Hardy-space function theory, operator model theory, and dissipative linear systems: the multivariable, free-noncommutative, weighted Bergman-space setting

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CHAPTER 1

Introduction

1.1. Overview

For $\mathcal{X}$ and $\mathcal{Y}$ any pair of Hilbert spaces, we use the notation $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ to denote the space of bounded, linear operators from $\mathcal{X}$ to $\mathcal{Y}$, shortening the notation $\mathcal{L}(\mathcal{X}, \mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. We start with the classical discrete-time linear system

$$\Sigma(U) : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \quad (1.1)$$

with $x(k)$ taking values in the state space $\mathcal{X}$, $u(k)$ taking values in the input-space $\mathcal{U}$ and $y(k)$ taking values in the output-space $\mathcal{Y}$, where $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{X}$ are given Hilbert spaces and where the connection matrix (sometimes also called colligation matrix or system matrix of the system $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : [\mathcal{X} U] \to [\mathcal{X} \mathcal{Y}]$ is a given bounded linear operator. If we let the system evolve on the nonnegative integers $n \in \mathbb{Z}_+$, then the whole trajectory $\{u(n), x(n), y(n)\}_{n \in \mathbb{Z}_+}$ is determined from the input signal $\{u(n)\}_{n \in \mathbb{Z}_+}$ and the initial state $x(0) = x$ according to the formulas

$$x(k) = A^k x + \sum_{j=0}^{k-1} A^{k-1-j} B u(j),$$

$$y(k) = C A^k x + \sum_{j=0}^{k-1} C A^{k-1-j} B u(k) + D u(k). \quad (1.2)$$

Application of the $Z$-transform

$$\{f(k)\}_{k \in \mathbb{Z}_+} \mapsto \hat{f}(\lambda) = \sum_{k=0}^{\infty} f(k) \lambda^k$$

to the system equations (1.1) converts the expressions (1.2) to the so-called frequency-domain formulas

$$\hat{x}(\lambda) = (I - \lambda A)^{-1} x + \lambda (I - \lambda A)^{-1} B \hat{u}(\lambda),$$

$$\hat{y}(\lambda) = C(I - \lambda A)^{-1} x + [D + \lambda C(I - \lambda A)^{-1} B] \hat{u}(\lambda) = O_{C,A} x + \Theta_U(\lambda) \hat{u}(\lambda),$$

where

$$O_{C,A} : x \mapsto \sum_{k=0}^{\infty} (CA^k x) \lambda^k = C(I - \lambda A)^{-1} x \quad (1.3)$$

is the observability operator and where

$$\Theta_U(\lambda) = D + \lambda C(I - \lambda A)^{-1} B \quad (1.4)$$
is the transfer function of the system $\Sigma$ given by (1.1). In particular, if the input signal $\{u(n)\}_{n \in \mathbb{Z}_+}$ is taken to be zero, the resulting output $\{y(n)\}_{n \in \mathbb{Z}_+}$ is given by $y = \mathcal{O}_{C,A} x(0)$. If $\mathcal{O}_{C,A}$ is injective, i.e., if $(C, A)$ satisfies the so-called observability condition

$$\bigcap_{k=0}^{\infty} \ker CA^k = \{0\},$$

we say that the output pair $(C, A)$ is observable. In case $\mathcal{O}_{C,A}$ is bounded as an operator from $\mathcal{X}$ into the standard vector-valued Hardy space of the unit disk

$${H^2_{\mathcal{Y}}} = \{ f(\lambda) = \sum_{k \geq 0} f_k \lambda^k : \sum_{k \geq 0} \|f_k\|_{\mathcal{Y}}^2 < \infty \},$$

we say that the pair $(C, A)$ is output-stable.

The case where the operator connection matrix $U$ is isometric, or more generally just contractive, is of special interest. In system-theoretic terms the isometric property of $U$ has the interpretation that the system $\Sigma(U)$ is conservative in the sense that the energy stored by the state at time $k$ ($\|x(k+1)\|^2 - \|x(k)\|^2$) is exactly compensated by the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$). In case $U$ is contractive the system $\Sigma(U)$ is said to be dissipative in the sense that the net energy stored by the state at time $k$ ($\|x(k+1)\|^2 - \|x(k)\|^2$) is no more than the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$) at time $k$. In case the system is dissipative (i.e., $\|U\| \leq 1$), the transfer function $\Theta_U$ is in the Schur class $S(U, \mathcal{Y})$ (i.e., analytic on the open unit disk $\mathbb{D}$ and such that $\Theta(z)$ is a contraction in $L(U, \mathcal{Y})$ for every $z \in \mathbb{D}$), and moreover the observability operator $\mathcal{O}_{C,A} : \mathcal{X} \rightarrow H^2_{\mathcal{Y}}$ is contractive. Conversely, if $\Theta$ is in the Schur class, then $\Theta$ has a realization as $\Theta = \Theta_U$ as in (1.1) with $\Sigma(U)$ dissipative (in fact, even conservative).

If $U$ is isometric and in addition the state space operator $A$ is strongly stable in the sense that $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$, then the observability operator is a partial isometry (even an isometry in case $(C, A)$ is observable) and the transfer function $\Theta_U$ is inner (the boundary values $\Theta_U(\zeta)$ existing as strong radial limits from inside $\mathbb{D}$ for almost every $\zeta$ on the unit circle $\mathbb{T}$ are isometric operators from $U$ to $\mathcal{Y}$), and conversely: any inner function $\Theta$ arises in this way as $\Theta = \Theta_U$ with $U = [A \ B]$ isometric with $A$ strongly stable.

We say that a subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is shift-invariant if $f \in \mathcal{M} \Rightarrow S_{\mathcal{Y}} f \in \mathcal{M}$ where $S_{\mathcal{Y}}$ is the shift operator given as the coordinate multiplication operator on $H^2_{\mathcal{Y}}$

$$S_{\mathcal{Y}} = M_{\lambda} : f(\lambda) \mapsto \lambda f(\lambda).$$

Note that if $\Theta$ is inner, then $\mathcal{M} := M_{\Theta} H^2_{\mathcal{Y}} = \Theta \cdot H^2_{\mathcal{U}}$ is a shift-invariant subspace for $S_{\mathcal{Y}}$; the content of the Beurling-Lax theorem is that conversely, any such invariant subspace can be represented in this way. Similarly we say that the subspace $\mathcal{N} \subset H^2_{\mathcal{Y}}$ is backward shift-invariant if $f \in \mathcal{N} \Rightarrow S^*_{\mathcal{Y}} f \in \mathcal{N}$ where the backward-shift operator $S^*_{\mathcal{Y}}$, the Hilbert-space adjoint of the forward-shift operator $S_{\mathcal{Y}}$, works out to be

$$S^*_{\mathcal{Y}} : f(\lambda) \mapsto [f(\lambda) - f(0)]/\lambda.$$

The computation

$$S^*_{\mathcal{Y}} : C(I - \lambda A)^{-1} x \mapsto z^{-1}(C(I - \lambda A)^{-1} - C)x = C(I - \lambda A)^{-1} Ax$$

is in the transfer function of the system $\Sigma$ given by (1.1). In particular, if the input signal $\{u(n)\}_{n \in \mathbb{Z}_+}$ is taken to be zero, the resulting output $\{y(n)\}_{n \in \mathbb{Z}_+}$ is given by $y = \mathcal{O}_{C,A} x(0)$. If $\mathcal{O}_{C,A}$ is injective, i.e., if $(C, A)$ satisfies the so-called observability condition

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If $U$ is isometric and in addition the state space operator $A$ is strongly stable in the sense that $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$, then the observability operator is a partial isometry (even an isometry in case $(C, A)$ is observable) and the transfer function $\Theta_U$ is inner (the boundary values $\Theta_U(\zeta)$ existing as strong radial limits from inside $\mathbb{D}$ for almost every $\zeta$ on the unit circle $\mathbb{T}$ are isometric operators from $U$ to $\mathcal{Y}$), and conversely: any inner function $\Theta$ arises in this way as $\Theta = \Theta_U$ with $U = [A \ B]$ isometric with $A$ strongly stable.

We say that a subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is shift-invariant if $f \in \mathcal{M} \Rightarrow S_{\mathcal{Y}} f \in \mathcal{M}$ where $S_{\mathcal{Y}}$ is the shift operator given as the coordinate multiplication operator on $H^2_{\mathcal{Y}}$

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$$S^*_{\mathcal{Y}} : f(\lambda) \mapsto [f(\lambda) - f(0)]/\lambda.$$

The computation

$$S^*_{\mathcal{Y}} : C(I - \lambda A)^{-1} x \mapsto z^{-1}(C(I - \lambda A)^{-1} - C)x = C(I - \lambda A)^{-1} Ax$$
shows that, for any output-stable pair $(C, A)$, Ran $O_{C,A}$ is $S_Y^*$-invariant. Conversely, if $M^\perp \subset H^2_Y$ is $S_Y^*$-invariant, then there is an output pair $(C, A)$ (with $C^*C = I - A^*A$) so that $M^\perp = \text{Ran} \ O_{C,A}$ (see [1] for additional background and generalizations). Moreover the Sz.-Nagy–Foias characteristic function for a completely nonunitary contraction operator $T$ on a Hilbert space $\mathcal{H}$

$$\Theta_T(\lambda) = [-T + \lambda D_T, (I - \lambda T^*)^{-1}D_T]: D_T \to D_T^*,$$

(1.7)

where we use the standard notation

$$D_T = (I - T^*T)^{\frac{1}{2}}, \quad D_{T*} = (I - TT^*)^{\frac{1}{2}}, \quad D_T = \text{Ran} \ D_T, \quad D_{T*} = \overline{\text{Ran}} \ D_{T*},$$

amounts to the transfer function $\Theta_U$ associated with the unitary connection matrix

$$U = \begin{bmatrix} T^* & D_T \\ D_{T*} & -T \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ D_T \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ D_{T*} \end{bmatrix}.$$ 

In summary the following themes have developed and matured over the last several decades connecting vectorial Hardy-space function theory, theory of Hilbert-space contraction operators, and conservative discrete-time linear systems:

1. A backward-shift invariant subspace of $H^2_Y$ arises as the range of some observability operator.
2. A forward-shift invariant subspaces of $H^2_Y$ has Beurling-Lax inner-function representations.
3. In the case of a conservative linear system with strongly stable state operator $A$ (i.e., $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is isometric and $\|A^nx\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$), the observability operator $O_{C,A} : \mathcal{X} \to H^2_Y$ and the transfer-function multiplier operator $M_{\Theta_U}$ are isometric, and one has the orthogonal decomposition

$$H^2_Y = \text{Ran} \ O_{C,A} \oplus \text{Ran} \ M_{\Theta_U},$$

and conversely: if $M \subset H^2_Y$ is $S_Y$-invariant (and hence $M^\perp \subset H^2_Y$ is $S_Y^*$-invariant), then there is an isometric $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A$ strongly stable such that $M^\perp = \text{Ran} \ O_{C,A}$ and $M = \Theta_U \cdot H^2_Y$.

4. Inner (and more generally, contractive) multipliers $M_A : H^2_U \to H^2_Y$ arise as the Sz.-Nagy–Foias characteristic function for some completely nonunitary Hilbert-space contraction operator $T$ which in turn induces a canonical functional model for the operator $T$.

Much work has been done to extend this set of ideas, particularly themes #2 and #4 above (the operator-model theory aspects without the system-theoretic connections) to more general settings, e.g.,

(i) to Bergman-spaces and hypercontraction operators; see Agler [2], Müller [71], Mülller-Vasilescu [72], Hedenmalm-Korenblum-Zhu [60], Duren-Schuster [46]),

(ii) to the Drury-Arveson space and commutative row-contractive operator tuples; see Bhattacharyya-Eschmeier-Sarkar [30, 31], Bhattacharyya-Sarkar [32], Ball-Bolotnikov [13],

(iii) to more general domains in $\mathbb{C}^d$ than the ball and associated more general commutative operator tuples; see Athavale [12], Curto-Vasilescu [40, 41], Timotin [100], Pott [91], Ambrozic-Engliš-Müller [9], Arazy-Engliš [10],

(iv) to the full Fock space and freely noncommutative row-contractive operator tuples, possibly also constrained to lie in a prescribed noncommutative operator variety; see Bunce [37], Frazho [52], Popescu [78, 79, 80, 81, 84, 85],
(v) to a more general formalism of representations of certain operator algebras based on tensor-algebra constructions; see Muhly-Solel [67, 69, 68, 70], and
(vi) to noncommutative hypercontractive operator tuples modeled on noncommutative varieties (see Popescu [88, 89, 90]) as well as a weighted version of the tensor-algebra context (see Muhly-Solel [70]).

Identification of a characteristic function defined by a formula of the Sz.-Nagy–Foias type [17] (the main thrust of theme #4 above) can be found (i) for the Bergman space setting only recently first in the work of Olofsson [73, 74, 75] and then followed up by the authors [14, 15], (ii) for the Drury-Arveson setting earlier in the work of Bhattacharyya-et-al [50, 51, 32], (iv) for the full Fock space in the work of Popescu [80], Ball-Bolotnikov-Fang [20, 21] and Ball-Vinnikov [28], for the tensor-algebra context in Muhly-Solel [68].

Let us note at this stage that theme #4 simplifies considerably in case the completely nonunitary contraction operator $T$ is in the Sz.-Nagy–Foias class $C_0$ (i.e., $T^*$ is strongly stable or equivalently, the contraction operator $T$ is pure in the terminology of some authors). Indeed, $T \in C_0$ is equivalent to $\Theta_T$ being inner, i.e., we are in the setting of item #3 in the list of themes above: $H^2_{D_T^*} = \mathcal{N} \oplus \mathcal{M}$ where $\mathcal{N} = \text{Ran} \mathcal{O}_{D_T^*, T^*}$ is backward-shift invariant and $\mathcal{M} = \Theta_T \cdot H^2_{D_T^*}$ is forward-shift invariant. It then turns out that $T$ is unitarily equivalent to its Sz.-Nagy–Foias model operator $\mathbf{T}$ given simply by $\mathbf{T} := P_{\mathcal{N}} M_{\lambda}|_{\mathcal{N}}$ where $M_{\lambda}: f(\lambda) \mapsto \lambda f(\lambda)$ is the coordinate multiplication operator on $H^2_{D_T^*}$.

In the multivariable context, with only a couple of exceptions (see [13, 28]) the full completely nonunitary Sz.-Nagy–Foias model theory has proved to be elusive. A common compromise choice for moving past the $C_0$ (or pure) case is to invoke an analogue of the assumption that $T$ is completely non-coisometric (see [31, 80, 68]). In this case the observability operator $\mathcal{O}_{D_T^*, T^*}$ is no longer isometric but is still injective (i.e. the output pair $(D_T^*, T^*)$ is observable) and one can equip $\text{Ran} \mathcal{O}_{D_T^*, T^*}$ with the lifted norm from $\mathcal{X}$ (rather than the induced norm from containment in the ambient Hardy space $H^2_{D_T^*}$) to view $T^*$ as unitarily equivalent to the restriction of the backward shift $S_{D_T^*}$ to the subspace $\text{Ran} \mathcal{O}_{D_T^*, T^*}$ which is now only contractively included in $H^2_{D_T^*}$. Many authors construct an additional defect space $\Delta_{\Theta_T} H^2_{D_T^*}$ so that $\text{Ran} \mathcal{O}_{D_T^*, T^*}$ (with lifted norm from $\mathcal{X}$) can be identified isometrically with a subspace of the two-component space $\text{Ran} \mathcal{O}_{D_T^*, T^*} \oplus \Delta_{\Theta_T} H^2_{D_T^*}$ (with the first component now taken with $H^2_{D_T^*}$-norm), but from our point of view this is not necessary. Then there is a Cholesky-factorization algorithm for constructing an operator $[B]: \mathcal{U} \rightarrow [D_T^*]$ so that $\mathbf{U} = [B]: [D_T^*] \rightarrow [D_T^*]$ is unitary (for the classical case one simply takes $\mathcal{U} = D_T, B = D_T: D_T \rightarrow \mathcal{H}$ and $D = -T|_{D_T}$, and we can define the transfer function $\Theta_\mathbf{U}$ associated with connection matrix $\mathbf{U}$ to be the characteristic function of $T$. Then the backward shift $S_{D_T^*}$ restricted to the subspace $\mathcal{N} := \text{Ran} \mathcal{O}_{D_T^*, T^*}$ considered with lifted norm is the de Branges-Rovnyak model for the original contraction operator $T$. The link with the characteristic function $\Theta_T$ is due to the validity of the formula

$$D_T^* (I - \lambda T)^{-1} (I - \zeta T)^{-1} D_T^* + \frac{\Theta_T(\lambda) \Theta_T(\zeta)^*}{1 - \lambda \zeta} = \frac{I_{D_T^*}}{1 - \zeta \lambda}$$
1.2. STANDARD WEIGHTED BERGMAN SPACES

(Where \( k(\lambda, \zeta) = \frac{1}{1 - \lambda \zeta} \) is the reproducing kernel for \( H^2 \)) which can be interpreted as saying that the subspace \( \mathcal{M} := \Theta \cdot H^2_0 \subset H^2_\mathcal{D} \), with lifted norm is the Brangesian complement of the backward-shift invariant subspace \( \mathcal{N} = \text{Ran} \mathcal{O}_{\mathcal{D}, T} \cdot T \) (also contractively included in \( H^2_\mathcal{D} \)); see Section 3.1.1 below. The idea of the generalized Beurling-Lax Theorem whereby one represents contractively included forward-shift invariant subspaces as being of the form \( \Theta \cdot H^2_0 \) with \( \Theta \) being in the Schur class rather than inner appears already in the work of de Branges-Rovnyak [35] and indeed the de Branges-Rovnyak model space \( \mathcal{H}(\Theta) \) for the non-inner case can be considered as a contractively included backward-shift invariant subspace \( \mathcal{N} \) as above.

Here we focus on a general setting of model spaces, forward and backward shift-operator tuples, and their joint invariant subspaces which simultaneously contain as special cases the Bergman setting (i) and the full Fock space setting (iv) mentioned above. Before plunging into the most general setting, we next sketch how the system theory setup (1.1), (1.2), (1.3), (1.4), (1.5) adapts to these motivating special cases.

1.2. Standard weighted Bergman spaces

For a Hilbert space \( Y \) and an integer \( n \geq 1 \), we denote by \( A_n, Y \) the Hilbert space of \( Y \)-valued functions \( f \) analytic in the open unit disk \( D \) and with finite norm

\[
\|f\|_{A_n, Y}^2 = \sum_{j \geq 0} \mu_{n,j} \|f_j\|_Y^2 < \infty,
\]

where the weights \( \mu_{n,j} \)'s are defined by

\[
\mu_{n,j} = \frac{j!(n-1)!}{(j+n-1)!}.
\]

The space \( A_n, Y \) can be alternatively characterized as the reproducing kernel Hilbert space with reproducing kernel \( k_n(\lambda, \zeta)I_Y \) where

\[
k_n(\lambda, \zeta) = (1 - \lambda \zeta)^{-n}.
\]

We introduce the function \( R_n \) and its shifted counterparts \( R_{n,k} \) by the formulas

\[
R_n(\lambda) := (1 - \lambda)^{-n} = \sum_{j=0}^{\infty} \mu_{n,j}^{-1} \lambda^j \quad \text{and} \quad R_{n,k}(\lambda) = \sum_{j=0}^{\infty} \mu_{n,j+k}^{-1} \lambda^j,
\]

so that \( R_{n,0} = R_n \) and \( k_n(\lambda, \zeta) = R_n(\lambda \zeta) \). Observe that functions (1.10) satisfy the following relations:

\[
R_{n,k}(\lambda) = \binom{n+k-1}{k} + \lambda R_{n,k+1}(\lambda),
\]

\[
R_{n,k}(\lambda) = \sum_{\ell=1}^{n} \binom{\ell+k-3}{\ell-1} R_{n-\ell+1}(\lambda) \quad \text{for} \quad k \geq 1.
\]

Identity (1.11) follows directly from definition (1.10) while the proof of (1.12) can be found in [14] Section 2. We also record that for any operator \( A \in \mathcal{L}(X) \) with spectral radius \( \rho_A \), the operator-valued functions

\[
R_{n,k}(\lambda A) = \sum_{j=0}^{\infty} \mu_{n,k+j}^{-1} A^j \lambda^j
\]

are analytic on the disk \( \{ \lambda : |\lambda| < 1/\rho_A \} \) for any \( k \in \mathbb{Z}_+ \).
In [14], we considered the following discrete-time time-varying linear system:

\[
\Sigma_n \left( \left\{ \begin{bmatrix} A & B_j \\ C & D_j \end{bmatrix} \right\}_{j \in \mathbb{Z}_+} \right) : \quad x(j + 1) = \frac{j + n}{j + 1} \cdot Ax(j) + \frac{(j + n)}{j + 1} \cdot B_j u(j), \\
y(j) = Cx(j) + (j + n - 1) \cdot D_j u(j)
\]  

(1.14)

where \( A \in \mathcal{L}(X) \), \( C \in \mathcal{L}(X, \mathcal{Y}) \), \( B_k \in \mathcal{L}(U_k, X) \), \( D_k \in \mathcal{L}(U_k, \mathcal{Y}) \) are given bounded linear operators acting between given Hilbert spaces \( X, \mathcal{Y} \) and \( U_k \) \((k \geq 0)\). We note that the case where \( n = 1 \) and where the operators \( B_k = B \) and \( D_k = D \) are taken independent of the time parameter \( k \in \mathbb{Z}_+ \) reduces to the classical time-invariant case [1.1]. If we let the system (1.14) evolve on \( \mathbb{Z}_+ \), then the whole trajectory \( \{u(j), x(j), y(j)\}_{j \in \mathbb{Z}_+} \) is determined from the input signal \( \{u(j)\}_{j \in \mathbb{Z}_+} \) and the initial state \( x(0) \) according to the formulas

\[
x(j) = \mu_{n,j}^{-1} \cdot (A^j x(0) + \sum_{\ell=0}^{j-1} A^{j-\ell-1} B_\ell u(\ell)),
\]

(1.15)

\[
y(j) = \mu_{n,j}^{-1} \cdot (CA^j x(0) + \sum_{\ell=0}^{j-1} CA^{j-\ell-1} B_\ell u(\ell) + D_j u(j)).
\]

(1.16)

Formula (1.15) is established by simple induction arguments, while (1.16) is obtained by substituting (1.15) into the second equation in (1.14).

To write the \( Z \)-transformed version of the system-trajectory formula (1.15), we multiply both sides of (1.15) by \( \lambda^j \) and sum over \( j \geq 0 \) to get, on account of (1.13),

\[
\hat{x}(\lambda) = \sum_{j=0}^{\infty} x(j) \lambda^j = \left( \sum_{j=0}^{\infty} \mu_{n,j}^{-1} A^j \lambda^j \right) x(0) + \sum_{k=1}^{\infty} \left( \sum_{j=k}^{\infty} \mu_{n,j-k}^{-1} A^{j-k} \lambda^j \right) B_{k-1} u(k - 1)
\]

\[
= (I - \lambda A)^{-n} x(0) + \sum_{k=1}^{\infty} \lambda^k \left( \sum_{j=0}^{\infty} \mu_{n,j+k}^{-1} A^j \lambda^j \right) B_{k-1} u(k - 1)
\]

\[
= (I - \lambda A)^{-n} x(0) + \sum_{k=0}^{\infty} \lambda^{k+1} R_{n,k+1}(\lambda A) B_k u(k).
\]

The same procedure applied to (1.16) gives

\[
\hat{y}(\lambda) = C(I - \lambda A)^{-n} x(0) + \sum_{k=0}^{\infty} \lambda^k \left( \mu_{n,k}^{-1} D_k + \lambda CR_{n,k+1}(\lambda A) B_k \right) u(k)
\]

\[
= \mathcal{O}_{n,C,A} x(0) + \sum_{k=0}^{\infty} \lambda^k \Theta_{n,k}(\lambda) u(k),
\]

(1.17)

where

\[
\mathcal{O}_{n,C,A} : \quad x \mapsto \sum_{j=0}^{\infty} \left( \mu_{n,j}^{-1} CA^j x \right) \lambda^j = C(I - \lambda A)^{-n} x
\]

(1.18)

is the \( n \)-observability operator and where

\[
\Theta_{n,k}(\lambda) = \mu_{n,k}^{-1} D_k + \lambda CR_{n,k+1}(\lambda A) B_k \quad (k = 0, 1, \ldots)
\]

is the family of transfer functions.

Note that observability of the output pair \((C, A)\) in the classical sense (1.5) is also equivalent to the injectivity of the \( n \)-observability operator \( \mathcal{O}_{n,C,A} \) (1.18).
Following [14], we say that the output pair \((C, A)\) is \(n\)-output stable if \(\mathcal{O}_{n, C, A}\) is bounded as an operator from \(\mathcal{X}\) into \(\mathcal{A}_{n, \mathcal{Y}}\).

We note next that the transfer function \(\Theta_{n,k}(z)\) encodes the result of a pulse input-vector \(u\) being applied at time \(j = k\):

\[
\hat{y}(\lambda) = \Theta_{n,k}(\lambda) \cdot \lambda^k u \quad \text{if} \quad x(0) = 0 \quad \text{and} \quad u(j) = \delta_{j,k} u
\]

(where \(\delta_{jk}\) stands for the Kronecker symbol). In fact the functions \(\Theta_{n,k}(\lambda)\) could have been derived in this way and then one could arrive at input-output relation (1.17) via superposition of all these time-\(k\) impulse responses. There is a notion of conservative for a system of the form (1.14) involving the connection matrix \(\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix}\) being unitary with respect to an appropriate choice of weights (see formulas (6.7) and (6.21) in [14]). When these metric constraints are satisfied, the associated transfer-function family \(\{\Theta_{n,k}\}\) serves as a representer of a shift-invariant subspace in the weighted Bergman space while the image space of an observability operator \(\mathcal{O}_{n, C, A}\) is the model for a backward shift invariant subspace in \(\mathcal{A}_{n, \mathcal{Y}}\) (see [14], [15]).

### 1.3. The Fock space setting

The classical results on the system (1.1) admit nice extensions to a number of multivariable settings, both commutative and noncommutative. In this section we recall the case where the Hardy space \(H^2_\mathcal{Y}\) is replaced by the Fock space \(H^2_\mathcal{Y}(\mathbb{F}^+_{d})\).

To define the Fock space, we let \(\mathbb{F}^+_{d}\) denote the unital free semigroup (i.e., monoid) generated by the set of \(d\) letters \(\{1, \ldots, d\}\). Elements of \(\mathbb{F}^+_{d}\) are words of the form \(i_N \cdots i_1\) where \(i_\ell \in \{1, \ldots, d\}\) for each \(\ell \in \{1, \ldots, N\}\) with multiplication given by concatenation. The unit element of \(\mathbb{F}^+_{d}\) is the empty word denoted by \(\emptyset\). For \(\alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{F}^+_{d}\), we let \(|\alpha|\) denote the number \(N\) of letters in \(\alpha\) and we let \(\alpha^\dagger := i_1 \cdots i_{N-1} i_N\) denote the transpose of \(\alpha\). We let \(z = (z_1, \ldots, z_d)\) to be a collection of \(d\) formal noncommuting variables and let \(\mathcal{Y}(\langle z \rangle)\) denote the set of noncommutative formal power series \(\sum_{\alpha \in \mathbb{F}^+_{d}} f_\alpha z^\alpha\) where \(f_\alpha \in \mathcal{Y}\) and where

\[
z^\alpha = z_{i_N} z_{i_{N-1}} \cdots z_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \cdots i_1.
\]

The Fock space \(H^2_\mathcal{Y}(\mathbb{F}^+_{d})\) is then defined as

\[
H^2_\mathcal{Y}(\mathbb{F}^+_{d}) = \left\{ \sum_{\alpha \in \mathbb{F}^+_{d}} f_\alpha z^\alpha \in \mathcal{Y}(\langle z \rangle) : \sum_{\alpha \in \mathbb{F}^+_{d}} \|f_\alpha\|^2_\mathcal{Y} < \infty \right\}.
\]

The Fock-space counterpart of (1.11) is the system

\[
\Sigma(\mathbf{U}) : \begin{cases}
  x(1\alpha) &= A_1 x(\alpha) + B_1 u(\alpha) \\
  \vdots &= \vdots \\
  x(d\alpha) &= A_d x(\alpha) + B_d u(\alpha) \\
  y(\alpha) &= C x(\alpha) + D u(\alpha)
\end{cases}
\]

which evolves along the free semigroup \(\mathbb{F}^+_{d}\), and, for each \(\alpha \in \mathbb{F}^+_{d}\), the state vector \(x(\alpha)\), input signal \(u(\alpha)\) and output signal \(y(\alpha)\) take values in the state space \(\mathcal{X}\),
input space \( \mathcal{U} \) and output space \( \mathcal{Y} \). The connection matrix \( \mathbf{U} \) has the form

\[
\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.
\]

(1.22)

Such systems were introduced in [28] and with further elaboration in [22] and [23]; following [22] we call this type of system a noncommutative Fornasini-Marchesini linear system.

We extend the noncommutative functional calculus \((1.10)\) from noncommuting indeterminates \( z = (z_1, \ldots, z_d) \) to a \( d \)-tuple of operators \( \mathbf{A} = (A_1, \ldots, A_d) \) by letting

\[
\mathbf{A}^\alpha := A_{i_1} A_{i_2} \cdots A_{i_d} \text{ if } \alpha = i_1i_2 \cdots i_1 \in \mathbb{F}_d^+, \tag{1.23}
\]

where the multiplication is now operator composition. Letting

\[
Z(z) = [z_1, \ldots, z_d] \otimes I_X, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}, \tag{1.24}
\]

we next observe that

\[
(Z(z)A)^j = \left( \sum_{i=1}^{d} z_i A_i \right)^j = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = j} \mathbf{A}^\alpha z^\alpha \text{ for all } j \geq 0 \tag{1.25}
\]

and therefore,

\[
(I - Z(z)A)^{-1} = \sum_{j=0}^{\infty} (Z(z)A)^j = \sum_{j=0}^{\infty} \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = j} \mathbf{A}^\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} \mathbf{A}^\alpha z^\alpha.
\]

Application of the formal noncommutative \( Z \)-transform

\[
\{f_\alpha\}_{\alpha \in \mathbb{F}_d^+} \mapsto \hat{f}(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha
\]

(1.26)

to the system \((1.21)\) then gives

\[
\hat{x}(z) = (I - Z(z)A)^{-1} x(\emptyset) + (I - Z(z)A)^{-1} Z(z) B \hat{u}(z),
\]

\[
\hat{y}(z) = \mathcal{O}_{C,A} x(\emptyset) + \Theta_{U}(z) \hat{u}(z),
\]

(1.27)

where

\[
\mathcal{O}_{C,A} : x \mapsto C(I - Z(z)A)^{-1} x = \sum_{\alpha \in \mathbb{F}_d^+} (CA^\alpha x) z^\alpha
\]

(1.28)

is the observability operator of \((C, \mathbf{A})\) and where \( \Theta_{U}(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle \) is given by

\[
\Theta_{U}(z) = D + C(I - Z(z)A)^{-1} Z(z) B = D + \sum_{\alpha \in \mathbb{F}_d^+} \sum_{j=1}^{d} CA^\alpha B_j z^\alpha z_j.
\]

(1.29)

Thus the initial state \( x = x_0 \) is uniquely determined by the output signal \( \hat{y}(z) \) when the input signal \( \hat{u}(z) \) is taken to be zero exactly when \( \mathcal{O}_{C,A} \) is injective; when this is the case, we say that the output pair \((C, \mathbf{A})\) is observable. The pair \((C, \mathbf{A})\) is called output-stable if \( \mathcal{O}_{C,A} \) is bounded as an operator from \( \mathcal{X} \) into \( H^2(\mathbb{F}_d^+) \), and exactly observable if \( \mathcal{O}_{C,A} \) is bounded and bounded below. As in the single-variable
1.4. Weighted Bergman-Fock spaces

We introduce a family of weighted Bergman-Fock spaces as a multivariable noncommutative counterpart of standard weighted Bergman spaces; the system-theoretic point of view presented here combines the single-variable setting handled in [14, 15] with the unweighted multivariable setting from section 1.3. Given an integer \( n \geq 1 \), the free semigroup \( \mathbb{F}_d^+ \), and the coefficient Hilbert space \( \mathcal{Y} \), we let

\[
A_{n, \mathcal{Y}}(\mathbb{F}_d^+) = \left\{ \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}(\langle z \rangle) : \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|} \| f_\alpha \|_\mathcal{Y}^2 < \infty \right\}
\]  

(1.31)

where, according to (1.8), \( \mu_{n,|\alpha|} = \frac{|\alpha|^n (n-1)!}{(n+|\alpha|-1)!} \). We propose to consider the following multidimensional system with evolution along the free semigroup \( \mathbb{F}_d^+ \):

\[
\Sigma_{\{U_\alpha, n\}} : \begin{cases} 
  x(1\alpha) &= \frac{n+|\alpha|}{|\alpha|+1} A_1 x(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{1,\alpha} u(\alpha) \\
  \vdots & \vdots \\
  x(d\alpha) &= \frac{n+|\alpha|}{|\alpha|+1} A_d x(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{d,\alpha} u(\alpha) \\
  y(\alpha) &= C x(\alpha) + \binom{n+|\alpha|-1}{|\alpha|} D_\alpha u(\alpha)
\end{cases}
\]  

(1.32)

with the \( d \)-tuple of state space operators \( A = (A_1, \ldots, A_d) \) and the state-output operator \( C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \). Here in addition we have a family of connection matrices and the family of input spaces indexed by \( \alpha \in \mathbb{F}_d^+ \):

\[
U_\alpha = \begin{bmatrix} A & \tilde{B}_\alpha \\ C & D_\alpha \end{bmatrix} : \mathcal{X} \to \mathcal{X}^{d} \mathcal{Y}, \quad \text{where} \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad \tilde{B}_\alpha = \begin{bmatrix} B_{1,\alpha} \\ \vdots \\ B_{d,\alpha} \end{bmatrix}
\]  

(1.33)

together with additional \( \alpha \)-dependent weights in the system equations indexed by the natural number \( n \). Upon running the system (1.32) with a fixed initial condition.
x(∅) = x ∈ X we get recursively

\[ x(α) = μ_{n,|α|}^{-1} \left( A^α x + \sum_{α'' | α''α = α} A^{α''} B_{j,α'} u(α') \right), \]  

(1.34)

\[ y(α) = μ_{n,|α|}^{-1} \left( CA^α x + \sum_{α'' | α''α = α} CA^{α''} B_{j,α'} u(α') + D_{α} u(α) \right). \]  

(1.35)

Making use of notation (1.24) and equality (1.25) we observe that

\[ (I - Z(z)A)^{-n} = \sum_{j=0}^{∞} \mu_{n,j}^{-1} \sum_{α \in \mathbb{F}_d^+ | |α| = j} A^α z^α = \sum_{α \in \mathbb{F}_d^+} \mu_{n,|α|}^{-1} A^α z^α, \]  

(1.36)

and then define \( R_{n,k}(Z(z)A) \) via formal power series

\[ R_{n,k}(Z(z)A) = \sum_{α \in \mathbb{F}_d^+} \mu_{n,|α|+k}^{-1} A^α z^α. \]  

(1.37)

We next apply the noncommutative \( Z \)-transform (1.26) to (1.34) and then invoke (1.36), (1.37) to get

\[ \hat{x}(z) = \sum_{α \in \mathbb{F}_d^+} \mu_{n,|α|}^{-1} \left( A^α x + \sum_{α'' | α''α = α} A^{α''} B_{j,α'} u(α') \right) z^α \]

\[ = \sum_{α \in \mathbb{F}_d^+} \left( \mu_{n,|α|}^{-1} A^α x \right) z^α \]

\[ + \sum_{α' \in \mathbb{F}_d^+} \left( \sum_{α'' | α''α' = α'} A^{α''} z^{α''} \right) \left( \sum_{j=1}^{d} z_j B_{j,α'} \right) z^{α'} u(α') \]

\[ = (I - Z(z)A)^{-n} x + \sum_{α \in \mathbb{F}_d^+} R_{n,|α|+1}(Z(z)A)Z(z)B_α z^α u(α). \]

The same procedure applied to (1.35) now gives

\[ \hat{y}(z) = C(I - Z(z)A)^{-n} x \]

\[ + \sum_{α \in \mathbb{F}_d^+} \left( CR_{n,|α|+1}(Z(z)A)Z(z)B_α + \mu_{n,|α|}^{-1} D_{α} \right) z^α u(α) \]

\[ = O_{n,C,A}(z)x + \sum_{α \in \mathbb{F}_d^+} Θ_{n,U_A}(z) z^α u(α), \]  

(1.38)

where the first term on the right presents the \( n \)-observability operator

\[ O_{n,C,A}(z)x = C(I - Z(z)A)^{-n} x = \sum_{α \in \mathbb{F}_d^+} \mu_{n,|α|}^{-1} (CA^α x) z^α \]  

(1.39)

associated with the state space \( d \)-tuple \( A \) and the state-output operator \( C \) and where

\[ Θ_{n,U_A}(z) = \mu_{n,|α|}^{-1} D_{α} + CR_{n,|α|+1}(Z(z)A)Z(z)B_α \]  

(1.40)

is the family of transfer functions indexed by \( α \in \mathbb{F}_d^+ \). One can see that the notion of the \( n \)-observability operator (1.39) generalizes the single-variable notion (1.18) as well as the unweighted multivariable one in (1.28). We say that the output pair \( (C, A) \) is \( n \)-observable if \( O_{n,C,A} \) is injective; from (1.39) we see that this is equivalent
to \((C, A)\) being observable when viewed as an output pair for an unweighted system as in \((1.28)\), i.e., observability is equivalent to

\[
\bigcap_{\alpha\in\mathbb{F}^+_{d}} \text{Ker } CA^\alpha = \{0\}. \tag{1.41}
\]

We say that the output pair \((C, A)\) is \(n\)-output stable if \(O_{n,C,A}\) is bounded as an operator from \(X\) into \(A_{n,Y}(\mathbb{F}^+_{d})\) and exactly \(n\)-observable if also \(O_{n,C,A}\) is bounded below.

In parallel with the discussion at the end of Section 1.2 we use the formula \((1.38)\) to view the transfer function \(\Theta_{n,U_n}(z)\) as encoding the result of a pulse input vector \(u\) being applied at position \(\alpha\in\mathbb{F}^+_{d}\) with zero initial state:

\[
\hat{y}(z) = \Theta_{n,U_n}(z) \cdot z^\alpha u \text{ if } x_0 = 0 \text{ and } u(\beta) = \delta_{\alpha,\beta} u.
\]

A preliminary notion of noncommutative \(n\)-Bergman conservative system for systems of the form \((1.32)\) will be developed in Section 7.1 below. The associated \(n\)-Bergman inner family is the main ingredient for one version of a Beurling-Lax representation for forward-shift invariant subspaces for this setting (see Section 7.2 below). We shall see that backward shift-invariant subspaces in this setting arise as the range of an \(n\)-observability operator form \(O_{n,C,A}\) for an appropriate choice of a state-space \(d\)-tuple \(A\) and an state-output operator \(C\).

The main goal of this paper is to carry out the program outlined above in themes \#1–\#4 (including the refinements where \(\Theta_T\) is allowed to be a non-inner Schur-class function and \(T\) is not required to be pure but only completely noncoisometric) for the setting where the system \(\Sigma(U)\) \((1.1)\) is replaced by the \((\text{time-varying})\) system \(\Sigma\{U_n\}_n\) (for a fixed \(n\in\mathbb{N}\)), where the Hardy space \(H^2_\Sigma\) is replaced by the weighted Bergman-Fock space \(A_{n,Y}(\mathbb{F}^+_{d})\) \((1.31)\), where the observability operator \(O_{C,A}\) \((1.3)\) becomes the \(n\)-observability operator \(O_{n,C,A}\) \((1.39)\), where the transfer function \(\Theta_U(\lambda)\) \((1.4)\) becomes the family of formal-power-series transfer functions \(\{\Theta_{n,U_n}(\lambda)\}_{\alpha\in\mathbb{F}^+_{d}}\) \((1.40)\), where the shift operator \(S_Y\) \((1.6)\) becomes the right-shift operator tuple \(S_{\mu_n} = (S_{\mu_{n,R,1}}, \ldots, S_{\mu_{n,R,d}})\) on \(A_{n,Y}(\mathbb{F}^+_{d})\) \((1.41)\), and where the contraction operator \(T\) becomes a \(\alpha\)-\(n\)-hypercontractive operator tuple, i.e., an operator tuple \(T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^d\) such that

\[
(I - B_T^\alpha)^m[I] = \sum_{\alpha\in\mathbb{F}^+_{2^d}: |\alpha| \leq m} (-1)^{|\alpha|} \left[ \begin{bmatrix} m \\ |\alpha| \end{bmatrix} \right] T^{\alpha^\top} T^{\alpha} \geq 0 \tag{1.42}
\]

for \(1 \leq m \leq n\), where in general \(B_T[X] = \sum_{j=1}^d T_j X T_j\).

Our results are actually more general than the setting based on the reciprocal binomial coefficient weights \(\mu_n\). In section 2.2 we introduce a class of weights \(\omega = \{\omega_j\}_{j\geq 0}\) satisfying certain natural admissibility conditions (automatically satisfied by \(\omega = \mu_n\) for all \(n = 1, 2, \ldots\)) and eventually show how the whole program \#1–\#4 extends to this level of generality. Let us point out that the setting \(\omega = \mu_n\) is also part of the setting studied in the recent paper of Popescu \([90]\). Many pieces of the program \#1–\#4 for the setting with \(\omega = \mu_n\) are also handled in Popescu’s paper, but many of our results along with our point of view and approach are complementary. The setting with a general admissible \(\omega\) not equal to \(\mu_n\) however appears to be a strict generalization and is not covered by Popescu’s results.
We also discuss a generalization of the setting $\omega = \mu_n$ whereby one fixes a free noncommutative function given by a formal power series in freely noncommuting arguments $z = (z_1, \ldots, z_d)$

$$p(z) = \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha z^\alpha \quad (1.43)$$

which is regular, meaning that

$$p_\emptyset = 0, \quad p_\alpha > 0 \text{ if } |\alpha| = 1, \quad p_\alpha \geq 0 \text{ for all } \alpha \in \mathbb{F}_d^+,$$

the series (1.43) has positive radius of convergence $\rho > 0$: $p(A) = \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha A^\alpha$ converges absolutely whenever $\| [A_1 \cdots A_d] \| < \rho$. We then consider the class of operator tuples $T = (T_1, \ldots, T_d)$ which are $\ast (p, n)$-hypercontractive in the sense that condition (1.42) is replaced by

$$(1 - p)^m (B_{T_i}) [I] \succeq 0 \text{ for } m = 1, \ldots, n. \quad (1.44)$$

In Chapter 9 we sketch how to carry out the program #1–#4 for this setting. This generalization originates in the work of Pott [91] for the univariate case and is handled by Popescu in [87] for the case $\omega = \mu_1$ and in [90] for the case $\omega = \mu_n$. The setting in [90] has an additional layer of flexibility: there is incorporated a constraint on the freeness of the lack of commutativity in the components $T_1, \ldots, T_d$ of the tuple $T$; in particular the set of constraints $T_i T_j = T_j T_i$ implies that one can pick up the commutative version of the setting as a special case. Let us mention that the related Popescu paper [89] has still another layer of flexibility which allows one to pick up the commutative polydisk and the freely noncommutative ball as special cases.

This paper is organized as follows:

After the present Introduction, Chapter 2 recalls the notion of the noncommutative formal reproducing kernel Hilbert space (NFRKHS) from [27, 28] which will be a substantial tool for the subsequent analysis here.

The notion of contractive multiplier between two NFRKHSs works out well and is the topic of Chapter 3. We specialize the general setting to the case of contractive multipliers from one Fock space into another as well as from a Fock space to a more general weighted Bergman-Fock spaces, with particular focus on inner multipliers (taken in various distinct senses). Some of these results are obtained by using a noncommutative version of Leech theorem, [65], two different proofs of which are presented in Section 3.3.

Chapter 4 provides an elaboration of theme #1 (backward-shift invariant subspaces and observability operators) in these more general settings. Much of the analysis hinges on the study of multivariable Stein equations and connections with stability for discrete-time linear systems—a standard topic in classical linear system theory in the univariate setting (see e.g. the book of Dullerud–Paganini [47]). Included here is an exposition of the basic properties of the model shift-operator tuple on $H^2_{\omega, \lambda} (\mathbb{F}_d^+)$ and a converse characterization of which $\omega$-isometric operator tuples are unitarily equivalent to such a shift-operator tuple, as the shift part in the Wold decomposition for a more general class of operator tuples which we call $\mathcal{C} (\omega)$, thereby generalizing results of Olofsson, Giselsson, and Wennman [55, 76] and Eschmeier-Langendörfer [51]. The analysis in this chapter builds on and clarifies the material from [28, 20, 21] treating the Fock-space case.
Chapter 5 moves on to an elaboration of theme #2 (Beurling-Lax representation theorems for shift-invariant subspaces). For these more general settings Chapter 5 has results for (i) shift-invariant subspaces $M$ isometrically included in the noncommutative weighted Hardy space $H^2_{\omega,Y}(F^+)$ in terms of a McCullough-Trent (McCT) inner multiplier $\Theta$ (a formal power series $\Theta$ for which the multiplication operator is a partial isometry), thereby generalizing the result of McCullough-Trent \[66\] to the multivariable noncommutative setting, (ii) for contractively-included shift-invariant subspaces $M$ in terms of a contractive multiplier $\Theta$, thereby generalizing results of de Branges-Rovnyak \[35, 36\] for the univariate case. For both of these representations, we use as the the Fock space $H^2_{\omega,Y}(F^+)$ as the input space for the multiplier (so $M_{\Theta}: H^2_{\omega,Y}(F^+) \to H^2_{\omega,Y}(F^+)$) rather than the same weighted Bergman-Fock space but with a different coefficient space as the model for the input space (so $M_{\Theta}: H^2_{\omega,Y}(F^+) \to H^2_{\omega,Y}(F^+)$) as in some previous treatments (see \[14, 15, 89\]); this enables one to have a more natural minimal set of hypotheses, as has already been seen in the nice abstract setup for commutative setting by Sarkar and coauthors \[95, 96, 29\] and in the noncommutative multivariable setting as here in work of the authors and Popescu \[16, 90\].

Chapter 6 also is concerned with theme #2 but of a different flavor. Namely, here we give our noncommutative multivariable version of the quasi-wandering subspace of Izuchi and others \[62, 63, 39\].

Chapter 7 is concerned exclusively with Beurling-Lax representation results for isometrically included subspaces $M$, but now in terms of a Bergman-inner multiplier $\Theta$ (i.e., $\Theta$ maps the constants isometrically onto a wandering subspace for $M$—generalizing work of Aleman, Duren, Khavinson, Richter, Shapiro, Sundberg \[6, 48, 49\] to the noncommutative multivariable setting. By using a whole family of Bergman inner functions rather than a single Bergman inner function, we get a more orthogonal Beurling-Lax representation, closer to but more complicated than, the classical case. We also obtain analogues of the expansive multiplier property and some results on characterizations of Bergman-inner multipliers as extremal solutions of interpolation problems analogous to results of Duren, Hedenmalm, Khavinson, Shapiro, Sundberg, Vukotić \[58, 48, 101\] for the univariate case.

Chapter 8 uses the results of Chapters 5 and 7 to flesh out themes #3 and #4 for our general noncommutative multivariable setting. There are two distinct types of model theory depending on whether one uses the Beurling-Lax representation results of Chapter 5 or the Beurling-Lax representation results of Chapter 7. The former includes model theory results for at least some classes of completely noncoisometric $*$-$\omega$-hypercontractive tuples while the latter to this point is only for pure (or $*$-strongly stable) $*$-$\omega$-hypercontractive operator tuples. Our use of contractively included subspaces and Brangesian complementary spaces for the former gives some complementary results and an alternative perspective to the results of Popescu in \[90\].

Chapter 9, as already mentioned, deals with the $(p, n)$ setting where the weighted Bergman-Fock space is adjusted to handle the model theory for $*$-$(p, n)$-hypercontractive operator tuples $T$ as in \[144\]. It turns out that, with the proper (not always expected) adjustments, all the results for the $\mu_n$ setting extend to this more general setting. In this way we recover the operator-model theory results of Popescu \[90\] for this setting.
CHAPTER 2

Formal Reproducing Kernel Hilbert Spaces

As was mentioned in Section 1.2, the standard weighted Bergman space $A_n(Y)$ can be viewed as a reproducing kernel Hilbert space with reproducing kernel given by (1.9). It is useful to have a similar point of view for the weighted Bergman-Fock spaces discussed in Section 1.4.

2.1. Basic definitions

In this section we review the notion of formal reproducing kernel Hilbert space developed in [27, Section 3].

Given a collection of freely noncommuting indeterminates $z = (z_1, \ldots, z_d)$, we suppose that we are given a Hilbert space $H$ whose elements are formal power series

$$f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}(z), \quad f_\alpha \in \mathcal{Y},$$

with coefficients from a coefficient Hilbert space $\mathcal{Y}$. We say that $H$ is a NFRKHS (noncommutative formal reproducing kernel Hilbert space) if, for each $\beta \in \mathbb{F}_d^+$, the linear operator $\Phi_\beta : H \rightarrow \mathcal{Y}$ defined by

$$\Phi_\beta : f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_\beta,$$

is continuous. As any such power series is completely determined by the list of its coefficients $\alpha \mapsto f_\alpha$ for $\alpha \in \mathbb{F}_d^+$, equivalently we can view the elements $f(z)$ as the functions $\alpha \mapsto f_\alpha$ on $\mathbb{F}_d^+$. Hence, by the noncommutative Aronszajn theory of reproducing kernel Hilbert spaces (see e.g. [27, Theorem 1.1]), there is a positive kernel $K : \mathbb{F}_d^+ \times \mathbb{F}_d^+ \rightarrow \mathcal{L}(\mathcal{Y})$ so that $H$ is the reproducing kernel Hilbert space associated with $K$. To spell this out, in the present context we denote the value of $K$ at $(\alpha, \beta)$ by $K_{\alpha, \beta} \in \mathcal{L}(\mathcal{Y})$ rather than $K(\alpha, \beta)$. Since we view an element $f \in H$ as a formal power series (2.1) rather than as a function $\alpha \mapsto f_\alpha$ on $\mathbb{F}_d^+$, we write, for a given $\beta \in \mathbb{F}_d^+$ and $y \in \mathcal{Y}$, the element $\Phi_\beta y \in \mathcal{H}$ as $\Phi_\beta y = K_\beta(\cdot) y$, where

$$K_\beta(z)y = \sum_{\alpha \in \mathbb{F}_d^+} K_{\alpha, \beta} y z^\alpha. \quad (2.3)$$

Then the reproducing kernel property can be written as

$$\langle f, K_\beta(\cdot)y \rangle_H = \langle f, \Phi_\beta y \rangle_H = \langle \Phi_\beta f, y \rangle_Y = \langle f_\beta, y \rangle_Y. \quad (2.4)$$

We can make the notation more suggestive of the classical case as follows. Let $\zeta = (\zeta_1, \ldots, \zeta_d)$ be a second $d$-tuple of noncommuting indeterminates. Given a coefficient Hilbert space $\mathcal{C}$, we can use the $\mathcal{C}$-inner product to define pairings

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}(\zeta)} \mapsto \mathcal{C}(\zeta) \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathcal{C}(\zeta) \times \mathcal{C}} \mapsto \mathcal{C}(\zeta)$$

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(where \( C(\langle \zeta \rangle) \) is the space of formal power series in the set of formal conjugate indeterminates \( \zeta = (\zeta_1, \ldots, \zeta_d) \) with coefficients in \( C \)) by
\[
\langle c, \sum f_\alpha \zeta^\alpha \rangle_{C \times C(\langle \zeta \rangle)} = \sum \langle c, f_\alpha \rangle_c \zeta^\alpha, \quad \langle \sum f_\alpha \zeta^\alpha, c \rangle_{C(\langle \zeta \rangle) \times C} = \langle f_\alpha, c \rangle_c \zeta^\alpha.
\]
These pairings can be seen as special cases of the more general pairing
\[
\langle \sum f_\alpha \zeta^\alpha, \sum g_\beta \zeta^{\beta\top} \rangle_{C(\langle \zeta \rangle) \times C(\langle \zeta \rangle)} = \sum \sum \langle f_\alpha, g_\beta \rangle_c \zeta^\alpha, \tag{2.5}
\]
which can be written more suggestively as
\[
\langle f(\zeta), g(\bar{\zeta}) \rangle_{C(\langle \zeta \rangle) \times C(\langle \zeta \rangle)} = \langle \sum f_\alpha \zeta^\alpha, \sum g_\beta \zeta^{\beta\top} \rangle_{C(\langle \zeta \rangle) \times C(\langle \zeta \rangle)} = g(\bar{\zeta})^* f(\zeta) \tag{2.6}
\]
if we set
\[
g(\bar{\zeta})^* = \left( \sum g_\beta \zeta^{\beta\top} \right)^* = \sum g_\beta^* \zeta^{\beta\top},
\]
where we view \( g_\beta^* \in \mathcal{L}(C, \mathbb{C}) \) as a linear functional on \( C \) so that
\[
g_\beta^* f_\alpha = \langle f_\alpha, g_\beta \rangle_c \quad \text{for any} \quad f_\alpha \in C.
\]
Then, if \( S(\zeta) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle \zeta \rangle) \), \( f(\zeta) \in \mathcal{U}(\langle \zeta \rangle) \) and \( g(\bar{\zeta}) \in \mathcal{Y}(\langle \bar{\zeta} \rangle) \), we see that
\[
\langle S(\zeta) f(\zeta), g(\bar{\zeta}) \rangle_{\mathcal{Y}(\langle \zeta \rangle) \times \mathcal{Y}(\langle \bar{\zeta} \rangle)} = (g(\bar{\zeta})^* (S(\zeta) f(\zeta))) = (g(\bar{\zeta})^* f(\zeta)) = \langle f(\zeta), S(\zeta)^* g(\bar{\zeta}) \rangle_{\mathcal{U}(\langle \zeta \rangle) \times \mathcal{U}(\langle \bar{\zeta} \rangle)}. \tag{2.7}
\]
The reproducing kernel property \(2.4\) can be written more suggestively as
\[
\langle f, K(\cdot, \zeta) y \rangle_{\mathcal{H}(\langle \zeta \rangle)} = \langle f(\zeta), y \rangle_{\mathcal{Y}(\langle \zeta \rangle) \times \mathcal{Y}}, \tag{2.8}
\]
where we set
\[
K(z, \zeta) = \sum_{\alpha, \beta \in \mathbb{F}_d^+} K_{\alpha, \beta} z^\alpha \bar{\zeta}^{\beta\top} \in \mathcal{L}(\mathcal{Y})(\langle z, \bar{\zeta} \rangle). \tag{2.9}
\]
We note that, for each \( y \in \mathcal{Y} \), the formal power series
\[
K(z, \zeta) y = \sum_{\alpha, \beta \in \mathbb{F}_d^+} K_{\alpha, \beta} y z^\alpha \bar{\zeta}^{\beta\top} = \sum_{\beta \in \mathbb{F}_d^+} \left[ \sum_{\alpha \in \mathbb{F}_d^+} K_{\alpha, \beta} y z^\alpha \right] \bar{\zeta}^{\beta\top}
\]
is an element of \( \mathcal{H}(\langle \zeta \rangle) \).

The formal kernel \(2.9\) is of the so-called hereditary form, i.e., the formal power series \(2.9\) involves only noncommutative monomials in \( z \) and \( \zeta \) of the form \( z^\alpha \bar{\zeta}^{\beta\top} \) (a monomial in \( z \) followed by a monomial in \( \zeta \)). For our purposes here we define the space \( \mathcal{L}(\mathcal{Y})(\langle z, \bar{\zeta} \rangle) \) to consist only of such hereditary formal power series in the freely noncommuting indeterminates. Here we are assuming that \( z = (z_1, \ldots, z_d) \) and \( \zeta = (\zeta_1, \ldots, \zeta_d) \) are each separately freely noncommuting indeterminates. It will be convenient to assume in addition that \( z_k \) commutes with each \( \zeta_j \)
\[
\zeta_j z_k = z_k \zeta_j \quad \text{for} \quad k, j = 1, \ldots, d. \tag{2.10}
\]

The following summary (see Theorem 3.1 in \(27\)) of the structure of such NFRKHSs among other things characterizes which formal power series \( K(z, \zeta) \in \mathcal{L}(\mathcal{Y})(\langle z, \bar{\zeta} \rangle) \) arise in this way from a NFRKHS.
Theorem 2.1. Let $K(z, \zeta)$ of the form \[(2.19)\text{ be a given element of } \mathcal{L}(\mathcal{Y})(\langle z, \zeta \rangle)\] where $z = (z_1, \ldots, z_d)$ and $\zeta = (\zeta_1, \ldots, \zeta_d)$ are $d$-tuples of noncommuting indeterminates which pairwise commute with each other. Then the following are equivalent:

1. $K$ is the reproducing kernel for a unique NFRKHS denoted as $\mathcal{H}(K)$.
2. There is an auxiliary Hilbert space $\mathcal{H}_0$ and a noncommutative formal power series $H(z) \in \mathcal{L}(\mathcal{H}_0, \mathcal{Y})(\langle z \rangle)$ such that
   \[K(z, \zeta) = H(z)H(\zeta)^*\] (2.11)
   (Kolmogorov decomposition for $K$). Here we use the conventions
   \[(\zeta^\alpha)^* = \zeta^{\alpha^T}, \quad H(\zeta)^* = \left(\sum \beta H_\beta \zeta^\beta\right)^* = \sum H_\beta^* \zeta^{\beta^T}.
   \]
3. $K(z, \zeta)$ is a positive formal kernel in the sense that for all finitely supported $\mathcal{Y}$-valued functions $\alpha \mapsto y_\alpha$ on $\mathbb{F}_d^n$,
   \[
   \sum_{\alpha, \beta \in \mathbb{F}_d^n} \langle K_{\alpha, \beta} y_\alpha, y_\beta \rangle_{\mathcal{Y}} \geq 0. \tag{2.12}
   \]
   Moreover, in this case the space $\mathcal{H}(K)$ can be defined directly in terms of the formal power series $H(z)$ appearing in condition (2) via
   \[
   \mathcal{H}(K) = \{ H(z)h_0 : h_0 \in \mathcal{H}_0 \}
   \]
   with norm taken to be the “lifted norm”
   \[
   \|H(\cdot)h_0\|_{\mathcal{H}(K)} = \|Qh_0\|_{\mathcal{H}_0}
   \]
   where $Q$ is the orthogonal projection of $\mathcal{H}_0$ onto the orthogonal complement of the kernel of the map $M_H : \mathcal{H}_0 \to \mathcal{Y}(\langle z \rangle)$ given by $M_H : h_0 \mapsto H(z) \cdot h_0$.

The more general pairings (2.5) and (2.6) are involved in the formulation of the following useful more general form of the reproducing kernel property.

Proposition 2.2. Let $\mathcal{H}(K)$ be the NFRKHS associated with a positive formal kernel $K \in \mathcal{L}(\mathcal{Y})(\langle z, \zeta \rangle)$. Then, for all $f \in \mathcal{H}(K)$ and $g(\zeta) \in \mathcal{Y}(\langle \zeta \rangle)$, we have
   \[
   \langle f, K(\cdot, \zeta)g(\zeta) \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)} = \langle f(\zeta), g(\zeta) \rangle_{\mathcal{Y}(\langle \zeta \rangle) \times \mathcal{Y}(\langle \zeta \rangle)}.
   \tag{2.13}
   \]

Proof. Note first that the reproducing property (2.8) can be rewritten as
   \[
   \langle K(\cdot, \zeta)g, f \rangle = y^*f(\zeta) \quad \text{for all } y \in \mathcal{Y},
   \]
   from which we deduce that $K(\cdot, \zeta)^*f = f(\zeta)$ for any $f \in \mathcal{H}(K)$. Then we have
   \[
   \langle f(\zeta), g(\zeta) \rangle_{\mathcal{Y}(\langle \zeta \rangle) \times \mathcal{Y}(\langle \zeta \rangle)} = \langle K(\cdot, \zeta)^*f, g(\zeta) \rangle_{\mathcal{Y}(\langle \zeta \rangle) \times \mathcal{Y}(\langle \zeta \rangle)}
   = \langle f, K(\cdot, \zeta)g(\zeta) \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)}
   \]
   for any $g(\zeta) \in \mathcal{Y}(\langle \zeta \rangle)$ which completes the proof. \hfill $\square$

The following general principle (a variation of the last statement of Theorem 2.1) will be useful in the sequel.

Proposition 2.3. Suppose that $H(z) = \sum_{\alpha \in \mathbb{F}_d^n} H_\alpha z^\alpha \in \mathcal{L}(\mathcal{X}, \mathcal{Y})(\langle z \rangle)$ is a formal power series such that the set
   \[
   \mathcal{M} = \{ H(\cdot)x : x \in \mathcal{X} \} \subset \mathcal{Y}(\langle z \rangle)
   \]
   is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ satisfying
   \[
   \langle H(\cdot)x, H(\cdot)x' \rangle_{\mathcal{M}} = \langle Gx, x' \rangle_{\mathcal{X}} \tag{2.14}
   \]
   where $G = \sum_{\alpha, \beta \in \mathbb{F}_d^n} H_\alpha H_\beta^* z^{\alpha^T} z^{\beta^T}$.
where $G$ is an invertible positive definite operator on $\mathcal{X}$. Then $\mathcal{M}$ is a NFRKHS with formal reproducing kernel $K_\mathcal{M}(z, \zeta)$ given by

$$K_\mathcal{M}(z, \zeta) = H(z)G^{-1}H(\zeta)^*.$$  

**Proof.** For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we compute

$$\langle (H(\zeta)x, y)_{\mathcal{Y}}(\zeta) \rangle_{\mathcal{X}} = \langle (Gx, G^{-1}H(\zeta)^*y)_{\mathcal{X}}(\zeta) \rangle_{\mathcal{X}} = \langle (\cdot x, H(\cdot)G^{-1}H(\zeta)^*y)_{\mathcal{M}}(\zeta) \rangle_{\mathcal{M}}$$

where the last step follows from the hypothesis (2.14). The result follows. \hfill \Box

**Remark 2.4.** Just as in the classical case, if $K$ is the reproducing kernel for the NFRKHS $\mathcal{H}(K)$, then there is a canonical choice of $H(z)$ which yields the factorization (2.11) in part (2) of Theorem 2.1 namely: take $\mathcal{H}_0$ equal to the space $\mathcal{H}(K)$ itself and set $H(z) = \sum_{\beta \in \mathbb{F}_d} \Phi_\beta z^\beta$ where $\Phi_\beta$ is the coefficient-evaluation functional (2.2).

**Example 2.5.** The Fock space $\mathcal{H}^2_d(\mathbb{F}_d^+)$ introduced in Section 1.3 is the NFRKHS $\mathcal{H}(k_{nc,Sz}I_d)$ where $k_{nc,Sz}$ is the noncommutative Szegő kernel

$$k_{nc,Sz}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} z^\alpha\overline{\zeta}^{\alpha^\top}. \quad (2.15)$$

**Example 2.6.** More generally, the weighted Bergman-Fock space $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ is the NFRKHS $\mathcal{H}(k_{nc,n}I_d)$ where the noncommutative weight-$n$ Bergman kernel $k_{nc,n}(z, \zeta)$ is defined by

$$k_{nc,n}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|} z^\alpha\overline{\zeta}^{\alpha^\top}, \quad \mu_{n,|\alpha|} = |\alpha|!(n-1)! / (|\alpha| + n - 1)! \quad (2.16)$$

The reproducing kernel property will be verified in Section 2.2 for a more general setting. We next point out two identities relating the kernels (2.13) and (2.16):

$$k_{nc,n}(z, \zeta) = \sum_{j=1}^d \overline{\zeta}_j k_{nc,n}(z, \zeta)z_j = k_{nc,n-1}(z, \zeta), \quad (2.17)$$

$$k_{nc,n}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n-1,|\alpha|} \overline{\zeta}^{\alpha^\top} k_{nc,Sz}(z, \zeta)z^\alpha. \quad (2.18)$$

To verify (2.17) and (2.18), it suffices to show that the coefficients of $z^\alpha\overline{\zeta}^{\alpha^\top}$ on both sides are equal for each $\alpha \in \mathbb{F}_d^+$. Due to (2.16) and (2.15), the latter amounts to the respective equalities

$$\mu_{n,|\alpha|} - \mu_{n,|\alpha|-1} = \mu_{n-1,|\alpha|} \quad \text{and} \quad \mu_{n,|\alpha|} = \mu_{n-1,0} + \mu_{n-1,1} + \cdots + \mu_{n-1,|\alpha|}.$$  

By the definition (1.8) of $\mu_{n,|\alpha|}$, the first equality is the binomial-coefficient identity

$$\binom{|\alpha|+n-1}{n-1} - \binom{|\alpha|+n-2}{n-1} = \binom{|\alpha|+n-2}{n-2},$$

while the second follows from the first by an induction argument. We finally note that letting $n = 1$ in (2.18) gives

$$k_{nc,Sz}(z, \zeta) = \sum_{k=1}^d \overline{\zeta}_k k_{nc,Sz}(z, \zeta)z_k = 1. \quad (2.19)$$
We shall use formal positive kernels as a tool for obtaining our multivariable Beurling-Lax theorem for the space $A_{\infty, \mathcal{Y}}(F_+^d)$ and the operator model theory for freely noncommuting $n$-hypercontractive operator $d$-tuples below.

**Definition 2.7.** We say that the nc positive kernel $K(z, \zeta)$ in free indeterminates $z = (z_1, \ldots, z_d)$ and $\zeta = (\zeta_1, \ldots, \zeta_d)$ is a contractive kernel if the space $\mathcal{H}(K)$ is invariant under the right coordinate-variable multipliers

$$M^*_y: f(z) \mapsto f(z) \cdot z_j \quad \text{for} \quad j = 1, \ldots, d$$

and the tuple $M^*_y = (M^*_{y_1}, \ldots, M^*_{y_d})$ is a row contraction, i.e., the block row matrix $M^*_y \in \mathcal{L}(Z, \mathcal{H}(K))$ is a contraction operator from $\mathcal{H}(K)^d$ to $\mathcal{H}(K)$:

$$\|M^*_y f_1 + \cdots + M^*_y f_d\| \leq \|f_1\|^2 + \cdots + \|f_d\|^2_M \quad \text{for all} \quad f_1, \ldots, f_d \in \mathcal{H}(K). \quad (2.20)$$

We have the following characterization of contractive formal positive kernels $K$.

**Proposition 2.8.** A given positive formal kernel $K$ is a contractive formal positive kernel if and only if $K$ has the form

$$K(z, \zeta) = G(z)(k_{nc, Sz}(z, \zeta) \otimes I_{\mathcal{U}})G(\zeta)^*$$

for some $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle)$ and some auxiliary Hilbert space $\mathcal{U}$.

**Proof.** We claim that condition (2.20) along with shift invariance of $\mathcal{H}(K)$ is equivalent to the condition

$$L(z, \zeta) = K(z, \zeta) - \sum_{j=1}^d \zeta_j K(z, \zeta) z_j \quad \text{is a positive kernel}, \quad (2.22)$$

i.e., that $L(z, \zeta)$ has a Kolmogorov decomposition

$$L(z, \zeta) = G(z)G(\zeta)^*$$

for some $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle)$. As a first step toward verifying (2.23), we wish to compute the adjoint of $M^*_y$ acting on a kernel element:

$$\left( (M^*_y)^* \otimes I_{\mathcal{C}(\zeta_j)} \right) K(z, \zeta) = \overline{\zeta}_j K(z, \zeta) \quad \text{for} \quad j = 1, \ldots, d. \quad (2.24)$$

To this end, for $y \in \mathcal{Y}$, write out $K(z, \zeta)y$ as $K(z, \zeta)y = \sum_{\alpha} K_\alpha y \zeta^\alpha$ with $K_\alpha y \in \mathcal{H}(K)$ for each $\alpha \in \mathcal{F}_+^d$, and then, for $f(\zeta) = \sum_{\alpha} f_\alpha \zeta^\alpha \in \mathcal{H}(K)$, compute

$$\langle f, \overline{\zeta}_j K(\cdot, \zeta) y \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)} = \langle f, \sum_{\alpha \in \mathcal{F}_+^d} K_\alpha y \zeta^\alpha \zeta_j \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)}$$

$$= \sum_{\alpha \in \mathcal{F}_+^d} \langle f, K_\alpha y \rangle_{\mathcal{H}(K)} \zeta^\alpha \zeta_j$$

$$= \sum_{\alpha \in \mathcal{F}_+^d} \langle f_\alpha, y \rangle_{\mathcal{Y}} \zeta^\alpha \zeta_j = \langle f(\cdot) \cdot \zeta_j, y \rangle_{\mathcal{Y}}(\langle \zeta \rangle) \times \mathcal{Y}. \quad (2.24)$$

Since on the other hand,

$$\langle f, (M^*_y)^* K(\cdot, \zeta) y \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)} = \langle M^*_y f, K(\cdot, \zeta) y \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)}$$

$$= \langle f(\cdot) \cdot \zeta_j, y \rangle_{\mathcal{Y}}(\langle \zeta \rangle) \times \mathcal{Y},$$

$$\langle f, K(\cdot, \zeta) y \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)} = \langle f, \overline{\zeta}_j K(\cdot, \zeta) y \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)(\langle \zeta \rangle)}$$
Let us define $G$ and we arrive at the Kolmogorov decomposition (2.23) with the factor $K$ formula (2.21) for $N$. Taking the limit as $N \to \infty$ in this expression then leaves us with the desired formula (2.21) for $K$.

Conversely, if $K$ is of the form (2.21), then any element of $\mathcal{H}(K)$ is of the form $G(z)g(z)$ for some $g \in \mathcal{H}(\mathcal{K}, \mathcal{H}(\mathcal{K})) = H^2(\mathbb{D})$ and since $H^2(\mathbb{D})$ is invariant under $M^*_j$ for $j = 1, \ldots, d$, it follows that $\mathcal{H}(K)$ is $M^*_j$-invariant as well. Representation (2.22) follows from (2.19), (2.21), and (2.22). Inequality (2.20) follows by combining (2.22) and (2.24).
Remark 2.9. Given a formal power series \( f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}(\langle z \rangle) \) in \( d \) free indeterminates \( z = (z_1, \ldots, z_d) \) with coefficients \( f_\alpha \) coming from a Hilbert space \( \mathcal{Y} \), one may substitute in a \( d \)-tuple \( Z = (Z_1, \ldots, Z_d) \) of \( n \times n \) matrices for the formal indeterminates \( z = (z_1, \ldots, z_d) \) to obtain a \( n \times n \) matrix
\[
f(Z) = \lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+: 0 \leq |\alpha| \leq N} f_\alpha \otimes Z^\alpha \in \mathcal{Y} \otimes \mathbb{C}^{N \times N} \cong \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) \tag{2.26}
\]
on a domain corresponding to the set of \( d \)-tuples \( Z \) when the series (2.26) converges (here \( Z^\alpha = Z_{i_N} \cdots Z_{i_1} \in \mathbb{C}^{N \times N} \) if \( \alpha \) is the word of the form \( \alpha = i_N \cdots i_1 \) with each \( i_j \in \{1, \ldots, d\} \)). This is a particular setting for what has come to be called a noncommutative function (see \cite{64, 71, 82, 25}). Similarly, if
\[
K(z, w) = \sum_{\alpha, \beta \in \mathbb{F}_d^+: 0 \leq |\alpha|, |\beta| \leq N} K_{\alpha, \beta} z^\alpha w^{\beta^T} \in \mathcal{L}(\mathcal{Y} \langle \langle z, w \rangle \rangle)
\]
is a formal kernel, one may substitute in a \( d \)-tuple \( Z = (Z_1, \ldots, Z_d) \) along with a \( d \)-tuple \( W = (W_1, \ldots, W_d) \) of \( n \times n \) complex matrices to get an evaluation
\[
K(Z, W) = \lim_{N \to \infty} \sum_{\alpha, \beta \in \mathbb{F}_d^+: 0 \leq |\alpha|, |\beta| \leq N} K_{\alpha, \beta} \otimes Z^\alpha (W^*)^{\beta^T} \in \mathcal{L}(\mathcal{Y} \otimes \mathbb{C}^{n \times n} \cong \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^m),
\]
where the limit exists for \( (Z, W) \) in some domain \( \bigcup_{n=0}^{\infty} D_n \times D_n \subset \bigcup_{n=0}^{\infty} \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \).

More generally, if \( Z \) is a \( d \)-tuple of \( n \times n \) matrices, \( W \) is a \( d \)-tuple of \( m \times m \) matrices, and \( P \) is \( n \times m \) matrix, one may define an evaluation
\[
K(Z, W)(P) = \sum_{\alpha, \beta \in \mathbb{F}_d^+: 0 \leq |\alpha|, |\beta| \leq N} K_{\alpha, \beta} \otimes Z^\alpha PW^{\beta^T} \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)
\]
wherever the series converges. In case \( K \) is a positive formal kernel, the result is an object satisfying the definition of a completely positive noncommutative kernel in the sense of \cite{25}. Thus the theory of formal noncommutative power series and formal positive kernels can be identified as a particular setting for the theory of noncommutative functions and completely positive noncommutative kernels; the precise connections are discussed in \cite{25} Section 3.3]. We leave open the question as to how one should define noncommutative Bergman spaces over more general noncommutative domains.

2.2. Weighted Hardy-Fock spaces

Let us say that the formal kernel \( K(z, \zeta) = \sum_{\alpha, \beta} K_{\alpha, \beta} z^\alpha \zeta^{\beta^T} \) is a scalar radial kernel if \( K_{\alpha, \beta} = 0 \) for \( \alpha \neq \beta \) and furthermore \( K_{\alpha, \alpha} = k_\alpha I_\mathcal{Y} \) for some scalar-valued function \( k: \mathbb{F}_d^+ \to \mathbb{C} \) \( (k: \alpha \mapsto k_\alpha) \). i.e., if \( K(z, \zeta) \) has the form
\[
K(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} (k_\alpha I_\mathcal{Y}) z^\alpha \zeta^{\alpha^T}
\]
where \( \mathcal{Y} \) is a coefficient Hilbert space. Often it is the case that \( k_\alpha \neq 0 \) for all \( \alpha \) and it is more convenient to write \( k_\alpha \) as \( k_\alpha = \omega^{-1}_\alpha \) for a nonzero-valued scalar function \( \omega: \mathbb{F}_d^+ \to \mathbb{C} \setminus \{0\} \). In many examples the sequence \( k_\alpha \) in fact depends only the length \( |\alpha| \) of \( \alpha \), in which case we say that \( K \) is a symmetrized radial kernel.
Specifically we shall consider symmetrized radial kernels arising in the following way. Starting with a positive sequence \( \omega = \{ \omega_j \}_{j \geq 0} \) and a Hilbert space \( \mathcal{Y} \), we denote by \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \) be the weighted Hardy-Fock space

\[
H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) = \left\{ \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in \mathcal{Y}(\{z\}) : \|f\|^2 := \sum_{\alpha \in \mathbb{F}^+_d} \omega|\alpha| \cdot \|f_\alpha\|^2_{\mathcal{Y}} < \infty \right\}.
\]

The space \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \) is the NFRKHS with noncommutative reproducing kernel \( k_\omega(z, \zeta) \mathcal{I}_\mathcal{Y} \) with

\[
k_\omega(z, \zeta) = \sum_{\alpha \in \mathbb{F}^+_d} \omega|\alpha| z^\alpha \zeta^\alpha
\]

which is verified by the standard computation for \( f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \) and \( y \in \mathcal{Y} \):

\[
\langle f, k_\omega(\cdot, \zeta) \rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \times H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}(\{\zeta\}) = \sum_{\alpha \in \mathbb{F}^+_d} (f(z), \omega|\alpha| y z^\alpha)_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)} \zeta^\alpha
\]

\[
= \sum_{\alpha \in \mathbb{F}^+_d} (f_\alpha, y)_{\mathcal{Y}} \zeta^\alpha = \langle f(\zeta), y \rangle_{\mathcal{Y}(\{\zeta\}) \times \mathcal{Y}}.
\]

In the specific context of the weighted Hardy-Fock space \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \), the tuple of the right coordinate-variable multipliers \( M_\alpha \) (see (2.20)) will be denoted by \( S_{\omega,R} \):

\[
S_{\omega,R} = (S_{\omega,R,1}, \ldots, S_{\omega,R,d}), \quad S_{\omega,R,j} : f(z) \mapsto f(z)z_j.
\]

The standard inner-product computation gives the formula for adjoint operators

\[
S^*_{\omega,R,j} : \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}^+_d} \frac{\omega|\alpha|+1}{\omega|\alpha|} f_{\alpha j} z^\alpha.
\]

We say that the weight sequence \( \omega = \{ \omega_j \}_{j \geq 0} \) is admissible if it is subject to the following conditions:

\[
\omega_0 = 1 \quad \text{and} \quad 1 \leq \frac{\omega_j}{\omega_{j+1}} \leq M \quad \text{for all} \quad j \geq 0 \quad \text{and some} \quad M \geq 1.
\]

The first condition says that \( \mathbb{C} \) embeds isometrically into the space of constant nc functions in \( H_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \), the second condition means that the shift-operator tuple \( S_{\omega,R} \) is a row contraction and \( S_{\omega,R,j} \) is left-invertible for \( j = 1, \ldots, d \). Indeed, it follows from (2.30) that

\[
\|S^*_j f\|^2_{H^2_{\omega, \mathcal{Y}}} = \sum_{\alpha \in \mathbb{F}^+_d} \frac{\omega^2|\alpha|+1}{\omega|\alpha|} \|f_\alpha\|^2
\]

and hence, for \( f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \), we have

\[
\sum_{j=1}^d \|S^*_j f\|^2_{H^2_{\omega, \mathcal{Y}}} = \sum_{\alpha \neq \emptyset} \frac{\omega^2|\alpha|-1}{\omega|\alpha|} \|f_\alpha\|^2 = \sum_{\alpha \neq \emptyset} \omega|\alpha| \left( \frac{\omega|\alpha|}{\omega|\alpha|-1} \right) \|f_\alpha\|^2
\]

\[
\leq \sum_{\alpha \in \mathbb{F}^+_d} \omega|\alpha| \|f_\alpha\|^2 = \|f\|^2_{H^2_{\omega, \mathcal{Y}}},
\]
and similarly,
\[ \|S_{\omega,R,j}f\|_{H^2_\omega}^2 = \sum_{\alpha \in \mathbb{Z}_+^d} \omega_{|\alpha|+1}\|f_{\alpha}\|^2 \geq \frac{1}{M} \sum_{\alpha \in \mathbb{Z}_+^d} \omega_{|\alpha|}\|f_{\alpha}\|^2 = \frac{1}{M} \|f\|_{H^2_\omega}^2 \]
for \( j = 1, \ldots, d \). Combining the row-contractive property of \( S_{\omega,R} \) with Definition 2.7 we arrive at the following.

**Remark 2.10.** The kernel (2.28) based on an admissible weight sequence \( \omega \) is a contractive nc positive kernel. Furthermore, the sequence \( \mu = \{\mu_{\alpha}\} \) (1.8) of reciprocal binomial coefficients is an admissible weight sequence which is strictly decreasing for \( n > 1 \).

It is worth mentioning the following example \( \mu_\rho \) with a continuous real parameter \( \rho > 1 \) which interpolates the weights \( \mu_n \) with discrete parameter \( n = 1, 2, \ldots \). Namely, for \( \rho > 1 \) real, we define \( \omega = \mu_\rho = \{\mu_{\rho,k}\}_{k \geq 0} \) where we set
\[ \mu_{\rho,k} = \frac{k!}{\rho(\rho + 1) \cdots (\rho + k - 1)} = \frac{k!\Gamma(\rho)}{\Gamma(\rho + k)} \] (2.32)
One can check that \( \mu_\rho \) meets the admissibility conditions (2.31) and that indeed \( \mu_{\rho,k} = \mu_{n,k} \) in case \( \rho = n \) is a positive integer. Therefore the weight sequence (2.32) generates a formal nc reproducing kernel Hilbert space \( \mathcal{H}(k_{\mu_\rho}) \) with formal kernel given by
\[ k_{\mu_\rho}(z,\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \mu_{\rho,|\alpha|}^{-1} z_\alpha^\top \zeta_\alpha \] (2.33)
The associated weighted Hardy spaces \( H^2_{\mu_\rho}(\mathbb{F}_d^+) \) are mentioned as examples of the more general spaces \( H^2_{\omega}(\mathbb{F}_d^+) \) for the single-variable case \( d = 1 \), in which case we simply write \( H^2_{\omega} \) rather than \( H^2_{\omega}(\mathbb{F}_1^+) \).

In particular, the choice \( \omega = \mu_n \) leads to the identification of a weighted Hardy-Fock space with a weighted Bergman-Fock space: \( H^2_{\mu_n,Y}(\mathbb{F}_d^+) = \mathcal{A}_{n,Y}(\mathbb{F}_d^+) \). In short, formulas (2.15), (2.16), (2.33), (2.28) give examples of increasing generality of symmetrized radial kernels which are also contractive kernels. As for shift operator tuples on weighted Bergman-Fock spaces, we shall write \( S_{n,R} \) rather than \( S_{\mu_n,R} \) and we shall also write \( S_{1,R} \) for the shift operator tuple on the unweighted Fock space \( H^2_\omega(\mathbb{F}_d^+) \).

We conclude this section by introducing a multidimensional system with evolution along \( \mathbb{F}_d^+ \) and based on an admissible weight \( \omega = \{\omega_j\}_{j \geq 0} \)
\[ \Sigma(\mathcal{U}_\alpha,\omega): \begin{cases} x(1\alpha) &= \frac{\omega_{|\alpha|}}{\omega_{|\alpha|+1}} A_1 x(\alpha) + \frac{1}{\omega_{|\alpha|+1}} B_{1,\alpha} u(\alpha) \\ \vdots &= \vdots \\ x(d\alpha) &= \frac{\omega_{|\alpha|}}{\omega_{|\alpha|+1}} A_d x(\alpha) + \frac{1}{\omega_{|\alpha|+1}} B_{d,\alpha} u(\alpha) \\ y(\alpha) &= C x(\alpha) + \frac{1}{\omega_{|\alpha|}} D_\alpha u(\alpha) \end{cases} \] (2.34)
with the same family of connection matrices \( \{\mathcal{U}_\alpha\}_{\alpha \in \mathbb{F}_d^+} \) as in (1.33) but with additional \( \alpha \)-dependent weights in the system equations determined by the weight sequence \( \omega \). Upon running the system (2.34) with a fixed initial condition \( x(0) = x \in \mathcal{X} \) we get recursively the formulas for \( x(\alpha) \) and \( y(\alpha) \) (the same as (1.34), (1.35))
but with \( \omega_{[\alpha]} \) instead of \( \mu_{n,[\alpha]} \). Subsequent application of the noncommutative \( Z \)-transform eventually leads us to the formulas

\[
\hat{y}(z) = \sum_{\alpha \in \mathbb{F}_{d}^{+}} \omega_{[\alpha]}^{-1} \left( C \mathbf{A}^{\alpha} x + \sum_{\alpha' + \alpha'' = \alpha} A^{\alpha''} B_{j,\alpha'} u(\alpha') + D_{\alpha} u(\alpha) \right) z^{\alpha} = \\
\sum_{\alpha \in \mathbb{F}_{d}^{+}} (\omega_{[\alpha]}^{-1} C \mathbf{A}^{\alpha} x) z^{\alpha} + \sum_{\alpha \in \mathbb{F}_{d}^{+}} \left( \omega_{[\alpha]}^{-1} D_{\alpha} + \sum_{j=1}^{d} \sum_{\alpha' \in \mathbb{F}_{d}^{+}} \omega_{[\alpha'+\alpha'_{1}+1]}^{-1} C \mathbf{A}^{\alpha'} B_{j,\alpha' z^{\alpha'_{j}}} \right) z^{\alpha} u(\alpha) = \\
\mathcal{O}_{\omega,\mathbf{A}x} + \sum_{\alpha \in \mathbb{F}_{d}^{+}} \Theta_{\omega,\mathbf{U}_{\alpha}}(z) z^{\alpha} u(\alpha),
\]

where the first term on the right presents the \( \omega \)-observability operator

\[
\mathcal{O}_{\omega,\mathbf{A}x} = \sum_{\alpha \in \mathbb{F}_{d}^{+}} \omega_{[\alpha]}^{-1} (C \mathbf{A}^{\alpha} x) z^{\alpha}
\]

associated with output pair \((\mathbf{C}, \mathbf{A})\) and where

\[
\Theta_{\omega,\mathbf{U}_{\alpha}}(z) = \omega_{[\alpha]}^{-1} D_{\alpha} + \sum_{j=1}^{d} \sum_{\alpha' \in \mathbb{F}_{d}^{+}} \omega_{[\alpha'+\alpha'_{1}+1]}^{-1} C \mathbf{A}^{\alpha'} B_{j,\alpha' z^{\alpha'_{j}}}
\]

is the family of transfer functions indexed by \( \alpha \in \mathbb{F}_{d}^{+} \). Making use of power series extending those in \((1.10)\) and operators \( \mathbf{A} \) and \( \hat{\mathbf{B}}_{\alpha} \) as in \((1.33)\), one can write the formulas for \((2.36)\) and \((2.37)\) as

\[
\mathcal{O}_{\omega,\mathbf{A}x} = C \mathcal{R}_{\omega}(Z(z)\mathbf{A}),
\]

\[
\Theta_{\omega,\mathbf{U}_{\alpha}}(z) = \omega_{[\alpha]}^{-1} D_{\alpha} + C \mathcal{R}_{\omega,[\alpha]+1}(Z(z)\mathbf{A}) Z(z) \hat{\mathbf{B}}_{\alpha},
\]

i.e., in the form similar to \((1.39)\) and \((1.40)\), respectively.

Now we may introduce an \( \omega \)-output stable pair \((\mathbf{C}, \mathbf{A})\) as one for which the \( \omega \)-observability operator \((2.36)\) is bounded from \( \mathcal{X} \) to \( H^{2}_{\omega}(\mathbb{F}_{d}^{+}) \). In this case, we see from \((1.4)\) and \((2.27)\) that for every \( x \in \mathcal{X} \),

\[
\langle \mathcal{G}_{\omega,\mathbf{A}x}, x \rangle_{\mathcal{X}} = \| \mathcal{O}_{\omega,\mathbf{A}x} \|^{2}_{H^{2}_{\omega}(\mathbb{F}_{d}^{+})} = \sum_{\alpha \in \mathbb{F}_{d}^{+}} \omega_{[\alpha]}^{-1} \cdot \| C \mathbf{A}^{\alpha} x \|^{2}_{H^{2}_{\omega}(\mathbb{F}_{d}^{+})},
\]

which justifies the representation of the \( \omega \)-observability gramian

\[
\mathcal{G}_{\omega,\mathbf{A}x} := \mathcal{O}_{\omega,\mathbf{A}x} \mathcal{O}_{\omega,\mathbf{A}x} = \sum_{\alpha \in \mathbb{F}_{d}^{+}} \omega_{[\alpha]}^{-1} \mathbf{A}^{\alpha} \mathbf{A}^{\alpha^{\top}} C^{*} C \mathbf{A}^{\alpha}
\]

by the weakly (and therefore, strongly) convergent series. Observability operators and their range spaces will be studied in detail in Chapter 4.
CHAPTER 3

Contractive multipliers

3.1. Contractive multipliers in general

Let $K'$ and $K$ be noncommutative reproducing kernels with values in $\mathcal{L}(\mathcal{U})$ and $\mathcal{L}(\mathcal{Y})$ respectively, and let $\mathcal{H}(K')$ and $\mathcal{H}(K)$ be the associated NFRKHSs. As we shall have many notions of "inner multiplier", we always use the term with some adjective to specify the type of inner under consideration. We note that the term "inner" has been used in the literature for what we are here calling "strictly inner" as well as "McCT-inner".

**Definition 3.1.** A formal power series $\Theta \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z, \zeta\rangle\rangle$ is called a multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if the operator $M_\Theta$ of multiplication by $\Theta$

$$M_\Theta : f(z) \mapsto \Theta(z) \cdot f(z) = \sum_{v \in F^+_\mathcal{J}} \left( \sum_{\alpha, \beta : \alpha\beta = v} \Theta_{\alpha} f_{\beta} \right)z^v$$

is bounded from $\mathcal{H}(K')$ to $\mathcal{H}(K)$. The multiplier $\Theta$ is called contractive, McCT-inner (suggesting the authors of the seminal paper [66]) or strictly inner if the operator $M_\Theta : \mathcal{H}(K') \to \mathcal{H}(K)$ is a contraction, a partial isometry or an isometry, respectively.

The following result for the special case where $K' = k_{nc,Sz} \otimes I_\mathcal{U}$ and $K = k_{nc,Sz} \otimes I_\mathcal{Y}$ appears in [27, Theorem 3.15] and in [21, Theorem 3.1]. We give a simple direct proof based on an adaptation of the proof of [26, Proposition 2.10] where a more complicated two-sided commutative setting is studied.

**Proposition 3.2.** Let $K \in \mathcal{L}(\mathcal{Y})\langle\langle z, \zeta\rangle\rangle$ and $K' \in \mathcal{L}(\mathcal{U})\langle\langle z, \zeta\rangle\rangle$ be two positive formal kernels and let $\Theta$ be a formal power series in $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$. Then:

1. $\Theta$ is a contractive multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if and only if

$$K_{\Theta}(z, \zeta) = K(z, \zeta) - \Theta(z)K'(z, \zeta)\Theta(\zeta)^* \in \mathcal{L}(\mathcal{Y})\langle\langle z, \zeta\rangle\rangle \quad (3.1)$$

is a positive formal kernel.

2. $\Theta$ is a coisometric multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if and only if $K_{\Theta} = 0$.

**Proof.** If $\Theta$ is a contractive multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$, then for all $f \in \mathcal{H}(K')$ and $y \in \mathcal{Y}$, we have

$$\langle f, \left( M_{\Theta} \otimes I_{\mathcal{Y}(\langle\langle \zeta\rangle\rangle)} \right) K(\cdot, \zeta) y \rangle_{\mathcal{H}(K') \times \mathcal{H}(K)(\langle\langle \zeta\rangle\rangle)}$$

$$= \langle \Theta(\cdot) f(\cdot), K(\cdot, \zeta) y \rangle_{\mathcal{H}(K') \times \mathcal{H}(K)(\langle\langle \zeta\rangle\rangle)}$$

$$= \langle \Theta(\zeta) f(\zeta), y \rangle_{\mathcal{Y}(\langle\langle \zeta\rangle\rangle) \times \mathcal{Y}} \quad \text{(by (2.8))}$$

$$= \langle f(\zeta), \Theta(\zeta)^* y \rangle_{\mathcal{H}(\langle\langle \zeta\rangle\rangle) \times \mathcal{H}(\langle\langle \zeta\rangle\rangle)} \quad \text{(by (2.7))}$$

$$= \langle f, K'(\cdot, \zeta) \Theta(\zeta)^* y \rangle_{\mathcal{H}(K') \times \mathcal{H}(K)(\langle\langle \zeta\rangle\rangle)} \quad \text{(by (2.13))}$$

$$= \langle f, K'(\cdot, \zeta) \Theta(\zeta)^* y \rangle_{\mathcal{H}(K') \times \mathcal{H}(K)(\langle\langle \zeta\rangle\rangle)} \quad \text{(by (2.13))}$$

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from which we conclude that
\[ (M_\Theta \otimes I_{C(\zeta)}) K(\cdot, \zeta)y = K'(\cdot, \zeta)\Theta(\zeta)^*y. \] (3.2)

Equating coefficients of $\zeta^{\beta^\top}$ in (3.2) and using the notation (2.3) then gives
\[ M_\Theta^* K_\beta(\cdot)y = \sum_{\gamma, \gamma', \gamma = \beta} K'_\gamma(\cdot)\Theta^*_\gamma y. \] (3.3)

The fact that $\|M_\Theta^*\| \leq 1$ then implies that
\[ \| \sum_{j=1}^N K_\beta_j(\cdot)y_j \|^2_{H(\gamma)} - \| \sum_{j=1}^N M_\Theta^* K_\beta_j(\cdot)y_j \|^2_{H(K')} \geq 0, \]
or, in terms of notation (2.9),
\[ \sum_{i,j=1}^N \left( (K_\beta_i(\cdot)y_i, y_j)_{\gamma} - \left( \sum_{\gamma_j, \gamma'_j, \gamma = \beta} \Theta_{\gamma_j} K'_{\gamma'_j} \Theta^*_{\gamma_j} y_j, y_i \right) \right) \geq 0 \] (3.4)
for all choices of $N \in \mathbb{N}$ and $\beta_1, \ldots, \beta_N \in \mathbb{F}^+_d$, $y_1, \ldots, y_N \in \mathcal{Y}$. On the other hand, the formal kernel $K_\Theta$ has power series expansion
\[ K_\Theta(z, \zeta) = \sum_{\alpha, \beta} \left( K_{\alpha, \beta} - \sum_{\gamma, \gamma', \delta, \delta', \gamma = \alpha, \delta = \beta} \Theta_{\gamma, \gamma'} K'_{\delta, \delta'} \Theta^*_{\gamma, \gamma'} \right) z^\alpha \zeta^{\beta^\top}. \] (3.5)

It now becomes clear that the condition (3.4) is just the criterion (2.12) for the positivity of the formal kernel $K_\Theta$. Note that $M_\Theta$ being coisometric means that $M_\Theta^*$ is isometric, so (3.3) holds with equality. This then forces the kernel (3.5) to be zero. This completes the proof of necessity in Proposition 3.2.

To prove sufficiency in part (1) of Proposition 3.2 proceed as follows. The formula (3.3) for the action of $M_\Theta^*$ on kernel functions suggests that we define an operator $T_0$ mapping the span of kernel functions in $H(K)$ to the span of kernel functions in $H(K')$ via
\[ T_0: \sum_{j=1}^N K_\beta_j y_j \mapsto \sum_{j=1}^N \sum_{\gamma, \gamma'_j, \gamma = \beta} K'_{\gamma'_j} \Theta^*_{\gamma'_j} y_j. \]
The computation in the first part of the proof read backwards tells us that the assumption $K_\Theta$ is a positive formal kernel implies that $T_0$ is contractive. Hence $T_0$ extends uniquely by continuity to a well-defined contraction operator from $H(K)$ into $H(K')$. Furthermore, the action of $T := T_0 \otimes I_{C(\zeta)}$ is given by (3.2). We then compute, for $f \in H(K')$ and $y \in \mathcal{Y}$,
\[ \langle T_0^* f, K(\cdot, \zeta)y \rangle_{H(K') \times H(K)(\zeta)} = \langle f, TK(\cdot, \zeta)y \rangle_{H(K') \times H(K)(\zeta)} \]
\[ = \langle f, K'(\cdot, \zeta)\Theta(\zeta)^*y \rangle_{H(K') \times H(K)(\zeta)} \]
\[ = \langle f(\cdot), \Theta(\zeta)^*y \rangle_{\mathcal{U}(\zeta) \times H(\zeta)} \quad \text{(by (2.13))} \]
\[ = \langle \Theta(\zeta)f(\cdot), y \rangle_{\mathcal{U}(\zeta) \times \mathcal{Y}} \quad \text{(by (2.7))} \]
from which we conclude that $T_0^* = M_\Theta$ and hence $\|M_\Theta\| \leq 1.$
We note that a homological-algebra Hilbert-module proof of part (2) in Proposition 3.2 has been given by Douglas-Misra-Sarkar for the classical (non-formal) setting [45, Theorem 1].

**Proposition 3.3.** Let \( \mathcal{H}(K_j) \) be the NFRKHS with \( \mathcal{L}(\mathcal{U}_j) \)-valued formal reproducing kernel \( K_j(z, \zeta) \) for \( j = 1, \ldots, n \) and let

\[
K(z, \zeta) = \sum_{j=1}^{n} F_j(z)K_j(z, \zeta)F_j(\zeta)^* \tag{3.6}
\]

where \( F_j \in \mathcal{L}(\mathcal{U}_j, \mathcal{X})(\{z\}) \). Then \( F(z) = [F_1(z) \cdots F_n(z)] \) is a coisometric multiplier from \( \bigoplus_{j=1}^{n} \mathcal{H}(K_j) \) to \( \mathcal{H}(K) \).

**Proof.** We view \( \bigoplus_{j=1}^{n} \mathcal{H}(K_j) \) as a NFRKHS with block-diagonal reproducing kernel

\[
K(z, \zeta) = \begin{bmatrix}
K_1(z, \zeta) & 0 \\
& \ddots & 0 \\
0 & & K_n(z, \zeta)
\end{bmatrix}.
\]

As an application of part (2) of Proposition 3.2 we see that the identity (3.6) is the precise condition required for the map

\[
M_F: \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{bmatrix} \mapsto [F_1(z) \cdots F_n(z)] \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{bmatrix}
\]

to be a partial isometry from \( \bigoplus_{j=1}^{n} \mathcal{H}(K_j) \) onto \( \mathcal{H}(K) \). \( \square \)

Given a contractive multiplier \( F \) from \( \mathcal{H}(K') \) to \( \mathcal{H}(K) \), we let \( M_F \in \mathcal{L}(\mathcal{H}(K'), \mathcal{H}(K)) \) denote the multiplication operator

\[
M_F: f(z) \mapsto F(z) \cdot f(z).
\]

We shall often view the linear space \( \mathcal{M} := \text{Ran } M_F \subset \mathcal{H}(K) \) as a Hilbert space in its own right with its own norm

\[
\|M_F\|_\mathcal{M} = \inf\{\|g\| : g \in \mathcal{H}(K') \text{ such that } M_Fg = M_Ff\} = \|Qf\|_{\mathcal{H}(K')},
\]

where \( Q \) is the orthogonal projection of \( \mathcal{H}(K') \) onto \( (\ker M_F)^\perp := \mathcal{H}(K') \oplus \ker M_F \).

### 3.1.1. Lifted-norm and pullback spaces and their Brangesian complements

More generally, if \( X : \mathcal{K}' \to \mathcal{K} \) is any contraction operator, let us write \( \mathcal{H}(X) \) for the *lifted norm space*

\[
\mathcal{H}(X) = \text{Ran } X \quad \text{with} \quad \|Xf\|_{\mathcal{H}(X)} := \inf\{\|g\|_{\mathcal{K}'} : Xg = Xf\}.
\]

It then follows that the inclusion map \( \iota : \mathcal{M} \to \mathcal{H}(K) \) is contractive since

\[
\|M_F\|_\mathcal{M} = \|M_FQf\|_\mathcal{M} = \|Qf\|_{\mathcal{H}(K')} \geq \|M_FQf\|_{\mathcal{H}(K)} = \|M_F\|_{\mathcal{H}(K)}.
\]

Conversely, if \( \mathcal{M} \) is any Hilbert space contractively included in an ambient Hilbert space \( \mathcal{H} \), then the inclusion map \( \iota : \mathcal{M} \to \mathcal{K} \) is contractive and we may view \( \mathcal{M} \) as equal to \( \text{Ran } \iota \). Let \( X : \mathcal{K}' \to \mathcal{K} \) be any operator such that \( XX^\ast = \iota^\ast \). Then \( \text{Ran } X = \text{Ran } \iota = \mathcal{M} \) and we have

\[
\|Xf\|_{\mathcal{M}} = \inf\{\|g\|_{\mathcal{K}'} : Xg = Xf\} \geq \|Xf\|_{\mathcal{K}}.
\]
In this case there is a complementary space, denoted by $\mathcal{M}^{[\perp]}$ and called the Brangesian complementary space (to $\mathcal{M}$ relative to $\mathcal{K}$) defined as $\mathcal{H}^p((I - XX^*)^{\perp})$, i.e.,

$$\mathcal{M}^{[\perp]} = \text{Ran}(I - XX^*)^{\perp}$$

with lifted norm

$$\|(I - XX^*)^{\perp}f\| = \inf\{\|g\|_\mathcal{K}: (I - XX^*)^{\perp}g = (I - XX^*)^{\perp}f\}.$$  

We note that the space $\mathcal{M}^{[\perp]}$ is independent of the choice of operator $X$ used to represent the Hilbert space $\mathcal{M} = \mathcal{H}^p(X)$ since $\mathcal{M}^{[\perp]}$ can be characterized intrinsically in terms of $\mathcal{M}$ via the formula

$$\mathcal{M}^{[\perp]} = \{g \in \mathcal{K}: \|g\|_{\mathcal{M}^{[\perp]}} := \sup\{\|Xf + g\|_\mathcal{K}^2 - \|f\|_\mathcal{M}^2: f \in \mathcal{M}\} < \infty\}.$$  

It is sometimes more convenient to view a lifted-norm space $\mathcal{H}^p(X)$ as a pullback space $H^p(\Pi)$ based on the positive-semidefinite operator $\Pi = XX^*$ and defined as follows. Given any positive semidefinite operator $\Pi$ on $\mathcal{X}$ (e.g., $\Pi = XX^*$), we define $H^p(\Pi)$ as the completion of Ran $\Pi$ in the inner product

$$\langle \Pi x, \Pi y \rangle_{H^p(\Pi)} = \langle x, y \rangle_{\mathcal{X}}.$$  

It is then not hard to show that the completion can be identified explicitly as

$$H^p(\Pi) = \text{Ran} \Pi^{\perp} = \text{Ran} X \ (\text{if } \Pi = XX^*)$$

with Ran $\Pi$ as a dense subset of $H^p(\Pi)$. In particular for the computation

$$\langle XX^*f, XX^*g \rangle_{\mathcal{H}^p(X)} = \langle X^*f, X^*g \rangle_{\mathcal{X}} = \langle XX^*f, XX^*g \rangle_{\mathcal{X}} = \langle XX^*f, XX^*g \rangle_{H^p(XX^*)}$$

shows that $H^p(XX^*)$ is isometrically equal to $H^p(X)$. In particular, if $\mathcal{M} = \mathcal{H}^p(X)$ is a contractively included subspace of $\mathcal{X}$, then its Brangesian complement $\mathcal{M}^{[\perp]} = \mathcal{H}^p((I - XX^*)^{\perp})$ is conveniently alternatively characterized as $\mathcal{M}^{[\perp]} = H^p(I - XX^*)$. Note that if $X$ is a partial isometry and $\Pi = XX^*$ is a projection, then $\mathcal{M}$ is isometrically included in $\mathcal{X}$ and the Brangesian complement $\mathcal{M}^{[\perp]} = H^p(I - \Pi)$ collapses to the standard Hilbert-space orthogonal complement $\mathcal{M}^{[\perp]} = \mathcal{M}^\perp$. For more complete details on this material, we refer to [19, 94] which has its origins in the work of de Branges-Rovnyak [35, 36].

Let us now return to the case where $\mathcal{K}' = \mathcal{H}(K')$ and $\mathcal{K} = \mathcal{H}(K)$ are noncommutative formal reproducing kernel Hilbert spaces and the operator $X: \mathcal{H}(K') \to \mathcal{H}(K)$ is a contractive multiplication operator $X = M_F$. Then we may view $\mathcal{M} = \mathcal{H}^p(M_F)$ as being the lifted-norm Hilbert space induced by the contraction operator $M_F$. In particular, $\mathcal{M}$ is contractively included in $\mathcal{H}(K)$, and we have the following result:

**Theorem 3.4.** Given a contractive multiplier $F \in \mathcal{L}(U, \mathcal{Y})((z))$ from $\mathcal{H}(K')$ to $\mathcal{H}(K)$, set $\mathcal{M}$ equal to the lifted norm space $\mathcal{M} = \mathcal{H}^p(M_F)$. Then:

1. $\mathcal{M}$ is itself a NFRKHS with reproducing kernel $K_{\mathcal{M}}(z, \zeta)$ given by

$$K_{\mathcal{M}}(z, \zeta) = F(z)K'(z, \zeta)F(\zeta)^*.$$

2. The Brangesian complement $\mathcal{M}^{[\perp]}$ is also a NFRKHS with reproducing kernel $K_{\mathcal{M}^{[\perp]}}$ given by

$$K_{\mathcal{M}^{[\perp]}}(z, \zeta) = K(z, \zeta) - K_{\mathcal{M}}(z, \zeta) = K(z, \zeta) - F(z)K'(z, \zeta)F(\zeta)^*.$$
Proof. As the elements of \( \mathcal{M} \) consist of formal power series, in order to show that \( \mathcal{M} \) is a NFRKHS it suffices to verify that the evaluation functional \( \Phi_\alpha : f(z) \to f(\alpha) \) is continuous (i.e., bounded as a linear operator from \( \mathcal{M} \) to \( \mathcal{Y} \)) for each \( \alpha \in \mathbb{F}_d^+ \). Since \( \Phi_\alpha \) is defined on all of \( \mathcal{M} \) which is a complete space in its norm, it is enough to show that \( \Phi_\alpha \) is a closed operator. To this end, we take an arbitrary sequence \( \{ f_n \} \subset \mathcal{M} \) converging to \( f \in \mathcal{M} \) and assume that \( \Phi_\alpha f_n \) converges to \( y \in \mathcal{Y} \). Since \( \mathcal{M} \) is contained contractively in \( \mathcal{H}(K) \) (since \( \mathcal{M} = \mathcal{H}^\mathcal{N}(M_F) \) and \( F \) is a contractive multiplier), we have

\[
\| f_n - f \|_{\mathcal{H}(K)} \leq \| f_n - f \|_{\mathcal{M}}.
\]

Since \( \{ f_n \} \) converges to \( f \) in \( \mathcal{M} \), it also converges to \( f \) in \( \mathcal{H}(K) \). As \( \mathcal{H}(K) \) is a NFRKHS, it follows that \( \Phi_\alpha f_n \) converges to \( f \in \mathcal{Y} \). By uniqueness of limits in \( \mathcal{Y} \) it now follows that \( y = f \), meaning that the operator \( \Phi_\alpha : \mathcal{M} \to \mathcal{Y} \) is closed. The fact that \( \mathcal{M}^{[\mathcal{N}]} \) is also a NFRKHS follows from the fact that \( \mathcal{M}^{[\mathcal{N}]} \) is also a lifted-norm space induced by a contraction operator, namely

\[
\mathcal{M}^{[\mathcal{N}]} = \mathcal{H}^\mathcal{N}(\mathcal{M}^{[\mathcal{N}]}),
\]

together with the general principle established in the preceding paragraph: any Hilbert space \( \mathcal{N} \) contractively included in a NFRKHS is itself a NFRKHS.

It remains to verify the expressions (3.7) and (3.8) for the reproducing kernels of \( \mathcal{M} \) and \( \mathcal{M}^{[\mathcal{N}]} \). As a consequence of the formula (3.2) and the discussion of the pullback space preceding the statement of the theorem and making use of the fact that we also have \( \mathcal{M} = \mathcal{H}^\mathcal{N}(M_F^*) \), we see that

\[
\begin{align*}
\langle ( (\mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y')(\zeta), y \rangle_{\mathcal{Y}(\mathcal{M}^\mathcal{N})} \\
= \langle ( (\mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y', K(\cdot, \zeta) y \rangle_{\mathcal{H}(K)(\mathcal{M}^\mathcal{N})}, \mathcal{H}(K)(\mathcal{M}^\mathcal{N}) \rangle \\
= \langle ( (\mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y', K(\cdot, \zeta) y \rangle_{\mathcal{H}(K)(\mathcal{M}^\mathcal{N})}, ( \mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y \rangle_{\mathcal{M}(\mathcal{M}^\mathcal{N}) \times \mathcal{M}(\mathcal{M}^\mathcal{N})}.
\end{align*}
\]

Since elements of the form \( (\mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y \) \( (\beta \in \mathbb{F}_d^+, y' \in \mathcal{Y}) \) span a dense set in \( \mathcal{M} \), we may then conclude by (3.7) again that

\[
K_{\mathcal{M}}(z, \zeta) y = ( (\mathcal{M}_F^\mathcal{N} \ast I_{\mathcal{L}(\mathcal{M}^\mathcal{N})}) K(\cdot, \zeta) y ) (z) = F(z) K(z, \zeta) y = \mathcal{F}(\cdot)^* y
\]

in agreement with (3.7). The formula (3.8) for \( K_{\mathcal{M}^{[\mathcal{N}]}} \) now follows by exactly the same argument as for \( K_{\mathcal{M}}(z, \zeta) \) with the substitution that \( \mathcal{M}^{[\mathcal{N}]} = \mathcal{H}^\mathcal{N}(I - M_F M_F^*) \), rather than \( \mathcal{M} = \mathcal{H}^\mathcal{N}(M_F^*) \). \( \square \)

### 3.2. Contractive Multipliers between Fock Spaces

Contractive multipliers from \( \mathcal{H}^\alpha_d(\mathbb{F}_d^+) \) to \( \mathcal{H}^\beta_d(\mathbb{F}_d^+) \) are well known. To place these into the context of related results from [27, 21], recall that an operator-tuple \( \mathbf{A} = (A_1, \ldots, A_d) \in \mathcal{L}^d \) is called strongly stable if

\[
\lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| = N} \| \mathbf{A}^\alpha x \|^2 \to 0 \quad \text{for all} \quad x \in \mathcal{X},
\]

where \( \mathbf{A}^\alpha \) is defined by \( \mathbf{A}^\alpha \) is defined according to [20]. Furthermore, a pair \( (C, \mathbf{A}) \) consisting of an operator \( C \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) and a operator-tuple \( \mathbf{A} = (A_1, \ldots, A_d) \)
Conversely, if $G(z)$ has a realization as in (3.10) with a contractive connection matrix $U$, then $G$ is a contractive multiplier from $H^2(U)$ to $H^2(U)$. The next result gives some finer system-theoretic structure associated with a contractive connection matrix $U$.

**Theorem 3.6.** Assume that the operator connection matrix $U$ (1.22) is contractive. Then (a) the pair $(C, \mathbf{A})$ is output-stable, (b) the observability operator $O_{C, \mathbf{A}} : \mathcal{X} \to H^2(U)$ (see (1.21)) is a contraction, and (c) the transfer function $\Theta_U$ (see (1.29)) is a contractive multiplier from $H^2(U)$ to $H^2(U)$. Moreover:

1. The operator $O_{C, \mathbf{A}} : \mathcal{X} \to H^2(U)$ is isometric if and only if $\mathbf{A}$ is strongly stable and the pair $(C, \mathbf{A})$ is isometric in the sense that
   \[ A_i^*A_1 + \ldots + A_d^*A_d + C^*C = I_\mathcal{X}. \] (3.11)

2. If $\mathbf{A}$ is strongly stable and $U$ is unitary, then $\Theta_U$ is strictly inner (i.e., $M_{\Theta_U} : H^2(U) \to H^2(U)$ is isometric) and the representation (1.27) is orthogonal in $H^2(U)$:
   \[ \|\tilde{\mathbf{y}}\|^2_{H^2(U)} = \|O_{C, \mathbf{A}}x(0)\|^2_{H^2(U)} + \|\Theta_U\tilde{\mathbf{y}}\|^2_{H^2(U)} = \|x(0)\|^2_{\mathcal{X}} + \|\tilde{\mathbf{y}}\|^2_{H^2(U)}. \] (3.12)

**Proof.** Items (a) and (b) follow from [20] Proposition 2.3 while item (c) follows from [20] Theorem 3.1. Item (1) is a consequence of Theorem 2.10 in [20].

We now address item (2) and assume now that $\mathbf{A}$ is strongly stable and that $U$ is unitary. To show that $M_{\Theta_U} : H^2(U) \to H^2(U)$ is isometric, it is convenient to return to the system equations (1.21). From the second of equations (1.27) with $x(\emptyset) = 0$, we see that $M_{\Theta_U}$ being an isometry is equivalent to the following: whenever $(u, x, y)$ is a system trajectory of $\Sigma_U$ (1.21) initialized with $x(\emptyset) = 0$, then necessarily

\[ \|y\|^2_{H^2(U)} = \|u\|^2_{H^2(U)}. \] (3.13)

(Here we use the notation $u = \{u(\alpha)\}_{\alpha \in \mathcal{F}^+_d}$, $x = \{x(\alpha)\}_{\alpha \in \mathcal{F}^+_d}$, $y = \{y(\alpha)\}_{\alpha \in \mathcal{F}^+_d}$.) The following argument shows that this conclusion requires only that $U$ is isometric. Indeed, if $U$ is isometric, from the system equations we see that for each $N \geq 0$,

\[ \sum_{\alpha : |\alpha| = N+1} \|x(\alpha)\|^2 + \sum_{\alpha : |\alpha| = N} \|y(\alpha)\|^2 = \sum_{\alpha : |\alpha| = N} \|x(\alpha)\|^2 + \sum_{\alpha : |\alpha| = N} \|u(\alpha)\|^2. \]
As a first approximation, let us restrict to input signals \( \sum_{\alpha: |\alpha|=N} \|x(\alpha)\|^2 = \sum_{\alpha: |\alpha|=N} \|u(\alpha)\|^2 - \sum_{\alpha: |\alpha|=N} \|y(\alpha)\|^2 \). (3.14)

If we sum from \( N = 0 \) to \( N = M \) and use that \( x(\emptyset) = 0 \), we arrive at

\[
\sum_{\alpha: |\alpha|=N} \|x(\alpha)\|^2 = \sum_{\alpha: |\alpha|\leq N} \|u(\alpha)\|^2 - \sum_{\alpha: |\alpha|\leq N} \|y(\alpha)\|^2.
\]

As a first approximation, let us restrict to input signals \( u \) such that \( u(\alpha) = 0 \) once \( |\alpha| > K \) for some large \( K \). After running the system up to words of length \( K \), we may consider the trajectory generated on words of length larger than \( K \) as determined by initializing the state \( x \) at words of length \( K \) and then continuing to run the system with zero inputs for words of length larger than \( K \). The strong-stability assumption on \( A \) implies that

\[
\sum_{\alpha: |\alpha|=N+1} \|x(\alpha)\|^2 \to 0 \quad \text{as} \quad N \to \infty.
\]

Hence we may take the limit as \( N \to \infty \) in (3.14) to conclude that (3.13) holds for the case where \( u \) has finite support. An approximation argument then implies that the same result holds for a general \( u \in L_2^U(\mathbb{F}_d^+) \).

We next address the string of equalities (3.12). Since \( U \) is unitary, in particular \((C, A)\) is an isometric output-pair (i.e. (3.11) holds) and hence by item (1) \( \mathcal{O}_{C, A} \) is isometric. By Proposition 2.3 (see also Theorem 2.10 in [20]) the space \( \mathcal{M} := \text{Ran} \mathcal{O}_{C, A} \) is then isometrically equal to the formal reproducing kernel Hilbert space \( \mathcal{H}(K_{C, A}) \) with kernel

\[
K_M(z, \zeta) = K_{C, A}(z, \zeta) = C(I - Z(z)A)^{-1}(I - A^*Z(\zeta)^*)^{-1}C^*.
\]

As \( U \) is unitary, it follows from Proposition 3.2 in [21] that

\[
K_{C, A}(z, \zeta) = k_{nc, Sz}(z, \zeta)I_Y - \Theta_U(z)(k_{nc, Sz}(z, \zeta)I_U)\Theta_U^*(\zeta).
\]

An application of Theorem 3.8 (for the case where \( \mathcal{M} \) and \( \mathcal{M}^{\perp} \) are isometrically contained in \( \mathcal{H}(k_{nc, Sz}I_Y) \)) then gives us that \( \mathcal{M}^{\perp} = \Theta_U \cdot H_2^U(\mathbb{F}_d^+) \). Thus the decomposition in the second equation in (172) is orthogonal and we have verified the first equality in (3.12). The second follows from the fact that we have already observed that both \( \mathcal{O}_{C, A} \) and \( M_{\Theta U} \) are isometric.

Theorem 3.8 below provides a useful procedure for constructing a contractive multiplier from \( H_2^U(\mathbb{F}_d^+) \) to \( H_2^C(\mathbb{F}_d^+) \) with prescribed output pair \((C, A)\) in its contractive realization, and furthermore identifying when this contractive multiplier is McCT-inner or even strictly inner. For future convenient reference let us first set forth our precise terminology concerning operator inequalities.

**Definition 3.7.** Let \( X \) be a selfadjoint operator on a Hilbert space \( \mathcal{X} \). Then we say that

1. \( X \) is **positive-semidefinite**, written as \( X \succeq 0 \) or \( -X \preceq 0 \), if \( \langle x, x \rangle_X \geq 0 \) for all \( x \in \mathcal{X} \),
2. \( X \) is **positive-definite** if \( \langle x, x \rangle_X > 0 \) for all nonzero \( x \in \mathcal{X} \), and
3. \( X \) is **strictly positive-definite**, written as \( X > 0 \) or \( -X < 0 \), if there is \( \epsilon > 0 \) so that \( \langle x, x \rangle_X \geq \epsilon^2 \|x\|^2 \) for all \( x \in \mathcal{X} \). Note that **positive-definite** and **strictly positive-definite** are equivalent in case \( \mathcal{X} \) is finite-dimensional.
In case $X - Y \geq 0$ (respectively $X - Y > 0$), we write $X \succeq Y$ or $-Y \preceq -X$ (respectively $X > Y$ or $-Y < -X$).

Let us also mention the following general fact: whenever $C \in \mathcal{L}(X, Y)$, $A \in \mathcal{L}(X', X'')$, $0 < H \in \mathcal{L}(X')$ and $0 < H' \in \mathcal{L}(X'')$, then

$$H - A^* H' A \succeq C^* C \iff \begin{bmatrix} H^{-1} & 0 \\ 0 & I_Y \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} H^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix} \succeq 0. \quad (3.16)$$

Indeed, the inequality $H - A^* H' A \succeq C^* C$ means that $\left\| \begin{bmatrix} H^{-\frac{1}{2}} A H'^{-\frac{1}{2}} \\ C H^{-\frac{1}{2}} \end{bmatrix} \right\| \leq 1$ which implies that $\left\| \begin{bmatrix} H^{-\frac{1}{2}} A^* H'^{-\frac{1}{2}} & H^{-\frac{1}{2}} C^* \end{bmatrix} \right\| \leq 1$, which in turn implies that

$$\begin{bmatrix} H^{-\frac{1}{2}} A H'^{-\frac{1}{2}} \\ C H^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} H^{-\frac{1}{2}} A^* H'^{-\frac{1}{2}} & H^{-\frac{1}{2}} C^* \end{bmatrix} \leq \begin{bmatrix} I_Y & 0 \\ 0 & I_Y \end{bmatrix},$$

which finally leads to the inequality on the right side of the implication (3.16), and conversely.

**Theorem 3.8.** Given a tuple $A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d$ and $C \in \mathcal{L}(X, Y)$, let $H \in \mathcal{L}(X)$ be a strictly positive definite operator such that

$$H - \sum_{j=1}^d A_j^* H A_j \succeq C^* C. \quad (3.17)$$

Let $A$ and $Z(z)$ be defined as in (1.24). As a consequence of (3.16) we can choose a solution $[B, D] : \mathcal{U} \to \mathcal{X}^d$ of the Cholesky factorization problem

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B & D^* \\ I & 0 \end{bmatrix} = \begin{bmatrix} H^{-1} \otimes I_d & 0 \\ 0 & I_Y \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} H^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix}. \quad (3.18)$$

1. Then the pair $(C, A)$ is output-stable and the power series

$$G(z) = D + C(I - Z(z)A)^{-1} Z(z)B \quad (3.19)$$

is a contractive multiplier from $H_Z^3(\mathbb{F}_d^2)$ to $H_Z^3(\mathbb{F}_d^2)$. Moreover,

$$k_{nc, Sz}(z, \zeta) Y - G(z)(k_{nc, Sz}(z, \zeta) I_d) G(\zeta)^* = C(I - Z(z) A)^{-1} H^{-1}(I - A^* Z(\zeta)^*)^{-1} C^*. \quad (3.20)$$

2. If (3.17) holds with equality and $A$ is strongly stable, then $G$ is McCT-inner. Conversely, any McCT-inner multiplier arises in this way.

3. If (3.17) holds with equality, $A$ is strongly stable, and the solution $[B, D]$ of (3.18) is normalized to be injective, then $G$ is strictly inner as a multiplier from $H_{Ud}^3(\mathbb{F}_d^2)$ to $H_{Ud}^3(\mathbb{F}_d^2)$. Conversely, any such strictly inner multiplier arises in this way.

The proof relies on the following general observations which will also come up in later applications.

**Proposition 3.9.** Suppose that $U : \mathcal{H} \to \mathcal{K}$ is a Hilbert-space contraction operator. Then there exists a Hilbert space $\mathcal{J}$ and an operator $V : \mathcal{J} \to \mathcal{K}$ so that

$$VV^* = I_\mathcal{K} - UU^* \quad (3.21)$$

and then $[V \quad V]$ is a coisometry from $[\mathcal{J}]$ to $\mathcal{K}$. Moreover,
(1) If $V: \mathcal{J} \to \mathcal{K}$ is normalized to be injective and if $V^* : \mathcal{J}' \to \mathcal{K}$ is another injective operator satisfying (3.21), then there is a unitary operator $W$ from $\mathcal{J}'$ onto $\mathcal{J}$ so that $V' = VW$. 

(2) The operator $[U \ V]$ is unitary if and only if the originally given operator $U$ is isometric and $V$ is normalized to be injective.

**Proof.** If $\|U\| \leq 1$, then $I - UU^* \succeq 0$ and we can solve for $V$ so that $VV^* = I - UU^*$. Then it follows immediately that $[U \ V]$ is a coisometry. As this is the only freedom in the construction, the uniqueness statement follows as well for the case where $V$ is required also to be injective. If $[U \ V]$ is unitary, then

$$
\begin{bmatrix}
U^* \\
V^*
\end{bmatrix}
\begin{bmatrix}
U & V
\end{bmatrix} = 
\begin{bmatrix}
I_H & 0 \\
0 & I_{\mathcal{J}'}
\end{bmatrix},
$$

so in particular $U^*U = I_H$ or $U$ is an isometry, and $V^*V = I_{\mathcal{J}'}$ so $V$ is injective. Conversely, if $U$ is an isometry, then $UU^* = P_{\text{Ran} U}$ is the orthogonal projection of $\mathcal{K}$ onto the range of $U$. Then $VV^* = I_{\mathcal{K}} - UU^* = P_{(\text{Ran} U)^{\perp}}$ is the orthogonal projection onto the orthogonal complement $(\text{Ran} U)^{\perp}$ of the range of $U$ in $\mathcal{K}$. Then $V$ is a partial isometry with final space equal to $(\text{Ran} U)^{\perp}$. If $V$ is injective, it follows that the initial space of $V$ is the whole space $\mathcal{J}$ and $V$ is an isometric embedding of $\mathcal{J}$ onto $(\text{Ran} U)^{\perp}$. It then follows that $[U \ V]$ is a unitary transformation from $\mathcal{H} \oplus \mathcal{J}'$ onto $\mathcal{K}$. \[\square\]

Other ingredients needed for the proof Theorem 3.8 are the shift operator-tuple $\mathbf{S}_{1,R} = (S_{1,R,1}, \ldots, S_{1,R,d})$, $S_{1,R,j} : f(z) \mapsto f(z^{j+1})$ (3.22) on $H^2_d(\mathbb{F}_d^+)$ (already mentioned in Section 2.2) and the empty-word-coefficient evaluation operator $E : \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \to f_\emptyset$ on $H^2_d(\mathbb{F}_d^+)$. The operators $S_{1,R,1}, \ldots, S_{1,R,d}$ are isometries with mutually orthogonal ranges, while the pair $(S_{1,R,E})$ is isometric in the sense that

$$
\sum_{j=1}^d S_{1,R,j}S^*_1 R,j + EE^* = I_{H^2_d(\mathbb{F}_d^+)}. 
$$

**Proof of Theorem 3.8 (1)** By replacing $A_j$ with $H^2_d A_j H^2_d^{-\frac{1}{2}}$ and $C$ with $CH^{-\frac{1}{2}}$, we may assume without loss of generality that $H = I_X$. With $G(z)$ defined as in (3.19), the inequality (3.17) says that $(C, A)$ is a contractive output-pair; it then follows that $(C, A)$ is output-stable by Proposition 2.3 in [20]. Once $G$ is defined by (3.19) with $[A \, B \, C \, D]$ isometric, the identity (3.20) follows again by Proposition 3.2 of [21]. The fact that the construction of $B, D$ via (3.18) makes $[A \, B \, C \, D]$ a coisometry is a simple consequence of Proposition 3.9 with $U = [\frac{A}{C}]$ and $V = [\frac{B}{D}]$ which then implies the identity (3.20). The validity of the identity (3.20) in turn implies that $G$ is a contractive multiplier (see [27] Theorem 3.15).

(2) If (5.17) holds with equality and $A$ is strongly stable, then the observability operator $\mathcal{O}_{C,A} : \mathcal{X} \to H^2_d(\mathbb{F}_d^+)$ is isometric (see [20] Proposition 2.3)), and then $\text{Ran} \mathcal{O}_{C,A}$ has reproducing kernel

$$
K_{C,A}(z, \zeta) = C(I - Z(z)A)^{-1}(I - A^*Z(\zeta)A)^{-1}C^* 
$$

as a subspace of $H^2_d(\mathbb{F}_d^+)$ (see [20] Theorem 2.10)). From the identity (3.20), we see that this kernel has the de Branges-Rovnyak form

$$
K_{C,A}(z, \zeta) = k_{nc,Sz}I_Y - G(z)(k_{nc,Sz}(z\zeta)I_Y)G(\zeta)^*. 
$$
As a consequence of Theorem 3.4, it follows that $\mathcal{M}^\perp = \mathcal{M}^{[\perp]}$ is equal to the pullback space $\mathcal{H}(M_GM_G^*)$. Since $\mathcal{M}^\perp$ is isometrically contained in $H^2_\mathcal{U}(\mathbb{F}_d^+)$, it follows that $M_GM_G^*$ is the projection onto $\mathcal{M}^\perp$. In particular, $M_G$ is a partial isometry, so $G$ is a McCT-inner multiplier.

Finally, suppose that $G$ is any McCT-inner multiplier from $H^2_\mathcal{U}(\mathbb{F}_d^+)$ to $H^2_\mathcal{U}(\mathbb{F}_d^+)$, and set $\mathcal{N} = \text{Ran } M_G$. Then $\mathcal{N}$ is isometrically contained in $H^2_\mathcal{U}(\mathbb{F}_d^+)$ and also we can identify $\mathcal{N}$ with the pullback space $\mathcal{H}(M_GM_G^*)$ having reproducing kernel

$$K_N(z, \zeta) = G(z)(k_{nc,Sx}(z, \zeta)I_\mathcal{U})G(\zeta)^*.$$  

As $\mathcal{N} = M_G \cdot H^2_\mathcal{U}(\mathbb{F}_d^+)$ is clearly $S_{1,R}$-invariant, it follows that $\mathcal{N}^\perp$ is $S_{1,R}^*$-invariant. Then we can represent $\mathcal{N}^\perp$ as the range of an observability operator $\mathcal{N}^\perp = \text{Ran } O_{C,A}$ with $(C, A)$ the restricted model output-pair

$$C = E|_{\mathcal{N}^\perp}, \ e.g., E: \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_0, \quad A = S_{1,R}^*|_{\mathcal{N}^\perp}.$$  

Since the model output pair $(E, S_{1,R})$ is an isometric pair with $S_{1,R}^*$ strongly stable, it follows that the same properties hold for $(C, A)$. If we construct $[\tilde{B}^\tau]$ from $\mathcal{U}$ to $\mathcal{N}^\perp$ according to the Cholesky factorization procedure (3.18) and define

$$G'(z) = D + C(I - Z(z)A)^{-1}B,$$

then by the first part of the proof it follows that we also recover $\mathcal{N}$ as $\mathcal{N} = \mathcal{H}(M_GM_G^*)$ and hence $M_G \cdot M_G^* = M_GM_G^*$, or in kernel form,

$$G(z)(k_{nc,Sx}(z, \zeta)I_\mathcal{U})G(\zeta)^* = G'(z)(k_{nc,Sx}(z, \zeta)I_\mathcal{U})G'(\zeta)^*.$$  

As a consequence of Proposition 3.17 to come, it follows that there is a partial isometry $W: \mathcal{U} \to \mathcal{U}'$ so that $G(z) = G'(z)W$ and $G'(z) = G(z)W^*$. Thus $G(z)$ has a realization of the form (3.19) but with connection matrix $\tilde{U}$ given by

$$\tilde{U} := \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} A & BW \\ C & DW \end{bmatrix}.$$  

To show that the matrix $\tilde{U}$ is coisometric (and hence $[\tilde{B}^\tau]$ arises as another solution of the Cholesky factorization procedure (3.18)), we need to show that

$$\begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} [\tilde{B}^* \tilde{D}^*] = \begin{bmatrix} B \\ D \end{bmatrix} [B^* D^*],$$

or equivalently,

$$\begin{bmatrix} B \\ D \end{bmatrix} W W^* [B^* D^*] = \begin{bmatrix} B \\ D \end{bmatrix} [B^* D^*].$$  

The equality $G(z) = G'(z)W$ when expressed in terms of power-series coefficients gives us

$$\tilde{D} = DW, \quad CA^\alpha \tilde{B}_j = CA^\alpha B_j W \quad \text{for all } \alpha \in \mathbb{F}_d^+ \text{ and } j = 1, \ldots, d.$$

As $(C, A)$ is isometric and $A$ is strongly stable by the setup for item (2) in the theorem, it follows from item (1) in Theorem 3.6 that $O_{C,A}$ is isometric, so in particular $(C, A)$ is observable. Thus the second equality in (3.24) implies that in fact $\tilde{B}_j = B_j W$ for $j = 1, \ldots, d$, or $\tilde{B} = BW$. This combined with the first equality in (3.24) gives us the desired equality (3.23).
A close relation between McCT-inner and strictly inner multipliers is pointed out in Remark 3.11 below. We start with a preliminary observation.

**Proposition 3.10.** Given any McCT-inner multiplier $G$ from $H^2_d(B^+_d)$ to $H^2_2(B^+_d)$, the space $\text{Ker } M_G$ is reducing for $S_{1,R}$.

**Proof.** Since $M_G$ intertwines $S_{1,R,j}$ on $H^2_d(B^+_d)$ with $S_{1,R,j}$ on $H^2_2(B^+_d)$, the computation

$$M_G S_{1,R,j} f = S_{1,R,j} M_G f = 0 \quad \text{for } f \in \text{Ker } M_G$$

shows that $\text{Ker } M_G$ is invariant for $S_{1,R}$ on $H^2_d(B^+_d)$. To show that $(\text{Ker } M_G)^{\perp}$ is also invariant for $S_{1,R}$, we use the fact that each $S_{1,R,j}$ is an isometry and that $M_G$ being a partial isometry implies that $(\text{Ker } M_G)^{\perp}$ is characterized by the property

$$f \in (\text{Ker } M_G)^{\perp} \iff \|M_G f\| = \|f\|.$$

Thus $f \in (\text{Ker } M_G)^{\perp}$ implies that

$$\|M_G S_{1,R,j} f\| = \|S_{1,R,j} M_G f\| = \|M_G f\| = \|f\|,$$

from which we conclude that $S_{1,R,j} f \in (\text{Ker } M_G)^{\perp}$ as well for $j = 1, \ldots, d$. □

**Remark 3.11.** For any McCT-inner multiplier $G$ from $H^2_d(B^+_d)$ to $H^2_2(B^+_d)$, there is an orthogonal decomposition $U = [U_0, U_\delta]$ of the input space so that with respect to this decomposition, $G(z)$ has the form $G(z) = [G_{\delta i}(z) \ 0]$ with $G_{\delta i}$ strictly inner. One way to see this statement is as an application of Proposition 3.10 given a McCT-inner multiplier $G$, set $U_0 = \text{Ker } M_G$ and $U_\delta = U \cap U_0$. An alternative state-space proof can be derived from Theorem 3.8 as follows. By item (2) in Theorem 3.8, McCT-inner multipliers $G$ are characterized by having a realization (3.19) with $(C,A)$ isometric and observable, $A$ strongly stable, and $B,D$ solving the Cholesky factorization problem (3.18) (so $[A \ B]$ is coisometric). By item (3), strictly inner multipliers are characterized via the same conditions with the additional property that $[A \ B]$ be injective. We may therefore set $U_\delta = (\text{Ker } [A \ B])^{\perp}$ and $U_0 = \text{Ker } [B \ B]$ to recover the decomposition $G = [G_{\delta i} \ 0]$ of $G$ with $G_{\delta i}$ strictly inner.

This reduction of a McCT-inner $G$ to a strictly inner $G_{\delta i}$ fails in the more general context of McCT-inner multipliers between weighted Bergman spaces, causing the Beurling-Lax representations for shift-invariant subspaces isometrically included in the ambient weighted Bergman space to be discussed in Section 3.14 to be formulated only with McCT-inner (rather than strictly inner) multipliers.
This same phenomenon is behind the incorrect statement of Theorem 4.6 in [21] where the additional hypothesis the output pair $(C, A)$ is isometric should be inserted and the conclusion $S$ is (strictly) inner should be changed to $S$ is MCD-inner. Also the conclusion of part (2) of Theorem 3.8 is stated incorrectly in part (4) of Proposition 3.3 in [21]: the hypothesis the output pair $(C, A)$ is a contractive should be changed to the output pair $(C, A)$ is isometric.

We recall that multiplication operators $M_F: H^2_0(F^+_d) \to H^2_0(F^+_d)$ are characterized as the operators intertwining $S_{1,R} \otimes I_d$ with $S_{1,R} \otimes I_d$: if the operator $X$ in $L(H^2_0(F^+_d), H^2_0(F^+_d))$ satisfies the intertwining equalities

$$X S_{1,R,j} = S_{1,R,j} X \quad \text{for} \quad j = 1, \ldots, d,$$

then $X = M_F: u(z) \mapsto F(z) u(z)$ for a multiplier $F \in \mathcal{L}(U, Y)\langle \langle z \rangle \rangle$ (see [82]). We shall have use for the following Commutant Lifting Theorem for this setting due to Popescu (see [79, Theorem 3.2]).

**Theorem 3.12.** Suppose that the subspaces $M \subset H^2_0(F^+_d)$ and $N \subset H^2_0(F^+_d)$ are invariant for the backward right-shift tuple $(S_{1,R})^*$ acting on $H^2_0(F^+_d)$ and $H^2_0(F^+_d)$ respectively, and that the operator $X \in \mathcal{L}(M, N)$ satisfies

$$\|X\| \leq 1, \quad X (S_{1,R,j|M})^* = (S_{1,R,j|N})^* X \quad \text{for} \quad j = 1, \ldots, d.$$ 

Then there is a contractive multiplier $F \in \mathcal{L}(U, Y)\langle \langle z \rangle \rangle$ so that $(M_F)^*|_M = X$.

Strictly inner multipliers between Fock spaces are of particular interest as they serve as Beurling-Lax representatives of shift invariant subspaces (see Chapter 5 for more details). The following Beurling-Lax type theorem was originally given by Popescu [81]: see also [21].

**Theorem 3.13.** Let $M$ be a closed $S_{1,R}$-invariant subspace of $H^2_0(F^+_d)$. Then there exist a Hilbert space $U$ and a strictly inner multiplier $G$ from $H^2_0(F^+_d)$ to $H^2_0(F^+_d)$ such that $M = G \cdot H^2_0(F^+_d)$.

Characterization of strictly inner multipliers between two Fock spaces in terms of state space realizations (3.19) was given in Theorem 3.8. Another characterization of strictly inner multipliers is as follows.

**Lemma 3.14.** Let $F$ be a contractive multiplier from $H^2_0(F^+_d)$ to $H^2_0(F^+_d)$. Then $F$ is a strictly inner multiplier if and only if $\|F u\|_{H^2_0(F^+_d)} = \|u\|_U$ for all $u \in U$.

**Proof.** The “only if” part is self-evident.

To prove the “if” part, we recall that the operators $S_{1,R,1, \ldots, S_{1,R,J}}$ are isometries on $H^2_0(F^+_d)$ with mutually orthogonal ranges, and therefore,

$$\|S_{1,R}^\alpha M_F u\|_{H^2_0(F^+_d)} = \|M_F u\|_{H^2_0(F^+_d)} = \|u\|_U$$

for all $u \in U$ and $\alpha \in F^+_d$. Moreover,

$$\|S_{1,R}^\alpha M_F u + S_{1,R}^\beta M_F v\|_{H^2_0(F^+_d)}^2 = \|M_F u\|_U^2 + \|v\|_U^2 + 2 \text{Re} \langle S_{1,R}^\alpha M_F u, S_{1,R}^\beta M_F v \rangle_{H^2_0(F^+_d)}$$

for any $u, v \in U$ and $\alpha, \beta \in F^+_d$. On the other hand, as $M_F: H^2_0(F^+_d) \to H^2_0(F^+_d)$ is a contraction, we have for $\alpha \neq \beta$,

$$\|S_{1,R}^\alpha M_F u + S_{1,R}^\beta M_F v\|_{H^2_0(F^+_d)}^2 \leq \|S_{1,R}^\alpha u + S_{1,R}^\beta v\|_{H^2_0(F^+_d)}^2 = \|u\|_U^2 + \|v\|_U^2.$$
Combining the two latter relations we conclude that
\[ 2\mathrm{Re}(S^\alpha_{1,R}MFu, S^\beta_{1,R}MFv)_{H^2_\mathcal{U}(\mathbb{F}_d^+)} \leq 0 \quad \text{for all } u, v \in \mathcal{U} \text{ and } \alpha \neq \beta, \]
which is possible (since \( u \) can be replaced by \(-u\) and by \( \pm iu \)) only if

\[ \langle S^\alpha_{1,R}MFu, S^\beta_{1,R}MFv \rangle_{H^2_\mathcal{U}(\mathbb{F}_d^+)} = 0. \]

If this is the case, then \( \|Fp\|_{H^2_\mathcal{U}(\mathbb{F}_d^+)} = \|p\|_{H^2_\mathcal{U}(\mathbb{F}_d^+)} \) for every \( \mathcal{U} \)-valued “polynomial” \( p \in \mathcal{U}(\langle z \rangle) \) and therefore, for any \( p \in H^2_\mathcal{U}(\mathbb{F}_d^+) \). Therefore, \( M_F \) is an isometry from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) and hence, \( F \) is strictly inner. \( \Box \)

The next result shows that the class of coisometric multipliers from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) is quite narrow.

**Lemma 3.15.** Let \( F(z) = \sum_{\alpha \in \mathbb{F}_d^+} F_\alpha z^\alpha \) be a coisometric multiplier from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \). Then \( F(z) \) is a coisometric constant: \( F_0 F_0^* = I_Y \) and \( F_\alpha = 0 \) for all \( \alpha \in \mathbb{F}_d^+ \setminus \{0\} \).

**Proof.** By specializing the second part of Proposition 3.2 to the case where \( K(z, \zeta) = k_{nc, Sz}(z, \zeta)I_Y \) and \( K'(z, \zeta) = k_{nc, Sz}(z, \zeta)I_U \) we conclude that \( F(z) \) is a coisometric multiplier from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) if and only if

\[ k_{nc, Sz}(z, \zeta)I_Y = F(z) k_{nc, Sz}(z, \zeta) F(\zeta)^* = 0. \]

Substituting explicit power series formulas for \( F \) and \( \langle z \rangle \) for \( k_{nc, Sz} \) into the latter equality and equating the coefficients of \( z^\alpha \zeta^\beta \) on both sides we get the desired conclusion. \( \Box \)

### 3.3. A Noncommutative Leech Theorem

The next result is the noncommutative version of the Leech theorem (see [65]) solving the problem of producing a contractive analytic operator function \( G \) which solves the factorization problem \( F = GS \) for two given meromorphic operator functions \( G \) and \( F \).

**Theorem 3.16.** Given power series \( G \in \mathcal{L}(\mathcal{V}, \mathcal{X})(\langle z \rangle) \) and \( F \in \mathcal{L}(\mathcal{U}, \mathcal{X})(\langle z \rangle) \), the formal kernel

\[ K_{G,F}(z, \zeta) := G(z)(k_{nc, Sz}(z, \zeta) \otimes I_Y)G(\zeta)^* - F(z)(k_{nc, Sz}(z, \zeta) \otimes I_U)F(\zeta)^* \] (3.25)

is positive if and only if there exists a contractive multiplier \( S \) from \( H^2_{\mathcal{U}}(\mathbb{F}_d^+) \) to \( H^2_{\mathcal{U}}(\mathbb{F}_d^+) \) such that \( F(z) = G(z)S(z) \).

**Proof.** If \( F(z) = G(z)S(z) \) for a contractive multiplier \( S \), then

\[ K_{G,F}(z, \zeta) = G(z)(k_{nc, Sz}(z, \zeta) \otimes I_Y - S(z)(k_{nc, Sz}(z, \zeta) \otimes I_U)S(\zeta)^*)G(\zeta)^* \]

is a positive kernel as a consequence of the characterization of contractive multipliers in Proposition 3.2. We provide two proofs of the converse which illustrate two distinct approaches to metric-constrained interpolation theory for this setting. The second proof is more general in that the first proof requires an additional hypothesis.

**Proof I** (via Commutant Lifting Theorem): For this proof we impose the additional hypothesis:

**Additional Hypothesis:** Assume that both \( G \) and \( F \) are themselves bounded multipliers from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) and from \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) to \( H^2_\mathcal{U}(\mathbb{F}_d^+) \) respectively.
Suppose that $K_{G,F}$ is a positive formal kernel. By the computations in the proof of Proposition 3.2, we see that the positivity of $K_{G,F}$ is equivalent to

$$
\| \sum_{j=1}^{N} M_{G}^k k_{\beta_j} f_{\beta_j} \|_2^2 - | \sum_{j=1}^{N} M_{G}^k k_{\beta_j} x_j |_2^2 \geq 0
$$

for all choices of $\beta_1, \ldots, \beta_N \in \mathbb{F}_d^+$ and $x_1, \ldots, x_N \in \mathcal{X}$. Thus, the operator

$$
T_0: M_{G}^* f \mapsto M_{F}^* f \quad \text{for} \quad f \in H_\mathcal{X}^2(\mathbb{F}_d^+)
$$

extends to a contraction from $\mathcal{M} = \overline{\text{Ran} M_{G}^k} \subset H_\mathcal{X}^2(\mathbb{F}_d^+)$ into $\mathcal{N} = \overline{\text{Ran} M_{F}^k} \subset H_\mathcal{X}^2(\mathbb{F}_d^+)$. Since multiplication operators intertwine the right shift operators $S_{1,R,j} \otimes I_H$ and $S_{1,R,j} \otimes I_Y$, it is easily verified that

$$
T_0 ((S_{1,R,j} \otimes I_Y)^* |_{\mathcal{M}}) = ((S_{1,R,j} \otimes I_H)^* |_{\mathcal{N}}) T_0 \quad \text{for} \quad j = 1, \ldots, d.
$$

By the Commutant Lifting Theorem 3.12, we conclude that there is a contractive multiplier $S \in \mathcal{L}(\mathcal{H},\mathcal{Y})((z))$ such that $M_S^* |_{\mathcal{M}} = T_0$, i.e., $M_S^* M_{G}^k = M_{F}^k$.

**Proof II (via Lurking Isometry method):** Assume that $K_{G,F}$ is a positive formal kernel. Then by (3) $\Rightarrow$ (2) in Theorem 2.1, $K_{G,F}$ has a formal Kolmogorov decomposition

$$
K_{G,F}(z,\zeta) = H(z)H(\zeta)^*
$$

for a formal power series $H(z) \in \mathcal{L}(H_0,\mathcal{X})$ ($H_0$ is an auxiliary Hilbert space). Combining (3.25) and (2.19) we see that the kernel $K_{G,F}(z,\zeta)$ satisfies

$$
K_{G,F}(z,\zeta) - \sum_{k=1}^{d} \tilde{\zeta}_k K_{G,F}(z,\zeta) \zeta_k = G(z)G(\zeta)^* - F(z)F(\zeta)^*.
$$

Applying the operation $X \mapsto X - \sum_{k=1}^{d} \tilde{\zeta}_k X \zeta_k$ to both sides of (3.25) and taking into account the last equality then yields

$$
G(z)G(\zeta)^* - F(z)F(\zeta)^* = H(z)H(\zeta)^* - \sum_{k=1}^{d} H(z) \zeta_k \tilde{\zeta}_k H(\zeta)^*.
$$

Equate coefficients of $z^\alpha \zeta^\beta$ to get

$$
G_\alpha G_\beta^* - F_\alpha F_\beta^* = H_\alpha H_\beta^* - \sum_{k=1}^{d} H_{\alpha k^{-1}} H_{\beta k^{-1}}^*
$$

where we use the convention

$$
\alpha k^{-1} = \begin{cases} \alpha', & \text{if } \alpha = \alpha' k, \\ \text{undefined,} & \text{otherwise,} \end{cases}
$$

and where by convention, $H_{\text{undefined}} = 0$. Rewrite (3.27) as

$$
\sum_{k=1}^{d} H_{\alpha k^{-1}} H_{\beta k^{-1}}^* + G_\alpha G_\beta^* = H_\alpha H_\beta^* + F_\alpha F_\beta^*.
$$
Then the map

$$V: \begin{bmatrix} H_{\beta 1-1}^* \\ \vdots \\ H_{\beta d-1}^* \\ G_{\beta}^* \end{bmatrix} x \mapsto \begin{bmatrix} H_{\beta}^* \\ F_{\beta}^* \end{bmatrix} x$$

extends to a well-defined isometry from the subspace

$$D = \overline{\text{span}}_{\beta \in \mathbb{F}_d^+, x \in X} \left\{ \begin{bmatrix} H_{\beta 1-1}^* \\ \vdots \\ H_{\beta d-1}^* \\ G_{\beta}^* \end{bmatrix} x \right\} \subset \begin{bmatrix} X_0' \\ Y \end{bmatrix}$$

onto the subspace

$$R = \overline{\text{span}}_{\beta \in \mathbb{F}_d^+, x \in X} \left\{ \begin{bmatrix} H_{\beta}^* \\ F_{\beta}^* \end{bmatrix} x \right\} \subset \begin{bmatrix} X_0 \\ U \end{bmatrix}.$$ 

Extend $V$ to a contraction (or even to a unitary at the possible expense of enlarging the state space $X_0$) $U^*: \begin{bmatrix} X_0' \\ Y \end{bmatrix} \to \begin{bmatrix} X_0 \\ U \end{bmatrix}$. Decompose $U^*$ into block-matrix form

$$U^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} = \begin{bmatrix} A_1^* & \cdots & A_d^* \\ B_1^* & \cdots & B_d^* \end{bmatrix} : \begin{bmatrix} X_0' \\ Y \end{bmatrix} \to \begin{bmatrix} X_0 \\ U \end{bmatrix}.$$ 

Since $U^*$ is an extension of $V$, we get from (3.28) the system of equations

$$\sum_{k=1}^d A_k^* H_{\beta k-1}^* + C^* G_{\beta}^* = H_{\beta}^*, \quad \sum_{k=1}^d B_k^* H_{\beta k-1}^* + D^* G_{\beta}^* = F_{\beta}^*.$$ 

Actually the adjoint form of these equations will be more convenient:

$$\sum_{k=1}^d H_{\beta k-1} A_k + G_{\beta} C = H_{\beta}, \quad \sum_{k=1}^d H_{\beta k-1} B_k + G_{\beta} D = F_{\beta}.$$ 

Apply the formal $Z$-transform (1.26) to get the formal power series version of this system of equations:

$$H(z)Z(z)A + G(z)C = H(z), \quad (3.29)$$
$$H(z)Z(z)B + G(z)D = F(z) \quad (3.30)$$

with $Z(z)$ as in (1.24) (but with $X_0$ in place of $X$). Use (3.29) to solve for $H(z)$:

$$H(z) = G(z)C(I - Z(z)A)^{-1}.$$ 

Plug this expression for $H(z)$ back into (3.30) to get

$$G(z) \left( C(I - Z(z)A)^{-1} Z(z)B + D \right) = F(z). \quad (3.31)$$

This suggests that we set

$$S(z) = D + C(I - Z(z)A)^{-1} Z(z)B.$$ 

Since $U$ is contractive/unitary, then $S$ is a contractive multiplier from $H_0^2(\mathbb{F}_d^+)$ to $H_3^2(\mathbb{F}_d^+)$, by Theorem 3.3. Finally, it is seen from (3.31) that $F(z) = G(z)S(z)$ which completes the proof. □

The following particular case of the Leech theorem is worth noting.
Proposition 3.17. Let $G \in \mathcal{L}(\mathcal{Y}, \mathcal{X})\langle\langle z \rangle\rangle$ and $F \in \mathcal{L}(\mathcal{U}, \mathcal{X})\langle\langle z \rangle\rangle$ satisfy the kernel identity

$$G(z)(k_{nc,S}(z, \zeta) \otimes I_{\mathcal{Y}})G(\zeta)^* = F(z)(k_{nc,S}(z, \zeta) \otimes I_{\mathcal{U}})F(\zeta)^*. \quad (3.32)$$

Then there exists a partial isometry $W \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that $F(z) = G(z)W$ and $G(z) = F(z)W^*$. \hfill $\square$

Proof. The “if” part is clear. To prove the “only if” part, we follow the lines of the second proof of Theorem 3.14. Identity (3.32) means that the kernel $K_{G,F}$ (3.25) is zero, so that the formal power series $H(z)$ from the formal Kolmogorov decomposition (3.26) is also zero. Then the map $H(3.25)$ is zero, so that the formal power series $H(z)$ from the formal Kolmogorov decomposition (3.26) is also zero. Then the map $V : G_{\beta}^*x \to F_{\beta}^*x$ extends to a well-defined isometry from the subspace $G = \text{span}_{\beta \in \mathbb{F}_d^+, x \in \Lambda}G_{\beta}^*x \subset \mathcal{Y}$ onto the subspace $\mathcal{R} = \text{span}_{\beta \in \mathbb{F}_d^+, x \in \Lambda}F_{\beta}^*x \subset \mathcal{U}$. Extending $V$ to a partial isometry $W^* : \mathcal{Y} \to \mathcal{U}$ by letting $W^*|_{\mathcal{Y}\setminus G} = 0$, we get $W^*G_{\beta}^* = F_{\beta}^*$ and $WF_{\beta}^* = G_{\beta}^*$ or equivalently,

$$F_{\beta} = G_{\beta}W \quad \text{and} \quad G_{\beta} = F_{\beta}W^* \quad \text{for all } \beta \in \mathbb{F}_d^+.$$  

Now the equalities $F(z) = G(z)W$ and $G(z) = F(z)W^*$ follow. \hfill $\square$

3.4. Contractive multipliers from $H^2_{d}(\mathbb{F}_d^+)$ to $H^2_{\omega,j}(\mathbb{F}_d^+)$ for admissible $\omega$

We now suppose that $\omega = \{\omega_j\}_{j \geq 0}$ is an admissible weight sequence as in (2.30) with associated weighted Hardy-Fock space $H^2_{\omega}(\mathbb{F}_d^+)$. By Remark 2.10 the associated kernel $k_{nc,\omega}$ is a contractive kernel, i.e., the right shift operator tuple $S_{\omega,R} = (S_{\omega,R,1}, \ldots, S_{\omega,R,d})$ is a row contraction. By Proposition 2.8 the kernel $K = k_{\omega}$ admits a factorization (2.21) for an appropriately chosen $G$. Our next goal is to construct this $G$ explicitly. Toward this end, let us introduce the space

$$\ell^2_{\mathcal{Y}}(\mathbb{F}_d^+) = \left\{ \{f_\alpha\}_\alpha \in \mathbb{F}_d^+ : \sum_{\alpha \in \mathbb{F}_d^+} \|f_\alpha\|^2_{\mathcal{Y}} < \infty \right\} \quad (3.33)$$

which is isomorphic to the Fock space $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ via the noncommutative $Z$-transform (1.26). Let us define the operator $\Psi_{\omega} : \ell^2_{\mathcal{Y}}(\mathbb{F}_d^+) \to H^2_{\omega,j}(\mathbb{F}_d^+)$ (a weighted noncommutative $Z$-transform) by

$$\Psi_{\omega} = \text{Row}_{\alpha \in \mathbb{F}_d^+}[\omega^\frac{1}{2}_|\alpha|^z\alpha] : \{f_\alpha\}_\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \omega^\frac{1}{2}_|\alpha|^zf_\alpha^z\alpha.$$  

We identify $\Psi_{\omega}$ with an operator-valued power series:

$$\Psi_{\omega}(z) = \sum_{\alpha \in \mathbb{F}_d^+} \left( \text{Row}_{\beta \in \mathbb{F}_d^+}[\delta_{\alpha,\beta}(\omega^\frac{1}{2}_|\alpha|^z\alpha) \otimes I_{\mathcal{Y}}] \right) z^\alpha : \{f_\alpha\}_\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \omega^\frac{1}{2}_|\alpha|^zf_\alpha^z\alpha. \quad (3.34)$$

Note that $\Psi_{\omega}(z)$ is the factor for the Kolmogorov decomposition of the positive noncommutative kernel $k_{\omega}$ (2.25):

$$k_{\omega}(z, \zeta) = \Psi_{\omega}(z)\Psi_{\omega}(\zeta)^*. \quad (3.35)$$

Assume now that $\omega$ is strictly decreasing ($\omega_j < \Omega_{j-1}$ for $j \geq 1$), and define a new weight $\gamma = \{\gamma_j\}_{j \geq 0}$ by

$$\gamma_0 = 1, \quad \gamma_j = (\omega_{j-1} - \omega_{j-2})^{-1} = \frac{\omega_j\omega_{j-1}}{\omega_{j-1} - \omega_j} \quad \text{for } j \geq 1. \quad (3.36)$$
Since $\omega$ is strictly decreasing and positive, we see that $\gamma$ is a positive weight sequence. It then follows from (2.28) and (3.36) that the kernel
\[
\bar{k}_\omega(z, \zeta) := k_\omega(z, \zeta) - \sum_{k=1}^{d} \zeta^k k_\omega(z, \zeta) z_k = \sum_{\alpha \in \mathbb{F}_d^+} \gamma^{-1} |z|^\alpha \zeta^\top = k_\gamma(z, \zeta) \tag{3.37}
\]
is a positive noncommutative kernel. The latter can also be seen as a consequence of the fact that the right shift operator tuple $S_{\omega, R}$ is a positive noncommutative kernel. The latter can also be seen as a consequence of item (1) in Proposition 3.2 with
\[
K(z, \zeta) = k_\omega(z, \zeta), \quad \Theta(z) = [z_1 \cdots z_d], \quad \bar{K}(z, \zeta) = k_\omega(z, \zeta) \otimes I_d.
\]
One can iterate (3.37) to solve for $k_\omega$ in terms of $k_\gamma$ as follows:
\[
k_\omega(z, \zeta) = k_\gamma(z, \zeta) + \sum_{k=1}^{d} \zeta^k k_\omega(z, \zeta) z_k
\]
\[
= k_\gamma(z, \zeta) + \sum_{k=1}^{d} \zeta^k k_\gamma(z, \zeta) z_k + \sum_{\alpha : |\alpha| = 2} \zeta^\top k_\omega(z, \zeta) z^\alpha
\]
\[
= \cdots + \sum_{\alpha : |\alpha| = N+1} \zeta^\top k_\omega(z, \zeta) z^\alpha.
\]
Taking a limit as $N \to \infty$ then gives
\[
k_\omega(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \zeta^\top k_\gamma(z, \zeta) z^\alpha. \tag{3.38}
\]
If we now recall (3.35) (with $\gamma$ in place of $\omega$), we can rewrite (3.38) as
\[
k_\omega(z, \zeta) = \Psi_\gamma(z) \left(k_{nc, Sz}(z, \zeta) \otimes I_{\ell_2^d(F_d^+)}\right) \Psi_\gamma(z)^*,
\]
which is the expected factorization (2.24) for the noncommutative positive contractive kernel $k_\omega$.

**Remark 3.18.** In case $\omega = \mu_n$ (for $n > 1$), the sequence $\gamma$ defined via formulas (3.36) turns out to be $\gamma = \mu_{n-1}$. Then the formula (3.38) takes the form
\[
k_n(z, \zeta) = \Psi_{n-1}(z) \left(k_{nc, Sz}(z, \zeta) \otimes I_{\ell_2^d(F_d^+)}\right) \Psi_{n-1}(\zeta)^*, \tag{3.39}
\]
where $\Psi_{n-1}(z) = \Psi_\gamma(z)$ is given by
\[
\Psi_{n-1}(z) = \sum_{\alpha \in \mathbb{F}_d^+} \left(\text{Row}_{\beta \in \mathbb{F}_d^+} [\delta_{\alpha, \beta}(\mu_n^{-\frac{1}{2}}) \otimes I_y]\right) z^\alpha \in \mathcal{L} (\ell_2^d(F_d^+), y)(\langle z \rangle).
\]
The single-variable $(d = 1)$ case of the next theorem is the main result of [18].

**Theorem 3.19.** Let $M_{\Psi_\gamma} : H^2_{\ell_2^d(F_d^+)}(F_d^+) \to H^2_{\gamma, Y}(F_d^+)$ be the multiplication operator associated with $\Psi_\gamma(z)$ defined as in (3.34). Then
(1) $M_{\Psi_\gamma}|_{\ell_2^d(F_d^+)}$ is an isometry from the space of constant functions $\ell_2^d(F_d^+)$ in $H^2_{\ell_2^d(F_d^+)}(F_d^+)$ into $H^2_{\gamma, Y}(F_d^+)$.  
(2) $M_{\Psi_\gamma}$ is a coisometry from $H^2_{\ell_2^d(F_d^+)}(F_d^+)$ onto $H^2_{\omega, Y}(F_d^+)$. 

\[\int_{\mathbb{C}} \Psi_\gamma(z) \overline{\Psi_\gamma(\zeta)} d\gamma(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \left(\text{Row}_{\beta \in \mathbb{F}_d^+} [\delta_{\alpha, \beta}(\mu_n^{-\frac{1}{2}}) \otimes I_y]\right) z^\alpha = I_{\ell_2^d(F_d^+), y}(\langle z \rangle).\]
is coisometric if and only if it is of the form

$$F(z) = \Psi_\gamma(z)S(z)$$

(3.40)

for some contractive multiplier $S(z)$ from $H^2_\mu(\mathbb{D})$ to $H^2_\mu,\omega(\mathbb{D})^+$ if and only if it is of the form

$$F(z) = \Psi_\gamma(z)S(z)$$

(3.40)

for some contractive multiplier $S(z)$ from $H^2_\mu(\mathbb{D})$ to $H^2_\mu,\omega(\mathbb{D})^+$.

(4) Any contractive multiplier $S$ from $H^2_\mu(\mathbb{D})$ into $H^2_\mu,\omega(\mathbb{D})^+$ for which (3.40) holds has the form

$$S(z) = S_0(z) + \Theta(z)H(z)$$

where $S_0$ is any particular contractive multiplier for which (3.40) holds, where $\Theta(z) \in \mathcal{L}(\mathcal{U}, \ell^2_\nu(\mathbb{D})^+)$ is the Beurling-Lax representation for the $S_{1,R}$-invariant subspace $\ker M_{\Psi_\gamma} \subset H^2_\mu,\omega(\mathbb{D})^+$ and where $H(z)$ is a bounded multiplier from $H^2_\mu(\mathbb{D})$ into $H^2_\mu,\omega(\mathbb{D})^+$ chosen so that $S_0 + \Theta H$ is still a contractive multiplier.

PROOF. For $y = \{y_\alpha\}_\alpha \in \ell^2_\nu(\mathbb{D})^+$, note that

$$\|\Psi_\gamma(z)y\|_{H^2_\mu,\omega(\mathbb{D})^+}^2 = \sum_{\alpha \in \nu} 2^{-\nu} |\gamma_\alpha|^2 |y_\alpha|^2 = \|y\|_{\ell^2_\nu(\mathbb{D})}^2,$$

and statement (1) follows. Statement (2) is an immediate consequence of the identity (3.38) combined with part (2) of Proposition 3.2.

By Proposition 3.2 $F$ is a contractive multiplier from $H^2_\mu(\mathbb{D})$ to $H^2_\mu,\omega(\mathbb{D})^+$ if and only if the formal kernel

$$K_F(z, \zeta) := k_\omega(z, \zeta) \otimes I_y - F(z)(k_{nc, \omega}(z, \zeta) \otimes I_U)F(\zeta)^*$$

is positive. Due to (3.38) we have

$$K_F(z, \zeta) := \Psi_\gamma(z)(k_{nc, \omega}(z, \zeta) \otimes I_{\nu}(\mathbb{D})^+)^*\Psi_\gamma(\zeta)^*$$

$$- F(z)(k_{nc, \omega}(z, \zeta) \otimes I_U)F(\zeta)^*,$$

and by Theorem 3.16, the latter kernel is positive if and only if $F$ admits the representation (3.40) for some contractive multiplier $S(z)$ from $H^2_\mu(\mathbb{D})$ to $H^2_\mu,\omega(\mathbb{D})^+$. Statement (3) now follows.

Suppose that $S_0$ is any particular contractive multiplier satisfying (3.40) and $S$ is another such multiplier. Then $M_{\Psi_\gamma}M_{S-S_0} = M_{\Psi_\gamma}(M_S - M_{S_0}) = 0$, i.e., $	ext{Ran } M_S - S_0 \subset \ker M_{\Psi_\gamma} = M_\Theta H^2_\mu$. It follows that $S - S_0$ has a factorization $S - S_0 = \Theta H$ where $H$ is a multiplier from $H^2_\mu(\mathbb{D})$ to $H^2_\mu(\mathbb{D})^+$ such that $S_0 + \Theta H$ is still a contractive multiplier. As the converse direction is clear, this completes the proof of statement (4). 

PROPOSITION 3.20. A contractive multiplier $F(z)$ from $H^2_\mu(\mathbb{D})^+$ to $H^2_\mu,\omega(\mathbb{D})^+$ is coisometric if and only if it is of the form $F(z) = \Psi_\gamma(z)$ for some coisometric operator $W \in \mathcal{L}(\mathcal{U}, \ell^2_\nu(\mathbb{D})^+)$.

PROOF. By Proposition 3.2 $F(z)$ is a coisometric multiplier from $H^2_\mu(\mathbb{D})^+$ to $H^2_\mu,\omega(\mathbb{D})^+$ if and only if the associated kernel $K_F$ vanishes:

$$K_F(z, \zeta) := (k_{nc, \omega}(z, \zeta) \otimes I_y) - F(z)(k_{nc, \omega}(z, \zeta) \otimes I_U)F(\zeta)^* = 0,$$
or, equivalently by \((3.38)\),

\[
\Psi_\gamma(z) \left( k_{nc, Sa}(z, \zeta) \otimes I_{\ell^2_d(F^+_d)} \right) \Psi_\gamma(\zeta)^* = F(z) \left( k_{nc, Sa}(z, \zeta) \otimes I_{\ell^2_d(F^+_d)} \right) F(\zeta)^*. \tag{3.41}
\]

If \(F(z) = \Psi_\gamma(z)W\) with \(WW^* = I_{\ell^2_d(F^+_d)}\), use the fact that \(k_{nc, Sa}(z, \zeta)\) has scalar coefficients to get

\[
W \left( k_{nc, Sa}(z, \zeta) \otimes I_{\ell^2_d(F^+_d)} \right) W^* = \left( k_{nc, Sa}(z, \zeta) \otimes I_{\ell^2_d(F^+_d)} \right) WW^* = k_{nc, Sa}(z, \zeta) \otimes I_{\ell^2_d(F^+_d)}
\]
to arrive at \((3.41)\) and the sufficiency direction in Proposition 3.20 follows.

To verify the necessity direction, assume that \((3.41)\) holds. Use Proposition 3.17 to deduce that there is a partial isometry \(W \in \mathcal{L}(\mathbb{U}, \ell^2_d(F^+_d))\) such that

\[
F(z) = \Psi_\gamma(z)W \quad \text{and} \quad \Psi_\gamma(z) = F(z)W^*. \tag{3.42}
\]

It remains to show that \(W: \mathbb{U} \to \ell^2_d(F^+_d)\) is a coisometry. To this end, we compare the coefficients of \(z^\alpha\) in \((3.42)\) to get

\[
F_\alpha = \gamma_\alpha^{-\frac{1}{2}} W_\alpha, \quad F_\alpha W_\alpha^* = \gamma_\alpha^{-\frac{1}{2}} I_{\ell^2_d(F^+_d)}, \quad F_\alpha W_\beta^* = 0 ; (\alpha \neq \beta).
\]

Therefore,

\[
W_\alpha W_\alpha^* = \mu_{n-1,|\alpha|} F_\alpha W_\alpha^* = I_{\ell^2_d(F^+_d)}, \quad W_\alpha W_\beta^* = \mu_{n-1,|\alpha|} F_\alpha W_\beta^* = 0 \quad (\beta \neq \alpha),
\]

so that \(W = \{W_\alpha\}_{\alpha \in F_+^d}\) is indeed a coisometry from \(\mathbb{U}\) to \(\ell^2_d(F^+_d)\).

\[\square\]

**Proposition 3.21.** There are no isometric multipliers from \(H^2_{\omega, \mu}(F^+_d)\) to \(H^2_{\omega, \mu}(F^+_d)\) for any strictly decreasing admissible \(\omega\).

**Proof.** Let \(F(z) = \sum_{\alpha \in F_+^d} F_\alpha z^\alpha\) be an isometric multiplier from \(H^2_{\omega, \mu}(F^+_d)\) to \(H^2_{\omega, \mu}(F^+_d)\). Then in particular,

\[
\|Fu\|_{H^2_{\omega, \mu}(F^+_d)} = \sum_{\alpha \in F_+^d} \omega_{|\alpha|} \|F_\alpha u\|_{\ell^2_d(F^+_d)} = \|u\|_{\mathbb{U}}^2 \quad \text{for all} \quad u \in \mathbb{U}. \tag{3.43}
\]

On the other hand, for the polynomial \(p(z) = z_1 u\), we have \(\|p\|_{H^2_{\omega, \mu}} = \|u\|_{\mathbb{U}}\) and therefore,

\[
\|Fp\|_{H^2_{\omega, \mu}(F^+_d)} = \sum_{\alpha \in F_+^d} \omega_{|\alpha|+1} \|F_\alpha u\|_{\ell^2_d(F^+_d)}^2 = \|p\|_{H^2_{\omega, \mu}} = \|u\|_{\mathbb{U}}^2. \tag{3.44}
\]

We then conclude from \((3.43)\) and \((3.44)\) that

\[
0 = \sum_{\alpha \in F_+^d} (\omega_{|\alpha|} - \omega_{|\alpha|+1}) \|F_\alpha u\|_{\ell^2_d(F^+_d)}^2,
\]

which is possible (since by assumption \(\omega_{|\alpha|} > \omega_{|\alpha|+1}\) for all \(\alpha \in F_+^d\)) only if \(F_\alpha = 0\) for all \(\alpha \in F_+^d\). The latter contradicts \((3.43)\). \[\square\]

In the sequel we shall be interested in more general contractive multipliers \(M_\theta\) from \(H^2_{\omega, \mu}(\mathbb{U})\) to \(H^2_{\omega, \mu}(\mathbb{U})\); note that Propositions 3.20 and 3.21 are concerned with the case \(\omega' = 1\). The following characterization of such contractive multiplication operators \(M_\theta\) among all operators between two weighted Bergman spaces, well known for the classical Hardy-space case where \(d = 1\), \(\omega_k' = \omega_k = 1\) for all \(k\) (see e.g. \([92]\)), will be useful in the sequel. Note also that for the case of a general \(d\) with \(\omega_k' = \omega_k = 1\) for all \(k\), the result can be seen as the special case of Theorem 3.12 where \(\mathcal{M} = H^2_{\mu}(F^+_d)\) and \(\mathcal{N} = H^2_{\mu}(F^+_d)\).
Proposition 3.22. Suppose that $G$ is an operator from $H^2_{\Omega, \Lambda}(F_d^+)$ to $H^2_{\Omega, \Lambda}(F_d^+)$. Then $G$ has the form $G = M_\Theta$ for a contractive multiplier $\Theta$ between $H^2_{\Omega, \Lambda}(F_d^+)$ and $H^2_{\Omega, \Lambda}(F_d^+)$ if and only if $\|G\| \leq 1$ and $G$ satisfies the intertwining conditions:

$$GS_{\omega, R,j} = S_{\omega, R,j}G \quad \text{for } j = 1, \ldots, d.$$  \hfill (3.45)

Proof. If $G = M_\Theta$, then the intertwining conditions (3.45) hold since $M_\Theta$ is a left-multiplication operator while each $S_{\omega, R,j}$ is a right multiplication operator.

Conversely, suppose that the operator $G : H^2_{\Omega, \Lambda}(F_d^+) \to H^2_{\Omega, \Lambda}(F_d^+)$ satisfies the intertwining conditions (3.45). Define a power series $\Theta(z) \in \mathcal{L}(U, Y)(\langle z \rangle)$ by

$$\Theta(z)u = (Gu)(z)$$

for $u \in U$, where here we view $u = u \cdot z^0$ as an element of $H^2_{\Omega, \Lambda}(F_d^+)$. From the intertwining conditions (3.45), we deduce that

$$(Gp)(z) = \Theta(z) \cdot p(z) = (M_\Theta p)(z)$$

for any noncommutative polynomial $p \in U(z)$. Thus $G$ is a multiplication operator when restricted to the dense set of polynomials in $H^2_{\Omega, \Lambda}(F_d^+)$. By assumption $G$ is bounded; an approximation argument then gives us that $\Theta$ is a multiplier (contractive multiplier if $\|G\| \leq 1$ and necessarily $G = M_\Theta$ as an operator from $H^2_{\Omega, \Lambda}(F_d^+)$ to $H^2_{\Omega, \Lambda}(F_d^+)$).

\hfill \Box

3.5. $H^2_{\Omega, \Lambda}(F_d^+)$-Bergman-inner multipliers

Definition 3.23. We say that a formal power series $\Theta \in \mathcal{L}(U, Y)(\langle z \rangle)$ is a $H^2_{\Omega, \Lambda}(F_d^+)$-Bergman-inner multiplier if

\begin{itemize}
  \item[(1)] $\|\Theta u\|_{H^2_{\Omega, \Lambda}(F_d^+)} = \|u\|_U$ for all $u \in U$, and
  \item[(2)] $\Theta \cdot u \perp \Theta \cdot S^*_{\omega, R,j}v$ in $H^2_{\Omega, \Lambda}(F_d^+)$ for all $u, v \in U$ and all nonempty $\alpha \in F_d^+$.
\end{itemize}

The next result says in particular that $H^2_{\Omega, \Lambda}(F_d^+)$-Bergman-inner multipliers are automatically also contractive multipliers. For the case of single-variable weighted Bergman spaces, this result goes back to Richter-et-al?. More generally, this result has been obtained for the commutative setting by Bhattacharjee-Eschmeier-Keshari-Sarkar [29] Theorem 6.2 and for the general setting by Popescu [90] Theorem 4.2 by different proofs which do not lead to the estimate (3.46).

Theorem 3.24. Let $\Theta \in \mathcal{L}(U, Y)(\langle z \rangle)$ be such that

\begin{itemize}
  \item[(1)] $\|M_\Theta u\|_{H^2_{\Omega, \Lambda}(F_d^+)} \leq \|u\|_U$ for all $u \in U$, and
  \item[(2)] $M_\Theta u \perp M_\Theta S^*_{\omega, R,j}v$ for all $u, v \in U$ and all nonempty $\alpha \in F_d^+$.
\end{itemize}

Then $\Theta$ is a contractive multiplier from $H^2_{\Omega}(F_d^+)$ to $H^2_{\Omega, \Lambda}(F_d^+)$, as shown by the following general estimate:

$$\|M_\Theta f\|_{H^2_{\Omega, \Lambda}(F_d^+)}^2 \leq \|f\|_{H^2_{\Omega}(F_d^+)}^2 \hfill (3.46)$$

$$- \sum_{\alpha \in F_d^+} \sum_{j=1}^d \left\| (I - S^*_{\omega, R,j} S_{\omega, R,j})^{\frac{1}{2}} M_\Theta (S^*_{\omega, R,j})^{\frac{1}{2}} f \right\|_{H^2_{\Omega, \Lambda}(F_d^+)}^2$$

for all $f \in H^2_{\Omega}(F_d^+)$. Moreover, if $\|M_\Theta u\|_{H^2_{\Omega, \Lambda}(F_d^+)} = \|u\|_U$ for all $u \in U$ (so $\Theta$ is $H^2_{\Omega, \Lambda}(F_d^+)$-Bergman-inner), then (3.46) holds with equality for all $f \in H^2_{\Omega}(F_d^+)$. 

Proof. We first verify (3.46) for any $U$-valued “polynomial”

$$f(z) = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m} f_{\alpha} z^{\alpha}. \quad (3.47)$$

Let $S_{1,R}^* = (S_{1,R,1}^*, \ldots, S_{1,R,d}^*)$ be the right backward shift tuple on $H_d^U(\mathbb{F}_d^+)$ so that for the polynomial $f$ as in (3.47),

$$S_{1,R,j}^* : \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m} f_{\alpha} z^{\alpha} \mapsto \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| < m} f_{\alpha j} z^{\alpha} \quad (3.48)$$

for $j = 1, \ldots, d$. By the assumptions of the theorem, and since the ranges of $S_{\omega,R,i}$ and $S_{\omega,R,j}$ are orthogonal whenever $i \neq j$, we have

$$\|M_\Theta f\|^2 = \left\| \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m} S_{\omega,R}^{\alpha^T} M_\Theta f_{\alpha} \right\|^2 \quad (3.49)$$

$$= \|M_\Theta f_\emptyset\|^2 + \left\| \sum_{j=1}^d \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m-1} S_{\omega,R}^{\alpha^T} M_\Theta f_{\alpha j} \right\|^2$$

$$\leq \|f_\emptyset\|^2 + \left\| \sum_{j=1}^d \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m-1} S_{\omega,R}^{\alpha^T} M_\Theta f_{\alpha j} \right\|^2$$

$$= \|f_\emptyset\|^2 + \left\| S_{\omega,R,j} M_\Theta \left( \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq m-1} f_{\alpha j} z^{\alpha} \right) \right\|^2$$

$$= \|f_\emptyset\|^2 + \sum_{j=1}^d \left\| S_{\omega,R,j} M_\Theta (S_{1,R,j}^* f) \right\|^2$$

$$= \|f_\emptyset\|^2 + \sum_{j=1}^d \left\| M_\Theta (S_{1,R,j}^* f) \right\|^2 - \sum_{j=1}^d \left\| (I - S_{\omega,R,j} S_{\omega,R,j}) \frac{1}{2} M_\Theta (S_{1,R,j}^* f) \right\|^2. \quad (3.50)$$

Observe that for any $\alpha \in \mathbb{F}_d^+$ with $|\alpha| \leq m$,

$$S_{1,R}^{\alpha^T} f = \sum_{\beta \in \mathbb{F}_d^+ : |\beta| \leq m-|\alpha|} f_{\beta} z^{\beta} \quad (3.50)$$

which can be seen upon iterating formula (3.48). It follows from (3.50) that

$$(S_{1,R}^{\alpha^T} f)_\emptyset = f_{\alpha}. \quad (3.49)$$

Replacing $f$ by $S_{1,R}^{\alpha^T} f$ in (3.49) now gives

$$\|M_\Theta S_{1,R}^{\alpha^T} f\|^2 \leq \|f_\emptyset\|^2 + \sum_{j=1}^d \left\| M_\Theta (S_{1,R}^{\alpha j^T} f) \right\|^2$$

$$- \sum_{j=1}^d \left\| (I - S_{\omega,R,j} S_{\omega,R,j}) \frac{1}{2} M_\Theta (S_{1,R}^{\alpha j^T} f) \right\|^2. \quad (3.51)$$
for any $\alpha \in \mathbb{F}_d^+$. Iterating the inequality (3.49) using (3.51) then gives, for any $m' = 1, 2, \ldots$,

$$
\|M_\alpha f\|_{H_{\omega, Y}^2(\mathbb{F}_d^+)}^2 \leq \sum_{|\alpha| < m'} \|f_\alpha\|^2_U + \sum_{|\alpha| = m'} \left\|M_\alpha (S_1, R)^{\alpha^T} f\right\|^2_{H_{\omega, Y}^2(\mathbb{F}_d^+)} - \sum_{|\alpha| < m'} \sum_{j=1}^d \left\|(I - S_{\omega, R,j}^* S_{\omega, R,j})^{\frac{1}{2}} M_\alpha (S_1, R)^{j\alpha^T} f\right\|^2_{H_{\omega, Y}^2(\mathbb{F}_d^+)}.
$$

Since $f_\alpha = 0$ as well as $(S_1^*)^{\alpha^T} f = 0$ once $|\alpha| \geq m$, it follows that, once $m' \geq m$, this last inequality collapses to

$$
\|M_\alpha f\|^2 \leq \|f\|^2_{H_{\omega, Y}^2(\mathbb{F}_d^+)} - \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| < m'} \sum_{j=1}^d \left\|(I - S_{\omega, R,j}^* S_{\omega, R,j})^{\frac{1}{2}} M_\alpha (S_1, R)^{j\alpha^T} f\right\|^2_{H_{\omega, Y}^2(\mathbb{F}_d^+)} (3.52)
$$

Letting $m' \to \infty$ in (3.52) now implies the validity of (3.46) for every $U$-valued polynomial $f \in H_{\omega, Y}^2(\mathbb{F}_d^+)$. We then get the result for a general $f$ in $H_{\omega, Y}^2(\mathbb{F}_d^+)$ by approximating $f$ by finite truncations of its series representation $f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha$.

If equality holds in condition (1) of the theorem (i.e., if $\|M_\alpha u\|_{H_{\omega, Y}^2(\mathbb{F}_d^+)} = \|u\|_U$ holds for all $u \in U$), then we have equalities throughout (3.49), (3.51) and therefore, in (3.36) as well.

Part of the content of the next result is a converse of the previous result: a contractive multiplier from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$ which acts isometrically on the coefficient space $U$ is in fact a Bergman-inner multiplier.

**Theorem 3.25.** Let $\Theta \in \mathcal{L}(U, Y)\langle \langle z \rangle \rangle$ be a contractive multiplier from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$ which is isometric on constants:

$$
\|M_\Theta u\|_{H_{\omega, Y}^2(\mathbb{F}_d^+)} = \|u\|_U \quad \text{for all } u \in U.
$$

Then:

1. There is a unique contractive multiplier $S$ from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$ such that

$$
\Theta(z) = \Psi_{\gamma}(z) S(z) \quad (3.53)
$$

where $\Psi_{\gamma}(z)$ is defined as in (3.34). Moreover, this unique $S$ is a strictly inner multiplier from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$. (2) $\Theta u \perp \Theta S_\alpha v$ for all $u, v \in U$ and all nonempty $\alpha \in \mathbb{F}_d^+$, i.e., $\Theta$ is a Bergman-inner multiplier.

**Proof.** Since $\Theta$ is, in particular, a contractive multiplier from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$, it is (by Theorem 3.19 (3)) of the form (3.53) for some contractive multiplier $S(z)$ from $H_{\omega, Y}^2(\mathbb{F}_d^+)$ to $H_{\omega, Y}^2(\mathbb{F}_d^+)$. By assumption we have

$$
\|u\|_U = \|M_\Theta u\|_{H_{\omega, Y}^2(\mathbb{F}_d^+)} = \|M_\Psi S u\|_{H_{\omega, Y}^2(\mathbb{F}_d^+)} \quad (3.54)
$$

for all $u \in U$. Since the operator

$$
M_S : U \to H_{\omega, Y}^2(\mathbb{F}_d^+)
$$
is contractive while the operator
\[ M_{\Psi} : \mathcal{H}^2_{\ell_2^2}(\mathcal{F}^*_d) \to \mathcal{H}^2_{\omega,\psi}(\mathcal{F}^*_d) \]

is coisometric (by Theorem 3.14 (2)), equality (3.54) implies that
\[ \| M_{\Psi} u \|_{\mathcal{H}^2_{\ell_2^2}(\mathcal{F}^*_d)} = \| u \|_U \] and \( M_{\Psi} u \perp \ker M_{\Psi} \) for all \( u \in U \). (3.55)

The first relation in (3.55) implies that \( S \) is a strictly inner multiplier (by Lemma 3.13). Let us rewrite the factorization (3.53) in multiplication-operator form
\[ M_\Theta = M_{\Psi} \gamma M_S : \mathcal{H}^2_{\ell_2^2}(\mathcal{F}^*_d) \to \mathcal{H}^2_{\omega,\psi}(\mathcal{F}^*_d). \] (3.56)

As \( M_{\Psi} \gamma \) is a coisometry, the second relation in (3.55) implies that
\[ M_S = M_{\Psi} \gamma M_{\Psi} M_S. \] (3.57)

Then multiplication of relation (3.56) on the left by \( M_{\Psi} \gamma \) and making use of (3.57) leaves us with
\[ M_{\Psi} \gamma M_\Theta = M_{\Psi} \gamma M_{\Psi} M_S = M_S, \]
i.e., \( S \) is uniquely determined from \( \Theta \) via the relation
\[ S(z) u = (M_{\Psi} \gamma M_\Theta u)(z) \quad \text{for all} \quad u \in U. \]

This completes the proof of part (1) of the theorem.

Again making use of relations (3.56) and (3.57), we see that, for \( u, v \in U \) and \( \alpha \) any nonempty word in \( \mathcal{F}_d^* \),
\[ (M_\Theta u, M_\Theta(S_{1,R}^\alpha v)_{A\nu,\gamma}(\mathcal{F}_d^*)) = (M_{\Psi} \gamma M_S u, M_{\Psi} \gamma S_{1,R}^\alpha M_S v)_{A\nu,\gamma}(\mathcal{F}_d^*) = (M_{\Psi} \gamma M_{\Psi} \gamma M_S u, M_{\Psi} \gamma S_{1,R}^\alpha M_S v)_{H^2_{\ell_2^2}(\mathcal{F}_d^*)} (\mathcal{F}_d^*) = (M_S u, M_S S_{1,R}^\alpha v)_{H^2_{\ell_2^2}(\mathcal{F}_d^*)} = 0 \]

where we use the fact that \( S \) is strictly inner for the last step. This completes the proof of part (2) of the theorem.

**Remark 3.26.** Bergman-inner multipliers and McCT-inner multipliers are closely related in still other ways besides that revealed by Theorem 3.19. Indeed, given a shift-invariant subspaces \( \mathcal{M} \subset \mathcal{H}^2_{\omega,\psi}(\mathcal{F}_d^*) \) with Beurling-Lax representation \( \mathcal{M} = \Theta \cdot \mathcal{H}^2_{\ell_2^2}(\mathcal{F}_d^*) \) for a McCT-inner multiplier \( \Theta \), if we set \( \mathcal{F} := U \cap \text{Ran} \Theta^* \) and \( \Psi(z) = \Theta(z)|_{\mathcal{F}} \) and view \( \Psi \) as an element of \( \mathcal{L}(\mathcal{F},Y)(\langle z \rangle) \), then \( \Psi \) is Bergman inner and the wandering subspace \( \mathcal{E} = \mathcal{M} \ominus \left( \bigoplus_{j=1}^\infty S_{\omega,j} \mathcal{M} \right) \) for \( \mathcal{M} \) is given by \( \mathcal{E} = \Psi \cdot \mathcal{F} \) (see [29] Theorem 6.6] for the commutative setting and [90] Theorem 4.3] for the general setting).

**Remark 3.27.** If \( S \) is a strictly inner multiplier from \( \mathcal{H}^2_{\ell_2^2}(\mathcal{F}_d^*) \) to \( \mathcal{H}^2_{\ell_2^2}(\mathcal{F}_d^*) \), then \( \Theta \) of the form (3.53) is a contractive multiplier from \( \mathcal{H}^2_{\ell_2^2}(\mathcal{F}_d^*) \) to \( \mathcal{H}^2_{\omega,\psi}(\mathcal{F}_d^*) \) by Theorem 3.19. However, \( \Theta \) does not have to be a Bergman-inner multiplier. For example, let \( d = 1 \) and \( \omega = \mu_2 = \{ \frac{1}{j+1} \}_{j \geq 0} \). The single-variable function
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\[ S(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ 0 \\ \vdots \end{bmatrix} \] is \( H^2_{\ell^2} \)-inner and therefore, the function

\[ \Theta(z) = \Psi_1(z) S(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z & z^2 & \cdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} (z + z^2) \]

is a contractive multiplier from \( H^2 \) to \( A_2 \). However, \( \Theta \) is not \( A_2 \)-Bergman inner. One reason is that \( \|\Theta\|^2_{A_2} = \frac{1}{2} (\frac{1}{2} + \frac{1}{3}) \neq 1 \), and another reason is that \( M_{\Theta} \cdot 1 = \frac{1}{\sqrt{2}} (z + z^2) \) and \( M_{\Theta} \cdot z = \frac{1}{\sqrt{2}} (z^2 + z^3) \) are not orthogonal in \( A_2 \).

**Remark 3.28.** Due to Theorems 3.24 and 3.25, \( H^2_{Y,\bar{Y}}(\mathbb{F}^+_d) \)-Bergman-inner multipliers can be equivalently defined as noncommutative formal power series \( \Theta \in \mathcal{L}(U,Y)\langle\langle z \rangle\rangle \) (for some Hilbert space \( U \)) which are contractive multipliers from \( H^2_{Y}(\mathbb{F}^+_d) \) to \( H^2_{\omega Y, \bar{Y}}(\mathbb{F}^+_d) \) which are isometric on \( U \).
CHAPTER 4

Stein relations and observability-operator range spaces

In this chapter we flesh out Theme 1 from the Introduction for the $\omega$-setting. In the classical setting of the discrete-time linear system (1.1) (see e.g. [47]), the output stability of the pair $(C, A)$, that is, the boundedness of the observability operator $O_{C,A} : X \rightarrow H^2_Y$ (see (1.3)), can be expressed in terms of certain Stein inequality, the minimal positive-semidefinite solution of which turns out to be equal to the observability gramian

$$G_{C,A} = \sum_{n=0}^{\infty} A^* C A^*.$$

Precise statements are recorded next; here the notational conventions from Definition (3.7) are in place.

**Theorem 4.1.** An output pair $(C, A) \in \mathcal{L}(X, Y) \times \mathcal{L}(X)$ is output-stable (i.e., the observability operator $O_{C,A}$ is bounded as an operator from $X$ into $H^2_Y$) if and only if the Stein inequality

$$H - A^* HA \succeq C^* C$$

(4.1) has a positive semidefinite solution $H \in \mathcal{L}(X)$. In this case,

1. The observability gramian $H = G_{C,A}$ is the minimal positive semidefinite solution of (4.1) and satisfies the Stein equation

$$H - A^* HA = C^* C.$$  

(4.2)

2. $G_{C,A}$ is a unique solution to this equality if $A$ is strongly stable.

3. If $A$ is a contraction, then the Stein equation (4.2) has a unique positive semidefinite solution if and only if $A$ is strongly stable.

The range of the observability operator of an output-stable pair turns out to be a generic backward-shift invariant subspace of $H^2_Y$ that in turn serves as the functional-model space for a Hilbert space contraction operator and on the other hand is used to fairly explicitly construct the Beurling-Lax representer of a shift-invariant subspace of $H^2_Y$.

Observability operators (1.18) appearing in the context of the time-varying linear system (1.14) and multivariable observability operators (1.28) arising from a noncommutative Fornasini-Marchesini linear system (1.21) produce two quite different generalizations of the above observations and eventually lead to the functional-model theory of $n$-hypercontractions and $d$-row contractions, respectively. Both generalizations naturally merge into the framework of the linear system (1.32) which in turn leads to the functional-model theory for row $n$-hypercontractions and to Beurling-Lax type theorems in the setting of weighted Bergman-Fock spaces.
In this chapter, we consider stability questions and observability operator range spaces in the setting of the weighted Hardy-Fock space $H^2_{\omega,Y}(F^+_d)$ with an admissible weight sequence $\omega$; in addition we take care to spell out the special case $\omega = \mu_n$ (i.e., of the standard weighted Bergman-Fock space) whenever the results can be simplified or detailed to a larger extent.

4.1. Observability operators and gramians

Let $\omega = \{\omega_j\}_{j \geq 0}$ be an admissible weight sequence (in the sense of (2.31)). Let us make the following definition.

**Definition 4.2.** We say that the output pair $(C, A) \in \mathcal{L}(X,Y) \times \mathcal{L}(X)^d$ is $\omega$-output stable if the $\omega$-observability operator

$$O_{\omega,C,A} : x \mapsto \sum_{\alpha \in F^+_d} (\omega^{-1}_{|\alpha|} C A^\alpha x) z^\alpha \quad (4.3)$$

is bounded from $X$ to $H^2_{\omega,Y}(F^+_d)$ or equivalently, the $\omega$-observability gramian

$$G_{\omega,C,A} := O_{\omega,C,A}^* O_{\omega,C,A} = \sum_{\alpha \in F^+_d} \omega^{-1}_{|\alpha|} A^\alpha C^\alpha C^\alpha \quad (4.4)$$

is bounded on $X$.

We next recall the power series

$$R_{\omega}(\lambda) = R_{\omega,0}(\lambda) = \sum_{j=0}^{\infty} \omega^{-1}_j \lambda^j \quad \text{and} \quad R_{\omega,k}(\lambda) = \sum_{j=0}^{\infty} \omega^{-1}_{j+k} \lambda^j \quad (k \geq 1) \quad (4.5)$$

used in realization formulas (2.38) and (2.39) for $O_{\omega,C,A}$ and $\Theta_{\omega,U}$. Due to conditions (2.31), the series (4.5) converge in the open unit disk $D$ and do not vanish there. Hence, the reciprocal power series

$$\sum_{j=0}^{\infty} c_j \lambda^j := \frac{1}{R_{\omega}(\lambda)} = \left( \sum_{j=0}^{\infty} \omega^{-1}_j \lambda^j \right)^{-1} \quad (4.6)$$

converges on $D$; for reasons that will become clear shortly, we assume that the latter power series belongs to the Wiener class $W^+$, that is, the coefficients $\{c_j\}_{j \geq 0}$ appearing in (4.6) are absolutely summable:

if $c_0 = 1$ and recursively $c_k = -\sum_{j=0}^{k-1} c_j \omega^{-1}_{k-j}$, then $\sum_{j=0}^{\infty} |c_j| < \infty. \quad (4.7)$

Meanwhile we record a useful identity satisfied by the sequences $\{\omega_j\}$ and $\{c_j\}$.

**Lemma 4.3.** The sequence $\omega$ and the derived sequence $\{c_j\}_{j \geq 0}$ satisfy the equality

$$\sum_{r=0}^{j} \frac{1}{\omega_{j-r}} \sum_{\ell=1}^{k} \frac{c_{\ell+r}}{\omega_{k-\ell}} = -\frac{1}{\omega_{k+j}} \quad \text{for} \quad j, k \geq 0. \quad (4.8)$$

**Proof.** By the recursion (4.7), we have

$$0 = \sum_{\ell=0}^{k+r} \frac{c_\ell}{\omega_{k+r-\ell}} = \sum_{\ell=0}^{r} \frac{c_\ell}{\omega_{k+r-\ell}} + \sum_{\ell=1}^{k} \frac{c_{\ell+r}}{\omega_{k-\ell}} \quad (k + r \geq 1). \quad (4.9)$$
4.1. Observability Operators and Gramians

Substituting the latter equality into the left side of (4.8), changing the order of summation and then again using (4.9) (with \( r \) instead of \( k + r \)) gives

\[
\sum_{r=0}^{j} \left( \frac{1}{\omega_{j-r}} \sum_{\ell=1}^{k} \frac{c_{\ell+r}}{\omega_{k-\ell}} \right) = -\sum_{r=0}^{j} \left( \frac{1}{\omega_{j-r}} \sum_{\ell=0}^{r} \frac{c_{\ell}}{\omega_{k+r-\ell}} \right) = -\sum_{r=0}^{j} \left( \frac{1}{\omega_{k+j-r}} \sum_{\ell=0}^{r} \frac{c_{\ell}}{\omega_{r-\ell}} \right) = -\frac{1}{\omega_{k+j}} \cdot \frac{c_{0}}{\omega_{0}} = -\frac{1}{\omega_{k+j}}.
\]

and identity (4.8) follows.

We remark that the weights (1.8) meet the assumption (4.7) as in this case, \( R_{\mu_{n}}(\lambda) := R_{n}(\lambda) = (1 - \lambda)^{-n} \), and hence the power series (4.6) amounts to a polynomial. The assumption (4.7) is imposed in order to define the operatorial maps (4.15) and (4.16) below, which in turn, will allow us to introduce the notion of a \( \omega \)-hypercontraction that extends the notion of a row-hypercontraction. It is not clear at the moment how much the assumption (4.7) can be weakened to still produce meaningful extensions of a hypercontractive fashion.

As suggested by the Agler hereditary functional calculus (in Ambrozie-Enől-Müller transcription [9]) we introduce the operator

\[
B_{A} : L(X) \to L(X) \quad \text{via} \quad B_{A} : X \mapsto \sum_{j=1}^{d} A_{j}^{*} X A_{j}.
\] (4.10)

One easily sees from the definition that \( B_{A} \) is a positive map, i.e.,

\[
X \succeq 0 \Rightarrow B_{A}[X] \succeq 0.
\] (4.11)

In fact, \( B_{A} \) satisfies the stronger property of being a completely positive map (see e.g. [77]), but we shall not have need of this fact. Iterating (4.10) gives

\[
B_{A}^{k} : X \mapsto \sum_{\alpha \in \mathbb{F}_{d}^{+} ; |\alpha| = k} A^{*\alpha} X A_{\alpha} \quad \text{for} \quad k \geq 1.
\] (4.12)

The following general principle will be useful in the sequel.

**Lemma 4.4.** Given a function \( f(\lambda) = \sum_{j \geq 0} f_{j} \lambda^{j} \in W^{+} \), the operatorial map

\[
f(B_{A}) := \sum_{j=0}^{\infty} f_{j} B_{A}^{j} : X \mapsto \sum_{j=0}^{\infty} f_{j} \left( \sum_{|\alpha| = j} A^{*\alpha} X A_{\alpha} \right) = \sum_{\alpha \in \mathbb{F}_{d}^{+}} f_{|\alpha|} A^{*\alpha} X A_{\alpha}
\] (4.13)

is well defined for any \( X \in L(X) \) subject to inequalities

\[
X \succeq \sum_{j=1}^{d} A_{j}^{*} X A_{j} \succeq 0.
\] (4.14)

**Proof.** Indeed, iterating (4.13) yields that

\[
X \succeq \sum_{\alpha \in \mathbb{F}_{d}^{+} ; |\alpha| = j} A^{*\alpha} X A_{\alpha} \succeq 0 \quad \text{for all} \quad j \geq 0,
\]

which together with (4.13) implies \( \|f(B_{A})[X]\| \leq \sum_{j=0}^{\infty} |f_{j}| \|X\| = \|f\|_{w^{+}} \cdot \|X\|. \) \( \square \)
As a consequence of Lemma 4.4, the operatorial map

\[ \Gamma_{\omega,A} := \frac{1}{R_\omega}(B_A)[X] = \sum_{\alpha \in F_+^d} c_{|\alpha|} A^{\alpha^T} X A^\alpha \]  

(4.15)

is well defined for any \( X \in \mathcal{L}(X) \) subject to inequalities (4.14). Note that the assumption (2.31) implies that \( 1 \leq \omega_j^{-1} \leq \omega_{j+1}^{-1} \) forcing that \( R_\omega(\lambda) \) (see (4.5)) cannot be in the Wiener class \( W^+ \); nevertheless the assumption (4.7) guarantees that for each \( k \geq 0 \), the function

\[ \frac{R_{\omega,k}(\lambda)}{R_\omega}(\lambda) = \left( \sum_{j=0}^{\infty} \frac{\lambda^j}{\omega_{k+j}} \right) \cdot \left( \sum_{j=0}^{\infty} c_j \lambda^j \right) \]

does belong to \( W^+ \); furthermore one can make use of the recursion (4.7) to see that

\[ \left( \sum_{j=0}^{\infty} \frac{\lambda^j}{\omega_{k+j}} \right) \cdot \left( \sum_{j=0}^{\infty} c_j \lambda^j \right) = \sum_{j=0}^{\infty} c_{\omega,k}^j B_j A[j] \]

(4.16)

is well defined for any operator \( X \in \mathcal{L}(X) \) subject to inequalities (4.14).

The next result establishes connections between \( \omega \)-output stability, \( \omega \)-observability gramians, and solutions of associated Stein equations and inequalities. Here we refer to the relations in (4.18) and (4.19) as the Stein inequalities and the Stein equation respectively.

**Theorem 4.5.** Let us assume that the weight sequence \( \omega \) meets conditions (2.31), (4.7) and let \((C,A) \in \mathcal{L}(X,Y) \times \mathcal{L}(X))^d\) be an output pair. Then:

1. The pair \((C,A)\) is \( \omega \)-output-stable if and only if there is an \( H \in \mathcal{L}(X) \) subject to inequalities

\[ H \succeq \sum_{j=1}^{d} A_j^* H A_j \succeq 0, \quad (4.17) \]

and the Stein inequality

\[ \Gamma_{\omega,A}[H] \succeq C^* C. \]

(4.18)

2. If \((C,A)\) is a \( \omega \)-output-stable pair, then the gramian \( H = G_{\omega,C,A} \) is the minimal positive semidefinite solution of the system (4.17), (4.18) and it also satisfies the Stein equation

\[ \Gamma_{\omega,A}[H] = C^* C. \]

(4.19)
converges strongly to the operator $H$ which implies, since $X$ the series (4.15), (4.16) (with $\omega \times 93$ to Mertens (see [127, 654] if the series where the last step follows from the recursion in (4.7).

Applying this General Principle due to Mertens (see Theorem 3.50) for the case of numerically-valued sequences),

If the series $\sum a_n$ converges absolutely and if $\sum a_n = a$ and $\sum b_n = b$, then

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) = \left( \sum_{n=0}^{\infty} a_n \right) \cdot \left( \sum_{n=0}^{\infty} b_n \right) = a \cdot b.$$  \hfill (4.21)

Applying this General Principle with the choice $a_n = c_n B_A^n \in L(L(x))$, $b_n = \omega_n B_A^n [C^* C] \in L(x)$ then leads to

$$\Gamma_{\omega, A} [G_{\omega, C, A}] = \left( \sum_{n=0}^{\infty} \omega^{-1}_n B_A^n \right) \left( \sum_{n=0}^{\infty} \omega^{-1}_n B_A^n \right) = \left( \sum_{n=0}^{\infty} \omega^{-1}_n B_A^n \right) \left( \sum_{n=0}^{\infty} \omega^{-1}_n B_A^n \right) = C^* C$$  \hfill (4.22)

where the last step follows from the recursion in (4.17).

Similarly, identity (4.3) together with (4.4) and (4.14) leads us to

$$\Gamma^{(k)}_{\omega, A} [G_{\omega, C, A}] = \left( \sum_{n=0}^{\infty} B_A^n \right) \left( \sum_{n=0}^{\infty} B_A^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_{\omega^{-1}_n B_A^n} \right) \left( \sum_{k=0}^{n} c_{\omega^{-1}_n B_A^n} \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_{\omega^{-1}_n B_A^n} \right) \left( \sum_{k=0}^{n} c_{\omega^{-1}_n B_A^n} \right) = C^* C$$  \hfill (4.23)
We now see from (4.20), (4.22) and the last inequality that $H = G_{\omega,C,A}$ satisfies relations (4.17), (4.19) and hence, also (4.18).

Conversely, suppose that (4.17), (4.18) hold for some $H \in \mathcal{L}(\mathcal{X})$. From definitions (4.13), it follows that for every $j \geq 0$,

$$R_{\omega,j}(\lambda) = \lambda R_{\omega,j+1}(\lambda) + \omega_j^{-1}. \quad (4.24)$$

Dividing both sides by $R_{\omega}(\lambda)$ and applying the $B_A$ functional calculus to the resulting identity and to the chosen operator $H$ (this can be done thanks to Lemma 4.4) gives the operator equality

$$\Gamma^{(j)}_{\omega,A}[H] = B_A \Gamma^{(j+1)}_{\omega,A}[H] + \omega^{-1}_j \Gamma_{\omega,A}[H] = \sum_{k=1}^{d} A_k^* \Gamma^{(j+1)}_{\omega,A}[H] A_k + \omega_j^{-1} \Gamma_{\omega,A}[H]$$

where $\Gamma^{(j)}_{\omega,A}[H]$ is simply $H$ for the case $j = 0$. Application of $(B_A)^j$ to both sides of this last identity then gives us

$$B_A^{j} \Gamma^{(j)}_{\omega,A}[H] - B_A^{j+1} \Gamma^{(j+1)}_{\omega,A}[H] = \omega_j^{-1} B_A^{j} \Gamma_{\omega,A}[H]$$

or, more explicitly,

$$\sum_{\alpha \in F_j^+:|\alpha| = j} A^{*\alpha} \Gamma^{(j)}_{\omega,A}[H] A^\alpha - \sum_{\alpha \in F_j^+:|\alpha| = j+1} A^{*\alpha} \Gamma^{(j+1)}_{\omega,A}[H] A^\alpha$$

$$= \omega_j^{-1} \sum_{\alpha \in F_j^+:|\alpha| = j} A^{*\alpha} \Gamma_{\omega,A}[H] A^\alpha \geq 0. \quad (4.25)$$

Define a sequence of operators $\{\Omega_k\}$ ($k = 0, 1, 2, \ldots$) by

$$\Omega_k = \sum_{\alpha \in F_j^+:|\alpha| = k} A^{*\alpha} \Gamma^{(k)}_{\omega,A}[H] A^\alpha.$$

From (4.25) and the fact that $\Gamma_{\omega,A}[H] \geq 0$ by hypothesis (4.18), we see that $\Omega_k$ is a decreasing operator sequence. As a consequence of hypothesis (4.17), we also have $\Omega_k \geq 0$ for all $k$. Therefore, there exists the strong limit

$$\Delta_{A,H} = \lim_{k \to \infty} \Omega_k. \quad (4.26)$$

Summing up equalities in (4.25) for $j = 0, \ldots, k - 1$ and taking into account that $\Gamma^{(0)}_{\omega,A}[H] = H$, we get

$$\sum_{\alpha \in F_j^+:|\alpha| < k} \omega^{-1}_j A^{*\alpha} \Gamma_{\omega,A}[H] A^\alpha = H - \sum_{\alpha \in F_j^+:|\alpha| = k} A^{*\alpha} \Gamma^{(k)}_{\omega,A}[H] A^\alpha$$

$$= H - \Omega_k. \quad (4.27)$$

Combining (4.20) with (4.18) gives

$$\sum_{\alpha \in F_j^+:|\alpha| \leq k} \omega^{-1}_j A^{*\alpha} C^* C A^\alpha \leq \sum_{\alpha \in F_j^+:|\alpha| < k} \omega^{-1}_j A^{*\alpha} \Gamma_{\omega,A}[H] A^\alpha = H - \Omega_k \quad (4.28)$$

for all $k \geq 1$. By letting $k \to \infty$ in (4.28) we conclude that the sum on the left-hand side converges to a bounded positive-semidefinite operator, which is $G_{\omega,C,A}$ by (4.3). Thus, passing to the limit in (4.28) as $k \to \infty$ gives

$$G_{\omega,C,A} \leq H - \Delta_{A,H} \leq H, \quad (4.29)$$
where $\Delta_{A,H} \geq 0$ is the limit defined in (4.20). Therefore, the pair $(C,A)$ is $\omega$-output-stable and $G_{\omega,C,A}$ is indeed the minimal positive-semidefinite solution to the system (4.17), (4.18).

REMARK 4.6. Let us consider Theorem 4.5 for the classical Hardy-space case where $d = 1$ and $\omega_j = 1$ for all $j \geq 0$. Then $\omega$-output stability amounts to classical output stability for the output pair $(C,A)$. Furthermore we have

$$\Gamma_{\omega,A}[H] = H - A^*HA \quad \text{and} \quad \Gamma_{\omega,A}^{(k)}[H] = I_{X} \quad \text{for all } k \geq 1.$$ Therefore, relations (4.18) and (4.19) amount to (4.1) and (4.2), respectively, while the only non-redundant inequality in the system of equations in the second part of (4.17) is $H \succeq 0$. Therefore, Theorem 4.5 can be viewed as a canonical extension of Theorem 4.1 to the $\omega$-setting, apart from the uniqueness statements (2) and (3) in Theorem 4.1. Observe that Theorem 4.5 does not address these uniqueness issues as the latter require some suitable notion of strong stability. See Remark 4.17 below for further discussion on this point.

By Lemma 4.4, the operatorial maps $\Gamma_{\omega,A}^{(k)}$ and $\Gamma_{\omega,A}$ are well defined on operators $X \in \mathcal{L}(X)$ subject to inequalities (4.14). In particular (by choosing $X = I_X$), the operators $\Gamma_{\omega,A}[I_X]$ and $\Gamma_{\omega,A}^{(k)}[I_X]$ are well defined if the tuple $A$ is contractive in the sense that

$$A_1^*A_1 + \cdots + A_d^* A_d \preceq I_X, \quad (4.30)$$

that is, $A^* = (A_1^*, \ldots, A_d^*)$ is a row-contraction.

DEFINITION 4.7. The tuple $A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d$ is called $\omega$-contractive if it is contractive (i.e., (4.30) holds) and

$$\Gamma_{\omega,A}[I_X] = \sum_{\alpha \in \mathbb{F}^d_+} c_{|\alpha|} A^{*\alpha^\top} A^\alpha \succeq 0.$$ The tuple $A$ is called $\omega$-hypercontractive if it is $\omega$-contractive and

$$\Gamma_{\omega,A}^{(k)}[I_X] := - \sum_{\alpha \in \mathbb{F}^d_+} \left( \sum_{\ell=1}^{k} \frac{c_{|\alpha|+\ell}}{\omega|k-\ell|} \right) A^{*\alpha^\top} A^\alpha \succeq 0 \quad \text{for all } k \geq 1.$$ 

DEFINITION 4.8. An $\omega$-hypercontractive operator tuple $A \in \mathcal{L}(X)^d$ will be called $\omega$-strongly stable if the limit (4.20) with $H = I_X$ equals zero, i.e.,

$$\Delta_{A,I_X} = \lim_{k \to \infty} \sum_{\alpha \in \mathbb{F}^d_+} A^{*\alpha^\top} \Gamma_{\omega,A}^{(k)}[I_X] A^\alpha = 0,$$

or, equivalently (see (4.16)),

$$\lim_{k \to \infty} \sum_{\alpha \in \mathbb{F}^d_+} \left( \sum_{\ell=1}^{k} \frac{c_{|\alpha|-k+\ell}}{\omega|k-\ell|} \right) \| A^\alpha x \|^2 = 0 \quad \text{for all } x \in X.$$ 

As we will see in Remark 4.38 below, $\omega$-strong stability of an $\omega$-hypercontraction implies its strong stability in the usual sense (1.30).

Taking advantage of relations (4.17)–(4.19) characterizing $\omega$-output stability, we introduce the notions of weighted contractive and isometric pairs.
DEFINITION 4.9. A pair \((C, A)\) (with \(C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\) and \(A \in \mathcal{X}^d\)) will be called \(\omega\)-contractive output pair if inequalities (4.17), (4.18) hold with \(H = I_X\), i.e., if \(A\) is \(\omega\)-hypercontractive and
\[
\Gamma_{\omega,A}[I_X] := \sum_{\alpha \in \mathbb{F}_d^+} c_{|\alpha|} A^{\alpha^\top} A^\alpha \succeq C^* C.
\]

The pair \((C, A)\) will be called \(\omega\)-isometric if \(A\) is \(\omega\)-hypercontractive and
\[
\Gamma_{\omega,A}[I_X] = \sum_{\alpha \in \mathbb{F}_d^+} c_{|\alpha|} A^{\alpha^\top} A^\alpha = C^* C. \quad (4.31)
\]

It is clear from (4.3) that the gramian \(G_{\omega,C,A}\) is positive-definite if and only if the pair \((C, A)\) is observable in the sense of (4.11). We will say that the pair \((C, A)\) is exactly \(\omega\)-observable if the \(\omega\)-gramian \(G_{\omega,C,A}\) is bounded and is strictly positive-definite.

LEMMA 4.10. (1) Suppose \((C, A) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d\) is a \(\omega\)-contractive pair. Then \((C, A)\) is \(\omega\)-output stable and \(G_{\omega,C,A} \leq I\) and hence \(G_{\omega,C,A} : \mathcal{X} \to H^2_{\omega,C}(\mathbb{F}_d^+)\) is a contraction.

(2) Suppose \((C, A)\) is a \(\omega\)-isometric pair. Then \(G_{\omega,C,A} = I_X\) if and only if \(A\) is \(\omega\)-strongly stable. In particular, if \((C, A)\) is \(\omega\)-isometric and \(A\) is \(\omega\)-strongly stable, then \(G_{\omega,C,A} : \mathcal{X} \to H^2_{\omega,C}(\mathbb{F}_d^+)\) is isometric, and hence also the pair \((C, A)\) is exactly \(\omega\)-observable.

PROOF. Suppose that \((C, A)\) is an \(\omega\)-contractive pair. Then by Definition 4.9 \(H = I_X\) satisfies (4.17) and (4.18). Hence by part (1) of Theorem 4.5 we conclude that \((C, A)\) is \(\omega\)-output stable, and the inequality (4.29) specified for the case \(H = I_X\) gives us also that \(G_{\omega,C,A}\) is contractive. This completes the proof of (1).

We now suppose that \((C, A)\) is \(\omega\)-isometric and \(A\) is \(\omega\)-strongly stable. Specifying (4.26) to the present case where \(H = I_X\), we conclude that the limit
\[
\Delta_{A,I} = \lim_{k \to \infty} \sum_{\alpha \in \mathbb{F}_d^+} A^{\alpha^\top} \Gamma_{\omega,A}^{(k)} [I_X] A^\alpha
\]
exists. Setting \(H = I_X\) in (4.27) and letting \(k \to \infty\) we see that the series below converges in the strong operator topology and satisfies
\[
\sum_{\alpha \in \mathbb{F}_d^+} \omega^{-1}_{|\alpha|} A^{\alpha^\top} \Gamma_{\omega,A}[I_X] A^\alpha = I_X - \Delta_{A,I}. \quad (4.32)
\]

Combining (4.32) and (4.31) now gives
\[
\sum_{\alpha \in \mathbb{F}_d^+} \omega^{-1}_{|\alpha|} A^{\alpha^\top} C^* C A^\alpha = I - \Delta_{A,I}.
\]
Since \(\Delta_{A,I} = 0\) and since the series on the left converges to \(G_{\omega,C,A}\), we conclude \(G_{\omega,C,A} = I_X\) from which it follows by definitions that \(G_{\omega,C,A}\) is isometric and \((C, A)\) is exactly \(\omega\)-observable.

Conversely, suppose that \((C, A)\) is an \(\omega\)-isometric pair such that \(G_{\omega,C,A} = I_X\). Then we still arrive at the identity (4.32), where the series on the left is just the definition of \(G_{\omega,C,A}\). The assumption that \(G_{\omega,C,A} = I_X\) then forces \(\Delta_{A,I} = 0\), i.e., that \(A\) is \(\omega\)-strongly stable. This completes the proof. \(\square\)
**Definition 4.11.** Let us say that the pair \((C, A)\) is similar to the pair \((\tilde{C}, \tilde{A})\) if there is an invertible operator \(T : X \to \tilde{X}\) so that
\[
\tilde{C} = CT^{-1}, \quad \tilde{A}_j = TA_jT^{-1} \quad \text{for } j = 1, \ldots, d.
\]
Then we have the following characterization of pairs \((C, A)\) which are similar to an \(\omega\)-contractive or to an \(\omega\)-isometric pair.

**Proposition 4.12.** The pair \((C, A)\) is similar to an \(\omega\)-contractive (to an \(\omega\)-isometric) pair \((\tilde{C}, \tilde{A})\) if and only if there exists a bounded, strictly positive-definite and satisfies inequalities (4.17), (4.18). Then we have the following characterization of pairs \((C, A)\) which are similar to an \(\omega\)-contractive or to an \(\omega\)-isometric pair.

**Proof.** Suppose that \(H > 0\) is a solution to the system (4.17), (4.18). Factor \(H\) as \(H = T^*T\) with \(T\) invertible and define \(\tilde{C}\) and \(\tilde{A}_j\) by formulas (4.33). Then the pair \((\tilde{C}, \tilde{A})\) is similar to \((C, A)\) and it is readily seen from (4.15), (4.16) that
\[
T^*\Gamma_{\omega, \tilde{A}}[I_X]T = \Gamma_{\omega, \tilde{A}}[I_X] \quad \text{and} \quad T^*\Gamma_{\omega, \tilde{A}}^{(k)}[I_X]T = \Gamma_{\omega, \tilde{A}}^{(k)}[I_X] \quad \text{for } k \geq 1.
\]
Thus, inequalities (4.17), (4.18) can be equivalently written in terms of \(\tilde{C}\) and \(\tilde{A}\) as
\[
I \geq \tilde{A}_1^*\tilde{A}_1 + \ldots + \tilde{A}_d^*\tilde{A}_d, \quad \Gamma_{\omega, \tilde{A}}^{(k)}[I_X] \geq 0 \quad (k \geq 1), \quad \Gamma_{\omega, \tilde{A}}[I_X] \geq \tilde{C}^*\tilde{C},
\]
and hence, the pair \((\tilde{C}, \tilde{A})\) is \(\omega\)-contractive.

Conversely, if \((\tilde{C}, \tilde{A})\) given by \((4.33)\) is \(\omega\)-contractive, then \(H = T^*T\) is bounded and strictly positive-definite and satisfies inequalities (4.17), (4.18). The statement concerning \(\omega\)-isometric pairs follows in a similar way. \(\square\)

**4.1.1. \(\mu_n\)-hypercontractions.** We now examine how the above general results specialize to the Bergman weights \(\mu_n\). Within this setting, we will often write \(n\) instead of \(\omega = \mu_n\). Since \(R_{\mu_n}(\lambda) = R_n(\lambda) = (1 - \lambda)^{-n}\), the formula (4.15) amounts to
\[
\Gamma_{n, A} := \Gamma_{\mu_n, A} : X \mapsto (I - B_A)^n[X] = \left( \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} B_A^\ell \right)[X]
\]

or more explicitly, on account of (4.12), to
\[
\Gamma_{n, A} : X \mapsto \sum_{\alpha \in \mathcal{P}_n^+ : |\alpha| \leq n} (-1)^{|\alpha|} \binom{n}{|\alpha|} A_\alpha A^\alpha X A^\alpha,
\]

and makes sense for all \(n \geq 0\). Upon applying the identity
\[(I - B_A)^k = (I - B_A)^{k-1} - B_A(I - B_A)^{k-1}\]
to an operator \(H \in \mathcal{L}(X)\) and making use of definition (4.34) we get
\[
\Gamma_{k, A}[H] = \Gamma_{k-1, A}[H] - \sum_{j=1}^{d} A_j^* \Gamma_{k-1, A}[H] A_j
\]
for all \(k \geq 1\). Dividing both sides of (1.12) by \(R_N\) and applying the \(B_A\) calculus to the resulting identity we specialize \((4.15)\) to the case \(\omega = \mu_n\):
\[
\Gamma_{n, A}^{(k)} = \frac{R_{n,k}}{R_n}(B_A) = \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} R_{n-k+1} R_n (B_A) = \sum_{\ell=1}^{n} \binom{\ell+k-2}{\ell-1} (I - B_A)^{\ell-1}.
\]
Making use of formula (4.34) for all \( \ell = 0, \ldots, n - 1 \), we conclude:

\[
\Gamma^{(k)}_{n,A} = \sum_{\ell=0}^{n-1} (\ell+k-1) \Gamma_{\ell,A}.
\]

(4.37)

Before turning to \( \mu_n \)-contractions and \( \mu_n \)-hypercontractions (obtained via specializing Definition 4.7 to the case \( \omega = \mu_n \)) let us recall a more elementary notion of contractivity and hypercontractivity indexed by a discrete parameter \( n = 1, 2, \ldots \) defined as follows.

**Definition 4.13.** The tuple \( A = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{H})^d \) is called \( n \)-contractive if \( \Gamma_{n,A}[I_H] \geq 0 \), and it is called \( n \)-hypercontractive if \( \Gamma_{k,A}[I_H] \geq 0 \) for all \( 1 \leq k \leq n \).

It was shown in [2] (see also [71] as well as [72] for a commutative multivariable version) that inequalities \( \Gamma_{1,A}[I_H] \geq 0 \) and \( \Gamma_{n,A}[I_H] \geq 0 \) imply that \( A \) is an \( n \)-hypercontraction. This result extends to our free noncommutative setting, even with \( I_H \) replaced by an arbitrary \( H \geq 0 \), as follows.

**Lemma 4.14.** Let us assume that the operators \( H \) and \( A_1, \ldots, A_d \) in \( \mathcal{L}(\mathcal{H}) \) are such that

\[
H \geq \sum_{j=1}^d A_j^* H A_j \geq 0 \quad \text{and} \quad \Gamma_{n,A}[H] \geq 0
\]

for some integer \( n \geq 3 \). Then

\[
\Gamma_{k,A}[H] \geq 0 \quad \text{for all} \quad k = 1, \ldots, n - 1.
\]

(4.39)

**Proof.** Observe that the leftmost inequalities in (4.38) mean that \( H \) and \( \Gamma_{1,A}[H] \) are both positive-semidefinite. In light of the positivity property (4.11), applying the map \( B_A \) to both sides of the first inequality in (4.38) and subsequent iterating lead us to

\[
H \geq \sum_{\alpha \in F_d^+ : |\alpha| = j} A^{\alpha \top} H A^\alpha \quad \text{for all} \quad j \geq 0.
\]

(4.40)

Let us introduce the Hermitian operators

\[
S_{m,k} := \sum_{\alpha \in F_d^+ : |\alpha| = k} A^{\alpha \top} \Gamma_{m,A}[H] A^\alpha = B^k_A (I - B_A)^m[H]
\]

(4.41)

for \( k \in \mathbb{Z}_+ \) and \( m = 0, 1, \ldots, n \) (observe that the second equality in (4.41) follows from (1.4) and (4.3)). Let us show that

\[
-2^m \cdot H \leq S_{m,k} \leq 2^m \cdot H \quad \text{for all} \quad k \in \mathbb{Z}_+ \quad \text{and} \quad m = 0, 1, \ldots, n.
\]

(4.42)

Indeed, since \( H \) is positive semidefinite, combining (4.35) and (4.40) gives

\[
S_{m,k} \leq \sum_{\alpha \in F_d^+ : |\alpha| = k} A^{\alpha \top} \left( \sum_{\beta \in F_d^+ : |\beta| \leq m} (-1)^{|\beta|} \left( \frac{m}{|\beta|} \right) A^{\beta \top} H A^{\beta} \right) A^\alpha
\]

\[
= \sum_{\alpha \in F_d^+ : |\alpha| \leq m + k} (-1)^{|\alpha| - k} \left( \frac{m}{|\alpha| - k} \right) A^{\alpha \top} H A^\alpha
\]

\[
= \sum_{j=k}^{m+k} (-1)^{j-k} \left( \frac{m}{j-k} \right) \sum_{\alpha \in F_d^+ : |\alpha| = k} A^{\alpha \top} H A^\alpha \leq \sum_{j=k}^{m+k} \left( \frac{m}{j-k} \right) \cdot H = 2^m \cdot H
\]

(4.43)
thus proving the right inequality in (4.42). The left inequality follows in much the same way. We next observe that due to (4.36) and the second inequality in (4.38),

\[ \Gamma_{n-1,\mathbf{A}}[H] \geq \sum_{j=1}^{d} A_j^* \Gamma_{n-1,\mathbf{A}}[H] A_j = B_{\mathbf{A}}[\Gamma_{n-1,\mathbf{A}}[H]]. \]

Therefore, on account of (4.11), we have

\[ S_{n-1,j} = B_{\mathbf{A}}^j[\Gamma_{n-1,\mathbf{A}}[H]] \geq B_{\mathbf{A}}^{j+1}[\Gamma_{n-1,\mathbf{A}}[H]] = S_{n-1,j+1}. \tag{4.43} \]

Also, it follows from definitions (4.41) and (4.34) that for any \( N \geq 1, \)

\[ \sum_{j=0}^{N} S_{n-1,j} = \sum_{j=0}^{N} B_{\mathbf{A}}^j(I - B_{\mathbf{A}})^{n-1}[H] \]
\[ = (I - B_{\mathbf{A}}^{N+1})(I - B_{\mathbf{A}})^{n-2}[H] \]
\[ = \Gamma_{n-2,\mathbf{A}}[H] - \sum_{\alpha \in \mathbb{F}^+_j: |\alpha| = N+1} \mathbf{A}^\alpha \Gamma_{n-2,\mathbf{A}}[H] \mathbf{A}^\alpha \]
\[ = S_{n-2,0} - S_{n-2,N+1}. \]

Taking the inner product of both parts in the latter equality against \( x \in \mathcal{X} \) and then making use of (4.42) gives

\[ \left| \sum_{j=0}^{N} \langle S_{n-1,j} x, x \rangle \right| = \left| \langle S_{n-2,0} x x \rangle - \langle S_{n-2,N+1} x, x \rangle \right| \leq 2^{n-1} \langle H x, x \rangle. \tag{4.44} \]

Due to (4.39), on the left-hand side of (4.44) we have the partial sum of a non-increasing sequence and, since the partial sums are uniformly bounded (by \( 2^{n-1} \)), it follows that all the terms in the sequence are nonnegative. In particular,

\[ \langle S_{n-1,0} x, x \rangle = \langle \Gamma_{n-1,\mathbf{A}}[H] x, x \rangle \geq 0 \quad \text{for every } x \in \mathcal{X}, \]

and hence, \( \Gamma_{n-1,\mathbf{A}}[H] \geq 0. \) We then obtain recursively all the desired inequalities in (4.39).

**Proposition 4.15.** For Bergman weights \( \mathbf{\omega} = \mu_n = \{ \mu_{n,j} \}_{j \geq 0} \), the classes of \( \mu_n \)-hypercontractions, of \( \mu_n \)-contractions, and of \( n \)-hypercontractions are identical.

**Proof.** The statement follows from three implications below.

1. **If a tuple \( \mathbf{A} \in \mathcal{L}(\mathcal{X})^d \) is \( \mu_n \)-contractive, then it is \( n \)-hypercontractive.** Indeed, if \( \mathbf{A} \) is \( \mu_n \)-contractive, then (see (4.30))

\[ \Gamma_{1,\mathbf{A}}[I_{\mathcal{X}}] = I_{\mathcal{X}} - A_1^* A_1 - \ldots - A_d^* A_d \geq 0 \]

and \( \Gamma_{n,\mathbf{A}}[I_{\mathcal{X}}] := \mu_{n,\mathbf{A}}[I_{\mathcal{X}}] \geq 0. \) Then it follows by Lemma 4.1.13 that \( \Gamma_{\ell,\mathbf{A}}[I_{\mathcal{X}}] \geq 0 \) for \( \ell = 0, \ldots, n-1 \), so that \( \mathbf{A} \) is \( n \)-hypercontractive.

2. **If a tuple \( \mathbf{A} \in \mathcal{L}(\mathcal{X})^d \) is \( n \)-hypercontractive, then it is \( \mu_n \)-hypercontractive.** Indeed, if \( \Gamma_{\ell,\mathbf{A}}[I_{\mathcal{X}}] \geq 0 \) for \( \ell = 0, \ldots, n-1 \), then upon applying the formula (4.37) to the identity operator we see that

\[ \Gamma_{\mu_n,\mathbf{A}}^{(k)}[I_{\mathcal{X}}] = \sum_{\ell=0}^{n-1} \frac{\ell+k-1}{\ell} \Gamma_{\ell,\mathbf{A}}[I_{\mathcal{X}}] \geq 0. \]
for all \( k \geq 1 \). The latter inequalities along with \( \Gamma_{1,A}[I \chi] \geq 0 \) and \( \Gamma_{n,A}[I \chi] \geq 0 \) imply that \( A \) in \( \mu_n \)-hypercontractive. Since any \( \mu_n \)-hypercontraction is clearly \( \mu_n \)-contractive, the statement follows. \( \Box \)

4.1.2. \( \mu_n \)-output-stability. A crucial feature of the output-stability indexed by integers, as opposed to the general \( \omega \)-output stability, is that \( n \)-output-stability implies the \( k \)-output-stability for any \( 0 \leq k < n \). We shall say that an output pair \((C,A)\) is \( n \)-output-stable if it is the case that the series \( \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|}^{-1} A^{\alpha^\top} C^* C A^\alpha \) is convergent in the strong (or equivalently in the weak) operator topology of \( \mathcal{L}(\chi) \), in which case we define the \( n \)-observability gramian \( G_{n,C,A} \) as the sum of the series:

\[
G_{n,C,A} := G_{\mu_n,C,A} = \sum_{\alpha \in \mathbb{F}_d^+} \mu_{n,|\alpha|}^{-1} A^{\alpha^\top} C^* C A^\alpha =: (I - B_A)^{-n}[C^* C].
\] (4.45)

For the case where \( n = 0 \) there are no convergence issues and we define

\[
G_{0,C,A} = C^* C.
\] (4.46)

**Proposition 4.16.** If the output pair \((C,A)\) is \( n \)-output stable, then it is also \( k \)-output stable for \( k = 1, \ldots, n - 1 \) and

\[
G_{k,C,A} = \sum_{j=1}^d A_j^* G_{k,C,A} A_j = G_{k-1,C,A},
\] (4.47)

\[
\Gamma_{k,A}[G_{n,C,A}] = G_{n-k,C,A} \text{ for } k = 0, \ldots, n.
\] (4.48)

Moreover we then have the chain of inequalities

\[
C^* C \preceq G_{1,C,A} \preceq \cdots \preceq G_{n,C,A}.
\] (4.49)

**Proof.** Note that \( \Gamma_{k,A} = (I - B_A)^k = \sum_{j=0}^k \binom{k}{j} B_A^j \) is a polynomial in \( B_A \) (and hence absolutely convergent) while by the stability assumption

\[
G_{k,C,A} = (I - B_A)^{-k}[C^* C] = \sum_{j=0}^{\infty} \mu_{n,j+k}^{-1} B_A^j [C^* C]
\]

is convergent. Therefore by the general principle \((1221)\) applied with \( a_j \in \mathcal{L}(\mathcal{L}(\chi)) \) and \( b_j \in \mathcal{L}(\chi) \) given by

\[
a_j = \begin{cases} (-1)^j \binom{k}{j} B_A^j & \text{for } 0 \leq j \leq k, \\ 0 & \text{for } j \geq k, \end{cases} \quad b_j = \mu_{n,j+k}^{-1} B_A^j [C^* C] \text{ for } j \geq 0,
\]

it follows (the general principle justifies the second equality in the computation below) that

\[
\Gamma_{k,A}[G_{n,C,A}] = (I - B_A)^k[(I - B_A)^{-n}[C^* C]] = (I - B_A)^{-n+k}[C^* C] = G_{n-k,C,A}
\]

which verifies \((4.43)\). By making use of \((4.45)\) and \((4.10)\), a similar application of the general principle \((1221)\) can be used to justify the calculation

\[
G_{k,C,A} - \sum_{j=1}^d A_j^* G_{k,C,A} A_j = (I - B_A)[G_{k,C,A}] = (I - B_A)[(I - B_A)^{-k}[C^* C]]
\]

\[
= (I - B_A)^{-(k-1)}[C^* C] = G_{k-1,C,A},
\]
for \( k = 1, 2, \ldots, n \), which verifies (4.47). As a consequence of (4.47) and the identity (4.46), we then get the chain of inequalities (4.49) which in turn, implies that \( n \)-output stability for \((C, A)\) implies \( k \)-output stability for \((C, A)\) for \( 0 \leq k < n \). □

**Remark 4.17.** As we pointed out in Remark 4.6, Theorem 4.5 is a partial extension of Theorem 4.1 to the general \( \omega \)-setting. Although Definition 4.8 provides a useful notion of \( \omega \)-strong stability in some contexts (see its role in the proof of statement (2) of Lemma 4.10), we do not know how to make use of this notion to prove the uniqueness of a solution to the system (4.17), (4.19). It is still quite remarkable that Theorem 4.1 admits a full-extent generalization to the standard-weight multivariable Bergman-Fock setting—as given in the next result.

**Theorem 4.18.** Given \( C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) and \( A = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{X})^d \), the pair \((C, A)\) is \( n \)-output-stable (see (4.45)) if and only if there exists an operator \( H \in \mathcal{L}(\mathcal{X}) \) satisfying the system of inequalities

\[
H \succeq \sum_{j=1}^d A_j^* A_j \succeq 0 \quad \text{and} \quad \Gamma_{n, A}[H] \succeq C^* C. \tag{4.50}
\]

If this is the case (i.e., if \((C, A)\) is \( n \)-output-stable), then:

1. The gramian \( H = G_{n, C, A} \) is the minimal positive semidefinite solution of the system (4.50) and actually solves (4.50) in the stronger form

\[
H \succeq \sum_{j=1}^d A_j^* A_j \succeq 0 \quad \text{and} \quad \Gamma_{n, A}[H] = C^* C. \tag{4.51}
\]

2. If \( A \) is strongly stable (see (1.30)), then \( H = G_{n, C, A} \) is the unique solution of the system (4.51).

3. If \( A \) is contractive (see (4.30)), the solution of the Stein equation in (4.51) is unique if and only if \( A \) is strongly stable.

**Proof.** If \( H \) satisfies (4.50), then \( \Gamma_{\ell, A}[H] \succeq 0 \) for \( \ell = 1, \ldots, n - 1 \), by Lemma 4.13. Applying identity (4.37) to the operator \( H \) then implies

\[
\Gamma^{(k)}_{\mu_n, A}[H] = \sum_{\ell=0}^{n-1} {\ell+k-1 \choose \ell} \Gamma_{\ell, A}[H] \succeq 0 \quad \text{for all} \quad k \geq 1.
\]

Therefore, the inequalities \( \Gamma^{(k)}_{\mu_n, A}[H] \succeq 0 \) in (4.17) are redundant when \( \omega = \mu_n \). Hence, all statements in Theorem 4.18 except parts (2) and (3) follow by specializing Theorem 4.15 to the case \( \omega = \mu_n \).

To prove part (2), suppose that \( A \) is strongly stable and that \( H \) solves the system (4.50). It is known from [20, Theorem 2.2] that if \( A = (A_1, \ldots, A_d) \) is strongly stable and \( Q \in \mathcal{L}(\mathcal{X}) \) is positive semidefinite, then, for \( P \in \mathcal{L}(\mathcal{X}) \),

\[
P - \sum_{j=1}^d A_j^* P A_j = Q \iff P = \sum_{\alpha \in \mathbb{F}_d^+} A^{* \alpha^\top} QA^\alpha
\]

In terms of the operator \( B_A : H \mapsto \sum_{j=1}^d A_j^* H A_j \) on \( \mathcal{L}(\mathcal{X}) \), we can rephrase the latter statement as saying that the operator \( I_{\mathcal{L}(\mathcal{X})} - B_A \) is invertible with the inverse
given by
\[
(I_{\mathcal{L}(\mathcal{X})} - B_{\mathbf{A}})^{-1} : Q \mapsto \sum_{\alpha \in \mathbb{F}_+^+} A^{*\alpha^T} Q A^\alpha.
\]

If \( I_{\mathcal{L}(\mathcal{X})} - B_{\mathbf{A}} \) is invertible, so also is \( (I_{\mathcal{L}(\mathcal{X})} - B_{\mathbf{A}})^n \) with inverse given by
\[
((I_{\mathcal{L}(\mathcal{X})} - B_{\mathbf{A}})^n)^{-1} = ((I_{\mathcal{L}(\mathcal{X})} - B_{\mathbf{A}})^{-1})^n.
\]

Indeed that formula for \( (I - B_{\mathbf{A}})^{-n} \) works out to be
\[
(I - B_{\mathbf{A}})^{-n}[Q] = \sum_{\alpha \in \mathbb{F}_+^+} \mu_{n,|\alpha|}^{-1} A^{*\alpha^T} Q A^\alpha,
\]
i.e., the series (4.45) converges with \( C^*C \) replaced by any \( Q \in \mathcal{L}(\mathcal{X}) \), not just for a special \( Q = C^*C \). Next note that the Stein equation in (4.51) can be viewed as the equation
\[
(I - B_{\mathbf{A}})^n[H] = C^*C.
\]
Hence \( H = (I - B_{\mathbf{A}})^{-n}[C^*C] \) necessarily is the unique solution. As \( \mathcal{G}_{n,C,\mathbf{A}} \) is given by the same formula (4.45) we conclude that \( \mathcal{G}_{n,C,\mathbf{A}} \) is the unique solution.

To complete the proof of part (3) of the theorem it remains to show that if \( \mathbf{A} \) is a contraction and the system (4.51) admits a unique solution (which necessarily is \( H = \mathcal{G}_{n,C,\mathbf{A}} \)), then the tuple \( \mathbf{A} \) is strongly stable. We prove the contrapositive: if \( \mathbf{A} \) is not strongly stable, then the solution of (4.51) is not unique.

Due to assumption (4.30), the sequence of operators
\[
\Delta_N = B_{\mathbf{A}}^N[I_N] = \sum_{\alpha \in \mathbb{F}_+^+ : |\alpha| = N} A^{*\alpha^T} A^\alpha, \quad N = 1,2,\ldots
\]
is decreasing and therefore has a strong limit \( \Delta = \lim_{N \to \infty} \Delta_N \geq 0 \). Since \( \mathbf{A} \) is assumed not to be strongly stable, this limit \( \Delta \) is not zero. Observe that
\[
\sum_{j=0}^{k} \left( \sum_{\alpha \in \mathbb{F}_+^+ : |\alpha| = j} (-1)^j \binom{k}{j} A^{*\alpha^T} \Delta_{N+k-j} A^\alpha \right) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \Delta_{N+k}
\]
for all \( N \geq 0 \) and \( k \in \{1,\ldots,n\} \). Taking the limit in the last equality as \( N \to \infty \) for a fixed \( k \) we get zero on the right side (as \( \sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0 \), by the binomial theorem), while the left side expression tends to
\[
\sum_{j=0}^{k} \left( \sum_{\alpha \in \mathbb{F}_+^+ : |\alpha| = j} (-1)^j \binom{k}{j} A^{*\alpha^T} \Delta A^\alpha \right) = \sum_{\alpha \in \mathbb{F}_+^+ : |\alpha| \leq k} (-1)^j \binom{k}{|\alpha|} A^{*\alpha^T} \Delta A^\alpha = \Gamma_{k,\mathbf{A}}[\Delta],
\]
by (4.35). Thus, \( \Gamma_{k,\mathbf{A}}[\Delta] = 0 \) for \( k = 1,\ldots,n \) and therefore, the operator \( H = \mathcal{G}_{n,C,\mathbf{A}} + \Delta \) (as well as \( \mathcal{G}_{n,C,\mathbf{A}} \)) satisfies the system (4.51) which therefore has more than one positive-semidefinite solution. \( \square \)

In Lemma (4.10) we saw that a \( \omega \)-isometric pair \((C,\mathbf{A})\) with \( \omega \)-strongly stable tuple \( \mathbf{A} \) is necessarily exactly \( \omega \)-observable. In the case of \( \omega = \mu_n \) this statement is immediate: if \((C,\mathbf{A})\) is \( n \)-isometric, then relations (4.51) hold for \( H = I_X \) (by the definition of an \( n \)-isometric pair) as well as for \( H = \mathcal{G}_{n,C,\mathbf{A}} \) by part (1) in Theorem (4.18). By the uniqueness assertion in part (2) of the same theorem, \( \mathcal{G}_{n,C,\mathbf{A}} = I_X \) and the pair \((C,\mathbf{A})\) is exactly \( n \)-observable. We do not know whether, conversely, the
exact observable of a \( \omega \)-isometric pair \((C, A)\) implies the \( \omega \)-strong stability of \( A \) in the general case. However, for \( \omega = \mu_n \) it does. We have the following (even stronger) result.

**Proposition 4.19.** Suppose that the pair \((C, A)\) is \(n\)-output-stable and exactly \(n\)-observable. Then \( A \) is strongly stable.

**Proof.** We first verify that for any operator \( H \in \mathcal{L}(\mathcal{X}) \) and any integers \( n \geq 1 \) and \( N \geq 0 \),

\[
H = \sum_{\alpha \in \mathbb{F}^+_n : |\alpha| \leq N} \left( \frac{n+|\alpha|-1}{|\alpha|} \right) A^* \Gamma_{n,A}[H] A^\alpha \\
+ \sum_{\alpha \in \mathbb{F}^+_n : N+1 \leq |\alpha| \leq N+n} \left( \frac{N+n}{|\alpha|} \right) A^* \Gamma_{n+|\alpha|-N,A}[H] A^\alpha.
\]  
(4.52)

To this end, we use the identity

\[
\sum_{j=0}^N \left( \frac{n+j-1}{j} \right) z^j = \frac{1}{(1-z)^n} - \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) \frac{z^{N+j}}{(1-z)^j}
\]

giving the explicit formula for the truncation of the infinite series representation for \((1-z)^{-n}\) (see [14] Section 2) for the proof). Multiplying both parts in the latter identity by \((1 - z)^n\) we get

\[
1 = \sum_{j=0}^N \left( \frac{n+j-1}{j} \right) z^j (1-z)^n + \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) z^{N+j} (1-z)^{n-j},
\]

which in turn implies the operator identity

\[
I_{\mathcal{L}(\mathcal{X})} = \sum_{j=0}^N \left( \frac{n+j-1}{j} \right) B^*_A (I - B_A)^n + \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) B^*_A (I - B_A)^{n-j}.
\]  
(4.53)

Upon applying this latter identity to an operator \( H \in \mathcal{L}(\mathcal{X}) \) and making use of (4.12) and of definition (4.35) we get (4.52):

\[
H = \sum_{j=0}^N \left( \frac{n+j-1}{j} \right) B^*_A \Gamma_{n,A}[H] + \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) B^*_A \Gamma_{n-j,A}[H]
\]

\[
= \sum_{j=0}^N \left( \frac{n+j-1}{j} \right) \left( \sum_{\alpha \in \mathbb{F}^+_n : |\alpha| = j} A^* \Gamma_{n,A}[H] A^\alpha \right)
\\
+ \sum_{j=1}^n \left( \frac{N+n}{N+j} \right) \left( \sum_{\alpha \in \mathbb{F}^+_n : |\alpha| = N+j} A^* \Gamma_{n-j,A}[H] A^\alpha \right)
\]

\[
= \sum_{\alpha \in \mathbb{F}^+_n : |\alpha| \leq N} \left( \frac{n+|\alpha|-1}{|\alpha|} \right) A^* \Gamma_{n,A}[H] A^\alpha
\\
+ \sum_{\alpha \in \mathbb{F}^+_n : N+1 \leq |\alpha| \leq N+n} \left( \frac{N+n}{|\alpha|} \right) A^* \Gamma_{n+N-|\alpha|,A}[H] A^\alpha.
\]

Since the pair \((C, A)\) is \(n\)-output-stable, the \(k\)-observability gramians \( \mathcal{G}_{k,C,A} \) are bounded operators for all \( 0 \leq k \leq n \), by Proposition 4.10. Plugging \( H = \mathcal{G}_{n,C,A} \)?
into (4.52) and making use of (4.48) (we note that \( \Gamma_{n,A}[\mathcal{G}_{n,C,A}] = \mathcal{G}_{0,C,A} = C^*C \), by (4.46)) we get

\[
\mathcal{G}_{n,C,A} = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq N} \left( \frac{n+|\alpha|-1}{|\alpha|} \right) A^{\alpha^\top} C^*C A^\alpha \\
+ \sum_{\alpha \in \mathbb{F}_d^+ : N+1 \leq |\alpha| \leq N+n} \left( \frac{N+n}{|\alpha|} \right) A^{\alpha^\top} \Gamma_{n+N-|\alpha|,A}[\mathcal{G}_{n,C,A}] A^\alpha \\
= \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq N} \left( \frac{n+|\alpha|-1}{|\alpha|} \right) A^{\alpha^\top} C^*C A^\alpha \\
+ \sum_{k=1}^n \left( \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+k} \left( \frac{N+n}{N+k} \right) A^{\alpha^\top} \mathcal{G}_{k,C,A} A^\alpha \right).
\]

(4.54)

From the representation (4.45) for \( \mathcal{G}_{n,C,A} \), taking limits as \( N \to \infty \) in (4.54) gives

\[
\lim_{N \to \infty} \sum_{k=1}^n \left( \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+k} \left( \frac{N+n}{N+k} \right) A^{\alpha^\top} \mathcal{G}_{k,C,A} A^\alpha \right) = 0
\]

which is equivalent, since all the terms on the left are positive semidefinite, to the system of equalities

\[
\lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+k} \left( \frac{N+n}{N+k} \right) A^{\alpha^\top} \mathcal{G}_{k,C,A} A^\alpha = 0 \quad \text{for} \quad k = 1, \ldots, n.
\]

Letting \( k = n \) gives

\[
\lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+n} A^{\alpha^\top} \mathcal{G}_{n,C,A} A^\alpha = 0 \quad \text{for} \quad k = 1, \ldots, n.
\]

(4.55)

The strict positive-definiteness of \( \mathcal{G}_{n,C,A} \) tells us that there is an \( \varepsilon > 0 \) so that

\[
\varepsilon \|x\|^2 \leq \langle \mathcal{G}_{n,C,A} x, x \rangle \quad \text{for all} \quad x \in \mathcal{X}.
\]

(4.56)

In particular, from (4.56) with \( A^\alpha x \) in place of \( x \) combined with (4.55) we get

\[
\varepsilon \cdot \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+n} \|A^\alpha x\|^2 \leq \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N+n} \langle \mathcal{G}_{n,C,A} A^\alpha x, A^\alpha x \rangle \to 0 \quad \text{as} \quad N \to \infty
\]

for all \( x \in \mathcal{X} \), and we conclude that \( A \) is strongly stable as asserted. \( \square \)

As we will see Section 4.5.2 below (specifically Remark 4.42), for the weight sequence \( \omega = \mu_n = \{ \mu_{n,j} \}_{j \geq 0} \), strong stability of a \( \mu_n \)-hypercontraction \( A \) is equivalent to the usual strong stability of \( A \) (see (1.30)).

### 4.2. Shifted \( \omega \)-gramians

We now formally introduce the shifted versions of the observability operator \( O_{\omega,C,A} \) (4.13) and observability gramian \( \mathcal{G}_{\omega,C,A} \) (4.14) which will be needed in the sequel. Observe that the operators \( O_{\omega,C,A} \) and \( \mathcal{G}_{\omega,C,A} \) can be expressed as

\[
O_{\omega,C,A} : \mathcal{X} \to CR_\omega(Z(z)A)x \quad \text{and} \quad \mathcal{G}_{\omega,C,A} = R_\omega(B_A)[C^*C]
\]

where \( R_\omega \) is the power series (1.25) associated with a given admissible weight \( \omega \), where \( B_A \) is the operator on \( L(\mathcal{X}) \) given by (4.10) and where \( Z(z) \) and \( A \) are defined
as in (1.24). One gets the shifted variants of these objects by simply replacing the
function \( R_\omega \) in the two formulas above by its shifts \( R_{\omega,k} \) (see (1.5)):

\[
\mathcal{O}_{\omega,k,C,A} : x \to CR_{\omega,k}(Z(z)A)x = \sum_{\alpha \in \mathbb{F}_d^+} \left( \omega_{|\alpha|}^{-1} + k \right) C \mathbf{A}^\alpha x, \quad (4.57)
\]

\[
\mathcal{G}_{\omega,k,C,A} := R_{\omega,k}(B_A)[C^* C] = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|}^{-1} A^\alpha C^* \mathbf{A}^\alpha \quad (4.58)
\]

for \( k \geq 1 \). Letting \( k = 0 \) in (4.57), (4.58) we conclude

\[
\mathcal{O}_{\omega,0,C,A} = \mathcal{O}_{\omega,C,A} \quad \text{and} \quad \mathcal{G}_{\omega,0,C,A} = \mathcal{G}_{\omega,C,A}.
\]

Note that as a consequence of identity (4.23) we have the identity

\[
\Gamma_{\omega,A}^{(k)}[\mathcal{G}_{\omega,C,A}] = \mathcal{G}_{\omega,k,C,A}. \quad (4.59)
\]

**Remark 4.20.** It is instructive to note the following elementary verification of
the identity (4.59) for the case where \( \omega = \mu_n \) for some \( n = 1, 2, 3, \ldots \). In this case,
by making use of the identity (4.37) along with its more fundamental companion (1.12)
and the definitions of the various quantities involved, we have

\[
\Gamma_{\omega,A}^{(k)}[\mathcal{G}_{n,C,A}] = \sum_{\ell=0}^{n-1} \left( \ell + k - 1 \right) (I - B_A)^\ell (I - B_A)^{-n} [C^* C]
\]

\[
= \sum_{\ell=0}^{n-1} (I + k - 1) (I - B_A)^{-n-1} [C^* C]
\]

\[
= \sum_{\ell=0}^{n-1} (I + k - 1) R_{n-\ell}(B_A)[C^* C]
\]

\[
= R_{n,k}(B_A)[C^* C] = \mathcal{G}_{n,k,C,A},
\]

and (4.59) follows for the case \( \omega = \mu_n \).

**Proposition 4.21.** If the weight sequence \( \omega \) is admissible and the pair \((C, A)\)
is \( \omega \)-output-stable, then the operator \( \mathcal{G}_{\omega,k,C,A} \) is bounded for all \( k \geq 1 \), and the weighted Stein identity

\[
\sum_{j=1}^{d} A_j^* \mathcal{G}_{\omega,k+1,C,A} A_j + \omega_k^{-1} C^* C = \mathcal{G}_{\omega,k,C,A} \quad (4.60)
\]

holds for all \( k \geq 0 \). Furthermore,

\[
\mathcal{G}_{\omega,k+1,C,A} \geq \mathcal{G}_{\omega,k,C,A} \geq \mathcal{G}_{\omega,0,C,A} = \mathcal{G}_{\omega,C,A} \quad \text{for all} \quad k \geq 1. \quad (4.61)
\]

**Proof.** Since \( \frac{\omega_j}{\omega_{j+1}} \leq M \) for all \( j \geq 0 \) (see (2.31)), we have \( \frac{\omega_j}{\omega_{j+k}} \leq M^k \) for all \( k, j \geq 0 \), and then it follows from (4.58) and (1.3) that

\[
\mathcal{G}_{\omega,k,C,A} = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|}^{-1} A^\alpha C^* \mathbf{A}^\alpha \leq M^k \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|}^{-1} A^\alpha C^* \mathbf{A}^\alpha = M^k \cdot \mathcal{G}_{\omega,C,A}.
\]
Since the pair \((C, A)\) is \(\omega\)-output-stable, the \(\omega\)-gramian \(G_{\omega, C, A}\) is bounded and hence, \(G_{\omega, k, C, A}\) is bounded as well.

We next apply the \(B_A\) calculus to the identity (4.24) to get the operator equality
\[
R_{\omega, k}(B_A) - B_A \circ R_{\omega, k+1}(B_A) = \omega^{-1}_k \cdot I_{L(x)}
\]
which in turn being applied to the operator \(C^*C\) gives, on account of (4.58),
\[
(R_{\omega, k}(B_A) - B_A \circ R_{\omega, k+1}(B_A)) \cdot [C^*C] = G_{\omega, k, C, A} - \sum_{j=1}^d A_j^* G_{\omega, k+1, C, A} A_j
= \omega^{-1}_k \cdot C^*C,
\]
and we arrive at (4.60) as wanted. Inequalities (4.61) follow from power series representation (4.58) due to the fact that the sequence \(\omega\) is non-increasing. □

For our future purposes, it is convenient to identify conditions which guarantee that all the shifted graminians are invertible, i.e., bounded below.

**Proposition 4.22.** Suppose that the \(\omega\)-output stable pair \((C, A)\) is furthermore exactly \(\omega\)-observable (as is the case for example if \((C, A)\) is \(\omega\)-isometric and \(A\) is \(\omega\)-strongly stable by part (2) of Lemma 4.10). Then the shifted graminians \(G_{\omega, k, C, A}\) are all bounded below, and hence invertible.

**Proof.** Exact \(\omega\)-observability of \((C, A)\) by definition means that \(G_{\omega, C, A}\) is bounded below. The result now follows from the chain of inequalities (4.61). □

### 4.3. The model shift-operator tuple on \(H^2_{\omega, \gamma}(F^+_d)\)

In this section we pursue a deeper study of the model operator tuples
\[
S_{\omega, R} = (S_{\omega, R, 1}, \ldots, S_{\omega, R, d}) \quad \text{and} \quad S^*_{\omega, R} = (S^*_{\omega, R, 1}, \ldots, S^*_{\omega, R, d})
\]
on the Hardy-Fock space \(H^2_{\omega, \gamma}(F^+_d)\) introduced in Section 2.2 by the formulas
\[
S_{\omega, R, j}: f(z) \mapsto f(z) z^j, \quad S^*_p, \omega, R, j: \sum_{\omega \in F^+_d} f_\omega z^\omega \mapsto \sum_{\omega \in F^+_d} \frac{\omega_\omega + 1}{\omega_\omega} f_\omega z^\omega. \quad (4.62)
\]
We define \(E: H^2_{\omega, \gamma}(F^+_d) \to \gamma\) to be the free-coefficient evaluation operator and we say that the pair
\[
(E, S^*_{\omega, R}) \quad \text{with} \quad S^*_{\omega, R} = (S^*_{\omega, R, 1}, \ldots, S^*_{\omega, R, d}), \quad E: \sum_{\omega \in F^+_d} f_\omega z^\omega \mapsto f_\emptyset \quad (4.63)
\]
is the \(\omega\)-model output pair.

**Proposition 4.23.** Let \(\omega\) be an admissible weight sequence (in the sense of (2.31)) and let \((E, S^*_{\omega, R})\) be the \(\omega\)-model output defined as in (4.63).

1. \(S^*_{\omega, R}\) is strongly stable in the sense that
\[
\lim_{N \to \infty} \sum_{|\omega| = N} \|S^*_{\omega, R, f}\|^2 = 0 \quad \text{for each} \quad f \in H^2_{\omega, \gamma}(F^+_d).
\]
2. The \(\omega\)-observability operator \(O_{\omega, E, S^*_{\omega, R}}\) equals \(I_{H^2_{\omega, \gamma}(F^+_d)}\) and hence the pair \((E, S^*_{\omega, R})\) is \(\omega\)-output stable and exactly \(\omega\)-observable.
and hence

We now conclude from this last identity combined with the observation (4.69) that

\[
\sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N} \| S^*_{\omega,R} f \|^2_{H^2_{\omega,Y}(\mathbb{F}_d^+)} \leq \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N} \left( \sum_{\beta \in \mathbb{F}_d^+ : |\beta| = N} \omega_{|\beta|+N} \| f_{\beta \alpha} \|^2_{\mathbb{F}_d^+} \right) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty,
\]
which proves the strong stability of $S_{\omega,R}^*$. 

**Proof of (2).** To verify that the $\omega$-observability operator $\mathcal{O}_{\omega,E} S_{\omega,R}$ is the identity operator on $H^2_{\omega,Y}(\mathbb{F}_d^+)$, note that by the formulas [4.70] and [4.63],
\[
E(S_{\omega,R})^\alpha f = \omega|_{\alpha}| f_\alpha
\]
and therefore,
\[
\mathcal{O}_{\omega,E} S_{\omega,R}^\alpha f = \sum_{\alpha \in \mathbb{F}_d^+} (\omega|_{\alpha})^{-1} E S_{\omega,R}^\alpha f z^\alpha = \sum_{\alpha \in \text{free}} \omega|_{\alpha}| f_\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha = f
\]
for all $f \in H^2_{\omega,Y}(\mathbb{F}_d^+)$, i.e., $\mathcal{O}_{\omega,E} S_{\omega,R}^* = I_{H^2_{\omega,Y}(\mathbb{F}_d^+)}$ as asserted. 

**Proof of (3).** To verify (4.64), we first combine (4.57) and (4.71) to compute
\[
\mathcal{O}_{\omega,E} S_{\omega,R}^* f = \sum_{\alpha \in \mathbb{F}_d^+} \omega|_{\alpha}| (E S_{\omega,R}^\alpha f) z^\alpha = \sum_{\alpha \in \text{free}} \omega|_{\alpha}| f_\alpha z^\alpha.
\]
Since $\omega_0 = 1$, the adjoint operator $E^* : \mathcal{Y} \to H^2_{\omega,Y}(\mathbb{F}_d^+)$ (see [4.63]) amounts to the inclusion of $\mathcal{Y}$ into $H^2_{\omega,Y}(\mathbb{F}_d^+)$ which along with [4.71] leads us to
\[
S_{\omega,R}^\alpha E^* E S_{\omega,R}^\alpha f = S_{\omega,R}^{\alpha^T} E^*(\omega|_{\alpha}| f_\alpha) = \omega|_{\alpha}| f_\alpha z^\alpha.
\]
Combining the latter equality with [4.58] we get
\[
\mathcal{O}_{\omega,E} S_{\omega,R}^\alpha E^* E S_{\omega,R}^\alpha f = \sum_{\alpha \in \mathbb{F}_d^+} \omega|_{\alpha}| f_\alpha z^\alpha,
\]
which completes the verification of [4.64]. By iterating formulas [4.62] we have
\[
S_{\omega,R}^{\alpha^T} S_{\omega,R}^\alpha : f_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega|_{\alpha}| + \beta}{\omega|_{\alpha}|} f_\alpha z^\alpha
\]
for any $\beta \in \mathbb{F}_d^+$ which being combined with [4.64] (with $k = |\beta|$) leads us to equalities [4.65]. We note that specialization of the equalities [4.65] to the case where $\beta = \emptyset$ amounts to the content of part (2) of the proposition.

**Proof of (4).** Combining [4.62] and [4.64] (with $k = 1$) gives
\[
S_{\omega,R,j} \mathcal{O}_{\omega,1,E} S_{\omega,R}^* S_{\omega,R,j}^* f(z) = \sum_{\beta \in \mathbb{F}_d^+} f_\beta z^\beta \mapsto \sum_{\beta \in \mathbb{F}_d^+} f_\beta^j z^\beta^j
\]
for $j = 1, \ldots, d$. Therefore, we have for any $f \in H^2_{\omega,Y}(\mathbb{F}_d^+)$,
\[
\left( \sum_{j=1}^d S_{\omega,R,j} \mathcal{O}_{\omega,1,E} S_{\omega,R}^* S_{\omega,R,j}^* \right) f = \sum_{j=1}^d \sum_{\beta \in \mathbb{F}_d^+} f_\beta^j z^\beta^j = \sum_{\beta \in \mathbb{F}_d^+: \beta \neq \emptyset} f_\beta^j z^\beta = f(z) - f_\emptyset
\]
which verifies the identity [4.66].
Theorem 4.23. Let us assume that the weight sequence $\omega$ meets conditions (4.7). Then the operator tuple $S_{\omega,R}$ on $H^{2}_{\omega,Y}(\mathbb{F}^+_d)$ is a $\omega$-strongly stable $\omega$-hypercontraction, while the $\omega$-model output pair $(E,S_{\omega,R}^*)$ is $\omega$-isometric. Moreover, for all $k \geq 1$, or equivalently,

$$\sum_{\alpha \in \mathbb{F}^+_d} c_{1|\alpha|} \| S_{\omega,R}^* f \|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2 = \| f_0 \|_{\mathbb{F}^+_d}^2,$$

for all $f \in H^2_{\omega,Y}(\mathbb{F}^+_d)$, where $c_j$'s are given in (4.7).

Proof. By part (2) in Proposition 4.23, $G_{\omega,E,S_{\omega,R}} = I$ and hence, equalities (4.7) follow from general formulas (4.22) and (4.39). Since $EE^* \geq 0$ and $G_{\omega,k,E,S_{\omega,R}} \geq 0$. We conclude from (4.70) that $S_{\omega,R}$ is $\omega$-hypercontractive. Next, we note that the first equality in (4.70) is equivalent to the quadratic-form identity $$(\Gamma_{\omega,S_{\omega,R}^*}^k | I_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^k | f, f \rangle_{H^2_{\omega,Y}(\mathbb{F}^+_d)} = (E^* Ef, f \rangle_{H^2_{\omega,Y}(\mathbb{F}^+_d)}$$ for all $f \in H^2_{\omega,Y}(\mathbb{Y})$, which in turn is equivalent to (4.77). The equivalence of the second identity in (4.76) with the quadratic-form identity (4.78) follows from the equality $$(\Gamma_{\omega,S_{\omega,R}^*}^k | I_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^k | f, f \rangle_{H^2_{\omega,Y}(\mathbb{F}^+_d)} = - \sum_{\alpha \in \mathbb{F}^+_d} \left( \sum_{\ell=1}^{k} \frac{c_{1|\alpha|+|\ell|}}{\omega_{k-\ell}} \right) \| S_{\omega,R}^* f \|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2,$$

for all $f \in H^2_{\omega,Y}(\mathbb{F}^+_d)$.
which is immediate from the definitions, and the identity
\begin{equation}
\langle \mathbf{G}_{\omega,k,E} \mathbf{S}_{\omega,R} f, f \rangle_{H_{\omega,Y}^2(\mathbb{F}_d^+)} = \sum_{\alpha \in \beta} \frac{\omega_{|\alpha|}^2}{\omega_{k+|\alpha|}} \|f_\alpha\|^2_{Y}\end{equation}
which follows from (4.72) and the definition of the inner product in \(H_{\omega,Y}^2(\mathbb{F}_d^+)\). Combining (4.79) and (4.70) gives
\begin{equation}
\sum_{\alpha \in \beta} \langle \mathbf{G}_{\omega,k,E} \mathbf{S}_{\omega,R} \mathbf{S}_{\omega,R} f, \mathbf{S}_{\omega,R} f \rangle_{H_{\omega,Y}^2(\mathbb{F}_d^+)} = \sum_{\alpha \in \beta} \frac{\omega_{|\alpha|}^2}{\omega_{k+|\alpha|}} \|f_\alpha\|^2_{Y}\end{equation}
This together with (4.69) and the second equality in (4.76) implies that
\begin{equation}
\omega \in \mathcal{O}\end{equation}
This finally verifies \(\omega\)-strong stability of \(\mathbf{S}_{\omega,R}\) and completes the proof. \(\square\)

### 4.4. A characterization of the model shift-operator tuple on \(H_{\omega,Y}^2(\mathbb{F}_d^+)\)

Let us note from the results of Section 4.3 that the model shift-operator tuple \(\mathbf{S}_{\omega,R}\) on \(H_{\omega,Y}^2(\mathbb{F}_d^+)\) enjoys the following properties:

\begin{equation}
(S_{\omega,R,R}^{-1} = \Gamma_{\omega,R} \mathbf{S}_{\omega,R} [I_{H_{\omega,Y}^2(\mathbb{F}_d^+)}] \quad \text{for} \quad j = 1, \ldots, d, \quad (4.80)
\end{equation}
\begin{equation}
\text{Ran} \mathbf{S}_{\omega,R,j} \subseteq \text{Ran} \mathbf{S}_{\omega,R,k} \quad \text{for} \quad j \neq k, \quad (4.81)
\end{equation}
\begin{equation}
\bigcap_{N \geq 0} \bigvee_{\alpha \in \beta} \mathbf{S}_{\omega,R} H_{\omega,Y}^2(\mathbb{F}_d^+) = \{0\} \quad (4.82)
\end{equation}
(where we use the notation \(\bigvee\) to denote \textit{closed linear span}). Indeed (4.80) follows by combining (4.75) and the second equality in (4.76) (for \(k = 1\)), (4.81) is clear from the form of the inner product on \(H_{\omega,Y}^2(\mathbb{F}_d^+)\), and (4.82) is clear from the observation that
\begin{equation}
\bigcap_{N \geq 0} \bigvee_{\alpha \in \beta} \mathbf{S}_{\omega,R} H_{\omega,Y}^2(\mathbb{F}_d^+) \subseteq \bigcap_{N \geq 0} \bigvee_{\alpha \in \beta} \mathcal{Y}(z) \cdot z^\alpha = \{0\}.
\end{equation}
The goal of this section is to prove a remarkable converse: \(\text{if} \ T = (T_1, \ldots, T_d) \text{ is any operator d-tuple on a Hilbert space } \mathcal{X} \text{ satisfying conditions} \ (4.80), (4.81), (4.82), \)
then $T$ is unitarily equivalent to $S_{\omega,R}$ on $H^2_{\omega,Y}(\mathbb{F}_d^+)\emptyset$, where the coefficient Hilbert space $Y$ is chosen so that $\dim Y = \dim (X \ominus \bigvee_{j=1,\ldots,d} \text{Ran} T_j)$.

We therefore let $\omega$ to be the weight sequence subject to conditions (2.31), (4.7). Let us define a class of operators $\mathcal{C}(\omega)$ to consist of all operator-tuples $T = (T_1, \ldots, T_d)$ of left-invertible operators $T_j \in \mathcal{L}(X)$ with mutually orthogonal ranges and satisfying the additional identity

$$ (T_j^* T_j)^{-1} = \Gamma_{\omega,T^*}[I_X]. $$

(4.83)

One can check that for the case $\omega = 1$ (the weight sequence consisting of all 1’s), the class $\mathcal{C}(1)$ consists of the set of all row isometries; we resist using the term $\omega$-isometries in general for the class $\mathcal{C}(\omega)$ since this term is reserved for the class of all operator tuples $T = (T_1, \ldots, T_d)$ such that $\Gamma_{\omega,T}(I_X) = 0$ (compatible with the terminology $\omega$-contraction for the class of all operator tuples $T$ with $\Gamma_{\omega,T}(I_X) \geq 0$). Given an operator-tuple $T \in \mathcal{C}(\omega)$ we associate the subspaces

$$ \mathcal{E} = X \otimes \left( \bigoplus_{j=1}^d T_j X \right) \quad \text{and} \quad \mathcal{X}_0 = \bigcap_{N \geq 0} \bigvee_{\alpha \in \mathbb{F}_d^+; |\alpha| = N} T^\alpha \mathcal{X}. $$

(4.84)

of $X$. Letting $A_j = T_j^*$, $X = I_X$ and $k = 1$ in (4.15) and (4.10) and taking into account that $\omega_0 = c_0 = 1$, we get explicit formulas

$$ \Gamma_{\omega,T^*}[I_X] = \sum_{\alpha \in \mathbb{F}_d^+} c_{|\alpha|} T^\alpha T^{*\alpha}, \quad \Gamma_{\omega,T^*}[I_X] = -\sum_{\alpha \in \mathbb{F}_d^+} c_{|\alpha|+1} T^\alpha T^{*\alpha}, $$

(4.85)

from which we see that

$$ I_X - \sum_{j=1}^d T_j \Gamma_{\omega,T^*}[I_X] T_j^* = \Gamma_{\omega,T^*}[I_X], $$

(4.86)

which, on account of (4.83), can be written as

$$ I_X - \sum_{j=1}^d T_j (T_j^* T_j)^{-1} T_j^* = \Gamma_{\omega,T^*}[I_X]. $$

(4.87)

Since the ranges of $T_1, \ldots, T_d$ are mutually orthogonal, it follows from (4.87) that $\Gamma_{\omega,T^*}[I_X]$ is an orthogonal projection. From (4.84) we see that the range space of this projection is the space $\mathcal{E}$ given by (4.84) and hence we arrive at the more complete version of (4.87):

$$ I_X - \sum_{j=1}^d T_j (T_j^* T_j)^{-1} T_j^* = \Gamma_{\omega,T^*}[I_X] = P_{\mathcal{E}}. $$

(4.88)

The Cauchy dual tuple $L = (L_1, \ldots, L_d)$ of $T$ is defined by

$$ L_j = T_j (T_j^* T_j)^{-1} \quad \text{for} \quad j = 1, \ldots, d, $$

(4.89)

and it is readily seen from (4.83) that

$$ L_j^* T_j = I_X \quad \text{and} \quad L_j^* T_i = 0 \quad \text{for all} \quad i \neq j. $$

(4.90)

**Proposition 4.25.** Let $T$ be a tuple of left-invertible operators satisfying conditions (4.83), let $L$ be its Cauchy dual, and let $\mathcal{E}$ be defined as in (4.84). Then

$$ P_{\mathcal{E}} L^\beta = \omega_{|\beta|}^{-1} P_{\mathcal{E}} T^\beta \quad \text{for all} \quad \beta \in \mathbb{F}_d^+. $$

(4.91)
Proof. We prove (4.91) by induction in $|\beta|$. The basis case $|\beta| = 0$ is trivial. Assuming that (4.91) holds for all $\beta$ with $|\beta| \leq N$ we will verify (4.91) for $\beta = \alpha j$ for a fixed $\alpha$ ($|\alpha| = N$). To this end, we first observe from (4.89) and (4.83) that

$$P_\xi L^{*\alpha j} = P_\xi L^{*\alpha} L_j^* = P_\xi L^{*\alpha}(T_j^* T_j) - 1 T_j^*$$

$$= P_\xi L^{*\alpha} T_1^{(1)} \omega[T_j^*] T_j^*$$

$$= -P_\xi L^{*\alpha} \left( \sum_{\beta \in \mathbb{F}_d^+} c_{|\beta|+1} T^{\beta^T} T^{*\beta} \right) T_j^*, \quad (4.92)$$

where we used the second formula from (4.85) for the last step. We next observe from (4.84) and (4.90) that $L^{*\alpha} T^{*\beta} \neq 0$ if and only if either $\beta = \delta \alpha$ (for some non-empty $\delta \in \mathbb{F}_d^+$) or $\alpha = \gamma \beta$ (for some $\gamma \in \mathbb{F}_d^+$). In the first case, $P_\xi L^{*\alpha} T^{*\beta} = P_\xi L^{*\alpha} T^{*\alpha} = P_\xi \gamma = 0$, while in the second case, we have

$$P_\xi L^{*\alpha} T^{*\gamma} T^{*\beta} = P_\xi L^{*\alpha} T^{*\gamma} T^{*\beta} = \omega_{|\gamma|}^{-1} P_\xi T^{*\gamma} T^{*\beta} = \omega_{|\beta|-|\gamma|}^{-1} P_\xi T^{*\alpha},$$

where we have used the induction hypothesis for the third equality. Thus, we have only $N + 1$ (nonzero) terms on the right side of (4.92):

$$P_\xi L^{*\alpha j} = -P_\xi \left( \sum_{i=0}^N c_{i+1} \omega_{N-i}^{-1} T^{*\alpha} T_j^* \right).$$

By the equality (4.8) (for $k = 1$), $\sum_{i=0}^N c_{i+1} \omega_{N-i}^{-1} = -\omega_N^{-1}$, which being combined with the last equality gives

$$P_\xi L^{*\alpha j} = \omega_N^{-1} P_\xi T^{*\alpha} T_j^* = \omega_N^{-1} P_\xi T^{*\alpha j},$$

and the induction argument completes the proof. \qed

Proposition 4.26. For the tuples $T$ and $L$ as above,

$$X_0 := \bigcap_{N \geq 0} \bigvee_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N} T^{*\alpha} X = \bigcap_{\alpha \in \mathbb{F}_d^+} \text{Ker } P_\xi L^{*\alpha} = \bigcap_{\alpha \in \mathbb{F}_d^+} \text{Ker } P_\xi T^{*\alpha}. \quad (4.93)$$

Proof. Since $P_\xi = I_X - T_1 L_1^* - \cdots - T_d L_d^*$, it follows that for any $N \geq 0$,

$$\sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq N-1} T^{*\alpha} P_\xi L^{*\alpha} = I_X - \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = N} T^{*\alpha} L^{*\alpha}.$$
Hence, if \( x \in \mathcal{X} \) belongs to \( \text{Ker} \, P_\xi L^{*\alpha} \) for all \( \alpha \in \mathbb{F}^+_0 \), then for each \( N \geq 0 \),

\[
x = \sum_{\alpha \in \mathbb{F}^+_0:|\alpha| \leq N-1} T^\alpha T_\xi L^{*\alpha} x + \sum_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} T^\alpha L^{*\alpha} x
\]

\[
= \sum_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} T^\alpha L^{*\alpha} x \in \bigvee_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} T^\alpha \mathcal{X},
\]

which verifies the inclusion

\[
\bigcap_{\alpha \in \mathbb{F}^+_0} \text{Ker} \, P_\xi L^{*\alpha} \subseteq \mathcal{X}_0.
\]  \hspace{1cm} (4.95)

For the converse inclusion, take arbitrary \( x \in \mathcal{X}_0 \) and \( \alpha \in \mathbb{F}^+_0 \). In particular, \( x \) can be represented as

\[
x = \sum_{\beta \in \mathbb{F}^+_0:|\beta| = |\alpha|+1} T^{\beta\top} x_\beta \quad \text{for some} \quad x_\beta \in \mathcal{X}.
\]

As we have seen in the proof of Proposition 4.25, the inequality \( L^{*\alpha} T^{\beta\top} \neq 0 \) may occur only if \( \beta = j\alpha \). Hence, from the latter representation for \( x \) we conclude that

\[
P_\xi L^{*\alpha} x = P_\xi \left( \sum_{j=1}^d T_j x_{j\alpha} \right) = 0
\]

for any \( \alpha \in \mathbb{F}^+_0 \) confirming the reverse inclusion in (4.95) and hence, verifying the second equality in (4.93). The third equality holds due to (4.91). \( \square \)

**Proposition 4.27.** Let \( T \) be a tuple of left-invertible operators satisfying conditions (4.83) and let \( E \) and \( \mathcal{X}_0 \) be defined as in (4.84). Then \( \mathcal{X}_0 \) reduces \( T \) (i.e., \( \mathcal{X}_0 \) is invariant under \( T \) and \( T^\top \)) and furthermore,

\[
\mathcal{X}_0 \oplus \mathcal{X}_s = \mathcal{X} \quad \text{where} \quad \mathcal{X}_s := \bigvee_{\alpha \in \mathbb{F}^+_0} T^\alpha \mathcal{E}.
\]  \hspace{1cm} (4.96)

**Proof.** If \( L \) is the Cauchy dual of \( T \), then \( T \) is the Cauchy dual of \( L \). Then it follows from representations (4.93) that \( \mathcal{X}_0 \) can be represented as

\[
\mathcal{X}_0 = \bigcap_{N \geq 0} \bigvee_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} L^{\alpha} \mathcal{X}.
\]  \hspace{1cm} (4.97)

Let \( x \) be an arbitrary vector in \( \mathcal{X}_0 \). Then for any \( N \geq 1 \), we can represent \( x \) as

\[
x = \sum_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} L^{\alpha} x_\alpha \quad \text{for some} \quad x_\alpha \in \mathcal{X}.
\]

Due to relations (4.90) and the assumption that \( |\alpha| \geq 1 \), for a fixed letter \( j \), the inequality \( T_j^* L^{\alpha} \neq 0 \) holds only if \( \alpha = j\beta \). Applying \( T_j^* \) to the vector \( x \) as above, we therefore have

\[
T_j^* x = \sum_{\alpha \in \mathbb{F}^+_0:|\alpha| = N} T_j^* L^{\alpha} x_{j\alpha} = \sum_{\beta \in \mathbb{F}^+_0:|\beta| = N-1} L^{\beta} x_{j\beta}.
\]

We now conclude that \( T_j^* x \) belongs to \( \bigvee_{\alpha \in \mathbb{F}^+_0:|\beta| = N-1} L^{\alpha} \mathcal{X} \) for every \( N \geq 1 \) and therefore, it belongs to \( \mathcal{X}_0 \), due to representation (4.97). Thus, \( \mathcal{X}_0 \) is \( T^* \)-invariant. Its \( T \)-invariance follows from the very definition (4.84) of \( \mathcal{X}_0 \).
To prove (4.97), note that \( x \in \mathcal{X} \) is orthogonal to \( \mathbf{T}^* \mathcal{E} \) for all \( \alpha \in \mathbb{F}_d^+ \) if and only if \( \mathbf{T}^* x \) is orthogonal to \( \mathcal{E} \) for all \( \alpha \in \mathbb{F}_d^+ \), i.e., that \( P_\alpha \mathbf{T}^* x = 0 \) for all \( \alpha \in \mathbb{F}_d^+ \). By (4.89), the latter means that \( x \in \mathcal{X}_0 \). \( \square \)

It is straightforward to check that the operator-theoretic properties defining the class \( \mathcal{C}(\omega) \) are invariant upon restriction to a reducing subspace; hence if \( \mathbf{T} \in \mathcal{C}(\omega) \) on \( \mathcal{X} \) with \( \mathcal{X} \) having the decomposition \( \mathcal{X}_0 \oplus \mathcal{X}_s \) as in (4.90), then the restricted operator-tuples \( \mathbf{T}_0 := \mathbf{T}|_{\mathcal{X}_0} \) and \( \mathbf{T}_s := \mathbf{T}|_{\mathcal{X}_s} \) are again in the class \( \mathcal{C}(\omega) \). This discussion suggests that we introduce two subclasses of the class \( \mathcal{C}(\omega) \), namely:

\[
\mathcal{C}_0(\omega) = \{ \mathbf{T} \in \mathcal{C}(\omega) : \mathcal{X}_0 = \mathcal{X} \} \quad \text{and} \quad \mathcal{C}_s(\omega) = \{ \mathbf{T} \in \mathcal{C}(\omega) : \mathcal{X}_s = \mathcal{X} \},
\]

(4.98)

where the subscript \( s \) is to suggest a shift operator tuple. From the definitions and Proposition 4.27 it follows that the general element \( \mathbf{T} \in \mathcal{C}(\omega) \) has the form \( \mathbf{T} = \mathbf{T}_s \oplus \mathbf{T}_0 \) where \( \mathbf{T}_s \in \mathcal{C}_s(\omega) \) and \( \mathbf{T}_0 \in \mathcal{C}_0(\omega) \). Our next goal is to get a more intrinsic characterization of each of the classes \( \mathcal{C}_0(\omega) \) and \( \mathcal{C}_s(\omega) \).

**Theorem 4.28.** The operator-tuple \( \mathbf{T} = (T_1, \ldots, T_d) \) in the class \( \mathcal{C}(\omega) \) is in the subclass \( \mathcal{C}_0(\omega) \) if and only if the subspace \( \bigoplus_{j=1}^d T_j \mathcal{X} \) is the whole space \( \mathcal{X} \). The class \( \mathcal{C}_0(\omega) \) can be equivalently characterized as consisting of all operator-tuples \( \mathbf{T} = (T_1, \ldots, T_d) \) of left-invertible operators with orthogonal ranges such that \( \mathbf{T}^* \) is an \( \omega \)-isometry, i.e., such that

\[
\Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] = 0.
\]

**Proof.** By (4.96) and the definition (4.98), a tuple \( \mathbf{T} \in \mathcal{C}(\omega) \) is in the subclass \( \mathcal{C}_0(\omega) \) exactly when the subspace \( \mathcal{E} := \mathcal{X} \ominus \bigoplus_{j=1}^d T_j \mathcal{X} \) is trivial, which is the same as to say that \( \mathcal{X} = \bigoplus_{j=1}^d T_j \mathcal{X} \). From the identity (4.88) we also see that \( \mathcal{E} = \{0\} \) implies that \( \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] = 0 \), i.e., \( \mathbf{T}^* \) is a \( \omega \)-isometry. Hence any operator-tuple \( \mathbf{T} \) in \( \mathcal{C}(\omega) \) meets the criteria in the second characterization.

Conversely, suppose that \( \mathbf{T} \) is a operator-tuple of left-invertible operators with pair-wise orthogonal ranges such that \( \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] = 0 \). To show that \( \mathbf{T} \) is in the class \( \mathcal{C}_0(\omega) \), by the first characterization of \( \mathcal{C}_0(\omega) \) it remains only to show that

(i) \( \bigvee_{j=1}^d \operatorname{Ran} T_j = \mathcal{X} \), and

(ii) the identity (4.83) holds for \( j = 1, \ldots, d \).

The hypothesis that \( \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] = 0 \) combined with identity (4.86) gives us

\[
\mathcal{X} = \sum_{j=1}^d T_j \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] T_j^*.
\]

From this identity we read off that indeed item (i) holds. Multiply this same identity on the left and on the right by the projection \( P_{\mathcal{X}} \) onto \( T_k \mathcal{X} \) and use the pairwise-orthogonality of the ranges of the operators \( T_k \) to deduce that

\[
I_{\mathcal{X}} = \sum_{j=1}^d T_j \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] T_j^*.
\]

Now use that the operators \( T_k \) is invertible as an operator from \( \mathcal{X} \) to \( \operatorname{Ran} T_k \), that \( T_k^*|_{\operatorname{Ran} T_k} \) is invertible as an operator from \( \operatorname{Ran} T_k \) to \( \mathcal{X} \), and that \( \Gamma_{\omega, \mathbf{T}^*}[\mathcal{X}] \) is an operator on \( \mathcal{X} \). Multiplying the last identity on the left by \( T_k^{-1} \) (viewed as an
operator from $\text{Ran} T_k$ to $X$) and on the right by $T_k^{*-1}$ (viewed as an operator from $X$ to $\text{Ran} T_k$) leads us to the identity
\[
T_k^{-1} T_k^{*-1} = \Gamma^{(1)}_{\omega,T^*}[I_X].
\]
It is now a matter of verifying that
\[
T_k^{-1} T_k^{*-1} (T_k^* T_k) = T_k^{-1} (T_k^{*-1} T_k^*) T_k = T_k^{-1} (I_{\text{Ran} T_k}) T_k = T_k^{-1} T_k = I_X
\]
to conclude that item (ii) holds as well. □

Now we will take a closer look at the second component in the orthogonal decomposition (4.96). It turns out that the observability gramian
\[
O_{P E,L^*} : x \rightarrow P E (I_X - z_1 L_1^* - \ldots - z_d L_d^*)^{-1} x
\]
and the $\omega$-observability gramian
\[
O_{\omega,P E,T^*} : x \rightarrow P E (I_X - z_1 L_1^* - \ldots - z_d L_d^*)^{-1} x
\]
are equal. Indeed, combining (4.91) and (4.5) gives
\[
O_{P E,L^*} = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{\alpha}^{-1} P E T^{*a} x z^a = O_{\omega,P E,T^*}. \tag{4.99}
\]
It follows from (4.99) and (4.93) that
\[
\text{Ker} O_{\omega,P E,T^*} = \bigcap_{\alpha \in \mathbb{F}_d^+} \text{Ker} P E T^{*a} = \mathcal{X}_0, \tag{4.100}
\]
so the pair $(P E, T^*)$ is observable if and only if $\mathcal{X}_0 = \{0\}$.

**Proposition 4.29.** Let $T$ be a tuple of left-invertible operators satisfying conditions (4.88) and let $L$ be its Cauchy dual. Then

1. The following intertwining relations hold for $j = 1, \ldots, d$:
   \[
   S_{\omega,R,j} O_{\omega,P E,T^*} = O_{\omega,P E,T^*} T_j, \quad S_{1,R,j}^* O_{\omega,P E,T^*} = O_{\omega,P E,T^*} L_j^*. \tag{4.101}
   \]
2. $O_{\omega,P E,T^*}$ is a partial isometry from $X$ into the Hardy-Fock space $H^2_{\omega,E}(\mathbb{F}_d^+)$ with initial space equal to $\bigvee_{\alpha \in \mathbb{F}_d^+} T^{*a} E$.

**Proof.** Recalling that $L^{*a} T_j \neq 0$ only if $\alpha = \beta j$, we have
\[
O_{\omega,P E,T^*} T_j x = O_{P E,L,T^*} T_j x = \sum_{\alpha \in \mathbb{F}_d^+} P E L^{*a} T_j x z^a = \sum_{\beta \in \mathbb{F}_d^+} P E L^{*\beta} L_j^* T_j x z^{\beta j} = \sum_{\beta \in \mathbb{F}_d^+} P E L^{*\beta} x z^{\beta j} = (O_{P E,L^*} x) z_j = S_{\omega,R,j} O_{\omega,P E,T^*} x.
\]
Making use of the explicit formula (3.38) for $S_{1,R,j}^*$, we get
\[
S_{1,R,j}^* O_{\omega,P E,T^*} x = S_{1,R,j}^* O_{P E,L^*} x = \sum_{\alpha \in \mathbb{F}_d^+} P E L^{*a j} x z^a = O_{P E,L^*} L_j^* x = O_{\omega,P E,T^*} L_j^* x,
\]
which completes the proof of (4.101).
By (4.100), $\mathcal{O}_\alpha, P_\alpha \cdot T^* : X_0 \to X$. Due to (4.100), statement (2) in Proposition 4.29 follows once we show that $\mathcal{O}_\alpha, P_\alpha \cdot T^*$ maps $\mathcal{X} \cap X_0 = \bigvee_{\alpha \in F_+^d} T^* \mathcal{E}$ isometrically into $H^2_{\omega, \mathcal{E}}(F_+^d)$. To this end, take $x \in \mathcal{X} \cap X_0$ which is of the form

$$x = \sum_{\alpha \in F_+^d : |\alpha| \leq M} T^\alpha x_\alpha, \quad x_\alpha \in \mathcal{E},$$

(4.102)

for some $M < \infty$. Then we have

$$\|x\|^2_\mathcal{X} = \sum_{\alpha, \beta \in F_+^d} \langle T^\alpha x_\alpha, T^\beta x_\beta \rangle_\mathcal{X} = \sum_{\alpha, \beta \in F_+^d} \langle T^\alpha T^\beta T^\alpha x_\alpha, x_\beta \rangle_\mathcal{X}
= \sum_{\alpha, \beta \in F_+^d} \langle P_\alpha T^\beta T^\alpha x_\alpha, x_\beta \rangle_\mathcal{X}
= \sum_{\alpha, \beta \in F_+^d} \omega_{|\beta|} \langle P_\alpha L^{\alpha \beta^T} T^\alpha x_\alpha, x_\beta \rangle_\mathcal{X},$$

(4.103)

where the two last equalities hold due to (4.101) since $x_\beta \in \mathcal{E}$. As we have seen in the proof of Proposition 4.25, the inequality $P_\alpha L^{\alpha \beta^T} T^\alpha \neq 0$ may occur only if $\beta = \alpha \gamma$ for some $\gamma \in F_+^d$, in which case

$$P_\alpha L^{\alpha \beta^T} T^\alpha x_\alpha = P_\alpha L^{\alpha \gamma^T} T^\alpha x_\alpha.$$ 

Since $\mathcal{E} \perp \text{Ran} T_j$ for all $j \in \{1, \ldots, d\}$ we have for any $e \in \mathcal{E}$ and $x \in \mathcal{X}$,

$$\langle L_j^* e, x \rangle = \langle e, L_j x \rangle = \langle e, T_j (T_j^* T_j)^{-1} x \rangle = 0$$

and therefore, $L_j^*|\mathcal{E} = 0$ (the same computation shows that $T_j^*|\mathcal{E} = 0$ as well) for $j = 1, \ldots, d$. Hence, if $\gamma \neq \emptyset$, then $L^{\alpha \gamma^T} x_\alpha = 0$. Therefore, all the terms on the right side of (4.103), with $\alpha \neq \beta$ are equal zero, and hence,

$$\|x\|^2_\mathcal{X} = \sum_{\alpha \in F_+^d} \omega_{|\alpha|} \langle P_\alpha L^{\alpha \alpha^T} T^\alpha x_\alpha, x_\alpha \rangle_\mathcal{X}
= \sum_{\alpha \in F_+^d} \omega_{|\alpha|} \langle P_\alpha x_\alpha, x_\alpha \rangle_\mathcal{X} = \sum_{\alpha \in F_+^d} \omega_{|\alpha|} \|x_\alpha\|^2_\mathcal{X},$$

(4.104)

On the other hand, since $T_j^*|\mathcal{E} = 0$, we have for any $e \in \mathcal{E}$

$$\mathcal{O}_{\alpha, P_\alpha} T^* e = \sum_{\alpha \in F_+^d} \omega_{|\alpha|}^{-1} P_\alpha T^{* \alpha} e z^\alpha = \omega_{0}^{-1} P_\alpha e = e.$$

Combining this latter observation with the first intertwining relation in (4.101), we have for $x$ of the form (4.102),

$$\mathcal{O}_{\alpha, P_\alpha} T^* x = \sum_{\alpha \in F_+^d} x_\alpha z^{\alpha^T}$$

and hence,

$$\|\mathcal{O}_{\alpha, P_\alpha} T^* x\|^2_{H^2_{\omega, \mathcal{E}}(F_+^d)} = \sum_{\alpha \in F_+^d} \omega_{|\alpha|} \cdot \|x_\alpha\|^2_\mathcal{X}.$$

Comparing the latter equality with (4.104) we conclude that $\|\mathcal{O}_{\alpha, P_\alpha} T^* x\|_{H^2_{\omega, \mathcal{E}}(F_+^d)} = \|x\|_\mathcal{X}$ for all $x \in \mathcal{X} \cap X_0$ of the form (4.102). As the set of all such elements form
a dense subset of \(X \ominus X_0\) (by (4.96)), we conclude by continuity that \(O_{\omega, P^*} T^*\) is isometric on all of \(X \ominus X_0\).

**Theorem 4.30.** The operator tuple \(T = (T_1, \ldots, T_d)\) in the class \(D(\omega)\) acting on the Hilbert space \(X\) is in the subclass \(C_*^s(\omega)\) if and only if \(T\) satisfies any one of the following additional conditions:

1. \(\bigcap_{N \geq 0} \bigvee_{|\alpha| = N} T^\alpha X = \{0\}\);
2. \(\bigvee_{|\alpha| = N} T^\alpha E = X\) where \(E = X \ominus \bigoplus_{1 \leq j \leq d} T_j X\);
3. There is a coefficient Hilbert space \(E\) so that \(T\) is unitarily equivalent to the model shift operator-tuple \(S_{\omega, R}\) acting on \(H_0^2(\mathbb{F}_d^+)(\mathbb{F}_d^+)\).

**Proof.** By the general decomposition (4.96) and the definition (1.99), the tuple \(T \in C(\omega)\) is in the subclass \(T \in C_*^s(\omega)\) if and only if \(X_\omega = X\), which is the same as condition (2). That this is equivalent to condition (1) is seen from the first formula for \(X_\omega\) in (4.93). Next note that by Proposition 4.29 \(T|\chi_i\) is unitarily equivalent to \(S_{\omega, R}\).

In the case \(\omega = 1\) and \(d = 1\), the class \(C(1)\) consists of the isometries and the decomposition (4.96) for \(T\) is referred to as the Wold decomposition for \(T\) (see [92]).

The next result gives an indication of the extent to which this criterion generalizes to the general setting \(C_*^s(\omega)-\)operator tuples.

**Theorem 4.31.** Suppose that \(T = (T_1, \ldots, T_d)\) is in the class \(C(\omega)\) for some admissible weight \(\omega\).

1. If \(T\) is in the class \(C_*^s(\omega)\), then \(T^*\) is strongly stable.
2. Suppose that \(\|T_1 \cdots T_d\| \leq 1\) and \(T \in C_0(\omega)\). Then for every \(x \in X\), it is the case that

\[
\liminf_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| = N} \|L^{*\alpha} x\|^2 > 0,
\]

where \(L\) is the Cauchy dual of \(T\).

**Proof.** (1) If \(T\) is in \(C_*^s(\omega)\), then \(T\) is unitarily equivalent to \(S_{\omega, R}\) on some weighted Hardy space \(H_0^2(\mathbb{F}_d^+)\). We have seen that \(S_{\omega, R}\) is strongly stable (item (1) in Proposition 4.29). Hence \(T^*\) is strongly stable. In particular \(L^*\) (rather than \(T^*\)) is not strongly stable.

(2) Suppose \(T\) is in \(C_0(\omega)\), so \(X_\omega = \{0\}\). Choose \(0 \neq x \in X_0\). Then \(x\) belongs to \(\text{Ker} P_2 L^{*\alpha}\) for all \(\alpha \in \mathbb{F}_d^+\) (by characterization (4.93)) and then the computation (4.94) shows that \(x = \sum_{\alpha: |\alpha| = N} T^\alpha L^* \alpha^\top x\) for all \(N = 0, 1, 2, \ldots\). As \(T\) is a row contraction, it follows that each \(T_j\) is a contraction and hence, for \(\alpha = i_N \cdots i_1 \in \mathbb{F}_d^+\), \(T^\alpha = T_{i_N} \cdots T_{i_1}\) is a contraction, and we can compute

\[
\|x\|^2 = \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| = N} \|T^\alpha L^* \alpha^\top x\|^2 \leq \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| = N} \|L^* \alpha^\top x\|^2.
\]

It follows that

\[
\liminf_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| = N} \|L^* \alpha^\top x\|^2 \geq \|x\|^2 > 0.
\]
In particular, when $T \in \mathcal{C}(\omega_0)$, it is $L^*$ (rather than $T^*$) that is guaranteed to be not strongly stable.

**Remark 4.32.** In case $\omega = 1$, it is automatic that $T$ is a contraction and that the Cauchy dual $L$ of $T$ is equal to $T$. Thus Theorem 4.30 recovers the result for the case $\omega = 1$, $d = 1$ mentioned immediately preceding the theorem. Let us also mention that it is known that, in the single-variable case ($d = 1$), there are many surjective $n$-isometries which are not contractions once $n \geq 3$ (see [3, 4, 5]).

### 4.5. Observability-operator range spaces

The main character of this section is the range of the $\omega$-observability operator

$$\text{Ran } O_{\omega,C,A} = \{ CR_\omega(Z(z)A)x : x \in \mathcal{X} \}. $$

We recall that $R_\omega$ is the power series (4.5) and $Z(z)$ and $A$ are defined as in (1.24). Throughout this section, we assume that the weight sequence $\omega$ is admissible and meets the condition (4.7). This material fleshes out theme #1 mentioned at the end of Section 1.1 for the general $\omega$-setting.

**Theorem 4.33.** Suppose that $(C,A)$ is a $\omega$-output-stable pair. Then:

1. The intertwining relation
   $$S^*_{\omega,R_\omega}O_{\omega,C,A}x = O_{\omega,C,A}A_jx \quad (x \in \mathcal{X})$$
   holds for every backward-shift operator $S^*_{\omega,R_\omega}$ defined in (2.30) and hence $\text{Ran } O_{\omega,C,A}$ is $S^*_{\omega,R}$-invariant.

2. Let $H \in \mathcal{L}(\mathcal{X})$ be a solution of the system (4.17), (4.18) and let $\mathcal{X}'$ be the completion of $\mathcal{X}$ with $H$-inner product $\|x\|_{\mathcal{X}'}^2 = \langle Hx, x \rangle_\mathcal{X}$. Then $A_j$ and $C$ extend to define bounded operators
   $$A_j' : \mathcal{X}' \to \mathcal{X}' \quad (j = 1, \ldots, d) \quad \text{and} \quad C' : \mathcal{X}' \to \mathcal{Y},$$
   so that the $\omega$-observability operator $O_{\omega,C',A'} : \mathcal{X}' \to \mathcal{H}_{\omega,C',A'}(\mathbb{F}^+_d)$ is a contraction.

3. If $H$ is subject to (4.17), (4.18) and the linear manifold $\mathcal{M} := \text{Ran } O_{\omega,C,A}$ is given the lifted norm
   $$\|O_{\omega,C,A}x\|_{\mathcal{M}}^2 = \inf_{y \in \mathcal{X}} \|O_{\omega,C,A}y\|_{\mathcal{M}} \langle Hy, y \rangle_{\mathcal{X}},$$
   then:

   a. $\mathcal{M}$ can be completed to $\mathcal{M}' = \text{Ran } O_{\omega,C',A'}$ with contractive inclusion in $\mathcal{H}_{\omega,C',A'}(\mathbb{F}^+_d)$:
      $$\|f\|_{\mathcal{H}_{\omega,C',A'}(\mathbb{F}^+_d)} \leq \|f\|_{\mathcal{M}'} \quad \text{for all } f \in \mathcal{M}'. $$
      Furthermore, $\mathcal{M}'$ is isometrically equal to the FNRKHS with reproducing kernel
      $$K_{C',A'}(z, \zeta) = C' R_\omega(Z(z)A') R_\omega(Z(\zeta)A')^* C'^*.$$  
      In case $H$ is invertible, the reproducing kernel can be written directly in terms of $(C,A)$:
      $$K_{C,A,H}(z, \zeta) = CR_\omega(Z(z)A) H^{-1} R_\omega(Z(\zeta)A)^* C^*.$$  
      (b) The restriction $(E|_{\mathcal{M}'},S^*_{\omega,R}|_{\mathcal{M}'})$ of the $\omega$-model output pair (4.63) to $\mathcal{M}$ is $\omega$-contractive. Moreover, it is $\omega$-isometric if and only if (4.19) holds.
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(4) Conversely, let $M$ be a Hilbert space included in $H^2_{\omega,Y}(\mathbb{F}^+_d)$ (not necessarily isometrically or even contractively) such that

(i) $M$ is invariant under the backward-shift tuple $S^*_{\omega,R}$,

(ii) the pair $(E_M, S^*_{\omega,R}|M)$ is an $\omega$-contractive output pair, i.e., the inequalities

\[
\sum_{k=1}^d \|S^*_{\omega,R,k}f\|_M^2 \leq \|f\|_M^2, \quad \sum_{\alpha \in \mathbb{F}^+_d} c_{|\alpha|} \|S^*_{\omega,R}f\|_M^2 \geq \|f_0\|^2, \tag{4.110}
\]

\[
\sum_{\alpha \in \mathbb{F}^+_d} \left( \sum_{\ell=1}^k \frac{c_{|\alpha|+\ell}}{\omega_{k-\ell}} \right) \|S^*_{\omega,R}f\|_M^2 \geq 0 \quad \text{for all} \quad k \geq 1
\]

hold for all $f \in M$.

Then it follows that $M$ is contractively included in $H^2_{\omega,Y}(\mathbb{F}^+_d)$ and there exists an $\omega$-contractive pair $(C,A)$ such that

\[M = \mathcal{H}(KC,A) = \text{Ran} \mathcal{O}_{\omega,C,A}\]

isometrically. One can take $(C,A)$ to be an $\omega$-isometric output pair if and only if the second of the inequalities (4.110) holds with equality. For example, one may take $\mathcal{X}' = M$, $C = E|_M$ and $A = S^*_{\omega,R}|M$.

PROOF OF (1): Making use of (2.30) and (4.3) we get for any $j \in \{1, \ldots, d\}$,

\[S^*_{\omega,R,j} \mathcal{O}_{\omega,C,A}x = S^*_{\omega,R,j} \sum_{\alpha \in \mathbb{F}^+_d} \omega^{-1}_{|\alpha|}(C^\alpha A)x\]

\[= \sum_{\alpha \in \mathbb{F}^+_d} \omega^{-1}_{|\alpha|}(C^\alpha A_j)x = \mathcal{O}_{\omega,C,A}A_jx\]

which proves (4.105). \hfill \Box

PROOF OF (2): The Stein system (4.17), (4.18) amounts to the statement that $(C,A)$ is $\omega$-contractive and well-defined on the dense subset $[X]$ of $\mathcal{X}'$ (where $[x]$ is the equivalence class containing $x$) and hence extends to a $\omega$-contractive pair $(C',A')$ on all of $\mathcal{X}'$. By part (2) in Theorem 4.5, $\mathcal{O}_{\omega,C,A} \preceq H$ and therefore,

\[\|\mathcal{O}_{\omega,C,A}x\|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2 = (\mathcal{O}_{\omega,C,A}x,x)_{\mathcal{X}} \leq (Hx,x)_{\mathcal{X}}\]

\[\quad \text{for all} \quad x \in \mathcal{X}, \tag{4.111}
\]

so $\mathcal{O}_{\omega,C,A}$ is contractive from $\mathcal{X}$ (with the $H$-pseudo-inner product) to $H^2_{\omega,Y}(\mathbb{F}^+_d)$ and hence also $\mathcal{O}_{\omega,C',A'}$ is contractive from $\mathcal{X}'$ to $H^2_{\omega,Y}(\mathbb{F}^+_d)$. \hfill \Box

PROOF OF (3a): The definition of the $M$-norm makes $\mathcal{O}_{\omega,C,A}$ a partial isometry from $[X]$ (with the $M$-norm (4.100)) onto $M$. In case $M$ is not complete in its norm, we work instead with $\mathcal{O}_{\omega,C',A'}$ which is a partial isometry from $\mathcal{X}'$ onto $M' = \text{Ran} \mathcal{O}_{\omega,C',A'}$. It suffices to verify the inequality (4.107) for the special case where $f = \mathcal{O}_{\omega,C,A}x$ for some $x \in \mathcal{X}$. In this case, (4.107) amounts to

\[\|\mathcal{O}_{\omega,C,A}x\|^2 \leq (Hx,x)_{\mathcal{X}}\]

which holds true, by (4.111).
To identify the formal reproducing kernel for $\mathcal{M}'$, we compute, for $x \in \mathcal{X}$,
\begin{align*}
\langle (O_\omega C^*Ax)(\zeta), y \rangle_{Y(\langle \zeta \rangle) \times Y} &= \langle C^*R_\omega (Z(\zeta)A')x, y \rangle_{Y(\langle \zeta \rangle) \times Y} \\
&= (x, R_\omega (Z(\zeta)A')^*C^*y)_{X' \times X''(\langle \zeta \rangle)} \\
&= \langle (O_\omega C^*A'x)(z), C^*R_\omega (Z(z)A')R_\omega (Z(\zeta)A')^*C^*y \rangle_{M' \times M'(\langle \zeta \rangle)}.
\end{align*}
This verifies that the kernel (4.108) is the formal reproducing kernel for $\mathcal{M}'$. If $H$ is invertible, then $\mathcal{X} = \mathcal{X}'$ as sets and we can simply quote Proposition 2.3 to conclude that $\mathcal{M}$ has formal reproducing kernel $K_{C,A,H}(z, \zeta)$ defined as in (4.109).

**Proof of (3b):** It follows from the definition of the range norm and from the intertwining relations (4.105) that for $f$ of the form $f = O_\omega C^*Ax$,
\[
\|f\|_M^2 = \langle Hx, x \rangle_X \quad \text{and} \quad S^*_{\omega,R,j}f = O_\omega C^*A_jx \quad (j = 1, \ldots, d).
\]
Therefore,
\[
\|f\|_M^2 = \sum_{j=1}^d \|S^*_{\omega,R,j}f\|_M^2 = \langle Hx, x \rangle_X - \sum_{j=1}^d \|HA_jx, A_jx\rangle_X
\]
\[
= \langle (H - \sum_{j=1}^d A_j^*HA_j)x, x \rangle_X.
\]

Due to the first relation in (4.117), the expression on the right side of (4.112) is nonnegative for every $x \in \mathcal{X}$. Therefore, the expression on the left side of (4.112) is nonnegative for every $f \in \mathcal{M}$ which means that $S^*_{\omega,R}|_\mathcal{M}$ is a contractive tuple. Iterating (4.105) gives
\[
S^*_{\omega,R}O_\omega C^*Ax = O_\omega C^*A^{\alpha^\top}x \quad \text{for all} \quad x \in \mathcal{X} \quad \text{and} \quad \alpha \in F_d^+.
\]
For $f = O_\omega C^*Ax$ we then have
\[
\|S^*_{\omega,R,f}\|_M^2 = \langle HA^{\alpha^\top}x, A^{\alpha^\top}x \rangle_X \quad \text{and} \quad f_0 = Ef = Cx.
\]
With these substitutions, we see that
\[
\langle \Gamma_{\omega,S^*_{\omega,R}|\mathcal{M}}[I_\mathcal{M}]f, f \rangle_{\mathcal{M}} = \sum_{\alpha \in F_d^+} c_{|\alpha|}\|S^*_{\omega,R,f}\|_M^2 \quad \text{(by definition (4.14))}
\]
\[
= \sum_{\alpha \in F_d^+} c_{|\alpha|}\langle HA^{\alpha^\top}x, A^{\alpha^\top}x \rangle_X \quad \text{(by (4.13))}
\]
\[
= \langle \sum_{\alpha \in F_d^+} c_{|\alpha|}A^{\alpha^\top}HA^{\alpha^\top}x, x \rangle_X \quad \text{(by substitution $\alpha \mapsto \alpha^\top$)}
\]
\[
= \langle \Gamma_{\omega,A}[H]x, x \rangle_{\mathcal{M}} \quad \text{(by definition (4.15))},
\]
(the substitution $\alpha \mapsto \alpha^\top$ is justified by the fact that the latter sum is taken over all elements of $F_d^+$ and that $|\alpha| = |\alpha^\top|$) and subsequently, in view of the second equality in (4.114)
\[
\langle \Gamma_{\omega,S^*_{\omega,R}|\mathcal{M}}[I_\mathcal{M}]f, f \rangle_{\mathcal{M}} - \|Ef\|_X^2 = \langle \Gamma_{\omega,A}[H]x, x \rangle_{\mathcal{M}} - \|Cx\|_X^2
\]
\[
= \langle \Gamma_{\omega,A}[H] - C^*C)x, x \rangle_{\mathcal{X}}.
\]
A similar computation relying on the definition (4.16) and relations (4.111) shows that for any fixed $k \geq 1$,

$$
\langle \Gamma^{(k)}_{\omega}S_{\omega,R}|M|f, f \rangle_M = - \sum_{\alpha \in F_d^+} \left( \sum_{\ell=1}^k \frac{c_{|\alpha|+\ell}}{\omega_{k-\ell}} \right) \|S^*_{\omega,R}f\|^2_M \\
= - \sum_{\alpha \in F_d^+} \left( \sum_{\ell=1}^k \frac{c_{|\alpha|+\ell}}{\omega_{k-\ell}} \right) \langle H A^\alpha x, A^\alpha x \rangle_X \\
= \left\langle - \sum_{\alpha \in F_d^+} \left( \sum_{\ell=1}^k \frac{c_{|\alpha|+\ell}}{\omega_{k-\ell}} \right) A^* A A^\alpha x, x \right\rangle_X \\
= \langle \Gamma^{(k)}_{\omega}H|f, f \rangle_X.
$$

(4.116)

Since $H$ satisfies inequalities (4.17), (4.18), the right hand side expressions in (4.115) and (4.116) are nonnegative for all $x \in X$. Hence, the left sides of (4.115) and (4.116) are nonnegative for all $f \in M$ meaning that $S^*_{\omega,R}|M$ is an $\omega$-hypercontraction and that the pair $(E|_M, S^*_{\omega,R}|M)$ is $\omega$-contractive. Equality (4.19) is equivalent to both sides in (4.115) vanish, i.e., that the pair $(E|_M, S^*_{\omega,R}|M)$ is $\omega$-isometric. This completes the verification of part (3). \(\square\)

**Proof of (4):** Suppose that $M$ is a Hilbert space included in $H^2_{\omega,Y}(F_d^+)$ which is invariant under $S^*_{\omega,R}$ and satisfies the inequalities (4.110) for all $f \in M$. Set $X = M$ and let $C = E|_M$ and $A = S^*_{\omega,R}|M$. The import of conditions (4.110) is that then the output pair $(C,A)$ is a $\omega$-contractive output pair (recall Definitions 4.39 and 4.7). Since $O_{\omega,C,A} = I_M$ by the first part of statement (2) in Proposition 4.23 (a purely algebraic statement independent of the choice of norm on $M$), it follows that for each $f \in M$ we have

$$
\|f\|^2_{H^2_{\omega,Y}(F_d^+)} = \|O_{\omega,C,A}f\|^2_{H^2_{\omega,Y}(F_d^+)} = \langle G_{\omega,C,A}f, f \rangle_M \leq \|f\|^2_M
$$

where the last step follows from part (1) of Lemma 4.10 since we have already observed that $(C,A)$ is a $\omega$-contractive output pair. We conclude that $M$ is contractively included in $H^2_{\omega,Y}(F_d^+)$ as claimed.

Since $O_{\omega,C,A}$ is just the identity operator on $M$, we also have

$$
\|O_{\omega,C,A}f\|^2_M = \|f\|^2_M.
$$

Thus we can view $M$ as $M = \text{Ran} \ O_{\omega,C,A}$ (with $O_{\omega,C,A}$ viewed as an operator from $M$ into $H^2_{\omega,Y}(F_d^+)$). We can then follow the same argument as used in the proof of part (3) of Theorem 4.33 to identify the reproducing kernel $K_M$ for $M$ as $K_M(z,\zeta) = K_{C,A}(z,\zeta)$, as claimed. \(\square\)

**Definition 4.34.** Let us call a subspace $M$ contained in $H^2_{\omega,Y}(F_d^+)$ (not necessarily isometrically or even contractively) an $\omega$-model subspace if it is $S^*_{n,R}$-invariant and the restricted $\omega$-model pair $(E|_M, S^*_{\omega,R}|M)$ is $\omega$-contractive.
As explained by part (4) of Theorem 4.33, for purposes of study of contractively included \( \omega \)-model subspaces of \( H^2_{\omega,Y}(F^+_d) \) without loss of generality we may suppose at the start that we are working with \( \mathcal{X} \) as the original state space and with the solution \( H \) of Stein inequalities (4.17), (4.18) to be normalized to \( H = I_M \). Then certain simplifications occur in parts (1)-(3) of Theorem 4.33 as explained in the hypotheses and indeed gets \( H \) as a conclusion. Note that part (4) of Theorem 4.33 does not involve a choice of \( H \) in the hypotheses and indeed gets \( H = I_M \) as a conclusion.

**Theorem 4.35.** Suppose that \( (C, A) \) is an \( \omega \)-contractive pair with state space \( \mathcal{X} \) and output space \( \mathcal{Y} \). Then:

1. \( (C, A) \) is \( \omega \)-output-stable and \( \text{Ran} \, O_{\omega,C,A} \) is \( S^*_{\omega,R,M} \)-invariant (by 4.103).
2. The operator \( O_{\omega,C,A} \) is a contraction from \( \mathcal{X} \) into \( H^2_{\omega,Y}(\mathbb{F}^+_d) \).
3. If the linear manifold \( M := \text{Ran} \, O_{\omega,C,A} \) is given the lifted norm \( \|O_{\omega,C,A}x\|_M = \|Qx\|_X \) (4.117) where \( Q \) is the orthogonal projection of \( \mathcal{X} \) onto \( (\text{Ker} \, O_{\omega,C,A})^\perp \), then
   - \( O_{\omega,C,A} \) is a coisometry of \( \mathcal{X} \) onto \( M \), and implements a unitary equivalence between \( S^*_{\omega,R,M} \) and \( QA|_{\text{Ran} \, Q} \).
   - \( M \) is contained contractively in \( H^2_{\omega,Y}(\mathbb{F}^+_d) \) and is isometrically equal to the FNRKHS with reproducing kernel given by
     \[ K_{C,A}(z, \zeta) = CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^* \]
   - The pair \( (E|_M, S^*_{\omega,R,M}) \) is \( \omega \)-contractive and the orthogonal projection \( Q \) in (4.117) satisfies Stein inequalities (4.17), (4.18). Moreover, \( (E|_M, S^*_{\omega,R,M}) \) is \( \omega \)-isometric if and only if \( Q \) satisfies (4.19).
   - In particular, if \( (C, A) \) is \( \omega \)-observable, then \( A \) in unitarily equivalent to \( S^*_{\omega,R,M} \) and \( (C, A) \) is an \( \omega \)-contractive pair. If \( Q = I \) satisfies (4.19), then \( (C, A) \) is an \( \omega \)-isometric output pair.
   - If \( (C, A) \) and \( \tilde{C}, \tilde{A} \) are two \( \omega \)-output-stable, observable pairs (with \( C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) and \( \tilde{C} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y}) \) realizing the same positive kernel
     \[ K_{C,A}(z, \zeta) := CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^* = \tilde{C}R_\omega(Z(z)\tilde{A})R_\omega(Z(\zeta)\tilde{A})^*\tilde{C}^* =: K_{\tilde{C},\tilde{A}}(z, \zeta) \]
     then \( (C, A) \) and \( (\tilde{C}, \tilde{A}) \) are unitarily equivalent, i.e., equalities (4.33) hold for a unitary operator \( T : \mathcal{X} \to \tilde{\mathcal{X}} \).

**Proof.** As for parts (1), (2), (3a), (3b), (3c) of the theorem, all statements but the last part of statement (3c) concerning \( Q \) are direct specializations to the case \( H = I_M \) of the corresponding results in Theorem 4.33. To complete the verification of the last part of (3c), observe from the intertwining relations (4.105) and (4.113) that inequalities

\[
\sum_{j=1}^d \|S^*_{\omega,R,M} f\|_M^2 \leq \|f\|_M^2, \quad \sum_{\alpha \in \mathbb{F}_d^+} c_{|\alpha|} \|S^*_{\omega,R,M} f\|_M^2 \geq \|f\|_M^2, \quad (4.119)
\]

\[
- \sum_{\alpha \in \mathbb{F}_d^+} \left( \frac{1}{\omega_{k+\ell}} \sum_{\ell=1}^k c_{|\alpha|+\ell} \right) \|S^*_{\omega,R,M} f\|_M^2 \geq 0 \quad \text{for all } f \in \mathcal{M} \text{ and } k \geq 1
\]
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By definition (4.117) of the operator \( f = O_{\omega,C,A} x \in \mathcal{M} \) mean that

\[
\sum_{j=1}^{d} \|O_{\omega,C,A} A_j x\|_{\mathcal{M}}^2 \leq \|O_{\omega,C,A} x\|_{\mathcal{M}}^2 \quad \text{for all } x \in \mathcal{X}
\]

\[
\sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{k=1}^{d} c_{[\alpha]} \cdot \|O_{\omega,C,A} A^\alpha x\|_{\mathcal{M}}^2 \right) \geq \|C x\|_{\mathcal{M}}^2,
\]

\[
0 \geq \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{k=1}^{d} c_{[\alpha]} \cdot \|O_{\omega,C,A} A^\alpha x\|_{\mathcal{M}}^2 \right)
\]

By definition (4.117) of the \( \mathcal{M} \)-norm, the latter relations can be written as

\[
\|Q x\|_{\mathcal{M}}^2 \geq \sum_{j=1}^{d} \|QA_j x\|_{\mathcal{M}}^2,
\]

\[
\|C x\|_{\mathcal{M}}^2 \leq \sum_{\alpha \in \mathbb{F}_d^+} c_{[\alpha]} \cdot \|QA^\alpha x\|_{\mathcal{M}}^2 = \sum_{\alpha \in \mathbb{F}_d^+} c_{[\alpha]} \cdot \|QA^\alpha x\|_{\mathcal{M}}^2,
\]

\[
0 \geq \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{k=1}^{d} c_{[\alpha]} \cdot \|QA^\alpha x\|_{\mathcal{M}}^2 \right)
\]

where we used the substitution \( \alpha \mapsto \alpha^\top \) in the two last formulas. Since \( Q = Q^\frac{1}{2} \) and since \( x \in \mathcal{X} \) is arbitrary, the latter relations can be written as

\[
Q \geq \sum_{j=1}^{d} A_j^\top QA_j, \quad \Gamma_{\omega,A}[Q] \geq C^* C, \quad \Gamma_{\omega,A}^{(k)}[Q] \geq 0 \quad (k \geq 1)
\]

telling us that \( Q \) satisfies Stein inequalities (4.117), (4.118). Note next that the pair \((E|_{\mathcal{M}}, S_{\omega,R}|_{\mathcal{M}})\) being \( \omega \)-isometric is equivalent to equality in the second relation in (4.119) (for all \( f \in \mathcal{M} \)) which is equivalent to equality in (4.120) (for all \( x \in \mathcal{X} \)), which in turn is equivalent to the equality \( \Gamma_{\omega,A}[Q] = C^* C \), which completes the proof of parts (1)-(3c) of the theorem. Statement (3d) amounts to the specialization of the last part of (3a) to the case where \( Q = I_\mathcal{X} \).

As for statement (3e), suppose that the output pairs \((C,A)\) and \((\tilde{C}, \tilde{A})\) generate the same kernels as in (4.118). Equating coefficients of \( z^{\alpha} \tilde{z}^{\beta} \) gives us the system of equations

\[
\omega_{[\alpha]}^{-1} C A^\alpha A^{\beta \top} C^* = \omega_{[\beta]}^{-1} C \tilde{A}^\alpha \tilde{A}^{\beta \top} \tilde{C}^* \quad \text{for all } \alpha, \beta \in \mathbb{F}_d^+,
\]

or more simply, after cancellation of the common factor \( \omega_{[\alpha]}^{-1} \omega_{[\beta]} \),

\[
C A^\alpha A^{\beta \top} C^* = \tilde{C} \tilde{A}^\alpha \tilde{A}^{\beta \top} \tilde{C}^* \quad \text{for all } \alpha, \beta \in \mathbb{F}_d^+.
\]

We conclude that the operator \( U \) defined by

\[
U: A^{\beta \top} C^* y \mapsto \tilde{A}^{\beta \top} \tilde{C}^* y \quad (4.121)
\]

extends by linearity and continuity to an isometry from its domain space

\[
\mathcal{D}_U = \sqrt{\{A^{\beta \top} C^* y: \beta \in \mathbb{F}_d^+, y \in \mathcal{Y}\}}
\]

onto its range space

\[
\mathcal{R}_U = \sqrt{\{\tilde{A}^{\beta \top} \tilde{C}^* y: \beta \in \mathbb{F}_d^+, y \in \mathcal{Y}\}}.
\]
The observability assumptions imply that \( D_U \) is all of \( \mathcal{X} \) and \( R_U \) is all of \( \overline{\mathcal{X}} \), and hence \( U: \mathcal{X} \to \overline{\mathcal{X}} \) is unitary. From the formula (4.121) we can read off the intertwining relations
\[
UC^* = C^*, \quad UA_j^* = A_j^* U \quad \text{for} \quad j = 1, \ldots, d.
\]
Since \( U \) is unitary we then also get
\[
\overline{CU} = C, \quad \overline{A_j}U = U A_j \quad \text{for} \quad j = 1, \ldots, d
\]
and we conclude the pairs \( (C, A) \) and \( (\overline{C}, \overline{A}) \) are unitarily equivalent as claimed.

We remark that this proof is essentially the same as that of Theorem 2.13 in [20] where the special case \( \omega = \mu_1 \) is handled.

We now turn to \( S_{\omega,R}^* \)-invariant subspaces of \( H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) that are isometrically included in \( H^2_{\omega,\gamma}(\mathbb{F}_d) \).

**Theorem 4.36.** If the pair \( (C, A) \) is an \( \omega \)-isometric pair with \( A \) strongly \( \omega \)-stable, then the observability operator \( O_{\omega,C,A}: \mathcal{X} \to H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) is an isometry onto a backward-shift-invariant subspace \( \mathcal{N} \subset H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) and \( A \) is unitarily equivalent to \( S_{\omega,R}|\mathcal{N} \). Moreover \( \mathcal{N} \) is the NFRKHS with reproducing kernel
\[
K_{\omega,C,A}(z, \zeta) = CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^*.
\] (4.122)
Conversely, if \( \mathcal{N} \) is a Hilbert space isometrically contained in \( H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) which is invariant under \( S_{\omega,R}^* \), then there is an \( \omega \)-isometric output pair \( (C, A) \) with \( A \) strongly \( \omega \)-stable so that \( \mathcal{N} = \text{Ran} O_{\omega,C,A} \) and \( A \) is unitarily equivalent to \( S_{\omega,R}|\mathcal{N} \), and the space \( \mathcal{N} \) is isometrically equal to the NFRKHS with reproducing kernel as in (4.122). In fact, one can choose
\[
C = E|\mathcal{N}, \quad A = S_{\omega,R}|\mathcal{N},
\] (4.123)
where \( E \) is the model output map on \( H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) given by \( E: \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_0 \).

**Proof.** Assume that \( (C, A) \) is an \( \omega \)-isometric output pair with \( A \) strongly \( \omega \)-stable. Then part (2) of Lemma (4.10) tells us that \( O_{\omega,C,A}: \mathcal{X} \to H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) is isometric, and hence \( O_{\omega,C,A} \) can be viewed as a unitary operator from \( \mathcal{X} \) onto its range spaces \( \mathcal{N} := \text{Ran} O_{\omega,C,A} \). From the general intertwining relation (4.105) we see that \( \mathcal{N} \) is \( S_{\omega,R}^* \)-invariant and that \( A \) is unitarily equivalent to \( S_{\omega,R}|\mathcal{N} \) via the unitary transformation \( O_{\omega,C,A}: \mathcal{X} \to \mathcal{N} \). The fact that \( \mathcal{N} \) has reproducing kernel of the form (4.122) is a direct consequence of Proposition (2.3).

Conversely, suppose that \( \mathcal{N} \) is a backward-shift-invariant subspace of \( H^2_{\omega,\gamma}(\mathbb{F}_d^+) \) isometrically contained in \( H^2_{\omega,\gamma}(\mathbb{F}_d) \). It suffices to check that \( (C, A) \) as in (4.123) meets all the requirements of the theorem. Proposition (4.23) part (2) tells us that \( O_{E,S_{\omega,R}} = I_{H^2_{\omega,\gamma}(\mathbb{F}_d)} \) and hence \( O_{C,A} = O_{E,S_{\omega,R}|\mathcal{N}} = I_{\mathcal{N}} \) is an isometry from \( \mathcal{X} \) to \( \mathcal{N} \subset H^2_{\omega,\gamma}(\mathbb{F}_d) \). To show that \( A \) is a \( \omega \)-strongly stable \( \omega \)-hypercontraction, it suffices to note that \( S_{\omega,R}^* \) is (by Proposition (4.24)), and then observe that restriction of \( S_{\omega,R}^* \) to an invariant subspace preserves these properties. Similarly, by Proposition (4.24) the pair \( (E, S_{\omega,R}) \) is \( \omega \)-isometric; it then suffices to observe that the \( \omega \)-isometric property is preserved upon restriction of \( (E, S_{\omega,R}) \) to a \( S_{\omega,R}^* \)-invariant subspace. \( \square \)
Theorem 4.37. Suppose that the Hilbert-space operator tuple $A \in \mathcal{L}(X)^d$ is an $\omega$-strongly stable $\omega$-hypercontraction. Let $\mathcal{Y}$ be a coefficient Hilbert space with $\dim \mathcal{Y} = \text{rank} \Gamma_{\omega,A}[I_X]$. Then there is a subspace $N \subset H^2_{\omega,\mathcal{Y}}(F_d^+)$ invariant under $S^*_{\omega,R}$ so that $A$ is unitarily equivalent to $S^*_{\omega,R}|N$.

Proof. Since $A$ is $\omega$-hypercontractive, $\Gamma_{\omega,A}[I_X]$ is positive semidefinite. We choose $C \in \mathcal{L}(X,\mathcal{Y})$ so that $C^*C = \Gamma_{\omega,A}[I_X]$. Then $(C,A)$ is an $\omega$-isometric pair. Since $A$ is $\omega$-strongly stable, the operator $O_{\omega,C,A}: \mathcal{X} \to H^2_{\omega,\mathcal{Y}}(F_d^+)$ is isometric, by Lemma 4.10. We set $N = \text{Ran} O_{\omega,C,A} \subset H^2_{\omega,\mathcal{Y}}(F_d^+)$. Then the intertwining property (4.105) leads to the conclusion that $A$ is unitarily equivalent to $S^*_{\omega,R}|N$ via the unitary similarity transformation $O_{\omega,C,A}: \mathcal{X} \to N$.

Remark 4.38. Since unitary similarity preserves stability properties of operator tuples, as a consequence of Theorem 4.37 combined with the fact that $S^*_{\omega,R}$ is strongly stable (by part (1) in Proposition 4.23), we conclude that $\omega$-strong stability of an $\omega$-hypercontraction $A$ implies its strong stability in the usual sense (1.30). The converse direction (strong stability in the usual sense implies $\omega$-strong stability for a $\omega$-hypercontraction) is true at least for the special case where $\omega = \mu_n$ for some $n \in \mathbb{N}$ (see Remark 4.42 below).

4.5.1. Shifted $\omega$-observability operator range spaces. For our subsequent constructions, we will need the range spaces associated with shifted $\omega$-observability operators (4.57):

$$\text{Ran} O_{\omega,k,C,A} = \{CR_{\omega,k}(Z(z)A)x : \ x \in \mathcal{X}\}.$$  

We first observe a factorization of the shifted $\omega$-gramian (4.58) in terms of the corresponding shifted $\omega$-observability operator.

Proposition 4.39. Let $(C, A)$ be a $\omega$-output-stable pair and let $O_{\omega,k,C,A}$ and $O_{\omega,k,C,A}$ be defined as in (4.57), (4.58). Then

$$\|S^*_{\omega,R} O_{\omega,[\alpha],C,A} x\|^2_{H^2_{\omega,\mathcal{Y}}(F_d^+)} = \langle O_{\omega,[\alpha],C,A} x, x \rangle_{\mathcal{X}}$$

(4.124)

for every $\alpha \in F_d^+$ and $x \in \mathcal{X}$, and hence we have the operator factorization

$$O_{\omega,[\alpha],C,A} = O_{\omega,[\alpha],C,A} S^\alpha_{\omega,R} S^\alpha_{\omega,R} O_{\omega,[\alpha],C,A}.$$  

Proof. By definition (4.57), we have

$$S^\alpha_{\omega,R} O_{\omega,[\alpha],C,A} x = \sum_{\alpha' \in F_d^+} \omega^{-1}_{[\alpha'] + [\alpha]} (CA\alpha') z^{\alpha'}.$$  

Making use of the latter formula and invoking the definition of the inner product in $H^2_{\omega,\mathcal{Y}}(F_d^+)$, we verify (4.124) as follows:

$$\|S^*_{\omega,R} O_{\omega,[\alpha],C,A} x\|^2_{H^2_{\omega,\mathcal{Y}}(F_d^+)} = \sum_{\alpha' \in F_d^+} \omega^{-1}_{[\alpha'] + [\alpha]} \langle CA\alpha', CA\alpha' \rangle_{\mathcal{X}}$$

$$= \langle \sum_{\alpha' \in F_d^+} \omega^{-1}_{[\alpha'] + [\alpha]} A^* \alpha' T^* CA\alpha', x \rangle_{\mathcal{X}}$$

$$= \langle O_{\omega,[\alpha],C,A} x, x \rangle_{\mathcal{X}},$$

where the last equality is clear from the definition (4.58) of $O_{\omega,[\alpha],C,A}$.  

We next represent the range of a $k$-shifted observability operator as a noncommutative formal reproducing kernel Hilbert space.

**Proposition 4.40.** Suppose that the pair $(C, A)$ is a $\omega$-output-stable and exactly $\omega$-observable. Then, for any $\beta \in \mathbb{F}_d^+$ the subspace

$$S_{\omega,R}^{\beta^T} \text{Ran} \mathcal{O}_{\omega,|\beta|,C,A} \subset H_\omega^2(\mathbb{F}_d^+)$$

(with inner product induced by $H_\omega^2(\mathbb{F}_d^+)$) is a NFRKHS with reproducing kernel $\mathcal{R}_\beta(z, \zeta)$ given by

$$\mathcal{R}_\beta(z, \zeta) = CR_{\omega,|\beta|}(Z(z)A)(z^{\beta^T}\mathbf{G}_{\omega,|\beta|,C,A}^{-1}R_{\omega,|\beta|}(Z(\zeta)A)^*C^*).$$

**Proof.** Due to equality (4.124), a direct application of Proposition 2.3 shows that the formal reproducing kernel for the space (4.125) is given by

$$\left(S_{\omega,R}^{\beta^T} \mathcal{O}_{\omega,|\beta|,C,A}(z)\mathbf{G}_{\omega,|\beta|,C,A}^{-1}(S_{\omega,R}^{\beta^T} \mathcal{O}_{\omega,|\beta|,C,A}(\zeta))\right)^*.$$  

4.5.2. **Range spaces of $n$-observability operators.** In this section we will specialize the previous general results to the case $\omega = \mu_n$, where certain simplifications occur. The only result that will be established independently (and, presumably, does not hold for general weights) is the characterization of $S_{n,R}^*$-invariant spaces isometrically included in $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$. Specializing part (4) of Theorem 4.33 to the present case, we see that without loss of generality we may start with the solution $H$ of Stein inequalities (4.50) to be normalized to $H = I_X$. Recall from [14] Definition 4.7 that we say that the output pair $(C, A)$ is $n$-contractive if (i) $A$ is $n$-hypercontractive and in addition $\Gamma_{n,A}[I_X] \geq C^*C$.

**Theorem 4.41.** Suppose that $(C, A)$ is an $n$-contractive pair with state space $\mathcal{X}$ and output space $\mathcal{Y}$. Then:

1. $(C, A)$ is $n$-output-stable and $\text{Ran} \mathcal{O}_{n,C,A} = S_{n,R}^*$-invariant.
2. The operator $\mathcal{O}_{n,C,A}$ is a contraction from $\mathcal{X}$ into $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$. Moreover, $\mathcal{O}_{n,C,A} : \mathcal{X} \to \mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ is an isometry if and only if $(C, A)$ is an $n$-isometric pair and $A$ is strongly stable.
3. If the manifold $\mathcal{M} := \text{Ran} \mathcal{O}_{n,C,A}$ is given the lifted norm $\|\mathcal{O}_{n,C,A}x\|_\mathcal{M} = \|Qx\|_\mathcal{X}$ where $Q$ is the orthogonal projection of $\mathcal{X}$ onto $(\text{Ker} \mathcal{O}_{n,C,A})^\perp$, then
   a. $\mathcal{M}$ is contained contractively in $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ and is isometrically equal to the FNRKHS with reproducing kernel $K_{C,A}(z, \zeta) = C(I - Z(z)A)^{-n}(I - A^*Z(\zeta)^*)^{-n}C^*$.
   b. The $d$-tuple $S_{n,R}^*|_\mathcal{M}$ is contractive on $\mathcal{M}$ and the inequality
      $$\sum_{\alpha \in \mathbb{F}_d^+: |\alpha| \leq n} (-1)^{|\alpha|} \binom{n}{|\alpha|} \cdot \|S_{n,R}^*f\|_\mathcal{M}^2 \geq \|f_0\|_\mathcal{Y}^2$$
      holds for all $f \in \mathcal{M}$. Moreover, (4.127) holds with equality if and only the orthogonal projection $Q$ of $\mathcal{X}$ onto $(\text{Ker} \mathcal{O}_{n,C,A})^\perp$ is subject to relations
      $$Q \geq \sum_{j=1}^d A_j^*QA_j \quad \text{and} \quad \Gamma_{n,A}[Q] = C^*C.$$
4.5. OBSERVABILITY-OPERATOR RANGE SPACES

(4) Conversely, let $\mathcal{M}$ be a Hilbert space included in $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ such that

(i) $\mathcal{M}$ is invariant under the backward shift $S^*_{n,R}$.
(ii) the restricted tuple $S^*_{n,R}|_{\mathcal{M}}$ is contractive, and the inequality (4.127) holds for all $f \in \mathcal{M}$.

Then it follows that $\mathcal{M}$ is contractively included in $\mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ and there exists an $n$-contractive pair $(C, A)$ such that

$\mathcal{M} = \mathcal{H}(K_{C,A}) = \text{Ran} \mathcal{O}_{n,C,A}$

isometrically. If (4.127) holds with equality, then $(C, A)$ can be taken to be an $n$-isometric pair. For example, one may take $\mathcal{X} = \mathcal{M}$, $A = S^*_{n,R}|_{\mathcal{M}}$, $C = E|_{\mathcal{M}}$.

Proof. We will show that $\mathcal{O}_{n,C,A} : \mathcal{X} \to \mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ is an isometry if and only if $(C, A)$ is an $n$-isometric pair and $A$ is strongly stable. Indeed, since the pair $(C, A)$ is $n$-contractive, the operator $H = I_X$ satisfies the Stein system (4.49). Thus, equalities (4.52) hold for all $N \geq 0$ and for $H = I_X$. It follows from (4.52) that operators

$$\Lambda_{N,A} = \sum_{\alpha \in F^+_d : |\alpha| \leq N} (N+n+1) \alpha A^\alpha \Gamma_{n+N-|\alpha|,A} |I_X| A^\alpha$$

(4.128)

satisfy $\Lambda_{N,A} \geq \Lambda_{N+1,A} \geq 0$ for all $N \geq 0$ and therefore, the limit

$$\Delta_A = \lim_{N \to \infty} \Lambda_{N,A} \geq 0$$

(4.129)

exists in the strong sense. Since $C^*C \leq \Gamma_{n,A} |I_X|$ and $B_A$ is a positive map, we have

$$\sum_{\alpha \in F^+_d : |\alpha| \leq N} \left( \binom{n+|\alpha|-1}{|\alpha|} C^*CA^\alpha \right) \leq \sum_{\alpha \in F^+_d : |\alpha| \leq N} \left( \binom{n+|\alpha|-1}{|\alpha|} \alpha \right) \Gamma_{n,A} |I_X| A^\alpha$$

$$= I_X - \Lambda_{N,A},$$

where the last equality holds due to (4.52). The latter relations are equivalent to

$$0 \leq \sum_{\alpha \in F^+_d : |\alpha| \leq N} \left( \binom{n+|\alpha|-1}{|\alpha|} \alpha \right) \Gamma_{n,A} |I_X| - C^*C A^\alpha$$

$$= I_X - \Lambda_{N,A} - \sum_{\alpha \in F^+_d : |\alpha| \leq N} \left( \binom{n+|\alpha|-1}{|\alpha|} \alpha \right) C^*CA^\alpha.$$  (4.130)

Letting $N \to \infty$ in (4.130) we get, on account (4.45) and (4.129),

$$0 \leq \sum_{\alpha \in F^+_d : |\alpha| = N+n} \alpha \Gamma_{0,A} |I_X| A^\alpha = \lim_{N \to \infty} \sum_{\alpha \in F^+_d : |\alpha| = N+n} \alpha A^\alpha = 0.$$  (4.131)

By definition, $\mathcal{O}_{n,C,A} : \mathcal{X} \to \mathcal{A}_{n,Y}(\mathbb{F}_d^+)$ being an isometry means that $G_{n,C,A} = I_X$ in which case (4.131) implies $\Delta_A = 0$ and equalities throughout (4.128). Since all the terms on the right side of (4.128) are positive semidefinite, condition $\Delta_A = 0$ implies, in particular, that

$$\lim_{N \to \infty} \sum_{\alpha \in F^+_d : |\alpha| = N+n} \alpha \Gamma_{0,A} |I_X| A^\alpha = \lim_{N \to \infty} \sum_{\alpha \in F^+_d : |\alpha| = N+n} \alpha A^\alpha = 0$$

which just means that $A$ is strongly stable. Since all the terms in the series in (4.131) are positive semidefinite, we conclude that each term equals zero. The term
corresponding to the index $\alpha = \emptyset$ is $\Gamma_{n,A}[I_X] = C^*C$. Hence $\Gamma_{n,A}[I_X] = C^*C$ and thus the Stein equation in (4.51) holds.

Conversely, by reversing the steps of the preceding argument, we see that $A$ being strongly stable and the Stein equation in (4.51) holding leads to $G_{n,C,A} = I_X$, i.e., to $\mathcal{O}_{n,C,A} : X \to A_{n,Y}(F_d^+)$ being an isometry. \hfill \square

As we know from Remark 4.38, a $\omega$-strongly stable $\omega$-hypercontraction is necessarily strongly stable in the usual sense. We do not know if the converse holds true in general. However, in the case $\omega = \mu_n$ it does!

**Remark 4.42.** We note here that strong stability and $\mu_n$-strong stability are equivalent for $\mu_n$-hypercontractions. Indeed, one direction ($\mu_n$-strong stability implies the usual strong stability) holds even for a general admissible weight $\omega$ by Remark 4.38. Thus it remains only to verify that any strongly stable $\mu_n$-hypercontraction $A$ (which is an $n$-hypercontraction by Proposition 4.15) is also $\mu_n$-strongly stable. To this end, note that by part (2) in Theorem 4.41, the strong stability of an $n$-hypercontraction $A$ implies that $A$ is unitarily equivalent to the restriction of $S_{n,R}^*$ to an invariant subspace $N \subset H^2_{\mu_n,Y}(F_d^+)$ for a suitable coefficient Hilbert space $Y$ as in Theorem 4.37. But by Proposition 4.24, $S_{n,R}^*$ is $\mu_n$-strongly stable. Putting all this together, we see that strong stability for a $\mu_n$-hypercontraction implies $\mu_n$-strong stability.

**Remark 4.43.** For the standard-weight case, Theorem 4.33 avoided the exact observability statement in part (2) of the statement and had the conclusion that the space $M$ with lifted norm is contractively included in the Bergman space $A_{n,Y}(F_d^+)$ rather than being isometrically included in $A_{n,Y}(F_d^+)$. For this reason Theorem 4.36 is only a partial $\omega$-analog of Theorem 4.33.
CHAPTER 5

Beurling-Lax theorems based on contractive multipliers

In Section 4.3 we characterized $S^\ast_{\omega,R}$-invariant spaces which are contractively or, more specifically, isometrically included in $H^2_{\omega,Y}(F^+_d)$. In this and in the next two chapters we focus on the spaces that are invariant under the forward-shift operator-tuple $S_{\omega,R}$, thereby fleshing out theme #2 from the end of Section 4.1 for the $\omega$-setting. The material from Chapter 4 on backward-shift-invariant subspaces and ranges of observability operators is relevant here in at least two respects:

- Given $\mathcal{M}$ contractively included in $H^2_{\omega,Y}(F^+_d)$ which is invariant under the forward shift-tuple $S_{\omega,R}$, we may set $A = (S_{\omega,R}|_{\mathcal{M}})^\ast$ and (under appropriate hypotheses) choose $C: \mathcal{M} \rightarrow \mathcal{U}$ so that $(C,A)$ is an $\omega'$-isometric output pair (for some weight sequence $\omega'$; interesting special cases include $\omega' = \omega$ and $\omega' = \mu_1$). If $A$ is also $\omega'$-strongly stable, then the observability operator $O_{\omega',C,A}$ is isometric from $\mathcal{M}$ into $H^2_{\omega,Y}(F^+_d)$ and implements a unitary equivalence between $A$ and $S^\ast_{\omega,R}|_{\text{Ran}O_{\omega',C,A}}$, since

$$S^\ast_{\omega,R,j}O_{\omega',C,A} = O_{\omega',C,A}A_j \quad \text{for } j = 1, \ldots, d$$

(see Theorem 4.36). For our application here we prefer to view the codomain of $O_{\omega',C,A}$ as all of $H^2_{\omega',C,A}$ rather than only $\text{Ran}O_{\omega',C,A}$. Taking adjoints in the above intertwining relation gives

$$O^\ast_{\omega',C,A}S^\ast_{\omega,R,j} = A^\ast_j O^\ast_{\omega',C,A}, \quad \text{where} \quad A^\ast_j = S_{\omega,R,j}|_{\mathcal{M}}.$$  

Thus $(O_{\omega',C,A})^\ast$ intertwines the respective shift tuples forcing it to have the form $M_\Theta$ for $\Theta$ a contractive multiplier from $H^2_{\omega,Y}(F^+_d)$ to $H^2_{\omega,Y}(F^+_d)$. The conclusion is that $\Theta$ so constructed serves as the Beurling-Lax representer for the forward-shift-invariant subspace $\mathcal{M}$. A finer analysis shows that $M_\Theta$ considered as an operator from $H^2_{\omega,Y}(F^+_d)$ into $H^2_{\omega,Y}(F^+_d)$ is a partial isometry in case $\mathcal{M}$ is isometrically included in $H^2_{\omega,Y}(F^+_d)$. For complete details, see Theorems 5.1, 5.3 and 5.8 below.

- If $\mathcal{M}$ is a forward-shift-invariant subspace isometrically contained in $H^2_{\omega,Y}(F^+_d)$, then its orthogonal complement $\mathcal{M}^\perp$ is a backward-shift-invariant subspace isometrically contained in $H^2_{\omega,Y}(F^+_d)$ for which results of Chapter 4 apply. Specifically, if we choose $(C,A)$ to be an $\omega$-isometric output pair with $A$ $\omega$-strongly stable such that $\mathcal{M}^\perp = \text{Ran}O_{\omega,C,A}$ as in the converse direction of Theorem 4.36 then the reproducing kernel $k_{\mathcal{M}^\perp}$ for $\mathcal{M}^\perp$ has the explicit form

$$k_{\mathcal{M}^\perp}(z,\zeta) = CR_\omega(Z(z)A)(R_\omega(Z(\zeta)A))^\ast C^\ast.$$ 

Then by standard calculus of reproducing kernels, we see that the reproducing kernel for $\mathcal{M}$ is given by

$$k_{\mathcal{M}}(z,\zeta) = k_{\omega,nc}(z,\zeta) \otimes I_Y - CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^\ast C^\ast. \quad (5.1)$$
In the case where $\omega = \mu_1$ (the Fock-space case), the Cholesky-factorization construction yields a pair of operators $B: \mathcal{U} \to \mathcal{X}$ and $D: \mathcal{Y} \to \mathcal{Y}$ so that $U := \begin{bmatrix} D & B \\ \mu_1 \end{bmatrix}$ is unitary from which it follows that we get the factorization

$$k_M(z, \zeta)I_Y = \Theta(z)(k_{\mathcal{X}, \mathcal{X}}(z, \zeta)I_\mathcal{U})\Theta(\zeta)^*$$

with $\Theta(z) = D + C(I - zA)^{-1}B$.

It then can be shown that $\Theta$ is a strictly inner multiplier which serves as the Beurling-Lax representer: $M = \Theta \cdot H^2_0(\mathbb{F}_d^+)$. In the case where $\omega = \mu_1$ (the standard weighted Bergman-Fock-space case), a certain iterative procedure leads to the representation

$$k_M(z, \zeta) = \sum_{j=1}^{n} F_j(z)k_{\mathcal{X}, \mathcal{X}}(z, \zeta)F_j(\zeta)^*$$

with $F_j$ being contractive multipliers from $A_{j;\mathcal{U},\mathcal{X}}(\mathbb{F}_d^+)$ to $A_{\mathcal{X},\mathcal{Y}}(\mathbb{F}_d^+)$ giving rise to Beurling-Lax representations for $S_{\omega, R}$-invariant subspaces of $A_{\mathcal{X},\mathcal{Y}}(\mathbb{F}_d^+)$ with model space equal to the direct sum of $A_{j;\mathcal{U},\mathcal{X}}(\mathbb{F}_d^+)$ for $j = 1, \ldots, n$; details are given in Section 5.2. Further elaboration of all these ideas will lead us (in Chapter 7 below) to the state-space realization formulas for the $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$-Bergman-inner family (see Definition 7.11 to come) representing a given forward-shift-invariant subspace of $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$.

### 5.1. Beurling-Lax theorem via McCT-inner multipliers and more general contractive multipliers

Beurling-Lax representations based on partially isometric multipliers (for the case of isometric inclusion) or more general contractive multipliers (for the case of contractive inclusion) appeared first in the work of Arveson [11] and in a more systematic operator-theoretic framework in McCullough-Trent [66] in the context of the Drury-Arveson space (the commutative version of $H^2_{\mu_1}$). In this section, we present noncommutative versions of such representations for shift-invariant subspaces in a weighted Hardy-Fock space $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ for an admissible weight $\omega$ as in (2.31). We shall in fact prove a more general version for contractively included (rather than isometrically included) subspaces of $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ due in the classical setting to de Branges (see [35]) and appearing in the noncommutative Fock-space setting (i.e., for $n = 1$ and $\omega = \mu_1$) in [20].

**Theorem 5.1.** A Hilbert space $\mathcal{M}$ is such that

1. $\mathcal{M}$ is contractively included in $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$,
2. $\mathcal{M}$ is $S_{\omega, R}$-invariant,
3. the $d$-tuple $T = (T_1, \ldots, T_d)$ where $T_j = (S_{\mathcal{X}, \mathcal{Y}}M)_{\mathcal{Y}}$ for $j = 1, \ldots, d$ is a row contraction

if and only if there is a coefficient Hilbert space $\mathcal{U}$ and a contractive multiplier $\Theta$ from $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ to $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ so that

$$\mathcal{M} = M_{\Theta}H^2_{\mathcal{U}}(\mathbb{F}_d^+)$$

(5.2)

with lifted norm

$$\|M_\Theta f\|_{\mathcal{M}} = \|Qf\|_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)}$$

(5.3)

where $Q$ is the orthogonal projection onto $(\text{Ker} M_\Theta)^\perp$. 
The latter relations also imply
\[ 0 \preceq \Pi \preceq \mathbb{I}, \quad \Pi - \sum_{j=1}^{d} S_{\omega, R, j} \Pi S_{\omega, R, j}^* \succeq 0. \]  
(5.4)

**Proof.** To verify sufficiency, let us suppose that \( \mathcal{M} \) is of the form \((5.2)\) for a contractive multiplier \( \Theta \) with \( \mathcal{M} \)-norm given by \((5.3)\). Since \( \| M_0 \| \leq 1 \), it follows that for any \( f \in H^2_\mathcal{G}(\mathbb{F}_d^+) \),

\[ \| M_0 f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)} = \| M_0 Q f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)} \leq \| Q f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)} = \| M_0 f \|_\mathcal{M}, \]

i.e., (1) holds. Property (2) follows from intertwining relations

\[ S_{\omega, R, j} M_0 = M_0 S_{1, R, j} \quad \text{for} \quad j = 1, \ldots, d. \]

The latter relations also imply \( M_0 S_{1, R, j}|_{\text{Ker} M_0} = 0 \), which can be written equivalently as \( QS_{1, R, j}(I - Q) = 0 \) where \( Q \) the orthogonal projection from \((5.3)\). Thus,

\[ QS_{1, R, j} = QS_{1, R, j} Q \quad \text{and} \quad S_{1, R, j}^* Q = QS_{1, R, j}^* Q \quad \text{for} \quad j = 1, \ldots, d. \]

(5.6)

Furthermore, for every \( f, g \in H^2_\mathcal{G}(\mathbb{F}_d^+) \) and \( j \in \{1, \ldots, d\} \), we have

\[ \langle M_0 g, T^*_j M_0 f \rangle_{\mathcal{M}} = (S_{\omega, R, j} M_0 g, M_0 f)_{\mathcal{M}} \\
= (M_0 S_{1, R, j} g, M_0 f)_{\mathcal{M}} \\
= (Q S_{1, R, j} Q g, f)_{H^2_\mathcal{G}(\mathbb{F}_d^+)} \\
= (Q g, S_{1, R, j}^* Q f)_{H^2_\mathcal{G}(\mathbb{F}_d^+)} = \langle M_0 g, M_0 S_{1, R, j}^* Q f \rangle_{\mathcal{M}}, \]

which implies that \( T^*_j : M_0 f \mapsto M_0 S_{1, R, j}^* Q f \). Iterating the latter formula and making use of relations \((5.6)\) we get

\[ T^{*\alpha} : M_0 f \mapsto M_0 (S_{1, R}^*)^\alpha Q f \quad \text{for} \quad \alpha \in \mathbb{F}_d^+. \]

(5.7)

It then follows that, for \( f \in H^2_\mathcal{G}(\mathbb{F}_d^+) \),

\[ \sum_{j=1}^{d} \| T^*_j M_0 f \|^2_{\mathcal{M}} = \sum_{j=1}^{d} \| M_0 S_{1, R, j}^* Q f \|_{\mathcal{M}}^2 \\
= \sum_{j=1}^{d} \| Q S_{1, R, j}^* Q f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)}^2 \\
\leq \sum_{j=1}^{d} \| S_{1, R, j}^* Q f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)}^2 \leq \| Q f \|_{H^2_\mathcal{G}(\mathbb{F}_d^+)}^2 \ |
\]

(5.8)

(inequalities above hold since \( Q \) is a projection and \((S_{1, R}^*) \) is a row contraction) which shows that \( T \) is a row contraction. This completes the proof of sufficiency.

Suppose now that the Hilbert space \( \mathcal{M} \) satisfies conditions (1), (2), (3) in the statement of the theorem. By Theorem 5.3, it follows that \( \mathcal{M} \) is a NFRKHS (noncommutative formal reproducing kernel Hilbert space) in its own right. As \( \mathcal{M} \subset H^2_\mathcal{G}(\mathbb{F}_d^+) \), the elements of \( \mathcal{M} \) are formal power series of the form \( f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \) with \( f_\alpha \in \mathcal{Y} \).
Thus $\mathcal{M}$ is a NFRKHS which is invariant under the right coordinate multipliers $T_j = S_{\omega, R, j}|_{\mathcal{M}}$ ($1 \leq j \leq d$) by assumption (2) and moreover, the tuple $T = (T_1, \ldots, T_d)$ is a row contraction, by assumption (3). We are therefore in a position to apply Proposition 2.8 to conclude that the reproducing kernel $k_{\mathcal{M}}(z, \zeta) \in \mathcal{L}(\mathcal{Y})\langle \langle z, \zeta \rangle \rangle$ for $\mathcal{M}$ has a factorization

$$k_{\mathcal{M}}(z, \zeta) = G(z)(k_{\text{nc, Sz}}(z, \zeta) \otimes I_U)G(\zeta)^*$$

for some coefficient Hilbert space $U$ and some power series $G(z) \in \mathcal{L}(U, \mathcal{Y})\langle \langle z \rangle \rangle$. Item (2) in Proposition 3.2 now tells us that $G$ is a coisometric multiplier from $H^2_d(F_d^+) \rightarrow \mathcal{M}$. Let us set $\Theta = G$ but consider $\Theta$ as a multiplier from $H^2_d(F_d^+) \rightarrow H^2_d(F_d^+)$ into $H^2_d(F_d^+)$. Then $M_\Theta = iM_\Theta$ where $i: \mathcal{M} \rightarrow H^2_d(F_d^+)$ is the inclusion map. Since $\mathcal{M}$ is contained in $H^2_d(F_d^+)$ contractively, $\|\| \leq 1$ and hence $\|M_\Theta\| \leq 1$, i.e., $\Theta$ is a contractive multiplier from $H^2_d(F_d^+)$ to $H^2_d(F_d^+)$.

Moreover, as $M_GH^2_d(F_d^+) = \mathcal{M}$, it is still the case that $M_GH^2_d(F_d^+) = \mathcal{M}$. The fact that $M_G: H^2_d(F_d^+) \rightarrow \mathcal{M}$ is a coisometry can be interpreted as saying that the $\mathcal{M}$-norm is given by the lifted-norm criterion (5.3).

We now suppose that the subspace $\mathcal{M}$ is presented as the pullback space $\mathcal{H}^p(\Pi)$ for some operator $\Pi$ on $H^2_d(F_d^+)$ satisfying conditions (5.4). The first condition in (5.3) then implies that $\mathcal{M} = \mathcal{H}^p(\Pi)$ is contractively included in $H^2_d(F_d^+)$. By the Douglas lemma (44), the second condition (5.4) implies that there is a row contraction $X = (X_1, \ldots, X_d)$ on $H^2_d(F_d^+)$ so that

$$S_{\omega, R, j}\Pi^\perp = \Pi^\perp X_j \quad \text{for } j = 1, \ldots, d. \quad (5.9)$$

In particular it follows that $\mathcal{H}^p(\Pi) = \text{Ran} \Pi^\perp$ is invariant under the tuple $S_{\omega, R}$. Furthermore we may redo the computations leading up to (5.6) and (5.7) for the present context as follows. From (5.3) we have $\Pi^\perp X_j|_{\text{Ker} \Pi} = 0$ which can be written as $QX_j(I - Q) = 0$, where $Q = P_{(\text{Ker} \Pi)^\perp}$. Thus $QX_j = QX_jQ$ and then for $T_j = S_{\omega, R, j}\mathcal{H}^p(\Pi)$ and $g, f \in H^2_d(F_d^+)$, we have (again due to (5.9))

$$\langle \Pi^\perp g, T_j^*\Pi^\perp f \rangle_{\mathcal{H}^p(\Pi)} = \langle S_{\omega, R, j}\Pi^\perp g, \Pi^\perp f \rangle_{\mathcal{H}^p(\Pi)} = \langle P^\perp X_jg, P^\perp f \rangle_{\mathcal{H}^p(\Pi)} = \langle QX_jg, f \rangle_{H^2_d(F_d^+)} = \langle QX_jQg, f \rangle_{H^2_d(F_d^+)} = \langle Qg, X_j^*Qf \rangle_{H^2_d(F_d^+)} = \langle \Pi^\perp g, \Pi^\perp X_j^*Qf \rangle_{\mathcal{H}^p(\Pi)}$$

from which we conclude that $T_j^*\Pi^\perp f = \Pi^\perp X_j^*Qf$. Hence the computations as in the derivation of (5.8) adapt to the present context as follows: for $f \in H^2_d(F_d^+)$, we have

$$\sum_{j=1}^d \|T_j^*\Pi^\perp f\|_{\mathcal{H}^p(\Pi)}^2 = \sum_{j=1}^d \|\Pi^\perp X_j^*Qf\|_{\mathcal{H}^p(\Pi)}^2 = \sum_{j=1}^d \|QX_j^*Qf\|_{H^2_d(F_d^+)}^2.$$
If we apply the transformation $X \in \mathbb{K}$, then we can compute the kernel function $\Theta(z) \in \mathbb{S}_{\omega,n}$ for a $\Sigma$ so that

$$\Theta(\alpha, \beta) = \Theta'_{\alpha, \beta} \Theta''_{\alpha, \beta} \quad \text{for all } \beta \in \mathbb{F}_d^+,$$

(5.10)

or equivalently, if and only if there is a partial isometry $U : \mathcal{U} \to \mathcal{U}'$ such that

$$\Theta'(z)U = \Theta(z), \quad \mathbf{K}(\Theta(z)^* \mathbf{K} \subseteq \mathbf{K}(\Theta(z)^* U)^*.$$

(5.11)

**Proof.** If $\Theta$ and $\Theta'$ are both Beurling-Lax representations for $\mathcal{M}$ as in Theorem 5.1, then we can compute the kernel function $K_M(z, \zeta) = \Theta(z) (k_{nc}, s_{\omega}(z, \zeta) I) \Theta(z)^* = \Theta'(z) (k_{nc}, s_{\omega}(z, \zeta) I) \Theta'(z)^*$. If we apply the transformation $X \mapsto X - \sum_{j=1}^d x_j z_j$ to both sides of the last equality, we get simply $\Theta(z) \Theta(z)^* = \Theta'(z) \Theta'(z)^*$ which, using the condition of $z_{\omega}^{\mathbb{F}_d^+}$, implies equalities (5.10). Hence there is a unitary map

$$U_0 : \mathcal{R} \to \mathcal{R}', \quad \mathbf{R} = \bigvee_{\beta \in \mathbb{F}_d^+} \mathbf{Ran} \Theta_{\beta}^*, \quad \mathbf{R}' = \bigvee_{\beta \in \mathbb{F}_d^+} \mathbf{Ran} \Theta''_{\beta}$$

so that

$$\Theta_{\alpha} y = U_0 \Theta_{\alpha} y \quad \text{for all } \beta \in \mathbb{F}_d^+, \quad y \in \mathcal{Y}.$$

Let $U : \mathcal{U} \to \mathcal{U}'$ be any partially isometric extension of $U_0$. Then $U$ meets the last two conditions in (5.11). From

$$\Theta_{\alpha} u = \Theta_{\alpha} u_0 = \Theta_{\alpha} \quad \text{for all } \alpha \in \mathbb{F}_d^+,$$

we see that $\Theta'(z)U = \Theta(z)$, i.e., the first condition in (5.11) holds as well.

Conversely, if $U : \mathcal{U} \to \mathcal{U}'$ is a partial isometry satisfying all of (5.11), one can work back to (5.10) and conclude that $\mathcal{M} = \Theta \cdot H^2_{\omega}(\mathbb{F}_d^+)$ and $\mathcal{M}' = \Theta' \cdot H^2_{\omega}(\mathbb{F}_d^+)$, both considered with lifted norm as contractively included subspaces of $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$, have the same kernel function $K_M(z, \zeta) = K_{M'}(z, \zeta)$ and hence $\mathcal{M} = \mathcal{M}'$. The latter uniqueness criterion has been stated with only the first condition in (5.11) as a necessary condition for $\Theta$ and $\Theta'$ to be representatives of the same contractively included subspace $\mathcal{M}$ (see [66, Theorem 4.1] and [29, Corollary 4.4]).

The special case of Theorem 5.1 where $\mathcal{M}$ is isometrically contained in $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ is of particular interest.

**Theorem 5.3.** A Hilbert space $\mathcal{M}$ is isometrically included in $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ and is $\mathbb{S}_{\omega, \mathcal{R}}$-invariant if and only if there is a Hilbert space $\mathcal{U}$ and a McCT-inner multiplier $\Theta$ from $H^2_{\omega}(\mathbb{F}_d^+)$ to $H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)$ so that

$$\mathcal{M} = \Theta \cdot H^2_{\omega}(\mathbb{F}_d^+).$$

(5.12)
PROOF. The "if" direction is immediate. To get the "only if" part as a consequence of Theorem 5.1 let us show that condition (3) in Theorem 5.1 is automatic if a $S_{\omega, R}$-invariant subspace $M$ is isometrically included in $H^2_{\omega, Y}(F^+_d)$. Indeed, if this is the case, then

$$T^*_j = (S_{\omega, R, j}|_M)^* = P_M S_{\omega, R, j}|_M \quad \text{for} \quad j = 1, \ldots, d,$$

and therefore, we have

$$\sum_{j=1}^d \|T^*_j f\|^2_M = \sum_{j=1}^d \|P_M (S_{\omega, R, j})^* f\|^2_{H^2_{\omega, Y}(F^+_d)} \leq \sum_{j=1}^d \|S_{\omega, R, j})^* f\|^2_{H^2_{\omega, Y}(F^+_d)} \leq \|f\|^2_{H^2_{\omega, Y}(F^+_d)}$$

which confirms that condition (3) in the statement of Theorem 5.1 holds and we indeed have (5.12). Furthermore, the fact that the operator $M_G$ in the proof of Theorem 5.1 is a coisometry translates to $M_\Theta$ is a partial isometry, i.e., $\Theta$ is a McCT-inner multiplier from $H^2_{\omega, Y}(F^+_d)$ to $H^2_{\omega, Y}(F^+_d)$. \hfill \Box

REMARK 5.4. Let us consider the special case of Theorem 5.3 where $\omega = \mu_1$ so $H^2_{\omega, Y}(F^+_d)$ is equal to the Hardy-Fock space $H^2_{\omega, Y}(F^+_d)$ as in Section 1.3. Then Theorem 5.3 gives the existence of a McCT-inner multiplier from some $H^2_{\omega, Y}(F^+_d)$ to $H^2_{\omega, Y}(F^+_d)$ to represent a closed $S_{1, R}$ invariant subspace. However it is known that in this case one can implement such a representation via a strictly inner multiplier (see [51] and also [20] Theorem 2.14 and [21] Theorems 5.1 and 5.2 for later proofs from other points of view). We note here that this finer result also follows easily from Theorem 5.3 if we recall Remark 5.11 to the effect that any McCT-inner multiplier $\Theta$ between Hardy-Fock spaces decomposes as $\Theta = [\Theta_{si}]$ with respect to an appropriate decomposition $U = [U_{si}]$ of the input space. Replacing the McCT-inner $\Theta$ with the strictly inner $\Theta_{si}$ generates the same shift-invariant subspace $M = \Theta \cdot H^2_{\omega, Y}(F^+_d) = \Theta_{si} \cdot H^2_{\mu_1}(F^+_d)$.

We now present an explicit formula for the Beurling-Lax representor for at least a class of shift-invariant subspaces $M$ contained only contractively in $H^2_{\omega, Y}(F^+_d)$ for the case where the weight sequence $\omega$ is strictly decreasing. In this case we can define the associated weight sequence $\gamma$ as in (3.30). If an output pair $(C, A)$ is $\omega$-output stable, it is also $\gamma$-output stable, since $\gamma_j^{-1} < \omega_j^{-1}$ for all $j \geq 1$ (by (3.30)) and therefore,

$$G_{\gamma, C, A} = \sum_{\alpha \in F^+_d} \gamma_{[\alpha]}^{-1} A^{\alpha^T} C^* C A^\alpha \leq \sum_{\alpha \in F^+_d} \omega_{[\alpha]}^{-1} A^{\alpha^T} C^* C A^\alpha = G_{\omega, C, A}.$$

It is readily seen from the above power series representations that the gramians $G_{\omega, C, A}$ and $G_{\gamma, C, A}$ satisfy the Stein equation

$$G_{\omega, C, A} - \sum_{j=1}^d A_j^* G_{\omega, C, A} A_j = G_{\gamma, C, A}. \quad (5.13)$$

Alternatively, we can apply the $B_A$ calculus to the identity

$$(1 - \lambda) R_\omega(\lambda) = R_\gamma(\lambda) \quad (5.14)$$
and then apply the resulting operator equality to the operator $C^*C$. Another
consequence of (5.14) which we shall have several occasions to use is the identity
\[ R_\omega(Z(z)A)(I - Z(z)A) = R_\gamma(Z(z)A). \]  
(5.15)

We next recall the space $\ell_2^2(F_d^+)$ defined in (3.33) and the formal power series $\Psi_\omega(z)$
(as well as $\Psi_\gamma(z)$) defined by (3.34) which implement the factorizations

\[ k_\omega(z, \zeta) = \Psi_\omega(z)\Psi_\omega(\zeta)^* = \Psi_\gamma(z) \left( k_{\text{uc-Sa}}(z, \zeta) I_{F_d^+} \right) \Psi_\gamma(\zeta)^*. \]  
(5.16)

Let us also introduce a normalized time-domain version of the $\gamma$-observability operator
$\hat{\Gamma}_{\gamma,C,A} : \mathcal{X} \to \ell_2^2(F_d^+)$ defined by

\[ \hat{\Gamma}_{\gamma,C,A} : x \mapsto \{ \gamma_\alpha^\frac{1}{2} C A^\alpha x \}_{\alpha \in \mathbb{F}_d^+}. \]

The equality $\| \hat{\Gamma}_{\gamma,C,A} x \|^2_{\ell_2^2(F_d^+)} = \langle \mathcal{G}_{\gamma,C,A} x, x \rangle_{\mathcal{X}}$ holds for all $x \in \mathcal{X}$ and justifies the factorization

\[ \mathcal{G}_{\gamma,C,A} = \hat{\Gamma}_{\gamma,C,A}^* \hat{\Gamma}_{\gamma,C,A}. \]  
(5.17)

Yet another easily verified useful identity is

\[ \Psi_\gamma(z) \hat{\Gamma}_{\gamma,C,A} = CR_\gamma(Z(z)A) = CR_\omega(Z(z)A)(I - Z(z)A). \]  
(5.18)

**Theorem 5.5.** Suppose that the admissible weight sequence $\omega$ is strictly decreasing.

1. Let $\mathcal{M}$ be a Hilbert space contractively included in $H^2_{\omega,\gamma}(F_d^+)$. Assume
that the Brangesian complement $\mathcal{M}^{[1]}$ (defined as in Section 3.1.1) satisfies the hypotheses of part (4) Theorem 4.33 and hence has a representation as

\[ \mathcal{M}^{[1]} = \text{Ran} \mathcal{O}_{\omega,C,A} \]

for some $\omega$-contractive output pair $(C, A)$. Let us impose the additional hypothesis that $(C, A)$ satisfies the inequality

\[ \sum_{j=1}^d A_j^* (I - \mathcal{G}_{\omega,C,A}) A_j \preceq I - \mathcal{G}_{\omega,C,A}. \]  
(5.19)

Then the related inequality

\[ \left[ A^* \quad \hat{\Gamma}_{\gamma,C,A}^* \hat{\Gamma}_{\gamma,C,A} \quad A \right] \preceq I_{\mathcal{X}}. \]  
(5.20)

holds and as a consequence of the observation (3.10) we can choose a solution

\[ \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 & B \\ \text{col}_{\alpha \in \mathbb{F}_d^+} [D_\alpha] \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{X}^d \\ \ell_2^2(F_d^+) \end{bmatrix} \]  
(5.21)

of the Cholesky factorization problem

\[ \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} 0 & I_{\ell_2^2(F_d^+)} \\ I_{\mathcal{X}^d} & 0 \end{bmatrix} - \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A^* \quad \hat{\Gamma}_{\gamma,C,A}^* \hat{\Gamma}_{\gamma,C,A} \end{bmatrix}. \]  
(5.22)

Define formal power series

\[ \mathcal{D}(z) = \sum_{\alpha \in \mathbb{F}_d^+} \gamma_\alpha^\frac{1}{2} D_\alpha z^\alpha \quad \text{and} \quad \Theta(z) = \mathcal{D}(z) + CR_\omega(Z(z)A)Z(z)B. \]  
(5.23)

Then $\Theta$ is a Beurling-Lax representer for $\mathcal{M}$, i.e., $\Theta$ is a contractive multiplier
from $H^2_{\alpha}(F_d^+)$ to $H^2_{\omega,\gamma}(F_d^+)$ such that $\mathcal{M} = \Theta \cdot H^2_{\alpha}(F_d^+)$. 

(2) Suppose that $\mathcal{M}$ is isometrically included subspace of $H^2_{\omega,y}(\mathbb{F}_d^+)\mathcal{M}$ which is $S_{\omega,R}$-invariant with $S_{\omega,R}^\perp$-invariant orthogonal complement $\mathcal{M}^\perp$ represented as $\mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{\omega,C,A}$ for a $\omega$-isometric pair $(C,A)$ with $A$ being $\omega$-strongly stable as in Theorem 4.36. Then the additional hypothesis (5.19) is automatic and a McCT-inner Beurling-Lax representer $\Theta$ for $\mathcal{M}$ can be constructed explicitly via the procedure given by (5.21), (5.22), (5.23).

PROOF. To see that (5.19) implies (5.20), rewrite (5.19) as
\[ A^* A + \left( \mathcal{G}_{\omega,C,A} - \sum_{j=1}^d A_j^* \mathcal{G}_{\omega,C,A} A_j \right) \leq I \]
which easily converts into (5.20) upon noting (5.13) and (5.17).

To verify part (1), i.e., to show that $\Theta$ (5.23) is a Beurling-Lax representer for $\mathcal{M}$, by Theorem 3.24 it suffices to show that $\mathcal{M}$ has kernel function $K_{\mathcal{M}}$ given by
\[ K_{\mathcal{M}}(z,\zeta) = \Theta(z)(k_{nc,Sz}(z,\zeta)I_d)\Theta(\zeta)^* \]
and furthermore, that the kernel function for the Brangesian complement $\mathcal{M}^{[1]}$ is given by
\[ K_{\mathcal{M}^{[1]}}(z,\zeta) = k_\omega(z,\zeta)I_{\gamma} - K_{\mathcal{M}}(z,\zeta). \]

On the other hand, by part (4) of Theorem 4.33 we know that the kernel function for the Brangesian complement $\mathcal{M}^{[1]}$ is given by
\[ K_{\mathcal{M}^{[1]}}(z,\zeta) = CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^* \]

Thus, to show that $\Theta$ is a Beurling-Lax representer for $\mathcal{M}$ it suffices to demonstrate the kernel decomposition
\[ k_\omega(z,\zeta)I_{\gamma} - \Theta(z)(k_{nc,Sz}(z,\zeta)I_d)\Theta(\zeta)^* = CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^* \] (5.24)
To simplify notation let us set $C_0 = \tilde{O}_{\gamma,C,A}$. With $[\tilde{B}]$ constructed as in (5.22) we set
\[ \Theta_0(z) = D + C_0(I - Z(z)A)^{-1}Z(z)B. \]

Then as a consequence of Theorem 3.8 (specifically the identity (3.29)), we have the kernel decomposition
\[ k_{nc,Sz}(z,\zeta)I_{\gamma} - \Theta_0(z)(k_{nc,Sz}(z,\zeta)I_d)\Theta_0(\zeta)^* = C_0(I - Z(z)A)^{-1}(I - A^* Z(\zeta)^*)^{-1}C_0^* \] (5.25)

Next we verify that the formula
\[ \Theta(z) = \Psi_\gamma(z)\Theta_0(z). \] (5.26)
defines the same power series $\Theta$ as in (5.23). Indeed, $\Psi_\gamma(z)D = \mathcal{O}(z)$, by the formula (5.34) for $\Psi_\gamma(z)$, and on the other hand,
\[ \Psi_\gamma(z)C_0 = \Psi_\gamma(z)\tilde{O}_{\gamma,C,A} = CR_\gamma(Z(z)A), \]
by (5.18). With these two latter substitutions combined with (5.13) we have
\[ \Theta(z) = \mathcal{O}(z) + CR_\gamma(Z(z)A)(I - Z(z)A)^{-1}Z(z)B = \mathcal{O}(z) + CR_\omega(Z(z)A)Z(z)B .\]
in agreement with (5.23). Then also
\[
q(z, \zeta)I_Y - \Theta(z)(q(z, \zeta)I_d)\Theta(\zeta)^* \\
= \Psi_\gamma(z)\left(q(z, \zeta)\left(I_{I_2^*O}\right)^* - \Theta(z)(q(z, \zeta)I_d)\Theta(\zeta)^*\right)\Psi_\gamma(\zeta)^* \\
(\text{by (5.10) and (5.20)}) \\
= \Psi_\gamma(z)C_0(I - Z(z)A)^{-1}(I - A^*Z(\zeta)^*)^{-1}C_0^*\Psi_\gamma(\zeta)^* \quad (\text{by (5.25)}) \\
= CR_\gamma(Z(z)A)(I - Z(z)A)^{-1}(I - A^*Z(\zeta)^*)^{-1}R_\gamma(Z(\zeta)A)^*C^* \\
= CR_\omega(Z(z)A)R_\omega(Z(\zeta)A)^*C^* \quad (\text{by (5.15)})
\]
and (5.24) follows. This completes the verification of part (1).

With regard to part (2), since $M$ is isometrically contained in $H^2_{\omega', \gamma}(F_d^\gamma)$, the Brangesian complement is just the standard Hilbert-space orthogonal complement $M^\perp$. As $M$ is $S_{\omega, R}$-invariant, then orthogonal complement $M^\perp$ is $S^*_{\omega, R}$-invariant and we may use Theorem 4.39 to represent $M^{\perp}$ as Ran $O_{C,A}$ with $(C,A)$ a $\omega$-isometric pair with in addition $A$ strongly $\omega$-stable. Furthermore in this case the observability operator $O_{C,\omega, C,A}$ is isometric, the observability gramian $G_{\omega, C,A}$ is equal to the identity $I_X$, and the additional hypothesis (6.19) is valid in the trivial form $0 \leq 0$. Hence condition (5.20) holds with equality and the procedure given in part (1) constructs the McC-T-inner Beurling-Lax repreenser for $M$.

\begin{remark}
Let us observe that the Stein equation (5.13) can be equivalently rewritten as
\[
\begin{bmatrix}
A^* & (O_{\gamma, C,A})^* \\
\end{bmatrix}
\begin{bmatrix}
G_{\omega, C,A} \otimes I_d & 0 \\
0 & I_{I^*_2 O}
\end{bmatrix}
\begin{bmatrix}
A \\
(O_{\gamma, C,A})
\end{bmatrix}
= G_{\omega, C,A}.
\]
Then the same calculations as in the derivation of the kernel decomposition (5.24) in the proof of Theorem 5.5 leads to the construction of a formal power series $\Theta(z)$ so that
\[
k_\omega(z, \zeta)I_Y - \Theta(z)(k_{\omega, S_0}(z, \zeta)I_d)\Theta(\zeta)^* = CR_\omega(Z(z)A)G_{\omega, C,A}^{-1}R_\omega(Z(\zeta)A)^*C^*.
\]
This kernel decomposition however appears to have no application to proving results about Beurling-Lax representations (unless of course $G_{\omega, C,A} = I_X$).
\end{remark}

\begin{remark}
Note that if $M$ is a contractively-included subspace of $H^2_{\omega', \gamma}(F_d^\gamma)$ such that its Brangesian complement satisfies the hypotheses of item (4) in Theorem 4.33 it follows that $M$ satisfies conditions (2) and (3) in Theorem 5.1 i.e., $M$ is $S_{\omega, R}$-invariant and $(S_{\omega, R}|M)^*$ is a row contraction. In general it need not be the case (unlike the classical case for $d = 1$) that $M$ contractively included in $H^2_{\omega', \gamma}(F_d^\gamma)$ and invariant for $S_{\omega, R}$ implies that $M^{\perp_1}$ is $S^*_{\omega, R}$-invariant (unless $M$ is isometrically contained in $H^2_{\omega', \gamma}(F_d^\gamma)$); see e.g., the end of the report [16].

There is a more flexible Beurling-Lax representation theorem in the same spirit but which appears to require an additional hypothesis (see Remark 5.9 below).
\end{remark}

\begin{theorem}
Let $\omega$ and $\omega'$ be two admissible weights satisfying condition (4.7). Then a Hilbert space $M$ is such that
\begin{enumerate}
\item $M$ is contractively included in $H^2_{\omega', \gamma}(F_d^\gamma)$,
\item $M$ is $S_{\omega, R}$-invariant,
\end{enumerate}
\end{theorem}
(3) the $d$-tuple $A = (A_1, \ldots, A_d)$ where $A_j = (S_{\omega, R, j}|\mathcal{M})^*$ for $j = 1, \ldots, d$, is a strongly $\omega'$-stable $\omega'$-hypercontraction

if and only if there is a coefficient Hilbert space $\mathcal{U}$ and a contractive multiplier $\Theta$ from $H^2_{\omega', \mathcal{U}}(F^+_d)$ to $H^2_{\omega, \mathcal{Y}}(F^+_d)$ so that

$$\mathcal{M} = M_{\Theta}H^2_{\omega', \mathcal{U}}(F^+_d) \text{ with lifted norm } \|M_{\Theta}f\|_{\mathcal{M}} = \|Qf\|_{H^2_{\omega', \mathcal{U}}(F^+_d)},$$

(5.27)

where $Q$ is the orthogonal projection onto $(\text{Ker } M_{\Theta})^\perp$.

Moreover, if it is the case that $\mathcal{M}$ is isometrically contained in $H^2_{\omega, \mathcal{Y}}(F^+_d)$, then $\Theta$ can be taken to be a McCT-inner multiplier from $H^2_{\omega', \mathcal{U}}(F^+_d)$ to $H^2_{\omega, \mathcal{Y}}(F^+_d)$.

**Proof.** **Sufficiency:** Suppose that $\mathcal{M}$ is of the form (5.27) for a contractive multiplier from $H^2_{\omega', \mathcal{U}}(F^+_d)$ to $H^2_{\omega, \mathcal{Y}}(F^+_d)$. The inequality (5.5) takes the form

$$\|M_{\Theta}f\|_{H^2_{\omega, \mathcal{Y}}(F^+_d)} \leq \|Qf\|_{H^2_{\omega', \mathcal{U}}(F^+_d)} \leq \|M_{\Theta}f\|_{\mathcal{M}}$$

and hence (1) holds. Next, relations (5.6), which take the form

$$QS_{\omega', R, j} = QS_{\omega', R, j} Q, \quad S_{\omega', R, j}^* Q = QS_{\omega', R, j} Q \quad \text{for} \quad j = 1, \ldots, d, \quad \text{(5.28)}$$

and mimicking the preceding computation (with $T_j$, $S_{\omega', R, j}$ and $H^2_{\omega', \mathcal{U}}(F^+_d)$ instead of $T^*$, $S_{1, R, j}$ and $H^2_{\omega, \mathcal{U}}(F^+_d)$, respectively) we get a more explicit formula for $A_j = (S_{\omega, R, j}|\mathcal{M})^*$:

$$A_j : M_{\Theta}f \mapsto M_{\Theta}S_{\omega', R, j} Qf \quad \text{for} \quad j = 1, \ldots, d. \quad \text{(5.29)}$$

Making use of the same substitutions as in computation (5.8) we conclude that $A^*$ is a row contraction, i.e., that the tuple $A$ is contractive in the sense of (4.34).

Iterating the formula (5.28) gives, on account of relations (5.28),

$$A^\alpha : M_{\Theta}f \mapsto M_{\Theta}(S_{\omega', R}^\alpha)Qf \quad \text{for} \quad \alpha \in F^+_d.$$ Combining the latter with the definition of the lifted norm (5.3), we get

$$\|A^\alpha \Theta f\|_{\mathcal{M}} = \|S_{\omega', R}^\alpha Qf\|_{H^2_{\omega', \mathcal{U}}(F^+_d)} \quad \text{for} \quad \alpha \in F^+_d.$$ Making use of all these relations along with the definitions (4.15), (4.16), we get

$$\langle (\Gamma_{\omega', A}|I_{\mathcal{M}}) \Theta f, \Theta f \rangle_{\mathcal{M}} = \langle (\Gamma_{\omega', A}|I_{\mathcal{M}}) \Theta Qf, \Theta Qf \rangle_{H^2_{\omega', \mathcal{U}}(F^+_d)},$$

\begin{align*}
(\Gamma_{\omega', A}|I_{\mathcal{M}}) \Theta f, \Theta f \rangle_{\mathcal{M}} = (\Gamma_{\omega', A}|I_{\mathcal{M}}) \Theta Qf, \Theta Qf \rangle_{H^2_{\omega', \mathcal{U}}(F^+_d)}
\end{align*}

for $k = 1, 2, \ldots$, and also

$$\sum_{\alpha \in F^+_d, \vert \alpha \vert = k} \langle A^\alpha \Gamma_{\omega', A}|I_{\mathcal{M}}|A^\alpha \Theta f, \Theta f \rangle_{\mathcal{M}}$$

$$= \sum_{\alpha \in F^+_d, \vert \alpha \vert = k} \langle \Gamma_{\omega', A}|I_{\mathcal{M}}|A^\alpha \Theta f, \Theta f \rangle_{\mathcal{M}}
= \sum_{\alpha \in F^+_d, \vert \alpha \vert = k} \langle \Gamma_{\omega', A}|I_{\mathcal{M}}|\Theta S_{\omega, R}^\alpha Qf, \Theta S_{\omega, R}^\alpha Qf \rangle_{\mathcal{M}}
= \sum_{\alpha \in F^+_d, \vert \alpha \vert = k} \langle (\Gamma_{\omega', A}|I_{\mathcal{M}}|\Theta S_{\omega, R}^\alpha |I_{H^2_{\omega', \mathcal{U}}(F^+_d)})Qf, QS_{\omega, R}^\alpha Qf \rangle_{H^2_{\omega', \mathcal{U}}(F^+_d)}
= \sum_{\alpha \in F^+_d, \vert \alpha \vert = k} \langle S_{\omega', R}^\alpha \Gamma_{\omega', A}|I_{\mathcal{M}}|\Theta S_{\omega, R}^\alpha |I_{H^2_{\omega', \mathcal{U}}(F^+_d)})Qf, \Theta S_{\omega, R}^\alpha Qf \rangle_{H^2_{\omega', \mathcal{U}}(F^+_d)}.$$
Since $S^*_{\omega',R}$ is an $\omega'$-strongly stable $\omega'$-hypercontraction on $H^2_{\omega',\mathcal{Y}}(F^+_d)$ (by Proposition 4.24), we conclude from the latter computations that

$$\langle (\Gamma_{\omega',\mathcal{A}}[I_M])\theta f, \theta f \rangle_M \geq 0, \quad \langle (\Gamma^{(k)}_{\omega',\mathcal{A}}[I_M])\theta f, \theta f \rangle_M \geq 0$$

for all $f \in H^2_{\omega',\mathcal{U}}(F^+_d)$ and all $k \geq 1$, and that moreover,

$$\lim_{k \to \infty} \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| = k} \langle A^{\alpha \top} \Gamma^{(k)}_{\omega',\mathcal{A}}[I_M]A^\alpha \theta f, \theta f \rangle_M = 0.$$

We conclude that $A$ is an $\omega'$-strongly stable $\omega'$-hypercontraction on $M$ and thereby complete the verification of condition (3). The verification of conditions (1) and (2) proceeds as in the proof of sufficiency in Theorem 5.1.

**Necessity.** Suppose that $M$ is a subspace of $H^2_{\omega',\mathcal{Y}}(F^+_d)$ satisfying conditions (1), (2), (3). By condition (3), the operator $\Gamma_{\omega',\mathcal{A}}[I_M]$ is positive semidefinite. Define $C : M \to \mathcal{U} := \text{Ran}(\Gamma_{\omega',\mathcal{A}}[I_M])^{1/2}$ so that $C^*C = \Gamma_{\omega',\mathcal{A}}[I_M]$. Then the pair $(C, A)$ is a $\omega'$-isometric output pair, and the $\omega'$-observability operator $O^*_{\omega',C,\mathcal{A}} : M \to H^2_{\omega',\mathcal{U}}(F^+_d)$ is isometric (by part (2) in Lemma 4.10) and satisfies the intertwining condition

$$S^*_{\omega',R,j}O^*_{\omega',C,A} = O^*_{\omega',C,A}A_j$$

for $j = 1, \ldots, d$.

Taking adjoints then gives

$$O^*_{\omega',C,A}S_{\omega',R,j} = A^*_jO^*_{\omega',C,A}$$

for $j = 1, \ldots, d$.

The inclusion map $\iota : M \to H^2_{\omega',\mathcal{Y}}(F^+_d)$ is a contraction due to condition (1). Due to condition (2) and the definition of $A_j$, we also have $\iota \circ A^*_j = S_{\omega',R,j} \circ \iota$. Therefore, the operator

$$X = \iota \circ O^*_{\omega',C,A} : H^2_{\omega',\mathcal{U}}(F^+_d) \to H^2_{\omega',\mathcal{Y}}(F^+_d)$$

is contractive (as $\iota$ is a contraction and $O^*_{\omega',C,A}$ is an isometry) and satisfies the intertwining relations

$$XS_{\omega',R,j} = \iota \circ O^*_{\omega',C,A}S_{\omega',R,j} = \iota \circ A^*_jO^*_{\omega',C,A} = S_{\omega',R,j} \circ \iota \circ O^*_{\omega',C,A} = S_{\omega',R,j}X$$

for $j = 1, \ldots, d$. By Proposition 5.22 there is a contractive multiplier $\Theta$ from $H^2_{\omega',\mathcal{U}}(F^+_d)$ to $H^2_{\omega',\mathcal{Y}}(F^+_d)$ so that $X = M_\Theta$. In case $M$ is isometrically included in $H^2_{\omega',\mathcal{Y}}(F^+_d)$, one can verify that $M_\Theta$ is a partial isometry so that $\Theta$ is McCT-inner just as in the proof of Theorem 5.1.

**Remark 5.9.** Note that the situation in Theorem 5.1 is the special case of Theorem 5.8 where in Theorem 5.8 one specializes $\omega'$ to be $\mu_1$ (i.e., $\omega'_j = 1$ for all $j \geq 0$). The results differ somewhat in that in Theorem 5.1 it was not necessary to assume that $A$ is $\omega'$-strongly stable, or equivalently for the case $\omega' = \mu_1$ as a consequence of Remark 4.42 $A$ is strongly stable in the usual sense. On the other hand, the proof of sufficiency in Theorem 5.8 shows that in the end necessarily $A$ is strongly stable after all. In summary we can say that specializing Theorem 5.8 to the case $\omega' = \mu_1$ leads us to the weaker version of Theorem 5.1 where one adds to hypothesis (3) the (in the end unneeded) additional hypothesis that $T^*$ is strongly stable.

An observation concerning Theorem 5.8 versus the isometric-inclusion version of 5.8 is the following. When $M$ is contained in $H^2_{\omega',\mathcal{Y}}(F^+_d)$ isometrically and $\omega' = \mu_1$,
it is automatic that $A$ is a $\omega'$-contraction; for the case of a general $\omega'$ this is not the case in general (see [14] Remark 7.6).

Finally let us note that the classical de Branges result in [36] amounts to the case $d = 1, \omega = \mu_1$ in Theorem 5.1.

5.2. Representations with model space of the form $\bigoplus_{j=1}^{n} A_j U_j(\mathbb{F}_d^+)$

A Beurling-Lax representation as in Theorem 5.8 lacks the precision of the classical version in several respects: (1) the operator $M_G$ is only a partial isometry rather than an isometry (the distinction between McCT-inner and strictly inner) with the consequence that in general $M_G$ has a huge kernel, and (2) the input coefficient space $U$ is almost always infinite dimensional. For the case of standard weighted Bergman-Fock spaces we will give a more structured version of the Beurling-Lax theorem which should carry more information about the $S_{n,R}$-invariant subspace $M$ of $A_{n,Y}(\mathbb{F}_d^+)$, at least under a strong additional assumption; we leave as an open question whether a version of the following result still holds without imposition of this extra assumption. The extra assumption on the $S_{n,R}$-invariant subspace $M$ of $A_{n,Y}(\mathbb{F}_d^+)$ is the following:

- The output pair $(C, A)$ with $C = E|_{M+}$ and $A = (S_{n,R,1}|_{M+}, \ldots, S_{n,R,d}|_{M+})$ is exactly observable, i.e.,

$$G_{1,C,A} := \sum_{\alpha \in \mathbb{F}_d^+} A^*\alpha C^* C A^{\alpha} > 0 \text{ (strictly positive definite).} \quad (5.30)$$

Since $G_{k,C,A} \geq G_{1,C,A}$ for all $k > 1$ (see the proof of Proposition 4.10), the assumption (5.30) implies that $G_{k,C,A}$ is strictly positive definite for all $k \geq 1$.

**Theorem 5.10.** Let $M$ be a closed $S_{n,R}$-invariant subspace of $A_{n,Y}(\mathbb{F}_d^+)$ satisfying the assumption (5.30). Then there exist Hilbert spaces $U_1, \ldots, U_d$ and a McCT-inner multiplier $F(z) = [F_1(z) \cdots F_n(z)]$ from $\bigoplus_{j=1}^{n} A_j U_j(\mathbb{F}_d^+)$ to $A_{n,Y}(\mathbb{F}_d^+)$ so that $M = M_F\left(\bigoplus_{j=1}^{n} A_j U_j(\mathbb{F}_d^+)\right)$.

**Proof.** We start as at the beginning of Chapter 5 by representing $M$ as $(\text{Ran}O_{n,C,A})^\perp$ for some $n$-isometric pair $(C, A)$ with $A$ strongly stable, but with the additional assumption (5.30) imposed.

By Proposition 5.8 it suffices to produce coefficient Hilbert spaces $U_1, \ldots, U_n$ and power series $F_j$ in $L(U_j, Y)(\langle z \rangle)$ so that

$$k_M(z, \zeta) = \sum_{j=1}^{n} F_j(z) k_{nc,j}(z, \zeta) F_j(\zeta)^*, \quad (5.31)$$

where $k_M$ is given in (5.11) and where $k_{nc,j}(z, \zeta)$ is the noncommutative Bergman kernel defined in (2.16). The construction proceeds via an iterated application of Theorem 5.8.
We first construct inner multipliers $\Psi_j(z) \ (j = 1, \ldots, n)$ between Fock spaces as follows. For $j = 1$ we use the identity

$$G_{1,C,A} - \sum_{\ell=1}^{d} A_\ell^* G_{1,C,A} A_\ell = C^* C$$

(i.e., the identity (4.47) for $k = 1$) and find $[B_1^1 : D_1^1] : U_n \rightarrow \chi^d$ such that

$$\left[ B_1^1 \ D_1^1 \right] = \left[ I_{\chi^d} \ 0 \ 0 \ I_y \right] - \left[ A \right] G_{1,C,A}^{-1} \left[ A^* \ C^* \right].$$

By Theorem 5.8 the power series

$$\Psi_1(z) = D_1 + C(I - Z(z)A)^{-1} Z(z)B_1$$

is a strictly inner multiplier from $H^2_{U_n}(\mathbb{F}_d)$ to $H^2\mathbb{D}(\mathbb{F}_d)$ and

$$k_{nc,Sz}(z, \zeta) I_y - \Psi_1(z) k_{nc,Sz}(z, \zeta) \Psi_1(\zeta)^*$$

$$= C(I - Z(z)A)^{-1} G_{1,C,A}^{-1} (I - A^* Z(\zeta)^*)^{-1} C^*.$$ (5.33)

For $j = 2, \ldots, n$ we do a similar construction based on the identity (see (4.47))

$$G_{j,C,A} - \sum_{\ell=1}^{d} A_\ell^* G_{j,C,A} A_\ell = G_{j-1,C,A} \ (j = 2, \ldots, n).$$

We find operators $B_j : U_{n+1-j} \rightarrow \chi^d$ and $D_j : U_{n+1-j} \rightarrow \gamma^d$ so that

$$\begin{bmatrix} A & B_j \\ G_{j-1,C,A}^{-1} & D_j \end{bmatrix} \begin{bmatrix} G_{j,C,A}^{-1} & 0 \\ 0 & I_{U_{n+1-j}} \end{bmatrix} \begin{bmatrix} A^* & G_{j-1,C,A}^* \\ B_j^* & D_j^* \end{bmatrix} = \begin{bmatrix} G_{j,C,A}^{-1} \otimes I_d & 0 \\ 0 & I_y \end{bmatrix}$$

and

$$\begin{bmatrix} A^* & G_{j-1,C,A}^* \\ B_j^* & D_j^* \end{bmatrix} \begin{bmatrix} G_{j,C,A} \otimes I_d & 0 \\ 0 & I_y \end{bmatrix} \begin{bmatrix} A & B_j \\ G_{j-1,C,A} & D_j \end{bmatrix} = \begin{bmatrix} G_{j,C,A} \otimes I_d & 0 \\ 0 & I_{U_{n+1-j}} \end{bmatrix}. $$ (5.34)

In fact, we claim that the latter equalities determine $B_j$ and $D_j$ uniquely up to a common isometric factor $W_j : U_{n+1-j} \rightarrow \chi^d$ on the right:

$$B_j = (G_{j,C,A}^{-1} \otimes I_d - A G_{j,C,A}^{-1} A^*)^{1/2} W_j, \quad W_j^* W_j = I_{U_{n+1-j}};$$

$$D_j = -G_{j-1,C,A}^* (G_{j,C,A} \otimes I_d) B_j.$$ (5.35)

To verify this claim, first note that comparison of (1,2)-block entries in (5.34) yields (5.36). Equating the bottom corner blocks in (5.34) then gives

$$B_j^* (G_{j,C,A} \otimes I_d) B_j + D_j^* D_j = I_{U_{n+1-j}},$$

which in view of (5.33) can be written as

$$B_j^* \left( G_{j,C,A} \otimes I_d + (G_{j,C,A} \otimes I_d) A G_{j-1,C,A}^{-1} A^* (G_{j,C,A} \otimes I_d) \right) B_j = I_{U_{n+1-j}}.$$ (5.37)

We claim that (5.37) can be simplified to

$$B_j^* \left( G_{j,C,A}^{-1} \otimes I_d - A G_{j-1,C,A}^{-1} A^* \right)^{-1} B_j = I_{U_{n+1-j}},$$ (5.38)

from which (5.35) will then follow.
To see this latter claim, we may use the operator-valued version of the Sherman-Morrison formula: given any strictly positive definite \(2 \times 2\) operator matrix \(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\) with \(X\) and \(Z\) invertible, then the Schur complement \(Z - Y^*X^{-1}Y\) is invertible if and only if the other Schur complement \(X - YZ^{-1}Y^*\) is invertible, and then their inverses are connected via the Sherman-Morrison formula (see e.g. [56]):

\[
(Z - Y^*X^{-1}Y)^{-1} = Z^{-1} + Z^{-1}Y^*(X - YZ^{-1}Y^*)^{-1}YZ^{-1}.
\]  

(5.39)

We apply the identity (5.39) to the case where

\[
\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} = \begin{bmatrix} G_{j,c,a} & A^* \\ A & G_{j,c,a}^{-1} \otimes I_d \end{bmatrix}
\]

to conclude that

\[
(G_{j,c,a}^{-1} \otimes I_d - AG_{j,c,a}^{-1}A^*)^{-1} = G_{j,c,a} \otimes I_d + (G_{j,c,a} \otimes I_d)A\Delta^{-1}A^*(G_{j,c,a} \otimes I_d)
\]

(5.40)

where

\[
\Delta = G_{j,c,a} - A^*(G_{j,c,a} \otimes I_d)A.
\]

However, a consequence of the \((1,1)\)-entry of the identity (5.34) is that \(\Delta = G_{j-1,c,a}\) which is invertible as a consequence of assumption (5.30). Substituting this expression for \(\Delta\) into (5.40) gives us the equivalence between (5.37) and (5.38), and hence also the verification of (5.35), as desired.

We now define the power series

\[
\Psi_j(z) = D_j + g_{j-1,c,a}^z(I - Z(z)A)^{-1}Z(z)B_j
\]

(5.41)

for \(j = 2, \ldots, n\), which are strictly inner multipliers from \(H^2_{U_{n-j+1}}(\mathbb{F}_d)\) to \(H^2_{A}(\mathbb{F}_d)\) and satisfy the identities

\[
k_{nc,Sa}(z, \zeta)I_Y - \Psi_j(z)k_{nc,Sa}(z, \zeta)\Psi_j(\zeta)^* = G_{j-1,c,a}^z(I - Z(z)A)^{-1}G_{j,c,a}^{-1}(I - A^*Z(\zeta)^*)^{-1}G_{j-1,c,a}^z.
\]

(5.42)

We finally define the series \(F_\ell \in \mathcal{L}(U_\ell, Y)(\langle z \rangle)\) via formulas

\[
F_\ell(z) = C(I - Z(z)A)^{-(n-\ell)}G_{n-\ell,c,a}^{-1}\Psi_{n+1-\ell}(z) \quad \text{for} \quad \ell = 1, 2, \ldots, n - 1,
\]

\[
F_n(z) = \Psi_1(z) = D_1 + C(I - Z(z)A)^{-1}Z(z)B_1,
\]

(5.43)

and claim that this choice of \(F_1, \ldots, F_n\) satisfies (5.31) (with coefficient spaces \(U_j = A^j\) for \(j = 1, \ldots, n - 1\)). Indeed, substituting equality (5.42) (for \(j = n\)) into the formula (5.1) for \(k_M\) and then making use of formula (5.43) for \(F_1\), we have

\[
k_{M}(z, \zeta) = k_{nc,n}(z, \zeta)I_Y - C(I - Z(z)A)^{-n+1}G_{n-1,c,a}^{-1}k_{nc,n}(z, \zeta)I_X - \Psi_n(z)k_{nc,Sa}(z, \zeta)\Psi_n(\zeta)^*
\]

\[
= k_{nc,n}(z, \zeta)I_Y + F_1(z)k_{nc,Sa}(z, \zeta)F_1(\zeta)^* - C(I - Z(z)A)^{-n+1}G_{n-1,c,a}^{-1}k_{nc,Sa}(z, \zeta)(I - A^*Z(\zeta)^*)^{-n+1}C^*.
\]

(5.44)
We next use equality (5.42) (for \( j \)) and the last equality follows from (2.18). Plugging (5.46) into the case of (5.45) leaves us with (5.31) as wanted. □

Remark 5.11. We remark that the entry \( F_n(z) = \Psi_1(z) \) in (5.32) is a strictly inner multiplier from \( H^2_1(\mathbb{F}_d^+ \rightarrow \mathbb{F}_d^+) \) to \( H^2_1(\mathbb{F}_d^+) \). Therefore, if \( n = 1 \), Theorem 5.10 amounts to Theorem 3.13.

To conclude this section, we present an alternate formula for \( \Psi_j \) (and hence also for \( F_\ell \)) which may prove useful in applications.
Proposition 5.12. The function $\Psi_j$ in (5.41) for $j = 2, \ldots, n$ can alternatively be given by

$$
\Psi_j(z) = G_{j-1, C, A}^\dagger (I - Z(z)A)^{-1} G_{j-1, C, A}^{-1} (Z(z) - A^*) \times (G_{j, C, A}^{-1} \otimes I_d - AG_{j, C, A}^{-1} A^*)^{-\frac{1}{2}} W_j.
$$

(5.47)

Hence $F_\ell(z)$ can alternatively be given for $\ell = 1, 2, \ldots, n - 1$ by

$$
F_\ell(z) = C(I - Z(z)A)^{-(n-\ell)} G_{n+1-\ell, C, A}^{-\frac{1}{2}} \Psi_{n+1-\ell}(z)
= C(I - Z(z)A)^{-(n+1-\ell)} G_{n+1-\ell, C, A}^{-1} (Z(z) - A^*) \times (G_{n+1-\ell, C, A}^{-1} \otimes I_d - AG_{n+1-\ell, C, A}^{-1} A^*)^{-\frac{1}{2}} W_{n+1-\ell}.
$$

(5.48)

Proof. In the proof, we shorten notation $G_{j, C, A}$ to $G_j$. We substitute (5.36) into (5.41) to get

$$
\Psi_j(z) = -G_{j-1}^{-\frac{1}{2}} A^*(G_j \otimes I_d) B_j + G_{j-1}^\dagger (I - Z(z)A)^{-1} Z(z) B_j
= G_{j-1}^\dagger (I - Z(z)A)^{-1} \left[ -(I - Z(z)A) G_{j-1}^{-1} A^*(G_j \otimes I_d) + Z(z) \right] B_j
= G_{j-1}^\dagger (I - Z(z)A)^{-1} \left[ Z(z)(I + AG_{j-1}^{-1} A^*(G_j \otimes I_d)) - G_{j-1}^{-1} A^*(G_j \otimes I_d) \right] B_j.
$$

(5.49)

To further simplify the latter expression, we make use of equality (5.47) in the form

$$
G_j - A^*(G_j \otimes I_d) A = G_{j-1}.
$$

We first observe that

$$
G_{j-1}^{-1} A^*(G_j \otimes I_d) (I + AG_{j-1}^{-1} A^*(G_j \otimes I_d)) = G_{j-1}^{-1} A^*(G_j \otimes I_d) + G_{j-1}^{-1} A^*(G_j \otimes I_d) AG_{j-1}^{-1} A^*(G_j \otimes I_d)
= G_{j-1}^{-1} A^*(G_j \otimes I_d) + G_{j-1}^{-1} (G_j - G_{j-1}) G_{j-1}^{-1} A^*(G_j \otimes I_d)
= G_{j-1}^{-1} A^*(G_j \otimes I_d),
$$

which can be written equivalently as

$$
G_{j-1}^{-1} A^*(G_j \otimes I_d) (I + AG_{j-1}^{-1} A^*(G_j \otimes I_d))^{-1} = G_{j-1}^{-1} A^*(G_j \otimes I_d),
$$

and, being combined with (5.49) gives

$$
\Psi_j(z) = G_{j-1}^{\dagger} (I - Z(z)A)^{-1} \left[ Z(z) - G_{j-1}^{-1} A^*(G_j \otimes I_d) \right] (I + AG_{j-1}^{-1} A^*(G_j \otimes I_d)) B_j
= G_{j-1}^{\dagger} (I - Z(z)A)^{-1} G_{j-1}^{-1} (Z(z) - A^*) \times (G_j \otimes I_d + (G_j \otimes I_d)AG_{j-1}^{-1} A^*(G_j \otimes I_d)) B_j
= G_{j-1}^{\dagger} (I - Z(z)A)^{-1} G_{j-1}^{-1} (Z(z) - A^*) (G_j^{-1} \otimes I_d - AG_{j-1}^{-1} A^*)^{-1} B_j.
$$

Substituting (5.35) into the latter formula implies (5.47). Formula (5.48) is a straightforward consequence of (5.47). \qed
CHAPTER 6

Non-orthogonal Beurling-Lax representations based on wandering subspaces

Nonorthogonal representations of a closed shift-invariant subspace $\mathcal{M}$ of $\mathcal{A}_n, Y$ based on the wandering subspace $E = \mathcal{M} \ominus \mathcal{S}_n \mathcal{M}$ appears in [6] (for $n = 2$) and [97] (for $n = 3$). Another version of a non-orthogonal representation of $\mathcal{M}$ is that of Izuchi-Izuchi-Izuchi [63] (see also [39]) is based on the notion of quasi-wandering subspace $Q = P_M S_n |_{\mathcal{M}^\perp}$. Fleshing out of this approach for the $\omega$-setting is the topic of Section 6.1.

With any $S_{\omega, R}$-invariant space $\mathcal{M}$ isometrically included in $H^2_{\omega, Y}(F^+_d)$, one can associate the wandering subspace (the notion goes back to [102, 57])

$$E = \mathcal{M} \ominus \left( \bigoplus_{j=1}^d S_{\omega, R, j} \mathcal{M} \right)$$  \hspace{1cm} (6.1)

and the quasi-wandering subspace (introduced more recently in [63])

$$Q = P_M \left( \bigoplus_{j=1}^d S_{\omega, R, j} \mathcal{M}^\perp \right),$$  \hspace{1cm} (6.2)

where $P_M$ denotes the orthogonal projection of $H^2_{\omega, Y}(F^+_d)$ onto $\mathcal{M}$ and where $\mathcal{M}^\perp := H^2_{\omega, Y}(F^+_d) \ominus \mathcal{M}$. In the case of the Fock space $H^2_{\omega, Y}(F^+_d)$ (i.e., when $\omega = \mu_1$) these two subspaces coincide and moreover,

$$E = Q = G \cdot U := \{ Gu : u \in U \},$$  \hspace{1cm} (6.3)

where $G$ is a strictly inner multiplier appearing in the Beurling-Lax representation $\mathcal{M} = GH^2_{\omega, Y}(F^+_d)$ of $\mathcal{M}$ (see Theorem 3.13). Furthermore, since $S_{1, R, 1}, \ldots, S_{1, R, d}$ are isometries (by formula (4.74) with $\omega = \mu_1$) with mutually orthogonal ranges, the subspaces $S_{1, R}^\alpha \mathcal{M}$ and $S_{1, R}^\beta \mathcal{M}$ are mutually orthogonal for all $\alpha \neq \beta$ in $F^+_d$ which eventually leads to the Wold-type representation

$$\mathcal{M} = \bigoplus_{\alpha \in F^+_d} S_{1, R}^\alpha E = \bigoplus_{\alpha \in F^+_d} S_{1, R}^\alpha Q.$$  \hspace{1cm} (6.4)

In the case of a general admissible weight $\omega$, the operators $S_{\omega, R, 1}, \ldots, S_{\omega, R, d}$ are contractions (rather than isometries) with still mutually orthogonal ranges. Therefore, the formulas (6.1) and (6.2) still make sense although (even in the case of the Bergman-Fock spaces $\mathcal{A}_{n, Y}(F^+_d)$ for $n > 1$) they define different subspaces $E$ and $Q$ of a closed $S_{\omega, R}$-invariant subspace $\mathcal{M}$ of $H^2_{\omega, Y}(F^+_d)$.

In general, the quasi-wandering subspace $Q$ and its shifts $S_{\omega, R}^\alpha Q$ are not orthogonal to each other. As we will see later, the wandering subspace $E$ is orthogonal to all its shifts $S_{\omega, R}^\alpha E$ ($\alpha \neq \emptyset$). However, it is not true anymore that the subspaces...
and \( S^\alpha \omega, R \mathcal{E} \) and \( S^\beta \omega, R \mathcal{E} \) are orthogonal for \( \alpha \neq \beta \). Thus, the best one can hope for is to recover a closed \( S^\omega, R \)-invariant subspace \( \mathcal{M} \subset H^2_{\omega, \mathbb{F}_d^+} \) from its wondering subspace \( \mathcal{E} \) or the quasi-wandering subspace \( \mathcal{Q} \) via the closed linear span of the respective (not-orthogonal) shifted subspaces \( S^\alpha \omega, R \mathcal{E} \) and \( S^\omega, R \mathcal{Q} \). In the rest of this chapter, we will see to which extent this is the case. A secondary question arising from representation formulas (6.3) is whether it is possible to represent (isometrically or at least as sets) the subspaces (6.1) and (6.2) in the form \( \mathcal{E} = \Theta \cdot \mathcal{U} \) and \( \mathcal{Q} = F \cdot \mathcal{U} \) for some Hilbert coefficient space \( \mathcal{U} \) and power series \( \Theta \) and \( F \), will be also discussed below.

### 6.1. Beurling-Lax representations based on quasi-wandering subspaces

Given a closed \( S^\omega, R \)-invariant subspace \( \mathcal{M} \) of \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \), let \( (C, A) \) (with \( C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \)) be an \( \omega \)-isometric pair such that \( \mathcal{M} = (\text{Ran} \mathcal{O}_{\omega, C, A})^\perp \). Let us consider the power series

\[
F(z) = CR\omega(z)A(z) - A^*.
\]

By (2.38), (2.36), the formula for \( F \omega (x) \in \mathcal{X}^d \) can be written more explicitly as

\[
F(z) \omega (x) = \sum_{\alpha \in \mathbb{F}_d^+, j = 1}^d \omega_{/\alpha} C A^\omega \omega (z_j x_j - A^\omega \omega (z_j x_j)) \quad (x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \in \mathcal{X}^d),
\]

which also can be equivalently written in terms of operators \( \mathcal{O}_{\omega, C, A} \) and \( S^\omega, R, j \) as

\[
F \omega (x) + \mathcal{O}_{\omega, C, A} \sum_{j = 1}^d A^\omega \omega _j x_j = \sum_{j = 1}^d S^\omega, R, j \mathcal{O}_{\omega, C, A} x_j.
\]

Due to the intertwining relations (4.105) and since \( \mathcal{O}_{\omega, C, A} \) is an isometry from \( \mathcal{X} \) into \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \), we have for any \( x, x_j \in \mathcal{X} \) and \( j = 1, \ldots, d \),

\[
\left\langle (S^\omega, R, j \mathcal{O}_{\omega, C, A} - \mathcal{O}_{\omega, C, A} A^\omega _j x_j, \mathcal{O}_{\omega, C, A} x) \right\rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} = \left\langle \mathcal{O}_{\omega, C, A} x_j, S^\omega, R, j \mathcal{O}_{\omega, C, A} x \right\rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} - \left\langle \mathcal{O}_{\omega, C, A} A^\omega _j x_j, \mathcal{O}_{\omega, C, A} x \right\rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} = \left\langle \mathcal{O}_{\omega, C, A} x_j, \mathcal{O}_{\omega, C, A} A^\omega _j x_j \right\rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} - \left\langle \mathcal{O}_{\omega, C, A} A^\omega _j x_j, \mathcal{O}_{\omega, C, A} x \right\rangle_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} = \left\langle x_j, A^\omega _j x_j \right\rangle_{\mathcal{X}} = 0.
\]

The latter equalities imply that the \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-inner product of the power series (6.8) against the vector \( \mathcal{O}_{\omega, C, A} x \) equals zero. Therefore, \( F \omega \perp \text{Ran} \mathcal{O}_{\omega, C, A} = \mathcal{M}^\perp \) for any \( x \in \mathcal{X}^d \), meaning that \( F \omega \in \mathcal{M} \) for any \( x \in \mathcal{X}^d \). Therefore, the sum on the left side of (6.7) is orthogonal in \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-metric (as the first term belongs to \( \mathcal{M} \) and the second is in \( \text{Ran} \mathcal{O}_{\omega, C, A} = \mathcal{M}^\perp \)). Therefore,

\[
F \omega (x) = P_\mathcal{M} \left( \sum_{j = 1}^d S^\omega, R, j \mathcal{O}_{\omega, C, A} x_j \right).
\]

and since the expression in the parentheses presents the generic element of the direct sum \( \bigoplus_{j = 1}^d S^\omega, R, j \mathcal{M}^\perp \), we conclude from (6.2) that

\[
F(z) \omega (x) = P_\mathcal{M} \left( \bigoplus_{j = 1}^d S^\omega, R, j \mathcal{M}^\perp \right) = \mathcal{Q}
\]
as sets. The following result is the quasi-wandering version of the Beurling-Lax theorem; the proof is adapted from the commutative versions \([63, 39]\).

**Theorem 6.1.** Let us assume that the weight \(\omega\) satisfies conditions \([2, 31]\), \((4.7)\). Let \(\mathcal{M}\) be a closed \(S_{\omega,R}\)-invariant subspace of \(H^2_{\omega,Y}(\mathbb{F}^d_+)\) containing no nontrivial reducing subspaces for \(S_{\omega,R}\) i.e., \(\mathcal{M}\) is such that

\[
\mathcal{M} \supset H^2_{\omega,Y}(\mathbb{F}^d_+) \otimes y
\]

for no nonzero vector \(y \in \mathcal{Y}\). Then

\[
\mathcal{M} = \bigvee_{\alpha \in \mathbb{F}^+_d} S_{\omega,R}^\alpha \mathcal{Q},
\]

where \(\mathcal{Q}\) is the quasi-wandering subspace of \(\mathcal{M}\) defined as in \([62]\).

**Proof.** By definition \([6.2]\), \(\mathcal{Q} \subset \mathcal{M}\). Since \(\mathcal{M}\) is a closed \(S_{\omega,R}\)-invariant subspace of \(H^2_{\omega,Y}(\mathbb{F}^d_+)\), the inclusion \(\bigvee_{\alpha \in \mathbb{F}^+_d} S_{\omega,R}^\alpha \mathcal{Q} \subset \mathcal{M}\) follows (regardless condition \((6.9)\)). To show the reverse containment, we take \(\mathcal{Q}\) in the form \((6.8)\) with \(F(z)\) defined by the formula \((6.5)\) in terms of an \(\omega\)-isometric pair \((C,A)\) such that \(\mathcal{M} = (\text{Ran} \ O_{\omega,C,A})^\perp\), and we suppose that there is a nonzero \(f \in \mathcal{M}\) which is orthogonal (in the metric of \(H^2_{\omega,Y}(\mathbb{F}^d_+)\)) to \(S_{\omega,R}^\alpha \mathcal{Q}\) for all \(\alpha \in \mathbb{F}^+_d\):

\[
\langle f , O_{\omega,C,A} x \rangle_{H^2_{\omega,Y}(\mathbb{F}^d_+)} = 0 = \langle f , S_{\omega,R}^\alpha F \mathcal{Q} \rangle_{H^2_{\omega,Y}(\mathbb{F}^d_+)} \quad (6.11)
\]

for all \(x \in \mathcal{X} \), \(x \in \mathcal{X}^d\) and \(\alpha \in \mathbb{F}^+_d\). Thus \(f\) is orthogonal to \(\text{Ran} \ O_{\omega,C,A}\) as well as to \(\bigvee_{\alpha \in \mathbb{F}^+_d} S_{\omega,R}^\alpha F \mathcal{X}^d\).

It follows from \((6.7)\) that \(f\) is orthogonal to \(\sum_{j=1}^d S_{\omega,R,j} O_{\omega,C,A} x_j\) for arbitrary \(x_j \in \mathcal{X}\) \((j = 1, \ldots , d)\) and furthermore, since the vectors \(x_j\) can be chosen arbitrarily, it follows that \(f\) is orthogonal to \(S_{\omega,R,j} O_{\omega,C,A} \mathcal{X}\) for all \(j = 1, \ldots , d\). Therefore, \(S_{\omega,R,j}^* f\) is orthogonal to \(O_{\omega,C,A} \mathcal{X} = \mathcal{M}^\perp\), that is,

\[
S_{\omega,R,j}^* f \in \mathcal{M} \quad \text{for} \quad j = 1, \ldots , d.
\]

Now it follows that conditions \((6.11)\) hold for the element \(S_{\omega,R,j}^* f\) (rather than \(f\) itself) for any fixed \(j \in \{1, \ldots , d\}\). Repeating the preceding arguments, we conclude that \(S_{\omega,R,j}^* S_{\omega,R,j}^\alpha f\) belongs to \(\mathcal{M}\) for any \(i , j \in \{1, \ldots , d\}\). The induction argument verifies that

\[
S_{\omega,R}^\alpha f \in \mathcal{M} \quad \text{for all} \quad \alpha \in \mathbb{F}^+_d. \quad (6.12)
\]

Since \(f \neq 0\), there is a \(\beta \in \mathbb{F}^+_d\) such that \(f_\beta \neq 0\) \((f_\beta \in \mathcal{Y})\). Recalling the evaluation operator \(E : H^2_{\omega,Y}(\mathbb{F}^d_+) \rightarrow \mathcal{Y}\) defined by \(E f = f_0\) whose adjoint \(E^*\) is just the inclusion of \(\mathcal{Y}\) into \(H^2_{\omega,Y}(\mathbb{F}^d_+)^{\perp}\), and invoking formula \((4.7)\), we get

\[
E^* ES_{\omega,R,j} f = (S_{\omega,R,j}^* f)_0 = \omega_{|0} f_\beta := y \neq 0.
\]

By the first operator equality from \((4.7)\) applied to the function \(S_{\omega,R}^\alpha f\) we have

\[
\Gamma_{\omega, S_{\omega,R,j}^*} |I_{H^2_{\omega,Y}(\mathbb{F}^d_+)}| S_{\omega,R,j}^* f = E^* ES_{\omega,R,j}^* f = y \neq 0. \quad (6.13)
\]

Due to \((6.12)\) and since \(\mathcal{M}\) is \(S_{\omega,R}\)-invariant, it follows that \(S_{\omega,R}^\alpha S_{\omega,R}^\alpha S_{\omega,R}^\beta f\) belongs to \(\mathcal{M}\) for any \(\alpha \in \mathbb{F}^+_d\) as well as the power series

\[
g_N = \sum_{\alpha \in \mathbb{F}^+_d ; |\alpha| \leq N} c_{|\alpha|} S_{\omega,R}^\alpha S_{\omega,R}^\alpha S_{\omega,R}^\beta f \quad \text{for all} \quad N \geq 1. \quad (6.14)
\]
Since $S_{\omega, R}$ is a row contraction, it follows that for each $N \geq 1$,
\[
\sum_{\alpha \in \mathbb{F}^+_d : |\alpha| \leq N} c_{|\alpha|} S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha \preceq \sum_{\alpha \in \mathbb{F}^+_d : |\alpha| \leq N} |c_{|\alpha|}| S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha = \sum_{j=0}^{N} |c_j| \sum_{\alpha \in \mathbb{F}^+_d : |\alpha| = j} S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha \preceq \sum_{j=0}^{N} |c_j| I_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)},
\]
and therefore,
\[
\left\| \sum_{\alpha \in \mathbb{F}^+_d : |\alpha| \leq N} c_{|\alpha|} S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha \right\| \leq \sum_{j=0}^{\infty} |c_j| = \| R^{-1}_\omega \|_{W^+} < \infty,
\]
due to the assumption \ref{4.17}. Thus, the sequence $\{g_N\}_{N \geq 1} \subset \mathcal{M}$ defined in \ref{6.14} is uniformly bounded (in the metric of $H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)$) and admits a subsequential weak limit (which is in fact the strong limit)
\[
\sum_{\alpha \in \mathbb{F}^+_d} c_{|\alpha|} S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha f = \Gamma_{\omega, S_{\omega, R}} [I_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}] S_{\omega, R}^\beta f
\]
which therefore, also belongs to $\mathcal{M}$. We now conclude from \ref{6.13} that $y$ belongs to $\mathcal{M}$. Then $H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d) \otimes y$ is a subspace of $\mathcal{M}$ and is reducing for $S_{\omega, R}$ which is a contradiction.

**Remark 6.2.** Let us consider the gramian-type operator
\[
\Upsilon_{\omega, C, A} = \mathcal{O}_{\omega, C, A} S_{\omega, R}^{\alpha^T} S_{\omega, R}^\alpha \mathcal{O}_{\omega, C, A} = \sum_{\alpha \in \mathbb{F}^+_d} \frac{\omega_{|\alpha|+1}}{\omega^2_{|\alpha|}} A^{\alpha^T \omega^\alpha} C A^\alpha
\]
which is independent of the choice of $j \in \{1, \ldots, d\}$. Since $S_{\omega, R, j}$ is a contraction, it is clear that $\Upsilon_{\omega, C, A} \preceq \mathcal{G}_{\omega} C, A$ and so $\Upsilon_{\omega, C, A}$ is a positive semidefinite contraction if the pair $(C, A)$ is $\omega$-isometric. Since the sum on the left side of \ref{6.7} is orthogonal, we have
\[
\| F \|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}^2 = \sum_{j=1}^{d} \| S_{\omega, R, j} \mathcal{O}_{\omega, C, A} x_j \|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}^2 - \| \mathcal{O}_{\omega, C, A} A^* x \|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}^2 = \sum_{j=1}^{d} \| S_{\omega, R, j} \mathcal{O}_{\omega, C, A} x_j \|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}^+_d)}^2 - \| A^* x \|_{\mathcal{X}}^2 = \langle (\Upsilon_{\omega, C, A} \otimes I_d - A A^*) x, x \rangle_{\mathcal{X}}.
\]
Therefore, the operator $\Delta_{\omega, C, A} := \Upsilon_{\omega, C, A} \otimes I_d - A A^*$ is positive semidefinite, and in representation \ref{5.33} we can take $F$ of the form
\[
F(z) = C R_{\omega}(Z(z) A)(Z(z) - A^*) B
\]
where $B$ is any operator from an auxiliary Hilbert space $\mathcal{U}$ onto $(\text{Ker}(\Delta_{\omega, C, A}))^\perp$. Furthermore, $F : \mathcal{U} \rightarrow \mathcal{Q}$ isometrically (contractively) if and only if $B^* \Delta_{\omega, C, A} B = I_{\mathcal{U}}$ (respectively, $B^* \Delta_{\omega, C, A} B \preceq I_{\mathcal{U}}$).

**Remark 6.3.** In case $\omega = \mu_n$ it is possible to make a connection with the power series $F = F_1$ from the representation \ref{5.31}. We suppose that $(C, A)$ is an $n$-isometric pair representing the $S_{n, \mathcal{V}}$-invariant subspace $\mathcal{M} \subset \mathcal{A}_{n, \mathcal{Y}}(\mathbb{F}^+_d)$ via formula $\mathcal{M} = (\text{Ran} \mathcal{O}_{n, C, A})^\perp$. We again impose the assumption \ref{5.30}. Thus,
\[ G_{n,c,A} = I_X \] and according to the formula (5.35) (for \( j = n \)), \( B_n = (I - AA^*)^{\frac{1}{2}}W_n \).

Combining the latter equalities with (5.48) (for \( \ell = 1 \)) gives

\[
F(z) = C(I - Z(z)A)\bar{A}^{-1}(z - A^*)(I - AA^*)^{\frac{1}{2}}.
\]

(6.15)

As it follows from (5.31), \( F \) is a contractive multiplier from \( H^2_{A,d} (\mathbb{F}_d^+) \) to \( A_{n,Y} (\mathbb{F}_d^+) \).

**Remark 6.4.** In case \( n = 1 \), the power series (6.15) is a strictly inner multiplier from \( H^2_{A,d} (\mathbb{F}_d^+) \) to \( H^2_{A,d} (\mathbb{F}_d^+) \), since

\[
F(z) = C(I - Z(z)A)^{-1}(z - A^*)(I - AA^*)^{\frac{1}{2}}
\]

\[
= -CA^*(I - AA^*)^{-\frac{1}{2}}
\]

\[
+ C(I - Z(z)A)^{-1} [Z(z) - A^* + (I - Z(z)A)A^*](I - AA^*)^{-\frac{1}{2}}
\]

\[
= -CA^*(I - AA^*)^{-\frac{1}{2}} + C(I - Z(z)A)^{-1}Z(z)(I - AA^*)^{\frac{1}{2}}
\]

and the connecting operator \( \begin{pmatrix} A & (I-AA^*)^{\frac{1}{2}} \\ C & -CA^*(I-AA^*)^{-\frac{1}{2}} \end{pmatrix} \) is isometric. Then the representation (6.10) amounts to \( \mathcal{M} = F \cdot H^2_{A,d} (\mathbb{F}_d^+) \) and hence, Theorem 6.1 reduces to Theorem 3.13.

**Remark 6.5.** If we substitute formulas (5.35), (5.36) (for \( \ell = 1 \)) into (5.41), we get

\[
\Psi_n(z) = -(I - A^*A)^{-\frac{1}{2}}A^*(I - AA^*)^{\frac{1}{2}}W_n
\]

\[
+ (I - A^*A)^{\frac{1}{2}}(I - Z(z)A)Z(z)(I - AA^*)^{\frac{1}{2}}W_n
\]

\[
= -A^*W_n + (I - A^*A)^{\frac{1}{2}}(I - Z(z)A)Z(z)(I - AA^*)^{\frac{1}{2}}W_n,
\]

i.e., \( \Psi_n \) is the characteristic function of the row-contraction \( A \). To get \( F_n \), we multiply \( \Psi_n \) by \( C(I - Z(z)A)^{-n+1} \) on the left. If \( n = 1 \), we multiply \( F_n \) just by \( C \). Note that in this case the final formula makes sense even without the imposition of the assumption (5.36).

### 6.2. Non-orthogonal Beurling-Lax representations based on wandering subspaces

We now take a look at the wandering subspace \( \mathcal{E} \) (see (6.11)) of a closed \( S_{\omega,R} \)-invariant space \( \mathcal{M} \) isometrically included into \( H^2_{A,Y} (\mathbb{F}_d^+) \). As we will see later, the isometric representation \( \mathcal{E} = \Theta \cdot \mathcal{U} \) indeed exists and gives rise to the (essentially unique) \( H^2_{A,Y} (\mathbb{F}_d^+) \)-Bergman inner multiplier \( \Theta \). In this subsection we will focus on the analog of the Beurling-Lax representation (6.1) which cannot be orthogonal as the subspaces \( S_{\omega,R}^0 \mathcal{E} \) and \( S_{\omega,R}^\beta \mathcal{E} \) (for nonempty \( \alpha \neq \beta \)) are not orthogonal in general. Thus, the best one can hope for is to recover a \( S_{\omega,R} \)-invariant subspace \( \mathcal{M} \) of its wandering subspace \( \mathcal{E} \) via the closed linear span \( \mathcal{M} = \bigvee_{n \in \mathbb{F}_d^+} S_{\omega,R}^0 \mathcal{E} \).

In the single-variable Bergman space \( A_{2,Y} \) this hope was realized in the seminal work of Aleman, Richter and Sundberg [6]. Later elaborations due to Shimorin [97, 98] showed that the same result holds in the space \( A_{3,Y} \) but in general, not in \( A_{n,Y} \) for \( n > 3 \). In [62], the result from [6] was recaptured via a substantial simplification of Shimorin’s approach. In this section, we adapt the Izuchis’ approach [62] to the noncommutative setting of \( A_{2,Y} (\mathbb{F}_d^+) \). The main result of the section is Theorem 6.8 below. We start with some needed preliminaries.
Theorem 6.6. Let \( T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{H})^d \) be a \(*\)-strongly stable \(d\)-tuple:

\[
\lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}^+_d, |\alpha| = N} ||T^{\alpha}h||^2 = 0 \quad \text{for all} \quad h \in \mathcal{H}.
\] (6.16)

Assume that the operators \( T_1, \ldots, T_d \) are left-invertible (that is, there is \( c > 0 \) such that \( ||T_jh|| \geq c ||h|| \) for all \( h \in \mathcal{H} \) and \( j = 1, \ldots, d \)) and have orthogonal ranges:

\[
\langle T_ih, T_jh' \rangle_{\mathcal{H}} = 0 \quad \text{for all} \quad h, h' \in \mathcal{H} \quad \text{and} \quad i \neq j.
\] (6.17)

Let us also assume that

\[
||T_jh||^2 + \sum_{i=1}^d ||T_i^*T_jh||^2 \leq 2||T_j^*T_jh||^2 \quad \text{for all} \quad h \in \mathcal{H}
\] (6.18) and \( j \in \{1, \ldots, d\} \). Then \( \mathcal{H} = \bigvee_{\alpha \in \mathbb{F}^+_d} T^\alpha \mathcal{E} \) where \( \mathcal{E} = \mathcal{H} \ominus \left( \bigoplus_{j=1}^d T_j \mathcal{H} \right) \).

Proof. Since \( T_j \) is bounded below, \( T_j \mathcal{H} \) is a closed subspace of \( \mathcal{H} \). Since \( T_j \mathcal{H} \) is orthogonal to \( T_i \mathcal{H} \) for \( j \neq i \), the formula for \( \mathcal{E} \) makes sense. Furthermore, the operator \( (T_j^*T_j)^{-1} \) is bounded and letting \( h = (T_j^*T_j)^{-1}y \) in (6.18) we conclude that

\[
\langle (T_j^*T_j)^{-1}y, y \rangle_{\mathcal{H}} + \sum_{i=1}^d ||T_i^*y||^2 \leq 2||y||^2_{\mathcal{H}}
\]

for all \( y \in \mathcal{H} \), which can be written in the operator form as

\[
(T_j^*T_j)^{-1} + \sum_{i=1}^d T_iT_i^* \preceq 2I_{\mathcal{H}}.
\] (6.19)

Therefore, \( (T_j^*T_j)^{-1} + T_jT_j^* \preceq 2I_{\mathcal{H}} \), which implies (see [62] p. 444) for the proof \( (T_j^*T_j)^{-1} \succeq I_{\mathcal{H}} \). Substituting the latter inequality into (6.19) we conclude that \( T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{H})^d \) is a row-contraction:

\[
\sum_{j=1}^d ||T_j^*h||^2 \leq ||h||^2 \quad \text{for all} \quad h \in \mathcal{H}.
\] (6.20)

Let us assume that \( \bigvee_{\alpha \in \mathbb{F}^+_d} T^\alpha \mathcal{E} \) is properly contained in \( \mathcal{H} \). Then there is \( h \in \mathcal{H} \) that is orthogonal to \( T^\alpha \mathcal{E} \) for each \( \alpha \in \mathbb{F}^+_d \). For this element, \( T^{\alpha}h \) is orthogonal to \( \mathcal{E} \) for each \( \alpha \in \mathbb{F}^+_d \). Hence,

\[
T^{\alpha}h \in \mathcal{H} \ominus \mathcal{E} = \bigoplus_{j=1}^d T_j \mathcal{H},
\]

so that for every fixed \( \alpha \in \mathbb{F}^+_d \), there exist \( g_{a1}, \ldots, g_{ad} \in \mathcal{H} \) such that

\[
T^{\alpha}h = \sum_{\ell=1}^d T_\ell g_{a\ell}.
\] (6.21)

Due to the orthogonality condition (6.17),

\[
||T^{\alpha}h||^2 = \sum_{j=1}^d ||T_jg_{a_j}||^2.
\]
On the other hand, combining (6.21) with (6.17) gives
\[ T_j^* T^{*\alpha} h = \sum_{\ell=1}^{d} T_j^* T_\ell g_\alpha \ell = T_j^* T_j g_\alpha j \text{ for } j = 1, \ldots, d. \]
Therefore, for each fixed \( \alpha \in \mathbb{F}_d^+ \),
\[ \| T^{*\alpha} h \|^2 - 2 \sum_{j=1}^{d} \| T_j^* T^{*\alpha} h \|^2 + \sum_{i,j=1}^{d} \| T_i^* T_j^* T^{*\alpha} h \|^2 \]
\[ = \sum_{j=1}^{d} \| T_j g_\alpha \|^2 - 2 \sum_{j=1}^{d} \| T_j^* T_j g_\alpha j \|^2 + \sum_{i,j=1}^{d} \| T_i^* T_j^* T_j g_\alpha j \|^2 \]
\[ = \sum_{j=1}^{d} \left( \| T_j g_\alpha \|^2 - 2 \| T_j^* T_j g_\alpha \|^2 + \sum_{i=1}^{d} \| T_i^* T_j^* T_j g_\alpha j \|^2 \right) \leq 0, \quad (6.22) \]
where the last inequality follows from (6.18) (applied to \( g_\alpha j \) instead of \( h \)). If we let
\[ R_N = \sum_{\alpha \in \mathbb{F}_d^+:|\alpha|=N} \| T^{*\alpha} h \|^2, \]
then (6.10) tells us that \( R_N \) tends to zero as \( N \to \infty \). Making use of the inequality (6.20) (applied to \( T^{*\alpha} h \) rather than \( h \)) we see that this convergence is monotone:
\[ R_{N+1} = \sum_{\alpha \in \mathbb{F}_d^+:|\alpha|=N+1} \| T^{*\alpha} h \|^2 = \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+:|\alpha|=N} \| T_j^* T^{*\alpha} h \|^2 \]
\[ = \sum_{\alpha \in \mathbb{F}_d^+:|\alpha|=N} \sum_{j=1}^{d} \| T_j^* T^{*\alpha} h \|^2 \]
\[ \leq \sum_{\alpha \in \mathbb{F}_d^+:|\alpha|=N} \| T^{*\alpha} h \|^2 = R_N. \]
Summing up inequalities (6.22) over all elements \( \alpha \in \mathbb{F}_d^+ \) with \( |\alpha| = N \) we get
\[ R_N - 2R_{N+1} + R_{N+2} \leq 0 \]
which together with the preceding inequality implies
\[ R_{N+1} - R_{N+2} \geq R_N - R_{N+1} \geq 0. \]
Since \( R_N \to 0 \) as \( N \to \infty \), it follows that \( R_N = R_{N+1} \) for all \( N \geq 0 \). In particular,
\( R_N = R_0 = \| T^{*0} h \|^2 = \| h \|^2 \) as \( N \to \infty \) so that \( \| h \| = 0. \)

**Theorem 6.7.** The shift \( d \)-tuple \( S_{n,R} \) satisfies all assumptions of Theorem 6.6 but (6.18). The tuple \( S_{2,R} \) satisfies equalities
\[ \sum_{i=1}^{d} S_{2,R,i} S_{2,R,i}^* S_{2,R,j} S_{2,R,j} f = 2S_{2,R,j}^* S_{2,R,j} f - f, \quad (6.23) \]
\[ \| S_{2,R,j} f \|^2 + \sum_{i=1}^{d} \| S_{2,R,i} S_{2,R,i}^* S_{2,R,j} S_{2,R,j} f \|^2 = 2\| S_{2,R,j}^* S_{2,R,j} f \|^2 \quad (6.24) \]
for all \( f \in A_{2,Y}(\mathbb{F}_d^+) \) and \( j = 1, \ldots, d. \)
PROOF. Strong stability of $S^*_{n,R}$ was established in Proposition 4.23. Orthogonality of Ran $S_{n,R,j}$ and Ran $S_{n,R,i}$ for $i \neq j$ is evident. Boundedness below follows from the estimate
\[ \|S_{n,R,j}f\|^2 = \sum_{\alpha \in \mathbb{F}^+_d} \mu_{n,|\alpha|+1} \|f_\alpha\|^2 \geq \left( \inf_{\alpha \in \mathbb{F}^+_d} \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|}} \right) \cdot \sum_{\alpha \in \mathbb{F}^+_d} \mu_{n,|\alpha|} \|f_\alpha\|^2 = \frac{1}{n} \|f\|^2. \]
To verify (6.24), take $f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in A_n,\mathcal{Y}(\mathbb{F}^+_d)$ and observe that by (4.74),
\[ S^*_{n,R,j}S_{n,R,j}f = \sum_{\alpha \in \mathbb{F}^+_d} \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|}} f_\alpha z^\alpha, \quad (6.25) \]
so that
\[ \|S^*_{n,R,j}S_{n,R,j}f\|^2 = \sum_{\alpha \in \mathbb{F}^+_d} \mu_{n,|\alpha|} \left( \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|}} \right)^2 = \sum_{\alpha \in \mathbb{F}^+_d} \frac{\mu_{n,|\alpha|+1}^2}{\mu_{n,|\alpha|}^2} \|f_\alpha\|^2. \]
From the formulas (4.62) specialized to the present setting of $\omega = \mu_n$, we have
\[ S_{n,R,i}S^*_{n,R,i}f = \sum_{\alpha \in \mathbb{F}^+_d} \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|}} f_\alpha z^{\alpha_i} \]
for $i = 1, \ldots, d$, and therefore,
\[ \sum_{i=1}^d S_{n,R,i}S^*_{n,R,i}f = \sum_{i=1}^d \sum_{\alpha \in \mathbb{F}^+_d} \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|}} f_\alpha z^{\alpha_i} = \sum_{\alpha \in \mathbb{F}^+_d: \alpha \neq \emptyset} \frac{\mu_{n,|\alpha|-1}}{\mu_{n,|\alpha|}} f_\alpha z^{\alpha}, \]
which being combined with (6.25), gives
\[ \sum_{i=1}^d S_{n,R,i}^* S_{n,R,i} S^*_{n,R,i} S_{n,R,j} f = \sum_{\alpha \in \mathbb{F}^+_d: \alpha \neq \emptyset} \frac{\mu_{n,|\alpha|-1}}{\mu_{n,|\alpha|}} f_\alpha z^{\alpha} = \sum_{\alpha \neq \emptyset} \frac{\mu_{n,|\alpha|+1}}{\mu_{n,|\alpha|-1}} f_\alpha z^{\alpha}. \quad (6.26) \]
Making use of the latter equality and (6.25), we get
\[ \sum_{i=1}^d \|S^*_{n,R,i}S_{n,R,i}S_{n,R,j}f\|^2 = \sum_{i=1}^d \langle S_{n,R,i}^* S_{n,R,i} S^*_{n,R,i} S_{n,R,j} f, S^*_{n,R,j} S_{n,R,j} f \rangle \]
\[ = \sum_{\alpha \neq \emptyset} \frac{\mu_{n,|\alpha|+1}^2}{\mu_{n,|\alpha|-1}} \|f_\alpha\|^2. \quad (6.27) \]
We finally observe that for each $j \in \{1, \ldots, d\}$,
\[ \|S_{n,R,j}f\|^2 = \sum_{\alpha \in \mathbb{F}^+_d} \mu_{n,|\alpha|+1} \|f_\alpha\|^2 \]
which together with (6.26) and (6.27) brings us to
\[ \|S_{n,R,j}f\|^2 - 2 \|S^*_{n,R,j} S_{n,R,j} f\|^2 + \sum_{i=1}^d \|S_{n,R,i}^* S_{n,R,i} S_{n,R,j} f\|^2 = \sum_{\alpha \neq \emptyset} \mu_{n,|\alpha|+1} \left( \frac{1}{\mu_{n,|\alpha|+1}} - 2 \frac{1}{\mu_{n,|\alpha|} + \mu_{n,|\alpha|-1}} \right) \|f_\alpha\|^2 + \left( \mu_{n,1} - 2 \mu_{n,1}^2 \right) \|f_\emptyset\|^2. \quad (6.28) \]
Making use of the binomial identity
\[ \mu_{n,|\alpha|+1}^{n,|\alpha|+1} - 2\mu_{n,|\alpha|}^{n,|\alpha|} + \mu_{n-2,|\alpha|+1}^{n-2,|\alpha|+1} (\alpha \neq \emptyset) \]
which is easily verified using the definition (1.8) (or by equating the coefficients in
the power-series identity \((1-\lambda)^{-n}(1-\lambda)^2 = (1-\lambda)^{-(n-2)}\) and taking into account
that \(\mu_{n,1} = \frac{1}{n}\) we write (6.28) as
\[ \|S_{n,R,j}^* f\|^2 - 2\|S_{n,R,j}^* S_{n,R,j}^* f\|^2 + \sum_{i=1}^{d} \|S_{n,R,i}^* S_{n,R,j}^* S_{n,R,j}^* f\|^2 \]
\[ = \frac{n-2}{n^2}\|f_0\|^2 + \sum_{\alpha \neq \emptyset} \mu_{n-2,|\alpha|+1}^{n-2,|\alpha|+1} \|f_\alpha\|^2 \]
\[ = \frac{n-2}{n^2}\|f_0\|^2 + \sum_{\alpha \neq \emptyset} (n-1)(n-2)\mu_{n,|\alpha|+1}^{n,|\alpha|+1} \|f_\alpha\|^2. \]
Letting \(n = 2\) in the latter equality we get (6.24). Letting \(n = 2\) in (6.26) we get
\[ \sum_{i=1}^{d} S_{2,R,i}^* S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j}^* f = \sum_{\alpha \in \mathbb{F}^+_{d}} \frac{|\alpha|}{|\alpha| + 2} f_\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}^+_{d}} \frac{|\alpha|}{|\alpha| + 2} f_\alpha z^\alpha, \]
while (6.25) (for \(n = 2\)) implies
\[ (2S_{2,R,j}^* S_{2,R,j} - I)f = \sum_{\alpha \in \mathbb{F}^+_{d}} \left( \frac{2|\alpha|}{|\alpha| + 2} - 1 \right) f_\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}^+_{d}} \frac{|\alpha|}{|\alpha| + 2} f_\alpha z^\alpha. \]
The two latter equalities imply (6.27).

Now let \(\mathcal{H} = \mathcal{M}\) be an \(S_{2,R}\)-invariant closed subspace of \(\mathcal{A}_{2,Y}(\mathbb{F}^+_d)\) and let \(T\)
be the restriction of \(S_{2,R}\) to \(\mathcal{M}\):
\[ T = (T_1, \ldots, T_d), \text{ where } T_j = S_{2,R,j}|_{\mathcal{M}} \text{ for } j = 1, \ldots, d. \] (6.29)
By Theorem 6.7, \(T\) is a strongly stable row contraction and the operators \(T_1, \ldots, T_d\)
are bounded below and their ranges are pair-wise orthogonal. Therefore, \(T\) meets
all the assumptions of Theorem 6.6 but, perhaps, (6.18). Let us show that it does
satisfy (6.18). To this end, we first observe that \(T_j^* = P_M S_{2,R,j}^*\). We next observe that for \(f \in \mathcal{M}\),
\[ S_{2,R,j}^* S_{2,R,j} f = (P_M + P_{M^\perp}) S_{2,R,j}^* S_{2,R,j} f \]
\[ = T_j^* T_j f + P_{M^\perp} S_{2,R,j}^* S_{2,R,j} f, \] (6.30)
\[ S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f = (P_M + P_{M^\perp}) S_{2,R,i}^* (P_M + P_{M^\perp}) S_{2,R,j}^* S_{2,R,j} f \]
\[ = P_M S_{2,R,i}^* P_M S_{2,R,j}^* S_{2,R,j} f + P_{M^\perp} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f \]
\[ = T_i^* T_j^* T_j f + P_{M^\perp} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f, \] (6.31)
where we used the equality \(P_M S_{2,R,i}^*|_{\mathcal{M}^\perp} = 0\) to get (6.31). Since the terms on the
right sides of (6.30) and (6.31) are orthogonal and since \(\mathcal{M}\) is included in \(A_{2,Y}(\mathbb{F}^+_d)\)
isometrically, we have
\[ \|S_{2,R,j}^* S_{2,R,j} f\|^2_\mathcal{A}_{2,Y}(\mathbb{F}^+_d) = \|T_j^* T_j f\|^2_\mathcal{M} + \|P_{M^\perp} S_{2,R,j}^* S_{2,R,j} f\|^2_\mathcal{A}_{2,Y}(\mathbb{F}^+_d), \]
\[ \|S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f\|^2_\mathcal{A}_{2,Y}(\mathbb{F}^+_d) = \|T_i^* T_j^* T_j f\|^2_\mathcal{M} + \|P_{M^\perp} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f\|^2_\mathcal{A}_{2,Y}(\mathbb{F}^+_d). \]
Substituting the two latter equalities into (6.24) gives
\[
\|T_j f\|_M^2 + \sum_{i=1}^d \|T_i^* T_j^* T_j f\|_M^2 - 2 \|T_j^* T_j f\|_M^2
\]
\[
= 2 \|P_{M+} S_{2,R,j}^* S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2 - \sum_{i=1}^d \|P_{M+} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2. \tag{6.32}
\]

In the next calculation, the first equality follows from (6.23), the second one holds since \( f \in M \) and since \( P_{M+} S_{2,R,j}\)|\(M = 0 \), the third inequality holds since we drop the projection, the fourth equality holds since the ranges of \( S_{2,R,i}^* \)'s are orthogonal, and the last inequality holds since \( S_{2,R,i}^* \) is a contraction:
\[
2 \|P_{M+} S_{2,R,j}^* S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2
\]
\[
= \frac{1}{2} \|P_{M+} (I + \sum_{i=1}^d S_{2,R,i} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j}) f\|_{A_{2,Y}(F^+_d)}^2
\]
\[
= \frac{1}{2} \|P_{M+} \sum_{i=1}^d S_{2,R,i} P_{M+} S_{2,R,i}^* S_{2,R,j}^* S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2
\]
\[
\leq \frac{1}{2} \|\sum_{i=1}^d S_{2,R,i} P_{M+} S_{2,R,i}^* S_{2,R,j} S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2
\]
\[
= \frac{1}{2} \|S_{2,R,i} P_{M+} S_{2,R,i}^* S_{2,R,j} S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2
\]
\[
\leq \frac{1}{2} \|P_{M+} S_{2,R,i} S_{2,R,j} S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2.
\]

Combining this result with (6.32) gives us
\[
\|T_j f\|_M^2 + \sum_{i=1}^d \|T_i^* T_j^* T_j f\|_M^2 - 2 \|T_j^* T_j f\|_M^2
\]
\[
\leq - \frac{1}{2} \sum_{i=1}^d \|P_{M+} S_{2,R,i} S_{2,R,j} S_{2,R,j} f\|_{A_{2,Y}(F^+_d)}^2 \leq 0.
\]

We conclude that the tuple (6.29) satisfies (6.18), that is, all assumptions of Theorem 6.6. We arrive at the following version of the Beurling-Lax theorem.

**Theorem 6.8.** Let \( M \) be an \( S_{2,R} \)-invariant closed subspace of \( A_{2,Y}(F^+_d) \). Then
\[
M = \bigvee_{\alpha \in F^+_d} S_{2,R} \mathcal{E}, \quad \text{where} \quad \mathcal{E} = M \ominus \left( \bigoplus_{j=1}^d S_{2,R,j} M \right).
\]
CHAPTER 7

Orthogonal Beurling-Lax representations based on wandering subspaces

In this chapter we present the most comprehensive (in our opinion) and precise version of the Beurling-Lax theorem for weighted Hardy-Fock spaces based on the notion of a Bergman-inner family and relying on realization formulas for such a family. The realization results are then applied to study expansive multiplier properties of Bergman-inner multipliers.

7.1. Transfer functions $\Theta_{\omega, U_\beta}$ and metric constraints

In this section we take a closer look at the transfer functions $\Theta_{\omega, U_\beta}$ introduced in Section 2.2 by the realization formula (2.39). Let us suppose now that $(C, A)$ is an $\omega$-output stable pair and that we construct the formal power series $\Theta_{\omega, U_\beta}$ according to (2.39), where the operators $B_{j,\beta}$ and $D_\beta$ are still to be determined:

$$\Theta_{\omega, U_\beta}(z) = \omega_{|\beta|}^{-1} D_\beta + \sum_{j=1}^{d} \sum_{\beta' \in \mathbb{F}_+^j} \omega_{|\beta|+|\beta'|+1}^{-1} C A^{\beta'} B_{j,\beta} z^{\beta'}. \quad (7.1)$$

In any case, $\Theta_{\omega, U_\beta}$ induces the bounded operator $M_{\Theta_{\omega, U_\beta}} : U_\beta \to H^2_{\omega, Y}(\mathbb{F}_+^d)$. We next impose some additional metric relations on the connection matrix (1.33), specifically one or more of the relations

$$\sum_{j=1}^{d} A_j^* \Theta_{\omega, |\beta|+1, C, A} B_{j,\beta} + \omega_{|\beta|}^{-1} C^* D_\beta = 0, \quad (7.2)$$

$$\sum_{j=1}^{d} B_{j,\beta}^* \Theta_{\omega, |\beta|+1, C, A} B_{j,\beta} + \omega_{|\beta|}^{-1} D_\beta^* D_\beta \preceq I_{U_\beta}, \quad (7.3)$$

$$\sum_{j=1}^{d} B_{j,\beta}^* \Theta_{\omega, |\beta|+1, C, A} B_{j,\beta} + \omega_{|\beta|}^{-1} D_\beta^* D_\beta = I_{U_\beta}, \quad (7.4)$$

and show how these lead to boundedness and orthogonality properties for the associated multiplication operator $M_{\Theta_{\omega, U_\beta}}$. Observe that the above relations can be written in terms of the column operators $A$ and $\widehat{B}_\beta$ in (1.33) as

$$A^* (\Theta_{\omega, |\beta|+1, C, A} \otimes I_{\mathbb{F}}) \widehat{B}_\beta + \omega_{|\beta|}^{-1} C^* D_\beta = 0,$$

$$B_{\beta}^* (\Theta_{\omega, |\beta|+1, C, A} \otimes I_{\mathbb{F}}) \widehat{B}_\beta + \omega_{|\beta|}^{-1} D_\beta^* D_\beta \preceq I_{U_\beta},$$

$$\widehat{B}_\beta^* (\Theta_{\omega, |\beta|+1, C, A} \otimes I_{\mathbb{F}}) \widehat{B}_\beta + \omega_{|\beta|}^{-1} D_\beta^* D_\beta = I_{U_\beta}.$$
orthogonal to the matrix inequality
\[
\begin{bmatrix}
A^* & C^* \\
\hat{B}_{\beta}^* & D_{\beta}^*
\end{bmatrix}
\begin{bmatrix}
\mathbf{G}_{\omega,|\beta|+1,\mathcal{C},\mathcal{A}} \otimes I_d \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
I_{\omega_{|\beta|}} & I_y
\end{bmatrix}
\begin{bmatrix}
A \hat{B}_{\beta} \\
C & D_{\beta}
\end{bmatrix}
\begin{bmatrix}
\mathbf{G}_{\omega,|\beta|,\mathcal{C},\mathcal{A}} \\
0 \\
0
\end{bmatrix}
I_{U_{\beta}}, \quad (7.5)
\]
while the equalities (7.2) and (7.4) are equivalent to the matrix equality
\[
\begin{bmatrix}
A^* & C^* \\
\hat{B}_{\beta}^* & D_{\beta}^*
\end{bmatrix}
\begin{bmatrix}
\mathbf{G}_{\omega,|\beta|+1,\mathcal{C},\mathcal{A}} \otimes I_d \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
A \hat{B}_{\beta} \\
C & D_{\beta}
\end{bmatrix}
\begin{bmatrix}
\mathbf{G}_{\omega,|\beta|,\mathcal{C},\mathcal{A}} \\
0 \\
0
\end{bmatrix}
I_{U_{\beta}}. \quad (7.6)
\]
The latter two conditions are of metric nature; they express the contractivity or isometric property of the connection operator \(U_{\beta} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) with respect to certain weights.

**Lemma 7.1.** Let \((C,A)\) be an \(\omega\)-output stable pair and let \(\Theta_{\omega, U_{\beta}}\) be defined as in (7.30) for some \(\beta \in \mathbb{F}_d^+\) and operators \(B_{1,\beta}, \ldots, B_{d,\beta} \in \mathcal{L}(U_{\beta}, \mathcal{X})\) and \(D_{\beta} \in \mathcal{L}(U_{\beta}, \mathcal{Y})\).

(1) If equality (7.2) holds, then
(a) \(O_{\omega, C, A, x}\) is orthogonal to \(S_{\omega, R}^{(\beta)} \Theta_{\omega, U_{\beta}} u\) for all \(\beta \in \mathbb{F}_d^+, x \in \mathcal{X}\) and \(u \in U_{\beta}\).
(b) \(S_{\omega, R}^{(\beta)} \Theta_{\omega, U_{\beta}} u\) is orthogonal to \(S_{\omega, R}^{(\beta, \gamma)} \Theta_{\omega, U_{\beta}} u'\) for all \(\beta \in \mathbb{F}_d^+\), \(u, u' \in U_{\beta}\), and all \(\beta \neq \gamma\) in \(\mathbb{F}_d^+\).
(c) \(S_{\omega, R}^{(\gamma, \beta)} \Theta_{\omega, U_{\beta}} u\) and \(S_{\omega, R}^{(\gamma, \beta)} \Theta_{\omega, U_{\beta}} u'\) are both orthogonal to \(S_{\omega, R}^{(\beta, \gamma)} \Theta_{\omega, U_{\beta}} u'\) for all \(\gamma \neq \emptyset\) and \(\beta \) in \(\mathbb{F}_d^+\) and for any \(u, u' \in U_{\beta}\).

(2) Moreover:
(a) If inequality (7.3) holds, then the operator \(S_{\omega, R}^{(\beta)} M_{\Theta_{\omega, U_{\beta}}}\) is a contraction from \(U_{\beta}\) into \(H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)\).
(b) If both (7.2) and (7.3) hold, i.e., if (7.5) holds, then the operator \(S_{\omega, R}^{(\beta)} M_{\Theta_{\omega, U_{\beta}}}\) is a contraction from the Fock space \(H_{U_{\beta}}^2(\mathbb{F}_d^+)\) into \(H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)\).

(3) Similarly:
(a) If equality (7.4) holds, then the operator \(S_{\omega, R}^{(\beta)} M_{\Theta_{\omega, U_{\beta}}}\) is an isometry from \(U_{\beta}\) into \(H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)\).
(b) If (7.2) and (7.4) hold, i.e., if (7.6) holds, then for every \(f \in H_{U_{\beta}}^2(\mathbb{F}_d^+)\),
\[
\|S_{\omega, R}^{(\beta)} \Theta_{\omega, U_{\beta}} f\|_{H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)}^2 = \|f\|_{H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)}^2
\]
\[
- \sum_{\gamma \in \mathbb{F}_d^+} \sum_{j=1}^d \|(I - S_{\omega, R_{\gamma}} S_{\omega, R_{\gamma}}) \Theta_{\omega, U_{\beta}} (S_{\omega, R_{\gamma}}) f\|_{H_{\omega, \mathcal{Y}}^2(\mathbb{F}_d^+)}^2.
\]
(c) If (7.6) holds, then
\[
\omega_{|\beta|} I_{U_{\beta}} - \Theta_{\omega, U_{\beta}} (z) \Theta_{\omega, U_{\beta}} (\zeta) = - \omega_{|\beta|} \hat{B}_{\beta} R_{\omega, |\beta|} (A Z(z))^* (\mathbf{G}_{\omega, |\beta|+1, \mathcal{C}, \mathcal{A}} \otimes I_d) R_{\omega, |\beta|} (A Z(\zeta)) \hat{B}_{\beta} (7.8)
\]
We then make use of expansions (4.3), (7.9) and the definition of the inner product where the fourth and the fifth equality follow from (4.58) and (7.2), respectively. The latter computation verifies part (1a).

Remark 7.2. In the terminology of Section 3.3, the content of parts (2b) and (3a) of Lemma 7.1 can be phrased as follows: if (3a) holds, then the formal power series \( \Theta_{\omega, u_\beta}(z)z^\beta \) is \( H^2_{\omega, Y}(F^+_d) \)-Bergman-inner.

Proof of (1): From the power series expansion (7.1) we have

\[
S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta}(z) = \omega_{|\beta|}^{-1} D_\beta z^\beta + \sum_{j=1}^{d} \sum_{\beta' \in F^+_d} \omega_{|\beta'| + |\beta| + 1}^{-1} C A^{\beta'} B_{j, \beta} z^{\beta' j \beta}. \quad (7.9)
\]

We then make use of expansions (4.3), (7.9) and the definition of the inner product in \( H^2_{\omega, Y}(F^+_d) \) to get

\[
\langle S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta}, C_{\omega, C} A x \rangle_{H^2_{\omega, Y}(F^+_d)} = \omega_{|\beta|}^{-1} \langle D_\beta u, C A^T x \rangle_D + \sum_{j=1}^{d} \sum_{\beta' \in F^+_d} \omega_{|\beta'| + |\beta| + 1}^{-1} \langle C A^{\beta'} B_{j, \beta} u, C A^{\beta' j \beta} x \rangle_D
\]

\[
= \left( \omega_{|\beta|}^{-1} C^T D_\beta + \sum_{j=1}^{d} A_j^T \left( \sum_{\beta' \in F^+_d} \omega_{|\beta'| + |\beta| + 1}^{-1} A^{\beta' T} C A^{\beta'} B_{j, \beta} \right) \right) u, A^T x
\]

\[
= \left( \omega_{|\beta|}^{-1} C^T D_\beta + \sum_{j=1}^{d} A_j \Theta_{\omega, |\beta| + 1, C} B_{j, \beta} \right) u, A^T x
\]

where the fourth and the fifth equality follow from (4.58) and (7.2), respectively. The latter computation verifies part (1a).

The verification of (1b) goes through several cases.

If \( |\beta| = |\beta'| \) and \( \beta \neq \beta' \), then any word of the form \( \alpha \beta \) is distinct from any work of the form \( \alpha' \beta \), where \( \alpha \) and \( \alpha' \) are independently arbitrary in \( F^+_d \). Note that the power series representations of \( S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta} u \) and \( S_{\omega, R}^{\beta' T} \Theta_{\omega, u_\beta'} u' \) have the form

\[
S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta} = \sum_{\alpha \in F^+_d} [\Theta_{\omega, u_\beta}]_{\alpha} u z^\alpha, \quad S_{\omega, R}^{\beta' T} \Theta_{\omega, u_\beta'} = \sum_{\alpha' \in F^+_d} [\Theta_{\omega, u_\beta'}]_{\alpha'} u' z^{\alpha' \beta},
\]

so the orthogonality \( S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta} u \perp S_{\omega, R}^{\beta' T} \Theta_{\omega, u_\beta'} u' \) holds due to the pairwise orthogonality of the monomials in the respective power series expansions. Hence we now need only consider the case where \( |\beta| \neq |\beta'| \).

Without loss of generality we may assume that \( |\beta| > |\beta'| \), so \( \beta = \delta \beta'' \) where \( \delta \neq 0 \) and \( |\beta''| = |\beta'| \). In case \( \beta' \neq \beta'' \), then every word of the form \( \alpha \beta = \alpha \delta \beta'' \) is distinct from any word of the form \( \alpha' \beta' \) for independently arbitrary words \( \alpha \) and \( \alpha' \) in \( F^+_d \), since the respective right tails of length \( |\beta''| = |\beta'| \) disagree. Again we have pairwise orthogonality of the monomials in the respective power series expansions for \( S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta} u \) and \( S_{\omega, R}^{\beta' T} \Theta_{\omega, u_\beta'} u' \), so the desired orthogonality \( S_{\omega, R}^{\beta^T} \Theta_{\omega, u_\beta} u \perp S_{\omega, R}^{\beta' T} \Theta_{\omega, u_\beta'} u' \) again holds.

It remains only to consider the case where \( \beta \) has the form \( \beta = \delta \beta' \) where \( \delta \neq 0 \). In this case the pairwise orthogonal of the monomials in the respective power series expansion fails and we must do a detailed calculation of the inner product which is
supposed to be zero. Toward this end, we use the formula (7.23) to see that

\[
(S^{\beta^*}_{\omega, R} \Theta_{\omega, u, \beta^*} u') (z) = \omega_{|\beta|}^{-1} D_{\beta^*} u' z^{\beta^*} + \sum_{j=1}^d \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\beta^*|+1}^{-1} C A^{\gamma} B_{j', \beta^*} u z^{\gamma j' \beta^*} u',
\]

\[
(S^{\beta^*}_{\omega, R} \Theta_{\omega, u, \beta^*} u) (z) = (S^{(\delta^* \beta')}_{\omega, R} \Theta_{\omega, u, \delta^* \beta^*} u) (z)
\]

\[
= \omega_{|\delta|+|\beta^*|}^{-1} D_{\delta^* \beta^*} u z^{\delta^* \beta^*} + \sum_{j=1}^d \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\delta^*|+|\beta^*|+1}^{-1} C A^{\gamma} B_{j, \delta^* \beta^*} z^{\gamma j \delta^* \beta^*} u.
\]

It is convenient to write the word \( \delta \) in the form \( \delta = \tilde{\delta} \bar{j} \) where \( \tilde{\delta} \) is a (possibly empty) word and \( \bar{j} \) is the right-most letter in the nonempty word \( \delta \). Using the latter representations we can now compute

\[
\langle S^{\delta^*}_{\omega, R} \Theta_{\omega, u, \delta^* \beta^*} u, S^{\beta^*}_{\omega, R} \Theta_{\omega, u, \beta^*} u' \rangle_{H^2_{\omega, \gamma}(\mathbb{F}_d^+)}
\]

\[
= \omega_{|\delta|+|\beta^*|}^{-1} \langle D_{\delta^* \beta^*} u, C A^{\delta^* \tilde{\delta} \beta^*} B_{j, \delta^* \beta^*} u' \rangle_Y
\]

\[
+ \sum_{j=1}^d \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\delta|+|\beta^*|+1}^{-1} \langle C A^{\gamma} B_{j', \delta^* \beta^*} u, C A^{\gamma} B_{j, \delta^* \beta^*} u' \rangle_Y
\]

\[
= \langle (\omega_{|\delta|+|\beta^*|}^{-1} C^* D_{\delta^* \beta^*} u + \sum_{j=1}^d A_j^* \left( \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\delta|+|\beta^*|+1}^{-1} A^{\gamma \gamma^*} C A^{\gamma^*} \right) B_{j', \delta^* \beta^*} u, A^{\delta^* \tilde{\delta} \beta^*} B_{j, \delta^* \beta^*} u' \rangle_Y
\]

\[
= \langle (\omega_{|\delta|+|\beta^*|}^{-1} C^* D_{\delta^* \beta^*} u + \sum_{j=1}^d A_j^* \Theta_{\omega,|\delta|+|\beta^*|+1, \mathcal{C}_A} B_{j, \delta^* \beta^*} u, A^{\delta^* \tilde{\delta} \beta^*} B_{j, \delta^* \beta^*} u' \rangle_X = 0,
\]

where the two last steps follow respectively from the definition (4.58) of \( \Theta_{\omega, k, \mathcal{C}; A} \) and equality (7.2) with \( \beta = \delta^* \tilde{\beta} \). This completes the verification of (1b).

Verification of part (1c) is quite similar: let us denote by \( \bar{j} \in \{1, \ldots, d\} \) the rightmost letter in the given \( \gamma \neq \emptyset \) so that \( \gamma = \bar{j} \gamma \). We then see from (7.23) and a similar expansion for \( S^{(\gamma \bar{j})^*}_{\omega, R} \Theta_{\omega, u, \gamma} \) that

\[
\langle S^{(\gamma \bar{j})^*}_{\omega, R} \Theta_{\omega, u} u, S^{(\beta^* \gamma\bar{j})^*}_{\omega, R} \Theta_{\omega, u, \beta^*} u' \rangle_{H^2_{\omega, \gamma}(\mathbb{F}_d^+)}
\]

\[
= \omega_{|\gamma|+|\bar{j}|}^{-1} \langle D_{\gamma \bar{j}} u, C A^{\gamma \bar{j} \beta^*} B_{\bar{j}, \gamma \bar{j} \beta^*} u' \rangle_Y
\]

\[
+ \sum_{j=1}^d \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\bar{j}|+|\beta^*|+1}^{-1} \langle C A^{\gamma \bar{j} \gamma^*} B_{\bar{j}, \gamma \bar{j} \beta^*} u, C A^{\gamma \gamma^*} B_{j, \gamma \bar{j} \beta^*} u' \rangle_Y
\]

\[
= \omega_{|\gamma|+|\bar{j}|}^{-1} \langle D_{\gamma \bar{j}} u, C A^{\gamma \bar{j} \beta^*} B_{\bar{j}, \gamma \bar{j} \beta^*} u' \rangle_Y + \sum_{j=1}^d \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\bar{j}|+|\beta^*|+1}^{-1} \langle C A^{\gamma \bar{j} \gamma^*} B_{\bar{j}, \gamma \bar{j} \beta^*} u, C A^{\gamma \gamma^*} B_{j, \gamma \bar{j} \beta^*} u' \rangle_Y
\]

\[
= \langle (\omega_{|\gamma|+|\bar{j}|}^{-1} C^* D_{\gamma \bar{j}} u + \sum_{j=1}^d A_j^* \left( \sum_{j' \in \mathbb{F}_d^+} \omega_{|\gamma|+|\bar{j}|+|\beta^*|+1}^{-1} A^{\gamma \gamma^*} C A^{\gamma^*} \right) B_{\bar{j}, \gamma \bar{j} \beta^*} u, A^{\gamma \gamma^*} B_{j, \gamma \bar{j} \beta^*} u' \rangle_X
\]

\[
= \langle (\omega_{|\gamma|+|\bar{j}|}^{-1} C^* D_{\gamma \bar{j}} u + \sum_{j=1}^d A_j^* \Theta_{\omega,|\gamma|+1, \mathcal{C}_A} B_{\bar{j}, \gamma \bar{j} \beta^*} u, A^{\gamma \gamma^*} B_{j, \gamma \bar{j} \beta^*} u' \rangle_X = 0,
\]
where the two last equalities follow again from (4.58) and (7.2). To prove the orthogonality of \( S^T_{\omega,R} \Theta_{\omega} u, u' \) and \( S^T_{\omega,R} \Theta_{\omega} u, u' \), we first observe that if \( \beta \) does not divide \( \gamma \) on the right, that is, \( \gamma \) is not of the form \( \gamma = \gamma' \beta \), then every monomial in \( S^T_{\omega,R} \Theta_{\omega} u, u' \) is orthogonal to every monomial in \( S^T_{\omega,R} \Theta_{\omega} u, u' \) and therefore, the desired orthogonality follows. On the other hand, if \( \gamma = \gamma_j \beta \) for some \( \gamma_j \in \mathbb{F}_d^+ \) (the case \( \gamma = \emptyset \) is not excluded), then the calculation similar to the previous one gives

\[
\langle S^T_{\omega,R} \Theta_{\omega} u, u', S^T_{\omega,R} \Theta_{\omega} u, u' \rangle_{H^2_{\omega,Y}(\mathbb{F}_d^+)} = \omega_{|\beta|}^{-1} \cdot \langle D_{\beta} u, C A^{\beta:j} B_j \beta u \rangle_Y + \sum_{j=1}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} \langle C A^{\beta'} B_j, B_j u, A^{\beta:j} B_j, \beta u \rangle_X
\]

\[
= \langle \omega_{|\beta|}^{-1} C^* D_{\beta} u, A^{\beta:j} B_j u \rangle_Y + \sum_{j=1}^d A_j^\top \left( \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} A^{\beta:j} C^* A^{\beta'} \right) B_j, \beta u, \beta u \rangle_X
\]

\[
= \langle (\omega_{|\beta|}^{-1} C^* D_{\beta} + \sum_{j=1}^d A_j^\top \Theta_{\omega,|\beta|+1,C} B_j, \beta u), A^{\beta:j} B_j, \beta u \rangle_Y = 0
\]

and completes the proof of part (1c).

**Proof of (2):** By (7.9) we have

\[
\| S^T_{\omega,R} \Theta_{\omega} u, u \|_{H^2_{\omega,Y}(\mathbb{F}_d^+)}^2 = \omega_{|\beta|}^{-1} \cdot \| D_{\beta} u \|_Y^2 + \sum_{j=1}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} \| C A^{\beta'} B_j, \beta u \|_Y^2.
\]

(7.10)

Since due to (4.58), the second term on the right side can be written as

\[
\sum_{j=1}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} \langle C A^{\beta'} B_j, \beta u, A^{\beta:j} B_j, \beta u \rangle_X
\]

\[
= \sum_{j=1}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} \langle C A^{\beta:j} + C A^{\beta:j} \beta C^{\beta:j} \beta u, u \rangle_X
\]

\[
= \sum_{j=1}^d \sum_{\beta' \in \mathbb{F}_d^+} \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} \langle C A^{\beta:j} B_j, \beta u \rangle_X
\]

and since, according to (7.3),

\[
\omega_{|\beta|}^{-1} \cdot \langle D_{\beta} u \rangle_Y^2 + \sum_{j=1}^d \langle \Theta_{\omega,|\beta|+1,C} B_j, \beta u \rangle_X \leq \| u \|^2_{U_\beta}
\]

(7.11)

for all \( u \in U_\beta \), it now follows from (7.10) that

\[
\| S^T_{\omega,R} \Theta_{\omega} u, u \|_{H^2_{\omega,Y}(\mathbb{F}_d^+)}^2 \leq \| u \|^2_{U_\beta}
\]
Thus, \( S_{\omega,R}^{\beta T} M_{\Theta} u_\beta \) is a contraction from \( U_\beta \) to \( H^2_{\omega,Y}(\mathbb{F}^+_d) \) and part (2a) follows. By statements (1c) and (2a) of the lemma, Theorem 3.24 applies to the series \( S_{\omega,R}^{\beta T} \Theta \omega, u_\beta \) which is therefore, a contractive multiplier from \( H^2_{\Delta \beta} \) to \( H^2_{\omega,Y}(\mathbb{F}^+_d) \). This completes the proof of (2b) of the lemma.

**Proof of (3):** In case (7.4) holds, then (7.10) and (7.11) hold with equalities and part (a) of (3) follows. If also (7.2) holds, then Theorem 3.24 applies to the series \( \Theta = S_{\omega,R}^{\beta T} \Theta \omega, u_\beta \) and equality (7.4) follows from (3.46).

It remains to verify the formula (7.8) under assumption (7.6) which is equivalent to the system of equations (7.2), (7.3) and (1.60). We use these relations to compute

\[
\begin{align*}
\omega_{[\beta]}^{-1} I u_\beta - \Theta \omega, u_\beta (z)^* \Theta \omega, u_\beta (\zeta) &= \omega_{[\beta]}^{-1} I u_\beta - \left[ \omega_{[\beta]}^{-1} D^*_\beta + \hat{B}^*_\beta Z(z)^* R_{\omega, [\beta] + 1} (Z(\zeta) A)^* C^* \right] \\
& \quad \times \left[ \omega_{[\beta]}^{-1} D^*_\beta + C R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) \hat{B}^*_\beta \right] \\
& = \omega_{[\beta]}^{-1} I u_\beta - \omega_{[\beta]}^{-2} D^*_\beta D^*_\beta - \omega_{[\beta]}^{-1} \hat{B}^*_\beta Z(z)^* R_{\omega, [\beta] + 1} (Z(\zeta) A)^* C^* D^*_\beta \\
& \quad - \hat{B}^*_\beta Z(z)^* R_{\omega, [\beta] + 1} (Z(\zeta) A)^* C^* C R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) \hat{B}^*_\beta \\
& = \omega_{[\beta]}^{-1} I u_\beta \left( \omega_{[\beta]}^{-1} I X^d + Z(z)^* R_{\omega, [\beta] + 1} (Z(\zeta) A)^* A^* \left( \Theta \omega, u_\beta, C, A \right) \otimes I_d \right) \\
& \quad \times \left( \omega_{[\beta]}^{-1} I X^d + A R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) \right) \hat{B}^*_\beta \\
& \quad - \omega_{[\beta]}^{-1} \hat{B}^*_\beta Z(z)^* R_{\omega, [\beta] + 1} (Z(\zeta) A)^* D^*_\beta \Theta \omega, u_\beta, \zeta + 1, C, A R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) \hat{B}^*_\beta.
\end{align*}
\]

We now observe that

\[
R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) = Z(\zeta) R_{\omega, [\beta] + 1} (AZ(\zeta)),
\]

which together with (1.11) implies

\[
\omega_{[\beta]}^{-1} I X^d + A R_{\omega, [\beta] + 1} (Z(\zeta) A) Z(\zeta) = \omega_{[\beta]}^{-1} I X^d + AZ(\zeta) R_{\omega, [\beta] + 1} (AZ(\zeta)) = R_{\omega, [\beta]} (AZ(\zeta)).
\]

Substituting the two latter formulas into (7.12) verifies formula (7.8). □

**Corollary 7.3.** Let us assume that the pair \((C,A)\) is \(\omega\)-output stable and that relations (7.2), (7.3) hold for all \(\beta \in \mathbb{F}^+_d\). Then the representation (2.35) of the function \(\tilde{y}\) is orthogonal in the metric of \(H^2_{\omega,Y}(\mathbb{F}^+_d)\) and

\[
\|\tilde{y}\|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2 = \|Q_{\omega,C} x\|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2 + \sum_{\beta \in \mathbb{F}^+_d} \|S_{\omega,R}^{\beta T} \Theta \omega, u_\beta \|_{H^2_{\omega,Y}(\mathbb{F}^+_d)}^2 \\
\leq \|\tilde{g}\|_{H^2_{\omega,C} \mathbb{R} x}^2 + \sum_{\beta \in \mathbb{F}^+_d} \|u_\beta\|_{U_\beta}^2.
\]

If relations (7.3) hold with equalities for all \(\beta \in \mathbb{F}^+_d\), then equality holds in (7.13).
Let us now impose the additional hypothesis that \((C, A)\) is exactly \(\omega\)-observable, so that we can make use of the invertibility of all the shifted gramians \(\mathfrak{G}_{\omega, k, C, A}\) guaranteed to us by Proposition 7.22. Then the inequality (7.5) can equivalently be expressed as \(||\Xi|| \leq 1\) where the operator \(\Xi : \left[ \begin{array}{c} x \\ \lambda \end{array} \right] \rightarrow \left[ \begin{array}{c} y \\ \beta \end{array} \right] \) is given by

\[
\Xi = \begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A} \otimes I_d & 0 \\
0 & \omega_{|\beta|}^{-1/2} I_y
\end{bmatrix}
\begin{bmatrix}
A & \hat{B}_\beta \\
C & D_\beta
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} & 0 \\
0 & \omega_{|\beta|}^{1/2} I_y
\end{bmatrix}.
\]  
(7.14)

Another equivalent condition is that \(||\Xi^*|| \leq 1\) which in turn can be expressed as

\[
\begin{bmatrix}
A & \hat{B}_\beta \\
C & D_\beta
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} & 0 \\
0 & I_{|\beta|}
\end{bmatrix}
\begin{bmatrix}
A^* & C^* \\
\hat{B}_\beta^* & D_\beta^*
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} & 0 \\
0 & \omega_{|\beta|}^{-1/2} I_y
\end{bmatrix} \preceq \begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|, C, A} & 0 \\
0 & I_{|\beta|}
\end{bmatrix}
\begin{bmatrix}
A & \hat{B}_\beta \\
C & D_\beta
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|, C, A}^{-1} & 0 \\
0 & \omega_{|\beta|}^{1/2} I_y
\end{bmatrix}. 
\]  
(7.15)

Note that equality (7.6) means that the operator \(\Xi\) is isometric. Of particular interest is the case where \(\Xi\) is coisometric, that is, where the connection matrix

\[
U_{|\beta|} = \begin{bmatrix}
A & \hat{B}_\beta \\
C & D_\beta
\end{bmatrix}
\]

is coisometric with respect to the weights indicated below:

\[
\begin{bmatrix}
A & \hat{B}_\beta \\
C & D_\beta
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} & 0 \\
0 & I_{|\beta|}
\end{bmatrix}
\begin{bmatrix}
A^* & C^* \\
\hat{B}_\beta^* & D_\beta^*
\end{bmatrix}
\begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} & 0 \\
0 & \omega_{|\beta|}^{-1/2} I_y
\end{bmatrix} = \begin{bmatrix}
\mathfrak{G}_{\omega, |\beta|+1, C, A} & 0 \\
0 & \omega_{|\beta|}^{1/2} I_y
\end{bmatrix}.
\]  
(7.15)

**Lemma 7.4.** Let \((C, A)\) be an exactly \(\omega\)-observable \(\omega\)-output stable pair and let \(\Theta_{\omega, U_{|\beta|}}\) be defined as in (7.1) for some operators \(B_{|\beta|, \beta} \in \mathcal{L}(U_{|\beta|}, X)\) and \(D_{|\beta|} \in \mathcal{L}(U_{|\beta|}, Y)\) subject to equality (7.15). Then

\[
\omega_{|\beta|}^{-1} I_y - \Theta_{\omega, U_{|\beta|}}(z)\Theta_{\omega, U_{|\beta|}}(\varepsilon)^* 
= CR_{\omega, |\beta|}(Z(z)A)\mathfrak{G}_{\omega, |\beta|, C, A}^{-1}R_{\omega, |\beta|}(Z(\varepsilon)A)^*C^* 
- CR_{\omega, |\beta|+1}(Z(z)A)Z(z)(\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} \otimes I_d)Z(\varepsilon)^*R_{\omega, |\beta|+1}(Z(\varepsilon)A)^*C^*. 
\]  
(7.16)

**Proof.** The proof parallels the verification of the identity (7.8) done above. The weighted-coisometry condition (7.15) gives us the set of equations

\[
\begin{align*}
A\mathfrak{G}_{\omega, |\beta|, C, A}^{-1} + \hat{B}_\beta^* & = \mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} \otimes I_d, \\
C\mathfrak{G}_{\omega, |\beta|, C, A}^{-1} + D_\beta & = 0, \\
C\mathfrak{G}_{\omega, |\beta|, C, A}^{-1}C^* + D_\beta D_\beta^* & = \omega_{|\beta|}^{-1} I_y.
\end{align*}
\]  
(7.17)

We then compute (using relations (7.17) at the third step below)

\[
\omega_{|\beta|}^{-1} I_y - \Theta_{\omega, U_{|\beta|}}(z)\Theta_{\omega, U_{|\beta|}}(\varepsilon)^* 
= \omega_{|\beta|}^{-1} I_y - \left[ \omega_{|\beta|}^{-1} D_\beta + CR_{\omega, |\beta|+1}(Z(z)A)Z(z)\hat{B}_\beta \right] 
\times \left[ \omega_{|\beta|}^{-1} D_\beta^* + \hat{B}_\beta^* Z(\varepsilon)^*R_{\omega, |\beta|+1}(Z(\varepsilon)A)^*C^* \right] 
= \omega_{|\beta|}^{-1} I_y - \omega_{|\beta|}^{-2} D_\beta D_\beta^* - \omega_{|\beta|}^{-1} CR_{\omega, |\beta|+1}(Z(z)A)Z(z)\hat{B}_\beta D_\beta^* 
- \omega_{|\beta|}^{-1} D_\beta \hat{B}_\beta^* Z(\varepsilon)^*R_{\omega, |\beta|+1}(Z(\varepsilon)A)^*C^* 
- CR_{\omega, |\beta|+1}(Z(z)A)Z(z)\hat{B}_\beta^* Z(\varepsilon)^*R_{\omega, |\beta|+1}(Z(\varepsilon)A)^*C^* 
= \omega_{|\beta|}^{-2} C\mathfrak{G}_{\omega, |\beta|, C, A}^{-1} + \omega_{|\beta|}^{-1} CR_{\omega, |\beta|+1}(Z(z)A)Z(z)A\mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1}C^* 
+ \omega_{|\beta|}^{-1} C\mathfrak{G}_{\omega, |\beta|, C, A}^{-1}Z(\varepsilon)^*R_{\omega, |\beta|+1}(Z(\varepsilon)A)^*C^* 
- CR_{\omega, |\beta|+1}(Z(z)A)Z(z) \left( \mathfrak{G}_{\omega, |\beta|+1, C, A}^{-1} \otimes I_d \right) - A\mathfrak{G}_{\omega, |\beta|, C, A}^{-1}A^* 
\]
\[ \times Z(\zeta)^* R_{\omega|\beta|+1}(Z(\zeta)A)^* C^* \]
\[ = C \left( \omega_{[\beta]}^{-1} I_X + R_{\omega|\beta|+1}(Z(z)A)Z(z)A \right) \mathcal{G}_{\omega|\beta|+1,C,A}^{-1} \]
\[ \times \left( \omega_{[\beta]}^{-1} I_X + A^* Z(\zeta)^* R_{\omega|\beta|+1}(Z(\zeta)A)^* \right) C^* \]
\[ - CR_{\omega|\beta|+1}(Z(z)A)Z(z) \left( \mathcal{G}_{\omega|\beta|+1,C,A}^{-1} \otimes I_d \right) Z(\zeta)^* R_{\omega|\beta|+1}(Z(\zeta)A)^* C^* \]
which implies (7.10), due to equality
\[ \omega_{[\beta]}^{-1} I_X + R_{\omega|\beta|+1}(Z(z)A)Z(z)A = R_{\omega|\beta|}(Z(z)A), \]
which in turn, is a consequence of (7.24).

Since equality (7.15) implies inequality (7.5), it follows that under assumptions of Lemma 7.4 all the conclusions of parts (1) and (2) in Lemma 7.1 are true. To have all conclusions true, we need the operator (7.14) to be unitary. The next result amounts to a more structured version of Proposition 3.9.

**Lemma 7.5.** Suppose that we are given an exactly \( \omega \)-observable \( \omega \)-output-stable pair \( (C, A) \) with \( A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d \) and \( C \in \mathcal{L}(X, Y) \). Then for every \( \beta \in \mathbb{F}_d^+ \), there exist operators \( B_{1, \beta}, \ldots, B_{d, \beta} \in \mathcal{L}(U_\beta, X') \) and \( D_\beta \in \mathcal{L}(U_\beta, Y) \) such that equalities (7.15) and (7.6) hold with \( A \) and \( B_\beta \) defined as in (1.33). Explicitly, such \( B_\beta \) and \( D_\beta \) are uniquely determined up to a common unitary right factor by solving the Cholesky factorization problem:

\[
\begin{bmatrix}
\hat{B}_\beta \\
D_\beta
\end{bmatrix} \begin{bmatrix}
\hat{B}_\beta & D_\beta
\end{bmatrix} = \left[ \mathcal{G}_{\omega|\beta|+1,C,A}^{-1} \otimes I_d \quad 0 \\
0 & \omega_{[\beta]}^{-1} I_Y
\right] - \left[ \frac{A}{C} \right] \mathcal{G}_{\omega|\beta|,C,A}^{-1} \left[ A^* \quad C^* \right]
\]
subject to the additional constraint that the coefficient space \( U_\beta \) be chosen so that \( \left[ \frac{\hat{B}_\beta}{D_\beta} \right] : U_\beta \to X^d \oplus Y \) is injective.

**Proof.** By Proposition 4.21, the weighted Stein identity (4.60) holds for each \( k \geq 0 \). Since \( (C, A) \) is exactly \( \omega \)-observable, the gramian \( \mathcal{G}_{\omega|\beta|,C,A} \) is strictly positive definite and then it follows from (4.60) that the operator

\[ U := \left[ \left( \mathcal{G}_{\omega|\beta|+1,C,A}^{-1} \otimes I_d \right) A \mathcal{G}_{\omega|\beta|,C,A}^{-1} \quad \omega_{[\beta]}^{-1} C \mathcal{G}_{\omega|\beta|,C,A}^{-1} \right] : X' \to X^d \oplus Y \]

is an isometry. We then find an injective \( V : U_\beta \to X^d_Y \) such that \( VV^* = I_{X^d \oplus Y} - UU^* \) as in Proposition 3.9. This translates to \( V \) having the form

\[ V = \begin{bmatrix}
\mathcal{G}_{\omega|\beta|+1,C,A}^{-1} \otimes I_d & 0 \\
0 & \mu_{\omega|\beta|}^{-1} I_Y
\end{bmatrix} \begin{bmatrix}
\hat{B}_\beta \\
D_\beta
\end{bmatrix}
\]
where \( \begin{bmatrix}
\hat{B}_\beta \\
D_\beta
\end{bmatrix} \) is a solution of the factorization problem (7.18). Uniqueness of the solution \( \begin{bmatrix}
\hat{B}_\beta \\
D_\beta
\end{bmatrix} \) of the factorization problem (7.18) corresponds exactly to the uniqueness of the partial isometry \( V \) in the factorization problem \( VV^* = I_{X^d \oplus Y} - UU^* \). The fact that \( \begin{bmatrix}
U \\
V
\end{bmatrix} \) is unitary means that both equalities (7.15) and (7.6) hold. \( \square \)
Remark 7.6. The operators $\hat{B}_\beta$ and $D_\beta$ depend on $|\beta|$ rather than on $\beta$; thus, in the construction above we can always take $U_\beta = U_{\beta'}$ and also $\hat{B}_\beta = \hat{B}_{\beta'}$ and $D_\beta = D_{\beta'}$ whenever $|\beta| = |\beta'|$.

7.2. Beurling-Lax representations based on Bergman-inner families

In this section we present our most elaborate version of the Beurling-Lax theorem. Arguably this version, while more complicated than the previous versions which we have discussed and the classical case, is the most compelling in that the representer is determined up to a unitary change of basis on the input-space sequence, closer to the classical case where the representer is determined up to a unitary change of basis on the input space. There results a unitary invariant for a shift invariant subspace.

In the classical case $(d = 1$ and $n = 1)$, if $\mathcal{M}$ is a subspace of the Hardy space $H^2$ invariant for the shift operator $S$: $f(z) \mapsto zf(z)$, the Beurling-Lax representer $\Theta$ for $\mathcal{M}$ can be constructed by choosing the coefficient space $\mathcal{U}$ to have the same dimension as the wandering subspace $\mathcal{E} := M \oplus zM$ for $\mathcal{M}$, and letting $\Theta$ be any unitary identification map of $\mathcal{U}$ to $\mathcal{E}$ (so $\mathcal{E} = \Theta \mathcal{U}$) and then extending $\Theta$ to $M_\Theta: H^2 \rightarrow H^2_\Theta$ by demanding shift-invariance: $M_\Theta S = SM_\Theta$. The subspace $M_\beta$ given below is the time-varying multidimensional analog of the wandering subspace, as explained in the following lemma.

Lemma 7.7. Suppose that $\mathcal{M} \subset H^2_{\omega,\gamma}(F^+_d)$ is an $S_{\omega,R}$-invariant subspace. For each word $\beta \in F^+_d$, define the subspace

$$\mathcal{M}_\beta = S^\omega_{\omega,R} \mathcal{M} \ominus \left( \bigoplus_{j=1}^d S^{\beta_j}_{\omega,R} S_{\omega,R,j} \mathcal{M} \right).$$

Then $\mathcal{M}$ has the orthogonal direct-sum decomposition

$$\mathcal{M} = \bigoplus_{\beta \in F^+_d} \mathcal{M}_\beta.$$  \hspace{1cm} (7.19)

More generally, for any $\alpha \in F^+_d$, we have the orthogonal direct-sum decomposition

$$S^\omega_{\omega,R} \mathcal{M} = \bigoplus_{\beta \in F^+_d} \mathcal{M}_{\beta \gamma}.$$ \hspace{1cm} (7.20)

Proof. We first show that

$$\mathcal{M}_\beta \perp \mathcal{M}_\gamma \quad \text{if} \quad \beta \neq \gamma,$$ \hspace{1cm} (7.21)

i.e., that $f \perp g$ in $H^2_{\omega,\gamma}(F^+_d)$ for any $f \in \mathcal{M}_\beta$ and $g \in \mathcal{M}_\gamma$. Without loss of generality we take $|\gamma| \geq |\beta|$. We then write $\gamma = \delta \beta'$ with $|\beta'| = |\beta|$ and $\delta \in F^+_d$ (possibly equal to 0).

Case 1: Suppose that $\beta' \neq \beta$. Then any $f \in S^\omega_{\omega,R} \mathcal{M}$, as an element of $S^\omega_{\omega,R} H^2_{\omega,\gamma}(F^+_d)$, has the form $f(z) = \sum_{\alpha \in F^+_d} f_{\alpha} z^\alpha \beta^\gamma$. Similarly, any $g \in S^\gamma_{\omega,R} \mathcal{M}$ has the form

$$g(z) = \sum_{\alpha' \in F^+_d} g_{\alpha'} z^{\alpha' \gamma} = \sum_{\alpha' \in F^+_d} g_{\alpha' \delta \beta'} z^{\alpha' \delta \beta'}.$$  \hspace{1cm} (7.22)

As $\beta$ and $\beta'$ are different and of the same length, every word of the form $\alpha \beta$ is distinct from every word of the form $\alpha' \delta \beta'$ and since the monomial subspaces $z^\alpha \gamma$...
satisfy orthogonality relations $z^\alpha y \perp z'^\alpha y'$ for all $y, y' \in \mathcal{Y}$ and $\alpha \neq \alpha'$ in $\mathcal{F}_d^+$, we have $f_{\alpha\beta}z'^{\alpha\beta} \perp g_{\alpha\delta\beta'}z'^{\alpha\delta\beta'}$ in $H^2_{\omega', \mathcal{Y}}(\mathcal{F}_d^+)$ and hence, $f \perp g$ in this case.

**Case 2:** Suppose that $\beta' = \beta$, so $\gamma = \delta\beta$. The assumption that $\beta \neq \gamma$ implies that $\delta \neq 0$. Then

$$g \in \mathcal{M}_\gamma \subset S^\gamma_{\omega, R} \mathcal{M} = S^\beta_{\omega, R} S^\delta_{\omega, R} \mathcal{M} \subset \bigoplus_{j=1}^d S^\beta_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M} \quad (\text{note that } S^\beta_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M} \perp S^\beta_{\omega, R} S^\omega_{\omega, R, i} \mathcal{M} \text{ for } i \neq j \text{ by Case 1}).$$

But $f \in \mathcal{M}_\beta$, and $\mathcal{M}_\beta$ is orthogonal to $\bigoplus_{j=1}^d S^\omega_{\omega, R, j} \mathcal{M}$ by definition (7.19). We conclude that $f \perp g$ in this case as well. This completes the verification of (7.22).

As (7.20) is the special case of (7.21) with $\alpha = 0$, it remains only to prove (7.21). First note that

$$\mathcal{M}_\beta = S^\omega_{\omega, R} S^\beta_{\omega, R} \mathcal{M} \subset (\bigoplus_{j=1}^d S^\omega_{\omega, R} S^\beta_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M}) \subset S^\omega_{\omega, R} \mathcal{M},$$

and it follows that

$$S^\omega_{\omega, R} \mathcal{M} \subset \bigoplus_{\beta \in \mathcal{F}_d^+} \mathcal{M}_\beta.$$

To show the reverse inclusion, it suffices to show that $h = 0$ whenever $h \in S^\omega_{\omega, R} \mathcal{M}$ is orthogonal to $\mathcal{M}_\beta$ for all $\beta \in \mathcal{F}_d^+$. Suppose therefore that $h \in S^\omega_{\omega, R} \mathcal{M}$ is orthogonal to $\mathcal{M}_\beta$ for all $\beta \in \mathcal{F}_d^+$. In particular, the special case $\beta = 0$ gives us that $h \perp \mathcal{M}_0 := S^\omega_{\omega, R} \mathcal{M} \subset \bigoplus_{j=1}^d S^\omega_{\omega, R, j} \mathcal{M}$; thus

$$h \in \text{closure } \bigoplus_{j=1}^d S^\omega_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M} \subset (S^\omega_{\omega, R} \mathcal{M})^\perp. \quad (7.23)$$

Since $h \in S^\omega_{\omega, R} \mathcal{M}$ and $\bigoplus_{j=1}^d S^\omega_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M} \subset S^\omega_{\omega, R} \mathcal{M}$, it follows from (7.23) (by taking the orthogonal projection of $H^2_{\omega, \mathcal{Y}}(\mathcal{F}_d^+)$ onto $S^\omega_{\omega, R} \mathcal{M}$) that

$$h \in \bigoplus_{j=1}^d S^\omega_{\omega, R} S^\omega_{\omega, R, j} \mathcal{M} = \bigoplus_{\beta \in \mathcal{F}_d^+ : |\beta| = 1} S^\omega_{\omega, R} S^\beta_{\omega, R} \mathcal{M}. \quad (7.24)$$

Inductively suppose that

$$h \in \bigoplus_{\beta \in \mathcal{F}_d^+ : |\beta| = N} S^\omega_{\omega, R} S^\beta_{\omega, R} \mathcal{M}. \quad (7.25)$$

Choose any $\beta_0 \in \mathcal{F}_d^+$ with $|\beta_0| = N$. Then the condition $h \perp \mathcal{M}_{\beta_0}$ leads to

$$h \in \text{closure } \bigoplus_{j=1}^d S^\omega_{\omega, R} S^\beta_{\omega, R, j} \mathcal{M} \subset (S^\omega_{\omega, R} S^\beta_{\omega, R} \mathcal{M})^\perp. \quad (7.26)$$

Combining the assumption (7.25) with the inclusion

$$\bigoplus_{j=1}^d S^\omega_{\omega, R} S^\beta_{\omega, R, j} \mathcal{M} \subset \bigoplus_{\beta \in \mathcal{F}_d^+ : |\beta| = N} S^\omega_{\omega, R} S^\beta_{\omega, R} \mathcal{M}$$
we conclude from (7.20) (by taking the orthogonal projection of $H^2_{\omega,Y}(F_d^+)$ onto $\bigoplus_{\beta \in F_d^+ : |\beta|=N} S^{\alpha^T}_{\omega,R} S^\beta_{\omega,R} \mathcal{M}$) that
\[
h \in \left( \bigoplus_{j=1}^d S^{\alpha^T}_{\omega,R} S^\beta_{\omega,R,j} \mathcal{M} \right) \oplus \left( \bigoplus_{\beta \in F_d^+ : |\beta|=N, \beta \neq \beta_0} S^{\alpha^T}_{\omega,R} S^\beta_{\omega,R} \mathcal{M} \right).
\]

We now use that $\beta_0$ is an arbitrary element of $F_d^+$ with $|\beta_0| = N$. Intersecting the right-hand side of the latter formula over all such $\beta_0$ leads to
\[
h \in \bigoplus_{\beta \in F_d^+ : |\beta|=N+1} S^{\alpha^T}_{\omega,R} S^\beta_{\omega,R} \mathcal{M}.
\]
Recalling that we established (7.24) and that the present argument started with the assumption (7.25), we conclude by the induction principle that (7.25) holds for all $N = 1, 2, \ldots$. Thus
\[
h \in \bigcap_{N=0}^\infty \bigoplus_{\beta \in F_d^+ : |\beta|=N} S^{\alpha^T}_{\omega,R} S^\beta_{\omega,R} \mathcal{M} \subset S^{\alpha^T}_{\omega,R} \left( \bigcap_{N=0}^\infty \bigoplus_{\beta \in F_d^+ : |\beta|=N} S^\beta_{\omega,R} H^2_{\omega,Y}(F_d^+) \right) = \{0\}
\]
and the orthogonal decomposition (7.21) is verified. \hfill \Box

In the discussion to follow, given our $S_{\omega,R}$-invariant subspace $\mathcal{M}$ (isometrically included in $H^2_{\omega,Y}(F_d^+)$), we fix a choice of $\omega$-isometric output pair $(C,A)$ with $A$ strongly stable so that $\mathcal{M}^\perp = \text{Ran} \mathcal{O}_{\omega,C,A}$ and so that $M$ has reproducing kernel $k_M$ defined as in (5.1):
\[
k_M(z,\zeta) = k_{\text{nc},\omega}(z,\zeta)1_Y - C R_{\omega}(Z(z)A) R_{\omega}(Z(\zeta)A)^* C^*.
\]
(7.27)

We remind the reader that a canonical choice of $(C,A)$ is the model output pair
\[
C = E|_{M^\perp}, \quad A = S_{\omega,R}|_{M^\perp}
\]

With any such choice, the observability operator $\mathcal{O}_{C,A} : \mathcal{M} \to H^2_{\omega,Y}(F_d^+)$ is an isometric inclusion and the gramian operator $G_{\omega,C,A}$ is the identity operator on $\mathcal{M}$. Furthermore, as a consequence of Proposition 4.22 we are assured that all the shifted gramiens $G_{\omega,k,C,A}$ are bounded and boundedly invertible. To get the reproducing kernel for $S^\alpha_{\omega,R} \mathcal{M}$ consistent with the metric of $H^2_{\omega,Y}(F_d^+)$, we start with the following characterization of the space $(S^\alpha_{\omega,R} \mathcal{M})^\perp$ in terms of the shifted observability operator $\mathcal{O}_{\omega,|\beta|,C,A}$ introduced in (4.37).

**PROPOSITION 7.8.** The space $(S^\alpha_{\omega,R} \mathcal{M})^\perp$ is characterized as
\[
(S^\alpha_{\omega,R} \mathcal{M})^\perp = \left( S^\beta_{\omega,R} H^2_{\omega,Y}(F_d^+) \right)^\perp \bigoplus_{\beta \in F_d^+ : |\beta|=N} S^\beta_{\omega,R} \mathcal{O}_{\omega,|\beta|,C,A}.
\] (7.28)

In particular,
\[
(S_{\omega,R,k} \mathcal{M})^\perp = \left( S_{\omega,R,k} H^2_{\omega,Y}(F_d^+) \right)^\perp \bigoplus_{\beta \in F_d^+ : |\beta|=N} S^\beta_{\omega,R,k} \mathcal{O}_{\omega,1,C,A}
\] (7.29)
for $k = 1, \ldots, d$. Furthermore,
\[
\left( \bigoplus_{j=1}^d S_{\omega,R,j} \mathcal{M} \right)^\perp = Y \bigoplus_{k=1}^d S_{\omega,R,k} \text{Ran} \mathcal{O}_{\omega,1,C,A}.
\] (7.30)
Orthogonal Beurling-Lax Representations

We wish to characterize all functions \( f(z) = \sum_{\alpha \in F_d^+} f_\alpha z^\alpha \) which are orthogonal to \( S_{\omega,R}^T \mathcal{M} \) in \( H^2_{\omega,Y}(\mathbb{F}_d^+) \). We may write

\[
\tilde{f}(z) = p(z) + \tilde{f}(z)z^\beta \quad \text{where} \quad p \in \left( S_{\omega,R}^T H^2_{\omega,Y}(\mathbb{F}_d^+) \right)^\perp, \quad \tilde{f} \in H^2_{\omega,Y}(\mathbb{F}_d^+).
\]

Since \( S_{\omega,R}^T \mathcal{M} \subset S_{\omega,R}^T H^2_{\omega,Y}(\mathbb{F}_d^+) \), it follows that \( (S_{\omega,R}^T H^2_{\omega,Y}(\mathbb{F}_d^+))^\perp \subset (S_{\omega,R}^T \mathcal{M})^\perp \), so it suffices to characterize which functions of the form \( \tilde{f}(z)z^\beta \) are orthogonal to \( S_{\omega,R}^T \mathcal{M} \). To this end, observe that \( S_{\omega,R}^T \tilde{f} \) is orthogonal to \( S_{\omega,R}^T \mathcal{M} \) if and only if the power series \( S_{\omega,R}^T \tilde{f} \) belongs to \( \mathcal{M}^\perp = \text{Ran} \Omega_{\omega,C,A} \). Since by (4.73),

\[
S_{\omega,R}^T \mathcal{M} = \left\{ \sum_{\alpha \in F_d^+} \tilde{f}_\alpha z^\alpha \right\} = \left\{ \sum_{\alpha \in F_d^+} \frac{\omega_{|\alpha|+|\beta|}}{\omega_{|\alpha|}} \tilde{f}_\alpha z^\alpha \right\} = \left\{ \sum_{\alpha \in F_d^+} (\omega_{|\alpha|}^{-1} \cdot CA^\alpha x) z^\alpha \right\}
\]

we thus conclude that \( S_{\omega,R}^T \tilde{f} \) is orthogonal to \( S_{\omega,R}^T \mathcal{M} \) if and only if there exists a vector \( x \in X \) such that

\[
\sum_{\alpha \in F_d^+} \frac{\omega_{|\alpha|+|\beta|}}{\omega_{|\alpha|}} \tilde{f}_\alpha z^\alpha = (\Omega_{\omega,C,A} x)(z) = \sum_{\alpha \in F_d^+} (\omega_{|\alpha|}^{-1} \cdot CA^\alpha x) z^\alpha.
\]

Equating the corresponding Taylor coefficients in the latter equality gives

\[
\tilde{f}_\alpha = \omega_{|\alpha|+|\beta|}^{-1} \cdot CA^\alpha x
\]

for all \( \alpha \in F_d^+ \), and therefore,

\[
\tilde{f}(z) = \sum_{\alpha \in F_d^+} \tilde{f}_\alpha z^\alpha = \sum_{\alpha \in F_d^+} (\omega_{|\alpha|+|\beta|}^{-1} \cdot CA^\alpha x) z^\alpha = \Omega_{\omega,|\beta|,C,A} x,
\]

by (4.31). Thus, \( \tilde{f} \in \text{Ran} \Omega_{\omega,|\beta|,C,A} \). As the analysis is necessary and sufficient, the formula (7.29) follows. Note that formula (7.29) is just the special case of (7.28) where the word \( \beta \) is taken to consist of the single letter \( k \).

To verify (7.30), we note first that

\[
\left( \bigoplus_{j=1}^d S_{\omega,R,j} \mathcal{M} \right)^\perp = \bigcap_{j=1}^d (S_{\omega,R,j} \mathcal{M})^\perp. \tag{7.31}
\]

We next introduce the subspaces

\[
N_j := S_{\omega,R,j} H^2_{\omega,Y}(\mathbb{F}_d^+) \ominus S_{\omega,R,j} \text{Ran} \Omega_{\omega,1,C,A}
\]

and observe the equalities

\[
(S_{\omega,R,k} H^2_{\omega,Y}(\mathbb{F}_d^+))^\perp = Y \bigoplus \left( \bigoplus_{j: j \neq k} S_{\omega,R,j} H^2_{\omega,Y}(\mathbb{F}_d^+) \right) = Y \bigoplus \left( \bigoplus_{j: j \neq k} S_{\omega,R,j} \text{Ran} \Omega_{\omega,1,C,A} \right) \bigoplus \left( \bigoplus_{j: j \neq k} N_j \right)
\]

which, when substituted into the formula (7.29) for \( (S_{\omega,R,k} \mathcal{M})^\perp \), give

\[
(S_{\omega,R,k} \mathcal{M})^\perp = Y \bigoplus \left( \bigoplus_{j=1}^d S_{\omega,R,j} \text{Ran} \Omega_{\omega,1,C,A} \right) \bigoplus \left( \bigoplus_{j: j \neq k} N_j \right). \tag{7.32}
\]
Since $N_j \perp N_i$ for $i \neq j$, it follows that
\[
\bigcap_{k=1}^{d} \left( \bigoplus_{j: j \neq k} N_j \right) = \{0\}. \tag{7.33}
\]
Making use of representation (7.32) and taking into account (7.33) we get
\[
\bigcap_{k=1}^{d} (S_{\omega,R,k}\mathcal{M})^\perp = \mathcal{Y} \bigoplus \left( \bigoplus_{k=1}^{d} S_{\omega,R,k}\text{Ran} \mathcal{D}_{\omega,1,C,A} \right)
\]
and now the formula (7.30) follows from (7.31). \hfill \Box

We reproduce kernel for the space $S$ which is, as the second equality in (7.34) indicates, the formal noncommutative $N_j \perp N_i$ and Theorem 4.40. Hence (7.35) follows.

It is convenient to introduce, in analogy with shifted observability operators and gramians, the shifted positive kernel
\[
k_{nc,\omega,\beta}(z, \zeta) \coloneqq \sum_{\alpha \in \mathbb{C}} \omega^{-1}_{|\alpha|+|\beta|} z^\alpha \zeta reversed kernel for the space $S$ of reproducing kernels for the subspaces $S$ and Theorem 4.40. Hence (7.35) follows.

With characterization (7.28) and positive kernels (7.34) in hand, it is straightforward to derive the kernel function for the space $S^T_{\omega,R}\mathcal{M}$ with respect to the metric inherited from $H^2_{\omega,Y}(\mathbb{F}^+_d)$.\[\[\]

**Proposition 7.9.** Let $\mathcal{M}$ be a closed $S_{\omega,R}$-invariant subspace of $H^2_{\omega,Y}(\mathbb{F}^+_d)$ with reproducing kernel $k_{\mathcal{M}}$ given by (7.24). Then for every $\beta \in \mathbb{F}^+_d$, the formal reproducing kernel functions for $S^T_{\omega,R}\mathcal{M}$ and for $\mathcal{M}_\beta$ (defined in (7.20)) in the metric of $H^2_{\omega,Y}(\mathbb{F}^+_d)$ are given by
\[
k_{S^T_{\omega,R}\mathcal{M}}(z, \zeta) = k_{nc,\omega,\beta}(z, \zeta) I_Y - \mathcal{R}_{\beta}(z, \zeta), \tag{7.35}
\]
\[
k_{\mathcal{M}_\beta}(z, \zeta) = z^{\beta} \zeta^T \omega^{-1}_{|\beta|} R_{\beta}(z, \zeta) + \sum_{j=1}^{d} \mathcal{R}_{j,\beta}(z, \zeta), \tag{7.36}
\]
where $\mathcal{R}_{\beta}$ is the positive kernel defined as in (4.126):/n\[
\mathcal{R}_{\beta}(z, \zeta) = CR_{\omega,|\beta|}(Z(z)A)(z^\beta \zeta^T S^{-1}_{\omega,|\beta|,C,A} R_{\omega,|\beta|}(Z(\zeta)A)^* C^*.
\]

**Proof.** From the formula (7.28) for $(S^T_{\omega,R}\mathcal{M})^\perp$ we deduce that in metric of $H^2_{\omega,Y}(\mathbb{F}^+_d)$,
\[
S^T_{\omega,R}\mathcal{M} = (S^T_{\omega,R} H^2_{\omega,Y}(\mathbb{F}^+_d)) \bigcap (S^T_{\omega,R} \text{Ran} \mathcal{D}_{\omega,|\beta|,C,A})^\perp = (S^T_{\omega,R} H^2_{\omega,Y}(\mathbb{F}^+_d)) \bigoplus (S^T_{\omega,R} \text{Ran} \mathcal{D}_{\omega,|\beta|,C,A}).
\]
Therefore, the reproducing kernel for the subspace $S^T_{\omega,R}\mathcal{M}$ is equal to the difference of reproducing kernels for the subspaces $S^T_{\omega,R} H^2_{\omega,Y}(\mathbb{F}^+_d)$ and $S^T_{\omega,R} \text{Ran} \mathcal{D}_{\omega,|\beta|,C,A}$. But these kernels are equal to $k_{nc,\omega,\beta}(z, \zeta) I_Y$ and $\mathcal{R}_{\beta}(z, \zeta)$ respectively, by (7.34) and Theorem 4.40. Hence (7.35) follows.
It remains to verify the formula (7.36). Toward this end, we first observe that replacing $\beta$ by $j\beta$ in (7.34) gives

$$k_{\text{nc},\omega,(j\beta)}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+|j\beta|+1}^{-1} z^{\alpha j\beta} \zeta^{(\alpha j\beta)^T}$$

for $j = 1, \ldots, d,$ and consequently,

$$k_{\text{nc},\omega,\beta}(z, \zeta) - \sum_{j=1}^d k_{\text{nc},\omega,(j\beta)}(z, \zeta)$$

$$= \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+|\beta|}^{-1} z^{\alpha \beta} \zeta^{(\alpha \beta)^T} - \sum_{j=1}^d \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+|j\beta|+1}^{-1} z^{\alpha j\beta} \zeta^{(\alpha j\beta)^T}$$

$$= \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+|\beta|}^{-1} z^{\alpha \beta} \zeta^{(\alpha \beta)^T} - \sum_{\alpha \in \mathbb{F}_d^+ : \alpha \neq \beta} \omega_{|\alpha|+|\beta|}^{-1} z^{\alpha \beta} \zeta^{(\alpha \beta)^T}.$$

For the next computation we use (7.35), (7.37) and the orthogonal representation for $\mathcal{M}_\beta$ in (7.19), according to which

$$k_{\mathcal{M}_\beta}(z, \zeta) = k_{\mathcal{S}_{\omega,R,\mathcal{M}}}(z, \zeta) - \sum_{j=1}^d k_{\mathcal{S}_{\omega,R,\mathcal{M}}}(z, \zeta)$$

$$= (k_{\text{nc},\omega,\beta}(z, \zeta) - \sum_{j=1}^d k_{\text{nc},\omega,(j\beta)}(z, \zeta)) I_\mathcal{Y} - \mathcal{R}_\beta(z, \zeta) + \sum_{j=1}^d \mathcal{R}_{j\beta}(z, \zeta).$$

This completes the proof of (7.36). \qed

**Lemma 7.10.** Given $\beta \in \mathbb{F}_d^+$ and an exactly $\omega$-observable $\omega$-output stable pair $(C, A),$ construct operators $\hat{B}_\beta = \begin{bmatrix} B_{1,\beta}^* \\ \vdots \\ B_{d,\beta}^* \end{bmatrix} \in \mathcal{L}(\mathcal{U}_\beta, \mathcal{X}^d)$ and $D_\beta \in \mathcal{L}(\mathcal{U}_\beta, \mathcal{Y})$ as in Lemma 7.5. For each $\beta \in \mathbb{F}_d^+$ form the connection $\mathcal{U}_\beta = \begin{bmatrix} A & B_\beta \\ C & D_\beta \end{bmatrix}$ and let $\Theta_{\omega,\mathcal{U}_\beta}$ be the associated transfer-function formal power series as in (7.11). Then the kernel (7.36) can be factored as

$$k_{\mathcal{M}_\beta}(z, \zeta) = \Theta_{\omega,\mathcal{U}_\beta}(z) \left( z^{\beta \zeta^T} I_{\mathcal{U}_\beta} \right) \Theta_{\omega,\mathcal{U}_\beta}^*(\zeta).$$

Moreover, if we consider $\mathbf{S}_{1,\beta}^0 = z^\beta \mathcal{U}_\beta$ as a Hilbert space with norm

$$\|z^\beta u\|_{z^\beta \mathcal{U}_\beta} = \|u\|_{\mathcal{U}_\beta} \text{ for all } u \in \mathcal{U}_\beta,$$

lifted from $\mathcal{U}_\beta,$ then the operator $M_{\Theta_{\omega,\mathcal{U}_\beta}} : z^\beta u \mapsto \Theta_{\omega,\mathcal{U}_\beta}(z) z^\beta u$ is unitary from $z^\beta \mathcal{U}_\beta$ onto $\mathcal{M}_\beta$ for each $\beta \in \mathbb{F}_d^+.$

**Proof.** By Lemma 7.1 identity (7.10) holds. Multiplying both parts of (7.10) by $z^\beta$ on the right and by $\zeta^{\beta^T}$ on the left, normalizing to a hereditary kernel, and then combining the obtained equality with (7.36) easily leads to (7.38).
We note next that the factorization \[(7.38)\] rewritten as
\[k_{\mathcal{M}_\beta}(z, \zeta) = \Theta_{\omega, \mathcal{U}_\beta}(z) z^\beta \cdot \zeta^{\mathcal{T}} \Theta_{\omega, \mathcal{U}_\beta}(\zeta)^*\]
amounts to a Kolmogorov decomposition for auxiliary Hilbert spaces \(\beta\). From the last statement in Theorem 2.1, we can infer that \(u\) is a coisometry from \(\mathcal{N}\) series (with \(\Theta\) forces \(M\) as an operator on \(\omega\)).
\[\begin{align*}
\text{From the last statement in Theorem 2.1, we can infer that} & \\
& \text{is a coisometry from } z^\beta U_\beta \text{ onto } M_\beta. \text{ Moreover, from the identity \[(7.13)\] (specialized to the case where } u_{\beta'} = 0 \text{ for } \beta' \neq \beta \text{ and } x = 0\), we see that } \tilde{\mu}(z) := \Theta_{\omega, \mathcal{U}_\beta}(z) u_{\beta} = 0 \text{ forces } u_{\beta} = 0, \text{ and hence also } \Theta_{\omega, \mathcal{U}_\beta}(z) \cdot z^\beta u_{\beta} = 0 \text{ forces } u_{\beta} = 0, \text{ so } M_{\Theta_{\omega, \mathcal{U}_\beta}} \text{ as an operator on } z^\beta U_\beta \text{ has no kernel. Putting the pieces together, we see that} \\
& \text{for each } \alpha \in \mathbb{F}_d^+, \\
& \text{we have:} \\
& \text{(1) The operator } M_{\Theta_{\omega, \beta}}: z^\beta U_\beta \to H^2_{\omega, \gamma}(\mathbb{F}_d^+) \text{ is isometric,} \\
& \text{(2) } M_{\Theta_{\omega, \beta}}(z^\beta U_\beta) \text{ is orthogonal to } M_{\Theta_{\omega, \gamma}}(z^\gamma U_\gamma) \text{ for all } \alpha, \beta, \gamma \in \mathbb{F}_d^+ \text{ with } \beta \neq \gamma, \text{ and} \\
& \text{(3) For each } \alpha \in \mathbb{F}_d^+, \\
& S_{\omega, \mathcal{R}} \left( \bigoplus_{\beta \in \mathbb{F}_d^+} \Theta_{\omega, \beta} z^\beta U_\beta \right) = \bigoplus_{\beta \in \mathbb{F}_d^+} \Theta_{\omega, \mathcal{R}} z^\beta U_\beta. \\
\end{align*}\]

We note that, for each \(u_{\beta} \in \mathcal{U}_\beta\), \(M_{\Theta_{\omega, \beta}}(z^\beta U_\beta) = S_{\omega, \mathcal{R}} M_{\Theta_{\omega, \beta}} u_{\beta} \in \text{ Ran } S_{\omega, \mathcal{R}}\) and in general \(\text{ Ran } S_{\omega, \mathcal{R}} \perp \text{ Ran } \tilde{S}_{\omega, \mathcal{R}}\) if \(|\beta| = |\gamma|\) and \(\beta \neq \gamma\) or if \(\gamma = \beta'\) with \(|\beta'| = |\beta|\) and \(\beta' \neq \beta\), so the content of condition (2) in Definition 7.11 is that the orthogonality condition holds for the special case where \(\gamma = \beta\) and \(\delta \neq \emptyset\).

Given any collection \(\{\mathcal{U}_\beta\}_{\beta \in \mathbb{F}_d^+}\) of coefficient Hilbert spaces indexed by \(\mathbb{F}_d^+\), we set
\[H^2_{\{\mathcal{U}_\beta\}}(\mathbb{F}_d^+) := \bigoplus_{\beta \in \mathbb{F}_d^+} S_{\omega, \mathcal{R}}^T z^\beta U_\beta = \bigoplus_{\beta \in \mathbb{F}_d^+} z^\beta U_\beta \]
(7.39)
equal to the \textit{time-varying Fock space} (compare with the standard Fock space \[(1.20)\]) where every vector
\[\mathbf{u} = \bigoplus_{\beta \in \mathbb{F}_d^+} z^\beta u_{\beta} \in H^2_{\{\mathcal{U}_\beta\}}(\mathbb{F}_d^+)\]
is assigned the Fock-space norm \(\|\mathbf{u}\|^2 = \sum_{\beta \in \mathbb{F}_d^+} \|u_{\beta}\|_{U_\beta}^2\) as in \[(1.20)\], the difference here being that the coefficient space \(U_\beta\) is allowed to depend on the index \(\beta \in \mathbb{F}_d^+\).

Given a \(H^2_{\omega, \gamma}(\mathbb{F}_d^+)-\text{Bergman-inner family} \{\Theta_{\omega, \beta}\}_{\beta \in \mathbb{F}_d^+}\), let us set
\[\mathcal{M} = \text{ row } \bigoplus_{\beta \in \mathbb{F}_d^+} \{\Theta_{\omega, \beta} : H^2_{\{\mathcal{U}_\beta\}}(\mathbb{F}_d^+) = \bigoplus_{\beta \in \mathbb{F}_d^+} \Theta_{\omega, \beta} S_{\omega, \mathcal{R}}^T U_\beta \subset H^2_{\omega, \gamma}(\mathbb{F}_d^+)\}. \quad (7.40)\]
Then conditions (1) and (2) in Definition 7.11 imply that the “multiplication” operator $M_{\Theta \omega}: H^2_{(U_\omega)}(F^+_d) \to H^2_{\omega',\mathcal{Y}}(F^+_d)$ given by
\[ M_{\Theta \omega} = \text{row}_{\beta \in F^+_d}[\Theta_{\omega,\beta}]: u = \bigoplus_{\beta \in F^+_d} z^\beta u_\beta \mapsto \sum_{\beta \in F^+_d} \Theta_{\omega,\beta}(z) z^\beta u_\beta \] (7.41)
maps $H^2_{(U_\omega)}(F^+_d)$ isometrically onto the subspace $\mathcal{M}$ given by (7.40). Furthermore, condition (3) in Definition 7.11 implies that $\mathcal{M}$ so defined is $S_{\omega,R}$-invariant. The next result is the converse: given any $S_{\omega,R}$-invariant closed subspace of $H^2_{\omega',\mathcal{Y}}(F^+_d)$, there is a $H^2_{\omega',\mathcal{Y}}(F^+_d)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in F^+_d}$ so that $\mathcal{M} = M_{\Theta \omega} H^2_{(U_\omega)}$, our next analog of the Beurling-Lax theorem for the freely noncommutative multivariable Hardy-Fock-space setting.

**Theorem 7.12.** Let $\mathcal{M}$ be a closed $S_{\omega,R}$-invariant subspace of $H^2_{\omega',\mathcal{Y}}(F^+_d)$. Define formal power series $\Theta_{\omega,\beta} \in \mathcal{L}(\mathcal{U}_\beta,\mathcal{Y})$ so that $M_{\Theta_{\omega,\beta}}$ maps $\mathcal{L}\mathcal{U}_\beta$ isometrically onto the subspace $\mathcal{M}_\beta$ given by (7.19). Then $\Theta_{\omega} = \{\Theta_{\omega,\beta}\}_{\beta \in F^+_d}$ is a $H^2_{\omega',\mathcal{Y}}(F^+_d)$-Bergman-inner family giving rise to a Beurling-Lax representation for the $S_{\omega,R}$-invariant subspace $\mathcal{M}$ (using the notations (7.39) and (7.41)):
\[ \mathcal{M} = M_{\Theta \omega} H^2_{(U_\omega)}(F^+_d). \] (7.42)
If $\Theta_{\omega}' = \{\Theta_{\omega',\beta}\}_{\beta \in F^+_d}$ is another such $H^2_{\omega',\mathcal{Y}}(F^+_d)$-Bergman-inner family, then for each $\beta \in F^+_d$ there is a unitary operator $U_\beta: \mathcal{U}_\beta \to \mathcal{U}_\beta'$ so that $\Theta_{\omega,\beta}'(z) U_\beta = \Theta_{\omega,\beta}(z)$.

Furthermore, one can construct such a $H^2_{\omega',\mathcal{Y}}(F^+_d)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in F^+_d}$ representing $\mathcal{M}$ via transfer-function realizations as follows:

1. Set $X = \mathcal{M}^\perp$ and define $A \in \mathcal{L}(X,X^G)$ and $C \in \mathcal{L}(X,\mathcal{Y})$ by
   \[ A = S_{\omega,R}^{*}|_{\mathcal{M}^\perp}, \quad Cf = f_\emptyset \quad \text{for} \quad f \in \mathcal{M}^\perp. \]

2. Construct injective $\begin{bmatrix} \hat{B}_\beta \\ D_\beta \end{bmatrix}$ by solving the Cholesky factorization problem (7.18) in Lemma 7.5.

3. Define $\Theta_{\omega,\beta}(z)$ by
   \[ \Theta_{\omega,\beta}(z) = z^{-1}|_\beta D_\beta + CR_{\omega,|\beta|+1}(Z(z)A)Z(z)\hat{B}_\beta. \] (7.43)

**Proof.** By Lemma 7.7 we know that $\mathcal{M}$ has the orthogonal decomposition (7.20) with $\mathcal{M}_\beta$ given by (7.19). We define $\Theta_{\omega,\beta}$ so that $M_{\Theta_{\omega,\beta}}$ maps $\mathcal{L}\mathcal{U}_\beta$ isometrically onto $\mathcal{M}_\beta$. We thus have the orthogonal decomposition
\[ \mathcal{M} = \bigoplus_{\beta \in F^+_d} M_{\Theta_{\omega,\beta}} z^\beta \cdot \mathcal{U}_\beta. \]
This leads to the operator $M_{\Theta \omega}: H^2_{(U_\omega)}(F^+_d) \to H^2_{\omega',\mathcal{Y}}(F^+_d)$ being a unitary map from $H^2_{(U_\omega)}$ onto $\mathcal{M}$. From the orthogonal decomposition (7.20) for $\mathcal{M}$, we see that any $\{\Theta_{\omega,\beta}\}$ constructed from $\mathcal{M}$ in this way necessarily is a $H^2_{\omega',\mathcal{Y}}(F^+_d)$-Bergman-inner family.

The only constraint on the choice of $\Theta_{\omega,\beta}$ is that $M_{\Theta_{\omega,\beta}}$ maps $z^\beta \mathcal{U}_\beta$ isometrically onto the subspace $\mathcal{M}_\beta$. Hence, any other choice $\{\Theta_{\omega,\beta}'\}$ with respective coefficient spaces $U_\beta'$ necessarily has the form $\Theta_{\omega,\beta}' z^\beta U_\beta' = \Theta_{\omega,\beta} z^\beta$ for a unitary operator $U: \mathcal{U}_\beta \to \mathcal{U}_\beta'$. 

7.2. BEURLING-LAX REPRESENTATIONS BASED ON BERGMAN-INNER FAMILIES

Suppose now that we are given the closed $S_{\omega,R}$-invariant subspace $M$ of $H^2_{\omega,Y}(\mathbb{F}_d^+)$ and we construct $\Theta_{\omega,\beta}$ according to the recipe (1), (2), (3) in the statement of the theorem. By Theorem 7.43 part (4), the pair $(C,A)$ is the canonical model $\omega$-isometric output pair for which the range $\text{Ran} O_{\omega,C,A}$ of the observability operator $O_{\omega,C,A}$ is exactly $M^\perp$. As a result of Lemma 7.3 we see that the metric constraints (7.6) and (7.15) hold. By Lemma 7.4, the $S_{\omega,R}$-invariant subspace $M$ has the orthogonal decomposition (7.20) with $\mathcal{M}_\beta$ as in (7.19). The kernel function $k_M(z,\zeta)$ for the subspace $M$ is given by (5.1) (also (7.27)). Then Proposition 7.9 applies to tell us that the kernel function for $\mathcal{M}_\beta$ is given by the formula (7.30). Then Lemma 7.10 applies to tell us that $k_{\mathcal{M}_\beta}(z,\zeta)$ has the factorization as in (7.38), and that the operator $M_{\Theta_{\omega,\beta}} : z^\beta u \mapsto \Theta_{\omega,\beta}(z)z^\beta u$ is unitary from $z^\beta U_\beta$ onto $\mathcal{M}_\beta$ for each $\beta \in \mathbb{F}_d^+$ as required.

As a corollary we see that any $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ has a realization as in (7.43).

**Corollary 7.13.** suppose that $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ is a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family. Then there is a $\omega$-isometric output pair $(C,A)$ with $A$ $\omega$-strongly stable embedded in a family of connection matrices $\left\{ \begin{bmatrix} A & B_{\alpha} \\ C & D_{\alpha} \end{bmatrix} : \alpha \in \mathbb{F}_d^+ \right\}$ satisfying the metric constraints (7.6) and (7.15) so that $\Theta_{\omega,\beta}(z)$ has the transfer-function realization (7.43).

**Proof.** In the discussion preceding the statement of Theorem 7.12 we saw that any $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ generates a $S_{n,R}$-invariant subspace $\mathcal{M}$ of $H^2_{\omega,Y}$ having $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ as its Beurling-Lax representer as in (7.38). We may then use the results of Theorem 7.12 to generate a transfer-function realization (7.43) for $\Theta_{\omega,\beta}$ meeting all the desired requirements.

Note that, by Definition 5.25, the content of conditions (1) and (2) in Definition 7.11 is that each $\Theta_{\omega,\beta}(z) \cdot z^\beta$ coming from a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ is itself $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman inner. It is condition (3) in Definition 7.11 which specifies how all these $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner power series fit together to form a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family. In particular, the first Bergman-inner power series $\Theta_{\omega,\beta}$ of the $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ has a transfer function realization

$$\Theta(z) = D + CR_{\omega,1}(Z(z)A)Z(z)\hat{B}$$

with connection matrix

$$U = \begin{bmatrix} A & \hat{B} \\ C & \hat{D} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \end{bmatrix}$$

(7.44)

satisfying the metric constraints of the form (7.6) and (7.15). Our next result is a converse to these observations.

**Theorem 7.14.** Suppose that the formal power series $\Theta(z) \in \mathcal{L}(U,Y)(\{z\})$ is $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner. Then:

1. There is a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{\omega,\beta}\}_{\beta \in \mathbb{F}_d^+}$ so that $\Theta = \Theta_{\omega,\beta}$. 

(2) \( \Theta \) has the form

\[
\Theta(z) = D + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+1}^{-1} C \alpha B_j \alpha_j \tag{7.45}
\]

where \((C, A)\) is \( \omega \)-isometric with \( A \), and the connection matrix \((7.44)\) satisfies the metric constraints

\[
\begin{bmatrix}
A^* & C^*
\end{bmatrix}
\begin{bmatrix}
\mathcal{G}_{\omega, 1, C_1} & 0 & I_d
\end{bmatrix}
= \begin{bmatrix}
A & B
\end{bmatrix}
\begin{bmatrix}
C & D
\end{bmatrix}
= \begin{bmatrix}
\mathcal{G}_{\omega, C_1} & 0
\end{bmatrix}
\begin{bmatrix}
I_d & I_U
\end{bmatrix}
= \begin{bmatrix}
A^* & C^*
\end{bmatrix}
\begin{bmatrix}
\mathcal{G}_{\omega, 1, C_1} & 0 & I_d
\end{bmatrix}
= \begin{bmatrix}
B^* & D^*
\end{bmatrix}
\begin{bmatrix}
C & D
\end{bmatrix}
= \begin{bmatrix}
\mathcal{G}_{\omega, C_2} & 0 & I_d
\end{bmatrix}
\begin{bmatrix}
I_d & I_U
\end{bmatrix}
\tag{7.46}
\]

**Proof.** Suppose that \( \Theta(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle \langle z \rangle \rangle) \) is \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-Bergman-inner. Define \( \mathcal{E} \subset H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \) by \( \mathcal{E} = M_{\Theta} \cdot \mathcal{U} \). Then condition (1) in Definition \( 3.23 \) tells us that \( M_{\Theta} \) maps \( \mathcal{U} \) isometrically into \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \) and hence \( \mathcal{E} \) is a closed subspace of \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \). We next set

\[
\mathcal{M} = \bigvee_{\alpha \in \mathbb{F}_d^+} \mathcal{S}_{\omega, R}^\alpha \mathcal{E}.
\tag{7.47}
\]

Then \( \mathcal{M} \) is a closed \( S_{\omega, R} \)-invariant subspace of \( H^2_{\omega, \mathcal{Y}} \). By Theorem \( 7.12 \) it follows that \( \mathcal{M} \) has a representation of the form \((7.42)\) in terms of a \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-Bergman-inner family \( \{\Theta_{\omega, \beta}\}_{\beta \in \mathbb{F}_d^+} \). By the second condition in Definition \( 3.23 \) we see that, for \( \beta \) a nonempty word in \( \mathbb{F}_d^+ \), the subspace \( \mathcal{E} = \Theta \cdot \mathcal{U} \) is orthogonal to the subspace

\[
\mathcal{S}_{\omega, R}^\beta = \bigvee_{\alpha \in \mathbb{F}_d^+} \mathcal{S}_{\omega, R}^\alpha \mathcal{E} = \bigvee_{\alpha \in \mathbb{F}_d^+} \mathcal{S}_{\omega, R}^\alpha \Theta \cdot \mathcal{U} = \bigvee_{\alpha \in \mathbb{F}_d^+} \Theta \cdot z^\alpha \cdot \mathcal{U}.
\]

In particular, \( \mathcal{E} \) is orthogonal to \( \mathcal{S}_{\omega, R} \mathcal{M} \) for \( k = 1, \ldots, d \) and hence

\[
\mathcal{E} \subseteq \mathcal{M} \otimes \left( \bigoplus_{k=1}^{d} \mathcal{S}_{\omega, R} \mathcal{M} \right) =: \mathcal{M}_\emptyset.
\tag{7.48}
\]

To show that the containment \((7.48)\) actually holds with equality, we verify that the assumption that \( f \in \mathcal{M}_\emptyset \) is orthogonal to \( \mathcal{E} \) forces \( f = 0 \). Indeed, if \( f \in \mathcal{M}_\emptyset \), then \( f \) is orthogonal to \( \mathcal{S}_{\omega, R, k} \mathcal{M} \) for \( k = 1, \ldots, d \) and hence, on account of \((7.47)\), \( f \) is orthogonal to \( \mathcal{S}_{\omega, R} \mathcal{E} \) for any nonempty \( \alpha \in \mathbb{F}_d^+ \). If in addition, \( f \) is orthogonal to \( \mathcal{E} \), then, again by \((7.47)\), it is orthogonal to the whole \( \mathcal{M} \). Since \( f \in \mathcal{M}_\emptyset \subset \mathcal{M} \), it follows that \( f = 0 \), and hence the containment \((7.48)\) holds with equality. This means that without loss of generality we may take \( \Theta_{\omega, \emptyset} \) to be \( \Theta \) in the construction in Theorem \( 7.14 \). Consequently, \( \Theta = \Theta_{\omega, \emptyset} \) has a realization as in \((7.45)\) and \((7.46)\) by specialization of the general state-space formulas for the whole Bergman-inner family in the second part of Theorem \( 7.12 \) to the case \( \beta = \emptyset \). \( \square \)

**Remark 7.15.** System-theoretic interpretation of Theorem \( 7.12 \). In systems theory language, the result of Theorem \( 7.12 \) can be expressed as follows. Associated with any time-varying noncommutative multidimensional linear system \( \Sigma(\mathcal{U}) \) of the form \( 7.32 \) is a well-defined input-output map

\[
T_{\mathcal{U}}: \{u_{\beta}\}_{\beta \in \mathbb{F}_d^+} \rightarrow \{y_{\beta}\}_{\beta \in \mathbb{F}_d^+},
\]
where the output string \( \{y_\beta\}_{\beta \in \mathbb{F}_d^+} \) is determined from the input string \( \{u_\beta\}_{\beta \in \mathbb{F}_d^+} \) (\( u_\beta \in \mathcal{U} \) for each \( \beta \in \mathbb{F}_d^+ \)) by solving the system equations (1.32) recursively with initial condition \( x(\emptyset) \) set equal to zero. The formal noncommutative Z-transform of \( T_U \) is the map \( \hat{T}_U \) defined by
\[
\hat{T}_U : \sum_{\beta \in \mathbb{F}_d^+} u_\beta z^\beta \mapsto \sum_{\beta \in \mathbb{F}_d^+} y_\beta z^\beta
\]
effectively when \( T_U : \{u_\beta\}_{\beta \in \mathbb{F}_d^+} \mapsto \{y_\beta\}_{\beta \in \mathbb{F}_d^+} \). Let us say that a time-varying formal noncommutative multidimensional linear system \( \Sigma \) is conservative if the operators \( A, B_\beta, C, D_\beta \) in the connection matrix (7.44) satisfy the metric conditions (7.6) and (7.15). As a consequence of the identity (2.35) (with initial condition \( x = \hat{x}(\emptyset) = 0 \)), we see that the multiplication operator \( M_\Theta \) has transfer function realization \( M_\Theta = \hat{T}_U \).

### 7.3. Expansive multiplier property

In this section we take advantage of the realization formula (7.35) to discuss another property of \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-inner multipliers (which, however, occurs only for particular weights). The question goes back to [58, 48, 49, 34], where it was shown that canonical divisors in the Bergman space \( \mathcal{A}_2 \) (also in the non-Hilbert Bergman spaces \( \mathcal{A}_2^{\prime} \)) are expansive multipliers in the sense that \( \|Gf\|_{\mathcal{A}_2} \geq \|f\|_{\mathcal{A}_2} \) for all \( f \in \mathcal{A}_2 \). Canonical divisors form a subclass of Bergman-inner functions for which the expansive multiplier property holds only in a weaker version: for any Bergman-inner function \( G \), the inequality \( \|Gf\|_{\mathcal{A}_2} \geq \|f\|_{\mathcal{A}_2} \) holds for all \( f \in H^2 \) (but not for all \( f \in \mathcal{A}_2 \), in general). The latter weak form of the expansive multiplier property holds for weighted Bergman spaces \( \mathcal{A}_n \) [59] but fails in \( \mathcal{A}_n \) for \( n > 3 \).

In the present noncommutative setting, we define the expansive multiplier (in the weak form) keeping in mind the Fock space \( H^2(\mathbb{F}_d^+) \) as a suitable substitute of the Hardy space \( H^2 \) of the unit disk.

Let us say that the space \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \) possesses the expansive multiplier property if for any \( L(\mathcal{U}, \mathcal{Y}) \)-valued \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-inner multiplier \( \Theta \),
\[
\|\Theta f\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} \geq \|f\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} \quad \text{for all} \quad f \in H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+).
\] (7.49)

**Remark 7.16.** If \( \Theta \in L(\mathcal{U}, \mathcal{Y})(\mathcal{U}(\mathcal{Z})) \) is \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-inner, then for every noncommutative polynomial \( p(z) = p_0 + \hat{p}(z) \) where \( \hat{p}(z) = \sum_{1 \leq |\alpha| \leq K} p_\alpha z^\alpha \), we have
\[
\|\Theta p\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} = \|\Theta p_0\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)}^2 + \|\Theta \hat{p}\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)}^2 = \|p_0\|_{\mathcal{U}}^2 + \|\Theta \hat{p}\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)}^2.
\]
Since \( \|p\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} = \|p_0\|_{\mathcal{U}}^2 + \|\hat{p}\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)}^2 \), we conclude that the inequality
\[
\|\Theta p\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)} \geq \|p\|_{H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+)}
\]
holds for all polynomials in \( \mathcal{U}(\mathcal{Z}) \) if and only if it holds for all polynomials \( p \) with \( p_0 = 0 \).

**Lemma 7.17.** Let \( \Theta \) be an \( L(\mathcal{U}, \mathcal{Y}) \)-valued \( H^2_{\omega, \mathcal{Y}}(\mathbb{F}_d^+) \)-inner power series realized as in Theorem 7.14. Then for every polynomial
\[
p(z) = \sum_{\alpha \in \mathbb{F}_d^+, 1 \leq |\alpha| \leq k} p_\alpha z^\alpha,
\] (7.50)
the following equality holds:

\[
\|\Theta p\|_{H_{2,Y}^2(F_+^d)}^2 - \|p\|_{H_{2,Y}^2(F_+^d)}^2 = \sum_{v \in F_+^d} \sum_{|v| \geq 2} \omega_{|v|} \left( \frac{\omega_{|v|}}{\omega_{|\alpha|+1} \omega_{|\beta|+1}} - \frac{\omega_{\max(|\alpha'|,|\beta'|)}}{\omega_{|\alpha'|+1} \omega_{|\beta'|}} \right) \cdot \langle CA^{\alpha}B_{2p_{\alpha'}}, CA^{\beta}B_{2p_{\beta'}} \rangle_Y
\]

where, according to (2.27), \( \|p\|_{H_{2,Y}^2(F_+^d)}^2 = \sum_{\alpha \in F_+^d : 1 \leq |\alpha| \leq k} \omega_{|\alpha|} \|p_{\alpha'}\|_Y^2 \).

PROOF. Making use of (7.54) and (7.55), we write the power series expansion

\[
\Theta(z)p(z) = D \sum_{j=1}^d p_j z^j + \sum_{v \in F_+^d : 2 \leq |v| \leq k} \left( Dp_v + \sum_{\alpha \in v} \omega_{|\alpha|+1} CA^{\alpha} B_{p_{\alpha'}} \right) z^v
\]

By the definition (2.27) of the \( H_{2,Y}^2(F_+^d) \)-norm, we have

\[
\|\Theta p\|_{H_{2,Y}^2(F_+^d)}^2 = \sum_{i=1}^d \omega_i \cdot \|Dp_i\|_Y^2
\]

\[
+ \sum_{v \in F_+^d : 2 \leq |v| \leq k} \omega_{|v|} \cdot \|Dp_v + \sum_{\alpha \in v} \omega_{|\alpha|+1} CA^{\alpha} B_{p_{\alpha'}}\|_Y^2
\]

\[
+ \sum_{v \in F_+^d : |v| > k} \omega_{|v|} \cdot \left( \sum_{\alpha \in v} \omega_{|\alpha|+1} CA^{\alpha} B_{p_{\alpha'}} \right)^2
\]

\[
= I + II + III,
\]

where

\[
I = \sum_{v \in F_+^d : |v| \leq k} \omega_{|v|} \cdot \|Dp_v\|_Y^2,
\]

\[
II = 2\text{Re} \left( \sum_{v \in F_+^d : 2 \leq |v| \leq k} \omega_{|v|} \cdot \left\langle Dp_v, \sum_{\alpha \in v} \omega_{|\alpha|+1} CA^{\alpha} B_{p_{\alpha'}} \right\rangle_Y \right),
\]

\[
III = \sum_{v \in F_+^d : |v| \geq 2} \omega_{|v|} \cdot \left( \sum_{\alpha \in v} \omega_{|\alpha|+1} CA^{\alpha} B_{p_{\alpha'}} \right)^2 \|\|_Y^2.
\]

To simplify the notation we set \( p_v = 0 \) for \( |v| > k \) and for any two representations

\[
v = \alpha \alpha' = \beta \beta' \quad (|\alpha'| \leq k, \; |\beta'| \leq k)
\]

of a word \( v \in F_+^d \), we let

\[
d_{v,\alpha',\beta'} = \frac{\omega_{|v|}}{\omega_{|\alpha'|+1} \cdot \omega_{|\beta'|+1}} \cdot \langle CA^{\alpha} B_{p_{\alpha'}}, CA^{\beta} B_{p_{\beta'}} \rangle_Y.
\]

Observe that the elements \( \alpha, \beta, i \) and \( j \) in representations (7.54) are completely determined by \( v, \alpha' \) and \( \beta' \). We also note that if \( |\alpha'| = |\beta'| \), then it follows from
For the second term on the right side of (7.58), we have, on account of \((7.57)\), \((4.58)\),

\[
\text{Substituting (7.59) and (7.60) into (7.58) gives}
\]

\[
\text{For the first term on the right side, we have}
\]

\[
\sum_{v \in \mathbb{F}_d^+} \sum_{\alpha \beta} d_v,\alpha',\beta' = \sum_{v \in \mathbb{F}_d^+} \sum_{\alpha \beta} \frac{\omega_v}{\omega_{|v|+1}} \cdot \|CA^\alpha B_{i \bar{v}}p_{\alpha'}\|^2_Y
\]

\[= \sum_{|\alpha'| < |\alpha|} \sum_{i=1}^d \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1}} \cdot \langle CA^\alpha B_{i \bar{v}}p_{\alpha'}, CA^\beta B_{j \bar{v}}p_{\beta'} \rangle_Y. \]

For the second term on the right side of (7.58), we have, on account of (7.57),

\[
\sum_{v \in \mathbb{F}_d^+} \sum_{\alpha \beta} d_v,\alpha',\beta' = \sum_{|\alpha'| < |\alpha|} \sum_{i=1}^d \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1}} \cdot \langle CA^\alpha B_{i \bar{v}}p_{\alpha'}, CA^\beta B_{j \bar{v}}p_{\beta'} \rangle_Y.
\]

Substituting (7.59) and (7.60) into (7.58) gives

\[
\text{III} = \sum_{|\alpha'| < |\alpha|} \sum_{i=1}^d \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1}} \cdot \|CA^\alpha B_{i \bar{v}}p_{\alpha'}\|^2_Y
\]

\[+ 2\text{Re} \left( \sum_{|\alpha'| < |\alpha|} \sum_{i=1}^d \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1}} \cdot \langle CA^\alpha B_{i \bar{v}}p_{\alpha'}, CA^\beta B_{j \bar{v}}p_{\beta'} \rangle_Y \right). \]

By equality of block (2,2)-entries in the first matrix equality in (7.46) and on account of (4.58),

\[
\|Dp_v\|^2_Y = \|p_v\|^2_{\mathcal{U}} - \sum_{i=1}^d \langle \mathbf{G}_{\omega,1,C,A} B_{i \bar{v}}p_v, B_{i \bar{v}}p_v \rangle_X
\]

\[= \|p_v\|^2_{\mathcal{U}} - \sum_{i=1}^d \left( \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+1}^{-1} A^{*\alpha} C^* CA^\alpha B_{i \bar{v}}p_v, B_{i \bar{v}}p_v \right)_X
\]

\[= \|p_v\|^2_{\mathcal{U}} - \sum_{i=1}^d \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+1}^{-1} \cdot \|CA^\alpha B_{i \bar{v}}p_v\|^2_X \]
and therefore (see (7.53)),

\[
I = \sum_{v \in \mathbb{F}_d^+ : 1 \leq |v| \leq k} \omega_{|v|} \cdot \| p_v \|^2 - \sum_{v \in \mathbb{F}_d^+ : 1 \leq |v| \leq m} \sum_{\alpha} \sum_{\alpha'}^{d} \frac{\omega_{|v|}}{\omega_{|\alpha|+1}} \cdot \| C \alpha B_{i} p_{v} \|^2_X
\]

\[
= \| p \|^2_{H_{2,d}(\mathbb{F}_d^+)} - \sum_{\alpha, \alpha' \in \mathbb{F}_d^+}^{d} \frac{\omega_{|\alpha'|}}{\omega_{|\alpha|+1}} \cdot \| C \alpha B_{i} p_{\alpha'} \|^2_X.
\]  \hspace{1cm} (7.62)

We next combine the equality of the (2,1)-block entries in the first matrix equation in \([7.30]\) with \([7.58]\) to get

\[
\langle D p_{\alpha'}, C \alpha B_{i} p_{\beta'} \rangle_Y = \left\langle p_{\alpha'}, D^* C \alpha B_{i} p_{\beta'} \right\rangle_U
\]

\[
= - \left\langle p_{\alpha'}, \sum_{i=1}^{d} B_i^* \Theta_{\omega,1,C} \alpha \gamma B_{i} p_{\beta'} \right\rangle_U
\]

\[
= - \left\langle p_{\alpha'}, \sum_{i=1}^{d} \sum_{\alpha} \omega_{|\alpha|+1}^{-1} B_i^* A^{\alpha} \gamma C^{*} \alpha^\gamma B_{i} p_{\beta'} \right\rangle_U
\]

\[
= - \sum_{\alpha} \sum_{i=1}^{d} \omega_{|\alpha|+1}^{-1} \cdot \langle C \alpha B_{i} p_{\alpha'}, \alpha^\gamma B_{i} p_{\beta'} \rangle_Y
\]

which together with \([7.54]\) and \([7.57]\) gives

\[
\| \Theta p \|^2_{H_{2,d}(\mathbb{F}_d^+)} - \| p \|^2_{H_{2,d}(\mathbb{F}_d^+)} =\]

\[
= 2 \text{Re} \left( \sum_{\alpha, \alpha' \in \mathbb{F}_d^+ : |\alpha'| \leq k} \omega_{|\alpha'|} \cdot \left\langle D p_{\alpha'}, \sum_{\alpha' = \gamma \beta'}^{d} \omega_{|\gamma|+1}^{-1} C \alpha \gamma B_{i} p_{\beta'} \right\rangle_Y \right).
\]  \hspace{1cm} (7.63)

We now substitute \([7.61]\), \([7.62]\) and \([7.63]\) into \([7.62]\):

\[
\| \Theta p \|^2_{H_{2,d}(\mathbb{F}_d^+)} - \| p \|^2_{H_{2,d}(\mathbb{F}_d^+)}
\]

\[
= \sum_{\alpha, \alpha' \in \mathbb{F}_d^+ : |\alpha'| \leq k} \sum_{i=1}^{d} \left( \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1}^2} - \frac{\omega_{|\alpha'|}}{\omega_{|\alpha|+1}^2} \right) \cdot \| C \alpha B_{i} p_{\alpha'} \|^2_Y
\]

\[
+ 2 \text{Re} \left( \sum_{\alpha = \gamma \beta', |\alpha'| \leq k} \sum_{i=1}^{d} \left( \frac{\omega_{|\alpha|+|\alpha'|+1}}{\omega_{|\alpha|+1} \cdot \omega_{|\alpha|+|\gamma|+1} + \omega_{|\alpha'|} \cdot \omega_{|\alpha|+|\gamma|+1}} - \frac{\omega_{|\alpha'|}}{\omega_{|\alpha|+1} \cdot \omega_{|\alpha|+|\gamma|+1}} \right) \cdot \langle C \alpha B_{i} p_{\alpha'}, \alpha^\gamma B_{i} p_{\beta'} \rangle_Y \right).\]
Making use of representations (7.56) and (7.57) we rewrite the latter equality as

\[
\| \Theta p \|^2_{H^2_{\omega,Y}(\mathbb{F}_d^+) - \| p \|^2_{H^2_{\omega,Y}(\mathbb{F}_d^+)}} = \sum_{v \in \mathbb{F}_d^+: |v| \geq 2} \sum_{\alpha' = v} \left( \frac{\omega_{|v|}}{\omega_{|\alpha| + 1}} - \frac{\omega_{|\alpha'|}}{\omega_{|\alpha'| + 1}} \right) \cdot \| CA^n B p_{\alpha'} \|^2_Y + 2 \text{Re} \left( \sum_{v \in \mathbb{F}_d^+: |v| \geq 2} \sum_{\alpha' = v, \alpha = \gamma_j \beta'} \left( \frac{\omega_{|v|}}{\omega_{|\alpha| + 1} \cdot \omega_{\beta'} + 1} - \frac{\omega_{|\alpha'|}}{\omega_{|\alpha'| + 1} \cdot \omega_{|\beta'|} + 1} \right) \right) \cdot \langle CA^n B p_{\alpha'}, CA^{\beta'} B p_{\beta'} \rangle_Y,
\]

which can be written, since \( \omega_0 = 1 \), in a more compact form as in (7.51). \( \square \)

**Remark 7.18.** There is an alternative way of writing the right-hand side of (7.51) which will be more convenient for our purposes. Fix a word \( v \in \mathbb{F}_d^+ \) with \( |v| \geq 2 \) and consider a splitting of \( v \) as

\[
v = \alpha \alpha',
\]

where \( 1 \leq |\alpha'| \leq |v| - 1 \). We note that once the length \( i = |\alpha'| \) is specified then each of the three factors \( \alpha, 1, \alpha' \) in the decomposition (7.64) is uniquely determined. We may therefore write \( \alpha = \alpha_{v,i} \), \( 1 = i_{v,i} \), \( \alpha' = \alpha'_{v,i} \). Similarly, in the decomposition \( v = \beta \beta' \), we may write \( \beta = \alpha_{v,j}, j = i_{v,j}, \beta' = \alpha'_{v,j} \). Then we also have

\[
|\alpha'_{v,i}| = i, \quad |\alpha_{v,i}| + 1 = |v| - i.
\]

Then the right-hand side of (7.51) has the form

\[
\sum_{v \in \mathbb{F}_d^+: |v| \geq 2} \sum_{i,j=1}^{\max(|v|)} \left( \frac{\omega_{|v|}}{\omega_{|v| - i} \omega_{|v| - j}} - \frac{\omega_{\max(i,j)}}{\omega_{\max(|v| - i, |v| - j)}} \right) \cdot \langle CA^n B_{i_{v,i},p_{\alpha'_{v,i}}}, CA^{\alpha_{v,j}} B_{i_{v,j},p_{\alpha'_{v,j}}} \rangle_Y.
\]

The following theorem presents sufficient conditions (in terms of the weight sequence \( \omega \)) for the space \( H^{2}_{\omega,Y}(\mathbb{F}_d^+) \) to possess the expansive multiplier property.

**Theorem 7.19.** Let \( \omega = \{ \omega_j \}_{j \geq 0} \) be the weight sequence satisfying conditions (7.31). For every pair \( (k, r) \) of positive integers \( k < r \), define the symmetric real matrix

\[
M^{(k,r)} = \left[ M_{ij}^{(k,r)} \right]_{i,j=1}^{k}
\]

with the entries \( M_{ij}^{(k,r)} \) given by

\[
M_{ij}^{(k,r)} = M_{ji}^{(k,r)} = 1 - \frac{\omega_j \omega_{r-i}}{\omega_i \omega_{j-i}} \quad \text{for} \quad 1 \leq i \leq j \leq k.
\]

If \( \det M^{(k,r)} \geq 0 \) for all \( 1 \leq k < r \), then \( H^{2}_{\omega,Y}(\mathbb{F}_d^+) \) possesses the expansive multiplier property, i.e., condition (7.49) holds for any \( H^{2}_{\omega,Y}(\mathbb{F}_d^+) \)-inner power series \( \Theta \).
7. ORTHOGONAL BEURLING-LAX REPRESENTATIONS

PROOF. For fixed $0 < k < r$, let $\Omega$ be the $k \times k$ diagonal matrix with the weights $\omega_{r-1}, \omega_{r-2}, \ldots, \omega_{r-k}$ on the main diagonal. It is readily checked that

$$N^{(r,k)} := \left[ \frac{\omega_r}{\omega_{r-j} \omega_r} - \frac{\omega_{\max(i,j)}}{\omega_{r-\max(i,j)} \omega_{|j-i|}} \right]_{i,j=1}^k = \omega_r \Omega^{-1} M^{(r,k)} \Omega^{-1},$$

and hence, the assumption $M^{(r,k)} \geq 0$ implies $N^{(r,k)} \geq 0$. Note that the right-hand side of (7.63), rewritten as in (7.65), becomes

$$\sum_{v \in \mathbb{F}^+_3: |v| \geq 2} \sum_{i,j=1}^{|v|-1} [N^{(v,i,v-1)}]_{i,j} \cdot \langle CA^{\alpha_{v,i}} B_{v,i} p_{\alpha_{v,j}} C A^{\alpha_{v,j}} B_{v,j} p_{\alpha_{v,j}} \rangle.$$

Since $N^{(r,k)}$ is positive semidefinite, we see immediately that this expression is nonnegative for any choice of operators $A, B_j, C$ and vectors $p_a \in \mathcal{L}$. Therefore, inequality (7.73) holds for all polynomials vanishing at the origin and now Remark 7.10 completes the proof. \hfill \square

If $\omega_j = \mu_{2,j} = \frac{1}{j+1}$, then (7.66) takes the form $M^{(k,r)}_{ij} = \frac{((r-j))(j+1)(r+1)!}{(j+1)(r+1)!}$. It was shown in (17) that

$$\det M^{(k,r)} = \frac{2^k(r-k)(r+1)^{k-1}(r-k+1)!}{r^2(k+1)(k+1)!}$$

for all $2 \leq k < r$.

Thus, $\det M^{(k,r)} > 0$ for all $1 \leq k < r$ and thus, the expansive multiplier property holds in the space $\mathcal{A}_{2, \mathcal{Y}}(\mathbb{F}^+_3)$ by Theorem 7.19.

If $\omega_j = \mu_{3,j} = \frac{2}{(j+1)(j+2)}$, then (7.66) takes the form

$$M^{(k,r)}_{ij} = M^{(k,r)}_{ji} = \frac{(r+1)(r+2)(j-i+1)(j-i+2)}{(j+1)(j+2)(r-i+1)(r-i+2)}.$$

It was shown in (17) that also in this case, $\det M^{(k,r)} > 0$ for all $1 \leq k < r$. By Theorem 7.19 we conclude that the Bergman-Fock space $\mathcal{A}_{3, \mathcal{Y}}(\mathbb{F}^+_3)$ also possesses the expansive multiplier property.

7.4. Bergman-inner multipliers as extremal solutions of interpolation problems

The Bergman-inner function associated with a sequence of nonzero points $\{z_j\}$ in the unit disk $\mathbb{D}$ appeared in [58, 58] as a unit-norm element $f_0$ of the subspace $\mathcal{M}$ of the Bergman space consisting of all functions $f$ with zero set containing $\{z_j\}$ which achieves the maximal possible value of $|f(0)|$. In this section we show $H^2_{\mathcal{Y}}(\mathbb{F}^+_3)$-Bergman-inner multipliers (see Definition 3.23) appear as extremal solutions of certain operator-argument interpolation problems in the space $H^2_{\mathcal{Y}} := H^2_{\mathcal{Y}}(\mathbb{F}^+_3) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

Being equipped with the $\mathcal{L}(\mathcal{U})$-valued inner product

$$[F, G] = \sum_{\alpha \in \mathbb{F}^+_3} \omega_{\alpha} G^*_a F_a$$

for $F(z) = \sum_{\alpha \in \mathbb{F}^+_3} F_a z^\alpha, G(z) = \sum_{\alpha \in \mathbb{F}^+_3} G_a z^\alpha$, (7.67)

the space $H^2_{\mathcal{Y}}(\mathbb{F}^+_3) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$ becomes a $C^*$-module (sometimes also called a Hilbert module) over the $C^*$-algebra $\mathcal{A} = \mathcal{L}(\mathcal{U})$ (see e.g. 33). Note that here we take the inner product to be linear in the first argument (as is standard for Hilbert spaces for the case where the $C^*$-algebra $\mathcal{A}$ is equal to the complex numbers $\mathbb{C}$ but is the
reverse of the standard convention for $C^*$-modules). It is readily seen from (7.67) and the definition (7.27) of the norm in $H^2_F(\mathbb{F}_d^+)$ that

$$[F,F]_{H^2_{\omega,E}(\mathbb{F}_d^+)} = I_U \iff \|Fu\|_{H^2_{\omega,E}(\mathbb{F}_d^+)} = \|u\|_U \text{ for all } u \in U.$$  (7.68)

For an $\omega$-output-stable pair $(E,T)$ with $T = (T_1, \ldots, T_d)$, we define a left-tangential functional calculus $F \to (E^*F)^{\mathcal{L}}(T^*)$ on $H^2_{\omega,E}(\mathbb{F}_d^+)$ by

$$(E^*F)^{\mathcal{L}}(T^*) = \sum_{\alpha \in \mathbb{F}_d^+} T^\alpha F \alpha$$  (7.69)

for $F(z) \in H^2_{\omega,E}(\mathbb{F}_d^+)$ as in (7.67). The computation

$$\left\langle \sum_{\alpha \in \mathbb{F}_d^+} T^\alpha F \alpha u, x \right\rangle_X = \sum_{\alpha \in \mathbb{F}_d^+} \langle F \alpha u, ET^\alpha x \rangle_Y$$

$$= \sum_{\alpha \in \mathbb{F}_d^+} \omega(\alpha) \langle F \alpha u, \omega^{-1}(\alpha) ET^\alpha x \rangle_Y$$

$$= (Fu, O_{\omega,E,T}x)_{H^2_{\omega,E}(\mathbb{F}_d^+)}$$

shows that the $\omega$-output-stability of the pair $(E,T)$ is exactly what is needed to verify that the infinite series in the definition (7.69) of $(E^*F)^{\mathcal{L}}(T^*)$ converges in the weak topology on $X$. In fact, the left-tangential evaluation with operator argument $f \to (E^*F)^{\mathcal{L}}(T^*)$ amounts to the adjoint of the $\omega$-observability operator:

$$(E^*F)^{\mathcal{L}}(T^*) = O_{\omega,E,T}^* F \text{ for } F \in H^2_{\omega,E}(\mathbb{F}_d^+)$$

and suggests the interpolation problem with operator argument $\text{OAP}(T,E,N)$ whose data set consists of a $d$-tuple $T = (T_1, \ldots, T_d)$ and operators $E \in \mathcal{L}(X,Y)$ and $F \in \mathcal{L}(X,U)$ such that the pair $(E,T)$ is $\omega$-output stable. We assume in addition that $(E,T)$ is exactly $\omega$-observable so that the gramian $G_{\omega,E,T}$ is strictly positive definite,

$\text{OAP}(T,E,N)$: Given the data set $\{T,E,N\}$ as above, find all $F \in H^2_{\omega,E}(\mathbb{F}_d^+)$ such that

$$(E^*F)^{\mathcal{L}}(T^*) := O_{\omega,E,T}^* M_F|_U = N^*.$$  (7.70)

**Theorem 7.20.** All solutions $F \in H^2_{\omega,E}(\mathbb{F}_d^+)$ of the problem (7.70) are parametrized by the formula

$$F(z) = F_{\min}(z) + G(z),$$  (7.71)

where

$$F_{\min}(z) = \sum_{\alpha \in \mathbb{F}_d^+} \omega^{-1}(\alpha) ET^\alpha G_{\omega,E,T}^{-1} N^* z^\alpha = ER_{\omega}(Z(z)T)G_{\omega,E,T}^{-1} N^*$$  (7.72)

and where $G(z) \in H^2_{\omega,E}(\mathbb{F}_d^+)$ subject to $M_G|_U \subset (\text{Ran} O_{\omega,E,T})^\perp$ is a free parameter. Furthermore, the representation (7.72) is orthogonal with respect to the inner product (7.67). Therefore,

$$[F,F]_{H^2_{\omega,E}(\mathbb{F}_d^+)} = [F_{\min},F_{\min}]_{H^2_{\omega,E}(\mathbb{F}_d^+)} + [G,G]_{H^2_{\omega,E}(\mathbb{F}_d^+)}$$

so that $F_{\min}$ has the minimal $\mathcal{L}(U)$-valued self-inner-product (and hence also the minimum possible norm) among all solutions to the problem (7.70).
The converse direction holds.

\[ C_G L_G \]

Calculation, we see that this is equivalent to the condition that the assumption that the pair \((E, F)\) and given

\[ T \sum_{\alpha \in F_d^+} T^{\omega \alpha} E^* E T^\alpha G_{E, F} \]

One of Beurling-Lax type theorems presented above describing the

\[ \omega \left( \begin{array}{c} \omega \alpha \\ \omega \alpha \end{array} \right) \in L(Y \oplus X) \]

Observe that by (7.74) and the power series representations (4.45) and (1.36),

\[ \sum_{\alpha \in F_d^+} T^{\omega \alpha} E^* E T^\alpha G_{E, F} \]

Showing that \( F_{\min}(z) \) belongs to \( H^2_{\omega, \mathcal{L}(U, Y)}(F_d^+) \) and satisfies condition (7.70). Next observe that \( G \in H^2_{\omega, \mathcal{L}(U, Y)}(F_d^+) \) satisfies the homogeneous condition \( O_{E, F} G(z) u = 0 \) if and only if \( G u \) belongs to \( (\text{Ran } O_{E, F})^\perp \). Therefore,

\[ (E^* G)^{L}(T^*) := O_{E, F} M_G|U = 0 \iff M_G|U \subset (\text{Ran } O_{E, F})^\perp, \tag{7.73} \]

and representation (7.74) follows. The representation is orthogonal with respect to the inner product (7.37), since \( F_{\min}(z) u \) belongs to \( \text{Ran } O_{E, F} \) for any \( u \in U \). \( \square \)

A more detailed parametrization in (7.71) can be obtained by invoking any one of Beurling-Lax type theorems presented above describing the \( S_{\omega, H} \)-invariant subspace \((\text{Ran } O_{E, F})^\perp \) of \( H^2_{\omega, \mathcal{L}(U, Y)}(F_d^+) \).

We next consider a more structured interpolation problem in \( H^2_{\omega, \mathcal{L}(U, Y)}(F_d^+) \). Given an \( \omega \)-isometric pair \((C, A)\) with \( C \in \mathcal{L}(X, Y) \) and \( A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d \), and given \( D \in \mathcal{L}(U, Y) \), let

\[ T_j = \begin{bmatrix} 0 & 0 \\ 0 & A_j \end{bmatrix} \in \mathcal{L}(Y \oplus X) \]

For \( F(z) = \sum_{\alpha \in F_d^+} F_\alpha z^\alpha \in H^2_{\omega, \mathcal{L}(U, Y)}(F_d^+) \),

\[ (E^* F)^{L}(T^*) = \begin{bmatrix} F_\emptyset \\ C^* F_\emptyset \end{bmatrix} + \sum_{\alpha \in F_d^+: |\alpha| > 0} \begin{bmatrix} 0 & 0 \\ 0 & A^{\star \alpha} \end{bmatrix} \begin{bmatrix} I_Y \\ C^* \end{bmatrix} F_\alpha = \begin{bmatrix} F_\emptyset \\ (C^* F)^{L}(A^*) \end{bmatrix} \]

and then the interpolation condition (7.70) amounts to

\[ (C^* F)^{L}(A^*) = 0 \quad \text{and} \quad F_\emptyset = D. \tag{7.75} \]

Observe that by (7.71) and the power series representations (4.45) and (1.36),

\[ G_{E, F} = \begin{bmatrix} I_Y & C \\ C^* & \tilde{G}_{E, C, A} \end{bmatrix}, \quad E R_{\omega}(Z(z) T) = I_Y + C R_{\omega}(Z(z) A). \tag{7.76} \]

The assumption that the pair \((E, F)\) be exactly \( \omega \)-observable means that \( G_{E, F} \) be invertible in \( \mathcal{L}(X) \); from the first formula in (7.76) and a Schur-complement calculation, we see that this is equivalent to the condition that \( G_{E, C, A} > C^* C \). In particular, \( G_{E, C, A} > 0 \), i.e., \((C, A)\) is exactly \( \omega \)-observable. We next clarify when the converse direction holds.

**Proposition 7.21.** Let \((C, A)\) be an \( \omega \)-isometric output pair. Then \( G_{E, C, A} > C^* C \) if and only if \((C, A)\) is exactly \( \omega \)-observable (so \( G_{E, C, A} > 0 \)) and \( \sum_{j=1}^d A_j^* A_j > 0 \).
7.4. Extremal Solutions of Interpolation Problems

Proof. If $\mathcal{G}_{\omega,C,A} \succeq C^*C$, then $\mathcal{G}_{\omega,1,C,A} \succeq \mathcal{G}_{\omega,C,A} \succ 0$, by (4.60). From the identity (4.60) applied with $k = 0$, we have
\[
\sum_{j=1}^{d} A_j^* \mathcal{G}_{\omega,1,C,A} A_j = \mathcal{G}_{\omega,C,A} - C^*C > 0
\]
and therefore,
\[
\sum_{j=1}^{d} A_j^* A_j \geq \frac{1}{\| \mathcal{G}_{\omega,1,C,A} \|} \sum_{j=1}^{d} A_j^* \mathcal{G}_{\omega,1,C,A} A_j > 0
\]
which verifies the "only if" part of the statement.

Conversely, if $\mathcal{G}_{\omega,C,A} \succeq \delta I_X$ for some $\delta > 0$ and $\sum_{j=1}^{d} A_j^* A_j > 0$, then $\mathcal{G}_{\omega,1,C,A} \succeq \mathcal{G}_{\omega,C,A} \succeq \delta I_X$, and hence,
\[
\mathcal{G}_{\omega,C,A} - C^*C = \sum_{j=1}^{d} A_j^* \mathcal{G}_{\omega,1,C,A} A_j \geq \delta \sum_{j=1}^{d} A_j^* A_j > 0
\]
which completes the proof. \qed

We note that the condition $\sum_{j=1}^{d} A_j^* A_j > 0$ can be viewed as a version of the assumption that none of the specified zero locations $\{z_j: j = 1, 2, \ldots \}$ for the shift-invariant subspace $\mathcal{M}$ are at the origin in the classical case mentioned above.

We now consider the following extremal problem:

**Extremal Interpolation Problem (EP):** Given an exactly $\omega$-observable pair $(C,A)$ with $\mathcal{G}_{\omega,1,C,A}$ positive definite, find an injective operator $D \in L(U,Y)$ acting from an auxiliary Hilbert space $U$ into $Y$ so that the minimal norm solution $F_{\text{min}}$ to the problem (7.75) satisfies $\| F_{\text{min}} \| = I_U.$

The solution of the Extremal Interpolation Problem is as follows.

**Theorem 7.22.** Suppose that $(C,A)$ is an exactly $\omega$-observable output pair. Then any solution $F_{\text{min}}$ of the EP is $H^2_{\omega^*} -$Bergman-inner. Moreover, any such solution $F_{\text{min}}$ is given by
\[
F_{\text{min}}(z) = D + CR_{\omega,1}(Z(z)A)Z(z)B
\]
where
\[
B = -A(\mathcal{G}_{\omega,C,A} - C^*C)^{-1}C^*D \quad \text{and} \quad D = (I - C\mathcal{G}_{\omega,C,A}^{-1}C^*)^{1/2}V,
\]
with $V$ be any isometry from an auxiliary Hilbert space $\mathcal{U}$ onto $\text{Ran}(I - C\mathcal{G}_{\omega,C,A}^{-1}C^*)^{1/2}$.

Proof. The solution set for the homogeneous problem (7.75) (i.e., with $D = 0$) consists of $G \in H^2_{\omega,C(U,Y)}$ with $\text{Ran} M_G|_{\mathcal{U}} \subset (\text{Ran} \mathcal{O}_{\omega,E,T})^\perp$, so by Theorem 7.20 the minimal norm solution $F_{\text{min}}$ of the non-homogeneous problem is such that $M_F|_{\mathcal{U}} \subset \text{Ran} \mathcal{O}_{\omega,E,T} \otimes \mathcal{U}$. In other words, for every $u \in \mathcal{U}$, the series $F_{\text{min}}(z)u$ belongs to $\text{Ran} \mathcal{O}_{\omega,E,T}$, and according to the second formula in (7.76), $F_{\text{min}}(z)u$ is of the form
\[
F_{\text{min}}(z)u = y + C(I - Z(z)A)^{-n}x
\]
for some $y \in Y$ and $x \in \mathcal{X}$. Since $\text{Ran} \mathcal{O}_{\omega,C,A}$ is $S_{\omega,R}^*$-invariant, $S_{\omega,R}^* F_{\text{min}}u$ belongs to $\text{Ran} \mathcal{O}_{\omega,C,A}$ for any non-empty $\alpha \in \mathbb{F}_d^+$. On the other hand, due to the first (homogeneous) condition in (7.75), $F_{\text{min}}u$ belongs to $(\text{Ran} \mathcal{O}_{\omega,C,A})^\perp$ for
each \( v \in \mathcal{U} \). Therefore, \( S_{\omega,R}^\alpha F_{\min} v \) is orthogonal to \( F_{\min} v \) or, equivalently, \( F_{\min} u \) is orthogonal to \( S_{\omega,R}^\alpha F_{\min} v \) for all \( u, v \in \mathcal{U} \) and non-empty \( \alpha \in \mathbb{F}_d^\times \). Besides, \( \|F_{\min} u\|_{H_{\omega,R}^2(\mathbb{F}_d^\times)} = \|u\|_{\mathcal{U}} \) for all \( u \in \mathcal{U} \), by (7.68). Then \( F_{\min} \) is \( H_{\omega,R}^2(\mathbb{F}_d^\times) \)-Bergman-inner, by Definition 3.23.

We now calculate the extremal series \( F_{\min} \). By the first formula in (7.70),

\[
G^{-1}_{\omega,E,T} = \begin{bmatrix} I_Y & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -C \\ I_X \end{bmatrix} (G_{\omega,C,A} - C^* C)^{-1} \begin{bmatrix} -C^* \\ I_X \end{bmatrix}
\]

and substituting the latter equality into (7.72) gives

\[
F_{\min}(z) = \begin{bmatrix} I_Y & C \\ 0 & R_\omega(z)A \end{bmatrix} \begin{bmatrix} I_Y & 0 \\ 0 & 0 \end{bmatrix} G^{-1}_{\omega,E,T} \begin{bmatrix} D \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} I_Y & CR_\omega(z)A \end{bmatrix} \begin{bmatrix} [D] \\ [I_X] \end{bmatrix} (G_{\omega,C,A} - C^* C)^{-1} C^* D
\]

\[
= D + C(R_\omega(z)A - I_X)(G_{\omega,C,A} - C^* C)^{-1} C^* D
\]

The latter formula can be written in the form (7.77) with \( B \) defined as in (7.78) and \( D \) to be yet determined. By formula (4.60) (for \( k = 0 \)),

\[
A^*(G_{\omega,1,C,A} \otimes I_d)A = \sum_{j=1}^d A_j^* G_{\omega,1,C,A} A_j + C^* C = G_{\omega,C,A}.
\]

Then for \( B \) of the form (7.78) we have

\[
A^*(G_{\omega,1,C,A} \otimes I_d)B = -A^*(G_{\omega,1,C,A} \otimes I_d)A(G_{\omega,C,A} - C^* C)^{-1} C^* D
\]

\[
= -C^* D,
\]

\[
B^*(G_{\omega,1,C,A} \otimes I_d)B = -D^* C(G_{\omega,C,A} - C^* C)^{-1} A^*(G_{\omega,1,C,A} \otimes I_d)B^*
\]

\[
= D^* C(G_{\omega,C,A} - C^* C)^{-1} C^* D.
\]

For \( F_{\min} \) of the form (7.77) with the entry \( B \) given as in (7.78), we have

\[
[F_{\min}, F_{\min}] H_{\omega,E(U,Y)}^2 = D^* D + \sum_{j=1}^d \sum_{\alpha \in \mathbb{F}_d^\times} \omega_{\alpha - 1}^{-1} B_j^* A^* \alpha \otimes C^* C A \alpha B_j
\]

\[
= D^* D + B^*(G_{\omega,1,C,A} \otimes I_d)B.
\]

Equalities (7.79), (7.80), (7.82) can be written in the matrix form

\[
\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} G_{\omega,1,C,A} \otimes I_d & 0 \\ 0 & I_Y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} G_{\omega,C,A} & 0 \\ 0 & [F_{\min}, F_{\min}] \end{bmatrix}
\]

Since \( F_{\min} \) satisfies the condition \([F_{\min}, F_{\min}] = I_{\mathcal{U}}\), we conclude from the latter equality by Remark 7.2 that \( F_{\min} \) is \( H_{\omega,E(U,Y)}^2(\mathbb{F}_d^\times) \)-Bergman-inner. To find \( D \) explicitly, we combine (7.81) and (7.82) and use the Sherman-Morrison formula as follows:

\[
I_Y = D^* D + D^* C(G_{\omega,C,A} - C^* C)^{-1} C^* D
\]

\[
= D^* (I + C(G_{\omega,C,A} - C^* C)^{-1} C^*) D = D^* (I - CG_{\omega,C,A}^{-1} C^* D)
\]

We then conclude from the latter equality that \( D \) is of the form as in (7.78). \( \square \)
CHAPTER 8

Model theory for $\omega$-hypercontractive operator $d$-tuples

Let us say that a commutative operator $d$-tuple $T = (T_1, \ldots, T_d)$ is an abstract $\omega$-shift if $T$ satisfies conditions (4.83) and $\mathcal{X}_0 = \{0\}$ where $\mathcal{X}_0$ is the space displayed in (4.53). Then, according to Theorem 4.30 any such $T$ is unitarily equivalent to the concrete $\omega$-shift $S_{\omega,R}$ on $H^2_{\omega,\mathcal{Y}}(\mathbb{F}_d^+)\uparrow$ for an appropriate coefficient Hilbert space $\mathcal{Y}$, i.e., $S_{\omega,R}$ on $H^2_{\omega,\mathcal{Y}}(\mathbb{F}_d^+)$ (with an appropriate multiplicity) serves as the functional model for any abstract $\omega$-shift (up to unitary equivalence and some multiplicity).

In this chapter we will characterize abstract operator tuples $T = (T_1, \ldots, T_d)$ such that $A = T^* = (T_1^*, \ldots, T_d^*)$ is unitarily equivalent to the restriction of the adjoint shift-tuple $S_{\omega,R}^*$ on $H^2_{\omega,\mathcal{Y}}(\mathbb{F}_d^+)$ to an invariant subspace $\mathcal{N}$, where $\mathcal{N}$ is isometrically included, or more generally only contractively included, in the ambient space $H^2_{\omega,\mathcal{Y}}(\mathbb{F}_d^+)$. We already have some results in this direction, namely Theorem 4.37.

In the classical case one can go further by arriving at an explicitly computable characteristic function $\Theta_T$ which serves as a complete unitary invariant and from which one can define an explicit functional model for the operator $T$. Our goal in this chapter is to enhance Theorems 4.37 by identifying a characteristic multiplier $\Theta_T$ which serves as a complete unitary invariant for the original operator tuple $T$, just as in the Sz.-Nagy–Foias theory [99].

In this chapter we present two approaches to this problem for the class of $\ast$-$\omega$-hypercontractions $T$ which are completely non-coisometric (c.n.u.)—corresponding to the case for $d = 1$ where the contraction operator $T$ has no coinvariant subspace $\mathcal{N}$ such that $T^*|_{\mathcal{N}}$ is isometric.

The second approach, developed for the case $d = 1$ in our earlier work [14, 15], restricts the analysis to the so-called pure case ($T^*$ is $\omega$-strongly stable) with the payoff that one gets a more explicit model theory by using the Beurling-Lax representation via a Bergman-inner family $\{\Theta_{\alpha} : \alpha \in \mathbb{F}_d^+\}$ for the $S_{\omega,R}$-invariant subspace $\mathcal{M} = \mathcal{N}^\perp$, including an explicit formula for each member $\Theta_{\alpha}$ of the Bergman-inner family in terms of the original operator $d$-tuple $T$. This is the topic of Section 8.2. We also show how to recover the original operator-tuple $T$ directly from the characteristic Bergman-inner family and verify that the characteristic Bergman-inner family is a complete unitary invariant for $T$.

8.1. Model theory based on contractive-multiplier/McCT-inner multiplier as characteristic function

Recall Definition 4.77 of a $\omega$-hypercontractive operator $d$-tuple $A = (A_1, \ldots, A_d)$: an operator $d$-tuple $A = (A_1, \ldots, A_d)$ on a Hilbert space $\mathcal{X}$ is $\omega$-hypercontractive if
it is contractive (in the sense of (4.3)) and is subject to inequalities
\[
\Gamma_{\omega;A}[I_N] := - \sum_{\alpha \in F^+_d} \left( \sum_{\ell=1}^k \frac{c_{|\alpha|+\ell}}{\omega_{k-\ell}} A^{*\alpha T} A^\alpha \right) \geq 0 \quad \text{for all } k \geq 1,
\]
\[
\Gamma_{\omega;A}[I_N] = \sum_{\alpha \in F^+_d} c_{|\alpha|} A^{*\alpha T} A^\alpha \geq 0.
\]
For consistency with the Sz.-Nagy-Foias model theory, we consider an operator tuple. Let us introduce the \( \omega \)-tuple so that
\[
A \text{ has trivial kernel, we can make } N \text{ c.n.c.}
\]
Then by construction the pair \((D, O)\) where we set \( T \) is contractive (in the sense of (4.30)) and is subject to inequalities
\[
\text{We shall use the convention that non-bold}\ T \text{ denotes the operator}
\]
\[
T = [T_1 \cdots T_d] : X^d \to X
\]
so that \( A = T^* \) is the column operator
\[
A = T^* = \begin{bmatrix} T^*_1 \\ \vdots \\ T^*_d \end{bmatrix} : X \to \begin{bmatrix} X \\ \vdots \\ X \end{bmatrix}.
\]

We consider an operator \( d \)-tuple \( T = (T_1, \ldots, T_d) X \) such that \( T^* \) is an \( \omega \)-c.n.c. \( \omega \)-hypercontractive operator tuple, i.e., we assume that \( T \) is a \( * \omega \)-hypercontractive operator tuple. Let us introduce the \( \omega \)-defect operator \( D_{\omega;T^*} \) defined by
\[
D_{\omega;T^*} = \Gamma_{\omega;A}[I_N]^{1/2} : X \to D_{\omega;T^*}
\]
where we set
\[
D_{\omega;T^*} = \overline{\text{Ran} D_{\omega;T^*}}.
\]
Then by construction the pair \((D_{\omega;T^*}, T^*)\) is an \( \omega \)-isometric output pair. We define the \( \omega \)-observability operator \( O_{\omega,D_{\omega;T^*},T^*} \) as in (1.30). We impose the condition that the output-pair \((D_{\omega;T^*}, T^*)\) be observable, i.e., that the associated observability operator \( O_{\omega,D_{\omega;T^*},T^*} \) have only trivial kernel. In terms of the original \( d \)-tuple \( T \), we say that \( T \) is \( \omega \)-completely non-coisometric (\( \omega \)-c.n.c. for short). As \( O_{\omega,D_{\omega;T^*}} \) has trivial kernel, we can make \( N := \text{Ran} O_{\omega,D_{\omega;T^*}} \) a Hilbert space by assigning to it the lifted norm
\[
\|O_{\omega,D_{\omega;T^*}} \cdot x\|_N = \|x\|_X.
\]
From Theorem 1.35 we see that \( N = H(K_N) \) is in fact a NKRKHS with reproducing kernel \( K_N \) given by
\[
K_N(z, \zeta) = D_{\omega;T^*} R_{\omega}(Z(z)T^*) R_{\omega}(Z(\zeta)T^*)^* D_{\omega;T^*},
\]
that \( N \) is contained contractively in \( H^2_{\omega,D_{\omega;T^*}}(\mathbb{F}^+_d) \), and moreover, \( N \) is \( S_{\omega,R} \)-invariant with \( O_{\omega,D_{\omega;T^*}} \) implementing a unitary equivalence between \( T^* \) and \( S_{\omega,R} \). Let us make the following formal definition.

**Definition 8.1.** We shall say that the \( \omega \)-c.n.c. \( * \omega \)-hypercontractive tuple \( T \) admits a characteristic multiplier \( \Theta_T \) if the Brangesian complement \( M = N^{1/2} \) of \( N = \text{Ran} O_{\omega,D_{T^*}} \) (equipped with the lifted norm) admits a Beurling-Lax representation \( M = \Theta \cdot H^2_{\omega}(\mathbb{F}_{d}^+) \) as in Theorem 5.1. We then say that \( \Theta \) is a characteristic multiplier \( \Theta_T \) for \( T \).

We then have the following characterization as to which \( \omega \)-c.n.c. \( * \omega \)-hypercontractive tuples have a characteristic function \( \Theta_T \) in the sense of Definition 8.1.
8.1. Contractive Multipliers as Characteristic Functions

**Theorem 8.2.** Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a $\omega$-c.n.c. $\ast\omega$-hypercontractive tuple.

1. $\mathbf{T}$ admits a characteristic multiplier if and only if
   \[
   I - \mathcal{O}_{\omega,D_{\omega,T},T^*}(\mathcal{O}_{\omega,D_{\omega,T},T^*})^* \succeq \sum_{j=1}^d S_{\omega,R,j}(I - \mathcal{O}_{\omega,D_{\omega,T},T^*}(\mathcal{O}_{\omega,D_{\omega,T},T^*})^*) S_{\omega,R,j}^*.
   \] (8.2)

2. In particular, (8.2) holds if it is the case that
   \[
   I - (\mathcal{O}_{\omega,D_{\omega,T},T^*})^* \mathcal{O}_{\omega,D_{\omega,T},T^*} \succeq \sum_{j=1}^d T_j(I - (\mathcal{O}_{\omega,D_{\omega,T},T^*})^* \mathcal{O}_{\omega,D_{\omega,T},T^*}) T_j^*,
   \] (8.3)
   in which case an explicit version of a characteristic multiplier $\Theta_\mathbf{T}$ for $\mathbf{T}$ can be constructed by implementing the algorithm given in Theorem 5.5 with $\mathbf{A} = \mathbf{T}^*$, $\mathbf{C} = D_{\omega,T}^*$.

3. In case $\mathcal{O}_{\omega,D_{\omega,T},T^*}$ is isometric (or equivalently, in case $\mathbf{T}^*$ is $\omega$-strongly stable), $\mathcal{N}$ and $\mathcal{M} = \mathcal{N}^*$ are isometrically contained in $H^2_{\omega,D_{\omega,T}}(\mathbb{F}_d^+)$,
   \[
   I - P_M = \mathcal{O}_{\omega,D_{\omega,T},T^*} \mathcal{O}_{\omega,D_{\omega,T},T^*} = P_N,
   \]
   and both conditions (8.2) and (8.3) are automatic.

**Proof.** Statement (1) follows as an application of Theorem 5.1 (see the last statement there) to the case where $\Pi = I - \mathcal{O}_{\omega,D_{\omega,T},T^*}(\mathcal{O}_{\omega,D_{\omega,T},T^*})^*$. Statements (2) and (3) follows as an application of Theorem 8.1 to the case $(\mathbf{C}, \mathbf{A}) = (D_{\omega,T}^*, \mathbf{T}^*)$.

Not all contractive multipliers can be characteristic multipliers; there are contractive multipliers $\Theta$ form $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ to $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ so that the space $\mathcal{N} = \mathcal{H}^\mathcal{P}(I - M_{\Theta} M_{\Theta}^*)$ is not $S_{\omega,R}^*$-invariant (see the end of 16). The following is a characterization (albeit not easily verifiable) of which contractive multipliers can arise as a characteristic multipliers for some $\omega$-c.n.c. $\ast\omega$-hypercontractive operator tuple $\mathbf{T}$.

**Theorem 8.3.** Let $\Theta$ be a contractive multiplier form $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ to $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$. Then there is an $\omega$-c.n.c. $\ast\omega$-hypercontractive tuple $\mathbf{T} = (T_1, \ldots, T_d)$ and a unitary identification map $\iota: \mathcal{Y} \to D_{\omega,T}$ so that $\iota \cdot \Theta = \Theta_\mathbf{T}$ if and only if the space $\mathcal{N} = \mathcal{H}^\mathcal{P}(I - M_{\Theta} M_{\Theta}^*) \subset H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ satisfies the conditions of item (4) in Theorem 8.1 in the stronger form where the second of the inequalities (4.110) is required to hold with equality. Explicitly, we may take $\mathbf{T} = (S_{\omega,R}^* | \mathcal{H}^\mathcal{P}(I - M_{\Theta} M_{\Theta}^*))^*$.

If $\Theta = \Theta_\mathbf{T}$ for an $\omega$-c.n.c. $\ast\omega$-hypercontractive tuple $\mathbf{T}'$, then $\mathbf{T}'$ is unitarily equivalent to the model $\mathbf{T}$ as constructed in the previous paragraph.

**Proof.** If $\Theta = \Theta_\mathbf{T}$, then by construction $\mathcal{N} = \text{Ran} \mathcal{O}_{\omega,D_{\omega,T},T^*}$ has the required form (with $\iota = I_{D_{\omega,T}}$). Conversely, if $\mathcal{N}$ satisfies the stronger version of the conditions in item (4) of Theorem 8.1, then we may take $\mathbf{C} = E|_N$, $\mathbf{T}' = S_{\omega,R}^*|_N$ and find a unitary identification map $\iota: \mathcal{Y} \to D_{\omega,T}$ so that $\iota \mathbf{C} = D_{\omega,T}$. Then
   \[
   D_{\omega,T} \mathcal{O}_{\omega}(Z(\zeta)T^*) \mathcal{O}_{\omega}(Z(\zeta)T^*)^* D_{\omega,T} = \iota \mathcal{C} \mathcal{O}_{\omega}(Z(\zeta)T^*) \mathcal{O}_{\omega}(Z(\zeta)T^*)^* D_{\omega,T} \iota^*
   \]
   \[
   = k_{\omega,\mathcal{C}}(\zeta, \zeta) I_{D_{\omega,T}} - \iota \Theta_{\omega}(k_{\mathcal{C},\mathcal{C}}(\zeta, \zeta) I_{\mathcal{U}}) \Theta_{\omega}^* \iota^*
   \]
   so $\iota \Theta_{\omega}$ serves as a characteristic function for $\mathbf{T}$. \qed
Remark 8.4. In the classical setting, while any pure contractive analytic function arises as the characteristic function of a completely nonunitary contraction operator $T$, those which arise as the characteristic function of a completely noncoisometric contraction operator are characterized as those contractive analytic functions $\Theta(z) : U \to \mathcal{Y}$ for which $I - \Theta(\zeta)^* \Theta(\zeta)$ has zero maximal factorable minorant, i.e., such that the only $L(U, F)$-valued analytic function $A(z)$ such that $A(\zeta)^* A(\zeta) \preceq I_{U} - \Theta(\zeta)^* \Theta(\zeta)$ for a.e. $\zeta \in \mathbb{T}$ is $A(z) \equiv 0$. (see [24] Theorem 6). See [28] for a formulation of all this in the Fock space setting. There does not appear to be in results in this direction for the weighted Bergman space setting.

8.2. Model theory for --strongly stable hypercontractions via characteristic Bergman-inner families

We now specialize the discussion to the case where $T = (T_1, \ldots, T_d)$ is such that $T^*$ is $\omega$-strongly stable as well as a $\omega$-hypercontractive operator tuple—we then say simply that $T$ is a $\omega$ strongly stable, hypercontraction. The significance of the $\omega$-strong stability hypothesis is that then the observability operator $\mathcal{O}_{\omega, D_{\omega, T^*}, T^*}$ is isometric (not just injective), so $\mathcal{N} := \text{Ran} \mathcal{O}_{\omega, D_{\omega, T^*}, T^*}$ with lifted norm is isometrically (not just contractively) contained in $H^2_{\omega, D_{\omega, T^*}}$ (see Theorem 4.36). Thus Theorem 4.35 specialized to this setup tells us that $T^*$ is unitarily equivalent to $S^*_{\omega, \mathcal{H}}|_{\mathcal{N}}$ where $\mathcal{N} := \text{Ran} \mathcal{O}_{\omega, D_{\omega, T^*}, T^*}$ with lifted norm is a $S^*_{\omega, \mathcal{H}}$-invariant subspace isometrically contained in $H^2_{\omega, D_{\omega, T^*}}(\mathbb{F}^+_d)$. We define the characteristic Bergman-inner family $\{\Theta_\beta\}_{\beta \in \mathbb{F}^+_d}$ to be the Bergman-inner Beurling-Lax representer for the subspace $\mathcal{M}$ having orthogonal complement equal to the range of the observability operator $\mathcal{O}_{\omega, D_{\omega, T^*}, T^*}$.

In detail, define the shifted observability operator $\mathcal{D}_{\omega, k, D_{\omega, T^*}, T^*}$ and the shifted gramian $G_{\omega, k, D_{\omega, T^*}, T^*}$ via formulas (4.57) and (4.58); then Proposition 4.21 gives us the validity of the weighted Stein identity

$$\sum_{j=1}^{d} T_j G_{\omega, k+1, D_{\omega, T^*}, T^*} T_j^* + \omega^{-1}_{k} D_{\omega, T^*} D_{\omega, T^*} = G_{\omega, k, D_{\omega, T^*}, T^*}.$$

As explained in the proof of Lemma 4.3, it follows that the operator

$$X_{\omega, k} := \begin{bmatrix} G_{\omega, k+1, D_{\omega, T^*}, T^*}^{-1} \otimes I_d & 0 \\ 0 & \omega_k I_{D_{\omega, T^*}} \end{bmatrix} - \begin{bmatrix} T^* \\ D_{\omega, T^*} \end{bmatrix} G_{\omega, k, D_{\omega, T^*}, T^*}^{-1} \begin{bmatrix} T \\ D_{\omega, T^*} \end{bmatrix}$$

is positive semidefinite. We let $D_{\omega, k, T}$ denote the positive semidefinite square root of $X_{\omega, k}$, and let $\mathcal{D}_{\omega, k, T} := \text{Ran} D_{\omega, k, T}$. We then view $D_{\omega, k, T}$ as an operator from $\mathcal{D}_{\omega, k, T}$ into $\mathcal{X}/\mathcal{D}_{\omega, T^*}$. For any $\beta \in \mathbb{F}^+_d$, we decompose $D_{\omega, |\beta|, T}$ as

$$D_{\omega, |\beta|, T} = \begin{bmatrix} \hat{B}_\beta \\ D_\beta \end{bmatrix} : \mathcal{D}_{\omega, |\beta|, T} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_{\omega, T^*} \end{bmatrix}$$

(8.4)

(so $\begin{bmatrix} \hat{B}_\beta \\ D_\beta \end{bmatrix}$ depends on $\beta$ only through $|\beta|$). We finally introduce the family of connection matrices

$$U_\beta = \begin{bmatrix} T^* \\ D_{\omega, T^*} \end{bmatrix} \begin{bmatrix} \hat{B}_\beta \\ D_\beta \end{bmatrix} : \mathcal{H} \to \mathcal{H}/\mathcal{D}_{\omega, T^*}$$
and define the characteristic Bergman-inner family for $T$ by $\Theta_T = \{\Theta_{T,\beta}\}_{\beta \in \mathbb{F}_d^+}$ where

$$\Theta_{T,\beta}(z) = \omega_{|\beta|}^{-1}D_{\beta} + D_{\beta}T R_{\omega,|\beta|+1}(Z(z)T^*)Z(z)\bar{B}_{\beta}$$

where $[\bar{B}_{\beta} D_{\beta}] = D_{\omega,|\beta|,T}$ (see (8.3)). (8.5)

The following statement summarizes the results from Chapter 7 when applied to the present model-theory context.

**Theorem 8.5.** Suppose that $T = (T_1, \ldots, T_d)$ is a $\ast$-$\omega$ strongly stable, hypercontractive operator-tuple on a Hilbert space $\mathcal{X}$. Let $\Theta_T = \{\Theta_{T,\beta}\}_{\beta \in \mathbb{F}_d^+}$ be the characteristic Bergman-inner family of $T$ as defined in (8.5). Then $\{\Theta_{T,\beta}\}$ is a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family and $T$ is unitarily equivalent to the compressed shift operator-tuple $P_{\mathcal{M}_{\omega,Y}}S_{\omega,R|\mathcal{M}^+}$, where

$$\mathcal{M} = M_{\Theta_T} \cdot H^2_{(D_{\omega,|\beta|,T})_{\beta \in \mathbb{F}_d^+}}(\mathbb{F}_d^+)$$

is the $S_{\omega,R}$-invariant subspace of $H^2_{\omega,Y}(\mathbb{F}_d^+)$ associated with the $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\{\Theta_{T,\beta}\}_{\beta \in \mathbb{F}_d^+}$.

Furthermore, if $T'$ is another $\ast$-$\omega$ strongly stable, hypercontractive operator-tuple on a Hilbert space $\mathcal{X}$ having $\{\Theta_{T',\beta}\}_{\beta \in \mathbb{F}_d^+}$ as its characteristic Bergman-inner family, then $T$ and $T'$ are unitarily equivalent if and only if the characteristic Bergman-inner families $\{\Theta_{T,\beta}\}_{\beta \in \mathbb{F}_d^+}$ and $\{\Theta_{T',\beta}\}_{\beta \in \mathbb{F}_d^+}$ coincide in the following sense: for each $k \in \mathbb{Z}_+$ there is a unitary operator $v_k: D_{\omega,k,T} \rightarrow D_{\omega,k,T'}$ and there is a unitary operator $u: D_{\omega,T} \rightarrow D_{\omega,T'}$ so that $u\Theta_{T,\beta}(z) = \Theta_{T',\beta}(z)v_{|\beta|}$.

**Proof.** The characteristic Bergman-inner family $\Theta_T = \{\Theta_{T,\beta}\}_{\beta \in \mathbb{F}_d^+}$ for a $\ast$-$\omega$ strongly stable, hypercontractive operator tuple $T$ is by definition the $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family constructed in Theorem 7.12 for the case where the $S_{\omega,R}$-invariant subspace $\mathcal{M}$ is chosen with $\mathcal{M}^+ = \text{Ran} O_{\omega,T^*,T^*}$.

The fact that $\Theta_T$ is an $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family is now a consequence of Theorem 7.12. The fact that $T$ is unitarily equivalent to the compressed shift operator-tuple $P_{\mathcal{M}_{\omega,Y}}S_{\omega,R|\mathcal{M}^+}$ is a simple consequence of the intertwining relations

$$S_{\omega,R,j}O_{\omega,T^*}T^* = O_{\omega,T^*}T^*T_j$$

(see (4.105)) with the observability operator $O_{\omega,T^*}$ implementing the unitary equivalence.

If $T$ and $T'$ are two $\ast$-$\omega$ strongly stable, hypercontractive operator tuples which are unitarily equivalent via a unitary operator $u: \mathcal{X} \rightarrow \mathcal{X}'$ (so $T_j = uT_j$ for $j = 1, \ldots, d$), then it is a matter of checking that the unitary operator $u$ can be passed through the definition of the defect operators $D_{\omega,T^*}$ (8.1) and $D_{\omega,|\beta|,T}$ (8.2) to arrive at the coincidence of the respective Bergman-inner families. The converse fact that coincidence of the respective Bergman-inner families implies unitary equivalence of the respective model $\omega$-$\ast$ strongly stable, hypercontractive operator-tuples $T$ and $T'$ is straightforward. □

Corollary 7.13 tells us that any $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\Theta = \{\Theta_{\beta}\}_{\beta \in \mathbb{F}_d^+}$ can be realized so that $M_{\Theta} = \text{row}_{\beta \in \mathbb{F}_d^+}[M_{\Theta_{\beta}}] = T_\mathcal{U}$ for a $\omega$-Bergman-conservative
linear system $U$. One way of constructing $\{U_\beta\}_{\beta \in \mathbb{F}_d^+}$ is via the construction outlined in Theorem 8.6, based on the $S_{\omega,R}$-invariant subspace $\mathcal{M} = M_{\Theta} H^2_{\{U\ell\}}(\mathbb{F}_d^+)$. A second (closely related) way is to view the family for the operator $d$-tuple $T = P_M S_{\omega,R} \mid_{M^+}$ where $\mathcal{M}$ is again taken equal to $\mathcal{M} = M_{\Theta} H^2_{\{U\ell\}}(\mathbb{F}_d^+)$. We now give a third, more direct way which expresses the operator components $A, B_\beta, C, D_\beta$ more directly as operators on function spaces without involving solving Cholesky factorization problems. Here we use the conventions that

$$S_{\omega,R} = (S_{\omega,R,1}, \ldots, S_{\omega,R,d}) \quad (8.6)$$

is the shift operator $d$-tuple on $H^2_{\omega,Y}(\mathbb{F}_d^+)$ for which we have the functional calculus notation

$$S^*_\omega R = S_{\omega,R,i_1} \cdots S_{\omega,R,i_N} \quad \text{if} \quad \alpha = i_N \cdots i_1 \in \mathbb{F}_d^+,$$

while

$$S_{\omega,R} = [S_{\omega,R,1} \cdots S_{\omega,R,d}] : (H^2_{\omega,Y}(\mathbb{F}_d^+))^d \to H^2_{\omega,Y}(\mathbb{F}_d^+) \quad (8.7)$$

and hence

$$S^*_\omega R = \begin{bmatrix} S^*_{\omega,R,1} \\ \vdots \\ S^*_{\omega,R,d} \end{bmatrix} : H^2_{\omega,Y}(\mathbb{F}_d^+) \to (H^2_{\omega,Y}(\mathbb{F}_d^+))^d. \quad (8.8)$$

**THEOREM 8.6.** Suppose that we are given a $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\Theta = \{\Theta_\beta\}_{\beta \in \mathbb{F}_d^+}$ as in Definition 7.11 generating the $S_{\omega,R}$-invariant subspace

$$\mathcal{M} = M_{\Theta} H^2_{\{U\ell\}}(\mathbb{F}_d^+).$$

Then a Bergman conservative realization $\{U_\beta\}_{\beta \in \mathbb{F}_d^+}$ for $\Theta$ can be constructed as follows: take $\mathcal{X} = \mathcal{M}^+ \subset H^2_{\omega,Y}(\mathbb{F}_d^+)$ and set

$$U_\beta = \begin{bmatrix} S^*_{\omega,R} \mid_{\mathcal{M}^+} \\ \vdots \\ S^*_{\omega,R} \mid_{\mathcal{M}^+} \end{bmatrix} [S^*_{\omega,R} \mid_{\mathcal{M}^+} S_{\omega,R}^T M_{\Theta_\beta} \mid_{\mathcal{M}^+}] : [\mathcal{M}^+]^{d} \to [\mathcal{M}^+]^{d}. \quad (8.9)$$

In all these formulas the conventions (8.6), (8.7), (8.8) apply.

**PROOF.** Given the $H^2_{\omega,Y}(\mathbb{F}_d^+)$-Bergman-inner family $\Theta = \{\Theta_\beta\}_{\beta \in \mathbb{F}_d^+}$, we set

$$\mathcal{M} = \bigoplus_{\beta \in \mathbb{F}_d^+} S^T_{\omega,R} M_{\Theta_\beta} U_\beta.$$

Then as a consequence of analysis in the proof of Theorem 7.12 we know that $\mathcal{M}$ is $S_{\omega,R}$ invariant, and due to property (3) in Definition 7.11 we have

$$S^T_{\omega,R} \mathcal{M} \subset \bigoplus_{j=1}^d S^T_{\omega,R} S_{\omega,R,j} \mathcal{M}$$

$$= \left( \bigoplus_{\alpha \in \mathbb{F}_d^+} \Theta_{\alpha_\beta} z^{\alpha_\beta} U_{\alpha_\beta} \right) \otimes \left( \bigoplus_{\alpha \in \mathbb{F}_d^+, j=1}^d \Theta_{\alpha j} \beta_{\alpha j} z^{\alpha_\beta} U_{\alpha j} \right)$$

$$= \left( \bigoplus_{\alpha \in \mathbb{F}_d^+} \Theta_{\alpha_\beta} z^{\alpha_\beta} U_{\alpha_\beta} \right) \otimes \left( \bigoplus_{\alpha \in \mathbb{F}_d^+: \alpha \neq \beta} \Theta_{\alpha_\beta} z^{\alpha_\beta} U_{\alpha_\beta} \right) = \Theta_{\beta} z^{\beta} U_{\beta}.
8.2. BERGMAN-INNER FAMILY AS CHARACTERISTIC MULTIPLIER-FAMILY

Thus,
\[ S_{\omega,R}^{T} M \ominus \left( \bigoplus_{k=1}^{d} S_{\omega,R}^{T} S_{\omega,R} \Theta_{\beta} \right) = S_{\omega,R}^{T} \Theta_{\beta} u_{\beta}. \]  

Let us set \( A = S_{\omega,R} \perp M \) and \( \hat{B}_{\beta} = S_{\omega,R}^{*} S_{\omega,R}^{T} M \Theta_{\beta} \perp u_{\beta} \), i.e.,

\[
A = \begin{bmatrix} A_{1} \\ \vdots \\ A_{d} \end{bmatrix} = \begin{bmatrix} S_{\omega,R}^{*} S_{\omega,R}^{T} M \perp \Theta_{\beta} \perp u_{\beta} \end{bmatrix}, \quad \hat{B}_{\beta} = \begin{bmatrix} B_{1,\beta} \\ \vdots \\ B_{d,\beta} \end{bmatrix} = \begin{bmatrix} S_{\omega,R}^{*} S_{\omega,R}^{T} M \Theta_{\beta} \perp u_{\beta} \end{bmatrix},
\]

and let us write the connection matrix \([8.9]\) as

\[ U_{\beta} = \begin{bmatrix} A & \hat{B}_{\beta} \\ C & D_{\beta} \end{bmatrix}, \quad \text{where} \quad C = E_{|M \perp}, \quad D_{\beta} = \omega_{\beta}^{-1} | \Theta_{\beta} |_{\delta}. \]  

**Step 1:** The connection matrix \( U_{\beta} = \begin{bmatrix} A & \hat{B}_{\beta} \\ C & D_{\beta} \end{bmatrix} \) maps \( \begin{bmatrix} M_{\perp} \\ \Theta_{\beta} \end{bmatrix} \) into \( \begin{bmatrix} \Theta_{\beta} \end{bmatrix} \).

**Proof:** Since \( M \) is \( S_{\omega,R} \)-invariant, its orthogonal complement \( M_{\perp} \) is \( S_{\omega,R}^{*} \)-invariant and hence \( A_{j} : M_{\perp} \rightarrow M_{\perp} \) for \( j = 1, \ldots, d \). As \( C : M_{\perp} \rightarrow \mathcal{Y} \) and \( D_{\beta} : U_{\beta} \rightarrow \mathcal{Y} \) by construction, it remains to show that \( B_{j,\beta} \) maps \( U_{\beta} \) into \( M_{\perp} \) for \( j = 1, \ldots, d \).

To this end, we apply formula \([8.73]\) to the power series representation for \( \Theta_{\beta} \):

\[ \Theta_{\beta}(z) = \sum_{\alpha \in \mathbb{F}_{+}^{d}} |\Theta_{\beta}|_{\alpha} z^{\alpha} \]

to conclude that for any \( u \in U_{\beta}, \)

\[ \left( S_{\omega,R}^{*} S_{\omega,R}^{T} M \Theta_{\beta} u \right)(z) = \sum_{\alpha \in \mathbb{F}_{+}^{d}} \frac{\omega_{|\alpha|+|\beta|}}{\omega_{|\alpha|}} |\Theta_{\beta}|_{\alpha} u z^{\alpha}, \]  

from which we get

\[ (B_{j,\beta}u)(z) = \left( S_{\omega,R}^{*} S_{\omega,R}^{T} M \Theta_{\beta} u \right)(z) = \sum_{\alpha \in \mathbb{F}_{+}^{d}} \frac{\omega_{|\alpha|+|\beta|+1}}{\omega_{|\alpha|}} |\Theta_{\beta}|_{\alpha} u z^{\alpha}. \]

For any \( f \in M \), we then have

\[ (f, B_{j,\beta}u)_{H_{\omega,R}^{2}(\mathbb{F}_{+}^{d})} = (f, S_{\omega,R}^{*} S_{\omega,R}^{T} M \Theta_{\beta} \cdot u)_{H_{\omega,R}^{2}(\mathbb{F}_{+}^{d})} \]

\[ = (S_{\omega,R}^{*} f, S_{\omega,R}^{T} \Theta_{\beta} \cdot u)_{H_{\omega,R}^{2}(\mathbb{F}_{+}^{d})} = 0, \]  

by \([8.10]\). Thus, for any \( u \in U_{\beta}, \) the vector \( B_{j,\beta}u \) is orthogonal to any \( f \in M \) and hence, \( B_{j,\beta} : U_{\beta} \rightarrow M_{\perp} \) as required.

**Step 2:** \( U_{\beta} \) satisfies the weighted isometry property \([7.6]\).

**Proof:** To verify the property \([7.6]\), it suffices to verify the three pieces:

1. \[ A^{*} \left( S_{\omega,|\beta|+1,C,A} \otimes I_{\delta} \right) A + \omega_{|\beta|}^{-1} C^{*} C = S_{\omega,|\beta|+1,C,A}. \]
2. \[ A^{*} \left( S_{\omega,|\beta|+1,C,A} \otimes I_{\delta} \right) \hat{B}_{\beta} + \omega_{|\beta|}^{-1} C^{*} D_{\beta} = 0, \]
3. \[ \hat{B}_{\beta}^{*} \left( S_{\omega,|\beta|+1,C,A} \otimes I_{\delta} \right) \hat{B}_{\beta} + \omega_{|\beta|}^{-1} D_{\beta}^{*} D_{\beta} = I_{\delta}. \]
We first note that the identity (8.14) is a consequence of the general identity (4.60). Note next that (8.15) is equivalent to the validity of

\[
\left\langle \left[ (\mathcal{E}_{\omega,|\beta|+1,C} \otimes I_d) A \right] f, \left[ \hat{B}_\beta \right] u \right\rangle_{H_{2,\gamma}(Y)} = 0 \quad (8.17)
\]

for all \( f \in \mathcal{M} \) and \( u \in U_\beta \). Due to the facts that \( A = S_{\omega,R}^* \) and \( \mathcal{M} \) is \( S_{\omega,R}^* \)-invariant, we can rewrite the left-hand side of (8.17) as

\[
\left\langle (\mathcal{E}_{\omega,|\beta|+1,E} S_{\omega,R}^* \otimes I_d) S_{\omega,R}^* f, S_{\omega,R}^* S_{\omega,R}^* \theta \right\rangle_{H_{2,\gamma}(Y)} + \left\langle E f, [\Theta_\beta]_0 u \right\rangle_Y
\]

where we made use of the general identity (4.60) applied with \( A = S_{\omega,R}^* \) and \( C = E \):

\[
S_{\omega,R} \left( \mathcal{E}_{\omega,|\beta|+1,E} S_{\omega,R} \otimes I_d \right) S_{\omega,R}^* = \mathcal{E}_{\omega,|\beta|,E} S_{\omega,R}^* - \omega^{-1}_|\beta| E^* E.
\]

From the formula (8.12), we can write (8.19) as

\[
E S_{\omega,R}^* \theta = \omega_0 [\Theta_\beta]_0
\]

and hence the last two terms in (8.18) cancel. Thus verification of the identity (8.15) collapses to verification that the first term in the last expression in (8.18) is zero, i.e., that

\[
\left\langle \mathcal{E}_{\omega,|\beta|,E} S_{\omega,R}^* f, S_{\omega,R}^* \theta \right\rangle_{H_{2,\gamma}(Y)} = 0 \quad (8.20)
\]

Making use of the second equality in (4.65), we can write (8.20) as

\[
\left\langle \mathcal{E}_{\omega,|\beta|,E} S_{\omega,R}^* f, S_{\omega,R}^* \theta \right\rangle_{H_{2,\gamma}(Y)} = \left\langle S_{\omega,R}^* \theta, \mathcal{E}_{\omega,|\beta|,E} S_{\omega,R} f, \Theta_\beta u \right\rangle_{H_{2,\gamma}(Y)} = \left\langle f, \Theta_\beta u \right\rangle_{H_{2,\gamma}(Y)}.
\]

As \( f \in \mathcal{M} \) and \( \Theta_\beta : \mathcal{U}_\beta \to \mathcal{M} \), we now can conclude that (8.20) holds, and hence the verification of (8.15) is complete.

As for (8.16), note that an equivalent condition is

\[
\left\langle (\mathcal{E}_{\omega,|\beta|+1,E} S_{\omega,R} \otimes I_d) \hat{B}_\beta u, \hat{B}_\beta u \right\rangle_{H_{2,\gamma}(Y)} + \omega^{-1}_|\beta| \cdot \| D_\beta u \|^2_Y = \| u \|^2_Y \quad (8.21)
\]
for all \( u \in \mathcal{U}_\beta \). From (8.13) and (8.61) we see that the first term on the left side is

\[
\left\langle \left( \mathcal{S}_{\omega,|\beta|+1, E, \Sigma_{\omega, R}} \otimes I_d \right) \bar{B}_\beta u, \bar{B}_\beta u \right\rangle_{H^2_{\omega, Y}(\mathbb{F}_d^+)}
\]

\[
= \sum_{j=1}^{d} \left( \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega_{|\alpha|}}{\omega_{|\alpha|+|\beta|+1}} \omega_{|\alpha|+|\beta|+1} \left[ \Theta_{\beta} \right]_{\alpha j} u z^\alpha \right) \left. \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega_{|\alpha|+|\beta|+1}}{\omega_{|\alpha|}} \left[ \Theta_{\beta} \right]_{\alpha j} u z^\alpha \right)_{H^2_{\omega, Y}(\mathbb{F}_d^+)}
\]

\[
= \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{|\alpha|+|\beta|+1} \left\| \left[ \Theta_{\beta} \right]_{\alpha j} u \right\|_{\mathcal{Y}}^2
\]

Combining the latter computation with equalities

\[
\left\| S^{\beta^T}_{\omega, R} \Theta_{\beta} u \right\|_{H^2_{\omega, Y}(\mathbb{F}_d^+)}^2 = \left\| u \right\|_{\mathcal{U}_\beta}^2 \quad \text{and} \quad \omega_{|\beta|} \left\| \left[ \Theta_{\beta} \right]_{\beta} u \right\|_{\mathcal{Y}}^2 = \omega_{|\beta|}^{-1} \left\| D_{\beta} u \right\|_{\mathcal{Y}}^2,
\]

that hold by property (1) of an \( H^2_{\omega, Y}(\mathbb{F}_d^+) \)-Bergman-inner family (Definition 7.11) and the definition of \( D_{\beta} \) in (8.11), respectively, we arrive at (8.21) as wanted.

**Step 3:** \( \mathcal{U}_\beta \) satisfies the weighted coisometry property (7.15).

**Proof:** To verify the weighted coisometry property (7.15), we note that, in view of the validity of the weighted isometry property (7.6) already checked, it suffices to show that the connection matrix \( \mathcal{U}_\beta \) (8.11) maps \( \mathcal{M}^\perp \oplus \mathcal{U}_\beta \) onto \( \mathcal{M}^\perp \oplus \mathcal{Y} \). To show that \( \mathcal{U}_\beta \) is onto, we suppose that a vector \( \left[ \begin{array}{c} y \\ \theta \end{array} \right] \in \left( \mathcal{M}^\perp \right)^d \) is orthogonal to \( \mathcal{U}_\beta \left[ \begin{array}{c} \mathcal{M}^\perp \\ \mathcal{U}_\beta \end{array} \right] \) in the \( \left[ \begin{array}{c} \mathcal{S}_{\omega,|\beta|+1, C, A} \otimes I_d \\ 0 \\ I_y \\ 0 \end{array} \right] \)-metric, i.e., we suppose that vectors \( g \in (\mathcal{M}^\perp)^d \) and \( y \in \mathcal{Y} \) are such that

\[
\left\langle \left[ \begin{array}{c} \mathcal{S}_{\omega,|\beta|+1, C, A} \otimes I_d \\ 0 \\ I_y \\ 0 \end{array} \right] \mathcal{U}_\beta \left[ \begin{array}{c} f \\ u \\ g \\ y \end{array} \right] \right\rangle_{(\mathcal{M}^\perp)^d \oplus \mathcal{Y}} = 0 \quad \text{(8.22)}
\]

for all \( f \in \mathcal{M}^\perp \), \( u \in \mathcal{U}_\beta \), and \( y \in \mathcal{Y} \) and we wish to show that this forces \( g = 0 \) and \( y = 0 \). From the identity (8.19) we see that the connection matrix \( \mathcal{U}_\beta \) (8.11) has the factorization

\[
\mathcal{U}_\beta = \left[ \begin{array}{c} S^{\beta^T}_{\omega, R} \\ E \end{array} \right] \left[ \begin{array}{c} I_{\mathcal{M}^\perp} \\ S^{\beta^T}_{\omega, R} \Sigma_{\omega, R} \Theta_{\beta} \end{array} \right] : \left[ \begin{array}{c} \mathcal{M}^\perp \\ \mathcal{U}_\beta \end{array} \right] \rightarrow \left[ \begin{array}{c} (\mathcal{M}^\perp)^d \\ \mathcal{Y} \end{array} \right].
\]

Substituting this factorization into (8.22) we get

\[
\left\langle \left[ \begin{array}{c} \mathcal{S}_{\omega,|\beta|+1, E, \Sigma_{\omega, R}} \\ E \end{array} \right] S^{\beta^T}_{\omega, R} \left[ f + S^{\beta^T}_{\omega, R} \Sigma_{\omega, R} \Theta_{\beta} u \right], \left[ \begin{array}{c} g \\ y \end{array} \right] \right\rangle_{(\mathcal{M}^\perp)^d \oplus \mathcal{Y}} = 0.
\]

Breaking out \( g \in (\mathcal{M}^\perp)^d \) into its components \( g_j \in \mathcal{M}^\perp \): \( g = \left[ \begin{array}{c} g_1 \\ \vdots \\ g_d \end{array} \right] \), rewrite the last equality in the form

\[
\left\langle f + \sum_{j=1}^{d} S^{\beta^T}_{\omega, R} \Sigma_{\omega, R} \Theta_{\beta} u, \sum_{j=1}^{d} S_{\omega, R, j} \left[ \mathcal{S}_{\omega,|\beta|+1, E, \Sigma_{\omega, R}} g_j + E^* y \right] \right\rangle_{H^2_{\omega, Y}(\mathbb{F}_d^+)} = 0 \quad \text{(8.23)}
\]
for all \( f \in \mathcal{M}^\perp \) and \( u \in \mathcal{U}_\beta \). For ease of notation let us set
\[
h_{g,y} := \sum_{j=1}^d S_{\omega,R,j} \Theta_{\omega,|\beta|+1,E,S_{\omega,R}^*g_j} + E^*y. \tag{8.24}\]
Setting \( u = 0 \in \mathcal{U}_\beta \) and varying \( f \) arbitrarily in \( \mathcal{M}^\perp \) we conclude from \( \text{(8.23)} \) that
\[
h_{g,y} \in \mathcal{M}. \tag{8.25}\]
If we set \( f = 0 \) and vary \( u \) arbitrarily in \( \mathcal{U}_\beta \) in \( \text{(8.23)} \), we get
\[
(\mathcal{S}^{\beta^T}_{\omega,R} M_{\Theta_\beta} u, (\mathcal{S}^{\beta^T}_{\omega,R} h_{g,y}) H_{\omega,R}^2(\mathbb{F}_d^2)) = 0 \quad \text{for all } u \in \mathcal{U}_\beta.
\]
When we combine this with \( \text{(8.25)} \) we arrive at
\[
(\mathcal{S}^{\beta^T}_{\omega,R} h_{g,y}) \in \mathcal{S}^{\beta^T}_{\omega,R} M \cap (\mathcal{S}^{\beta^T}_{\omega,R} M_{\Theta_\beta})^\perp. \tag{8.26}\]
As a consequence of the identity \( \text{(8.10)} \), we know that
\[
\mathcal{S}^{\beta^T}_{\omega,R} M \cap (\mathcal{S}^{\beta^T}_{\omega,R} M_{\Theta_\beta})^\perp = \bigoplus_{j=1}^d \mathcal{S}^{\beta^T}_{\omega,R,j} \mathcal{M}
\]
and hence condition \( \text{(8.26)} \) gives us
\[
\mathcal{S}^{\beta^T}_{\omega,R} h_{g,y} \in \bigoplus_{j=1}^d \mathcal{S}^{\beta^T}_{\omega,R,j} \mathcal{M}. \tag{8.27}\]
Applying the operator \( \mathcal{S}^{\beta^T}_{\omega,R} \) to both parts of \( \text{(8.24)} \) and taking into account that
\( \Theta_{\omega,k,E,S_{\omega,R}^*} = \Theta_{\omega,k,E,S_{\omega,R}} \) (by explicit formulas \( 1.64 \)), we get
\[
\mathcal{S}^{\beta^T}_{\omega,R} h_{g,y} = \sum_{j=1}^d \mathcal{S}^{\beta^T}_{\omega,R,j} \mathcal{S}_{\omega,R,j} \mathcal{M} \quad \text{and hence from } \text{(8.26)} \quad \mathcal{S}_{\omega,R} \mathcal{S}_{\omega,R,j} \mathcal{M} \quad \text{satisfies}\]
\[
\mathcal{S}_{\omega,R} \mathcal{S}_{\omega,R,j} \mathcal{M} \quad \text{for each } j = 1, \ldots, d. \tag{8.30}\]
But for \( j \neq k \), \( \mathcal{S}^{(j,k)^T}_{\omega,R} \perp \mathcal{S}^{(k,j)^T}_{\omega,R} \) and hence \( \text{(8.30)} \) actually implies
\[
\mathcal{S}^{(j,k)^T}_{\omega,R} \mathcal{S}_{\omega,R,j} \mathcal{M} \quad \text{for each } j = 1, \ldots, d. \tag{8.31}\]
Note next that \( \mathcal{S}^{\beta^T}_{\omega,R} E^*y \), the last term on the right hand side of \( \text{(8.26)} \), satisfies the orthogonality property
\[
(\mathcal{S}^{\beta^T}_{\omega,R} E^*y, (\mathcal{S}^{(j,k)^T}_{\omega,R} f) H_{\omega,R}^2(\mathbb{F}_d^2)) = 0
\]
for all \( f \in H^2_{\omega,Y}(\mathbb{F}_d^+) \) and for all \( j = 1, \ldots, d \); in other words,
\[
S_{\omega,R}^\beta E^y \in \left( \bigoplus_{j=1}^d S_{\omega,R}^{(j\beta)^\top} H^2_{\omega,Y}(\mathbb{F}_d^+) \right)^\perp.
\]
In particular, we have
\[
S_{\omega,R}^\beta E^y \in \left( \bigoplus_{j=1}^d S_{\omega,R}^{(j\beta)^\top} \mathcal{M} \right)^\perp.
\]
(8.32)

Adding up (8.31) over all \( j = 1, \ldots, d \) together with (8.32) and recalling (8.28) and (8.27), we are left with
\[
S_{\omega,R}^\beta h g, y \in \left( \bigoplus_{j=1}^d S_{\omega,R}^{\beta} \mathcal{M} \right)^\perp,
\]
and hence \( S_{\omega,R}^\beta h g, y = 0 \). As \( S_{\omega,R}^\beta \) is injective, it then follows that \( h g, y = 0 \). Note that the sum in the definition (8.24) is orthogonal, and hence \( y = 0 \) and
\[
S_{\omega,R,j} \mathcal{G}_{\omega,|\beta|+1,E,S_{\omega,R,j}^*} g_j = 0 \quad \text{for each} \quad j = 1, \ldots, d.
\]
As \( S_{\omega,R,j} \mathcal{G}_{\omega,|\beta|+1,E,S_{\omega,R,j}^*} \) is injective, we conclude that each \( g_j = 0 \) as well, and the coisometric property of \( U_\beta \) now follows.

Step 4: Verification of the transfer-function realization formula (2.37) for \( \Theta_\beta \).

It remains now only to check that we recover \( \Theta_\beta \) via the realization formula (2.37) (or equivalently the representation (7.1)) explicitly in terms of power-series coefficients:
\[
\Theta_\beta(z) = \omega_{|\beta|}^{-1} D_\beta + \sum_{j=0}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1}^{-1} CA^{\beta'} B_{j,\beta} z^{\beta' j}.
\]

We compute, using (8.13) and (4.71),
\[
CA^{\beta'} B_{j,\beta} = E S_{\omega,R}^{\beta'} \left( \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega_{|\alpha|+|\beta|+1}}{\omega_{|\alpha|}} [\Theta_\beta]_{|\alpha|} z^\alpha \right)
= \omega_{|\beta'|} \cdot \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega_{|\alpha|+|\beta|+1}}{\omega_{|\alpha|}} [\Theta_\beta]_{|\alpha|} z^\alpha
= \omega_{|\beta'|} \cdot \frac{\omega_{|\beta'|+|\beta|+1}}{\omega_{|\beta'|}} \cdot [\Theta_\beta]_{|\beta'|} = \omega_{|\beta'|+|\beta|+1} [\Theta_\beta]_{|\beta'|}.
\]

We combine this with the identity \( D_\beta = \omega_{|\beta|}[\Theta_\beta]_0 \) to get
\[
\omega_{|\beta|}^{-1} D_\beta + \sum_{j=0}^d \sum_{\beta' \in \mathbb{F}_d^+} \omega_{|\beta'|+|\beta|+1} CA^{\beta'} B_{j,\beta} z^{\beta' j}
= [\Theta_\beta]_0 + \sum_{j=0}^d \sum_{\beta' \in \mathbb{F}_d^+} [\Theta_\beta]_{|\beta'|} z^{\beta' j} = \Theta_\beta(z)
\]
as wanted. \( \square \)
8.3. Model theory for $n$-hypercontractions

Recall Definition 4.13 of an $n$-hypercontractive operator $d$-tuple $\mathbf{A} = (A_1, \ldots, A_d)$: an operator $d$-tuple $\mathbf{A} = (A_1, \ldots, A_d)$ on a Hilbert space $\mathcal{X}$ is $n$-hypercontractive if the defect operators

$$
\Gamma_{m,A}[I_X] = (I - B_A)^m[I_X] = \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| \leq m} (-1)^{|\alpha|} \left( \binom{m}{|\alpha|} A^{\alpha^\top} \right) A^\alpha
$$

are all positive semidefinite for $1 \leq m \leq n$. As in the general $\omega$-setting, we consider an operator $d$-tuple $\mathbf{T} = (T_1, \ldots, T_d)$ for which the adjoint tuple $\mathbf{A} := \mathbf{T}^* := (T_1^*, \ldots, T_d^*)$ is an $n$-hypercontraction. Note that while it is the case that $\Gamma_{m,A} = \Gamma_{\mu_n,A}$, the definition of $\mu_n$-hypercontraction requires that the shifted defect operators

$$
\Gamma_{\mu_n,A}^{(k)}[I_X] = \frac{R_{\mu_n,k}}{R_{\mu_n}} (B_A)
$$

be positive-semidefinite for all $k \geq 0$. However the linking identity (4.37) enables one to show that these two classes are identical. One can check that in all the formulas in the preceding sections of this Chapter, while shifts of observability gramian operators are ubiquitous, shifted defect operators ($\Gamma_{\mu_n}$ versus $\Gamma^{(k)}$ with $k \geq 1$) do not occur. Furthermore, by Remark 4.42, in case $\omega = \mu_n$, $\mu_n$-strong stability may be replaced by ordinary strong stability. Thus all the results concerning model theory for $*\omega$-hypercontractive operator tuples apply mutatis mutandis for the particular case of $n$-hypercontractions by simply specializing the weight $\omega$ to the case $\omega = \mu_n$, and with $\omega$-strong stability replaced by the standard notion of strong stability.
Hardy-Fock spaces built from a regular formal power series

9.1. Introduction

We let \( p \) be a regular noncommutative formal power series; by this we mean that \( p \) has a formal power series representation
\[
p(z) = \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha z^\alpha \tag{9.1}
\]
with scalar coefficients \( p_\alpha \in \mathbb{C} \) such that
\[
p_\emptyset = 0, \quad p_\alpha > 0 \text{ if } |\alpha| = 1, \quad p_\alpha \geq 0 \text{ for all } \alpha \in \mathbb{F}_d^+.
\]
We assume that this series has a positive radius of convergence, i.e., there is a number \( \rho > 0 \) such that, whenever \( A = (A_1, \ldots, A_d) \) is a Hilbert-space operator \( d \)-tuple such that \( \| [A_1, \ldots, A_d] \| < \rho \), then the series
\[
p(A) := \sum_{k=1}^{\infty} \sum_{|\alpha| = k} p_\alpha A^\alpha
\]
converges in the strong operator topology (see [83] for additional background on free holomorphic functions defined on multivariable operator balls via formal power series in freely noncommutative indeterminates as we have here). We also apply this functional calculus to operator \( d \)-tuples in \( \mathcal{L}(\mathcal{L}(X))^d \), in particular to the operator \( d \)-tuple \( B_A = (B_{A_1}, \ldots, B_{A_d}) \) where \( B_{A_j} : X \to A_j^* X A_j \) (not to be confused with the Chapter 4 notation [4.10] appropriate for the case where \( p_\alpha = p_{|\alpha|} \)): a standing assumption throughout this chapter is that
\[
\| [B_{A_1}, \ldots, B_{A_d}] \| = \| [A_1, \ldots, A_d] \| < \rho \tag{9.3}
\]
so that the series
\[
p(B_A) : X \mapsto \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha A^{*\alpha^\top} X A^\alpha
\]
converges in the norm topology of \( \mathcal{L}(\mathcal{L}(X)) \). Returning to viewing \( p \) as a formal power series (9.1), for any \( n \in \mathbb{Z}_+ \) we may form the formal power series \( (1 - p)^n \) given by
\[
(1 - p)^n(z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \sum_{\alpha \neq \emptyset} p_\alpha z^\alpha \right)^k = \sum_{\alpha \in \mathbb{F}_d^+} c_{p,n;\alpha} z^\alpha \tag{9.4}
\]
where
\[
c_{p,n;\alpha} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{\beta_1, \ldots, \beta_k \neq \emptyset : \beta_1 \cdots \beta_k = \alpha} p_{\beta_1} \cdots p_{\beta_k}.
\]
Applying the functional calculus described above, given an operator $d$-tuple $A$ with the assumption $\text{(9.3)}$, we may form the operator $(1 - p)^n(B_A) \in \mathcal{L}(\mathcal{L}(X))$:

$$\Gamma_{p,n;A}[X] := n(B_A)[X] = \sum_{\alpha \in \mathbb{F}^+_d} c_{p,n;\alpha} A^{\alpha^T} X A^\alpha. \quad (9.5)$$

**Definition 9.1.** We shall say that the operator-tuple $A = (A_1, \ldots, A_d)$ is $(p, n)$-contractive if, in addition to $\text{(9.3)}$, it is the case that $\Gamma_{p,n;A}[I] \geq 0$. We shall say that $A$ is $(p, n)$-hypercontractive if

$$\Gamma_{p,k;A}[I] \geq 0 \quad \text{for} \quad 1 \leq k \leq n.$$

Below, many of the results of Chapters 3–6 concerning $n$-hypercontractions (which are $(p, n)$-hypercontractions for the choice of $p(z) = z_1 + \cdots + z_d$) will be extended to the general $(p, n)$-setting for the general $p(z)$ subject to $\text{(9.1)}$, $\text{(9.2)}$.

In addition to the auxiliary power series $\text{(9.4)}$, we shall have need of the $(p, n)$-analog of the resolvent-power function $R_n \text{(1.10)}$:

$$R_{p,n}(z) := R_n(p(z)) = (1 - p(z))^{-n}$$

$$= \sum_{j=0}^\infty \binom{n+j-1}{j} \left( \sum_{\alpha \neq \emptyset} p_\alpha z^\alpha \right)^j := \sum_{\alpha \in \mathbb{F}^+_d} \omega_{p,n;\alpha}^{-1} \omega_{p,n;\alpha} \left( \sum_{\beta_1, \ldots, \beta_j \neq \emptyset: \beta_1 + \cdots + \beta_j = \alpha} p_{\beta_1} \cdots p_{\beta_j} \right) \quad (9.6)$$

where $\omega_{p,n;\emptyset} = 1$ and

$$\omega_{p,n;\alpha}^{-1} := \sum_{j=0}^{\lfloor \alpha \rfloor} \binom{n+j-1}{j} \sum_{\beta_1, \ldots, \beta_j \neq \emptyset: \beta_1 + \cdots + \beta_j = \alpha} p_{\beta_1} \cdots p_{\beta_j} \quad \text{for} \quad |\alpha| \geq 1. \quad (9.7)$$

Due to assumptions $\text{(9.2)}$, all the terms on the right side of $\text{(9.7)}$ are nonnegative and at least one term (corresponding to $j = |\alpha|$) is positive. Thus, the reciprocal notation on the left side of $\text{(9.2)}$ makes sense and it is clear that $\omega_{p,n;\alpha} > 0$ for all $\alpha \in \mathbb{F}^+_d$. We next introduce the shifted counterparts of $\text{(9.6)}$

$$R_{p,n;\beta}(z) := \sum_{\alpha \in \mathbb{F}^+_d} \omega_{p,n;\alpha}^{-1} \omega_{p,n;\alpha}^z \beta \quad \text{for all} \quad \beta \in \mathbb{F}^+_d \quad (9.8)$$

(the $(p, n)$-analog of $\text{(1.10)}$) and observe that for any $\beta \in \mathbb{F}^+_d$,

$$R_{p,n;\beta}(z) = \omega_{p,n;\beta}^{-1} + \sum_{j=1}^d R_{p,n;j;\beta}(z) z_j \quad (9.9)$$

which can be seen as the noncommutative $(p, n)$-analog of the relation $\text{(1.11)}$.

Given an operator $d$-tuple $A = (A_1, \ldots, A_d)$ and $z = (z_1, \ldots, z_d)$ our set of free noncommutative indeterminates, we set $zA$ equal to the $d$-tuple of monomials with operator coefficients

$$zA = (z_1 A_1, \ldots, z_d A_d)$$

and then define $R_{p,n}(zA)$ and $R_{p,n;\beta}(zA)$ via formal power series

$$R_{p,n}(zA) = \sum_{\alpha \in \mathbb{F}^+_d} \omega_{p,n;\alpha}^{-1} A^\alpha z^\alpha, \quad R_{p,n;\beta}(zA) = \sum_{\alpha \in \mathbb{F}^+_d} \omega_{p,n;\alpha}^{-1} A^\alpha z^\alpha. \quad (9.10)$$

Note that the assumption that $p_\alpha \geq 0$ for all $\alpha \in \mathbb{F}^+_d$ and $p_\alpha > 0$ when $|\alpha| = 1$ guarantees that the term $j = |\alpha|$ in the sum $\text{(9.7)}$ is strictly positive while all other
terms are nonnegative; hence $\omega_{p,n,\alpha}^{-1} > 0$, justifying the inverse notation. We now have arrived at a weight function $\omega_{p,n} = \{\omega_{p,n,\alpha}\}_{\alpha \in F_+}$ (where $\omega_{p,n,\alpha} = (\omega_{p,n,\alpha}^{-1})^{-1}$) of positive real numbers. The $(p, n)$-analog of (13.32) is the weighted system

$$
\Sigma_{\{U_n\}}\{p,n\}:
\begin{cases}
x(1\alpha) = \frac{\omega_{p,n,\alpha}}{\omega_{p,n,1\alpha}} A_1 x(\alpha) + \frac{1}{\omega_{p,n,1\alpha}} B_{1,\alpha} u(\alpha) \\
\vdots \\
x(d\alpha) = \frac{\omega_{p,n,\alpha}}{\omega_{p,n,d\alpha}} A_d x(\alpha) + \frac{1}{\omega_{p,n,d\alpha}} B_{d,\alpha} u(\alpha) \\
y(\alpha) = C x(\alpha) + \omega_{p,n,1}^{-1} D_{\alpha} u(\alpha).
\end{cases}
$$

Upon running the latter system with the fixed initial condition $x_0 = x$ we get

$$
y(\alpha) = \omega_{p,n,1}^{-1} \left( C A^\alpha x + D_{\alpha} u(\alpha) + \sum_{\alpha'j\alpha' = \alpha} C A^{\alpha''} B_{j,\alpha'} u(\alpha') \right).
$$

Applying the noncommutative $Z$-transform (12.20) to the latter formula and taking into account (9.10) we have

$$
\hat{y}(z) = \sum_{\alpha \in F_+^d} \omega_{p,n,\alpha}^{-1} \left( C A^\alpha x + D_{\alpha} u(\alpha) + \sum_{\alpha'j\alpha' = \alpha} C A^{\alpha''} B_{j,\alpha'} u(\alpha') \right) z^\alpha \\
= C R_{p,n}(zA)x + \sum_{\alpha \in F_+^d} \omega_{p,n,\alpha}^{-1} D_{\alpha} z^\alpha u(\alpha) \\
+ \sum_{\alpha' \in F_+^d} \sum_{j=1}^{d} \left( \sum_{\alpha'' \in F_+^d} \omega_{p,n,\alpha''j\alpha}\left( C A^{\alpha''} z^\alpha '' \right) z_j B_{j,\alpha'} \right) z^\alpha u(\alpha') \\
= C R_{p,n}(zA)x + \sum_{\alpha \in F_+^d} \left( \omega_{p,n,\alpha}^{-1} D_{\alpha} + C \sum_{j=1}^{d} R_{p,n,j\alpha}(zA) z_j B_{j,\alpha} \right) z^\alpha u(\alpha) \\
= \Theta_{p,n;U_{\alpha}}(z) + \sum_{\alpha \in F_+^d} \Theta_{p,n;A_{\alpha}}(z) z^\alpha u(\alpha),
$$

(9.11)

where the first term represents the $(p, n)$-observability operator

$$
\Theta_{p,n;C;A} : x \mapsto C R_{p,n}(zA)x = \sum_{\alpha \in F_+^d} (\omega_{p,n,\alpha}^{-1} C A^\alpha x) z^\alpha
$$

(9.12)

associated with the output pair $(C, A)$ and where

$$
\Theta_{p,n;U_{\alpha}}(z) = \omega_{p,n,\alpha}^{-1} D_{\alpha} + C \sum_{j=1}^{d} R_{p,n;j\alpha}(zA) z_j B_{j,\alpha} \\
= \omega_{p,n,\alpha}^{-1} D_{\alpha} + \sum_{\gamma \in F_+^d} \sum_{j=1}^{d} \omega_{p,n,\gamma j\alpha} C A^\gamma B_{j,\alpha} z^\gamma j
$$

(9.13)

is a family of transfer functions. Observe that except for the input-to-state operator $B_{\alpha}$, the transfer function $\Theta_{p,n;U_{\alpha}}$ in (2.37) depends on $|\alpha|$ rather than $\alpha$. In contrast to this basic case ($p(z) = z_1 + \cdots + z_d$), the dependence of $\Theta_{p,n;U_{\alpha}}$ in (9.13) on $\alpha$
in general is more substantial. We point out, however, that the formula (9.13) can be written in the form resembling the basic case (1.40) as follows:

\[
\Theta_{p,n}(z) = \omega_{p,n}^{-1} D_{\beta} + C \hat{R}_{p,n;\beta}(z A) \hat{B}_{\beta},
\]

(9.14)

where \( \hat{B}_{\beta} = \begin{bmatrix} B_{1,\beta} & \vdots & B_{d,\beta} \end{bmatrix} \in \mathcal{L}(\mathcal{H}_{\beta}, \mathcal{X}^d) \) is defined as in (1.33) and where

\[
\hat{R}_{p,n;\beta}(z) = \begin{bmatrix} R_{p,n;1;\beta}(z) & \ldots & R_{p,n;d;\beta}(z) \end{bmatrix}, \quad \hat{Z}(z) = \begin{bmatrix} z_1 I_X & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z_d I_X \end{bmatrix}.
\]

(9.15)

### 9.2. Contractive multipliers from \( H^2_{\hat{p},\hat{d}}(\mathbb{F}_d^+) \) to \( H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) \)

Starting with a power series (9.11) with nonnegative coefficients and composing it with \( R_n \) in (9.6) we came up with the weight function \( \hat{\omega}_{p,n} := \{ \omega_{p,n;\alpha} \}_{\alpha \in \mathbb{F}_d^+} \). The corresponding Hardy-Fock space \( H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) \) is now defined as in (2.27):

\[
H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) = \left\{ \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha \hat{z}_\alpha \in \mathcal{Y}(z) : \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n;\alpha} \cdot \| f_\alpha \|_{\mathcal{Y}}^2 < \infty \right\},
\]

(9.16)

and it follows, as in (2.28), that \( H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) \) is a NFRKS with noncommutative reproducing kernel

\[
k_{\hat{\omega}_{p,n}}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n;\alpha} z^\alpha \zeta^\top.
\]

(9.17)

However, it is no longer the case that the sequence \( \omega = \hat{\omega}_{p,n} \) satisfies the conditions (2.31). We therefore must develop some of the results from Chapter 3 from scratch.

To get an analog of Theorem 3.24 for the \((p, n)\)-setting, we need an adjustment of the Hardy-Fock space \( H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) \) serving as the input space for \( M_\Theta \) in the statement of the theorem. Given \( p(z) \) as in (9.11), we let \( \tilde{p} \) be the linear part of \( p \), or, more precisely, define \( \tilde{p} \) according to the following recipe:

\[
p(z) = \tilde{p}(z) + \sum_{\alpha \in \mathbb{F}_d^+: |\alpha| \geq 2} \ p_\alpha z^\alpha, \quad \text{so} \quad \tilde{p}(z) = \sum_{j=1}^d p_j z_j.
\]

Then the generalized Hardy space \( H^2_{\hat{\omega}_{\tilde{p},1,\mathcal{U}}}(\mathbb{F}_d^+) \) is the adjustment of the Hardy-Fock space \( H^2_{\hat{\omega}_{p,n,\mathcal{Y}}}(\mathbb{F}_d^+) \) which we seek. Note that if \( \tilde{p}(z) = z_1 + \cdots + z_d \) (i.e., \( p_\alpha = 1 \) for each word \( \alpha \) with \( |\alpha| = 1 \)), then \( H^2_{\hat{\omega}_{\tilde{p},1,\mathcal{U}}}(\mathbb{F}_d^+) \) is just the same space as Hardy-Fock space \( H^2_{\hat{\omega}_{\hat{d}}}(\mathbb{F}_d^+) \) and we have the redundant notation \( k_{\omega_{\tilde{p},1}} = k_1 \) for the noncommutative Szegö kernel (where \( 1 \) is the weight sequence consisting of all 1’s).

For the case of a general \( p \), for each \( \alpha = i_1 i_2 \ldots i_N \in \mathbb{F}_d^+ \), we let

\[
d_{\tilde{p},\alpha} = d_{\tilde{p},i_1 i_2 \ldots i_N} := p_{i_1} p_{i_2} \cdots p_{i_N} \quad \text{and} \quad d_{\tilde{p},\emptyset} = 1.
\]

(9.18)

Then the kernel

\[
k_{\omega_{\tilde{p},1}}(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} d_{\tilde{p},\alpha} z^\alpha \zeta^\top
\]

(9.19)
Lemma 3.14. Note that the shift operators $S_\alpha$ and therefore, for any $u, v \in H_{\omega_{p,1}^\alpha}^2(\mathbb{F}_d^+)$, have the following analog of Lemma 3.14.

The spaces $H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$ have many properties in common with the Fock space $H_{\omega_{p,1}^\alpha}^2(\mathbb{F}_d^+)$, as in the proof of Lemma 3.14. Therefore, for any $u, v \in H_{\omega_{p,1}^\alpha}^2(\mathbb{F}_d^+)$, we have

$$
\| \sum_{\alpha \in F_d^+} f_\alpha z^\alpha \|_{H_{\omega_{p,1}^\alpha}^2(\mathbb{F}_d^+)}^2 = \sum_{\alpha \in F_d^+} d^{-1}_{\bar{p},\alpha} \| f_\alpha \|^2.
$$

(9.20)

Lemma 9.2. Let $F$ be a contractive multiplier from $H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$ to $H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$. Then $F$ is a strictly inner if and only if $\|Fu\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)} = \|u\|_{\mathcal{U}}$ for all $u \in \mathcal{U}$.

Proof. The proof of the non-trivial "if" part follows the lines of the proof of Lemma 3.14. Note that the shift operators $S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta}$ satisfy

$$
S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} = p_j^{-1} I_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)} (j = 1, \ldots, d)
$$

and have mutually orthogonal ranges. Therefore,

$$
\|S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)} = \frac{\|F u\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}^2}{\sqrt{d_{\bar{p},\alpha}}} = \frac{\|u\|_{\mathcal{U}}^2}{\sqrt{d_{\bar{p},\alpha}}} \quad \text{for all } u \in \mathcal{U}, \alpha \in \mathbb{F}_d^+.
$$

Therefore, for any $u, v \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{F}_d^+$, we have

$$
\|S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u + S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} v\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}^2 = \frac{\|u\|_{\mathcal{U}}^2}{d_{\bar{p},\alpha}} + \frac{\|v\|_{\mathcal{U}}^2}{d_{\bar{p},\beta}} + 2\text{Re}\langle S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u, S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} v \rangle_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}.
$$

On the other hand, as $M_F : H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+) \to H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$ is a contraction, we have

$$
\|S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u + S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} v\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}^2 \leq \|S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}^2 + \|S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} v\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}^2
$$

for all $\alpha \neq \beta$. From the two latter relations, we conclude (as in the proof of Lemma 3.14) that

$$
\langle S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} u, S_{\omega_{p,1}^\alpha,\omega_{p,1}^\beta} v \rangle_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)} = 0,
$$

and therefore, $\|F h\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)} = \|h\|_{H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)}$ for every $\mathcal{U}$-valued “polynomial” $h$ and therefore, for any $h \in H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$. Therefore, $F$ is a strictly inner multiplier from $H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$ to $H_{\omega_{p,1}^\alpha,\omega}(\mathbb{F}_d^+)$.

We present next the $\omega_{p,1}^\alpha$-analog of Theorems 3.5 and 3.8. To reduce this more general result to the setting already covered in Chapter 3.2, we use the change of variable

$$
z = (z_1, \ldots, z_d) \mapsto \iota z := (p_1^{-\frac{1}{2}} z_1, \ldots, p_d^{-\frac{1}{2}} z_d)$$
together with the induced map on formal power series
\[ f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \rightarrow (\iota f)(z) := f(\iota z) = \sum_{\alpha \in \mathbb{F}_d^+} (\iota f_\alpha)(z) = \sum_{\alpha \in \mathbb{F}_d^+} d_{\iota \alpha}^{-\frac{1}{2}} f_\alpha z^\alpha. \] (9.21)

Then using (9.20), it is easily checked that \( \iota \) is unitary as a map from \( H^2_{\omega_{p,1},U}(\mathbb{F}_d^+) \) onto the Fock space \( H^2_{\iota}(\mathbb{F}_d^+) \):
\[ ||f||^2_{H^2_{\iota}(\mathbb{F}_d^+)} = ||f||^2_{H^2_{\omega_{p,1},U}(\mathbb{F}_d^+)} \] for all \( f \in H^2_{\omega_{p,1},U}(\mathbb{F}_d^+) \),

while in terms of kernel functions we have the identity
\[ k_{\omega_{p,1}}(\iota z, \iota \zeta) = k_{nc,Sz}(z, \zeta). \] (9.22)

**Theorem 9.3.** Define \( \tilde{P} \) and \( Z_{\tilde{p}}(z) \) as
\[ \tilde{P} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & p_{d-1} & 0 \\ 0 & \cdots & 0 & p_d \end{bmatrix}, \quad Z_{\tilde{p}}(z) = [p_1 z_1 \cdots p_1 z_1] \otimes IX \] (9.23)
(where the Hilbert space \( \mathcal{X} \) is determined by the context).

1. Any contractive multiplier \( G \) from \( H^2_{\omega_{p,1},U}(\mathbb{F}_d^+) \) to \( H^2_{\omega_{p,1},Y}(\mathbb{F}_d^+) \) admits a \( \tilde{p} \)-unitary realization in the following sense: there exists a Hilbert space \( \mathcal{X} \) and a connection matrix \( U = [A B] \) of the form (9.22) and subject to constraints
\[ U^* \begin{bmatrix} \tilde{P} \otimes IX & 0 \\ 0 & I_Y \end{bmatrix} U = \begin{bmatrix} IX & 0 \\ 0 & I_d \end{bmatrix}, \quad UU^* = \begin{bmatrix} \tilde{P}^{-1} \otimes IX & 0 \\ 0 & I_Y \end{bmatrix}, \] (9.24)
so that
\[ G(z) = D + \sum_{j=1}^d d_{\iota \alpha} C A^\alpha B_j z^{\alpha j} = D + C(I - Z_{\tilde{p}}(z)A)^{-1} Z_{\tilde{p}}(z)B. \] (9.25)

Conversely, if \( G \in \mathcal{L}(U,Y)(\langle z \rangle) \) has a realization as in (9.20) with the connection matrix \( U = [A B] \), such that \( U^* \begin{bmatrix} \tilde{P} \otimes IX & 0 \\ 0 & I_Y \end{bmatrix} U \preceq \begin{bmatrix} IX & 0 \\ 0 & I_d \end{bmatrix} \), then \( G \) is a contractive multiplier from \( H^2_{\omega_{p,1},U}(\mathbb{F}_d^+) \) to \( H^2_{\omega_{p,1},Y}(\mathbb{F}_d^+) \).

2. Given a tuple \( A = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{X})^d \) and \( C \in \mathcal{L}(\mathcal{X},Y) \), let \( H \in \mathcal{L}(\mathcal{X}) \) be a strictly positive definite operator such that
\[ H - \sum_{j=1}^d p_j A_j^* H A_j \succeq C^* C. \] (9.26)

Let \( A \) be defined as in (1.24) and let \( [\begin{smallmatrix} B \\ D \end{smallmatrix}] : U \rightarrow \begin{bmatrix} X^d \\ Y \end{bmatrix} \) be a solution of the Cholesky factorization problem
\[ \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} (\tilde{P} \otimes H)^{-1} & 0 \\ 0 & I_Y \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} H^{-1} [A^* C^*]. \] (9.27)

(i) Then the power series
\[ G(z) = D + C(I - Z_{\tilde{p}}(z)A)^{-1} Z_{\tilde{p}}(z)B \]
is a contractive multiplier from $H^2_{\omega_{p,1},\mathcal{U}(\mathbb{F}^+_d)}$ to $H^2_{\omega_{p,1},\mathcal{Y}(\mathbb{F}^+_d)}$. Moreover,
\[ k_{\omega_{p,1}}(z,\zeta)I_Y - G(z)(k_{\omega_{p,1}}(z,\zeta)I_U)G(\zeta)^* = C(I - Z_\tilde{p}(z)A)^{-1}H^{-1}(I - A^*Z_\tilde{p}(\zeta)^*)^{-1}C^*. \]

(ii) If \( (9.20) \) holds with equality and \( A \) is \( \tilde{p} \)-strongly stable in the sense that
\[ \lim_{N \to \infty} \tilde{p}(B_\alpha)^N[I_X] := \lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| = N} d_{\alpha,\alpha} A^{**T} A^* = 0 \]
in the strong operator topology, then \( G \) is McCT-inner.

Conversely, any contractive, McCT-inner and strictly inner multiplier from $H^2_{\omega_{p,1},\mathcal{U}(\mathbb{F}^+_d)}$ to $H^2_{\omega_{p,1},\mathcal{Y}(\mathbb{F}^+_d)}$ can be expressed in terms of the positivity of the associated formal kernel
\[ K^G_{\tilde{p}}(z,\zeta) := k_{\omega_{p,1}}(z,\zeta)I_Y - G(z)(k_{\omega_{p,1}}(z,\zeta)I_U)G(\zeta)^*. \]
Replace \( z \) with \( \iota z \) and \( \zeta \) with \( \iota\zeta \) to see that
\[ K^G_{\tilde{p}}(\iota z,\iota\zeta) = k_{nc,Sa}(z,\zeta)I_Y - G(\iota z)(k_{nc,Sa}(z,\zeta)I_U)G(\iota\zeta)^* \]
is also a positive formal kernel. Again by Proposition 3.2 this in turn means that the formal power series \( G(\iota z) \) is a contractive multiplier from \( H^2_{\mathcal{U}(\mathbb{F}^+_d)} \) into \( H^2_{\mathcal{Y}(\mathbb{F}^+_d)} \).

By Theorem 3.5 there is a unitary connection matrix as in \( (1.22) \) which we write in the form \( \tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ C & D \end{bmatrix} \) so that
\[ G(\iota z) = D + C(I - Z(\iota z)\tilde{A})^{-1}Z(\iota z)\tilde{B}. \]

If we let \( A := (\tilde{P}^{-\frac{1}{2}} \otimes I_X)\tilde{A} \) and \( B := \tilde{P}^{-\frac{1}{2}}B \), then the operator
\[ \mathbf{U} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \tilde{P}^{-\frac{1}{2}} \otimes I_X & 0 \\ 0 & I_{U^2} \end{bmatrix} \tilde{U} \]
meets the constraints \( (9.24) \), since \( \tilde{U} \) is unitary. Furthermore, we see from \( (9.23) \) and \( (1.24) \) that \( Z_{\tilde{p}}(z) = Z(z)\tilde{P} \otimes I_X \) and hence,
\[ Z(\iota^{-1}z)\tilde{A} = Z(z)(\tilde{P}^{-\frac{1}{2}} \otimes I_X)\tilde{A} = Z_\tilde{p}(z)(\tilde{P}^{-\frac{1}{2}} \otimes I_X)\tilde{A} = Z_{\tilde{p}}(z)A, \]
\[ Z(\iota^{-1}z)\tilde{B} = Z(z)(\tilde{P}^{-\frac{1}{2}} \otimes I_X)\tilde{B} = Z_\tilde{p}(z)(\tilde{P}^{-\frac{1}{2}} \otimes I_X)\tilde{B} = Z_{\tilde{p}}(z)B. \]
Replacing \( z \) with \( \iota^{-1}z \) in \( (9.28) \) and making use of the latter equalities we get
\[ G(z) = D + C(I - Z(\iota^{-1}z)\tilde{A})^{-1}Z(\iota^{-1}z)\tilde{B} = D + C(I - Z_p(z)A)^{-1}Z_p(z)B. \]

Thus, \( G(z) \) admits a realization \( (9.25) \) with the connection matrix \( \tilde{U} \) subject to conditions \( (9.21) \). In this way part (1) in Theorem 53 reduces to the content of Theorem 3.5. Furthermore, replacement of \( z \) by \( \iota z \) and \( \zeta \) by \( \iota\zeta \) in the kernel identity \( (9.22) \) (with \( \tilde{A} \) in place of \( A \) and \( \tilde{B} \) in place of \( B \)) implies that replacement of \( z \) by \( \iota z \) and \( \zeta \) by \( \iota\zeta \) leads to the identity
\[ k_{nc,Sa}(z,\zeta)I_Y - G(z)(k_{nc,Sa}(z,\zeta)I_U)G(\zeta)^* = C(I - Z(z)A)^{-1}(I - A^*Z(\zeta))^{-1}C^*. \]
Continuing along these lines, one can reduce all the assertions of the second statement (2) of Theorem 9.3 to the corresponding assertions in Theorem 3.8.
An application of the lurking isometry argument (or more simply of the change-
of-variable map \(9.21\)) to reduce the statement to that of Theorem 9.14 leads us to the Leech theorem in the present setting.

**Theorem 9.4.** Given power series \(G \in \mathcal{L}(\mathcal{Y}, \mathcal{X})(\langle z \rangle)\) and \(F \in \mathcal{L}(\mathcal{U}, \mathcal{X})(\langle z \rangle)\), the formal kernel

\[
K_G, F(z, \zeta) := G(z)(k_{\omega, 1}(z, \zeta) \odot I_Y)G(\zeta)^* - F(z)(k_{\omega, 1}(z, \zeta) \odot I_U)F(\zeta)^*
\]

is positive if and only if there exists a contractive multiplier \(S\) from \(H^2_{\omega, 1, \mathcal{U}}(\mathbb{F}_d^+)\) to \(H^2_{\omega, 1, \mathcal{Y}}(\mathbb{F}_d^+)\) such that \(F(z) = G(z)S(z)\).

Now we will establish the \(p\)-version of Theorem 9.19. Given \(p(z)\) as in \([9.11]\)–\([9.22]\) and the associated weight function \([9.7]\), let us define another non-negative weight \(\gamma_{p, n}\) associated with \(p\), namely, \(\gamma_{p, n} = \{\gamma_{p, n; \alpha}\}_{\alpha \in \mathbb{F}_d^+}\), where

\[
\gamma_{p, n; \alpha}^{-1} = \omega_{p, n; \alpha}^{-1} - \omega_{p, n; \alpha}^{-1}p_j (\alpha \in \mathbb{F}_d^+, j \in \{1, \ldots, d\}).
\]

If \(n = 1\), then we let \(\gamma_{p, 0; \alpha} = 1\) if \(\alpha = 0\) and \(\gamma_{p, 0; \alpha} = 0\), otherwise. The positivity of \(\gamma_{p, n}\) is clear from the formula

\[
\omega_{p, n; \alpha}^{-1} - \omega_{p, n; \alpha}^{-1}p_j = \omega_{p, n; \alpha}^{-1} + \sum_{\alpha'' = \alpha; \alpha'' \neq 0} \omega_{p, n; \alpha''}p_{\alpha''} > 0,
\]

which in turn, is verified by equating the coefficients of \(z^{\alpha_j}\) in the identity

\[
(1 - p(z))^{-n} - (1 - p(z))^{-n+1} = (1 - p(z))^{-n}p(z)
\]

combined with \([9.6]\) and \([9.7]\) as follows:

\[
\omega_{p, n; \alpha}^{-1} - \omega_{p, n; \alpha}^{-1}p_j = \sum_{\alpha'' = \alpha} \omega_{p, n; \alpha''}p_{\alpha''} = \omega_{p, n; \alpha}^{-1}p_j + \sum_{\alpha'' = \alpha; \alpha'' \neq 0} p_{\alpha''}.
\]

Let us define the weighted \(Z\)-transform \(\Psi_{p, n}\) by

\[
\Psi_{p, n} = \text{Row}_{\alpha \in \mathbb{F}_d^+}[\gamma_{p, n; \alpha}^{-1} z^{\alpha}] : \{f_\alpha\}_{\alpha \in \mathbb{F}_d^+} \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \gamma_{p, n; \alpha}^{-1} f_\alpha z^{\alpha}
\]

(9.30)

and let us identify \(\Psi_{p, n}\) with operator-valued power series

\[
\Psi_{p, n}(z) = \sum_{\alpha \in \mathbb{F}_d^+} \left( \text{Row}_{\beta \in \mathbb{F}_d^+}[\delta_{\alpha, \beta}(\gamma_{p, n; \alpha}^{-1} \odot I_Y)] \right) z^{\alpha}
\]

(9.31)

where \(\delta_{\alpha, \beta}\) is the Kronecker symbol. Then we have the following analog of item (2) in Theorem 3.19.

**Theorem 9.5.** Let \(M_{\Psi_{p, n}}\) be the multiplication operator associated with the operator-valued function \(\Psi_{p, n}\) (9.30). Then:

(1) \(M_{\Psi_{p, n}}|_{\ell^2(\mathbb{F}_d^+)}\) is an isometry from the space of constant functions \(\ell^2(\mathbb{F}_d^+))\) in \(H^2_{\omega, 1, \mathcal{U}}(\mathbb{F}_d^+)\) into \(H^2_{\omega, 1, \mathcal{Y}}(\mathbb{F}_d^+)\).

(2) \(M_{\Psi_{p, n}}\) is a coisometry from \(H^2_{\omega, 1, \mathcal{U}}(\mathbb{F}_d^+)\) to \(H^2_{\omega, 1, \mathcal{Y}}(\mathbb{F}_d^+)\).

(3) A given formal power series \(F(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle)\) is a contractive multiplier from \(H^2_{\omega, 1, \mathcal{U}}(\mathbb{F}_d^+)\) to \(H^2_{\omega, 1, \mathcal{Y}}(\mathbb{F}_d^+)\) if and only if it is of the form

\[
F(z) = \Psi_{p, n}(z)S(z)
\]

(9.32)

for some contractive multiplier \(S(z)\) from \(H^2_{\omega, 1, \mathcal{U}}(\mathbb{F}_d^+)\) to \(H^2_{\omega, 1, \mathcal{Y}}(\mathbb{F}_d^+)\).
9.2. Contractive Multipliers from $H^2_{\mathcal{F},L} (\mathcal{F}_d^+) \text{ to } H^2_{\omega_p,n} (\mathcal{F}_d^+)$

Proof. For $y = \{y_\alpha\}_{\alpha \in \mathcal{F}_d^+}$ in $\ell^2_{\mathcal{F},L} (\mathcal{F}_d^+)$, note that

$$\| \Psi_{\gamma_p,n} (z) y \|_{\ell^2_{\mathcal{F},L} (\mathcal{F}_d^+)}^2 = \sum_{\alpha \in \mathcal{F}_d^+} \gamma_{p,n;\alpha} \| \gamma_{p,n;\alpha} y_\alpha \|^2 = \| y \|_{\ell^2_{\mathcal{F},L} (\mathcal{F}_d^+)}^2.$$

This verifies statement (1). To verify statement (2), by Proposition 3.2, it suffices to verify the identity (the analog of identity (3.39))

$$k_{\omega_p,n} (z, \zeta) I_{L^2_{\mathcal{F},L} (\mathcal{F}_d^+)} = \Psi_{\gamma_p,n} (z) (k_{\omega\gamma_p,n} (z, \zeta) I_{L^2_{\mathcal{F},L} (\mathcal{F}_d^+)} \Psi_{\gamma_p,n} (\zeta)^*).$$

(9.33)

Indeed, since $\omega_{p,n;\emptyset}^{-1} = 1$, we can write (9.17) as

$$k_{\omega_p,n} (z, \zeta) = 1 + \sum_{j=1}^{d} \sum_{\alpha \in \mathcal{F}_d^+} \omega_{p,n;\alpha_j} \zeta^{\omega_{p,n;\alpha_j} \alpha^\top}$$

and then, upon taking into account the formula (9.29) for $\gamma_{p,n;\alpha_j}$, we see that

$$k_{\omega_p,n} (z, \zeta) - \sum_{j=1}^{d} p_j \zeta_j k_{\omega_p,n} (z, \zeta) z_j = 1 + \sum_{j=1}^{d} \sum_{\alpha \in \mathcal{F}_d^+} (\omega_{p,n;\alpha_j} - p_j \omega_{p,n;\alpha}) \zeta^{\omega_{p,n;\alpha_j} \alpha^\top}$$

$$= 1 + \sum_{j=1}^{d} \sum_{\alpha \in \mathcal{F}_d^+} \gamma_{p,n;\alpha_j} \zeta^{h_{p,n;\alpha_j} \alpha^\top} = \sum_{\alpha \in \mathcal{F}_d^+} \gamma_{p,n;\alpha} \zeta^{h_{p,n;\alpha} \alpha^\top}$$

which is to say that

$$k_{\omega_p,n} (z, \zeta) = \sum_{j=1}^{d} p_j \zeta_j k_{\omega_p,n} (z, \zeta) z_j + k_{\gamma_p,n} (z, \zeta).$$

Iteration of this last identity and recalling the definition of $d_{\tilde{\gamma}_{p,\alpha}}$ (9.18) then gives

$$k_{\omega_p,n} (z, \zeta) = \sum_{\alpha: |\alpha| = N+1} d_{\tilde{\gamma}_{p,\alpha}} \zeta^\top k_{\omega_p,n} (z, \zeta) z^\alpha + \sum_{\alpha: |\alpha| \leq N} d_{\tilde{\gamma}_{p,\alpha}} \zeta^\top k_{\gamma_p,n} (z, \zeta) z^\alpha$$

for $N = 1, 2, \ldots$. Taking the limit as $N \to \infty$ then leads us to

$$k_{\omega_p,n} (z, \zeta) = \sum_{\alpha} d_{\tilde{\gamma}_{p,\alpha}} \zeta^\top k_{\gamma_p,n} (z, \zeta) z^\alpha$$

$$= \sum_{\alpha \in \mathcal{F}_d^+} d_{\tilde{\gamma}_{p,\alpha}} \zeta^\top \Psi_{\gamma_p,n} (z) \Psi_{\gamma_p,n} (\zeta)^* z^\alpha$$

$$= \Psi_{\gamma_p,n} (z) \left( \sum_{\alpha \in \mathcal{F}_d^+} d_{\tilde{\gamma}_{p,\alpha}} z^\alpha \zeta^\top \right) \Psi_{\gamma_p,n} (\zeta)^*$$

$$= \Psi_{\gamma_p,n} (z) (k_{\omega\gamma_p,n} (z, \zeta) \otimes I_{L^2_{\mathcal{F},L} (\mathcal{F}_d^+)} \Psi_{\gamma_p,n} (\zeta)^*$$

where we used (9.19) for the last step, and (9.33) is thus verified.

The proof of (3) parallels the proof of statement (3) in Theorem 3.19. Let $F (z) \in \mathcal{L} (\mathcal{U}, \mathcal{Y}) (\|z\|)$ be a contractive multiplier from $H^2_{\omega_p,n} (\mathcal{F}_d^+)$ to $H^2_{\omega_p,n} (\mathcal{F}_d^+)$. By Proposition 3.4, this amounts to the statement that the kernel

$$K_{\omega_p,n} (z, \zeta) = k_{\omega_p,n} (z, \zeta) \otimes I_{\mathcal{Y}} - F (z) (k_{\omega\gamma_p,n} (z, \zeta) \otimes I_{L^2_{\mathcal{F},L} (\mathcal{F}_d^+)}) F (\zeta)^*$$
be a positive formal kernel. On the other hand, by the Leech theorem (Theorem 3.4), the existence of the factorization (3.32) with \( S(z) \) a contractive multiplier from \( H^2_{\omega p,1, U}(F^+_d) \) to \( H^2_{\omega p,1, \ell^2_j(F^+_d)}(F^+_d) \) is equivalent to the positivity of the kernel

\[
K^F_{\omega p,1} (z, \zeta) = \Psi_{\tau_{p,n}}(z)(k_{\omega p,1}(z, \zeta) \otimes I_{\ell^2_j(F^+_d)}) \Psi_{\tau_{p,n}}(\zeta)^* - F(z)(k_{\omega p,1}(z, \zeta) \otimes I_U) F(\zeta)^*.
\]

But as a consequence of the identity (9.33), the kernel \( K^F_{\omega p,1} \) is the same as the kernel \( K^F_{\omega p,n} \) and (3) now follows. \( \square \)

In parallel with Definitions 3.23, we now introduce the following definition of \((p,n)\)-Bergman-inner.

**Definition 9.6.** We say that a power series \( \Theta \in L(U, Y) \langle \langle z \rangle \rangle \) is \((p,n)\)-Bergman inner if

1. \( \| M_\Theta u \|_{H^2_{\omega p,n, Y}(F^+_d)} = \| u \|_U \) for all \( u \in U \), and
2. \( M_\Theta u \perp S^*_{\omega p,n,R} M_\Theta v \) in \( H^2_{\omega p,n}(F^+_d) \) for all \( u, v \in U \) and \( \emptyset \neq \alpha \in F^+_d \).

The next two theorems (the \((p,n)\)-analogs of Theorems 3.24 and 3.25) show that \((p,n)\)-Bergman-inner power series can be alternatively defined as contractive multipliers from \( H^2_{\omega p,1, U}(F^+_d) \) to \( H^2_{\omega p,n, Y}(F^+_d) \) that act isometrically on constants.

**Theorem 9.7.** Let \( \Theta \in L(U, Y) \langle \langle z \rangle \rangle \) be such that

1. \( \| M_\Theta u \|_{H^2_{\omega p,n, Y}(F^+_d)} \leq \| u \|_U \) for all \( u \in U \), and
2. \( M_\Theta u \perp S^*_{\omega p,n,R} M_\Theta v \) for all \( u, v \in U \) and all nonempty \( \alpha \in F^+_d \).

Then \( \Theta \) is a contractive multiplier from \( H^2_{\omega p,1, U}(F^+_d) \) to \( H^2_{\omega p,n, Y}(F^+_d) \) and

\[
\| M_\Theta f \|_{H^2_{\omega p,n,Y}(F^+_d)}^2 \leq \| f \|_{H^2_{\omega p,1, U}(F^+_d)}^2 - \sum_{\alpha \in F^+_d} \sum_{j=1}^d d_{\alpha,j} \| D_j M_\Theta S^*_{\omega p,1,R} \rangle_j \langle \alpha \|_{H^2_{\omega p,n,Y}(F^+_d)}^2.
\]

where \( d_{\alpha,j} \) is defined in (9.18) and where we let

\[
D_j = (I - p_jS^*_{\omega p,n,R_j} S_{\omega p,n,R_j}) \frac{\hat{z}}{z}
\]

for short. Moreover, if \( \| M_\Theta u \|_{H^2_{\omega p,n,Y}(F^+_d)} = \| u \|_U \) for all \( u \in U \), then equality holds in (9.34) and \( \Theta \) is \((p,n)\)-Bergman inner.

**Proof.** For any \( U \)-valued polynomial \( f \) as in (3.47) and the right backward shift tuple \( S^*_{\omega p,R} \) on \( H^2_{\omega p,1, U}(F^+_d) \), we have

\[
S^*_{\omega p,R} f = p_j^{-1} \sum_{\alpha \in F^+_d : |\alpha| < m} f_{\alpha j} z^\alpha.
\]

Due to assumptions of the theorem (the same as in Theorem 3.24), and since the ranges of \( S_{\omega p,n,R,i} \) and \( S_{\omega p,n,R,j} \) are orthogonal whenever \( i \neq j \), we can repeat the
calculation (3.49) adjusting the formula (9.35) instead of (3.48) as follows:

\[
\|M_\Theta f\|^2 \leq \|f_0\|^2 + \left\| \sum_{j=1}^d \sum_{|\alpha| \leq m-1} S_{\omega_{p,n,R}M_\Theta f_\alpha}^{j\alpha} \right\|^2 \tag{9.36}
\]

\[
= \|f_0\|^2 + \sum_{j=1}^d \left\| S_{\omega_{p,n,j}M_\Theta \left( \sum_{|\alpha| \leq m-1} f_\alpha z^\alpha \right)} \right\|^2 
= \|f_0\|^2 + \sum_{j=1}^d \left\| S_{\omega_{p,n,R,j}M_\Theta (p_j S_{\omega_{p,1,R_j}f})} \right\|^2 \quad \text{(by (9.35))}
\]

\[
= \|f_0\|^2 + \sum_{j=1}^d \left\| (I - p_j S_{\omega_{p,n,R,j}S_{\omega_{p,n,R_j}}}^{1/2} M_\Theta (p_j S_{\omega_{p,1,R_j}f}) \right\|^2 
- \| (I - p_j S_{\omega_{p,n,R,j}S_{\omega_{p,n,R_j}}}^{1/2} M_\Theta (p_j S_{\omega_{p,1,R_j}f}) \|^2 \right) 
\]

\[
= \|f_0\|^2 + \sum_{j=1}^d \left( \left\| M_\Theta (p_j^{1/2} S_{\omega_{p,1,R_j}f}) \right\|^2 - \| \mathcal{D}_{j,M_\Theta (p_j^{1/2} S_{\omega_{p,1,R_j}f})} \|^2 \right) .
\]

Replace \( f \) by \( d_{p,\alpha} S_{\omega_{p,1,R_j}f} \) in (9.36) and take into account that \( (S_{\omega_{p,1,R_j}})^{\alpha} = d_{p,\alpha} f_{\alpha} \) (as a consequence of (9.34)) to arrive at

\[
\|M_\Theta f\|^2 \leq \|f_0\|^2 + \sum_{j=1}^d \left\| M_\Theta (d_{p,j}^{1/2} S_{\omega_{p,1,R_j}}^{\alpha}) f \right\|^2 
- \sum_{j=1}^d \left\| \mathcal{D}_{j,M_\Theta (d_{p,j}^{1/2} S_{\omega_{p,1,R_j}}^{\alpha})} f \right\|^2 .
\tag{9.37}
\]

Iterating the inequality (9.36) and using (9.37) then gives, for any \( m' = 1, 2, \ldots, \)

\[
\|M_\Theta f\|^2 \leq \sum_{|\alpha| < m'} d_{p,\alpha} \|f_{\alpha}\|^2 + \sum_{|\alpha| = m'} \left\| M_\Theta (d_{p,j}^{1/2} S_{\omega_{p,1,R_j}}^{\alpha}) f \right\|^2 
- \sum_{|\alpha| < m'} \sum_{j=1}^d \left\| \mathcal{D}_{j,M_\Theta (d_{p,j}^{1/2} S_{\omega_{p,1,R_j}}^{\alpha})} f \right\|^2 .
\]

Since \( f_{\alpha} = 0 \) once \( |\alpha| \geq m \) it follows that \( (S_{\omega_{p,1,R_j}})^{\alpha} f = 0 \) for any \( \alpha \) with \( |\alpha| \geq m \) and we see that, once \( m' \geq m \), this last inequality collapses to

\[
\|M_\Theta f\|^2 \leq \|f\|^2 - \sum_{|\alpha| < m'} \sum_{j=1}^d \left\| \mathcal{D}_{j,M_\Theta (d_{p,j}^{1/2} S_{\omega_{p,1,R_j}}^{\alpha})} f \right\|^2 .
\]

Letting \( m' \to \infty \) in the last inequality then gives us (9.34) for any \( \mathcal{U} \)-valued non-commutative polynomial \( f \). We then get the result for a general \( f \) in \( H^2_{\omega_{p,1,R}}(\mathbb{F}_d^+) \) by approximating \( f \) by finite truncations of its power series representation.
If equality \( \| M_\Theta u \|_{H^{\infty}_{\omega_{p,n},Y}(F^+_d)} = \| u \|_U \) holds for all \( u \in U \), then we have equalities throughout (9.36), (9.37), and therefore, in (9.34) as well. Moreover as a consequence of (9.34), we have that \( \Theta \) is a contractive multiplier from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,n},Y}(F^+_d) \). As we are also assuming the equality \( \| M_\Theta u \|_{H^{\infty}_{\omega_{p,n},Y}(F^+_d)} = \| u \|_U \) for all \( u \in U \), it follows from Definition 9.6 that \( \Theta \) is \((p,n)\)-Bergman inner.

**Theorem 9.8.** Let \( \Theta \in \mathcal{L}(U,Y)(\langle z \rangle) \) be a contractive multiplier from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,n},Y}(F^+_d) \) which is isometric on constants: \( \| M_\Theta u \|_{H^{\infty}_{\omega_{p,n},Y}(F^+_d)} = \| u \|_U \) for all \( u \in U \). Then \( \Theta \) is \((p,n)\)-Bergman inner. Furthermore, there is a unique contractive multiplier \( S \) from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,n},Y}(F^+_d) \) such that

\[
\Theta(z) = \Psi_{\gamma_{p,n}}(z)S(z)
\]  

(9.38)

where \( \Psi_{\gamma_{p,n}}(z) \) is defined as in (9.29) and (9.31). Moreover, this unique \( S \) is a strict inner multiplier from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,1},\ell^2(F^+_d)}(F^+_d) \).

**Proof.** Since \( \Theta \) is a contractive multiplier from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,n},Y}(F^+_d) \), it is (by Theorem 9.5 (3)) of the form (9.33) for some contractive multiplier \( S(z) \) from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,1},\ell^2(F^+_d)}(F^+_d) \). By assumption we have

\[
\| u \|_U = \| M_\Theta u \|_{H^{\infty}_{\omega_{p,n},Y}(F^+_d)} = \| M_{\Psi_{\gamma_{p,n}}}Su \|_{H^{\infty}_{\omega_{p,n},Y}(F^+_d)}
\]

for all \( u \in U \). To repeat verbatim the arguments from the proof of Theorem 9.5 regarding uniqueness and strict-inner property of \( S \) as well as the orthogonality relations \( \Theta u \perp S_{\omega_{p,n}}v \) (required to conclude that \( \Theta \) is \((p,n)\)-Bergman inner) we need the operators \( M_{S^*} : U \to H^2_{\omega_{p,1},\ell^2(F^+_d)}(F^+_d) \) and \( M_{\Psi_{\gamma_{p,n}}^*} : H^2_{\omega_{p,n},Y}(F^+_d) \to H^2_{\omega_{p,n},Y}(F^+_d) \) to be respectively contractive and coisometric. But we have this by part (2) in Theorem 9.5 and the fact that \( S(z) \) is a contractive multiplier from \( H^2_{\omega_{p,1},U}(F^+_d) \) to \( H^2_{\omega_{p,1},\ell^2(F^+_d)}(F^+_d) \).

### 9.3. Output stability, Stein equations and inequalities

For convenience of future reference, let us collect the following definitions.

**Definition 9.9.** Given \( C \in \mathcal{L}(X,Y) \) and \( A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d \), we say that:

1. \((C, A)\) is \((p,n)\)-output stable if the \((p,n)\)-observability operator \( O_{p,n:C,A} \) defined in (9.12) is bounded from \( X \) into \( H^2_{\omega_{p,n},Y}(F^+_d) \). When this is the case, it makes sense to introduce the \((p,n)\)-observability gramian

\[ G_{p,n:C,A} := O^*_{p,n:C,A} \theta_{p,n:C,A} \in \mathcal{L}(X), \]

whose representation in terms of a strongly convergent series

\[
G_{p,n:C,A} = R_{p,n}(B_A)[C^*C] = \sum_{\alpha \in F^+_d} \omega^{-1}_{p,n,\alpha} A^{*\alpha} C^{*\alpha} \tag{9.39}
\]

follows from (9.6) and (9.16) and suggests to let \( G_{p,0:C,A} = C^*C \).

In addition we say that the output pair \( C, A \) is \((p,n)\)-observable if the observability operator \( O_{p,n:C,A} \) has trivial kernel. If moreover, the observability operator
\( \mathcal{O}_{p,n,C,A} \) is bounded below (equivalently, the graman \( \mathcal{G}_{p,n,C,A} \) is strictly positive definite), we say that \( (C,A) \) is \textit{exactly observable}.

(2) \( A = (A_1, \ldots, A_d) \) is \( p \)-strongly stable if it holds that
\[
\lim_{N \to \infty} p(B_A)^N[I_X] = 0 \quad \text{in the strong operator topology.} \tag{9.40}
\]
Since \( p(B_A)^N[I_X] \geq 0 \) for all \( N \geq 0 \), the convergence in the strong operator topology is equivalent to convergence in the weak operator topology in (9.40).

(3) \( (C,A) \) is \( (p,n) \)-contractive if \( A \) is \( (p,n) \)-hypercontractive (as in Definition 9.11) and in addition \( \Gamma_{p,n;A}[I_X] \geq C^*C \), where the \( (p,n) \)-defect operator \( \Gamma_{p,n;A} \) is given by (9.5).

(4) \( (C,A) \) is \( (p,n) \)-isometric if \( A \) is \( (p,n) \)-hypercontractive and in addition \( \Gamma_{p,n;A}[I_X] = C^*C \).

**Proposition 9.10.** Let \( \Gamma_{p,n;A} \) be the operatorial map defined in (9.5) \( (\text{defined as long as } \|B_A\| < \rho) \). Then for all \( H \in \mathcal{L}(X) \) and any integers \( k \geq 1 \) and \( N \geq 0 \),
\[
\Gamma_{p,k;A}[H] = \Gamma_{p,k-1;A}[H] - p(B_A)[\Gamma_{p,k-1;A}[H]], \tag{9.41}
\]
\[
H = \left( \sum_{j=0}^{N} \binom{k+j-1}{j} p^j(B_A)\Gamma_{p,k;A} + \sum_{j=1}^{k} \binom{N+k}{N+j} p(B_A)^{N+j}\Gamma_{p,k-j;A} \right)[H]. \tag{9.42}
\]

Furthermore, if \( (C,A) \) is \( (p,n) \)-output stable, then it is also \( (p,k) \)-output stable, and the equalities
\[
\Gamma_{p,k;A}[\mathcal{G}_{p,n;C,A}] = \mathcal{G}_{p,n-k;C,A} \quad \text{for } k = 0, 1, \ldots, n. \tag{9.43}
\]
hold, as well as the chain of inequalities:
\[
C^*C \leq \mathcal{G}_{p,1;C,A} \preceq \cdots \preceq \mathcal{G}_{p,n;C,A}. \tag{9.44}
\]

**Proof.** Formula (9.41) follows from substitution of the operator argument \( H \) into the identity
\[
(1 - p)^k = (1 - p)^{k-1} - p(1 - p)^{k-1}. \tag{9.43}
\]

By substituting \( p(B_A) \) for \( B_A \) and \( (1 - p)^k(B_A) \) for \( (I - B_A)^k \) in the derivation of (4.53) we arrive at the following \( p \)-analog of (4.53):
\[
I_{\mathcal{L}(X)} = \sum_{j=0}^{N} \binom{k+j-1}{j} p^j(B_A)(1-p)^k(B_A) + \sum_{j=1}^{k} \binom{N+k}{N+j} p^{N+j}(B_A)(1-p)^{k-j}(B_A). \tag{9.45}
\]
Applying this operator identity to \( H \) leads to (9.42).

To verify (9.43), we shall make use of the following extension of (4.21) which we shall call the **Free Noncommutative Mertens Theorem:**
If the series \( \sum_{\alpha \in \mathcal{F}_d^+} a_\alpha \) converges absolutely and if \( \sum_{\alpha \in \mathcal{F}_d^+} a_\alpha = a \) and \( \sum_{\alpha \in \mathcal{F}_d^+} b_\alpha = b \), then
\[
\sum_{\alpha \in \mathcal{F}_d^+} \left( \sum_{\beta, \gamma \in \mathcal{F}_d^+} a_{\beta} b_{\gamma} \right) = \left( \sum_{\alpha \in \mathcal{F}_d^+} a_\alpha \right) \cdot \left( \sum_{\alpha \in \mathcal{F}_d^+} b_\alpha \right) = a \cdot b. \tag{9.46}
\]

The proof follows by an adaptation of the proof of [93, Theorem 3.50], the details of which we leave to the reader for lack of space.

We apply this Mertens theorem as follows. As \( (C,A) \) is \( (p,n) \)-output stable, the series representation for \( \mathcal{G}_{p,n;C,A} = (1-p)^{-n}(B_A)[C^*C] \) is convergent. By
the assumption that \( \| [A_1 \cdots A_d] \| < \rho \), the series representation for \( \Gamma_{p,k:A} = (1-p)^k(B_A) \) is absolutely convergent. Hence by the free noncommutative general principle (9.45), we can compute

\[
\Gamma_{p,k,A}[\mathcal{G}_{p,n;C,A}] = (1-p)^k(B_A)[(1-p)^-n(B_A)[C^*C]]
= ((1-p)^k \cdot (1-p)^-n) (B_A)[C^*C]
= (1-p)^-(n-k) (B_A)[C^*C] =: \mathcal{G}_{p,n;C,A}.
\]

In particular it falls out that \((C,A)\) being \((p,n)\)-output stable implies that \((C,A)\) is \((p,k)\)-output stable for \(1 \leq k \leq n\). The special case \(k = 1\) in (9.43) tells as that

\[
\mathcal{G}_{p,n-1;C,A} = \Gamma_{p,1;A}[\mathcal{G}_{p,n;C,A}]
= (1-p)(B_A)[\mathcal{G}_{p,n;C,A}]
= \mathcal{G}_{p,n;C,A} - p(B_A)[\mathcal{G}_{p,n;C,A}] \preceq \mathcal{G}_{p,n;C,A}
\]

since \(p(B_A)\) is a positive map. Iteration of this argument then gives us the chain of inequalities (9.44). As in Proposition 4.16 for the special case \((p,n)\)-output stability implies \((p,k)\)-output stability for \(k = 0,1,\ldots,n-1\) can also be seen as a consequence of the chain of inequalities (9.44). Finally one can also see these inequalities as a consequence of verifying directly from the formula (9.43) the coefficient inequalities \(\omega_{p,k-1,\alpha} \preceq \omega_{p,k,\alpha}^{-1}\) for \(k = 1,\ldots,n\). \(\square\)

We next verify that contractivity implies hyper-contractivity for the \((p,n)\)-setting.

**Lemma 9.11.** Let us assume that operators \(H\) and \(A_j\) in \(\mathcal{L}(X)\) are such that

\[
H \succeq p(B_A)[H] \succeq 0 \quad \text{and} \quad \Gamma_{p,n;A}[H] \succeq 0
\]

for some integer \(n \geq 3\). Then \(\Gamma_{p,k;A}[H] \succeq 0\) for all \(k = 1,\ldots,n-1\).

**Proof.** Since the map \(X \mapsto p(B_A)[X]\) is positive, iterating the first condition in (9.46) gives

\[
H \succeq p^j(B_A)[H] \quad \text{for all} \quad j \geq 0.
\]

Then we introduce Hermitian operators

\[
S_{m,k} := p^k(1-p)^m(B_A)[H] \quad \text{for} \quad k \in \mathbb{Z}_+ \quad \text{and} \quad m = 0,1,\ldots,n.
\]

and verify inequalities (4.42) for these operators. Indeed, by (9.47),

\[
S_{m,k} = p^k(1-p)^m(B_A)[H]
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} p^{k+j}(B_A)[H] \preceq \sum_{j=0}^{m} \binom{m}{j} H = 2^m \cdot H,
\]

thus proving the right inequality in (4.42). The left inequality follows by a similar argument using lower estimates. We next observe that due to (9.41) and the second inequality in (9.46),

\[
\Gamma_{p,n-1;A}[H] \succeq p(B_A)[\Gamma_{p,n-1;A}[H]].
\]

Since the map \(X \mapsto p(B_A)[X]\) is positive, we then have

\[
p^j(B_A)[\Gamma_{p,n-1;A}[H]] \succeq p^{j+1}(B_A)[\Gamma_{p,n-1;A}[H]],
\]

which, on account of (9.49) can be written as

\[
S_{n-1,j} \succeq S_{n-1,j+1}.
\]
Also, it follows from definitions (9.49) that for any $N \geq 1$,
\[
\sum_{j=0}^{N} S_{n-1,j} = \sum_{j=0}^{N} p^j (1-p)^{n-1}(B_A)[H] = (1-p^{N+1})(1-p)^{n-2}(B_A)[H] = S_{n-2,0} - S_{n-2,N+1}.
\]
Using the same arguments as in the proof of Lemma 4.14 we conclude that the operator $S_{n-1,0} = (1-p)^{n-1}(B_A)[H] = \Gamma_{p,n-1,A}[H]$ is positive semidefinite. We then obtain recursively that $\Gamma_{p,k,A}[H] \geq 0$ for all $k = 1, \ldots, n-1$. \hfill \Box

Note that for $H = I_X$, the left condition in (9.46) amounts to $I_X \geq p(B_A[I_X])$ or equivalently, to
\[
(1-p)(B_A[I_X]) = \Gamma_{p,1,A}[I_X] \geq 0.
\]
Hence, specializing Lemma 9.11 to the case where $H = I_X$ and recalling Definition 9.1 we arrive at the following result.

**Corollary 9.12.** If the operator tuple $A = (A_1, \ldots, A_d)$ is $(p, 1)$-contractive and $(p, n)$-contractive for some $n \geq 3$, then it is also $(p, n)$-hypercontractive.

We next present the $(p, n)$-analogue of Theorem 4.5 and Theorem 4.18.

**Theorem 9.13.** Let $C \in \mathcal{L}(X,Y)$ and $A = (A_1, \ldots, A_d) \in \mathcal{L}(X)^d$. Then:

1. The pair $(C, A)$ is $(p, n)$-output stable if and only if there exists an operator $H \in \mathcal{L}(X)$ satisfying the inequalities

   \[
   H \succeq p(B_A)[H] \geq 0 \quad \text{and} \quad \Gamma_{p,n,A}[H] \succeq C^*C. \tag{9.50}
   \]

2. If $(C, A)$ is $(p, n)$-output stable, then the observability gramian $H = G_{p,n;A}$ satisfies

   \[
   H \succeq p(B_A)[H] \geq 0 \quad \text{and} \quad \Gamma_{p,n,A}[H] = C^*C \tag{9.51}
   \]

   and is the minimal positive semidefinite solution of the system (9.50).

3. There is a unique solution $H$ of the system (9.51) with $H = G_{p,n;A}$ if $A$ is $p$-strongly stable. Moreover, in case $A$ is $p$-contractive in the sense that $p(B_A[I_X]) \leq I_X$, then the solution of the $(p, n)$-Stein equation in (9.51) is unique if and only if $A$ is $p$-strongly stable.

Note that (9.50) and (9.51) holding with $H = I_X$ is just the definition of the output pair $(C, A)$ being $(p, n)$-contractive or $(p, n)$-isometric, respectively (see Definition 9.9).

**Proof.** If $(C, A)$ is $(p, n)$-output stable, then the series in (9.39) converges in the strong operator topology to the operator $H = G_{p,n;A} \succeq 0$. By formulas (9.5) and (9.43) (with $k = 1$), we have
\[
G_{p,n;A} - p(B_A)[G_{p,n;A}] = (1-p)(B_A)[G_{p,n;A}] = \Gamma_{p,1,A}[G_{p,n;A}] = G_{p,n-1;A} \succeq 0.
\]
By formula (9.43) (for $k = n$), we have
\[
\Gamma_{p,n,A}[G_{p,n;A}] = G_{p,0;A} = C^*C.
\]
Thus, the operator $H = G_{p,n;A} \succeq 0$ satisfies relations (9.51) and hence also inequalities (9.50).
Suppose now that $H$ is any solution of the inequalities (9.50). Let us note that the analog of (4.52) (based on identity (9.42) and Lemma 9.11 rather than identity (4.52) and Lemma 4.14) is

$$
\sum_{j=0}^{N} \left( \frac{n+j-1}{j} \right) p^j (B_A) [C^* C] \preceq \sum_{j=0}^{N} \left( \frac{n+j-1}{j} \right) p^j (B_A) \Gamma_{p,n;A} [H]
$$

$$
= H - \sum_{j=1}^{n} \left( \frac{N+n}{N+j} \right) p^{N+j} (B_A) [\Gamma_{p,n-j;A} [H]] \preceq H. \tag{9.52}
$$

Thus the nondecreasing operator sequence of positive semidefinite operators

$$
S_N = \sum_{j=0}^{N} \left( \frac{n+j-1}{j} \right) p^j (B_A)[C^* C], \quad N = 1, 2, \ldots
$$

is bounded above (by $H$) and hence converges strongly to a positive semidefinite operator. From the computation (9.6) we see that the limit of this series is exactly $R_{p,n}(A) [C^* C] = \mathcal{G}_{p,n;C,A}$. Passing to the limit in (9.52) as $N \to \infty$ now gives $\mathcal{G}_{p,n;C,A} \preceq H$. In particular, $\mathcal{G}_{p,n;C,A}$ is bounded (since $H$ is) and therefore the pair $(C, A)$ is $(p, n)$-output stable. Besides, as $H$ is an arbitrary positive semidefinite solution to the system (9.50), the latter inequality tells us that $\mathcal{G}_{p,n;C,A}$ is the minimal such solution. This completes the proof of parts (1) and (2).

Now suppose that $A$ is $p$-strongly stable, and that $H$ solves the system (9.50). We shall show that necessarily $H = \mathcal{G}_{p,n;C,A}$. We first observe that if positive semidefinite operators $P, Q \in \mathcal{L}(X)$ satisfy the Stein equation

$$
\Gamma_{p,1;A} [P] := P - p (B_A) [P] = Q, \tag{9.53}
$$

then $P$ is uniquely recovered from (9.53) via the strongly convergent series

$$
P = \sum_{j=0}^{\infty} p^j (B_A) [Q]. \tag{9.54}
$$

Indeed, iterating (9.53) gives

$$
P = \sum_{j=0}^{N} p^j (B_A) [Q] + p^{N+1} (B_A) [P], \tag{9.55}
$$

and since all the terms in the latter equality are positive semidefinite, the strong convergence of the series on the right side of (9.55) follows. The $p$-strong stability of $A$ guarantees that $p^{N+1} (B_A) [P]$ tends to zero (strongly) as $N \to \infty$, and then, upon letting $N \to \infty$ in (9.55), we arrive at (9.54). In terms of the operator $B_A$, the latter uniqueness means that that the operator $I_{\mathcal{L}(X)} - p (B_A)$ is invertible with inverse given by

$$
(I_{\mathcal{L}(X)} - p (B_A))^{-1} : Q \mapsto \sum_{j=0}^{\infty} p^j (B_A) [Q].
$$

Taking the $n$-the power of the latter operator and making use of the formula (9.6) we get

$$
(I - p (B_A))^{-n} [Q] = \sum_{j=0}^{\infty} \left( \frac{n+j-1}{j} \right) p^j (B_A) [Q] = \sum_{\alpha \in F_{d}^+} \omega_{p,n;\alpha}^{-1} A^{\alpha \top} Q A^{\alpha}
$$
Next note that the Stein equation in (9.51) can be viewed as the equation

\[(I - p(B_A))^n[H] = C^*C.\]

Hence

\[H = (I - B_A)^{-n}[C^*C] = \sum_{\alpha \in F^+} \omega_{p,n,\alpha} A^{\alpha^\top} Q A^\alpha\]

necessarily is the unique solution. As \(G\) is assumed not to be strongly stable, this limit \(\Delta\) is not zero. Let us assume for a moment that \(\Delta\) is not uniquely determined. As in the proof of Theorem 4.18, we prove the contrapositive: if \(A\) is not p-strongly stable, then the solution of (9.51) is not unique.

It remains to show that if \(A\) is a p-contraction and the system (9.51) admits a unique solution (which necessarily is \(H = G_{n,C,A}\)), then the tuple \(A\) is p-strongly stable. As in the proof of Theorem 4.18 we prove the contrapositive: if \(A\) is not p-strongly stable, then the solution of (9.51) is not unique.

Due to assumption \(p(B_A)[I_X] \leq I_X\), the sequence of operators

\[\Delta_N = p^N(B_A)[I_X] = \sum_{\beta_1, \ldots, \beta_N \neq \emptyset: \beta_1 \cdots \beta_N = \alpha} p_{\beta_1} \cdots p_{\beta_N} A^{\alpha^\top} A^\alpha, \quad N = 1, 2, \ldots\]

(9.56)
is decreasing and therefore has a strong limit \(\Delta = \lim_{N \to \infty} \Delta_N \geq 0\). Since \(A\) is assumed not to be p-strongly stable, this limit \(\Delta\) is not zero. Let us assume for a moment that

\[p(B_A)[\Delta] = \Delta.\]

(9.57)

Then it follows from (9.5) and (9.56) that

\[\Gamma_{p,n;A}[\Delta] = (1 - p)^n(B_A)[\Delta] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} p^j(B_A)[\Delta] = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \Delta = 0,\]

and therefore, the operator \(H = G_{n,c,A} + \Delta\) (as well as \(G_{n,c,A}\)) satisfies the system (9.51) which therefore has more than one positive-semidefinite solution.

It remains to verify (9.57). To this end, we first observe that since \(\Delta \leq \Delta_N\) for all \(N\) and since all the coefficients of \(p\) are non-negative,

\[\sum_{|\alpha| \leq k} p_{\alpha} A^{*\alpha^\top} \Delta A^\alpha \leq p(B_A)[\Delta_N] = \Delta_{N+1}.\]

Letting \(k, N \to \infty\) in the latter inequality we conclude that \(p(B_A)[\Delta]\) is well-defined and satisfies

\[p(B_A)[\Delta] \leq \Delta.\]

(9.58)

We now fix a nonzero \(x \in X\) and \(\varepsilon > 0\). Since the series representing \(p(B_A)[I_X] \leq I_X\) converges, we can find \(k \geq 0\) such that

\[\sum_{\alpha \in F^+_d:|\alpha| > k} p_{\alpha} \|A^\alpha x\|_X^2 < \varepsilon.\]

Then for any operator \(Q \in L(X)\) such that \(0 \leq Q \leq I_X\), we also have

\[\sum_{\alpha \in F^+_d:|\alpha| > k} p_{\alpha} \langle QA^\alpha x, A^\alpha x\rangle_X \leq \sum_{\alpha \in F^+_d:|\alpha| > k} p_{\alpha} \|A^\alpha x\|_X^2 < \varepsilon.\]

(9.59)

Since the sequence \(\Delta_N\) converges to \(\Delta\) (weakly or strongly), for every \(\alpha\), we can find \(n_\alpha\) such that

\[p_{\alpha} \cdot \langle (\Delta_n - \Delta) A^\alpha x, A^\alpha x\rangle_X < d^{-k} \varepsilon \quad \text{for all} \quad n > n_\alpha.\]
Then for any $n > \max(n_\alpha : |\alpha| \leq k)$, we have

\[ \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq k} p_\alpha \cdot \langle (\Delta_n - \Delta)A^\alpha x, A^\alpha x \rangle_x < \varepsilon. \]  

(9.60)

Combining (9.59) (with $Q = \Delta_n - \Delta$) and (9.60) gives, for the same $n$ as above,

\[ \langle p(B_A)[\Delta_n - \Delta]x, x \rangle_x = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq k} p_\alpha \cdot \langle (\Delta_n - \Delta)A^\alpha x, A^\alpha x \rangle_x + \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha| > k} p_\alpha \cdot \langle (\Delta_n - \Delta)A^\alpha x, A^\alpha x \rangle_x < 2\varepsilon. \]

Therefore, for any fixed $x \in X$ and any $\varepsilon > 0$, there is $n$ such that

\[ \langle \Delta x, x \rangle \leq \langle \Delta_{n+1} x, x \rangle = \langle p(B_A)[\Delta_n]x, x \rangle = \langle p(B_A)[\Delta]x, x \rangle + \langle p(B_A)[\Delta_n - \Delta]x, x \rangle \leq \langle p(B_A)[\Delta]x, x \rangle + 2\varepsilon. \]

Letting $\varepsilon \to 0^+$ in the latter inequality we get $\langle \Delta x, x \rangle \leq \langle p(B_A)[\Delta]x, x \rangle$. By polarization, $\Delta \leq p(B_A)[\Delta]$, which together with (9.58) implies (9.57), thus completing the proof of the theorem. \hfill \Box

The following analog of Theorem 4.18 is a direct consequence of Definition 9.9 (part (3)) applied to the statements in Theorem 9.13 with $H = I_X$.

**Proposition 9.14.** (1) Suppose that $(C, A)$ is a $(p, n)$-contractive pair. Then $(C, A)$ is $(p, n)$-output-stable with $\mathcal{G}_{p,n,C,A} \preceq I_X$ and the gramian $\mathcal{G}_{p,n,C,A}$ is the unique positive semidefinite solution of the system (9.51) if and only if $A$ is $p$-strongly stable.

(2) Suppose that $(C, A)$ is a $(p, n)$-isometric pair. Then $(C, A)$ is $(p, n)$-output-stable. Moreover $H = I_X$ is the unique solution of the system (9.51) if and only if $A$ is $p$-strongly stable. In this case $\mathcal{O}_{p,n,C,A}$ is isometric and hence also $(C, A)$ is exactly $(p, n)$-observable.

### 9.4. The $\omega_{p,n}$-shift model operator tuple $S_{\omega_{p,n},R}$

We define the shift operator tuple $S_{\omega_{p,n},R}$ on $H^2_{\omega_{p,n},H}(\mathbb{F}_d^+)$ as in (2.29) and observe that it is not a row-contraction, in general. However, it is a $(p, 1)$-contraction (see Proposition 9.11 below), i.e., $\sum_{\alpha \in \mathbb{F}_d^+} p_\alpha S^\alpha_{\omega_{p,n},R} \mathbf{S}_{\omega_{p,n},R}^\alpha \preceq I$. In particular, $p_j S_{\omega_{p,n},R,j} S_{\omega_{p,n},R,j}^* \preceq I$ where $p_j$ is the coefficient of $z_j$ in the formal power series representation (9.1), and therefore,

\[ \|S_{\omega_{p,n},R,j}\| \leq p_j^{1/2} < \infty \quad \text{for} \quad 1 \leq j \leq d. \]

The formula for $S^\alpha_{\omega_{p,n},R,j}$ is again given by (2.30)

\[ S^\alpha_{\omega_{p,n},R,j} : \sum_{\gamma \in \mathbb{F}_d^+} f_\gamma z^\gamma \mapsto \sum_{\gamma \in \mathbb{F}_d^+} \frac{\omega_{p,n,\gamma}}{\omega_{p,n,\gamma}} f_{\gamma j} z^\gamma \quad \text{for} \quad j = 1, \ldots, d \]  

(9.61)

(with $\omega_{p,n}$ in place of $\omega$) while its iteration gives

\[ (S^\alpha_{\omega_{p,n},R})^\alpha : \sum_{\gamma \in \mathbb{F}_d^+} f_\gamma z^\gamma \mapsto \sum_{\gamma \in \mathbb{F}_d^+} \frac{\omega_{p,n,\gamma \alpha}}{\omega_{p,n,\gamma}} f_{\gamma \alpha} z^\gamma \quad \text{for all} \quad \alpha \in \mathbb{F}_d^+. \]  

(9.62)
PROPOSITION 9.15. The operator tuple $S_{\omega,p,n,R}^* = (S_{\omega,p,n,R,1}^*, \ldots, S_{\omega,p,n,R,d}^*)$ is a $p$-strongly stable $(p,n)$-hypercontraction. Furthermore, the model pair $(E, S_{\omega,p,n,R})$ is $(p,n)$-isometric, where $E$ is the operator defined by $Ef = f_0$ for $f \in H^2_{\omega,p,n}(\mathbb{F}_d^+)$. 

PROOF. Since $p_\beta \geq 0$ for all $\beta \in \mathbb{F}_d^+$, we have for any $\alpha, \gamma \in \mathbb{F}_d^+$, and $N \leq |\alpha|$, 

$$
\sum_{j=0}^{|\gamma|} \left( \sum_{\phi_1, \ldots, \phi_j \neq \emptyset: \phi_1 \cdots \phi_j = \gamma_n} p_{\phi_1} \cdots p_{\phi_j} \right) \geq 0.
$$

Note that this condition may be viewed as the more general $p$-version of the inequality $1 \leq \frac{\omega_{p,n,\gamma}}{\omega_{p,n,\gamma+1}}$ appearing in assumption (2.31) for the weighted case discussed in Chapter 2.2.

For an arbitrary element $f \in H^2_{\omega,p,n}(\mathbb{F}_d^+)$, the series representing $\|f\|_2^2$ converges and hence, 

$$
\lim_{N \to \infty} \sum_{\gamma \in \mathbb{F}_d^+: |\gamma| \geq N} \omega_{p,n,\gamma} \|f_\gamma\|_Y^2 = 0, \quad \text{if } f(z) = \sum_{\gamma \in \mathbb{F}_d^+} f_\gamma z^\gamma.
$$

Making use of (9.62), (9.16) and (9.63), we have 

$$
\left\langle p^N(Bs_{\omega,p,n,R})[I_{H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)}, f] p^N(Bs_{\omega,p,n,R}) \right\rangle = \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{\beta_1, \ldots, \beta_N \neq \emptyset: \beta_1 \cdots \beta_N = \alpha} p_{\beta_1} \cdots p_{\beta_N} \right) \|S_{\omega,p,n,R}^{*\alpha}\|_{H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)}^2 \
\leq \sum_{\alpha, \gamma \in \mathbb{F}_d^+: |\gamma| \geq N} \omega_{p,n,\gamma} \|f_\gamma\|_Y^2 \
\leq \sum_{\alpha, \gamma \in \mathbb{F}_d^+: |\gamma| \geq N} \omega_{p,n,\gamma} \|f_\gamma\|_Y^2 \
\leq \sum_{\beta \in \mathbb{F}_d^+: |\beta| \geq N} \omega_{p,n,\beta} \|f_\beta\|_Y^2.
$$

Combining the latter estimate with (9.64), we conclude that $p^N(Bs_{\omega,p,n,R})[I_{H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)}, f]$ tends to zero in the strong operator topology meaning that the operator tuple $S_{\omega,p,n,R}^*$ is $p$-strongly stable.
We next make use of (9.5) and (9.62) to compute for the same $f$ as in (9.64),
\[
\left\langle \Gamma_{p,k}S_{\omega_{p,n,R}}^*[I_{H_{p,n}}^2](f)H_{p,n}^2(F_d^+)f, f \right\rangle_{H_{p,n}^2(F_d^+)} = \sum_{\alpha \in F_d^+} c_{p,k;\alpha} \| S_{\omega^{\alpha}}^*[f]H_{p,n}^2(F_d^+) \|
\]
\[
= \sum_{\alpha \in F_d^+} c_{p,k;\alpha} \sum_{\gamma \in F_d^+} \omega_{p,n;\gamma}^2 \| f_{\gamma} \|^2_{Y^2}
\]
\[
= \sum_{\alpha, \beta \in F_d^+: \beta \alpha = \gamma} c_{p,k;\alpha} \omega_{p,n;\beta}^{-1} \omega_{p,n;\gamma}^{-1} \| f_{\gamma} \|^2_{Y^2}. \tag{9.65}
\]
for all $f \in H_{p,n}^2(F_d^+)$ and $k = 1, \ldots, n$.

Equating the coefficients of $z^\gamma$ in the power series identity
\[(1 - p(z))^{-n} \prod_{i = 1}^{n-k} (1 - p(z))^{-1} = (1 - p(z))^{-k}
\]
for a fixed $k \in \{1, \ldots, n\}$, we conclude, on account of (9.64), (9.60) and (9.16) that
\[
\sum_{\alpha, \beta \in F_d^+: \beta \alpha = \gamma} c_{p,k;\alpha} \omega_{p,n;\beta}^{-1} = \omega_{p,n;\gamma}^{-1} \quad \text{for} \quad k = 0, \ldots, n - 1,
\]
and that
\[
\sum_{\alpha, \beta \in F_d^+: \beta \alpha = \gamma} \omega_{p,n;\beta}^{-1} = \left\{ \begin{array}{ll}
1 & \text{if } |\gamma| = 0, \\
0 & \text{if } |\gamma| > 0.
\end{array} \right.
\]
Combining the two latter formulas with (9.65) leads us to equalities
\[
\left\langle \Gamma_{p,k}S_{\omega_{p,n,R}}^*[I_{H_{p,n}}^2](f)H_{p,n}^2(F_d^+)f, f \right\rangle_{H_{p,n}^2(F_d^+)} = \sum_{\gamma \in F_d^+} \frac{\omega_{p,n;\gamma}^2}{\omega_{p,n;\gamma}^{-1}} \| f_{\gamma} \|^2_{Y^2}
\]
for all $k = 1, \ldots, n - 1$, and also
\[
\left\langle \Gamma_{p,n}S_{\omega_{p,n,R}}^*[I_{H_{p,n}}^2](f)H_{p,n}^2(F_d^+)f, f \right\rangle_{H_{p,n}^2(F_d^+)} = \| f_{0} \|^2_{Y^2} = \| Ef \|^2_{Y^2}. \tag{9.66}
\]
Since the latter relations hold for all $f \in H_{p,n}^2(F_d^+)$, it follows that
\[
\Gamma_{p,k}S_{\omega_{p,n,R}}^*[I_{H_{p,n}}^2](f)H_{p,n}^2(F_d^+) \geq 0 \quad \text{for all} \quad k = 1, \ldots, n,
\]
and in addition, $\Gamma_{p,n}S_{\omega_{p,n,R}}^*[I_{H_{p,n}}^2](f)H_{p,n}^2(F_d^+) = E^*E$. Hence, the tuple $S_{\omega_{p,n,R}}^*$ is $(p, n)$-hypercontractive and the pair $(E, S_{\omega_{p,n,R}}^*)$ is $(p, n)$-isometric.

**Remark 9.16.** The results of Proposition 9.15 tell us that $S_{\omega_{p,n,R}}^*$ is $p$-strongly stable and that the output-pair $(E, S_{\omega_{p,n,R}}^*)$ is $(p, n)$-isometric. Hence, as a consequence of part (2) of Proposition 9.14 it follows that $O_{p,n;E;S_{\omega_{p,n,R}}^*}$ is isometric, i.e.,
\[
(O_{p,n;E;S_{\omega_{p,n,R}}^*})^*O_{p,n;E;S_{\omega_{p,n,R}}^*} = I_{H_{p,n}^2(F_d^+)^*}.
\]

\(^1\)If $f$ is a polynomial, then the sums in (9.65) are finite, which justifies rearrangements used in that calculation. Then the general case follows by approximation arguments.
9.5. Observability operator range spaces in $H^2_{\omega,p,n}(\mathbb{F}_d^+)$

In fact, as a simple consequence of the formula (9.62) for $(S_{\omega,p,n,R})^\alpha$ and the definition (9.12) of $\mathcal{O}_{p,n;C,A}$ (applied with $(C,A) = (E,S^*_{p,n,R})$, one can see that in fact already

$$\mathcal{O}_{p,n;E,S^*_{p,n,R}} = I_{H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)}.$$  

Specifying Proposition 9.15 to the linear case we arrive at the following result.

**Corollary 9.17.** For $\tilde{p}(z) = p_1 z_1 + \cdots + p_d z_d$ with $p_1, \ldots, p_d > 0$, the operator tuple $S^*_{\omega,p,1,R}$ is a $\tilde{p}$-strongly stable $(\tilde{p},1)$-contraction on $H^2_{\omega,p,1,Y}(\mathbb{F}_d^+)$. Furthermore, the model pair $(E,S^*_{p,R})$ is a $(\tilde{p},1)$-isometric.

9.5. Observability operator range spaces in $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$

Although it is possible to work out a $(p,n)$-analogue of Theorem 9.18 for the sake of simplicity we present here only an abridged version of its simplified version (the $(p,n)$-analogue of Theorem 9.15) along with the $(p,n)$-version of the converse statement (part (4)) in Theorem 9.18 as well as Theorem 9.16.

**Theorem 9.18.** Let $(C,A)$ be a $(p,n)$-contractive pair with state space $\mathcal{X}$ and output space $\mathcal{Y}$. Then:

1. $(C,A)$ is a $(p,n)$-output-stable and the intertwining relation

$$S^*_{p,n,R,j}\mathcal{O}_{p,n;C,A}x = \mathcal{O}_{p,n;C,A}A_jx \quad (x \in \mathcal{X})$$

holds for all $j = 1, \ldots, d$. Hence $\text{Ran} \mathcal{O}_{p,n;C,A}$ is $S^*_{p,n,R}$-invariant.

2. The operator $\mathcal{O}_{p,n;C,A}$ is a contraction from $\mathcal{X}$ into $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$. Moreover, $\mathcal{O}_{p,n;C,A}$ is isometric if and only if $(C,A)$ is a $(p,n)$-isometric pair and $A$ is p-strongly stable.

3. If the linear manifold $\mathcal{M} := \text{Ran} \mathcal{O}_{p,n;C,A}$ is given the lifted norm

$$\|\mathcal{O}_{p,n;C,A}x\|_\mathcal{M} = \|Qx\|_\mathcal{X}$$

where $Q$ is the orthogonal projection of $\mathcal{X}$ onto $(\text{Ker} \mathcal{O}_{p,n;C,A})^\perp$, then:

a. $\mathcal{O}_{p,n;C,A}$ is a coisometry from $\mathcal{X}$ onto $\mathcal{M}$ and implements a unitary equivalence between $S^*_{p,n,R}|_{\mathcal{M}}$ and $QA|_{\text{Ran} Q}$.

b. $\mathcal{M}$ is contained contractively in $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$ $\mathcal{M}$ is isometrically equal to the FNRKHS $H(K_{p,n;C,A})$ with reproducing kernel $K_{p,n;C,A}(z,\zeta)$ given by $K_{p,n;C,A}(z,\zeta) = CR_{p,n}(Z(z)A)R_{p,n}(Z(\zeta)A)^*C^*$.  

(c) The pair $(E|_{\mathcal{M}},S^*_{p,n,R}|_{\mathcal{M}})$ is a $(p,n)$-contractive output pair, or explicitly (in view of Corollary 9.12), $S^*_{p,n,R}|_{\mathcal{M}}$ is $(p,1)$-contractive on $\mathcal{M}$, i.e.,

$$\langle (\Gamma_{p,1}S^*_{p,n,R}|_{\mathcal{M}})[I_{\mathcal{M}}]f,f\rangle_{\mathcal{M}} = \|f\|_{\mathcal{M}}^2 - \sum_{\alpha \in F_d^+} p_\alpha \cdot \|S^*_{p,n,R}f\|_{\mathcal{M}}^2 \geq 0, \quad \text{for all } f \in \mathcal{M}.$$  

Moreover,

$$\langle (\Gamma_{p,n}S^*_{p,n,R}|_{\mathcal{M}})[I_{\mathcal{M}}] - E^*E \rangle f,f\rangle_{\mathcal{M}} = \sum_{\alpha \in F_d^+} \epsilon_{p,n,\alpha} \cdot \|S^*_{p,n,R}f\|_{\mathcal{M}}^2 - \|f\|_{\mathcal{Y}}^2 \geq 0$$

(9.70)

for all $f \in \mathcal{M}$. Moreover, (9.70) holds with equality if and only the orthogonal projection $Q$ of $\mathcal{X}$ onto $(\text{Ker} \mathcal{O}_{p,n;C,A})^\perp$ is subject to relations

$$Q \succeq \sum_{\alpha \in F_d^+} p_\alpha A^{*\alpha} \cdot Q A^\alpha \quad \text{and} \quad \Gamma_{p,n;A}[Q] = C^*C.$$
(d) In particular, if $(C, A)$ is observable, then $\mathcal{O}_{p,n,C,A}$ implements a unitary equivalence between $A$ and $S_{p,n,R}^*|\mathcal{M}$. Furthermore, \((9.70)\) holds with equality if and only if $(C, A)$ is a $(p, n)$-isometric pair.

(e) If $A$ is $p$-strongly stable and $(C, A)$ is $(p, n)$-isometric, then $\mathcal{O}_{p,n,C,A}$ is isometric (and hence in particular $(C, A)$ is $(p, n)$-observable). Hence $\mathcal{M}$ is contained in $H^2_{p,n,Y}(\mathbb{F}_d^+)$ isometrically and $\mathcal{O}_{p,n,C,A}$ implements a unitary equivalence between $A$ and $S_{p,n,R}^*|\mathcal{M}$, where $\mathcal{M}$ is the orthogonal complement of a subspace $\mathcal{N} = \mathcal{M}^\perp$ isometrically included in $H^2_{p,n,Y}(\mathbb{F}_d^+)$ which is invariant under the forward shift tuple $S_{p,n,R}$.

(4) Conversely, let $\mathcal{M}$ be a Hilbert space contractively included in $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$ (not necessarily isometrically or even contractively) such that

(i) $\mathcal{M}$ is invariant under the backward shift $S_{\omega,p,n,R}^*$,

(ii) $(E|\mathcal{M}, S_{\omega,p,n,R}^*|\mathcal{M})$ is a $(p, n)$-contractive output pair.

Then it follows that $\mathcal{M}$ is contractively included in $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$ and there exists an $(n, p)$-contractive pair $(C, A)$ so that $\mathcal{M} = \mathcal{H}(K_{p,n,C,A})$ with $K_{p,n,C,A}$ as in \((9.69)\).

If $\mathcal{M}$ is isometrically included in $H^2_{p,n,Y}(\mathbb{F}_d^+)$ and invariant under $S_{p,n,R}$, then $\mathcal{M}$ has the form $\mathcal{M} = \text{Ran} \mathcal{O}_{p,n,C,A}$ with $(C, A)$ a $(p, n)$-isometric output pair and $A$ $p$-strongly stable. In fact, one can take $(C, A) = (E|\mathcal{M}, S_{p,n,R}^*|\mathcal{M})$.

**Proof.** With regard to statement (1), making use of the power series expansion \((9.12)\) and of formula \((9.61)\) to get for any fixed $x \in \mathcal{X}$,

$$ S_{\omega,p,n,R,j}^* \mathcal{O}_{p,n,C,A} x = S_{\omega,p,n,R,j}^* \left( \sum_{\gamma \in \mathbb{F}_d^+} (\omega_{p,n,\gamma}^{-1} C A \gamma_j x) \ z^{\gamma} \right) $$

$$ = \sum_{\gamma \in \mathbb{F}_d^+} \left( \omega_{p,n,\gamma} \cdot \omega_{p,n,\gamma}^{-1} C A \gamma_j x \right) z^{\gamma} $$

$$ = \sum_{\gamma \in \mathbb{F}_d^+} \omega_{p,n,\gamma}^{-1} (C A \gamma_j x) z^{\gamma} = \mathcal{O}_{p,n,C,A} A_j x, $$

and \((9.67)\) follows. Statement (2) is a direct consequence of Proposition \((9.14)\).

Statement (3a) is a direct consequence of the intertwining relation \((9.67)\) and the definition of the $\mathcal{M}$-norm \((9.68)\).

Statement (3b) is a consequence of the definition of the norm \((9.68)\); the calculation of the associated kernel \((9.69)\) follows from Proposition \((9.25)\).

Statement (3c) is a consequence of the assumed condition that $(C, A)$ is a $(p, n)$-contractive pair and the definition \((4.117)\) of the $\mathcal{M}$-norm, analogous to the proof of (3b) in Theorem \((4.33)\).

Statement (3d) is just the specialization (3a) to the case where $C = I_X$.

Statement (3e) is a consequence of statement (2) in Proposition \((9.14)\).

Statement (4) can be seen as a consequence of the previous statements combined with the observation of Remark \((9.10)\) that $\mathcal{O}_{E,S_{p,n,R}^*} = I_{H^2_{p,n,Y}(\mathbb{F}_d^+)}$.

### 9.6. Beurling-Lax theorems: $(p, n)$-versions

In this section we present Beurling-Lax representation theorems for shift-invariant subspaces of the space $H^2_{\omega,p,n,Y}(\mathbb{F}_d^+)$ which can be considered as $(p, n)$-versions of
the theorems presented in Sections 5.1 and 6.1. We start with the Beurling-Lax representation for contractively included shift-invariant subspaces.

**Theorem 9.19.** Let \( \tilde{p} \) be the linear part of a regular noncommutative formal power series \( p \) as in (9.1), (9.2). A Hilbert space \( \mathcal{M} \) is such that

1. \( \mathcal{M} \) is contractively included in \( H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d) \),
2. \( \mathcal{M} \) is \( S_{\omega_{p,n}, \mathcal{R}} \)-invariant,
3. the \( d \)-tuple \( \mathbf{T} = (T_1, \ldots, T_d) \) where \( T_j = S_{\omega_{p,n,R,j}}|_{\mathcal{M}} \) for \( j = 1, \ldots, d \), is \((\tilde{p}, 1)\)-contractive if and only if there is a coefficient Hilbert space \( \mathcal{U} \) and a contractive multiplier \( \Theta \) from \( H_{\omega_{\tilde{p},1}, \mathcal{M}}^{2}(\mathbb{F}^+_d) \) to \( H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d) \) so that

\[
\mathcal{M} = \Theta \cdot H_{\omega_{\tilde{p},1}, \mathcal{U}}^{2}(\mathbb{F}^+_d), \quad \text{with lifted norm} \quad \|\Theta \cdot f\|_{\mathcal{M}} = \|Qf\|_{H_{\omega_{\tilde{p},1}, \mathcal{U}}^{2}(\mathbb{F}^+_d)} \tag{9.71}
\]

where \( Q \) is the orthogonal projection onto \( \ker \Theta_{\mathcal{M}} \).

If \( \mathcal{M} \) is represented as a pullback space \( \mathcal{M} = \mathcal{H}^{p}(\Pi) \) for some positive semi-definite operator \( \Pi \in \mathcal{L}(H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d)) \), then conditions (1), (2), (3) above can be expressed more succinctly simply as

\[
0 \leq \Pi \leq I_{H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d)}, \quad \Pi - \sum_{j=1}^{d} p_j S_{\omega_{p,n,R,j}} M_{\mathcal{S}_{\omega_{p,n,R,j}}^{\times}} \geq 0. \tag{9.72}
\]

**Proof.** If \( \Theta \) is a contractive multiplier from \( H_{\omega_{\tilde{p},1}, \mathcal{U}}^{2}(\mathbb{F}^+_d) \) to \( H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d) \) and \( \mathcal{M} \) is of the form (9.71), then it follows as in (5.5) that \( \|\Theta f\|_{H_{\omega_{\tilde{p},1}, \mathcal{U}}^{2}(\mathbb{F}^+_d)} \leq \|\Theta f\|_{\mathcal{M}} \) which verifies property (1). Property (2) follows from the intertwining equalities

\[
S_{\omega_{p,n,R,j}} M_{\Theta} = M_{\Theta} S_{\omega_{p,n,R,j}}
\]

where \( M_{\Theta} \) is multiplication by \( \Theta \) on the left. Furthermore, the formulas (5.6) and (5.7) still hold true (with \( S_{\tilde{p},R} \) instead of \( S_{1,R} \)). Since \( Q \) is a projection and the tuple \( S_{\tilde{p},R} \) is \((\tilde{p}, 1)\)-contractive on \( H_{\tilde{p}, \mathcal{U}}^{2}(\mathbb{F}^+_d) \) (by Corollary 9.17), we can mimic the computation (5.8) to get

\[
\sum_{j=1}^{d} p_j \|T_j^{*} M_{\Theta} f\|_{\mathcal{M}}^{2} = \sum_{j=1}^{d} p_j \|Q S_{\omega_{\tilde{p}, R,j}} f\|_{H_{\tilde{p}, \mathcal{U}}^{2}(\mathbb{F}^+_d)}^{2} \leq \|Qf\|_{H_{\tilde{p}, \mathcal{U}}^{2}(\mathbb{F}^+_d)}^{2} = \|M_{\Theta} f\|_{\mathcal{M}}^{2},
\]

which shows that \( \mathbf{T} \) is \((\tilde{p}, 1)\)-contractive on \( \mathcal{M} \) and completes the proof of sufficiency.

Conversely, let us assume that the Hilbert space \( \mathcal{M} \) satisfies conditions (1), (2), (3), i.e. (on account of Theorem 3.4), \( \mathcal{M} \) is a NFRKHS contractively included in \( H_{\omega_{p,n}, \mathcal{Y}}^{2}(\mathbb{F}^+_d) \), which is invariant under the right coordinate multipliers \( T_j = S_{\omega_{R,j}}|_{\mathcal{M}} \) for \( 1 \leq j \leq d \) and moreover, the tuple \( \mathbf{T} = (T_1, \ldots, T_d) \) is \((\tilde{p}, 1)\)-contractive:

\[
\Gamma_{\tilde{p},1, \mathbf{T}} \cdot [I_{\mathcal{M}}] \geq 0.
\]

Let \( k_{\mathcal{M}}(z, \zeta) \) denote the reproducing kernel for \( \mathcal{M} \). Computations similar to those in the proof of Proposition 2.8 show that the inequality

\[
\left\langle \left( \Gamma_{\tilde{p},1, \mathbf{T}} \cdot [I_{\mathcal{M}}] \otimes I_{C(\mathbb{T})} \right) k_{\mathcal{M}}(\cdot, \zeta) \right\rangle_{\mathcal{M}_1(\zeta) \times \mathcal{M}_1(\bar{\zeta})} \geq 0
\]

holding for all \( y, y' \in \mathcal{Y} \) is equivalent to the kernel

\[
L(z, \zeta) := k_{\mathcal{M}}(z, \zeta) - \sum_{j=1}^{d} p_j \overline{z_j} k_{\mathcal{M}}(z, \zeta) z_j \tag{9.73}
\]
being positive, that is, to having a Kolmogorov decomposition \( L(z, \zeta) = G(z)G(\zeta)^* \) for some \( G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle z \rangle) \). Write \((9.73)\) in the form

\[
 k_M(z, \zeta) = G(z)G(\zeta)^* + \sum_{j=1}^{d} p_j \bar{\zeta}_j k_M(z, \zeta) z_j
\]

and then iterate the latter identity while making use of the notation \((9.18)\) to arrive at

\[
 k_M(z, \zeta) = \sum_{\alpha: |\alpha| \leq N} d_{p,\alpha} \bar{\zeta}_\alpha^T G(z)G(\zeta)^* z_\alpha + \sum_{\alpha: |\alpha| = N+1} d_{p,\alpha} \bar{\zeta}_\alpha^T k_M(z, \zeta) z_\alpha
\]

for all \( N = 0, 1, 2, \ldots \). Taking the limit as \( N \to \infty \) then gives us

\[
 k_M(z, \zeta) = \sum_{\alpha \in \mathbb{F}_d^+} d_{p,\alpha} \bar{\zeta}_\alpha^T G(z)G(\zeta)^* z_\alpha
\]

\[
 = G(z) \left( \sum_{\alpha \in \mathbb{F}_d^+} d_{p,\alpha} z_\alpha^* \bar{\zeta}_\alpha^T \right) G(\zeta)^* = G(z)k_p(z, \zeta)G(\zeta)^*.
\]

By part (2) in Proposition \(9.22\), the latter identity tells us that \( G \) is a coisometric multiplier from \( H^2_{\mathcal{U}, \mathcal{Y}}(\mathbb{F}_d^+) \) onto \( \mathcal{M} \). The rest of the proof is the same as in Theorem \(5.1\).

We consider \( G \) as a multiplier from \( H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) \) into \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \). Since the inclusion map \( \iota: \mathcal{M} \to H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \) is a contraction (by condition (1)), we have \( ||M_\Theta|| = ||\iota M_G|| \leq 1 \), i.e., \( \Theta \) is a contractive multiplier from \( H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) \) to \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \). Moreover, as \( M_G H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) = \mathcal{M} \), we have \( \Theta H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) = \mathcal{M} \).

The fact that \( M_G: H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) \to \mathcal{M} \) is a coisometry can be interpreted as saying that the \( \mathcal{M} \)-norm is given by \((9.71)\).

Finally, the reformulation of conditions (1), (2), (3) in the form \((9.72)\) follows via a straightforward \( p \)-adaptation of the argument in Theorem \(5.4\). \( \square \)

We next derive from Theorem \(9.19\) the \((p, n)\)-analog of Theorem \(9.3\) on isometrically included shift invariant subspaces \( \mathcal{M} \) of \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \).

**Theorem 9.20.** A Hilbert space \( \mathcal{M} \) is isometrically included in \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \) and is \( \mathbf{S}_{\omega_{p,n}, R} \)-invariant if and only if there is a Hilbert space \( \mathcal{U} \) and a McCT-inner multiplier \( \Theta \) from \( H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+) \) to \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \) so that

\[
 \mathcal{M} = \Theta \cdot H^2_{\omega_{p,1}, \mathcal{U}}(\mathbb{F}_d^+).
\]

**Proof.** As the ”if” direction is immediate, it suffices to consider only the ”only if” direction. As \( \mathcal{M} \) is assumed to be isometrically included in \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \), a Beurling-Lax representation of the form \((9.74)\) with \( \Theta \) a contractive multiplier forces \( \Theta \) to actually be a McCT-inner multiplier. Thus it remains only to verify condition (3) in Theorem \(9.19\). Toward this end we note that

\[
 T_j^* = (S_{\omega_{p,n}, R_j}|_{\mathcal{M}})^* = P_M S_{\omega_{p,n}, R_j}|_{\mathcal{M}} \quad \text{for} \quad j = 1, \ldots, d,
\]

for the adjoints of \( T_j = S_{\omega_{p,n}, R_j}|_{\mathcal{M}} \) (which holds true if \( \mathcal{M} \) is a closed subspace of \( H^2_{\omega_{p,n}, \mathcal{Y}}(\mathbb{F}_d^+) \)) and use the fact that the tuple \( \mathbf{S}_{\omega_{p,n}, R} \) is \((p, 1)\)-contractive to
compute
\[
\sum_{j=1}^{d} p_j \| T_j f \|_{H_d}^2 = \sum_{j=1}^{d} p_j \| \mathcal{P}_M(S_{\omega,R,j})^* f \|_{H_{\omega_{\tilde{p}},\gamma}(\mathbb{F}_d^+)}^2
\]
\[
\leq \sum_{j=1}^{d} p_j \| (S_{\omega,R,j})^* f \|_{H_{\omega_{\tilde{p}},\gamma}(\mathbb{F}_d^+)}^2
\]
\[
\leq \sum_{\alpha \in \mathbb{F}_d^+} p_\alpha \| (S_{\omega,R,j})^* \alpha f \|_{H_{\omega_{\tilde{p}},\gamma}(\mathbb{F}_d^+)}^2 \leq \| f \|_{H_{\omega_{\tilde{p}},\gamma}(\mathbb{F}_d^+)}^2
\]
which confirms that condition (3) in the statement of Theorem 5.1 holds and we indeed have (5.12). Furthermore, the fact that the operator \( M_\Theta \) in the proof of Theorem 9.19 is a coisometry translates to \( M_\Theta \) is a partial isometry, i.e., \( \Theta \) is a McCT-inner multiplier from \( H^2_{\omega_{\tilde{p}},1}(\mathbb{F}_d^+) \) to \( H^2_{\omega_{\tilde{p}},Y}(\mathbb{F}_d^+) \).

To get more explicit formulas for the McCT-inner multiplier \( \Theta \) in Theorem 9.20, we use the weight \( \gamma_{p,n} = \{ \gamma_{p,n,\alpha} \} \) constructed from \( p(z) \) via formulas (9.29), the time-domain \( \gamma \)-observability operator \( \hat{\mathcal{O}}_{\gamma_{p,n};C,A}: \mathcal{X} \to \ell^2_{\beta}(\mathbb{F}_d^+) \) defined by
\[
\hat{\mathcal{O}}_{\gamma_{p,n};C,A}: x \mapsto \{ \gamma_{p,n,\alpha} C \alpha^* x \}_{\alpha \in \mathbb{F}_d^+},
\]
and the \( \gamma \)-gramian
\[
\mathcal{G}_{\gamma_{p,n};C,A} = \hat{\mathcal{O}}_{\gamma_{p,n};C,A}^* \hat{\mathcal{O}}_{\gamma_{p,n};C,A} = \sum_{\alpha \in \mathbb{F}_d^+} \gamma_{p,n,\alpha}^{-1} C^* C \alpha^*.
\]
Straightforward power series computation based on formulas (9.12), (9.76) and (9.29) verifies the Stein equation
\[
\mathcal{G}_{p,n;C,A} - \sum_{j=1}^{d} p_j A_j^* \mathcal{G}_{p,n;C,A} A_j = \mathcal{G}_{\gamma_{p,n};C,A}.
\]
We next apply the operator-valued power series (9.31) to the operator \( \hat{\mathcal{O}}_{\gamma_{p,n};C,A} \) and show that, in terms of notation from (9.12) and (9.23),
\[
\Psi_{\gamma_{p,n}}(z) \hat{\mathcal{O}}_{\gamma_{p,n};C,A} = C R_{p,n}(zA)(I - Z_{\tilde{p}}(z)A).
\]
Indeed, from the explicit formulas (9.31), (9.75), we have, on account of (9.29),
\[
\Psi_{\gamma_{p,n}}(z) \hat{\mathcal{O}}_{\gamma_{p,n};C,A} = \sum_{\alpha \in \mathbb{F}_d^+} \gamma_{p,n,\alpha}^{-1} C \alpha^* z^\alpha
\]
\[
= C + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \gamma_{p,n,\alpha}^{-1} C \alpha^* z^\alpha
\]
\[
= C + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} (\omega_{p,n,\alpha j} - \omega_{p,n,\alpha j}^{-1}) C \alpha^* z^\alpha
\]
Then conditions (i) and (ii) in part (4) of Theorem 9.18, and hence has a representation for some $(H)$ of the Cholesky factorization problem $S$ from which (9.77) follows, since

$$C + \sum_{j=1}^{d} \sum_{\alpha \in F_d^+} \omega_{p,n;\alpha} C^{\alpha} \tilde{z}^{\alpha} = \sum_{\alpha \in F_d^+} \omega_{p,n;\alpha} C^{\alpha} \tilde{z}^{\alpha} = CR_{p,n}(zA),$$

$$\sum_{j=1}^{d} \sum_{\alpha \in F_d^+} \omega_{p,n;\alpha} p_j C^{\alpha} \tilde{z}^{\alpha} = \left( \sum_{\alpha \in F_d^+} \omega_{p,n;\alpha} C^{\alpha} \tilde{z}^{\alpha} \right) \left( \sum_{j=1}^{d} p_j A_j \tilde{z}_j \right) = CR_{p,n}(zA)Z_p(z)A.$$

We now present the $(p,n)$-version of Theorem 9.5.

**Theorem 9.21.** (1) Let us assume that a Hilbert space $M$ is contractively contained in $H^2_{p,n;Y}(F_d^+)$ and that its Brangesian complement $M^[1\perp]$ satisfies conditions (i) and (ii) in part (4) of Theorem 9.18 and hence has a representation $M^[1\perp] = Ran C_{n,p;C,A}$ for some $(p,n)$-contractive output pair $(C,A)$. Let us impose the additional hypothesis that $(C,A)$ satisfies the inequality

$$\sum_{j=1}^{d} p_j A_j^\ast (I - G_{n,p;C,A}) A_j \preceq I - G_{p,n;C,A}. \quad (9.78)$$

Then the related inequality

$$\begin{bmatrix} A^\ast & \hat{O}_{\gamma,C,A} \end{bmatrix} \begin{bmatrix} P \otimes I_X & 0 \\ 0 & I_Y \end{bmatrix} \begin{bmatrix} A \\ \hat{O}_{\gamma,p,n;C,A} \end{bmatrix} \preceq I_X \quad (9.79)$$

holds and as a consequence of (3.16) we may choose a solution

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} \text{col}_n \in F_d^+ \mid D_n \end{bmatrix} : U \rightarrow \begin{bmatrix} X^d \\ F_d^+ \end{bmatrix} \quad (9.80)$$

of the Cholesky factorization problem

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^\ast & D^\ast \end{bmatrix} = \begin{bmatrix} \tilde{P}^{-1} \otimes I_X & 0 \\ 0 & I_Y \end{bmatrix} - \begin{bmatrix} A \\ \hat{O}_{\gamma,p,n;C,A} \end{bmatrix} \begin{bmatrix} A^\ast & (\hat{O}_{\gamma,p,n;C,A})^\ast \end{bmatrix}. \quad (9.81)$$

Define formal power series

$$\varnothing(z) = \sum_{\alpha \in F_d^+} \gamma_{p,n;\alpha} D_{\alpha} z^\alpha, \quad \Theta(z) = \varnothing(z) + CR_{p,n}(zA)Z_p(z)B. \quad (9.82)$$

Then $\Theta$ is a Beurling-Lax representer for $M$, i.e., $\Theta$ is a contractive multiplier from $H^2_{\omega_{p,n,Y}(F_d^+)}$ to $H^2_{\omega_{p,n,Y}(F_d^+)}$ such that $M = \Theta \cdot H^2_{\omega_{p,n,Y}(F_d^+)}$.

(2) Suppose that $M$ is isometrically-included subspace of $H^2_{\omega_{p,n,Y}(F_d^+)}$ which is $S_{p,n;R}$-invariant with $S^\ast_{p,n;R}$-invariant orthogonal complement $M^[\perp]$ represented as $M^[\perp] = Ran C_{p,n;C,A}$ with $A$ being $p$-strongly stable and $(C,A)$ a $(p,n)$-isometric output pair as in the last part of item (4) in Theorem 9.18. Then the additional hypothesis (9.78) in part (1) above is automatic and a McCT-inner Beurling-Lax representer $\Theta$ for $M$ can be constructed explicitly via the procedure given by (9.80), (9.81), (9.82).
PROOF. The proof of statement (1) follows the outline of the proof of statement (1) in Theorem 5.5 with suitable adjustments, specifically:

(i) Use the fact that, by item (3b) in Theorem 9.18

\[ K_{(\mathcal{M},)}(z, \zeta) = CR_{p,n}(zA)R_{p,n}(\zeta A)^* C^*, \]  

and the argument as in the first step in the proof of Theorem 5.5 that we are only required to produce a formal power series \( \Theta(z) \) so that the following kernel decomposition holds:

\[ k_{\omega_{p,n}}(z, \zeta)I_Y - \Theta(z)(k_{\tilde{p},1}(z, \zeta)I_d)\Theta(z)^* = CR_{p,n}(zA)R_{p,n}(\zeta A)^* C^*. \]  

(ii) Verify that (9.78) implies (9.79) (in fact they are equivalent) in view of the factorization (9.76). Use Theorem 9.3 (in place of Theorem 3.8) to arrive at the kernel decomposition

\[ \Theta_{0}(z) = D + C_{0}(I - Z_{\tilde{p}}(z)A)^{-1}Z_{\tilde{p}}(z)B. \]

(iii) With \( C_{0} = \tilde{O}_{p,n,c,A} \) and \( \frac{B}{D} \) constructed as in (9.80), introduce

\[ \Psi_{p,n,p}(z)\Theta_{0}(z) = \Theta(z) \]

where \( \Theta(z) \) is as in (9.82). Use Theorem 9.3 (in place of Theorem 3.8) to verify that \( \Theta_{0} \) satisfies (in place of (5.25)) the kernel decomposition

\[ k_{\omega_{p,1}}(z, \zeta)I_{(\mathcal{F}_{d}^{+})} - \Theta_{0}(z)(k_{\omega_{p,1}}(z, \zeta)I_{d})\Theta_{0}(\zeta)^* = C_{0}(I - Z_{\tilde{p}}(z)A)^{-1}(I - A^* Z_{\tilde{p}}(\zeta)^*)^{-1}C_{0}. \]

(iv) Use the definition of \( \mathcal{D}(z) \) and the identity (9.77) to verify that

\[ (9.77) \]

As for statement (2), note that, as a consequence of part (3c) of Theorem 9.18 \( A \) being \( p \)-strongly stable and \( (C, A) \) being \( (p, n) \)-isometric implies that \( \tilde{O}_{p,n, c, A} \) is isometric, i.e., that \( \tilde{O}_{p,n, c, A} = I_{\mathcal{K}} \). Then condition (9.78) holds trivially (in the form \( o \leq 0 \)) and hence all the analysis of part (1) applies. As \( \mathcal{M} \) is isometrically included in \( H_{\omega_{p,n}}^{2}(\mathcal{F}_{d}^{+}) \), the resulting contractive-multiplier Beurling-Lax representer \( \Theta \) is in fact McCT-inner. 

We now present the \((p, n)\)-version of Theorem 5.8.

**Theorem 9.22.** Let \( p \) and \( q \) be two noncommutative regular power series and let \( n \) and \( m \) be two positive integers. A Hilbert space \( \mathcal{M} \) is such that

1. \( \mathcal{M} \) is contractively included in \( H_{\omega_{p,n, Y}}^{2}(\mathcal{F}_{d}^{+}) \),
2. \( \mathcal{M} \) is \( S_{\omega_{p,n, Y}} \)-invariant,
3. the \( d \)-tuple \( \mathbf{A} = (A_1, \ldots, A_d) \) where \( A_j = (S_{\omega_{p,n, Y}}(j)_{\mathcal{M}})^{j} \) for \( j = 1, \ldots, d \), is a \( q \)-strongly stable \((q, m)\)-hypercontraction

if and only if there is a coefficient Hilbert space \( \mathcal{U} \) and a contractive multiplier \( \Theta \) from \( H_{\omega_{p,n, Y}}^{2}(\mathcal{F}_{d}^{+}) \) to \( H_{\omega_{p,n, Y}}^{2}(\mathcal{F}_{d}^{+}) \) so that

\[ \mathcal{M} = \Theta \cdot H_{\omega_{p,n, Y}}^{2}(\mathcal{F}_{d}^{+}) \]

with lifted norm \( \| \Theta \cdot f \|_{\mathcal{M}} = \| Qf \|_{H_{\omega_{p,n, Y}}^{2}(\mathcal{F}_{d}^{+})} \) (9.86)

where \( Q \) is the orthogonal projection onto \((\text{Ker} M_{\Theta})^{\perp}\).
Sketch of the proof. If $\Theta$ is a contractive multiplier from $H^2_{\omega_{q,m}, \mathcal{M}}(\mathbb{F}_d^+)$ to $H^2_{\omega_{q,n}, \mathcal{Y}}(\mathbb{F}_d^+)$ and $\mathcal{M}$ is of the form $[\mathcal{A}]$, then it follows as in the proof of Theorem 5.8 that $\|\Theta f\|_{H^2_{\omega_{q,n}, \mathcal{Y}}(\mathbb{F}_d^+)} \leq \|\Theta f\|_{\mathcal{M}}$ which verifies property (1). Property (2) follows from the intertwining equalities $S_{\omega_{p,n}, \mathcal{R}j} M_\Theta = M_\Theta S_{\omega_{q,m}, \mathcal{R}j}$. Furthermore, the formulas (5.28) and (5.29) still hold true (with $S_{\omega_{q,m}, \mathcal{R}}$ instead of $S_{\omega', \mathcal{R}}$) and therefore,

$$\|A^\alpha \Theta f\|_{\mathcal{M}}^2 = \|S_{\omega_{q,n}, \mathcal{R}} Q f\|_{H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+)}^2 \text{ for all } \alpha \in \mathbb{F}_d^+, f \in H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+).$$

Since $S_{\omega_{q,m}, \mathcal{R}}$ is a $q$-strongly stable $(q, m)$-hypercontraction on $H^2_{\omega_{q,m}}(\mathbb{F}_d^+)$, we have

$$\langle q^N (B_A) | I_\mathcal{M} | \Theta f, \Theta f \rangle_{\mathcal{M}} = \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{\beta_1, \ldots, \beta_N \neq \emptyset, \beta_1 \cdot \beta_N = \alpha} q_{\beta_1} \cdots q_{\beta_N} \right) \|A^\alpha \Theta f\|_{\mathcal{M}}^2$$

$$= \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{\beta_1, \ldots, \beta_N \neq \emptyset, \beta_1 \cdot \beta_N = \alpha} q_{\beta_1} \cdots q_{\beta_N} \right) \|S_{\omega_{q,m}, \mathcal{R}} Q f\|_{H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+)}^2$$

$$= \langle q^N (BS_{\omega_{q,m}, \mathcal{R}} | I_\mathcal{H}_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+)) | Q f, Q f \rangle_{H^2_{\omega_{q,m}, \mathcal{Y}}(\mathbb{F}_d^+)} \to 0 \text{ (as } N \to \infty)$$

and

$$\langle \Gamma_{q,k;A} | I_\mathcal{M} | \Theta f, \Theta f \rangle_{\mathcal{M}} = \sum_{\alpha \in \mathbb{F}_d^+} c_{q,k;\alpha} \|A^\alpha \Theta f\|_{\mathcal{M}}^2$$

$$= \sum_{\alpha \in \mathbb{F}_d^+} c_{q,k;\alpha} \|S_{\omega_{q,m}, \mathcal{R}} Q f\|_{H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+)}^2$$

$$= \langle \Gamma_{q,k;A} | I_{H^2_{\omega_{q,m}, \mathcal{Y}}(\mathbb{F}_d^+)} | Q f, Q f \rangle_{H^2_{\omega_{q,m}, \mathcal{Y}}(\mathbb{F}_d^+)} \geq 0$$

for $k = 1, \ldots, m$, which shows that $A$ is a $q$-strongly stable $(q, m)$-hypercontraction on $\mathcal{M}$ and therefore completes the proof of sufficiency.

For the necessity part, given a Hilbert space $\mathcal{M}$ subject to conditions (1), (2), (3), we start with the $(q, m)$-hypercontraction tuple $A = (A_1, \ldots, A_d)$ where $A_j = (S_{\omega_{q,m}, \mathcal{R}j})_\mathcal{M}$ and factor the positive semidefinite operator $\Gamma_{q,m;A} | I_\mathcal{M} |$ as $\Gamma_{q,m;A} | I_\mathcal{M} | = C^* C$ for an appropriately chosen operator $C: \mathcal{M} \to \mathcal{U}$ from $\mathcal{M}$ into a Hilbert space $\mathcal{U}$ subject to condition $\dim \mathcal{U} = \rank \Gamma_{q,m;A} | I_\mathcal{M} |$. Then the pair $(C, A)$ is $(q, m)$-isometric and, since $A$ is $q$-strongly stable by hypothesis (3), it follows from part (2) of Theorem 5.15 that the observability operator $O_{q,m;C,A}$ is an isometry from $\mathcal{M}$ into $H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+)$. Since the inclusion map $\iota: \mathcal{M} \to H^2_{\omega_{q,n}, \mathcal{Y}}(\mathbb{F}_d^+)$ is a contraction by hypothesis (1), the operator

$$G = \iota \circ O_{q,m;C,A}^* | H^2_{\omega_{q,m}, \mathcal{U}}(\mathbb{F}_d^+) \to H^2_{\omega_{q,n}, \mathcal{Y}}(\mathbb{F}_d^+)$$

is a contraction. Making use of intertwining relations (5.28) and (5.29) and taking into account that $\iota \circ A_j = S_{\omega_{q,n}, \mathcal{R}j} \circ \iota$, we conclude as in the proof of Theorem 5.8 that

$$GS_{\omega_{q,m}, \mathcal{R}j} = S_{\omega_{q,n}, \mathcal{R}j} G \text{ for } j = 1, \ldots, d.$$

By the adaptation of Proposition 5.22 to the setting of weighted spaces of the form $H^2_{\omega_{q,n}, \mathcal{Y}}(\mathbb{F}_d^+)$, it follows that $G$ is a multiplication operator, i.e., there is a contractive multiplier $\Theta$ so that $G = M_\Theta$. 


Since \( \mathcal{O}_{q,m,C,A} : \mathcal{M} \rightarrow H^2_{\omega_{q,m,C,A}}(\mathbb{F}^+_d) \) is an isometry, it follows that \( \text{Ran} \mathcal{O}^*_{q,m,C,A} = \mathcal{M} \) and also that \( \mathcal{M} = \Theta : H^2_{\omega_{q,m,C,A}}(\mathbb{F}^+_d) \) with \( \mathcal{M} \)-norm given by (9.84). This completes the proof. \( \square \)

The next result shows that the quasi-wandering-subspace version of the Beurling-Lax theorem extends verbatim to the \((p,n)\)-setting.

**Theorem 9.23.** Let \( \mathcal{M} \) be a closed \( S_{\omega_{p,n,R}} \)-invariant subspace of \( H^2_{\omega_{p,n,R}}(\mathbb{F}^+_d) \) containing no nontrivial reducing subspaces for \( S_{\omega_{p,n,R}} \). Then

\[
\mathcal{M} = \bigvee_{\alpha \in \mathbb{F}^+_d} S_{\omega_{p,n,R}}^\alpha \mathcal{Q}, \quad \text{where} \quad \mathcal{Q} = P_{\mathcal{M}} \left( \bigoplus_{j=1}^d S_{\omega_{p,n,R},j} \mathcal{M}^{\perp} \right). \tag{9.87}
\]

**Proof.** The proof follows the lines of that of Theorem 6.1. We let \((C,A)\) be an \((p,n)\)-isometric pair such that \( \mathcal{M} = (\text{Ran} \mathcal{O}_{p,n,C,A})^{\perp} \), and then observe that the quasi-wandering subspace \( \mathcal{Q} \) in (9.87) is equal (as a set) to

\[
\mathcal{Q} = F(z)A^d, \quad \text{where} \quad F(z) = CR_{p,n}(zA)(Z(z) - A^*) . \tag{9.88}
\]

In more details, for a fixed \( z \in A^d \), we represent \( F(z)x \) as in (9.6) (with \( \omega_{p,n,\alpha} \) instead of \( \omega_{[\alpha]} \)), and then write this representation in terms of operators \( \mathcal{O}_{p,n,C,A} \) and \( S_{\omega_{p,n,R},j} \) as

\[
Fx + \mathcal{O}_{p,n,C,A} \sum_{j=1}^d A_j x_j = \sum_{j=1}^d S_{\omega_{p,n,R},j} \mathcal{O}_{p,n,C,A} x_j . \tag{9.89}
\]

Making use of intertwining relations (9.67), it is not hard to verify that the two terms on the right side of (9.89) are orthogonal in \( H^2_{\omega_{p,n,Y}}(\mathbb{F}^+_d) \)-metric, from which (9.88) follows, since \( \mathcal{M}^{\perp} = \text{Ran} \mathcal{O}_{p,n,C,A} \).

Then, assuming that there is a nonzero \( f \in \mathcal{M} = (\text{Ran} \mathcal{O}_{p,n,C,A})^{\perp} \) which is orthogonal to \( S_{\omega_{p,n,R}}^\alpha Fx^d \) for all \( \alpha \in \mathbb{F}^+_d \), we use the decomposition (9.7) (adjusted to the present setting) to conclude that \( S_{\omega_{p,n,R}}^\alpha \mathcal{M} \) belongs to \( \mathcal{M} \) for all \( \alpha \in \mathbb{F}^+_d \).

Since \( \mathcal{M} \) is \( S_{\omega_{p,n,R}} \)-invariant, it now follows that

\[
S_{\omega_{p,n,R}}^\gamma \mathcal{M} = \mathcal{M} \quad \text{for all} \quad \alpha, \gamma \in \mathbb{F}^+_d. \tag{9.90}
\]

Since \( S_{\omega_{p,n,R}} \) is a \((p,1)\)-contraction, we have (as in (9.17)) \( p^j(B_{S_{\omega_{p,n,R}}})(I) \leq I \) for all \( j \geq 0 \). It then follows, as in the computation (9.48), that

\[
\sum_{\alpha \in \mathbb{F}^+_d} c_{p,n,\alpha} S_{\omega_{p,n,R}}^\alpha S_{\omega_{p,n,R}}^\alpha \leq (1 + p)^n (B_{S_{\omega_{p,n,R}}})(I)_{H^2_{\omega_{p,n,R}}(\mathbb{F}^+_d)}
\]

\[
= \sum_{j=0}^n \binom{n}{j} p^j (B_{S_{\omega_{p,n,R}}})(I)_{H^2_{\omega_{p,n,R}}(\mathbb{F}^+_d)}
\]

\[
\leq n \sum_{j=0}^n \binom{n}{j} I_{H^2_{\omega_{p,n,R}}(\mathbb{F}^+_d)} = 2^n \cdot I_{H^2_{\omega_{p,n,R}}(\mathbb{F}^+_d)}
\]

and hence,

\[
\sum_{\alpha \in \mathbb{F}^+_d} c_{p,n,\alpha} S_{\omega_{p,n,R}}^\alpha S_{\omega_{p,n,R}}^\alpha \leq 2^n \quad \text{for each} \quad N \geq 1. \tag{9.91}
\]
Since \( f \neq 0 \), there is a \( \beta \in \mathbb{F}_d^+ \) such that \( f_\beta \neq 0 \). If \( E : f \to f_\theta \) is the evaluation operator on \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \), then its adjoint \( E^* \) is the inclusion of \( Y \) into \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \), and from the formula (9.62) we have

\[
E^* E S_{\omega_{p,n},R}^\beta f = (S_{\omega_{p,n},R}^\beta f)_\theta = \omega_{p,n;\beta} f_\beta := y \neq 0.
\]

By the equality (9.66) applied to the power series \( S_{\omega_{p,n},R}^\beta f \) we have

\[
\Gamma_{p,n} S_{\omega_{p,n},R}^\beta [I_{H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+)}] S_{\omega_{p,n},R}^\beta f = E^* E S_{\omega_{p,n},R}^\beta f = y \neq 0. \tag{9.92}
\]

Due to (9.90), the power series

\[
g_N = \sum_{\alpha \in \mathbb{F}_d^+ | |\alpha| \leq N} c_{p,n;\alpha} S_{\omega,R}^\alpha S_{\omega,R}^\beta f
\]

belongs to \( \mathcal{M} \) for all \( N \geq 1 \), and the sequence \( \{g_N\}_{N \geq 1} \subset \mathcal{M} \) is uniformly bounded in metric of \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \), due to (9.91). Therefore, it admits a subsequential weak limit

\[
\sum_{\alpha \in \mathbb{F}_d^+} c_{p,n;\alpha} S_{\omega,R}^\alpha S_{\omega,R}^\beta f = \Gamma_{p,n} S_{\omega_{p,n},R}^\beta [I_{H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+)}] S_{\omega_{p,n},R}^\beta f
\]

which belongs to \( \mathcal{M} \) and which is equal to \( y \), by (9.92). Then \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \otimes y \) is a subspace of \( \mathcal{M} \) and is reducing for \( S_{\omega_{p,n},R} \) which is a contradiction. \( \square \)

### 9.7. \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \)-Bergman-inner families

We next present the \((p,n)\)-analog of the most elaborate version of the Beurling-Lax theorem presented in Theorem 7.12. To get such an analog we need to make the extra assumption that the right shift operators \( S_{\omega_{p,n},R,j} \) on \( H^2_{\omega_{p,n},Y}(\mathbb{F}_d^+) \) are left invertible for \( j = 1, \ldots, d \). As follows from (9.61),

\[
S_{\omega_{p,n},R,j}^* S_{\omega_{p,n},R,j} : \sum_{\alpha \in \mathbb{F}_d^+} f_\gamma z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \frac{\omega_{p,n;\alpha;j}}{\omega_{p,n;\alpha}} f_\alpha j z^\alpha \quad (j = 1, \ldots, d)
\]

and hence our assumption can be expressed in terms of the numbers (9.7) as follows.

- The regular power series \( p \) is such that the associated positive numbers \( \omega_{p,n;\alpha} \) are subject to inequalities

\[
\frac{\omega_{p,n;\alpha}}{\omega_{p,n;\alpha;j}} \leq M \quad \text{for some} \quad M > 0 \quad \text{and all} \quad \alpha \in \mathbb{F}_d^+ \quad \text{and} \quad j = 1, \ldots, d. \tag{9.93}
\]

The latter assumption is not very restrictive – it holds true if the coefficients \( p_\alpha \) (9.1) are uniformly bounded (in particular, if \( p(z) \) is a noncommutative polynomial) and even more generally, if \( p_\alpha \) does not grow too fast with respect to \( |\alpha| \).

**Proposition 9.24.** Let \( p(z) \) be a regular noncommutative power series as in (9.1), (9.2) and let us assume that \( p_{\alpha;j} \leq \varphi p_\alpha \) for some \( \varphi > 0 \) and all \( \alpha \in \mathbb{F}_d^+ \) and \( j = 1, \ldots, d \). Then (9.93) holds.

**Proof.** For a fixed \( \alpha \in \mathbb{F}_d^+ \) and \( j \in \{1, \ldots, d\} \), we have by (9.7),

\[
\omega_{p,n;\alpha;j}^{-1} = \sum_{\ell=0}^{n-1} \binom{n+\ell-1}{\ell} \sum_{\beta_1,\ldots,\beta_{\ell}\neq 0} p_{\beta_1} \cdots p_{\beta_{\ell}}.
\]
For each fixed $\ell \geq 0$, we consider separately the terms $p_{\beta_1} \cdots p_{\beta_{\ell}}$ with $\beta_{\ell} = j$ and with $\beta_{\ell} = \beta'_{\ell}$ for some $\beta'_{\ell} \neq 0$. We therefore have $\omega_{p,n;\alpha_j}^{-1} = \Pi_1 + \Pi_2$, where

$$
\Pi_1 = \sum_{\ell=1}^{[\alpha]+1} \binom{n+\ell-1}{n-1} \sum_{\beta_1,\ldots,\beta_{\ell-1} \neq 0; \beta_1 \cdots \beta_{\ell-1} = \alpha} p_{\beta_1} \cdots p_{\beta_{\ell-1}} p_j,
$$

$$
\Pi_2 = \sum_{\ell=0}^{[\alpha]} \binom{n+\ell-1}{n-1} \sum_{\beta_1,\ldots,\beta_{\ell} \neq 0; \beta_1 \cdots \beta_{\ell} = \alpha} p_{\beta_1} \cdots p_{\beta_{\ell-1}} p_{\beta_{\ell}}.
$$

Note that in (9.95) we changed notation $\beta'_{\ell}$ back to $\beta_{\ell}$. Since $p_{\beta(j)} \leq \varphi p_{\beta_{\ell}}$ for all $\beta \in \mathbb{F}_d^+$ (by the assumption), we have from (9.95) and (9.7)

$$
\Pi_2 \leq \varphi \sum_{\ell=0}^{[\alpha]} \binom{n+\ell-1}{n-1} \sum_{\beta_1,\ldots,\beta_{\ell} \neq 0; \beta_1 \cdots \beta_{\ell} = \alpha} p_{\beta_1} \cdots p_{\beta_{\ell-1}} p_{\beta_{\ell}} = \varphi \omega_{p,n;\alpha}^{-1}.
$$

Shifting the index $\ell$ in (9.94) and taking into account that $\binom{n+\ell}{n-1} \leq n \binom{n+\ell-1}{n-1}$ for all $\ell \geq 0$, we get

$$
\Pi_1 = p_j \sum_{\ell=0}^{[\alpha]} \binom{n+\ell}{n-1} \sum_{\beta_1 \cdots \beta_{\ell} = \alpha} p_{\beta_1} \cdots p_{\beta_{\ell}}
$$

$$
\leq np_j \sum_{\ell=0}^{[\alpha]} \binom{n+\ell-1}{n-1} \sum_{\beta_1 \cdots \beta_{\ell} = \alpha} p_{\beta_1} \cdots p_{\beta_{\ell}} = np_j \omega_{p,n;\alpha}^{-1}.
$$

From the last two inequalities, $\omega_{p,n;\alpha_j}^{-1} \leq \omega_{p,n;\alpha}^{-1} (\varphi + np_j)$ and therefore (9.93) holds with $M = \varphi + n \max\{p_1, \ldots, p_d\}$. \hfill $\square$

To consider the family of transfer functions (9.13) under suitable metric conditions imposed on the system matrices (1.33), we first need to introduce the shifted $(p, n)$-observability operators and gramians which can be done upon making use of the power series (9.8) as follows. Given an output pair $(C, A)$ with $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $A \in \mathcal{L}(\mathcal{X})^d$ we formally define for each $\beta \in \mathbb{F}_d^+$,

$$
\Sigma_{p,n;\beta;C,A} : x \mapsto CR_{p,n;\beta}(zA) = \sum_{\alpha \in \mathbb{F}_d^+} (\omega_{p,n;\alpha\beta}^{-1} C A^\alpha) x^\alpha,
$$

$$
\mathcal{G}_{p,n;\beta;C,A} = R_{p,n;\beta}(B A) [C^* C] = \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n;\alpha\beta}^{-1} A^{*\alpha} C^* C A^\alpha.
$$

Letting $\beta = \emptyset$ in (9.96) and (9.97) we conclude from (9.12) and (9.39)

$$
\Sigma_{p,n,\emptyset;C,A} = \mathcal{O}_{p,n;C,A} \quad \text{and} \quad \mathcal{G}_{p,n,\emptyset;C,A} = \mathcal{G}_{p,n;C,A}.
$$

The $(p, n)$-output stability of the pair $(C, A)$ is exactly what is needed for convergence of the series in (9.39). In general, this condition does not guarantee the convergence in (9.97) for $\beta \neq \emptyset$. Now the assumption (9.93) comes into play.

**Lemma 9.25.** Let us assume that a regular noncommutative power series $p$ satisfies the assumption (9.93) and let the pair $(C, A)$ as above be $(p, n)$-output stable. Then for each $\beta \in \mathbb{F}_d^+$, the formulas (9.96) and (9.97) define bounded operators $\Sigma_{p,n,\beta;C,A} \in \mathcal{L}(\mathcal{X}, H_{p,n;\beta}^2(\mathbb{F}_d^+))$ and $\mathcal{G}_{p,n,\beta;C,A} \in \mathcal{L}(\mathcal{X})$.

If in addition, the pair $(C, A)$ is exactly $(p, n)$-observable, then
(1) The operator $\mathfrak{G}_{p,n,\beta;C,A}$ is strictly positive definite.

(2) The space $S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A}$ (with $H^2_{\omega_{p,n,Y}(F_d)}$-inner product) is a NFRKHS with reproducing kernel given by

$$\mathfrak{R}_{p,n,\beta}(z, \zeta) = CR_{p,n,\beta}(zA)(z^\beta \zeta^\beta)^{-1} R_{p,n,\beta}(\zeta A)^{\ast}.$$  \hfill (9.98)

**Proof.** By (8.79), $\omega_{p,n,\alpha} \omega_{p,n,\beta} \leq M|\beta|$ for all $\alpha \in F_d$. Then we have from (9.97) and (9.39),

$$\mathfrak{G}_{p,n,\beta;C,A} = \sum_{\alpha \in F_d} \frac{\omega_{p,n,\alpha}}{\omega_{p,n,\alpha \beta}} \cdot \omega_{p,n,\alpha}^{\ast} \alpha \top C^\ast C^\alpha \leq M|\beta| \cdot \mathfrak{G}_{p,n;C,A}$$

Since the pair $(C, A)$ is $(p, n)$-output-stable, the gramian $\mathfrak{G}_{p,n;C,A}$ is bounded and hence, $\mathfrak{G}_{p,n,\beta;C,A}$ is bounded as well. A similar computation based on formulas (9.37) and (9.72) verifies the inequality

$$\|\Delta_{p,n,\beta;C,A} x\|_{H^2_{\omega_{p,n,Y}(F_d)}} \leq M|\beta| \cdot \|\mathfrak{G}_{p,n;C,A} x\|_{H^2_{\omega_{p,n,Y}(F_d)}}$$

for all $x \in X$ which implies that the operator $\Delta_{p,n,\beta;C,A} : X \rightarrow H^2_{\omega_{p,n,Y}(F_d)}$ is bounded. On the other hand, by virtue of (9.63) with $\beta = i_1 \ldots i_N$,

$$\omega_{p,n,\alpha \beta}^{-1} \geq \omega_{p,n,\alpha}^{-1} \cdot \varepsilon_{p,\beta}, \quad \text{where} \quad \varepsilon_{p,\beta} = p_{i_1} \ldots p_{i_N} > 0$$

which on account of (9.97) and (9.39), implies

$$\mathfrak{G}_{p,n,\beta;C,A} = \sum_{\alpha \in F_d} \frac{\omega_{p,n,\alpha}}{\omega_{p,n,\alpha \beta}} \cdot \omega_{p,n,\alpha}^{\ast} \alpha \top C^\ast C^\alpha \geq \varepsilon_{p,\beta}^{-1} \cdot \mathfrak{G}_{p,n;C,A}.$$  \hfill (9.97)

If the pair $(C, A)$ is exactly $(p, n)$-observable, then $\mathfrak{G}_{p,n;C,A} \approx 0$ and hence, $\mathfrak{G}_{p,n,\beta;C,A}$ is strictly positive definite as well. This proves part (1) in the last statement. To prove part (2), we write for a fixed $x \in X$,

$$S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A} x = \sum_{\alpha \in F_d} \omega_{p,n,\alpha \beta}^{\ast} (CA^\alpha x) z^\alpha \beta$$

and then invoke the definition of the inner product in $H^2_{\omega_{p,n,Y}(F_d)}$ to get

$$\|S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A} x\|_{H^2_{\omega_{p,n,Y}(F_d)}}^2 = \sum_{\alpha \in F_d} \omega_{p,n,\alpha \beta}^{\ast} (CA^\alpha x, CA^\alpha x)_X$$

which can be written, by the definition (9.97), as

$$\|S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A} x\|_{H^2_{\omega_{p,n,Y}(F_d)}}^2 = (\mathfrak{G}_{p,n,\beta;C,A} x, x)_X.$$  \hfill (9.98)

Now a direct application of Proposition 2.3 shows that the formal reproducing kernel for the space $S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A}$ is given by

$$(S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A})(z) \mathfrak{G}_{p,n,\beta;C,A}^{-1} ((S_{\omega_{p,n,R}}^\alpha \operatorname{Ran} \Delta_{p,n,\beta;C,A})(\zeta))^\ast,$$

which agrees exactly with $\mathfrak{R}_{p,n,\beta}(z, \zeta)$ given by (9.98). \hfill \square
A power-series computation based on the formula (9.97) and similar formulas for \( \mathbf{G}_{p,n,j;\beta,C,A} \) \((j = 1, \ldots, d)\) confirms the weighted Stein identity

\[
\sum_{j=1}^{d} A_j^* \mathbf{G}_{p,n,j;\beta,C,A} A_j + \omega^{-1}_{p,n;\beta} \cdot C^* C = \mathbf{G}_{p,n,\beta;C,A} \quad \text{for all } \beta \in \mathbb{F}_d^+, \tag{9.99}
\]

which can be written in a more compact form as

\[
A^* \mathbf{G}_{p,n,\beta;C,A} + \omega^{-1}_{p,n;\beta} \cdot C^* C = \mathbf{G}_{p,n,\beta;C,A},
\]

where we have set

\[
\mathbf{G}_{p,n,\beta;C,A} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{G}_{p,n,d;\beta,C,A} \end{bmatrix} \quad \text{for all } \beta \in \mathbb{F}_d^+. \tag{9.100}
\]

We now introduce the \((p,n)\)-analog of the metric relation (7.6):

\[
\begin{bmatrix} A^* & C^* \\ B_\beta & D_\beta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{G}}_{p,n,\beta;C,A} & 0 \\ 0 & \omega^{-1}_{p,n;\beta} \cdot I_Y \end{bmatrix} \begin{bmatrix} A & B_\beta \\ C & D_\beta \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{G}}_{p,n,\beta;C,A} & 0 \\ 0 & \omega^{-1}_{p,n;\beta} I_Y \end{bmatrix} \tag{9.101}
\]

and note that equality (9.99) corresponds to the \((1,1)\)-entry of (9.101).

**Remark 9.26.** We next remark that if we are given only a \((p,n)\)-output stable pair \((C,A)\) which is exactly \((p,n)\)-observable, then by solving a suitable Cholesky factorization problem (i.e., following the construction in Lemma 7.5) we can construct operators \(B_1,\ldots,B_d,\beta \in \mathcal{L}(U_\beta)\) and \(D_\beta \in \mathcal{L}(U_\beta,Y)\) satisfying not only equality (9.101) but also the weighted coisometry condition

\[
\begin{bmatrix} A & \hat{B}_\beta \\ C & D_\beta \end{bmatrix} \begin{bmatrix} \mathbf{G}^{-1}_{p,n,\beta;C,A} & 0 \\ 0 & \omega^{-1}_{p,n;\beta} I_Y \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B_\beta & D_\beta \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{-1}_{p,n,\beta;C,A} & 0 \\ 0 & \omega^{-1}_{p,n;\beta} I_Y \end{bmatrix}. \tag{9.102}
\]

**Lemma 9.27.** Let \((C,A)\) be a \((p,n)\)-output stable pair and let \(\Theta_{p,n,\beta;U} \) be defined as in (9.13) for some \(\beta \in \mathbb{F}_d^+\) and some operators \(B_1,\ldots,B_d,\beta \in \mathcal{L}(U_\beta)\) and \(D_\beta \in \mathcal{L}(U_\beta,Y)\) subject to equality (9.101). Then \(\Theta_{p,n,\beta;U}(z)^{\top}\) is \((p,n)\)-Bergman inner. Moreover,

1. \(\mathcal{O}_{p,n,C,A}x\) is orthogonal to \(S_{\omega_{p,n;\beta}^R}^T \Theta_{p,n,\beta;U} u\) for all \(x \in X\) and \(u \in U_\beta\).
2. \(S_{\omega_{p,n;\beta}^R}^T \Theta_{p,n,\beta;U} u\) and \(S_{\omega_{p,n;\beta}^R}^T \Theta_{p,n,\beta;U} u\) are both orthogonal to \(S_{\omega_{p,n;\beta}^R}^T \Theta_{p,n,\beta;U} u\) for all \(\gamma \neq \emptyset\) and for any \(u, u' \in U_\beta\).
3. With notation as in (9.14), the following power-series identity holds:

\[
\omega^{-1}_{p,n;\beta} I_{U_\beta} - \Theta_{p,n,\beta;U}(z)^{\top} \Theta_{p,n,\beta;U}(\zeta) = \omega^{-1}_{p,n;\beta} I_{U_\beta} + \hat{R}_{p,n;\beta}(zA)^{\top} \hat{\mathbf{G}}_{p,n,\beta;C,A} (\omega^{-1}_{p,n;\beta} I_{U_\beta} + A \hat{R}_{p,n;\beta}(\zeta A)) \hat{B}_\beta \\
- \omega^{-1}_{p,n;\beta} \hat{B}_\beta \hat{R}_{p,n;\beta}(zA)^{\top} \hat{\mathbf{G}}_{p,n,\beta;C,A} \hat{R}_{p,n;\beta}(\zeta A) \hat{B}_\beta. \tag{9.103}
\]

**Proof.** To lighten the notation in the proof, we write simply \(\Theta_{p,n,\beta}\) rather than \(\Theta_{p,n,\beta;U}\). Statements (1) and (2) rely on the equality

\[
\omega^{-1}_{p,n;\beta} C^* D_\beta + \sum_{j=1}^{d} A_j^* \mathbf{G}_{p,n,j;\beta,C,A} B_{j,\beta} = 0 \tag{9.104}
\]
which occurs as the (1,2)-entry in $[U,10]$. Making use of expansions $[U,12]$, $[U,13]$ and the definition of the inner product in $H^2_{\omega_{p,n},\gamma}(\mathbb{F}_d^+)$ we get
\[
\langle S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u, O_{p,n,c} A x \rangle_{H^2_{\omega_{p,n},\gamma}(\mathbb{F}_d^+)} = \langle D_{\beta} u, C A ^x \rangle_{Y} + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} \cdot \langle C A^\alpha B_{j,\beta} u, C A^{\alpha j} u \rangle_{Y}
\]
\[
= \left( \left( \omega_{p,n,\beta} \cdot C^* D_{\beta} + \sum_{j=1}^{d} A^j \left( \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} A^\alpha C A^\alpha \right) B_{j,\beta} \right) u, A^\beta x \right)_{X} = 0
\]
thus proving statement (1). Letting $\tilde{j} \in \{1, \ldots, d\}$ to denote the rightmost letter in the given $\gamma \neq \emptyset$ (i.e., $\gamma = \tilde{\gamma}_{\tilde{j}}$) we compute (as in the proof of Lemma $[U,14]$),
\[
\langle S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u, S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u' \rangle_{H^2_{\omega_{p,n},\gamma}(\mathbb{F}_d^+)} = \omega_{p,n,\gamma} \cdot \langle \omega_{p,n,\beta} D_{\beta} u, \omega_{p,n,\gamma} C A \tilde{\gamma} B_{\tilde{j},\beta} u' \rangle_{Y}
\]
\[
+ \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} \cdot \langle \omega_{p,n,\alpha j} C A^\alpha B_{j,\beta} u, \omega_{p,n,\alpha j} C A^{\alpha j} B_{j,\beta} u' \rangle_{Y}
\]
\[
= \omega_{p,n,\beta} \cdot \langle D_{\beta} u, C A \tilde{\gamma} B_{\tilde{j},\beta} u' \rangle_{Y} + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} \cdot \langle C A^\alpha B_{j,\beta} u, C A^{\alpha j} B_{j,\beta} u' \rangle_{Y}
\]
\[
= \left( \left( \omega_{p,n,\beta} \cdot C^* D_{\beta} + \sum_{j=1}^{d} A^j \left( \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} A^\alpha C A^\alpha \right) B_{j,\beta} \right) u, A^\beta B_{\tilde{j},\beta} u' \right)_{X} = 0,
\]
where the last equality follows from $[U,10]$. Thus, $S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u$ is orthogonal to $S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u'$. Orthogonality of $S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u$ to $S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u'$ is verified in much the same way. Yet another inner-product computation,
\[
||S^\gamma_{\omega_{p,n},R} \Theta_{p,n,\beta} u||^2_{H^2_{\omega_{p,n},\gamma}(\mathbb{F}_d^+)} = \omega_{p,n,\beta} \cdot ||D_{\beta} u||_Y^2 + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} \cdot ||C A^\alpha B_{j,\beta} u||_{X}^2
\]
\[
= \omega_{p,n,\beta} \cdot ||D_{\beta} u||_Y^2 + \sum_{j=1}^{d} \left( B^j_{\beta} \left( \sum_{\alpha \in \mathbb{F}_d^+} \omega_{p,n,\alpha j} A^\alpha C A^\alpha \right) B_{j,\beta} u, u \right)_U = ||u||_U^2
\]
(where the last equality is justified by the equality of (2, 2)-entries in (9.101) shows that the operator $S_{\omega_{p,n}, R}^{\beta \top} \Theta_{p,n, \beta}$ is an isometry from $H_{\omega_{p,n}, Y}(F_d^+)$ into $H_2^{(p,n), Y}(F_2^+)$. Combining this fact with part (2) of the lemma, we conclude by Definition 9.6 that $\Theta_{p,n, \beta}(z)z^\beta$ is $(p,n)$-Bergman inner.

Finally, substituting the representation (9.14) into the left hand side of (9.103) and making use of (9.101) we get

$$
\omega_{p,n,\beta}^{-1} I_{\beta} - (\omega_{p,n,\beta}^{-1} D_{\beta} + \hat{B}_{\beta}^* \hat{Z}(z)^* \hat{R}_{p,n,\beta} (zA)^* C^*) \omega_{p,n,\beta}^{-1} D_{\beta} + C \hat{R}_{p,n,\beta} (zA)^* \hat{Z}(z) \hat{B}_{\beta}
$$

$$
= \omega_{p,n,\beta}^{-1} I_{\beta} - \omega_{p,n,\beta}^{-2} D_{\beta}^* D_{\beta} - \omega_{p,n,\beta}^{-1} \hat{B}_{\beta}^* \hat{Z}(z)^* \hat{R}_{p,n,\beta} (zA)^* C^* D_{\beta}
$$

$$
- \omega_{p,n,\beta}^{-1} D_{\beta}^* C \hat{R}_{p,n,\beta}(zA) \hat{Z}(z) \hat{B}_{\beta} - \hat{B}_{\beta}^* \hat{Z}(z)^* \hat{R}_{p,n,\beta}(zA)^* C \hat{R}_{p,n,\beta}(zA) \hat{Z}(z) \hat{B}_{\beta}
$$

$$
= \omega_{p,n,\beta}^{-1} \hat{B}_{\beta}^* \hat{G}_{p,n,\beta;C,A} \hat{B}_{\beta} + \hat{B}_{\beta}^* \hat{Z}(z)^* \hat{R}_{p,n,\beta}(zA)^* A^* \hat{G}_{p,n,\beta;C,A} \hat{B}_{\beta}
$$

$$
- \omega_{p,n,\beta} \hat{B}_{\beta}^* \hat{Z}(z)^* \hat{R}_{p,n,\beta}(zA)^* (\hat{G}_{p,n,\beta;C,A} - A^* \hat{G}_{p,n,\beta;C,A}) \hat{R}_{p,n,\beta}(zA) \hat{Z}(z) \hat{B}_{\beta},
$$

which is the same as the expression on the right side of (9.103). □

As a corollary we obtained the following analogue of Corollary 7.3

**Corollary 9.28.** Assume that $(C,A)$ and $B_{1,\beta}, \ldots, B_{d,\beta} \in \mathcal{L}(U_{\beta}, X)$, $D_{\beta} \in \mathcal{L}(U_{\beta}, Y)$ are as in the hypotheses of Lemma 9.27 with associated family of transfer functions $\Theta_{p,n, U_{\beta}}$ as in (9.13). Then the representation (9.11) is orthogonal in the metric of $H_2^{(p,n), Y}(F)$ and we have

$$
\|\hat{g}\|^2_{H_2^{(p,n), Y}(F_2^+)} = \|\Theta_{p,n, C,A} x\|^2_{H_2^{(p,n), Y}(F_2^+)} + \|S_{\omega_{p,n}, R} \Theta_{p,n, U_{\beta}} u_{\beta}\|^2_{H_2^{(p,n), Y}}
$$

$$
= \|G_{\omega_{p,n}, C,A} x\|^2_{Y} + \sum_{\beta \in F_d^+} \|u_{\beta}\|^2_{\beta_1},
$$

We now get back to closed $S_{\omega_{p,n}, R}$-invariant subspaces of $H_2^{(p,n), Y}(F_2^+)$. A careful inspection of the proof of Lemma 7.7 shows that the same proof goes through for any closed T-invariant subspace of a Hilbert space $H$ for any operator tuple $T = (T_1, \ldots, T_d)$ of bounded operators with orthogonal ranges. Since the right coordinate-variable multipliers $S_{\omega_{p,n}, R,1}, \ldots, S_{\omega_{p,n}, R,d}$ on $H_2^{(p,n), Y}(F_2^+)$ have mutually orthogonal ranges, we have, in particular, the following result.

**Lemma 9.29.** If a closed subspace $M \subset H_2^{(p,n), Y}(F_2^+)$ is $S_{\omega_{p,n}, R}$-invariant, then

$$
M = \bigoplus_{\beta \in F_d^+} M_{\beta}, \quad \text{where} \quad M_{\beta} := S_{\omega_{p,n}, R}^{\beta \top} M \cap \left( \bigoplus_{j=1}^d S_{\omega_{p,n}, R}^{\beta \top} S_{\omega_{p,n}, R,j} M \right).
$$

Furthermore, for any $\alpha \in F_d^+$, we have the orthogonal direct-sum decomposition

$$
S_{\omega_{p,n}, R}^{-1} M = \bigoplus_{\beta \in F_d^+} M_{\beta \alpha}.
$$

Given a $S_{\omega_{p,n}, R}$-invariant closed subspace $M$ of $H_2^{(p,n), Y}(F_2^+)$, we let $(C,A)$ be a $(p,n)$-isometric pair such that $M^\perp = \operatorname{Ran} \Theta_{p,n, C,A}$ (by Theorem 9.18 we can let $(C,A) = (E|M, S_{\omega_{p,n}, R}|M)$). Thus, $M^\perp$ is the NFRKHS with reproducing kernel
\[ k_M(z, \zeta) = k_{nc, \omega_{p,n}}(z, \zeta) I_Y - CR_{p,n}(zA) G_{p,n,C,A}^{-1} R_{p,n}(\zeta A)^* C^*. \] (9.106)

The next step is to characterize the spaces \( S^{\beta^T}_{\omega_{p,n},R} M \) (or their orthogonal complements) in terms of the chosen pair \((C, A)\).

**Proposition 9.30.** The space \((S^{\beta^T}_{\omega_{p,n},R} M)^\perp\) is characterized as
\[
(S^{\beta^T}_{\omega_{p,n},R^*} M)^\perp = (S^{\beta^T}_{\omega_{p,n},R} H^2_{\omega_{p,n},Y}(F^+_d))^\perp \bigoplus S^{\beta^T}_{\omega_{p,n},R} \text{Ran} \Omega_{p,n,\beta;C,A} \quad (9.107)
\]
where the shifted gramian \( \text{Ran} \Omega_{p,n,\beta;C,A} \) is defined as in (9.96).

**Proof.** Following the strategy of the proof of Proposition 9.28, we write a power series \( f \in H^2_{\omega_{p,n},Y}(F^+_d) \) as
\[
f(z) = p(z) + \tilde{f}(z)z^\beta \quad \text{with} \quad p \in \left(S^{\beta^T}_{\omega_{p,n},R} H^2_{\omega_{p,n},Y}(F^+_d)\right)^\perp \subset S^{\beta^T}_{\omega_{p,n},R} H^2_{\omega_{p,n},Y}(F^+_d)
\]
and then characterize power series \( \tilde{f} \in H^2_{\omega_{p,n},Y}(F^+_d) \) such that \( S^{\beta^T}_{\omega_{p,n},R^*} \tilde{f} \) is orthogonal to \( S^{\beta^T}_{\omega_{p,n},R} M \), or equivalently, such that \( S^{\beta^T}_{\omega_{p,n},R} S^{\beta^T}_{\omega_{p,n},R^*} \tilde{f} \) is orthogonal to \( M \), that is, belongs to \( M^\perp = \text{Ran} \Omega_{p,n,C,A} \). Since by (9.92),
\[
S^{\beta^T}_{\omega_{p,n},R} S^{\beta^T}_{\omega_{p,n},R} : \sum_{\alpha \in F^+_d} \tilde{f}_\alpha z^\alpha \to \sum_{\alpha \in F^+_d} \frac{\omega_{p,n,\alpha \beta}}{\omega_{p,n,\alpha}} \tilde{f}_\alpha z^\alpha,
\]
we conclude that \( S^{\beta^T}_{\omega_{p,n},R} \tilde{f} \) is orthogonal to \( S^{\beta^T}_{\omega_{p,n},R^*} M \) if and only if
\[
\sum_{\alpha \in F^+_d} \omega_{p,n,\alpha \beta} \tilde{f}_\alpha z^\alpha = (\Omega_{p,n,C,A} x)(z) = \sum_{\alpha \in F^+_d} (\omega_{p,n,\alpha}^{-1} \cdot CA^\alpha x)z^\alpha.
\]
for some vector \( x \in X \). Equating the corresponding Taylor coefficients in the latter equality gives \( \tilde{f}_\alpha = \omega_{p,n,\alpha}^{-1} \cdot CA^\alpha x \) for all \( \alpha \in F^+_d \), and therefore,
\[
\tilde{f}(z) = \sum_{\alpha \in F^+_d} \tilde{f}_\alpha z^\alpha = \sum_{\alpha \in F^+_d} (\omega_{p,n,\alpha}^{-1} \cdot CA^\alpha x)z^\alpha = \Omega_{p,n,\beta;C,A} x,
\]
by (1.57). Thus, \( \tilde{f} \in \text{Ran} \Omega_{p,n,\beta;C,A} \). As the analysis is necessary and sufficient, the formula (9.107) follows. \( \square \)

**Lemma 9.31.** Let \( M \) be a closed \( S_{\omega_{p,n},R} \)-invariant subspace of \( H^2_{\omega_{p,n},Y}(F^+_d) \) with reproducing kernel \( k_M \) given by (9.106). Then for every \( \beta \in F^+_d \), the formal reproducing kernel for the space \( M_\beta \) (defined in (9.105)) in the metric of \( H^2_{\omega_{p,n},Y}(F^+_d) \) is given by
\[
k_{M_\beta}(z, \zeta) = z^\beta \omega_{p,n;\beta}^1 I_Y - \mathcal{K}_{p,n;\beta}(z, \zeta) + \sum_{j=1}^d \mathcal{K}_{p,n;j;\beta}(z, \zeta), \quad (9.108)
\]
where \( \mathcal{K}_{p,n;\beta} \) is the kernel defined as in (9.98).
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Proof. Taking orthogonal complements in the equality \((9.107)\) gives
\[
S_{\omega_{p,n,R}^\beta}^\top M = (S_{\omega_{p,n,R}^\beta}^\top H^2_{p,n,y}(F^+_d) ) \ominus (S_{\omega_{p,n,R}^\beta}^\top \text{Ran} \, \mathcal{O}_{p,n,\beta;C,A}). \tag{9.109}
\]
Since for any $\beta \in F^+_d$, the reproducing kernel for the space $S_{\omega_{p,n,R}^\beta}^\top H^2_{p,n,y}(F^+_d)$ equals
\[
k_{nc,p,n;\beta}(z, \zeta) := \sum_{\alpha \in F^+_d} \omega_{p,n,\alpha\beta}^\top z^{\alpha} \zeta^{\beta}(\alpha\beta)^\top = k_{S_{\omega_{p,n,R}^\beta}^\top H^2_{p,n,y}(F^+_d)}(z, \zeta), \tag{9.110}
\]
and since the reproducing kernel for the space $S_{\omega_{p,n,R}^\beta}^\top \text{Ran} \, \mathcal{O}_{p,n,\beta;C,A}$ equals $\mathcal{R}_{p,n;\beta}$ (by Lemma 9.23), we conclude from the formula \((9.109)\) that the reproducing kernel for the space $S_{\omega_{p,n,R}^\beta}^\top M$ (in the metric of $H^2_{p,n,y}(F^+_d)$) equals
\[
k_{S_{\omega_{p,n,R}^\beta}^\top M}(z, \zeta) = k_{nc,p,n;\beta}(z, \zeta)I_y - \mathcal{R}_{p,n;\beta}(z, \zeta). \tag{9.111}
\]
Replacing $\beta$ by $j\beta$ in \((9.111)\) gives
\[
k_{S_{\omega_{p,n,R}^\beta}^\top S_{\omega_{p,n,R}^\beta}^\top M}(z, \zeta) = k_{nc,p,n;j\beta}(z, \zeta)I_y - \mathcal{R}_{p,n;j\beta}(z, \zeta). \tag{9.112}
\]
Using the reproducing kernels \((9.111)\) and \((9.112)\), we conclude from the orthogonal representation \((9.106)\) for $\mathcal{M}_\beta$ that the reproducing kernel for $\mathcal{M}_\beta$ equals
\[
k_{\mathcal{M}_\beta}(z, \zeta) = k_{S_{\omega_{p,n,R}^\beta}^\top M}(z, \zeta) - \sum_{j=1}^d k_{S_{\omega_{p,n,R}^\beta}^\top S_{\omega_{p,n,R}^\beta}^\top M}(z, \zeta)
\]
\[
= k_{nc,p,n;\beta}(z, \zeta)I_y - \mathcal{R}_{p,n;\beta}(z, \zeta) - \sum_{j=1}^d (k_{nc,p,n;j\beta}(z, \zeta)I_y - \mathcal{R}_{p,n;j\beta}(z, \zeta)).
\]
We finally observe from \((9.110)\) that
\[
k_{nc,p,n;\beta}(z, \zeta) = \sum_{j=1}^d k_{nc,p,n;j\beta}(z, \zeta)
\]
\[
= \sum_{\alpha \in F^+_d} \omega_{p,n,\alpha\beta}^\top z^{\alpha} \zeta^{\beta}(\alpha\beta)^\top - \sum_{j=1}^d \sum_{\alpha \in F^+_d} \omega_{p,n,\alpha j\beta}^\top z^{\alpha} \zeta^{j\beta}(\alpha j\beta)^\top
\]
\[
= \sum_{\alpha \in F^+_d} \omega_{p,n,\alpha\beta}^\top z^{\alpha} \zeta^{\beta}(\alpha\beta)^\top - \sum_{\alpha \in F^+_d : \alpha \neq \emptyset} \omega_{p,n,\alpha\beta}^\top z^{\alpha} \zeta^{\beta}(\alpha\beta)^\top = \omega_{p,n,\beta}^\top z^{\beta}(\beta)^\top.
\]
Combining the two latter equalities completes the proof of \((9.108)\). \qed

We next factor the kernel \((9.108)\) as follows.

Lemma 9.32. Given an exactly $(p,n)$-observable and $(p,n)$-output-stable pair $(C, A)$ with $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $A = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{X})^d$, let $[\hat{B}_\beta] \in \mathcal{U}_\beta \rightarrow \mathcal{X}^d \oplus \mathcal{Y}$ (with $\hat{B}_\beta = \begin{bmatrix} B_{1,\beta} & \vdots & B_{d,\beta} \end{bmatrix}$) be an injective solution to the Cholesky factorization problem:
\[
[\hat{B}_\beta^* \ D_{\beta}] = \begin{bmatrix} \hat{G}_{p,n,\beta;C,A}^{-1} I_y & 0 \\ 0 & \hat{G}_{p,n,\beta;C,A} \end{bmatrix} \begin{bmatrix} \hat{A}^* & \hat{C}^* \end{bmatrix}, \tag{9.113}
\]
(where $\mathbf{G}_{p,n,\beta;C,A}$ and $\tilde{\mathbf{G}}_{p,n,\beta;C,A}$ are given by (9.97) and (9.100)) and let $\Theta_{p,n,U_\beta}(z)$ be defined as in (9.14). Then

1. The equality (9.101) holds and hence, $\Theta_{p,n,U_\beta}(z)$ is $(p,n)$-Bergman inner.
2. Then the kernel (9.108) can be factored as

$$k_{M_\beta}(z, \zeta) = \Theta_{p,n,U_\beta}(z)(z^{d-1} I_{U_\beta}) \Theta_{p,n,U_\beta}(\zeta^*). \quad (9.114)$$

3. The operator $M_{\Theta_{p,n,U_\beta}} : z^\beta u \mapsto \Theta_{p,n,U_\beta}(z)z^\beta u$ is unitary from $z^\beta U_\beta$ (considered as a Hilbert space with lifted norm $\|z^\beta u\|_{z^\beta U_\beta} = \|u\|_{U_\beta}$) onto $M_\beta$.

\textbf{Proof.} The existence of an injective solution to the factorization problem (9.113) and verification of the equality (9.101) goes through by the arguments used in the proof of Lemma (7.5). The multiplier $\Theta_{p,n,U_\beta}$ is $(p,n)$-Bergman inner, by Lemma (9.27). Making use of the formula (9.14) and taking into account equalities of the corresponding blocks in (9.102), we get

\begin{align*}
\omega^{-1}_{p,n;\beta} I_Y - \Theta_{p,n,U_\beta}(z)\Theta_{p,n,U_\beta}(\zeta)^* & = \omega^{-1}_{p,n;\beta} I_Y - (\omega^{-1}_{p,n;\beta} D_\beta + C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \tilde{B}_\beta) (\omega^{-1}_{p,n;\beta} D_\beta^* + \tilde{B}_\beta^* \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^*) \\
& = \omega^{-1}_{p,n;\beta} I_Y - \omega^{-1}_{p,n;\beta} D_\beta D_\beta^* - \omega^{-1}_{p,n;\beta} C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \tilde{B}_\beta D_\beta^* \\
& \quad - \omega^{-1}_{p,n;\beta} D_\beta B_\beta^* \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* - C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \tilde{B}_\beta B_\beta^* \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \\
& = \omega^{-1}_{p,n;\beta} C \mathbf{G}_{p,n,\beta;C,A} C^* + \omega^{-1}_{p,n;\beta} C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) A \mathbf{G}_{p,n,\beta;C,A}^{-1} \\
& \quad + \omega^{-1}_{p,n;\beta} C \mathbf{G}_{p,n,\beta;C,A}^{-1} A^* \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \\
& \quad - C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \left[ \mathbf{G}_{p,n,\beta;C,A}^{-1} - A \mathbf{G}_{p,n,\beta;C,A}^{-1} A^* \right] \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \\
& = C \left( \omega^{-1}_{p,n;\beta} I_X + \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) A \mathbf{G}_{p,n,\beta;C,A}^{-1} \omega^{-1}_{p,n;\beta} I_X + A^* \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \\
& \quad - C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \mathbf{G}_{p,n,\beta;C,A}^{-1} \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \\
& \quad - C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \mathbf{G}_{p,n,\beta;C,A}^{-1} \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^* \right) \\
& = CR_{p,n;\beta}(zA) \mathbf{G}_{p,n,\beta;C,A}^{-1} R_{p,n;\beta}(\zeta A)^* C^* \\
& \quad - C \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \mathbf{G}_{p,n,\beta;C,A}^{-1} \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^* C^*, \quad (9.115)
\end{align*}

where for the last step we used the equality

$$\omega^{-1}_{p,n;\beta} I + \tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) A = \omega^{-1}_{p,n;\beta} I_X + \sum_{j=1}^{d} R_{p,n;\beta}(zA) z_j A_j = R_{p,n;\beta}(zA),$$

which follows from (9.9) by definitions (9.15). We next observe from definitions (9.100) and again (9.15) that

$$\tilde{R}_{p,n;\beta}(zA) \tilde{Z}(z) \mathbf{G}_{p,n,\beta;C,A}^{-1} \tilde{Z}(\zeta)^* \tilde{R}_{p,n;\beta}(\zeta A)^*$$

$$= \sum_{j=1}^{d} R_{p,n;\beta}(zA) z_j \mathbf{G}_{p,n,\beta;C,A}^{-1} z_j R_{p,n;\beta}(\zeta A)^*$$
Substituting the latter equality into (9.115) and then multiplying both sides in (9.115) by \( z^\beta \) on the right and by \( \bar{z}^{\beta^\top} \) on the left, we get

\[
\omega_{p,n;\beta}^{-1} z^{\beta^\top} I_{Y} - \Theta_{n,p;U_{\beta}}(z) (z^{\beta^\top} I_{U_{\beta}}) \Theta_{n,p;U_{\beta}}(\zeta)^* \\
= CR_{p,n;\beta}(zA)(z^{\beta^\top} \bar{z}^{\beta^\top} \Theta_{p,n,\beta;C,A}^{-1}) R_{p,n;\beta}(\zeta A)^* C^* \\
- \sum_{j=1}^{d} R_{p,n,j;\beta}(zA)(z^{\beta^\top} \bar{z}^{\beta^\top}) \Theta_{p,n,j;\beta;C,A}^{-1} R_{p,n,j;\beta}(\zeta A)^*
\]

\[
= \mathcal{R}_{p,n;\beta}(z, \zeta) - \sum_{j=1}^{d} \mathcal{R}_{p,n,j;\beta}(z, \zeta),
\]

where we used the definition (9.95) for the last step. Comparing the latter formula with (9.108) we conclude (9.114). The last statement of the lemma is justified exactly as in Lemma 7.10. \( \square \)

### 9.8. Operator model theory for c.n.c. \( *(p,n) \)-hypercontractive operator tuples \( T \)

In this brief final section we consider a model theory for \( *(p,n) \)-hypercontractive operator tuples analogous to that developed in Chapter 8 for the case of \( *\omega \)-hypercontractive tuples. As in Chapter 8 there are two flavors: (i) model theory with characteristic function equal to a contractive multiplier, and (ii) model theory based on Bergman-inner families.

#### 9.8.1. \( (p,n) \)-model theory based on contractive multipliers

We now consider operator \( d \)-tuples \( T = (T_1, \ldots, T_d) \) on a Hilbert space \( X \) such that the operator \( d \)-tuple \( A = T^* = (T_1^*, \ldots, T_d^*) \) is a \( (p,n) \)-hypercontractive, or equivalently, that \( T^* \) is \( (p,1) \)- and \( (p,n) \)-contractive (see Definition 9.1 and Corollary 9.12). Thus in particular \( \Gamma_{p,n;T^*}[I_X] \geq 0 \). We define the \( (p,n) \)-defect operator \( D_{p,n;T^*} \) of \( T^* \) and the \( (p,n) \)-defect space \( D_{p,n;T^*} \) by

\[
D_{p,n;T^*} = \Gamma_{p,n}[I_X] - \{ \text{Defect space} \}.
\]

Then by construction the pair \( (D_{p,n;T^*}, T^*) \) is a \( (p,n) \)-isometric output pair (see Definition 9.19). We next form the \( (p,n) \)-observability operator \( O_{p,n;D_{p,n;T^*}} \) as in (9.12). As a consequence of part (2) of Theorem 9.18 \( O_{p,n;D_{p,n;T^*}} \) maps \( \mathcal{X} \) contractively into \( H^2_{\omega,p,n;D_{p,n;T^*}}(\mathbb{F}_d^+) \), so in particular the output pair \( (D_{p,n;T^*}, T^*) \) is also \( (p,n) \) stable in the sense of the definition given at the beginning of Section 9.3.

We shall assume in addition that \( T \) is \( (p,n) \)-completely noncoisometric (or \( (p,n) \)-c.n.c. for short), by which we mean that the observability operator \( O_{p,n;D_{p,n;T^*}} \) has trivial kernel. Thus we may define a norm on the space \( \mathcal{N} := \text{Ran} O_{\omega,D_{\omega;T^*}} \) as the lifted norm defined by

\[
\|O_{\omega,D_{\omega;T^*}} \cdot x\|_n = \|x\|_X.
\]

Then as a consequence of Theorem 9.18 \( \mathcal{N} \) has the structure of a NFRKHS \( \mathcal{N} = \mathcal{H}(K_{\mathcal{N}}) \) with kernel \( K_{\mathcal{N}} \) given by

\[
K_{\mathcal{N}}(z, \zeta) = D_{p,n;T^*} R_{p,n}(Z(z)T^*) R_{p,n}(Z(\zeta)T^*)^* D_{p,n;T^*}.
\]
\( N \) is \( S^*_{p,n,R} \)-invariant, and the observability operator \( O_{\omega,D,T^*} \) implements a unitary equivalence between \( T^* \) and \( S^*_{\omega,n,R}|_N \).

Let us make the following formal definition.

**Definition 9.33.** We shall say that the \((p,n)\)-c.n.c. \(*-(p,n)\)-hypercontractive tuple \( T \) admits a characteristic multiplier \( \Theta_T \) if the Brangesian complement \( M = N^\perp \) of \( N = \text{Ran} O_{p,n,D,T^*} \) (equipped with the lifted norm) admits a Beurling-Lax representation \( M = \Theta \cdot H^2_d(\mathbb{P}_d^+) \) as in Theorem 9.19. We then say that \( \Theta \) is a characteristic multiplier \( \Theta_T \) for \( T \).

With this definition in hand it is possible to formulate \((p,n)\)-analogues of Theorem 8.2 and 8.3. We leave the details to the interested reader.

**9.8.2.** \((p,n)\)-model theory based on Bergman-inner families. This flavor of model theory for the moment handles only the class of \(*-(p,n)\)-hypercontractive tuples \( T = (T_1, \ldots, T_d) \) with \( T^* \) \( p \)-strongly stable. In this case we define the \((p,n)\)-defect operator \( D_{p,n,T^*} \) as in (9.116) and form the observability operator \( O_{p,n,D_{p,n,T^*},T^*} \). As a consequence of part (3e) in Theorem 9.18, \( O_{p,n,D_{p,n,T^*},T^*} \) is isometric and \( N = \text{Ran} O_{p,n,D_{p,n,T^*},T^*} \) sits isometrically in \( H^2_{\omega_{p,n},D_{p,n,T^*}}(\mathbb{P}_d^+) \) as a closed \( S^*_{p,n,R} \)-invariant subspace. Then \( M := N^\perp \) is \( S_{p,n,R} \)-invariant and, assuming that \((p,n)\) satisfy the mild restriction (9.93), we can represent \( M \) via a \((p,n)\)-Bergman-inner family \( \{\Theta_T, \beta \in \mathbb{F}_d^+\} \) as in Section 9.7. This \((p,n)\)-Bergman-inner family we declare to be the characteristic Bergman-inner family for \( T \) since it serves as a complete unitary invariant (up to natural identifications) for the original \( p \)-strongly stable, \(*-(p,n)\)-hypercontractive \( d \)-tuple \( T \). With the Beurling-Lax representation theorems of Section 9.6 replacing those of Section 7.2, we get a direct analog of Theorem 8.5. Furthermore one can check that the structures are sufficiently parallel that one can derive a \((p,n)\)-analog of the realization formula (8.9) for a given \((p,n)\)-Bergman-inner family \( \{\Theta_\beta : \beta \in \mathbb{F}_d^+\} \).
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