Edge Metric Dimension on Some Families of Tree

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Abstract. Metric Dimension as a graph invariant has various applications in real life. One of the metric dimension application is for the navigation system in transportation. In this Paper, we continue to develop the study of edge metric dimension. Let $G = (V, E)$ be a connected graph with $e \in V$ and $e = uw \in E$. The distance between the vertex $v$ and the edge $e$ is given by $d_G(e, v) = \min\{d(u, v), d(w, v)\}$. A vertex $w \in V$ distinguishes two edges $e_1, e_2 \in E$ if $d_G(w, e_1) \neq d_G(w, e_2)$. A set $S$ of vertices in a connected graph $G$ is an edge metric generator for $G$ if every two edges of $G$ are distinguished by some vertices of $S$. The edge metric dimension of $G$, denoted by $\dim_{\text{EQ}}(G)$, is the minimum cardinality of edge metric generator for $G$. As a main result, we provide some results of edge metric dimension on some families of tree graph, namely star graph, broom graph, double broom graph, and banana tree graph.

1. Introduction

A tree is an undirected graph in which any two vertices are connected by exactly one path. Every acyclic connected graph is a tree. Tree graph is a graph which does not contain any cycle. Metric Dimension as a graph invariant has various applications in real life. One of the metric dimension’s application is for the navigation system in autonomous car navigation system at PG. Semboro, East Java. Based on the observation, by observing the resolving set of each car stop station in PG. Semboro, author discovered the optimum coordinate of some car stop stations in PG. Semboro that can be applied in autonomous car navigation system.

The research about metric dimension has been carried out extensively in numerous types of metric dimension. Chartrand, et al. in [2] define the distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest path between these two vertices. Suppose that $W = \{s_1, s_2, \ldots, s_k\}$ is an ordered set of vertices of a nontrivial connected graph $G$. The metric representation of $v$ with respect to $W$ is the $k$-vector $r(v|W) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$. In this Paper, we continue to develop the study of edge metric dimension. This study is the development of metric dimension topic which was independently introduced by Slater [11], Hararry and Melter [7, 8]. For further definition and terminology of graph, see [1, 2, 3, 4, 5, 6]. Some results about edge metric dimension can be seen in [9, 10]. In this paper, we mainly discuss about edge metric dimension on some families of tree graph namely star graph, broom graph, double broom graph, and Banana Tree graph. The distance between the vertex $v$ and the edge $e$ is given by $d_G(e, v) = \min\{d(u, v), d(w, v)\}$, where $e = uw$. A vertex $w \in V$ distinguishes two
edges $e_1, e_2 \in E$ if $(d_G(w, e_1) \neq d_G(w, e_2))$. A set $S$ of vertices in a connected graph $G$ is an edge metric generator for $G$ if every two edges of $G$ are distinguished by some vertices of $S$. The edge metric dimension of $G$, denoted by $dim_E(G)$, is the minimum cardinality of edge metric generator for $G$.

2. Main Results

In this paper, we investigate the edge metric dimension of some families of tree graph namely star graph, broom graph, double broom graph and banana tree graph. We start this section with new result of edge metric dimension of the star graph in the following theorem.

**Theorem 2.1** Let $S_n$ be a star graph with $n \geq 4$. The edge metric dimension of $S_n$ is $dim_E(S_n) = n - 1$.

**Proof.** The star graph is a tree graph with $n + 1$ vertices. The vertex set $\{c\} \cup \{x_i; 1 \leq i \leq n\}$ and edge set $E(S_n) = \{cx_i; 1 \leq i \leq n\}$. Vertex $c$ is a central vertex and $x_i$ are pendant vertex. In order to prove the edge metric dimension of $S_n$, we prove that the lower bound and upper bound of edge metric dimension of star graph respectively are $dim_E(S_n) \leq n - 1$ and $dim_E(S_n) \geq n - 1$.

In this section we proposed the proof of lower and upper bound edge metric dimension on star graph. We prove that the upper bound of edge metric dimension of $S_n$ is $n - 1$, which is $dim_E(S_n) \leq n - 1$. Choose the edge metric generator $S = \{x_i, 1 \leq i \leq n - 1\}$ so the representation of all edges $e \in E(S_n)$ respect to $S$ can be seen in 1. Based on Table 1, we can see that all representation of $S_n$ with respect to $S$ are distinct, so $S$ is the edge metric generator of $S_n$ with the cardinality of $S$ namely $|S| = n - 1$. Thus, the upper bound of edge metric dimension of $S_n$ is $dim_E(S_n) \leq n - 1$.

Furthermore, we will prove that the lower bound of edge metric dimension of $S_n$ is $n - 1$, which is $dim_E(S_n) \geq n - 1$. Assume that $dim_E(S_n) < n - 1$. We take $|w| = n - 2$ where $w_i \in S$ such that there are $m - 2$ vertices in pendant as the element of edge metric generator in $S_n$. Hence, we have have two pendant vertex in $S_n$ which is not belong to edge metric generator $S$, $u, v \notin S$. Let we consider the distance of $uc$ and $vc$ to $w_i$. We will have $d(uc, w_1) = \min\{d(u, w_1), d(c, w_1)\} = \min\{d(u, c) + d(c, w_1), d(c, w_1)\} = d(c, w_1)$, $d(uc, w_2) = \min\{d(u, w_2), d(c, w_2)\} = \min\{d(u, c) + d(c, w_2), d(c, w_2)\} = d(c, w_2), \ldots, d(uc, w_i) = \min\{d(u, w_i), d(c, w_i)\} = \min\{d(u, c) + d(c, w_i), d(c, w_i)\} = d(c, w_i), 1 \leq i \leq n - 1$.

Then, the metric representation of two pendant vertex $u$ and $v$ respect to $S$ can be written as $r(uS) = (d(uc, w_1), d(uc, w_2), \ldots, d(uc, w_{n-2})) = (d(c, w_1), d(c, w_2), \ldots, d(c, w_{n-2}))$ and $r(vS) = (d(vc, w_1), d(vc, w_2), \ldots, d(vc, w_{n-2})) = (d(c, w_1), d(c, w_2), \ldots, d(c, w_{n-2}))$. It can clearly be seen that $r(uS) = r(vS)$. Thus, it is a contradiction. Furthermore, we should have $n - 1$ edge metric generator in star’s pendant. Because we have already proved that $dim_E(S_n) \leq n - 1$ and $dim_E(S_n) \geq n - 1$, we can conclude that $dim_E(S_n) = n - 1$.

**Theorem 2.2** Let $B_{n,m}$ be $(n, m)$–Broom graph with order $n \geq 1$ and $m \geq 2$, the edge metric dimension of $B_{n,m}$ is $dim_E(B_{n,m}) = m$.

**Proof.** The Broom graph is tree graph on $n + m$ vertices with vertex set $\{x_i; 1 \leq i \leq m\} \cup \{y_j; 1 \leq j \leq n\}$ and edge set $E(B_{n,m}) = \{x_iy_{i+1}; 1 \leq i \leq n - 1\} \cup \{x_1y_j; 1 \leq j \leq m\}$. In order to prove the edge metric dimension of $B_{n,m}$ is $dim_E(B_{n,m}) = m$, we will prove the lower bound of edge metric dimension is $m$ thus $dim_E(B_{n,m}) \geq m$ and the upper bound of edge metric dimension of
that there are three condition for placement of the vertices in edge metric generator of graph

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Based on Table 2, we can see that all edges representation of $B_{n,m}$ with respect to $S$ are distinct. So, $S$ is the edge metric generator of $B_{n,m}$ with the cardinality of edge metric generator is $|S| = m$. Thus, the upper bound of the edge metric dimension of $B_{n,m}$ is $m$, which is $\dim_E(B_{n,m}) \leq m$.

Furthermore, we will prove that the lower bound of edge metric dimension of $B_{n,m}$ is $m$, which is $\dim_E(B_{n,m}) \geq m$. Assume that $\dim_E(B_{m,n}) < m$. We take $|S| = m-1$ where $w_i \in S$ such that there are three condition for placement of the vertices in edge metric generator of graph $B_{n,m}$ as follows:

- The first condition, we put all of the vertices $w_i$ in the path such that we have $m-1$ vertices as the element of edge metric generator in path. If we have all of the vertices $w_i$ in the path, we will have two pendants vertex in $B_{n,m}$ which is not belong to edge resolving set $S$, $u, v \notin S$. Let we consider the distance of $ux_1$ and $vx_1$ respect to $w_i$. We will have $d(u, w_i) = \min \{\{d(u, v), 1\} + d(x_1, w_i)\} = \min \{d(u, x_1) + 0, d(x_1, w_i)\} = d(x_1, w_1), d(u, w_2) = \min \{d(u, x_2), d(x_1, w_1)\} = \min \{d(u, x_1) + 0, d(x_1, w_2)\} = d(x_1, w_2), ..., d(u, w_i) = \min \{d(u, w_i), d(x_1, w_i)\} = \min \{d(u, x_1) + 0, d(x_1, w_i)\} = d(x_1, w_i)$. In this section we proposed the proof of lower and upper bound edge metric dimension on Broom graph. First, We prove that the upper bound of edge metric dimension of $B_{n,m}$ is $m$, which is $\dim_E(B_{n,m}) \leq m$. Choose the element of edge metric generator $w_i = \{y_j, 1 \leq j \leq m-1\} \cup \{x_i, i = 2\}$ so the representation of all edges $e \in E(B_{m,n})$ respect to $S$ can be seen in the Table 2.

| $e$  | $r(e|S)$             | condition |
|------|----------------------|-----------|
| $cx_1$ | $(0, 1, \ldots, 1)$ |            |
| $cx_i$ | $(1, \ldots, 1, 0, 1, \ldots, 1)$ | $i \geq 2$ |
| $cx_n$ | $(1, \ldots, 1)$ | $n \geq 2$ |

**Table 1.** Edges Representation respect to $S$

Broom graph is $m$ which is $\dim_E(B_{n,m}) \leq m$.

**Figure 1.** Edge Metric Dimension of $B_{7,4}$
Based on the explanation above, we can see that the lower bound of edge metric dimension of $B_{n,m}$ is $m$, which is $\dim_E(B_{n,m}) \geq m$. Because we have already proved that $\dim_E(B_{n,m}) \leq m$ and $\dim_E(B_{n,m}) \geq m$, we can conclude that $\dim_E(B_{n,m}) = m$. The illustration of edge metric dimension of broom graph $B_{7,4}$ can be seen in figure 1.

**Theorem 2.3** Let $B_{tn}$ be Banana Tree graph with order $n \geq 2$, the edge metric dimension of $B_{tn}$ is $\dim_E(B_{tn}) = 2n - 2$.

**Proof.** The Banana Tree graph is tree graph with vertex set $\{x_i; 1 \leq i \leq 3\} \cup \{y_j; 1 \leq j \leq n\}$ and edge set $E(B_{tn}) = \{x_ix_{i+1}; 1 \leq i \leq 2\} \cup \{x_1y_1; 1 \leq j \leq n\} \cup \{x_3y_1^2; 1 \leq j \leq n\}$.
Choose the edge metric generator $S$ is the edge metric generator of $B_{t}$.

Based on Table 3, we can see all edges representation of $B_{t}$ are $2$-bounded, so the lower bound and upper bound of edge metric dimension of banana tree graph respectively are $2n - 2$. In order to prove the edge metric dimension of $B_{t}$ is $\dim E(B_{t}) = 2n - 2$, we will show the lower bound and upper bound of edge metric dimension of banana tree graph respectively are $2n - 2$, which is $\dim E(B_{t}) \geq 2n - 2$ and $\dim E(B_{t}) \leq 2n - 2$.

First, we prove that the upper bound of edge metric dimension of $B_{t}$ is $\dim E(B_{t}) \leq 2n - 2$. Choose the edge metric generator $S = \{y_{j}^{1},\ 1 \leq j \leq n - 1\} \cup \{y_{j}^{2},\ 1 \leq j \leq n - 1\}$, so the representation of all edges $e \in E(B_{t})$ respect to $S$ can be seen in Table 3.

Based on Table 3, we can see all edges representation of $B_{t}$ with respect to $S$ are distinct, so $S$ is the edge metric generator of $B_{t}$ with the cardinality of $S$ is $|S| = 2n - 2$. Thus, the upper bound of the edge metric dimension of $B_{t}$ is $\dim E(B_{t}) \leq 2n - 2$.

### Table 2. Edges Representation respect to $S$

| $e$   | $r(e|S)$         | condition |
|-------|------------------|-----------|
| $x_{i}y_{1}$ | $(1, 0, 1, \ldots, 1)_{m-k}$ | $m \geq 2$ |
| $x_{1}y_{j}$ | $(1, \ldots, 1, 0, 1, \ldots, 1)_{j-1}$ | $j \geq 2$ |
| $x_{1}x_{2}$ | $(1, 0, 1, \ldots, 1)_{n-3}$ | $n \geq 4$ |
| $x_{i}x_{i+1}$ | $(i - 2, i, \ldots, i)_{m-n}$ | $i \geq 2$ |
| $x_{1}y_{j}$ | $(1, \ldots, 1)_{j}$ | $j = m$ |
| $x_{2}x_{3}$ | $(0, 2, \ldots, 2)_{m-1}$ | $m \geq 2$ |

### Table 3. Edges Representation respect to $S$

| $e$   | $r(e|S)$         | condition |
|-------|------------------|-----------|
| $x_{1}y_{1}^{1}$ | $(0, 1, \ldots, 1, 3, \ldots, 3)_{m-j-1}$ | $m \geq 2$ |
| $x_{1}y_{j}^{1}$ | $(1, \ldots, 1, 0, 1, \ldots, 1, 3, \ldots, 3)_{j-1}$ | $j \geq 2$ |
| $x_{1}x_{2}$ | $(1, \ldots, 1, 2, \ldots, 2)_{m-1}$ | $m \geq 2$ |
| $x_{3}y_{1}^{2}$ | $(3, \ldots, 3, 0, \ldots, 0, 1, \ldots, 1)_{m-1}$ | $m \geq 2$ |
| $x_{3}y_{j}^{2}$ | $(3, \ldots, 3, 1, \ldots, 1, 0, 1, \ldots, 1)_{j-1}$ | $j \geq 2$ |
| $x_{2}x_{3}$ | $(2, \ldots, 2, 1, \ldots, 1)_{m-1}$ | $m \geq 2$ |
| $x_{1}y_{m}^{1}$ | $(1, \ldots, 1, 3, \ldots, 3)_{m-1}$ | $m \geq 2$ |
| $x_{3}y_{m}^{2}$ | $(3, \ldots, 3, 1, \ldots, 1)_{m-1}$ | $m \geq 2$ |
Next, we will prove that the lower bound of edge metric dimension of $B_{t_n}$ is $2n - 2$, which is $\text{dim}_E(B_{t_n}) \geq 2n - 2$. Assume that $\text{dim}_E(B_{t_n}) < 2n - 2$. We take $|u| = 2n - 3$ such that there are $2n - 3$ vertices as the element of edge metric generator in $B_{t_n}$. If we have $|S| = 2n - 3$, we will have two pendant’s vertex in $B_{t_n}$ which is not belong to edge metric generator $S$, $u, v \notin S$.

Let consider the distance of $ux_i$ and $vx_i$ to $w_i$. Vertex $x_i$ can be in $x_1$ or in $x_3$. We will have $d(ux_i, w_i) = \min\{d(u, w_i), d(x_i, w_i)\} = \min\{d(u, c) + d(x_i, w_i), d(c, w_i)\} = d(x_i, w_i)$, $d(ux_i, w_2) = \min\{d(u, w_2), d(x_i, w_2)\} = \min\{d(u, x_i) + d(x_i, w_2), d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(ux_i, w_j) = \min\{d(u, w_j), d(x_i, w_j)\} = \min\{d(u, x_i) + d(x_i, w_j), d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq j \leq |S|$ and $d(ux_i, w_1) = \min\{d(v, x_i), d(x_i, w_1)\} = d(x_i, w_1)$, $d(ux_i, w_2) = \min\{d(v, w_2), d(x_i, w_2)\} = \min\{d(v, x_i) + d(x_i, w_2), d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(ux_i, w_j) = \min\{d(v, w_j), d(x_i, w_j)\} = \min\{d(v, x_i) + d(x_i, w_j), d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq i \leq |S|$.

Then, the metric representation of two pendant edges $ux_i$ and $vx_i$ respect to $S$ can be written as $r(ux_i | S) = (d(ux_i, w_1), d(ux_i, w_2), \ldots, d(ux_i, w_{2n-2})) = (d(x_i, w_1), d(x_i, w_2), \ldots, d(x_i, w_{2n-3})))$ and $r(vx_i | S) = (d(vx_i, w_1), d(vx_i, w_2), \ldots, d(vx_i, w_{2n-3})) = (d(x_i, w_1), d(x_i, w_2), \ldots, d(x_i, w_{2n-3})$. It can clearly be seen that $r(ux_i | S) = r(vx_i | S)$. Thus, it is a contradiction. Furthermore, we have condition that lower bound of edge metric dimension of banana tree graph respectively are $2n - 2$, which is $\text{dim}_E(B_{t_n}) \leq 2n - 2$. Based on the observation above, we get $\text{dim}_E(B_{t_n}) \leq 2n - 2$ and $\text{dim}_E(B_{t_n}) \geq 2n - 2$. Thus, we can conclude that the edge metric dimension of $B_{t_n}$ is $\text{dim}_E(B_{t_n}) = 2n - 2$.

**Theorem 2.4** Let $B_{2n,m}$ be $(n, m)$- Double Broom graph with order $n \geq 2$ and $m \geq 2$, the edge metric dimension of $B_{2n,m}$ is $\text{dim}_E(B_{2n,m}) = 2m - 2$.

**Proof.** The double broom graph is tree graph on $n + 2m$ vertices with vertex set $\{x_i: 1 \leq i \leq n\} \cup \{y_{ij}: 1 \leq j \leq m\}$ and edge set $E(B_{2n,m}) = \{x_i, x_{i+1}: 1 \leq i \leq n - 1\} \cup \{x_i, y_{ij}: 1 \leq j \leq m\}$ and $\{x_i, y_{ij}: 1 \leq j \leq m\}$ in edge set $E(B_{2n,m}) = \{x_i, x_{i+1}: 1 \leq i \leq n - 1\} \cup \{x_i, y_{ij}: 1 \leq j \leq m\}$. In order to prove the edge metric dimension of $B_{2n,m}$ is $\text{dim}_E(B_{2n,m}) = 2m - 1$, we will prove the lower bound of edge metric dimension is $2m - 1$, which is $\text{dim}_E(B_{2n,m}) \leq 2m - 2$ and the upper bound of edge metric dimension of borem graph is $2m - 2$, which is $\text{dim}_E(B_{2n,m}) \geq 2m - 2$.

In this section we proposed the proof of lower and upper bound edge metric dimension on double broom graph. First, We prove that the upper bound of edge metric dimension of $B_{2n,m}$ is $2m - 2$, $\text{dim}_E(B_{2n,m}) \leq 2m - 2$. Choose the element of edge metric generator $S = \{y_{ij}: 1 \leq j \leq n - 1\} \cup \{y_{ij}: 1 \leq j \leq n - 1\}$, such that the representation of all edges $e \in E(B_{2n,m})$ with respect to $S$ can be seen in 4.

Based on the Table 2, we can see that all edges representation of $B_{2n,m}$ with respect to $S$ are distinct, so $S$ is the edge metric generator of $B_{2n,m}$ with the cardinality of $S$ is $|S| = 2m - 2$. Thus, the upper bound of the edge metric dimension of $B_{2n,m}$ is $2m - 2$ or in another hand we can write $\text{dim}_E(B_{2n,m}) \leq 2m - 2$. The illustration of edge metric dimension of double broom graph $B_{2m,n}$ can be seen in figure2.

Furthermore, we will prove that the lower bound of edge metric dimension of double broom graph is $2m - 1$, which is $\text{dim}_E(B_{2n,m}) \geq 2m - 1$. Assume that $\text{dim}_E(B_{2n,m}) < 2m - 2$. We take $|S| = 2m - 3$ where $w_i \in S$ such that there are three conditions for placement of the vertex in edge metric generator of graph $B_{2n,m}$ as follows:

- The first condition, we put all of the vertices $w_i$ in the path such that we have $2m - 3$ edges as the element of edge metric generator in path. If we have all of the vertices $w_i$ in the path, we will have two pendants in $B_{2n,m}$ either in first pendant or second pendant which is not belong to edge resolving set $S$, $u, v \notin S$. Let we consider the distance of $ux_i$ and $vx_i$ respect to $w_i$. $x_i$ can be put either in $x_1$ or $x_3$. We will have $d(ux_i, w_1) = \min\{d(u, w_1), d(x_i, w_1)\} = \min\{d(u, x_i) + d(x_i, w_1), d(x_i, w_1)\} = d(x_i, w_1)$, $d(ux_i, w_2) = \min\{d(u, w_2), d(x_i, w_2)\} = \min\{d(u, x_i) + d(x_i, w_2), d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(ux_i, w_j) = \min\{d(u, w_j), d(x_i, w_j)\} = \min\{d(u, x_i) + d(x_i, w_j), d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq i \leq |S|$ and $d(ux_i, w_1) = \min\{d(v, x_i), d(x_i, w_1)\} = d(x_i, w_1)$, $d(ux_i, w_2) = \min\{d(v, w_2), d(x_i, w_2)\} = \min\{d(v, x_i) + d(x_i, w_2), d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(ux_i, w_j) = \min\{d(v, w_j), d(x_i, w_j)\} = \min\{d(v, x_i) + d(x_i, w_j), d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq i \leq |S|$.
Table 4. Edges Representation respect to S

| e               | r(c|S)         | condition |
|-----------------|---------------|-----------|
| $x_iy_j$        | $(1, \ldots, 1, 0, 1, \ldots, 1, n, \ldots, n)$ | $j \geq 1$ |
| $x_iy_{i+1}$   | $(i, \ldots, i, n - i, \ldots, n - i)$ | $i \geq 1$ |
| $x_ny_j^2$     | $(n, \ldots, n, 1, \ldots, 1, 0, 1, \ldots, 1)$ | $j \geq 1$ |
| $x_{1y_m}$     | $(1, \ldots, 1, n, \ldots, n)$ | $m \geq 2$ |
| $x_{1y_m}$     | $(n, \ldots, n, 1, \ldots, 1)$ | $m \geq 2$ |

Figure 2. Edge Metric Dimension of $B_{2,4}$

$d(x_i, w_1), d(u\bar{x}_i, w_2) = \min \{d(u, w_2), d(x_i, w_2)\} = \min \{d(v, x_i) + d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(u\bar{x}_i, w_j) = \min \{d(u, w_j), \min \{d(u, x_i) + d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq j \leq |S|$ and $d(v\bar{x}_i, w_1) = \min \{d(v, w_1), d(x_i, w_1)\} = \min \{d(v, x_i) + d(x_i, w_1)\}$ and $d(v\bar{x}_i, w_2) = \min \{d(v, w_2), d(x_i, w_2)\} = \min \{d(v, x_i) + d(x_i, w_2)\}$, with $1 \leq j \leq |S|$. Then, the metric representation of two pendant edges $u\bar{x}_i$ and $v\bar{x}_i$ respect to $S$ can be written as $r(u\bar{x}_i|S) = (d(u\bar{x}_i, w_1), d(u\bar{x}_i, w_2), \ldots, d(u\bar{x}_i, w_{2m-3}))$ and $r(v\bar{x}_i|S) = (d(v\bar{x}_i, w_1), d(v\bar{x}_i, w_2), \ldots, d(v\bar{x}_i, w_{2m-3}))$. We can see that $r(u\bar{x}_i|S) = r(v\bar{x}_i|S)$. Thus, it is a contradiction.

- The second condition, we put the vertex $w_i$ both in the path and pendant. In this condition, the maximum edges we can take in pendant edges metric generator are $m - 2$ edges and the other edges $w_i$ are in path and in the other pendant such that we have $2m - 3$ edges as the element of edge metric generator. If we have these condition, we will have two pendant’s vertex $v\bar{x}_i$ in $B_{2,m}$ either in first pendant or second pendant which is not belong to edge resolving set $S$. Let us consider the distance of $u\bar{x}_i$ and $v\bar{x}_i$ respect to $w_i$. The $d(u\bar{x}_i, w_1) = \min \{d(u, w_1), d(x_i, w_1)\} = \min \{d(u, x_i) + d(x_i, w_1)\} = d(x_i, w_1)$, $d(u\bar{x}_i, w_2) = \min \{d(u, w_2), d(x_i, w_2)\} = \min \{d(u, x_i) + d(x_i, w_2)\} = d(x_i, w_2)$.

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Table 4. Edges Representation respect to S

| e               | r(c|S)         | condition |
|-----------------|---------------|-----------|
| $x_iy_j$        | $(1, \ldots, 1, 0, 1, \ldots, 1, n, \ldots, n)$ | $j \geq 1$ |
| $x_iy_{i+1}$   | $(i, \ldots, i, n - i, \ldots, n - i)$ | $i \geq 1$ |
| $x_ny_j^2$     | $(n, \ldots, n, 1, \ldots, 1, 0, 1, \ldots, 1)$ | $j \geq 1$ |
| $x_{1y_m}$     | $(1, \ldots, 1, n, \ldots, n)$ | $m \geq 2$ |
| $x_{1y_m}$     | $(n, \ldots, n, 1, \ldots, 1)$ | $m \geq 2$ |

Figure 2. Edge Metric Dimension of $B_{2,4}$

$d(x_i, w_1), d(u\bar{x}_i, w_2) = \min \{d(u, w_2), d(x_i, w_2)\} = \min \{d(v, x_i) + d(x_i, w_2)\} = d(x_i, w_2)$, $\ldots$, $d(u\bar{x}_i, w_j) = \min \{d(u, w_j), \min \{d(u, x_i) + d(x_i, w_j)\} = d(x_i, w_j)$, with $1 \leq j \leq |S|$ and $d(v\bar{x}_i, w_1) = \min \{d(v, w_1), d(x_i, w_1)\} = \min \{d(v, x_i) + d(x_i, w_1)\}$ and $d(v\bar{x}_i, w_2) = \min \{d(v, w_2), d(x_i, w_2)\} = \min \{d(v, x_i) + d(x_i, w_2)\}$, with $1 \leq j \leq |S|$. Then, the metric representation of two pendant edges $u\bar{x}_i$ and $v\bar{x}_i$ respect to $S$ can be written as $r(u\bar{x}_i|S) = (d(u\bar{x}_i, w_1), d(u\bar{x}_i, w_2), \ldots, d(u\bar{x}_i, w_{2m-3}))$ and $r(v\bar{x}_i|S) = (d(v\bar{x}_i, w_1), d(v\bar{x}_i, w_2), \ldots, d(v\bar{x}_i, w_{2m-3}))$. Thus, it is a contradiction.

- The second condition, we put the vertex $w_i$ both in the path and pendant. In this condition, the maximum edges we can take in pendant edge metric generator are $m - 2$ edges and the other edges $w_i$ are in path and in the other pendant such that we have $2m - 3$ edges as the element of edge metric generator. If we have these condition, we will have two pendant’s vertex $v\bar{x}_i$ in $B_{2,m}$ either in first pendant or second pendant which is not belong to edge resolving set $S$. Let us consider the distance of $u\bar{x}_i$ and $v\bar{x}_i$ respect to $w_i$. The $d(u\bar{x}_i, w_1) = \min \{d(u, w_1), d(x_i, w_1)\} = \min \{d(u, x_i) + d(x_i, w_1)\} = d(x_i, w_1)$, $d(u\bar{x}_i, w_2) = \min \{d(u, w_2), d(x_i, w_2)\} = \min \{d(u, x_i) + d(x_i, w_2)\} = d(x_i, w_2)$.
... , \ d(u_{x_i},w_j) = \min\{d(u,w_j),d(x_i,w_j)\} = \min\{d(u,x_i) + d(x_i,w_j),d(x_i,w_j)\} = d(x_i,w_j), with 1 \leq j \leq |S| and d(v_{x_i},w_j) = \min\{d(v,w_j),d(v_{x_i},w_j)\} = \min\{d(v,x_i) + d(x_i,w_j),d(x_i,w_j)\} = d(x_i,w_j), d(x_i,w_j) = d(x_i,w_j), \ \ \ ... , d(v_{x_i},w_j) = \min\{d(v,w_j),d(x_i,w_j)\} = \min\{d(v,x_i) + d(x_i,w_j),d(x_i,w_j)\} = d(x_i,w_j), with 1 \leq j \leq |S|. Then, the metric representation of two pendant edges \ ux_i and \ vx_i respect to \ S can be written as \ r(ux_i|S) = (d(ux_i,w_1),d(ux_i,w_2),...,d(ux_i,w_{2m-3})) and \ r(vx_i|S) = (d(vx_i,w_1),d(vx_i,w_2),...,d(vx_i,w_{2m-3})) = (d(x_i,w_1),d(x_i,w_2),...,d(x_i,w_{2m-3})). \ x_i can be put either in \ x_1 or \ x_n. It can clearly be seen that \ r(ux_i|S) = r(vx_i|S). Thus, it is contradiction.

- The third condition, we put all of the vertex \ w_i in the pendant such that we have \ 2m - 3 \ vertex as the element of edge metric generator in pendant. If we have all of the vertex \ w_i in the pendant, we will have one vertex in pendant and one vertex in path which is not belong to edge metric generator \ S. Let we consider the first pendant which adjacent to vertex \ x_1. Let we take \ uz \ not \ S where vertex \ u in pendant and vertex \ z in path (vertex \ z is incident with vertex \ x_1 as the central pendant). Let we consider the distance of \ uz_1 and \ xz_1 \ respect to \ w_1. We will have \ d(uz_1,w_1) = \min\{d(u,w_1),d(x_1,w_1)\} = \min\{d(u,x_1) + d(x_1,w_1),d(x_1,w_1)\} = d(x_1,w_1), \ d(uz_1,w_2) = \min\{d(u,w_2),d(x_1,w_2)\} = \min\{d(u,x_1) + d(x_1,w_2),d(x_1,w_2)\} = d(x_1,w_2), \ \ \ ... , d(uz_1,w_i) = \min\{d(u,w_i),d(x_1,w_i)\} = \min\{d(u,x_1) + d(x_1,w_i),d(x_1,w_i)\} = d(x_1,w_i), with 1 \leq i \leq |S| and d(xz_1,w_1) = \min\{d(x_1,w_1),d(x_1,w_1)\} = \min\{d(x_1,x_1) + d(x_1,w_1),d(x_1,w_1)\} = d(x_1,w_1), d(x_1,w_i) = d(x_1,w_1), d(xz_1,w_2) = \min\{d(z,w_2),d(x_1,w_2)\} = \min\{d(z,x_1) + d(x_1,w_2),d(x_1,w_2)\} = d(x_1,w_2), \ \ \ ... , d(xz_1,w_i) = \min\{d(z,w_i),d(x_1,w_i)\} = \min\{d(z,x_1) + d(x_1,w_i),d(x_1,w_i)\} = d(x_1,w_i), with 1 \leq i \leq |S|. Then, the metric representation of two pendant edges \ uz_1 \ and \ xz_1 \ respect to \ S can be written as \ r(uz_1|S) = (d(uz_1,w_1),d(uz_1,w_2),...,d(uz_1,w_{2m-3})) and \ r(xz_1|S) = (d(xz_1,w_1),d(xz_1,w_2),...,d(xz_1,w_{2m-3})) = (d(x_1,w_1),d(x_1,w_2),...,d(x_1,w_{n-1})). It can clearly be seen that \ r(uz_1|S) = r(xz_1|S). Thus, it is contradiction.

Based on the explanation above, it can be seen that the lower bound of edge metric dimension of \ B_{2m,n} in is \ \dim_E(B_{2m,n}) \geq 2m + 2. Because we have already proved that \ \dim_E(B_{2m,n}) \leq 2m + 2 and \ \dim_E(B_{2m,n}) \geq 2m + 2, we can conclude that \ \dim_E(B_{2m,n}) = 2m + 2.

3. Concluding Remarks
In this paper, we have investigated the exact values of edge metric dimension on some families of tree, namely star graph \ (S_n), broom graph \ (B_{n,n}), double broom graph \ (B_{2n,m}) and banana tree graph \ (B_{n,m}). Since this research is a new topic in the metric dimension study, many problems related to this topic have not been discovered yet. Consequently, we propose the following open problem.

Open Problem 1 Let \ G be any connected graph, determine the upper bound of edge metric dimension of any graph \ G.

Open Problem 2 Determine the exact value of edge metric dimension of another special graphs and its operations.

4. Acknowledgments
We gratefully acknowledge the support from "Hibah Penelitian Dosen Pemula"- University of Jember of year 2018.
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