Kac–Moody symmetries of IIB supergravity

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Abstract

We formulate the bosonic sector of IIB supergravity as a non-linear realisation. We show that this non-linear realisation contains the Borel subalgebras of $SL(11)$ and $E_7$ and argue that it can be enlarged so as to be based on the rank eleven Kac–Moody algebra $E_{11}$.

One of the most remarkable features of supergravity theories is that the scalars they contain always occur in a coset structure. While this can be viewed as a consequence of supersymmetry, the groups that occur in these cosets are rather mysterious. The two most studied examples are perhaps the $E_7/SU(8)$ [7] of the maximal supergravity in four dimensions and the $SU(1,1)/U(1)$ [4] of the ten-dimensional IIB theory. It has been conjectured [8] that the symmetries found in these cosets are symmetries of the associated non-perturbative string theory.

The coset construction was extended [9] to include the gauge fields of supergravity theories. This method used generators that were inert under Lorentz transformations and, as such, it is difficult to extend this method to include either gravity or the fermions. However, this construction did include the gauge and scalar fields as well as their duals, and as a consequence the equations of motion for these fields could be expressed as a generalised self-duality condition. This formulation was given for eleven-dimensional supergravity, all its reductions to four dimensions as well as for the IIB theory [9].

Recently [10], it was shown that the entire bosonic sectors of eleven-dimensional and ten-dimensional IIA supergravity theories could be formulated as non-
linear realisations. In this way of proceeding gravity and the gauge fields appeared on an equal footing and one could hope to see the full symmetries of supergravity theories.

Here we shall confirm the conjecture in references [10,11] that the entire bosonic sector of ten-dimensional IIB supergravity can also be formulated as a non-linear realisation. The formulation of the IIB theory we find is one in which all the degree of freedom of the theory, except for the graviton and the four form gauge field, whose field strength satisfies a type of self-dual condition. The complete field content is then given by the set

\[ h_a^b, A^1, A^c_{11,2}, A^2_{11,3}, A^c_{11,4}, A^c_{11,5}, A^c_{11,6}, \] (2.1)

where \( s \) can take value 1 or 2 corresponding to the sectors. Each of the above fields is to be a Goldstone boson and as such we introduce a corresponding set of generators which is then given by

\[ K^a_b, R_s, R^c_{s1-cp}, R^c_{s1-cp}, R^c_{s1-cp}, s = 1, 2. \] (2.2)

We also include the momentum generator \( P_c \) which introduces space–time into the group element.

We take the generators to obey the following relations:

\[
\begin{align*}
[K^a_b, K^c_d] &= \delta^b_d K^a_c - \delta^a_c K^b_d, \\
[K^a_b, P_c] &= -\delta^a_c P_b, \\
[K^a_b, R_s] &= \delta^c_b R^c_{s1-cp} + \cdots, \\
[R^c_{s1-cp}, R^c_{s1-cp}] &= \epsilon_{p,q} R^c_{p,q}, \\
[R^c_{s1-cp}, R^c_{s1-cp}] &= \epsilon_{p,q} R^c_{p,q}. \\
\end{align*}
\] (2.3)

where \( + \cdots \) means the appropriate anti-symmetrisations. The generators \( K^a_b \) satisfy the commutation relations of \( GL(10, \mathbb{R}) \). In the third line the superscript \( s \) depends on the fields in the commutator and we have therefore written \( s = s(s_1, s_2) \). This function satisfies the properties \( s(1, 1) = s(2, 2) = 1, s(1, 2) = s(2, 1) = 2 \). In the last line we have split the scalar commutators into those for the dilaton (superscript 1) with coefficient \( d^p_{s1} \), and the axion (subscript 2) with coefficient \( d^p_{s2} \). One can see that in the commutator the dilaton is sector preserving while the axion changes the sector of the other generator. The Jacobi identity implies the following relations among the constants:

\[
\begin{align*}
\epsilon_{q,z}^{s_2} \epsilon_{p,q}^{s_1} x(s_2, s_3) &\equiv \epsilon_{p,q}^{s_1} \epsilon_{q,z}^{s_2} x(s_1, s_3), \\
&\quad + \epsilon_{p,z}^{s_1} \epsilon_{q,p}^{s_2} x(s_1, s_3), \\
&\quad \epsilon_{q,z}^{s_2} \epsilon_{p,q}^{s_1} x(s_2, s_3), \\
&\quad \epsilon_{p,z}^{s_1} \epsilon_{q,p}^{s_2} x(s_1, s_3), \\
&\quad \epsilon_{q,z}^{s_2} \epsilon_{p,q}^{s_1} x(s_2, s_3), \\
&\quad \epsilon_{p,z}^{s_1} \epsilon_{q,p}^{s_2} x(s_1, s_3), \\
\end{align*}
\] (2.4)

where

\[
\epsilon_{0,q}^{s_2} = \epsilon_{0,q}^{s_2},
\quad \text{and} \quad \epsilon_{0,q}^{s_2} + \epsilon_{p,q}^{s_1} \epsilon_{q,p}^{s_2} = 0. \] (2.5)
The constants in the above commutation relations are taken to be:
\begin{align}
\delta_{ab}^1 &= \delta_{ab}^2 = -\delta_{ab}^3 = -\frac{1}{2}, \\
\delta_{ab}^0 &= -\delta_{ab}^8 = -1, \\
\epsilon_{1,2,2}^1 &= -\epsilon_{1,2,2}^2 = -1, \\
\epsilon_{2,4}^1 &= -\epsilon_{2,4}^2 = -4, \\
\epsilon_{2,6}^1 &= 1, \\
\epsilon_{2,6}^2 &= -\frac{1}{2}, \\
\delta_{ab}^1 &= \delta_{ab}^2 = -\delta_{ab}^3 = 1, \\
\delta_{ab}^0 &= \delta_{ab}^8 = \delta_{ab}^3 = 0. 
\end{align} 
(2.6)
All not mentioned coefficients are zero. One can verify that they do indeed satisfy the Jacobi relations. We denote the above algebra by \( G_{IIB} \).

The algebra \( G_{IIB} \) possesses the Lorentz algebra as a subalgebra. The generators \( J_{ab} \) of the latter are given by the anti-symmetric part of the \( K^a_b \) generators, i.e., \( J_{ab} = K_{ab} - K_{ba} \), where the indices are lowered and raised with the Minkowski metric. We will show that IIB supergravity can be described as a nonlinear representation of the group \( G_{IIB} \) taking the Lorentz group as the local subgroup. The general element of \( G_{IIB} \) can be written as
\[
g = \exp(x^a P_a) \exp(h_a b K^a_b) g_A \equiv g h g_A, 
\tag{2.8}
\]
where
\[
g_A = \exp\left(\frac{1}{8!} A_{21}^2 A_{21}^1 - \frac{1}{8!} A_4 \cdot K^4_1 - \frac{1}{8!} K_{21}^1 \cdot K_1 - \frac{1}{8!} K_{21}^2 \cdot K_2 \right) e^{(1/8) A_{21}^1 A_{21}^2} e^{(1/8! A_4 \cdot K^4_1} + A_{21}^1 \cdot K^2_1 + A_{21}^2 \cdot K^1_1 \right) e^{(1/4) A_{21}^1 A_{21}^2} e^{(1/2) A_{21}^1 A_{21}^2} e^{(1/2) A_{21}^1 A_{21}^2}. \tag{2.9}
\]
For easier identification with the known literature in what follows below we will often relabel \( A^1 = \sigma \) and \( A^2 = \chi \).

Following the standard procedure of non-linear realisations we demand that the theory be invariant under
\[
g \to g_0 g h^{-1}, \tag{2.10}
\]
where \( g_0 \) is an element from the whole group \( G_{IIB} \) and is a rigid transformation while \( h \) is a local Lorentz transformation.

We now calculate the Maurer–Cartan form
\[
\mathcal{V} = g^{-1} dg - \omega \tag{2.11}
\]
in the presence of the Lorentz connection \( \omega = \frac{1}{2} \eta^\mu \kappa_\mu \) \( J^a \), which transforms as
\[
\omega \to h \omega h^{-1} + h d h^{-1}. \tag{2.12}
\]
As a result, \( \mathcal{V} \) transforms as
\[
\mathcal{V} \to h \mathcal{V} h^{-1}. \tag{2.13}
\]
Writing \( \mathcal{V} \) in the form
\[
\mathcal{V} = (g_h^{-1} d g_h) + (g_A^{-1} d g_A + g_A^{-1} (g_h^{-1} d g_h) g_A - g_h^{-1} d g_h), \tag{2.14}
\]
using the relations
\[
e^{-A} d e^A = d A - \frac{1}{2} [A, d A] + \frac{1}{6} [A, [A, d A]]
\]
\[
- \frac{1}{24} [A, [A, [A, d A]]] + \cdots,
\]
\[
e^{-A} B e^A = B - [A, B] + \frac{1}{2} [A, [A, B]] + \cdots, \tag{2.15}
\]
and the commutation relations of the \( G_{IIB} \) algebra and we find that
\[
\mathcal{V} \equiv d x^\mu \left(e_\mu^a P_a + d x^\mu \Omega_{\mu a b} K^a_b \right) + d x^\mu \left( \sum_{p=1}^8 \frac{1}{p!} e^{-d^p - \sigma} \tilde{D}_\mu A_{a_1 \cdots a_p} R_{a_2 a_3 \cdots a_p} \right), \tag{2.16}
\]
where
\[
e^a_\mu \equiv (e^b_\mu)^a_b, \tag{2.17}
\]
\[
\Omega_{ab}^c \equiv \left(e^{-1}_a \right)^\mu_\mu \left(e^{-1}_b \sigma \right)^c_c - \omega_{ab} e^c \tag{2.18}
\]
and the definition of \( \tilde{D}_\mu A_{a_1 \cdots a_p} \) will be given below.

The IIB supergravity theory is the non-linear realisation of the group that is the closure of the \( G_{IIB} \) algebra given above with the ten-dimensional conformal algebra. However, rather than working with this infinite dimensional group we first construct the Cartan forms of the \( G_{IIB} \) algebra, as above, and then take only such combinations of these that can be rewritten in terms of Cartan forms of the conformal group. This procedure was described in detail in reference [10] and we will simply state the results of this method when applied to the \( G_{IIB} \) algebra. In the gravity sector we adopt the unique constraint
\[
\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0, \tag{2.19}
\]
which gives the usual expression for the spin connection in terms of the vielbein. The only objects which are Lorentz covariant and therefore covariant under the full non-linear realisation composed out of the closure of the conformal and G\(\mathfrak{g}_\text{III}\) algebras are the Riemann tensor composed out of the spin-connection in the usual way and the completely anti-symmetrised derivatives \(e^{-d^a}_\mu \sigma \tilde{D}_{a1}A_{a2...a_9}\).

The latter are denoted by
\[
\tilde{F}^{\sigma}_{a1...a_9} = pe^{-d^a}_\mu \sigma \tilde{D}_{a1}A_{a2...a_9}.
\]
(2.20)

We observe that these expressions begin with the field strength of the gauge fields as they should. The explicit expressions for these objects, whose calculation was explained above, are then given for the scalars by
\[
\begin{align*}
\tilde{F}^1_a &= \tilde{D}_a \sigma, \\
\tilde{F}^2_a &= e^\sigma \tilde{D}_a \chi,
\end{align*}
\]
(2.21)

for the 2-index fields:
\[
\begin{align*}
\tilde{F}^1_{a_1a_2} &= 3e^{-\frac{1}{2}}(\tilde{D}_{a_1}A_{a_2a_3} - \chi \tilde{D}_{a_1}A_{a_3a_2}), \\
\tilde{F}^2_{a_1a_2} &= 3e^{\frac{1}{2}}(\tilde{D}_{a_1}A_{a_2a_3} - \chi \tilde{D}_{a_1}A_{a_3a_2}).
\end{align*}
\]
(2.22)

for the 4-index field
\[
\begin{align*}
\tilde{F}_{a_1...a_5} &= 5(\tilde{D}_{a_1}A_{a_2...a_5} + 3A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5} \\
&\quad - 3A_{a_1a_2}^2 \tilde{D}_{a_1}A_{a_3a_4a_5}),
\end{align*}
\]
(2.23)

for the 6-index fields
\[
\begin{align*}
\tilde{F}_{a_1...a_7} &= 7e^{-\frac{1}{4}}(\tilde{D}_{a_1}A_{a_2...a_7} \\
&\quad + 60A_{a_1a_2} \tilde{D}_{a_1}A_{a_3...a_7} + A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7} \\
&\quad - A_{a_1a_2}^2 \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7}),
\end{align*}
\]
(2.24)

and finally for the 8-index fields
\[
\begin{align*}
\tilde{F}_{a_1...a_9} &= 9(\tilde{D}_{a_1}A_{a_2...a_9} \\
&\quad - 7 \cdot 2A_{a_1a_2} \tilde{D}_{a_1}A_{a_3...a_9} \\
&\quad - 6 \cdot 5A_{a_1a_2} A_{a_3a_4a_5a_6a_7a_8a_9} \\
&\quad + \frac{1}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3...a_9} - \frac{3}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3...a_9}),
\end{align*}
\]
\[
\begin{align*}
&\quad + 7 \cdot 2A_{a_1a_2} (\tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9}) \\
&\quad + 6 \cdot 5A_{a_1a_2} (\tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9}) \\
&\quad + \frac{1}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9} \\
&\quad + \frac{1}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9})
\end{align*}
\]
\[
\begin{align*}
&\quad + 9 \chi (\tilde{D}_{a_1}A_{a_2...a_9} \\
&\quad - 7 \cdot 4A_{a_1a_2} (\tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9}) \\
&\quad + 6 \cdot 5A_{a_1a_2} (\tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9}) \\
&\quad + \frac{1}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9} - \frac{3}{2} A_{a_1a_2} \tilde{D}_{a_1}A_{a_3a_4a_5a_6a_7a_8a_9}).
\end{align*}
\]
(2.26)

The equations of motion can only be constructed from the spin connection and the covariant objects \(\tilde{F}^{\sigma}_{a_1...a_9}\) and can only be
\[
\begin{align*}
\tilde{F}^{(1)\mu\nu} &= \frac{1}{7!} \epsilon^{\mu\nu\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}^{\sigma}_{\rho\sigma\tau\delta\epsilon\zeta}, \\
\tilde{F}^{(2)\mu\nu} &= \frac{1}{7!} \epsilon^{\mu\nu\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}^{\sigma}_{\rho\sigma\tau\delta\epsilon\zeta},
\end{align*}
\]
(2.28)

The remaining equation of motion is that for the vielbein and is given by
\[
\begin{align*}
R_{\mu\nu} &= -\frac{1}{2} g_{\mu\nu} R - \frac{1}{2} g_{\mu\nu} (\partial_{\rho} \sigma \partial_{\rho} \sigma + \frac{1}{2} \epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \partial_{\rho} \chi \partial_{\rho} \chi) \\
&\quad - \frac{1}{4} g_{\mu\nu} (\partial_{\rho} \sigma \partial_{\rho} \sigma + \epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \partial_{\rho} \chi \partial_{\rho} \chi) \\
&\quad - \frac{1}{6} \tilde{F}_{\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}^{\rho\sigma\tau\delta\epsilon\zeta} - \frac{1}{16} \epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}_{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta} \tilde{F}^{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta}} \\
&\quad - \frac{1}{16} \epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}_{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta} \tilde{F}^{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta}} \\
&\quad + \frac{1}{96} g_{\mu\nu} (\epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}_{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta} \tilde{F}^{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta}} \\
&\quad + \epsilon^{\rho\sigma\tau\delta\epsilon\zeta} \tilde{F}_{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta} \tilde{F}^{(\rho}^{\mu\nu)_{\sigma\tau\delta\epsilon\zeta}}) = 0.
\end{align*}
\]
(2.30)
The value of the constants in front of the field strength squared terms can only be fixed by considering the full non-linear realisation of the IIB theory that includes the fermionic sector of the theory or the Kac–Moody groups considered later in this paper. In the above equation we have fixed the values of these constants to their correct values.

We can obtain the more standard second order equations of IIB supergravity in terms of the original fields without their duals by differentiating the equations (2.28) and (2.29) and using the Bianchi identities of the dual field strengths. For example, we can rewrite the second equation in equation (2.28) as

$$\partial_\mu (\partial_\mu \sigma) = e_\sigma \epsilon_{\mu \nu \rho \lambda} \tilde{G}^{\nu \rho \lambda}_{\mu \nu \rho \lambda}. \quad (2.31)$$

Similarly we find that the other equations imply that

$$\partial_\mu (e^{\sigma} \tilde{F}_{\mu \nu \rho\lambda}^2) = \frac{2}{3} e^{\sigma \rho \mu} \tilde{F}_{\nu \rho \lambda} G_{\mu \nu \rho \lambda, \mu \rho \lambda}. \quad (2.32)$$

Since the field strength of the four form gauge field is self-dual we leave this equation to be first order in derivatives.

The IIB supergravity theory was first discovered [4–6] in a formulation in which the scalars belong to the coset $SU(1, 1)/U(1)$. In this formulation the field content consists of the graviton, a complex two form gauge field $A_{\mu \nu}$, a real four form $A_{a_1 ... a_4}$ and two scalars denoted by the complex field $\phi$ which belong to the coset $SU(1, 1)/U(1)$. The rewriting of the IIB supergravity theory in terms of an $SL(2, \mathbb{R})/SO(2)$ coset was given in Ref. [12] and was reviewed in Ref. [13]. The equations of motion given above are precisely those found in this latter formulation.

### 3. E\textsubscript{11} and IIB supergravity

In reference [11] was argued that eleven-dimensional supergravity was invariant under a Kac–Moody algebra that was identified to be $E_{11}$. We refer the reader to this paper for a discussion of how the non-linear realisation of eleven-dimensional supergravity given in Ref. [10] might be extended to incorporate such a large algebra by using an alternative formulation of eleven-dimensional supergravity and increasing the size of the local subgroup. The same group was identified as a symmetry of the IIA supergravity theory. In this section, we will assume that the non-linear realisation of the IIB supergravity theory given above can be similarly enlarged to a non-linear realisation of a Kac–Moody algebra. We will show that this algebra is also $E_{11}$.

The proposed Kac–Moody algebra of the IIB theory must contain the algebra denoted $G_{\text{IIB}}$ above and given in Eqs. (2.3)–(2.7). In the non-linear realisation discussed above the local subgroup is taken to be the ten-dimensional Lorentz group and so all the remaining generators in $G_{\text{IIB}}$ are coset generators and as such correspond to fields in IIB supergravity. In the enlarged non-linear realisation based on a Kac–Moody algebra, the local subgroup is taken to be that invariant under the Cartan involution and as a result the coset representatives can be written as exponentials of the Cartan subalgebra and positive root generators of the Kac–Moody algebra. Consequently, all the generators of $G_{\text{IIB}}$, except the negative root generators of $SL(10)$, must be included in the Cartan subalgebra and positive root generators of the Kac–Moody algebra. A set of commuting generators of $G_{\text{IIB}}$ can be taken to be

$$K_{a}^{a}, \quad a = 1, \ldots, 10, \quad \text{and} \quad R_{1}, \quad (3.1)$$

and these may be taken to belong to the Cartan subalgebra of the Kac–Moody algebra. We also observe that the remaining generators of $G_{\text{IIB}}$, except the negative root generators of $SL(10)$, can be generated by taking multiple commutators of the generators

$$K_{a+1}^{a}, \quad a = 1, \ldots, 9, \quad R_{2} \quad \text{and} \quad R_{10}^{910}. \quad (3.2)$$

We may identify these as positive simple root generators of the Kac–Moody algebra. Thus we are seeking a rank eleven Kac–Moody algebra.

Calculating the commutator of the positive simple root and Cartan sub-algebra generators leads to the
Cartan matrix from which we can uniquely identify the Kac–Moody algebra. However, working with only the Cartan sub-algebra and simple positive root generators — that is without the negative simple root generators — does not automatically encode the particular basis for the Cartan sub-algebra that satisfies the Chevalley relations and hence produces the correct Cartan matrix. However, we must use a basis that leads to an acceptable Cartan matrix, that is one which satisfies the correct properties to be associated with a Kac–Moody algebra. Even taking this into account, the choice of the basis is not free from ambiguity. As in Ref. [11] this ambiguity may be resolved by identifying appropriate subgroups and so we first carry out this step.

The non-linear realisation of the IIB supergravity theory given above is obviously invariant under appropriate subgroups and so we first carry out this step.

The simple positive root generators are given by

\[ E_{a} = K^{a}_{a+1}, \quad a = 1, \ldots, 8, \]
\[ E_{9} = R_{10}^{9}, \quad E_{10} = R_{2}, \quad E_{11} = K^{9}_{10}. \]

These agree with the simple positive root generators of the SL(11) and E7 groups in the proposed rank eleven Kac–Moody algebra we can finally identify this Kac–Moody Lie algebra. The simple positive root generators are given by

\[ E_{a} = K^{a}_{a+1}, \quad a = 1, \ldots, 8, \]
\[ E_{9} = R_{10}^{9}, \quad E_{10} = R_{2}, \quad E_{11} = K^{9}_{10}. \]

These agree with the simple positive root generators of the SL(11) and E7 subgroups found above. The basis of the Cartan subalgebra that leads to an acceptable Cartan matrix and agrees with the above identifications of the Cartan subalgebra elements of SL(11) and E7 is given by

\[ H_{a} = K^{a}_{a} - K^{a+1}_{a+1}, \quad a = 1, \ldots, 8, \]
\[ G = R_{10}^{9}, \quad G_{11} = K^{9}_{10}. \]
\[ H_0 = K^9 + K^{10} + R_1 - \frac{1}{4} \sum_{a=1}^{11} K^a a. \]

\[ H_{10} = -2R_1, \quad H_{11} = K^9 - K^{10}. \]  

(3.10)

One can verify that

\[ [H_a, E_b] = A_{ab} E_b \]  

(3.11)

where \( A_{ab} \) is the Cartan matrix for \( E_{11} \). Hence, we identify \( E_{11} \) as the Kac–Moody algebra that underlies IIB supergravity.

In Ref. [11] it was explained how one might enlarge the non-linear realisation of eleven-dimensional supergravity such that it contained the \( E_{11} \) Kac–Moody algebra. In particular, it was shown how by adopting a first order formulation of gravity involving the usual metric and a dual field one could extend the non-linear realisation to include the Borel subgroup of \( E_8 \). We refer the reader to this reference for the details of this procedure and we now outline the analogous steps for IIB supergravity.

We consider the restriction of the \( G_{IIB} \) algebra resulting from only considering generators with the indices \( i, j = 4, \ldots, 10 \). We will find the formulation of the \( E_8 \) algebra with the \( SL(8) \) symmetry manifest and so the generators will carry the indices \( i, j = 4, \ldots, 11 \). The Borel subgroup of this \( SL(8) \) is provided by the generators \( \tilde{K}^{i}_{j}, \tilde{1} < j \). Since this coincides with only the \( SL(6) \) subgroup of the obvious \( SL(8) \) we will have to treat the indices from \( i, j = 10, 11 \) differently from \( i, j = 4, \ldots, 9 \). The \( E_8 \) algebra possesses the commutation relations

\[ [X^{i_1 i_2 i_3}, X^{j_1 j_2 j_3}] = \epsilon^{i_1 i_2 i_3 j_1 j_2 j_3} \epsilon^{k_1 k_2} S_{k_1 k_2} \]  

(3.12)

and

\[ [S_{k_1 k_2}, X^{i_1 i_2 i_3}] = 3 \epsilon^{i_1 i_2 i_3} S^{k_1 k_2}. \]  

(3.13)

Reevaluating the commutators of Eqs. (3.12) and (3.13) with these modified relations we now find the missing generators of \( E_8 \) which are given by

\[ S_{kl} = \frac{1}{4!} \epsilon_{k l i_1 \cdots i_4} R_{1}^{i_1 \cdots i_4}, \]

\[ S_{k10} = \frac{2}{5!} \epsilon_{k i_1 \cdots i_5} R_{1}^{i_1 \cdots i_5 10}, \]

\[ S_{k11} = -\frac{2}{5!} \epsilon_{k i_1 \cdots i_5} R_{1}^{i_1 \cdots i_5 10}, \]

\[ S_{1011} = -\frac{1}{6!} \epsilon_{i_1 \cdots i_6} \tilde{R}_{1}^{i_1 \cdots i_6 10}, \]  

(3.18)

and

\[ S^k = -\frac{4}{5!} \epsilon_{n i_1 \cdots i_5} \tilde{R}_{1}^{i_1 \cdots i_5 [k, i_6]}, \]

\[ S^{10} = -\frac{4}{6!} \epsilon_{i_1 \cdots i_6} R_{1}^{i_1 \cdots i_6}, \]

\[ S^{11} = -\frac{4}{6!} \epsilon_{i_1 \cdots i_6} \tilde{R}_{1}^{i_1 \cdots i_6}. \]  

(3.19)

For the restriction of the restriction of the \( G_{IIB} \) algebra considered above that there are no other generators except those considered so far and these generate the Borel subalgebra of \( E_8 \). The 248 adjoint of \( E_8 \) decomposes into \( SL(8, R) \) representations as

\[ 248 = 1 \left( \sum_{j} K^j \right) + 63 \left( K^j \right) + 56 \left( X^{i_1 i_2 i_3} \right) \]

\[ + 28 \left( S^k \right) + 8 \left( S^{10} \right) \]  

(3.20)

as well as the negative roots \( \bar{56} + 28 + 8 \).

As explained in Ref. [11], the introduction of the extra generator \( \tilde{R}_{1}^{i_1 \cdots i_6 [a_7, a_8]} \) in the \( G_{IIB} \) algebra implies
the presence of an additional field $h_{a_1...a_7,d}$ which together with $h_{a,b}$ provides a first order formulation of gravity.

4. Discussion

In this Letter we have shown that the bosonic sector of the IIB supergravity theory can be formulated as a non-linear realisation of an infinite dimensional algebra which is the closure of the conformal algebra and the algebra denoted above by $G_{11}$. This formulation includes the Borel subalgebra of $SL(11)$ and $E_7$, but we argue that the non-linear realisation can be enlarged to include the Kac–Moody algebra $E_{11}$. We carry out the first step in this enlargement and show, by using a first order formulation of gravity involving two fields which are related by duality, that the algebra contains the Borel subalgebra of $E_8$.

It was perhaps not too surprising that the IIA supergravity theory has the same Kac–Moody algebra underlying it as the eleven-dimensional supergravity theory as they are related by a reduction on a circle. However, IIB supergravity can not be obtained from eleven-dimensional supergravity in a simple way and so the appearance of the same algebra is perhaps surprising. It is consistent with the idea expressed in [10, 11] that M theory has an underlying $E_{11}$ symmetry and that the maximal supergravity theories in eleven and ten dimensions appear as different manifestations of this symmetry. It is instructive to recall that non-linear realisations typically arise when a symmetry is spontaneously broken and it describes the theory controlling the low energy excitations. The local subgroup in the non-linear realisation corresponds to that part of the original symmetry that is preserved in the symmetry breaking. If a theory has different possible vacua one finds corresponding different low energy theories based on the same symmetry group, but with different local symmetry groups. This is precisely the picture we find with the maximal supergravities, they possess the same underlying group $E_{11}$, but have different local subgroups and so can be interpreted as different vacua of one theory, namely, M theory. Indeed, from this perspective what distinguishes the IIA and IIB theory is the way the $SL(11)$ subgroup is embedded in $E_{11}$ and correspondingly what parts of it are in the local subgroups. The embedding is fixed by the occurrence of the momentum generator which in turn gives rise to the space–time coordinates in the theory.

The different $SL(11)$ embeddings have an $SL(9)$ subgroups in common. This is consistent with the fact that IIA and IIB supergravity theories are the same when reduced to nine dimensions where the different embeddings give rise to the T duality transformations [12] between these two theories.

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References

[1] W. Nahm, Supersymmetries and their representations, Nucl. Phys. B 135 (1978) 149.
[2] E. Cremmer, B. Julia, J. Scherk, Supergravity theory in 11 dimensions, Phys. Lett. B 76 (1978) 409–412.
[3] I.C.G. Campbell, P.C. West, $N = 2, d = 10$ nonchiral supergravity and its spontaneous compactification, Nucl. Phys. B 243 (1984) 112; M. Huq, M. Namazie, Kaluza–Klein supergravity in ten dimensions, Class. Quantum Grav. 2 (1985); F. Giani, M. Perini, $N = 2$ supergravity in ten dimensions, Phys. Rev. D 30 (1984) 325.
[4] J.H. Schwarz, P.C. West, Symmetries and transformations of chiral $N = 2, D = 10$ supergravity, Phys. Lett. B 126 (1983) 301.
[5] P.S. Howe, P.C. West, The complete $N = 2, d = 10$ supergravity, Nucl. Phys. B 238 (1984) 181.
[6] J.H. Schwarz, Covariant field equations of chiral $N = 2, D = 10$ supergravity, Nucl. Phys. B 226 (1983) 269.
[7] E. Cremmer, B. Julia, The $N = 8$ supergravity theory. I. The Lagrangian, Phys. Lett. B 80 (1978) 48.
[8] C.M. Hull, P.K. Townsend, Unity of superstring dualities, Nucl. Phys. B 438 (1995) 109, hep-th/9410167.
[9] E. Cremmer, B. Julia, H. Lü, C.N. Pope, Dualisation of dualities. II: Twisted self-duality of doubled fields and superdualities, Nucl. Phys. B 535 (1998) 242, hep-th/9806106.
[10] P.C. West, Hidden superconformal symmetry in M theory, JHEP 08 (2000) 007, hep-th/0005270.
[11] P. West, E(11) and M theory, hep-th/0104081.
[12] E. Bergshoeff, C. Hull, T. Ortin, Duality in the type II superstring effective action, Nucl. Phys. B 451 (1995) 547–578, hep-th/9504081.
[13] P. West, Supergravity, brane dynamics and string duality, hep-th/9811110.