A NOTE ON \( p \)-ADIC INVARIANT INTEGRAL
IN THE RINGS OF \( p \)-ADIC INTEGERS

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ABSTRACT. In [2], I constructed the \( p \)-adic \( q \)-integral \( I_q(f) \) on \( \mathbb{Z}_p \). In this paper, we consider the properties of \( \lim_{q \to -1} I_q(f) = I_{-1}(f) \). Finally we give the some applications of \( I_{-1}(f) \) and integral equations for \( I_{-1}(f) \). These properties are useful and worthwhile to study Euler numbers and polynomials.

1. Introduction

Let \( p \) be a fixed odd prime. Throughout \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will respectively denote the rings of \( p \)-adic integers, the fields of \( p \)-adic numbers and the completion of of the algebraic closure of \( \mathbb{Q}_p \). For a fixed positive integer with \((p, d) = 1\), let

\[
X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N, \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp\mathbb{Z}_p,
\]

\[
a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N}\},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \), (cf. [1], [2]).

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The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let $q$ be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume $|q - 1|_p < 1$ for $|x|_p \leq 1$. Throughout this paper, we use the following notation:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}.$$ 

We say that $f$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$– and denote this property by $f \in UD(\mathbb{Z}_p)$– if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y},$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$, cf. [1, 2]. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p),$$

representing $q$-analogue of Riemann sums for $f$.

The integral of $f$ on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of these sums, when it exists. An invariant $p$-adic $q$-integral of a function $f \in UD(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} f(j) q^j,$$

Note that if $f_n \to f$ in $UD(\mathbb{Z}_p)$; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

The classical Euler polynomials are defined by

$$F(t, x) = \frac{2}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$ 

Note that $E_n = E_n(0)$ are called $n$-th Euler numbers. In [3], an analogue of Bernoulli number is defined by $\frac{t}{we^{t-1}} = \sum_{n=0}^{\infty} B_n(w) \frac{t^n}{n!}$, where $w$ is the element of locally constant functions’ space [3]. For $f \in UD(\mathbb{Z}_p)$, we define a $p$-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q \to -1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

The purpose of this paper is to give the integral equations related to $I_{-1}(f)$ and to investigate some properties for $I_{-1}(f)$. From these integral equations, we derive the interesting formulae related to Euler numbers.
§2. \( p \)-adic invariant integrals on \( \mathbb{Z}_p \)

For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral was defined by

\[
I_q(f) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} q^x f(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} f(x) \mu_q(x + p^N \mathbb{Z}_p) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x),
\]

representing \( p \)-adic \( q \)-analogue of Riemann integral for \( f \) (see [2]). In the meaning of fermionic, we now define \( I_{-1}(f) \)-integral as

\[
I_{-1}(f) = \lim_{q \to -1} I_q(f) = \lim_{q \to -1} \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).
\]

From (2), we derive the below Eq.(3):

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x).
\]

Let \( f_1(x) = f(x+1) \). Then we note that

\[
I_{-1}(f_1) = - \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x + 2f(0) = -I_{-1}(f) + 2f(0).
\]

Therefore we obtain the below theorem:

**Theorem 1.** For \( f \in UD(\mathbb{Z}_p) \), we have

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0),
\]

where \( f_1(x) \) is translation with \( f_1(x) = f(x+1) \).

From Theorem 1, we can derive the below theorem:

**Theorem 2.** For \( f \in UD(\mathbb{Z}_p) \), \( n \in \mathbb{N} \), we have

\[
I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{x=0}^{n-1} f(x)(-1)^{n-1-x},
\]

where \( f_n(x) = f(x+n) \).
By using Theorem 1 and Theorem 2, we can consider the new extension of Euler numbers and polynomials. If we take $f(x) = \lambda^x e^{xt}$, $(\lambda \in \mathbb{Z}_p)$, then we have

$$\frac{2}{\lambda e^t + 1} = \int_{\mathbb{Z}_p} \lambda^x e^{xt} d\mu_1(x).$$

Define the new extension of Euler numbers as follows:

$$\frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} E_n(\lambda) \frac{t^n}{n!}.$$

By (5) and (6), we obtain the below theorem:

**Theorem 3. (Witt’s formula)** For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = E_n(\lambda), \lambda \in \mathbb{Z}_p,$$

where $E_n(\lambda)$ are called analog of Euler numbers.

By using $I_1$-integral, we define the new extension of classical Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} \lambda^y (x + y)^n d\mu_1(y) = E_n(\lambda : x).$$

From (7), we can derive the below:

$$E_n(\lambda : x) = \sum_{k=0}^{n} \binom{n}{k} E_n(\lambda) x^{n-k} = f^n \sum_{a=0}^{f-1} (-1)^a \lambda^a E_n(\lambda^f : \frac{x + a}{f}),$$

where $f$ is odd positive integer.

By (8), we easily see that

$$\frac{2e^{xt}}{\lambda e^t + 1} = \int_X e^{(x+y)t} d\mu_1(y) = \frac{2 \sum_{a=0}^{f-1} (-1)^a \lambda^a e^{(x+a)t}}{\lambda^f e^f t + 1}.$$
Remark 1. In [3], it was known that

\[
\lim_{q \to 1} \int_X w^y e^{(x+y)t} d\mu_q(t) = \frac{te^t}{w e^t - 1} = \sum_{m=0}^{\infty} B_m(w) \frac{t^m}{m!},
\]

where \(B_m(w)\) are called an analogue of Bernoulli numbers. In the viewpoint of (9-1), we gave the new extension \(E_n(\lambda)\) of classical Euler number.

Remark 2. From using multivariate \(p\)-adic invariant integral for \(I_{-1}(f)\), we can easily derive the Euler polynomials of higher order as follows:

\[
e^t \int_{Z_p} \cdots \int_{Z_p} \lambda^{y_1+\cdots+y_r} e^{(y_1+\cdots+y_r)t} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) = \left(\frac{2}{\lambda e^t + 1}\right)^r e^t.
\]

From this, we can define the extension of classical Euler polynomials of order \(r\) as follows:

\[
\left(\frac{2}{\lambda e^t + 1}\right)^r e^t = \sum_{n=0}^{\infty} E_n^{(r)}(\lambda : x).
\]

Let \(\chi\) be the Dirichlet’s character with conductor \(f(= \text{odd}) \in \mathbb{N}\) and let us take \(f(x) = \chi(x) e^t x\). From Theorem 2, we derive the below formula:

\[
\int_X e^{tx} \chi(x) d\mu_{-1}(x) = \frac{\sum_{a=0}^{f-1} e^{ta} (-1)^a \chi(a)}{e^{ft} + 1}.
\]

The generalized Euler numbers attached to \(\chi\) were defined by

\[
2 \sum_{a=0}^{f-1} \frac{e^{ta} (-1)^a \chi(a)}{e^{ft} + 1} = \sum_{n=0}^{\infty} E_n^{(r)}(\lambda : x), \text{ cf. [4]}. \]

From (10) and (11), we derive the below Witt’s formula for the generalized Euler numbers attache to \(\chi\) as follows:

**Theorem 4.** Let \(f\) be an odd positive integer and let \(\chi\) be the Dirichlet’s character with conductor \(f\). Then we have

\[
\int_X x^n \chi(x) d\mu_{-1}(x) = E_{n,\chi}.
\]
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