Unbiased time-average estimators for Markov chains

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Abstract

We consider a time-average estimator $f_k$ of a functional of a Markov chain. Under a coupling assumption, we show that the expectation of $f_k$ has a limit $\mu$ as the number of time-steps goes to infinity. We describe a modification of $f_k$ that yields an unbiased estimator $\hat{f}_k$ of $\mu$. It is shown that $\hat{f}_k$ is square-integrable and has finite expected running time. Under certain conditions, $\hat{f}_k$ can be built without any precomputations, and is asymptotically at least as efficient as $f_k$, up to a multiplicative constant arbitrarily close to 1. Our approach provides an unbiased estimator for the bias of $f_k$. We study applications to volatility forecasting, queues, and the simulation of high-dimensional Gaussian vectors. Our numerical experiments are consistent with our theoretical findings.

Keywords: multilevel Monte Carlo, unbiased estimator, steady-state, Markov chain, time-average estimator

1 Introduction

Markov chains arise in a variety of fields such as queuing networks, machine learning and healthcare. The steady-state of certain Markov chains is accurately determined via analytical tools. For instance, in a $M/M/m$ queue with $m$ servers and exponentially distributed interarrival and service times, the steady-state distribution of customers in the system is given by a simple analytical formulae. On the other hand, the steady-state behavior of queuing networks with generally distributed interarrival and service times is intractable (see (Bandi, Bertsimas and Youssef 2015) for a detailed discussion). Monte Carlo simulation can be used to study the steady-state of intractable systems. In general, Monte Carlo simulation has a high computation cost, but its performance can be improved via variance reduction techniques such as the control variate technique, moment matching, stratified sampling, importance sampling (Glasserman 2004, Asmussen and Glynn 2007) and multilevel Monte Carlo (MLMC) (Giles 2015). The related Quasi-Monte Carlo method often outperforms standard Monte Carlo simulation in low-dimensional problems and in pricing of financial derivatives (Glasserman 2004). Another issue with Monte Carlo simulation is that it sometimes produces biased estimators. For instance, the price of a financial derivative obtained by standard Monte Carlo simulation and discretization of a stochastic differential equation is usually biased. Randomized Multilevel Monte Carlo methods (RMLMC) that provide unbiased estimators for expectations of functionals associated with stochastic differential equations are given in (McLeish 2011, Rhee and Glynn 2015). Jacob and Thiery (2015) study the existence of unbiased nonnegative estimators. Unbiased estimators have been used in diverse settings including Markov chain Monte Carlo methods (Bardenet, Doucet and Holmes 2017, Agapiou, Roberts and Vollmer 2018, Middleton, Deligiannidis, Doucet and Jacob 2020, Jacob, O’Leary and Atchade 2020), estimating the expected cumulative discounted cost (Cui, Fu, Peng and Zhu 2020), pricing of discretely monitored Asian options (Kahalé 2020a), inference for hidden Markov model diffusions (Chada, Franks, Jasra, Law and Vihola 2021), and

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estimating the gradient of the log-likelihood (Jasra, Law and Lu 2021). Vihola (2018) describes stratified versions of RMLMC methods that, under general conditions, are asymptotically as efficient as MLMC. Unbiased estimators have the following advantages. First, a confidence interval is easily calculated from independent replications of an unbiased estimator. Second, taking the average of \( m \) independent copies of an unbiased estimator produces an unbiased estimator with a variance equal to that of the original estimator divided by \( m \). This leads to an efficient parallel computation of an unbiased estimator.

This paper considers a Markov chain \((X_i, i \geq 0)\) with state-space \( F \) and deterministic initial value \( X_0 \). Let \( f \) be a deterministic real-valued measurable function on \( F \) such that \( f(X_i) \) is square-integrable for \( i \geq 0 \). For \( k \geq 1 \), define the time-average estimator

\[
f_k := \frac{1}{k - b(k)} \sum_{i=b(k)}^{k-1} f(X_i),
\]

where \( b(k) \geq 0 \) is a burn-in period that may depend on \( k \). The estimator \( f_k \) is often used to estimate the limit \( \mu \) of \( E(f(X_m)) \) as \( m \) goes to infinity, when such a limit exists. Whitt (1991) studies the performance of time-average estimators in a continuous-time framework. He provides evidence that, in general, one long time-average estimator is more efficient than several independent replications of time-average estimators of shorter length. He finds that, if the simulation length is large enough to obtain reasonable estimates of \( \mu \), then several independent replications are almost as efficient as one longer run. He also shows that, in general, it is not efficient to run a very large number of independent replications with very short length.

Time-average estimators have been used in various contexts such as the sampling from a posterior distribution (Tierney 1994), computing the volume of a convex body (Cousins and Vempala 2016), and estimating the steady-state performance metrics of time-dependent queues (Whitt and You 2019). For general Markov chains, however, time-average estimators have the following drawbacks (Asmussen and Glynn 2007, p. 96). First, they are usually biased because, in general, the distribution of the \( X_i \)'s is not the steady-state distribution. Second, because of the bias and since the \( f(X_i) \)'s are usually correlated, calculating confidence intervals from time-average estimators is challenging. The method of batch means (BM) divides a single time-average into several consecutive batches, and calculates an asymptotic confidence interval from the averages over each batch. The quality of this confidence interval depends on the extent to which these averages are independent, identically distributed and Gaussian (Asmussen and Glynn 2007, p. 110). A confidence interval can also be calculated via the method of independent replications (IR), that simulates independent copies of \( f_k \), but the quality of this confidence interval depends on the bias \( E(f_k) - \mu \) of \( f_k \). Argon, Andradóttir, Alexopoulos and Goldsman (2013) study variants of the BM and IR methods.

Following Whitt (1991) and assuming \( \mu \neq 0 \), the bias of \( f_k \) can be reduced by setting \( b(k) \) equal to the smallest integer \( s \) such that \( |E(f(X_i)) - \mu| \leq |\mu|\epsilon \) for \( i \geq s \), where \( \epsilon \) is a small constant such as 0.01 or 0.001 (see also (Asmussen and Glynn 2007, p. 102)). In other words, the relative absolute bias is at most \( \epsilon \) at any time-step \( i \) larger than or equal to \( s \). Such \( s \) is closely related to the relaxation time (Asmussen and Glynn 2007, Chap. IV), and an analytic expression or approximation for \( s \) or for the bias has been calculated for certain Markov chain functionals. For instance, Whitt (1991) calculates \( s \) analytically for the number of busy servers in an \( M/G/\infty \) queue, and provides an analytic approximation for \( s \) for the \( M/M/1 \) queue length process. Asmussen and Glynn (2007, Chap. IV) show that, for general Markov chains and under suitable conditions, the bias of \( f_k \) is of order \( 1/k \) when \( b(k) = 0 \), and give an analytic approximation for the bias at a given time for the \( GI/G/1 \) queue waiting time process. While explicit convergence rates to the steady-state distribution have been established in the previous literature for many Markov chains (e.g. (Diaconis and Stroock 1991, Sinclair 1992, Cousins and Vempala 2016, Kahalé 2019, Barkhagen, Chau, Moulines, Résonyi, Sabnis and Zhang 2021, Blanca, Sinclair and Zhang 2022)), the mixing time of other Markov chains that arise in practice
is not formally known (Diaconis 2009). Furthermore, the dependence of the bias $E(f(X_i)) - \mu$ on $i$ does not always follow the same pattern: in the $M/G/\infty$ queue example, the bias can decay polynomially or exponentially in $i$, depending on the service time distribution. In the absence of knowledge on the relaxation time, Asmussen and Glynn (2007, p. 102) suggest to select $b(k)$ in an ad-hoc manner, by setting $b(k) = \lfloor k/10 \rfloor$ for instance.

The previous discussion shows that the bias makes it difficult to ascertain the quality of time-average estimators for general Markov chains. This paper provides a randomized multilevel framework for estimating and correcting the bias in time-average estimators. Under suitable conditions, we first construct a RMLMC unbiased estimator of the bias of a time-average estimator. Combining this estimator with a conventional time-average estimator yields an unbiased estimator $\hat{f}_k$ of $\mu$, that is, $E(\hat{f}_k) = \mu$. Our construction is based on a coupling assumption and a time-reversal transformation inspired from Glynn and Rhee (2014), and a RMLMC estimator introduced by Rhee and Glynn (2015). A similar coupling is used in (Kahalé 2020b, Kahalé 2022) to design and analyse variance reduction algorithms for time-varying Markov chains with finite horizon. The main contributions of our paper are as follows:

1. Our approach constructs an unbiased square-integrable estimator, that can be simulated in finite expected time, of the bias of a time-average estimator. This allows to estimate the bias and to determine the number of time-steps needed to substantially reduce it.

2. $\hat{f}_k$ is an unbiased estimator of $\mu$, is square-integrable and can be computed in finite expected time. For a suitable choice of parameters and under certain assumptions, the work-normalized variance of $\hat{f}_k$ is at most equal to that of $f_k$, up to a multiplicative factor that can be made arbitrarily close to 1 as $k$ goes to infinity. As shown by Glynn and Whitt (1992), the efficiency of an unbiased estimator can be measured through the work-normalized variance, i.e., the product of the variance and expected running time. The smaller the work-normalized variance, the higher the efficiency. The performance of a biased estimator such as $f_k$ incorporates its bias, in addition to its variance (Glasserman 2004, p. 16). Thus, for an appropriate choice of parameters, $\hat{f}_k$ is at least as efficient as $f_k$ as $k$ goes to infinity, up to a multiplicative factor arbitrarily close to 1.

3. Under suitable conditions, $\hat{f}_k$ can be constructed without any precomputations or knowledge of the relaxation time or related properties of the chain. Furthermore, our approach does not require any recurrence properties of the chain. In our numerical experiments, that use a conservative choice for the parameters, $f_k$ is about twice as efficient as $\hat{f}_k$ for large values of $k$.

For general Markov chains, we are not aware of a previous construction of an efficient unbiased estimator of the bias of a time-average estimator, or of efficient unbiased estimators for $\mu$ based on time-averaging. Assuming that $f$ is Lipschitz and that $X$ is ‘contractive on average’, Glynn and Rhee (2014) construct square-integrable unbiased RMLMC estimators for the steady-state expectation of Markov chain functionals. In view of the time-reversal transformation and RMLMC estimator used, our techniques are closely related to theirs. However, their method is not based on time-averaging, and our approach does not require $f$ to be Lipschitz or $X$ to be contractive on average. We provide several examples where $f$ is discontinuous and our method is provably efficient. Glynn and Rhee (2014) also describe another unbiased estimator for positive recurrent Harris chains. Jacob, O’Leary and Atchadé (2020) study unbiased Markov Chain Monte Carlo methods that use time-averaging.

The rest of the paper is organised as follows. Section 2 presents the coupling assumption and studies conventional time-average estimators under this assumption. In particular, it shows that the mean square error $E((f_k - \mu)^2)$ is of order $1/k$. Section 3 describes and analyses an unbiased estimator of the bias of a time-average estimator. It also constructs and studies $\hat{f}_k$ as well as a stratified version of $\hat{f}_k$ under the coupling assumption. Section 4 provides examples
Assumption 1 (A1). There is a positive decreasing sequence (i.e., $(\nu_i)_{i \geq 0}$) such that (2.5) holds with a small $\nu_i$, that is, if $f_k$ is $k$ units of time, we assume that there are independent and identically distributed (i.i.d.) random variables $U_i$, $i \geq 0$, that take values in a measurable space $F'$, and a measurable function $g$ from $F \times F'$ to $F$ such that, for $i \geq 0$,

$$X_{i+1} = g(X_i, U_i).$$

(1.1)

2 Conventional time-average estimators

We introduce the coupling assumption in Subsection 2.1 and use it in Subsection 2.2 to establish bounds on the bias, standard deviation and mean square error of conventional time-average estimators. Subsection 2.3 describes an example showing the sharpness of the standard deviation and mean square error bounds.

2.1 The coupling assumption

Extend the random sequence $(U_i, i \geq 0)$ to all $i \in \mathbb{Z}$, so that $U_i$, $i \in \mathbb{Z}$, are i.i.d. random variables taking values in $F'$. For $i \geq 0$, define recursively the measurable function $G_i$ from $F \times F^i$ to $F$ by setting $G_0(x) := x$ and

$$G_{i+1}(x; u_0, \ldots, u_i) := g(G_i(x; u_0, \ldots, u_{i-1}), u_i),$$

for $x \in F$ and $u_0, \ldots, u_i \in F'$. It can be shown by induction that, for $i \geq 0$,

$$X_i = G_i(X_0; U_0, \ldots, U_{i-1}).$$

(2.1)

For $m \in \mathbb{Z}$ and $i \geq -m$, let

$$X_{i,m} := G_{i+m}(X_0; U_{-m}, U_{-m+1}, \ldots, U_{i-1}).$$

(2.2)

Thus $X_{i,0} = X_i$ for $i \geq 0$. By (2.1), $X_{i,m} \sim X_{i+m}$ for $m \in \mathbb{Z}$ and $i \geq -m$, where ‘∼’ denotes equality in distribution. Furthermore,

$$X_{-m,m} = X_0, \text{ and } X_{i+1,m} = g(X_{i,m}, U_i).$$

(2.3)

In other words, $(X_{i,m}, i \geq -m)$ is a Markov chain that is a copy of $(X_i, i \geq 0)$, and is driven by $(U_i, i \geq -m)$. For $i, m \geq 0$, the last $i$ random variables driving the calculation of $X_i$ and $X_{i,m}$, i.e., $U_0, \ldots, U_{i-1}$, are the same. This leads us to state the following assumption.

Assumption 1 (A1). There is a positive decreasing sequence $(\nu(i), i \geq 0)$ such that

$$\sum_{i=0}^{\infty} \frac{\nu(i)}{i+1} < \infty,$$

(2.4)

and, for $i, m \geq 0$,

$$E((f(X_{i,m}) - f(X_i))^2) \leq \nu(i).$$

(2.5)

Intuitively speaking, (2.5) holds with a small $\nu(i)$ if, for $h \geq i$, $f(X_h)$ is mainly determined by $(U_{h-i}, \ldots, U_{h-1})$, that is, if $f(X_k)$ depends to a large extent on the last $i$ copies of $U_0$ driving the Markov chain $(X_k : 0 \leq k \leq h)$. Proposition 2.1 shows that Assumption A1 holds under a condition similar to (2.5). Note that (2.5) and (2.6) are identical if $X_0 = x$. 

and Section 5 presents numerical experiments. Omitted proofs are in the appendix. Throughout the paper, the running time refers to the number of arithmetic operations. For simplicity, it is supposed that $b(k) \leq k/2$ for $k \geq 1$, and that the expected time to simulate $f_k$ is $k$ units of time. The discussion follows the same line of thought as in Section 4.
Proposition 2.1. Suppose there is \( x \in F \) and a positive decreasing sequence \((\nu'(i), i \geq 0)\) that satisfies (2.4) and, for \( i, m \geq 0 \),
\[
E((f(G_i(x;U_0, \ldots, U_{i-1}))) - f(X_{i,m})))^2) \leq \nu'(i).
\] (2.6)
Then Assumption A1 holds with \( \nu(i) = 4\nu'(i) \) for \( i \geq 0 \).

Proof. Applying (2.6) with \( m = 0 \) shows that, for \( i \geq 0 \),
\[
E((f(G_i(x;U_0, \ldots, U_{i-1}))) - f(X_i)))^2) \leq \nu'(i).
\]
Together with (2.6), and since \( E((Z + Z')^2) \leq 2(E(Z^2) + E(Z'^2)) \) for square-integrable random variables \( Z \) and \( Z' \), this implies that, for \( i, m \geq 0 \),
\[
E((f(X_{i,m}) - f(X_i)))^2) \leq 4\nu'(i).
\]

Assumption A2 stated below is stronger than Assumption A1 and says that the \( \nu(i) \)'s decay exponentially with \( i \).

Assumption 2 (A2). Assumption A1 holds with \( \nu(i) \leq ce^{-\xi i} \) for \( i \geq 0 \), where \( c \) and \( \xi \) are positive constants with \( \xi \leq 1 \).

Proposition 2.2 shows that Assumption A1 holds under certain conditions. As \( \eta \leq e^{\gamma - 1} \) for \( \eta \in \mathbb{R} \), Assumption A2 holds as well under the same conditions.

Proposition 2.2. Assume that \( F \) is a metric space with metric \( \rho : F \times F \rightarrow \mathbb{R}_+ \) and there are positive constants \( \eta, \kappa, \kappa' \) and \( \gamma \) with \( \eta < 1 \) such that, for \( i, m \geq 0 \),
\[
E((f(X_{i,m}) - f(X_i)))^2) \leq \kappa^2(E(\rho^2(X_{i,m}, X_i)))^\gamma,
\] (2.7)
and
\[
E(\rho^2(X_0, X_i)) \leq \kappa',
\] (2.8)
and, for \( x, x' \in F \),
\[
E(\rho^2(g(x, U_0), g(x', U_0))) \leq \eta \rho^2(x, x').
\] (2.9)
Then Assumption A1 holds with \( \nu(i) = \kappa^2\kappa'^\eta^\gamma i \) for \( i \geq 0 \).

The generalized Lipschitz condition (2.7) obviously holds for Lipschitz functions. Examples of non-Lipschitz functions, including discontinuous functions, that satisfy (2.7), are given by Kahalé (2019) in the context of simulating high-dimensional Gaussian vectors. Condition (2.8) says that the expected square distance between \( X_0 \) and \( X_i \) is bounded. The contractivity condition (2.9) is used by Glynn and Rhee (2014) to obtain unbiased estimators for Markov chains.

2.2 Convergence properties

Given a non-negative sequence \((\omega(i), i \geq 0)\) such that \( \sum_{i=0}^{\infty} \sqrt{\omega(i)/(i+1)} \) is finite, set
\[
\omega(j) := \sum_{i=j}^{\infty} \sqrt{\omega(i)/(i+1)},
\]
for \( j \geq 0 \). Note that \( \omega(j) \) is finite and is a decreasing function of \( j \), and that \( \omega(j) \) goes to 0 as \( j \) goes to infinity. Theorem 2.1 shows that, under Assumption A1, the sequence \( (E(f(X_h)), h \geq 0) \) is convergent and examines the convergence properties of \( f(X_h) \) and of standard time-average estimators.
Theorem 2.1. Suppose that Assumption A1 holds. Then \( E(f(X_h)) \) has a finite limit \( \mu \) as \( h \) goes to infinity. For \( h \geq 0 \),
\[
|E(f(X_h)) - \mu| \leq \sqrt{\nu(h)}. \tag{2.10}
\]
For \( h \geq 0 \) and \( k > 0 \),
\[
|E\left( \frac{1}{k} \sum_{i=h}^{h+k-1} f(X_i) \right) - \mu| \leq \frac{\varpi(h/2)}{\sqrt{k}}, \tag{2.11}
\]
and
\[
E\left( \left( \frac{1}{k} \sum_{i=h}^{h+k-1} f(X_i) - \mu \right)^2 \right) \leq \frac{26(\varpi(0))^2}{k}. \tag{2.12}
\]

Equations (2.10), (2.11) and (2.12) provide upper-bounds on the absolute bias of \( f(X_h) \), and on the absolute bias and mean square error of time-average estimators of \( \mu \). Under Assumption A1, Theorem 2.1 implies that \( f_k \) is an estimator of \( \mu \) with mean square error \( E((f_k - \mu)^2) = O(1/k) \). Furthermore, if \( b(k) \) goes to infinity with \( k \), then \( |E(f_k) - \mu| = o(1/\sqrt{k}) \), i.e., \( \sqrt{k}E(f_k) - \mu \) goes to 0 as \( k \) goes to infinity. Also, if \( \sum_{i=0}^{\infty} \sqrt{\nu(i)} < \infty \), then (2.10) implies immediately a bound of order \( 1/k \) on \( |E(f_k) - \mu| \). In both cases, the absolute bias \( |E(f_k) - \mu| \) is asymptotically negligible, as \( k \) goes to infinity, in comparison with the bound of order \( 1/\sqrt{k} \) on \( \text{Std}(f_k) \) implied by Lemma 2.1 below. The proof of Theorem 2.1 relies on Lemma 2.1.

Lemma 2.1. Suppose that A1 holds. Then, for \( h \geq 0 \) and \( k > 0 \),
\[
\text{Std}\left( \frac{1}{k} \sum_{i=h}^{h+k-1} f(X_i) \right) \leq \frac{5\varpi(0)}{\sqrt{k}}. \tag{2.13}
\]

The mean square error bound (2.12) is proportional to \((\varpi(0))^2\). Proposition 2.3 provides bounds on \( \varpi(0) \). Under Assumption A2, the bound on \( \varpi(0) \) is inversely proportional to \( \sqrt{\xi} \). Under an additional decay assumption on \( \nu \), it is a polylogarithmic function of \( \xi \).

Proposition 2.3. Suppose that Assumption A2 holds. Then
\[
\varpi(0) \leq 9\sqrt{\frac{c}{\xi}}. \tag{2.14}
\]
Moreover, if \( \nu(i) \leq c/(i + 1) \) for \( i \geq 0 \) then
\[
\varpi(0) \leq 14\sqrt{c} \ln \left( \frac{2}{\xi} \right). \tag{2.15}
\]

2.3 Sharpness of bounds

This subsection gives an example proving the optimality of (2.12), (2.13) and (2.14), up to a multiplicative constant. Consider the real-valued autoregressive sequence \((X_i, i \geq 0)\) given by the recursion
\[
X_{i+1} = \sqrt{\eta}X_i + U_i,
\]
for \( i \geq 0 \), with \( X_0 = 0 \), where \( \eta \in [0,1) \) and \( U_i, i \geq 0 \), are real-valued i.i.d. with \( E(U_i) = 0 \) and \( \text{Var}(U_i) = 1 \). In this example, \( F = F' = \mathbb{R} \) and \( g(x, u) = \sqrt{\eta}x + u \). Assume that \( f \) is the identity function on \( \mathbb{R} \) and that \( b(k) = 0 \). It is easy to verify by induction that \( E(X_i) = 0 \) and \( \text{Var}(X_i) \leq 1/(1 - \eta) \) for \( i \geq 0 \). The conditions in Proposition 2.2 hold for the Euclidean distance \( \rho(x, x') = \|x - x'\| \) for \( (x, x') \in \mathbb{R}^2 \), with \( \kappa = \gamma = 1 \) and \( \kappa' = 1/(1 - \eta) \). Thus, Assumption A1 holds with \( \nu(i) = \eta^i/(1 - \eta) \) for \( i \geq 0 \), and Assumption A2 holds with \( c = 1/(1 - \eta) \) and
\[ \xi = 1 - \eta. \] Applied (2.12) with \( h = 0 \) and noting that \( \mu = 0 \) shows, in combination with (2.14), that
\[ \text{Var}(f_k) \leq \frac{2106}{k(1-\eta)^2}. \]
On the other hand, it can be shown by induction that, for \( i \geq 0, \)
\[ X_i = \sum_{j=0}^{i-1} (\sqrt{\eta})^{i-1-j} U_j, \]
and, for \( k \geq 0, \)
\[ \sum_{i=0}^{k} X_i = \sum_{j=0}^{k} \frac{1 - (\sqrt{\eta})^{k-j}}{1 - \sqrt{\eta}} U_j. \]
Consequently,
\[ \text{Var} \left( \sum_{i=0}^{k} X_i \right) = \sum_{j=0}^{k} \alpha_j^2, \]
where \( \alpha_j := \frac{(1 - (\sqrt{\eta})^{j})}{(1 - \sqrt{\eta})}. \)
By standard calculations, \( 2\alpha_j \geq 1/(1-\eta) \) for \( j \geq j_0, \) where \( j_0 := \lceil 2/\log_2(1/\eta) \rceil. \)
Thus, for \( k \geq 2j_0, \) we have
\[ \text{Var}(f_k) \geq \frac{1}{8k(1-\eta)^2}. \]
This implies that (2.12) as well as (2.14) are tight, up to an absolute multiplicative constant. The same calculations show that (2.13) is tight, as well.

3 Unbiased time-average estimators

Subsection 3.1 recalls the single term estimator, a RMLMC estimator introduced by Rhee and Glynn (2015). Subsection 3.2 uses this estimator to construct an unbiased time-average estimator \( \hat{f}_k. \) Subsection 3.3 shows how to choose the parameters used to construct \( \hat{f}_k \) in order to ensure that \( \hat{f}_k \) has good convergence properties. Some of these parameters are calculated in terms of \( \tilde{\nu}, \) though. Under additional assumptions, Subsection 3.4 provides choices for these parameters without explicit knowledge of \( \tilde{\nu}. \) Subsection 3.5 describes a stratified version of \( \hat{f}_k. \) Subsection 3.6 gives implementation details.

3.1 The single term estimator

Let \( (Y_l, l \geq 0) \) be a sequence of square-integrable random variables such that \( E(Y_l) \) has a limit \( \mu_Y \) as \( l \) goes to infinity. Consider a probability distribution \( (p_l, l \geq 0) \) such that \( p_l > 0 \) for \( l \geq 0. \) Let \( N \in \mathbb{N} \) be an integral random variable independent of \( (Y_l, l \geq 0) \) such that \( \text{Pr}(N = l) = p_l \) for \( l \geq 0. \) Theorem 3.1 due to Rhee and Glynn (2015) (see also (Vihola 2018, Theorem 2)), describes the single term estimator \( Z \) and shows that, under suitable conditions, it has expectation equal to \( \mu_Y. \)

**Theorem 3.1 ((Rhee and Glynn 2015)).** Set \( Z := (Y_N - Y_{N-1})/p_N, \) with \( Y_{-1} := 0. \) If \( \sum_{l=0}^{\infty} E((Y_l - Y_{l-1})^2)/p_l \) is finite then \( Z \) is square-integrable, \( E(Z) = \mu_Y, \) and
\[ E(Z^2) = \sum_{l=0}^{\infty} \frac{E((Y_l - Y_{l-1})^2)}{p_l}. \]
3.2 Construction of $\hat{f}_k$

This subsection supposes that Assumption A1 holds and constructs $\hat{f}_k$ along the following steps:

1. Build a random sequence $(f_{k,l}, l \geq 0)$ such that $E(f_{k,l}) \to \mu$ as $l$ goes to infinity and $f_{k,0}$ is a standard time-average estimator with burn-in period $b'(k) \in [b(k), k/2]$.

2. Use the sequence $(f_{k,l}, l \geq 0)$ to construct a RMLMC estimator $Z_k$ with $E(Z_k) = \mu - E(f_{k,0})$.

3. Combine $f_{k,0}$ and $Z_k$ to produce $\hat{f}_k$.

First, we detail Step 1. For $k \geq 1$, let $b'(k)$ be a burn-in period with $b(k) \leq b'(k) \leq k/2$. Different choices for $b'(k)$ will be studied in Subsections 3.3 and 3.4. For $k \geq 1$ and $l \geq 0$, let

$$f_{k,l} := \frac{1}{k - b'(k)} \sum_{i=b'(k)}^{k-1} f(X_{i,k(2^l-1)}).$$  \hspace{1cm} (3.1)

In particular,

$$f_{k,0} = \frac{1}{k - b'(k)} \sum_{i=b'(k)}^{k-1} f(X_i).$$  \hspace{1cm} (3.2)

As $X_{i,k(2^l-1)} \sim X_{i+k(2^l-1)}$, Theorem 2.1 implies that $E(f_{k,l}) \to \mu$ as $l$ goes to infinity. By (2.2), for $0 \leq l < l'$ and $i \geq 0$, the last $i + m$ copies of $U_0$ used to calculate $f_{k,l}$ and $f_{k,l'}$ are the same, where $m = k(2^l - 1)$. Thus, intuitively speaking, $f_{k,l'}$ should be close to $f_{k,l}$ for large values of $l$, and increasing $b'(k)$ should make $f_{k,l'}$ closer to $f_{k,l}$ even for small $l$. For simplicity, it is assumed that the expected time to simulate $f_{k,l}$ is equal to $k2^l$. This assumption is justified by the fact that $f_{k,l}$ is calculated by generating $U_{-m}, \ldots, U_{k-2}$, and using (2.3) to calculate $X_{-m,m}, \ldots, X_{k-1,m}$. Lemma 3.1 gives an upper bound on the variance of $f_{k,0}$ in terms of that of $f_k$.

**Lemma 3.1.** For $k \geq 1$,

$$\text{Var}(f_{k,0}) \leq \frac{796(\varpi(0))^2}{k^{3/2}} \sqrt{b'(k) - b(k)} + \text{Var}(f_k).$$

Next, we describe Step 2. Let $(p_l, l \geq 0)$ be a probability distribution on $\mathbb{N}$ with $p_l > 0$ for $l \geq 0$. For $k \geq 1$, let

$$Z_k^{(b'(k))} := \frac{f_{k,N+1} - f_{k,N}}{PN},$$  \hspace{1cm} (3.3)

where $N \in \mathbb{N}$ is an integer-valued random variable independent of $(U_i, i \in \mathbb{Z})$ such that $\text{Pr}(N = l) = p_l$ for $l \geq 0$. For simplicity, we will often denote $Z_k^{(b'(k))}$ by $Z_k$. Let $T_k$ be the expected time required to simulate $Z_k$. Lemma 3.2 provides bounds on $T_k$ and on the second moment of $Z_k$ and shows that, under certain conditions, $Z_k$ is an unbiased estimator for the negated bias $\mu - E(f_{k,0})$.

**Lemma 3.2.** For $k \geq 1$, we have $T_k \leq 3k \sum_{l=0}^{\infty} 2^l p_l$, and

$$kE(Z_k^2) \leq 2(\varpi([b'(k)/2]))2(\frac{1}{p_0} + \frac{1}{p_1} + \sum_{l=2}^{\infty} \frac{2^{2-l}(\varpi(k2^{l-2}) - \varpi(k2^{l-1}))^2}{p_l}).$$  \hspace{1cm} (3.4)

If the right-hand side of (3.3) is finite, then $Z_k$ is square-integrable and $E(f_{k,0} + Z_k) = \mu$. 

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We now detail Step 3. Given \( q \in (0, 1] \), let \( Z_k' \) be a copy of \( Z_k \) independent of \( f_{k,0} \) and let \( Q \) be a binary variable independent of \((f_{k,0}, Z_k')\) such that \( \Pr(Q = 1) = q \). Set
\[
\hat{f}_k := f_{k,0} + q^{-1}QZ_k'.
\] (3.5)

In other words, \( \hat{f}_k \) is constructed by sampling \( f_{k,0} \) once and sampling a copy of \( Z_k \) with frequency \( q \). By Lemma 3.2, if the right-hand side of (3.3) is finite, then \( E(\hat{f}_k) = E(f_{k,0}) + E(Z_k) = \mu \), and \( \hat{f}_k \) is an unbiased estimator of \( \mu \). When \( q = 1 \), copies of \( f_{k,0} \) and of \( Z_k \) are sampled with the same frequency. When \( q < 1 \), \( Z_k \) is sampled less often than \( f_{k,0} \), which can improve the efficiency of \( \hat{f}_k \), in the same spirit as the Multilevel Monte Carlo Method (MLMC) (Giles 2008) and the randomized dimension reduction algorithm (Kahalé 2020b). Selecting the \( p_l \)'s and \( q \) is studied in Subsections 3.3 and 3.4. Let \( T_k \) be the expected time to simulate \( \hat{f}_k \). As the expected time to simulate \( f_{k,0} \) is equal to \( k \), we have \( T_k = k + qT_k \). Note that the estimator \( Z_k \) is interesting by itself as it provides an unbiased estimator for the bias of \( f_k \) if we set \( b'(k) = b(k) \). We now state the following assumption:

**Assumption B.** There is a positive real number \( w_0 \) such that \( k\text{Var}(f_k) \geq w_0 \) for sufficiently large \( k \).

When \( b(k) = 0 \), Assumption B can be shown under certain correlation hypotheses (Asmussen and Glynn 2007, p. 99).

### 3.3 \( \nu \)-dependent parameters

This subsection gives a construction of \((p_l, l \geq 0)\) and of \( q \) in terms of \( \nu \). For \( l \geq 2 \), set
\[
p_l = \frac{\nu(k2^{l-2}) - \nu(k2^{l-1})}{2\nu(k)},
\] (3.6)
and
\[
p_1 = (1 - \sum_{l=2}^{\infty} p_l)/3 \quad \text{and} \quad p_0 = 2p_1.
\] (3.7)

Note that \( p_l > 0 \) for \( l \geq 2 \) since \((\nu(i), i \geq 0)\) is a strictly decreasing sequence. Furthermore,
\[
\sum_{l=2}^{\infty} 2^l p_l = \sum_{l=2}^{\infty} \frac{\nu(k2^{l-2}) - \nu(k2^{l-1})}{\nu(k)} = 1.
\] (3.8)

Hence \( \sum_{l=2}^{\infty} p_l \leq 1/4 \). Consequently, \( p_1 \geq 1/4 \), \( p_0 \geq 1/2 \), and \((p_l, l \geq 0)\) is a probability distribution. The \( p_l \)'s have been chosen so that the summands in the bounds on \( T_k \) and \( E(Z_k^2) \) in Lemma 3.2 are proportional for \( l \geq 2 \). Lemma 3.3 shows that \( Z_k \) is an unbiased estimator of \( \mu - E(f_{k,0}) \) and provides bounds on its second moment and expected running time. Note that the bound on \( E(Z_k^2) \) is, up to a multiplicative constant, the square of the bound on \( |E(Z_k)| \) that follows from (2.11) and the equality \( E(Z_k) = \mu - E(f_{k,0}) \).

**Lemma 3.3.** Suppose that A1 holds and that \((p_l, l \geq 0)\) are given by (3.6) and (3.7). For \( k \geq 1 \), we have \( E(f_{k,0} + Z_k) = \mu \), \( T_k \leq 9k \), and
\[
kE(Z_k^2) \leq 20(\nu([b'(k)/2]))^2.
\] (3.9)

Set
\[
q = \frac{\nu([b'(k)/2])}{\nu(0)}.
\] (3.10)

Section H gives a motivation for (3.10).
\textbf{Theorem 3.2.} Suppose that A1 holds, that $k \geq 1$, and that $(p_l, l \geq 0)$ and $q$ are given by (3.6), (3.7) and (3.10). Then $f_k$ is square-integrable and $E(f_k) = \mu$. Moreover, $T_k \leq k + 9qk$, and

$$T_k \text{Var}(f_k) \leq k \text{Var}(f_k) + 8610(\overline{\nu}(0))^2 \max \left( q, \sqrt{\frac{b'(k) - b(k)}{k}} \right).$$

(3.11) gives a bound on the work-normalized variance of $\hat{f}_k$ in terms of the work-normalized variance of $f_k$. The constant 8610 is an artifact of our calculations. By setting $b'(k) = \max(b(k), \sqrt{k})/2$, it is easy to check that the second term in the RHS of (3.11) goes to 0 as $k$ goes to infinity. Consequently, under Assumption B, for any given $\epsilon > 0$, we have

$$T_k \text{Var}(\hat{f}_k) \leq (1 + \epsilon)k \text{Var}(f_k)$$

for sufficiently large $k$. In other words, the work-normalized variance of $\hat{f}_k$ is at most equal to that of $f_k$, up to the multiplicative factor $1 + \epsilon$. Thus, $\hat{f}_k$ is asymptotically at least as efficient as $f_k$, as $k$ goes to infinity, up to a multiplicative constant arbitrarily close to 1.

### 3.4 Oblivious parameters

When the sequence $\overline{\nu}$ is known or can be estimated, the choices of $(p_l, l \geq 0)$, of $q$ and of $b'(k)$ in Subsection 3.3 yield an $\hat{f}_k$ that is asymptotically at least as efficient as $f_k$. Under certain assumptions and without explicit knowledge of $\nu$, this subsection provides choices of $(p_l, l \geq 0)$, of $q$ and of $b'(k)$ so that the work-normalized variance of $\hat{f}_k$ is at most equal to that of $f_k$, up to a multiplicative factor arbitrarily close to 1. We first state the following assumption.

\textbf{Assumption 3 (A3).} For $l \geq 0$,

$$p_l = \frac{1}{\theta(l)2^l} - \frac{1}{\theta(l + 1)2^{l+1}},$$

(3.12)

where $\theta$ is an increasing function on $[0, \infty)$, with $\theta(x) = 1$ for $x \in [0, 1]$,

$$\sum_{l=0}^{\infty} \frac{1}{\theta(l)} < \infty.$$  

(3.13)

Furthermore, Assumption A1 holds and

$$\sum_{i=0}^{\infty} \sqrt{\frac{\nu(i)\theta(\log_2(4i + 1))}{i + 1}} < \infty.$$  

(3.14)

Observe that the $p_l$’s given in (3.12) depend only on $\theta$, and that (3.14) is a stronger version of (2.4). Standard calculations show the following.

\textbf{Example 3.1.} Suppose that, for some positive constants $c$, $\xi$ and $\delta$ with $\delta < \xi - 1$, Assumption A1 holds with $\nu(i) = c(i + 1)^{-\xi}$ for $i \geq 0$, and that the $p_l$’s are given by (3.12), with $\theta(x) = 1$ for $x \in [0, 1]$, and $\theta(x) = 2^{\delta(x-1)}$ for $x \geq 1$. Then Assumption A3 holds and $p_l$ is of order $2^{-(\delta+1)l}$.

Distributions with exponentially decreasing tails have been previously used in RMLMC pricing of financial derivatives (Rhee and Glynn 2015, Kahale 2020a). In Example 3.1 the choice of the $p_l$’s depends on $\xi$ because of the condition $\delta < \xi - 1$. Example 3.2 shows that the $p_l$’s can chosen without any knowledge on $\xi$.

\textbf{Example 3.2.} Suppose that, for some positive constants $c$ and $\xi$ with $\xi > 1$, Assumption A1 holds with $\nu(i) = c(i + 1)^{-\xi}$ for $i \geq 0$, and that the $p_l$’s are given by (3.12), with $\theta(x) = \max(1, x)^{\delta}$ for $x \geq 0$, where $\delta > 1$. Then Assumption A3 holds and $p_l$ is of order $l^{-\delta}2^{-l}$.
When Assumption A2 holds, Example 3.3 shows that the $p_j$’s can be chosen as in Example 3.1 without any further knowledge on $\nu$.

**Example 3.3.** Suppose that Assumption A2 holds and that the $p_j$’s are given by (3.12), with $\theta(x) = 1$ for $x \in [0, 1]$, and $\theta(x) = 2^k(x - 1)$ for $x \geq 1$, where $\delta$ is a positive constant. Then Assumption A3 holds.

Suppose now that Assumption A3 holds. For $j \geq 0$, let

$$\mathfrak{p}_\theta(j) := \sum_{i=j}^{\infty} \frac{\nu(i)\theta(\log_2(4i + 1))}{i + 1}.$$ 

Assumption A3 shows that $\mathfrak{p}_\theta(j)$ is finite and goes to 0 as $j$ goes to infinity, and that $\mathfrak{p} (j) \leq \mathfrak{p}_\theta(j)$ for $j \geq 0$. Lemma 3.4 shows that, under Assumption A3, $Z_k$ is an unbiased estimator of $\mu - E(f_{k,0})$, and provides a bound on its second moment and on $T_k$.

**Lemma 3.4.** Suppose that Assumption A3 holds. Then, for $k \geq 1$, we have $E(f_{k,0} + Z_k) = \mu$ and

$$kE(Z_k^2) \leq 28(\mathfrak{p}_\theta(|b'(k)/2|))^2.$$  \hspace{1cm} (3.15)

Furthermore, $T_k \leq 3k \sum_{l=0}^{\infty} 1/\theta(l)$.

Theorem 3.3 shows that, under Assumption A3, $\hat{f}_k$ is an unbiased estimator of $\mu$ and gives bounds on its running time and variance.

**Theorem 3.3.** Suppose that Assumption A3 holds. Then, for $k \geq 1$, $\hat{f}_k$ is square-integrable, $E(\hat{f}_k) = \mu$, and

$$k\text{Var}(\hat{f}_k) \leq k\text{Var}(f_k) + 796(\mathfrak{p}(0))^2 \sqrt{\frac{b'(k) - b(k)}{k}} + \frac{28}{k} (\mathfrak{p}_\theta(|b'(k)/2|))^2.$$  \hspace{1cm} (3.16)

Moreover, $T_k \leq k + 3(\sum_{l=0}^{\infty} 1/\theta(l))qk$.

Observe that the second (resp. last) term in the right-hand side of (3.16) is an increasing (resp. decreasing) function of $b'(k)$. Likewise, the bound on the variance (resp. running time) of $\hat{f}_k$ is a decreasing (resp. increasing) function of $q$. Theorem 3.3 shows that setting

$$q = \frac{\epsilon}{3 \sum_{l=0}^{\infty} 1/\theta(l)},$$

where $\epsilon \in (0, 1)$, ensures that $\hat{T}_k \leq k(1 + \epsilon)$. Furthermore, if $b'(k) = \max(b(k), \sqrt{k})/2$, then $k\text{Var}(\hat{f}_k) \leq k\text{Var}(f_k) + \epsilon$ for sufficiently large $k$. This is because $\mathfrak{p}_\theta(j)$ goes to 0 as $j$ goes to infinity. Then, under Assumption B, for any given $\epsilon' > 0$, if $\epsilon$ is sufficiently small and $k$ sufficiently large, we have

$$\hat{T}_k \text{Var}(\hat{f}_k) \leq (1 + \epsilon')k\text{Var}(f_k).$$

Here again, $\hat{f}_k$ is asymptotically at least as efficient as $f_k$, as $k$ goes to infinity, up to a multiplicative factor arbitrarily close to 1. In practice, in the absence of precise knowledge on the behavior of the chain, setting $b'(k) = \max(b(k), |\epsilon''k|)$, where $\epsilon'' \in (0, 1/2]$, e.g., $\epsilon'' = 0.1$, would make $b'(k)$ reasonably large without deleting too many observations.

Under Assumption A2, and for specific values of the $p_j$’s, Theorem 3.4 gives a bound on the variance of $\hat{f}_k$ that depends explicitly on $c$ and $\xi$. It also provides an improved variance bound under an additional decay assumption on $\nu$. 

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Theorem 3.4. Suppose that Assumption A2 holds, and that the $p_i$’s are given by (3.12), with $	heta(x) = \max(1,x)^\delta$ for $x \geq 0,$ where $\delta \in (1,2]$. Then, for $k \geq 1$, Assumption A3 holds,

$$\tilde{T}_k \leq k(1 + \frac{9q}{\delta - 1}), \quad \text{(3.17)}$$

and

$$k\text{Var}(\hat{f}_k) \leq k\text{Var}(f_k) + \frac{Ac}{\xi} \sqrt{\frac{b'(k) - b(k)}{k}} + \frac{Ac}{q\xi} \min \left( \ln^\delta \left( \frac{3}{\xi} \right), \frac{e^{-\xi b'(k)/2}}{\xi} \right), \quad \text{(3.18)}$$

where $A$ is an absolute constant. Moreover, if $\nu(i) \leq c/(i+1)$ for $i \geq 0$, then

$$k\text{Var}(\hat{f}_k) \leq k\text{Var}(f_k) + A'c \ln^2 \left( \frac{3}{\xi} \right) \sqrt{\frac{b'(k) - b(k)}{k}} + \frac{A'c}{q} \min \left( \ln^\delta \left( \frac{3}{\xi} \right), \frac{e^{-\xi b'(k)/2}}{\xi^2} \right), \quad \text{(3.19)}$$

where $A'$ is an absolute constant.

The second term in the RHS of (3.18) is of order $1/\xi$, while the second term in the RHS of (3.19) has a logarithmic dependence on $\xi$. Both terms can be made arbitrarily small by setting $b'(k) = \max(b(k), \lfloor\epsilon k\rfloor)$, with $\epsilon \in (0, 1/2]$ sufficiently small. For fixed $q$, the last term in the RHS of (3.18) is uniformly bounded by a term of order $1/\xi$, up to a polylogarithmic factor, while the last term in the RHS of (3.19) is uniformly bounded by a term with a logarithmic dependence on $\xi$. When $b'(k)$ is proportional to $k$, both terms decrease exponentially with $k$.

Remark 3.1. The results of Theorem 3.4 are still valid if the constraint $\delta \in (1,2]$ is replaced with $\delta \in (1, \delta_0]$, for any fixed $\delta_0 > 1$, and if $A$ and $A'$ and the constant $9$ in (3.17) are replaced with constants that depend on $\delta_0$.

3.5 A stratified unbiased estimator

Given $n,k \geq 1$ and $q \in (0,1)$, let

$$\tilde{f}_{k,n} := \tilde{f}_k + \tilde{Z}_k,$$

where $\tilde{f}_k$ is the average of $n$ independent copies of $f_{k,0}$ and $\tilde{Z}_k$ is the average of $\lceil nq \rceil$ independent copies of $Z_k$. The estimator $\tilde{f}_{k,n}$ is a stratified version of $\hat{f}_k$ and has similar properties. By Lemma 3.4 under Assumption 3,

$$E(\tilde{f}_{k,n}) = E(f_{k,0}) + E(Z_k) = \mu.$$

Furthermore,

$$\text{Var}(\tilde{f}_{k,n}) = \frac{\text{Var}(f_{k,0})}{n} + \frac{\text{Var}(Z_k)}{\lceil nq \rceil}.$$

On the other hand, it follows from the definition of $\tilde{f}_k$ that

$$\text{Var}(\tilde{f}_k) = \text{Var}(f_{k,0}) + q^{-2}\text{Var}(QZ_k^2) \geq \text{Var}(f_{k,0}) + q^{-1}\text{Var}(Z_k).$$

Thus, $n\text{Var}(\tilde{f}_{k,n}) \leq \text{Var}(\tilde{f}_k)$. The expected time to simulate $\tilde{f}_{k,n}$ is $\tilde{T}_{k,n} = nk + \lceil nq \rceil \tilde{T}_k$. Thus $\tilde{T}_{k,n} \leq n\tilde{T}_k + \tilde{T}_k$ and $\tilde{T}_{k,n}\text{Var}(\tilde{f}_{k,n}) \leq (\tilde{T}_k + T_k/n)\text{Var}(\tilde{f}_k)$. Consequently, as $n$ goes to infinity, the estimator $\tilde{f}_{k,n}$ is asymptotically at least as efficient as $\tilde{f}_k$. 

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3.6 Implementation details

Algorithm 1 Procedure LongRun

1: procedure LR($B, K, X[0], \ldots, X[h], S[0], \ldots, S[h]$)
2:     for $j \leftarrow 0, h$ do
3:         $S[j] \leftarrow 0$
4:     end for
5:     for $i \leftarrow 0, K - 1$ do
6:         Simulate $V \sim U_0$
7:         for $j \leftarrow 0, h$ do
8:             if $i \geq B$ then
9:                 $S[j] \leftarrow S[j] + f(X[j])$
10:             end if
11:         end for
12:     end for
13: end procedure

Algorithm 2 Procedure Bias

1: procedure BIAS($k, b'(k), (p_l, l \geq 0))$
2:     Simulate a random variable $N$ such that $\Pr(N = l) = p_l$ for $l \in \mathbb{N}$
3:     $X[0] \leftarrow X_0$
4:     LR($0, k2^N, X[0], S[0]$)
5:     $X[1] \leftarrow X_0$
6:     LR($k(2^N - 1) + b'(k), k2^N, X[0], X[1], S[0], S[1]$)
7:     return \[ \frac{1}{k - b'(k)} (S[1] - S[0]) \]
8: end procedure

Algorithm 3 Procedure UnbiasedLongRun

1: procedure ULR($k, b'(k), (p_l, l \geq 0), q$)
2:     $X[0] \leftarrow X_0$
3:     LR($b'(k), k, X[0], S[0]$)
4:     $f_{k,0} \leftarrow \frac{1}{k - b'(k)} S[0]$
5:     Sample $W$ uniformly from $[0, 1]$
6:     if $W > q$ then
7:         return $f_{k,0}$
8:     else
9:         return $f_{k,0} - \frac{1}{q} \text{Bias}(k, b'(k), (p_l, l \geq 0))$
10:     end if
11: end procedure
Algorithm 4 Procedure StratifiedUnbiasedLongRun

1: procedure SULR(n, k, b’(k), (p_l, l ≥ 0), q)
2: S’ ← 0
3: for i ← 1, n do
4: X[0] ← X_0
5: LR(b’(k), k, X[0], S[0])
6: S’ ← S’ + \frac{1}{k-b(k)} S[0]
7: end for
8: S'' ← 0
9: for i ← 1, \lfloor nq \rfloor do
10: S'' ← S'' + Bias(k, b’(k), (p_l, l ≥ 0))
11: end for
12: return S’/n - S''/\lfloor nq \rfloor
13: end procedure

Algorithm 4 assumes that B and K are integers with 0 ≤ B < K. The arguments X[0], . . . , X[h] and S[0], . . . , S[h] of LR are real numbers passed by reference, i.e., modifications made to these arguments in LR have effect in any procedure that calls LR. For 0 ≤ j ≤ h, denote by X_0[j], . . . , X_K[j] the successive values of X[j] during the execution of Algorithm 4 where X_0[j] is the value of X[j] at the beginning of LR. It is assumed that the V’s generated in LR and (X_0[0], . . . , X_0[h]) are independent random variables.

Under Assumption A3, Lemma 3.1 shows that E(Z_k^{(b'(k))}) = µ - E(f_k,0) for k ≥ 1. Thus -Z_k^{(b'(k))} is an unbiased estimator of the bias of f_k,0. Algorithm 2 provides a detailed implementation of -Z_k^{(b'(k))} based on (3.1) and (3.3) and on the procedure LR. Algorithm 3 gives a detailed implementation for \hat{f}_k based on LR and on the procedure BIAS in Algorithm 2. Proposition 3.1 shows that the outputs of Algorithms 2 and 3 are consistent with the definitions of Z_k^{(b'(k))} and of \hat{f}_k.

Proposition 3.1. The random variable output by the procedure BIAS (resp. ULR) has the same distribution as -Z_k^{(b'(k))} (resp. \hat{f}_k).

Note that the procedure BIAS can be used to estimate the bias of f_k by setting b’(k) = b(k). The procedure SULR in Algorithm 4 provides an implementation of f_k,n.

4 Examples

4.1 GARCH volatility model

In the GARCH(1,1) volatility model (see (Hull 2014, Ch. 23)), the daily volatility σ_i of an index or exchange rate, calculated at the end of day i, satisfies the following recursion:

σ_{i+1}^2 = w + ασ_i^2U_i^2 + βσ_i^2,

i ≥ 0, where w, α and β are positive constants with α + β < 1, and (U_i, i ≥ 0) are independent standard Gaussian random variables. At the end of day 0, given σ_0 ≥ 0 and a real number z, we want to estimate lim_{i→∞} Pr(σ_i^2 > z), if such a limit exists. In this example, F = F’ = R, with X_i = σ_i^2 and g(x, u) = w + αxu^2 + βx, and f(u) = 1{u > z} for u ∈ R. The proof of (Kahalé 2020b, Proposition 9) implies (2.6) with x = 0 and \nu’(i) = c(α + β)^i/2 for i ≥ 0, for some constant c. Thus, Assumption A2 holds.
4.2 GI/G/1 queue

Consider a GI/G/1 queue where customers are served by a single server in order of arrival. For \( n \geq 0 \), let \( A_n, V_n \) and \( X_n \) be the arrival time, service time and waiting time (exclusive of service time) of customer \( n \). For \( n \geq 0 \), define the interarrival time \( D_n := A_{n+1} - A_n \). Assume that the system starts empty at time 0, that \( (D_n, V_n), n \geq 0 \), are identically distributed, and that the random variables \( \{D_n, V_n, n \geq 0\} \) are independent. The waiting times satisfy the Lindley recursion (Asmussen and Glynn 2007, p.1)

\[
X_{i+1} = \max(0, X_i + U_i),
\]

where \( U_i := V_i - D_i \) for \( i \geq 0 \), with \( X_0 = 0 \). We want to estimate \( \lim_{i \to \infty} E(X_i) \), if such a limit exists. Here, we have \( F = F' = \mathbb{R} \), with \( g(x,u) = \max(0, x + u) \), and \( f \) is the identity function. Proposition 4.1 below shows that Assumption A2 holds under suitable conditions.

Proposition 4.1. If there are constants \( \gamma > 0 \) and \( \eta < 1 \) such that

\[
E(e^{\gamma U_i}) \leq \eta
\]

for \( i \geq 0 \), then Assumption A1 holds when \( f \) is the identity function, with \( \nu(i) = \gamma \eta^i \) for \( i \geq 0 \), where \( \gamma' \) is a constant.

The proof of Proposition 4.1 is very similar to that of (Kahalé 2020b, Proposition 10), and is omitted. The condition (4.1) is related to the stability condition \( E(U_i < 0) \), and is justified in (Kahalé 2020b).

Our approach can also estimate \( \lim_{i \to \infty} \Pr(X_i > z) \), where \( z \) is a real number, under suitable conditions. In this case, \( F, F' \) and \( g \) are the same as above, and \( f(u) = 1\{u > z\} \) for \( u \in \mathbb{R} \).

Proposition 4.2. If \( E(U_i < 0) \) and \( E(U_i^0) \) is finite, then Assumption A1 holds when \( f(u) = 1\{u > z\} \), with \( \nu(i) = c(i+1)^{-2} \) for \( i \geq 0 \), where \( c \) is a constant.

4.3 High-dimensional Gaussian vectors

Let \( V \) be a \( d \times d \) positive definite matrix with all diagonal entries equal to 1. The standard algorithm to generate a Gaussian vector with covariance matrix \( V \) is based on the Cholesky decomposition, that takes \( O(d^3) \) time. Kahalé (2019) describes an alternative method that approximately simulates a centered \( d \)-dimensional Gaussian vector \( X \) with covariance matrix \( V \). Let \( j \) be a random integer uniformly distributed in \( \{1, \ldots, d\} \), and let \( e \) be the \( d \)-dimensional random column vector whose \( j \)-th coordinate is 1 and remaining coordinates are 0. Let \( (e_i, i \geq 0) \) be a sequence of independent copies of \( e \), and let \( (g_i, i \geq 0) \) be a sequence of independent standard Gaussian random variables, independent of \( (e_i, i \geq 0) \). Define the Markov chain of \( d \)-dimensional column vectors \( (X_i, i \geq 0) \) as follows. Let \( X_0 = 0 \) and, for \( i \geq 0 \), let

\[
X_{i+1} = X_i + (g_i - e_i^T X_i)(V e_i).
\]

In this example, \( F \) is the set of \( d \)-dimensional column vectors, \( F' = \mathbb{R} \times F \), with \( U_i = (g_i, e_i) \) and \( g(x,g',e') = x + (g' - e^T x)(Ve') \) for \( (g',e') \in F' \).

Theorem 4.1. Let \( \hat{\kappa} \) and \( \hat{\gamma} \) be two positive constants with \( \hat{\gamma} \leq 1 \). Consider a real-valued Borel function \( f \) of \( d \) variables such that

\[
E((f(X) - f(X'))^2) \leq \hat{\kappa}^2 E(||X - X'||^2)^{\hat{\gamma}} \tag{4.2}
\]

for any centered Gaussian column vector \( \begin{pmatrix} X \\ X' \end{pmatrix} \) with \( \text{Cov}(X) \leq V \) and \( \text{Cov}(X') \leq V \), where \( X \) and \( X' \) have dimension \( d \). Then Assumption A1 holds for the function \( f \), with

\[
\nu(i) = \hat{\kappa}^2 \min((\lambda_{\max} d)^{\hat{\gamma}} (1 - \lambda_{\min}^i d)^{\hat{\gamma}} (d^2 i + 1)^{\hat{\gamma}}),
\]

\( i \geq 0 \), where \( \lambda_{\max} \) (resp. \( \lambda_{\min} \)) is the largest (resp. smallest) eigenvalue of \( V \).
Assumption A2 holds with \( c \) replications and burn-in period \( b \) limit \( \mu \).

Figure 1: Absolute bias and standard deviation of time-average estimators with \( 10^6 \) independent replications and burn-in period \( b(k) = \lfloor k/10 \rfloor \).

Since \( \text{tr}(V) = d \), we have \( \lambda_{\text{max}} \leq d \). As \( 1 + x \leq e^x \) for \( x \in \mathbb{R} \), Theorem 4.1 shows that Assumption A2 holds with \( c = \hat{k}^2d^2 \gamma \) and \( \xi = \lambda_{\text{min}}\hat{\gamma}/d \). By Theorem 2.1, \( E(f(X_k)) \) has a finite limit \( \mu \) as \( h \) goes to infinity. It follows from (Kahalé 2019, Theorem 4) that \( \mu = E(f(X)) \), where \( X \) is a \( d \)-dimensional column vector with \( X \sim N(0, V) \).

Assume now that \( \hat{\gamma} = 1 \) and that \( b(k) = 0 \). Theorem 4.1 shows that \( \nu(i) \leq c/(i + 1) \) for \( i \geq 0 \). Combining (2.12) and (2.15) yields \( k E((f_k - \mu)^2) \leq 5096c \ln^2(2/\xi) \), which is weaker than the bound

\[
  k E((f_k - \mu)^2) \leq 18c
\]

of (Kahalé 2019, Theorem 2) by a polylogarithmic factor. Note that Theorem 3.4 is applicable in this example.

5 Numerical experiments

The codes in our simulation experiments were written in the C++ programming language. We assume that the \( p_l \)'s are determined as in Example 3.1 with \( \delta = 1/2 \). We set \( q = (3 \sum_{l=0}^{\infty} 2^l p_l)^{-1} \).

By Lemma 3.2 and the discussion thereafter, for \( k \geq 1 \), we have \( \hat{T}_k \leq 2k \). Thus, on average, at most half of the running time of \( f_k \) is devoted to the simulation of \( Z_k \). Fig. 4 shows the standard deviation per replication and the absolute value of the bias of \( f_k \), i.e., \( |E(f_k) - \mu| \), estimated from \( 10^6 \) independent replications of \( f_k \) and of \( Z_k \), respectively, with \( b(k) = b'(k) = \lfloor k/10 \rfloor \). The absolute value of the bias is also reported in Tables 1 and 2. In these tables, “Std” and “Cost” refer to the standard deviation and running time of a single iteration of \( Z_k \), respectively. Tables 2, 3, 4, 5, 6, 7, and 9 compare the methods LR, ULR, and SULR with burn-in periods \( b(k) = b'(k) = \lfloor k/10 \rfloor \) and \( b(k) = b'(k) = \lfloor k/2 \rfloor \). The methods LR and ULR were implemented with \( 10^6 \) independent replications, and the method SULR was implemented with parameter \( n = 10^6 \). A 95% confidence interval was calculated for \( \mu \) by the methods ULR and SULR.

For the methods LR and ULR, “Std” refers to the standard deviation per replication, whereas “Std” refers to the standard deviation of the output for the method SULR. For the method LR, the root mean square error “RMSE” is calculated using the formula \( \text{RMSE} = \sqrt{\text{Std}^2 + \text{Bias}^2} \), where the bias is estimated from \( 10^5 \) independent replications of \( Z_k \).

For the methods ULR and SULR, RMSE = Std. For the methods LR and ULR, the variable “Cost” is the average number of times the Markov chain is simulated per replication, that is, the average number of calls to the function \( g \). For the methods SULR, the variable “Cost” is the total number of calls to the function \( g \). Finally, the mean square error is \( \text{MSE} = \text{RMSE}^2 \). In other words, the mean square error is equal to the sum of bias squared and variance. For a fixed computing budget, the mean square error is a standard measure of the performance of a biased estimator (Glasserman 2004). Following (Rhee and Glynn 2015), we measure the performance of a method through the product \( \text{Cost} \times \text{MSE} \): this product is low when the performance is high.
values of $k$ the diminishing contribution of the bias to the work-normalized variance of ULR and of SULR as $k$ (Whitt 1991), and the impact of the bias on the performance of ULR and of SULR diminishes for large values of $k$.

Table 2: Estimation of $\lim_{k \to \infty} \Pr(\sigma_k^2 > z)$ in a GARCH volatility model with $b(k) = [k/10]$.

| $k$ | burn-in | Method | $\mu$ | Std | RMSE | Cost | $\text{Cost} \times \text{MSE}$ |
|-----|---------|--------|-------|-----|------|------|-------------------------------|
| 25  | 2       | LR     | 0.1126| 1.8 $\times$ 10^{-1} | 3.4 $\times$ 10^{-1} | 5.00 $\times$ 10^{4} | 5.8 |
|     |         | ULR    | 0.398 ± 0.003 | 1.4 $\times$ 10^{0} | 1.4 $\times$ 10^{0} | 1.02 $\times$ 10^{2} | 200 |
|     |         | SULR   | 0.400 ± 0.002 | 1.2 $\times$ 10^{-3} | 1.2 $\times$ 10^{-3} | 1.01 $\times$ 10^{6} | 140 |
| 50  | 5       | LR     | 0.3319| 2.1 $\times$ 10^{-1} | 2.2 $\times$ 10^{-1} | 2.00 $\times$ 10^{2} | 9.5 |
|     |         | ULR    | 0.3991 ± 0.0008 | 4.2 $\times$ 10^{-1} | 4.2 $\times$ 10^{-1} | 4.03 $\times$ 10^{2} | 71 |
|     |         | SULR   | 0.3993 ± 0.0007 | 3.8 $\times$ 10^{-4} | 3.8 $\times$ 10^{-4} | 3.96 $\times$ 10^{8} | 58 |
| 200 | 20      | LR     | 0.3969| 1.2 $\times$ 10^{-1} | 1.2 $\times$ 10^{-1} | 8.00 $\times$ 10^{2} | 11 |
|     |         | ULR    | 0.3996 ± 0.0002 | 1.2 $\times$ 10^{-1} | 1.2 $\times$ 10^{-1} | 1.59 $\times$ 10^{3} | 23 |
|     |         | SULR   | 0.3996 ± 0.0002 | 1.2 $\times$ 10^{-4} | 1.2 $\times$ 10^{-4} | 1.59 $\times$ 10^{9} | 23 |
| 800 | 80      | LR     | 0.39963| 6.1 $\times$ 10^{-2} | 6.1 $\times$ 10^{-2} | 3.20 $\times$ 10^{3} | 12 |
|     |         | ULR    | 0.39970 ± 0.0001 | 6.1 $\times$ 10^{-2} | 6.1 $\times$ 10^{-2} | 6.34 $\times$ 10^{3} | 23 |
|     |         | SULR   | 0.39963 ± 0.0001 | 6.1 $\times$ 10^{-5} | 6.1 $\times$ 10^{-5} | 6.28 $\times$ 10^{9} | 23 |

5.1 GARCH volatility model

Tables 2 and 3 estimate $\lim_{k \to \infty} \Pr(\sigma_k^2 > z)$, with $\alpha = 0.05$, $\beta = 0.92$, $\sigma_0^2 = 2 \times 10^{-5}$, $w = 1.2 \times 10^{-6}$ and $z = 4 \times 10^{-5}$. The left panel in Fig. 1 shows that, for small values of $k$, the bias and standard deviation of $f_k$ are of the same order of magnitude, while for large values of $k$, the bias is much smaller than the standard deviation and decays at a faster rate. This is consistent with the discussion preceding Lemma 2.1. In Tables 2 and 3 because of the choice of $q$, the total running time of ULR and of SULR is about twice that of LR. For small values of $k$, the product Cost $\times$ MSE is much smaller for LR than for ULR and SULR but LR exhibits a strong bias. For large values of $k$, the product Cost $\times$ MSE is twice as large for ULR and SULR as for LR. This is due to the fact that, because of the choice of $q$, about half the running time of ULR and SULR is devoted to estimating the bias, that is negligible for large values of $k$. The performance of ULR and of SULR tends to increase with $k$. This can be explained by the diminishing contribution of the bias to the work-normalized variance of ULR and of SULR as $k$ increases. Finally, ULR and SULR perform better in Table 2 than in Table 3 for large values of $k$, and the reverse effect is observed for small values of $k$. This can be explained by the fact that the variance (resp. bias) of $f_k$ tends to be low (resp. high) when the burn-in is small (Whitt 1991), and the impact of the bias on the performance of ULR and of SULR diminishes as $k$ increases.


\[ \text{Pr}(D_\alpha = 0) \times \text{product Cost} \]

A strong bias for small values of total running time of ULR and of SULR is about twice that of LR. Once again, LR exhibits the bias decays at a rate faster than that of the standard deviation and, in Tables 5 and 6, the i-th customer has a hyperexponential distribution with \( \Pr(V_i > z) = \frac{1}{\alpha} \cdot \frac{(1 + z)^{-\alpha}}{\Gamma(\alpha)} \) for \( z \geq 0 \), with \( \alpha = 0.8 \). The service-time parameters are taken from (Whitt 1991). Tables 5 and 6 estimate \( \lim_{i \to \infty} \text{E}(X_i) \). The center panel of Fig. 1 shows that the bias decays at a rate faster than that of the standard deviation and, in Tables 5 and 6, the total running time of ULR and of SULR is about twice that of LR. Once again, LR exhibits a strong bias for small values of \( k \). When \( k \) is sufficiently large so that the bias is small, the product \( \text{Cost} \times \text{MSE} \) is twice as large for ULR and SULR as for LR. Here again, the performance of ULR and of SULR tends to increase with \( k \), is higher in Table 5 than in Table 6 for large values of \( k \), while the reverse effect is true for small values of \( k \). By the Pollaczek–Khinchine formula,

\[
\lim_{i \to \infty} \text{E}(X_i) = \frac{\lambda \text{E}(S^2)}{2(1 - \lambda \text{E}(S))} = \frac{\lambda}{4p(1-p)(1-\lambda)} \approx 7.51174,
\]

which is consistent with the results in Tables 5 and 6.

### 5.2 \( M/H_k/1 \) queue

Consider a single-server queue with Poisson arrivals at rate \( \lambda = 0.75 \), where the service time \( V_n \) for the \( n \)-th customer has a hyperexponential distribution with \( \Pr(V_n > z) = pe^{-pz} + (1-p)e^{-2(1-p)z} \) for \( z \geq 0 \), with \( p = 0.8875 \). The service-time parameters are taken from (Whitt 1991). Tables 5 and 6 estimate \( \lim_{i \to \infty} \text{E}(X_i) \). The center panel of Fig. 1 shows that the bias decays at a rate faster than that of the standard deviation and, in Tables 5 and 6, the total running time of ULR and of SULR is about twice that of LR. Once again, LR exhibits a strong bias for small values of \( k \). When \( k \) is sufficiently large so that the bias is small, the product \( \text{Cost} \times \text{MSE} \) is twice as large for ULR and SULR as for LR. Here again, the performance of ULR and of SULR tends to increase with \( k \), is higher in Table 5 than in Table 6 for large values of \( k \), while the reverse effect is true for small values of \( k \). By the Pollaczek–Khinchine formula,

\[
\lim_{i \to \infty} \text{E}(X_i) = \frac{\lambda \text{E}(S^2)}{2(1 - \lambda \text{E}(S))} = \frac{\lambda}{4p(1-p)(1-\lambda)} \approx 7.51174,
\]

which is consistent with the results in Tables 5 and 6.

### 5.3 \( GI/G/1 \) queue

Assume that the interarrival time \( D_n \) and service time \( V_n \) for the \( n \)-th customer have Pareto distributions with \( \Pr(D_n \geq z) = (1 + z)^{-\alpha} \) and \( \Pr(V_n \geq z) = (1 + z/\alpha)^{-\alpha} \) for \( z \geq 0 \), with \( \alpha = 0.8 \). Tables 5 and 6 estimate \( \lim_{i \to \infty} \text{Pr}(X_i > 1) \). The right panel Fig. 1 shows that the bias decays at a rate faster than that of the standard deviation, the total running time of ULR

\[
\frac{\text{E}(S^2)}{2(1 - \text{E}(S))} = \frac{\lambda}{4p(1-p)(1-\lambda)} \approx 7.51174,
\]

which is consistent with the results in Tables 5 and 6.

### 5.3 \( GI/G/1 \) queue

Assume that the interarrival time \( D_n \) and service time \( V_n \) for the \( n \)-th customer have Pareto distributions with \( \Pr(D_n \geq z) = (1 + z)^{-\alpha} \) and \( \Pr(V_n \geq z) = (1 + z/\alpha)^{-\alpha} \) for \( z \geq 0 \), with \( \alpha = 0.8 \). Tables 5 and 6 estimate \( \lim_{i \to \infty} \text{Pr}(X_i > 1) \). The right panel Fig. 1 shows that the bias decays at a rate faster than that of the standard deviation, the total running time of ULR

\[
\frac{\text{E}(S^2)}{2(1 - \text{E}(S))} = \frac{\lambda}{4p(1-p)(1-\lambda)} \approx 7.51174,
\]

which is consistent with the results in Tables 5 and 6.
Table 5: Estimation of $E(X_i)$ in an $M/H_k/1$ queue with $b(k) = b'(k) = \lfloor k/10 \rfloor$.

| $k$ | burn-in | Method | $\mu$   | Std     | RMSE    | Cost    | $\text{Cost} \times \text{MSE}$ |
|-----|---------|--------|---------|---------|---------|---------|----------------------------------|
| 50  | 5       | LR     | 4.35    | $4.9 \times 10^1$ | $5.9 \times 10^6$ | $5.00 \times 10^4$ | $1.7 \times 10^5$ |
|     |         | ULR    | 7.53 ± 0.07 | $3.8 \times 10^1$ | $3.8 \times 10^1$ | $1.00 \times 10^2$ | $1.5 \times 10^5$ |
|     |         | SULR   | 7.53 ± 0.07 | $3.8 \times 10^{-2}$ | $3.8 \times 10^{-2}$ | $9.87 \times 10^7$ | $1.4 \times 10^5$ |
| 200 | 20      | LR     | 6.547   | $5.8 \times 10^0$ | $5.8 \times 10^0$ | $2.00 \times 10^2$ | $6.8 \times 10^3$ |
|     |         | ULR    | 7.51 ± 0.03 | $1.7 \times 10^1$ | $1.7 \times 10^1$ | $4.00 \times 10^2$ | $1.1 \times 10^5$ |
|     |         | SULR   | 7.50 ± 0.03 | $1.6 \times 10^{-2}$ | $1.6 \times 10^{-2}$ | $4.00 \times 10^8$ | $1.1 \times 10^5$ |
| 800 | 80      | LR     | 7.412   | $4.2 \times 10^0$ | $4.2 \times 10^0$ | $8.00 \times 10^2$ | $1.4 \times 10^4$ |
|     |         | ULR    | 7.503 ± 0.01 | $5.3 \times 10^0$ | $5.3 \times 10^0$ | $1.59 \times 10^3$ | $4.4 \times 10^4$ |
|     |         | SULR   | 7.516 ± 0.01 | $5.4 \times 10^{-3}$ | $5.4 \times 10^{-3}$ | $1.60 \times 10^9$ | $4.7 \times 10^4$ |
| 3200| 320     | LR     | 7.512   | $2.3 \times 10^0$ | $2.3 \times 10^0$ | $3.20 \times 10^3$ | $1.6 \times 10^4$ |
|     |         | ULR    | 7.511 ± 0.004 | $2.3 \times 10^0$ | $2.3 \times 10^0$ | $6.29 \times 10^3$ | $3.2 \times 10^4$ |
|     |         | SULR   | 7.513 ± 0.004 | $2.3 \times 10^{-3}$ | $2.3 \times 10^{-3}$ | $6.70 \times 10^9$ | $3.4 \times 10^4$ |

Table 6: Estimation of $E(X_i)$ in an $M/H_k/1$ queue with $b(k) = b'(k) = \lfloor k/2 \rfloor$.

| $k$ | burn-in | Method | $\mu$   | Std     | RMSE    | Cost    | $\text{Cost} \times \text{MSE}$ |
|-----|---------|--------|---------|---------|---------|---------|----------------------------------|
| 50  | 25      | LR     | 5.15    | $6.5 \times 10^0$ | $6.9 \times 10^0$ | $5.00 \times 10^1$ | $2.4 \times 10^3$ |
|     |         | ULR    | 7.53 ± 0.07 | $3.5 \times 10^1$ | $3.5 \times 10^1$ | $1.00 \times 10^2$ | $1.2 \times 10^5$ |
|     |         | SULR   | 7.52 ± 0.07 | $3.5 \times 10^{-2}$ | $3.5 \times 10^{-2}$ | $9.87 \times 10^7$ | $1.2 \times 10^5$ |
| 200 | 100     | LR     | 7.11    | $7.6 \times 10^0$ | $7.6 \times 10^0$ | $2.00 \times 10^2$ | $1.2 \times 10^4$ |
|     |         | ULR    | 7.51 ± 0.03 | $1.4 \times 10^1$ | $1.4 \times 10^1$ | $4.00 \times 10^2$ | $7.7 \times 10^4$ |
|     |         | SULR   | 7.51 ± 0.03 | $1.4 \times 10^{-2}$ | $1.4 \times 10^{-2}$ | $4.00 \times 10^8$ | $7.4 \times 10^4$ |
| 800 | 400     | LR     | 7.508   | $5.5 \times 10^0$ | $5.5 \times 10^0$ | $8.00 \times 10^2$ | $2.4 \times 10^4$ |
|     |         | ULR    | 7.508 ± 0.01 | $5.6 \times 10^0$ | $5.6 \times 10^0$ | $1.59 \times 10^3$ | $5.0 \times 10^4$ |
|     |         | SULR   | 7.514 ± 0.01 | $5.7 \times 10^{-3}$ | $5.7 \times 10^{-3}$ | $1.60 \times 10^9$ | $5.2 \times 10^4$ |
| 3200| 1600    | LR     | 7.512   | $3.0 \times 10^0$ | $3.0 \times 10^0$ | $3.20 \times 10^3$ | $2.9 \times 10^4$ |
|     |         | ULR    | 7.509 ± 0.006 | $3.0 \times 10^0$ | $3.0 \times 10^0$ | $6.29 \times 10^3$ | $5.6 \times 10^4$ |
|     |         | SULR   | 7.512 ± 0.006 | $3.0 \times 10^{-3}$ | $3.0 \times 10^{-3}$ | $6.70 \times 10^9$ | $6.0 \times 10^4$ |
Table 7: Absolute value of bias in estimating of \( \lim_{i \to \infty} \Pr(X_i > 1) \) in a \( GI/G/1 \) queue with burn-in period \( b(k) = \lfloor k/10 \rfloor \) and \( 10^6 \) independent replications.

| \( k \) | burn-in | 95% confidence interval | Std | Cost |
|--------|---------|-------------------------|-----|------|
| 25     | 2       | \( 2.1 \times 10^{-4} \pm 1 \times 10^{-4} \) | \( 6.3 \times 10^{-1} \) | \( 2.11 \times 10^2 \) |
| 50     | 5       | \( 1.3 \times 10^{-1} \pm 8 \times 10^{-4} \) | \( 3.9 \times 10^{-1} \) | \( 4.03 \times 10^2 \) |
| 100    | 10      | \( 7.0 \times 10^{-2} \pm 4 \times 10^{-4} \) | \( 2.3 \times 10^{-1} \) | \( 8.12 \times 10^2 \) |
| 200    | 20      | \( 3.0 \times 10^{-2} \pm 2 \times 10^{-4} \) | \( 1.2 \times 10^{-1} \) | \( 1.60 \times 10^3 \) |
| 400    | 40      | \( 9.9 \times 10^{-3} \pm 1 \times 10^{-4} \) | \( 5.1 \times 10^{-2} \) | \( 3.20 \times 10^3 \) |
| 800    | 80      | \( 2.3 \times 10^{-3} \pm 3 \times 10^{-5} \) | \( 1.8 \times 10^{-2} \) | \( 6.47 \times 10^3 \) |
| 1600   | 160     | \( 3.4 \times 10^{-4} \pm 1 \times 10^{-5} \) | \( 4.9 \times 10^{-3} \) | \( 1.31 \times 10^4 \) |
| 3200   | 320     | \( 2.1 \times 10^{-5} \pm 2 \times 10^{-6} \) | \( 8.7 \times 10^{-4} \) | \( 2.58 \times 10^4 \) |

Table 8: Estimation of \( \lim_{i \to \infty} \Pr(X_i > 1) \) in a \( GI/G/1 \) queue with \( b(k) = b'(k) = \lfloor k/10 \rfloor \).

| \( k \) | burn-in | Method | \( \mu \) | Std | RMSE | Cost | Cost \times MSE |
|--------|---------|--------|--------|-----|------|------|---------------|
| 50     | 5       | LR     | 0.2004 | \( 2.4 \times 10^{-1} \) | \( 2.7 \times 10^{-1} \) | \( 5.00 \times 10^1 \) | 3.8 |
|        |         | ULR    | \( 0.33 \pm 0.002 \) | \( 1.2 \times 10^0 \) | \( 1.2 \times 10^0 \) | \( 1.00 \times 10^2 \) | 135 |
|        |         | SULR   | \( 0.334 \pm 0.002 \) | \( 1.1 \times 10^{-3} \) | \( 1.1 \times 10^{-3} \) | \( 1.00 \times 10^3 \) | 131 |
| 200    | 20      | LR     | 0.302  | \( 2.0 \times 10^{-1} \) | \( 2.1 \times 10^{-1} \) | \( 2.00 \times 10^2 \) | 8.4 |
|        |         | ULR    | \( 0.3327 \pm 0.0008 \) | \( 4.0 \times 10^{-1} \) | \( 4.0 \times 10^{-1} \) | \( 3.93 \times 10^2 \) | 64 |
|        |         | SULR   | \( 0.3323 \pm 0.0008 \) | \( 4.0 \times 10^{-4} \) | \( 4.0 \times 10^{-4} \) | \( 3.97 \times 10^3 \) | 63 |
| 800    | 80      | LR     | 0.3298 | \( 1.2 \times 10^{-1} \) | \( 1.2 \times 10^{-1} \) | \( 8.00 \times 10^2 \) | 12 |
|        |         | ULR    | \( 0.3321 \pm 0.0003 \) | \( 1.3 \times 10^{-1} \) | \( 1.3 \times 10^{-1} \) | \( 1.64 \times 10^3 \) | 29 |
|        |         | SULR   | \( 0.3321 \pm 0.0003 \) | \( 1.3 \times 10^{-4} \) | \( 1.3 \times 10^{-4} \) | \( 1.63 \times 10^3 \) | 28 |
| 3200   | 320     | LR     | 0.33222 | \( 6.3 \times 10^{-2} \) | \( 6.3 \times 10^{-2} \) | \( 3.20 \times 10^3 \) | 13 |
|        |         | ULR    | \( 0.33221 \pm 0.0001 \) | \( 6.3 \times 10^{-2} \) | \( 6.3 \times 10^{-2} \) | \( 6.35 \times 10^3 \) | 25 |
|        |         | SULR   | \( 0.33224 \pm 0.0001 \) | \( 6.3 \times 10^{-5} \) | \( 6.3 \times 10^{-5} \) | \( 6.33 \times 10^9 \) | 25 |

and of SULR is about twice that of LR, and the bias of LR is strong for small values of \( k \). When \( k \) is large enough so that the bias is small, the product Cost \times MSE is twice as large for ULR and SULR as for LR. Here again, the performance of ULR and of SULR tends to increase with \( k \), is higher in Table 7 than in Table 8 for large values of \( k \), while the reverse is true for small values of \( k \).

6 Conclusion

Under a coupling assumption, we have established bounds on the bias, variance and mean square error of standard time-average estimators, and shown the sharpness of the variance and mean square error bounds. We have built an unbiased RMLMC estimator for the bias of a conventional time-average estimator. Combining this unbiased estimator with a conventional time-average estimator yields an unbiased estimator \( \hat{f}_k \) of \( \mu \). Both unbiased estimators are square-integrable and have finite expected running time. Under certain conditions, they can be built without any precomputations. For a suitable choice of parameters, we have shown that \( \hat{f}_k \) is asymptotically at least as efficient as \( f_k \), up to a multiplicative factor arbitrarily close to 1. We have also constructed an efficient stratified version \( \hat{f}_{k,n} \) of \( \hat{f}_k \). Building more refined stratified versions of \( \hat{f}_k \), such as those in (Vihola 2018), is left for future research. Our approach permits to estimate the bias of \( f_k \) and to determine the number of time-steps needed to substantially reduce it. It can be implemented in a parallelized fashion and allows the robust construction of confidence intervals for \( \mu \). We have provided examples in volatility forecasting, queues, and the simulation of high-dimensional Gaussian vectors where our approach is provably efficient, even when \( f \) is discontinuous. Our numerical experiments are consistent with our theoretical findings. In
our experiments, the value of $q$ is fixed and $f_k$ is about twice as efficient as $\hat{f}_k$ and $\tilde{f}_{k,m}$ when $k$ is sufficiently large. As per the discussion following Theorem 3.3 for large values of $k$, the performance of $\hat{f}_k$ and of $\tilde{f}_{k,m}$ should increase if $q$ decreases. In practice, though, determining the optimal value of $q$ for a given $k$ may require a large amount of pre-computations. For $\tilde{f}_{k,m}$, for instance, it can be shown that the optimal value of $q$ depends on the variance of $Z_k$, that is not easy to estimate accurately for large values of $k$.

### A Proof of Proposition 2.2

We show by induction on $i$ that, for $i, m \geq 0$, 
\[
E(\rho^2(X_i, X_{i,m})) \leq \kappa' \eta^i.
\] (A.1)

As $X_{0,m} \sim X_{m}$, (A.1) holds for $i = 0$. Assume now that (A.1) holds for $i$. It follows from the definitions of $G_i$ and of $X_{i,m}$ that 
\[
X_{i+1,m} = g(X_{i,m}, U_i).
\]
Together with (1.1) and (2.9), this implies that 
\[
E(\rho^2(X_{i+1}, X_{i+1,m})) \leq \eta E(\rho^2(X_i, X_{i,m})).
\]
Thus (A.1) holds for $i + 1$. Combining (2.7) and (A.1) shows that $E((f(X_{i,m}) - f(X_i))^2) \leq \kappa^2 \kappa'^2 \eta^{2i}$ for $i, m \geq 0$. This concludes the proof.

### B Proof of Lemma 2.1

We first prove the following.

**Proposition B.1.** For $n, m, m' \in \mathbb{Z}$ with $n \geq -m$ and $n \geq -m'$, we have 
\[
E((f(X_{n,m}) - f(X_{n,m'}))^2) = E((f(X_{n+m,m'-m}) - f(X_{n+m}))^2).
\]

**Proof.** We have $X_{n,m} = G_{n+m}(X_0; U_{-m}, \ldots, U_{n-1})$, and $X_{n,m'} = G_{n+m'}(X_0; U_{-m'}, \ldots, U_{n-1})$. Also, $X_{n+m} = G_{n+m}(X_0; U_0, \ldots, U_{n+m-1})$, and $X_{n+m,m'-m} = G_{n+m'}(X_0; U_{-m'}, \ldots, U_{n+m-1})$. As 
\[
(U_{-m}, \ldots, U_{n-1}) \sim (U_0, \ldots, U_{n+m-1}, (U_{m'-m}, \ldots, U_{n+m-1})),
\]
the pair $(X_{n,m}, X_{n,m'})$ has the same distribution as $(X_{n+m}, X_{n+m,m'-m})$. This concludes the proof.
We now prove the lemma. For \( l \geq 0 \), let
\[
\sigma_l := 2^{l/2} \left( \sqrt{\nu(0)} + (\sqrt{2} + 1) \sum_{i=1}^{2^l} \sqrt{\nu(i)}(i^{-1/2} - 2^{-l/2}) \right).
\]
In particular, we have \( \sigma_0 = \sqrt{\nu(0)} \). We show by induction on \( l \) that
\[
\text{Std}(\sum_{i=h}^{h+k-1} f(X_i)) \leq \sigma_l \text{ for } h \geq 0 \text{ and } 0 \leq k \leq 2^l. \tag{B.1}
\]
If \( k = 0 \), the summation in the left-hand side of (B.1) is null by convention, and (B.1) trivially holds. Applying (2.5) with \( i = 0 \) shows that, for \( m \geq 0 \),
\[
E((f(X_m) - f(X_0))^2) \leq \nu(0). \tag{B.2}
\]
Hence (B.1) holds for \( l = 0 \). Assume now that (B.1) holds for \( l \). We show that it holds for \( l + 1 \). Fix non-negative integers \( k \) and \( h \), with \( 0 \leq k \leq 2^{l+1} \). If \( k \leq 1 \) then (B.1) holds for \( l + 1 \) as a consequence of (B.2). Assume now that \( k > 1 \) and let \( j = \lfloor k/2 \rfloor \). For \( i \geq 0 \), let \( X'_i = X_{h+j+i-1,h-j} \). By (2.3) and the remark that follows it, the sequences \((X_i, 0 \leq i \leq k-j)\) and \((X'_i, 0 \leq i \leq k-j)\) have the same distribution. Set \( V_1 = \sum_{i=h}^{h+j-1} f(X_i), V_2 = \sum_{i=h+j}^{h+k-1} f(X_i), \) and \( V'_2 := \sum_{i=h+j}^{h+k-1} f(X'_i) \). Then
\[
\text{Std}(\sum_{i=h}^{h+k-1} f(X_i)) = \text{Std}(V_1 + V_2) \leq \text{Std}(V_1 + V'_2) + \text{Std}(V_2 - V'_2). \tag{B.3}
\]
The second equation follows from the sub-linearity of the standard deviation, i.e., \( \text{Std}(V + V') \leq \text{Std}(V) + \text{Std}(V') \) for any square-integrable random variables \( V \) and \( V' \). Note that \( V'_2 \) has the same distribution as \( \sum_{i=1}^{k-j} f(X_i) \). As \( j \) and \( k-j \) are upper-bounded by \( 2^l \), the induction hypothesis implies that \( \text{Std}(V_1) \) and \( \text{Std}(V'_2) \) are both upper-bounded by \( \sigma_l \). Furthermore, by construction, \( V_1 \) (resp. \( V'_2 \)) is a deterministic measurable function of \((U_0, \ldots, U_{h+j-2})\) (resp. \((U_{h+j-1}, \ldots, U_{h+k-2})\)). Thus, \( V_1 \) and \( V'_2 \) are independent. Hence
\[
\text{Var}(V_1 + V'_2) = \text{Var}(V_1) + \text{Var}(V'_2) \leq 2\sigma_l^2. \tag{B.4}
\]
Let \( i \in [1, k-j] \). Applying Proposition (B.1) with \( m' = 0 \), \( n = h + j + i - 1 \) and \( m = 1 - h - j \) shows that
\[
E((f(X_{h+j+i-1}) - f(X'_i))^2) = E((f(X_{h+j+i-1}) - f(X_i))^2).
\]
Since \( j > 0 \), together with (2.5), this implies that \( E((f(X_{h+j+i-1}) - f(X'_i))^2) \leq \nu(i) \). Thus
\[
\text{Std}(V_2 - V'_2) = \text{Std} \left( \sum_{i=1}^{k-j} (f(X_{h+j+i-1}) - f(X'_i)) \right) \leq \sum_{i=1}^{k-j} \sqrt{\nu(i)}, \tag{B.5}
\]
where the second equation follows from the sub-linearity of the standard deviation. Combining (B.3), (B.4) and (B.5) yields
\[
\text{Std}(\sum_{i=h}^{h+k-1} f(X_i)) \leq \sqrt{2} \sigma_l + \sum_{i=1}^{2^l} \sqrt{\nu(i)}.
\]
By the definition of $\sigma_l$, 
\[
\sqrt{2} \sigma_l + \sum_{i=1}^{2^l} \sqrt{\nu(i)} = 2^{(l+1)/2} \left( \sqrt{\nu(0)} + (\sqrt{2} + 1) \sum_{i=1}^{2^l} \sqrt{\nu(i)(i^{-1/2} - 2^{-l/2})} \right) + \sum_{i=1}^{2^l} \sqrt{\nu(i)} 
\]
\[
= 2^{(l+1)/2} \left( \sqrt{\nu(0)} + (\sqrt{2} + 1) \sum_{i=1}^{2^l} \sqrt{\nu(i)(i^{-1/2} - 2^{-(l+1)/2})} \right) 
\]
\[
\leq \sigma_{l+1},
\]
where the second equation follows from standard calculations. Thus, the induction hypothesis holds for $l + 1$.

Given $h \geq 0$ and $k \geq 1$, let $l := \lceil \log_2(k) \rceil$. By (B.1),
\[
\text{Std} \left( \sum_{i=h}^{h+k-1} f(X_i) \right) \leq 2^{l/2} \left( \sqrt{\nu(0)} + (\sqrt{2} + 1) \sum_{i=1}^{2^l} \sqrt{\nu(i)} \right),
\]
\[
\leq 2k \sqrt{2} \left( \sqrt{\nu(0)} + (2 + \sqrt{2}) \sum_{i=1}^{2^l} \sqrt{\nu(i)} \right),
\]
\[
\leq 2(\sqrt{2} + 1) \sqrt{k} \sum_{i=0}^{2^l} \sqrt{\nu(i)} \sqrt{i + 1},
\]
where the second equation follows from the inequalities $l \leq \log_2(k) + 1$ and $2i \geq i + 1$ for $i \geq 1$. As $2(\sqrt{2} + 1) \leq 5$, this completes the proof.

C Proof of Theorem 2.1

The following proposition gives a bound on the tail of the sequence $\nu$.

**Proposition C.1.** For non-negative integers $h, h'$ with $2h \leq h'$, we have 
\[
\sum_{i=2h}^{h'} \nu(i) \leq (\overline{\nu}(h))^2.
\]

**Proof.** We have 
\[
\left( \sum_{i=h}^{h'} \sqrt{\frac{\nu(i)}{i + 1}} \right)^2 = \sum_{i=h}^{h'} \frac{\nu(i)}{i + 1} + 2 \sum_{i=h}^{h'} \sqrt{\nu(i)} \left( \sum_{j=h}^{i-1} \sqrt{\nu(j)} \right) 
\]
\[
\geq \sum_{i=h}^{h'} (2i - 2h + 1) \frac{\nu(i)}{i + 1} 
\]
\[
\geq \sum_{i=2h}^{h'} \nu(i).
\]
The second equation follows by observing that $\nu(i)/(i + 1) \leq \nu(j)/(j + 1)$ for $0 \leq j < i$, which implies that $\sum_{j=h}^{i-1} \sqrt{\nu(j)/(j + 1)} \geq (i - h) \sqrt{\nu(i)/(i + 1)}$. 

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We now prove Theorem 2.1. By Proposition C.1, \( \sum_{i=0}^{\infty} \nu(i) < \infty \). Hence \( \nu(i) \) goes to 0 as \( i \) goes to infinity. Since \( (E(V))^2 \leq E(V^2) \) for any square-integrable random variable \( V \), it follows from (2.6) that, for \( i, m \geq 0 \),

\[
|E(f(X_{i,m}) - f(X_i))| \leq \sqrt{\nu(i)}.
\]

As \( X_{i,m} \sim X_{i+m} \), we have \( E(f(X_{i,m})) = E(f(X_{i+m})) \). Therefore,

\[
|E(f(X_{i,m}) - f(X_i))| \leq \sqrt{\nu(i)}. \tag{C.1}
\]

Thus \( (E(f(X_h)), h \geq 0) \) is a Cauchy sequence and has a finite limit \( \mu \) as \( h \) goes to infinity. Letting \( m \) go to infinity in (C.1) implies (2.10).

By convexity of the square norm function,

\[
(E\left(\frac{1}{k} \sum_{i=h}^{h+k-1} f(X_i)\right) - \mu)^2 \leq \frac{1}{k} \sum_{i=h}^{h+k-1} (E(f(X_i)) - \mu)^2,
\]

\[
\leq \frac{1}{k} \sum_{i=h}^{h+k-1} \nu(i)
\]

\[
\leq \frac{1}{k} \sum_{i=2\lfloor h/2 \rfloor}^{h+k-1} \nu(i)
\]

\[
\leq \frac{1}{k} (\nu(\lfloor h/2 \rfloor))^2.
\]

The third equation follows from the inequality \( 2\lfloor h/2 \rfloor \leq h \), and the last one from Proposition C.1. This implies (2.11).

As \( \nu(\lfloor h/2 \rfloor) \leq \nu(0) \) for \( h \geq 0 \), it follows from (2.11) that, for \( k > 0 \),

\[
(E\left(\frac{1}{k} \sum_{i=h}^{h+k-1} f(X_i)\right) - \mu)^2 \leq \frac{\nu(0)^2}{k}. \tag{C.2}
\]

Since the mean square error is related to the bias and standard deviation via the equation

\[
E((V - \mu)^2) = (E(V) - \mu)^2 + (\text{Std}(V))^2,
\]

for any square-integrable random variable \( V \), (2.12) follows by combining (C.2) with Lemma 2.1.

\( \square \)

**D  Proof of Proposition 2.3**

We first prove the following propositions.

**Proposition D.1.** For \( u, v > 0 \) and \( \delta \geq 0 \), we have

\[
(u + v)^\delta \leq 2^\delta (u^\delta + v^\delta).
\]

**Proof.** Assume without loss of generality that \( u \leq v \). Then

\[
(u + v)^\delta \leq (2v)^\delta \leq 2^\delta (u^\delta + v^\delta),
\]

as desired. \( \square \)
Proposition D.2 gives bounds on \( \varpi \) under an exponential decay assumption on \( \omega \). Up to a polylogarithmic factor, the bound on \( \varpi(0) \) is inversely proportional to \( \sqrt{\xi} \), where \( \xi \) is the decay rate of \( \omega \). The bound on \( \varpi(j) \) is exponentially decaying with decay rate \( \xi/2 \).

**Proposition D.2.** Let \((\omega(i), i \geq 0)\) be a non-negative sequence such that \( \omega(i) \leq c \ln^\delta(i+2)e^{-\xi i} \) for \( i \geq 0 \), where \( c \) and \( \xi \) are positive constants, with \( \xi \leq 1 \) and \( \delta \geq 0 \). Then

\[
\varpi(0) \leq \sqrt{\frac{ec}{\xi}} \left( 2^{\delta+1/2}\Gamma\left(\frac{\delta+1}{2}\right) + 2^{\delta/2}\sqrt{2\pi} \ln^{\delta/2}\left(\frac{2}{\xi}\right) \right), \tag{D.1}
\]

and, for \( j \geq 0 \),

\[
\varpi(j) \leq 2\sqrt{c} \left( \frac{\delta}{e} \right)^{\delta/2} \frac{e^{-\delta j/2}}{\xi}, \tag{D.2}
\]

where \( 0^0 = 1 \) by convention.

**Proof.** By replacing \( \omega \) with \((1/c)\omega\), it can be assumed without loss of generality that \( c = 1 \). We have

\[
\varpi(0) \leq \sum_{i=0}^{\infty} \ln^{\delta/2}(i+2)e^{-\xi i/2} \frac{1}{\sqrt{i+1}} \leq \sqrt{e} \int_0^\infty \ln^{\delta/2}(x+2)e^{-\xi x/2} \frac{1}{\sqrt{x}} \, dx \leq 2\sqrt{\frac{e}{\xi}} \int_0^\infty \ln^{\delta/2}(y^2/\xi + 2)e^{-y^2/2} \, dy,
\]

where the second equation follows from the inequality \( e^{-\xi i/2} \leq \sqrt{e}e^{-\xi x/2} \) for \( x \in [i, i+1] \), and the third equation follows from the change of variables \( y = \sqrt{\xi}x \). On the other hand, for \( y > 0 \),

\[
\ln^{\delta/2}(y^2/\xi + 2) \leq (\ln(y^2 + 1) + \ln(2/\xi))^{\delta/2} \leq 2^{\delta/2}(\ln^{\delta/2}(y^2 + 1) + \ln^{\delta/2}(2/\xi)) \leq 2^{\delta/2}(y^\delta + \ln^{\delta/2}(2/\xi)),
\]

where the second equation follows from Proposition [D.1] and the last one from the inequality \( \ln(1+z) \leq z \) for \( z \geq 0 \). Thus

\[
\int_0^\infty \ln^{\delta/2}(y^2/\xi + 2)e^{-y^2/2} \, dy \leq 2^{\delta/2} \int_0^\infty (y^\delta + \ln^{\delta/2}(2/\xi))e^{-y^2/2} \, dy = 2^{\delta-1/2}\Gamma\left(\frac{\delta+1}{2}\right) + 2^{\delta/2}\ln^{\delta/2}\left(\frac{2}{\xi}\right)\sqrt{\pi}.
\]

This implies (D.1).

We now prove (D.2). For \( j \geq 0 \), we have

\[
\varpi(j) \leq \sum_{i=j}^{\infty} \ln^{\delta/2}(i+2)e^{-\xi i/2} \frac{1}{\sqrt{i+1}} \leq \sqrt{2} \left( \frac{\delta}{e} \right)^{\delta/2} \sum_{i=j}^{\infty} e^{-\xi i/2} \leq \sqrt{2} \left( \frac{\delta}{e} \right)^{\delta/2} \frac{e^{-\xi j/2}}{1 - e^{-\xi/2}} \leq 2\sqrt{2\pi}e^{-\xi j/2} \frac{\ln^{\delta/2}(2/\xi)}{\xi},
\]

This implies (D.2).
where the second equation follows from the inequality \(\ln x/x \leq (\delta/e)^\delta\) for \(x > 1\), which implies that \(\ln(x+1)/x \leq 2(\delta/e)^\delta\) for \(x \geq 1\), and the last equation follows from the inequality \(1-e^{-x} \geq x/\sqrt{e}\) for \(x \in [0,1/2]\).

Assuming a bound on \(\omega\) with an exponential decay rate \(\xi\) combined with an additional decay assumption, Proposition D.3 shows that \(\omega(0)\) is bounded by a polylogarithmic function of \(\xi\).

**Proposition D.3.** Let \((\omega(i), i \geq 0)\) be a non-negative sequence such that, for \(i \geq 0\), \(\omega(i) \leq c \ln^\delta(i+2) \min(\delta/e, 1/(i+1))\), where \(\delta \geq 0\) and \(c\) and \(\xi\) are positive constants with \(\xi \leq 1\). Then

\[
\omega(0) \leq 7 \sqrt{c} \ln^{\delta/2+1} \left( \frac{\delta^2 + 4}{\xi^2} \right).
\]

**Proof.** By replacing \(\omega\) with \((1/c)\omega\), it can be assumed without loss of generality that \(c = 1\). Set \(j = \lceil (\delta^2 + 2)\xi^{-2} \rceil\). We have

\[
\omega(0) = \sum_{i=0}^{j-1} \sqrt{\frac{\omega(i)}{i+1}} + \omega(j).
\]

Since \(\xi^{-1} \geq \ln(\xi^{-1})\), we have \(\xi j \geq \delta^2 + 2 \ln(\xi^{-1})\). Hence \(e^{-\xi j/2} \leq e^{-\delta^2/2}\xi\), Proposition D.2 and the inequality \(\delta \leq e^\delta\) show that \(\omega(j) \leq 5\). On the other hand,

\[
\sum_{i=0}^{j-1} \sqrt{\frac{\omega(i)}{i+1}} \leq \ln^{\delta/2}(j+1) \sum_{i=1}^{j} \frac{1}{i} \leq (1 + \ln(j)) \ln^{\delta/2}(j+1).
\]

Hence

\[
\omega(0) \leq 5 + (1 + \ln(j)) \ln^{\delta/2}(j+1) \leq 7 \ln^{\delta/2+1}(j+1),
\]

where the second equation follows from the inequality \(1 \leq \ln(j+1)\). As \(j+1 \leq (\delta^2 + 4)\xi^{-2}\), this concludes the proof.

We now prove Proposition 2.3. The sequence \(\nu\) satisfies the conditions of Proposition D.2 with \(\delta = 0\). As \(\Gamma(1/2) = \sqrt{\pi}\), (D.1) shows that

\[
\nu(0) \leq 2 \sqrt{\frac{2\pi ec}{\xi}}.
\]

This implies (2.14). Similarly, (2.15) follows by applying Proposition D.3 with \(\delta = 0\).

**E Proof of Lemma 3.1**

We first give an upper bound on the standard deviation of \(f_{k,0} - f_k\).

**Lemma E.1.** For \(k \geq 1\),

\[
\text{Std}(f_{k,0} - f_k) \leq \frac{25\nu(0)}{k} \sqrt{b'(k) - b(k)}.
\]
Proof. We have
\[
\begin{align*}
 f_{k,0} - f_k &= \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} - \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} \\
 &= \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} - \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} + \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} - \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)} \\
 &= \frac{b'(k) - b(k)}{(k-b(k))(k-b(k))} \sum_{i=b(k)}^{k-1} f(X_i) - \frac{\sum_{i=b(k)}^{k-1} f(X_i)}{k-b(k)}.
\end{align*}
\]

By sub-linearity of the standard deviation and the inequalities \( b(k) \leq b'(k) \leq k/2 \), this implies that
\[
\text{Std}(f_{k,0} - f_k) \leq 4 \frac{b'(k) - b(k)}{k^2} \text{Std}(\sum_{i=b(k)}^{k-1} f(X_i)) + \frac{2}{k} \text{Std}(\sum_{i=b(k)}^{b(k)-1} f(X_i)).
\]

Using Lemma 2.1 it follows that
\[
\text{Std}(f_{k,0} - f_k) \leq 2(\overline{\nu}(0)) \frac{b'(k) - b(k)}{k^{3/2}} + \frac{10\overline{\nu}(0)}{k} \sqrt{b'(k) - b(k)}.
\]

As \( b'(k) - b(k) \leq k/2 \), this concludes the proof.

We now prove Lemma 3.1. By Lemma E.1 and the sub-linearity of the standard deviation,
\[
\text{Var}(f_{k,0}) \leq \left( \frac{25\overline{\nu}(0)}{k} \sqrt{b'(k) - b(k)} + \text{Std}(f_k) \right)^2 \leq \frac{625\overline{\nu}(0)^2}{k^2} (b'(k) - b(k)) + \frac{50\overline{\nu}(0)}{k} \sqrt{b'(k) - b(k)} \text{Std}(f_k) + \text{Var}(f_k).
\]

We bound the first term by noting that
\[
b'(k) - b(k) \leq \sqrt{\frac{k(b'(k) - b(k))}{2}},
\]
and the second term using the relation
\[
\text{Std}(f_k) \leq \frac{5\sqrt{2}\overline{\nu}(0)}{\sqrt{k}}, \tag{E.1}
\]
which is a consequence of Lemma 2.1. The lemma follows after some simplifications.

\section{Proof of Lemma 3.2}

For \( l \geq 0 \), let
\[
\nu_{k,l} := \frac{\sum_{i=b'(k)+k(2^l-1)}^{k2^l-1} \nu(i)}{k-b'(k)}.
\]

Lemma F.1 gives bounds on the \( \nu_{k,l} \)'s.

\textbf{Lemma F.1.} For \( l \geq 0 \),
\[
k\nu_{k,l} \leq 2(\overline{\nu}(\lfloor b'(k)/2 \rfloor))^2, \tag{F.1}
\]
and, for \( l \geq 2 \),
\[
k\nu_{k,l} \leq 2^{3-l}(\overline{\nu}(k2^{l-2}) - \overline{\nu}(k2^{l-1}))^2. \tag{F.2}
\]
Proof. Applying Proposition [C.1] with $h = |b'(k)/2|$ and $h' = k2^l - 1$ shows that $(k - b'(k))\nu_{k,l} \leq (\mathfrak{p}([b'(k)/2]))^2$. Since $b'(k) \leq k/2$, this implies (F.1).

We now prove (F.2). Let $l \geq 2$. As $k(2^l - 1) \geq k2^l - 1$, for any $i \geq k(2^l - 1)$, we have $\nu(i) \leq \nu(k2^l - 1)$. Since $\nu_{k,l}$ is the average of $\nu(i)$, where $i$ ranges in $[b'(k) + k(2^l - 1), k2^l - 1]$, it follows that $\nu_{k,l} \leq \nu(k2^l - 1)$. Consequently,

$$\mathfrak{p}(k2^l - 2) - \mathfrak{p}(k2^l - 1) = \sum_{i=k2^l - 2}^{k2^l - 1} \sqrt{\frac{\nu(i)}{i + 1}} \geq k2^l - 2 \frac{\nu(k2^l - 1)}{k2^l - 1} = \sqrt{\nu(k2^l - 1)k2^l - 3} \geq \sqrt{\nu_{k,l}k2^l - 3},$$

where the second equation follows from the inequality $\nu(i) \geq \nu(k2^l - 1)$ for $i \leq k2^l - 1$. Hence (F.2).

For $l \geq 0$, set $Y_i = f_{k,l+1} - f_{k,0}$, with $Y_{-1} = 0$. By (3.3), $Z_k = (Y_N - Y_{N-1})/p_N$.

Lemma F.2. For $l \geq 0$,

$$E((Y_i - Y_{i-1})^2) \leq \nu_{k,l}.$$  

Proof. It follows from Proposition [3.1] and [2.5] that $E((f(X_i,m') - f(X_i,m))^2) \leq \nu(i + m)$ for $i \geq 0$ and $0 \leq m \leq m'$. Consequently, $E((f(X_i,k(2^l+1) - 1)) - f(X_i,k(2^l-1)))^2) \leq \nu(i + k(2^l - 1))$ for $i,l \geq 0$. For $l \geq 0$, we have

$$Y_i - Y_{i-1} = f_{k,l+1} - f_{k,l} = \frac{\sum_{i=b'(k)}^{k-1} f(X_i,k(2^l+1) - 1) - f(X_i,k(2^l-1))}{k - b'(k)}.$$  

By the Cauchy-Schwarz inequality,

$$E((Y_i - Y_{i-1})^2) \leq \frac{\sum_{i=b'(k)}^{k-1} E((f(X_i,k(2^l+1) - 1)) - f(X_i,k(2^l-1)))^2}{k - b'(k)} \leq \frac{\sum_{i=b'(k)}^{k-1} \nu(i + k(2^l - 1))}{k - b'(k)} = \nu_{k,l}.\]  

We now prove Lemma [3.2]. The expected cost of computing $f_{k,l+1} - f_{k,l}$ is at most $3k2^l$. Thus $T_k \leq 3k \sum_{l=0}^{\infty} 2^l p_l$. For any $i \geq 0$, as $l$ goes to infinity, $f(X_i,k(2^l-1))$ converges to $\mu$ by Theorem [2.1]. Hence, by the definitions of $f_{k,l}$ and of $Y_i$, as $l$ goes to infinity, $E(f_{k,l})$ converges to $\mu$ and $E(Y_i)$ converges to $\mu' := \mu - E(f_{k,0})$. If the right-hand side of (3.4) is infinite, then (3.4) clearly holds. Assume now that the right-hand side of (3.4) is finite. Combining Lemmas [F.2] and [2.2] shows that, for $0 \leq l \leq 1$,

$$kE((Y_i - Y_{i-1})^2) \leq 2(\mathfrak{p}([b'(k)/2]))^2,$$

and, for $l \geq 2$,

$$kE((Y_i - Y_{i-1})^2) \leq 2^{3-l}(\mathfrak{p}(k2^l - 2) - \mathfrak{p}(k2^l - 1))^2.$$

Hence $k \sum_{l=0}^{\infty} E((Y_i - Y_{i-1})^2)/p_l$ is upper-bounded by the right-hand side of (3.4). Theorem [3.1] shows that $Z_k$ is square-integrable, that (3.4) holds, and that $E(Z_k) = \mu'$. Consequently, $E(f_{k,0} + Z_k) = \mu$. 

\[\square\]
G Proof of Lemma 3.3

Denote by $M$ the right-hand side of (3.4). As $p_1 \geq 1/4$ and $p_0 \geq 1/2$, we have
\[
M \leq 12(\nu([b/(k)]/2))^2 + 8\nu(k) \sum_{l=2}^{\infty} (\nu(k2^{l-2}) - \nu(k2^{l-1})) \\
= 12(\nu([b/(k)]/2))^2 + 8(\nu(k))^2 \\
\leq 20(\nu([b/(k)]/2))^2.
\]
Hence $M$ is finite. By Lemma 3.2, this implies (3.9) and that $E(f_{k,0} + Z_k) = \mu$. By (3.8),
\[
\sum_{l=0}^{\infty} 2^lp_l = p_0 + 2p_1 + 1 \\
\leq 2p_0 + 2p_1 + 1 \\
\leq 3.
\]
Using Lemma 3.2, it follows that $T_k \leq 9k$.

H Motivation for (3.10)

We first show the following.

**Proposition H.1.** Let $V_1$ and $V_2$ be independent square integrable random variables with finite expected running times $\tau_1$ and $\tau_2$. Let $Q$ be a binary random variable independent of $(V_1, V_2)$, with $\Pr(Q = 1) = q$, where $q \in (0, 1]$. Set $V = V_1 + q^{-1}QV_2$. Let $\tau$ be the expected time to simulate $V$. Then
\[ Var(V)\tau \leq (Var(V_1) + q^{-1}E(V_2^2))((\tau_1 + q\tau_2). \]  
If $0 < E(V_2^2)\tau_1 \leq Var(V_1)\tau_2$ then the RHS of (H.1) is minimized when
\[ q = \sqrt{\frac{E(V_2^2)\tau_1}{Var(V_1)\tau_2}}. \]  

**Proof.** We have
\[
Var(QV_2) \leq E((QV_2)^2) \\
= qE(V_2^2).
\]
Hence
\[
Var(V) = Var(V_1) + q^{-2}Var(QV_2) \\
\leq Var(V_1) + q^{-1}E(V_2^2). \]  
Simulating $V$ requires to simulate $V_1$ and, when $Q = 1$, to simulate $V_2$. Thus $\tau = \tau_1 + q\tau_2$. A standard calculation implies (H.2).

By Lemma 3.2 and (3.8), the expected running times of $Z_k$ and of $f_{k,0}$ are of order $k$. Since the length of the time-averaging period in $f_{k,0}$ is at least $k/2$, it follows from Lemma 2.1 that
\[ Var(f_{k,0}) \leq \frac{50\nu(0)^2}{k}. \]
Similarly, Lemma 3.3 gives an upper bound on $E(Z_k^2)$. Applying Proposition H.1 with $V_1 = f_{k,0}$ and $V_2 = Z_k$ and replacing $Var(f_{k,0})$ and $E(Z_k^2)$ with their upper bounds yields (3.10), up to a multiplicative factor.
I Proof of Theorem 3.2

It follows from Lemma 3.3 and the definition of \( \hat{f}_k \) that \( E(\hat{f}_k) = E(f_{k,0}) + E(Z_k) = \mu \). Also, by (3.3),

\[
\text{Var}(\hat{f}_k) \leq \text{Var}(f_{k,0}) + q^{-1}E(Z_k^2).
\]

By Lemma 3.1,

\[
\text{Var}(f_{k,0}) \leq \frac{796(\nu(0))^2 \beta(k)}{k} + \text{Var}(f_k),
\]

where

\[
\beta(k) := \max\left(q, \sqrt{b'(k) - b(k)} \right).
\]

Furthermore, by Lemma 3.3

\[
q^{-1}E(Z_k^2) \leq \frac{20(\nu([b'(k)/2]))^2}{q^k} = \frac{20(\nu(0))^2 q}{k} \leq \frac{20(\nu(0))^2 \beta(k)}{k}.
\]

Thus,

\[
\text{Var}(\hat{f}_k) \leq \text{Var}(f_k) + \frac{816(\nu(0))^2 \beta(k)}{k}.
\]

Lemma 3.3 shows that \( \hat{T}_k \leq k + 9qk \leq k(1 + 9\beta(k)) \). Hence,

\[
\hat{T}_k \text{Var}(\hat{f}_k) \leq (k \text{Var}(f_k) + 816(\nu(0))^2 \beta(k))(1 + 9\beta(k)) = k \text{Var}(f_k) + 9k \text{Var}(f_k) \beta(k) + 816(\nu(0))^2 \beta(k) + 7344(\nu(0))^2(\beta(k))^2.
\]

As, by (3.1), \( k \text{Var}(f_k) \leq 50(\nu(0))^2 \), and \( \beta(k) \leq 1 \), this implies (3.11) after some calculations.

\[\square\]

J Proof of Lemma 3.4

Denote by \( M \) the right-hand side of (3.3). By (3.12) and the monotonicity of \( \theta \), we have \( p_l \geq 2^{l-1}/\theta(l) \) for \( l \geq 0 \). As \( \theta(0) = \theta(1) = 1 \), it follows that

\[
M \leq 12(\nu([b'(k)/2]))^2 + 16 \sum_{l=2}^{\infty} \theta(l)(\nu(k2^{l-2}) - \nu(k2^{l-1}))^2.
\]

For \( l \geq 2 \),

\[
\sqrt{\theta(l)(\nu(k2^{l-2}) - \nu(k2^{l-1}))} = \sqrt{\theta(l)} \sum_{i=k2^{l-2}}^{k2^{l-1}-1} \frac{\nu(i)}{i+1} \leq \sum_{i=k2^{l-2}}^{k2^{l-1}-1} \frac{\nu(i)\theta(\log_2(4i+1))}{i+1} = \nu_\theta(k2^{l-2}) - \nu_\theta(k2^{l-1}),
\]

\[30\]
where the second equation follows from the monotonicity of $\theta$. Hence,
\[
\sum_{l=2}^{\infty} \theta(l)(\varphi(k2^{l-2}) - \varphi(k2^{l-1}))^2 \leq \sum_{l=2}^{\infty} (\varphi(\theta(k2^{l-2}) - \varphi(\theta(k2^{l-1})))
\leq \varphi(\theta) \sum_{l=2}^{\infty} (\varphi(\theta(k2^{l-2}) - \varphi(\theta(k2^{l-1})))
= (\varphi(\theta))^2
\leq (\varphi(\theta(b(k)/2)))^2,
\]
where the second and the last equation follow from the fact that $\varphi(\theta)$ is a decreasing function of $\theta$. Thus $M \leq 28(\varphi(\theta(b(k)/2)))^2$. Together with Lemma 3.2, this implies (3.15) and that $E(f_{k,0} + Z_k) = \mu$. As $p_l \leq 2^{-l}/\theta(l)$, Lemma 3.2 implies the desired bound on $T_k$.

**Proof of Theorem 3.3**

It follows from Lemma 3.4 and the definition of $\hat{f}_k$ that $E(\hat{f}_k) = E(f_{k,0}) + E(Z_k) = \mu$. Also, by definition of $\hat{f}_k$ and (H.3),
\[
\text{Var}(\hat{f}_k) \leq \text{Var}(f_{k,0}) + q^{-1}E(Z_k^2).
\]
Lemma 3.1 implies that
\[
\text{Var}(f_{k,0}) \leq 796(\varphi(0))^2 \sqrt{b'(k)/k} + \text{Var}(f_k).
\]
Together with (3.15), this implies (3.16). The desired bound on $\hat{T}_k$ follows from the bound on $T_k$ in Lemma 3.4.

**Proof of Theorem 3.4**

A standard calculation shows that
\[
\sum_{l=2}^{\infty} \frac{1}{\theta(l)} \leq \frac{1}{\delta - 1},
\]
which implies (3.13). Also, (3.14) follows from Assumption A2. Thus, Assumption A3 holds. By the inequality $\delta \leq 2$, Theorem 3.3 implies (3.17). Set $\omega(i) = \nu(i)\theta(\log_2(4i + 1))$ for $i \geq 0$. For $j \geq 0$, we have $\omega(j) = \varphi(\theta(j))$. For $i \geq 0$,
\[
\omega(i) \leq ce^{-\xi i} \max(1, \log_2(4i + 1))^{-\delta}
\leq ce^{-\xi i}(\log_2(4i + 2))^{-\delta}
\leq ce^{-\xi i}(2\log_2(i + 2))^{-\delta}
\leq \frac{4c}{\ln^2(2)} \ln^{\delta}(i + 2)e^{-\xi i},
\]
where the third equation follows from the inequality $4i + 2 \leq (i + 2)^2$, and the last equation follows from the inequality $\delta \leq 2$. Using the inequality $\delta \leq 2$ once again, (3.11) implies that
\[
\varphi(\theta(0)) \leq \frac{2}{\ln(2)} \sqrt{\frac{ec}{\xi}} \left(2^{3/2} \Gamma \left(\frac{3}{2}\right) + 2\sqrt{2\pi} \ln^{\delta/2} \left(\frac{2}{\xi}\right)\right)
= \frac{4\sqrt{2\pi}}{\ln(2)} \sqrt{\frac{ec}{\xi}} \left(1 + \ln^{\delta/2} \left(\frac{2}{\xi}\right)\right)
\leq 8\sqrt{2\pi} \ln^{\delta/2} \left(\frac{3}{\xi}\right),
\]
where the second equation follows from the equality $\Gamma(3/2) = \sqrt{\pi}/2$ and the last one from the fact that 1 and $\ln(2/\xi)$ are upper-bounded by $\ln(3/\xi)$. On the other hand, (D.2) and the inequality $\delta \leq 2$ show that, for $j \geq 0$,

$$\nu_\theta(j) \leq \frac{20 \sqrt{c} e^{-\xi j/2}}{e \ln(2) \xi}.$$  

As $\nu_\theta(j) \leq \nu_\theta(0)$, it follows that

$$\nu_\theta(j) \leq \min \left( \frac{8 \sqrt{2 c \pi}}{\ln(2) \sqrt{\xi} \ln^{6/2} \left( \frac{3}{\xi} \right)} \frac{20 \sqrt{c} e^{-\xi j/2}}{e \ln(2) \xi} \right).$$  

(L.1)

Combining (2.13), (L.1) and (3.16) yields (3.18).

Assume now that $\nu(j) \leq c/(i+1)$ for $i \geq 0$. A calculation similar to the one above shows that, for $i \geq 0$,

$$\omega(i) \leq \frac{4 c \ln^6(i+2)}{\ln^2(2)} \frac{1}{i+1}.$$  

As $\delta \leq 2$, by Proposition D.3

$$\nu_\theta(0) \leq \frac{14 \sqrt{c}}{\ln(2)} \ln^{6/2+1} \left( \frac{8}{\xi^2} \right) \leq \frac{56 \sqrt{c}}{\ln(2)} \ln^{6/2+1} \left( \frac{3}{\xi} \right).$$  

Hence, for $j \geq 0$,

$$\nu_\theta(j) \leq \min \left( \frac{56 \sqrt{c}}{\ln(2)} \ln^{6/2+1} \left( \frac{3}{\xi} \right) \frac{20 \sqrt{c} e^{-\xi j/2}}{e \ln(2) \xi} \right).$$  

(L.2)

Combining (2.15) and (L.2) with (3.16) yields (3.19).

**M Proof of Proposition 3.1**

We first prove the correctness of Algorithm 2. Denote by $S_K[0], \ldots, S_K[h]$ the values of $S[0], \ldots, S[h]$ at the end of LR. It can be shown by induction that if $X_0[0] = X_0$ then $X_i[0] \sim X_i$ for $0 \leq i \leq K$ and

$$S_K[0] \sim \sum_{i=B}^{K-1} f(X_i).$$  

(M.1)

Hence, at the end of line 4 of Algorithm 2, $X[0] \sim X_{k2^N}$.

Assume now that $h = 1$ in Algorithm 1 and that $X_0[0] \sim X_m$ for some non-negative integer $m$, and that $X_0[1] = X_0$. Denote by $V_0, V_1, \ldots, V_{K-1}$ the successive copies of $U_0$ generated by Algorithm 1. We assume that $X_0[0], V_0, \ldots, V_{K-1}$ are independent. We show by induction on $i$ that, for $0 \leq i \leq K$,

$$((X_j[0], X_j[1]), 0 \leq j \leq i) \sim ((X_{j-m',m+m'}, X_{j-m',m'}), 0 \leq j \leq i).$$  

(M.2)

The base case holds since $X_{-m',m+m'} \sim X_m$. Assume that (M.2) holds for $i$. Step 11 in Algorithm 1 shows that $X_{i+1}[0] = g(X_i[0], V_i)$ and $X_{i+1}[1] = g(X_i[1], V_i)$. Similarly, (2.3) shows that $X_{i+1-m',m+m'} = g(X_{i-m',m+m'}, U_{i-m'})$ and $X_{i+1-m',m'} = g(X_{i-m',m'}, U_{i-m'})$. Since $V_i \sim U_{i-m'}$
and \( V_i \) is independent of \((X_i[0], X_i[1])\), and \( U_{i-m'} \) is independent of \((X_{i-m', m+m'}, X_{i-m', m'})\), and both sides of \((M.2)\) are Markov chains, this implies that \((M.2)\) holds for \( i + 1 \). Thus,

\[
(S_K[0], S_K[1]) = \left( \sum_{i=B}^{K-1} f(X_i[0]), \sum_{i=B}^{K-1} X_i[1] \right) \\
\sim \left( \sum_{i=B}^{K-1} f(X_{i-m', m+m'}), \sum_{i=B}^{K-1} f(X_{i-m', m'}) \right).
\]

(M.3)

Applying \((M.3)\) with \( B = k(2^N - 1) + b'(k) \), \( m = K = k2^N \) and \( m' = k(2^N - 1) \), and using \((3.1)\), shows after some simplifications that, at the end of line 6 of Algorithm 2,

\[
\frac{1}{k - b'(k)} (S[0], S[1]) \sim (f_{k,N+1}, \hat{f}_{k,N}).
\]

By \((3.3)\), line 7 of Algorithm 2 outputs a random variable with the same distribution as \(-Z_k^{b'(k)}\).

We now prove the correctness of Algorithm 3. By \((M.1)\), at the end of line 3 of Algorithm 3,

\[
S[0] \sim \sum_{i=b'(k)}^{k-1} f(X_i),
\]

and so line 4 is consistent with \((3.2)\). As line 7 is executed with probability \( 1 - q \) and line 9 is executed with probability \( q \), \((5.5)\) shows that Algorithm 3 outputs a random variable with the same distribution as \( \hat{f}_k \).

\[\square\]

N Proof of Proposition 4.2

We have \( X_n = \max_{0 \leq j \leq n} \{ S_n - S_j \} \) for \( n \geq 0 \), where \( S_n := \sum_{k=0}^{n-1} U_k \), with \( S_0 := 0 \) (Asmussen and Glynn 2007, §I, Eq. (1.4))). For \( i, m \geq 0 \), using \((2.3)\), it can be shown by induction on \( i \) that \( X_{i+m, -m} = \max_{0 \leq j \leq m-1} \{ S_{i+m} - S_j \} \).

\[
X_{i+m} = \max(X_{i+m, -m}, \max_{0 \leq j \leq m-1} \{ S_{i+m} - S_j \}).
\]

(N.1)

By applying Proposition \((3.1)\) with \( m' = 0 \) and \( n = i \), it follows that

\[
E((f(X_{i,m}) - f(X_i))^2) = E((f(X_{i+m}) - f(X_{i+m, -m}))^2) \\
\leq \Pr(X_{i+m} \neq X_{i+m, -m}) \\
\leq \sum_{j=0}^{m-1} \Pr(S_{i+m} - S_j > 0) \\
= \sum_{j=0}^{m-1} \Pr(S_{i+m-j} > 0) \\
= \sum_{j=1}^{m} \Pr(S_{i+j} > 0),
\]

where the third equation follows from \((N.1)\), and the fourth equation follows by noting that \( S_{i+m} - S_j \sim S_{i+m-j} \). Set \( \mu' = E(U_0) \) and \( \mu'' = E(U_0^6) \), where \( U_i' = U_i - \mu' \) for \( i \geq 0 \). For \( n \geq 0 \), let \( S_{n}' := \sum_{j=0}^{n-1} U_j' \). Since \( U_n' \) and \( S_{n}' \) are centered and independent, the binomial theorem shows that, for \( n \geq 0 \),

\[
E(S_{n+1}')^6 = E(S_n'^6) + 15E(U_n'^2)E(S_n'^4) + 15E(U_n'^4)E(S_n'^2) + E(U_n'^6) \\
\leq E(S_n'^6) + 15(\mu'^1/3)(E(S_n'^6))^{2/3} + 15(\mu''^2/3)(E(S_n'^6))^{1/3} + \mu'',
\]

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Thus, (2.7) holds with \( \rho \) as follows from (Kahalé 2019, Lemma 1) that, conditional on \( e \), we first prove the exponential bound by showing that Proposition 2.2 holds with
\[
E((f(X_{i,m}) - f(X_i))^2) \leq 125 \frac{\mu''}{(\mu')^6} \sum_{j=1}^{\infty} (i + j)^{-3}.
\]
where the second equation follows from Markov’s inequality and the fact that \( \mu' < 0 \). Hence,
\[
E((f(X_{i,m}) - f(X_i))^2) \leq 125 \frac{\mu''}{(\mu')^6} (i + 1)^{-3} + \int_{i+1}^{\infty} x^{-3} dx.
\]
where the second equation follows from Jensen’s inequality and the fact that \( \mu' = 0 \). Hence,
\[
E((f(X_{i,m}) - f(X_i))^2) \leq \frac{375 \mu''}{2(\mu')^6}(i + 1)^{-2}.
\]
\[
\Pr(S_n > 0) = \Pr(S_n > -\mu'n) 
\leq E\left(\frac{S_n^6}{(\mu'n)^6}\right) 
\leq 125 \frac{\mu''}{(\mu')^6} n^{-3},
\]
where the second equation follows from (Kahalé 2019, Lemma 1) that, conditional on \( e \),

### O Proof of Theorem 4.1

We first prove the exponential bound by showing that Proposition 2.2 holds with \( \rho(x, x') = ||\sqrt{V^{-1}}(x - x')|| \) for \( (x, x') \in F \times F \). Fix non-negative integers \( i \) and \( m \) with \( i + m \geq 0 \). It follows from (Kahalé 2019, Lemma 1) that, conditional on \( e_{-m}, \ldots, e_{i-1} \), \( \left(\frac{X_i - X_{i,m}}{\sqrt{V^{-1}}}\right) \) is a centered Gaussian vector with \( \text{Cov}(X_i) \leq V \) and \( \text{Cov}(X_{i,m}) \leq V \). Consequently, by (4.2),
\[
E((f(X_{i,m}) - f(X_i))^2) \leq \hat{\kappa}^2 \left(E(||X_{i,m} - X_i||^2|e_{-m}, \ldots, e_{i-1})\right)^\hat{\gamma}.
\]
Taking expectations and using the tower law and Jensen’s inequality implies that
\[
E((f(X_{i,m}) - f(X_i))^2) \leq \hat{\kappa}^2 \left(E(||X_{i,m} - X_i||^2)\right)^\hat{\gamma}.
\]
As \( \rho^2(x, x') = (x - x')^T V^{-1} (x - x') \), we have \( \rho^2(x, x') \geq \lambda_{\max}^2 ||x - x'||^2 \). Hence
\[
E((f(X_{i,m}) - f(X_i))^2) \leq \hat{\kappa}^2 \lambda_{\max}^\hat{\gamma} (E(\rho^2(X_{i,m}, X_i)))^\hat{\gamma}.
\]
Thus, (2.2) holds with \( \kappa = \hat{\kappa} \lambda_{\max}^{\hat{\gamma}/2} \) and \( \gamma = \hat{\gamma} \).

For \( m \geq 0 \), we have \( \rho(X_0, X_m) = ||\sqrt{V^{-1}}X_m|| \). It follows from (Kahalé 2019, Lemma 2) that \( E(X_m) = 0 \) and \( \text{Cov}(\sqrt{V^{-1}}X_m) \leq I \). As the variance of each entry of \( \sqrt{V^{-1}}X_m \) is at most 1 and its expectation is 0, we have \( E(||\sqrt{V^{-1}}X_m||^2) \leq d \). Consequently, (2.8) holds with \( \kappa' = d \).

Furthermore, for \( x, x' \in F \), we have
\[
g(x, g_0, e_0) - g(x', g_0, e_0) = (I - Ve_0e_0^T)(x - x').
\]
Thus
\[
\rho(g(x, g_0, e_0), g(x', g_0, e_0)) = ||Py||,
\]
where \( P = I - \sqrt{V} e_0 e_0^T \sqrt{V} \) and \( y = \sqrt{V^{-1}}(x - x') \). A standard calculation (e.g., Kahalé (2019)) shows that \( P^2 = P \) and \( E(P) = I - d^{-1} V \). Hence,
\[
E(\rho^2(g(x, g_0, e_0), g(x', g_0, e_0))) = E(y^T Py)
\leq y^T(I - d^{-1} V)y
\leq (1 - \lambda_{\min}/d)||y||^2,
\]

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where the last equation follows from the fact that the largest eigenvalue of $I - d^{-1}V$ is $1 - \lambda_{\text{min}}/d$. As $\rho(x, x') = \|y\|$, (2.9) holds for $\eta = 1 - \lambda_{\text{min}}/d$. By Proposition 2.2

$$E(\|f(X_{i,m}) - f(X_i)\|^2) \leq \kappa^2(\lambda_{\text{max}}d)^i (1 - \lambda_{\text{min}}/d)^{\frac{i}{\hat{\gamma}_i}}.$$

We now prove the geometric bound. For $n \geq 0$, set $P_n := I - \sqrt{V}e_ne_n^T\sqrt{V}$, and $M_n := P_{n-1}P_{n-2} \cdots P_0$, with $M_0 := I$. By (2.3), for $i, m \geq 0$,

$$X_{i+1,m} = X_{i,m} + (g_i - e_i^TX_{i,m})(V e_i). \quad \text{(O.2)}$$

As (O.2) also holds for $m = 0$, it follows that

$$X_{i+1,m} - X_{i+1} = (I - V e_i e_i^T)(X_{i,m} - X_i).$$

Consequently, it can be shown by induction on $i$ that $X_{i,m} - X_i = \sqrt{V}M_i\sqrt{V^{-1}}X_{0,m}$ for $i, m \geq 0$. Hence

$$E(\|X_{i,m} - X_i\|^2) = E(\|\sqrt{V}M_i\sqrt{V^{-1}}X_{0,m}\|^2)$$
$$= E(\text{tr}(\sqrt{V}M_i\sqrt{V^{-1}}X_{0,m}X_{0,m}^T\sqrt{V^{-1}}M_i^T\sqrt{V}))$$
$$= E(\text{tr}(\sqrt{V}M_i\sqrt{V^{-1}}E(X_{0,m}X_{0,m}^T)\sqrt{V^{-1}}M_i^T\sqrt{V})).$$

The second equation follows from the equality $\|Z\|^2 = \text{tr}(ZZ^T)$, whereas the last equation follows from the independence of $M_i$ and $X_{0,m}$. On the other hand,

$$E(X_{0,m}X_{0,m}^T) = E(X_mX_m^T) \leq V,$$

where the last equation follows from (Kahalé 2019, Lemma 2). Consequently,

$$E(\|X_{i,m} - X_i\|^2) \leq E(\text{tr}(\sqrt{V}M_iM_i^T\sqrt{V}))$$
$$= E(\text{tr}(M_i^TVM_i))$$
$$\leq \frac{\mu^2}{i+1},$$

where the second equation follows from the equality $\text{tr}(AB) = \text{tr}(BA)$, and the last equation follows from (Kahalé 2019, Theorem 1). Applying (O.1) concludes the proof. □

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