On two channel flavor anisotropic and one channel compactified
Kondo models

Jinwu Ye
Physics Laboratories, Harvard University, Cambridge, MA, 02138
and Department of Physics and Astronomy, Johns Hopkins University, Baltimore, MD, 21218
(January 21, 2022)

Abstract

We reinvestigate the two channel flavor anisotropic model (2CFAK) and one
channel compactified Kondo model (1CCK). For these two models, all the possible fixed points and their symmetries are identified; the finite size spectra, the electron conductivity and pairing susceptibility are calculated. It is shown that the only non-fermi liquid (NFL) fixed point of the 2CFAK is the NFL of the two channel Kondo model (2CK) with the symmetry $O(3) \times O(5)$. Any flavor anisotropies between the two channels drive the system to the fermi-liquid (FL) fixed point with the symmetry $O(4) \times O(4)$ where one of the two channels suffers the phase shift $\pi/2$ and the other remains free. The NFL fixed point of the 1CCK has the symmetry $O(3) \times O(1)$ and has the same thermodynamics as the NFL fixed point of the 2CK. However, in contrast to the 2CK, its conductivity shows $T^2$ behavior and there is no pairing susceptibility divergence. Any anisotropies between the spin and isospin sectors drive the system to the FL fixed point with the symmetry $O(4)$ where the electrons suffer the phase shift $\pi/2$. The connection and differences between the two models are explicitly demonstrated. The recent conjectures and claims on the NFL behaviors of the two models are commented.
I. INTRODUCTION

Extensive attention has been lavished on the overscreened multichannel Kondo model after the discovery of its non-fermi liquid (NFL) behavior by Nozières and Blandin (NB)\(^1\). NB also pointed out that lattice effects in real metals will cause the anisotropy between the two flavor channels and that in the low temperature limit, the impurity is totally screened by the strong coupling channel with the weak coupling channel unaffected. Using Numerical Renormalization Group (NRG), Ref.\(^2\) confirmed NB’s conjecture. Using Conformal Field Theory (CFT), Ref.\(^3\) found a relevant dimension 1/2 operator in the flavor sector near the 2 channel Kondo (2CK) fixed point and suggested the system flows to the Fermi-liquid (FL) fixed point pointed out by NB. Using Yuval-Anderson’s approach, Ref.\(^4\) found a solvable line and calculated the exact crossover free energy function from the 2CK fixed point to the FL fixed point along this solvable line.

It is known that in the large \(U\) limit, the ordinary one channel symmetric Anderson impurity model (AIM) can be mapped to the one channel Kondo model. However, as shown by Ref.\(^5\), \(^6\), if the original \(O(4)\) symmetry of the AIM is broken to \(O(3) \times O(1)\), in the strong coupling limit, the AIM is mapped to the one channel compactified Kondo model (1CCK) where the impurity spin couples to both the spin and the isospin(charge) currents of the one channel conduction electrons.

Recently, Andrei and Jerez\(^5\), using Bethe Ansatz, reinvestigated the 2CFAK and conjectured that the 2CFAK flow to some new NFL fixed points. Coleman and Schofield\(^7\), using strong coupling method, reinvestigated the 1CCK and claimed the system flows to another kind of non-Fermi liquid fixed point which, similar to 1-dim Luttinger liquid, has the same thermodynamics as fermi liquid but different excitation spectrum. Moreover, they claimed that the 1CCK has exactly the same low energy excitations as those of the 2CFAK, therefore concluded that their results also apply to the 2CFAK.

So far, Bethe Ansatz can only calculate thermodynamic quantities of multichannel Kondo models, the correlation functions are needed to resolve if the fixed points are NFL or FL. It is important to point out that the charge degrees of freedom of the original model being removed, the 1CCK in Ref.\(^8\)\(^,\)\(^9\) has completely different transport properties, correlation functions and excitation spectrums than the original 2CFAK, although it do share the same thermodynamic properties as the 2CFAK.

As emphasized by AL\(^10\), although the boundary interactions only happen in the spin sector; the spin, flavor and charge degree of freedoms are not totally decoupled, there is a constraint( or gluing condition) to describe precisely how these degree of freedoms are combined at different boundary fixed points, the finite size spectrum is determined by this gluing condition. The boundary operator contents and the scaling dimensions of all the boundary operators are also given by the gluing condition. However, in order to find the gluing conditions at the intermediate coupling fixed points, the fusion rules should be identified which are usually difficult in Non-Abelian bosonization approach. For 4 pieces of bulk fermions, the non-interacting theory possesses \(SO(8)\) symmetry, Maldacena and Ludwig (MS)\(^11\) showed that finding the gluing conditions at the fixed points are exactly equivalent to finding the boundary conditions of the fermions at the fixed points; the CFT describing the fixed points are simply free chiral bosons with the boundary conditions. In Ref.\(^12\), the author developed a simple and powerful method to study certain class of quantum impurity
models. The method can quickly identify all the possible boundary fixed points and their maximum symmetry, therefore avoid the difficulty of finding the fusion rules, it can also demonstrate the physical picture at the boundary explicitly. In this paper, we apply the method to study the two models. All the possible fixed points and their symmetries are identified; the finite size spectra, the electron conductivity and pairing susceptibility are calculated. All the leading and subleading irrelevant operators are identified, their corrections to the correlation functions are evaluated. In section II, Taking all the degrees of freedom into account, We show that the only NFL fixed point of the 2CFAK is the NFL fixed point of the 2CK with the symmetry $O(3) \times O(5)$. Any flavor anisotropies between the two channels drive the system to the fermi-liquid (FL) fixed point with the symmetry $O(4) \times O(4)$ where one of the two channels suffers the phase shift $\pi/2$ and the other remains free. The conventional wisdom about the 2CFAK is rigorously shown to be correct. In section III, we repeat the same program to the 1CCK. We find that the NFL fixed point of the 1CCK has the symmetry $O(3) \times O(1)$ and has the same thermodynamics as the NFL fixed point of the 2CK. The finite size spectrum is listed and compared with that of the 2CK. However, in contrast to the 2CK, its conductivity shows $T^2$ behaviour and there is no pairing susceptibility enhancement. Any anisotropies between the spin and isospin sectors drive the system to the FL fixed point with the symmetry $O(4)$ where the electrons suffer the phase shift $\pi/2$. The finite size spectrum of this FL fixed point is also listed and compared with that of the 2CFAK. In section IV, we conclude and propose some open questions. Finally, in the appendix, we study the stable FL fixed point of the 2CFAK using Non-Abelian bosonization and compare with the Abelian bosonization calculations done in section II.

II. THE TWO CHANNEL FLAVOR ANISOTROPIC KONDO MODEL

The Hamiltonian of the 2CFAK is:

$$H = iv_F \int_{-\infty}^{\infty} dx \psi_{i\alpha}^\dagger(x) \frac{d\psi_{i\alpha}(x)}{dx} + \sum_{a=x,y,z} \lambda^a(J_1^a(0) + J_2^a(0))S^a + \sum_{a=x,y,z} \alpha^a(J_1^a(0) - J_2^a(0))S^a + h(\int dx J_1^a(x) + S^z)$$

(1)

where $J_i^a(x) = \frac{1}{2} \psi_{i\alpha}^\dagger(x) \sigma_{\alpha\beta}^a \psi_{i\beta}(x)$ are the spin currents of the channel $i = 1, 2$ conduction electrons respectively. $\alpha^a = 0, \pm \lambda^a$ correspond to the 2CK and the one channel Kondo model respectively. If $\lambda^a = \lambda, \alpha^a = \alpha \neq 0$, the above Hamiltonian breaks $SU(2)_s \times SU(2)_f \times U(1)_c$ symmetry of the 2CK to $SU(2)_s \times U(1)_f \times U(1)_c$ ( or equivalently $SU(2)_s \times U(1)_{c1} \times U(1)_{c2}$, because we have two independent $U(1)$ charge symmetries in the channel 1 and the channel 2 ).

In this section, for simplicity, we take $\lambda^x = \lambda^y = \lambda, \lambda^z \neq \lambda; \alpha^x = \alpha^y = \alpha, \alpha^z \neq \alpha$. The symmetry in the spin sector is reduced to $U(1)_1 \times Z_2 \sim O(2)$. In the following, we closely follow the notations of Emery-Kivelson.\[11\] Abelian-bosonizing the four bulk Dirac fermions separately:

$$\psi_{i\alpha}(x) = \frac{P_{i\alpha}}{\sqrt{2\pi a}} e^{-i\Phi_{i\alpha}(x)}$$

(2)

Where $\Phi_{i\alpha}(x)$ are the real chiral bosons satisfying the commutation relations...
\[ [\Phi_{i\alpha}(x), \Phi_{j\beta}(y)] = \delta_{ij} \delta_{\alpha\beta} i\pi \text{sgn}(x - y) \] (3)

The cocycle factors have been chosen as: \( P_1^\uparrow = P_{1\downarrow} = e^{i\pi N_{1\uparrow}} \), \( P_2^\uparrow = P_{2\downarrow} = e^{i\pi (N_{1\uparrow} + N_{1\downarrow} + N_{2\uparrow})} \).

It is convenient to introduce the following charge, spin, flavor, spin-flavor bosons:

\[
\begin{align*}
\Phi_c &= \frac{1}{2}(\Phi_{1\uparrow} + \Phi_{1\downarrow} + \Phi_{2\uparrow} + \Phi_{2\downarrow}) \\
\Phi_s &= \frac{1}{2}(\Phi_{1\uparrow} - \Phi_{1\downarrow} + \Phi_{2\uparrow} - \Phi_{2\downarrow}) \\
\Phi_f &= \frac{1}{2}(\Phi_{1\uparrow} + \Phi_{1\downarrow} - \Phi_{2\uparrow} - \Phi_{2\downarrow}) \\
\Phi_{sf} &= \frac{1}{2}(\Phi_{1\uparrow} - \Phi_{1\downarrow} - \Phi_{2\uparrow} + \Phi_{2\downarrow})
\end{align*}
\]

The spin currents \( J^a(x) = J_1^a(x) + J_2^a(x) \) and \( \tilde{J}^a(x) = J_1^a(x) - J_2^a(x) \) can be expressed in terms of the above chiral bosons

\[
\begin{align*}
J_x &= \frac{1}{\pi a} \cos \Phi_s \cos \Phi_{sf}, \quad J_y = \frac{1}{\pi a} \sin \Phi_s \cos \Phi_{sf}, \quad J_z = \frac{1}{2\pi} \frac{\partial \Phi_s}{\partial x} \\
\tilde{J}_x &= -\frac{1}{\pi a} \sin \Phi_s \sin \Phi_{sf}, \quad \tilde{J}_y = \frac{1}{\pi a} \cos \Phi_s \sin \Phi_{sf}, \quad \tilde{J}_z = -\frac{1}{2\pi} \frac{\partial \Phi_{sf}}{\partial x}
\end{align*}
\]

After making the canonical transformation \( U = \exp[iS^z\Phi_s(0)] \) and the following refermionization

\[
\begin{align*}
S^x &= \frac{a}{\sqrt{2}} e^{i\pi N_{sf}}, \quad S^y = \frac{b}{\sqrt{2}} e^{i\pi N_{sf}}, \quad S^z = -i\tilde{a}\tilde{b} \\
\psi_{sf} &= \frac{1}{\sqrt{2}} (a_{sf} - ib_{sf}) = \frac{1}{\sqrt{2\pi a}} e^{i\pi N_{sf}} e^{-i\Phi_{sf}} \\
\psi_{s,i} &= \frac{1}{\sqrt{2}} (a_{s,i} - ib_{s,i}) = \frac{1}{\sqrt{2\pi a}} e^{i\pi (d_{s,i} + N_{sf})} e^{-i\Phi_s}
\end{align*}
\]

The transformed Hamiltonian \( H' = UHU^{-1} = H_{sf} + H_s + \delta H \) can be written in terms of the Majorana fermions:

\[
\begin{align*}
H_{sf} &= \frac{iv_F}{2} \int dx (a_{sf}(x) \frac{\partial a_{sf}(x)}{\partial x} + b_{sf}(x) \frac{\partial b_{sf}(x)}{\partial x}) - i \frac{\lambda}{2\pi a} \tilde{a} b_{sf}(0) + i \frac{\alpha}{2\pi a} \tilde{a} a_{sf}(0) \\
H_s &= \frac{iv_F}{2} \int dx (a_{s}(x) \frac{\partial a_{s}(x)}{\partial x} + b_{s}(x) \frac{\partial b_{s}(x)}{\partial x}) - ih \int dx a_{s}(x) b_{s}(x) \\
\delta H &= -\lambda_z \tilde{a} b_{s}(0) b_{s}(0) - \alpha_z \tilde{a} a_{sf}(0) b_{sf}(0)
\end{align*}
\]

where \( \lambda_z = \lambda^2 - 2\pi v_F \).

It is instructive to compare the above equation with Eq.3 in Ref. They looks very similar: half of the impurity spin coupled to half of the spin-flavor electrons, another half of the impurity spin coupled to another half of the spin-flavor electrons. However, the two canonical transformations employed in the two models are different. This fact make the boundary conditions of this model rather different from that of the two channel spin-flavor Kondo model (2CSFK) discussed in Ref.
The above Hamiltonian was first derived by Ref.4 using Anderson-Yuval’s approach. They found the solvable line $\lambda^z = 2\pi v_F, \alpha^z = 0$ and calculated the exact crossover function of free energy along this solvable line. Using EK’s method, we rederived this Hamiltonian16.

The huge advantage of EK’s method over Anderson-Yuval’s approach is that the boundary conditions at different boundary fixed points can be identified17.

By using the Operator Product Expansion (OPE) of the various operators in Eq.20, we get the RG flow equations near the weak coupling fixed point $\lambda^z = 2\pi v_F, \lambda = \alpha = \alpha^z = 0$

$$\frac{d\lambda}{dl} = \frac{1}{2}\lambda + \alpha \alpha^z$$

$$\frac{d\alpha}{dl} = \frac{1}{2}\alpha - \lambda \alpha^z$$

$$\frac{d\alpha^z}{dl} = -\lambda \alpha$$ (8)

The fact that we find two relevant operators in the above equations may indicate there are two intermediate coupling fixed points. However, in the following, the two intermediate coupling fixed points are shown to be the same.

The original impurity spin in $H$ are related to those in $H'$ by

$$S^H_x = US_xU^{-1} = S_x \cos \Phi_s(0) - S_y \sin \Phi_s(0)$$

$$S^H_y = US_yU^{-1} = S_x \sin \Phi_s(0) + S_y \cos \Phi_s(0)$$

$$S^H_z = US_zU^{-1} = S_z$$ (9)

Using the refermionization Eq.3, the original impurity spin in $H$ can be written in terms of fermions

$$S^H_x = i(\hat{b}a_{s,i} + \hat{a}b_{s,i})$$

$$S^H_y = i(\hat{b}b_{s,i} - \hat{a}a_{s,i})$$

$$S^H_z = -i\hat{a}\hat{b}$$ (10)

At $\lambda^z = 0$, the spin boson $\Phi_s$ completely decouples from the impurity in $H'$, therefore $\chi_{imp} = 0$. Because the canonical transformation $U$ is a boundary condition changing operator1617, at $\lambda^z = 0$, this leads to

$$a^s_L(0) = -a^s_R(0), \quad b^s_L(0) = -b^s_R(0)$$ (11)

Following Ref.3, in order to identify the fixed points along the solvable line $\lambda^z = 0, \alpha^z = 0$ (we also set $h = 0$), we write $H_{sf}$ in the action form

$$S = S_0 + \frac{\gamma_1}{2} \int d\tau \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} + \frac{\gamma_2}{2} \int d\tau \hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau}$$

$$- i \frac{\lambda}{\sqrt{2\pi a}} \int d\tau \hat{a}(\tau)b_{sf}(0, \tau) + i \frac{\alpha}{\sqrt{2\pi a}} \int d\tau \hat{b}(\tau)a_{sf}(0, \tau)$$ (12)

When performing the RG analysis of the action $S$, we keep13 1: $\gamma_2 = 1, \lambda$ fixed, 2: $\gamma_1 = 1, \alpha$ fixed, 3: $\lambda, \alpha$ fixed; three fixed points of Eq.4 can be identified.
A. Fixed point 1

This fixed point is located at $\gamma_1 = 0, \gamma_2 = 1$ where $\hat{b}$ decouples, but $\hat{a}$ loses its kinetic energy and becomes a Grassmann Lagrangian multiplier. Integrating $\hat{a}$ out leads to the following boundary conditions:

$$b^f_L(0) = -b^f_R(0)$$  \hspace{1cm} (13)

Eqs.11,13 can be expressed in terms of bosons:

$$\Phi_{s,L}(0) = \Phi_{s,R}(0) + \pi, \quad \Phi_{sf,L}(0) = -\Phi_{sf,R}(0) + \pi$$  \hspace{1cm} (14)

This is just the non-fermi liquid fixed point of the 2CK. The three Majorana fermions in the spin sector being twisted, this fixed point possesses the symmetry $O(3) \times O(5)$. The finite size spectrum of this fixed point was listed in Ref.17.

The local correlation functions at the 2CK fixed point are:

$$\langle \hat{a}(\tau)\hat{a}(0) \rangle = \frac{1}{\tau}, \quad \langle b_{sf}(\tau)b_{sf}(0) \rangle = \frac{\gamma_1^2}{\tau^3}$$  \hspace{1cm} (15)

From the above equation, we can read the scaling dimensions of the various fields $[\hat{b}] = 0, [\hat{a}] = [a_s] = [b_s] = [a_{sf}] = 1/2, [b_{sf}] = 3/2$.

As shown in Ref.17, at the fixed point, the impurity degree of freedoms completely disappear: $\hat{b}$ decouples and $\hat{a}$ turns into the non-interacting scaling field at the fixed point:

$$\hat{a} \sim b_{sf}(0, \tau)$$  \hspace{1cm} (16)

Using Eq.10, the impurity spin turns into

$$S_x^H(\tau) = i(\hat{b}a_{s,i}(0, \tau) + b_{sf}(0, \tau)b_{s,i}(0, \tau))$$
$$S_y^H(\tau) = i(\hat{b}b_{s,i}(0, \tau) - b_{sf}(0, \tau)a_{s,i}(0, \tau))$$
$$S_z^H(\tau) = i\hat{b}b_{sf}(0, \tau)$$  \hspace{1cm} (17)

Using the relation

$$\psi_s^H(x) = U\psi_s(x)U^{-1} = isgnx\psi_{s,i}(x)$$  \hspace{1cm} (18)

We get

$$S_x(\tau) = i(-\hat{b}b_s(0, \tau) + b_{sf}(0, \tau)a_s(0, \tau))$$
$$S_y(\tau) = i(\hat{b}a_s(0, \tau) + b_{sf}(0, \tau)b_s(0, \tau))$$
$$S_z(\tau) = i(\hat{b}b_{sf}(0, \tau) + a_s(0, \tau)b_s(0, \tau))$$  \hspace{1cm} (19)

The impurity spin-spin correlation function $\langle S^a(\tau)S^a(0) \rangle = \frac{1}{\tau}$. The above equations are consistent with the CFT identifications

$$\tilde{S} \sim \tilde{\phi} + \tilde{J} + \cdots$$  \hspace{1cm} (20)
The 2CK fixed point is unstable, because there is a dimension 1/2 relevant operator $\hat{b} a_{sf}$, the OPE of $a_{sf}$ with itself will generate the dimension 2 energy momentum tensor of this Majorana fermion $T(\tau) = \frac{1}{2} a_{sf}(0, \tau) \frac{\partial a_{sf}(0, \tau)}{\partial \tau}$. The OPE of the energy momentum tensor with the primary field $a_{sf}$ is

$$T(\tau_1) a_{sf}(\tau_2) = \frac{1}{(\tau_1 - \tau_2)^2} + \frac{L_{-1} a_{sf}(\tau_2)}{\tau_1 - \tau_2} + L_{-2} a_{sf}(\tau_2) + \cdots$$

(21)

First order descendant field of this primary field $L_{-1} a_{sf}(0, \tau) = \partial a_{sf}(0, \tau) / \partial \tau$ with dimension 3/2 is generated. The $\chi'$ term in $\delta H$ has scaling dimension 3/2, it will generate a dimension 2 operator $a_{sf}(0, \tau) \partial a_{sf}(0, \tau) / \partial \tau + b_{sf}(0, \tau) \partial b_{sf}(0, \tau) / \partial \tau$. The $\gamma_2$ term has dimension 2 also.

From Eq.15, we can see $\alpha_z$ term has scaling dimension 5/2, it can be written as

$$: \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} : a_{sf}(0, \tau) = : b_{sf}(0, \tau) \frac{\partial b_{sf}(0, \tau)}{\partial \tau} : a_{sf}(0, \tau)$$

(22)

The bosonized form of this operator is

$$: (\cos 2\Phi_{sf}(0, \tau) - \frac{1}{2} (\partial \Phi_{sf}(0, \tau))^2) : \sin \Phi_{sf}(0, \tau)$$

(23)

Using CFT, Ref.3 predicted a dimension 1/2 relevant operator $\phi^3_f$ in the flavor sector. Ref.3 classified all the first order descendants of the primary operator in the spin sector. In the flavor sector, the same classification apply, $\vec{J}_{-1} \cdot \phi_f$ is Charge-Time Reversal (CT) odd, therefore is not allowed, but $L_{-1} \phi^3_f$ is CT even. The CFT analysis is completely consistent with the above EK’s solution.

In order to make this fixed point stable, we have to tune $\alpha = \alpha_z = 0$, namely the channel anisotropy is strictly prohibited. If $\alpha = 0$, but $\alpha_z \neq 0$, because $\alpha_z$ is highly irrelevant, it seems the 2CK fixed point is stable. However, this is not true. From the RG flow Eq.8 it is easy to see that even initialy $\alpha = 0$, it will be generated, $\alpha_z$ is 'dangerously' irrelevant.

B. Fixed point 2

This fixed point is located at $\gamma_1 = 1, \gamma_2 = 0$ where $\hat{a}$ decouples, but $\hat{b}$ loses its kinetic energy and becomes a Grassmann Lagrangian multiplier. Integrating $\hat{b}$ out leads to the following boundary conditions:

$$a_{sf}^L(0) = -a_{sf}^R(0)$$

(24)

Eqs.11,24 can be expressed in terms of bosons:

$$\Phi_{s,L}(0) = \Phi_{s,R}(0) + \pi, \quad \Phi_{sf,L}(0) = -\Phi_{sf,R}(0)$$

(25)

This fixed point also possesses the symmetry $O(3) \times O(5)$. In fixed points 1 and 2, $\hat{a}$ and $\hat{b}$, $b_{sf}$ and $a_{sf}$ exchange roles.

As discussed in the fixed point 1, $\alpha_z$ is 'dangerously' irrelevant. In order to make this fixed point stable, we have to tune $\lambda = \alpha_z = 0$. This fixed point is actually the same with
the 2CK fixed point. This can be seen most clearly from the original Eq. 1: if \( \lambda = \alpha_z = 0 \), under the \( SU(2) \) transformation on the channel 2 fermions \( \psi_{2\uparrow} \rightarrow i\psi_{2\uparrow}, \psi_{2\uparrow} \rightarrow -i\psi_{2\uparrow} \), the spin currents of channel 2 transform as \( J_x^{2} \rightarrow -J_x^{2}, J_y^{2} \rightarrow -J_y^{2}, J_z^{2} \rightarrow J_z^{2} \), Eq.1 is transformed back to the 2 channel flavor symmetric Kondo model. This can also be seen from Eq.5, \( \tilde{J}_x, \tilde{J}_y, \tilde{J}_z \) also satisfy the \( \hat{SU}_2(2) \) algebra.

C. Fixed point 3

This fixed point is located at \( \gamma_1 = \gamma_2 = 0 \) where both \( \hat{a} \) and \( \hat{b} \) lose their kinetic energies and become two Grassmann Lagrangian multipliers. Integrating them out leads to the following boundary conditions:

\[
\begin{align*}
    b_{sf}^L(0) &= -b_{sf}^R(0), & a_{sf}^L(0) &= -a_{sf}^R(0) \\
    b_{sf}^R(0) &= b_{sf}^L(0), & a_{sf}^R(0) &= a_{sf}^L(0)
\end{align*}
\]

Eqs.14, 26 can be expressed in term of bosons:

\[
\begin{align*}
    \Phi_L^s &= \Phi_R^s + \pi, & \Phi_L^{sf} &= \Phi_R^{sf} + \pi
\end{align*}
\]

Substituting the above equation to Eqs. 2 4 and paying attention to the spinor nature of the representation, it is easy to see that depending on the sign of \( \alpha \), one of the two channels suffer \( \pi/2 \) phase shift, the other remains free. The four Majorana fermions being twisted, this fixed point has the symmetry \( O(4) \times O(4) \) with \( g = 1 \). The finite size spectrum of this fixed point is listed in Table I, it is the sum of that with phase shift \( \pi/2 \) and that of free electrons. This scenario is completely consistent with NRG results of Ref. 2.

The local correlation functions at the FL fixed point are:

\[
\begin{align*}
    \langle \hat{a}(\tau)\hat{a}(0) \rangle &= \frac{1}{\tau}, & \langle b_{sf}(\tau)b_{sf}(0) \rangle &= \frac{\gamma_1^2}{\tau^3} \\
    \langle \hat{b}(\tau)\hat{b}(0) \rangle &= \frac{1}{\tau}, & \langle a_{sf}(\tau)a_{sf}(0) \rangle &= \frac{\gamma_2^2}{\tau^3}
\end{align*}
\]

From the above equation, We can read the scaling dimensions of the various fields:

\( [\hat{a}] = [\hat{b}] = [a_s] = [b_s] = 1/2, [a_{sf}] = [b_{sf}] = 3/2 \).

At the fixed point, the impurity degree of freedoms completely disappear: \( \hat{a}, \hat{b} \) turn into the non-interacting scaling fields at the fixed point

\[
\hat{a} \sim b_{sf}(0, \tau), \quad \hat{b} \sim a_{sf}(0, \tau)
\]

Using Eqs.10, 18, the impurity spin turns into

\[
\begin{align*}
    S_x(\tau) &= i(-a_{sf}(0, \tau)b_s(0, \tau) + b_{sf}(0, \tau)a_s(0, \tau)) \\
    S_y(\tau) &= i(a_{sf}(0, \tau)a_s(0, \tau) + b_{sf}(0, \tau)b_s(0, \tau)) \\
    S_z(\tau) &= i(a_{sf}(0, \tau)b_{sf}(0, \tau) + a_s(0, \tau)b_s(0, \tau))
\end{align*}
\]

The impurity spin-spin correlation function show typical FL behavior

\[
\langle S^z(\tau)S^z(0) \rangle = \frac{1}{\tau^2}
\]
Using the fermionized form of the Eq. 5 and paying attention to the spinor nature of the representation, it is easy to see the impurity spin renormalizes into either $\vec{J}_1(0, \tau)$ or $\vec{J}_2(0, \tau)$ depending on the sign of $\alpha$. This is consistent with the CFT analysis in the Appendix.

There are 4 leading irrelevant operators with dimension 2 in the action $\hat{S}$: $\gamma_1$ and $\gamma_2$ terms, $\lambda'_z$ term and $a_s(0, \tau) \frac{\partial a_s(0, \tau)}{\partial \tau} + b_s(0, \tau) \frac{\partial b_s(0, \tau)}{\partial \tau}$ which will be generated by the $\lambda'_z$ term.

The $\alpha_z$ term has dimension 4, it can be written as: $\hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} = \hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau}$.

The bosonized forms of the 4 leading irrelevant operators are:

\[
\hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} = \cos 2\Phi_{sf} - \frac{1}{2} (\partial \Phi_{sf}(0))^2 \\
\hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau} = -\cos 2\Phi_{sf} - \frac{1}{2} (\partial \Phi_{sf}(0))^2 \\
\hat{a}^b a_s(0) b_s(0) = \partial \Phi_{sf}(0, \tau) \partial \Phi_{sf}(0, \tau) \\
a_s(0, \tau) \frac{\partial a_s(0, \tau)}{\partial \tau} + b_s(0, \tau) \frac{\partial b_s(0, \tau)}{\partial \tau} = (\partial \Phi_{sf}(0, \tau))^2
\]  

(32)

Following the method developed in Ref. 17, their contributions to the single particle Green functions can be calculated. The first order correction to the single particle L-R Green function ($x_1 > 0, x_2 < 0$) due to the first operator in the above Eq. is

\[
\langle \psi_1^\dagger(x_1, \tau_1) \psi_1^\dagger(x_2, \tau_2) \rangle = \int d\tau \langle e^{-\Phi_{sf}(x_1, \tau_1)} e^{\Phi_{sf}(x_2, \tau_2)} \rangle \\
\times \langle e^{-\frac{\Phi_{sf}(x_1, \tau_1)}{2}} e^{\frac{\Phi_{sf}(x_2, \tau_2) + \pi}{2}} \rangle \langle e^{-\frac{\Phi_{sf}(x_1, \tau_1)}{2}} e^{\frac{\Phi_{sf}(x_2, \tau_2) + \pi}{2}} \rangle \\
\times \langle e^{-\frac{\Phi_{sf}(x_1, \tau_1)}{2}} (\cos 2\Phi_{sf}(0, \tau) : - \frac{1}{2} : (\partial \Phi_{sf}(0, \tau))^2 : e^{\frac{\Phi_{sf}(x_2, \tau_2) + \pi}{2}} \rangle \\
\sim (z_1 - \bar{z}_2)^{-2}
\]  

(33)

Where $z_1 = \tau_1 + i x_1$ is in the upper half plane, $\bar{z}_2 = \tau_2 + i x_2$ is in the lower half plane.

By using the following OPE:

\[
: e^{-\frac{\Phi_{sf}(z_1)}{2}} : e^{\frac{\Phi_{sf}(z_2)}{2}} := (z_1 - z_2)^{-1/4} - \frac{i}{2} (z_1 - z_2)^{3/4} : \Phi_{sf}(z_2) : \\
- \frac{i}{4} (z_1 - z_2)^{7/4} : \Phi_{sf}(z_2) : - \frac{1}{8} (z_1 - z_2)^{7/4} : (\partial \Phi_{sf}(z_2))^2 : + \cdots
\]  

(34)

It is easy to see that the primary field $: \cos 2\Phi_{sf}(0, \tau) :$ makes no contributions to the three point function. It was shown by the detailed calculations in Ref. 27 that only the part of the self-energy which is both imaginary and even function of $\omega$ contributes to the conductivity. Although the energy momentum tensor $: (\partial \Phi_{sf}(0, \tau))^2 :$ do make $\sim \omega$ contribution to the self-energy in the first order, because it is an odd function, it does not contribute to the electron conductivity in this order. Same arguments apply to the other operators in Eq. 32. Second order perturbations in these operators lead to the generic $T^2$ fermi liquid behaviour of the electron conductivity.

The results of this section were applied to a two level tunneling system with slight modifications in Ref. 28. The universal scaling functions in the presence of external magnetic field which breaks the channel symmetry were also discussed there.
III. COMPACTIFIED ONE CHANNEL KONDO MODEL

Assuming Particle-Hole symmetry, the non-interacting one channel Kondo model has two commuting \( SU(2) \) symmetry, one is the usual spin symmetry with the generators \( J^a(a = x, y, z) \) another is the isospin symmetry with the generators \( I^a(a = x, y, z) \).

\[
J_x = \frac{1}{2}(\psi^\dagger \psi_\uparrow + \psi_\downarrow), \quad J_y = \frac{1}{2i}(\psi^\dagger \psi_\downarrow - \psi_\uparrow), \quad J_z = \frac{1}{2}(\psi^\dagger \psi_\uparrow - \psi_\downarrow)
\]

\[
I_x = \frac{1}{2}(\psi^\dagger \psi_\uparrow + \psi_\downarrow), \quad I_y = \frac{1}{2i}(\psi^\dagger \psi_\downarrow - \psi_\uparrow), \quad I_z = \frac{1}{2}(\psi^\dagger \psi_\uparrow + \psi_\downarrow)
\] (35)

The diagonal and off-diagonal components of the isospin currents represent respectively the charge and pairing density at the site \( x \).

The one channel compactified model proposed by Ref.7 is a model where the impurity spin couples to both the spin and the isospin currents of the one channel conduction electrons

\[
H_c = iv_F \int_{-\infty}^{\infty} dx \psi^\dagger_\alpha(x) \frac{d\psi_\alpha(x)}{dx} + \sum_{a=x,y,z} \lambda^a (I^a(0) + J^a(0)) S^a + \sum_{a=x,y,z} \alpha^a(I^a(0) - J^a(0)) S^a
\]

\[
+ h(\int dx (I^z(x) + J^z(x)) + S^z) \] (36)

The ordinary symmetric Anderson impurity model in a one dimensional lattice is

\[
H = it \sum_{n,\alpha} (\psi^\dagger_\alpha(n + 1)\psi_\alpha(n) - h.c.)
\]

\[
+ iV \sum_\alpha (\psi^\dagger_\alpha(0)d_\alpha - h.c.) + U(n_{d\uparrow} - \frac{1}{2})(n_{d\downarrow} - \frac{1}{2}) \] (37)

The \( O(4) \) symmetry of the AIM can be clearly displayed in terms of the Majorana fermions

\[
\psi_\uparrow(n) = \frac{1}{\sqrt{2}}(\chi_1(n) - i\chi_2(n)), \quad d_\uparrow = \frac{1}{\sqrt{2}}(d_1 - id_2)
\]

\[
\psi_\downarrow(n) = \frac{1}{\sqrt{2}}(-\chi_3(n) - i\chi_0(n)), \quad d_\downarrow = \frac{1}{\sqrt{2}}(-d_3 - id_0) \] (38)

Breaking the symmetry from \( O(4) \) to \( O(3) \times O(1) \) in the hybridization, the Hamiltonian becomes:

\[
H = it \sum_n \sum_{\alpha=0}^{3} \chi_\alpha(n + 1)\chi_\alpha(n) + iV_0\chi_0(0)d_0
\]

\[
+ iV \sum_{\alpha=1}^{3} \chi_\alpha(0)d_\alpha + Ud_\uparrow d_\downarrow d_3 d_0 \] (39)

In the large \( U \) limit, projecting out the excited impurity states, we can map the Hamiltonian to the 1CCK Hamiltonian with

\[
\lambda = \frac{2V^2}{U}, \quad \alpha = -\frac{2V_0V}{U} \] (40)
If \( V_0 = V \), Eq.33 comes back to the original \( O(4) \) symmetric AIM. In the strong coupling limit, it becomes the one channel Kondo model where the impurity only couples to the spin currents (or isospin currents) of the conduction electrons. If \( V_0 = 0 \), then \( \alpha = 0 \), Eq.39 becomes the isotropic 1CCK where the impurity couples to the spin and isospin currents with equal strength. If we define the P-H transformation \( \psi_\uparrow \rightarrow \psi_\uparrow, \psi_\downarrow \rightarrow \psi_\downarrow^\dagger \), then spin and isospin currents transform to each other \( I^a \rightarrow J^a, J^a \rightarrow I^a \).

The Hamiltonian 36 has the P-H symmetry if \( \alpha = 0 \). In the following, parallel to the discussions on the 2CFAK, we take \( \lambda_x = \lambda_y = \lambda, \lambda_z \neq \lambda \); \( \alpha_x = \alpha_y = \alpha, \alpha_z \neq \alpha \). We bosonize the spin \( \uparrow \) and spin \( \downarrow \) electrons separately

\[
\psi_\alpha(x) = \frac{P_\alpha}{\sqrt{2\pi a}} e^{-i\Phi_\alpha(x)}
\]

(41)

The cocyle factors have been chosen as \( P_\uparrow = P_\downarrow = e^{i\pi N_c} \).

The bosonized form of the spin and isospin currents in Eq.35 are

\[
J_x = \frac{1}{2\pi a} \cos \sqrt{2}\Phi_s, \quad J_y = \frac{1}{2\pi a} \sin \sqrt{2}\Phi_s, \quad J_z = -\frac{1}{4\pi} \frac{\partial}{\partial x} \sqrt{2}\Phi_s
\]

\[
I_x = \frac{1}{2\pi a} \cos \sqrt{2}\Phi_c, \quad I_y = \frac{1}{2\pi a} \sin \sqrt{2}\Phi_c, \quad I_z = -\frac{1}{4\pi} \frac{\partial}{\partial x} \sqrt{2}\Phi_c
\]

(42)

where \( \Phi_c, \Phi_s \) are charge and spin bosons:

\[
\Phi_c = \frac{1}{\sqrt{2}}(\Phi_\uparrow + \Phi_\downarrow), \quad \Phi_s = \frac{1}{\sqrt{2}}(\Phi_\uparrow - \Phi_\downarrow)
\]

(43)

The sum \( J^a_s(x) = I^a(x) + J^a(x) \) and the difference \( J^a_d(x) = I^a(x) - J^a(x) \) can be expressed in terms of the chiral bosons

\[
J^s_x = \frac{1}{\pi a} \cos \Phi_\uparrow \cos \Phi_\downarrow, \quad J^s_y = \frac{1}{\pi a} \sin \Phi_\uparrow \cos \Phi_\downarrow, \quad J^s_z = -\frac{1}{2\pi} \frac{\partial \Phi_\uparrow}{\partial x}
\]

\[
J^d_x = -\frac{1}{\pi a} \sin \Phi_\uparrow \sin \Phi_\downarrow, \quad J^d_y = \frac{1}{\pi a} \cos \Phi_\uparrow \sin \Phi_\downarrow, \quad J^d_z = -\frac{1}{2\pi} \frac{\partial \Phi_\downarrow}{\partial x}
\]

(44)

Compare Eq.36 with Eq.44, we immediately realize that the mapping between the 2CFAK and the 1CCK is \( \Phi_s \rightarrow \Phi_\uparrow, \Phi_{sf} \rightarrow \Phi_\downarrow \), therefore \( \psi_s \rightarrow \psi_\uparrow, \psi_{sf} \rightarrow \psi_\downarrow \). The following fixed point structure can be immediately borrowed from the corresponding discussions on the 2CFAK.

**A. Fixed point 1**

The boundary conditions are

\[
\psi_{\uparrow,L} = -\psi_{\uparrow,R}, \quad \psi_{\downarrow,L} = \psi_{\downarrow,R}^\dagger
\]

(45)

It is easy to see that the above boundary conditions respect the P-H symmetry, they can be expressed in terms of bosons

\[
\Phi_{\uparrow,L} = \Phi_{\uparrow,R} + \pi, \quad \Phi_{\downarrow,L} = -\Phi_{\downarrow,R} + \pi
\]

(46)
Spin $\uparrow$ electrons suffer a $\frac{\pi}{2}$ phase shift, however, spin $\downarrow$ electrons are scattered into holes and vice-versa. The one particle S-matrix are $S_\uparrow = -1, S_\downarrow = 0$. The residual conductivity of the spin $\uparrow$ electron takes unitary limit, but that of the spin $\downarrow$ is half of the unitary limit. The isotropic 1CCK has the same thermodynamic behaviors as the 2CK, but its fixed point has the local KM symmetry $\hat{O}_1(3) \times \hat{O}_1(1)$. The finite size spectrum of this NFL fixed point is listed in Table III. Comparing this finite size spectrum with that of the NFL fixed point of the 2CK listed in Ref. 17, it is easy to see that it has the same energy levels as those of the 2CK, but the corresponding degeneracy is much smaller. This is within the expectation, because the central charge $c = 2$ and the fixed point symmetry of the isotropic 1CCK is smaller than that of the 2CK.

This fixed point is stable only when $\alpha = \alpha_z = 0$ where the Hamiltonian $36$ has P-H symmetry.

Away from the fixed point, there is only one dimension 3/2 operator

$$\hat{a}\hat{b}\partial\Phi_\uparrow(0) \sim \cos\Phi_\downarrow(0)\partial\Phi_\uparrow(0)$$ (47)

The first order correction to the single particle L-R Green function ($x_1 > 0, x_2 < 0$) due to this operator is

$$\int d\tau \langle e^{-i\Phi_\uparrow(x_1,\tau)}\partial\Phi_\uparrow(0,\tau)e^{i(\Phi_\uparrow(x_2,\tau) + \pi)} \rangle \langle \cos\Phi_\downarrow(0,\tau) \rangle = 0$$

$$\int d\tau \langle e^{-i\Phi_\uparrow(x_1,\tau)}\cos\Phi_\downarrow(0,\tau)e^{-i\Phi_\downarrow(x_2,\tau)} \rangle \langle \partial\Phi_\uparrow(0,\tau) \rangle = 0$$ (48)

By Wick theorem, it is easy to see that any odd order corrections vanish.

Second order correction goes as $\sim \omega$ which is a odd function, therefore does not contribute to the electron conductivity. The fourth order makes $T^2$ contributions.

There are two dimension 2 operators:

$$a_\uparrow(0,\tau)\frac{\partial a_\uparrow(0,\tau)}{\partial\tau} + b_\uparrow(0,\tau)\frac{\partial b_\uparrow(0,\tau)}{\partial\tau} = (\partial\Phi_\uparrow(0,\tau))^2$$

$$\hat{a}(\tau)\frac{\partial\hat{a}(\tau)}{\partial\tau} = \cos 2\Phi_\downarrow - \frac{1}{2}(\partial\Phi_\downarrow(0))^2$$ (49)

The first order correction to the spin $\uparrow$ electron L-R Green function due to the first operator in Eq.49 is

$$\int d\tau \langle e^{-i\Phi_\uparrow(x_1,\tau)} : (\partial\Phi_\uparrow(0,\tau))^2 : e^{i(\Phi_\uparrow(x_2,\tau) + \pi)} \rangle \sim (z_1 - \bar{z}_2)^{-2}$$ (50)

As pointed out in the last section, the energy momentum tensor : $(\partial\Phi_\uparrow(0,\tau))^2 :$ makes $\sim \omega$ contribution to the self-energy in the first order, therefore does not contribute to the electron conductivity. Second order perturbation in this operator leads to $T^2$ contributions.

Adding the contributions from all the leading irrelevant operators, we get

$$\sigma_\uparrow(T) \sim \sigma_u(1 + T^2 + T^4 + \cdots)$$ (51)

The first order correction to the spin $\downarrow$ electron L-R Green function due to the 2nd operator in Eq. 49 is
\[
\int d\tau \langle e^{-i\Phi_\downarrow(x_1,\tau_1)} \cos 2\Phi_\downarrow(0,\tau)e^{-i\Phi_\downarrow(x_2,\tau_2)} \rangle \\
-\frac{1}{2} \int d\tau \langle e^{-i\Phi_\downarrow(x_1,\tau_1)}(\partial\Phi_\downarrow(0,\tau))^2e^{-i\Phi_\downarrow(x_2,\tau_2)} \rangle
\]

By using the following OPE:

\[
e^{-i\Phi_\downarrow(z_1)} : e^{-i\Phi_\downarrow(z_2)} := (z_1 - z_2) : e^{-i2\Phi_\downarrow(z_2)} : -i(z_1 - z_2)^2 : e^{-i2\Phi_\downarrow(z_2)}\partial\Phi_\downarrow(z_2) : + \cdots
\]

It is easy to see that the second integral vanishes, but the first becomes

\[
\frac{1}{(z_1 - \bar{z}_2)^{1}} \int d\tau \frac{1}{(z_1 - \tau)^2(\tau - \bar{z}_2)^2} \sim (z_1 - \bar{z}_2)^{-2}
\]

Putting \(\Delta = 1\) in Eq. (3.52) of Ref. 27, we find the imaginary and real parts of self-energy go as \(\text{Im} \Sigma(\omega, T = 0) = 0, \text{Re} \Sigma(\omega, T = 0) \sim \omega\), therefore the first order perturbation does not contribute to the spin \(\downarrow\) electron conductivity. Second order perturbation yields a \(T^2\) contributions.

Adding the contributions from all the leading irrelevant operators, we get

\[
\sigma_\downarrow(T) \sim 2\sigma_u(1 + T^2 + T^4 + \cdots)
\]

The total conductivity is the summation of the two spin components

\[
\sigma(T) = \sigma_\uparrow(T) + \sigma_\downarrow(T) \sim 3\sigma_u(1 + T^2 + T^4 + \cdots)
\]

The boundary OPE of the spin and density of the conduction electrons are

\[
\psi_\uparrow^\dagger(z_1)\psi_\uparrow(\bar{z}_2) = (z_1 - \bar{z}_2)^{-1} + i\partial\Phi_\uparrow + \cdots
\]

\[
\psi_\downarrow^\dagger(z_1)\psi_\downarrow(\bar{z}_2) = 0 + \cdots
\]

\[
\psi_\uparrow^\dagger(z_1)\psi_\downarrow(\bar{z}_2) = e^{i\sqrt{2}\Phi_u(0)} + \cdots
\]

\[
\psi_\downarrow^\dagger(z_1)\psi_\uparrow(\bar{z}_2) = e^{-i\sqrt{2}\Phi_u(0)} + \cdots
\]

The boundary OPE of the spin singlet and triplet pairing operators are

\[
\psi_\uparrow(z_1)\psi_\downarrow(\bar{z}_2) = 0 + \cdots
\]

\[
\psi_\downarrow(z_1)\psi_\downarrow(\bar{z}_2) = (z_1 - \bar{z}_2)^{-1} - i\partial\Phi_\downarrow + \cdots
\]

\[
\psi_\uparrow(z_1)\psi_\uparrow(\bar{z}_2) = e^{-i\sqrt{2}\Phi_u(0)} + \cdots
\]

\[
\psi_\downarrow(z_1)\psi_\uparrow(\bar{z}_2) = -e^{-i\sqrt{2}\Phi_u(0)} + \cdots
\]

The P-H symmetry interchanges the pairing and spin operators in the \(\uparrow\downarrow\) and \(\downarrow\uparrow\) channels. From Eq. 58, we can identify the pairing operators

\[
\mathcal{O}_s = e^{-i\sqrt{2}\Phi_u(0)}, \quad \mathcal{O}_c = e^{-i\sqrt{2}\Phi_u(0)}, \quad \mathcal{O}_\downarrow = \partial\Phi_\downarrow(0)
\]

The paring operators in all the channels except in the \(\uparrow\uparrow\) channel have scaling dimension 1, therefore their correlation functions decay as \(\tau^{-2}\). Comparing these pairing operators with those at the FL fixed point ( Eq. 71 ) to be discussed in the following, we find the
pairing susceptibility in $\downarrow\downarrow$ channel is enhanced. However, in contrast to the 2CK fixed point $^4_{17}$, the enhancement is so weak that there is no pairing susceptibility divergence at the impurity site in any spin channel. This result is somewhat surprising. Naively, we expect pairing susceptibility divergence because the impurity interacts with the pairing density of the conduction electrons at the impurity site. However, the above explicit calculations showed that this is not true if there is only one channel of conduction electrons. Naively, we do not expect pairing susceptibility divergence in the 2CK, because the impurity spin interacts only with the total spin currents of channel 1 and 2, no isospin currents of channel 1 and 2 are involved in the interaction. However, the explicit calculation of the 2CK showed that the pairing operator in the spin and flavor singlet channel has dimension 1/2 (however, the pairing operators in flavor singlet and spin triplet channel has dimension 3/2), therefore the spin and flavor singlet pairing susceptibility at the impurity site is divergent $^4_{17}$. This indicates that we can achieve the pairing susceptibility divergence without a pairing source term. We conclude that more than one channel of conduction electrons are needed to achieve the pairing susceptibility divergence.

**B. Fixed pointed 2**

The boundary conditions are

$$\psi_{\uparrow,L} = -\psi_{\uparrow,R}, \quad \psi_{\downarrow,L} = -\psi_{\downarrow,R}$$

(60)

The above boundary conditions can be expressed in terms of bosons

$$\Phi_{\uparrow,L} = \Phi_{\uparrow,R} + \pi, \quad \Phi_{\downarrow,L} = -\Phi_{\downarrow,R}$$

(61)

This fixed point is stable only when $\lambda = \alpha_z = 0$. If we define the P-H transformation $\psi_{\uparrow} \to \psi_{\uparrow}, \psi_{\downarrow} \to -\psi_{\downarrow}^\dagger$, then the spin and isospin currents transform as $I^x \to -J^z, I^y \to -J^y, I^z \to J^z, J^x \to -I^x, J^y \to -I^y, J^z \to I^z$. The Hamiltonian $^3_{36}$ has this P-H symmetry if $\lambda = \alpha_z = 0$.

This is the same fixed point as fixed point 1.

**C. Fixed pointed 3**

The boundary conditions are

$$\psi_{\uparrow,L} = -\psi_{\uparrow,R}, \quad \psi_{\downarrow,L} = -\psi_{\downarrow,R}$$

(62)

The above boundary conditions can be expressed in terms of bosons

$$\Phi_{\uparrow,L} = \Phi_{\uparrow,R} + \pi, \quad \Phi_{\downarrow,L} = \Phi_{\downarrow,R} + \pi$$

(63)

Both spin $\uparrow$ and $\downarrow$ electrons suffer $\frac{\pi}{2}$ phase shift. The physical picture is that the impurity spin is either totally screened by the spin current or the isospin current of conduction electrons depending on which coupling is stronger. This is a FL fixed point with $O(4)$ symmetry. The finite size spectrum is listed in Table $^3_{37}$.
The bosonized forms of the 4 leading irrelevant operators are

\[ \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} = \cos 2\Phi \downarrow - \frac{1}{2}(\partial \Phi_{\downarrow}(0))^2 \]

\[ \hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau} = -\cos 2\Phi \downarrow - \frac{1}{2}(\partial \Phi_{\downarrow}(0))^2 \]

\[ \hat{a} \hat{b} \tau(0) b\tau(0) = \Phi_{\downarrow}(0, \tau) \Phi_{\tau}(0, \tau) \]

\[ a\tau(0, \tau) \frac{\partial a\tau(0, \tau)}{\partial \tau} + b\tau(0, \tau) \frac{\partial b\tau(0, \tau)}{\partial \tau} = (\partial \Phi_{\tau}(0, \tau))^2 \]  

(64)

The first order correction to the spin \( \uparrow \) electron L-R Green function due to the 4th operator in Eq. 64 is also given by Eq. 50. The correction due to the 3rd operator in Eq. 64 can be similarly evaluated. We get the low temperature expansion of the spin \( \uparrow \) electron conductivity

\[ \sigma_{\uparrow}(T) \sim \sigma_u(1 + T^2 + T^4 + \cdots) \]  

(65)

The first order correction to the spin \( \downarrow \) electron L-R Green function due to the first operator in Eq. 64 is

\[ \int d\tau \langle e^{-i\Phi_{\downarrow}(z, \tau)} \cos 2\Phi_{\downarrow}(0, \tau) e^{i(\Phi_{\uparrow}(0, \tau) + \pi)} \rangle \]

\[ -\frac{1}{2} \int d\tau \langle e^{-i\Phi_{\downarrow}(z, \tau)} (\partial \Phi_{\downarrow}(0, \tau))^2 e^{i(\Phi_{\uparrow}(0, \tau) + \pi)} \rangle \]  

(66)

By using the following OPE:

\[ : e^{-i\Phi_{\downarrow}(z_1)} : \cdot : e^{i\Phi_{\downarrow}(z_2)} := (z_1 - z_2)^{-1} - i : \partial \Phi_{\downarrow}(z_2) : \]

\[ + \frac{z_1 - z_2}{2} : \partial^2 \Phi_{\downarrow}(z_2) : - \frac{z_1 - z_2}{2} : (\partial \Phi_{\downarrow}(z_2))^2 : + \cdots \]  

(67)

It is easy to see that the first integral vanishes and the second are the same as Eq. 50. The corrections due to the 2nd and the 3rd operators in Eq. 64 can be similarly evaluated, the low temperature expansion of the spin \( \downarrow \) electron conductivity follows

\[ \sigma_{\downarrow}(T) \sim \sigma_u(1 + T^2 + T^4 + \cdots) \]  

(68)

Note that only at the FL fixed point, the spin \( SU(2) \) symmetry is restored, therefore the expansion coefficients in Eqs. 65, 68 are different.

The total conductivity is the summation of the two spin components

\[ \sigma(T) = \sigma_{\uparrow}(T) + \sigma_{\downarrow}(T) \sim 2\sigma_u(1 + T^2 + T^4 + \cdots) \]  

(69)

The boundary OPE of the spin and density of the conduction electrons are

\[ \psi_{\uparrow}^\dagger(z_1) \psi_{\uparrow}(\bar{z}_2) = (z_1 - \bar{z}_2)^{-1} + i\partial \Phi_{\uparrow} + \cdots \]

\[ \psi_{\downarrow}^\dagger(z_1) \psi_{\downarrow}(\bar{z}_2) = (z_1 - \bar{z}_2)^{-1} + i\partial \Phi_{\downarrow} + \cdots \]

\[ \psi_{\uparrow}^\dagger(z_1) \psi_{\downarrow}(\bar{z}_2) = e^{i\sqrt{2}\Phi_{\downarrow}(0)} + \cdots \]

\[ \psi_{\downarrow}^\dagger(z_1) \psi_{\uparrow}(\bar{z}_2) = e^{-i\sqrt{2}\Phi_{\downarrow}(0)} + \cdots \]  

(70)
The boundary OPE of the spin singlet and triplet pairing operators are

\[
\psi_\uparrow(z_1)\psi_\uparrow(\bar{z}_2) = 0 + \cdots \\
\psi_\downarrow(z_1)\psi_\downarrow(\bar{z}_2) = 0 + \cdots \\
\psi_\uparrow(z_1)\psi_\downarrow(\bar{z}_2) = -e^{-i\sqrt{2}\Phi_c(0)} + \cdots \\
\psi_\downarrow(z_1)\psi_\downarrow(\bar{z}_2) = e^{-i\sqrt{2}\Phi_c(0)} + \cdots
\]  

(71)

The above equations should be compared with the corresponding Eqs\(^57\) and \(^58\) at the NFL fixed point.

**IV. CONCLUSIONS**

By the detailed discussions on the low temperature properties of the two related, but different single impurity models, we clarify the confusing conjectures and claims made on these two models. In evaluating the single particle Green functions and pairing susceptibilities, all the degree of freedoms have to be taken into account, even though some of them decouple from the interactions with the impurity. We explicitly demonstrate that different quantum impurity models are simply free chiral bosons with different boundary conditions. In Ref.\(^24\), the author studied another single impurity model where the impurity couples to both the spin and the flavor currents of the two channel electrons (2CSFK). In Ref.\(^28\), the author solved a two level tunneling model which can also mapped to a single impurity model. As shown in Ref.\(^28\), finite number of impurity models can always mapped to a single impurity model. From the results of this paper and Refs.\(^24\),\(^28\) we conclude that in clean, finite number of impurity models (1) FL behaviors are extremely robust, any perturbation in the flavor sectors will destroy the NFL behaviors.(2) due to the phase space arguments given in this paper and in Refs.\(^24\),\(^33\),\(^28\) it is very unlikely to find the NFL linear \(T\) behaviour of the electron conductivity which was observed in the certain heavy fermion systems\(^3\) and in the normal state of high-\(T_c\) cuprate superconductors. There are three possible ways to explain this experimental observation (1) disorder effect\(^3\) (1) Kondo lattice model\(^3\) (3) near to some quantum phase transitions\(^3\), for example, near the phase transition between the metallic spin-glass and disordered metal\(^3\).

**ACKNOWLEDGMENTS**

We thank D. S. Fisher, B. Halperin, A. Millis, N. Read for helpful discussions. This research was supported by NSF Grants Nos. DMR 9630064, DMR9416910 and Johns Hopkins University.

**APPENDIX A: CFT ANALYSIS OF THE STABLE FIXED POINT OF THE 2CFAK**

We can also analysis the stable fixed point from CFT. Without losing generality, supposing at this fixed point, the impurity is totally absorbed by the channel 1 conduction electrons \(\vec{J}_1(x) = \vec{J}_1(x) + 2\pi\delta(x)\vec{S}\). Slightly away the fixed point, the channel 2 conduction electrons also couple to the impurity. It has been shown by the author there is only one
leading irrelevant operator $Q_0^0 = \vec{J}_1(0) \cdot \vec{J}_1(0)$ even in the $O(2)$ one channel Kondo model\textsuperscript{30}. The Hamiltonian $H = H_1 + H_2 + H_{12}$ is

\begin{align*}
H_1 &= \frac{1}{3} \int dx \vec{J}_1(x) \cdot \vec{J}_1(x) + h \int dx J_3^1(x) + \lambda_1 \vec{J}_1(0) \cdot \vec{J}_1(0) + \cdots \\
H_2 &= \frac{1}{3} \int dx \vec{J}_2(x) \cdot \vec{J}_2(x) + h \int dx J_3^2(x) \\
H_{12} &= \lambda_2^z J_z^1(0) S_z + \lambda_2^x (J_x^1(0) S_x + J_y^1(0) S_y) \tag{A1}
\end{align*}

At the fixed point $\vec{S} \sim \vec{J}_1(0) + \cdots$, it is easy to show that the following 4 leading irrelevant operators with dimension 2 will be generated\textsuperscript{39}

\begin{align*}
O_1 &= \frac{1}{3} (\vec{J}_1(0) \cdot \vec{J}_1(0) + \vec{J}_2(0) \cdot \vec{J}_2(0)) \\
O_2 &= \frac{1}{3} (\vec{J}_1(0) \cdot \vec{J}_1(0) - \vec{J}_2(0) \cdot \vec{J}_2(0)) \\
O_3 &= \frac{2}{3} \vec{J}_1(0) \cdot \vec{J}_2(0) \\
O_4 &= \frac{2}{3} \vec{J}_1^3(0) J_2^2(0) - \frac{1}{3} (\vec{J}_1^1(0) J_2^1(0) + \vec{J}_1^2(0) J_2^2(0)) \tag{A2}
\end{align*}

This four operators are consistent with those listed in Eq.\textsuperscript{32}. The first order perturbations in the leading irrelevant operators yield the typical fermi liquid behaviors: $C_{\text{imp}} \sim \lambda_1 \frac{\pi^2}{3} T, \chi_{\text{imp}} \sim \frac{1}{2} (\lambda_1 + \lambda_3 + \lambda_4)$. In contrast to one channel spin anisotropic Kondo model, $R = 2(1 + \frac{\lambda_3 + \lambda_4}{3 \lambda_1})$ is non-universal.

The second order perturbations yield

$$
\sigma_1(T) \sim \sigma_u (1 + T^2 + \cdots), \tag{A3}
$$

The residual conductivity of channel 2 comes from the potential scattering which is neglected in this appendix\textsuperscript{38}. The total conductivity is the summation of those from the two channels. Similar CFT analysis can be applied to 1CCK.

The CFT analysis in this appendix can only show that the fixed point examined here is stable, but cannot rule out the possible existence of the other fixed points. The Abelian Bosonization analysis in Sec. II identified all the possible fixed points and showed this fixed point is the only stable fixed point.
### TABLE I.
The finite size spectrum at the stable FL fixed point of the 2CFAK with the symmetry $O(4) \times O(4)$. See Ref. for explanations. It is the superposition of the finite size spectrum of free electrons and that of electrons with phase shift $\pi/2$.

| $O(4)$ | $O(4)$ | $\frac{l}{v_F \pi}(E - \frac{1}{4})$ | Degeneracy |
|--------|--------|----------------------------------|------------|
| R      | NS     | 0                                | 4          |
| NS     | R      | 0                                | 4          |
| R      | NS+1st | $\frac{1}{2}$                    | 16         |
| NS+1st | R      | $\frac{1}{2}$                    | 16         |
| R+1st  | NS     | 1                                | 16         |
| R      | NS+2nd | 1                                | 24         |
| NS     | R+1st  | 1                                | 16         |
| NS+2nd | R      | 1                                | 24         |

### TABLE II.
The finite size spectrum at the FL fixed point of the 1CCK with the symmetry $O(4)$.

| $O(4)$ | $\frac{l}{v_F \pi}(E - \frac{1}{4})$ | Degeneracy |
|--------|----------------------------------|------------|
| R      | 0                                | 4          |
| R+1st  | 1                                | 16         |
| R+2nd  | 2                                | 40         |

### TABLE III.
The finite size spectrum at the NFL fixed point of the 1CCK with the symmetry $O(1) \times O(3)$.

| $O(1)$ | $O(3)$ | $\frac{l}{v_F \pi}(E - \frac{1}{4})$ | Degeneracy |
|--------|--------|----------------------------------|------------|
| R      | NS     | 0                                | 2          |
| NS     | R      | $\frac{1}{5}$                    | 2          |
| R      | NS+1st | $\frac{3}{5}$                    | 6          |
| NS+1st | R      | $\frac{2}{5}$                    | 2          |
| R+1st  | NS     | 1                                | 2          |
| R      | NS+2nd | 1                                | 6          |
| NS     | R+1st  | $1 + \frac{1}{5}$                | 6          |
| NS+2nd | R      | $1 + \frac{1}{5} + \frac{1}{2}$  | 2          |
REFERENCES

1 Noziéres and Blandin, J. Phys. (Paris) 41, 193 (1980)
2 H. B. Pang and D. L. Cox, Phys. Rev. B44, 9454(1991)
3 I. Affleck, A. W. W. Ludwig, H. B. Pang and D. L. Cox, Phys. Rev. B45, 7918(1992)
4 M. Fabrizio and A. O. Gogolin, Ph. Noziéres, Phy. Rev. Lett. 74, 4503 (1995), Phys. Rev. B 51, 16088 (1995)
5 N. Andrei and A. Jerez, Phy. Rev. Lett. 74, 4507 (1995)
6 P. Coleman, L. B. Ioffe and A. M. Tsvelik, Phys. Rev. B52, 6611 (1995)
7 P. Coleman and A. J. Schofield, Phy. Rev. Lett. 75, 2184 (1995)
8 For a review, see A. W. W. Ludwig, Int. J. Mod. Phys. B 8, 347 (1994)
9 J. M. Maldacena and A. W. W. Ludwig, to be published in Nucl. Phys. B
10 Strictly speaking, it should be a semi-product.
11 V. J. Emery and S. Kivelson, Phys. Rev. B46, 10812(1992).
12 A. Georges and A. M. Sengupta, Phys. Rev. Lett. 74, 2808 (1995).
13 A. W. W. Ludwig, Int. J. Mod. Phys. B 8, 347 (1994)
14 The difference between \( \psi_{s,i} \) and \( \psi_s \) are neglected in the following
15 By \( O(1) \), we mean the Ising sector with \( c = 1/2 \).
16 Schofield ( Phy. Rev. B55, 5627 (1997)) also derived the Hamiltonian using Abelian Bosonization.
17 Jinwu Ye, Phys. Rev. Lett. 79, 1385(1997)
18 see \ref{eq:19} for the discussion of this boundary condition.
19 \( b_{sf}(0, \tau) \) in this equation is the non-interacting field, therefore has scaling dimension 1/2.
   However, that in Eq.15 is \( \frac{1}{2}(b_{sf,L}(0, \tau) + b_{sf,R}(0, \tau)) \), therefore has scaling dimension 3/2.
   For simplicity, we use the same notation.
20 J. Cardy’s lecture in fields, strings and critical phenomena.
21 If keeping \( \gamma_1 = \gamma_2 = 1 \), we get the weak coupling RG flow Eq.8.
22 The superscript \( H \) is omitted in the following without causing confusions.
23 In \( S^2 \) of equation 19, the \( SU(2) \) invarance at the 2CK fixed pint dictates the existence of the second (current) term with dimension 2, this term is subleading at the 2CK fixed point, but is crucial in the FL fixed point ( Fixed point 3) to be discussed in the following.
24 Jinwu Ye, Phys. Rev. B 56, R489 (1997).
25 The 3rd operator in the following equation will also generate dimension 2 operator :
   \( (\partial \Phi_{sf}(0, \tau))^2 \), but this operator is just the linear combination of the first two.
26 When transforming to finite temperature and considering only connected correlation function, we could treat the energy momentum tensor as if it is primary operator, even it is only quasi-primary.
27 I. Affleck and A. W. W. Ludwig, Phys. Rev. B48, 7297 (1993)
28 Jinwu Ye, Phys. Rev. B 56, 1316 (1997).
29 I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. 68, 1046 (1992) ;I. Affleck, A. W. W. Ludwig and B. A. Jones, Phys. Rev. B 52, 9528 (1995).
30 Jinwu Ye, Phys. Rev. Lett. 77, 3224 (1996).
31 Under the P-H transformation \( \psi_\uparrow \rightarrow \psi_\uparrow, \psi_\downarrow \rightarrow \psi_\downarrow \), the spin currents and isospin currents transform to each other.
32 N. B. Bhatt and D. S. Fisher, Phys. Rev. Lett. 68, 3072 (1992). They discussed the metallic
phase of disordered metal and showed that the low temperature thermodynamic quantities show NFL behaviours, but the charge transport still shows FL behaviours. This is due to residual local moments which are essentially decoupled from the conduction electron fluids at any low temperature.

33 C.L. Seaman et al., Phy. Rev. Lett. 67, 2882 (1991).
34 M. Jarrell et al., Phy. Rev. Lett. 77, 1612 (1996).
35 A. J. Millis, Phys. Rev. B 48, 7183 (1993);
36 B. Andraka and A. M. Tsvelik, Phy. Rev. Lett. 67, 2886 (1991).
37 S. Sachdev, N. Read and R. Oppermann, Phys. Rev. B 52, 10286 (1995); S. Sachdev and N. Read, J. Phys. 9723 (1996).
38 At low enough temperature, the single impurity model breaks down anyway due to the interactions between the impurities.
39 Operator $O_5 = J_1^1(0)J_2^2(0) - J_1^2(0)J_2^1(0)$ violates $Z_2$ symmetry.