Danzer’s problem, effective constructions of dense forests and digital sequences

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Abstract
A 1965 problem due to Ludwig Danzer asks whether there exists a set in Euclidean space with finite density intersecting any convex body of volume 1. A recent approach to this problem is concerned with the construction of dense forests and is obtained by a suitable weakening of the volume constraint. A dense forest is a discrete point set of finite density getting uniformly close to long enough line segments. The distribution of points in a dense forest is then quantified in terms of a visibility function. Another way to weaken the assumptions in Danzer’s problem is by relaxing the density constraint. In this respect, a new concept is introduced in this paper, namely that of an optical forest. An optical forest in $\mathbb{R}^d$ is a point set with optimal visibility but not necessarily with finite density. In the literature, the best constructions of Danzer sets and dense forests are not deterministic. The goal of this paper is to provide deterministic constructions of dense and optical forests which yield the best known results in any dimension $d \geq 2$ in terms of visibility and density bounds, respectively. Namely, there are three main results in this work: (1) the construction of a dense forest with the best known visibility bound which, furthermore, enjoys the property of being deterministic; (2) the deterministic construction of an optical forest with a density failing to be finite only
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up to a logarithm and (3) the construction of a planar Peres-type forest (that is, a dense forest obtained from a construction due to Peres) with the best known visibility bound. This is achieved by constructing a deterministic digital sequence satisfying strong dispersion properties.

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1 | INTRODUCTION

In what follows, the cardinality of a set $A$ will be denoted by $|A|$ or $\#A$. The norms $|| \cdot ||_2$, $|| \cdot ||_\infty$ stand for the Euclidean and supremum norm in $\mathbb{R}^d$, respectively. Similarly, given $x \in \mathbb{R}^d$ and $r > 0$, the balls $B_2(x, r)$ and $B_\infty(x, r)$ with centre $x$ and radius $r$ are taken with respect to the Euclidean and supremum norm, respectively. The unit torus is denoted by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given $a, b \in \mathbb{R}$, let $[a, b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$ be the integer interval with endpoints $a$ and $b$. If $a = 1$, we write $[b] = [1, b]$. The set of positive integers is denoted by $\mathbb{N}$ and the set of non-negative integers by $\mathbb{N}_0$. Given two functions $f, g : \mathbb{R} \to \mathbb{R}^+$, the asymptotic notation $f(x) \ll g(x)$ (equivalently $f(x) = O(g(x))$) denotes the existence of a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for every $x$. If the constant $C$ depends on some parameter, say $t$, then this is denoted by indexing the notation, for instance, $f(x) = O_t(g(x))$. Finally, given two subsets $A, B \subseteq \mathbb{R}^{d+1}$, define the distance between $A$ and $B$ as $\text{dist}(A, B) = \inf\{||a - b||_2 : a \in A, b \in B\}$. If one of the sets contains only one element, say $A = \{a\}$, then one may write $\text{dist}(a, B)$.

Let $d \geq 2$ be a natural number which, throughout the paper, stands for a dimension. A function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a growth rate bound for a discrete subset $Y \subseteq \mathbb{R}^d$ if

$$\#(B_2(0, T) \cap Y) = O(g(T)).$$

Moreover, a subset $Y \subseteq \mathbb{R}^d$ has finite density if it admits $O(T^d)$ as a growth rate bound.

A subset $\mathfrak{S} \subseteq \mathbb{R}^d$ is a Danzer set if $\mathfrak{S}$ intersects every convex body of volume 1. In this case, $\mathfrak{S}$ is said to satisfy the Danzer property. In 1965, Danzer asked whether there exists a Danzer set of finite density in $\mathbb{R}^d$. This problem is still open (see the survey of Adiceam [1] for more details). There are several similar versions of the Danzer problem such as Conway’s dead fly problem [1, Section 2.2], which also posed independently by Boshernitzan. In this version, it is asked for a Danzer set which enjoys the extra property of being Delone, that is, a set with both lower and upper bounds on the spacing of the points. Another is the version of Gowers [1, Section 2.3] which asks whether there exist a Danzer set $\mathfrak{S}$ for which there is a finite bound $C$ on the number of points of intersection between $\mathfrak{S}$ and any convex body of unit volume. A negative answer to the latter problem has been given by Solan–Solomon–Weiss [12].

Two of the main approaches to tackle the Danzer problem focus, on the one hand, onto the construction of point sets satisfying the Danzer property with growth rate bound as close as possible to the optimal $O(T^d)$ and, on the other, onto the study of the weaker concept of dense forests obtained by a suitable relaxation of the volume constraint.
Definition 1.1 (Dense forest). A set $\mathcal{F} \subseteq \mathbb{R}^d$ is a dense forest if it has finite density and if it satisfies the following property: there exists a decreasing function $V : (0, 1) \rightarrow \mathbb{R}^+$ tending to $+\infty$ as $\epsilon \rightarrow 0^+$ such that for any $\epsilon \in (0, 1)$ and any line segment $L \subseteq \mathbb{R}^d$ with length $V(\epsilon)$, there is a point $x = x(L) \in \mathcal{F}$ such that $\text{dist}(x, L) \leq \epsilon$. The function $V$ is said to be a visibility function for the dense forest $\mathcal{F}$.

The problem of constructing dense forests is about the existence of a dense forest in $\mathbb{R}^d$ with visibility function $V(\epsilon) = O(\epsilon^{-(d-1)})$. This bound is the best one can hope for since, as will be proved in detail in Section 2, given a dense forest $\mathcal{F} \subseteq \mathbb{R}^d$, it always holds that

$$V(\epsilon) \gg \epsilon^{-(d-1)}.$$  \hspace{1cm} (1)

In the definition of a dense forest in $\mathbb{R}^d$, one fixes the density of the point set and allows its visibility to grow to infinity faster than $O(\epsilon^{-(d-1)})$ as $\epsilon \rightarrow 0^+$. In the definition of an optical forest introduced below, one fixes the visibility of the forest to be optimal and allows its growth rate bound to be ‘larger’ than $O(T^d)$; that is, the forest has not necessarily finite density.

Definition 1.2 (Optical forest). Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be such that $g(T) \gg T^d$. A set $\mathcal{F} \subseteq \mathbb{R}^d$ is an optical forest if it has growth rate bound $g(T)$ and if for every $\epsilon \in (0, 1)$ and every line segment of length $O(\epsilon^{-(d-1)})$, there is a point $x = x(L) \in \mathcal{F}$ such that $\text{dist}(x, L) \leq \epsilon$.

The problem of constructing optical forests is concerned with the existence of such a point set in $\mathbb{R}^d$ with growth rate bound as close as possible to the optimal bound $O(T^d)$.

It is easy to see that a Danzer set of finite density is a dense forest with optimal visibility. Similarly, a Danzer set of finite density is an optical forest with optimal growth rate bound $g(T) = O(T^d)$. For more details regarding the connection between Danzer’s problem and that of dense forests, see [1, Sections 3 & 4]. In the planar case, the three notions are equivalent; however, this is not true in higher dimensions [1, p. 12].

The first goal of this paper is to provide effective constructions of (1) dense forests with almost optimal visibility bounds (in a suitable sense) and of (2) optical forests with almost optimal growth rate bounds (in a suitable sense). As far as the construction of dense forests is concerned, the best known result is a purely probabilistic planar construction due to Alon [4] with visibility $V(\epsilon) = O(\epsilon^{-1} \cdot 2^{O(\sqrt{\ln(\epsilon^{-1})})})$. Alon’s forest enjoys the extra property of being uniformly discrete; that is, there exists a uniform positive lower bound for the distance between any two distinct points in the forest.

The main result of this paper yields a completely effective construction of dense forests with almost optimal visibility in all dimensions $d \geq 2$.

Theorem 1.1. Let $V : (0, 1) \rightarrow \mathbb{R}^+$ be a decreasing function such that $V(\epsilon) \rightarrow +\infty$ as $\epsilon \rightarrow 0^+$. Assume that there exists a decreasing sequence $(e_j)_{j \geq 1}$ in $(0, 1)$ with $e_j \rightarrow 0^+$ such that

$$\sum_{j=1}^{+\infty} \frac{e_j^{-(d-1)}}{V(e_j)} < +\infty.$$
Then, there exists a deterministic construction of a dense forest in \( \mathbb{R}^d \) with visibility function \( V(\varepsilon) = W(\varepsilon) = 2\sqrt{d} \cdot V(e_1) \), where \( i = i(\varepsilon) \) is the unique index such that \( e_i \leq \varepsilon < e_{i-1} \).

In [4], Alon claims that by optimising his probabilistic construction one could prove the existence of a planar dense forest with visibility bound \( V(e) = O(e^{-1} \cdot \ln(e^{-1}) \cdot \ln(\ln(e^{-1}))) \). However, the author could not verify this claim and further discussions with Prof. N. Alon confirm that its validity is doubtful.

By applying Theorem 1.1 to the function \( V(\varepsilon) = e^{-(d-1) \cdot \ln(e^{-1}) \cdot \ln(\ln(\ln(e^{-1})))} \) for an arbitrary \( \eta > 0 \), one obtains the following corollary, which stands as the best known visibility bound for a dense forest, in any dimension \( d \geq 2 \). Furthermore, the corresponding construction is deterministic.

**Corollary 1.1.** Given \( d \geq 2 \), for every \( \eta > 0 \), there exists a deterministic dense forest in \( \mathbb{R}^d \) with visibility

\[
V(\varepsilon) = O\left(e^{-(d-1) \cdot \ln(e^{-1}) \cdot \ln(\ln(e^{-1})^{1+\eta})}\right).
\]

When relaxing the density constraints, the best known construction of a Danzer set is due to Solomon and Weiss [13] who prove the existence of a Danzer set in \( \mathbb{R}^d \) with growth rate bound \( O(T^d \cdot \ln(T)) \) for every \( d \geq 2 \). The concept of an optical forest is a weakening of that of a Danzer set in the sense that a Danzer set with growth rate bound \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) is an optical forest with growth rate bound \( g \). The second result of this paper shows that the growth rate bound obtained by Solomon and Weiss can be achieved by a deterministic construction if one considers optical forests instead of Danzer sets.

To this end, the definition of range spaces is introduced. A range space is a pair \((S, R)\) where \( S \) is a set and \( R \) is a family of subsets of \( S \). The members of \( S \) are called points and those of \( R \) ranges. In our context \( S \) will always be the \( d \)-dimensional box \( I^d \), where \( I = [-\frac{1}{2}, \frac{1}{2}] \). The set \( R \) will be either the family of ranges \( B \) consisting of all boxes in \( I^d \) or the family \( B' \) of boxes in \( I^d \) with side lengths \( s_1, ..., s_d \) such that \( s_1 = \cdots = s_{d-1} \leq s_d \). Obviously, it holds that \( B' \subseteq B \). Given \( \varepsilon > 0 \), a subset \( N'_\varepsilon \subseteq I^d \) is an \( \varepsilon \)-net if \( N'_\varepsilon \) intersects non-trivially any range \( R \in R \) as soon as \( \mu_d(R) \geq \varepsilon \), where \( \mu_d \) is the Lebesgue measure in \( I^d \).

The existence of \( \varepsilon \)-nets with growth rate bound \( O(\varepsilon^{-1}) \) is known as the Danzer–Rogers problem and it is the combinatorial analogue to Danzer’s problem. Indeed, let \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) be a function of polynomial growth \(^\dagger\) such that \( x \to g(x)/x \) is non-decreasing. In [13, Theorem 1.4], Solomon and Weiss prove that there exists a Danzer set \( \mathcal{D} \subseteq \mathbb{R}^d \) with growth rate bound \( O(g(T)) \) if and only if for every \( \varepsilon > 0 \), there exists an \( \varepsilon \)-net \( N'_\varepsilon \) in the range space \((I^d, B)\) which has a finite cardinality growing like \#\( N'_\varepsilon = O(g(\varepsilon^{-\frac{1}{d}})) \). The following theorem shows that the same claim is true if one replaces, on the one hand, the ranges \( B \) with \( B' \) and, on the other hand, Danzer sets with optical forests.

**Theorem 1.2.** Given \( d \geq 2 \) and an increasing function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

\[
g(x) \gg x^d \quad \text{and for every } x > 0, \quad 1 + c \leq \frac{g(2x)}{g(x)} \leq C
\]

\(^\dagger\) A function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) has polynomial growth rate bound if there exists a real number \( \alpha > 0 \) such that \( g(x) \ll x^{\alpha} \).
for some positive constants $c, C > 0$, the following statements are equivalent.

(1) There exists an optical forest $\mathcal{F} \subseteq \mathbb{R}^d$ with growth rate bound $g(T)$.

(2) For every $\epsilon > 0$ there exists $\mathcal{N}_\epsilon \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ such that $\# \mathcal{N}_\epsilon = O(g(\epsilon^{-\frac{1}{d}}))$ and such that $\mathcal{N}_\epsilon$ intersects every box in $B'$ of volume $\epsilon$. In other words, $\mathcal{N}_\epsilon$ is an $\epsilon$-net in the range space $(I^d, B')$.

By establishing the analogue of Theorem 1.2 for Danzers sets and the range space $(I^d, B)$, Solomon and Weiss [13, Theorem 1.4] exploit a probabilistic argument due to Haussler and Welzl [9] to show the existence of $\epsilon$-nets in $(I^d, B)$ of growth rate bound $O(\epsilon \cdot \ln(\epsilon^{-1}))$, and thus, equivalently, the existence of a Danzer set with growth rate bound $G(T) = O(T^d \cdot \ln(T))$ [13, Theorem 1.6].

In view of these results, it is asked in [1, Problem 8] if one can construct deterministic $\epsilon$-nets in $(I^d, B)$ with growth rate bound $O(\epsilon^{-1} \cdot \ln(\epsilon^{-1}))$. Our result yields an affirmative answer if one replaces the range space $(I^d, B)$ by $(I^d, B')$.

**Theorem 1.3.** Given $d \geq 2$ and the range space $(I^d, B')$ defined above, for every $\epsilon > 0$, one can construct a deterministic $\epsilon$-net $\mathcal{N}_\epsilon$ with cardinality $\# \mathcal{N}_\epsilon = O(\epsilon^{-1} \cdot \ln(\epsilon^{-1}))$. Equivalently, one can construct a deterministic optical forest in $\mathbb{R}^d$ with growth rate bound $O(T^d \cdot \ln(T))$.

Since a Danzer set is in particular an optical forest, the result of Solomon and Weiss [13, Theorem 1.6] yields an optical forest with a same growth rate bound as the one provided by Theorem 1.3. The main feature of Theorem 1.3 is that the construction is deterministic; however, for $d \geq 3$, it is not a Danzer set. This will be justified in detail after the proof of Theorem 1.3 in Section 3. Note that, in the case $d = 2$, it holds that $B = B'$ and thus an optical forest is also a Danzer set.

The idea underlying the proof of Theorem 1.3 is known but is reproduced here as the author could not find any proper reference. Similar constructions are given in the work of Bambah and Woods [5] who proved the existence of a Danzer set in $\mathbb{R}^d$ with growth rate bound $O(T \cdot \log(T)^{d-1})$.

The last result in this paper implies the construction of the best known deterministic planar Peres-type dense forest; that is, a dense forest obtained from a construction due to Peres [6] described as follows: given a sequence $a = (a_n)_{n \in \mathbb{N}}$ in the unit torus $\mathbb{T}$, define the set

$$
\mathcal{F}(a) = \mathcal{F}_1(a) \cup \mathcal{F}_2(a),
$$

where

$$
\mathcal{F}_1(a) = \{(k, a_{|k|} + l) : k \in \mathbb{Z} \setminus \{0\}, l \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{F}_2(a) = R_{\frac{\pi}{2}}(\mathcal{F}_1).
$$

Here, $R_{\frac{\pi}{2}}(\cdot)$ is the $\frac{\pi}{2}$-rotation around the origin $(0,0)$. Peres [6] specialises construction (3) to the case where

$$
a_n = \begin{cases} 
\frac{n}{2} \cdot \phi, & \text{if } n \in 2\mathbb{N} \\
0 & \text{if } n \in 2\mathbb{N} - 1
\end{cases}
$$

The author owes this remark to the referee.
with \( \phi = \frac{1 + \sqrt{5}}{2} \) the golden ratio. He then proves that the resulting dense forest \( \mathcal{F}(a) \) has visibility \( O(\varepsilon^{-4}) \), providing this way the first example of a deterministic dense forest in the literature (although this construction was introduced in [6] in a problem of rectifiability of curves). A more careful analysis carried out in [3] shows that the same forest has visibility \( O(\varepsilon^{-3}) \), yielding this way the best known deterministic Peres-type forest in the plane.

In [2, 3], the authors refine construction (3) and generalise it to higher dimensions. In particular, in [3], the authors exploit these higher dimensional constructions to show the existence of dense forests in \( \mathbb{R}^d \) with (almost optimal) visibility \( O(\varepsilon^{-(d-1) - \eta}) \) for every \( d \geq 2 \), where \( \eta > 0 \) can be chosen arbitrarily small. However, these constructions are almost-deterministic in the sense that they still depend on the probabilistic choice of a set of parameters.

The construction of dense forests of the form (3) is of independent interest, as the visibility properties of the point set \( \mathcal{F}(a) \) depend on the properties of the sequence \( a \) in \( \mathbb{T} \). This also holds for the corresponding constructions in higher dimensions. This allows one to use tools and techniques from the theory of distribution of sequences modulo 1 and Diophantine approximation to study the problem of dense forests.

Recall that, given a sequence \( a = (a_n)_{n \in \mathbb{N}} \) in \( \mathbb{T} \), the dispersion is a measure of the density of the sequence \( a \). To be more precise, the dispersion of the first \( N \) terms of \( a \) is the quantity

\[
d_a(N) = \sup_{\gamma \in \mathbb{T}} \min_{j \in \llbracket N \rrbracket} ||a_j - \gamma||.
\]

Here, \( ||t|| \) denotes the distance from \( t \in \mathbb{R} \) to the nearest integer; that is, \( ||t|| = \min_{n \in \mathbb{Z}} |t - n| \).

In particular, the properties of \( a \) related to the visibility in \( \mathcal{F}(a) \) are captured by the following definition, which is a strengthening of the concept of dispersion (see, for instance, [7, 10, 11] for more details on this concept). Namely, given a sequence \( a \) in \( \mathbb{T} \), the forest \( \mathcal{F}(a) \) defined in (3) has visibility bound \( O(V) \) wherever \( a \) is \( O(V) \)-super uniformly dispersed in the following sense—see also [1, p. 18, Theorem 8].

**Definition 1.3** (Super Uniformly Dispersion). Let \( a = (a_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{T} \). The super-uniform dispersion of the first \( N \) terms of the sequence \( a \) is defined as

\[
\Delta_a(N) = \sup_{m \in \mathbb{N}_0} \sup_{\xi \in \mathbb{T}} d_a(N, m, \xi),
\]

where

\[
d_a(N, m, \xi) = \sup_{\gamma \in \mathbb{T}} \min_{j \in \llbracket N \rrbracket} ||(a_{j+m} - j\xi) - \gamma||.
\]

If \( \Delta_a(N) \xrightarrow{N \to +\infty} 0 \), then the sequence \( a \) is said to be super-uniformly dispersed. Moreover, given a function \( V : (0, 1] \to \mathbb{R}^+ \) such that \( V(\varepsilon) \to +\infty \) when \( \varepsilon \to 0^+ \), the sequence \( a \) is \( V \)-super-uniformly dispersed, if for every \( \varepsilon \in (0, 1] \) it holds that \( \Delta_a(V(\varepsilon)) \leq \varepsilon \), that is, for any \( m \in \mathbb{N}_0 \) and \( \gamma, \xi \in \mathbb{T} \), there exists \( j \in \llbracket V(\varepsilon) \rrbracket \) such that \( ||(a_{j+m} - j\xi) - \gamma|| \leq \varepsilon \).

The quantities (4) and (5) in the definition of super-uniform dispersion impose uniformity both in the index parameter \( m \) and in the parameter \( \xi \) of the linear perturbation of the sequence. The definition of a \( V \)-super-uniformly dispersed sequence is a quantitative refinement of the concept of (just) being super-uniformly dispersed.
Digital sequences are integer sequences defined from the digits in the expansion of a real number in a given integer base (see, for instance, [8, Chapter 1.4.3]). The following result is concerned with the effective construction of a digital $V$-super-uniformly dispersed sequence with $V(\varepsilon) = O_\eta(\varepsilon^{2-\eta})$ for every $\eta > 0$. In view of [1, p. 18, Theorem 8], one thus obtains the best known deterministic planar Peres-type forest.

**Theorem 1.4.** There exists a deterministic $V$-super-uniformly dispersed sequence in $\mathbb{T}$ with

$$V(\varepsilon) = O\left(\varepsilon^{-2} \cdot 2^{O(\sqrt{-\ln(\varepsilon)})}\right).$$

As a consequence, there exists a deterministic dense forest of the form (3) with visibility $O(\varepsilon^{-2} \cdot 2^{O(\sqrt{-\ln(\varepsilon)})})$ in the plane.

The paper is organised as follows. In Section 2 the proofs of Theorems 1.1, 1.3 and of Corollary 1.1 are given. Theorem 1.2 is proved in Section 3. Theorem 1.4 is proved in Section 4.

## 2 | PROOF OF THEOREMS 1.1 AND 1.3

**Proof (Visibility Bound (1)).** To prove the lower bound for the visibility of a dense forest, first note that one can replace the Euclidean norm in the definition of the dense forest by the sup norm. This change affects the visibility of a given forest only up to a constant. A similar remark can be made for the definition of a growth rate bound of a set, where one can replace the Euclidean ball of radius $T$ centred at the origin with the ball of radius $T$ with respect to the sup norm centred at the origin.

Now, assume that a given dense forest $\mathcal{F}$ in $\mathbb{R}^d$ has visibility $V$. Fix $\varepsilon > 0$ and set $C_\varepsilon$ to be the hypercube centred at the origin $\mathbf{0}$ with side length $V(\varepsilon)$; that is, $C_\varepsilon = B_{\infty}(\mathbf{0}, V(\varepsilon)/2)$. Decompose the hypercube $C_\varepsilon$ into axes-parallel boxes which have $d-1$ sides of length $\varepsilon$ and one side of length $V(\varepsilon)$. From the definition of a dense forest, any such box contains at least one point from $\mathcal{F}$. This yields that

$$\varepsilon^{-(d-1)} \cdot V(\varepsilon)^{d-1} \leq \#(B_{\infty}(\mathbf{0}, V(\varepsilon)/2) \cap \mathcal{F}) \ll V(\varepsilon)^d,$$

where the left-hand side quantity stands for the number of boxes that the cube $C_\varepsilon$ is decomposed into and where the middle quantity is, by definition, the total number of points of $\mathcal{F}$ belonging to $C_\varepsilon$. The right-hand side holds from the assumption that the forest $\mathcal{F}$ has finite density. Therefore, one obtains that $V(\varepsilon) \gg \varepsilon^{-(d-1)}$.\hfill $\square$

**Proof (Theorem 1.1).** Fix a natural number $d \geq 2$ and a sequence $(e_j)_{j \geq 1}$ satisfying the assumptions of Theorem 1.1. For every $j \in \mathbb{N}$, define the sets

$$S_j = \left\{ \left( k \cdot V(e_j), \frac{e_j}{\sqrt{d-1}}, \ldots, \frac{e_j}{\sqrt{d-1}} \right) : k \in \mathbb{Z}\setminus\{0\}, \quad l_2, \ldots, l_d \in \mathbb{Z} \right\}. \quad (6)$$

For every $l \in \{1, \ldots, d\}$, let $R_l : \mathbb{R}^d \mapsto \mathbb{R}^d$ be the map which permutes the first and the $l$th coordinate of a point; that is,

$$R_l(x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_d) = (x_l, \ldots, x_{l-1}, x_1, x_{l+1}, \ldots, x_d). \quad (7)$$
Define also the sets

\[ \mathcal{F}_j = \bigcup_{l=1}^{d} R_j(S_j) \quad \text{and} \quad \mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j \]  

(see Figure 1). We prove that the point set \( \mathcal{F} \) is a dense forest with visibility \( W : (0, 1) \mapsto \mathbb{R}^+ \), where \( W(\epsilon) = 2\sqrt{d} \cdot V(e_i) \) and \( i = i(\epsilon) \) is the unique index such that \( e_i \leq \epsilon < e_{i-1} \). To this end, fix \( \epsilon \in (0, 1) \) and set \( i = i(\epsilon) \). It is easy to check that every line segment \( L \) of length \( 2\sqrt{d} \cdot V(e_i) \) is such that the distance \( \text{dist}(L, \mathcal{F}_i) \) from \( L \) to the set \( \mathcal{F}_i \) is smaller than \( e_i \). Indeed, if the line segment \( L \) has length \( 2\sqrt{d} \cdot V(e_i) \), then \( L \) contains at least one point which has at least one coordinate equal to \( k \cdot V(e_i) \), for some \( k \in \mathbb{Z} \). Thus, from the definition of the set \( \mathcal{F}_i \), one obtains that \( \text{dist}(L, \mathcal{F}_i) \leq e_i \leq \epsilon \). This implies the claim regarding the visibility function of the forest \( \mathcal{F} \) in Theorem 1.1.

As for the density of the forest \( \mathcal{F} \), it is enough to show that

\[ \limsup_{T \to +1} \left( \frac{\#(\mathcal{F} \cap B_2(0, T))}{T^d} \right) < +\infty. \]

Indeed, given \( j \in \mathbb{N} \) and \( T_j \geq V(e_j) \) with \( kV(e_j) \leq T_j < (k+1)V(e_j) \) for some \( k \in \mathbb{N} \), one has that

\[ \frac{\#(\mathcal{F}_j \cap B_2(0, T_j))}{T_j^d} \leq \frac{\#(\mathcal{F}_j \cap B_\infty(0, (k+1)V(e_j)))}{k^d \cdot V(e_j)^d} \leq \frac{d2^d \cdot (d-1)^{d-1} \cdot (k+1)^d \cdot V(e_j)^{d-1} \cdot e_j^{-(d-1)}}{k^d \cdot V(e_j)^d} \leq C_d \cdot \frac{e_j^{-(d-1)}}{V(e_j)}. \]
where \( C_d = 2^{d+1} \cdot d(d-1)^{d-1} \) and the second inequality follows from the construction of \( \mathfrak{S}_j \) as a union of \( d \) rotations of the set \( S_j \). Fix \( T \geq 1 \) and set \( i_T = i(T) \) to be the unique index such that \( V(e_{i_T}) \leq T < V(e_{i_T+1}) \). Notice that \( \#(\mathfrak{S}_j \cap B_2(0, T)) = 0 \) for every \( j > i_T \). Therefore, one has that

\[
\frac{\#(\mathfrak{S}_j \cap B_2(0, T))}{T^d} = \frac{\sum_{j=1}^{i(T)} \#(\mathfrak{S}_j \cap B_2(0, T))}{T^d} \leq C_d \cdot \sum_{j=1}^{+\infty} \frac{e_j^{-(d-1)}}{V(e_j)}.
\]

The right-hand side of inequality (11) converges by assumption. The choice of \( T > 0 \) was arbitrary; therefore, inequality (9) is proved and the forest \( \mathfrak{S} \) has thus finite density. The proof is complete.

**Proof (Corollary 1.1).** Fix \( \eta > 0 \). Applying Theorem 1.1 with \( e_j = \frac{1}{2^j} \) and \( V_0(\varepsilon) = \varepsilon^{-(d-1)} \cdot \ln(V_0(\varepsilon)) \) yields the result. More explicitly, the deterministic construction of the corresponding dense forest is as follows: set

\[
S_j = \left\{ \left( k \cdot V_0(2^{-j}), \frac{l_2}{\sqrt{d-1} \cdot 2^{-j}}, \ldots, \frac{l_d}{\sqrt{d-1} \cdot 2^{-j}} \right) : k \in \mathbb{Z} \setminus \{0\}, l_2, \ldots, l_d \in \mathbb{Z} \right\}
\]

and \( \mathfrak{S} = \bigcup_{j=1}^{+\infty} (\bigcup_{l=1}^{d} R_l(S_j)) \) with \( R_l \) defined in (7). Given \( \varepsilon > 0 \), let \( i = i(\varepsilon) \) be the unique index such that \( e_i \leq \varepsilon < e_{i-1} \). Then, it holds that

\[
V(\varepsilon) = O(V_0(e_i)) = \frac{V_0(e_i)}{V_0(e_{i-1})} \cdot O(V_0(\varepsilon)) = O(V_0(\varepsilon)),
\]

since, given \( i \in \mathbb{N} \), one has that \( \frac{V_0(e_i)}{V_0(e_{i-1})} \leq 2^{d+1} \). The proof of the corollary is complete.

**Proof (Theorem 1.3).** Fix a natural number \( d \geq 2 \). It is enough to prove the existence of an \( \varepsilon \)-net \( \mathcal{N}_\varepsilon \) with \( \# \mathcal{N}_\varepsilon = \varepsilon^{-1} \cdot \ln(\varepsilon^{-1}) \) in \((I^d, B')\) for every \( \varepsilon \in \{2^{-(d-1)n} : n \in \mathbb{N}\} \). For every \( j \in \mathbb{N} \), define the sets

\[
S_j = \left\{ \left( k \cdot 2^{(d-1)j}, \frac{l_2}{\sqrt{d-1} \cdot 2^{j+1}}, \ldots, \frac{l_d}{\sqrt{d-1} \cdot 2^{j+1}} \right) : k \in \mathbb{Z} \setminus \{0\}, l_2, \ldots, l_d \in \mathbb{Z} \right\}.
\]

Given \( l \in \{1, \ldots, d\} \), let \( R_l : \mathbb{R}^d \to \mathbb{R}^d \) be the rotation defined in Equation (7). Set also

\[
\mathfrak{S}_j = \bigcup_{l=1}^{d} R_l(S_j) \quad \text{and} \quad \mathfrak{S} = \bigcup_{j \in \mathbb{N}} \mathfrak{S}_j.
\]

The goal is to prove that every box in \( \mathbb{R}^d \) with side lengths \( s_1, \ldots, s_d \) such that

\[
s_1 = \ldots = s_{d-1} \leq s_d
\]

and with volume \( 2^d \cdot \sqrt{d} \) intersects \( \mathfrak{S} \). To this end, fix such a box \( B' \) in \( \mathbb{R}^d \). Then, there exists \( \varepsilon > 0 \) such that the sides of \( B' \) have lengths \( 2^d \cdot \sqrt{d} \cdot e^{-(d-1)} \cdot \varepsilon, \ldots, \varepsilon \). Define \( j = j(\varepsilon) \) to be the smallest
natural number such that $2^{j-1} < \varepsilon^{-1} < 2^j$. Then, the box $B'$ contains a box $B$ with sides of lengths $2 \cdot \sqrt{d} \cdot 2^{(d-1)j}, \frac{1}{2^j}, \ldots, \frac{1}{2^j}$. Define $L$ to be the line segment connecting the middle points of the two faces of $B$ which have sides of length $1/2^j$. Obviously, $L$ has length $2 \cdot \sqrt{d} \cdot 2^{j(d-1)}$. Therefore, $L$ contains at least one point $x = x(L)$ which has at least one coordinate equal to $k \cdot 2^{(d-1)j}$ for some $k = k(L) \in \mathbb{Z}$. By construction of the set $S_j$, there exists at least one point $y \in S_j$ such that $||x - y||_2 \leq \frac{1}{2^{j+1}} \leq \varepsilon$. Therefore, $y \in B \subseteq B'$. Thus, the claim is proved.

It is left to prove that the optical forest $\mathfrak{G}$ admits $O(T^d \cdot \ln(T))$ as a growth rate bound. Applying inequality (10) to $e_j = 2^{-j}$ and $V(\varepsilon) = \varepsilon^{-d} \cdot \ln(\varepsilon^{-1})$ (that is, to $\mathfrak{F}_j = \mathfrak{G}_j$) yields that for every $T \geq 2^{(d-1)j}$,

$$\frac{\#(\mathfrak{G}_j \cap B_2(0, T))}{T^d} \leq C_d,$$

where $C_d = 2^{d+1} \cdot d(d - 1)^{\frac{d-1}{2}}$. Fix $T \geq 1$ and set $i_T = i(T)$, which is the unique natural number such that $2^{(d-1)j} \leq T < 2^{(i+1)(d-1)}$. Notice that $\#(\mathfrak{G}_j \cap B_2(0, T)) = 0$ for every $j > i_T$. Therefore, one has that

$$\frac{\#(\mathfrak{G} \cap B_2(0, T))}{T^d} = \frac{\sum_{j=1}^{i_T} \#(\mathfrak{G}_j \cap B_2(0, T))}{T^d} \leq \frac{C_d}{d-1} \cdot \log_2(T).$$

Thus, this shows that $\#(\mathfrak{G} \cap B_2(0, T)) = O(T^d \cdot \ln(T))$.

As for the construction of an $\varepsilon$-net $\mathcal{N}_\varepsilon$ in $(T^d, B')$ with growth rate bound $O(\varepsilon^{-d} \cdot \ln(\varepsilon^{-1}))$, it is enough to construct it only in the case $\varepsilon = \varepsilon_n$, where $\varepsilon_n = 2^{-(d-1)dn}, n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$ and set $Q_n = \mathfrak{G} \cap [0, \tau_d(n)]^d$, where $\tau_d(n) = 2 \cdot d^{1/2d} \cdot 2^{(d-1)n}$. Furthermore, set

$$\mathcal{N}_{\varepsilon_n} = \left\{ \left( \frac{x_1}{\tau_d(n)} - \frac{1}{2}, \ldots, \frac{x_d}{\tau_d(n)} - \frac{1}{2} \right) : (x_1, \ldots, x_d) \in Q_n \right\} \subseteq T^d.$$

From the construction of the set $Q_n$, it follows easily that the set $\mathcal{N}_{\varepsilon_n}$ intersects every box $B'$ of volume larger than $\varepsilon_n$ and, moreover, that $\# \mathcal{N}_{\varepsilon_n} \ll 2^{(d-1)dn} \cdot \ln(2^{(d-1)dn})$.

The proof is complete. \qed

We conclude this section by showing that the optical forest defined in the proof of Theorem 1.3 is not a Danzer set for $d \geq 3$. Namely, fix $d \geq 3$ and let $\mathfrak{G} \subseteq \mathbb{R}^d$ be the optical forest defined in (12). The goal is to show that there are arbitrary large boxes in $\mathbb{R}^d$ which do not intersect $\mathfrak{G}$. To this end, for every $j \in \mathbb{N}$ set the box

$$B_j = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \frac{2^{-j-2}}{\sqrt{d} - 1} \leq x_1 \leq \frac{2^{-j-2}}{\sqrt{d} - 1} + \frac{2^{-j-3}}{\sqrt{d} - 1}, 0 \leq x_2, \ldots, x_d \leq 2^{(d-1)j-1} \right\}.$$

It is readily checked that for any $j \in \mathbb{N}$ the box $B_j$ does not intersect the point set $\mathfrak{G}$. Moreover, for the volume of $B_j$ it holds that

$$Vol(B_j) = \frac{2^{-j-3}}{\sqrt{d} - 1} \cdot \left(2^{(d-1)j-1}\right)^{d-1} = 2^{-(d+2)} \cdot \frac{2^{(d-1)^2j-j}}{\sqrt{d} - 1}.$$

Since it was assumed that $d \geq 3$, the claim follows by noticing that $Vol(B_j) \to +\infty$ when $j \to +\infty$.\n
3 | PROOF OF THEOREM 1.2

Proof (Theorem 1.2). 1. ⇒ 2.: Given \( d \geq 2 \), assume that there exists an optical forest \( \mathcal{F} \subseteq \mathbb{R}^d \) with growth rate bound \( g \). Without loss of generality, one can take \( V(\varepsilon) = \varepsilon^{-(d-1)} \) as a visibility function \( V \) for the forest \( \mathcal{F} \). Indeed, in order to meet this condition, one can work with the set \( c \cdot \mathcal{F} = \{ c \cdot x : x \in \mathcal{F} \} \) if necessary, where \( c > 0 \) a sufficiently small constant.

Fix \( \varepsilon \in (0, 1) \) and let \( B_\infty(\mathbf{0}, (\varepsilon^{-(d-1)}/2)) \) be the box centred at the origin with side-length \( \varepsilon^{-(d-1)} \). Set

\[ Q_\varepsilon = \mathcal{F} \cap B_\infty(\mathbf{0}, \left( \frac{\varepsilon^{-(d-1)}}{2} \right)) \quad \text{and} \quad \mathcal{N}_\varepsilon = \varepsilon^{d-1} \cdot Q_\varepsilon \subseteq I^d. \]

By assumption, the set \( Q_\varepsilon \) contains \( O(g(\varepsilon^{-(d-1)})) \) points and so does the set \( \mathcal{N}_\varepsilon \). By assumption and the way that \( \mathcal{N}_\varepsilon \) is constructed, \( \mathcal{N}_\varepsilon \) intersects every box in \( B' \) with volume \( \varepsilon^{(d-1)d} \). Setting \( \eta = \varepsilon^{(d-1)d} \) yields that the set \( \mathcal{N}_\varepsilon \) is an \( \eta \)-net in \((I^d, B')\) such that \( \# \mathcal{N}_\varepsilon = g(\eta^{-\frac{1}{d}}) \). The choice of \( \varepsilon \in (0, 1) \) (and thus of \( \eta \in (0, 1) \)) is arbitrarily. Therefore, the claim is proved.

2. ⇒ 1.: Assume that for every \( \varepsilon \in (0, 1) \), there exists an \( \varepsilon \)-net \( \mathcal{N}_\varepsilon \) in \((I^d, B')\) such that \( \# \mathcal{N}_\varepsilon \leq g(\varepsilon^{-\frac{1}{d}}) \). In particular, by abusing slightly the notation, for every \( i \in \mathbb{N} \), let \( \mathcal{N}_i = \mathcal{N}_{\varepsilon_i} \) be an \( \varepsilon_i \)-net with \( \varepsilon_i = 1/2^{i(d-1)d} \). Moreover, for every \( i \in \mathbb{N} \), set \( B_i = B_\infty(\mathbf{0}, 2^{i(d-1)-1}) \), \( D_i = B_\infty(\mathbf{0}, 2^{i(d-1)}) \) and

\[ Q_i = 2^{i(d-1)} \cdot \mathcal{N}_i \subseteq B_i, \quad (14) \]

where, by assumption, \( \#Q_i = g(2^{i(d-1)}) \). By construction of the set \( \mathcal{N}_i \), one has that any line segment \( L \subseteq B_i \) with length \( \varepsilon^{-(d-1)} \leq 2^{i(d-1)} \) is \( O(\varepsilon) \)-close to the point set \( Q_i \).

Given any \( i \in \mathbb{N} \), the set \( D_{i+1} \setminus D_i \) can be tiled with the use of \( 2^{d^2} - 2^d \) hypercubes where each hypercube has side length \( 2^{i(d-1)} \). Let \( \{ C_j \}_{j=1}^{2^{d^2}-2^d} \) be such a tiling. Each hypercube \( C_j^{(i)} \) can be identified with the hypercube \( B_i \) through a translation, that is,

\[ C_j^{(i)} = B_i + a_j^{(i)}, \quad (15) \]

for some vector \( a_j^{(i)} \in \mathbb{R}^d \). Set

\[ Q_j^{(i)} = Q_i + a_j^{(i)}, \]

where the vector \( a_j^{(i)} \) is defined in Equation (15). Thus, for every \( j \in \{1, \ldots, 2^{d^2} - 2^d\} \), \( Q_j^{(i)} \) is a copy of the set \( Q_i \) inside the set \( C_j^{(i)} \).

The goal is to prove that the set

\[ \mathcal{F} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{d^2}-2^d} Q_j^{(i)} \quad (16) \]

is an optical forest with growth rate bound \( O(g(T)) \). To this end, fix a line segment \( L \) with length \( C \cdot \varepsilon^{-(d-1)} \), where \( C = 2^{d+1}(d + 1)^2 \sqrt{d} \). Then, \( L \) contains a line segment \( L' \subseteq L \) with length
which is contained in $C(i)$ for some $i \in \mathbb{N}$ such that $\epsilon^{- (d-1)} \leq 2^{i \cdot (d-1)}$. Note here that the choice of the natural number $i$ depends on the choice of the line segment $L$, that is, its position and length, and not only on the choice of the real number $\epsilon \in (0, 1)$. In other words, the natural number $i$ is not necessary the smallest integer such that $\epsilon^{- (d-1)} \leq 2^{i \cdot (d-1)}$. From the construction of the sets $Q_i$ and $Q^{(i)}_j$, the point set $\mathcal{F}$ is $O(\epsilon)$-close to the line segment $L'$ and thus to the line segment $L$. Therefore, this establishes that $\mathcal{F}$ is an optical forest.

It is left to prove that the point set $\mathcal{F}$ admits $O(g(T))$ as a growth rate bound. To this end, fix $i \in \mathbb{N}$. Upon setting $T_i = 2^{i \cdot (d-1)}$, it holds that

$$
\#(\mathcal{F} \cap B_\infty (0, T_i)) \leq (16) \left(2^{d^2} - 2^d\right) \cdot \sum_{k=1}^{i-1} \#Q_k
$$

$$
\leq (14) \left(2^{d^2} - 2^d\right) \cdot \sum_{k=1}^{i-1} g \left(2^{k \cdot (d-1)}\right)
$$

$$
\ll_d \sum_{k=1}^{i-1} \frac{g \left(2^{i \cdot (d-1)}\right)}{(1 + c)(d-1)(i-k)}
$$

$$
\ll_{c,T} g \left(2^{i \cdot (d-1)}\right) \ll C \cdot g(T_i).
$$

Finally, from the upper bound of the right-hand inequality of (2), it follows easily that a growth rate bound for the point set $\mathcal{F}$ is $O(g(T))$. The proof is complete. □

### 4 | CONSTRUCTION OF SUPER-UNIFORMLY DISPERSED SEQUENCES

The goal in this section is to prove Theorem 1.4. The sequence which satisfies the statement of the theorem, denote it by $\mathbf{u} = (u_n)_{n \geq 1}$, is defined as follows: decompose (throughout this section) the integer $n \geq 1$ as

$$
n = k \cdot 2^i + 2^{i-1} - 2 \quad \text{with} \quad i \geq 1 \quad \text{and} \quad k \geq 0,
$$

(17)

and the integer $k$ as

$$
k \equiv r \cdot 2^{i^2} + s \pmod{2 \cdot 2^{i^2}} \quad \text{with} \quad 0 \leq r \leq 2 \cdot 2^{i^2} - 1 \quad \text{and} \quad 1 \leq s \leq 2^{i^2}.
$$

The existence and uniqueness of decomposition (17) is guaranteed by the dyadic decomposition of $n + 2$. Then, $\mathbf{u}$ is given for all $n \geq 1$ by

$$
u_n = \begin{cases} 
\frac{rs}{2^{i^2}} & \text{if } 0 \leq r \leq 2^{i^2} - 1, \\
\frac{rs}{2^{i^2}} + \frac{s}{2^i} & \text{if } 2^{i^2} \leq r \leq 2 \cdot 2^{i^2} - 1.
\end{cases}
$$

(18)

The proof of Theorem 1.4 is achieved in two parts. Firstly, Lemma 4.1 introduces a general technique to construct super-uniformly dispersed sequence by using simpler sequences. In the
second step, one exploits Lemma 4.1 to infer Theorem 1.4. This work builds on [1, Theorem 8] which relates the visibility of a Peres’ type dense forest $\mathcal{G}(\mathbf{a})$, defined in (3), with the super-uniform dispersion of the toral sequence $\mathbf{a}$ which generates the forest.

**Lemma 4.1.** Let $V : (0, 1] \to \mathbb{R}^+$ be a decreasing function such that $V(\xi) \geq \frac{1}{\xi}$ for every $\xi \in (0, 1]$. Let $c^{(i)} = (c^{(i)}_k)_{k \in \mathbb{N}} \in \mathbb{T}^\mathbb{N}$, $i \in \mathbb{N}$, be a family of sequences in $\mathbb{T}$ such that upon setting $V_i = V(\frac{1}{2^i}) \in \mathbb{R}^+$, it holds that

$$\Delta_{c^{(i)}}(V_i) \leq \frac{1}{2^{i^2}} \text{ for all } i \geq 1$$

(here, the quantities $\Delta_{c^{(i)}}(V_i)$ are defined in Definition 1.3). Then, the sequence $(b_n)_{n \in \mathbb{N}}$ with formula

$$b_n = c^{(i)}_k,$$

where the integers $n, k, i$ are related by relation (17), is $W$-super-uniformly dispersed. Here,

$$W(\varepsilon) = 2^{i+2} \cdot \frac{V_i}{V_{i-1}} \cdot V(\varepsilon)$$

with $i = i(\varepsilon)$ the unique index such that $2^{-i^2} \leq \varepsilon < 2^{-(i+1)^2}$.

**Proof.** Set $(b_n)_{n \in \mathbb{N}}$ as in the statement and $\varepsilon_i = \frac{1}{2^{i^2}}$ for every $i \in \mathbb{N}_0$, that is, $i = \sqrt{-\log_2 \varepsilon_i}$. The goal is to show that the sequence $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ is $W$-super-uniformly dispersed. Fix $\varepsilon > 0$, $\xi, \gamma \in \mathbb{T}$ and $m \in \mathbb{N}_0$. There exists a unique $i = i(\varepsilon) \in \mathbb{N}$ such that $\varepsilon_i \leq \varepsilon < \varepsilon_{i-1}$ and a minimal natural number $m_0 \in \mathbb{N}$ such that $m_0 \cdot 2^i + 2^{i-1} - 2 \geq m$. By assumption (19), there exists $j \in \llbracket 1, V_i \rrbracket$ such that

$$\left\| c^{(i)}_{j+m_0} - \xi \cdot j - \gamma_0 \right\| \leq \varepsilon_i,$$

where $\xi_0 = \xi \cdot 2^i$ and $\gamma_0 = \xi \cdot m_0 2^i + \xi 2^{i-1} - 2 \xi - m \xi + \gamma$. In turn, by setting $j' = j \cdot 2^i + 2^{i-1} - 2 + m_0 \cdot 2^i - m$, from the way that the sequence $(b_n)_{n \in \mathbb{N}}$ and the quantities $\xi_0, \gamma_0, m_0$ are defined, one infers that

$$\left\| b_{j'+m} - \xi \cdot j' - \gamma \right\| \leq \varepsilon_i.$$

Since the choice of $\xi, \gamma \in \mathbb{T}$ and $m \in \mathbb{N}_0$ is arbitrary, it is left to prove that $j' \leq W(\varepsilon)$. To this end, notice that

$$1 \leq j' = j \cdot 2^i + 2^{i-1} - 2 + m_0 \cdot 2^i - m \leq 2^i \cdot V_i + 2^{i-1} + 2^i \leq 2^i \cdot V_i + \frac{3}{2} \cdot 2^i \leq 4 \cdot 2^i \cdot V_i,$$

since $m_0 \cdot 2^i - m \leq 2^i$ and $2^{i^2} \leq V_i$. Thus, $j' \in \llbracket 2^i \cdot V_i + 3 \cdot 2^{i-1} \rrbracket \subseteq \llbracket 2^{i+2} \cdot V_i \rrbracket$. Considering the monotonicity of the function $V$, it follows that

$$W(\varepsilon) \leq 2^{i+2} \cdot V_i \leq 2^{i+2} \cdot \frac{V_i}{V_{i-1}} \cdot V(\varepsilon).$$

The proof is complete. \qed
The sequence \( u = (u_n)_{n \in \mathbb{N}} \) defined in (18) and its respective super-uniform dispersion bound given from the statement of the Theorem 1.4 are obtained as an application of the Lemma 4.1, for a suitable choice of the sequences \( c^{(i)} \).

**Proof (Theorem 1.4).** The proof is obtained by applying Lemma 4.1 in the following way. Instead of finding a family of sequences \( \{c^{(i)}\}_{i \in \mathbb{N}} \) satisfying \( \Delta_{c^{(i)}}(V_i') \leq \frac{1}{2^i} \) for some properly chosen naturals \( V_i' \), the goal will be to find a family of finite sequences \( \{c^{(i)}\}_{i \in \mathbb{N}} = \{(c^{(i)}_k)_{k \in [V_i]}\}_{i \in \mathbb{N}} \) satisfying \( \sup_{\xi' \in \mathbb{T}} d_{c^{(i)}}(V_i, 0, \xi) \leq \frac{1}{2^i} \), for some properly chosen naturals \( V_i \). This is sufficient in order to apply Lemma 4.1 as, for instance, given a finite sequence \( c^{(i)} = (c^{(i)}_k)_{k \in [V_i]} \) such that \( \sup_{\xi' \in \mathbb{T}} d_{c^{(i)}}(V_i, 0, \xi) \leq \frac{1}{2^i} \), by concatenating the terms of \( c^{(i)} \), one can construct a sequence \( \tilde{c}^{(i)} \) such that \( \Delta_{\tilde{c}^{(i)}}(2V_i') \leq \frac{1}{2^i} \). Recall that given two finite sequences \( \alpha = \{\alpha_i\}_{i=1}^{a} \) and \( \beta = \{\beta_j\}_{j=1}^{b} \), the concatenation of \( \alpha \) with \( \beta \) is the finite sequence \( \gamma = \{\gamma_k\}_{k=1}^{a+b} \) where, for every \( \gamma_k \),

\[
\gamma_k = \begin{cases} 
\alpha_k & \text{if } k \in [1, a] \\
\beta_{k-a} & \text{if } k \in [a+1, a+b].
\end{cases}
\]

Indeed, assume that \( c = (c_k)_{k \in [V]} \) is a finite sequence such that \( \sup_{\xi' \in \mathbb{T}} d_{c}(|V|, 0, \xi) \leq \varepsilon \) for some \( \varepsilon > 0 \) and \( V \geq 1 \). Decompose a natural number \( n \in \mathbb{N} \) as \( n = u \cdot \lfloor V \rfloor + k \) with \( u \in \mathbb{N}_0 \) and \( k \in \lfloor V \rfloor \) and define the sequence \( a_n = c_k \) for every \( n \in \mathbb{N} \). Then, setting \( a = (a_n)_{n \in \mathbb{N}} \) easily yields that \( \Delta_a(2 \cdot |V|) \leq \varepsilon \).

Fix now \( i \in \mathbb{N} \) and decompose every \( k \in [1, 2 \cdot 2^{2i^2}] \) as \( k = r \cdot 2^{i^2} + s \) with \( 0 \leq r \leq 2 \cdot 2^{i^2} - 1 \) and \( 1 \leq s \leq 2^{i^2} \). Also, set \( V(\varepsilon) = 2 \cdot \varepsilon^{-2} \) and \( V_i = V(2^{-i^2}) = 2 \cdot 2^{i^2} \).

In view of Lemma 4.1 and of the remark above, it is enough to prove that the finite sequence \( c^{(i)} = (c_k)_{k \in [V_i]} \), where

\[
c_k = \begin{cases} 
\frac{rs}{2^{i^2}} & \text{if } 0 \leq r \leq 2^{i^2} - 1, \\
\frac{rs}{2^{i^2}} + \frac{s}{2^{i^2}} & \text{if } 2^{i^2} \leq r \leq 2 \cdot 2^{i^2} - 1 \end{cases}
\]

is such that \( d_{c^{(i)}}(V_i, 0, \xi') \leq \frac{1}{2^i} \) for every \( \xi' \in \mathbb{T} \). If this is the case, then one can easily check that, for every \( \varepsilon \in (0, 1) \), it holds that

\[
W(\varepsilon) \leq 4 \cdot 2^{i' \cdot i' + 1} \cdot \frac{V_i'+1}{V_i'} \cdot V(\varepsilon) = O \left( \varepsilon^{-2} \cdot 2^{O(\sqrt{-\ln(\varepsilon)})} \right),
\]

where \( i' = i'(\varepsilon) \) is the unique index such that \( 1 - \frac{1}{2^{i'2}} \leq \varepsilon < \frac{1}{2^{(i' - 1)^2}} \).

Let us then prove that for every \( \xi' \in \mathbb{T} \), one has that \( d_{c}(V_i, 0, \xi') \leq \frac{1}{2^{i^2}} \). Set \( u = i^2 \). The first step is to show that for every \( \xi \in \mathbb{T} \) of the form

\[
\xi = \frac{l}{2u} + \frac{l'}{2^{2u}}, \quad l \in [0, 2^{u} - 1], \quad l' \in [0, 2^{u} - 1],
\]

(20)
it holds that \( d_{cV}(V_1, 0, \xi) \leq \frac{1}{2^{2u}} \). To see this, fix such \( \xi \in \mathbb{T} \) and fix also \( \gamma \in \mathbb{T} \). If \( l \) is odd, then for every \( k' \) of the form \( k' = l' \cdot 2^u + j \in \left[ l' \cdot 2^u + 1, (l' + 1) \cdot 2^u \right] \), where \( j \in \mathbb{Z} \), one has
\[
\|k' \xi + \gamma - c_k'\| = \left\| \frac{l^2}{2^u} + \gamma + \frac{j l}{2^u} \right\|.
\]
Since \( l \) is odd, one can find \( j_0 \in \mathbb{Z} \) such that
\[
\left\| \frac{l^2}{2^u} + \gamma + \frac{j_0 \cdot l}{2^u} \right\| \leq \frac{1}{2^{u+1}}.
\]
Similarly, if \( l \) is even, then for every \( k' \) of the form
\[
k' = (2^u + l') \cdot 2^u + j \in \left[ (2^u + l') \cdot 2^u + 1, (2^u + l' + 1) \cdot 2^u \right],
\]
where \( j \in \mathbb{Z} \), one has
\[
\|k' \xi + \gamma - c_k'\| = \left\| \frac{l^2 - 1}{2^u} + \gamma + \frac{j \cdot (l - 1)}{2^u} \right\|.
\]
Since \( l - 1 \) is odd, there is a choice of \( j_0 \in \mathbb{Z} \) such that
\[
\left\| \frac{l^2 - 1}{2^u} + \gamma + \frac{j_0 \cdot (l - 1)}{2^u} \right\| \leq \frac{1}{2^{u+1}}.
\]
Fix now any \( \xi', \gamma \in \mathbb{T} \). Then, there exists \( \xi = \frac{l}{2^u} + \frac{l'}{2^{2u}} \) such that \( \|\xi' - \xi\| \leq 2^{-(2u+1)} \). Therefore, upon setting \( m_0 = l' \) if \( l \) is odd and \( m_0 = 2^u + l' \) if \( l \) is even, there exists \( j_0 \in \mathbb{Z} \) such that the integer \( k = m_0 \cdot 2^u + j_0 \) satisfies the relation
\[
\|c_k - k \xi - \gamma_0\| \leq \frac{1}{2^{u+1}},
\]
where \( \gamma_0 = m_0 \cdot (\xi' - \xi) + \gamma \). From the Triangle Inequality,
\[
\|c_k - k \xi' - \gamma\| = \|c_k - k \xi - k(\xi' - \xi) - \gamma\|
\leq \|c_k - k \xi - \gamma - m_0 \cdot 2^u \cdot (\xi' - \xi)\| + j_0 \cdot \|\xi' - \xi\|
\leq \|c_k - k \xi - \gamma_0\| + j_0 \cdot \|\xi' - \xi\|
\leq \frac{1}{2^{u+1}} + \frac{2^u}{2^{2u+1}} \leq \frac{1}{2^u}.
\]
Thus, \( d_{cV}(V_1, 0, \xi) \leq \frac{1}{2^u} \). Consequently, the sequence \( u = (u_n)_{n \in \mathbb{N}} \) defined in (18) is \( W \)-superuniformly dispersed. The resulting explicit construction of a dense forest with the claimed visibility bound follows from [1, p. 18, Theorem 8], which furthermore implies that the forest \( F(u) \) defined in Equation (3) has visibility \( O(W) \). The proof of the theorem is complete. \( \square \)
Some questions arise naturally from this work.

1. Theorem 1.1 provides a strong sufficient condition for the existence of dense forests with a given visibility. Is this condition necessary? A positive answer to this question immediately implies a negative answer to Danzer’s problem.

2. Can one construct a deterministic $V$-super-uniformly dispersed sequence with $V(\varepsilon) = O(\varepsilon^{-2})$? In [14], the author uses a probabilistic argument in order to prove the existence of $V$-super-uniformly dispersed sequences with $V(\varepsilon) = O(\varepsilon^{-1} \cdot 2^{O(\sqrt{\ln(\varepsilon)})})$.

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