Symmetry aspects of the pion leptoproduction and the upper limit of the Levelt-Mulders asymmetry*

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We examined the symmetry aspect of the semi-inclusive one-pion production in the deep inelastic scattering of a lepton beam off an unpolarized nucleon target, with an emphasis on the positivity restrictions on the corresponding structure functions. In combination with the Callan-Gross-type relation between two twist-two structure functions \( W_1 \) and \( W_2 \), we derived an upper bound on the Levelt-Mulders asymmetry, which occurs when the lepton beam is longitudinally polarized.

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* This work was supported in part by the National Science Foundation of China.
I. INTRODUCTION

At present, one irritating fact is that one has no way to make precise predictions for the properties of hadron-involved processes from the first principles, due to lack of reliable handle on the QCD soft interactions. On the other hand, experiences tell us that many bulk properties of particle interactions are determined by the symmetry of the system, irrelevant of the dynamical details. Hence, it is desirable to learn as much as possible about the process of interest from its symmetries. In practice, such analyses are of important guidance both for theoretical attempts and for experimental researches.

During the past decade, much theoretical attention [1-12] has been paid to the semi-inclusive pion production in the deep inelastic lepton-nucleon scattering. Moreover, some experimental efforts [13,14] have already appeared. In studying the deep inelastic scattering of the charged lepton off the nucleon, one usually adopts the so-called one-photon exchange approximation. Accordingly, the response of the target nucleon to the photon probe is characterized by a hadronic tensor (see its definition in Section II). One of its symmetry properties is reflected in its decomposition into Lorentz invariant structure functions. To our knowledge, Mulders [4] was first to suggest the decomposition of the hadronic tensor for the unpolarized pion leptoproduction. In the Mulders parameterization, there are four independent structure functions. Recently, Levelt and Mulders [9] worked out a QCD factorization approach to the hadronic tensor, whose results contain an imaginary part, which cannot be accommodated into the early Mulders decomposition. At the same time, Kotzinian [11] counted the number of independent structure functions in the case of a polarized nucleon target, by working in a specific frame and treating the hadronic tensor as a matrix in the Lorentz space. According to Ref. [11], there are five independent structure functions. Hence, we feel it necessary to clarify the question how many independent structure functions there are in the pion leptoproduction with an unpolarized nucleon target.

Furthermore, there is another aspect of the symmetry properties of the pion leptoproduction, i.e., the hadronic tensor is subject to some positivity constraints. We anticipate that
the positivity of the hadronic tensor for the pion leptoproduction can also provide us some useful bounds to the structure functions, just like in the case of the deeply inelastic scattering \cite{16}. Historically, the positivity restrictions to the pion leptoproduction was investigated by Gourdin \cite{16} as early as in 1972. However, the structure functions were parameterized in Ref. \cite{16} in terms of the cross sections for the associated photon-nucleon reactions with the same inclusive-pion final states. Since it is now a prevailing practice for the particle physics community to decompose directly the hadronic tensors into the Lorentz invariant structure functions, we recast the positivity restrictions to the pion leptoproduction, with the hadronic tensor parameterized in terms of Lorentz invariant structure functions.

In Ref. \cite{9}, Levelt and Mulders \cite{9} identified a $\langle \sin \phi \rangle$-type asymmetry, which comes about when one collides a longitudinally polarized lepton beam on an unpolarized nucleon target. Here $\phi$ is the azimuthal angle of the detected pion about the lepton scattering plane. As these authors showed, the measurement of such a single spin asymmetry can allow for a determination of some naively-time-reversal-odd quark fragmentation functions. This kind of experiments can be expected to be done in the near future at the high luminosity facilities, such as DESY HERA \cite{17}, CERN LHC \cite{18}, and the proposed Electron Laboratory for Europe (ELFE) \cite{19}. Since single spin asymmetries are of their own relevance in our understanding of the hadron structure and dynamics, it is preferable to learn more about the Levelt-Mulders asymmetry before the relevant experiments come true.

The purpose of this paper is to examine systematically the symmetry aspect of the semi-inclusive one-pion production in the deep inelastic scattering of a lepton beam off an unpolarized nucleon target. By applying Mueller’s generalized optical theorem, we show that there are five independent structure functions for the process considered, which is consistent with the counting by Kotzinian. Then, we study the positivity constraints to these structure functions, which are essentially due to the symmetry of the hadronic tensor. An important new ingredient imbedded in our positivity analysis is to combine the positivity restrictions with the Callan-Gross relation among the structure functions, which yields an upper bound to the Levelt-Mulders asymmetry. The upper limits derived eliminate the possibility of
observing large Levelt-Mulders asymmetries in certain kinematical regions and serve as a
judgement for the reliability of the experimental data in the future.

The paper is organized as follows. In Sect. II we define our kinematics and examine
the general symmetry constraints to the Lorentz structure of the hadronic tensor, with an
emphasis on the fact that time-reversal invariance of electromagnetic interactions does not
exert any constraints on the hadronic tensor, owing to the existence of the final-state inter-
actions in the inclusively-detected final state. In Sect. III, we employ the Mueller theorem
to enumerate the number of independent structure functions for the process considered. In
Sect. IV, we discuss several scenarios for decomposing the hadronic tensor. Sect. V is
devoted to the derivation of various positivity constraints among the structure functions.
In Sect. VI, we derive an upper limit for the Levelt-Mulders asymmetry by combining the
positivity constraints and the Callan-Gross relation. Sect. VII contains our concluding
remarks.

II. HADRONIC TENSOR AND ITS GENERAL SYMMETRY CONSTRAINTS

The process we will consider is the semi-inclusive pion production on an unpolarized
ucleon target

\[ l(k, s_l) + N(P) \rightarrow l(k') + \pi(P_\pi) + X, \]

where the particle momenta are self-explanatory and \( s_l \) is the spin four-vector of the incident
lepton. We normalize the spin vector of the lepton as \( s_l \cdot s_l = -1 \) for a pure state. In the
one-photon exchange approximation, the differential cross section can be put into a Lorentz
contraction of the leptonic and hadronic tensors:

\[
\frac{d\sigma(s_l)}{dx dy dz d\phi} \left| P_{\pi\perp} \right|^2 \propto \frac{\alpha^2 y}{32 M^2 \pi^2 Q^2 \left| P_{\pi\parallel} \right|^2} L_{\mu\nu}(k, s_l; k') W^{\mu\nu}(q, P, P_\pi),
\]

where \( q = k - k' \) is the virtual photon momentum, \( P_{\pi\parallel} \) is the longitudinal momentum of the
pion along the direction of the motion of the virtual photon, and \( P_{\pi\perp} \) is the corresponding
transverse pion momentum. In our presentation, we will employ the virtuality of the probe
photon $Q = \sqrt{-q^2}$, and the energy loss of the lepton in the target rest frame $\nu = P \cdot q / M$ with $M$ the nucleon mass. Moreover, we adopt the the scalar variables defined as

$$x = -\frac{q^2}{2P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot k}, \quad z = \frac{P \cdot P_\pi}{P \cdot q}. \quad (2)$$

$|P_{\pi\parallel}|$ is related to $|P_{\pi\perp}|$ by

$$|P_{\pi\parallel}| \frac{zQ^2}{2Mx} \sqrt{1 - \left(\frac{2Mx}{zQ^2}\right)^2 (|P_{\pi\perp}|^2 + M_{\pi}^2)}. \quad (3)$$

In Eq. (1), the leptonic tensor is defined as

$$L^{\mu\nu}(k, s; k') \equiv \text{Tr} \left[ (\not{q}' + M_l)\gamma_\mu (\not{k}' + M_l)\gamma_\nu \frac{1 + \gamma_5 k'}{2} \right] = -q^2(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) + 4(k^{\mu} - \frac{q^\mu}{2})(k^{\nu} - \frac{q^\nu}{2}) + 2iM_l \epsilon^{\mu\nu\alpha\beta} q_\alpha s_{\beta}, \quad (4)$$

where $M_l$ is the lepton mass. Accordingly, the hadronic tensor in this paper is defined as

$$W^{\mu\nu}(q, P, P_\pi) = \frac{1}{4\pi} \sum_X \int d^4 \xi \exp(ik \cdot \xi) \langle P | J^\mu(0) | \pi(P_\pi), X \rangle \langle \pi(P_\pi), X | J^\nu(\xi) | P \rangle, \quad (5)$$

where the summation over $X$ exhausts all the possible final states that contain the chosen pion. In our work, the electromagnetic quark current is defined as $J^\mu = \sum_f e_f \bar{\psi_f} \gamma^\mu \psi_f$, with $f$ the quark flavor index and $e_f$ the electric charge of the quark in unit of the electron charge. Throughout we normalize the one-particle state in a relativistic way that $\langle P | P' \rangle = (2\pi)^3 2E_\delta^3(P - P')$. Our conventions are different from those in Refs. [4,9,10,12], but there is no principal difference.

Because the fundamental interaction vertex is electromagnetic in the one-photon exchange approximation, the Lorentz structure of the hadronic tensor is subject to all the symmetries that electromagnetic interactions observe. Now we examine these symmetry constraints.

First, the electromagnetic interaction is gauge invariant, which is reflected as the following current conservation conditions:

$$q_\mu W^{\mu\nu}(q, P, P_\pi) = q_\nu W^{\mu\nu}(q, P, P_\pi) = 0. \quad (6)$$
Second, the electromagnetic current is Hermitian, which leads to

\[ [W^{\mu\nu}(q, P, P_\pi)]^* = W^{\nu\mu}(q, P, P_\pi). \] (7)

Thirdly, the electromagnetic interaction is parity conserved. For a generic Lorentz vector \( x^\mu \), we define \( \tilde{x}^\mu = x_\mu \) following Itzykson and Zuber [20]. Then, the parity conservation of the electromagnetic interaction informs us that

\[ W^{\mu\nu}(q, P, P_\pi) = W^{\nu\mu}(\tilde{q}, \tilde{P}, \tilde{P}_\pi). \] (8)

Fourthly, the fundamental electromagnetic vertex is invariant under time-reversal transformation. We recall that time-reversal transformation includes making a complex conjugation and changing the in-state into its corresponding out-state, or vice versa. In general, an in-state is related to its corresponding out-state by \( S \) matrix (operator):

\[ |\rangle_{in} = S |\rangle_{out}, \] (9)

with \( S = 1 + iT \). The difference between the in-state and its associated out-state is essentially due to the final-state interactions described by \( T \), the transition matrix (operator). Unless the state is composed of an individual particle or a set of non-interactive particles, the in-state differs from its corresponding out-state. Hence, time-reversal invariance can only tell us

\[ W^{\mu\nu}(q, P, P_\pi) = \left[ \frac{1}{4\pi} \sum_X \int d^4\xi e^{iq\cdot\xi} \langle \tilde{P}|j_\mu(0)|\pi(\tilde{P}_\pi), X \rangle_{in} \langle \pi(\tilde{P}_\pi), X|j_\nu(\xi)|\tilde{P} \rangle \right]^*. \] (10)

If one does not distinguish the in-state from its corresponding out-state, then there is the so-called naive time reversal transformation. Under such a simplified time-reversal transformation, there will be

\[ W^{\mu\nu}(q, P, P_\pi) = [W_{\mu\nu}(\tilde{q}, \tilde{P}, \tilde{P}_\pi)]^*. \] (11)

From Eqs. (8) and (11), it can be seen that it is more convenient to use the adjoint parity-time-reversal transformation instead of the individual parity and time-reversal transformations. For our hadronic tensor, the adjoint parity-time-reversal transformation gives rise to
$W^{\mu\nu}(q, P, P_{\pi}) = \left[ \frac{1}{4\pi} \sum_X d^4\xi e^{iq\cdot\xi} \langle P| j^{\mu}(0)|\pi(P_{\pi}), X \rangle_{\text{in}} \langle \pi(P_{\pi}), X|j^{\nu}(\xi)|P \rangle \right]^*.$ \hspace{1cm} (12)

Substituting Eq. (9) into (12), we can decompose the hadronic tensor into two parts:

$$W^{\mu\nu}(q, P, P_{\pi}) = W^{(S)\mu\nu}(q, P, P_{\pi}) + W^{(A)\mu\nu}(q, P, P_{\pi}),$$ \hspace{1cm} (13)

where $W^{(S)\mu\nu}(q, P, P_{\pi})$ survives the naive time-reversal transformation but $W^{(A)\mu\nu}(q, P, P_{\pi})$ does not. The occurrence of $W^{(A)\mu\nu}(q, P, P_{\pi})$ is completely due to the difference between the in-state and out-state, i.e., the final-state interactions. By turning off the final-state interactions, one can show from Eqs. (7), (8), and (11) that $W^{(S)\mu\nu}(q, P, P_{\pi})$ is symmetric with respect to indices $\mu$ and $\nu$. In principle, the final-state-interaction-caused contributions to $W^{\mu\nu}(q, P, P_{\pi})$ are asymmetric under the exchange $\mu \leftrightarrow \nu$. However, one can partition those symmetric contributions from the final-state interactions into $W^{(S)\mu\nu}(q, P, P_{\pi})$. Therefore, $W^{(A)\mu\nu}(q, P, P_{\pi})$ will be antisymmetric with respect to $\mu$ and $\nu$, or equivalently, odd under the naive parity-time-reversal transformation.

At this stage, we have clarified all the symmetry constraints of the hadronic tensor. It seems straightforward to write down its general Lorentz decomposition, in the complete basis constructed by the Lorentz vectors associated with the probe photon, target nucleon, and the inclusive pion, along with the metric tensor $g_{\mu\nu}$ and the completely antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. However, if we set about immediately this task, great risk is taken of overcounting or undercounting the number of structure functions as many terms can be constructed satisfying Eqs. (8), (7), and (11). Therefore, it is imperative to know in advance the number of the independent terms before setting about the general Lorentz expansion of hadronic tensor.

III. NUMBER OF INDEPENDENT STRUCTURE FUNCTIONS

For counting the number of independent structure functions, the Mueller theorem \[21\] supplies us with a very convenient method. Let us truncate the leptonic scattering part of the pion leptoproduction and consider equivalently the inclusive process $\gamma^*(q, \epsilon) + N(P) \rightarrow \pi(P_{\pi}) + X$, where $\epsilon^\nu$ is the polarization vector of the virtual photon $\gamma^*$. Obviously, its cross
section is proportional to $\epsilon^\mu \epsilon^\nu W_{\mu\nu}(q, P, P_\pi)$ and can be parameterized in terms of a set of independent structure functions. On the other hand, the Mueller theorem tells us that this cross section can be related to the helicity amplitudes for the forward three-body scattering $\gamma^* + N + \pi \rightarrow \gamma^* + N + \pi$. Therefore, the number of the structure functions is equal to that of the independent forward three-body scattering amplitudes. In such a helicity amplitude analysis, the unpolarized nucleon can be replaced by a spin-zero particle \[22\]. Then, the helicity amplitude for the above forward scattering process is characterized by $f_{\lambda_{\gamma^*}; \lambda'_{\gamma^*}}$, where $\lambda_{\gamma^*}$ and $\lambda'_{\gamma^*}$ are the helicities of the virtual photon before and after the scattering. Since the virtual photon has three helicity states, there are $3 \times 3 = 9$ helicity amplitudes for the forward three-body scattering considered. Obviously, not all of them are independent and they are subject to the following parity conservation constraints:

\[ f_{-\lambda_{\gamma^*}; -\lambda'_{\gamma^*}} = (-1)^{\lambda_{\gamma^*} - \lambda'_{\gamma^*}} f_{\lambda_{\gamma^*}; \lambda'_{\gamma^*}}. \]  

(14)

Hence there are only five independent helicity amplitudes. Correspondingly, there are five structure functions in the decomposition of the hadronic tensor. Although time reversal invariance does not yield any further constraints, we can still learn some useful information about the naive-parity-time-reversal properties of structure functions. If there were no final-state interactions, there would be the relation like $f_{\lambda_{\gamma^*}; \lambda'_{\gamma^*}} = f_{\lambda'_{\gamma^*}; \lambda_{\gamma^*}}$, which leads to one more restriction among the five independent helicity amplitudes. Therefore, we conclude that $W^{(S)\mu\nu}(q, P, P_\pi)$ and $W^{(A)\mu\nu}(q, P, P_\pi)$ contain four and one structure functions, respectively.

IV. SEVERAL LORENTZ DECOMPOSITIONS OF THE HADRONIC TENSOR

Nowadays, it is a common practice for the particle physics community to decompose the hadronic tensor into Lorentz invariant structure functions. On the basis of the discussion in the last two sections, one can easily construct the following most general Lorentz decomposition for our hadronic tensor:

\[ W^{\mu\nu}(q, P, P_\pi) = \frac{1}{P \cdot q} (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) w_1 + \frac{1}{q^2 (P \cdot q)} (P^\mu - \frac{P \cdot q}{q^2} q^\mu) (P^\nu - \frac{P \cdot q}{q^2} q^\nu) w_2 \]
where \( w_1, w_2, w_3, w_4 \) and \( \hat{w} \) are dimensionless structure functions, dependent on \( q^2, P \cdot q, \) and \( P \cdot P_\pi \). This decomposition is irrelevant of the frame in which one works. Here we note that the hadronic tensor for inclusive one-particle leptoproduction has an energy dimension two lower than its deep inelastic scattering counterpart.

In practice, one usually chooses a specific frame in which to work. If one lets the \( \hat{z} \)-axis be along the direction of the motion of the probe photon and puts the \( \hat{x} \)-axis in the lepton scattering plane, one can build another decomposition of the hadronic tensor. In this case, one can introduce an auxiliary four-momentum,

\[
P_{\pi \perp}^\mu = (0, |P_{\pi \perp}| \cos \phi, |P_{\pi \perp}| \sin \phi, 0),
\]

where \( P_{\pi \perp} \) is the pion transverse momentum with respect to the travelling direction of virtual photon and \( \phi \) is the azimuthal angle of the detected pion. Obviously, \( P_{\pi \perp}^\mu \) satisfies \( P_{\pi \perp} \cdot q = 0 \). By substituting \( P_{\pi \perp} \) for \( (P_\pi - \frac{P \cdot q}{q^2} q) \) in Eq. (13), one arrives at the following decomposition:

\[
W^{\mu \nu}(q, P, P_\pi) = \frac{1}{P \cdot q} (-g^{\mu \nu} + \frac{q^\mu q^\nu}{q^2}) W_1 + \frac{1}{q^2 (P \cdot q)} (P^\mu - \frac{P \cdot q}{q^2} q^\mu) (P^\nu - \frac{P \cdot q}{q^2} q^\nu) W_2
\]

\[
+ \frac{1}{q^2 (P \cdot q)} \left[ (P^\mu - \frac{P \cdot q}{q^2} q^\mu) P_{\pi \perp}^\nu + P_{\pi \perp}^\mu (P^\nu - \frac{P \cdot q}{q^2} q^\nu) \right] W_3 + \frac{1}{q^2 (P \cdot q)} P_{\pi \perp}^\mu P_{\pi \perp}^\nu W_4
\]

\[
+ \frac{i}{q^2 (P \cdot q)} \left[ (P^\mu - \frac{P \cdot q}{q^2} q^\mu) P_{\pi \perp}^\nu - P_{\pi \perp}^\mu (P^\nu - \frac{P \cdot q}{q^2} q^\nu) \right] \hat{W}
\]

(17)

where \( W_1, W_2, W_3, W_4 \) and \( \hat{W} \) are dimensionless structure functions, dependent on \( q^2, P \cdot q, \) and \( P_{\pi \perp}^2 \). The advantage of this decomposition is that the dependence of cross section on the transverse momentum of the detected pion can be easily worked out in analytical calculations. Nevertheless, one has to be aware that this decomposition is frame-dependent.
The QCD factorization results in Ref. [9] can be tailored into the above decomposition, Eq. (17). In the literature, it is Mulders [4] who first worked out the terms associated with $W_1$, $W_2$, $W_3$, $W_4$. Indeed, the $\hat{W}$ term, because of its antisymmetric property, does not make contributions to cross section when the incident lepton beam is unpolarized, which is the very case discussed in Ref. [4,10]. As has been explained in Sects. I and II, however, the term associated with $\hat{W}$ incorporates the final-state interactions in the inclusively detected state, so its existence does not depend upon whether the initial-state beam is polarized or not. Hence, we claim that it was inappropriate in Refs. [4,10] to ignore the final-state effects without precautions.

Since the hadronic tensor is a $4 \times 4$ matrix in the Lorentz space, one can also parameterize it in terms of its specific matrix elements. Such an analysis has already been done by Kotzinian [11], who discussed the more complicated case with a polarized nucleon target. However, the Lorentz invariance of the structure functions in such parameterizations is not manifest. For comparison, we note that five spin-independent structure functions, under distinct disguises, were also identified in Ref. [11]. Among them, the imaginary part of a matrix element, $\text{Im}H_{01}^{(0)}$, corresponds to $\hat{w}$ in (15) and $\hat{W}$ in (17).

In fact, one can also construct other Lorentz decompositions. Because adopting different conventions, different authors usually have different decompositions. However, the number of independent structure functions should always be fixed because it is a reflection of the symmetries of the hadronic tensor. In principle, one can establish the connections among his own structure functions, on the one hand, and those by other authors, on the other hand.

V. POSITIVITY CONSTRAINTS TO THE HADRONIC TENSOR

In the rest of this paper, we work in the target rest frame, with the direction of the motion of the photon probe in the $\hat{z}$-axis and the lepton scattering plane in the $\hat{x} - \hat{z}$ plane. Correspondingly, we adopt Eq. (17) as our decomposition of the hadronic tensor.

The starting point for our positivity analysis is the Hermiticity of the electromagnetic
current, $J^\mu = J^\mu$. For an arbitrary Lorentz vector $a^\mu$, one can show

$$W_{\mu\nu}(q, P, P_\pi) a^\mu a^\nu \propto \sum_X \delta^4(q - P - P_\pi - P_X)|\langle \pi(P_\pi), X|a \cdot J|P\rangle|^2,$$

(18)

so $W^{\mu\nu}(q, P, P_\pi)$ is a semi-positive definite form:

$$W_{\mu\nu}(q, P, P_\pi) a^\mu a^\nu \geq 0.$$  

(19)

As a consequence, the relevant structure functions are constrained by some positivity conditions.

For a generic Lorentz vector, one can always expand it over a complete set of bases constructed by four other independent vectors. Of course, one can choose one vector of the bases to be proportional to the momentum of the virtual photon and the other three as the three polarization vectors of the probe photon:

$$e_1^\mu = -\frac{1}{\sqrt{2}}(0, 1, +i, 0),$$

(20)

$$e_2^\mu = +\frac{1}{\sqrt{2}}(0, 1, -i, 0),$$

(21)

$$e_3^\mu = \frac{1}{Q}(\nu, q^2 + Q^2, 0, 0, 0).$$

(22)

Notice that these three polarization vectors are orthonormal, namely,

$$e_1^* \cdot e_1 = e_2^* \cdot e_2 = -e_3^* \cdot e_3 = -1,$$

(23)

$$e_i^* \cdot e_j = 0, \text{ with } i \neq j.$$  

(24)

In addition, they satisfy the Lorentz condition

$$e_i \cdot q = 0, \text{ } i = 1, 2, 3.$$  

(25)

Obviously, by letting $a^\mu = q^\mu$ one can gain only an identity $0 \equiv 0$, which reflects the current conservation of the electromagnetic interaction. Taking either $a^\mu = e_1^\mu$ or $a^\mu = e_2^\mu$, however, one can obtain the following restrictions:
\[ M\nu e_1^\mu e_1^\nu W_{\mu\nu} = M\nu e_2^\mu e_2^\nu W_{\mu\nu} = W_1 - \frac{|P_{\pi\perp}|^2}{2Q^2}W_4 \geq 0, \]

(26)

On the other hand, one has with \(a^\mu = e_3^\mu\)

\[ M\nu e_3^\mu e_3^\nu W_{\mu\nu} = -W_1 - \frac{M^2(\nu^2 + Q^2)}{Q^4}W_2 \geq 0. \]

(27)

However, both (26) and (27) are only the direct consequences of the positivity of \(W^{\mu\nu}(q, P, P_\pi)\). In other words, they are only necessary conditions.

As a matter of fact, the sufficient and necessary conditions for the positivity of a matrix are that all of its submatrices have semi-positively finite determinants [23]. Note that the hadronic tensor considered is a matrix in the Lorentz space,

\[ W^{\mu\nu}(q, P, P_\pi) = \begin{pmatrix}
W^{00} & W^{01} & W^{02} & W^{03} \\
W^{10} & W^{11} & W^{12} & W^{13} \\
W^{20} & W^{21} & W^{22} & W^{23} \\
W^{30} & W^{31} & W^{32} & W^{33}
\end{pmatrix}. \]

(28)

In order to investigate the positivity restrictions on the pion leptoproduction, we write out explicitly the elements of \(W^{\mu\nu}(q, P, P_\pi)\) in our coordinate system:

\[ W^{00} = -\frac{\nu^2 + Q^2}{M\nu Q^2}W_1 - \frac{M(\nu^2 + Q^2)}{\nu Q^6}W_2, \]

(29)

\[ W^{11} = \frac{W_1}{M\nu} - \frac{|P_{\pi\perp}|^2}{M\nu Q^2} \cos^2 \phi W_4, \]

(30)

\[ W^{22} = \frac{W_1}{M\nu} - \frac{|P_{\pi\perp}|^2}{M\nu Q^2} \sin^2 \phi W_4, \]

(31)

\[ W^{33} = -\frac{\nu}{MQ^2}W_1 - \frac{M\nu(\nu^2 + Q^2)}{Q^6}W_2, \]

(32)

\[ W^{01} = W^{*10} = -\frac{(\nu^2 + Q^2)|P_{\pi\perp}|}{\nu Q^4} \cos \phi (W_3 + i\tilde{W}), \]

(33)

\[ W^{02} = W^{*20} = -\frac{(\nu^2 + Q^2)|P_{\pi\perp}|}{\nu Q^4} \sin \phi (W_3 + i\tilde{W}), \]

(34)

\[ W^{03} = W^{30} = -\frac{\sqrt{\nu^2 + Q^2}}{MQ^2}W_1 - \frac{M\sqrt{(\nu^2 + Q^2)^3}}{Q^6}W_2, \]

(35)

\[ W^{12} = W^{21} = -\frac{|P_{\pi\perp}|^2}{M\nu Q^2} \cos \phi \sin \phi W_4, \]

(36)
\[ W^{31} = W^{*13} = -\frac{\sqrt{\nu^2 + Q^2} |\mathbf{P}_{\pi\perp}|}{Q^4} \cos \phi(W_3 - i\hat{W}), \quad (37) \]
\[ W^{32} = W^{*23} = -\frac{\sqrt{\nu^2 + Q^2} |\mathbf{P}_{\pi\perp}|}{Q^4} \sin \phi(W_3 - i\hat{W}). \quad (38) \]

Now we are in the position to examine the necessary and sufficient positivity conditions for our hadronic tensor.

First, the determinant of \( W^{\mu\nu}(q, P, P_\pi) \) itself must be semi-definitely positive. However, our explicit calculation shows that \( \text{Det}[W^{\mu\nu}(q, P, P_\pi)] = 0 \). This occurs by no means accidentally for it reflects the electromagnetic gauge invariance of the hadronic tensor. Because \( q_\mu W^{\mu\nu}(q, P, P_\pi) = q_\nu W^{\mu\nu}(q, P, P_\pi) = 0 \), \( W^{\mu\nu}(q, P, P_\pi) \) is at most at rank three. Correspondingly, the determinant of \( W^{\mu\nu}(q, P, P_\pi) \) vanishes identically.

Second, two \( 3 \times 3 \) submatrices of \( W^{\mu\nu}(q, P, P_\pi) \) must be semi-definitely positive, i.e.,
\[
\begin{vmatrix}
W^{00} & W^{01} & W^{02} \\
W^{10} & W^{11} & W^{12} \\
W^{20} & W^{21} & W^{22}
\end{vmatrix} \geq 0, \quad (39)
\]
\[
\begin{vmatrix}
W^{11} & W^{12} & W^{13} \\
W^{21} & W^{22} & W^{23} \\
W^{31} & W^{32} & W^{33}
\end{vmatrix} \geq 0. \quad (40)
\]

Herewith we obtain a restriction among five structure functions
\[
-W_1 \left[ W_1 + \frac{M^2(\nu^2 + Q^2)}{Q^4} W_2 \right] \left[ W_1 - \frac{|\mathbf{P}_{\pi\perp}|^2}{Q^2} W_4 \right] - \frac{M^2(\nu^2 + Q^2)|\mathbf{P}_{\pi\perp}|^2}{Q^6} W_1 (W_3^2 + \hat{W}^2) \geq 0. \quad (41)
\]

Thirdly, the determinants of three \( 2 \times 2 \) submatrices are semi-definitely positive, i.e.,
\[
\begin{vmatrix}
W^{00} & W^{01} \\
W^{10} & W^{11}
\end{vmatrix} \geq 0, \quad (42)
\]
\[
\begin{vmatrix}
W^{11} & W^{12} \\
W^{21} & W^{22}
\end{vmatrix} \geq 0. \quad (43)
\]
In our parameterization, these three inequalities assume the following forms:

\[
-W_1 \left[ W_1 + \frac{M^2(v^2 + Q^2)}{Q^4} W_2 \right] \left[ W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} \cos^2 \phi W_4 \right] - \frac{M^2(v^2 + Q^2)}{Q^6} |P_{\pi\perp}|^2 \cos^2 \phi W_1 (W_3^2 + \hat{W}^2) \geq 0, \quad (45)
\]

\[
W_1 \left( W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} W_4 \right) \geq 0, \quad (46)
\]

\[
-W_1 \left[ W_1 + \frac{M^2(v^2 + Q^2)}{Q^4} W_2 \right] \left[ W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} \sin^2 \phi W_4 \right] - \frac{M^2(v^2 + Q^2)}{Q^6} |P_{\pi\perp}|^2 \sin^2 \phi W_1 (W_3^2 + \hat{W}^2) \geq 0. \quad (47)
\]

Simply letting \( \cos \phi = 1 \) in (45) or \( \sin \phi = 1 \) in (47), two corresponding inequalities reduce to (41). However, combining (45) with (47) will give rise to

\[
-W_1 \left[ W_1 + \frac{M^2(v^2 + Q^2)}{Q^4} W_2 \right] \left[ W_1 - \frac{|P_{\pi\perp}|^2}{2Q^2} W_4 \right] - \frac{M^2(v^2 + Q^2)|P_{\pi\perp}|^2}{2Q^6} W_1 (W_3^2 + \hat{W}^2) \geq 0. \quad (48)
\]

If taking \( \cos \phi = 0 \) in (45) or \( \sin \phi = 0 \) in (47), one will regain (27).

Last, each of the diagonal elements of \( W^{\mu\nu}(q, P, P_\pi) \) has to be semi-definitely positive, i.e.,

\[
W^{00} \geq 0, \; W^{11} \geq 0, \; W^{22} \geq 0, \; W^{33} \geq 0. \quad (49)
\]

Accordingly, we obtain (27) as well as the following two inequalities:

\[
W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} \cos^2 \phi W_4 \geq 0, \quad (50)
\]

\[
W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} \sin^2 \phi W_4 \geq 0. \quad (51)
\]

Either by letting \( \cos \phi = 0 \) in (50) or \( \sin \phi = 0 \) in (51), one has
At this stage, we observe that (41) through (48) can be divided safely by \( W_1 \) without changing the direction of the inequality sign. Furthermore, (27) in combination with (52) implies that

\[
W_2 \leq 0.
\] (53)

As one sets \( \cos \phi = 1 \) in (50) or \( \sin \phi = 1 \) in (51), it will lead to

\[
W_1 - \frac{|P_\pi \perp|^2}{Q^2} W_4 \geq 0.
\] (54)

When adding (50) to (51), however, one recovers (26).

VI. UPPER LIMITS OF THE LEVELT-MULDERS ASYMMETRY

Now we discuss the phenomenological implications of the above positivity constraints. In principle, structure functions \( W_i \) (\( i = 1, \cdots, 4 \)) can be measured with an unpolarized lepton beam while the measurement of \( \hat{W} \) requires the polarization of the incident beam. As Levelt and Mulders have clarified [10], the determination of \( \hat{W} \) can be done by measuring a \( \langle \sin \phi \rangle \) asymmetry of the considered process in the case that the lepton beam is polarized longitudinally.

Substituting Eqs. (4) and (17) into (1) and completing the Lorentz contractions, one has

\[
\frac{d\sigma(s_l)}{dxdydzd|P_\pi \perp|^2d\phi} = \frac{\alpha^2 y}{32M\pi^2 xQ^2P_\pi ||} \left[ 4xW_1 - \frac{2\kappa^2}{xy^2} W_2 + \frac{4\kappa(2-y)}{yy^2} \frac{|P_\pi \perp|}{Q} \cos \phi W_3 
- 2x \left( 1 + \frac{4\kappa^2}{\eta^2 y^2} \cos^2 \phi \right) \left( \frac{|P_\pi \perp|}{Q} \right)^2 W_4 - 8xM \tilde{M}_l \frac{s_l \cdot q \times P_\pi \perp \hat{W}}{Q^4} \right],
\] (55)

where

\[
\kappa = \sqrt{1 - y - \frac{M^2 x^2 y^2}{Q^2}}, \quad \eta = \sqrt{1 + \frac{4M^2 x^2}{Q^2}}.
\] (56)

Now we define a single spin asymmetry as
\[
A = \frac{d\sigma(s_l)}{dx dy dz |P_{\pi\perp}|^2 d\phi} \frac{d\sigma(-s_l)}{dx dy dz |P_{\pi\perp}|^2 d\phi} + \frac{d\sigma(s_l)}{dx dy dz |P_{\pi\perp}|^2 d\phi} \frac{d\sigma(-s_l)}{dx dy dz |P_{\pi\perp}|^2 d\phi}. \quad (57)
\]

Substituting Eq. (55) into (57), one has

\[
A = -8xMMls_l \cdot q \times P_{\pi\perp} \hat{W}.
\]

In the case of longitudinal polarization, the spin four-vector of the beam lepton is related to its momentum via

\[
\lim_{M_l \to 0} M_l s_l^\mu = 2\lambda_l k^\mu, \quad (59)
\]

with \(\lambda_l\) being the lepton helicity. Correspondingly, there is the following Leveilt-Mulders asymmetry

\[
A_L \equiv \frac{d\sigma(\lambda_l = +\frac{1}{2})}{dx dy dz |P_{\pi\perp}|^2 d\phi} \frac{d\sigma(\lambda_l = -\frac{1}{2})}{dx dy dz |P_{\pi\perp}|^2 d\phi} + \frac{d\sigma(\lambda_l = +\frac{1}{2})}{dx dy dz |P_{\pi\perp}|^2 d\phi} \frac{d\sigma(\lambda_l = -\frac{1}{2})}{dx dy dz |P_{\pi\perp}|^2 d\phi}. \quad (60)
\]

After a little algebra, we have

\[
A_L = \frac{-4\kappa |P_{\pi\perp}|}{y Q} \sin \phi \hat{W}.
\]

Because \(A_L\) is essentially proportional to \(s_l \cdot q \times P_{\pi\perp}\), it has the largest values at \(\sin \phi = 1\), i.e., when the pion momentum has no transverse components in the lepton scattering plane.

Now we bound \(A_L\) by use of the results derived from the positivity analysis. From (11) and (52), we have by letting \(W_3 = 0\)

\[
|P_{\pi\perp}| \hat{W} \leq \sqrt{\left[ \frac{4x^2 Q^2}{Q^2 + 4x^2 M^2} W_1 + W_2 \right] \left[ W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} W_4 \right]}.
\]

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Correspondingly, there will be

\[
|A_L| \leq \frac{4\kappa}{y} \sqrt{- \left[ \frac{4x^2Q^2}{Q^2 + 4x^2M^2} W_1 + W_2 \right] \left[ W_1 - \frac{|P_{\pi\perp}|^2}{Q^2} W_4 \right] \sin \phi}
\]

\[
\left| 4x W_1 - \frac{2\kappa^2}{xy^2} W_2 + \frac{4\kappa(2 - y)}{\eta y^2} \frac{|P_{\pi\perp}|}{Q} \cos \phi W_3 - 2x \left( 1 + \frac{4\kappa^2}{\eta^2 y^2} \cos^2 \phi \right) \left( \frac{|P_{\pi\perp}|}{Q} \right)^2 W_4 \right|.
\]

\[(63)\]

As a rough approximation, we may neglect all the \(Q\)-power suppressed effects as compared to \(O(Q^0)\) quantities. More concretely, we drop out all \(W_3\)- and \(W_4\)-terms and substitute \(\sqrt{1 - y}\) for \(\kappa\) in (63). As a result,

\[
|A_L| \leq \frac{2xy \sqrt{1 - y} \left[ W_1 - \frac{4x^2Q^2}{Q^2 + 4x^2M^2} W_1 + W_2 \right] \sin \phi}{|2x^2y^2 W_1 - (1 - y)W_2|}. \tag{64}
\]

At this stage, we employ the Callan-Gross-type relation between \(W_1\) and \(W_2\)

\[
W_1 + \frac{1}{4x^2} W_2 = 0 \tag{65}
\]

to simplify further the right-hand side of (64). The verification of Eq. (65) can be very easily done in the naive quark-parton model without intrinsic transverse parton momentum. Actually, it can also be obtained simply from the fact that the virtual photon probe tends to be transversely polarized in the high energy limit. Put it in another way,

\[
e_3^{\mu} e_3^{\nu} W_{\mu\nu}(q, P, P_\pi) \rightarrow 0 \text{ as } Q \rightarrow \infty \text{ with } x \text{ fixed.} \tag{66}
\]

Inserting Eq. (65) into (64), we arrive at the following upper limit for the considered asymmetry

\[
|A_L| \leq \frac{4xy \sqrt{1 - y} \frac{M}{Q} \sin \phi}{(y - 1)^2 + 1} \tag{67}
\]

Since both \(W_3\) and \(\hat{W}\) contribute at one-power suppressed level, i.e., at twist three, it should be stressed that (65) is an amplified upper bound for \(A_L\). The reason is that in deriving (62) from (11) and (52), we have assumed \(W_3 = 0\). Because experiments on spin asymmetries
are usually subject to large statical errors, this positivity constraint can be taken as a very useful guide to judge the reliability of the experimental results.

On the other hand, (67) simply informs us that the chance to measure the Levelt-Mulders asymmetry is very faint in some kinematical domain. To be illustrative, we draw in Fig. 1 the derived upper limit of the Levelt-Mulders asymmetry $A_L$ versus the fraction of the lepton energy loss $y$, with $x = 0.5$, $\sin \phi = 1$ and $Q = 5M$ at which the perturbative QCD can be applicable so that the Callan-Gross relation is reliable.

**VII. CONCLUDING REMARKS**

Obviously, our discussion can be generalized to the case of the lepton beam being transversely polarized. However, Eq. (59) will no longer hold and the corresponding spin asymmetry will be $M_t/Q$-suppressed. We do not carry out such an extension for it will not supply us with any practical experimental guidance. In principle, our discussion can also be generalized to the case in which the nucleon is polarized and even to the case the semi-inclusively detected hadron is a baryon, to say, a $\Lambda$ hyperon with its spin state monitored. However, such a generalization will be of less relevance because a large number of structure functions will be involved in decomposing the corresponding hadronic tensors.

In summary, we have examined systematically the symmetry properties of the semi-inclusive one-pion production induced by a charged lepton beam on an unpolarized nucleon target, with an emphasis on the positivity constraints to the structure functions. We found that due to the positivity of the hadronic tensor, the signs of two twist-two structure functions $W_1$ and $W_2$ can be determined. Moreover, there exists an inequality restricting five structure functions. This restriction, in connection to the Callan-Gross relation between $W_1$ and $W_2$, yields an upper bound on the Levelt-Mulders asymmetry.

*Acknowledgement* One of the authors (W.L) would like to thank A. Kotzinian, P. J. Mulders, and J. Soffer for useful correspondence.
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Figure Caption

Figure 1. The upper bound of the Levelt-Mulders asymmetry ($A_L$) versus the fraction of the lepton energy loss ($y$). The Bjorken $x$ is taken to be 0.5, the pion azimuthal angle ($\phi$) to be 90° with respect to the lepton scattering plane, and the momentum transfer ($Q$) to be five times the nucleon mass ($M$).