NON-VANISHING OF VECTOR-VALUED POINCARÉ SERIES

SONJA ŽUNAR

ABSTRACT. We prove a vector-valued version of Muić’s integral non-vanishing criterion for Poincaré series on the upper half-plane $\mathcal{H}$. Moreover, we give an accompanying result on the construction of vector-valued modular forms in the form of Poincaré series. As an application of these results, we construct and study the non-vanishing of the classical and elliptic vector-valued Poincaré series.

1. Introduction

Vector-valued modular forms (VVMFs) have prominent applications in analytic number theory [4, 5, 9, 34] and the theory of vertex operator algebras [8, 23, 25, 35]. Although their usefulness was noticed already in the 1960s by A. Selberg [34], the theory of VVMFs was established in a systematic way only in the early 2000s by M. Knopp and G. Mason [14–16]. Ever since, the theory has been steadily developing [1, 3, 12, 19–22], with many new results in the recent years [2, 6, 10, 11, 33].

Let us fix a multiplier system $v : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ of weight $k \in \mathbb{R}$. We recall that a VVMF of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$ with respect to a representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_p(\mathbb{C})$ is a $p$-tuple $F = (F_1, \ldots, F_p)$ of holomorphic functions on $\mathcal{H} := \mathbb{C} \setminus \{z \mid \Im(z) > 0\}$ that have suitable Fourier expansions, such that

$$F|_k \gamma = \rho(\gamma) F, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

where elements of $\mathbb{C}^p$ are regarded as column-vectors, and $|_k$ denotes the standard right action (depending on $k$ and $v$) of $\mathrm{SL}_2(\mathbb{Z})$ on the set $(\mathbb{C}^p)^\mathcal{H}$ of functions $\mathcal{H} \to \mathbb{C}^p$ (see (2-4)).

Similarly as in the theory of classical modular forms (see, e.g., [24, §2.6]), one of the simplest ways to construct a VVMF is to define it as the sum of a Poincaré series

$$P_{\Lambda \setminus \mathrm{SL}_2(\mathbb{Z}), \rho} f := \sum_{\gamma \in \Lambda \setminus \mathrm{SL}_2(\mathbb{Z})} \rho(\gamma)^{-1} f|_k \gamma,$$

where $\Lambda$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $f$ is a suitable function $\mathcal{H} \to \mathbb{C}^p$ (cf. [16, §3]). In the case when the constructed VVMF is cuspidal, the question whether it vanishes identically is non-trivial. In fact, it has no complete answer even in the scalar-valued case, although in...
that case it was recognized as interesting as early as H. Poincaré [31, p. 249]. In the scalar-valued case, most known approaches to addressing this question aim at individual families of Poincaré series and are based on estimates of their Fourier coefficients [17,18,32]. A different, more general approach was discovered by G. Muić in 2009, when he proved an integral non-vanishing criterion for Poincaré series on unimodular locally compact Hausdorff groups [26, Theorem 4.1], with applications in the theory of automorphic forms and automorphic representations (see, e.g., [13]). As a corollary, he obtained a criterion for the non-vanishing of Poincaré series of integral weight on $H$ [28, Lemma 3.1], applied it to several families of cusp forms [27–29], and we extended his results to the half-integral weight case [36–38].

In this paper, we prove a vector-valued version (Theorem 5.2) of Muić’s integral non-vanishing criterion for Poincaré series on $H$ and use it to study the non-vanishing of two families of cuspidal VVMFs, which we call, respectively, the classical and elliptic vector-valued Poincaré series, in analogy with their scalar-valued versions studied by H. Petersson [30]. We also prove an accompanying result (Proposition 4.1) on the construction of VVMFs in the form of vector-valued Poincaré series (VVPSs). Let us emphasize that our results apply only to the case when the representation $\rho$ is unitary. Namely, the unitarity of $\rho$ is indispensable in the computations, involving integrals and Poincaré series, that are at the heart of our proofs (see, e.g., the third equality in (5-3)). On the other hand, a careful reader will notice that in the special case when $p = 1$ and $\rho$ is the trivial representation, we obtain results on scalar-valued Poincaré series on $H$ of arbitrary real weight, while in the previous work on integral non-vanishing criteria only the integral and half-integral weights were considered.

The paper is organized as follows. After introducing some basic notation in Section 2, in Section 3 we introduce vector spaces of VVMFs to be studied in this paper. In this, we essentially follow [16], the only difference being that whereas in [16] only VVMFs for $\text{SL}_2(\mathbb{Z})$ are considered, we work with VVMFs for a general subgroup $\Gamma$ of finite index in $\text{SL}_2(\mathbb{Z})$. As is well known, this slight generalization does not enlarge the class of considered VVMFs in a substantial way (see Lemma 3.1), but it greatly simplifies the notation in subsequent sections.

In Section 4, we prove a result on the construction of VVMFs in the form of VVPSs (Proposition 4.1). In Section 5, we prove our integral non-vanishing criterion for VVPSs (Theorem 5.2). We end the paper by Sections 6 and 7, in which we apply our results to the classical and elliptic VVPSs, respectively.

I would like to thank Marcela Hanzer and Goran Muić for their support and useful comments. The work on this paper was in part conducted while I was a visitor at the Faculty of Mathematics, University of Vienna. I would like to thank Harald Grobner and the University of Vienna for their hospitality.

2. Basic notation

Throughout the paper, let $i := \sqrt{-1} \in \mathbb{C}$ and

$$z^k := |z|^k e^{ik \arg(z)}, \quad z \in \mathbb{C}^\times, \quad \arg(z) \in (-\pi, \pi], \quad k \in \mathbb{R}.$$
Let \( \mathcal{H} := \mathbb{C}_{\Im(z) > 0} \). The group \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathcal{H} \cup \mathbb{R} \cup \{\infty\} \) by linear fractional transformations:

\[
g \cdot \tau := \frac{a\tau + b}{c\tau + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad \tau \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}.
\]

Defining \( j : \text{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C} \),

\[
j(g, \tau) := c\tau + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad \tau \in \mathcal{H},
\]
we recall that

\[
\Im(g \cdot \tau) = \frac{\Im(\tau)}{|j(g, \tau)|^2}, \quad g \in \text{SL}_2(\mathbb{R}), \quad \tau \in \mathcal{H}.
\]

Throughout the paper, we fix \( p \in \mathbb{Z}_{>0}, k \in \mathbb{R} \) and a unitary multiplier system \( v \) for \( \text{SL}_2(\mathbb{Z}) \) of weight \( k \), i.e., a function \( v : \text{SL}_2(\mathbb{Z}) \to \mathbb{C} | z | = 1 \) such that the function \( \mu : \text{SL}_2(\mathbb{Z}) \times \mathcal{H} \to \mathbb{C}, \)

\[
\mu(\gamma, \tau) := v(\gamma) j(\gamma, \tau)^k,
\]
is an automorphic factor, in the sense that

\[
\mu(\gamma_1 \gamma_2, \tau) = \mu(\gamma_1, \gamma_2, \tau) \mu(\gamma_2, \tau), \quad \gamma_1, \gamma_2 \in \text{SL}_2(\mathbb{Z}), \quad \tau \in \mathcal{H}.
\]

Following [16], we impose on \( v \) the nontriviality condition

\[
v(-I_2) = (-1)^{-k}
\]

and, writing \( T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), define \( \kappa \in [0, 1[ \) by the condition

\[
v(T) = e^{2\pi i \kappa}.
\]

The group \( \text{SL}_2(\mathbb{Z}) \) acts on the right on the space \( (\mathbb{C}^p)^\mathcal{H} \) of functions \( \mathcal{H} \to \mathbb{C}^p \) as follows:

\[
(F | k)_\gamma(\tau) := v(\gamma)^{-1} j(\gamma, \tau)^{-k} F(\gamma, \tau), \quad F \in (\mathbb{C}^p)^\mathcal{H}, \quad \gamma \in \text{SL}_2(\mathbb{Z}), \quad \tau \in \mathcal{H}.
\]

We note that due to the nontriviality condition (2-3), we have

\[
(F | k)(-I_2) = F, \quad F \in (\mathbb{C}^p)^\mathcal{H}.
\]

Next, we recall that the group

\[
K := \text{SO}_2(\mathbb{R}) = \left\{ \kappa_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}
\]
is a maximal compact subgroup of \( \text{SL}_2(\mathbb{R}) \) and the stabilizer of \( i \) under the action (2-1). Let us denote

\[
n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a_y := \begin{pmatrix} y^2 & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}, \quad h_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}
\]
for \( x \in \mathbb{R}, y \in \mathbb{R}_{>0} \) and \( t \in \mathbb{R}_{>0} \). By the Iwasawa (resp., Cartan) decomposition of \( \text{SL}_2(\mathbb{R}) \), every \( g \in \text{SL}_2(\mathbb{R}) \) can be written in the form

\[
g = n_x a_y \kappa_\theta = \kappa_\theta h_t \kappa_{\theta_2}
\]
for some $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\geq 0}$ and $\theta, \theta_1, \theta_2 \in \mathbb{R}$, and then we have $g.i = x + iy$. Denoting by $v$ the standard $\text{SL}_2(\mathbb{R})$-invariant Radon measure on $\mathcal{H}$ given by $d\nu(x+iy) := \frac{dx\,dy}{y^2}$, we have the following Haar measure on $\text{SL}_2(\mathbb{R})$:

\begin{equation}
(2-6) \quad \int_{\text{SL}_2(\mathbb{R})} \varphi(g)\,dg = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\mathcal{H}} \varphi(n_xa_y\kappa_\theta)\,d\nu(x+iy)\,d\theta
\end{equation}

\begin{equation}
(2-7) \quad = \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{0}^{\infty} \varphi(\kappa_\theta_1h_t\kappa_\theta_2)\,\sinh(2t)\,d\theta_1\,dt\,d\theta_2, \quad \varphi \in C_c(\text{SL}_2(\mathbb{R})).
\end{equation}

Moreover, for every discrete subgroup $\Gamma$ of $\text{SL}_2(\mathbb{R})$, we have an $\text{SL}_2(\mathbb{R})$-invariant Radon measure on $\Gamma \backslash \text{SL}_2(\mathbb{R})$ defined by the condition

\begin{equation}
\int_{\Gamma \backslash \text{SL}_2(\mathbb{R})} \varphi(\gamma g)\,dg = \int_{\text{SL}_2(\mathbb{R})} \varphi(g)\,dg, \quad \varphi \in C_c(\text{SL}_2(\mathbb{R})),
\end{equation}

or equivalently by the condition

\begin{equation}
(2-8) \quad \int_{\Gamma \backslash \text{SL}_2(\mathbb{R})} \varphi(g)\,dg = \frac{1}{2\pi |\Gamma \cap (-I_2)|} \int_{0}^{2\pi} \int_{\Gamma \backslash \mathcal{H}} \varphi(n_xa_y\kappa_\theta)\,d\nu(x+iy)\,d\theta
\end{equation}

for all $\varphi \in C_c(\Gamma \backslash \text{SL}_2(\mathbb{R}))$.

Finally, let us mention that throughout the paper, for every $n \in \mathbb{Z}_{>0}$ we regard the elements of $\mathbb{C}^n$ as column-vectors. Moreover, we equip $\mathbb{C}^n$ with the standard inner product

\begin{equation}
\langle x, y \rangle_{\mathbb{C}^n} := \sum_{j=1}^{n} x_j\overline{y}_j, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n,
\end{equation}

and denote the induced norm on $\mathbb{C}^n$ by $\| \cdot \|$. We use the same notation for the Frobenius norm on the space $M_n(\mathbb{C})$ of complex square matrices of order $n$:

\begin{equation}
\|X\| := \sqrt{\sum_{r=1}^{n} \sum_{s=1}^{n} |x_{r,s}|^2}, \quad X = (x_{r,s})_{r,s=1}^{n} \in M_n(\mathbb{C}).
\end{equation}

### 3. Vector-valued modular forms

Let $\text{Hol}(\mathcal{H})$ denote the space of holomorphic functions $\mathcal{H} \rightarrow \mathbb{C}$. We define its subspace $\mathcal{F}(k)$ to consist of the functions $f \in \text{Hol}(\mathcal{H})$ with the following property: for every $\sigma \in \text{SL}_2(\mathbb{Z})$, the function $f|_k \sigma$ has a Fourier expansion of the form

\begin{equation}
(f|_k \sigma)(\tau) = \sum_{n=h_\sigma}^{\infty} a_n(\sigma) e^{2\pi i \frac{nx_s}{y_\sigma} \tau}, \quad \tau \in \mathcal{H},
\end{equation}

where $h_\sigma \in \mathbb{Z}$, $a_n(\sigma) \in \mathbb{C}$, and $N_\sigma \in \mathbb{Z}_{>0}$. We note that the space $\mathcal{F}(k)$ is an $\text{SL}_2(\mathbb{Z})$-submodule of $\text{Hol}(\mathcal{H})$ with respect to the action $|_k$. Its submodule of functions $f \in \mathcal{F}(k)$ such that $h_\sigma \geq 0$ (resp., $h_\sigma > 0$) for all $\sigma \in \text{SL}_2(\mathbb{Z})$ will be denoted by $\mathcal{M}(k)$ (resp., $\mathcal{S}(k)$).
Let $\Gamma$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$. We say that $F = (F_1, \ldots, F_p) \in \mathcal{F}(k)^p$ is a vector-valued modular form (VVMF) of weight $k$ for $\Gamma$ (with multiplier system $v$) with respect to a representation $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ if
\begin{equation}
F|_{k\gamma} = \rho(\gamma)F, \quad \gamma \in \Gamma.
\end{equation}

We will denote the space of all such $F$ by $\mathcal{F}(k, \rho, \Gamma)$. We also define its subspaces
\begin{equation}
\mathcal{M}(k, \rho, \Gamma) := \mathcal{F}(k, \rho, \Gamma) \cap \mathcal{M}(k)^p
\end{equation}
of entire VVMFs and
\begin{equation}
\mathcal{S}(k, \rho, \Gamma) := \mathcal{F}(k, \rho, \Gamma) \cap \mathcal{S}(k)^p
\end{equation}
of cuspidal VVMFs.

It will prove useful to note that (3-1) is equivalent to the condition
\begin{equation}
F|_{k, \rho \gamma} = F, \quad \gamma \in \Gamma,
\end{equation}
where $|_{k, \rho}$ is the right action of $\Gamma$ on $(\mathbb{C}^p)^\mathcal{H}$ defined by
\begin{equation}
F|_{k, \rho \gamma} := \rho(\gamma)^{-1}F|_{k \gamma}, \quad F \in (\mathbb{C}^p)^\mathcal{H}, \quad \gamma \in \Gamma.
\end{equation}

We note that in most standard texts on VVMFs (see, e.g., [16, §2]), only the spaces
\begin{equation}
\mathcal{F}(k, \rho) := \mathcal{F}(k, \rho, \text{SL}_2(\mathbb{Z})), \quad \mathcal{M}(k, \rho) := \mathcal{M}(k, \rho, \text{SL}_2(\mathbb{Z})) \quad \text{and} \quad \mathcal{S}(k, \rho) := \mathcal{S}(k, \rho, \text{SL}_2(\mathbb{Z}))
\end{equation}
are studied. Our slightly more general definition of spaces of VVMFs serves only to facilitate the construction of VVMFs in Section 4 and does not enlarge the class of studied VVMFs in a substantial way. The latter observation is elementary and well-known (see, e.g., [34, §2], [4, §1–2] and [12, §1]). For convenience of the reader, we provide its details in the following lemma.

**Lemma 3.1.** Let $\Gamma$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, and let $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ be a representation. Denoting $d := |\text{SL}_2(\mathbb{Z})/\Gamma|$, let us fix $\gamma_1, \ldots, \gamma_d \in \text{SL}_2(\mathbb{Z})$ such that $\text{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^d \Gamma \gamma_j$. Then, the rule
\begin{equation}
F \mapsto (F|_{k \gamma_j})_{j=1}^d
\end{equation}
defines embeddings
\begin{align*}
\mathcal{F}(k, \rho, \Gamma) &\hookrightarrow \mathcal{F}(k, \rho_0), \\
\mathcal{M}(k, \rho, \Gamma) &\hookrightarrow \mathcal{M}(k, \rho_0), \\
\mathcal{S}(k, \rho, \Gamma) &\hookrightarrow \mathcal{S}(k, \rho_0),
\end{align*}
where $\rho_0 : \text{SL}_2(\mathbb{Z}) \to \text{GL}_pd(\mathbb{C})$ is a representation equivalent to the induced representation $\text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho)$ and defined as follows: for every $\gamma \in \text{SL}_2(\mathbb{Z})$, defining a permutation $\ell \in \mathcal{S}_d$ by the rule
\begin{equation}
\Gamma \gamma_j \gamma^{-1} = \Gamma \gamma_{\ell(j)}, \quad j \in \{1, \ldots, d\},
\end{equation}
we put

\[
\rho_0(\gamma) := \begin{pmatrix}
\delta_{1,1} I_p & \cdots & \delta_{1,d} I_p \\
\vdots & \ddots & \vdots \\
\delta_{d,1} I_p & \cdots & \delta_{d,d} I_p
\end{pmatrix}
\begin{pmatrix}
\rho(\gamma_1^1 \gamma_1^{1-1}) \\
\vdots \\
\rho(\gamma_1^d \gamma_1^{d-1})
\end{pmatrix},
\]

where \(\delta\) is the Kronecker delta. If \(\rho\) is unitary, then so is \(\rho_0\).

**Proof.** Let us prove the only two non-obvious parts of the claim:

1. \((F|k\gamma_j)_{j=1}^d|_k = \rho_0(\gamma) (F|k\gamma_j)_{j=1}^d\) for all \(\gamma \in \text{SL}_2(\mathbb{Z})\).
2. \(\rho_0 \cong \text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho)\).

(1) For \(\gamma \in \text{SL}_2(\mathbb{Z})\) and \(\ell\) as in the statement of the lemma, we have

\[
\rho_0(\gamma) (F|k\gamma_j)_{j=1}^d \overset{(3-3)}{=} \left(\delta_{r,\ell(s)} I_p\right)_{r,s=1}^d \text{diag} \left(\rho(\gamma_{\ell(j)} \gamma_{\ell(j)}^{-1})\right)_{j=1}^d (F|k\gamma_j)_{j=1}^d
\]

\[
= \left(\rho(\gamma_j \gamma_{\ell-1}(j)) F|k\gamma_{\ell-1}(j)\right)_{j=1}^d
\]

\[
= \left(F|k\gamma_j \gamma_{\ell-1}(j) |k\gamma_{\ell-1}(j)\right)_{j=1}^d
\]

\[
= (F|k\gamma_j)_{j=1}^d.
\]

(2) Denoting \(\delta_j := \gamma_j^{-1}\), we recall the following standard realization of \(\text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho)\): for every \(j \in \{1, \ldots, d\}\), let \(\delta_j \mathbb{C}^p = \{\delta_j u : u \in \mathbb{C}^p\}\) be a complex vector space isomorphic to \(\mathbb{C}^p\) via \(\delta_j u \mapsto u\); then, \(\text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho)\) can be defined as a representation of \(\text{SL}_2(\mathbb{Z})\) on \(\bigoplus_{j=1}^d \delta_j \mathbb{C}^p\) given by the formula

\[
\left(\text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho)\right)(\gamma) \left(\sum_{j=1}^d \delta_j u_j\right) := \sum_{j=1}^d \delta_{\ell(j)} \rho(\delta_{\ell(j)}^{-1} \gamma \delta_j) u_j
\]

\[
= \sum_{j=1}^d \delta_j \rho(\gamma_j \gamma_{\ell-1}(j)) u_{\ell-1}(j)
\]

for all \(\gamma \in \text{SL}_2(\mathbb{Z})\) and \(u_1, \ldots, u_d \in \mathbb{C}^p\). On the other hand, from (3-3) we see that

\[
\rho_0(\gamma) (u_j)_{j=1}^d = \left(\rho(\gamma_j \gamma_{\ell-1}(j)) u_{\ell-1}(j)\right)_{j=1}^d, \quad (u_j)_{j=1}^d \in (\mathbb{C}^p)^d.
\]

Thus, the rule

\[
(u_j)_{j=1}^d \mapsto \sum_{j=1}^d \delta_j u_j
\]

defines an \(\text{SL}_2(\mathbb{Z})\)-equivalence

\[
(\rho_0, \mathbb{C}^{pd}) \cong \left(\text{Ind}_{\Gamma}^{\text{SL}_2(\mathbb{Z})}(\rho), \bigoplus_{j=1}^d \delta_j \mathbb{C}^p\right).
\]

\(\Box\)
The proof of the following lemma is identical to that of [15, §2, Proposition], so we omit it.

**Lemma 3.2.** Let $\Gamma$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$. Let $F = (F_1, \ldots, F_p) \in \mathcal{F}(k)^p$ such that $\mathbb{C}F_1 + \ldots + \mathbb{C}F_p$ is a $\Gamma$-submodule of $\mathcal{F}(k)$ with respect to the action $|_k$. Then, there exists a representation $\rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C})$ such that $F \in \mathcal{F}(k, \rho, \Gamma)$.

By (2-5), for every subgroup $\Gamma \not\ni -I_2$ of finite index in $\text{SL}_2(\mathbb{Z})$ and representation $\rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C})$, we have

$$\mathcal{F}(k, \rho, \Gamma) = \mathcal{F}(k, \rho', \langle -I_2 \rangle \Gamma)$$

and analogously for the subspaces of entire (resp., cuspidal) VVMFs, where $\rho'$ is the unique extension of $\rho$ to $\langle -I_2 \rangle \Gamma$ satisfying $\rho'(-I_2) = I_p$. This shows that we may restrict our study of VVMFs to the case when $-I_2 \in \Gamma$.

**Lemma 3.3.** Let $\Gamma \ni -I_2$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, and let $\rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C})$ be a representation. Suppose that there exists $F = (F_1, \ldots, F_p) \in \mathcal{F}(k, \rho, \Gamma)$ such that the functions $F_1, \ldots, F_p$ are linearly independent. Then:

(N1) $\rho(-I_2) = I_p$.

(N2) Let $\sigma \in \text{SL}_2(\mathbb{Z})$ and $M \in \mathbb{Z}_{>0}$ such that $\sigma T^M \sigma^{-1} \in \Gamma$. Then, there exists $N \in \mathbb{Z}_{>0}$ such that

$$(e^{2\pi i k M} \rho(\sigma T^M \sigma^{-1}))^N = I_p.$$

**Proof.** (N1) We note that

$$\rho(-I_2)F \overset{(3-1)}{=} F|_k (-I_2) \overset{(2-5)}{=} F,$$

hence by the linear independence of $F_1, \ldots, F_p$ it follows that $\rho(-I_2) = I_p$.

(N2) Since $F \in \mathcal{F}(k)^p$, there exists $N \in \mathbb{Z}_{>0}$ such that $F|_k \sigma$ is $N$-periodic. We have

$$\left(F|_k \sigma \right)(\tau) = \left(F|_k \sigma \right)(\tau + MN) = e^{2\pi i k MN} \left(F|_k \sigma T^{MN} \right)(\tau) \overset{(3-1)}{=} (e^{2\pi i k M} \rho(\sigma T^M \sigma^{-1}))^N \left(F|_k \sigma \right)(\tau), \quad \tau \in \mathcal{H},$$

so by the linear independence of $F_1|_k \sigma, \ldots, F_p|_k \sigma$ it follows that

$$(e^{2\pi i k M} \rho(\sigma T^M \sigma^{-1}))^N = I_p. \quad \Box$$

From now until the end of this section, let $\Gamma \ni -I_2$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, and let $\rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C})$ be a representation. We say that $\rho$ is a normal representation of $\Gamma$ if it satisfies the conditions (N1) and (N2) of Lemma 3.3.

Next, applying Lemma 3.1, it follows from [16, Lemma 2.4 and Theorem 2.5] that the complex vector space $\mathcal{M}(k, \rho, \Gamma)$ is finite-dimensional for every $k \in \mathbb{R}$ and is trivial if $k \ll 0$. Moreover, by [16, §7] we have the following lemma.

**Lemma 3.4.** If $\rho$ is unitary, then $\mathcal{M}(k, \rho, \Gamma) = 0$ for $k < 0$.

The proof of the following lemma is analogous to [24, proof of Theorem 2.1.5], and we leave it as an exercise to the reader.
Lemma 3.5. Suppose that $\rho$ is unitary. Let $F \in \mathcal{S}(k, \rho, \Gamma)$. Then, the function $\mathcal{H} \to \mathbb{C}, \quad \tau \mapsto \|F(\tau)\| \Im(\tau)^{\frac{k}{2}},$

is $\Gamma$-invariant and bounded on $\mathcal{H}$.

Lemma 3.6. Suppose that $\rho$ is unitary. Then, $\mathcal{S}(k, \rho, \Gamma)$ is a finite-dimensional Hilbert space under the Petersson inner product

$$\langle F, G \rangle_{\mathcal{S}(k, \rho, \Gamma)} := \int_{\Gamma \backslash \mathcal{H}} \langle F(\tau), G(\tau) \rangle_{\mathbb{C}^p} \Im(\tau)^{\frac{k}{2}} d(\tau), \quad F, G \in \mathcal{S}(k, \rho, \Gamma),$$

and the embedding $\mathcal{S}(k, \rho, \Gamma) \hookrightarrow \mathcal{S}(k, \rho_0)$ of Lemma 3.1 is an isometry.

Proof. The space $\mathcal{S}(k, \rho, \Gamma)$ is finite-dimensional by our comments before Lemma 3.4. One shows that the integrand in (3-4) is $\Gamma$-invariant as in [16, proof of Lemma 5.1]. Moreover, we have

$$\int_{\Gamma \backslash \mathcal{H}} |\langle F(\tau), G(\tau) \rangle_{\mathbb{C}^p}| \Im(\tau)^{\frac{k}{2}} d(\tau) \leq v(\Gamma \backslash \mathcal{H}) \left( \sup_{\tau \in \mathcal{H}} \|F(\tau)\| \Im(\tau)^{\frac{k}{2}} \right) \left( \sup_{\tau \in \mathcal{H}} \|G(\tau)\| \Im(\tau)^{\frac{k}{2}} \right) \overset{\text{Lem. 3.5}}{<} \infty,$$

for all $F, G \in \mathcal{S}(k, \rho, \Gamma)$, so the inner product (3-4) is well-defined.

Using the notation of Lemma 3.1, the second claim of the lemma follows from the equality

$$\left\langle \left. \left( F \big|_{\gamma_j} \right)_{j=1}^d, \left. \left( G \big|_{\gamma_j} \right)_{j=1}^d \right) \right\rangle_{\mathcal{S}(k, \rho_0)} = \int_{\Gamma \backslash \mathcal{H}} \sum_{j=1}^d \langle \langle F \big|_{\gamma_j} \rangle(\tau), \langle G \big|_{\gamma_j} \rangle(\tau) \rangle_{\mathbb{C}^p} \Im(\tau)^{\frac{k}{2}} d(\tau)$$

$$= \int_{\Gamma \backslash \mathcal{H}} \sum_{j=1}^d \langle \langle F \rangle_{\gamma_j}, \langle G \rangle_{\gamma_j} \rangle_{\mathbb{C}^p} \Im(\gamma_j, \tau)^{\frac{k}{2}} d(\tau)$$

$$= \langle \langle F, G \rangle_{\mathcal{S}(k, \rho, \Gamma)} \rangle_{\mathbb{C}^p} \Im(\gamma_j, \tau)^{\frac{k}{2}} d(\tau)$$

$$= \langle \langle F, G \rangle_{\mathcal{S}(k, \rho, \Gamma)} \rangle_{\mathbb{C}^p} \Im(\gamma_j, \tau)^{\frac{k}{2}} d(\tau)$$

where the second equality follows from (2-4) and (2-2) using the unitarity of $v$. \qed

4. Construction of vector-valued Poincaré series

Let $\Gamma$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, let $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ be a representation, and let $\Lambda$ be a subgroup of $\Gamma$. The defining property (3-2) of VVMFs suggests that, as in the classical theory (see, e.g., [24, §2.6]), interesting elements of $\mathcal{F}(k, \rho, \Gamma)$ may be constructed in the form of a vector-valued Poincaré series (VVPS)

$$P_{\Lambda \backslash \Gamma, \rho} f := \sum_{\gamma \in \Lambda \backslash \Gamma} f|_{\gamma \rho},$$
where $f : \mathcal{H} \to \mathbb{C}^p$ is a suitable function invariant under the $|_{k,\rho}$-action of $\Lambda$. The following proposition, based on this idea, is a vector-valued version of [38, Lemmas 3 and 5] (see also [29, first part of Lemma 2.3]).

**Proposition 4.1.** Let $\Gamma \supset -I_2$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, and let $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ be a unitary representation. Let $\Lambda \supset -I_2$ be a subgroup of $\Gamma$. Let $f : \mathcal{H} \to \mathbb{C}^p$ be a measurable function with the following two properties:

(f1) $f_{|_{k,\rho}} \lambda = f$ for all $\lambda \in \Lambda$.

(f2) $\int_{\Lambda \backslash \mathcal{H}} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}} \, dv(\tau) < \infty$.

Then, we have the following:

(1) The Poincaré series $P_{\Lambda \backslash \Gamma, \rho} f$ converges absolutely a.e. on $\mathcal{H}$, satisfies

$$\left. \left( P_{\Lambda \backslash \Gamma, \rho} f \right) \right|_{k,\rho} \gamma = P_{\Lambda \backslash \Gamma, \rho} f, \quad \gamma \in \Gamma,$$

and we have

$$\int_{\Gamma \backslash \mathcal{H}} \left\| \left( P_{\Lambda \backslash \Gamma, \rho} f \right) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} \, dv(\tau) \leq \int_{\Lambda \backslash \mathcal{H}} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}} \, dv(\tau).$$

(2) Suppose additionally that $\rho$ is normal and that $f \in \text{Hol}(\mathcal{H})^p$. Then, the series $P_{\Lambda \backslash \Gamma, \rho} f$ converges absolutely and uniformly on compact sets in $\mathcal{H}$ and defines an element of

$$\begin{cases} 
\mathcal{S}(k, \rho, \Gamma), & \text{if } k \geq 2 \\
\mathcal{M}(k, \rho, \Gamma), & \text{if } 0 \leq k < 2 \\
0, & \text{if } k < 0.
\end{cases}$$

**Proof.** (1) One checks easily that the integral in (f2) is well-defined, i.e., the integrand is $\Lambda$-invariant, using (f1), (2-2) and the unitarity of $\rho$ and $v$. The terms of the series $P_{\Lambda \backslash \Gamma, \rho} f$ are also well-defined by (f1). All claims in (1) now easily follow from the estimate

$$\int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \left\| \left( f_{|_{k,\rho}} \right) (\gamma) \right\| \Im(\tau)^{\frac{k}{2}} \, dv(\tau)$$

$$= \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \|v(\gamma)^{-1} \rho(\gamma)^{-1} f(\gamma, \tau)\| |j(\gamma, \tau)|^{-k} \Im(\tau)^{\frac{k}{2}} \, dv(\tau)$$

$$= \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Lambda \backslash \Gamma} \|f(\gamma, \tau)\| \Im(\gamma, \tau)^{\frac{k}{2}} \, dv(\tau)$$

$$= \int_{\Lambda \backslash \mathcal{H}} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}} \, dv(\tau) \overset{\text{(f2)}}{< \infty},$$

where the second equality holds by (2-2) and the unitarity of $\rho$ and $v$.

(2) It follows easily from the estimate (4-3) and [24, Corollary 2.6.4] that the series $P_{\Lambda \backslash \Gamma, \rho} f$ converges absolutely and uniformly on compact sets in $\mathcal{H}$ and defines a function $F \in \text{Hol}(\mathcal{H})^p$. By (4-1), $F$ satisfies (3-2).
Next, let \( \sigma \in \text{SL}_2(\mathbb{Z}) \). Denoting \( x := \sigma \infty \), by [24, Theorem 1.5.4(2)] there exists \( M \in \mathbb{Z}_{>0} \) such that \( \Gamma_x = \langle \pm \sigma^M \sigma^{-1} \rangle \). In particular, \( \sigma^M \sigma^{-1} \in \Gamma \), hence by the unitarity of \( \rho \) and (N2) there exist a unitary matrix \( U \in \mathbb{U}(p) \) and \( m_1, \ldots, m_p \in ]0, 1[ \cap \mathbb{Q} \) such that
\[
e^{2\pi i M \rho(\sigma^M \sigma^{-1})} = U^{-1} \text{diag}(e^{2\pi i m_1}, \ldots, e^{2\pi i m_p}) U.
\]
By (4-1),
\[
F|_k \sigma^M \sigma^{-1} = \rho(\sigma^M \sigma^{-1}) F,
\]
hence
\[
UF|_k \sigma^M = e^{-2\pi i M} \text{diag}(e^{2\pi i m_j}) F|_k \sigma,
\]
so for every \( j \in \{1, \ldots, p\} \) we have
\[
((UF)_j|_k \sigma)(\tau + M) = e^{2\pi i m_j} ((UF)_j|_k \sigma)(\tau), \quad \tau \in \mathcal{H},
\]
which implies that the (holomorphic) function \( \mathcal{H} \to \mathbb{C} \),
\[
\tau \mapsto e^{-2\pi i M \tau} ((UF)_j|_k \sigma)(\tau),
\]
is \( M \)-periodic, hence the function \( (UF)_j|_k \sigma \) has a Fourier expansion of the form
\[
((UF)_j|_k \sigma)(\tau) = \sum_{n \in \mathbb{Z}} b_n(j) e^{2\pi i \frac{n + m_j}{M} \tau}, \quad \tau \in \mathcal{H},
\]
where \( b_n(j) \in \mathbb{C} \) are given by
\[
b_n(j) = \frac{1}{M} \int_0^M ((UF)_j|_k \sigma)(x + iy) e^{-2\pi i \frac{n + m_j}{M} (x + iy)} dx, \quad y \in \mathbb{R}_{>0}.
\]
We have
\[
\int_{-\infty}^{\infty} |b_n(j)| e^{-2\pi \frac{n + m_j}{M} y} y^{k-2} dy \leq \frac{1}{M} \int_{[0, M] \times [0, \infty[} |((UF)_j|_k \sigma)(\tau)| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]
\[
= \frac{1}{M} \int_{\sigma([0, M] \times [0, \infty[)} |(UF)_j(\sigma, \tau)| \Im(\sigma, \tau)^{\frac{k}{2}} d\nu(\tau)
\]
\[
= \frac{1}{M} \int_{\Gamma \setminus \mathcal{H}} \| (UF)(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]
\[
= \frac{1}{M} \int_{\Gamma \setminus \mathcal{H}} \| F(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]
\[
< \infty,
\]
where the second inequality holds because by \cite[Corollary 1.7.5]{24} no two different points of \( (0, M] \times [M, \infty] \) are mutually \( \Gamma \)-equivalent. The estimate (4-6) implies that \( b_n(j) = 0 \) if either \( n + m_j = 0 \) and \( k \geq 2 \) or \( n + m_j < 0 \). This means that the functions \((UF)_j\) satisfy

\[
(UF)_j \in \begin{cases} 
S(k), & \text{if } k \geq 2 \\
M(k), & \text{if } k < 2,
\end{cases} \quad j \in \{1, \ldots, p\},
\]

so the same holds for their linear combinations \( F_j \). It follows that

\[
F \in \begin{cases} 
S(k, \rho, \Gamma), & \text{if } k \geq 2 \\
M(k, \rho, \Gamma), & \text{if } k < 2.
\end{cases}
\]

Finally, the claim in the case when \( k < 0 \) follows by Lemma 3.4. \( \square \)

5. A non-vanishing criterion for vector-valued Poincaré series

Let \( \Gamma \cong \mathbb{I}_2 \) be a subgroup of finite index in \( \text{SL}_2(\mathbb{Z}) \), and let \( \rho : \Gamma \to \text{GL}_p(\mathbb{C}) \) be a unitary representation. Moreover, let \( \Lambda \cong \mathbb{I}_2 \) be a subgroup of \( \Gamma \).

We start this section with a technical lemma.

**Lemma 5.1.** Let \( f : \mathcal{H} \to \mathbb{C}^p \) be a measurable function satisfying (f1), and let \( A \) be a Borel-measurable subset of \( \mathcal{H} \). Then,

\[
\int_{\Lambda \setminus \Lambda \setminus A} \| f(\tau) \| \mathcal{S}(\tau)^{\frac{k}{2}} \, d\nu(\tau) = 2 \int_{\Lambda \setminus \Lambda \setminus A} \| f(g.i) \| \| j(g, i) \|^{-k} \, dg,
\]

where we use the notation

\[
(5-1) \quad \mathcal{S} := \{ n_xa_y : x + iy \in S \} K = \{ g \in \text{SL}_2(\mathbb{R}) : g.i \in S \}, \quad S \subseteq \mathcal{H}.
\]

**Proof.** Denoting by \( \mathbb{I}_{\Lambda \setminus \Lambda} \) (resp., \( \mathbb{I}_{\Lambda \setminus \Lambda \setminus A} \)) the characteristic function of \( \Lambda \setminus \Lambda \) (resp., \( \Lambda \setminus \Lambda \setminus A \)) in \( \mathcal{H} \) (resp., \( \text{SL}_2(\mathbb{R}) \)), we have

\[
\int_{\Lambda \setminus \Lambda \setminus A} \| f(\tau) \| \mathcal{S}(\tau)^{\frac{k}{2}} \, d\nu(\tau)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_{\Lambda \setminus \mathcal{H}} \| f(x + iy) \| y^{\frac{k}{2}} \mathbb{I}_{\Lambda \setminus \Lambda}(x + iy) \, d\nu(x + iy) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_{\Lambda \setminus \mathcal{H}} \| f(n_xa_y\kappa_\theta, i) \| \| j(n_xa_y\kappa_\theta, i) \|^{-k} \mathbb{I}_{\Lambda \setminus \Lambda}(n_xa_y\kappa_\theta) \, d\nu(x + iy) \, d\theta
\]

\[
(2-8) \quad = 2 \int_{\Lambda \setminus \Lambda \setminus A} \| f(g.i) \| \| j(g, i) \|^{-k} \, dg. \quad \square
\]

The following theorem may be regarded as a vector-valued version of the integral non-vanishing criterion \cite[Lemma 3.1]{28} for Poincaré series of integral weight on \( \mathcal{H} \) (see also \cite[Theorem 2]{38} for the half-integral weight version, and \cite[Theorem 4.1]{26} for the original version of the criterion, in which Poincaré series on unimodular locally compact Hausdorff groups are considered).
**Theorem 5.2.** Let $\Gamma \ni -I_2$ be a subgroup of finite index in $\text{SL}_2(\mathbb{Z})$, and let $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ be a unitary representation. Let $\Lambda \ni -I_2$ be a subgroup of $\Gamma$, and let $f : \mathcal{H} \to \mathbb{C}^p$ be a measurable function with the following properties:

(f1) $f|_{E,\rho} \lambda = f$ for all $\lambda \in \Lambda$.

(f2’) The series $P_{\Lambda \setminus \Gamma, \rho} f$ converges absolutely a.e. on $\mathcal{H}$.

Then, we have that

$$\int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} f)(\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau) > 0$$

if one of the following holds:

(i) There exists a Borel-measurable set $A \subseteq \mathcal{H}$ with the following properties:

(A1) No two points of $A$ are mutually $\Gamma$-equivalent.

(A2) Denoting $(\Lambda.A)^c := \mathcal{H} \setminus \Lambda.A$, we have

$$\int_{\Lambda \setminus \Lambda.A} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau) > \int_{\Lambda \setminus (\Lambda.A)^c} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau).$$

(ii) There exists a Borel-measurable set $C \subseteq \text{SL}_2(\mathbb{R})$ with the following properties:

(C1) $CK = C$.

(C2) $CC^{-1} \cap \Gamma \subseteq \langle -I_2 \rangle$.

(C3) Denoting $(\Lambda C)^c := \text{SL}_2(\mathbb{R}) \setminus \Lambda C$, we have

$$\int_{\Lambda \setminus \Lambda.C} \| f(g.i) \| |j(g, i)|^{-k} dg > \int_{\Lambda \setminus (\Lambda C)^c} \| f(g.i) \| |j(g, i)|^{-k} dg.$$

**Remark 5.3.** By Proposition 4.1(1), Theorem 5.2 remains true if we replace the property (f2’) in it by (f2).

**Proof of Theorem 5.2.** Suppose that (i) holds. First, we recall that by [24, Theorem 1.7.8], the set of elliptic points for $\Gamma$ in $\mathcal{H}$ is countable, hence of measure zero. Next, we note that if $\tau \in \mathcal{H}$ is not an elliptic point for $\Gamma$, i.e., if $\Gamma_\tau = \langle -I_2 \rangle$, then

$$\# \{ \gamma \in \Lambda \setminus \Gamma : 1_{\Lambda.A}(\gamma.\tau) \neq 0 \} \leq 1.$$ 

Namely, if $\gamma.\tau, \gamma'.\tau \in \Lambda.A$ for some $\gamma, \gamma' \in \Gamma$, then there exist $\lambda, \lambda' \in \Lambda$ such that $\lambda.\gamma.\tau, \lambda'.\gamma'.\tau \in \Lambda$, hence by (A1) we have $\lambda.\gamma.\tau = \lambda'.\gamma'.\tau$, which by the non-ellipticity of $\tau$ implies that $\lambda'.\gamma' \in \{ \pm \lambda \gamma \}$, hence $\Lambda.\gamma' = \Lambda.\gamma$. 


Denoting by \(1_S\) the characteristic function of a set \(S \subseteq \mathcal{H}\), we have

\[
\int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} (1_{\Lambda, A} f)) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
= \int_{\Gamma \setminus \mathcal{H}} \left\| \sum_{\gamma \in \Lambda \setminus \Gamma} 1_{\Lambda, A}(\gamma, \tau) \rho(\gamma)^{-1} (f|_{k \gamma}) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
= \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Lambda \setminus \Gamma} 1_{\Lambda, A}(\gamma, \tau) \| \rho(\gamma)^{-1} (f|_{k \gamma}) (\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
= \int_{\Lambda \setminus \Lambda, A} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau),
\]

where the third equality holds by (2-2), (2-4) and the unitarity of \(\rho\) and \(v\).

On the other hand, we have

\[
\int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} (1_{\Lambda, A})^c f)) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
\leq \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Lambda \setminus \Gamma} 1_{\Lambda, A}(\gamma, \tau) \| \rho(\gamma)^{-1} (f|_{k \gamma}) (\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
= \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Lambda \setminus \Gamma} 1_{\Lambda, A}(\gamma, \tau) \| f(\gamma, \tau) \| \Im(\gamma, \tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
= \int_{\Lambda \setminus \Lambda, A} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau).
\]

Thus,

\[
\int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} f) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
\geq \int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} (1_{\Lambda, A} f)) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
- \int_{\Gamma \setminus \mathcal{H}} \left\| (P_{\Lambda \setminus \Gamma, \rho} (1_{\Lambda, A})^c f)) (\tau) \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
\geq (5-3) \int_{\Lambda \setminus \Lambda, A} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau) - \int_{\Lambda \setminus \Lambda, A} \| f(\tau) \| \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
> 0.
\]
Next, suppose that (ii) holds. To finish the proof of the theorem, it suffices to prove that the set
\[ (5-5) \quad A := C.i = \{ x + iy : n_x a_y K \subseteq C \} \]
has the properties (A1) and (A2).

(A1) Suppose that \( \gamma_n(x + iy) = x' + iy' \) for some \( \gamma \in \Gamma \) and \( x + iy, x' + iy' \in A \). Equivalently, \( \gamma n_x a_y i = n_x a_y' i \), i.e., \( a_y^{-1} n_x^{-1} \gamma n_x a_y i = i \), hence \( a_y^{-1} n_x^{-1} \gamma n_x a_y \in K \), so
\[
\gamma \in (n_x a_y K) (n_x a_y)^{-1} \cap \Gamma \subseteq CC^{-1} \cap \Gamma \subseteq \langle -I_2 \rangle,
\]
which implies that \( x + iy = x' + iy' \).

(A2) Using the notation (5-1), by (C1) and (5-5) we have that \( C = A \), \( \Lambda C = \Lambda A \), and \( (\Lambda C)^c = (\Lambda A)^c \), so (A2) follows from (C3) by applying Lemma 5.1. \( \square \)

6. Classical vector-valued Poincaré series

As a first example application of our results, in this section we construct and study the non-vanishing of the cuspidal VVMFs that are vector-valued analogues of the classical Poincaré series (for details on the latter cusp forms, see, e.g., [24, Theorems 2.6.9(1) and 2.6.10]). We note that in the case when \( \Gamma = \text{SL}_2(\mathbb{Z}) \), these VVMFs have already been studied in [16, §3].

We will need the following lemma.

Lemma 6.1. Let \( \Gamma \ni -I_2 \) be a subgroup of finite index in \( \text{SL}_2(\mathbb{Z}) \), let \( \rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C}) \) be a unitary representation, and let \( \Lambda \ni -I_2 \) be a subgroup of \( \Gamma \). Let \( f : \mathcal{H} \rightarrow \mathbb{C}^p \) be a measurable function satisfying (f1) and (f2), such that \( P_{\Lambda \setminus \Gamma, \rho} f \in \mathcal{S}(k, \rho, \Gamma) \). Then,
\[ (6-1) \quad \langle F, P_{\Lambda \setminus \Gamma, \rho} f \rangle_{\mathcal{S}(k, \rho, \Gamma)} = \int_{\Lambda \setminus \mathcal{H}} \langle F(\tau), f(\tau) \rangle_{\mathbb{C}^p} \mathfrak{Z}(\tau)^k d\nu(\tau), \quad F \in \mathcal{S}(k, \rho, \Gamma). \]

Proof. We have
\[
\langle F, P_{\Lambda \setminus \Gamma, \rho} f \rangle_{\mathcal{S}(k, \rho, \Gamma)} = \int_{\Lambda \setminus \mathcal{H}} \langle F(\tau), (P_{\Lambda \setminus \Gamma, \rho} f)(\tau) \rangle_{\mathbb{C}^p} \mathfrak{Z}(\tau)^k d\nu(\tau)
\]
\[ = \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Lambda \setminus \Gamma} \langle \rho(\gamma)^{-1} (F|_{|k} \gamma)(\tau), \rho(\gamma)^{-1} (f|_{|k} \gamma)(\tau) \rangle_{\mathbb{C}^p} \mathfrak{Z}(\tau)^k d\nu(\tau)
\]
\[ = \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Lambda \setminus \Gamma} \langle F(\gamma \tau), f(\gamma \tau) \rangle_{\mathbb{C}^p} \mathfrak{Z}(\gamma \tau)^k d\nu(\tau)
\]
\[ = \int_{\Lambda \setminus \mathcal{H}} \langle F(\tau), f(\tau) \rangle_{\mathbb{C}^p} \mathfrak{Z}(\tau)^k d\nu(\tau), \quad F \in \mathcal{S}(k, \rho, \Gamma), \]
where in the third equality we used (2-2), (2-4) and the unitarity of \( \rho \) and \( v. \) \( \square \)

Let \( \Gamma \ni -I_2 \) be a subgroup of finite index in \( \text{SL}_2(\mathbb{Z}) \), let \( \rho : \Gamma \rightarrow \text{GL}_p(\mathbb{C}) \) be a normal unitary representation, and let \( M \in \mathbb{Z}_{>0} \) such that \( \Gamma_\infty = \langle \pm T^M \rangle \) (see [24, Theorem 1.5.4(2)]). By the unitarity of \( \rho \) and (N2), there exist \( U \in \text{U}(p) \) and \( m_1, \ldots, m_p \in [0, 1] \cap \mathbb{Q} \) such that
\[ (6-2) \quad \rho(T^M) = e^{-2\pi i m_1} U^{-1} \text{diag}(e^{2\pi i m_1}, \ldots, e^{2\pi i m_p}) U. \]
Proposition 6.2. Let $k \in \mathbb{R}_{>2}$, $\nu \in \mathbb{Z}_{\geq 0}$, and $j \in \{1, \ldots, p\}$. Denoting by $e_j$ the $j$th vector of the canonical basis for $\mathbb{C}^p$, we have the following:

1. The Poincaré series

$$\Psi_{k,\rho,\nu,U,j} := P_{\Gamma_{\infty}\setminus \Gamma,\rho} \left( e^{2\pi i \frac{\nu + m_j}{M}} U^{-1} e_j \right)$$

converges absolutely and uniformly on compact sets in $\mathcal{H}$ and defines an element of $\mathcal{S}(k, \rho, \Gamma)$.

2. For every $F \in \mathcal{S}(k, \rho, \Gamma)$, we have

$$\langle F, \Psi_{k,\rho,\nu,U,j} \rangle_{\mathcal{S}(k,\rho,\Gamma)} = b_{\nu}(j) \frac{M^k \Gamma(k-1)}{(4\pi(\nu + m_j))^{k-1}},$$

where $b_{\nu}(j) \in \mathbb{C}$ are coefficients in the Fourier expansion

$$\langle UF \rangle_j(\tau) = \sum_{n=0}^{\infty} b_n(j) e^{2\pi i \frac{n + m_j}{M} \tau}, \quad \tau \in \mathcal{H},$$

and $\Gamma$ on the right-hand side of (6-3) denotes the gamma function $\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt$, $\Re(s) > 0$.

Proof. (1) By Proposition 4.1, it suffices to prove that the function $f : \mathcal{H} \to \mathbb{C}^p$,

$$f(\tau) := e^{2\pi i \frac{\nu + m_j}{M}} U^{-1} e_j,$$

satisfies (f1) and (f2) with $\Lambda = \Gamma_{\infty}$. The property (f1) is satisfied by (N1), (2-5) and the equality

$$\left( f \otimes_{k,\rho} T^{M} \right)(\tau) = e^{-2\pi i n M} \rho(T^{-M}) f(\tau + M) \left( f(\tau) \right)_{(6-5)}, \quad \tau \in \mathcal{H},$$

and (f2) holds by the following estimate: since $k > 2$,

$$\int_{\Gamma_{\infty}\setminus \mathcal{H}} \|f(\tau)\| \Im(\tau)^{\frac{\nu}{2}} d\nu(\tau) \overset{(6-5)}{=} \int_0^{M} \int_0^{\infty} e^{-2\pi i \frac{\nu + m_j}{M} y} \|U^{-1} e_j\| y^{\frac{\nu}{2} - 2} dy dx$$

$$\overset{(6-5)}{=} \frac{M^{\frac{\nu}{2}}}{(2\pi(\nu + m_j))^{\frac{\nu}{2} - 1}} \int_0^{\infty} e^{-y} y^{\frac{\nu}{2} - 2} dy < \infty.$$
First, we note that the Fourier expansion (6-4) exists by the same argument as the Fourier expansion (4-4). Now we have

\[
\langle F, \Psi_{k, \rho, \Gamma, \nu, U, j} \rangle_{S(k, \rho, \Gamma)}^{(6-1)} = \int_{\Gamma \setminus \mathcal{H}} \left( UF(\tau), e^{2\pi i \nu + mj} \frac{U^{-1} e_j}{M} \right)_{C^p} \mathcal{Z}(\tau) d\nu(\tau)
\]

\[
= \lim_{R \to 0^+} \int_0^M \int_{-\infty}^\infty \sum_{n=0}^\infty b_n(j) e^{2\pi i \nu + mj} e^{-2\pi \nu x + 2\pi \nu y} y^{-2} dy dx
\]

\[
= b_\nu(j) M^k \Gamma(k - 1) \frac{\Gamma(k - 1)}{(4\pi(\nu + mj))^k - 1},
\]

where the second equality holds because \( U \) is a unitary matrix, and the fourth one by the dominated convergence theorem.

We note the following direct consequence of Proposition 6.2.

**Corollary 6.3.** Let \( k \in \mathbb{R}_{>2} \). Then,

\[
\mathcal{S}(k, \rho, \Gamma) = \text{span}_{\mathbb{C}} \{ \Psi_{k, \rho, \Gamma, \nu, U, j} : \nu \in \mathbb{Z}_{\geq 0}, \ j \in \{1, \ldots, p\} \}.
\]

Finally, applying our non-vanishing criterion (Theorem 5.2), we obtain the following result on the non-vanishing of VVMFs \( \Psi_{k, \rho, \Gamma, \nu, U, j} \).

**Theorem 6.4.** Let \( k \in \mathbb{R}_{>2} \) and \( N \in \mathbb{Z}_{>0} \). Let \( \Gamma \in \{ \Gamma_0(N), \langle -I_2 \rangle \Gamma_1(N), \langle -I_2 \rangle \Gamma(N) \} \) and

\[
M := \begin{cases} 
1, & \text{if } \Gamma \in \{ \Gamma_0(N), \langle -I_2 \rangle \Gamma_1(N) \} \\
N, & \text{if } \Gamma = \langle -I_2 \rangle \Gamma(N). 
\end{cases}
\]

Let \( \rho : \Gamma \to \text{GL}_p(\mathbb{C}) \) be a normal unitary representation, and fix \( U \in \text{U}(p) \) and \( m_1, \ldots, m_p \in [0, 1] \cap \mathbb{Q} \) such that

\[
\rho(T^M) = e^{-2\pi i m U^{-1} \text{ diag}(e^{2\pi i m_1}, \ldots, e^{2\pi i m_p}) U}.
\]

Then, \( \Psi_{k, \rho, \Gamma, \nu, U, j} \neq 0 \) if

\[
\nu + mj \leq \frac{MN}{4\pi} \left( k - \frac{8}{3} \right).
\]

**Proof.** We apply Theorem 5.2(i) with

\[
A := [0, M] \times \left[ \frac{1}{N}, \infty \right].
\]
Let us prove that the set $A$ defined in this way satisfies (A1). Let $\tau \in A$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\gamma.\tau \in A$. Then $c = 0$, because otherwise we would have $|c| \geq N$ and consequently
\[
\frac{1}{N} < \Im(\gamma.\tau) = \frac{1}{(c\Re(\tau) + d)^2 + (c\Im(\tau))^2} \leq \frac{\Im(\tau)}{(c\Im(\tau))^2} < \frac{1}{N^2} \cdot \frac{1}{N} = \frac{1}{N}.
\]
Thus, $\gamma \in \Gamma_\infty = \langle \pm T^M \rangle$, hence $\gamma.\tau = \tau + nM$ for some $n \in \mathbb{Z}$. The fact that $\Re(\tau), \Re(\gamma.\tau) \in [0, M]$ implies that $n = 0$, hence $\gamma.\tau = \tau$, which proves (A1).

On the other hand, our set $A$ satisfies (A2) if and only if
\[
\int_{\Gamma_\infty \setminus \Gamma_\infty \cdot A} \left\| e^{2\pi i \frac{\nu + mj}{M} \tau} U^{-1} e_j \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau) > \int_{\Gamma_\infty \setminus (\Gamma_\infty \cdot A)^c} \left\| e^{2\pi i \frac{\nu + mj}{M} \tau} U^{-1} e_j \right\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau),
\]
i.e., recalling that $U \in U(p)$ and that $]0, M[ \times ]0, \infty[$ is a fundamental domain for $\Gamma_\infty$ in $\mathcal{H}$, if and only if we have
\[
\int_{0}^{M} \int_{0}^{\infty} e^{-\frac{2\pi}{M} \nu \frac{\nu + mj}{M} y} y^{k-2} dy \, dx > \int_{0}^{M} \int_{0}^{\infty} e^{-\frac{2\pi}{M} \nu \frac{\nu + mj}{M} y} y^{k-2} dy \, dx
\]
or equivalently
\[
\int_{0}^{\infty} \left( \frac{2\pi}{M} (\nu + mj) \right) \frac{k}{2} e^{-t} dt > \int_{0}^{\frac{2\pi}{M} (\nu + mj)} \left( \frac{2\pi}{M} (\nu + mj) \right) \frac{k}{2} e^{-t} dt,
\]
i.e., if and only if
\[
\frac{2\pi (\nu + mj)}{MN} < M_{\Gamma(a, b)},
\]
where $M_{\Gamma(a, b)} \in \mathbb{R}_{>0}$ is the median of the gamma distribution $\Gamma(a, b)$ with parameters $a, b \in \mathbb{R}_{>0}$, determined by the condition
\[
\int_{0}^{M_{\Gamma(a, b)}} x^{a-1} e^{-bx} \, dx = \int_{M_{\Gamma(a, b)}}^{\infty} x^{a-1} e^{-bx} \, dx.
\]
Applying Chen and Rubin’s estimate [7, Theorem 1], stating that
\[
a - \frac{1}{3} < M_{\Gamma(a, 1)} < a, \quad a \in \mathbb{R}_{>0},
\]
it follows that (A2) holds if
\[
\frac{2\pi (\nu + mj)}{MN} \leq \frac{k}{2} - \frac{4}{3},
\]
which is equivalent to (6-6). □
In this section, we use Proposition 4.1 and Theorem 5.2 to construct and study the non-vanishing of the vector-valued analogues of elliptic Poincaré series. The latter cusp forms were studied already by Petersson [30, (8)].

Let \( \Gamma \supset -I_2 \) be a subgroup of finite index in \( SL_2(\mathbb{Z}) \), and let \( \rho : \Gamma \to GL_p(\mathbb{C}) \) be a normal unitary representation.

**Proposition 7.1.** Let \( k \in \mathbb{R}_{>2}, \nu \in \mathbb{Z}_{\geq 0}, \xi \in \mathcal{H} \). Then, we have the following:

1. Let \( u \in \mathbb{C}^p \). The Poincaré series

\[
\Phi_{k,\rho,\Gamma,\nu,\xi,\Gamma} := P_{(-I_2)\backslash \Gamma,\rho} \left( \frac{(\cdot - \xi)^\nu}{(\cdot - \xi)^{\nu+k}} u \right)
\]

converges absolutely and uniformly on compact sets in \( \mathcal{H} \) and defines an element of \( \mathcal{S}(k,\rho,\Gamma) \).

2. For every \( j \in \{1, \ldots, p\} \), we have

\[
\langle F, \Phi_{k,\rho,\Gamma,\nu,\xi,\Gamma} \rangle_{\mathcal{S}(k,\rho,\Gamma)} = \frac{4\pi}{(4\Im(\xi))^k} \frac{\nu!}{(k-1)k \cdots (k+\nu-1)} b_{\nu,\xi}(j)
\]

for every \( F = (F_1, \ldots, F_p) \in \mathcal{S}(k,\rho,\Gamma) \), where \( b_{\nu,\xi}(j) \in \mathbb{C} \) are coefficients in the expansion

\[
(\tau - \bar{\xi})^k F_j(\tau) = \sum_{n=0}^{\infty} b_{n,\xi}(j) \left( \frac{\tau - \xi}{\tau - \bar{\xi}} \right)^n, \quad \tau \in \mathcal{H}.
\]

**Proof.** (1) The claim follows from Proposition 4.1 as soon as we prove that \( f : \mathcal{H} \to \mathbb{C}^p \),

\[
f(\tau) := \frac{(\tau - \xi)^\nu}{(\tau - \bar{\xi})^{\nu+k}} u,
\]

satisfies (f2). Applying the change of variables \( \tau \mapsto n\Re(\xi)a\Im(\xi)\tau \), we obtain

\[
\int_{(-I_2)\backslash \mathcal{H}} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}} d\nu(\tau) = \frac{\|u\|}{\Im(\xi)^{\frac{k}{2}}} \int_{\mathcal{H}} \left| \frac{\tau - i}{\tau + i} \right|^\nu \Im(\tau)^{\frac{k}{2}} d\nu(\tau)
\]

\[
\leq \frac{\|u\|}{\Im(\xi)^{\frac{k}{2}}} \int_{\mathcal{H}} \frac{\Im(\tau)^{\frac{k}{2}}}{|\tau + i|^k} d\nu(\tau)
\]

\[
= \frac{\|u\|}{\Im(\xi)^{\frac{k}{2}}} \int_{0}^{\infty} \int_{\mathbb{R}} \frac{y^{k-2}}{(x^2 + (y + 1)^2)^{\frac{k}{2}}} dx dy
\]

\[
= \frac{\|u\|}{\Im(\xi)^{\frac{k}{2}}} \int_{\mathbb{R}} \frac{dx}{(x^2 + 1)^{\frac{k}{2}}} \int_{0}^{\infty} \frac{y^{k-2}}{(y + 1)^{k-1}} dy,
\]

introducing the change of variables \( x \mapsto (y + 1)x \) in the last equality. Since \( k > 2 \), the right-hand side is finite, which proves (f2).
(2) Let \( F \in S(k, \rho, \Gamma) \). By Lemma 6.1, we have
\[
\langle F, \Phi_{k, \rho, \nu, \xi, e_j} \rangle_{S(k, \rho, \Gamma)} = \int_{\mathcal{H}} \left\langle F(\tau), \frac{(\tau - \xi)^{\nu}}{(\tau - \xi)^{\nu+k}} e_j \right\rangle_{\mathbb{C}^p} \mathfrak{S}(\tau)^k \, dv(\tau).
\]
Using (7-2) and introducing the change of variables \( \tau \mapsto n_{\mathbb{R}(\xi)\mathcal{A}(\xi)\cdot \tau} \), we see that the right-hand side equals
\[
\mathfrak{S}(\xi)^{-k} \int_{\mathcal{H}} \sum_{n=0}^{\infty} b_{n, \xi}(j) \left( \frac{\tau - i}{\tau + i} \right)^{n-\nu} \left| \frac{\tau - i}{\tau + i} \right|^{2\nu} \mathfrak{S}(\tau)^k \frac{1}{|\tau + i|^{2k}} \, dv(\tau),
\]
which, introducing the substitution \( w = \frac{\tau - i}{\tau + i} \) (so \( dx \, dy = \frac{4}{|1-w|^2} \, dw \)) and denoting \( \mathcal{D} := \{ w \in \mathbb{C} : |w| < 1 \} \), equals
\[
\frac{4}{(4\mathfrak{S}(\xi))^k} \int_{\mathcal{D}} \sum_{n=0}^{\infty} b_{n, \xi}(j) w^{n-\nu} |w|^{2\nu} (1 - |w|^2)^{k-2} \, dw.
\]
Going over to polar coordinates, we obtain
\[
\frac{4}{(4\mathfrak{S}(\xi))^k} \lim_{R \to 1} \int_0^R \int_0^{2\pi} \sum_{n=0}^{\infty} b_{n, \xi}(j) r^{n+\nu+1} (1 - r^2)^{k-2} \, e^{i(n-\nu)t} \, dt \, dr,
\]
i.e., applying the dominated convergence theorem,
\[
\frac{4}{(4\mathfrak{S}(\xi))^k} \lim_{R \to 1} \sum_{n=0}^{\infty} b_{n, \xi}(j) \int_0^R r^{n+\nu+1} (1 - r^2)^{k-2} \, dr \int_0^{2\pi} e^{i(n-\nu)t} \, dt
\]
\[
= \frac{8\pi}{(4\mathfrak{S}(\xi))^k} b_{\nu, \xi}(j) \int_0^1 r^{2\nu+1} (1 - r^2)^{k-2} \, dr
\]
\[
= \frac{4\pi}{(4\mathfrak{S}(\xi))^k} b_{\nu, \xi}(j) \frac{\nu!}{(k-1)k \cdots (k + \nu - 1)},
\]
where the last equality is obtained by \( \nu \)-fold partial integration after substituting \( t = r^2 \). \( \square \)

As a direct consequence of Proposition 7.1, we obtain the following corollary.

**Corollary 7.2.** Let \( k \in \mathbb{R}_{>2} \) and \( \xi \in \mathcal{H} \). Then,
\[
S(k, \rho, \Gamma) = \text{span}_{\mathbb{C}} \{ \Phi_{k, \rho, \nu, \xi, e_j} : \nu \in \mathbb{Z}_{\geq 0}, \ j \in \{1, \ldots, p\} \}.
\]

In the following theorem, we give a result on the non-vanishing of elliptic VVPSs. To state it, we need the notion of the median \( M_{B(a,b)} \in [0,1] \) of the beta distribution \( B(a,b) \) with parameters \( a, b \in \mathbb{R}_{>0} \), defined by the condition
\[
\int_0^{M_{B(a,b)}} x^{a-1} (1 - x)^{b-1} \, dx = \int_{M_{B(a,b)}}^1 x^{a-1} (1 - x)^{b-1} \, dx.
\]
Theorem 7.3. Let $N \in \mathbb{Z}_{\geq 2}$, and let $\Gamma \ni -I_2$ be a subgroup of finite index in $\langle -I_2 \rangle \Gamma(N)$. Let $\rho : \Gamma \to \text{GL}_p(\mathbb{C})$ be a normal unitary representation. Let $k \in \mathbb{R}_{>2}$, $\nu \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^p \setminus \{0\}$. If

\begin{equation}
N > \frac{4 \left(M_B\left(\frac{\nu + 1}{2}, \frac{k}{2} - 1\right)\right)^{\frac{1}{2}}}{1 - M_B\left(\frac{\nu + 1}{2}, \frac{k}{2} - 1\right)},
\end{equation}

then

\[ \Phi_{k,\rho,\Gamma,\nu,u} \not\equiv 0. \]

Proof. We recall that $\Phi_{k,\rho,\Gamma,\nu,u} = P_{\langle -I_2 \rangle \backslash \Gamma} f$ for $f : \mathcal{H} \to \mathbb{C}^p$,

\begin{equation}
 f(\tau) := \frac{(\tau - i)^\nu}{(\tau + i)^{\nu+k}} u.
\end{equation}

To apply Theorem 5.2, it suffices to find a Borel-measurable set $C \subseteq \text{SL}_2(\mathbb{R})$ satisfying (C1)–(C3) with $\Lambda = \langle -I_2 \rangle$. Following the idea of [27, Lemma 6-5], let us look for such a set $C$ of the form

\[ C_r := K \{ h_t : t \in [0, r] \} K \]

with $r \in \mathbb{R}_{>0}$. For every $r \in \mathbb{R}_{>0}$, the set $C_r$ obviously satisfies (C1). Next, by [27, Lemma 6-20] we have

\[ \max_{g \in C_r} \|g\| = \sqrt{2 \cosh(4r)}, \]

and on the other hand, obviously

\[ \min_{\gamma \in \Gamma \setminus \langle -I_2 \rangle} \|\gamma\| \geq \sqrt{N^2 + 2}, \]

so $C_r$ satisfies (C2) if

\begin{equation}
\sqrt{2 \cosh(4r)} < N^2 + 2.
\end{equation}

Next, one checks easily that (for every $f : \mathcal{H} \to \mathbb{C}^p$)

\[ \|f(\kappa t h_t \kappa h_t^{-1}, i)\| \cdot |j(\kappa t h_t \kappa h_t^{-1}, i)|^{-k} \]

\[ = \|f\left(\frac{e^t i \cos \theta_1 - e^{-t} \sin \theta_1}{e^t i \sin \theta_1 + e^{-t} \cos \theta_1}\right)\| \cdot |e^t i \sin \theta_1 + e^{-t} \cos \theta_1|^{-k} \]

for all $t \in \mathbb{R}_{\geq 0}$ and $\theta_1, \theta_2 \in \mathbb{R}$. From this, it follows by an elementary computation, using (7-4), that

\[ \|f(\kappa t h_t \kappa h_t^{-1}, i)\| \cdot |j(\kappa t h_t \kappa h_t^{-1}, i)|^{-k} = \frac{\tanh^\nu t}{(2 \cosh t)^k} \|u\| \]

for all $t \in \mathbb{R}_{\geq 0}$ and $\theta_1, \theta_2 \in \mathbb{R}$. Thus, using (2-7), $C_r$ satisfies (C3) if and only if

\begin{equation}
\int_0^r \frac{\tanh^\nu t}{\cosh^k t} \sinh(2t) dt > \int_r^\infty \frac{\tanh^\nu t}{\cosh^k t} \sinh(2t) dt.
\end{equation}

The computation from [37, proof of Proposition 6.7] shows that there exists $r \in \mathbb{R}_{>0}$ satisfying both (7-5) and (7-6) if and only if (7-3) holds. This proves the theorem. \qed
Remark 7.4. For concrete values of $\nu$ and $k$, it is easy to compute the value of the right-hand side in (7-3) explicitly using mathematical software (e.g., in R 3.3.2, $M_{B(a,b)}$ is implemented as \texttt{qbeta(0.5,a,b)}). Moreover, [37, Corollary 6.18] lists a few elementary sufficient conditions on $\nu$, $k$ and $N$ for the inequality (7-3) to hold.

References

1. Bantay, P.: The dimension of spaces of vector-valued modular forms of integer weight. Lett. Math. Phys. 103(11), 1243–1260 (2013)
2. Bantay, P.: A trace formula for vector-valued modular forms. Commun. Contemp. Math. 17(6) (2015)
3. Bantay, P., Gannon, T.: Vector-valued modular functions for the modular group and the hypergeometric equation. Commun. Number Theory Phys. 1(4), 651–680 (2007)
4. Borcherds, R. E.: Automorphic forms with singularities on Grassmannians. Invent. Math. 132(3), 491–562 (1998)
5. Borcherds, R. E.: The Gross-Kohnen-Zagier theorem in higher dimensions. Duke Math. J. 97(2), 219–233 (1999)
6. Candelori, L., Franc, C.: Vector-valued modular forms and the modular orbifold of elliptic curves. Int. J. Number Theory 13(1), 39–63 (2017)
7. Chen, J., Rubin, H.: Bounds for the difference between median and mean of gamma and Poisson distributions. Statist. Probab. Lett. 4(6), 281–283 (1986)
8. Dong, C., Li, H., Mason, G.: Modular-invariance of trace functions in orbifold theory and generalized Moonshine. Comm. Math. Phys. 214, 1–56 (2000)
9. Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in Mathematics 55, Birkhäuser Boston, Inc., Boston, MA (1985)
10. Franc, C., Mason, G.: Fourier coefficients of vector-valued modular forms of dimension 2. Canad. Math. Bull. 57(3), 485–494 (2014)
11. Franc, C., Mason, G.: On the structure of modules of vector-valued modular forms. Ramanujan J. 47(1), 117–139 (2018)
12. Gannon T.: The theory of vector-valued modular forms for the modular group. Conformal field theory, automorphic forms and related topics, Contrib. Math. Comput. Sci. 8, Springer, Heidelberg, 247–286 (2014)
13. Grobner, H.: Smooth automorphic forms and smooth automorphic representations. Book in preparation, to appear in Series on Number Theory and Its Applications, WorldScientific.
14. Knopp, M., Mason, G.: Generalized modular forms. J. Number Theory 99(1), 1–28 (2003)
15. Knopp, M., Mason, G.: On vector-valued modular forms and their Fourier coefficients. Acta Arith. 110(2), 117–124 (2003)
16. Knopp, M., Mason, G.: Vector-valued modular forms and Poincaré series. Illinois J. Math. 48(4), 1345–1366 (2004)
17. Kohnen, W.: Nonvanishing of Hecke L-functions associated to cusp forms inside the critical strip. J. Number Theory 67(2), 182–189 (1997)
18. Lehner, J.: On the non-vanishing of Poincaré series. Proc. Edinb. Math. Soc. (2) 23(2), 225–228 (1980)
19. Marks, C.: Irreducible vector-valued modular forms of dimension less than six. Illinois J. Math. 55(4), 1267–1297 (2011)
20. Marks, C., Mason, G.: Structure of the module of vector-valued modular forms. J. Lond. Math. Soc. (2) 82(1), 32–48 (2010)
21. Mason, G.: Vector-valued modular forms and linear differential operators. Int. J. Number Theory 3(3), 377–390 (2007)
22. Mason, G.: 2-dimensional vector-valued modular forms. Ramanujan J. 17(3), 405–427 (2008)
23. Milas, A.: Virasoro algebra, Dedekind $\eta$-function and specialized Macdonald identities. Transform. Groups 9(3), 273–288 (2004)

24. Miyake, T.: Modular forms. Translated from the 1976 Japanese original by Yoshitaka Maeda. Reprint of the first 1989 English edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2006)

25. Miyamoto, M.: Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness. Duke Math. J. 122(1), 51–91 (2004)

26. Mučić, G.: On a construction of certain classes of cuspidal automorphic forms via Poincaré series. Math. Ann. 343(1), 207–227 (2009)

27. Mučić, G.: On the cuspidal modular forms for the Fuchsian groups of the first kind. J. Number Theory 130(7), 1488–1511 (2010)

28. Mučić, G.: On the non-vanishing of certain modular forms. Int. J. Number Theory 7(2), 351–370 (2011)

29. Mučić, G.: On the analytic continuation and non-vanishing of L-functions. Int. J. Number Theory 8(8), 1831–1854 (2012)

30. Petersson, H.: Einheitliche Begründung der Vollständigkeitssätze für die Poincaréschen Reihen von reeller Dimension bei beliebigen Grenzkreisgruppen von erster Art. Abh. Math. Sem. Hansischen Univ. 14, 22–60 (1941)

31. Poincaré, H.: Mémoire sur les fonctions fuchsiennes. Acta Math. 1(1), 193–294 (1882)

32. Rankin, R. A.: The vanishing of Poincaré series. Proc. Edinb. Math. Soc. (2) 23(2), 151–161 (1980)

33. Saber, H., Sebbar, A.: On the existence of vector-valued automorphic forms. Kyushu J. Math. 71(2), 271–285 (2017)

34. Selberg, A.: On the estimation of Fourier coefficients of modular forms. Theory of Numbers. Proc. Sympos. Pure Math. VIII, Providence, RI, Amer. Math. Soc., 1–15 (1965)

35. Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. 9(1), 237–302 (1996)

36. Žunar, S.: On Poincaré series of half-integral weight, Glas. Mat. Ser. III 53(2), 239–264 (2018)

37. Žunar, S.: On the non-vanishing of Poincaré series on the metaplectic group. Manuscripta Math. 158(1–2), 1–19 (2019)

38. Žunar, S.: On the non-vanishing of L-functions associated to cusp forms of half-integral weight. Ramanujan J. 51(3), 455–477 (2020)