SEARCHING THROUGH THE REALS

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Abstract. It is a commonplace to say that one can search through the natural numbers, by which is meant the following: For a property, decidable in finite time and which is not false for all natural numbers, checking said property starting at zero, then for one, for two, and so on, one will eventually find a natural number which satisfies the property, assuming no resource bounds. By contrast, it seems one cannot search through the real numbers in any similarly ‘basic’ fashion: The reals are not countable, and their well-orders carry extreme logical strength compared to the basic notions involved in ‘searching through the natural numbers’. In this paper, we study two principles (PB) and (TB) from Nonstandard Analysis which essentially state that one can search through the reals. These principle are basic in that they involve only constructive objects of type zero and one, and the associated ‘search through the reals’ amounts to nothing more than a bounded search involving nonstandard numbers as upper bound, but independent of the choice of this number. We show that (PB) and (TB) are equivalent to known systems from the foundational program Reverse Mathematics, namely respectively the existence of the hyperjump and $\Delta^1_1$-comprehension. We also show that (PB) and (TB) exhibit remarkable similarity to, respectively, the Turing jump and recursive comprehension. In particular, we show that Nonstandard Analysis allows us to treat number quantifiers as ‘one-dimensional’ bounded searches, and set quantifiers as ‘two-dimensional’ bounded searches.

1. Introduction

1.1. Searching through the naturals and the reals. It is a commonplace to say that one can search through the natural numbers, by which is meant the following: For a property $Q(n)$, decidable in finite time and which is not false for all natural numbers, one successively checks if $Q(0), Q(1), Q(2), \ldots$ holds, and one will eventually find a natural number $n$ such that $Q(n)$, assuming no further resource bounds.

In fact, Kleene defines the class of partial recursive functions as those obtained via primitive recursion plus the axiom Unbounded search, and the latter exactly formalises the aforementioned informal description of ‘searching through the natural numbers’; We refer to [23, Def. 2.2, p. 10] for more details. Furthermore, the semi-constructive Markov’s principle has a similar interpretation (See [21, 1.11.5]).

In contrast to the case of the natural numbers, it seems one cannot search through the real numbers in any remotely ‘basic’ fashion: The reals are not countable, and the existence of a well-order requires the axiom of choice. Even fragments of the latter carry tremendous logical strength compared to the basic notions involved in ‘searching through the natural numbers’; See [22, Table 4] for a detailed overview of the strength of small fragments of the axiom of choice.

In this paper, we show that the framework of stratified Nonstandard Analysis (See [8]) allows one to ‘search through the reals’ in a rather basic fashion. In particular, we formulate a nonstandard principle (PB) which essentially states that

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one can search through the reals. The principle (PB) is basic in that it involves only constructive objects of type 0 and 1, and the ‘search through the reals’ amounts to nothing more than a bounded search through the natural numbers involving a nonstandard number as an upper bound. It should be noted that the bounded search is independent of the choice of the nonstandard number. We show that (PB) is equivalent to a known principle, namely the Suslin functional (See e.g. [3][4]). Similarly, we formulate an analogous principle (TB) and prove equivalence to $\Delta^1_1$-comprehension in functional form (See e.g. [22] I.11.8 for the latter).

As to the structure of this paper, we provide some more detailed motivation in Section 1.2. We introduce a suitable weak ‘base theory’ in Section 2 and recall known results. In Section 3.1, we formulate the principle (PB) and prove its equivalence to the Suslin functional ($S^2$) over our base theory. In Section 3.2, we obtain similar results for $\Delta^1_1$-comprehension and the principle (TB).

As to background information, ($S^2$) is the functional version of the strongest ‘Big Five’ system $\Pi_1^1$-CA$_0$ studied in the foundational program Reverse Mathematics. The principle $\Delta^1_1$-comprehension is also studied in the latter program. We refer to [3][11][21][22] for more details. We do point out the following quote by Simpson:

From the above it is clear that the five basic systems RCA$_0$, WKL$_0$, ACA$_0$, ATR$_0$, $\Pi_1^1$-CA$_0$ arise naturally from investigations of the Main Question. The proof that these systems are mathematically natural is provided by Reverse Mathematics. ([22] p. 43). Hence, the principle (PB) is mathematically natural due to its equivalence to ($S^2$), the functional version of $\Pi_1^1$-CA$_0$. Finally, we urge the reader to first consult Remark 5 so as to clear up a common misconception regarding Nelson’s approach to Nonstandard Analysis.

1.2. Motivation. In this section, we discuss the background of, and more detailed motivation for, the topic of this paper. We first study the notion of ‘searching through the naturals’ in Nonstandard Analysis. We only require very basic familiarity with Nelson’s internal set theory, also introduced in Section 2.

Firstly, we show that ‘searching through the naturals’ amounts to a bounded search in Nonstandard Analysis. To this end, recall that by Post’s theorem a computable set can be described by a $\Delta^0_1$-formula, and vice versa ([23] Theorem 2.2, p. 64]). Thus, consider the $\Delta^0_1$-formula, relative to ‘st’, given by:

$$\forall n, k \left( \exists x \right) \left( f(n, k) = 0 \leftrightarrow \forall m \left( g(n, m) \neq 0 \right) \right).$$

(1)

Now define $p(n, h, M)$ as $(\mu k \leq M) h(n, k) = 0$, if such exists and $M$ otherwise. Then it is easy to show that for any infinite number $M^0$:

$$\forall n, k \left( \exists x \right) \left( f(n, k) = 0 \leftrightarrow p(n, f, M) \leq p(n, g, M) \right).$$

(2)

Hence, to decide if a $\Delta^0_1$-formula (relative to ‘st’) holds, one need only perform a bounded search, where the upper bound is any nonstandard number.

Secondly, we show that relative to the Turing jump ‘searching through the naturals’ also amounts to an explicit bounded search in Nonstandard Analysis, in contrast to the Turing jump’s ‘oracle status’. To this end, consider the Turing jump functional:

$$\exists \varphi \left( \exists x \right) \left( \varphi \left( x \right) = 0 \leftrightarrow \varphi \left( x \right) = 0 \right),$$

($\exists^2$)

which by [5] Cor. 12 is equivalent over a version of EFA to

$$\forall f \left( \exists x \right) \left( f(x) \neq 0 \leftrightarrow \exists x \left( f(x) \neq 0 \right) \right).$$

($\Pi_1^0$-TRANS)
The latter is the Transfer principle from Nonstandard Analysis limited to $\Pi^0_1$-formulas. As it turns out, $\Pi^0_1$-TRANS provides a straightforward way to turn $\Pi^0_1$-formulas into bounded formulas: For standard $f^1$ possibly involving standard parameters and infinite $M^0$, $\Pi^0_1$-TRANS implies that

$$\forall x^0(f(x) \neq 0) \leftrightarrow \forall x^0(M(f(x) \neq 0)$$

(3)

Hence, to find a (standard) zero for standard $f^1$, one need only perform the bounded search $(\mu k \leq M)f(k) = 0$. Furthermore, the latter (resp. the right-hand side of (3)) is elementary computable (resp. decidable) in terms of $f$ and $M$, and involves only objects of type zero besides $f$. This explicit nature, and the similarity to a $\Pi^0_1$-formula, should be contrasted to the right-hand side of (3). In other words, the right-hand side of (3) is much less of a ‘black box’ than that of the ‘oracle’ (3).

In short, the two previous examples suggest that ‘searching through the naturals’ amounts to nothing more than a bounded search (involving an arbitrary nonstandard number) in Nonstandard Analysis. This search is ‘basic’ in that it is given by an explicit formula, and is closely connected to the original formula.

The aim of this paper is to show that a similarly basic ‘bounded search’ in Nonstandard Analysis allows us to ‘search through the real numbers’ using the algorithm (3) defined in Section 3.1. We follow Kohlenbach (11) in assuming that any sequence of type one can be viewed as a real using his ‘hat function’. Intuitively speaking, we shall establish that Nonstandard Analysis allows us to treat the bounded formula (equivalent to the $\Pi^0_1$-TRANS) as a similar ‘bounding result’ as in generalised to $\Pi^1_0$-formulas. In the same way as $\Pi^0_1$-TRANS is essential to (3), the principle $\Pi^1_1$-TRANS is essential in establishing (PB):

$$(\forall x^1)[(\forall x^0)(\exists x^0) f(\overline{x}) \neq 0] \leftrightarrow (\forall g^1)(\exists x^0) f(\overline{x}) = 0)$$

(II)

What is more, by [3 Cor. 15] and Theorem 11, $\Pi^1_1$-TRANS is equivalent to (PB), and to the Suslin functional, defined as follows:

$$(\exists S^2)(\forall f^1)[S(f) = 0] \leftrightarrow (\exists g^1)(\forall x^0)(f(\overline{x}) = 0)$$

(S2)

The Suslin functional is the ‘hyperjump’ functional and corresponds to $\Pi^1_1$-CA0, the strongest so-called Big Five system from Reverse Mathematics (See [22 VI]).

Inspired by the results regarding the Suslin functional, we obtain a similar bounding result (TB) for $\Delta^1_1$-comprehension using $\Delta^1_1$-TRANS, i.e. the Transfer principle limited to $\Delta^1_1$-formulas. In particular, we obtain a version of (2) for $\Delta^1_1$-formulas to underline the analogy between standard sets and nonstandard numbers.

As to methodology, inspired by the bounding result (3), we shall require that the bounded formula (equivalent to the $\Pi^1_1$ or $\Delta^1_1$-formula at hand) in (PB), (TB), and related principles, is basic, by which we mean that it satisfies the following:

(I) Only type 0 and constructive type 1 objects occur in the bounded formula.

(II) The syntactic structure of the bounded formula is similar to that of the original $\Pi^1_1$ or $\Delta^1_1$-formula.

With regard to condition (II), $\Pi^1_1$-formulas can be brought into the Kleene normal form (See [22 V.1.4]). The latter can be gleaned from the Suslin functional and we will directly work with this normal form. Furthermore, it is clear that the well-known practice of ‘coding sets of numbers as nonstandard numbers’ (See e.g. [13]) is not basic in our sense, as we deal with equivalent bounded formulas.
Finally, to obtain the aforementioned results, we need to adopt the richer framework of *stratified* Nonstandard Analysis developed by Hrbacek ([8–11]) and pioneered by Peráire ([17]). We briefly introduce this framework in Section 2.

### 2. Nonstandard Analysis

In this section, we define the system in which we shall prove the equivalences mentioned in the previous section. We first introduce Nelson’s *internal set theory* and a suitable subsystem $\text{RCA}_0^\Omega$ thereof in Section 2.1. We then introduce *stratified* Nonstandard Analysis and $\text{RCA}_0^{\Omega^+}$, a suitable extension of $\text{RCA}_0^\Omega$, in Section 2.2.

#### 2.1. Nelson’s syntactic Nonstandard Analysis.

In Nelson’s *internal set theory* ([16]), a syntactic approach to Nonstandard Analysis as opposed to Robinson’s semantic one ([18]), a new predicate ‘st($x$)’, read as ‘$x$ is standard’ is added to the language of ZFC. The notations $(\forall x)(\text{st}(x) \to \ldots)$ and $(\exists y)(\text{st}(y) \land \ldots)$. The three axioms *Idealization*, *Standard Part*, and *Transfer* govern the new predicate ‘st’ and give rise to a conservative extension of ZFC.

Nelson’s approach has been studied in the context of higher-type arithmetic in e.g. [2, 4, 5], and we single out one particular system, called $\text{RCA}_0^\Omega$. In two words, the system $\text{RCA}_0^\Omega$ is a conservative extension of Kohlenbach’s base theory $\text{RCA}_0^\omega$ from [14] with certain axioms from Nelson’s Internal Set Theory based on the approach from [4, 5]. This conservation result is proved in [5], while certain partial results are implicit in [4]. In turn, the system $\text{RCA}_0^\omega$ is a conservative extension of the base theory of Reverse Mathematics $\text{RCA}_0$ for the second-order language by [17, Prop. 3.1]. Following Nelson’s approach in arithmetic, we define $\text{RCA}_0^\omega$ as the system $\text{E-PRA}_0^{\omega^*} + \text{QF-AC}^{1,0} + \text{HAC}_{\text{int}} + I + \text{PF-TP}_{\omega}$.

On a technical note, the language $\text{RCA}_0^\Omega$ actually involves a predicate st$_\rho$ for every finite type $\rho$, but the subscript is always omitted.

#### Theorem 1.

The system $\text{RCA}_0^\Omega$ is a conservative extension of $\text{RCA}_0^\omega$. The system $\text{RCA}_0^\omega$ is a $\Pi^0_1$-conservative extension of PRA.

*Proof.* See [5, Cor. 9]. □

The conservation result for $\text{E-PRA}_0^{\omega^*} + \text{QF-AC}^{1,0}$ is trivial. Furthermore, omitting $\text{PF-TP}_\omega$, the theorem is implicit in [4, Cor. 7.6] as the proof of the latter goes through as long as EFA is available.

The following theorem of $\text{RCA}_0^\Omega$ is important. Note that the abbreviation ‘$M \in \Omega$’ for ‘$\neg \text{st}(M^0)$’ is used. The statement that (1) $\to$ (5) for all such standard functionals is abbreviated $\Omega$-CA. If for a standard functional $F$, the functional $F(\cdot, M)$ satisfies (1), we say the latter is $\Omega$-invariant.

#### Theorem 2.

In $\text{RCA}_0^\Omega$, we have for all standard $F^{(\sigma \times 0) \to 0}$ that

\[(\forall x^\sigma)(\forall M, N \in \Omega)[F(x, M) = F(x, N)] \quad (4)\]

\[\rightarrow (\exists \sigma^G^{\sigma \to 0})(\forall x^\sigma)(\forall N_0 \in \Omega)[G(x) =_0 F(x, N)]. \quad (5)\]

*Proof.* See [20, §2]. □
The ‘base theory’ $\text{RCA}_0^\Omega$ is quite useful in establishing equivalences, as is clear from the following theorem, which also establishes that the omission of parameters in $\text{PF-TP}_v$ is necessary (for obtaining a conservative extension as in Theorem 1).

**Theorem 3.** The system $\text{RCA}_0^\Omega$ proves $\Pi^1_1\text{-TRANS} \iff (\exists^2)$. Adding QF-AC$^{1,1}$, we obtain $(\exists^2) \iff \Pi^1_1\text{-TRANS}$. 

**Proof.** By [1, Cor. 12 and 15].

Finally, the following theorem establishes that restricting the Standard Part principle as in $\text{HAC}_{\text{int}}$ is necessary (for obtaining a conservative extension as in Theorem 1). Let $\text{WKL}$ be weak König’s lemma as in [22, IV] and let $(\text{STP})$ be

$$(\forall X^1) (\exists^0 Y^1)(\forall^0 x^0)(x \in X \iff x \in Y).$$

(\text{STP})

Note that $\Omega\text{-CA}$ for $\sigma = 1$ is a version of the Standard Part Principle $(\text{STP})$.

**Theorem 4.** In $\text{RCA}_0^\Omega + (\text{STP})$, we have $\text{WKL}$.

**Proof.** See [20, §5]. By way of a sketch, a standard binary tree with sequences of any standard length also contains a sequence of nonstandard length (by overspill or induction). Apply $(\text{STP})$ to the latter sequence to obtain a standard path through the tree, and hence $\text{WKL}^\text{st}$. Rewrite $\text{WKL}^\text{st}$ as its contraposition, sometimes called fan theorem, and apply QF-AC$^{1,1}$ relative to ‘st’ (which follows from $\text{HAC}_{\text{int}}$) to the antecedent. Drop all ‘st’ in the antecedent and consequent of the innermost implication, and apply $\text{PF-TP}_v$ to yield $\text{WKL}$. □

By [6, 13], $(\text{STP})$ actually yields a conservative extension of $\text{WKL}_0$ from [22, IV].

### 2.2. Stratified Nonstandard Analysis

The framework of Stratified Nonstandard Analysis (SINT) is a refinement of Nelson’s where the unary standardness predicate ‘st’ is replaced by the binary predicate ‘$x \subseteq y$’ read as ‘$x$ is standard relative to $y$’ and $x \subseteq 0$ is still read ‘$x$ is standard’ or ‘st$(x)$’. We denote $\neg(y \subseteq x)$ by $x \nsubseteq y$ and say that ‘$y$ is nonstandard relative to $x$’.

In the same way, extend the language of $\text{RCA}_0^\Omega$ with new predicates $\subseteq_{\rho,\tau}$, one for each pair of finite types. We will often omit the subscript as is common for the standardness predicate of $\text{RCA}_0^\Omega$. The axioms of $\text{RCA}_0^\Omega$ govern the predicate st$(x)$, whereas the following basic axioms govern $x \nsubseteq y$.

**Axiom 5** (BASIC).

(i) $(\forall x)[\text{st}(x) \iff x \subseteq 0]$.

(ii) $(\forall x)(0 \subseteq x \land x \subseteq x)$.

(iii) $(\forall x, y, z)[(x \subseteq y \land y \subseteq z) \rightarrow x \subseteq z]$.

(iv) $(\forall x)(\exists^0 y^0, z^0)(x \subseteq y \land x \subseteq z)$.

(v) $(\forall x^{\sigma \rightarrow \tau}, y^\sigma, z)(x, y \subseteq z \rightarrow x(y) \subseteq z)$.

The BASIC axioms are rather elementary and express the following facts:

(i) Being standard is the same as being standard relative to zero.

(ii) All objects are standard relative to themselves. Zero is the ‘least’ level of standardness.

(iii) Transitivity holds for ‘being standard relative to’.

(iv) Every level of standardness is inhabited by a number. There always exists a similarly inhabited higher level.

(v) Functional application preserves relative standardness.

Denote $\text{RCA}_0^\dagger$ as $\text{RCA}_0^\Omega + \text{BASIC}$ in the language extended by ‘$\subseteq$’. Clearly $\text{RCA}_0^\dagger$ is only a definitional extension of $\text{RCA}_0^\Omega$, i.e. the former is also a conservative extension.
of RCA₀ and PRA similar to Theorem [1] We also require the following Standard part principle, not stronger than [STP].

\[(\forall X^1, z)(\exists Y^1 \subseteq z)(\forall x^0 \subseteq z)(x \in Y \iff x \in X).\]  

(STP2)

We finish this section with an important remark about the internal framework.

**Remark 6.** Tennenbaum’s theorem ([12, §11.3]) ‘literally’ states that any non-standard model of PA is not computable. What is meant is that for a non-standard model \(\mathcal{M}\) of PA, the operations \(+_\mathcal{M}\) and \(\times_\mathcal{M}\) cannot be computably defined in terms of the operations \(+_\mathbb{N}\) and \(\times_\mathbb{N}\) of the standard model \(\mathbb{N}\) of PA.

While Tennenbaum’s theorem is of interest to the semantic approach to Nonstandard Analysis involving nonstandard models, RCA₀ is based on Nelson’s syntactic framework, and therefore Tennenbaum’s theorem does not apply: Any attempt at defining the (external) function ‘+’ limited to the standard numbers’ is an instance of illegal set formation, forbidden in Nelson’s internal framework ([16, p. 1165]).

To be absolutely clear, lest we be misunderstood, Nelson’s internal set theory IST forbids the formation of external sets \(\{x \in A : st(x)\}\) and functions \(f(x)\) limited to standard \(x\). Therefore, any appeal to Tennenbaum’s theorem to claim the ‘non-computable’ nature of \(+\) and \(\times\) from RCA₀ is blocked, for the simple reason that the functions ‘+’ and \(\times\) limited to the standard numbers’ simply do not exist. On a related note, we recall Nelson’s dictum from [16, p. 1166] as follows:

*Every specific object of conventional mathematics is a standard set.*

It remains unchanged in the new theory [IST].

In other words, the operations ‘+’ and ‘\(\times\)’, but equally so primitive recursion, in (subsystems of) IST, are exactly the same familiar operations we know from (subsystems of) ZFC. Since the latter is a first-order system, we however cannot exclude the presence of nonstandard objects, and internal set theory just makes this explicit, i.e. IST turns a supposed bug into a feature.

### 3. Main results

#### 3.1. Stratified bounding and the Suslin functional.**

In this section, we formulate the bounding principle (PB) and prove its equivalence to the Suslin functional. We also prove that (PB) gives rise to the algorithm (3§) for finding witnesses to \(\Sigma^1\)-formulas relative to the standard world.

First of all, we prove some ‘relative’ versions of the Transfer principle.

**Theorem 7.** In RCA₀, \(\Pi^1_1\)-TRANS is equivalent to

\[(\forall z)(\forall f^1 \subseteq z)[(\forall x^0 \subseteq z)f(x) \neq 0 \rightarrow (\forall x)f(x) \neq 0],\]

and \(\Pi^1_1\)-TRANS is equivalent to

\[(\forall z)(\forall f^1 \subseteq z)[(\forall g^1 \subseteq z)(\exists x^0)f(\neg x) \neq 0 \leftrightarrow (\forall g^1)(\exists x^0)f(\neg x) \neq 0] \]

**Proof.** Clearly, the reverse implications follow from BASIC and taking \(z = 0\). By [5 Cor. 12], \(\Pi^1_1\)-TRANS is equivalent to the following sentence:

\[(\exists \mu^2)(\forall f^1)[(\exists x^0)f(x) = 0 \rightarrow f(\mu(f)) = 0].\]  

(\(\mu^2\))

Since \((\mu^2)\) does not involve parameters, we may apply (the contraposition of) PF-TP₂ to \((\mu^2)\), and hence assume that \(\mu\) as in \((\mu^2)\) is standard. Thus, we have \(\mu \subseteq z\) for any \(z\) by axiom [31] in BASIC. By axiom [3] in the latter, for any \(f \subseteq z\), we have \(\mu(f) \subseteq z\). By the definition of \((\mu^2)\), if \((\exists x^0)f(x) = 0\), then \((\exists x^0 \subseteq z)f(x) = 0\),
which is what we needed to prove for $\Pi^0_1$-TRANS. One proceeds in exactly the same way for $\Pi^1_1$-TRANS, as the latter is equivalent to
\[(\exists \nu^{\rightarrow 1})(\forall f^1)(\exists g^0)(f(f(x)) = 0) \rightarrow (\forall x^0)(f(x)) = 0),\] by [5 Cor. 15], and we are done.

The functional $\mu$ is called ($\mu_1$) in [4], and is equivalent to $(S^2)$ assuming QF-AC$^1$. It is clear that $(\mu_1)$ and $(S^2)$ do not satisfy either of the conditions [1] and [11].

Secondly, we consider an important consequence of the idealization axiom 1.

Theorem 8. In RCA$^0_0$, there is a (nonstandard) function $h_0$ which dominates all standard $f^1$ everywhere, i.e. $(\forall^* f^1)(\forall^0 n)f(n) \leq h_0(n))$ or $(\forall^* f^1)(f \leq_1 h_0).

Proof. Note that the following formula is trivially true:
\[(\forall^* g^1)(\exists h^1)(\forall k^1 \in g^1)(\forall x^0)(k(x) \leq_0 h(x))\]
where $1^*$ is the type of sequences (with length of type 0) of type 1 objects. The formula in square brackets in (7) is internal and applying idealization 1 yields:
\[(\exists h^1)(\exists^* g^1)(\forall x^0)(g(x) \leq_0 h(x))\]
The function $h$ is as required for the theorem.

Remark 9 (Constructive idealization). We shall refer to the function $h_0$ from Theorem 8 as ‘constructive’ for the following reason: The proof of Theorem 8 trivially goes though in the system $H$ from [4 §5.2], which is a conservative extension of Heyting arithmetic with among other axioms 1 (See [4 Cor. 5.6]). Hence, the existence of $h_0$ is constructively acceptable, as the axiom 1 included in the system $H$ results in a conservative extension of Heyting arithmetic (in the original language). Heyting arithmetic in all finite types is only a small fragment of the usual systems providing a foundation for Bishop’s Constructive Analysis (7).

Hence, we may refer to the function $h_0$ as ‘constructive (in the sense of Bishop)’.

Thirdly, we formulate our long-awaited bounding principle (PB). The function $h_0$ therein is intended to be the one from the previous theorem. Recall also the definition of $\nu^0 \leq_0 \sigma^0$ as $|\sigma| = |\nu| \land (\forall i < |\sigma|)(\nu(i) \leq_0 \sigma(i))$.

Principle 10 (PB). There is $h_0 \oplus 0$ such that for $f^1 \subset 0$, $h \geq_1 h_0$ and $M \supset h$,
\[(\forall^* y^1)(\exists^1 x^0)(f(x) \neq 0) \leftrightarrow (\forall^* y^0 \leq_0 \tau M)(\exists x^0 \leq M)(f(x) \neq 0).\]

Fourth, we prove the following theorem. By [19 Theorem 2.2] and [13], the base theory is weak, i.e. certainly not stronger than ACA$^0_0$ and WKL$^0$ respectively.

Theorem 11. In RCA$^1_0 + (STP2)$, we have $(\mu_1) \leftrightarrow (PB) \leftrightarrow \Pi^1_1$-TRANS.
In RCA$^1_0 + (STP2) +$ QF-AC$^1$-1, we have $(S^2) \leftrightarrow (PB) \leftrightarrow \Pi^1_1$-TRANS.

Proof. We establish $\Pi^1_1$-TRANS $\leftrightarrow (PB)$ in RCA$^1_0 + (STP2)$ using Theorem 7 and the theorem is then immediate by [5 Cor. 15].

In order to prove $\Pi^1_1$-TRANS $\rightarrow (PB)$, consider $h_0$ from Theorem 8 assume $(\forall^* y^1)(\exists^1 x^0)(f(x) \neq 0)$ for $f \subset 0$ and apply $\Pi^1_1$-TRANS to obtain $(\forall y^1 \leq h_0)(\exists x^0 \leq h_0)(f(x) \neq 0)$. Now consider $g_0 \leq h_0$ (which may or may not satisfy $g_0 \subset h_0$) and apply $\Pi^1_1$-TRANS to obtain $g_1 \subset h_0$ such that:
\[(\forall x^0 \leq h_0)g_0(x) = g_1(x).\]
Since we already proved $(\forall g^1 \subseteq h_0)(\exists x^0 \subseteq h_0)(f(\overline{g}x) \neq 0)$, we obtain $(\exists x_0 \subseteq h_0)(f(\overline{g}x_0) \neq 0)$. By (9), we also get $(\exists x_0 \subseteq h_0)(f(\overline{g}x_0) \neq 0)$, as $\overline{g}z = 0 \overline{g}z \subseteq h_0$ for any $z \subseteq h_0$. Hence, we have proved that

$$(\forall g^1 \leq h_1 h_0)(\exists x^0 \subseteq h_0)(f(\overline{g}x_0) \neq 0),$$

and for $h_0 \subseteq M^0$, we obtain:

$$(\forall g^1 \leq h_0)(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0).$$

By definition, (10) now yields:

$$(\forall g^0 \leq_0 \overline{h}_0 M)(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0).$$

Indeed, for $g^0 \leq_0 \overline{h}_0 M$, define $l^1 := g \ast 00 \ldots$ and apply (10) in light of $l \leq_1 h_0$.

Now repeat the above steps for any $h \geq_1 h_0$ instead of $h_0$ to obtain the forward implication in (8).

Now assume the formula (II) for $M \supseteq h_0$ and $h_0$ as in the first paragraph of this proof, and consider standard $g^3$. By the definition of $h_0$, we have $g \leq_1 h_0$, implying $\overline{g}M \subseteq_0 \overline{h}_0 M$. By assumption, we have $(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0)$, which immediately yields $(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0)$ and also $(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0)$.

Applying $\Pi^1_1$-TRANS yields $(\exists x^0 \subseteq 00)(f(\overline{g}x_0) \neq 0)$, and we have proved $(\forall g^1)(\exists x^0 \subseteq 00)(f(\overline{g}x_0) \neq 0)$. The equivalence (9) now follows.

For the implication (PB) → $\Pi^1_1$-TRANS, note that (PB) implies $\Pi^0_1$-TRANS, which immediately yields the reverse direction in $\Pi^1_1$-TRANS. To prove the remaining implication in the latter, assume $(\forall g^1)(\exists x^0 \subseteq 00)(f(\overline{g}x_0) \neq 0)$ for standard $f$, and let $h_0$ be the function from (PB). Fix $g^1$ and define $h^1$ by $h(n) := \max(h_0(n), g_1(n))$.

Clearly, $h \geq_1 h_0$, yielding $(\forall g^0 \leq_0 \overline{h}_0 M)(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0)$ by (PB) for $M \supseteq h$. Hence, for $g^0_1 = g_1 M$, we obtain $(\exists x^0 \subseteq M)(f(\overline{g}x_0) \neq 0)$, implying $(\exists x^0)(f(\overline{g}x) \neq 0)$. Then $(\forall g^1)(\exists x^0 \subseteq 00)(f(\overline{g}x) \neq 0)$ by $\Pi^0_1$-TRANS and the forward implication in $\Pi^1_1$-TRANS also holds. □

Comparing (3) and (8), we note that Nonstandard Analysis allows us to treat type zero quantifiers as ‘one-dimensional’ bounded searches, and type one quantifiers as ‘two-dimensional’ bounded searches.

It is then a natural question, originally due to Dag Normann, if (8) allows one to find a standard $g$ such that $(\forall x^0 \subseteq M)(f(\overline{g}x) = 0)$, assuming such exists? Now, the formula (8) from (PB) suggests the following algorithm to solve this question.

As above, we fix $h_0$ as in Theorem (8) and $M \supseteq h_0$.

**Algorithm 12 (A).** Check in lexicographical order starting with $\sigma = 00 \ldots 00$ the formula $A(\sigma) \equiv (\forall x^0 \subseteq M)(f(\overline{g}x) = 0)$ for all $\sigma$ of length $M$ and bounded above by $\overline{h}_0 M$. Output the lexicographically first $\sigma_0$ satisfying $A(\sigma_0)$ if such there is, and $M_0 \ldots M_0$ otherwise, where $M_0 = \max_{\leq_M} h_0(i)$.

Let us denote by $A(f)$ the output of the previous algorithm on input $f^1$. Note that there is no a priori reason why $A(\sigma)$ even outputs a sequence with a standard part, i.e. such that $(\forall x^0 \subseteq M)(\exists f)(\overline{g}x \subseteq M)$, as the lexicographical order places lots of sequences without a standard part before those with one.

The previous observation notwithstanding, the following corollary shows that for standard $f^1$ and assuming (PB), the algorithm (A) always outputs a witness to $(\exists x^0 \subseteq 00)(f(\overline{g}x) = 0)$ if and only if the latter formula holds. In other words, (A) finds the required standard witness if such exists.

\footnote{For instance, the sequence $0M00 \ldots 000$ comes before $n00 \ldots 00$ for any infinite $M^0$ and standard $n^0$ in the lexicographical order. In general, there does not seem to be an internal ordering in which all the type zero sequences with standard part come first.}
Corollary 13. In $\text{RCA}_0^1 + \text{STP}_2 + (\text{PB})$, we have

\[(\forall^x f^1)[\forall^x n^0] \text{st}(\mathfrak{A}(f)(n)) \leftrightarrow (\exists^x g^1)(\forall^x x^0) f(\overline{x}) = 0]. \tag{12}\]

Proof. The forward direction in (12) is immediate due to $\text{STP}$. For the reverse direction, by the theorem, we may use $\Pi^1_1$-TRANS: Now suppose that for some standard $f_0$, we have the right-hand side of (12) and the sequence $\mathfrak{A}(f_0)$ is such that $(\exists^x n^0) \neg \text{st}(\mathfrak{A}(f_0)(n_0 + 1))$. For now, we assume that $n_0$ is the least such number, and later prove that such a least number indeed exists using $\text{STP}$. By our assumption, the sequence $\mathfrak{A}(f_0)n_0$ is standard and we have the following formula:

\[(\exists \sigma^n_0 \leq M \exists h_0 M)(\forall x^0 \leq M) f_0(\overline{x}) = 0 \land \overline{x} n_0 \leq 0 \land \mathfrak{A}(f_0)n_0]. \tag{13}\]

Since $(\forall n^0 \leq h_0)(\sigma(n) \leq h_0(n))$, we can apply $\text{STP}_2$ for $z = h_0$ and obtain $g \subseteq h_0$ which is the standard part of $\sigma_1$ in the previous formula. Hence, (13) yields:

\[\exists g^1 \subseteq h_0)(\forall x^0 \leq h_0) f_0(\overline{x}) = 0 \land \overline{x} n_0 \leq 0 \land \mathfrak{A}(f_0)n_0]. \tag{14}\]

By (a trivial variation of) Theorem 7, the previous formula and $\Pi^1_1$-TRANS yield:

\[(\forall^x g^1)(\forall x^0) f_0(\overline{x}) = 0 \land \overline{x} n_0 \leq 0 \land \mathfrak{A}(f_0)n_0]. \tag{15}\]

Note that we are allowed to apply Transfer as $\mathfrak{A}(f_0)n_0$ is a standard parameter in (13) by the axioms $\text{T}_a'$ of $\text{RCA}_0^1$ (See [3] Def. 2.2 and Lemma 2.8). Alternatively, use $\text{STP}$ to obtain the standard part of $\mathfrak{A}(f_0)n_0 * 00 \ldots$ and replace the latter by the former in (13).

However, for the standard $g^1$ as in (15), we also have $g_1(n_0 + 1) < \mathfrak{A}(f_0)(n_0 + 1)$, as the former number is finite, and the latter infinite. Hence, it is clear that $\overline{y} n M$ comes before $\mathfrak{A}(f_0)$ in the lexicographical ordering, and $(\mathfrak{A})$ should have output $\overline{y} n M$ by (15). This contradiction proves the theorem, modulo our assumption on $n_0$. To prove the latter assumption, we will prove the following version of external induction using $\text{STP}$:

\[(\forall f^1)[\text{st}(f(0)) \land (\forall^x m)(\text{st}(f(m)) \rightarrow \text{st}(f(m + 1))) \rightarrow (\forall^x n)\text{st}(f(n))]. \tag{16}\]

To this end, fix $f^1$ satisfying the antecedent of (16) and define the set $X^1$ as $\{(n, f(n)) : n = n\}$ (using the well-known (standard) coding of pairs). Using $\text{STP}$, let $Y^1$ be the standard part of $X$, and let $Z$ be the projection of $Y$ onto the first coordinate. Then $0 \in Y$ and $(\forall^x m^n)[m \in Y \rightarrow m + 1 \in Y]$ by the antecedent of (16). By (quantifier-free) induction, we have $(\forall^x n^0)(n \in Y)$, implying the consequent of (16). Finally, note that our assumption on $n_0$ from the previous paragraph of this proof, follows from the contrapositive of (16).

In light of the above, it is straightforward to formulate a version of (12) equivalent to (PB) using $h \geq h_0$ and $M \supseteq h$ as in the latter.

In conclusion, we have proved that (PB) gives rise to the algorithm $(\mathfrak{A})$ for finding witnesses to $\Sigma^1_1$-formulas relative to the standard world. In particular, Corollary 13 establishes that (PB) expresses that one can search through the reals. We end this section with a highly relevant note on generalisations of (PB).

Remark 14 (Generalisations). Above, we proved (PB) for standard functions $f^1$, but the proof of Theorem 11 easily generalises to any $f$ using the following ‘relativised idealisation’:

\[(\forall z)[(\forall x^0) \subseteq z](\exists y^n)(\forall x^0 \in x)\phi(x', y) \rightarrow (\exists y^n)(\forall x^0 \subseteq z)\phi(x, y)]. \tag{17}\]

Indeed, in the same way as in the proof of Theorem 8 the axiom (17) easily yields a function $h^1 \supseteq z$ which dominates all functions $f^1 \subseteq z$ everywhere, i.e. $(\forall f^1 \subseteq z)(\forall n)^0 f(n) \leq h(n))$. It is then easy to obtain a version of (PB) for ‘$\subseteq 0$’ replaced by ‘$\subseteq z$’ following the proof of Theorem 11. This version would be as follows:
Principle 15 (rPB). For all $z$, there is $h_0$ s.t. $z \ni h_0$ such that for $f^1 \subseteq z$, $M \ni h \geq_1 h_0$:

$$(\forall q^1 \subseteq z)(\exists x^0 \subseteq z) [f^1(\overline{x}) \neq 0] \iff (\forall q^0 \leq_0 \overline{h}M)(\exists x^0 \leq M) [f^1(\overline{x}) \neq 0].$$  \hspace{1cm} (17)

A generalised version of the algorithm (3) can now be read off from (rPB). Finally, in the same way as for [2] Cor. 7.8, one proves that (h) yields a conservative extension of Peano Arithmetic (and the same for fragments at least EFA). These generalisations are straightforward and we therefore do not go into details.

3.2. Stratified bounding and $\Delta^1_1$-CA$_0$. In this section, we establish a bounding result like (PB) for the system $\Delta^1_1$-CA$_0$ (See [22] 1.11.8)). In light of the similarities between the nonstandard treatment of $\Pi^0_1$ and $\Delta^0_1$-formulas in Section 1.2 such a result is expected. We use the abbreviation $D(f,g)$ for the following formula, expressing that $f,g$ give rise to a $\Delta^1_1$-formula:

$$(\exists h^1)(\forall x^0)[f(\overline{x}) = 0] \iff (\forall k^1)(\exists y^0)[g(\overline{y}) \neq 0].$$

First of all, consider the following comprehension and transfer principle:

$$(\forall f^1,g^1) D(f,g) \rightarrow [D(f,g) \leftrightarrow (\exists h^1)(\forall x^0)(f(\overline{x}) = 0)], \hspace{1cm} (\Delta^1_1\text{-TRANS})$$

$$(\exists \Phi^0(\overline{1}x^0)(\forall f^1,g^1)(D(f,g) \rightarrow [\Phi(f,g) = 0 \leftrightarrow (\exists h^1)(\forall x^0)(f(\overline{x}) = 0)]). \hspace{1cm} (D_2)$$

Clearly, $D_2$ is the functional version of $\Delta^1_1$-CA, and we have the following theorem.

Theorem 16. In RCA$_0^\Omega + [\text{STP}] + \text{QF-AC}^{1,1}$, we have $\Delta^1_1\text{-TRANS} \leftrightarrow (D_2)$.

Proof. Clearly, both principles imply $\Pi^0_1\text{-TRANS}$ and (32). Assume $(D_2)$ and drop the reverse implication in the consequent. In the resulting formula, bring all type one quantifiers to the front, which results in a formula of the form $(\exists \Phi)(\psi(\Phi)$ where $\psi \in \Pi^0_1$. The existential set-quantifiers in $\psi(\Phi)$ originate from the $\exists h^1$ in the consequent of $(D_2)$, and the reverse implication in $D(f,g)$. The universal quantifiers in $\psi(\Phi)$ originate from $(\forall f^1,g^1)$ and from the forward implication in $D(f,g)$. Now use $(32)$ to remove arithmetical quantifiers in $\psi(\Phi)$ and apply QF-AC$^{1,1}$ to obtain $\exists \Phi$ witnessing the existential set-quantifiers in $\psi(\Phi)$. One of the components of $\exists \Phi$, say the first one, witnesses the existential quantifier which originated from the $(\exists h^1)$ quantifier in the consequent of $(D_2)$; We ignore the other components of $\exists \Phi$, and obtain the following, thanks to the definition of $D(f,g)$:

$$(\exists \Phi, \Xi)(\forall f^1,g^1)[D(f,g) \rightarrow [\Phi(f,g) = 0 \rightarrow (\forall x^0)(f(\overline{x}) = 0)]]. \hspace{1cm} (18)$$

Since the previous formula is parameter-free, we may assume $\Phi$ and $\Xi$ are standard by PF-TP$_\forall$. Hence, if $D(f,g)$ for standard $f^1,g^1$, $\Xi(f,g,f,g)$ is a standard witness for the left-hand side of $D(f,g)$, if this side holds, and $\Delta^1_1$-TRANS follows.

Now assume $\Delta^1_1$-TRANS and note that the latter and $\Pi^0_1\text{-TRANS}$ imply:

$$(\forall^*=f^1,g^1)(\exists^*=1)(D(f,g) \rightarrow [(\exists h^1)(\forall x^0)(f(\overline{x}) = 0) \rightarrow (\exists^* h^1 \leq^1 t^1)(\forall x^0)(f(\overline{x}) = 0)]].$$

Apply HAC$_\text{int}$ to obtain standard $\Psi^{(1\times 1)\rightarrow 1}$ such that $(\exists \Psi \in (f,g))$. Define $\Phi^{(1\times 1)\rightarrow 1}$ as follows: $\Phi(f,g)(n) := \max_i (\forall \Psi(f,g)(i)(n)).$ Clearly, we have for all standard $f^1,g^1$ that if $D(f,g)$ then

$$(\exists h^1)(\forall x^0)(f(\overline{x}) = 0) \rightarrow (\exists h^1 \leq^1 f(\overline{g})(\forall x^0)(f(\overline{x}) = 0)], \hspace{1cm} (19)$$

and the reverse implication is trivial. By [STP] and $(32)$, the consequent of $(19)$ is equivalent to $(\exists h^0 \leq_0 \overline{f}(\overline{g})M)(\forall x^0 \leq M)(f(\overline{x}) = 0)$ for any infinite $M$. Hence, with the same assumptions in place, we obtain:

$$(\exists h^1)(\forall x^0)(f(\overline{x}) = 0) \iff (\exists h^0 \leq_0 \overline{f}(\overline{g})M)(\forall x^0 \leq M)(f(\overline{x}) = 0).$$

Using $\Omega$-CA, we obtain the functional as in $(D_2)^\Pi$. The latter implies $(D_2)$ in the same way that $(S^2) \leftrightarrow (S^2)^\Pi$ in the proof of [2] Cor. 15.

$\Box$
We now prove a result similar to Theorem 11 for $\Delta^1_1$-TRANS.

**Corollary 17.** In RCA$_0^1$, $\Delta^1_1$-TRANS is equivalent to its relativised version:

$$(\forall z)(\forall f^1, g^1 \subseteq z)[D(f, g) \rightarrow [D(f, g) \leftrightarrow (\exists h^1 \subseteq z)((\forall x^0 \subseteq z)(f(\overline{f}x) = 0))]].$$

**Proof.** Immediate from (18). \qed

Now define the following versions of (PB) as follows:

**Principle 18 (SB).** There is $h_0 \supseteq 0$ such that for $f^1, g^1 \subseteq 0$ with $D(f, g)$, $l_1 \geq 1$, $h_0$, and $M^0 \supseteq l$, we have

$$(\forall^* k^1)(\exists^* x^0)(f(\overline{f}x) \neq 0) \leftrightarrow (\forall k^0 \leq_{0^*} 7M)(\exists x^0 \leq M)(f(\overline{f}x) \neq 0).$$

Let $P(k^1, h^1, M^0)$ be the lexicographically least sequence $\sigma^0 \leq_{0^*} 7M$ of length $M$ such that $([\sigma(\overline{x}) = 0])$, if such exists and $M_0 \ldots M_0$ (of length M) otherwise, where $M_0$ is the maximum of $h(i) + 1$ for $i \leq M$. The following principle should be compared to (2) in Section 3.2.

**Principle 19 (TB).** There is $h_0 \supseteq 0$ such that for all $f^1, g^1 \subseteq 0$ with $D(f, g)$, all $l_1 \geq 1$, $h_0$, and $M^0 \supseteq l$, we have

$$(\exists^* k^1)(\forall x^0>(f(\overline{f}x) = 0) \leftrightarrow P(f, l, M) <_{0^*} P(g, l, M).$$

**Theorem 20.** In RCA$_0^1$ + STP + QF-$\AC^{1,1}$, $(D_2) \Leftrightarrow (SB) \Leftrightarrow (TB) \Leftrightarrow \Delta^1_1$-TRANS.

**Proof.** The proof of Theorem 11 can easily be adapted to yield the equivalence to (SB). For the implication $\Delta^1_1$-TRANS $\rightarrow$ (TB), if the left-hand side of $D(f, g)$ holds, there is a standard such $h^1$ and $P(f, l, M)$ will be the initial segment of such a standard function. Since the right-hand side of $D(f, g)$ holds, we can prove $P(k^1, h^1, M^0)$ holds, if there exists $M^0 \ldots M^0$ (of length $M$) otherwise, where $M_0$ is the maximum of $h(i) + 1$ for $i \leq M$. The following principle should be compared to (2) in Section 3.2.

**Principle 19 (TB).** There is $h_0 \supseteq 0$ such that for all $f^1, g^1 \subseteq 0$ with $D(f, g)$, all $l_1 \geq 1$, $h_0$, and $M^0 \supseteq l$, we have

$$(\exists^* k^1)(\forall x^0>(f(\overline{f}x) = 0) \leftrightarrow P(f, l, M) <_{0^*} P(g, l, M).$$

Comparing (2) and (21), we note that stratified Nonstandard Analysis allows us to treat type zero quantifiers as ‘one-dimensional’ bounded searches, and type one quantifiers as ‘two-dimensional’ bounded searches. Furthermore, as discussed in Remark 14, the search can be adapted to allow any parameter. Note that the right-hand side of (21) now plays the role of the algorithm (3) from Section 3.1.

**3.3. Conclusion.** We now formulate the conclusion of this paper. In particular, we exhibit the similarity between the notions ‘computable’ (in the form $\Delta^0_1$) and ‘$\Delta^1_1$’, and ‘Turing jump’ and ‘hyperjump’.

First of all, in light of Theorem 11, the principle (PB) implies that for standard $f^1$, $h_0$ as in Theorem 11 and $M^0 \supseteq h_0$, we have

$$(\exists g^1)(\forall x^0)(f(\overline{f}x) = 0) \leftrightarrow (\exists g^1 \leq_{h_0} 7M)(\forall x^0 \leq M)(f(\overline{f}x) = 0).$$

Hence, if we know that $P(f, l, M)$ holds, then $\Delta^0_1$-TRANS follows in the same way as in the proof of Theorem 11. The implication (TB) $\rightarrow$ $\Delta^1_1$-TRANS follows in the same way as for (SB), i.e. as in the proof of Theorem 11 by noting that $P(f, l, M)$ in the right-hand side of (21) must be a sequence other than $M_0 \ldots M_0$, implying that $P(f, l, M) <_{0^*} P(g, l, M)$ follows. The reverse implication in (TB) follows similarly from $\Delta^1_1$-TRANS. The implication (TB) $\rightarrow$ $\Delta^1_1$-TRANS follows in the same way as for (SB), i.e. as in the proof of Theorem 11 by noting that $P(f, l, M)$ in the right-hand side of (21) must be a sequence other than $M_0 \ldots M_0$, implying that $P(f, l, M) <_{0^*} P(g, l, M)$ follows.

Hence, if we know that $P(f, l, M)$ holds, then $\Delta^0_1$-TRANS follows in the same way as in the proof of Theorem 11. The implication (TB) $\rightarrow$ $\Delta^1_1$-TRANS follows in the same way as for (SB), i.e. as in the proof of Theorem 11 by noting that $P(f, l, M)$ in the right-hand side of (21) must be a sequence other than $M_0 \ldots M_0$, implying that $P(f, l, M) <_{0^*} P(g, l, M)$ follows.

Comparing (2) and (21), we note that stratified Nonstandard Analysis allows us to treat type zero quantifiers as ‘one-dimensional’ bounded searches, and type one quantifiers as ‘two-dimensional’ bounded searches. Furthermore, as discussed in Remark 14, the search can be adapted to allow any parameter. Note that the right-hand side of (21) now plays the role of the algorithm (3) from Section 3.1.

**3.3. Conclusion.** We now formulate the conclusion of this paper. In particular, we exhibit the similarity between the notions ‘computable’ (in the form $\Delta^0_1$) and ‘$\Delta^1_1$’, and ‘Turing jump’ and ‘hyperjump’.

First of all, in light of Theorem 11, the principle (PB) implies that for standard $f^1$, $h_0$ as in Theorem 5 and $M^0 \supseteq h_0$, we have

$$(\exists g^1)(\forall x^0)(f(\overline{f}x) = 0) \leftrightarrow (\exists g^1 \leq_{h_0} 7M)(\forall x^0 \leq M)(f(\overline{f}x) = 0).$$

Hence, if we know that $P(f, l, M)$ holds, then $\Delta^0_1$-TRANS tells us that a ‘two-dimensional’ bounded search (including the bounds $h_0$ and $M \supseteq h_0$) will yield a sequence $g^0$ of length $M$ such that $P(f, l, M) = 0$. By Corollary 13 we can find such a sequence with a standard part using the algorithm (3). By STP and $\Pi^0_1$-TRANS, the output of (3) then has a unique standard part $g^1$, which is such that $P(g^1)(\overline{f}x) = 0)$. The search performed by the algorithm (3) is similar to that associated to (3), i.e. the Turing (resp. hyper-) jump corresponds to a one-
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dimensional bounded search. In both cases, an instance of the Transfer principle derives from the Turing- and hyperjump, and this principle is needed to certify that the associated search provides the correct output.

Secondly, by Theorem 20, a similar result is available for $\Delta^1_1$-formulas, analogous to the case of $\Delta^0_1$-formulas. Indeed, to verify if a $\Delta^0_1$-formula as in (1) (with ‘st’ removed) holds for some $n_0$, one checks, one by one, the following sequence:

\[
f(n_0,0) = 0, g(n_0,0) \neq 0, f(n_0,1) = 0, g(n_0,1) \neq 0, \ldots,
\]

which by definition yields a terminating search, and gives rise to $p(\cdot, M)$ in (2).

The latter is similar to $P(\cdot, l, M)$ from (21), and one can perform a search similar to (23) for a $\Delta^1_1$-formula as in $D(f,g)$ by checking $(\forall \mathbf{x} \leq M) f(\overline{\mathbf{x}}) = 0$ and $(\exists \mathbf{y} \leq M) g(\overline{\mathbf{x}}) = 0$ for $\rho^0$ equal to $00 \ldots 00, 00 \ldots 01, 00 \ldots 02$, and so forth, where all sequences have length $M$ (and are below $h_0 M$ from Theorem 8). Thus, verifying if a $\Delta^1_1$-formula holds corresponds to a double one- (resp two-) dimensional bounded search involving $p(\cdot, M)$ (resp. $P(\cdot, h_0, M)$).

Thirdly, we should stress that the right-hand sides of (8), (20), (21), and (22) do satisfy our conditions (I) and (II). Indeed, as pointed out above, the function $h_0$ from Theorem 8 is constructively acceptable, and there is a clear similarity between the Kleene normal form and the bounded formulas.

In conclusion, stratified Nonstandard Analysis allows us to treat type zero quantifiers as ‘one dimensional’ bounded searches, and type one quantifiers as ‘two dimensional’ bounded searches. In particular, in light of Nelson’s dictum from Remark 6, that every specific object of conventional mathematics is a standard set, it seems that (PB), (SB), and (TB) allow us to search through the reals for internal properties involving parameters from conventional mathematics, which is quite a rich world. By Remark 13, the search can be adapted to allow any parameter.

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