Thermodynamic Bethe Ansatz for $N=1$ Supersymmetric Theories

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Abstract

We study a series of $N=1$ supersymmetric integrable particle theories in $d = 1 + 1$ dimensions. These theories are represented as integrable perturbations of specific $N=1$ superconformal field theories. Starting from the conjectured $S$-matrices for these theories, we develop the Thermodynamic Bethe Ansatz (TBA), where we use that the 2-particle $S$-matrices satisfy a free fermion condition. Our analysis proves a conjecture by E. Melzer, who proposed that these $N=1$ supersymmetric TBA systems are “folded” versions of $N=2$ supersymmetric TBA systems that were first studied by P. Fendley and K. Intriligator.
I. INTRODUCTION

Integrable quantum field theories in $d = 1 + 1$ dimensions are more than just “theoretical laboratories”. Apart from being very rich mathematical structures, they have proven to be of interest for a variety of problems in theoretical physics, ranging from string theory to statistical mechanics lattice models and problems in condensed matter theory. At the technical level, the magic of integrable field theories can be traced back to the existence of an infinite number of non-trivial conserved charges [1]. These imply the conservation of individual momenta in many-particle collisions and the factorizability of the $S$-matrix. The two-body $S$-matrix satisfies the Yang-Baxter equation, in addition to the usual properties of unitarity, crossing symmetry and analyticity. In many cases, these properties of the $S$-matrix are so restrictive that one may set up a bootstrap program and determine the exact $S$-matrix, up to the famous “CDD ambiguity” [2].

One special class of integrable field theories, that we will be studying in this paper, are the integrable massive perturbations of conformal field theories. Such theories are obtained by starting with a conformal field theory (CFT) and perturbing it by a relevant operator that preserves integrability. We will see the technicalities later. The ultraviolet (UV) properties of the CFT will not be affected by the perturbation, but the infrared (IR) behavior will change. The modified IR behavior can be conformal or massive (finite correlation length). In the latter case, the IR behavior is given by a massive particle theory with factorizable scattering and one may employ the above-mentioned bootstrap program to identify the exact $S$-matrix.

Once the $S$-matrix of an integrable field theory is known, one may study its thermodynamics by following a procedure called Thermodynamic Bethe Ansatz (TBA) [8,9]. In the context of applications in condensed matter problems (such as the scattering of edge currents in the fractional quantum Hall effect, [3]), the TBA results offer concrete predictions that can be tested in experiments or simulations. Another nice feature is that, among the quantities that can be computed by the TBA are the central charge and the scaling dimen-
sions of the UV limit of the theory. In this way, the TBA may be used as a check on the validity of a conjectured $S$-matrix for a perturbed CFT. The name of the TBA technique, which will be reviewed briefly in section III, is a little misleading, since the TBA is an exact non-perturbative procedure and not a trial-and-error method.

The theories that we study in this paper are examples of integrable field theories with supersymmetry. Adding supersymmetry is of course natural in the context of string theory applications, but even in the context of statistical mechanics lattice models supersymmetry is meaningful. For example, the tricritical Ising model in two dimensions \cite{10} is an example of a physical system that realizes superconformal symmetry. If we start from a superconformal field theory and choose a perturbation that preserves supersymmetry we end up with a massive theory with supersymmetry. In such a case supersymmetry may be added to the bootstrap ingredients, and one may try to identify exact $S$-matrices by exploiting both supersymmetry and factorizability.

In this paper we will study the TBA for a series \cite{4} of $N=1$ supersymmetric scattering theories with $2n$ particles, $n$ some positive integer, arranged in $n$ supermultiplets of one bosonic and one fermionic particle each, with masses labeled by an integer $k$ and given by

$$m_k = \frac{\sin(k\beta\pi)}{\sin(\beta\pi)},$$

(1.1)

where $\beta = 1/(2n+1)$ and $k = 1, 2, \cdots, n$. It has been conjectured \cite{4} that these scattering theories correspond to an integrable perturbation of specific superconformal field theories of central charge $c_n = -3n(4n+3)/(2n+2)$. Note that this is a series of non-unitary theories.

One important observation is that the $S$-matrix for these particular theories is non-diagonal, since we can have, for example, a scattering process of the form $b_i b_j \rightarrow f_i f_j$, where $b_i (f_i)$ is a boson (fermion) in the $i$-th supermultiplet. This makes the TBA analysis more difficult than in the case of diagonal scattering, where the thermodynamic limit can be studied via a set of coupled integro-differential equations. In the non-diagonal case we will have to diagonalize a transfer matrix in order to obtain a tractable system of equations.

In general, the transfer matrix that features in the TBA for non-diagonal scattering
theories is identical to the transfer matrix of a corresponding statistical mechanics lattice model, which is obtained by interpreting the two-body $S$-matrix elements as Boltzmann weights. In the case of the above $N=1$ supersymmetric theories, this lattice model is of eight-vertex type, and one might expect that the analysis would be quite involved. However, it has been observed that in a rather general setting the eight-vertex models that correspond to $N=1$ scattering matrices satisfy a technical condition called the free fermion condition. This case is no exception, and we shall use this special property to determine the eigenvalues of the transfer matrix and to complete the TBA analysis.

One of the motivations for us for deriving our $N=1$ supersymmetric TBA systems has been a proposal made by E. Melzer in [6]. He conjectured that the $N=1$ TBA systems can be derived by starting from scattering theories with $N=2$ supersymmetry, and twice as many particles, and applying a procedure called “folding” on these $N=2$ TBA system. These $N=2$ TBA systems were first studied by P. Fendley and K. Intriligator in [7]. Melzer checked that the “folded” TBA systems had all the right properties to be related to the $N=1$ superconformal theories that underly the $N=1$ $S$-matrices, but was unable to give a direct derivation starting from the $N=1$ $S$-matrices. In this paper we fill in this missing link and confirm Melzer’s conjecture by showing that the true $N=1$ TBA systems are indeed identical to the folded $N=2$ systems of [6].

In earlier work, the bosonic scattering theories that underly the $N=1$ and $N=2$ supersymmetric scattering theories that we just discussed have been related by a similar folding, both at the level of the TBA and at the level of the $S$-matrices [8,9]. Our results in this paper clearly suggest that it should be possible to find a “folding relation” between the $N=1$ and $N=2$ supersymmetric theories, directly at the level of the $S$-matrices. We will address this issue in a forthcoming publication [11].

This paper is organized as follows. In section II we review the formalism for supersymmetric particle theories with factorizable scattering and we introduce a specific series of $N=1$ supersymmetric theories. In section III we discuss the TBA technique and highlight some aspects that will be relevant later. In sections IV and V we obtain the TBA system
for the $N=1$ supersymmetric theories and verify the UV and IR limits. In section VI we present Melzer’s folded $N=2$ systems and demonstrate that they agree with our $N=1$ systems. Our conclusions are formulated in section VII.

II. SUPERSYMMETRIC PARTICLE THEORIES IN 1+1 DIMENSIONS WITH FACTORIZABLE SCATTERING

In this section we briefly introduce our $n$-component supersymmetric scattering theories and establish some notation. For a more detailed discussion, see [4].

We are interested in perturbed superconformal field theories described by an action of the form

$$S_\lambda = S + \lambda \int (\mathcal{G}_{-\frac{1}{2}} \mathcal{G}_{-\frac{1}{2}} \phi_{h,h}) d^2x ,$$

(2.1)

where $S$ is the action of the unperturbed theory, which we take to be the minimal superconformal model $\mathcal{M}(2,4n+4)$ of central charge $c_n = -3n(4n+3)/(2n+2)$. The perturbation contains $\phi_{h,h}$, which is a primary field in the Neveu-Schwarz sector of the left and right-chiral superconformal algebras. The perturbations in (2.1) are manifestly supersymmetric. If we consider the case with $h = h_{(1,3)}$ we obtain an integrable theory. This is what we meant by massive deformations of CFT’s in the introduction. It can be shown by means of a slight variation of Zamolodchikov’s counting argument [12] that for this perturbation there is at least one non-trivial integral of motion, which ensures the factorizability of the $S$-matrix. Since the perturbation has the dimension of action, we are automatically introducing a scale in our model and the resulting theory is not conformal invariant. On the other hand, in the ultra-violet (UV) limit we obtain a (super)conformal theory. This UV limit is simply the theory described by $S$ (the unperturbed theory) since in the limit of extremely high energy scattering, the particles do not “see” any finite energy scale. The infra-red (IR) regime of the theory is completely described by a factorizable scattering theory, which, according to the conjecture in [4] can be described as follows.
The IR scattering theory contains $2n$ massive particles, arranged in $n$ supermultiplets $(b_i, f_i)$, with masses $m_i$ as given in [1.1]. We denote by $A_i(\theta_i)$ be any particle, boson or fermion, from any multiplet, with asymptotic momentum $p_i^0 = m_i \cosh(\theta_i)$ and $p_i^1 = m_i \sinh(\theta_i)$. Multi-particle states are then written as

$$|A_{i_1}(\theta_{i_1})A_{i_2}(\theta_{i_2})...A_{i_n}(\theta_{i_n})\rangle_{in(out)} , \quad (2.2)$$

where $\theta_{i_1} \geq \theta_{i_2} \geq ... \geq \theta_{i_n}$ for in-states and the other way around for out-states.

The two-body $S$-matrix will carry two labels, $i$ and $j$, that tell us which supermultiplets the particles in the scattering belong to. From Lorentz invariance it is easy to see that the two-body $S$-matrix can depend only on the rapidity difference $\theta = \theta_i - \theta_j$. The complete two-body $S$-matrix is written as a product of two pieces

$$S^{[ij]}(\theta) = S^{[ij]}_{BF}(\theta) S^{[ij]}_B(\theta) , \quad (2.3)$$

where $S^{[ij]}_{BF}$ is a piece of the $S$-matrix that mixes bosons and fermions and $S^{[ij]}_B$ is a bosonic $S$-matrix that acts only in the bosonic sector.

For the model at hand, the bosonic piece $S^{[ij]}_B$ is the $S$-matrix found by P. Freund, T. Klassen and E. Melzer in [13] and given by

$$S^{[ij]}_B(\theta) = -F_{[i-j]B}(\theta) \left[ F_{(i-j+2)B}(\theta) \right] F_{(i+j+2)B}(\theta) , \quad (2.4)$$

with $F_{\alpha}(\theta) = \frac{\sinh(\theta) + i \sin(\alpha \theta)}{\sinh(\theta) - i \sin(\alpha \theta)}$. In a basis given by $|b_i b_j\rangle$, $|b_i f_j\rangle$, $|f_i b_j\rangle$, $|f_i f_j\rangle$ the $S^{[ij]}_{BF}$ piece is of the form

$$S^{[ij]}_{BF}(\theta) = f^{[ij]}(\theta) \left( \begin{array}{cccc} 1 - t\bar{t} & 0 & 0 & -i(t + \bar{t}) \\ 0 & -t - \bar{t} + 1 + t\bar{t} & 0 & 0 \\ 0 & 1 + t\bar{t} & t - \bar{t} & 0 \\ -i(t + \bar{t}) & 0 & 0 & 1 - t\bar{t} \end{array} \right) + g^{[ij]}(\theta) \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) , \quad (2.5)$$

where $t = \tanh((\theta + \log(m_i/m_j))/4)$ and $\bar{t} = \tanh((\theta - \log(m_i/m_j))/4)$. The functions $f^{[ij]}(\theta)$ and $g^{[ij]}(\theta)$ are related by

$$6$$
\begin{align}
\tilde{f}^{ij}(\theta) &= \frac{\alpha}{4\pi} \sqrt{m_i m_j} \left[ 2 \cosh(\theta/2) + (\rho^2 + \rho^{-2}) \right] \tilde{g}^{ij}(\theta),
\end{align}

\text{where } \rho = (m_i/m_j)^{1/4} \text{ and } \alpha = -\sin(\beta \pi). \text{ This specific form of the Bose-Fermi } S\text{-matrix } S^{ij}_{BF} \text{ is almost completely fixed by } N=1 \text{ supersymmetry (which allows only two free functions } f^{ij}(\theta) \text{ and } g^{ij}(\theta)) \text{ and by the Yang-Baxter equation, which fixes the ratio of } f^{ij}(\theta) \text{ and } g^{ij}(\theta) \text{ up to one free constant } \alpha. \text{ The value of } \alpha \text{ is then dictated by consistency with the bound-state structure induced by the bosonic factor } S_i^B \text{ in (2.4).}

The function } g^{ij}(\theta) \text{ is constrained by the conditions of analyticity, crossing symmetry and unitarity,

\text{analyticity : } g^{ij}(-\theta) = g^{ij\ast}(\theta),
\text{crossing symmetry : } g^{ij}(i\pi - \theta) = g^{ij}(\theta),
\text{unitarity : } g^{ij}(\theta) g^{ij}(-\theta) =
\left[ 1 + \alpha^2 m_i m_j \left( \frac{\sinh^2(\theta/2) + \frac{1}{4}(\rho^2 + \rho^{-2})^2}{\cosh^2(\theta/2) \sinh^2(\theta/2)} \right) \right]^{-1}.

\text{An explicit integral expression for } g^{ij}(\theta), \text{ which will be important for the TBA analysis, can be found as follows. If we define the angles } \delta_1 = \frac{1}{2}(i + j)\beta \pi \text{ and } \delta_2 = \frac{1}{2}(\pi - (i - j)\beta \pi) \text{ we can write the unitarity equation as}
\begin{align}
g^{ij}(\theta) g^{ij}(-\theta) &= \frac{\sin(i\theta/2 - \pi/2) \sin(i\theta/2 + \pi/2) \sin^2(i\theta/2)}{\sin(i\theta/2 - \delta_1) \sin(i\theta/2 + \delta_1) \sin(i\theta/2 - \delta_2) \sin(i\theta/2 + \delta_2)}.
\end{align}

\text{The solution for the one component case was found in [14] where the equation for } g(\theta) \text{ is:}
\begin{align}
g(\Delta) g(-\theta) &= \frac{\sinh^2(i\theta/2)}{\sinh^2(i\theta/2) + \sin^2(\Delta \pi)},
\end{align}

\text{where } \Delta \text{ is some parameter. The solution is then}
\begin{align}
g(\theta) &= \frac{\sinh(i\theta/2)}{\sinh(i\theta/2) + i \sin(\Delta \pi)} \exp \left( i \int_0^\infty dt \frac{\sinh(\Delta t) \sinh((1 - \Delta) t)}{\cosh(\Delta t) \cosh(t) \sin(\theta \pi)} \right). \tag{2.10}
\end{align}

We can now find a solution for (2.8) by writing a product of three functions of the form (2.10).
\[ g^{[ij]}(\theta) = \frac{g_{\Delta_1}(\theta)g_{\Delta_2}(\theta)}{g_{\Delta_3}(\theta)}, \quad (2.11) \]

where \( \Delta_1 = \frac{1}{2}(i+j)\beta, \Delta_2 = \frac{1}{2}(1-(i-j)\beta) \) and \( \Delta_3 = \frac{1}{2} \). This is the form of \( g^{[ij]}(\theta) \) that we shall use in later sections.

Before closing this section, we remark that the \( S \)-matrix \( S_B^{[ij]} \) in (2.5) is related to the hatted matrix \( \hat{S}_B^{[ij]} \) of reference [4] by \( S_B = \Pi \hat{S}_B^{[ij]} \), where \( \Pi \) is an ordinary (not graded) permutation. Making this choice (rather than working with \( \Pi_{\text{graded}} \)) effectively compensates for minus signs that would be induced by the fermionic nature of the particles \( f_i \), and it will allow us to treat all particles as bosons when performing the TBA. Note however that this choice breaks the manifest supersymmetry of our formalism.

### III. SOME ASPECTS OF THE TBA

In the previous section we have recalled the conjecture, made by one of us some five years ago, that the supersymmetric and factorizable \( S \)-matrices (2.3)–(2.5) are actually the true \( S \)-matrices for the perturbed conformal field theories (2.1). This claim was made on the basis of the compatibility of the set of conserved quantities (which can be derived in the perturbed CFT framework) and the bound-state structure that is implied by the proposed \( S \)-matrix. One way to obtain further support for the conjectured identification is by using the TBA. In brief, the TBA makes it possible to express some of the UV CFT data (such as the central charge) in terms of the \( S \)-matrix data, providing a non-trivial check on the correctness of the proposed \( S \)-matrix. Before we come to this analysis, we shall in this section briefly review the TBA for diagonal and non-diagonal \( S \)-matrices.

If we want to know the energy levels of a free particle on a circle of radius \( R \), we just have to send the particle on a round trip around the circle and impose a matching condition for the wave function. This will constrain the possible momenta and from that we can compute the energy levels. In an equation this reads simply

\[ e^{ip2\pi R} = 1. \quad (3.1) \]
From here we obtain $p_n = n/R$ for some integer $n$ and energy levels given by $E_n = p_n^2/2m = n^2/2mR^2$.

Integrable models are not free of course, but due to the fact that multi-particle scattering can be studied two particles at a time, and that this scattering will be purely elastic, we obtain enormous simplifications and the parallel with a free particle on a circle is very appropriate. The only input we need is the exact $S$-matrix and the mass spectrum of our theory. Let us explain the case where the $S$-matrix is diagonal first, just to make the concepts more clear. A more complete discussion of these matters can be found in [8,9].

### 3.1 The diagonal case

The idea of the TBA is very simple physically. We consider a situation where $M$ particles are put on a circle of length $L = 2\pi R$. We then take one particle, of mass $m_i$ and rapidity $\theta_i$, and send it on a round trip around the circle. Since we the $S$-matrix is diagonal we will simply pick up a phase every time our particle meets one of the remaining $M - 1$ particles. The wave function should come back to its original value after the particle has completed its trip and this implies

$$e^{im_i \sinh(\theta_i)L} \prod_{j \neq i} S^{ij}(\theta_{ij}) = 1. \quad (3.2)$$

Taking the logarithm of this equation and going to the thermodynamic limit ($M \to \infty$, $L \to \infty$, with $M/L$ constant) we obtain the following equation

$$2\pi P_i(\theta) = m_i L \cosh(\theta) + \sum_j \int d\theta' \rho_j(\theta') \phi_{ij}(\theta - \theta_j), \quad (3.3)$$

where $P_i(\theta)$ is the density of available rapidities for particles of type $i$, and $\rho_i(\theta)$ is number of such rapidities that are actually occupied. The phase shift $\phi_{ij}(\theta)$ is given by

$$\phi_{ij}(\theta) = -i \frac{\partial \ln S^{[ij]}(\theta)}{\partial \theta}. \quad (3.4)$$

Let us define the pseudo-energy levels $\epsilon_i(\theta)$ by

$$\frac{\rho_i(\theta)}{P_i(\theta)} = \frac{e^{-\epsilon_i(\theta)}}{1 + e^{-\epsilon_i(\theta)}}. \quad (3.5)$$
The $\rho_i(\theta)$ are such as to minimize the free energy. From this, we get an equation for the $\epsilon_i(\theta)$

$$
\epsilon_i(\theta) = m_i R \cosh(\theta) - \sum_j \int \frac{d\theta'}{2\pi} \phi_{ij}(\theta - \theta') \ln(1 + e^{-\epsilon_j(\theta')}) , \quad (3.6)
$$

The total energy of our system can be written in terms of $\epsilon_i(\theta)$ according to

$$
E(R) = - \sum_i \frac{m_i}{2\pi} \int d\theta \cosh(\theta) \ln(1 + e^{-\epsilon_i(\theta)}) , \quad (3.7)
$$

We shall call the equations (3.6), (3.7) the TBA equations of the scattering theory.

From the TBA equations we can find numerically the Casimir energy for any $R$. In the UV limit we can do even better and use the amazing Roger’s dilogarithm identities to derive closed-form expressions (an explicit formula will be given in the next subsection).

### 3.2 The non-diagonal case

In this case we have conceptually the same situation. The difference is that now a particle, when taking a round trip, scatters non-diagonally with all the others so that the periodic boundary conditions lead to an equation involving a transfer matrix of size $2^M \times 2^M$.

This transfer matrix can be written as

$$
\left(T^{(jj_1j_2\ldots j_M)}_{st}\right)_{s_1s_2\ldots s_M}^{t_1t_2\ldots t_M} (\theta|\theta_1 \theta_2 \ldots \theta_M) = 
\sum_{\{r_l\}} (S^{[jj_1]}_{ss_1})^{r_1t_1} (\theta - \theta_1) (S^{[jj_2]}_{s_1s_2})^{r_2t_2} (\theta - \theta_2) \ldots (S^{[jj_M]}_{s_{M-1}s_M})^{t_{M-1}t_M} (\theta - \theta_M) , \quad (3.8)
$$

In this notation, the indices $j$ and $j_l$ denote the different multiplets ($j, j_l = 1, 2, \ldots, n$) and the indices $r_t$, $s_t$ and $t_t$ denote the two states $b$ and $f$ within each multiplet. The periodic boundary condition is now expressed as (see, for example, [7])

$$
e^{im_l \sinh(\theta_l)L} \sum_s \left(T^{(jj_1j_2\ldots j_M)}_{ss}\right) (\theta_l|\theta_1, \ldots \theta_M) \psi = -\psi , \quad l = 1, 2, \ldots, M , \quad (3.9)
$$

where $\psi$ is a $2^M$-component wave-function.

To make further use of this equation, we have to diagonalize the transfer matrix, or at least find some relationship among the eigenvalues that will allow us to write the TBA
equations. In general, the problem of diagonalizing a transfer matrix is extremely difficult and many techniques have been developed to handle such problems \[1,3\]. We will, in the next section, use a method developed by B. Felderhof \[10\] to diagonalize the transfer matrix for the eight-vertex model in the case where the Boltzmann weights satisfy the free fermion condition. Remarkably, the resulting TBA equations can be cast in a form identical to \(3.6, 3.7\), this time for a set of particles that the contains the original set plus an additional particle of mass zero (see section V). This shows that the form \(3.6, 3.7\) of the TBA equations is universal, applying both to diagonal and non-diagonal cases.

Returning to the Casimir energy, it can be shown that the UV limit is given by

\[
E_0 = E(mR \to 0) \sim -\frac{1}{\pi R} \sum_i \left[ L\left(\frac{x_i}{1+x_i}\right) - L\left(\frac{y_i}{1+y_i}\right) \right],
\]

where \(L(x)\) is the Roger’s dilogarithm function, which is defined by

\[
L(x) = -\frac{1}{2} \int_0^x dt \left[ \frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t} \right],
\]

and \(x_i = e^{-\epsilon_i(0)}\) and \(y_i = e^{-\epsilon_i(\infty)}\). The \(x_i, y_i\) are the solutions of

\[
x_i = \prod_j (1 + x_j)^{N_{ij}}, \quad y_{i'} = \prod_j' (1 + y_{j'})^{N_{i'j'}},
\]

where the \(N_{i,j}\) are given by

\[
N_{i,j} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \phi_{ij}(\theta).
\]

The primed indices \(i'\) refer to the massless particles only.

**IV. DIAGONALIZING THE TRANSFER MATRIX**

As we have seen in the previous section, we need to gain control over the eigenvalues of the trace of the transfer matrix \(T(\theta|\theta_1, ..., \theta_N)\). Notice that this is a inhomogeneous transfer matrix, depending on all the differences \((\theta - \theta_l)\). Due to the Yang-Baxter equation, \(\text{tr}[T(\theta)]\) and \(\text{tr}[T(\theta')]\) commute for any \(\theta\) and \(\theta'\). This means that the eigenvectors of \(\text{tr}[T(\theta)]\) are
\( \theta \)-independent. The eigenvalues will of course depend on \( \theta \) and this dependence is precisely what we need to know to do the TBA.

In section II, we mentioned that the non-diagonal part of the \( S \)-matrix, called \( S_{BF}^{ij} \), can be derived by imposing supersymmetry and the Yang-Baxter equation. A remarkable property is that the elements of \( S_{BF}^{ij} \), when interpreted as the Boltzmann weights of a lattice statistical mechanics model, satisfy the free fermion condition. Explicitly this means that, for \( S_{BF}^{ij} \) given by

\[
S_{BF}^{ij} = \begin{pmatrix}
    a_+ & 0 & 0 & d \\
    0 & b_+ & c & 0 \\
    0 & c & b_- & 0 \\
    d & 0 & 0 & a_-
\end{pmatrix},
\]  
(4.1)

the \( a_+ \), \( a_- \), \( b_+ \), \( b_- \), \( c \) and \( d \) satisfy

\[
a_+ a_- + b_+ b_- = c^2 + d^2. 
\]  
(4.2)

It can be proved that an eight-vertex model that satisfies this condition can be written as an \( XY \) model with a magnetic field. The magnetic field is given in tems of the Boltzmann weights

\[
H = \frac{a_+^2 + b_+^2 - a_-^2 - b_-^2}{2(a_+b_- + a_-b_+)}.
\]  
(4.3)

Using the results from section II we find that \( H = -1 \) for all \( i, j \). This is the critical point of the \( XY \) model.

For the determination of the eigenvalues of \( \text{tr}[T(\theta)] \) we will follow a method given by Felderhof [16]. We shall explain the idea of this method and apply it to our case. The strategy will be to find an inversion relation for \( \text{tr}[T(\theta)] \) which can then be used to write a functional equation for the eigenvalues. It will turn out that, rather miraculously, the analysis for general \( n \) can be done by extending the analysis for \( n = 1 \), which was presented in [17]. Our presentation in this section therefore follows [17] rather closely.

It is not difficult to see that the \( S \)-matrix (2.5) satisfies the following relation
\[ S_{BF}^{ij}(\theta + i\pi) = -\frac{g^{ij}(\theta + i\pi)}{g^{ij}(\theta)} \begin{pmatrix} a_- & 0 & 0 & c \\ 0 & -b_- & -d & 0 \\ 0 & -d & -b_+ & 0 \\ c & 0 & 0 & a_+ \end{pmatrix}, \]  \tag{4.4}

where we are using a obvious notation. If we make a similarity transformation by means of a matrix \( M \) (acting on the indices \( r = b, f \) given by

\[ M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  \tag{4.5}

we obtain the following \( \tilde{S}_{BF}^{ij} \)

\[ \tilde{S}_{BF}^{ij}(\theta) = -\frac{g^{ij}(\theta + i\pi)}{g^{ij}(\theta)} \begin{pmatrix} -b_+ & 0 & 0 & -d \\ 0 & a_+ & c & 0 \\ 0 & c & a_- & 0 \\ -d & 0 & 0 & -b_+ \end{pmatrix}. \]  \tag{4.6}

It is easy to see that \( \tilde{S}_{BF}^{ij} \) again satisfies the free fermion condition.

Returning to the transfer matrix \( (3.8) \), let us write \( S_l = S_{BF}^{jj}((\theta - \theta_l) \), etc., so that \( \text{tr}[T(\theta)] = \text{tr}[S_1S_2\ldots S_M] \) and similarly \( \text{tr}[T(\theta + i\pi)] = \text{tr}[\tilde{T}(\theta)] = \text{tr}[\tilde{S}_1\tilde{S}_2\ldots \tilde{S}_M] \). The product \( \text{tr}[T(\theta)]\text{tr}[T(\theta + i\pi)] \) can then be written as

\[ \text{tr}[T(\theta)]\text{tr}[T(\theta + i\pi)] = \text{tr}[(S_1 \otimes \tilde{S}_1)(S_2 \otimes \tilde{S}_2)\ldots (S_N \otimes \tilde{S}_N)], \]  \tag{4.7}

where the notation \( \otimes \) denotes a matrix product of the \( s \)-indices and a tensor product on the \( r \)-indices. It can be proved that there is a similarity transformation with a constant (\( \theta \)-independent) \( 4 \times 4 \) matrix \( X \) (acting on the indices \( r_l, \tilde{r}_l \)) that puts \( S_l \otimes \tilde{S}_l \) in a triangular

\[ \text{1} \text{ Notice that the indices } r \text{ and } s \text{ on the two-body } S \text{-matrix } (S_{BF}^{ij})_{rs} \text{ are not on equal footing in the transfer matrix } (3.8). \text{ In the usual terminology, the } r \text{ are called matrix indices and the } s \text{ are quantum indices.} \]
form. We do not give the explicit form of $X$ here, but mention the important fact that it
depends on $j, j_l$ through a parameter $\phi$, given by

$$\tanh(\phi) = \frac{2cd}{a_+b_+ + a_-b_-}. \quad (4.8)$$

In terms of the elements of $S_{BF}^{[jj_l]}$ we get $\tanh(\phi) = -\alpha m_j$, i.e, it only depends on the index $j$, which is common to all factors in the transfer matrices. We can thus use one and the
same $X$ to put all the $S_l \otimes \tilde{S}_l$ in triangular form, and the total trace $\text{tr}[T(\theta)]\text{tr}[T(\theta + i\pi)]$ is
invariant under this operation. We find

$$-S_{BF}^{[jj_l]}(\theta - \theta_l) \otimes S_{BF}^{[jj_l]}(\theta - \theta_l) \sim \begin{pmatrix}
M_+ & * & * \\
0 & \pm F_- & * \\
0 & 0 & \pm F_+
\end{pmatrix}, \quad (4.9)$$

where $M_+, M_-, F_+$ and $F_-$ are given by

$$M_+ = a_+a_- - d^2, \quad M_- = a_+a_- - c^2,$$

$$F_\pm = \pm \left( \sinh^2(\phi)a_\pm b_\mp + \cosh^2(\phi)a_\pm b_\mp \mp 2 \sinh(\phi) \cosh(\phi)cd \right), \quad (4.10)$$

which can be rewritten as

$$M_+(\theta) = \frac{-4g_2(\theta)}{\sinh^2(\theta)} \cosh\left(\frac{\theta + i\beta_-\pi}{2}\right) \cosh\left(\frac{\theta - i\beta_-\pi}{2}\right) \sinh\left(\frac{\theta + i\beta_+\pi}{2}\right) \sinh\left(\frac{\theta - i\beta_+\pi}{2}\right),$$

$$M_-(\theta) = \frac{-4g_2(\theta)}{\sinh^2(\theta)} \sinh\left(\frac{\theta + i\beta_-\pi}{2}\right) \sinh\left(\frac{\theta - i\beta_-\pi}{2}\right) \cosh\left(\frac{\theta + i\beta_+\pi}{2}\right) \cosh\left(\frac{\theta - i\beta_+\pi}{2}\right),$$

$$F_+(\theta) = \frac{-4g_2(\theta)}{\sinh^2(\theta)} \sinh\left(\frac{\theta - i\beta_-\pi}{2}\right) \cosh\left(\frac{\theta + i\beta_-\pi}{2}\right) \sinh\left(\frac{\theta - i\beta_+\pi}{2}\right) \cosh\left(\frac{\theta + i\beta_+\pi}{2}\right),$$

$$F_-(\theta) = \frac{-4g_2(\theta)}{\sinh^2(\theta)} \cosh\left(\frac{\theta - i\beta_-\pi}{2}\right) \sinh\left(\frac{\theta + i\beta_-\pi}{2}\right) \cosh\left(\frac{\theta - i\beta_+\pi}{2}\right) \sinh\left(\frac{\theta + i\beta_+\pi}{2}\right), \quad (4.11)$$

where $\beta_\pm = (j_l \pm j)\beta$ and $g_2(\theta) = g^{[jj_l]}(\theta)g^{[jj_l]}(\theta + i\pi)$. The entries of the matrix $(4.9)$ act
diagonally on the indices $s_l$, with the $\pm$ sign equal to $+1$ ($-1$) if total in-state is bosonic (fermionic).
Denoting by \( x \) the eigenvalues of the product \( \text{tr}[T(\theta)] \text{tr}[T(\theta + i\pi)] \). Denoting the eigenvalues of \( \text{tr}[T(\theta)] \) by \( \Lambda(\theta) \), we find

\[
\Lambda(\theta)\Lambda(\theta + i\pi) = (-1)^M \left( \prod_{l=1}^{M} M_+(\theta - \theta_l) + \prod_{l=1}^{M} M_- (\theta - \theta_l) + F \left[ \prod_{l=1}^{M} F_+(\theta - \theta_l) + \prod_{l=1}^{M} F_-(\theta - \theta_l) \right] \right), \tag{4.12}
\]

where \( F \) is +1 for a bosonic state and -1 for a fermionic eigenstate. Notice that the dependence of \( M_\pm \) and \( F_\pm \) on \( j, j_l \) has been suppressed in the notation. In terms of reduced eigenvalues \( \lambda(\theta) \), defined by

\[
\Lambda(\theta) = \prod_{l=1}^{M} 2 \frac{g^{ij\beta\pi}(\theta - \theta_l)}{\sinh(\theta - \theta_l)} \lambda(\theta), \tag{4.13}
\]

the equation (4.12) can be rewritten in the following factorized form

\[
\lambda(\theta)\lambda(\theta + i\pi) = \left[ F \prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \cosh(\frac{\theta - \theta_{l_k} - i(k-j)\beta\pi}{2}) \sinh(\frac{\theta - \theta_{l_k} + i(k+j)\beta\pi}{2}) \right. \\
+ \left. \prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \sinh(\frac{\theta - \theta_{l_k} - i(k-j)\beta\pi}{2}) \cosh(\frac{\theta - \theta_{l_k} + i(k+j)\beta\pi}{2}) \right] \\
\times \left[ F \prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \cosh(\frac{\theta - \theta_{l_k} + i(k-j)\beta\pi}{2}) \sinh(\frac{\theta - \theta_{l_k} - i(k+j)\beta\pi}{2}) \right. \\
+ \left. \prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \sinh(\frac{\theta - \theta_{l_k} + i(k-j)\beta\pi}{2}) \cosh(\frac{\theta - \theta_{l_k} - i(k+j)\beta\pi}{2}) \right], \tag{4.14}
\]

where we assumed that out of the \( M \) particles on our circle, \( M_k \) are of type \( k \), \( \sum_k M_k = M \). Recall that the auxiliary particle that makes the roundtrip is of type \( j \).

The \( \lambda(\theta) \) are \( 2\pi i \) periodic meromorphic functions, and as such they are completely determined by their zeros and poles. Let us denote by \( z^+ \) the zeros of the first factor in the rhs of (4.14) and by \( z^- \) those of the second factor. They satisfy the equations

\[
\prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \frac{\tanh(z^+ - \theta_{l_k} - i(k-j)\beta\pi)}{\tanh(z^+ - \theta_{l_k} + i(k+j)\beta\pi)} = -F, \quad \prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \frac{\tanh(z^- - \theta_{l_k} + i(k-j)\beta\pi)}{\tanh(z^- - \theta_{l_k} - i(k+j)\beta\pi)} = -F. \tag{4.15}
\]

Denoting by \( x_m \) the real solutions of the equation

\[
\prod_{k=1}^{n} \prod_{l_k=1}^{M_k} \frac{\tanh(x - \theta_{l_k} - i k \beta\pi)}{\tanh(x - \theta_{l_k} + i k \beta\pi)} = -F, \tag{4.16}
\]
one may check that the following $z^{\pm}$ satisfy (4.13)

$$z^{+,m}_e = x_m - ij\beta\pi + i\frac{1 - \epsilon^+}{2}\pi, \quad z^{-,m}_e = x_m + ij\beta\pi + i\frac{1 - \epsilon^-}{2}\pi,$$  \hspace{1cm} (4.17)

Out of each pair each of zero’s $z^{\pm}_{e,m}, \epsilon^{\pm} = -1, 1$, one will come from $\lambda(\theta)$ and the other from $\lambda(\theta + \pi i)$, which gives us two choices. It turns out that the choices to be made for the two factors (i.e. the choices for $\epsilon^+$ and $\epsilon^-$) are correlated (see, for example, [17]), so that for each $m$ there are two possible factors contributing to $\lambda(\theta)$

$$\sinh(\frac{\theta - z^{+,m}_e}{2}) \sinh(\frac{\theta - z^{-,m}_e}{2}), \quad \epsilon = -1, 1.$$  \hspace{1cm} (4.18)

We thus arrive at the following expression for the eigenvalues

$$\lambda_{\{x_m,\epsilon_m\}}(\theta) = \text{Constant} \times \prod_m \sinh(\frac{\theta - x_m}{2} + i\epsilon_m j\beta\pi/2) \cosh(\frac{\theta - x_m}{2} - i\epsilon_m j\beta\pi/2),$$  \hspace{1cm} (4.19)

The counting of the eigenvalues (there should be $2^M$ of them), works out as follows. There are $M - 1$ solutions $x_m$ in the sector $F = -1$, which, with freely chosen $\epsilon_m$, gives the correct number of $2^{M-1}$ eigenvalues in that sector. In the sector $F = 1$ there are $M$ solutions $x_m$, and in order to obtain the correct number of $2^{M-1}$ eigenvalues there, we need one constraint on the allowed values of $\epsilon_m$. Explicit results for some small values of $M$ suggest that this constraint takes the form $\prod_m \epsilon_m = 1$.

The expressions (4.13) and (4.19) for the eigenvalues $\Lambda(\theta)$ will be important ingredients in the TBA analysis, which we present in the next section.

V. THE THERMODYNAMIC BETHE ANSATZ

5.1 The TBA system

In order to do the thermodynamics we have a large number $M$ of particles on a circle of length $L$. We send one of the particles, of type $j$ and rapidity $\theta$, on a round trip and scatter it off as it moves, one particle at a time due to integrability. When it comes back to the original point, we impose a periodic boundary condition. From this condition we may derive the allowed rapidity configurations, and from there the thermodynamics.
As we have seen before, the full $S$-matrix can be written as a scalar factor $S^{ij}_B(\theta)$, given by (2.4), times the $4 \times 4$ matrix $S^{ij}_{BF}(\theta)$. We will denote the product $S^{ij}_B(\theta)S^{ij}_{BF}(\theta)$ by $Z^{ij}(\theta)$.

The periodic boundary condition (3.9) then takes the following form

$$e^{i \text{Im} j \sinh(\theta) L} \prod_{k=1}^n \prod_{l_k=1}^{M_k} Z^{[jk]}(\theta - \theta_{lk}) \frac{1}{\sinh(\theta - \theta_{lk})} \chi_{x_{m},\epsilon_m}(\theta) = -1.$$  \hspace{1cm} (5.1)

We denote by $\rho_k(\theta)$ the density of occupied states and by $P_k(\theta)$ the density of available states for particles of type $k$, $k = 1, 2, \ldots, n$. Similarly, we introduce the symbol $P_0(\theta)$ for the density distribution of solutions $x_m$ to the equation (4.16). This is further split as $P_0(\theta) = \rho_0(\theta) + \bar{\rho}_0(\theta)$ where the densities $\rho_0$, $\bar{\rho}_0$ refer to those $x_m$ for which the corresponding $\epsilon_m = +1, -1$, respectively. Notice that $\rho_0(\theta)$ can be viewed as a density distribution for a new type of “particle” with label 0. This explains that the final TBA system will be in terms of $n + 1$ rather than $n$ types of particles.

From (4.16) we obtain the density distribution of the available rapidities for the “particles” of type 0

$$2\pi P_j(\theta) = \sum_{k=1}^n \int d\theta' \rho_k(\theta') \frac{\partial}{\partial \theta} \text{Im} \ln \left( \frac{\tanh\left(\frac{1}{2}(\theta - \theta' - ik\beta\pi)\right)}{\tanh\left(\frac{1}{2}(\theta - \theta' + ik\beta\pi)\right)} \right).$$  \hspace{1cm} (5.2)

Notice that the “mass term”, which in general occurs on the r.h.s. of TBA equations of this sort, is absent. This means that the auxiliary particles of type 0 should be viewed as massless.

Taking the derivative of the imaginary part of the logarithm of (5.1) leads to

$$2\pi P_j(\theta) = m_j \cosh(\theta) L + \int d\theta' \left[ \sum_{k=1}^n \left( \rho_k(\theta') \frac{\partial}{\partial \theta} \text{Im} \ln \left( \frac{Z^{[jk]}(\theta - \theta')}{\sinh(\theta - \theta')} \right) \right) 
+ \rho_0(\theta') \frac{\partial}{\partial \theta} \text{Im} \ln(\lambda^j_0(\theta - \theta')) + \bar{\rho}_0(\theta') \frac{\partial}{\partial \theta} \text{Im} \ln(\bar{\lambda}^j_0(\theta - \theta')) \right],$$  \hspace{1cm} (5.3)

where $\lambda^j_0$, $\bar{\lambda}^j_0$ are defined by

$$\lambda^j_0 = \sinh\left(\frac{1}{2}(\theta + ij\beta\pi)\right) \cosh\left(\frac{1}{2}(\theta - ij\beta\pi)\right)$$

$$\bar{\lambda}^j_0 = \sinh\left(\frac{1}{2}(\theta - ij\beta\pi)\right) \cosh\left(\frac{1}{2}(\theta + ij\beta\pi)\right),$$  \hspace{1cm} (5.4)
Let us define some quantities that will make the notation lighter. Define $\Phi_{z}^{jk}(\theta)$ and $\Phi^{j}(\theta)$ as

$$
\Phi_{z}^{jk}(\theta) = \frac{\partial}{\partial \theta} \text{Im} \ln \left( \frac{Z^{jk}(\theta)}{\sinh(\theta)} \right)
$$

$$
\Phi^{j}(\theta) = 2 \frac{\partial}{\partial \theta} \text{Im} \ln(\lambda_{0}^{j}(\theta)) = -2 \frac{\partial}{\partial \theta} \text{Im} \ln(\bar{\lambda}_{0}^{j}(\theta)).
$$

(5.5)

Notice that the same quantity $\Phi^{k}(\theta)$ occurs in the r.h.s. of the equation (5.2).

Using $P_{0}(\theta) = \rho_{0}(\theta) + \bar{\rho}_{0}(\theta)$ we can now eliminate $\bar{\rho}_{0}(\theta)$ from (5.2), (5.3) to obtain

$$
2\pi P_{j}(\theta) = m_{j} \cosh(\theta) L + \sum_{k=1}^{n} \int d\theta' \left( \rho_{k}(\theta') \left( \Phi_{z}^{jk}(\theta - \theta') - \frac{1}{2}(\Phi^{j} \ast \Phi^{k})(\theta - \theta') \right) + \rho_{0}(\theta') \Phi^{j}(\theta - \theta') \right),
$$

(5.6)

where we introduced the convolution

$$
(\Phi^{j} \ast \Phi^{k})(\theta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\theta' \Phi^{j}(\theta - \theta') \Phi^{k}(\theta').
$$

(5.7)

We can now vary the $\rho_{i}(\theta), \rho_{0}(\theta)$ so as to minimize the free energy. In terms of the quasi-particle energies $\epsilon_{a}(\theta), a = 0, 1, 2, \ldots, n,$ defined as

$$
\frac{\rho_{a}(\theta)}{P_{a}(\theta)} = \frac{e^{-\epsilon_{a}(\theta)}}{1 + e^{-\epsilon_{a}(\theta)}},
$$

(5.8)

this gives the equations

$$
\epsilon_{j}(\theta) = m_{j} L \cosh(\theta) - \sum_{k=1}^{n} \left( (\Phi_{z}^{jk} - \frac{1}{2}(\Phi^{j} \ast \Phi^{k})) \ast \ln(1 + e^{-\epsilon_{k}}) \right) (\theta)
$$

$$
+ (\Phi^{j} \ast \ln(1 + e^{-\epsilon_{0}}))(\theta)
$$

$$
\epsilon_{0}(\theta) = -\sum_{k=1}^{n} (\Phi^{k} \ast \ln(1 + e^{-\epsilon_{k}}))(\theta),
$$

(5.9)

which have the general form (3.6). The total energy is given by (3.7) with the sum running over the index $a = 0, 1, 2, \ldots, n$.

Now that we have the TBA equations we can explore the UV limit and obtain analytical solutions for the ground state energy. This will be done in the next subsection.
5.2 The UV limit

The UV limit provides an important test for the conjectured S-matrix since we can compute the ground state energy analytically in this limit and compare it with the value predicted by CFT: $E_0 = -\frac{\pi}{6} c_{\text{eff}}$, with $c_{\text{eff}} = c - 12(h_{\text{min}} + \bar{h}_{\text{min}})$, where $h_{\text{min}}$ ($\bar{h}_{\text{min}}$) is the smallest conformal dimension in the theory. In the case of unitary theories $h_{\text{min}} = \bar{h}_{\text{min}} = 0$, corresponding to the identity operator, which is always present in the operator content of a unitary CFT. In the case of non-unitary theories we have operators with negative dimensions and the effective central charge picks up the term $-12(h_{\text{min}} + \bar{h}_{\text{min}})$.

In the CFT’s that we started from in section II we have $c = c_n = -3n(4n + 3)/(2n + 2)$ and $h_{\text{min}} = \bar{h}_{\text{min}} = -\frac{n}{4}$ so that $c_{\text{eff}} = 3n/(2n + 2)$.

To obtain the ground state energy in the UV limit from the TBA developed in the previous section we shall work out the general result (3.10)-(3.13) for these particular theories.

We denote by $A_{jk}$ and $B_k$ the following basic integrals

$$
A_{jk} = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \Phi_{jk}^z(\theta), \quad B_j = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \Phi^j(\theta).
$$

(5.10)

Using the explicit form of $\Phi_{jk}^z(\theta)$ it is easy to compute $A_{jk}$. We have

$$
\Phi_{jk}^z(\theta) = \Im \frac{\partial}{\partial \theta} \ln \left( \frac{Z^{[jk]}(\theta)}{\sinh(\theta)} \right) = \Im \frac{\partial}{\partial \theta} \ln \left( \frac{[\hat{g}_{\Delta_1}(\theta)/\sinh(\theta)][\hat{g}_{\Delta_2}(\theta)/\sinh(\theta)]}{[\hat{g}_{\Delta_3}(\theta)/\sinh(\theta)]} e^{i \int_{-\infty}^{\infty} \frac{\Phi^t}{\pi} \sin(t\theta) F(t) S_B^{\alpha}(\theta)} \right),
$$

(5.11)

where $F(t)$ is a smooth function that falls exponentially to 0 at $t \to \pm \infty$ and $\hat{g}_\Delta$ is the prefactor of formula (2.10). From equation (2.11) we see that the “$g_\Delta$” piece gives $\frac{1}{2} + \frac{1}{2} - \frac{1}{2}$. The exponential gives zero since after the derivative with respect to $\theta$ we get an integral of a smooth function $F(t)$ times $\sin(t\theta)$ and by the Riemann-Lebesgue lemma this will be zero in the limit $\theta \to \pm \infty$. Each factor $F_\alpha$ (with $\alpha \neq 0$) from the bosonic S-matrix (2.4) can be easily seen to give $-1$, and therefore we obtain

$$
\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{\partial}{\partial \theta} \Im \ln(S_B^{[jk]}(\theta)) = -2k + \delta_{j,k}, \quad j \geq k.
$$

(5.12)
Collecting these results we see that \( A_{jk} = -2j + \delta_{j,k} + \frac{1}{2} \) for \( j \geq k \). The computation of \( B_j \) is straightforward and gives \( B_j = 1 \). This is all we need to write the equations (3.12) for the pseudo-energies in the UV limit

\[
x_j = (1 + x_0)^{N_{0,j}} \prod_{k=1}^{n} (1 + x_k)^{N_{j,k}}, \quad x_0 = \prod_{k=1}^{n} (1 + x_k)^{N_{0,k}}, \quad y_0 = 1.
\]

with \( x_j = e^{-\epsilon_j(0)} \), \( x_0 = e^{-\epsilon_0(0)} \), \( y_j = e^{-\epsilon_j(\infty)} \) and \( y_0 = e^{-\epsilon_0(\infty)} \). Note that due to the presence of the mass term, in the limit \( \theta \to \infty \) the \( \epsilon_j(\theta) \) diverge and this implies directly that \( y_j = 0 \).

The \( N_{a,b} \) are given by \( N_{j,k} = A_{jk} - \frac{1}{2} B_j B_k \) and \( N_{0,j} = B_j \) so that

\[
N_{j,k} = -2k + \delta_{j,k} , \quad N_{0,j} = 1 , \quad N_{0,0} = 0 .
\]

for \( j \geq k \). The other \( N_{a,b} \) are obtained using the symmetry \( N_{a,b} = N_{b,a} \).

The (far from trivial) solutions for equation (5.13) are

\[
x_j = \frac{\sin^2\left(\frac{\pi}{2n+2}\right)}{\sin\left(\frac{(2j+3)\pi}{4n+4}\right) \sin\left(\frac{(2j-1)\pi}{4n+4}\right)}, \quad x_0 = \frac{\sin\left(\frac{3\pi}{4n+4}\right)}{\sin\left(\frac{\pi}{4n+4}\right)}, \quad y_j = 0 , \quad y_0 = 1 .
\]

We can now use the Roger’s dilogarithm technology and compute, using equation (3.10), the effective central charge \( c_{\text{eff}} \)

\[
c_{\text{eff}} = \frac{6}{\pi^2} \left[ \sum_{j=1}^{n} \mathcal{L} \left( \frac{\sin^2\left(\frac{\pi}{2n+2}\right)}{\sin^2\left(\frac{(2j+1)\pi}{4n+4}\right)} \right) - \mathcal{L} \left( \frac{1}{2} \right) \right] = \frac{3n}{2n+2} ,
\]

in agreement with the value in the original conformal field theory.

We found the solutions (5.13) for \( x_j, x_0 \) and the dilogarithm identity (5.16) in the second paper of [7], which deals with TBA systems for \( N=2 \) supersymmetric theories. The relation between the \( N=1 \) and \( N=2 \) TBA systems is the topic of our next section.

VI. MELZER’S FOLDING

In a very beautiful paper [8], E. Melzer studied a series of new TBA systems with some mathematical applications in mind, namely, the study of number theoretical identities of the Gordon-Andrews type. He developed some TBA systems that provide such identities. Melzer’s obtained his “first TBA system” by a folding in half a TBA system derived by
P. Fendley and K. Intriligator [7] for specific $N = 2$ supersymmetric theories. He then conjectured that this folded TBA system should correspond to the $N = 1$ supersymmetric scattering theories of [4], which we discussed in section II. Now that we have derived explicitly the $N = 1$ TBA systems we are in a position to check, and confirm, Melzer’s proposal.

Let us briefly summarize Melzer’s discussion. We consider a general TBA system

$$E(R) = -\sum_a \frac{m_a}{2\pi} \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \cosh(\theta) \ln(1 + e^{-\epsilon_a(\theta)})$$

(6.1)

with

$$\epsilon_a(\theta) = m_a R \cosh(\theta) - \sum_b \left( \Phi_{a,b} \ast \ln(1 + e^{-\epsilon_b}) \right)(\theta),$$

(6.2)

where the kernels $\Phi_{a,b}$ are symmetric in $a, b$. Let us now assume that we have $2n$ types of particles, $a = 1, 2, \ldots, 2n$ and that the TBA data $m_a, \Phi_{a,b}$ possess the following symmetry

$$m_a = m_{2n+1-a}, \quad \Phi_{a,b} = \Phi_{2n+1-a,2n+1-b}.$$  

(6.3)

In such a situation we can define a folded TBA system for half the number of particles, $a = 1, 2, \ldots, n$, by defining a folded kernel

$$\Phi_{a,b}^{\text{folded}} = \Phi_{a,b} + \Phi_{a,2n+1-b}, \quad a, b = 1, 2, \ldots, n.$$  

(6.4)

The $N = 2$ supersymmetric TBA systems of [4] contain $2k - 2$ massive particles and auxiliary massless species labeled as $0, \bar{0}$. They correspond to massive scattering theories that are obtained by starting from the $N = 2$ minimal superconformal field theories and perturbing by the most relevant supersymmetry preserving operator. This corresponds to the Landau-Ginzburg superpotential $\frac{X^{2k}}{2k} - \lambda X$, $k = 2, 3, \ldots$. The masses are given by

$$m_a = \frac{\sin(\frac{a\pi}{2k-1})}{\sin(\frac{\pi}{2k-1})}, \quad a = 1, 2, \ldots, 2k - 2,$$

(6.5)

so that $m_a = m_{2k-1-a}$, and the kernel $\Phi_{a,b}^{N=2}$ has the symmetry property $\Phi_{a,b}^{N=2} = \Phi_{2k-1-a,2k-1-b}^{N=2}$ (we identify the labels $\bar{0} = 2k - 1$). We can thus use (6.4) to define a folded version of the $N=2$ TBA system. According to Melzer’s conjecture, this folded $N=2$ system will be the same as the $N=1$ system that we derived in this paper.
Identifying \( n = k - 1 \), we see that the mass spectra (1.1) and (1.5) indeed agree. We also checked that

\[
\Phi_{a,b}^{N=1}(\theta) = \Phi_{a,b}^{N=2}(\theta) + \Phi_{a,2n+1-b}^{N=2}(\theta), \quad a, b = 0, 1, 2, \ldots, n, \quad (6.6)
\]

where \( \Phi_{a,b}^{N=2}(\theta) \) are the kernels obtained in [7] and \( \Phi_{a,b}^{N=1}(\theta) \) are the kernels in our TBA system (5.9). For the kernels \( \Phi_{k,0}(\theta) \) this result is elementary while for the kernels \( \Phi_{j,k}(\theta) \) some tedious cosine Fourier transforms are needed.

In the UV limit the folding relation can be expressed as

\[
N_{a,b}^{N=1} = N_{a,b}^{N=2} + N_{a,2n+1-b}^{N=2}, \quad (6.7)
\]

and we can use the equation (4.20) from Fendley and Intriligator’s later paper [7] to check that this is indeed an identity.

**VII. CONCLUSIONS**

In this paper we have studied the thermodynamic limit of a family of perturbed non-unitary superconformal field theories with \( N = 1 \) and central charge \( c = -3(4n+3)/(2n+2) \). We identified the two-body \( S \)-matrices of those theories with the Boltzmann weights of an eight-vertex model and we have seen that supersymmetry implies the free fermion condition and the value \( H = -1 \) for the magnetic field of the associated \( XY \) model. This allowed us to diagonalize the transfer matrix by Felderhof’s method and to develop its TBA. We found the algebraic equations that the TBA leads to in the UV limit. Solving those equations led to the correct value for the central charges. This analysis has thus confirmed the validity of the \( S \)-matrices (2.3)–(2.5) and, in particular, ruled out possible “CDD” correction factors for those \( S \)-matrices.

Another interesting result was the verification that the \( N = 1 \) TBA systems obtained here can be obtained by folding in half a series of \( N = 2 \) supersymmetric TBA systems. This idea was first proposed by Melzer in [3].
There are several possibilities for further work. We have seen that Melzer’s proposal works and that the folding idea can indeed be extended to TBA systems for non-diagonal $S$-matrices. It should be possible to understand this folding directly at the level $S$-matrices, possibly as some generalization of the non-critical orbifold construction of P. Fendley and P. Ginsparg [18]. This will be subject of a future publication [11].

Recently a lot of (deserved) interest has been given to $1+1$ field theories with a boundary, i.e., defined on a half-line. It is not very difficult to find a boundary $S$-matrix or reflection matrix $R(\theta)$ for the theories that we discussed in this paper. In particular, we can easily find the boundary analogue of the Bose-Fermi $S$-matrix $S_{BF}(\theta)$, see (2.3). In order to do that we have to solve the equation

$$Q_B(\theta)R_{BF}(\theta) = R_{BF}(\theta)Q_B(-\theta), \quad (7.1)$$

where $Q_B(\theta)$ is the supersymmetry charge on the half-line. In terms of the bulk supersymmetric charges $Q(\theta)$ and $\tilde{Q}(\theta)$ [4] we have $Q_B(\theta) = Q(\theta) + \tilde{Q}(\theta)$. This gives

$$Q_B(\theta) = e^{\frac{\theta}{2}\sigma_x} + e^{-\frac{\theta}{2}\sigma_y}. \quad (7.2)$$

Solving the condition (7.1) we obtain

$$R_{BF}(\theta) = Z(\theta) \begin{pmatrix} \cosh\left(\frac{\theta}{2} + i\frac{\pi}{4}\right) & 0 \\ 0 & \cosh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) \end{pmatrix}, \quad (7.3)$$

where $Z(\theta)$ is a normalization factor. We have verified that this boundary matrix $R_{BF}(\theta)$, together with the bulk $S$-matrix $S_{BF}(\theta)$, satisfies the boundary Yang-Baxter equation. Clearly, we can extend the entire analysis performed in [4] and in this paper to the boundary case and explore the physical meaning of such results. This will be the subject of future work.

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