Some problems in the theory of Hardy spaces *†

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Abstract

This paper provides a study of several problems related to Hardy spaces left by G. Weiss in [4]. First, we will prove that the Hardy spaces $H^p(\mathbb{R}^n)(0 < p \leq 1)$ can be characterized by the Lipschitz function. Second, we will also prove that there exists a $H^1_\gamma(\mathbb{R})$ such that $H^p(\mathbb{R}) \subset H^1_\gamma(\mathbb{R})$ for $0 < p \leq 1$.

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1 Introduction

Fefferman and Stein in [1] showed that $H^p(\mathbb{R}^n)$ can be defined as follows:

**Theorem 1.1.** For $0 < p \leq \infty$, let $f$ be a distribution, then the following conditions are equivalent:

1. There is a $\phi \in S(\mathbb{R}^n)$ with $\int \phi(x)dx \neq 0$ so that $M_\phi f \in L^p(\mathbb{R}^n)$
2. The distribution $f$ is a bounded distribution and $\sup_{|u-x|<1} (f \ast P_t)(u) \in L^p(\mathbb{R}^n)$.
3. $M f(x) = \sup_{\phi \in \mathcal{S}_p, \sup_{\phi>0} (f \ast \phi_t)(x) \in L^p(\mathbb{R}^n)$, where $\mathcal{F} = \{\|\cdot\|_{a,b}\}$ is any finite collection of semi-norms on $S(\mathbb{R}^n)$, and $S_F$ is a subset of $S(\mathbb{R}^n)$ controlled by this collection of semi-norms: $S_F = \{\phi \in S(\mathbb{R}^n): \|\phi\|_{a,b} \leq 1 \text{ for any } \|a\| \cdot \|b\| \in F\}$.

They also discussed the minimal conditions on $\phi$ so that $M_\phi f \in L^p(\mathbb{R}^n)$ with $\|M_\phi f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}$ whenever $f \in H^p_p(\mathbb{R})$.

(a) For $\phi$ that have compact support, it suffices to have $\phi \in \Lambda_\gamma$ for some $\gamma > n(p-1) - 1$.

(b) For $\phi$ not having compact support but vanishing at infinity and satisfying $|\partial_\gamma^\alpha \phi(x)| \lesssim (1 + |x|)^{-N}$ for $|\gamma| = n(p-1) + 1$, it suffice to have $N_p > n$.

In [3] for $n = 1$ and in [10] for $n > 1$, the Lipschitz spaces $\Lambda_\gamma$ can be paired with $H^p$ if $\gamma = n(p-1)$ and $0 < p < 1$. That is for any $f \in H^p_p(\mathbb{R})$, the following holds:

$$\|f\|_{H^p(\mathbb{R}^n)} = \sup_{\|\phi\|_{\Lambda_\gamma} \leq 1} \left| \int f(x)g(x)dx \right|.$$ (1)

Notice that $H^p_p(\mathbb{R}^n)$ $(1 > p \leq 1)$ spaces is also a kind of Homogeneous spaces where the topological spaces $X$ is $\mathbb{R}^n$ and the distance $d(x,y)$ is $|x-y|^n$. Thus from [8], we could also see that the $H^p_p(\mathbb{R}^n)$ $(1 > p \leq 1)$ spaces can also be defined as following:

$$\|f\|_{H^p_p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{0 \neq \phi} \left\{ \int_{\mathbb{R}^n} f(y)\phi(y)dy \right\}/r^n : r > 0, \text{supp } \phi \subset B(x,r),$$ (2)

$$H^\gamma(\phi) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\} dx,$$

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where \( f \in (\Lambda^\gamma)' \).

From Theorem 1.1, Formula (1) and Formula (2), we could see that the smoothness of the kernel in Theorem 1.1 may be reduced. Thus the problem reducing smoothness of \( \phi \) in Hardy spaces was proposed by some mathematicians. The example that \( \phi = \chi_{\{ x \mid |x| > 1 \}} \) shows that the assumption of smoothness of \( \phi \) can not be removed in the definition of \( H^p \). This shows that the Hardy-Littlewood maximal function \( M f \) can not characterize any \( H^p \) for \( 0 < p \leq 1 \). Thus we wish to replace the \( \phi \) in Theorem 1.1 which is a Schwartz function with the \( \phi \) which is only a Lipschitz function.

Our first main result is Theorem 2.8. We will prove in Theorem 2.8 that the following holds for \( \frac{1}{1+\gamma} < p \leq 1 \):

\[
\| f \|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \| (f * \phi) \varphi \|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \| (f * \phi)_+ \|_{L^p(\mathbb{R}^n)},
\]

where the \( \phi \) is a fixed Lipschitz function with compact support satisfying \( \int \phi(x)dx \sim 1 \). From the above Formula, we could also see that the norm of radial maximal function \( (f * \phi)_+ \) and the norm of non-tangential maximal function \( (f * \phi)^\ast \) are equivalent when \( \phi \in \Lambda^\gamma \) and \( \phi \) is a function with compact support. The Formula (3) is different to Formula (2), because the \( \phi \) in Formula (2) is not a fixed Lipschitz function.

G. Weiss also asked whether there was an \( H^1_0(\mathbb{R}) \) that was neither \{0\} nor \( H^1(\mathbb{R}) \). In 1983, Uchiyama and Wilson in [12] proved that there exists \( \phi(x) \) such that \( H^1_0(\mathbb{R}) \neq H^1(\mathbb{R}) \) and \( H^1_0(\mathbb{R}) \neq \{0\} \). Thus a question whether there was an \( H^p_0(\mathbb{R}) \) that was neither \{0\} nor \( H^p(\mathbb{R}) \) for \( 0 < p \leq 1 \) is still an open problem that is unknown. From Formula (3), we could know that \( H^p_0(\mathbb{R}) = H^p(\mathbb{R}) \) for \( \frac{1}{1+\gamma} < p \leq 1 \) when \( \phi \) is a Lipschitz function with compact support. In 1983, Han in [5] give another characterization of \( H^1_0(\mathbb{R}) \) with the Carleson measure. He proved the following inequality:

\[
\| f \|_{H^1} \sim \sup_{\phi \in \text{BMO} \cap \mathcal{S}(\mathbb{R}^n), \| \phi \|_{\text{BMO}} \leq 1} \left| \int_{\mathbb{R}^n} f(x) \phi(x)dx \right|.
\]

Han also proved that \( (f * \phi) \varphi \in L^1(\mathbb{R}) \Rightarrow f(x) = 0 \ a.e. x \in \mathbb{R} \) for \( \phi \) which a Lipschitz function without compact support satisfying the following:

\[
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{1+\gamma}},
\]

\[
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{1+2\gamma}}, \quad \text{if } |h| \lesssim |x|/2.
\]

when \( 1 \geq \gamma > 0 \). However, when we replace the Formulas (4) and (5) with Formulas (43) and (44), we could deduce the Proposition 3.1: \( f \in H^p(\mathbb{R}^n) \Rightarrow (f * \phi) \varphi \in L^p(\mathbb{R}^n) \) for \( \frac{1}{1+\gamma} < p \leq 1 \).

Our second main result is Theorem 4.5. We will prove that the following holds for \( 0 < p \leq 1 \):

\[
H^p(\mathbb{R}) \subset H^p_0(\mathbb{R}),
\]

where \( \phi(y) \in S^m(\mathbb{R}) \) for \( m \geq |p^{-1} - 1| \), with \( \supp \phi(y) \subseteq I(0, 10) \cap I(0, 5)^c \).

## 2 Lipschitz function with compact support in \( \mathbb{R}^n \)

Let \( \mathcal{S}(\mathbb{R}^n) \) be the space of \( C^\infty \) functions on \( \mathbb{R}^n \) with the Euclidean distance rapidly decreasing together with their derivatives (Schwartz Class), \( \mathcal{S}'(\mathbb{R}^n) \) the tempered distributions. In this paper, \( 0 < \gamma \leq 1, \beta \in \mathbb{R}^+ \), \( \alpha \in \mathbb{N}^n \) satisfying:

\[
\alpha = (\alpha_i)_{i=1}^n, \quad \text{where } \alpha_i \in \mathbb{N}, \quad \text{and } |\alpha| = \sum_{i=1}^n \alpha_i.
\]

We use \( \mathbb{S}^n \) to denote the unit sphere in \( \mathbb{R}^{n+1} \), \( B(x, r) \) or \( I(x, r) \) to denote the set:

\[
B(x, r) = I(x, r) = \{ y : |x - y| < r \}.
\]

\( B(x, r_1) \setminus B(y, r_2) \) is denoted as the set:

\[
B(x, r_1) \cap B(y, r_2)^c.
\]
Definition 2.1 (H^\gamma(\phi), [\phi]_\beta). For \phi \in C(\mathbb{R}^n), n \in \mathbb{N}, \alpha, we use \{\beta\}, [\beta], H^\gamma(\phi) to denote as:

\{\beta\} = \beta - [\beta]; \quad [\beta] = \max\{m : m \in \mathbb{Z}; m \leq \beta\};

H^\gamma(\phi) = \sup_{x,y \in \mathbb{R}^n, x \neq y} |\phi(x) - \phi(y)|/|x - y|^\gamma;

|\phi|_\beta = \sup_{\alpha \in \mathbb{N}_0, |\alpha| \leq |\beta|} H^{(\beta)}(\partial^\alpha \phi) where |\alpha| \leq |\beta|.

For \( f \in L^1(\mathbb{R}^n) \), the maximal function \( f^*_\gamma(x) \) in \( \mathbb{R}^n \) is defined as:

\[
f^*_\gamma(x) = \sup_{\phi, \gamma > 0} \left\{ \left( \int_{\mathbb{R}^n} f(y) \phi(y) dy \right)^{1/\gamma} \bigg| r^n : r > 0, \text{supp} \phi \subset B(x, r), H^\gamma(\phi) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\}.
\]

For \( f \in S'(\mathbb{R}^n) \), we use \( f^*_\gamma(x) \) to denote as:

\[
f^*_\gamma(x) = \sup_{\phi, \gamma > 0} \left\{ \left( \int_{\mathbb{R}^n} f(y) \phi(y) dy \right)^{1/\gamma} \bigg| r^n : r > 0, \text{supp} \phi \subset B(x, r), H^\gamma(\phi) \leq r^{-\gamma}, \phi \in S(\mathbb{R}^n), \|\phi\|_{L^\infty} \leq 1 \right\}.
\]

Definition 2.2 (SS_\beta). We use SS_\beta (\beta > 0) to denote as

\[
SS_\beta = \left\{ \phi : \phi \in S(\mathbb{R}^n), \text{supp} \phi \subset B(0, 1), \|\phi\|_{L^\infty} \leq 1, [\phi]_\beta \leq 1 \right\}.
\]

For \( f \in S'(\mathbb{R}^n) \), we could also define \( f^*_\gamma \) and \( f^*_\beta \) as following:

\[
f^*_\gamma(x) = \sup_{\phi, \gamma > 0} \left\{ \left( \int_{\mathbb{R}^n} f(y) \phi(y) dy \right)^{1/\gamma} \bigg| r^n : r > 0, \psi(t) \in S(\mathbb{R}^n), \text{supp} \psi(t) \subset B(0, 1), \|\psi\|_{L^\infty} \leq 1, H^\gamma \psi \leq 1 \right\};
\]

\[
f^*_\beta(x) = \sup_{\phi, \gamma > 0} \left\{ \left( \int_{\mathbb{R}^n} f(y) \phi(y) dy \right)^{1/\gamma} \bigg| r^n : r > 0, \psi(t) \in SS_\beta \right\}.
\]

Definition 2.3 (M_\alpha f(x)). For \( f \in S'(\mathbb{R}^n) \), \( M_\alpha f(x) \) is defined as

\[
M_\alpha f(x) = \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} f(y) \phi(x - y) dy \bigg| r^n : r > 0, \phi(t) \in S(\mathbb{R}^n) \right\}.
\]

The Lipschitz function \( L^\gamma \) is defined as follow

\( L^\gamma = \{ f : \sup_{x \in \mathbb{R}^n} |f(x - y) - f(x)| \leq C|y|^\gamma \}. \)

(\( L^\gamma \))' is denoted as the dual space of \( L^\gamma \). We use \((f * \phi)_\gamma(x)\) and \((f * \phi)_+(x)\) to denote as:

\[
(f * \phi)_\gamma(x) = \sup_{|x - u| < y} |f * \phi_y(u)|, \quad (f * \phi)_+(x) = \sup_{y > 0} |f * \phi_y(x)|,
\]

where \( \phi_y(x) = \frac{1}{y^n} \phi\left(\frac{x}{y}\right) \).

Proposition 2.4. [9] For fixed numbers \( a \geq b > 0 \), \( F(x, r) \) is a function defined on \( \mathbb{R}^{n+1} \), its nontangential maximal function \( F^*_\alpha(x) \) is defined as

\[
F^*_\alpha(x) = \sup_{|x - y| < ar} |F(y, r)|.
\]

If \( F^*_\alpha(x) \in L^1(\mathbb{R}^n) \) or \( F^*_\alpha(x) \in L^1(\mathbb{R}^n) \), then we could obtain the following inequality for \( p > 0 \):

\[
\int_{\mathbb{R}^n} |F^*_\alpha(x)|^p dx \leq c \left( \frac{a + b}{b} \right)^n \int_{\mathbb{R}^n} |F^*_\alpha(x)|^p dx,
\]

where \( c \) is a constant independent on \( F, a, b \).
Proposition 2.5. For $f \in L^1(\mathbb{R}^n)$, $\infty > p > 0$ we could obtain

$$f_{\gamma}^*(x) = f_\gamma^*(x) \quad a.e. x \in \mathbb{R}^n.$$  

Further more, if $\int_{\mathbb{R}^n} |f_\gamma^*(x)|^p dx \leq \infty$ or $\int_{\mathbb{R}^n} |f_{\gamma}^*(x)|^p dx \leq \infty$, then we could obtain

$$\int_{\mathbb{R}^n} |f_\gamma^*(x)|^p dx \sim \int_{\mathbb{R}^n} |f_{\gamma}^*(x)|^p dx < \infty.$$  

Proof. We will prove the following Formula (11) first:

$$f_{\gamma}^*(x) = f_\gamma^*(x) \quad a.e. x \in \mathbb{R}^n. \quad (11)$$

By the definition of $f_{\gamma}^*(x)$ and $f_\gamma^*(x)$, it is clear that $f_{\gamma}^*(x) \leq f_\gamma^*(x)$. If $\phi$ satisfies $L(\phi, \gamma) \leq r^{-\gamma}$ and $\text{supp} \phi \subset B(x, r)$, then $\phi$ is a continuous function with compact support. Thus there exists sequence $\{\psi_k\}_k \subset S(\mathbb{R}^n)$ with $\lim_{k \to \infty} \|\psi_k(t) - \phi(t)\|_\infty = 0$, $\|\psi_k(t) - \phi(t)\|_\infty \neq 0$. Denote $\delta_k(x)$ as

$$\delta_k(x) = \left| \int_{B(x, r)} f(y) (\phi(y) - \psi_k(y)) dy/r^n \right|.$$  

Then we could conclude:

$$\delta_k(x) \leq M f(x) \|\psi_k(y) - \phi(y)\|_\infty.$$  

We use $i_k$ to denote as $i_k = \|\psi_k(y) - \phi(y)\|_\infty$, thus we could obtain that:

$$\{x : \delta_k(x) > \alpha\} \subseteq \left\{x : M f(x) > \frac{\alpha}{i_k}\right\}.$$  

Notice that $M$ is weak-(1, 1) bounded, thus the following inequality holds for any $\alpha > 0$:

$$|\{x : \delta_k(x) > \alpha\}| \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)} \|\psi_k(y) - \phi(y)\|_\infty.$$  

Thus

$$\lim_{k \to +\infty} |\{x : \delta_k(x) > \alpha\}| = 0.$$  

Then there exists a sequence $\{k_j\} \subseteq \{k\}$ such that

$$\int_{\mathbb{R}^n} f(y)\phi(y)dy/r^n = \lim_{k_j \to \infty} \int_{\mathbb{R}^n} f(y)\psi_{k_j}(y)dy/r^n, \quad a.e. x \in \mathbb{R}^n$$

for $f \in L^1(\mathbb{R}^n)$. Thus we could obtain:

$$\int_{\mathbb{R}^n} f(y)\phi(y)dy/r^n \leq f_{\gamma}^*(x) \quad a.e. x \in \mathbb{R}^n$$

for any $\phi$ satisfies $L(\phi, \gamma) \leq r^{-\gamma}$ and $\text{supp} \phi \subset B(x, r)$. We could then deduce

$$\sup_{\phi, r > 0} \left| \int_{\mathbb{R}^n} f(y)\phi(y)dy/r^n \right| \leq f_{\gamma}^*(x) \quad a.e. x \in \mathbb{R}^n.$$  

Thus

$$f_{\gamma}^*(x) = f_\gamma^*(x) \quad a.e. x \in \mathbb{R}^n.$$  

Let $E$ denote a set defined as $E = \{x : f_{\gamma}^*(x) = f_\gamma^*(x)\}$. Next we will prove that for any $x_0 \in \mathbb{R}^n$, there is a point $\pi_0 \in E$ such that

$$f_{\gamma}^*(x_0) \lesssim f_{\gamma}^*(\pi_0). \quad (12)$$

Notice that for $x_0 \in \mathbb{R}^n$, there exist $r_0 > 0$ and $\phi_0$ satisfying: $\text{supp} \phi_0 \subset B(x_0, r_0)$, $\phi_0 \in S(\mathbb{R}^n)$, $L(\phi_0, \gamma) \leq r_0^{-\gamma}$, $\|\phi_0\|_{L^\infty} \leq 1$. Then the following inequality could be obtained:

$$\frac{1}{r_0} \int f(y)\phi_0(y)dy \geq \frac{1}{2} f_{\gamma}^*(x_0).$$
We use $S$ to denote as $\{x \in \mathbb{R}^n : |x| < 1\}$, $\infty > p > \frac{1}{1+\gamma}$, $|\phi(x)| \leq 1$ if $x \in S$, then we could deduce the following inequality for any $f \in H^p(\mathbb{R}^n)$:

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \gamma \phi \|f * \phi\|_{L^p(\mathbb{R}^n)}.$$  \hfill (15)

**Proof.** Let $f \in L^1(\mathbb{R}^n)$, $\psi \in S(\mathbb{R}^n)$ with $\int \psi(x)dx \sim 1$. There exists sequence $\{\phi^m(x) : \phi^m(x) \in S(\mathbb{R}^n)\}_{m \in \mathbb{N}}$ satisfying:

$$\|\phi^m(x) - \phi(x)\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{1}{m}, \quad \int \phi^m(x)dx \sim 1.

We use $\varphi \in S(\mathbb{R}^n)$ to denote as:

$$\begin{aligned}
\varphi(\xi) &= 0 \text{ for } |\xi| \geq 1 \\
\varphi(\xi) &= 1 \text{ for } |\xi| \leq 1/2.
\end{aligned}

We use $\varphi^k \in S(\mathbb{R}^n)$ to denote as:

$$\begin{aligned}
\varphi^k(\xi) &= \varphi(\xi) \text{ for } k = 0, \\
\varphi^k(\xi) &= \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi) \text{ for } k \geq 1.
\end{aligned}

Then we could have that

$$1 = \sum_{k=0}^{\infty} \varphi^k(\xi).$$

Notice that $\int \phi^m(x)dx \sim 1$, thus $(\mathcal{F}\phi^m)(2^{-k_0}\xi) \geq C$ for $|\xi| \leq 1$, where $C$ and $k_0$ are independent on $m$. Then $\eta^k_m$ can be defined as

$$(\mathcal{F}\eta^k_m)(\xi) = \frac{\varphi^k(\xi)(\mathcal{F}\psi)(\xi)}{(\mathcal{F}\phi^m)(2^{-k-k_0}\xi)}.$$
where \( \mathcal{F} \) denotes the Fourier transform. Then we could obtain that:

\[
(\mathcal{F}\psi)(\xi) = \sum_{k=0}^{\infty} \frac{\phi^k(\xi)(\mathcal{F}\psi)(\xi)}{(2^{-k-k_0} \xi)} (\mathcal{F}\phi^m)(2^{-k-k_0} \xi)
\]

\[
= \sum_{k=0}^{\infty} (\mathcal{F}\eta^k_m)(\xi)(\mathcal{F}\phi^m)(2^{-k-k_0} \xi).
\]

Thus

\[
\psi(x) = \sum_{k=0}^{\infty} \eta^k_m \ast \phi^m_{2^{-k-k_0}}(x).
\]

(16)

By the fact that \( \sup_{\xi \in \mathbb{R}^n} |\partial^\alpha \mathcal{F}^m(\phi^m)(\xi)| \leq C_\alpha \) and

\[
\sup_{\xi \in \mathbb{R}^n} \left| \partial^\alpha \mathcal{F}^m \left((\mathcal{F}\psi)(\xi)\right) \right| \leq C_{\alpha,M} 2^{-kM} \text{ for any } M > 0,
\]

where \( C_\alpha \) is a constant independent on \( m \), we could deduce that

\[
\sup_{\xi \in \mathbb{R}^n} \left| \partial^\alpha \mathcal{F}^m \left((\mathcal{F}\eta^k_m)(\xi)\right) \right| \leq C_{\alpha,M,k_0} 2^{-kM} \text{ for any } M > 0,
\]

where \( C_{\alpha,M,k_0} \) is a constant independent on \( m \) and \( k \). Thus we could have:

\[
\left| \int_{\mathbb{R}^n} \eta^k_m (u) \left(1 + 2^{k+k_0} |u|^N\right) du \right| \leq C_{k_0,N} 2^{-k_0},
\]

(19)

where \( C_{k_0,N} \) is a constant independent on \( m \). Then by Formulas (16) with the fact that \( f \in L^1(\mathbb{R}^n) \) we have

\[
M_{\psi} f(x) = \sup_{r > 0} \left| \int_{B(x,r)} f(y) \frac{1}{r^n} \psi \left(\frac{x-y}{r}\right) dy \right| \leq C \sup_{r > 0} \left| \int_{B(x,r)} f(y) \frac{1}{r^n} \phi^m \left(\frac{x-y-s}{2^{-k-k_0} r}\right) ds \right| dy
\]

\[
\leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \eta^k_m \left(\frac{s}{r}\right) \left(1 + \frac{|s|}{2^{-k-k_0} r}\right)^N ds \sup_{r > 0, s \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left(\frac{x-y-s}{r}\right) \left(1 + \frac{|s|}{r}\right)^{-N} ds \right| dy.
\]

(20)

where \( C \) is a constant independent on \( m \). From Formula (19) and Formula (20) we could obtain:

\[
M_{\psi} f(x) \leq \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left(\frac{x-y-s}{r}\right) \left(1 + \frac{|s|}{r}\right)^{-N} dy \right| \leq \left( \sup_{0 \leq |s| < r} \sum_{k=0}^{\infty} \int_{B(x,r)} f(y) \phi^m \left(\frac{x-y-s}{r}\right) dy \right) \left(1 + \frac{|s|}{r}\right)^{-N} dy
\]

\[
\leq \sum_{k=0}^{\infty} 2^{-(k-1)N} \sup_{0 \leq |s| < 2^k r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left(\frac{x-y-s}{r}\right) dy \right|.
\]

(21)

Formula (10) leads to

\[
\int_{\mathbb{R}^n} \sup_{0 \leq |s| < 2^k r} \left| \int_{B(x,r)} f(y) \phi^m \left(\frac{x-y-s}{r}\right) dy \right|^p dx
\]

\[
\leq C (1 + 2^k)^n \int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{B(x,r)} f(y) \phi^m \left(\frac{x-y-s}{r}\right) dy \right|^p dx.
\]

(22)

For \( N > n/p \), Formulas (21), (22) lead to

\[
\int_{\mathbb{R}^n} \left| M_{\psi} f(x) \right|^p dx \leq C \int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left(\frac{x-y-s}{r}\right) dy \right|^p dx,
\]

(23)
Thus we could have:

\[(F^m f)(x, r) = \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y}{r} \right) \frac{dy}{r^n}, \quad (24)\]

and

\[(F f)(x, r) = \int_{\mathbb{R}^n} f(y) \phi \left( \frac{x - y}{r} \right) \frac{dy}{r^n}. \quad (25)\]

Thus we could have:

\[| (F^m f)(u, r) - (F f)(u, r) | \leq \int_{\mathbb{R}^n} |f(y)| \phi^m \left( \frac{u - y}{r} \right) - \phi \left( \frac{u - y}{r} \right) \frac{dy}{r^n} \]

\[\leq C \frac{1}{m} |Mf(u)|, \quad (26)\]

where C is dependent on \( \gamma \) and M is the Hardy-Littlewood Maximal Operator. Let us set:

\[\delta_m(u) = |(F^m f)(u, r) - (F f)(u, r)|.\]

Thus we could deduce the following:

\[\{ x : \delta_m(x) > \alpha \} \subseteq \left\{ x : Mf(x) > \frac{1}{c} m \alpha \right\} \quad \text{for some constant } c.\]

Notice that \( M \) is weak-(1, 1) bounded, thus the following holds for any \( \alpha > 0 \):

\[|\{ x : \delta_m(x) > \alpha \}| \leq \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^n)} \frac{1}{m}.\]

Thus we could obtain:

\[\lim_{m \to +\infty} |\{ x : \delta_m(x) > \alpha \}| = 0.\]

Thus there exists a sequence \( \{m_j\} \subseteq \{m\} \) such that the following holds:

\[\lim_{m_j \to +\infty} (F^{m_j} f)(u, r) = (F f)(u, r), \quad \text{a.e. } u \in \mathbb{R}^n\]

for \( f \in L^1(\mathbb{R}^n) \). Let us set \( E \) as:

\[E = \{ u \in \mathbb{R}^n : \lim_{m_j \to +\infty} (F^{m_j} f)(u, r) = (F f)(u, r) \}.\]

Thus it is clear that \( E \) is dense in \( \mathbb{R}^n \). For any \( x_0 \in \mathbb{R}^n \), there exists a \( (u_0, r_0) \) with \( r_0 > 0 \), \( u_0 \in \mathbb{R}^n \), \( |u_0 - x_0| < r_0 \) such that the following holds:

\[| (F^{m_j} f)(u_0, r_0) | \geq \frac{1}{2} \sup_{|u - u_0| < r} | (F^{m_j} f)(u, r) |.\]

Notice that \( (F^{m_j} f)(u, r_0) \) is a continuous function in \( u \) variable and \( E \) is dense in \( \mathbb{R}^n \). There exists a \( \tilde{u}_0 \in E \) with \( |\tilde{u}_0 - x_0| < r_0 \) such that

\[| (F^{m_j} f)(\tilde{u}_0, r_0) | \geq \frac{1}{2} \sup_{|u - \tilde{u}_0| < r} | (F^{m_j} f)(u, r) |.\]

Thus we could deduce that

\[\sup_{\{ u \in E : |u - u_0| < r \}} |(F^{m_j} f)(u, r)| \sim \sup_{\{ u \in \mathbb{R}^n : |u - u_0| < r \}} |(F^{m_j} f)(u, r)|. \quad (27)\]

Formula (27) together with the dominated convergence theorem, we could conclude:

\[\lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|u - u_0| < r} |(F^{m_j} f)(u, r)|^p \, dx \sim \lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{u \in E : |u - x| < r} |(F^{m_j} f)(u, r)|^p \, dx \]

\[\leq C \int_{\mathbb{R}^n} \sup_{|u - u_0| < r} |(F^{m_j} f)(u, r)|^p \, dx \]

\[\leq C \int_{\mathbb{R}^n} \sup_{|u - u_0| < r} |(F f)(u, r)|^p \, dx \]

\[\leq C \int_{\mathbb{R}^n} \sup_{|u - u_0| < r} |(F f)(u, r)|^p \, dx. \quad (28)\]
Also it is clear that the following inequality holds for $f \in L^1(\mathbb{R}^n)$:

$$
\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \lesssim \|f^*_\sigma\|_{L^p(\mathbb{R}^n)} = \|f\|_{H^p(\mathbb{R}^n)}.
$$

(29)

Notice that $H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$. Then by Formula (28) and Formula (29), we could deduce that the following inequality holds for $f \in H^p(\mathbb{R}^n)$

$$
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)}.
$$

(30)

This proves our proposition.

Proposition 2.7. For $\phi(x) \in \Lambda^\gamma$, supp $\phi(x) \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$, $\infty > p > \frac{1}{1+\gamma}$, $|\phi(x)| \leq 1$ \int \phi(x)dx = 1$, then we could obtain the following inequality for $f \in H^p(\mathbb{R}^n)$:

$$
\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma} \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)}.
$$

(31)

Proof. Let us set $0 < \alpha < \gamma \leq 1$, $f \in L^1(\mathbb{R}^n)$ and $p > \frac{1}{1+\gamma - \alpha}$ first. We use $F(a,b,x,y,r)$ with $|a-b| < r$ to denote as:

$$
F(a,b,y,z,r) = \left( \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right) - \left( \phi \left( \frac{a-z}{r} \right) - \phi \left( \frac{b-z}{r} \right) \right).
$$

We use $T(a,b,y,r)$ with $|a-b| < r$ to denote as:

$$
T(a,b,y,r) = \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right).
$$

Thus it is clear to see that the following inequalities hold for $0 < \alpha < \gamma \leq 1$:

$$
|F(a,b,y,z,r)| \leq C \left( \frac{a-b}{r} \right)^\alpha \left( \frac{y-z}{r} \right)^{\gamma-\alpha};
$$

(32)

$$
|T(a,b,y,r)| \leq C;
$$

(33)

$$
\text{supp} T(a,b,y,r) \subseteq B(x,2r) \text{ when } a \in B(x,r).
$$

(34)

For $p > \frac{1}{1+\gamma - \alpha}$, let $F$ denote as:

$$
F = \left\{ x \in \mathbb{R}^n : f^*_\gamma(x) \leq \sigma(f \ast \phi)\varpi(x) \right\}.
$$

It is clear that the following inequality holds for $f \in L^1(\mathbb{R}^n)$:

$$
\|f\|_{L^\gamma(\mathbb{R}^n)} \sim_{\gamma,\alpha} \|f^*_\gamma\|_{L^\gamma(\mathbb{R}^n)} \sim_{\gamma,\alpha} \|f^*_\gamma\|_{L^\gamma(\mathbb{R}^n)} \sim_{\gamma,\alpha} \|f\|_{L^\gamma(\mathbb{R}^n)}.
$$

Then it is clear that

$$
\int_{\mathbb{R}^n} |(f \ast \phi)\varpi(x)|^p dx \leq C_{\gamma,\sigma} \int_{\mathbb{R}^n} |f^*_\gamma(x)|^p dx \leq C_{\gamma,\sigma} \int_{\mathbb{R}^n} |f(x)|^p dx.
$$

$\sigma \geq 2C$, we could have

$$
\int_{\mathbb{R}^n} |(f \ast \phi)\varpi(x)|^p dx \lesssim \int_{\mathbb{R}^n} |f(x)|^p dx.
$$

(35)

(36)

Denote $Df(x)$ and $F(x,r)$ as:

$$
Df(x) = \sup_{\tau > 0} |f \ast \phi_\tau(x)|, \quad F(x,t) = f \ast \phi_\tau(x).
$$
Next, we will show that for any $q > 0$,

$$
(f * \phi)\psi(x) \leq C \left| M(Df)^q(x) \right|^{1/q} \quad \text{for } x \in F, 
$$

(37)

where $M$ is the Hardy-Littlewood maximal operator. Fix any $x_0 \in F$, then there exists $(u_0, r_0)$ satisfying $|u_0 - x_0| < r_0$ such that the following inequality holds:

$$
|F(u_0, r_0)| > \frac{1}{2} (f * \phi)\psi(x_0).
$$

(38)

Choosing $\delta < 1$ small enough and $u$ with $|u - u_0| < \delta r_0$, we could deduce that

$$
|F(u, r_0) - F(u_0, r_0)| = \left| \int_{\mathbb{R}^n} f(y)\phi \left( \frac{u - y}{r_0} \right) \frac{dy}{r_0^n} - \int_{\mathbb{R}^n} f(y)\phi \left( \frac{u_0 - y}{r_0} \right) \frac{dy}{r_0^n} \right|
$$

$$
\leq \left| \int_{\mathbb{R}^n} f(y) T(u, u_0, y, r_0) \frac{dy}{r_0^n} \right|.
$$

We could consider $T(u, u_0, y, r_0)$ as a new kernel. By Formulas (32) (33) (34) we could obtain:

$$
|F(u, r_0) - F(u_0, r_0)| \leq C \delta^n f_{-\alpha}^*(x_0) \leq C \delta^n (f * \phi)\psi(x_0) \quad \text{for } x_0 \in F.
$$

Taking $\delta$ small enough such that $C \delta^n \sigma \leq 1/4$, we could obtain

$$
|F(u, r_0)| \geq \frac{1}{4} (f * \phi)\psi(x_0) \quad \text{for } u \in I(u_0, \delta r_0).
$$

Thus the following inequality holds for any $x_0 \in F$,

$$
\|(f * \phi)\psi(x_0)\|^q \leq \left| \frac{1}{B(u_0, \delta r_0)} \right| \int_{B(u_0, \delta r_0)} \left| B(x_0, (1+\delta) r_0) \right| \left| \frac{1}{B(x_0, (1+\delta) r_0)} \right| \int_{I(x_0, (1+\delta) r_0)} 4^n |F(u, r_0)|^q du
$$

$$
\leq \left( \frac{1+\delta}{\delta} \right)^n \left| \frac{1}{B(x_0, (1+\delta) r_0)} \right| \int_{B(x_0, (1+\delta) r_0)} 4^n |F(u, r_0)|^q du
$$

$$
\leq C M[(Df)^q](x_0)
$$

C is independent on $x_0$. Finally, using the maximal theorem for $M$ when $q < p$ leads to

$$
\int_{\mathbb{R}^n} \|(f * \phi)\psi(x)^p \| dx \leq C \int_{\mathbb{R}^n} \left| M[(Df)^q](x) \right|^{p/q} dx \leq C \int_{\mathbb{R}^n} \|(f * \phi)\psi(x)^p \| \|Df\|_{L^p} dx.
$$

(39)

Thus for any fixed $\alpha$ satisfying $0 < \alpha < \gamma$ and $p > \frac{1}{1+\gamma - \alpha}$, the above Formula (39) combined with Formula (36) lead to

$$
\|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)} \leq C \|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)} ,
$$

(40)

where $C$ is dependent on $p$ and $\alpha$. Next we will remove the number $\alpha$. For any $p > \frac{1}{1+\gamma}$, let $p_0 = \frac{1}{2} \left( p + \frac{1}{1+\gamma} \right)$ with $p > p_0 > \frac{1}{1+\gamma}$ and let $\alpha = 1 + \gamma - \frac{1}{p_0}$. By Formula (40), we could obtain the following inequality holds for $p > \frac{1}{1+\gamma}$ and $f \in L^1(\mathbb{R}^n)$

$$
\|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)} \leq C \|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)},
$$

where $C$ is dependent on $p$ and $\gamma$. Thus by the fact that $L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$, we could deduce Formula (21) holds for any $f \in H^p(\mathbb{R}^n)$. This proves the Proposition.

Thus from Proposition 2.6 and Proposition 2.7, we could obtain the following theorem:

**Theorem 2.8.** For $\phi(x) \in \Lambda^\gamma$, supp $\phi(x) \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$, $\infty > p > \frac{1}{1+\gamma}$, $|\phi(x)| \leq 1$, $\int \phi(x) dx = 1$, then we could obtain the following inequality for any $f \in H^p(\mathbb{R}^n)$:

$$
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|(f * \phi)\psi\|_{L^p(\mathbb{R}^n)}.
$$

(41)
Proposition 2.9. For $\phi(x) \in S(\mathbb{R}^n)$, with $|\mathcal{F}\phi(\xi)| \geq C$ for $\xi \in I(\xi_0, r_0)$, we could obtain the following inequality for $f \in H^p(\mathbb{R}^n)$:
\[
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p, \xi_0, r_0} (f * \tilde{\phi})_\mathcal{F} \|_{L^p(\mathbb{R}^n)} \sim_{p, \xi_0, r_0} \|f * \tilde{\phi}_+\|_{L^p(\mathbb{R}^n)} \quad \text{when } 0 < p < \infty,
\]  
(42)
where $\tilde{\phi}(x) = \phi(r_0x)e^{-2\pi i(\xi_0, x)}$.

Proof. Notice that $\mathcal{F}\tilde{\phi}(\xi) = \mathcal{F}\phi\left(\frac{\xi + r_0\xi_0}{r_0}\right)$, thus $|\mathcal{F}\tilde{\phi}(\xi)| \geq C$, when $\xi \in B(0, 1)$. This proves our proposition. \[\Box\]

Thus in a way similar to Theorem 2.8, by Proposition 2.9, we could obtain the following corollary:

Corollary 2.10. For $\phi(x) \in \Lambda^\gamma$, supp $\phi(x) \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$, \(\infty > p > \frac{1}{1+\gamma}\}, |\phi(x)| \leq 1, \) with $|\mathcal{F}\phi(\xi)| \geq C$, when $\xi \in I(\xi_0, r_0)$, then we could obtain the following inequality for any $f \in H^p(\mathbb{R}^n)$:
\[
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p, \xi_0, r_0, \phi} (f * \tilde{\phi})_\mathcal{F} \|_{L^p(\mathbb{R}^n)} \sim_{p, \xi_0, r_0, \phi} \|f * \tilde{\phi}_+\|_{L^p(\mathbb{R}^n)},
\]
where $\tilde{\phi}(x) = \phi(r_0x)e^{-2\pi i(\xi_0, x)}$.

3 Lipschitz function without compact support in $\mathbb{R}^n$

Proposition 3.1. $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$) without compact support in $\mathbb{R}^n$ satisfying the following:
\[
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{n+\gamma}},
\]  
(43)
\[
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}}, \text{ if } |h| \lesssim 1 + |x|.
\]  
(44)

Then we could deduce the following inequality for $\frac{1}{1+\gamma} < p \leq 1 \ (0 < \gamma \leq 1)$:
\[
\|f * \phi\|_{L^p(\mathbb{R}^n)} \lesssim_{p, n, \gamma} \|f\|_{H^p(\mathbb{R}^n)}.
\]

Proof. Choose positive $\varphi(t) \in S(\mathbb{R}^n)$ satisfying supp $\varphi(t) \subseteq B(0, 1)$, and $\varphi(t) = 1$ for $t \in B(0, 1/2)$. Let the functions $\psi_{k,x}(t)$ be defined as follows:
\[
\psi_{0,x}(t) = \varphi\left(\frac{x - t}{r}\right),
\]
\[
\psi_{k,x}(t) = \varphi\left(\frac{x - t}{2^k r}\right) - \varphi\left(\frac{x - t}{2^{k+1} r}\right), \text{ for } k \geq 1.
\]

Thus $\psi_{k,x}(t) \in S(\mathbb{R}^n)$ for $k \geq 0$, with supp $\psi_{0,x}(t) \subseteq B(x, r)$, supp $\psi_{k,x}(t) \subseteq B(x, 2^{k+1} r) \setminus B(x, 2^{k-2} r)$ for $k \geq 1$. It is clear that
\[
\sum_{k=0}^{\infty} \psi_{k,x}(t) = 1.
\]

Then we could write $(f * \phi)_\mathcal{F}(x)$ as following:
\[
(f * \phi)_\mathcal{F}(x) = \sup_{r > 0, |x - s| \leq r} \left| \int_{\mathbb{R}^n} \phi\left(\frac{s-y}{r}\right) \sum_{k=0}^{\infty} \psi_{k,x}(y) f(y) dy / r^n \right|
\]
\[
\leq \sum_{k=0}^{\infty} \sup_{r > 0, |x - s| \leq r} \left| \int_{\mathbb{R}^n} \phi\left(\frac{s-y}{r}\right) \psi_{k,x}(y) f(y) dy / r^n \right|.
\]
It is clear that the function \( y \to (1 + 2^k)^{1+\gamma} \psi \left( \frac{x-y}{r} \right) \psi_{k,x}(y) \) with \( |s - x| < r \) satisfies the following:

\[
\left\{ \begin{array}{l}
\| (1 + 2^k)^{n+\gamma} \phi \left( \frac{x-y}{r} \right) \psi_{k,x}(y) \| \lesssim 1 \\
H^\gamma \left( (1 + 2^k)^{n+\gamma} \phi \left( \frac{x-y}{r} \right) \psi_{k,x}(y) \right) \lesssim (2^k r)^{-\gamma} \\
\text{supp} (1 + 2^k)^{n+\gamma} \phi \left( \frac{x-y}{r} \right) \psi_{k,x}(y) \subseteq B(x, 2^k+1)r \setminus B(x, 2^{k-2}r) \text{ for } k \geq 1.
\end{array} \right.
\]

Then we could deduce that:

\[
(f \ast \nabla) \varphi(x) = \sup_{r > 0, \|s-x\| \leq r} \left| \int_{\mathbb{R}^n} \phi \left( \frac{s-y}{r} \right) f(y) dy / r^n \right|
\leq \sum_{k=0}^{+\infty} \frac{(2k)^n}{(1 + 2^k)^{n+\gamma}} \sup_{r > 0, \|s-x\| \leq r} \left| \int_{\mathbb{R}^n} (1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) f(y) dy / (2^k r)^n \right|
\lesssim \sum_{k=0}^{+\infty} \frac{(2k)^n}{(1 + 2^k)^{n+\gamma}} f_k^\gamma(x).
\]

Thus the following inequality holds for \( \frac{1}{1+\gamma} < p \leq 1 \ (0 < \gamma \leq 1)\):

\[
\| (f \ast \nabla) \varphi \|_{L^p(\mathbb{R}^n)} \lesssim_{n, \gamma} \| f \|_{H^\gamma(\mathbb{R}^n)}.
\]

This proves the proposition. \(\square\)

**Proposition 3.2.** \( \phi(x) \) is a Lipschitz function \( (\phi(x) \in \Lambda^\gamma) \) without compact support in \( \mathbb{R}^n \) satisfying the following:

\[
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{n+\gamma}}, \quad (45)
\]

\[
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}}, \quad \text{if } |h| \lesssim 1 + |x|.
\]

(46)

Then we could deduce the following inequalities for any fixed \( \alpha \) with \( 0 < \alpha < \gamma \leq 1 \), and \( r > 0 \):

\[
0 \leq \left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\alpha (1 + \frac{|x-y|}{r})^{-(\gamma-\alpha)-n},
\]

and

\[
\left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) - \phi \left( \frac{a-z}{r} \right) + \phi \left( \frac{b-z}{r} \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\alpha \left( \frac{|y-z|}{r} \right)^{\gamma-\alpha} \left( \frac{x-y}{r} \right)^{2(\gamma-\alpha)-n},
\]

for \( |a-b| \lesssim r, \frac{|a-y|}{r} \leq C_3 \min \left\{ 1 + \frac{|a-y|}{r}, 1 + \frac{|a-z|}{r} \right\}, x \in B(a, 2r) \cap B(b, 2r). \)

**Proof.** From the fact that \( |a-b| \lesssim r, \frac{|a-y|}{r} \leq C_3 \min \left\{ 1 + \frac{|a-y|}{r}, 1 + \frac{|a-z|}{r} \right\}, \) the following relations could be obtained:

\[
1 + \frac{|a-y|}{r} \sim 1 + \frac{|b-y|}{r}, \quad 1 + \frac{|a-z|}{r} \sim 1 + \frac{|b-z|}{r}, \quad \text{and } 1 + \frac{|a-z|}{r} \sim 1 + \frac{|a-y|}{r}.
\]

(47)

First, we will consider the case when \( |a-b| \leq |y-z| \). Then from Formula (46), we could get

\[
\left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\gamma \left( 1 + \frac{|a-y|}{r} \right)^{-2\gamma-2n}
\leq C \left( \frac{|a-b|}{r} \right)^\gamma \left( 1 + \frac{|a-y|}{r} \right)^{-\gamma-\alpha} \left( 1 + \frac{|a-y|}{r} \right)^{-(\gamma-\alpha)-n}
\leq C \left( \frac{|a-b|}{r} \right)^\alpha \left( 1 + \frac{|a-y|}{r} \right)^{-(\gamma-\alpha)-n}.
\]

(48)
Also we could obtain
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{b - y}{r}\right)| \leq C\left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}, \]
and
\[ |\phi\left(\frac{a - z}{r}\right) - \phi\left(\frac{b - z}{r}\right)| \leq C\left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - z|}{r}\right)^{-2\alpha - n}. \]

Together with Formula (47), we could conclude
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{b - y}{r}\right) - \left(\phi\left(\frac{a - z}{r}\right) - \phi\left(\frac{b - z}{r}\right)\right)| \leq C\left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}. \]

By the fact \(|a - b| \leq |y - z|\) and \(1 \leq 1 + \frac{|a - y|}{r}\), we could obtain:
\[ \left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n} \leq \left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}. \]

Then for \(|a - b| \leq |y - z|\), the Formula
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{a - z}{r}\right)| \leq C\left(\frac{|y - z|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n} \]
holds. In a similar way, we will obtain the Formula (49) for the case when \(|a - b| \geq |y - z|\). Notice that by Formula (47),
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{a - z}{r}\right)| \leq C\left(\frac{|y - z|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}, \]
and
\[ |\phi\left(\frac{b - y}{r}\right) - \phi\left(\frac{b - z}{r}\right)| \leq C\left(\frac{|y - z|}{r}\right)\gamma \left(1 + \frac{|b - y|}{r}\right)^{-2\alpha - n} \]
hold. Then we could obtain
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{b - y}{r}\right) - \left(\phi\left(\frac{a - z}{r}\right) - \phi\left(\frac{b - z}{r}\right)\right)| \leq C\left(\frac{|y - z|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}. \]

By the fact \(|a - b| \geq |y - z|\) and \(1 \leq 1 + \frac{|a - y|}{r}\), the following holds:
\[ \left(\frac{|y - z|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n} \leq \left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}. \]

Then for \(|a - b| \geq |y - z|\), we could get
\[ |\phi\left(\frac{a - y}{r}\right) - \phi\left(\frac{b - y}{r}\right) - \left(\phi\left(\frac{a - z}{r}\right) - \phi\left(\frac{b - z}{r}\right)\right)| \leq C\left(\frac{|a - b|}{r}\right)\gamma \left(1 + \frac{|a - y|}{r}\right)^{-2\alpha - n}. \]

By the fact that \(x \in B(a, 2r) \cap B(b, 2r)\), we could deduce that:
\[ 1 + \frac{|a - y|}{r} \sim 1 + \frac{|x - y|}{r}. \]

Formulas (48) (49) (50) (51) yeald the Proposition.
Some problems in the theory of Hardy spaces

Proposition 3.3. For any $0 < \gamma \leq 1$, $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$) without compact support in $\mathbb{R}^n$ satisfying Formulas (45) and (46). For $f \in L^1(\mathbb{R}^n)$, if the following inequality holds

$$\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \sim \|f^*_\gamma\|_{L^p(\mathbb{R}^n)}$$

then for $p > \frac{1}{1 + \gamma}$, we could deduce that:

$$\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \leq C\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)},$$

where $C$ is dependent on $p$ and $\gamma$.

Proof. By Proposition 2.5, we could deduce that the following holds for $f \in L^1(\mathbb{R}^n)$:

$$\|f^*_{\gamma - \alpha}\|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \alpha} \|f^*_{\gamma, \alpha}\|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \alpha} \|f^*_\gamma\|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \alpha} \|f^*_\gamma\|_{L^p(\mathbb{R}^n)}.$$ 

For any fixed $\alpha$ satisfying $0 < \alpha < \gamma$ and $p > \frac{1}{1 + \gamma - \alpha}$, Let $F$ denote as:

$$F = \left\{ x \in \mathbb{R}^n : f^*_{\gamma - \alpha}(x) \leq \sigma(f \ast \phi)(x) \right\}.$$ 

Then it is clear that

$$\int_{F^c} |(f \ast \phi)(x)|^p dx \leq \frac{C}{\sigma^p} \int_{F^c} f^*_{\gamma - \alpha}(x)\|dx \leq \frac{C'}{\sigma^p} \int_{\mathbb{R}^n} |f^*_\gamma(x)|^p dx \leq \frac{C}{\sigma^p} \int_{\mathbb{R}^n} |(f \ast \phi)(x)|^p dx.$$ \hspace{1cm} (52)

Choosing $\sigma^p \geq 2C'_\gamma$, we could have

$$\int_{\mathbb{R}^n} |(f \ast \phi)(x)|^p dx \leq \int_F |(f \ast \phi)(x)|^p dx.$$ \hspace{1cm} (53)

We use $Df(x)$ and $F(x, r)$ to denote as:

$$Df(x) = \sup_{t > 0} |f \ast \phi_t(x)|, \quad F(x, t) = f \ast \phi_t(x).$$

Next, we will show that for any $q > 0$,

$$(f \ast \phi)(x) \leq C[M(Df)^q(x)]^{1/q} \quad \text{for } x \in F,$$ \hspace{1cm} (54)

where $M$ is the Hardy-Littlewood maximal operator. For any fixed $x_0 \in F$, there exists $(u_0, r_0)$ satisfying $|u_0 - x_0| < r_0$ such that the following inequality holds:

$$|F(u_0, r_0)| > \frac{1}{2} |(f \ast \phi)(x_0)|.$$ \hspace{1cm} (55)

Choosing $\delta < 1$ small enough and $u$ satisfying $|u - u_0| < \delta r_0$, we could deduce that

$$|F(u, r_0) - F(u_0, r_0)| = \left| \int_{\mathbb{R}^n} f(y) \phi \left( \frac{u - y}{r_0} \right) \frac{dy}{r_0^n} - \int_{\mathbb{R}^n} f(y) \phi \left( \frac{u_0 - y}{r_0} \right) \frac{dy}{r_0^n} \right|$$

$$\leq \left| \int_{\mathbb{R}^n} f(y) \phi \left( \frac{u - y}{r_0} \right) - \phi \left( \frac{u_0 - y}{r_0} \right) \right| \frac{dy}{r_0^n}.$$ 

We could consider $\left( \phi \left( \frac{u - y}{r_0} \right) - \phi \left( \frac{u_0 - y}{r_0} \right) \right)$ as a new kernel. By Proposition 3.2 and Proposition 3.1, we could obtain:

$$|F(u, r_0) - F(u_0, r_0)| \leq C\delta^\alpha f^*_{\gamma - \alpha}(x_0) \leq C\delta^\alpha \sigma(f \ast \phi)(x_0) \quad \text{for } x_0 \in F.$$ 

Taking $\delta$ small enough such that $C\delta^\alpha \sigma \leq 1/4$, we obtain

$$|F(u, r_0)| \geq \frac{1}{4} |(f \ast \phi)(x_0)| \quad \text{for } u \in B(u_0, \delta r_0).$$
Thus the following inequality holds: for any $x_0 \in F$,

$$
|\langle f \ast \phi \rangle_\gamma(x_0)|^q \leq \frac{1}{B(u_0, \delta r_0)} \int_{B(u_0, \delta r_0)} 4^q |F(u, r_0)|^q du
$$

$$
\leq \frac{B(x_0, (1 + \delta)r_0)}{B(u_0, \delta r_0)} \frac{1}{B(x_0, (1 + \delta)r_0)} \int_{B(x_0, (1 + \delta)r_0)} 4^q |F(u, r_0)|^q du
$$

$$
\leq \frac{1 + \delta}{\delta} \frac{1}{B(x_0, (1 + \delta)r_0)} \int_{B(x_0, (1 + \delta)r_0)} 4^q |F(u, r_0)|^q du
\leq CM|\gamma|\Vert \gamma \Vert_{L^p(\mathbb{R}^n)}
C$$

is independent on $x_0$. Finally, using the maximal theorem for $M$ when $q < p$ leads to

$$
\int_F |\langle f \ast \phi \rangle_\gamma(x)|^p dx \leq C \int_{\mathbb{R}^n} M|\gamma|\Vert \gamma \gamma \Vert_{L^p(\mathbb{R}^n)} dx \leq C \int_{\mathbb{R}^n} |\langle f \ast \phi \rangle_\gamma(x)|^p dx.
$$

(56)

Thus for any fixed $\alpha$ satisfying $0 < \alpha < \gamma$ and $p > \frac{1}{1 + \gamma - \alpha}$, the above Formula (56) combined with Formula (53) leads to

$$
\|\langle f \ast \phi \rangle_\gamma\|_{L^p(\mathbb{R}^n)} \leq C\|\langle f \ast \phi \rangle_\gamma\|_{L^p(\mathbb{R}^n)},
$$

(57)

where $C$ is dependent on $p$ and $\alpha$. Next we will remove the number $\alpha$. For any $p > \frac{1}{1 + \gamma}$, let

$$
p_0 = \frac{1}{2} \left( p + \frac{1}{1 + \gamma} \right)
\text{ with } p > p_0 > \frac{1}{1 + \gamma}
\text{ and let } \gamma = 1 + \gamma - \frac{1}{p_0}.
$$

Thus it is clear that

$$
p_0 = \frac{1}{1 + \gamma - \alpha},
\text{ } p > p_0.
$$

Thus by Formula (57), we could obtain the following inequality holds for $p > \frac{1}{1 + \gamma}$

$$
\|\langle f \ast \phi \rangle_\gamma\|_{L^p(\mathbb{R}^n)} \leq C\|\langle f \ast \phi \rangle_\gamma\|_{L^p(\mathbb{R}^n)},
$$

where $C$ is dependent on $p$ and $\gamma$. This proves the Proposition.

\[\square\]

**Proposition 3.4.** $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$) without compact support in $\mathbb{R}^n$ satisfying the following:

$$
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{n + N}},
$$

(58)

$$
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{n + 2\gamma}}, \text{ if } |h| \lesssim 1 + |x|,
$$

(59)

where $N > \frac{2}{p} + 1$ ($0 < p \leq 1$).

Then there exists sequence $\{\phi^k(x) : \phi^k(x) \in C_c(\mathbb{R}^n)\}_{k=1}^\infty$ satisfying the following:

(i) $\text{supp } \phi^k(x) \subseteq B(0, k)$;

(ii) $\lim_{k \to \infty} \|\phi^k(x) - \phi(x)\|_\infty = 0$;

(iii) $|\phi^k(x)| \leq C_1 \frac{1}{(1 + |x|)^{n + N}}$;

(iv) If $|h| \lesssim 1 + |x|$, then

$$
|\phi^k(x + h) - \phi^k(x)| \leq C_2 \frac{|h|^\gamma}{(1 + |x|)^{n + 2\gamma}};
$$

(v) The following two inequalities hold:

$$
|\phi^k(x) - \phi(x)| \leq C_3 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n + N + \frac{\gamma}{2}}},
$$

where $C_1, C_2, C_3$ are constants.

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where $C_1$, $C_2$, and $C_3$ are constants independent on $k$.

Proof. Choose a nonnegative function $\psi(t) \in S(\mathbb{R}^n)$ with $\psi(t) \leq 1$, $\|H^\gamma \psi\|_{L^\infty} \leq C$, $\text{supp} \, \psi(t) \subseteq B(0,1)$, $\psi(t) = 1$ when $t \in B(0,1/2)$. We use $\phi^k(x)$ to denote as:

$$\phi^k(x) = \phi(x) \psi \left( \frac{x}{k} \right), \quad k = 1, 2, \ldots, \infty.$$ 

Then we could check that sequence $\{\phi^k(x)\}_{k=1}^\infty$ satisfies parts (i), (ii), (iii), (iv).

When $|x| \leq \frac{1}{2}$, it is clear that $|\phi^k(x) - \phi(x)| = 0$.

When $|x| \geq \frac{1}{2}$, we could have

$$|\phi^k(x) - \phi(x)| \leq C \frac{1}{(1 + |x|)^{n+N}} \leq C \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n+N - \frac{\gamma}{2}}}.$$ 

Thus we could conclude that

$$|\phi^k(x) - \phi(x)| \leq C_3 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n+N - \frac{\gamma}{2}}}.$$ 

Thus it is clear that

$$\int_{\mathbb{R}^n} |\phi^k(x) - \phi(x)| \, dx \lesssim \gamma \left( \frac{1}{k} \right)^{\gamma/2}.$$ 

(61)

By Proposition 3.4, we could obtain the following Proposition:

**Proposition 3.5.** $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma (0 < \gamma \leq 1)$) without compact support in $\mathbb{R}^n$ satisfying Formulas (58) and (59).

Then there exists sequence $\{\psi^k(x) : \psi^k(x) \in S(\mathbb{R}^n)\}_{k=1}^\infty$ satisfying the following:

(i) $\text{supp} \, \psi^k(x) \subseteq B(0,2k)$;

(ii) $\lim_{k \to \infty} \|\psi^k(x) - \psi(x)\|_{L^\infty} = 0$;

(iii) For $N \geq \left[ \frac{n}{p} \right] + 1$ ($0 < p \leq 1$),

$$|\psi^k(x)| \leq C_1 \frac{1}{(1 + |x|)^{n+N}};$$

(iv) If $|h| \lesssim 1 + |x|$, then

$$|\psi^k(x + h) - \psi^k(x)| \leq C_2 \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}};$$

(v) The following two inequalities hold:

$$|\psi^k(x) - \phi(x)| \leq C_4 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n+\gamma + \frac{\gamma}{2}}}.$$ 

$$\int_{\mathbb{R}^n} |\psi^k(x) - \phi(x)| \, dx \lesssim \gamma \left( \frac{1}{k} \right)^{\gamma/2}.$$ 

where $C_1$, $C_2$, and $C_4$ are constants independent on $k$. 

This proves our proposition.

Proof. Let \( \{ \phi^k(x) : \phi^k(x) \in C_c(\mathbb{R}^n) \}_{k=1}^\infty \) to be the sequence in Proposition 3.4. Let

\[
\rho(x) = \begin{cases} 
\vartheta \exp \left( \frac{|x|^2}{|x|^2 - 1} \right), & \text{for } |x| < 1 \\
0, & \text{for } |x| \geq 1,
\end{cases}
\]

where \( \vartheta \) is a constant such that \( \int \rho(x) dx = 1 \). Let

\[
\phi^{k,\tau}(x) = \int_{\mathbb{R}^n} \phi^k(x-t) \rho \left( \frac{t}{\tau} \right) \frac{dt}{\tau^n}.
\]

It is clear that (i) (ii) and (iii) hold. Next we prove (iv). Notice that \( \text{supp } \rho(x) \subseteq \{ x : |x| < 1 \} \). Let us set \( \tau = \frac{1}{k} \) for \( k \in \mathbb{Z}, k \geq 1 \), thus it is clear that \( |h| \lesssim 1 + |x-t| \) holds if \( |h| \lesssim 1 + |x| \). Thus the following holds for \( |h| \lesssim 1 + |x| \):

\[
\left| \phi^{k,\tau}(x+h) - \phi^{k,\tau}(x) \right| = \left| \int_{\mathbb{R}^n} \phi^k(x+h-t) \rho(kt) k^n dt - \int_{\mathbb{R}^n} \phi^k(x-t) \rho(kt) k^n dt \right| \overset{(62)}{\leq} C_2 \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}}.
\]

We could also deduce the following inequality:

\[
|\phi^{k,\tau}(x) - \phi(x)| \leq |\phi^{k,\tau}(x) - \phi^k(x)| + |\phi^k(x) - \phi(x)| \leq C_2 \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}} + C_3 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n+N-\gamma}} \leq C_4 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n+N-\gamma}}.
\]

Thus it is clear that

\[
\int_{\mathbb{R}^n} |\phi^{k,\tau}(x) - \phi(x)| dx \lesssim \left( \frac{1}{k} \right)^{\gamma/2}.
\]

At last, we could set \( \psi^k(x) \) as

\[
\psi^k(x) = \phi^{k,\tau}(x).
\]

This proves our proposition.

\[\square\]

**Proposition 3.6.** \( \phi(x) \) is a Lipschitz function \( (\phi(x) \in \Lambda^\gamma, \int \phi(x) dx \sim 1), (0 < \gamma \leq 1), \phi(x) > 0 \) without compact support in \( \mathbb{R}^n \) satisfying Formulas (58) and (59), then we could deduce the following inequality for \( \frac{1}{1+\gamma} < p \leq 1 \):

\[
\|f\|_{H_p(\mathbb{R}^n)} \lesssim_{p,n,\gamma,\phi} \|f \ast \phi \|_{L_p(\mathbb{R}^n)}.
\]

Proof. Let \( f \in L^1(\mathbb{R}^n) \) first. There exists sequence \( \{ \phi^m(x) : \phi^m(x) \in S(\mathbb{R}^n) \}_{m \in \mathbb{N}} \) as in Proposition 3.5. We use \( \varphi \in S(\mathbb{R}^n) \) and \( \varphi^k \in S(\mathbb{R}^n) \) to denote as:

\[
\begin{cases} 
\varphi(\xi) = 0 & \text{for } |\xi| \geq 1 \\
\varphi(\xi) = 1 & \text{for } |\xi| \leq 1/2,
\end{cases}
\]

\[
\varphi^k(\xi) = \varphi(\xi) \text{ for } k = 0,
\]

\[
\varphi^k(\xi) = \varphi(2^{-k} \xi) - \varphi(2^{1-k} \xi) \text{ for } k \geq 1.
\]

Thus

\[
1 = \sum_{k=0}^{\infty} \varphi^k(\xi).
\]

It is clear that \( \int \phi^m(x) dx \sim C \), where \( C \) is a constant independent on \( k \), thus \( (\mathcal{F} \phi^m)(2k_0 \xi) \geq C_p \) for \( |\xi| \leq 1 \) where \( C_p \) is a constant independent on \( m \), and \( k_0 \) is independent on \( m \). Also by Proposition 3.5, we could deduce that the following inequality holds:

\[
|\partial_\xi^\alpha (\mathcal{F} \phi^m)(\xi)| \leq C_0
\]

(63)
for $0 \leq |\alpha| \leq N$ ($N \geq \frac{a}{p} + 1$), and where $C_0$ is a constant independent on $m$. Let $\eta_m^k$ to be defined as

$$(\mathcal{F} \eta_m^k)(\xi) = \frac{\varphi^k(\xi)(\mathcal{F} \psi)(\xi)}{(\mathcal{F} \phi^m)(2^{-k - k_0} \xi)}.$$ 

where $\mathcal{F}$ denotes the Fourier transform. Then we could obtain that:

$$(\mathcal{F} \psi)(\xi) = \sum_{k=0}^{\infty} \frac{\varphi^k(\xi)(\mathcal{F} \psi)(\xi)}{(\mathcal{F} \phi^m)(2^{-k - k_0} \xi)} (\mathcal{F} \phi^m)(2^{-k - k_0} \xi)$$

$$= \sum_{k=0}^{\infty} (\mathcal{F} \eta_m^k)(\xi) (\mathcal{F} \phi^m)(2^{-k - k_0} \xi).$$

Thus

$$\psi(x) = \sum_{k=0}^{\infty} \eta_m^k \ast \phi^m_{2^{-k - k_0}}(x).$$

(64)

By Formula (63) and

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^\alpha \varphi^m((\mathcal{F} \psi)(\xi)) \leq C_{\alpha, \alpha', M} 2^{-kM}$$

for any $M > 0$, (65)

where $C_{\alpha}$ is a constant independent on $m$, we could deduce that

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^\alpha \varphi^m((\mathcal{F} \eta_m^k)(\xi)) \leq C_{\alpha, \alpha', k_0, N} 2^{-kN}$$

where $C_{\alpha, \alpha', k_0, N}$ is a constant independent on $m$ and $k$. Thus we could have:

$$\left| \int_{\mathbb{R}^n} \eta_m^k(u) \left(1 + 2^{k+k_0}|u|\right)^N \, du \right| \leq C_{k_0, N} 2^{-k},$$

(67)

where $C_{k_0, N}$ is a constant independent on $m$. Then by Formulas (64) with the fact that $f \in L^1(\mathbb{R}^n)$ we have

$$M_{\varphi}f(x) = \sup_{r > 0, s \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) \frac{1}{r^n} \psi \left( \frac{x - y}{r} \right) \, dy \right|$$

(68)

$$= C \sup_{r > 0, s \in \mathbb{R}^n} \int_{\mathbb{R}^n} f(y) \eta_m^k \left( \frac{s}{r} \right) \frac{1}{r^n} \phi^m \left( \frac{x - y - s}{2^{-k - k_0} r} \right) \frac{ds}{(2^{-k - k_0} r)^n} \, dy$$

$$\leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \eta_m^k \left( \frac{s}{r} \right) \left(1 + \frac{|s|}{2^{-k - k_0} r}\right)^N \frac{ds}{r^n} \sup_{r > 0, s \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y - s}{r} \right) \left(1 + \frac{|s|}{r}\right)^{-N} \, dy \right|,$$

where $C$ is a constant independent on $m$. From Formula (67) and Formula (68) we could obtain:

$$M_{\varphi}f(x) \leq \sup_{r > 0, s \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y - s}{r} \right) \left(1 + \frac{|s|}{r}\right)^{-N} \, dy \right|$$

(69)

$$\leq C \left( \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y - s}{r} \right) \left(1 + \frac{|s|}{r}\right)^{-N} \, dy \right| \right)$$

$$\leq \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y - s}{r} \right) \, dy \right|^p.$$
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Notice that \( N \geq \lceil \frac{2}{p} \rceil + 1 \), thus Formulas (69), (70) lead to

\[
\int_{\mathbb{R}^n} |M_n f(x)|^p dx \leq C \int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y - s}{r} \right) \frac{dy}{r^n} \right|^p dx,
\]

where \( C \) is a constant independent on \( m \). We use \((F^m f)(x, r)\) and \((F f)(x, r)\) to denote as following:

\[
(F^m f)(x, r) = \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y}{r} \right) \frac{dy}{r^n},
\]

and

\[
(F f)(x, r) = \int_{\mathbb{R}^n} f(y) \phi \left( \frac{x - y}{r} \right) \frac{dy}{r^n}.
\]

Thus by Proposition 3.5(v), we could deduce the following inequality:

\[
|(F^m f)(u, r) - (F f)(u, r)| \leq \int_{\mathbb{R}^n} |f(y)| \left| \phi^m \left( \frac{u - y}{r} \right) - \phi \left( \frac{u - y}{r} \right) \right| \frac{dy}{r^n}
\]

\[
\leq C \int_{\mathbb{R}^n} \left| f(y) \right| \left( \frac{1}{m} \right)^{\gamma/2} \left( 1 + \frac{|u - y|}{r} \right)^{-n + \frac{\gamma}{2}} \frac{dy}{r^n}
\]

\[
\leq C \sum_{k=0}^{+\infty} \left( \frac{2}{m} \right)^{-n + \frac{\gamma}{2}} 2^{nk} |Mf(u)| \left( \frac{1}{m} \right)^{\gamma/2}
\]

\[
\leq C |Mf(u)| \left( \frac{1}{m} \right)^{\gamma/2}
\]

where \( C \) is dependent on \( \gamma \) and \( M \) is the Hardy-Littlewood Maximal Operator. Let us set:

\[
\delta_m(u) = |(F^m f)(u, r) - (F f)(u, r)|.
\]

Thus we could deduce the following:

\[
\{ x : \delta_m(x) > \alpha \} \subseteq \left\{ x : Mf(x) > \frac{1}{\alpha} m^{\gamma/2} \right\}
\]

for any \( \alpha > 0 \):

\[
|\{ x : \delta_m(x) > \alpha \}| \leq \frac{c}{\alpha} \| f \|_{L^1(\mathbb{R}^n)} \left( \frac{1}{m} \right)^{\gamma/2}.
\]

Thus we could obtain:

\[
\lim_{m \to +\infty} |\{ x : \delta_m(x) > \alpha \}| = 0.
\]

Thus there exists a sequence \( \{m_j\} \subseteq \{m\} \) such that the following holds:

\[
\lim_{m_j \to +\infty} (F^{m_j} f)(u, r) = (F f)(u, r), \quad a.e. u \in \mathbb{R}^n
\]

for \( f \in L^1(\mathbb{R}^n) \). Let us set \( E \) as:

\[
E = \{ u \in \mathbb{R}^n : \lim_{m_j \to +\infty} (F^{m_j} f)(u, r) = (F f)(u, r) \}.
\]

Thus it is clear that \( E \) is dense in \( \mathbb{R}^n \). For any \( x_0 \in \mathbb{R}^n \), there exists a \( (u_0, r_0) \) with \( r_0 > 0, u_0 \in \mathbb{R}^n, |u_0 - x_0| < r_0 \) such that the following holds:

\[
|(F^{m_j} f)(u_0, r_0)| \geq \frac{1}{2} \sup_{|x_0 - u| < r} |(F^{m_j} f)(u, r)|.
\]

Notice that \((F^{m_j} f)(u, r_0)\) is a continuous function in \( u \) variable and \( E \) is dense in \( \mathbb{R}^n \). There exists a \( \tilde{u}_0 \in E \) with \( |\tilde{u}_0 - x_0| < r_0 \) such that

\[
|\tilde{(F^{m_j} f)}(\tilde{u}_0, r_0)| \geq \frac{1}{4} \sup_{|x_0 - u| < r} |(F^{m_j} f)(u, r)|.
\]
Thus we could deduce that
\[
\sup_{\{u \in E : |x - u| < r\}} |(F_{m_j} f)(u, r)| \lesssim \sup_{\{u \in \mathbb{R}^n : |x - u| < r\}} |(F_{m_j} f)(u, r)|. \tag{75}
\]
Formula (75) together with the dominated convergence theorem, we could conclude:
\[
\lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|x - u| < r} |(F_{m_j} f)(u, r)|^p \, dx \sim \lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|u - x| < r} |(F_{m_j} f)(u, r)|^p \, dx \\
\leq C \int_{\mathbb{R}^n} \lim_{m_j \to +\infty} \sup_{|u - x| < r} |(F_{m_j} f)(u, r)|^p \, dx \\
\leq C \int_{\mathbb{R}^n} \sup_{|u - x| < r} |(F f)(u, r)|^p \, dx.
\tag{76}
\]
By Formulas (71) and (76), we could deduce that the following inequality holds for \( f \in L^1(\mathbb{R}^n) \)
\[
\|f\|_{H^p(\mathbb{R}^n)} \lesssim_{p, n, \gamma, \phi} \|f * \phi\|_{L^p(\mathbb{R}^n)}. \tag{77}
\]
Notice that \( H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) is dense in \( H^p(\mathbb{R}^n) \). Then we could deduce that the Formula (77) holds for \( f \in H^p(\mathbb{R}^n) \). This proves our proposition.

Thus from Proposition 3.3, Proposition 3.6 and Proposition 3.1, we could obtain the following theorem:

**Theorem 3.7.** \( \phi(x) \) is a Lipschitz function \( \phi(x) \in \Lambda^\gamma, \int \phi(x) \, dx \sim 1 \), \( 0 < \gamma \leq 1 \), \( \phi(x) > 0 \) without compact support in \( \mathbb{R}^n \) satisfying Formulas (58) and (59), then we could deduce the following inequality for \( \frac{1}{1 + \gamma} < p \leq 1 \), \( f \in H^p(\mathbb{R}^n) \):
\[
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|f * \phi\|_{L^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|f * \phi\|_{L^p(\mathbb{R}^n)}. \tag{78}
\]

Similar to the Possion kernel, we could obtain the following inequality:

**Proposition 3.8.** For some fixed \( \epsilon > 0 \), let
\[
\phi(x) = \frac{1}{(1 + |x|)^{n+\epsilon}}.
\]
then we could deduce the following inequality for \( 0 < p \leq 1 \), \( f \in H^p(\mathbb{R}^n) \):
\[
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|f * \phi\|_{L^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \|(f * \phi)_+\|_{L^p(\mathbb{R}^n)}. \tag{79}
\]

### 4 Other kernels related to Hardy spaces in \( \mathbb{R} \)

In this section 4, we will discuss a problem left by G. Weiss in [4]. Weiss proposed the following problem associate with \( H^1(\mathbb{R}) \):

**Question 4.1.** There exists a function \( \phi \in L^\infty(\mathbb{R}) \) such that
\[
H^1_\phi(\mathbb{R}) = \{ f \in L^1(\mathbb{R}) : (f * \phi)\varphi(x) \in L^1(\mathbb{R}) \} \neq \emptyset,
\]
and
\[
H^1_\phi(\mathbb{R}) \neq H^1(\mathbb{R}).
\]

In 1983, Uchiyama and Wilson proved the above Question 4.1 in [12]. Han gave another prove of the Question 4.1 in [6]. In this section, we will prove the following problem:

**Question 4.2.** There exists a function \( \phi \in L^\infty(\mathbb{R}) \) such that for \( 0 < p \leq 1 \)
\[
H^p_\phi(\mathbb{R}) = \{ f \in S'(\mathbb{R}) : (f * \phi)\varphi(x) \in L^p(\mathbb{R}) \} \neq \emptyset,
\]
and
\[
H^p(\mathbb{R}) \nsubseteq H^p_\phi(\mathbb{R}).
Definition 4.3 \( (S^m(\mathbb{R})) \). \( S^m(\mathbb{R}) \) can be defined as follows:

\[
S^m(\mathbb{R}) = \left\{ \phi \in S(\mathbb{R}) : \int x^s \phi(x) dx = 0, \ 0 \leq s \leq m, s \in \mathbb{N}, \text{with sup} \phi \subseteq I(0,10) \cap I(0,5)^c \right\}.
\]

Proposition 4.4. There exists \( \phi \in S^m(\mathbb{R}) \), satisfying \( \text{supp} \phi \subseteq I(0,10) \cap I(0,5)^c \), \( f_{x} = \int x^m \phi(x) dx \sim 1 \).

Proof. Let \( \gamma(x) \in S(\mathbb{R}) \) satisfying \( \text{supp} \gamma(x) \subseteq I(0,10) \cap I(0,5)^c \) and \( \int \gamma(x) dx \sim 1 \). Let \( P^\kappa \) to be the \( \kappa \)-order polynomials where \( \kappa \geq m+1 \) with its Hilbert norm:

\[
\|f\|^\kappa = \left( \int \frac{|f(x)|^2 \gamma(x) dx}{\gamma(x) dx} \right)^{\frac{1}{\kappa}}.
\]

In addition, let \( \{\pi^s\}_{s=0}^\kappa \) to be the orthonormal basis associated with the above norm and \( \pi^s \) is an \( s \)-order polynomial. Thus it is clear to see that

\[
\int x^k \pi^m(x) \gamma(x) dx = 0 \quad \text{for} \quad k = 0, 1, 2 \ldots m
\]

\[
\int x^m \pi^m(x) \gamma(x) dx = c \neq 0.
\]

Thus we could let \( \phi(x) = \frac{1}{c} \pi^m(x) \gamma(x) \). This proves our Proposition.

Theorem 4.5. For \( m \geq [p^{-1} - 1] \), \( 0 < p \leq 1 \), \( m \in \mathbb{N} \), \( \phi(y) \in S^m(\mathbb{R}) \), with \( \text{supp} \phi(y) \subseteq I(0,10) \cap I(0,5)^c \), we could deduce that

\[
H^p(\mathbb{R}) \subsetneq H^p_\phi(\mathbb{R}).
\]

Proof. It is clear that

\[
H^p(\mathbb{R}) \subsetneq H^p_\phi(\mathbb{R}).
\]

Thus we need to find a \( f(y) \notin H^p(\mathbb{R}) \) with \( f(y) \in H^p_\phi(\mathbb{R}) \) to show that

\[
H^p(\mathbb{R}) \subsetneq H^p_\phi(\mathbb{R}).
\]

By Proposition 4.4, there exists \( \phi(y) \in S^m(\mathbb{R}) \), with \( \text{supp} \phi(y) \subseteq I(0,10) \cap I(0,5)^c \) and \( \int y^m \phi(y) dy \sim 1 \). For fixed \( x, t \in \mathbb{R} \), we could see that the function \( y \rightarrow \phi \left( \frac{x-y}{t} \right) \) satisfies \( \text{supp} \phi \left( \frac{x-y}{t} \right) \subseteq I(x,10t) \cap I(x,5t)^c \). Let \( P^\kappa \) to be the \( \kappa \)-order polynomials with its Hilbert norm:

\[
\|g\|^\kappa = \left( \int_{I(x,10t) \cap I(x,5t)^c} \frac{|g(y)|^2 dy}{dy} \right)^{\frac{1}{\kappa}},
\]

where \( \kappa \geq m+1 \). In addition, let \( \{\pi^s\}_{s=0}^\kappa \) be the orthonormal basis associated with the above norm and \( \pi^s \) an \( s \)-order polynomial. Thus it is clear to see that the following holds:

\[
\int_{I(x,10t) \cap I(x,5t)^c} y^k \pi^s(y) dy = \int_{I(x,10t) \cap I(x,5t)^c} \pi^s(y) \pi^s(y) dy = 0 \quad \text{for} \quad k \in \mathbb{Z}, \ 0 \leq k < s,
\]

\[
\int_{I(x,10t) \cap I(x,5t)^c} y^s \pi^s(y) dy \neq 0,
\]

\[
\int_{I(x,10t) \cap I(x,5t)^c} \pi^s(y) \pi^s(y) dy \frac{dy}{t} = 1.
\]

We could see that

\[
1 \sim \|\pi^s\|^\kappa \geq \int_{I(x,10t) \cap I(x,5t)^c} \pi^s(y) \pi^s(y) dy \frac{dy}{t}
\]

\[
\geq \int_{I(0,10) \cap I(0,5)^c} \pi^s(x+tu) \pi^s(x+tu) du.
\]
For any $P \in P^\infty$, by using the fact that any two norms on a finite dimensional spaces are equivalent, we obtain
\[
\left( \int_{I(0,10) \cap I(0,5)^c} |P(u)|^2 du \right)^{1/2} \geq C_I \sup_{u \in I(0,10)} |P(u)|,
\]
where $C_I$ is independent of $P$ and $x,t$. Taking $P(u) = \pi^s(x+tu)$, from the above two inequalities, it follows that:
\[
\sup_{u \in I(x,10t) \cap I(x,5t)^c} |\pi^s(u)| \leq C, \tag{80}
\]
where $C$ is a constant independent on $x,t$.

Let $f(y)$ with $\text{supp } f(y) \subseteq I(0,1)$ and $f(y) > 0$ and $\|f(y)\|_\infty \leq 1$. Thus it is clear that $f(y) \notin H^p(\mathbb{R})$. We will show that $f(y) \in H^p(\mathbb{R})$ next. We use $C_i$ to denote as following:
\[
C_i = \int_{I(x,10t) \cap I(x,5t)^c} f(y)\pi^i(y) \frac{dy}{t} \quad (i = 0, 1, 2 \cdots m).
\]
Thus by Formula (80), we could deduce that:
\[
\|C_i\|_\infty \lesssim 1 \quad (i = 0, 1, 2 \cdots m). \tag{81}
\]
Thus by the vanish property of $\phi(y)$, we could deduce that
\[
\int_{I(x,10t) \cap I(x,5t)^c} f(y)\phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \tag{82}
\]
\[
= \int_{I(x,10t) \cap I(x,5t)^c} \left( f(y) - \sum_{i=0}^{m} C_i \pi^i(y) \right) \phi \left( \frac{x-y}{t} \right) \frac{dy}{t}.
\]
It is also clear that for any $j \in \{0, 1, \cdots m\}$, the following holds:
\[
\int_{I(x,10t) \cap I(x,5t)^c} \left( f(y) - \sum_{i=0}^{m} C_i \pi^i(y) \right) \pi^j(y) \frac{dy}{t} = C_j - C_j = 0. \tag{83}
\]
Thus from Formula (83) we could deduce that
\[
\int_{I(x,10t) \cap I(x,5t)^c} \left( f(y) - \sum_{i=0}^{m} C_i \pi^i(y) \right) g^j \frac{dy}{t} = 0 \text{ for any } j \in \{0, 1, \cdots m\}. \tag{84}
\]
Thus by Formula (84), we could write Formula (82) as:
\[
\int_{I(x,10t) \cap I(x,5t)^c} f(y)\phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \tag{85}
\]
\[
= \int_{I(x,10t) \cap I(x,5t)^c} \left( f(y) - \sum_{i=0}^{m} C_i \pi^i(y) \right) \left( \phi \left( \frac{x-y}{t} \right) - q_{x,t}(y) \right) \frac{dy}{t},
\]
where $q_{x,t}(y)$ is the degree $m$ Taylor polynomial of the function $y \to \phi \left( \frac{x-y}{t} \right)$ expanded about origin. By the usual estimate of the remainder term in Taylor expansion, we could obtain that:
\[
\left| \phi \left( \frac{x-y}{t} \right) - q_{x,t}(y) \right| \leq C_{\frac{m+1}{m+1}} \leq C \frac{1}{m+1}, \tag{86}
\]
where the last inequality of Formula (86) is deduced from the fact that $\text{supp } f(y) \subseteq I(0,1)$. Thus by Formula (85) Formula (86), Formula (80) and Formula (81) we could deduce that
\[
\left| \int_{I(x,10t) \cap I(x,5t)^c} f(y)\phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \right| \leq C \frac{1}{m+2}. \tag{87}
\]
Case 1: $u \in (I(0,3))^c$, we will consider the following formula:

$$\sup_{|u-x|<t} \left| \int_{I(x,10t) \cap I(x,5t)^c} f(y) \phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \right|.$$ 

By the fact that $|y| < 1$ and $5 \leq \frac{|x-y|}{t} \leq 10$, we could deduce the following:

$$0 < t \leq \frac{|x| + 1}{5}. \quad (88)$$

By Formula (88), we could deduce that:

$$|u| - |x| < |u-x| < \frac{|x| + 1}{5}.$$ 

Thus we could obtain

$$\frac{5}{3} \leq |x| \leq |u|.$$

Together with the fact that

$$t \geq \frac{|x-y|}{10},$$

we could obtain

$$t \geq |x| \geq |u|.$$ 

Thus we could obtain from Formula (87) that

$$\left| \int_{I(x,10t) \cap I(x,5t)^c} f(y) \phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \right| \leq C \frac{1}{|u|^{m+2}}.$$ 

Let $m$ be the smallest integral with $m > p^{-1} - 2$ (i.e., $m = \lfloor p^{-1} - 1 \rfloor$), we could deduce that

$$\int_{I(0,3)^c} \sup_{|u-x|<t} \left| \int_{I(x,10t) \cap I(x,5t)^c} f(y) \phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \right|^p du \leq C. \quad (89)$$

Case 2: $u \in I(0,3)$, it is clear that:

$$\int_{I(0,3)} \sup_{|u-x|<t} \left| \int_{I(x,10t) \cap I(x,5t)^c} f(y) \phi \left( \frac{x-y}{t} \right) \frac{dy}{t} \right|^p du \leq C. \quad (90)$$

Thus by Formula (89) and Formula (90), we could deduce that $f(y) \in H^p_R(\mathbb{R})$. This proves the theorem. 

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