Dynamical origin of primordial black holes in spatially flat gauge with inflationary $\alpha$-attractor potentials

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We consider primordial black hole (PBH) formations in the framework of linear cosmological perturbations in the spatially flat gauge with the $\alpha$-attractor potentials. Our formalism comprises the first-order perturbations in the quantized single inflaton field, in the classical background metric and their coupled non-linear differential equations, in the space of the mode momentum $k$. We first obtain the slow-roll horizon-crossing density contrasts at various momenta in the high momentum limit and their values during the horizon re-entry in the radiation dominated era, using a transfer function. The density contrasts for the black hole formations are located in the regions of $k$-space where the perturbative framework breaks down and the slow-roll feature is violated. These locations are observed to be associated with large enhancement in the scalar power spectra and large negative Bardeen potentials. In our formalism, we find that the same set of $k$-space evolution equations can conform to the Planck data in the $(n_s - r)$ plane in the $k \to 0$ limit and create the scenario of PBH formations in the $k \to \infty$ limit, when we use the traditional (unmodified) $\alpha$-attractor $T$-model and $E$-model potentials. We have obtained the masses ranging from $1.35 \times 10^{-13} M_\odot$ to $2.93 \times 10^{-16} M_\odot$ in the $k$-range $0.43 \times 10^{13} - 9.25 \times 10^{13} \text{Mpc}^{-1}$, which are consistent with the LISA and the BBO forecasts of PBH detections through gravitational waves.

I. INTRODUCTION

The concept of primordial black holes (PBHs), associated with comparatively large density fluctuations at the early stages of the evolution of the universe, was introduced by Zel’dovich and Novikov [1] and later theorized in details by S. W. Hawking and B. J. Carr in [2–4]. Subsequently, the hydrodynamics of PBH formation [5], accretion of matter around PBHs [6] and PBH formations in the contexts of Grand Unified Theories [7] and in a double inflation in supergravity [8] were studied. It is conjectured that these PBHs have either evaporated by Hawking radiation or have evolved into supermassive black holes [9] or remain as dark matter in the present universe [9–23]. The relations between PBHs and primordial gravitational waves have been studied in [19–21, 30]. Formation of PBH during a first order phase transition in the inflationary period has also been studied [31,32]. Studies of the influence of PBHs on the CMB $\mu$ and $y$ distortions [33, 34] and $\mu T$ [35] cross-correlations have been carried out. Detection of signals from stochastic gravitational wave background, connected with PBH formations, in present and future experiments, has been discussed in Refs. [30, 37].

In some of the studies of the inflationary scenarios, PBHs have been identified as massive compact halo objects with mass $\sim 0.5 M_\odot$ in the work by J. Yokoyama [38], who has also examined the formation of PBHs in the framework of a chaotic new inflation [39]. Josan and Green [40], studied constraints on the models of inflation through the formation of PBHs, using a modified flow analysis. Harada, Yoo and Kohri [41] examined the threshold of PBH formation, both analytically and numerically. R. Arya [42] has considered the PBH formation as a result of enhancement of power spectrum during the thermal fluctuations in a warm inflation. Formation of PBHs in density perturbations was studied in two-field Hybrid inflationary models [43, 44], Starobinsky model including dilaton [45], multi-field inflation models [46], isocurvature fluctuation and chaotic inflation models [48], inflection-point models [24, 47], quantum diffusion model [48], model with smoothed density contrast in the super-horizon limit [49] and with the collapse of large amplitude metric perturbation [50] and large-density perturbation [51] upon horizon re-entry. PBH abundance in the framework of non-perturbative stochastic inflation has been studied by F. Kühnel and K. Freese [52]. Relation between the constraints of primordial black hole abundance and those of the primordial curvature power spectrum has also been studied [53, 54]. Recently, PBHs solutions have been obtained in the framework of non-linear cosmological perturbations and non-linear effects arising at horizon crossing [55].

PBH production has recently been studied, in the framework of $\alpha$-attractor polynomial super-potentials and modulated chaotic inflaton potentials models [56]. Mahbub [57] utilized the superconformal inflationary $\alpha$-attractor potentials with a high level of fine tuning to produce an ultra-slow-roll region, where the enhancement for curvature power spectra giving rise to massive PBHs was found at $k \sim 10^{14} \text{Mpc}^{-1}$. In a subsequent work [58], this author re-examined the earlier work using the optimised peak theory. The ultra-slow-roll process along with a non-Gaussian Cauchy probability distribu-
tion has been applied in Ref. [59] to obtain large PBH masses. The constant-rate ultra-slow-roll-like inflation [60] has also been applied to obtain the enhancement in the power spectra, triggered by entropy production, resulting in PBH formation. Ref. [61] has simulated the onset of PBHs formation by adding a term to the non-canonical $\alpha$-attractor potential, which enhances the curvature perturbations at some critical values of the field. The enhancement of the power spectrum by a limited period of strongly non-geodesic motion of the inflationary trajectory and consequent PBH production has been studied by J. Fumagalli et al. [62].

In almost all of the above references, PBH formation has been studied in terms of the curvature perturbation $R$ and the curvature power spectrum $P_{R}(k)$, which are usually obtained from the Mukhanov-Sasaki equation [63–66] in the co-moving gauge, characterised by a zero inflaton perturbation ($\delta \phi = 0$). This way of analysis, obviously ignores the role of the inflaton field ($\phi$) in the inflationary scenario of PBH formation. In the present work, we shall take an alternative route. We shall use the spatially flat gauge, thereby including the role of $\phi$ in the mechanism of PBH formation. In this respect, we shall follow the formalism developed in our previous work [67], comprising a set of linear perturbative evolution equations which could explain the Planck-2018 data [68] in low $k$ limit. We shall show, here, that the same equations can yield PBH-like solutions in the high $k$ regime with the conventional chaotic $T$ and $E$ model potentials without any modifications. Thus, we find that a salient feature of the present approach is that we can explore different regions of $k$-space evolution in the inflationary period under suitable initial conditions for the same differential equations. We believe that this work will open up an avenue for the dynamical origin of PBH formation in the deep sub-horizon $k$-space. In fact, we shall highlight here the important role played by the Bardeen potential ($\Phi_B$) in building up the density contrasts and the associated PBH formations, which, to our knowledge, has not been done so far in the literature.

The paper is organised as follows. In Sec. II A we briefly describe the basic formalism of the linear perturbation theory, leading to the setting up of the three coupled non-linear differential equations which play the central role of our study of the PBH formations. In Sec. II B we write about the $\alpha$-attractor $T$ and $E$ model potentials which have been used in the present study. The expression of the transfer function is written and its plot is shown in Sec. II C. Results and discussion are presented in Sec. III. Finally, in Sec. IV we make some concluding remarks.

### II. FORMALISM

#### A. Linear perturbations in the metric and the inflaton field

The Einstein-Hilbert action with minimal coupling between quantised inflaton field,

$$\phi(t, \vec{X}) = \int \frac{d^3k}{(2\pi)^3} [\phi(k, t)\hat{a}(k)e^{i\vec{k}\cdot\vec{x}} + \phi^*(k, t)\hat{a}^\dagger(k)e^{-i\vec{k}\cdot\vec{x}}],$$

and the background linearly-perturbed metric, in spatially flat gauge, with no anisotropic stress,

$$ds^2 = -(1 + 2\Phi)dt^2 + 2a(t)\partial_iBdx^i dt + a^2(t)\delta_{ij}dx^i dx^j,$$

is

$$S = \int d^4x\sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} g^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi - V(\phi) \right).$$

The linear perturbation in the inflaton field is written as

$$\phi(t, \vec{X}) = \phi^{(0)}(t) + \delta \phi(t, \vec{X}).$$

The perturbation can be translated, through the energy-momentum tensor to that in the density as,

$$\rho(t, \vec{X}) = \rho^{(0)}(t) + \delta \rho(t, \vec{X}),$$

where,

$$\rho^{(0)}(t) = \frac{\dot{\phi}^{(0)}(t)^2}{2} + V(\phi^{(0)})$$

and

$$\delta \rho(t, \vec{X}) = \frac{dV(\phi^{(0)})}{d\phi^{(0)}} \delta \phi + \dot{\phi}^{(0)}(t) \delta \dot{\phi} - \Phi B(\phi^{(0)})^2.$$ 

(Note: The last term in Eq. 7 was neglected in Ref. [67] as $\Phi B(\phi^{(0)})^2$ is small in slow-roll approximation. In the present paper, we retain this term as we expect the metric perturbation $\Phi$ to play a major role in the PBH formation.) Using the solutions in [69] of the unperturbed Einstein’s equations,

$$H^2 = \frac{\rho^{(0)}}{3},$$

$$\dot{H} + H^2 = -\frac{1}{6}(\rho^{(0)} + 3\rho^{(0)})$$

and the perturbed Einstein’s equations,

$$3H^2\Phi + \frac{k^2}{a^2}(-aHB) = -\delta \rho,$$

$$H\Phi = -\frac{1}{2}\delta q,$$
\[
H \Phi + (3H^2 + 2\dot{H}) \Phi = \frac{\delta p}{2},
\]
(12)

\[
(\partial_t + 3H) \frac{B}{a} + \frac{\Phi}{a^2} = 0
\]
(13)

and the Bardeen potentials \[70\]

\[
\Phi_B = \Phi - \frac{d}{dt} \left[ a^2 \left( -\frac{B}{a} \right) \right],
\]
(14)

\[
\Psi_B = a^2 H \left( -\frac{B}{a} \right)
\]
(15)

we obtain a relation between \(\Phi\) and \(\Phi_B\) as,

\[
\Phi = \Phi_B + \partial_t \left( \frac{\Phi_B}{H} \right).
\]
(16)

In Eqs. \[11\] and \[12\] \(\delta q\) and \(\delta p\) are the magnitudes of the momentum perturbation and pressure perturbation, respectively. Eqs. \[7\] and \[16\] show that the density perturbation \(\delta p\) contains \(\phi^{(0)}\), \(\delta \phi\) as well as the Bardeen potential \(\Phi_B\).

Using the slow roll dynamical horizon crossing condition, \(k = aH\), we go from the space of the cosmic time \(t\) to that of the mode momentum \(k\) and set up three nonlinear coupled differential equations in the \(k\)-space and solve for the quantities, \(\phi^{(0)}(k), \delta \phi(k)\) and \(\Phi_B(k)\). The equations, similar to those in \[67\], are,

\[
\delta \phi(k^2 \phi'' + k^2 G_1 \phi' + 4k \phi' + 6G_1) + \delta \phi'(-2k^3 G_1 \phi'^2) = 0,
\]
(17)

\[
\delta \phi(1 + 12G_1^2 + 6G_2) + \delta \phi'(4k + k^2 G_1 \phi') + k^2 \delta \phi'' + \Phi_B(-k \phi' + 12G_1 + k^2 G_1 \phi'^2 + k^3 G_1 \phi'' - 12k G_1^2 \phi') + k^3 G_2 \phi'^3 + \Phi_B'(-2k^2 \phi' + 12k G_1 + k^3 G_1 \phi'^2) + \Phi_B''(-3k \phi') = 0
\]
(18)

where, \(\Omega_n = \frac{\rho_n}{\rho_{\text{crit}}} \ln \sqrt{V(\phi^{(0)})}\), \(n = 1, 2\).

In the slow-roll approximation, the density contrast in \(k\)-space, which is useful for the study of the PBH formation, is given by,

\[
\frac{\delta \rho(k)}{\rho^{(0)}(k)} = \frac{\delta V(k) + V^{(0)}(k) \left( \frac{k^2}{3} \phi' \delta \phi' + \Phi_B \left( -\frac{k^2}{3} \phi'^2 + \frac{k^2}{3} G_1 \phi'^3 \right) + \Phi_B' \left( -\frac{k^2}{3} \phi'^2 \right) \right)}{V^{(0)}(k)}
\]
(20)

where, \(\delta V(k) = 2G_1 V^{(0)}(k) \delta \phi(k)\) and \(V^{(0)}(k) \equiv V(\phi^{(0)}(k))\). In first line of Eq. \[20\], we have used the Fourier-space version of the slow-roll approximation and, therefore, following Eq. \[4\], we have written, \(\rho^{(0)}(k) \approx V^{(0)}(k)\). We will show that, both \(\delta \phi(k)\) and \(\Phi_B(k)\) in Eq. \[20\] will play a significant role in making \(\frac{\delta \rho(k)}{\rho^{(0)}(k)} > \delta_c(t \approx 0.41)\), i.e. in the formation of PBHs in the early universe, \(\delta_c\) being the density contrast (see Sec. \[3\] for details).

B. The \(\alpha\)-attractor E-model and T-model potentials

We use here, the \(\alpha\)-attractor potentials \[71, 72\].

(I) the T-model potential

\[
V(\phi) = V_0 \tanh^\alpha \frac{\phi}{\sqrt{6} \alpha},
\]
(21)

(II) the E-model potential

\[
V(\phi) = V_0 (1 - e^{-\sqrt{\frac{2}{3\pi}} \phi})^n
\]
(22)
where \( \alpha \) is the inverse curvature of \( SU(1,1)/U(1) \) Kähler manifold \([72]\).

C. Transfer Function

We have related the quantities at horizon crossing during inflation to those at horizon re-entry in the radiation-dominated era using the transfer function given in Ref. \([54]\):

\[
T(k, \eta) = 3\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3},
\]

(23)

where \( \eta \) is the conformal time.

![Figure 1: Transfer function for radiation dominated period at a very small conformal time \( \eta = 10^{-12} \). It has a very small oscillating behaviour around \( k \sim 10^{13} \) Mpc\(^{-1} \), where \( k\eta > 1 \), although it dies down in the high \( k \) limit, \( k \to \infty \). The momentum scale reflects the range of sub-horizon momenta which will be studied in this paper.](image)

![Figure 2: Unperturbed part \( V(0)(k) \) of the chaotic \( \alpha \)-attractor \( T \)-model potential at high \( k \) limit.](image)

![Figure 3: Perturbation \( \delta V(k) \) of chaotic \( \alpha \) attractor \( T \)-model potential in high \( k \) limit. The perturbation in the potential, \( \delta V(k) \gg V(0)(k) \) (see Figure 2). It blows up as \( k \to \infty \), signifying the breakdown of the slow roll in the linear perturbation formalism and creating a situation, favourable for the PBH formation.](image)

![Figure 4: Unperturbed inflaton field \( \phi(0)(k) \) for chaotic \( \alpha \) attractor \( T \)-model potential at high \( k \) limit.](image)

III. RESULTS AND DISCUSSION

At the outset, let us give an overall perspective of the evolution of the modes during the inflationary period. As shown and stated in Ref. \([67]\), the higher \( k \) modes undergo less number of e-folds and thus, while the Hubble sphere shrinks, they exit the horizon at later times. As a consequence, the very high \( k \) modes remain in the deep subhorizon region during the major part of the inflationary period. When the Hubble sphere starts expanding after the end of inflation, the high \( k \) modes re-enter the horizon first in small positive conformal times. The smaller \( k \) modes re-enter the horizon at later conformal times.

Our interest, here, is mainly in the high \( k \) regions where, it will be shown that, the density contrast shoots up, at a number of momentum values, signalling the breakdown of the perturbative framework and creating a condition favourable for the formation of PBHs, when the modes re-enter the horizon. Interestingly, this condition is achieved in our study quite naturally by solving the \( k \)-space evolution equations and not by artificially preparing the inflaton potential to meet this goal.
of parameters which has been shown to be efficacious in fitting the Planck data in the \((n_s - r)\) plane 67 68. In Figure 2, we show the \(k\)-space behaviour of the unperturbed chaotic \(\alpha\)-attractor \(T\)-model potential in the high \(k\) limit, during inflation. In Figure 3, we demonstrate the corresponding behaviour of the perturbation in the potential, where it is shown that \(\delta V(k)\) becomes very large in the high \(k\) limit, signaling breakdown of perturbation in this limit. It may be noted that this does not happen in the low \(k\) limit, where \(\delta V(k)\) remains very small 67. In Figure 4, we have plotted the unperturbed inflaton field for the chaotic \(\alpha\)-attractor \(T\)-model potential in the high \(k\) limit and in Figure 5, the corresponding perturbation, \(\delta \phi(k)\), in the same limit. Here also we observe the breakdown of perturbation at high \(k\) values.

In Figures 6 and 7, we plotted the density contrast \((\delta \rho(k))/\rho(k)\) for chaotic \(\alpha\)-attractor \(T\)-model potential in high \(k\) limit. The perturbation in the inflaton field, \(\delta \phi(k) \gg \phi'(0)(k)\) (see Figure 1). It blows up as \(k \to \infty\), signifying the breakdown of the slow-roll in the linear perturbation formalism and creating a situation, favourable for the PBH formation.

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Figure 9: Comparison of the uses of the $\alpha$-attractor chaotic $T$-model potential (upper figure) and a power law potential, $\phi^2$ (lower figure) in calculating the density contrast $\left(\frac{\delta \rho}{\rho}\right)_{\text{inf}}$ during inflation, in the high momentum regime. While the density contrast has very large positive values in the former case, it shows large negative values in the latter case. The unphysical nature of the density contrast in high momentum regime, in the case of the power law potential, clearly indicates its unsuitability in the formation of PBHs. Same inference can be drawn during radiation dominated era also.

Figure 9 illustrates the fact that the requirement of large positive inflationary density contrasts at high $k$ values for the PBH formations, is satisfied only by the $\alpha$-attractor $T$-model potential and not by the power law type $\phi^2$ potential, for example. We have also found in Ref. [67] that, such type of potential does constitute an experimentally-favourable model for inflation at low $k$ limit within some specified range of parameters. Therefore, the $\alpha$-attractor potential in its pristine form has the capability of explaining the inflationary paradigm at the low $k$ limit and the PBH formation at the high $k$ limit, simultaneously.

Although the calculations from Figure 2 through Figure 9 are with the chaotic $\alpha$-attractor $T$ model potential, the results will be similar for the corresponding $E$ model potential. Therefore, we are not displaying those results here.

To further endorse the identification of the peaks with the PBH formation events, in Table I we have shown our calculations regarding some properties of these PBHs. The momentum-dependent PBH masses in the unit of the solar mass ($M_\odot$) have been presented in column 5. The calculation of the masses is based on the formula given in Ref. [61]. Table I suggests that, massive PBHs are formed at those values of $k$, where the density contrast shows a large positive peak and the Bardeen potential shows a large negative value, which reflects the efficacy of the Bardeen potential in the PBH formation. Thus it is right to say that the Bardeen potential acts as a source of intense gravitational field, where the tiny quantum nuggets are amplified into massive PBHs.

The mass-dependent evaporation times and the Hawking temperatures [73] are presented in columns 5 and 6 respectively. The calculations of these quantities are based on the formulae given in Ref. [54]. The evaporation time scale of the PBHs in our calculations is found in the range $10^{25} - 10^{28}$ s, which is very large in comparison to the age of the universe ($\sim 10^{17}$ s). Also, the Hawking temperatures ($\sim 10^{-5} - 10^{-8}$ GeV) are very small, showing that the possibility of extinction of such PBHs by Hawking radiation is negligibly small. Thus, these PBHs may exist in the present universe, thereby making themselves potential candidates for the dark matter.
Table I: PBH properties corresponding to the peaks in Figure 6. These properties are consistent with the LISA and BBO observations [19].

| Peak No. | $k_{\text{peak}}$ (in Mpc$^{-1}$) | $\left(\frac{\alpha^2}{\rho_{\text{rad}}}\right)_{\text{peak}}$ | $M_{\text{PBH}}$ (in $M_{\odot}$) | $t_{\text{evap}}$ (in sec.) | $T_H$ (in GeV) |
|---------|-------------------------------|--------------------------|-----------------|-----------------|--------------|
| 1       | $0.43 \times 10^{13}$         | 8.77                     | $1.35 \times 10^{-14}$ | $7.72 \times 10^{33}$ | $3.72 \times 10^{-8}$ |
| 2       | $1.05 \times 10^{13}$         | 3.49                     | $2.27 \times 10^{-14}$ | $3.67 \times 10^{31}$ | $2.21 \times 10^{-7}$ |
| 3       | $1.60 \times 10^{13}$         | 2.29                     | $9.79 \times 10^{-15}$ | $2.95 \times 10^{30}$ | $5.13 \times 10^{-7}$ |
| 4       | $2.16 \times 10^{13}$         | 1.70                     | $5.37 \times 10^{-15}$ | $4.86 \times 10^{29}$ | $9.36 \times 10^{-7}$ |
| 5       | $2.71 \times 10^{13}$         | 1.37                     | $3.41 \times 10^{-15}$ | $1.24 \times 10^{29}$ | $1.47 \times 10^{-6}$ |
| 6       | $3.25 \times 10^{13}$         | 1.14                     | $2.37 \times 10^{-15}$ | $4.18 \times 10^{28}$ | $2.12 \times 10^{-6}$ |
| 7       | $3.79 \times 10^{13}$         | 0.97                     | $1.74 \times 10^{-15}$ | $1.65 \times 10^{28}$ | $2.89 \times 10^{-6}$ |
| 8       | $4.30 \times 10^{13}$         | 0.84                     | $1.35 \times 10^{-15}$ | $7.72 \times 10^{27}$ | $3.72 \times 10^{-6}$ |
| 9       | $4.80 \times 10^{13}$         | 0.79                     | $1.09 \times 10^{-15}$ | $4.06 \times 10^{26}$ | $4.61 \times 10^{-6}$ |
| 10      | $5.44 \times 10^{13}$         | 0.71                     | $8.47 \times 10^{-16}$ | $1.91 \times 10^{27}$ | $5.93 \times 10^{-6}$ |
| 11      | $5.97 \times 10^{13}$         | 0.64                     | $7.03 \times 10^{-16}$ | $1.09 \times 10^{27}$ | $7.15 \times 10^{-6}$ |
| 12      | $6.50 \times 10^{13}$         | 0.58                     | $5.93 \times 10^{-16}$ | $6.54 \times 10^{26}$ | $8.48 \times 10^{-6}$ |
| 13      | $7.07 \times 10^{13}$         | 0.52                     | $5.01 \times 10^{-16}$ | $3.95 \times 10^{26}$ | $1.00 \times 10^{-5}$ |
| 14      | $7.60 \times 10^{13}$         | 0.49                     | $4.34 \times 10^{-16}$ | $2.57 \times 10^{26}$ | $1.15 \times 10^{-5}$ |
| 15      | $8.16 \times 10^{13}$         | 0.44                     | $3.76 \times 10^{-16}$ | $1.66 \times 10^{26}$ | $1.33 \times 10^{-5}$ |
| 16      | $8.70 \times 10^{13}$         | 0.42                     | $3.31 \times 10^{-16}$ | $1.14 \times 10^{26}$ | $1.51 \times 10^{-5}$ |
| 17      | $9.25 \times 10^{13}$         | 0.41                     | $2.93 \times 10^{-16}$ | $7.89 \times 10^{25}$ | $1.72 \times 10^{-5}$ |

IV. CONCLUSIONS

In conclusion, we have examined in this paper the possibility of the PBH formation in the inflationary perturbation theory in the spatially flat gauge with the $\alpha$-attractor inflaton potentials. The implications of the present study may be summarized in the following points.

(i) Despite the various means of enhancing the power spectrum, that has come up in the literature, such as the modification and the fine-tuning of the potential [24, 47, 57], introduction of the two-field [43, 44] and multi-field [46] scenarios (although the Planck data [68] support the single-field inflation) etc., we have shown, in the present work, that we can obtain PBH solutions in the natural way of $k$-space evolution, if we incorporate the interaction of the classically-perturbed gravitational field with the quantum fluctuations of the inflaton field and solve the resulting non-linear coupled differential equation. The PBH formations occur whenever the large fluctuations in the inflaton field meet the large negative potentials in the background gravitational field in some regions of the $k$-space.

(ii) One of the striking results, here, is that the Bardeen potential $\Phi_B(k)$ in the spatially flat gauge manifests as a driving force for accumulating mass around the inflaton perturbation, which leads to dynamical PBH formation in radiation dominated era when large sub-horizon modes re-enter the Hubble horizon in small conformal time.

(iii) The range of $k$-space viz., $0.43 \times 10^{13} - 9.25 \times 10^{13}$ Mpc$^{-1}$ for the PBH formation where density contrast exceeds the critical density (see Table [1] and Figure [6]), and Bardeen potential becomes largely negative (see Figure 7), is naturally found through self-consistent solutions of the Eqs. [17] - [19], and it is checked that no other ranges of $k$-values meet this requirement. Therefore, we claim that, this is an important outcome of the mutual interaction among $\phi^{(0)}(k)$, $\delta\phi(k)$ and $\Phi_B(k)$ in the background of $\alpha$-attractor potential in the framework of linear cosmological perturbation theory with the spatially flat gauge.

(iv) In our formalism, we consider that, the PBHs are the result of large over-density in the inflaton field under strong gravitational environment and there are no other ingredients which can influence PBH formation. For example, we are neither taking into account the PBH formation from dark matter collapse or considering PBHs as dark matter - we are simply suspecting from the large evaporation times and the small Hawking temperatures that our PBHs could be dark matter. Therefore some other prospects regarding PBHs like abundance and mass fraction should be considered as outside the scope of the present paper. Our sole aim, here, is to show, how a single formalism can explain both inflation and PBH formation from single, unmodified $\alpha$-attractor-inflaton perturbation only. Thus, some related future works under this scheme could be inclusion of dark matter and dark energy (or quintessence) in our formalism, which may tighten the $k$-range or may constrain more stringently the PBH properties, useful for upcoming experiments.

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