An exact solution of the slow-light problem

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Abstract

We investigate propagation of a slow-light soliton in atomic vapors and Bose-Einstein condensates described by the nonlinear Λ-model. We show that the group velocity of the soliton monotonically decreases with the intensity of the controlling laser field, which decays exponentially after the laser is switched off. The shock wave of the vanishing controlling field overtakes the slow soliton and stops it, while the optical information is recorded in the medium in the form of spatially localized polarization. We find an explicit exact solution describing the whole process within the slowly varying amplitude and phase approximation. Our results point to the possibility of addressing spatially localized memory formations and moving these memory bits along the medium in a controllable fashion.

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In this paper we study the interaction of light with a gaseous active medium whose working energy levels are well approximated by the Λ-scheme. The model is a very close prototype for a gas of alkali atoms, whose interaction with the light is approximated by the structure of levels of the Λ-type given in Fig. 1. We consider the case when the atoms are cooled down to microkelvin temperatures in order to suppress the Doppler shift and increase the coherence life-time for the ground levels. Typically, in experiments [1, 2, 3, 4, 5, 6] the pulses have the length of microseconds, which is much shorter than the coherence life-time and longer than the optical relaxation times. The gas cell is illuminated by two circularly polarized optical beams co-propagating in the $z$-direction. One $\sigma^-$-polarized field is denoted as $a$, and the other $\sigma^+$-polarized field is denoted as $b$. We study dynamics of the fields within the slowly varying amplitude and phase approximation (SVEPA). The total superposition of the fields is represented in the form
\[
\vec{E} = \vec{e}_a E_a e^{i(k_a x - \omega_a t)} + \vec{e}_b E_b e^{i(k_b x - \omega_b t)} + c.c.
\] (1)

Here, $k_{a,b}$ are the wave numbers, while the vectors $\vec{e}_a, \vec{e}_b$ describe polarizations of the fields with the amplitudes $E_{a,b}$. We introduce two corresponding Rabi frequencies
\[
\Omega_a = \frac{2\mu_a E_a}{\hbar}, \quad \Omega_b = \frac{2\mu_b E_b}{\hbar},
\] (2)

where $\mu_{a,b}$ are dipole moments of corresponding transitions in the atom.

In the interaction picture and within the SVEPA the Hamiltonian describing the interaction of a three-level atom with the fields assumes the form
\[
H_\Lambda = H_0 + H_I, \quad H_0 = -\frac{\Delta}{2} D,
\] (3)
\[
H_I = -\frac{1}{2} (\Omega_a |3\rangle \langle 1| + \Omega_b |3\rangle \langle 2|) + h.c.,
\]
where

\[ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

We set \( \hbar = 1 \).

Dynamics of the fields is described by the Maxwell equations, which within the SVEPA take the form

\[ \partial_t H_I = \frac{i\nu_0}{4} [D, \rho]. \]  

(4)

Here we have introduced new variables \( \zeta = (z - z_0)/c, \tau = t - (z - z_0)/c \), \( \rho \) is the density matrix in the interaction representation, \( n_A \) is the density of atoms, and \( \epsilon_0 \) is the vacuum susceptibility. For many experimental situations it is typical that the coupling constants \( \nu_{a,b} = (n_A|\mu_{a,b}|^2\omega_{a,b})/\epsilon_0 \) are almost the same. Therefore we assume that \( \nu_a \approx \nu_b = \nu_0 \).

Together with the Liouville equation,

\[ \partial_\tau \rho = -i [H_\Lambda, \rho], \]  

(5)

we obtain a system of equations Eqs.(4),(5), which is exactly solvable in the framework of the Inverse Scattering (IS) method [7, 8, 9, 10].

At this point it is worth discussing the initial and boundary conditions underlying the physical problem in question. We consider a semi-infinite \( \zeta \geq 0 \) active medium with a pulse of light incident at the point \( \zeta = 0 \) (initial condition). This means that the evolution is considered with respect to the space variable \( \zeta \), while the boundary conditions should be specified with respect to the variable \( \tau \). In this paper we assume the atom-field system to be prepared in an initial state corresponding to the typical experimental setup (see e.g. [1, 2, 4]):

\[ \Omega_a(0) = 0, \quad \Omega_b(0) = \Omega(\tau), \quad |\psi_{\text{at}}\rangle = e^{-i\frac{\Delta}{2}\tau}|1\rangle. \]  

(6)

The state satisfies the Maxwell-Bloch system of equations Eqs.(4),(5). The field \( \Omega(\tau) \) plays the role of the controlling background field generated by a laser. In [10] (see also [8]) for the constant background field \( \Omega(\tau) = \Omega_0 = \Omega_0^* \) we found the slow-light soliton

\[ \Omega_a = \frac{-i\sqrt{2\epsilon_0}\Omega_0}{\sqrt{\epsilon_0} + \sqrt{\epsilon_0^2 - \Omega_0^2}} \text{sech} \phi_s, \quad \Omega_b = \Omega_0 \tanh \phi_s, \]  

(7)

where \( \phi_s = \frac{i\nu_0}{2\epsilon_0} - \frac{i}{2} \left( \epsilon_0 - \sqrt{\epsilon_0^2 - \Omega_0^2} \right) + \phi_0 \). In these expressions, for simplicity we set \( \Delta = 0, \epsilon_0 > \Omega_0 \). The meaning of the parameter \( \epsilon_0 \) is explained below. It can be readily seen
that the speed of the slow-light soliton sent into the system depends on the intensity of the controlling background field. In the simplifying approximation $\frac{\Omega^2}{\epsilon_0} \ll 1$, the group velocity reads

$$v_g \approx c \frac{\Omega^2}{2 \nu_0}. \quad (8)$$

This expression immediately suggests a plausible conjecture that when the controlling field is switched off the soliton stops propagating while the information borne by the soliton remains in the medium in the form of an imprinted polarization flip. In what follows we refer to this flip as a memory cell or memory imprint. For the time-dependent background field some approximate solutions based on the methods of scaled time are studied in [4, 11, 12, 13].

In this paper we provide strong evidence substantiating the dynamical mechanism formulated above as a hypothesis. This evidence is based on the exact solution described below.

We envisage the following dynamics scenario. We assume that the slow-light soliton was created in the system before the moment of time $t = 0$ and is propagating on the background of the constant controlling field $\Omega_0$. This is the property of the soliton, supported by the exact solution [10] that, after the soliton has passed, the medium returns to the initial quantum state Eq.(6). Suppose that at the moment in time $t = 0$ the laser source of the controlling field is switched off. We assume that after this moment the background field will exponentially decay with some characteristic rate $\alpha$. The exponential front of the vanishing controlling field will then propagate into the medium, starting from the point $z = z_0$, where the laser is placed. The state of the quantum system Eq.(6) is dark for the controlling field. Therefore the medium is transparent for the spreading front of the vanishing field, which then propagates with the speed of light, eventually overtaking the slow-light soliton and stopping it.

To realize the mechanism described above we define the time dependence of $\Omega(\tau)$ as follows:

$$\Omega(\tau) = \begin{cases} 
\Omega_0, & \tau < 0 \\
\Omega_0 \exp(-\alpha \tau), & \tau \geq 0 
\end{cases} \quad (9)$$

This regime of switching the field off is quite realistic. An experimental setup, where the parameter $\alpha$ becomes experimentally adjustable can be easily envisaged. Greater values of $\alpha$ correspond to steeper fronts of the incoming background wave. In what follows we also discuss this limit.
Using the methods of our previous work\cite{10}, we construct the exact analytical single-soliton solution of the problem. For the fields the solutions are
\begin{equation}
\tilde{\Omega}_a = \frac{(\lambda^* - \lambda)w(\tau, \lambda)}{\sqrt{1 + |w(\tau, \lambda)|^2}} e^{i\tilde{\phi}_s} \sech \tilde{\varphi}_s, \tag{10}
\end{equation}
\begin{equation}
\tilde{\Omega}_b = \frac{(\lambda - \lambda^*)w(\tau, \lambda)}{1 + |w(\tau, \lambda)|^2} e^{i\tilde{\phi}_s} \sech \tilde{\varphi}_s - \Omega(\tau),
\end{equation}
and for the atomic medium
\begin{equation}
|\tilde{\psi}_{\text{at}}\rangle = e^{-i\tilde{\Phi}_s} \left[ \left( \frac{\lambda^* - \Delta}{|\lambda - \Delta|} - \frac{\tilde{\Omega}_a e^{i\left(\frac{\nu_0}{2\omega} - \tilde{\theta}_0\right) - z(\tau, \lambda)}}{2|\lambda - \Delta|w(\tau, \lambda)} \right) |1\rangle + \frac{\tilde{\Omega}_a}{2|\lambda - \Delta|w(\tau, \lambda)} |2\rangle - \frac{\tilde{\Omega}_a}{2|\lambda - \Delta|} |3\rangle \right]. \tag{11}
\end{equation}
The parameters of the slow-light soliton are
\begin{equation}
\tilde{\phi}_s = \tilde{\varphi}_0 - \frac{\nu_0 \Im(\lambda)\zeta}{2|\Delta - \lambda|^2} + \Re(z(\tau, \lambda)) + \frac{1}{2} \ln(1 + |w(\tau, \lambda)|^2),
\end{equation}
\begin{equation}
\tilde{\theta}_s = \tilde{\theta}_0 - \frac{\nu_0 \zeta}{2} \Re\left( \frac{1}{\lambda - \Delta} \right) + \Im(z(\tau, \lambda)), \; \lambda \in \mathbb{C}. \tag{12}
\end{equation}
We find the functions $w, z$. For $\tau < 0$ these functions are:
\begin{equation}
w = w_0 = \frac{\Omega_0}{\lambda + \sqrt{\lambda^2 + \Omega_0^2}}, \; z(\tau, \lambda) = \frac{i}{2} \Omega_0 w_0 \tau, \tag{13}
\end{equation}
while for $\tau \geq 0$ the functions $w, z$ take the form:
\begin{equation}
z(\tau, \lambda) = -\alpha \gamma \tau + \ln \frac{C J_{-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right) + J_{\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}{C J_{-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right) + J_{\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}, \tag{14}
\end{equation}
\begin{equation}
w(\tau, \lambda) = i\frac{C J_{1-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right) - J_{\gamma-1}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}{C J_{1-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right) + J_{\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}, \tag{15}
\end{equation}
where $\gamma = \frac{\alpha + i\lambda}{2\alpha}$ and $J_\nu$ are Bessel functions. The constant $C$ is uniquely defined by the condition $w(0, \lambda) = w_0$
\begin{equation}
C = \frac{-iw_0 J_\gamma\left(-\frac{\Omega(\tau)}{2\alpha}\right) + J_{\gamma-1}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}{J_{1-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right) + iw_0 J_{-\gamma}\left(-\frac{\Omega(\tau)}{2\alpha}\right)}. \tag{16}
\end{equation}
To identify a physically relevant solution we require that $w(\infty, \lambda) = 0$. This requirement places a restriction on the parameter $\lambda$, such that $\Im(\lambda) < 0$ and hence $\Re(\gamma) > 1/2$.

We now discuss the physical properties of the exact solution given above. The group velocity of the slow-light soliton reads
\begin{equation}
\frac{v_g}{c} = \frac{|w(\tau, \lambda)|^2}{\nu_0(1 + |w(\tau, \lambda)|^2) + |w(\tau, \lambda)|^2}. \tag{17}
\end{equation}
Notice that in the case of the constant background field, i.e. in the case $\alpha = 0$, the conventional expressions for the slow-light soliton Eq. (7) along with the expression for the group velocity Eq. (8) - the main motivational quantity for this paper - can be readily recovered from Eqs. (10), (17).

We calculate the distance $L_s(\alpha)$ that the slow-light soliton will propagate from the moment $t = 0$, when the laser is switched off, until the full stopping of the signal. This distance is

$$L_s(\alpha) = \frac{2c|\Delta - \lambda|^2}{\nu_0 \operatorname{Im}(\lambda)} \frac{\tau=\infty}{\phi_s|\tau=0} = \frac{2c|\Delta - \lambda|^2}{\nu_0 \operatorname{Im}(\lambda)} \left[ \frac{1}{2} \ln \left( 1 + |w_0|^2 \right) - \operatorname{Re}(z(\infty, \lambda)) \right]. \quad (18)$$

It is clear that $\operatorname{Re} z(\infty, \lambda) \leq 0$ and hence $L_s(\alpha) > 0$. The case when the controlling field is instantly switched off corresponds to the limit $\alpha \to \infty$. In this limit the profile of the background field approaches the Heaviside step-function, and we find that $\lim_{\alpha \to \infty} \operatorname{Re} z(\infty, \lambda) = 0$. In this case, as is intuitively evident, the soliton will still propagate over some finite distance. Notice that the limits $\tau \to \infty$ and $\alpha \to \infty$ do not commute and the latter should be taken after the former.

The half-width of the polarization flip written into the medium after the soliton is completely stopped is

$$W_s = 4c \ln(2 + \sqrt{3}) \frac{|\Delta - \lambda|^2}{\nu_0 \operatorname{Im}(\lambda)}. \quad (19)$$

It is important to notice that the width Eq. (19) of the imprinted memory cells do not depend on the rate $\alpha$. In other words, the width of the memory cell is not sensitive to how rapidly the controlling field is switched off. This leads to an important conclusion. Indeed, through the variation of the experimentally adjustable parameter $\alpha$ it is possible to control the location of the memory cell, while the characteristic size of the imprinted memory cell remains intact. This result is strongly supported by recent experiments [14].

In what follows we assume the parameter $\lambda$ to be imaginary, i.e. $\lambda = -i\varepsilon_0$, with $\varepsilon_0 > \Omega_0$. We demonstrate the slow-light dynamics in Fig. 2 where we show how the slow-light soliton stops and disappears. In Fig. 3 we demonstrate the behavior of the intensity in the channel $b$. The groove in the constant background field corresponds to the slow-light soliton. This groove is complementary to the peak in the channel $a$. The shock wave, whose front has
FIG. 2: The slow-light soliton is stopped. Contour plot of the intensity $I_a$ of the field $\tilde{\Omega}_a$. The parameters used for all the plots in this report are the same: $\Delta = 0$, $\Omega_0 = 2$, $\varepsilon_0 = 2.1$, $\nu_0 = 10$, $\alpha = 1$.

FIG. 3: The intensity $I_b$ of the field $\tilde{\Omega}_b$ as a function of time $t$ and space $z$. The parameters are defined in Fig. 2. The groove corresponds to the slow-light soliton. The soliton collides with the shock wave of the vanishing control field, whose front has an exponential profile.

an exponential profile, propagates with the speed of light, reaches the slow-light soliton and stops it. Notice that after the collision with the slow-light soliton the front of the shock wave shows some short peak at a level higher than the background intensity. This effect reflects an essentially nonlinear nature of interactions inherent to the considered dynamical system. In Fig. 4 we plot the dynamics of the polarization flip in the atomic medium, which bears the information and stores it in the medium. Before the shock wave has approached, the spatially localized population of the level $|2\rangle$ moves together with the slow-light soliton as a composite whole. This is a nonlinear analog of the dark-state polariton. After the
FIG. 4: Imprinting the information. We plot the dynamics of the population $P_1$ of the level $|1\rangle$ and $P_2$ of the level $|2\rangle$ as functions of time $t$ and space $z$. It is clear that after the collision the populations freeze.

It is important to notice that, before the slow-light soliton is stopped, the population on the level $|2\rangle$ does not reach unity, because there is a remnant population in the upper state $|3\rangle$. However, after the polariton is completely stopped, the level $|3\rangle$ depopulates, while the population of the level $|2\rangle$ reaches unity at the maximum of the signal. Before and after the collision, the population of the level $|1\rangle$ at the minimum of the groove always vanishes. The result is valid for zero detuning. From our solution Eq. (11) it is not difficult to estimate the maximum population of the level $|2\rangle$ for finite detuning. After the soliton is completely stopped the population at the maximum is $\varepsilon_0^2/(\varepsilon_0^2 + \Delta^2)$.

Any destructive influence of the relaxation processes on the overall picture of dynamics described above is negligible. Indeed, as we show in Ref. 10, the population of the upper level $|3\rangle$ is proportional to the intensity of the background field $\Omega_0$, which is required to be small,
to ensure a small velocity of the signal (cf. Eq. (8)).

Discussion. In this report we have investigated the dynamics of a slow-light soliton whose group velocity explicitly depends on the background (controlling) field. Taking advantage of an explicit exactly solvable example, we demonstrate that the soliton can indeed be stopped, provided that the background field vanishes. After the signal is stopped, the information borne by the soliton is imprinted into the medium in the form of a spatially localized polarization flip. The width of the imprinted memory, in our example, does not depend on how rapidly the field vanishes. However, the position of the spatially localized optical memory imprint is controlled by the experimentally adjustable parameter \( \alpha \). Therefore, we have shown that our approach allows addressing of optical memory recorded into an atomic medium at an exact location without changing the characteristic size of the spatial domain occupied by the memory. The imprinted memory can be subsequently read. Moreover, our results point to the possibility of switching the controlling field on and off repeatedly, and by virtue of this mechanism move the imprinted memory through the medium in a strictly controlled fashion. The localized polarization will follow the sequence of step-like laser pulses and therefore hop over finite distances in a prescribed manner.

Another experimentally interesting application of our results is to inject multi-soliton packets into the medium, creating solitonic ‘trains’, whose ‘carriages’ are subsequently stopped by the shock wave of the vanishing background. In this way a multi-bit optical memory can be realized.

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[1] L. N. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Lett. to Nature 397, 594 (1999).
[2] C. Liu, Z. Dutton, C. H. Behroozi, and L. V. Hau, Lett. to Nature 409, 490 (2001).
[3] D. F. Phillips, A. Fleischhauer, A. Mair, R. L. Walsworth, and M. D. Lukin, Phys. Rev. Lett. 86, 783 (2001).
[4] M. Bajcsy, A. S. Zibrov, and M. D. Lukin, Lett. to Nature 426, 638 (2003).
[5] D. A. Braje, V. Balic, G. Y. Yin, and S. E. Harris, Phys. Rev. A 68, 041801(R) (2003).
[6] E. E. Mikhailov, V. A. Sautenkov, Y. V. Rostovtsev, and G. R. Welch, J. Opt. Soc. Am. B
21, 425 (2004).

[7] L. D. Faddeev and L. A. Takhtadjan, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, 1987).

[8] Q.-H. Park and H. J. Shin, Phys. Rev. A *57*, 4643 (1998).

[9] J. A. Byrne, I. R. Gabitov, and G. Kovačić, Physica D *186*, 69 (2003).

[10] A. V. Rybin and I. P. Vadeiko, Journal of Optics B: Quantum and Semiclassical Optics *6*, 416 (2004).

[11] R. Grobe, F. T. Hioe, and J. H. Eberly, Phys. Rev. Lett. *73*, 3183 (1994).

[12] T. N. Dey and G. S. Agarwal, Phys. Rev. A *67*, 033813 (2003).

[13] U. Leonhardt, Preprint ArXiv: quant-ph/0408046 (2004).

[14] Z. Dutton and L. V. Hau, Phys. Rev. A *70*, 053831 (2004).

[15] M. Fleischhauer and M. D. Lukin, Phys. Rev. Lett. *84*, 5094 (2000).