Some results on the higher Abel Jacobi map for open varieties

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Abstract

In this article, we study the infinitesimal invariant of the relative higher Abel Jacobi map of a smooth open morphism. We give a generalization of a theorem of Voisin to open algebraic varieties and higher Chow groups and as a corollary a non vanishing criterion for the higher Abel Jacobi map of an general open smooth hypersurface section of high degree of a smooth projective variety $Y$. On the other side by Nori connectness theorem, the image of the primitive part of the higher Abel Jacobi map of a general open smooth hypersurface of high degree of $Y$ is, modulo torsion, generated by the restriction to this open smooth hypersurface of a closed Bloch cycle in the corresponding affine subset of $Y$ whose cohomology class in $Y$ is primitive.

1 Introduction

Notations:

- We denote by $\text{Var}(\mathbb{C})$ the category of algebraic varieties over $\mathbb{C}$, $\text{SmVar}(\mathbb{C}) \subset \text{Var}(\mathbb{C})$ the full subcategory of smooth algebraic varieties, $\text{PVar}(\mathbb{C}) \subset \text{Var}(\mathbb{C})$, the full subcategory of projective varieties, $\text{PSmVar}(\mathbb{C}) \subset \text{Var}(\mathbb{C})$ the full subcategory of smooth projective varieties.

- For $V \in \text{Var}(\mathbb{C})$, we denote by $V^{an}$ the complex analytic space associated to $V$ with the usual topology induced by $\mathbb{C}^N$. By $V' \subset V$ an open subset, we mean an open subset of $V^{an}$ (i.e. an open subset for the usual topology).

- For a sheaf $\mathcal{F}$ of abelian group on a locally compact Hausdorff topological space $V$, we denote by $D^\vee(\mathcal{F})$ the (Verdier dual) sheaf: for $V' \subset V$ an open subset, $\Gamma(V', D^\vee(\mathcal{F})) = \Gamma_c(V', \mathcal{F})^\vee$.

- For $V \in \text{SmVar}(\mathbb{C})$, we denote by $O_V$ the sheaf of holomorphic function on $V^{an}$ and by $(\Omega^\bullet_V, \partial)$ the complex of sheaf of holomorphic forms on $V^{an}$. We denote by $(\mathcal{A}^\bullet_V, \partial, \bar{\partial})$ the bicomplex of sheaf of differential forms on $V^{an}$. The filtration $F$ associated to its total complex $(\mathcal{A}^\bullet_V, d)$ is the Frölicher filtration. We denote by $D^\vee_V = D^\vee(\mathcal{A}^\bullet_V)$ the complex of sheaf of currents on $V^{an}$ which is filtered by the Frölicher filtration $F$.

- For $V \in \text{Var}(\mathbb{C})$ and $\mathcal{F}$ a sheaf of $O_V$ module on $V^{an}$, we denote by $D^\vee_{O_V}(\mathcal{F}) = \mathcal{H}om_{O_V}(\mathcal{F}, O_V)$ the dual sheaf of $O_V$ module on $V^{an}$: for $V' \subset V$ an open subset, $\Gamma(V', D^\vee_{O_V}(\mathcal{F})) = \mathcal{H}om(\mathcal{F}_{|V'}, O_{V'})$.

- For a complex $A^\bullet$ in an abelian category, we denote by $F_b$ the filtration bête on it: $F^p_b A^\bullet = A^\bullet_{\geq p}$.

- We denote by $\Box^n = (\mathbb{P}^1 \setminus \{1\})^n \subset (\mathbb{P}^1)^n$ and by $Z^p(X, n) \subset Z^p(X \times \Box^n)$ the subgroup of $p$ codimensional cycle in $X \times \Box^n$ meeting all faces of $\Box^n$ properly. We denote by $\pi_X : X \times (\mathbb{P}^1)^n \to X$ and $\pi_{(\mathbb{P}^1)^n} : X \times (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$ the projections.

- For $V \in \text{Top}$ a topological space, we denote by $C^\bullet_{\text{sing}}(V, \mathbb{Z}) = Z \text{Hom}_{\text{Top}}(\Delta^\bullet, V)$ the complex of singular chains, $\Delta^p \subset \mathbb{R}^p$ being the standard simplex. For $V \in \text{Diff}(\mathbb{R})$ a differential manifold, we have an the inclusion of complexes $C^\bullet_{\text{diff}}(V, \mathbb{Z}) = Z \text{Hom}_{\text{Diff}(\mathbb{R})}(\Delta^\bullet, V) \subset C^\bullet_{\text{sing}}(V, \mathbb{Z})$ which is a quasi-isomorphism.
The Abel Jacobi map and normal functions associated to a family of algebraic cycles has been studied a lot for projective varieties, but few appears in the literature for open varieties. By an open variety, we mean a non complete algebraic variety, or most specifically in our case a non projective quasi-projective variety. In this article we give generalization of classical result for projective varieties to the case of open varieties.

Every smooth open variety is the complementary subset of a normal crossing divisor in a smooth projective variety. For an open variety \( U = X \setminus D \), with \( X \in \text{PSmVar}(\mathbb{C}) \) and \( D \subset X \) a normal crossing, we have \( (D^\bullet_X(\log D), F) = (D^\bullet(A^\bullet_X(\text{null} D)), F) \) the complex of sheaves of log \( D \) currents on \( X^{\text{an}} \) defined by King [4] and \( F \) is the Frölicher filtration. A log \( D \) current on an open subset \( V \subset X \) is a linear form on the null \( D \) differential forms with compact support on \( V \). The complex sheaves of null \( D \) differential forms on \( X^{\text{an}} \) is the subcomplex \( A^\bullet_X(\text{null} D) \subset A^\bullet_X \) of differential forms on \( X^{\text{an}} \) consisting of those which vanishes holomorphically on \( D \).

The main goal of the first section is to note the \( E_1 \) degenerescence of the filtered complex \( (\Gamma(X, A^\bullet_X(\text{null} D)), F) \) where \( F \) is the Frölicher filtration and to reinterpret the Poincare duality paring

\[
\left< \cdot, \cdot \right>_{\text{ev}} : (H^k(U, \mathbb{C}), F) \otimes (H^{2d-k}(X, D, \mathbb{C}), F) \to \mathbb{C} \quad \lambda \otimes \mu \mapsto \langle \lambda, \mu \rangle([X])
\]

which is a morphism of mixed Hodge structure, as the one induced in cohomology by the pairing

\[
\left< \cdot, \cdot \right>_{\text{ev}} : (\Gamma(X, D^\bullet_X(\log D)), F) \otimes (\Gamma(X, A^{2d-k}_X(\text{null} D)), F) \to \mathbb{C} \quad T \otimes \eta \mapsto \langle T, \eta \rangle
\]

To see the \( E_1 \) degenerescence of \( (\Gamma(X, A^\bullet_X(\text{null} D)), F) \), we prove (c.f. proposition 6) that the inclusion map of filtered complexes

\[
\tau : (\Gamma(X, A^\bullet_X(\text{null} D)), F) \to (\Gamma(X, A^\bullet_{X,D}(F)), F), \quad \tau(\omega) = (\omega, 0, \cdots, 0)
\]

is a filtered quasi-isomorphism and use the \( E_1 \) degenerescence of \( (\Gamma(X, A^\bullet_{X,D}(F)), F) \), where \( A^\bullet_{X,D} = \text{Cone}(i_{D^\bullet} : A^\bullet_X \to i_{D^\bullet} A^\bullet_{D^\bullet})[-1], D^\bullet \) is the simplicial variety associated to \( D \) together with the canonical morphism \( i_{D^\bullet} : D^\bullet \to D \to X \).

For an open variety \( U = X \setminus D \), with \( X \in \text{PSmVar}(\mathbb{C}) \) and \( D \subset X \) a normal crossing divisor, we have (c.f. [4]) the classical realization map

\[
R^p(X, D) : \mathbb{Z}^p(U, \bullet)^{pr/X} \to C^p(X, D, \mathbb{Z}), \quad Z \mapsto (T_Z, \Omega_Z, R_Z) := r_{X,D}^p(T_Z, \Omega_Z, R_Z),
\]

where \( \bar{Z} \in \mathbb{Z}^p(X, n) \) is the closure of \( Z \) in \( X \times (\mathbb{P}^1)^n \), which take naturally value in the relative Deligne homology complex,

\[
C^p(X, D, \mathbb{Z}) = \text{Cone}(C^\text{diff}_{2d-2p+\bullet}(X, D, \mathbb{Z}) \oplus \Gamma(X, F^p D^{2p+\bullet}_X(\log D)) \to \Gamma(X, D^{2p+\bullet+1}_X(\log D))).
\]

This leads to the higher Abel Jacobi map for \( U = X \setminus D \) :

\[
AJ_U : \mathbb{Z}^p(U, n)^{pr/X, \text{hom}} \to \text{CH}^p(U, n)^{\text{hom}} \to J^{p, 2p-n-1}(U), \quad Z \mapsto AJ(Z) = [R'_Z],
\]

where the abelian group \( \mathbb{Z}^p(U, n)^{pr/X, \text{hom}} \) consist of the closed Bloch cycle on \( U \), whose closure \( \bar{Z} \in \mathbb{Z}^p(X \times \square^n) \) in \( X \times \square^n \) is still a Bloch cycle, i.e. meet all the faces of \( X \in \square^n \) properly \( (\partial Z = 0 \text{ is then equivalent to } \partial \bar{Z} \in i_{D^\bullet} \mathbb{Z}^{p-1}(D, n)) \), and whose cohomology class \( \Omega_Z = 0 \in H^{2p-n}(U, \mathbb{C}) \) vanishes, and the complex variety

\[
J^{p,k}(U) = H^k(U, \mathbb{C})/(F^p H^k(U, \mathbb{C}) \oplus H^k(U, \mathbb{Z})) \simeq (F^{d-k} H^{d-k}(X, D, \mathbb{C}))^\vee / H^{d-k}(X, D, \mathbb{Z})
\]

is the intermediate jacobian. We show in proposition 9 that

- \( AJ_U \) for \( U \in \text{SmVar}(\mathbb{C}) \) is independent of the choise of a compactification \( (X, D) \), \( U = X \setminus D \), \( X \in \text{PSmVar}(\mathbb{C}) \), \( D \subset X \) n.c.d ;
- \( AJ_U \) is covariantly functorial in \( U \in \text{SmVar}(\mathbb{C}) \) for proper morphisms.
• \( AJ_U \) is contravariantly functorial in \( U \in \text{SmVar}(\mathbb{C}) \) for all morphisms.

In the second section (section 3), we study the relative case. Let \( f_U : U \to S \) an open morphism which is the restriction to the complementary of a divisor \( D \subset X \) of a smooth projective morphism \( f : X \to S, X, S \in \text{SmVar}(\mathbb{C}) \), such that \( D \) restrict on each fiber \( X_s \) of \( f \) to a normal crossing divisor \( D_s \subset X_s \). We then introduce the (holomorphic) Leray filtration on the complexes of log \( D \) currents and \( \text{mul} \ D \) differential forms giving rise to the commutative diagram of inclusion of bifiltered complexes of sheaves on \( X^{an} \) (cf proposition 11 and proposition 13):

\[
\begin{align*}
(\Omega^*_X(\text{null} \ D), F_b, L)^{\circ} & \quad \longrightarrow \quad (\mathcal{A}^*_X(\text{null} \ D), F, L) \\
(\Omega^*_X, F_b, L)^{\circ} & \quad \longrightarrow \quad (\mathcal{A}^*_X, F, L) \\
(\Omega^*_X(\log \ D), F_b, L)^{\circ} & \quad \longrightarrow \quad (\mathcal{A}^*_X(\log \ D), F, L) \longrightarrow (\mathcal{D}_X(\log \ D), F, L)
\end{align*}
\]

whose rows are bifiltered quasi-isomorphisms of sheaves. As in the first section, we note the \( E_1 \) degeneration of the filtered complex \( (f_\ast \mathcal{A}^*_X(\text{null} \ D), F) \) where \( F \) is the Frölicher filtration and we reinterpret the Poincaré duality paring

\[
< \cdot, \cdot >_{ev_f} : (\mathcal{H}^k(f_U), F) \otimes (\mathcal{H}^{2d_X - k}(f_X, D), F) \to \mathcal{O}_S \lambda \otimes \mu \to (\lambda, \mu)([X])
\]

which is a morphism of variation of mixed Hodge structure, as the one induced in cohomology by the pairing

\[
< \cdot, \cdot >_{ev_f} : (f_\ast \mathcal{D}^k_X(\log \ D), F) \otimes (\Gamma(X, f_\ast \mathcal{A}^{2d_X - k}(\text{null} \ D)), F) \to \mathbb{C} T \otimes \eta \to f_\ast (T \wedge \eta)
\]

Here,

- \( \mathcal{H}^k(f_U) = \mathcal{H}^k f_\ast \mathcal{D}^k_X(\log \ D) \simeq Rf_U! \mathcal{C} \otimes \mathcal{C} \mathcal{O}_S \) and
- \( \mathcal{H}^k(f_X, D) = \mathcal{H}^k f_\ast \mathcal{A}^*_X(\text{null} \ D) \simeq Rf_{X,D}^! \mathcal{C} \otimes \mathcal{C} \mathcal{O}_S \)

are sheaves of \( \mathcal{O}_S \) modules on \( S^{an} \) whose evaluations on \( s \in S \) are \( H^k(U_s, \mathcal{C}) \) and \( H^k(X_s, D_s, \mathcal{C}) \) respectively, and the filtration \( F \) is the one induced by the Frölicher filtration (see definition 11). For \( s \in S \), since the fiber \( U_s \subset U \) is closed in \( U^{an} \) and \( U^{an} \) is paracompact, we have \((R^k f_{U_s})_s \xrightarrow{\sim} H^k(U_s, \mathcal{C})\). We have the canonical quasi isomorphism \( Rf_{X,D,s}^! \mathcal{C} \otimes \mathcal{C} \mathcal{O}_S \to \text{cone}(Rf_{U,s}^! \mathcal{C} \to Rf_{D,s}^! \mathcal{C})[-1] \). On the other hand, \((R^k f_{X,s})_s \xrightarrow{\sim} H^k(X_s, \mathcal{C}) \) and \((R^k f_{D,s})_s \xrightarrow{\sim} H^k(D_s, \mathcal{C}) \) since the fibers \( X_s \subset X \) and \( D_s \subset D \) are closed in \( X^{an} \) and \( D^{an} \) respectively and \( X^{an} \) and \( D^{an} \) are compact (hence paracompact). Hence, for \( s \in S \), \((R^k f_{X,D,s})_s \xrightarrow{\sim} H^k(X_s, D_s, \mathcal{C})\)

To see the \( E_1 \) degenerescence of \((f_\ast \mathcal{A}^*_X(\text{null} \ D), F)\), we prove (c.f. proposition 10 and corollary 4) that map of filtered complexes of sheaves on \( S^{an} \)

\[
< \tau > : (f_\ast \mathcal{A}^*_X(\text{null} \ D), F) \to (f_\ast \mathcal{A}^*_X(\text{null} \ D), F), \quad < \tau > (\omega) = (\omega, 0, \cdots, 0)
\]

is a filtered quasi-isomorphism and use the \( E_1 \) degenerescence of \((f_\ast \mathcal{A}^*_X(D), F)\). The commutative diagram of bicomplexes of sheaves on \( X^{an} \) (c.f. proposition 11 see also remark 4)

\[
\begin{align*}
\phi^{r,p} : \text{Gr}^r_L \mathcal{A}^*_X(\text{null} \ D) & \xrightarrow{\sim} \mathcal{A}^{r,p}_X(\text{null} \ D) \otimes \mathcal{O}_X f^* \Omega^*_S \\
\phi^{r,p,q} : \text{Gr}^r_L \mathcal{A}^*_X(\log \ D) & \xrightarrow{\sim} \mathcal{A}^{r,p,q}_X(\log \ D) \otimes \mathcal{O}_X f^* \Omega^*_S
\end{align*}
\]
Theorem 6. We prove using the duality and the proper generalization a classical result: \( \nu_Z : s \in S \mapsto [R'_Z] = ev_{X,S}[R_{Z,s}] \in J^{p,2p-n-1}(Us) \) (11)

In theorem 6 we prove the duality and the \( E_1 \) degenerescence of \((f,A_X^{\bullet}(\text{mul}D),F)\), the following generalization a classical result: \( \nu_Z \in NS(f_U) \subset \Gamma(S,J^{p,2p-n-1}(f_U)) \) is a normal function, that is is holomorphic and horizontal. Here

\[
J^{p,2p-n-1}(f_U) = H^{2p-n-1}(S,f_U)/(H^{p,2p-n-1}(f_U) \oplus H^{2p-n-1}(f_U))
\]

is the relative intermediate jacobian. As a normal function \( \nu_Z \) has an infinitesimal invariant \( \delta \nu_Z \in \Gamma(S,H^{p,2p-n-1}(f_U)/\text{Im}(\nabla)) \) (c.f. the end of the subsection 3.3). On the other hand the class \( \Omega_Z|_{Us} = 0 \in H^{2p-n}(Us,C) \) of the current \( R_Z \) restrict to zero on the fiber by hypothesis leading to a class \( \Omega_Z \in \Gamma(S,L^1R^{p-n}(f,U,\Omega^p_X(\log D))) \), which has an infinitesimal invariant:

\[
\delta[\Omega_Z] = \nabla^1(\Omega_Z/L^2) = \nabla^1([\Omega_Z/L^2] \in \Gamma(S,H^{p,2p-n}(f_U) \otimes OS,OS/\text{Im}(\nabla))
\]

where, c.f. subsection 3.3, 3.4 and 3.5

- \( \psi_L^1 : Gr^r_L R^{p-n}f_*\Omega^p_X(\log D) = E^{p-n}_r \rightarrow R^{p-n}f_*(Gr^r_L\Omega^p_X(\log D)) = E^{p-n}_r \) is the inclusion of sheaves on \( S^{an} \) induced by the spectral sequence associated to the complex \( (\Omega^p_X(\log D),L) \): for degree reason no arrow \( d_r, r \geq 2 \) can lead to \( E^{p-n}_2 \).
- \( \nabla : OS \otimes H^{p,2p-n}(f_U)/\text{Im}(\nabla) \rightarrow R^{p-n}f_*(Gr^r_L\Omega^p_X(\log D)) \) is the isomorphism induced by
  - the morphism of sheaves on \( S^{an} \) \( \nabla : R^{p-n}f_*Gr^r_L\Omega^p_X(\log D) \rightarrow R^{p-n}f_*(\Omega^p_X(\log D)/L^2) \) (induced in relative cohomoloy by the morphism of sheaves on \( X^{an} \) \( \nabla : Gr^r_L\Omega^p_X(\log D) \rightarrow \Omega^p_X(\log D)/L^2) \),
  - the isomorphism of sheaves on \( S^{an} \) \( \delta^{1,p} : R^{p-n}f_*Gr^r_L\Omega^p_X(\log D) \rightarrow OS \otimes H^{p,2p-n}(f_U) \) (induced in the \( f \) direct image cohomoloy by the isomorphism of complexes of sheaves on \( X^{an} \) \( \delta^{1,p} : Gr^r_L\mathcal{A}^{p,\bullet}(\log D)/X \rightarrow \mathcal{A}^{p,\bullet}(\log D) \otimes OS \)

In theorem 7 (c.f. subsection 3.5), we prove using the duality and the \( E_1 \) degenerescence of \((f,A_X^{\bullet}/S(\text{mul}D),F)\), the following generalization a result of Voisin (5 theorem 19.14), which is one of the main result of this paper:

**Theorem 1.** Let \( Z = \sum_i n_i Z_i \in \mathcal{Z}(U,n)_{\text{pr}/X}^{\text{pr}/X} \) such that \( \pi_X(Z_i) \subset X \) is a local complete intersection for all \( i \). Then \( \delta \nu_Z = \delta[\Omega_Z] \in \Gamma(S,OS \otimes H^{p,2p-n}(f_U)/\text{Im}(\nabla)) \).
In the last section (section 4), we give two results on the relative higher Abel Jacobi map for families of ample open hypersurface section of high degree of smooth projective variety \( Y \in \text{PSmVar}(\mathbb{C}) \). In subsection 4.1, we give (c.f. theorem [9]) the following application of theorem [7]. Let \( Y \in \text{PSmVar}(\mathbb{C}) \) together with an embedding \( Y \subset (\mathbb{P}^1)^n \). Consider the commutative diagram [51] of families of hypersurface sections of degree \( d \) and \( e \), whose squares are cartesians:

\[
\begin{array}{cccc}
  f_D : D = X \cap Z \overset{k_D}{\longrightarrow} Z \\
  \downarrow \scriptstyle{i_D} & & & \downarrow \scriptstyle{i_x} \\
  f : X \overset{i_X}{\longrightarrow} Y \times S_d \times S_e & \longrightarrow & S_d \times S_e \\
  \downarrow \scriptstyle{j_u} & & & \downarrow \scriptstyle{p_d,e} \\
  f_U : U = X \setminus D \overset{i_u}{\longrightarrow} (Y \times S_d) \times S_e \\
\end{array}
\]

and denote by \( p_Y : Y \times S_d \times S_e \rightarrow Y \) the other projection. Note that \( X, Z, D \in \text{PSmVar}(\mathbb{C}) \), since \( \pi_{Y/X} : X \rightarrow Y, \pi_{Y/Z} : Z \rightarrow Y, \pi_{Y/D} : D \rightarrow Y \) are projective bundles and \( Y \) is smooth. For \( 0 \in S_e \), consider the pullback of this diagram:

\[
\begin{array}{cccc}
  f'_D : D = X \cap (Z_0 \times S_d) \overset{k_0}{\longrightarrow} Z_0 \times S_d \\
  \downarrow \scriptstyle{i_D} & & & \downarrow \scriptstyle{i_X} \\
  f' : X = X_{S_d \times 0} \overset{i_X}{\longrightarrow} Y \times S_d \times 0 & \longrightarrow & S_d \\
  \downarrow \scriptstyle{j_u} & & & \downarrow \scriptstyle{p_d} \\
  f'_U : U = X \setminus D \overset{i_u}{\longrightarrow} (Y \setminus Z_0) \times S_d \\
\end{array}
\]

where \( Z_0 = \pi_Y(\mathcal{Z}_{S_d \times 0}) \subset Y \), \( \pi_0^Y = \pi_{Y/X} \times \pi_{Y/Z} : Y \times S_d \times 0 \rightarrow Y \) being the projection, so that we have \( \mathcal{Z}_{S_d \times 0} = Z_0 \times S_d \). Then \( Y \setminus Z_0 \) is an affine variety. We have \( H^{dv}(Y \setminus Z_0, \mathbb{C}) = H^{dv}(Y, \mathbb{C}) \) (see subsection 4.1 for the definition of the primitive cohomology of a smooth quasi-projective variety \( V \) as the kernel of the action of \( \Delta(V_H) \subset V \times V \) where \( V_H \subset V \) is an ample hypersurface section). For a morphism \( T \rightarrow S_d \), we consider the pullback of the diagram (12):

\[
\begin{array}{cccc}
  f''_D : D_T = X_T \cap (Z_0 \times T) \overset{k_0^T}{\longrightarrow} Z_0 \times T \\
  \downarrow \scriptstyle{i_D^T} & & & \downarrow \scriptstyle{i_X^T} \\
  f^T : X_T = X_{S_d \times 0} \overset{i_X^T}{\longrightarrow} Y \times T \times 0 & \longrightarrow & T \\
  \downarrow \scriptstyle{j_u^T} & & & \downarrow \scriptstyle{p_T} \\
  f''_U : U_T = X_T \setminus D_T \overset{i_u^T}{\longrightarrow} (Y \setminus Z_0) \times T \\
\end{array}
\]

where \( X_T = X \times S_d, U_T = U \times S_d, T, D_T = D \times S_d, T \). Then we have a version of Nori connectness theorem for families of ample open hypersurfaces of \( Y \in \text{PSmVar}(\mathbb{C}) \) (c.f. theorem [5]).

**Theorem 2.** Assume \( d_Y \geq 4 \) Let \( 0 \in S_e \) sufficiently general and \( S \subset S_d \) the open subset over which such that the morphisms \( f^0 : X \rightarrow S_d \) and \( f''_D : D \rightarrow S_d \) are smooth projective. Then, if \( d, e >> 0 \), for all smooth morphism \( T \rightarrow S_d \),

(i) \( i^*_X : H^{dv-p}(Y \times T, \Omega^p_{X_T}(\log(Z_0 \times T))) \sim H^{dv-p}(X_T, \Omega^p_{X_T}(\log D_T)) \) is an isomorphism,

(ii) \( i^*_U : H^{dv}((Y \setminus Z_0) \times T, \mathbb{C}) \sim H^{dv}(U_T, \mathbb{C}) \) is an isomorphism of mixed hodge structure.
Theorem 3. Assume \( d_Y \geq 4 \). Let \( 0 \in S_e \) sufficiently general and \( S \subset S_d \) the open subset over which such that the morphisms \( f^0 : X \to S_d \) and \( f^0_D : D \to S_d \) are smooth projective. Let \( Z \in \mathbb{Z}^p(Y \setminus Z_0, 2p - d_Y)_{p=0}^{\mathbb{Q}} \) such that \( [\Omega_x] \neq 0 \in H^{d_Y}(Y \setminus Z_0, \mathbb{C}) \). Then for \( s \in S \) general, \( AJ_{U_s}(Z_s) := [R_{Z_s}^s] \neq 0 \in J^p, d_Y^{-1}(U_s) \).

Finally, we note that this version of Nori connectness theorem implies the following (c.f. theorem [10] which is a version of a result of Green and Müller-Stach [2] for open ample hypersurface of a smooth projective variety:

Theorem 4. Assume \( d_Y \geq 4 \). Let \( 0 \in S_e \) sufficiently general and \( S \subset S_d \) the open subset over which such that the morphisms \( f^0 : X \to S_d \) and \( f^0_D : D \to S_d \) are smooth projective. Consider the commutative diagram

\[
\begin{array}{ccc}
\text{CH}^p(Y \setminus Z_0, 2p - d_Y, \mathbb{Q}) & \xrightarrow{i_{U_s}^*} & \text{CH}^p(U_s, 2p - d_Y, \mathbb{Q}) \\
\downarrow{\mathcal{R}^p(Y, Z_0)} & & \downarrow{\mathcal{R}^p(X_s, D_s)} \\
H^d_Y(D_Y, Y, Z_0, \mathbb{Q}) & \xrightarrow{i_{U_s}^*} & H^d_Y(D_Y, X_s, D_s, \mathbb{Q})/J^p, d_Y^{-1}(Y \setminus Z_0, \mathbb{Q})
\end{array}
\]

Then for a general point \( s \in S \), \( \text{Im}((\mathcal{R}(X_s, D_s)) = \text{Im}((\mathcal{R}(X_s, D_s) \circ i_{U_s}^*). \)

That is if \( d_Y \geq 4 \), \( 0 \in S_e \) and \( s \in S \subset S_e \) are general, the image of the primitive part of the Abel Jacobi map:

\[
AJ_{U_s}^0(Z_s) \in \mathbb{Z}^p(U_s, 2p - d_Y, \mathbb{Q})_{p=0}^{\mathbb{Q}} \to J^p, d_Y^{-1}(U_s) \cap i_{U_s}^* J^p, d_Y^{-1}(Y \setminus Z_0, \mathbb{Q})
\]

is modulo torsion generated by the \( AJ_{U_s}^0(Z_s) \) for \( Z \in \mathbb{Z}^p(Y \setminus Z_0, 2p - d_Y, \mathbb{Q})_{p=0}^{\mathbb{Q}}. \)

2 Higher Abel Jacobi map for open varieties

Let \( X \in \text{SmVar}(\mathbb{C}) \) and \( D = \cup_{j=1}^s D_j \subset X \) a normal crossing divisor with smooth components \( D_j \). Let \( U = X \setminus D \). Denote by

- \( j : U \hookrightarrow X \) the open inclusion,
- \( i_D : D \hookrightarrow X \) and \( i_{D_j} : D_j \hookrightarrow X \), for \( j \in \{1, \ldots, s\} \), the closed inclusions.
- \( \pi_X : X \times (\mathbb{P}^1)^n \to X \) and \( \pi_{(\mathbb{P}^1)^n} : X \times (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n \) the projections.

Denote by \( D_* \) the simplicial algebraic variety associated to \( D : D_j = \cap_{j \in J} D_j \) for \( J \subset \{1, \ldots, s\} \) with morphisms the alternate sum of the inclusion maps \( i_{D_j, D_j} : D_j \hookrightarrow D_j \) for \( J \subset J' \). Let

\[
i_{D_*} : D_* \xrightarrow{a_{D_*}} D' \xrightarrow{i_D} X, \quad i_{D_j} : D_j \xrightarrow{a_{D_j}} D_j \xrightarrow{i_D} X,
\]

be the morphism of simplicial algebraic varieties given by the inclusions \( i_{D_j} : D_j \hookrightarrow X \) of the smooth varieties \( D_j \) in \( X \).

The adjunction morphism of complexes of sheaves on \( D^{an} \) \( \text{ad}(a_D) : C_D \to Ra_{D_*} a_{D_*}^* C_D \) is a quasi-isomorphism and \( Ra_{D_*} a_{D_*}^* C_D \) is \( a_{D_*} A_{D_*}^* \). By definition, \( a_{D_*} A_{D_*}^* \) is the total complex of sheaves on \( D^{an} \) associated to the double complex \( (a_{D_j} A_{D_j}^*, d, D_r) \), where \( D_r = \sum_{j \subset J} \sum_{J' \subset J} (-1)^{r+1} i_{D_j, D_j}. \)

Denote by \( \omega |_{D_j} := i_{D_j}^* \omega \). We have the adjunction morphism of complexes of sheaves on \( X^{an} \):

\[
i_{D_*}^* : A_{D_*}^* \to i_{D_*} A_{D_*}^*, \quad \omega \in \Gamma(V, A_{D_*}^*) \mapsto i_{D_*}^* \omega \in \Gamma(V, A_{D_*}^*) \mapsto \Gamma(D \cap V, (A_{D_j}^*)^k) := \oplus \Gamma(D \cap V, A_{D_j}^*)_{n \cap J}^{n \cap J+1}.
\]

6
2.1 The relative complex of differential forms for the pair \((X, D)\)

**Definition 1.** The relative complex of sheaves of holomorphic forms for the pair \((X, D)\) is \(\Omega_{X,D}^\bullet := \text{Cone}(i_{D*}^*: \Omega_X \to i_{D*}^*\Omega_D)\), that is, for \(V \subset X\) an open subset,
\[
\Gamma(V, \Omega_{X,D}^\bullet) = \text{Cone}(i_{D*}^*: \Gamma(V, \Omega_X^\bullet) \to \Gamma((D \cap V)^\bullet, \Omega_D^\bullet))[−1]
\]
- \(\Gamma(V, \Omega_{X,D}^p) = \Gamma(V, \Omega_X^p) \oplus (\oplus J \Gamma(D_j \cap V, \Omega_D^{p−\text{card}\ J}))\)
- \(\partial(\omega, \eta_j) = (\partial \omega, \omega|_{D_j \cap V} - \partial \eta_1, \ldots, \omega|_{D_j \cap V} - \partial \eta_s, \ldots, \eta_2, \ldots, s|_{D_1, \ldots, s \cap V} + \cdots + (−1)^{s−1} \eta_1, \ldots, s−1|_{D_1, \ldots, s \cap V})− \partial \eta_{1, \ldots, s} \]

There is the filtration induced by the filtration bête \(F_b:\)
\[
F^p_{X,D} = \Omega_X^{\geq p} \ominus i_{D*}^*\Omega_D^{\geq p}[−1] : \text{Gr}^p_{F_b} \Omega_{X,D}^\bullet = \Omega_X^p[−p] \oplus (\oplus J i_{D*}^*\Omega_D^{p−\text{card}\ J}]
\]

**Definition 2.** The relative complex of sheaves of differential forms for the pair \((X, D)\) is \(\mathcal{A}_{X,D}^\bullet = \text{Cone}(i_{D*}^*: \mathcal{A}_X^\bullet \to i_{D*}^*\mathcal{A}_D^\bullet)[−1]\), that is, for \(V \subset X\) an open subset,
\[
\Gamma(V, \mathcal{A}_{X,D}^\bullet) = \text{Cone}(i_{D*}^*: \Gamma(V, \mathcal{A}_X^\bullet) \to \Gamma((D \cap V)^\bullet, \mathcal{A}_D^\bullet))[−1]
\]
- \(\Gamma(V, \mathcal{A}_{X,D}^k) = \Gamma(V, \mathcal{A}_X^k) \oplus (\oplus J \Gamma(D_j \cap V, \mathcal{A}_D^{k−\text{card}\ J}))\)
- \(d(\omega, \eta_j) = (d \omega, \omega|_{D_j \cap V} - \partial \eta_1, \ldots, \omega|_{D_j \cap V} - \partial \eta_s, \ldots, \eta_2, \ldots, s|_{D_1, \ldots, s \cap V} + \cdots + (−1)^{s−1} \eta_1, \ldots, s−1|_{D_1, \ldots, s \cap V})− \partial \eta_{1, \ldots, s} \]

It is a filtered complex of sheaves on \(X_{an}\) by the Frölicher filtration \(F\); there is also the weight filtration \(W\) with respect to the sequence \(D_1, \ldots, \cup_{k=1}^p D_j, X\) : for \(V \subset X\) an open subset,
\[
\Gamma(V, \mathcal{A}_{X,D}^k) = \Gamma(V, \mathcal{A}_X^k) \oplus (\oplus J \Gamma(D_j \cap V, \mathcal{A}_D^{k−\text{card}\ J})); \quad \Gamma(V, \mathcal{A}_{X,D}^k) = \oplus J \subseteq \Gamma(D_j \cap V, \mathcal{A}_D^{k−\text{card}\ J}).
\]

If \(X \in \text{PSmVar}(\mathbb{C})\) is smooth projective, it is clear that \(\Gamma(X, \mathcal{A}_{X,D}^\bullet, F)\) is a mixed hodge complex [7] so that the spectral sequence given by the Frölicher filtration \(F\) is \(E^1\) degenerate.

**Proposition 1.** (i) The wedge product induces an isomorphism of complexes of sheaves on \(X_{an}\)
\[
w_X : \text{Gr}^p_{F_b} \Omega_{X,D}^\bullet \otimes_{\mathcal{O}_X} (\mathcal{A}_X^\bullet, \partial) \cong (\mathcal{A}_{X,D}^\bullet, \partial)
\]
(ii) The inclusion of filtered complexes of sheaves on \(X_{an}\)
\[
(\Omega_{X,D}^\bullet, F_b) \hookrightarrow (\mathcal{A}_{X,D}^\bullet, F)
\]
is a filtered quasi-isomorphism.

**Proof.** (i): We check that it define a morphism of complex. The fact that it is an isomorphism is clear. Assume for simplicity that \(D_1 = D\). Let \(V \subset X\) an open subset, \(\omega \in (V, \Omega_X^p)\) and \(\gamma \in \Gamma(V, \Omega_D^{p−1})\). Then
\[
d(\omega \wedge \gamma, 0) = (d(\omega \wedge \gamma), (\omega \wedge \gamma)|_D) = (\partial \omega \wedge \gamma + (−1)^p \omega \wedge d\gamma, \omega|_D \wedge \gamma|_D)
\]
\[
= (\partial \omega \wedge \gamma + (−1)^p \omega \wedge d\gamma, \omega|_D \wedge \gamma|_D) \in \Gamma(V, \mathcal{A}_X^{p+1})
\]
Thus taking the quotient by \(F^{p+1}\), we obtain
\[
\tilde{\partial}(\omega \wedge \gamma, 0) = (\omega \wedge \partial \gamma, \omega|_D \wedge \gamma|_D) \in \Gamma(V, \mathcal{A}_{X,D}^{p+1})
\]
(ii): This comes from (i). We can also see (ii) directly: we have the commutative diagram
\[
\begin{array}{ccc}
(\Omega_{X,D}^\bullet, F_b) & \xrightarrow{i_{D*}^\bullet} & (\mathcal{A}_{X,D}^\bullet, F) \\
\downarrow & & \downarrow \\
(\mathcal{A}_X^\bullet, F) & \xrightarrow{i_{D*}^\bullet} & (\mathcal{A}_D^\bullet, F)
\end{array}
\]
whose column are filtered quasi-isomorphism, thus the morphism \((\Omega_{X,D}^\bullet, F_b) \hookrightarrow (\mathcal{A}_{X,D}^\bullet, F)\) is a filtered quasi-isomorphisms.

□
2.2 Complex of differential forms whose restriction on $D$ vanishes and log currents for the pair $(X,D)$

Definition 3. [B],[E]

The bicomplex $(\mathcal{A}^\bullet_X(\log D), \partial, \bar{\partial})$ of sheaf on $X^{an}$ for the pair $(X,D)$ is:

$$\mathcal{A}^{p,q}_X(\log D) := \Omega^p_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{A}^{0,q}_X \hookrightarrow \Omega^p_X(\log D) \wedge \mathcal{A}^{0,q}_X,$$

(15)

together with the holomorphic and anti-holomorphic differential $\partial$ and $\bar{\partial}$ respectively. The induced filtration on the total complex $(\mathcal{A}^\bullet_X(\log D), d) = \text{Tot}(\mathcal{A}^\bullet_X(\log D), \partial, \bar{\partial}),$ with differential $d = \partial + \bar{\partial},$ is the Frölicher Filtration.

Definition 4. [D]

- Denote by

$$\Omega^p_X(\text{nul } D) := \cap_{i=1}^r \ker(i_{D,i}^*: \Omega^p_X \to \text{Tot}(\mathcal{A}^{0,q}_X)) \subset \Omega^p_X,$$

the locally free sheaf of $\mathcal{O}_X$ module on $X^{an}$ consisting of holomorphic $p$ forms whose restriction to $D$ vanishes.

- The bicomplex $(\mathcal{A}^\bullet_X(\text{nul } D), \partial, \bar{\partial})$ of sheaf on $X^{an}$ for the pair $(X,D)$ is:

$$\mathcal{A}^{p,q}_X(\text{nul } D) := \Omega^p_X(\text{nul } D) \otimes_{\mathcal{O}_X} \mathcal{A}^{0,q}_X \hookrightarrow \Omega^p_X(\text{nul } D) \wedge \mathcal{A}^{0,q}_X \subset \mathcal{A}^{p,q}_X,$$

(15)

together with the holomorphic and anti-holomorphic differential $\partial$ and $\bar{\partial}$ respectively. The induced filtration on the total complex $(\mathcal{A}^\bullet_X(\text{nul } D), d) = \text{Tot}(\mathcal{A}^\bullet_X(\text{nul } D), \partial, \bar{\partial}),$ with differential $d = \partial + \bar{\partial},$ is the Frölicher Filtration.

- The bicomplex $(\mathcal{A}^\bullet_X(\text{log } D_\infty), \partial, \bar{\partial})$ of sheaf on $X^{an}$ for the pair $(X,D)$ is:

$$\mathcal{A}^{p,q}_X(\text{log } D_\infty) := \cap_{j=1}^r \ker(i_{D,j}^*: \mathcal{A}^{p,q}_X \to \text{Tot}(\mathcal{A}^{0,q}_X)) \subset \mathcal{A}^{p,q}_X,$$

(15)

together with the holomorphic and anti-holomorphic differential $\partial$ and $\bar{\partial}$ respectively. The induced filtration on the total complex $(\mathcal{A}^\bullet_X(\text{log } D_\infty), d) = \text{Tot}(\mathcal{A}^\bullet_X(\text{log } D_\infty), \partial, \bar{\partial}),$ with differential $d = \partial + \bar{\partial},$ is the Frölicher Filtration.

By definition we have inclusion of bicomplexes $\mathcal{A}^\bullet_X(\text{nul } D) \subset \mathcal{A}^\bullet_X(\text{nul } D_\infty) \subset \mathcal{A}^\bullet_X.$ Denote by $t_{X,D} : \mathcal{A}^\bullet_X(\text{nul } D) \hookrightarrow \mathcal{A}^\bullet_X(\text{nul } D_\infty) \hookrightarrow \mathcal{A}^\bullet_X$ the inclusion of bicomplexes of sheaves on $X^{an}.$

Proposition 2. (i) The subcomplex of sheaves on $X^{an}$ $\mathcal{I}_D \mathcal{O}^{\bullet}_X(\log D) \subset \mathcal{O}^{\bullet}_X(\log D)$ and

the subbicomplex of sheaves on $X^{an}$ $\mathcal{I}_D \mathcal{A}^{\bullet}_X(\text{nul } D) \subset \mathcal{A}^{\bullet}_X(\text{nul } D)$ are a graded, respectively bigraded, ideal for the wedge product.

(ii) The sheaves of $\mathcal{O}_X$ modules $\Omega^p_X(\log D)$ and $\Omega^p_X(\text{nul } D)$ are locally free of rank $C^{p}_{dX}.$ Moreover, the wedge product $w_X$ induces an isomorphism of sheaves of $\mathcal{O}_X$ modules $\Omega^{dX-p}_X(\text{nul } D) \hookrightarrow D^{\partial}_D(\Omega^p_X(\log D)) \otimes_{\mathcal{O}_X} K_X.$

Proof. (i): This is proved in [E].

(ii): The fact that these sheaves are locally free is proved in [D]. The the wedge product induces an isomorphism of sheaves of $\mathcal{O}_X$ modules on $X^{an}:

$$w_X : \Omega^p_X(\log D) \otimes_{\mathcal{O}_X} \Omega^{dX-p}_X(\text{nul } D) \hookrightarrow K_X.$$ 

Indeed, for $V \subset X$ an open subset such that $V \subset \mathbb{C}^{dX}$ and $D \cap V = V(z_1 \cdots z_r),$ $w_X$ put together terms of the form

- $(\wedge_{i \in I \subset \{1, \cdots, r\}} \frac{dz_i}{z_i}) \wedge (\wedge_{j \in J \subset \{r+1, \cdots, dX\}} dz_j)$ and
\[ \Pi_{i \in I} z_i (\bigwedge_{k \in \{1, \ldots, r\}} d z_k) \wedge (\bigwedge_{l \in \{r+1, \ldots, d\}} d z_l), \]

with card \( I + \text{card } J = p \).

**Proposition 3.**  
(i) The wedge product induces an isomorphism of complexes of sheaves on \( X^{\text{an}} \)
\[ w_X : \Omega^p_X (\text{nul } D) \otimes_{O_X} (A^0_X, \bar{\partial}) \simto (A^p_X (\text{nul } D), \bar{\partial}). \]

(ii) The inclusion of filtered complexes of sheaves on \( X^{\text{an}} \)
\[(\Omega^k_X (\text{nul } D), F_k) \hookrightarrow (A^*_X (\text{nul } D), F),\]
is a filtered quasi-isomorphism.

**Proof.** (i): It is clear that it is a morphism of complex since for \( V \subset X \) an open subset, \( \omega \in \Gamma(V, \Omega^p(\text{nul } D)) \) and \( \gamma \in \Gamma(V, A^{0,p}_X) \) we have \( \bar{\partial}(\omega \wedge \gamma) = \omega \wedge \bar{\partial} \gamma \). It is an isomorphism by definition.

(ii): This comes from (i): we have the Dolbeau resolutions
\[ 0 \rightarrow \Omega^p_X (\text{nul } D) \rightarrow \Omega^p_X (\text{nul } D) \otimes_{O_X} (A^0_X, \bar{\partial}) \simto (A^p_X (\text{nul } D), \bar{\partial}). \quad (16) \]

We now give the definition of the complex of sheaves of currents:

**Definition 5.** \[ [6] \] The logaritmic complex \( (D^*_X (\log D), d) := D^v (A^*_X (\text{nul } D), d) \) of sheaf on \( X^{\text{an}} \) of currents for the pair \( (X, D) \) is the Verdier dual of \( A^*_X (\text{nul } D) \):
\[ V \subset X \text{ an open subset } \rightarrow \Gamma(V, D^k_X (\log D)) = \Gamma_c (V, A^{2d_X - k}_{X} (\text{nul } D))^v \]
It is a filtered complex by the Frölicher filtration \( F \). Indeed we get a bifiltered complex of sheaves on \( X^{\text{an}} \): for \( V \subset X \) an open subset
\[ \Gamma(V, D^p_X (\log D)) = \left \{ T \in \Gamma(V, D^p_X (\log D)), \text{ s.t. } T|_{\Gamma_c (V, A^{r,s}_X (\text{nul } D))^v} = 0, \text{ for } (r, s) \neq (d_X - p, d_X - q) \right \} \rightarrow \Gamma_c (V, A^{d_X - p, d_X - q}_X (\text{nul } D))^v \]
together with the holomorphic and anti-holomorphic differential \( \partial \) and \( \bar{\partial} \) respectively. The induced filtration on the total complex \( D^*_X (\log D) = \text{Tot} (D^r_X (\log D)) \) with differential \( d = \partial + \bar{\partial} \) is the Frölicher Filtration.

- We have the restriction map of filtered bicomplexes of sheaves on \( X^{\text{an}} \)
  \( r_{X, D} = t^v_{X, D} : D^*_X (\log D) \rightarrow \text{Int} \Gamma(V, D^r_X (\log D)) \)
  which is the (Verdier) dual to the inclusion \( t_{X, D} : \text{for } V \subset X \text{ an open subset,} \)
  \[ T \in \Gamma(V, D^r_X (\log D)) \rightarrow r_{X, D} (T) : \left( \eta \in \Gamma_c (V, A^{d_X - p, d_X - q}_X (\text{nul } D)) \rightarrow T(\eta) \right) \]

- The morphism of complexes of abelian groups
  \[ \text{Int} : C^\text{diff}_{2d_X - r} (X, \mathcal{Z}) \rightarrow C^\text{diff, BM}_{2d_X - r} (X, \mathcal{Z}) \]
given by integration factors through the quotient map \( r_{X, D} : C^\text{diff}_{2d_X - r} (X, \mathcal{Z}) \rightarrow C^\text{diff}_{2d_X - r} (X, D, \mathcal{Z}) \) to the embedding of complexes of abelian groups:
  \[ C^\text{diff}_{2d_X - r} (X, D, \mathcal{Z}) \rightarrow C^\text{diff, BM}_{2d_X - r} (X, D, \mathcal{Z}) \rightarrow \Gamma(X, D^r_X (\log D)) \rightarrow \int \gamma \]
• We have the wedge product which is the morphism of bicomplexes of presheaves on $X^{an}$

$$w_X : D^{\bullet, \bullet}_{X, r,s} \otimes_{O_X} A^{\bullet, \bullet}_{X, r,s} \to D^{\bullet, \bullet}_{X, r,s}$$ for $V \subset X$ an open subset

$$T \otimes \omega \in \Gamma(V, D^{p,q}_{X, r,s}) \otimes \Gamma(V, A^{r,s}_{X, r,s}) \mapsto T \wedge \omega : (\eta \in \Gamma_c(V, A^{d_{X, r,s}^{p,q} + d_{X, s} - s + q}_X)) \mapsto T(\omega \wedge \eta).$$ (17)

It restricts to the morphism of bicomplexes of presheaves on $X^{an}$

$$w_X : D^{\bullet, \bullet}_{X, r,s} \otimes_{O_X} A^{\bullet, \bullet}_{X, r,s} \to D^{\bullet, \bullet}_{X, r,s}$$ for $V \subset X$ an open subset

$$T \otimes \omega \in \Gamma(V, D^{p,q}_{X, r,s}(\log D)) \otimes \Gamma(V, A^{r,s}_{X, r,s}(\log D)) \mapsto T \wedge \omega : (\eta \in \Gamma_c(V, A^{d_{X, r,s}^{p,q} + d_{X, s} - s + q}_X(\log D))) \mapsto T(\omega \wedge \eta).$$ (19)

and also to the morphism of bicomplexes of presheaves on $X^{an}$

$$w_X : D^{\bullet, \bullet}_{X, r,s}(\log D) \otimes_{O_X} A^{\bullet, \bullet}_{X, r,s}(\log D) \to D^{\bullet, \bullet}_{X, r,s}(\log D)$$ for $V \subset X$ an open subset

$$T \otimes \omega \in \Gamma(V, D^{p,q}_{X, r,s}(\log D)) \otimes \Gamma(V, A^{r,s}_{X, r,s}(\log D)) \mapsto T \wedge \omega : (\eta \in \Gamma_c(V, A^{d_{X, r,s}^{p,q} + d_{X, s} - s + q}(\log D))) \mapsto T(\omega \wedge \eta).$$ (20)

• We have embeddings of sheaves on $X^{an}$ $int : A^{p,q}_X(\log D) \to D^{p,q}_X$ given by integration : for $V \subset X$ an open subset,

$$\omega \in \Gamma(V, A^{p,q}_X(\log D)) \mapsto (\eta \in \Gamma_c(V, A^{d_{X, r,s}^{p,q} + d_{X, s} - s + q}_X)) \mapsto int(\omega)(\eta) = \int_V \omega \wedge \eta.$$

These integrals are convergent because $D$ is a normal crossing divisor. Note that they do not define an embedding of bicomplexes (they do not commute with the differentials).

Denote by $\iota^r : A^{p,q}_X(\log D) \xrightarrow{\text{int}} D^{p,q}_X \xrightarrow{r} D^{p,q}_X(\log D)$ the composition : for $V \subset X$ an open subset

$$\omega \in \Gamma(V, A^{p,q}_X(\log D)) \mapsto (\eta \in \Gamma_c(V, A^{d_{X, r,s}^{p,q} + d_{X, s} - s + q}(\log D))) \mapsto \iota(\omega)(\eta) = \int_V \omega \wedge \eta.$$

We then have the following

**Theorem 5.** [R Theorem 1.3.11]

The compositions $\iota^r = r_X, D \circ int : A^{p,q}_X(\log D) \to D^{p,q}_X(\log D)$ define an embedding of bicomplexes of sheaves on $X^{an}$

$$\iota : A^{\bullet, \bullet}_X(\log D) \hookrightarrow D^{\bullet, \bullet}_X(\log D).$$

The bicomplex of sheaves on $X^{an}$ $D^{\bullet, \bullet}_X(\log D)$ is a bigraded $A^{\bullet, \bullet}_X(\log D)$ module by the map of sheaves on $X^{an}$ :

$$\alpha^{r,s}_{r,s} : A^{r,s}_X(\log D) \otimes_{O_X} D^{p,q}_X \to D^{r+s,q+r}_X(\log D).$$

If $(r, s) = (1, 0)$, this map is given by, for $V \subset X$ open subset such that $V \subset \mathbb{C}^{d_X}$ as an open subset and $D \cap V = V(z)$, $T \in \Gamma(V, D^{p,q}_{X, r,s})$, $\alpha((\frac{dz}{z} \otimes T) = r_X, D(\frac{T}{z} \wedge dz)$, where $\frac{T}{z} \in \Gamma(V, D^{p,q}_{X, r,s})$ is a current such that $\frac{zT}{z} = T$.

This bigraded module structure induces, an isomorphism of sheaves on $X^{an}$

$$\alpha^{p,q} := \alpha^{p,0,0} : \Omega^{p,q}_X(\log D) \otimes_{O_X} D^{p,q}_X \to D^{p,q}_X(\log D).$$ (21)

**Proposition 4.** We have the following exact sequences of sheaves on $X^{an}$, they are the Dolbeau resolution of locally free sheaves of $O_X$ modules $\Omega^{p,q}_X(\log D)$ and $\Omega^{d_{X, r,s}^{p,q} - p}(\log D)$ respectively :

$$0 \to \Omega^{p,q}_X(\log D) \to Gr^r_F D^{p+q}_X(\log D) = (D^{p+q, \bullet}_X(\log D), \partial)$$

$$0 \to \Omega^{d_{X, r,s}^{p,q} - p}(\log D) \to Gr^r_F D^{d_{X, r,s}^{p,q} - p+\bullet}_X(\log D) = (A^{d_{X, r,s}^{p,q} - p, \bullet}_X(\log D), \partial).$$

**Proof.** The second resolution is given by proposition [3] (ii). The first one follows from the isomorphisms (21) of theorem [5] : $\Omega^{p,q}_X(\log D) \otimes_{O_X} D^{p,q}_X \cong X D^{p,q}_X(\log D)$. \qed
Proposition 5. \[\text{The embeddings of filtered complexes of sheaves on } X^\text{an}, \text{ where } F \text{ is the Fr"{o}licher filtration and } F_b \text{ the filtration b"{e}te :}\]

\[\begin{align*}
(\Omega^\bullet_X (\log D), F_b) &\longrightarrow (A^\bullet_X (\log D), F) \\
(\Omega^\bullet_X (\log D), F) &\longrightarrow (D^\bullet_X (\log D), F)
\end{align*}\]

are filtered quasi-isomorphism.

Proof. It comes from the Dolbeau resolution of the sheaf $\Omega^p(\log D)$ (proposition 4). \qed

2.3 Degenerescence in $E_1$ of the Fr"{o}licher filtration for complex of differential forms whose restriction to $D$ vanish and duality

Consider the following inclusion of filtered complexes of sheaves on $X^\text{an}$, where $F$ is the Fr"{o}licher filtration,

\[\tau : (A^\bullet_X (\text{nul } D), F) \hookrightarrow (A^\bullet_{X,D}, F),\]

\[\omega \in \Gamma(V, A^p_X (\text{nul } D)) \mapsto (\omega, 0, \cdots, 0) \in \Gamma(V, A^p_{X,D}), \quad \text{for } V \subset X \text{ an open subset.} \]

Then, have the following :

Proposition 6. (i) The restriction $\tau : (\Omega^\bullet_X (\text{nul } D), F_b) \hookrightarrow (\Omega^\bullet_{X,D}, F_b)$ of $\tau$ is a filtered quasi-isomorphism of complexes of sheaves.

(ii): Consider the embeddings of filtered complex of sheaves on $X^\text{an}$ :

\[\begin{align*}
(j_i^\bullet, A^\bullet_U, F) &\hookrightarrow (\pi^\bullet_U, A^\bullet_{X,D}, F) \\
(j_i^\bullet, A^\bullet_U, F) &\hookrightarrow (\pi^\bullet_U, A^\bullet_{X,D}, F)
\end{align*}\]

Then $\tau$ is a filtered quasi-isomorphism. The inclusion $t^\bullet_U$ is quasi-isomorphism but NOT a filtered quasi-isomorphism.

(iii) The inclusion map $\tau : (A^\bullet_X (\text{nul } D), F) \hookrightarrow (A^\bullet_{X,D}, F)$, is a filtered quasi-isomorphism of complexes of presheaves, that is for all open subset $V \subset X$, and all integer $p$ the restriction

\[\tau : \Gamma(V, F^p A^\bullet_X (\text{nul } D)) \hookrightarrow \Gamma(V, F^p A^\bullet_{X,D}),\]

of $\tau$ are quasi-isomorphisms.

Proof. (i):The sequence of complexes of sheaves on $X^\text{an}$

\[0 \rightarrow \Omega^p_X (\text{nul } D) \xrightarrow{t^\bullet_X} \Omega^p_X \xrightarrow{D_{i,j}} \bigoplus_{j=1}^s i_{D_j}^* \Omega^p_{D_j} \xrightarrow{D_{i_j}} \cdots \xrightarrow{i_{D_1}} \Omega^p_{D_1} \rightarrow 0,\]

is exact. This prove (i).

(ii): By (i),

\[\tau : (\Omega^\bullet_X (\text{nul } D), F_b) \hookrightarrow (\Omega^\bullet_{X,D}, F_b)\]

of is a filtered quasi-isomorphism of complexes of sheaves. On the other side,

- the inclusion $(\Omega^\bullet_X (\text{nul } D), F_b) \hookrightarrow (A^\bullet_X (\text{nul } D), F)$ is a filtered quasi-isomorphism of complexes of sheaves by proposition \[\Box \text{(ii)} \]

- the inclusion $(\Omega^\bullet_{X,D}, F_b) \hookrightarrow (A^\bullet_{X,D}, F)$ is a filtered quasi-isomorphism of complexes of sheaves by proposition \[\Box \text{(ii)}.\]

Hence,

\[\tau : (A^\bullet_X (\text{nul } D), F) \hookrightarrow (A^\bullet_{X,D}, F)\]

is a filtered quasi-isomorphism of complexes of sheaves. The fact that two complexes of sheaves are quasi-isomorphic to $j_i U$ by \[\Box.\] This prove (ii).
(ii): By (ii), the inclusion maps of complexes of sheaves on $X^{an}$
\[ \tau : F^pA_X^\bullet (\text{nul } D) \hookrightarrow F^pA_{X,D}^\bullet \]
are quasi-isomorphism of complexes of sheaves. Thus, for every open subset $j_V : V \hookrightarrow X$, $j_V^* \tau : j_V^* F^pA_X^\bullet (\text{nul } D) \hookrightarrow j_V^* F^pA_{X,D}^\bullet$ are quasi-isomorphism of complexes of sheaves. Hence, for every open subset $V \subset X$, the maps
\[ \tau : \mathbb{H}^k(V,F^pA_X^\bullet (\text{nul } D)) \hookrightarrow \mathbb{H}^k(V,F^pA_{X,D}^\bullet) \]
are quasi-isomorphism of complexes of $\mathbb{C}$-vector spaces. The sheaves $F^pA_X^k (\text{nul } D)$, $F^pA_X^k$ and $i_{D,j}^* F^pA_{D,j}^k$ are sheaves of $O_X^\infty$ modules on $X^{an}$, so are c-soft (because the existence of partition of unity) and thus acyclic for the global section functor on each open subset $V \subset X$ ($X^{an}$ is a denombable union of compact subsets). Hence, for every open subset $V \subset X$,
\[ H^k \Gamma (V,F^pA_X^\bullet (\text{nul } D)) = \mathbb{H}^k(V,F^pA_X^\bullet (\text{nul } D)) \quad \text{and} \quad H^k \Gamma (V,F^pA_{X,D}^\bullet) = \mathbb{H}^k(V,F^pA_{X,D}^\bullet). \]
This proves (iii).

**Corollary 1.** The following embeddings complexes of sheaves on $X^{an}$ :

- $j_* C_U^\xi \hookrightarrow \Omega_X^\xi (\text{log } D)^{\xi \leftarrow l} \hookrightarrow j_* A_U^\bullet$, and

- $j! C_U^\xi \hookrightarrow j!* A_U^\bullet \hookrightarrow A_X^\bullet (\text{nul } D)^{\xi \leftarrow \tau} \hookrightarrow A_{X,D}^\bullet$

are quasi-isomorphisms.

**Proof.** The fact that the first sequence of inclusion are quasi-isomorphism comes from the resolution $0 \rightarrow C_U \rightarrow A_U^\bullet$ and the proposition ii). The fact that the second sequence of inclusion are quasi-isomorphism is given by proposition iii).

**Remark 1.** Note that the embedding of filtered complexes of sheaves on $X^{an}$ is: $(\Omega_X (\text{log } D), F_b) \hookrightarrow (j_* A_U^\bullet, F)$ is NOT a filtered quasi-isomorphism.

**Corollary 2.** Suppose $X \in \text{PSmVar}(\mathbb{C})$ is smooth projective, then

(i) the spectral sequence associated to the filtered complex $(\Gamma (X,A_X^\bullet (\text{nul } D)), F)$ by Frölicher filtration $F$ is $E_1$ degenerate.

(ii) for all integer $k, p$, the map induced on hypercohomology of the quotient map
\[ H^k \Gamma (X,F^pA_X^\bullet (\text{nul } D)) \rightarrow H^k \Gamma (X,\text{Gr}^p_F A_X^\bullet (\text{nul } D)) = H^{k-p} \Gamma (X,A_X^{p,k} (\text{nul } D)) = H^{k-p} (X,\Omega_X^{p,k} (\text{nul } D)) \]

for $\omega \rightarrow [\omega^{p,k-p}]$ is surjective.

**Proof.** (i) By proposition ii(iii), the inclusion map of complexes of $\mathbb{C}$-vector spaces $\tau : (\Gamma (X,A_X^\bullet (\text{nul } D)), F) \rightarrow (\Gamma (X,A_X^{\infty,D})$, $F)$ is a filtered quasi-isomorphism. On the other hand the spectral sequence associated to $(\Gamma (X,A_X^{\infty,D}), F)$ is $E_1$ degenerate (see definition ii). Thus the spectral sequence associated to $(\Gamma (X,A_X^{\infty,D}), F)$ is $E_1$ degenerate.

(ii) This is a classical fact on spectral sequence that (ii) is equivalent to (i) see for example ii).

**Definition 6.** If $X \in \text{PSmVar}(\mathbb{C})$ is smooth projective, the hodge filtration on the $\mathbb{C}$ vector spaces $H^k (U, \mathbb{C}) = H^k \Gamma (X,D_X^\bullet (\text{log } D))$ and $H^k (X, D, \mathbb{C}) = H^k \Gamma (X,A_X^\bullet (\text{nul } D))$ are given by the Frölicher filtration $F$ of the filtered complexes of sheaves on $X^{an}$ $(D_X^\bullet (\text{log } D), F)$ and $(A_X^\bullet (\text{nul } D), F)$ respectively. The $E_1$ degeneracy of the Frölicher filtration (corollary ii) for the complex $(\Gamma (X,A_X^\bullet (\text{nul } D), F)$), say that the following canonical surjective maps are isomorphisms:
Remark 2. \( H^k \Gamma(X, F^p D_X^\bullet \{\log D\}) \xrightarrow{\sim} F^p H^k(U, \mathbb{C}) \)

and their \( F \) graded pieces are

\[
H^{p,k-p}(U, \mathbb{C}) := F^p H^k(U, \mathbb{C})/F^{p+1} H^k(U, \mathbb{C}) \xrightarrow{\sim} H^{k-p} \Gamma(X, D_X^p D_X^\bullet \{\log D\}) = H^{k-p}(X, \Omega_X^p(\log D))
\]

\[
H^{p,k-p}(X, D, \mathbb{C}) := F^p H^k(X, D, \mathbb{C})/F^{p+1} H^k(X, D, \mathbb{C}) \xrightarrow{\sim} H^{k-p} \Gamma(X, D_X^p A_X^\bullet(\null D)) = H^{k-p}(X, \Omega_X^p(\null D))
\]

(see also corollary [2] (ii)).

The wedge product \( w_X \) of bicomplexes of presheaves on \( X^{an} \) gives the morphism of filtered complex of presheaves on \( X^{an} \)

\[
w_X : (D_X^\bullet(\log D), F) \otimes_{O_X} (A_X^{2d_x-\bullet}(\null D), F) \rightarrow D_X^{2d_x}.
\]

We have then the following:

**Proposition 7.** If \( X \in \text{PSmVar}(\mathbb{C}) \), the pairing of filtered complexes of \( \mathbb{C} \) vector spaces:

\[
ev_X = a_{X*} w_X = \langle \cdot, \cdot \rangle_{\ev_X} : (\Gamma(X, D_X^\bullet(\log D)), F) \otimes_{\mathbb{C}} (\Gamma(X, A_X^{2d_x-\bullet}(\null D)), F) \rightarrow (\Gamma(X, D_X^{2d_x}), F),
\]

\[
T \otimes \omega \mapsto T(\omega) = a_{X*}(T \wedge \omega)
\]

induces on cohomology isomorphisms

- \( \ev_X : H^k(U, \mathbb{C})/F^p H^k(U, \mathbb{C}) \xrightarrow{\sim} (F^{d_x-p+1} H^{2d_x-k}(X, D, \mathbb{C}))^\vee \) and
- \( \ev_X : H^k(X, \Omega_X^p(\log D)) \xrightarrow{\sim} H^{d_x-k}(X, \Omega_X^{d_x-p}(\null D))^\vee \).

Note that for \( \omega \in \Gamma(X, A_X^\bullet(\log D))^d=0 \) a closed log form and \( \eta \in \Gamma(X, A_X^{2d_x-k}(\null D))^d=0 \), we have

\[
<|\omega|, |\eta|>_{\ev_X} = \int_X \omega \wedge \eta.
\]

**Proof.** The fact that the pairing induced in cohomology is non degenerated is Poincare duality for the pair \((X, D)\) which is a morphism of mixed hodge structures since the class of the wedge product of a closed log current by a closed nul form is the cup product of the two classes (c.f. [7] for example).

\[\square\]

**Remark 2.** If \( X \in \text{PSmVar}(\mathbb{C}) \), the Frölicher filtration of \( (\Gamma(X, A_X^\bullet(\log D)), F^\bullet) \) is \( E_1 \) degenerate because it is a mixed hodge complex. On the other hand \( \bar{\ell} \) is a filtered quasi-isomorphism (proposition [2]).

Thus the Frölicher filtration of \( (\Gamma(X, D_X^\bullet(\log D)), F^\bullet) := (\Gamma(X, A_X^{2d_x-\bullet}(\null D)), F^\bullet)^\vee \) is \( E_1 \) also degenerate. But the Föllinger filtration on \( j_* A^\bullet_X \) is not \( E_1 \) degenerate and the hypercohomogogy of his graded piece \( \mathbb{H}^k(X, j_* \text{Gr}^p_0 A_X^\bullet) = H^k(U, \Omega^p_U) \) vanishes for \( k > 0 \) if \( X \) is affine.

### 2.4 The higher Abel Jacobi map for \( U \)

Recall that for any \( V \in \text{SmVar}(\mathbb{C}) \) quasi-projective there exist \( Y \in \text{PSmVar}(\mathbb{C}) \) such that \( Y \setminus V \) is a normal crossing divisor with smooth components. In this subsection, we assume that \( Y \in \text{PSmVar}(\mathbb{C}) \) is smooth projective.

Denote by \( \mathcal{Z}^p(U, \bullet)^{pr/X} \subset \mathcal{Z}^p(U, \bullet) \) the subcomplex consisting of closed cycles on \( U \times \Box^* \) such that their closure on \( X \times \Box^* \) intersect all face properly. By Bloch, the latter is quasi-isomorphic to the former.

By definition, there is an exact sequence of complexes of abelian groups

\[
0 \rightarrow \mathcal{Z}^p(D, \bullet) \xrightarrow{iD} \mathcal{Z}^p(X, \bullet) \xrightarrow{\delta} \mathcal{Z}^p(U, \bullet)^{pr/X} \rightarrow 0.
\]

(26)

For \( Z \in \mathcal{Z}^p(U, \bullet)^{pr/X} \) denote by \( \bar{Z} = \sum n_i Z_i \in \mathcal{Z}^p(X \times (\mathbb{P}^1)^n) \) the closure of \( Z = \sum n_i Z_i \in \mathcal{Z}^p(U \times \Box^n) \).

For \( Z \in \mathcal{Z}^p(U, \bullet)^{pr/X} \), we have, by (26)

\[
\partial\bar{Z} \in iD \mathcal{Z}^{p-1}(D, \bullet) \subset \mathcal{Z}^p(X, \bullet).
\]

Let

\[
C^p(D, X, Z) = \text{Cone}(C^{\text{diff}}_{2d_x-2p+\bullet}(X, D, Z) \oplus \Gamma(X, F^p D_X^{2p+\bullet})) \rightarrow \Gamma(X, D_X^{2p+\bullet})
\]

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be the Deligne homology complex of $X$,

$$C^k_D(X, D, Z) = \text{Cone}(C^\text{diff}_{2d - 2p + 1}(X, D, Z) \oplus \Gamma(X, F^p D^{2p - 1}_X (\log D))) \to \Gamma(X, D^{2p - 1}_X (\log D)))$$

be relative homology complex of $(X, D)$, and

$$r^D_{X,D} : C^k_D(X, D, Z) \to C^k_D(X, D, Z), \quad (T, \Omega, R) \mapsto (r_{X,D}(T), r_{X,D}(\Omega), r_{X,D}(R))$$

be the quotient map.

There is the classical realization maps

$$R^p(X, D) : Z^p(U, \bullet)^{pr/X} \to C^p_D(X, D, Z), \quad Z \mapsto (T_Z, \Omega_Z, R_Z) := r^D_{X,D}(T_Z, \Omega_Z, R_Z)$$

where, c.f. [4],

- $T_Z = r_{X,D}(T_Z) = \sum_i n_i \pi_X((X \times T_{\overline{\Gamma}}) \cap \overline{\zeta_i}) \in C^\text{diff}_{2d - 2p + n}(X, D, Z)$, we have $dT_Z = T_{\partial Z}$
- $\Omega_Z = r_{X,D}(\Omega_Z) : \omega \in \Gamma(X, A^{d - 2p + n}_X (\log D)) \mapsto \Omega_Z(\omega) = \sum_i n_i \int_{\overline{\zeta_i}} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n} := \lim_{\epsilon \to 0} \sum_i \int_{\overline{\zeta_i}} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n}$, it is a current of type $(p, p - n)$, i.e. $\Omega_Z \in \Gamma(X, D^{2p - n}(\log D))$.
- $R_Z = r_{X,D}(R_Z) : \omega \in \Gamma(X, A^{2d - 2p + n + 1}(\log D)) \mapsto R_Z(\omega) = \sum_i n_i \int_{\overline{\zeta_i}} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n} := \lim_{\epsilon \to 0} \sum_i \int_{\overline{\zeta_i}} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n}$, we have $d\Omega = \partial \Omega_Z$ since we have $d\Omega_{\square^n} = 2i\pi \sum_{i=0}^n (-1)^i \Omega_{\square^n}(z_0, \ldots, \hat{z}_i, \ldots, z_n) \delta(z_i)$, and we have $dR_Z = \Omega - (2i\pi) R_{\partial Z} - (2i\pi)^n T_Z$ since we have $dR_{\square^n} = \Omega_{\square^n} - 2i\pi (1)^i \sum_{i=0}^n \Omega_{\square^n}(z_0, \ldots, \hat{z}_i, \ldots, z_n) \delta(z_i) - (2i\pi)^n T_{\square^n}$.

The currents $T_Z$ and $\Omega_Z$ are closed if $\partial Z = 0$ that is if $\partial Z \in i_{D^*} Z^{p-1}(D, n)$. For $Z \in Z^p(U, n)^{pr/X}$, the equality $dR_Z = \Omega - (2i\pi)^n T_Z$ shows that $[\Omega_Z] = (2i\pi)[T_Z] \in H^{2n-p}(U, \mathbb{C})$.

Denote by $Z^p(U, n)^{pr/X, \text{hom}} \subset Z^p(U, n)^{pr/X}$ the subspace consisting of $Z \in Z^p(U, n)^{pr/X}$ such that $\partial Z = 0$ and $[\Omega_Z] = 0 \in H^{2p-n}(U, \mathbb{C})$, that is $\Omega_Z \in \Gamma(X, D_X (\log D))$ is exact. Let $Z \in Z^p(U, n)^{pr/X, \text{hom}}$. Then, for a choice of $d^{-1} \Omega_Z \in \Gamma(X, D_X (\log D))$ and of $d^{-1} T_Z \in C^\text{diff}_{2d - 2p}(X, D)$, the current

$$R_Z' = R_Z - d^{-1} \Omega_Z - (2i\pi)^n d^{-1} T_Z \in \Gamma(X, A^{2d - 2p + n + 1}_X (\log D))^{\vee}$$

is closed, that is $R_Z' \in \Gamma(X, A^{2d - 2p + n + 1}_X (\log D))^{d=0}$.

**Definition 7.** The complex analytic variety

$$J^{p-k}(U) = H^k(U, \mathbb{C})/F^p H^k(U, \mathbb{C}) \oplus H^k(U, \mathbb{Z})$$

is the intermediate jacobian. By proposition [2], ev_X induces an isomorphism of complex varieties ev_X : $J^{p,k}(U) \to (F^p D^{2p-k}(X, D, \mathbb{C}))^{\vee}/H^{2d-k}(X, D, Z)$. The map

$$AJ_U : Z^p(U, n)^{pr/X, \text{hom}} \to CH^p(U, n)^{\text{hom}} \to J^{p,2p-n-1}(U), \quad Z \mapsto AJ(Z) = [R_Z']$$

is the higher Abel Jacobi map

**Proposition 8.** For $Z \in Z^p(U, n)^{pr/X, \text{hom}}$, there exist a topological cycle $\Gamma_Z \in C_{2d - 2p + 1}(X, D, Z)$ such that $\partial \Gamma_Z = \zeta$ for $0 < \epsilon << 1$. This gives, for $\omega \in \Gamma(X, A^{2d - 2p + n + 1}(\log D))^{d=0}$,

$$R_Z(\omega) = \lim_{\epsilon \to 0} \sum_{i} n_i \int_{\overline{\zeta_i}} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n} = \lim_{\epsilon \to 0} \int_{\Gamma_Z} \pi^*_X \omega \wedge \pi^*_i \Omega_{\square^n}$$
In particular, $R_Z$ restrict to a closed current on the subspace $\Gamma(X,F^{d_X-p+1}A_{X}^{d_X-2p+n+1}(\text{null}D)) \subset \Gamma(X,A_{X}^{d_X-2p+n+1}(\text{null}D))$, that is $R_Z \in \Gamma(X,F^{d_X-p+1}A_{X}^{d_X-2p+n+1}(\text{null}D))^{\vee,d=0}$ and we have

$$AJ_{\nu}(Z) = [R_Z^\nu] = ev_X([R_Z]).$$  \hfill (27)

**Proof.** It a straightforward generalization of [3] proposition 5.1 : for $\omega \in \Gamma(X,A_{X}^{d_X-2p+n+1}(\text{null}D))^{d=0}$, we have

$$\sum_i n_i \int_{\bar{Z}_i} \pi_X^*\omega \wedge \pi_{(p_1)^*}\Omega_{\square^n} = \int_{\bar{Z}_2} d(\pi_X^*\omega \wedge \pi_{(p_1)^*}\Omega_{\square^n}) \text{ by Stokes formula}$$

$$= \sum_i n_i \int_{\bar{Z}_i} \pi_X^*\omega \wedge \pi_{(p_1)^*}\Omega_{\square^n}$$

since $d\omega = 0, \omega|_D = 0, d\Omega_{\square^n} = \Omega_{\square^n} - (2i\pi) \sum_{i=0}^{n} (-1)^i R(z_0, \cdots, \hat{z}_i, \cdots, z_n) \delta(z_i) - (2i\pi)^n \Omega_{\square^n}$ and $\partial \bar{Z} \in i_{i_D}Z^p(D,n)$.

\hfill \square

**Proposition 9.** (i) The higher Abel Jacobi map of a smooth quasi-projective variety $V \in \text{SmVar}(\mathbb{C})$ is independent of the choice of a compactification $(Y,Y\setminus V), Y \in \text{PSmVar}(\mathbb{C})$, with $E = Y\setminus V$ a normal crossing divisor.

(ii) The higher Abel Jacobi map is functorial in $V \in \text{SmVar}(\mathbb{C})$ covariantly for proper morphisms.

(iii) The higher Abel Jacobi map is functorial in $V \in \text{SmVar}(\mathbb{C})$ contravariantly for all morphism.

**Proof.** (i): Let $(Y,E)$ and $(Y',E')$ be two such compactification of $V$. Then exist another compactification $(Y'',E'')$ together with two morphism of pairs $g : (Y'',E'') \rightarrow (Y,E), g' : (Y'',E'') \rightarrow (Y',E')$ such that $g \circ j'' = j \circ I_Y$ and $g' \circ j'' = j' \circ I_Y$. One can take $Y'' \rightarrow A_Y \subset Y \times Y'$ a desingularisation of the closure of the diagonal of $V$ inside $Y \times Y'$.

(ii): Let $f : U \rightarrow V$ be proper morphism. Then there exists a compactification $\tilde{f} : X \rightarrow Y$ of $f$ such that $\tilde{f}(D) \subset E$. That is $\tilde{f}$ induces a morphism of pair $\tilde{f} : (X,D) \rightarrow (Y,E)$ and $f \circ j = j \circ f$. Then, for $Z \in \mathcal{Z}^p(U,n)^{pr/X}$, we have

$$f_*(T_Z,\Omega_Z,\Omega_{\square^n}) = r_{Y,E}(T_{\tilde{f},Z},\Omega_{\tilde{f}},\Omega_{\tilde{f},Z}).$$

(iii): Let $h : U \rightarrow V$ be any morphism and $\tilde{h} : X \rightarrow Y$ be any compactification of $h$. That is $\tilde{h} \circ j = j \circ h$. Let

$$\mathcal{Z}^p(V,\bullet)^{pr/Y,pr/h} \subset \mathcal{Z}^p(V,\bullet)^{pr/Y}$$

be the submodule of abelian group consisting of cycles $Z = \sum_i n_i Z_i$ such that codim$(\tilde{h}^{-1}(Z_i),X) = p$ for all $i$ and such that $\tilde{h}^{-1}(Z) := \sum_i n_i \tilde{h}^{-1}(Z_i) \in \mathcal{Z}^p(U,n)^{pr/X}$ (that is whose closure in $X$ intersect all faces of $X \times \square^n$ properly). By Bloch this inclusion is a quasi-isomorphism. Then, for $Z \in \mathcal{Z}^p(V,n)^{pr/Y,pr/h}$, considering $\tilde{h}^{-1}(Z) := \sum_i n_i \tilde{h}^{-1}(Z_i) \in \mathcal{Z}^p(X,n)$, we have

- $\text{supp}(\tilde{h}^{-1}(Z)) \subset \text{supp}(h^{-1}(Z))$ and $(T_{\tilde{h}^{-1}(Z)},\Omega_{\tilde{h}^{-1}(Z)},R_{\tilde{h}^{-1}(Z)}) = r_{X,D}(T_{h^{-1}(Z)},\Omega_{h^{-1}(Z)},R_{h^{-1}(Z)}),$ \hfill (28)

- $\tilde{h}^*(T_Z,\Omega_Z,\Omega_{\square^n}) = (T_{\tilde{h}^{-1}(Z)},\Omega_{\tilde{h}^{-1}(Z)},R_{\tilde{h}^{-1}(Z)}),$ see [3] for the definition of the pullback or Gysin map for current, and $\tilde{h}^*[T_Z,\Omega_Z,\Omega_{\square^n}] = [\tilde{h}^*[T_Z,\Omega_Z,\Omega_{\square^n}]] = [(T_{\tilde{h}^{-1}(Z)},\Omega_{\tilde{h}^{-1}(Z)},R_{\tilde{h}^{-1}(Z)})].$

Hence, for $Z \in \mathcal{Z}^p(V,n)^{pr/Y,pr/h}$,

$$h^*[T_Z,\Omega_Z,\Omega_{\square^n}] = r_{X,D}\tilde{h}^*[T_Z,\Omega_Z,\Omega_{\square^n}] = [r_{X,D}(T_{\tilde{h}^{-1}(Z)},\Omega_{\tilde{h}^{-1}(Z)},R_{\tilde{h}^{-1}(Z)})]$$

$$=[(T_{\tilde{h}^{-1}(Z)},\Omega_{\tilde{h}^{-1}(Z)},R_{\tilde{h}^{-1}(Z)})] \in H^p_{2d_X-2p+n}(X,D).

\hfill \square
3 Relative Higher Abel Jacobi map for open morphism and infinitesimal invariants

Let \( X, S \in \text{SmVar}(\mathbb{C}) \) and \( f : X \to S \) be a smooth projective morphism. Consider \( U \subset X \) an open subset such that \( D = X \setminus U \) has the property that \( D_s \subset X_s \) is a normal crossing divisor (with smooth components) for all \( s \in S \). Denote by \( j : U \hookrightarrow X \) the inclusion and \( f_U = f \circ j : U \to S \). Let \( d = d_X - d_S \).

3.1 The Leray filtration on the complexes of sheaves \( \mathcal{A}_X(\log D), \mathcal{A}_X(\text{nul } D) \) and \( D_X(\log D) \) on \( X^{an} \)

The exact sequence of sheaves on \( X^{an} \): \( 0 \to f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0 \) gives the following exact sequences of sheaves on \( X^{an} \):

\[
0 \to \mathcal{I}_D \otimes \Omega_X \to f^*\Omega^1_S \to \Omega^1_X (\text{nul } D) \to \Omega^1_{X/S} (\text{nul } D) \to 0
\]

(28)

**Definition 8.** The complex of sheaves on \( X^{an} \) \( \Omega^*_X(\text{nul } D) \) is clearly a graded ideal of \( \Omega^*_X \) (see also [2] for a stronger result). The Leray filtration on the complexes of sheaves on \( X^{an} \): \( \Omega^*_X(\text{nul } D) \subset \Omega^*_X \subset \Omega^*_X(\log D) \) is then defined by:

- \( L^r \Omega^*_X(\text{nul } D) := f^*\Omega^*_S \wedge \Omega^r_X(\text{nul } D), \quad \Omega^*_X(\text{nul } D) := \text{Gr}_L^0 \Omega^*_X(\text{nul } D) \)
- \( L^r \Omega^*_X(\log D) := f^*\Omega^*_S \wedge \Omega^r_X(\log D), \quad \Omega^*_X(\log D) := \text{Gr}_L^0 \Omega^*_X(\log D) \)

The (holomorphic) Leray filtrations on the bicomplexes \( (\mathcal{A}^*_X(\text{nul } D), \partial, \bar{\partial}) \subset (\mathcal{A}^*_X, \partial, \bar{\partial}) \subset (\mathcal{A}^*_X(\log D), \partial, \bar{\partial}) \) of sheaves on \( X^{an} \) are then defined by:

- \( L^r \mathcal{A}^{p,q}_X(\text{nul } D) := L^r \Omega^*_X(\text{nul } D) \wedge \mathcal{A}^{0,q}_X(\text{nul } D), \quad L^r \mathcal{A}^{p,q}_X := L^r \Omega^*_X \wedge \mathcal{A}^{0,q}_X \)
- \( L^r \mathcal{A}^{p,q}_X(\log D) := L^r \Omega^*_X(\log D) \wedge \mathcal{A}^{0,q}_X(\log D) \).

We denote \( \mathcal{A}^{p,q}_X(\text{nul } D) := \text{Gr}_L^0 \mathcal{A}^{p,q}_X(\text{nul } D), \quad \mathcal{A}^{p,q}_X := \text{Gr}_L^0 \mathcal{A}^{p,q}_X \) and \( \mathcal{A}^{p,q}_X(\log D) := \text{Gr}_L^0 \mathcal{A}^{p,q}_X(\log D) \).

Their first graded pieces.

This gives the (holomorphic) Leray filtration on its total complex \( (\mathcal{A}^*_X(\log D), d) \).

**Remark 3.** Note that the holomorphic Leray filtrations \( L^r \mathcal{A}^*_X(\text{nul } D) = f^*\Omega^*_S \wedge \mathcal{A}^{r,-}_X(\text{nul } D) \subset f^*\mathcal{A}^*_S \wedge \mathcal{A}^{r,-}_X(\text{nul } D) \) \( L^r \mathcal{A}^*_X := f^*\Omega^*_S \wedge \mathcal{A}^{r,-}_X \subset f^*\mathcal{A}^*_S \wedge \mathcal{A}^{r,-}_X \) \( L^r \mathcal{A}^*_X(\log D) = f^*\Omega^*_S \wedge \mathcal{A}^{r,-}(\log D) \subset f^*\mathcal{A}^*_S \wedge \mathcal{A}^{r,-}(\log D) \) are in the differential Leray filtration but not equal since we only pullback from \( S \) forms with zero anti-holomorphic part.

**Proposition 10.** We get the following inclusions of bifiltered complexes of sheaves on \( X^{an} \):

\[
\begin{array}{ccc}
(\Omega^*_X(\text{nul } D), L) & \subset & (\Omega^*_X, L) \\
\downarrow & & \downarrow \\
(\mathcal{A}^*_X(\text{nul } D), F, L) & \subset & (\mathcal{A}^*_X, F, L) \\
\downarrow & & \downarrow \\
(\Omega^*_X(\log D), L) & \subset & (\Omega^*_X, L) \\
\downarrow & & \downarrow \\
(\mathcal{A}^*_X(\log D), F, L) & \subset & (\mathcal{A}^*_X, F, L).
\end{array}
\]

**Proof.** By definition, the inclusion of complexes and bicomplexes of sheaves on \( X^{an} \):

\( \Omega^*_X(\text{nul } D) \subset \Omega^*_X \subset \Omega^*_X(\log D) \) and \( \mathcal{A}^{\bullet,\bullet}_X(\text{nul } D) \subset \mathcal{A}^{\bullet,\bullet}_X \subset \mathcal{A}^{\bullet,\bullet}_X(\log D) \).

are by definition compatible with the Leray filtration (even strictly compatible). \( \square \)
Proposition 11. Taking interior product gives the following identifications of sheaves on $X^{an}$: for $0 \leq r \leq d_S$ and $0 \leq r \leq 2d_S$ respectively
\[
\phi^r_{p,q} : \Gr_L^r \Omega^r_X(nul D) \xrightarrow{\sim} \Omega_{X/S}^r(nul D) \otimes_{\mathcal{O}_X} f^*\Omega^r_S, \ \phi^r_{p,q} : \Gr_L^r \mathcal{A}_X^{p,q}(nul D) \xrightarrow{\sim} \mathcal{A}_{X/S}^{p-r,q}(nul D) \otimes_{\mathcal{O}_X} f^*\Omega^r_S
\]
which are induced by, for $V \subset X$ an open subset,
\[
\omega \in \Gamma(V, \Omega^r_X \ominus (\log D)) \mapsto (u \in \Gamma(V, f^*(\wedge^r T_S)) \mapsto \iota(\tilde{u})\omega \in \Gamma(V, \Omega_{X/S}^{r-\nabla}(\log D)), \omega \in \Gamma(V, \mathcal{A}_X^{p,q}(\log D)) \mapsto (u \in \Gamma(V, f^*(\wedge^r T_S)) \mapsto \iota(\tilde{u})\omega \in \Gamma(V, \mathcal{A}_{X/S}^{p-r,q}(\log D)),
\]
where $\tilde{u} \in \Gamma(V, \wedge^r T_X)$ is a replement of $u$, that is satisfy $df(\tilde{u}) = u$ and $< \cdot , > : L^r \mathcal{A}_X(\log D) \to \Gr_L^r \mathcal{A}_X(\log D)$ denote the quotient class map for the Leray filtration. These maps are independent of the choice of a replement $\tilde{u}$ since $\omega$ is in $L^r$ (thus the interior product by a wedge product of vector fields tangent to the fibers of $f$ vanishes).

Proof. The only thing that is perhaps non trivial is that for $\omega = \omega^p \wedge \omega^{0,q} \in \Gamma(V, \mathcal{A}_X^{p,q}(\log D))$ and $u \in \Gamma(V, f^*T_S)$, $< \iota(\tilde{u})\omega > \subseteq \Gamma(V, \mathcal{A}_{X/S}^{p-1,q}(nul D))$. We have, since $\iota(\tilde{u})\omega^{0,q} = 0$ for type reason,
\[
\iota(\tilde{u})\omega = \iota(\tilde{u})\omega^p \wedge \omega^{0,q} \in \Gamma(V, \mathcal{A}_X^{p,q}).
\]
We have to prove that $\iota(\tilde{u})\omega^p \in \Gamma(V, \Omega_{X/S}^{p-1})$ vanishes on the fibers $D_s = X_s \cap D \subseteq X$ of $f_D : D \to S$. This comes from the fact that $D$ is transversal to the fibers of $f$. Indeed, let $x \in D$ and $u_{D,f}^{-1} \in \wedge^{p-1} T_{D,x}$ with $s = f(x)$. Since $D$ is transversal to the fibers of $f$, $T_x X = \text{Vect}(T_x D, T_x X_f(x))$. Hence, there exist $\lambda_D, \lambda_f \in \mathbb{C}$ such that $\tilde{u}(x) = \lambda_D \tilde{u}_D + \lambda_f \tilde{u}_f$ with $\tilde{u}_D \in T_x X$ and $\tilde{u}_f \in T_x X_f(x)$. This gives
\[
\tilde{u}(x) \wedge u_{D,f}^{-1} = \lambda_D \tilde{u}_D \wedge u_{D,f}^{-1} + \lambda_f \tilde{u}_f \wedge u_{D,f}^{-1}
\]
Now,
\begin{itemize}
  \item since $\omega^p \in \Gamma(V, L^1 \Omega^p_X \ominus \mathcal{O}_{X_f(x)} = 0$, hence $\omega^p(x)(\lambda_f \tilde{u}_f \wedge u_{D,f}^{-1}) = 0$ (this says that $\iota(\tilde{u})\omega$ does not depends of the choice of the replement $\tilde{u}$ of $u$).
  \item since $\omega^p_D = 0$, $\omega^p(x)(\lambda_D \tilde{u}_D \wedge u_{D,f}^{-1}) = 0$.
\end{itemize}
Thus, $\iota(\tilde{u})\omega^p(x)(u_{D,f}^{-1}) = \omega^p(x)(\tilde{u}_D \wedge u_{D,f}^{-1}) = 0$. This shows that $\iota(\tilde{u})\omega^p|_{D_s} = 0$. Hence, $\iota(\tilde{u})\omega^p|_{D} = f^*\gamma \land \eta^p_{D} = \mathcal{A}_X^{p,q}|_{D} \subseteq \Gamma(V \cap D, L^1 \Omega^p_D)$, where $\gamma \in \Gamma(f(V), \Omega^p_S)$ and $\eta_D \in \Gamma(V \cap D, \Omega^p_D)$, and thus
\[
\iota(\tilde{u})\omega^p|_{D} = f^*\gamma \land \eta^p_{D} \land \gamma^p_{D} \subseteq \Gamma(V \cap D, L^1 \mathcal{A}_X^{p,q}).
\]
Now, shrinking $V \subset X$ if necessary, there exist $\eta \in \Gamma(V, \Omega^p_{X/S})$ such that $\eta_D = \eta_D$. Take $\omega' = f^*\gamma \land \eta^{p-1} \subseteq \Gamma(V, L^1 \mathcal{A}_X^{p,q}).$ Then $\iota(\tilde{u})\omega - \omega' \in \Gamma(V, \mathcal{A}_{X/S}^{p-1,q}(nul D))$
\[
< \iota(\tilde{u})\omega > = < \iota(\tilde{u})\omega - \omega' > \in \Gamma(V, \mathcal{A}_{X/S}^{p-1,q}(nul D)).
\]

Remark 4. The maps $\phi^r_{p,q}$ and $\phi^r_{p,q}$ define morphism of complexes $\phi^r_{p,q}$ and $\phi^r_{p,q}$. Indeed, recall that for $\eta \in \Gamma(V, \mathcal{A}_X^{p,q}(\log D))$ and $\nu \in \Gamma(V, \wedge^r T_X)$, we have $d\nu(\eta) = \iota(\nu)\eta + L_\nu \eta$, where $L_\nu$ is the Lie derivative. Now if $\omega \in \Gamma(V, L^r \mathcal{A}_X^{p,q}(\log D))$, we have
\[
\phi(d_{X/S} \omega > = \phi(d \omega > = < \iota(\tilde{u})\omega > = < \iota(\tilde{u})\omega > = d_{X/S} \omega > = \omega >, \text{since } L_{\tilde{u}} \omega \in \Gamma(V, L^r \mathcal{A}_X^{p,q}(\log D)).
\]

\[\Box\]
The Leray filtration is compatible with proposition 3:

**Proposition 12.** (i) The wedge product induces an isomorphism of filtered complexes of sheaves on \( X^{an} \)

\[
  w_X : (\Omega_X^p (\text{null } D), L) \otimes_{O_X} A_X^{0,\bullet} \sim (\Lambda^p \cdot (\text{null } D), L)
\]

(ii) The inclusion of bifiltered complexes of sheaves on \( X^{an} \)

\[
  (\Omega_X^p (\text{null } D), F^\bullet, L) \hookrightarrow (A_X^\bullet (\text{null } D), F^\bullet, L),
\]

is a bifiltered quasi-isomorphism.

**Proof.** (i): By proposition 3(ii), it is a morphism of complex. It is an isomorphism by definition.

(ii): This comes from (i) : we have the Dolbeau resolutions

\[
  0 \to L^r \Omega_X^p (\text{null } D) \to L^r \Omega_X^p (\text{null } D) \otimes_{O_X} (A_X^{0,\bullet}, \partial) \sim (L^r A^p \cdot (\text{null } D), \partial).
\]

We now give the definition of the Leray filtration on complexes of currents:

**Definition 9.** The Leray filtration the logarithmic complex of sheaves of currents \((\mathcal{D}_X^{\bullet,\bullet} (\log D), \partial, \bar{\partial})\) on \( X^{an} \) is:

\[
  L^r \mathcal{D}_X^{p,q} (\log D) := \alpha^{p,q} (L^r \Omega_X^p (\log D) \otimes_{O_X} \mathcal{D}_X^{0,d}) \subset \mathcal{D}_X^{p,q} (\log D).
\]

By definition, the wedge product \( w_X \) is compatible with the Leray filtration on gives the morphism of filtered complexes of presheaves on \( X^{an} \):

\[
  w_X : (\mathcal{D}_X^{\bullet,\bullet} (\log D), L^r) \otimes_{O_X} (A_X^{\bullet,\bullet} (\text{null } D), L^r) \to (\mathcal{D}_X^{\bullet,\bullet}, L^{r+s})
\]

In particular it induces on the first graded piece the morphism of presheaves on \( X^{an} \)

\[
  < w_X > : \mathcal{D}_{X/S}^{\bullet,\bullet} (\log D)) \otimes_{O_X} A_X^{0,\bullet} (\text{null } D) \to \mathcal{D}_{X/S}^{\bullet,\bullet}.
\]

**Remark 5.** For \( j_V : V \hookrightarrow X \) an open subset, the pairing

\[
  f_{V \cap} j_V^* w_X : (\Gamma(V, \mathcal{D}_X^{p,q} (\log D))/L^1) \otimes_{\mathbb{C}} \Gamma(V, \mathcal{D}_X^{d-p,d-q}(\text{null } D)/L_1) \to \mathbb{C}, \ T \otimes \omega \to f_{V \cap} (T \wedge \omega)
\]

shows that \( \Gamma(V, \mathcal{D}_{X/S}^{p,q} (\log D)) = \Gamma(V, \mathcal{D}_X^{d-p,d-q}(\text{null } D))/L_1 \). That is, \( \mathcal{D}_{X/S}^{p,q} (\log D) \) is the Verdier dual \( \mathcal{A}_{X/S}^{d-p,d-q}(\text{null } D) : \mathcal{A}_X^{p,q} (\text{null } D)/L_1 \).

**Proposition 13.** For all integer \( 0 \leq r \leq d_X \), the Dolbeau resolutions of proposition 4 induces resolutions

- (i) \( 0 \to L^r \Omega_X^p (\log D) \to L^r \mathcal{D}_X^{p,\bullet} (\log D) \)
- (ii) \( 0 \to L^r \Omega_X^p (\text{null } D) \to L^r A_X^{0,\bullet} (\text{null } D) \)

**Proof.** The second resolution is given by proposition 12(ii). The first one follows from the isomorphisms \( \alpha^{p,q} : L^r \Omega_X^p (\log D) \otimes_{O_X} \mathcal{D}_X^{0,d} \sim L^r \mathcal{D}_X^{p,\bullet} (\log D) \).

**Proposition 14.** The following embeddings of bifiltered complexes of sheaves on \( X^{an} \):

\[
  (\Omega_X^\bullet (\log D), F^\bullet, L^\bullet) \hookrightarrow (A_X^\bullet (\log D), F^\bullet, L^\bullet) \hookrightarrow (\mathcal{D}_X^\bullet (\log D), F^\bullet, L^\bullet)
\]

is a bifiltered quasi-isomorphism of complexes of sheaves. In particular,

\[
  (\Omega_X^{\bullet,\bullet} (\log D), F^\bullet) \hookrightarrow (A_X^{\bullet,\bullet} (\log D), F^\bullet) \hookrightarrow (\mathcal{D}_X^{\bullet,\bullet} (\log D), F^\bullet),
\]

where < \iota > is the morphism induced by \( \iota \) on \( \text{Gr}_L^0 \), are filtered quasi-isomorphism.
Proof. This comes from proposition 13(i).

Definition 10. Leray filtration on the complex of sheaves on $X^{\text{an}}$ $\mathcal{A}_{X,D}^\bullet$ is given by $L^r\mathcal{A}_{X,D}^\bullet := \text{Cone}(i_D^p : L^r\mathcal{A}_X^\bullet \to L^r\mathcal{A}_{D_s}^\bullet)[-1] \subset \mathcal{A}_{X,D}^\bullet$, that is for $V \subset X$ an open subset

$$\Gamma(V, L^r\mathcal{A}_{X,D}^\bullet) = \text{Cone}(i_{D_s}^p : \Gamma(V, L^r\mathcal{A}_X^\bullet) \to \Gamma((V \cap D)_\bullet, L^r\mathcal{A}_{D_s}^\bullet) \subset \Gamma(V, \mathcal{A}_{X,D}^\bullet)$$

is the subcomplex whose terms are $\Gamma(V, L^r\mathcal{A}_{X,D}^\bullet) = \Gamma(V, L^r\mathcal{A}_X^\bullet) \oplus (\oplus_j \Gamma(V \cap D_j, L^r\mathcal{A}_{D_j}^{k-\text{card} J})) \subset \Gamma(V, \mathcal{A}_{X,D}^\bullet)$.

We will consider the complex of sheaf on $X^{\text{an}} \times \mathcal{A}_{X,D}^\bullet$ := $\text{Gr}^0 L^D \mathcal{A}_{X,D}^\bullet$ Since the morphisms $f : X \to S$ and $f_D : D \to S$ are smooth projective, the spectral sequence associated to the Frölicher filtration $(f_* \mathcal{A}_{(X,D)/S}^\bullet, F)$ on this complex is $E_1$ degenerate. of sheaves on $S$ is $E_1$ degenerate.

Proposition 15. (i) The wedge product induces an isomorphism of filtered complexes of sheaves on $X^{\text{an}}$

$$w_X : (\Omega_{X,D}^p, L) \otimes_{O_X} A_X^\bullet \xrightarrow{\sim} (A_{X,D}^\bullet, L)$$

(ii) The inclusion of bifiltered complexes of sheaves on $X^{\text{an}}$

$$(\Omega_{X,D}^\bullet, F, L) \hookrightarrow (A_{X,D}^\bullet, F, L),$$

is a bifiltered quasi-isomorphism.

Proof. (i): This is a morphism of complexes by proposition 11(i).

(ii): It follow from (i). We can also see (ii) directly : we have the commutative diagram

$$
\begin{array}{ccc}
(\Omega_{X}^\bullet, F_b, L) & \xrightarrow{(a_D, \Omega_D^\bullet)} & (A_{X,D}^\bullet, F_b, L) \\
\downarrow & & \downarrow \\
(A_X^\bullet, F, L) & \xrightarrow{(a_D, A_D^\bullet)} & (A_{X,D}^\bullet, F, L)
\end{array}
$$

where the columns are bifiltered quasi-isomorphisms.

3.2 $E_1$ degenerescence and duality in the relative case

The inclusion (22) of filtered complexes of sheaves on $X^{\text{an}}$, $\tau : (A_X^\bullet(\text{nul } D), F) \hookrightarrow (A_{X,D}^\bullet, F)$ is by definition compatible with the Leray filtration. Hence $\tau$ is an inclusion of bi filtered complexes of sheaves on $X^{\text{an}}$

$$\tau : (A_X^\bullet(\text{nul } D), F, L) \hookrightarrow (A_{X,D}^\bullet, F, L).$$

Denote by $< \tau > : (A_{X/S}^\bullet(\text{nul } D), F) \hookrightarrow (A_{(X,D)/S}^\bullet, F)$ the map induced on $\text{Gr}_0 L^D$. Similarly $t_D^p : (j_* A^\bullet_U, L) \hookrightarrow (A_{X}^\bullet(\text{nul } D), L)$ (c.f proposition 9) and $l : (\Omega_X^\bullet(\log D), L) \hookrightarrow (j_* A^\bullet_U, L)$ (c.f corollary 1) are inclusions of filtered complexes of sheaves on $X^{\text{an}}$. Then,

Proposition 16. (i) The restriction $\tau : (\Omega_{X}^\bullet(\text{nul } D), L) \hookrightarrow (\Omega_{X,D}^\bullet, L)$ of $\tau$ is a filtered quasi-isomorphism of sheaves.

(ii): Consider embeddings of bifiltered complex of sheaves on $X^{\text{an}}$ given by (24):

$$
(j_* A^\bullet_U, F, L) \xrightarrow{t_D^p} (A_X^\bullet(\text{nul } D), F) \xrightarrow{\tau} (A_{X,D}^\bullet, F, L)
$$

Then $\tau$ is a bifiltered quasi-isomorphism of sheaves. It induces the maps of filtered complex of sheaves on $X^{\text{an}}$:

$$
(j_* A^\bullet_U/S, F) \xleftarrow{< t_D^p >} (A_{X/S}^\bullet(\text{nul } D), F) \xleftarrow{< \tau >} (A_{(X,D)/S}^\bullet, F)
$$
where $<\tau_V>$ are the morphism induced by $\tau_V$ on $Gr^0_L$ and $A^*_U/S = j^* A^*_X/S = Gr^0_L A^*_U$. In particular, $<\tau>$ is a filtered quasi-isomorphism. The inclusion $<\tau_V>$ is quasi-isomorphism but NOT a filtered quasi-isomorphism.

(iii) The inclusion map $\tau : (A^*_X(nul D), F, L) \hookrightarrow (A^*_{X,D}, F, L)$, is a bi-filtered quasi-isomorphism of complexes of presheaves, that is for all open subset $V \subset X$, and for all integers $p, r$ the restriction

$$\tau : \Gamma(V, L^r F^p A^*_X(nul D)) \hookrightarrow \Gamma(V, L^r F^p A^*_{X,D})$$

of $\tau$ are quasi-isomorphisms.

Proof. (i): The sequence of complexes of sheaves on $X^an$

$$0 \to L^r \Omega^p_X(nul D) \xrightarrow{t_X,D} L^r \Omega^p_X \xrightarrow{D_1} \bigoplus_{j=1}^s i_{D_j} \to L^r \Omega^p_{D_{j+1}} \to 0$$

is exact. This proves (i).

(ii): By (i),

$$\tau : (\Omega^* X(nul D), F_b, L) \hookrightarrow (\Omega^*_{X,D}, F_b, L)$$

of is a bifiltered quasi-isomorphism of complexes of sheaves. On the other side,

- the inclusion $(\Omega^*_X(nul D), F_b, L) \hookrightarrow (A^*_X(nul D), F, L)$ is a bifiltered quasi-isomorphism of complexes of sheaves by proposition 12 (ii)
- the inclusion $(\Omega^*_{X,D}, F_b, L) \hookrightarrow (A^*_{X,D}, F, L)$ is a bifiltered quasi-isomorphism of complexes of sheaves by proposition 13 (ii).

Hence,

$$\tau : (A^*_X(nul D), F, L) \hookrightarrow (A^*_{X,D}, F, L)$$

is a bifiltered quasi-isomorphism of complexes of sheaves. This proves (ii).

(ii): By (ii), the inclusion maps of complexes of sheaves on $X^an$

$$\tau : L^r F^p A^*_X(nul D) \hookrightarrow L^r F^p A^*_{X,D}$$

are quasi-isomorphism of complexes of sheaves. Thus, for all every open subset $j_V : V \hookrightarrow X$, $j^*_V \tau : j^*_V L^r F^p A^*_X(nul D) \to j^*_V L^r F^p A^*_{X,D}$ are quasi-isomorphism of complexes of sheaves. Hence, for every open subset $V \subset X$, the maps

$$\tau : \mathbb{H}^*(V, L^r F^p A^*_X(nul D)) \hookrightarrow \mathbb{H}^*(V, L^r F^p A^*_{X,D})$$

are quasi-isomorphism of complexes of $\mathbb{C}$-vector spaces. The sheaves $L^r F^p A^*_X(nul D)$, $L^r F^p A^*_{X,D}$ and $i_{D_j} L^r F^p A^*_X(S)$ are sheaves of $O_X$ modules on $X^an$, so are c-soft (because the existence of partition of unity) and thus acyclic for the global section functor on each open subset $V \subset X$ ($X^an$ is a denumerable union of compact subsets). Hence, for every open subset $V \subset X$,

$$H^k \Gamma(V, L^r F^p A^*_X(nul D)) = \mathbb{H}^k(V, L^r F^p A^*_X(nul D)) \text{ and } H^k \Gamma(V, L^r F^p A^*_{X,D}) = \mathbb{H}^k(V, L^r F^p A^*_{X,D})$$

This proves (iii).

**Corollary 3.** The following maps of complexes of sheaves on $X^an$:

- $j_* f^*_U O_S \to \Omega^*_{X/S}(log D) \xrightarrow{<\tau_V>} j_* A^*_U/S$, and
- $j_* f^*_U O_S \xrightarrow{<\tau_U>} A^*_U/S(nul D) \xrightarrow{<\tau_V>} A^*_{X,D/S}$
are quasi-isomorphisms.

Proof. The fact that the maps of the first sequence are quasi-isomorphism comes from the resolution $0 \to f_U^* O_S \to A_{U/S}^\bullet$. The fact that the maps of the second sequence are quasi-isomorphism is given by proposition 11(11). \hfill \square

Corollary 4. (i) The spectral sequence associated to the filtered complex of sheaves on $S^{an}$ $(f_* A^\bullet_{X/S}(\text{null } D), F)$ by Frölicher filtration $F$ is $E^1$ degenerate.

(ii) For all integer $k, p$, the map induced on relative hypercohomology of the quotient map $F^p A^\bullet_{X/S}(\text{null } D) \to \text{Gr}_{F^p} A^\bullet_{X/S}(\text{null } D)$

$$
\mathcal{H}^k f_* F^p A^\bullet_{X/S}(\text{null } D) \to \mathcal{H}^k f_* \text{Gr}_{F^p} A^\bullet_{X/S}(\text{null } D) = \mathcal{H}^{k-p} f_* A^p_{X/S}(\text{null } D)) = R^{k-p} f_* \Omega^p_{X/S}(\text{null } D)),
$$
given by for $W \subset S$ an open subset and $\omega \in \Gamma(X_W, F^p A^\bullet_{X/S}(\text{null } D))^{d_{X/S}=0}$,

$$
[\omega] \in \Gamma(W, \mathcal{H}^k F^p A^\bullet_{X/S}(\text{null } D)) \mapsto [\omega^{p-k}] \in \Gamma(W, \mathcal{H}^{k-p} A^p_{X/S}(\text{null } D))
$$
is surjective.

Proof. (i) By proposition 11(iii), the map of complexes of sheaves on $X^{an} < \tau > : (A^\bullet_{X/S}(\text{null } D), F) \to (A^\bullet_{(X,D)/S}, F)$ is a filtered quasi-isomorphism of complexes of sheaves. Hence, the map of complexes of sheaves on $S^{an}$ $f_* < \tau > : (f_* A^\bullet_{X/S}(\text{null } D), F) \to (f_* A^\bullet_{(X,D)/S}, F)$ is a filtered quasi-isomorphism of complexes of presheaves, hence a filtered quasi-isomorphism of complexes of sheaves. On the other hand the spectral sequence associated to the complex of sheaves $(f_* A^\bullet_{(X,D)/S}, F)$ is $E_1$ degenerate (see definition 13). Thus the spectral sequence associated to $(f_* A^\bullet_{X/S}(\text{null } D), F)$ is $E_1$ degenerate.

(ii) This is a classical fact on spectral sequence that (ii) is equivalent to (i) see for example [7]. \hfill \square

Denote by $H^k_Z(f_U) := R^k f_U Z_U$, $H^k_C(f_U) := R^k f_U C_U$, and by $H^k_Z(f_{X,D}) := R^k f_{U,D} Z_U$, $H^k_C(f_{X,D}) := R^k f_{U,D} C_U$. For $s \in S$, since the fiber $U_s \subset U$ is closed in $U^{an}$ and $U^{an}$ is paracompact, we have $(R^k f_U)_{U_s} \cong H^k(U_s, C)$. We have the canonical quasi isomorphism $Rf_{X,D,c} = Rf_{U,D} \to \text{Cone}(Rf_{U,D} \to Rf_{U,D})[-1]$. On the other hand, $(R^k f_{X,D,c})_{U_s} \cong H^k(X_s, C)$ and $(R^k f_{D,c})_{U_s} \cong H^k(D_s, C)$ since the fibers $X_s \subset X$ and $D_s \subset D$ are closed in $X^{an}$ and $D^{an}$ respectively and $X^{an}$ and $D^{an}$ are compact (hence paracompact). Hence, for $s \in S$, $(R^k f_{X,D,c})_{U_s} \cong H^k(X_s, D_s, C)$.

In our situation, the $H^k_Z(f_U)$ and the $H^k_Z(f_{X,D})$ are local systems on $S^{an}$ because the maps $f : X \to S$ and $f_{D,J} : D_J \to S$ are smooth projective. For $0 \leq k \leq 2d$ (otherwise the sheaves are zero), the sheaves of $O_S$ modules $H^k_Z(f_U) := H^k(f_U) \otimes_{C} O_S = R^k f_U C_U \otimes_{C} O_S$ are locally free and we will denote again $H^k_Z(f_U)$ the corresponding holomorphic vector bundles on $S$. For $0 \leq k \leq 2d$ (otherwise the sheaves are zero), the sheaves of $O_S$ modules $H^k_Z(f_{X,D}) := H^k_Z(f_{X,D}) \otimes_{C} O_S = R^k f_{U,D} C_U \otimes_{C} O_S$ are locally free and we will denote again $H^k_Z(f_{X,D})$ the corresponding holomorphic vector bundles on $S$.

Proposition 17. We have the following isomorphisms of sheaves on $S^{an}$:

- $H^k_Z(f_U) \cong R^k f_{U,*}(f^* O_S) = \mathcal{H}^k f_{U,*} A_{U/S}^\bullet = \mathcal{H}^k f_* A_{X/S}^\bullet(\log D) = \mathcal{H}^k f_* D_{X/S}(\log D)$

- $R^k f_{U!}(f^* O_S) = \mathcal{H}^k f_{U!} A_{U/S}^\bullet = \mathcal{H}^k f_* A_{X/S}^\bullet(\text{null } D) \cong H^k_Z(f_{X,D})$.

Proof. These two isomorphism are given by the two projection formula. The equalities comes from corollary 3. \hfill \square

Remark 6. In our situation, since $H^k_Z(f_U)$ and $H^k_Z(f_{X,D})$ are local systems, these isomorphisms can be explicit in common local trivialisations of the differentially locally trivial maps $f : X^{an} \to S^{an}$, $f_{D,J} : D_J^{an} \to S^{an}$.
Definition 11. The Hodge filtrations on the vector bundles $\mathcal{H}^k_S(f_U)$ and $\mathcal{H}_S^{k}(f_{X,D})$ is the one given by the Fröhlicher filtration $F$ on the complexes of sheaves on $S^an$ $f_*A^*_X/S(log D)$ and $f_*A^*_X/S(nul D)$ respectively. By the $E_1$ degenerescence of the spectral sequences associated to $(f_*A^*_X/S(log D), F)$ and $(f_*A^*_X/S(nul D), F)$ (corollary [3](ii)), the following canonical surjective maps of sheaves on $S^an$ are isomorphisms

- $\mathcal{H}^k f_* F_\bullet A^*_X/S(log D) = \mathcal{H}^k f_* F_\bullet D^\bullet_X/S(log D) \sim \to F_\bullet \mathcal{H}^k_S(f_U)$
- $\mathcal{H}^k f_* F_\bullet A^*_X/S(nul D) \sim \to F_\bullet \mathcal{H}^k_S(f_{X,D})$

and their graded pieces are

- $\mathcal{H}^{k-p}_S(f_U) := F_\bullet \mathcal{H}^k_S(f_U)/F_\bullet \mathcal{H}^{k+1}_S(f_U) \sim \to \mathcal{H}^{k-p} f_* A^\bullet_X/S(log D) = \mathcal{H}^{k-p} f_* D^\bullet_X/S(log D) = R^{k-p} f_* \Omega^p_S(log D)$
- $\mathcal{H}^{k-p}_S(f_{X,D}) := F_\bullet \mathcal{H}^k_S(f_{X,D})/F_\bullet \mathcal{H}^{k+1}_S(f_{X,D}) \sim \to \mathcal{H}^{k-p} f_* A^\bullet_X/S(nul D) = R^{k-p} f_* \Omega^p_S(nul D)$ (see also corollary [3](ii)).

The wedge product (17) is a bifiltered morphism of complexes of presheaves on $X^an$:

$$w_X : (D^\bullet_X(log D), F, L) \otimes_{O_X} (A^\bullet_X(nul D), F, L) \to D^\bullet_X$$

and induces the pairings of filtered complexes of presheaves on $S^an$:

- $ev_f = f_* < w_X > = < \cdot, >_{ev_f} : (f_* D^\bullet_X/S(log D), F) \otimes_{O_S} (f_* A^{2d-\bullet}_X/S(nul D), F) \to (f_* D^{2d}_X/S, F)$, given by, for $W \subset S$, $T \otimes \omega \in \Gamma(X_W, D^\bullet_X/S(log D)) \otimes \Gamma(X_W, A^{2d-\bullet}_X/S(nul D)) \mapsto < T, \omega >_{ev_f} = f_{X_W}* (T \wedge \omega)$
- $f_* ev_X = f_* w_X = < \cdot, >_{f_* ev_X} : (f_* D^\bullet_X(log D)/L^2, F) \otimes_{O_S} (f_* L^{2d-1} A^{2d-\bullet}_X/S(nul D), F) \sim \to (f_* D^{2d}_X, F)$, given by, for $W \subset S$, $T \otimes \omega \in \Gamma(X_W, D^\bullet_X/S(log D)) \otimes \Gamma(X_W, A^{2d-\bullet}_X/S(nul D)) \mapsto < T, \omega >_{f_* ev_X} = f_{X_W}* (T \wedge \omega)$.

Proposition 18. (i) The pairing of filtered complexes of presheaves on $S^an$:

$$ev_f = < \cdot, >_{ev_f} : (f_* D^\bullet_X/S(log D), F) \otimes_{O_S} (f_* A^{2d-\bullet}_X/S(nul D), F) \to (f_* D^{2d}_X/S, F),$$

induces on cohomology isomorphisms of sheaves on $S^an$ (see definition [17]):

- $- ev_f : H^k_S(f_U)/F_\bullet H^k_S(f_U) \sim \to D^\bullet_{O_S}(H^k_S(f_{X,D}))$
- $- ev_f : H^{k-p}_S(f_U) \sim \to D^\bullet_{O_S}(H^{d-p, d-k}_S(f_{X,D}))$

(ii) The pairing of filtered complexes of presheaves on $S^an$:

$$f_* ev_X = < \cdot, >_{f_* ev_X} : (f_* (D^\bullet_X(log D)/L^2), F) \otimes_{O_S} (f_* L^{2d-1} A^{2d-\bullet}_X/S(nul D), F) \to (f_* D^{2d}_X, F)$$

induces on cohomology isomorphisms of sheaves on $S^an$:

$$f_* ev_X : R^q f_* (\Omega_X^p(log D)/L^2) \sim \to R^{d-q} f_* (L^{2d-1} \Omega^{d-k}_X)^p(nul D)).$$

Proof. (i): As these sheaves on $S^an$ are locally free sheaves of $O_S$ modules, it suffices to show that the evaluation of the induced maps at every point $s \in S$ are isomorphisms. But this is Poincare duality for the pair $(X_* , D_*)$ (c.f proposition [1]).

(ii): As in (i), since these sheaves on $S^an$ are locally free sheaves of $O_S$ modules, it suffices to show that the evaluation of the induced maps at every point $s \in S$ are isomorphisms. But this is Serre duality for $X_*$ since

$$\Omega^p_X(log D)|_{X_*} = D^\bullet_{O_{X_*}}(\Omega^{d-k}_X)^p(nul D)|_{X_*} \otimes K_{X_*} = D^\bullet_{O_{X_*}}(\Omega^{d-k}_X)^p(nul D)|_{X_*} \otimes K_{X_*} \quad (33)$$

by proposition [2](ii) for $X$, and the fact that $K_{X_*} \simeq K_{X_*}$. 

\[ \square \]
Proposition 19. For \( s \in S \) and

- \( T \in \Gamma(W(s), f_* D^\bullet_{X/S}(log D))^{d_X/s=0} \) and \( T' \in \Gamma(W(s), f_* D^\bullet_{X/S}(log D))^{d_X/s=0} \), whose restriction to the fibers of \( f \) is proper (c.f. [D] for the definition of the pullback or Gysin map for currents)

- \( \eta \in \Gamma(W(s), f_* A^{d-d-*}_{X/S}(mul D))^{d_X/s=0} \) and \( \eta' \in \Gamma(W(s), f_* A^{d-d-*}_{X/S}(mul D))^{d_X/s=0} \),

where \( W(s) \subset S \) is a neighborhood of \( s \) in \( S \), we have

\[ \langle T, \eta \rangle_{ev_f} = \langle T|_{X_s}, \eta|_{X_s} \rangle_{ev_{X_s}} \quad \text{and} \quad \langle T', \eta' \rangle = \langle T'|_{X_s}, \eta'|_{X_s} \rangle_{ev_{X_s}}. \]

This gives on cohomology

\[ \langle [T], [\eta] \rangle_{ev_f} = \langle [T](s), [\eta](s) \rangle_{ev_{X_s}} \quad \text{and} \quad \langle [T'], [\eta'] \rangle = \langle [T'](s), [\eta'](s) \rangle_{ev_{X_s}}. \]

In particular, if \( \omega \in \Gamma(W(s), f_* A^{d-d-*}_{X/S}(log D))^{d_X/s=0} \) and \( \omega' \in \Gamma(W(s), f_* A^{d-d-*}_{X/S}(log D))^{d_X/s=0} \) are log forms, then \( \langle [\omega], [\eta] \rangle > (s) = \int_{X_s} \omega \wedge \eta \) and \( \langle [\omega'], [\eta'] \rangle > (s) = \int_{X_s} \omega' \wedge \eta' \).

Proof. See [D] proposition 3.2.2.

3.3 The Gauss-Manin connexion

We have the the commutative diagram of filtered complexes of sheaves on \( X^{an} \), \( F \) being the Frölicher filtration on \( F_b \) the filtration bte,

\[
\begin{array}{cccccc}
0 & \rightarrow & (Gr^1_L \Omega^\bullet_X(log D), F_b) & \overset{r^\vee}{\rightarrow} & (\Omega^\bullet_X(log D)/L^2, F_b) & \overset{q}{\rightarrow} & (\Omega^\bullet_{X/S}(log D), F_b) & \rightarrow & 0 \\
0 & \rightarrow & (Gr^1_L A^\bullet_X(log D), F) & \overset{r^\vee}{\rightarrow} & (A^\bullet_X(log D)/L^2, F) & \overset{q}{\rightarrow} & (A^\bullet_{X/S}(log D), F) & \rightarrow & 0 \\
0 & \rightarrow & (Gr^1_L D^\bullet_X(log D), F) & \overset{r^\vee}{\rightarrow} & (D^\bullet_X(log D)/L^2, F) & \overset{q}{\rightarrow} & (D^\bullet_{X/S}(log D), F) & \rightarrow & 0
\end{array}
\]

where the row are by definition exact sequences of filtered complexes (the embedding \( r^\vee = Gr^1_L \rightarrow L^0/L^2 \) is the quotient of the inclusion \( L^1 \subset L^0 \) by \( L^2 \) and \( q : L^0/L^2 \rightarrow Gr^1_L \) is the projection ) and the column are filtered quasi-isomorphisms by proposition [14].

Consider also the commutative diagram of filtered complexes of sheaves on \( X^{an} \) whose rows are exact :

\[
\begin{array}{cccccc}
0 & \rightarrow & (Gr^1_L A^\bullet_X(mul D), F) & \overset{r^\vee}{\rightarrow} & (A^\bullet_X(mul D)/L^2, F) & \overset{q}{\rightarrow} & (A^\bullet_{X/S}(mul D), F) & \rightarrow & 0 \\
0 & \rightarrow & (Gr^1_L A^\bullet_X, F) & \overset{r^\vee}{\rightarrow} & (A^\bullet_X/L^2) & \overset{q}{\rightarrow} & (A^\bullet_{X/S}, F) & \rightarrow & 0 \\
0 & \rightarrow & (Gr^1_L A^\bullet_X(log D), F) & \overset{r^\vee}{\rightarrow} & (A^\bullet_X(log D)/L^2, F) & \overset{q}{\rightarrow} & (A^\bullet_{X/S}(log D), F) & \rightarrow & 0
\end{array}
\]

Definition 12. The Gauss Manin connections of the bundles \( \mathcal{H}^k_S(f_U), \mathcal{H}^k_S(f_{x,D}) \) respectively, are induced by the connecting morphism associated to the long cohomological exact sequence of last, respectively first, row of the diagram [23].

- \( \nabla : F^{p} \mathcal{H}^k_S(f_U) \rightarrow \mathcal{H}^{k+1} f_* Gr^1_L F^{p} A^\bullet_X(log D) = F^{p-1} \mathcal{H}^k_S(f_U) \otimes_{\mathcal{O}_S} \Omega_S, \)
Definition-Proposition 1. Let \( \nabla : F \mathcal{H}^k_S(f_{X,D}) \to \mathcal{H}^{k+1}f_*\text{Gr}^1_L F^p\mathcal{A}^k_{X,S}(\text{null } D) = \mathcal{H}^{p-1}f^*_S(f_{X,D}) \otimes O_S \Omega_S \), where the above equalities are given by the identifications \( \text{(22)} \) of proposition \( \text{(11)} \) (see also remark \( \text{(4)} \) and by the projection formula \((X^\text{an})\) being paracompact the canonical map of sheaves on \( S^\text{an} \) \( f_*F \otimes_{O_X} G \to f_*(F \otimes f^*G) \) is an isomorphism. 

Hence, for \( W \subset S \) an open subset, \( \omega \in \Gamma(W, f_* F^p\mathcal{A}^k_{X/S}(\log D)) \mathcal{H}^{d_{X/S}=0} = \Gamma(X^\omega, F^p\mathcal{A}^k_{X/S}(\log D)) \mathcal{H}^{d_{X/S}=0} \), \( \eta \in \Gamma(W, f_* F^p\mathcal{A}^k_{X/S}(\text{null } D)) \mathcal{H}^{d_{X/S}=0} = \Gamma(X^\eta, F^p\mathcal{A}^k_{X/S}(\text{null } D)) \mathcal{H}^{d_{X/S}=0} \) and \( u \in \Gamma(W,T_S) \),

\[
\nabla_u([\omega]) = \phi^{1,**}(d\omega) = [\langle \iota(\bar{u})d\omega \rangle \text{ and } \nabla_u([\eta]) = \phi^{1,**}(d\eta) = [\langle \iota(\bar{u})d\eta \rangle],
\]

where \( \bar{u} \in \Gamma(X^\omega, T_X) \) is a replement of \( u \) (i.e. \( df(\bar{u}) = u \)).

Remark 7. The diagram \( \text{(34)} \) and the identifications \( \text{(22)} \) of proposition \( \text{(11)} \) induces the commutative diagrams

\[
\begin{align*}
\nabla : F \mathcal{H}^k_S(f_{X,D}) &\to F \mathcal{H}^{k+1}f_*\text{Gr}^1_L F^p\mathcal{A}^k_{X,s}(\text{null } D) = \mathcal{H}^{p-1}f^*_S(f_{X,D}) \otimes O_S \Omega_S \\
\nabla : F \mathcal{H}^k_S(f) &\to F \mathcal{H}^{k+1}_S(f) \otimes O_S \Omega_S \\
\nabla : F \mathcal{H}^k_S(f_U) &\to F \mathcal{H}^{k+1}_S(f_U) \otimes O_S \Omega_S
\end{align*}
\]

Definition-Proposition 1. Let \( \nabla \) the morphism induced by \( \nabla \) on graded pieces.

\[
\begin{align*}
\nabla : \mathcal{H}^{p-k-p}_S(f_U) &\to \mathcal{H}^{p-1,k-p+1}_S(f_U) \otimes O_S \Omega_S \\
\nabla : \mathcal{H}^{p-k-p}_S(f_{X,D}) &\to \mathcal{H}^{p-1,k-p+1}_S(f_{X,D}) \otimes O_S \Omega_S,
\end{align*}
\]

Then, for \( W \subset S \) an open subset, \( \omega' \in \Gamma(W, f_* F^p\mathcal{A}^{p,k-p}_{X/S}(\log D)) \mathcal{H}^{d_{X/S}=0} = \Gamma(X^\omega, F^p\mathcal{A}^{p,k-p}_{X/S}(\log D)) \mathcal{H}^{d_{X/S}=0} \), \( \eta' \in \Gamma(W, f_* F^p\mathcal{A}^{p,k-p}_{X/S}(\text{null } D)) \mathcal{H}^{d_{X/S}=0} = \Gamma(X^\eta, F^p\mathcal{A}^{p,k-p}_{X/S}(\text{null } D)) \mathcal{H}^{d_{X/S}=0} \) and \( u \in \Gamma(W,T_S) \),

\[
\nabla((\omega')) = \phi^{1,**}(d\omega') = [\langle \iota(\bar{u})d\omega' \rangle \text{ and } \nabla((\eta')) = \phi^{1,**}(d\eta') = [\langle \iota(\bar{u})d\eta' \rangle],
\]

Proof. This follows from corollary \( \text{(4)(ii)} \) and the description of the morphism \( \nabla \). \( \square \)

Proposition 20. For simplicity of notation denote by \( <\cdot,\cdot> = <\iota,\iota>_{eq} \).

(i) We have, for \( s \in S, u \in T_{S,s}, \lambda \in \Gamma(W(s), \mathcal{H}^k_S(f_U)) \) and \( \mu \in \Gamma(W(s), \mathcal{H}^{2d-k}_S(f_{X,D})) \), where \( W(s) \subset S \) is an open neighborhood of \( s \) in \( S \):

\[
\nabla_u \lambda, \mu > (s) = d_u < \lambda, \mu > (s) - < \lambda, \nabla_u \mu > (s)
\]

(ii) The pairing \( ev_f = <\iota,\iota> \) induces isomorphisms

\[
ev_f : \mathcal{H}^{p,k}_S(f_U) \otimes \Omega_S/\text{Im}(\nabla) \sim (\mathcal{H}^{d-p,d-q}_{S}(f_{X,D}) \otimes T_S) \mathcal{H} = 0
\]

where \( \iota(\nabla(\mu \otimes u) = \nabla_{u,\mu} \).

Proof. (i): Shrink the \( W(s) \) if necessary, there exist closed forms \( \omega \in \Gamma(W(s), f_* A^k_{X/S}(\log D)) \mathcal{H}^{d_{X/S}=0} \) and \( \eta \in \Gamma(W(s), f_* A^{2d-k}_{X/S}(\text{null } D)) \mathcal{H}^{d_{X/S}=0} \) such that \( [\omega] = \lambda \) and \( [\eta] = \mu \). Then,

\[
d_u < \lambda, \mu > (s) = \int_{X^\omega} \omega \wedge (\iota(\bar{u})d\eta) = \int_{X^\omega} \iota(\bar{u})d(\omega \wedge \eta)
\]

\[
= \int_{X^\omega} \iota(\bar{u})d\omega \wedge \eta + \int_{X^\omega} \omega \wedge (\iota(\bar{u})d\eta) = <\nabla_u[\omega], [\eta] > (s) + < [\omega], \nabla_u[\eta] > (s)
\]
(ii): If \( \lambda \in \Gamma(W(s), F^p \mathcal{H}_S^k(\nu_U)) \) and \( \mu' \in \Gamma(W(s), F^{k-p+1} \mathcal{H}_S^k(\nu_U)) \) we have \( < \lambda', \mu'> = 0 \) as Poincaré duality for the pair \((X_s, D_s)\) is a morphism of mixed hodge structures. Hence by (i), \( < \nabla_u \lambda', \mu' > < \lambda', \nabla_u \mu' > \). Thus, \( < \nabla_u \lambda, \mu' > = < \lambda, \nabla_u \mu' > \). Point (ii) follows from this equality.

\[\begin{array}{ccc}
0 & \rightarrow & \Omega_{X/S}^{p-1}(\log D) \otimes f^* \Omega_S \rightarrow \Omega_{X}^{p}(\log D)/L^2 \rightarrow \Omega_{X/S}^{p}(\log D) \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & D^{p-1}_{X/S}(\log D) \otimes f^* \Omega_{S} \rightarrow D^{p}_{X}(\log D)/L^2 \rightarrow D^{p}_{X/S}(\log D) \rightarrow 0,
\end{array}\]

The \( F \) graded piece of first and last rows of the diagram (34) is the commutative diagram

\[0 \rightarrow \Omega_{X/S}^{d-p}(\text{null } D) \rightarrow L^{d-1} \Omega_{X}^{d-p}(\text{null } D) \rightarrow \Omega_{X/S}^{d-p+1}(\text{null } D) \otimes f^* T_S \rightarrow 0 \]

\[0 \rightarrow A_{X/S}^{d-p}(\text{null } D) \rightarrow L^{d-1} A_{X}^{d-p}(\text{null } D) \rightarrow A_{X/S}^{d-p+1}(\text{null } D) \otimes f^* T_S \rightarrow 0,
\]

whose rows are by definition exact sequence of complexes of sheaves, where

- \( q^\vee : \omega \mapsto \omega \wedge f^* \kappa \) is the dual of \( q \),
- \( r : L^{d-1} \Omega_{X}^{d-p}(\text{null } D) \rightarrow \mathcal{G}_{L}^{d-1} \Omega_{X}^{d-p}(\text{null } D) \rightarrow \Omega_{X/S}^{d-p+1}(\text{null } D) \otimes f^* \Omega_{S}^{d-1} \) is the dual of \( r^\vee \),

and whose columns are quasi-isomorphism by proposition 13 (the Dolbeau resolutions).

The maps \( ev_f = f_* < w_X > \) and \( f_\ast w_X \) induces a pairing between the images by \( f_* \) of the second rows of the diagramms (37) and (38) of sheaves on \( X^{an} \). Thus, by proposition 13, it induces an isomorphism between the two long cohomological exact sequences of these two exact sequences:

\[\begin{array}{ccc}
\bar{\mathcal{H}}_{S}^{p-1,q}(\nu_U) \otimes f^* \Omega_S \rightarrow R^d f_* (\Omega_X^p(\log D)/L^2) \rightarrow \mathcal{H}^{p-1,q}(\nu_U) \\
\downarrow \sim \quad \quad \quad \downarrow \sim \\
D_{O_S}^{(r \vee)}(\mathcal{H}_{S}^{p,d-1,q}(f_X,D) \otimes T_S) \rightarrow D_{O_S}^{(r \vee)}(R^{d-q} f_* L^{d-1} \Omega_{X}^{d-p}(\text{null } D)) \rightarrow D_{O_S}^{(r \vee)}(\mathcal{H}_{S}^{p,d-1,d-1,q}(f_X,D))
\end{array}\]

3.4 Normal functions and infinitesimal invariants

**Definition 13.** The relative intermediate jacobian of \( f \) is the of the fibration of complex analytic varieties

\[J^{p,k}(f_U) = \mathcal{H}_S^k(\nu_U)/(F^{p} \mathcal{H}_S^k(\nu_U) \oplus H^k_{2}(f_U)) \rightarrow S.\]

By proposition 13, the map \( ev_f \) induces an isomorphism over \( S \)

\[ev_f : J^{p,k}(f_U) \cong D_{O_S}^{(r \vee)}(F^{d-p+1} \mathcal{H}_S^{d-1}(f_X,D))/H_{2d-k,Z}(f_X,D)
\]

A normal function is a holomorphic section \( \nu \in \Gamma(S, J^{p,k}(f_U)) \) of the fibration \( J^{p,k}(f_U) \rightarrow S \), such that every local relevation \( \nu_W \in \Gamma(W, \mathcal{H}_S^k(\nu_U)) \) of \( \nu \) over an open subset \( W \subset S \) is horizontal, i.e. \( \nabla_W \nu \) is holomorphic and satisfy \( \nabla_W \nu \in \Gamma(W, F^{d-1} \mathcal{H}_S^k(\nu_U) \otimes \Omega_S D) \). Denote by \( NF(f_U)(S) \subset \Gamma(S, J^{p,k}(f_U)) \) the subspace of normal functions.
Definition-Proposition 2. Let $\nu \in NF(f_U)(S)$. Then for $W \subset S$ and $\tilde{\nu}_W \in \Gamma(W, H^k_S(f_U))$, the class
$$[\nabla \nu_W] \in \Gamma(W, H^{k-1,-p+1}_S(f_U) \otimes_{O_S} \Omega_S/\text{Im} \nabla)$$
of the projection $\nabla \nu_W \in \Gamma(W, H^{k-1,-p+1}_S(f_U) \otimes_{O_S} \Omega_S)$ of $\nabla \nu_W$ modulo the image of $\nabla$ does not depend on the choice of a relevation. Thus the local sections $[\nabla \nu_W]$ patches together to get the infinitesimal invariant of $\mu$
$$\delta \nu \in \Gamma(S, (H^{k-1,-p+1}_S(f_U) \otimes_{O_S} \Omega_S)/\text{Im} \nabla), \ \delta \nu_W = [\nabla \nu_W] \text{ for all local relevements.}$$

- Let $\nu \in \Gamma(S, J^{p,k}(f_U))$, then using the exact sequence of sheaves on $S^{an} 0 \to H^k_S(f_U) \to H^k_S(f_U)/F^p H^k_S(f_U) \to J^{p,k}(f_U) \to 0$ by definition of $J^{p,k}(f_U)$, $\nu$ has a cohomology class $[\nu] \in H^1(S, H^k_S(f_U))$.

Proof. Standard. \qed

3.5 Relative Abel jacobi map and infinitesimal invariants

Denote by

- $Z^p(U, \bullet)^{pr/X/S} \subset Z^p(U, \bullet)^{pr/X}$ be the subcomplex consisting of $Z = \sum_i n_i Z_i \in Z^p(U, n)$ such that their closures $\bar{Z} = \sum_i n_i \bar{Z}_i \in Z^p(X, n)$ intersect all the fibers $X_s \hookrightarrow X$ of $f : X \to S$ properly. By Bloch this inclusion of complexes of abelian group is a quasi-isomorphism: consider the generic fiber of $f$ and go on by a decreasing induction on the dimension of subvarieties of $S$.

- $Z^p(U, n)^{pr/X/hom/S} \subset Z^p(U, n)^{pr/X/S}$ the subspace such that $\partial Z = 0$ and $[\Omega_{Z_s}] = 0 \in H^{2p-n}(U_s, \mathbb{C}) \xrightarrow{\sim} H^{2d-2p+n}(X_s, D_s, \mathbb{C})$ for all $s \in S$.

Let $Z \in Z^p(U, n)^{pr/X/hom/S}$. Recall that its closure $\bar{Z} \in Z^p(X, n)$ satisfy $\partial \bar{Z} \in i_{D_s} Z^{p-1}(D, n)$. Let
$$R_{Z/S} = q(R_{Z}) \in \Gamma(X, D_{X/S}^{2p-n-1})) = \Gamma(S, f^* D_{X/S}^{2p-n-1})$$
where $q : D_X(\log D) \to D_{X/S}(\log D)$ is the quotient map of sheaves on $X^{an}$. By hypothesis, for all $s \in S$,
$$[\Omega_{Z_s}]|_{X_s} = [\Omega_{Z_s}] = [\Omega_{Z_s}] = 0 \in H^{2p-n}(U_s, \mathbb{C}).$$
Hence, by proposition for all $s \in S$, $R_{Z,s} = R_{Z,s} \cap X_s$ restrict to a closed current on $F^{d-1}A_{X,s}^{2d-2p+n+1}(\text{nul} D)$. That is, $R_{Z,s}$ restrict to a closed current on $F^{d-1}A_{X/s}^{2d-2p+n+1}(\text{nul} D)$, and choice of $\Gamma_{Z,s} \in C_{2d-2p+n+1}(X_s, D_s, \mathbb{Z})$ such that
$$\partial \Gamma_{Z,s} \subset C_{2d-2p+n}(X_s, D_s, \mathbb{Z}),$$
for each $s \in S$ gives the following section $\tilde{\nu}_Z \in \Gamma(S, D_{O_S}^{\nu}(F^{d-1}H_{X/s}^{2d-2p+n+1}(f, D)))$ of the dual vector bundle of the mixed hodge subbundle:
$$\tilde{\nu}_Z(s) := ev_f(R_{Z/s})(s) := (\eta \in \Gamma(X_{W(s)}, F^{d-1}H_{X/s}^{2d-2p+n+1}(\text{nul} D)))_{X/s=0}$$
$$\mapsto [R_{Z/s}]|_{\eta} > ev_f(s) = R_{Z,s}(\eta, X_s) = \int_{\Gamma_{Z,s}} \pi_X^\eta \eta \land \pi^{(p+1)p}(\Omega \cap^n)$$
where $W(s) \subset S$ is an open neighborhood of $s$ in $S$, the first equality follows from proposition and the last equality from proposition.

Theorem 6. Let $Z = \sum n_i Z_i \in Z^p(U, n)^{pr/X/hom/S}$. Then,
$$\nu_Z \in \Gamma(S, D_{O_S}^{\nu}(F^{d-1}H_{X/s}^{2d-2p+n+1}(f, D))) / H_{Z,2d-2p+n+1}(f, D), \ \nu_Z(s) = [\tilde{\nu}_Z(s)]$$
is a normal function (i.e. holomorphic and horizontal), the higher normal function associated to $Z$.
Proof. For simplicity of the notation denote $<\cdot,\cdot>=<\cdot,\cdot>_{ev_f}$.

Let $s \in S$. There exists a diffeomorphism

$$T: (X_{W(s)}, D_{W(s)}) \to (X_s, D_s) \times W(s), \quad T(x) = (T_x, f),$$

over a sufficiency small open neighborhood $W(s) \subset S$ of $s$ in $S$. As $[\Omega^1_{\Omega W(s)}] = 0 \in H_{2d-2p+n}^{BM}(X_{W(s)}, D_{W(s)}, \mathbb{C})$, there exist $\Gamma \in C^{BM}(X_{W(s)}, D_{W(s)}, \mathbb{C})$, intersecting properly the fibers of $f : X \to S$, such that $\partial \Gamma_s = Z_s$. Then we can choose $\Gamma = \Gamma_s \cap X_s$ for $s \in W(s)$ and we see that $\nu_Z|\Omega W(s) = \nu_{ZW(s)}$ is $C^\infty$ This shows that $\nu_Z$ is $C^\infty$ and in particular continuous on $S^0$. Hence, to prove the holomorphicity and the horizontality of $\nu_Z$, it is enough by continuity of $\nu_Z$ on $S^0$, to prove it on a Zariski analytic open subset of $S$ since it is dense in $S^0$. Thus, we can restrict to the Zariski open subset of $S^0 \subset S$ over which the families $f|\tilde{Z}_s : \tilde{Z}_s \to S$ are isosingular.

Let $s \in S^0$. There exists a diffeomorphism

$$T: (X_{W(s)}, D_{W(s)}) \to (X_s, D_s) \times W(s), \quad T(x) = (T_x, f),$$

over a sufficiency small open neighborhood $W(s) \subset S$ of $s$ in $S^0$ such that $T$ induces on $Z_i$ trivialisations :

$$T = T_i Z_i : (Z_i, D \cap Z_i) \to (Z_i, D \cap Z_i) \times S^0$$

and such that $T^{-1}(x \times S^0)$ are complex subvarieties of $X$. We can choose $\Gamma_{Z,s} = T^{-1}(\Gamma_s \times s')$ for $s' \in W(s)$ Then, for $u \in \Gamma(W(s), T^0{(s)}, u \in \Gamma(X_{W(s)}, T^0{(s)})$ a relevation of $u$ of type $(0,1)$ i.e. $d_f(u) = u$, and $u \in \Gamma(X_{W(s)}, F^{d-p+1}A^2d-2p+n+1(\nu D))_{d/s=0}$, $(\iota(\tilde{u})d\eta) \in \Gamma(X_{W(s)}, F^{d-p+1}A^2(\nu D))$ is $d_{X/S}$ exact. Hence,

$$d_u < \nu_Z, [\eta] > < s > = d_u(s') \to \int_{\Gamma_{Z,s}} \pi_X^* \eta \cap \pi_{(p1)}^* \Omega_{\square}^n(s) = \int_{\Gamma_{Z,s}} \pi_X^* \iota(\tilde{u})d\eta \cap \pi_{(p1)}^* \Omega_{\square}^n = 0$$

since the form $(\iota(\tilde{u})d\eta)|_X \in \Gamma(X_s, F^{d-p+1}A^2(\nu D))$ is exact and in $F^{d-p+1}$. This proves that $\nu_Z$ is holomorphic. Now let $\omega \in \Gamma(X_{W(s)}, F^{d-p+2}A^2d-2p+n+1(\nu D))_{d/s=0}$ and $u \in \Gamma(W(s), T_S)$. Then $(\iota(\tilde{u})d\omega) \in \Gamma(X_{W(s)}, F^{d-p+2}A^2d-2p+n+1(\nu D))_{d/s=0}$ and $\nabla u[\omega] = [\iota(\tilde{u})d\omega] \in \Gamma(W(s), F^{d-p+1}H_S^{d-2p+n+1}(f_{X,D}))$.

Hence, by proposition 21 (i),

$$< \nabla u \tilde{\nu}_Z, [\omega] > = < \nu_Z, \nabla u[\omega] > < s > - d_u < \nu_Z, [\omega] > < s > = d_u(s') \to \int \pi_X^* \omega \cap \pi_{(p1)}^* \Omega_{\square}^n(s) - \int \pi_X^* \iota(\tilde{u})d\omega \cap \pi_{(p1)}^* \Omega_{\square}^n$$

$$= \int \pi_X^* \iota(\tilde{u})d\omega \cap \pi_{(p1)}^* \Omega_{\square}^n = 0.$$

This proves that $\nu_Z$ is horizontal. 

\[ \square \]

**Definition 14. The map**

$$AJ_{f_U} : Z^p(U, n)_{\text{hom}}^{pr,X/S} \to CH^p(U, n)^{\text{hom}}/S \to NF_S(f_U) \subset \Gamma(S, J^{n, 2p+n-1}(f_U)), \quad Z \mapsto \nu_Z = [\tilde{\nu}_Z]$$

is the relative higher Abel Jacobi map. The image of $AJ(f_U)$ lies in the subspace $NF_S(f_U) \subset \Gamma(S, J^{n, 2p+n-1}(f_U))$ by theorem 6.

**Proposition 21.** (i) We have decompositions

$$m_{Uk} = \left(m_{Uk}^0, \cdots, m_{Uk}^k\right) : H^k(U, \mathbb{Z}) \to \bigoplus_{r=0}^k H^{k-r}(S, H_Z^2(f_U))$$

$$m_{(X,D)k} = \left(m_{(X,D)k}^0, \cdots, m_{(X,D)k}^k\right) : H^k(X, D, \mathbb{Z}) \to \bigoplus_{r=0}^k H^{k-r}(S, H_Z^2(f_{X,D}))$$
(ii) For $Z \in \mathcal{Z}^p(U,n)_{\theta=0}^{pr/X \hom/S}$, we have $[\nu_Z] = m_{1/(2p-n)}([T_Z]) \in H^1(S,\mathbb{Z}^{2p-n-1}(f_U))$

Proof. (i): This follows from the fact that the morphisms $f : X \to S$ and $f_{Dj} : X \to S$ are smooth projectif. Indeed, assume for simplicity that $D = D_1 \subset X$ is smooth. Then we have decompositions

$$
\begin{align*}
Rf_*Z &\to Rf_{D1,*}Z &\to Rf_{D,*}Z[-1] &\to Rf_*Z[1] \\
m_X &\downarrow & m_U &\downarrow m_D & m_X[1]
\end{align*}
$$

$$
\oplus_{r=0}^{2d} H^r_S(f)[r] \oplus_{r=0}^{2d} H^r_D(f_U)[r] \oplus_{r=0}^{2d} H^r_D(f)[r-1] \oplus_{r=0}^{2d} H^r_D(f)[r-1]
$$

and the map $m_{U1}$ is the one induced by $m_X$ and $m_D$. Taking hypercohomology gives the decompositions featuring in the commutative diagram whose rows are long exact sequence:

$$
\begin{align*}
\cdots &\to H^k(X,\mathbb{Z}) &\to H^k(U,\mathbb{Z}) &\to H^k(D,\mathbb{Z}) &\to \cdots \\
&\downarrow m_{Xk} & m_{Un} & m_{Dk} & m_{X1}
\end{align*}
$$

$$
\cdots \to \oplus_{r=0}^{k-r} H^r(S,\mathbb{Z}^r_S(f)) \to \oplus_{r=0}^{k-r} H^r(S,\mathbb{Z}^r_D(f)) \to \oplus_{r=0}^{k-r} H^r(S,\mathbb{Z}^r_D(f)) \to \cdots
$$

The map $m_{(X,D)}$ is defined similarly: we have decompositions

$$
\begin{align*}
Rf_*Z &\to Rf_{D1,*}Z &\to Rf_{X,D,*}Z[-1] &\to Rf_*Z[1] \\
m_X &\downarrow & m_{XD} &\downarrow m_{X,D}[1]
\end{align*}
$$

$$
\oplus_{r=0}^{2d} H^r_S(f)[r] \oplus_{r=0}^{2d} H^r_D(f_U)[r] \oplus_{r=0}^{2d} H^r_D(f)[r-1] \oplus_{r=0}^{2d} H^r_D(f)[r-1]
$$

and the map $m_{(X,D)}$ is the one induced by $m_X$ and $m_D$.

(ii): Standard.

Let $\mathcal{Z}^p(U,n)_{\theta=0}^{pr/X \hom/S}$. Denote again by $[\Omega_Z] = [\Omega_Z^{p,n}] \in H^{p-n}(X,\Omega_X^p(\log D))$ its class, recall that $\Omega_Z = \Omega_Z^{p,n}$ is of type $(p,p-n)$. Denote again by $[\Omega_Z] \in \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D)))$ its image by the canonical map $H^{p-n}(X,\Omega_X^p(\log D)) \to \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D)))$. Since $[\eta(\Omega_Z)] = 0$,

$$
[\Omega_Z] \in \ker(\Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))) \to \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))) = \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))).
$$

Denote by

- $[\Omega_Z/L^2] \ker(\Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))/L^2)) \to \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D)))$, the image of $[\Omega_Z]$ by the projection $\Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))/L^2)$.)

- $[\Omega_Z]/L^2 \in \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D)))$, the image of $[\Omega_Z]$ by the projection $\Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D)) \to \Gamma(S,\mathcal{S},R^{p-n} f_*(\Omega_X^p(\log D))/L^2)$.

We have the following commutative diagram of sheaves on $S_{en}$:

$$
\begin{align*}
\begin{array}{c}
\Omega_S \otimes \mathcal{H}_{S}^{p-n}(f_U)/\text{Im}(\nabla) \\
\end{array}
\xrightarrow{\psi_1} \xrightarrow{\psi_2}
\begin{array}{c}
\ker (R^{p-n} f_*(\Omega_X^p(\log D))/L^2) \to R^{p-n} f_*(\Omega_X^p(\log D))
\end{array}
\end{align*}
$$

(46)

where

- $\psi_1 : \Omega_S \otimes \mathcal{H}_{S}^{p-n}(f_U)/\text{Im}(\nabla) \to R^{p-n} f_*(\Omega_X^p(\log D))$ is the isomorphism induced by
– the morphism of sheaves on $S^{an}$ $r^\vee : Rp^n f_* \mathcal{O}_{\mathcal{X}}(log D) \to Rp^n f_*(\mathcal{O}_{\mathcal{X}}(log D)/L^2)$ (induced in relative cohomology by the morphism of sheaves on $X^{an}$ $r^\vee : \mathcal{O}_X(\mathcal{O}) \to \mathcal{O}_X(\mathcal{O})/L^2$),

– the isomorphism of sheaves on $S^{an}$ $\phi^{p,p} : Rp^n f_* \mathcal{O}_{\mathcal{X}}(log D) \sim \mathcal{O}_S \otimes \mathcal{H}^{p-1,p-n}_{S}(f_U)$ (induced in $f$ direct image cohomology by the isomorphism of complexes of sheaves on $X^{an}$ $\phi^{p,p,*} : \mathcal{A}_X^p(\mathcal{O}) \sim \mathcal{A}_X^{p-1,p-n}(f_U) \otimes \mathcal{O}_S$, c.f. proposition 11 and remark 1).

- $\psi^0_L : Gr_1^1 Rp^n f_* \mathcal{O}_{\mathcal{X}}(log D) = E_1^{0,p-n} \to Rp^n f_*(\mathcal{O}_{\mathcal{X}}(log D)) = E_1^{1,p-n}$ is the inclusion of sheaves on $S^{an}$ induced by the spectral sequence associated to the complex $(\mathcal{O}_{\mathcal{X}}(log D),L)$ : for degree reason no arrow $d_r, r \geq 2$ can lead to $E_r^{1,p-n}$. We have $\psi^0_L([\Omega_Z/L^2]) = [\Omega_Z]/L^2$. 

- $\psi^1_L = r^{\vee-1} \circ \psi^0_L$ is the inclusion given by composition.

The infinitesimal invariant associated to the class $[\Omega_Z] \in H^{p,p-n}(U,\mathbb{C})$ is

$$\delta[\Omega_Z] := \psi^1_L([\Omega_Z]/L^2) = r^{\vee-1}([\Omega_Z]/L^2) \in \Gamma(S,\mathcal{O}_S \otimes \mathcal{H}^{p-1,p-n}_{S}(f_U)/\text{Im}(\nabla))$$

**Lemma 1.** Let $Z = \sum_i n_i Z_i \in \mathbb{Z}^n(U,n)^{pr/X hom/S}$ such that $\pi_X(Z_i) \subset X$ is a local complete intersection for all $i$. Then, for $s \in S$ and $\gamma \in \Gamma(W(s), R^{d-p+n} f_! L^{d-s-1} \mathcal{O}^{d-s-p}_X(\text{null } D))$, 

$$< [\Omega_Z/L^2], \gamma >_{\text{ev}_X} (s) = < [\Omega_Z/L^2](s), \gamma(s) >_{\text{ev}_X(s)} = \sum_i \int_{Z_i} \pi^{*}_s \tilde{\gamma}(s)^{N_i}_s \wedge \pi_{(p-1)} \Omega^{\partial n},$$

where,

- $[\Omega_Z/L^2](s) \in O_s \otimes \mathcal{O}_{S,s}(R^{p-n} f_!(\mathcal{O}_{\mathcal{X}}(log D)/L^2))_s = H^{p-n}(X_s, (\mathcal{O}_{\mathcal{X}}(log D)/L^2))_{X,s}$

- $\gamma(s) \in O_s \otimes \mathcal{O}_{S,s}(R^{d-p+n} f_! L^{d-s-1} \mathcal{O}^{d-s-p}_X(\text{null } D))_s = H^{d-p+n}(X_s, (L^{d-s-1} \mathcal{O}^{d-s-p}_X(\text{null } D)))_{X,s}$

- $\tilde{\gamma}(s)^{N_i}_s = \pi^{*}_s (\tilde{\gamma}_s) \in H^{d-p+n}(\pi_X(Z_i), L^{d-s-1} \mathcal{O}^{d-s-p}_Z(\text{null } D))$

are the evaluation in $s$ of the respective sheaves on $S^{an}$ of $O_S$ module, and $\tilde{\gamma}(s) \in \Gamma(X_s, \mathcal{O}^{d-s-p}_X(\text{null } D))_{X,s} \otimes O_{X,s}$, $A^{0,d-p+n}(\mathcal{O})_{\partial \delta = 0}$ is a closed form such that $[\tilde{\gamma}(s)] = [\gamma(s)]$.

**Proof.** We can assume that $\pi_X|Z_i : Z_i \to \pi_X(Z_i)$ is generically finite, otherwise $\Omega_Z = 0$. It is then a straightforward generalization of the description given in 8 section 19.2.2. and the remark that the description is still correct in the case the $\pi_X(Z_i)$ are not smooth but only local complete intersection in $X$ : The class $[\Omega_Z/L^2](s) \in H^{d-p+n}(X_s, L^{d-s-1} \mathcal{O}^{d-s-p}_X(\text{null } D))_{X,s}$ is given by the composite

$$H^{d-p+n}(X_s, L^{d-s-1} \mathcal{O}^{d-s-p}_X(\text{null } D))_{X,s} \to H^{d-p+n}(X_s, \mathcal{O}^{d-s-p}_X(\text{null } D))_{X,s} \to H^{d-p+n}(\pi_X(Z_i_s), \mathcal{O}^{d-s-p}_{\pi_X(Z_i_s)}(\text{null } D \cap \pi_X(Z_i_s)))$$

Note that $\dim \pi_X(Z) = \dim Z = d_X - p + n$ and $\dim \pi_X(Z_s) = \dim Z_s = d - p + n$.

We have then one of the main result of this paper :

**Theorem 7.** Let $Z = \sum_i n_i Z_i \in \mathbb{Z}^n(U,n)^{pr/X hom/S}$ such that $\pi_X(Z_i) \subset X$ is a local complete intersection for all $i$. Then $\delta[s] = \delta[\Omega_Z] \in \Gamma(S,\mathcal{O}_S \otimes \mathcal{H}^{p-1,p-n}_{S}(f_U)/\text{Im}(\nabla))$. 

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Proof. For simplicity of notation, denote by \( < \cdot, \cdot > = < \cdot, \cdot >_{\nu_f} \). By proposition \([20]\) (ii), we have to prove that for all \( s \in S \), and all \( \mu \otimes u \in \Gamma(W(s), \mathcal{H}^{d-p+1, d-p+n}_S(f_X, D)) \nabla = 0 \), where \( W(s) \subset S \) is an open neighborhood of \( s \) in \( S \),

\[
< (\delta \nu_Z)_{|W(s)}, \mu \otimes u > (s) = < (\delta [\Omega_Z])_{|W(s)}, \mu \otimes u > (s)
\]

So, let \( s \in S \), and \( \mu \otimes u \in \Gamma(W(s), \mathcal{H}^{d-p+1, d-p+n}_S(f_X, D)) \nabla = 0 \). Shrinking \( W(s) \) if necessary, there exist \( \eta \in \Gamma(W(s), f_* \mathcal{A}^{d-p+1, d-p+n}_X/(\text{null } D)) \nabla = 0 \) such that

\[
[\eta] = \mu \in \Gamma(W(s), \mathcal{H}^{d-p+1, d-p+n}_S(f_X, D))
\]

(see definition \([14]\)). By corollary \([14]\)(ii), there exist \( \tilde{\eta} \in \Gamma(W(s), f_* \mathcal{A}^{d-p+1, d-p+n}_X/(\text{null } D)) \nabla = 0 \) such that

\[
[\tilde{\eta}] = [\eta] \in \Gamma(W(s), \mathcal{H}^{d-p+1, d-p+n}_S(f_X, D)).
\]

By definition,

\[
\nabla_u[\tilde{\eta}] = (\tilde{\eta} \cdot \mu \cdot \eta) \in \Gamma(W(s), F^{d-p} \mathcal{H}^{2d-2p+n+1}_S(f_X, D)),
\]

with \( \iota(\tilde{u})d\tilde{\eta} \in \Gamma(W(s), f_* \mathcal{A}^{d-p+1, d-p+n}_X/(\text{null } D)) \nabla = 0 \). By hypothesis,

\[
\nabla_u[\mu] = \nabla_u[\tilde{\eta}] = \nabla_u[\tilde{\eta}] = \nabla_u(\mu) \in \Gamma(W(s), \mathcal{H}^{d-p+1, d-p+n}_S(f_X, D)),
\]

that is

\[
\nabla_u[\tilde{\eta}] = [\iota(\tilde{u})d\tilde{\eta}] \in \Gamma(W(s), F^{d-p+1} \mathcal{H}^{2d-2p+n+1}_S(f_X, D)).
\]

Thus, using again the \( E_1 \) degenerescence of \((f_* \mathcal{A}_X/(\text{null } D), F)\) (corollary \([14]\)(i)), there exist

- \( \alpha \in \Gamma(W(s), f_* F^{d-p} \mathcal{A}^{2d-2p+n+1}_X/(\text{null } D)) \nabla = 0 \),
- \( \beta \in \Gamma(W(s), f_* F^{d-p+1} \mathcal{A}^{2d-2p+n+1}_X/(\text{null } D)) \nabla = 0 \)

such that

\[
\iota(\tilde{u})d\tilde{\eta} = \alpha + d\alpha \in \Gamma(W(s), f_* F^{d-p} \mathcal{A}^{2d-2p+n+1}_X/(\text{null } D)) \nabla = 0.
\]

Let us now compute the first term \(< (\delta \nu_Z)_{|W(s)}, \mu \otimes u > (s) = < (\delta \nu_Z)(s), \mu(s) \otimes u(s) >_{\nu_f} \). We have

\[
< (\delta \nu_Z)_{|W(s)}, \mu \otimes u > (s) = < \nabla_u[\tilde{\nu}\omega], [\tilde{\eta}] > (s)
\]

\[
= d_u < [\tilde{\nu}\omega], [\tilde{\eta}] > (s) - < [\tilde{\nu}\omega], \nabla_u[\tilde{\eta}] > (s) \text{ by proposition } [20](i)
\]

\[
= d_u(s' \mapsto \int_{\Gamma_{Z,s}} \pi_X^* \tilde{\eta} \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1} - \int_{\Gamma_{Z,s}} \pi_X^* \beta \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1})
\]

We have \( d_u(s' \mapsto \int_{\Gamma_{Z,s}} \pi_X^* \tilde{\eta} \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1}) = \int_{\Gamma_{Z,s}} \iota(\tilde{u}(\square^n))d(\pi_X^* \tilde{\eta} \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1}) = \int_{\Gamma_{Z,s}} \pi_X^* \iota(\tilde{u})d\tilde{\eta} \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1} \]

where \( \tilde{u}(\square^n) \in \Gamma(X_{W(s)} \times \square^n, T_X \otimes \square^n) \) is a releved of \( \tilde{u} \) hence a releved of \( u \) for \( f(\square^n) : X \times \square^n \to S \), \( f(\square^n)(x,t) = f(x) \), since \( \iota(\tilde{u}(\square^n)) \pi^*_\nu \ell = 0 \) for all differential form \( \ell \in \Gamma((\square^n)^n, \mathcal{A}_{(p_1)}\rightarrow) \) (hence in particular \( \iota(\tilde{u}(\square^n)) \pi^*_\nu \ell = 0 \)). Hence,

\[
< (\delta \nu_Z)_{|W(s)}, \mu \otimes u > (s) = \int_{\Gamma_{Z,s}} \pi_X^* (\iota(\tilde{u})d\tilde{\eta} - \beta) \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1} = \int_{\Gamma_{Z,s}} \pi_X^* d\alpha - \pi^*_{(p_1)} \Omega^\alpha_{p_1}
\]

\[
= \sum_i n_i \int_{Z_i} \pi_X^* \alpha \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1} = \sum_i n_i \int_{Z_i} \pi_X^* \alpha^{d-p, d-p+n} \wedge \pi^*_{(p_1)} \Omega^\alpha_{p_1}
\]

where the third equality follows by Stoke formula and the last equality for type reason (\( \Omega_{Z,s} \) is of type \( (p, p-n) \)).
Let us compute the second term. Shrinking \( W(s) \subset S \) if necessary, there exist, by the exactness of the first row of the diagram of sheaves on \( S^{an} \),
\[
\gamma \in \Gamma(W(s), R^{d-p+n} f_\ast L^{d_s-1} \Omega_X^{d_X-p}(\text{null}D))
\]
such that \( r(\gamma) = \mu \otimes u \). By commutativity of this diagram \( \delta \),
\[
< \delta[\Omega_Z]|_{W(s)}, \mu \otimes u > =< \delta[\Omega_Z]|_{W(s)}, r(\gamma) > =< [\Omega_Z/L^2]|_{W(s)}, \gamma > f_{evX}
\]
Hence, by lemma \( \Box \)
\[
< \delta[\Omega_Z]|_{W(s)}, \mu \otimes u > (s) =< \delta[\Omega_Z|(s), \mu(s) \otimes u(s) >_{ev_X} = < \delta[\Omega_Z|(s), r(s)(\gamma(s)) >_{ev_X} = < [\Omega_Z/L^2](s), \gamma > f_{evX(s)} (48) = \sum_i n_i \int_{Z_i} \pi_i^\ast \gamma(s)^{\mathfrak{N}} \wedge \pi_p \gamma_{\Omega_D} (50)
\]
Hence, we have to find a form \( \xi \in \Gamma(X, L^{d_s-1} A_X^{d_X-p,d-p+n}(\text{null}D)|_{X_i}) \) \( \partial_X = 0 \) such that
\[
\rho (s)(\xi) = \mu(s) \otimes u(s) \in H^{d-p+n}(X, \Omega_X^{d-p}(\text{null}D)).
\]
Consider the form
\[
\chi := \eta^{d-p+1,d-p+n} \wedge f^\ast(u) \kappa + \alpha^{d-p,d-p+n} \wedge f^\ast \kappa \in \Gamma(W(s), L^{d_s+1} A_X^{d_X-p,d-p-n}(\text{null}D)).
\]
We have \( r(\chi) = \eta^{d-p+1,d-p+n} \otimes u \). Taking the component of type \( (d-p,d-p+n) \) in the relation
\[
(i(\bar{u})d\eta)|_{X_i} = \beta_{X_i} + d\alpha|_{X_i}
\]
which is the restriction of \( \Box \) to \( X_i \), we find that the form
\[
\xi := \chi|_{X_i} \in \Gamma(X_i, L^{d_s-1} A_X^{d_X-p,d-p-n}(\text{null}D)|_{X_i}) \partial_X = 0
\]
is closed. Moreover, since \( r(\chi) = \eta^{d-p+1,d-p+n} \otimes u \), we have
\[
r(\chi)(\xi) = r(\chi)|_{X_i} = \eta^{d-p+1,d-p+n} \otimes u.
\]
Hence, on cohomology \( r(\chi)(\xi) = [\eta|_{X_i}] \otimes u(s) = \mu(s) \otimes u(s) \). We have the desired form. Then \( \Box \) gives
\[
< \delta[\Omega_Z]|_{W(s)}, \mu \otimes u > (s) =< \delta[\Omega_Z]|_{s}, \mu(s) \otimes u(s) >_{ev_X} = < \delta[\Omega_Z]|_{s}, r(s)(\xi) >_{ev_X} = < [\Omega_Z/L^2](s), \xi > f_{evX(s)} = \sum_i n_i \int_{Z_i} \pi_i^\ast \omega^{d-p,d-p+n} \wedge \pi_p \omega_{\Omega_D},
\]
where the last equality follows again from the fact that \( \Omega_{Z_i} \in \Gamma(X, D_X^{d_X-p,d-p-n}(\text{null}D)) \) is of type \( (d-p,d-p+n) \).

Remark 8. Note that the form \( \omega^{d-p,d-p-n} \in \Gamma(X, A_X^{d_X-p,d-p-n}(\text{null}D)) \) is not \( \partial_X/S \) closed, hence not \( d_X/S \) closed since it is of single type \( (d-p,d-p+n) \), that is \( \omega_X|_{X_i} \) is not \( \partial_X \) and not \( d_X \) closed. But \( \pi^\ast_X \omega^{d-p,d-p-n} \in \tilde{Z}_{i,reg}/S \) closed, that is \( \pi^\ast_X \omega^{d-p,d-p-n} \in \tilde{Z}_{i,reg} \) closed, where, \( Z_{i,reg} \subset Z_i \) the smooth locus of \( Z_i \). Denote by \( \tilde{i}_Z : \tilde{Z}_i \rightarrow X \times (\mathbb{P}^1)^n \) the closed embedding. On the other side, the current \( \Omega_Z = \sum_i n_i \pi_X \ast i_{\tilde{Z}_i} \ast \omega^{Z_{i,reg}} \in \Gamma(X, D_X^{p-p-n}(\text{null}D)) \) is \( d_X/S \) closed, hence \( \tilde{Z}_{i,reg} \) closed since it is of single type \( (p,p-n) \), that is \( \Omega_{Z_{i,reg}} \) is \( d_{Z_{i,reg}} \) closed, since \( \partial \tilde{Z}_{i,reg} = 0 \). But the currents \( \Omega_{Z_{i,reg}}^{Z_{i,reg}} \in \Gamma(Z_i, D_{Z_{i,reg}}^{p-p-n}(\text{null}D \cap Z)) \) are not \( \partial \tilde{Z}_{i,reg}/S \) closed.

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4 Higher Abel Jacobi map for open complete intersection

Let $Y \in \text{PSmVar}(\mathbb{C})$ together with an embedding $Y \subset \mathbb{P}^N$. For $d, e \gg 0$, the morphisms of $\mathbb{C}$ vector spaces

- $\Gamma(\mathbb{P}^N, O(d)) \to \Gamma(Y, O_Y(d)) = S_d$,
- $\Gamma(\mathbb{P}^N, O(e)) \to \Gamma(Y, O_Y(e)) = S_e$, and
- $\Gamma(\mathbb{P}^N, O(d)) \to \Gamma(Z, O_Z(d)) = S_d$ for $Z \subset Y$ such that $Z \in \Gamma(Y, O_Y(e))$,

are surjective. Denote by $p_{d,e} : Y \times S_d \times S_e \to S_d \times S_e$ and $p_Y : Y \times S_d \times S_e \to Y$ the projections. Consider the commutative diagram of families of hypersurface sections of degree $d$ and $e$, whose squares are cartesians :

$$
\begin{array}{ccc}
D & \xrightarrow{k_D} & Z \\
\downarrow{\iota_D} & & \downarrow{\iota} \\
\mathcal{X} & \xrightarrow{i_X} & Y \times S_d \times S_e \\
\downarrow{j_Y} & & \downarrow{p_{d,e}} \\
U & \xrightarrow{i_U} & Y \times S_d \times S_e \\
\end{array}
$$

Note that $\mathcal{X}, Z, D \in \text{PSmVar}(\mathbb{C})$, since $p_Y|_X : \mathcal{X} \to Y$, $p_Y|_Z : Z \to Y$, $p_Y|_D : D \to Y$ are projective bundles and $Y$ is smooth.

For $0 \in S_e$, denote by $p_{d,e}^0 = p_{Y|Y \times S_d \times 0} : Y \times S_d \times 0 \to Y$ and $p_{Y\setminus Z_0} = p_{Y|(Y\setminus Z_0) \times S_d \times 0} : (Y\setminus Z_0) \times S_d \times 0 \to Y\setminus Z_0$, $p_{d,e}^0 = p_{d,e}|_{Y \times S_d \times 0} : Y \times S_d \times 0 \to S_d$, the projections, and consider the pullback of the diagram (51) :

$$
\begin{array}{ccc}
D & \xrightarrow{k_D} & Z \\
\downarrow{\iota_D} & & \downarrow{\iota} \\
\mathcal{X} & \xrightarrow{i_X} & Y \times S_d \times S_e \\
\downarrow{j_Y} & & \downarrow{p_{d,e}} \\
U & \xrightarrow{i_U} & Y \times S_d \times S_e \\
\end{array}
$$

where $Z_0 = p_{Y}(Z_{S_d \times 0}) \subset Y$ so that we have $Z_{S_d \times 0} = Z_0 \times S_d$. Then $Y \setminus Z_0$ is an affine variety. For $s \in S_d$, consider the correspondence $\Delta(U_s) \subset (Y\setminus Z_0) \times (Y\setminus Z_0)$ which is the diagonal of $U_s$. Since the projection $\Delta(U_s) \to (Y\setminus Z_0)$ is proper there is a well defined action of this correspondence on cohomology. We denote by

$$
H^k(Y\setminus Z_0, \mathbb{C})^0 := \ker(\Delta(U_s)*) \subset H^k(Y\setminus Z_0, \mathbb{C})
$$

the primitive cohomology of $Y\setminus Z_0$, that is kernel of this action. For $s \in S_d$ such that $U_s \subset Y\setminus Z_0$ is smooth, we have the equality (by Poincare duality for $U_s$)

$$
H^k(Y\setminus Z_0, \mathbb{C})^0 := \ker(\Delta(U_s)*) = \ker(i_{U_s}^*) \subset H^k(Y\setminus Z_0, \mathbb{C}),
$$

that is the primitive cohomology coincide with the kernel of pullback by the inclusion of an ample smooth hypersurface section. Since $Y\setminus Z_0$ is affine, $H^k(Y\setminus Z_0, \mathbb{C}) = 0$ for $k \geq d_Y + 1$ and $H^{d_Y}(Y\setminus Z_0, \mathbb{C})^0 = H^{d_Y}(Y\setminus Z_0, \mathbb{C})$. 

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For a morphism $T \to S_d$, we consider the pullback of the diagram \((55)\):

\[
\begin{array}{ccc}
0 & \longrightarrow & \Omega_{Y \times T}^{p} \\
\downarrow i_{X_T} & & \downarrow i_{X_T}
\end{array} \quad \begin{array}{ccc}
\Omega_{Y \times T}^{p} & \longrightarrow & \Omega_{Y \times T}^{p}(\log(Z_0 \times T)) \\
\downarrow & & \downarrow
\end{array} \quad \begin{array}{ccc}
i_{(Z_0 \times T)} \circ \Omega_{Z_0 \times T}^{p-1} & \longrightarrow & 0 \\
\downarrow k_{D_T} & & \downarrow k_{D_T}
\end{array}
\]

where $X_T = X \times S_d, T, U_T = U \times S_d, D_T = D \times S_d, T$.

We now give a version of Nori connectivity theorem for families of ample open hypersurfaces of $Y \in \text{PSmVar}(\mathbb{C})$.

**Theorem 8.** Assume $d_Y \geq 4$ Let $0 \in S_d$ sufficiently general and $S \subset S_d$ the open subset over which such that the morphisms $f^0 : X \to S_d$ and $f_D^0 : D \to S_d$ are smooth projective. Then, if $d,e >> 0$, for all smooth morphism $T \to S_d$ and all $0 \leq k \leq d_Y$, 

(i) $\iota_{X_T}^* : H^{k-p}(Y \times T, \Omega_{X_T}^{p}(\log(Z_0 \times T))) \sim H^{k-p}(X_T, \Omega_{X_T}^{p}(\log D_T))$ is an isomorphism,

(ii) $\iota_{U_T}^* : H^{k}(Y \setminus Z_0) \times T, C) \sim H^{k}(U_T, C)$ is an isomorphism of mixed hodge structure.

**Proof.** (i): Consider the commutative diagram of sheaves on $Y \times T$:

\[
\cdots \longrightarrow H^{k-p}(Y \times T, \Omega_{Y \times T}^{p}) \longrightarrow H^{k-p}(Y \times T, \Omega_{Y \times T}^{p}(\log(Z_0 \times T))) \longrightarrow H^{k-p}(Z_0 \times T, \Omega_{Z_0 \times T}^{p-1}) \longrightarrow \cdots
\]

\[
\cdots \longrightarrow H^{k-p}(X_T, \Omega_{X_T}^{p}) \longrightarrow H^{k-p}(X_T, \Omega_{X_T}^{p}(\log(D_T))) \longrightarrow H^{k-p}(D_T, \Omega_{D_T}^{p-1}) \longrightarrow \cdots
\]

Now,

- by Nori connectivity theorem for the pair $(Y \times S_d, X)$, since $d_Y \geq 4$ (hence $d_Y < 2d_Y - 2$), $d_Y >> 0$ and $T \to S_d$ is smooth, the map $\iota_{X_T}^* : H^{k-p}(Y \times T, \Omega_{Y \times T}^{p}) \sim H^{k-p}(X_T, \Omega_{X_T}^{p})$ is an isomorphism for all $0 \leq k \leq d_Y$,

- by Nori connectivity theorem for the pair $(Z_0 \times S_d, D)$, since $d_Y \geq 4$ (hence $d_Y - 1 < 2d_Y - 4$), $e >> 0$, and $T \to S_d$ is smooth, the map $k_{D_T}^* : H^{k-p}(Z_0 \times T, \Omega_{Z_0 \times T}^{p-1}) \sim H^{k-p}(D_T, \Omega_{D_T}^{p-1})$ is an isomorphism for all $0 \leq k \leq d_Y$.

Hence, by the diagramm \((57)\) $\iota_{X_T}^* : H^{k-p}(Y \times T, \Omega_{Y \times T}^{p}(\log(Z_0 \times T))) \sim H^{k-p}(X_T, \Omega_{X_T}^{p}(\log(D_T)))$ is an isomorphism for all $0 \leq k \leq d_Y$. 

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(ii): It follows from (i). We can also prove (ii) directly. Indeed, we have the commutative diagram whose rows are long exact sequences:

\[
\begin{array}{ccccccccc}
\cdots & H^k(Y \times T, \mathbb{C}) & \xrightarrow{\delta^j_{Y \times Z_0}} & H^k((Y \setminus Z_0) \times T, \mathbb{C}) & \xrightarrow{\text{Res}} & H^{k-1}(Z_0 \times T, \mathbb{C}) & \xrightarrow{i^*_T} & \cdots \\
\downarrow{i^*_Y} & \downarrow{i^*_Y} & \downarrow{i^*_Y} & \downarrow{i^*_Y} & \downarrow{i^*_Y} & \downarrow{i^*_Y} & \downarrow{k^*_T} & \cdots \\
\cdots & H^{dv}(X_T, \mathbb{C}) & \xrightarrow{\delta^j_{U_T}} & H^k(U_T, \mathbb{C}) & \xrightarrow{\text{Res}} & H^{k-1}(D_T, \mathbb{C}) & \xrightarrow{i^*_{D_T}} & \cdots
\end{array}
\]

Now,

- by Nori connectivity theorem for the pair \((Y \times S_d, X)\), since \(d_Y \geq 4\) (hence \(d_Y < 2d_Y - 2\)), \(d > 0\) and \(T \to S_d\) is smooth, the map \(i^*_X : H^k(Y \times T, \mathbb{C}) \xrightarrow{\sim} H^k(X_T, \mathbb{C})\) is an isomorphism of mixed hodge structures for all \(0 \leq k \leq d_Y\),

- by Nori connectivity theorem for the pair \((Z_0 \times S_d, D)\), since \(d_Y \geq 4\) (hence \(d_Y - 1 < 2d_Y - 4\)), \(e \gg 0\), and \(T \to S_d\) is smooth, the map \(k^*_D : H^{k-1}(Z_0 \times T, \mathbb{C}) \xrightarrow{\sim} H^{k-1}(D_T, \mathbb{C})\) is an isomorphism of mixed hodge structures for all \(0 \leq k \leq d_Y - 1\).

Hence, by the diagram (58) \(i^*_U : H^k((Y \setminus Z_0) \times T, \mathbb{C}) \xrightarrow{\sim} H^k(U_T, \mathbb{C})\) an isomorphism of mixed hodge structures for all \(0 \leq k \leq d_Y\).

\[\square\]

A non vanishing criterion for an ample hypersurface of \(Y \setminus Z_0\)

We will prove theorem \[\square\] We begin by a lemma:

**Lemma 2.** Let \(0 \in S_e\) sufficiently general and \(S \subset S_d\) the open subset over which such that the morphisms \(f^0 : X \to S_d\) and \(f^0_D : D \to S_d\) are smooth projective. The map of filtered complexes of sheaves on \((Y \times S_d)^{an}\)

\[i^*_X : (\Omega^*_{Y \times S_d}(\log(Z_0 \times S_d)), L) \to (i_X \Omega^*_X(\log D), L)\]

induces a surjection of sheaves on \(S^{an}\)

\[i^*_s : L^2 R^{dv,-p} f^0 \Omega^p_{Y \times S}(\log(Z_0 \times S)) \to L^2 R^{dv,-p} f^0 \Omega^p_{X_S}(\log D)\]

**Proof.** By Lefschetz theorem, the restriction morphism \(i^*_Y : H^k(Y \setminus Z_0, \mathbb{C}) \to H^k(U_s, \mathbb{C})\) is an isomorphism for \(0 \leq k < d_Y - 1\) and is injective for \(k = d_Y - 1\) \((Y \setminus (Z_0 \cup U_s)\) is a smooth affine variety). Moreover it is a morphism of mixed hodge structures. Hence, since the Frölicher filtration is \(E_1\) degenerate,

\[i^*_s : H^{1,m}_S(pY \setminus Z_0) = R^m f^0 \Omega^1_{Y \times S}(\log(Z_0 \times S)) = E^1 \to H^{1,m}_S(pY \setminus Z_0) = R^m f^0 \Omega^1_{X_S}(\log D) = E^1\]

is an isomorphism for \(0 \leq l + m < d_Y - 1\) and is injective for \(l + m = d_Y - 1\).

\[\square\]

**Theorem 9.** Assume \(d_Y \geq 4\). Let \(0 \in S_e\) sufficiently general and \(S \subset S_d\) the open subset over which such that the morphisms \(f^0 : X \to S_d\) and \(f^0_D : D \to S_d\) are smooth projective. Let \(Z \in \mathcal{Z}^p(Y \setminus Z_0, 2p - d_Y)_{\text{pr/Y}}\) such that \(\Omega_Z \neq 0 \in H^{d_Y}(Y \setminus Z_0, \mathbb{C})\). Then for \(s \in S\) general, \(AJ_{U_s}(Z) := [R^0_{Z_s}] \neq 0 \in J_{p,d_Y-1}(U_s)\).

**Proof.** Consider the cycle \(Z = i^*_U f^0 \times Y \setminus Z_0 \in \mathcal{Z}^p(U, 2p - d_Y)_{\text{pr/X, hom/S}}\). We want to show that

\[\nu_Z \neq 0 \in \Gamma(S, J_{p,d_Y-1}(f^0_U)) = \Gamma(S, D^*_S(f^{d_Y-p} H^{d_Y-1}_S(f^0_U)) H_{d_Y-1,Z}(f^0_U)).\]

Since for all \(s \in S\),

\[\nu^*_Z(s) = [R_{Z_s}] = e v_{X_s}(AJ_{U_s}(Z_s)) \in F^{d_Y-p} H^{d_Y-1}_S(X_s, D_s, \mathbb{C})^\vee / H_{d_Y-1}(X_s, D_s, Z),\]

this will give the result because then \(V(\nu_Z) \subset S\) will be a proper analytic subset, even a proper algebraic subset by a result of Brossman, Pearlstein and Schnell. By theorem \[\square\]

\[\delta \mu_Z = \delta(\Omega_Z) \in \Gamma(S, H^{p-1,d_Y-p}(f^0_U)^{\otimes S} \Omega_S / \text{Im } \nabla).\]
Then for a general point $i \in H$, it suffice to show that $\delta|_{Z} \neq 0$. Since, by the commutativity of $\delta|_{Z}$, we have, for all $s \in S$, $W(s) \subset S$ an open neighborhood of $s$ in $S$ and $\mu \in \Gamma(W(s), H_{\mathcal{O}_S}^{d^{2}+p} \cap \mathcal{O}_S T_S)^{\tilde{\gamma}} = 0$,

$$< \delta|_{Z}|_{W(s), \mu > c_{W} = < \Omega^2/Z, \mu >_{f_{L,c_{W}}}$$

where $\gamma \in \Gamma(W(s), R^{n-1} F^0 \mathcal{O}_X^{d^{2}+p} \cap \mathcal{O}_S (\mu D))$ is such that $r(\gamma) = \mu$, it suffice to show that $\Omega^2/Z, \mu >_{f_{L,c_{W}}}$. Hence, the map from the Leray spectral sequence of associated to the filtered complex $(\Omega^p_X (\log D), L)$ (c.f. diagramm (46))

$$\psi^*_L : \Gamma(S, (L^1 R^{d^{2}+p} f^0 \mathcal{O}_X (\log D))/L^2) = E_{\infty}^{d^{2}+p} \rightarrow \Gamma(S, R^{d^{2}+p} f^0 \mathcal{O}_X (\log D))/L^2)$$

is injective, it suffice to show that $\Omega^2/Z, \mu >_{f_{L,c_{W}}}$. So, suppose that $\Omega^2/Z, \mu >_{f_{L,c_{W}}}$. By the lemma 2 since $S \subset S_d$ is affine, there exist

$$\alpha \in \Gamma(S, L^1 R^{d^{2}+p} f^0_0 \mathcal{O}_Y \cdot S (\log (Z_0 \times S)))$$

such that $i^*_X \alpha = [\Omega^2/Z]$. Since $S$ is affine, the canonical maps

- $L^2 H^{d^{2}+p} (Y \times S, \mathcal{O}_Y \cdot S (\log (Z_0 \times S))) \xrightarrow{\sim} \Gamma(S, L^2 R^{d^{2}+p} f^0_0 \mathcal{O}_X (\log (Z_0 \times S)))$
- $L^2 H^{d^{2}+p} (X_S, \mathcal{O}_X (\log D)) \xrightarrow{\sim} \Gamma(S, L^2 R^{d^{2}+p} f^0 \mathcal{O}_X (\log D))$

are isomorphisms. Hence, seeing $\alpha \in L^2 H^{d^{2}+p} (Y \times S, \mathcal{O}_Y \cdot S (\log (Z_0 \times S)))$

$$i^*_X \alpha = [\Omega^2/Z] = i^*_X p^*_Y \mathcal{O}_Z \in H^{d^{2}+p} (X_S, \mathcal{O}_X (\log D)),$n

that is $i^*_X \alpha = p^*_Y [\Omega^2/Z] = 0 \in H^{d^{2}+p} (X_S, \mathcal{O}_X (\log D))$. But since $p^*_Y [\Omega^2/Z] \notin L^2 H^{d^{2}+p} (Y \times S, \mathcal{O}_Y \cdot S (\log (Z_0 \times S)))$, $\alpha = p^*_Y [\Omega^2/Z] \neq 0 \in H^{d^{2}+p} (Y \times S, \mathcal{O}_X (\log (Z_0 \times S)))$. But by the theorem 8 (i), since $S \rightarrow S_d$ is smooth,

$$i^*_X : H^{d^{2}+p} (Y \times S, \mathcal{O}_Y \cdot S (\log (Z_0 \times S))) \xrightarrow{\sim} H^{d^{2}+p} (X_S, \mathcal{O}_X \cdot S (\log D))$$

is an isomorphism. We get a contradiction. $\square$

**The image of the Abel Jacobi map of an ample hypersurface of $Y \setminus Z_0$**

**Theorem 10.** Assume $d^{2} > 4$. Let $0 \in S$, sufficiently general and $S \subset S_d$ the open subset over which such that the morphisms $f^0 : X \rightarrow S_d$ and $f^0 : D \rightarrow S_d$ are smooth projective. Consider the commutative diagram

$$\begin{array}{ccc}
\text{CH}^2(Y \setminus Z_0, 2p - d_Y, \mathbb{Q}) & \xrightarrow{i^*_U} & \text{CH}^2(U_s, 2p - d_Y, \mathbb{Q}) \\
\downarrow r(Y, Z_0) & & \downarrow r(X, D_s) \\
H^{d^{2}y}_{dy}(Y, Z_0, \mathbb{Q}) & \cong & H^{d^{2}y}_{dy}(X, D_s, \mathbb{Q})/j^{p, dy}(Y \setminus Z_0)\mathbb{Q}
\end{array}$$

Then for a general point $s \in S$, $\text{Im}(r(Y, Z_0)) = \text{Im}(r(X, D_s) \circ i^*_U)$.

**Proof.** We follow 2. Let $s \in S$ a general point and $Z_s = \sum_{i=1}^{k} n_i Z_{s_i} \in Z^P(U_s, n, \mathbb{Q})|_{\partial = 0}/X_s$. Then, there exists a branched covering $h : T \rightarrow S_d$, $t \in t^{-1}(s)$, and $Z \in Z^P(T, n, \mathbb{Q})$ such that

- $h^{-1}(s) = \{t, t_1, \ldots , t_r\} \subset T_0$, where $T_0 \subset T$ is the Zariski open subset such that $h : T_0 \rightarrow S_d$ is smooth,
- $\partial Z \in i_{D_t} Z^P(D_T, n - 1, \mathbb{Q})$,
- $Z \cdot (X_t \times \{t\}) = Z_s$,
with $X_T = X \times_{S_d} T$ and $D_T = D \times_{S_d} T$. For this, consider, for each $1 \leq i \leq k$, the relative Hilbert scheme $h_i : H_i \to S_d$ of $f(\mathbb{P}^n) : \mathcal{X} \times \mathbb{P}^n \to S_d$, such that $\bar{Z}_i$ belongs to and $h : H \to H_1 \times_{S_d} \cdots \times_{S_d} H_k \to S_d$ defining the condition $\partial^i G_{h_i} \in \iota_D^p \mathcal{Z}_{n,2}(P^1(D,n))$. Note that $H \to S_d$ is surjective since there always exist such a cycle in a fiber and $s \in S_d$ is general. Take a multisection $T \to H \to S_d$ of such that $h^{-1}(s) \cap T \cap \text{sing}(h) = \emptyset$, where $\text{sing}(h)$ is the singular locus of $h$, and such that the intersection $h^{-1}(s) \cap T \subset H$ is transversal.

Denote by $C = \{t, t_1, \ldots, t_r\} \subset h^{-1}(s)$, with $1 \leq r' \leq r$, the subset such that $Z_{t_i} \subset X_s$ is not included in $D_s$. By theorem $[\text{S}(\text{ii})]$, \[ i_{U_T}^* : H^3_T \otimes (X \setminus Z_0) \times T, \mathbb{C} \to H^3_T (U_T, \mathbb{C}) \]
is an isomorphism and in particular surjective. Hence, there exist $\gamma \in H^3_T \otimes (X \setminus Z_0) \times T, \mathbb{C}$ such that $\mathcal{R}(\mathbb{C} \otimes (X \setminus Z_0) \times T, \mathbb{C}) = i_{U_T}^* \gamma$. Hence, for $t_i, t_j \in C$,

$$\mathcal{R}(X_s, D_s)(Z_{t_i}) - \mathcal{R}(X_s, D_s)(Z_{t_j}) \in i_{U_T}^* J^{p, d_Y - 1}(X \setminus Z_0)$$

This gives the equality

$$\mathcal{R}(X_s, D_s)(\sum_{t_i \in C} Z_{t_i}) = \mathcal{R}(X_s, D_s)(\sum_{t_i \in h^{-1}(s)} Z_{t_i}) = r' \mathcal{R}(X_s, D_s)(\bar{Z}_s) \quad \text{(60)}$$

Consider now a pencil $\Lambda_d \subset S_d$ such that $s \in \Lambda_d$, and $\bar{T} = h^{-1}(\Lambda_d) \subset T$.

- In $Y \times \bar{T}$ we have $(X_s \times \bar{T}, X_{\bar{T}} = \sum_{i=1}^r X_s \times \{t_i\} + B(\Lambda_d) \times \bar{T}$.
- In $(Y \setminus Z_0) \times \bar{T}$ we have $(U_s \times \bar{T}, U_{\bar{T}} = \sum_{i=1}^r U_s \times \{t_i\} + (B(\Lambda_d) \cap (Y \setminus Z_0)) \times \bar{T}$

where $X_{\bar{T}} = X \times_{S_d} \bar{T}$, $U_{\bar{T}} = U \times_{S_d} \bar{T}$ and $B(\Lambda_d) = X_s \cap X_{s'} \subset Y, s' \neq s \in S_d$ is the base locus of the pencil. Consider

- $Z_{\bar{T}} = Z \cdot X_{\bar{T}} \in Z^p(X_{\bar{T}}, 2p - d_Y, \mathbb{Q})$ and
- $Z_{\bar{T} \mid U_{\bar{T}}} := j_{U_{\bar{T}}}^* Z_{\bar{T}} = (j_{U_{\bar{T}}}^* Z) U_{\bar{T}} \in Z^p(U_{\bar{T}}, 2p - d_Y, \mathbb{Q})$ its restriction.

We may assume, adding a boundary if necessary, that \[(j_{U_{\bar{T}}}^* Z) \cap ((B(\Lambda_d) \cap (Y \setminus Z_0)) \times \bar{T}) := (j_{U_{\bar{T}}}^* Z)(U_s \times \bar{T}) \cap (U_{s'} \times \bar{T}) \in Z^{p+3}(Y \setminus Z_0) \times \bar{T}, 2p - d_Y), \]
that is the intersection is a Bloch cycle of the appropriate codimension. By the projection formula, we have, denoting $p_{Y \setminus Z_0}^\bar{T} : (Y \setminus Z_0) \times \bar{T} \to Y \setminus Z_0$ the projection (which is proper since $\bar{T}$ is projective),

$$(p_{Y \setminus Z_0}^\bar{T}(Z_{\bar{T} \mid U_{\bar{T}}})) U_s = p_{Y \setminus Z_0}^\bar{T}(j_{U_{\bar{T}}}^* Z)(U_s \times \bar{T})) = \sum_{i \in C} j_{U_s}^* Z_{t_i} \cup (p_{Y \setminus Z_0}^\bar{T}(j_{U_{\bar{T}}}^* Z)(U_{s'} \times \bar{T})) \cdot U_s \quad \text{(61)}$$

Finally, we obtain,

$$\mathcal{R}(X_s, D_s)(\bar{Z}_s) = \frac{1}{r'} \mathcal{R}(X_s, D_s)(\sum_{t_i \in C} j_{U_s}^* Z_{t_i}) \quad \text{by (60)}$$

$$= \frac{1}{r'} \mathcal{R}(X_s, D_s) \circ i_{U_T}^*(p_{Y \setminus Z_0}^\bar{T}(Z_{\bar{T} \mid U_{\bar{T}}})) - p_{Y \setminus Z_0}^\bar{T}(j_{U_{\bar{T}}}^* Z)(U_{s'} \times \bar{T})) \quad \text{by (61)}$$

This gives the theorem.
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