Comparison of the Cost Metrics for Reversible and Quantum Logic Synthesis*

Dmitri Maslov and D. Michael Miller
Department of Computer Science
University of Victoria
Victoria, BC, V8W 3P6, Canada
dmaslov@uvic.ca, mmiller@cs.uvic.ca

January 19, 2022

Abstract

A breadth-first search method for determining optimal 3-line circuits composed of quantum NOT, CNOT, controlled-V and controlled-V+ (NCV) gates is introduced. Results are presented for simple gate count and for technology motivated cost metrics. The optimal NCV circuits are also compared to NCV circuits derived from optimal NOT, CNOT and Toffoli (NCT) gate circuits. The work presented here provides basic results and motivation for continued study of the direct synthesis of NCV circuits, and establishes relations between function realizations in different circuit cost metrics.

1 Introduction

Reversible and quantum logic synthesis have attracted recent attention as a result of advances in quantum and nano technologies. Many of the proposed synthesis methods, especially in the area of reversible logic synthesis, assume large gate libraries where implementation costs of the gates may vary significantly. However, these methods most often target minimization of the gate count. Use of such a circuit cost metric is likely to result in seemingly small circuits, which are in fact expensive to construct.

The synthesis of circuits composed of NOT, CNOT and Toffoli (NCT) gates [14] and multiple control Toffoli gates [9, 16, 8, 1, 4] has recently been extensively studied. Converting an NCT or a multiple control Toffoli gate circuit into one composed of NOT, CNOT, controlled-V and controlled-V+ (NCV) gates has also been considered [2, 7]. Further simplification of NCV circuits has also been well studied [7].

It is an important question how close the circuits found by the above approaches are to optimal. The direct synthesis of NCV circuits is also of considerable interest since intuitively one would expect that to produce better results than the indirect route via NCT circuits. Finally, we note that observing optimal circuits for small cases often will shed light on good (if not optimal) synthesis approaches.

For these reasons, we here present an approach to finding optimal 3-line NCV circuits using various cost metrics. We compare these results to those found through mapping optimal NCT circuits. The advantage of direct NCV synthesis will be clear even for the 3-line case. Also, the results clearly demonstrate the difference between using simple gate count and technology motivated cost metrics.

The necessary background is reviewed in Section 2. A breadth-first search procedure to find optimal 3-line NCV circuits is given in Section 3 and properties identified in those circuits are discussed in Section 4. Section 5 presents comparative results to circuits derived from NCT circuits and for various cost metrics. The paper concludes with remarks and suggestions for ongoing research in Section 7.

2 Background

We here provide a brief review of the basic concepts required for this paper. For a more detailed and formal introduction we refer the reader to [11].

*This work was supported in part by PDF and Discovery Grants from the Natural Sciences and Engineering Research Council of Canada.
A single quantum bit (qubit) has two values, 0 or 1, traditionally depicted as \( |0 \rangle \) and \( |1 \rangle \) respectively. The state of a single qubit is a linear combination \( \alpha |0 \rangle + \beta |1 \rangle \) (also written as a vector \( (\alpha, \beta) \)) in the basis \( \{ |0 \rangle, |1 \rangle \} \), where \( \alpha \) and \( \beta \) are complex numbers called the amplitudes, and \( |\alpha|^2 + |\beta|^2 = 1 \). Real numbers \( |\alpha|^2 \) and \( |\beta|^2 \) represent the probabilities \( (p \text{ and } q) \) of reading the values \( 0 \) and \( 1 \) upon physical measurement of the qubit. The state of a quantum system with \( n > 1 \) qubits is described as an element of the tensor product of the single state spaces yielding a normalized vector of length \( 2^n \) called the state vector. Quantum system evolution results in changes of the state vector expressible as products of \( 2^n \times 2^n \) unitary matrices. This formulation characterizes a transformation but provides no indication of its implementation cost.

In one common approach, small gates are used as elementary building blocks with unit cost \([\underline{1} \quad 2 \quad 3 \quad 4 \quad 5]\). Among them are NOT \( (x \rightarrow \bar{x}) \) and CNOT \( ((x, y) \rightarrow (x, x \oplus y)) \) gates, the 2-bit controlled-V gate which changes the target line according to the transformation given by matrix \( V = \frac{1+i}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \) if, and only if, the control line is 1. Controlled-V\(^+\) analogously applies the transformation \( V^+ = V^{-1} \). Consult \([\underline{11}]\) for the case where the control line assumes a quantum (non-Boolean) value. Gates controlled-V and controlled-V\(^+\) are also referred to as controlled-sqrt-of-NOT gates since \( V^2 = (V^+)^2 = \text{NOT} \). In order to implement a Boolean specification, and assuming an auxiliary line is available in addition to the minimal set of lines needed for reversibility \([\underline{6}]\), the set of gates NOT, CNOT, controlled-V and controlled-V\(^+\) is complete. We call this set the NCV gates.

**Definition 1.** The NCV-111 cost of a circuit composed of NCV gates is the number of gates in the circuit.

In an alternative approach \([\underline{10} \quad 13]\), it is observed that any circuit can be composed with single qubit and CNOT gates and the circuit cost is calculated based on the number of CNOT gates required. With regards to the NCV library, we next define NCV-012 cost and motivate our definition by the fact that controlled-V and controlled-V\(^+\) gates require at most 2 CNOT gates when decomposed into a circuit with single qubit and CNOT gates \((\underline{11}, \text{page 181})\).

**Definition 2.** The NCV-012 cost of an NCV circuit is linear with weights 0, 1, 2, and 2 associated with the gates NOT, CNOT, controlled-V and controlled-V\(^+\), respectively.

The last cost metric that we consider in this paper is motivated by a recent investigation of the technological (liquid NMR) costs of quantum and reversible primitives \([\underline{5}]\).

**Definition 3.** The NCV-155 cost of an NCV circuit is linear with weights 1, 5, 5, and 5 associated with the gates NOT, CNOT, controlled-V and controlled-V\(^+\), respectively.

Finally, we define the multiple control Toffoli gate \([\underline{15}]\).

**Definition 4.** For the set of Boolean variables \( \{x_1, x_2, \ldots, x_n\} \) the Toffoli gate has the form \( \text{TOF}(C; T) \), where \( C = \{x_{t_1}, x_{t_2}, \ldots, x_{t_k}\} \), \( T = \{x_j\} \) and \( C \cap T = \emptyset \). It maps the Boolean pattern \( (x_1^+, x_2^+, \ldots, x_n^+) \) to \( (x_1^+, x_2^+, \ldots, x_{t_1}^+, x_{t_2}^+, \ldots, x_{t_k}^+, x_{t_{k+1}}^+, \ldots, x_n^+) \). \( C \) will be called the control set and \( T \) will be called the target.

The Toffoli gate and its generalizations with more than two controls form a good basis for synthesis purposes and have been used by many authors \([\underline{9} \quad 10 \quad 13 \quad 14]\). However, Toffoli gates are not simple transformations. Rather they are composite gates themselves and Toffoli gates with a large set of controls can be quite expensive \([\underline{3} \quad 7]\). As a result, the NCV-111 cost of a 100-gate Toffoli circuit with 10 input/output lines can be as low as 100, or may be as high as 15,200. These are, of course, extreme numbers, however, in Section \([\underline{5}]\) we show a better analysis of how the costs may differ.

### 3 The Search Procedure

Our main tool for the investigation of relations amongst optimal NCV-111, NCV-012 and NCV-155 implementations and their relation to the most commonly used reversible circuit cost metric, multiple control Toffoli gate count, is a search procedure for the optimal synthesis of NCV circuits for all 40,320 size 3 reversible Boolean functions. For this problem, we use a prioritized, pruned breadth-first search. This is because the search tree for simple full breadth-first search grows too fast for a search to be accomplished in feasible time and space. This is
in contrast to using a full breadth-first search for NCT optimal synthesis [14], which is tractable for 3-line circuits with no search optimizations.

To see the size of the problem, note that the number of base transformations for NCV gates is 21: 3 transformations involve a NOT gate, and the use of the gates CNOT, controlled-V and controlled-V+ each result in 6 transformations. The results in Table 4 show that the length of the optimal implementations can be up to 16. Thus, the number of nodes at one level of the search tree can be as high as $21^{16} \approx 1.4 \times 10^{21}$.

Breadth first search was previously applied to the synthesis of optimal circuits for the size 3 reversible functions using NOT, CNOT, and Toffoli gate library [14]. The size of the bottom level of their search tree (branching factor to the power of tree depth) is $9^8 \approx 4.3 \times 10^7$ showing that such a search is significantly simpler than the one we are pursuing and requires no techniques to reduce the search space or make the search efficient. In addition, Toffoli gates are not simple transformations, while synthesis of optimal circuits makes more sense in terms of simple transformations. Our search procedure can find optimal circuits in any weighted gate count metric, not just in the simple gate count metric used in [14].

An earlier attempt to synthesize optimal NCV circuits [17] was capable of synthesizing optimal implementations of the maximal cost 7 in one specific cost metric (NCV-011?). The maximal number of synthesized functions is 10136 [17], which is about a quarter of the pool size. Our program synthesizes optimal implementations for all 40,320 size 3 reversible functions, and is not tied to a specific costing metric. Further, our search procedure is significantly faster: it completes the entire search in the lesser time than the search of all optimal 5-gate implementations in [14].

In our work, we do not allow a quantum gate (controlled-V and controlled-V+) to have a control line that may at that point in the circuit take a quantum (non-Boolean) value. We do not have a mathematical proof that deleting this restriction will not result in construction of smaller circuits. However, numerous experiments indicate that use of a quantum gate with a quantum control line does not lead to a more efficient circuit than the one we construct. The above restriction allows us to work with quaternary logic instead of a continuous values.

The techniques used to reduce the size of the search include:

- Based on the observation above, we can view a quantum function as a base reversible Boolean function with a quantum factor added. In other words, each of the values that may occur in the truth table is stored as a 2-bit number. Values 0, 1, V and $V^+$ are represented by 00, 10, 01 and 11, respectively.

- While our final goal is to synthesize all size three reversible functions, and none of the quantum (non-Boolean) functions, the latter still have to be stored and referred to during the search process. Each such function is stored in a queue associated with its base reversible Boolean function that can be uniquely identified from the first digit in the above 2-bit encoding. The quantum function itself is then identified by a 24-bit number composed of the second bits, that we call the quantum signature. A quantum function is uniquely defined by the base Boolean function and its quantum signature.

- When we assign a new gate to the existing optimal cascade we never choose a gate with the same set of controls and targets as the immediately previous one in the circuit. This is because such a sequence can always be reduced by template application [7], and thus will not be part of an optimal circuit. This cuts down the number of gates that one must consider at each step.

- We note that in an NCV implementation of a reversible function one can interchange controlled-V with controlled-$V^+$ gates without changing the function realized, provided all such gates are interchanged. In our search procedure, this is accounted for by never using a controlled-$V^+$ gate as the first quantum gate during construction of an NCV circuit. In this context, the first quantum gate is the one that transforms a Boolean line to one that can take on quantum values.

- Once an optimal implementation of a function is found, we have also found an optimal implementation for all functions that differ from this one only by their input-output labeling. Potentially, this accounts for at most 6 different functions.

- Once $G_1G_2...G_k$ is a circuit for a reversible function $f$, $G_k^{-1}G_{k-1}^{-1}...G_1^{-1}$ is a valid circuit for $f^{-1}$ [7]. It can be shown that for each metric considered in this paper, as well as in any weighted linear type metric if $G_1G_2...G_k$ is optimal for $f$, then $G_k^{-1}G_{k-1}^{-1}...G_1^{-1}$ will be an optimal implementation for $f^{-1}$. From the
point of view of the search for optimal circuits, this means that once an optimal circuit for $f$ is found, so is
an optimal circuit for $f^{-1}$. This observation would further help to cut down the search space, however, we
have not yet implemented it since our program is fast enough at present. It takes approximately 1 minute to
synthesize optimal 3-line NCV circuits in each metric on a single 750 MHz processor Sun Blade 1000.

Due to the potentially differing costs of the basis gates, our procedure maintains several queues of functions,
each corresponding to the cost associated with the circuits in it. During the search, new gates are assigned to the
circuits with smallest cost not yet considered thereby yielding new circuits to be considered. Cheaper gates are
applied first. However, due to varying gate costs, the first circuit found realizing a Boolean function may not be
optimal. To see this, consider the example illustrated in Figure 1. Our program finds the non-optimal circuit with
NCV-155 cost 7 before the optimal implementation with NCV-155 cost 6. This is because the procedure generates
a circuit with two NOT gates before a circuit with a single CNOT gate, and consequently finds the three gate
circuit in advance of the cheaper two gate alternative. Note that for gate count these two circuits are generated in
the opposite order.

Figure 1: (a) Non-optimal, but first found and (b) optimal in NCV-155 cost metric circuits.

4 Observations on the optimal 3-line NCV circuits

Using the above search procedure we found several interesting properties of the optimal 3-line NCV circuits:

- Optimal implementations found with cost metrics NCV-111 and NCV-155 are interchangeable, and optimal
NCV-111 implementations have optimal costs in NCV-012 cost metric. Diagram in Figure 2 shows which
optimal implementations can be substituted without losing the property of optimality in the corresponding
metric. Practically, this means that the set of optimal NCV-111 circuits contains circuits optimal in other
(NCV-012 and NCV-155) metrics. In Section 5 we make some observations with regard to the NCV-111
metric optimal circuits. The same comparisons apply for optimal NCV-012 and optimal NCV-155
implementations. Further, our experiments with different metrics suggest that the set of optimal NCV-111 circuits
will contain optimal implementations in NCV-xyz cost metric as long as non-negative integer numbers $x$, $y$, and $z$
which represent costs of the gates NOT, CNOT, and controlled-V (and assuming the cost of the controlled-$V^+$ equals the cost of the controlled-$V$) satisfy inequality $y < 2z$.

Figure 2: Interchangeability of the optimal implementations.

- In every circuit cost metric that we have considered (NCV-111, NCV-155, and NCV-012), the total number
of controlled-$V$ and controlled-$V^+$ gates in any single circuit is divisible by 3 and is never more than 9. We
conjecture that the overall number of controlled-$V$ and controlled-$V^+$ gates in any NCV implementation of
any reversible function is divisible by three.
• A 3-bit Toffoli gate and a 3-bit Toffoli gate with one negative control have the same cost in each of the metrics NCV-111, NCV-155, and NCV-012. Some optimal circuits are illustrated in Figure 3. An optimal implementation of a 3-bit Toffoli gate with two negative controls has 6 gates.

Figure 3: (a) optimal NCV realization of the Toffoli gate (b),(c) optimal NCV realizations of the Toffoli gate with a single negative control.

This observation allows us to generalize the famous result by Barenco et al. [2] and improve [6] on the implementation costs of the generalized multiple controlled Toffoli gates by suggesting that large Toffoli gates with some, but not all, negative controls can be simulated with the same cost as the same size Toffoli gate with all positive controls. Illustrated in Figure 4 is a Toffoli gate with 5 controls, and a circuit simulating it, analogous to the one from [2]. Since the cost of a size 3 Toffoli gate with one negative control equals to the cost of size 3 Toffoli gate with both positive controls, large Toffoli gate illustrated in Figure 4 will have a cost equal to the cost of the same size Toffoli gate but having only positive controls. Application of the local optimization techniques like [7] may choose different scenarios for simplification of the circuits for large Toffoli gates with all positive and some but not all negative controls. However, results of such optimization are out of the scope of the present paper.

Figure 4: Construction of a large Toffoli gate with some but not all negative controls.

• Observing all optimal implementations in the NCV-111 metric we came to the conclusion that every Boolean function \( f(x_1, x_2) \) of two variables can be computed by a quantum circuit with no more than 5 gates. If this function is from Boolean class 0 (when \( f(0, 0) = 0 \)), only 4 quantum gates are required.

5 Comparisons of the Sets of Optimal Implementations

In this section, we compare the NCV-111 (NCV-012 and NCV-155) costs for the optimal synthesis of size 3 reversible functions using Toffoli gates up to size 3 (NCT library, [14]) with the costs of optimal NCV circuits. We note that our implementation of the breadth first search for the size 3 reversible NCT circuits may differ from the original as discussed in [14]. Due to the large number of optimal NCT circuits for some functions, the results shown below may vary slightly depending on the actual program implementation. Further, the comparison is done through substitution of every Toffoli gate in an optimal NCT circuit with a 5-gate NCV circuit (11, page 182), and thus assigning a cost of 5 in the NCV-111 metric. Gates NOT and CNOT are present in both libraries, NCT and NCV, and thus require no specific attention.
5.1 Optimal NCV-111 vs. Optimal NCT Circuits

The leftmost column of Table 1 refers to the cost of an implementation of a 3-bit reversible function. The second and third columns refer to the number of optimal NCT circuits reported in [14]; the fourth column refers to the number of optimal NCV circuits in the NCV-111 metric found by our search procedure. The circuit costs used in the last three columns are NCT gate count, NCV-111 and NCV-111 correspondingly. Table also reports the weighted average (WA) for the given cost metric. The third and fourth columns compare the NCV-111 costs of the optimal NCT and optimal NCV-111 implementations in cost metric NCV-111 for the size 3 reversible functions. We observed that the maximal ratio of the cost of one optimal implementation over the other is $3.375 = \frac{27}{8}$. That is, even for circuits with a small number of inputs/outputs an optimal Toffoli circuit transformed to NCV can be a factor of 3.375 off its optimal NCV realization. We also determined that on average, the optimal NCT circuit is 1.3902 times more expensive (for the NCV-111 metric) than the corresponding optimal NCV circuit found by our search procedure. An important question of whether the NCT and NCV costs are related is addressed by the computing the correlation between the vectors of NCV-111 costs of the optimal NCT and optimal (in NCV-111 metric) NCV circuits. The correlation equals 0.896. We conclude that the costs are reasonably correlated. Finally, we found it interesting to determine how many optimal NCT circuits have optimal NCV-111 cost. There are 1,610.

The results are illustrated in the cost comparison chart in Figure 5.

Table 1: Optimal NCT and optimal NCV-111 NCV circuits.

| Cost | Opt. NCT | Opt. NCV-111 | Opt. NCV-111 |
|------|----------|--------------|--------------|
|      | GC [14]  | NCV-111      | NCV-111      |
| 0    | 1        | 1            | 1            |
| 1    | 12       | 9            | 9            |
| 2    | 102      | 51           | 51           |
| 3    | 625      | 187          | 187          |
| 4    | 2780     | 392          | 417          |
| 5    | 8921     | 475          | 714          |
| 6    | 17049    | 259          | 1373         |
| 7    | 10253    | 335          | 3176         |
| 8    | 577      | 1300         | 4470         |
| 9    | 0        | 3037         | 4122         |
| 10   | 0        | 3394         | 10008        |
| 11   | 0        | 793          | 5036         |
| 12   | 0        | 929          | 1236         |
| 13   | 0        | 4009         | 8340         |
| 14   | 0        | 8318         | 1180         |
| 15   | 0        | 4385         | 0            |
| 16   | 0        | 255          | 0            |
| 17   | 0        | 1297         | 0            |
| 18   | 0        | 4626         | 0            |
| 19   | 0        | 4804         | 0            |
| 20   | 0        | 475          | 0            |
| 21   | 0        | 106          | 0            |
| 22   | 0        | 503          | 0            |
| 23   | 0        | 357          | 0            |
| 24   | 0        | 4            | 0            |
| 27   | 0        | 17           | 0            |
| 28   | 0        | 2            | 0            |
| WA   | 5.8655   | 14.0548      | 10.0319      |

Table 1: Optimal NCT and optimal NCV-111 NCV circuits.
5.2 Optimal NCV-012 vs. Optimal NCT Circuits

We considered the set of optimal NCV-012 circuits as created by our program. Optimal NCV-111 implementations were not substituted for those optimal NCV-012 circuits whose NCV gate count is not optimal.

A comparison of the NCV-012 costs of optimal NCV-012 NCV circuits and NCT circuits is made in Table 2 (this table is organized similarly to the previous table) and Figure 6. In this metric, the maximum ratio of NCT optimal circuit NCV-012 cost over NCV-012 optimal circuit NCV-012 cost equals $8 = \frac{16}{2}$. The optimal NCT and optimal NCV-012 circuits for one such function are illustrated in Figure 7. On average, however, this ratio is 1.2728. Correlation between the vectors of costs is 0.8999, which is almost identical to the correlation between optimal NCT and optimal NCV-111 circuits in the NCV-111 cost metric. The number of functions where the NCV-012 cost of an optimal NCT circuit equals the NCV-012 cost of an optimal NCV-012 circuit equals 1,774.

6 Optimal implementations with restricted qubit-to-qubit interactions

In the discussion above we assumed that a direct interaction between any two qubit can be established (e.g. a 2-qubit gate may be built on any two qubits). However, due to the specifics of a particular physical realization, this may not always be the case. Some of the qubit-to-qubit interactions may only be available indirectly. On the other hand, in every $n$-bit quantum computation it must always be possible to construct a connected graph with vertices representing qubits and edges representing possibility of the direct interaction between qubits. In the case $n = 3$, there are only two non-isomorphic connected graphs. One is the complete graph, and the remaining is a star (all vertices are connected to one). In this section we report results for the optimal synthesis assuming direct interactions are allowed between qubits $a$ and $b$, and $b$ and $c$. We assume NCV-111 costing metric, however, the results can be calculated for other metrics as well. Assuming the implementation costs (numbers of gates) are 0..23, the numbers of functions requiring this many gates are 1, 7, 29, 82, 181, 334, 334, 337, 753, 1652, 2654, 2482, 1674, 1350, 3236, 6304, 6028, 1508, 1302, 2566, 4314, 2804, and 14. There are no functions requiring more than 23 gates.

We found it interesting to notice that in the case of non-restricted qubit interactions one of the cheapest non-linear (with respect to EXOR) reversible gates is the Peres gate [12] defined by the transformation $(a, b, c) \mapsto (a, b \oplus a, c \oplus ab)$. It can be composed with 4 quantum NCV gates only. An analogy (smallest non-linear reversible gate with a similar transformation) of this gate in the case of restricted qubit interactions is the gate defined by the transformation $(a, b, c) \mapsto (b, a, c \oplus ab)$. It requires 6 quantum NCV gates as illustrated in Figure 8(a). Toffoli gates $TOF(a, b; c)$ and $TOF(b, c; a)$ require 9 elementary quantum gates each and can be thought of as a SWAP gate (3 CNOTs) followed by the $(a, b, c) \mapsto (b, a, c \oplus ab)$ transformation. An optimal implementation of $TOF(a, b; c)$ is illustrated in Figure 8(b). Toffoli gate $TOF(a, c; b)$ is somewhat more expensive. It requires 13 elementary operations in its optimal implementation (see Figure 8(c)).
| Cost | Opt. NCT | Opt. NCT | Opt. NCT | Opt. NCT | Opt. NCT |
|------|----------|----------|----------|----------|----------|
|      | GC       | NCV-012  | NCV-012  | NCV-012  | NCV-111  |
| 0    | 1        | 8        | 8        | 1        |          |
| 1    | 12       | 48       | 48       | 9        |          |
| 2    | 102      | 183      | 192      | 45       |          |
| 3    | 625      | 398      | 408      | 142      |          |
| 4    | 2780     | 486      | 480      | 315      |          |
| 5    | 8921     | 201      | 192      | 585      |          |
| 6    | 17049    | 16       | 16       | 1169     |          |
| 7    | 10253    | 0        | 192      | 2286     |          |
| 8    | 577      | 47       | 1056     | 3414     |          |
| 9    | 0        | 352      | 3168     | 4790     |          |
| 10   | 0        | 1347     | 4320     | 6744     |          |
| 11   | 0        | 3130     | 672      | 6420     |          |
| 12   | 0        | 3340     | 0        | 4328     |          |
| 13   | 0        | 561      | 0        | 4360     |          |
| 14   | 0        | 3        | 2880     | 4032     |          |
| 15   | 0        | 0        | 11520    | 1568     |          |
| 16   | 0        | 162      | 4416     | 112      |          |
| 17   | 0        | 1219     | 0        | 0        |          |
| 18   | 0        | 4435     | 0        | 0        |          |
| 19   | 0        | 8029     | 0        | 0        |          |
| 20   | 0        | 3872     | 0        | 0        |          |
| 21   | 0        | 128      | 9856     | 0        |          |
| 22   | 0        | 0        | 896      | 0        |          |
| 23   | 0        | 341      | 0        | 0        |          |
| 24   | 0        | 1946     | 0        | 0        |          |
| 25   | 0        | 4482     | 0        | 0        |          |
| 26   | 0        | 3977     | 0        | 0        |          |
| 27   | 0        | 609      | 0        | 0        |          |
| 28   | 0        | 6        | 0        | 0        |          |
| 29   | 0        | 289      | 0        | 0        |          |
| 30   | 0        | 489      | 0        | 0        |          |
| 31   | 0        | 194      | 0        | 0        |          |
| 32   | 0        | 3        | 0        | 0        |          |
| 33   | 0        | 16       | 0        | 0        |          |
| 34   | 0        | 3        | 0        | 0        |          |
| 35   | 0        | 3        | 0        | 0        |          |
| 36   | 0        | 3        | 0        | 0        |          |
| 37   | 0        | 3        | 0        | 0        |          |
| 38   | 0        | 3        | 0        | 0        |          |
| 39   | 0        | 3        | 0        | 0        |          |
| 40   | 0        | 3        | 0        | 0        |          |
| 41   | 0        | 3        | 0        | 0        |          |

Table 2: Optimal size 3 reversible circuit NCV-012 costs in NCT and NCV bases.
Figure 6: Optimal NCT (X-coordinate) VS Optimal NCV-012 (Y-coordinate) circuits.

Figure 7: (a) optimal NCT and (b) optimal NCV-012 circuits for the function $[7, 6, 4, 5, 2, 3, 1, 0]$.

7 Conclusion

The results in Section 5 lead us to the following conclusion. Minimization of Toffoli gate count as a criterion for a reversible synthesis method is not optimal and even for small parameters may result in a seemingly small circuit which may be as far off a technologically favorable implementation as a factor of 8. It is natural to expect that for circuits with more lines the difference will grow. We suggest that the commonly used gate count metric should be replaced with a metric that accounts for the different costs of large building block (i.e., Toffoli gates), such as the weighted gate cost presented here. Using such a metric would lead to the technologically favorable circuit (b) in Figure 7.

Finally, we observe that for some small parameters minimizing the cost in any of the technology oriented metrics that we considered in this paper (NCV-111, NCV-155 or NCV-012) should result in a reasonably small if
not optimal solution if another metric is applied to the cascade of gates. Thus, at the present time it is sufficient to use any of the suggested technology oriented cost metrics. NCV-111 and NCV-155 are preferred over the NCV-012 metric, since all such realizations (assuming reversible functions with no more than 3 variables) have optimal NCV-012 cost.

References

[1] A. Agrawal and N. K. Jha. Synthesis of reversible logic. In DATE, pages 21384–21385, Paris, France, February 2004.

[2] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVinchenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter. Elementary gates for quantum computation. Phys. Rev. A, 52:3457–3467, 1995.

[3] W. N. N. Hung, X. Song, G. Yang, J. Yang, and M. A. Perkowski. Quantum logic synthesis by symbolic reachability analysis. In DAC, pages 838–841, 2004.

[4] P. Kerntopf. A new heuristic algorithm for reversible logic synthesis. In DAC, pages 834–837, June 2004.

[5] S. Lee, S. Lee, T. Kim, J. Lee, J. Biamonte, and M. Perkowski. The cost of quantum gates. IEEE International Journal of Multi Valued Logic, 2005. In review.

[6] D. Maslov and G. W. Dueck. Reversible cascades with minimal garbage. IEEE Transactions on CAD, 23(11):1497–1509, November 2004.

[7] D. Maslov, C. Young, D. M. Miller, and G. W. Dueck. Quantum circuit simplification using templates. In DATE, pages 1208–1213, March 2005.

[8] D. M. Miller, D. Maslov, and G. W. Dueck. A transformation based algorithm for reversible logic synthesis. In DAC, pages 318–323, June 2003.

[9] A. Mishchenko and M. Perkowski. Logic synthesis of reversible wave cascades. In IWLS, pages 197–202, June 2002.

[10] M. Mottonen, J. J. Vartiainen, V. Bergholm, and M. M. Salomaa. Quantum circuits for general multiqubit gates. Physical Review Letters, 93(130502), 2004.

[11] M. Nielsen and I. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[12] A. Peres. Reversible logic and quantum computers. Physical Review A, 32:3266–3276, 1985.

[13] V. V. Shende, I. L. Markov, and S. S. Bullock. Synthesis of quantum logic circuits. In ASP-DAC, January 2005.

[14] V. V. Shende, A. K. Prasad, I. L. Markov, and J. P. Hayes. Synthesis of reversible logic circuits. IEEE Transactions on CAD, 22(6):723–729, June 2003.

[15] T. Toffoli. Reversible computing. Tech memo MIT/LCS/TM-151, MIT Lab for Comp. Sci, 1980.

[16] I. M. Tsai and S. Y. Kuo. Quantum boolean circuit construction and layout under locality constraint. In IEEE Conference on Nanotechnology, pages 111–116, 2001.

[17] G. Yang, W. N. N. Hung, X. Song, and M. A. Perkowski Exact Synthesis of 3-Qubit Quantum Circuits from Non-Binary Quantum Gates Using Multiple-Valued Logic and Group Theory. In DATE, pages 434–435, 2005.