On Addition Theorems Related to Elliptic Integrals

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Abstract—We present formulas for the components of the Buchstaber formal group law and its exponent over $\mathbb{Q}[p_1,p_2,p_3,p_4]$. This leads to an addition theorem for the general elliptic integral $\int_0^x dt/R(t)$ with $R(t) = \sqrt{1 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4}$. The study is motivated by Euler’s addition theorem for elliptic integrals of the first kind.

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1. INTRODUCTION AND STATEMENTS

The Jacobi elliptic sine is the elliptic version of the circular sine. In traditional notation, $\text{sn } u$ is the inversion of the elliptic integral of the first kind:

$$\text{sn } u = x, \quad u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}},$$

where $k$ is some parameter called a modulus.

Addition theorems offer a means of determining the value of the function for the sum of two quantities as an argument when the value of the function for each quantity as an argument is known.

The function $f(u) = \text{sn } u$ satisfies Euler’s addition theorem

$$f(u + v) = \frac{f(u)R(f(v)) + f(v)R(f(u))}{1 - k^2 f(u)^2 f(v)^2}, \quad \text{where } R(u) = \sqrt{(1 - u^2)(1 - k^2 u^2)}. \quad (1.1)$$

Let us rewrite (1.1) in the form of the addition law due to Cayley [16] (see also [9, (1.4)]):

$$f(u + v) = \frac{f^2(u) - f^2(v)}{f(u)f'(v) - f(v)f'(u)}, \quad f(0) = 0, \quad f'(0) = 1. \quad (1.2)$$

The further solutions of (1.2) are as follows. The exponent series of the Ochanine elliptic genus $f = \exp(\phi_{\text{Oc}})$ over $\mathbb{Q}[\delta, \epsilon]$ is defined as the inversion of the elliptic integral

$$\int_0^x \frac{dt}{\sqrt{1 + \delta t^2 + \epsilon t^4}} \quad (1.3)$$

(see [13]).
Let \( f_0(u) = 1 / \Phi(u) \), where \( \Phi(u) = \Phi(u, \alpha) \) is the simplest Baker–Akhiezer function \([11]\). One has the following addition formula \([5]\):

\[
f_0(u + v) = \frac{f_0^2(u) - f_0^2(v)}{f_0(u) f_0(v) - f_0(v) f_0(u)}. \tag{1.4}
\]

For other interesting examples see \([5, 7, 8]\).

Consider the universal Buchstaber formal group law, the universal example of the formal group law of the form

\[
F(x, y) = \frac{x^2 A(y) - y^2 A(x)}{xB(y) - yB(x)}, \tag{1.5}
\]

also specializing to the Euler formal group law \((1.1)\) for \( A(x) = 1 \) and \( B(x) = R(x) \).

It is known from \([5]\) that the exponent series of \((1.5)\) gives the universal example of \( f(u) \) such that \( f(0) = 0, f'(0) = 1 \) and such that \( f \) has an addition theorem of the form

\[
f(u + v) = \frac{f(u)^2 \xi_1(v) - f(v)^2 \xi_1(u)}{f(u) \xi_2(v) - f(v) \xi_2(u)} \tag{1.6}
\]

for some series \( \xi_1(u) = A(f(u)) \) and \( \xi_2(u) = B(f(u)) = f'(u) \) such that \( \xi_1(0) = \xi_2(0) = 1 \).

By \([10]\), \( f \) is defined over the polynomial ring \( \mathbb{Q}[p_1, p_2, p_3, p_4] \) on four variables of degrees 2, 4, 6 and 8.

Our main result is Theorem 1.1. It provides new explicit formulas \((1.8)\) and \((1.11)\) for the series \( \mathcal{B} \) and \( \mathcal{A} \) in \((1.5)\) after tensoring the coefficient ring of \( F \) with rationals, i.e., over \( \Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, p_3, p_4] \). It also provides the differential equation \((1.10)\) with the general solution \( \mathcal{B} \).

Let us define the following formal power series over the polynomial ring \( \mathbb{Q}[p_1, p_2, p_3, p_4] \), where \( p_1, p_2, p_3 \) and \( p_4 \) are variables of degrees 2, 4, 6 and 8, respectively:

\[
\mathcal{R}(x) := \sqrt{1 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4}, \tag{1.7}
\]

\[
\mathcal{B}(x) := \frac{\mathcal{R}(\mu(x))}{\mu'(x)}, \quad \text{i.e.,} \quad \mathcal{R}(x) = \frac{\mathcal{B}(\nu(x))}{\nu'(x)} \tag{1.8}
\]

where

\[
\mu(x) := \frac{x}{\mathcal{B}(x)}, \quad \nu(x) := \mu^{-1}(x). \tag{1.9}
\]

Thus, \( \mathcal{B}(x), \mathcal{B}(0) = 1 \), is a general solution of the differential equation

\[
\mathcal{B}(x)^2 (\mathcal{B}(x) - x \mathcal{B}'(x))^2 = \mathcal{B}(x)^4 + p_1 x \mathcal{B}(x)^3 + p_2 x^2 \mathcal{B}(x)^2 + p_3 x^3 \mathcal{B}(x) + p_4 x^4. \tag{1.10}
\]

We also define

\[
\mathcal{A}(x) := \mathcal{B}^2(x) - \frac{1}{2} x \mathcal{B}(x) \mathcal{B}'(x) + \frac{1}{4} p_1 x \mathcal{B}(x) - \left( \frac{1}{16} p_1^2 - \frac{1}{4} p_2 \right) x^2. \tag{1.11}
\]

Note that all the above series are defined in terms of \( \mathcal{R}(x) \).

**Theorem 1.1.** In the notation \((1.5)\)–\((1.9)\) and \((1.11)\), over the polynomial ring \( \mathbb{Q}[p_1, p_2, p_3, p_4] \) one has

\[
\mathcal{A}(x) = \mathcal{A}(x), \quad \mathcal{B}(x) = \mathcal{B}(x) \quad \text{and} \quad \log_F(x) = \int_{0}^{x} \frac{dt}{\mathcal{B}(t)}.
\]

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Theorem 1.1 leads to the following

**Corollary 1.1.** Let \( f(u) \) be a formal power series over a \( \mathbb{Q} \)-algebra with \( f(0) = 0 \) and \( f'(0) = 1 \). Then \( f \) has an addition theorem of the form (1.6) if and only if it has one of the following properties:

(i) \( f \) is the inversion of \( u = \int_0^x dt/B(t) \), or, equivalently, \( f \) is a solution of the equation \( f' = B(f) \);

(ii) the series

\[
h(x) := \frac{f'(x)}{f(x)}
\]

satisfies the differential equation

\[
(h'(x))^2 = S(h(x)),
\]

where \( S \) is the generic monic polynomial of degree 4:

\[
S(t) = p_4 + p_3 t + p_2 t^2 + p_1 t^3 + t^4.
\]

Taking into account the remarks made after (1.6), one has the following

**Corollary 1.2.** The following series \( \xi_1 \) and \( \xi_2 \) satisfy (1.6):

\[
\begin{align*}
\xi_1(u) &= A(f(u)) = f^2(u) - \frac{1}{2} (f''(u) + f''(0)f'(u)) f(u) + \frac{1}{2} (f''(0) - f'''(0)) f^2(u), \\
\xi_2(u) &= B(f(u)) = f'(u).
\end{align*}
\]

In the proof of Theorem E.5.4 in [8], formulas which agree with (1.13) and (1.14) were derived. In particular, the coefficient \( b_1 \) of \( x \) in \( B(x) \) does not affect (1.5) and can therefore be chosen arbitrarily. Similarly, the series \( \xi_1(u) = A(f(u)) \) and \( \xi_2(u) = B(f(u)) \) in (1.6) are defined up to terms \( k_1 f(u)^2 \) and \( k_2 f(u) \), respectively, for any constants \( k_1 \) and \( k_2 \). We derive (1.11) and (1.13) using formula (2.3) of Lemma 2.2 (see below).

The following is the addition theorem for the general elliptic integral and its inversion \( \text{SN}(u) \), which specializes to the elliptic sine. In particular, it is defined by

\[
\text{SN}(u) = x, \quad u = \int_0^x \frac{dt}{R(t)},
\]

where \( R \) is as in (1.7). The function \( \text{SN} \) is the exponent series of a genus \( \psi \) introduced in [15].

**Theorem 1.2.** One has the addition formula

\[
\int_0^x \frac{dt}{R(t)} + \int_0^y \frac{dt}{R(t)} = \int_0^{G(x,y)} \frac{dt}{R(t)},
\]

where

\[
G(x,y) = \mu \left( P_1 + \sigma P_1 + \frac{1}{2} \nu(x)\nu(y) \frac{P_2 - \sigma P_2}{P_1 - \sigma P_1} \right);
\]

here \( \sigma \) is the transposition of \( x \) and \( y \) and the series \( P_i = P_i(x,y) \) are defined by

\[
P_1 = \nu(x)R(y)\nu'(y),
\]

\[
P_2 = -\nu(x)\nu'(y)(R(y)(R'(y) - R'(0)) - \nu(x)R^2(y)\nu''(y)).
\]
Corollary 1.3. Let \( SN(x) \) be the inversion of \( \int_0^x dt/R(t) \). Then

\[
SN(x + y) = G(SN(x), SN(y)).
\]

To prove Theorem 1.2, we first reduce the problem of an explicit addition theorem to the Kričhever–Höhn genus [10], which is defined over \( \mathbb{Q}[p_1, p_2, p_3, p_4] \) by (1.12). For this purpose we use the explicit strict isomorphism of Lemma 2.1. From [1] we know that the universal Buchstaber formal group law \( F_B \) (1.5) can be alternatively defined by the Nadiradze genus \( \varphi \) (see (2.2) below). In Theorem 2.1 we prove that the Kričhever–Höhn genus is identical to the Nadiradze genus \( \varphi \) after rationalizing its coefficient ring \( \Lambda \), i.e., over \( \Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, p_3, p_4] \). This reduces the problem to the universal Buchstaber formal group law \( F_B \), and we apply our explicit formulas for the components of \( F_B \) obtained in Theorem 1.1.

2. PRELIMINARIES

It is convenient to present the proofs of our results in terms of formal group laws. We give here the necessary definitions and facts. We refer the reader to [6] as a detailed survey on the subject.

A formal group law over a commutative ring \( R \) is a formal power series \( F(x, y) \) in \( R[[x, y]] \) satisfying the following conditions:

1. \( F(x, 0) = F(0, x) = x; \)
2. \( F(x, y) = F(y, x); \)
3. \( F(x, F(y, z)) = F(F(x, y), z). \)

Let \( F \) and \( G \) be formal group laws. A homomorphism from \( F \) to \( G \) is a power series \( h(x) \in R[[x]] \) with constant term 0 such that

\[
h(F(x, y)) = G(h(x), h(y)).
\]

It is an isomorphism if \( h'(0) \) (the coefficient of \( x \)) is a unit in \( R \), and a strict isomorphism if the coefficient of \( x \) is 1.

If \( F \) is a formal group law over a commutative \( \mathbb{Q} \)-algebra \( R \), then it is strictly isomorphic to the additive formal group law \( x + y \). In other words, there is a strict isomorphism \( l(x) \) from \( F \) to the additive formal group law. The series \( l(x) \) is called the logarithm of \( F \), so we have \( F(x, y) = l^{-1}(l(x) + l(y)) \). The inverse of the logarithm is called the exponential of \( F \). The logarithm \( \log_F(x) \) of a formal group law \( F \) is given by

\[
\log(x) = \int_0^x \frac{dt}{\omega(t)}, \quad \omega(x) = \frac{\partial F(x, y)}{\partial y}(x, 0).
\]

We will often use the following consequence of the above definitions: If \( h(x) \) is a strict isomorphism from the formal group law \( F \) to \( G \), then

\[
\log_F = \log_G(h), \quad \text{i.e.,} \quad \log_G = \log_F(h^{-1}).
\]

There is a ring \( \mathbf{L} \), called the universal Lazard ring, and a universal formal group law \( F(x, y) = \sum a_{ij}x^i y^j \) defined over \( \mathbf{L} \). This means that for any formal group law \( F' \) over any commutative ring with unit \( R \) there is a unique ring homomorphism \( r: \mathbf{L} \to R \) such that \( F'(x, y) = \sum r(a_{ij})x^i y^j \).

The formal group law of geometric cobordism was introduced in [12]. Following Quillen, we will identify it with the universal Lazard formal group law, since it is proved in [14] that the coefficient ring of complex cobordism \( \mathbf{MU}^* = \mathbb{Z}[x_1, x_2, \ldots], \left[ x_i \right] = 2i \), is naturally isomorphic as a graded ring to the universal Lazard ring.
The coefficients of the formal group law of geometric cobordism $F_U$ and its logarithm can be described geometrically by the following results.

**Theorem** (Buchstaber [4]).

\[ F_U(u, v) = \sum_{i,j \geq 0} [H_{ij}] u^i v^j \left( \sum_{r \geq 0} [\mathbb{CP}_r] u^r \right) \left( \sum_{s \geq 0} [\mathbb{CP}_s] v^s \right), \]

where $\mathbb{CP}_r$ are the complex projective spaces, $H_{ij}$ ($0 \leq i \leq j$) are Milnor hypersurfaces and $H_{ji} = H_{ij}$.

**Theorem** (A. S. Mishchenko, see [12]). The logarithm of the formal group law of geometric cobordism is given by the series

\[ \log(u) = u + \sum_{k \geq 1} [\mathbb{CP}_k] u^{k+1} \in \text{MU}^* \otimes \mathbb{Q}[[u]]. \]

The addition formula (1.1) corresponds to Euler’s addition formula for the elliptic integrals of the first kind

\[ \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} + \int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^T(x,y) \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \]

where

\[ T(x, y) = \frac{x \sqrt{(1 - y^2)(1 - k^2 y^2)} + y \sqrt{(1 - x^2)(1 - k^2 x^2)}}{1 - k^2 x^2 y^2} \]

is the Euler formal group law. In terms of the logarithm of the formal group law, this means

\[ \log_T(x) + \log_T(y) = \log_T(T(x, y)), \]

or, equivalently,

\[ T(x, y) = \exp_T(\log_T(x) + \log_T(y)). \]

The following two ideas arise naturally.

First, one can replace $\sqrt{(1 - x^2)(1 - k^2 x^2)}$ by the more general expression (1.7) and consider the corresponding formal group law $G(x, y)$ with logarithm

\[ \log_G(x) = \int_0^x \frac{dt}{\mathcal{R}(t)}, \quad (2.1) \]

specializing to the elliptic integral of the first kind. The corresponding formal group law was recently studied in [15].

Second, one can consider the universal Buchstaber formal group law, the universal example of the formal group law of the form (1.5).

The definition of the general complex elliptic genus (also called the Krichever–Höhn genus) was motivated by string theory: in [11] Krichever wrote down its characteristic power series $Q(x)$ using the Baker–Akhiezer function. In [10], Höhn defined a four-variable elliptic genus $\phi_{KH}$ determined by the following property: if one denotes by $f = f_{KH}$ the exponent of the corresponding formal group $F_{KH}$, then the series

\[ h(x) := \frac{f'(x)}{f(x)}. \]
satisfies the differential equation
\[(h'(x))^2 = S(h(x)),\]
where \(S\) is the generic monic polynomial of degree 4,
\[S(t) = p_4 + p_3t + p_2t^2 + p_1t^3 + t^4.\]

He also showed that \(\phi_{KR}\) takes values in the ring \(\mathbb{Q}[p_1, p_2, p_3, p_4]\) and that this agrees with Krichever’s definition.

We use the observation that these two formal groups \(G\) and \(F = F_{KH}\) are related by an explicit isomorphism. In particular, the following statement holds.

**Lemma 2.1.** The series \(x/B(x)\) is a strict isomorphism from \(F_{KH}\) to \(G\).

For a proof see [2] or [3]. Note that in [2, p. 13] the value of \(\psi(\mathbb{CP}_4)\) should read
\[
\frac{35}{128}p_1^4 - \frac{15}{16}p_1^2p_2 + \frac{3}{8}p_2^2 + \frac{3}{4}p_1p_3 - \frac{1}{2}p_4.
\]

We will need some results of [1, 2] and we collect them together as follows.

Recall here how the universal Nadiradze formal group law \(F_N\) is constructed (see [1]). Let \(F_U\) be the universal formal group law over \(\mathbb{MU}^*\). Define the series
\[
\sum A_{ij}x^iy^j = F_U(x, y)(x\omega(y) - y\omega(x)).
\]
Now we set all \(A_{ij}\) with \(i, j \geq 3\) equal to zero. Then the Nadiradze formal group law is classified by the quotient map
\[
\phi = \phi_N: \mathbb{MU}^* \to \mathbb{MU}^*/(A_{ij}, i, j \geq 3) := \Lambda.
\]
In other words, \(F_N\) is the universal formal group law whose invariant differential \(\omega(x) \in \Lambda[[x]]\) is
\[
\frac{\partial F(x, y)}{\partial y}(x, 0) = \frac{1}{\phi(\mathbb{CP}(x))} = \frac{1}{1 + \sum \phi(\mathbb{CP}_i)x^i}.
\]

Let \(F_B\) be the universal Buchstaber formal group law, the universal example among the formal group laws of the form (1.5).

**Lemma 2.2** [1]. (i) The universal Nadiradze formal group law \(F_N\) over \(\Lambda\) is identical to the universal Buchstaber formal group law \(F_B\); i.e., \(F_B\) is a formal group of the form (1.5) with
\[B(x) = \frac{1}{\phi(\mathbb{CP}(x))}.
\]
(ii) The series \(A(x) \in \Lambda[[x]]\) can be defined by
\[A(x) = -x^2B(x)\beta(x) - b_1xB(x) + B^2(x) - b_2x^2,
\]
where
\[
\beta(x) = \frac{B'(x) - b_1}{2x} \in \Lambda[[x]]
\]
and \(b_1, b_2\) are the coefficients of \(B(x) = 1 + \sum_{i \geq 1} b_i x^i\).

Note that to prove formula (2.3), we should use the series \(\sum A_{ij}x^iy^j\) in the proof of Proposition 2 in [1] and rewrite \(A(x) = \sum A_{i2}x^i\) and \(B(x) = \omega(x)\).

One also has the following

**Theorem 2.1.** Let \(\Lambda\) be the coefficient ring of the universal Nadiradze formal group \(F_N\). Then \(F_N\) over \(\Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, p_3, p_4]\) is identical to the Krichever–Höhn formal group law \(F_{KH}\).
3. PROOFS

**Proof of Theorem 2.1.** By [10] the coefficient ring \( \Lambda \) of the universal Buchstaber formal group law tensored by rationals is

\[
\Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, p_3, p_4];
\]

therefore, it suffices to consider the formal group laws over \( \mathbb{Q}[p_1, p_2, p_3, p_4] \).

Let \( l = \log G \) be the logarithm of the formal group law \( G \) in Lemma 2.1 and let \( j(x) = 1/l' \). Lemma 2.1 says that for a formal group law \( F \), the Höhn condition (1.12) is satisfied if and only if \( j(x)^2 \) is a polynomial of degree 4 with constant term 1.

Let \( \phi \) be the classifying map of a Buchstaber formal group law, and let

\[
C(x) = \sum C_i x^i = \phi(\mathbb{C}P(x)) = 1 + \sum_{i \geq 1} \phi([\mathbb{C}P_1]) x^i.
\]

By Lemma 2.2(i) we necessarily have

\[
B(x) = \frac{1}{C(x)},
\]

so that we obtain

\[
A(x, y) := F(x, y) \left( \frac{x}{C(y)} - \frac{y}{C(x)} \right) = x^2 A(y) - y^2 A(x).
\]

It follows that

\[
\frac{\partial^3 A}{\partial y^3}(x, 0) = cx^2
\]

for some constant \( c \).

Taking into account that \( F(x, y) = f(g(x) + g(y)) \), that \( g(0) = 0 \) and \( g'(0) = 1 \), that \( f(g(x)) = x \) and that \( f'(g(x)) = 1/g'(x) = 1/C(x) \), one obtains

\[
\frac{\partial^3 A}{\partial y^3}(x, 0) = -\frac{x C''(x)}{C(x)^4} + \frac{3 x C'(x)^2}{C(x)^5} + \frac{3 C'(x)}{C(x)^4} - \frac{3 C_1}{C(x)^2} + \frac{(3 C_1^2 - 4 C_2)x}{C(x)}
\]

\[
+ (-6 C_3^3 + 12 C_1 C_2 - 4 C_3) x^2,
\]

so that the series \( C(x) \) satisfies the differential equation

\[
x C(x) C''(x) - 3 x C'(x)^2 - 3 C(x) C'(x) + 3 C_1 C(x)^3 - (3 C_1^2 - 4 C_2) x C(x)^4 + k x^2 C(x)^5 = 0
\]

for some constant \( k \).

Let us now substitute \( \nu(x) \) for \( x \) in this equation, where \( \nu(x) \) is the inversion of \( xC(x) \), i.e., \( \nu(x) C(\nu(x)) = x \), so that

\[
C(\nu(x)) = \frac{x}{\nu(x)}, \quad C'(\nu(x)) = \nu'(x)^{-1} \frac{\nu(x) - x \nu'(x)}{\nu(x)^2},
\]

\[
\nu(x) C(\nu(x)) C''(\nu(x)) = -\nu'(x)^{-3} \frac{x}{\nu(x)} \nu''(x) + 2 \nu'(x)^{-1} \frac{x}{\nu(x)} \frac{x \nu'(x) - \nu(x)}{\nu(x)^2}.
\]

We then find that \( \nu(x) \) satisfies the differential equation

\[
2 x \nu(x)^2 \nu''(x) + (k x^3 + (6 C_1^2 - 8 C_2) x^2 - 6 C_1 x - 1) x^2 \nu'(x)^3 - 2 x \nu(x) \nu'(x)^2 + 6 \nu(x)^2 \nu'(x) = 0.
\]
Next let us consider
\[ j(x) = \frac{1}{p'(x)} \quad \text{and} \quad l = \log_G \]
mentioned above.

By the strict isomorphism we have \( l(x) = g(\nu(x)) \). Hence
\[ f(l(x)) = f(g(\nu(x))) = \nu(x), \quad f'(l(x))l'(x) = \nu'(x), \]
and, taking into account the equalities
\[ \frac{f(l(x))}{f'(l(x))} = \frac{1}{h((\frac{1}{y})^{-1}(x))} = x, \]
we have
\[ j(x) = \frac{\nu(x)}{x \nu'(x)}, \quad \nu'(x) = \frac{\nu(x)}{xj(x)}, \quad \nu''(x) = \frac{\nu(x)(1 - j(x) - xj'(x))}{x^2j(x)^2}. \]

Thus, for the series \( j \) we obtain the differential equation
\[ 2xj(x)j'(x) = 4j(x)^2 + kx^3 + (6C_1^2 - 8C_2)x^2 - 6C_1x - 1. \]
Then, further substituting \( j(x) = \sqrt{p(x)} \), we obtain
\[ xp'(x) = 4p(x) + kx^3 + (6C_1^2 - 8C_2)x^2 - 6C_1x - 1, \]
and it is easy to see that the general solution of this equation is a fourth-degree polynomial.

Conversely, by [5], \( F_{KH} \) is identical to \( F_B \) over \( \mathbb{Q}[p_1, p_2, p_3, p_4] \), and by Lemma 2.2(i) \( F_B \) is identical to \( F_N \). ∎

One can prove that \( F_{KH} \) is a Buchstaber formal group law by Lemma 2.1 again. For \( l = \log_G \) we have \( l' = 1/R(x) \); therefore, for \( \mu(x) = x/B(x), \nu = \mu^{-1} \) and \( g := \log_{F_{KH}} = l(\mu) \) we have
\[ g' = (l(\mu))' = \frac{1}{R(\mu)}\mu' := \frac{1}{B}. \]

Therefore, differentiating the inverse function, we get
\[ f' = B(f). \]

By Corollary 1.1(i) the corresponding formal group law is the Buchstaber formal group law over \( \Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, p_3, p_4] \) classified by the obvious inclusion \( \Lambda \to \Lambda \otimes \mathbb{Q} \).

**Proof of Theorem 1.1.** Let \( l_G \) and \( l_{F_{KH}} \) be the logarithms of the formal group laws \( G \) and \( F_{KH} \), respectively, in Lemma 2.1. Then, because of the strict isomorphism \( \mu(x) = x/B(x) \) of Lemma 2.1, one has \( l_{F_{KH}}(x) = l_G(\nu(x)) \). By the definition (2.1) of \( l_G \), we have the equalities (1.8) and \( l_{F_{KH}}(x) = \int_0^x dt/B(t) \). But the latter coincides with the logarithm of the universal Buchstaber formal group law by Theorem 2.1.

Now let us prove (1.11) for the series \( A \) and \( B \) over \( \mathbb{Q}[p_1, p_2, p_3, p_4] \). By the note after Lemma 2.2 one has
\[ A(x) := -x^2B(x)\beta(x) - b_1xB(x) + B^2(x) - b_2x^2, \quad \beta(x) = \frac{B'(x) - B'(0)}{2x} = \frac{B'(x) - b_1}{2x}, \]
or
\[ A(x) := B^2(x) - \frac{1}{2}xB(x)B'(x) - \frac{1}{2}b_1xB(x) - b_2x^2. \]
Therefore, it suffices to see that

\[ b_1 = -\frac{1}{2}p_1, \quad b_2 = \frac{1}{16}p_1^2 - \frac{1}{4}p_2. \]  

(3.1)

By Lemma 2.2(i) and Theorem 2.1,

\[
B(x) = \phi \left( \frac{1}{\mathbb{C}P(x)} \right) = \frac{1}{1 + \phi([\mathbb{C}P_1])x + \phi([\mathbb{C}P_2])x^2 + \ldots} = 1 - \phi([\mathbb{C}P_1])x + \phi([\mathbb{C}P_1]^2 - [\mathbb{C}P_2])x^2 + \ldots.
\]

Then there is a formula in [3] for calculating the values of \( \phi = \phi_{KH} \) on \([\mathbb{C}P_i]\), the generators of \(\text{MU}^* \otimes \mathbb{Q}\). In particular,

\[
\phi([\mathbb{C}P_1]) = \frac{1}{2}p_1, \quad \phi([\mathbb{C}P_1]^2 - [\mathbb{C}P_2]) = \frac{1}{4}p_1^2 - \frac{3}{16}p_1^2 - \frac{1}{4}p_2 = \frac{1}{16}p_1^2 - \frac{1}{4}p_2.
\]

This proves (1.11). \( \square \)

**Proof of Corollary 1.1.** Property (i) follows directly from Theorem 1.1. To prove property (ii), we apply Theorem 2.1. Here are some comments.

Buchstaber’s formal group law \( F \) is of the form (1.5). Hence

\[
f(u + v) = F(f(u), f(v)) = \frac{f(u)^2A(f(v)) - f(v)^2A(f(u))}{f(u)B(f(v))} = f(v)B(f(u)).
\]

Thus, \( f \) has an addition theorem of the required form and

\[
\xi_1(v) = A(f(u)), \quad \xi_2(v) = B(f(u)).
\]

Let \( f(u) \) be a series over any \( \mathbb{Q} \)-algebra. If it satisfies the addition theorem (1.6) and \( \log = f^{-1} \), then the corresponding formal group law is as follows:

\[
F(u, v) = f(\log(u) + \log(v)) = \frac{f(\log(u))^2\xi_1(\log(v)) - f(\log(v))^2\xi_1(\log(u))}{f(\log(u))\xi_2(\log(v)) - f(\log(v))\xi_2(\log(u))} = \frac{u^2\xi_1(\log(v)) - u^2\xi_1(\log(u))}{u\xi_2(\log(v)) - v\xi_2(\log(u))}.
\]

By its form, \( F \) is a Buchstaber formal group law. \( \square \)

**Proof of Corollary 1.2.** Formula (1.14) is just a differentiation of the inverse function in Theorem 1.1.

Let us prove (1.13). We have \( \xi_1(u) = A(f(u)) \). Let

\[
f(x) = x + f_1x^2 + f_2x^3 + \ldots.
\]

Then by (1.14) we have \( B(f(x)) = f'(x) \), that is,

\[
1 + b_1(x + f_1x^2 + f_2x^3 + \ldots) + b_2(x + f_1x^2 + f_2x^3 + \ldots)^2 + \ldots = 1 + 2f_1x + 3f_2x^2 + \ldots.
\]

Hence

\[
b_1 = 2f_1 = f''(0), \quad b_2 = 3f_2 - 2f_1^2 = \frac{1}{2}f'''(0) - \frac{1}{2}f''(0).
\]
Now, since $B(f(u)) = f'(u)$ and $B'(f(u)) = f''(u)/f'(u)$, we get by (1.11)

$$
\xi_1(u) = A(f(u)) = B(f(u))^2 - \frac{1}{2} f(u)B(f(u))B'(f(u)) - \frac{1}{2} b_1 f(u)B(f(u)) - b_2 f(u)^2
$$

Then (1.8) and $\mu = \nu^{-1}$ imply

$$
B(\nu(x)) = \frac{\mathcal{R}(\mu(\nu(x)))}{\mu'(\nu(x))} = \mathcal{R}(x)\nu'(x),
$$

$$
B'(\nu(x)) = \frac{B(\nu(x))'}{\nu'(x)} = \frac{\mathcal{R}(x)\nu'(x) + \mathcal{R}(x)\nu''(x)}{\nu'(x)}.
$$

By (1.11) we have

$$
A(\nu(x)) = \mathcal{R}^2(x)(\nu'(x))^2 - \frac{1}{2} \nu(x)\mathcal{R}(x)\nu'(x) \frac{\mathcal{R}'(x)\nu'(x) + \mathcal{R}(x)\nu''(x)}{\nu'(x)}
$$

$$
+ \frac{1}{4} p_1 \nu(x)\mathcal{R}(x)\nu'(x) - \left( \frac{1}{16} p_1^2 - \frac{1}{4} p_2 \right) \nu^2(x)
$$

$$
= \mathcal{R}^2(x)(\nu'(x))^2 - \frac{1}{2} \nu(x)\mathcal{R}(x)\mathcal{R}'(x)\nu'(x) + \frac{1}{4} p_1 \nu(x)\mathcal{R}(x)\nu'(x)
$$

$$
- \frac{1}{2} \nu(x)\mathcal{R}^2(x)\nu''(x) - \left( \frac{1}{16} p_1^2 - \frac{1}{4} p_2 \right) \nu^2(x)
$$

$$
= \mathcal{R}^2(x)(\nu'(x))^2 - \frac{1}{2} \nu(x)\mathcal{R}(x)\nu'(x) \left[ \mathcal{R}'(x) - \mathcal{R}'(0) \right]
$$

$$
- \frac{1}{2} \nu(x)\mathcal{R}^2(x)\nu''(x) - \left( \frac{1}{16} p_1^2 - \frac{1}{4} p_2 \right) \nu^2(x),
$$

as $\mathcal{R}'(0) = p_1/2$.

Now note that

$$
\frac{A(\nu(y))\nu^2(x) - A(\nu(x))\nu^2(y)}{\nu(x)\mathcal{R}(y)\nu'(y) - \nu(y)\mathcal{R}(x)\nu'(x)} = \nu(x)\mathcal{R}(y)\nu'(y) + \nu(y)\mathcal{R}(x)\nu'(x) + \frac{1 + \Pi}{\mathcal{R}'(0)},
$$

where

$$
I = -\frac{1}{2} \nu^2(x)\nu(y)\nu'(y)\mathcal{R}(y) \left[ \mathcal{R}'(y) - \mathcal{R}'(0) \right] + \frac{1}{2} \nu^2(y)\nu(x)\nu'(x)\mathcal{R}(x) \left[ \mathcal{R}'(x) - \mathcal{R}'(0) \right],
$$

$$
\Pi = -\frac{1}{2} \nu^2(x)\nu(y)\mathcal{R}^2(y)\nu''(y) + \frac{1}{2} \nu^2(y)\nu(x)\mathcal{R}^2(x)\nu''(x),
$$

$$
\mathcal{R} = \nu(x)\mathcal{R}(y)\nu'(y) - \nu(y)\mathcal{R}(x)\nu'(x).
$$
Thus we get
\[ \nu(G(x, y)) = \nu(x)\mathcal{R}(y)\nu'(y) + \nu(y)\mathcal{R}(x)\nu'(x) + \frac{1}{2} \nu(x)\nu(y)\frac{P - \sigma P}{\nu(x)\mathcal{R}(y)\nu'(y) - \nu(y)\mathcal{R}(x)\nu'(x)}, \]
where \( \sigma \in S_2 \) is the transposition of \( x \) and \( y \) and
\[ P = -\nu(x)\nu'(y)(\mathcal{R}(y)(\mathcal{R}'(y) - \mathcal{R}'(0)) - \nu(x)\mathcal{R}^2(y)\nu''(y)). \]
This proves Theorem 1.2. \( \Box \)

Proof of Corollary 1.3. As usual, the formal group law \( F = G \) gives the addition formula for its exponent \( f = SN \):
\[ f(u + v) = F(f(u), f(v)), \]
and the claim follows. \( \Box \)

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REFERENCES

1. M. Bakuradze, “The formal group laws of Buchstaber, Krichever, and Nadiradze coincide,” Russ. Math. Surv. 68 (3), 571–573 (2013) [transl. from Usp. Mat. Nauk 68 (3), 189–190 (2013)].
2. M. Bakuradze, “On the Buchstaber formal group law and some related genera,” Proc. Steklov Inst. Math. 286, 1–15 (2014) [transl. from Tr. Mat. Inst. Steklova 286, 7–21 (2014)].
3. M. Bakuradze, “Computing the Krichever genus,” J. Homotopy Relat. Struct. 9 (1), 85–93 (2014).
4. V. M. Bukhshtaber, “The Chern–Dold character in cobordisms. I,” Math. USSR, Sb. 12 (4), 573–594 (1970) [transl. from Mat. Sb. 83 (4), 575–595 (1970)].
5. V. M. Bukhshtaber, “Functional equations associated with addition theorems for elliptic functions and two-valued algebraic groups,” Russ. Math. Surv. 45 (3), 213–215 (1990) [transl. from Usp. Mat. Nauk 45 (3), 185–186 (1990)].
6. V. M. Bukhshtaber, “Complex cobordism and formal groups,” Russ. Math. Surv. 67 (5), 891–950 (2012) [transl. from Usp. Mat. Nauk 67 (5), 111–174 (2012)].
7. V. M. Bukhshtaber and E. Yu. Bunkova, “Krichever formal groups,” Funct. Anal. Appl. 45 (2), 99–116 (2011) [transl. from Funkts. Anal. Prilozh. 45 (2), 23–44 (2011)].
8. V. M. Bukhshtaber and T. E. Panov, Toric Topology (Am. Math. Soc., Providence, RI, 2015), Math. Surv. Monogr. 204.
9. V. M. Bukhshtaber and A. V. Ustinov, “Coefficient rings of formal group laws,” Sb. Math. 206 (11), 1524–1563 (2015) [transl. from Mat. Sb. 206 (11), 19–60 (2015)].
10. G. Höhn, “Komplexe elliptische Geschlechter und \( S^1 \)-äquivariante Kobordismustheorie,” Diplomarbeit (Bonn Univ., Bonn, 1991); arXiv:math/0405232 [math.AT].
11. I. M. Krichever, “Generalized elliptic genera and Baker–Akhiezer functions,” Math. Notes 47 (2), 132–142 (1990) [transl. from Mat. Zametki 47 (2), 34–45 (1990)].
12. S. P. Novikov, “The methods of algebraic topology from the viewpoint of cobordism theory,” Math. USSR, Izv. 1 (4), 827–913 (1967) [transl. from Izv. Akad. Nauk SSSR, Ser. Mat. 31 (4), 855–951 (1967)].
13. S. Ochanine, “Sur les genres multiplicatifs définis par des intégrales elliptiques,” Topology 26 (2), 143–151 (1987).
14. D. Quillen, “On the formal group laws of unoriented and complex cobordism theory,” Bull. Am. Math. Soc. 75, 1293–1298 (1969).
15. S. Schreieder, “Dualization invariance and a new complex elliptic genus,” J. Reine Angew. Math. 692, 77–108 (2014); arXiv:1109.5394v3 [math.AT].
16. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions (Univ. Press, Cambridge, 1927).

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