{E_1E_2}^{CE_2} M_{E_1E_2}^{A_1A_2} M_{E_1E_2}^{B_1B_2} \theta = \frac{1}{28} \left( \Gamma^C \hat{\Theta}_7^{A_1A_2;B_1B_2} + 2 \Gamma^A_1 \hat{\Theta}_7^{A_2B_1;B_2C} - 2 \Gamma^A_1 \hat{\Theta}_7^{B_1A_2;C_1B_2} - \Gamma A_1 \Gamma B_1 \Theta_7^{A_2B_2C} \right) \]  
(B.3)

\[ \hat{\Theta}_7^{BA_1A_2C} = \frac{3}{2} \Theta_7^{(B;C)_1A_2} \]  
(B.4)

\[ \Gamma_E M_{E_1E_2}^{EBF} \Theta_7^{C;A_1A_2} = M_{E_1E_2}^{A_1A_2E} \Theta_7^{BC_E} \]  
(B.5)

\[ \Theta_9^{ABC} = \frac{1}{24} \Gamma_{E_1E_2E_3} M_{E_1E_2E_3} \Theta_7^{A_1A_2B_1B_2C_1C_2} \]  
(B.6)

\[ M_{E_1E_2}^{AD} M_{D_1E_1}^{B_1B_2} \Theta_7^{C_1E_1} = \hat{\Theta}_7^{C_1AB} \]  
(B.7)

\[ M_{E_1E_2}^{AE} \Theta_9^{B;CE_1E_2} = \frac{2}{3} \hat{\Theta}_7^{B;AC} \]  
(B.8)

\[ \Theta_9^{(C;D)_1A_1A_2} = \frac{2}{3} M_{E_1E_2}^{EA_1A_2} \Theta_7^{CD_E} + \frac{1}{3} \Gamma A_1 \Theta_9^{A_2CD} \]  
(B.9)

\[ M_{E_1E_2}^{AE} M_{E_2E_3}^{BE} M_{E_3E_4}^{CE} M_{E_4E_5}^{DE} \Theta_7^{E_5E_1} = \frac{1}{42} \left( 2 \Gamma^A \hat{\Theta}_7^{B;CD} + \Gamma^A \hat{\Theta}_7^{A;C;BD} - 4 \Gamma^B \hat{\Theta}_7^{A,C;BD} - \Gamma^B \hat{\Theta}_7^{C;AD} - 4 \Gamma^C \hat{\Theta}_7^{A,B;D} - 5 \Gamma^C \hat{\Theta}_7^{B;A;D} + \Gamma^D \hat{\Theta}_7^{B;AC} - 2 \Gamma^D \hat{\Theta}_7^{C;AB} \right) \]  
(B.10)

\[ \Gamma_E M_{E_1E_2}^{EA} \hat{\Theta}_7^{F;BC} = \frac{1}{3} \Gamma^{(B;C)_1A} \]  
(B.11)

\[ \Gamma_E M_{E_1E_2}^{EA} \hat{\Theta}_7^{B;CF} = \frac{1}{2} \Gamma^A \hat{\Theta}_7^{B;BC} + \frac{1}{3} \Gamma^B \hat{\Theta}_7^{B;CA} + \frac{1}{6} \Gamma^C \hat{\Theta}_7^{B;AB} \]  
(B.12)
Appendix C. Young Projector Method

Let us consider a Young diagram $R$ with $n$ rows having $m_i$ boxes in the $i^{th}$ row ($m_1 \geq m_2 \geq \ldots \geq m_n$) and having $\lambda_j$ boxes in the $j^{th}$ column ($n = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$). The Young projector corresponding to a particular $(R_I)$ standard tableau is given by

$$Y(R_I) = \alpha(R)QP$$ \hspace{1cm} (C.1)

where $P_j$ is the (normalized) operator that fully symmetrizes over the entries of the $j^{th}$ row and $Q_i$ is the (normalized) one that fully antisymmetrizes over the entries of the $i^{th}$ column. For operators so normalized, the normalization factor $\alpha$ needed for $Y$ to be idempotent $Y^2 = Y$, is

$$\alpha(R) = \frac{\text{dim}(R)}{m!} \left( \prod_{j=1}^{n} m_j! \right) \left( \prod_{i=1}^{m_1} \lambda_i! \right)$$ \hspace{1cm} (C.2)

where $m = \sum_{j=1}^{n} m_j = \sum_{i=1}^{\lambda_1}$ is the total number of boxes in the Young diagram and $\text{dim}(R)$ the dimension of the irreducible representation of the symmetric group $S_m$ corresponding to the diagram $R$ \cite{21}. The products of factorials in (C.2) appear because we considered normalized $Q_i$ and $P_j$ in (C.1) ($Q_i^2 = Q_i, P_j^2 = P_j$).

There are 14 standard tableaux associated with the diagram \hspace{1cm} (C.3), however, due to identity (2.6) many of them do not contribute. The tableaux that give non-vanishing results are

$$\begin{array}{cccccccc}
1 & 4 & 1 & 4 & 1 & 4 & 1 & 5 \\
2 & 5 & 2 & 6 & 2 & 7 & 2 & 6 \\
3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \\
6 & 5 & 6 & 7 & 6 & 7 & 6 & 5 \\
\end{array}$$

and the results for all the tableaux can be inferred from the first two

$$Y \left( \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array} \right) M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2}D =$$

$$= \frac{\alpha}{4} (M^{[A_1A_2A_3M^{C_1C_2}]}DM^{B_1B_2D} + M^{[B_1A_2A_3M^{C_1C_2}]}DM^{A_1B_2D} + M^{[A_1B_2A_3M^{C_1C_2}]}DM^{B_1A_2D} + M^{[B_1B_2A_3M^{C_1C_2}]}DM^{A_1A_2D})$$ \hspace{1cm} (C.4)
\[
\begin{aligned}
\alpha &= \frac{\alpha}{8} (M^{[A_1 A_2 A_3 M B_2 C_2]} D M^{C_1 B_1 D} + M^{[B_1 A_2 A_3 M B_2 C_2]} D M^{A_1 C_1 D} \\
&+ M^{[A_1 C_1 A_3 M B_2 C_2]} D M^{B_1 A_2 D} + M^{[B_1 C_1 A_3 M B_2 C_2]} D M^{A_1 A_2 D}),
\end{aligned}
\]
(C.5)

the letter convention has been momentarily suspended in (C.4) and (C.5).

So, to obtain the total projection corresponding to the diagram we add the contributions of all the standard tableaux in (C.3)

\[
Y \left( \begin{array}{c} \text{Y} \\ \text{Y} \end{array} \right) M^{A_1 A_2 A_3 M B_1 B_2 D M^{C_1 C_2 D}} =
\]
\[
= \frac{\alpha}{4} (M^{[A_1 A_2 A_3 M B_1 B_2]} D M^{C_1 C_2 D} + M^{[A_1 A_2 A_3 M C_1 C_2]} D M^{B_1 B_2 D} \\
+ 2 M^{[A_1 A_2 A_3 M B_1 C_1]} D M^{B_2 C_2 D})
\]
(C.6)

\[
\alpha = \frac{\dim \left( \begin{array}{c} \text{Y} \\ \text{Y} \end{array} \right)}{7!} (2!2!)(5!2!) = \frac{8}{3}
\]

A comment is in order here. In projecting an arbitrary tensor one obtains a different irreducible representation for each standard tableau \([20]\). The same is not true here, of course, because of the nilpotency of the \(\theta\)-tensors. Each irreducible representation appears only once at each level in Table 1. The number of degrees of freedom are dramatically reduced by the nilpotency of these structures and that is why the problem becomes manageable. For instance, the product \(M^{A_1 A_2 A_3 M^{B_1 B_2 B_3}}\) instead of having \(\binom{10}{3} \times \binom{10}{3} = 120^2 = 14400\) degrees of freedom, it has only \(\binom{16}{4} = 770 + 1050 = 1820\). But doing the counting explicitly by subtracting the number of independent constraints implied by the conditions on the irreducible pieces and otherwise derivable identities, can be an extremely painful task. However, one does not need to dwell into all that detail, fortunately, but rather proceed to add all the projectors for the different standard tableaux corresponding to a Young diagram in order to consistently extract the unique representation involved in all the cases.

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